GR "cheat sheet"

Midterm #1

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PH 335

- A small, non-rotating, freely-falling frame in a grav. field is an inertial fr
- Strong EQV principle → all physics reduces to SR in a freely falling frame
- Weak EQV principle → all point particles fall @ same rate in g field → good for GR, not QM
  ↳ we use this

Gauss $\oint \vec{F}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{F}\, d^3r$ **Stokes** $\oint \vec{F}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$

Maxwell $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t}$, $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

Theorem $dx^1 dx^2 \dots dx^n = \det(\underset{\sim}{U})\, dy^1 dy^2 \dots dy^n$ U is the Jacobian!

Basis vectors $\vec{e}_i = \frac{\partial\vec{r}}{\partial u^i}$ (natural), $\vec{e}^i = \vec{\nabla}u^i$ (dual), $\vec{e}^i\vec{e}_j = \delta^i_j$

Properties $\vec{\lambda}\cdot\vec{\mu} = \lambda^i\mu_i = \lambda_i\mu^i = g_{ij}\lambda^i\mu^j = g^{ij}\lambda_i\mu_j$ $\Big\}$ $\vec{e}^i\vec{e}^j = g^{ij}$, $\vec{e}_i\vec{e}_j = g_{ij}$

and

$\boxed{\text{Inverse metric tensor}} \to g^{ij}g_{jk} = \delta^i_k$, $\lambda^i = g^{ij}\lambda_j$, $\lambda_j = g_{ij}\lambda^i$ ← $\boxed{\text{metric tensor}}$ In Cartesian, $[g_{ij}] = I$

$i,j,k : 1\,2\,3$
$A\,B\,C : 1,2$
$\mu,\nu : 0,1,2,3$
$a,b,c : 1..N$

"length" $L = \int_a^b \sqrt{g_{ij}\,du^i du^j}$

Line element

$ds^2 = g_{ij}\,du^i du^j$ $= \int_a^b \sqrt{g_{ij}\dot{u}^i\dot{u}^j}\, dt$

Derivation $\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{\vec{e}_i\,du^i\cdot\vec{e}_j\,du^j} = \sqrt{g_{ij}\,du^i du^j} = ds$

In matrices $\vec{\lambda}\cdot\vec{\mu} = \lambda^i\mu_i = g_{ij}\lambda^i\mu^j = [\lambda^i]^T[g_{ij}][\mu^j] = \underset{\sim}{L}^T\underset{\sim}{G}\underset{\sim}{M}$

↳ lowering of indices $\underset{\sim}{L}^* = \underset{\sim}{G}\underset{\sim}{L}$
raising of indices $\underset{\sim}{L} = \hat{\underset{\sim}{G}}\underset{\sim}{L}^*$

$\boxed{[g_{ij}] = [g^{ij}]^{-1}}$

$\boxed{\text{Coordinate Transforms}}$ $\boxed{\vec{e}_j = \frac{\partial\vec{r}}{\partial u^j} = \frac{\partial\vec{r}}{\partial u^{i'}}\frac{\partial u^{i'}}{\partial u^j} = U^{i'}_j\,\vec{e}_{i'}}$

Properties

$\vec{\lambda} = \lambda^{i'}\vec{e}_{i'} = \lambda^{i'}U^j_{i'}\vec{e}_j = \lambda^j\vec{e}_j$ So $\boxed{\lambda^j = U^j_{i'}\lambda^{i'}}$

So $\boxed{U^k_{i'}U^{i'}_j = \delta^k_j,\ U^{k'}_i U^i_{j'} = \delta^k_j}$ (same for covariants & contravariants)

EM field strength

$$\boxed{[F^{\mu\nu}] = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & B^3 & -B^2 \\ -E^2/c & -B^3 & 0 & B^1 \\ -E^3/c & B^2 & -B^1 & 0 \end{pmatrix}} \rightarrow \boxed{F^{\mu\nu} = -F^{\nu\mu}}$$

$$\boxed{\left\{ \begin{array}{l} \partial_\nu F^{\mu\nu} = \mu_0 j^\mu \\ \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \end{array} \right\}}$$

$\boxed{j^\mu = (\rho c, \vec{J})}$ $\boxed{\frac{1}{\mu_0 \varepsilon_0} = c^2}$

components, not coordinates

But partials of coords

(p. 38)

$\mu\nu] = [\eta_{\mu\alpha}]$

$[F^{\alpha\beta}]$

$[\eta_{\beta\gamma}]$

Def **Vector**: obj whose components transform as $\lambda^{i} = U^{i}_{i'} \lambda^{i'}$, $u^i = u^i(u^{i'})$

Tensor: obj whose components transform as vector components (multi-dim)

$$\boxed{g_{i'j'} = (U^k_{i'} \vec{e}_k) \cdot (U^l_{j'} \vec{e}_l) = U^k_{i'} U^l_{j'} g_{kl}}$$

& $g^{i'j'} = U^{i'}_k U^{j'}_l g^{kl}$ | Type (r,s) → r contravariant, s covariant.

As matrices $[g^{i'j'}] = [U^{i'}_k][g^{kl}][U^{j'}_l]^T$

and $[g_{i'j'}] = [U^k_{i'}]^T [g_{kl}][U^l_{j'}]$

Scalars

(0,0) tensor, invariant.

Show line element = scalar:

⊛ $g_{ij} du^i du^j = g_{k'l'} U^{k'}_i U^{l'}_j U^j_{m'} du^{m'} U^i_{n'} du^{n'} = g_{k'l'} du^{m'} du^{n'} \delta^l_m \delta^k_n = g_{k'l'} du^{k'} du^{l'}$

Summary $T^{i'j'}{}_{k'} = U^{i'}_l U^{j'}_m U^n_{k'} T^{lm}{}_n$ → invariant.

(SR) $[\eta_{\mu\nu}] = \mathrm{diag}(1,-1,-1,-1) = [\eta^{\mu\nu}]$ (minkowski metric tensor)

$= [\eta_{\mu'\nu'}] = [\eta^{\mu'\nu'}]$ $\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$

(lowering/raising) $\lambda^\mu = \eta^{\mu\nu} \lambda_\nu$ | $\lambda^0 = \lambda_0$ but $\lambda^i = -\lambda_i$

$(x = -x)$ $(t = -t)$

Lorentz Transform → Poincaré Transf (1) Boost (2) Translate (3) Spatial Rotate (4) Space parity (5) Time reverse

→ **Boost** $[\Lambda^{\mu'}_\nu] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ → constant → X^μ form components of vectors under LTs.

Inhomogen (translate)
Homogen (no translate)
improper (reverse)
proper (boost)
+ rotate

$\left\{ \begin{array}{l} X_\mu = \eta_{\mu\nu} X^\nu \\ X^\mu = \eta^{\mu\nu} X_\nu \end{array} \right\}$ → coordinates. $[\Lambda^\nu_{\mu'}] = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$\boxed{\text{Boost along } y = \text{rotate } \tfrac{\pi}{2} \rightarrow \text{boost } x \rightarrow \text{rotate } -\tfrac{\pi}{2}}$

Poincaré $\boxed{X^{\mu'} = \Lambda^{\mu'}_\nu X^\nu + a^{\mu'}}$ (+ translate + rotate + boost)

→ properties $\boxed{\eta_{\mu'\nu'} = \Lambda^\alpha_{\mu'} \Lambda^\beta_{\nu'} \eta_{\alpha\beta}}$

invariant!

Summary $T^{\mu'\nu'}{}_{\sigma'} = \Lambda^{\mu'}_\alpha \Lambda^{\nu'}_\beta \Lambda^\gamma_{\sigma'} T^{\alpha\beta}{}_\gamma$

X^μ not vector if $a^{\mu'} \neq 0$ ($X^{\mu'} = \Lambda^{\mu'}_\nu X^\nu$ need)

But dX^μ, $\frac{\partial \varphi}{\partial X^\mu}$ are vectors → $\frac{\partial \varphi}{\partial x^\mu} = \partial_\mu = (\partial_0, \vec{\nabla}_i)$

$\left\{ \begin{array}{l} \partial_\mu = \eta_{\mu\nu} \partial^\nu \\ \partial^\mu = \eta^{\mu\nu} \partial_\nu \end{array} \right\}$

Timelike: ~~∃ simultaneous~~ $|\vec{\lambda}| > 0$
Spacelike $|\vec{\lambda}| < 0$
Null $|\vec{\lambda}| = 0$

N.B. $V^{\mu'} \neq \Lambda^{\mu'}_\nu V^\nu$

Rather $u^{\mu'} = \Lambda^{\mu'}_\nu u^\nu$

where $V^\mu = \frac{dX^\mu}{dt}$, $u^\mu = \frac{dX^\mu}{d\tau}$

$u^\mu u_\mu = c^2$ (inv), $\frac{dt}{d\tau} = \gamma$

$u^\mu = \gamma V^\mu$, $p^\mu p_\mu = m^2 c^2$, $p^\mu = (\gamma mc, \gamma m u^\mu)$

For light

$u^\mu u_\mu = 0 \rightarrow \sigma$

Exam Practice

1. Consider flat 3-dimensional Euclidean space. The transformation matrix $U^{i'}_{j}$ from Cartesian coordinates $u^j = (x, y, z)$ to spherical coordinates $u^{j'} = (r, \theta, \phi)$ is

$$[U^{i'}_{j}] = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\phi & \frac{1}{r}\cos\theta\sin\phi & -\frac{1}{r}\sin\theta \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} = A$$

Using that the metric with upper indices in the Cartesian frame is

$$g^{i'j'} = U^{i'}_{k} U^{j'}_{\ell} g^{k\ell}$$

$$[g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

find the metric $g^{i'j'}$ in the spherical-coordinate system (where i', j' denote r, θ, ϕ) as a transformation with $U^{i'}_{j}$.

~~$g^{i'j'}$ [illegible] $U^{i'}_{j}$ [illegible] $U^{j'}_{i}$~~

$$g^{i'j'} = U^{i'}_{k} U^{j'}_{\ell} \underset{\substack{\uparrow \\ I}}{g^{k\ell}} = \cancel{U^{i'}_{k} U^{j'}_{k}} = [U^{i'}_{k}][g][U^{j'}_{\ell}]^T$$

$$= A A^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 1/r^2\sin^2\theta \end{pmatrix}$$

2. Consider a tensor $T^{\mu\nu}$ in Minkowski spacetime using Cartesian coordinates. The components of $T^{\mu\nu}$ defined in matrix form are

~~$T^{\mu\nu} = AA^T$~~

$= T^{\mu\nu} T_{\nu\gamma} = \delta^{\mu}_{\gamma}$

$$[T^{\mu\nu}] = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & -2 \end{pmatrix}$$

~~$2a$ [illegible] $-3n = 1$~~

~~[illegible]~~

$c + 3g = 0$

$2a + i - n = 1$

$-a + 3i + 2h = 0$

$e = 0$

Also consider a vector V^μ with contravariant components

$$V^\mu = (-1, 2, 0, -2)$$

$T_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}T^{\alpha\beta}$

Find the following:

(a) the components of $[T_{\mu\nu}] = [T^{\mu\nu}]^{-1} = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & h & l & m \\ n & o & 1 & q \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$

(b) $V^\mu V_\mu = 1 - 4 - 0 - 4 = \cancel{7}\ \boxed{-7}$

not $T^{\mu\nu}$

(c) $V^\mu V^\nu T_{\mu\nu} = V^0 V^\nu T_{0\nu} + V^1 V^\nu T_{1\nu} + V^2 V^\nu T_{2\nu} + V^3 V^\nu T_{3\nu}$

got it

$(-1)(-1)2 + \cancel{(-1)(2)0} + \cancel{(-1)(0)} ~~+ (-1)(-2)(-1)~~$

$+ 2(-1)(-1) + \cancel{(2)(2)0} + \cancel{(2)}~~(-2)(2) + (-2)(-1)(-2) + (3)(2) + (-2)(-2)(-2)~~$

$$[T^{\mu\nu}] = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}$$

$2a + b - c = 1$
$-a + 3b + 2c = 0$
$2a + 2c = 0$

~~$[T_{\mu\nu}] = ? = [T^{\mu\nu}]^{-1} = \begin{pmatrix} a & d & g & j \\ 0 & 0 & 1 & 0 \\ b & e & h & k \\ c & f & i & m \end{pmatrix} = \begin{pmatrix} 1/4 & -1/12 & -5/24 & 5/24 \\ 0 & 0 & 1 & 0 \\ 1/4 & 1/4 & 1/8 & -1/8 \\ -1/4 & 1/12 & -7/24 & 7/24 \end{pmatrix}$~~

or

$$[T_{\mu\nu}] = \eta_{\mu\alpha}\eta_{\nu\beta}T^{\alpha\beta} = [\eta_{\mu\alpha}][T^{\alpha\beta}][\eta_{\nu\beta}]^T$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & -1 & 1 \\ -1 & 0 & -3 & -2 \\ 0 & -1 & 0 & 0 \\ 2 & -1 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & -2 \end{pmatrix}$$

$$V^{\mu}V^{\nu}T_{\mu\nu} = [V^{\mu}]^T[T_{\mu\nu}][V^{\nu}] = -14$$

$x = \gamma(x' + \beta ct)$
$ct = \gamma(ct' + \beta x)$
$y = y'$
$z = z'$

$\begin{pmatrix} \gamma\beta & \cancel{\gamma} & 0 & 0 \\ \gamma & \gamma\beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cancel{ct'} \\ x' \\ y' \\ z' \end{pmatrix}$

$\begin{matrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$

Practice #4

True or False (in Minkowski spacetime)?

1. $\lambda \cdot \lambda \geq 0$ F spacelike

~~$\begin{pmatrix} \gamma\beta & \gamma \\ \gamma & \gamma\beta \end{pmatrix} \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \delta$~~

2. $\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^\mu_\sigma$ T

3. $\Lambda^\mu_{\nu'}\Lambda^{\nu'}_\sigma = \delta^\mu_\sigma$ T

4. $[\eta_{\mu'\nu'}] = [\eta_{\alpha\beta}] = [\eta^{\rho'\sigma'}] = [\eta^{\lambda\varsigma}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ T

5. $\Lambda^{\alpha'}_\mu \Lambda^{\beta'}_\nu \eta_{\alpha'\beta'} = \eta_{\mu\nu}$ T

6. $\vec{e}_\mu \cdot \vec{e}_\nu = \eta_{\mu\nu}$ T

7. $\eta_{\mu\nu} a^\mu b_\sigma c^\sigma d^\nu = a_\alpha d^\alpha b_\beta c^\beta$ T
   $a_\nu b_\sigma c^\sigma d^\nu$

8. $L = \int \sqrt{|\eta_{\mu\nu} dx^\mu dx^\nu|}$ T

9. $\Lambda^{\mu'}_\alpha \Lambda^{\nu'}_\beta = \eta^{\mu'\nu'}\eta_{\alpha\beta} \rightarrow$ gibberish [illegible] Ⓕ

10. $\underbrace{\eta^{\mu\nu}\eta_{\nu\sigma}}\underbrace{\eta^{\sigma\rho}\eta_{\rho\mu}} = 4$ T
    $\delta^\mu_\sigma \cdot \delta^\sigma_\mu$

Practice #3

Connect the items on the left with the ones on the right.

4x4 $\Lambda^{\mu'}_{\nu}$

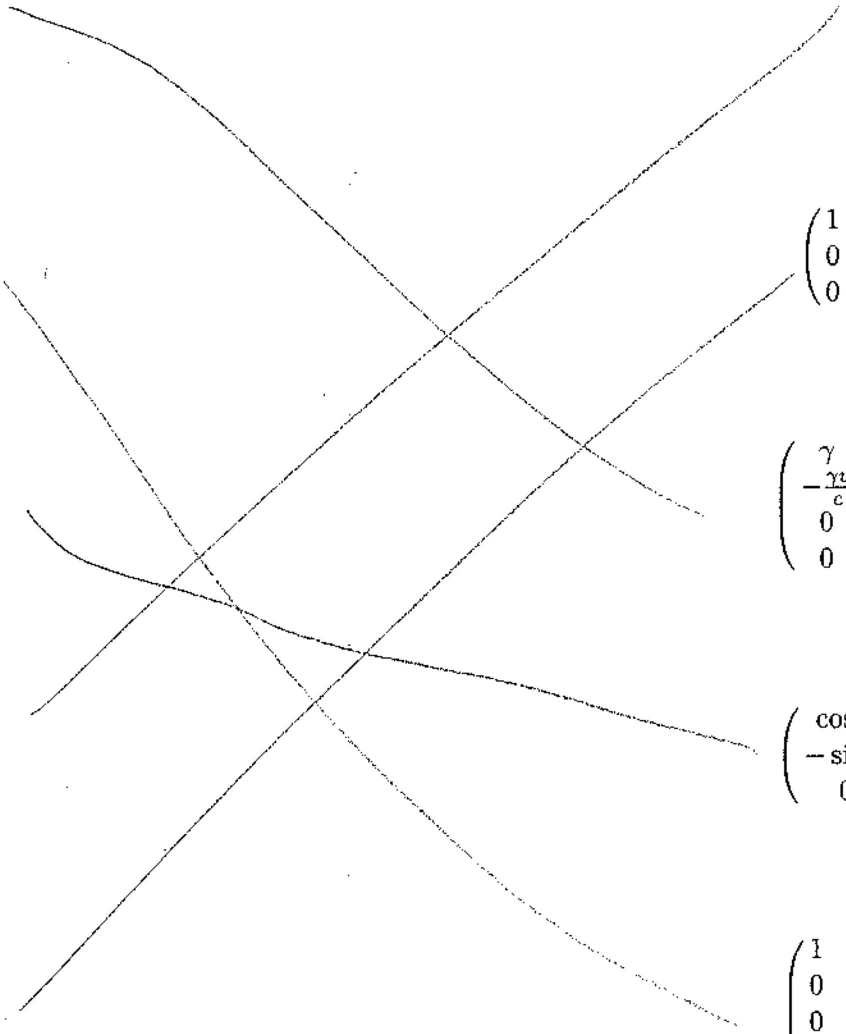
4x4 $\eta_{\mu\nu}$

$U^{i'}_{j}$

3x3

4x4 δ^{ν}_{μ}

3x3 g_{ij}



$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$$\begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Practice #2

State in words what each of the following is, does, and/or means:

(natural) 1. $\vec{e}_i \Rightarrow$ unit vec corr. to contravariant component

(dual) ~~(unit)~~ 2. $\vec{e}^j \Rightarrow$ unit vec corr. to covariant components

3. $\delta^i_j \Rightarrow$ kronecker delta = 1 if $i=j$, 0 if $j \neq i$

4. $\lambda^i \Rightarrow$ contravariant vector component

5. $\lambda_k \Rightarrow$ covariant vector component

6. $g_{ij} \Rightarrow$ $\vec{e}_i \cdot \vec{e}_j$ metric tensor in general coords

7. $g^{ij} \Rightarrow$ $\vec{e}^i \cdot \vec{e}^j$ inverse metric tensor

8. $\nabla u^i \Rightarrow$ ~~$\vec{e}$~~ (dual basis) $\{\vec{e}^i\}$

9. $\frac{\partial \vec{r}}{\partial u^j} \Rightarrow$ ~~[illegible]~~ $\vec{e}_j$ [natural basis vector]

10. $L = \int |\dot{\vec{r}}(\sigma)| d\sigma \Rightarrow$ arc length

11. $ds^2 = g_{ij} du^i du^j \Rightarrow$ line element in general coords

12. $ds^2 = dx^2 + dy^2 + dz^2 \Rightarrow$ line element in Cartesian

13. $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \Rightarrow$ line element in spherical coords

coord transform'n } ← 14. $u^{i'} = u^{i'}(u^j) \Rightarrow$ parametrization of $u^{i'}$ with u^j

15. $\lambda^{i'} = U^{i'}_j \lambda^j \Rightarrow$ defines a vector, then components transform $j \rightarrow i'$

16. $U^{i'}_j \Rightarrow$ ~~Jacobian~~, transforms component $j \Leftrightarrow i'$ for covariant)

~~[illegible]~~) inverses

17. $U^j_{i'} \Rightarrow$ ~~Jacobian~~ transform component $i' \leftrightarrow j$ for covariant

~~[illegible]~~)

[16] 18. $\left[\frac{\partial u^{i'}}{\partial u^j}\right] \Rightarrow [U^{i'}_j] \rightarrow$ Jacobian

flat space

19. $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$ metric tensor w/ matrix rep. (for Cartesian)

20. $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \Rightarrow$ metric tensor in matrix, for spherical

Practice #1

1. Write out each of the following sums $(i, j, \ldots = 1, 2, 3)$. Simplify the resulting expressions where appropriate.
   - (a) $\lambda^i\lambda_i =$ $\lambda^1\lambda_1 + \lambda^2\lambda_2 + \lambda^3\lambda_3 = \sum_{i=1}^{3} \lambda^i\lambda_i = \vec{\lambda}\cdot\vec{\lambda} = \|\vec{\lambda}\|^2$
   - (b) $\lambda^j\lambda_j =$ $\lambda^1\lambda_1 + \lambda^2\lambda_2 + \lambda^3\lambda_3 = \sum_{j=1}^{3} \lambda^j\lambda_j = \|\vec{\lambda}\|^2$ //
   - (c) $\delta^i_j a^j =$ a^i
   - (d) $a_k\delta^k_1 =$ a_1
   - (e) $\vec{e}^i\cdot\vec{e}_i =$ $\sum_{i=1}^{3} \vec{e}^i\vec{e}_i = \vec{e}^1\vec{e}_1 + \vec{e}^2\vec{e}_2 + \vec{e}^3\vec{e}_3 = 3 = \delta^i_i$

2. How do you write the following using the suffix notation?

$$(a_1b^1 + a_2b^2 + a_3b^3)(f_1g^1 + f_2g^2 + f_3g^3) =$$

$$a_ib^i \cdot f_jg^j$$

$$\vec{\lambda} = \sum_i^3 \lambda^i\vec{e}_i = \lambda^i e_i$$

$$\vec{\mu} = \sum_j^3 \mu^j\vec{e}_j = \mu^j e_j$$

$$\vec{\lambda}\cdot\vec{\mu} = \sum\lambda^i e_i \sum \mu^j e_j$$

$$= \cancel{\sum\lambda^i e_i \mu^i e_i}$$

$(i, j = 1, 1, 3 \ldots)$

3. How many equations are each of the following?
   - (a) $a_ib_jc^k = \Gamma^k_{ij}$ 27
   - (b) $a_ib^i = 5$ 1
   - (c) $\vec{e}^i\cdot\vec{e}_j = \delta^i_j$ 9
   - (d) $a_ib_j\delta^j_k = c_id_k$ 9

     $a_ib_k = c_id_k$

4. State whether the following are valid or invalid equations:
   - (a) $g^{ij}a_j = a^i$ (valid)
   - (b) $a^kb_k = g^{ij}a_ib_j$ $= a^jb_j$ (valid)
   - g_{jk} — (c) $\delta^i_jg_{ik} = g_{jk}$ = ~~[illegible]~~ (valid)
   - (d) $g^{ij}g_{ij} = 1$ (NOT valid)

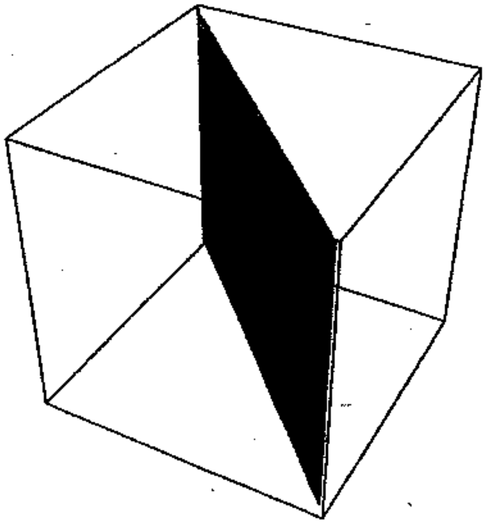
$\downarrow$

$g^{ij}g_{ij} =$ ~~3~~ ? ✓

$$\vec{\lambda} = \lambda^i\vec{e}_i = \lambda_i\vec{e}^i$$

$$\vec{e}^i\cdot\vec{e}_j = \delta^i_j$$

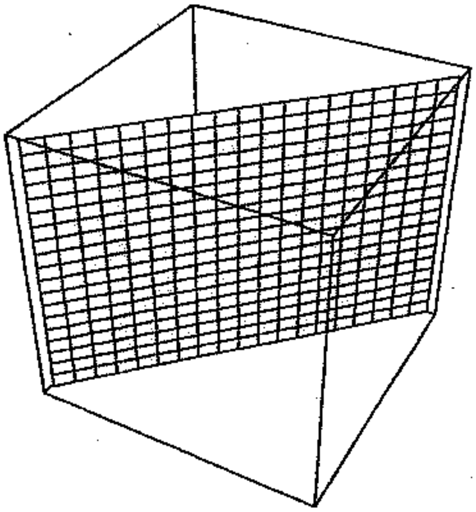
```
ParametricPlot3D[{x, 2 - x, z}, {x, -25, 25}, {z, -25, 25}, Ticks → None]
```



$u = \frac{1}{2}(x+y)$

$u = \text{const}$

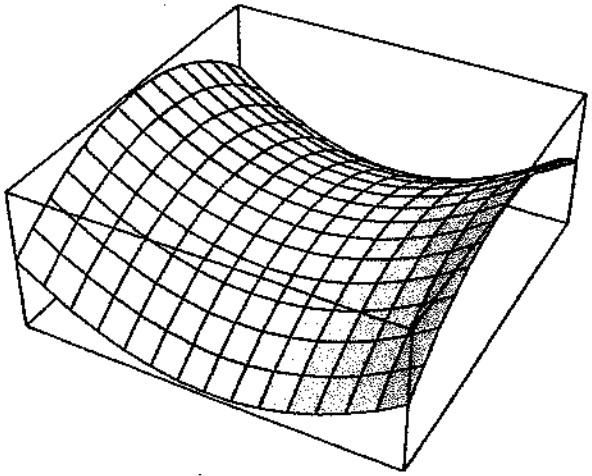
```
ParametricPlot3D[{x, x - 2, z}, {x, -25, 25}, {z, -25, 25}, Ticks → None]
```



$v = \frac{1}{2}(x-y)$

$v = \text{Const}$

```
Plot3D[(1 / 2) * (x^2 - y^2), {x, -25, 25}, {y, -25, 25}, Ticks → None]
```



$w = z - \frac{1}{2}(x^2 - y^2)$

$w = \text{const}$

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Review of Vector Calculus

Scalar functions:

$$f = f(x, y, z)$$

Partial derivatives:

$\frac{\partial f}{\partial x} \Rightarrow$ gives the rate of change of f along x, with y and z fixed

Chain rules:

1. For a function of a single variable $f = f(x)$ where $x = x(t)$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

2. For a function $f = f(x, y)$ with $x = x(s)$, $y = y(s)$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}$$

3. For a function $f = f(x, y, z)$ with $x = x(s, t)$, $y = y(s, t)$, $z = z(s, t)$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

Gradients:

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}$$

$\vec{\nabla} f \Rightarrow$ points along direction of maximum increase in f

$\vec{\nabla} f \cdot \hat{v} \Rightarrow$ directional derivative (rate of change of f along direction $\hat{v}$)

Position vector:

$$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$$

Parameterized curve or trajectory (t = parameter) in 3D space:

$$\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$$

Tangent vector (velocity if t = time):

$$\frac{d\vec{r}}{dt} = \dot{\vec{r}}(t) = \frac{dx}{dt}\,\hat{\imath} + \frac{dy}{dt}\,\hat{\jmath} + \frac{dz}{dt}\,\hat{k}$$

$\frac{d\vec{r}}{dt} = \dot{\vec{r}}(t) \Rightarrow$ vector tangent to the curve $\vec{r}(t)$

Length of a curve along $\vec{r}(t)$ for $a \leq t \leq b$:

$$L = \int_a^b |d\vec{r}| = \int_a^b |\frac{d\vec{r}}{dt}|\,dt$$

Vector functions:

$$\vec{F}(\vec{r}) = F_x(x,y,z)\,\hat{\imath} + F_y(x,y,z)\,\hat{\jmath} + F_z(x,y,z)\,\hat{k}$$

Divergence:

$$\vec{\nabla} \cdot \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Curl:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = \hat{\imath}(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y) - \hat{\jmath}(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x) + \hat{k}(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x)$$

Line integral of $\vec{F}$ along curve $\vec{r}(s)$ for $a \leq s \leq b$:

$\int_a^b \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(s)) \cdot \frac{d\vec{r}}{ds}\,ds \quad \Rightarrow \quad$ sum of components of $\vec{F}$ along curve $\vec{r}(s)$

Surface integral of $\vec{F}$:

$\int_A \vec{F} \cdot d\vec{a} \quad \Rightarrow \quad$ flux of $\vec{F}$ through surface A

Gauss' theorem:

$$\int_A \vec{F} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{F}\,d^3r$$

Stoke's theorem:

$$\oint \vec{F} \cdot d\vec{s} = \int_A (\vec{\nabla} \times \vec{F}) \cdot d\vec{a}$$

Sep 5

GR FORMULAS

Metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Connection:

$$\Gamma^\mu_{\nu\lambda} = \frac{1}{2} g^{\mu\rho} \left(\partial_\nu g_{\rho\lambda} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda} \right)$$

Geodesic Equation:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\tau} \frac{dx^\sigma}{d\tau} = 0$$

Spacetime Covariant Derivatives:

$$\phi_{;\mu} = \partial_\mu \phi$$

$$A^\nu{}_{;\mu} = \partial_\mu A^\nu + \Gamma^\nu{}_{\mu\sigma} A^\sigma$$

$$A_{\nu;\mu} = \partial_\mu A_\nu - \Gamma^\sigma{}_{\mu\nu} A_\sigma$$

$$B^{\nu\lambda}{}_{\sigma;\mu} = \partial_\mu B^{\nu\lambda}{}_\sigma + \Gamma^\nu{}_{\mu\rho} B^{\rho\lambda}{}_\sigma + \Gamma^\lambda{}_{\mu\rho} B^{\nu\rho}{}_\sigma - \Gamma^\rho{}_{\mu\sigma} B^{\nu\lambda}{}_\rho$$

Curvature:

$$R^\mu{}_{\nu\lambda\sigma} = \partial_\lambda \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\lambda} + \Gamma^\rho_{\nu\sigma} \Gamma^\mu_{\rho\lambda} - \Gamma^\rho_{\nu\lambda} \Gamma^\mu_{\rho\sigma}$$

$$R_{\mu\nu} = R^\lambda{}_{\mu\nu\lambda}$$

$$R = R^\lambda_\lambda$$

Einstein's Equations (without and with Λ):

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} = -\frac{8\pi G}{c^2} T^{\mu\nu}$$

$$R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu} = -\frac{8\pi G}{c^2} T^{\mu\nu}$$

Schwarzshild metric:

$$c^2 d\tau^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2$$

FRW metric:

$$d\tau^2 = dt^2 - R(t)^2 \left((1 - kr^2)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right)$$

Sep 5

Review of Special Relativity

Postulates of special relativity:

1. The laws of physics are the same in all inertial reference frames.
2. The speed of light (in a vacuum) is the same in all inertial reference frames.

