Introductory Topics in Complex Analysis

- A Quick Guide -

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Let $w \in C^0([a,b],\mathbb{C})$ then

$$\left|\int_a^b w(t)\,dt\right| \leq \int_a^b |w(t)|\,dt. \tag{1}$$

Proof. This is essentially the triangle inequality.
add proof here □

3 Bound on Modulus of Contour Integrals

Let C be a contour and let $f : \mathrm{Dom}(f) \to \mathbb{C}$ be piecewise continuous on C. If $|f(z)| \leq M \forall z \in \mathbb{C}$, then

$$\left|\int_C f(z)\,dz\right| \leq M\mathcal{L}(C) \tag{2}$$

where $\mathcal{L}(C)$ is the arclength of C.

Proof. **add proof here** □

4 TFAE

Let f be continuous on $\mathcal{D}$. The following are equivalent (TFAE):

1. $f(z)$ has an antiderivative $F(z)$ throughout $\mathcal{D}$.
2. Given any $z_1, z_2 \in \mathcal{D}$ and contours $C_1, C_2 \subset \mathcal{D}$ both going from z_1 to z_2,

   $$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz. \tag{3}$$

   In other words, the integral is independent of contour.
3. Given any close contour $C \subset \mathcal{D}$,

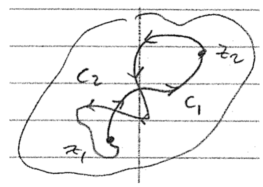
   $$\int_C f(z)\,dz = 0. \tag{4}$$

In the case that one (and hence every) condition is satisfied, we have that for any $z_1, z_2 \in \mathcal{D}$ and contour C from $z_1 \to z_2 \subset \mathcal{D}$,

$$\int_C f(z)\,dz = F(z_2) - F(z_1) \tag{5}$$

where F's existence is guaranteed by (1).

Proof. ($2 \iff 3$) Suppose (2) is valid and let C be a closed contour in $\mathcal{D}$. Then C contains 2 points z_1, z_2 and we can divide C into 2 pieces $C_1 + C_2$ where $C_1 : z_1 \to z_2$ and $C_2 : z_2 \to z_1$.



Note that by reversing the direction of C_2, we ave both C_1 and $-C_2$ go from z_1 to z_2 and stay inside of $\mathcal{D}$. Thus,

$$\oint_C f\,dz = \int_{C_1} f\,dz - \int_{-C_2} f\,dz. \tag{6}$$

By (2), we have that

$$\int_{C_1} f\,dz = \int_{C_2} f\,dz. \tag{7}$$

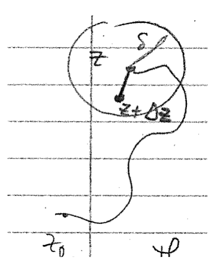
This means

$$\oint_C f(z)\,dz = 0. \tag{8}$$

So (2) $\implies$ (3).

Now, assume (3) is true and let $z_0, z_1 \in \mathcal{D}$. Let $C_1, C_2 \subset \mathcal{D}$ be contours going from z_0 to z_1. We observe that $C := C_1 - C_2$ is a s.c.c. in $\mathcal{D}$. So by (3),

$$0 = \oint_C f\,dz = \int_{C_1 - C_2} f\,dz = \int_{C_1} f\,dz - \int_{C_2} f\,dz. \tag{9}$$



($1 \iff 2$) Assume (1) is true. Let $z_0, z_1 \in \mathcal{D}$ and let C be a contour from $z_0 \to z_1$, i.e., $C : z(t) \in C([a,b], \mathbb{C})$ piecewise differentiable, $z(a) = z_0$ and $z(b) = z_1$. As F is an antiderivative of f, for all $t \in [a,b]$ for which $z'(t)$ exists the chain rule gives

$$\frac{d}{dt} F(z(t)) = F'(z(t))z'(t) = f(z(t))z'(t). \tag{10}$$

So,

$$\oint_C f\,dz = \sum_{k=1}^{n} \int_{a_k}^{b_k} f(z(t))z'(t)\,dt = \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{d}{dt}F(z(t))\,dt \tag{11}$$

where a_k, b_k are points at which z fails to be differentiable, $a_1 = a, b_n = b$. By the fundamental theorem of calculus,

$$\begin{aligned}\oint_C f\,dz &= \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{d}{dt}F(z(t))\,dt \\ &= \sum_{k=1}^{n} F(z(b_k)) - F(z(a_k)) \\ &= F(b) - F(a) = F(z_1) - F(z_0).\end{aligned} \tag{12}$$

So, given any 2 contours $C_1, C_2 \in\subset \mathcal{D}$ from $z_0 \to z_1$, we have

$$\int_{C_1} f\,dz = F(z_1) - F(z_0) = \int_{C_2} f\,dz. \tag{13}$$

Now, assume (2) is true. We need to construct an antiderivative F. Let $z_0 \in \mathcal{D}$ and define $F : \mathcal{D} \to \mathbb{C}$ by

$$F(z) = \int_{C_z} f(w)\,dw \tag{14}$$

where C_z is a contour from $z_0 \to z_1$. Since $\mathcal{D}$ is a domain, it is a path connected, and so for each z, a path C_z exists. By (2) this is not dependent on the choice of contour C_z. So F is well-defined. We wish to show that $F(z)$ is differentiable and its derivative is f.

Let $z \in\subset \mathcal{D}$ and choose $\epsilon > 0$. Given th continuity of f, let δ be chosen so that

1. $$|f(w) - f(z)| < \frac{\epsilon}{2} \forall |w - z| < \delta \tag{15}$$

2. $\mathcal{B}_\delta(z) \subset \mathcal{D}$ (or $\mathcal{D}$ is open.)

