Physics 8.321, Fall 2020
Midterm

You have **75 minutes** to complete the exam and a grace period of an additional 15 minutes to get your solutions uploaded to canvas. You may use your books and notes including the notes on canvas from the course, and you may use symbolic manipulation tools like mathematica and matlab, but you may not consult other online resources, and you may not communicate with other people in any way while doing the midterm. You also may not communicate any information about the exam to anyone after you have completed it until the exam period is over at the end of the day on 10/23/20.

Note: you are expected to upload your completed exam to the canvas website immediately after completing the exam and within 75 minutes or less of downloading it. You have a few extra minutes in case of technical complications. The system will log your download and upload times. If for some reason you have difficulty uploading your exam after completion, please email it immediately to one of the course staff.

1. (30 points)

A *spin-1* particle is associated (in a specific choice of basis) with angular momentum operators

$$S_x = \hbar/\sqrt{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S_y = \hbar/\sqrt{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad S_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

A particle is in a state that can be described in this basis as $|\psi\rangle = (1/2, \sqrt{2}/2, 1/2)^{\mathrm{T}}$ (the T just denotes the transpose to make this a column vector, not a row vector).

(a) What are the expectation values $\langle O\rangle = \langle\psi|O|\psi\rangle$ in this state of the three spin operators S_x, S_y, S_z?

(b) What are the expectation values $\langle S_x^2\rangle$ and $\langle S_z^2\rangle$?

(c) Compute both sides of the uncertainty relation

$$\langle\Delta S_x^2\rangle\langle\Delta S_z^2\rangle \geq \frac{1}{4}|\langle[S_x, S_z]\rangle|^2$$

and confirm that the inequality holds.

Answer:

(a) It is easy to compute directly by simply writing out the matrices and vectors that

$$\langle S_x\rangle = \hbar, \langle S_y\rangle = \langle S_z\rangle = 0$$

Note that $S_x|\psi\rangle = \hbar S_x|\psi\rangle$, so the state is an eigenstate of S_x.

(b) Similarly, we can compute

$$\langle S_x^2\rangle = \hbar^2, \langle S_z^2\rangle = \hbar^2/2\,.$$

(c) We use $[S_x, S_z] = -i\hbar S_y$

$$(0)(\hbar^2/2) \geq 0\,.$$

2. (30 points)

Consider a combined system of one spin-1/2 particle as described in class and one spin-1 particle as described above.

(a) What are the possible values in the combined system of the observable associated with the operator

$$S_z = S_z^{(1/2)} + S_z^{(1)}?$$

Identify the states associated with each eigenvalue in a basis of your choice.

(b) A particle is in the state $|\psi\rangle = |+, 1\rangle$, where $|\pm, s^{(1)}\rangle$ denotes the state with eigenvalue of $S_z^{(1/2)}$ given by $\pm\hbar/2$ and eigenvalue of $S_z^{(1)}$ given by $s^{(1)}\hbar$. The spin of the spin-1/2 particle along the x axis is measured. What are the possible values of this spin and what is the probability of each outcome?

(c) After performing the measurement in the previous part, the value of $S_x^{(1)}$ is measured. What are the possible outcomes and probabilities?

(d) How does your answer for (c) differ from what you would get if the initial state was again the state $|\psi\rangle = |+, 1\rangle$, i.e. the measurement from part (b) had not been performed first?

Answer:

(a) Possible eigenvalues are $3\hbar/2, \hbar/2, -\hbar/2, -3\hbar/2$, and they are associated with the following states

$$\begin{aligned} 3\hbar/2 &: \ |+,1\rangle \\ \hbar/2 &: \ |+,0\rangle, |-,1\rangle \\ -\hbar/2 &: \ |+,-1\rangle, |-,0\rangle \\ -3\hbar/2 &: \ |-,-1\rangle\,. \end{aligned}$$

This can be easily seen by noting that the states above form a complete basis and hitting them with the operator $S_z = S_z^{(1/2)} + S_z^{(1)}$. Note that we **can't** ever get $S_z = 0$ by combining spin-1 and spin-1/2 systems, there would be always some non-zero spin, pointing either up or down some amount.

