8.(3)09 Final Exam Solutions 2019

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1. (a) Poisson Brackets
(3)

$$[Q_1, Q_2] = 0$$
$$[P_1, P_2] = 0$$
$$[Q_i, P_j] = \delta_{ij}$$

(b) Take $G = P_z$. This generates translation in z direction
(2)

or $G = L_z$ " " rotation about $\hat{z}$

or $G = H$ " " evolution in time

(c) Kaplan-Yorke: $d_F = 1 + \frac{\lambda_1}{|\lambda_2|}$ for $\lambda_2 < 0 < \lambda_1$ $(\lambda_3 = 0)$
(4)

d_F = fractal dimension, λ_i describe contraction / expansion of phase space trajectories that start out differing by δ_{i0}

$$\delta_i \simeq \delta_{i0}\, e^{\lambda_i t} \quad \text{for } i = 1, 2 \text{ directions}$$

(d) Two of:
(3)

- chaotic
- larger dissipation of energy
- vortices at all scales
- effectively transports energy (or momentum or temperature)
- reduces drag on boundary layer of an object

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2. 27

(a) (15)

$$H = \frac{1}{2\mu}\left(p_r^2 + \frac{p_\psi^2}{r^2}\right) - k/r = \alpha_1 \quad \text{constant} \qquad 1.$$

ψ cyclic: $P_\psi = \alpha_\psi$ = constant, $W = W_1(r) + \alpha_\psi \psi$ 3.

H.J eqtn: $$\frac{1}{2\mu}\left[\left(\frac{\partial W_1}{\partial r}\right)^2 + \frac{\alpha_\psi^2}{r^2}\right] - k/r = \alpha_1$$ 5.

$$\frac{\partial W_1}{\partial r} = \left[2\mu\left(\alpha_1 + \frac{k}{r}\right) - \frac{\alpha_\psi^2}{r^2}\right]^{1/2} \Rightarrow \boxed{W = \alpha_\psi \psi + \int dr \left[2\mu\left(\alpha_1 + \frac{k}{r}\right) - \frac{\alpha_\psi^2}{r^2}\right]^{1/2}}$$

Transfm Eqtns: 3.

$$\boxed{t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \int \frac{\mu\, dr}{\left[2\mu(\alpha_1 + k/r) - \alpha_\psi^2/r^2\right]^{1/2}} = t(r)}$$ radial motion

constants

3.

$$\boxed{\beta_2 = \frac{\partial W}{\partial \alpha_\psi} = \psi - \int \frac{\alpha_\psi\, dr}{r^2\left[2\mu(\alpha_1 + k/r) - \alpha_\psi^2/r^2\right]^{1/2}} = \psi(r)}$$ orbital equation

2. (b) (12)

$$\phi(r,\theta,z) = c\, r^a \cos(b\theta + d)$$

Need $\nabla^2\phi = 0 = \left(\frac{1}{r}\frac{\partial}{\partial r}\underbrace{r\frac{\partial}{\partial r} r^a}_{a r^a}\right) c\cos(b\theta+d) + r^{a-2}\, c\, \frac{\partial^2}{\partial\theta^2}\cos(b\theta+d)$

$$= a^2\left(r^{a-2} c\cos(b\theta+d)\right) - b^2\left(r^{a-2} c\cos(b\theta+d)\right)$$

$\therefore$ $\boxed{a^2 = b^2}$ 3.

Origin $r=0$ stagnation point. $\vec{v}(r=0) = 0$

$\vec{v} = \nabla\phi = \hat{r}\frac{\partial\phi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \hat{r}\, c\, a\, r^{a-1}\cos(b\theta+d)$

$- \hat{\theta}\, c\, r^{a-1}\, b\sin(b\theta+d)$

4.

$\therefore$ $\boxed{a>1}$ so $a = b > 1$

Require $\hat{\theta}\cdot\vec{v} = 0$ for $\theta = \alpha$ and $\theta = -\alpha$

$\sin(b\alpha + d) = 0$

$\sin(-b\alpha + d) = 0$

$\therefore$ take $b\alpha + d = \pi$, $-b\alpha + d = 0$ $\Rightarrow$ $\boxed{d = \pi/2,\quad b = \frac{\pi}{2\alpha}}$ 5.

or $b\alpha + d = 0$, $-b\alpha + d = \pi$ $\Rightarrow$ $d = \pi/2$, $b = -\frac{\pi}{2\alpha}$

or $\pi \to -\pi$ in one of above solutions is also OK

or $b\alpha + d = n\pi$, $-b\alpha + d = m\pi$ $\quad d = \frac{(n+m)\pi}{2}$, $b = \frac{(n-m)\pi}{2\alpha}$ (general solution)

