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Course: **8.421 - AMO I**
Problem set: **#7**
Due: Friday, April 1, 2022.

1. Spherical Harmony. We want to evaluate matrix elements

$$\langle J' m_J' | Y_{LM} | J m_J \rangle = \int d\Omega \, Y^*_{J' m_J'} Y_{LM} Y_{J m_J}.$$

To do this, we consider two particles with angular momenta $\boldsymbol{j}_1$ and $\boldsymbol{j}_2$. The total angular momentum is $\boldsymbol{J} = \boldsymbol{j}_1 + \boldsymbol{j}_2$. We can go between the coupled and uncoupled basis via

$$|(j_1 j_2) JM\rangle = \sum_{m_1, m_2} |j_1 m_1\rangle |j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | JM\rangle$$
$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{J,M} |(j_1 j_2) JM\rangle \langle JM | j_1 m_1 j_2 m_2\rangle.$$

The sum over M has only one nonzero term $M = m_1 + m_2$, and $|j_1 - j_2| < J < j_1 + j_2$. We also have the wavefunction of each particle at polar angle $\Omega_i = (\theta_1, \phi_i)$ is

$$\langle \Omega_i | j_i m_i \rangle = Y_{j_i m_i}(\Omega_i).$$

For the state of definite total angular momentum, we have

$$\Phi_{JM}(\Omega_1, \Omega_2) = \langle \Omega_1, \Omega_2 | (j_1 j_2) JM\rangle.$$

Now consider the function

$$F_{JM}(\Omega) \equiv \langle \Omega, \Omega | (j_1 j_2) JM \rangle$$

where $\Omega_1 = \Omega_2 = \Omega$. This is a wavefunction of an effective particle with angular momentum quantum numbers J, M. Indeed, it inherits its eigenvalues $\boldsymbol{J}^2$ and J_z from $\Phi_{JM}(\Omega_1, \Omega_2)$. We conclude that $F_{JM}(\Omega)$ must be proportional to the spherical harmonic $Y_{JM}(\Omega)$. Let us call

$$F_{JM}(\Omega) = A_{(j_1 j_2)J} Y_{JM}(\Omega).$$

The factor $A_{(j_1 j_2)J}$ cannot depend on M as F_{JM} must behave exactly like Y_{JM}, in particular when acted upon by $J_\pm$ which changes M. From here we have that

$$A_{(j_1 j_2)J} Y_{JM}(\Omega) = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle Y_{j_1 m_1}(\Omega) Y_{j_2 m_2}(\Omega).$$

(a) To find $A_{(j_1 j_2)J}$ we consider the special case where $\Omega = (\theta = 0, \phi)$. In this case, we have that

$$Y_{j_i m_i}(\Omega) = Y_{j_i m_i}(\theta = 0, \phi) = \sqrt{\frac{2j_i + 1}{4\pi}} \delta_{m_i 0}.$$

From the equation above we find that

$$A_{(j_1 j_2)J} \sqrt{\frac{2J+1}{4\pi}} \delta_{M0} = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle \sqrt{\frac{2j_1 + 1}{4\pi}} \delta_{m_1 0} \sqrt{\frac{2j_2 + 1}{4\pi}} \delta_{m_2 0}.$$

This equation is nontrivial if $M = m_1 = m_2 = 0$, in which case we can solve for $A_{(j_1 j_2)J}$:

$$\boxed{A_{(j_1, j_2)J} = \sqrt{\frac{(2j_1 + 1)(2j_2 + 1)}{4\pi(2J+1)}} \langle j_1 0 j_2 0 | J0 \rangle}$$

(b) By applying $\langle\Omega,\Omega|$ to the LHS of

$$|j_1m_1\rangle|j_2m_2\rangle = \sum_{J,M}|(j_1j_2)JM\rangle\langle JM|j_1m_1j_2m_2\rangle$$

we find that

$$\begin{aligned}
\boxed{Y_{j_1m_1}(\Omega)Y_{j_2m_2}(\Omega)} &= \sum_{J,M}F_{JM}(\Omega)\langle JM|j_1m_1j_2m_2\rangle \\
&= \sum_{J,M}A_{(j_1j_2)J}Y_{JM}(\Omega)\langle JM|j_1m_1j_2m_2\rangle \\
&= \boxed{\sum_{J,M}\sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2J+1)}}\langle j_10j_20|J0\rangle\langle JM|j_1m_1j_2m_2\rangle Y_{JM}(\Omega)}
\end{aligned}$$

(c) It remains to find the matrix element given at the top. To do this, we simply plug things in and use orthonormality of spherical harmonics:

$$\begin{aligned}
\boxed{\langle j_3m_3|Y_{j_2m_2}|j_1m_1\rangle} &= \int d\Omega\, Y^*_{j_3m_3}(\Omega)Y_{j_2m_2}(\Omega)Y_{j_1m_1}(\Omega) \\
&= \int d\Omega\, Y^*_{j_3m_3}(\Omega)\sum_{J,M}\sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j_3+1)}}\langle j_10j_20|j_30\rangle\langle j_3m_3|j_1m_1j_2m_2\rangle Y_{JM}(\Omega) \\
&= \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j_3+1)}}\langle j_10j_20|j_30\rangle\langle j_3m_3|j_1m_1j_2m_2\rangle \underbrace{\int d\Omega Y^*_{j_3m_3}(\Omega)Y_{j_3m_3}(\Omega)}_{1} \\
&= \boxed{\sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j_3+1)}}\langle j_10j_20|j_30\rangle\langle j_3m_3|j_1m_1j_2m_2\rangle}
\end{aligned}$$

2. Dipole Operator. A symmetric top molecule has a Hamiltonian $\mathcal{H} = B\boldsymbol{J}^2$, with B the rotational constant. The dipole moment operator is $\hat{\boldsymbol{d}} = d\hat{\boldsymbol{r}}$, with d the value of the "permanent dipole moment" (in the molecular frame).

