Sep 5

PH 335 General Relativity & Cosmology - Course Outline

I. Overview and review
- Principle of equivalence

II. Review of multi-variable calculus

III. Flat 3-dimensional space (chapter 1 - first half)
- Basis vectors
- Contravariant and covariant vectors
- Metric tensor
- Coordinate transformations
- Tensors

IV. Flat spacetime (appendix A)
- Special relativity
- Relativistic electrodynamics

V. Curved spaces (chapter 1 - last half)
- 2 dimensional curved spaces
- Manifolds
- Tensors on manifolds

VI. Gravitation and curvature (chapter 2)
- Geodesics & affine connection $\Gamma^{\sigma}{}_{\mu\nu}$
- Parallel transport
- Covariant differentiation
- Newtonian limit

VII. Einstein's field equations (chapter 3)
- Stress-energy tensor $T^{\mu\nu}$
- Curvature tensor $R^{\lambda}{}_{\mu\nu\sigma}$
- Einstein's equations
- Schwarzschild solution

VIII. Predictions and tests of general relativity (chapter 4)
- Gravitational redshift
- Radar time-delay experiments
- Black Holes

IX. Cosmology (chapter 6)
- Friedman-Robertson-Walker solution
- Hubble's "constant" H(t)
- Recent Discoveries in Cosmology
- Cosmological constant

GR "cheat sheet" Midterm #1 Oct 10, 2018

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- A small, non-rotating, freely-falling frame in a grav. field is an inertial fr
- Strong EQV principle → all physics reduces to SR in a freely falling frame
- Weak EQV principle → all point particles fall @ same rate in g field → good for GR, not QM
  ↳ we use this

Gauss $\oint \vec{F}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{F}\, d^3r$ **Stokes** $\oint \vec{F}\cdot d\vec{s} = \int (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$

Maxwell $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

Theorem $dx^1 dx^2 \ldots dx^n = \det(\underset{\sim}{U})\, dy^1 dy^2 \ldots dy^n$ U is the Jacobian!

Basis vectors $\vec{e}_i = \frac{\partial\vec{r}}{\partial u^i}$ (natural), $\vec{e}^{\,i} = \vec{\nabla}u^i$ (dual), $\vec{e}^{\,i}\vec{e}_j = \delta^i_j$

Properties $\vec{\lambda}\cdot\vec{\mu} = \lambda^i\mu_i = \lambda_i\mu^i = g_{ij}\lambda^i\mu^j = g^{ij}\lambda_i\mu_j$ $\Big\}$ $\vec{e}^{\,i}\vec{e}^{\,j} = g^{ij}$, $\vec{e}_i\vec{e}_j = g_{ij}$

and $\boxed{\text{Inverse metric tensor}}$ ↗ $g^{ij}g_{jk} = \delta^i_k$, $\lambda^i = g^{ij}\lambda_j$, $\lambda_j = g_{ij}\lambda^i$. In Cartesian, $[g_{ij}] = I$ $\boxed{\text{metric tensor}}$

$\boxed{\begin{array}{l} i,j,k : 123 \\ ABC : 1,2 \\ \mu,\nu : 0,1,2,3 \\ a,b,c : 1 \ldots N \end{array}}$

"length" $L = \int_a^b \sqrt{g_{ij}\,du^i du^j} = \int_a^b \sqrt{g_{ij}\dot{u}^i\dot{u}^j}\, dt$

Line element $ds^2 = g_{ij}\,du^i du^j$

Derivation $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{\vec{e}_i du^i\cdot\vec{e}_j du^j} = \sqrt{g_{ij}\,du^i du^j} = ds$

In matrices $\vec{\lambda}\cdot\vec{\mu} = \lambda^i\mu_i = g_{ij}\lambda^i\mu^j = [\lambda^i]^T[g_{ij}][\mu^j] = \underset{\sim}{L}^T\underset{\sim}{G}\,\underset{\sim}{M}$

↳ lowering of indices $\underset{\sim}{L}^* = \underset{\sim}{G}\,\underset{\sim}{L}$

raising of indices $\underset{\sim}{L} = \underset{\sim}{\hat{G}}\,\underset{\sim}{L}^*$

$\boxed{[g_{ij}] = [g^{ij}]^{-1}}$

$\boxed{\text{Coordinate Transforms}}$ $\boxed{\vec{e}_j = \frac{\partial\vec{r}}{\partial u^j} = \frac{\partial\vec{r}}{\partial u^{i'}}\frac{\partial u^{i'}}{\partial u^j} = U^{i'}_j\,\vec{e}_{i'}}$

Properties

$\vec{\lambda} = \lambda^{i'}\vec{e}_{i'} = \lambda^{i'}U^j_{i'}\vec{e}_j = \lambda^j\vec{e}_j$ So $\boxed{\lambda^j = U^j_{i'}\lambda^{i'}}$

So $\boxed{U^k_{i'}U^{i'}_j = \delta^k_j,\quad U^{k'}_i U^i_{j'} = \delta^k_j}$ (same for covariants & contravariants)

EM field strength

$$[F^{\mu\nu}] = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & B^3 & -B^2 \\ -E^2/c & -B^3 & 0 & B^1 \\ -E^3/c & B^2 & -B^1 & 0 \end{pmatrix} \to F^{\mu\nu} = -F^{\nu\mu}$$

$$\left\{ \begin{array}{l} \partial_\nu F^{\mu\nu} = \mu_0 j^\mu \\ \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \end{array} \right\}$$

$j^\mu = (\rho c, \vec{J})$ $\qquad \frac{1}{\mu_0 \varepsilon_0} = c^2$

components, not coordinates
But partials of coords

(p. 38) Def Vector: obj whose components transform as $\partial^{i} = U^{i}_{i'} \partial^{i'}$, $u^i = u^i(u^{i'})$

Tensor: obj whose components transform as vector components (multi-lin)

$$\boxed{g_{i'j'} = (U^k_{i'} \vec{e}_k) \cdot (U^l_{j'} \vec{e}_l) = U^k_{i'} U^l_{j'} g_{kl}}$$

$[F_{\mu\nu}] = [\eta_{\mu\alpha}]\,[F^{\alpha\beta}]\,[\eta_{\beta\nu}]$

So $g^{i'j'} = U^{i'}_k U^{j'}_l g^{kl}$ | Type $(r,s) \to r$ contravariant, s covariant.

As matrices $[g^{i'j'}] = [U^{i'}_k][g^{kl}][U^{j'}_l]^T$

and $[g_{i'j'}] = [U^k_{i'}]^T [g_{kl}] [U^l_{j'}]$

Scalars

(0,0) tensor, invariant.

Show line element = scalar:

(★) $g_{ij} du^i du^j = g_{kl} U^k_{i} U^l_{j} U^i_{m'} du^{m'} U^j_{n'} du^{n'} = g_{k'l'} du^{m'} du^{n'} \cdot \delta^l_m \delta^k_n = g_{k'l'} du^{k'} du^{l'}$

Summary $T^{i'j'}{}_{k'} = U^{i'}_l U^{j'}_m U^n_{k'} T^{lm}{}_n$ $\to$ invariant.

(SR) $[\eta_{\mu\nu}] = \mathrm{diag}(1,-1,-1,-1) = [\eta^{\mu\nu}]$ (minkowski metric tensor)

$= [\eta_{\mu'\nu'}] = [\eta^{\mu'\nu'}]$ $\qquad \boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$ Lorentz/covariant $\qquad \partial^\mu = \eta^{\mu\nu} \partial_\nu$ | $\partial^0 = \partial_0$ but $\partial^i = -\partial_i$

$(x = -x)$ $(t = -t)$

Lorentz Transform $\to$ Poincaré Transf (1) Boost (2) Translate (3) Spatial Rotate (4) Space parity (5) Time reverse

inhomogen (translate)
homogen (no translate)
improper (reversal)
proper (boost)
+ rotate

Boost $\quad [\Lambda^{\mu'}{}_\nu] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to$ constant

$\to X^\mu$ form components of vectors under LTs.

$$\left\{ \begin{array}{l} X_\mu = \eta_{\mu\nu} X^\nu \\ X^\mu = \eta^{\mu\nu} X_\nu \end{array} \right\} \to \text{coordinates.}$$

$$[\Lambda^\nu{}_{\mu'}] = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$\boxed{\text{Boost along } y = \text{rotate } \tfrac{\pi}{2} \to \text{boost } x \to \text{rotate } -\tfrac{\pi}{2}}$

Poincare $\boxed{X^{\mu'} = \Lambda^{\mu'}{}_\nu X^\nu + a^{\mu'}}$ (+ translate + rotate + boost)

$\to$ properties $\boxed{\eta_{\mu'\nu'} = \Lambda^\alpha{}_{\mu'} \Lambda^\beta{}_{\nu'} \eta_{\alpha\beta}}$

invariant!

Summary $T^{\mu'\nu'}{}_{\sigma'} = \Lambda^{\mu'}{}_\alpha \Lambda^{\nu'}{}_\beta \Lambda^\gamma{}_{\sigma'} T^{\alpha\beta}{}_\gamma$

X^μ not vector if $a^{\mu'} \neq 0$ $\quad (X^{\mu'} = \Lambda^{\mu'}{}_\nu X^\nu)$ need

But dX^μ, $\frac{\partial \phi}{\partial X^\mu}$ are vectors

$\frac{\partial \phi}{\partial u^\mu} = \Phi_\mu = (\partial_0, \vec{\nabla}_i)$

$$\left\{ \begin{array}{l} \partial_\mu = \eta_{\mu\nu} \partial^\nu \\ \partial^\mu = \eta^{\mu\nu} \partial_\nu \end{array} \right\}$$

Timelike: ~~simultaneous~~ $|\vec{\lambda}| > 0$
Spacelike $|\vec{\lambda}| < 0$
Null $|\vec{\lambda}| = 0$

Note $V^{\mu'} \neq \Lambda^{\mu'}{}_\nu V^\nu$
Rather $u^{\mu'} = \Lambda^{\mu'}{}_\nu u^\nu$
where

$V^\mu = \frac{dX^\mu}{dt}$

$u^\mu = \frac{dX^\mu}{d\tau}$

$u^\mu u_\mu = c^2$ (inv), $\frac{dt}{d\tau} = \gamma$

$u^\mu = \gamma V^\mu$, $p^\mu p_\mu = m^2 c^2$, $p^\mu = (\gamma mc, \gamma m u^\mu)$

For light

$u^\mu u_\mu = 0 \to \sigma$

Sep 5

GR FORMULAS

Metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Connection:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda} \right)$$

Geodesic Equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

Spacetime Covariant Derivatives:

$$\phi_{;\mu} = \partial_\mu \phi$$
$$A^\nu{}_{;\mu} = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\sigma} A^\sigma$$
$$A_{\nu;\mu} = \partial_\mu A_\nu - \Gamma^\sigma{}_{\mu\nu} A_\sigma$$
$$B^{\nu\lambda}{}_{\sigma;\mu} = \partial_\mu B^{\nu\lambda}{}_\sigma + \Gamma^\nu{}_{\mu\rho} B^{\rho\lambda}{}_\sigma + \Gamma^\lambda{}_{\mu\rho} B^{\nu\rho}{}_\sigma - \Gamma^\rho{}_{\mu\sigma} B^{\nu\lambda}{}_\rho$$

Curvature:

$$R^\mu{}_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\lambda} + \Gamma^\rho_{\nu\sigma} \Gamma^\mu_{\rho\lambda} - \Gamma^\rho_{\nu\lambda} \Gamma^\mu_{\rho\sigma}$$
$$R_{\mu\nu} = R^\lambda{}_{\mu\nu\lambda}$$
$$R = R^\lambda_\lambda$$

Einstein's Equations (without and with Λ):

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} = -\frac{8\pi G}{c^2} T^{\mu\nu}$$

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^2} T^{\mu\nu}$$

Schwarzshild metric:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

FRW metric:

$$d\tau^2 = dt^2 - R(t)^2 \left((1 - kr^2)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right)$$

Mini Review of GR (so far)

⇒ In GR gravity is a bending of spacetime, not a force. Mass and energy warp the spacetime around it.

⇒ If we are given the metric $g_{\mu\nu}$, we can figure out the geometry of the spacetime, physical lengths and distances, and the trajectories of particles in the presence of gravity.

⇒ Ultimately, however, we will use Einstein's equations to solve for the metric for a given distribution of mass and energy.

⇒ But for now, let's assume we are just given the metric tensor $g_{\mu\nu}$.

⇒ With the metric, the line element gives infinitesimal distances in spacetime

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu$$

⇒ While the squared norm of a vector λ^μ is given as

$$|\lambda|^2 = g_{\mu\nu}\lambda^\mu\lambda^\nu = \lambda_\mu\lambda^\mu$$

⇒ Inner products between two vectors can always be written in four ways

$$\lambda \cdot \mu = g_{\mu\nu}\lambda^\mu\mu^\nu = \lambda_\mu\mu^\mu = \lambda^\mu\mu_\mu = g^{\mu\nu}\lambda_\mu\mu_\nu$$

⇒ The metric raises and lowers indices on vectors and tensors

$$\lambda_\mu = g_{\mu\nu}\lambda^\nu \qquad \tau^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}\tau_{\alpha\beta}$$

⇒ The Christoffel connection is computed from the metric as

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2}g^{\mu\rho}\left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\right)$$

⇒ The geodesic equation describes the trajectory of a free particle (or geodesic)

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

⇒ The solution of the geodesic equation $x^\mu(\tau)$ therefore gives the trajectory of a particle in a gravitational field (described by the metric $g_{\mu\nu}$)

⇒ Parallel transporting a vector $\vec{\lambda}$ along a curve $x^\mu(t)$ means moving it without altering it. So it obeys $\frac{d\vec{\lambda}}{dt} = 0$. However, in a curved spacetime its direction can change, and the components λ^μ must obey the parallel-transport equation:

$$\frac{d\lambda^\mu}{dt} + \Gamma^\mu_{\nu\sigma}\lambda^\nu\dot{x}^\sigma = 0$$

⇒ In curved spacetime, the derivatives $\frac{d}{dt}$ or $\partial_\mu = \frac{\partial}{\partial x^\mu}$ acting on tensors do not give tensors. For this reason absolute and covariant derivatives must be introduced.

⇒ Absolute Derivatives:

$$\frac{D\phi}{dt} = \frac{d\phi}{dt}$$

$$\frac{DA^\mu}{dt} = \frac{dA^\mu}{dt} + \Gamma^\mu{}_{\nu\sigma} A^\nu \dot{x}^\sigma$$

$$\frac{DA_\mu}{dt} = \frac{dA_\mu}{dt} - \Gamma^\rho{}_{\mu\sigma} A_\rho \dot{x}^\sigma$$

$$\frac{D\tau^\mu{}_\nu}{dt} = \frac{d\tau^\mu{}_\nu}{dt} + \Gamma^\mu{}_{\rho\sigma}\, \tau^\rho{}_\nu\, \dot{x}^\sigma - \Gamma^\rho{}_{\nu\sigma} \tau^\mu{}_\rho\, \dot{x}^\sigma$$

⇒ Covariant Derivatives:

$$\phi_{;\mu} = \partial_\mu \phi$$

$$A^\nu{}_{;\mu} = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\sigma} A^\sigma$$

$$A_{\nu;\mu} = \partial_\mu A_\nu - \Gamma^\sigma{}_{\mu\nu} A_\sigma$$

$$B^{\nu\lambda}{}_{\sigma;\mu} = \partial_\mu B^{\nu\lambda}{}_\sigma + \Gamma^\nu{}_{\mu\rho} B^{\rho\lambda}{}_\sigma + \Gamma^\lambda{}_{\mu\rho} B^{\nu\rho}{}_\sigma - \Gamma^\rho{}_{\mu\sigma} B^{\nu\lambda}{}_\rho$$

⇒ Using absolute and covariant derivatives, the derivative of a tensor is then a tensor

⇒ Principle of General Covariance: If an equation is true in special relativity and it is a tensor equation, then it is true in GR.

⇒ Prescription for finding physics equations in GR:

1. Write down the equation in special relativity
2. Change all derivatives to absolute or covariant derivatives (it should then be a tensor equation)
3. By the principle of general covariance, the resulting equation should hold in GR

⇒ Newtonian gravitational force in terms of the gravitational potential V:

$$\vec{F} = m\frac{d^2\vec{x}}{dt^2} = -m\vec{\nabla}V \qquad \text{which implies} \qquad \frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_j V$$

⇒ Newtonian potential for a point mass

$$V = -\frac{GM}{r}$$

⇒ In the Newtonian limit (weak static fields), with $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$, where the corrections $h_{\mu\nu}$ are small, the geodesic equation must match the Newtonian force law equation

⇒ This results in the correspondence that

$$g_{00} \simeq 1 + \frac{2V}{c^2} \qquad \text{or} \qquad h_{00} \simeq \frac{2V}{c^2}$$

in the weak static Newtonian limit

Cosmology – Expanded Outline

(1) Large-scale geometry of the universe
- cosmological principle
- Robertson-Walker (flat, open, closed) geometries
- expansion of the universe
- distances and speeds
- redshifts

(2) Dynamical evolution of the universe
- Friedmann equations
- cosmological constant Λ
- equations of state
- matter-dominated universe ($\Lambda = 0$) [Friedmann models]
- flat matter-dominated universe ($\Lambda = 0$) [old favorite model]

(3) Observational cosmology
- Hubble law
- acceleration of the universe
- matter densities & dark matter
- flatness & horizon problems
- CMB anisotropy

(4) Modern Cosmology
- inflation
- dark energy (cosmological constant?)
- concordance model [new favorite model]
- open questions

Practice #5

1. Write down in words only what each of the following is and/or does:

   (LT) (a) $\Lambda^{\mu'}_{\nu}$ transforms $\nu \to \mu'$ contravariant in Minkowski space
   (b) $g_{\mu\nu}$ metric tensor
   (c) $U^{i'}_{j}$ Jacobian $j \to i'$ contravariant (3D)
   (d) $X^{a'}_{b}$ transforms $b \to a'$ contravariant (N-D) (GCT matrix N-D)
   (e) Γ^{k}_{ij} connection, represents curvature of space?
   Christoffel connection (in geodesic equation, parallel transport, $\frac{D\lambda^a}{dt}$)

2. Define each of the following in words only:

   (a) geodesic in curved space path of free particle
   (b) scalar type (0,0) tensor, invariant
   (c) parallel transport moving a vector without altering it
   (d) equivalence principle in freely falling frame → physics obey SR
   gravity = acceleration are equivalent
   (e) principle of general covariance
   ↳ ~~laws of physics take the same form in freely falling frame~~?
   eq. true in GR if
   {(1) true in SR
   (2) tensor equation}

3. Which of the following are expressions the book uses to denote the tangent vector in 3-dimensional space (pick all that apply)

   (a) $\vec{\lambda}$
   (b) $\lambda^i \vec{e}_i$
   (c) $\frac{d\vec{r}}{ds}$
   (d) $\frac{\partial \vec{r}}{\partial u^i}\frac{du^i}{ds}$
   (e) $\frac{du^j}{ds}\vec{e}_j$
   (f) $\dot{u}^i \vec{e}_i$
   (g) all of the above

Exam Practice

1. Consider flat 3-dimensional Euclidean space. The transformation matrix $U^{i'}_j$ from Cartesian coordinates $u^j = (x, y, z)$ to spherical coordinates $u^{j'} = (r, \theta, \phi)$ is

$$[U^{i'}_j] = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\phi & \frac{1}{r}\cos\theta\sin\phi & -\frac{1}{r}\sin\theta \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} = A$$

Using that the metric with upper indices in the Cartesian frame is

$g^{i'j'} = U^{i'}_k U^{j'}_l g^{kl}$

$$[g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

find the metric $g^{i'j'}$ in the spherical-coordinate system (where i', j' denote r, θ, ϕ) as a transformation with $U^{i'}_j$.

$$g^{i'j'} = U^{i'}_k U^{j'}_l \underbrace{g^{kl}}_{I} = [U^{i'}_k][U^{j'}_l]^T$$

$$= AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

2. Consider a tensor $T^{\mu\nu}$ in Minkowski spacetime using Cartesian coordinates. The components of $T^{\mu\nu}$ defined in matrix form are

$$[T^{\mu\nu}] = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & -2 \end{pmatrix}$$

$= T^{\mu\nu} T_{\nu\gamma} = \delta^{\mu}_{\gamma}$

$2a + i + n = 1$

$-a + 3i + 2h = 0$

$e = 0$

Also consider a vector V^μ with contravariant components

$$V^\mu = (-1, 2, 0, -2)$$

$T_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}T^{\alpha\beta}$

Find the following:

(a) the components of $T_{\mu\nu} = (T^{\mu\nu})^{-1}$ $= \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & h & l & m \\ n & o & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$

(b) $V^\mu V_\mu = 1 - 4 - 0 - 4 = -7$

not $T^{\mu\nu}$

(c) $V^\mu V^\nu T_{\mu\nu} = V^0 V^\nu T_{0\nu} + V^1 V^\nu T_{1\nu} + V^2 V^\nu T_{2\nu} + V^3 V^\nu T_{3\nu}$

got $T_{\mu\nu}$

$(-1)(-1)\,2 + \cancel{(-1)(2)\,0} + \cancel{(-1)\,0} + (-1)(-2)(-1)$

$+\ 2(-1)(-1) + \cancel{(2)(2)\,0} + \cancel{(2)(-2)(2)} + \cancel{(-2)(-1)(-2)} + (-2)(-2)(-2)$

$$2a + b - c = 1$$
$$-a + 3b + 2c = 0$$
$$2a + 2c = 0$$

$$[T^{\mu\nu}] = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}$$

$$\cancel{[T_{\mu\nu}] = ? = [T^{\mu\nu}]^{-1} = \begin{pmatrix} a & d & g & j \\ 0 & 0 & 1 & 0 \\ b & e & h & k \\ c & f & i & m \end{pmatrix} = \begin{pmatrix} 1/4 & -1/12 & -5/24 & 5/24 \\ 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/8 & -1/8 \\ -1/4 & 1/12 & -7/24 & 7/24 \end{pmatrix}}$$

or

$$[T_{\mu\nu}] = \eta_{\mu\alpha}\eta_{\nu\beta}T^{\alpha\beta} = [\eta_{\mu\alpha}][T^{\alpha\beta}][\eta_{\nu\beta}]^T$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 & 1 \\ -1 & 0 & -3 & -2 \\ 0 & -1 & 0 & 0 \\ 2 & -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & -2 \end{pmatrix}$$

$$V^\mu V^\nu T_{\mu\nu} = [V^\mu]^T[T_{\mu\nu}][V^\nu] = -14$$

$x = \gamma(x' + \beta ct)$

$ct = \gamma(ct' + \beta x)$

$y = y'$

$z = z'$

$$\begin{pmatrix} \gamma\beta & \gamma & 0 & 0 \\ \gamma & \gamma\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{matrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

Practice #4

True or False (in Minkowski spacetime)?

1. $\lambda \cdot \lambda \geq 0$ F spacelike

2. $\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^{\mu}_{\sigma}$ T

3. $\Lambda^{\mu}_{\nu'}\Lambda^{\nu'}_{\sigma} = \delta^{\mu}_{\sigma}$ T

4. $[\eta_{\mu'\nu'}] = [\eta_{\alpha\beta}] = [\eta^{\rho'\sigma'}] = [\eta^{\lambda\varsigma}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ T

5. $\Lambda^{\alpha'}_{\mu}\Lambda^{\beta'}_{\nu}\eta_{\alpha'\beta'} = \eta_{\mu\nu}$ T

6. $\vec{e}_{\mu} \cdot \vec{e}_{\nu} = \eta_{\mu\nu}$ T

7. $\eta_{\mu\nu}a^{\mu}b_{\sigma}c^{\sigma}d^{\nu} = a_{\alpha}d^{\alpha}b_{\beta}c^{\beta}$ T

   $a_{\nu}b_{\sigma}c^{\sigma}d^{\nu}$

8. $L = \int \sqrt{|\eta_{\mu\nu}dx^{\mu}dx^{\nu}|}$ T

9. $\Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta} = \eta^{\mu'\nu'}\eta_{\alpha\beta}$ → gibberish lmao F

10. $\underbrace{\eta^{\mu\nu}\eta_{\nu\sigma}}\underbrace{\eta^{\sigma\rho}\eta_{\rho\mu}} = 4$ T

    $\delta^{\mu}_{\sigma} \cdot \delta^{\sigma}_{\mu}$

Practice #3

Connect the items on the left with the ones on the right.

4×4	$\Lambda^{\mu'}_{\nu}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
4×4	$\eta_{\mu\nu}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$
3×3	$U^{i'}_{j}$	$\begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
4×4	δ^{ν}_{μ}	$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$
3×3	g_{ij}	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

Practice #2

State in words what each of the following is, does, and/or means:

(natural) 1. $\vec{e}_i \Rightarrow$ unit vec cor. to contravariant component

(dual) (~~unit~~) 2. $\vec{e}^j \Rightarrow$ unit vec cor. to covariant components

3. $\delta^i_j \Rightarrow$ kronecker delta = 1 if $i=j$, 0 if $j \neq i$

4. $\lambda^i \Rightarrow$ contravariant vector component

5. $\lambda_k \Rightarrow$ covariant vector component

6. $g_{ij} \Rightarrow \vec{e}_i \cdot \vec{e}_j$ metric tensor in general words

7. $g^{ij} \Rightarrow \vec{e}^i \cdot \vec{e}^j$ inverse metric tensor

8. $\nabla u^i \Rightarrow$ ~~$\vec{e}$~~ (dual basis) $\{\vec{e}^i\}$

9. $\frac{\partial \vec{r}}{\partial u^j} \Rightarrow$ $\vec{e}_j$ [natural basis vector]

10. $L = \int |\dot{\vec{r}}(\sigma)| d\sigma \Rightarrow$ arc length

11. $ds^2 = g_{ij} du^i du^j \Rightarrow$ line element in general coords

12. $ds^2 = dx^2 + dy^2 + dz^2 \Rightarrow$ line element in Cartesian

13. $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \Rightarrow$ line element in spherical coords

coord transform 14. $u^{i'} = u^{i'}(u^j) \Rightarrow$ parametrization of $u^{i'}$ with u^j

15. $\lambda^{i'} = U^{i'}_j \lambda^j \Rightarrow$ defines a vector. Its components transform $j \to i'$

16. $U^{i'}_j \Rightarrow$ ~~Jacobian~~, transforms component $j \leftrightarrow i'$ for covariant

17. $U^j_{i'} \Rightarrow$ ~~Jacobian~~ transform component $i' \leftrightarrow j$ for covariant ~~for contravariant~~ ~~$j \to i'$ for contravariant~~ inverses

[16] 18. $\left[\frac{\partial u^{i'}}{\partial u^j}\right] \Rightarrow [U^{i'}_j] \to$ Jacobian

flat space

19. $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$ metric tensor w/ matrix rep. (for Cartesian)

20. $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \Rightarrow$ metric tensor in matrix, for spherical

Practice #1

1. Write out each of the following sums ($i, j, \ldots = 1, 2, 3$). Simplify the resulting expressions where appropriate.

   (a) $\lambda^i\lambda_i =$ $\lambda^1\lambda_1 + \lambda^2\lambda_2 + \lambda^3\lambda_3 = \sum_{i=1}^{3}\lambda^i\lambda_i = \vec{\lambda}\cdot\vec{\lambda} = \|\vec{\lambda}\|^2$

   (b) $\lambda^j\lambda_j =$ $\lambda^1\lambda_1 + \lambda^2\lambda_2 + \lambda^3\lambda_3 = \sum_{j=1}^{3}\lambda^j\lambda_j = \|\vec{\lambda}\|^2$ //

   (c) $\delta^i_j a^j =$ a^i

   (d) $a_k\delta^k_1 =$ a_1

   (e) $\vec{e}^i\cdot\vec{e}_i =$ $\sum_{i=1}^{3}\vec{e}^i\vec{e}_i = \vec{e}^1\vec{e}_1 + \vec{e}^2\vec{e}_2 + \vec{e}^3\vec{e}_3 = 3 = \delta^i_i$

2. How do you write the following using the suffix notation?

$$(a_1b^1 + a_2b^2 + a_3b^3)(f_1g^1 + f_2g^2 + f_3g^3) =$$

   $a_ib^i \cdot f_jg^j$

$\vec{\lambda} = \sum_i^3 \lambda^i\vec{e}_i = \lambda^i e_i$

$\vec{\mu} = \sum_j^3 \mu^j\vec{e}_j = \mu^j e_j$

$\vec{\lambda}\cdot\vec{\mu} = \sum\lambda^i e_i \sum\mu^j e_j$

$= \cancel{\sum\lambda^i e_i \mu^i e_i}$

$(i, j = 1, 2, 3 \ldots)$

3. How many equations are each of the following?

   (a) $a_ib_jc^k = \Gamma^k_{ij}$ 27

   (b) $a_ib^i = 5$ 1

   (c) $\vec{e}^i\cdot\vec{e}_j = \delta^i_j$ 9

   (d) $a_ib_j\delta^j_k = c_id_k$ 9

   $a_ib_k = c_id_k$

4. State whether the following are valid or invalid equations:

   (a) $g^{ij}a_j = a^i$ (valid)

   (b) $a^kb_k = g^{ij}a_ib_j$ $= a^jb_j$ (valid)

   δ_{jk} — (c) $\delta^i_j g_{ik} = g_{jk}$ $=$ ~~(NOT valid)~~ (valid)

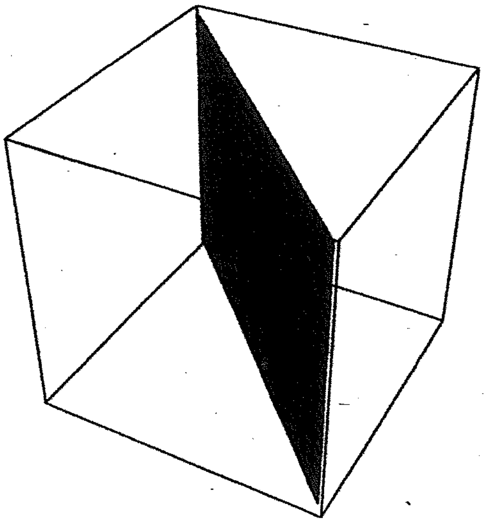
   (d) $g^{ij}g_{ij} = 1$ (NOT valid)

   $g^{ij}g_{ij} = 3$ ✓

$\vec{\lambda} = \lambda^i\vec{e}_i = \lambda_i\vec{e}^i$

$\vec{e}^i\cdot\vec{e}_j = \delta^i_j$

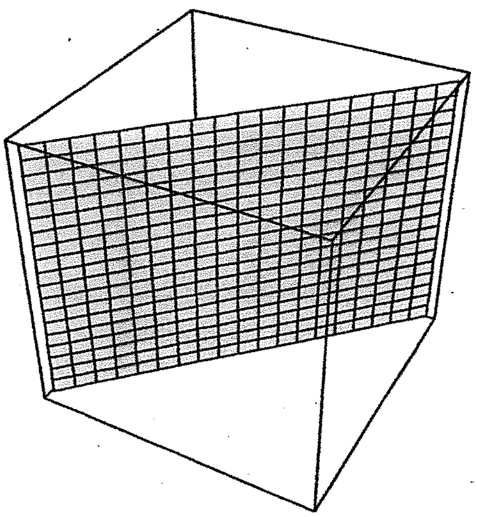
```
ParametricPlot3D[{x, 2 - x, z}, {x, -25, 25}, {z, -25, 25}, Ticks → None]
```



$u = \frac{1}{2}(x+y)$

$u = \text{const}$

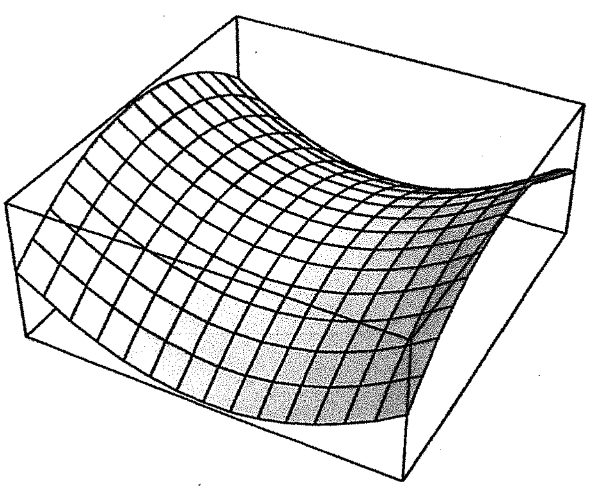
```
ParametricPlot3D[{x, x - 2, z}, {x, -25, 25}, {z, -25, 25}, Ticks → None]
```



$v = \frac{1}{2}(x-y)$

$v = \text{Const}$

```
Plot3D[(1/2)*(x^2 - y^2), {x, -25, 25}, {y, -25, 25}, Ticks → None]
```



$w = z - \frac{1}{2}(x^2-y^2)$

$w = \text{const}$

Sep 7, 2018

Review of Vector Calculus

Scalar functions:

$$f = f(x, y, z)$$

Partial derivatives:

$$\frac{\partial f}{\partial x} \Rightarrow \text{gives the rate of change of } f \text{ along } x\text{, with } y \text{ and } z \text{ fixed}$$

Chain rules:

1. For a function of a single variable $f = f(x)$ where $x = x(t)$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

2. For a function $f = f(x, y)$ with $x = x(s)$, $y = y(s)$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

3. For a function $f = f(x, y, z)$ with $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Gradients:

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\vec{\nabla} f \Rightarrow \text{points along direction of maximum increase in } f$$

$$\vec{\nabla} f \cdot \hat{v} \Rightarrow \text{directional derivative (rate of change of } f \text{ along direction } \hat{v})$$

Position vector:

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

Parameterized curve or trajectory (t = parameter) in 3D space:

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$$

Tangent vector (velocity if t = time):

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}}(t) = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath} + \frac{dz}{dt}\,\hat{k}$$

$$\tfrac{d\vec{r}}{dt} = \dot{\vec{r}}(t) \Rightarrow \text{vector tangent to the curve } \vec{r}(t)$$

Length of a curve along $\vec{r}(t)$ for $a \le t \le b$:

$$L = \int_a^b |d\vec{r}| = \int_a^b |\frac{d\vec{r}}{dt}|\,dt$$

Vector functions:

$$\vec{F}(\vec{r}) = F_x(x,y,z)\,\hat{\imath} + F_y(x,y,z)\,\hat{\jmath} + F_z(x,y,z)\,\hat{k}$$

Divergence:

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Curl:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{\imath}(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y) - \hat{\jmath}(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x) + \hat{k}(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x)$$

Line integral of $\vec{F}$ along curve $\vec{r}(s)$ for $a \le s \le b$:

$\int_a^b \vec{F}(\vec{r})\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(s))\cdot\frac{d\vec{r}}{ds}\,ds \quad \Rightarrow \quad$ sum of components of $\vec{F}$ along curve $\vec{r}(s)$

Surface integral of $\vec{F}$:

$\int_A \vec{F}\cdot d\vec{a} \quad \Rightarrow \quad$ flux of $\vec{F}$ through surface A

Gauss' theorem:

$$\int_A \vec{F}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{F}\,d^3r$$

Stoke's theorem:

$$\oint \vec{F}\cdot d\vec{s} = \int_A (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$$

Sep 5

Review of Special Relativity

Postulates of special relativity:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light (in a vacuum) is the same in all inertial reference frames.

Time dilation and length contraction (Δt_0 = proper time, L_0 = proper length):

$$\Delta t = \gamma \Delta t_0 \qquad L = \frac{L_0}{\gamma}$$

Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

Lorentz transformations (for relative motion along x):

$$\begin{aligned}
x' &= \gamma(x - vt) & x &= \gamma(x' + vt') \\
y' &= y & y &= y' \\
z' &= z & z &= z' \\
t' &= \gamma(t - \frac{v}{c^2}x) & t &= \gamma(t' + \frac{v}{c^2}x')
\end{aligned}$$

Spacetime coordinates:

$$\begin{gathered}
(x^0, x^1, x^2, x^3) = \text{position 4-vector} \\
x^0 = ct \\
x^1 = x \\
x^2 = y \\
x^3 = z
\end{gathered}$$

Invariant spacetime interval ($\Delta x \to \Delta x'$, etc. under a Lorentz transformation):

$$\begin{aligned}
c^2\,(\Delta\tau)^2 &= c^2\,(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\
&= c^2\,(\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2
\end{aligned}$$

Velocity transformations (for relative motion along x):

$$u' = \frac{u-v}{1-\frac{vu}{c^2}} \qquad\qquad u = \frac{u'+v}{1+\frac{vu'}{c^2}}$$

Relativistic definitions of energy, momentum, and kinetic energy:

$$E = \gamma mc^2$$
$$p = \gamma mv$$
$$K = (\gamma - 1)mc^2$$

Relativistic relation between energy and momentum:

$$E^2 = c^2\vec{p}^2 + m^2c^4$$

Lorentz transformations for energy-momentum (for relative motion along x):

$$\begin{aligned} p'_x &= \gamma(p_x - \frac{v}{c^2}E) & \qquad p_x &= \gamma(p'_x + \frac{v}{c^2}E') \\ p'_y &= p_y & p_y &= p'_y \\ p'_z &= p_z & p_z &= p'_z \\ E' &= \gamma(E - vp_x) & E &= \gamma(E' + vp'_x) \end{aligned}$$

Spacetime energy-momentum:

$$(p^0, p^1, p^2, p^3) = \text{energy}-\text{momentum 4}-\text{vector}$$
$$p^0 = \frac{E}{c}$$
$$p^1 = p_x$$
$$p^2 = p_y$$
$$p^3 = p_z$$

Invariant energy-momentum ($p_x \longrightarrow p'_x$, etc. under a Lorentz transformation):

$$\begin{aligned} (mc)^2 &= (\frac{E}{c})^2 - (p_x)^2 - (p_y)^2 - (p_z)^2 \\ &= (\frac{E'}{c})^2 - (p'_x)^2 - (p'_y)^2 - (p'_z)^2 \end{aligned}$$

Sep 5

PH 335 General Relativity & Cosmology

Robert Bluhm
414 Mudd Building
859-5862
e-mail address: robert.bluhm@colby.edu

Office Hours: Mondays 1:00 – 2:00
Thursdays 3:00 – 4:30
or by appointment.

Required Texts: A Short Course in General Relativity, 3rd Ed.,
by J. Foster and J.D.Nightingale
(Springer, 2006)

Was Einstein Right? 2nd Ed.,
by C. Will
(Basic Books, 1993)

Recommended: Spacetime and Geometry,
by Sean M. Carroll
(Addison Wesley, 2004)

Reading: There will be regular reading assignments. A lot of effort in this course must go into reading the book. You need to stay current with the reading assignments or you risk becoming lost.

Problems Sets: Problem sets will be due most weeks. Late problem sets without prior excuse will not be accepted. You may work together and discuss problems with others before writing your solutions, but what you hand in must be your own work.

Exams: There will be two mid-term exams and a final exam. The mid-term exams will be untimed, closed book, and individually administered take-home exams on an honor system. The final exam will be a three-hour in-class exam during finals week and will also be closed book. However, you will be allowed to bring one sheet of paper with formulas on it to each of the exams. You may use a calculator. The midterms will be due back within two days.

Midterm #1 - Wednesday Oct. 10th (due Friday Oct. 12th)
Midterm #2 - Wednesday Nov. 28th (due Friday Nov. 30th)
Final Exam - Thursday Dec. 13th at 9:00 AM (3 hours)

Attendance: You are expected to come to class. If you have an unexcused absence, you will need to make up the material on your own.

Electronics: You can use a tablet to take notes if you want. But please do not use laptops or other electronic devices such as cell phones in class unless you have written permission from a dean or a doctor.

Goals: The primary objectives of the course are for you to learn the subject of general relativity and to apply it to the study of cosmology. The class is roughly 80% general relativity and 20% cosmology. For a more specific list of topics, please see the course outline handout. In addition to learning these subjects you will develop your skills in:

- Listening and concentration
- Appreciating the development of a new theory
- Mathematics of general coordinate systems
- Mathematical descriptions of curved spaces
- Mathematics of vectors and tensors
- Using symbolic notation
- Problem solving at an advanced level
- Persevering with long computations (not giving up)
- Understanding conceptually difficult material
- Reading and studying the textbook
- Working both independently and collaboratively

Academic Honesty: Honesty, integrity, and personal responsibility are cornerstones of a Colby education. The values stated in the Colby Affirmation are central to this course. Students are expected to demonstrate academic honesty in all aspects of this course.

Religious Holidays: If you need to change an exam date or the due date for an assignment in order to observe a religious holiday, please let me know in advance and we will work something out.

Assessment: Your grade for the course will be the average of your grades on the problem sets, mid-term exams, and final exam with the following weights:

Problem sets	30%
Mid-term exams	40% (20% each)
Final Exam	30%

GENERAL RELATIVITY & Cosmology
PH 335 Prof. Bluhm

Sept 5, 2018

I. OVERVIEW & REVIEW

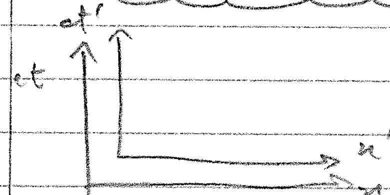
General Relativity? → Theory of Gravity

Replaces Newton's gravity law for heavy masses or at high precision

Keep in mind... expected that GR isn't compatible with QM

↳ Question in Physics → how to reconcile GR & QM

Special Relativity (SR) → involves moving inertial frames



Use Lorentz transformation

$$\begin{cases} x' = \gamma(x - vt) = \gamma(x - \beta ct) \\ y' = y \,, \quad z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

$\boxed{\text{Minkowski space}}$ → flat 4D spacetime of SR

↳ Invariant spacetime interval

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

$$= (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = (\Delta s')^2$$

↳ Invariant under Lorentz transformation.

- What is (Δs) physically? → go to a rest frame

↳ $\Delta z' = \Delta z = \Delta y' = \Delta x' = 0$

→ $\Delta t = \Delta \tau$ proper time

So $\boxed{(\Delta s)^2 = (c\Delta \tau)^2}$

In Minkowski spacetime → 4-vectors. Ex (ct, x, y, z)

Position (ct, x, y, z)
Momentum $(E/c, p_x, p_y, p_z)$ → Energy-momentum

→ these transform under Lorentz Transformations

$$\left\{ \begin{array}{l} p_x' = \gamma\left(p_x - \beta \frac{E}{c}\right) \\ p_y' = p_y, \quad p_z' = p_z \\ E' = \gamma(E - \beta c p_x) \end{array} \right\}$$

▨ E/c transform like ct, p_x transform like x ...

Also get an invariant for E-p:

$$\frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = \frac{E^2}{c^2} - \vec{p}^{\,2}$$

• Recall $E^2 = c^2\vec{p}^{\,2} + m^2c^4$

→ $\boxed{\frac{E^2}{c^2} - \vec{p}^{\,2} = (mc)^2}$ Invariant under Lorentz transformations...

Go to a rest frame $E = mc^2$, $p_x = p_y = p_z = 0$

So $\boxed{\frac{(mc^2)^2}{c^2} - \underbrace{\cancel{\vec{p}^{\,2}}}_{0} = (mc)^2}$ (true)

Notice → have 2 types of objects

(1) Proper time, Mass } → called SCALARS

(same in all Lorentz frames)

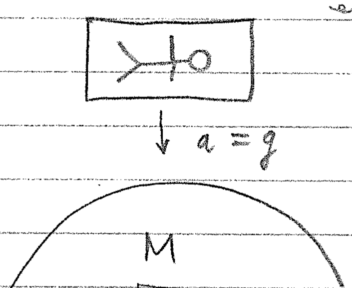
(2) 4-vectors (ct, x, y, z) and $(E/c, p_x, p_y, p_z)$ → 4-vectors all transform the same way under Lorentz transf

"

Now, want to look at the principle that got Einstein started on GR

↳ the **Equivalence Principle** (EP)

- <u>1907</u> → Einstein's happiest thought of his life
  ↳ realized that in a freely falling frame, the effects of gravity go away



⇒ freely falling frame (non rotating) (accelerating)

⇒ inside, it's an inertial frame

> Einstein realized there's an equivalence between gravity & acceleration
> ⇒ they can undo each other

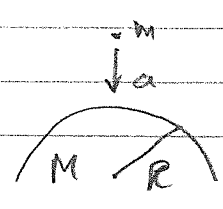
<u>Statement</u> **A small, non-rotating, freely falling frame in a gravitational field is an inertial frame**

{ This is a direct result of Galileo's discovery that all obj have the same acceleration due to gravity. }

<u>BUT</u> • This is a result of a coincidence!

↳ Mass has 2 roles: → causing gravitational force (like charge)
→ measure of inertia...

• Why are those the same?



(mass as "charge")

$$F = \frac{GMm}{R^2} = mg$$

but $F = ma$ mass as "inertia"

- $ma = mg$ so $a = g$ for all objects...

But it could have been that

$$\left\{ \begin{array}{l} m_g = \text{grav. mass} \\ m_I = \text{inertial mass} \end{array} \right\} \begin{array}{l} \to F = m_g g \\ \to F = m_I a \end{array}$$

so $m_I a = m_g g \to \boxed{a = \left(\frac{m_g}{m_I}\right) g}$

This ratio $\frac{m_g}{m_I}$ determines whether $a = g$

$\boxed{\text{The Equivalence Principle wouldn't hold if } m_g \neq m_I}$

Exp. show $\frac{|m_g - m_I|}{m_I} \lesssim 10^{-10}$ (Eötvos expt)

Sep 7, 2018

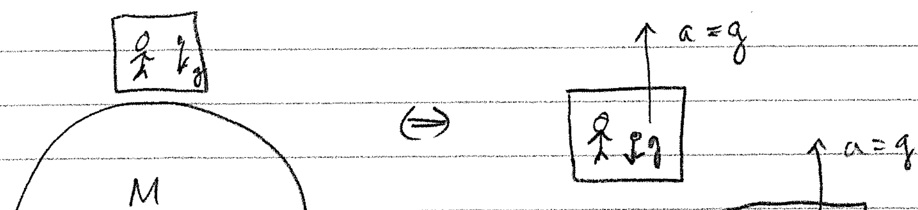
$\boxed{GR}$ → gravity is not a force

⇒ mass/energy cause curving/warping of spacetime

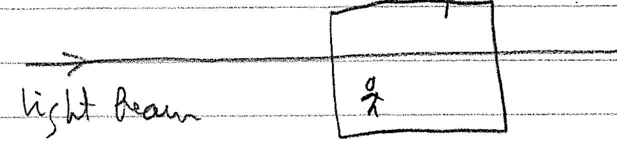
It was the equivalence principle that caused Einstein to think about curved spacetime.

