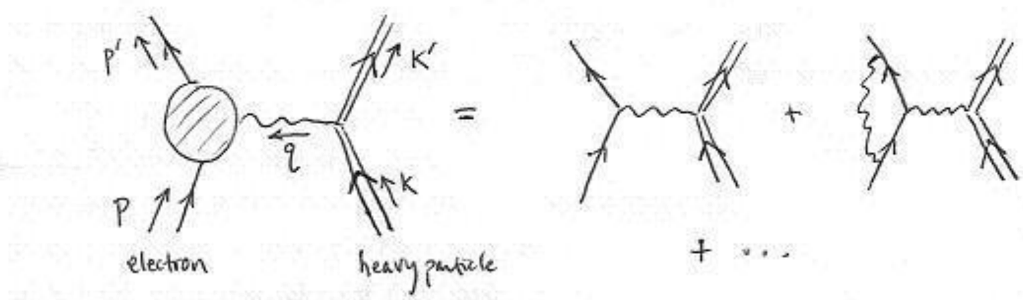
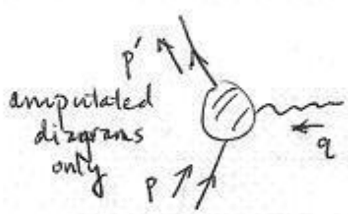
Electron Vertex Function



We are interested in the "formal" structure of the unknown function



$= -ie\,\Gamma^{\mu}(p',p)$

If we were to add a classical field A^{cl}_{μ} to our interaction Hamiltonian

$$\Delta H_{int} = \int d^3\vec{x}\; e\, A^{cl}_{\mu}(x)\, j^{\mu}(x)$$

where $j^{\mu}(x) = \bar{\Psi}(x)\,\gamma^{\mu}\,\Psi(x)$, then we find the electron scattering matrix amplitude

$$i\mathcal{M}\,(2\pi)\,\delta(p'^0 - p^0) = -ie\,\bar{u}(p')\,\gamma^{\mu}\,u(p)\,\tilde{A}^{cl}_{\mu}(p'-p)$$
$$+ \cdots$$

for time-independent $A^{cl}_{\mu}(x)$

$$= -ie\,\bar{u}(p')\,\Gamma^\mu(p',p)\,u(p)\,\tilde{A}^{cl}_\mu(p'-p)$$

Clearly to lowest order, $\Gamma^\mu(p',p) = \gamma^\mu + \mathcal{O}(e^2)$
The corrections to Γ^μ is some function of p, p',
the gamma matrices, m, and e.

Since the lowest order term is γ^μ, which is a Lorentz vector, we to maintain the same Lorentz properties...

$$\Gamma^\mu = \gamma^\mu A + (p'^\mu + p^\mu) B + (p'^\mu - p^\mu) C$$
$$+ \cancel{\gamma^\mu\gamma^5 D} \quad \text{parity is wrong}$$

where A, B, C are still functions of p, p' but have no uncontracted indices (i.e., $\not{p}$ or $p_\mu p^\mu$).

But notice that $\not{p}\,u(p) = m\,u(p)$ and $\bar{u}(p')\,\not{p}' = m\,\bar{u}(p')$.
So we can replace any $\not{p}$ or $\not{p}'$ by anticommuting to reach the spot next to $u(p)$ or $\bar{u}(p')$. Therefore the only Lorentz scalars left are $p^2 = m^2$, $p'^2 = m^2$, $p\cdot p'$, e, m.

We have $q^2 = (p'-p)^2 = p'^2 + p^2 - 2p'\cdot p = 2m^2 - 2p'\cdot p$
Also by the Ward identity $q_\mu\,\bar{u}(p')\,\Gamma^\mu\,u(p) = 0$. This

$$0 = \bar{u}(p')\left[\not{q}A + (p'+p)\cdot q\, B + q^2 C\right]u(p) = \bar{u}(p')\left[\underbrace{(\not{p}'-\not{p})}_{=0}A + \underbrace{(p'^2-p^2)}_{=0}B + q^2 C\right]u(p)$$

So we can say $\underline{\underline{C = 0}}$.

We now prove something called "Gordon identity" $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$$

Proof.

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\frac{\not{p}'}{m}\gamma^\mu u(p) = \bar{u}(p')\gamma^\mu \frac{\not{p}}{m} u(p)$$

So

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{4m}\bar{u}(p')(\{\not{p}',\gamma^\mu\} + [\not{p}',\gamma^\mu])u(p)$$
$$+ \frac{1}{4m}\bar{u}(p')(\{\gamma^\mu,\not{p}\} + [\gamma^\mu,\not{p}])u(p)$$
$$= \bar{u}(p')\left(\frac{p'^\mu}{2m} + \frac{p'_\nu}{4m}[\gamma^\nu,\gamma^\mu]\right)u(p)$$
$$+ \bar{u}(p')\left(\frac{p^\mu}{2m} + \frac{p_\nu}{4m}[\gamma^\mu,\gamma^\nu]\right)u(p)$$
$$= \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$$

We can use the Gordon identity to replace the $(p'^\mu + p^\mu)B$ term. So now we have

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2)$$

where F_1 and F_2 are called the Dirac + Pauli form factors. They are unknown functions of q^2.

We know that to lowest order $F_1 = 1$ and $F_2 = 0$.

It is rather interesting how much we have been able to say about the unknown function $\Gamma^\mu(p', p)$ just by using Lorentz invariance, parity invariance, and gauge invariance.

