Midterm quiz

1 Magnetic versus electric dipolar energy (10 pts.)

a) By any method you like, e.g. considering kinetic and potential energy plus Heisenberg's uncertainty, find the Bohr radius a_0 of hydrogen (assuming a fixed proton) in terms of the electron's mass m and charge e, and $\hbar$.

b) Write down a typical "atomic value" for an electric dipole moment d.

c) Find a typical "atomic" value for the electric field strength E.

d) Considering the electron as a current loop of radius a_0, find a typical atomic value for the magnetic moment μ (ignore numerical factors).

e) Find the magnetic field strength B created by this current loop, again no prefactors required.

f) Find the ratio of typical magnetic dipolar energy μB to electric dipolar energy dE in terms of the fine structure constant $\alpha = e^2/\hbar c$ (in cgs units. SI will make your life harder).

2 Fine structure in a magnetic field (20 pts)

An atom has orbital angular momentum L, spin angular momentum S, and no nuclear spin. The fine structure Hamiltonian is $H_{\mathrm{FS}} = A\boldsymbol{L} \cdot \boldsymbol{S}$. It is interacting with a magnetic field according to the Zeeman Hamiltonian $H_{\mathrm{Z}} = -\mu_B \left(g_S \boldsymbol{S} + \boldsymbol{L}\right) \cdot \boldsymbol{B}$. The total angular momentum is denoted $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

a) What are the "good" quantum numbers at low magnetic field ($\mu_B B \ll A$)?

b) What are the "good" quantum numbers at high magnetic fields ($\mu_B B \gg A$)?

c) What are the energy levels of the atom in zero magnetic field $B = 0$?

d) What are the energy levels at very high field?

e) At low magnetic fields one observes a linear Zeeman effect whose description involves the Landé g-factor. Derive the Landé g-factor in terms of g_s, S, L and other good quantum number(s) at low field. In terms of g, and relevant quantum number(s), write down the expression for the Zeeman energies at low field.

3 Microwave dressing of molecules (20 pts)

The rotational Hamiltonian of a molecule is $H = B\boldsymbol{J}^2$, where B is the rotational constant. In the molecule frame, the "permanent" dipole moment of the molecule is d. We would like to induce a strong dipole moment in the lab frame, but instead of using a large static electric field we will use microwaves tuned to rotational transitions. For simplicity, we will consider π-polarized microwaves, i.e. the electric field will oscillate along the z-direction, our quantization axis. So the perturbation is

$$V = -\boldsymbol{d} \cdot \boldsymbol{E}(t) = -dE \cos\vartheta \cos(\omega t)$$

a) Starting with the molecule in the rotational ground state $|J = 0, m_J = 0\rangle$, which excited state can the microwave reach and what is the resonant frequency ω_0 for this transition?

b) Derive the coupling matrix element between $|0, 0\rangle \equiv |{\downarrow}\rangle$ and the excited state (denoted $|{\uparrow}\rangle$). This can be written as $\hbar\omega_R \cos\omega t$ with Rabi frequency ω_R that you will find. Note that $\cos\vartheta = \sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta, \phi)$.

c) Using the rotating wave approximation, derive a time-independent 2×2 Hamiltonian matrix describing the evolution in the presence of the microwave drive.

d) Find the eigenstates (in the rotating frame, i.e. time-independent) in the case of a resonant drive $\omega = \omega_0$.

e) Find the dipole moment in these two eigenstates.

f) Starting in the state $|0, 0\rangle$, sketch the dipole moment versus time. Choose a scale where dE is a good fraction of ω_0, so that all relevant behavior is visible. At what times is the dipole moment at its largest magnitude? *Note: Just a proper sketch suffices, you don't need to give the wavefunction as a function of time, although you might want to write that down if you are stuck, and derive the dipole moment vs. time.*

g) Instead of the behavior in f), we would like to find a microwave dressing scheme that smoothly "turns on" the molecule's dipole moment permanently. Can you propose a method that achieves this? Is there an adiabaticity criterion you need to fulfill, and if so, what is it?