EP ⇒ says that the effects of gravity & acceleration are equivalent

Means these 2 situations are the same



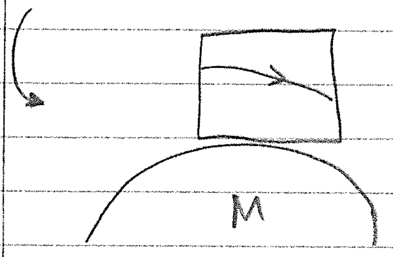
$\boxed{\text{Imagine a light ray}}$



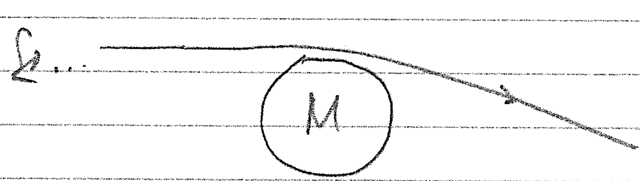
Inside

light bent

Now, according to the equivalence principle (postulate)

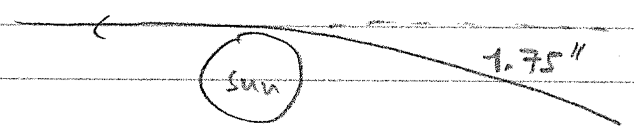


Got a prediction that light bends around massive object



GR predicts that light going 1 km past Earth's surface, will fall by 1 Å. (not observable)

But for Sun, GR predicts bending by 1.75" (arcsec) of light (Eddington)



Note

Could argue as well from Newtonian mechs that light falls with a = g
But to get 1.75" prediction, the spacetime must actually be curved

assumes spacetime is flat... assumes NOT

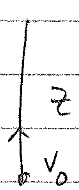
$\boxed{\text{Falling objects on Earth}}$ → how do we view this as due to curvature?

Let's compare 2 cases, each with initial velocity

$v_0 = 4.9\ m/s$

$t = 1s$

With no gravity



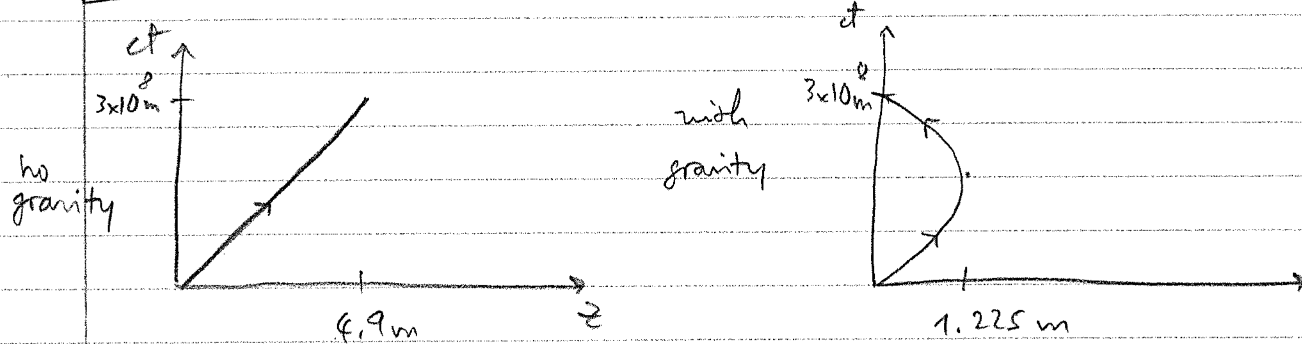
$a = 0$

Find $z = 4.9m = v_0 t$

With gravity

Find $z = 1.225m$ (turns around)

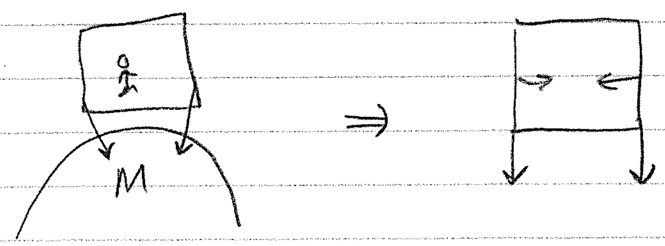
Must view this in spacetime (not to scale)



If drawn to scale, both would look like vertical lines
$\Rightarrow$ curvature of spacetime @ earth surface is very weak...

A few notes on the EP $\rightarrow$ freely falling frames are infinitesimal + instantaneous...

Why? because otherwise get tidal effects



(inward component $\rightarrow$ squeezing effect)

$\rightarrow$ If fall into black hole $\rightarrow$ turn into spaghetti!
(sphagettification)

There are also different versions of the EP

Strong equivalence principle $\rightarrow$ all of physics reduces to special relativity in a freely falling frame...

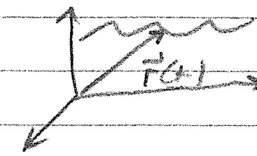
Weak EP $\rightarrow$ all point particles fall @ the same rate in a gravitational field ($m_g = m_I$) $\rightarrow$ applies to gravity only
$\rightarrow$ sufficient to develop GR, but not for QM

we use this

Sep 10, 2018

<u>Review</u> Curves in 3D space, parametrized by t, σ, s

$$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

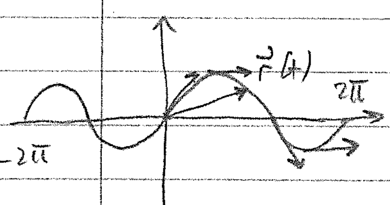


<u>tangent</u> $\dot{\vec{r}} = \dfrac{d\vec{r}}{dt}$

<u>Length of a curve</u> $\quad |d\vec{r}| = \left|\dfrac{d\vec{r}}{dt}\right| dt = \dot{\vec{r}}\,dt$

$$\Rightarrow \ell = \int_a^b |d\vec{r}| = \int_a^b |\dot{\vec{r}}|\,dt$$

<u>Ex</u> Consider $\vec{r}(t) = (t, \sin t) \qquad (-2\pi \leq t \leq 2\pi)$



$\dfrac{d\vec{r}}{dt} = ? \qquad \dot{\vec{r}} = (1, \cos t)$

At $t = 0 \quad \dot{\vec{r}} = (1, 1)$

$t = \pm\frac{\pi}{2} \quad \dot{\vec{r}} = (1, 0)$

$t = \pm\pi \quad \dot{\vec{r}} = (1, -1)$

$t = \pm\frac{3\pi}{2} \quad \dot{\vec{r}} = (1, 0)$

<u>Find length ℓ of curve</u>

$$\ell = \int_a^b |\dot{\vec{r}}|\,dt = \int_{-2\pi}^{2\pi} \|(1, \cos t)\|\,dt = \int_{-2\pi}^{2\pi} \sqrt{1 + \cos^2 t}\,dt$$
(elliptic int)

Use Mathematica ... $\ell \simeq 15.28$

<u>Can consider vector functions</u> $\boxed{\vec{F}(\vec{r}) = \begin{pmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{pmatrix}}$

Act with $\vec{\nabla} = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}, \dfrac{\partial}{\partial z}\right)$ by dotting or crossing

<u>Dot</u> (div?) $\quad \vec{\nabla}\cdot\vec{F} = \dfrac{\partial F_x}{\partial x} + \dfrac{\partial F_y}{\partial y} + \dfrac{\partial F_z}{\partial z}$

<u>Note</u> $\vec{\nabla} f$ gives gradient if f <u>scalar-valued</u>

<u>Cross</u> (curl?) $\quad \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F & F & F \end{vmatrix}$

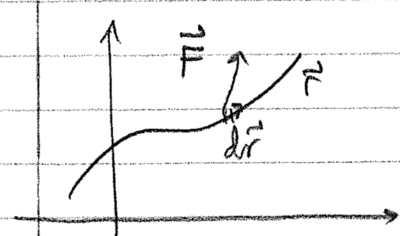
In E&M, can introduce potentials...

$$\boxed{\vec{E} = -\vec{\nabla}\phi}$$ where ϕ is electric potential (volts) (scalar)

$\hookrightarrow$ $\vec{E} \perp$ surfaces of constant ϕ (equipotentials)

$$\boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$ where $\vec{A}$ is vector potential

Line integrals $\rightarrow$ of a vector along a curve



$$\boxed{\int_a^b \vec{F} \cdot d\vec{r}}$$ $\Rightarrow$ sum of components of F along the curve...

e.g. $\vec{F}$ = force $\rightarrow$ $W = \int_a^b \vec{F} \cdot d\vec{r}$

e.g. $\vec{F} = \vec{E}$ = e field

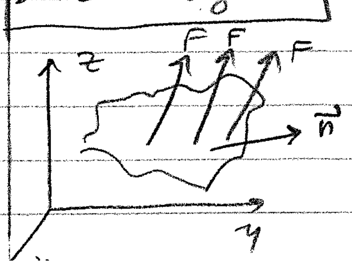
$-\int \vec{E} \cdot d\vec{r}$ = potential = $\Delta\phi$ change in E potential

To do line integral $\rightarrow$ parameterize...

let $\vec{r} = \vec{r}(s)$, then $\vec{F}(\vec{r}) = \vec{F}(\vec{r}(s))$

$$\boxed{\int_a^b \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(s)) \cdot \frac{d\vec{r}}{ds}\, ds}$$

Surface integrals $\rightarrow$ give flux of a vector field thru a surface



$\int \vec{F} \cdot d\vec{a}$ = flux thru surface

normal area $d\vec{a} = da\,\vec{n}$

e.g. $\vec{F} = \vec{E}$ electric field $\quad \int \vec{E} \cdot d\vec{a} =$ electric flux $= \Phi_E$

Gauss's Law $\quad \int_A \vec{E} \cdot d\vec{a} = \frac{q}{\varepsilon_0} \rightsquigarrow$ enclosed charge ...

Two famous theorems

Gauss' theorem

$$\oint \vec{F} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{F} \, d^3\vec{r}$$

flux $\quad$ vol int $\quad$ div

curl

Stokes' Theorem

$$\oint \vec{F} \cdot d\vec{s} = \int_A (\vec{\nabla} \times \vec{F}) \cdot d\vec{a}$$

flux

Ex Find the differential form of Maxwell's Eqn

Gauss's law ... $\oint \vec{E} \cdot d\vec{a} = \frac{q}{\varepsilon_0}$ $\qquad \oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = \frac{-\partial}{\partial t} \int_A \vec{B} \cdot d\vec{a}$ (Faraday law)

No magnetic monopole ... $\oint \vec{B} \cdot d\vec{a} = 0$ $\qquad \oint \vec{B} \cdot d\vec{s} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \int_A \vec{E} \cdot d\vec{a}$

(Ampere - Maxwell's law ...)

Use Gauss theorem on Gauss' law ... also that

$q = \int_V \rho \, d^3r$ where ρ = volume density

$$\oint \vec{E} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{E} \, d^3\vec{r} = \frac{1}{\varepsilon_0} \int_V \rho \, d^3r$$

So $\quad \varepsilon_0 \vec{\nabla} \cdot \vec{E} = \rho \quad \Rightarrow \quad \boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}}$

Immediately $\rightarrow$ $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

Use Stokes' theorem for the next two ...

closed loop

$$\oint \vec{E}\cdot d\vec{s} = \int_A (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = \frac{-\partial}{\partial t}\int_A \vec{B}\cdot d\vec{a}$$

So $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$

current density

$$\oint \vec{B}\cdot d\vec{s} = \int_A (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 I + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\int_A \vec{E}\cdot d\vec{a} \qquad \text{Let } I = \int_A \vec{J}\cdot d\vec{a}$$

$$= \mu_0 \int_A \vec{J}\cdot d\vec{a} + \mu_0\varepsilon_0 \int \frac{\partial \vec{E}}{\partial t}\cdot d\vec{a}$$

So $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}}$

So

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \end{array}}$$

We'll see how to make these eqn fully relativistic ...

#

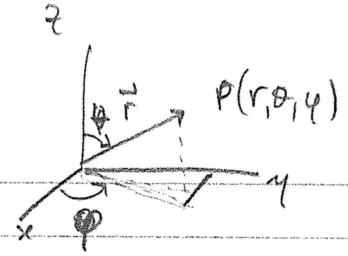
Coordinate Systems

In 3D space .. (there are lots of coordinate systems ...)

- Cartesian Coordinates (x, y, z)
- Spherical Coordinates (r, θ, φ)
- Cylindrical Coordinate (ρ, φ, z)

⋮

$\boxed{\text{Spherical Coordinate}}$



$$\begin{cases} 0 \le r < \infty \\ 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases}$$

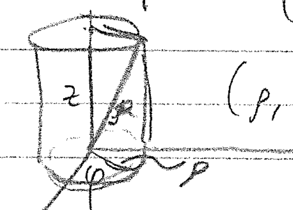
OR

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

$\boxed{\text{Cylindrical Coordinate}}$



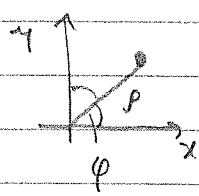
(ρ, φ, z)

$$\left\{\begin{matrix} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{matrix}\right\} \quad \text{or} \quad \left\{\begin{matrix} \rho = \sqrt{x^2+y^2} \\ z = z \\ \varphi = \tan^{-1}\left(\frac{y}{x}\right) \end{matrix}\right\}$$

How do we do integrals?

In Cartesian $\boxed{dA = dx\,dy}$ □

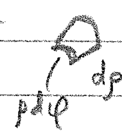
$\boxed{\text{2D polar}}$



$$\left\{\begin{matrix} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{matrix}\right\}$$

$$\rho = \sqrt{x^2+y^2}, \quad \varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

In Polar



$\boxed{dA = \rho\,d\rho\,d\varphi}$

extra function

Is there a systematic way to find this extra part?

↳ Use the Jacobian!

Sep 11, 2018

We can find the extra factor using Jacobian
↳ matrix of partial derivatives

e.g. Polar ⇌ Cartesian

$$\underset{\sim}{J} = \left[\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right] = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{bmatrix}$$

Theorem → $\boxed{dx\,dy = \det(\underset{\sim}{J})\,d\rho\,d\varphi}$

For 2D polar coordinates: $x = \rho\cos\varphi \rightarrow \frac{\partial x}{\partial \rho} = \cos\varphi,\ \frac{\partial x}{\partial \varphi} = -\rho\sin\varphi$

$y = \rho\sin\varphi \rightarrow \frac{\partial y}{\partial \rho} = \sin\varphi,\ \frac{\partial y}{\partial \varphi} = \rho\cos\varphi$

so $\det(\underset{\sim}{U}) = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho$ so $\boxed{dx\,dy = \rho\,d\rho\,d\varphi}$

In 3D relate $dx\,dy\,dz$ to spherical coordinates

$$dx\,dy\,dz = \det(\underset{\sim}{U})\,dr\,d\theta\,d\varphi$$

Now

$$\underset{\sim}{U} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix}$$

OR we could go to cylindrical coordinate $dx\,dy\,dz = \det(\underset{\sim}{U})\,d\rho\,d\varphi\,dz$

Now

$$\underset{\sim}{U} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{pmatrix}$$

We can also write a Jacobian for going from spherical to cylindrical

$$\underset{\sim}{U} = \begin{pmatrix} \frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \varphi} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial \rho} & \frac{\partial \theta}{\partial \varphi} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial \rho} & \frac{\partial \varphi}{\partial \varphi} & \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

Note: in this case $dr\,d\theta\,d\varphi$ & $d\rho\,d\varphi\,dz$ are not proper volume element.

But Jacobian like this will still be useful to us.

Ex Find Jacobian for $dx\,dy\,dz \rightarrow$ spherical

$$\underset{\sim}{U} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

So $\det(\underset{\sim}{U}) = ?$

So $\det(\underset{\sim}{V}) = \sin\theta\cos\varphi\left[+r\sin^2\theta\cos\varphi\right] - r\cos\theta\cos\varphi\left[-r\sin\theta\cos\theta\cos\varphi\right]$
$+ (-r)\sin\theta\sin\varphi\left[-r\sin^2\theta\sin\varphi - r\cos^2\theta\sin\varphi\right]$

$$= r^2\sin^3\theta\cos^2\varphi + r^2\sin\theta\cos^2\theta\cos^2\varphi + r^2\sin^3\theta\sin^2\varphi + r^2\sin\theta\cos^2\theta\sin^2\varphi$$
$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$$
$$= \boxed{r^2\sin\theta} \quad \text{as expected} \ldots$$

As a check we can integrate over a region of radius r

$$\int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin\theta\, dr\, d\theta\, d\varphi = \frac{4}{3}\pi R^3$$

III. Flat 3D space (called Euclidean space)

↳ "flat" means "no curvature". We want to see how to use arbitrary coordinates... All coordinate systems in 3D specify points as intersection of 3 surfaces...

Cartesian $\{x = \text{const},\ y = \text{const},\ z = \text{const}\}$ 3 planes!

Spherical $\{\underset{\text{sphere}}{r = \text{const}},\ \underset{\text{cone}}{\theta = \text{const}},\ \underset{\text{plane}}{\varphi = \text{const}}\}$ 3 surfaces

Cylindrical $\{\underset{\text{cylinder}}{\rho = \text{const}},\ \underset{\text{ver. plane}}{\varphi = \text{const}},\ \underset{\text{hor. plane}}{z = \text{const}}\}$

Curvilinear Coordinates (arbitrary coordinates in 3D)

↳ Call (u, v, w) = arbitrary coordinates
Specify a point by $u = \text{const}$, $v = \text{const}$, $w = \text{const}$

Note Coordinates are curvy, but the spaces are still flat...

→ Can find relations with (x, y, z)

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases} \text{ or } \begin{cases} u = u(x, y, z) \\ v = v(x, y, z) \\ w = w(x, y, z) \end{cases}$$

Basis Vectors Want to be able to describe vectors using curvilinear coordinates

$\Rightarrow$ need a basis set that span the space...

In Cartesian ... $\{\hat{i}, \hat{j}, \hat{k}\}$ span 3D space (Euclidean)

What set $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ would give a basis in curvilinear coordinates

Well, how do we get $\{\hat{i}, \hat{j}, \hat{k}\}$ in Cartesian coordinates?

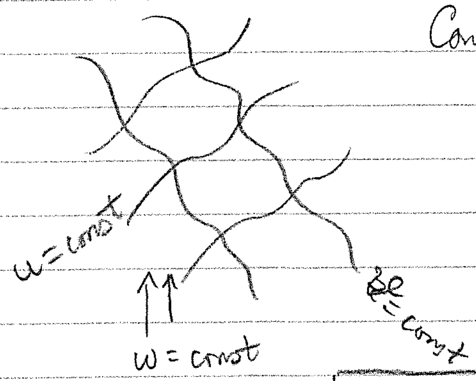
$\hat{i}$: vector that follows change in $\vec{x}$ with, y, z fixed ...
↳ a tangent vector along change in $\vec{x}$.

$$\hat{i} = \frac{\partial \vec{r}}{\partial x} \rightarrow \text{gives a tangent vector along } x$$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \rightarrow \frac{\partial \vec{r}}{\partial x} = \hat{i}$

Likewise $\hat{j} = \frac{\partial \vec{r}}{\partial y}$, $\hat{k} = \frac{\partial \vec{r}}{\partial z}$

Now, Consider (u, v, w)



Consider $\frac{\partial \vec{r}}{\partial u}$ (how u changing with v, w const)
↳ tangent vector along the changing u direction

Let $\boxed{\vec{e}_u = \frac{\partial \vec{r}}{\partial u}}$

Likewise, call $\boxed{\vec{e}_v = \frac{\partial \vec{r}}{\partial v}}$, $\boxed{\vec{e}_w = \frac{\partial \vec{r}}{\partial w}}$ } form a natural basis set ...

The set $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ can then be used as a basis for any vector in the space

Sep 12, 2018

Recall Curvilinear Coordinates $\rightarrow (u, v, w)$
Natural basis $\rightarrow \{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$

where $\vec{e}_u = \dfrac{\partial \vec{r}}{\partial u}$, $\vec{e}_v = \dfrac{\partial \vec{r}}{\partial v}$, $\vec{e}_w = \dfrac{\partial \vec{r}}{\partial w}$ } tangent vectors.

To calculate these in terms of $\{\hat{i}, \hat{j}, \hat{k}\}$ use
$$\vec{r} = x(u,v,w)\hat{i} + y(u,v,w)\hat{j} + z(u,v,w)\hat{k}$$

Notes $\rightarrow$ directions of these basis vectors can change as you move around (unlike $\{\hat{i}, \hat{j}, \hat{k}\}$)

$\rightarrow$ the set $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ need not be orthogonal. They only need to be linearly independent (to span the space).
They also don't need to be unit vectors...

Can make unit vectors: $\hat{e}_u = \dfrac{\vec{e}_u}{\|\vec{e}_u\|}$ (but NOT as useful...)

What, then, is "natural" about this set? $\Rightarrow$ They will lead us to the METRIC TENSOR ...

Last note $\rightarrow$ will often use $\{\hat{i}, \hat{j}, \hat{k}\}$ as a reference basis.
$\rightarrow$ Can express $\vec{e}_u, \vec{e}_v, \vec{e}_w$ in terms of these

e.g. $\vec{e}_u = (e_u)_x \hat{i} + (e_u)_y \hat{j} + (e_u)_z \hat{k}$

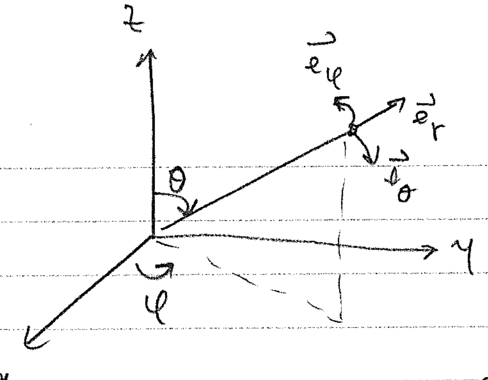
$\boxed{\text{Example}}$ find $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ for spherical coordinates.

$(u, v, w) \rightarrow (r, \theta, \varphi) \rightarrow$ So... $\vec{r} = (x, y, z) = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}$

So • $\vec{e}_r = \dfrac{\partial \vec{r}}{\partial r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

• $\vec{e}_\theta = \dfrac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{pmatrix}$

• $\vec{e}_\varphi = \dfrac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -r\sin\theta\sin\varphi \\ +r\sin\theta\cos\varphi \\ 0 \end{pmatrix}$



orientation depends on where you are...

Note this set is orthogonal, but not normaliz

Now

$$\vec e_r \cdot \vec e_r = \sin^2\theta\left(\cos^2\varphi + \sin^2\varphi\right) + \cos^2\theta = 1$$

$$\vec e_r \cdot \vec e_\theta = r\cos^2\varphi \sin\theta\cos\theta + r\sin^2\varphi\sin\theta\cos\theta - r\sin\theta\cos\theta = 0$$

$$\vec e_r \cdot \vec e_\varphi = -r\sin^2\theta\sin\varphi\cos\varphi + r\sin^2\theta\sin\varphi\cos\varphi = 0$$

$$\vec e_\varphi \cdot \vec e_\theta = r^2\cos^2\varphi\cos^2\theta + r^2\cos^2\varphi\sin^2\theta + r^2\sin^2\theta = r^2$$

$$\vec e_\theta \cdot \vec e_\varphi = 0$$

$$\vec e_\varphi \cdot \vec e_\varphi = r^2\sin^2\varphi\sin^2\theta + r^2\sin^2\theta\cos^2\varphi = r^2\sin^2\theta$$

- **See that** $\{\vec e_u, \vec e_v, \vec e_w\}$ orthogonal, but not unit vectors.

$\{\,|\vec e_r| = 1,\ |\vec e_\theta| = r,\ |\vec e_\varphi| = r\sin\theta\,\}$

$\boxed{\text{Dual basis}}$ → there's an alternative basis $\{\vec e^{\,u}, \vec e^{\,v}, \vec e^{\,w}\}$
Instead of using tangent vectors, we could use perpendicular of surfaces of constant, (u, v, w)

Recall that $\vec\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ gives $\vec\nabla f \perp$ surfaces of $f =$ const

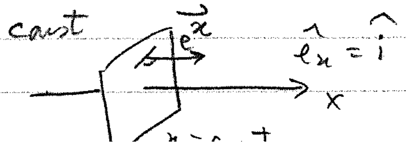
Since curvilinear coord are given by $u =$ const, $v =$ const, $w =$ const
this suggests using $\perp$'s to these...

Ex $\left\{\begin{array}{l} \vec e^{\,u} = \vec\nabla u \\ \vec e^{\,v} = \vec\nabla v \\ \vec e^{\,w} = \vec\nabla w \end{array}\right\}$ ($\perp$ to surface $u =$ const)

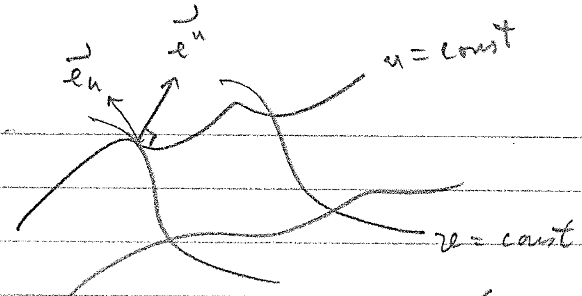
- What's the dual basis in Cartesian coord?

$$\left.\begin{array}{l} \vec e^{\,x} = \vec\nabla x = (1,0,0) = \hat i = \hat e_x \\ \vec e^{\,y} = \vec\nabla y = (0,1,0) = \hat j = \hat e_y \\ \vec e^{\,z} = \vec\nabla z = (0,0,1) = \hat k = \hat e_z \end{array}\right\}$$

Why? Because direction along x is the same as the direction $\perp$ $x =$ const



<u>But in curvilinear...</u>



<u>To compute</u> $\{\vec{e}^u, \vec{e}^v, \vec{e}^w\}$ $\rightarrow$ use $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and the inverted relations

$$u(x,y,z),\ \varphi(x,y,z),\ \theta(x,y,z)$$

Find $\vec{e}^u = \vec{\nabla} u$ in Cartesian in $\vec{i}, \vec{j}, \vec{k}$, then replace (x,y,z) with (u,v,w)

<u>Ex</u> find local basis set for spherical... $(u,v,w) \rightarrow (r,\theta,\varphi)$

$\longrightarrow$ use inverted expression.

$r = (x^2+y^2+z^2)^{1/2}$

$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

$$\vec{e}^r = \vec{\nabla} r = \vec{\nabla}(x^2+y^2+z^2)^{1/2} = \begin{pmatrix} x(x^2+y^2+z^2)^{-1/2} \\ y(x^2+y^2+z^2)^{-1/2} \\ z(x^2+y^2+z^2)^{-1/2} \end{pmatrix}$$

So $\boxed{\vec{e}^r = \vec{\nabla} r = (\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta)}$ $(= \vec{e}_r)$

$$\vec{e}^\theta = \vec{\nabla}\theta = \vec{\nabla}\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \left(\frac{-1}{\sqrt{1-\frac{z^2}{(\ )}}}\right)\left[\frac{-zx}{(\ )^{3/2}}, \frac{-zy}{(\ )^{3/2}}, \frac{1}{(\ )^{1/2}} + \frac{-z^2}{(\ )^{3/2}}\right]$$

$$= \frac{-1}{r\sin\theta}\left(\frac{-r^2\cos\theta\sin\theta\cos\varphi}{r^3}, \frac{-r^2\cos\theta\sin\theta\sin\varphi}{r^3}, \left(\frac{r^2}{r^3} - \frac{r^2\cos^2\theta}{r^3}\right)\right)$$

So $\boxed{\vec{e}^\theta = \left(\frac{1}{r}\cos\theta\cos\varphi,\ \frac{1}{r}\cos\theta\sin\varphi,\ -\frac{\sin\theta}{r}\right)}$

<u>Next</u>, $\vec{e}^\varphi = \vec{\nabla}\varphi = \vec{\nabla}\tan^{-1}\left(\frac{y}{x}\right) = (\ \cdots\ )$

get $\boxed{\vec{e}^\varphi = \left(-\frac{\sin\varphi}{r\sin\theta},\ \frac{\cos\varphi}{r\sin\theta},\ 0\right)}$

Compare $\{\vec{e}^r, \vec{e}^\theta, \vec{e}^\varphi\}$ to $\{\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi\}$

$\vec{e}^r = \vec{e}_r$, but $\vec{e}_\theta \neq \vec{e}^\theta$, and $\vec{e}^\varphi \neq \vec{e}_\varphi$

Sep 14, 2018 Recall Natural basis $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ $\rightarrow$ tangent vectors $\left(\frac{\partial \vec{r}}{\partial u}\right)$

Dual basis $\{\vec{e}^u, \vec{e}^w, \vec{e}^v\}$ $\rightarrow$ $\perp$ to surface of const u, v (∇)

$\boxed{\text{Ex}}$ Paraboloidal Surfaces (u, v, w) (non-orthogonal set)

$$\left.\begin{aligned} x &= u+v \\ y &= u-v \\ z &= 2uv+w \end{aligned}\right\} \rightarrow \left\{\begin{aligned} u &= \tfrac{1}{2}(x+y) \\ v &= \tfrac{1}{2}(x-y) \\ w &= z - \tfrac{1}{2}(x^2-y^2) \end{aligned}\right.$$

Surfaces: u = const $\rightarrow$ plane
v = const $\rightarrow$ plane
w = const $\rightarrow$ hyperbolic paraboloid

Now $\vec{r} = (x, y, z) = (u+v,\ u-v,\ 2uv+w)$ (in $\hat{i}, \hat{j}, \hat{k}$)

Natural basis:

$$\left.\begin{aligned} \vec{e}_u &= \frac{\partial \vec{r}}{\partial u} = (1, 1, 2v) \\ \vec{e}_v &= \frac{\partial \vec{r}}{\partial v} = (1, -1, 2u) \\ \vec{e}_w &= \frac{\partial \vec{r}}{\partial w} = (0, 0, 1) \end{aligned}\right\}$$

Non orthogonal!

$\vec{e}_u \cdot \vec{e}_v = 4uv \neq 0$
$\vec{e}_u \cdot \vec{e}_w = 2v \neq 0$
$\vec{e}_v \cdot \vec{e}_w = 2u \neq 0$

$$\vec{e}^u = \vec{\nabla} u = \vec{\nabla}\left(\tfrac{1}{2}(x+y)\right) = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

$$\vec{e}^v = \vec{\nabla} v = \vec{\nabla}\left(\tfrac{1}{2}(x-y)\right) = \left(\tfrac{1}{2}, -\tfrac{1}{2}, 0\right)$$

$$\vec{e}^w = \vec{\nabla} w = \vec{\nabla}\left(z - \tfrac{1}{2}(x^2-y^2)\right) = (-x, +y, 1) = (-u-v, +u-v, 1)$$

Note $\vec{e}^u \cdot \vec{e}^w = -v$, $\vec{e}^u \cdot \vec{e}^v = 0$, $\vec{e}^v \cdot \vec{e}^w = -u$

$\boxed{\text{Suffix notation}}$ → convenient to change notation

upper indices

For the coordinates, we use $(u, v, w) \longmapsto (u^1, u^2, u^3) = \{u^i\}$

$(i = 1, 2, 3)$

Similar things for basis vectors

$\{\vec{e}_u, \vec{e}_v, \vec{e}_w\} \longrightarrow \{\vec{e}_i\} \quad i = 1, 2, 3$ (natural)

$\{\vec{e}^u, \vec{e}^v, \vec{e}^w\} \longrightarrow \{\vec{e}^i\} \quad i = 1, 2, 3$ (dual)

Since both span a space, any vector $\vec{\lambda}$ can be written in terms of either

$$\vec{\lambda} = \lambda^1 \vec{e}_1 + \lambda^2 \vec{e}_2 + \lambda^3 \vec{e}_3$$

(upper index for coords)
(for natural basis)

$$\boxed{\vec{\lambda} = \sum_{i=1}^{3} \lambda^i \vec{e}_i}$$

Coordinates = components of natural basis

But also

$$\vec{\lambda} = \lambda_1 \vec{e}^1 + \lambda_2 \vec{e}^2 + \lambda_3 \vec{e}^3$$

$$\boxed{\vec{\lambda} = \sum_{i=1}^{3} \lambda_i \vec{e}^i}$$

(lower ~~index for coords~~ for ~~dual basis~~)

$\boxed{\text{Einstein summation convention}}$

→ any index that appears once (up) & once (down) is automatically summed

So $\boxed{\vec{\lambda} = \lambda^i \vec{e}_i}$ (instead of $\sum_{i=1}^{3} \lambda^i \vec{e}_i$)

Since i is dummy index, it can be any letter

So... $a^i b_i = a^k b_k = a^j b_j = \sum_{n=1}^{3} a^n b_n$ ...

But $a_i b_i$ makes no sense } → not defined ...
→ need to put in $\sum_i a_i b_i$

Otherwise $a_i b^i c^i$ → doesn't make sense either...

↳ only "1 up, 1 down" allowed

Note $\boxed{\text{Certain letters are reserved for special cases}}$

$i, j, k, l, \ldots$	$= 1, 2, 3$	3D space
$\mu, \nu, \alpha, \beta, \gamma, \rho \ldots$	$= 0, 1, 2, 3$	4D spacetime
$A, B, C, \ldots$	$= 1, 2, \ldots$	2D spaces
$a, b, c, \ldots$	$= 1, 2, \ldots N$	N-D manifold

$\boxed{\text{Now, any vector is then}}$ $\quad \vec{\lambda} = \lambda^i \vec{e}_i = \lambda_i \vec{e}^{\,i}$

call λ^i a "contravariant component"

and

λ_i = "covariant component" $\qquad$ "co" is low.

Note λ_i, λ^i → are components

But $\vec{e}^{\,i}, \vec{e}_i$ → are vectors... (have 3 components themselves with respect to some other basis)

$\boxed{\text{So... what does this get us?}}$

$\boxed{\text{Dot products...}}$

not summed $(i \neq j)$. This is 9 diff. objects... $i = 1, 2, 3$, $j = 1, 2, 3 \ldots$

Consider $\vec{e}^{\,i} \cdot \vec{e}_j$

Use def. $\quad \vec{e}^{\,i} = \vec{\nabla} u^i = \dfrac{\partial u^i}{\partial x}\hat{i} + \dfrac{\partial u^i}{\partial y}\hat{j} + \dfrac{\partial u^i}{\partial z}\hat{k}$

$$\vec{e}_j = \frac{\partial \vec{r}}{\partial u^j} = \frac{\partial x}{\partial u^j}\hat{i} + \frac{\partial y}{\partial u^j}\hat{j} + \frac{\partial z}{\partial u^j}\hat{k}$$

So $\vec{e}^{\,i} \cdot \vec{e}_j = \dfrac{\partial u^i}{\partial x}\dfrac{\partial x}{\partial u^j} + \dfrac{\partial u^i}{\partial y}\dfrac{\partial y}{\partial u^j} + \dfrac{\partial u^i}{\partial z}\dfrac{\partial z}{\partial u^j}$ looks like a chain rule...

Suppose $u^i = u^i(x,y,z)$

where $x = x(u^j)$

$y = y(u^j)$

$z = z(u^j)$

$$\Rightarrow \boxed{\frac{\partial u^i}{\partial u^j} = \frac{\partial u^i}{\partial x}\frac{\partial x}{\partial u^j} + \frac{\partial u^i}{\partial y}\frac{\partial y}{\partial u^j} + \frac{\partial u^i}{\partial z}\frac{\partial z}{\partial u^j} = \vec{e}^i \cdot \vec{e}_j}$$

But $\{u^i\} = \{u^1, u^2, u^3\}$ independent variables

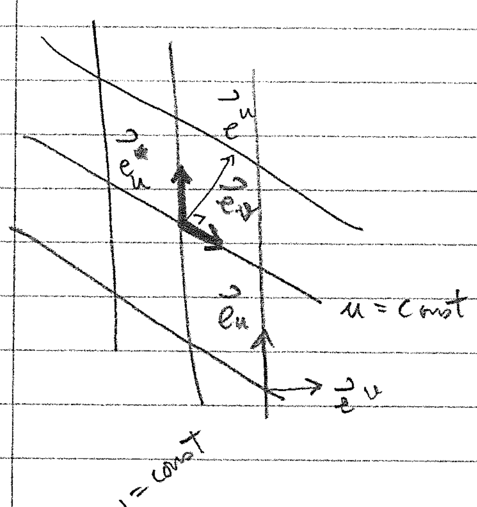
$$\frac{\partial u^1}{\partial u^1} = 1, \quad \frac{\partial u^1}{\partial u^2} = 0, \quad \frac{\partial u^1}{\partial u^3} = 0$$

So

Introduce
$$\boxed{\delta^i_j = \begin{cases} 1 & i \neq j \\ 0 & i \neq j \end{cases} \quad \text{kronecker delta}}$$

So $\boxed{\vec{e}^i \cdot \vec{e}_j = \delta^i_j}$ ⤳ 9 eqns (6 answers = 0, 3 = 0)

Notice $\boxed{\vec{e}^u \perp \vec{e}_v^*}$ $(u \neq v)$ why? (by definition)



what about inner products $\{\vec{e}_i\}$ with themselves, likewise $\{\vec{e}^i\}$

Define
$$\boxed{\left\{ \begin{aligned} g_{ij} &= \vec{e}_i \cdot \vec{e}_j \\ g^{ij} &= \vec{e}^i \cdot \vec{e}^j \end{aligned} \right\}}$$

Since $\vec{e}_i \cdot \vec{e}_j = \vec{e}_j \cdot \vec{e}_i$ (commute), $\boxed{g_{ij} = g_{ji}}$

So $\boxed{\begin{aligned} g_{ij} &= g_{ji} \\ g^{ji} &= g^{ij} \end{aligned}}$ (Symmetric) in matrix → symmetric

Ex Cartesian g_{ij} = unit matrix

$\boxed{g_{ij} \rightarrow \text{called the metric tensor}}$

a quantity that tells ~~use~~ how to find length, distance in arbitrary coords

Consider $\vec{\lambda}, \vec{\mu}$

Then $\vec{\lambda} = \lambda^i \vec{e}_i = \lambda_i \vec{e}^{\,i}$

Likewise $\vec{\mu} = \mu^i \vec{e}_i = \mu_i \vec{e}^{\,i}$

There are 4 ways to get $\vec{\lambda} \cdot \vec{\mu}$, and they all give the same ans

Now $\vec{\lambda} \cdot \vec{\mu} = \cancel{\lambda^i \vec{e}_i \cdot \lambda_i \vec{e}^{\,i}}$ → different index

Rather (correctly) $\boxed{\vec{\lambda} \cdot \vec{\mu} = \lambda^i \vec{e}_i \cdot \mu^j \vec{e}_j}$

So $\boxed{\vec{\lambda} \cdot \vec{\mu} = \lambda^i \vec{e}_i \cdot \mu^j \vec{e}_j = g_{ij} \lambda^i \mu^j}$

Sept 17, 2018

Showed $\boxed{\begin{aligned} &\vec{e}^{\,i} \cdot \vec{e}_j = \delta^i_j \\ &\vec{e}_i \cdot \vec{e}_j = g_{ij}, \quad \vec{e}^{\,i} \cdot \vec{e}^{\,j} = g^{ij} \end{aligned}}$

Consider $\vec{\lambda} = \lambda^i \vec{e}_i = \lambda_i \vec{e}^{\,i}$

$\vec{\mu} = \mu^i \vec{e}_i = \mu_i \vec{e}^{\,i}$

dot the two → get 4 equivalent expressions for $\vec{\lambda} \cdot \vec{\mu}$

$$\boxed{\begin{aligned}\vec{\lambda}\cdot\vec{\mu} &= \lambda^i\vec{e}_i \cdot \mu^j\vec{e}_j = g_{ij}\lambda^i\mu^j \\ &= \lambda_i\vec{e}^{\,i}\cdot\mu_j\vec{e}^{\,j} = g^{ij}\lambda_i\mu_j \\ &= \lambda_i\vec{e}^{\,i}\cdot\mu^j\vec{e}_j = \lambda_i\mu^j\delta_j^i = \lambda_i\mu^i \\ &= \lambda^i\vec{e}_i\cdot\mu_j\vec{e}^{\,j} = \lambda^i\mu_j\delta_i^j = \lambda^i\mu_i\end{aligned}}$$ (✱)

<u>Note</u> $\mu^j\delta_j^i$ $\quad \delta = 0$ if $j \neq i$
$\quad \delta = 1$ if $j = i$

<u>So</u> $\mu^j\delta_j^i = \mu^i$

<u>Note</u> $\boxed{\sum_{i=1}^{3}\lambda^i\vec{e}_i\cdot\mu^i\vec{e}_i \neq \sum_{i=1}^{3}\lambda^i\vec{e}_i\cdot\sum_{j=1}^{3}\mu^j\vec{e}_j}$

3 terms — 9 terms (correct)

we have 4 equivalent expressions ...

$$\boxed{\vec{\lambda}\cdot\vec{\mu} = g_{ij}\lambda^i\mu^j = g^{ij}\lambda_i\mu_j = \lambda^i\mu_i = \lambda_i\mu^i}$$

double sum — single sum

<u>These imply</u>

$$\boxed{g_{ij}\cdot\mu^j = \mu_i} \quad \text{and} \quad \boxed{g^{ij}\cdot\mu_j = \mu^i}$$

→ Can use metric tensor to go back n forth between contravariants - covariants

$$\boxed{\begin{cases} g^{ij} \rightarrow \text{raises an index} \\ g_{ij} \rightarrow \text{lower an index}\end{cases}}$$

Can also write $\quad \mu^i = g^{ij}\mu_j = g^{ij}(g_{jk}\mu^k)$

It's also true that $\quad \mu^i = \delta^i_k \mu^k$

So $\quad \boxed{g^{ij} g_{jk} = \delta^i_k}$

We can also do: $\quad \mu_i = g_{ij}\mu^j = g_{ij}(g^{jk}\mu_k) = \delta_i^k \mu_k$

So $\quad \boxed{g_{ij} g^{jk} = \delta_i^k} \longrightarrow$ identity matrix

These show that g_{ij} is the inverse of g^{ij} $\qquad$ Note g = matrix

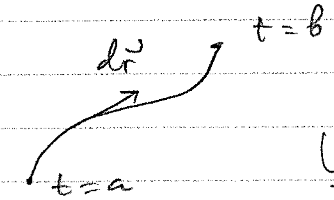
Call

$$\begin{cases} g_{ij} \longrightarrow \text{metric tensor} \\ g^{ij} \longrightarrow \text{inverse metric tensor} \end{cases}$$

The METRIC TENSOR

g^{ij} = metric tensor in 3D space. $\Rightarrow$ contains info about physical lengths, geometry of the space

Consider a curve in 3D flat space with param t.



$\vec{r} = \vec{r}(t)$

Length $= \int_a^b \|\dot{\vec{r}}\| \, dt$

Originally, $\vec{r} = \vec{r}(x, y, z)$

But, we can change to curvilinear coordinates $\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$

Then, for curve $\begin{cases} u = u(t) \\ v = v(t) \\ w = w(t) \end{cases} \to \vec{r} = \vec{r}(u(t), v(t), w(t))$

So $$\frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial \vec{r}}{\partial w}\frac{\partial w}{\partial t}$$
$$= \vec{e}_u \frac{\partial u}{\partial t} + \vec{e}_v \frac{\partial v}{\partial t} + \vec{e}_w \frac{\partial w}{\partial t}$$

$$\boxed{\frac{d\vec{r}}{dt} = \vec{e}_i \frac{du^i}{dt}}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{\vec{e}_i \frac{du^i}{dt}\cdot \vec{e}_j \frac{du^j}{dt}} = \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}$$

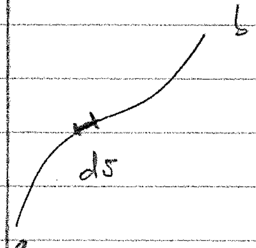
So $$\boxed{L = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{g_{ij}\,\dot{u}^i \dot{u}^j}\, dt = \int_a^b \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\, dt}$$

Sep 18, 2018 $\boxed{\text{Length of a curve}}$ in curvilinear coordinates

<u>Note</u> parametrization can be used e.g. σ = param

$$L = \int_a^b \sqrt{g_{ij}\frac{du^i}{d\sigma}\frac{du^j}{d\sigma}}\, d\sigma$$

We can introduce an <u>infinitesimal line element</u>



ds – In 3D space $ds = |d\vec{r}|$

So $L = \int_a^b |d\vec{r}| = \int_a^b ds \rightsquigarrow$ but this is still parameterized in t (NOT $b-a$)

However, we can compare this with

$$L = \int_a^b \sqrt{g_{ij}\,\dot{u}^i \dot{u}^j}\, dt = \int_a^b ds$$

so $ds = \sqrt{g_{ij}\,\dot{u}^i \dot{u}^j}\; dt = \sqrt{g_{ij} \frac{du^i}{dt} \frac{du^j}{dt}}\; dt$

Square this $ds^2 = g_{ij} \frac{du^i}{dt} \frac{du^j}{dt} dt^2$

So $\boxed{ds^2 = g_{ij}\, du^i du^j}$ → line element

metric gives length changes in terms of coordinate changes...

Example 1

$\boxed{\text{Cartesian coordinates}}$ $\{\vec{e}_i\} = \{\hat{i}, \hat{j}, \hat{k}\}$

So $g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

As a matrix $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

row column

So the line element $ds^2 = g_{ij}\, du^i du^j =$ 9 terms

$= 1\, du^1 du^1 + 0\, du^1 du^2 + \dots$

$\Rightarrow ds^2 = du^{1\,2} + du^{2\,2} + du^{3\,2}$
And $u^1 = x,\ u^2 = y,\ u^3 = z$

So $\boxed{ds^2 = dx^2 + dy^2 + dz^2}$ (Cartesian, flat 3D space)

looks Pythagorean

comes from the form of the metric

Example 2

$\boxed{\text{Spherical Coordinates}}$ (r, θ, φ)

$\vec{e}_r \cdot \vec{e}_r = 1,\ \vec{e}_\theta \cdot \vec{e}_\theta = r^2,\ \vec{e}_\varphi \cdot \vec{e}_\varphi = r^2 \sin^2\theta$ (others are zero)
$\vec{e}_r \cdot \vec{e}_\theta = \vec{e}_\theta \cdot \vec{e}_\varphi = \vec{e}_\varphi \cdot \vec{e}_r = 0$

These give $[g_{ij}] = \vec{e}_i \cdot \vec{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$ flat space metric in spherical coords...

So the line element $(u^1, u^2, u^3) = (r, \theta, \varphi)$

$$\boxed{ds^2 = g_{ij}\,du^i du^j = (1)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$

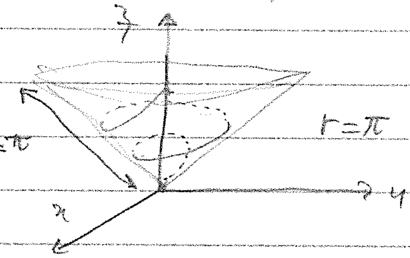
line element in flat 3D space in Spherical coords.

