Problem Set 12 – Optional, but awesome!

Due (to get extra credit): Monday 5pm, May 9th, a Canvas upload or in envelope outside 26-255
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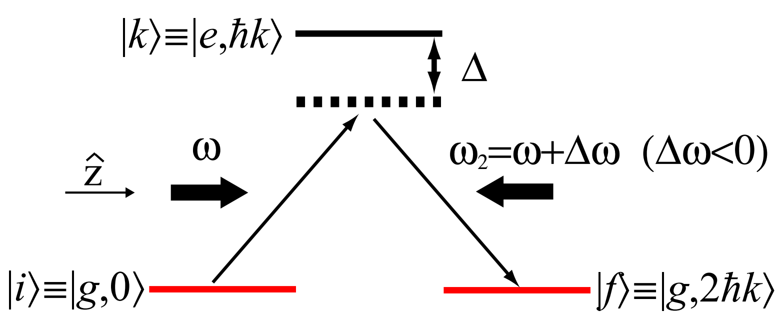
1 Bragg Scattering (10 points)



Consider the above energy level diagram of an atom where the states are product states of an internal state and an external (momentum) state which can be written $|\text{internal, external}\rangle \equiv |\text{internal}\rangle \otimes |\text{external}\rangle$. In this problem the internal state is either $|g\rangle$ or $|e\rangle$, and the momentum state is $|\mathbf{p}\rangle$, where $\mathbf{p}$ is as indicated in the figure for each state. If two counterpropagating lasers are tuned as indicated, recoil momentum will be transferred to the atoms by redistributing photons between the beams. We want to look at this "Bragg scattering" in two ways:

- by describing it as a two-photon stimulated Raman process, and
- by considering the mechanical effect of the AC Stark shift potential seen by an atom.

Such an arrangement is a grating for matter waves and is heavily used in atom interferometers (see, for example, D. M. Giltner, R. W. McGowan, S. A. Lee, Phys. Rev. Lett. **75**, 2638 (1995) and a recent measurement of atomic recoil to refine the fine structure constant, R. H. Parker et al., and H. Müller, Science 360, 191, 2018). The first observation of Bragg scattering of atoms by light was accomplished by a group at MIT: P. J. Martin, B. G. Oldaker, A. H. Miklich, D. E. Pritchard, Phys. Rev. Lett. **60**, 515 (1988). A comprehensive review of atom interferometry is given in the article by A. D. Cronin, J. Schmiedmayer, and DE Pritchard, Rev. Mod. Phys. **81**, 1051 (2009).

1. Two-Photon Stimulated Raman Process

(a) In the above figure, what should $\Delta\omega$ be in order to realize resonance condition for the Raman process?

(b) Assume the beams are counterpropagating along the z axis, have the same polarization, and can be expressed

$$\begin{aligned} E_1 &= E_0 \cos(kz - \omega_1 t) \\ E_2 &= E_0 \cos(-kz - (\omega_1 + \Delta\omega)t). \end{aligned}$$

Write down the interaction Hamiltonian. Don't forget to include the spatial dependence of the electric field. Although the dipole approximation essentially allows you to neglect the spatial dependence of the electric field in evaluating the matrix elements for the internal states, it is necessary for properly evaluating the matrix elements for the external states.

(c) Write down the wavefunction of a particle with definite momentum $p = \hbar k$ in position space. Assume periodic boundary conditions in a 1D box of length L.

(d) Calculate the two-photon Rabi frequency for the Raman process shown above in the figure. Assume that $|i\rangle$, $|k\rangle$, and $|f\rangle$ have external wavefunctions of the form you wrote down above with appropriate momenta. The dipole matrix element of the internal states is D_{eg}. Also, note that only the excited state $|k\rangle \equiv |e, \hbar k\rangle$ has nonvanishing matrix elements with the initial and final states. It is possible to consider this an effective two-level system involving only $|i\rangle$, $|f\rangle$, and the coupling between them.

(e) If H' is the perturbation due to E_1 and E_2, what is $\langle i|H'|f\rangle$ in terms of the given parameters? Hint: use the analogy between one-photon and two-photon transitions and the meaning of the corresponding Rabi frequencies.

