8.09/8.309, Classical Mechanics III, Fall 2018
MIDTERM
Tuesday October 30, 7:30-9:30pm
You have 120 minutes.

Answer all problems in the white books provided. Write YOUR NAME on EACH book you use.

There are four problems, totalling 100 points. You should do all four. The problems are worth 15, 25, 30, 30 points.

None of the problems requires extensive algebra. If you find yourself lost in a calculational thicket, stop and think.

No books, notes, or calculators allowed.

Some potentially useful information

- Euler-Lagrange equations for generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j} \ , \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot{q}_j}$$

  constraints: holonomic $f_\alpha(q,t) = 0$ or semiholonomic $g_\beta = \sum_j a_{\beta j}(q,t)\dot{q}_j + a_{\beta t}(q,t) = 0$

- Generalized forces: $d/dt(\partial L/\partial \dot{q}_j) - \partial L/\partial q_j = R_j$

  Friction forces: $\vec{f}_i = -h(v_i)\vec{v}_i/v_i$, $\vec{v}_i = \dot{\vec{r}}_i$ gives $R_j = -\partial \mathcal{F}/\partial \dot{q}_j$, $\mathcal{F} = \sum_i \int_0^{v_i} dv_i' \, h(v_i')$

- Hamilton's equations for canonical variables (q_j, p_j): $\quad \dot{q}_j = \dfrac{\partial H}{\partial p_j}$, $\quad \dot{p}_j = -\dfrac{\partial H}{\partial q_j}$

- Hamiltonian for a Lagrangian quadratic in velocities

$$L = L_0(q,t) + \dot{\vec{q}}^T \cdot \vec{a} + \frac{1}{2}\dot{\vec{q}}^T \cdot \hat{T} \cdot \dot{\vec{q}} \quad \Rightarrow \quad H = \frac{1}{2}(\vec{p} - \vec{a})^T \cdot \hat{T}^{-1} \cdot (\vec{p} - \vec{a}) - L_0(q,t)$$

- The Moment of Inertia Tensor and its relations:

$$I_{ab} = \int dV \, \rho(\vec{r})[\vec{r}^2\delta_{ab} - r_a r_b] \qquad \text{or} \qquad I^{ab} = \sum_i m_i[\delta^{ab}\vec{r}_i^{\,2} - r_i^a r_i^b]$$

$$I_{ab}^{(Q)} = M(\delta_{ab}\,\vec{R}^2 - R_a R_b) + I_{ab}^{(\mathrm{CM})} \ , \qquad \hat{I}' = \hat{U}\,\hat{I}\,\hat{U}^T$$

- Euler's Equations:

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= \tau_3 \end{aligned}$$

- Vibrations: $L = \frac{1}{2}\dot{\vec{\eta}}^T \cdot \hat{T} \cdot \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T \cdot \hat{V} \cdot \vec{\eta}$ has Normal modes $\vec{\eta}^{(k)} = \vec{a}^{(k)} \exp(-i\omega^{(k)} t)$

$$\det(\hat{V} - \omega^2\hat{T}) = 0 \ , \qquad \left(\hat{V} - [\omega^{(k)}]^2\,\hat{T}\right) \cdot \vec{a}^{(k)} = 0 \ , \qquad \vec{\eta} = \mathrm{Re}\sum_k C_k\,\vec{\eta}^{(k)}$$

- Generating functions for Canonical Transformations: $K = H + \partial F_i/\partial t$ and

$$F_1(q,Q,t): \quad p_i = \frac{\partial F_1}{\partial q_i} \ , \ P_i = -\frac{\partial F_1}{\partial Q_i} \ , \qquad F_2(q,P,t): \quad p_i = \frac{\partial F_2}{\partial q_i} \ , \ Q_i = \frac{\partial F_2}{\partial P_i}$$

- Poisson Brackets: $\quad [u,v]_{q,p} = \displaystyle\sum_j \left[\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j}\right]$, $\quad \dfrac{du}{dt} = [u,H] + \dfrac{\partial u}{\partial t}$

