Problem Set 8

Due: Friday 5pm, April 8th, via Canvas upload or in envelope outside 26-255
TA: Evgenii Kniazev, knyazev@mit.edu

1 Optical Traps and Scattering

The AC Stark effect has been used to trap neutral atoms at the focus of one or more laser beams. This obviates the necessity of magnetic traps, which can be bulky and power guzzling. Scientifically more important is the fact that many interesting phenomena occur between states that are not necessarily all magnetically trappable. This has led to studies of, for example, spin domains in spinor Bose-Einstein condensates (Sadler *et al.*, *Nature* **443**, 312 (2006)), among many other examples. In the following exercise, you will explore the proper laser power and wavelength needed to trap an ultracold atomic gas.

Let us consider an alkali atom with resonance frequency ω_0 on the principal $nS \to nP$ transition. A sample of atoms in the ground state nS are exposed to monochromatic radiation of intensity I and frequency $\omega_L < \omega_0$ (red-detuned from resonance). Use the fact that essentially all of the oscillator strength out of the ground state comes from the $nS \to nP$ transition.

a) AC Stark Shift

i. Calculate the AC Stark shift U_i using time-dependent perturbation theory for the dynamic polarizability $\alpha(\omega_L)$.

We derived in class a perturbative expression for the wavefunction and thus the polarizability of an atom in an oscillating electric field. The result we obtained was straightforward except perhaps for the neglect of a "-1" term. This term was dropped because we chose to neglect transient effects, that is we sought to describe the atom after a long exposure to monochromatic light. Another way to view our neglect of transient terms is to consider that rather than suddenly switching on the the light field, we had ramped it on "adiabatically," meaning that the atom would be oscillating in a steady state manner. Other than an overall phase factor, this would yield exactly the result that we had obtained in class, namely that if we consider an electric field $\epsilon(t) = \epsilon\hat{\mathbf{z}}\cos(\omega t)$ which couples the ground state $|g\rangle$ to a single excited state $|e\rangle$, and represent our wavefunction in the form

$$|\psi(t)\rangle = a_g|g\rangle + a_e e^{-i\omega_0 t}|e\rangle \tag{1}$$

then

$$a_e(t) = \frac{\epsilon}{2\hbar}\langle e|D_z|g\rangle \left(\frac{e^{i(\omega_0-\omega)t}}{\omega_0-\omega} + \frac{e^{i(\omega_0+\omega)t}}{\omega_0+\omega}\right) \tag{2}$$

The atomic dipole moment of the atom is given by

$$\mathbf{d}(t) = \frac{1}{\hbar}|\langle e|D_z|g\rangle|^2\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right)\epsilon(t) \tag{3}$$

$$= \alpha(\omega)\,\epsilon(t). \tag{4}$$

From the fact that the oscillator strength is unity we obtain

$$|\langle e|D_z|g\rangle|^2 = \frac{\hbar e^2}{2m\omega_0} \tag{5}$$

and thus,

$$\alpha(\omega) = \frac{e^2}{m}\frac{1}{\omega_0^2-\omega^2} \tag{6}$$

The AC Stark shift is then given by

$$U_{AC}(i) = -\frac{1}{2}\alpha(\omega)\epsilon^2\overline{\cos^2\omega t} \tag{7}$$

$$= -\frac{1}{4}\frac{e^2}{m}\frac{1}{\omega_0^2-\omega^2}\,\epsilon^2 \tag{8}$$

$$\boxed{U_{AC}(i) = -\frac{\hbar\omega_R^2}{4}\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right) = -\frac{2\pi e^2 I}{mc(\omega_0^2-\omega^2)}} \tag{9}$$

where the intensity I is given by $I = c\epsilon^2/8\pi$.

ii. Now calculate the AC Stark shift U_{ii} using the rotating wave approximation *i.e.* $|\omega_L-\omega_0| \ll \omega_L+\omega_0$, and approximated for small field strengths *i.e.* Rabi frequency much less than the detuning.

