Classical Mechanics III (8.09)
Assignment 4: Solutions

October 11, 2021

1. A Heavy Symmetric Top [10 points]

(a) [3 points] In the inertial frame, the gravitational force on the top is $\vec{F} = -mg\hat{z}$. To convert to body coordinates we use the matrix $A = BCD$ given in p.153 of Goldstein,

$$\vec{F}_{body} = \begin{pmatrix} \cdots & \cdots & \sin\psi\sin\theta \\ \cdots & \cdots & \cos\psi\sin\theta \\ \cdots & \cdots & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} = -mg \begin{pmatrix} \sin\psi\sin\theta \\ \cos\psi\sin\theta \\ \cos\theta \end{pmatrix}$$

or $\vec{F} = -mg(\sin\psi\sin\theta\hat{x}' + \cos\psi\sin\theta\hat{y}' + \cos\theta\hat{z}')$, where the ellipsis indicate entries that we do not need. Now in body coordinates the center of mass is located at $\vec{R} = R\hat{z}'$ (assuming the CM lies above the fixed point), and hence

$$\vec{\tau} = \vec{r} \times \vec{F} = mgR(\cos\psi\sin\theta\hat{x}' - \sin\psi\sin\theta\hat{y}').$$

The components of the torque are $\tau_1 = mgR\cos\psi\sin\theta$, $\tau_2 = -mgR\sin\psi\sin\theta$, and $\tau_3 = 0$.

(b) [2 points] The conditions specified in the problem are that θ is constant and $\dot{\phi} = \Omega$ is constant. Hence reading off the components of the angular velocity from Problem Set 3, problem 3(a) (or Eq. (4.87) of Goldstein) we have

$$\begin{aligned} \omega_1 &= \Omega\sin\theta\sin\psi \\ \omega_2 &= \Omega\sin\theta\cos\psi \\ \omega_3 &= \Omega\cos\theta + \dot{\psi}. \end{aligned}$$

Note the Euler equation of motion for the z'-axis gives $\tau_3 = I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2)$, and (since $\tau_3 = 0$ and $I_1 = I_2$) we obtain $\dot{\omega}_3 = 0$. This immediately gives $\dot{\psi} = \omega'$ is constant.

(c) [5 points] The first Euler equation of motion is $I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = \tau_1$, or in this case

$$I_1\Omega\dot{\psi}\sin\theta\cos\psi - \Omega\sin\theta\cos\psi(\Omega\cos\theta + \omega')(I_1 - I_3) = mgR\cos\psi\sin\theta$$

and after dividing by $\sin\theta\cos\psi$ and rearranging terms we get

$$(I_3 - I_1)\cos\theta\Omega^2 + I_3\omega'\Omega - mgR = 0.$$

We can treat this as a quadratic equation in Ω; for there to be a real solution for Ω, the discriminant of the equation must be non-negative, i.e.

$$I_3^2\omega'^2 + 4mgR(I_3 - I_1)\cos\theta \geq 0$$

or

$$\omega'^2 \geq \frac{4(I_1 - I_3)mgR\cos\theta}{I_3^2}.$$

For a top with $I_1 < I_3$ (e.g. a top shaped like a disk) this condition always holds, and there is no minimum value for $\omega' = \dot{\psi}$. For a top shaped more like a rod $I_1 > I_3$ and there is a minimum ω'.

Note: There is a discussion on this problem in pg. 218-221, Goldstein (3rd edition). The minimum condition (5.79) in Goldstein is different from our condition because they find a minimum condition for $\omega_3 = \Omega\cos\theta + \dot{\psi}$, while we find a minimum condition for $\dot{\psi}$.

2. Three Point Masses on a Circle [16 points]

(a) [6 points] Let θ_i be the displacement of the masses from equilibrium, as shown in the figure. Then $\alpha = \theta_1 - \theta_2 + 2\pi/3$, $\beta = \theta_2 - \theta_3 + 2\pi/3$, $\gamma = \theta_3 - \theta_1 + 2\pi/3$. Let $\vec{q} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$. Assuming the displacements from equilibrium are small ($\theta_i \ll 1$), we have, using $e^x = 1 + x + \frac{x^2}{2} + O(x^3)$, that

$$\begin{aligned}
V &= V_0(e^{-2\alpha} + e^{-2\beta} + e^{-2\gamma}) = V_0 e^{-4\pi/3}(e^{2(\theta_2-\theta_1)} + e^{2(\theta_3-\theta_2)} + e^{2(\theta_1-\theta_3)}) \\
&= V_0 e^{-4\pi/3}[3 + 2(\theta_2-\theta_1)^2 + 2(\theta_3-\theta_2)^2 + 2(\theta_1-\theta_3)^2] + O(\theta^3) \\
&= \frac{3}{4}A + A(\theta_1^2 + \theta_2^2 + \theta_3^2 - \theta_1\theta_2 - \theta_2\theta_3 - \theta_3\theta_1), \qquad \text{where } A = 4V_0 e^{-4\pi/3} \\
&= \text{const.} + \frac{1}{2}\vec{q}^T \begin{pmatrix} 2A & -A & -A \\ -A & 2A & -A \\ -A & -A & 2A \end{pmatrix} \vec{q} \equiv \text{const.} + \frac{1}{2}\vec{q}^T \hat{V}\vec{q}, \qquad \text{where } \hat{V} = \begin{pmatrix} 2A & -A & -A \\ -A & 2A & -A \\ -A & -A & 2A \end{pmatrix}.
\end{aligned}$$

The kinetic energy is much more straightforward:

$$\begin{aligned}
T &= \frac{ma^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) = \frac{B}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2), \qquad \text{where } B = ma^2 \\
&= \frac{1}{2}\dot{\vec{q}}^T \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix} \dot{\vec{q}} = \frac{1}{2}\dot{\vec{q}}^T \hat{T}\dot{\vec{q}}, \qquad \hat{T} = \begin{pmatrix} B & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & B \end{pmatrix}.
\end{aligned}$$

Now taking the ansatz $\vec{q} = \vec{a}e^{-i\omega t}$, we get $(\hat{V} - \lambda\hat{T})\vec{a} = 0$ and $\det(\hat{V} - \lambda\hat{T}) = 0$, with $\lambda = \omega^2$. Writing this out,

$$\begin{vmatrix} 2A - \lambda B & -A & -A \\ -A & 2A - \lambda B & -A \\ -A & -A & 2A - \lambda B \end{vmatrix} = (2A - \lambda B)^3 - 2A^3 - 3A^2(2A - \lambda B) = 0$$

or

$$\lambda B(3A - \lambda B)^2 = 0.$$

This admits the roots $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \frac{3A}{B}$ (double root); or $\omega_1 = 0$, $\omega_2 = \omega_3 = \sqrt{\frac{3A}{B}} = \sqrt{\frac{12V_0e^{-4\pi/3}}{ma^2}}$. Now we need to find the corresponding eigenvectors.

