

# FINAL EXAM

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Problem	Earned	Total
1		20
5		20
6		20
8		20
10		20
11		20
<b>Total</b>	/100	120

## Problem 1 (10 pts)

Show that the affine variety in  $\mathbb{A}^2$  defined by  $xy = 1$  is not isomorphic to  $\mathbb{A}^1$ .

Solution: Let  $V = V(xy - 1) \subset \mathbb{A}^2$  denote the affine variety defined by  $xy = 1$ . We know that  $V$  is isomorphic to  $\mathbb{A}^1$  if and only if the coordinate rings  $k[V]$  and  $k[\mathbb{A}^1] = k[t]$  are also isomorphic. It suffices to show  $k[V] = k[x, y]/\langle xy - 1 \rangle$  is **not** isomorphic to  $k[t]$ . To this end, consider any ring homomorphism  $\Phi : k[x, y] \rightarrow k[t]$  with  $\ker \Phi = \langle xy - 1 \rangle$ . This is possible because every ideal  $I$  of a ring  $R$  is the kernel of some ring homomorphism of  $R$ .

Let  $\Phi(x) = \alpha(t) \in k[t]$  and  $\Phi(y) = \beta(t) \in k[t]$ , then

$$0 = \Phi(xy - 1) = \Phi(x)\Phi(y) - 1 = \alpha(t)\beta(t) - 1.$$

This means

$$\alpha(t)\beta(t) = 1.$$

Since  $\alpha(t)$  and  $\beta(t)$  must be polynomials in  $t$ , the equation holds only if  $\alpha(t)$  and  $\beta(t)$  are elements of  $k$ , which are just constants. Now, since  $\Phi$  is a ring homomorphism, for any  $f \in k[V]$ ,  $\Phi(f)$  must also be a constant (in  $k$ ). So,  $\Phi$  is not surjective, i.e.,  $\Phi(k[x, y])$  is not isomorphic to  $k[V]$ .

The first isomorphism theorem for rings says  $k[x, y]/\langle xy - 1 \rangle$  is isomorphic to  $\Phi(k[x, y])$ . But we just showed  $\Phi(k[x, y])$  is not isomorphic to  $k[x]$ , so  $k[V]$  is not isomorphic to  $k[x]$ . This implies  $V$  is not isomorphic to  $\mathbb{A}^1$ .  $\square$

## Problem 2 (20 pts)

Given two affine varieties  $V$  and  $W$  and a polynomial map  $f : V \rightarrow W$ , we get a ring homomorphism  $f^* : k[W] \rightarrow k[V]$ . When is  $f^*$  injective?

Solution:

## Problem 4 (20 pts)

Suppose that  $f$  is a rational function on  $\mathbb{P}^1$ .

- (a) Show that if  $f$  is regular at every point of  $\mathbb{P}^1$  then it is constant. (Hint: consider two affine pieces  $\mathbb{A}_{(0)}^1$  and  $\mathbb{A}_{(1)}^1$ .)
- (b) Show that there are no non-constant morphisms  $\mathbb{P}^1 \rightarrow \mathbb{A}^m$ .

Solution:

- (a) We know that

$$\mathbb{P}^1 = \bigcup_{i=1}^{1+1} \mathbb{A}_{(i)}^1 = \mathbb{A}_{(0)}^1 \cup \mathbb{A}_{(1)}^1$$

where

$$\mathbb{A}_{(i)}^1 \equiv \{P = [x_0 : x_1] \in \mathbb{P}^1 \mid x_i \neq 0\}.$$

Let  $f$  a rational function on  $\mathbb{P}^1$  be given such that it is regular at any  $P \in \mathbb{P}^1$ , i.e.,  
Then we have

$$\text{dom}(f) = \mathbb{P}^1 = \mathbb{A}_{(0)}^1 \cup \mathbb{A}_{(1)}^1.$$

Theorem 4.8. says that  $\text{dom}(f) = V \iff f \in k[V]$  where  $V$  is an affine variety. We have  $\mathbb{P}^1$  a projective variety, so we can't quite use this result yet. But since we can write  $\mathbb{P}^1$  as a union of two copies of  $\mathbb{A}^1$  which are affine varieties, this will be helpful. Indeed,

