QUANTUM FIELD THEORY

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Preface. These notes come from Prof. Patton's QFT notes, Peskin-Schröder QFT, and "Quantum field theory & Condensed matter" - Shankar.

Enjoy!

Conventions

- $\hbar = c = 1$
- $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$
- $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1\ -1\ -1\ -1)$
- $p^2 = p_\mu p^\mu = E^2 - |p|^2 = mc^2 = m$
- $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$
- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$
- $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{\partial x^0}, \nabla\right)$
- $\varepsilon^{0123} = +1 \ ; \ \varepsilon_{0123} = -1$

  $\varepsilon^{1230} = -1$
- $E = i\dfrac{\partial}{\partial x^0}, \quad \vec{p} = -i\vec{\nabla}$
- $p^\mu = i\partial^\mu$

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $$\sigma^i \sigma^j = \delta^{ij} + i \varepsilon^{ijk} \sigma^k$$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \cancel{<} 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \dfrac{d}{dx} \theta(x)$

- n-dimensional Dirac δ-fn:

$$\int d^n x \, \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(x)$$

$$\tilde{f}(x) = \int d^4 k \, e^{ik\cdot x} f(x)$$

- $$\int d^4 x \, e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k)$$

- <u>EM</u> $\Phi = \dfrac{Q}{4\pi r}$ $\leftarrow$ Coulomb potential

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \quad ; \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\Phi, \vec{A}) \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of classical Field Theory

Lagrangian Field Theory:

$$S = \int \underline{L}\, dt = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

Lag.-density

$$\left(\underline{L} = \mathcal{L}\, d^3x\right)$$

(Lagrangian)

Principle of least action:

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right] \right\}$$

(E-L eqn)

$$\Rightarrow \boxed{0 = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi}}$$

FTC $\rightarrow$ term vanishes @ boundary

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

Ex

$$\mathcal{L} = \phi^2 : \qquad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi \; ; \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow 2\phi = 0$$

$$\left.\begin{array}{l} \mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) \\ \Rightarrow \dfrac{\partial \mathcal{L}}{\partial \phi} = 0 \; ; \; \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2\partial^\mu \phi \end{array}\right\} \Rightarrow \partial^\mu \phi = 0.$$

Ex

Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\cdot \; \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\cdot \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi .$$

$$\text{E-L} \Rightarrow \; -m^2 \phi = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$\rightarrow \boxed{(\Box + m^2)\phi = 0}$$

Klein - Gordon Eqn.

relativistic particle of mass m.

Ex $\phi = e^{-ip\cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

Noether's Thm

Thm

$$\boxed{\text{For every continuous symmetry, there exists a conserved current } j^\mu \text{ which implies a local conservation law}}$$

$$\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j} \qquad \Leftrightarrow \qquad \partial_\mu j^\mu = 0$$

Ex Conservation of charge ...

(gauss' law)

$$Q = \int j^0 \, d^3x$$

$$\Rightarrow \frac{\partial Q}{\partial t} = \int \frac{dj^0}{dt} d^3x = -\int \vec{\nabla}\cdot\vec{j}\, d^3x$$

$$= -\oint \vec{j}\, d^2 s$$

#

Idea Consider continuous transf. $\to$ infinitesimally...

(local)

$$(\star) \qquad \phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

small

$(\star)$ is a symmetry if EOM invariant under $(\star)$.

$\Leftrightarrow$ S is invariant.

$\Rightarrow$ $\mathcal{L}$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$.

for some J^μ.

Let us compare this expectation for $\Delta\mathcal{L}$ to the result obtained by varying the fields ...

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi)$$

$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right\}\right)}_{0}\Delta\phi$$

$$\Rightarrow \quad \Delta\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right\}$$

So $\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi$ is the desired J^μ.

So that $\quad \partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu.$$

Ex massless KG field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$$

Consider transformation:

$$\phi(x) \longrightarrow \phi(x) + \alpha \quad , \Delta\phi = 1$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = \partial^\mu\phi.$$

Check $\quad \partial_\mu j^\mu = \partial_\mu\partial^\mu\phi = \Box\phi = 0$ since

$$(m^2 + \Box)\phi = 0 \;\;\&\; m = 0$$

Ex **Complex KG field**

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi .$$

again, EOM is

$$(m^2 + \Box)\phi = 0.$$

Symmetry: $\phi \to e^{i\alpha}\phi$.

