# Matrices in Quantum Computing

Huan Q. Bui

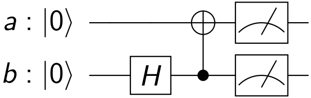
Matrix Analysis

Professor Leo Livshits

CLAS, May 2, 2019

# Presentation layout

# Background



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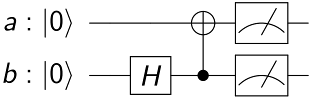


Components:

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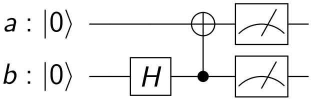


Components:

1. Quantum bits - Qubits

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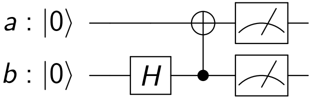


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1. Quantum bits - Qubits
2. Quantum gates: single and multiple-qubit gates

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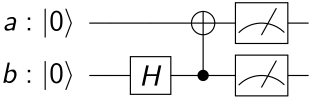
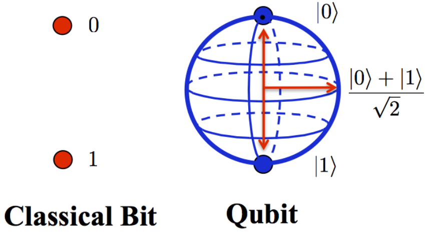



Components:

1. Quantum bits - Qubits
2. Quantum gates: single and multiple-qubit gates
3. Measurement

# Quantum Bits - Qubits



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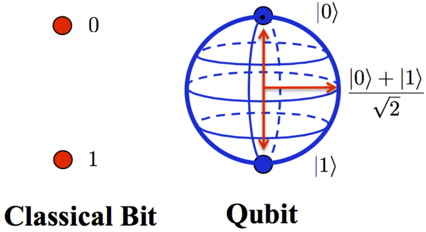


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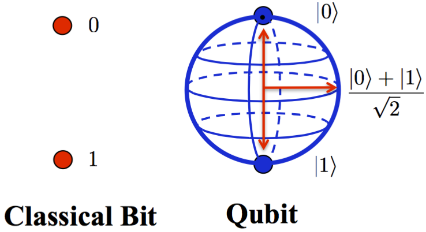


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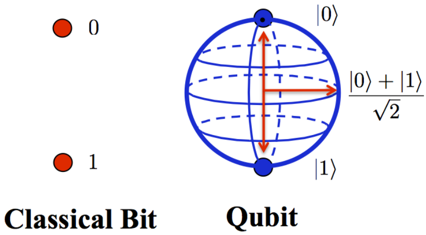


$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$\boxed{a\,|0\rangle + b\,|1\rangle}$$

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$\rightarrow$ linear transformations on one or many qubits.

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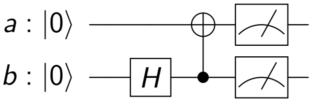

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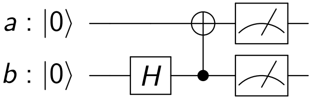


What does $H$ do to, say $|0\rangle$?

$$H|0\rangle = H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle .$$

# Multiple Qubits

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Qubit 1: $a|0\rangle + b|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ Qubit 2: $c|0\rangle + d|1\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$

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$$\begin{bmatrix} a \\ b \end{bmatrix} \boxtimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}.$$

# Multiple Qubits

Do this for the basis states

$$|0\rangle \boxtimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |0\rangle \boxtimes |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |1\rangle \boxtimes |0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |1\rangle \boxtimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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Notation:

$$|00\rangle = |0\rangle \boxtimes |0\rangle \qquad |01\rangle = |0\rangle \boxtimes |1\rangle$$
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Can see that we have a basis for describing the combined state.

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxtimes \begin{bmatrix} c \\ d \end{bmatrix} = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle\,.$$

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Ex: $p(x) \cdot q(y)$ is a "combined state." But there are NO $p(x), q(y)$ s.t.

$$p(x) \cdot q(y) = xy + 1,$$

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Back to qubits. Consider this combined state:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \longrightarrow \textbf{Entangled}$$

# Kronecker Product

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If

$$\mathcal{A} = \begin{bmatrix} m & n \\ o & p \end{bmatrix} \qquad \text{and } \mathcal{B} = \begin{bmatrix} q & r & s \\ t & u & v \\ w & x & y \end{bmatrix}$$

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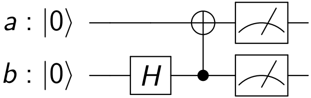
$$= \begin{bmatrix} mq & mr & ms & nq & nr & ns \\ mt & mu & mv & nt & nu & nv \\ mw & mx & ms & nw & nx & ny \\ oq & or & os & pq & pr & ps \\ ot & ou & ov & pt & pu & pv \\ ow & ox & oy & pw & px & pys \end{bmatrix}$$