## Problem 5 (20 pts)

Below are three formulas that possibly define rational maps  $f : \mathbb{P}^2 \rightarrow \mathbb{P}^2$ . Decided whether the formulas do define rational maps. If they do, determine  $\text{dom}(f)$  and decide whether  $f$  is birational.

$$(i) \ f([x : y : z]) = [1/x : 1/y : 1/z]$$

$$(ii) \ f([x : y : z]) = [x : y : 1]$$

$$(iii) \ f([x : y : z]) = [(x^3 + y^3)/z^3 : y^2/z^2 : 1].$$

Rational maps must be ratio(s) of homogeneous polynomials of the same degree. On first glance we see that (ii) does not define a rational map because there is no way to write its output into ratios of homogeneous polynomials of the same degree:

$$[x : y : 1] = [1 : y/x : 1/x] = [x/y : 1 : 1/y].$$

On the other hand, the outputs in (i) and (iii) can be written in the desired forms:

$$\left[ \frac{1}{x} : \frac{1}{y} : \frac{1}{z} \right] = \left[ 1 : \frac{x}{y} : \frac{x}{z} \right] = \left[ \frac{y}{x} : 1 : \frac{y}{z} \right] = \left[ \frac{z}{x} : \frac{z}{y} : 1 \right] = \dots$$

and

$$\left[ \frac{x^3 + y^3}{z^3} : \frac{y^2}{z^2} : 1 \right] = \left[ \frac{x^3 + y^3}{zy^2} : 1 : \frac{z^2}{y^2} \right] = \left[ 1 : \frac{y^2z}{x^3 + y^3} : \frac{z^3}{x^3 + y^3} \right] = \dots$$

Now we want to find  $\text{dom}(f)$  for (i) and (iii). By definition,

$$\text{dom}(f) = \{P \in \mathbb{P}^2 \mid f \text{ is regular at } P\}.$$

**Find the domain:**

- For (i), clearly  $f$  is regular at all points with  $x, y, z \neq 0$ . Without loss of generality, suppose  $x = 0$  and  $y, z \neq 0$  then we write the output as  $[1 : x/y : x/z] = [1 : 0 : 0]$ . So  $f$  is also regular there. Similarly, we can see  $f$  is also regular at  $[x : y : z]$  where only  $z = 0$  and only  $y = 0$ . However, when two of  $x, y, z$  are zero,  $f([x, y, z])$  is no longer defined. So, for (i),

$$\text{dom}(f_{(i)}) = \mathbb{P}^2 - \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$$

- For (iii), we are interested in cases where  $z = 0$ ,  $y = 0$ , and  $x^3 + y^3 = 0$ . By writing the output of  $f$  in different forms above, we see that  $f$  is still regular at  $[x : y : z]$  where only **one** of the possibilities  $z = 0$ ,  $y = 0$ , or  $x^3 + y^3 = 0$  occurs, or if only  $z = y = 0$ ,  $x = y = x^3 + y^3 = 0$  occurs. However, since we have the factor  $[(x^3 + y^3)/z]^{\pm 1}$  in all of the three representations of the output of  $f$ , we see that  $f$  fails to be regular when  $z = 0$  and  $x^3 + y^3 = 0$ . So, for (iii),

$$\text{dom}(f_{(iii)}) = \mathbb{P}^2 - \{[-1 : 1 : 0]\}$$

**Birational?** Next,  $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  is *birational* if there exists a rational (inverse) map  $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  such that  $f \circ g = \text{id}_{\mathbb{P}^2}$  and  $g \circ f = \text{id}_{\mathbb{P}^2}$ .

