Name: **Huan Q. Bui**
Course: **8.321 - Quantum Theory I**
Problem set: **#2**

1. Let A be a skew-Hermitian operator, i.e., $A^\dagger = -A$.

(a) Let λ and $|\lambda\rangle$ be an eigenvalue and eigenvector of A, respectively. Then we have

$$A|\lambda\rangle = \lambda|\lambda\rangle \implies \lambda\langle\lambda|\lambda\rangle = \langle\lambda|A|\lambda\rangle = -\langle\lambda|A^\dagger|\lambda\rangle = -\lambda^*\langle\lambda|\lambda\rangle \implies -\lambda = \lambda^*.$$

If λ is real, then the only solution is $\lambda = 0$. So, up to degeneracy A has at most one real eigenvalue which is 0.

(b) Let A, B be Hermitian operators. Then

$$[A,B] = AB - BA = A^\dagger B^\dagger - B^\dagger A^\dagger = (BA - AB)^\dagger = -(AB - BA)^\dagger = -[A,B]^\dagger.$$

Thus $[A,B]$ is skew-Hermitian.

2. Let H, K be Hermitian operators with non-negative eigenvalues and assume that the trace is defined throughout this problem. Since H, K are Hermitian we may assume that there exist complete orthonormal bases $\{|h_i\rangle\}$ and $|k_i\rangle$ for H, K respectively with $H|h_i\rangle = h_i|h_i\rangle$ and $K|k_i\rangle = k_i|k_i\rangle$, and $h_i, k_i \geq 0$ for all i. With this, we can spectrally decompose H, K in their product as follows

$$HK = \sum_n h_n |h_n\rangle\langle h_n| \sum_m k_m |k_m\rangle\langle k_m| = \sum_{n,m} h_n k_m |h_n\rangle\langle h_n|k_m\rangle\langle k_m|.$$

Since $\mathrm{tr}(A) = \sum_i \langle\phi_i|A|\phi_i\rangle$ for any matrix A and orthonormal basis $\{\phi_i\}$, we have

$$\begin{aligned}
\mathrm{tr}(HK) &= \sum_j \langle h_j| \left[\sum_{n,m} h_n k_m |h_n\rangle\langle h_n|k_m\rangle\langle k_m|\right] |h_j\rangle \\
&= \sum_{n,m} h_n k_m \langle h_n|k_m\rangle\langle k_m|h_n\rangle, \quad \text{by orthonormality} \\
&= \sum_{n,m} h_n k_m |\langle h_n|k_m\rangle|^2.
\end{aligned}$$

Since $h_i, k_i \geq 0$ for all i, and that the modulus square is always nonnegative, we see that $\mathrm{tr}(HK) \geq 0$, as desired. Moreover, suppose $\mathrm{tr}(HK) = 0$, then by the nonnegativity of each term in the sum above we must have $h_n k_m |\langle h_n|k_m\rangle|^2 = 0$ for all n, m, or equivalently $h_n k_m \langle h_n|k_m\rangle = 0$ for all n, m, i.e., $HK = 0$.

3. Let a Hermitian operator H be given with positive spectrum and a complete orthonormal basis.

(a) We want to prove that for any two vectors $|\alpha\rangle, |\beta\rangle$

$$\left|\langle\alpha|H|\beta\rangle\right|^2 \leq \langle\alpha|H|\alpha\rangle\langle\beta|H|\beta\rangle.$$

There are two ways to go about this proof, but both approaches are actually the same and only differ by appearance. I will present the notationally "light" version first. This goes as follows: Since H is Hermitian with positive spectrum, we may find a complete orthonormal basis in which H is diagonal. The transformation between H and its diagonalization D is given by a unitary operator U as $H = U^\dagger D U$. Since D is diagonal with positive entries, we can define its square root $\sqrt{D}$. From here, we can also define the square root of H, denoted $\sqrt{H}$, by $U^\dagger\sqrt{D}U$. We can check:

$$\sqrt{H}\sqrt{H} = U^\dagger\sqrt{D}UU^\dagger\sqrt{D}U = U^\dagger\sqrt{D}\sqrt{D}U = U^\dagger DU = H.$$

