PY 711 Fall 2010
Homework 3: Due Tuesday, September 14

1. (8 points) Consider a Lorentz boost along the x^1-direction with relative speed β (in units where $c = 1$). In order to get our conventions the same, the explicit transformation from the original frame to the primed frame is

$$\begin{bmatrix} x^{0\prime} \\ x^{1\prime} \\ x^{2\prime} \\ x^{3\prime} \end{bmatrix} = B^1 \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \qquad B^1 = \begin{bmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{1}$$

where

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\eta, \qquad \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}} = \sinh\eta. \tag{2}$$

The parameter γ is the Lorentz factor, and η is called the rapidity. Let K^1 be the infinitesimal boost generator for the x^1-direction,

$$K^1 = i\frac{\partial B^1}{\partial\eta}\bigg|_{\eta=0}. \tag{3}$$

We define analogous boosts and infinitesimal boost generators for the other spatial directions,

$$B^2 = \begin{bmatrix} \cosh\eta & 0 & \sinh\eta & 0 \\ 0 & 1 & 0 & 0 \\ \sinh\eta & 0 & \cosh\eta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad B^3 = \begin{bmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{bmatrix}, \tag{4}$$

$$K^2 = i\frac{\partial B^2}{\partial\eta}\bigg|_{\eta=0}, \qquad K^3 = i\frac{\partial B^3}{\partial\eta}\bigg|_{\eta=0}.$$

Calculate the matrix commutators $[K^1, K^2]$, $[K^2, K^3]$, and $[K^3, K^1]$. For each case describe the type of spacetime transformation the commutator generates.

2. (7 points) Show that $J^{\mu\nu} = i\left(x^\mu\partial^\nu - x^\nu\partial^\mu\right)$, a differential operator acting on functions of the spacetime variable x, satisfies the Lorentz algebra,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho}\right). \tag{5}$$

1. CONSIDER A LORENTZ BOOST ALONG THE x^1- DIRECTION WITH RELATIVE SPEED β. (IN UNITS WHERE $c=1$). IN ORDER TO GET OUR CONVENTIONS THE SAME, THE EXPLICIT TRANSFORMATION FROM THE ORIGINAL FRAME TO THE PRIMED FRAME IS

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$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = B^1 \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad B^1 = \begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

WHERE

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh(\eta) \qquad \beta\gamma = \frac{\beta}{\sqrt{1-\beta^2}} = \sinh(\eta).$$

THE PARAMETER γ IS THE LORENTZ FACTOR, AND η IS CALLED THE RAPIDITY. LET K^1 BE THE INFINITESIMAL BOOST GENERATOR FOR THE x^1- DIRECTION.

$$K^1 = i \left.\frac{\partial B^1}{\partial \eta}\right|_{\eta=0}.$$

WE DEFINE ANALOGOUS BOOSTS AND INFINITESIMAL BOOST GENERATORS FOR THE OTHER SPATIAL DIRECTIONS

$$B^2 = \begin{pmatrix} \cosh(\eta) & 0 & \sinh(\eta) & 0 \\ 0 & 1 & 0 & 0 \\ \sinh(\eta) & 0 & \cosh(\eta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B^3 = \begin{pmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{pmatrix}$$

✓

$$K^2 = i \left.\frac{\partial B^2}{\partial \eta}\right|_{\eta=0} \qquad K^3 = i \left.\frac{\partial B^3}{\partial \eta}\right|_{\eta=0}$$

CALCULATE THE MATRIX COMMUTATORS $[K^1, K^2]$, $[K^2, K^3]$ AND $[K^3, K^1]$. FOR EACH CASE DESCRIBE THE TYPE OF SPACETIME TRANSFORMATION THE COMMUTATOR GENERATES.

$$\frac{\partial}{\partial \eta} (\cosh(\eta)) \Big|_{\eta=0} = 0 \qquad \frac{\partial}{\partial \eta} (\sinh(\eta)) \Big|_{\eta=0} = 1$$

$$K^1 = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

✓

$$K^3 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

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$$K^1 \cdot K^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^2 \cdot K^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$K^1 \cdot K^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^3 \cdot K^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$K^2 \cdot K^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad K^3 \cdot K^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} [K^1, K^2] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ [K^2, K^3] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ [K^3, K^1] &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \end{aligned}}$$

