Lecture 3 - From Maxwell to QED

2.1 Maxwell's Equations

$$ec{
abla} \cdot ec{E} = rac{1}{\epsilon_0}
ho \ (M1)$$
 $ec{
abla} \cdot ec{B} = 0 \ (M2)$
 $ec{
abla} imes ec{E} = -rac{\partial}{\partial t} ec{B} \ (M3)$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{M2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \tag{M3}$$

$$ec{
abla} imesec{B}=rac{1}{c^2}rac{\partial}{\partial t}ec{E}+\mu_0ec{j} \quad (M4)$$

• Lorentz equation ($v \ll c$):

$$m_{lpha}\ddot{ec{r}}_{lpha}=q_{lpha}\left(ec{E}+ec{v}_{lpha} imesec{B}
ight)$$

Local conservation of charge (from M1 and M4)

$$rac{\partial}{\partial t}
ho + \vec{
abla}\cdot\vec{j} = 0$$

$$ho(ec{r},t) = \sum_{lpha} q_{lpha} \, \delta(ec{r} - ec{r}_{lpha}(t))$$

$$ec{j}(ec{r},t) = \sum_{lpha} q_{lpha} ec{v}_{lpha}(t) \delta(ec{r} - ec{r}_{lpha}(t))$$

Constants o Motion:

Total energy:
$$H=\sum_lpha rac{1}{2} m_lpha v_lpha^2 + rac{\epsilon_0}{2} \int \mathrm{d}^3 r \left(ec{E}^2 + ec{B}^2
ight)$$

Total Momentum:
$$ec{P} = \sum_{lpha} m_{lpha} ec{v}_{lpha} + \epsilon_0 \int \mathrm{d}^3 r ec{E} imes ec{B}$$

Total angular momentum:
$$ec{J}=\sum_lphaec{r}_lpha imes m_lphaec{v}_lpha+\epsilon_0\int\mathrm{d}^3rec{r} imes\left(ec{E} imesec{B}
ight)$$

2.2 Vector Potential

(M2):
$$ec{
abla} \cdot ec{B} = 0 \Leftrightarrow \boxed{ec{B} = ec{
abla} imes ec{A}}$$

(M3):
$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = -\vec{\nabla} \times \left(\frac{\partial}{\partial t} \vec{A} \right)$$

$$\rightarrow \vec{\nabla} \times (\vec{E} + \frac{\partial}{\partial t} \vec{A}) = 0 \Leftrightarrow \vec{E} + \frac{\partial}{\partial t} \vec{A} = -\vec{\nabla} U$$

Given vector potential \vec{A} and scalar potential U, (M2) and (M3) are automatically verified.

$$\begin{array}{l} \text{(M1)} \Rightarrow \Delta U = -\frac{1}{\epsilon_0} \rho - \vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{A} \\ \\ \text{(M4)} \Rightarrow \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{A} = \mu_0 \vec{j} - \vec{\nabla} \left(\vec{\nabla} \cdot A + \frac{1}{c^2} \frac{\partial}{\partial t} U \right) \end{array}$$

State of the field given by $\vec{A}(\vec{r}\,,t_0)$ and $rac{\partial}{\partial t}\vec{A}(\vec{r}\,,t_0)$ $orall \vec{r}$

Gauge invariance:

$$ec{A}
ightarrow ec{A}' = ec{A} + ec{
abla} f \ U
ightarrow U' = U - rac{\partial}{\partial t} f \ .$$

Lorentz gauge: $ec{
abla}\cdotec{A}+rac{1}{c^2}rac{\partial}{\partial t}U=0 \qquad ext{ or } \qquad \partial_{\mu}A^{\mu}=0$

(M1)+(M4):

$$\Box U = rac{1}{\epsilon_0}
ho$$
 $\Box ec{A} = \mu_0 ec{j}$ or $\partial_
u \partial^
u A^\mu = \mu_0 j^\mu$

Here
$$\Box\equivrac{1}{c^2}rac{\partial^2}{\partial t^2}-\Delta$$
 , $\partial_\mu=\left(rac{1}{c}rac{\partial}{\partial t},ec{
abla}
ight)$, $A^\mu=\left(rac{1}{c}U,ec{A}
ight)$, $j^\mu=\left(c
ho,ec{j}
ight)$

