Quantization of Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

The canonical momentum conjugate to ψ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)}\partial_0\psi - \mathcal{L} = i\bar{\psi}\gamma^0\partial_0\psi - i\bar{\psi}\gamma^0\partial_0\psi - i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi$$

$$= -i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi$$

$$H = \int d^3\vec{x}\ \bar{\psi}\left(-i\vec{\gamma}\cdot\vec{\nabla} + m\right)\psi$$

Let's try to figure out the commutators to make this quantum field theory.

Our first try (this will not work) ...

Guess $\left[\psi_a(\vec{x}),\ i\psi_b^\dagger(\vec{y})\right] = i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$

spinor components $\quad a, b = 1,2,3,4$

or $\left[\psi_a(\vec{x}),\ \psi_b^\dagger(\vec{y})\right] = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})$

In matrix notation,

$$\left[\psi(\vec{x}),\ \psi^\dagger(\vec{y})\right] = \mathbb{1}\cdot \delta^{(3)}(\vec{x}-\vec{y})$$

column vector $\quad$ row vector $\quad$ 4×4 identity

$$a\downarrow\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} \qquad \underset{b\ \longrightarrow}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \end{bmatrix}} \qquad a\downarrow\underset{b\ \longrightarrow}{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}}$$

We also guess
$$\left[\psi_a(\vec{x}),\ \psi_b(\vec{y})\right] = 0$$
$$\left[\psi_a^\dagger(\vec{x}),\ \psi_b^\dagger(\vec{y})\right] = 0$$

Note that
$$\left[\psi(\vec{x}),\ \bar{\psi}(\vec{y})\right] = \left[\psi(\vec{x}),\ \psi^\dagger(\vec{y})\right]\gamma^0 = \gamma^0\,\delta^{(3)}(\vec{x}-\vec{y})$$

Recall for a free boson we could write
$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left\{a_{\vec{p}} + a^\dagger_{-\vec{p}}\right\} e^{i\vec{p}\cdot\vec{x}}$$

For a complex field we get
$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left\{a_{\vec{p}} + b^\dagger_{-\vec{p}}\right\} e^{i\vec{p}\cdot\vec{x}}$$

In the case of a Dirac field, we also need spin degrees of freedom. We try

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a^r_{\vec{p}}\, u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}}\, v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

spin degrees of freedom

$$\psi^\dagger(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[a^{r\dagger}_{\vec{p}}\, u^{r\dagger}(\vec{p}) + b^r_{-\vec{p}}\, v^{r\dagger}(-\vec{p}) \right] e^{-i\vec{p}\cdot\vec{x}}$$

Recall that $u^r(\vec{p})$ satisfies $p^\mu \gamma_\mu u^r(\vec{p}) = m\, u^r(\vec{p})$

$v^r(\vec{p})$ satisfies $p^\mu \gamma_\mu v^r(\vec{p}) = -m\, v^r(\vec{p})$

$(p^0 = E_{\vec{p}})$

try the commutators

$[a^r_{\vec{p}}, a^{s\dagger}_{\vec{p}'}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$

$[b^r_{\vec{p}}, b^{s\dagger}_{\vec{p}'}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$ (minus needed to get things right)

$[a^r_{\vec{p}}, a^s_{\vec{p}'}] = 0$

$\vdots$ (all other commutators zero)

We find that

$[\psi_a(\vec{x}), \psi_b(\vec{y})]$ and $[\psi^\dagger_a(\vec{x}), \psi^\dagger_b(\vec{y})]$ vanish

as desired.

We also find $[\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})$, as desired. (Skipping some steps here which are similar to the boson calculation).

The Hamiltonian is

$$H = \int d^3\vec{x}\,\left[-i\bar{\psi}\,\vec{\gamma}\cdot\vec{\nabla}\,\psi + m\bar{\psi}\psi\right]$$

$$= \int d^3\vec{x}\,\left\{\psi^\dagger\gamma^0\left[-i\vec{\gamma}\cdot\vec{\nabla} + m\right]\psi\right\}$$

Since $$\underbrace{(p^0\gamma^0 - \vec{p}\cdot\vec{\gamma})}_{p^\mu\gamma_\mu}\,u^r(\vec{p}) = m\,u^r(\vec{p})$$

$$\Rightarrow (\vec{p}\cdot\vec{\gamma} + m)\,u^r(\vec{p}) = p^0\gamma^0 u^r(\vec{p}) = E_{\vec{p}}\,\gamma^0 u^r(\vec{p})$$

Also $$\underbrace{(p^0\gamma^0 + \vec{p}\cdot\vec{\gamma})}_{p'^\mu\gamma_\mu \quad (p' = (E_{\vec{p}}, -\vec{p}))}\,v^r(-\vec{p}) = -m\,v^r(-\vec{p})$$

$$\Rightarrow (\vec{p}\cdot\vec{\gamma} + m)\,v^r(-\vec{p}) = -E_{\vec{p}}\,\gamma^0 v^r(\vec{p})$$