It is easy to show that $\sqrt{H}$ is also Hermitian:

$$\sqrt{H}^{\dagger} = \left(U^{\dagger}\sqrt{D}U\right)^{\dagger} = U^{\dagger}\sqrt{D}^{\dagger}U = U^{\dagger}\sqrt{D}U = \sqrt{H},$$

where we have used the fact that $\sqrt{D}$ is strictly diagonal and positive, thus Hermitian. The rest of the proof is now a simple application of the Cauchy-Schwarz inequality for inner products:

$$\begin{aligned}
\left|\langle\alpha| H \left|\beta\right\rangle\right|^2 &= \left|\langle\alpha| \sqrt{H}\sqrt{H} \left|\beta\right\rangle\right|^2 = \left|\langle\alpha| \sqrt{H}^{\dagger}\sqrt{H} \left|\beta\right\rangle\right|^2 = \left|\left\langle\sqrt{H}\alpha\middle|\sqrt{H}\beta\right\rangle\right|^2 \\
&\leq \left\langle\sqrt{H}\alpha\middle|\sqrt{H}\alpha\right\rangle \left\langle\sqrt{H}\beta\middle|\sqrt{H}\beta\right\rangle \\
&= \langle\alpha| \sqrt{H}^{\dagger}\sqrt{H} \left|\alpha\right\rangle \left\langle\beta\right| \sqrt{H}^{\dagger}\sqrt{H} \left|\beta\right\rangle = \langle\alpha| H \left|\alpha\right\rangle \left\langle\beta\right| H \left|\beta\right\rangle
\end{aligned}$$

as desired.

The more notationally heavy approach is to consider a complete orthonormal eigenbasis for H, which we may call $\{|\lambda_i\rangle\}$ where $\{\lambda_i\}$ are the eigenvalues of H. Under this basis, we have

$$|\alpha\rangle = \sum_i a_i |\lambda_i\rangle \qquad |\beta\rangle = \sum_i b_i |\lambda_i\rangle$$

and so

$$\left|\langle\alpha| H \left|\beta\right\rangle\right|^2 = \left|\sum_{i,j} a_i^* \langle\lambda_i| \lambda_j b_j |\lambda_j\rangle\right|^2 = \left|\sum_i a_i^* \lambda_i b_i\right|^2 = \left|\sum_i \left(a_i\sqrt{\lambda_i}\right)^* \left(b_i\sqrt{\lambda_i}\right)\right|^2.$$

Note that $\sqrt{\lambda_i} \in \mathbb{R}^+$, which is possible because $\lambda_i > 0$. Now, call

$$|\alpha'\rangle = \sum_i a_i\sqrt{\lambda_i} |\lambda_i\rangle \qquad |\beta'\rangle = \sum_i b_i\sqrt{\lambda_i} |\lambda_i\rangle.$$

It is clear that

$$\left|\langle\alpha| H \left|\beta\right\rangle\right|^2 = \left|\langle\alpha'|\beta'\rangle\right|^2.$$

On the other hand, we have

$$\begin{aligned}
\langle\alpha| H |\alpha\rangle &= \sum_{i,j} a_i^* a_j \lambda_j \langle\lambda_i|\lambda_j\rangle = \sum_i |a_i|^2\lambda_i = \langle\alpha'|\alpha'\rangle \\
\langle\beta| H |\beta\rangle &= \sum_{i,j} b_i^* b_j \lambda_j \langle\lambda_i|\lambda_j\rangle = \sum_i |b_i|^2\lambda_i = \langle\beta'|\beta'\rangle.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality,

$$\left|\langle\alpha| H \left|\beta\right\rangle\right|^2 = \left|\langle\alpha'|\beta'\rangle\right|^2 \leq \langle\alpha'|\alpha'\rangle \langle\beta'|\beta'\rangle = \langle\alpha| H |\alpha\rangle \langle\beta| H |\beta\rangle$$

and we're done.

