Problem Set 11

Due: Monday 5pm, May 2, via Canvas upload or in envelope outside 26-255
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Office hours Thu Apr 28, 1-2pm in 26-201 (QIS room)

1 Scattering of light by a trapped atom

a) To show that

$$\langle\psi_f| e^{ikx} |\psi_i\rangle = \langle\phi_f(p+k)|\phi_i(p)\rangle = \int \frac{dp}{2\pi}\phi_f^*(p+k)\phi_i(p) \tag{1}$$

where

$$\phi_{i,f}(p) = \int dx\ e^{-ipx}\psi_{i,f}(x) := \mathcal{F}_x[\psi_{i,f}](p) \tag{2}$$

is the Fourier transform of $\psi_{i,f}(x)$, we make use of properties of Fourier transforms. The multiplication property says

$$\mathcal{F}_x[fg](p) = \mathcal{F}_x[f](p) * \mathcal{F}_x[g](p) \tag{3}$$

where $f * g$ indicates a convolution of f and g

$$\{f * g\}(x) := \int_{-\infty}^{\infty} dx' f(x')g(x-x') \tag{4}$$

The matrix element is therefore

$$\begin{aligned}\langle\phi_f(p+k)|\phi_i(p)\rangle &= \int \frac{dp}{2\pi}\phi_f^*(p+k)\phi_i(p)\\ &= \{\phi_f^*(p) * \phi_i(p)\}(-k)\\ &= \{\mathcal{F}_x[\psi_f^*](p) * \mathcal{F}_x[\psi_i](p)\}(-k)\\ &= \mathcal{F}_x[\psi_f^*\psi_i](-k)\\ &= \int dx\ \psi_f^*(x)e^{-i(-k)x)}\psi_i(x)\\ &= \langle\psi_f(x)| e^{ikx} |\psi_i(x)\rangle\,.\end{aligned} \tag{5}$$

You can also plug in Eq. 2 explicitly into the first line of Eq. 5 to get the same result.

b) Using the Baker-Campbell-Hausdorff (BCH) formula for when $[A,[A,B]] = 0$ and $[B,[A,B]] = 0$:

$$\exp(ikx) = \exp[ikx_0(a + a^\dagger)] = \exp\left[-(kx_0)^2/2\right]\exp(ikx_0a^\dagger)\exp(ikx_0a) \tag{6}$$

c) Recall that $a\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle$. Applying the annihilation operator j times yields

$$a^j\left|n\right\rangle = \begin{cases} \sqrt{\frac{n!}{(n-j)!}}\left|n-j\right\rangle & j \leq n \\ 0 & j > n. \end{cases} \tag{7}$$

Then

$$\exp(ikx_0 a)\left|n\right\rangle = \sum_{j=0}^{\infty}\frac{(ikx_0)^j}{j!}\sqrt{\frac{n!}{(n-j)!}}\left|n-j\right\rangle \tag{8}$$

d) The matrix elements are

$$\left\langle m\right|\exp(ikx)\left|n\right\rangle = \exp\left[-(kx_0)^2/2\right]\left\langle m\right|\exp(ikx_0 a^\dagger)\exp(ikx_0 a)\left|n\right\rangle. \tag{9}$$

Operate on the right with $\exp(ikx_0 a)$ and on the left with $\exp(ikx_0 a^\dagger)$ and use orthogonality to get

$$= \exp\left[-(kx_0)^2/2\right]\sqrt{\frac{n!}{m!}}(ikx_0)^{\Delta n}\sum_{j=0}^{n_<}\frac{(-1)^j(kx_0)^{2j}}{j!(j+\Delta n)!(n_<-j)!} \tag{10}$$

where $\Delta n = |m-n|$ and $n_<$ is the lesser of m and n. Using the explicit form of the generalized Laguerre polynomial

$$L_m^\alpha(X) = \sum_{m=0}^{n}(-1)^m\binom{n+\alpha}{n-m}\frac{X^m}{m!} \tag{11}$$

we obtain

$$\left\langle m\right|\exp(ikx)\left|n\right\rangle = \exp\left[-(kx_0)^2/2\right]\sqrt{\frac{n_<!}{n_>!}}(ikx_0)^{\Delta n}L_{n_<}^{\Delta n}\left[(kx_0)^2\right] \tag{12}$$

e) If $n, m \gg 1$, we retrieve the classical result from lecture. For fixed α and $n \gg 1$ we can approximate the Laguerre polynomial in the matrix element by its asymptotic form

