·109. What is the minimum number of vertices of degree one in a finite tree? What is it if the number of vertices is bigger than one? Prove that you are correct. See if you can find (and give) more than one proof.

Solution: The minimum is zero, which happens with a tree with one vertex. If the tree has more than one vertex, the minimum number of vertices of degree one is two. To prove this, we prove that every tree with two or more vertices has at least two vertices of degree two. Note that a tree with two vertices has exactly two vertices of degree 2. Now take a tree with more than two vertices. Remove an edge e without removing its endpoints. As in the solution to Problem 108 this gives two trees. We may assume inductively that each has at least two vertices of degree 1, or else is a single vertex. When we put e back in, it connects one vertex in one tree to one in the other. If both these vertices have degree 1 in their trees, there will be at least one vertex of degree 1 remaining in each tree, so there will be at least two vertices of degree 1 in the tree we get. If exactly one of these vertices is a tree with one vertex after the removal of e, when we connect it to the other tree, we will increase the degree of at most one vertex of degree 1 and will create a new vertex of degree 1, so the tree that results still has at least two vertices of degree 1. Therefore by the strong principle of mathematical induction, every tree with more than two vertices has at least two vertices of degree 1. Since a two-vertex tree has two vertices of degree 1, the minimum number of vertices of degree 1 in a tree with two or more vertices is two. (In fact a path with n vertices is a tree and it has exactly two vertices of degree one also.)

Alternately, the number of edges in a n vertex tree is $n-1$, and so the sum of the degrees of the vertices is $2n-2$. If we have more than one vertex, we can have no vertices of degree zero, and if all or all but one vertex had degree at least two, the sum of the degrees would have to be more than $2n-2$. ▮

➔·110. In a tree on any number of vertices, given two vertices, how many paths can you find between them? Prove that you are correct.

Solution: Exactly one. Suppose there were two distinct paths P_1 and P_2 from x to y. As they leave x, they might leave on the same edge or on different edges. However, since they are different, there must be some first vertex x' on both paths so that when leave x' (as we go from x to y), they leave on different edges. Then since they must both enter y, there must be some first vertex y', following x' on both paths as we go from x to y, such that the two paths enter y' on two different edges. Then the portion of path 1 from x' to y' followed by the portion of path 2 from y' to x' will be a cycle. This is impossible in a tree, so the supposition that there were two distinct paths is impossible. ▮

112. (a) How long will the sequence of b_is be if it is computed from a tree with n vertices (labelled with 1 through n)?

Solution: On a tree with n vertices, the sequence b will have length $n-1$. ▮

(b) What can you say about the last member of the sequence of b_is?

Solution: The last member of the sequence b will be n. To see why, note that vertex n can not be in the sequence a, because the tree that remains after we delete an a_i will have at least two vertices of degree 1, so the one of smaller degree will be a_{i+1}. Thus we never delete the vertex n from the tree. Therefore when we choose the last b, we have vertex n and one other vertex, so the other vertex is our a-vertex and n is the vertex adjacent to it. ▮

(c) Can you tell from the sequence of b_is what a_1 is?

Solution: a_1 will be the smallest number that is not in the sequence of b's. ▮

(d) Find a bijection between labelled trees and something you can "count" that will tell you how many labelled trees there are on n labelled vertices.

Solution: Once we know a_1, we know one edge of the tree, namely the edge between a_1 and b_1. In general, when we know a_i, this will tell us that the edge form a_i to b_i is in the tree. The vertex a_2 will be the smallest number different from a_1 not in the sequence b_2 through b_{n-1}. In general, a_i will be the smallest vertex different from a_1 through a_{i-1} not in the sequence b_i through b_{n-1}, which gives us all $n-1$ edges of the tree (edge i goes from a_i to b_i). Thus there is a bijection between trees and the sequences b_1 through b_{n-1}. But since $b_{n-1}=n$, there is also a bijection between trees and the sequences b_1 through b_{n-2}. But given a sequence of numbers $c_1, c_2, \ldots, c_{n-2}, c_{n-1}$, all between 1 and n and with $c_{n-1}=n$, there is always a smallest number a_1 not in the sequence, and given $a_1, a_2, \ldots a_{i-1}$, there is always a smallest number not in the sequence c_i through c_{n-1} and different from the a_is already chosen, so we can construct the edges from a_i to c_i. Further, if we start with the edge from a_{n-1} to c_{n-1} and work backwards, we will always have a connected graph and will always be adding a vertex of degree 1 to it, so we will have no cycles. Therefore we will get a tree. Thus we have a bijection between labelled trees on n vertices and sequences of length $n-2$ consisting of members of $[n]$. There are n^{n-2} such sequences, and thus n^{n-2} labelled trees on n vertices. ▮

➔116. What is the number of (labelled) trees on n vertices with three vertices of degree 1? (Assume they are labelled with the integers 1 through n.) This problem will appear again in the next chapter after some material that will make it easier.