For $\lambda_1 = 0$,

$$\begin{pmatrix} 2A & -A & -A \\ -A & 2A & -A \\ -A & -A & 2A \end{pmatrix} \begin{pmatrix} a_1^1 \\ a_2^1 \\ a_3^1 \end{pmatrix} = A \begin{pmatrix} 2a_1^1 - a_2^1 - a_3^1 \\ -a_1^1 + 2a_2^1 - a_3^1 \\ -a_1^1 - a_2^1 + 2a_3^1 \end{pmatrix} = 0$$

and the corresponding solution is $a_1^1 = a_2^1 = a_3^1$, so we can take $\vec{a}^{\,1} = N_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ for some N_1; we want to normalize $\vec{a}^1$ such that

$$1 = \vec{a}_1^T\hat{T}\vec{a}_1 = 3N_1^2 B \qquad \Rightarrow N_1 = \frac{1}{\sqrt{3B}}.$$

For $\lambda_2 = \lambda_3 = \frac{3A}{B}$, and hence for both $\vec{a}^2$ and $\vec{a}^3$, we have

$$\begin{pmatrix} -A & -A & -A \\ -A & -A & -A \\ -A & -A & -A \end{pmatrix} \begin{pmatrix} a_1^{2,3} \\ a_2^{2,3} \\ a_3^{2,3} \end{pmatrix} = A \begin{pmatrix} -a_1^{2,3} - a_2^{2,3} - a_3^{2,3} \\ -a_1^{2,3} - a_2^{2,3} - a_3^{2,3} \\ -a_1^{2,3} - a_2^{2,3} - a_3^{2,3} \end{pmatrix} = 0$$

and hence $a_1^{2,3} + a_2^{2,3} + a_3^{2,3} = 0$. In these equations the superscript $2,3$ means either everything is 2 or everything is 3. We can take then, say, $\vec{a}^{\,2} = N_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\vec{a}^{\,3} = N_3 \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ (other choices are possible, but we must have $\vec{a}^{\,2} \cdot \hat{T} \cdot \vec{a}^{\,3} = B\vec{a}^{\,2} \cdot \vec{a}^{\,3} = 0$ since $\hat{T}$ is diagonal). For these to be a set of normal basis (using lower indices on the vectors just for simplicity),

$$1 = \vec{a}_2^T\hat{T}\vec{a}_2 = 2(N_2)^2 B \qquad \Rightarrow N_1 = \frac{1}{\sqrt{2B}}$$

$$1 = \vec{a}_3^T\hat{T}\vec{a}_3 = 6(N_3)^2 B \qquad \Rightarrow N_3 = \frac{1}{\sqrt{6B}}.$$

All together then, our normal modes are given by

$$\omega_1 = 0, \vec{a}^{\,1} = \frac{1}{\sqrt{3ma^2}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \ \omega_2 = \omega_3 = \sqrt{\frac{12V_0 e^{-4\pi/3}}{ma^2}}, \vec{a}^{\,2} = \frac{1}{\sqrt{2ma^2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{a}^{\,3} = \frac{1}{\sqrt{6ma^2}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}.$$

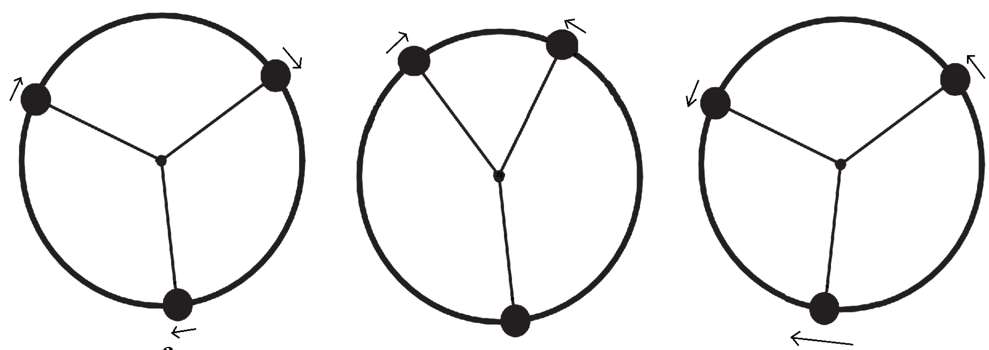
(b) [3 points] Let the corresponding normal coordinates be $\zeta_1, \zeta_2, \zeta_3$. We can pretty much read off our results from (a) (along with the equations of motion $\ddot{\zeta}_i + \omega_i^2 \zeta_i = 0$):

$$\zeta_1 = \frac{1}{\sqrt{3ma^2}}(\theta_1 + \theta_2 + \theta_3), \qquad \ddot{\zeta}_1 = 0$$

$$\zeta_2 = \frac{1}{\sqrt{2ma^2}}(\theta_1 - \theta_2), \qquad \ddot{\zeta}_2 - \frac{12V_0 e^{-4\pi/3}}{ma^2}\zeta_2 = 0$$

$$\zeta_3 = \frac{1}{\sqrt{6ma^2}}(\theta_1 + \theta_2 - 2\theta_3), \qquad \ddot{\zeta}_3 - \frac{12V_0 e^{-4\pi/3}}{ma^2}\zeta_3 = 0$$

(c) [3 points] The mode $\vec{a}^{\,1}$ corresponds to a uniform rotation; the potential energy is unchanged under such a rotation, and hence this mode is not actually an oscillation. The mode $\vec{a}^{\,2}$ corresponds to the first two masses oscillating in opposite directions while the third mass is fixed. The mode $\vec{a}^{\,3}$ corresponds to the first two masses oscillating in the same direction, while the third mass oscillates in the opposite direction at twice the amplitude, twice the instantaneous speed, and same frequency.