✓

$$[K^1, K^2] \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 0 \\ -x^2 \\ x^1 \\ 0 \end{bmatrix} \qquad [K^2, K^3] \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -x^3 \\ x^2 \end{bmatrix} \qquad [K^3, K^1] \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} 0 \\ x^3 \\ 0 \\ -x^1 \end{bmatrix}$$

These commutators are generators of infinitesimal rotations. ✓

$[K^1, K^2] \rightarrow$ clockwise rotation about x^3

$[K^2, K^3] \rightarrow$ clockwise rotation about x^1

$[K^3, K^1] \rightarrow$ clockwise rotation about x^2

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2. SHOW THAT $J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$, A DIFFERENTIAL OPERATOR ACTING ON FUNCTIONS OF THE SPACETIME VARIABLE x, SATISFIES THE LORENTZ ALGEBRA,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho})$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = \Big((i(x^\mu \partial^\nu - x^\nu \partial^\mu))(i(x^\rho \partial^\sigma - x^\sigma \partial^\rho))$$

$$- (i(x^\rho \partial^\sigma - x^\sigma \partial^\rho))(i(x^\mu \partial^\nu - x^\nu \partial^\mu)) \Big) \phi(x)$$

$$= \Big(- \big(x^\mu \partial^\nu (x^\rho \partial^\sigma \phi) - x^\mu \partial^\nu (x^\sigma \partial^\rho \phi) - x^\nu \partial^\mu (x^\rho \partial^\sigma \phi)$$

$$+ x^\nu \partial^\mu (x^\sigma \partial^\rho \phi) \big)$$

$$+ \big(x^\rho \partial^\sigma (x^\mu \partial^\nu \phi) - x^\rho \partial^\sigma (x^\nu \partial^\mu \phi)$$

$$- x^\sigma \partial^\rho (x^\mu \partial^\nu \phi) + x^\sigma \partial^\rho (x^\nu \partial^\mu \phi) \big) \Big)$$

Look at one of these terms.

$$x^\mu \partial^\nu (x^\rho \partial^\sigma \phi) = x^\mu (\partial^\nu x^\rho)(\partial^\sigma \phi) + x^\mu x^\rho \partial^\nu \partial^\sigma \phi$$

$$\partial^\nu x^\rho = \frac{\partial x^\rho}{\partial x_\nu} \qquad \frac{\partial x^\rho}{\partial x^\nu} = \delta^\rho_\nu$$

since $\nabla^\ell = -\nabla_\ell = -\frac{\partial}{\partial x^\ell}$

$\rightarrow \quad \partial^\nu x^\rho = g^{\nu\rho}$ to account for the negative signs in spatial terms.

$$x^\mu \partial^\nu (x^\rho \partial^\sigma \phi) = x^\mu g^{\nu\rho} \partial^\sigma \phi + x^\mu x^\rho \partial^\nu \partial^\sigma \phi$$

Now we can expand all terms and drop the ϕ since we know how the terms expand.

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$$
\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] = -\Big(& g^{\nu\rho} x^\mu \partial^\sigma + \underline{x^\mu x^\rho \partial^\nu \partial^\sigma} \\
& - g^{\nu\sigma} x^\mu \partial^\rho - \underline{x^\mu x^\sigma \partial^\nu \partial^\rho} \\
& - g^{\mu\rho} x^\nu \partial^\sigma - \underline{x^\nu x^\rho \partial^\mu \partial^\sigma} \\
& + g^{\mu\sigma} x^\nu \partial^\rho + \underline{x^\nu x^\sigma \partial^\mu \partial^\rho} \Big) \\
+ \Big(& g^{\mu\sigma} x^\rho \partial^\nu + \underline{x^\rho x^\mu \partial^\sigma \partial^\nu} \\
& - g^{\nu\sigma} x^\rho \partial^\mu - \underline{x^\rho x^\nu \partial^\sigma \partial^\mu} \\
& - g^{\mu\rho} x^\sigma \partial^\nu - \underline{x^\sigma x^\mu \partial^\rho \partial^\sigma} \\
& + g^{\nu\rho} x^\sigma \partial^\mu + \underline{x^\sigma x^\nu \partial^\rho \partial^\mu} \Big)
\end{aligned}
$$

