MA439: Functional Analysis
Tychonoff Spaces: Exercises 2.1 - 2.6, Ben Mathes

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Due: Wed, Sep 2, 2020

Exercise 1 (2.1)**.** *Let d denote the Euclidean metric on $\mathbb{R}^3$. Prove that d actually is a metric.*

Proof. Let $u, v \in \mathbb{R}^3$ be given.

$$d(u,v) = \sqrt{\sum_{i=1}^{3}(u_i - v_i)^2}.$$

It is clear that $d(u,v) \geq 0$ and $d(u,v) = 0$ if and only if $u_i = v_i$ for $i = 1,2,3$, i.e., $u = v$. Next, because $(u_i - v_i)^2 = (v_i - u_i)^2$ for any pair of numbers u_i, v_i, $d(u,v) = d(v,u)$. Finally, consider $w \in \mathbb{R}^3$:

$$\begin{aligned}
(d(u,v) + d(v,w))^2 &= \left(\sqrt{\sum_{i=1}^{3}(u_i - v_i)^2} + \sqrt{\sum_{i=1}^{3}(v_i - w_i)^2}\right)^2 \\
&= \sum_{i=1}^{3}(u_i - v_i)^2 + 2\sqrt{\sum_{i=1}^{3}(u_i - v_i)^2 \cdot \sum_{i=1}^{3}(u_i - v_i)^2} + \sum_{i=1}^{3}(v_i - w_i)^2 \\
&\geq \sum_{i=1}^{3}\left[(u_i - v_i)^2 + 2(u_i - v_i)(v_i - w_i) + (v_i - w_i)^2\right], \quad \text{C-S inequality} \\
&= \sum_{i=1}^{3}(u_i - w_i)^2 \\
&= (d(u,w))^2.
\end{aligned}$$

Since $d(u,v) \geq 0$ for all u, v, we can take the square root on both sides and obtain the desired triangle inequality. Thus, d is a bona-fide metric on $\mathbb{R}^3$. $\square$

Exercise 2 (2.2)**.** *Let $\mathcal{X} = \{a,b,c\}$, and suppose d is a symmetric function with $d(a,b) = 1$, $d(b,c) = 1$, $d(a,c) = \sqrt{2}$, and $d(a,a) = d(b,b) = d(c,c) = 0$. Show that d is a metric, and find a subset of $\mathcal{B}(\mathcal{X})$ that is isometric to $(\mathcal{X}, d)$.*

Proof. By definition, $d(u,v) \geq 0$ for all $u, v \in \mathcal{X}$ and $d(u,v) = 0 \iff u = v$. Next, since d is a symmetric function, $d(u,v) = d(v,u)$ for any $u, v \in \mathcal{X}$. Finally, consider $u, v, w \in \mathcal{X}$. If $u = v = w$ then $d(u,v) + d(v,w) = d(v,w) = 0$. Else, assume $u \neq v$, then $d(u,v) + d(v,w) \geq \sqrt{2} = \max\{d(u,v) : u, v \in \mathcal{X}\}$. Thus, the triangle inequality property is satisfied. Therefore, d is a metric on $\mathcal{X}$. Now, consider the set $\mathcal{F} = \{f_a, f_b, f_c\}$ where

$$f_a(x) = d(x,a) - d(x,a), \quad f_b(x) = d(x,b) - d(x,a), \quad f_c(x) = d(x,c) - d(x,a)$$

We want to show that $\mathcal{F}$ is isometrically isomorphic to $(\mathcal{X}, d)$. Since $d(x,y) < \infty$ for all $x, y \in \mathcal{X}$, we have that $\|f_x\|_\infty = \sup_{x'}|f_x(x')| = \sup_{x'}|d(x',x) - d(x',a)| < \infty$ for all $x \in \mathcal{X}$, i.e., f_x is bounded for all $x \in \mathcal{X}$. So, $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X})$. Further, the map $x \mapsto f_x$ is (clearly) bijective and is distance-preserving:

$$\begin{aligned}\|f_{x_1} - f_{x_2}\|_\infty &= \sup_{x'} \big|(d(x', x_1) - d(x', a)) - (d(x', x_2) - d(x', a))\big| \\ &= \sup_{x'} \big|d(x', x_1) - d(x', x_2)\big| \\ &= d(x_1, x_2)\end{aligned}$$

because

$$\begin{aligned}\sup_{x'} \big|d(x', a) - d(x', b)\big| &= 1 = d(a, b) \\ \sup_{x'} \big|d(x', b) - d(x', c)\big| &= 1 = d(b, c) \\ \sup_{x'} \big|d(x', c) - d(x', a)\big| &= \sqrt{2} = d(c, a).\end{aligned}$$

