MA439: Functional Analysis
Tychonoff Spaces: Extra Problem & 1, 2, 5, 7 pg. 51, Ben Mathes

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Exercise 1. *$C(X) = \{f : X \to \mathbb{C} : f$ unif. cont., bdd$\}$ and uniform norm $||f|| = \sup_{x \in X} |f(x)|$. Consider $B(X) = \{f : X \to \mathbb{C},$ bdd$\}$. Show that $C(X) \subseteq B(X)$ is closed, i.e. a uniform limit of uniformly continuous function is uniformly continuous.*

Proof. Let us prove the "easier" case on metric spaces first. Suppose that $f_n \to f$ uniformly where $\{f_n\}$ is a sequence of uniformly continuous functions. We claim that f must also be uniformly continuous. To see this, let $\epsilon > 0$. We first have that

$$||f_n(x), f(x)|| < \frac{\epsilon}{3}$$

for all x whenever n is sufficiently large. Now, each f_n is uniformly continuous, so there is a δ for which $d(x, y) < \delta$ implies $|f_n(x) - f_n(y)| < \frac{\epsilon}{3}$. Finally, consider $x, y \in X$ such that $d(x, y) < \delta$, then

$$||f(x) - f(y)|| \le ||f(x) - f_n(x)|| + ||f_n(x) - f_n(y)|| + ||f_n(y) - f(y)|| < \epsilon.$$

so f is uniformly continuous. This implies that the space $C(X)$ of all uniformly continuous functions from X to $\mathbb{C}$ is a closed subset of $B(X)$. □

Exercise 2 (Ex. 1, pg. 51)**.** *Prove that a closed subset of a complete uniform space is complete*

Proof. Let a C be a closed subset of $(\mathcal{X}, \mathcal{U})$ a complete uniform space be given. Consider a Cauchy net $\{x_i\}$ in C. Since $C \subseteq X$, $\{x_i\}$ is also a Cauchy net in $\mathcal{X}$. Thus, $\{x_i\}$ converges because $\mathcal{X}$ is complete. This limit must belong to the closed set C, so C is complete. □

Exercise 3 (Ex. 2, pg. 51)**.** *If $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{F}_0$, prove that $\mathcal{F}_0$ is a Cauchy.*

Proof. Let $\mathcal{F}$ be a Cauchy filter in $(\mathcal{X}, \mathcal{U})$ and $\mathcal{F} \subseteq \mathcal{F}_0$. It is clear that $\mathcal{F}_0$ is also a filter. Now, let $\epsilon > 0$ and $d \in \mathcal{U}$ be given. $\mathcal{F}$ is Cauchy, so there exists an $x \in \mathcal{X}$ for which $B_d(x, \epsilon) \in \mathcal{F} \subset \mathcal{F}_0$. So $\mathcal{F}_0$ is also Cauchy. □

Exercise 4 (Ex. 5, pg. 51)**.** *An element x of a Tychonoff space is a cluster point of a net $\{x_i\}$ if the net is frequently in every neighborhood of x. Prove that a Cauchy net converges to any of its cluster points.*

Proof. Let a Cauchy net $\{x_i\}_{i \in I}$ be given. Consider a cluster point $x \in \{x_i\}_{i \in I}$. We want to show that $\{x_i\}_{i \in I} \to x$. By the hypothesis, any neighborhood $\mathcal{U}$ containing x must also contain infinitely many elements of $\{x_i\}_{i \in I}$. The Cauchyness of $\{x_i\}_{i \in I}$ guarantees that $\mathcal{U}$ contains a tail of $\{x_i\}_{i \in I}$. This implies the convergence of $\{x_i\}_{i \in I}$. □

Exercise 5 (Ex. 7, pg. 51)**.** *Any filter $\mathcal{F}$ is a directed set, and if, for $F \in \mathcal{F}$ we choose $x_F \in F$, we obtain a **net based on the filter** (there are many of them). Prove that the filter converges to x if and only if the net $\{x_F\}_{F \in \mathcal{F}}$ converges to x.*

Proof. ($\Longrightarrow$) Suppose that a filter $\mathcal{F} \to x$. Consider a net based on $\mathcal{F}$ denoted by $\{x_F\}_{F\in\mathcal{F}}$. $\mathcal{F} \to x$ iff $\mathcal{F}_x \subseteq \mathcal{F}$. Consider a neighborhood $F_x \in \mathcal{F}_x$ of x. We have that F_x contains a tail of $\{x_F\}_{F\in\mathcal{F}}$. To see this, fix an F_x. Because any F in $\mathcal{F}$ meets F_x, $F' = F \cap F_x \subseteq F_x \in \mathcal{F}$. It is now clear that $\{x_F\}_{F\in\mathcal{F}}$ for all $F \geq F'$ is contained in F_x. Therefore, $\{x_F\}_{F\in\mathcal{F}} \to x$.

($\Longleftarrow$) Let a filter $\mathcal{F}$ and $\{x_F\}_{F\in\mathcal{F}} \to x$ be given. Assume to get a contradiction that $\mathcal{F} \not\to x$. This means that there is some set $O \subseteq \mathcal{X}$ containing an open set $F_x \ni x$ such that $O \notin \mathcal{F}$. This means that $F_x \notin \mathcal{F}$. Now, consider the net $\{y_F\}_{F\in\mathcal{F}}$ which does not intersect F_x. Clearly, this net cannot converge to x, which contradicts the fact that all nets $\{x_F\}_{F\in\mathcal{F}} \to x$. $\square$