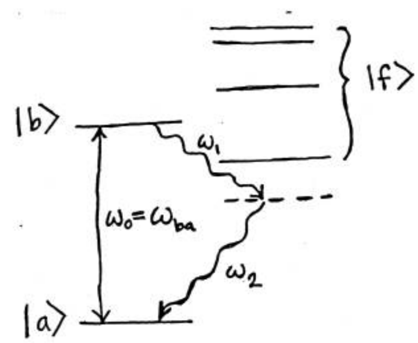
2. AC Stark Shift

(a) Calculate the AC Stark shift $U(z,t)$ of an atom in the ground state $|g\rangle$ due to the total electric field $E_1 + E_2$. For now, ignore the external state of the atom, but keep the spatial dependence of the electric field (while making the dipole approximation, of course). Assume the weak-field limit holds, and average over the oscillation at optical frequencies.

(b) What is the coupling $\langle i|U(z,t)|f\rangle$ due to the mechanical potential presented by the AC Stark shift? Note that now we take into account the external states of the states $|i\rangle$ and $|f\rangle$, where these have the form as you wrote down in 1c) Compare this with the perturbation matrix element obtained in 1e).

This problem illustrates that forces due to the AC Stark effect, *i.e.*, the stimulated light forces, correspond in the photon picture to a stimulated Raman process which redistributes photons between laser beams.

2 Spontaneous Two-Photon Emission (10 pts)



In this problem you will first derive a very general formula for calculating the rate and fluorescence spectrum of an atomic transition from an upper state $|b\rangle$ to a lower state $|a\rangle$ by spontaneous emission of two photons. Next, you will apply this formula to estimate the lifetime of the metastable $2S$ state of hydrogen. Throughout this problem, difference frequencies are labeled according to the convention $\omega_{mn} \equiv \omega_m - \omega_n$.

1. The Göppert-Mayer formula.

   In order to find $A(\omega_1)\mathrm{d}\omega_1$, the rate for spontaneous emission of two photons with one photon having frequency between ω_1 and $\omega_1 + \mathrm{d}\omega_1$, we can follow an approach similar to the one used in class for calculating the single-photon spontaneous emission rate. We will start with the two-photon excitation rate from $|a\rangle$ to $|b\rangle$ for monochromatic beams with frequencies ω_1 and ω_2.

   If we neglect absorption of two photons with the same frequency, then second order perturbation theory gives the following amplitude for state $|b\rangle$ after excitation for a time t:

$$a_b^{[2]} = \frac{1}{4\hbar^2} \sum_f \left\{ \frac{H_{bf,2} H_{fa,1}}{\omega_1 - \omega_{fa}} \frac{e^{i(\omega_{ba} - \omega_1 - \omega_2)t} - 1}{\omega_{ba} - \omega_1 - \omega_2} + \frac{H_{bf,1} H_{fa,2}}{\omega_2 - \omega_{fa}} \frac{e^{i(\omega_{ba} - \omega_1 - \omega_2)t} - 1}{\omega_{ba} - \omega_1 - \omega_2} \right\}$$

   (As will be clear later, the contribution to the spontaneous emission rate from processes involving two photons of the same frequency is completely negligible because there are so many other probable ways to emit two photons.)

   i) Find the excitation rate $\Gamma_{ab}(\omega_1)$ when one beam has frequency ω_1. Your expression should include a resonance condition which constrains ω_2.

ii) Show that your expression is mathematically equivalent to

$$\begin{aligned}\Gamma_{ab}(\omega_1) &= \Gamma_{ba}(\omega_1)\\ &= \frac{\pi E_1^2 E_2^2 e^4}{8\hbar^4}\left|\sum_f \left\{\frac{\langle a|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{1}|f\rangle\langle f|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{2}|b\rangle}{\omega_2+\omega_{fb}} + \frac{\langle a|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{2}|f\rangle\langle f|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{1}|b\rangle}{\omega_1+\omega_{fb}}\right\}\right|^2\\ &\quad\times\delta(\omega_{ba}-(\omega_1+\omega_2)).\end{aligned}$$

Note that we can also obtain this expression by considering two-photon *emission*—one needs only to swap a and b and change the signs of ω_1 and ω_2 in the expression for the absorption rate.

iii) Give a physical interpretation for the terms in the sum.

iv) Obtain the expression of $A(\omega_1)$. To do so, first express E_1^2 and E_2^2 in terms of the number of photons n_1, n_2 occupying some specific modes with frequencies ω_1, ω_2. Hint: write the electric field operator E in terms of photon number raising/lowering operators $a^\dagger$ and a. Given a coherent mode $|\alpha\rangle$, i.e. an eigenstate of the lowering operatore a with eigenvalue α, consider the expectation value of the photon number operator $\langle n\rangle = \langle a^\dagger a\rangle$ and the amplitude of the expectation value of the electric field $\langle E_0\rangle$.

