THE QUANTUM ISING CHAIN FOR BEGINNERS

Sep 28, 2020

1 The Jordan-Wigner transformation

There are techniques to deal with large assemblies of bosons & fermions. But not with spin systems.

→ need a way to "map" the hard problem to easy!

→ Consider single spin-$\frac{1}{2}$ ⇒ 3 components of spin operator

- σ^x, σ^y, σ^z. Hilbert space is $\{|\uparrow\rangle, |\downarrow\rangle\}$
- Eigenstates: $\begin{cases} \sigma^z|\uparrow\rangle = |\uparrow\rangle \\ \sigma^z|\downarrow\rangle = -|\downarrow\rangle \end{cases}$
- Commutation relation (from angular momentum $J^i \leftrightarrow \sigma^i$)

index

$$\boxed{[\sigma_j^i, \sigma_{j'}^{i'}] = 0}, \quad \boxed{[\sigma_j^x, \sigma_j^y] = 2i\sigma_j^z}$$

site

cyclic

same site, obey usual comm. relation (can be written with ε^{ijk} ...)

- Define $\sigma^{\pm} = \dfrac{\sigma_j^x \pm i\sigma_j^y}{2}$

gives $\boxed{\sigma^+|\downarrow\rangle = |\uparrow\rangle, \quad \sigma^-|\uparrow\rangle = |\downarrow\rangle}$

& $\boxed{\{\sigma_j^+, \sigma_j^-\} = \mathbb{1}}$ → anticommutator, typical of rules for fermions

→ Should we describe spins w/ bosons or fermions?

→ Let's start w/ bosons... (hard...)

Suppose have single boson $\hat{b}^+$ with associated vacuum state $|0\rangle$ s.t. $\hat{b}|0\rangle = |0\rangle$

then because $[\hat{b}, \hat{b}^+] = 1$, can have

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^+)^n|0\rangle \qquad n = 0, 1, \dots, \infty$$

→ Now, since we want only spin-$\frac{1}{2}$ ⇒ truncate Hilbert space so that $(\hat{b}^+)^2|0\rangle = 0$ ⇒ get sth like Hilbert space of single spin-$\frac{1}{2}$.

(!) this kind of truncation is called "hard-core boson"

Now, how do we relate $\hat{b}^+$, $\hat{b}$, to the Pauli matrices?

→ Observe that if we identify $\begin{cases} |0\rangle \leftrightarrow |\uparrow\rangle \\ |1\rangle \leftrightarrow |\downarrow\rangle \end{cases}$

then $\hat{b}^+|0\rangle = |1\rangle \longleftrightarrow |\downarrow\rangle = \hat{\sigma}^-|\uparrow\rangle$
and so on...

so
$$\boxed{\begin{cases} \hat{\sigma}_j^+ \equiv \hat{b}_j \\ \hat{\sigma}_j^- \equiv \hat{b}_j^+ \\ \hat{\sigma}_j^z \equiv 1 - 2\hat{b}_j^+\hat{b}_j \end{cases} \rightarrow \begin{cases} \hat{\sigma}_j^x = \hat{b}_j^+ + \hat{b}_j \\ \hat{\sigma}_j^y = i(\hat{b}_j^+ - \hat{b}_j) \\ \hat{\sigma}_j^z = 1 - 2\hat{b}_j^+\hat{b}_j \end{cases}}$$

• Note that $[\hat b_j, \hat b_i] = 0 = [\hat b_j^\dagger, \hat b_i^\dagger]$ $(j \neq i)$

(like how $\vec\sigma$ commutes @ different sites)

→ But $\hat b_j^\dagger, \hat b_j$ are not ordinary bosonic operators.

• Also ~~Note~~ that b/c of the truncation, $(\hat b_j^\dagger)^2|0\rangle = 0$, and that

$$\{\hat b_j, \hat b_j^\dagger\} = \mathbb{1}.$$

⇒ $\boxed{\text{At most one boson is allowed at one site}}$

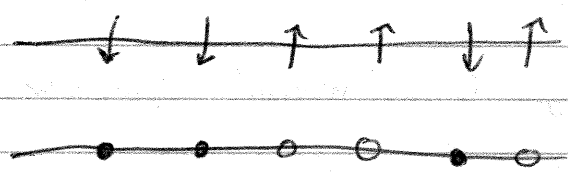
Now, if we pay close attention... the hard-core bosons representation is calling out "spinless fermions" $\hat c_j^\dagger$

↳ Why? b/c the absence of double occupancy is actually enforced by the Pauli Exclusion Princip

& that the anti-commutation rule comes for free!

h.c. bosons

spinless fermions



$$= \hat b_1^\dagger \hat b_2^\dagger \hat b_5^\dagger |0\rangle = \hat c_1^\dagger \hat c_2^\dagger \hat c_5^\dagger |0\rangle$$

There's a difficulty, however. The mapping $\hat\sigma_j \to \hat b_j^\dagger$ can be done in any dimension.