(b) The trace of H is simply the sum of its eigenvalues, so $\mathrm{tr}(H) > 0$. To show explicitly, we use the orthonormal basis introduced in Part (a). Since $\lambda_i > 0$ for all i, we have

$$\mathrm{tr}(H) = \sum_i \langle\lambda_i| H |\lambda_i\rangle = \sum_i \lambda_i \langle\lambda_i|\lambda_i\rangle = \sum_i \lambda_i > 0.$$

4. Let a unitary operator U be given which satisfies the eigenvalue equation $U|\lambda\rangle = \lambda|\lambda\rangle$.

(a) Since $\langle\lambda|\lambda\rangle \neq 0$ (because $|\lambda\rangle$ is an eigenvector), we have

$$\langle\lambda|\lambda\rangle = \langle\lambda| U^\dagger U |\lambda\rangle = |\lambda|^2 \langle\lambda|\lambda\rangle \implies |\lambda|^2 = 1.$$

Since $\lambda \in \mathbb{C}$, it must be of the form $\lambda = e^{i\theta}$ where $\theta \in \mathbb{R}$.

(b) Let distinct eigenvectors $|\mu\rangle$ and $|\lambda\rangle$ be given with corresponding (distinct) eigenvalues $e^{i\theta_\mu}$ and $e^{i\theta_\lambda}$. We have

$$\langle\mu|\lambda\rangle = \langle\mu| U^\dagger U |\lambda\rangle = e^{-i\theta_\mu} e^{i\theta_\lambda} \langle\mu|\lambda\rangle .$$

Since $e^{-i\theta_\mu} e^{i\theta_\lambda} \neq 1$, equality holds only if $\langle\mu|\lambda\rangle = 0$.

5.

(a) First, we will show that the set of $N \times N$ complex matrices form a vector space (over the complex numbers).

- The zero matrix $\mathcal{O}$ is the identity for vector (matrix) addition.
- For every matrix A, the matrix $-A$ exists and $A + (-A) = \mathcal{O}$, so every matrix has an additive inverse.
- Matrix addition is associative.
- Matrix addition is commutative.
- Scalar multiplication: For $a, b \in \mathbb{C}$ and a matrix A, we have $a(bA) = (ab)A = (ab)A$, as usual.
- The number $1 \in \mathbb{C}$ is the identity for scalar multiplication.
- Scalar multiplication is distributive with respect to matrix addition. Given $a \in \mathbb{C}$ and matrices A, B we have $a(A + B) = aA + aB$.
- Finally, for $a, b \in \mathbb{C}$ and a matrix A, we have $(a + b)A = aA + bA$.

Basically, the rules for matrix addition show that the set of $N \times N$ complex matrices form a vector space. To show that the dimension of this space is N^2, we consider the following set of N^2 matrices $\{M(ij)\}_{i,j=1}^N$ where each $M(ij)$ is an $N \times N$ matrix whose entries are all zeros except for a 1 in the ij position. It is clear that there exists no non-trivial linear combination of the $M(ij)$'s that gives the zero matrix. Thus, $\{M(ij)\}_{i,j=1}^N$ is a linearly independent set. Moreover, it is also obvious that any $N \times N$ matrix can be written as a linear combination of the $M(ij)$ matrices (i.e., given a matrix $A = [a_{ij}]$ we have $A = \sum a_{ij} M(ij)$). Therefore, the vector space of $N \times N$ complex matrices is N^2-dimensional.

(b) Let $(A, B) = \mathrm{Tr}(A^\dagger B)$. We will show that $(\cdot, \cdot)$ defines an inner product over the vector space $\mathcal{V}$ above.

- Positive semidefinite: Given $A \in \mathcal{V}$. Then

$$\mathrm{Tr}\left(A^\dagger A\right) = \left(A^\dagger A\right)_{ii} = A^\dagger_{ij} A_{ji} = A^*_{ji} A_{ji} = \sum_{i,j=1}^N \left|A_{ij}\right|^2 \geq 0,$$

  with equality occurring if and only if $A_{ij} = 0$ for all i, j, i.e., $A = 0$.
- Linear in the second argument: For $\beta \in \mathbb{C}$ and $A, B \in \mathcal{V}$, we have, by the linearity of the trace function, $\mathrm{Tr}(A^\dagger \beta B) = \beta \,\mathrm{Tr}(A^\dagger B)$. Moreover, given $C \in \mathcal{V}$, we have

$$\mathrm{Tr}\left(A^\dagger (B + C)\right) = \mathrm{Tr}\left(A^\dagger B + A^\dagger C\right) = \mathrm{Tr}\left(A^\dagger B\right) + \mathrm{Tr}\left(A^\dagger C\right).$$