Example 3

Find the length of a curve in spherical coordinates by the param

$$\vec{r}(t) = (r(t), \theta(t), \varphi(t)) = \left(t, \frac{\pi}{4}, 4t\right) \quad 0 \le t \le \pi$$

What does this look like? (wraps around twice)



Use that

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \quad \text{with param}$$

$$dr = dt, \quad d\theta = 0, \quad d\varphi = 4\,dt$$

So $ds^2 = \left[1 + 0 + 4^2 t^2 \sin^2\left(\frac{\pi}{4}\right)\right] dt^2 = \left(1 + 8t^2\right) dt^2$

So

$$\boxed{L = \int_0^{\pi} \sqrt{1 + 8t^2}\, dt \simeq 14.55}$$

Note we've only seen diagonal metric.

↳ BUT not all metrics are diagonal

Ex Paraboloidal coordinates have non-diagonal $[g_{ij}]$

We found

$$\vec{e}_u = (1, 1, 2v)$$
$$\vec{e}_v = (1, -1, 2u)$$
$$\vec{e}_w = (0, 0, 1)$$

So

$$\boxed{[g_{ij}] = \begin{pmatrix} 2 + 4v^2 & 4uv & 2v \\ 4uv & 2 + 4u^2 & 2u \\ 2v & 2u & 1 \end{pmatrix}}$$

Then, $ds^2 = g_{ij}\,du^i du^j$ → get all 9 terms, which then reduce to 6, since $du\,dv = dv\,du$

$= g_{11}\,du^1 du^1 + g_{12}\,du^1 du^2 + \dots$

The metric also gives norms of vectors + inner products of vectors

(norm) $|\vec{\lambda}|^2 = \vec{\lambda}\cdot\vec{\lambda} = g_{ij}\,\lambda^i\lambda^j$ → 9 terms

(inner prod) $\vec{\lambda}\cdot\vec{\mu} = g_{ij}\,\lambda^i\mu^j = g_{11}\lambda^1\mu^1 + g_{12}\lambda^1\mu^2 + \dots\ g_{33}\lambda^3\mu^3$

In Cartesian ⇒ $g_{ij} = \delta_{ij}$ $\qquad [g_{ij}] = I$

↳ $\vec{\lambda}\cdot\vec{\mu} = \lambda^1\mu^1 + \lambda^2\mu^2 + \lambda^3\mu^3$

Now, can we turn these summations into Matrix Products?

↳ Convenient to write vectors and 2-component tensors using matrices

Note ⇒ more general tensors can't be written using matrices TÜ

First, remember how to multiply matrices...

i	j
row	column

Suppose $\underset{\sim}{A} = [a_{ij}]$ and $\underset{\sim}{B} = [b_{ij}]$

and $\underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{B} = [c_{ij}]$

$$C = \begin{pmatrix} & \vdots & \\ \dots & c_{ij} & \dots \\ & \vdots & \end{pmatrix} = \begin{pmatrix} \\ a_{i1} & a_{i2} & \dots \\ \\ \end{pmatrix}\begin{pmatrix} & b_{1j} & \\ & b_{2j} & \\ & \vdots & \end{pmatrix}$$

So $\boxed{c_{ij} = \sum_k a_{ik}\, b_{kj}}$ (summed index is in the middle - goes column-row)

column row

Can also multiply vectors

e.g. $\underset{\sim}{F} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} \qquad \underset{\sim}{G} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \end{pmatrix}$

So $\boxed{\underset{\sim}{F} \cdot \underset{\sim}{G} = \underset{\sim}{F}^T \underset{\sim}{G} = \sum_k f_k g_k}$

Sept 18, 2018

Metric → line element $ds = g_{ij}\, du^i du^j$

→ inner products. $\vec{\lambda}\cdot\vec{\mu} = g_{ij}\lambda^i\mu^j = g^{ij}\lambda_i\mu_j = \lambda_i\mu^i = \lambda^i\mu_i$

→ raising / lowering indices

Flat spacetime Cartesian $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow a_i = g_{ij} a^j \Rightarrow a_1 = a^1,\ a_2 = a^2,\ a_3 = a^3$

But in spherical coords

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$\Rightarrow$ if $\vec{a} = (1)\vec{e}_\theta$ → $\boxed{\begin{matrix} a^1 = 0 \\ a^2 = 1 \\ a^3 = 0 \end{matrix}}$ $\qquad (a^1, a^2, a^3) = (0, 1, 0)$

(contravariant)

So what are $a_1 = g_{1j} a^j = 0$

$a_2 = g_{2j} a^j = g_{22} a^2 = r^2$ → (covariant)

$a_3 = g_{3j} a^j = 0$

Norm? $|\vec{a}|^2 = a^i a_i = a^2 a_2 = r^2$ (makes sense)

or $|\vec{a}|^2 = g_{ij} a^i a^j = g_{22} a^2 a^2 = r^2 \cdot 1 \cdot 1 = r^2$

How do we write these things using matrices?

Can represent contravariant vectors as columns

$$\boxed{\underset{\sim}{L} = [\lambda^i] = \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix}}$$

Similarly,

$$\boxed{\underset{\sim}{M} = [\mu^i] = \begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix}}$$

contravariant

How can we write

$$\tilde{\lambda}\cdot\tilde{\mu} = g_{ij}\lambda^i\mu^j \quad \text{using matrices?}$$

$$\boxed{\underset{\sim}{G} = [g_{ij}]}$$

Now, must be careful with ordering + need transposes...

$$\boxed{\tilde{\lambda}\cdot\tilde{\mu} = \lambda^i g_{ij}\mu^j} \Rightarrow \boxed{\tilde{\lambda}\cdot\tilde{\mu} = \underset{\sim}{L}^T \underset{\sim}{G} \underset{\sim}{M}} \quad (1\times3\times3\times1)$$

need transpose

So $$\boxed{\tilde{\lambda}\cdot\tilde{\mu} = (\lambda^1\ \lambda^2\ \lambda^3)\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}\begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix}}$$

For COVARIANT (acc. to books)

* : covariant
^ : inverse

$$\underset{\sim}{L}^* = [\lambda_i] = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \qquad \hat{\underset{\sim}{G}} = [g^{ij}]$$

$$\underset{\sim}{M}^* = [\mu_i] = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

$\boxed{\text{Could write}}$

$$\boxed{\underset{\sim}{L}^* = \underset{\sim}{G} \cdot \underset{\sim}{L}}$$ (covariant indices)

since
$$\boxed{\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}\begin{pmatrix}\lambda^1\\ \lambda^2\\ \lambda^3\end{pmatrix} = \lambda_i = g_{ij}\lambda^j}$$

with
$$\boxed{\underset{\sim}{I} = \underset{\sim}{\hat{G}}^* \underset{\sim}{G} = [\delta^i_j] = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}}$$

then $\boxed{g^{ij}g_{jk} = \delta^i_k \; , \; g_{ik}g^{kj} = \delta_i^j \rightarrow \underset{\sim}{\hat{G}} \cdot \underset{\sim}{G} = [\delta^i_j]}$

Now, want to find $[g^{ij}]$ in spherical coords...

Could use def. $\boxed{g^{ij} = \vec{e}^i \cdot \vec{e}^j}$ with $\begin{cases}\vec{e}^r = \nabla r\\ \vec{e}^\theta = \nabla\theta\\ \vec{e}^\varphi = \nabla\varphi\end{cases}$

We found those... BUT there's another way

$$\boxed{[g^{ij}] = [g_{ij}]^{-1}}$$ so $$\boxed{[g^{ij}] = \begin{pmatrix}1&0&0\\0&r^2&0\\0&0&r^2\sin^2\theta\end{pmatrix}^{-1}}$$

Easy for diagonal matrix

$$\boxed{[g^{ij}] = \begin{pmatrix}1&0&0\\0&1/r^2&0\\0&0&1/r^2\sin^2\theta\end{pmatrix}}$$ (easy for diagonal matrix)

COORDINATE TRANSFORMATION in EUCLIDEAN SPACE

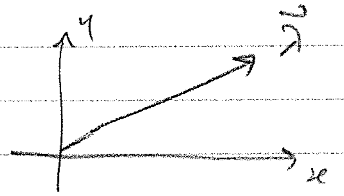
want to learn how to transform between arbitrary coords

$$\boxed{(u, v, w) \longleftrightarrow (u', v', w')}$$ → important in relativity

Note no moving frames here. We also want to learn how vectors and tensors transform, as well as what these are...

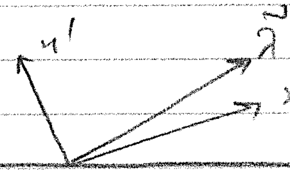
What is a vector? → has magnitude & direction... $\vec{\lambda}$ = vector

Suppose



Same $\vec{\lambda}$ → not changed
But can now give it components w.r.t. to a basis set ... (λ^i)

Suppose



The same $\vec{\lambda}$, but different comp. coordinates (since different basis set)

Under coordinate transforms, vectors don't change, but their components change, since their basis set changes

Using Hoh's notation,

we'll use this... ⇒

$$\left\{ \begin{array}{l} \lambda^{i'} = \text{component of } \vec{\lambda} \text{ in } (x', y') \text{ frame} \\ \lambda'^{i} = \text{same thing} \end{array} \right\}$$

$\lambda^{i'}$ is weird, because i' no longer a dummy. We can't change it to $l, u, m, \ldots$

But, we can change i' to l' or k', ...

<u>Suppose</u> $\vec{\lambda}$ = vector <u>and</u> have 2 coords system

$\{u^i\}$ and $\{u^{i'}\}$

<u>e.g</u> $u^i = \{r, \theta, \varphi\}$, and $u^{i'} = \{\rho, \varphi, z\}$

These are related, $\boxed{u^{i'} = u^{i'}(u^j)}$

We also have <u>basis sets</u> with respect to each coord. system

<u>Unprimed</u> : $\vec{e}_i = \dfrac{\partial \vec{r}}{\partial u^i}$, $\vec{e}^{\,i} = \nabla u^i$, $g_{ij} = \vec{e}_i \cdot \vec{e}_j$

<u>Primed</u> : $\vec{e}_{i'} = \dfrac{\partial \vec{r}}{\partial u^{i'}}$, $\vec{e}^{\,i'} = \nabla u^{i'}$, $g_{i'j'} = \vec{e}_{i'} \cdot \vec{e}_{j'}$

A vector $\vec{\lambda}$ can have components in <u>either</u> frame

$$\boxed{\vec{\lambda} = \lambda^i \vec{e}_i \equiv \lambda^{i'} \vec{e}_{i'}}$$

<u>So</u> $\lambda^i \vec{e}_i$, $\lambda^{i'} \vec{e}_{i'}$ must transform in a way that leaves $\vec{\lambda}$ alone.

<u>Use chain rule</u> $\boxed{\vec{r} = \vec{r}(u^{i'}) = \vec{r}(u^{i'}(u^j))}$

$$\boxed{\vec{e}_j = \frac{\partial \vec{r}}{\partial u^j} = \frac{\partial \vec{r}}{\partial u^{i'}} \frac{\partial u^{i'}}{\partial u^j} = \vec{e}_{i'} \frac{\partial u^{i'}}{\partial u^j} = \frac{\partial u^{i'}}{\partial u^j} \vec{e}_{i'}}$$

<u>Call</u> $\boxed{U^{i'}_j = \dfrac{\partial u^{i'}}{\partial u^j}}$ $\longrightarrow$ 9 partial derivatives...

$$\text{Matrix } [U^{i'}_j] = \text{Jacobian} = \begin{pmatrix} \dfrac{\partial u^{1'}}{\partial u^1} & \dfrac{\partial u^{2'}}{\partial u^1} & \dfrac{\partial u^{3'}}{\partial u^1} \\[2ex] \dfrac{\partial u^{1'}}{\partial u^2} & \dfrac{\partial u^{2'}}{\partial u^2} & \dfrac{\partial u^{3'}}{\partial u^2} \\[2ex] \dfrac{\partial u^{1'}}{\partial u^3} & \dfrac{\partial u^{2'}}{\partial u^3} & \dfrac{\partial u^{3'}}{\partial u^3} \end{pmatrix}$$

We have that $\boxed{\vec{e}_j = U^{i'}_j \vec{e}_{i'}}$

Now

$$\vec{\lambda} = \lambda^{j'} \vec{e}_{j'} = \lambda^j \vec{e}_j = \lambda^j U^{i'}_j \vec{e}_{i'}$$

So

$$\boxed{\lambda^{i'} = \lambda^j U^{i'}_j = U^{i'}_j \lambda^j}$$

→ transformation rule for contravariant vector components

Jacobian...

We can also define Jacobian...

$$\boxed{U^j_{i'} = \left[\frac{\partial u^j}{\partial u^{i'}}\right]}$$

$[U^j_{i'}]$ = Jacobian...

Ex 1.4.1 → show that

$$\boxed{\begin{aligned} U^k_{i'} U^{i'}_j &= \delta^k_j \\ U^{k'}_i U^i_{j'} &= \delta^k_j \end{aligned}}$$

Note

$$\delta^{k'}_{j'} = 1 \quad \text{if } k=j \rightarrow \text{it's same as } \delta^k_j$$
$$= 0 \quad \text{if } k \neq j$$

→ Kronecker delta doesn't depend on Basis set / components...

Sept 21, 2018 Under $u^j \rightarrow u^{i'}(u^j)$ We found $\vec{e}_j = U^{i'}_j \vec{e}_{i'}$

where $U^{i'}_j = \dfrac{\partial u^{i'}}{\partial u^j}$ (Jacobian matrix)

also found

$$\lambda^{i'} = U^{i'}_j \lambda^j$$

and

$$U^i_{j'} = \frac{\partial u^i}{\partial u^{j'}}$$

which obey

$$\begin{cases} U^k_{i'} U^{i'}_j = \delta^k_j \\ U^{k'}_j U^j_{i'} = \delta^k_i \end{cases} \quad \Bigg| \quad \delta^{k'}_{j'} = \delta^k_j$$

Next can insert $\lambda^{i'} = U_j^{i'} \lambda^j$

$\Rightarrow$ mult. by $U_{i'}^k$ + sum

$$\boxed{U_{i'}^k \lambda^{i'} = U_j^{i'} U_{i'}^k \lambda^j}$$

So $$\boxed{U_{i'}^k \lambda^{i'} = \delta_j^k \lambda^j = \lambda^k}$$

Can let $k = i$, $i' \to j'$ $\Rightarrow$ $\boxed{\lambda^i = U_{j'}^i \lambda^{j'}}$

So $$\boxed{\lambda^{i'} = U_j^{i'} \lambda^j \quad \text{and} \quad \lambda^i = U_{j'}^i \lambda^{j'}}$$ (swapping primes & unprime)

Can also transform Covariant components

$$\vec{\lambda} = \lambda_{i'} \vec{e}^{\,i'} = \lambda_j \vec{e}^{\,j}$$

where $$\boxed{\vec{e}^{\,j} = \nabla u^j = \frac{\partial u^j}{\partial x}\hat{i} + \frac{\partial u^j}{\partial y}\hat{j} + \frac{\partial u^j}{\partial z}\hat{k}}$$

if $u^j = u^j(u^{i'}(x,y,z))$ $\to$ need chain rule...

$$\to \frac{\partial u^j}{\partial x} = \frac{\partial u^j}{\partial u^{i'}} \frac{\partial u^{i'}}{\partial x}$$

So $$\boxed{\vec{e}^{\,j} = \frac{\partial u^j}{\partial u^{i'}} \frac{\partial u^{i'}}{\partial x}\hat{i} + \frac{\partial u^j}{\partial u^{i'}} \frac{\partial u^{i'}}{\partial y}\hat{j} + \frac{\partial u^j}{\partial u^{i'}} \frac{\partial u^{i'}}{\partial z}\hat{k}} \to 9 \text{ terms}$$

rearrange these 9 terms... How, separate the $1', 2', 3'$ sums...

$$\vec{e}^{\,j} = \left[\frac{\partial u^j}{\partial u^{1'}} \frac{\partial u^{1'}}{\partial x}\hat{i} + \frac{\partial u^j}{\partial u^{1'}} \frac{\partial u^{1'}}{\partial y}\hat{j} + \frac{\partial u^j}{\partial u^{1'}} \frac{\partial u^{1'}}{\partial z}\hat{k}\right] + 2' \text{ terms} + 3' \text{ terms}$$

$$= \frac{\partial u^j}{\partial u^{1'}}\left(\frac{\partial u^{1'}}{\partial x}\hat{i} + \frac{\partial u^{1'}}{\partial y}\hat{j} + \frac{\partial u^{1'}}{\partial z}\hat{k}\right) + \frac{\partial u^j}{\partial u^{2'}}\left(\quad 2' \quad\right) + \frac{\partial u^j}{\partial u^{3'}}\left(3'\right)$$

$$= \frac{\partial u^j}{\partial u^{1'}} \nabla u^{1'} + \frac{\partial u^j}{\partial u^{2'}} \nabla u^{2'} + \frac{\partial u^j}{\partial u^{3'}} \nabla u^{3'}$$

So $$\boxed{\vec{e}^{\,j} = \frac{\partial u^j}{\partial u^{1'}}\vec{e}^{\,1'} + \frac{\partial u^j}{\partial u^{2'}}\vec{e}^{\,2'} + \frac{\partial u^j}{\partial u^{3'}}\vec{e}^{\,3'} = \frac{\partial u^j}{\partial u^{i'}}\vec{e}^{\,i'}}$$

Note $$\frac{\partial u^j}{\partial u^{i'}} = U^j_{i'} \Rightarrow \boxed{\vec{e}^{\,j} = U^j_{i'}\vec{e}^{\,i'}}$$ (analogous form ...)

okay... what about covariant components...?

$$\boxed{\vec{\lambda} = \lambda_{i'}\vec{e}^{\,i'} = \lambda_j\vec{e}^{\,j} = \lambda_j U^j_{i'}\vec{e}^{\,i'}}$$

Therefore $\boxed{\lambda_{i'} = U^j_{i'}\lambda_j}$ Similarly $\boxed{\lambda_j = U^{i'}_j \lambda_{i'}}$

Note, we can introduce matrices

$$\boxed{\underset{\sim}{U} = [U^{i'}_j] = \begin{pmatrix} \frac{\partial u^{1'}}{\partial u^1} & \frac{\partial u^{1'}}{\partial u^2} & \cdots \\ \frac{\partial u^{2'}}{\partial u^1} & \cdots & \end{pmatrix}}$$

(row i', col j)

and the inverse $\boxed{\hat{\underset{\sim}{U}} = [U^j_{i'}]}$

And $\boxed{\underset{\sim}{U}\,\hat{\underset{\sim}{U}} = I}$

Summarize Under a coordinate transform $u^j \to u^{i'}$ or $u^{j'} \to u^i$

$$\boxed{\vec{\lambda} = \lambda^i\vec{e}_i = \lambda^{j'}\vec{e}_{j'} = \lambda_i\vec{e}^{\,i} = \lambda_{j'}\vec{e}^{\,j'}}$$

These are all related by

$$\vec{e}_j = U^{i'}_j\vec{e}_{i'}, \quad \vec{e}^{\,j} = U^j_{i'}\vec{e}^{\,i'}, \quad \vec{e}^{\,i'} = U^{i'}_j\vec{e}^{\,j}$$

$$\vec{e}^{\,j} = U^j_{i'}\vec{e}^{\,i'}$$

Contravariant $\lambda^{i'} = U^{i'}_j\lambda^j, \quad \lambda^j = U^j_{i'}\lambda^{i'}$

Covariant $\lambda_{i'} = U^j_{i'}\lambda_j, \quad \lambda_j = U^{i'}_j\lambda_{i'}$

notice the patterns!

The components of a vector must transform this way under general coordinate transformation.

→ We can turn this around to define a vector...

<u>Def</u>: $\boxed{\begin{array}{l}\text{A vector is a quantity whose components transform as} \\ \qquad \lambda^{i'} = U^{i'}_{j} \lambda^{j} \quad \text{(contravariant way)} \\ \text{under a general coordinate transformation } u^{i'} = u^{i'}(u^{j})\end{array}}$

<u>Remarks</u> We're often interested in vector fields (collection of vectors at different points)

(i) ⇒ components depend on coordinates

$$\lambda^{j} = \lambda^{j}(u^{i})$$

At each point P, we would need $\lambda^{i'} = U^{i'}_{j} \lambda^{j}$ to hold for this to be a vector field...

(ii) Not all 3-tuples of functions are vectors...

↳ e.g. Consider 3-tuple of coordinates

$$\left.\begin{array}{l}\lambda^{i} = u^{i} \\ \lambda^{j'} = u^{j'}\end{array}\right\} \quad \text{linked by } u^{j'} = u^{j'}(u^{i})$$

To be a vector field under general coordinate transforms, it must be true that

$\lambda^{j'} = U^{j'}_{i} \lambda^{i}$. In this case becomes

↳ $u^{j'} = U^{j'}_{i} u^{i}$ with $U^{j'}_{i} = \frac{\partial u^{j'}}{\partial u^{i}}$

But in general this is <u>NOT</u> true $u^{j'} \neq \frac{\partial u^{j'}}{\partial u^{i}} u^{i}$ ← instead $u^{j'} = u^{j'}(u^{i})$

So coordinates do not make a vectors & As components they don't transform correctly

→ This is why we never lower u^i, i.e. $u^j \neq g^{ij} u_i$

BUT $\boxed{\text{there are special case exceptions}}$

e.g → restrict to linear transformation

$$\boxed{u^{j'} = u^{j'}(u^i) = C^{j'}_i u^i} \quad \text{where } C^{j'}_i \text{ constant}$$

new coords are just linear comb. of old ...

So $$\frac{\partial u^{j'}}{\partial u^k} = C^{j'}_i \frac{\partial u^i}{\partial u^k} = C^{j'}_i \delta^i_k = C^{j'}_k$$

Let $k = i \Rightarrow \boxed{C^{j'}_i = \frac{\partial u^{j'}}{\partial u^i} = U^{j'}_i}$

→ Get $u^{j'} = u^{j'}(u^i)$ get $\boxed{u^{j'} = U^{j'}_i u^i}$ under linear transforms → so they

So coordinates do form a vector under linear coords transformation (but not general coord. transf)

(iii) { Properly speaking we can define vectors with respect to a particular class of transformation... }

{ It is possible for sth to be a vector w.r.t one class of transformation, but NOT a vector under another

Default → under general coordinate transform }

Sep 24, 2018 $\boxed{\text{Example}}$

Recall Coordinate transform $u^j \to u^{i'}$

there $U^{i'}_j = \frac{\partial u^{i'}}{\partial u^j}$, $U^i_{j'} = \frac{\partial u^i}{\partial u^{j'}}$

obey $U^{i}_{k'} U^{k'}_j = \delta^i_j$ and $\lambda^{i'} = U^{i'}_j \lambda^j$

Can define a vector as a quantity whose components transform this way..

Note $\Rightarrow$ coordinates do not form a vector since $u^{i'} \neq \frac{\partial u^{i'}}{\partial u^j} u^j$ in general

But $\rightarrow$ differentials of coordinates do make a vector (they are displacements)

$du^i = \{du^1, du^2, du^3\}$ From the chain rule $du^{i'} = \frac{\partial u^{i'}}{\partial u^j} du^j$

$\rightarrow du^{i'} = U^{i'}_j du^j \rightarrow (du^i)$ make a vector...

$\boxed{\text{Example}}$ Find $U^{i'}_j$ for a coordinate transform from Cartesian to spherical in flat 3D space...

$u^j \mapsto u^{i'}$ with $u^j = \{x, y, z\}$, $u^{i'} = \{r, \theta, \varphi\}$

So
$$[U^{i'}_j] = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

Get
$$[U^{i'}_j] = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\varphi & \frac{1}{r}\cos\theta\sin\varphi & -\frac{1}{r}\sin\theta \\ \frac{-\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix}$$

Note this is the inverse of the Jacobian found previously
$$dx\,dy\,dz = \det[U^j_{i'}]\, dr\, d\theta\, d\varphi$$

Call $[U^{i'}_j] = \hat{U}$, and $[U^j_{i'}] = U$

We can show $\boxed{\hat{U}U = U\hat{U} = I}$

Example Suppose $\lambda^i = (1,0,0)$ in Cartesian coordinates. So $\vec{\lambda} = \hat{i} + 0\hat{j} + 0\hat{k}$

What are the components of $\vec{\lambda}$ in spherical coordinates? well...

$$\vec{\lambda} = \lambda^{i'} \vec{e}_{i'} \Rightarrow \text{where } \vec{e}_{i'} = \{\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi\}$$

Now
$$\lambda^{i'} = \begin{pmatrix} \lambda^{1'} \\ \lambda^{2'} \\ \lambda^{3'} \end{pmatrix} = O_j^{i'} \lambda^j = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\varphi & \frac{1}{r}\cos\theta\sin\varphi & -\frac{1}{r}\sin\theta \\ \frac{-\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So
$$\begin{pmatrix} \lambda^{1'} \\ \lambda^{2'} \\ \lambda^{3'} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \frac{1}{r}\cos\theta\cos\varphi \\ \frac{-\sin\varphi}{r\sin\theta} \end{pmatrix} \leftarrow \text{components with respect to spherical coordinates...}$$

Now have
$$\vec{\lambda} = \lambda^{1'}\vec{e}_r + \lambda^{2'}\vec{e}_\theta + \lambda^{3'}\vec{e}_\varphi$$

$$\boxed{\vec{\lambda} = \sin\theta\cos\varphi\,\vec{e}_r + \frac{1}{r}\cos\theta\cos\varphi\,\vec{e}_\theta - \frac{\sin\varphi}{r\sin\theta}\,\vec{e}_\varphi}$$

We have $|\vec{\lambda}| = 1$ in Cartesian. Is this still true in spherical...

$$\cancel{|\vec{\lambda}| = \sin^2\theta\cos^2\varphi + \frac{1}{r^2}\cos^2\theta\cos^2\varphi + \frac{\sin^2\varphi}{r^2\sin^2\theta} = \lambda^{i'}\lambda^{j'}g_{ij}}$$

Now
$$|\vec{\lambda}| = \sqrt{\vec{\lambda}\cdot\vec{\lambda}} \Rightarrow \text{where } \boxed{\vec{\lambda}\cdot\vec{\lambda} = g_{i'j'}\lambda^{i'}\lambda^{j'}}$$

with the metric
$$g_{i'j'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Note metric tensor

$g_{ij} \neq I$ in general...

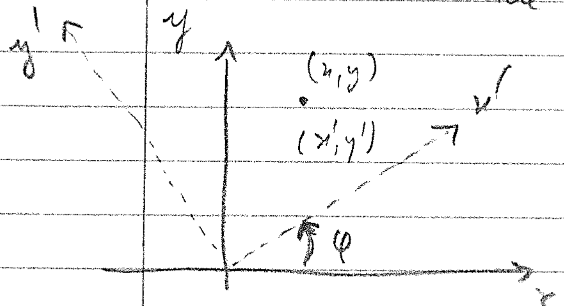
(exception is in Cartesian)

So
$$\vec{\lambda}\cdot\vec{\lambda} = (\lambda^{1'})^2 g_{1'1'} + (\lambda^{2'})^2 g_{2'2'} + (\lambda^{3'})^2 g_{3'3'}$$

$$= \sin^2\theta\cos^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\varphi = 1$$

So $\boxed{|\vec{\lambda}| = 1}$

<u>Example</u> Find $U^{j'}_i$ for a rotation of Cartesian coords by φ ~~...~~ about the z axis ...



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

<u>More completely</u>

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{\partial u^{i'}}{\partial u^j} = [U^{i'}_j] = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Same thing ...)

<u>Note</u> (φ is fixed)

<u>So</u> <u>note</u> $[U^{i'}_j]$ is a constant matrix → <u>linear transformation</u>.

⇒ coordinates transform like vectors ... which is what we showed

$$\boxed{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow u^{i'} = U^{i'}_j u^j}$$

This is <u>NOT</u> true in general. True only if components are fixed ...

Any vector $\vec{\lambda}$ will have components that transform under rotation given by (generally)

$$\boxed{\lambda^{i'} = U^{i'}_j \lambda^j}$$

rotated unrotated

Suppose $(x, y, z) = (1, 1, 0)$ what is (x', y', z')? after rotation by φ.

Well

$$\vec{\lambda}\cdot\vec{\lambda} = g_{ij}\lambda^i\lambda^j = \delta^i_j\lambda^i\lambda^j = \lambda^i\cdot\lambda^i \quad (g_{ij} = \delta^i_j \text{ in Cartesian})$$
$$= 2$$

In (x', y', z')

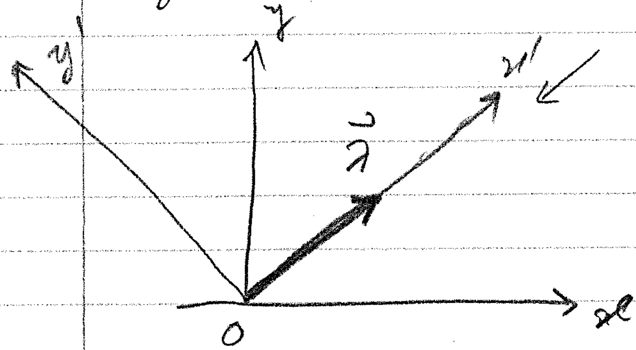
$$\lambda^{i'} = U^{i'}_j\lambda^j = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\varphi + \sin\varphi \\ -\sin\varphi + \cos\varphi \\ 0 \end{pmatrix}$$

↑ w.r.t. (x', y', z')

So $\boxed{\vec{\lambda} = (\cos\varphi + \sin\varphi)\hat{i}' + (-\sin\varphi + \cos\varphi)\hat{j}' + 0\hat{k}'}$

e.g. if $\varphi = 45^\circ$, then $\vec{\lambda} = \sqrt{2}\,\hat{i}' + 0\hat{j}' + 0\hat{k}'$ (makes sense)

$= \hat{i} + \hat{j} + 0\hat{k}$



Note $|\vec{\lambda}|$ still $= \sqrt{2}$

But we need to know what $g_{i'j'}$ is...

$|\vec{\lambda}|^2 = g_{i'j'}\lambda^{i'}\lambda^{j'}$ does this $= (\sqrt{2})^2$

↑ what is $g_{i'j'}$?

Question	How does the metric tensor transform. But first what is a tensor?

Vector ⇒ has magnitude & direction (one direction + one length)

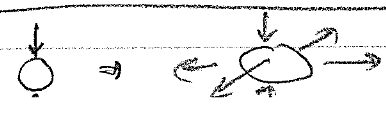
$\boxed{\text{Tensors} \rightarrow \text{generalization of vectors, but they're multi-directional}}$

Ex

Vector: force $\vec{F}$ ↗ $\vec{F} = m\vec{a}$ ($\vec{a}$ follows $\vec{F}$)

$\boxed{\text{But now consider a balloon + squeeze it in 1 direction}}$

(response in all directions...)



→ $\boxed{\text{Stress tensor}}$ $F_{xx}, F_{xy}, F_{xz}, F_{yx}, F_{yy}, F_{yz}, F_{zx}, F_{zy}, F_{zz}$

☆ Mathematically, generalize the def of a vector.

⇒ $\boxed{\text{Give a definition based on how their components transform}}$

Sept 25, 2018

$\boxed{\text{TENSORS}}$ → generalizations of vectors, but multidirectional.
⇒ can't represent them as an arrow...

Can generalize def. of a vector to say...

Def $\boxed{\text{A tensor is a multicomponent quantity whose components transform as contravariant or covariant vector components}}$

e.g. $\tau^{ij}{}_{k}{}^{l}$ is a tensor if

$$\tau^{i'j'}{}_{k'}{}^{l'} = U^{i'}_{m} U^{j'}_{n} U^{p}_{k'} \tau^{mn}{}_{p}{}^{q} U^{l'}_{q}$$

Under a general coordinate transformation $u^{i'} = u^{i'}(u^{j})$

Show $\boxed{g_{ij} \text{ is a tensor}}$ $g_{ij} = \vec{e}_i \cdot \vec{e}_j$
$g_{i'j'} = \vec{e}_{i'} \cdot \vec{e}_{j'}$

We can use $\vec{e}_{i'} = U^{k}_{i'} \vec{e}_{k}$

$$\rightarrow \boxed{g_{i'j'} = U^{k}_{i'} \vec{e}_{k} \cdot U^{l}_{j'} \vec{e}_{l} = U^{k}_{i'} U^{l}_{j'} g_{kl}}$$

So g_{ij} is a tensor

Similarly $\boxed{g^{i'j'} = U^{i'}_{k} U^{j'}_{l} g^{kl}}$

$$\boxed{\text{A tensor } T^{ijk}{}_{mnp\ldots} \text{ is said to be of type } (r,s) \text{ when it has } r \text{ contravariants and } s \text{ covariants.}}$$

<u>Ex</u> $g_{ij} \to$ type $(0,2)$ tensors $\quad$ | $\quad \lambda^i \to$ type $(1,0)$ tensor

$g^{ij} \to$ type $(2,0)$ tensor $\quad$ | $\quad \lambda_i \to$ type $(0,1)$ tensor

<u>Note</u> $U^{i'}_j$ is <u>NOT</u> a tensor. Rather, it's a transformation matrix

↳ take components $j \leftrightarrow i'$

<u>Ex</u>

<u>write</u> $g_{i'j'} = U^k_{i'} U^l_{j'} g_{kl}$ as matrix eqn

<u>Let</u> $\underset{\sim}{G} = [g_{ij}]$, and $\underset{\sim}{G}' = [g_{i'j'}]$

$$\underset{\sim}{\hat{U}} = \underset{\sim}{U}^{-1} = \left[\frac{\partial u^k}{\partial u^{i'}}\right]$$

<u>Put metric in the middle</u>

$$g_{i'j'} = \overset{\text{row}}{U^k_{i'}}\, \underset{\text{row}\ \ \text{col}}{g_{kl}}\, \overset{\text{row}}{U^l_{j'}}$$

→ not gonna work. Need to transpose 1st matrix

$$\boxed{\underset{\sim}{G}' = \hat{U}^T G \hat{U}}$$

<u>Note</u> only tensors of type (r,s) with $r+s \leq 2$ can be written as matrices multiplications ... Can't write $T^{ij}{}_{kl}$ as a matrix

$\boxed{\text{Ex}}$ <u>Look at rotation by ϕ about z again</u>

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \to [U^{i'}_j] = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Recall, in xyz frame, $g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, what is $g_{i'j'}$ in (x', y', z')?

Here

$$[g_{i'j'}] = [U^k_{\ i'} U^l_{\ j'} g_{kl}] = \underset{\sim}{\hat{U}}^T \underset{\sim}{G} \underset{\sim}{\hat{U}} = \underset{\sim}{G}'$$

Recall $\underset{\sim}{\hat{U}} = \underset{\sim}{U}^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\frac{\partial u^k}{\partial u^{i'}}$ (rotation by $-\varphi$)

This gives

transpose...

$$\underset{\sim}{G}' = [g_{i'j'}] = \underset{\sim}{\hat{U}}^T \underset{\sim}{G} \underset{\sim}{\hat{U}} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

I

$$= I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Metric is the same in rotated Cartesian frame...

Notice in this case $\boxed{\underset{\sim}{\hat{U}} = \underset{\sim}{U}^{-1} = \underset{\sim}{U}^T \Rightarrow \underset{\sim}{U} \text{ is orthogonal}}$

//

Scalars $\rightarrow$ invariant quantities under general coordinate transformation
- $\rightarrow$ have no open indices
- $\rightarrow$ type $(0,0)$ tensors
- $\rightarrow$ just numbers... $\rightarrow$ same in all coords system...

Ex: Show that the magnitude of a vector is a scalar

Let $\vec{\lambda} = \{\lambda^i\} = \{\lambda^{i'}\}$

$\|\vec{\lambda}\| = \sqrt{\vec{\lambda} \cdot \vec{\lambda}} = \sqrt{\lambda^i \lambda_i}$ this has no open indices (it's a sum)

$|\vec{\lambda}|$ is a scalar if $\vec{\lambda} \cdot \vec{\lambda} = \lambda^{i'} \lambda_{i'}$ $\rightarrow$ same number. Need to show $\lambda^i \lambda_i = \lambda^{i'} \lambda_{i'}$ (INVARIANT)

Use $\lambda^{j'}\lambda_{j'} = (U^{j'}_{j}\lambda^{j})(U^{k}_{j'}\lambda_k) = \underbrace{U^{j'}_{j}U^{k}_{j'}}\lambda^{j}\lambda_k$

$$= \delta^{k}_{j}\lambda^{j}\lambda_k = \lambda^k\lambda_k$$

So $\lambda^{j'}\lambda_{j'} = \lambda^k\lambda_k \Rightarrow$ **$|\vec{\lambda}|$ is a scalar**

\#

Example $\boxed{\text{Show } ds^2 = g_{ij}du^i du^j \text{ is a scalar}}$

Need to show $g_{i'j'}du^{i'}du^{j'} = g_{ij}du^i du^j$

↳ Use $g_{i'j'} = U^{k}_{i'}U^{l}_{j'}g_{kl}$, $du^{i'} = \frac{\partial u^{i'}}{\partial u^{j}}du^{j} = U^{i'}_{j}du^{j}$

So $g_{i'j'}du^{i'}du^{j'}$

$$= (U^{k}_{i'}U^{l}_{j'}g_{kl})(U^{i'}_{m}du^m)(U^{j'}_{n}du^n)$$

$$= (U^{k}_{i'}U^{i'}_{m})(U^{l}_{j'}U^{j'}_{n})g_{kl}du^m du^n$$

$$= \delta^{k}_{m}\delta^{l}_{n}g_{kl}du^m du^n$$

$$= g_{kl}du^k du^l = g_{ij}du^i du^j$$

Therefore ds^2 is a scalar

\#

$\boxed{\text{Summarize}}$ 3 classes of objects ... **Scalars**: $\lambda \to$ no open indices (invariant)..

Vectors $\to$ upper/lower index
$\to$ transform as

$$\boxed{\lambda^{i'} = U^{i'}_{j}\lambda^{j}, \quad \lambda_{i'} = U^{j}_{i'}\lambda_{j}}$$

Tensors $\tau^{ij}{}_{k} \to$ type (r,s)

type $(2,1)$

$$\boxed{\tau^{i'j'}{}_{k'} = U^{i'}_{l}U^{j'}_{m}U^{n}_{k'}\tau^{lm}{}_{n}}$$

Components transform, but tensors themselves don't transform...

Sept 26, 2018

IV - Flat Spacetime

$(ct, x, y, z) \rightarrow$ spacetime coords. Let $\mu, \nu, \ldots = 0,1,2,3$

So $\quad X^\mu = \{X^0, X^1, X^2, X^3\} = (ct, x, y, z)$

$$X^\mu = (X^0, \vec{x}) = (X^0, X^i) \qquad (i = 1,2,3)$$

Coordinate transformation in special relativity are Lorentz Transformation

Note Under LT there's an invariant spacetime interval...

$$\left\{\begin{aligned} ds^2 &= c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \end{aligned}\right\} \leftarrow \text{line element in Cartesian coordinates in flat spacetime}$$

gives "distances" in spacetime

$\boxed{\text{Can read off the metric}}$

$$\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu}$$

where

$$[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \leftarrow \text{Minkowski metric}$$

Since in any other frame connected by a LT

$$\rightarrow (ds')^2 = (dx^{0'})^2 - (dx^{1'})^2 - (dx^{2'})^2 - (dx^{3'})^2 = (ds)^2$$

$\boxed{\text{Says that}}$

$$[\eta_{\mu'\nu'}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = [\eta_{\mu\nu}] \rightarrow \text{same metric (Cartesian)}$$

So

$$\boxed{ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = \eta_{\mu'\nu'}\, dx^{\mu'} dx^{\nu'} = \eta_{\mu\nu}\, dx^\mu dx^\nu}$$

Note $[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & -g_{ij} & \\ 0 & & & \end{pmatrix}$ can change w/to spherical

Generally, in non-Cartesian coordinates or when there's curvature, we use

$$\boxed{g_{\mu\nu} = \text{metric} \qquad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

convention → But when using Cartesian coords in flat spacetime, let $\boxed{g_{\mu\nu} \to \eta_{\mu\nu}}$

$\boxed{\text{With metric, we can raise/lower tensor indices}}$

if $\lambda^\mu = (\lambda^0, \lambda^1, \lambda^2, \lambda^3) = (\lambda^0, \vec{\lambda})$ → contravariant

then $\lambda_\mu = \eta_{\mu\nu} \lambda^\nu = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ covariant

$= (\lambda^0, -\lambda^1, -\lambda^2, -\lambda^3)$

⇒ In flat spacetime in Cartesian coords, $\boxed{\lambda^0 = \lambda_0}$

But spatial component → $\boxed{\lambda^i = -\lambda_i}$

Rename $[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

How to get $[\eta^{\mu\nu}]$? Take inverse. Must satisfy $\eta_{\mu\nu}\eta^{\nu\sigma} = \delta_\mu^\sigma$

Not hard to see that $[\eta^{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = [\eta_{\mu\nu}]$

Then $$\boxed{\lambda^\mu = \eta^{\mu\nu} \lambda_\nu}$$

As before there are 4 ways to take inner product...

$$\boxed{a \cdot b = a^\mu \cdot b_\mu = a_\mu \cdot b^\mu = \eta_{\mu\nu} a^\mu b^\nu = \eta^{\mu\nu} a_\mu b_\nu}$$

inner product of two 4-vectors.

Notice that $a^\mu b_\mu = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3$ (sum. notations...)

But $\eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^\mu b_\mu$

Why? → simply because $\boxed{b_\mu = -b^\mu}$ (by $\eta_{\mu\nu}$)

Note The metric contains info on how to calculate lengths and intervals in spacetime...

Note We've skipped introducing basis vectors. Could define a set $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ ~~{...}~~

so $\boxed{\lambda = \lambda^0 \vec{e}_0 + \lambda^1 \vec{e}_1 + \lambda^2 \vec{e}_2 + \lambda^3 \vec{e}_3}$

However, $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are not $\hat{i}, \hat{j}, \hat{k}$

why? $\vec{e}_1 \cdot \vec{e}_1 = \eta_{11} = -1$, where $\hat{i} \cdot \hat{i} = 1$

note index start @ 0

So $\vec{e}_\mu$ could have imaginary parts

Basically, won't use basis vectors going forward!

$\boxed{\text{Lorentz Transformation}}$ → is a coordinate transform from one <u>inertial</u> frame to another $K \to K'$

<u>Most general LT's include</u>

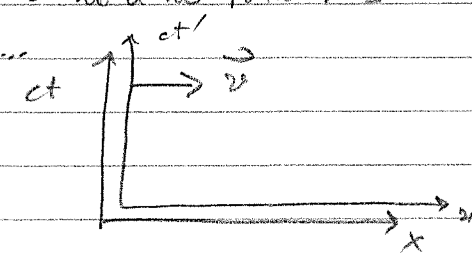
(1) Lorentz boost (relative motion w/ const. v)
(2) Translation (origins don't coincide at $t' = t = 0$)
(3) spatial rotation $x \not\parallel x'$, ...
(4) spatial inversion (parity transform ...) $(x' = -x)$
(5) Time reversal $(t' = -t)$

Usually called collectively
↓
"<u>Poincaré transformations</u>"

<u>other distinctions</u>

→ inhomogeneous LT's → ~~can~~ have translation
→ homogeneous → no translation (same origin)
→ improper LT's → (parity / time reversal)
→ proper LT's → <u>NO</u> parity / time reversal ...

We can first look at homogeneous, proper LT's with no rotations
⇒ these are the ~~KCE~~ Lorentz boosts ...
<u>e.g.</u> A boost along x



<u>Lorentz boost</u>

$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

$\boxed{\text{In flat 3D space}}$

$$U^{i'}_{j} = \frac{\partial u^{i'}}{\partial u^{j}} \quad \to U$$

$\boxed{\text{In 4D spacetime, in general}}$

$$X^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \quad \to \text{big } X : \overline{X}$$

$\boxed{\text{But for Lorentz transformations}}$ use ~~A~~, Λ

$$X^{\mu'}_{\nu} = \Lambda^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \qquad \Lambda^{\mu'}_{\nu} \to \text{LT's only}$$

$\boxed{\text{For a Lorentz boost}}$

$$\left[\Lambda^{\mu'}{}_{\nu}\right] = \left[\frac{\partial x^{\mu'}}{\partial x^{\nu}}\right] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Sept 28, 2018

Recall Lorentz Transformation $x^{\nu} \rightarrow x^{\mu'}$

$$\Lambda^{\mu'}{}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}$$ e.g. for a boost along x

$$\left[\Lambda^{\mu'}{}_{\nu}\right] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note $\Lambda^{\mu'}{}_{\nu}$ constant

This means LTs are linear ~~trans~~ transformations

This means Cartesian coords x^{μ} form the components of a vector under LTs

$$\boxed{x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu} \text{ is obeyed}} \rightarrow$$

This gives back $x^{0'} = \gamma(x^0 - \beta x^1)$, $x^{1'} = \gamma(x^1 - \beta x^0)$

This also means that in SR we can lower index of x^{μ}

$$\boxed{\begin{aligned} x_{\mu} &= \eta_{\mu\nu} x^{\nu} \\ x^{\mu} &= \eta^{\mu\nu} x_{\nu} \end{aligned}}$$

But we never do this in general, e.g. in curved spacetime)

But remember $x^{\mu} = (ct, x, y, z)$

while $x_{\mu} = (ct, -x, -y, -z)$

these obey

$$\underbrace{\Lambda^{\mu'}{}_{\nu} \Lambda^{\nu}{}_{\lambda'} = \delta^{\mu}_{\lambda}}$$

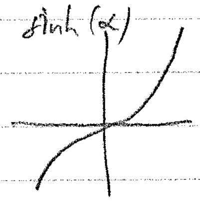
$\boxed{\text{To find inverse}}$

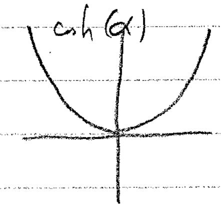
$$\Lambda^{\mu}{}_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\nu'}}$$. Just let $v = -v$ & $\left[\Lambda^{\mu}{}_{\nu'}\right] = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A Curiosity about Lorentz boosts

→ can make them look like rotation using hyperbolic functions...

