MA338 (S'20): Final Exam

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1. Differentiation

(a) Assume that

$$f(x) = \begin{cases} \frac{g(x)}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and assume that $g(0) = 0$ and $g''(0) = 17$. With no further assumptions, find $f'(0)$, justify everything.

Solution: The answer is $\boxed{f'(0) = 17/2}$. The key is using L'Hôpital's rule twice. By definition,

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{g(x)}{x^2}.$$

Since $g''(0)$ exists, $g'(x)$ must be differentiable at 0. It follows that $g(x)$ must be continuous at 0. Now, $g(0) = 0$, so $\lim_{x\to 0} g(x) = g(0) = 0$ by continuity. It is also clear that $\lim_{x\to 0} x^2 = 0$. L'Hôpital's rule says that

$$\lim_{x\to 0} \frac{g(x)}{x^2} = \lim_{x\to 0} \frac{g'(x)}{2x},$$

provided the limit on the right hand side exists. The evaluate the limit on the right hand side, we apply L'ôpital's rule again: Clearly $\lim_{x\to 0} 2x = 0$. It remains to show $\lim_{x\to 0} g'(x) = g'(0) = 0$. The first equality follows from the fact that $g'(x)$ is differentiable (hence continuous) at $x = 0$. We want to justify the second equality. By definition,

$$g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x\to 0} \frac{g(x)}{x} = f(0) = 0.$$

So, because $\lim_{x\to 0} g'(x) = 0$ and $\lim_{x\to 0} 2x = 0$, L'Hôpital's rule says

$$\lim_{x\to 0} \frac{g'(x)}{2x} = \lim_{x\to 0} \frac{g''(x)}{2} = \frac{17}{2}.$$

Thus,

$$f'(0) = \frac{17}{2}.$$

□

(b) Assuming only that $f'(0) > 0$ and f' continuous at 0, prove that there exists an interval containing 0 on which f is increasing. (This f is in no way related to the previous f in part (a).)

Proof: Since f' is continuous at 0 and $f'(0) > 0$ there exists a neighborhood $(-\delta, \delta) \subset \mathbb{R}$ on which $f' > 0$. This makes sense, because by definition, for $\epsilon = f'(0)/2 > 0$, there exists $\delta > 0$ such that whenever $|y - x| < \delta$, $|f'(y) - f'(x)| < \epsilon = f'(0)/2 < f'(0)$. The triangle inequality says that $f'(t) > 0$ for all $t \in (-\delta, \delta)$. With this, take $x, y \in (-\delta, \delta)$ such that $x < 0 < y$.

The mean value theorem says that there exists $t \in [x, y]$ such that

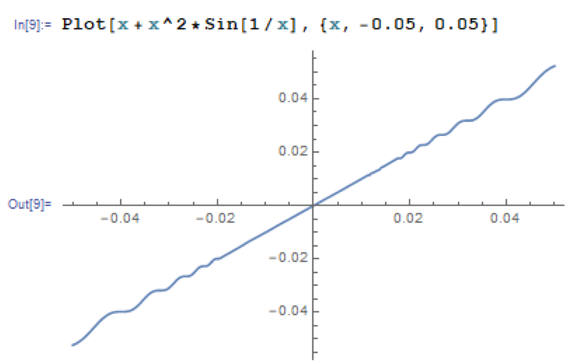
$$\frac{f(y) - f(x)}{y - x} = f'(t) > 0, \text{ since } t \in [x, y] \subset (-\delta, \delta).$$

Rearranging gives $f(y) - f(x) > (y - x)f'(t)$ for any $x, y \in (-\delta, \delta)$ such that $y > 0 > x$. We have demonstrated that it is possible to find an interval containing 0 on which f is increasing. □

(c) Show that there exists a continuous function f with $f'(0) > 0$, but f is not increasing on any interval containing 0.

