8.311 : EM theory

Pset 3

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(1) $\boxed{\text{2D Green's function for free space}}$

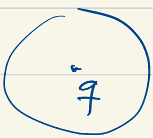
- Before doing this problem, we have to find the appropriate 2D Green's function, generically

- Well, $G(\vec{r}, \vec{r}\,')$ solves

$$\boxed{\nabla^2 G = +2\pi\,\delta(\vec{r}-\vec{r}\,')}$$

This is because in 2D... we have

$$\oint \vec{E}\cdot\vec{n}\,dl = \frac{q}{\varepsilon_0} = E(2\pi R)$$



$$\Rightarrow E \sim \frac{q}{2\pi R\varepsilon_0} \Rightarrow \phi \sim -\frac{\ln R}{2\pi\varepsilon_0}$$

From here, we can guess that

$$\boxed{G(\vec{r},\vec{r}\,') = A \ln |\vec{r}-\vec{r}\,'|}$$

where A is a constant, which depends on the convention...

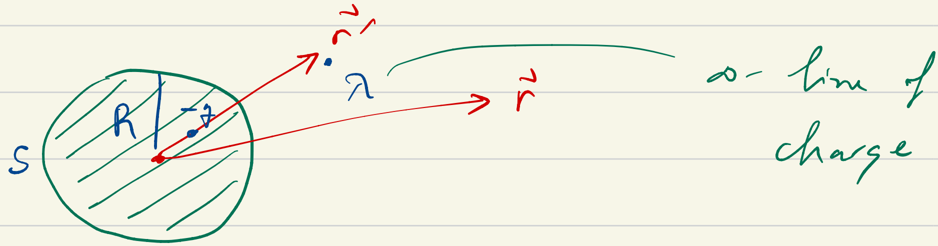
Here we will choose $A = -1$, so that

$$\boxed{G(\vec{r},\vec{r}\,') = -\ln |\vec{r}-\vec{r}\,'|}$$

With this we can solve the problem by making $G(\vec{r},\vec{r}\,')$ satisfy various boundary conditions...

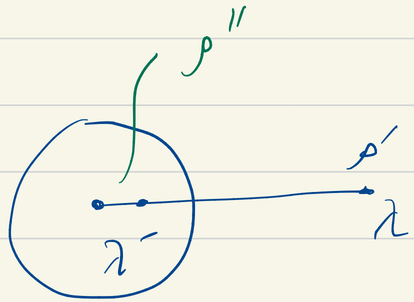
The prefactor A varies by convention I think, so it's not super important...

(i) outside a round conducting cylinder...



Want $G(\vec{r},\vec{r}\,')\Big|_{\vec{r}\in S} = 0$

By method of images...



$$\boxed{G(\vec{r},\vec{r}\,') = -\ln|\vec{r}-\vec{r}\,'| - \frac{\lambda'}{\lambda}\ln|\vec{r}-\vec{r}\,''|}$$

<u>What is λ'/λ?</u> <u>What is ρ''?</u>

First, the potentials due to the lines of charge don't vanish @ ∞, since

$E \sim \lambda/r \Rightarrow \phi \sim \ln r$

so that $\phi_{total} = 0$ @ ∞

- Therefore we require that $\boxed{\phi_\lambda = -\phi_{\lambda'} \text{ at } \infty}$

which implies $\boxed{\lambda' = -\lambda}$.

- It remains to find ρ'' in terms of ρ', R.

Let the potential @ $\rho = R$ be V, then

$$\phi_{total} \sim -\ln(R^2 + \rho'^2 - 2R\rho'\cos\phi) + \ln(R^2 + \rho''^2 - 2R\rho''\cos\phi) = V$$

So $\quad V = \ln \dfrac{R^2 + \rho''^2 - 2R\rho''\cos\phi}{R^2 + \rho'^2 - 2R\rho'\cos\phi}$

This holds particularly for $\phi = \pi/2$ and 0, so

$$\frac{(R-\rho'')^2}{(R-\rho')^2} = \frac{R^2 + \rho''^2}{R^2 + \rho'^2} \Rightarrow \boxed{\rho'' = \frac{R^2}{\rho'}}$$

