HW #4 : COMBINATORICS

(65) Consider a person X out of the 6 people.

⇒ X knows at least 3 or at most 2 people.

$\Downarrow$ if any of these know each other then there's nothing to prove

↓ But if no one knows each other, then we have ≥ 3 people who don't know each other

$\Downarrow$ don't know at least 3

$\Downarrow$ if any of these don't know each other then we're done

↓ But if no one are strangers then ≥ 3 people who know each other.

(66) ~~Let total # be n, where n odd. Suppose no one knows an even # of people ⇒ Everyone knows an odd # of people.~~

We prove this by drawing the graph with n ~~edges~~ vertices, where n is odd. Suppose we draw a line between 2 ~~edges~~ vertices if the 2 know each other, and do nothing otherwise.

* → If Everyone knows an odd # of people, then # ~~edges~~ lines from each vertex must also be odd. This is not possible if n odd

→ Consider $n = 3$ (counter example) → no way to do this.

⇒ ~~At least one does not know an even # of people~~

⇒ At least one knows an even number of people ⇒ Stranger to an odd # of people.

(69) Proof by drawing lines. With n vertices, we will draw a line between 2 people if they know each other & do nothing otherwise.

We notice that ~~x~~ if for some n odd, everyone knows an odd # of people, then the following happens.

@ each vertex) Every additional line connects two vertices.
→ with an odd # lines going down one vertex,

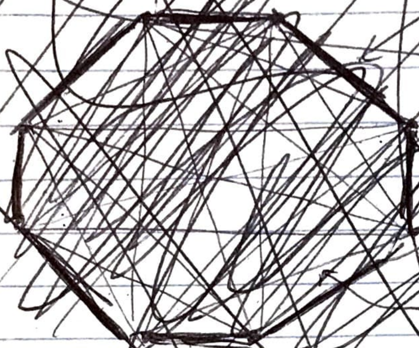
→ each vertex on the graph must be connected an odd # of times

But this is not possible with n odd, since there will always be one vertex that is not paired an odd # of times. → (contradiction..). $\times$

⇒ ~~this vertex~~

So for all n odd, at least one person must know an even # of people, and so doesn't know an even # of people as well.

70



~~green~~ ~~red~~

~~we see that there's no green K_3 (green is exclusively "outside") and no red K_4 since to make a K_4 a green is needed~~

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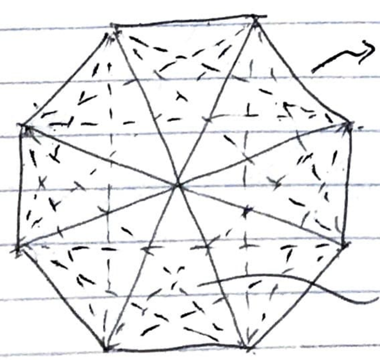
From 70, we know that $R(4,3) > 8$.
From 68, we know that $R(4,3) \leq 10$

$\Rightarrow R(4,3) = 9$ or 10

__Claim__: $\boxed{R(4,3) = 9}$. If there are ≥ 4 green lines from this extra vertex to the original 8, then there is guaranteed 2 adjacent vertices or opposite which connect to this extra vertex with 2 green lines $\rightarrow$ ~~forms~~ a green K_3.

Else, < 4 green lines $\Rightarrow$ ≥ 5 red lines from this extra vertex to the original 8. But we can also check that we always form a red K_4 ~~or a green K_3~~ or a green K_3.

70



$\rightarrow$ green, No green K_3

red, no red K_4

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with $4 \cdot 1 = 4$ people, there are 3 ways.

$$ABCD \to \begin{cases} AB \text{ v } CD \\ AC \text{ v } BD \\ AD \text{ v } BC \end{cases}$$

with $n > 1$, we need to sequentially pick 4, then for each court of 4 people, there are 3 possible arrangements of the $n!$

$$\text{Total } \# = \frac{\binom{4n}{4}\binom{4n-4}{4}\cdots\binom{4}{4}\cdot 3^n}{n!}$$

If on each team, we designate 1st server, then

$$\text{Total } \# = \frac{\binom{4n}{4}\binom{4n-4}{4}\cdots\binom{4}{4}\cdot 3^n \cdot 2^{2n}}{n!}$$

$2n$ teams in total

$$= \frac{(4n)!\,3^n}{(4!)^n n!}$$

$$= \frac{(4n)!\,3^n \cdot 2^{2n}}{(4!)^n n!}$$

if we don't care about the arrangements within the groups of 4, then we can just divide out the factors of 3.

But we do care --- so keep them.

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$$\text{Total} = \sum_{k=0}^{\lfloor n/2 \rfloor} \left\{ \binom{n}{2k} \sum_{m=0}^{n-2k} \binom{n-2k}{m} \cdot 1 \right\}$$

choose an even # of green

choose m out of n−2k to paint red

paint the rest blue

$$\text{So } \# = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 2^{(n-2k)} \cdot 1^{2k}$$

$$= \sum_{k=0}^{n} \binom{n}{2k} 2^{n-2k} \cdot 1^{2k}$$ (okay if $2k > n$ → just add 0's)

$$= \sum_{j=0}^{n} \binom{n}{j} \frac{2^{n-j}(1+(-1)^j)}{2}$$ { if j even → keep, odd → = 0

$$= \frac{1}{2} \sum_{j=0}^{n} \binom{n}{j} 2^{n-j} \left[1+(-1)^j\right]$$

$$= \frac{1}{2} \left\{ (1+2)^n + (2-1)^n \right\}$$

$$= \boxed{\frac{1}{2} \left\{ 1 + 3^n \right\}}$$