Given a $\Delta z \in \mathbb{C}$ such that $\Delta z < \delta$, we consider a path $C_{z,\Delta z}$ defined by $w(t) = z + t\Delta z$, $t \in [0, 1]$. We have that $C_z + C_{z,\Delta z}$ is a contour in $\mathcal{D}$ from

$z_0 \to z + \Delta z$. Then,

$$\begin{aligned}
\frac{1}{\Delta z}\left(F(z+\Delta z) - F(z)\right) &= \frac{1}{\Delta z}\left(\int_{C_z + C_{z,\Delta z}} f(w)\,dw - \int_{C_z} f(w)\,dw\right) \\
&= \frac{1}{\Delta z}\int_{C_{z,\Delta z}} f(w)\,dw \\
&= \frac{1}{\Delta z}\int_0^1 f(z+t\Delta z)(z+t\Delta z)'\,dt \\
&= \int_0^1 f(z+t\Delta z)\,dt. \qquad (16)
\end{aligned}$$

So, for $|\Delta z| < \delta$,

$$\begin{aligned}
\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| &= \left|\int_0^1 f(z+t\Delta z)\,dt - f(z)\right| \\
&= \left|\int_0^1 \left[f(z+t\Delta z) - f(z)\right]\,dt\right| \\
&\le \int_0^1 |f(z+t\Delta z) - f(z)|\,dt \\
&\le \int_0^1 \frac{\epsilon}{2}\,dt \\
&\le \frac{\epsilon}{2} \\
&< \epsilon \qquad (17)
\end{aligned}$$

by choice of δ. So, we have shown that given $z \in \mathcal{D}$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| < \epsilon \qquad (18)$$

whenever $|\Delta z| < \delta$. So, F is differentiable at z and $F'(z) = f(z)$. □

5 Cauchy-Goursat Theorem

Suppose that C is a simple closed contour and f is analytic on the interior of C and all points of C then

$$\oint_C f(z)\,dz = 0. \qquad (19)$$

Proof. The proof involves slicing the interior of C into squares and partial squares. I won't try to reproduce it here. □

6 Simply-connected domain

A domain $\mathcal{D}$ is called simply-connected if every simple closed contour $C \subset \mathcal{D}$ contains only points of $\mathcal{D}$ and its interior, i.e., every simple closed contour is contractible to a point.

7 Multiply-connected domain

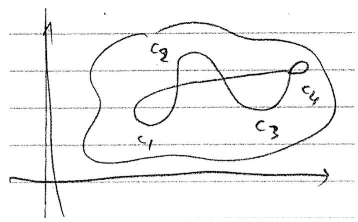
A multiply-connected domain $\mathcal{D}$ is a dmain which is not simply-connected. (very imaginative)

8 Cauchy-Goursat Theorem for simply-connected domain

Let $\mathcal{D}$ be a simply connected domain. f is analytic in $\mathcal{D}$. For all closed contour $C \subset \mathcal{D}$,

$$\oint_C f(z)\, dz = 0. \tag{20}$$

Proof. Notice that we C need not be simple. Consider the figure



Let C be a closed contour in $\mathcal{D}$ with a finite number of self-intersections. Given that C only has n interactions, we can split C into a finite number m of simple closed contour C_j. Also, given $\mathcal{D}$ is simply connected, the interior of each C_j lives in $\mathcal{D}$. By the previous theorem, we have

$$\oint_{C_j} f(z)\, dz = 0 \forall j = 1, 2, 3, \cdots \implies \oint_C f(z)\, dz = \oint_{\sum C_j} f(z)\, dz = 0. \tag{21}$$

□

9 Corollary to Cauchy-Goursat for simply-connected domain

If f is analytic on a simply connected domain in $\mathcal{D}$ then f has an antiderivative F everywhere in $\mathcal{D}$.

Proof. TFAE. □

10 Cauchy-Goursat Theorem for multiply-connected regions

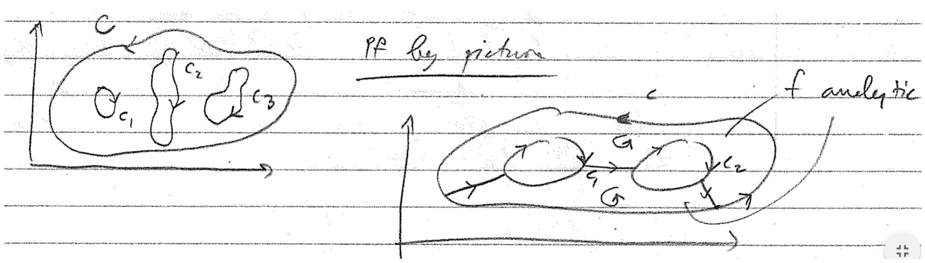
Suppose that

1. C is a s.c.c.(+).
2. C_j, $j = 1, 2, \ldots, n$ are s.c.c.(-), all disjoint and all live in the interior of C.

If f is analytic on $C, C_j \forall j$ and the region between C, C_j (enclosed by C but outside of C_j) then

$$\oint_C f(z)\,dz + \sum_{j=1}^{n} \oint_{C_j} f(z)\,dz = 0. \tag{22}$$

Proof. The proof follows from the this figure



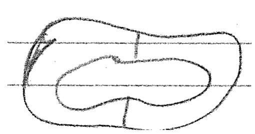
□

11 Principle of Path Deformation (Corollary to Cauchy-Goursat)

Let C_1 and C_2 be simple closed curves and C_2 encloses C_1. Both are (+) oriented. Then if f is analytic on the region between C_1, C_2 then

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz. \tag{23}$$

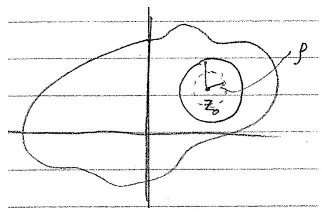
Proof. Consider the following suggestive figure: □



12 Cauchy's Integral Formula

Let C be a s.c.c.(+) and let f be analytic on C and its interior. If z_0 lives interior to C then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz. \tag{24}$$



Proof. Let $\delta < 1$ be small enough such that $|z - z_0| < \delta$ so that C encloses z. Since the quotient $f(z)/(z - z_0)$ is analytic in the region exterior to $\mathcal{B}_\delta(z_0)$ and interior to C, we have that

$$\oint_C \frac{f(z)}{z - z_0}\,dz = \oint_{C_\rho} \frac{f(z)}{z - z_0}\,dz \tag{25}$$

where $\rho < \delta$ and C_ρ is a (+) circle centered at z_0 of radius ρ. The equality is guaranteed by the principle of deformation of path.