Coulomb gauge: $ec{
abla} \cdot ec{A} = 0$

(M1):
$$\Delta U = -rac{1}{\epsilon_0}
ho$$

Laplace equation for U. $U(\vec{r},t)$ is completely specified by $ho(\vec{r},t)!$

$$\Box \vec{A} = \mu_0 \vec{j} - \frac{1}{c^2} \vec{\nabla} \frac{\partial}{\partial t} U$$

2.3 Electrodynamics in reciprocal space

$$ec{\mathcal{E}}(ec{k},t) = rac{1}{(2\pi)^{3/2}} \int \mathrm{d}^3 r \, ec{E}(ec{r},t) e^{-iec{k}\cdotec{r}} \ ec{E}(ec{r},t) = rac{1}{(2\pi)^{3/2}} \int \mathrm{d}k^3 \, ec{\mathcal{E}}(ec{k},t) e^{iec{k}\cdotec{r}}$$

same for
$$ec{E}\leftrightarrowec{\mathcal{E}}$$
 , $ec{B}\leftrightarrowec{\mathcal{B}}$, $ec{A}\leftrightarrowec{\mathcal{A}}$, $U\leftrightarrow\mathcal{U}$, $\rho\leftrightarrow\rho$, $ec{j}\leftrightarrowec{j}$

• Field equations become <u>local</u> in reciprocal space:

$$i\vec{k}\cdot\vec{\mathcal{E}} = \frac{1}{\epsilon_0}\rho$$
 (R1)

$$i\vec{k}\cdot\vec{\mathcal{B}} = 0$$
 (R2)

$$i\vec{k} imes \vec{\mathcal{E}} = -\dot{\vec{\mathcal{B}}}$$
 (R3)

$$iec{k} imesec{\mathcal{B}}=rac{1}{c^2}\dot{ec{\mathcal{E}}}+\mu_0ec{j} \hspace{1.5cm} (R4)$$

Continuity equation:

$$i ec{k} \cdot ec{j} + \dot{
ho} = 0$$

• Fields and potentials:

$$ec{\mathcal{B}}=iec{k} imesec{A}$$

$$ec{\mathcal{E}}=\dot{-ec{\mathcal{A}}}-iec{k}\,\mathcal{U}$$

• Gauge transformation:

$$ec{\mathcal{A}}
ightarrow ec{\mathcal{A}}' = ec{\mathcal{A}} + i ec{k} f \ \mathcal{U}
ightarrow \mathcal{U}' = \mathcal{U} - \dot{f}$$

Equations for potentials:

$$egin{align} k^2 \mathcal{U} &= rac{1}{\epsilon_0}
ho + i ec{k} \cdot \dot{ec{\mathcal{A}}} \ &rac{1}{c^2} \ddot{\mathcal{A}} + k^2 ec{\mathcal{A}} &= \mu_0 ec{j} - i ec{k} (i ec{k} \cdot ec{\mathcal{A}} + rac{1}{c^2} \dot{\mathcal{U}}) \end{split}$$

- Longitudinal and transverse fields:
 - Longitudinal field:

$$ec{
abla} imesec{V}_{||}(ec{r})=0 \ iec{k} imesec{\mathcal{V}}_{||}(ec{k})=0$$

Transverse field:

$$ec{
abla} \cdot ec{V}_{\perp}(ec{r}) = 0 \ i ec{k} \cdot ec{\mathcal{V}}_{\perp}(ec{k}) = 0$$

in reciprocal space, decomposition is simple: (not so in real space!)

$$egin{aligned} ec{\mathcal{V}}(ec{k}) &= ec{\mathcal{V}}_{||}(ec{k}) + ec{\mathcal{V}}_{\perp}(ec{k}) \ ec{\mathcal{V}}_{||}(ec{k}) &= ec{\kappa}(ec{\kappa} \cdot ec{\mathcal{V}}(ec{k})) \qquad ext{parallel to } ec{k} \, orall \, ec{k} \ ec{\mathcal{V}}_{\perp}(ec{k}) &= ec{\mathcal{V}}(ec{k}) - ec{\mathcal{V}}_{||}(ec{k}) &= (ec{\kappa} imes ec{\mathcal{V}}) imes ec{\kappa} & \perp ext{ to } ec{k} \, orall \, ec{k} \end{aligned}$$

Return to Maxwell:

(M2)
$$\Rightarrow ec{\mathcal{B}}_{||} = ec{0} = ec{B}_{||}$$

 \Rightarrow The magnetic field is purely transverse!