Thus, $\mathcal{F}$ is isometrically isomorphic to $(\mathcal{X}, d)$. □

Exercise 3 (2.3)**.** *If d is obtained from a norm via $d(s,t) = \|s - t\|$, prove that d is a metric.*

Proof. Let s, t, u be given. First, $d(s,t) = \|s - t\| \geq 0$, and $d(s,t) = \|s - t\| = 0$ if and only if $s = t$. Next, $d(s,t) = \|s - t\| = \|t - s\| = d(t,s)$. Finally, $d(s,t) + d(t,u) = \|s - t\| + \|t - u\| \geq \|s - u\| = d(s,u)$. Thus, d is a metric. □

Exercise 4 (2.4)**.** *On $\mathbb{R}^2$ define a function $\|\cdot\|_3 : \mathbb{R}^3 \to \mathbb{R}$ by $\|(x,y)\|_3 = (|x|^3 + |y|^3)^{1/3}$. Prove this is a norm.*

Proof. Before showing $\|\cdot\|_3$ is a norm, we treat some special cases of known inequalities[1]:

Lemma 0.1 (Young's Inequality)**.** *For positive numbers p, q such that $1/p + 1/q = 1$ and $a, b \geq 0$:*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Proof. If $a = 0$ or $b = 0$ then the result is clear. Thus, assume that $a, b \neq 0$, we have

$$\begin{aligned}ab &= \exp\left(\ln a + \ln b\right) \\ &= \exp\left(\frac{1}{p}\ln a^p + \frac{1}{q}\ln b^q\right) \\ &\leq \frac{1}{p}e^{\ln a^p} + \frac{1}{q}e^{\ln b^q} \\ &= \frac{a^p}{p} + \frac{b^q}{q}\end{aligned}$$

where the last inequality follows because the exponential function is convex and $1/q + 1/p = 1$. //

Lemma 0.2 (Hölder's Inequality)**.** *For positive numbers p, q such that $1/p + 1/q = 1$, and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$*

$$|\mathbf{a} \cdot \mathbf{b}| = \sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=0}^{n} |b_k|^q\right)^{1/q} = \|\mathbf{a}\|_p \|\mathbf{b}\|_q.$$

[1] Stein & Shakarchi, *Functional Analysis, Princeton Lectures in Analysis IV*, Princeton University Press 2011.

Proof. If $\mathbf{a} = 0$ or $\mathbf{b} = 0$ then the result follows directly. Thus, assume that $\mathbf{a} \neq 0, \mathbf{b} \neq 0$. Let $\mathbf{u} = \mathbf{a}/\|\mathbf{a}\|_p$ and $\mathbf{v} = \mathbf{b}/\|\mathbf{b}\|_q$, so that $\|\mathbf{u}\|_p = \|\mathbf{v}\|_q = 1$. It follows from Young's inequality that for all $n \in \mathbb{N}_+$,

$$|u_n v_n| \leq \frac{|u_n|^p}{p} + \frac{|v_n|^q}{q}.$$

Thus, from the triangle inequality we have

$$|\mathbf{u} \cdot \mathbf{v}| \leq \frac{1}{p}\sum_{k=1}^{n} |u_k|^p + \frac{1}{q}\sum_{k=1}^{n} |v_k|^q = \frac{1}{p}\|\mathbf{u}\|_p + \frac{1}{q}\|\mathbf{v}\|_p = \frac{1}{p} + \frac{1}{q} = 1.$$