• With these, we can find the Green's function for this problem...

$$G(\vec{r},\vec{r}') = -\ln(\rho^2+\rho'^2-2\rho\rho'\cos\phi)$$

$$+\ln\left(\rho^2+R^4/\rho'^2-2\rho\frac{R^2}{\rho'}\cos\phi\right)$$

$$\Rightarrow \boxed{G(\vec{r},\vec{r}') = \ln\left[\frac{\rho^2\rho'^2+R^4-2\rho\rho' R^2\cos\phi}{\rho'^2(\rho^2+\rho'^2-2\rho\rho'\cos\phi)}\right]}$$

up to some constant factor depending how the boundary condition is selected.

Ex Jackson EM has $\phi(\infty) = \frac{\lambda}{2\pi\varepsilon_0}\ln\left(R/\rho\right)$ ... so

$$G(\vec{r},\vec{r}') = \ln\left(\frac{\rho^2\rho'^2+R^4-2\rho\rho' R^2\cos\phi}{R^2(\rho^2+\rho'^2-2\rho\rho'\cos\phi)}\right)$$

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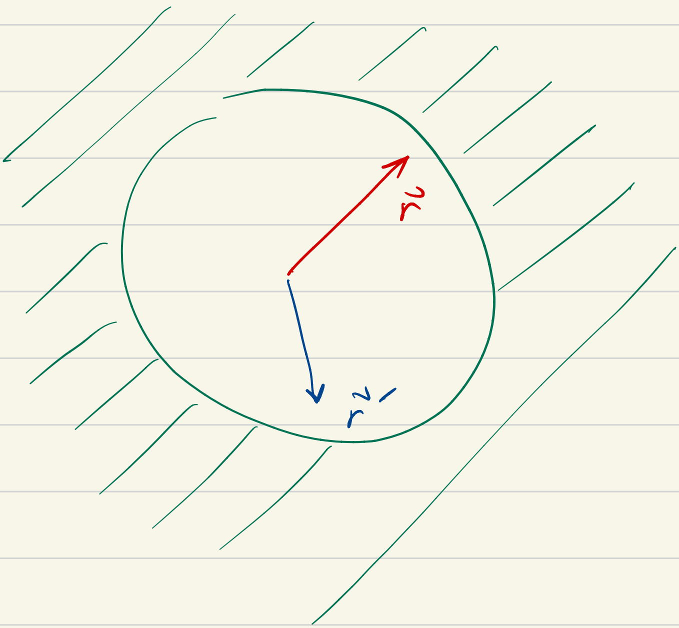
(ii) inside a round cylindrical hole in a conductor ...



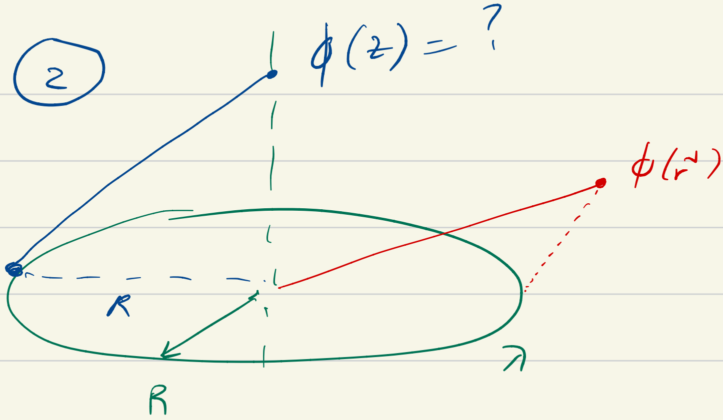
Here we see that the situation is reversed ... But also notice that the choice of "conductor" and "free space" is arbitrary (sort of)

So we conclude that the Green's function has the same functional form!

$$\boxed{G(\tilde{r},\tilde{r}') = \ln\left[\frac{\rho^2\rho'^2 + R^4 - 2\rho\rho' R^2\cos\phi}{\rho^{-2}(\rho^2+\rho'^2 - 2\rho\rho'\cos\phi)}\right]}$$

(the roles of ρ', ρ'' are reversed twice, so G remains the same) $\rho' = \frac{R^2}{\rho''} \Rightarrow \rho'' = \frac{R^2}{\rho'}$ ✓