Time dilation and length contraction (Δt_0 = proper time, L_0 = proper length):

$$\Delta t = \gamma \Delta t_0 \qquad L = \frac{L_0}{\gamma}$$

Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

Lorentz transformations (for relative motion along x):

$$\begin{aligned} x' &= \gamma(x - vt) \qquad & x &= \gamma(x' + vt') \\ y' &= y & y &= y' \\ z' &= z & z &= z' \\ t' &= \gamma(t - \frac{v}{c^2}x) & t &= \gamma(t' + \frac{v}{c^2}x') \end{aligned}$$

Spacetime coordinates:

$$\begin{gathered} (x^0, x^1, x^2, x^3) = \text{position 4}-\text{vector} \\ x^0 = ct \\ x^1 = x \\ x^2 = y \\ x^3 = z \end{gathered}$$

Invariant spacetime interval ($\Delta x \rightarrow \Delta x'$, etc. under a Lorentz transformation):

$$\begin{aligned} c^2 (\Delta\tau)^2 &= c^2 (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= c^2 (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 \end{aligned}$$

Velocity transformations (for relative motion along x):

$$u' = \frac{u - v}{1 - \frac{vu}{c^2}} \qquad u = \frac{u' + v}{1 + \frac{vu'}{c^2}}$$

Relativistic definitions of energy, momentum, and kinetic energy:

$$E = \gamma mc^2$$

$$p = \gamma mv$$

$$K = (\gamma - 1)mc^2$$

Relativistic relation between energy and momentum:

$$E^2 = c^2\vec{p}^2 + m^2c^4$$

Lorentz transformations for energy-momentum (for relative motion along x):

$$\begin{array}{ll} p'_x = \gamma(p_x - \frac{v}{c^2}E) & p_x = \gamma(p'_x + \frac{v}{c^2}E') \\ p'_y = p_y & p_y = p'_y \\ p'_z = p_z & p_z = p'_z \\ E' = \gamma(E - vp_x) & E = \gamma(E' + vp'_x) \end{array}$$

Spacetime energy-momentum:

$$(p^0, p^1, p^2, p^3) = \text{energy}-\text{momentum 4}-\text{vector}$$

$$p^0 = \frac{E}{c}$$

$$p^1 = p_x$$

$$p^2 = p_y$$

$$p^3 = p_z$$

Invariant energy-momentum ($p_x \rightarrow p'_x$, etc. under a Lorentz transformation):

$$\begin{aligned} (mc)^2 &= (\frac{E}{c})^2 - (p_x)^2 - (p_y)^2 - (p_z)^2 \\ &= (\frac{E'}{c})^2 - (p'_x)^2 - (p'_y)^2 - (p'_z)^2 \end{aligned}$$

Sep 5

PH 335 General Relativity & Cosmology - Course Outline

I. Overview and review
  - Principle of equivalence

II. Review of multi-variable calculus

III. Flat 3-dimensional space (chapter 1 - first half)
  - Basis vectors
  - Contravariant and covariant vectors
  - Metric tensor
  - Coordinate transformations
  - Tensors

IV. Flat spacetime (appendix A)
  - Special relativity
  - Relativistic electrodynamics

V. Curved spaces (chapter 1 - last half)
  - 2 dimensional curved spaces
  - Manifolds
  - Tensors on manifolds

VI. Gravitation and curvature (chapter 2)
  - Geodesics & affine connection $\Gamma^{\sigma}{}_{\mu\nu}$
  - Parallel transport
  - Covariant differentiation
  - Newtonian limit

VII. Einstein's field equations (chapter 3)
  - Stress-energy tensor $T^{\mu\nu}$
  - Curvature tensor $R^{\lambda}{}_{\mu\nu\sigma}$
  - Einstein's equations
  - Schwarzschild solution

VIII. Predictions and tests of general relativity (chapter 4)
  - Gravitational redshift
  - Radar time-delay experiments
  - Black Holes

IX. Cosmology (chapter 6)
  - Friedman-Robertson-Walker solution
  - Hubble's "constant" H(t)
  - Recent Discoveries in Cosmology
  - Cosmological constant

1 of 5

PH 335 General Relativity & Cosmology

Robert Bluhm
414 Mudd Building
859-5862
e-mail address: robert.bluhm@colby.edu

Office Hours:	Mondays 1:00 - 2:00 Thursdays 3:00 - 4:30 or by appointment.
Required Texts:	A Short Course in General Relativity, 3rd Ed., by J. Foster and J.D.Nightingale (Springer, 2006) Was Einstein Right? 2nd Ed., by C. Will (Basic Books, 1993)
Recommended:	Spacetime and Geometry, by Sean M. Carroll (Addison Wesley, 2004)
Reading:	There will be regular reading assignments. A lot of effort in this course must go into reading the book. You need to stay current with the reading assignments or you risk becoming lost.
Problems Sets:	Problem sets will be due most weeks. Late problem sets without prior excuse will not be accepted. You may work together and discuss problems with others before writing your solutions, but what you hand in must be your own work.
Exams:	There will be two mid-term exams and a final exam. The mid-term exams will be untimed, closed book, and individually administered take-home exams on an honor system. The final exam will be a three-hour in-class exam during finals week and will also be closed book. However, you will be allowed to bring one sheet of paper with formulas on it to each of the exams. You may use a calculator. The midterms will be due back within two days. Midterm #1 - Wednesday Oct. 10th (due Friday Oct. 12th) Midterm #2 - Wednesday Nov. 28th (due Friday Nov. 30th) Final Exam - Thursday Dec. 13th at 9:00 AM (3 hours)

Attendance: You are expected to come to class. If you have an unexcused absence, you will need to make up the material on your own.

Electronics: You can use a tablet to take notes if you want. But please do not use laptops or other electronic devices such as cell phones in class unless you have written permission from a dean or a doctor.

Goals: The primary objectives of the course are for you to learn the subject of general relativity and to apply it to the study of cosmology. The class is roughly 80% general relativity and 20% cosmology. For a more specific list of topics, please see the course outline handout. In addition to learning these subjects you will develop your skills in:

- Listening and concentration
- Appreciating the development of a new theory
- Mathematics of general coordinate systems
- Mathematical descriptions of curved spaces
- Mathematics of vectors and tensors
- Using symbolic notation
- Problem solving at an advanced level
- Persevering with long computations (not giving up)
- Understanding conceptually difficult material
- Reading and studying the textbook
- Working both independently and collaboratively

Academic Honesty: Honesty, integrity, and personal responsibility are cornerstones of a Colby education. The values stated in the Colby Affirmation are central to this course. Students are expected to demonstrate academic honesty in all aspects of this course.

Religious Holidays: If you need to change an exam date or the due date for an assignment in order to observe a religious holiday, please let me know in advance and we will work something out.

Assessment: Your grade for the course will be the average of your grades on the problem sets, mid-term exams, and final exam with the following weights:

Problem sets	30%
Mid-term exams	40% (20% each)
Final Exam	30%

GENERAL RELATIVITY & Cosmology
PH 335 Prof. Bluhm

Sept 5, 2018

I. OVERVIEW & REVIEW

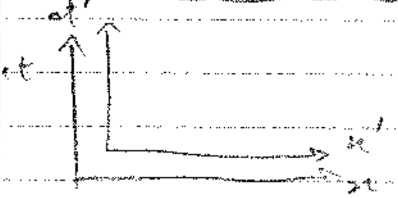
General Relativity? → Theory of Gravity

Replaces Newton's gravity law for heavy masses or at high precision

keep in mind... expected that GR isn't compatible with QM

↳ Question in Physics → how to reconcile GR & QM

Special Relativity (SR) → involves moving inertial frames



Use Lorentz transformation

$$\begin{cases} x' = \gamma(x - vt) = \gamma(x - \beta ct) \\ y' = y, \quad z' = z \\ t' = \gamma\left(t - \frac{v}{c^2}x\right) \end{cases}$$

Minkowski space → flat 4D spacetime of SR

↳ Invariant spacetime interval

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$$

$$= (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2 = (\Delta s')^2$$

↳ Invariant under Lorentz transformation.

- What is (Δs) physically? → go to a rest frame

↳ $\Delta z' = \Delta z = \Delta y' = \Delta x' = 0$

→ $\Delta t = \Delta\tau$ proper time

So $\boxed{(\Delta s)^2 = (c\Delta\tau)^2}$

In Minkowski spacetime → 4-vectors: Ex (ct, x, y, z)

<u>Position</u> (ct, x, y, z)
<u>Momentum</u> $(E/c, p_x, p_y, p_z)$ → Energy-momentum

→ these transform under Lorentz Transformations

$$\left\{ \begin{array}{l} p_x' = \gamma\left(p_x - \beta \frac{E}{c}\right) \\ p_y' = p_y, \quad p_z' = p_z \\ E' = \gamma\left(E - \beta c p_x\right) \end{array} \right\}$$

▨ E/c transform like ct, p_x transform like x ...

Also get an invariant for E-p:

$$\left. \begin{array}{l} \frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = \frac{E^2}{c^2} - \vec{p}^{\,2} \\ \bullet \text{ Recall } E^2 = c^2\vec{p}^{\,2} + m^2c^4 \end{array} \right\}$$

→ $\boxed{\frac{E^2}{c^2} - \vec{p}^{\,2} = (mc)^2}$ Invariant under Lorentz transformations...

Go to a rest frame $E = mc^2$, $p_x = p_y = p_z = 0$

<u>So</u> $\boxed{\frac{(mc^2)^2}{c^2} - \underset{0}{\cancel{\vec{p}^{\,2}}} = (mc)^2}$ (true)

$\boxed{\text{Notice}}$ → have 2 types of objects

(1) Proper time, Mass $\}$ → called <u>SCALARS</u>

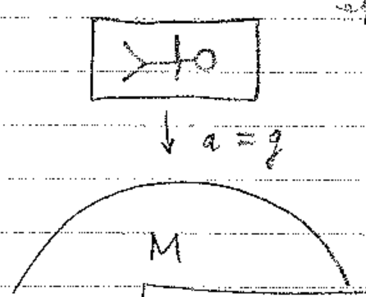
(same in all Lorentz frames)

(2) 4-vectors (ct, x, y, z)] → 4-vectors
$(E/c, p_x, p_y, p_z)$] all transform the same way under Lorentz transf

Now, want to look at the principle that got Einstein started on GR

↳ the $\boxed{\text{Equivalence Principle}}$ (EP)

- 1907 → Einstein's happiest thought of his life
  ↳ realized that in a freely falling frame, the effects of gravity go away



⇒ freely falling frame (non rotating)
(accelerating)

⇒ inside, it's an inertial frame

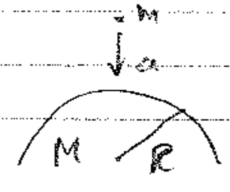
$\boxed{\text{Einstein realized there's an equivalence between gravity \& acceleration ⇒ they can undo each other}}$

Statement $\boxed{\text{A small, non-rotating, freely falling frame in a gravitational field is an inertial frame}}$

{ This is a direct result of Galileo's discovery that all obj have the same acceleration due to gravity. }

BUT
- This is a result of a coincidence!
  ↳ Mass has 2 roles; → causing gravitational force (like charge)
  → Measure of inertia...

∘ Why are these the same?



$F = \dfrac{GMm}{R^2} = mg$ ← (mass as "charge")

but $F = ma$ → mass as "inertia"

- $ma = mg$ so $a = g$ for all objects...

But it could have been that

$$\begin{Bmatrix} m_g = \text{grav. mass} \\ m_I = \text{inertial mass} \end{Bmatrix} \begin{matrix} \to F = m_g g \\ \to F = m_I a \end{matrix}$$

$$\text{so} \quad m_I a = m_g g \to \boxed{a = \left(\frac{m_g}{m_I}\right) g}$$

thisratio $\frac{m_g}{m_I}$ determines whether $a = g$

$\boxed{\text{The Equivalence Principle wouldn't hold if } m_g \neq m_I}$

Exp. show $\frac{|m_g - m_I|}{m_I} \lesssim 10^{-11}$ (Eötvos expt)

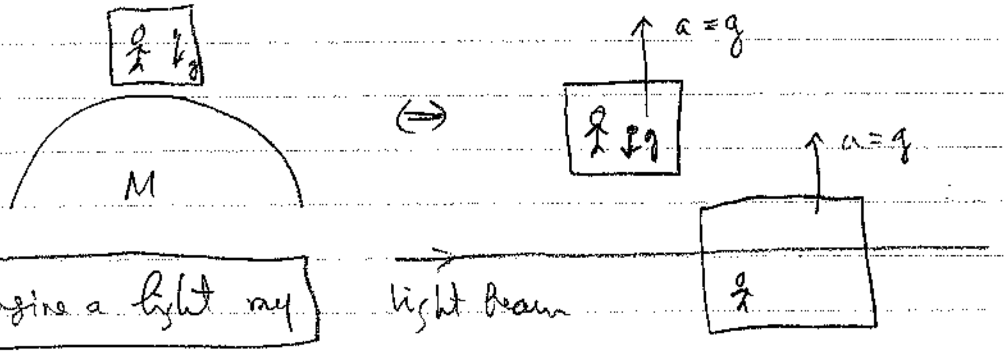
Sep 7, 2018

$\boxed{GR}$ → gravity is not a force
⟹ mass/energy cause curving/wrapping of spacetime

It was the equivalence principle that caused Einstein to think about curved spacetime.

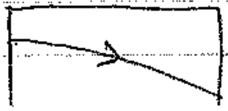
EP ⟹ says that the effects of gravity & acceleration are equivalen

Means these 2 situations are the same



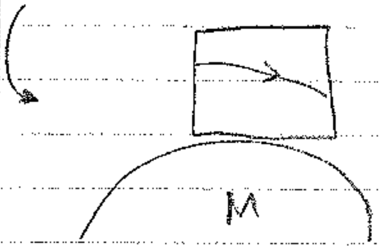
$\boxed{\text{Imagine a light ray}}$

Inside

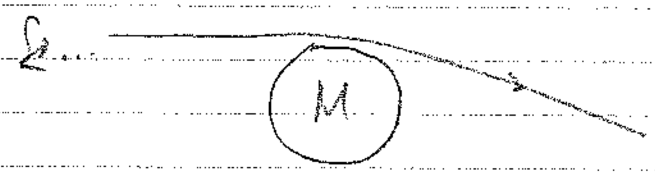


light bent

Now, according to the equivalence principle (postulate)

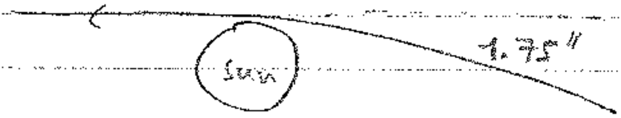


Got a prediction that light bends around massive object



GR predicts that light going 1 km past Earth's surface, will fall by 1 Å. (not observable)

But for Sun, GR predicts bending by 1.75" (arcsec) of light (Eddington)



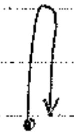
Note

Could argue as well from Newtonian mechs that light falls with $a = g$

But to get 1.75" prediction, the spacetime must actually be curved

assumes spacetime is flat... assumes NOT

Falling objects on Earth → how do we view this as due to curvature?

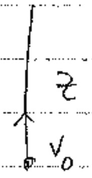


Let's compare 2 cases, each with initial velocity

$v_0 = 4.9$ m/s

$t = 1$s

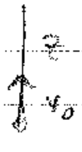
With no gravity



$a = 0$

Find $z = 4.9$m $= v_0 t$

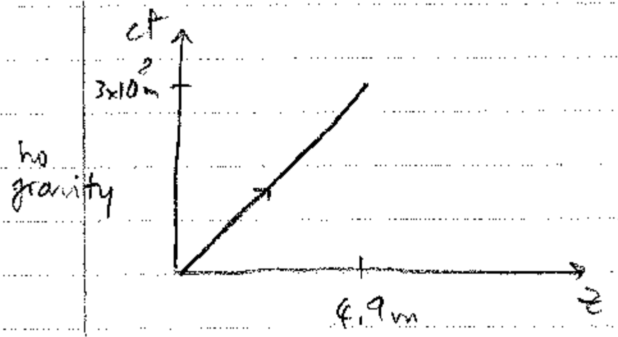
With gravity

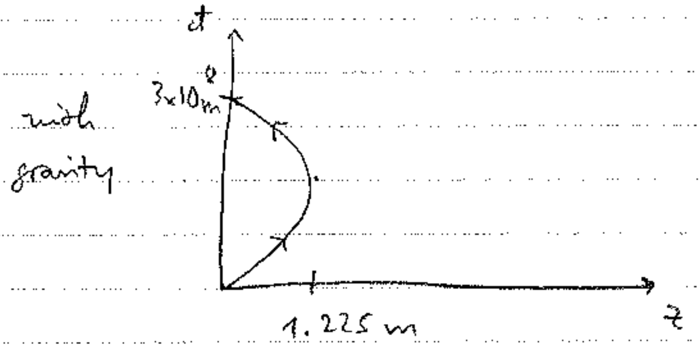


Find $z = 1.225$m (turns around)

Must view this in spacetime

(not to scale)

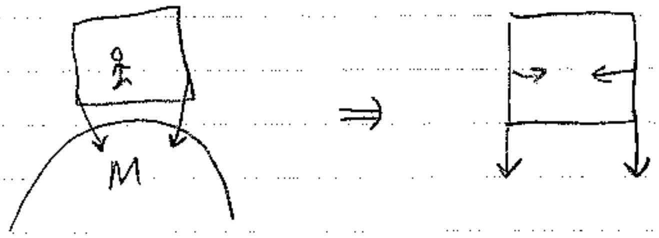




If drawn to scale, both would look like vertical lines
⇒ curvature of spacetime @ earth surface is very weak ...

A few notes on the EP → freely falling frames are infinitesimal
& instantaneous...

Why? because otherwise get tidal effects



(inward component
→ squeezing effect)

→ If fall into black hole → turn into sphagetti!
(sphagetification)

There are also different versions of the EP

Strong equivalence principle → all of physics reduces to special relativity in a freely falling frame...

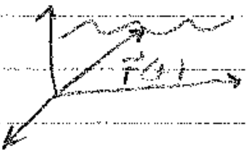
Weak EP → all point particles fall @ the same rate in a gravitational field ($m_g = m_I$) → applies to gravity only
→ sufficient to develop GR, but not for QM

↑ we use this

Sep 10, 2018

Review Curves in 3D space, parametrized by t, σ, s

$$\vec r(t) = x(t)\,\hat i + y(t)\,\hat j + z(t)\,\hat k$$

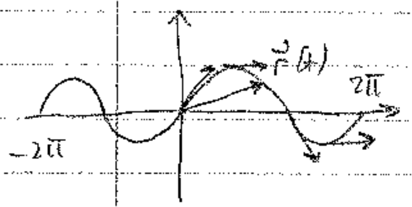


tangent $\dot{\vec r} = \frac{d\vec r}{dt}$.

Length of a curve $|d\vec r| = \left|\frac{d\vec r}{dt}\right| dt = \dot{\vec r}\,dt$

$$\Rightarrow \ell = \int_a^b |d\vec r| = \int_a^b |\dot{\vec r}|\,dt$$

Ex Consider $\vec r(t) = (t, \sin t)$ $(-2\pi \leq t \leq 2\pi)$



$\frac{d\vec r}{dt} = ?$ $\dot{\vec r} = (1, \cos t)$

At $t = 0$ $\dot{\vec r} = (1, 1)$
$t = \pm\frac{\pi}{2}$ $\dot{\vec r} = (1, 0)$
$t = \pm\pi$ $\dot{\vec r} = (1, -1)$
$t = \pm\frac{3\pi}{2}$ $\dot{\vec r} = (1, 0)$

Find length ℓ of curve

$$\ell = \int_a^b |\dot{\vec r}|\,dt = \int_{-2\pi}^{2\pi} \|(1, \cos t)\|\,dt = \int_{-2\pi}^{2\pi} \sqrt{1 + \cos^2 t}\,dt$$ (elliptic int)

Use Mathematica ... $\ell \doteq 15.28$

Can consider vector functions $\boxed{\vec F(\vec r) = \begin{pmatrix} F_x(x,y,z) \\ F_y(x,y,z) \\ F_z(x,y,z) \end{pmatrix}}$

Act with $\vec\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ by dotting or crossing

Dot (div?) $\vec\nabla\cdot\vec F = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$

Note $\vec\nabla f$ gives gradient if f scalar-valued

Cross (curl?) $\vec\nabla\times\vec F = \begin{vmatrix} \hat i & \hat j & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix}$

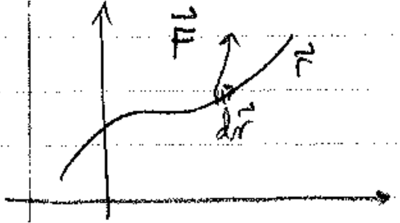
In E&M, can introduce potentials...

$\boxed{\vec{E} = -\vec{\nabla}\phi}$ where ϕ is electric potential (volts) (scalar)

↳ $\vec{E} \perp$ surfaces of constant ϕ (equipotentials)

$\boxed{\vec{B} = \vec{\nabla}\times\vec{A}}$ where $\vec{A}$ is vector potential

Line integrals → of a vector along a curve



$\boxed{\int_a^b \vec{F}\cdot d\vec{r}}$ ⇒ sum of components of F along the curve...

e.g. $\vec{F}$ = force → $W = \int_a^b \vec{F}\cdot d\vec{r}$

e.g. $\vec{F} = \vec{E}$ = e field

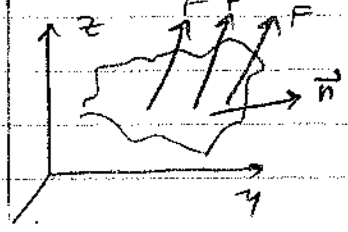
$-\int \vec{E}\cdot d\vec{r}$ = potential = $\Delta\phi$ change in E potential

To do line integral → parametrize...

let $\vec{r} = \vec{r}(s)$, then $\vec{F}(\vec{r}) = \vec{F}(\vec{r}(s))$

$$\boxed{\int_a^b \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(s))\cdot\frac{d\vec{r}}{ds}\,ds}$$

Surface integrals → give flux of a vector field thru a surface



$\int \vec{F}\cdot d\vec{a}$ = flux thru surface

$\uparrow$ normal area $d\vec{a} = da\,\vec{n}$

e.g $\vec{F}=\vec{E}$ electric field $\int \vec{E}\cdot d\vec{a}$ = electric flux = Φ_E

Gauss's law $\int_A \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_0} \rightsquigarrow$ enclosed charge ...

$\boxed{\text{Two famous theorems}}$

Gauss' theorem

$$\oint \vec{F}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{F}\, d^3\vec{r}$$

flux vol int div

curl

Stokes' Theorem

$$\oint \vec{F}\cdot d\vec{s} = \int_A (\vec{\nabla}\times\vec{F})\cdot d\vec{a}$$

flux

Ex Find the differential form of Maxwell's Eqn

Gauss's law ... $\oint \vec{E}\cdot d\vec{a} = \frac{q}{\varepsilon_0}$ $\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = \frac{-\partial}{\partial t}\int_A \vec{B}\cdot d\vec{a}$ (Faraday law)

No magnetic monopole ... $\oint \vec{B}\cdot d\vec{a} = 0$ $\oint \vec{B}\cdot d\vec{s} = \mu_0 I + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\int_A \vec{E}\cdot d\vec{a}$

(Ampere - Maxwell's law ...)

Use Gauss theorem on Gauss' law ... also that

$q = \int_V \rho\, d^3r$ where ρ = volume density

$$\oint \vec{E}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{E}\, d^3\vec{r} = \frac{1}{\varepsilon_0}\int_V \rho\, d^3r$$

So $\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho \Rightarrow \boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$

Immediately → $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

Use Stokes' theorem for the next two...

closed loop

$$\oint \vec{E}\cdot d\vec{s} = \int_A (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = \frac{-\partial}{\partial t}\int_A \vec{B}\cdot d\vec{a}$$

So $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}}$

current density

$$\oint \vec{B}\cdot d\vec{s} = \int_A (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0 I + \mu_0\epsilon_0\frac{\partial}{\partial t}\int_A \vec{E}\cdot d\vec{a} \qquad \text{Let } I = \int_A \vec{J}\cdot d\vec{a}$$

$$= \mu_0\int_A \vec{J}\cdot d\vec{a} + \mu_0\epsilon_0\int \frac{\partial\vec{E}}{\partial t}\cdot d\vec{a}$$

So $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}}$

So
$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t} \end{array}}$$

We'll see how to make these eqn fully relativistic...

#

Coordinate Systems

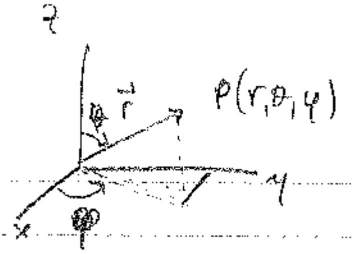
In 3D space.. (there are lots of coordinate systems ...)

- Cartesian Coordinates (x, y, z)
- Spherical Coordinates (r, θ, φ)
- Cylindrical Coordinate (ρ, φ, z)

⋮

⑪

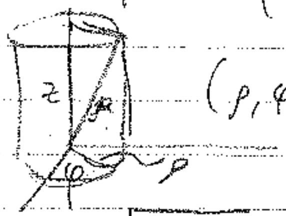
Spherical Coordinate



$$\begin{cases} 0 \le r < \infty \\ 0 \le \theta \le \pi \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \quad \underline{\text{OR}} \quad \begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

Cylindrical Coordinate



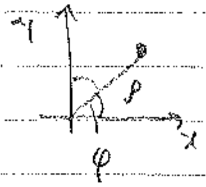
(ρ, φ, z)

$$\left\{\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= z \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} \rho &= \sqrt{x^2+y^2} \\ z &= z \\ \varphi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}\right\}$$

<u>How do we do integrals?</u>

<u>In Cartesian</u> $\boxed{dA = dxdy}$ ☐

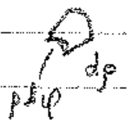
2D polar



$$\left\{\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \end{aligned}\right\}$$

$\rho = \sqrt{x^2+y^2}, \ \varphi = \tan^{-1}\left(\frac{y}{x}\right)$

<u>In Polar</u>



$$\boxed{dA = \rho d\rho d\varphi}$$

↳ extra function

Is there a systematic way to find this extra part?

↳ <u>Use the Jacobian!</u>

ep 11, 2018

We can find the extra factor using Jacobian

↳ matrix of partial derivatives

e.g. <u>polar ↔ Cartesian</u>

$$\underset{\sim}{J} = \left[\frac{\partial(x,y)}{\partial(\rho,\varphi)}\right] = \begin{bmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} \end{bmatrix}$$

<u>Theorem</u> → $\boxed{dxdy = \det(\underset{\sim}{J})\, d\rho d\varphi}$

For 2D polar coordinates: $x = \rho\cos\varphi \rightarrow \frac{\partial x}{\partial \rho} = \cos\varphi,\ \frac{\partial x}{\partial \varphi} = -\rho\sin\varphi$

$y = \rho\sin\varphi \rightarrow \frac{\partial y}{\partial \rho} = \sin\varphi,\ \frac{\partial y}{\partial \varphi} = \rho\cos\varphi$

so $\det(\underset{\sim}{U}) = \rho\cos^2\varphi + \rho\sin^2\varphi = \rho$ so $\boxed{dxdy = \rho\, d\rho\, d\varphi}$

In 3D relate $dxdydz$ to spherical Coordinates

$$dxdydz = \det(\underset{\sim}{U})\, dr\, d\theta\, d\varphi$$

Now

$$\underset{\sim}{U} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix}$$

Or we could go to cylindrical coordinate $dxdydz = \det(\underset{\sim}{U})\, d\rho\, d\varphi\, dz$

Now

$$\underset{\sim}{U} = \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{pmatrix}$$

We can also write a Jacobian for going from Spherical to Cylindrical

$$\underset{\sim}{U} = \begin{pmatrix} \frac{\partial r}{\partial \rho} & \frac{\partial r}{\partial \varphi} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial \rho} & \frac{\partial \theta}{\partial \varphi} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial \rho} & \frac{\partial \varphi}{\partial \varphi} & \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

Note: in this case $dr\,d\theta\,d\varphi$ & $d\rho\,d\varphi\,dz$ are not proper volume element

But Jacobian like this will still be useful to us.

Ex Find Jacobian for $dxdydz \rightarrow$ spherical

$$\underset{\sim}{U} = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

So $\det(\underset{\sim}{U}) = ?$

$$\text{So } \det(\underset{\sim}{U}) = \sin\theta\cos\varphi\left[+r^2\sin^2\theta\cos\varphi\right] - r\cos\theta\cos\varphi\left[-r\sin\theta\cos\theta\cos\varphi\right]$$
$$+ (-r)\sin\theta\sin\varphi\left[-r\sin^2\theta\sin\varphi - r\cos^2\theta\sin\varphi\right]$$
$$= r^2\sin^3\theta\cos^2\varphi + r^2\sin\theta\cos^2\theta\cos^2\varphi$$
$$+ r^2\sin^3\theta\sin^2\varphi + r^2\sin\theta\cos^2\theta\sin^2\varphi$$
$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta$$
$$= \left[r^2\sin\theta\right] \quad \text{as expected ...}$$

As a check we can integrate over a region of radius r

$$\int_0^R\int_0^\pi\int_0^{2\pi} r^2\sin\theta\, dr\, d\theta\, d\varphi = \frac{4}{3}\pi R^3$$

$\boxed{\text{III. Flat 3D space}}$ (called Euclidean space)

↳ "Flat" means "no curvature". We want to see how to use arbitrary coordinates... All coordinate systems in 3D specify points as intersection of 3 surfaces...