Thus,

$$|\mathbf{a} \cdot \mathbf{b}| \leq \|\mathbf{a}\|_p \|\mathbf{b}\|_q$$

as desired. //

Lemma 0.3 (Minkowski's Inequality for sums)**.** *Let $p > 1$ and $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$,*

$$\|\mathbf{a} + \mathbf{b}\|_p \leq \|\mathbf{a}\|_p + \|\mathbf{b}\|_p$$

Proof. If $\mathbf{a} + \mathbf{b} = 0$ then the result follows directly. Thus, assume that $\mathbf{a} + \mathbf{b} \neq 0$. Let $q = p/(p-1)$ so that $1/p + 1/q = 1$. Then,

$$\begin{aligned}
(\|\mathbf{a} + \mathbf{b}\|_p)^p &= \sum_{k=1}^{n} |a_k + b_k|^p = \sum_{k=1}^{n} |a_k + b_k||a_k + b_k|^{p-1} \\
&\leq \sum_{k=1}^{n} (|a_k| + |b_k|)|a_k + b_k|^{p-1} \\
&= \sum_{k=1}^{n} |a_k||a_k + b_k|^{p-1} + \sum_{k=1}^{n} |b_k||a_k + b_k|^{p-1} \\
\text{(Hölder's ineq.)} \quad &\leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |a_k + b_k|^{q(p-1)}\right)^{1/q} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |a_k + b_k|^{q(p-1)}\right)^{1/q} \\
&= \left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/q} \left(\|\mathbf{a}\|_p + \|\mathbf{b}\|_p\right) \\
&= (\|\mathbf{a} + \mathbf{b}\|_p)^{p/q} \left(\|\mathbf{a}\|_p + \|\mathbf{b}\|_p\right).
\end{aligned}$$

Since $p - p/q = p(1 - 1/q) = p/p = 1$, we have

$$\|\mathbf{a} + \mathbf{b}\|_p \leq \|\mathbf{a}\|_p + \|\mathbf{b}\|_p.$$

as desired. //

Now are ready to prove the statement of Exercise 2.4. Let $(x_1, x_2) \in \mathbb{R}^2$ be given. Since $|x| \geq 0$ for all $x \in \mathbb{R}$ with equality occurring if and only if $x = 0$, $\|(x_1, x_2)\|_3 = (|x_1|^3 + |x_2|^3)^{1/3} \geq 0$ for all $x_1, x_2 \in \mathbb{R}$ and $\|(x_1, x_2)\|_3 = 0$ if and only if $(x_1, x_2) = 0$. Next, let $\alpha \in \mathbb{R}$. We have $\|(x_1, x_2)\|_3 = (|\alpha x_1|^3 + |\alpha x_2|^3)^{1/3} = |\alpha|(|x_1|^3 + |x_2|^3)^{1/3} = |\alpha|\|(x_1, x_2)\|_3$. Finally, let $x, y \in R^2$. By Minkowski's inequality for sums,

$$\|x + y\|_3 \leq \|x\|_3 + \|y\|_3$$

Thus, $\|\cdot\|_3$ is a norm. □

Exercise 5 (2.5)**.** *Provide the details in the proof of Theorem 2:*

Theorem 0.4. *Every metric space $(\mathcal{X}, d)$ is isometrically isomorphic to a subset of $\mathcal{B}(\mathcal{X})$.*

Proof. Fix an element $x_0 \in \mathcal{X}$ and for each $x \in \mathcal{X}$ define a real valued function f_x by

$$f_x(x') = d(x', x) - d(x', x_0).$$

Let $\mathcal{F}$ denote the collection $\{f_x : x \in \mathcal{X}\}$. We first verify that $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X})$. To this end, we verify that f_x is bounded:

$$\begin{aligned}
\|f_x\|_\infty &= \sup_{x'} \big|f_x(x')\big| \\
&= \sup_{x'} \big|d(x', x) - d(x', x_0)\big| \\
&\le \sup_{x'} |d(x, x_0)|, \quad \text{triangle inequality, since } d \text{ is a metric} \\
&= d(x, x_0) \\
&< \infty.
\end{aligned}$$