②



Well... in general ... $\phi(\vec{r}) = \int \frac{\lambda\, dl}{|\vec{r}-\vec{r}\,'|} \frac{1}{4\pi\varepsilon_0}$

But on the symmetry axis ... $|\vec{r}-\vec{r}\,'| = \sqrt{z^2+R^2}$

So $\phi(z) = \int \frac{\lambda\, dl}{\sqrt{z^2+R^2}} \frac{1}{4\pi\varepsilon_0} = \frac{\lambda\, 2\pi R}{4\pi\varepsilon_0 \sqrt{z^2+R^2}}$

$$\Rightarrow \boxed{\phi(z) = \frac{\lambda R}{2\varepsilon_0 \sqrt{z^2+R^2}}}$$

Now look at $z \gg R$ ... then can expand the solution about $R/z \simeq 0$

$$\phi(z) = \frac{\lambda R}{2\varepsilon_0 \sqrt{z^2 + R^2}} = \frac{\lambda R / z}{2\varepsilon_0 \sqrt{1 + R^2/z^2}}$$

$$\frac{Q}{4\pi\varepsilon_0} \longleftarrow \approx \frac{\lambda R}{2\varepsilon_0} \frac{1}{z} \left[1 - \frac{1}{2} \frac{R^2}{z^2} + \frac{3}{8} \frac{R^4}{z^4} + \cdots \right]$$

- Leading term: (monopole) $\quad \phi^0(z) = \frac{Q}{4\pi\varepsilon_0} \frac{1}{z}$ ✓

- Next term: (quadrupole) $\quad \frac{Q}{4\pi\varepsilon_0} \left(\frac{-1}{2}\right) \frac{R^2}{z^3}$

$$\sim \left(\frac{1}{z^3}\right) \frac{1}{2} \left(\frac{-QR^2}{4\pi\varepsilon_0} \right)$$

Let's check ... $\quad -\frac{1}{2z^3} R^2 \cdot Q$ ✓ (correct :))

$$\frac{1}{2z^5} zz\, Q_{zz} = \frac{1}{2z^3} \int \rho(\vec{r}\,') \big(3\underbrace{z'z'}_{0} - \underbrace{r'^2}_{R} \delta_{zz} \big) d^3 r'$$

- The dipole term vanishes ...

$$\frac{1}{z^3}\, z \int \rho(\vec{r}\,')\, \underbrace{z'}_{0}\, d^3r' = 0$$

- The octopole term is

$$\frac{1}{z^4}\,\frac{1}{2} \int \rho(\vec{r}\,') \left[\underbrace{5z'^3}_{0} - \underbrace{3z'r'^2}_{0} \right] d^3r' = 0$$

$\Rightarrow$ octopole vanishes as well

- I couldn't find a formula for the next term (hexapole?)

- But okay, behavior is as expected by Eqns 3.3 & 3.4 in the book.

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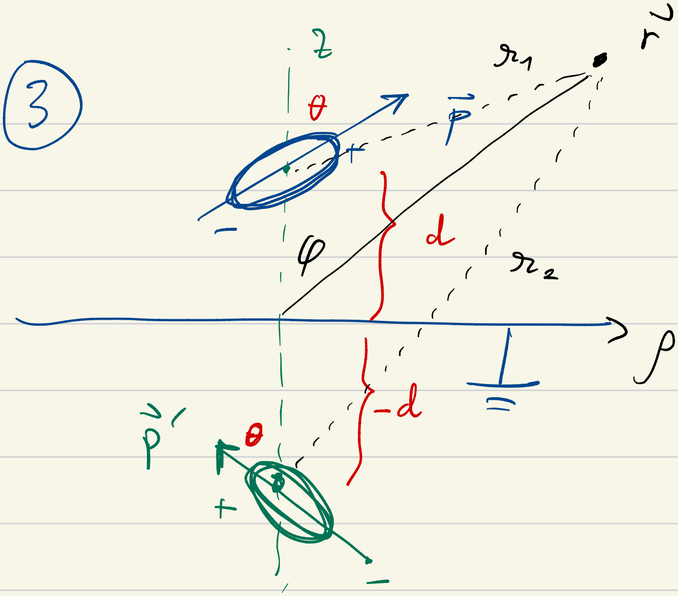