Now, consider

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0} - f(z_0) \\
&= \frac{1}{2\pi i}\oint_{C_\rho} \frac{f(z)}{z - z_0} - \frac{f(z_0)}{2\pi i}\oint_{C_\rho} \frac{1}{z - z_0}\,dz \\
&= \frac{1}{2\pi i}\left(\oint_{C_\rho} \frac{f(z) - f(z_0)}{z - z_0}\,dz\right).
\end{aligned} \tag{26}$$

Given that $f(z)$ is continuous at z_0, $\forall \epsilon > 0, \exists \rho > 0$ s.t. $|f(z) - f(z_0)| < \epsilon$ whenever $|z - z_0| < 2\rho < \delta$. Since $|z - z_0| = \rho < 2\rho$ on C_ρ, we have

$$\left|\frac{f(z) - f(z_0)}{z - z_0}\right| = \frac{1}{\rho}|f(z) - f(z_0)| < \frac{\epsilon}{\rho} \text{ on } C_\rho. \tag{27}$$

So,

$$|\mathcal{E}| \leq \frac{1}{2\pi}\frac{\epsilon}{\rho}\mathcal{L}(C_\rho) = \epsilon. \tag{28}$$

So, given any $\epsilon > 0$, $|\mathcal{E}| \leq \epsilon$. This says that

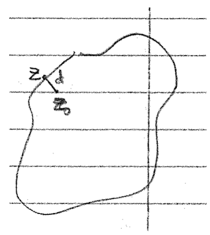
$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz = f(z_0). \tag{29}$$

$\square$

13 Cauchy's Integral Formula for First-Order Derivative

Let C s.c.c.(+) and let f be analytic on the interior of C and on C. Then if $z_0 \in \mathrm{int}(C)$ then

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz. \tag{30}$$



Proof. Let $M = \max |f(z)|$ where $z \in C$. Given $z_0 \in \mathrm{int}(C)$, let $d = \min |z - z_0| > 0$ where $z \in C$. Let $h = \Delta z$ is such that $|h| = |\Delta z| < d$. Using Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz. \tag{31}$$

Because $|h| < d$, $z_0 + h \in \mathrm{int}(C)$. So,

$$f(z_0+h) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-(z_0+h)}\, dz. \tag{32}$$

Now, observe that

$$\begin{aligned}
\mathcal{E} &= \frac{f(z_0+h) - f(z_0)}{h} - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz \\
&= \frac{1}{h}\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-(z_0+h)}\, dz - \frac{1}{h}\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0}\, dz - \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2}\, dz \\
&= \dots \\
&= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^2} \frac{h}{z-(z_0+h)}\, dz
\end{aligned} \tag{33}$$

for all $z \in \mathrm{int}(C), d \leq |z - z_0|$. So,

$$\frac{1}{|z-z_0|^2} \leq \frac{1}{d^2}. \tag{34}$$

Also, $0 \leq d - |h| \leq |z - (z_0 + h)| \forall |h| < d$. So for all $z \in C$, whenever $|h| < d$,

$$\left| \frac{f(z)}{(z-z_0)^2} \frac{h}{z-(z_0+h)} \right| \leq \frac{M|h|}{d^2(d-|h|)}. \tag{35}$$

So, whenever $|h| < d$, we have

$$|\mathcal{E}| \leq \frac{1}{2\pi} \frac{M|h|}{d^2(d - |h|)} \mathcal{L}(C) = \frac{M|h|}{2\pi d^2(d - |h|)} \mathcal{L}(C). \tag{36}$$

Let $\epsilon > 0$ be given and choose

$$\delta = \min \left[\frac{d}{2}, \frac{\pi d^3}{M\mathcal{L}(C)} \right] \tag{37}$$

then whenever $|h| < \delta \leq \frac{d}{2} < d$,

$$\frac{1}{d - |h|} \leq \frac{1}{d/2}. \tag{38}$$

With this,

$$\mathcal{E} \leq \frac{M|h|}{2\pi d^3/2} \mathcal{L}(C) < \frac{M\mathcal{L}(C)}{\pi d^3} \frac{\pi d^3 \epsilon}{M\mathcal{L}(C)} = \epsilon. \tag{39}$$

So,

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, dz. \tag{40}$$

□

14 Cauchy's Integral Formula for Higher-Order Derivatives

Let C be s.c.c.(+) and f analytic on C and its interior. Then $\forall z_0 \in \text{int}(C)$, and $n \in \mathbb{N}$, f is n-times differentiable at z_0 and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz. \tag{41}$$

15 Analyticity of Derivatives

If f is analytic at z_0 then f has derivatives of all orders which are also analytic at z_0.

Proof. We simply applying the preceding theorem. □

16 Analyticity of Derivatives on a Domain

If $\mathcal{D}$ is a domain and f is analytic on $\mathcal{D}$ then f has derivatives of all orders and each derivative is analytic on $\mathcal{D}$. This means f is infinitely differentiable on $\mathcal{D}$.

17 Infinite Differentiability

Let $f(z) = u(x, y) + iv(x, y)$ be analytic at $z_0 = (x_0, y_0)$. Then u, v have continuous partial derivatives of all orders at z_0. Further, if $f = u + iv$ is analytic on $\mathcal{D}$, then u, v are infinitely differentiable in $\mathcal{D}$, i.e., $u, v \in C^\infty(\mathcal{D})$.

Proof. The proof follows from Cauchy-Riemann theorem and equations. □

18 Hörmander's Theorem

If u is harmonic in a domain $\mathcal{D}$ then u is smooth $\iff u \in C^\infty(\mathcal{D})$.

