n prev
Next: An integral for the **Up:** No Title **Previous:** Dirac's delta function and

A guide to the saddle point method

The saddle point method is discussed in the book, pp. 82-90. Here we give a slightly more general account of the real case, leading to a simple recipe. Consider the integral

$$I = \int_a^b dt\ e^{-xf(t)} g(t). \tag{123}$$

We shall attempt to compute this integral for large values of the parameter x. It is understood that x does not enter in the real functions $f(t)$ and $g(t)$. Often the starting point is different from (123), but can nevertheless be brought into this form after a change of variable.

A necessary condition for the use of the saddle point method is that

$$f'(t_0)=0 \tag{124}$$

for one or more values of t_0. We expand

$$f(t) = f(t_0) + \frac{1}{2} f''(t_0)(t-t_0)^2 + O((t-t_0)^2), \quad g(t) = g(t_0) + O(t-t_0), \tag{125}$$

and insert this in (123). We then obtain

$$I = e^{-xf(t_0)} \int_a^b\ dt\ e^{-\frac{1}{2}xf''(t_0)(t-t_0)^2 + O((t-t_0)^3)} [g(t_0) + O(t-t_0)]. \tag{126}$$

The crucial point is now that the saddle point method works *only if* $f''(t_0)$ is *positive*. In this case we can change the variable in the integral according to $t = t_0 + y\sqrt{2/f''(t_0)}$, and we then get

$$I = e^{-xf(t_0)} \sqrt{\frac{2}{f''(t_0)}}\ \int_{a'}^{b'} dy\ e^{-xy^2 + O(y^3)} [g(t_0) + O(y)]. \tag{127}$$

Here $a' = (a - t_0)\sqrt{f''(t_0)/2}$ and similarly for *b*'. In the integral we should remember that *x* is large. Therefore the Gaussian exp($-xy^2$) is extremely narrow with center at *y*=0. When *x* increases the Gaussian becomes a peak with width of order $1/x \to 0$.

Therefore we can to a good approximation replace the limits *a*',*b*' by $-\infty, +\infty$. Thus,

$$I \approx e^{-xf(t_0)}\sqrt{\frac{2}{f''(t_0)}}\int_{-\infty}^{\infty} dy\ e^{-xy^2+O(y^3)}[g(t_0)+O(y)]. \qquad (128)$$

Changing variable according to $z = \sqrt{x}y$ we obtain

$$I \approx e^{-xf(t_0)}\sqrt{\frac{2}{xf''(t_0)}}\int_{-\infty}^{\infty} dz\ e^{-z^2+O(z^3/\sqrt{x})}[g(t_0)+O(z/\sqrt{x})]. \qquad (129)$$

Ignoring the small $O(z^3/\sqrt{x})$ and $O(z/\sqrt{x})$ terms in the integral, it reduces to a Gaussian, and the final result is

$$I \approx e^{-xf(t_0)}\ g(t_0)\ \sqrt{\frac{2\pi}{xf''(t_0)}}\left(1+O(1/\sqrt{x})\right). \qquad (130)$$

In the case of two or more saddle points one has to sum over these.

In many cases the relevant integral may not be given in as in (123), but by a simple transformation it can be brought to this form. For example, the Gamma function is given by $\Gamma(x+1) = \int_0^\infty dt \exp(-t + x\ln t)$, which is not of the form (123). The derivative of the exponent becomes zero for $-1+x/t=0$, i.e. for $t=x$. One can therefore make the change of variable $t=xy$, leading to

$$\Gamma(x+1) = x^{x+1}\int_0^\infty dy\ e^{-x(y-\ln y)}. \qquad (131)$$

This has the canonical form (123), and the reader can easily check that the result (130) leads to Stirling's formula $\Gamma(x+1) \approx x^x e^{-x}\sqrt{2\pi x}$.

For the case of a complex function *f*(*t*) the reader should consult the book. However, in the case where the function is purely imaginary, one can easily extend the results given above. Starting from

$$J = \int_a^b dt\ e^{ixf(t)} g(t), \tag{132}$$

with a *real* function $f(t)$, we proceed as before by finding stationary point(s) t_0 of the exponent satisfying $f'(t_0)=0$. As before we obtain

$$J = e^{if(t_0)} \int_a^b dt\ e^{ix\frac{1}{2}f''(t_0)(t-t_0)^2 + O(x(t-t_0)^3)}\ [g(t_0) + O(t-t_0)]. \tag{133}$$

Again we can change the variable. Depending on the sign of $f''(t_0)$ we get

$$J \approx e^{if(t_0)}\ g(t_0) \sqrt{\pm\frac{2}{xf''(t_0)}} \int_{-\infty}^{\infty} dz\, e^{-(\mp i)z^2},\ \text{ with } z = \sqrt{\pm\frac{xf''(t_0)}{2}}\,(t-t_0), \tag{134}$$

where the signs $\pm$ are selected according to whether $f''(t_0)$ is positive or negative, respectively (i.e. $\pm f''(t_0)$ is by definition always positive). It only remains to do the integral

$$\int_{-\infty}^{\infty} dz\ e^{-(\mp i)z^2} = \sqrt{\pi}\ e^{\pm i\frac{\pi}{4}}, \tag{135}$$

which can be obtained by analytic continuation from an ordinary Gaussian integral[2]. The final result therefore depends on the sign of $f''(t_0)$ through the factor $e^{\pm i\pi/4}$. Finally we then get

$$J \approx e^{if(t_0)\pm i\frac{\pi}{4}}\ g(t_0) \sqrt{\frac{\pm 2\pi}{xf''(t_0)}}. \tag{136}$$

The $\pm$ signs in this equation are correlated. The above equations are of use e.g. when we discuss the asymptotic behavior of the Bessel functions.

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n prev

Next: An integral for the **Up:** No Title **Previous:** Dirac's delta function and

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