We will use method of images

(i) To find σ on conductor ... we need to find $\phi(r)$

then take $\sigma = -\varepsilon_0 \dfrac{\partial \phi}{\partial z}\Big|_{z=0}$

Okay, now find $\phi(\vec{r})$ simply by superposition ...

$$\phi_{total}(\vec{r}) = \phi_{dipole}(\vec{r}) + \phi_{dipole}(\vec{r})$$

- $$\phi_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}_1)\cdot\vec{p}}{(\vec{r}-\vec{r}_1)^3}$$

$$\vec{p} = (p\sin\theta, \; 0, \; p\cos\theta)$$
$$\vec{r} = (x, \; y, \; z)$$
$$\vec{r}_1 = (0, \; 0, \; d)$$

So

$$\boxed{\phi_{dipole}(\vec{r}) = \left(\frac{p}{4\pi\varepsilon_0}\right) \frac{x\sin\theta + (z-d)\cos\theta}{(x^2+y^2+(z-d)^2)^{3/2}}}$$

- $$\phi_{dipole}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{(\vec{r}-\vec{r}_2)\cdot\vec{p}'}{|\vec{r}-\vec{r}_2|^2}$$

$$\vec{p}' = (-p\sin\theta, \; 0, \; p\cos\theta)$$
$$\vec{r} = (x, \; y, \; z)$$
$$\vec{r}_2 = (0, 0, -d)$$

So

$$\boxed{\phi_{dipole}(\vec{r}) = \frac{p}{4\pi\varepsilon_0} \frac{-x\sin\theta + (z+d)\cos\theta}{(x^2+y^2+(z+d)^2)^{3/2}}}$$

Now we add them up to get the total potential

$$\boxed{\phi_{total}(\rho, z) = \frac{p}{4\pi\varepsilon_0}\left\{\frac{x\sin\theta + (z-d)\cos\theta}{(x^2+y^2+(z-d)^2)^{3/2}} + \frac{-x\sin\theta + (z+d)\cos\theta}{(x^2+y^2+(z+d)^2)^{3/2}}\right\}}$$

$\downarrow$

$\phi = 0$ @ $z = 0$ ✓

Now can calculate σ...

$$\sigma = -\varepsilon_0 \frac{\partial \phi}{\partial z}\bigg|_{z=0} = \text{mathematica} \ldots$$

$$\boxed{\sigma = \frac{+p}{2\pi(x^2+y^2+d^2)^{5/2}}\left((2d^2-x^2-y^2)\cos\theta - 3dx\sin\theta\right)}$$

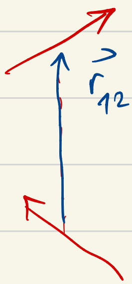
Just for fun... total charge induced is...

$$Q = \int_{\mathbb{R}^2} \sigma \, dx\, dy = \frac{-p\sin\theta}{2d\pi}$$

$$\boxed{\text{(ii) dipole-to-plane interaction energy ...}}$$

Dipole interacts with the field due to its image

. So $\quad \mathcal{U} = -\left(\vec{p}\cdot\vec{E}_2\right)\frac{1}{2}$



$$= \frac{1}{2}\,\frac{-\vec{p}\,\cdot}{4\pi\varepsilon_0}\left\{\frac{3(\vec{p}'\cdot\vec{r}_{12})\,\vec{r}_{12}}{r_{12}^5} - \frac{\vec{p}'}{r_{12}^3}\right\}$$

Here $\quad \vec{r}_{12} = (0, 2d)$

$\vec{p} = (p\sin\theta,\ p\cos\theta)$

$\vec{p}' = (-p\sin\theta,\ p\cos\theta)$

So Using Mathematica ...