Use $\sinh(\alpha) = \dfrac{e^{\alpha} - e^{-\alpha}}{2}$, $\cosh(\alpha) = \dfrac{e^{\alpha} + e^{-\alpha}}{2}$





$\tanh(\alpha) = \dfrac{\sinh\alpha}{\cosh\alpha}$ $\quad$ $\operatorname{sech}(\alpha) = \dfrac{1}{\cosh(\alpha)}$

$\operatorname{csch}(\alpha) = \dfrac{1}{\sinh(\alpha)}$, $\quad$ $\coth(\alpha) = \dfrac{1}{\tanh(\alpha)}$

OBEY

$$\boxed{\begin{gathered}\cosh^2\alpha - \sinh^2\alpha = 1 \\ 1 - \tanh^2(\alpha) = \operatorname{sech}^2(\alpha)\end{gathered}}$$

Look at

$$[\Lambda^{\mu'}_{\ \nu}] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Introduce $\tanh\varphi = \dfrac{v}{c}$ where φ = rapidity

So $\gamma = \dfrac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \dfrac{1}{\sqrt{1 - \tanh^2\varphi}} = (\operatorname{sech}\alpha)^{-1} = \cosh\varphi$

So $\gamma \dfrac{v}{c} = \beta\gamma = \sinh\varphi$

So

$$\boxed{[\Lambda^{\mu'}_{\ \nu}] = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

form of hyperbolic rotation between ct & x

Proper Homogeneous Lorentz Transform

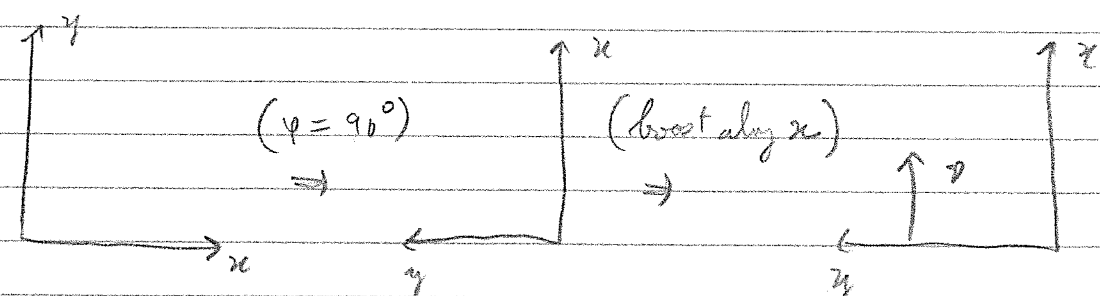
↳ boost + rotation. These still leave form $X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^{\nu}$
But now $\Lambda^{\mu'}_{\ \nu}$ can be a boost or rotation

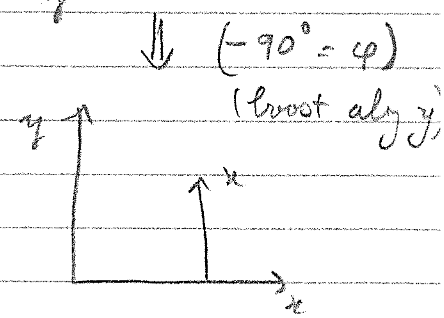
Can look at a <u>rotation about z by φ</u>

$$[\Lambda^{\mu'}_{\ \nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A Lorentz boost along an arbitrary direction can be formed as a combination of a boost along x & spatial rotation

e.g.





<u>So</u> the end result is <u>boost along y</u>

<u>So matrix multiply</u>

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{rotate by } -90°} \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{boost along } x} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{rotate by } 90°} = \underbrace{\begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{boost along } y}$$

Poincare Transformations

↳ boosts, rotation, translations, time / spatial inversions ...

Here $\boxed{x^{\mu'} = \Lambda^{\mu'}_{\ \nu} x^{\nu} + a^{\mu'}}$ ⟵ general form

(the rest) (translation) (constant), so $\frac{\partial a^{\mu'}}{\partial x^{\nu}} = 0$

These are "affine" transformations: Linear transformation with a shift

Suppose we take $\frac{\partial}{\partial x^{\nu}}$ of $x^{\mu'}$

↳ $$\frac{\partial x^{\mu'}}{\partial x^{\nu}} = \frac{\partial}{\partial x^{\nu}} x^{\mu'} = \Lambda^{\mu'}_{\ \nu} = \Lambda^{\mu'}_{\ \nu} \text{ for LTs}$$

⇒ Get the usual definition $\Lambda^{\mu'}_{\ \nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}}$. With chain rule, still get $\Lambda^{\mu'}_{\ \nu} \Lambda^{\nu}_{\ \sigma} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \frac{\partial x^{\nu}}{\partial x^{\sigma}} = \delta^{\mu}_{\sigma}$

⇒ Still holds for Poincare transformation

Note → The defining features of a Lorentz Transform is that

$$\boxed{\begin{aligned} ds^2 &= \eta_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} \end{aligned}} \quad (\star)$$

LT's preserve the Minkowski metric (with Cartesian)

where
$$[\eta_{\mu\nu}] = [\eta_{\mu'\nu'}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

From $x^{\mu'} = \Lambda^{\mu'}_{\ \sigma} x^{\sigma} + a^{\mu'}$, take differential

$dx^{\mu'} = \Lambda^{\mu'}_{\ \sigma} dx^{\sigma}$, plug into $(\star)$

So $\eta_{\mu'\nu'} dX^{\mu'} dX^{\nu'} = \eta_{\sigma\rho} dX^{\sigma} dX^{\rho}$

$\rightarrow \eta_{\mu'\nu'} dX^{\mu'} dX^{\nu'} = \eta_{\mu'\nu'} \left(\Lambda^{\mu'}_{\sigma} dX^{\sigma}\right) dX^{\nu'}$

$\qquad \| \qquad = \eta_{\mu'\nu'} \left(\Lambda^{\mu'}_{\sigma} dX^{\sigma}\right)\left(\Lambda^{\nu'}_{\rho} dX^{\rho}\right)$

So $\Longrightarrow \eta_{\sigma\rho} dX^{\sigma} dX^{\rho} = \eta_{\mu'\nu'} \Lambda^{\mu'}_{\sigma} \Lambda^{\nu'}_{\rho} dX^{\sigma} dX^{\rho}$

Let $\sigma \to \mu$, $\rho \to \nu$, $\mu' = \alpha'$, $\nu' = \beta'$

$\hookrightarrow$ $$\boxed{\eta_{\mu\nu} = \Lambda^{\alpha'}_{\mu} \Lambda^{\beta'}_{\nu} \eta_{\alpha'\beta'}}$$

Metric obeys this under Poincare transforms. This shows 2 things

① $\eta_{\mu\nu}$ is a tensor $\rightarrow$ transforms correctly

② $\eta_{\mu\nu}$ is unchanged under Lorentz transformation.

For other vectors & tensors under LTs, must have:

Contravariant	$\lambda^{\mu'} = \Lambda^{\mu'}_{\nu} \lambda^{\nu}$
Covariant	$\lambda_{\mu'} = \Lambda^{\nu}_{\mu'} \lambda_{\nu}$
Tensor	$\tau^{\mu'\nu'}{}_{\sigma'} = \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\sigma'} \tau^{\alpha\beta}{}_{\gamma}$

general case these will be different

Scalars $\Rightarrow$ invariants under Lorentz transformations (same in all inertial frames)

Oct 1, 2018 4-vector under Lorentz-Transformation

→ must obey $\boxed{\lambda^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, \lambda^{\nu}}$

Scalars → invariant under LT's.

e.g. Show inner products are scalars...

$$\left\{\begin{aligned} a^{\mu'} b_{\mu'} &= \Lambda^{\mu'}_{\ \nu} a^{\nu} \Lambda^{\sigma}_{\ \mu'} b_{\sigma} \\ &= \Lambda^{\mu'}_{\ \nu} \Lambda^{\sigma}_{\ \mu'} a^{\nu} b_{\sigma} \\ &= \delta^{\sigma}_{\nu} a^{\nu} b_{\sigma} = a^{\nu} b_{\nu} \end{aligned}\right.$$

invariant, same in all frames ← → scalars. ↑ scalars

This shows that the norm of every 4-vector is invariant

$$\boxed{\lambda \cdot \lambda = \lambda^{\mu}\lambda_{\mu} = \lambda^{\mu'}\lambda_{\mu'}}$$

Therefore the sign of the norm is invariant as well

$$\lambda^2 = (\lambda \cdot \lambda) = (\lambda^0)^2 - (\lambda^1)^2 - (\lambda^2)^2 - (\lambda^3)^2 \quad \text{Can be } (-, 0, +)$$

There are 3 cases

$$\left\{\begin{aligned} &\lambda^2 > 0 \rightarrow \text{time-like} \\ &\lambda = 0 \rightarrow \text{light-like / null} \\ &\lambda^2 < 0 \rightarrow \text{space-like} \end{aligned}\right.$$

→ These labels do not change under Lorentz Transformations

- For time like vectors, there is always a frame where $\lambda^{\mu} = (\lambda^0, 0, 0, 0)$
  → always rotate + boost to get this...

- For space like, can always find a frame where $\lambda^{\mu} = (0, \lambda^1, 0, 0)$
  or a frame where $\lambda^{\mu} = (0, 0, \lambda^2, 0)$, etc..

- For null vectors, can always find a frame where $\lambda^{\mu} = (\lambda^0, \lambda^0, 0, 0)$
  or $(\lambda^0, 0, \lambda^0, 0)$, etc... More generally, $\lambda^{\mu} = (\lambda^0, \vec{\lambda})$
  so that $\lambda^{\mu}\lambda_{\mu} = 0$ with $|\vec{\lambda}| = \lambda^0$

$\boxed{Ex\ 1}$ Is $X^\mu = (ct, x, y, z)$ a contravariant vector under Poincaré transformation?

- If so, then $X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^\nu$ would need to hold

Note Poincaré transform: $\boxed{X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^\nu + a^{\mu'}}$

↳ See that X^μ is not a vector if $a^{\mu'} \neq 0$. (Can't allow translations). Under LT's ($a^{\mu'} = 0$), then X^μ is a vector

$\boxed{Ex\ 2}$ Is $dX^\mu = (c\,dt, dx, dy, dz)$ a vector under Poincaré transform?

Note Poincaré transform: $X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^\nu + a^{\mu'}$

$$\text{So} \quad dX^{\mu'} = \Lambda^{\mu'}_{\ \nu} dX^\nu + 0$$

So $dX^{\mu'}$ is a vector $\Rightarrow$ dX^μ is a vector under Poincaré transform

$\boxed{Ex\ 3}$ Suppose we take $\frac{\partial}{\partial X^\mu}$ of a scalar a vector? Is $\frac{\partial \varphi}{\partial X^{\mu'}}$ a vector? What type?

Chain rule: $\varphi = \varphi(X^\nu(X^{\mu'}))$

$$\rightarrow \frac{\partial \varphi}{\partial X^{\mu'}} = \frac{\partial \varphi}{\partial X^\nu} \frac{\partial X^\nu}{\partial X^{\mu'}} = \Lambda^{\nu}_{\ \mu'} \frac{\partial \varphi}{\partial X^\nu} \quad ✓$$

So $\frac{\partial \varphi}{\partial X^{\mu'}}$ is a vector. Note It's a covariant vector, because there's the upper indices cancel out...

↳ Use notation to show this better:

$\boxed{\frac{\partial}{\partial X^\mu} = \partial_\mu}$ → Then $\partial_\mu \varphi = \frac{\partial \varphi}{\partial X^\mu}$ is a covariant vector

Also $\boxed{\vec{\nabla} = \partial_i = (\partial_1, \partial_2, \partial_3)}$

So $\boxed{\partial_\mu = (\partial_0, \partial_i) = (\partial_0, \vec{\nabla})}$

Now, in Minkowski spacetime with Cartesian coordinates, that we can also define a lower coordinate

$$X_\mu = \eta_{\mu\nu} X^\nu \text{. Call } \partial^\mu = \frac{\partial}{\partial X_\mu}$$

From $X^\mu = \eta^{\mu\nu} x_\nu \rightarrow \frac{\partial X^\mu}{\partial X_\nu} = \eta^{\mu\nu}$

$$\partial^\mu = \frac{\partial}{\partial X_\mu} = \frac{\partial X^\nu}{\partial X_\mu} \frac{\partial}{\partial X^\nu} = \eta^{\mu\nu} \partial_\nu$$

gives a contravariant vector

So we get $\boxed{\partial^\mu = \eta^{\mu\nu} \partial_\nu \quad \therefore \quad \partial^\mu \varphi = \eta^{\mu\nu} \partial_\nu \varphi}$

But $\partial^i \neq \vec{\nabla}$. Instead $\partial^i = -\partial_i = -\vec{\nabla}$

Can write $\boxed{\partial^\mu = (\partial^0, \partial^i) = (\partial^0, -\vec{\nabla})}$

VELOCITY, MOMENTUM, FORCE What are these as 4-vectors? → Must transform correctly!

Consider again $X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^\nu + a^{\mu'}$

Velocity $\frac{d}{dt} X^{\mu'} = \frac{d}{dt} \Lambda^{\mu'}_{\ \nu} X^\nu + \frac{d}{dt} a^{\mu'}$ ← constant translation

So $\frac{dX^{\mu'}}{dt} = \Lambda^{\mu'}_{\ \nu} \frac{dX^\nu}{dt} + 0$ → Note, same t on both sides

with $X^\mu = (ct, \vec{x})$ → take t derivative

$\frac{d}{dt} X^\mu = (c, \vec{v})$ with $\vec{v} = \frac{d\vec{x}}{dt}$. Can call $\boxed{V^\mu = \frac{dX^\mu}{dt} = (c, \vec{v})}$ ← coordinate velocity

<u>But in a primed frame</u> $V^{\mu'} = \frac{dX^{\mu'}}{dt'} = (c, \vec{v}')$

<u>Note</u> $\frac{dX^{\mu'}}{dt'} \neq \frac{dX^{\mu'}}{dt}$ <u>so</u> $V^{\mu'} = \frac{dX^{\mu'}}{dt'} \neq \frac{dX^{\mu'}}{dt} = \Lambda^{\mu'}_{\nu} V^{\nu}$

<u>So</u> $\boxed{V^{\mu'} \neq \Lambda^{\mu'}_{\nu} V^{\nu} \text{ so it's not a 4-vector}}$

- However, we <u>CAN</u> find an actual 4-vector velocity. Consider object with mass <u>and</u> $v<c$ (no photons yet)

In this case $\boxed{ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} dX^\mu dX^\nu > 0}$

timelike product

proper time.

<u>Divide by $d\tau^2$</u> $\longrightarrow$ $\boxed{c^2 = \eta_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau}}$

<u>Call</u> $\boxed{u^\mu = \frac{dX^\mu}{d\tau}}$ $\rightarrow$ <u>world velocity</u>

<u>Chain rule</u>

$$\boxed{u^{\mu'} = \frac{dX^{\mu'}}{d\tau} = \left(\frac{dX^{\mu'}}{dX^\nu}\right)\frac{dX^\nu}{d\tau} = \Lambda^{\mu'}_{\nu} u^{\mu}}$$

invariant

This shows that u^μ is a <u>contravariant</u> 4-vector under LT's.

<u>Also see that</u> $\boxed{u^\mu u_\mu = \eta_{\mu\nu} \frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = c^2}$ (invariant inner product)

massive objects. invariant!

Can relate u^μ to V^μ by: $c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2$

<u>So</u> $\frac{d\tau^2}{dt^2} = 1 - \frac{1}{c^2}\frac{d\vec{x}^2}{dt^2} = 1 - \frac{1}{c^2}\left|\frac{d\vec{x}}{dt}\right|^2 = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$

So $\boxed{\frac{dt}{d\tau} = \gamma}$ ← time dilation

So then $\boxed{u^\mu = \frac{dx^\mu}{d\tau} = \left(\frac{dt}{d\tau}\right)\frac{dx^\mu}{dt} = \gamma v^\mu}$

with $v^\mu = (c, \vec{v})$

still obeys $u^\mu u_\mu = c^2$

So $\boxed{u^\mu = (\gamma c, \gamma \vec{v}) = \gamma(c, \vec{v}) = \gamma v^\mu}$

In the object's rest frame, $\vec{v} = 0$, $\gamma = 1$ $\Rightarrow$ $\boxed{u^\mu = (c, 0, 0, 0)}$ in rest frame

↳ object at rest moves at speed c in time direction.

. And moving objects $\boxed{u^\mu u_\mu = c^2}$

Oct 2, 2018 $\boxed{\text{Recall}}$. Velocities "coordinate velocity" $v^\mu = \frac{dx^\mu}{dt} = (c, \vec{v})$

↑ NOT a 4 vector

"world velocity" → $u^\mu = \frac{\partial x^\mu}{\partial \tau} = (\gamma c, \gamma \vec{v})$ → for massive objects

↳ is a 4-vector

also obeys that $u^\mu u_\mu = c^2$

and $\boxed{u^\mu = \gamma v^\mu = \gamma(c, \vec{v})}$

$\boxed{\text{Now, momentum}}$ 4-momentum can be defined as $\boxed{p^\mu = m u^\mu}$

See that $p^\mu = \gamma m v^\mu = m\gamma(c, \vec{v}) = (m\gamma c, m\gamma \vec{v})$

or $p^\mu = \left(\frac{\gamma m c^2}{c}, m\gamma \vec{v}\right)$ But note $E = \gamma m c^2$, $\vec{p} = \gamma m \vec{v}$

So $\boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right)}$

p^0 $\vec{p}$

Norm^2 : p^μ has invariant $|p^\mu|^2$

$$\boxed{p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2}$$

But also $\boxed{p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^{\,2}} \Rightarrow \boxed{E^2 = c^2|p|^2 + m^2c^4}$

But what about massless particles (light)?

$\hookrightarrow$ massless photons $v = c$ always. $\rightarrow$ No proper-time $d\tau$ DNE

$\rightarrow$ The def $u^\mu = \frac{dx^\mu}{d\tau}$ is undefined for light?

For light: $ds^2 = c^2dt^2 - |d\vec{x}|^2$

$$= c^2dt^2\left(1 - 1/c^2\left|\frac{d\vec{x}}{dt}\right|^2\right) \quad \rightarrow c^2$$

$\boxed{ds^2 = 0}$ $\rightarrow$ for photons $\rightarrow$ photon travels on null trajectory (zero norm)

For light, can't use τ = proper time. But we can still parametrize their trajectory $x^\mu(\sigma) \rightarrow$ some parameter

Can define $\boxed{u^\mu = \frac{\partial x^\mu}{\partial \sigma}}$

$$\boxed{\begin{array}{l} \Rightarrow u^\mu u_\mu = \eta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma} = \frac{\eta_{\mu\nu}\,dx^\mu dx^\nu}{d\sigma^2} = \frac{ds^2}{d\sigma^2} = 0 \\ \rightarrow u^\mu \text{ is light-like (zero norm)} \end{array}}$$

But light has energy + momentum

$p^\mu = \left(\frac{E}{c}, \vec{p}\right) = \left(p^0, \vec{p}\right)$ Recall, $E = h\nu$, $|\vec{p}| = \frac{h}{\lambda}$

Note $\lambda\nu = c$

$\rightarrow E = c|\vec{p}|$

For light $\boxed{p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = 0}$ $\quad (E = c|\vec{p}|)$

↳ momentum is also light like vector (makes sense)

Also use wave vectors

$$\vec{p} = \hbar \vec{k} = \frac{h}{2\pi}\vec{k} \Rightarrow \boxed{|\vec{k}| = \frac{2\pi}{\lambda}}$$

Can define a 4-vector $\boxed{p^\mu = \hbar k^\mu}$

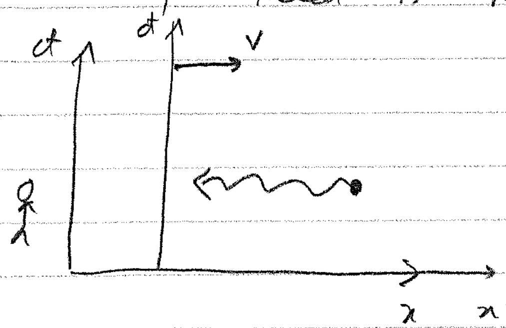
$$\boxed{k^\mu = (k^0, \vec{k})}$$

where $k^0 = \frac{p^0}{\hbar} = \frac{h}{\lambda} \cdot \frac{1}{\hbar} = \frac{2\pi}{\lambda} = |\vec{k}|$

$\Rightarrow$ Both $|k^0| = |\vec{k}| = \frac{2\pi}{\lambda}$

So $\boxed{k^\mu k_\mu = (k^0)^2 - (\vec{k})^2 = 0}$ (again, since $k^\mu \propto p^\mu$)

Example Find λ for light emitted from a source (where λ_0) that is receding



$$k^{\mu\prime} = (k^{0\prime}, \vec{k}') = \left(\frac{2\pi}{\lambda_0}, \frac{-2\pi}{\lambda_0}, 0, 0\right)$$

In stationary frame

$$k^\mu = (k^0, \vec{k}) = \left(\frac{2\pi}{\lambda}, \frac{-2\pi}{\lambda}, 0, 0\right)$$

But $\boxed{k^\mu = \Lambda^\mu_{\ \nu'} k^{\nu'}}$ (inverse LT)

↳ where $[\Lambda^\mu_{\ \nu'}] = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Let $\mu = 0$

$$k^0 = \Lambda^0_{\ \nu'} k^{\nu'}$$

$$\frac{2\pi}{\lambda} = \Lambda^0_{\ 0'} k^{0'} + \Lambda^0_{\ 1'} k^{1'} + \overbrace{\Lambda^0_{\ 2'} k^{2'} + \Lambda^0_{\ 3'} k^{3'}}^{0} = \gamma \frac{2\pi}{\lambda_0} + \gamma\beta\left(\frac{-2\pi}{\lambda_0}\right)$$

So $\frac{2\pi}{\lambda} = \gamma \frac{2\pi}{\lambda_0} - \gamma\beta \frac{2\pi}{\lambda_0} = \gamma \frac{2\pi}{\lambda_0}(1-\beta)$

So
$$\boxed{\frac{1}{\lambda} = \frac{\gamma}{\lambda_0}(1-\beta) = \frac{1}{\lambda_0}\sqrt{\frac{1-\beta}{1+\beta}}}$$

So $$\boxed{\lambda = \lambda_0 \sqrt{\frac{1+\beta}{1-\beta}}}$$ (red shifted)

For light emitted from a source moving toward, $v \to -v$

$$\boxed{\lambda = \lambda_0 \sqrt{\frac{1-\beta}{1+\beta}}}$$ (blue shifted)

Note: There are Doppler shifts due to relative motion.
Later we'll look at gravitational spectral shifts + cosmological redshift

$\boxed{\text{Can define a 4-force vector } f^\mu}$ (back to dealing w/ massive obj)

$$\boxed{f^\mu = \frac{dp^\mu}{d\tau}}$$ (only for massive objects)

where $p^\mu = m u^\mu = m \frac{dx^\mu}{d\tau}$

Get $$\boxed{f^\mu = m \frac{d^2 x^\mu}{d\tau^2}}$$ (relativistic 2nd law)

with

$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ + chain rule $\frac{dp^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dp^\mu}{dt}$

we showed $\frac{dt}{d\tau} = \gamma$

$\Rightarrow \frac{dp^\mu}{d\tau} = \gamma \frac{dp^\mu}{dt} \Rightarrow \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$ $\vec{F}$ constant

power $\frac{dE}{dt} = \frac{d}{dt}(\vec{F}\cdot\vec{r}) = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$

So $\boxed{f^\mu = \gamma\left(\frac{1}{c}\vec{F}\cdot\vec{v}, \vec{F}\right)}$ for a constant force $\vec{F}$.

Oct 2, 2018

Recall $f^\mu = \frac{\partial p^\mu}{\partial \tau} = m\frac{\partial^2 x^\mu}{\partial \tau^2}$ where $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$

and for constant force $\frac{dE}{dt} = \vec{F}\cdot\vec{v}$

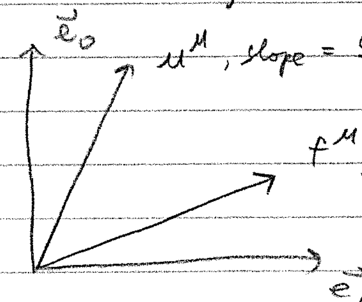
$\Rightarrow f^\mu = \gamma\left(\frac{1}{c}\vec{F}\cdot\vec{v}, \vec{F}\right) \rightarrow \boxed{u^\mu f_\mu = 0} \rightarrow$ orthogonal in 4D spacetime

$\boxed{\text{Can look in 1D}}$

$$f^\mu = \left(\frac{\gamma v}{c}F, \gamma F, 0, 0\right)$$

and $u^\mu = \left(\gamma c, \gamma v, 0, 0\right)$

So plot these in spacetime

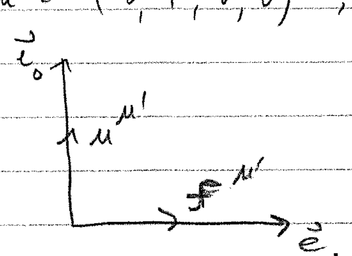


Well, we can also look in instantaneous rest frame.

$\rightarrow v = 0, \gamma = 1$

$\rightarrow$

$f^{\mu'} = (0, F, 0, 0)$, and $u^{\mu'} = (c, 0, 0, 0)$



} look orthogonal!

What we have is an inner product $u^\mu f_\mu = 0$. It's a scalar and therefore same in all frames $\rightarrow$ only takes one frame for them to be orthogonal $\rightarrow u^\mu f_\mu = 0 \ \forall$ frames.

Relativistic Electromagnetism

↳ We previously found Maxwell's Eqns in differential form

$$\boxed{\begin{aligned} \vec\nabla\cdot\vec E &= \frac{\rho}{\epsilon_0} & \vec\nabla\times\vec E &= -\frac{\partial\vec B}{\partial t} \\ \vec\nabla\cdot\vec B &= 0 & \vec\nabla\times\vec B &= \mu_0\vec J + \mu_0\epsilon_0\frac{\partial\vec E}{\partial t} \end{aligned}}$$

$\vec E = (E^1, E^2, E^3) = E^i$, likewise $\vec B = (B^1, B^2, B^3) = B^j$

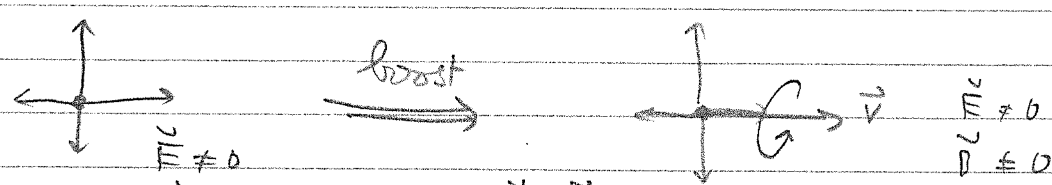
charge density $\longrightarrow \rho$; current density $\rightarrow \vec J$

<u>N.B.</u> $q = \int \rho\, dV$, $I = \int \vec J \cdot d\vec A$, and $\frac{1}{\mu_0\epsilon_0} = c^2$

<u>Note</u> $\vec E$, $\vec B$ are 3D, What are they in 4D?

↳ Together have 6 components which <u>mix</u> under Lorentz Transform

<u>ex</u> Boost a rest charge into moving frame $\Rightarrow$ from $\vec E$ to $\vec E + \vec B$



$\vec B = 0$ $\Rightarrow$ says $\vec E$ & $\vec B$ join relativistically

Find that $\vec E$ & $\vec B$ combine to give tensor

<u>Define</u> electromagnetic field strength $F^{\mu\nu}$

$$\boxed{[F^{\mu\nu}] = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & B^3 & -B^2 \\ -E^2/c & -B^3 & 0 & B^1 \\ -E^3/c & B^2 & -B^1 & 0 \end{pmatrix}}$$

<u>Note</u> $F^{\mu\nu} = -F^{\nu\mu}$
$\Rightarrow$ has only 6 components

$F^{\mu\nu} = 0$ if $\mu = \nu$

Can also define $\boxed{F_{\mu\nu} = \eta_{\mu\alpha}\,\eta_{\nu\beta}\,F^{\alpha\beta}}$

As matrices $\boxed{[F_{\mu\nu}] = [\eta_{\mu\alpha}][F^{\alpha\beta}][\eta_{\beta\beta}]}$

$$= \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & B^3 & -B^2 \\ E^2/c & -B^3 & 0 & B^1 \\ E^3/c & B^2 & -B^1 & 0 \end{pmatrix}$$

Now, can form vectors out of ρ and $\vec{J}$

$j^\mu = (\rho c, \vec{J})$ define the 4-vector current density

In terms of these, Maxwell's eqn become.

$$\boxed{\begin{gathered} \partial_\nu F^{\mu\nu} = \mu_0 j^\mu \\ \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \end{gathered}}$$

e.g look at $\partial_\nu F^{\mu\nu} = \mu_0 j^\mu$

Look at $\boxed{\mu = 0}$ $\rightarrow$ $\partial_\nu F^{0\nu} = \mu_0 j^0 = \mu_0 \rho c$

$$\rightarrow \underbrace{\partial_0 F^{00}}_{0} + \underbrace{\partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03}}_{\frac{1}{c}\partial_i E^i} = \mu_0 c \rho$$

$$\frac{1}{c}\partial_i E^i = \rho c \mu_0$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho c^2 \mu_0 = \frac{\rho}{\varepsilon_0}}$$

Next, let $\mu = k$, $k = \{1, 2, 3\}$

$$F^{k0} = -E^k/c$$

So $\partial_\nu F^{k\nu} = \mu_0 j^k = \mu_0 J^k = \partial_0 F^{k0} + \partial_i F^{ki}$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$

So $\partial_0 F^{k0} = \frac{-1}{c^2}\frac{\partial E^k}{\partial t}$

For $\partial_i F^{ki}$, let $k=1$

$\hookrightarrow \partial_i F^{1i} = \underbrace{\partial_1 F^{11}}_{0} + \partial_2 F^{12} + \partial_3 F^{13} = \partial_2 B^3 + \partial_3(-B^2)$

$= (\vec{\nabla}\times\vec{B})^1$

Similarly, $k=1 \Rightarrow \partial_i F^{ki} = (\vec{\nabla}\times\vec{B})^2$

$k=2 \Rightarrow \partial_i F^{ki} = (\vec{\nabla}\times\vec{B})^3$ So $\partial_i F^{ki} = (\vec{\nabla}\times\vec{B})^k$

So $\frac{-1}{c^2}\frac{\partial E^k}{\partial t} + (\vec{\nabla}\times\vec{B})^k = \mu_0 J^k$

So $\boxed{(\vec{\nabla}\times\vec{B}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}}$ (Ampere - Maxwell)

Similarly, can look at

$\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0$

Can show that for various values of σ, ν, μ $\Rightarrow \begin{cases} \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \end{cases}$

e.g. $(\mu=0,\ \nu=1,\ \sigma=2) \Rightarrow (\vec{\nabla}\times\vec{E})^3 = -\left(\frac{\partial\vec{B}}{\partial t}\right)^3$

To summarize, in SR, all physical properties are some sort of tensors with scalar = m, τ, ds^2, c

Vectors $\rightarrow u^\mu, p^\mu, f^\mu$. $f^\mu = \frac{\partial p^\mu}{\partial\tau} = m\frac{\partial^2 x^\mu}{\partial\tau^2}$

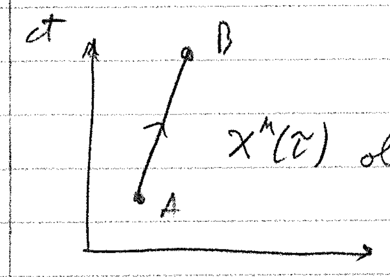
Tensors $\eta_{\mu\nu}$, $F^{\mu\nu}$ $(\mathcal{E}\cdot m)$

$\boxed{\text{All transforms in definite ways under Lorentz Transformation}}$

Geodesics In 3D, flat space, can think of these as shortest distance between 2 points → straight line → path of free particle. <u>Free particles follow geodesics</u>

<u>But</u> in 4D spacetime, Minkowski. Now, free particle, ⇒ $f^\mu = 0$

So $\dfrac{\partial^2 x^\mu}{\partial \tau^2} = 0$ has a solution $X^\mu(\tau)$ that is a <u>straight line in spacetime</u>



$X^\mu(\tau)$ obeying $\dfrac{\partial^2 x^\mu}{\partial \tau^2} = 0$ gives a straight line ⇒ can call this a geodesic

Geodesics are solutions of $\dfrac{\partial^2 x^\mu}{\partial \tau^2} = 0$

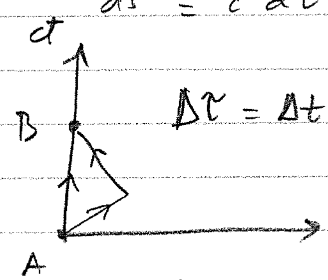
<u>BUT</u> geodesics in Minkowski spacetime are not the shortest 'distance'

We calc. distance using $ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu$.

Moving massive particles

$ds^2 = c^2 d\tau^2 > 0$ → timelike

<u>Consider</u> $A \to B$



$\Delta\tau = \Delta t$ (particle at rest in space)

For moving path $c\Delta\tau' = 2\sqrt{\left(\tfrac{1}{2}c\Delta t\right)^2 - (\Delta x)^2}$

Find that $\Delta\tau' < \Delta\tau$ → $\boxed{\text{geodesics has maximal propertime}}$

not a geodesics (time slows in moving frame)

So we won't think in terms of shortest distance. We'll use that

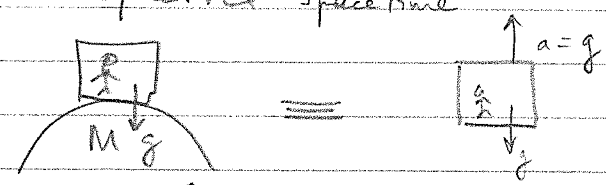
$$\boxed{\text{geodesic} \Rightarrow \text{path of free particle} \quad \frac{\partial^2 x^\mu}{\partial \tau^2} = 0}$$

Oct 5, 2018

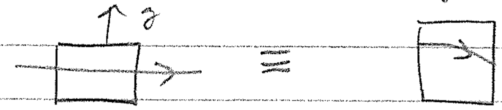
V. CURVED SPACES

↳ Recall: Equivalence principle (EP) leads us towards the idea of curved spacetime

Ep.



For light,



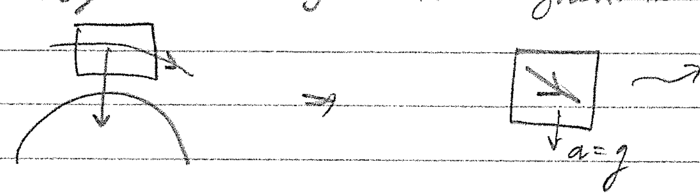
$$\boxed{\text{In GR gravity is not a force. Instead, massive objects curve or warp spacetime around them. Light travels as a free particle along a "geodesic" through curved spacetime.}}$$

Q: how to find equation for geodesic?

Two ways to go

One uses that we know the geodesic eq in an inertial frame $\Rightarrow \dfrac{\partial^2 x^\mu}{\partial \tau^2} = 0$

EP says for an object in a gravitational field...



get straight line in local frame.

The geodesics in the freely falling frame ... with $x^{\mu'}$ coords obey

$$\frac{d^2 x^{\mu'}}{d\tau^2} = 0$$

Coord. transform μ' back to μ. Get

$$\boxed{\frac{\partial^2 x^\mu}{\partial \tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0}$$

geodesic eqn

$\Gamma^\mu_{\nu\sigma}$ = Christoffel symbol or affine connection

Also transform

↳ $ds^2 = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = g_{\mu\nu} dx^\mu dx^\nu$

$g_{\mu\nu}$ (curved space)

We could also find $\Gamma^{\mu}_{\nu\sigma}$ in terms of $g_{\mu\nu}$.

→ But we won't take this route!

Instead, we'll see how to describe curved spaces & spacetimes directly
We'll find the same geodesic equation

$$\frac{d^2X^\mu}{d\tau^2} + \Gamma^{\mu}_{\nu\sigma}\frac{dX^\nu}{d\tau}\frac{dX^\sigma}{d\tau} = 0$$

We'll see how $g_{\mu\nu}$, $\Gamma^{\mu}_{\nu\sigma}$, and the Riemann curvature tensor

$R^{\lambda}_{\mu\nu\sigma}$ are related.

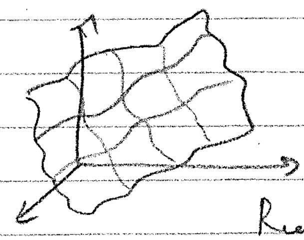
Then we'll look at the Einstein eqn that'll let us solve for $g_{\mu\nu}$ for a given distribution of matter (mass/energy)

#

$\boxed{\text{Curved Spaces}}$

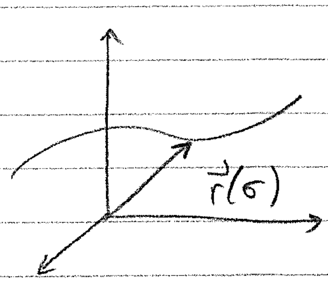
According to GR we live in a curved 4-D spacetime ⇒ hard to visualize. To start off simpler, can look at 2D spaces that we can embed in 3D.

$\boxed{\text{Curved 2D spaces}}$ → can embed in flat 3D spaces.



← can be closed / open
← can't flatten it if it's curved.

Recall that 1D curve thru 3D space is a set of parametrized points $\sigma, t, \ldots$

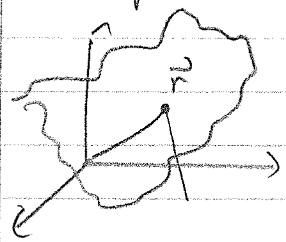


$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad x = x(\sigma)$

$y = y(\sigma)$

$z = z(\sigma)$

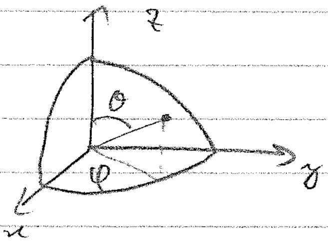
In a similar way, can parameterized 2D surface in 3D space using 2 params. $\rightarrow (u,v)$



$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$x = x(u,v)$
$y = y(u,v)$
$z = z(u,v)$

e.g. Sphere of radius a.



radius $= a$ $\quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

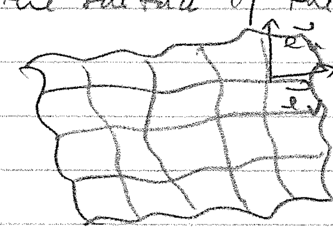
$x = a\sin\theta\cos\varphi$ $\quad (u,v) = (\theta,\varphi)$
$y = a\sin\theta\sin\varphi$
$z = a\cos\theta$

Can also think about staying entirely with in the 2D surface, what knowing about the 3rd direction

In this case $(u,v) \rightarrow$ become coordinates of the curved space.

Note $\rightarrow$ can't put Cartesian words over the surface of the whole space

We can then generate tangent vectors...



With embedding $\vec{e}_u = \frac{d\vec{r}}{du}$ $\quad \vec{e}_v = \frac{d\vec{r}}{dv}$

$\rightarrow$ these are tangent to the surface. They don't lie in the space!
$\Rightarrow$ still give the directions along the curve

$\rightarrow$ vector lives in tangent space T_P at each point P.

Look at a little displacement $ds^2 = d\vec{r}\cdot d\vec{r}$

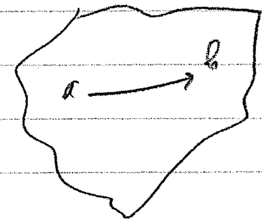
$r = r(u,v) \rightarrow dr = \frac{\partial \vec{r}}{\partial u}\partial u + \frac{\partial \vec{r}}{\partial v}\partial v = \vec{e}_u\, \partial u + \vec{e}_v\, dv$

Call $u^A = (u^1, u^2) = (u,v)$, $A = 1,2$ $\}$ $\boxed{d\vec{r} = \vec{e}_A du^A}$

Then $ds^2 = d\vec{r} \cdot d\vec{r} = (\vec{e}_A du^A) \cdot (\vec{e}_B \cdot du^B) = \vec{e}_A \cdot \vec{e}_B \, du^A du^B$

so $\boxed{ds^2 = g_{AB} \, du^A du^B}$ $\qquad [g_{AB}] \to$ 2×2 matrix in 2D

just as before ~~but~~ in 2D and with a curved space ... Can then calculate the length of the curve in curved 2D space.



Have a line in the surface → must param. the coords

$u = u(\sigma)$, $v = v(\sigma)$) gives the line

Length of curve $L = \int ds$

where $ds^2 = g_{AB} du^A du^B = g_{AB} \frac{du^A(\sigma)}{d\sigma} \frac{du^B(\sigma)}{d\sigma} d\sigma^2$

Call $\dot{u}^A(\sigma) = \frac{du^A(\sigma)}{d\sigma}$

$\Rightarrow \quad ds = \sqrt{g_{AB} \dot{u}^A(\sigma) \dot{u}^B(\sigma)} \, d\sigma$ and so $\boxed{L = \int_a^b \sqrt{g_{AB} \dot{u}^A(\sigma) \dot{u}^B(\sigma)} \, d\sigma}$

This is same as before, but now in curved space.

What about the dual basis $\vec{e}^A$? → not well-defined as $\vec{e}^A = \vec{\nabla} u^A$ as before. Why? with 3 coords in 3D → $\vec{\nabla} u$ is ⊥ to surface, u = constant.

But here u = const is a line → there are many normals to u = const. We can't use the gradient of u^A.

Instead, what we do is first, define $\vec{e}_A$ as tangent vectors along u^A then find $g_{AB} = \vec{e}_A \cdot \vec{e}_B$. Then find g^{AB} (the inverse)

($g_{AB} g^{BC} = \delta_A^C$). Then use g^{AB} to raise index of $\vec{e}_A$

$\boxed{\vec{e}^A = g^{AB} \vec{e}_B}$ → then we'll have both sets ...

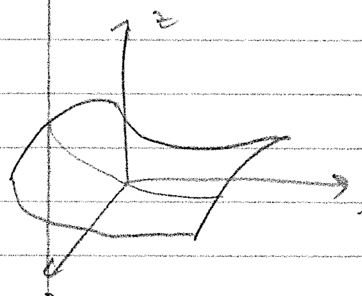
Oct 8, 2018

$\boxed{\text{Curved spaces}}$ $(u, v) \to$ coords $\to u^A$ $A = 1, 2$

$\vec{e}_A$ Tangents and $g_{AB} = \vec{e}_A \cdot \vec{e}_B$

Dual basis $\vec{e}^A = g^{AB} \vec{e}_B$

$\boxed{\text{Ex}}$ $\to$ Consider a saddle embedded in 3D flat space



Use paraboloidal coords with $w =$ constant

$x = u + v$
$y = u - v$
$z = 2uv$

$\vec{r} = (u+v,\ u-v,\ 2uv)$

$\vec{e}_1 = \vec{e}_u = \frac{\partial \vec{r}}{\partial u} = (1, 1, 2v)$. $\vec{e}_v = (1, -1, 2u)$

So $[g_{AB}] = [\vec{e}_A \cdot \vec{e}_B] = \begin{pmatrix} 2+4v^2 & 4uv \\ 4uv & 2+4u^2 \end{pmatrix}$

So $[g^{AB}] = [g_{AB}]^{-1} = \begin{pmatrix} 1+2u^2 & -2uv \\ -2uv & 1+2v^2 \end{pmatrix} \cdot \dfrac{1}{2(1+2u^2+2v^2)}$

So $\vec{e}^A = g^{AB} \vec{e}_B = \ ?$ (See p. 33 in book) (but easy to compute)

{ Ultimately, we won't use basis vects much going forward. The important info is contained in metric }

Ex $\boxed{ds^2 = g_{AB}\, du^A du^B}$

knowing this enough!

E.g. flat 2D space $g_{AB} = \delta^A_B \to \boxed{ds^2 = dx^2 + dy^2}$

In GR, we'll use the Einstein eqn to find $g_{\mu\nu}$

Oct 9, 2018 $\boxed{\text{Manifolds}}$ $\rightarrow$ $\boxed{\text{An arbitrary curved N-D space is called a manifold}}$

Assume we know the metric. Can write coords

$$X^a = (X^1, X^2, \ldots X^N)$$

with more than one coord system. We assume differentiable functions

$$X^{a'} = X^{a'}(X^b), \text{ and that these are invertible}$$
$$\Rightarrow X^a = X^a(X^{b'})$$

Call M a differentiable manifold with defined Jacobian

$$\left.\begin{array}{l} X^{a'}_b = \dfrac{\partial X^{a'}}{\partial X^b} \\ X^a_{b'} = \dfrac{\partial X^a}{\partial X^{b'}} \end{array}\right\} \Rightarrow \begin{array}{l} X^{a'}_b X^b_{c'} = \delta^a_c \\ X^a_{b'} X^{b'}_c = \delta^a_c \end{array}$$

We've seen flat Euclidean space $\rightarrow \left\{ \begin{array}{l} v^a \leftarrow X^a \\ O^{i'}_j \leftarrow X^{a'}_b \end{array}\right.$

and flat 4D spacetime $\rightarrow \left\{ \begin{array}{l} X^\mu \leftarrow X^a \\ \Lambda^{\mu'}_\nu \leftarrow X^{a'}_b \end{array}\right.$

We define vectors, tensors, scalars by how they transform.

$$\left\{ \begin{array}{l} \lambda^{a'} = X^{a'}_b \lambda^b \rightarrow \text{contravariant vector} \\ \mu_{a'} = X^b_{a'} \mu_b \rightarrow \text{covariant vector} \\ \tau^{a'b'}{}_{c'd'} = X^{a'}_e X^{b'}_f X^g_{c'} X^h_{d'} \tau^{ef}{}_{gh} \leftarrow \text{tensor} \\ \text{Metric lowers/raises} \quad \lambda_a = g_{ab} \lambda^b + \text{has an inverse} \\ \qquad g^{ab} g_{bc} = \delta^a_c \end{array} \right\}$$

In general, the metric need not be positive definite

$$ds^2 = g_{ab}dx^a dx^b \rightarrow \text{can be } (+, 0, -)$$

Signature of g_{ab} = (# positive) − (# negative) down the diagonal

↳ $\eta_{\mu\nu}$ has signature -2. $(\text{sig}(g_{ab}) = 1-3 = -2)$

Note $\boxed{\text{All metrics in GR have signature} = -2}$ (local SR)

Two classes of manifold: Riemannian manifolds (positive def. metric)

pseudo-Riemannian manifold
↳ can have neg inner products

Note Spacetime ⇒ pseudo Riemannian manifold

Recall There are 4 ways to compute inner products

$$\lambda \cdot \mu = \lambda^i \mu_i = \lambda_i \mu^i = g_{ij} \lambda^i \mu^j = g^{ij} \lambda_i \mu_j$$

These are scalars under general coord. transforms.

$$\lambda \cdot \mu = \lambda^a \mu_a = \lambda^{a'} \mu_{a'}$$

To define lengths + distances as real numbers, need abs. values

Distance $\quad ds = \sqrt{|g_{ab}\, dx^a dx^b|}$

Length of curve $\quad L = \int_a^b ds = \int_a^b \sqrt{|g_{ab}\, dx^a dx^b|}$

Length of vector

$$|\lambda| = \sqrt{|\lambda^a \lambda_a|} \rightarrow \text{can still be null}$$

For non-null vectors, we can define "angle" between them

$$\cos\theta = \frac{\lambda \cdot \mu}{|\lambda||\mu|}$$ → have to be non null to avoid div. by 0

$$= \frac{g_{ab}\lambda^a \mu^b}{|\lambda||\mu|}$$

→ Works well for positive def. metrics. But become weird for spacetime.