(a) We will prove the spherical tensor decomposition:

$$\sum_m C^*_{1m}\hat{\boldsymbol{e}}_m = \sum_m C_{1m}\hat{\boldsymbol{e}}_m = \hat{\boldsymbol{r}}$$

where $C_{1m}(\theta,\phi) = \sqrt{4\pi/3}Y_{1m}(\theta,\phi)$,

$$\hat{\boldsymbol{e}}_\pm = \mp\frac{\hat{\boldsymbol{e}}_x \pm i\hat{\boldsymbol{e}}_y}{\sqrt{2}} \qquad \hat{\boldsymbol{e}}_0 = \hat{\boldsymbol{e}}_z$$

To this end, we simply write everything out explicitly. We will show that the left-most term is equal to $\hat{\boldsymbol{r}}$. Once done, the other equality follows immediately from the fact that $\hat{\boldsymbol{r}}$ is real (and therefore the second term is equal to the (conjugate of) the first term).

$$\begin{aligned}
&C^*_{1-}\hat{\boldsymbol{e}}_- + C^*_{10}\hat{\boldsymbol{e}}_0 + C^*_{1+}\hat{\boldsymbol{e}}_+ \\
&= \frac{1}{2}e^{+i\phi}\sqrt{\frac{3}{2\pi}}\sqrt{\frac{4\pi}{3}}\sin\theta\frac{\hat{\boldsymbol{e}}_x - i\hat{\boldsymbol{e}}_y}{\sqrt{2}} + \frac{1}{2}\sqrt{\frac{3}{\pi}}\sqrt{\frac{4\pi}{3}}\cos\theta\hat{\boldsymbol{e}}_z + \frac{1}{2}e^{-i\phi}\sqrt{\frac{3}{2\pi}}\sqrt{\frac{4\pi}{3}}\sin\theta\frac{\hat{\boldsymbol{e}}_x + i\hat{\boldsymbol{e}}_y}{\sqrt{2}} \\
&= \sin\theta\cos\phi\,\hat{\boldsymbol{e}}_x + \sin\theta\sin\phi\,\hat{\boldsymbol{e}}_y + \cos\theta\,\hat{\boldsymbol{e}}_z \\
&= \hat{\boldsymbol{r}}. \qquad \checkmark
\end{aligned}$$

(b) Now we will show that

$$\hat{\boldsymbol{e}}_m^* \cdot \hat{\boldsymbol{e}}_n = \sum_p \delta_{mp}\delta_{np} = \delta_{mn}.$$

It suffices to demonstrate the following cases:

$$\hat{\boldsymbol{e}}_+^* \cdot \hat{\boldsymbol{e}}_- = -\frac{\hat{\boldsymbol{e}}_x - i\hat{\boldsymbol{e}}_y}{\sqrt{2}} \cdot \frac{\hat{\boldsymbol{e}}_x - i\hat{\boldsymbol{e}}_y}{\sqrt{2}} = 0 \iff \hat{\boldsymbol{e}}_-^* \cdot \hat{\boldsymbol{e}}_+ = 0$$

and

$$\hat{\boldsymbol{e}}_\pm^* \cdot \hat{\boldsymbol{e}}_\pm = \frac{\hat{\boldsymbol{e}}_x \mp i\hat{\boldsymbol{e}}_y}{\sqrt{2}} \cdot \frac{\hat{\boldsymbol{e}}_x \mp i\hat{\boldsymbol{e}}_y}{\sqrt{2}} = \frac{2}{2} = 1.$$

With these we are done.

(c) Suppose we have two unit vectors $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{r}}'$ pointing in the direction of solid angle (θ, ϕ) and (θ', ϕ'). Let us call Θ the angle between the vectors, then we have

$$\begin{aligned}
\cos\Theta &= \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}' \\
&= \sum_{m,n} C_{1m}(\theta,\phi)C_{1n}^*(\theta',\phi')\,\hat{\boldsymbol{e}}_m^* \cdot \hat{\boldsymbol{e}}_n \\
&= \sum_{m,n} C_{1m}(\theta,\phi)C_{1n}^*(\theta',\phi')\delta_{mn} \\
&= \sum_{m} C_{1m}(\theta,\phi)C_{1m}^*(\theta',\phi') \\
&= \cos\theta\cos\theta' + \frac{1}{2}e^{-i\phi-i\phi'}\sin\theta\sin\theta' + \frac{1}{2}e^{i\phi+i\phi'}\sin\theta\sin\theta' \\
&= \cos\theta\cos\theta' + \cos\big(\phi-\phi'\big)\sin\theta\sin\theta',
\end{aligned}$$

as expected from standard geometry. A generalization of this result (for which $l = 1$) is

$$P_l(\cos\Theta) = \sum_m C_{lm}^*(\theta,\phi)C_{lm}(\theta',\phi')$$

where

$$C_{lm}(\theta,\phi) = \sqrt{\frac{4\pi}{2l+1}}Y_{lm}(\theta,\phi).$$

The proof is done by setting one of the unit vectors the z-axis, and the angles simplify.