(d) [4 points] The general solution of the problem is (recalling that $\omega_1 = 0$ corresponds to a uniform rotation of the masses):

$$\vec{q}(t) = \vec{a}^{\,1}(vt + c_1) + \mathrm{Re}[c_2 \vec{a}^{\,2} e^{-i\omega_2 t} + c_3 \vec{a}^{\,3} e^{-i\omega_3 t}], \qquad v, c_1 \text{ real, } c_2, c_3 \text{ complex}$$

We now plug in the initial conditions. At $t = 0$ we have $\vec{q}(t = 0) = \vec{0}$:

$$0 = \mathrm{Re}[c_1 \vec{a}^{\,1} + c_2 \vec{a}^{\,2} + c_3 \vec{a}^{\,3}] = \frac{1}{\sqrt{6B}} \mathrm{Re} \begin{pmatrix} \sqrt{2}c_1 + \sqrt{3}c_2 + c_3 \\ \sqrt{2}c_1 - \sqrt{3}c_2 + c_3 \\ \sqrt{2}c_1 - 2c_3 \end{pmatrix}$$

so we can take $c_1 = \mathrm{Re}(c_2) = \mathrm{Re}(c_3) = 0$. We also have $\dot{\vec{q}}(t=0) = \begin{pmatrix} 2\omega_0 \\ -\omega_0 \\ -\omega_0 \end{pmatrix}$:

$$\begin{pmatrix} 2\omega_0 \\ -\omega_0 \\ -\omega_0 \end{pmatrix} = v\vec{a}_1 + \mathrm{Re}[-i\omega_2 c_2 \vec{a}_2 - i\omega_2 c_3 \vec{a}_3] = \frac{1}{\sqrt{6B}} \begin{pmatrix} \sqrt{2}v + \sqrt{3}\omega_2 \mathrm{Im}c_2 + \omega_2 \mathrm{Im}c_3 \\ \sqrt{2}v - \sqrt{3}\omega_2 \mathrm{Im}c_2 + \omega_2 \mathrm{Im}c_3 \\ \sqrt{2}v - 2\omega_2 \mathrm{Im}c_3 \end{pmatrix}$$

Here we used $\mathrm{Re}[-ic_i] = \mathrm{Im}c_i$. This has the solution $v = 0$, $\mathrm{Im}c_2 = \sqrt{\frac{9B}{2}}\frac{\omega_0}{\omega_2} = \sqrt{\frac{9B^2}{6A}}\omega_0$, and $\mathrm{Im}c_3 = \frac{\sqrt{6B}}{2}\frac{\omega_0}{\omega_2} = \sqrt{\frac{B^2}{2A}}\omega_0$. We then have

$$\begin{aligned}
\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} &= \mathrm{Re}\left[i\sqrt{\frac{9B^2}{6A}}\omega_0 \frac{e^{-i\omega_2 t}}{\sqrt{2B}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + i\sqrt{\frac{B^2}{2A}}\omega_0 \frac{e^{-i\omega_2 t}}{\sqrt{6B}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right] \\
&= (\sqrt{\frac{9B}{12A}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \sqrt{\frac{B}{12A}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}) \, [\omega_0 \sin(\omega_2 t)] \\
&= \omega_0 \frac{1}{2}\sqrt{\frac{ma^2}{12V_0 e^{-4\pi/3}}} \sin\left(\sqrt{\frac{12V_0 e^{-4\pi/3}}{ma^2}}t\right) \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} \\
&= \frac{\omega_0}{2\omega_2}\sin(\omega_2 t) \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix}, \qquad \omega_2 = \sqrt{\frac{12V_0 e^{-4\pi/3}}{ma^2}}.
\end{aligned}$$

3. Small Oscillations of the Double Pendulum [14 points]

(a) [4 points] Let us define $\vec{q} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$. Our results from problem set #1 give (we use the small angle approximation $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \theta^2/2$, and ignore all terms higher than second order):

$$\begin{aligned}
T &= \frac{1}{2}(m_1 + m_2)\ell_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2 \ell_2^2 \dot{\theta}_2^2 + m_2 \ell_1 \ell_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\
&= \frac{1}{2}\dot{\vec{q}}^T \begin{pmatrix} 2m\ell_1^2 & m\ell_1\ell_2 \\ m\ell_1\ell_2 & m\ell_2^2 \end{pmatrix} \dot{\vec{q}} \equiv \frac{1}{2}\dot{\vec{q}}^T \hat{T} \dot{\vec{q}}.
\end{aligned}$$

and

$$\begin{aligned}
V &= (m_1 + m_2) g\ell_1 (1 - \cos\theta_1) + m_2 g \ell_2 (1 - \cos\theta_2) \\
&= mg\ell_1 \theta_1^2 + \frac{1}{2} mg\ell_2 \theta_2^2 \\
&= \frac{1}{2}\vec{q}^T \begin{pmatrix} 2mg\ell_1 & 0 \\ 0 & mg\ell_2 \end{pmatrix} \vec{q} \equiv \frac{1}{2}\vec{q}^T \hat{V} \vec{q}.
\end{aligned}$$

(Note that in our formula for V, we've added a constant to the answer from problem set #1 to cancel off the constant term.)

(b) [4 points] Taking the ansatz $\vec{q} = \vec{a}e^{-i\omega t}$, the normal mode frequencies ω are given by the equation $\det(\hat{V} - \omega^2 \hat{T}) = 0$, that is,

$$\begin{aligned} 0 &= \begin{vmatrix} 2m\ell_1(g-\omega^2\ell_1) & -m\ell_1\ell_2\omega^2 \\ -m\ell_1\ell_2\omega^2 & m\ell_2(g-\omega^2\ell_2) \end{vmatrix} \\ &= 2m^2\ell_1\ell_2(g-\omega^2\ell_1)(g-\omega^2\ell_2) - m^2\ell_1^2\ell_2^2\omega^4 \\ &= \ell_1^2\ell_2^2\omega^4 - 2g\ell_1\ell_2(\ell_1+\ell_2)\omega^2 + 2g^2\ell_1\ell_2 \end{aligned}$$

which has the roots

$$\omega_\pm^2 = \frac{g}{\ell_1\ell_2}\left[\ell_1 + \ell_2 \pm \sqrt{\ell_1^2+\ell_2^2}\right]$$

We can easily see that the term in the square root $(\ell_1^2+\ell_2^2)$ is less than $(\ell_1+\ell_2)^2$. Hence the squares of the normal mode frequencies $\omega_\pm^2$ are real and positive, as they should be.