Proof. If u is harmonic then u has a harmonic conjugate v. Then $f = u + iv$ is analytic, etc. □

19 Morera's Theorem

Let f be continuous on $\mathcal{D}$. If for all closed $C \subset \mathcal{D}$,

$$\oint_C f(z)\,dz = 0, \tag{42}$$

then f is analytic on $\mathcal{D}$.

Proof. The proof follows from TFAE. By TFAE, f has an antiderivative F throughout $\mathcal{D}$. But F is analytic because $f' = F$. This means F's derivatives are analytic throughout $\mathcal{D}$ as well. So, f is analytic throughout $\mathcal{D}$. □

20 Cauchy's Inequality

Let f be analytic on and inside a (+) circle C with center z_0 and radius R. Let $M_R = \max\left[|f(z)|\right], z \in C_R$. Then $\forall n \in \mathbb{N}$,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n! M_R}{R^n}. \tag{43}$$

Proof. This follows from Cauchy's integral formula and the triangle inequality:

$$\begin{aligned} \left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i}\oint_{C_R} \frac{f(z)}{(z - z_0)^{n+1}}\,dz\right| \\ &\leq \frac{n!}{2\pi}\frac{M_R}{R^{n+1}}(2\pi R) \\ &= \frac{n! M_R}{R^n}. \end{aligned} \tag{44}$$

□

21 Liouville's Theorem

If f is bounded and entire and f is constant.

Proof. Let $M \geq 0$ for which $|f(z)| \leq M \forall z \in \mathbb{C}$. Given any $z_0 \in \mathbb{C}$, f is analytic on every neighborhood of z_0 and so $\forall R > 0$,

$$|f'(z_0)| \leq \frac{1! M_R}{R} \tag{45}$$

where $M_R = \max |f(z)| \leq M$ where $z \in C_R(z_0)$. So, for any $z_0 \in \mathbb{C}$, $R > 0$,

$$|f'(z_0)| \leq \frac{M}{R}. \tag{46}$$

This shows $f'(z_0) = 0 \forall z_0 \in \mathbb{C}$. So, f is constant because $\mathbb{C}$ is a domain. $\square$

22 The Fundamental Theorem of Algebra

If $P(z)$ is a non constant polynomial, i.e.,

$$P(z) = a_0 + a_1 z^1 + \cdots + a_n z^n \tag{47}$$

where $a_n \neq 0, n = \deg(P)$, then $\exists z_0 \in \mathbb{C}$ at which $P(z_0) = 0$.

Proof. Let

$$w = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} \tag{48}$$

and note that

$$P(z) = (w + a_n) z^n. \tag{49}$$

We observe that z^k from $k \in \{1, 2, 3, \dots\}$ has $1/z^k \to 0$ has $z \to \infty$. So, given $\epsilon = |a_n|/2$, there exists $R > 0$ for which

$$|w| \leq \frac{|a_n|}{2} \forall |z| > R. \tag{50}$$

So, for $|z| > R$,

$$|w + a_n| \geq ||w| - |a_n|| = |a_n| - |w| \geq \frac{|a_n|}{2}. \tag{51}$$

So,

$$\left| \frac{1}{P(z)} \right| = \frac{1}{|w + a_n||z^n|} \leq \frac{2}{|a_n|} \frac{1}{|z^n|} \leq \frac{2}{|a_n|} \frac{1}{R^n} \tag{52}$$

where $|z| > R$. Now, suppose that $P(z) \neq 0 \forall z \in \mathbb{C}$ to get a contradiction. Since $P(z)$ is never vanishes, $f(z) = 1/P(z)$ is entire. Since, in particular, $f(z)$

is continuous, it is bounded on all closed bounded set. So, $\exists M > 0$ such that $|f(z)| \leq M \forall z, |z| \leq R$. So, by what we've just shown

$$\left|\frac{1}{P(z)}\right| \leq \max\left[M, \frac{2}{|a_n|R^n}\right]. \tag{53}$$

So, we have $f(z)$ is bounded and entire. By Liouville's theorem, $1/P(z)$ must be constant. This is a contradiction. □

23 Corollary to The Fundamental Theorem of Algebra

If $P(z)$ has degree n, then there exists $c \in \mathbb{C}$ and $z_1, z_2, \ldots, z_n \in \mathbb{C}$ such that

$$P(z) = c(z - z_1)\ldots(z - z_n). \tag{54}$$

24 The Maximum Modulus Principle 1

Suppose that an analytic function f has $|f(z)|$ maximized at z_0 in some nbh $\mathcal{B}_\epsilon(z_0)$ for some $\epsilon > 0$. Then $f(z)$ is constant on $\mathcal{B}_\epsilon(z_0)$.

Proof. Take $0 < \rho < \epsilon$ and by invoking Cauchy's integral formula, we have

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi i}\oint_{C_\rho} \frac{f(z)}{z - z_0}\,dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0 + \rho e^{it})}{z_0 + \rho e^{it} - z_0} i\rho e^{it}\,dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + \rho e^{it})\,dt. \end{aligned} \tag{55}$$

So

$$\begin{aligned} |f(z_0)| &= \frac{1}{2\pi}\left|\int_0^{2\pi} f(z_0 + \rho e^{it})\,dt\right| \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\left|f(z_0 + \rho e^{it})\right|}_{\leq |f(z_0)|}\,dt \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|\,dt = |f(z_0)|. \end{aligned} \tag{56}$$

This says

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + \rho e^{it})\right|\,dt \tag{57}$$

so

$$\frac{1}{2\pi}\int_0^{2\pi} \underbrace{|f(z_0)| - \left|f(z_0+\rho e^{it})\right|}_{\geq 0} dt. \tag{58}$$

This says $\forall t \in [0, 2\pi]$ and $\forall \rho < \epsilon$

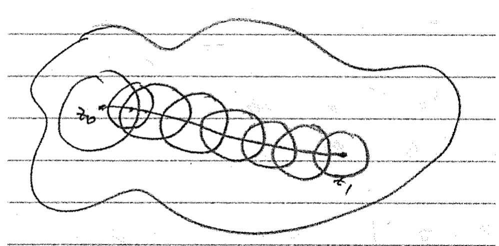
$$|f(z_0)| = \left|f(z_0+\rho e^{it})\right|. \tag{59}$$

This is true for all $\rho < \epsilon$, so $|f(z)| = |f(z_0)|$ for all $z \in \mathcal{B}_\epsilon(z_0)$. $\square$

25 The Maximum Modulus Principle 2

Let f be analytic and non-constant on a domain $\mathcal{D}$ (open and connected), then $|f(z)|$ cannot be maximized in $\mathcal{D}$.