(b) The probability is 50% each for measuring $S_x^{(1/2)} = \pm\hbar/2$, since the particle starts in an eigenstate of $S_z^{(1/2)}$ with eigenvalue $\hbar/2$ and $S_x^{(1/2)}$ only acts on that degree of freedom, so the calculation is just like for a single particle of spin 1/2.

(c) Again, it's just like a single particle of spin 1 starting in the state where $S_z^{(1)} = \hbar$. In the $S_z^{(1)}$ basis, we can compute the eigenvalues and eigenstates of $S_x^{(1)}$:

$$\begin{aligned} \hbar &: \ (1/2, 1/\sqrt{2}, 1/2) \\ 0 &: \ (1/\sqrt{2}, 0, 1/-\sqrt{2}) \\ -\hbar &: \ (1/2, -1/\sqrt{2}, 1/2). \end{aligned}$$

In terms of this basis we have

$$|1_z\rangle = \frac{1}{2}|1_x\rangle + \frac{1}{2}|-1_x\rangle + \frac{1}{\sqrt{2}}|0_x\rangle\,.$$

It follows that we have the results $\pm\hbar$ each with probability 25% and 0 with probability 50%. This can also be seen immediately from symmetry and the answer to problem 1, since $\langle S_x^2\rangle = \hbar^2/2$ and by symmetry the probability of $\pm\hbar$ must be equal. Note that probabilities are **not** equally distributed among spins for spin-1 system.

(d) No difference, since the operators commute

$$[S_x^{(1/2)}, S_x^{(1)}] = 0\,,$$

and that is because they act different factors of the Hilbert space of this combined system.

3. (40 points)

In this problem we choose units where $\hbar = m = 1$. Consider a particle in a 2D potential

$$V(x,y) = \begin{cases} \frac{1}{2}x^2, & \text{when } |y| < 1 \\ \infty, & \text{when } |y| > 1. \end{cases}$$

Describe the eigenvalue spectrum of the Hamiltonian and the associated eigenstates. What are the energy eigenvalues for the lowest-lying 5 states of the system?

Answer: The potential is a sum $V(x,y) = f(x) + g(y)$, where f is the SHO potential and g is an infinite square well. We can therefore use separation of variables and write the eigenstates as product eigenstates

$$\psi_{n,m}(x,y) = \psi_n^{\text{SHO}}(x)\psi_m^{\text{square}}(y), \quad n \geq 0, m > 0\,.$$

Formally, this corresponds to writing the Hamiltonian as $H = H^{\text{SHO}} \otimes \mathbb{1} \oplus \mathbb{1} \otimes H^{\text{square}}$ on the tensor product space of the Hilbert spaces of functions on x, y, and the eigenvalues of the energy eigenstates are $E_{n,m} = E_n^{\text{SHO}} + E_m^{\text{square}}$. The SHO eigenstates are those described in class (Gaussian times Hermite polynomials) and the square well eigenstates are $\psi_m(y) = \sin(m\pi(y-1)/2)$. The SHO eigenvalues are $(n+1/2)\hbar$ and the square well eigenvalues are $m^2\pi^2/8$ (note that the width of the square well is $a = 2$, which gives a factor of $a^2 = 4$ in the energy denominator), so the energy eigenvalues for the full system are

$$E_{n,m} = (n + 1/2) + \frac{m^2\pi^2}{8}\,.$$

The lowest lying five eigenvalues are

$$\begin{aligned} E_{0,1} &= 1/2 + \pi^2/8 \cong 1.73, \\ E_{1,1} &= 3/2 + \pi^2/8 \cong 2.73, \\ E_{2,1} &= 5/2 + \pi^2/8 \cong 3.73, \\ E_{3,1} &= 7/2 + \pi^2/8 \cong 4.73, \\ E_{0,2} &= 1/2 + \pi^2/2 \cong 5.43\,. \end{aligned}$$

Note that exciting the SHO adds less energy since the wave function can spread out more broadly and the approximate wavelength can be longer.