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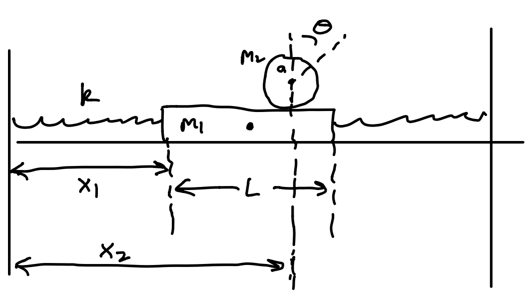
3. 26



$I = M_2 a^2$ for hoop

3 (a) constraint $\dot{x}_2 - \dot{x}_1 = a\dot{\theta}$ ($x_2 - x_1 = a\theta + K$ also OK)

$g = \dot{x}_2 - \dot{x}_1 - a\dot{\theta} = 0$

5 (b) $T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{m_2 a^2}{2}\dot{\theta}^2$ 3.

length of 2nd spring $= 3L - L - x_1$

$V = \frac{k}{2}x_1^2 + \frac{k}{2}(2L - x_1)^2$ 2.

$$\boxed{L = T - V = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{m_2 a^2}{2}\dot{\theta}^2 - \frac{k}{2}x_1^2 - \frac{k}{2}(2L - x_1)^2}$$

3c. 6

x_1 // $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = \lambda\frac{\partial g}{\partial \dot{x}_1} \Rightarrow m_1\ddot{x}_1 + kx_1 + k(x_1 - 2L) = -\lambda$

$m_1\ddot{x}_1 + 2k(x_1 - L) = -\lambda$ 2.

x_2 // $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = \lambda\frac{\partial g}{\partial \dot{x}_2} \Rightarrow m_2\ddot{x}_2 = \lambda$ 2.

θ // $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial g}{\partial \theta} \Rightarrow m_2 a^2\ddot{\theta} = -a\lambda \quad \therefore \quad m_2 a\ddot{\theta} = -\lambda$ 2.

(d.) 6 solve with $g = 0 = \dot{x}_2 - \dot{x}_1 - a\dot{\theta} \Rightarrow \ddot{x}_2 = \ddot{x}_1 + a\ddot{\theta}$ 2.

$$\Rightarrow \frac{\lambda}{m_2} = \left(\frac{-\lambda - 2k(x_1 - L)}{m_1}\right) + \left(\frac{-\lambda}{m_2}\right)$$

$$\frac{2\lambda}{m_2} + \frac{\lambda}{m_1} = -\frac{2k}{m_1}(x_1 - L) = \lambda\left(\frac{2m_1 + m_2}{m_1 m_2}\right)$$

$$\therefore \quad \boxed{\lambda = \frac{-2km_2}{(2m_1 + m_2)}(x_1 - L)}$$ 4.

(e) 5 block $\ddot{x}_1 = -\frac{2k}{m_1}(x_1 - L) - \frac{\lambda}{m_1} = -\frac{2k}{m_1}(x_1 - L)\left[1 - \frac{m_2}{2m_1 + m_2}\right]$

$= \frac{-4k}{2m_1 + m_2}(x_1 - L)$ Harmonic Osc in $x_1' = x_1 - L$

freq. $\boxed{\omega = \sqrt{\frac{4k}{2m_1 + m_2}}}$ 4.

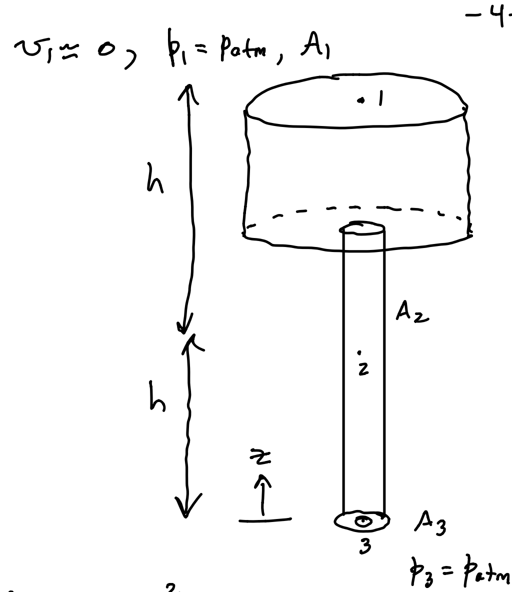
hoop $\ddot{x}_2 + a\ddot{\theta} = 0$ & $\ddot{x}_2 = \ddot{x}_1 + a\ddot{\theta}$

so $2\ddot{x}_2 = \ddot{x}_1 = -2a\ddot{\theta}$

hoop will have same oscillation frequency as block 2.

only amplitude change.