Proof. Assume to reach a contradiction that f is maximized at $z_0 \in \mathcal{D}$. Let $z_1 \in \mathcal{D}$ be arbitrary. Then by the following figure



we get a contradiction, using the maximum modulus principle 1, as desired. $\square$

26 Convergence of Series

Consider a sequence $\{z_n\} = (z_0, z_1, \dots)$ of complex numbers. Write $\{z_n\} \in \mathbb{C}$. We say that the sequence converges if $\exists z \in \mathbb{C}$ for which the following holds: $\forall \epsilon > 0, \exists N = N_\epsilon \in \mathbb{N}$ s.t.

$$|z - z_n| < \epsilon \, \forall n \geq N. \tag{60}$$

In this sense, we also say that $\{z_n\}$ converges to z and call z the limit of the sequence:

$$z = \lim_{n\to\infty} z_n. \tag{61}$$

27 Real and Imaginary parts of a convergent sequence

Let $z_n = x_n + iy_n$ be a sequence, then $z_n \to z = x + iy$ if and only if $x_n \to x$ and $y_n \to y$ in the sense of real numbers.

28 Cauchy sequences

A sequence $\{z_n\}$ is called a Cauchy sequence if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ such that

$$|z_n - z_m| < \epsilon \,\forall n, m \geq N. \tag{62}$$

29 Cauchy and Convergence

A sequence is convergent if and only if it is Cauchy.

30 Series

Consider a sequence $\{z_n\}_{n=0}^{\infty}$ and the series formed with the sequential elements as its terms:

$$\sum_{n=0}^{\infty} z_k = z_0 + z_1 + z_2 + \dots \tag{63}$$

where, a priori, we don't assume they add to anything. This series convergences if $\{S_N\}$ where

$$S_N = \sum_{n=0}^{N} z_k \tag{64}$$

is a convergent sequence, i.e.,

$$S = \lim_{N \to \infty} S_N \tag{65}$$

exists.

31 Convergence of Series

32 Taylor's Theorem

Let $f(z)$ be analytic on a disk $\mathcal{B}_{R_0}(z_0)$, then for any $z \in \mathcal{B}_{R_0}(z_0)$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n. \tag{66}$$

Remarks:

1. In particular, the series $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ converges.

2. The sum is f.

3. For real functions $h : \mathbb{R} \to \mathbb{R}$. If h is differentiable on an open set containing x_0, it might not be twice differentiable.

4. For infinitely differentiable functions, now the series makes sense, but we might have h being representable by a Taylor series that is infinitely differentiable, but not equal to its Maclaurin series. For example:

$$h(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases} . \tag{67}$$

Proof. Without loss of generality, assume that $z_0 = 0$ and consider $\mathcal{B}_{R_0}(z_0)$ on which f is analytic. Let $z \in \mathcal{B}_{R_0}(z_0)$. Let $|z_0| < |z| < R_0$, and define a s.c.c.(+) C centered at $z_0 = 0$ of radius R_0. Since z lives in the interior of C, Cauchy integral formula says

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z}\, dw. \tag{68}$$

Since $w \neq 0$, we write

$$\frac{1}{w-z} = \frac{1}{w}\frac{1}{1-z/w} = \sum_{n=0}^{N} \frac{z^n}{w^{n+1}} + \frac{1}{w-z}\left(\frac{z}{w}\right)^{N+1}, \tag{69}$$

which is made possible by the fact that

$$\frac{1}{1-a} = \frac{1-a^{N+1}}{1-a} + \frac{a^{N+1}}{1-a} = \sum_{n=0}^{N} a^n + \frac{a^{N+1}}{1-a}. \tag{70}$$

Next, by Cauchy's derivative formula,

$$f^{(n)}(0) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-0)^{n+1}}\, dw. \tag{71}$$

So we have

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-0)^{n+1}}\, dw. \tag{72}$$

Next, let the error be

$$\begin{aligned}
\rho_N &= f(z) - \sum_{n=0}^{N} a_n z^n \\
&= \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}\,dw - \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\oint_C \frac{f(w)}{(w-0)^{n+1}} z^n\,dw \\
&= \frac{1}{2\pi i}\oint_C f(w)\left[\frac{1}{w-z} - \sum_{n=0}^{N}\frac{z^n}{w^{n+1}}\right] dw \\
&= \frac{1}{2\pi i}\oint_C f(w)\frac{(z/w)^{N+1}}{w-z}\,dw. \qquad (73)
\end{aligned}$$

Set

$$d = \min |w-z| \quad z \in C \qquad (74)$$

and

$$M = \max |f(z)| \quad z \in \mathcal{B}_{R_0}(z_0 = 0) \qquad (75)$$

then

$$\begin{aligned}
|\rho_N| &= \frac{1}{2\pi}\left|\oint_C f(w)\frac{(z/w)^{N+1}}{w-z}\,dw\right| \\
&\le \frac{1}{2\pi}\frac{|z/w|^{N+1}}{d} M\mathcal{L}(C) \\
&= \frac{M|z/w|^{N+1}}{d} r_0 \qquad (76)
\end{aligned}$$