Cartesian $\{x = \text{const},\ y = \text{const},\ z = \text{const}\}$ 3 planes!

Spherical $\{r = \text{const},\ \theta = \text{const},\ \varphi = \text{const}\}$ 3 surfaces
sphere cone plane

Cylindrical $\{\rho = \text{const},\ \varphi = \text{const},\ z = \text{const}\}$
cylinder ver. plane hor. plane

$\boxed{\text{Curvilinear Coordinates}}$ (arbitrary coordinates in 3D)

↳ Call (u, v, w) = arbitrary coordinates
Specify a point by $u = \text{const}$, $v = \text{const}$, $w = \text{const}$

Note Coordinates are curvy, but the spaces are still flat ...

→ Can find relations with (x, y, z)

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases} \text{ or } \begin{cases} u = u(x, y, z) \\ v = v(x, y, z) \\ w = w(x, y, z) \end{cases}$$

Basis Vectors Want to be able to describe vectors using curvilinear coordinates

⇒ need a basis set that span the space...

<u>In Cartesian</u> ... $\{\hat{i}, \hat{j}, \hat{k}\}$ span 3D space (Euclidean)

<u>What set</u> $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ would give a basis in curvilinear coordinates

Well, how do we get $\{\hat{i}, \hat{j}, \hat{k}\}$ in Cartesian coordinates?

$\hat{i}$: vector that follows change in $\vec{x}$ with, y, z fixed ...

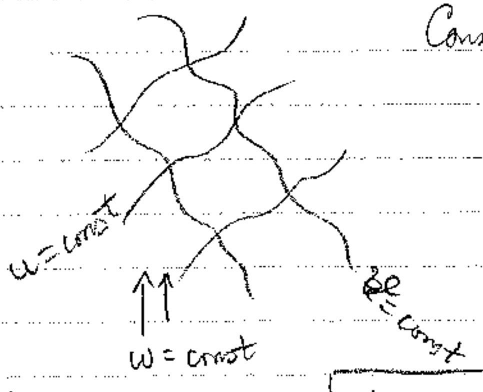
↳ a tangent vector along change in $\vec{x}$.

$$\hat{i} = \frac{\partial \vec{r}}{\partial x} \rightarrow \text{gives a tangent vector along } x$$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \rightarrow \frac{\partial \vec{r}}{\partial x} = \hat{i}$

<u>Likewise</u> $\hat{j} = \frac{\partial \vec{r}}{\partial y}$, $\hat{k} = \frac{\partial \vec{r}}{\partial z}$

Now, <u>Consider (u, v, w)</u>



Consider $\frac{\partial \vec{r}}{\partial u}$ (how u changing with v, w const)

↳ tangent vector along the changing u direction

Let $\boxed{\vec{e}_u = \frac{\partial \vec{r}}{\partial u}}$ } form a natural basis set ...

<u>Likewise</u>, call $\boxed{\vec{e}_v = \frac{\partial \vec{r}}{\partial v}}$, $\boxed{\vec{e}_w = \frac{\partial \vec{r}}{\partial w}}$

The set $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ can then be used as a basis for any vector in the space

Sep 12, 2018

Recall Curvilinear Coordinates → (u, v, w)
Natural basis → $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$

where $\vec{e}_u = \frac{\partial \vec{r}}{\partial u}$, $\vec{e}_v = \frac{\partial \vec{r}}{\partial v}$, $\vec{e}_w = \frac{\partial \vec{r}}{\partial w}$ } tangent vectors.

To calculate these in terms of $\{\hat{i}, \hat{j}, \hat{k}\}$ use

$$\vec{r} = x(u,v,w)\hat{i} + y(u,v,w)\hat{j} + z(u,v,w)\hat{k}$$

Notes → directions of these basis vectors can change as you move around (unlike $\{\vec{i}, \vec{j}, \vec{k}\}$)

→ the set $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ need not be orthogonal. They only need to be linearly independent (to span the space).
They also don't need to be unit vectors...

Can make unit vectors: $\hat{e}_u = \frac{\vec{e}_u}{\|\vec{e}_u\|}$ (but NOT as useful...)

What, then, is "natural" about this set? ⇒ They will lead us to the METRIC TENSOR...

Last note → will often use $\{\vec{i}, \vec{j}, \vec{k}\}$ as a reference basis.
→ Can express $\vec{e}_u$, $\vec{e}_v$, $\vec{e}_w$ in terms of these

e.g. $\vec{e}_u = (e_u)_x \hat{i} + (e_u)_y \hat{j} + (e_u)_z \hat{k}$

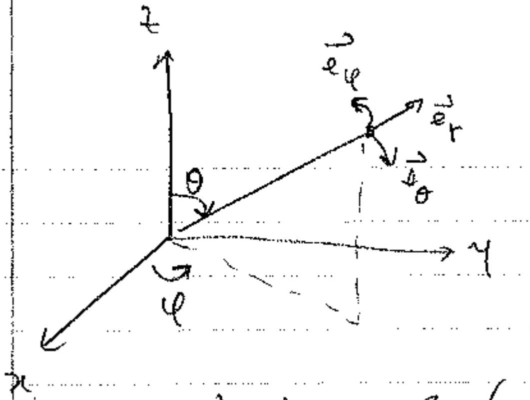
$\boxed{\text{Example}}$ Find $\{\vec{e}_u, \vec{e}_v, \vec{e}_w\}$ for spherical coordinates...

$(u, v, w) \to (r, \theta, \varphi)$ → So... $\vec{r} = (x, y, z) = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix}$

So, • $\vec{e}_r = \frac{\partial \vec{r}}{\partial r} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$

• $\vec{e}_\theta = \frac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} r\cos\theta\cos\varphi \\ r\cos\theta\sin\varphi \\ -r\sin\theta \end{pmatrix}$

• $\vec{e}_\varphi = \frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -r\sin\theta\sin\varphi \\ +r\sin\theta\cos\varphi \end{pmatrix}$



orientation depends on where you are...

Note this set is orthogonal, but not normaliz

Now
$$\vec e_r\cdot\vec e_r = \sin^2\theta\left(\cos^2\varphi+\sin^2\varphi\right)+\cos^2\theta = 1$$
$$\vec e_\varphi\cdot\vec e_\theta = r\cos^2\varphi\sin\theta\cos\theta + r\sin^2\varphi\sin\theta\cos\theta - r\sin\theta\cos\theta = 0$$
$$\vec e_r\cdot\vec e_\varphi = -r\sin^2\theta\sin\varphi\cos\varphi + r\sin^2\theta\sin\varphi\cos\varphi = 0$$
$$\vec e_\theta\cdot\vec e_\theta = r^2\cos^2\varphi\cos^2\theta + r^2\cos^2\varphi\sin^2\theta + r^2\sin^2\theta = r^2$$
$$\vec e_\theta\cdot\vec e_\varphi = 0$$
$$\vec e_\varphi\cdot\vec e_\varphi = r^2\sin^2\varphi\sin^2\theta + r^2\sin^2\theta\cos^2\varphi = r^2\sin^2\theta$$

- See that $\{\vec e_u, \vec e_v, \vec e_w\}$ orthogonal, but not unit vectors...

$\{|\vec e_r|=1,\ |\vec e_\theta| = r,\ |\vec e_\varphi| = r\sin\theta\}$

$\boxed{\text{Dual basis}}$ → there's an alternative basis $\{\vec e^u, \vec e^v, \vec e^w\}$
Instead of using tangent vectors, we could use perpendicular of surfaces of constant (u, v, w)

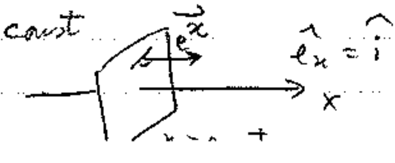
Recall that $\vec\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ gives $\vec\nabla f \perp$ surfaces of $f =$ const

Since curvilinear coord are given by $u =$ const, $v =$ const, $w =$ const this system uses ⊥'s to these...

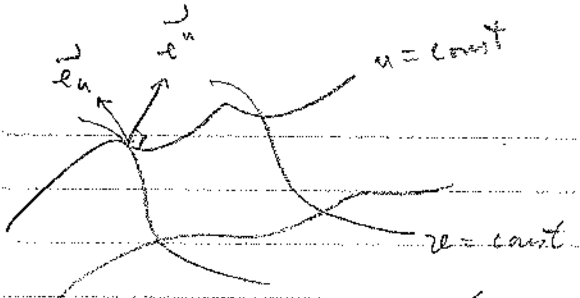
Ex
$$\left\{\begin{array}{l}\vec e^u = \vec\nabla u \\ \vec e^v = \vec\nabla v \\ \vec e^w = \vec\nabla w\end{array}\right\}\quad (\perp \text{ to surface } u = \text{const})$$

- What's the dual basis in Cartesian coord?

$$\left.\begin{array}{l}\vec e^x = \vec\nabla x = (1,0,0) = \hat i = \hat e_x \\ \vec e^y = \vec\nabla y = (0,1,0) = \hat j = \hat e_y \\ \vec e^z = \vec\nabla z = (0,0,1) = \hat k = \hat e_z\end{array}\right\}$$ Why? Because direction along x is the same as the direction $\perp$ $x = $ const



But in curvilinear



To compute $\{\vec{e}^u, \vec{e}^v, \vec{e}^w\}$ → use $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and the inverted relations

$$u(x,y,z),\ v(x,y,z),\ w(x,y,z)$$

Find $\vec{e}^u = \vec{\nabla} u$ in Cartesian in $\vec{i}, \vec{j}, \vec{k}$, then replace (x,y,z) with (u,v,w)

Ex find dual basis set for spherical ... $(u,v,w) \to (r,\theta,\varphi)$

→ use inverted expression

$r = (x^2+y^2+z^2)^{1/2}$

$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$

$$\vec{e}^r = \vec{\nabla} r = \vec{\nabla}(x^2+y^2+z^2)^{1/2} = \begin{pmatrix} x(x^2+y^2+z^2)^{-1/2} \\ y(x^2+y^2+z^2)^{-1/2} \\ z(x^2+y^2+z^2)^{-1/2} \end{pmatrix}$$

So $\boxed{\vec{e}^r = \vec{\nabla} r = \left(\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\ \cos\theta\right)}$ $(= \vec{e}_r)$

$$\vec{e}^\theta = \vec{\nabla}\theta = \vec{\nabla}\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) = \left(\frac{-1}{\sqrt{1-\frac{z^2}{(\ )}}}\right)\left[\frac{-zx}{(\ )^{3/2}}, \frac{-zy}{(\ )^{3/2}}, \frac{1}{(\ )^{1/2}} + \frac{-z^2}{(\ )^{3/2}}\right]$$

$$= \frac{-1}{r\sin\theta}\left(\frac{-r^2\cos\theta\sin\theta\cos\varphi}{r^3}, \frac{-r^2\cos\theta\sin\theta\sin\varphi}{r^3}, \left(\frac{r^2}{r^3} - \frac{r^2\cos^2\theta}{r^3}\right)\right)$$

So $\boxed{\vec{e}^\theta = \left(\frac{1}{r}\cos\theta\cos\varphi,\ \frac{1}{r}\cos\theta\sin\varphi,\ -\frac{\sin\theta}{r}\right)}$

Next, $\vec{e}^\varphi = \vec{\nabla}\varphi = \vec{\nabla}\tan^{-1}\left(\frac{y}{x}\right) = \left(\ \cdots\ \right)$

get $\boxed{\vec{e}^\varphi = \left(-\frac{\sin\varphi}{r\sin\theta},\ \frac{\cos\varphi}{r\sin\theta},\ 0\right)}$

$\boxed{\text{Compare } \{\vec{e}^{r}, \vec{e}^{\theta}, \vec{e}^{\varphi}\} \text{ to } \{\vec{e}_{r}, \vec{e}_{\theta}, \vec{e}_{\varphi}\}}$

$\vec{e}^{r} = \vec{e}_{r}$, but $\vec{e}_{\theta} \neq \vec{e}^{\theta}$, and $\vec{e}^{\varphi} \neq \vec{e}_{\varphi}$

Sep 14, 2018 **Recall** Natural basis $\{\vec{e}_u^{*}, \vec{e}_v^{*}, \vec{e}_w^{*}\} \rightarrow$ tangent vectors $\left(\frac{\partial \vec{r}}{\partial u}\right)$

Dual basis $\{\vec{e}^{u}, \vec{e}^{w}, \vec{e}^{v}\} \rightarrow \perp$ to surface of const u, v (∇)

$\boxed{\text{Ex}}$ Paraboloidal Surfaces (u, v, w) (non-orthogonal set)

$$\left.\begin{aligned} x &= u+v \\ y &= u-v \\ z &= 2uv+w \end{aligned}\right\} \rightarrow \left\{\begin{aligned} u &= \tfrac{1}{2}(x+y) \\ v &= \tfrac{1}{2}(x-y) \\ w &= z-\tfrac{1}{2}(x^2-y^2) \end{aligned}\right.$$

__Surfaces__: $u = \text{const} \rightarrow$ plane
$v = \text{const} \rightarrow$ plane
$w = \text{const} \rightarrow$ hyperbolic paraboloid

__Now__ $\vec{r} = (x, y, z) = (u+v,\ u-v,\ 2uv+w)$ $(\hat{i}, \hat{j}, \hat{k})$

Natural basis

$$\left.\begin{aligned} \vec{e}_u &= \frac{\partial \vec{r}}{\partial u} = (1, 1, 2v) \\ \vec{e}_v &= \frac{\partial \vec{r}}{\partial v} = (1, -1, 2u) \\ \vec{e}_w &= \frac{\partial \vec{r}}{\partial w} = (0, 0, 1) \end{aligned}\right\}$$

__Non orthogonal!__

$\vec{e}_u \cdot \vec{e}_v = 4uv \neq 0$
$\vec{e}_u \cdot \vec{e}_w = 2v \neq 0$
$\vec{e}_v \cdot \vec{e}_w = 2u \neq 0$

$$\vec{e}^{u} = \vec{\nabla} u = \vec{\nabla}\left(\tfrac{1}{2}(x+y)\right) = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right)$$

$$\vec{e}^{v} = \vec{\nabla} v = \vec{\nabla}\left(\tfrac{1}{2}(x-y)\right) = \left(\tfrac{1}{2}, \tfrac{-1}{2}, 0\right)$$

$\vec{e}^{w} = \vec{\nabla} w = \vec{\nabla}\left(z-\tfrac{1}{2}(x^2-y^2)\right) =$ (~~[illegible]~~)

$= (-x, +y, 1) = (-u-v, +u+v, 1)$

__Note__ $\vec{e}^{u} \cdot \vec{e}^{w} = -v$, $\vec{e}^{u} \cdot \vec{e}^{v} = 0$, $\vec{e}^{v} \cdot \vec{e}^{w} = -u$

Suffix notation

→ convenient to change notation (upper indices)

For the coordinates, we use $(u, v, w) \longmapsto (u^1, u^2, u^3) = \{u^i\}$
$(i = 1, 2, 3)$

Similar things for basis vectors

$\{\vec{e}_u, \vec{e}_v, \vec{e}_w\} \longrightarrow \{\vec{e}_i\}$ $i = 1, 2, 3$ (natural)

$\{\vec{e}^u, \vec{e}^v, \vec{e}^w\} \longrightarrow \{\vec{e}^i\}$ $i = 1, 2, 3$ (dual)

Since both span a space, any vector $\vec{\lambda}$ can be written in terms of either

$$\vec{\lambda} = \lambda^1 \vec{e}_1 + \lambda^2 \vec{e}_2 + \lambda^3 \vec{e}_3$$

(upper index for coords)
(for natural basis)

$$\boxed{\vec{\lambda} = \sum_{i=1}^{3} \lambda^i \vec{e}_i}$$

Coordinates = components of natural basis

But also

$$\vec{\lambda} = \lambda_1 \vec{e}^1 + \lambda_2 \vec{e}^2 + \lambda_3 \vec{e}^3$$

(lower ~~index for coords~~ for dual ~~basis~~)

$$\boxed{\vec{\lambda} = \sum_{i=1}^{3} \lambda_i \vec{e}^i}$$

Einstein summation convention

↳ any index that appears $\overset{\text{once}}{\text{up}}$ & $\overset{\text{once}}{\text{down}}$ is automatically summed

So $\boxed{\vec{\lambda} = \lambda^i \vec{e}_i}$ (instead of $\sum_{i=1}^{3} \lambda^i \vec{e}_i$)

Since i is dummy index, it can be any letter

So... $a^i b_i = a^k b_k = a^j b_j = \sum_{n=1}^{3} a^n b_n$

But $a_i b_i$ makes no sense } → not defined
→ need to put in $\sum_i a_i b_i$

Likewise $a_i b^i c^i$ → doesn't make sense either...

↳ only "1 up, 1 down" allowed

Note $\boxed{\text{Certain letters are reserved for special cases}}$

$i, j, k, l, \ldots = 1, 2, 3$ 3D space

$\mu, \nu, \alpha, \beta, \sigma, \rho \ldots = 0, 1, 2, 3$ 4D spacetime

$A, B, C, \ldots = 1, 2, \ldots$ 2D spaces

$a, b, c, \ldots = 1, 2, \ldots N$ N-D manifold

$\boxed{\text{Now, any vector is then}}$ $\vec{\lambda} = \lambda^i \vec{e}_i = \lambda_i \vec{e}^i$

call λ^i a "contravariant component"

and λ_i = "covariant component" "co" is low.

Note λ_i, λ^i → are components

But $\vec{e}^i, \vec{e}_i$ → are vectors... (have 3 components themselves with respect to some other basis)

$\boxed{\text{So... what does this get us?}}$

$\boxed{\text{Dot products...}}$

Consider $\vec{e}^i \cdot \vec{e}_j$ ⤴ not summed ($i \neq j$). This is 9 diff. objects... $i = 1, 2, 3$, $j = 1, 2, 3$...

Use def. $\vec{e}^i = \vec{\nabla} u^i = \frac{\partial u^i}{\partial x} \hat{i} + \frac{\partial u^i}{\partial y} \hat{j} + \frac{\partial u^i}{\partial z} \hat{k}$

$$\vec{e}_j = \frac{\partial \vec{r}}{\partial u^j} = \frac{\partial x}{\partial u^j} \hat{i} + \frac{\partial y}{\partial u^j} \hat{j} + \frac{\partial z}{\partial u^j} \hat{k}$$

So $\vec{e}^i \cdot \vec{e}_j = \frac{\partial u^i}{\partial x}\frac{\partial x}{\partial u^j} + \frac{\partial u^i}{\partial y}\frac{\partial y}{\partial u^j} + \frac{\partial u^i}{\partial z}\frac{\partial z}{\partial u^j}$ looks like a chain rule...

Suppose $u^i = u^i(x,y,z)$

where $x = x(u^j)$

$y = y(u^j)$

$z = z(u^j)$

$$\Rightarrow \boxed{\frac{\partial u^i}{\partial u^j} = \frac{\partial u^i}{\partial x}\frac{\partial x}{\partial u^j} + \frac{\partial u^i}{\partial y}\frac{\partial y}{\partial u^j} + \frac{\partial u^i}{\partial z}\frac{\partial z}{\partial u^j} = \vec{e}^{\,i}\cdot\vec{e}_j}$$

But $\{u^i\} = \{u^1, u^2, u^3\}$ independent variables

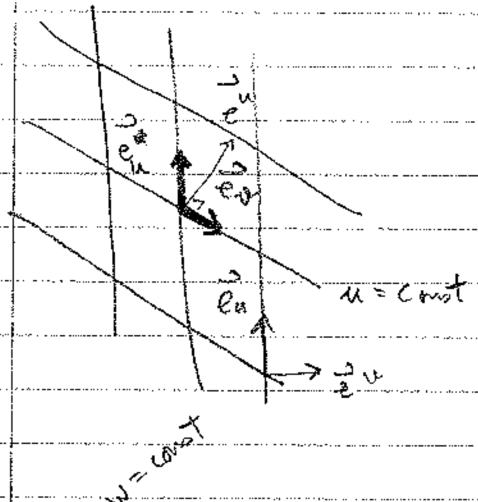
$$\frac{\partial u^1}{\partial u^1} = 1, \quad \frac{\partial u^1}{\partial u^2} = 0, \quad \frac{\partial u^1}{\partial u^3} = 0$$

So

Introduce $\boxed{\delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{Kronecker delta}}$

So $\boxed{\vec{e}^{\,i}\cdot\vec{e}_j = \delta^i_j}$ ⤳ 9 eqns (6 answers = 0, $3 \neq 0$)

Notice $\boxed{\vec{e}^{\,u} \perp \vec{e}_v}$ $(u \neq v)$ why? (by definition)



what about inner products $\{\vec{e}_i\}$ with themselves, likewise $\{\vec{e}^{\,i}\}$

Define $\boxed{\begin{Bmatrix} g_{ij} = \vec{e}_i\cdot\vec{e}_j \\ g^{ij} = \vec{e}^{\,i}\cdot\vec{e}^{\,j} \end{Bmatrix}}$

Since $\vec{e}_i \cdot \vec{e}_j = \vec{e}_j \cdot \vec{e}_i$ (commute), $\boxed{g_{ij} = g_{ji}}$

So $\boxed{\begin{array}{c} g_{ij} = g_{ji} \\ g^{ji} = g^{ij} \end{array}}$ (Symmetric) in matrix → Symmetric

Ex Cartesian g_{ij} = unit matrix

$\boxed{g_{ij} \rightarrow \text{called the metric tensor}}$

a quantity that tells ~~use~~ how to find length, distances in arbitrary coords

Consider $\vec{\lambda}, \vec{\mu}$

Then $\vec{\lambda} = \lambda^i \vec{e}_i = \lambda_i \vec{e}^i$

likewise $\vec{\mu} = \mu^i \vec{e}_i = \mu_i \vec{e}^i$

} There are 4 ways to get $\vec{\lambda} \cdot \vec{\mu}$, and they all give the same ans

Now $\vec{\lambda} \cdot \vec{\mu} = \cancel{\lambda^i \vec{e}_i \cdot \lambda_i e^i}$ → different index

Rather (correctly) $\boxed{\vec{\lambda} \cdot \vec{\mu} = \lambda^i \vec{e}_i \cdot \mu^j \vec{e}_j}$

So $\boxed{\vec{\lambda} \cdot \vec{\mu} = \lambda^i \vec{e}_i \cdot \mu^j \vec{e}_j = g_{ij} \lambda^i \mu^j}$

Sept 17, 2018

Showed $\boxed{\begin{array}{l} \vec{e}^i \cdot \vec{e}_j = \delta^i_j \\ \vec{e}_i \cdot \vec{e}_j = g_{ij} , \quad \vec{e}^i \cdot \vec{e}^j = g^{ij} \end{array}}$

Consider $\vec{\lambda} = \lambda^i \vec{e}_i = \lambda_i \vec{e}^i$

$\vec{\mu} = \mu^i \vec{e}_i = \mu_j \vec{e}^j$

} dot the two → get 4 equivalent expressions for $\vec{\lambda} \cdot \vec{\mu}$

$$\boxed{\begin{aligned}\vec{\lambda}\cdot\vec{\mu} &= \lambda^i\vec{e}_i\cdot\mu^j\vec{e}_j = g_{ij}\lambda^i\mu^j\\ &= \lambda_i\vec{e}^{\,i}\cdot\mu_j\vec{e}^{\,j} = g^{ij}\lambda_i\mu_j\\ &= \lambda_i\vec{e}^{\,i}\cdot\mu^j\vec{e}_j = \lambda_i\mu^j\delta^i_j = \lambda_i\mu^i\\ &= \lambda^i\vec{e}_i\cdot\mu_j\vec{e}^{\,j} = \lambda^i\mu_j\delta^j_i = \lambda^i\mu_i\end{aligned}} \qquad (☆)$$

Note $\mu^j\delta^i_j$ $\quad\delta = 0$ if $j\neq i$

$\delta = 1$ if $j = i$

So $\mu^j\delta^i_j = \mu^i$

Note $$\boxed{\sum_{i=1}^{3}\lambda^i\vec{e}_i\cdot\mu^i\vec{e}_i \neq \sum_{i=1}^{3}\lambda^i\vec{e}_i\cdot\sum_{j=1}^{3}\mu^j\vec{e}_j}$$

3 terms $\qquad$ 9 terms (correct)

We have 4 equivalent expressions

$$\boxed{\vec{\lambda}\cdot\vec{\mu} = g_{ij}\lambda^i\mu^j = g^{ij}\lambda_i\mu_j = \lambda^i\mu_i = \lambda_i\mu^i}$$

double sum $\qquad$ single sum

These imply

$$\boxed{g_{ij}\cdot\mu^j = \mu_i} \quad \text{and} \quad \boxed{g^{ij}\cdot\mu_j = \mu^i}$$

→ Can use metric tensor to go back and forth between contravariants & covariants

$$\boxed{\begin{cases} g^{ij} \rightarrow \text{raises an index}\\ g_{ij} \rightarrow \text{lower an index}\end{cases}}$$

Can also write $$\boxed{\mu^i = g^{ij}\mu_j = g^{ij}(g_{jk}\mu^k)}$$

It's also true that $$\boxed{\mu^i = \delta^i_k \mu^k}$$

So $$\boxed{g^{ij} g_{jk} = \delta^i_k}$$

We can also do: $\mu_i = g_{ij}\mu^j = g_{ij}(g^{jk}\mu_k) = \delta_i^k\mu_k$

So $\boxed{g_{ij}\, g^{jk} = \delta_i^k} \rightarrow$ identity matrix

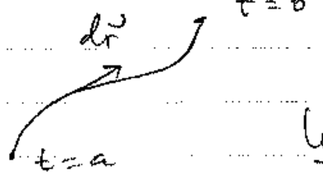
These show that g_{ij} is the inverse of g^{ij} Note g = matrix

Call
$$\begin{cases} g_{ij} \rightarrow \text{metric tensor} \\ g^{ij} \rightarrow \text{inverse metric tensor} \end{cases}$$

The METRIC TENSOR

g^{ij} = metric tensor in 3D space. ⇒ contains info about physical lengths. geometry of the space

Consider a curve in 3D flat space with param t.



$\vec{r} = \vec{r}(t)$

$$\text{length} = \int_a^b \|\dot{\vec{r}}\|\, dt$$

Originally, $\vec{r} = \vec{r}(x,y,z)$

But, we can change to curvilinear coordinates $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

Then, for curve $\begin{Bmatrix} u = u(t) \\ v = v(t) \\ w = w(t) \end{Bmatrix} \to \vec{r} = \vec{r}(u(t), v(t), w(t))$

So $$\frac{d\vec{r}}{dt} = \frac{\partial \vec{r}}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \vec{r}}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial \vec{r}}{\partial w}\frac{\partial w}{\partial t}$$
$$= \vec{e}_u \frac{\partial u}{\partial t} + \vec{e}_v \frac{\partial v}{\partial t} + \vec{e}_w \frac{\partial w}{\partial t}$$

$$\boxed{\frac{d\vec{r}}{dt} = \vec{e}_i \frac{du^i}{dt}}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}} = \sqrt{\vec{e}_i \frac{du^i}{dt}\cdot \vec{e}_j \frac{du^j}{dt}} = \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}$$

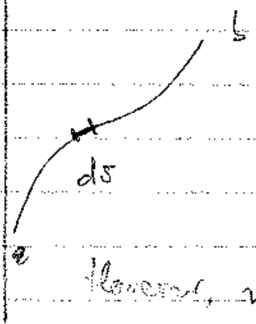
So $$\boxed{L = \int_a^b \left|\frac{d\vec{r}}{dt}\right| dt = \int_a^b \sqrt{g_{ij}\,\dot{u}^i \dot{u}^j}\, dt = \int_a^b \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\, dt}$$

Sep 18, 2018 $\boxed{\text{Length of a curve}}$ in curvilinear coordinates

Note parametrization can be any e.g. σ = param.

$$L = \int_a^b \sqrt{g_{ij}\frac{du^i}{d\sigma}\frac{du^j}{d\sigma}}\, d\sigma$$

We can introduce an infinitesimal line element



ds — In 3D space $ds = |d\vec{r}|$

So $L = \int_a^b |d\vec{r}| = \int_a^b ds \rightsquigarrow$ but this is still parametrized in t (NOT b−a)

However, we can compare this with

$$L = \int_a^b \sqrt{g_{ij}\,\dot{u}^i \dot{u}^j}\, dt = \int_a^b ds$$

$\Rightarrow \quad ds = \sqrt{g_{ij}\,\dot{u}^i \dot{u}^j}\; dt = \sqrt{g_{ij}\,\frac{du^i}{dt}\,\frac{du^j}{dt}}\; dt$

Square this $\quad ds^2 = g_{ij}\,\frac{du^i}{dt}\,\frac{du^j}{dt}\,dt^2$

So $\quad \boxed{ds^2 = g_{ij}\,du^i du^j} \longrightarrow$ line element

metric gives length changes in terms of coordinate changes...

Example 1 Cartesian coordinates $\quad \{\vec{e}_i\} = \{\hat{i}, \hat{j}, \vec{k}\}$

So $\quad g_{ij} = \vec{e}_i \cdot \vec{e}_j = \delta_{ij}$

As a matrix $\quad [g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

row column

So the line element $\quad ds^2 = g_{ij}\,du^i du^j = \quad$ 9 terms

$= 1\,du^1 du^1 + 0\,du^1 du^2 + \ldots$

$\Rightarrow ds^2 = du^{1\,2} + du^{2\,2} + du^{3\,2}$

And $\quad u^1 = x, \quad u^2 = y, \quad u^3 = z$

So $\quad \boxed{ds^2 = dx^2 + dy^2 + dz^2} \quad$ (Cartesian, flat 3D space)

looks Pythagorean $\quad$ comes from the form of the metric

Example 2 Spherical Coordinates $\quad (r, \theta, \varphi)$

$\vec{e}_r \cdot \vec{e}_r = 1, \quad \vec{e}_\theta \cdot \vec{e}_\theta = r^2, \quad \vec{e}_\varphi \cdot \vec{e}_\varphi = r^2 \sin^2\theta$ $\quad$ (others are zero)

$\vec{e}_r \cdot \vec{e}_\theta = \vec{e}_\theta \cdot \vec{e}_\varphi = \vec{e}_\varphi \cdot \vec{e}_r = 0$

These give $\quad [g_{ij}] = \vec{e}_i \cdot \vec{e}_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \end{pmatrix}$ $\quad$ flat space metric in spherical coords...