# Kronecker Product

Check that $I\,|0\rangle \boxtimes H\,|0\rangle = (I \otimes H)(|0\rangle \boxtimes |0\rangle)$:



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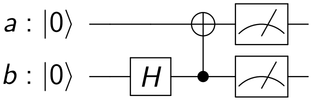

LHS:

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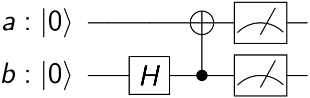


LHS:

$$I\left|0\right\rangle \boxtimes H\left|1\right\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boxtimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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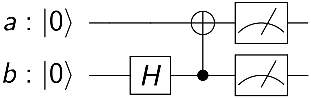


LHS:

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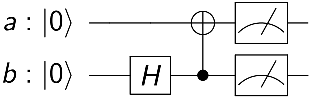


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&= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boxtimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
$$

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4. NOT commutative. Ex: $\left|01\right\rangle \neq \left|10\right\rangle$.

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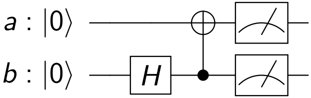
5. Elementariness.

# Some properties and Elementariness revisited

Ex:



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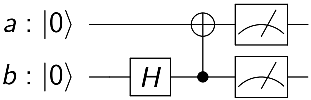
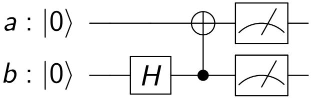



The Control-NOT gate:

# Some properties and Elementariness revisited

Ex:




The Control-NOT gate:

$$CNOT_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \longrightarrow \begin{cases} |00\rangle \to |00\rangle \\ |10\rangle \to |10\rangle \\ |01\rangle \to |11\rangle \\ |11\rangle \to |01\rangle \end{cases}$$

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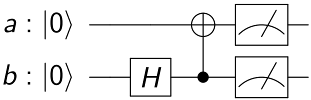


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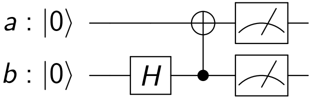
Also called "entangled."

# Entanglement Circuit

Time to decode:

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# Entanglement Circuit

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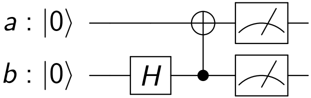


1 Step 1:

$$
\begin{aligned}
a &: |0\rangle \to |0\rangle \\
b &: |0\rangle \to \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\
\left|a'\right\rangle &\boxtimes \left|b'\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0 & 0\end{bmatrix}^{\top}
\end{aligned}
$$

# Entanglement Circuit

2 Step 2:

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$$CNOT_b \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

# Entanglement Circuit

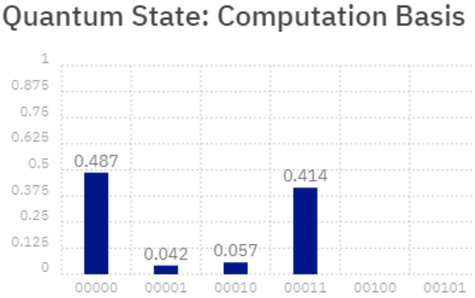
2 Step 2:

$$CNOT_b \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which is:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \leftarrow \textbf{Entangled}$$

# Simulation on IBM-Q



## Tensor Product

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**Postulate (QM):**

The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

# Tensor Product

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\hat{f}", "\mathit{linear}"'] & & \mathbf{X} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[urr, "f", "\mathit{bilinear}"'] & &
\end{tikzcd}
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\end{tikzcd}
$$

## Roughly speaking...

Giving the $\hat{f} : \mathbf{V} \otimes \mathbf{W} \stackrel{\text{linear}}{\longrightarrow} \mathbf{X}$ is the same as giving $f : \mathbf{V} \times \mathbf{W} \stackrel{\text{bilinear}}{\longrightarrow} \mathbf{X}$.
$f = \hat{f} \circ \phi$

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by uniqueness

$$(\mathcal{L} \otimes \mathcal{M})(v \otimes w) = \mathcal{L}[v] \otimes \mathcal{M}[w].$$

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\mathbb{C}^{nm} \arrow[r, "{[\mathcal{L} \otimes \mathcal{M}]_{\tau \leftarrow \tau}}", "linear"'] & \mathbb{C}^{nm} \arrow[u, "\mathcal{A}_\tau"]
\end{tikzcd}$$

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$\nu$ a basis for **V**, $\omega$ for **W** $\rightarrow$ can make a basis $\tau$ for **V** $\otimes$ **W**

$$\begin{tikzcd}[column sep=huge, row sep=huge]
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$\rightarrow$ Can calculate this via the Kronecker product

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# References


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