- Conjugate-linear in the first argument (optional since the previous condition suffices): For $\alpha \in \mathbb{C}$ and $A, B \in \mathcal{V}$, we have, by the linearity of the trace function, $\mathrm{Tr}((\alpha A)^\dagger B) = \mathrm{Tr}(\alpha^* A^\dagger B) = \alpha^* \,\mathrm{Tr}(A^\dagger B)$. Similarly, given $C \in \mathcal{V}$,

$$\mathrm{Tr}\left((A + C)^\dagger B\right) = \mathrm{Tr}\left(A^\dagger B + C^\dagger B\right) = \mathrm{Tr}\left(A^\dagger B\right) + \mathrm{Tr}\left(C^\dagger B\right).$$

- Conjugate symmetry: Given $A, B \in \mathcal{V}$, we have $\operatorname{Tr}\big(B^\dagger A\big) = \operatorname{Tr}\big((A^\dagger B)^\dagger\big) = \overline{\operatorname{Tr}((A^\dagger B)^\top)} = \overline{\operatorname{Tr}(A^\dagger B)}$, using the fact that $\operatorname{Tr}(X) = \operatorname{Tr}(X^\top)$ for a square matrix X.

(c) By inspection, we can see that the collection $\{\mathbb{I}, \sigma_1, \sigma_2, \sigma_3\}$ is linearly independent. It remains to show that it spans the space of 2×2 complex matrices. To this end, let a 2×2 matrix A be given.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We may write A as a linear combination of $\{\mathbb{I}, \sigma_1, \sigma_2, \sigma_3\}$ as

$$A = \left(\frac{a+d}{2}\right)\mathbb{I} + \left(\frac{b+c}{2}\right)\sigma_1 + \left(\frac{c-b}{2i}\right)\sigma_3 + \left(\frac{a-d}{2}\right)\sigma_3.$$

Therefore, $\{\mathbb{I}, \sigma_1, \sigma_2, \sigma_3\}$ also spans the space and so it is a basis for this space. Now, we claim that this basis under the normalization factor $1/\sqrt{2}$

$$B = \left\{\frac{1}{\sqrt{2}}\mathbb{I}, \frac{1}{\sqrt{2}}\sigma_1, \frac{1}{\sqrt{2}}\sigma_2, \frac{1}{\sqrt{2}}\sigma_3\right\}$$

is orthonormal with respect to the inner product defined in Part (b). To see this, we observe that each element of the basis B is already Hermitian and that $\sigma_i^2 = \mathbb{I} = \mathbb{I}^2$ for $i = 1, 2, 3$. So, we have that $\operatorname{Tr}\Big((\sigma_i^\dagger/\sqrt{2})(\sigma_i/\sqrt{2})\Big) = \operatorname{Tr}\big(\sigma_i^2\big)/2 = \operatorname{Tr}(\mathbb{I})/2 = \operatorname{Tr}\big(\mathbb{I}^2\big)/2 = 1$, as desired. Moreover, since each of $\sigma_1, \sigma_2, \sigma_3$ is traceless, and that

$$\sigma_1\sigma_2 = i\sigma_3 \qquad \sigma_2\sigma_3 = i\sigma_1 \qquad \sigma_3\sigma_1 = i\sigma_2$$

all of which are traceless, we have $\operatorname{Tr}\big(\sigma_i^\dagger \sigma_j\big) = \operatorname{Tr}\big(\sigma_i\sigma_j\big) \propto \operatorname{Tr}(\sigma_k) = \operatorname{Tr}(\mathbb{I}\sigma_k) = 0$ for all $i \neq j$. Therefore, B is mutually orthogonal collection of unit norm. In view of the previous result, B is an orthonormal basis.