The AC Stark shift can also be obtained by nonperturbatively solving the time-dependent Schrödinger equation for the time-dependent eigenstates of the atom and field combined. This approach is suited for describing near-resonant driving of a two-level atom and is known as the dressed atom picture. In this approach, we make the rotating wave approximation. This requires us to solve the following Hamiltonian (see Homework 1, problem 2):

$$H = \frac{\hbar}{2}\begin{pmatrix} -\omega_0 & \omega_R e^{i\omega t} \\ \omega_R e^{i\omega t} & \omega_0 \end{pmatrix} \tag{10}$$

where ω_0 is the frequency separation between the S and P states , ω_R is the Rabi frequency, and ω is the radiation frequency, as usual. The time-dependent eigenstates are

$$|+\rangle = e^{-i\left(\omega'_R-\omega\right)t/2}\left[\cos\theta\,|S\rangle + \sin\theta e^{-i\omega t}\,|P\rangle\right] \tag{11}$$

$$|-\rangle = e^{-i\left(-\omega'_R+\omega\right)t/2}\left[-\sin\theta e^{i\omega t}\,|S\rangle + \cos\theta\,|P\rangle\right] \tag{12}$$

up to a shared, global (unimportant) phase. We again use the conventions $\delta = \omega - \omega_0$, $\omega'_R = \sqrt{\omega_R^2+\delta^2}$, $\sin\theta = \sqrt{\frac{\omega'_R+\delta}{2\omega'_R}}$, and $\cos\theta = \sqrt{\frac{\omega'_R-\delta}{2\omega'_R}}$.
Note, for $\delta < 0$ (red-detuning, as we are using here), in the weak field limit ($\omega_R = 0$), we identify the states

$$|+\rangle \to |P\rangle \tag{13}$$
$$|-\rangle \to |S\rangle \tag{14}$$

Therefore, if we again neglect transients, then the effect of slowly turning on the light is to place our once ground-state atom into the dressed state $|-\rangle$. The time-dependent phase factors in the equations above can be identified as energies. We obtain the AC Stark shift by considering the difference in energy with the light on and off, i.e.

$$U_{AC}(ii) = \hbar\left(\left.\frac{-\omega'_R}{2}\right|_{\omega_R} - \left.\frac{-\omega'_R}{2}\right|_{\omega_R=0}\right) \tag{15}$$

$$= -\hbar\frac{\sqrt{\omega_R^2+\delta^2}-|\delta|}{2} \tag{16}$$

$$= -\frac{\hbar\omega_R^2}{4|\delta|} \tag{17}$$

Using the definition of the oscillator strength and the identification

$$\frac{\hbar\omega_R}{2} = \frac{\epsilon\cdot\mathbf{D}}{2} \tag{18}$$

we obtain

$$\boxed{U_{AC}(ii) = -\frac{e^2}{8m\omega_0}\left(\frac{1}{\omega_0-\omega}\right)\epsilon^2 = -\frac{\hbar\omega_R^2}{4}\frac{1}{(\omega_0-\omega)} = -\frac{\pi e^2 I}{m\omega_0(\omega_0-\omega)}.} \tag{19}$$

We can cast these results in more useful quantities by introducing the saturation intensity I_{SAT} which for our ideal two-level atom is

$$I_{SAT} = \frac{m\omega_0\Gamma^2}{8\pi\alpha}. \tag{20}$$

$\alpha = \frac{e^2}{\hbar c}$ is the fine structure constant. This gives

$$U_{AC}(i) = -\frac{1}{4}\frac{\omega_0\Gamma}{\omega_0^2-\omega^2}\frac{I}{I_{SAT}}\hbar\Gamma. \tag{21}$$

and for small detunings, we get the dressed atom result, namely

$$U_{AC}(ii) = -\frac{1}{4}\frac{I}{I_{SAT}}\frac{1}{2|\delta|/\Gamma}\hbar\Gamma. \tag{22}$$

iii. Calculate the ratio U_i/U_{ii} as a function of ω_L. What is the ratio when $\omega_L \approx \omega_0$ and $\omega_L \approx 0$?
The ratio of results *(i)* and *(ii)* is

$$\boxed{\frac{\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}}{\frac{1}{\omega_0-\omega}} = 1+\frac{\omega_0-\omega}{\omega_0+\omega}} \tag{23}$$

On resonance, the two expressions are identical. At DC ($\omega = 0$), the perturbative (correct) result is twice the dressed atom result, since then the counter- and co-rotating terms give equal Stark shifts.

So we see that if the intensity has spatial structure (which we have not considered so far), with the appropriate detuning, the AC Stark shift can have energy minima where the atoms can be trapped.

b) What is the probability of finding the atom in the excited state P_e using both pictures *i* and *ii*? Neglect rapidly oscillating terms and time-average the remaining time-dependent terms.