(d) The electric field can be written

$$\begin{aligned}
\boldsymbol{E} &= E_z\hat{\boldsymbol{e}}_z + E_x\hat{\boldsymbol{e}}_x + E_y\hat{\boldsymbol{e}}_y \\
&= E_0\hat{\boldsymbol{e}}_0 + E_+\hat{\boldsymbol{e}}_+ + E_-\hat{\boldsymbol{e}}_- \\
&= \sum_m E_m^*\hat{\boldsymbol{e}}_m = \sum_m E_m\hat{\boldsymbol{e}}_m^*
\end{aligned}$$

where $E_0, E_\pm$ defined in terms of $E_{x,y,z}$ in a similar way as the $\hat{e}_m$'s are defined in terms of $\hat{\boldsymbol{e}}_{x,y,z}$. The dipole operator may be decomposed into spherical harmonics as

$$\begin{aligned}
-\hat{\boldsymbol{d}} \cdot \boldsymbol{E} &= -d\hat{\boldsymbol{r}} \cdot \mathbf{E} \\
&= -d\sum_{m,n} C_{1m}^*E_n\hat{\boldsymbol{e}}_m \cdot \hat{\boldsymbol{e}}_n^* = -d\sum_{m,n} C_{1m}E_n^*\hat{\boldsymbol{e}}_m^* \cdot \hat{\boldsymbol{e}}_n \\
&= -d\sum_m C_{1m}^*E_m = -d\sum_m C_{1m}E_m^*.
\end{aligned}$$

(e) (Extra credit) Take $\boldsymbol{E} = E\hat{\boldsymbol{e}}_z$. The matrix elements of the Hamiltonian $\mathcal{H} = B\boldsymbol{J}^2 - \hat{\boldsymbol{d}} \cdot \boldsymbol{E}$ in the $\{|Jm_J\rangle\}$ basis are given by

$$\begin{aligned}
\langle J'm_{J'}|\mathcal{H}|Jm_J\rangle &= BJ(J+1)\delta_{JJ'}\delta_{m_{J'}m_J} - dE\langle J'm_{J'}|C_{10}|Jm_J\rangle \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_{J'}m_J} - dE\sqrt{\frac{4\pi}{3}}\underbrace{\int d\Omega\, Y^*_{J'm_{J'}}Y_{10}Y_{Jm_J}} \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_{J'}m_J} - dE\sqrt{\frac{(2J+1)(2+1)}{3(2J'+1)}}\langle (J,0)(1,0)|(J',0)\rangle\langle J'm_{J'}|(Jm_J)(1,0)\rangle \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_{J'}m_J} - dE\sqrt{\frac{(2J+1)(2+1)}{3(2J'+1)}}\langle (J,0)(1,0)|(J',0)\rangle\langle J'm_{J'}|(Jm_J)(1,0)\rangle.
\end{aligned}$$

where we have used the fact that $C_{10} = C^*_{10}$ and remove the conjugation symbol. To get the matrix elements in the second term, we must use Wigner's 3-j symbols:

$$\langle j_1m_1j_2m_2|JM\rangle = (-1)^{-j_1+j_2-M}\sqrt{2J+1}\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M\end{pmatrix}$$

which with we write the Hamiltonian matrix elements as

$$\begin{aligned}
\langle J'm_{J'}|\mathcal{H}|Jm_J\rangle &= BJ(J+1)\delta_{JJ'}\delta_{m_{J'}m_J} \\
&\quad - dE\sqrt{\frac{(2J+1)(2+1)}{3(2J'+1)}}(-1)^{-J+1}(-1)^{-J+1-m_{J'}}\sqrt{2J'+1}\sqrt{2J'+1}\begin{pmatrix} J & 1 & J' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} J & 1 & J' \\ m_J & 0 & -m_{J'}\end{pmatrix} \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_{J'}m_J} - dE(-1)^{-m_{J'}}\sqrt{\frac{(2J+1)(2+1)(2J'+1)}{3}}\begin{pmatrix} J & 1 & J' \\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix} J & 1 & J' \\ m_J & 0 & -m_{J'}\end{pmatrix}.
\end{aligned}$$

Using MATLAB, we can generate this matrix and diagonalize to find the eigenstates and their energies. Since it is convenient, I actually generated the Hamiltonian and carried out exact diagonalization in MATLAB but then plotted the probabilities in using `SphericalPlot3D[]` in Mathematica. Perhaps the grader will tell me that this solution is *cursed*. Figure 1 shows the first six energy levels up to an electric field $E \approx 10B/d$. For this calculation, I have picked $J_{\max} = 10$.

Using Mathematica, we plot $\left|\langle\theta,\phi|\Psi\rangle\right|^2$ of the lowest state for $dE/B = 0, 1, 10$. To do this, I have used MATLAB to find the lowest energy eigenstate for each value of dE/B. Then, I express these states in terms of spherical harmonics by identifying each entry of the state vector with the correct $|Jm_J\rangle$ state in the basis. For this part of the problem, I have used $J_{\max} = 4$. See Figures 2a, 2b, 3 for the results.

