Homework 3 8.370/18.435/2.111 Due 10/5/2022

1. Find a matrix whose square is σ_x.

   Hint: There are lots of ways of doing this. One way is to first find a matrix whose square is σ_z.

2. **Making a SWAP gate**

   Show how to create a SWAP gate using 3 CNOT gates.

3. **Controlled Hadamard gate**

   Write down the 8×8 matrix that corresponds to a controlled Hadamard gete where the control is bit 3 and the target is bit 1.

4. **The Fredkin gate**

   The Fredkin gate is a classical reversible three-bit gate. If the input is (x, y, z), then the output is $(0, y, z)$ if $x = 0$ and $(1, z, y)$ if $x = 1$. In other words, it is a controlled SWAP: if x is 0, you do nothing, and if x is 1, you swap y and z.

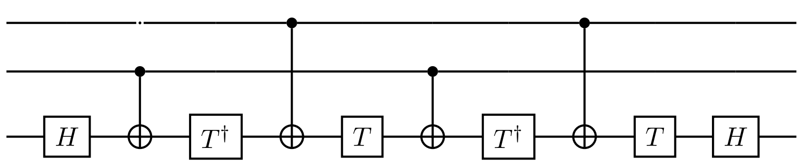
   (a) Show how you can obtain the gates AND, OR, NOT, and FANOUT by putting fixed values in one or two of the inputs of the Fredkin gate.

   (b) Show that the Fredkin gate preserves the number of 1s in the system. This is useful, for example, if you want to show how to build a circuit that is implemented by ideal colliding billiard balls.

   (c) A half-adder takes as input two bits x and y, and has two outputs $x \wedge y$ (x AND y) and $x \oplus y$ (x XOR y). Show how to build a half-adder using Fredkin gates (you will need more than two output bits).

5. **Constructing a quantum Toffoli gate**

   (a) Consider the following quantum circuit, made from one-qubit gates and four CNOT gates:

   where the T gate is

   $$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1+i}{2} \end{pmatrix}.$$

   Show that this nearly yields a Toffoli gate. What unitary transformation does it implement, and how does it differ from a Toffoli gate?

   (b) Explain how you can add one-qubit gates and CNOTs to the first two quantum wires to make this a true Toffoli gate.