Using the above results, one finds the likelihood of finding the atom in the excited state in the perturbative approach is

$$\begin{aligned} P_e &= |a_e^2| && (24)\\ &= \frac{\epsilon^2}{4\hbar^2}|\langle e|D_z|g\rangle|^2\left(\frac{1}{(\omega_0-\omega)^2}+\frac{1}{(\omega_0+\omega)^2}+\cdots\right) && (25)\\ &= \frac{\omega_R^2}{4}\left(\frac{1}{(\omega_0-\omega)^2}+\frac{1}{(\omega_0+\omega)^2}\right) && (26)\end{aligned}$$

where the $\cdots$ refer to rapidly oscillating terms that we neglect. From the definition of the oscillator strength, we thus obtain

$$\boxed{P_e(i) = \frac{\epsilon^2 e^2}{8m\hbar\omega_0}\left(\frac{1}{(\omega_0-\omega)^2}+\frac{1}{(\omega_0+\omega)^2}\right) = \frac{\pi\alpha I}{m\omega_0}\left(\frac{1}{(\omega_0-\omega)^2}+\frac{1}{(\omega_0+\omega)^2}\right).} \tag{27}$$

For the dressed atom result, we obtain

$$P_e(ii) = \cos^2\theta \tag{28}$$
$$= \frac{1+\cos 2\theta}{2} \tag{29}$$
$$= \frac{\omega_R' - |\delta|}{2\omega_R'} \tag{30}$$
$$\simeq \frac{\omega_R^2}{4\delta^2} \tag{31}$$

$$\boxed{P_e(ii) = \frac{\pi\alpha I}{m\omega_0}\frac{1}{(\omega_0-\omega)^2} = \frac{\omega_R^2}{4(\omega-\omega_0)^2}} \tag{32}$$

Equivalently, expressed in terms of the saturation intensity, we find

$$P_e = \frac{1}{2}\frac{I}{I_{SAT}}\frac{1}{(2|\delta|/\Gamma)^2}. \tag{33}$$

c) In this part you will calculate the photon scattering rate. Knowing the scattering rate for a particular atom and particular laser is important because it causes heating and may severely limit the holding time for an ultracold cloud. Here you are given a classical model to estimate the scattering rate, in which an atom can be treated as a dipole moment induced by the external light field, and it is oscillating at the same frequency. We only need to consider coherent scattering here, which means the scattered photons have the same frequency as the incident laser light. The scattered light can be regarded as radiation from the oscillating dipole. Scattering rate is defined as the total number of scattering events per second, and one scattering event corresponds to scattering one photon of $\hbar\omega$ in this case. Use the fact that, from classical electromagnetism, the power P radiated by an oscillating dipole of dipole moment $\mathbf{d}(t) = \mathbf{d}\cos\omega t$ is $P = ck^4|\mathbf{d}|^2/3$.

i. What is the photon scattering rate using *i* and *ii* as a function of ω_L?
Rayleigh scattering can be calculated several ways. Let us use results of classical electromagnetism (for a fully quantum-mechanical derivation, look up Exercize 3 in Cohen-Tannoudji's *Atom-Photon Interactions*). The power emitted by an oscillating dipole of moment $\mathbf{d}(t) = \mathbf{d}\cos\omega t$ is [Jackson, Chapter 9.2]

$$P = \frac{ck^4}{3}|\mathbf{d}|^2. \tag{34}$$

Since each photon carries $\hbar\omega$ of energy, the photon scattering rate is

$$\Gamma_{scat} = \frac{P}{\hbar\omega} = \frac{\omega^3}{3\hbar c^3}|\mathbf{d}|^2. \tag{35}$$

In part a) we calculated the dipole moments of the atoms in a monochromatic wave. So we obtain

$$\boxed{\Gamma_{scat}(i) = \frac{\omega^3}{3\hbar c^3} \times \left(\frac{e^2}{m}\frac{1}{\omega_0^2-\omega^2}\right)^2 \frac{8\pi I}{c}.} \tag{36}$$

We can also recast this equation into the useful form

$$\Gamma_{scat} = \frac{|U_{AC}|}{\hbar} \times \frac{4}{3}\alpha^3 \frac{\omega}{\omega_A}\frac{\omega^2}{\omega_0^2-\omega^2} \tag{37}$$