But writing $\hat b_j^\dagger$ in terms of $\hat c_j^\dagger$ is only useful in 1D.

b/c there's a natural ordering of the sites!

What is this mapping $\hat{b}_j \to \hat{c}_j$?

→ the Jordan - Wigner transformation!

$$\boxed{\begin{aligned}\hat{b}_j = \hat{K}_j \hat{c}_j = \hat{c}_j \hat{K}_j \quad \text{where } \hat{K}_j &= e^{i\pi \sum_{j'=1}^{j-1} \overset{\#\text{ operator}}{\hat{n}_{j'}}} \\ &= \prod_{j'=1}^{j-1} (1 - 2\hat{n}_{j'})\end{aligned}}$$

where we've introduced $\overset{\hat{K}_j}{\wedge}$ the "non-local" <u>string operator</u>

→ now, $\hat{K}_j$ is just a sign! $\hat{K}_j = \pm 1$.

→ Intuitively, $\boxed{\hat{K}_j \text{ counts the parity of \# of fermions \underline{before} site } j.}$

Now, $\hat{K}_j = e^{i\pi \sum_{j'=1}^{j-1} \hat{n}_{j'}}$, b/c $\hat{K}_j = \pm 1$

→ $\boxed{\hat{K}_j = \hat{K}_j^{\dagger} = \hat{K}_j^{-1} \text{ , and } \hat{K}_j^{2} = 1.}$

<u>Why?</u> $\hat{n}_j = \hat{c}_j^{\dagger} \hat{c}_j$ is a number operator.

→ We will now show that <u>if</u> we take c_j to be the fermionic operators with the anti-comm. rel.

$$\boxed{\{\hat{c}_j, \hat{c}_{j'}^{\dagger}\} = \delta_{j,j'} \quad , \quad \{\hat{c}_j, \hat{c}_{j'}\} = \{\hat{c}_j^{\dagger}, \hat{c}_{j'}^{\dagger}\} = 0}$$

then the expected properties of the $\hat{b}_j$, $\hat{b}_j^{\dagger}$ will follow...

<u>Same-site property:</u> (anti-commutation relation)

$$\{\hat{b}_j, \hat{b}_j^\dagger\} = 1$$

$$\{\hat{b}_j, \hat{b}_j\} = \{\hat{b}_j^\dagger, \hat{b}_j^\dagger\} = 0$$

<u>Different-site property:</u> (commutation relation)

$$[\hat{b}_j, \hat{b}_{j'}^\dagger] = 0$$

$$[\hat{b}_j, \hat{b}_{j'}] = 0 \quad , \quad [\hat{b}_j^\dagger, \hat{b}_{j'}^\dagger] = 0$$

@ different sites, always commute.

i.e. that $\hat{b}_j^\dagger$'s are hard-core bosons.

→ To show the same-site property, just use the fact that

$$\hat{b}_j^\dagger \hat{b}_j = (\hat{c}_j^\dagger \underbrace{\hat{K}_j^\dagger)(\hat{K}_j}_{\hat{K}_j^2} \hat{c}_j) = \hat{c}_j^\dagger (1) \hat{c}_j = \hat{c}_j^\dagger \hat{c}_j$$

⇒ $\hat{b}_j$'s follow the same anti-comm. relation as $\hat{c}_j$

Similarly … $\hat{b}_j \hat{b}_j^\dagger = \hat{c}_j \hat{c}_j^\dagger$.

Now, to show the different-site property … ~~use the~~

↳ Consider $[\hat{b}_{j_1}, \hat{b}_{j_2}]$, assuming $j_2 > j_1$.