(c) [6 points] We have

$$\begin{aligned} \omega_\pm^{-2} &= \frac{\ell_1\ell_2}{g}\left[\ell_1+\ell_2 \pm \sqrt{\ell_1^2+\ell_2^2}\right]^{-1} \\ &= \frac{1}{2g}\left[\ell_1+\ell_2 \mp \sqrt{\ell_1^2+\ell_2^2}\right] \end{aligned}$$

after rationalizing the denominator. Let us first compute

$$\frac{g}{\omega_\pm^2} - \ell_1 = \frac{\ell_2-\ell_1}{2} \mp \frac{1}{2}\sqrt{\ell_1^2+\ell_2^2} \equiv \frac{\Delta}{2} \mp \frac{s}{2}$$

$$\frac{g}{\omega_\pm^2} - \ell_2 = \frac{\ell_1-\ell_2}{2} \mp \frac{1}{2}\sqrt{\ell_1^2+\ell_2^2} \equiv -\frac{\Delta}{2} \mp \frac{s}{2},$$

where we define $\Delta \equiv \ell_2 - \ell_1$ and $s \equiv \sqrt{\ell_1^2+\ell_2^2}$. (Note that $s > |\ell_1 - \ell_2|$, so both $s-\Delta$ and $s+\Delta$ are positive.) Now to solve for the normal modes, we recall that $(\hat{V} - \omega_\pm^2\hat{T})\vec{a}_\pm = 0$, or

$$\begin{pmatrix} 4m\ell_1(g-\omega_\pm^2\ell_1) & -m\ell_1\ell_2\omega_\pm^2 \\ -m\ell_1\ell_2\omega_\pm^2 & m\ell_2(g-\omega_\pm^2\ell_2) \end{pmatrix} \begin{pmatrix} a_1^\pm \\ a_2^\pm \end{pmatrix} = 0$$

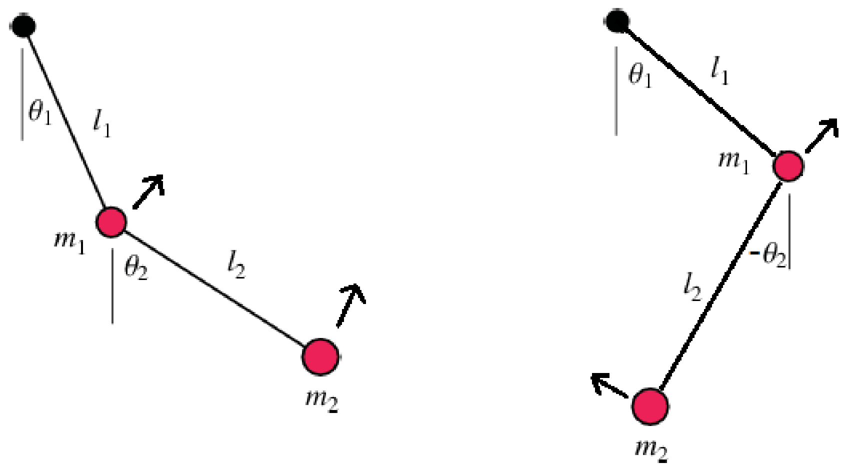
or using our result above,

$$\begin{aligned} 4(\frac{\Delta}{2} \mp \frac{s}{2})a_1^\pm &= \ell_2 a_2^\pm \\ \ell_1 a_1^\pm &= (-\frac{\Delta}{2} \mp \frac{s}{2})a_2^\pm \end{aligned}$$

and hence we can take as (unnormalized) eigenvectors

$$\vec{a}_+ = \begin{pmatrix} 1 \\ -\frac{2\ell_1}{s+\Delta} \end{pmatrix}, \qquad \vec{a}_- = \begin{pmatrix} 1 \\ \frac{2\ell_1}{s-\Delta} \end{pmatrix}.$$

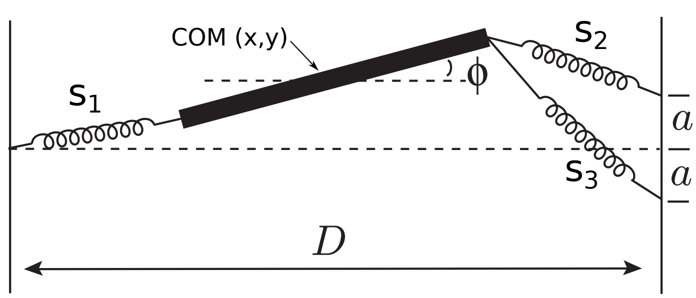
In the fast mode with ω_+, the two masses oscillate in opposite directions, while in the slow mode with ω_- they oscillate in the same direction. (It can be checked, with some algebra, that $\vec{a}_+^T T \vec{a}_- = 0$ and $\vec{a}_+^T V \vec{a}_- = 0$, as we'd expect. Note however that $\vec{a}_+ \cdot \vec{a}_- \neq 0$. The eigenvectors are only orthogonal when expressed in normal coordinates.)



slow mode **fast mode**

4. A Rigid Oscillating Bar [8.09 Only, 20 points]

(a) [6 points] The length of the bar is 2ℓ. Let s_1, s_2, s_3 be the stretched lengths of the left, top right, and bottom right springs, respectively. We can choose a set of coordinates as $\{x_{CM}, y_{CM}, \phi\}$, or alternatively another three variables. For the sake of concreteness (and simplicity) we'll take $\{x = x_{CM}, y = y_{CM}, \phi\}$.



Let's set up our coordinate system so that $+x$ points to the right, $+y$ points upwards, and the origin is located at the fixed point of the left spring, as in the figure shown. The potential energy is

$$V = \frac{1}{2}k\left(s_1^2 + s_2^2 + s_3^2\right) \tag{1}$$

and the kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\phi}^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m\ell^2}{6}\dot{\phi}^2 \tag{2}$$

where the moment of inertia of the rod about the CM, I, is given by

$$I = \frac{m}{2\ell}\int_0^{2\ell}(x' - \ell)^2 dx' = \frac{m\ell^2}{3}.$$

The Lagrangian is just $L = T - V$. We now need to put s_1, s_2, and s_3 in terms of our chosen variables. We have

$$s_1^2 = (x - \ell\cos\phi)^2 + (y - \ell\sin\phi)^2 \tag{3}$$

$$s_2^2 = (D - x - \ell\cos\phi))^2 + (a - y - \ell\sin\phi)^2 \tag{4}$$

$$s_3^2 = (D - x - \ell\cos\phi))^2 + (a + y + \ell\sin\phi)^2 \tag{5}$$

Equations (1)-(5), together with $L = T - V$, give the Lagrangian of the system.