So, we have shown that given $z \in \mathcal{B}_{R_0}(0)$, $\exists |z| < r_0 < R_0$ for which

$$|\rho_N| \le M\frac{|z|^{N+1}}{d \cdot r_0^N} = \left(\frac{M|z|}{d}\right)\left(\frac{|z|}{r_0}\right)^N \ \forall N \in \mathbb{N}. \qquad (77)$$

Since we've chosen $|z| < r_0 < R_0$, $|z|/r_0 < 1$. Given $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ for which $\forall N \ge N_0$,

$$\left(\frac{|z|}{r_0}\right)^N < \frac{\epsilon d}{M|z|}. \qquad (78)$$

So, for all $N \ge N_0$,

$$|\rho_N| \le \frac{M|z|}{d}\left(\frac{|z|}{r_0}\right)^N < \epsilon. \qquad (79)$$

Thus,

$$f(z) = \lim_{N\to\infty} S_N = \lim_{N\to\infty}\sum_{n=0}^{N} a_n z^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!} z^n. \qquad (80)$$

$\square$

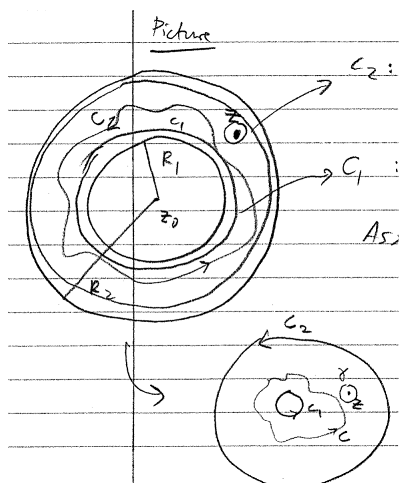
33 Laurent's Theorem

Let f be analytic on a region $\mathcal{D}$ defined by $R_1 < |z - z_0| < R_2$, and let a simple closed contour C endowed with a positive orientation in this annulus be given. Then, for each $z \in \mathcal{D}$,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^{-n+1}} \tag{81}$$

where

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}}\, dz \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}}\, dz. \tag{82}$$



Proof. Without loss of generality, assume $z_0 = 0$. Let C_1, C_2, s.c.c.(+) be given such that C_2 encloses C_1, z, C; C encloses C_1, and the exterior of C_1 contains z, C. Also, let γ be a s.c.c.(+) around z, exterior to C_1 but interior to C_2. An appeal to Cauchy-Goursat for multiply-connected domain shows that

$$\oint_{C_2} \frac{f(s)}{s - z}\, ds - \oint_{C_1} \frac{f(s)}{s - z}\, ds - \oint_{C_\gamma} \frac{f(s)}{s - z}\, ds = 0. \tag{83}$$

Next, by Cauchy integral formula,

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \oint_{C_\gamma} \frac{f(s)}{s - z}\, ds \\ &= \oint_{C_2} \frac{f(s)}{s - z}\, ds - \oint_{C_1} \frac{f(s)}{s - z}\, ds \\ &= \oint_{C_2} \frac{f(s)}{s - z}\, ds + \oint_{C_1} \frac{f(s)}{z - s}\, ds. \end{aligned} \tag{84}$$

For the first integral, we can make the following replacement

$$\begin{aligned}\frac{1}{s-z} &= \frac{1}{s}\left(\frac{1}{1-z/s}\right) \\ &= \sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}} + \frac{1}{s-z}\left(\frac{z}{s}\right)^N. \end{aligned} \tag{85}$$

For the second integral, we can make the following replacement (interchanging the role of s and z)

$$\begin{aligned}\frac{1}{z-s} &= \frac{1}{z}\left(\frac{1}{1-s/z}\right) \\ &= \sum_{n=0}^{N-1}\frac{s^n}{z^{n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N \\ &= \sum_{n=1}^{N}\frac{s^{n-1}}{z^n} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N \\ &= \sum_{n=1}^{N}\frac{z^{-n}}{s^{-n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N. \end{aligned} \tag{86}$$

And so we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_2} f(s)\left[\sum_{n=0}^{N-1}\frac{z^n}{s^{n+1}} + \frac{1}{s-z}\left(\frac{z}{s}\right)^N\right] z^n\, dz \\ &\quad + \frac{1}{2\pi i}\oint_{C_1} f(s)\left[\sum_{n=1}^{N}\frac{z^{-n}}{s^{-n+1}} + \frac{1}{z-s}\left(\frac{s}{z}\right)^N\right] z^{-n}\, dz \\ &= \sum_{n=0}^{N-1}\underbrace{\left[\frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s^{n+1}}\, ds\right]}_{\alpha_n} z^n + \sum_{n=1}^{N}\underbrace{\left[\frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{s^{-n+1}}\, ds\right]}_{\beta_n} z^{-n} + \rho_N + \sigma_N \end{aligned} \tag{87}$$

where

$$\rho_N = \frac{1}{2\pi i}\oint_{C_2}\frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds \tag{88}$$

$$\sigma_N = \frac{1}{2\pi i}\oint_{C_1}\frac{f(s)}{z-s}\left(\frac{s}{z}\right)^N ds. \tag{89}$$

Now, on C_2,

$$\frac{1}{|s-z|} \leq \frac{1}{R_2 - R}, \tag{90}$$

and on C_1,

$$\frac{1}{|z-s|} \leq \frac{1}{R-R_1}, \tag{91}$$

where $R = |z|$, $R_1 < R < R_2$. Setting $M = \max|f(s)|$ where $s \in C_1 \cap C_2$, by triangle inequality, we have that

$$|\rho_N| = \frac{1}{2\pi}\left|\oint_{C_2} \frac{f(s)}{s-z}\left(\frac{z}{s}\right)^N ds\right| \leq \frac{1}{2\pi}\frac{M}{R_2-R}\left(\frac{R}{R_2}\right)^N 2\pi R_2 = \frac{M}{1-R/R_2}\left(\frac{R}{R_2}\right)^N. \tag{92}$$