For infinitesimal transf we have:

$$\alpha \Delta\phi = i\alpha\phi \qquad \text{(Taylor expand)}$$
$$\alpha \Delta\phi^\dagger = -i\alpha\phi^\dagger$$

One can check that

$$j^\mu = i\left[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)\right]$$

is the conserved current.

$\boxed{\text{Ex}}$

Space-time translation:

Consider infi. translation:

$$x^\mu \to x^\mu - a^\mu$$

in field transforms this becomes:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Lagrangian is a scalar ⇒ must transform the same way:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Compare this to the eqn:

$$\mathcal{L} \longrightarrow \mathcal{L} + a \partial_\mu \mathcal{J}^\mu ,$$

we have $$\mathcal{J}^\mu = \delta^\mu_\nu \mathcal{L}$$

→ apply thm, we find:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\mu \phi) - \delta^\nu_\mu \mathcal{L}$$

Make μ explicit...

$$\boxed{T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L}}$$

→ STRESS-ENERGY TENSOR. (or Energy-Momentum tensor)

<u>Conserved charge</u> is the Hamiltonian...

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad (\text{time-translation})$$

momentum $\longrightarrow p^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi \, d^3x$ (spatial translation)

Ex Klein-Gordon field again...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 .$$

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - g^{00}\mathcal{L} = \dots$$

$$= \frac{1}{2}(\partial_0\phi)(\partial^0\phi) - \frac{1}{2}(\partial_i\phi)(\partial^i\phi) + \frac{1}{2}m^2\phi^2$$

So

$$H = \int T^{00}d^3x = \int\left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right\}d^3x$$

note: all terms are positive ... (sum of squares)

→ can't fall into arbitrary negative energy.

$\boxed{\text{THE KLEIN-GORDON FIELD} \; \infty \; \text{HARMONIC OSCILLATORS}}$

promote: ϕ, π to operators & impose suitable commutation relations...

Recall ...

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0 .$$

Harmonic oscillator: $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$

- $a^\dagger a = \frac{1}{\omega}H - \frac{1}{2} \Leftrightarrow H = \omega\left(a^\dagger a + \frac{1}{2}\right)$

  $\uparrow$

  # operator...

- $|0\rangle$, $a|0\rangle = 0$.

  $|n\rangle = \frac{1}{\sqrt{n!}}\underbrace{a^\dagger \ldots a^\dagger}_{n}|0\rangle$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -\omega a$; $[H, a^\dagger] = \omega a^\dagger$

  a lowers by ω

  $a^\dagger$ raises by ω

- $Ha^\dagger|n\rangle = (E_n + \omega)a^\dagger|n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

For continuum system ... commutation relations become ..

$$\boxed{\begin{aligned} [\phi(x), \pi(y)] &= i\delta^{(3)}(x-y) \\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0. \end{aligned}}$$

Next, the Hamiltonian is now also an operator ...
To find spec(H), Fourier transf $\phi(x)$

$$\rightarrow \quad \phi(x,t) = \int \frac{d^3p}{(2\pi)^3}\, e^{ip\cdot x}\, \phi(\vec{p},t)$$

Recall KG eqn: $(m^2 + \Box)\phi = 0$

$$\rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) \right\} \phi(\vec{p},t) = 0$$

$\Rightarrow$ This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}.$$

Now, $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2 \qquad (m=1)$

$\rightarrow$ know spectrum! $\quad (n+\frac{1}{2})\omega.$

$$\phi = \frac{1}{\sqrt{2\omega}}(a^\dagger + a) \quad ; \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$[a, a^\dagger] = 1.$$

Since it's more convenient to work in position space

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right) \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^\dagger e^{-ip x} \right) \end{aligned}}$$

note

$$\begin{cases} a_p \text{ goes with } e^{+ip\cdot x} \\ a_p^\dagger \text{ goes with } e^{-ip\cdot x}. \end{cases}$$

- Easy to show that

$$\begin{cases} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \end{cases}$$

- Can re-arrange...