(M1)
$$\Rightarrow ec{\mathcal{E}}_{||}(ec{k}) = -rac{i}{\epsilon_0}
ho(ec{k}) rac{ec{k}}{k^2}$$

That is a product of two functions of \vec{k} whose Fourier transforms are

$$ho(ec{k}) \leftrightarrow
ho(ec{r})$$

$$-rac{i}{\epsilon_0}rac{ec{k}}{k^2} \leftrightarrow rac{(2\pi)^{3/2}}{4\pi\epsilon_0}rac{ec{r}}{r^3}$$

The Fourier transform of a product is a convolution:

$$egin{aligned} \Rightarrow ec{E}_{||}(ec{r},t) &= rac{1}{4\pi\epsilon_0} \int \mathrm{d}^3 r' \,
ho(ec{r}',t) rac{ec{r} - ec{r}'}{|ec{r} - ec{r}'|^3} \ &= rac{1}{4\pi\epsilon_0} \sum_{lpha} q_lpha rac{ec{r} - ec{r}_lpha(t)}{|ec{r} - ec{r}_lpha(t)|^3} \end{aligned} \tag{1}$$

Longitudinal E-field = instantaneous Coulomb field of ρ .

Result independent of gauge!

Q: Does this mean perturbations can travel faster than light?

Answer: No! Only the total E-field matters, and E_{\perp} makes E_{total} purely retarded!

Note: i)

$$egin{align} ec{\mathcal{E}} &= -\dot{ec{\mathcal{A}}} - iec{k}\mathcal{U} \ &\Rightarrow ec{\mathcal{E}}_{\perp} &= -\dot{ec{\mathcal{A}}}_{\perp} \ &\Rightarrow ec{\mathcal{E}}_{\perp} &= -\dot{ec{\mathcal{A}}}_{\perp} \ ec{\mathcal{E}}_{||} &= -\dot{ec{\mathcal{A}}}_{||} - iec{k}\mathcal{U} \ &\Rightarrow ec{E}_{||} &= -\dot{ec{\mathcal{A}}}_{||} - ec{
abla}\mathcal{U} \end{split}$$

In the Coulomb gauge $ec{
abla}\cdotec{A}=0\Rightarrow iec{k}\cdotec{\mathcal{A}}=0\Rightarrow ec{\mathcal{A}}_{||}=0.$

$$ec{E}_{\perp} = - \dot{ec{A}}_{\perp} \qquad ext{transverse field given by} ec{A} \ ec{E}_{||} = - ec{
abla} U \qquad ext{longitudinal field given by} U$$

From $ec{E}_{||}$ we find $U=rac{1}{4\pi\epsilon_0}\int {
m d}^3r'rac{
ho(ec{r}',t)}{|ec{r}-ec{r}'|}$ Coulomb potential.

ii) gauge transformation does not change \vec{A}_{\perp} ! transverse vector potential \vec{A}_{\perp} is gauge invariant:

$$ec{A}'_{\perp} = ec{A}_{\perp}$$

iii) generally, in any gauge:

$$ec{\cal E}_{\perp} = -\dot{ec{\cal A}}_{\perp} \ ec{\cal B} = ec{\cal B}_{\perp} = iec{k} imesec{\cal A}_{\perp}$$

Transverse fields only depend on $\vec{\mathcal{A}}_{\perp}$.

iv) longitudinal part of (M4) gives:

$$egin{aligned} \dot{ec{\mathcal{E}}}_{||} + rac{1}{\epsilon_0}ec{j}_{||} &= 0 \ \Rightarrow iec{k}\cdot\dot{ec{\mathcal{E}}}_{||} + rac{1}{\epsilon_0}iec{k}\cdotec{j}_{||} &= iec{k}\cdot\dot{ec{\mathcal{E}}} + rac{1}{\epsilon_0}iec{k}\cdotec{j} &= rac{\dot{
ho}}{\epsilon_0} + rac{iec{k}\cdotec{j}}{\epsilon_0} &= 0 \ \Rightarrow \dot{
ho} + ec{
abla}\cdotec{j} &= 0 \end{aligned}$$

This is just the continuity equation, so this does not add anything new.