MATLAB code for calculating:

```
clear all
close all

J = 10;
strength = 0:0.05:10;
size = (J+1)^2;
H = zeros(size,size);

% creat a basis for the Hamiltonian
basis = [];

for j = 0:1:J
for mj = -j:1:j
basis = [basis; [j mj]];
end
end

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```

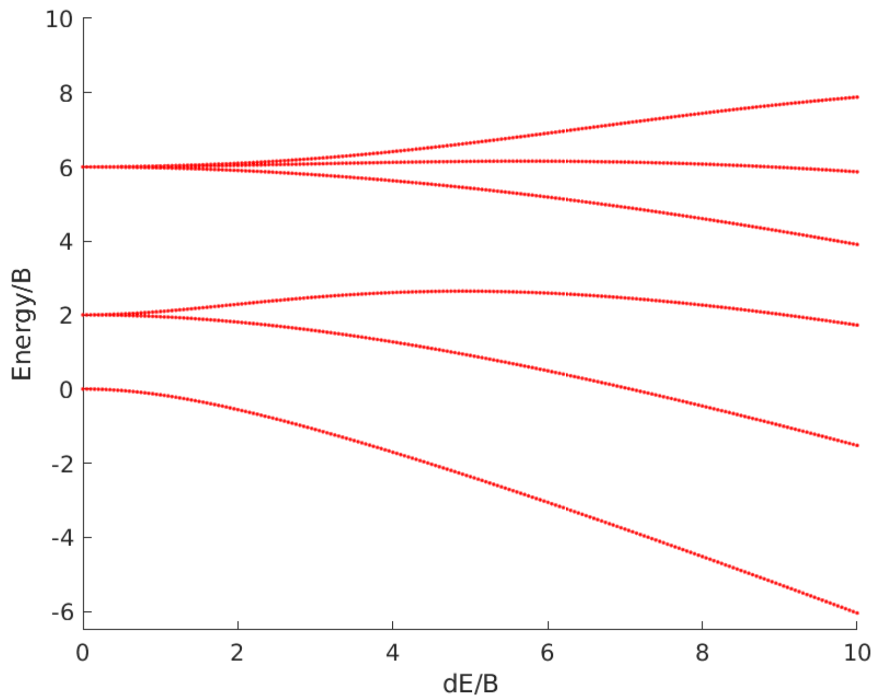
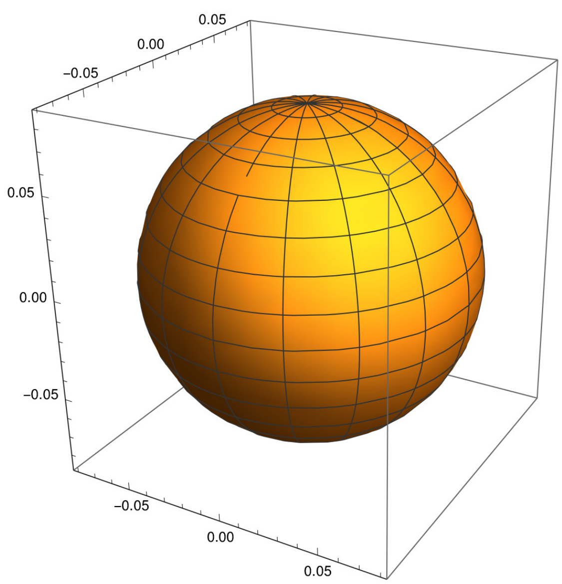
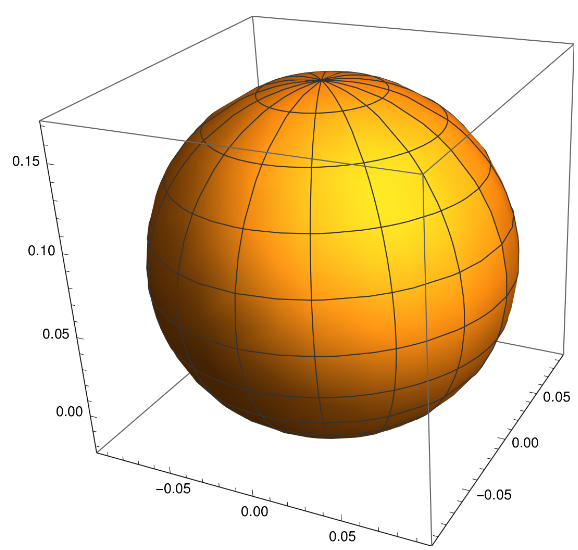


Figure 1: Energies of the first six energy levels p to an electric field $E \approx 10B/d$.



(a) $\left|\langle \theta, \phi | \Psi \rangle\right|^2$ for $dE/B = 0$.



(b) $\left|\langle \theta, \phi | \Psi \rangle\right|^2$ for $dE/B = 1$.

