Problem Set 6

For this problem set you have more time:
Due: Friday 11:59pm, March 31st via Canvas upload
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Office hour: TBA

1 Rayleigh and Thomson scattering using two different interaction Hamiltonians

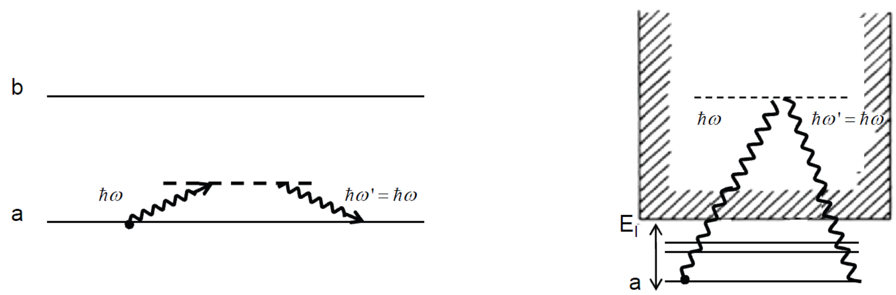

Figure 1: Rayleigh and Thomson scattering

Elastic light scattering takes the system from state $|i\rangle = |a; \mathbf{k}\epsilon\rangle$ to state $|f\rangle = |a; \mathbf{k}'\epsilon'\rangle$ where a is the (unchanged) state of the atom, and $\mathbf{k}(\mathbf{k}')$ and $\epsilon(\epsilon')$ the wavevector and polarization of the incident (scattered) photon.
In class, we introduced the scattering (S) matrix $S_{fi} = \delta_{fi} - 2\pi i\,\delta(E_f - E_i)\mathcal{T}_{fi}$ and found an expression up to second order for $\mathcal{T}_{fi}$ (the matrix element of the T matrix).

a) Calculate $\mathcal{T}_{fi}$ in leading (non-zero) order for both Rayleigh and Thomson scattering ($\hbar\omega$ smaller/larger than excitation energies), for both the electric-dipole Hamiltonian:

$$H_I' = -\mathbf{d} \cdot \mathbf{E} \tag{1}$$

and the Coulomb-gauge interaction Hamiltonian:

$$H_I = -\frac{q}{m}\mathbf{p} \cdot \mathbf{A}(\mathbf{0}) + \frac{q^2}{2m}\mathbf{A}^2(\mathbf{0}) \tag{2}$$

You will realize that Rayleigh scattering is more easily calculated with the dipole Hamiltonian, whereas the reverse is true for Thomson scattering. Of course, both Hamiltonians lead to the same answer.

b) Show that the total cross section for Thomson scattering is given by $\sigma_0 = \frac{8\pi}{3}r_0^2$ where r_0 is the classical electron radius.

Try to advance with this problem as far as you can by yourself. If you get stuck, you will find a discussion of this problem in Exercises 3 and 4 in our text book Atom-Photon Interactions by Cohen-Tannoudji, Dupont-Roc, Grynberg.

2 Long-range (Van der Waals) interaction between ground-state atoms

The electrostatic interaction between atoms a and b is described to first order by the dipole-dipole term:

$$H_{el}(R) = \frac{\mathbf{d}_a \cdot \mathbf{d}_b - 3(\mathbf{d}_a \cdot \hat{\mathbf{R}})(\mathbf{d}_b \cdot \hat{\mathbf{R}})}{R^3} \tag{3}$$

where $\mathbf{d}_a = e\mathbf{r}_a$ is the electric dipole operator of atom a, $\mathbf{d}_b = e\mathbf{r}_b$ is the electric dipole operator of atom b, and $\mathbf{R} = \mathbf{R}_{nb} - \mathbf{R}_{na}$ is a position vector pointing from the nuclei of a to the nuclei of b. We will use time-independent perturbation theory to calculate the effect of H_{el}.
Notation: Let $|g_a g_b\rangle$ denote atom a and atom b in the ground state.
Let $|i_a g_b\rangle$ denote atom a in an excited state i and atom b in the ground state.

a) What is the first non-vanishing term in the series for the perturbed ground state energy of the system?

b) Dipole matrix elements in atomic physics are often discussed in terms of "oscillator strength," $f_{ig} = \frac{2m\omega_{ig}}{\hbar}|\langle i|x|g\rangle|^2$. Note: $\omega_{ig} = \frac{E_i - E_g}{\hbar}$, so f_{ig} is positive for absorption and negative for emission. Also, $\sum_i f_{ig} = 1$, the Thomas-Reiche-Kuhn sum rule. Express your result from (a) in terms of oscillator strengths. You will have to make some arguments (non-mathematical if you prefer) about the symmetry of photon emission to get rid of annoying cross terms.

c) We can estimate C_6 using the approximation that the oscillator strength f_{ig} is large for only one transition, $|g\rangle \to |i\rangle$. The $|nS\rangle \to |(n+1)P\rangle$ transitions in alkali atoms are the classic examples, with $f \approx 0.98$. Use this in combination with the sum rule above and the definition of the static polarizability of the ground state:

$$\alpha_g = 2e^2 \sum_i \frac{|\langle i|z|g\rangle|^2}{E_i - E_g} \tag{4}$$

and express your result for C_6 from (b) in terms of polarizabilities $\alpha_g^{(a)}$ and $\alpha_g^{(b)}$. (You should not have any summation signs in your final answer.)

3 Long-range interaction between an excited atom and a ground-state atom

Consider the case where one atom is excited and the other atom is in its ground state. For simplicity model each atom as a two level system with one ground state and one excited state.

a) Assume you have two atoms a and b with almost (but not quite) degenerate ground $\leftrightarrow$ excited state transition energies $(E_i^{(a)} - E_g^{(a)}) \approx (E_i^{(b)} - E_g^{(b)})$. How does the energy of the state $|i_a g_b\rangle$ change as a function of the separation R for large distances? What about state $|g_a i_b\rangle$? For what separation does perturbation theory become invalid?

b) Now assume you have two identical (i.e. same transition energy) atoms. Calculate the long-range interaction potential curves for the case of one excited atom and one ground state atom.

c) For case (b) what is the relation between the spontaneous decay rate of the atom and its long-range interaction coefficient?