$$\to \boxed{\mathcal{U} = \frac{1}{4\pi\varepsilon_0}\left(\frac{-p^2(3+\cos 2\theta)}{16\,d^3}\right)\cdot\frac{1}{2}}$$

$$= \frac{-p^2}{8\pi\varepsilon_0}\left(\frac{1+\cos^2\theta}{8d^3}\right)$$

a little nicer

$1/2$ from the fact that $\vec{E} \propto \vec{p}$

(iii) Force and torque on $\vec{p}$

Force

$$\vec{F} = (\vec{p}\cdot\vec{\nabla})\vec{E} \longrightarrow$$

$\vec{E}$ field by $\vec{p}$ due to $\vec{p}'$

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\,\frac{3(\vec{r}-\vec{r}_2)\big((\vec{r}-\vec{r}_2)\cdot\vec{p}'\big) - \vec{p}'|\vec{r}-\vec{r}_2|^2}{|\vec{r}-\vec{r}_2|^5}$$

where
$$\vec{r} = (x, y, z)$$
$$\vec{r}_2 = (0, 0, -d)$$
$$\vec{p}' = (-p\sin\theta, 0, p\cos\theta)$$

$$\rightarrow \quad \vec{F} = (p\sin\theta\,\partial_x + p\cos\theta\,\partial_z)\vec{E}\Big|_{z=d,\,x=0}$$

$$= (\text{mathematica} \ldots)$$

$$\boxed{\vec{F} = \frac{-3p^2(3+\cos 2\theta)}{4\pi\varepsilon_0 \cdot 32 d^4}\,\hat{z}}$$

$2(1+\cos^2\theta)$ if you're looking for angle-reduction

- Sanity check ... when $\theta = 0$ aka the dipole and its image align, then

$$\vec{F} = \frac{-3p^2(3+1)}{4\pi\varepsilon_0 \cdot 32 d^4}\hat{z}$$

$$\vec{F} = \frac{-3p^2}{32\pi\varepsilon_0 d^4}\hat{z}$$

I believe this is the answer you'd get if the original dipole is perpendicular to the plane ...

- In any case, torque ...

$$U = -\vec{p}\cdot\vec{E} \Rightarrow \boxed{\vec{\tau} = \vec{p}\times\vec{E}}$$

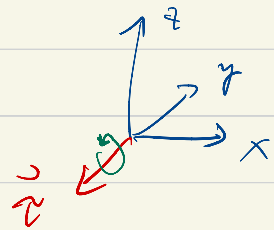
produced by $\vec{p}\,'$

- Same approach as before, but now we have <u>Mathematica</u>...

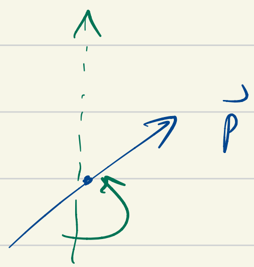
- we get $\vec{\tau} = \vec{p} \times \vec{E}\Big|_{z=d,\, x=0}$

$$\downarrow$$

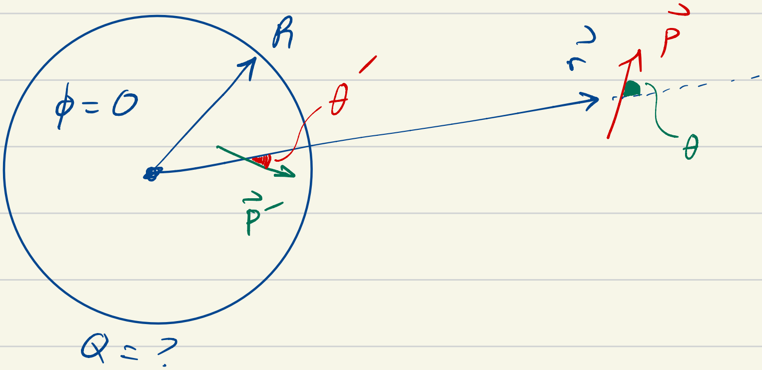
$$\boxed{\vec{\tau} = \frac{-p^2 \sin 2\theta}{64\pi\varepsilon_0 d^3}\,\hat{y}}$$



Basically the dipoles want to <u>align</u> themselves and <u>move closer</u> to each other.