Total Energy:

$$rac{\epsilon_0}{2}\int \mathrm{d}^3 r\, ec{E}\cdotec{E} = rac{\epsilon_0}{2}\int \mathrm{d}^3 k\, ec{\mathcal{E}}^*\cdotec{\mathcal{E}}$$

(Parseval-Plancherel Theorem)

$$ec{\mathcal{E}}
ightarrow ec{\mathcal{E}}_{\perp} + ec{\mathcal{E}}_{||}, \qquad ec{\mathcal{E}}_{\perp} \cdot ec{\mathcal{E}}_{||} = 0$$

$$ho \Rightarrow rac{\epsilon_0}{2} \int \mathrm{d}^3 r \, ec{E}^2 = rac{\epsilon_0}{2} \int \mathrm{d}^3 k \, \left| ec{\mathcal{E}}_{||}(ec{k})
ight|^2 + rac{\epsilon_0}{2} \int \mathrm{d}^3 k \, \left| ec{\mathcal{E}}_{\perp}(ec{k})
ight|^2$$

$$H_{
m long} = rac{\epsilon_0}{2} \int \mathrm{d}^3 k \left| ec{\mathcal{E}}_{||}(ec{k})
ight|^2 = rac{\epsilon_0}{2} \int \mathrm{d}^3 r \, ec{E}_{||}^2(ec{r})$$

$$egin{aligned} H_{ ext{trans}} &= rac{\epsilon_0}{2} \int \mathrm{d}^3 k \left(\left| ec{\mathcal{E}}_\perp(ec{k})
ight|^2 + c^2 \left| ec{\mathcal{B}}(ec{k})
ight|^2
ight) \ &= rac{\epsilon_0}{2} \int \mathrm{d}^3 r \left(\left| ec{E}_\perp(ec{r})
ight|^2 + c^2 ec{B}^2(ec{r})
ight) \end{aligned}$$

 $H_{
m long}$ is the Coulomb energy:

$$H_{
m long} = rac{1}{2\epsilon_0} \int {
m d}^3 k \,
ho^*(ec k) \, rac{
ho(ec k)}{k^2} = rac{1}{8\pi\epsilon_0} \iint {
m d}^3 r \, {
m d}^3 r' \, rac{
ho(ec r)
ho(ec r')}{(|ec r - ec r'|)}$$

For a system of charges:

$$ho(ec{r}) = \sum_{lpha} q_lpha \delta(ec{r} - ec{r}_lpha(t))$$

$$ho(ec{k}) = \sum_{lpha} rac{q_lpha}{(2\pi)^{3/2}} e^{-iec{k}\cdotec{r}_{lpha}}$$

$$egin{aligned} H_{
m long} &= V_{
m Coulomb} = \sum_{lpha} rac{q_{lpha}^2}{2\epsilon_0 (2\pi)^3} \int \mathrm{d}^3 k rac{1}{k^2} + \sum_{lpha
eq eta} rac{q_{lpha} q_{eta}}{2\epsilon_0 (2\pi)^3} \int \mathrm{d}^3 k rac{e^{-i ec{k} \cdot (ec{r}_{lpha} - ec{r}_{eta})}}{k^2} \ &= \sum_{lpha} \epsilon_{
m Coulomb}^{lpha} + rac{1}{8\pi\epsilon_0} \sum_{lpha
eq eta} rac{q_{lpha} q_{eta}}{|ec{r}_{lpha} - ec{r}_{eta}|} \end{aligned}$$

The first term

$$\epsilon_{ ext{Coulomb}}^{lpha} = rac{q_{lpha}^2}{2\epsilon_0(2\pi)^3} \int \mathrm{d}^3k rac{1}{k^2} = rac{q_{lpha}^2}{4\pi^2\epsilon_0} k_c$$

is the Coulomb self energy of the particle lpha, infinite unless we introduce a cut-off k_c . Physically, k_c should be the momentum scale at which our non-relativistic treatment breaks down, so $k_c \approx \frac{1}{\lambda_{\rm Compton}}$, with $\lambda_{\rm Compton} = \frac{\hbar}{m_e c}$.

The second term is the Coulomb interaction between particles.