(b) [6 points] We first find the equilibrium coordinates x_0, y_0, and ϕ_0. At equilibrium all of the first derivatives of V must be zero: $\frac{\partial V}{\partial x}\Big|_{eq.} = \frac{\partial V}{\partial y}\Big|_{eq.} = \frac{\partial V}{\partial \phi}\Big|_{eq.} = 0$. We evaluate the first derivatives of V:

$$\begin{aligned}
\frac{\partial V}{\partial x} &= k(-2D + 3x + \ell\cos\phi) \\
\frac{\partial V}{\partial y} &= k(3y + \ell\sin\phi) \\
\frac{\partial V}{\partial \phi} &= \ell k(y\cos\phi + (2D - x)\sin\phi)
\end{aligned}$$

We know $\phi_0 = 0$. Setting each derivative to 0, we get $y_0 = 0$ and $x_0 = \frac{2D-\ell}{3}$

We can continue by expanding V in terms of small displacements around equilibrium by evaluating the second derivatives of V. Let $V_{xx} = \frac{\partial^2 V}{\partial x^2}\Big|_{eq.}$, and define the other terms of $\hat{V}$ similarly. We'll evaluate term-by-term:

$$\begin{aligned}
V_{xx} &= 3k \\
V_{xy}\Big|_{eqbm} &= 0 \\
V_{x\phi}\Big|_{eqbm} &= 0 \\
V_{yy} &= 3k \\
V_{y\phi}\Big|_{eqbm} &= \ell k \\
V_{\phi\phi}\Big|_{eqbm} &= \frac{k}{3}\ell(\ell + 4D)
\end{aligned}$$

Therefore if we take $\vec{\eta} = \begin{pmatrix} x - x_0 \\ y \\ \phi \end{pmatrix}$ as the vector of small displacements from equilibrium, we get

$$V = \frac{1}{2}\vec{\eta}^T \hat{V} \vec{\eta}, \qquad T = \frac{1}{2}\dot{\vec{\eta}} \hat{T}\, \dot{\vec{\eta}}$$

where we've subtracted a constant $V_0 = V(x_0, 0, 0)$ from V (the potential energy at equilibrium), and

$$\hat{T} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \frac{m\ell^2}{3} \end{pmatrix}$$

$$\begin{aligned} \hat{V} &= \begin{pmatrix} V_{xx} & V_{xy} & V_{x\phi} \\ V_{xy} & V_{yy} & V_{y\phi} \\ V_{x\phi} & V_{y\phi} & V_{\phi\phi} \end{pmatrix} \\ &= \begin{pmatrix} 3k & 0 & 0 \\ 0 & 3k & \ell k \\ 0 & \ell k & \frac{k}{3}\ell(\ell + 4D) \end{pmatrix}. \end{aligned}$$

Note that although we've evaluated it with arbitrary D, full marks are given if you took $D = 3\ell$ everywhere as stated on the problem set.

(c) [8 points] To find the normal mode frequencies, we set $\lambda = \omega^2$ and solve the characteristic equation

$$\det(\hat{V} - \omega^2 \hat{T}) = \begin{vmatrix} V_{xx} - m\lambda & 0 & 0 \\ 0 & V_{yy} - m\lambda & V_{y\phi} \\ 0 & V_{y\phi} & V_{\phi\phi} - I\lambda \end{vmatrix} = (V_{xx} - m\lambda)[(V_{yy} - m\lambda)(V_{\phi\phi} - I\lambda) - V_{y\phi}^2] = 0$$

Using $\alpha \equiv D/\ell = 3$, the eigenvalues are then

$$\begin{aligned} \lambda_1 &= \omega_1^2 = \frac{3k}{m} \\ \lambda_2 &= \omega_2^2 = \frac{2k}{m}\left[1 + \alpha - \sqrt{1 - \alpha + \alpha^2}\right] = 2(4 - \sqrt{7})k/m \\ \lambda_3 &= \omega_3^2 = \frac{2k}{m}\left[1 + \alpha + \sqrt{1 - \alpha + \alpha^2}\right] = 2(4 + \sqrt{7})k/m \end{aligned}$$

Let us now solve for the eigenvectors:

$$(\hat{V} - \lambda_1 \hat{T})\vec{a}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \ell k \\ 0 & \ell k & \frac{k}{3}\ell(\ell + 4D) - \lambda_1 \frac{m\ell^2}{3} \end{pmatrix} \begin{pmatrix} a_x^1 \\ a_y^1 \\ a_\phi^1 \end{pmatrix} = 0$$

Thus these equations give us $a_y^1 = 0$ and $a_\phi^1 = 0$; setting $a_x^1 = 1$, we get

$$\vec{a}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

This corresponds to a mode with horizontal motion only.

For the second eigenvector, we have

$$(\hat{V} - \lambda_2 \hat{T})\vec{a}_2 = \begin{pmatrix} k(2\sqrt{7}-5) & 0 & 0 \\ 0 & k(2\sqrt{7}-5) & \ell k \\ 0 & \ell k & \frac{(2\sqrt{7}+5)}{3}\ell^2 k \end{pmatrix} \begin{pmatrix} a_x^2 \\ a_y^2 \\ a_\phi^2 \end{pmatrix} = 0$$

and so $a_x^2 = 0$. We can leave our eigenvector unnormalized:

$$\vec{a}_2 = \begin{pmatrix} 0 \\ -\frac{2\sqrt{7}+5}{3\ell} \\ 1 \end{pmatrix}.$$

In this mode the bar moves down as it rotates anticlockwise with x fixed.

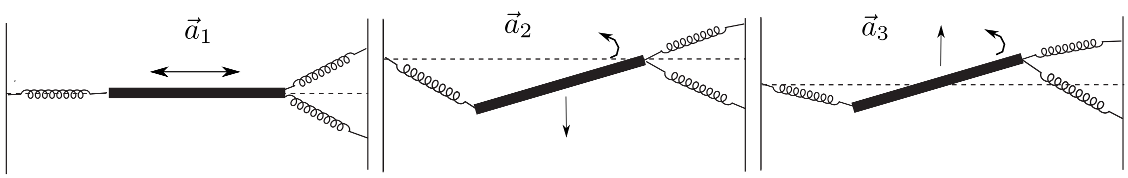
Finally, for the last eigenvector

$$(\hat{V} - \lambda_3 \hat{T})\vec{a}_3 = \begin{pmatrix} -k(2\sqrt{7}+5) & 0 & 0 \\ 0 & -k(2\sqrt{7}+5) & \ell k \\ 0 & \ell k & \frac{-2\sqrt{7}+5}{3}\ell^2 k \end{pmatrix} \begin{pmatrix} a_x^3 \\ a_y^3 \\ a_\phi^3 \end{pmatrix} = O(\theta_0^2)$$

so $a_x^3 = 0$. Taking $a_\phi^3 = 1$ and leaving the eigenvector unnormalized,

$$\vec{a}_3 = \begin{pmatrix} 0 \\ \frac{2\sqrt{7}-5}{3\ell} \\ 1 \end{pmatrix}.$$

In this mode the bar moves up as it rotates anticlockwise with x fixed. The three modes are drawn in the diagram below.