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p + a_{-p}^\dagger \right) e^{ip\cdot x} \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p - a_{-p}^\dagger \right) e^{+ip\cdot x}. \end{cases}$$

→ get commutation relation between a_p:

$$\boxed{[a_p ; a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

Now, can check that

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{w_{p'}}{w_p}} \times e^{i(p\cdot x + p'\cdot x')} \times$$

$$\left([a^\dagger_{-p}, a_{p'}] - [a_p, a^\dagger_{-p'}]\right)$$

$$= i\delta^{(3)}(x - x') \checkmark$$

- Now, can express Hamiltonian in terms of ladder ops

Recall that

KG field, lost time

$$H = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} \partial^0\phi - g^{00}\mathcal{L} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi(x) \rightarrow \left(\text{like } p = \frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

So ...

$$H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2}m^2\phi^2 \right\}$$

with $\pi(x) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{w_p}{2}}\,(a_p - a^\dagger_{-p})\, e^{ip\cdot x}$

we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}}(a_p + a^\dagger_{-p})\, e^{ip\cdot x}$$

results in $\delta(p-p')$ $\Rightarrow p = p'$

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} \underbrace{e^{i(p+p')\cdot x}} \Big\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4} (a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'})$$

$$+ \frac{-p\cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}} (a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) \Big\}$$

Some $\delta^{(3)}$ will appear...

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)$$

So

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)}$$

$\rightarrow$ can evaluate commutators...

$$\boxed{[H, a_p^\dagger] = \omega_p a_p^\dagger \ ; \ [H, a_p] = -\omega a_p}$$

With H, can find momentum operator...

KG field $\longrightarrow$ from $P^i = \int d^3x\, T^{0i} = -\int \pi \partial_i \phi\, d^3x$, we get

$$\boxed{\begin{aligned} \vec{P} &= -\int d^3x\, \pi(x) \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a_p^\dagger a_p . \end{aligned}}$$

$a_p^\dagger$ creates momentum $\vec{p}$ & energy $\underset{E_p}{\omega_p} = \sqrt{|p|^2 + m^2} \rightarrow 0$.

Excitation : $a_p^\dagger a_q^\dagger \ldots |0\rangle \equiv$ "particles".

$\hookrightarrow$ each excitation $a^\dagger_p$ is a particle.

$\Rightarrow$ get particle statistics...

Consider 2-particle state... $a_p^\dagger a_q^\dagger |0\rangle$.

Since $[a_p^\dagger, a_q^\dagger] = 0$, we have

$$\boxed{a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle}$$

$\rightarrow$ Klein Gordon particles follow Bose-Einstein stats.

- **Normalization** $\langle 0|0\rangle = 1$.

$$|p\rangle \propto a_p^\dagger |0\rangle.$$

This normalization $\longrightarrow$ $\langle q|p\rangle = (2\pi)^3 \delta^{(3)}(q-p)$ is NOT Lorentz inv

Pf Under a Lorentz boost $\rightarrow$ $p_3' = \gamma(p_3 + \beta E)$

$$E' = \gamma(E + \beta p_3)$$

using the identity $\rightarrow$

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

we compute :

$$\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \cdot \left(\frac{dp_3'}{dp_3}\right)$$

$\underbrace{\delta(p_1-q_1)\delta(p_2-q_2)}_{\text{same}}\ \underbrace{\delta(p_3-q_3)}_{\text{boosted}}$

$$= \delta^{(3)}(p'-q') \cdot \gamma\left(1 + \beta \frac{dE}{dp_3}\right)$$

$$= \delta^{(3)}(p'-q') \frac{\gamma}{E}(E + \beta p_3)$$

$$= \delta^{(3)}(p'-q')(E'/E)$$

So $\boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\left(\frac{E'}{E}\right)}$ → $1 \Leftrightarrow E = E_p = E'$

For normalization to work → use $\overset{\omega_p}{\overset{\shortparallel}{E_p}}$, not E.

→ define: $\boxed{|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$

So that $\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$

Completeness relation...

1 particle → $\boxed{1 = \int \frac{d^3p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$

Ex Interpret $\phi(x)|0\rangle$ .... we know that $a_p^\dagger$ creates momentum p & energy $E_p = \omega_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x} |p\rangle \quad , \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

Pf. By defn...

annihilates.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|0\rangle \quad \square$$

→ $\phi(x)|0\rangle$ is a lin. superposition of single-particle states

~~Note HW1, HW2 easy~~

that have well-defd momentum.

When nonrelativistic $\to$ $E_p \simeq$ constant!

$\Rightarrow$ $\phi(x)$ acting on the vacuum, "creates a particle at position x".

$\hookrightarrow$ Confirm this by computing.

$$\langle 0|\, \phi(x)\, |p\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \left(a_{p'} e^{ip'\cdot x} + a^\dagger_{p'} e^{-ip'\cdot x}\right) \sqrt{2E_p}\, a^\dagger_p |0\rangle$$

$$\leadsto \boxed{\langle 0|\, \phi(x)\, |p\rangle = e^{ip\cdot x}}\,.$$

$\hookrightarrow$ Interpretation: position-space representation of the single-particle wfn of the state $|p\rangle$, just like ...