So the line element $(u^1, u^2, u^3) = (r, \theta, \varphi)$

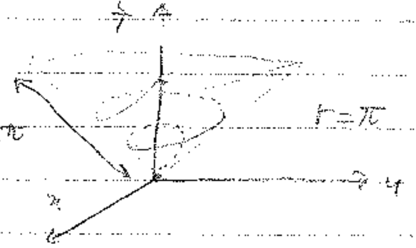
$$\boxed{ds^2 = g_{ij}\,du^i du^j = (1)\,dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2}$$ line element in flat 3D space in Spherical Coords

Example 3

Find the length of a curve in spherical coordinates by the param

$$\vec{r}(t) = (r(t), \theta(t), \varphi(t)) = \left(t, \frac{\pi}{4}, 4t\right) \quad 0 \leq t \leq \pi$$

What does this look like?



(wraps around twice)

Use this

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \quad \text{with param}$$

$$dr = dt, \quad d\theta = 0, \quad d\varphi = 4\,dt$$

So $$ds^2 = \left[1 + 0 + 4^2 t^2 \sin^2\left(\frac{\pi}{4}\right)\right] dt^2 = \left(1 + 8t^2\right) dt^2$$

So $$\boxed{L = \int_0^{\pi} \sqrt{1 + 8t^2 \; dt^2} \cong 14.55}$$

Note we've all seen diagonal metric.

↳ BUT not all metrics are diagonal.

Ex Paraboloidal coordinates have non-diagonal $[g_{ij}]$

We found $\vec{e}_u = (1, 1, 2v)$
$\vec{e}_v = (1, -1, 2u)$
$\vec{e}_w = (0, 0, 1)$

So $$\boxed{[g_{ij}] = \begin{pmatrix} 2 + 4v^2 & 4uv & 2v \\ 4uv & 2 + 4u^2 & 2u \\ 2v & 2u & 1 \end{pmatrix}}$$

<u>Then</u>, $ds^2 = g_{ij}\,du^i du^j$ → get all 9 terms, which then reduce to 6, since $du\,dv = dv\,du$

$= g_{11}\,du^1 du^1 + g_{12}\,du^1 du^2 + \dots$

The metric also gives <u>norms of vectors</u> + <u>inner products of vectors</u>

(norm) $|\vec{\lambda}|^2 = \vec{\lambda}\cdot\vec{\lambda} = g_{ij}\,\lambda^i\lambda^j$ → 9 terms

(inner prod) $\vec{\lambda}\cdot\vec{\mu} = g_{ij}\,\lambda^i\mu^j = g_{11}\lambda^1\mu^1 + g_{12}\lambda^1\mu^2 + \dots\ g_{33}\lambda^3\mu^3$

In Cartesian → $g_{ij} = \delta_{ij}$ $[g_{ij}] = I$

↳ $\vec{\lambda}\cdot\vec{\mu} = \lambda^1\mu^1 + \lambda^2\mu^2 + \lambda^3\mu^3$

Now, can we turn these summations into Matrix Products?

↳ Convenient to write vectors and 2-component tensors using matrices
<u>Note</u> ⇒ more general tensors can't be written using matrices TÜ

First, remember how to multiply matrices...

i j
row column

<u>Suppose</u> $\underset{\sim}{A} = [a_{ij}]$ and $\underset{\sim}{B} = [b_{ij}]$

<u>and</u> $\underset{\sim}{C} = \underset{\sim}{A}\underset{\sim}{B} = [c_{ij}]$

$$C = \begin{pmatrix} & \vdots & \\ \cdots & c_{ij} & \cdots \\ & \vdots & \end{pmatrix} = \begin{pmatrix} a_{i1} & a_{i2} & \cdots \end{pmatrix}\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \end{pmatrix}$$

<u>So</u> $\boxed{c_{ij} = \sum_k \underset{\text{column}}{a_{ik}}\ \underset{\text{row}}{b_{kj}}}$ (summed index is in the middle - goes column - row)

Can also multiply vectors

e.g. $\underset{\sim}{F} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} \quad \underset{\sim}{G} = \begin{pmatrix} g_1 \\ g_2 \\ \vdots \end{pmatrix}$

So $\boxed{\underset{\sim}{F}\cdot\underset{\sim}{G} = \underset{\sim}{F}^T\underset{\sim}{G} = \sum_k f_k g_k}$

Sept 18, 2018

$\boxed{\text{Metric}}$ → line element $ds = g_{ij}\,dx^i dx^j$
→ inner products $\vec{\lambda}\cdot\vec{\mu} = g_{ij}\lambda^i\mu^j = g^{ij}\lambda_i\mu_j = \lambda_i\mu^i = \lambda^i\mu_i$
→ raising / lowering indices

$\boxed{\text{Flat spacetime}}$ Cartesian $[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow a_i = g_{ij}a^j \Rightarrow a_1 = a^1,\ a_2 = a^2,\ a_3 = a^3$

But in spherical coords:

$$[g_{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$\Rightarrow$ if $\vec{a} = (1)\vec{e}_\theta \rightarrow \boxed{\begin{array}{l} a^1_{\cancel{\phi}} = 0 \\ a^2_{\cancel{\theta}} = 1 \\ a^3_{\cancel{\phi}} = 0 \end{array}}$ $\quad (a^1, a^2, a^3) = (0, 1, 0)$

$\uparrow$
(contravariant)

So what are $a_1 = g_{1j}a^j = 0$
$a_2 = g_{2j}a^j = \cancel{r^2}\ g_{22}a^2 = r^2 \rightarrow$ (covariant)
$a_3 = g_{3j}a^j = 0$

Norm? $\boxed{\begin{array}{l} |\vec{a}|^2 = a^i a_i = a^2 a_2 = r^2 \quad \text{(regular sense)} \\ \text{or} \quad |\vec{a}|^2 = g_{ij}a^i a^j = g_{22}a^2 a^2 = r^2 \cdot 1 \cdot 1 = r^2 \end{array}}$

$\boxed{\text{How do we write these things using matrices?}}$

Can represent contravariant vectors as columns

$$\boxed{\underset{\sim}{L} = [\lambda^i] = \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix}}$$

Similarly,

$$\boxed{\underset{\sim}{M} = [\mu^i] = \begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix}}$$

Contravariant

How can we write

$\vec{\lambda} \cdot \vec{\mu} = g_{ij} \lambda^i \mu^j$ using matrices?

$$\boxed{\underset{\sim}{G} = [g_{ij}]}$$

Now, must be careful with ordering + need transposes...

$$\boxed{\vec{\lambda} \cdot \vec{\mu} = \lambda^i g_{ij} \mu^j} \Rightarrow \boxed{\vec{\lambda} \cdot \vec{\mu} = \underset{\sim}{L}^T \underset{\sim}{G} \underset{\sim}{M}} \quad (1 \times 3 \times 3 \times 1)$$

need transpose

So $$\boxed{\vec{\lambda} \cdot \vec{\mu} = (\lambda^1 \lambda^2 \lambda^3) \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} \mu^1 \\ \mu^2 \\ \mu^3 \end{pmatrix}}$$

$\boxed{\text{For COVARIANT}}$ (acc. to books)

$\boxed{\begin{matrix} * : \text{covariant} \\ \wedge : \text{inverse} \end{matrix}}$

$$\underset{\sim}{L}^* = [\lambda_i] = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} \qquad \underset{\sim}{\hat{G}} = [g^{ij}]$$

$$\underset{\sim}{M}^* = [\mu_i] = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}$$

$\boxed{\text{Could write}}$ $\boxed{\underset{\sim}{L}^{*} = \underset{\sim}{G} . \underset{\sim}{L}}$ (lowering indices)

<u>since</u> $$\boxed{\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} \lambda^1 \\ \lambda^2 \\ \lambda^3 \end{pmatrix} = \lambda_i = g_{ij} \lambda^j}$$

<u>with</u> $$\boxed{\underset{\sim}{I} = \underset{\sim}{\hat{G}}^{*} \underset{\sim}{G} = [\delta^i_j] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

then $$\boxed{g^{ij} g_{jk} = \delta^i_k \ , \ g_{ik} g^{kj} = \delta_i^j \rightsquigarrow \underset{\sim}{\hat{G}} . \underset{\sim}{G} = [\delta^i_j]}$$

<u>Now</u>, want to find $[g^{ij}]$ in spherical coords..

Could use def. $\boxed{g^{ij} = \vec{e}^{\,i}_{\phi} \cdot \vec{e}^{\,j}}$ with $\left\{ \begin{array}{l} \vec{e}^{\,r} = \nabla r \\ \vec{e}^{\,\theta} = \nabla \theta \\ \vec{e}^{\,\varphi} = \nabla \varphi \end{array} \right.$

We found those... BUT there's another way

$$\boxed{[g^{ij}] = [g_{ij}]^{-1}} \quad \underline{So} \quad \boxed{[g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}^{-1}}$$

<u>Easy for diagonal matrix</u>

$$\boxed{[g^{ij}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/r^2 \sin^2\theta \end{pmatrix}}$$ (easy for diagonal matrix)

COORDINATE TRANSFORMATION in EUCLIDEAN SPACE

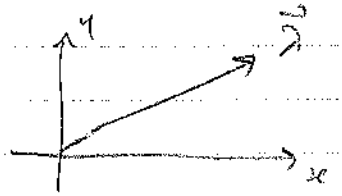
want to learn how to transform between arbitrary coords

$$\boxed{(x, y, z) \longleftrightarrow (x', y', z')} \rightarrow \text{important in relativity}$$

Note no moving frames here. We also want to learn how vectors and tensors transform, as well as what these are...

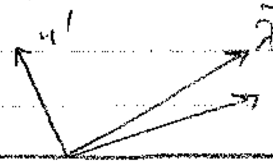
$\boxed{\text{What is a vector?}}$ → has magnitude & direction... $\vec{\lambda}$ = vector

Suppose



Same $\vec{\lambda}$ → not changed
But can now give it components w.r.t. to a basis set ... (λ^i)

Suppose



The same $\vec{\lambda}$, but different $\lambda^{i'}$ comp. coordinates (since different basis set

$\boxed{\text{Under coordinate transforms, vectors don't change, but their components change, since their basis set changes}}$

Using book's notation,

we'll use this... ⇒

$$\boxed{\left\{ \begin{aligned} \lambda^{i'} &= \text{component of } \vec{\lambda} \text{ in } (x', y') \text{ frame} \\ \lambda'^{i} &= \text{same thing} \end{aligned} \right\}}$$

$\lambda^{i'}$ is weird, because i' no longer a dummy. We can't change it to $l, n, m, \ldots$

But, we can change i' to l' or k', ...

<u>Suppose</u> $\vec{\lambda}$ = vector <u>and</u> have 2 coords system

$\{u^i\}$ and $\{u^{i'}\}$

<u>e.g</u> $u^i = \{r, \theta, \phi\}$, and $u^{i'} = \{\rho, \varphi, z\}$

These are related, $\boxed{u^{i'} = u^{i'}(u^j)}$

We also have <u>basis sets</u> with respect to each coord. system

<u>Unprimed</u> : $\vec{e}_i = \frac{\partial \vec{r}}{\partial u^i}$, $\vec{e}^i = \nabla u^i$, $g_{ij} = \vec{e}_i \cdot \vec{e}_j$

<u>Primed</u> : $\vec{e}_{i'} = \frac{\partial \vec{r}}{\partial u^{i'}}$, $\vec{e}^{i'} = \nabla u^{i'}$, $g_{i'j'} = \vec{e}_{i'} \cdot \vec{e}_{j'}$

A vector $\vec{\lambda}$ can have components in <u>either</u> frame

$$\boxed{\vec{\lambda} = \lambda^i \vec{e}_i \equiv \lambda^{i'} \vec{e}_{i'}}$$

<u>So</u> $\lambda^i \vec{e}_i$, $\lambda^{i'} \vec{e}_{i'}$ must transform in a way that leaves $\vec{\lambda}$ alone

<u>Use chain rule</u> $\boxed{\vec{r} = \vec{r}(u^{i'}) = \vec{r}(u^{i'}(u^j))}$

$$\boxed{\vec{e}_j = \frac{\partial \vec{r}}{\partial u^j} = \frac{\partial \vec{r}}{\partial u^{i'}} \frac{\partial u^{i'}}{\partial u^j} = \vec{e}_{i'} \frac{\partial u^{i'}}{\partial u^j} = \frac{\partial u^{i'}}{\partial u^j} \vec{e}_{i'}}$$

<u>Call</u> $\boxed{U^{i'}_j = \frac{\partial u^{i'}}{\partial u^j}}$ → 9 partial derivatives...

Matrix $[U^{i'}_j]$ = Jacobian = $\begin{pmatrix} \frac{\partial u^{1'}}{\partial u^1} & \frac{\partial u^{2'}}{\partial u^2} & \frac{\partial u^{3'}}{\partial u^1} \\ \frac{\partial u^{1'}}{\partial u^2} & \frac{\partial u^{2'}}{\partial u^2} & \frac{\partial u^{3'}}{\partial u^2} \\ \frac{\partial u^{1'}}{\partial u^3} & \frac{\partial u^{2'}}{\partial u^3} & \frac{\partial u^{3'}}{\partial u^3} \end{pmatrix}$

We have that $\boxed{\vec{e}_j = U^{i'}_j \vec{e}_{i'}}$

Now $\vec{\lambda} = \lambda^{i'}\vec{e}_{i'} = \lambda^j \vec{e}_j = \lambda^j U^{i'}_j \vec{e}_{i'}$

So $\boxed{\lambda^{i'} = \lambda^j U^{i'}_j = U^{i'}_j \lambda^j}$ → transformation rule for contravariant vector components

Jacobian...

We can also define Jacobian...

$\boxed{U^j_{i'} = \left[\frac{\partial u^j}{\partial u^{i'}}\right]}$ $[U^j_{i'}]$ = Jacobian...

Ex 1.4.1 → show that $\boxed{\begin{array}{c} U^k_{i'} U^{i'}_j = \delta^k_j \\ U^{k'}_i U^i_{j'} = \delta^k_j \end{array}}$

Note

$\delta^{k'}_{j'} = 1$ if $k=j$ → is same as δ^k_j

$= 0$ if $k \neq j$

→ Kronecker delta don't depend on basis set / components. –

Sept 21, 2018 Under $u^j \to u^{i'}(u^j)$ We found $\vec{e}_j = U^{i'}_j \vec{e}_{i'}$

where $U^{i'}_j = \frac{\partial u^{i'}}{\partial u^j}$ (Jacobian matrix)

also found $\lambda^{i'} = U^{i'}_j \lambda^j$

and $U^i_{j'} = \frac{\partial u^i}{\partial u^{j'}}$

which obey $\begin{cases} U^k_{i'} U^{i'}_j = \delta^k_j \\ U^{k'}_j U^j_{i'} = \delta^k_i \end{cases}$ | $\delta^{k'}_{j'} = \delta^k_j$

<u>Next</u> can invert $\lambda^{i'} = U^{i'}_{j}\lambda^{j}$

→ mult. by $U^{k}_{i'}$ + sum

$$\boxed{\begin{aligned} U^{k}_{i'}\lambda^{i'} &= U^{i'}_{j}U^{k}_{i'}\lambda^{j} \\ \text{So}\quad U^{k}_{i'}\lambda^{i'} &= \delta^{k}_{j}\lambda^{j} = \lambda^{k} \end{aligned}}$$

Can let $k = i$, $i' \to j'$ ⇒ $\boxed{\lambda^{i} = U^{i}_{j'}\lambda^{j'}}$

<u>So</u> $\boxed{\lambda^{i'} = U^{i'}_{j}\lambda^{j} \quad \text{and} \quad \lambda^{i} = U^{i}_{j'}\lambda^{j'}}$ (swapping primes & unprime)

Can also transform Covariant components

$$\vec{\lambda} = \lambda_{i'}\vec{e}^{\,i'} = \lambda_{j}\vec{e}^{\,j}$$

<u>where</u> $\boxed{\vec{e}^{\,j} = \nabla u^{j} = \frac{\partial u^{j}}{\partial x}\hat{i} + \frac{\partial u^{j}}{\partial y}\hat{j} + \frac{\partial u^{j}}{\partial z}\hat{k}}$

<u>if</u> $u^{j} = u^{j}(u^{i'}(x,y,z))$ → need chain rule ...

→ $\frac{\partial u^{j}}{\partial x} = \frac{\partial u^{j}}{\partial u^{i'}}\frac{\partial u^{i'}}{\partial x}$

<u>So</u> $\boxed{\vec{e}^{\,j} = \frac{\partial u^{j}}{\partial u^{i'}}\frac{\partial u^{i'}}{\partial x}\hat{i} + \frac{\partial u^{j}}{\partial u^{i'}}\frac{\partial u^{i'}}{\partial y}\hat{j} + \frac{\partial u^{j}}{\partial u^{i'}}\frac{\partial u^{i'}}{\partial z}\hat{k}}$ → 9 terms

rearrange these 9 terms ... Now, separate the 1', 2', 3' sums...

$$\vec{e}^{\,j} = \left[\frac{\partial u^{j}}{\partial u^{1'}}\frac{\partial u^{1'}}{\partial x}\hat{i} + \frac{\partial u^{j}}{\partial u^{1'}}\frac{\partial u^{1'}}{\partial y}\hat{j} + \frac{\partial u^{j}}{\partial u^{1'}}\frac{\partial u^{1'}}{\partial z}\hat{k}\right] + 2'\text{ terms} + 3'\text{ terms}$$

$$= \frac{\partial u^{j}}{\partial u^{1'}}\left(\frac{\partial u^{1'}}{\partial x}\hat{i} + \frac{\partial u^{1'}}{\partial y}\hat{j} + \frac{\partial u^{1'}}{\partial z}\hat{k}\right) + \frac{\partial u^{j}}{\partial u^{2'}}\left(\quad 2' \quad\right) + \frac{\partial u^{j}}{\partial u^{3'}}\left(3'\right)$$

$$= \frac{\partial u^{j}}{\partial u^{1'}}\nabla u^{1'} + \frac{\partial u^{j}}{\partial u^{2'}}\nabla u^{2'} + \frac{\partial u^{j}}{\partial u^{3'}}\nabla u^{3'}$$

So $$\boxed{\vec{e}^{\,j} = \frac{\partial u^j}{\partial u^{1'}}\vec{e}^{\,1'} + \frac{\partial u^j}{\partial u^{2'}}\vec{e}^{\,2'} + \frac{\partial u^j}{\partial u^{3'}}\vec{e}^{\,3'} = \frac{\partial u^j}{\partial u^{i'}}\vec{e}^{\,i'}}$$

Note $$\frac{\partial u^j}{\partial u^{i'}} = U^j_{i'} \Rightarrow \boxed{\vec{e}^{\,j} = U^j_{i'}\vec{e}^{\,i'}}$$ (analogous form ...)

okay... what about covariant components..?

$$\boxed{\vec{\lambda} = \lambda_{i'}\vec{e}^{\,i'} = \lambda_j\vec{e}^{\,j} = \lambda_j U^j_{i'}\vec{e}^{\,i'}}$$

Therefore $$\boxed{\lambda_{i'} = U^j_{i'}\lambda_j}$$ Similarly $$\boxed{\lambda_j = U^{i'}_j \lambda_{i'}}$$

Note, we can introduce matrices

$$\boxed{\underset{\sim}{U} = [U^{\overset{\text{row}}{i'}}_{\underset{\text{col}}{j}}] = \begin{pmatrix} \frac{\partial u^{1'}}{\partial u^1} & \frac{\partial u^{1'}}{\partial u^2} & \cdots \\ \frac{\partial u^{2'}}{\partial u^1} & \cdots & \end{pmatrix}}$$

and the inverse $$\boxed{\underset{\sim}{\hat{U}} = [U^j_{i'}]}$$

And $$\boxed{\underset{\sim}{U}\,\underset{\sim}{\hat{U}} = \underline{I}}$$

Summarize Under a coordinate transform $u^j \to u^{i'}$ or $u^{j'} \to u^i$

$$\boxed{\vec{\lambda} = \lambda^i \vec{e}_i = \lambda^{j'}\vec{e}_{j'} = \lambda_i\vec{e}^{\,i} = \lambda_{j'}\vec{e}^{\,j'}}$$

These are all related by

$$\vec{e}_j = U^{i'}_j \vec{e}_{i'} \quad , \quad \vec{e}^{\,j} = U^j_{i'}\vec{e}^{\,i'}$$

$$\vec{e}^{\,i'} = U^{i'}_j \vec{e}^{\,j}$$

$$\vec{e}^{\,j} = U^j_{i'}\vec{e}^{\,i'}$$

Contravariants $\lambda^{i'} = U^{i'}_j \lambda^j \quad , \quad \lambda^j = U^j_{i'}\lambda^{i'}$

Covariants $\lambda_{i'} = U^j_{i'}\lambda_j \quad , \quad \lambda_j = U^{i'}_j \lambda_{i'}$

↑

notice the patterns!

The components of a vector must transform this way under general coordinate transformation.

→ We can turn this around to define a vector...

Def A vector is a quantity whose components transform as

$$\lambda^{i'} = U^{i'}_{j} \lambda^{j} \quad \text{(contravariant way)}$$

under a general coordinate transformation $u^{i'} = u^{i'}(u^{j})$

Remarks We're often interested in vector fields (collection of vectors at different points)

(i) ⇒ components depend on coordinates

$$\lambda^{j} = \lambda^{j}(u^{i})$$

At each point P, we would need $\lambda^{i'} = U^{i'}_{j} \lambda^{j}$ to hold for this to be a vector field...

(ii) Not all 3-tuples of functions are vectors...

→ e.g. Consider 3-tuple of coordinates

$$\left\{ \begin{array}{l} \lambda^{i} = u^{i} \\ \lambda^{j'} = u^{j'} \end{array} \right. \quad \text{linked by } u^{j'} = u^{j'}(u^{i})$$

To be a vector field under general coordinate transforms, it must be true that

$$\lambda^{j'} = U^{j'}_{i} \lambda^{i} \text{. In this case becomes}$$

$$\hookrightarrow u^{j'} = U^{j'}_{i} u^{i} \quad \text{with } U^{j'}_{i} = \frac{\partial u^{j'}}{\partial u^{i}}$$

But in general this is **NOT** true $u^{j'} \neq \frac{\partial u^{j'}}{\partial u^{i}} u^{i}$ ← instead $u^{j'} = u^{j'}(u^{i})$

So coordinates do not make a vector + As components they don't transform

→ This is why we never lower u^i, i.e. $u^j \neq g^{ij} u_j$

BUT $\boxed{\text{there are special case exceptions}}$

e.g → restrict to linear transformation

$$\boxed{u^{j'} = u^{j'}(u^i) = C^{j'}_i u^i} \quad \text{where } C^{j'}_i \text{ constant}$$

↑ new coords are just linear comb. of old ...

So $\frac{\partial u^{j'}}{\partial u^k} = C^{j'}_i \frac{\partial u^i}{\partial u^k} = C^{j'}_i \delta^i_k = C^{j'}_k$

Let $k = i$ ⇒ $\boxed{C^{j'}_i = \frac{\partial u^{j'}}{\partial u^i} = U^{j'}_i}$

→ Got $u^{j'} = u^{j'}(u^i)$ get $\boxed{u^{j'} = U^{j'}_i u^i}$ under linear transforms → so they

So coordinates do form a vector under linear coords transformation (but not general coord. transf.)

(iii) { Properly speaking we can define vectors with respect to a particular class of transformation ... }

{ It is possible for sth to be a vector w.r.t one class of transformation, but NOT a vector under another

Default → under general coordinate transform }

Sep 24, 2018 $\boxed{\text{Example}}$

Recall Coordinate transform $u^j \to u^{j'}$
there $U^{j'}_j = \frac{\partial u^{j'}}{\partial u^j}$, $U^j_{j'} = \frac{\partial u^j}{\partial u^{j'}}$

obey $U^i_{k'} U^{k'}_j = \delta^i_j$ and $\lambda^{i'} = U^{i'}_j \lambda^j$

Can define a vector as a quantity whose components transform this way...

Note ⇒ coordinates do not form a vector since $u^{i'} \neq \frac{\partial u^{i'}}{\partial u^{j}} u^{j}$ in general

But → differentials of coordinates do make a vector (they are displacements)

$du^{i} = \{du^{1}, du^{2}, du^{3}\}$. From the chain rule $du^{i'} = \frac{\partial u^{i'}}{\partial u^{j}} du^{j}$

$\to du^{i'} = U^{i'}_{j} du^{j} \to (du^{i})$ make a vector...

$\boxed{\text{Example}}$ Find $U^{i'}_{j}$ for a coordinate transform from Cartesian to spherical in flat 3D space...

$u^{j} \mapsto u^{i'}$ with $u^{j} = \{x, y, z\}$, $u^{i'} = \{r, \theta, \varphi\}$

So
$$[U^{i'}_{j}] = \begin{pmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} & \frac{\partial r}{\partial z} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{pmatrix}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\theta = \cos^{-1}\left(\frac{z}{r}\right) = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$

$$\varphi = \tan^{-1}\left(\frac{y}{x}\right)$$

Cont
$$[U^{i'}_{j}] = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\varphi & \frac{1}{r}\cos\theta\sin\varphi & -\frac{1}{r}\sin\theta \\ \frac{-\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix}$$

Note this is the inverse of the Jacobian found previously

$$dx\,dy\,dz = \det[U^{j}_{i'}]\,dr\,d\theta\,d\varphi$$

Call $[U^{i'}_{j}] = \underset{\sim}{\hat{U}}$, and $[U^{j}_{i'}] = \underset{\sim}{U}$

We can show $\boxed{\underset{\sim}{\hat{U}}\,\underset{\sim}{U} = \underset{\sim}{U}\,\underset{\sim}{\hat{U}} = I}$

Example

Suppose $\lambda^i = (1,0,0)$ in Cartesian coordinates. So $\vec{\lambda} = \hat{i} + 0\hat{j} + 0\hat{k}$

What are the components of $\vec{\lambda}$ in spherical coordinates? well...

$$\vec{\lambda} = \lambda^{i'}\vec{e}_{i'} \Rightarrow \text{where } \vec{e}_{i'} = \{\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi\}$$

Now

$$\lambda^{i'} = \begin{pmatrix} \lambda^{1'} \\ \lambda^{2'} \\ \lambda^{3'} \end{pmatrix} = O^{i'}_{j}\lambda^{j} = \begin{pmatrix} \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\varphi & \frac{1}{r}\cos\theta\sin\varphi & -\frac{1}{r}\sin\theta \\ \frac{-\sin\varphi}{r\sin\theta} & \frac{\cos\varphi}{r\sin\theta} & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So

$$\begin{pmatrix} \lambda^{1'} \\ \lambda^{2'} \\ \lambda^{3'} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \frac{1}{r}\cos\theta\cos\varphi \\ \frac{-\sin\varphi}{r\sin\theta} \end{pmatrix} \leftarrow \text{components with respect to spherical coordinates...}$$

Now have

$$\vec{\lambda} = \lambda^{1'}\vec{e}_r + \lambda^{2'}\vec{e}_\theta + \lambda^{3'}\vec{e}_\varphi$$

$$\boxed{\vec{\lambda} = \sin\theta\cos\varphi\,\vec{e}_r + \frac{1}{r}\cos\theta\cos\varphi\,\vec{e}_\theta - \frac{\sin\varphi}{r\sin\theta}\,\vec{e}_\varphi}$$

We know $|\vec{\lambda}| = 1$ in Cartesian.. Is this still true in spherical ...

$$\hookrightarrow \cancel{|\vec{\lambda}| = \sin^2\theta\cos^2\varphi + \frac{1}{r^2}\cos^2\theta\cos^2\varphi + \frac{\sin^2\varphi}{r^2\sin^2\theta} = \lambda^{i'}\lambda^{j'}g_{ij}}$$

Now $|\vec{\lambda}| = \sqrt{\vec{\lambda}\cdot\vec{\lambda}} \Rightarrow$ where $\boxed{\vec{\lambda}\cdot\vec{\lambda} = g_{i'j'}\lambda^{i'}\lambda^{j'}}$

with the metric $g_{i'j'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$

NOTE metric tensor

$g_{ij} \neq I$ in general..

(exception is in Cartesian)

So

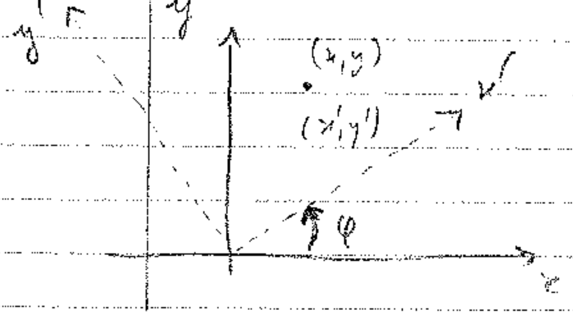
$$\vec{\lambda}\cdot\vec{\lambda} = (\lambda^{1'})^2 g_{1'1'} + (\lambda^{2'})^2 g_{2'2'} + (\lambda^{3'})^2 g_{33'}$$

$$= \sin^2\theta\cos^2\varphi + \cos^2\theta\cos^2\varphi + \sin^2\varphi = 1$$

So $\boxed{|\vec{\lambda}| = 1}$

㊶

Example Find $U^{i'}_j$ for a rotation of Cartesian coords by φ [illegible] about the z axis ...



$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

More completely $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\frac{\partial u^{i'}}{\partial u^j} = [U^{i'}_j] = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (Same thing ...)

Note (φ is fixed)

So note $[U^{i'}_j]$ is a constant matrix → linear transformation

⇒ coordinates transform like vectors ... which is what we showed

$$\boxed{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Leftrightarrow u^{i'} = U^{i'}_j u^j}$$

This is NOT true in general. True only if components are fixed ...

Any vector $\vec{\lambda}$ will have components that transform under rotation given by (generally)

$$\boxed{\lambda^{i'} = U^{i'}_j \lambda^j}$$

rotated   unrotated

Suppose $(x, y, z) = (1, 1, 0)$ what is (x', y', z') after rotation by φ.