We obtain the dressed atom result with the substitution

$$\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right) = \frac{2\omega_0}{\omega_0^2-\omega^2} \to \frac{1}{\omega_0-\omega} \tag{38}$$

so

$$\boxed{\Gamma_{scat}(ii) = \frac{\omega^3}{3\hbar c^3} \times \left(\frac{e^2}{m}\frac{1}{2\omega_0}\frac{1}{\omega_0-\omega}\right)^2 \frac{8\pi I}{c}.} \tag{39}$$

As one can see, when $|\omega-\omega_0| \ll \omega_0$, the scattered power is $\hbar\omega\Gamma_{scat}$ proportional to ω^4 – the blue-sky formula.

ii. Express the scattering rate for the case *ii* in terms of P_e that you found in part *b* and the natural scattering rate Γ and other factors. The natural scattering rate is this case is the spontaneous emission rate, and it is

$$\Gamma = \frac{4}{3}\frac{e^2\omega^3}{\hbar c^3}|\langle b|\mathbf{r}|a\rangle|^2$$

You can use the fact that for alkali atoms, all of the oscillator strength of the ground state comes from nS to nP transition. The significance of this will become clear if you think about the next optional part.

We can relate this to the results of part b) by

$$\Gamma_{scat}(ii) = \left(\frac{\pi e^2 I}{\hbar c m\omega_0\delta^2}\right) \times \left(\frac{4}{3}\frac{\omega_0^3}{c^3\hbar}\frac{\hbar e^2}{2m\omega_0}\right) \times \left(\frac{\omega}{\omega_0}\right)^3 \tag{40}$$

$$= P_e \times \Gamma \times \left(\frac{\omega}{\omega_0}\right)^3. \tag{41}$$

Thus, the Rayleigh scattering rate is given by the probability of being in the excited state, times the spontaneous emission rate from that state, corrected by a phase-space factor which guarantees that an atom in a DC field does not radiate.

iii. (Optional) One can describe scattering as spontaneous emission from a virtual energy level. Draw the energy level diagram (including the ground, excited, and virtual state) including the frequency spacing. What is the lifetime of this virtual state in terms of the frequency spacing and other constants?

The lifetime of the virtual state is just the natural lifetime of the excited state of the atom except that the resonant frequency ω_0 is replaced with the laser frequency ω, and the virtual state is ω away from the ground state, with all other parameters the same as for the real excited state.

d) Finally, let us apply our formulae to the optical trapping of sodium. Do this problem just for the case i which includes the counter-rotating term. Atoms are to be trapped at the focus of a single laser beam. At the focal plane, the irradiance distribution has the form

$$I(r) = \frac{2P}{\pi w^2} \exp(-2r^2/w^2) \tag{42}$$

where $w = 6\,\mu$m is called the beam waist radius, r is the distance away from the center of the beam, and P is the laser power. Such a beam forms a trap for which the trap depth is given by the AC Stark shift at the maximum intensity $I(0)$.

We wish to trap a gas of sodium atoms with a temperature of about $1\,\mu$K. Thus, let us plan for a trap depth of $k_B \times 10\ \mu$K. The principal transition of sodium is at 590 nm. consider two types of lasers:

1. **Yellow Laser:** a detuning of 1.7 GHz *i.e.* $\omega_L - \omega_0 = -2\pi \times 1.7$ GHz (light at such a detuning is used for cooling sodium in the first place).
2. **Infrared Laser:** the light of a diode laser at 985 nm.

i. If you used the expressions derived from *ii*, *i.e.* the rotating wave approximation in the low field limit, for which type of laser would the expressions be better suited? Why?

The expressions derived from *ii* are better suited for the yellow laser, since that is much closer to resonance than for the infrared laser.

ii. Calculate the required power and the scattering rate for each of the two types of lasers.

$$U_{AC} = -\frac{\hbar\omega_R^2}{4}\left(\frac{1}{\omega_0 - \omega} + \frac{1}{\omega_0 + \omega}\right) \tag{43}$$

The Rabi frequency is defined as $\hbar\omega_R = \langle e|D_z|g\rangle \cdot E$, where E is the electric field amplitude, and it's related to intensity of light by $I = \epsilon_0 E^2 c/2$. For a