Let us get some intuition for the form factors.
We suppose that there is a classical field

$$A_\mu^{cl}(x) = (\phi(\vec{x}), \vec{0})$$ time independent electrostatic potential ϕ

Then $\tilde{A}_\mu^{cl}(q) = (2\pi)\delta(q^0)\,(\tilde{\phi}(\vec{q}), \vec{0})$ and so

$$i\mathcal{M} = -ie\,\bar{u}(p')\,\Gamma^0(p', p)\,u(p)\,\tilde{\phi}(\vec{q})$$

If the electric field slowly varies in space then $\tilde{\phi}(\vec{q})$ is peaked at $\vec{q} = 0$, and we can approximately write

$$\bar{u}(p')\,\Gamma^0(p', p)\,u(p) \approx \bar{u}(p')\,\gamma^0 F_1(0)\,u(p)$$

(the F_2 term is proportional to $\vec{q}$ and so is $\mathcal{O}(|\vec{q}|)$)

$$\approx u^\dagger(p')\,u(p)\,F_1(0)$$

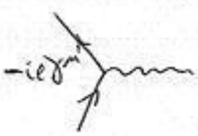
(non-relativistic p' and p) $\approx 2m\,\xi'^\dagger \xi\, F_1(0)$

Then the non-relativistic scattering amplitude is

$$i\mathcal{M} = -ie\, F_1(0)\, \tilde{\phi}(\vec{q}) \cdot 2m \xi'^{\dagger} \xi ,$$

and this corresponds with scattering in the Born approximation from a $e F_1(0)\, \phi(\vec{x})$ potential.

So $F_1(0)$ is the physically measured charge of the electron in units of e. But that should be 1 ($e = -1.602 \times 10^{-19}\,C$). So $F_1(0) = 1$ to all orders in perturbation theory. Since $F_1(0) = 1$ already when computing the "tree"-level lowest order diagram,



that means $F_1(0)$ must vanish for the loop corrections.

* This statement will require a properly normalized electron wave function... we say more on this later.

Now let us think about the magnetic moment of the electron. Consider the static vector potential

$$A^0_{cl} = 0\,, \quad \vec{A}_{cl}(\vec{x})$$

which produces a constant magnetic field (no electric field). Then

$$i\mathcal{M} = \sum_j ie\, \bar{u}(p')\left(\gamma^j F_1 + \frac{i\sigma^{j\nu} q_\nu}{2m} F_2\right) u(p)\, \tilde{A}^j_{cl}(\vec{q})$$

We will take the non-relativistic limit again. But we must go a little further than before since a magnetic field couples to moving charges...

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \approx \begin{pmatrix} \sqrt{m + (-\vec{p}\cdot\vec{\sigma})}\,\xi \\ \sqrt{m + (\vec{p}\cdot\vec{\sigma})}\,\xi \end{pmatrix} \approx \begin{pmatrix} \left(1 - \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \\ \left(1 + \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \end{pmatrix}\sqrt{m}$$

The F_1 term gives

$$\bar{u}(p')\gamma^i u(p) \approx 2m\, \xi'^\dagger\left(\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\sigma^i + \sigma^i \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \qquad \left(\text{terms without } \vec{p}' \text{ or } \vec{p} \text{ are zero}\right)$$

Since $\sigma^i\sigma^j = \underset{\frac{1}{2}\{\sigma^i,\sigma^j\}}{\underset{\uparrow}{\delta^{ij}}} + \underset{\frac{1}{2}[\sigma^i,\sigma^j]}{\underset{\uparrow}{i\varepsilon^{ijk}\sigma^k}}$ we have

$$(\vec{p}'\cdot\vec{\sigma})\,\sigma^i = p'^i + (-i\varepsilon^{ijk} p'^j \sigma^k)$$

$$\sigma^i(\vec{p}\cdot\vec{\sigma}) = p^i + i\varepsilon^{ijk} p^j \sigma^k$$

So we have

$$\bar{u}(p')\gamma^i u(p) = 2m\,\xi'^\dagger\left(\underbrace{\frac{p'^i+p^i}{2m}}_{\text{contributes to } \vec{p}\cdot\vec{A} \text{ interaction}} - \underbrace{\frac{i}{2m}\varepsilon^{ijk}q^j\sigma^k}_{\text{contributes to coupling with spin}}\right)\xi$$

For the F_2 term we have

$$\bar{u}(p')\left(\frac{i}{2m}\sigma^{i\nu}q_\nu\right)u(p)$$

$$\approx 2m\,\xi'^\dagger\left(\underbrace{-\frac{i}{2m}\varepsilon^{ijk}q^j\sigma^k}_{\text{contributes to coupling with spin}}\right)\xi$$

We are interested in the magnetic moment and we consider the $-\frac{i}{2m}\varepsilon^{ijk}q^j\sigma^k$ spin coupling ...

$$i\mathcal{M} = \vec{p}\cdot\vec{A} \text{ term} + 2mie\,\xi'^\dagger\left[-\frac{i}{2m}\varepsilon^{ijk}q^j\sigma^k\left(F_1(q^2)+F_2(q^2)\right)\right]\xi \cdot \tilde{A}^i_{cl}(\vec{q})$$

where we can take

$$F_1(q^2) \approx F_1(0) = 1$$
$$F_2(q^2) \approx F_2(0)$$

At weak magnetic field we have scattering due to