Proof: Intuitively, we want to construct a function f such that even though $f'(0) > 0$, it "wiggles" so much that f is never strictly increasing on any interval around zero, no matter how small. This idea suggests picking an f that oscillates faster near zero. To this end, consider $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x + 2x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$



We first show that f is continuous. When $x \neq 0$, f is clearly continuous. So we only focus on showing f is continuous at 0. Let $\epsilon > 0$ be given, then

$$|f(x) - f(0)| = |f(x)| \leq |x| + \left|2x^2 \sin \frac{1}{x}\right| \leq |x| + 2|x^2|.$$

Choose $\delta = \min\{1, \epsilon/3\}$. Then whenever $|x - 0| < \delta$, we have

$$\left|f(x) - f(0)\right| \le |x| + 2\left|x^2\right| = |x|(2|x| + 1) < \frac{\epsilon}{3} \cdot 3 < \epsilon.$$

This shows f is continuous. Next, we want to show $f'(0) > 0$. To this end, we just evaluate $f'(0)$. By definition,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \left(1 + 2x \sin \frac{1}{x}\right) = 1 > 0$$

since $x \sin(1/x) \to 0$ as $x \to 0$. Finally, we will show that f is not increasing on any interval containing 0. Assume (to get a contradiction) that f is increasing on some interval containing 0. Because f' is continuous and positive at 0, there exists an interval containing 0 on which $f' > 0$ (we proved this in the last item). Now, look at f again. For $x \neq 0$,

$$f'(x) = 1 - 2\cos\frac{1}{x} + 4x \sin\frac{1}{x}.$$

Let an interval containing 0 be given. It is possible to find a sufficiently small t in this interval such that $\cos(1/t) = 1$ and $|4t| < 1/2$ (This is possible because $1/t$ will be sufficiently large in magnitude and cos is periodic.) It follows that

$$\left|f'(t) + 1\right| = \left|1 - 2 + 4t \sin\frac{1}{t} + 1\right| = \left|4t \sin\frac{1}{t}\right| \le |4t| < 1/2,$$

which implies $-3/2 < f(t) < -1/2$. Clearly, $f(t) < 0$, which contradicts the fact that there exists an interval containing 0 on which $f'(t) > 0$ for all t on that interval. □

(d) Assume that $\left|f(x) - f(y)\right| \le (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

Proof: Let $\delta > 0$ be given. Pick $x, y \in \mathbb{R}$ such that $0 < x - y < \delta$. Because $\left|f(x) - f(y)\right| \le (x - y)^2$ for all $x, y \in \mathbb{R}$, we have

$$0 \le \frac{\left|f(x) - f(y)\right|}{x - y} = \left|\frac{f(x) - f(y)}{x - y}\right| \le x - y < \delta.$$

Since this holds for any $\delta > 0$, $f'(x) = 0$ for all $x \in \mathbb{R}$ (because $f'(x)$ is the limit of the difference quotient as $y \to x$). This means f is constant, by Theorem 5.11(b), Baby Rudin. □

2. Series

(a) Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then

$$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|.$$

Proof: Since $\sum_{n=1}^{\infty} a_n$ converges absolutely, $\sum_{n=1}^{\infty} a_n$ converges (Theorem 3.45). Let $C = \sum_{n=1}^{\infty} a_n$. Consider the sequence $\{|s_N|\}$ where each $s_N = \lim_{n=1}^{N} a_n$. Clearly,

$$||s_N| - |C|| \leq |s_N - C| \to 0, \text{ as } N \to \infty.$$

Thus, $\lim_{N\to\infty} |s_N| = |C|$. Now, for each N, we also have the triangle inequality:

$$|s_N| = \left|\sum_{n=1}^{N} a_n\right| \leq \sum_{n=1}^{N} |a_n|.$$

Taking $N \to \infty$ on both sides we have

$$\lim_{N\to\infty} |s_N| = |C| = \left|\sum_{n=1}^{\infty} a_n\right| \leq \lim_{N\to\infty} \sum_{n=1}^{N} |a_n| = \sum_{n=1}^{\infty} |a_n|.$$

□

(b) Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and b_n is a subsequence of a_n, then $\sum_{n=1}^{\infty} b_n$ is absolutely convergent. Given an example that shows this statement is false if $\sum_{n=1}^{\infty} a_n$ is assumed to be only conditionally convergent.

Proof: The absolute convergence of $\sum_{n=1}^{\infty} a_n$ implies the convergence of $\sum_{n=1}^{\infty} |a_n|$. Since b_n is a subsequence of a_n, we must have that

$$\sum_{n=1}^{\infty} |b_n| \leq \sum_{n=1}^{\infty} |a_n| < \infty,$$

where the first inequality follows because we are summing only nonnegative terms. Therefore, $\sum_{n=1}^{\infty} b_n$ is absolutely convergent.