Gaussian beam, the total power of light is related to the intensity in the center by $P = \pi I_0 w^2/2$. So

$$\text{Trap Depth} = U_{AC}(0) = -\frac{\hbar\omega_R^2(0)}{4}\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right) \tag{44}$$

$$= -\frac{|\langle e|D_z|g\rangle|^2 E^2(0)}{4\hbar}\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right) \tag{45}$$

for alkali atoms, the oscillation strength is unity, so $|\langle e|D_z|g\rangle|^2 = \hbar e^2/2m_e\omega_0$, so

$$\text{Trap Depth} = -\frac{e^2E^2(0)}{8m_e\omega_0}\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right) \tag{46}$$

$$= -\frac{e^2}{8m_e\omega_0}\frac{2I(0)}{\epsilon_0 c}\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right) \tag{47}$$

$$= -\frac{e^2}{8m_e\omega_0}\frac{4P}{\pi w^2\epsilon_0 c}\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right) \tag{48}$$

In the question, we have:

Trap Depth $= k_B \times 10\mu K$

beam waist $= w = 6\mu m$

$\omega_0 = 2\pi c/590nm = 3.19 \times 10^{15}$ rad/s

Detuning $\omega_L - \omega_0 = -2\pi \times 1.7$ GHz $= -1.068 \times 10^{10}$ rad/s

Yellow Option: For the small detuning

$$\boxed{P = 0.1\,\text{uW}} \tag{49}$$

Infrared Option:

$$\omega_L = 1.91 \times 10^{15}\,\text{s}^{-1} \tag{50}$$

$$I = 1.7 \times 10^7\,\frac{\text{mW}}{\text{cm}^2} \tag{51}$$

$$\boxed{P = 9.6\,\text{mW}} \tag{52}$$

The power emitted by oscillating dipole

$$P_{dipole} = \frac{ck^4}{12\pi\epsilon_0}|\mathbf{d}|^2 \tag{53}$$

The rate for coherent scattering

$$\Gamma_{scat} = \frac{P_{dipole}}{\hbar\omega} = \frac{\omega^3}{12\pi\epsilon_0 c^3\hbar}|\mathbf{d}|^2 \tag{54}$$

where with the fact the oscillation strength for alkali is unity,

$$|\mathbf{d}| = \frac{1}{\hbar}|\langle e|D_z|g\rangle|^2\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right)E \tag{55}$$

$$= \frac{e^2}{2m_e\omega_0}\left(\frac{1}{\omega_0-\omega}+\frac{1}{\omega_0+\omega}\right)E \tag{56}$$

$$= \frac{e^2}{m_e}\frac{1}{\omega_0^2-\omega^2}E \tag{57}$$

So

$$\Gamma_{scat} = \frac{\omega^3}{12\pi\epsilon_0 c^3\hbar}\frac{e^4}{m_e^2}\left(\frac{1}{\omega_0^2-\omega^2}\right)^2 E^2 \tag{58}$$

$$= \frac{\omega^3 e^4}{12\pi\hbar\epsilon_0 m_e^2 c^3}\frac{4P}{\pi\epsilon_0 c w^2}\left(\frac{1}{\omega_0^2-\omega^2}\right)^2 \tag{59}$$

$$= \frac{\omega^3 e^4 P}{3\pi^2\hbar\epsilon_0^2 m_e^2 c^4 w^2}\left(\frac{1}{\omega_0^2-\omega^2}\right)^2 \tag{60}$$

using the required laser power for trap depth of $k_B \times 10\mu K$ above, we get the corresponding scattering rate

Yellow Option: For the small detuning

$$\boxed{\Gamma_{scat} = 7.9\times10^3\,\mathrm{s}^{-1}} \tag{61}$$

Infrared Option:

$$\omega_L = 1.91\times10^{15}\,\mathrm{s}^{-1} \tag{62}$$

$$\boxed{\Gamma_{scat} = 1.7\times10^{-2}\,\mathrm{s}^{-1}} \tag{63}$$

Based on the above considerations, we chose to use infrared light — only handfuls of milliwatts are needed, and the optical traps are extremely long lived. As a result, we have been able to study optically trapped condensates for as long as $20 \sim 30$ seconds.

Many current experiments with Bose-Einstein condensates study phenomena that occur on the millisecond to second time scale, and so these considerations are important.

For reference:

Chu *et al.* "Experimental observation of optically trapped atoms," *Phys. Rev. Lett.* **57**, 314 (1986). First optical dipole trap for laser-cooled sodium atoms.