Finally we obtain the Hamiltonian:

$$H = \sum_{lpha} rac{1}{2} m_{lpha}^2 \dot{ec{r}}_{lpha}^2 + V_{
m Coulomb} + H_{
m trans}$$

Total Momentum:

Let's use again $ec E oec E_\perp+ec E_{||}$ in the definition of momentum, and find the parts:

$$ec{P}_{
m long} = \epsilon_0 \int {
m d}^3 r ec{E}_{||}(ec{r}) imes ec{B}(ec{r}) = \epsilon_0 \int {
m d}^3 k \, ec{\mathcal{E}}_{||}^*(ec{k}) imes ec{\mathcal{B}}(ec{k})$$

$$ec{P}_{
m trans} = \epsilon_0 \int {
m d}^3 r ec{E}_\perp(ec{r}) imes ec{B}(ec{r}) = \epsilon_0 \int {
m d}^3 k \, ec{\mathcal{E}}_\perp^*(ec{k}) imes ec{\mathcal{B}}(ec{k})$$

We plug in:
$$ec{\cal E}_{||} = -rac{i}{\epsilon_0}
ho(ec{k}) rac{ec{k}}{k^2}; \qquad ec{\cal B} = i ec{k} imes ec{\cal A}$$

$$egin{aligned} ec{P}_{
m long} &= \int \mathrm{d}^3 k \,
ho^* rac{ec{k}}{k^2} imes (i ec{k} imes ec{\mathcal{A}}) \ &= \int \mathrm{d}^3 k
ho^* [ec{\mathcal{A}} - ec{\kappa} (ec{\kappa} \cdot ec{\mathcal{A}})] \ &= \int \mathrm{d}^3 k \,
ho^* ec{\mathcal{A}}_{\perp} = \int \mathrm{d}^3 r \,
ho ec{A}_{\perp} = \sum_{lpha} q_lpha ec{A}_{\perp} (ec{r}_lpha) \end{aligned}$$

Independent of gauge, as $ec{A}_{\perp}$ is independent of gauge.

total momentum:

$$ec{P} = \sum_{lpha} \left(m_lpha \dot{ec{r}}_lpha + q_lpha ec{A}_ot (ec{r}_lpha)
ight) + ec{P}_{
m trans}$$

The three terms are 1) the mechanical momentum, 2) the longitudinal field momentum and 3) the transverse field momentum.

$$ec{P} = \sum_{lpha} ec{p}_{lpha} + ec{P}_{
m trans}$$

$$ec{p}_lpha = m_lpha \dot{ec{r}}_lpha + q_lpha ec{A}_ot (ec{r}_lpha)$$

In Coulomb gauge, \vec{p}_{lpha} is the canonical (or generalized) momentum to the coordinate \vec{r}_{α} of particle α .

Total energy:

$$H = \sum_{lpha} rac{1}{2m_lpha} \left[ec{p}_lpha - q_lpha ec{A}_ot (ec{r}_lpha)
ight]^2 + V_{
m Coulomb} + H_{
m trans}$$

This is precisely the Hamiltonian of the system in the coulomb gauge!

Total angular momentum:

$$ec{J} = \sum_lpha ec{r}_lpha imes ec{p}_lpha + \epsilon_0 \int \mathrm{d}^3 r \, ec{r} imes (ec{E}_ot imes ec{B})$$

2.4 Normal Variables

Maxwell's equations for transverse fields:

(M3):
$$\dot{ec{\mathcal{B}}} = -iec{k} imesec{\mathcal{E}}_{\perp}$$

$$\Rightarrow oxedsymbol{ec{k} imes \dot{ec{\mathcal{B}}} = ik^2 ec{\mathcal{E}}_{\perp}}$$

$$\Rightarrow egin{bmatrix} ec{k} imes \dot{ec{\mathcal{B}}} = ik^2 ec{\mathcal{E}}_ot \end{bmatrix}$$
 (M4): $egin{bmatrix} \dot{ec{\mathcal{E}}}_ot = ic^2 ec{k} imes ec{\mathcal{B}} - rac{1}{\epsilon_0} ec{j}_ot (ec{k},t) \end{bmatrix}$

These are two coupled equations for $\vec{\mathcal{E}}_{\perp}$, $\vec{k} imes \vec{\mathcal{B}}$, that are <u>local</u> in reciprocal space.