Ex Spacelike λ → $\theta = 180°$ between it and itself

Can also get $\cos\theta > 1$ → don't make sense

$\boxed{\text{Call vectors obeying}}$ → $\lambda \cdot \mu = 0$ orthogonal

↳ there exists a frame where they're perpendicular

$\boxed{\text{Combining Tensors}}$ Given that $\lambda^a, \mu_b, \tau^{ab}{}_c$ are tensors.

We can show → $\boxed{\text{adding tensors of the same type gives a tensor}}$

Ex $\xi^{ab}{}_c = \tau^{ab}{}_c + \sigma^{ab}{}_c$ is a tensor if τ and σ are tensors

Proof $\xi^{a'b'}{}_{c'} = \tau^{a'b'}{}_{c'} + \sigma^{a'b'}{}_{c'}$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \tau^{de}{}_f + X^{a'}_d X^{b'}_e X^f_{c'} \sigma^{de}{}_f$$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \left(\tau^{de}{}_f + \sigma^{de}{}_f\right)$$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \xi^{de}{}_f$$

$\Rightarrow \xi^{ab}{}_c$ is a tensor

$\boxed{\text{Multiplying a tensor by a scalar gives a tensor}}$

↳ Suppose $\sigma^a{}_b = \alpha \tau^a{}_b$

Proof $\sigma^{a'}{}_{b'} = \alpha \tau^{a'}{}_{b'} = \alpha X^{a'}_c X^d_{b'} \tau^c{}_d$

$$= X^{a'}_c X^d_{b'} \alpha \tau^c{}_d = X^{a'}_c X^d_{b'} \sigma^c{}_d$$

So $\sigma^a{}_b$ is a tensor.

$\boxed{\text{Multiplying tensors gives tensors}}$

Suppose $\sigma^{ab}{}_c = \lambda^a \tau^b{}_c$

Proof $\sigma^{a'b'}{}_{c'} = \lambda^{a'} \tau^{b'}{}_{c'} = \left(X^{a'}_d \lambda^d\right) X^{b'}_e X^f_{c'} \tau^e{}_f$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \lambda^d \tau^e{}_f$$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \sigma^{de}{}_f \quad \text{So } \sigma^{ab}{}_c \text{ tensor}$$

$\boxed{\text{Contracting a tensor of type } (r,s) \text{ gives a tensor of type } (r-1, s-1)}$

Suppose $\tau^{ab}{}_{cd}$ is a $(2,2)$ tensor

Call $\sigma^a{}_b = \tau^{ac}{}_{cb}$ is this a one-one $(1,1)$ tensor?

Proof $\sigma^{a'}{}_{b'} = \tau^{a'c'}{}_{c'b'} = X^{a'}_d X^{c'}_e X^f_{c'} X^g_{b'} \tau^{de}{}_{fg}$

$$= X^{a'}_d X^g_{b'} \tau^{de}{}_{fg} \delta^f_e$$

$$= X^{a'}_d X^g_{b'} \tau^{de}{}_{eg}$$

$$\sigma^{a'}{}_{b'} = X^{a'}_d X^g_{b'} \sigma^d{}_g \quad \text{So } \sigma^a{}_b = \tau^{ac}{}_{cb} \text{ is a } (1,1) \text{ tensor.}$$

We've used this already! $\lambda_a = g_{ab}\lambda^b \rightarrow$ gives a vector

So, as a consequence, $\boxed{\sigma^{ab}{}_c = \tau^{abe}\mu_e g_{cf}\lambda^f \text{ is a tensor}}$

Oct 10, 2018

Recall Combining tensors $\rightarrow$ adding, multiplying & contracting tensors gives new tensors

e.g. $\tau^{ab}{}_c \lambda^c \mu_b$ = type (1,0) (vector)

$\boxed{\text{Dividing : Quotient theorem}}$

$\boxed{\text{Suppose } \tau^a{}_{bc}\lambda^c \text{ transforms as a tensor } \forall \lambda^c. \text{ Then the quotient theorem says } \tau^a{}_{bc} \text{ is a tensor}}$

Proof $\tau^{a'}{}_{b'c'}\lambda^{c'} = X^{a'}_d X^e_{b'} \tau^d{}_{ef}\lambda^f$

We also know $\lambda^{c'} = X^{c'}_f \lambda^f$

So $\tau^{a'}{}_{b'c'} X^{c'}_f \lambda^f - X^{a'}_d X^e_{b'} \tau^d{}_{ef}\lambda^f = 0$ (true $\forall \lambda^f$)

So $\tau^{a'}{}_{b'c'} X^{c'}_f - X^{a'}_d X^e_{b'} \tau^d{}_{ef} = 0$

So $\tau^{a'}{}_{b'c'} X^{c'}_f = X^{a'}_d X^e_{b'} \tau^d{}_{ef}$

So $\tau^{a'}{}_{b'c'}\delta^{c'}_{g'} = X^{a'}_d X^e_{b'} X^f_{g'} \tau^d{}_{ef}$

So $\boxed{\tau^{a'}{}_{b'g'} = X^{a'}_d X^e_{b'} X^f_{g'} \tau^d{}_{ef}} \rightarrow \tau^a{}_{bc}$ tensor

$\boxed{\text{Special Tensors}}$ $\boxed{\text{Symmetric}}$ if $\boxed{\tau^{ab} = \tau^{ba}}$ (metric)

This is then true $\forall$ coord frames.

1 $\Rightarrow \tau^{a'b'} = \tau^{b'a'}$

(will show this in 1.8.2)

(2) $\boxed{\text{Anti symmetric tensors}}$. $\boxed{T^{ab} = -T^{ba}}$

→ also true for all frames

(3) $\boxed{\text{kronecker delta}}$ → coord. independent

$$\delta^a_b = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a\neq b \end{cases} \quad \text{(type (1,1) tensor)}$$

$$\delta^{a'}_{b'} = X^{a'}_e X^f_{b'} \delta^e_f = \underbrace{X^{a'}_e X^e_{b'}}_{} = \delta^{a'}_{b'}$$

because $X^{a'}_c X^c_{b'} = \frac{\partial x^{a'}}{\partial x^c}\frac{\partial x^c}{\partial x^{b'}} \xrightarrow{\text{inverses}} = \frac{\partial x^{a'}}{\partial x^{b'}} = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a\neq b \end{cases}$

(4) $\boxed{\text{For most tensors the order of indices matter}}$

<u>Ex</u> $T^a{}_b{}^c = g_{bd} T^{adc}$

<u>But</u> $T^a{}_b{}^c \neq g_{bd} T^{acd} = T^{ac}{}_b$

Don't write T^{ac}_b unless we know order don't matter

VI. GRAVITATION & CURVATURE

In GR gravity is not a force → mass & energy cause spacetime to be curved.

"Free particles" ≡ moving with no forces (other than gravity)
↳ follow <u>geodesics</u>

$\boxed{\text{We need to understand}}$
→ curvature (how to tell a space is curved?)
→ geodesic (what is the eqn for geodesic?)
→ motion in curved space: how do vectors behave? (parallel transport)
→ laws of physics e.g. $f^\mu = \frac{dp^\mu}{d\tau}$ in curved spacetime.

↑ absolute. covariant derivatives

(Chap 2. Assuming we know the metric)

(Chap 3. Solve for metric)

<u>Newtonian limit</u>

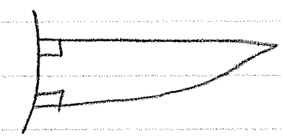
$F = \frac{GMm}{r^2}$

→ limit back to gravity as a force?

CURVATURE Imagine ants on globe. How can they tell it's a curved space? How do the ants walk "straight".

⇒ Left step must = right step to walk straight (without turning).

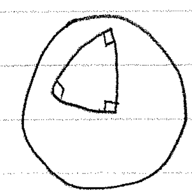
⇒ Start 2 ants walking parallel & straight



(1) Parallel lines cross ⇒ space is non Euclidean.
(2) These "straight" lines are geodesic.

On a sphere, the equator, longitudes, and great circles are all geodesics and hence "straight lines". Latitude lines are not geodesics

Another test is make a triangle of 3 straight lines



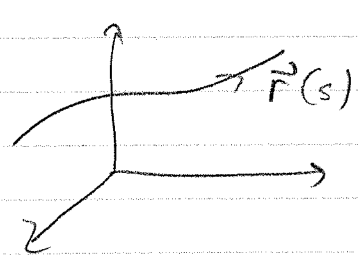
Sum of the angles = 270°, not 180°.
→ says space is curved.
→ Bugs can tell if a space is curved!

Geodesic equation Suppose we're in space or spacetime and we know what the metric is. How do we find a geodesic? → Follow a "straight" line!

Flat 3D space In Cartesian coord, a straight line obeys $\frac{\partial^2 \vec{r}}{\partial t^2} = \vec{0}$ (no forces)

Suppose we use curvilinear coords.

↳ what is the eqn of a straight line? → arc length param



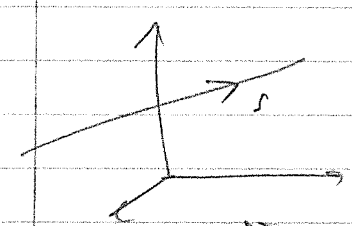
s = arc length as parameter

↳ $|\dot{\vec{r}}(s)| = \left|\frac{d\vec{r}}{ds}\right| = 1$ fixed length

let $\vec{T} = \frac{d\vec{r}}{ds}$ (tangent)

$\vec{\lambda} = \frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial u^i}\frac{\partial u^i}{\partial s} = \frac{du^i}{ds}\cdot\vec{e}_i = \lambda^i \vec{e}_i$ so $\boxed{\lambda^i = \frac{\partial u^i}{\partial s} = \dot{u}^i(s)}$

components of tangent vector in curvilinear coords



Since $\vec{\lambda}$ tangent, its direction does not change along a straight line. Also $|\vec{\lambda}(s)| = 1$

→ $\boxed{\vec{\lambda} \text{ has both fixed direction \& magnitude along straight line}}$

↳ "straightness" → derivative of tangent vector w.r.t arclength $= 0$

$\frac{d\vec{\lambda}}{ds} = 0$ → Tangent vector does not change (constant along a straight line)

Oct 17 2018

<u>Geodesics</u> → Path followed by a free particle → straight line in flat 3D space, obeys $\frac{\partial^2 x}{\partial t^2} = 0$. What about in curvilinear coords?

Use s as parameter $\vec{\lambda} = \frac{d\vec{r}}{ds}$ → tangent vector (fixed magnitude)

<u>Condition of straightness</u>: $\frac{d\vec{\lambda}}{ds} = 0 \Rightarrow \boxed{\frac{d}{ds}(\lambda^i \vec{e}_i) = 0}$

$\lambda^i = \frac{\partial u^i}{\partial s}$

So $\boxed{\dot{\lambda}^i \vec{e}_i + \lambda^i \dot{\vec{e}}_i = 0}$

In Cartesian $\{\vec{e}_i\} = \hat{i}, \hat{j}, \hat{k}$ constant → $\dot{\vec{e}}_i = 0$ <u>Get</u> $\dot{\lambda}^i = 0$ for straight line

But since $\lambda^i = \dot{u}^i = \dot{x}^i \Rightarrow \boxed{\frac{\partial^2 x^i}{\partial s^2} = 0 \text{ for a straight line in Cartesian coords}}$

<u>Note</u> $\frac{\partial^2 x^i}{\partial s^2} = 0 = \frac{\partial^2 x^i}{\partial t^2}$ as long as $s \propto t$, but <u>NOT</u> equivalent if $s \not\propto t$ → has acceleration.

<u>But</u> if coords are not Cartesian → $\frac{d}{ds}(\lambda^i \vec{e}_i)$ has <u>2 terms</u>!

$$\dot{\lambda}^i \breve{e}_i + \lambda^i \dot{\breve{e}}_i = 0 \iff \boxed{\frac{\partial \lambda^i}{\partial s} \breve{e}_i + \lambda^i \frac{\partial \breve{e}_i}{\partial s} = 0}$$

<u>where</u> $\frac{\partial \breve{e}_i}{\partial s} = \frac{\partial \breve{e}_i}{\partial u^j} \frac{\partial u^j}{\partial s} \neq 0$ in general

<u>Use</u> $\frac{\partial}{\partial u^j} = \partial_j \Rightarrow \boxed{\frac{\partial \breve{e}_i}{\partial s} = \left(\partial_j \breve{e}_i\right) \dot{u}^j}$

The derivative

↳ $\partial_j \breve{e}_i$ are vectors. We can expand them in terms of basis set

<u>Call</u> $\boxed{\partial_j \breve{e}_i = \Gamma^k_{ij} \breve{e}_k}$ → k^{th} component of the u^jth derivative of $\breve{e}_i$, called "affine connection" or "christoffel symbol"

<u>Note</u> $\boxed{\Gamma^k_{ij} \text{ is } \underline{\text{not}} \text{ a tensor} \rightarrow \text{they're a connection}}$

<u>With this</u> $\dot{\breve{e}}_i = (\partial_j \breve{e}_i) \dot{u}^j = \Gamma^k_{ij} \breve{e}_k \dot{u}^j$

<u>So, straightness condition is</u>

$k \to i$
$i \to j$
$j \to k$

$$\frac{d\vec{\lambda}}{ds} = \dot{\lambda}^i \breve{e}_i + \lambda^i \Gamma^k_{ij} \breve{e}_k \dot{u}^j = 0$$

$$= \dot{\lambda}^i \breve{e}_i + \lambda^j \Gamma^i_{jk} \breve{e}_i \dot{u}^k = 0$$

$$= \left(\dot{\lambda}^i + \Gamma^i_{jk} \lambda^j \dot{u}^k\right) \breve{e}_i = 0$$

or $\dot{\lambda}^i + \Gamma^i_{jk} \lambda^j \dot{u}^k = 0$ <u>But</u> $\lambda^i = \dot{u}^i = \frac{du^i}{ds}$

$\dot{\lambda}^i = \frac{d^2 u^i}{ds^2}$

$$\Rightarrow \boxed{\frac{d^2 u^i}{ds^2} + \Gamma^i_{jk} \frac{du^j}{ds} \frac{du^k}{ds} = 0}$$

→ gives the eq. of a straight line in flat 3D space.

<u>Note</u> Γ^k_{ij} has $3 \times 3 \times 3 = 27$ coefficients. → Want simpler relation!

<u>Note</u> $\partial_j \breve{e}_i = \Gamma^k_{ij} \breve{e}_k \Rightarrow$ dot with $\breve{e}^l$

So $(\partial_j \vec{e}_i)\vec{e}^{\,l} = \Gamma^k_{ij}\,\vec{e}_k\cdot\vec{e}^{\,l} = \Gamma^k_{ij}\,\delta^l_k$

So $\boxed{\vec{e}^{\,l}(\partial_j \vec{e}_i) = \Gamma^l_{ij}}$

But, note $\partial_j \vec{e}_i = \frac{\partial}{\partial u^j}\frac{\partial \vec{r}}{\partial u^i} = \frac{\partial}{\partial u^i}\frac{\partial \vec{r}}{\partial u^j} = \partial_i \vec{e}_j$

So $\boxed{\Gamma^l_{ij} = \Gamma^l_{ji}}$ (symmetric) $\rightarrow$ 18 independent comps

Next, want to find relation for connection in terms of the metric.

<u>Consider</u> $\partial_k g_{ij} = \partial_k(\vec{e}_i\cdot\vec{e}_j) = \vec{e}_j\,\partial_k\vec{e}_i + \vec{e}_i\,\partial_k\vec{e}_j$

$= \vec{e}_j\,\Gamma^m_{ik}\,\vec{e}_m + \vec{e}_i\,\Gamma^m_{kj}\,\vec{e}_m$

So $\boxed{\partial_k g_{ij} = \Gamma^m_{ik}\,g_{jm} + \Gamma^m_{jk}\,g_{im}}$

Use same tricks to get Γ^j_{ik}...

Let $k\to i,\ i\to j,\ j\to k \Rightarrow$
$$\boxed{\begin{aligned}\partial_i g_{jk} &= \Gamma^m_{ji}\,g_{mk} + \Gamma^m_{ki}\,g_{jm}\\ \partial_j g_{ik} &= \Gamma^m_{kj}\,g_{im} + \Gamma^m_{ij}\,g_{km}\end{aligned}}$$

So Add first two eqn, subtract 3rd

$\hookrightarrow \boxed{\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik} = 2\Gamma^m_{ik}\,g_{jm}}$

<u>Note</u> $g_{jm} = g_{mj}$ (symmetric)

$\hookrightarrow$ Next multiply by $g^{jl} \rightarrow g_{jm}g^{jl} = \delta^l_m$

So $\boxed{\Gamma^l_{ik} = \frac{1}{2}g^{jl}\left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}\right)}$

Let $l\to k$
$k\to i$
$i\to j$
$j\to l$

So $\boxed{\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ji}\right)}$

Note in Cartesian coords, $g_{ij} = \delta^i_j \rightarrow \partial_k g_{ij} = 0 \therefore \Gamma^k_{ij} = 0$

Note $\Gamma^k_{ij} \neq 0$ does not mean space is curved!

→ In fact, get $\Gamma^k_{ij} \neq 0$ in curvilinear coords in flat space whenever $\hat{e}_i$ are not constant.

How do we calculate Γ^k_{ij}? ⟹ By brute force ... (won't use book's short cut)

e.g $\Gamma^1_{23} = \Gamma^1_{32}$

$$= \frac{1}{2} g^{11} \left(\partial_2 g_{31} + \partial_3 g_{21} - \partial_1 g_{23} \right)$$
$$+ \frac{1}{2} g^{12} \left(\partial_2 g_{32} + \partial_3 g_{22} - \partial_2 g_{23} \right)$$
$$+ \frac{1}{2} g^{13} \left(\partial_2 g_{33} + \partial_3 g_{23} - \partial_3 g_{23} \right)$$

then repeat for remaining 25 cases ...

Now $\boxed{\dfrac{d^2 u^i}{ds^2} + \Gamma^i_{jk} \dfrac{du^j}{ds} \dfrac{du^k}{ds} = 0} \rightarrow 3$ eqns

↳ solution gives eqn of straightline (geodesics) curve u^i in flat space
But the same eqn carry into curved space!

Oct 19, 2018

Affine parameters We used arclength as a parameter in finding geodesic eqn

$\boxed{\dfrac{d^2 u^i}{ds^2} + \Gamma^i_{jk} \dfrac{du^j}{ds} \dfrac{du^k}{ds} = 0}$. What if we use a different parameters $t = f($

→ Modified eqn $\boxed{\dfrac{d^2 u^i}{dt^2} + \Gamma^i_{jk} \dfrac{du^j}{dt} \dfrac{du^k}{dt} = -\left(\dfrac{d^2 t}{ds^2}\right)\left(\dfrac{dt}{ds}\right)^{-2} \dfrac{du^i}{dt}}$

↳ this is different from the original unless the second derivative $\dfrac{d^2 t}{ds^2} = 0$, i.e,

$$t = As + B \quad (A, B \text{ constant}, A \neq 0)$$

→ A parameter of this form is called an <u>affine parameter</u>.
→ <u>key</u> t linearly related to s.

$$\frac{ds}{dt} = \frac{1}{A} = A^{-1} \neq 0 \quad \text{says } s \sim t \Rightarrow \text{no acceleration}$$

So we'll use affine parameters for geodesics in which case the eqn is

flat space! → $$\boxed{\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = -\left(\frac{d^2t}{ds^2}\right)\left(\frac{dt}{ds}\right)^{-2}\frac{du^i}{dt} = 0}$$

<u>Geodesics in Curved Spaces</u>

We've seen correspondences between flat 3D space in curvilinear coords + curved N-dim manifolds..

$\lambda^{i'} = U^{i'}_j \lambda^j$. $u^j \to x^a$. $ds^2 = g_{ij}\,du^i du^j$

$\lambda^{a'} = X^{a'}_b \lambda^b$. $g_{ij} \to g_{ab}$ $= g_{ab}\,du^a du^b$

. $U^{i'}_j \to X^{a'}_b$

| Same is true for geodesic eqn | → <u>Similar form</u>

geodesic eqn in curved space → $$\boxed{\frac{d^2x^a}{d\sigma^2} + \Gamma^a_{bc}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma} = 0}$$

<u>Note</u> σ is <u>an affine param</u>, ie, $\sigma \sim s$

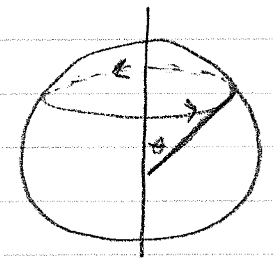
where the connection $$\boxed{\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)}$$

<u>where</u> $$\boxed{\Gamma^a_{bc} = \Gamma^a_{cb} = \vec{e}^{\,a}\left(\partial_c \vec{e}_b\right) = \vec{e}^{\,a}\left(\partial_b \vec{e}_c\right)}$$

Could show this holds in GR as a result of the EP
What we'll do is show that this gives the correct geodesic on a 2-sphere

<u>Ex</u> Determine if lines of constant latitude of a 2-sphere of radius a are geodesics

<u>Know</u> only Equator is!



Do these curves satisfy

$$\frac{d^2u^A}{d\sigma^2} + \Gamma^A_{BC}\frac{du^B}{d\sigma}\frac{du^C}{d\sigma} = 0 \, ?$$ (assume σ is an affine param)

<u>where</u> $\Gamma^A_{BC} = \frac{1}{2} g^{AD}\left(\partial_B g_{CD} + \partial_C g_{BD} - \partial_D g_{BC}\right)$

<u>Here</u> $u^A = (u^1, u^2)$. Use $u^A = (\theta, \varphi)$ $\quad A, B = 1, 2$

radius $= a$

The metric tensor of 2-sphere of radius a is

$$[g_{AB}] = \begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix}$$ (showed in 1.6.2)

<u>So</u> $$[g^{AB}] = \begin{pmatrix} a^{-2} & 0 \\ 0 & a^{-2}\sin^{-2}\theta \end{pmatrix}$$

<u>Connection</u> $\Gamma^A_{BC} = \frac{1}{2} g^{AD}\left(\partial_B g_{CD} + \partial_C g_{BD} - \partial_D g_{BC}\right)$

There are 8 of these → But can use symmetry ...

Will show (2.1.5) that <u>answer</u>
$$\begin{cases} \Gamma^1_{22} = -\sin\theta\cos\theta \\ \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta \\ \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0 \end{cases}$$

Look at $\Gamma^1_{12} = \Gamma^1_{21} \rightarrow A=1, B=1, C=2$

$$\rightarrow \Gamma^1_{12} = \frac{1}{2} g^{1D}\left(\partial_1 g_{2D} + \partial_2 g_{1D} - \partial_D g_{12}\right)$$

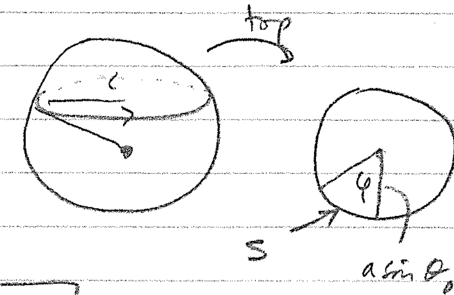
$$= \frac{1}{2} g^{11}\left(\partial_1 g_{21} + \partial_2 g_{11} - \partial_1 g_{12}\right) + \frac{1}{2} g^{12}\left(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}\right)$$

Note $[g_{AB}]$ is diagonal

$\rightarrow \quad \Gamma^1_{12} = \frac{1}{2} g^{11}\left(\partial_2 g_{11}\right) = \frac{1}{2} g^{11} \partial_\varphi g_{\theta\theta} = \frac{1}{2} g^{11} \partial_\varphi (a^2) = 0$

Next

Find affine param of latitude line



Coords $u^A = (u^1, u^2) = (\theta, \varphi)$, with $\theta = \theta_0$

need param in terms of s with $\boxed{s = \varphi (a \sin\theta_0)}$

or $\quad \varphi = s (a \sin\theta_0)^{-1} = As \quad$ so φ is an affine param!

Have $\quad u^A(s) = \left(\theta_0, \; s (a\sin\theta_0)^{-1}\right) \quad$ use s as param

Need $\quad \frac{du^A}{ds} = \left(0, \; (a\sin\theta_0)^{-1}\right) \quad$ and $\quad \frac{d^2u^A}{ds^2} = (0, 0)$

Now, check with geodesic eqn

2 eqns $\quad \rightarrow \quad \frac{d^2u^A}{ds} + \Gamma^A_{BC} \frac{du^B}{ds}\frac{du^C}{ds} =$

(A = 1)

$\bullet \; \frac{d^2u^1}{ds} + \Gamma^1_{BC}\frac{du^B}{ds}\frac{du^C}{ds} = 0 + (-\sin\theta_0 \cos\theta_0) \frac{du^2}{ds}\frac{du^2}{ds} \overset{?}{=} 0$

Use $\Gamma^1_{22} = -\sin\theta\cos\theta$, $\Gamma^2_{12} = \cot\theta = \Gamma^2_{21}$ $\quad \therefore \boxed{(-\sin\theta_0\cos\theta_0)(a\sin\theta_0)^{-2} \overset{?}{=} 0}$

only true if $\theta_0 = \frac{\pi}{2}$

$\rightarrow$ Only Equator works!

(A = 2) $\quad \bullet \; \frac{d^2u^2}{ds^2} + \Gamma^2_{BC}\frac{du^B}{ds}\frac{du^C}{ds} = 0 + \Gamma^2_{12}\frac{du^1}{ds}\frac{du^2}{ds} + \Gamma^2_{21}\frac{du^2}{ds}\frac{du^1}{ds}$

$= \cot\theta \left[0 + 0\right] = 0$ so this is satisfied

$\boxed{\text{only latitude line that is also a geodesic is the Equator}}$ $\rightarrow$ for sphere $\rightarrow$ geodesics = circles ~~with~~ with center

$\boxed{\text{Parallel Transport}}$ Our condition for geodesic was that the tangent vector $\tilde{\lambda} = \lambda^i \tilde{e}_i = \lambda_i \tilde{e}^i = \dot{u}^i \tilde{e}_i$ does not change as we move along the curve...

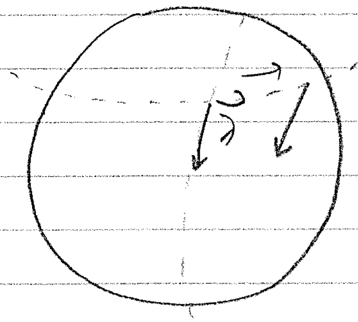
$$\frac{d\tilde{\lambda}}{ds} = 0 \quad \text{(condition of straightness)}$$

This leads to Geodesic eqn $\boxed{\ddot{u}^i + \Gamma^i_{jk} \dot{u}^j \dot{u}^k = 0}$

We can generalize this. Consider $\tilde{\lambda} = \lambda^i \tilde{e}_i$ that's an arbitrary vect. Want to transport $\tilde{\lambda}$ along a curve parametrized by t without altering it, $u^i(t)$. $\lambda = \lambda^i \tilde{e}_i$

Condition: $\dfrac{d\tilde{\lambda}}{dt} = 0$ (t = affine param) called parallel transport

Oct 22, 2018 In flat space, the vector does not change its direction. But in curved space, a vector that is parallel transported can change direction.



← effect of curvature. Note along the equator the direction does not change → holds for any geodesic!

We can derive the math of parallel transport

$$\frac{d\tilde{\lambda}}{dt} = 0 \quad \text{with } \tilde{\lambda} = \lambda^i \tilde{e}_i$$

$\Rightarrow \dot{\lambda}^i \tilde{e}_i + \lambda^i \dot{\tilde{e}}_i = 0$. We also know $\dot{\tilde{e}}_i = (\partial_j \tilde{e}_i)\dot{u}^j = \Gamma^k_{ij} \dot{u}^j \tilde{e}_k$

$\rightarrow \dot{\lambda}^i \tilde{e}_i + \lambda^i \Gamma^k_{ji} \dot{u}^j \tilde{e}_k = 0$ let $k \to i$, $i \to j$, $j \to k$

$$\Rightarrow \boxed{\dot{\lambda}^i + \lambda^j \Gamma^i_{kj} \dot{u}^k = 0}$$

↳ This says how the components λ^i change when the vector is parallel transported along the curve parametrized by t.

Ex If $\lambda^i = \dot{u}^i$ (tangent vector to curve)

$$\boxed{\ddot{u}^i + \Gamma^i_{kj}\dot{u}^j\dot{u}^k = 0} \rightarrow \text{geodesic eqn}$$

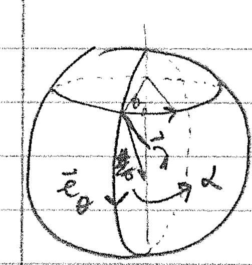
This says that to parallel transport tangent vectors, the curve must be a geodesic (so that it remains a tangent vector)

To go to an N-dim curve manifold, we can just change notation

$$\boxed{\dot{\lambda}^a + \Gamma^a_{bc}\lambda^b\dot{x}^c = 0}$$ (most general case) ($t \equiv As + B$ is affine param)

$\dot{\lambda}^a = \frac{d\lambda^a}{dt}$ and so on...

$\boxed{\text{Example}}$ Consider unit vector $\vec{\lambda}$ on surface of sphere of radius a which makes an angle α wrt. a longitude.



Show that parallel transport along line of constant latitude, the direction of $\vec{\lambda}$ changes by ~~an angle~~ angle $\chi = 2\pi\omega$
where $\omega = \cos\theta_0$ & θ_0 = polar angle of the latitude.

$\frac{s}{a\sin\theta_0}$ as

First, param the curve (2D)

$$u^A = (u^1, u^2) = (\theta, \varphi)$$

Here $\theta = \theta_0$ is fixed $\rightarrow u^A = (\theta_0, \varphi)$. Can let φ run from $0 \rightarrow 2\pi$
$\rightarrow \varphi = t$

$\Rightarrow u^A(t) = (\theta_0, t)$. Note: this is a <u>different param than before</u>

But before, $u^A(s) = (\theta_0, (a\sin\theta_0)^{-1}s)$
$= (\theta_0, \varphi)$

Here, $t = \varphi = \underbrace{(a\sin\theta_0)^{-1}}_{A}s$. And so <u>$t$ is affine</u> ($a\sin\theta_0$ <u>constant</u>)

Let $\vec{\lambda}(0)$ be initial vector ($t=0$) <u>and</u> χ = angle between these
$\vec{\lambda}(2\pi)$ be final vector ($t = 2\pi$) 2 vectors!

Next, want to find initial unit vector $\tilde{\lambda}(0)$ making an angle α w.r.t to latitude.

Claim
$$\boxed{\begin{aligned}\lambda^A(0) &= (\lambda^1(0), \lambda^2(0)) \\ &= \left(a^{-1}\cos\alpha, (a\sin\theta_0)^{-1}\sin\alpha\right)\end{aligned}}$$
is that initial vector

Verify it's correct

is this a unit vector? $\lambda^A(0)\lambda^B(0) \overset{?}{=} 1$

here $[g_{AB}] = \begin{pmatrix} a^2 & 0 \\ 0 & (a\sin\theta_0)^2 \end{pmatrix}$

$$[\lambda^A(0) g_{AB} \lambda^B(0)]$$
$$= \left(a^{-1}\cos\alpha, (a\sin\theta_0)^{-1}\sin\alpha\right)\begin{pmatrix} a^2 & 0 \\ 0 & (a\sin\theta_0)^2 \end{pmatrix}\begin{pmatrix} a^{-1}\cos\alpha \\ (a\sin\theta_0)^{-1}\sin\alpha \end{pmatrix}$$
$$= \cos^2\alpha + \sin^2\alpha = \boxed{1} \rightarrow \underline{\text{unit vector}}$$

Next, does it make angle α w.r.t longitude?

$$\text{Longitude} = \underline{(1)\vec{e}_\theta + (0)\vec{e}_\varphi}$$

Call

$$\mu^A_{long} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ (vector that points along longitude)

Check $[\mu^A_{long}\mu^A_{long}] = [\mu^A_{long} g_{AB} \mu^B_{long}] = \boxed{a^2}$ (not unit vector)

Next, find $$\cos(\alpha) = \frac{g_{AB}\mu^A_{long}\lambda^B(0)}{|\mu^A_{long}||\lambda^B_{(0)}|} = \frac{g_{11}\mu^1\lambda^1}{(a)(1)} = \frac{a^2(1)\, a^{-1}\cos\alpha}{a\cdot 1} = \boxed{\cos\alpha}$$

$\therefore$ $\tilde{\lambda}(0)$ is at angle α w.r.t a longitude!

Next, parallel transport $\tilde{\lambda}$ around the latitude line
$\Rightarrow$ want new components. Need to solve parallel transport eqn:

Need to solve $\dot{\lambda}^A + \Gamma^A_{BC}\lambda^B \dot{u}^C = 0$ (2 eqns)

Initial values $\vec{\lambda}(0) = \begin{pmatrix} a^{-1}\cos\alpha \\ (a\sin\theta_0)^{-1}\sin\alpha \end{pmatrix}$

Can use $\left\{ \begin{array}{l} \Gamma^1_{22} = -\sin\theta_0\cos\theta_0 \\ \Gamma^2_{21} = \Gamma^2_{12} = \cot\theta_0 \end{array} \right\}$ **and** $u^A(t) = (\theta_0, t)$

Since $u^A(t) = (\theta_0, t) \rightarrow \dot{u}^C = (0, 1)$

$\boxed{A=1}$ $\dot{\lambda}^1 + \Gamma^1_{22}\lambda^2\dot{u}^2 = 0 \Leftrightarrow$?

$\boxed{A=2}$ $\dot{\lambda}^2 + \Gamma^2_{12}\lambda^1\dot{u}^1 = 0 \Leftrightarrow$?
$+ \Gamma^2_{21}\lambda^2\dot{u}^2$

Will verify that the solution satisfying IVP is: (Exercise 2.2.1)

$$\boxed{\begin{array}{c} \lambda^A(t) = (\lambda^1(t), \lambda^2(t)) = \left(a^{-1}\cos(\alpha - \omega t),\ (a\sin\theta_0)^{-1}\sin(\alpha - \omega t)\right) \\ \text{with } \omega = \cos\theta_0 \quad \forall t \end{array}}$$

Oct 23, 2018 Next, go all the way around, to $t = 2\pi$

$$\Rightarrow \lambda^A(2\pi) = \left(a^{-1}\cos(\alpha - 2\pi\omega),\ (a\sin\theta_0)^{-1}\sin(\alpha - 2\pi\omega)\right)$$

Is this still a unit vector?

$$|\lambda^A(2\pi)|^2 = g_{AB}\lambda^A(2\pi)\lambda^B(2\pi) = a^2a^{-2}\cos^2(\alpha - 2\pi\omega) + (a\sin\theta_0)^2(a\sin\theta_0)^{-2}\sin^2(\alpha - 2\pi\omega)$$
$$= 1 \Rightarrow \text{still unit normal.}$$

Now, what's the angle χ between $\vec{\lambda}(0)$ & $\vec{\lambda}(2\pi)$

$$\cos\chi = \frac{\lambda^A(0)\cdot\lambda^B(2\pi)\cdot g_{AB}}{|\lambda^A(0)|\,|\lambda^B(2\pi)|} = g_{AB}\lambda^A(0)\lambda^B(2\pi) = a^2(a^{-1}\cos\alpha)(a^{-1}\cos(\alpha - \omega t)) + (a\sin\theta_0)^2\cdot(a\sin\theta_0)^{-2}\sin\alpha\cdot\sin(\alpha - \omega t)$$

$$= \cos\alpha\cos(\alpha - \omega t) + \sin\alpha\sin(\alpha - \omega t) = \cos(\alpha - \alpha + \omega t)$$

$\rightarrow \boxed{\chi = \omega t = 2\pi\omega}$ $(t = 2\pi)$

So $\boxed{\chi = 2\pi w = 2\pi \cos\theta_0}$

e.g. if $\theta_0 = \pi/2$ (equator) $\Rightarrow \chi = 0$ (along geodesic, direction does not change)

Curved Spacetime

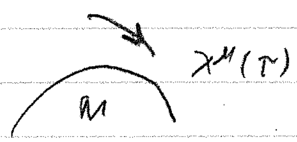
$\to$ The same equations hold. E.g., the geodesic eqn is

$$\boxed{\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0}$$

with
$$\boxed{\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\rho}\left(\partial_\nu g_{\sigma\rho} + \partial_\sigma g_{\nu\rho} - \partial_\rho g_{\nu\sigma}\right)}$$

$\Rightarrow$ gives the trajectory of free particle in curved spacetime $x^\mu(\tau)$

$\Rightarrow$ Gives the eqn for particle in gravitational field



For a massive particle, we can use proper time as parameter because

$$ds^2 = c^2 d\tau^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

Likewise, any vector λ^μ can be parallel transported along a curve $x^\mu(\tau)$ where the components obey 4D parallel transport eqn:

$$\boxed{\dot{\lambda}^\mu + \Gamma^\mu_{\nu\sigma}\lambda^\nu \dot{x}^\sigma = 0}$$

$\rightsquigarrow$ need this for sensible physics eqn in spacetime.

How to formulate the laws of physics in curved spacetime?

Covariance Recall that one of the postulates of SR is that the laws of physics are the same in all inertial frames

$\Rightarrow$ Eqns of physics are <u>invariant</u> under LT's.

e.g. $f^\mu = \frac{dp^\mu}{d\tau}$ in SR, after LT multiply each side with $\Lambda^{\nu'}_\mu$

$\Rightarrow \Lambda^{\nu'}_\mu f^\mu = \Lambda^{\nu'}_\mu \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}\left(\Lambda^{\nu'}_\mu p^\mu\right)$ get $\boxed{f^{\nu'} = \frac{dp^{\nu'}}{d\tau}}$ (same eqn)

Let $\nu' \to \mu \Rightarrow$ get back eqn $f^\mu = \frac{dp^\mu}{d\tau}$. At the same time, the metric remains $\eta_{\mu\nu} = \eta_{\mu'\nu'}$. Everything is the same $\Rightarrow$ INVARIANT eqns.

$\boxed{\text{In GR}}$

$\hookrightarrow$ the eqns should maintain the <u>same form</u> under general coord transformations $\Rightarrow$ ~~said~~ to be <u>covariant</u> (not as strict as in SR)

But in GR, eqns can include $g_{\mu\nu}$ (metric) and $\Gamma^{\lambda}_{\mu\nu}$ (connection)
$\rightarrow$ these are different in different circumstances

$\Rightarrow$ Eqns need to be <u>covariant</u> but not invariant.

<u>Note</u> invariance implies covariance.

- In trying to figure out how eqns hold in curved space time, Einstein introduced a principle...

$\hookrightarrow$ <u>Principle of Covariance</u>: Eqn is true iff GR in all coord systems if
(1) The eqn is <u>true in SR</u>
(2) The eqn is a <u>tensor eqn</u> that preserves its form under general coord. transf (covariant)

<u>Recall</u> Tensors of the same type all transform the same way

e.g. if $A^\mu = B^\mu$ for tensors A^μ, B^μ, then

$$X^{\nu'}_{\mu} A^\mu = A^{\nu'} = X^{\nu'}_{\mu} B^\mu = B^{\nu'} \Leftarrow \text{covariant form}$$

<u>Note</u> $\left\{ \begin{array}{l} \text{(1) stems from Equiv. principle. There is always a freely falling} \\ \text{coord where the laws of SR hold locally.} \end{array} \right\}$

As long as the SR laws involve tensors, the same eqns will hold in the presence of gravity.

↳ This gives prescription for finding the laws of physics in GR.

E.g.

We know $f^\mu = \dfrac{dp^\mu}{d\tau}$ holds in SR. Does this eqn also hold in curved spacetime?

$\Rightarrow$ <u>If both sides are tensors then yes.</u>

<u>BUT</u> $\dfrac{dp^\mu}{d\tau}$ is <u>not</u> a tensor under general coord transformation

<u>Why?</u> In a diff. frame
$$\frac{dp^{\mu'}}{d\tau} = \frac{d}{d\tau}\left(X^{\mu'}_{\ \nu} p^\nu\right)$$
$$= \underset{(v)}{X^{\mu'}_{\ \nu}} \frac{dp^\nu}{d\tau} + \underset{(x)}{\frac{dX^{\mu'}_{\ \nu}}{d\tau}} p^\nu$$

<u>Note</u> $\dfrac{dX^{\mu'}_{\ \nu}}{d\tau} \neq 0$ for general coord. transformation.

$\Rightarrow \dfrac{dA^\mu}{d\tau}$ is not a tensor in general coord. transf. (GCT)

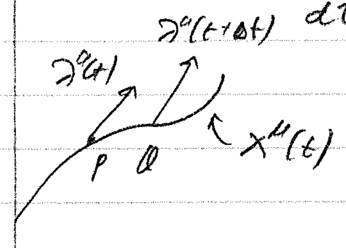
<u>So</u> $f^\mu = \dfrac{dp^\mu}{d\tau}$ is <u>not</u> covariant. Can't find eqn in new frame

$\rightarrow$ <u>The problem is with derivative!</u> $\dfrac{\partial}{\partial\tau}$, or $\partial_\mu = \dfrac{\partial}{\partial x^\mu}$

$\Rightarrow$ $\boxed{\text{Derivatives of tensors are NOT tensors in GCT}}$

$\Rightarrow$ Need to fix the def. of derivatives so that derivatives of tensors are tensors...

<u>Consider</u> $\dfrac{d\lambda^a}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{\lambda^a(t+\Delta t) - \lambda^a(t)}{\Delta t}$



But when we transform these we use

$X^{b'}_{\ a}(t)$ on $\lambda^a(t)$ at Q <u>and</u> $X^{b'}_{\ a}(t+\Delta t)$ on $\lambda^a(t+\Delta t)$ at P

But $\boxed{\text{Space is different at } P, Q!!}$ $\Rightarrow$ don't get the same factor of $X^{b'}_{a}$ at just one point

$\rightarrow$ Would be better to subtract $\lambda^a(t+\Delta t)$ and $\lambda^a(t)$ at the same point

$\Rightarrow$ To do that, we need to parallel transport $\lambda^a(t+\Delta t)$ from Q to P.

↳ Need to redefine differentiation for curved spaces.

Oct 24, 2018 $\boxed{\text{Derivatives of tensors are NOT tensors in general}}$

E.g. $g_{\mu\nu} \rightarrow$ tensor but $\partial_\lambda g_{\mu\nu}$ is not a tensor

$$\partial_{\lambda'} g_{\mu'\nu'} = \partial_{\lambda'} X^{\alpha}_{\mu'} X^{\beta}_{\nu'} g_{\alpha\beta} \neq X^{\gamma}_{\lambda'} X^{\alpha}_{\mu'} X^{\beta}_{\nu'} \partial_\gamma g_{\alpha\beta} \rightarrow \text{get extra terms}$$

For this reason $\Gamma^{\mu}_{\gamma\lambda} = \frac{1}{2} g^{\mu\beta}\left(\partial_\gamma g_{\beta\lambda} + \partial_\lambda g_{\gamma\beta} - \partial_\beta g_{\gamma\lambda}\right)$ is also not a tensor

But this relation is covariant. Go to a primed frame $\rightarrow$ get

$$\boxed{\Gamma^{\mu'}_{\gamma'\lambda'} = \frac{1}{2} g^{\mu'\beta'}\left(\partial_{\gamma'} g_{\beta'\lambda'} + \partial_{\lambda'} g_{\gamma'\beta'} - \partial_{\beta'} g_{\gamma'\lambda'}\right)} \text{ All}$$

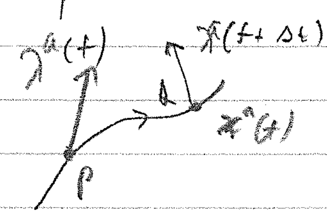
the extra terms cancel $\Rightarrow$ this relation is in fact covariant

But more generally, we have a problem with derivatives

$\boxed{\text{Absolute \& Covariant derivatives}}$

- Consider a manifold: contravariant vector λ^a parameterized by t, then

$$\frac{d\lambda^a}{dt} = \lim_{\Delta t \to 0} \frac{\lambda^a(\Delta t + t) - \lambda^a(t)}{\Delta t}$$



So $\lambda^a(t)$ @ P, $\lambda^a(t+\Delta t)$ @ Q $\}$ Problem arises because

$$\boxed{X^{a'}_{b}\big|_{@P} \neq X^{a'}_{b}\big|_{@Q}}$$

As $\Delta t \to 0$, we'll get extra terms of derivatives of $X^{a'}_{b'}$. To fix this, we change the def. of derivative $\Rightarrow$ Absolute derivative...

$$\boxed{\text{Define } \frac{D\lambda^a}{dt} = \lim_{\Delta t \to 0} \frac{\overset{@Q}{\lambda^a(t+\Delta t)} - \overset{@Q}{\bar{\lambda}^a}}{\Delta t} \quad \text{where } \bar{\lambda}^a = \lambda^a \text{ at } P \text{, parallel transported to } Q}$$

We want an expression for this... For the 1^{st} term, we can Taylor expand...

$$\boxed{\lambda^a(t+\Delta t) \simeq \lambda^a(t) + \frac{d\lambda^a}{dt}\Delta t = \lambda^a(P) + \frac{d\lambda^a}{dt}\Delta t} \quad (p = t)$$

Second term: parallel transport eqn: $\boxed{\dot{\lambda}^a + \Gamma^a_{bc}\lambda^b \dot{x}^c = 0}$

For small finite intervals, $\dot{\lambda}^a \simeq \frac{\Delta\lambda^a}{\Delta t}$ and $\dot{x}^c \simeq \frac{\Delta x^c}{\Delta t}$

So $\boxed{\Delta\lambda^a + \Gamma^a_{bc}\lambda^b \Delta x^c = 0}$ (parallel transport)

where $\Delta\lambda^a = \bar{\lambda}^a(Q) - \lambda^a(P)$

So $\boxed{\bar{\lambda}^a(Q) = \Delta\lambda^a + \lambda^a(P)}$ $\Rightarrow$ $\boxed{\bar{\lambda}^a(Q) \simeq \lambda^a(P) - \Gamma^a_{bc}\lambda^b \Delta x^c}$

plug into derivative $\rightarrow$ $\frac{D\lambda^a}{dt} = \lim_{\Delta t \to 0} \frac{(d\lambda^a/dt)\Delta t + \Gamma^a_{bc}\lambda^b \Delta x^c}{\Delta t}$

So

$$\frac{D\lambda^a}{dt} = \lim_{\Delta t \to 0}\left(\frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b \Delta x^c/\Delta t\right)$$

So $\boxed{\frac{D\lambda^a}{dt} = \underbrace{\frac{d\lambda^a}{dt}}_{\text{(usual)}} + \underbrace{\Gamma^a_{bc}\lambda^b \dot{x}^c}_{\text{(correction)}}}$ $\rightarrow$ Absolute derivative for a contravariant vector

$\Rightarrow$ Transforms as a tensor by contraction: $\boxed{\frac{D\lambda^{a'}}{dt} = X^{a'}_{b}\frac{D\lambda^b}{dt}}$

Notice that the RHS is the same as in the parallel transport eq.

$$\frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b \dot{x}^c = 0$$ $\Rightarrow$ If we parallel transport a vector λ^a its component are constant under absolute differentiation

$$\boxed{\frac{D\lambda^a}{dt} = 0 \text{ when parallel transported}}$$

→ What about taking absolute derivatives of scalar, covariant vector, or tensors

For scalars $\phi \to \phi$ as $x^a \to x^{a'}$ → no factor of $X^{a'}_{b}$ in derivative

$$\boxed{\frac{D\phi}{dt} = \frac{d\phi}{dt}}$$ → absolute deriv of scalar

Under a GCT $\Rightarrow$ $\frac{D\phi}{dt} \to \frac{d\phi}{dt}$

For covariant vectors

Consider $\lambda^a \mu_a$ is a scalar.

$$\hookrightarrow \frac{D\lambda^a \mu_a}{dt} = \frac{d}{dt}(\lambda^a \mu_a) = \frac{d\lambda^a}{dt}\mu_a + \lambda^a \frac{d\mu_a}{dt}$$

$$\Rightarrow \frac{D\lambda^a}{dt}\mu_a + \lambda^a \frac{D\mu_a}{dt} = \frac{d\lambda^a}{dt}\mu_a + \lambda^a \frac{d\mu_a}{dt}$$

$$\Rightarrow \left(\cancel{\frac{d\lambda^a}{dt}} + \Gamma^a_{bc}\lambda^b \dot{x}^c\right)\mu_a + \lambda^a\left[\frac{D\mu_a}{dt}\right] = \cancel{\frac{d\lambda^a}{dt}}\mu_a + \lambda^a \frac{d\mu_a}{dt}$$

$$\text{So } \left(\frac{D\mu_a}{dt}\right) = \frac{1}{\lambda^a}\left[\lambda^a \frac{d\mu_a}{dt} - \mu_a \Gamma^a_{bc}\lambda^b \dot{x}^c\right]$$

let $b \to a$
$a \to d$

$$\text{So } \frac{D\mu_a}{dt} = \frac{1}{\lambda^a}\left[\lambda^a \frac{d\mu_a}{dt} - \mu_d \Gamma^d_{ac}\lambda^a \dot{x}^c\right]$$

$$\text{So } \boxed{\frac{D\mu_a}{dt} = \frac{d\mu_a}{dt} - \Gamma^d_{ac}\mu_d \dot{x}^c}$$ ← Absolute deriv. of covariant vector, Note the (−) sign to connection.