5. A Rigid Oscillating Bar [8.309 Only, 20 points]

(a) [7 points] The length of the bar is 2ℓ. I will take the unstretched length of the springs to be b, and their stretched lengths at equilibrium to be a. There are many possible sets of independent coordinates we can take here, for example $\{x_{CM}, y_{CM}, \phi\}$; $\{s_1, s_2, \phi\}$, where s_1, s_2 are the stretched lengths of the left and right spring, resp.; and$\{\theta_1, \theta_2, \phi\}$ (see diagram). For the sake of concreteness we'll take $\{x = x_{CM}, y = y_{CM}, \phi\}$, but the other sets of coordinates are equally good.

Let's set up our coordinate system so that $+x$ points to the right, $+y$ points upwards, and the origin is located at the fixed point of the left spring. Let (x^0_{CM}, y^0_{CM}) be the position of the center of mass at equilibrium; then by symmetry, the fixed point of the right spring is at $(2x^0_{CM}, 0)$. If we take the gravitational potential to be zero at $y = 0$ (the location of the fixed points), then the potential energy is

$$V = \frac{1}{2}k(s_1 - b)^2 + \frac{1}{2}k(s_2 - b)^2 + mgy_{CM} \tag{6}$$

and the kinetic energy is

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{I}{2}\dot{\phi}^2 = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{m\ell^2}{6}\dot{\phi}^2 \tag{7}$$

where the moment of inertia of the rod about the CM, I, is given by

$$I = \frac{m}{2\ell}\int_0^{2\ell}(x' - \ell)^2 dx' = \frac{m\ell^2}{3}.$$

The Lagrangian is just $L = T - V$. We now need to put s_1, s_2 in terms of our chosen variables. We have

$$s_1 = [(x - \ell\cos\phi)^2 + (y - \ell\sin\phi)^2]^{1/2} \tag{8}$$

$$s_2 = [(2x^0_{CM} - (x + \ell\cos\phi))^2 + (y + \ell\sin\phi)^2]^{1/2} \tag{9}$$

Finally, we need to express b and x^0_{CM} in terms of our given constants θ_0, ℓ, and a. We can evaluate the first derivatives of V:

$$\begin{aligned}
\frac{\partial V}{\partial x} &= k(s_1 - b)\frac{\partial s_1}{\partial x} + k(s_2 - b)\frac{\partial s_2}{\partial x} \\
&= k(s_1 - b)\frac{x - \ell\cos\phi}{s_1} - k(s_2 - b)\frac{(2x^0_{CM} - x) - \ell\cos\phi}{s_2} \\
\frac{\partial V}{\partial y} &= k(s_1 - b)\frac{\partial s_1}{\partial y} + k(s_2 - b)\frac{\partial s_2}{\partial y} + mg \\
&= k(s_1 - b)\frac{y - \ell\sin\phi}{s_1} + k(s_2 - b)\frac{y + \ell\sin\phi}{s_2} + mg \\
\frac{\partial V}{\partial \phi} &= k(s_1 - b)\frac{\partial s_1}{\partial \phi} + k(s_2 - b)\frac{\partial s_2}{\partial \phi} \\
&= k\ell\frac{s_1 - b}{s_1}(x\sin\phi - y\cos\phi) + k\ell\frac{s_2 - b}{s_2}((2x^0_{CM} - x)\sin\phi + y\cos\phi).
\end{aligned}$$

At equilibrium all of these must be zero: $\frac{\partial V}{\partial x}\Big|_{eq.} = \frac{\partial V}{\partial y}\Big|_{eq.} = \frac{\partial V}{\partial \phi}\Big|_{eq.} = 0$. Plugging in the values at equilibrium ($x = x^0_{CM}$, $y = y^0_{CM} = -a\cos\theta_0$, $\phi = 0$) then gives the equations

$$a = b + \frac{mg}{2k\cos\theta_0} \tag{10}$$

$$x^0_{CM} = a\sin\theta_0 + \ell \tag{11}$$

Equations (1)-(6), together with $L = T - V$, give the Lagrangian of the system.

(b) [5 points] We can continue to expand V in terms of small displacements by evaluating the second derivatives of V. Let $V_{xx} = \left.\frac{\partial^2 V}{\partial x^2}\right|_{eq.}$, and define the other terms of $\hat{V}$ similarly. We'll evaluate term-by-term:

$$\begin{aligned}
V_{xx} &= \frac{\partial}{\partial x}\left[k(s_1-b)\frac{x-\ell\cos\phi}{s_1} - k(s_2-b)\frac{(2x^0_{CM}-x)-\ell\cos\phi}{s_2}\right]_{eq.} \\
&= \frac{\partial}{\partial x}\left[2k(x-x^0_{CM}) - kb\frac{x-\ell\cos\phi}{s_1} + kb\frac{(2x^0_{CM}-x)-\ell\cos\phi}{s_2}\right]_{eq.} \\
&= \left[2k - \frac{kb}{s_1} + kb\frac{(x-\ell\cos\phi)^2}{s_1^3} - \frac{kb}{s_2} + kb\frac{((2x^0_{CM}-x)-\ell\cos\phi)^2}{s_2^3}\right]_{eq.} \\
&= 2k(1 - \frac{b}{a} + \frac{b(x^0_{CM}-\ell)^2}{a^3}) = 2k(1 - \frac{b}{a}\cos^2\theta_0) \\
&= 2k\sin^2\theta_0 + \frac{mg}{a}\cos\theta_0
\end{aligned}$$

$$\begin{aligned}
V_{xy} &= \frac{\partial}{\partial y}\left[2k(x-x^0_{CM}) - kb\frac{x-\ell\cos\phi}{s_1} + kb\frac{(2x^0_{CM}-x)-\ell\cos\phi}{s_2}\right]_{eq.} \\
&= \left[kb\frac{(x-\ell\cos\phi)(y-\ell\sin\phi)}{s_1^3} - kb\frac{((2x^0_{CM}-x)-\ell\cos\phi)(y+\ell\sin\phi)}{s_2^3}\right]_{eq.} \\
&= \left[kb\frac{(x^0_{CM}-\ell)y^0_{CM}}{a} - kb\frac{(x^0_{CM}-\ell)y^0_{CM}}{a}\right] = 0
\end{aligned}$$