Next, replace n_1, n_2 with the densities of modes at frequencies ω_1, ω_2. Proceed to calculate $A(\omega_1)\mathrm{d}\omega_1$, the two-photon spontaneous emission rate with one photon at frequency ω_1 and the other photon at frequency $\omega_2 = \omega_{ba} - \omega_1$. (By analogy to the one-photon case, if the two-photon absorption rate is $\bar{n}_1\bar{n}_2 R$, then the two-photon spontaneous emission rate is R.)

At last, derive the following expression:

$$A(\omega_1)\mathrm{d}\omega_1 = \frac{8e^4}{\pi\hbar^2c^6}\omega_1^3\omega_2^3\left\langle\left|\sum_f \left\{\frac{\langle a|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{1}|f\rangle\langle f|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{2}|b\rangle}{\omega_2+\omega_{fb}} + \frac{\langle a|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{2}|f\rangle\langle f|\mathbf{r}\cdot\hat{\mathbf{e}}_\mathbf{1}|b\rangle}{\omega_1+\omega_{fb}}\right\}\right|^2\right\rangle_{\mathrm{avg}}\mathrm{d}\omega_1.$$

The angle brackets indicate an average over all possible polarizations and directions of propagation for the two photons. (Don't lose too much sleep if you can't get the factor of 8 in front.)

This result was first published in 1931 by Maria Göppert-Mayer, one of the first persons to investigate multi-photon processes using the new quantum mechanics.

You don't need to work this out, but the formula above can be simplified by evaluating the average and expressing the matrix elements in terms of the operator z:

$$A(\omega_1)\mathrm{d}\omega_1 = \frac{8e^4}{3\pi\hbar^2 c^6}\omega_1^3\omega_2^3\left|\sum_f z_{af}z_{fb}\left(\frac{1}{\omega_1+\omega_{fb}}+\frac{1}{\omega_2+\omega_{fb}}\right)\right|^2 \mathrm{d}\omega_1.$$

(G. Breit and E. Teller, Astrophysical Journal **91**, 215, (1940).)

2. $2S$ natural lifetime.

The $2S$ state in hydrogen decays almost exclusively by spontaneous two-photon emission. (FYI: The transition to the Lamb shifted $2\ ^2P_{1/2}$ level is allowed by electric dipole radiation, but owing to the small value of the Lamb shift, the transition probability is negligibly small, equivalent to a lifetime of 163 years. The magnetic dipole transition to the $1\ ^2S_{1/2}$ is forbidden in the non-relativistic approximation since the radial wavefunctions of the two states are orthogonal. However, after relativistic corrections, the matrix element is no longer exactly zero, corresponding to a lifetime of about five days (see Corney, Laser Spectroscopy)).

We will use the following approximations:

(a) Only the $2P$ level contributes as an intermediate state.

(b) Use the Bohr radius a_0 for both relevant matrix elements of z.

(c) Treat $2S$ and $2P$ as degenerate (namely, $\omega_{fb} \simeq 0$).

For an order of magnitude estimate of the hydrogen $2S$ state lifetime, we can use the Göppert-Mayer formula and use the above approximations.

i) Use the total spontaneous decay rate to obtain the lifetime τ of the $2S$ state:

$$\frac{1}{\tau} = A_\tau = \frac{1}{2}\int_0^{\omega_{ba}} A(\omega_1)d\omega_1.$$

The factor 1/2 is included since the same photon pair occurs twice when integrating from 0 to ω_{ba}.

ii) Express the $2S$ lifetime in seconds.

iii) Plot the spectrum $A(\omega_1)$ over its entire range.

The actual $2S$ state lifetime is 0.122 s. The higher P states as well as the continuum contribute significantly to the $2S$ decay rate. However, it turns out to be an arduous calculation to sum over the intermediate states accurately, even for the hydrogen atom. It was nearly three decades after Göppert-Mayer did her pioneering work that the $2S$ lifetime was calculated to more than one significant digit! (J. Shapiro and G. Breit, Phys. Rev. **113**, 179 (1959).)

The $2S$ lifetime has been directly measured in ultracold, magnetically trapped hydrogen. (C. L. Cesar *et al.*, Phys. Rev. Lett. **77**, 225 (1996).)