→ Contravariant $(+\Gamma)$ → covariant $(-\Gamma)$

For a tensor $\boxed{\tau^{ab}{}_{c} = \lambda^a \sigma^b \mu_c}$ ← multiplying vectors gives tensors

We can show that

usual ↗ ↗ correction (+, −)

$$\boxed{\frac{D\tau^{ab}{}_{c}}{dt} = \frac{d\tau^{ab}{}_{c}}{dt} + \Gamma^a_{de}\tau^{db}{}_{c}\dot{x}^e + \Gamma^b_{de}\tau^{ad}{}_{c}\dot{x}^e - \Gamma^d_{ce}\tau^{ab}{}_{d}\dot{x}^e}$$

This is a tensor, so under GCT

$$\rightarrow \boxed{\frac{D\tau^{a'b'}{}_{c'}}{dt} = X^{a'}_{d} X^{b'}_{e} X^{f}_{c'} \frac{D\tau^{de}{}_{f}}{dt}}$$

Note that In Cartesian coordinates, $\Gamma^a_{bc} = 0$ for SR (flat)

$$\hookrightarrow \boxed{\frac{D\tau^{ab}{}_{c}}{dt} = \frac{d\tau^{ab}{}_{c}}{dt}} \text{ in SR}$$

The absolute derivative is w.r.t a parameter (like t, σ, s ..).
We also need to take derivatives w.r.t coordinates.

$\partial_a = \frac{\partial}{\partial x^a}$ ⇒ need to introduce a derivative that transforms correctly.

→ $\boxed{\text{Covariant derivative}}$ → w.r.t coord x^a.

Since $x^a = x^a(t)$ along a curve ⇒ can think of chain rule where

$$\frac{D\lambda^a}{dt} = \frac{D\lambda^a}{dx^c}\frac{dx^c}{dt} \quad \text{(new type of derivative)}$$

$$= \frac{D\lambda^a}{dx^c}\dot{x}^c$$

But since $\frac{D\lambda^a}{dt} = \frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b\dot{x}^c$

$$\rightarrow \frac{D\lambda^a}{dx^c}\dot{x}^c = \frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b\dot{x}^c$$

chain rule $\frac{d\lambda^a}{dt} = \frac{\partial\lambda^a}{\partial x^c}\frac{dx^c}{dt} = \frac{\partial\lambda^a}{\partial x^c}\dot{x}^c$

So $\frac{D\lambda^a}{dx^c}\dot{x}^c = \frac{\partial\lambda^a}{\partial x^c}\dot{x}^c + \Gamma^a_{bc}\lambda^b\dot{x}^c$

Therefore $\boxed{\frac{D\lambda^a}{dX^c} = \underset{\uparrow\text{ usual}}{\frac{\partial\lambda^a}{\partial x^c}} + \underset{\uparrow\text{ correction}}{\Gamma^a_{bc}\lambda^b}}$ But we don't use this notation

Define
$$\boxed{\begin{aligned}\lambda^a_{;c} &= \frac{\partial\lambda^a}{\partial x^c} + \Gamma^a_{bc}\lambda^b \\ \text{or}\quad \lambda^a_{;c} &= \partial_c\lambda^a + \Gamma^a_{bc}\lambda^b\end{aligned}}$$
→ covariant derivative of contravariant vector

semi-colon

comma

We also write $\boxed{\lambda^a_{,c} = \frac{\partial\lambda^a}{\partial X^c} = \partial_c\lambda^a}$

So $\boxed{\lambda^a_{;c} = \lambda^a_{,c} + \Gamma^a_{bc}\lambda^b}$

Why do this? Because λ^a is a type (1,0) tensor but $\lambda^a_{;c}$ is a type (1,1) tensor

But other notations $\frac{D\lambda^a}{dx^c} = \nabla_c\lambda^a = D_c\lambda^a$

Oct 26, 2018

$\boxed{\text{Absolute derivatives}}$ $\frac{d}{d\sigma} \to \frac{D}{d\sigma}$ σ = ~~param~~

w.r.t to param σ, t, s

$\boxed{\frac{D\varphi}{d\sigma} = \frac{d\varphi}{d\sigma}}$ (φ - scalar) $\boxed{\frac{D\lambda^a}{d\sigma} = \frac{d\lambda^a}{d\sigma} + \Gamma^a_{bc}\lambda^b\dot{x}^c}$ (contravariant)

$\boxed{\frac{D\lambda_a}{d\sigma} = \frac{d\lambda_a}{d\sigma} - \Gamma^c_{ab}\lambda_c\dot{x}^b}$ Covariant ← w.r.t param

Note $\boxed{\text{Covariant derivatives}}$ (w.r.t to coordinate)

w.r.t coordinate x^c

$\to \partial_a = \frac{\partial}{\partial x^a} \Rightarrow D_a = ;a$
$$\left\{\begin{aligned}\frac{D\lambda^a}{dx^c} &= \frac{\partial\lambda^a}{\partial x^c} + \Gamma^a_{bc}\lambda^b \\ &= \partial_c\lambda^a + \Gamma^a_{bc}\lambda^b\end{aligned}\right.$$

Note $\lambda^a \to$ type $(1,0)$ tensor, whereas $\frac{D\lambda^a}{dx^c} = \lambda^a{}_{;c} \to$ type $(1,1)$ tensor

"semi-colon"

Under GCT

$\hookrightarrow \quad \lambda^{a'}{}_{;c'} = X^{a'}{}_d X^e{}_{c'} \lambda^d{}_{;e}$

Note $\boxed{\text{Derivative of a scalar}} \to \boxed{\varphi_{;a} = \partial_a \varphi} \leftarrow$ scalar

So $\boxed{\mu_{a;c} = \frac{D\mu_a}{dx^c} = \partial_c \mu_a - \Gamma^b_{ac}\mu_b} \leftarrow$ covariant vectors

$$\boxed{\underbrace{\tau^a{}_{b;c} = \partial_c \tau^a{}_b}_{\text{regular derivative}} + \underbrace{\Gamma^a_{dc}\tau^d{}_b}_{\text{contrav. correction}} - \underbrace{\Gamma^d_{bc}\tau^a{}_d}_{\text{covariant correction}}}$$

tensor, in general

$\boxed{\text{Example}}$ $\boxed{\text{Show that } g_{ab;c} = 0}$ "Metric is covariantly constant"

$g_{ab;c} = \cancel{\partial_c g_{ab} - \Gamma^d_a} \; ???$

Start with $\Gamma^a_{bc} = \frac{1}{2} g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)$

Call $\Gamma_{abc} = g_{ae}\Gamma^e_{bc}$ (lowering indices)

$= \frac{1}{2} g_{ae} g^{ed}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)$

$= \frac{1}{2}\left(\cancel{\partial_b g_{ac}} + \partial_c g_{ab} - \cancel{\partial_a g_{bc}}\right)$

Swap $a \leftrightarrow b$ so $\Gamma_{abc} \Rightarrow \Gamma_{bac} = \frac{1}{2}\left(\cancel{\partial_a g_{bc}} + \partial_c g_{ba} - \cancel{\partial_b g_{ac}}\right)$

So $\Gamma_{abc} + \Gamma_{bac} = \frac{1}{2}\left(\partial_c g_{ab} + \partial_c g_{ba}\right) = \partial_c g_{ab}$ ([g] symmetric)

So by def

$$\boxed{g_{ab;c} = \partial_c g_{ab} - \Gamma^d_{ac} g_{db} - \Gamma^d_{bc} g_{ad}}$$

So $g_{ab;c} = \Gamma_{abc} + \Gamma_{bac} - \Gamma^d_{ac} g_{bd} - \Gamma^d_{bc} g_{ad}$

$= \Gamma_{abc} + \Gamma_{bac} - \Gamma_{bac} - \Gamma_{abc}$ (lowering of indices)

$= 0$

So $\boxed{g_{ab;c} = 0}$

$\boxed{\text{Example}}$ Can also show $\boxed{\delta^a_{b;c} = 0}$

Likewise $\boxed{g^{ab}{}_{;c} = 0}$

Also $\boxed{\frac{D g_{ab}}{dt} = \frac{D g^{ab}}{dt} = \frac{D \delta^a_b}{dt} = 0}$

$\boxed{\text{Note}}$ We might have predicted ahead of time that $g_{ab;c} = 0$

Go to a local Lorentz frame $\leftrightarrow$ $g_{\mu\nu;c} = 0$ in 4D spacetime

There $g_{\mu\nu} = \eta_{\mu\nu} \Rightarrow \partial_\lambda g_{\mu\nu} = \partial_\lambda \eta_{\mu\nu} = 0$

So $\Gamma^\lambda_{\mu\nu} = 0$ as well (by definition)

So $\boxed{g_{\mu\nu;c} = 0}$ (tensor)

So under GCT to an arbitrary frame, we get $g_{\mu'\nu';\lambda'} = 0$
since it's covariant + drop primes.

$\boxed{\text{Important fact}}$

If a tensor is 0 in one frame, then it's 0 in all frames

$\rightarrow$ follows from the fact that tensor eqns are covariant.

With the principle of general covariance, we now have a prescription for finding physics eqn in GR

Step 1 : Write down eqn in SR (in an inertial frame)

Step 2 : Change all derivatives to absolute / covariant derivatives
($\Rightarrow$ turns into tensor eqns)

Step 3 : $\rightarrow$ transform to arbitrary frame where the eq doesn't change

Example . In SR : $f^\mu = \frac{dp^\mu}{d\tau}$

In GR : Let $\frac{dp^\mu}{d\tau} \rightarrow \frac{Dp^\mu}{d\tau}$ (turning p^μ into tensor)

So $f^\mu = \frac{Dp^\mu}{d\tau} \Rightarrow$ eqn that holds in all frames of GR

(Suppose) $f^\mu = 0 \rightarrow$ free particle

So $\frac{Dp^\mu}{d\tau} = 0$. But $p^\mu = m u^\mu \Rightarrow \frac{Du^\mu}{d\tau} = 0$

So $\frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\lambda\nu} u^\lambda \cancel{x}^\nu = 0$ (absolute deriv)

But then $u^\mu = \frac{dx^\mu}{d\tau}$

So $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\lambda\nu} \frac{dx^\lambda}{d\tau} \frac{dx^\nu}{d\tau} = 0$

$\Rightarrow$ free particle! (No force $\rightarrow$ geodesic eqn)

Oct 29, 2018

$\boxed{\text{Newtonian Limit of GR}}$

, In Newtonian physics, gravity is a force. $\breve{F} = \frac{-GmM}{r^2}\hat{r}$

- Eq of motion $\frac{d^2 x^\nu}{dt^2} = 0$

<u>But in GR</u> → the eqn of motion is the geodesic eqn

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

In some limit, these eqns have to match up

Something in GR links up with something in Newtonian physics - that thing is called the gravitational potential V.

$\boxed{\text{Gravitational Potential by analogy to E\&M}}$

$$\tilde{F} = \frac{1}{4\pi\epsilon_0}\cdot\frac{q_1 q_2}{r^2}\hat{r} \quad . \quad PE = \frac{+1}{4\pi\epsilon_0}\frac{q_1 q_2}{r} \text{ (joules)} \quad \left(= -\int F\,dr\right)$$

Define electric potential → $V = \frac{U}{q} = \frac{+1}{4\pi\epsilon_0}\frac{q}{r}$ (volts)

- Can do the same with gravity

$$\tilde{F} = \frac{-GMm}{r^2}\hat{r} \rightarrow U = \frac{-GMm}{r} \text{ (Joules)}$$

→ gravitational potential → $V = \frac{U}{m} = \frac{-GM}{r}$ (m^2/s^2 → gh units)

For a point mass → $\boxed{V = \frac{-GM}{r}}$ (grav. potential)

The relation between gravitational potential & force

$\boxed{\breve{F} = -m\vec{\nabla}V}$ How does V link up with the metric?

Newtonian eqn of motion : $\boxed{\vec{F} = m\vec{a} = -m\vec{\nabla}V}$

So $\dfrac{d^2X}{dt^2} = -\vec{\nabla}V$

with indices $\Rightarrow \vec{X} \to x^i$ while $\vec{\nabla} \to \partial_i$ } mismatched because not relativistic

$\to$ fix with a δ^{ij}

So $\boxed{\dfrac{d^2x^i}{dt^2} = -\delta^{ij}\partial_j V}$ $\to$ match with relativistic theory

- How does this match up with Geodesic Eqn? $\dfrac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\nu}\dfrac{dx^\sigma}{d\tau}\dfrac{dx^\nu}{d\tau} = 0$

in a non-relativistic limit?

Weak field limit of GR Effects of gravity near Earth or Sun are weak $\Rightarrow$ only a slight curvature is expected $\to$ can approximate.

$\Rightarrow$ Can approximate that $\boxed{g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}}$ with $|h_{\mu\nu}| \ll 1$

In Newtonian limit, spacetime is almost Minkowski (flat)
keeping only first order terms in $h_{\mu\nu}$, we can show in Ex 2.7.1 that $\boxed{g^{\mu\nu} \cong \eta^{\mu\nu} - h^{\mu\nu}}$ where $h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}$. To show this, verify that $g^{\mu\nu}g_{\nu\sigma} \simeq \delta^\mu_\sigma$ to 1^{st} order ($h \cdot h \to 0$)

- Once we have those $\to$ can find $\Gamma^\mu_{\nu\sigma}$ in terms of h

$$\boxed{\Gamma^\mu_{\nu\sigma} \simeq \frac{1}{2}\eta^{\mu\rho}\left(\partial_\nu h_{\sigma\rho} + \partial_\sigma h_{\nu\rho} - \partial_\rho h_{\nu\sigma}\right)}$$ to 1^{st} order in $h_{\mu\nu}$

$\hookrightarrow$ Use in geodesic eqn ...

We also want a non-relativistic (slow) limit $\Rightarrow$ use $\boxed{\frac{dx^0}{d\tau} \gg \frac{dx^i}{d\tau}}$

This follows since $\frac{dx^0}{d\tau} = \frac{d}{d\tau}(ct) = c\frac{dt}{d\tau}$

while $\frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau}$ and $\boxed{\underbrace{\frac{dx^i}{dt}}_{v^i} \ll c}$ for slow objects ...

So $\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$

with this, we can ignore $\frac{dx^i}{d\tau}$ in summing compared to $\frac{dx^0}{d\tau}$

↳ in slow limit $\Rightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau}$

$\simeq \boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} \simeq 0}$

Also, assume static gravitational field (not changing in time)
(assume stationary Earth...)

$$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\sigma}\underbrace{(\partial_0 h_{\sigma 0} + \partial_0 h_{\sigma 0}}_{\begin{subarray}{c}0 \quad\quad 0\end{subarray}} - \partial_\sigma h_{00}) \cong \frac{1}{2}\eta^{\mu\sigma}(-\partial_\sigma h_{00})$$

[time derivatives
vanish in static limit]

4 eqns

So $\boxed{\Gamma^\mu_{00} \simeq -\frac{1}{2}\eta^{\mu\sigma}\partial_\sigma h_{00}}$

So Geodesic becomes $\Rightarrow$
$$\boxed{\begin{aligned}\frac{d^2x^\mu}{d\tau^2} &\simeq \left(\frac{1}{2}\eta^{\mu\sigma}\partial_\sigma h_{00}\right)\left(\frac{dx^0}{d\tau}\right)^2 \\ &\simeq \left(\frac{1}{2}\eta^{\mu\sigma}\partial_\sigma h_{00}\right)\left(c^2\left(\frac{dt}{d\tau}\right)^2\right)\end{aligned}}$$

So ⟶

ct

$\boxed{\text{Let } \mu = 0}$ (time) $\Rightarrow \frac{d^2x^0}{d\tau^2} \cong c^2\frac{d^2t}{d\tau^2} \simeq \left(\frac{1}{2}\eta^{0\sigma}\partial_\sigma h_{00}\right)c^2\left(\frac{dt}{d\tau}\right)^2$

Must have $\sigma = 0$ since $\eta^{0i} = 0$
But $\partial_0 h_{00} = 0$ in static limit
$\Bigg\} \Rightarrow \frac{d^2x^0}{d\tau^2} = 0$ or $\boxed{\frac{d^2t}{d\tau^2} = 0}$

$\boxed{\text{Let } \mu = i} \rightarrow \dfrac{d^2x^i}{dt^2} \cong \dfrac{1}{2}\eta^{i\sigma}(\partial_\sigma h_{00})\, c^2 \left(\dfrac{dt}{d\tau}\right)^2$

Using the chain rule

$$\hookrightarrow \frac{d^2x^i}{d\tau^2} = \frac{d}{d\tau}\left(\frac{dx^i}{d\tau}\right) = \frac{d}{d\tau}\left(\frac{dx^i}{dt}\frac{dt}{d\tau}\right) = \frac{d^2x^i}{dt\,d\tau}\underbrace{\frac{dt}{d\tau}}_{\text{constant}} + \frac{dx^i}{dt}\overbrace{\frac{d^2t}{d\tau^2}}^{\nearrow 0}$$

But we also know that $\boxed{\dfrac{dt}{d\tau} = \text{no } \tau \text{ dep}}$ (from $\mu = 0$)

So

$$\frac{d^2x^i}{d\tau^2} = \left(\frac{dt}{d\tau}\right)\frac{d}{d\tau}\left(\frac{dx^i}{dt}\right) = \left(\frac{dt}{d\tau}\right)\frac{d}{dt}\left(\frac{dx^i}{dt}\right)\left(\frac{dt}{d\tau}\right) = \left(\frac{dt}{d\tau}\right)^2\frac{d^2x^i}{dt^2}$$

with this

$$\hookrightarrow \cancel{\left(\frac{dt}{d\tau}\right)^2}\frac{d^2x^i}{dt^2} \cong \frac{1}{2}\eta^{i\sigma}(\partial_\sigma h_{00})\, c^2 \cancel{\left(\frac{dt}{d\tau}\right)^2}$$

or $$\boxed{\frac{d^2x^i}{dt^2} \cong \frac{c^2}{2}\eta^{i\sigma}(\partial_\sigma h_{00})}$$

$\left\{ \begin{array}{l} \sigma = 0 \Rightarrow \eta^{i0} = 0 \\ \text{while } \sigma = j \text{ gives } \eta^{ij} = -1 = -\delta^{ij} \end{array} \right.$

So $$\boxed{\frac{d^2x^i}{dt^2} \simeq -\frac{c^2}{2}\delta^{ij}(\partial_j h_{00})}$$

Compare this with Newtonian eqn

$$\hookrightarrow \boxed{\frac{d^2x^i}{dt^2} = -\delta^{ij}\partial_j V}$$

In order for GR to go back to Newtonian, must have correspondence, that in the limit

$$\hookrightarrow \boxed{V \simeq \frac{c^2}{2}h_{00} + \text{constant}}$$

Since we want $V \rightarrow 0$ as $h_{\mu\nu} \rightarrow 0$ (no gravity)

$\rightarrow V = 0$

$\rightarrow \text{constant} = 0$

So $V \simeq \dfrac{c^2}{2}h_{00}$, or $\boxed{h_{00} = \dfrac{2V}{c^2}}$ to get Newtonian limit

So $h_{00} = \frac{2V}{c^2}$. But since $g_{00} = \eta_{00} + h_{00} = 1 + h_{00}$

Get that $\boxed{g_{00} \simeq 1 + \frac{2V}{c^2}}$ Correspondence between GR + Newtonian physics.

Einstein used this in coming up with the Einstein eqn...

For pt mass $\rightarrow V = \frac{-GM}{r} \rightarrow$ involves G

↳ Einstein eqn will include G as well!

Oct 30, 2018 Recall

Newton: $\frac{d^2X}{dt^2} = -\delta^{ij}\partial_j V$ where $V = \frac{-GM}{r}$

GR $\frac{d^2X}{d\tau^2} + \Gamma^{\mu}_{\sigma\nu}\frac{dX^\sigma}{d\tau}\frac{dX^\nu}{d\tau} = 0$ in weak static limit

Weak static limit $g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu}$ $|h_{\mu\nu}| << 1$

and

$\partial_0 h_{\mu\nu} = 0$ (static)

$\boxed{\text{Found correspondence}}$

$$h_{00} = \frac{2V}{c^2} \quad \text{or} \quad g_{00} = 1 + \frac{2V}{c^2}$$

where we have used $\frac{d^2t}{d\tau^2} \simeq 0$, or $\frac{dt}{d\tau}$ has no τ dependence

A more careful analysis shows $\boxed{\frac{d\tau}{dt} = (1+h_{00})^{1/2}}$, which follows from

$c^2 d\tau^2 = g_{\mu\nu}dX^\mu dX^\nu \Rightarrow$ this is independent of τ in static limit

But this gives a new type of time dilation which we'll look at later.

The exact solution outside a spherical mass M in GR is the Schwarzschild solution

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2GM}{c^2r} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2GM}{c^2r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}$$

$\rightarrow$ close to spherical Minkowski

Note $g_{00} = 1 + \frac{2V}{c^2}$

This means when $\frac{GM}{rc^2} \ll 1$ (small M or large r) then geodesic motion in GR will appear like motion due to a force in Newtonian physics

Curvature effects ~ force behavior

VII – The Einstein Eqns

- We've been assuming we know the metric + have looked at physics in curved spaces...
- Einstein knew he had to find an eq that lets you solve for the metric given a distribution of mass and energy
  - ⇒ Took him 8 years.
- Ultimately, he found the eqns:

$$\boxed{R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu}} \quad \text{Einstein Eqns}$$

Here $\boxed{T^{\mu\nu}}$ = energy – momentum stress tensor
= density of energy, mass, momentum
⇒ source of gravity (curvature of spacetime)

$\boxed{R^{\mu\nu}}$ ⇒ Ricci tensor → contraction of the Riemann curvature tensor $R^{\mu}{}_{\nu\lambda\sigma}$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\nu\sigma}$$

$\boxed{R}$ ⇒ curvature scalar ⇒ contraction of $R_{\mu\nu}$

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu}$$

- We'll see $\Rightarrow$ $R^{\mu}{}_{\nu\lambda\sigma}$ is a function of $g_{\mu\nu}$ and its derivative

  $\Rightarrow$ Einstein equations $\Rightarrow$ are a set of <u>coupled non-linear partial differential equations</u> for $g_{\mu\nu}$

- When Einstein looked at solutions for a gas of cosmic matter, he found evolving solution $\Rightarrow$ expanding / contracting universe

  But Einstein thought the universe is static $\Rightarrow$ he was doing this before Hubble's discovery (1929) that the universe is expanding

- To get solutions that describe static universe, he added an extra term

$$\boxed{R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{-8\pi G}{c^4} T^{\mu\nu}}$$

- Λ cosmological constant $\Rightarrow$ acts as a cosmic source of energy density

  $\Rightarrow$ an energy associate with the vacuum (dark energy)

- After Hubble's discovery (universe is expanding), Einstein set Λ to 0, and he called putting in Λ his "greatest blunder"...

- For decades, all cosmology was $\Lambda = 0$. Then in the 1990s it was discovered that the universe has accelerated expansion this brought back Λ

- Now, all cosmological models include the Λ term or some form of "dark energy"
  $\downarrow$
  (most)

- We'll study cosmology with & without Λ. Our plan is look at $T^{\mu\nu}$, $R^{\mu}{}_{\nu\lambda\sigma}$, $R_{\mu\nu}$, R, ... $\Rightarrow$ retrace some of Einstein's steps with coming up with his solutions. The eqns are very hard to solve

- Why? → Because they're nonlinear. Gravitational fields carry energy which affects themselves
→ gravitational fields interact with each other.

- In E+M → set linear equations → obey superposition principle
{ E+M waves don't carry charge & do not interact with each other...

- We won't attempt to solve Einstein's Eqns. Instead, we'll study 2 well-known solutions

(1) <u>Schwarzschild ~~solution~~</u> → gives $g_{\mu\nu}$ outside a spherical static mass M (Earth, Sun, black hole

(2) <u>Friedman-Robertson-Walker solution (FRW)</u>
→ gives $g_{\mu\nu}$ for a homogeneous & spatially isotropic universe (with $\Lambda = 0$ or $\Lambda \neq 0$)

FRW with Λ is the current best cosmological model

Oct 31, 2018

$$\boxed{R^{\mu\nu} - \frac{R}{2} g^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{-8\pi G}{c^4} T^{\mu\nu}}$$

How was Einstein guided to find this eqn?

<u>In Newtonian limit</u>

$\vec{F} = -m\vec{\nabla}V$ with $V = -\frac{GM}{r}$, <u>for point particle</u>

What about for a mass density ρ? For this ρ is given by

$$\boxed{\vec{\nabla}^2 V = 4\pi G\rho}$$ Poisson's eq.

How do we show this? Analogy with E+M

<u>in E+M</u> $\vec{\nabla}\cdot E = \frac{\rho}{\varepsilon_0}$ → charge per volume

<u>Potential in E+M</u> $E = -\vec{\nabla}V$

$$\Rightarrow \vec{\nabla}^2 V = -\frac{\rho}{\varepsilon_0}$$ Poisson's eq in E+M

We can simply map $F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}$ by $-\frac{GMm}{r^2}$

So $\frac{1}{\varepsilon_0} \rightarrow -4\pi G$ so $\boxed{\vec{\nabla}^2 V = 4\pi G \rho}$

Einstein used this as a guide

The Energy Momentum Stress Tensor

$T^{\mu\nu} \rightarrow$ density of energy & momentum

For a dist. of matter, $\rho = \frac{M}{V}$

$\Rightarrow$ $\boxed{\rho c^2 \text{ gives the mass-energy density}}$

We know that energy & momentum couple relativistically, what is the momentum type density? (that goes with a mass density?)

$\hookrightarrow$ It's the pressure P (force per area/~~volume~~)

If we look at units: $P = \frac{F}{A} = \frac{ma}{A} = \frac{mv}{V} = \frac{p}{V}$ ($p \rightarrow$ momentum, $V \rightarrow$ volume)

So $\boxed{\text{Pressure } (P) \text{ is the mean momentum transfer per area}}$

In relativity, pressure P acts as a source of energy-momentum density in GR.

<u>But</u> P is NOT a vector!

So $\boxed{P = \rho c^2 \text{ should be part of the tensor } T^{\mu\nu} \text{ for energy/mom density}}$

Also, since $g_{\mu\nu} = g_{\nu\mu}$ is curved by $T^{\mu\nu}$, expect $\boxed{T^{\mu\nu} = T^{\nu\mu}}$

For a simple gas of particles in rest frame, in flat spacetime,

$$[T^{\mu\nu}] = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$$

But to put $T^{\mu\nu}$ in covariant form that allows moving matter we use world velocity u^{μ}. This gives a form

$$\boxed{T_{\cancel{x}}^{\mu\nu} = \begin{pmatrix} & & \\ & ? & \\ & & \end{pmatrix} = \left(\rho + \frac{P}{c^2}\right) u^{\mu} u^{\nu} - P\eta^{\mu\nu}}$$

where $u^{\mu} = (\gamma c, \gamma \vec{v})$ for moving matter. And $\eta^{\mu\nu} = g^{\mu\nu}$ in flat spacetime

Einstein knew this was the quantity to use because it obeys conservation law

$$\boxed{T^{\mu\nu}{}_{,\mu} = 0 \Leftrightarrow \partial_{\mu} T^{\mu\nu} = 0}$$

This gives 2 well known eqns in fluid dynamics

→ <u>Continuity Eqn</u> $\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0}$ (expresses energy-matter conservation)

<u>Euler's Eqn</u> $\boxed{\rho\left(\frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla}\right)\vec{v} = -\vec{\nabla}p}$ (has to do with momentum flow)

<u>Main point</u> $T^{\mu\nu}$ depends on ρ & P

$$\boxed{T^{\mu\nu}{}_{,\mu} = \partial_{\mu} T^{\mu\nu} = 0}$$

<u>Note</u> pressure P is the source of gravity ...

Can also have energy density from electromagnetism
⇒ Electric & magnetic fields carry energy & momentum. Can def. a stress tensor for these as well

$T^{\mu\nu}_{EM}$ = energy-mom for EM fields

Relativistic form

$$\boxed{T^{\mu\nu}_{EM} = F^{\mu}{}_{\lambda} F^{\nu\lambda} + \frac{1}{4}\eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}$$ $F^{\mu\nu}$ = tensor for $\vec{E}$ & $\vec{B}$

Can show

$$\left\{ \begin{array}{l} T^{00}_{EM} \sim \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right) \sim \frac{\text{energy}}{\text{volume}} \\ T^{ij}_{EM} \sim \text{radiation pressure (Poynting vector)} \end{array} \right\}$$

☒ The total energy momentum tensor is the sum of all contributing parts

$$\boxed{T^{\mu\nu} = T^{\mu\nu}_{matter} + T^{\mu\nu}_{EM} + \ldots}$$

$\uparrow$ ρ, P, u^{μ} $\uparrow$ $\vec{E}, \vec{B}$

Lastly, to make the equations covariant (hold in curved spacetime)

$\rightarrow$ $\eta_{\mu\nu}$ replaced by $g_{\mu\nu}$

and , replaced by ;

} $\eta^{\mu\nu} \to g^{\mu\nu}$ and use covariant derivatives ...

☒ This gives matter for GR is

$$\boxed{T^{\mu\nu}_{matter} = \left(\rho + \frac{P}{c^2}\right) u^{\mu} u^{\nu} - P g^{\mu\nu}}$$

and

$$\boxed{T^{\mu\nu}{}_{;\mu} = 0 \quad \text{and} \quad T^{\mu\nu} = T^{\nu\mu}}$$

Next, how do we find eqn that lets us solve for $g_{\mu\nu}$ given a distribution of matter ($T^{\mu\nu}$)

An obvious first guess is $\boxed{g^{\mu\nu} = kT^{\mu\nu}}$ (k = constant)

If $g^{\mu\nu} = kT^{\mu\nu}$, then $g^{\mu\nu} = g^{\nu\mu}$, $T^{\mu\nu} = T^{\nu\mu}$,

$$g^{\mu\nu}{}_{;\mu} = 0 \quad (0 \text{ divergence})$$

$$T^{\mu\nu}{}_{;\mu} = 0 \quad (0 \text{ divergence})$$

Good, but it doesn't give Poisson Eqn... So... look for eqn involving <u>connection</u>. But here, $\Gamma^{\lambda}_{\mu\nu}$ is <u>NOT</u> a tensor.

- Also $\Gamma^{\mu}_{\lambda\nu} \neq 0$ does not mean spacetime is curved (<u>ex</u> spherical coords in Minkowski spacetime)

<u>So</u> Study $\Rightarrow$ quantity that describes curvature is the Riemann curvature tensor

$$\boxed{R^{\mu}{}_{\nu\lambda\sigma} = \partial_{\lambda}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\lambda} + \Gamma^{\rho}_{\nu\sigma}\Gamma^{\mu}_{\rho\lambda} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\mu}_{\rho\sigma}}$$

Riemann curvature tensor. Math fact.

A spacetime is <u>flat</u> if $R^{\mu}{}_{\nu\lambda\sigma} = 0$ at all points.
If $R^{\mu}{}_{\nu\lambda\sigma} \neq 0$ at ~~each~~ any (some) points, it's <u>curved</u> spacetime

<u>How to get $R^{\mu}{}_{\nu\lambda\sigma}$?</u>

By doing repeated covariant differentiation ...

- Usual derivatives obey $\dfrac{\partial^2}{\partial x^{\mu}\partial x^{\nu}} = \dfrac{\partial^2}{\partial x^{\nu}\partial x^{\mu}}$
- <u>But</u> this isn't true when there's curvature
- Suppose λ_{ν} is a covariant vector ..

λ_ν is a covariant vector...

$$\lambda_{\nu;\sigma} = \partial_\sigma \lambda_\nu - \Gamma^\mu_{\nu\sigma}\lambda_\mu$$

But then $\lambda_{\nu;\sigma\lambda} = (\lambda_{\nu;\sigma})_{;\lambda} = (\partial_\sigma \lambda_\nu - \Gamma^\mu_{\nu\sigma}\lambda_\mu)_{;\lambda}$

Find that $\boxed{\lambda_{\nu;\lambda\sigma} \neq \lambda_{\nu;\sigma\lambda}}$

Can show that

$$\boxed{\lambda_{\nu;\lambda\sigma} - \lambda_{\nu;\sigma\lambda} = R^\mu_{\nu\lambda\sigma}\lambda_\mu}$$

when $R^\mu_{\nu\lambda\sigma} \neq 0 \Rightarrow \lambda_{\nu;\lambda\sigma} \neq \lambda_{\nu;\sigma\lambda}$ (curved spacetime)

But when $R^\mu_{\nu\lambda\sigma} = 0 \Rightarrow \lambda_{\nu;\lambda\sigma} = \lambda_{\nu;\sigma\lambda}$ (no curvature)

- We also already found that parallel transporting around a closed curve gives $\Delta\lambda^\mu \neq 0$

- Can also show that when $R^\mu_{\nu\lambda\sigma} \neq 0$, this follows as well.

Nov 2, 2018

$\boxed{\text{Riemann Curvature Tensor}}$

$$R^\mu_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\lambda} + \Gamma^\rho_{\nu\sigma}\Gamma^\mu_{\rho\lambda} - \Gamma^\rho_{\nu\lambda}\Gamma^\mu_{\rho\sigma}$$

flat spacetime $\Rightarrow R^\mu_{\nu\lambda\sigma} = 0$ everywhere
curved $\Rightarrow R^\mu_{\nu\lambda\sigma} \neq 0$ somewhere...

$R^\mu_{\nu\lambda\sigma} \Rightarrow$ has $4^4 = 256$ components. But not all are independent

$\boxed{\text{You'll prove}}$ $R^\mu_{\nu\lambda\sigma} + R^\mu_{\lambda\sigma\nu} + R^\mu_{\sigma\nu\lambda} = 0$ (cyclic identity)

Also if we lower

$R_{\mu\nu\lambda\sigma} = g_{\mu\sigma} R^{\sigma}_{\nu\lambda\sigma}$. Can prove that

$$\begin{cases} R_{\mu\nu\lambda\sigma} = -R_{\nu\mu\lambda\sigma} & \text{(anti-sym first 2 indices)} \\ R_{\mu\nu\lambda\sigma} = -R_{\mu\nu\sigma\lambda} & \text{(anti-sym second 2 indices)} \\ R_{\mu\nu\lambda\sigma} = R_{\lambda\sigma\mu\nu} & \text{(symmetric, double swap)} \end{cases}$$

These all follow from definitions in terms of $\Gamma^{\mu}_{\nu\sigma}$ & $g_{\mu\nu}$. With all these relations, there are only 20 independent ~~indices~~ components in $R^{\mu}_{\nu\lambda\sigma}$

Still, $g_{\mu\nu}$ has only 10 independent components.

$\Rightarrow$ We can look at contraction of $R^{\mu}_{\nu\lambda\sigma}$

$\boxed{\text{Look at Contraction}}$ $\rightarrow$

$$R^{\mu}_{\mu\lambda\sigma} = g^{\mu\rho} R_{\rho\mu\lambda\sigma} = -g^{\mu\rho} R_{\mu\rho\lambda\sigma} = \boxed{-R^{\rho}_{\rho\lambda\sigma} = R^{\mu}_{\mu\lambda\sigma}}$$

This says that $R^{\mu}_{\mu\lambda\sigma} = 0$ (vanishes)

There is a contraction that doesn't vanish

$$\boxed{R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma}} \quad \text{Ricci tensor.}$$

Show that symmetric: $R_{\mu\nu} = R_{\nu\mu}$

$\hookrightarrow$ Start w/ cyclic identity: $R^{\mu}_{\nu\lambda\sigma} + R^{\mu}_{\lambda\sigma\nu} + R^{\mu}_{\sigma\nu\lambda} = 0$

$\hookrightarrow$ Contract μ & σ

So $R^{\mu}_{\nu\lambda\mu} + R^{\mu}_{\lambda\mu\nu} + \underbrace{R^{\mu}_{\mu\nu\lambda}}_{0} = 0$

So $R^{\mu}_{\nu\lambda\mu} - R^{\mu}_{\lambda\nu\mu} = 0 \Rightarrow R_{\nu\lambda} - R_{\lambda\nu} = 0$ so $\boxed{R_{\nu\lambda} = R_{\lambda\nu}}$

- This means that $R_{\mu\nu}$ has only 10 independent components. Same as $g_{\mu\nu}$
- Lastly, we can define $\boxed{R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu}}$ → curvature scalar
- Back to Einstein equation. Einstein looked at combining $g_{\mu\nu}$, $T^{\mu\nu}$, and $R_{\mu\nu}$ and R in various combinations.

• One he tried + published in 1915 was $R^{\mu\nu} = k T^{\mu\nu}$ k = coupling constant

But this doesn't work, since $T^{\mu\nu}{}_{;\mu} = 0$ for energy-momentum conservation, but $R^{\mu\nu}{}_{;\mu} \neq 0$ in general. → $\boxed{\text{divergence of } T^{\mu\nu}}$

• Ultimately, he found the combination involving

$$\boxed{G^{\mu\nu} = R^{\mu\nu} - \frac{R}{2} g^{\mu\nu}}$$

→ Einstein tensor, which has $G^{\mu\nu}{}_{;\mu} = 0$ as identity → covariant divergence

<u>Note</u> $G^{\mu\nu}{}_{;\mu}$ = covariant divergence

Einstein settled on $\boxed{G^{\mu\nu} = k T^{\mu\nu}}$ → consistent with $T^{\mu\nu}{}_{;\mu} = 0$

Einstein defined k from Newtonian limit, that for weak fields you get the Poisson eq.

$$\vec{\nabla}^2 V = 4\pi G \rho$$

Requires ⇒ $k = \frac{-8\pi G}{c^4}$

<u>So this gives</u>

$$\boxed{R^{\mu\nu} - \frac{R}{2} g^{\mu\nu} = \frac{-8\pi G}{c^4} T^{\mu\nu}}$$

Einstein eqns

⇒ 10 equations (all eqns)

→ coupled, nonlinear, partial differential eqns.

This looks like a lot of guessing. But it's shown that possibilities are very limited...

- One can show mathematically that a tensor

$t^{\mu\nu}$ = a function of $g_{\mu\nu}$ + at most 2 derivatives that obeys

$$t^{\mu\nu}{}_{;\mu} = 0$$

A, B, C constants

Can be written as $\boxed{t^{\mu\nu} = AR^{\mu\nu} + BRg^{\mu\nu} + Cg_{\mu\nu}}$

The only generalization is the cosmological constant term
$[C = \Lambda]$

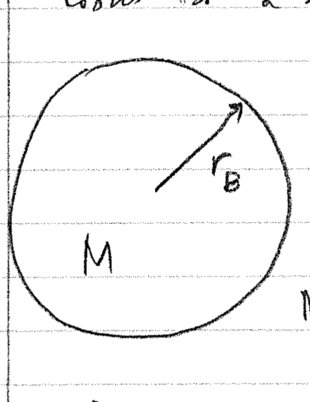
- Einstein's eqn with Λ is of the most general form

- We'll look at Einstein equations with + without Λ. We'll see that Λ is very important in cosmology
  → but in that context it's very small.

- On solar system scales $\Lambda \ll 1$ plays no role → Can ignore it for Earth, sun, etc...

$\boxed{\text{Schwarzschild Metric}}$ → 1916 → exact solution to Einstein eqn ($\Lambda = 0$)

Looks for a solution <u>outside</u> a static dist of mass



r_B → boundary radius

<u>Find</u> $g_{\mu\nu}$ for $r \geq r_B$. (outside)

<u>Note</u> Empty space for $r \geq r_B$. → $T^{\mu\nu} = 0$ (no E-Mom)

<u>Ex 3.5.1</u>, will show that $R = \frac{8\pi G}{c^4} T$, $T = T^{\mu}{}_{\mu} = g^{\mu\nu}T_{\mu\nu} = g_{\mu\nu}T^{\mu\nu}$

If $T^{\mu\nu} = 0$, then $T = 0$, so $R = 0$. So the Einstein eqn in empty space reduces to

$$\boxed{R^{\mu\nu} = 0}$$

Schwarzschild wrote down the general form of $g_{\mu\nu}$ for a static spherical symmetry, requiring that

$$\boxed{g_{\mu\nu} \to \eta_{\mu\nu} \text{ as } r \to \infty} \qquad (GR \to SR)$$

imposed $R^{\mu\nu} = 0$. requires agreement with Newtonian limit, where

$$\boxed{g_{00} = 1 + \frac{2V}{c^2} \quad \text{with } V = \frac{-GM}{r}} \quad \text{in weak, static limit}$$

Get Schwarzschild metric

$$\boxed{[g_{\mu\nu}] = \begin{pmatrix} \left(1 - \frac{2MG}{c^2 r}\right) & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}}$$

Note $M \to 0$

or $r \to \infty$

$$\to \boxed{[g_{\mu\nu}] \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}} = \eta_{\mu\nu} \text{ in spherical coordinates}$$

We want to study this metric. → Can apply it to Earth, Sun, or black hole.

For $r \geq r_B$, the line element ~~$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2$~~

$$\boxed{ds^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2dt^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2}$$

Resch used $m = \frac{GM}{c^2}$ → defines length

Can rewrite
$$\boxed{ds^2 = (1-2m/r)\,c^2dt^2 - (1-2m/r)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2}$$

Observe that
$$\boxed{g_{00} = \left(1-\frac{2GM}{c^2 r}\right)^{-1} \to \infty \text{ at } r = r_s = \frac{2GM}{c^2} = 2m}$$

Schwarzschild radius

Need to distinguish 2 types of object

- $r_B > r_s$ → no problem, since r_s is inside the object, while the solution is outside
  → planets
- $r_B < r_s$ → Black hole

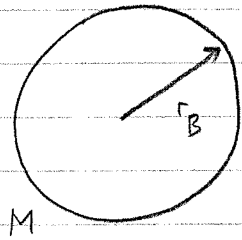
Note How big is r_s? → depends on mass ...

For $M = M_{earth}$ → $r_s = 0.009$ m
For Sun → $r_s = 3$ km

Nov 5, 2018 **VIII TESTS AND PREDICTIONS OF GR**

Want to investigate curvature near a planet or star

Consider Schwarzschild metric



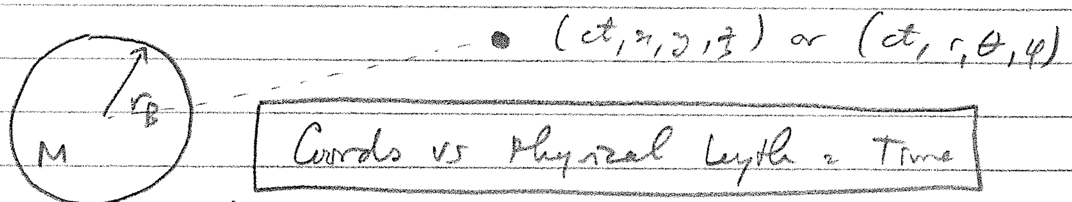
$r \geq r_B$ → no black holes

↳ How does GR differ from SR?

$\boxed{SR}$: a relativity theory. Coordinates x, t are physical lengths and times in frame (K), and x', t' are physical length and times in frame (K') $\Rightarrow$ measured by rulers and clocks. Related by LT's

$\boxed{GR}$ Theory of gravity $\Rightarrow$ can transform between frames but we generally don't do that.

<u>key difference</u> $\rightarrow$ coordinates do not give physical lengths and times



$\boxed{\text{Coords vs Physical Length + Time}}$

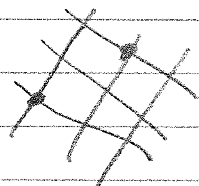
<u>Schwarzschild metric</u> is written in terms of coordinates.

(ct, r, θ, φ) $\rightarrow$ dimensional quantities that uniquely label points in spacetime. But they are not physical lengths, times.