$$\begin{aligned}
V_{x\phi} &= \frac{\partial}{\partial \phi}\left[2k(x-x^0_{CM}) - kb\frac{x-\ell\cos\phi}{s_1} + kb\frac{(2x^0_{CM}-x)-\ell\cos\phi}{s_2}\right]_{eq.} \\
&= kb\left[-\ell\frac{\sin\phi}{s_1} + \ell\frac{(x-\ell\cos\phi)(x\sin\phi - y\cos\phi)}{s_1^3} - \ell\frac{\sin\phi}{s_2}\right. \\
&\qquad \left. -\ell\frac{\sin\phi}{s_2} - \ell\frac{((2x^0_{CM}-x)-\ell\cos\phi)((2x^0_{CM}-x)\sin\phi + y\cos\phi)}{s_1^3}\right]_{eq.} \\
&= -\frac{2kb\ell(x^0_{CM}-\ell)y^0_{CM}}{a^3} = 2k\ell\cos\theta_0\sin\theta_0 - \frac{mg\ell}{a}\sin\theta_0
\end{aligned}$$

$$
\begin{aligned}
V_{yy} &= \frac{\partial}{\partial y}\left[k(s_1-b)\frac{y-\ell\sin\phi}{s_1}+k(s_2-b)\frac{y+\ell\sin\phi}{s_2}+mg\right]_{eq.}\\
&= \frac{\partial}{\partial y}\left[2ky-kb\frac{y-\ell\sin\phi}{s_1}-kb\frac{y+\ell\sin\phi}{s_2}\right]_{eq.}\\
&= \left[2k-\frac{kb}{s_1}+kb\frac{(y-\ell\sin\phi)^2}{s_1^3}-\frac{kb}{s_2}+kb\frac{(y+\ell\sin\phi)^2}{s_2^3}\right]_{eq.}\\
&= 2k(1-\frac{b}{a}+\frac{b(y^0_{CM})^2}{a^3}) = 2k(1-\frac{b}{a}\sin^2\theta_0)\\
&= 2k\cos^2\theta_0+\frac{mg}{a}\sin\theta_0\tan\theta_0
\end{aligned}
$$

$$
\begin{aligned}
V_{y\phi} &= \frac{\partial}{\partial \phi}\left[k(s_1-b)\frac{y-\ell\sin\phi}{s_1}+k(s_2-b)\frac{y+\ell\sin\phi}{s_2}\right]_{eq.}\\
&= kb\left[\ell\frac{\cos\phi}{s_1}+\ell\frac{(y-\ell\sin\phi)(x\sin\phi-y\cos\phi)}{s_1^3}-\ell\frac{\cos\phi}{s_2}+\ell\frac{(y+\ell\sin\phi)((2x^0_{CM}-x)\sin\phi+y\cos\phi)}{s_1^3}\right]_{eq.}\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
V_{\phi\phi} &= \frac{\partial}{\partial \phi}\left[k\ell\frac{s_1-b}{s_1}(x\sin\phi-y\cos\phi)+k\ell\frac{s_2-b}{s_2}((2x^0_{CM}-x)\sin\phi+y\cos\phi)\right]_{eq.}\\
&= \frac{\partial}{\partial \phi}\left[2k\ell x^0_{CM}\sin\phi-\frac{kb\ell}{s_1}(x\sin\phi-y\cos\phi)-\frac{kb\ell}{s_2}((2x^0_{CM}-x)\sin\phi+y\cos\phi)\right]_{eq.}\\
&= \Big[2k\ell x^0_{CM}\cos\phi-\frac{kb\ell}{s_1}(x\cos\phi+y\sin\phi)+\frac{kb\ell^2}{s_1^3}(x\sin\phi-y\cos\phi)^2\\
&\quad -\frac{kb\ell}{s_2}((2x^0_{CM}-x)\cos\phi-y\sin\phi)+\frac{kb\ell^2}{s_2^3}((2x^0_{CM}-x)\sin\phi+y\cos\phi)^2\Big]_{eq.}\\
&= 2k\ell(x^0_{CM}-\frac{b}{a}x^0_{CM}+\frac{2b\ell}{a^2}(y^0_{CM})^2) = 2k\ell^2+2k\ell\sin\theta_0(a-b)-\frac{2kb\ell^2}{a}\sin^2\theta_0\\
&= 2k\ell^2\cos^2\theta_0+mg\ell\tan\theta_0+\frac{4mg\ell^2}{a}\sin\theta_0\tan\theta_0.
\end{aligned}
$$

Therefore if we take $\vec{\eta}=\begin{pmatrix} x-x^0_{CM}\\ y-y^0_{CM}\\ \phi\end{pmatrix}$ as the vector of small displacements from equilibrium, we get

$$
V=\frac{1}{2}\vec{\eta}^T\hat{V}\vec{\eta},\qquad T=\frac{1}{2}\dot{\vec{\eta}}\hat{T}\dot{\vec{\eta}}
$$

where we've subtracted a constant $V_0 = k(b-a)^2+mgy^0_{CM}$ from V (the potential energy at equilibrium), and

$$
\hat{T}=\begin{pmatrix} m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & I\end{pmatrix}=\begin{pmatrix} m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & \frac{m\ell^2}{3}\end{pmatrix}
$$

$$
\begin{aligned}
\hat{V} &= \begin{pmatrix} V_{xx} & V_{xy} & V_{x\phi} \\ V_{xy} & V_{yy} & V_{y\phi} \\ V_{x\phi} & V_{y\phi} & V_{\phi\phi} \end{pmatrix} \\
&= \begin{pmatrix} 2k\sin^2\theta_0 + \frac{mg}{a}\cos\theta_0 & 0 & 2k\ell\cos\theta_0\sin\theta_0 - \frac{mg\ell}{a}\sin\theta_0 \\ 0 & 2k\cos^2\theta_0 + \frac{mg}{a}\sin\theta_0\tan\theta_0 & 0 \\ 2k\ell\cos\theta_0\sin\theta_0 - \frac{mg\ell}{a}\sin\theta_0 & 0 & 2k\ell^2\cos^2\theta_0 + mg\ell\tan\theta_0 + \frac{4mg\ell^2}{a}\sin\theta_0\tan\theta_0 \end{pmatrix}.
\end{aligned}
$$

If we ignore all terms of quadratic order and above in θ_0, this simplifies to

$$
\hat{V} = \begin{pmatrix} \frac{mg}{a} & 0 & \frac{(2ka-mg)\ell}{a}\theta_0 \\ 0 & 2k & 0 \\ \frac{(2ka-mg)\ell}{a}\theta_0 & 0 & 2k\ell^2 + mg\ell\theta_0 \end{pmatrix}.
$$