Stamper-Kurn *et al.* "Optical confinement of a Bose-Einstein condensate," *Phys. Rev. Lett.* **80**, 2027 (1998). First optical dipole trap for Bose condensed sodium atoms.

2 Magic Wavelength Optical Trap

The official time standard is currently based on a cesium microwave transition, but major research efforts are underway to improve on that by using an optical transition (see for example Ludlow *et al.*, Science **319**, 1805 (2008)). In particular, one path is to use neutral atoms trapped in optical lattices that have only one atom per lattice site. This can eliminate collisional effects that can limit the precision of such clocks. However, AC Stark shifts of the two states involved in the relevant transition due to the lattice lasers can cause systematic uncertainties. This can be alleviated by choosing a lattice laser wavelength where the AC Stark shift of the two states are exactly the same, thereby eliminating the systematic uncertainty. This wavelength is called the magic wavelength. You will find the magic wavelength for a model system in this problem.

Consider a laser of frequency ω and intensity I which creates a dipole trap for an atom.

a) Consider a two-level system with lower state $|S\rangle$ and upper state $|P\rangle$ with 'bare' energy separation $\hbar\omega_{PS}$. Also let the dipole moment be d_{PS}.

i. Find the AC Stark shift for the state $|S\rangle$ in terms of the dipole moment, the frequencies, and an overall constant factor. Do not make the rotating wave approximation!

The AC Stark shift for the S state, as you solved in problem 1, is

$$U_S = \frac{\hbar\omega_{\mathrm{R}}^2}{4}\left(\frac{1}{\omega-\omega_{PS}} - \frac{1}{\omega+\omega_{PS}}\right) \tag{64}$$

$$= \frac{I d_{PS}^2}{2\epsilon_0\hbar c}\left(\frac{1}{\omega-\omega_{PS}} - \frac{1}{\omega+\omega_{PS}}\right), \tag{65}$$

where $\hbar\omega_{\mathrm{R}} = d_{PS}E$, and $I = c\epsilon_0 E^2/2$, where E is the electric field.

ii. Find the AC Stark shift for the state $|P\rangle$ in the same way. What is the relationship between this and the AC Stark shift found in *i*?

Looking at the derivation for the AC Stark shift for the P state, you can see that

$$U_P = -U_S. \tag{66}$$

iii. Consider the difference in transition frequency $E_P - E_S$ with and without the laser. Is there any way this can be zero with the current setup (while still having a trapping laser, of course)?

The relevant parameter is given as

$$E_P^{\text{laser}} - E_S^{\text{laser}} - (E_P^{\text{nolaser}} - E_S^{\text{nolaser}}) = 2U_P. \tag{67}$$

This cannot be zero unless $I = 0$, in which case there will be no trapping laser.

b) Now consider adding a third, higher level to the system to make it a three-level system with the third, highest state $|D\rangle$ with energy separation between $|D\rangle$ and $|P\rangle$ as $\hbar\omega_{DP}$. Also let the dipole moment be d_{DP}. Assume that $\omega_{PS} = f\omega_{DP}$ and $d_{PS} = nd_{DP}$, where f and n are some numbers, and that the states $|S\rangle$ and $|D\rangle$ do not couple directly.

i. Find the AC Stark shift for the state $|P\rangle$ taking into account both the higher state $|D\rangle$ and lower state $|S\rangle$.

You just need to add the contributions from coupling to both the S and D states, so

$$U_P = -\frac{I}{2\epsilon_0\hbar c}\left[d_{PS}^2\left(\frac{1}{\omega-\omega_{PS}} - \frac{1}{\omega+\omega_{PS}}\right) - d_{DP}^2\left(\frac{1}{\omega-\omega_{DP}} - \frac{1}{\omega+\omega_{DP}}\right)\right] \tag{68}$$

$$= U_{PS} + U_{PD}, \tag{69}$$

where U_{PS} and U_{PD} should be clear.

ii. What is the laser frequency needed to make the $S \to P$ transition frequency independent of trap laser power? Express this in terms of ω_{DP} and d_{DP}. This is the so-called magic wavelength.