Using
$$ec{\mathcal{B}}=iec{k} imesec{\mathcal{A}}=iec{k} imesec{\mathcal{A}}_{\perp}$$
 we have

$$ec{k} imesec{\mathcal{B}}=-ik^2ec{\mathcal{A}}_{\perp}$$

Therefore we have, with $\omega=ck$

$$egin{aligned} \dot{ec{\mathcal{A}}}_{\perp} &= -ec{\mathcal{E}}_{\perp} \ \dot{ec{\mathcal{E}}}_{\perp} &= \omega^2 ec{\mathcal{A}}_{\perp} - rac{1}{\epsilon_0} ec{j}_{\perp} (ec{k},t) \end{aligned}$$

This is two coupled differential equations, just like

$$egin{aligned} \dot{ec{x}} &= rac{ec{p}}{m} \ rac{\dot{ec{p}}}{m} &= -\omega_0^2 ec{x} + rac{q}{m} ec{E}(t) \end{aligned}$$

So we read off the correspondence $ec{x}\equivec{\mathcal{A}}_{\perp}$ and $rac{ec{p}}{m}\equiv-ec{\mathcal{E}}_{\perp}.$

The goal now is to uncouple these equations, at least for the free field, $ec{j}_{\perp}=0.$

Normal variables for the position / momentum case:

$$egin{aligned} lpha &= \mathcal{N}(x+irac{p}{m\omega_0}) \ eta &= \mathcal{N}(x-irac{p}{m\omega_0}) = lpha^* \end{aligned}$$

equations of motion:
$$\dot{lpha}=\mathcal{N}(\dot{x}+irac{\dot{p}}{m\omega_0})=\mathcal{N}(rac{p}{m}-i\omega_0x+irac{q}{m\omega_0}E)=-i\omega_0\alpha+irac{q\mathcal{N}}{m\omega_0}E$$

$$\dot{lpha}=-i\omega_0lpha+irac{q\mathcal{N}}{m\omega_0}E$$

quantization:

$$\alpha \rightarrow a$$

$$eta
ightarrow a^\dagger$$

$$\left[a,a^{\dagger}
ight]=-\mathcal{N}^{2}rac{i}{m\omega_{0}}2\left[x,p
ight]=\mathcal{N}^{2}rac{2\hbar}{m\omega_{0}}\stackrel{!}{=}1$$

Therefore
$$\mathcal{N}=\sqrt{rac{m\omega_0}{2\hbar}}=rac{1}{\sqrt{2}}rac{1}{a_{
m h.o.}}$$

Using our correspondence $ec x\equiv ec{\cal A}_\perp$ and $rac{ec p}{m}\equiv -ec{\cal E}_\perp$ we find

$$egin{aligned} ec{lpha}(ec{k},t) &= \mathcal{N}(k) \left(ec{\mathcal{A}}_{\perp} - i rac{ec{\mathcal{E}}_{\perp}}{\omega_0}
ight) \ ec{eta}(ec{k},t) &= \mathcal{N}(k) \left(ec{\mathcal{A}}_{\perp} + i rac{ec{\mathcal{E}}_{\perp}}{\omega_0}
ight) \end{aligned}$$

In terms of $ec{\mathcal{B}}=iec{k} imesec{\mathcal{A}}_{\perp}\Rightarrowec{\mathcal{A}}_{\perp}=irac{ec{k}}{k^2} imesec{\mathcal{B}}=rac{i}{k}ec{\kappa} imesec{\mathcal{B}}.$

$$ec{lpha}(ec{k},t) = rac{i}{ck} \mathcal{N}(k) \left(c ec{\kappa} imes ec{\mathcal{B}} - ec{\mathcal{E}}_{\perp}
ight) \ ec{eta}(ec{k},t) = rac{i}{ck} \mathcal{N}(k) \left(c ec{\kappa} imes ec{\mathcal{B}} + ec{\mathcal{E}}_{\perp}
ight)$$

