Part 2: tensor product spaces

All of that was hopefully a review, though maybe from a slightly new perspective.

Now we're going to introduce a 2^{nd} spin:

A B

basis $= \{|\uparrow\rangle_A, |\downarrow\rangle_A\}$ basis $= \{|\uparrow\rangle_B, |\downarrow\rangle_B\}$

Classical intuition says total state is described by giving the state of A and the state of B

eg $A \to |\uparrow\rangle_A$, $B \to |\uparrow\rangle_B$ $\Rightarrow$ total state is $|\uparrow_A \uparrow_B\rangle$

While this is not the full story, it is the right start. This does work for basis states.

Here is the formal definition:

Let $\mathcal{H}_A$ be a Hilbert space w/ basis $\{|0\rangle_A, |1\rangle_A, \dots |n\rangle_A\}$
$\mathcal{H}_B$ " $\{|0\rangle_B, |1\rangle_B, \dots |m\rangle_B\}$

Then $\mathcal{H}_A \otimes \mathcal{H}_B$ is the Hilbert space w/ basis

$$\{|i\rangle_A \otimes |j\rangle_B \mid i \in 1, \dots, n,\ j \in 1, \dots, m\}$$

The combined state of 2 particles must be a vector in $\mathcal{H}_A \otimes \mathcal{H}_B$

Let's be concrete: $2 \times (\text{spin}-\frac{1}{2})$

$\mathcal{H}_A \to$ basis $\{|\uparrow\rangle_A, |\downarrow\rangle_A\}$

$\mathcal{H}_B \to$ basis $\{|\uparrow\rangle_B, |\downarrow\rangle_B\}$

Then the basis for the two-spin space is

$$|\uparrow\rangle_A \otimes |\uparrow\rangle_B, \quad |\uparrow\rangle_A \otimes |\downarrow\rangle_B, \quad |\downarrow\rangle_A \otimes |\uparrow\rangle_B, \quad |\downarrow\rangle_A \otimes |\downarrow\rangle_B$$

Of course, I haven't said yet what $\otimes$ actually means. Basically, A and B are still separate. Here are the key rules:

① Given $|\varphi_A\rangle \otimes |\varphi\rangle_B$ and $|\phi_A\rangle \otimes |\phi_B\rangle$, their inner product is

$$\left(\langle\varphi_A| \otimes \langle\varphi_B|\right)\left(|\phi_A\rangle \otimes |\phi_B\rangle\right) = \langle\varphi_A|\phi_A\rangle \cdot \langle\varphi_B|\phi_B\rangle$$

② Any operator can be written as

$$\mathcal{O}_{AB} = \sum_{ij} o_{ij}\, \mathcal{O}_i^A \otimes \mathcal{O}_j^B$$

where $\left(\mathcal{O}_i^A \otimes \mathcal{O}_j^B\right)\left(|\varphi_A\rangle \otimes |\varphi_B\rangle\right)$

$$= \left(\mathcal{O}_i^A |\varphi_A\rangle\right) \otimes \left(\mathcal{O}_j^B |\varphi_B\rangle\right)$$

③ commuting?

If $[\mathcal{O}_A, \tilde{\mathcal{O}}_A] = 0$ and $[\mathcal{O}_B', \tilde{\mathcal{O}}_B'] = 0$,

$[\mathcal{O}_A \otimes \mathcal{O}_B', \tilde{\mathcal{O}}_A \otimes \tilde{\mathcal{O}}_B'] = 0$

(for bosons)

* All operators must act on both A and B. If you want to act only on B, use $\mathcal{O}_A \otimes Id_B$

③ Born rule for probabilities:

Let $|\psi\rangle = a|\uparrow_A\rangle \otimes |\uparrow_B\rangle + b|\uparrow_A\rangle \otimes |\downarrow_B\rangle + c|\downarrow_A\rangle \otimes |\uparrow_B\rangle + d|\downarrow_A\rangle \otimes |\downarrow_B\rangle$

w/ $|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$

Then $P(\uparrow_A) = P(\uparrow_A, \uparrow_B) + P(\uparrow_A, \downarrow_B) = |a|^2 + |b|^2$