The metric tensor + line element gives physical lengths, times

$$\boxed{ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\varphi^2}$$

<u>Like a city</u>



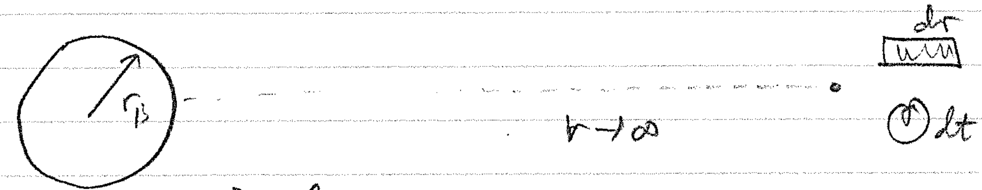
coords label points
<u>but</u> distances require more info (metric)

☒ Want to see how to use the metric to calculate lengths, times

(1) look at purely spatial + time separation (involves only ds^2)
(2) Consider moving in spacetime $\rightarrow$ this involves both ds^2 and the geodesic equations ...
$\rightarrow$ look at both massive + massless particles ...

{ Is there any situation where r, θ, ϕ, t become physical lengths & times?

↳ Yes! If we go far away $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$



why? because spacetime flattens → Minkowski space → coefficients of the metric go to 1 or −1

⇒ So, we will often talk about time & measurements made by far away observers → r, t.

$\boxed{\text{Lengths \& Times}}$

Note Schwarzschild metric has no t dep.

$$[g_{\mu\nu}] = \begin{pmatrix} \left(1-\frac{2GM}{rc^2}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^{+2} & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

⇒ Can separate space & time ... Take a $t = \text{const}$ slice of spacetime ... look at spatial geometry

So $dt = 0$ ⇒ 3D spatial geometry. We can also change the signs of the remaining components..

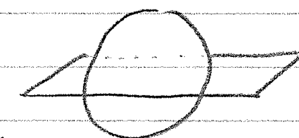
So $$[\gamma_{ij}] = \begin{pmatrix} +\left(1-\frac{2GM}{rc^2}\right)^{-1} & 0 & 0 \\ 0 & +r^2 & 0 \\ 0 & 0 & +r^2\sin^2\theta \end{pmatrix}$$

So, new line element:

$$\boxed{ds^2 = \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$

where $[\tilde{g}_{ij}] = [-g_{ij}]$ $\quad x^i = (r, \theta, \varphi)$

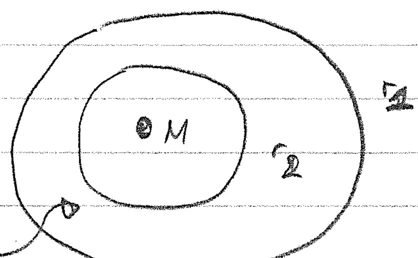
What is the geometry of this space? Consider $\theta = \frac{\pi}{2}$.

↳ Equitorial plane  . In fact, any slice thru the center will be the same ... What is the geometry of this?

↳ If $\theta = \frac{\pi}{2}$, find $\Rightarrow d\theta = 0$. So reduced to 2D surface

$$\boxed{ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\varphi^2} \qquad \left(\sin\frac{\pi}{2} = 1\right)$$

This describes a 2D sheet through equators. Consider 2 circles with coordinate radii r_1 & r_2



space is curved here so $R \geq \Delta r$

$\Delta r = r_1 - r_2$

For each we can find distance going around ...

$r = r_1 = \text{const}, \quad 0 \leq \varphi \leq 2\pi$
$r = r_2 = \text{const}, \quad 0 \leq \varphi \leq 2\pi$

Get $ds^2 = 0 + r^2 d\varphi^2 \qquad (dr = 0)$

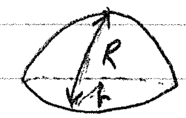
So $$\boxed{s = r\int_0^{2\pi} d\varphi = 2\pi r} \quad \text{(physical circumference)}$$

So $r = \frac{s}{2\pi}$. This suggests that r is the distance to the center.

To find radial distances → integrate r with φ fixed → dφ = 0

↳ $ds^2 = 0 + 0 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2$

⇒ $\boxed{s = R = \int_{r_2}^{r_1} \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} dr}$ R : physical distance

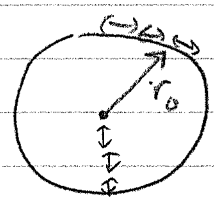


Notice: $R \geq \Delta r$, because $\frac{2GM}{rc^2} < 1$

Note To make measurements, we need calibrated rulers?

→ Open a factory at $r \to \infty$, build 1m sticks, then distribute them everywhere. Any measurements counts how many "1m" sticks are needed ...

So



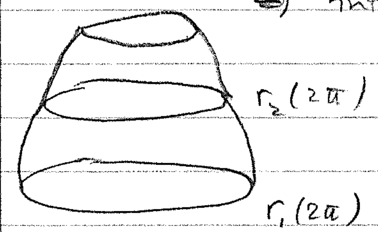
Find going around circumference need $2\pi r_0$ sticks

But going radial inward, we need more than r_0 sticks.

How do we visualize the geometry of this 2D sheet?

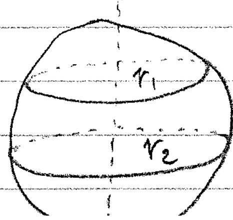
Use a hyperspace as an embedding space ...

⇒ introduce "fake" 3D in hyperspace ...

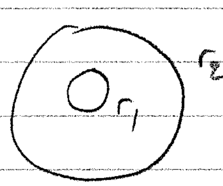


Find 2D sheet is the surface of a funnel in 3D hyperspace

Note Something happens on 2D sphere → curved 2D space



there's



$2\pi r_1$ $2\pi r_2$

But $R_{12} > \Delta r_{12}$

We see that the space near a static mass M is curved, but for the Earth & Sun the effects are small...

For Earth $\frac{2m}{r_B} = \frac{2GM}{c^2 r_B} \simeq 10^{-9}$ or $R \simeq r_1 - r_2$

For Sun $\frac{2m}{r_B} = \frac{2GM}{c^2 r_B} \simeq 10^{-6}$ or $R \simeq r_1 - r_2$

Small

Nov 6, 2018

Recall Schwarzschild solution $\rightarrow$ 2D shots $t = \text{const}, \theta = \frac{\pi}{2}$

$$\tilde{g}_{ij} = -g_{ij}$$

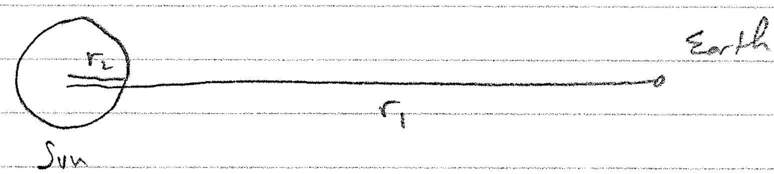
$$\Rightarrow ds^2 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\varphi^2$$

Circles $s = r \int_0^{2\pi} d\varphi = 2\pi r$

Radial distances: $\Delta R = \int \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} dr \geq \Delta R$

Embed sheet in 3D space

Ex Find ΔR & Δr between surface of the Sun & Earth



Use $r_2 = r_B = 7.0 \times 10^8\,m$
$r_1 = 1.5 \times 10^{11}\,m$

$$m = \frac{GM}{c^2} = 1482\,m \text{ for Sun}$$

So $\frac{2m}{r_1} \ll 1$, likewise $\frac{2m}{r_2} \ll 1$

So $\boxed{\left(1 - \frac{2m}{r}\right)^{-1/2} \simeq 1 + \frac{m}{r}}$ $\boxed{(1+x)^n \simeq 1 + nx,\ x \ll 1}$

So $\Delta R = \int_{r_2}^{r_1} \left(1 - \frac{2m}{r}\right)^{-1/2} dr = \int_{r_2}^{r_1} 1 + \frac{m}{r}\, dr = \Delta r_{12} + m \ln\left(\frac{r_1}{r_2}\right)$

$\Delta r \simeq 1.5 \times 10^{11}\,m$, $m \ln \frac{r_1}{r_2} \simeq 7.9 \times 10^3\,m \ll \Delta R - \Delta r \Rightarrow$ $\boxed{\Delta R \simeq \Delta r \simeq 1.5 \times 10^{11}}$

So $\boxed{\dfrac{\Delta R - \Delta r}{\Delta R} \simeq 5.3 \times 10^{-8}}$ → parts per 100 million

→ astronomers don't worry about this for our solar system

__Exact solution__ $\Delta R = \sqrt{r_1(r_1-2m)} - \sqrt{r_2(r_2-2m)} + 2m \ln\left(\dfrac{\sqrt{r_1}+\sqrt{r_1-2m}}{\sqrt{r_2}+\sqrt{r_2-2m}}\right)$

but answer still the same ...

$\boxed{\text{We also want to look at time intervals}}$

Clock @ rest in gravitational field → r, θ, φ constant

or $dr = d\theta = d\varphi = 0$

$s = c\tau$ $\quad t \neq \tau$ (proper time)

$$\Rightarrow ds^2 = c^2 d\tau^2$$
$$= \left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2$$

So $\boxed{d\tau = \left(1 - \dfrac{2MG}{c^2 r}\right)^{1/2} dt}$ → time dilation

τ = physical time on clock

t = coordinate time, or time of faraway clocks

Suppose 2 clocks at rest at 2 different location.

$$\Delta\tau_1 = \left(1 - \frac{2GM}{c^2 r_1}\right)^{\frac{1}{2}} \Delta t \qquad r = r_1$$

$$\Delta\tau_2 = \left(1 - \frac{2GM}{c^2 r_2}\right)^{1/2} \Delta t \qquad r = r_2$$

So $\boxed{\dfrac{\Delta\tau_1}{\Delta\tau_2} = \sqrt{\dfrac{1 - 2GM/c^2 r_1}{1 - 2GM/c^2 r_2}}}$ Gravitational time dilation

So that if $r_2 < r_1$, then $\Delta\tau_2 < \Delta\tau_1$

→ __time goes slower__ in __stronger gravitational field__

__But__ everything slows down together → don't notice anything

Because we'll measure <u>slowed down events</u> with <u>slowed down clocks</u>..

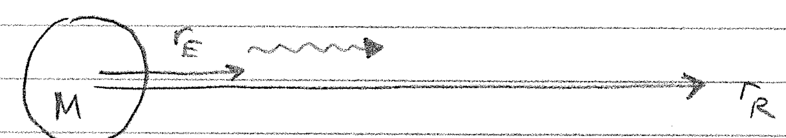
↳ Need 2 different locations to detect anything. Can compare clocks on the ground v. clocks on airplane / satellite.

→ Experiments agree with general relativity. (GPS)

<u>OR</u> send signals between 2 places ⇒ spectral shift (gravitational)

Gravitational Spectral Shift

Consider light emitted & received at 2 locations



E: emitted

R: received

Put clocks at r_E & r_R and time n cycles of light to pass

<u>Frequencies</u> $\quad \nu_E = \dfrac{n}{\Delta\tau_E}$

$$\nu_R = \frac{n}{\Delta\tau_R}$$

Each proper time is related to a coor. time.

$$\boxed{\begin{aligned} \Delta\tau_R &= \sqrt{1 - \frac{2GM}{c^2 r_R}}\,\Delta t_R \\ \Delta\tau_E &= \sqrt{1 - \frac{2GM}{c^2 r_E}}\,\Delta t_E \end{aligned}}$$

Δt_E = coord time for emission of n cycles $= t_E^{(n)} - t_E^{(0)}$

Δt_R is analogous

end of last wave

start of first wave

Light moves in null trajectory

↳ $$ds^2 = 0 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2$$

$$= \left(1 - \frac{2m}{r}\right) c^2 dt^2 + \tilde{g}_{ij}\, dx^i dx^j$$

So $$\boxed{dt = \frac{1}{c}\left[\left(1 - \frac{2GM}{rc^2}\right)^{-1} g_{ij}\, dx^i dx^j\right]^{\frac{1}{2}}}$$

Get
$$\boxed{t_R^{(0)} - t_E^{(0)} = \frac{1}{c}\int \left[\left(1-\frac{2GM}{c^2 r}\right)^{-1} \tilde{g}_{ij}\, dx^i dx^j\right]^{1/2}}$$

no t dependence. Get the same RHS for the end of n^{th} wave

$$t_R^{(n)} - t_E^{(n)} = t_R^{(0)} - t_E^{(0)}$$

so

$$\boxed{\Delta t_R = \Delta t_E}$$

So

$$\frac{\Delta \tau_R}{\sqrt{1-\frac{2GM}{c^2 r_R}}} = \frac{\Delta \tau_E}{\sqrt{1-\frac{2GM}{c^2 r_E}}}$$

So
$$\boxed{\frac{\nu_R}{\nu_E} = \frac{n/\Delta\tau_R}{n/\Delta\tau_E} = \frac{\Delta\tau_E}{\Delta\tau_R} = \sqrt{\frac{1-2GM/c^2 r_E}{1-2GM/c^2 r_R}}}$$

grav. spectral shift $m = \frac{GM}{c^2}$. For $\frac{2m}{r} = \frac{2GM}{c^2 r} \ll 1$,

we get $\frac{\nu_R}{\nu_E} \simeq \frac{1-m/r_E}{1-m/r_R}$

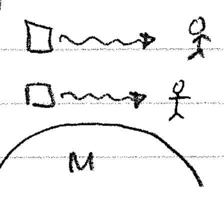
OR
$$\boxed{\frac{\Delta\nu}{\nu_E} = \frac{\nu_R - \nu_E}{\nu_E} \simeq \frac{GM}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right)}$$

For $r_R > r_E \Rightarrow \Delta\nu < 0 \rightarrow$ redshift (away from g)

$r_E > r_R \Rightarrow \Delta\nu > 0 \rightarrow$ blueshift (into g)

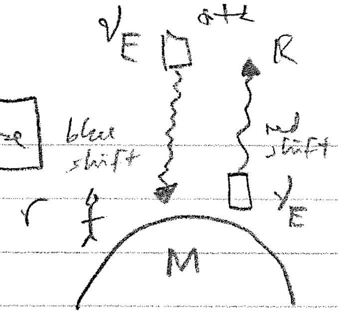
Suppose

For each observer, $\nu_R = \nu_E$, because $\Delta r = 0$ for each



Note ν_E at source is always the same because no one changed light with changed eyes (ν_E same for both)

$\boxed{\text{Suppose}}$



Note Same ν_E for both

But for $R \to$ see redshifted light
for $r \to$ see blueshifted light

Pound – Rebka experiment confirmed this (@Harvard)

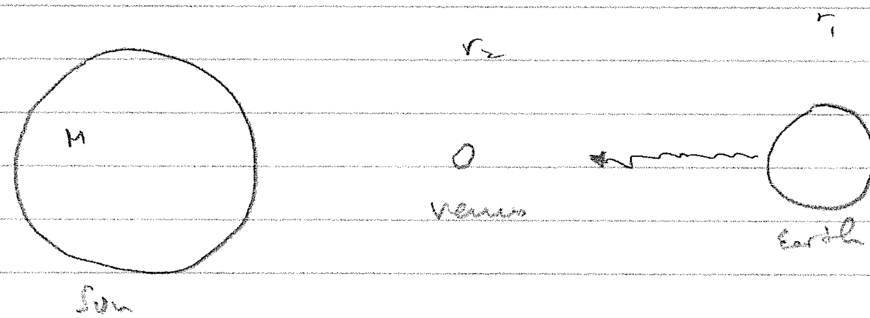
Nov 7, 2018 Recall $dR = \left(1 - \frac{2GM}{rc^2}\right)^{-1/2} dr$ for $t = $ const,

and $d\tau = \left(1 - \frac{2GM}{c^2 r}\right)^{1/2} dt$ time on clock at rest

$$\frac{\Delta \nu}{\nu_E} = \frac{\nu_R - \nu_E}{\nu_E} \simeq \frac{GM}{c^2}\left(\frac{1}{r_R} - \frac{1}{r_E}\right) \quad \text{spectral shift}$$

$\boxed{\text{Radar Time Delay Experiment}}$ → provides one of the last tests of GR

Consider



- bounce radar off Venus with Sun behind
  → Time the round trip using a clock at rest on Earth

Might expect $\Delta \tau = 2\frac{(r_2 - r_1)}{c} = \frac{2\Delta r}{c}$

But there's actually a time delay

For light → $ds^2 = 0$ (null line element)
Set θ const, φ const, ~~etc~~

So $ds^2 = 0 = \left(1 - \frac{2GM}{c^2 r}\right)^{-1} c^2 dt - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2$

So $\boxed{\dfrac{dr}{dt} = \pm c\left(1 - \dfrac{2GM}{c^2 r}\right)}$ → coordinate speed of light

So that $\boxed{\left|\dfrac{dr}{dt}\right| < c}$ → coordinate speed of light $< c$

But far away as $r \to \infty$ → $\left|\dfrac{dr}{dt}\right| = c$

Also, light has no proper time, and hence no world velocity $u^\mu = \dfrac{dx^\mu}{d\tau}$ not defined ...

How long does a round trip take, measured with a clock on Earth?

For r, t

$$dt = \frac{1}{c}\left(1 - \frac{2GM}{r c^2}\right)^{-1} dr \simeq \left(1 + \frac{2GM}{rc^2}\right) dr$$

So

$$dt = 2\int_{r_2}^{r_1} \frac{1}{c}\left(1 + \frac{2GM}{rc^2}\right) dr = \boxed{\frac{2}{c}\Delta r + \frac{4m}{c}\ln\left|\frac{r_1}{r_2}\right| = \cancel{\Delta t}}$$

round trip

↑ coordinate time

For clock on Earth $\boxed{\Delta\tau = \left(1 - \dfrac{2m}{r}\right)^{1/2}\Delta t =}$

here $m = \dfrac{GM}{c^2}$ where M is the Sun's mass. There is a gravitational effect due to Earth's mass, but it's smaller than due to that of the Sun

$\left(\dfrac{2m}{r_E}\right)_{earth} << \left(\dfrac{2m}{r_1}\right)_{Sun}$. This is even true classically for $V = \dfrac{1}{r}$ (Newtonian potential)

But not true for acc. $g = \dfrac{1}{r^2}$ → $\boxed{\text{Earth's } g \text{ wins, but Sun's potential wins}}$

Expand $\left(1 - \dfrac{2GM}{c^2 r_1}\right)^{1/2} \simeq 1 - \dfrac{GM}{c^2 r_1}$

So $\boxed{\Delta\tau = \left(1 - \dfrac{2m}{r_1}\right)^{1/2}\Delta t \simeq \left(1 - \dfrac{GM}{c^2 r_1}\right)\left[\dfrac{2}{c}\Delta r + \dfrac{4m}{c}\ln\left|\dfrac{r_1}{r_2}\right|\right]}$

So $\boxed{\Delta\tau_{GR} \simeq \frac{2}{c}\Delta r - \frac{2m}{r_1 c}\Delta r + \frac{4m}{c}\ln\left|\frac{r_1}{r_2}\right|}$ → (Measured...)

Compare this with expected result $\boxed{2\Delta R = c\Delta\tilde{\tau}}$ → expected time

physical distance to Venus (for a $t=$ const time)

So $\Delta\tilde{\tau} = \frac{2}{c}\Delta R$

$$= \frac{2}{c}\int_{r_2}^{r_1}\left(1-\frac{2GM}{c^2 r}\right)^{-1/2} dr \simeq \frac{2}{c}\int_{r_2}^{r_1}\left(1+\frac{GM}{c^2 r}\right) dr$$

$$\rightarrow \Delta\tilde{\tau} \simeq \frac{2}{c}\left[\Delta r + m\ln\left|\frac{r_1}{r_2}\right|\right]$$

$$\boxed{\Delta\tilde{\tau}_E = \frac{2}{c}\Delta r + \frac{2}{c}m\ln\left|\frac{r_1}{r_2}\right|} \longrightarrow \text{(Expected)}$$

We see that $\boxed{\Delta\tau_{GR} \neq \Delta\tilde{\tau}.}$

Note $\Delta\tau_{GR} - \Delta\tilde{\tau} \simeq \frac{2GM}{c^3}\left(\ln\left|\frac{r_1}{r_2}\right| - \frac{\Delta r}{r_1}\right) > 0$

So $\boxed{\Delta\tau_{GR} - \Delta\tilde{\tau} > 0}$ → GR predicted a time delay

What does this mean? → Up to interpretations...

issue → (1) Speed of light is slowed down in GR. True that $\left|\frac{dr}{dt}\right| < c$, but this is not the physical speed.
→ this interpretation seems misleading

(2) Different interpretation ⇒ (i) you can't use a clock on Earth and ΔR for a $t=$ const slice for light moving through different grav.
↳ clocks run differently all along the way

(ii) We're also using ΔR that assumes $t=$ constant. But the light is moving

→ The predicted GR result takes all of this into account and gives a different answer...

$\boxed{\text{Question}}$ What speed does light have in GR? Again, $u^\mu = \frac{dx^\mu}{dt}$ is NOT defined. But, we can also go to freely falling frames...

$$g_{\mu\nu} \to \eta_{\mu\nu}, \quad ds^2 = 0 = \eta_{\mu\nu} dx^\mu dx^\nu$$

This gives $c^2 dt^2 - dx^{i\,2} = 0$ or $\left|\frac{dx^i}{dt}\right| = c$

But what about in a non-inertial frame?
⇒ need to measure the speed locally (in a lab)
→ use local clocks
⟶ dR
Ⓑ $d\tau$ Use clock at rest in lab for light passing by

$$dR = \left(1 - \frac{2m}{r}\right)^{-1/2} dr, \quad d\tau = \left(1 - \frac{2m}{r}\right)^{1/2} dt$$

So $$\boxed{\frac{dR}{d\tau} = \left(1 - \frac{2m}{r}\right)^{-1} \frac{dr}{dt}}$$

physical speed — coord. speed

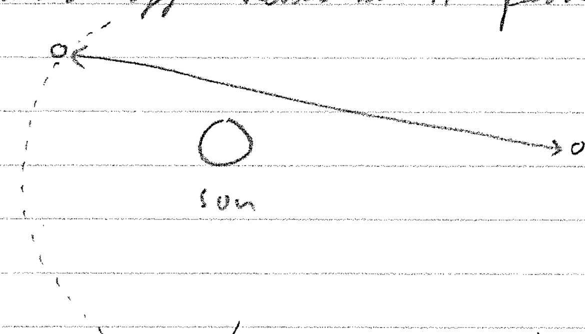
But $$\frac{dr}{dt} = \pm c\left(1 - \frac{2m}{r}\right)$$

So $$\boxed{\frac{dR}{d\tau} = \pm c}$$ → speed of light is still c.

But we can't ~~co~~ conclude that going a distance $2\Delta R$ gives $\tau = \frac{2\Delta R}{c}$ because instead, GR predicts an extra delay.

Experiments of Shapiro (1968–1971)

→ did radar delay experiments. Measured delays of radar bounces off Venus as it passes behind the Sun



Can't compute time delay accurately to test GR, but instead → look at change in delay, fit data to GR

Put

$$g_{\mu\nu} = \begin{pmatrix} \left(1-\frac{2M}{r}\right) & 0 & 0 & 0 \\ 0 & -\left(1-\gamma\frac{2M}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

where γ is a parameter. They fit γ to data to find the best value.

→ Shapiro et al. found that $\boxed{\gamma \simeq 1.03 \pm 0.01}$

→ Consistent with Schwarzschild metric that predicts $\gamma=1$

Improved tests have taken this below 1%

v 9, 2018 $\boxed{\text{Particle Motion in Schwarzschild geometry}}$

5 variables

Massive particle → has proper time $c^2 d\tau^2 = ds^2$

$$\boxed{c^2 d\tau^2 = ds^2 = \left(1-\frac{2m}{r}\right)dt^2c^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2}$$

We also have geodesic equation:

$$\boxed{\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{d\tau}\frac{dx^\lambda}{d\tau} = 0}$$

→ we have 4 more equations

→ can solve for 5 variables.

$\boxed{\text{We need connection}}$

$$\boxed{\Gamma^{\mu}_{\rho\sigma} = \frac{1}{2} g^{\mu\lambda}\left(\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\rho\lambda} - \partial_\lambda g_{\rho\sigma}\right)} \longrightarrow \text{refer to sheet for connection} \sim \text{metric}$$

We have 4 eqns with $\mu = 0, 1, 2, 3$. We can write using dot notation

$\hookrightarrow \dot{t} = \frac{dt}{d\tau}$, $\ddot{t} = \frac{d^2 t}{d\tau^2}$, etc

So geodesic eqn becomes $\boxed{\ddot{X}^\mu + \Gamma^\mu_{\nu\sigma}\dot{X}^\nu \dot{X}^\sigma = 0}$

So $\quad c\ddot{t} + 2\Gamma^0_{01}(c\dot{t})(\dot{r}) = 0 \qquad$ (only $\Gamma^0_{01} = \Gamma^0_{10} \neq 0$)

$$\rightarrow \boxed{\ddot{t} + \frac{2m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}\dot{t}\dot{r} = 0} \qquad m = \frac{GM}{c^2}$$

Repeat for $\mu = 1, 2, 3 \Rightarrow$ we get 3 more equations...

$$\mu = 1 \quad \boxed{\ddot{r} + \frac{mc^2}{r^2}\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(\frac{-m}{r^2}\right)\left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 + (-r + 2m)\dot{\theta}^2 - r\sin^2\theta\left(1 - \frac{2m}{r}\right)\dot{\varphi}^2 = 0} \quad \mu = 1$$

$$\mu = 2 \quad \boxed{\ddot{\theta} + \frac{2\dot{r}}{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = 0}$$

$$\mu = 3 \quad \boxed{\ddot{\varphi} + \frac{2\dot{r}}{r}\dot{\varphi} + \frac{2\cot\theta}{r}\dot{\theta}\dot{\varphi} = 0}$$

<u>Consider planar motion:</u> $\theta = \frac{\pi}{2} \Rightarrow 2^{\text{nd}}$ eqn goes away $(\dot{\theta} = \ddot{\theta} = 0)$

$\sin\theta = 1, \ \cos\theta = 0$

<u>So</u>

$\mu = 0 \quad \ddot{t} + \frac{2m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}\dot{t}\dot{r} = 0$

$\mu = 1 \quad \ddot{r} + \frac{mc^2}{r^2}\left(1 - \frac{2m}{r}\right)\dot{t}^2 - \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}\dot{r}^2 - r\left(1 - \frac{2m}{r}\right)\dot{\varphi}^2 = 0$

$\mu = 3 \quad \ddot{\varphi} + \frac{2\dot{r}}{r^2}\dot{\varphi} = 0$

For first eq, divide by $\dot{t}$

$\Rightarrow$ $\mu = 0$: $$\frac{1}{\dot{t}}\frac{d\dot{t}}{d\tau} = \frac{-2m/r^2}{\left(1-\frac{2m}{r}\right)}\frac{dr}{d\tau}$$

$$\Rightarrow \int \frac{d\dot{t}}{\dot{t}} = \int \frac{-2m/r^2}{1-2m/r}\,dr$$

So $$\ln \dot{t} = -\ln\left(1-\frac{2m}{r}\right) + C$$

So $$\boxed{\dot{t} = k\left(1-\frac{2m}{r}\right)}$$

For 3rd eqn, write it as $\frac{d}{d\tau}(r^2\dot{\varphi}) = 0$

Get $\boxed{r^2\dot{\varphi} = h, \quad h = \text{const}}$

We get
$$\boxed{\begin{aligned} &\left(1-\frac{2m}{r}\right)^{-1}\ddot{r} + \frac{mc^2}{r^2}\dot{t}^2 + \left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2 - r\dot{\varphi}^2 = 0 \\ &\left(1-\frac{2m}{r}\right)^{-1}\dot{t} = k \\ &r^2\dot{\varphi} = h \end{aligned}}$$

Line element $\theta = \frac{\pi}{2}$

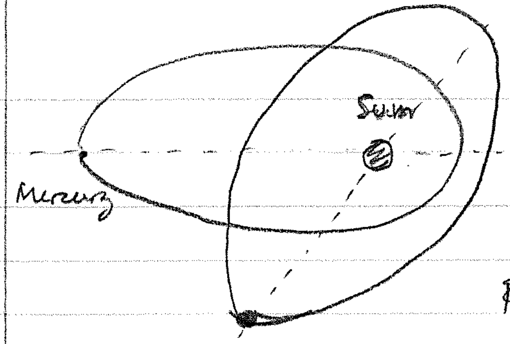
$$\boxed{c^2 = c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2}$$ massive

$\Rightarrow$ 4 eqns for 4 unknowns with k, h constants

$\rightarrow$ can solve for r, t, φ, τ

Ex

Using these eqns Einstein calculated the precession of Mercury's perihelion (point of closest approach)



In Newtonian physics, there's a precession rate of 532''/century caused by other planets...

But there was always an extra 43''/century that could not be explained...

Einstein did the calculation and find an extra 43''/century.
We're not going to worry about the calculations (Sec 4.5)

Light motion For light, we must use null line element..
→ can't use τ as parameter.

$ds^2 = 0$. Line element mid $\theta = \frac{\pi}{2}$ plane is

$$\boxed{\begin{aligned} 0 &= \left(1-\frac{2m}{r}\right)c^2dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\varphi^2 \\ &= \left(1-\frac{2m}{r}\right)c^2dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2 - r^2d\varphi^2 \end{aligned}}$$

Param the null trajectory with ω (not s or τ).

Have $\dot{x}^\mu = \frac{dx^\mu}{d\omega}$ and so on... Any ω is good as long as it gives lightlike trajectory $0 = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$

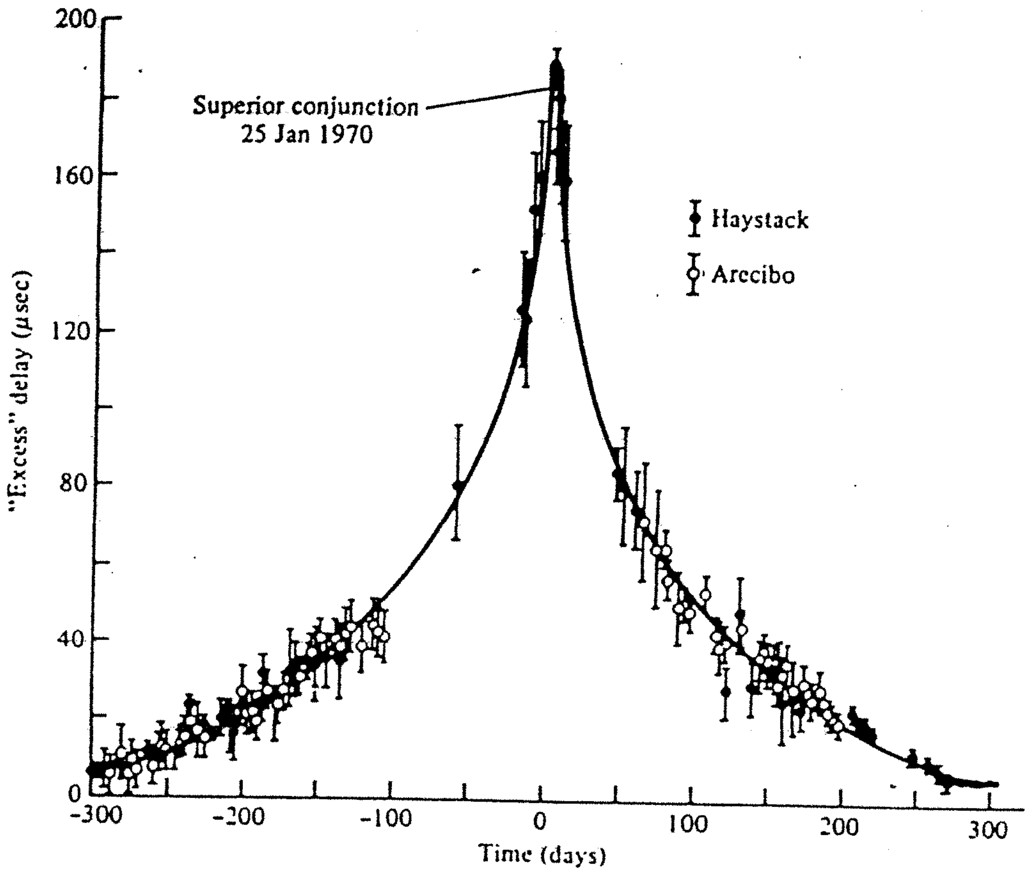
Can use ω in geodesic eqn → moves as free particle...

$$\boxed{\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0} \iff \frac{d^2x^\mu}{d\omega^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\omega}\frac{dx^\sigma}{d\omega} = 0$$

Let $\dot{t} = \frac{dt}{d\omega}$ and so on... → get the same eqns (1) (2) (3) on sheet...

Can also divide line element by $d\omega^2$

massless →
$$\Rightarrow \boxed{c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2 = 0} \quad \text{(null line element)}$$



Results of Earth-Venus time-delay measurements. The solid curve gives the theoretical prediction. (From Shapiro et al., 1971.)

Schwarzschild Solution

$$c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad\qquad m = \frac{GM}{c^2} \Rightarrow \text{a length}$$

$$c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\right)$$

With the Schwarzschild metric, we can compute the nonzero Christoffel symbols:

$$\Gamma^0_{01} = \frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1} \qquad \Gamma^1_{00} = \frac{mc^2}{r^2}\left(1 - \frac{2m}{r}\right) \qquad \Gamma^1_{11} = -\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1}$$

$$\Gamma^1_{22} = -(r - 2m) \qquad \Gamma^1_{33} = -r \sin^2\theta \left(1 - \frac{2m}{r}\right) \qquad \Gamma^2_{12} = \frac{1}{r}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \qquad \Gamma^3_{13} = \frac{1}{r} \qquad \Gamma^3_{23} = \cot\theta$$

Using these, we can write out the geodesic equations:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

If we restrict the solution to the plane ($\theta = \pi/2$), we get three equations for $\ddot{r}$, $\ddot{t}$, and $\ddot{\phi}$, where $\dot{r} = \frac{dr}{d\tau}$, etc. Two of these equations can be integrated once, which introduces integration constants k and h. The resulting three equations are:

$$\left(1 - \frac{2m}{r}\right)^{-1} \ddot{r} + \frac{mc^2}{r^2}\dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-2} \frac{m}{r^2}\dot{r}^2 - r\,\dot{\phi}^2 = 0 \tag{1}$$

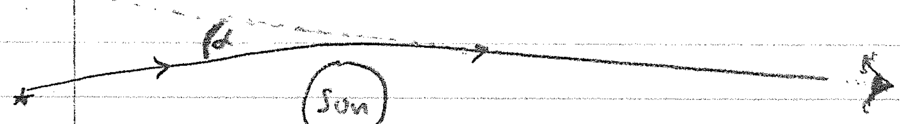
$$\left(1 - \frac{2m}{r}\right)\dot{t} = k \tag{2}$$

$$r^2\dot{\phi} = h \tag{3}$$

Eqs. (1), (2), and (3) are, respectively, Eqs. (4.21), (4.22), and (4.23) in the book. These equations along with the line element $c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ are used to study the motion of nonzero mass particles along geodesics in the Schwarzschild geometry.

Using these Einstein calculated deflection of light passing close by the Sun...

image ☆



$\Delta\alpha$: deflection angle.

Einstein predicted that $\Delta\alpha = 1.75''$. This was measured by Sir Eddington in 1919 (Sec 4.6)

Ex $\boxed{\text{Can light have circular orbit?}}$

Yes, but only for $r = 3m$ $\Rightarrow$ need solutions for $r_B < 3m$
$\Rightarrow$ Need either blackhole with $r_B < 2m$ or very close...

Nov 12, 2018

Look at a plane $\theta = \pi/2$ with line element

$$0 = c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 - r^2\dot{\varphi}^2$$

But for circular orbit $\Rightarrow$ $\dot{r} = \ddot{r} = 0$

$$\Rightarrow \boxed{0 = c^2\left(1-\frac{2m}{r}\right)\dot{t}^2 - r^2\dot{\varphi}^2} \qquad (1)$$

The r-geodesic eqn (sheet)

$$\hookrightarrow \left(1-\frac{2m}{r}\right)\ddot{r}^2 + \frac{mc^2}{r^2}\dot{t}^2 - \left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2 - r\dot{\varphi}^2 = 0$$

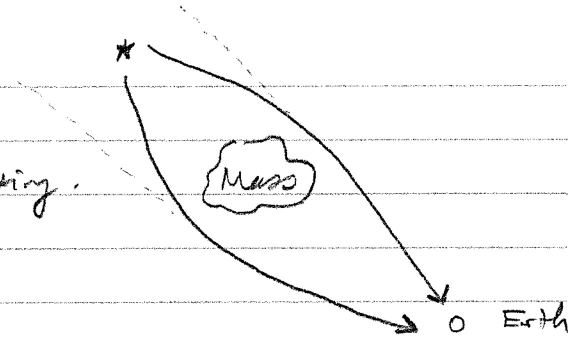
But with $\dot{r} = \ddot{r} = 0$

$$\hookrightarrow \frac{mc^2}{r^2}\dot{t}^2 - r\dot{\varphi}^2 = 0 \quad \text{or} \quad \boxed{\frac{mc^2}{r}\dot{t}^2 - r^2\dot{\varphi}^2 = 0} \qquad (2)$$

$$(1) \text{ \& } (2) \Rightarrow \boxed{r = 3m = \frac{3GM}{c^2}}$$ $\leftarrow$ radius of a circular orbit for light

Other Tests of GR

→ gravitational lensing.
See double images, rings, circles...



Binary pulsar → radiate gravitational wave, lose energy. slow down ⇒ rate of slowing down agrees with GR

Gravity Waves detected directly at LIGO 2015 - 2016

BLACK HOLES

For $r > r_B$ → Schwarzschild solution

$$ds^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2$$

See that for event horizon $r = 2m$ → the $(1,1)$ component of the metric → $\boxed{g_{11} \to \infty \text{ as } r \to 2m}$

Also, there's another singularity $\boxed{g_{00} \to -\infty, \ g_{11} \to \infty \text{ as } r \to 0}$

For Sun, Earth, etc $r_B \gg 2m$ → no problem

But for some objects, singularities matter ⇒ $r_B < r_s = 2m$ ↘ singularity

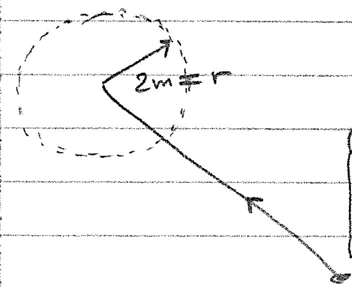
Such objects are blackholes

For the sun $2m \simeq 3\,km$.

Will look at blackholes for
- $r > 2m$, outside ...
- $r < 2m$, inside ...
- $r = 2m$, event horizon ...

$\boxed{\text{Consider radial trajectory (massive objects)}}$

↳ Fall radially from rest from $r = r_0$ into a black hole...



$\boxed{\begin{matrix}\dot{r} = 0 \\ r = r_0\end{matrix}}$, θ = constant, φ = constant

$\dot{\varphi} = \dot{\theta} = 0$

Start with line element

Put in terms of proper time

$$\boxed{c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2}$$

proper time (τ)

Divide by $d\tau^2$

↳ $$\boxed{c^2 = \left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2}$$

Apply $\dot{r} = 0$ at $r = r_0 \Rightarrow c^2 = \left(1 - \frac{2m}{r_0}\right) c^2 \dot{t}^2$ at r_0

So $$\boxed{\dot{t} = \left(1 - \frac{2m}{r_0}\right)^{-1/2}}$$

or $$\boxed{dt = \left(1 - \frac{2m}{r_0}\right)^{-1/2} d\tau}$$

Look also at $\ddot{t}$ geodesic equation

↳ $\left(1 - \frac{2m}{r}\right) \dot{t} - k = 0$, where k is a constant (Eq. 2 on sheet)

So, at $r = r_0$, to find k.

↳ $$\boxed{k = \dot{t}\left(1 - \frac{2m}{r_0}\right) = \left(1 - \frac{2m}{r_0}\right)^{-1/2}\left(1 - \frac{2m}{r_0}\right) = \left(1 - \frac{2m}{r_0}\right)^{1/2}}$$

Can we interpret this constant?

For $\frac{m}{r_0} \ll 1$, then $\boxed{k \simeq 1 - \frac{m}{r_0} = 1 - \frac{GM}{c^2 r_0}}$

Suppose our object has mass M_0, then its rest energy + potential energy

$$E = M_0 c^2 - \frac{GMM_0}{r_0} \quad \text{at } r = r_0$$

(no KE)

So $\boxed{\frac{E}{M_0 c^2} = 1 - \frac{GM}{c^2 r_0} \cong K}$

So $\boxed{K \text{ is a ratio between total energy vs. rest energy}}$

Can make another approximation. Let $r_0 \to \infty$, then $K \simeq 1$

→ Can use $K \simeq 1$ for falling from rest far away where $r_0 \to \infty$.

But we also don't want $r_0 = \infty$ exactly, just big enough.

→ We assume $\underline{r_0 \text{ is big enough}}$ so we can use $\underline{K = 1}$

Then $K = \left(1 - \frac{2m}{r}\right)\dot{t}$ holds $\forall r, t$

becomes $\boxed{\dot{t} = \left(1 - \frac{2m}{r}\right)^{-1} = \frac{dt}{d\tau}}$ → coordinate time, → proper time

or $\boxed{d\tau = \left(1 - \frac{2m}{r}\right) dt}$ → This is for massive object falling on a geodesic (with $\dot{r} = 0$, at $r \simeq \infty$)

Note this is different from the time dilation formula: $d\tau = \left(1 - \frac{2m}{r}\right)^{1/2} dt$ for clock at rest...

$\boxed{\text{What's the problem?}}$ Clock at rest doesn't follow geodesic! Clock at rest has a net force in g field

→ not free falling...

☑ Here $d\tau = \left(1 - \frac{2m}{r}\right) dt$ is the proper time of a falling object or observer... (their wristwatch time)

Go back to line element. $c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2$

→ plug in $d\tau^2 = \left(1 - \frac{2m}{r}\right)^2 dt^2$

So $\quad c^2\left(1-\frac{2m}{r}\right)^2 dt^2 = c^2\left(1-\frac{2m}{r}\right)dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2$

So $\quad \left(1-\frac{2m}{r}\right)dt^2 = dt^2 - \frac{1}{c^2}\left(1-\frac{2m}{r}\right)^{-2}dr^2$

So $\quad 1-\frac{2m}{r} = 1 - \frac{1}{c^2}\left(1-\frac{2m}{r}\right)^{-2}\left(\frac{dr}{dt}\right)^2$

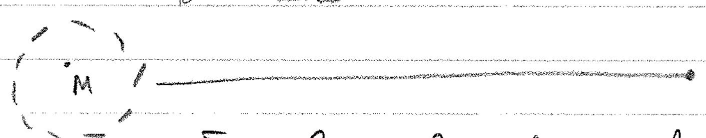
So $\quad \left(\frac{dr}{dt}\right)^2 = +\frac{2mc^2}{r}\left(1-\frac{2m}{r}\right)^2$ $\qquad$ falling into blackhole

So $\quad \frac{dr}{dt} = -\sqrt{\frac{2mc^2}{r}\left(1-\frac{2m}{r}\right)^2} = -c\sqrt{\frac{2m}{r}}\left(1-\frac{2m}{r}\right)$

falling into blackhole ... $\longrightarrow$ So $\quad \boxed{\frac{dr}{dt} = -c\sqrt{\frac{2m}{r}}\left(1-\frac{2m}{r}\right)}$ $\rightarrow$ coordinate velocity w.r.t clocks far away ...

v 13, 2018

Black hole



$\dot{r} = 0 \quad , r \gg$

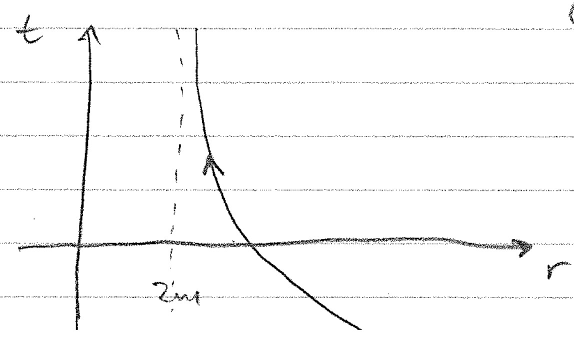
From line element $\Rightarrow \frac{dr}{dt} = -c\sqrt{\frac{2m}{r}}\left(1-\frac{2m}{r}\right)$

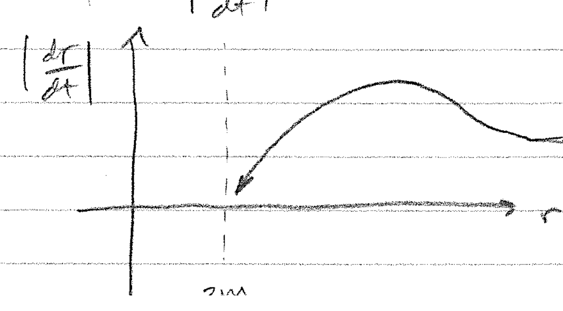
We can integrate to find ~~r(t)~~ t in terms of r

$$\rightarrow \boxed{t = \int_\infty^r \frac{dt}{dr}\,dr = -\int_\infty^r \frac{dr}{c}\sqrt{\frac{r}{2m}}\left(1-\frac{2m}{r}\right)^{-1} = \infty}$$

Must cut off integral at some larger r ... Can numerically evaluate & plot t vs. r

Can also plot $\left|\frac{dr}{dt}\right|$ vs. r





With t = time on far away clocks. Viewers at ∞ see the falling object <u>slowing</u> as $r \to 2m$ & it <u>never reaches</u> the horizon

$$\left|\frac{dr}{dt}\right| \to 0 \quad \text{as} \quad r \to 2m$$

What about for the falling observer with τ = their proper time.
Can look at $\frac{dr}{d\tau}$ <u>and</u> τ vs. r

$\to$ Use $\frac{dt}{d\tau} = \left(1 - \frac{2m}{r}\right)^{-1}$ $\to$ Eq. 2 on sheet with $K = 1$

<u>Chain rule</u>

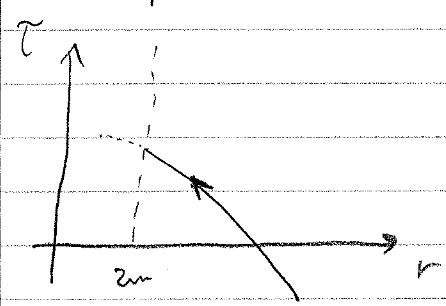
$$\frac{dt}{dr}\frac{dr}{d\tau} \Rightarrow \frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}$$

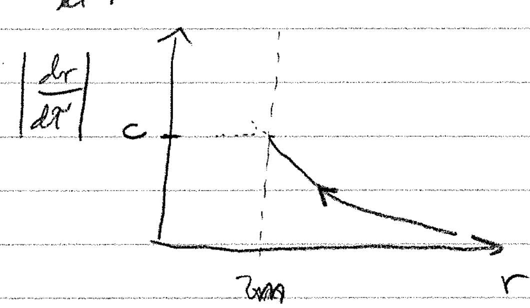
So $\frac{dr}{d\tau} = -c\sqrt{\frac{2m}{r}}\left(1 - \frac{2m}{r}\right)\left(1 - \frac{2m}{r}\right)^{-1} = -c\sqrt{\frac{2m}{r}}\left(1 - \frac{2m}{r}\right)^{0}$

So $\boxed{\frac{dr}{d\tau} = -c\sqrt{\frac{2m}{r}}}$

We can integrate to get $\boxed{\tau = -\int_{-\infty}^{r} \frac{dr}{c}\sqrt{\frac{r}{2m}} = -\int_{-\infty}^{r} \frac{dr}{c}\sqrt{\frac{r}{2m}}}$

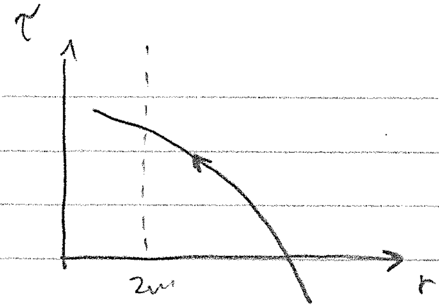
Can plot τ vs. r and $\left|\frac{dr}{d\tau}\right|$ vs. r





For a falling observer, he/she reaches event horizon in finite time in <u>faster & faster rate</u>. In fact, since nothing blows up, the observer passes right through event horizon

↳ If we do that we can complete the picture



The falling observer reaches $r=0$ in finite τ

$\boxed{\text{Ex}}$ Calculate the proper time to go from $r = 2m$ to $r=0$

$$\tau = -\int_{2m}^{0} \frac{dr}{c}\sqrt{\frac{r}{2m}} = \frac{4m}{3c} = \boxed{\frac{4GM}{3c^3}}$$

For $M = M_{sun}$ → $\boxed{\tau = 6.5\,\mu s}$

$\boxed{\text{Summarize}}$ → two different views.