4. 27



(a) Ideal fluid. $z=0$ at 3.

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<u>continuity</u>

$$\int dV\, \nabla\cdot(\rho \vec{v}) = \rho \int ds\, \hat{n}\cdot\vec{v} = 0$$

so $A_2 v_2 = A_3 v_3$

$$v_2 = \frac{A_3}{A_2} v_3 = \frac{v_3}{4} \qquad 2.$$

<u>Bernoulli</u>

$$\cancel{\frac{v_1^2}{2}}_{0} + 2hg + \frac{p_{atm}}{\rho} = \frac{v_2^2}{2} + hg + \frac{p_2}{\rho} = \frac{v_3^2}{2} + 0 + \frac{p_{atm}}{\rho}$$

So $v_3^2 = 4gh$

$$\boxed{\begin{aligned} v_3 &= 2\sqrt{gh} \\ v_2 &= v_3/4 = \frac{\sqrt{gh}}{2} \end{aligned}} \qquad 3.$$

and
$$\frac{p_2}{\rho} = \frac{p_{atm}}{\rho} + \frac{v_3^2 - v_2^2}{2} - hg$$
$$= \frac{p_{atm}}{\rho} + \frac{4gh - \frac{1}{4}gh}{2} - hg, \qquad 2 - \frac{1}{8} - 1 = \frac{7}{8}$$

$$\boxed{p_2 = p_{atm} + \frac{7}{8}\rho h g} \qquad 2.$$

(b) As z decreases in cylinder 2 we have same

2

v_2 (by continuity) but zg gravity term decreases

so <u>pressure must increase</u>

(c) (11) $\vec{v} = v_z(r)\,\hat{z}$ $\qquad \nu = \eta/\rho$

$$\underbrace{\frac{\partial \vec{v}}{\partial t}}_{0} + \underbrace{(\vec{v}\cdot\vec{\nabla})\vec{v}}_{v_z \frac{\partial}{\partial z} v_z(r) = 0} + \frac{1}{\rho}\vec{\nabla} p - \nu \nabla^2 \vec{v} = \vec{g} = -g\hat{z}$$

1.

$$\frac{1}{\rho}\vec{\nabla} p - \nu \nabla^2 v_z(r)\hat{z} = -g\hat{z}$$

so $\vec{\nabla} p$ must be in $\hat{z}$ $\therefore$ $p = p(z)$

$$\underbrace{\frac{1}{\rho}\frac{\partial}{\partial z} p(z)}_{\text{function of } z} = \underbrace{\nu \nabla^2 v_z(r) - g}_{\text{function of } r} = \text{constant} = K \qquad 2.$$

$\therefore$ $\dfrac{dp(z)}{dz} = \rho K$ $\qquad \boxed{p(z) = \rho K z + \underbrace{p_0}_{\text{constant}}}$ $\qquad$ 2.

$$\therefore \quad \nu \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} v_z(r) = g + K$$

$$\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} v_z(r)\right) = \frac{g+K}{\nu}\, r$$

wrong! chain rule needed

$$r \frac{\partial}{\partial r} v_z(r) = \left(\frac{g+K}{\nu}\right)\frac{r^2}{2} + a$$

$$\frac{\partial}{\partial r} v_z(r) = \left(\frac{g+K}{\nu}\right)\frac{r}{2} + \frac{a}{r}$$

$$v_z(r) = \left(\frac{g+K}{\nu}\right)\frac{r^2}{4} + a \ln r + b \qquad 3.$$

- nothing special happens at $r = 0$ (non-singular) so $\underline{a = 0}$ $\quad$ 1.
- $v_z(r = R) = 0 \quad \therefore \quad \left(\dfrac{g+K}{\nu}\right)\dfrac{R^2}{4} + b = 0$

$$\underline{b = -\left(\frac{g+K}{\nu}\right)\frac{R^2}{4}} \qquad 2.$$

$$\boxed{v_z(r) = \frac{(g+K)}{4\nu}\left(r^2 - R^2\right)} \qquad \text{where } \nu = \frac{\eta}{\rho}$$

(d) friction force / unit area $= (-\hat{r})_k \, \sigma_{ki} \, (\hat{z})_i$

(4)