$P(\uparrow_B) = |a|^2 + |c|^2$

$\vdots$

More generally, for

$$|\psi\rangle = \sum_{ij} a_{ij} \, |i_A\rangle \otimes |j_B\rangle$$

$$P(\ell_A) = \sum_j |a_{\ell j}|^2$$

Let's use this (Born rule) to compute an expectation value:

$\langle \sigma^z_A \rangle$ for our $2 \times (\text{spin-}\tfrac{1}{2})$ system w/ a, b, c, d

Expectation value is $1 \cdot P(\text{spin} \uparrow) + -1 \cdot P(\text{spin} \downarrow)$

$$= (|a|^2 + |b|^2) - (|c|^2 + |d|^2)$$

Now let's compute the same expectation value directly.

"σ^z_A" doesn't really exist → actually, it's $\sigma^z_A \otimes Id_B$

So "$\langle \sigma^z_A \rangle$" $= \langle \psi | \sigma^z_A \otimes Id_B | \psi \rangle$

16 total terms

$$|a|^2 \cdot \left(\langle \uparrow_A | \otimes \langle \uparrow_B | \right) \left(\sigma^z_A \otimes Id_B \right) \left(|\uparrow_A\rangle \otimes |\uparrow_B\rangle \right)$$
$$+ \; a^* b \left(\langle \uparrow_A | \otimes \langle \uparrow_B | \right) \left(\sigma^z_A \otimes Id_B \right) \left(|\uparrow_A\rangle \otimes |\downarrow_B\rangle \right)$$
$$+ \; \ldots$$

However Each term becomes just a product like

$$\langle \uparrow_A | \sigma^z_A | \uparrow_A \rangle \cdot \langle \uparrow_B | Id_B | \uparrow_B \rangle$$

Because each individual operator is diagonal, (σ^z, Id)

if A spin is not equal, or if B spin is not equal, get 0.

So only 4 terms are left:

$$\begin{aligned}
&|a|^2 \langle \uparrow_A | \sigma^z_A | \uparrow_A \rangle \cdot \langle \uparrow_B | Id_B | \uparrow_B \rangle \\
+\; &|b|^2 \langle \uparrow_A | \sigma^z_A | \uparrow_A \rangle \cdot \langle \downarrow_B | Id_B | \downarrow_B \rangle \\
+\; &|c|^2 \langle \downarrow_A | \sigma^z_A | \downarrow_A \rangle \cdot \langle \uparrow_B | Id_B | \uparrow_B \rangle \\
+\; &|d|^2 \langle \downarrow_A | \sigma^z_A | \downarrow_A \rangle \cdot \langle \downarrow_B | Id_B | \downarrow_B \rangle
\end{aligned}
= \begin{aligned}
&|a|^2 \cdot 1 \cdot 1 \\
+\; &|b|^2 \cdot 1 \cdot 1 \\
+\; &|c|^2 \cdot -1 \cdot 1 \\
+\; &|d|^2 \cdot -1 \cdot 1
\end{aligned}
= \begin{aligned}
&|a|^2 + |b|^2 \\
&-|c|^2 - |d|^2
\end{aligned}$$

The same!

Of course, doing all of this by hand is a huge pain.

