Path Integrals

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1 First-Order Term

By inserting the resolution of the identity

$$\mathbf{1} = \int \frac{dp}{2\pi\hbar} |p\rangle\langle p| \tag{1}$$

and using

$$\langle q|p\rangle = e^{ipq/\hbar} \tag{2}$$

show that the first-order term can be expressed as

$$-i\frac{\Delta t}{\hbar}\langle q_{j+1}|H|q_j\rangle = -i\frac{\Delta t}{\hbar}\int \frac{dp_j}{2\pi\hbar} e^{ip_j(q_{j+1}-q_j)/\hbar}\left(\frac{p_j^2}{2m} + V(q_j)\right). \tag{3}$$

What changes at second order?

2 Statistical Physics and Imaginary Time

Show that the partition function

$$Z[\beta] = \sum_j e^{-\beta E_j} \tag{4}$$

can be expressed as a sum over imaginary time propagators

$$Z[\beta] = \int dq K_E(q, \beta\hbar, q, 0). \tag{5}$$

3 Path Integral Derivation of Propagator

In the path integral formalism the propagator in imaginary time is

$$K_E(q, \beta\hbar, q, 0) = \lim_{N\to\infty} \left(\frac{m}{2\pi\Delta\tau\hbar}\right)^{N/2} \int dq_1 dq_2 \cdots dq_{N-1} \exp\left[-\Delta\tau\hbar \sum_{j=1}^{N} \frac{m}{2}\left(\frac{q_j - q_{j-1}}{\Delta\tau\hbar}\right)^2\right] . \tag{6}$$

(a) Show that the propagator can be rewritten as

$$K_E(q, \beta\hbar, q, 0) = \lim_{N\to\infty} \left(\frac{m}{2\pi\Delta\tau\hbar}\right)^{N/2} \left(\frac{2\Delta\tau\hbar}{m}\right)^{\frac{N-1}{2}} \int dy_1 dy_2 \cdots dy_{N-1} \exp\left[-\sum_{j=1}^{N} (y_j - y_{j-1})^2\right] . \tag{7}$$

(b) Use induction to show that

$$\int dy_1 dy_2 \cdots dy_{N-1} \exp\left[-\sum_{j=1}^{N} (y_j - y_{j-1})^2\right] = \left(\frac{\pi^{N-1}}{N}\right)^{1/2} e^{-(y_N - y_0)^2/N} . \tag{8}$$

(c) Take the continuum limit $N \to \infty$ to recover the result from lecture

$$K_E(q, \beta\hbar, q, 0) = \sqrt{\frac{m}{2\pi\beta}} \exp\left(-\frac{m(q_f - q_i)^2}{2\beta}\right) . \tag{9}$$