(d) Let $\Sigma(\cdot)$ denote the spectrum and $\mathcal{E}$ the set of eigenvectors for each matrix. Note: to avoid confusion, we use the capital Σ rather than the lowercase. Except for the case of $\mathbb{I}$, the characteristic polynomial for each of $\sigma_1, \sigma_2, \sigma_3$ is $\lambda^2 = 1$, so $\lambda = \pm 1$.

$$\begin{aligned}
&\mathbb{I}: \ \ \Sigma(\mathbb{I}) = \{1, 1\} && \mathcal{E}(\mathbb{I}) = \{\vec{v} : \vec{v} \in \mathbb{C}^2, \vec{v} \neq 0\} \\
&\sigma_1 : \Sigma(\sigma_1) = \{1, -1\} && \mathcal{E}(\sigma_1) = \left\{(1 \quad 1)^\top, (1 \quad -1)^\top\right\} \\
&\sigma_2 : \Sigma(\sigma_2) = \{1, -1\} && \mathcal{E}(\sigma_2) = \left\{(1 \quad i)^\top, (1 \quad -i)^\top\right\} \\
&\sigma_3 : \Sigma(\sigma_3) = \{1, -1\} && \mathcal{E}(\sigma_3) = \left\{(1 \quad 0)^\top, (0 \quad 1)^\top\right\},
\end{aligned}$$

where the eigenvectors are ordered to match their corresponding eigenvalues.

(e) Let $\mathbf{A} = (A_1, A_2, A_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$ be given such that $[A_j, \sigma_i] = 0$ and $[B_n, \sigma_m] = 0$ for all i, j, m, n. Then, using the following identities

$$\sigma_1\sigma_2 = i\sigma_3, \quad \sigma_2\sigma_3 = i\sigma_1, \quad \sigma_3\sigma_1 = i\sigma_2, \quad \sigma_2\sigma_1 = -i\sigma_3, \quad \sigma_3\sigma_2 = -i\sigma_1, \quad \sigma_1\sigma_3 = -i\sigma_2, \quad \sigma_i^2 = \mathbb{I},$$

we find

$$\begin{aligned}
(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) &= (\sigma_1 A_1 + \sigma_2 A_2 + \sigma_3 A_3)(\sigma_1 B_1 + \sigma_2 B_2 + \sigma_3 B_3) \\
&= (A_1 B_1 + A_2 B_2 + A_3 B_3) + i\sigma_1(A_2 B_3 - A_3 B_2) + i\sigma_2(A_3 B_1 - A_1 B_3) + i\sigma_3(A_1 B_2 - A_2 B_1) \\
&= (\mathbf{A} \cdot \mathbf{B})\,\mathbb{I} + i\sigma \cdot (\mathbf{A} \times \mathbf{B})
\end{aligned}$$

as desired.

(f) We claim:

$$\exp(i\theta\sigma\cdot\mathbf{n}) = \cos\theta\,\mathbb{I} + i\sigma\cdot\mathbf{n}\sin\theta.$$

In view of Part (e), we observe that $[\sigma\cdot\mathbf{n}]^{2n} = [(\mathbf{n}\cdot\mathbf{n})\mathbb{I}]^n = \mathbb{I}$ and thus $[\sigma\cdot\mathbf{n}]^{2n+1} = \sigma\cdot\mathbf{n}$. This will help with simplifying the power series expansion of $\exp(i\theta\sigma\cdot\mathbf{n})$ below by splitting up the odd-powered and even-powered terms:

$$\begin{aligned}
\exp(i\theta\sigma\cdot\mathbf{n}) &= \sum_{n=0}^{\infty}\frac{(i\theta)^n}{n!}[\sigma\cdot\mathbf{n}]^n \\
&= \sum_{k=0}^{\infty}\frac{(i\theta)^{2k}}{(2k)!}\mathbb{I} + [\sigma\cdot\mathbf{n}]\sum_{j=0}^{\infty}\frac{(i\theta)^{2j+1}}{(2j+1)!} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^k\theta^{2k}}{(2k)!}\mathbb{I} + i[\sigma\cdot\mathbf{n}]\sum_{j=0}^{\infty}\frac{(-1)^j\theta^{2j+1}}{(2j+1)!} \\
&= \cos\theta\,\mathbb{I} + i\sigma\cdot\mathbf{n}\sin\theta.
\end{aligned}$$

And we're done with the proof.