~~So let's go back and figure out the matrix representation.~~

First step: introduce shorthand, with the following rules:

meaning $\mapsto$ what's written

$$|\uparrow_A\rangle \otimes |\uparrow_B\rangle \longmapsto |\uparrow_A \uparrow_B\rangle \quad \text{or even } |\uparrow\uparrow\rangle$$

$$\mathcal{O}_A \otimes Id_B \mapsto \mathcal{O}_A$$

Then the last expectation value calc looks like $\left(|\psi\rangle = \sum_{\substack{\sigma\sigma' \\ =\uparrow\downarrow}} a_{\sigma\sigma'} |\sigma\sigma'\rangle \right)$

$$\langle \sigma^z_A \rangle = \sum_{\substack{\sigma\sigma' \\ ss'}} a_{\sigma\sigma'}{}^* a_{ss'} \langle \sigma\sigma' | \sigma^z_A | ss' \rangle$$

$$= \sum_{\substack{\sigma\sigma' \\ ss'}} a_{\sigma\sigma'}{}^* a_{ss'} \langle \sigma | \sigma^z | s \rangle \langle \sigma' | s' \rangle$$

$$= \sum_{\substack{\sigma\sigma' \\ ss'}} a_{\sigma\sigma'}{}^* a_{ss'} \langle \sigma | \sigma^z | s \rangle \delta_{\sigma' s'}$$

$$= \sum_{s'} \sum_{\sigma s} a_{\sigma s'}{}^* a_{ss'} \langle \sigma | \sigma^z | s \rangle$$

skip

This helps a little, but it will still be much better if we can again develop a matrix approach.

We use the same steps as before:

① Assign basis vectors $\left[\text{eg } |\uparrow\rangle \to \begin{pmatrix}1\\0\end{pmatrix},\ |\downarrow\rangle \to \begin{pmatrix}0\\1\end{pmatrix} \right]$

② Find what the operators look like in this representation.

$$\left[\begin{array}{c} \text{eg} \quad \sigma^z|\uparrow\rangle = |\uparrow\rangle, \ \sigma^z|\downarrow\rangle = -|\downarrow\rangle \\ \Downarrow \\ \sigma^z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \end{array} \right]$$

There is again flexibility / possible change of basis, but the standard choice would be

$$|\uparrow\uparrow\rangle \to \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\quad |\uparrow\downarrow\rangle \to \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\quad |\downarrow\uparrow\rangle \to \begin{pmatrix}0\\0\\1\\0\end{pmatrix},\quad |\downarrow\downarrow\rangle \to \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

Let's construct $\mathcal{O} = \sigma^z_A \otimes Id_B$:

$$\mathcal{O}|\uparrow\uparrow\rangle = |\uparrow\uparrow\rangle \qquad \mathcal{O}|\downarrow\uparrow\rangle = -|\downarrow\uparrow\rangle$$

$$\mathcal{O}|\uparrow\downarrow\rangle = |\uparrow\downarrow\rangle \qquad \mathcal{O}|\downarrow\downarrow\rangle = -|\downarrow\downarrow\rangle$$

so $\mathcal{O}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix},\ \mathcal{O}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},\ \mathcal{O}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = -\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\ \mathcal{O}\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = -\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$

ie $\quad \sigma^z_A \otimes Id_B = \begin{pmatrix}1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}$

Let's check the expectation value again:

$$(a^* \ b^* \ c^* \ d^*)\begin{pmatrix}1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

$$= (a^* \ b^* \ c^* \ d^*)\begin{pmatrix}a\\b\\-c\\-d\end{pmatrix} = |a|^2 + |b|^2 - |c|^2 - |d|^2 \ \checkmark$$

<u>Exercise</u>: ① construct a matrix in this basis for

$$\sigma^y_A \otimes \sigma^z_B$$

② construct a matrix for $\sigma^z_A \otimes \sigma^y_B$

~~Your answer should be the same but w/ the middle 2×2 block of the matrix rotated $180°$ like~~

$\cancel{\begin{pmatrix}a & b\\ c & d\end{pmatrix} \to \begin{pmatrix}d & c\\ b & a\end{pmatrix}}$

Explain using a change of basis matrix the relationship between the two

Answers: ① $\sigma^Y \otimes \sigma^Z |\uparrow\uparrow\rangle = i|\downarrow\uparrow\rangle$