So when we compute

$$[-i\vec{\gamma}\cdot\vec{\nabla}+m]\psi$$

and write

$$\psi(\vec{x}) = \sum_{r=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[a^r_{\vec{p}}\,u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}}\,v^r(-\vec{p})\right]e^{i\vec{p}\cdot\vec{x}}$$

we get

$$[-i\vec{\gamma}\cdot\vec{\nabla}+m]\psi(\vec{x})$$

$$= \gamma^0\sum_{r=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left[E_{\vec{p}}\,a^r_{\vec{p}}\,u^r(\vec{p}) - E_{\vec{p}}\,b^{r\dagger}_{-\vec{p}}\,v^r(-\vec{p})\right]e^{i\vec{p}\cdot\vec{x}}$$

If we now compute H we get (dropping terms that vanish)

$$H = \int d^3\vec{x}\,\left\{\psi^\dagger\gamma^0[-i\vec{\gamma}\cdot\vec{\nabla}+m]\psi\right\}$$

$$= \sum_r\int\frac{d^3\vec{p}}{(2\pi)^3}\,E_{\vec{p}}\left(a^{r\dagger}_{\vec{p}}\,a^r_{\vec{p}} - \underbrace{b^r_{-\vec{p}}\,b^{r\dagger}_{-\vec{p}}}_{=\,b^{r\dagger}_{-\vec{p}}\,b^r_{-\vec{p}}\,+\,\cancel{\text{const}}}\right)$$

This is no good! Energy is unbounded below!

How can we fix this?

Let us try some Fermi statistics.

We try anticommutators instead of commutators...

$$\{a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}\} = \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})\, \delta^{rs}$$

All other anticommutators zero.

When then find

$$\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\, \delta_{ab}$$

$$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi^\dagger_a(\vec{x}), \psi^\dagger_b(\vec{y})\} = 0$$

We again use

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \left[a^r_{\vec{p}}\, u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}}\, v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Everything goes through the same way, except we get

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{r=1,2} E_{\vec{p}} \left(a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} - \underbrace{b^r_{-\vec{p}} b^\dagger_{-\vec{p}}}_{-b^\dagger_{-\vec{p}} b_{-\vec{p}} + \cancel{\text{const}}} \right)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{r=1,2} E_{\vec{p}} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r} + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^{r} \right)$$

good!
bounded below

Also $\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3} \sum_{r=1,2} \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^{r} + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^{r} \right)$

We usually write

$$\psi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \left(a_{\vec{p}}^{r} u^r(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^{r\dagger} v^r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)$$

As a Heisenberg field

$$\psi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \left(a_{\vec{p}}^{r} u^r(\vec{p}) e^{-ip\cdot x} + b_{\vec{p}}^{r\dagger} v^r(\vec{p}) e^{+ip\cdot x} \right)$$

$p^0 = E_{\vec{p}}$ as usual

$a_{\vec{p}}^{s}$	annihilates	particles
$a_{\vec{p}}^{s\dagger}$	creates	particles
$b_{\vec{p}}^{s}$	annihilates	antiparticles
$b_{\vec{p}}^{s\dagger}$	creates	antiparticles

We define the vacuum as the state $|0\rangle$ where $a^s_{\vec{p}}|0\rangle = 0$

$$b^s_{\vec{p}}|0\rangle = 0$$

Define one-particle states with covariant normalization ...

$$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\; a^{s\dagger}_{\vec{p}}|0\rangle$$

$$\langle \vec{p}, s|\vec{q}, r\rangle = (2E_{\vec{p}})(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})\,\delta^{rs}$$

Under a Lorentz transformation

$$\psi(x) \to \psi'(x) = \Lambda_{\frac{1}{2}}\psi(\Lambda_4^{-1}x)$$

Let us compute the charge density for a rotation angle $|\vec{\theta}|$ about the $\hat{\theta}$ direction

$$\Lambda_{\frac{1}{2}} \approx 1 - \frac{i}{2}\vec{\theta}\cdot\underbrace{\vec{\Sigma}}_{\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}}$$

$$\psi(\Lambda_4^{-1}x) \approx (1 - i\vec{\theta}\cdot\vec{J})\,\psi(x)$$

$$\vec{J} = \vec{x}\times(-i\vec{\nabla})$$

So $\psi \to \psi + \delta\psi$ where

$$\delta\psi = -\tfrac{i}{2}\vec{\theta}\cdot\vec{\Sigma}\,\psi(x) - \vec{\theta}\cdot(\vec{x}\times\vec{\nabla})\psi(x)$$

So $\vec{J}_{total}$ (total spin) $= \int d^3\vec{x}\,\left[\psi^\dagger(-i\vec{x}\times\vec{\nabla})\psi + \tfrac{1}{2}\psi^\dagger\vec{\Sigma}\psi\right]$