Note: $ec{E}_{\perp}$ is real, so $ec{\mathcal{E}}^*(ec{k},t)=ec{\mathcal{E}}(-ec{k},t)$ and thus

$$\alpha^{*}(\vec{k},t) = -\frac{i}{ck} \mathcal{N}(k) \left(c\vec{\kappa} \times \vec{\mathcal{B}}^{*}(\vec{k}) - \vec{\mathcal{E}}_{\perp}^{*}(\vec{k}) \right)$$

$$= -\frac{i}{ck} \mathcal{N}(k) \left(c\vec{\kappa} \times \vec{\mathcal{B}}(-\vec{k}) - \vec{\mathcal{E}}_{\perp}(-\vec{k}) \right)$$

$$= -\frac{i}{ck} \mathcal{N}(k) \left(-c(-\vec{\kappa}) \times \vec{\mathcal{B}}(-\vec{k}) - \vec{\mathcal{E}}_{\perp}(-\vec{k}) \right)$$

$$= +\vec{\beta}(-\vec{k},t)$$
(2)

So $\vec{eta}(\vec{k},t)$ is not independent of $\vec{lpha}(\vec{k},t).$

Therefore knowing $\vec{lpha}(\vec{k},t) \forall \vec{k}$ is fully equivalent to knowing $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}$.

• Time evolution:

$$\dot{ec{lpha}}(ec{k},t)+i\omega_0ec{lpha}(ec{k},t)=rac{i\mathcal{N}(k)}{\epsilon_0\omega}ec{j}_{\perp}(ec{k},t)$$

This equation is strictly equivalent to Maxwell's equations!

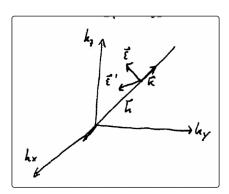
$$egin{aligned} ec{\mathcal{E}}_{\perp}(ec{k},t) &= rac{i}{2}rac{\omega_0}{\mathcal{N}(k)}\left(ec{lpha}(ec{k},t) - ec{lpha}^*(-ec{k},t)
ight) \ ec{\mathcal{B}}(ec{k},t) &= rac{i}{2}rac{k}{\mathcal{N}(k)}ec{\kappa} imes\left(ec{lpha}(ec{k},t) + ec{lpha}^*(ec{k},t)
ight) \end{aligned}$$

So what we have is equivalent to a driven harmonic oscillator with eigenfrequency $\omega=ck$, driven by a source term proportional to $\vec{j}_{\perp}(\vec{k},t)$.

- Comments:
- $ec{j}_{\perp}=0\Rightarrowec{lpha}(ec{k},t)=lpha(ec{k},0)e^{-i\omega_0t}$ free evolution \Rightarrow the $ec{lpha}(ec{k},t)$ are "normal variables"
- If $\vec{j}_{\perp}(\vec{k},t)$ is from an external source (i.e. independent of the $\vec{\alpha}(\vec{k},t)$), the $\vec{\alpha}(\vec{k},t)$ at the various \vec{k} still evolve independently from each other.

If $\vec{j}_\perp(\vec{k},t)$ are particles interacting with the field, then \vec{j}_\perp depends on $\vec{\alpha}$, so generally the various \vec{j}_\perp . We thus need to solve the coupled problem: Maxwell and Lorentz equation.

• Notation: $\vec{\alpha}$ is (like $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}$) transverse. So for each \vec{k} , one may expand $\vec{\alpha}$ into two unit vectors $\vec{\epsilon}$ and $\vec{\epsilon}'$, normal to each other and both located in the plane normal to $\vec{\kappa}$.



$$\vec{\epsilon} \cdot \vec{\epsilon} = \vec{\epsilon}' \cdot \vec{\epsilon}' = \vec{\kappa} \cdot \vec{\kappa} = 1$$

$$\vec{\epsilon} \cdot \vec{\epsilon}' = \vec{\epsilon} \cdot \vec{\kappa} = \vec{\epsilon}' \cdot \vec{\kappa} = 0$$

$$egin{aligned} ec{lpha}(ec{k},t) &= ec{\epsilon}lpha_\epsilon(ec{k},t) + ec{\epsilon}'lpha_{\epsilon'}(ec{k},t) \ \Rightarrow &= \sum_\epsilon ec{\epsilon}lpha_\epsilon(ec{k},t) \end{aligned}$$

with $lpha_\epsilon(ec k,t)=ec\epsilon\cdoteclpha(ec k,t)$ the component of eclpha along $ec\epsilon$.