$\sigma^Y \otimes \sigma^Z |\uparrow\downarrow\rangle = -i|\downarrow\downarrow\rangle$

$\sigma^Y \otimes \sigma^Z |\downarrow\uparrow\rangle = -i|\uparrow\uparrow\rangle$

$\sigma^Y \otimes \sigma^Z |\downarrow\downarrow\rangle = i|\uparrow\downarrow\rangle$

$$\rightarrow \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}$$

② $\sigma^Z \otimes \sigma^Y |\uparrow\uparrow\rangle = i|\uparrow\downarrow\rangle$

$|\uparrow\downarrow\rangle \mapsto -i|\uparrow\uparrow\rangle$

$|\downarrow\uparrow\rangle \mapsto -i|\downarrow\downarrow\rangle$

$|\downarrow\downarrow\rangle \mapsto i|\downarrow\uparrow\rangle$

$$\begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

~~Operator for A and B swapped, but~~

$\cancel{\mathcal{O}_A \otimes \mathcal{O}_B |\sigma_A\rangle \otimes |\sigma_B\rangle =}$

Swapping operators for A and B is the same as just writing states in opposite order:

Matrix of $\sigma^Z_A \otimes \sigma^Y_B$ in basis $|\uparrow_A\uparrow_B\rangle, |\uparrow_A\downarrow_B\rangle, |\downarrow_A\uparrow_B\rangle, |\downarrow_A\downarrow_B\rangle$

is the same as

$\sigma^Z_B \otimes \sigma^Y_A$ in basis $|\uparrow_B\uparrow_A\rangle, |\uparrow_B\downarrow_A\rangle, |\downarrow_B\uparrow_A\rangle, |\downarrow_B\downarrow_A\rangle$

is the same as

$\sigma^Y_A \otimes \sigma^Z_B$ in basis $|\uparrow_A\uparrow_B\rangle, |\downarrow_A\uparrow_B\rangle, |\uparrow_A\downarrow_B\rangle, |\downarrow_B\downarrow_A\rangle$

so the effect of swapping the operators on A and B is just to reorder the basis. Do this by swapping two center cols and 2 center rows.

Or by change of basis matrix $\begin{pmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{pmatrix}$

Now, notice something:

$\sigma^Y_A \otimes \sigma^Z_B$ can be written as $\left(\begin{array}{cc|cc} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ \hline i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{array}\right) = \begin{pmatrix} 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & -i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ i \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}$

Like σ^Y w/ σ^Z inserted as a multiplication at each elt.

$\sigma^z_A \otimes \sigma^y_B$ can likewise be written as $\begin{pmatrix} 1\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & 0\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ 0\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & -1\cdot\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \end{pmatrix}$

In fact, we can think of the basis states the same way:

$$|\uparrow\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$|\uparrow\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

This is called a <u>Kronecker product</u>

<u>Why does this work?</u>

① <u>Proof</u>: consider a basis state $|i\rangle_A \otimes |j\rangle_B$

and operator $O = O_A \otimes O_B$

Let matrix of O_A have elts $O_{ki} \longrightarrow O_A = \begin{array}{c} \\ \langle 0| \\ \langle 1| \end{array} \begin{array}{c} \begin{array}{cc} |0\rangle & |1\rangle \end{array} \\ \begin{pmatrix} o_{11} & o_{12} \\ o_{21} & o_{22} \end{pmatrix} \end{array}$

O_B $\qquad \tilde{o}_{lj} \longrightarrow O_B = \begin{pmatrix} \tilde{o}_{11} & \tilde{o}_{12} \\ \tilde{o}_{21} & \tilde{o}_{22} \end{pmatrix}$

Then $\displaystyle O|ij\rangle = \sum_{kl} o_{ki}\tilde{o}_{lj}\,|kl\rangle$

$$\langle kl|O|ij\rangle = O_{ki}\tilde{o}_{lj} \longrightarrow O = \begin{array}{c} \langle kl| \\ \langle 00| \\ \langle 01| \\ \\ \end{array} \begin{array}{c} \begin{array}{cc} |ij\rangle \\ |00\rangle & |01\rangle \end{array} \\ \begin{pmatrix} o_{11}\tilde{o}_{11} & o_{11}\tilde{o}_{12} & \\ o_{11}\tilde{o}_{21} & o_{11}\tilde{o}_{22} & \cdots \\ & \vdots & \end{pmatrix} \end{array}$$

So it definitely is correct.