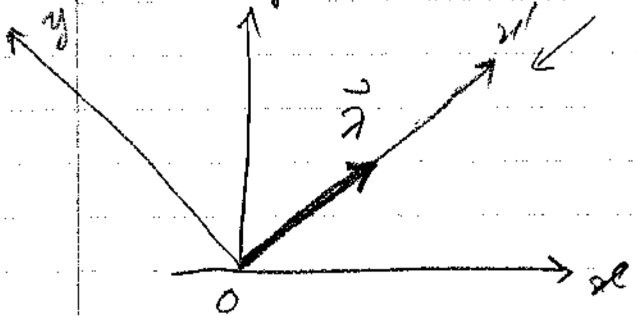
Well
$$\vec{\lambda}\cdot\vec{\lambda} = g_{ij}\lambda^i\lambda^j = \delta^i_j \lambda^i\lambda^j = \lambda^i\cdot\lambda^i \quad (g_{ij} = \delta^i_j \text{ in Cartesian})$$
$$= 2$$

In (x', y', z')

$$\lambda^{i'} = U^{i'}_j \lambda^j = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \underset{\substack{\uparrow \text{ w.r.t} \\ (x', y', z')}}{\begin{pmatrix} \cos\varphi + \sin\varphi \\ -\sin\varphi + \cos\varphi \\ 0 \end{pmatrix}}$$

So $$\boxed{\breve{\lambda} = (\cos\varphi + \sin\varphi)\,\hat{i}' + (-\sin\varphi + \cos\varphi)\,\hat{j}' + 0\,\hat{k}'}$$

eg if $\varphi = 45^\circ$, then $\breve{\lambda} = \sqrt{2}\,\hat{i}' + 0\,\hat{j}' + 0\,\hat{k}'$ (makes sense)
$$= \hat{i} + \hat{j} + 0\,\hat{k}$$



Note $|\breve{\lambda}|$ still $= \sqrt{2}$

But we need to know what $g_{i'j'}$ is...

$$|\breve{\lambda}|^2 = \underset{\substack{\uparrow \\ \text{what is } g_{i'j'}?}}{g_{i'j'}}\lambda^{i'}\lambda^{j'} \text{ does this } = (\sqrt{2})^2$$

Question | How does the metric tensor transform. But first what is a tensor?

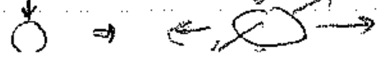
Vector ⇒ has magnitude + direction (one direction + one length)

Tensors → generalization of vectors, but they're multi-directional

Ex

Vector : force $\vec{F}$ ↗ $\vec{F} = m\vec{a}$ ($\vec{a}$ follows $\vec{F}$)

But now consider a balloon + squeeze it in 1 direction



(response in all directions...)

→ Stress tensor $F_{xx}, F_{xy}, F_{xz}, F_{yx}, F_{yy}, F_{yz}, F_{zx}, F_{zy}, F_{zz}$

☆ Mathematically, generalize the def of a vector.

⇒ Give a definition based on how their components transform

Sept 25, 2018

TENSORS → generalizations of vectors, but multidirectional

→ can't represent them as an arrow...

Can generalize def. of a vector to say...

Def A tensor is a multicomponent quantity whose components transform as contravariant or covariant vector components

e.g $\tau^{ij}{}_{k}{}^{l}$ is a tensor if

$$\tau^{i'j'}{}_{k'}{}^{l'} = U^{i'}_{m} U^{j'}_{n} U^{p}_{k'} \tau^{mn}{}_{p}{}^{q} U^{l'}_{q}$$

Under a general coordinate transformation $u^{i'} = u^{i'}(u^{j})$

Show $\boxed{g_{ij} \text{ is a tensor}}$ $g_{ij} = \vec{e}_i \cdot \vec{e}_j$

$g_{i'j'} = \vec{e}_{i'} \cdot \vec{e}_{j'}$

We can use $\vec{e}_{i'} = U^{k}_{i'} \vec{e}_{k}$,

$$\Rightarrow \boxed{g_{i'j'} = U^{k}_{i'} \vec{e}_{k} \cdot U^{l}_{j'} \vec{e}_{l} = U^{k}_{i'} U^{l}_{j'} g_{kl}}$$

So g_{ij} is a tensor

Similarly $\boxed{g^{i'j'} = U^{i'}_{k} U^{j'}_{l} g^{kl}}$

A tensor $\tau^{ijk}{}_{mnp\ldots}$ is said to be of type (r,s) when it has

r contravariants and s covariants.

__Ex__ $g_{ij} \rightarrow$ type $(0,2)$ tensor $\quad\Big\}\quad x^i \rightarrow$ type $(1,0)$ tensor

$g^{ij} \rightarrow$ type $(2,0)$ tensor $\quad\Big|\quad \partial_i \rightarrow$ type $(0,1)$ tensor

__Note__ $U^{i'}_j$ is __NOT__ a tensor. Rather, it's a transformation matrix

$\hookrightarrow$ take components $j \leftrightarrow i'$

__Ex__

__write__ $g_{i'j'} = U^k_{i'} U^l_{j'} g_{kl}$ as matrix eqn

__let__ $\underset{\sim}{G} = [g_{ij}]$, and $\underset{\sim}{G}' = [g_{i'j'}]$

$$\underset{\sim}{\hat{U}} = \underset{\sim}{U}^{-1} = \left[\frac{\partial u^k}{\partial u^{i'}}\right]$$

__Put metric in the middle__

$$g_{i'j'} = U^{\overset{\text{row}}{k}}_{i'}\, g_{\underset{\text{row}}{k}\underset{\text{col}}{l}}\, U^{\overset{\text{row}}{l}}_{j'} \quad \rightarrow \text{not gonna work. Need to transpose } 1^{st} \text{ matrix}$$

$$\boxed{\underset{\sim}{G}' = \hat{U}^T G \hat{U}}$$

__Note__ only tensors of type (r,s) with $r+s \leq 2$ can be written as matrices multiplications ... Can't write $\tau^{ij}{}_{kl}$ as a matrix

$\boxed{\text{Ex}}$ __Look at rotation by ϕ about z again__

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow [U^{i'}_j] = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

<u>Recall</u>, in xyz frame, $g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, what is $g_{i'j'}$ in (x', y', z')?

<u>Have</u>

$$[g_{i'j'}] = [U^k_{\ i'} U^l_{\ j'} g_{kl}] = \underset{\sim}{\hat{U}}^T \underset{\sim}{G} \underset{\sim}{\hat{U}} = \underset{\sim}{G}'$$

<u>Recall</u> $\underset{\sim}{\hat{U}} = \underset{\sim}{U}^{-1} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\frac{\partial u^k}{\partial u^{i'}}$

(rotation by $-\varphi$)

<u>This gives</u>

transpose...

$$\underset{\sim}{G}' = [g_{i'j'}] = \underset{\sim}{\hat{U}}^T \underset{\sim}{G} \underset{\sim}{\hat{U}} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

I

$$= I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Metric is the <u>same</u> in rotated Cartesian frame...

<u>Notice in this case</u> $\boxed{\underset{\sim}{\hat{U}} = \underset{\sim}{U}^{-1} = \underset{\sim}{U}^T \Rightarrow \underset{\sim}{U} \text{ is orthogonal}}$

Scalars $\rightarrow$ invariant quantities under general coordinate transformation
- $\Rightarrow$ have no open indices
- $\rightarrow$ type (0,0) tensors
- $\Rightarrow$ just numbers ... $\rightarrow$ same in all coords system...

<u>Ex</u> Show that the magnitude of a vector is a scalar

let $\vec{\lambda} = \{\lambda^i\} = \{\lambda^{i'}\}$

$\|\vec{\lambda}\| = \sqrt{\vec{\lambda} \cdot \vec{\lambda}} = \sqrt{\lambda^i \lambda_i}$ this has no open indices (it's a sum)

$|\vec{\lambda}|$ is a scalar if $\vec{\lambda} \cdot \vec{\lambda} = \lambda^{i'} \lambda_{i'}$ $\rightarrow$ same number. Need to <u>show</u> $\lambda^i \lambda_i = \lambda^{i'} \lambda_{i'}$ (INVARIANT)

Use $\lambda^{i'}\lambda_{j'} = (U^{i'}_{j}\lambda^{j})(U^{k}_{i'}\lambda_k) = \underbrace{U^{i'}_{j}U^{k}_{i'}}\lambda^{j}\lambda_k$

$$= \delta^k_j \lambda^j \lambda_k = \lambda^k \lambda_k$$

So $\lambda^{i'}\lambda_{i'} = \lambda^k\lambda_k \Rightarrow \underline{|\vec{\lambda}| \text{ is a scalar}}$

#

Example $\boxed{\text{Show } ds^2 = g_{ij}du^i du^j \text{ is a scalar}}$

Need to show $g_{i'j'}du^{i'}du^{j'} = g_{ij}du^i du^j$

↳ Use $g_{i'j'} = U^k_{i'}U^l_{j'}g_{kl}$, $du^i = \frac{\partial u^i}{\partial u^{j'}}du^{j'} = U^i_{j'}du^{j'}$

So $g_{i'j'}du^{i'}du^{j'}$

$$= (U^k_{i'}U^l_{j'}g_{kl})(U^{i'}_m du^m)(U^{j'}_n du^n)$$

$$= (U^k_{i'}U^{i'}_m)(U^l_{j'}U^{j'}_n)g_{kl}du^m du^n$$

$$= \delta^k_m \delta^l_n g_{kl} du^m du^n$$

$$= g_{kl}du^k du^l = g_{ij}du^i du^j$$

Therefore ds^2 is a scalar

#

$\boxed{\text{Summarize}}$ 3 classes of objects ... Scalars : λ → no open indices (invariant)..

Vectors → upper/lower index
→ transform as

$$\boxed{\lambda^{i'} = U^{i'}_j\lambda^j \ , \ \lambda_{i'} = U^j_{i'}\lambda_j}$$

Tensors $\tau^{ij}{}_k$ → type (r,s)

↖ type $(2,1)$

$$\boxed{\tau^{i'j'}{}_{k'} = U^{i'}_l U^{j'}_m U^n_{k'}\tau^{lm}{}_n}$$

Components transform, but tensors themselves don't transform...

Sept 26, 2018

IV – Flat Spacetime

$(ct, x, y, z) \rightarrow$ spacetime coords (lbl μ, ν), $\mu = 0, 1, 2, 3$

So $X^\mu = \{x^0, x^1, x^2, x^3\} = (ct, x, y, z)$

$$X^\mu = (X^0, \vec{x}) = (x^0, x^i) \qquad (i = 1, 2, 3)$$

Coordinate transformation in special relativity are Lorentz Transformation

Note Under LT there's an invariant spacetime interval.

$$\left.\begin{aligned} ds^2 &= c^2dt^2 - dx^2 - dy^2 - dz^2 \\ &= (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \end{aligned}\right\} \leftarrow \text{line element}$$

in Cartesian coordinates in flat spacetime

gives "distance" in spacetime

Can read off the metric

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu}$$

where

$$[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \leftarrow \text{Minkowski metric}$$

Since in any other frame connected by a LT

$$(ds')^2 = (dx^{0'})^2 - (dx^{1'})^2 - (dx^{2'})^2 - (dx^{3'})^2 = (ds)^2$$

Says that

$$[\eta_{\mu'\nu'}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = [\eta_{\mu\nu}] \rightarrow \text{same metric (Cartesian)}$$

So

$$\boxed{ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = \eta_{\mu\nu} dx^\mu dx^\nu}$$

Note $[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0\,0\,0 \\ 0 & \\ 0 & -g_{ij} \\ 0 & \end{pmatrix}$ → can change in/to spherical

Generally, in <u>non - Cartesian</u> coordinates or when there's <u>curvature</u>, we m

$$\boxed{g_{\mu\nu} = \text{metric} \qquad ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu}$$

<u>invention</u> → But when using Cartesian coords in flat spacetime, let $\boxed{g_{\mu\nu} \to \eta_{\mu\nu}}$

$\boxed{\text{With metric, we can raise/lower tensor indices}}$

if $\lambda^\mu = (\lambda^0, \lambda^1, \lambda^2, \lambda^3) = (\lambda^0, \vec{\lambda})$ → contravariant

then $\lambda_\mu = \eta_{\mu\nu}\lambda^\nu = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ covariant

$= (\lambda^0, -\lambda^1, -\lambda^2, -\lambda^3)$

⇒ In flat spacetime in Cartesian coords, $\boxed{\lambda^0 = \lambda_0}$

But spatial component → $\boxed{\lambda^i = -\lambda_i}$

Rename $[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

<u>How to get $[\eta^{\mu\nu}]$?</u> Take ↑ inverse that satisfy $\eta_{\mu\nu}\eta^{\nu\sigma} = \delta$

Not hard to see that $[\eta^{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = [\eta_{\mu\nu}]$

<u>Then</u> $\boxed{\lambda^\mu = \eta^{\mu\nu}\lambda_\nu}$

As before there are 4 ways to take <u>inner product</u>...

$$\boxed{\mathbf{a}\cdot\mathbf{b} = a^\mu \cdot b_\mu = a_\mu \cdot b^\mu = \eta_{\mu\nu} a^\mu b^\nu = \eta^{\mu\nu} a_\mu b_\nu}$$

↑ inner product of <u>two</u> 4-vectors.

<u>Notice that</u> $a^\mu b_\mu = a^0 b_0 + a^1 b_1 + a^2 b_2 + a^3 b_3$ (sum. notation...)

<u>But</u> $\eta_{\mu\nu} a^\mu b^\nu = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^\mu b_\mu$

<u>Why?</u> → simply because $\boxed{b_\mu = -b^\mu}$ (by $\eta_{\mu\nu}$)

<u>Note</u> The metric contains info on how to calculate lengths and intervals in spacetime...

Note We've skipped introducing basis vector. Could define a set $(\vec{e}_0, \vec{e}_1, \vec{e}_2, \vec{e}_3)$ {~~[illegible]~~}

<u>to</u> $\boxed{\boldsymbol{\lambda} = \lambda^0 \vec{e}_0 + \lambda^1 \vec{e}_1 + \lambda^2 \vec{e}_2 + \lambda^3 \vec{e}_3}$

However, $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are <u>not</u> $\hat{i}, \hat{j}, \hat{k}$

<u>Why?</u> $\vec{e}_1 \cdot \vec{e}_1 = \eta_{11} = -1$, where $\hat{i}\cdot\hat{i} = 1$

↑ <u>note</u> index start @ 0

<u>So</u> $\vec{e}_\mu$ would have <u>imaginary parts</u>

Basically, won't use basis vectors going forward!

Lorentz Transformation → is a coordinate transform from one ***inertial*** frame to another $K \to K'$

Most general LT's include
- (1) Lorentz boost (relative motion w/ const. v)
- (2) Translation (origins don't coincide at $t' = t = 0$)
- (3) Spatial rotation $x \not\parallel x'$, ...
- (4) spatial inversion (parity transform) $(x' = -x)$
- (5) Time reversal $(t' = -t)$

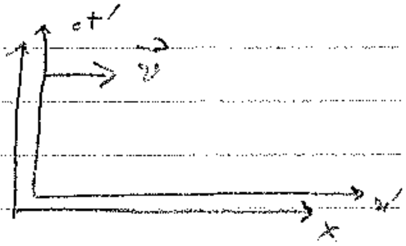
Usually called collectively ↓ "Poincaré transformations"

other distinctions
- inhomogeneous LT's → ~~case~~ have translation
- homogeneous → no translation (same origin)
- improper LT's → (parity / time reversal)
- proper LT's → NO parity / time reversal ...

We can first look at homogeneous, proper LT's with no rotations

⇒ these are the Lorentz boosts ...

e.g. A boost along x



Lorentz boost
$$\begin{pmatrix} x^{0'} \\ x^{1'} \\ x^{2'} \\ x^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^{0} \\ x^{1} \\ x^{2} \\ x^{3} \end{pmatrix}$$

In flat 3D space
$$U^{i'}_{j} = \frac{\partial u^{i'}}{\partial u^{j}} \to U$$

In 4D spacetime, in general,
$$\bar{X}^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \to \text{big } X : \bar{X}$$

But for Lorentz transformations use ~~A~~, Λ
$$\bar{X}^{\mu'}_{\nu} = \Lambda^{\mu'}_{\nu} = \frac{\partial x^{\mu'}}{\partial x^{\nu}} \qquad \Lambda^{\mu'}_{\nu} \to \text{LT's only}$$

For a Lorentz boost

$$[\Lambda^{\mu'}_{\nu}] = \left[\frac{\partial X^{\mu'}}{\partial X^{\nu}}\right] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

ept 28, 2018

Recall Lorentz Transformation $X^{\nu} \to X^{\mu'}$

$$\Lambda^{\mu'}_{\nu} = \frac{\partial X^{\mu'}}{\partial X^{\nu}}$$ e.g for a boost along x

$$[\Lambda^{\mu'}_{\nu}] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

N.B. $\Lambda^{\mu'}_{\nu}$ constant

This means LT's are linear transformations

This means Cartesian coords X^{μ} form the components of a vector under LTS

$$\boxed{X^{\mu'} = \Lambda^{\mu'}_{\nu} X^{\nu} \text{ is obeyed}}$$

This gives back $X^{0'} = \gamma(X^0 - \beta X^1)$, $X^{1'} = \gamma(X^1 - \beta X^0)$

This also means that in SR we can lower index of X^{μ}

$$\boxed{\begin{aligned} X_{\mu} &= \eta_{\mu\nu} X^{\nu} \\ X^{\mu} &= \eta^{\mu\nu} X_{\nu} \end{aligned}}$$

But we never do this in general, e.g. in curved spacetime)

Remember $X^{\mu} = (ct, x, y, z)$

while $X_{\mu} = (ct, -x, -y, -z)$

these obey

$$\underbrace{\Lambda^{\mu'}_{\nu}\Lambda^{\nu}_{\lambda'} = \delta^{\mu}_{\lambda}}$$

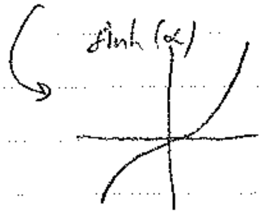
To find inverse

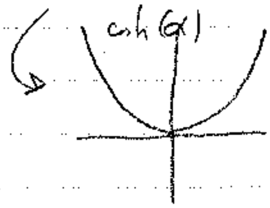
$$\Lambda^{\mu}_{\nu'} = \frac{\partial X^{\mu}}{\partial X^{\nu'}}$$ Just let $v = -v$ $\Rightarrow$ $$[\Lambda^{\mu}_{\nu'}] = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A Curiosity about Lorentz boosts

→ can make them look like rotation using hyperbolic functions...

Use $\sinh(\alpha) = \frac{e^{\alpha} - e^{-\alpha}}{2}$, $\cosh(\alpha) = \frac{e^{\alpha} + e^{-\alpha}}{2}$





$\tanh(\alpha) = \frac{\sinh\alpha}{\cosh\alpha}$ $\quad$ $\mathrm{sech}(\alpha) = \frac{1}{\cosh(\alpha)}$

$\mathrm{csch}(\alpha) = \frac{1}{\sinh(\alpha)}$ $\quad$ $\coth(\alpha) = \frac{1}{\tanh(\alpha)}$

OBEY

$$\boxed{\begin{array}{l} \cosh^2\alpha - \sinh^2\alpha = 1 \\ 1 - \tanh^2(\alpha) = \mathrm{sech}^2(\alpha) \end{array}}$$

Look at

$$[\Lambda^{\mu'}_{\ \nu}] = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Introduce $\tanh\varphi = \frac{v}{c}$ where φ = rapidity

So $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \tanh^2\varphi}} = (\mathrm{sech}\,\alpha)^{-1} = \cosh\varphi$

So $\gamma\frac{v}{c} = \beta\gamma = \sinh\varphi$

So

$$\boxed{[\Lambda^{\mu'}_{\ \nu}] = \begin{pmatrix} \cosh\varphi & -\sinh\varphi & 0 & 0 \\ -\sinh\varphi & \cosh\varphi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

form of hyperbolic rotation between ct & x

$\boxed{\text{Proper Homogeneous Lorentz Transform}}$

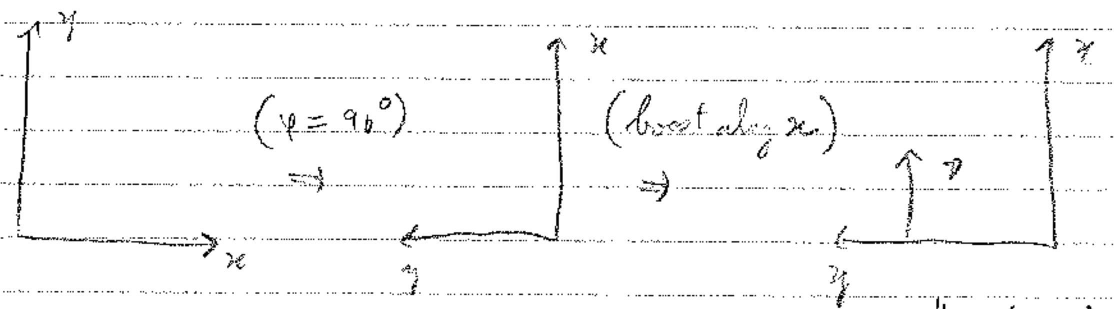
↳ boost + rotation. These still leave form $X^{\mu'} = \Lambda^{\mu'}_{\nu} X^{\nu}$
But now $\Lambda^{\mu'}_{\nu}$ can be a boost or rotation

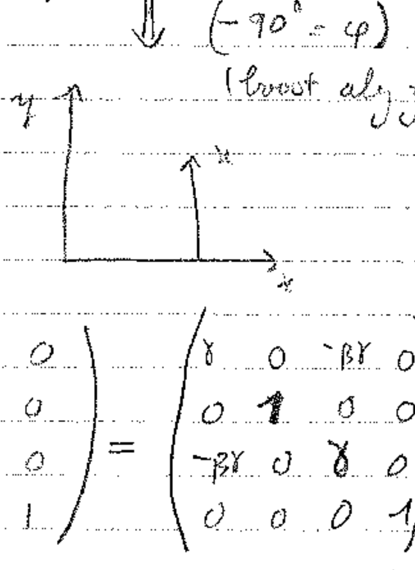
Can look at a rotation about z by φ

$$\left[\Lambda^{\mu'}_{\nu}\right] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

A Lorentz boost along an arbitrary direction can be found as a combination of a boost along x & spatial rotation

e.g.





So the end result is boost along y

So matrix multiply

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{rotate by } -90^\circ} \underbrace{\begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{boost along } x} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{rotate by } 90^\circ} = \underbrace{\begin{pmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{boost along } y}$$

Poincare Transformations

↳ boosts, rotation, translations, time / spatial inversions ...

Here $\boxed{X^{\mu'} = \Lambda^{\mu'}_{\nu} X^{\nu} + a^{\mu'}}$ ← general form

(the rest) (translation) (constant), so $\frac{\partial a^{\mu'}}{\partial X^{\nu}} = 0$

These are "affine" transformations: Linear transformation with a shift

Suppose we take $\frac{\partial}{\partial X^{\nu}}$ of $X^{\mu'}$

↳ $$\frac{\partial X^{\mu'}}{\partial X^{\nu}} = \frac{\partial}{\partial X^{\nu}} X^{\mu'} = \overline{\underline{X}}^{\mu'}_{\nu} = \Lambda^{\mu'}_{\nu} \text{ for LTs}$$

⇒ Get the usual definition $\Lambda^{\mu'}_{\nu} = \frac{\partial X^{\mu'}}{\partial X^{\nu}}$. With chain rule, still get $\Lambda^{\mu'}_{\nu}\Lambda^{\nu}_{\sigma} = \frac{\partial X^{\mu'}}{\partial X^{\nu}}\frac{\partial X^{\nu}}{\partial X^{\sigma}} = \delta^{\mu}_{\sigma}$

⇒ Still holds for Poincare transformation

Note → The defining features of a Lorentz Transform is that

$$\boxed{\begin{aligned} ds^2 &= \eta_{\mu\nu} dX^{\mu} dX^{\nu} \\ &= \eta_{\mu'\nu'} dX^{\mu'} dX^{\nu'} \end{aligned}} \quad (\star)$$

→ LT's preserve the Minkowski metric (with Cartesian)

where

$$[\eta_{\mu\nu}] = [\eta_{\mu'\nu'}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

From $X^{\mu'} = \Lambda^{\mu'}_{\sigma} X^{\sigma} + a^{\mu'}$, take differential

$dX^{\mu'} = \Lambda^{\mu'}_{\sigma} dX^{\sigma}$, plug into $(\star)$

So $\eta_{\mu'\nu'} dX^{\mu'} dX^{\nu'} = \eta_{\sigma\rho} dX^{\sigma} dX^{\rho}$

$\rightarrow \eta_{\mu'\nu'} dX^{\mu'} dX^{\nu'} = \eta_{\mu'\nu'} (\Lambda^{\mu'}_{\sigma} dX^{\sigma}) dX^{\nu'}$

$\qquad \| \qquad = \eta_{\mu'\nu'} (\Lambda^{\mu'}_{\sigma} dX^{\sigma}) (\Lambda^{\nu'}_{\rho} dX^{\rho})$

So $\Longrightarrow \eta_{\sigma\rho} dX^{\sigma} dX^{\rho} = \eta_{\mu'\nu'} \Lambda^{\mu'}_{\sigma} \Lambda^{\nu'}_{\rho} dX^{\sigma} dX^{\rho}$

Let $\sigma \to \mu$, $\rho \to \nu$, $\mu' = \alpha'$, $\nu' = \beta'$

$$\hookrightarrow \boxed{\eta_{\mu\nu} = \Lambda^{\alpha'}_{\mu} \Lambda^{\beta'}_{\nu} \eta_{\alpha'\beta'}}$$

Metric obeys this under Poincare transforms. This shows 2 things

{ ① $\eta_{\mu\nu}$ is a tensor → transforms correctly

② $\eta_{\mu\nu}$ is unchanged under Lorentz transformations }

For other vectors & tensors under LTs, must have:

Contravariant $\lambda^{\mu'} = \Lambda^{\mu'}_{\nu} \lambda^{\nu}$

Covariant $\lambda_{\mu'} = \cancel{\Lambda^{\mu'}_{\nu}} \; \Lambda^{\nu}_{\mu'} \lambda_{\nu}$

Tensor $T^{\mu'\nu'}{}_{\sigma'} = \Lambda^{\mu'}_{\alpha} \Lambda^{\nu'}_{\beta} \Lambda^{\gamma}_{\sigma'} T^{\alpha\beta}{}_{\gamma}$

general case these will be different

{ Scalars → invariants under Lorentz transformations (same in all inertial frames) }

⑸6

Oct 1, 2018

4-vector under Lorentz-Transformation

→ must obey $\boxed{\lambda^{\mu'} = \Lambda^{\mu'}_{\nu} \lambda^{\nu}_{x}}$

scalars ↑

Scalars → invariant under LT's.

e.g. Show inner products are scalars...

$$a^{\mu'} b_{\mu'} = \Lambda^{\mu'}_{\nu} a^{\nu} \Lambda^{\sigma}_{\mu'} b_{\sigma}$$

$$= \Lambda^{\mu'}_{\nu} \Lambda^{\sigma}_{\mu'} a^{\nu} b_{\sigma}$$

$$= \delta^{\sigma}_{\nu} a^{\nu} b_{\sigma} = a^{\nu} b_{\nu}$$

invariant, same ⇐ in all frames
→ scalars.

This shows that the norm of every 4-vector is invariant

$$\boxed{\lambda \cdot \lambda = \lambda^{\mu} \lambda_{\mu} = \lambda^{\mu'} \lambda_{\mu'}}$$

Therefore the sign of the norm is invariant as well

$$\lambda^2 = (\lambda \cdot \lambda) = (\lambda^0)^2 - (\lambda^1)^2 - (\lambda^2)^2 - (\lambda^0)^3 \quad \text{Can be } (-, 0, +)$$

There are 3 cases
- $\lambda^2 > 0$ → time-like
- $\lambda = 0$ → light-like / null
- $\lambda^2 < 0$ → space-like

→ These labels do not change under Lorentz Transformations

- For time like vectors, there is always a frame where $\lambda^{\mu} = (\lambda^0, 0, 0, 0)$
→ always rotate + boost to get this...

- For spacelike, can always find a frame where $\lambda^{\mu} = (0, \lambda^1, 0, 0)$ or a frame where $\lambda^{\mu} = (0, 0, \lambda^2, 0)$, etc..

- For null vectors, can always find a frame where $\lambda^{\mu} = (\lambda^0, \lambda^0, 0, 0)$ or $(\lambda^0, 0, \lambda^0, 0)$, etc... More generally, $\lambda^{\mu} = (\lambda^0, \vec{\lambda})$ with $|\vec{\lambda}| = \lambda^0$ so that $\lambda^{\mu} \lambda_{\mu} = 0$

Ex 1 Is $X^{\mu} = (ct, x, y, z)$ a contravariant vector under Poincaré transformation?

- If so, then $X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^{\nu}$ would need to hold

Note Poincaré transform $\boxed{X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^{\nu} + a^{\mu'}}$

↳ See that X^{μ} is not a vector if $a^{\mu'} \neq 0$. (Can't allow translations). Under LT's ($a^{\mu'} = 0$), then X^{μ} is a vector

Ex 2 Is $dX^{\mu} = (c\,dt, dx, dy, dz)$ a vector under Poincaré transform?

Note Poincaré transform $X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^{\nu} + a^{\mu'}$

So $dX^{\mu'} = \Lambda^{\mu'}_{\ \nu}\, dX^{\nu} + 0$

So $dX^{\mu'}$ is a vector ⇒ dX^{μ} is a vector under Poincaré transform

Ex 3 Suppose we take $\frac{\partial}{\partial X^{\mu}}$ of a scalar a vector? Is $\frac{\partial \varphi}{\partial X^{\mu'}}$ a vector? What type?