- Relations for Hamilton's Principle function, $S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n, t)$

$$K = 0 \ , \qquad P_i = \alpha_i \ , \qquad Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i} \ , \qquad p_i = \frac{\partial S}{\partial q_i}$$

- Relations for Hamilton's Characteristic function, $W = W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n)$

$$K = H = \alpha_1 \ , \qquad P_i = \alpha_i \ , \qquad \beta_1 + t = \frac{\partial W}{\partial \alpha_1} \ , \qquad \beta_{i>1} = \frac{\partial W}{\partial \alpha_i} \ , \qquad p_i = \frac{\partial W}{\partial q_i}$$

- Action Angle Variables: $\quad J = \oint p\,dq$, $\quad w = \dfrac{\partial W(q,J)}{\partial J}$, $\quad \dot{w} = \dfrac{\partial H(J)}{\partial J} = \nu(J)$

1. **Semi-short answer problems** [15 points]

These problems require less algebra, and a correct answer with no work shown will receive full credit. Your answers should be short.

(a) [3 points] For the Lagrangian $L(x, \dot{x}) = \frac{m}{2}\,\dot{x}^2 + b\,\dot{x} - c\,x^4$ with constants $\{m, b, c\}$, what is the Hamiltonian?

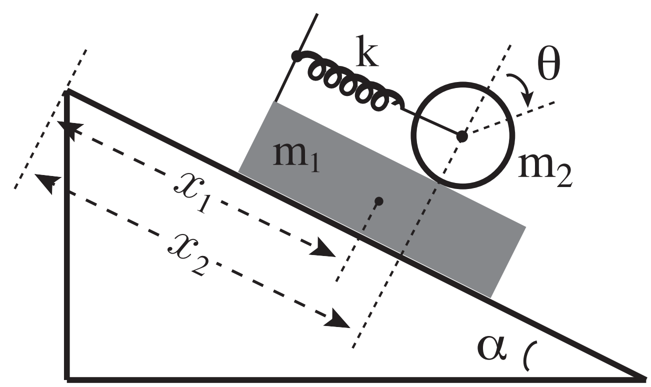
(b) [4 points] For a one-dimensional system with Hamiltonian $H(q, p)$ and rotational motion in phase space, argue that the action variable J is time independent.

(c) [3 points] The Coriolis fictitious force acts on winds that travel in from the outskirts to the low pressure eye of a hurricane. This causes the hurricane to spin in opposite directions in the northern and southern hemispheres. Viewed from space do they spin clockwise or counterclockwise in the northern hemisphere?

(d) [5 points] For a pendulum with energy $E = p_\theta^2/(2ma^2) - mga\cos\theta$ sketch a phase space diagram. Be sure to include one oscillatory trajectory, and a trajectory for rotational motion in each of the two possible directions.

2. **A Sliding Oscillator** [25 points]

Consider a block of mass m_1, which slides without friction on a fixed wedge. On top of this block is a spring attached to a hoop of mass m_2 and radius a (and hence it has a moment of inertia $I = m_2 a^2$). The spring constant is k and when the spring is at its relaxed length the hoop sits above the center of the block (indicated by the black dot). The spring is attached to a frictionless bearing at the center of the hoop, which enables the hoop to rotate freely. Gravity acts on the system.



(a) [8 points] Start by assuming that the contact point between the block and hoop is frictionless. What is the Lagrangian for this system? What are the equations of motion?

(b) [2 points] Now assume that the hoop rolls on the block without slipping. What is the no slip constraint?

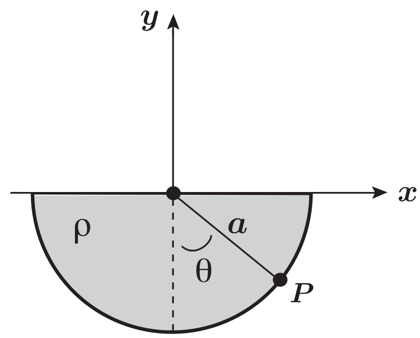
(c) [9 points] Derive a result for the force associated to your no slip constraint from (b) in terms of the coordinates x_1 and x_2. Your result should not involve time derivatives.