- For (i), we consider the rational function  $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  defined by  $g([u : v : w]) = [1/u : 1/v : 1/w]$ . So,  $g$  is just  $f$ . For  $[u : v : w] \in \text{dom}(g) = \text{dom}(f)$ , we have

$$f \circ g([u : v : w]) = f([1/u : 1/v : 1/w]) = [u : v : w].$$

for all  $[u : v : w]$  in  $\text{dom}(g) = \text{dom}(f)$ . Similarly,  $g \circ f$  is also the identity function on  $\text{dom}(f) = \text{dom}(g)$ . Finally, since  $f$  and  $g$  are really the same function, it remains to show  $f$  is dominant. By definition,  $f$  is dominant if  $f(\text{dom}(f))$  is dense in  $\mathbb{P}^2$ . This is equivalent to saying  $f(\text{dom}(f)) \cap \mathcal{O} \neq \emptyset$  for any nonempty open set  $\mathcal{O} \subset \mathbb{P}^2$ . It is clear that the output of  $f$  is not only all tuples  $[1/x : 1/y : 1/z]$  with  $x, y, z \neq 0$  but also  $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$ :

$$f([1 : 1 : 0]) = [0 : 0 : 1], \quad f([1 : 0 : 1]) = [0 : 1 : 0], \quad f([0 : 1 : 1]) = [1 : 0 : 0].$$

So,  $f(\text{dom}(f)) = \mathbb{P}^2$ . It follows that  $f$  is dominant. With the dominant rational inverse  $g$  (which is just  $f$  itself), we conclude that  $f$  is birational.

(†) Alternatively, we can see that the induced  $k$ -algebra homomorphism  $f^* : k[\mathbb{P}^2] \rightarrow k(\mathbb{P}^2)$  given by  $g \mapsto g \circ f$  is an isomorphism. This (I believe) is easy to see because for any  $g \in k[\mathbb{P}^2]$ , if  $g([1/x : 1/y : 1/z]) = 0$  then  $g = 0$  necessarily, which shows  $f^*$  is injective. Further, any element of  $k(\mathbb{P}^2)$  (which has the form of a ratio of two homogeneous polynomials of the same degree) can be put into the form  $g \circ f$  where  $g \in k[\mathbb{P}^2]$ . So  $f^*$  is an isomorphism. This combined with the fact that  $f$  is dominant is equivalent to  $f$  being birational.

- For (iii), we claim that  $f$  is not birational. This is because the induced  $k$ -algebra homomorphism  $f^* : k[\mathbb{P}] \rightarrow k(\mathbb{P}^2)$  is **not** onto (hence not an isomorphism). Consider the element  $x/y \in k(\mathbb{P}^2)$ . There is no  $g \in k[\mathbb{P}^2]$  such that  $g \circ f[x : y : z] = x/y$  because  $x$  always appears as  $x^3$  in the output of  $f$ . We conclude  $f$  is not birational.

□

## Problem 6 (10 pts)

Prove statements (i), (ii), (iii), (iv) from Example I from section 5.7 of *Undergraduate Algebraic Geometry*

Solution: Define  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^m$  by

$$[U : V] \mapsto [U^m : U^{m-1}V : \dots : V^m]$$

- (i)  $f$  is a rational map: We notice that while  $U^i V^j$  are rational functions (since they are not given by a ratio of homogeneous polynomials of the same degree), we can re-write the definition of  $f$  as

$$[U : V] \xrightarrow{f} \left[ \frac{U^m}{V^m} : \frac{U^{m-1}}{V^{m-1}} : \dots : 1 \right].$$

Now, each component  $f_i$  is a rational function, so we have a rational map.

- (ii)  $f$  is a morphism:  $f$  is a morphism if we can show  $\mathbb{P}^1 \subset \text{dom}(f)$ , i.e.,  $f$  is regular at every point of  $\mathbb{P}^1$ . If  $V \neq 0$  then there's nothing to prove because of the formula we just wrote down. If  $U \neq 0$  then we can just rewrite the definition of  $f$  as

$$[U : V] \xrightarrow{f} \left[ 1 : \frac{V}{U} : \dots : \frac{V^m}{U^m} \right]$$

which shows that  $f$  is also regular at these points. When  $U, V \neq 0$ , there's nothing to worry about. So,  $f$  is indeed regular at every point in  $\mathbb{P}^1$ , i.e.,  $f : \mathbb{P}^1 \rightarrow \mathbb{P}^m$  is a morphism.