← friction force in $+\hat{z}$

$$= -\sigma'_{rz} = -\eta \frac{\partial v_z}{\partial r}$$

$$= -\frac{\rho(g+k)}{4} \, 2r \Big|_{r=R} = -\frac{\rho(g+k)R}{2}$$

$$\boxed{\vec{F}_{friction} = -\frac{\rho(g+k)}{2} R \, \hat{z}}$$

(e) • Energy goes into heating fluid through friction force

(3) • or does work on walls if we don't hold them fixed

(2 of these) • or there is an increase in pressure potential energy as fluid drops

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5. $\boxed{34}$

$$\ddot{x} + a\dot{x}(x^2 + \dot{x}^2 - 1) + x = 0$$

(a) Let $\omega = \dot{x}$ then $\dot{x} = \omega$

(2)

$$\dot{\omega} = -a\omega(x^2 + \omega^2 - 1) - x$$

(b) Fixed Pts $\omega^* = 0$, $x^* = 0$ only

(10)

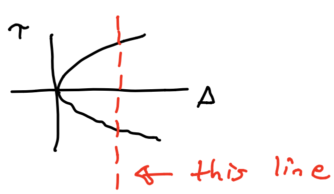
2.

Linear Analysis $\dot{x} = \omega$, $\dot{\omega} = a\omega - x$

$$\frac{d}{dt}\begin{pmatrix} x \\ \omega \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}}_{M} \begin{pmatrix} x \\ \omega \end{pmatrix}$$

$\Delta = \det M = +1$

$\tau = \operatorname{tr} M = a$



$\sqrt{\tau^2 - \Delta} = \sqrt{a^2 - 4}$

$$\lambda_\pm = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

1. $\boxed{a = 0}$ center $\dot{x} = \omega$, $\dot{\omega} = -x$

1. $\boxed{0 < a < 2}$ unstable spiral $\lambda_\pm$ complex

2. $\boxed{a=2}$ $\lambda_\pm = 1$ one eigenvalue

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \vec{a} = \vec{a}, \quad \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \vec{a} = 0$$

$\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ only 1-eigenvector $\therefore$ unstable degenerate node

1. $\boxed{a > 2}$ $\lambda_\pm$ real distinct unstable node

1. $\boxed{-2 < a < 0}$ stable spiral

1. $\boxed{a = -2}$ $\lambda_\pm = -1$ $\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \vec{a} = -\vec{a}$, $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{a} = 0$

so $\vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ 1 eigenvector stable degenerate node

1. $\boxed{a < -2}$ stable node

(c) limit cycle $x^2 + \omega^2 = 1$ $\dot{x} = \omega$, $\dot{\omega} = -x$ $x = \sin t$, $\omega = \cos t$

7

radial coords: $x = r\sin\theta$ $\omega = r\cos\theta$ $r^2 = x^2 + \omega^2$

$$r\dot{r} = x\dot{x} + \omega\dot{\omega} = x\omega + (-x\omega) - a\omega^2(r^2-1)$$

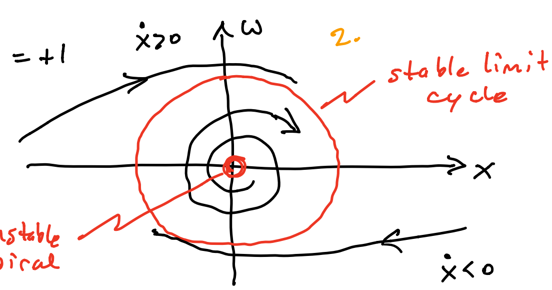
$$= -a\, r^2\cos^2\theta\,(r^2-1)$$ 3. $a \neq 0$

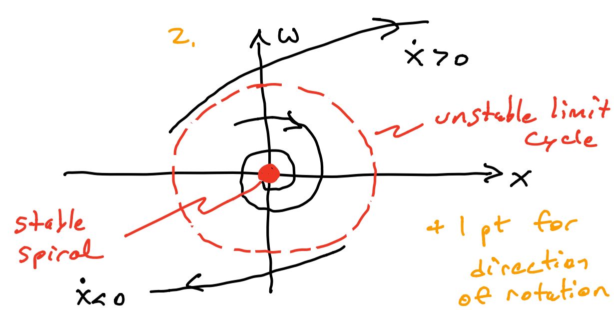
if $a > 0$ $r > 1$ $\dot{r} < 0$; $r < 1$ $\dot{r} > 0$ $\therefore$ stable limit cycle (& isolated) 2.

if $a < 0$ $r > 1$ $\dot{r} > 0$; $r < 1$ $\dot{r} < 0$ $\therefore$ unstable limit cycle 2.