$$L_n^\alpha(X) \to \left(\frac{n}{X}\right)^{\alpha/2}e^{X/2}J_\alpha\left(2\sqrt{nX}\right) \tag{13}$$

where $J_\alpha(x)$ is the Bessel function. This yields

$$\begin{aligned}\left\langle m\right|e^{ikx}\left|n\right\rangle &= \exp\left[-(kx_0)^2/2\right]\sqrt{\frac{n_<!}{n_>!}}(ikx_0)^{\Delta n}\left[\frac{n_<}{(kx_0)^2}\right]^{\Delta n/2} \\ &\qquad \times \exp\left[(kx_0)^2/2\right]J_{\Delta n}\left(2\sqrt{n_<(kx_0)^2}\right) \\ &= \sqrt{\frac{n_<!}{n_>!}}i^{\Delta n}(n_<)^{\Delta n/2}J_{\Delta n}\left(2\sqrt{n_<(kx_0)^2}\right).\end{aligned} \tag{14}$$

The modulus squared is

$$|\langle m| e^{ikx} |n\rangle|^2 = \frac{n_<!}{n_>!}(n_<)^{\Delta n} J_{\Delta n}\left(2\sqrt{n_<(kx_0)^2}\right). \tag{15}$$

The classical amplitude is given by

$$M\omega^2 x_{\rm cl}^2/2 = n\hbar\omega \to (kx_0)^2 = (kx_{\rm cl})^2/4n \tag{16}$$

Assuming $\Delta n \ll n, m$ the argument of the Bessel function is

$$2\sqrt{nk^2x_{\rm cl}^2/4n} = kx_{\rm cl} := \beta \tag{17}$$

We can further simplify the prefactor of the Bessel function by using Stirling's approximation

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \tag{18}$$

The ratio of the left and right hand sides goes to 1 as $n \to \infty$. Let us choose $m < n$ without loss of generality. Using this and Eq. 18, we can simplify the prefactor:

$$\begin{aligned}
\frac{m!}{n!}m^{n-m} &\approx \frac{\sqrt{2\pi m}(m/e)^m}{\sqrt{2\pi n}(n/e)^n}m^{n-m} \\
&= \frac{m^{m+1/2}}{n^{n+1/2}}e^{\Delta n}m^{\Delta n} \\
&= \frac{m^{m+1/2}}{(m+\Delta n)^{m+\Delta n+1/2}}e^{\Delta n}m^{\Delta n} \\
&= \frac{m^{m+\Delta n+1/2}}{m^{m+\Delta n+1/2}(1+\Delta n/m)^{m+\Delta n+1/2}}e^{\Delta n} \\
&= e^{\Delta n}(1+\Delta n/m)^{-(m+\Delta n+1/2)} \\
&\approx e^{\Delta n}(1+\Delta n/m)^{-m} \\
&= e^{\Delta n}(1-\Delta n/m')^{m'}
\end{aligned} \tag{19}$$

where $m' = -m$. Then, as $m' \to \infty$,

$$= e^{\Delta n}\lim_{m'\to\infty}(1-\Delta n/m')^{m'} = e^{\Delta n}e^{-\Delta n} = 1. \tag{20}$$

In the end we find

$$|\langle m| e^{ikx} |n\rangle|^2 = J_{\Delta n}^2(\beta). \tag{21}$$

2 Saturation spectroscopy

a) **Homogenous broadening**. The steady-state solution to the optical Bloch equations for a two-level system driven by a monochromatic field, with decay from state 2 to state 1 at rate Γ gives:

$$\rho_{22} = \frac{1}{2}\frac{s}{1+s+\Delta^2}, \tag{22}$$

where ρ_{22} is the probability for the system to be in the excited state 2, $\Delta = 2\delta/\Gamma$, and $s = I/I_{SAT}$. The densities are:

$$n_2 = \frac{n}{2}\frac{s}{1+s+\Delta^2}$$
$$n_1 = n - n_2 = \frac{n}{2}\frac{2+s+2\Delta^2}{1+s+\Delta^2}.$$

The absorption cross section σ is defined by the relation $P = I\sigma$, where P is the power scattered by the atom.