```
%%% first plot energies as a fn of dE/B

stark = figure(1);
for a = strength % loop over field strengths
% create the Hamiltonian, element-by-element
for r = 1:size
j = basis(r,1);
mj = basis(r,2);
```

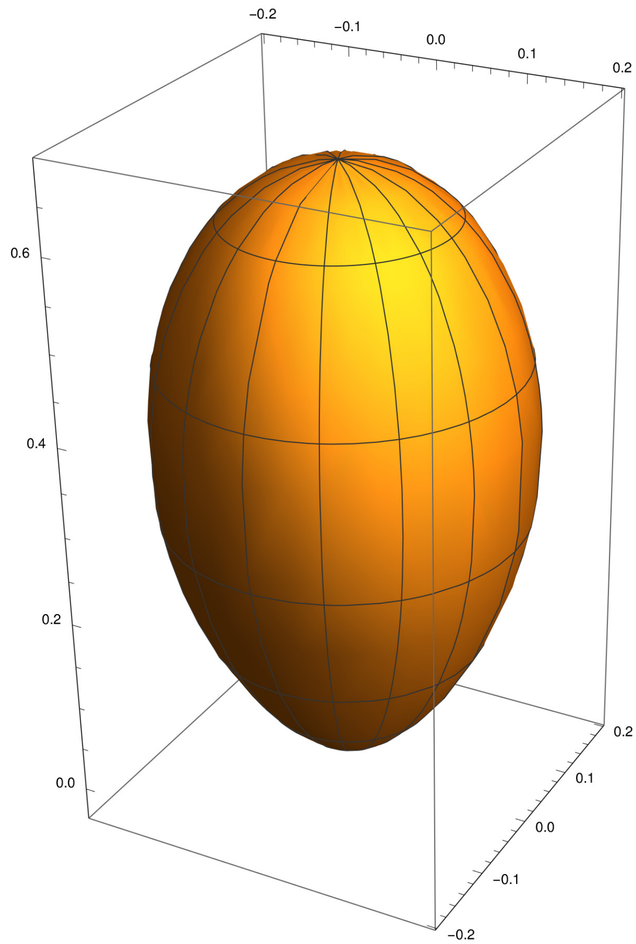


Figure 3: $\left|\langle\theta,\phi|\Psi\rangle\right|^2$ for $dE/B = 10$.

```
for c = 1:size
jj = basis(c,1);
mjj = basis(c,2);

H(r,c) = j*(j+1)*(j==jj)*(mj==mjj)...
-a*(-1)^(-mjj)*sqrt((2*j+1)*(2*1+1)*(2*jj+1)/3)...
*Wigner3j([j,1,jj],[0,0,0])*Wigner3j([j,1,jj],[mj,0,-mjj]);
end
end
% diag n plot eigenvalues associated with field strength a = dE/B
energies = eig(H);
hold on
plot(a*ones(size), energies, '.', 'Color', 'red', 'MarkerSize',4);
end

% disp(H)

% plot includes up to 6 lowest energies only

hold off
%title('Energy vs dE/B')
ylim([-6.5 10])
xlabel('dE/B')
ylabel('Energy/B')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```

```
% now find the lowest state and associated energy for various a = dE/B
% then take result over to mathematica to plot wavefunction^2

strength = [0 1 10];
wavefunction = 0;

for a = strength % loop over field strengths
% create the Hamiltonian, element-by-element
for r = 1:size
j = basis(r,1);
mj = basis(r,2);
for c = 1:size
jj = basis(c,1);
mjj = basis(c,2);

H(r,c) = j*(j+1)*(j==jj)*(mj==mjj)...
-a*(-1)^(-mjj)*sqrt((2*j+1)*(2*1+1)*(2*jj+1)/3)...
*Wigner3j([j,1,jj],[0,0,0])*Wigner3j([j,1,jj],[mj,0,-mjj]);
end
end
% diag n plot eigenvalues associated with field strength a = dE/B
[state,energy] = eigs(H,1,'SA');
disp('Ground state energy:')
disp(energy)
disp('Ground state:')
disp(state)

end
```

Mathematica code for plotting:

```
(*dE/B=0 ---> |0,0> state*)

Energy1 = 0;

State1 = SphericalHarmonicY[0, 0, \[Theta], \[Phi]];

SphericalPlot3D[
Conjugate[State1]*State1, {\[Theta], 0, Pi}, {\[Phi], 0, 2 Pi},
PlotRange -> All]


(*dE/B=1 ---> get superposition state*)

Jmax2 = 4;

Base2 = Flatten[
Table[SphericalHarmonicY[J, mJ, \[Theta], \[Phi]], {J, 0,
Jmax2}, {mJ, -J, J}], 1];

Energy2 = -0.1577;

State2 = {-0.9644, 0.0000, -0.2634, -0.0000, -0.0000,
0.0000, -0.0222, -0.0000, -0.0000, 0.0000, 0.0000,
0.0000, -0.0009, -0.0000, 0.0000, 0.0000, 0.0000, -0.0000, -0.0000,
0.0000, -0.0000, -0.0000, 0.0000, -0.0000, 0.0000};

State2 = State2/Norm[State2];

wfn2 = Dot[Base2, State2]

-0.27206 - 0.128702 Cos[\[Theta]] -
0.0070019 (-1 + 3 Cos[\[Theta]]^2) -
0.000335869 (-3 Cos[\[Theta]] + 5 Cos[\[Theta]]^3)

SphericalPlot3D[
Conjugate[wfn2]*wfn2, {\[Theta], 0, Pi}, {\[Phi], 0, 2 Pi},
AspectRatio -> Full]


(*dE/B=10 ---> get superposition state*)

Jmax3 = 4;

Base3 = Flatten[
Table[SphericalHarmonicY[J, mJ, \[Theta], \[Phi]], {J, 0,
Jmax2}, {mJ, -J, J}], 1];

Energy3 = -6.0448;
```

```
State3 = {-0.6477, 0.0000, -0.6782, -0.0000,
0.0000, -0.0000, -0.3323, -0.0000, 0.0000, -0.0000, 0.0000,
0.0000, -0.0987, -0.0000, -0.0000, 0.0000, 0.0000, 0.0000, -0.0000,
0.0000, -0.0191, -0.0000, 0.0000, -0.0000, 0.0000};

State3 = State3/Norm[State3];

wfn3 = Dot[Base3, State3]

-0.182713 - 0.33137 Cos[\[Theta]] -
0.104805 (-1 + 3 Cos[\[Theta]]^2) -
0.0368325 (-3 Cos[\[Theta]] + 5 Cos[\[Theta]]^3) -
0.0020205 (3 - 30 Cos[\[Theta]]^2 + 35 Cos[\[Theta]]^4)

SphericalPlot3D[
Conjugate[wfn3]*wfn3, {\[Theta], 0, Pi}, {\[Phi], 0, 2 Pi},
PlotRange -> All, AspectRatio -> Full]
```