Underlined terms cancel out.

$$
\begin{aligned}
= \; & g^{\nu\rho}(x^\sigma \partial^\mu - x^\mu \partial^\sigma) + g^{\mu\sigma}(x^\rho \partial^\nu - x^\nu \partial^\rho) \\
& + g^{\nu\sigma}(x^\mu \partial^\rho - x^\rho \partial^\mu) + g^{\mu\rho}(x^\nu \partial^\sigma - x^\sigma \partial^\nu)
\end{aligned}
$$

$$\left\{ iJ^{\mu\nu} = -(x^\mu \partial^\nu - x^\nu \partial^\mu) = (x^\nu \partial^\mu - x^\mu \partial^\nu) \right\}$$

So...

$$= g^{\nu\rho}(iJ^{\mu\sigma}) + g^{\mu\sigma}(iJ^{\nu\rho}) - g^{\nu\sigma}(iJ^{\mu\rho}) - g^{\mu\rho}(iJ^{\nu\sigma})$$

$$\Rightarrow \boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} + g^{\mu\sigma}J^{\nu\rho} - g^{\nu\sigma}J^{\mu\rho} - g^{\mu\rho}J^{\nu\sigma})}$$

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Solutions #3

1. $K_1 = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_2 = \begin{bmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad K_3 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix}$

$$[K_1, K_2] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

generates rotations about z-axis

$$[K_2, K_3] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

generates rotations about x-axis

$$[K_3, K_1] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

generates rotations about y-axis

This is the reason for "Thomas precession" of the electron spin in an atom.

2. Let $f^{\mu\nu} = i x^\mu \partial^\nu$. Then

$$J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu) = f^{\mu\nu} - f^{\nu\mu}$$

Then
$$\begin{aligned}
[f^{\mu\nu}, f^{\rho\sigma}] &= -x^\mu\partial^\nu x^\rho\partial^\sigma + x^\rho\partial^\sigma x^\mu\partial^\nu \\
&= -g^{\nu\rho}x^\mu\partial^\sigma + g^{\sigma\mu}x^\rho\partial^\nu \\
&= i g^{\nu\rho} f^{\mu\sigma} - i g^{\sigma\mu} f^{\rho\nu}
\end{aligned}$$

This gives
$$\begin{aligned}
[J^{\mu\nu}, J^{\rho\sigma}] &= [f^{\mu\nu} - f^{\nu\mu}, f^{\rho\sigma} - f^{\sigma\rho}] \\
&= i g^{\nu\rho} f^{\mu\sigma} - i g^{\sigma\mu} f^{\rho\nu} \quad \leftarrow [f^{\mu\nu}, f^{\rho\sigma}] \\
&\quad - i g^{\mu\rho} f^{\nu\sigma} + i g^{\sigma\nu} f^{\rho\mu} \quad \leftarrow -[f^{\nu\mu}, f^{\rho\sigma}] \\
&\quad - i g^{\nu\sigma} f^{\mu\rho} + i g^{\rho\mu} f^{\sigma\nu} \quad \leftarrow -[f^{\mu\nu}, f^{\sigma\rho}] \\
&\quad + i g^{\mu\sigma} f^{\nu\rho} - i g^{\rho\nu} f^{\sigma\mu} \quad \leftarrow [f^{\nu\mu}, f^{\sigma\rho}]
\end{aligned}$$

$$= i g^{\nu\rho}(f^{\mu\sigma} - f^{\sigma\mu}) + i g^{\mu\sigma}(f^{\nu\rho} - f^{\rho\nu}) - i g^{\mu\rho}(f^{\nu\sigma} - f^{\sigma\nu}) - i g^{\nu\sigma}(f^{\mu\rho} - f^{\rho\mu})$$

$$= i\left(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho}\right)$$