Thus, $\mathcal{F} \subseteq \mathcal{B}(\mathcal{X})$. Next, we verify that the map $x \mapsto f_x$ is a distance-preserving bijection. It is clear that the map is a bijection. Let $x_1, x_2 \in \mathcal{X}$ be given, by the previous argument, we find

$$\begin{aligned}
\|f_{x_1} - f_{x_2}\|_\infty &= \sup_{x'} \big|(d(x', x_1) - d(x', x_0)) - (d(x', x_2) - d(x', x_0))\big| \\
&= \sup_{x'} \big|d(x', x_1) - d(x', x_2)\big| \\
&= d(x_1, x_2), \quad \text{triangle inequality, and maximum attained at } x' = x_1 \text{ or } x_2
\end{aligned}$$

which implies that the map $x \mapsto f_x$ is distance-preserving, as desired. Therefore, $\mathcal{F}$ is isomorphically isometric to $(\mathcal{X}, d)$. □

Exercise 6 (2.6)**.** *Assume that $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfies the first two conditions for a metric, but does not satisfy the triangle inequality. Define a function $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ by*

$$d(x, y) = \inf\left\{\sum_{i=1}^{n} \rho(x_i, x_{i-1}) : \{x_0, x_1, \ldots, x_n\} \subset \mathcal{X}, x_0 = x, x_n = y\right\}.$$

Show that d is a metric.

Proof. Since $\rho(x, y)$ is a symmetric function, $d(x, y)$ is also a symmetric function, by construction. Next, since $\rho(x, y) \ge 0$ for all $x, y \in \mathcal{X}$, $d(x, y) \ge 0$ for all $x, y \in \mathcal{X}$. Now, suppose $x = y = 0$, because ρ is nonnegative, we have

$$d(0, 0) = \inf\left\{\sum_{i=1}^{n} \rho(x_i, x_{i-1}) : \{0, x_1, \ldots, x_{n-1}, 0\} \subset \mathcal{X}\right\} = 0$$

occurring when $x_0 = x_1 = \cdots = x_n = x = y = 0$. Conversely, if $d(x, y) = 0$ then because ρ is nonnegative and $\rho(x_i, x_{i-1}) = 0$ if and only if $x_i = x_{i-1} = 0$, we must have that $x_0 = x_1 = \cdots = x_n = 0$, or $x = y = 0$. Finally, to verify that d satisfies the triangle inequality, let $x, y, z \in \mathcal{X}$ be given. Fix $\{x, x_1', x_2', \dots, x_{n-1}', y\} \subset \mathcal{X}$ and $\{y, y_1', y_2', \dots, y_{n-1}', z\} \subset \mathcal{X}$. Because $\rho(a,b) \geq 0$ for all $a, b \in \mathcal{X}$, we have that

$$\begin{aligned}
d(x,y) + d(y,z) &= \inf\left\{\sum_{i=1}^{n} \rho(x_i, x_{i-1}) : \{x_0, \dots, x_n\} \subset \mathcal{X}, x_0 = x, x_n = y\right\} \\
&\qquad + \inf\left\{\sum_{i=1}^{n} \rho(x_i, x_{i-1}) : \{x_0, \dots, x_n\} \subset \mathcal{X}, x_0 = y, x_n = z\right\} \\
&\geq \left[\rho(x, x_1') + \rho(x_1', x_2') + \cdots + \rho(x_{n-1}', y)\right] + \left[\rho(y, y_1') + \rho(y_1', y_2') + \cdots + \rho(y_{n-1}', z)\right] \\
&\geq \rho(x, x_1') + \rho(x_1', x_2') + \cdots + \rho(x_{n-2}', x_{n-1}') + \rho(y_{n-1}', z) \\
&\geq \inf\left\{\sum_{i=1}^{n} \rho(x_i, x_{i-1}) : \{x_0, x_1, \dots, x_n\} \subset \mathcal{X}, x_0 = x, x_n = z\right\} \\
&= d(x,z).
\end{aligned}$$

So, d satisfies the triangle inequality as desired. Therefore, d is a bona-fide metric on $\mathcal{X}$. $\square$