6.

(a) To see that $R := (1/\sqrt{2})(\mathbb{I} + i\sigma_x)$ is a rotation by $-\pi/2$ around the x-axis, it suffices to show that (1) R keeps the σ_x eigenstates invariant and (2) R rotates clockwise the σ_z eigenstates into the σ_y eigenstates.

It is clear from the definition of R that $R\,|\pm,x\rangle = |\pm,x\rangle$ (since $|\pm,x\rangle$ is a simultaneous eigenket of both $\mathbb{I}$ and σ_x). So R keeps the x-axis the same. Now, the matrix representation of this operator in the z basis is

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix}$$

Consider $|+,z\rangle = (1\quad 0)^\top$. We see that $R\,|+,z\rangle = (1/\sqrt{2})(1\quad i)^\top = |+,y\rangle$. Applying R one more time, we find $R\,|+,y\rangle = R^2\,|+,z\rangle = i\sigma_x\,|+,z\rangle = i\,|-,z\rangle \equiv |-,z\rangle$. The total effect is that $+z$ gets rotated into $+y$ and $+y$ gets rotated into $-z$, all with x fixed. As a result, R is a rotation by $-\pi/2$ about the x-axis.

To see this even more clearly, plot the yz-plane with the x-axis pointing out of the paper. Let the state $|+,z\rangle$ represent the $+z$ direction and $|+,y\rangle$ represent the $+y$ direction. Because $R\,|+,z\rangle = |+,y\rangle$, $R\,|+,y\rangle \equiv |-,z\rangle$ and $R\,|\pm,x\rangle = |\pm,x\rangle$, we have that $+z$ gets sent to $+y$, and $+y$ gets sent to $-z$. So, the yz-plane gets rotated by $-\pi/2$ about the x-axis.

Alternatively, one could also do this problem by explicitly consider the x-rotation matrix $\exp(-i\theta\sigma_x/2)$ where $\theta = -\pi/2$. The matrix exponentiation is straightforward. One finds that

$$\exp\left(+\frac{i\pi}{4}\sigma_x\right) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \frac{1}{\sqrt{2}}(\mathbb{I} + i\sigma_x).$$

(b) We will set $\hbar/2 \equiv 1$ for convenience. The matrix elements of S_z in the y-basis are given by $\langle y_i|\,S_z\,|y_j\rangle$. So, in the y-basis, S_z is

$$S_z|_y = \begin{pmatrix}\langle +,y|\,S_z\,|+,y\rangle & \langle +,y|\,S_z\,|-,y\rangle\\ \langle -,y|\,S_z\,|+,y\rangle & \langle -,y|\,S_z\,|-,y\rangle\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \equiv \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

where we have used the fact that $S_z\,|+,y\rangle = |-,y\rangle$ and $S_z\,|-,y\rangle = |+,y\rangle$. Alternatively, we could calculate the matrix elements exclusively using known results in the z-basis (see Part (d)).

7. We want to construct a matrix which connects the z-basis to the x-basis. To do this, we must know how $|\pm, z\rangle$ appears in the x-basis:

$$
\begin{aligned}
|+, z\rangle &= \frac{1}{\sqrt{2}} |+, x\rangle + \frac{1}{\sqrt{2}} |-, x\rangle \\
|-, z\rangle &= \frac{1}{\sqrt{2}} |+, x\rangle - \frac{1}{\sqrt{2}} |-, x\rangle .
\end{aligned}
$$

To see what vectors in the z-basis look like in the x-basis, we apply the following matrix to those vectors:

$$
U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .
$$

Compare this with the given formula:

$$
\begin{aligned}
U &= \sum_r |x_r\rangle \langle z_r| \\
&= |+, x\rangle \langle +, z| + |-, x\rangle \langle -, z| \\
&= |+, x\rangle \left(\frac{1}{\sqrt{2}} \langle +, x| + \frac{1}{\sqrt{2}} \langle -, x| \right) + |-, x\rangle \left(\frac{1}{\sqrt{2}} \langle +, x| - \frac{1}{\sqrt{2}} \langle -, x| \right) \\
&= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} ,
\end{aligned}
$$

which is consistent with what we found before.