3. The Stark Effect in Hydrogen.

(a) **Stark quenching of the $2S$ state.** In hydrogen, the $2S$ state is metastable. In the absence of external electric fields, its lifetime is 1/8 seconds. When an external electric field is applied, the $2S$ becomes mixed with the $2P$ state, which is strongly coupled to the ground state. The $2P$ state lifetime is only 1.6 ns. Depending on the strength of the electric field, the lifetime of the $2S$ state can be shortened by many orders og magnitude. This process is known as "quenching." To see how this works, we will look at how the amplitude $a(t)$ of $|a\rangle$ ($2S$ state) evolves ocer time in the presenece of a DC Stark perturbation with matrix element $\hbar V = \langle b| e\boldsymbol{E}\cdot\boldsymbol{r}\,|a\rangle$ where $|b\rangle$ stands for the $2P$ state.

Assuming that the atom is initially in the $2S$ state, i.e., $a(0) = 1, b(0) = 0$. Working in the interaction picture, we can derive the following differential equations for $a(t)$ and $b(t)$:

$$\begin{aligned} i\dot{a} &= V^* e^{i\omega_0 t} b - i\frac{\Gamma_a}{2} a \\ i\dot{b} &= V e^{-i\omega_0 t} a - i\frac{\Gamma_b}{2} b \end{aligned}$$

where $\Gamma_a = 8\text{ s}^{-1}$ and $\Gamma_b = 6.3\times 10^8\text{ s}^{-1}$. Here, $\hbar\omega_0$ is the energy difference $E_a - E_b$. To solve these equations, we make the following ansatz

$$\begin{aligned} a(t) &= a_1 e^{-\mu_1 t} + a_2 e^{-\mu_2 t} \\ b(t) &= b_1 e^{-(\mu_1 + i\omega_0)t} + b_2 e^{-(\mu_2 + i\omega_0)t} \end{aligned}$$

where a_1, a_2, b_1, b_2 are constants. Applying the initial condition $a(0) = 1, b(0) = 0$ we find that

$$a_1 + a_2 = 1 \qquad b_1 + b_2 = 0.$$

With this, we may write our ansatz as

$$\begin{aligned} a(t) &= a_1 e^{-\mu_1 t} + (1 - a_1) e^{-\mu_2 t} \\ b(t) &= b_1 e^{-(\mu_1 + i\omega_0)t} - b_1 e^{-(\mu_2 + i\omega_0)t} \end{aligned}$$

From this point, we may proceed using Mathematica. Plugging this ansatz into the system of differential equations above and set $t = 0$, we can solve for μ_1 and μ_2 in terms of a_1, b_1. The result is

$$\mu_1 = \frac{\Gamma_a}{2} - \frac{i(a_1 - 1)V}{b_1} \qquad \mu_2 = \frac{\Gamma_a}{2} - \frac{i a_1 V}{b_1}$$

It remains to find a_1, b_1. To this end, we pick $t = 1/V$. By writing μ_1, μ_2 in terms of a_1, b_1, we can solve for a_1, b_1. The result is

$$a_1 = \frac{1}{2} + \frac{i(\Gamma_a - \Gamma_b) - 2\omega_0}{2\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)}}$$
$$b_1 = -\frac{2V}{\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)}}$$

Note that there is another solution (a_1, b_1), but it doesn't matter all that much what we pick to compute μ_1, μ_2 since the result is the same. So we will pick the solution above. We should simplify this even more by writing the denominator as

$$\begin{aligned}\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)} &= \sqrt{16V^2 - ((\Gamma_a - \Gamma_b) + 2i\omega_0)^2} \\ &= \sqrt{16V^2 - (\Gamma_a - \Gamma_b)^2 + 4\omega_0^2 - 4i\omega_0(\Gamma_a - \Gamma_b)}.\end{aligned}$$

Now recall that for $x, y \in \mathbb{R}$,

$$\sqrt{x + iy} = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} + i\,\mathrm{sgn}(y)\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}.$$