Chain rule: $\varphi = \varphi(X^{\nu}(X^{\mu'}))$

$$\rightarrow \frac{\partial \varphi}{\partial X^{\mu'}} = \frac{\partial \varphi}{\partial X^{\nu}} \frac{\partial X^{\nu}}{\partial X^{\mu'}} = \Lambda^{\nu}_{\ \mu'} \frac{\partial \varphi}{\partial X^{\nu}} \quad \checkmark$$

So $\frac{\partial \varphi}{\partial X^{\mu'}}$ is a vector. **Note** It's a covariant vector, because there's the upper indices cancel out

↳ Use notation to show this better:

$$\boxed{\frac{\partial}{\partial X^{\mu}} = \partial_{\mu}} \rightarrow \text{Then } \partial_{\mu}\varphi = \frac{\partial \varphi}{\partial X^{\mu}} \text{ is a covariant vector}$$

Also $\boxed{\vec{\nabla} = \partial_i = (\partial_1, \partial_2, \partial_3)}$

So $\boxed{\partial_\mu = (\partial_0, \partial_i) = (\partial_0, \vec{\nabla})}$

Now, in Minkowski spacetime with Cartesian coordinates that we can also define a lower coordinate

$$X_\mu = \eta_{\mu\nu} X^\nu \quad . \quad \text{Call } \partial^\mu = \frac{\partial}{\partial X_\mu} \Bigg\}$$

From $X^\mu = \eta^{\mu\nu} X_\nu \rightarrow \frac{\partial X^\mu}{\partial X_\nu} = \eta^{\mu\nu}$

$$\partial^\mu = \frac{\partial}{\partial X_\mu} = \frac{\partial X^\nu}{\partial X_\mu} \frac{\partial}{\partial X^\nu} = \eta^{\mu\nu} \partial_\nu$$

gives a contravariant vector

So we get $\boxed{\partial^\mu = \eta^{\mu\nu} \partial_\nu \quad \therefore \quad \partial^\mu \varphi = \eta^{\mu\nu} \partial_\nu \varphi}$

But $\partial^i \neq \vec{\nabla}$. Instead $\partial^i = -\partial_i = -\vec{\nabla}$

Can write $\boxed{\partial^\mu = (\partial^0, \partial^i) = (\partial^0, -\vec{\nabla})}$

VELOCITY, MOMENTUM, FORCE

What are these as 4-vectors? → Must transform correctly!

Consider again $X^{\mu'} = \Lambda^{\mu'}_{\ \nu} X^\nu + a^{\mu'}$

Velocity $\frac{d}{dt} X^{\mu'} = \frac{d}{dt} \Lambda^{\mu'}_{\ \nu} X^\nu + \frac{d}{dt} a^{\mu'}$ → constant translation

So $\frac{dX^{\mu'}}{dt} = \Lambda^{\mu'}_{\ \nu} \frac{dX^\nu}{dt} + 0$ → Note, same t on both sides

with $X^\mu = (ct, \vec{x})$ → take t derivative

$\frac{d}{dt} X^\mu = (c, \vec{v})$ with $\vec{v} = \frac{d\vec{x}}{dt}$. Can call $\boxed{V^\mu = \frac{dX^\mu}{dt} = (c, \vec{v})}$ ← coordinate velocity

Put in a primed frame $V^{\mu'} = \frac{dX^{\mu'}}{dt'} = (c, \vec{v}')$

Note $\frac{dX^{\mu'}}{dt'} \neq \frac{dX^{\mu'}}{dt}$ so $V^{\mu'} = \frac{dX^{\mu'}}{dt'} \neq \frac{dX^{\mu'}}{dt} = \Lambda^{\mu'}_{\nu} V^{\nu}$

So $\boxed{V^{\mu'} \neq \Lambda^{\mu'}_{\nu} V^{\nu} \text{ so it's not a 4-vector}}$

- However, we CAN find an actual 4-vector velocity. Consider objects with mass and $v < c$ (no photons yet)

In this case $\boxed{ds^2 = c^2 d\tau^2 = \eta_{\mu\nu} dX^{\mu} dX^{\nu} > 0}$

timelike product

proper time.

Divide by $d\tau^2$ $\longrightarrow$ $\boxed{c^2 = \eta_{\mu\nu} \frac{dX^{\mu}}{d\tau} \frac{dX^{\nu}}{d\tau}}$

Call $\boxed{u^{\mu} = \frac{dX^{\mu}}{d\tau}}$ $\rightarrow$ world velocity

Chain rule $\boxed{u^{\mu'} = \frac{dX^{\mu'}}{d\tau} = \left(\frac{dX^{\mu'}}{dX^{\nu}}\right) \frac{dX^{\nu}}{d\tau} = \Lambda^{\mu'}_{\nu} u^{\mu}}$

invariant

$\hookrightarrow$ This shows that u^{μ} is a contravariant 4-vector under LT's.

Also see that $\boxed{u^{\mu} u_{\mu} = \eta_{\mu\nu} \frac{dX^{\mu}}{d\tau} \frac{dX^{\nu}}{d\tau} = c^2}$ (invariant inner product)

massive objects. invariant!

Can relate u^{μ} to V^{μ} by: $c^2 d\tau^2 = c^2 dt^2 - d\vec{x}^2$

So $\frac{d\tau^2}{dt^2} = 1 - \frac{1}{c^2}\frac{d\vec{x}^2}{dt^2} = 1 - \frac{1}{c^2}\left|\frac{d\vec{x}}{dt}\right|^2 = 1 - \frac{v^2}{c^2} = \frac{1}{\gamma^2}$

So $\boxed{\frac{dt}{d\tau} = \gamma}$ ← time dilation

So then $\boxed{u^\mu = \frac{dX^\mu}{d\tau} = \left(\frac{dt}{d\tau}\right)\frac{dX^\mu}{dt} = \gamma v^\mu}$

with $V^\mu = (c, \vec{v})$

still obeys $u^\mu u_\mu = c^2$

$\downarrow$

So $\boxed{u^\mu = (\gamma c, \gamma \vec{v}) = \gamma(c, \vec{v}) = \gamma V^\mu}$

In the object rest frame, $\vec{v} = 0$, $\gamma = 1$ ⇒ $\boxed{u^\mu = (c,0,0,0) \text{ in rest frame}}$

↳ object at rest moves at speed c in time direction.

And moving objects $\boxed{u^\mu u_\mu = c^2}$

Oct 2, 2012 **Recall**. Velocities "coordinate velocity" $V^\mu = \frac{dX^\mu}{dt} = (c, \vec{v})$

↑ NOT a 4 vector

"world velocity" → $u^\mu = \frac{\partial X^\mu}{\partial \tau} = (\gamma c, \gamma \vec{v})$ → for massive object

↳ is a 4-vector

also obeys that $u^\mu u_\mu = c^2$

and $\boxed{u^\mu = \gamma V^\mu = \gamma(c, \vec{v})}$

Now, momentum 4-momentum can be defined as $\boxed{p^\mu = m u^\mu}$

See that $p^\mu = \gamma m v^\mu = m\gamma(c, \vec{v}) = (m\gamma c, m\gamma \vec{v})$

or $p^\mu = \left(\frac{\gamma m c^2}{c}, m\gamma \vec{v}\right)$ But note $E = \gamma m c^2$, $\vec{p} = \gamma m \vec{v}$

So $\boxed{p^\mu = \left(\frac{E}{c}, \vec{p}\right)}$

$\downarrow \qquad \downarrow$

$^0 \qquad \vec{p}$

<u>Norm²</u> : p^μ has invariant $|p^\mu|^2$

$$\boxed{p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2}$$

<u>But also</u> $\boxed{p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^{\,2}} \Rightarrow \boxed{E^2 = c^2|p|^2 + m^2c^4}$

<u>But what about massless particles (light)?</u>

↳ massless photons $v=c$ always. → <u>No</u> proper-time $d\tau$ DNE

→ The def $u^\mu = \frac{dx^\mu}{d\tau}$ is undefined for light?

For light: $ds^2 = c^2dt^2 - |d\vec{x}|^2$ $\longrightarrow c^2$

$$= c^2dt^2\left(1 - 1/c^2\left|\frac{d\vec{x}}{dt}\right|^2\right)$$

$\boxed{ds^2 = 0}$ → for photons ⇒ photon travels on <u>null</u> trajectory (zero norm)

For light, can't use τ = proper time. But we can still parametrize their trajectory $X^\mu(\sigma)$ → some parameter

<u>Can define</u> $\boxed{u^\mu = \frac{\partial X^\mu}{\partial \sigma}}$

$$\boxed{\begin{array}{l} \Rightarrow u^\mu u_\mu = \eta_{\mu\nu}\frac{dX^\mu}{d\sigma}\frac{dX^\nu}{d\sigma} = \frac{\eta_{\mu\nu}\,dX^\mu dX^\nu}{d\sigma^2} = \frac{ds^2}{d\sigma^2} = 0 \\ \quad \to u^\mu \text{ is light-like (zero norm)} \end{array}}$$

<u>But</u> light has energy & momentum

$p^\mu = \left(\frac{E}{c}, \vec{p}\right) = \left(p^0, \vec{p}\right)$ <u>Recall</u>, $E = h\nu$, $|\vec{p}| = \frac{h}{\lambda}$

<u>Note</u> $\lambda\nu = c$

→ $E = c|\vec{p}|$

For light $\boxed{p^\mu p_\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = 0}$ $(E = c|\vec{p}|)$

↳ momentum is also light like vector (makes sense)

Also use wave vectors

$$\vec{p} = \hbar\vec{k} = \frac{h}{2\pi}\vec{k} \Rightarrow \boxed{|\vec{k}| = \frac{2\pi}{\lambda}}$$

Can define a 4-vector $\boxed{p^\mu = \hbar k^\mu}$

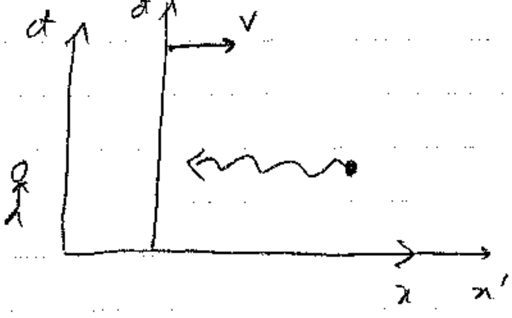
$$\boxed{k^\mu = (k^0, \vec{k})}$$

where $k^0 = \frac{p^0}{\hbar} = \frac{h}{\lambda}\cdot\frac{1}{\hbar} = \frac{2\pi}{\lambda} = |\vec{k}|$

⇒ Both $|k^0| = |\vec{k}| = \frac{2\pi}{\lambda}$

So $\boxed{k^\mu k_\mu = (k^0)^2 - (\vec{k})^2 = 0}$ (again, since $k^\mu \propto p^\mu$)

Example Find λ for light emitted from a source (where λ_0) that is receding



$$k^{\mu\prime} = (k^{0\prime}, \vec{k}') = \left(\frac{2\pi}{\lambda_0}, -\frac{2\pi}{\lambda_0}, 0, 0\right)$$

In stationary frame

$$k^\mu = (k^0, \vec{k}') = \left(\frac{2\pi}{\lambda}, -\frac{2\pi}{\lambda}, 0, 0\right)$$

But $\boxed{k^\mu = \Lambda^\mu{}_{\nu'} k^{\nu'}}$ (inverse LT)

↳ where $[\Lambda^\mu{}_{\nu'}] = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Let $\mu = 0$

$$k^0 = \Lambda^0{}_{\nu'} k^{\nu'}$$

$$\frac{2\pi}{\lambda} = \Lambda^0{}_{0'} k^{0'} + \Lambda^0{}_{1'} k^{1'} + \overbrace{\Lambda^0{}_{2'} k^{2'} + \Lambda^0{}_{3'} k^{3'}}^{0} = \gamma\frac{2\pi}{\lambda_0} + \gamma\beta\left(-\frac{2\pi}{\lambda_0}\right)$$

So $\frac{2\pi}{\lambda} = \gamma\frac{2\pi}{\lambda_0} - \gamma\beta\frac{2\pi}{\lambda_0} = \gamma\frac{2\pi}{\lambda_0}(1-\beta)$

So
$$\boxed{\frac{1}{\lambda} = \frac{\gamma}{\lambda_0}(1-\beta) = \frac{1}{\lambda_0}\sqrt{\frac{1-\beta}{1+\beta}}}$$

So $\boxed{\lambda = \lambda_0\sqrt{\frac{1+\beta}{1-\beta}}}$ (red shifted)

For light emitted from a source moving toward, $v \to -v$

$\boxed{\lambda = \lambda_0\sqrt{\frac{1-\beta}{1+\beta}}}$ (blue shifted)

Note: These are Doppler shifts due to relative motion.
Later we'll look at gravitational spectral shifts + cosmological redshift

Can define a 4-force vector f^μ (back to dealing w/ massive obj)

$\boxed{f^\mu = \frac{dp^\mu}{d\tau}}$ (only for massive objects)

where $p^\mu = mu^\mu = m\frac{dx^\mu}{d\tau}$

Get $\boxed{f^\mu = m\frac{d^2x^\mu}{d\tau^2}}$ (relativistic 2nd law)

with

$p^\mu = \left(\frac{E}{c}, \vec{p}\right)$ + chain rule $\frac{dp^\mu}{d\tau} = \frac{dt}{d\tau}\frac{dp^\mu}{dt}$

we showed $\frac{dt}{d\tau} = \gamma$

$\Rightarrow \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt} \Rightarrow \gamma\left(\frac{1}{c}\frac{dE}{dt}, \frac{d\vec{p}}{dt}\right)$ $\quad \vec{F}$ constant

power $\frac{dE}{dt} = \frac{d}{dt}(\vec{F}\cdot\vec{r}) = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v}$

So $\boxed{f^\mu = \gamma\left(\frac{1}{c}\vec{F}\cdot\vec{v}\ , \vec{F}\right)}$ for a constant force $\vec{F}$

Oct 2, 2018

Recall $f^\mu = \frac{\partial p^\mu}{\partial \tau} = m\frac{\partial^2 x^\mu}{\partial \tau^2}$ where $p^\mu = \left(\frac{E}{c}, \vec{p}\right)$
and for constant force $\frac{dE}{dt} = \vec{F}\cdot\vec{v}$ $\Big\}$

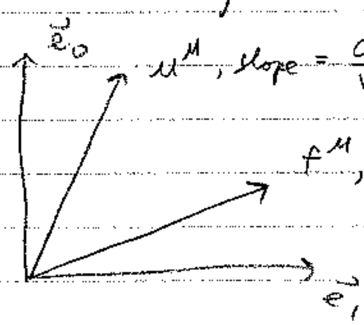
$\Rightarrow f^\mu = \gamma\left(\frac{1}{c}\vec{F}\cdot\vec{v}\ , \vec{F}\right) \rightarrow \boxed{u^\mu f_\mu = 0} \rightarrow$ orthogonal in 4D spacetime

$\boxed{\text{Can look in 1D}}$

$$f^\mu = \left(\frac{\gamma v}{c}F\ , \gamma F, 0, 0\right)$$

and $u^\mu = \left(\gamma c, \gamma v, 0, 0\right)$

So plot these in spacetime



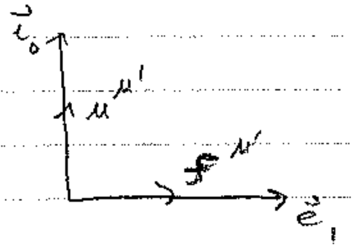
} don't look orthog.

Well, we can also look in instantaneous rest frame:

$\rightarrow v = 0,\ \gamma = 1$

$\rightarrow$

$f^{\mu'} = (0, F, 0, 0)$, and $u^{\mu'} = (c, 0, 0, 0)$



} look orthogonal!

What we have is an inner product $u^\mu f_\mu = 0$. It's a scalar and therefore same in all frames $\rightarrow$ only takes one frame for them to be orthogonal $\rightarrow u^\mu f_\mu = 0$ ∀ frames.

—//—

Relativistic Electromagnetism

↳ We previously found Maxwell's Eqns in differential form

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} & \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial \vec{E}}{\partial t} \end{array}}$$

$\vec{E} = (E^1, E^2, E^3) = E^i$, likewise $\vec{B} = (B^1, B^2, B^3) = B^j$

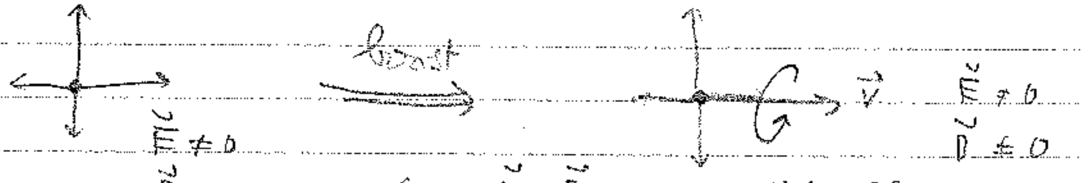
charge density → ρ & current density → $\vec{J}$

N.B. $q = \int \rho\, dV$, $I = \int \vec{J}\cdot d\vec{A}$, and $\frac{1}{\mu_0\epsilon_0} = c^2$

Note $\vec{E}$, $\vec{B}$ are 3D. What are they in 4D?

↳ Together have 6 components which mix under Lorentz transform

Ex Boost a rest charge into moving frame ⇒ from $\vec{E}$ to $\vec{E} + \vec{B}$



⇒ Says $\vec{E}$ & $\vec{B}$ join relativistically

Find that $\vec{E}$ & $\vec{B}$ combine to give tensor

Define electromagnetic field strength $F^{\mu\nu}$

$$\boxed{\left[F^{\mu\nu}\right] = \begin{pmatrix} 0 & E^1/c & E^2/c & E^3/c \\ -E^1/c & 0 & B^3 & -B^2 \\ -E^2/c & -B^3 & 0 & B^1 \\ -E^3/c & B^2 & -B^1 & 0 \end{pmatrix}}$$

Note $F^{\mu\nu} = -F^{\nu\mu}$ ⇒ has only 6 components

$F^{\mu\nu} = 0$ if $\mu = \nu$

<u>Can also define</u> $\boxed{F_{\mu\nu} = \eta_{\mu\alpha}\eta_{\nu\beta}F^{\alpha\beta}}$

As matrices $\boxed{[F_{\mu\nu}] = [\eta_{\mu\alpha}][F^{\alpha\beta}][\eta_{\beta\beta}]}$

$$= \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & B^3 & -B^2 \\ E^2/c & -B^3 & 0 & B^1 \\ E^3/c & B^2 & -B^1 & 0 \end{pmatrix}$$

Now, can form vector out of ρ and $\vec{J}$

$j^\mu = (\rho c, \vec{J})$ define the 4-vector current density

In terms of these, Maxwell's eqn become

$$\boxed{\begin{aligned} &\partial_\nu F^{\mu\nu} = \mu_0 j^\mu \\ &\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \end{aligned}}$$

e.g look at $\partial_\nu F^{\mu\nu} = \mu_0 j^\mu$

<u>Look at</u> $\boxed{\mu = 0}$ $\rightarrow \partial_\nu F^{0\nu} = \mu_0 j^0 = \mu_0 \rho c$

$$\rightarrow \underbrace{\partial_0 F^{00}}_{0} + \underbrace{\partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03}}_{} = \mu_0 c \rho$$

$$0 \qquad \frac{1}{c}\underbrace{\partial_i}_{} E^i = \rho c \mu_0$$

$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho c^2 \mu_0 = \frac{\rho}{\varepsilon_0}}$$

Next, let $\mu = k$, $k = \{1, 2, 3\}$

$F^{k0} = -E^k/c$

<u>So</u> $\partial_\nu F^{k\nu} = \mu_0 j^k = \mu_0 J^k = \partial_0 F^{k0} + \partial_i F^{ki}$, $\partial_0 = \frac{1}{c}\frac{\partial}{\partial t}$

So $\partial_0 F^{k0} = \frac{-1}{c^2}\frac{\partial E^k}{\partial t}$

For $\partial_i F^{ki}$ let $k=1$

$$\hookrightarrow \partial_i F^{1i} = \underbrace{\partial_1 F^{11}}_{0} + \partial_2 F^{12} + \partial_3 F^{13} = \partial_2 B^3 + \partial_3(-B^2)$$

$$= (\vec{\nabla}\times\vec{B})^1$$

Similarly, $k=1 \Rightarrow \partial_i F^{ki} = (\vec{\nabla}\times\vec{B})^2$

$k=2 \Rightarrow \partial_i F^{ki} = (\vec{\nabla}\times\vec{B})^3$ So $\partial_i F^{ki} = (\vec{\nabla}\times\vec{B})^k$

So $\frac{-1}{c^2}\frac{\partial E^k}{\partial t} + (\vec{\nabla}\times\vec{B})^k = \mu_0 J^k$

So $\boxed{(\vec{\nabla}\times\vec{B}) = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}}$ (Ampere-Maxwell)

Similarly, can look at

$$\left.\begin{array}{l}\partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0 \\ \text{Can show that for various values of } \sigma, \nu, \mu\end{array}\right\} \Rightarrow \left\{\begin{array}{l}\vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\end{array}\right.$$

e.g. $(\mu=0,\ \nu=1,\ \sigma=2) \Rightarrow (\vec{\nabla}\times\vec{E})^3 = -\left(\frac{\partial\vec{B}}{\partial t}\right)^3$

—//—

To summarize, in SR, all physical properties are some sort of tensors with scalar = m, τ, ds^2, c

Vectors → u^μ, p^μ, f^μ. $f^\mu = \frac{\partial p^\mu}{\partial\tau} = m\frac{\partial^2 x^\mu}{\partial\tau^2}$

Tensors $\eta_{\mu\nu}$, $F^{\mu\nu}$ (E·m)

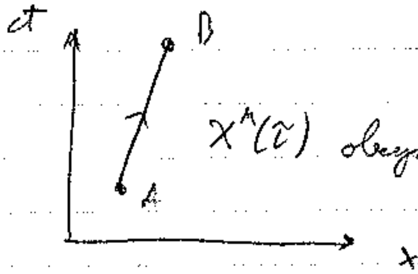
$\boxed{\text{All transforms in definite ways under Lorentz Transformation}}$

—//—

Geodesics In 3D, flat space, can think of these as shortest distance between 2 points → straight line → path of free particle. Free particles follow geodesics

But in 4D spacetime, Minkowski. Now, free particle, ⇒ $f^\mu = 0$

So $\frac{\partial^2 X^\mu}{\partial \tau^2} = 0$ has a solution $X^\mu(\tau)$ that is a straight line in spacetime



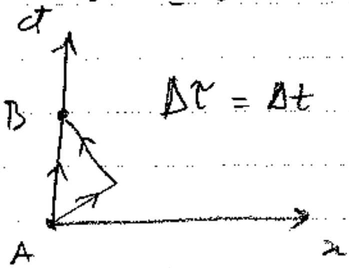
$X^\mu(\tau)$ obeying $\frac{\partial^2 X^\mu}{\partial \tau^2} = 0$ gives a straight line ⇒ can call this a geodesic

Geodesics are solutions of $\frac{\partial^2 X^\mu}{\partial \tau^2} = 0$

BUT geodesics in Minkowski spacetime are not the shortest 'distance'

We calc. distance using $ds^2 = \eta_{\mu\nu}\, dX^\mu dX^\nu$.
Moving massive particles $ds^2 = c^2 d\tau^2 > 0$ → timelike

Consider A → B



$\Delta\tau = \Delta t$ (particle at rest in space)

For moving path $c\Delta\tau' = 2\sqrt{\left(\frac{1}{2}c\Delta t\right)^2 - (\Delta x)^2}$

Find that $\Delta\tau' < \Delta\tau$ → $\boxed{\text{geodesics has maximal propertime}}$

↑ not a geodesics (time slows in moving frame)

So we won't think in terms of shortest distance. We'll use that

$$\boxed{\text{geodesic} \Rightarrow \text{path of free particle} \quad \frac{\partial^2 X^\mu}{\partial \tau^2} = 0}$$

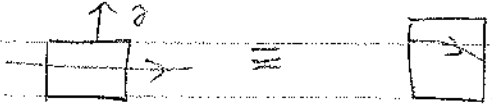
V. CURVED SPACES

Oct 5, 2018

↳ Recall: Equivalence principle (EP) leads us towards the idea of curved spacetime

Ep.

For light,



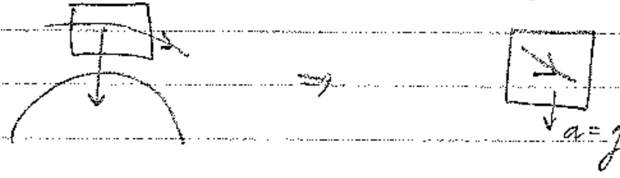
In GR gravity is not a force. Instead, massive objects curve or warp spacetime around them. Light travels as a free particle along a "geodesic" through curved spacetime.

Q: how to find equation for geodesic?

Two ways to go

One uses that we know the geodesic eq in an inertial frame $\Rightarrow \frac{\partial^2 x^\mu}{\partial \tau^2} = 0$

EP says for an object in a gravitational field..



→ get straight line in local frame.

The geodesic in the freely falling frame with $X^{\mu'}$ coords obey

$$\frac{d^2 X^{\mu'}}{d\tau^2} = 0$$

Coord. transform μ' back to μ. Get $\boxed{\frac{\partial^2 X^\mu}{\partial \tau^2} + \Gamma^\mu_{\nu\sigma} \frac{dX^\nu}{d\tau}\frac{dX^\sigma}{d\tau} = 0}$

geodesic eqn

$\Gamma^\mu_{\nu\sigma}$ = Christoffel symbol or affine connection

Also transform

↳ $ds^2 = \eta_{\mu'\nu'} dX^{\mu'} dX^{\nu'} = g_{\mu\nu} dX^\mu dX^\nu$

(curved space)

We could also find $\Gamma^{\mu}_{\nu\sigma}$ in terms of $g_{\mu\nu}$.

→ But we won't take this route!

Instead, we'll see how to describe curved spaces & spacetimes directly. We'll find the same geodesic equation

$$\frac{d^2X^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dX^\nu}{d\tau}\frac{dX^\sigma}{d\tau} = 0$$

We'll see how $g_{\mu\nu}$, $\Gamma^\mu_{\nu\sigma}$, and the Riemann curvature tensor $R^\lambda_{\mu\nu\sigma}$ are related.

Then we'll look at the Einstein eqn that'll let us solve for $g_{\mu\nu}$ for a given distribution of matter (mass/energy)

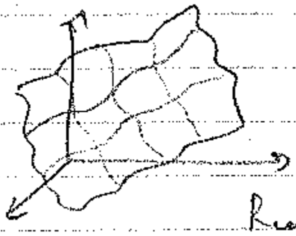
* * *

Curved Spaces

According to GR we live in a curved 4-D spacetime → hard to visualize. To start off simpler, can look at 2D spaces that we can embed in 3D.

Curved 2D spaces

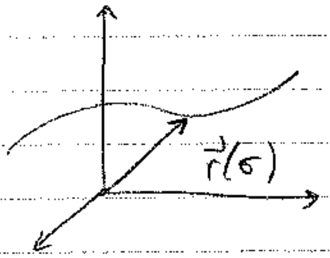
→ can embed in flat 3D space.



← can be closed / open

← can't flatten it if it's curved.

<u>Recall</u> that 1D curves thru 3D space is a set of parametrized points $\sigma, t, \ldots$



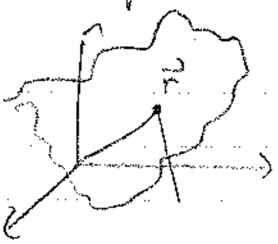
$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$x = x(\sigma)$

$y = y(\sigma)$

$z = z(\sigma)$

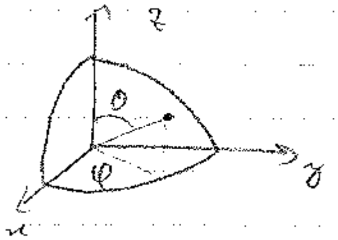
In a similar way, can parametrised 2D surface in 3D space using
2 params. → (u,v)



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$x = x(u,v)$$
$$y = y(u,v)$$
$$z = z(u,v)$$

e.g. Sphere of radius a.



radius $= a$ $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

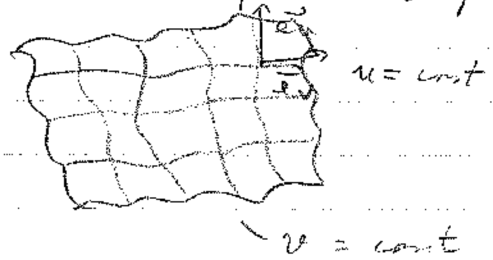
$x = a\sin\theta\cos\varphi$ $(u,v) = (\theta,\varphi)$
$y = a\sin\theta\sin\varphi$
$z = a\cos\theta$

Can also think about
staying entirely within the 2D surface, without knowing about the 3rd direction

In this case $(u,v) \rightarrow$ become coordinates of the curved space.

Note → can't put Cartesian coords over the surface of the whole space

We can then generate tangent vectors...



With embedding $\vec{e}_u = \frac{d\vec{r}}{du}$ $\vec{e}_v = \frac{d\vec{r}}{dv}$

→ these are tangent to the surface. They don't live in the space!
⇒ still give the directions along the curve

→ vector lives in tangent space T_p at each point P.

Look at a little displacement $ds^2 = d\vec{r} \cdot d\vec{r}$

$r = r(u,v) \rightarrow dr = \frac{\partial \vec{r}}{\partial u} \partial u + \frac{\partial \vec{r}}{\partial v} \partial v = \vec{e}_u \partial u + \vec{e}_v dv$

Call $u^A = (u^1, u^2) = (u,v)$ s, $A = 1, 2$ } $\boxed{d\vec{r} = \vec{e}_A du^A}$

Then $ds^2 = d\vec{r} \cdot d\vec{r} = (\vec{e}_A du^A) \cdot (\vec{e}_B du^B) = \vec{e}_A \cdot \vec{e}_B \, du^A du^B$

So $\boxed{ds^2 = g_{AB} \, du^A du^B}$ $[g_{AB}] \to 2\times2$ matrix in 2D

Just as before but in 2D and with a curved space ... Can then calculate the length of the curve in curved 2D space.