Similarly,

$$|\sigma_N| \leq \frac{M}{1-R_1/R}\left(\frac{R_1}{R}\right)^N. \tag{93}$$

We see that $\rho_N \to 0$, $\sigma \to 0$ as $N \to \infty$. It follows (with ϵ's and N's similar to those in the proof of Taylor's theorem) that

$$f(z) = \sum_{n=0}^{\infty} \alpha_n z^n + \sum_{n=1}^{\infty} \beta_n z^{-n}. \tag{94}$$

And by corollary to Cauchy-Goursat for multiply-connected regions,

$$\begin{aligned} \alpha_n &= \frac{1}{2\pi i}\int_C (\ )\, ds = a_n \\ \beta_n &= \frac{1}{2\pi i}\int_C (\ )\, ds = b_n \end{aligned} \tag{95}$$

for all n. $\square$

34 More results about series

Consider a power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n, \tag{96}$$

1. If $S(z)$ converges at some $z_1 \neq z_0$ the $S(z)$ converges on $\mathcal{B}_R(z_0)$ where $|z_0 - z_1| \leq R$.
2. The series converges uniformly and absolutely on every ball $\mathcal{B}$ properly contained in $\mathcal{B}_R(z_0)$.
3. On $\mathcal{B}_R(z_0)$, $S(z)$ is analytic, $S'(z) = \sum_{n=1}^{\infty} n a_n (z-z_0)^{n-1}$.

4. If C is a s.c.c.(+) and g is continuous on C and $C \subset \mathcal{B}_R(z_0)$ then

$$\oint_C fg\,dz = \sum_{n=0}^{\infty} \oint_C a_n g(z)(z-z_0)^n\,dz \tag{97}$$

5. Uniqueness of Laurent series: If $S(z) = \sum_{n\in\mathbb{Z}} c_n(z-z_0)^n$ converges on an annulus $R_1 \leq |z-z_0| \leq R_2$ then this is precisely the Laurent series of S at z_0.

35 Residues

For C a s.c.c.(+), let f have singularities at $z_1, z_2, \ldots, z_n$ enclosed by C. Then all the z_k's are isolated singularities, and there exist punctured disks $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_n$ inside C which are on-overlapping whose centers contains z_k's, respectively.

Next, suppose that f has an isolated singularity at z_0. Then f has a Laurent series expansion on an annulus $0 < |z-z_0| < R$ with

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}. \tag{98}$$

Further, for any s.c.c.(+) C_k,

$$b_n = \frac{1}{2\pi i}\oint_{C_k} \frac{f(z)}{(z-z_0)^{-n+1}}\,dz \forall n = 1,2,3,\ldots \tag{99}$$

In particular,

$$b_1 = \frac{1}{2\pi i}\oint_{C_k} f(z)\,dz. \tag{100}$$

We shall call this coefficient of $1/(z-z_0)$ in the Laurent series expansion the residue of f at z_0, denoted

$$b_1 := \operatorname{Res}_{z=z_0} f(z). \tag{101}$$

This gives us a way to compute integrals by finding Laurent series expansions.

36 The Residue Theorem

Let C be a s.c.c.(+) and suppose that f is analytic on C and the interior to C except at a finite number of points $z_1, z_2, \ldots, z_n$, all enclosed by C. Then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z). \tag{102}$$

Proof. Take $C_1, C_2, \ldots, C_n$ to be non-intersecting s.c.c.(+) inside C where each enclosed only the singular point z_k, respectively. Then f is analytic on $\text{Int}(C) \setminus \cup^n \text{Int} C_k$. By Cauchy-Goursat for multiply-connected region,

$$\oint_C f(z)\,dz = \sum_{k=1}^{n} \oint_{C_k} f(z)\,dz. \tag{103}$$

But for each k, we also have

$$\oint_{C_k} f(z)\,dz = 2\pi i\, \text{Res}_{z=z_k} f(z). \tag{104}$$

So,

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z). \tag{105}$$

□

37 Classification of Singularities

If the principal part of the Laurent series expansion of f is identically zero then z_0 is said to be a removable singularity.

If z_0 is an isolated removable singularity for f for $z \neq z_0$ but $0 < |z - z_0| < R$, then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + 0. \tag{106}$$

At $z = z_0$, the left-hand side is a_0. So if we define

$$f_{ext}(z) = \begin{cases} f(z) & 0 < |z - z_0| < R \\ a_0 & z = z_0 \end{cases} \tag{107}$$

then

$$f_{ext}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{108}$$

for all z such that $|z - z_0| < R$. This is called an extension of f. We note that $f_{ext}(z)$ is analytic on $\mathcal{B}_R(z_0)$. We have just removed the removable singularity.

When the principal part of f is nonzero and contains a finite number of summands

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} = \frac{b_1}{(z - z_0)} + \ldots \frac{b_m}{(z - z_0)^m} \tag{109}$$

and $b_k \neq 0 \forall k \geq m+1$ then z_0 is a pole of order m for f. When $m = 1$, z_0 is called a simple pole.

If the principal part of f is identically zero, then z_0 is a removable singularity for f, because f can be extended via its valid Taylor-Laurent series expansion to an analytic function on $\mathcal{B}_R(z_0)$.

z_0 is said to be an essential singularity of f it it is not removable or a pole, i.e., the principle part of the Laurent series of f contains an infinite number of non-zero terms.