We need the AC Stark shifts of both the P and S states to be the same, so

$$U_P = U_S \Rightarrow U_{PS} + U_{PD} = U_S \Rightarrow -U_S + U_{PD} = U_S. \tag{70}$$

Solving this for ω gives us

$$\omega = \sqrt{\frac{2n^2f - f^2}{2n^2f - 1}}\omega_{PD}. \tag{71}$$

3 Species-Dependent and Spin-Dependent AC Stark Shift

Now consider how multiple species or multiple sub-states of atoms can be selectively manipulated using AC Stark shifts. We can consider putting more than one type of atom in the laser field. Also, in reality atoms do have sub-states, and the different sub-states can couple differently to the laser field. We will explore these situations

in this problem. For these cases, the polarization of the light with respect to the quantization axis also begins to matter.

Consider a three-level system with fine structure, *i.e.* an alkali atom. The lowest level is $S_{1/2}$ and the upper P state splits into two levels, $P_{3/2}$ and $P_{1/2}$. Assume the state $P_{3/2}$ has higher energy than $P_{1/2}$. Let the energy difference be $\hbar\omega_1 = E_{P_{1/2}} - E_{S_{1/2}}$ and $\hbar\omega_2 = E_{P_{3/2}} - E_{S_{1/2}}$. (As a side note, the labels 1 and 2 for the frequencies correspond to the fact that the transitions are called the D1 and D2 lines, respectively.) Assume the atom is quantized along the z direction. You also need to know the dipole coupling of the form $\langle S_{1/2}, m_J|\hat{\epsilon}\cdot\mathbf{r}|P_{3/2,1/2}, m'_J\rangle$ between the relevant states, where $\hat{\epsilon}$ is the polarization vector of the electric field. The relevant dipole matrix elements are as follows:

$$|e\langle S_{1/2}, +1/2|\hat{\sigma}_+\cdot\mathbf{r}|P_{3/2}, +3/2\rangle|^2 = d_2^2 \tag{72}$$
$$|e\langle S_{1/2}, +1/2|\hat{\sigma}_-\cdot\mathbf{r}|P_{3/2}, -1/2\rangle|^2 = d_2^2/3 \tag{73}$$
$$|e\langle S_{1/2}, -1/2|\hat{\sigma}_+\cdot\mathbf{r}|P_{3/2}, +1/2\rangle|^2 = d_2^2/3 \tag{74}$$
$$|e\langle S_{1/2}, -1/2|\hat{\sigma}_-\cdot\mathbf{r}|P_{3/2}, -3/2\rangle|^2 = d_2^2 \tag{75}$$
$$|e\langle S_{1/2}, +1/2|\hat{\sigma}_-\cdot\mathbf{r}|P_{1/2}, -1/2\rangle|^2 = 2d_1^2/3 \tag{76}$$
$$|e\langle S_{1/2}, -1/2|\hat{\sigma}_+\cdot\mathbf{r}|P_{1/2}, +1/2\rangle|^2 = 2d_1^2/3, \tag{77}$$

where e is the electron charge. All other matrix elements are irrelevant for this problem (which should become clear if you consider the selection rules and the photon polarizations considered in this problem). You should also notice that the interference terms between the different polarizations vanish for this problem. Note that $\hat{\sigma}_\pm = (\hat{x} \pm i\hat{y})/\sqrt{2}$ correspond to $\sigma_\pm$ polarized light where absorbing a photon with such a polarization leads to $\Delta m_J = \pm 1$. The total AC Stark shift for a given sub-state is then just a sum over all of the AC Stark shifts due to the coupled states. Do not use the rotating wave approximation in this problem.

a) Consider a linearly polarized dipole trap laser with electric field polarization along the x direction propagating along the z direction.

i. At what laser wavelength will the atom in state $S_{1/2}$ not feel the dipole trap anymore? Use the following parameters to numerically find the wavelength specifically for rubidium: $\lambda_1 = 795$ nm, $\lambda_2 = 780$ nm, $d_1 = 3.0ea_0$, $d_2 = 4.2ea_0$, where a_0 is the Bohr radius.

