Exercises for Lecture 3,4,5
Foundations of Stat Mech

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1 Time averages by Hamiltonian evolution

Be $\mathcal{U}(X) = e^{iHt} X e^{-iHt} \equiv X_t$ the time evolution generated by a non degenerate Hamiltonian $H = \sum_n E_n P_n$ and $\phi_t = \mathcal{U}\phi = e^{1iHt}\phi e^{iHt}$ the time evolved state. Let p_n be the probability of being in the state P_n, namely $p_n = \mathrm{tr}(\phi P_n)$.

1.1 Time averaged state

Let ϕ be a pure state with amplitudes $|\phi\rangle = \sum_n a_n |E_n\rangle$. Show that

$$\overline{\phi} = \sum_n P_n \phi P_n = \sum_n p_n P_n = \sum_n |a_n|^2 P_n = D_H \phi \tag{1}$$

and compute its purity.

1.2 Average Loschmidt Echo

Define the Loschmidt echo as $\mathcal{L}_t = |\langle\phi|\,\phi_t\rangle|^2$. Show that $\overline{\mathcal{L}_t} = \mathrm{Tr}\overline{\phi}^2$.

1.3 Typicality in time (challenge)

Define $\omega = D_H\phi$. Be H non-degenerate and having non degenerate gaps. Show that

$$\overline{D(\phi_{St}, \omega_S)} \le \frac{1}{2}\sqrt{d_S^2 \mathrm{tr}\omega^2} \tag{2}$$

2 Averages in $\mathcal{H}$

2.1 Haar-average Loschmidt Echo

Show that

$$\langle \overline{\mathcal{L}_t} \rangle_\phi = \frac{2}{d+1} \tag{3}$$

2.2 Haar-average purity

Show that

$$\langle \mathrm{tr}\omega_A^2 \rangle_\phi = \frac{d_A + d_B}{d_A d_B + 1} \tag{4}$$

2.3 Purity of the dephased state

Let $\omega = D_H \phi$. Show that

$$\langle \mathrm{tr}\omega^2 \rangle_\phi < \frac{2}{d} \tag{5}$$

2.4 Typicality of canonical state

Let $\omega = D_H \phi$. Also define $\Omega_S \equiv \langle \mathrm{tr}_B \Omega \rangle_\phi = \langle \omega_S \rangle_\phi$. Be $d(\cdot, \cdot)$ the trace distance. Show that

$$\langle d\left(\omega_S, \frac{I_S}{d_S}\right) \rangle_\phi \le \frac{1}{2}\sqrt{\frac{1}{d_B}} \tag{6}$$

2.5 General canonical state

Consider a subspace $\mathcal{H}_R \subset \mathcal{H}$ and let Π_R the projector on it. If we take the Haar average on $\mathcal{H}_R$ one has

$$\langle \phi^{\otimes 2} \rangle_\phi = \frac{\Pi_R^{\otimes 2}\left(I^{\otimes 2} + T^{(2)}\right)}{d_R(d_R+1)} \tag{7}$$

Then show that, in this subspace, the bound of the previous exercise holds as

$$\langle d\left(\omega_S, \frac{\Omega_S}{d_S}\right) \rangle_\phi \le \frac{1}{2}\sqrt{\frac{d_S}{d_R}} \tag{8}$$

Also show that, in the same subspace, the bound on purity Eq.(5) reads $\langle \mathrm{tr}\omega^2 \rangle_\phi < \frac{2}{d_R}$.