When $\sum_{n=1}^{\infty} a_n$ is assumed to be only conditionally convergent, that is $\sum_{n=1}^{\infty} |a_n|$ is divergent, then the statement is false. Consider the conditionally convergent series $\sum_{n=1}^{\infty} (-1)^{n+1}/n$:

$$\sum_{n=1}^{\infty} (-1)^{n+1}/n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

We know that $\sum_{n=1}^{\infty} 1/n$ is divergent (harmonic series). Call $a_n = (-1)^{n+1}/n$. Clearly, $|a_1| \geq |a_2| \geq \ldots$; the sequence $\{a_n\}$ is alternating; and $\lim_{n\to\infty} a_n = 0$. Theorem 3.43 (alternating series test) tells us that $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is convergent. Hence, $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ is conditionally convergent.

Consider the subsequence $\{b_n\}$ of $\{a_n\}$ consisting only of the terms of a_n where n is odd:

$$\sum_{n=1}^{\infty} b_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$$

We want to show that the series $\sum_{n=1}^{\infty} b_n$ is NOT absolutely convergent. We notice that

$$\sum_{n=1}^{\infty} |b_n| = \sum_{n=1}^{\infty} b_n = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2n-1}$$

So this comes down to determining the convergence of $\sum_{n=1}^{\infty} b_n$. It turns out that $\sum_{n=1}^{\infty} b_n > \infty$ because it fails the integral test:

$$\int_1^{\infty} \frac{1}{2n-1}\, dn = \lim_{k\to\infty} \int_1^{k} \frac{1}{2n-1}\, dn = \frac{1}{2}\ln(1+2k) \longrightarrow \infty$$

as $k \longrightarrow \infty$. Thus, $\sum_{n=1}^{\infty} b_n$ is NOT absolutely convergent. Therefore, the statement is false when $\sum_{n=1}^{\infty} a_n$ is only conditionally convergent. □

(c) Assume a_n is a decreasing sequence of positive numbers, and that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} n a_n = 0$.

Proof: The key here is to put an upper bound on na_n and show that bound goes to zero as $n \longrightarrow \infty$. Consider the partial sum $S_n = \sum_{i=1}^{n} a_i$. We have

$$\begin{aligned} S_{2n} - S_n = \sum_{i=n+1}^{2n} &= a_{2n} + a_{2n-1} + \cdots + a_{n+1} && (1)\\ &\geq a_{2n} + a_{2n} + \cdots + a_{2n} && (2)\\ &= n a_{2n}. && (3)\end{aligned}$$

where we have used the condition that a_n is a decreasing sequence of positive numbers to get the inequality. Now, $\sum_{n=1}^{\infty} a_n$ is convergent, so the sequence of partial sums is convergent, hence Cauchy. It follows that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that whenever $n \geq N$,

$$|n a_{2n}| \leq |S_{2n} - S_n| = \left|\sum_{i=n+1}^{2n}\right| < \epsilon.$$

Thus $\lim_{n\to\infty} na_{2n} = 0$ and so $\lim_{n\to\infty} 2na_{2n} = 0$. Further,

$$(2n+1)a_{2n+1} \leq (2n+1)a_{2n} = \left(1 + \frac{1}{2n}\right)(2na_n) \leq 2 \cdot 2na_{2n} = 4na_{2n},$$

which also goes to 0 as $n \to \infty$. So, because both $na_n \to 0$ as $n \to \infty$ for all n (odd or even), $\lim_{n\to\infty} na_n = 0$. □

(d) Prove that every positive rational number can be written as a finite sum of *distinct* numbers of the form $1/k$ with $k \in \mathbb{N}$.

Proof: We will first show this is true for all rationals r such that $0 \leq r = p/q < 1$ where p, q are positive integers with no common factor. If $r = 0$ or $p = 1$ then the statement is true. Assume (an inductive hypothesis) that the statement holds for all rationals r above but with $p < P$. Consider the rational number $P/q < 1$. We can always find the least positive integer m such that $1/m \leq P/q$. Because $P/q < 1$ and m is an integer, we have

$$\frac{1}{m} \leq \frac{P}{q} < \frac{1}{m-1} \implies mP - P < q \leq mP \implies 0 \leq mP - q < P.$$

Let the residual $R = P/q - 1/m = (mP - q)/qm$. Because $mP - q < P$, R can be written as a finite sum of distinct $1/k$'s, $k \in \mathbb{N}$. We also have that

$$R < \frac{1}{m-1} - \frac{1}{m} == \frac{1}{m(m-1)} \leq \frac{1}{m}.$$

So, $1/m$ cannot appear in the finite sum for R. This means $r = P/q = R + 1/m$ can be written as a finite sum of distinct $1/k$'s. By induction, all rationals less than 1 can be written as a finite sum of distinct $1/k$'s, $k \in \mathbb{N}$.