The $\left\{ lpha_{\epsilon}(ec{k},t)
ight\}$ form a complete set of independent variables

$$\dot{lpha}_\epsilon + i\omegalpha_\epsilon = rac{i\mathcal{N}(k)}{\epsilon_0\omega}ec\epsilon\cdotec j$$

ullet The energy can be expressed in terms of the $lpha_\epsilon$ as

$$egin{aligned} H_{ ext{trans}} &= rac{\epsilon_0}{2} \int \mathrm{d}^3 r \left(ec{E}_{\perp}^2(ec{r}) + c^2 ec{B}^2(ec{r})
ight) \ &= rac{\epsilon_0}{2} \int \mathrm{d}^3 k \left(ec{\mathcal{E}}_{\perp}^*(ec{k}) ec{\mathcal{E}}_{\perp}(ec{k}) + c^2 ec{\mathcal{B}}^*(ec{k}) ec{\mathcal{B}}(ec{k})
ight) \ &= \epsilon_0 \int \mathrm{d}^3 k rac{\omega_0^2}{4 \mathcal{N}^2(k)} \left(ec{lpha}^*(ec{k}) \cdot ec{lpha}(ec{k}) + ec{lpha}(-ec{k}) \cdot ec{lpha}^*(-ec{k})
ight) \ &\stackrel{!}{=} \int \mathrm{d}^3 k \sum_{\epsilon} rac{\hbar \omega}{2} \left(lpha_{\epsilon}^*(ec{k},t) lpha_{\epsilon}(ec{k},t) + lpha_{\epsilon}(ec{k},t) lpha_{\epsilon}^*(ec{k},t)
ight) \end{aligned}$$

We choose therefore $\mathcal{N}^2=rac{\epsilon_0\omega^2}{2\hbar\omega}=rac{\epsilon_0\omega}{2\hbar}$ or

$$\mathcal{N}=\sqrt{rac{\epsilon_0\omega}{2\hbar}}$$

This will also ensure that $\left[a(ec{k}),a^{\dagger}(ec{k})
ight]=\delta(ec{k}-ec{k}')$

Momentum:

$$ec{P}_{ ext{trans}} = \int ext{d}^3 k \sum_{\epsilon} rac{\hbar ec{k}}{2} \left(lpha_{\epsilon}^* lpha_{\epsilon} + lpha_{\epsilon} lpha_{\epsilon}^*
ight)$$

Fields:

$$ec{E}_{\perp}(ec{r},t) = i \int \mathrm{d}^3k \sum_{\epsilon} {\cal E}_{\omega} \left(lpha_{\epsilon}(ec{k},t) ec{\epsilon} e^{iec{k}\cdotec{r}} - lpha_{\epsilon}^*(ec{k},t) ec{\epsilon} e^{-iec{k}\cdotec{r}}
ight)$$

$$ec{B}(ec{r},t) = i \int \mathrm{d}^3k \sum_{\epsilon} \mathcal{B}_{\omega} \left(lpha_{\epsilon}(ec{k},t) ec{\kappa} imes ec{\epsilon} \, e^{iec{k}\cdotec{r}} - lpha_{\epsilon}^*(ec{k},t) ec{\kappa} imes ec{\epsilon} \, e^{-iec{k}\cdotec{r}}
ight)$$

with
$${\cal E}_\omega=\sqrt{rac{\hbar\omega}{2\epsilon_0(2\pi)^3}}$$
 and ${\cal B}_\omega=rac{{\cal E}_\omega}{c}.$

For free fields $ec{j}_{\perp}=ec{0}$: $lpha_{\epsilon}(ec{k},t)=lpha_{\epsilon}(ec{k})e^{-i\omega t}$, so

$$ec{E}_{\perp}(ec{r},t) = i \int \mathrm{d}^3k \sum_{\epsilon} {\cal E}_{\omega} lpha_{\epsilon}(ec{k}) ec{\epsilon} e^{i(ec{k}\cdotec{r}-\omega t)} + \mathrm{c.c.}$$

Notation:

$$ec{E}_{ot}^{+}(ec{r},t)=i\int\mathrm{d}^{3}k\sum_{\epsilon}\mathcal{E}_{\omega}lpha_{\epsilon}(ec{k})ec{\epsilon}\,e^{iec{k}\cdotec{r}}$$

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