(d) $a = +1$

5





5e 10

$$x_{n+1} = r x_n - x_n^3 = f(x_n), \qquad f'(x) = r - 3x^2 \quad 2.$$

$$|f'(x^*)| < 1 \text{ stable}, \qquad |f'(x^*)| > 1 \text{ unstable}$$

<u>fixed points</u> $\quad x^* = r x^* - x^{*3} \quad \therefore \quad (r-1)x^* = x^{*3}$

so $\quad x^* = 0 \qquad f'(0) = r \quad$ so $\quad 0 < r < 1$ stable
$\qquad 1 < r < 3/2$ unstable $\qquad 4.$

$x^* = +\sqrt{r-1}$
$x^* = -\sqrt{r-1}$ $\quad$ <u>only if $r > 1$</u> $\qquad f'(x^*) = r - 3(r-1) = 3 - 2r$ for both $\qquad 4.$

so stable $\quad 1 < r < \frac{3}{2}$

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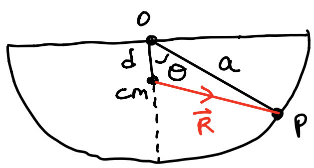
6. 24

a 10



Use // axes theorem for I_{zz}'s

$$\vec{R} = \hat{x}\, a \sin\theta + \hat{y}\,(d - a\cos\theta) \qquad 2.$$

$$I^P_{zz} = I^{cm}_{zz} + M\left(\delta_{zz}\vec{R}^2 - R_z R_z\right) \qquad 2.$$

$$\& \qquad I^0_{zz} = I^{cm}_{zz} + M d^2$$

$$I^0_{zz} = \frac{M}{\frac{1}{2}\pi a^2}\int_0^a dr\, r \int_\pi^{2\pi} d\theta\, (x^2+y^2) = \frac{2M}{\pi a^2}\,\pi\,\frac{a^4}{4} = \frac{Ma^2}{2}$$

$$I^{cm}_{zz} = M\left(\frac{a^2}{2} - d^2\right) \qquad 3.$$

$$I^P_{zz} = M\left(\frac{a^2}{2} - d^2\right) + M\left(a^2\sin^2\theta + (d - a\cos\theta)^2\right)$$

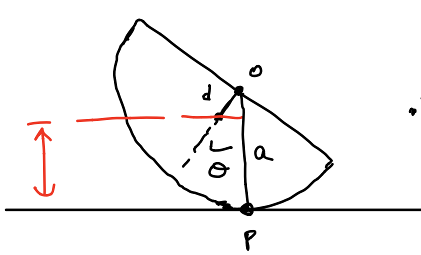
$$= M\frac{a^2}{2} - \cancel{Md^2} + Ma^2 + \cancel{Md^2} - 2Mad\cos\theta$$

$$= M\left(\frac{3}{2}a^2 - 2ad\cos\theta\right) \qquad 3.$$

← don't have to simplify

b 8 $\quad L = T - V$

height $= a - d\cos\theta$



$$\therefore \quad \boxed{V = Mg(a - d\cos\theta)} \qquad 4.$$

T: contact point P is instantaneously at rest
rotates about $\hat{z}'$ axis through P, angular velocity $\omega = \dot{\theta}\hat{z}'$

$$\boxed{T = \frac{M}{2}\dot{\theta}^2\left[\frac{3}{2}a^2 - 2ad\cos\theta\right]}$$ 4.

(c) At $t=0$ $\theta=0$ & $\dot{\theta} = \frac{v_0}{a}$ 1.

(6)

Use energy conservation $E = T + V$ 1.

at $t=0$, $\theta=0$: $E = \frac{M}{2}\frac{v_0^2}{a^2}\left[\frac{3}{2}a^2 - 2ad\right] + mg(a-d)$

at $\theta = \frac{\pi}{2}$ need $\dot{\theta}=0$ to not tip over

$$E = 0 + mga$$

$$\therefore \quad \frac{M}{2}v_0^2\left(\frac{3}{2} - 2\frac{d}{a}\right) + \cancel{mg}a - mgd = \cancel{mg}a$$ 4.

So $$\boxed{v_0 = \left[\frac{2gd}{\frac{3}{2} - \frac{2d}{a}}\right]^{\frac{1}{2}}} = \left[\frac{4gd}{3 - 4d/a}\right]^{1/2}$$

The END