Let $x = 16V^2 - (\Gamma_a - \Gamma_b)^2 + 4\omega_0^2$ and $y = -4\omega_0(\Gamma_a - \Gamma_b) > 0$, then we have

$$\begin{aligned}\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)} &= \sqrt{x + iy} \\ &= \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} + i\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}.\end{aligned}$$

From here we can calculate μ_1, μ_2:

$$\mu_1 = \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{i}{4}\sqrt{x + iy} - \frac{i\omega_0}{2}$$
$$\mu_2 = \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} - \frac{i}{4}\sqrt{x + iy} - \frac{i\omega_0}{2}$$

From here, we can find the real and imaginary parts of μ_1, μ_2:

$$\mathrm{Re}(\mu_1) = \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} - \frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}$$
$$\mathrm{Im}(\mu_1) = \frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} - \frac{\omega_0}{2}$$
$$\mathrm{Re}(\mu_2) = \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}$$
$$\mathrm{Im}(\mu_2) = -\frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} - \frac{\omega_0}{2}$$

Here, the real parts of μ_1, μ_2 give the decay rate of the $2S$ state, and the imaginary parts tell us the level shifts. With these expressions (for $\mu_1, \mu_2, a_1, a_2, b_1, b_2$) in terms of the known constants, we can write down the full solution for $a(t)$. While it is possible, I won't do that here since this is simply plugging things into the ansatz.

In the small V limit, we may Taylor expand these expressions about $V = 0$ to second order in V to find

$$\begin{aligned}
\mathrm{Re}(\mu_1) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{\Gamma_a}{4} - \frac{\Gamma_b}{4} - \frac{2(\Gamma_b - \Gamma_a)}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2}V^2 \\
\mathrm{Im}(\mu_1) &\approx \frac{\omega_0}{2} + \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2}V^2 - \frac{\omega_0}{2} = \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2}V^2 \\
\mathrm{Re}(\mu_2) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} - \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{2(\Gamma_b - \Gamma_a)}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2}V^2 \\
\mathrm{Im}(\mu_2) &\approx -\frac{\omega_0}{2} - \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2}V^2 - \frac{\omega_0}{2} = -\omega_0 + \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2}V^2.
\end{aligned}$$

Now let us make various assumptions to simplify things. Let us assume that $\Gamma_a \ll \Gamma_b$ and $\Gamma_a \ll \mu_1, \mu_2$. From these, we have

$$\begin{aligned}
\mathrm{Re}(\mu_1) &\approx -\frac{\Gamma_b/2}{(\Gamma_b/2)^2 + \omega_0^2}V^2 \\
\mathrm{Im}(\mu_1) &\approx \frac{\omega_0}{(\Gamma_b/2)^2 + \omega_0^2}V^2 \\
\mathrm{Re}(\mu_2) &\approx \frac{\Gamma_b}{2} + \frac{(\Gamma_b/2)}{(\Gamma_b/2)^2 + \omega_0^2}V^2 \\
\mathrm{Im}(\mu_2) &\approx -\omega_0 + \frac{\omega_0}{(\Gamma_b/2)^2 + \omega_0^2}V^2
\end{aligned}$$

On the other hand, we have in the large V limit, $x \longrightarrow 16V^2$ and $x \gg y$, and so

$$\begin{aligned}
\mathrm{Re}(\mu_1) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} \approx \frac{\Gamma_b}{4} \\
\mathrm{Im}(\mu_1) &\approx V - \frac{\omega_0}{2} \\
\mathrm{Re}(\mu_2) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} \approx \frac{\Gamma_b}{4} \\
\mathrm{Im}(\mu_2) &\approx -V - \frac{\omega_0}{2}
\end{aligned}$$

We see that these results agree well with perturbation theory results for Stark shift energies.

(b) **Effect of the Lamb shift on quenching.** Here we want to find the electric field in V/cm for which the $2S$ state lifetime is equal to 1 μs for two cases:

- **Case 1:** Weak coupling limit ($V^2 \ll \omega_0^2$) assuming that ω_0 is much smaler than the actual $2P$ linewidth Γ_b, so $\omega_0 \ll \Gamma_b$. First, we can identify from the solution above that the decay rate of the $2S$ state in the weak field limit is given by

  $$\gamma_{2S} = 2 \times \frac{\Gamma_b/2}{(\Gamma_b/2)^2 + \omega_0^2}V^2 \longrightarrow \frac{4V^2}{\Gamma_b}$$

  where the factor of 2 comes from the fact that $|a(t)|^2$ and $|b(t)|^2$ are the relevant quantities for population. The lifetime is therefore

  $$\tau_{2S} = \frac{\Gamma_b}{4V^2}.$$

  Since $V = 3ea_0\mathcal{E}/\hbar$, we can set τ_{2S} to 1 μs to find that, the desired electric field strength is

  $$\mathcal{E} = \sqrt{\frac{\Gamma_b}{4(1\ \mu\text{s})(3ea_0/\hbar)^2}} \approx \boxed{0.5203\ \text{V/cm}}$$

- **Case 2:** Weak field limit as above but with $\omega_0 = \omega_{\text{Lamb}} = 2\pi \times 1057.864$ MHz. In this case, we find that

$$\tau_{2S} = \frac{(\Gamma_b/2)^2 + \omega_0^2}{\Gamma_b V^2} \implies \mathcal{E} = \sqrt{\frac{(\Gamma_b/2)^2 + \omega_0^2}{\Gamma_b (1\ \mu\text{s})(3a_0 e/\hbar)^2}} \approx \boxed{1.82\ \text{V/cm}}$$

Including the Lamb shift increases the electric field required by a factor of ~3.5 on this time scale.