② <u>Intuition</u>

same action on A $\qquad$ same on A

$$\begin{array}{rl} & \begin{array}{cccc} |\uparrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\uparrow\rangle & |\downarrow\downarrow\rangle \end{array} \\ \text{same on A} \begin{array}{l} \langle\uparrow\uparrow| \\ \langle\uparrow\downarrow| \end{array} & \left(\begin{array}{cccc} & & & \\ & & & \\ & & & \\ & & & \end{array} \right) \\ \text{same on A} \begin{array}{l} \langle\downarrow\uparrow| \\ \langle\downarrow\downarrow| \end{array} & \end{array}$$

action on B should get repeated here

3) Intuition version 2

Keep the tensor product idea:

$$|\uparrow\rangle \otimes |\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{array}{c|cc} & |\uparrow_B\rangle & |\downarrow\rangle_B \\ \hline |\uparrow\rangle_A & 1 & 0 \\ |\downarrow\rangle_A & 0 & 0 \end{array} \quad \text{(outer product)}$$

operators are now not matrices, but <u>rank-4 tensors</u>

(shape is $2 \times 2 \times 2 \times 2$)

~~action is like~~

Normal vector is 2, matrix is like $2 \to 2$

Here vector is 2×2, operator is $2 \times 2 \to 2 \times 2$

$$\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \qquad\qquad \begin{pmatrix}1&0\\0&-1\end{pmatrix} \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0&-i\\i&0\end{pmatrix} \longrightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

2 actions in parallel

$$\begin{pmatrix} 1\begin{pmatrix}0&-i\\i&0\end{pmatrix} & 0\begin{pmatrix}0&-i\\i&0\end{pmatrix} \\ 0\begin{pmatrix}0&-i\\i&0\end{pmatrix} & -1\begin{pmatrix}0&-i\\i&0\end{pmatrix} \end{pmatrix} \begin{pmatrix} 1\begin{pmatrix}1\\0\end{pmatrix} \\ 0\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}$$

<u>separately</u>

big $\begin{pmatrix}1\\0\end{pmatrix}$ vector selects big 1st col

little $\begin{pmatrix}1\\0\end{pmatrix}$ selects little 1st col $\longrightarrow \begin{pmatrix} 1\cdot\begin{pmatrix}0\\i\end{pmatrix} \\ 0\cdot\begin{pmatrix}0\\i\end{pmatrix} \end{pmatrix}$

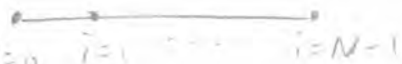
~~so in total the column we select is 2·0~~

in total we select a big column and a little column, result is one col of the 4×4 matrix. Which one? The same one as we would get by counting basis states since ordering is the same.

2-site Ising model

We will now use this formalism to write down more precisely, then solve, the Ising model on 2 spins.

Recall $H = -J \sum_i S_i^z \cdot S_{i+1}^z - h \sum_i S_i^x$



with only two spins, the terms are

$$H = -J \sum_{i=0}^{0} S_i^z S_{i+1}^z - h \sum_{i=0}^{1} S_i^x$$
$$= -J \left(S_0^z S_1^z\right) - h\left(S_0^x + S_1^x\right)$$

Two issues to take care of here:

① We've been working with σ^z, σ^x; but $S^z = \frac{\hbar}{2}\sigma^z$, $S^x = \frac{\hbar}{2}\sigma^x$