Have a line in the surface ⇒ must param the coords

$u = u(\sigma)$, $v = v(\sigma)$) gives the line

Length of curve $L = \int ds$

where $ds^2 = g_{AB} du^A du^B = g_{AB} \frac{du^A(\sigma)}{d\sigma} \frac{du^B(\sigma)}{d\sigma} d\sigma^2$ }

Call $\dot{u}^A(\sigma) = \frac{du^A(\sigma)}{d\sigma}$

$\Rightarrow$ $ds = \sqrt{g_{AB} u^A(\sigma) u^B(\sigma)} \, d\sigma$ and so $\boxed{L = \int_a^b \sqrt{g_{AB} \dot{u}^A(\sigma) \dot{u}^B(\sigma)} \, d\sigma}$

This is same as before, but now in curved space.

What about the dual basis $\vec{e}^A$? ⇒ not well-defined as $\vec{e}^A = \vec{\nabla} u^A$ as before. Why? with 3 coords in 3D ⇒ $\vec{\nabla} u$ is ⊥ to surface $u = \text{constant}$

But here $u = \text{const}$ is a line ⇒ there are many normals to $u = \text{const}$. We can't use the gradient of u^A.

Instead, what we do is first, define $\vec{e}_A$ as tangent vectors along u^A then find $g_{AB} = \vec{e}_A \cdot \vec{e}_B$. Then find g^{AB} (the inverse)

$(g_{AB} g^{BC} = \delta_A^C)$. Then use g^{AB} to raise index of $\vec{e}_A$

↳ $\boxed{\vec{e}^A = g^{AB} \vec{e}_B}$ → then we'll have both sets ...

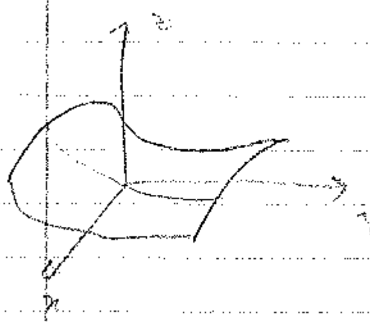
Oct 8, 2018

$\boxed{\text{Curved spaces}}$ $(u, v) \rightarrow$ coords $\rightarrow u^A$ $A = 1, 2$

$\vec{e}_A$ Tangents and $g_{AB} = \vec{e}_A \cdot \vec{e}_B$

Dual basis $\vec{e}^A = g^{AB} \vec{e}_B$

$\boxed{\text{Ex}}$ → Consider a saddle embedded in 3D flat space



Use paraboloidal coords with $w =$ constant

$$x = u + v$$
$$y = u + v \qquad \vec{r} = (u+v,\ u-v,\ 2uv)$$
$$z = 2uv$$

$$\vec{e}_1 = \vec{e}_u = \frac{\partial \vec{r}}{\partial u} = (1, 1, 2v) \quad \vec{e}_v = (1, -1, 2u)$$

So $$[g_{AB}] = [\vec{e}_A \cdot \vec{e}_B] = \begin{pmatrix} 2+4v^2 & 4uv \\ 4uv & 2+4u^2 \end{pmatrix}$$

So $$[g^{AB}]^{-1} = [g_{AB}]^{-1} = \begin{pmatrix} 1+2u^2 & -2uv \\ -2uv & 1+2v^2 \end{pmatrix} \cdot \frac{1}{2(1+2u^2+2v^2)}$$

So $\vec{e}^A = g^{AB} \vec{e}_B = \ ?$ (See p. 37 in book) (but easy to compute)

{ Ultimately, we won't use basis vecs much going forward. The important info is contained in metric }

Ex $\boxed{ds^2 = g_{AB}\, du^A du^B}$

↑ knowing this enough!

E.g. flat 2D space $g_{AB} = \delta^A_B \rightarrow \boxed{ds^2 = dx^2 + dy^2}$

In GR, we'll use the Einstein eqn ~~to~~ find $g_{\mu\nu}$

Oct 9, 2018 **Manifolds** → **An arbitrary curved N-D space is called a manifold**

↳ Assume we know the metric. Can write coords

$$X^a = (X^1, X^2, \dots X^N)$$

with more than one coor. system. We assume differentiable functions

$$X^{a'} = X^{a'}(X^b)$$, and that these are invertible

$$\Rightarrow X^a = X^a(X^{b'})$$

↳ Call M a differentiable manifold with defined Jacobian

$$\hookrightarrow \left.\begin{aligned} X^{a'}_b &= \frac{\partial X^{a'}}{\partial X^b} \\ X^a_{b'} &= \frac{\partial X^a}{\partial X^{b'}} \end{aligned}\right\} \Rightarrow \begin{aligned} X^{a'}_b X^b_{c'} &= \delta^a_c \\ X^a_{b'} X^{b'}_c &= \delta^a_c \end{aligned}$$

We've seen flat Euclidean space $\rightarrow \left\{ \begin{aligned} U^a &\leftarrow X^a \\ O^{i'}_j &\leftarrow X^{a'}_b \end{aligned} \right.$

and flat 4D spacetime $\rightarrow \left\{ \begin{aligned} X^\mu &\leftarrow X^a \\ \Lambda^{\mu'}_\nu &\leftarrow X^{a'}_b \end{aligned} \right.$

We define vectors, tensors, scalars by how they transform

$$\lambda^{a'} = X^{a'}_b \lambda^b \longrightarrow \text{contravariant vector}$$

$$\mu_{a'} = X^b_{a'} \mu_b \longrightarrow \text{covariant vector}$$

$$\tau^{a'b'}_{c'd'} = X^{a'}_e X^{b'}_f X^g_{c'} X^h_{d'} \tau^{ef}_{gh} \longleftarrow \text{tensor}$$

Metric lowers/raises $\lambda_a = g_{ab} \lambda^b$ + has an inverse

$$g^{ab} g_{bc} = \delta^a_c$$

In general, the metric need not be positive definite

$$ds^2 = g_{ab}dx^a dx^b \rightarrow \text{can be } (+, 0, -)$$

Signature of g_{ab} = (# positive) − (# negative) down the diagonal

↳ $\eta_{\mu\nu}$ has signature -2. $(\text{sig}\,(g_{ab}) = 1-3 = -2)$

Note $\boxed{\text{All metrics in GR have signature} = -2}$ (local SR)

Two classes of manifold: Riemannian manifolds (positive def. metric)

{ pseudo-Riemannian manifold

↳ can have neg inner products

Note Spacetime ⇒ pseudo Riemannian manifold

Recall there are 4 ways to compute inner products

$$\lambda\cdot\mu = \lambda^i\mu_i = \lambda_i\mu^i = g_{ij}\lambda^i\mu^j = g^{ij}\lambda_i\mu_j$$

These are scalars under general coord. transforms.

$$\lambda\cdot\mu = \lambda^a\mu_a = \lambda^{a'}\mu_{a'}$$

To define lengths & distances as real numbers, need abs. values

Distance $ds = \sqrt{|g_{ab}\,dx^a dx^b|}$

Length of curve $L = \int_a^b ds = \int_a^b \sqrt{|g_{ab}\,dx^a dx^b|}$

Length of vector $|\lambda| = \sqrt{|\lambda^a\lambda_a|}$ → can still be null

For non-null vectors, we can define "angle" between them

$$\cos\theta = \frac{\lambda\cdot\mu}{|\lambda||\mu|} \rightarrow \text{have to be non null to avoid div. by } 0$$

$$= \frac{g_{ab}\lambda^a\mu^b}{|\lambda||\mu|}$$

↳ works well for positive def. metrics. But become weird for spacetime.

Ex Spacelike λ → $\theta = 180°$ between it and itself
Can also get $\cos\theta > 1$ → don't make sense

$\boxed{\text{Call vectors obeying}}$ → $\lambda\cdot\mu = 0$ orthogonal
↳ there exists a frame where they're perpendicular

$\boxed{\text{Combining Tensors}}$ Given that $\lambda^a, \mu_b, \tau^{ab}{}_c$ are tensors.

We can show → $\boxed{\text{adding tensors of the same type gives a tensor}}$

Ex $\zeta^{ab}{}_c = \tau^{ab}{}_c + \sigma^{ab}{}_c$ is a tensor if τ and σ are tensors

Proof $\zeta^{a'b'}{}_{c'} = \tau^{a'b'}{}_{c'} + \sigma^{a'b'}{}_{c'}$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \tau^{de}{}_f + X^{a'}_d X^{b'}_e X^f_{c'} \sigma^{de}{}_f$$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \left(\tau^{de}{}_f + \sigma^{de}{}_f\right)$$

$$= X^{a'}_d X^{b'}_e X^f_{c'} \zeta^{de}{}_f$$

$$\Rightarrow \zeta^{ab}{}_c \text{ is a tensor}$$

Multiplying a tensor by a scalar gives a tensor

↳ Suppose $\sigma^a{}_b = \alpha \tau^a{}_b$

Proof $\sigma^{a'}{}_{b'} = \alpha \tau^{a'}{}_{b'} = \alpha X^{a'}{}_c X^d{}_{b'} \tau^c{}_d$

$= X^{a'}{}_c X^d{}_{b'} \alpha \tau^c{}_d = X^{a'}{}_c X^d{}_{b'} \sigma^c{}_d$

So $\sigma^a{}_b$ is a tensor.

Multiplying tensors gives tensors

Suppose $\sigma^{ab}{}_c = \lambda^a \tau^b{}_c$

Proof $\sigma^{a'b'}{}_{c'} = \lambda^{a'} \tau^{b'}{}_{c'} = \left(X^{a'}{}_d \lambda^d\right) X^{b'}{}_e X^f{}_{c'} \tau^e{}_f$

$= X^{a'}{}_d X^{b'}{}_e X^f{}_{c'} \lambda^d \tau^e{}_f$

$= X^{a'}{}_d X^{b'}{}_e X^f{}_{c'} \sigma^{de}{}_f$ So $\sigma^{ab}{}_c$ tensor

Contracting a tensor of type (r,s) gives a tensor of type $(r-1,s-1)$

Suppose $\tau^{ab}{}_{cd}$ is a $(2,2)$ tensor

Call $\sigma^a{}_b = \tau^{ac}{}_{cb}$ is this a one-one $(1,1)$ tensor?

Proof $\sigma^{a'}{}_{b'} = \tau^{a'c'}{}_{c'b'} = X^{a'}{}_d X^{c'}{}_e X^f{}_{c'} X^g{}_{b'} \tau^{de}{}_{fg}$

$= X^{a'}{}_d X^g{}_{b'} \tau^{de}{}_{fg} \delta^e{}_f$

$= X^{a'}{}_d X^g{}_{b'} \tau^{de}{}_{eg}$

$\sigma^{a'}{}_{b'} = X^{a'}{}_d X^g{}_{b'} \sigma^d{}_g$ So $\sigma^a{}_b = \tau^{ac}{}_{cb}$ is a $(1,1)$ tensor.

We've used this already! $\lambda_a = g_{ab}\lambda^b$ → gives a vector

So, as a consequence, $\boxed{\sigma^{ab}{}_{c} = \tau^{abe}\mu_e g_{cf}\lambda^f \text{ is a tensor}}$

Oct 10, 2018

Recall Combining tensors → adding, multiplying & contracting tensors gives new tensors

e.g. $\tau^{ab}{}_{c}\lambda^c\mu_b$ = type $(1,0)$ (vector)

$\boxed{\text{Dividing : Quotient theorem}}$

$\boxed{\text{Suppose } \tau^a{}_{bc}\lambda^c \text{ transforms as a tensor } \forall\ \lambda^c\text{. Then the quotient theorem says } \tau^a{}_{bc} \text{ is a tensor}}$

↳ Proof $\tau^{a'}{}_{b'c'}\lambda^{c'} = X^{a'}_{d}X^{e}_{b'}\tau^{d}{}_{ef}\lambda^f$

We also know $\lambda^{c'} = X^{c'}_{f}\lambda^f$

So $\tau^{a'}{}_{b'c'}X^{c'}_{f}\lambda^f - X^{a'}_{d}X^{e}_{b'}\tau^{d}{}_{ef}\lambda^f = 0$ (true $\forall\ \lambda^f$)

So $\tau^{a'}{}_{b'c'}X^{c'}_{f} - X^{a'}_{d}X^{e}_{b'}\tau^{d}{}_{ef} = 0$

So $\tau^{a'}{}_{b'c'}X^{c'}_{f} = X^{a'}_{d}X^{e}_{b'}\tau^{d}{}_{ef}$

So $\tau^{a'}{}_{b'c'}\delta^{c'}_{g'} = X^{a'}_{d}X^{e}_{b'}X^{f}_{g'}\tau^{d}{}_{ef}$

So $\boxed{\tau^{a'}{}_{b'g'} = X^{a'}_{d}X^{e}_{b'}X^{f}_{g'}\tau^{d}{}_{ef}} \to \tau^a{}_{bc}$ tensor

$\boxed{\text{Special Tensors}}$ $\boxed{\text{Symmetric}}$ if $\boxed{\tau^{ab} = \tau^{ba}}$ (metric)

This is then true ∀ coord frames.

① $\Rightarrow \tau^{a'b'} = \tau^{b'a'}$

(will show this in 1.8.2)

② Anti symmetric tensor $\boxed{T^{ab} = -T^{ba}}$

→ also true for all frames

③ Kronecker delta → coord. independent

$$\delta^a_b = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a \neq b \end{cases} \quad \text{(type (1,1) tensor)}$$

$$\delta^{a'}_{f'} = X^{a'}_{e} X^{f}_{b'} \delta^{e}_{f} = \underbrace{X^{a'}_{e} X^{e}_{f'}} = \delta^{a'}_{f'}$$

$$\text{because } X^{a'}_{c} X^{c}_{b'} = \frac{\partial x^{a'}}{\partial x^{c}} \frac{\partial x^{c}}{\partial x^{b'}} \xrightarrow{\text{inverses}} = \frac{\partial x^{a'}}{\partial x^{b'}} = \begin{cases} 1 & \text{if } a=b \\ 0 & \text{if } a \neq b \end{cases}$$

④ For most tensors the order of indices matter

Ex $T^{a\ c}_{\ b} = g_{bd} T^{adc}$

but $T^{a\ c}_{\ b} \neq g_{bd} T^{acd} = T^{ac}_{\ \ b}$

Don't write T^{ac}_{b} unless we know order doesn't matter

VI. GRAVITATION & CURVATURE

In GR gravity is not a force → mass & energy cause spacetime to be curved.

"Free particles" is moving with no forces (other than gravity)
↳ follow geodesics

We need to understand
→ curvature (how to tell a space is curved?)
→ geodesic (what is the eqn for geodesic?)
→ motion in curved space: how do vectors behave? (parallel transport)
→ laws of physics e.g. $f^\mu = \frac{dp^\mu}{d\tau}$ in curved spacetime

(Chap 2. assuming we know the metric)

(Chap 3 solve for metric)

Newtonian limit

$F = \frac{GMm}{r^2}$

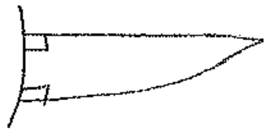
absolute, covariant derivatives

→ limit back to gravity as a force?

CURVATURE Imagine ants on globe. How can they tell it's a curved space? How do the ants walk "straight".

⇒ left step must = right step to walk straight (without turning).

⇒ Start 2 ants walking parallel & straight

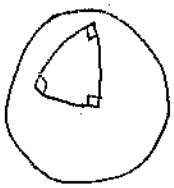


(1). Parallel lines cross ⇒ space is non Euclidean.

(2) These "straight" lines are geodesic.

On a sphere, the equator, longitudes, and great circles are all geodesics and hence "straight lines". Latitude lines are not geodesics

Another test is make a triangle of 3 straight lines



Sum of the angles = 270°, not 180°,

→ says space is curved.

→ Bugs can tell if a space is curved!

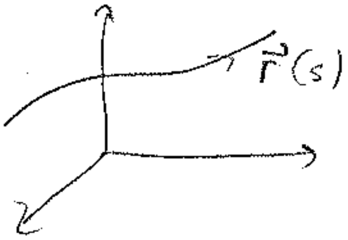
Geodesic equation Suppose we're in space or spacetime and we know what the metric is. How do we find a geodesic? → Follow a "straight" line!

Flat 3D space In Cartesian coord. a straight line obeys $\frac{\partial^2 \tilde{r}}{\partial t^2} = \tilde{0}$

(→ no force)

Suppose we use curvilinear coords.

↳ what is the eqn of a straight line? → arc length param



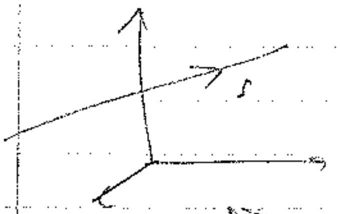
s = arc length as parameter

↳ $|\dot{\vec{r}}(s)| = \left|\frac{d\tilde{r}}{ds}\right| = 1$ fixed length

let $\vec{t} = \frac{d\tilde{r}}{ds}$ (tangent)

$$\vec{\lambda} = \frac{d\vec{r}}{ds} = \frac{\partial \vec{r}}{\partial u^i}\frac{\partial u^i}{\partial s} = \frac{du^i}{ds}\cdot\vec{e}_i = \lambda^i\vec{e}_i \quad \text{So} \quad \boxed{\lambda^i = \frac{\partial u^i}{\partial s} = \dot{u}^i(s)}$$

components of tangent → vector in curvilinear coords



Since $\vec{\lambda}$ tangent, its direction does not change along a straight line. Also $|\vec{\lambda}(s)| = 1$

→ $\boxed{\vec{\lambda} \text{ has both fixed direction + magnitude along straight line}}$

↳ "straightness" → derivative of tangent vector w.r.t arclength = 0

$\frac{d\vec{\lambda}}{ds} = 0$ → Tangent vector does not change (constant along a straight line)

Oct 17 2018

<u>Geodesics</u> → Path followed by a free particle → straight line in flat 3D space obeys $\frac{\partial^2 x}{\partial t^2} = 0$. What about in curvilinear coords?

Use s as parameter $\vec{\lambda} = \frac{d\vec{r}}{ds}$ → tangent vector (fixed magnitude)

<u>Condition of straightness</u> : $\frac{d\vec{\lambda}}{ds} = 0 \Rightarrow \boxed{\frac{d}{ds}(\lambda^i\vec{e}_i) = 0}$ ↳ $\lambda^i = \frac{\partial u^i}{\partial s}$

So $\boxed{\dot{\lambda}^i\vec{e}_i + \lambda^i\dot{\vec{e}}_i = 0}$

In Cartesian $\{\vec{e}_i\} = \hat{i}, \hat{j}, \hat{k}$ constant → $\dot{\vec{e}}_i = 0$. <u>Get</u> $\dot{\lambda}^i = 0$ for straight line

but since $\lambda^i = \dot{u}^i = \dot{x}^i \Rightarrow \boxed{\frac{\partial^2 x^i}{\partial s^2} = 0 \text{ for a straight line in Cartesian coords}}$

<u>Note</u> $\frac{\partial^2 x^i}{\partial s^2} = 0 = \frac{\partial^2 x^i}{\partial t^2}$ as long as $s \propto t$, but <u>NOT</u> equivalent if $s \not\propto t$ → has acceleration.

<u>But</u> if coords are not Cartesian → $\frac{d}{ds}(\lambda^i\vec{e}_i)$ has <u>2 terms</u>!

$$\dot\lambda^i \breve{e}_i + \lambda^i \dot{\breve{e}}_i = 0 \iff \boxed{\frac{\partial \lambda^i}{\partial s}\breve{e}_i + \lambda^i \frac{\partial \breve{e}_i}{\partial s} = 0}$$

<u>where</u> $\frac{\partial \breve{e}_i}{\partial s} = \frac{\partial \breve{e}_i}{\partial u^j}\frac{\partial u^j}{\partial s} \neq 0$ in general

<u>Use</u> $\frac{\partial}{\partial u^j} = \partial_j \Rightarrow \boxed{\frac{\partial \breve{e}_i}{\partial s} = \left(\partial_j \breve{e}_i\right)\dot{u}^j}$

The derivative

↳ $\partial_j \breve{e}_i$ are vectors. We can expand them in terms of basis set

<u>Call</u> $\boxed{\partial_j \breve{e}_i = \Gamma^k_{ij}\breve{e}_k}$ → k^{th} component of the u^jth derivative of $\breve{e}_i$, called "affine connection" or "Christoffel symbol"

<u>Note</u> $\boxed{\Gamma^k_{ij} \text{ is } \underline{\text{not}} \text{ a tensor} \to \text{they're a connection}}$

<u>With this</u> $\dot{\breve{e}}_i = (\partial_j \breve{e}_i)\dot{u}^j = \Gamma^k_{ij}\breve{e}_k \dot{u}^j$

<u>So, straightness condition is</u>

k → i
i → j
j → k

$$\frac{d\vec{\lambda}}{ds} = \dot\lambda^i \breve{e}_i + \lambda^i \Gamma^k_{ij}\breve{e}_k \dot{u}^j = 0$$

$$= \dot\lambda^i \breve{e}_i + \lambda^j \Gamma^i_{jk}\breve{e}_i \dot{u}^k = 0$$

$$= \left(\dot\lambda^i + \Gamma^i_{jk}\lambda^j \dot{u}^k\right)\breve{e}_i = 0$$

or $\dot\lambda^i + \Gamma^i_{jk}\lambda^j \dot{u}^k = 0$ <u>Put</u> $\lambda^i = \dot{u}^i = \frac{du^i}{ds}$

$\dot\lambda^i = \frac{d^2 u^i}{ds^2}$

$$\Rightarrow \boxed{\frac{d^2 u^i}{ds^2} + \Gamma^i_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0}$$

↳ gives the eq. of a straight line in flat 3D space.

<u>Note</u> Γ^k_{ij} has $3 \times 3 \times 3 = 27$ coefficients. → Want simpler relation

<u>Note</u> $\partial_j \breve{e}_i = \Gamma^k_{ij}\breve{e}_k \Rightarrow$ dot with $\breve{e}^l$

So $(\partial_j \vec{e}_i)\,\vec{e}^{\,l} = \Gamma^k_{ij}\,\vec{e}_k \cdot \vec{e}^{\,l} = \Gamma^k_{ij}\,\delta^l_k$

So $\boxed{\vec{e}^{\,l}(\partial_j \vec{e}_i) = \Gamma^l_{ij}}$

But, note $\partial_j \vec{e}_i = \frac{\partial}{u^j}\frac{\partial \vec{r}}{\partial u^i} = \frac{\partial}{\partial u^i}\frac{\partial \vec{r}}{\partial u^j} = \partial_i \vec{e}_j$

So $\boxed{\Gamma^l_{ij} = \Gamma^l_{ji}}$ (symmetric) → 18 independent comps

Next, want to find relation for connection in terms of the metric.

Consider $\partial_k g_{ij} = \partial_k(\vec{e}_i \cdot \vec{e}_j) = \vec{e}_j\,\partial_k \vec{e}_i + \vec{e}_i\,\partial_k \vec{e}_j$

$= \vec{e}_j\,\Gamma^m_{ik}\,\vec{e}_m + \vec{e}_i\,\Gamma^m_{kj}\,\vec{e}_m$

So $\boxed{\partial_k g_{ij} = \Gamma^m_{ik}\,g_{jm} + \Gamma^m_{jk}\,g_{im}}$

Use same tricks to get Γ^j_{ik} ...

Let $k \to i,\ i \to j,\ j \to k$ $\Rightarrow$ $\boxed{\begin{aligned}\partial_i g_{jk} &= \Gamma^m_{ji}\,g_{mk} + \Gamma^m_{ki}\,g_{jm}\\ \partial_j g_{ik} &= \Gamma^m_{kj}\,g_{im} + \Gamma^m_{ij}\,g_{km}\end{aligned}}$

So Add first two eqns, subtract 3rd

↳ $\boxed{\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik} = 2\Gamma^m_{ik}\,g_{jm}}$

Note $g_{jm} = g_{mj}$ (symmetric)

↳ Next multiply by g^{jl} ⇒ $g_{jm}\,g^{jl} = \delta^l_m$

So $\boxed{\Gamma^l_{ik} = \frac{1}{2} g^{jl}\left(\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}\right)}$

Let $l \to k$
$k \to i$
$i \to j$
$j \to l$

So $\boxed{\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ji}\right)}$

Note in Cartesian coords, $g_{ij} = \delta^i_j \to \partial_k g_{ij} = 0 \therefore \Gamma^k_{ij} = 0$

Note $\Gamma^k_{ij} \neq 0$ does not mean space is curved!

→ In fact, get $\Gamma^k_{ij} \neq 0$ in curvilinear coords in flat space whenever $\hat{e}_i$ are not constant.

How do we calculate Γ^k_{ij}? ⇒ By brute force ... (won't use book's short cut)

e.g. $\Gamma^1_{23} = \Gamma^1_{32}$

$$= \frac{1}{2} g^{11}\left(\partial_2 g_{31} + \partial_3 g_{21} - \partial_1 g_{23}\right)$$
$$+ \frac{1}{2} g^{12}\left(\partial_2 g_{32} + \partial_3 g_{22} - \partial_2 g_{23}\right)$$
$$+ \frac{1}{2} g^{13}\left(\partial_2 g_{33} + \partial_3 g_{23} - \partial_3 g_{23}\right)$$

then repeat for remaining 25 cases ...

Now $\boxed{\frac{d^2u^i}{ds^2} + \Gamma^i_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0} \to$ 3 eqns

↳ Solution gives eqn of straightline (geodesics) curve u^i in flat spa

But the same eqn carry into curved space!

ct 19, 2018

Affine parameters We used arclength as a parameter in finding geodesic eq

$\boxed{\frac{d^2u^i}{ds^2} + \Gamma^i_{jk}\frac{du^j}{ds}\frac{du^k}{ds} = 0}$. What if we use a different parameters $t = f($

→ Modified eqn $\boxed{\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = -\left(\frac{d^2t}{ds^2}\right)\left(\frac{dt}{ds}\right)^{-2}\frac{du^i}{dt}}$

↳ this is different from the original unless the second derivative $\frac{d^2t}{ds^2} = 0$, i.e.,

$$t = As + B \quad (A, B \text{ constant}, A \neq 0)$$

→ A parameter of this form is called an <u>affine parameter</u>.
→ <u>key</u> t linearly related to s.

$$\frac{ds}{dt} = \frac{1}{A} = A^{-1} \neq 0 \text{ says } s \propto t \Rightarrow \text{no acceleration}$$

So we'll use affine parameters for geodesics in which case the eqn is

flat space! →
$$\boxed{\frac{d^2u^i}{dt^2} + \Gamma^i_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = -\left(\frac{d^2t}{ds^2}\right)\left(\frac{dt}{ds}\right)^{-2}\frac{du^i}{dt} = 0}$$

<u>Geodesics in Curved Space</u>

We've seen correspondences between flat 3D space in curvilinear coords + curved N-dim manifolds.

$\lambda^{i'} = U^{i'}_j \lambda^j$ ↪ $\lambda^{a'} = X^{a'}_b \lambda^b$

. $u^j \to x^a$

. $g_{ij} \to g_{ab}$

. $U^{i'}_j \to X^{a'}_b$

. $ds^2 = g_{ij}\,du^i du^j$ ↪ $= g_{ab}\,dx^a dx^b$

Same is true for geodesic eqn → <u>Similar form</u>

geodesic eqn in curved space →
$$\boxed{\frac{d^2x^a}{d\sigma^2} + \Gamma^a_{bc}\frac{dx^b}{d\sigma}\frac{dx^c}{d\sigma} = 0}$$

<u>Note</u> σ is <u>an affine param</u>, i.e., $\sigma \sim s$

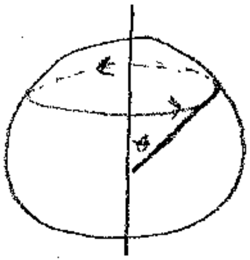
where the connection
$$\boxed{\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc}\right)}$$

<u>where</u>
$$\boxed{\Gamma^a_{bc} = \Gamma^a_{cb} = \vec{e}^{\,a}\left(\partial_c \vec{e}_b\right) = \vec{e}^{\,a}\left(\partial_b \vec{e}_c\right)}$$

Could show this holds in GR as a result of the EP
What we'll do is show that this gives the correct geodesic on a 2-sphere

__Ex__ Determine if lines of constant latitude of a 2-sphere of radius a are geodesics

__Know__ only Equator is!



Do these curves satisfy

$$\frac{d^2u^A}{d\sigma^2} + \Gamma^A_{BC}\frac{du^B}{d\sigma}\frac{du^C}{d\sigma} = 0 \,? \qquad (\text{assume } \sigma \text{ is an affine param})$$

__where__ $\Gamma^A_{BC} = \frac{1}{2} g^{AD}\left(\partial_B g_{CD} + \partial_C g_{BD} - \partial_D g_{BC}\right)$

__Here__ $u^A = (u^1, u^2)$. Use $u^A = (\theta, \varphi)$ $\quad A, B = 1, 2$

radius = a

The metric tensor of 2-sphere of radius a is

$$[g_{AB}] = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix} \qquad (\text{showed in } 1.6.2)$$

__So__ $[g^{AB}] = \begin{pmatrix} a^{-2} & 0 \\ 0 & a^{-2}\sin^{-2}\theta \end{pmatrix}$

__Connection__ $\Gamma^A_{BC} = \frac{1}{2} g^{AD}\left(\partial_B g_{CD} + \partial_C g_{BD} - \partial_D g_{BC}\right)$

There are 8 of these → But can use symmetry ...

Will show (2.1.5) that __answer__ $\begin{cases} \Gamma^1_{22} = -\sin\theta\cos\theta \\ \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta \\ \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{11} = \Gamma^2_{22} = 0 \end{cases}$

Look at $\Gamma^1_{12} = \Gamma^1_{21} \rightarrow \begin{matrix} A=1 \\ B=1 \\ C=2 \end{matrix}$

$$\rightarrow \Gamma^1_{12} = \frac{1}{2} g^{1D}\left(\partial_1 g_{2D} + \partial_2 g_{1D} - \partial_D g_{12}\right)$$

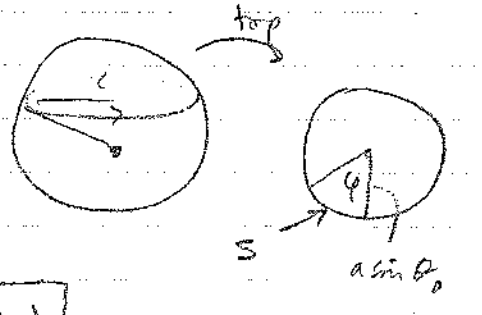
$$= \frac{1}{2} g^{11}\left(\partial_1 g_{21} + \partial_2 g_{11} - \partial_1 g_{12}\right) + \frac{1}{2} g^{12}\left(\partial_1 g_{22} + \partial_2 g_{12} - \partial_2 g_{12}\right)$$

<u>Note</u> $[g_{AB}]$ is diagonal

$$\rightarrow \quad \Gamma^1_{12} = \frac{1}{2} g^{1}\left(\partial_2\, g_{11}\right) = \frac{1}{2} g^{11}\, \partial_\varphi\, g_{\theta\theta} = \frac{1}{2} g^{11}\, \partial_\varphi\,(a^2) = 0$$

<u>Next</u>

Find affine param of lattitude line



<u>Coords</u> $u^A = (u^1, u^2) = (\theta, \varphi)$, with $\theta = \theta_0$

need param in terms of s with $\boxed{s = \varphi(a\sin\theta_0)}$

<u>or</u> $\varphi = s(a\sin\theta_0)^{-1} = As$ & φ is an affine param!