Mathemetica code:

```
(*Problem 3: Stark stuff*)

In[1]:= a[t_] := a1*Exp[-m1*t] + (1 - a1)*Exp[-m2*t];

In[2]:= b[t_] := b1*Exp[-(m1 + I*w0)*t] - b1*Exp[-(m2 + I*w0)*t];

In[3]:= FullSimplify[
I*a'[t] == V*Exp[I*w0*t]*b[t] - I*Ga/2*a[t]] /. {t -> 0}

Out[3]= I (a1 (Ga - 2 m1) + 2 I b1 V) -
I ((-1 + a1) (Ga - 2 m2) + 2 I b1 V) == 0

In[4]:= FullSimplify[
I*b'[t] == V*Exp[-I*w0*t]*a[t] - I*Gb/2*b[t]] /. {t -> 0}

Out[4]= I b1 (Gb - 2 m1) + 2 (-1 + a1) V - 2 a1 V -
I b1 (Gb - 2 (m2 + I w0)) + 2 b1 w0 == 0


(*Solve for m1, m2*)

In[5]:= Solve[{I (a1 (Ga - 2 m1) + 2 I b1 V) -
I ((-1 + a1) (Ga - 2 m2) + 2 I b1 V) == 0,
I b1 (Gb - 2 m1) + 2 (-1 + a1) V - 2 a1 V -
I b1 (Gb - 2 (m2 + I w0)) + 2 b1 w0 == 0}, {m1,
m2}] // FullSimplify

Out[5]= {{m1 -> Ga/2 - (I (-1 + a1) V)/b1, m2 -> Ga/2 - (I a1 V)/b1}}


(*Pick t=1/V, solve for a1, b1*)

In[6]:= eqn1 =
I*a'[t] == V*Exp[I*w0*t]*b[t] - I*Ga/2*a[t] /. {t -> 1/V,
m1 -> Ga/2 - (I (-1 + a1) V)/b1, m2 -> Ga/2 - (I a1 V)/b1} //
FullSimplify

Out[6]= (((-1 + a1) a1 + b1^2) E^((I (-1 + a1))/b1 - Ga/(
2 V)) (-1 + E^(I/b1)) V)/b1 == 0

In[8]:= eqn2 =
I*b'[t] == V*Exp[-I*w0*t]*a[t] - I*Gb/2*b[t] /. {t -> 1/V,
m1 -> Ga/2 - (I (-1 + a1) V)/b1, m2 -> Ga/2 - (I a1 V)/b1} //
FullSimplify

Out[8]= E^((I (-1 + a1))/b1 - (Ga + 2 I w0)/(
2 V)) (-1 + E^(I/b1)) (2 (-1 + 2 a1) V +
I b1 (Ga - Gb + 2 I w0)) == 0

(*Solve for a1, b1*)
In[9]:= Solve[{eqn1, eqn2}, {a1, b1}] // FullSimplify


Out[9]= {{a1 -> (
I Ga - I Gb +
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] -
2 w0)/(2 Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V +
2 I w0))]),
b1 -> -((2 V)/
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V +
2 I w0))])}, {a1 -> (-I Ga + I Gb +
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] +
2 w0)/(2 Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V +
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2 I w0))]),
b1 -> (2 V)/
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))]}}

In[135]:= (*Find mu1*)

In[30]:= Ga/2 - (I (-1 + a1) V)/b1 // Expand

Out[30]= Ga/4 + Gb/4 -
1/4 I Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] - (
I w0)/2

In[13]:= (*Find mu2*)

In[31]:= Ga/2 - (I a1 V)/b1 // Expand

Out[31]= Ga/4 + Gb/4 +
1/4 I Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] - (
I w0)/2

(*Weak field limit --> get V^2 scaling*)

In[26]:= Series[
Sqrt[(Sqrt[x^2 + y^2] - x)/2], {V, 0, 2}] // FullSimplify

Out[26]= SeriesData[V, 0, {
2^Rational[-1, 2] (
Gab^2 - 4 w0^2 + ((Gab^2 + 4 w0^2)^2)^Rational[1, 2])^Rational[
1, 2], 0, (-4)
2^Rational[1, 2] ((Gab^2 + 4 w0^2)^2)^Rational[-1, 2] (
Gab^2 - 4 w0^2 + ((Gab^2 + 4 w0^2)^2)^Rational[1, 2])^Rational[
1, 2]}, 0, 3, 1]

In[146]:=
Series[Sqrt[(Sqrt[x^2 + y^2] + x)/2], {V, 0, 2}] // FullSimplify

Out[146]= SeriesData[V, 0, {
2^Rational[-1,
2] (-Gab^2 + 4 w0^2 + ((Gab^2 + 4 w0^2)^2)^Rational[
1, 2])^Rational[1, 2],
0, (4 ((Gab^2 + 4 w0^2)^2)^Rational[-1, 2]) ((-2)
Gab^2 + 8 w0^2 + 2 ((Gab^2 + 4 w0^2)^2)^Rational[
1, 2])^Rational[1, 2]}, 0, 3, 1]
```