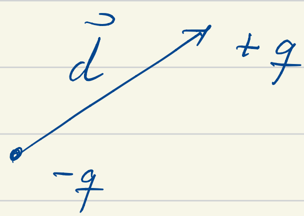


④ Q induced in gnd conducting sphere of radius R by $\vec{p}$ @ $\vec{r}$ $(r > R)$



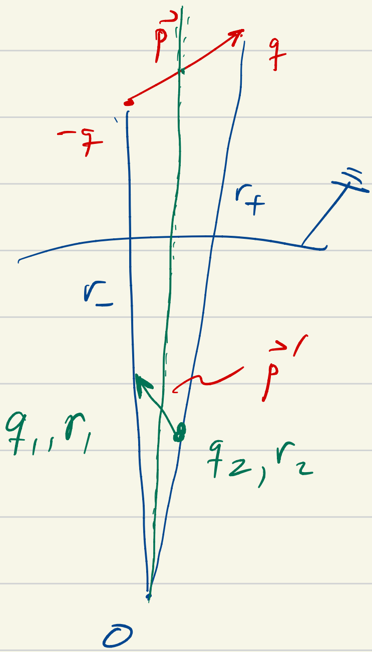
- Once again... we use method of images ... But it is more complicated this time

... We have to imagine that the dipole has 2 charges



$\vec{p} = q\vec{d}$

• Now we make image charges



$$\begin{cases} q_1 = q \cdot \dfrac{R}{r_-}, & r_1 = \dfrac{R^2}{r_-} \\[2ex] q_2 = -q\,\dfrac{R}{r_+}, & r_2 = \dfrac{R^2}{r_+} \end{cases}$$

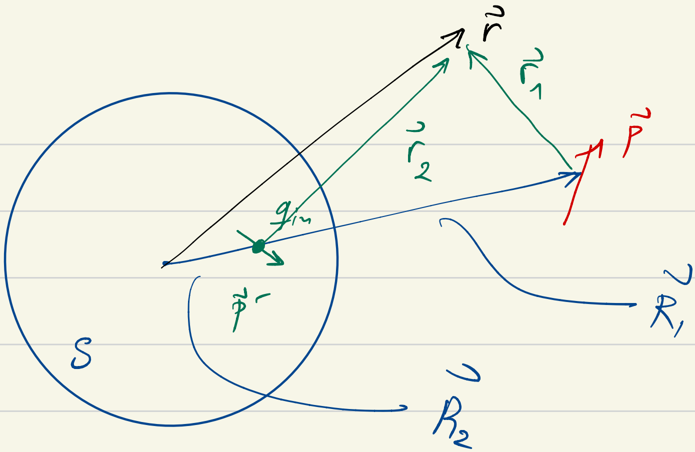
• Notice that there is a net charge inside the sphere ...

$$q_{in} = q_1 + q_2 = q\left(\frac{R}{r_-} - \frac{R}{r_+}\right)$$

• So, for simplicity, let's restart the problem, but now we will make ...

(1) image dipole $\vec{p}\,'$

(2) image charge q_{in}



- It makes sense to put q_{in} at the ~~image~~ dipole location as well...

- With this we can write, generically

$$\phi(\vec{r}) = \phi_{\vec{p}}(\vec{r}) + \phi_{\vec{p}\,'}(\vec{r}) + \phi_{q_{in}}(\vec{r})$$

$$= \frac{\vec{r}_1 \cdot \vec{p}}{4\pi\varepsilon_0 r_1^3} + \frac{\vec{r}_2 \cdot \vec{p}\,'}{4\pi\varepsilon_0 r_2^3} + \frac{q_{in}}{4\pi\varepsilon_0 r_2}$$

- We want $\phi(S) = 0$, so

$$\left. \frac{\vec{r}_1 \cdot \vec{p}}{r_1^3} + \frac{\vec{r}_2 \cdot \vec{p}\,'}{r_2^3} + \frac{q_{in}}{r_2} \right|_S = 0$$

- We now need to find q_{in}, $\vec{p}'$, $\vec{R}_2$.