We now want to extend this to all rationals greater than or equal to 1. We now use the fact that $\sum_{n=1}^{\infty} 1/n = \infty$. Let $S_n = \sum_{i=1}^{n} 1/i$, which is rational. Let a rational $r \geq 1$ be given. There exists $n \in \mathbb{N}$ such that

$$S_n \leq r < S_{n+1}.$$

Now let us write $r = (r - S_n) + S_n$, which is a sum of two rational numbers. By the choice of n,

$$r - S_n < S_{n+1} - S_n = \frac{1}{n+1} < 1,$$

which means $r - S_n$ can be written as a finite sum of distinct $1/k$'s, $k \in \mathbb{N}$. Further, none of the summands in the sum for $r - S_n$ is can be greater than $1/(n+1)$, which means no summand in the sum for $r - S_n$ can be a summand of S_n, which is a finite sum of distinct $1/k$'s, $k \in \mathbb{N}$. Therefore, r can be written as a finite sum of distinct numbers of the form $1/k$ with $k \in \mathbb{N}$. □

3. Hilbert Space

(a) Let V denote the set of continuous functions that map $[0,1]$ into the complex numbers $\mathbb{C}$. With $f \in V$, each complex number $f(x)$ can be written in terms of its real and imaginary parts

$$f(x) = \mathrm{Re}\{f(x)\} + i\,\mathrm{Im}\{f(x)\}.$$

The real valued functions $\mathrm{Re}\{f\}$ and $\mathrm{Im}\{f\}$ are called the real part of f and the imaginary part of f, respectively. We define the integral of a complex valued function by

$$\int_0^1 f(x)\,dx \equiv \int_0^1 \mathrm{Re}\{f(x)\}\,dx + i\int_0^1 \mathrm{Im}\{f(x)\}\,dx.$$

Show that the assignment

$$\langle f, g\rangle \equiv \int_0^1 f(x)\overline{g(x)}\,dx$$

satisfies the axioms of a complex inner product (find the axioms in a book or on the internet).

Proof: Let $f, g, h \in V$ and $c \in \mathbb{C}$ be given.

- $\boxed{\langle f, g\rangle = \overline{\langle g, f\rangle}}$. We have that

$$\langle f, g\rangle \equiv \int_0^1 f(x)\overline{g(x)}\,dx$$

and

$$\begin{aligned} g(x)\overline{f(x)} &= \left[\mathrm{Re}\{g(x)\} + i\,\mathrm{Im}\{g(x)\}\right]\left[\mathrm{Re}\{f(x)\} - i\,\mathrm{Im}\{f(x)\}\right] \\ &= \left[\mathrm{Re}\{f(x)\}\,\mathrm{Re}\{g(x)\} + \mathrm{Im}\{f(x)\}\,\mathrm{Im}\{g(x)\}\right] \\ &\quad + i\left[\mathrm{Im}\{g(x)\}\,\mathrm{Re}\{f(x)\} - \mathrm{Re}\{g(x)\}\,\mathrm{Im}\{f(x)\}\right], \end{aligned}$$

$$\begin{aligned} f(x)\overline{g(x)} &= \left[\mathrm{Re}\{f(x)\} + i\,\mathrm{Im}\{f(x)\}\right]\left[\mathrm{Re}\{g(x)\} - i\,\mathrm{Im}\{g(x)\}\right] \\ &= \left[\mathrm{Re}\{g(x)\}\,\mathrm{Re}\{f(x)\} + \mathrm{Im}\{g(x)\}\,\mathrm{Im}\{f(x)\}\right] \\ &\quad + i\left[\mathrm{Im}\{f(x)\}\,\mathrm{Re}\{g(x)\} - \mathrm{Re}\{f(x)\}\,\mathrm{Im}\{g(x)\}\right]. \end{aligned}$$