<u>Hence</u> $u^A(s) = \left(\theta_0,\ s(a\sin\theta_0)^{-1}\right)$ use s as param

<u>Need</u> $\frac{du^A}{ds} = \left(0,\ (a\sin\theta_0)^{-1}\right)$ <u>and</u> $\frac{d^2u^A}{ds^2} = (0, 0)$

Now, check with geodesic eqn

2 eqns $\longrightarrow \quad \frac{d^2u^A}{ds} + \Gamma^A_{BC}\frac{du^B}{ds}\frac{du^C}{ds} \; ⊄$

$\boxed{A=1}$

$$\cdot\ \frac{d^2u^1}{ds} + \Gamma^1_{BC}\frac{du^B}{ds}\frac{du^C}{ds} = 0 + (-\sin\theta_0\cos\theta_0)\frac{du^2}{ds}\frac{du^2}{ds} \overset{?}{=} 0$$

<u>Use</u> $\Gamma^1_{22} = -\sin\theta\cos\theta$, $\Gamma^2_{12} = \cot\theta$ $= \Gamma^2_{21}$ $\quad\therefore\ \boxed{(-\sin\theta_0\cos\theta_0)(a\sin\theta_0)^{-2} \overset{?}{=} 0}$

(only true if $\theta_0 = \frac{\pi}{2}$

→ <u>Only Equator works!</u>

$\boxed{A=2}$

$$\cdot\ \frac{du^2}{ds^2} + \Gamma^2_{BC}\frac{du^B}{ds}\frac{du^C}{ds} = 0 + \Gamma^2_{12}\frac{du^1}{ds}\frac{du^2}{ds} + \Gamma^2_{21}\frac{du^2}{ds}\frac{du^1}{ds}$$

$$= \cot\theta\left[0 + 0\right] = 0$$ so this is satisfied

$\boxed{\text{only latitude line that is also a geodesic is the Equator}}$ → for sphere → geodesics = circles ~~with~~ with center

Parallel Transport

Our condition for geodesic was that the tangent vector $\breve{\lambda} = \lambda^i \breve{e}_i = \lambda_i \breve{e}^i = \dot{u}^i \breve{e}_i$ does not change as we move along the curve...

$$\frac{d\breve{\lambda}}{ds} = 0 \quad \text{(condition of straightness)}$$

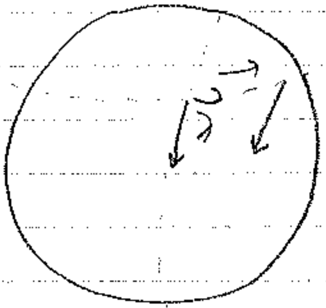
This leads to Geodesic eqn $\boxed{\ddot{u}^i + \Gamma^i_{jk} \dot{u}^j \dot{u}^k = 0}$

We can generalize this. Consider $\breve{\lambda} = \lambda^i \breve{e}_i$ that's an arbitrary vect. Want to transport $\breve{\lambda}$ along a curve parametrized by t without altering it, $u^i(t)$. $\lambda = \lambda^i \breve{e}_i$

Condition: $\frac{d\breve{\lambda}}{dt} = 0$ (t = affine param) called parallel transport

Oct 22, 2018

In flat space, the vector does not change its direction. But in curved space, a vector that is parallel transported can change direction.



← effect of curvature. Note along the equator the direction does not change → holds for any geodesic!

We can derive the math of parallel transport

$$\frac{d\breve{\lambda}}{dt} = 0 \quad \text{with } \breve{\lambda} = \lambda^i \breve{e}_i$$

$\Rightarrow \dot{\lambda}^i \breve{e}_i + \lambda^i \dot{\breve{e}}_i = 0$. We also know $\dot{\breve{e}}_i = (\partial_j \breve{e}_i)\, \dot{u}^j = \Gamma^k_{ij} \dot{u}^j \breve{e}_k$

$\Rightarrow \dot{\lambda}^i \breve{e}_i + \lambda^i \Gamma^k_{ji} \dot{u}^j \breve{e}_k = 0$ let $k \to i$, $i \to j$, $j \to k$

$$\Rightarrow \boxed{\dot{\lambda}^i + \lambda^j \Gamma^i_{kj} \dot{u}^k = 0}$$

↳ This says how the components λ^i change when the vector is parallel transported along the curve parametrized by t.

Ex If $\lambda^i = \dot{u}^i$ (tangent vector to curve)

$$\hookrightarrow \boxed{\ddot{u}^i + \Gamma^i_{kj} \dot{u}^j \dot{u}^k = 0} \rightarrow \text{geodesic eqn}$$

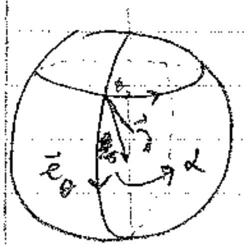
This says that to parallel transport tangent vectors, the curve must be a geodesic (so that it remains a tangent vector)

To go to an N-dim curved manifold, we can just change notation

$$\hookrightarrow \boxed{\dot{\lambda}^a + \Gamma^a_{bc} \lambda^b \dot{x}^c = 0} \text{ (most general case)} \quad \left(t \equiv As + B \text{ is affine param} \right)$$

$\hookrightarrow \dot{\lambda}^a = \frac{d\lambda^a}{dt}$ and so on...

$\boxed{\text{Example}}$ Consider Unit vector $\vec{\lambda}$ on surface of sphere of radius a which makes an angle α wrt. a longitude.



Show that parallel transport along line of constant latitude, the direction of $\vec{\lambda}$ changes by ~~angle~~ angle $\chi = 2\pi w$

where $w = \cos\theta_0$ & θ_0 = polar angle of the latitude.

First, param the curve (2D)

$$u^A = (u^1, u^2) = (\theta, \varphi)$$

Here $\theta = \theta_0$ is fixed $\rightarrow u^A = (\theta_0, \varphi)$ & can let φ run from $0 \rightarrow 2\pi$

$$\rightarrow \varphi = t$$

$\rightarrow u^A(t) = (\theta_0, t)$. Note: this is a <u>different param than before</u>

But before, $u^A(s) = (\theta_0, (a\sin\theta_0)^{-1}s)$

$$= (\theta_0, \varphi)$$

Here, $t = \varphi = \underbrace{(a\sin\theta_0)^{-1}}_{A} s$. And so <u>t is affine</u> ($a\sin\theta_0$ <u>constant</u>)

let $\vec{\lambda}(0)$ be initial vector $(t=0)$ <u>and</u> χ = angle between these
$\vec{\lambda}(2\pi)$ be final vector $(t=2\pi)$ 2 vectors!

Next, want to find initial unit vector $\tilde{\lambda}(0)$ making an angle α w.r.t to latitude.

<u>Claim</u>
$$\boxed{\begin{aligned}\lambda^A(0) &= (\lambda^1(0),\ \lambda^2(0))\\ &= \left(a^{-1}\cos\alpha,\ (a\sin\theta_0)^{-1}\sin\alpha\right)\end{aligned}}$$ is that initial vector

<u>Verify it's correct</u>

is this a unit vector? $\lambda^A(0)\lambda^B(0) \overset{?}{=} 1$

here $[g_{AB}] = \begin{pmatrix} a^2 & 0 \\ 0 & (a\sin\theta_0)^2 \end{pmatrix}$

$$\left[\lambda^A(0)\, g_{AB}\, \lambda^B(0)\right]$$
$$= \left(a^{-1}\cos\alpha,\ (a\sin\theta_0)^{-1}\sin\alpha\right)\begin{pmatrix} a^2 & 0 \\ 0 & (a\sin\theta_0)^2 \end{pmatrix}\begin{pmatrix} a^{-1}\cos\alpha \\ (a\sin\theta_0)^{-1}\sin\alpha \end{pmatrix}$$
$$= \cos^2\alpha + \sin^2\alpha = \boxed{1} \rightarrow \underline{\text{Unit vector}}$$

<u>Next</u>, does it make angle α w.r.t longitude?

$$\text{Longitude} = \underline{(1)\vec{e}_\theta + (0)\vec{e}_\varphi}$$

Call
$$\mu^A_{\text{long}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{vector that points along longitude})$$

<u>Check</u> $\left[\mu^A_{\text{long}}\, \mu^A_{\text{long}}\right] = \left[\mu^A_{\text{long}}\, g_{AB}\, \mu^B_{\text{long}}\right] = \boxed{a^2}$ (not unit vector)

Next, find
$$\cos(\alpha) = \frac{g_{AB}\,\mu^A_{\text{long}}\,\lambda^B(0)}{|\mu^A_{\text{long}}|\,|\lambda^B_{(0)}|} = \frac{g_{11}\,\mu^1\,\lambda^1}{(a)(1)} = \frac{a^2(1)\,a^{-1}\cos\alpha}{a\cdot 1} = \boxed{\cos}$$

$\therefore$ $\tilde{\lambda}(0)$ is at angle α w.r.t a longitude!

<u>Next</u>, parallel transport $\tilde{\lambda}$ around the latitude line
$\Rightarrow$ want new components. Need to solve parallel transport eqn.

<u>Need to solve</u> $\dot{\lambda}^A + \Gamma^A_{BC}\lambda^B \dot{u}^C = 0$ (2 eqns)

<u>Initial value</u> $\vec{\lambda}(0) = \begin{pmatrix} a^{-1}\cos\alpha \\ (a\sin\theta_0)^{-1}\sin\alpha \end{pmatrix}$

<u>Can use</u> $\left\{ \begin{matrix} \Gamma^1_{22} = -\sin\theta_0\cos\theta_0 \\ \Gamma^2_{21} = \Gamma^2_{12} = \cot\theta_0 \end{matrix} \right\}$ <u>and</u> $u^A(t) = (\theta_0, t)$

Since $u^A(t) = (\theta_0, t) \rightarrow \dot{u}^C = (0, 1)$

$\boxed{A=1}$ $\dot{\lambda}^1 + \Gamma^1_{22}\lambda^2\dot{u}^2 = 0 \Leftrightarrow ?$

$\boxed{A=2}$ $\dot{\lambda}^2 + \Gamma^2_{12}\lambda^1\dot{u}^1 = 0 \Leftrightarrow ?$
$+ \Gamma^2_{21}\lambda^2\dot{u}^2$

Will verify that the solution satisfying IVP is: (Exercise 2.2.1

$$\boxed{\begin{matrix} \lambda^A(t) = (\lambda^1(t), \lambda^2(t)) = \left(a^{-1}\cos(\alpha - \omega t), (a\sin\theta_0)^{-1}\sin(\alpha - \omega t)\right) \\ \text{with } \omega = \cos\theta_0 \quad \forall t \end{matrix}}$$

<u>Oct 23, 2018</u> Now, go all the way round, to $t = 2\pi$

$$\Rightarrow \lambda^A(2\pi) = \left(a^{-1}\cos(\alpha - 2\pi\omega), (a\sin\theta_0)^{-1}\sin(\alpha - 2\pi\omega)\right)$$

Is this still a unit vector?

$$\|\lambda^A(2\pi)\|^2 = g_{AB}\lambda^A(2\pi)\lambda^B(2\pi) = a^2 a^{-2}\cos^2(\alpha - 2\pi\omega) + (a\sin\theta_0)^2(a\sin\theta_0)^{-2}\sin^2(\alpha - 2\pi\omega)$$
$$= 1 \Rightarrow \text{still unit normal.}$$

Now, what's the angle χ between $\vec{\lambda}(0)$ and $\vec{\lambda}(2\pi)$

$$\cos\chi = \frac{\lambda^A(0)\cdot\lambda^B(2\pi)\cdot g_{AB}}{|\lambda^A(0)||\lambda^B(2\pi)|} = g_{AB}\lambda^A(0)\lambda^B(2\pi) = a^2(a^{-1}\cos\alpha)(a^{-1}\cos(\alpha - \omega t))$$
$$+ (a\sin\theta_0)^2(a\sin\theta_0)^{-2}\sin\alpha$$
$$\cdot \sin(\alpha - \omega t)$$

$$= \cos\alpha\cos(\alpha - \omega t) + \sin\alpha\sin(\alpha - \omega t) = \cos(\alpha - \alpha + \omega t)$$

$\hookrightarrow \boxed{\chi = \omega t = 2\pi\omega}$ $(t = 2\pi)$

So $\boxed{\chi = 2\pi w = 2\pi \cos\theta_0}$

e.g. if $\theta_0 = \pi/2$ (equator) → $\chi = 0$ (along geodesic, direction does not change)

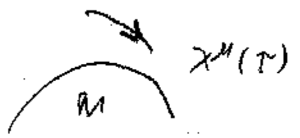
Curved Spacetime

↳ The same equations hold. E.g., the geodesic eqn is

$$\boxed{\frac{d^2 x^0}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\mu}{d\tau}\frac{dx^\sigma}{d\tau} = 0}$$

with $$\boxed{\Gamma^\mu_{\nu\sigma} = \frac{1}{2} g^{\mu\rho}\left(\partial_\nu g_{\sigma\rho} + \partial_\sigma g_{\nu\rho} - \partial_\rho g_{\nu\sigma}\right)}$$

⇒ gives the trajectory of free particle in curved spacetime $x^\mu(\tau)$
⇒ Gives the eqn for particle in gravitational field



For a massive particle, we can use proper time as parameter because

$$ds^2 = c^2 d\tau^2$$
$$= g_{\mu\nu}\, dx^\mu dx^\nu$$

Likewise, any vector λ^μ can be parallel transported along a curve $X^\mu(\tau)$ where the components obey 4D parallel transport eqn:

$$\boxed{\dot{\lambda}^\mu + \Gamma^\mu_{\nu\sigma}\lambda^\mu \dot{x}^\sigma = 0}$$ ⇝ need this for sensible physics eqn in spacetime.

How to formulate the laws of physics in curved spacetime?

↳ **Covariance** Recall that one of the postulates of SR is that the laws of physics are the same in all inertial frames
⇒ Eqns of physics are invariant under LT's,

e.g. $f^\mu = \frac{dp^\mu}{d\tau}$ in SR, after LT multiply a term with $\Lambda^{\nu'}_\mu$

$\Rightarrow \Lambda^{\nu'}_\mu f^\mu = \Lambda^{\nu'}_\mu \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}\left(\Lambda^{\nu'}_\mu p^\mu\right)$ get $\boxed{f^{\nu'} = \frac{dp^{\nu'}}{d\tau}}$ (same eqn) let $\nu' \rightarrow \mu \Rightarrow$ get back eqn $f^\mu = \frac{dp^\mu}{d\tau}$. At the same time, the metric remains $\eta_{\mu\nu} = \eta_{\mu'\nu'}$. Everything is the same $\Rightarrow$ INVARIANT eqns.

In GR

↳ the eqns should maintain the same form under general coord transformations $\Rightarrow$ said to be covariant (not as strict as in SR)

But in GR, eqns can include $g_{\mu\nu}$ (metric) and $\Gamma^{\lambda}_{\mu\nu}$ (connection) $\rightarrow$ these are different in different circumstances

$\Rightarrow$ Eqns need to be covariant but not invariant.

NB invariance implies covariance.

- In trying to figure out how eqns hold in curved space time, Einstein introduced a principle...

↳ Principle of Covariance: Eqn is true in GR in all coord systems if
(1) The eqn is true in SR
(2) The eqn is a tensor eqn that preserves its form under general coord. transf (covariant)

Recall Tensors of the same type all transform the same way

e.g. if $A^\mu = B^\mu$ for tensors A^μ, B^μ, then

$$X^{\nu'}_{\mu} A^\mu = A^{\nu'} = X^{\nu'}_{\mu} B^\mu = B^{\nu'} \Leftarrow \text{covariant form}$$

NB {(1) stems from equiv. principle: there is always a freely falling coord where the laws of SR hold locally.}

As long as the SR laws involve tensors, the same eqns will hold in the presence of gravity.

↳ This gives prescription for finding the laws of physics in GR.

E.g. We know $f^\mu = \frac{dp^\mu}{d\tau}$ holds in SR. Does this eqn also hold in curved spacetime?

⇒ <u>If both sides are tensors then yes.</u>

<u>BUT</u> $\frac{dp^\mu}{d\tau}$ is <u>not</u> a tensor under general coord transformation

<u>why?</u> In a diff. frame

$$\frac{dp^\mu}{d\tau} = \frac{d}{d\tau}\left(\Lambda^{\mu'}_{\nu} p^\nu\right)$$

$$= \underset{(\checkmark)}{\Lambda^{\mu'}_{\nu}\frac{dp^\nu}{d\tau}} + \underset{(\times)}{\frac{d\Lambda^{\mu'}_{\nu}}{d\tau}} p^\nu$$

<u>Note</u> $\frac{d\Lambda^{\mu}_{\nu}}{d\tau} \neq 0$ for general coord. transformation.

⇒ $\frac{dA^\mu}{d\tau}$ is not a tensor in general coord. transf. (GCT)

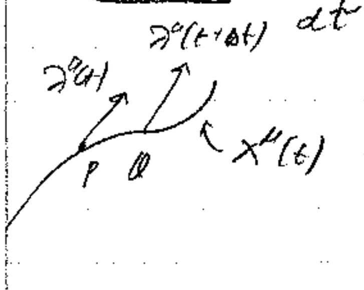
<u>So</u> $f^\mu = \frac{dp^\mu}{d\tau}$ is <u>not</u> covariant. Can't find eqn in new frame

→ <u>The problem is with derivative!</u> $\frac{\partial}{\partial\tau}$, or $\partial_\mu = \frac{\partial}{\partial x^\mu}$

⇒ $\boxed{\text{Derivatives of tensors are NOT tensors in GCT}}$

⇒ Need to fix the def. of derivatives so that derivatives of tensors are tensors ...

<u>Consider</u> $\frac{d\lambda^a}{dt} = \lim_{\Delta t \to 0} \frac{\lambda^a(t+\Delta t) - \lambda^a(t)}{\Delta t}$



<u>But</u> when we transform these, we use

$\Lambda^{b'}_{a}(t)$ on $\lambda^a(t)$ at Q <u>and</u> $\Lambda^{b'}_{a}(t+\Delta t)$ on $\lambda^a(t+\Delta t)$ at P

But Space is different at P, Q! ⇒ don't get the same factor of $X^{b'}_{a}$ at just one point

→ Would be better to subtract
$\lambda^a(t+\Delta t)$ and $\lambda^a(t)$ at the same point

→ To do that, we need to parallel transport $\lambda^a(t+\delta t)$ from Q to P. one

↳ Need to redefine differentiation for curved spaces.

Oct 24, 2018

Derivatives of tensors are NOT tensors in general

E.g. $g_{\mu\nu}$ → tensor but $\partial_\lambda g_{\mu\nu}$ is not a tensor

$$\partial_{\lambda'} g_{\mu'\nu'} = \partial_{\lambda'} X^{\alpha}_{\mu'} X^{\beta}_{\nu'} g_{\alpha\beta} \neq X^{\gamma}_{\lambda} X^{\alpha}_{\mu'} X^{\beta}_{\nu'} \partial_\gamma g_{\alpha\beta}$$ → get extra terms

For this reason $\Gamma^{\mu}_{\gamma\lambda} = \frac{1}{2} g^{\mu\rho}(\partial_\gamma g_{\rho\lambda} + \partial_\lambda g_{\gamma\rho} - \partial_\rho g_{\gamma\lambda})$ is also not a tensor

But this relation is covariant. Go to a primed frame
→ get

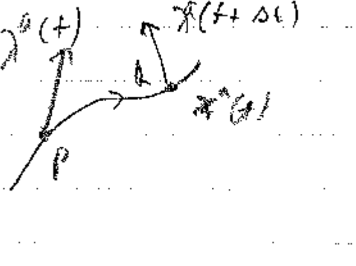
$$\boxed{\Gamma^{\mu'}_{\gamma'\lambda'} = \frac{1}{2} g^{\mu'\rho'}(\partial_{\gamma'} g_{\rho'\lambda'} + \partial_{\lambda'} g_{\gamma'\rho'} - \partial_{\rho'} g_{\gamma'\lambda'})}$$ Also

the extra terms cancel ⇒ this relation is in fact covariant
but more generally, we have a problem with derivatives

Absolute & Covariant derivatives

- Consider a manifold: contravariant vector λ^a parameterized by t, then

$$\frac{d\lambda^a}{dt} = \lim_{\Delta t \to 0} \frac{\lambda^a(\Delta t + t) - \lambda^a(t)}{\Delta t}$$



So $\lambda^a(t)$ @ P
$\lambda^a(t+\Delta t)$ @ Q } problem arises because

$$\boxed{X^{a'}_{b}\Big|_{@Q} \neq X^{a'}_{b}\Big|_{@P}}$$

As $\Delta t \to 0$, can't get extra terms of derivatives of $X^{a'}_{b'}$. To fix this, we change the def. of derivative ⇒ Absolute derivative...

$$\boxed{\underline{\text{Define}} \quad \frac{D\lambda^a}{dt} = \lim_{\Delta t \to 0} \frac{\overset{@Q}{\lambda^a(t+\Delta t)} - \overset{@Q}{\bar{\lambda}^a}}{\Delta t} \quad \text{where} \quad \bar{\lambda}^a = \lambda^a \text{ at } P\text{, parallel transported to } Q}$$

We want an expression for this... For the 1st term, we can Taylor expand..

$$\boxed{\lambda^a(t+\Delta t) \cong \lambda^a(t) + \frac{d\lambda^a}{dt}\Delta t = \lambda^a(P) + \frac{d\lambda^a}{dt}\Delta t} \quad (p = t)$$

Second term parallel transport eqn: $\boxed{\dot{\lambda}^a + \Gamma^a_{bc}\lambda^b\dot{x}^c = 0}$

For small finite intervals, $\dot{\lambda}^a \simeq \frac{\Delta\lambda^a}{\Delta t}$ and $\dot{x}^c \simeq \frac{\Delta x^c}{\Delta t}$

So $\boxed{\Delta\lambda^a + \Gamma^a_{bc}\lambda^b\Delta x^c = 0}$ (parallel transport)

where $\Delta\lambda^a = \bar{\lambda}^a(Q) - \lambda^a(P)$ ⇒ $\boxed{\bar{\lambda}^a(Q) \simeq \lambda^a(P) - \Gamma^a_{bc}\lambda^b\Delta x^c}$

So $\boxed{\bar{\lambda}^a(Q) = \Delta\lambda^a + \lambda^a(P)}$

plug into derivative → $\frac{D\lambda^a}{dt} = \lim_{\Delta t \to 0} \frac{(d\lambda^a/dt)\Delta t + \Gamma^a_{bc}\lambda^b\Delta x^c}{\Delta t}$

So

$$\frac{D\lambda^a}{dt} = \lim_{\Delta t \to 0}\left(\frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b\Delta x^c/\Delta t\right)$$

So $\boxed{\frac{D\lambda^a}{dt} = \underset{\text{(usual)}}{\frac{d\lambda^a}{dt}} + \underset{\text{(correction)}}{\Gamma^a_{bc}\lambda^b\dot{x}^c}}$ → Absolute derivative for a contravariant vector

⇒ Transforms as a tensor by construction: $\boxed{\frac{D\lambda^a}{dt} = X^{a'}_b\frac{D\lambda^b}{dt}}$

Notice that the RHS is the same as in the parallel transport eq.

$$\frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b\dot{x}^c = 0$$ ⇒ If we parallel transport a vector λ^a its components are constant under absolute Differentiation

$$\boxed{\frac{D\lambda^a}{dt} = 0 \text{ when parallel transported}}$$

→ What about taking absolute derivatives of scalar, covariant vector, or tensors

For scalars $\phi \to \phi$ as $x^a \to x^{a'}$ → no factor of $X^{a'}_{b}$ in derivative

$$\boxed{\frac{D\phi}{dt} = \frac{d\phi}{dt}}$$ → absolute deriv of scalar

Under a GCT ⇒ $\frac{D\phi}{dt} \to \frac{d\phi}{dt}$

For covariant vectors

Consider $\lambda^a\mu_a$ is a scalar.

$$\hookrightarrow \frac{D\lambda^a\mu_a}{dt} = \frac{d}{dt}(\lambda^a\mu_a) = \frac{d\lambda^a}{dt}\mu_a + \lambda^a\frac{d\mu_a}{dt}$$

$$\Rightarrow \frac{D\lambda^a}{dt}\mu_a + \lambda^a\frac{D\mu_a}{dt} = \frac{d\lambda^a}{dt}\mu_a + \lambda^a\frac{d\mu_a}{dt}$$

$$\Rightarrow \left(\cancel{\frac{d\lambda^a}{dt}} + \Gamma^a_{bc}\lambda^b\dot{x}^c\right)\mu_a + \lambda^a\left[\frac{D\mu_a}{dt}\right] = \cancel{\frac{d\lambda^a}{dt}}\mu_a + \lambda^a\frac{d\mu_a}{dt}$$

$$\underline{\text{So}} \quad \left(\frac{D\mu_a}{dt}\right) = \frac{1}{\lambda^a}\left[\lambda^a\frac{d\mu_a}{dt} - \mu_a\Gamma^a_{bc}\lambda^b\dot{x}^c\right]$$ let $b \to a$
$a \to d$

$$\underline{\text{So}} \quad \frac{D\mu_a}{dt} = \frac{1}{\lambda^a}\left[\lambda^a\frac{d\mu_a}{dt} - \mu_d\Gamma^d_{ac}\lambda^a\dot{x}^c\right]$$

$$\underline{\text{So}} \quad \boxed{\frac{D\mu_a}{dt} = \frac{d\mu_a}{dt} - \Gamma^d_{ac}\mu_d\dot{x}^c}$$ ← Absolute deriv. of covariant vector. Note the (−) sign to connection.

→ Contravariant (+Γ) → covariant (−Γ)

For a tensor $\boxed{\tau^{ab}{}_{c} = \lambda^a \sigma^b \mu_c}$ ← multiplying vectors gives tensors

We can show that

$$\boxed{\frac{D\tau^{ab}{}_{c}}{dt} = \overset{\text{usual}}{\frac{d\tau^{ab}{}_{c}}{dt}} + \Gamma^a_{de}\tau^{db}{}_{c}\dot{x}^e + \Gamma^b_{de}\tau^{ad}{}_{c}\dot{x}^e - \Gamma^d_{ce}\tau^{ab}{}_{d}\dot{x}^e}$$

correction (+, −)

This is a tensor, so under GCT

$$\rightarrow \boxed{\frac{D\tau^{a'b'}{}_{c'}}{dt} = X^{a'}_{d} X^{b'}_{e} X^{f}_{c'} \frac{D\tau^{de}{}_{f}}{dt}}$$

Note that In Cartesian coordinates, $\Gamma^a_{bc} = 0$ for SR (flat)

$$\hookrightarrow \boxed{\frac{D\tau^{ab}{}_{c}}{dt} = \frac{d\tau^{ab}{}_{c}}{dt}} \text{ in SR}$$

The absolute derivative is w.r.t a parameter (like t, σ, s..).
We also need to take derivatives w.r.t coordinates.

$\partial_a = \frac{\partial}{\partial x^a} \Rightarrow$ need to introduce a derivative that transforms correctly.

→ $\boxed{\text{Covariant derivative}}$ → w.r.t coord x^a.

Since $x^a = x^a(t)$ along a curve → can think of chain rule where

$$\frac{D\lambda^a}{dt} = \frac{D\lambda^a}{dx^c}\frac{dx^c}{dt} \quad \text{(new type of derivative)}$$

$$= \frac{D\lambda^a}{dx^c}\dot{x}^c$$

But since $\frac{D\lambda^a}{dt} = \frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b \dot{x}^c$

$$\Rightarrow \frac{D\lambda^a}{dx^c}\dot{x}^c = \frac{d\lambda^a}{dt} + \Gamma^a_{bc}\lambda^b\dot{x}^c$$

chain rule $\quad \frac{d\lambda^a}{dt} = \frac{\partial\lambda^a}{\partial x^c}\frac{dx^c}{dt} = \frac{\partial\lambda^a}{\partial x^c}\dot{x}^c$

So $\frac{D\lambda^a}{dx^c}\dot{x}^c = \frac{\partial\lambda^a}{\partial x^c}\dot{x}^c + \Gamma^a_{bc}\lambda^b\dot{x}^c$

Therefore $\boxed{\frac{D\lambda^a}{dx^c} = \underset{\substack{\uparrow\\ \text{usual}}}{\frac{\partial\lambda^a}{\partial x^c}} + \underset{\substack{\uparrow\\ \text{correction}}}{\Gamma^a_{bc}\lambda^b}}$ But we don't use this notation

Define

$$\boxed{\begin{aligned}\lambda^a_{;c} &= \frac{\partial\lambda^a}{\partial x^c} + \Gamma^a_{bc}\lambda^b \\ \text{or}\quad \lambda^a_{;c} &= \partial_c\lambda^a + \Gamma^a_{bc}\lambda^b\end{aligned}}$$

→ covariant derivative of contravariant vector

semi-colon

comma

We also write $\boxed{\lambda^a_{,c} = \frac{\partial\lambda^a}{\partial x^c} = \partial_c\lambda^a}$

So $\boxed{\lambda^a_{;c} = \lambda^a_{,c} + \Gamma^a_{bc}\lambda^b}$

Why do this? Because λ^a is a type (1,0) tensor but $\lambda^a_{;c}$ is a type (1,1) tensor

But other notations $\frac{D\lambda^a}{dx^c} = \nabla_c\lambda^a = D_c\lambda^a$