→ far away observer says you never reach the event horizon.

→ but as you fall, you find you cross the horizon + head into $r=0$ in a finite time

To understand this better, let's look ~~at~~ light signal.
Suppose the falling observer sends light signals outward

$\boxed{\text{Light Rays}}$ → follow null trajectories... $ds^2 = 0$

radial ⇒ $d\theta = d\varphi = 0$ ⇒ $ds^2 = 0 = \left(1-\frac{2m}{r}\right)c^2dt^2 - \left(1-\frac{2m}{r}\right)^{-1}dr^2$

So $\boxed{\frac{dr}{dt} = \sqrt{c^2\left(1-\frac{2m}{r}\right)^2} = \pm c\left(1-\frac{2m}{r}\right)}$

coord velocity of radial light waves...

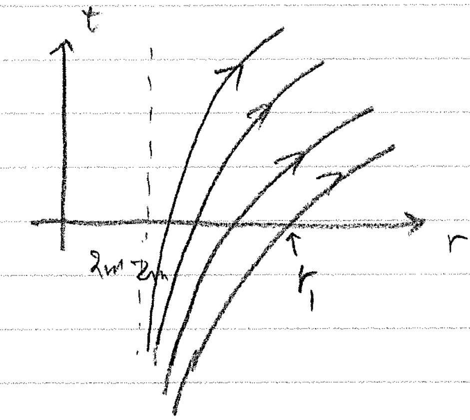
So, we can integrate $\int c\,dt = \pm\int\left(1-\frac{2m}{r}\right)^{-1}dr$

⇒ $ct = \pm\left[r + 2m\ln(r-2m) + C\right]$ }

impose initial condition @ $t=0$, $r=r_1$

Solve for constant & simplify result $\Rightarrow$ $ct = \pm\left[(r-r_1) + 2m\ln\frac{r-2m}{r_1-2m}\right]$
in going & outgoing light rays for $r > 2m$

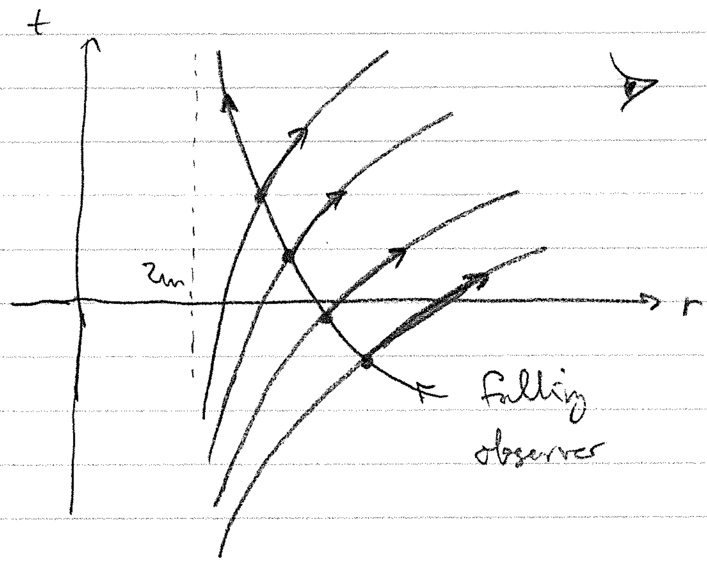
(+) Outgoing rays



Shows all possible outgoing light rays...

$r = r_1$ at $t = 0$

Suppose a falling observer sending outward light signal to r far away
→ they would originate from intersection point



observer sending sequence of signals from closer & closer to the event horizon

Note No outgoing rays from $r \leq 2m$. But at the same time, gravitational redshift of signal gets bigger & bigger...

We worked this out $\frac{\lambda_R}{\lambda_E} = \left(1-\frac{2m}{r_R}\right)^{1/2}\left(1-\frac{2m}{r_E}\right)^{-1/2}$

Observer is at $r = r_R$. But as $r_E \to 2m$, $\lambda_R \to \infty$
→ extremely redshifted...

The light signal gets redshifted away... $\lambda \to \infty \Rightarrow \nu = 0$
→ No light as $r \to 2m$

A black hole is "black" because light emitted from $r = 2m$ is redshifted away...

Nov 14, 2018 $\boxed{\text{Inside event horizon}}$ $r < 2m \Rightarrow \left(1 - \frac{2m}{r}\right) = -\left|1 - \frac{2m}{r}\right|$ negative

line element becomes $c^2 d\tau^2 = -\left|1-\frac{2m}{r}\right| c^2 dt^2 + \left|1 - \frac{2m}{r}\right| dr^2$ for radial motion

$\Rightarrow$ t & r switch roles

$\Rightarrow$ r becomes time like, t becomes space like

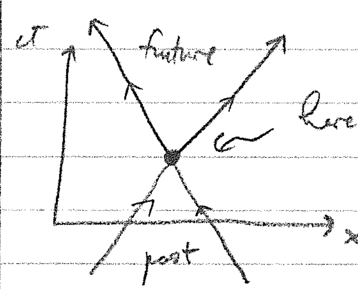
$\boxed{\text{For } r > 2m}$ $\Rightarrow$ t only has 1 direction $\rightarrow$ forward while we can go forwards / backwards in r

$\boxed{\text{For } r < 2m}$ $\Rightarrow$ r only has 1 direction $\rightarrow$ $\boxed{\text{decreasing}}$ (towards $r = 0$) But can go backwards / forward in t.

$\Rightarrow$ Everything moves to $r = 0$ $\Rightarrow$ have a singularity there...

$\Rightarrow$ point of infinite mass density

$\Rightarrow$ We can look at what light cones do... Light cones in SR



Can look at light cone from Schwarzschild geometry...

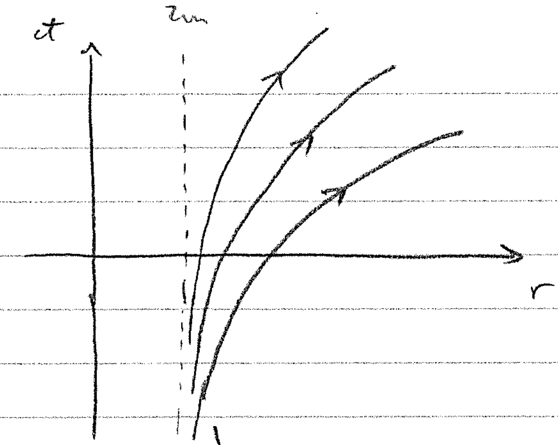
Null line element $0 = \left(1 - \frac{2m}{r}\right) c^2 dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2$

For $r > 2m$ $\quad \frac{dr}{dt} = \pm c \left(1 - \frac{2m}{r}\right)$

For $r < 2m$ $\quad \frac{dr}{dt} = \pm c \left|1 - \frac{2m}{r}\right| = \pm c \left(\frac{2m}{r} - 1\right)$

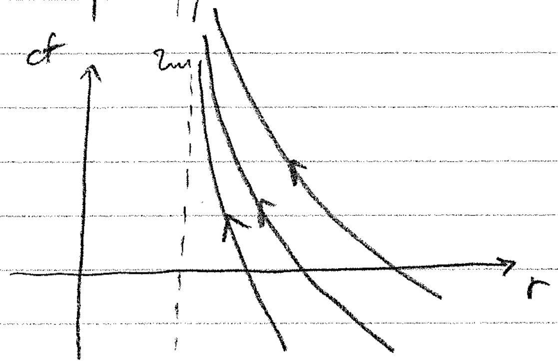
We can look at t & r for all 4 cases...

$\boxed{r > 2m}$



(+) sign $\frac{dr}{dt} > 0$

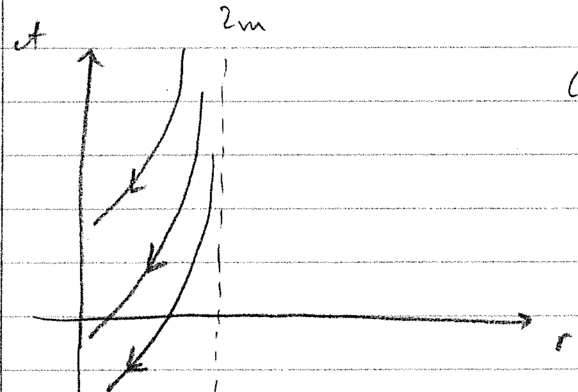
spacelike r outgoing, increasing



(−) sign $\frac{dr}{dt} < 0$

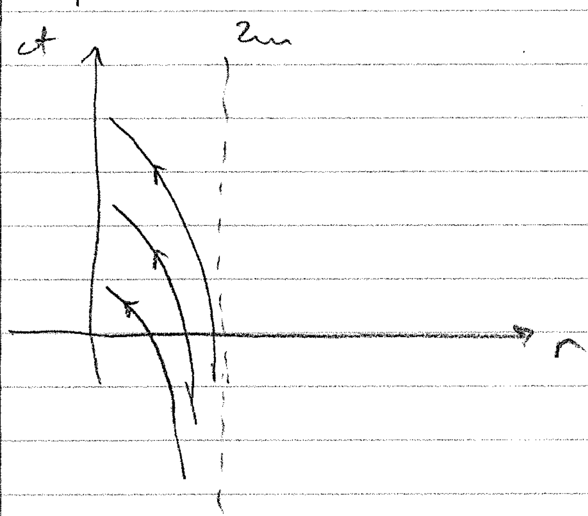
spacelike r ingoing, decreasing

$\boxed{r < 2m}$



(+) sign $\frac{dr}{dt} > 0$ → spacelike t decreasing

"incoming"

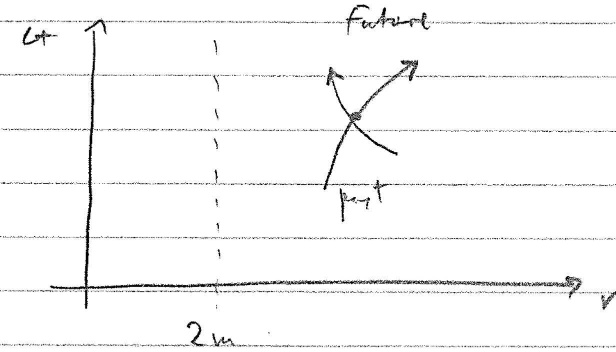


(−) sign $\frac{dr}{dt} < 0$ → spacelike t increasing

"outgoing"

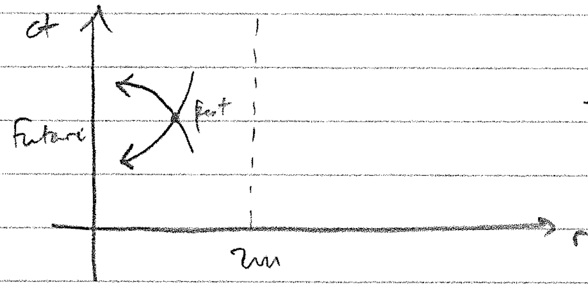
Now look at light cones

$\boxed{r > 2m}$



can escape...

$\boxed{r < 2m}$



future = singularity

Lightcones flip over at event horizon $\Rightarrow$ Inside, $r < 2m$, future takes to singularity at $r = 0$

Note $r = 0$ is a true singularity, but nothing $r = 2m$ is weird, but it's not a true singularity

$\Rightarrow$ the $r = 2m$ infinity is a coordinate infinite

$\Rightarrow$ artifact of coord - choice...

{ Can make a coord transformation that gets rid of singularity in $g_{\mu\nu}$ at $r = 2m$.

Ex Eddington - Finkelstein coordinates. Let $v = ct + r + 2m \ln\left(\frac{r}{2m} - 1\right)$.

rewrite $ds^2 = c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$ in v, r, θ, φ

So line element becomes $\boxed{c^2 d\tau^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2\,dv\,dr - r^2 d\theta^2 - r^2 \sin^2\theta\, d\varphi^2}$

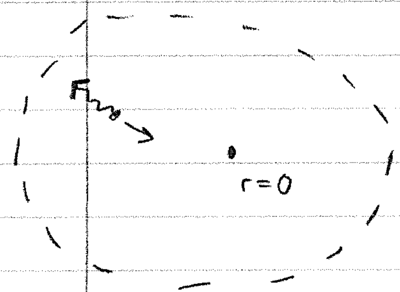
No ∞ in $g_{\mu\nu}$ in these coords... $r = 2m$ is a non-essential singularity

But $r = 0$ infinity remains...

Look at "outgoing rays" as a falling observer falls into $r=0$

claim light rays go inward, despite being shone outward

How?



Can look at coord. velocity of falling observer + light ray.

- $$\left|\frac{dr}{dt}\right|_{obs} = c\sqrt{\frac{2m}{r}}\left|\left(\frac{2m}{r}-1\right)\right| = c\underbrace{\sqrt{\frac{2m}{r}}}_{>1}\left|\frac{2m}{r}-1\right| \quad (r<2m)$$

- $$\left|\frac{dr}{dt}\right|_{light} = c\left|\frac{2m}{r}-1\right| < \left|\frac{dr}{dt}\right|_{observer}$$

So it's possible for both to go to $r=0$ without reversing flashlight direction

$\boxed{\text{How does it feel crossing the event horizon? } \Rightarrow \text{ tidal forces (big) that stretch you out.}}$

Look at $\ddot{r}$ geodesic eqn (1) on sheet ... with $\dot{\varphi}=0$

$$\Rightarrow \left(1-\frac{2m}{r}\right)^{-1}\ddot{r} + \frac{mc^2}{r^2}\dot{t}^2 \cancel{+} - \left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2 = 0$$

Then, can use line element + eq (2) with $K=1$ to eliminate $\dot{r}$, $\dot{t}$ in terms of r.

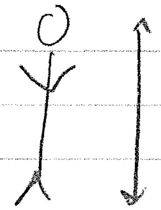
$$\Rightarrow \quad \ddot{r} + \frac{GM}{r^2} = 0 \quad \text{where} \quad \ddot{r} = \frac{d^2r}{d\tau^2}$$

if we multiply by m

$$m\ddot{r} = -\frac{GMm}{r^2} \rightarrow \text{introduce } f = m\ddot{r} = -\frac{GMm}{r^2}$$

$\Rightarrow$ coordinate label ... not physical length

We can use this to estimate a Newtonian-type force while r is not a legitimate length ...



z height ...

$$\Delta F = F_{head} - F_{feet}$$

$$dF = \frac{2GMm}{r^3}\, dr$$

we can approximate $\Delta F \simeq dF$ (very crudely)
$dr \simeq \Delta r = z$

$$\Rightarrow \Delta F \cong \frac{2GMm_0}{r^3}\, z$$

Suppose $r = 2m = \frac{2GM}{c^2}$ for $M = 10\, M_{sun} \simeq 2\times 10^{31}$ kg

$\Rightarrow r \cong 3\times 10^4$ m. Let $m_0 = 80$ kg, and $z = 2$ m

So $F_{stretch} = \Delta F \simeq 3\times 10^{10}$ N (big force ...)

$\boxed{\text{Note}}$ → This decreases for heavier blackholes ... because $r \sim M$ and $\frac{1}{r^3} \sim M^{-3}$. So $F_{stretch} \sim \frac{1}{M^2}$

IV. COSMOLOGY

⇒ Study of the structure of the universe
⇒ we will focus on large-scale geometry.
⇒ apply GR to the universe

(1) Large scale geometry of the universe

if we zoom out, view universe on the largest scale, it looks like a gas / fluid of galaxies

⇒ approximate universe as a perfect fluid

ρ = mass density. p = pressure. For perfect fluid

$$\boxed{T_{\mu\nu} = \left(\rho + \frac{p}{c^2}\right) u_\mu u_\nu - p\, g_{\mu\nu}}$$

stress tensor, becomes source in Einstein's eqn

Note: We're approximating the universe as a cosmological model

⇒ We solve Einstein's equations for the model

$$\boxed{R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \frac{-8\pi G}{c^4} T^{\mu\nu}}$$ (starting with $\Lambda = 0$)

Most models are based on the "Cosmological principle" — which hypothesize that the universe is spatially homogeneous and isotropic

Homogeneous → every point the same
Isotropy → every direction the same

Can treat ρ & p as uniform spatially ⇒ can have t dep. only

Historically, Friedmann → found eqn for how ρ and p evolve
→ solved for $p = 0$ case (no pressure)
(matter dominate universe)

Robertson & Walker studied the form of the metric for a spatially homogeneous + isotropic universe ...

→ They showed that there are only 3 possible geometries.
OPEN, CLOSED, FLAT.

Note "flat" means spatially flat, whereas 4D spacetime still has $R^{\mu}_{\lambda\sigma\nu} \neq 0$ (even with a flat 3D space)

→ Also "flat" only in average sense on largest scales ...

Collectively, these are called Friedmann–Robertson-Walker solns (FRW

RW metric We want to find $ds^2 = g_{\mu\nu}$ for a spatially isotropic & homogeneous universe

If space is homogeneous & isotropic, then all clocks must tell the same time. Can write

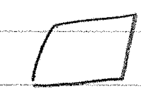
$$\boxed{ds^2 = c^2 d\tau^2 = c^2 dt^2 - g_{ij}\, dx^i dx^j}$$ where t = cosmic time, same for all clocks at rest.

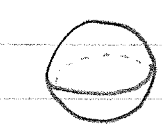
Spatial part. Can called $d\ell^2 = g_{ij}\, dx^i dx^j$
<u>where</u> $ds^2 = c^2 dt^2 - d\ell^2$

Robertson & Walker proved that there're only 3 possible geometries

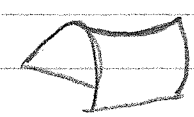
→ To visualize, we can start with 2D spaces & can embed 2D surfaces into 3D hyperspace to visualize

Only spatially homogeneous & isotropic 2D spaces are

(i) flat xy plane 

(ii) positively curved (closed) spherical space 

(iii) negatively curved (open) hyperbolic space

Idea: every point is like middle of saddle 
(can't embed this in flat 3D space)

<u>Claim</u>: These are the only spatially homogeneous + isotropic geometries, but proving this is <u>hard</u>.

Let's look at 2D sphere embed in 3D space. We'll use Cartesian coordinates...

<u>Let</u> x^1, x^2 be spatially coords. of surface (not θ, φ)
x^3 : fake 3rd dim

1 **Sphere** $(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2$

Also $dl^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$

But we can eliminate the fake 3rd dim... by solving x^3

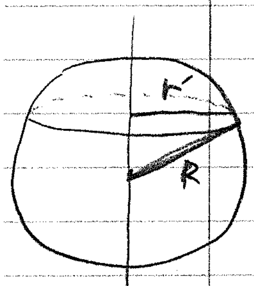
Take differential of $(x^1)^2 + (x^2)^2 + (x^3)^2 = R^2$

$$\Rightarrow \quad 2x^1 dx^1 + 2x^2 dx^2 + 2x^3 dx^3 = 0$$

So
$$\boxed{dx^3 = -\frac{x^1 dx^1 + x^2 dx^2}{x^3} = \frac{-(x^1 dx^1 + x^2 dx^2)}{\sqrt{R^2 - x^{1^2} - x^{2^2}}}}$$

Then, for line element...

$$\boxed{ds^2 = (dx^1)^2 + (dx^2)^2 + \frac{(x^1 dx^1 + x^2 dx^2)^2}{R^2 - (x^1)^2 - (x^2)^2}}$$



Can then introduce polar coordinates ... Can let $x^1 = r'\cos\varphi$
$x^2 = r'\sin\varphi$

Can verify that $\boxed{(x^1 dx^1 + x^2 dx^2)^2 = r'^2 dr'^2}$

It's also true that $(dx^1)^2 + (dx^2)^2 = dr'^2 + r'^2 d\varphi^2$

Then $$\boxed{dl^2 = (dr')^2 + r'^2 d\varphi^2 + \frac{r'^2 dr'^2}{R^2 - r'^2}}$$

OR

$$\boxed{dl^2 = \frac{R^2 dr'^2}{R^2 - r^2} + r'^2 d\varphi^2 \Rightarrow dl^2 = \frac{dr'^2}{1 - \frac{r'^2}{R^2}} + r'^2 d\varphi^2}$$

Near north pole, then $r' \ll R$ → get $\boxed{dl^2 = dr'^2 + r'^2 d\varphi^2}$

→ like flat space ... (locally)

But at greater distances, $dl^2 \neq$ flat

<u>Note</u> r' is not unbounded : $r'^2 + (x^3)^2 = R^2$ → $\boxed{r' \leq R}$

<u>Note</u> not a one-to-one mapping (different points with the same r', φ)

→ need to keep track of hemisphere we're in.

2 $\boxed{\text{Plane}}$ Can get flat plane by letting $R \to \infty$

$$\Rightarrow \boxed{dl^2 = dr'^2 + r'^2 d\varphi^2}$$ like Euclidean plane in polar coords

3 $\boxed{\text{Hyperbolic}}$ → these arguments don't hold, but letting $R \to iR$ gives the solution ($i = \sqrt{-1}$)

So $$\boxed{dl^2 = \frac{dr'^2}{1 + \frac{r'^2}{R^2}} + r'^2 d\varphi^2}$$ → hyperbolic geometry (still locally flat)

→ How can we generalize notation?

$$\boxed{dl^2 = \frac{dr'^2}{1 - k\frac{r'^2}{R^2}} + r'^2 d\varphi^2 \;, \quad k = \underset{\substack{\text{(sphere)}\\\text{(closed)}}}{1},\ \underset{\substack{\text{(flat)}\\\text{(unbound)}}}{0},\ \underset{\substack{\text{(hyperbolic)}\\\text{(open)}}}{-1}}$$

<u>Nov 19, 2018</u> $\boxed{\text{Robertson - Walker metric}}$

2D line element for homogeneous, isotropic space

$$\boxed{dl^2 = \frac{dr'^2}{1 - k\frac{r'^2}{R^2}} + r'^2 d\varphi^2}$$

with $k = \begin{cases} 1 & \text{spherical} \\ 0 & \text{flat} \\ -1 & \text{hyperbolic} \end{cases}$

With $k=0 \rightarrow$ no longer need $R \to \infty$ limit so we scale out R

Let $r = r'/R \Rightarrow r' = rR$

$$\rightarrow \boxed{dl^2 = R^2\left[\frac{dr^2}{1-kr^2} + r^2 d\varphi^2\right]}$$

Note $r \rightarrow$ dimensionless
$R \rightarrow$ length units

- For spherical $0 \le r \le 1$. For flat/hyperbolic $0 \le r \le \infty$
- R can't depend on r or ϕ but spatial homogeneity + isotropy still holds if R depends on t
  $\rightarrow R = R(t) \Rightarrow$ evolving scale factor
- For 3D, can follow similar procedure.

$$\boxed{r^2 d\varphi^2 \Rightarrow r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2}$$

The full 4D spacetime has $ds^2 = c^2 dt^2 - dl^2$

$\rightarrow$ RW metric

$$\boxed{ds^2 = c^2 dt^2 - R(t)^2\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right]}$$

with $k = \begin{cases} 1 & \text{spherical} \\ 0 & \text{flat} \\ -1 & \text{hyperbolic} \end{cases}$

Note many books use $a(t) = \frac{R(t)}{R_0}$ where $R_0 = R(t_0)$ $\rightarrow$ today

Then $a(t_0) = 1$. If $r' = R_0 r$

$$\boxed{ds^2 = c^2 dt^2 - a(t)^2\left[\frac{dr'^2}{1-k\frac{r'^2}{R_0^2}} + r'^2 d\theta^2 + r'^2 \sin^2\theta \, d\varphi^2\right]}$$

three $a(t)$ has no units $\rightarrow r'$ has length units

Note Don't confuse $R(t)$ with the curvature scalar $R = R^{\mu}_{\mu}$

→ Common to use units where $c = 1$. We'll mostly do this

→ Since every point is the same

⇒ doesn't matter where $r = 0$ is

Use $r = 0$ → location on Earth

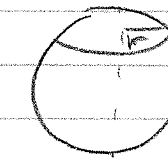
$t = t_0$ → today's cosmos time ($t = 0$ → Big Bang time)

Want to explore the 3 K cases → see what geometry is...

$\boxed{k = 1 \text{, Spherical}}$ → 3D space is a 3-sphere "surface" of a 4D ball

Put back $r' = rR$ $0 \le r \le R$ ⇒ $\boxed{dl^2 = \dfrac{dr'^2}{1 - \dfrac{r'^2}{R^2}} + r'^2 d\theta^2 + r'^2 \sin^2\theta \, d\varphi^2}$

For $r' \ll R$ ⇒ looks just like 3D spherical coords...

→ Analogous to 2D case w/ polar coords  → seems like a polar coord

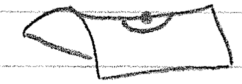
→ can wrap around & go back to starting point... 3D case is just like this.

⇒ head out straight radially and you'll eventually get back to starting point. r' is not a true spherical coord. on large enough scales...

$\boxed{\text{Flat } k = 0}$ Here r is unbounded $0 \le r \le \infty$

$c = 1$ $\boxed{ds^2 = dt^2 - R(t)^2 \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \right]}$

$r' = R(t) r$ → true spherical coordinate

$\boxed{\text{Hyperbolic } k = -1}$ Also has $0 \le r \le \infty$ 

Circumference $> 2\pi r$ → open geometry

$\boxed{\text{Summary}}$ For fixed cosmic time t, have 3 spatial geometry

$k=0 \rightarrow$ spherically flat $\rightarrow$ infinite

$k=1 \rightarrow$ positively curved $\rightarrow$ finite

$k=-1 \rightarrow$ negatively curved $\rightarrow$ infinite

$\boxed{\text{Scale factor } R(t)}$

Hubble param

$\rightarrow$ governs evolution of the universe. Introduce $\boxed{H(t) = \frac{\dot{R}(t)}{R(t)}}$

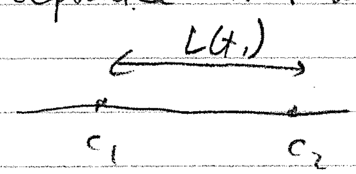
$\dot{R}(t) > 0 \Rightarrow$ expanding universe
$\dot{R}(t) < 0 \Rightarrow$ contracting universe

Over the recent past: $H(t) \simeq$ constant (slowly changing)
Units time^{-1}

Call $H_0 = H(t_0)$ today's value
Observation show that $H_0 > 0 \Rightarrow$ universe is expanding

$\boxed{\text{Comoving coord}}$ $\Rightarrow r, \theta, \varphi$ are comoving coords
$\Rightarrow$ galaxies have approximately constant r, θ, φ
$\Rightarrow$ yet they move apart

As universe expands, coords of galaxies do not change, but they move apart because $R(t)$ increase. Consider 2 galaxies separated in r only (θ, φ are the same)



At any fixed time ($\partial t = 0$)

$$ds^2 = -dl^2 = -R^2(t)\,\frac{\partial r^2}{1-kr^2}$$

$$\boxed{L(t) = \int_1^2 dl = \int_{r_1}^{r_2} R(t)\frac{dr}{\sqrt{1-kr^2}} = R(t)\int_{r_1}^{r_2}\frac{1}{\sqrt{1-kr^2}}\,dr}$$

Call $F(r_1, r_2) = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1-kr^2}}$ → has no t dependence

$L(t) = R(t) F(r_1, r_2)$ → coords are co-moving

Find $\frac{L(t_1)}{L(t_2)} = \frac{R(t_1)}{R(t_2)}$ → $L(t_2) = \frac{R(t_2)}{R(t_1)} L(t_1)$

co-moving r_1, r_2 don't change

$\boxed{\text{Big Bang}}$

→ look back in time at expanding universe
Allows $R(t) = 0$ in distant past
Suggest an initial singularity → Big Bang!

Nov 20, 2018

- $R(t) = 0$ in the past Big Bang Theory
  $t = 0$ Big Bang moment

Universe started in a gigantic explosion
best evidence ⇒ cosmic microwave background (CMB)
⇒ afterglow of an explosion
→ blackbody dist with $T_{now} = 2.7K$

- If the universe is finite $(k=1)$, then It should have begun at a single point.

- But we need to distinguish the "universe" and the "observable universe". With a finite age of the universe, can only see a limited distance due to travel time of light

- At the Big Bang, in all 3 cases $(k=1, 0, -1)$, the observable universe would have been a hot, dense singular point

- For the $k=0, -1$ model, the universe would be huge & well modeled as infinite. But we don't know what would be beyond the observable limit → might not even be homogeneous & isotropic beyond the observable region.

$\boxed{\text{Distance \& Speed}}$

how far away is a distant galaxy & how fast is it moving? Tricky to answer, because distances are continually changing → takes light time to travel

RW line element $\boxed{ds^2 = c^2dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta\, d\varphi^2\right]}$

- Set $r = 0$ at Earth location (does not change → coords are comoving)
- Call $r = r_g$ → position of galaxy (does not change → comoving)
- Can also have $\theta = \theta_g$, $\varphi = \varphi_g$ fixed
- Define proper distance

$L_G(t)$ = spatial distance to galaxy @ fixed time t

- With $dt = d\theta = d\varphi = 0 \Rightarrow$ $\boxed{ds^2 = -dl^2 = -R^2(t)\left[\frac{dr^2}{1-kr^2}\right]}$

So $\boxed{L_G(t) = \int_0^{r_G} dl = \int_0^{r_G} R(t)\frac{dr}{\sqrt{1-kr^2}} = R(t)\int_0^{r_G}\frac{dr}{\sqrt{1-kr^2}}}$

proper distance

Can take $\frac{d}{dt}$ of L

$\boxed{\dot{L}_G(t) = \dot{R}(t)\int_0^{r_G}\frac{dr}{\sqrt{1-kr^2}}} \Rightarrow \boxed{\frac{\dot{L}_G(t)}{L_G(t)} = \frac{\dot{R}_G(t)}{R_G(t)} = H(t)}$, Hubble param

Can call $\boxed{V = \dot{L}_G(t) = \text{speed of recession}}$

So $\boxed{V = H(t)\,L_G(t)}$ → form of the Hubble law. But astronomers don't measure directly V or $L_G(t)$. They like to measure redshift $= z$ & $d_L =$

$\boxed{\text{Cosmological Redshift}}$

As the universe expands, light waves get stretched
⇒ a new kind of redshift

Define $\boxed{z = \frac{\Delta\lambda}{\lambda_0}}$ → redshift param., with $\Delta\lambda = \lambda - \lambda_0$

⇒ $\boxed{\frac{\lambda}{\lambda_0} = 1+z}$

proper wavelength, value at source

λ: observed wavelength at a distance ...

Hubble measured redshift & found z increases with distance
⇒ He assumed Doppler shift (as in Special Relativity)

Spec. shift $\boxed{\frac{\lambda}{\lambda_0} = \sqrt{\frac{1+\beta}{1-\beta}}}$ v: recessional velocity
c: speed of light

For $v \ll c$ → expand ...

for $v \ll c$

$$\boxed{\frac{\lambda}{\lambda_0} \simeq 1 + \frac{v}{c} + \cdots \Rightarrow z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1 \simeq \frac{v}{c}}$$

Hubble made these measurements out to $z \lesssim 10^{-4}$ (tiny) $\simeq \frac{v}{c}$
and found, assuming $z \simeq v/c$

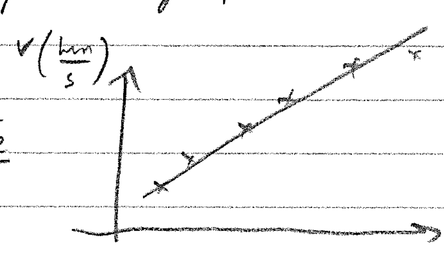
Found that $\boxed{v \propto \text{distance}}$

Hubble wrote down a law $\boxed{V = HD}$ → measured distance (mpc)

velocity

mpc: mega parsecond $1\,pc = 3.26$ lightyears
1 lightyears $\simeq 9.46 \times 10^{15}\,m$
$1\,mp = 3 \times 10^{22}\,m$

Hubble



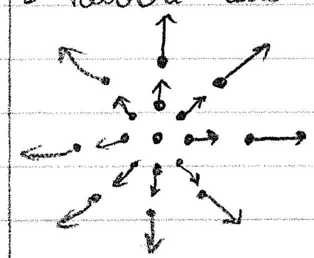
Recent values give $\boxed{H_0 \simeq 70 \pm 2 \ \frac{km/s}{mpc}}$. Hubble assumed $H = $ const, which is not true

But we see some agreement with $\boxed{V(t) = \dot{L}_G(t) = H(t)\, L_G(t)}$

but in GR, $z \neq \frac{v}{c}$, and H not constant

So we'll need to improve the original Hubble law.

- Hubble law says → "speed $\propto$ distance"



true for every point in universe

$\boxed{\text{Need Light Rays}}$ → consider light rays traveling towards us at $r=0$ from a distant galaxy ($r = r_G$)

$r = r_G$

Earth
$r=0$

Call t_E = time light is emitted
t_R = time received

For light rays $ds^2 = 0$ (null), $d\theta = d\varphi = 0$, $c = 1$

$$\boxed{0 = c^2 dt^2 - R^2(t)\,\frac{dr^2}{1-kr^2}}$$

So $\boxed{\frac{dr}{dt} = \frac{-\sqrt{1-kr^2}}{R(t)}}$ Coord. velocity of incoming light

(−) : incoming
+ hidden c

⇒ Need to look at relation between periods of light
δt_E → period when emitted
δt_R → period when received

Get these by integrating the coord. velocity

<u>Recall</u> $\boxed{\lambda = \frac{c}{\nu} = c\,\delta t}$ $\quad \delta t = \frac{1}{\nu} = $ period

If $c = 1 \to \boxed{\lambda = \delta t}$

<u>Will show</u> $\boxed{\frac{\lambda_R}{\lambda_E} = \frac{R(t_R)}{R(t_E)}}$ $\to$ light gets stretched.

Nov 26, 2018 <u>So far, we have</u>

RW $\longrightarrow$ $$\boxed{ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2\right]}$$

$k = \begin{cases} 1 & \text{spherical} \\ 0 & \text{flat} \\ -1 & \text{hyperbolic} \end{cases}$ $\quad H(t) = \frac{\dot R(t)}{R(t)}$. $H_0 = H(t_0) = 70\,\frac{\text{km/s}}{\text{Mpc}} \simeq (14 \times 10^9\,y)^{-1}$

$$\boxed{L_G = \int_0^{r_G} \frac{R(t)}{\sqrt{1-kr^2}}\,dr}$$ proper distance to galaxy G

<u>Recession rate</u> $\dot L_G = V = H_0 L_G \to$ today

originally, Hubble used $z \cong \frac{V}{c}$ in SR . $V = z = H_0 L_G$ $(c=1)$

Redshift $z = \frac{\Delta\lambda}{\lambda_0} = \frac{\lambda}{\lambda_0} - 1$ ... Want sth beyond linear relation

<u>Consider light rays</u>

O Earth $(r=0)$ $\longleftarrow$ 🌀 $r = r_G$ $\quad \boxed{ds^2 = 0}$

Call t_E = time emitted, t_R = time received

with $d\phi = d\theta = 0$ (radial) $\Rightarrow$ $\boxed{0 = dt^2 - R^2(t)\,\frac{dr^2}{1-kr^2}}$ $\quad (c=1)$

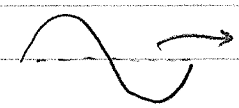
(−): ~~recede~~ light toward earth

So $\frac{dr}{dt} = \frac{-\sqrt{1-kr^2}}{R(t)}$ coord. velocity of light

Integrate ... ⇒ For the leading edge of light ray

$$\boxed{\int_{t_E}^{t_R} \frac{dt}{R(t)} = \int_{r_G}^{0} \frac{-dr}{\sqrt{1-kr^2}}}$$

A ray 1 period later goes from $t_E + \delta t_E \rightarrow t_R + \delta t_R$

 δt = period = δt

One period later ...

$$\Rightarrow \boxed{\int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{dt}{R(t)} = -\int_{r_G}^{0} \frac{dr}{\sqrt{1-kr^2}}}$$

δt = period of light

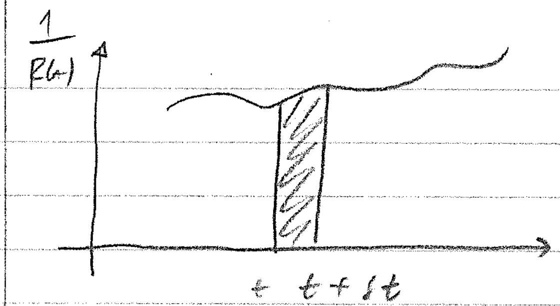
Therefore,
$$\int_{t_E}^{t_R} \frac{dt}{R(t)} = \int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{dt}{R(t)}$$

$$\Rightarrow \cancel{\int_{t_E}^{t_R} \frac{dt}{R(t)}} = \int_{t_E+\delta t_E}^{t_E} \frac{dt}{R(t)} + \cancel{\int_{t_E}^{t_R} \frac{dt}{R(t)}} + \int_{t_R}^{t_R+\delta t_R} \frac{dt}{R(t)} = \int_{t_E+\delta t_E}^{t_R+\delta t_R} \frac{dt}{R(t)}$$

So
$$0 = -\int_{t_E}^{t_E+\delta t_E} \frac{dt}{R(t)} + \int_{t_R}^{t_R+\delta t_R} \frac{dt}{R(t)}$$

OR
$$\boxed{\int_{t_E}^{t_E+\delta t_E} \frac{dt}{R(t)} = \int_{t_R}^{t_R+\delta t_R} \frac{dt}{R(t)}}$$

Here $\delta t_E \rightarrow$ period of light (frac. of sec)
whereas $t_E \rightarrow$ cosmological time (billions of years)



So $\displaystyle\int_t^{t+\delta t} \frac{dt}{R(t)} \simeq \frac{1}{R(t)} \cdot \delta t$

Hence, we get $\boxed{\frac{1}{R(t_E)}\,\delta t_E = \frac{1}{R(t_R)}\,\delta t_R}$ Rewrite $\boxed{\frac{\delta t_E}{R(t_E)} = \frac{\delta t_R}{R(t_R)}}$

So $\frac{R(t_R)}{R(t_E)} = \frac{\delta t_R}{\delta t_E}$

For light, $\lambda = \frac{c}{\nu} = \frac{c}{1/\text{period}} = c\,\delta t$. But if $c = 1$, then $\lambda \sim \delta t$

So $\boxed{\frac{\lambda_R}{\lambda_E} = \frac{R(t_R)}{R(t_E)}}$ → wavelength stretches with scale factor...
→ makes sense as a scaling with R(t)

↳ redshift due to stretching...

With $\lambda_E = \lambda_0$ → proper wavelength
and $\lambda_R = \lambda$ → observed wavelength

→ Redshift $\boxed{z = \frac{\lambda_R}{\lambda_E} - 1 = \frac{\lambda}{\lambda_0} - 1 = \frac{R(t_R)}{R(t_E)} - 1}$

redshift due to galaxies far away...

→ this gives z in terms of R(t). But we want z in terms of L_G, including quadratic contributions (2nd order approx.)

We want z in terms of higher order L's. So, get z in terms of δt, then get δt in terms of L_G.

→ more general Hubble law

Expand $R(t)$ as Taylor's series around t_R

$$R(t) \simeq R(t_R) + \dot{R}(t_R)(t-t_R) + \frac{1}{2}\ddot{R}(t_R)(t-t_R)^2 + \ldots$$

plug in t_E, and note that $t_E < t_R$

$$\Rightarrow R(t_E) \simeq R(t_R) + \dot{R}(t_R)(t_E - t_R) + \frac{1}{2}\ddot{R}(t_R)(t_E - t_R)^2 + \ldots$$

$$\rightsquigarrow \boxed{R(t_E) \cong R(t_R) - \dot{R}(t_R)(t_R - t_E) + \frac{1}{2}\ddot{R}(t_R)(t_R - t_E)^2 - \ldots}$$

Use Hubble param $= \frac{\dot{R}(t)}{R(t)} = H(t)$.

and define $\boxed{q(t) = -\frac{R(t)\ddot{R}(t)}{\dot{R}^2(t)}}$ ∞ deceleration term.

If we expected that $\ddot{R} < 0$ (decelerating) $\Rightarrow$ so $q > 0$ for deceleration -

$$\rightarrow R(t_E) \simeq R(t_R)\left[1 - H(t_R)(t_R - t_E) - \frac{1}{2}q(t_R)H^2(t_R)(t_R - t_E)^2 - \ldots\right]$$

OR call $\Delta t = t_R - t_E$, let t_R = today

$\rightarrow H(t_R) = H_0$

$\rightarrow q(t_R) = q_0$

This gives $\boxed{R(t_E) \simeq R(t_R)\left[1 - H_0\Delta t - \frac{1}{2}q_0 H_0^2 \Delta t^2 + \ldots\right]}$

then, $z = \frac{R(t_R)}{R(t_E)} - 1 \Rightarrow \frac{R(t_E)}{R(t_R)} = \frac{1}{z+1}$

so $\boxed{z = \left[1 - H_0\Delta t - \frac{1}{2}H_0^2 q_0 \Delta t^2 - \ldots\right]^{-1} - 1}$

Use $(1-x)^{-1} \simeq 1 + x + x^2 + \dots$ for small x

let $x = H_0 \Delta t + \frac{1}{2} q_0 H_0^2 \Delta t^2$

and

$x^2 \simeq H_0^2 \Delta t^2 + \dots$

So $z \simeq \left[1 + H_0 \Delta t + \frac{1}{2} q_0 H_0^2 \Delta t^2 + H_0^2 \Delta t^2 + \dots \right] - 1$

$$\Rightarrow \boxed{z \simeq H_0 \Delta t + H_0^2 \left(\frac{1}{2} q_0 + 1 \right) \Delta t^2 + \dots}$$

This gives z in terms of Δt. But now, we want L_G in terms of Δt

$\rightarrow$ go back to $L_G(t) = R(t) \int_0^{r_G} \frac{dr}{\sqrt{1-kr^2}}$. But we also found that

$$\int_{t_E}^{t_R} \frac{dt}{R(t)} = - \int_{r_G}^{0} \frac{dr}{\sqrt{1-kr^2}}$$

So

$$\boxed{L_G(t_R) = R(t_R) \int_{t_E}^{t_R} \frac{dt}{R(t)} = R(t_R) \int_{t_E}^{t_R} \frac{dt}{R(t)}}$$

$\rightarrow$ to do this integral, go back to original Taylor...

$$\frac{1}{R(t)} \simeq \frac{1}{R(t_R)} \left[1 - H(t_R)(t_R - t) + \frac{1}{2} \dots \right]^{-1}$$ use $(1-x)^{-1} \simeq 1 + x$

So $\frac{1}{R(t)} \simeq \frac{1}{R(t_R)} \left[1 + H(t_R)(t_R - t) + \dots \right] \cong \frac{1}{R(t_R)} \left(1 - H(t_R)(t - t_R) \right)$

So

$$\boxed{L_G(t_R) = \cancel{R(t_R)} \int_{t_E}^{t_R} dt \left(\frac{1}{\cancel{R(t_R)}} \left[1 - H(t_R)(t - t_R) + \dots \right] \right)}$$

so $L_G(t_R) \cong \Delta t - \frac{1}{2} H_0 (t - t_R)^2 \Big|_{t_E}^{t_R} + \ldots$

$$= \Delta t + \frac{1}{2} H_0 (t_E - t_R)^2 + \ldots$$

$$\rightarrow \boxed{L_G'(t_R) = \Delta t + \frac{1}{2} H_0 \Delta t^2 + \ldots}$$

now, need to solve this for Δt ... $\rightarrow$ $\boxed{\frac{1}{2} H_0 \Delta t^2 + \Delta t - L_G(t_R) = 0}$

$$\Delta t = \frac{-1 \pm \sqrt{1 + 4 L_G(t_R) \frac{1}{2} H_0}}{2 \cdot \frac{1}{2} H_0} \quad \leftarrow \text{keep (+) sign}$$

$\rightarrow \Delta t = \dfrac{-1 + \sqrt{1 + 2 L_G H_0}}{H_0}$ use $(1+x)^{1/2} \simeq 1 + \frac{1}{2}x - \frac{1}{8}x^2$

so

$$\Delta t \simeq H_0^{-1} \left(-1 + \left(1 + L_G(t_R) H_0 - \frac{1}{2}\left(L_G(t_R) H_0\right)^2\right)\right)$$

$$\boxed{\Delta t = L_G(t_R) - \frac{1}{2} H_0 L_G^2(t_R)}$$

so $z \cong H_0 L_G(t_R) + \dfrac{1 + q_0}{2} \left(H_0 L_G(t_R)\right)^2$

$L_G = L_G(t_R)$ $\boxed{z \simeq H_0 L_G + \frac{1 + q_0}{2} \left(H_0 L_G\right)^2}$

redshift - proper time relation