- (iii) The image of  $f$  is the set of points  $[X_0 : \dots : X_m] \in \mathbb{P}^m$  such that

$$[X_0 : X_1] = [X_1 : X_2] = \dots = [X_{m-1} : X_m]$$

that is

$$X_0 X_2 = X_1^2; \quad X_0 X_3 = X_1 X_2; \quad X_0 X_4 = X_1 X_3; \quad \text{etc.}$$

We notice that for every input  $[U : V]$ , the output looks like

$$[X_0 : X_1 : \dots : X_m] = [U^m : U^{m-1}V : \dots : V^m]$$

So, we have that

$$\begin{aligned} [X_0 : X_1] &= [U^m : U^{m-1}V] = [U : V] \\ [X_1 : X_2] &= [U^{m-1}V : U^{m-2}V^2] = [U : V] \end{aligned}$$

and so forth. So, we end up with

$$[X_0 : X_1] = [X_1 : X_2] = \dots = [X_{m-1} : X_m]$$

From here it is not hard to generalize:

$$[X_0 : X_1] = [X_{n-1} : X_n]$$

and so we have a chain of equalities  $X_0 X_n = X_1 X_{n-1}$  for different values of  $n$ . This means any  $2 \times 2$  matrix of the form

$$\begin{bmatrix} X_0 & X_{n-1} \\ X_1 & X_n \end{bmatrix}$$

has vanishing determinant. This leads to the condition

$$\text{rank} \begin{bmatrix} X_0 & X_1 & X_2 & \dots & X_{m-1} \\ X_1 & X_2 & X_3 & \dots & X_m \end{bmatrix} \leq 1.$$

This condition coincides exactly with the all-vanishing determinant condition above: If the matrix rank is zero, the matrix is the zero matrix, in which case there is nothing interesting (in fact this case won't happen because at least one  $X_i$  has to be nonzero –  $[X_0 : \dots : X_m] \in \mathbb{P}^m$ ). If the matrix has rank one, then one row is a constant multiple of the other. After writing, say, the first row as some multiple of the second row, we see that any  $2 \times 2$  minor has the form  $a(X_n X_m - X_m X_n)$ , which vanishes identically. When the matrix has rank 2, the both rows are linearly independent, and we no longer have the vanishing  $2 \times 2$  minor condition.

- (iv) There is an inverse morphism  $g : C \rightarrow \mathbb{P}^1$ . The inverse morphism takes a point of  $C$  into the common ratio:

$$[X_0 : \dots : X_m] \xrightarrow{g} [X_0 : X_1]$$

where  $[X_0 : X_1]$  is “common” in the sense of the previous item. We want to check that this is actually a morphism, i.e., it is a rational map that is regular at every point in  $C$ . Clearly, we can write

$$[X_0 : \dots : X_m] \xrightarrow{g} \left[1 : \frac{X_1}{X_0}\right] \text{ or } \left[\frac{X_{m-1}}{X_m} : 1\right]$$

depending on whether  $X_1 = 0$  or  $X_0 = 0$  (or both). In any case, we see that  $g$  is a rational function (as given by ratios of homogeneous polynomials of the same degree) that is regular at every point on  $C$ .  $\square$



**Problem 7** (20 pts)

Consider the rational map  $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  defined by

$$f([u : v : w]) = [uw : w^2 : vw].$$

Describe the domain and the image of  $f$ .