(d) [6 points] Show that the relative motion of the hoop and block is that of a simple harmonic oscillator and determine the corresponding frequency.

(continue)

3. **A Half Disk** [30 points]

Consider a thin half disk of radius a with uniform surface density ρ as shown. Here the z-axis is coming out of the page.



(a) [2 points] What is the mass m of the half disk?

(b) [8 points] What is the half disk's moment of inertia tensor for the (x,y,z) axes in terms of a and m? [Hint: exploit symmetry if you can.]

(c) [10 points] Let the center of mass of the disk be at $\vec{R} = -d\,\hat{y}$. (Do not calculate d.) What is the moment of inertia tensor for rotations about the point P on the edge of the disk with axes parallel to (x,y,z)? Write your answer using a, m, d, and θ.

(d) [6 points] Now take this half disk to carry out simple oscillations on a horizontal table, balancing on its rounded edge without slipping. It is acted upon by gravity g. Find its Lagrangian in terms of the angle θ and $\dot{\theta}$ and constant parameters. [Hint: There are two potential ways you might calculate the kinetic energy here, and one of them is much easier than the other.]

(e) [4 points] Would your answer in (a) change if you were to halve the density of the disk? What about your answer in (b)? Next, would your answer in (b) change if you were to take the original half disk of density ρ and attach a second half disk with density $\rho/2$, in order to obtain a full disk with non-uniform density? (Yes or No answers to these 3 questions suffice.)

You might find it useful to recall that $\int_{-\pi/2}^{\pi/2} d\theta\, \cos^2\theta = \int_{-\pi/2}^{\pi/2} d\theta\, \sin^2\theta = \pi/2$

(continue)

4. **A Strange Hamiltonian** [30 points]

In this problem we will find solutions for a one-dimensional system with a generalized coordinate $q = q(t)$ and canonical momentum $p = p(t)$ that are governed by the Hamiltonian

$$H(q,p) = \frac{q^4 p^2}{2\mu} + \frac{\lambda}{q^2}\,. \tag{1}$$

Here $\mu > 0$ and $\lambda > 0$ are positive constants.

(a) [2 points] What are the Hamilton equations of motion for q and p?

(b) [5 points] Consider the change of variable

$$Q = \gamma\, q^a\,, \qquad\qquad P = q^b\, p\,. \tag{2}$$

What relations between the three constants a, b, and γ are needed to guarantee that the new variables Q, P are canonical?

(c) [10 points] Find a choice for the constants in your transformation from part (b) so that you can write $H(q,p) = K(Q,P)$, where the new Hamiltonian $K(Q,P)$ has a harmonic oscillator type form. Find a solution for $Q = Q(t)$ and $P = P(t)$, and then obtain $q = q(t)$ and $p = p(t)$. Your answers should involve μ and λ, plus two constants you introduce that would be used to satisfy initial conditions.

Lets check the answer in (c) by solving the original Hamiltonian in Eq. (1) with a different method. [The questions below can be done without correctly answering the parts above.]

(d) [5 points] What is the time-independent Hamilton-Jacobi equation for the Hamiltonian in Eq. (1)? Solve for Hamilton's Characteristic function W, leaving your result in terms of an integral.

(e) [8 points] Use your result from (d) and the Hamilton-Jacobi method to solve for $q = q(t)$. Your answer should agree with part (c), up to perhaps different choices for the constants which could be fixed by initial conditions. [NOTE: You are not being asked to find $p = p(t)$ here, but if you decide to then you'll also have a check on your answer for $p(t)$ in part (c).]

You may find the following integral useful:
$\int dz\, z^{-1}(z^2 - z_0^2)^{-1/2} = (-1/z_0)\sin^{-1}(z_0/z)$.

(the end)