38 Residues with Φ theorem

Let z_0 be an isolated singularity of f. Then z_0 is a pole or order m if and only if $\exists$ a function $\phi(z)$ which is non zero at z_0, analytic at z_0 and for which

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \tag{110}$$

for $z \in$ a nbh of z_0. In this case,

$$\mathrm{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}. \tag{111}$$

Proof. ($\rightarrow$) Suppose that

$$f(z) = \frac{\phi(z)}{(z-z_0)^m} \tag{112}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Then we have that $\phi(z)$ has a valid Taylor series expansion in $\mathcal{B}_R(z_0)$:

$$\phi(z) = \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n. \tag{113}$$

With this, we can write $f(z)$ as

$$\begin{aligned}
f(z) &= \frac{1}{(z-z_0)^m} \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^n \\
&= \sum_{n=0}^{\infty} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} \\
&= \sum_{n=0}^{m-1} \frac{\phi^{(n)}(z_0)}{n!}(z-z_0)^{n-m} + (\text{Taylor}) \\
&= \sum_{k=1}^{m} \frac{\phi^{(n-k)}(z_0)}{(m-k)!}(z-z_0)^{k} + (\text{Taylor}), \quad (k = m-n).
\end{aligned} \tag{114}$$

And so z_0 is a pole of order m, since $\phi^{(0)}(z_0) \neq 0$. And of course, we get for free

$$\mathrm{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}. \tag{115}$$

($\leftarrow$) Conversely, assume that f has a pole at z_0 or order m. Then

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + 0 \dots \\ &= \frac{1}{(z-z_0)^m} \left[\sum_{n=0}^{\infty} a_n (z-z_0)^{n+m} + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^{n-m}} \right] \\ &:= \frac{\phi(z)}{(z-z_0)^m} \end{aligned} \tag{116}$$

where $\phi(z)$ is defined to be the expression in the square brackets. With this, we see that $\phi(z)$ is analytic at z_0 and $\phi(z_0) = 0 + b_m \neq 0$ by hypothesis. □

39 Residues with p-q theorem

Let p, q be analytic at z_0. If $p(z_0) \neq 0, q'(z_0) \neq 0$, and $p'(z_0) = 0$ then

$$f(z) = \frac{p(z)}{q(z)} \tag{117}$$

has a simple pole of z_0 and

$$\mathrm{Res}_{z=z_0} f(z) = \mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}. \tag{118}$$

Proof. hello □

40 What happens near singularities?

If z_0 is a pole of order m for f, then

$$\lim_{z \to z_0} f(z) = \infty. \tag{119}$$

41 Removable singularity - Boundedness - Analyticity (RBA)

If z_0 is a removable singularity for f then f is bounded and analytic on a punctured nbh of z_0.

42 The converse of RBA

Let f be analytic on $0 < |z - z_0| < \delta$ for some $\delta > 0$. If f is also bounded on $0 < |z - z_0| < \delta$, then if z_0 is a singularity for f, it must be removable.

Proof. By assumption, f has a Laurent series representation of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} \tag{120}$$

where b_n in particular is given by

$$b_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{-n+1}} \, dz \tag{121}$$

where C is a s.c.c.(+) in the annulus of the analyticity. In particular, if $0 < \rho < \delta$, and $C_\rho := \{z, |z - z_0| = \rho\}$, (+) then

$$|b_n| = \left| \frac{1}{2\pi i} \oint_{C_\rho} \frac{f(z)}{(z - z_0)^{-n+1}} \, dz \right| \tag{122}$$

and if M is such that $f(z) \leq M \forall 0 < |z - z_0| < \delta$ then

$$|b_n| \leq \frac{1}{2\pi} \frac{M}{\rho^{-n+1}} 2\pi\rho = M\rho^n. \tag{123}$$

Since this is valid $\forall \rho < \delta$, we must have that $b_n = 0 \forall n$. $\square$

43 Casorati-Weierstrass Theorem

Let f have an essential singularity at z_0. Then $\forall w_0 \in \mathbb{C}$ and $\epsilon > 0$,

$$|f(z) - w_0| < \epsilon \tag{124}$$

for some $z \in \mathcal{B}_\delta(z_0) \forall \delta 0$.
$\iff$ f is arbitrarily close to every complex number on every nbh of z_0.
$\iff$ $\forall \delta > 0, f(\mathcal{B}_\delta(z_0) \setminus \{z_0\})$ is dense on $\mathbb{C}$.
$\iff$ f gets close to every single point in a ball for any ball.
$\iff$ If z_0 is an essential singularity for f then f attains, except for at most one value, every complex number an infinite number of time on every nbh of z_0.

Proof. Assume to reach a contradiction that $\exists w_0 \in \mathbb{C}, \epsilon, \delta > 0$ s.t.

$$|f(z) - w_0| \geq \epsilon \forall 0 < |z - z_0| < \delta, \tag{125}$$

i.e., f does not get close to some value w_0 in some nbh of z_0 of radius δ. Then, consider

$$g(z) = \frac{1}{f(z) - w_0} \tag{126}$$

which is bounded and analytic on the punctured disk $0 < |z - z_0| < \delta$. At worst, z_0 is a removable singularity for g. Also note that $g(z)$ is not identically zero since f is not constant (as f has a singularity). With this,

$$g(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{127}$$

which allows us to extend g to z_0. Let $m = \min(k = 0, 1, 2, \dots)$ such that $a_k \neq 0$, which exists because $g \neq 0$. Then

$$g(z) = (z - z_0)^m \sum_{k=0}^{\infty} a_k (z - z_0)^{k-m} = (z - z_0)^m \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k. \tag{128}$$

Call the sum $h(z)$, which $h(z_0) = a_m \neq 0$. So, in $\mathcal{B}_\delta(z_0) \setminus \{z_0\}$, we have

$$f(z) = w_0 + \frac{1}{g(z)}. \tag{129}$$

If $g(z_0) \neq 0 \iff m = 0$, then this formula allows s to extend f to z_0, which is then analytic, which makes z_0 a removable singularity. This is a contradiction. If $g(z_0) = 0$, then because $m \geq 1$ (by definition) and

$$f(z) = w_0 + \frac{1}{g(z)} = \frac{w_0 g(z) + 1}{(z - z_0)^m h(z)} := \frac{\phi(z)}{(z - z_0)^m}. \tag{130}$$

We see that $\phi(z_0) \neq 0$, and $\phi(z)$ is analytic. So, z_0 is a pole of order m of f. This is also a contradiction. $\square$