Solution:

## Problem 11 (30 pts)

Given an invertible matrix  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  with complex coefficients, define a function  $f_A : \mathbb{P}_{\mathbb{C}}^1 \rightarrow \mathbb{P}_{\mathbb{C}}^1$  by

$$f_A([u : v]) = [au + bv : cu + dv].$$

- (a) Show that  $f_A$  is a morphism.
- (b) How does  $f_{AB}$  relate to  $f_A$  and  $f_B$ ?
- (c) Show that  $f_A$  has an inverse morphism, so that  $f_A$  defines an automorphism of  $\mathbb{P}_{\mathbb{C}}^1$ .
- (d) If we identify  $\mathbb{C}$  with the standard  $\mathbb{A}^1 \subset \mathbb{P}^1$  defined by  $v \neq 0$ , show that the restriction of  $f_A$  to  $\mathbb{C}$  is a rational function, and find its formula.

Solution:

- (a)  $f_A$  is a morphism if  $\mathbb{P}_{\mathbb{C}}^1 \subset \text{dom}(f)$ , i.e.,  $f$  is regular at every point in  $\mathbb{P}_{\mathbb{C}}^1$ , i.e.,  $au + bv$  and  $cu + dv$  are never simultaneously zero for any  $u, v$ . Now, we don't have the possibility  $u = v = 0$  because  $[u : v] \in \mathbb{P}_{\mathbb{C}}^1$ . So,  $au + bv = 0 = cu + dv$  if and only if  $\det(A) = 0$ . But this never happens because  $A$  is invertible. So,  $f$  is regular at every point  $[u : v] \in \mathbb{P}_{\mathbb{C}}^1$ , or  $f$  is a morphism.
- (b) We claim that  $f_{AB} = f_A \circ f_B$ . Let an invertible matrix  $B$  be given,

$$B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \implies AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

Then

$$\begin{aligned} f_{AB}([u : v]) &= [(aa' + bc')u + (ab' + bd')v : (ca' + dc')u + (cb' + dd')v] \\ &= [a(a'u + b'v) + b(c'u + d'v) : c(a'u + b'v) + d(c'u + d'v)] \\ &= f_A[a'u + b'v : c'u + d'v] \\ &= f_A \circ f_B([u : v]). \end{aligned}$$

- (c) To show that  $f_A$  has an inverse morphism, it suffices to construct one. Consider  $f_{A^{-1}}$  defined by  $A^{-1}$ , the matrix inverse of  $A$ :

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We know that  $\det(A) \neq 0$ , so  $A^{-1}$  exists. Also, since the scaling factor  $1/\det(A) \neq 0$  appears at every entry of  $A^{-1}$ , we can ignore it in the definition of  $f_{A^{-1}}$ :

$$\begin{aligned} f_{A^{-1}}([u : v]) &= \left[ \frac{d}{\det(A)}u + \frac{-b}{\det(A)}v : \frac{-c}{\det(A)}u + \frac{a}{\det(A)}v \right] \\ &= [du - bv : -cu + av]. \end{aligned}$$

Next we check that  $f_A$  and  $f_{A^{-1}}$  are inverses. By the previous item, we know that  $f_{A^{-1}A} = f_{A^{-1}A} = f_I$  where  $I$  is the  $2 \times 2$  identity matrix. Since

$$f_I([u : v]) = [u + 0v : 0u + v] = [u : v],$$

$f_A$  and  $f_{A^{-1}}$  are inverses. So,  $f_A$  is an isomorphism from (the entire)  $\mathbb{P}_{\mathbb{C}}^1$  to itself. This makes  $f_A$  an automorphism.

- (d) We want to look at  $f_A|_{\mathbb{C}}$  where  $\mathbb{C}$  is identified with the standard  $\mathbb{A}^1 \subset \mathbb{P}^1$  defined by  $v \neq 0$ . We want to show that  $f_A|_{\mathbb{C}}$  is a rational function and find its formula. Now, when  $v \neq 0$ , we can write the input  $[u : v]$  as  $[u/v : 1] = [t : 1]$  where  $t \in \mathbb{C}$ . With this,

$$f_A|_{\mathbb{C}}([t : 1]) = [at + b : ct + d] = \left[ \frac{at + b}{ct + d} : 1 \right].$$

We can identify this as rational function  $f : \mathbb{C} \rightarrow \mathbb{C}$  defined by

$$f(t) = \frac{at + b}{ct + d}.$$