Solution: There are $\binom{n}{3}$ ways to choose the three vertices of degree one. Each of the other $n-3$ vertices must appear in the Prüfer Code, so exactly one must appear twice. We have $n-3$ ways to choose that one vertex and $\binom{n-2}{2}\binom{n-4}{1}\binom{n-5}{1}\cdots\binom{1}{1} = \frac{(n-2)!}{2!}$ ways to choose which of the $n-2$ places to use for which vertices in the Prüfer code. Thus there are $\binom{n}{3}(n-3)\frac{(n-2)!}{2} = \frac{n!(n-2)(n-3)}{12}$ labelled trees with three vertices of degree one. ▮

➔ *9. How many labelled trees on n vertices have exactly four vertices of degree 1? (This problem also appears in the next chapter since some ideas in that chapter make it more straightforward.)

Solution: The vertices of degree 1 are the vertices that do not appear in the Prüfer code for the tree. So we first choose four vertices out of n in $\binom{n}{4}$ ways to be our vertices of degree 1, and then we use the remaining $n-4$ vertices to fill in our list of $n-2$ vertices, using each of the $n-4$ at least once. Thus we either use one of them 3 times and the rest once, or two of them twice and the rest once. There are $n-4$ ways to choose the one we use three times and $\binom{n-2}{3}\binom{n-5}{1}\binom{n-6}{1}\cdots\binom{1}{1} = \frac{(n-2)!}{3!}$ ways to label the $n-2$ places with the chosen vertices. There are $\binom{n-4}{2}$ ways to choose the two vertices we would use twice, and $\binom{n-2}{2}\binom{n-4}{2}\binom{n-6}{1}\binom{n-7}{1}\cdots\binom{1}{1}/2 = \frac{(n-2)!}{2!2!}$ ways to assign the chosen vertices to the $n-2$ places in the Prüfer Code. Thus we have

$$\begin{aligned}
&\binom{n}{4}\left((n-4)\frac{(n-2)!}{3!} + \frac{(n-4)(n-5)}{2}\frac{(n-2)!}{4}\right) \\
=\ & \frac{n!}{24}(n-2)^{\underline{3}}\left(\frac{1}{6} + \frac{n-5}{8}\right) \\
=\ & n!(n-2)(n-3)(n-4)(3n-11)/576
\end{aligned}$$

possible Prüfer codes and therefore the same number of labelled trees. ∎

➔ $*10.$ The *degree sequence* of a graph is a list of the degrees of the vertices in nonincreasing order. For example the degree sequence of the first graph in Figure 2.4 is $(4, 3, 2, 2, 1)$. For a graph with vertices labelled 1 through n, the *ordered degree sequence* of the graph is the sequence $d_1, d_2, \ldots d_n$ in which d_i is the degree of vertex i. For example the ordered degree sequence of the first graph in Figure 2.2 is $(1, 2, 3, 3, 1, 1, 2, 1)$.

(a) How many labelled trees are there on n vertices with ordered degree sequence $d_1, d_2, \ldots d_n$? (This problem appears again in the next chapter since some ideas in that chapter make it more straightforward.)

Solution: We are given that d_i is the degree of vertex i. The number of times i appears in the Prüfer code of a tree is one less than the degree of i, so vertex i appears $d_i - 1$ times. Thus the sum of the $d_i - 1$ should be $2n - 2 - n = n - 2$. Of the $n - 2$ places in the Prüfer code, we want to label $d_1 - 1$ of them with 1, $d_2 - 1$ of them with 2 and in general $d_i - 1$ of them with i. There are

$$\binom{n-2}{d_1-1}\binom{n-2-(d_1-1)}{d_2-1}\binom{n-2-(d_1-1+d_2-1)}{d_3-1}\cdots\binom{d_n-1}{d_n-1}$$

ways to do this, so the number of trees in which vertex i has degree d_i is $\frac{(n-2)!}{(d_1-1)!(d_2-1)!\cdots(d_n-1)!}$ ∎

$*$(b) How many labelled trees are there on n vertices with with the degree sequence in which the degree d appears i_d times?

Solution: Now we modify the solution of the previous part by observing that to count all graphs with a given degree sequence, the actual vertices which have the given degrees is irrelevant, so we must multiply the result of the easier problem by the number of ways to assign the degrees to the vertices. To assign the degrees, we can list the vertices in $n!$ ways, choose the first i_1 of these vertices to have degree 1, the next i_2 to have degree 2, and so on. But the order in which we list the vertices of a given degree is irrelevant. Thus the number of ways to assign the degrees is $\frac{n!}{i_1!i_2!\cdots i_n!}$. Once the degrees are assigned, there are $\frac{(n-2)!}{\prod_{d=1}^{n}(d-1)!^{i_d}}$, by translating our easier result. Thus the total number of trees with the degree sequence in which there are i_d vertices of degree d is

$$\frac{n!(n-2)!}{\prod_{j=1}^{n} i_j!(j-1)!^{i_j}}.$$

∎