So, $\mathrm{Re}\left\{g(x)\overline{f(x)}\right\} = \mathrm{Re}\left\{f(x)\overline{g(x)}\right\}$ and $\mathrm{Im}\left\{g(x)\overline{f(x)}\right\} = -\mathrm{Im}\left\{f(x)\overline{g(x)}\right\}$

$$\begin{aligned}
\overline{\langle g, f\rangle} &= \overline{\int_0^1 g(x)\overline{f(x)}\,dx} \\
&= \overline{\int_0^1 \mathrm{Re}\left\{g(x)\overline{f(x)}\right\}dx + i\int_0^1 \mathrm{Im}\left\{g(x)\overline{f(x)}\right\}dx} \\
&= \int_0^1 \mathrm{Re}\left\{g(x)\overline{f(x)}\right\}dx - i\int_0^1 \mathrm{Im}\left\{g(x)\overline{f(x)}\right\}dx \\
&= \int_0^1 \mathrm{Re}\left\{f(x)\overline{g(x)}\right\}dx + i\int_0^1 \mathrm{Im}\left\{f(x)\overline{g(x)}\right\}dx \\
&= \int_0^1 f(x)\overline{g(x)}\,dx \\
&= \langle f, g\rangle.
\end{aligned}$$

- $\boxed{\langle f+g, h\rangle = \langle f, h\rangle + \langle g, h\rangle}$. We have that

$$\begin{aligned}
\mathrm{Re}\{(f+g)h\} &= \mathrm{Re}\{fh+gh\} = \mathrm{Re}\{fh\} + \mathrm{Re}\{gh\} \\
\mathrm{Im}\{(f+g)h\} &= \mathrm{Im}\{fh+gh\} = \mathrm{Im}\{fh\} + \mathrm{Im}\{gh\}
\end{aligned}$$

So,

$$\begin{aligned}
\langle f+g, h\rangle &= \int_0^1 (f+g)\overline{h}\,dx \\
&= \\
&= \\
&=
\end{aligned}$$

- $\boxed{\langle f, g+h\rangle = \langle f, g\rangle + \langle f, h\rangle}$
- $\boxed{\langle cf, g\rangle = c\langle f, g\rangle}$
- $\boxed{\langle f, cg\rangle = \overline{c}\langle f, g\rangle}$
- $\langle f, f\rangle$ is a nonnegative real number and $\langle f, f\rangle = 0$ if and only if $f = 0$.

(b) Assume V is a complex inner product space with inner product $\langle x, y\rangle$ and its associated metric

$$d(x, y) = \sqrt{\langle x-y, x-y\rangle}$$

and let $\mathcal{H}$ denote the metric completion of V. Thus we may think of V as a dense subset of the metric space $\mathcal{H}$. The purpose of the following exercises is to show how one may extend the vector space structure of V to $\mathcal{H}$, and how to extend the inner product to $\mathcal{H}$, which shows that the metric completion of an inner product space is a Hilbert space.

- Given $x, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$, we define $\alpha x + \beta y$ to be the limit of the sequence $\alpha x_i + \beta y_i$, where x_i is any sequence in V converging to x, y_i is any sequence in V converging to y. Show that this definition is well-defined.
- Imitate the procedure above to show how to extend the inner product so that $\langle x, y\rangle$ is defined for all $x, y \in \mathcal{H}$. (Hint: extend one variable at a time.)

4. Isometries

(a) Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $\left|f(x) - f(y)\right| = \left|x - y\right|$ for all $x, y \in \mathbb{R}$. Prove that

$$f(x) = mx + b$$

with $m = 1$ or $m = -1$.

(b) Prove that there does not exist a function $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $\left|f(x) - f(y)\right| = \left\|x - y\right\|$ for all $x, y \in \mathbb{R}^2$.

(c) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\left\|f(x) - f(y)\right\| = \left\|x - y\right\|$ for all $x, y \in \mathbb{R}^n$ then f is onto.

(d) Let $\mathcal{H}$ denote an infinite dimensional (real or complex) Hilbert space. Given an example of a function $f : \mathcal{H} \to \mathcal{H}$ that satisfies

$$\left\|f(x) - f(y)\right\| = \left\|x - y\right\|$$

for all $x, y \in \mathcal{H}$ but f is *not* onto.