I choose to look at the $|S_{1/2}, -1/2\rangle$ state. The answer is the same for the $|S_{1/2}, +1/2\rangle$ by symmetry. The relevant transitions to consider are $|S_{1/2}, -1/2\rangle \to |P_{1/2}, +1/2\rangle$, $|S_{1/2}, -1/2\rangle \to |P_{3/2}, +1/2\rangle$, and $|S_{1/2}, -1/2\rangle \to |P_{3/2}, -3/2\rangle$. Up

to an overall constant C, the AC Stark shift is given by

$$U_S = C\left[\frac{2}{3}d_1^2\left(\frac{1}{\omega-\omega_1}-\frac{1}{\omega+\omega_1}\right)+\frac{1}{3}d_2^2\left(\frac{1}{\omega-\omega_2}-\frac{1}{\omega+\omega_2}\right)+d_2^2\left(\frac{1}{\omega-\omega_2}-\frac{1}{\omega+\omega_2}\right)\right], \tag{78}$$

where the transitions are in the order mentioned above. Setting this equal to zero, we get

$$\omega^2 = \frac{2d_2^2\omega_1+d_1^2\omega_2}{2d_2^2\omega_2+d_1^2\omega_1}\omega_1\omega_2. \tag{79}$$

Plugging in the numbers given, we get

$$\lambda = \frac{2\pi c}{\omega} = 792\text{nm} \tag{80}$$

ii. This has been used to create a system where one type of atom (^{41}K) feels a dipole trap, but the other type (^{87}Rb) does not, in order to demonstrate entropy exchange between the two species (Catani *et al.*, *Phys. Rev. Lett.* **103**, 140401 (2009)). They used a laser at a wavelength of 790 nm for this purpose. Does the wavelength you found agree with what was used in their paper? Is there anything that we neglected to take into account?

792 nm does not agree with the 790 nm laser used in the paper. This discrepancy can be accounted for by taking into account the degeneracy of the fine structure states due to hyperfine levels.

b) Now consider a σ_+ polarized light propagating along the z direction, with laser frequency much closer to the $S_{1/2} \to P_{3/2}$ transition than the $S_{1/2} \to P_{1/2}$ transition so that you can neglect coupling to the $P_{1/2}$ state.

i. What is the AC Stark shift for the two states $|S_{1/2}, m_J = +1/2\rangle$ and $|S_{1/2}, m_J = -1/2\rangle$?

For the $|S_{1/2}, m_J = +1/2\rangle$ state, the coupling that needs to be considered is $|S_{1/2}, +1/2\rangle \to |P_{3/2}, +3/2\rangle$. This gives us

$$U_{+1/2} = \frac{I}{2\epsilon_0\hbar c}d_2^2\left(\frac{1}{\omega-\omega_2}-\frac{1}{\omega+\omega_2}\right). \tag{81}$$

For the $|S_{1/2}, m_J = -1/2\rangle$ state, the coupling that needs to be considered is $|S_{1/2}, -1/2\rangle \to |P_{3/2}, +1/2\rangle$. This gives us

$$U_{-1/2} = \frac{1}{3}\frac{I}{2\epsilon_0\hbar c}d_2^2\left(\frac{1}{\omega-\omega_2}-\frac{1}{\omega+\omega_2}\right). \tag{82}$$

ii. Rewrite the AC Stark shift you found in i in the form $U = U_S + U_V$, where U_S is the spin-independent, scalar light shift, and $U_V = -\mu \cdot \mathbf{B}_{\text{eff}} = g_J m_J \mu_B B_{\text{eff}}$ is the spin-dependent, vector light shift. Note $g_J(S_{1/2}) = 2$. What is $\mathbf{B}_{\text{eff}}$? We see that the spin-dependence of the AC Stark shift can be expressed in terms of a fictitious magnetic field acting on the spin of the atoms.

We can rewrite what we found in i as

$$U_{m_J} = \frac{2}{3}\frac{I}{2\epsilon_0 \hbar c} d_2^2 \frac{2\omega_2}{\omega^2 - \omega_2^2}(1 + m_J). \tag{83}$$

The second term can be rewritten in the form $g_J m_J \mu_B B_{\text{eff}}$. By doing this we find

$$B_{\text{eff}} = \frac{1}{3}\frac{I}{2\epsilon_0 \hbar c}\frac{d_2^2}{\mu_B}\frac{2\omega_2}{\omega^2 - \omega_2^2} = \frac{1}{6}\frac{\epsilon^2 d_2^2}{\hbar \mu_B}\frac{\omega_2}{\omega^2 - \omega_2^2}. \tag{84}$$

This configuration is relevant to situations where you want to induce spin-dependent interactions. This can be the used to generate entanglement for quantum information processing (Mandel *et al.*, *Nature* **425**, 937 (2003)), or realizing, for example, anti-ferromagnetic many-body ground states to study quantum magnetism (Duan *et al.*, *Phys. Rev. Lett.* **91**, 090402 (2003)).