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Course: **8.309 - Classical Mechanics III**
Problem set: **#1**
Re-grade request: Problem 2(b)

2. Double Pendulum in a Plane with Gravity

Re-grade justification: In my original write-up, my Hamiltonian has a small typo, which is a missing parenthesis highlighted in red in the expression below. However, this missing parenthesis is a genuine typo because, as I will show below, the rest of my solution (equations of motion involving $\dot{p}_{\theta_1}$ and $\dot{p}_{\theta_2}$) is otherwise correct. Moreover, since the computations were carried out in Mathematica (the code is in my write-up), there not should not be inaccuracies beyond typos, so long as the setup is correct (and mine is).

First, I will simplify my Hamiltonian so that it matches the solution:

$$\begin{aligned}
\mathcal{H} &= -\frac{l_1^2\left(gl_2{}^2m^2[\cos(2(\theta_1-\theta_2))-3](2l_1\cos\theta_1+l_2\cos\theta_2)+2p_{\theta_2}^2\right)-2l_1l_2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)+l_2^2p_{\theta_1}^2}{l_1^2l_2^2m[\cos(2(\theta_1-\theta_2))-3]}\\
&= -2l_1gm\cos\theta_1-l_2gm\cos\theta_2-\frac{2l_1^2p_{\theta_2}^2-2l_1l_2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)+l_2^2p_{\theta_1}^2}{l_1^2l_2^2m[\cos(2(\theta_1-\theta_2))-3]}\\
&= -2l_1gm\cos\theta_1-l_2gm\cos\theta_2+\frac{2l_1^2p_{\theta_2}^2-2l_1l_2p_{\theta_1}p_{\theta_2}\cos(\theta_1-\theta_2)+l_2^2p_{\theta_1}^2}{2l_1^2l_2^2m(1+\sin^2(\theta_1-\theta_2))}.
\end{aligned}$$

□

Next, I will show that my equations of motion for the $\dot{p}$'s are also the same as the solution's.

$$\begin{aligned}
\dot{p}_{\theta_1} = -\frac{\partial\mathcal{H}}{\partial\theta_1} &= \frac{-2gl_1^3l_2^2m^2\sin\theta_1[\cos(2(\theta_1-\theta_2))-3]^2+2\sin(2(\theta_1-\theta_2))\left(2l_1^2p_{\theta_2}^2+l_2^2p_{\theta_1}^2\right)}{l_1^2l_2^2m[\cos(2(\theta_1-\theta_2))-3]^2}\\
&\quad+\frac{-2l_1l_2p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)[\cos(2(\theta_1-\theta_2))+5]}{l_1^2l_2^2m[\cos(2(\theta_1-\theta_2))-3]^2}\\
&= -2gl_1m\sin\theta_1+\frac{4\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)\left(2l_1^2p_{\theta_2}^2+l_2^2p_{\theta_1}^2\right)}{4l_1^2l_2^2m[1+\sin^2(\theta_1-\theta_2)]^2}-\frac{2l_1l_2p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)[2\cos^2(\theta_1-\theta_2)+4]}{4l_1^2l_2^2m[1+\sin^2(\theta_1-\theta_2)]^2}\\
&= -2gl_1m\sin\theta_1+\frac{\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)\left(2l_1^2p_{\theta_2}^2+l_2^2p_{\theta_1}^2\right)}{l_1^2l_2^2m[1+\sin^2(\theta_1-\theta_2)]^2}+\frac{-l_1l_2p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)[\cos^2(\theta_1-\theta_2)+2]}{l_1^2l_2^2m[1+\sin^2(\theta_1-\theta_2)]^2}\\
&= -2gl_1m\sin\theta_1+\frac{\sin(\theta_1-\theta_2)\left[\cos(\theta_1-\theta_2)\left(2l_1^2p_{\theta_2}^2+l_2^2p_{\theta_1}^2\right)-l_1l_2p_{\theta_1}p_{\theta_2}(\cos^2(\theta_1-\theta_2)+2)\right]}{l_1^2l_2^2m[1+\sin^2(\theta_1-\theta_2)]^2}\\
&= -2gl_1m\sin\theta_1+\frac{\sin(\theta_1-\theta_2)\left[l_2p_{\theta_1}\cos(\theta_1-\theta_2)-2l_1p_{\theta_2}\right]\left[l_2p_{\theta_1}-l_1p_{\theta_2}\cos(\theta_1-\theta_2)\right]}{l_1^2l_2^2m[1+\sin^2(\theta_1-\theta_2)]^2}.
\end{aligned}$$

□

This matches the solution. And finally,

$$
\begin{aligned}
\dot{p}_{\theta_2} = -\frac{\partial \mathcal{H}}{\partial \theta_2} &= -g l_2 m \sin\theta_2 \\
&+ \frac{2\sin(\theta_1-\theta_2)\left(-4l_1^2 p_{\theta_2}^2 \cos(\theta_1-\theta_2) + l_1 l_2 p_{\theta_1} p_{\theta_2}[\cos(2(\theta_1-\theta_2))+5] - 2l_2^2 p_{\theta_1}^2 \cos(\theta_1-\theta_2)\right)}{l_1^2 l_2^2 m[\cos(2(\theta_1-\theta_2))-3]^2} \\
&= -g l_2 m \sin\theta_2 + \frac{2\sin(\theta_1-\theta_2)\left[-4l_1^2 p_{\theta_2}^2 \cos(\theta_1-\theta_2) + l_1 l_2 p_{\theta_1} p_{\theta_2}[2\cos^2(\theta_1-\theta_2)+4] - 2l_2^2 p_{\theta_1}^2 \cos(\theta_1-\theta_2)\right]}{4l_1^2 l_2^2 m[1+\sin^2(\theta_1-\theta_2)]^2} \\
&= -g l_2 m \sin\theta_2 + \frac{\sin(\theta_1-\theta_2)\left[-2l_1^2 p_{\theta_2}^2 \cos(\theta_1-\theta_2) + l_1 l_2 p_{\theta_1} p_{\theta_2}[\cos^2(\theta_1-\theta_2)+2] - l_2^2 p_{\theta_1}^2 \cos(\theta_1-\theta_2)\right]}{l_1^2 l_2^2 m[1+\sin^2(\theta_1-\theta_2)]^2} \\
&= -g l_2 m \sin\theta_2 + \frac{-\sin(\theta_1-\theta_2)\left[l_2 p_{\theta_1}\cos(\theta_1-\theta_2) - 2l_1 p_{\theta_2}\right]\left[l_2 p_{\theta_1} - l_1 p_{\theta_2}\cos(\theta_1-\theta_2)\right]}{l_1^2 l_2^2 m[1+\sin^2(\theta_1-\theta_2)]^2} \\
&= -g l_2 m \sin\theta_2 + \frac{\sin(\theta_2-\theta_1)\left[l_2 p_{\theta_1}\cos(\theta_1-\theta_2) - 2l_1 p_{\theta_2}\right]\left[l_2 p_{\theta_1} - l_1 p_{\theta_2}\cos(\theta_1-\theta_2)\right]}{l_1^2 l_2^2 m[1+\sin^2(\theta_1-\theta_2)]^2},
\end{aligned}
$$

which also matches the solution. $\square$