By the JW transform, we find that

$$\boxed{\hat{b}^\dagger_{j_2}\hat{b}_{j_1}} = \hat{c}^\dagger_{j_2}\hat{k}^\dagger_{j_2}\hat{k}_{j_1}\hat{c}_{j_1}$$

$$= \hat{c}^\dagger_{j_2}\underbrace{\left\{e^{-i\pi\sum_{j'=1}^{j_2-1}\hat{n}_{j'}}\right\}\left\{e^{i\pi\sum_{j'=1}^{j_1-1}\hat{n}_{j'}}\right\}}\hat{c}_{j_1}$$

$$= \hat{c}^\dagger_{j_2}\, e^{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}}\,\hat{c}_{j_1}$$

$$= e^{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}}\,\hat{c}^\dagger_{j_2}\hat{c}_{j_1} \qquad K_{j_1} \leftrightarrow c_{j_2}$$

$$= -\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}_{j_1}\hat{c}^\dagger_{j_2} \qquad \leftarrow \text{by anti comm. relation}$$

$$= \boxed{+\hat{c}_{j_1}\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}^\dagger_{j_2}}$$

where the last eq comes from the fact that

$$-\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}_{j_1}\hat{c}^\dagger_{j_2}$$

annihilates site $j_1 \Rightarrow \hat{n}_{j_1} = 0$

whereas

$$\hat{c}_{j_1}\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}^\dagger_{j_2}$$ has $\hat{n}_{j_1} = 1$ since there is no $\hat{c}_{j_1}$ present.

→ differ by (−)

Similarly, can show that

$$\boxed{\hat{b}_{j_1}\hat{b}^\dagger_{j_2} = \hat{c}_{j_1}\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}^\dagger_{j_2}}$$

With these, ... can check that $\boxed{\{\hat{b}_{j_1}, \hat{b}_{j_2}^{\dagger}\} = 0}$

→ all other $\underset{\text{different-site}}{\wedge}$ relations are proven similarly.

Facts

$$\prod_{j'=1}^{j-1}(1-2\hat{n}_{j'})\prod_{j'=1}^{j}(1-2\hat{n}_{j'}) = 1-2\hat{n}_j$$

since $(1-2\hat{n}_{j'})^2 = 1$ & terms with different j's commute.

note true b/c $\hat{n}_j$ can only be 0 or 1.

With this relation, we get ...

- $\hat{b}_j^{\dagger}\hat{b}_j = \hat{c}_j^{\dagger}\hat{c}_j$
- $\hat{b}_j^{\dagger}\hat{b}_{j+1}^{\dagger} = \hat{c}_j^{\dagger}(1-2\hat{n}_j)c_{j+1}^{\dagger} = \hat{c}_j^{\dagger}\hat{c}_{j+1}^{\dagger}$
- $\hat{b}_j^{\dagger}\hat{b}_{j+1} = \hat{c}_j^{\dagger}(1-2\hat{n}_j)\hat{c}_{j+1} = \hat{c}_j^{\dagger}\hat{c}_{j+1}$
- $\hat{b}_j\hat{b}_{j+1} = \hat{c}_j(1-2\hat{n}_j)\hat{c}_{j+1} = \hat{c}_j[1-2(1-\hat{c}_j\hat{c}_j^{\dagger})]\hat{c}_{j+1}$
  $= -\hat{c}_j\hat{c}_{j+1}$
- $\hat{b}_j\hat{b}_{j+1}^{\dagger} = \hat{c}_j(1-2\hat{n}_j)\hat{c}_{j+1} = \hat{c}_j[1-2(1-\hat{c}_j\hat{c}_j^{\dagger})]\hat{c}_{j+1}^{\dagger}$
  $= -\hat{c}_j\hat{c}_{j+1}^{\dagger}$

To summarize, the JW transformation is given by

$$\begin{cases} \hat\sigma_j^x = K_j(\hat c_j^\dagger + \hat c_j) = \hat b_j^\dagger + \hat b_j \\ \hat\sigma_j^y = K_j i(\hat c_j^\dagger - \hat c_j) = i(\hat b_j^\dagger - \hat b_j) \\ \hat\sigma_j^z = 1 - 2\hat n_j = 1 - 2\hat c_j^\dagger \hat c_j = 1 - 2\hat b_j^\dagger \hat b_j \end{cases}$$

where

$$\hat K_j = \prod_{j'=1}^{j-1} (1 - 2\hat n_{j'})$$

Under this map, spin operators become local ferm. op.

$$\hat\sigma_j^z = 1 - 2\hat n_j = (\hat c_j^\dagger + \hat c_j)(\hat c_j^\dagger - \hat c_j)$$

$$\hat\sigma_j^x \hat\sigma_{j+1}^x = \left[\hat c_j^\dagger \hat c_{j+1}^\dagger + \hat c_j^\dagger \hat c_{j+1} + h.c.\right]$$

$$\hat\sigma_j^y \hat\sigma_{j+1}^y = -\left[\hat c_j^\dagger \hat c_{j+1}^\dagger - \hat c_j^\dagger \hat c_{j+1} + h.c.\right]$$

<u>Note</u> a longitudinal field term involving a single $\hat\sigma_j^x$ <u>cannot</u> be translated into a simple <u>local</u> fermionic operator