Note that we have defined

$$\vec{r} = \vec{R}_1 + \vec{r}_1 = \vec{R}_2 + \vec{r}_2 .$$

- From here we find that by eliminating $\vec{r}_1, \vec{r}_2 \ldots$ and define $r_1/r_2 = k$ we get

$$\frac{(\vec{r} - \vec{R}_1)\cdot\vec{p}}{k^3} + (\vec{r} - \vec{R}_2)\cdot\vec{p}' + q_{in}(\vec{r} - \vec{R}_2)^2 = 0$$

$$\Rightarrow \left(\frac{\vec{p}}{k^3} + \vec{p}' - 2q_{in}\vec{R}_2\right)\cdot\vec{r} - \frac{\vec{p}\cdot\vec{R}_1}{k^3}$$

$$- \vec{p}'\cdot\vec{R}_2 + q_{in}(r^2 + R_2^2) = 0$$

- This holds for any $\vec{r}$, so and must be independently zero!

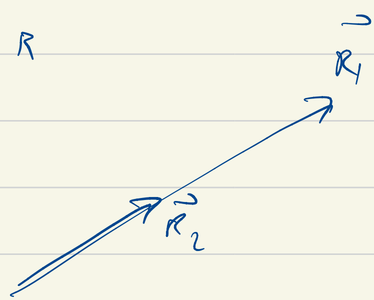
$$
\circ\ \text{So}\ \begin{cases} \dfrac{\vec{p}}{k^3} + \vec{p}\,' - 2\underline{q_{in}}\,\vec{R}_2 = 0 & (1) \\[2em] \dfrac{\vec{p}\cdot\vec{R}_1}{k^3} + \vec{p}\,'\cdot\vec{R}_2 - \underline{q_{in}}\left(r^2 + R_2^2\right) = 0 & (2) \end{cases}
$$

Solve this for q_{in} and $\vec{p}_2$ now... Do this by dotting (1) with R_2 ...

$$
\Rightarrow \quad \frac{\vec{p}\cdot\vec{R}_2}{k^3} + \cancel{\vec{p}\,'\cdot\vec{R}_2} - 2q_{in}R_2^2 = \frac{\vec{p}\cdot\vec{R}_1}{k^3} + \cancel{\vec{p}\,'\cdot\vec{R}_2} - q_{in}\left(r^2 + R_2^2\right)
$$

$$
\Rightarrow \quad q_{in} = \left.\frac{-\vec{p}\left(\vec{R}_2 - \vec{R}_1\right)}{k^3\left(r^2 - R_2^2\right)}\right|_{r=R}
$$

$$
= \frac{-\vec{p}\left(R_2 - R_1\right)\hat{R}_1}{k^3\left(R^2 - R_2^2\right)}
$$



- Now the image charge is at $\vec{R}_2$ as well, and (by design)

$$R_2 = R^2/R_1$$

- So we have

$$\boxed{q_{in}} = \frac{-\vec{p}\cdot\vec{R}_1\,(R^2/R_1 - R_1)}{h^3(R^2 - R^4/R_1^2)}$$

$$= \frac{-\vec{p}\cdot\vec{R}_1\,(R^2/R_1^2 - 1)}{h^3 R^2(1 - R^2/R_1^2)}$$

$$= \frac{\vec{p}\cdot\vec{R}_1}{h^3 R^2} = \boxed{\frac{\vec{p}\cdot\vec{R}_1}{R_1^3}\,R}$$

where we have used the fact that for $\vec{r} = \vec{R} \parallel \vec{R}_1$

$$h = \frac{r_1}{r_2} = \left.\frac{R_1 - r}{r - R_2}\right|_{r=R} = \frac{R_1 - R}{R - R_2} = \frac{R_1 - R}{R - R^2/R_1}$$

$$= \frac{R_1}{R}$$

• Now that we have solved for q_{in}, it remains to find $\vec{p}\,'$.