$\frac{\hbar}{2}$ is annoying. Solution:

$$-J\left(S_0^z S_1^z\right) = \underbrace{-J\left(\tfrac{\hbar}{2}\right)^2}_{= \tilde{J} \to \text{rename to } J} \sigma_0^z \sigma_1^z$$

$$-h S_0^x = \underbrace{-h \tfrac{\hbar}{2}}_{\tilde{h} \to \text{rename to } J} \sigma_0^x$$

So $H = -J\left(\sigma_0^z \sigma_1^z\right) - h\left(\sigma_0^x + \sigma_1^x\right)$

② Operators should be $\sum O_A \otimes O_B$, so how to rewrite?

$\to \sigma_0^z \sigma_1^z$ means $\sigma_0^z \otimes \sigma_1^z$

$\to \sigma_0^x$ means $\sigma_0^x \otimes Id_1$

σ_1^x means $Id_0 \otimes \sigma_1^x$

Now we have a useful operator:

$$H = -J\left(\sigma_0^z \otimes \sigma_1^z\right) - h\left(\sigma_0^x \otimes Id_1 + Id_0 \otimes \sigma_1^x\right)$$

In this case, it will not be hard to solve this and find the eigenstates, expectation values, etc. But let's practice good form, and start by understanding limiting cases.

Exercises

(1) In the limit $J \gg h$ (set h to 0), what are the eigenstates and eigenvalues. Conclude: what is the effect of the J term in the Hamiltonian?

Find $\sigma_0^z \otimes \sigma_1^z$ then this

Then find the expectation value of the h term in each of these states. If we turn on $h \ll J$, will it change which of these states are favored as the GS?

(2) Same story w/ $h \leftrightarrow J$. (For this one, remember that if two operators commute, they can be simultaneously diagonalized) (Use whatever basis you like)

Find $\sigma_0^x \otimes Id + Id \otimes \sigma_1^x$

(3) If neither term seems to have a "preference" between eigenstates of the other, can we satisfy both terms at once? (Hint: do they commute?)

(4) Find the 4x4 matrix for H.

Answers:

①

state	E
$\|\uparrow\uparrow\rangle$	$-J$
$\|\downarrow\downarrow\rangle$	$-J$
$\|\uparrow\downarrow\rangle$	J
$\|\downarrow\uparrow\rangle$	J

2 GS's

$\langle h(\sigma_0^x + \sigma_1^x)\rangle = 0$ in each case $\Rightarrow$ no

②

$\|\rightarrow\rightarrow\rangle$	$-2h \leftarrow GS$
$\|\rightarrow\leftarrow\rangle$	0
$\|\leftarrow\rightarrow\rangle$	0
$\|\leftarrow\leftarrow\rangle$	$2h$

$\langle J\sigma_0^z\sigma_1^z\rangle = 0$ in each case $\Rightarrow$ no

③ No, b/c they do not commute.

$$[\sigma_0^z\otimes\sigma_1^z, \sigma_0^x\otimes Id_1] = (\sigma_0^z\otimes\sigma_1^z)(\sigma_0^x\otimes Id_1) - (\sigma_0^x\otimes Id_1)(\sigma_0^z\otimes\sigma_1^z)$$
$$= i\sigma_0^y\otimes\sigma_1^z - -i\sigma_0^y\otimes\sigma_1^z$$
$$= 2i\sigma_0^y\otimes\sigma_1^z \neq 0$$

and likewise

④ $\sigma_0^z\otimes\sigma_1^z \rightarrow \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1\end{pmatrix}$

$\sigma_0^x\otimes Id_1 \rightarrow \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$, $Id_0\otimes\sigma_1^x \rightarrow \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$

$$\Rightarrow H = \begin{pmatrix} -J & -h & -h & \\ -h & J & & -h \\ -h & & J & -h \\ & -h & -h & -J\end{pmatrix}$$

Now a demo w/ Mathematica:

(turns out it can be solved analytically)