$$\sigma = P/I = \hbar\omega_L\Gamma\rho_{22}/I = \frac{\hbar\omega_L\Gamma}{2I_{SAT}}\frac{1}{1+s+\Delta^2} = \frac{\omega_L}{\omega_0}\frac{\sigma_0}{1+s+\Delta^2} \approx \frac{\sigma_0}{1+s+\Delta^2}, \tag{23}$$

where

$$\sigma_0 = \frac{\hbar\omega_0\Gamma}{2I_{SAT}} = \frac{6\pi}{k_0^2}$$

The fraction absorbed is

$$f = 1 - e^{-n\sigma L} \approx n\sigma L \tag{24}$$

The above approximation holds when the optical density $n\sigma L$ is small and uniform. For larger optical densities, the absorption coefficient $n\sigma$ varies as the beam propagates.

b) **The Bennett hole**. The Maxwell-Boltzmann velocity distribution is

$$P(v_z) = \sqrt{\frac{m}{2\pi k_B T}}e^{-\frac{1}{2}\beta m v_z^2} = \frac{1}{v\sqrt{2\pi}}e^{-\frac{1}{2}v_z^2/v^2},$$

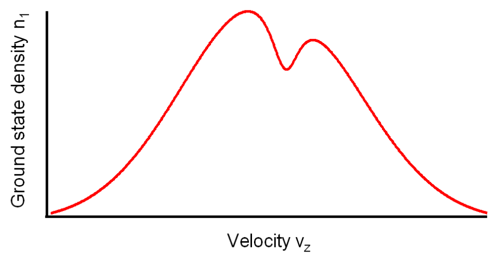
where $v^2 = k_B T/m$. The density of atoms in the ground state is then

$$n_1(v_z) = P(v_z)\frac{n}{2}\frac{2+s+2\Delta^2(v_z)}{1+s+\Delta^2(v_z)}$$
$$\Delta(v_z) = \frac{2}{\Gamma}(\omega_L - \omega_0 - k_L v_z)$$

The hole occurs where the Doppler shift equals the detuning so that $\Delta(v_z) = 0$:

$$v_z^{hole} = \frac{\omega_L - \omega_0}{k_L}$$

The width and depth of the hole increase with s. For high temperatures, the depth



3 b) Distribution $n1$ of atoms in the ground state. The Bennet hole appears as a dip in the density.

of the hole in n_1 as a fraction of the value when no light is present is

$$\frac{\Delta n_1}{n_1} = \frac{1}{2}\frac{s}{1+s}$$

and the width is

$$\Delta v_z = \sqrt{1+s}\frac{\Gamma}{k_L}$$

c) **Inhomogenous broadening**. The fraction of light absorbed is

$$\begin{aligned}
f &= \int_{-\infty}^{\infty} \mathrm{d}v_z n(v_z)\sigma(v_z)L \\
&= n\sigma_0 L \int_{-\infty}^{\infty} \mathrm{d}v_z \frac{1}{v\sqrt{2\pi}} e^{-\frac{1}{2}v_z^2/v^2} \frac{1}{1+s+(\delta - k_L v_z)^2/(\Gamma/2)^2} \\
&\approx n\sigma_0 L \frac{1}{v\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\delta}{k_L v})^2} \int_{-\infty}^{\infty} \mathrm{d}v_z \frac{1}{1+s+(\delta - k_L v_z)^2/(\Gamma/2)^2} \\
&= \frac{n\sigma_0 L}{v\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\delta}{k_L v})^2} \frac{\pi\Gamma}{2k_L\sqrt{1+s}}
\end{aligned}$$

The absorption line is a Gaussian with width $k_L v$. The width is independent of s provided that the above approximation is valid, i.e. $1+s \ll (\frac{k_L v}{\Gamma})^2$.

d) **Saturation spectroscopy**. This question requires thinking about the effect of light at two different frequencies on an atom. The absorption of light by a two-level atom at frequency ω is:

$$f = (n_1 - n_2)\sigma(\omega)L, \tag{25}$$

where $\sigma(\omega)$ is the cross section for absorption of a single photon by a ground state atom, and is equal to the stimulated emission cross section of an excited atom. When an atom interacts with light at a single frequency, the factor of $n_1 - n_2$ can be combined with $\sigma(\omega)$ into an expression in terms of the saturated cross section $\sigma(\omega, s)$ and the total density of atoms:

$$(n_1 - n_2)\sigma(\omega) = n\frac{1+\Delta^2}{1+s+\Delta^2}\frac{\sigma_0}{1+\Delta^2} = n\sigma(\omega, s).$$

When two frequencies excite the atom, equation (25) still holds but the value of $n_1 - n_2$ depends on the frequency and intensity of both beams, so the expression for the saturated absorption cross section changes.