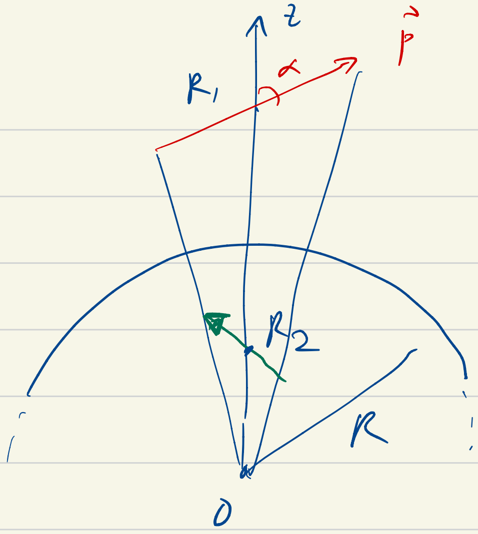
Well... from (1)

$$\vec{p}\,' = 2 q_{in} \vec{R}_2 - \vec{p}/k^3$$

$$= 2R \, \frac{\vec{p} \cdot \vec{R}_1}{R_1^3} \cdot \left(\frac{R^2}{R_1} \hat{R}_1 \right) - \vec{p} \cdot \frac{R^3}{R_1^3}$$

So $$\boxed{\vec{p}\,' = \frac{-R^3}{R_1^3} \left[\vec{p} - 2 (\vec{p} \cdot \hat{R}_1) \hat{R}_1 \right]}$$

• With q_{in} & $\vec{p}\,'$ let us redo the problem with everything <u>clean</u>.



Let us write

$$\boxed{\vec{p} = p\,(\sin\alpha,\ 0,\ \cos\alpha)}$$

$$\Downarrow$$

$$\boxed{q_{in} = \frac{\vec{p}\cdot\vec{R}_1}{R_1^3}\,R = \frac{pR}{R_1^2}\cos\alpha}$$

and finally ...

$$\vec{p}\,' = \frac{-R^3}{R_1^3}\left[\vec{p} - 2\,\overbrace{(\vec{p}\cdot\hat{R}_1)\,\hat{R}_1}^{(0,0,p\cos\alpha)}\right]$$

$$= \frac{-R^3}{R_1^3}\left\{ p\begin{pmatrix}\sin\alpha\\ 0\\ \cos\alpha\end{pmatrix} - 2p\begin{pmatrix}0\\ 0\\ \cos\alpha\end{pmatrix}\right\}$$

$$\boxed{\vec{p}\,' = p\left(\frac{R}{R_1}\right)^3 \left(-\sin\alpha,\ 0,\ \cos\alpha\right)}$$

$$\left(R_1 R_2 = R^2\right)$$

Now we have an expression for ϕ with only $\vec{r}, \vec{R}_1, \vec{p}$

$$\boxed{4\pi\varepsilon_0 \phi(r) = \frac{(\vec{r}-\vec{R}_1)\cdot\vec{p}}{|\vec{r}-\vec{R}_1|^3} + \frac{(\vec{r}-\vec{R}_2)\cdot\vec{p}\,'}{|\vec{r}-\vec{R}_2|^3} + \frac{\frac{pR}{R_1^2}\cos\alpha}{|\vec{r}-\vec{R}_2|}}$$

depend on R_1

We will now let Mathematica do the work

But we'll say that

$$\begin{cases} \vec{r} = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta) \\ \vec{R}_1 = (0, 0, R_1) \\ \vec{R}_2 = (0, 0, R^2/R_1) \\ \vec{p} = p(\sin\alpha, 0, \cos\alpha) \\ \vec{p}\,' = p\left(\frac{R}{R_1}\right)^3(-\sin\alpha, 0, \cos\alpha) \end{cases}$$

- Inserting into Mathematica ... and we calculate $\left.\frac{\partial \phi}{\partial r}\right|_{r=R}$, so that

$$\sigma = -\varepsilon_0 \left.\frac{\partial \phi}{\partial r}\right|_{r=R}$$

- Then calculate $Q \sim \int_S \sigma \, dS$ where S is the surface of the sphere

$$Q = \int_S \sigma \, dS = \int \sigma R^2 \sin\theta \, d\theta \, d\phi$$

- <u>The result is</u>

$$Q = \frac{2\pi\varepsilon_0}{4\pi\varepsilon_0} p \frac{R}{R_1^2} \left\{ \underbrace{\frac{R_1 - R}{|R_1 - R|}}_{1} + \underbrace{\frac{R_1 + R}{|R_1 + R|}}_{1} \right\} \cos\alpha$$

since $R_1 > R$

$$\boxed{Q = \frac{pR}{R_1^2} \cos\alpha}$$

which is exactly q_{in}