(c) [8 points] To find the normal mode frequencies, we set $\lambda = \omega^2$ and solve the characteristic equation

$$
\det(\hat{V} - \omega^2\hat{T}) = \begin{vmatrix} V_{xx} - m\lambda & 0 & V_{x\phi} \\ 0 & V_{yy} - m\lambda & 0 \\ V_{x\phi} & 0 & V_{\phi\phi} - I\lambda \end{vmatrix} = (V_{yy} - m\lambda)[(V_{\phi\phi} - I\lambda)(V_{xx} - m\lambda) - V_{x\phi}^2] = 0
$$

but since $V_{x\phi} = O(\theta_0)$, the $V_{x\phi}^2$ term in the determinant is quadratic in θ_0 and can be neglected. The eigenvalues are then obviously

$$
\lambda_1 = \frac{V_{xx}}{m} = \frac{g}{a}; \qquad \lambda_2 = \frac{V_{yy}}{m} = \frac{2k}{m}; \qquad \lambda_3 = \frac{V_{\phi\phi}}{I} = \frac{3(2k + \frac{mg}{\ell}\theta_0)}{m}.
$$

(The off-diagonal term does not affect the eigenvalues up to linear order.) Let us now solve for the eigenvectors:

$$
(\hat{V} - \lambda_1\hat{T})\vec{a}_1 = \begin{pmatrix} 0 & 0 & V_{x\phi} \\ 0 & 2k - m\lambda_1 & 0 \\ V_{x\phi} & 0 & V_{\phi\phi} - I\lambda_1 \end{pmatrix} \begin{pmatrix} a_x^1 \\ a_y^1 \\ a_\phi^1 \end{pmatrix} = O(\theta_0^2)
$$

Here we note that due to our expansion $V_{x\phi}a_\phi^1 = O(\theta_0^2)$ does not imply $a_\phi^1 = 0$, since this is accurate only to second order in θ_0! What this does imply is that a_ϕ^1 needs to be linear order in θ_0. Thus these equations give us $a_y^1 = 0$ and $a_\phi^1 = -\frac{V_{x\phi}}{V_{\phi\phi} - I\lambda_1}a_x^1$; setting $a_x^1 = 1$, we get

$$
\omega_1 = \sqrt{\frac{g}{a}}, \qquad \vec{a}_1 = \begin{pmatrix} 1 \\ 0 \\ -\frac{V_{x\phi}}{V_{\phi\phi} - I\omega_1^2} \end{pmatrix},
$$

where $V_{x\phi} = \frac{(2ka-mg)\ell}{a}\theta_0 = \frac{2kb\ell}{a} > 0$ and $V_{\phi\phi} = 2k\ell^2 + mg\ell\theta_0$. Note that $V_{\phi\phi} - I\omega_1^2 = 2k\ell^2 + mg\ell\theta_0 - \frac{m\ell^2}{3}\frac{g}{a} > 0$, and hence a_ϕ^1 and a_x^1 differ in sign: as the bar oscillates to the right, there is a small rotation of the bar in the clockwise direction.

For the second eigenvector, we have

$$(\hat{V} - \lambda_2 \hat{T})\vec{a}_2 = \begin{pmatrix} V_{xx} & 0 & V_{x\phi} \\ 0 & 0 & 0 \\ V_{x\phi} & 0 & V_{\phi\phi} \end{pmatrix} \begin{pmatrix} a_x^2 \\ a_y^2 \\ a_\phi^2 \end{pmatrix} = 0$$

and so $a_x^2 = a_\phi^2 = 0$. We can take

$$\omega_2 = \sqrt{\frac{2k}{m}}, \qquad \vec{a}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

This is a normal mode that consists simply of a translation in the y-direction.

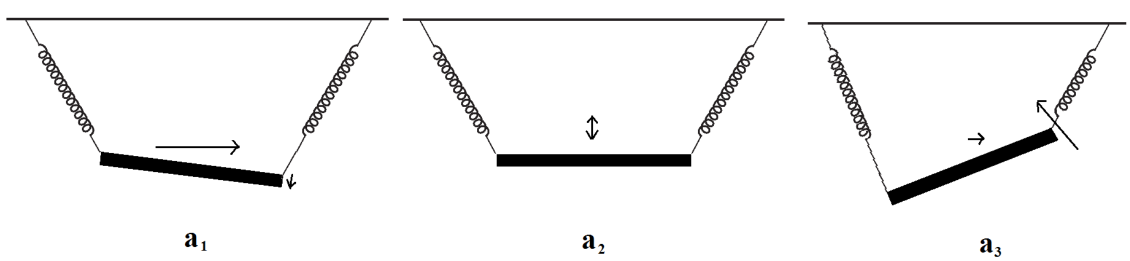
Finally, for the last eigenvector

$$(\hat{V} - \lambda_3 \hat{T})\vec{a}_3 = \begin{pmatrix} V_{xx} - m\lambda_3 & 0 & V_{x\phi} \\ 0 & 2k - m\lambda_3 & 0 \\ V_{x\phi} & 0 & 0 \end{pmatrix} \begin{pmatrix} a_x^3 \\ a_y^3 \\ a_\phi^3 \end{pmatrix} = O(\theta_0^2)$$

so $a_y^3 = 0$ and $a_x^3 = -\frac{V_{x\phi}}{V_{xx} - m\lambda_3} a_\phi^3$. Taking $a_\phi^3 = 1$,

$$\omega_3 = \sqrt{\frac{3(2k + \frac{mg}{\ell}\theta_0)}{m}}, \qquad \vec{a}_3 = \begin{pmatrix} -\frac{V_{x\phi}}{V_{xx} - m\omega_3^2} \\ 0 \\ 1 \end{pmatrix},$$

where $V_{x\phi} = \frac{(2ka - mg)\ell}{a}\theta_0 = \frac{2kb\ell}{a} > 0$ and $V_{xx} = \frac{mg}{a}$. This time $V_{xx} - m\omega_3^2 = \frac{mg}{a} - 3(2k + \frac{mg}{\ell}\theta_0) < 0$, so a_x^3 and a_ϕ^3 have the same sign. As the bar makes a counterclockwise rotation, a small translation to the right occurs.



Finally, notice that if $\theta_0 = 0$ then $V_{x\phi} = 0$, and $\hat{V}$ is now diagonal (the coordinates are decoupled). In this case the normal modes are just a translation in x, a translation in y, and a rotation.