i) The fraction of light absorbed from the weak second beam ($s_p \ll 1$) is:

$$\begin{aligned}
f_p &= \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{v}(n_1(\mathbf{v}) - n_2(\mathbf{v}))\sigma(\omega_p, \mathbf{v})L \\
&= n\sigma_0 L \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{v}P(\mathbf{v})\frac{1+\Delta_L^2}{1+s_L+\Delta_L^2}\frac{1}{1+\Delta_p^2} \\
&= n\sigma_0 L \int_{-\infty}^{\infty} \mathrm{d}^3\mathbf{v}P(\mathbf{v}) \underbrace{(1 - \frac{s_L}{1+s_L+\Delta_L^2})}_{\text{power broadened Lorentzian of Bennet hole}} \underbrace{\frac{1}{1+\Delta_p^2}}_{\text{Lorentzian of probe beam}}
\end{aligned}$$

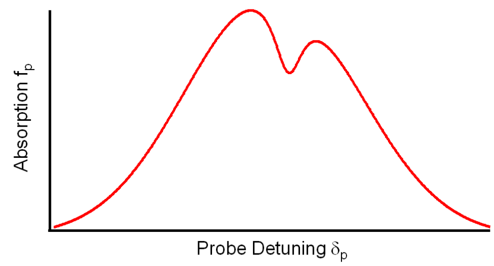
For

$$\Delta_L = \frac{2}{\Gamma}(\omega_L - \omega_0 - \mathbf{k_L}\cdot\mathbf{v}); \Delta_p = \frac{2}{\Gamma}(\omega_p - \omega_0 - \mathbf{k_p}\cdot\mathbf{v}).$$

When the laser beams are parallel (along the z-axis) a hole appears in the absorption spectrum when both beams address the same velocity class:

$$\delta_L = -\delta_p$$

The width of the hole is given by the convolution of the power broadened Lorentzian ($\sqrt{1+s_L}\Gamma$) of Bennet hole and the Lorentzian of probe beam (Γ)

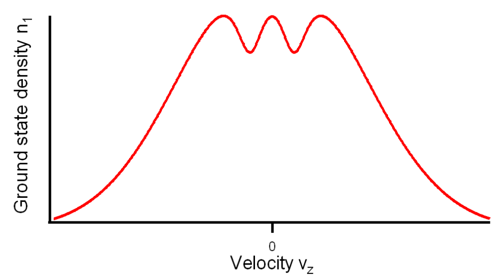


3 d) *i*)

$\Delta\delta_p = (1 + \sqrt{1 + s_L})\Gamma.$

This is the result for δ_L fixed while δ_p is swept.

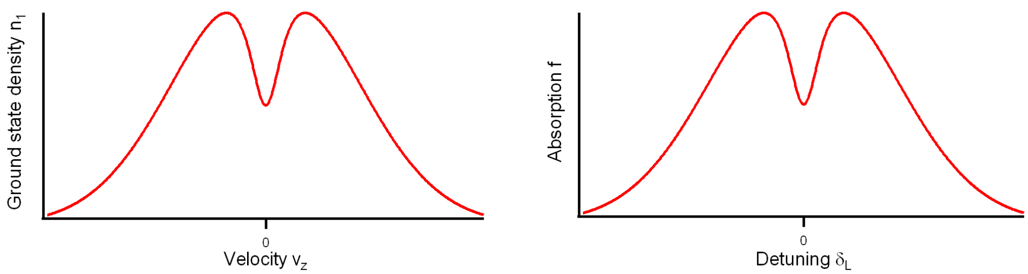
ii) In this case, $\mathbf{k_p} = -\mathbf{k_L}$ and $s_p \approx s_L \equiv s$.
For $\omega_L \neq \omega_0$, the two beams saturate different velocity classes.
The depths of the holes, as a fraction of the $s = 0$ density, are $\frac{1}{2}\frac{s}{1+s}$.



3 d) *ii*)

For $\omega_L = \omega_0$, the two beams saturate the same velocity classes.
The depth of the hole in n_1 is $\frac{s}{1+2s}$.

When the pump is retro-reflected, if the the reflectivity is small, the saturation is only due to the strong pump not the reflected beam. However, when we sweep the pump by δ, the Bennet hole will be simultaneously shifted by $-\delta$. Thus the width of the feature will be reduced by a factor of 2 compared with the case of i). Now, consider $s_p \approx s_L$. Both beams will contribute to the broadening resulting in a saturation parameter 2s.


Ground state density n1
0
Velocity vz
Absorption f
0
Detuning δL

3 d) *ii*)