$\boxed{\langle x|p\rangle \propto e^{ip\cdot x}}$ in QM!

$\langle 0|\, \phi(x) \sim \langle x|$ ... (don't take this literally, ofc).

Note HW1, HW2 are easy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time → we quantized KG field in the Schrödinger picture.

→ Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$\mathcal{U}(t) = e^{-iHt}$ is the time evolution.

$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle.$ → State evolve in time

→ In the Heisenberg picture, ... Operators evolve in time

$$\mathcal{O}(t) = \mathcal{U}^{\dagger}(t)\, \mathcal{O}(0)\, \mathcal{U}(t).$$

So that

$$\langle \psi_1 | \mathcal{O}(t) | \psi_1 \rangle = \langle \psi_1(t) | \mathcal{O} | \psi_1(t) \rangle$$

↓ Heisenberg ↓ Schrödinger.

→ make the operators ϕ & π time-dependent like this

$$\boxed{\begin{aligned} \phi(x) \to \phi(x,t) &= e^{iHt}\phi(x)e^{-iHt} \\ \pi(x) \to \pi(x,t) &= e^{iHt}\pi(x)e^{-iHt} \end{aligned}}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x'\left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\right]$$

$(\phi \leftrightarrow \phi)$
$\Rightarrow$ only nontrivial term is 1^{st}.

$$= \int d^3x' \left(i\delta^{(3)}(x-x')\,\pi'(x,t)\right)$$

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t}\phi(x,t) = \pi(x,t)}$$

and

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x'\left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')(-\nabla^2 + m^2)\phi(x',t)\right)$$

(integrate by parts here)

$$= -i(-\nabla^2 + m^2)\phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\pi(x,t) = (m^2 - \nabla^2)\phi(x,t)}$$

Combining these 2 results we get...

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) = (\nabla^2 - m^2)\phi(x,t)}$$

rearranging this gives

$$\boxed{(\Box + m^2)\phi(x,t) = 0} \longrightarrow \text{just the KG eqn...}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

<u>Recall</u>: $H a_p = a_p (H - E_p)$ → from comm. rule.

$$\Longrightarrow \begin{pmatrix}\text{proof by} \\ \text{induction}\end{pmatrix} \quad \boxed{H^n a_p = a_p (H - E_p)^n}$$

Similarly, $$\boxed{H^n a_p^\dagger = a_p^\dagger (H + E_p)^n}$$

→ So, we have

$$\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}}$$ → from above...

and

$$\boxed{e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{+iE_p t}}$$

→ Now... we want to write $\phi(x,t)$ in terms of these operators. (since $\underset{\pi(x)}{\phi(x)}$ is a combo of a & $a^\dagger$)

- We know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before...

$$\phi(x) = \phi(x,0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

Substitute this into $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$ we find ...

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-iE_pt} e^{ip\cdot x} + a_p^\dagger e^{iE_pt} e^{-ip\cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow \quad p\cdot x = E_p \cdot x^0 - \vec{p}\cdot\vec{x} \Big|_{p^0=E_p} = E_p t - \vec{p}\cdot\vec{x} \Big|_{p^0=E_p}$$

So...

$$\boxed{\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right\} \Big|_{p^0=E_p}}$$

Similarly, we can find

$$\boxed{\pi(x,t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left(a_p e^{-i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right) \Big|_{p^0=E_p}}$$

from defn or from $\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$.

Note: we can also do everything, but starting from P and not H. But we won't worry about that.

Causality: Note that causality is broken ~~then~~ without the presence of fields.

$\rightarrow$ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture...

...the amplitude for a particle to propagate from y to x is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

"Two-point correlation function"

Let $\boxed{\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \text{ in terms of } a_p \text{ \& } a_p^\dagger \ldots |0\rangle$$

$$= \langle 0| a_p a_q^\dagger |0\rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

More explicitly...

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

@ $p^0 = E_p$
$p^{0\prime} = E_{p'}$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0| e^{-ip\cdot x} e^{+ip'\cdot y} a_p a_{p'}^\dagger |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{\cancel{(2\pi)^6}} \left(\frac{1}{\sqrt{2E_p}}\right)\frac{1}{\sqrt{2E_{p'}}} e^{-ip\cdot(x-y)} \cancel{(2\pi)^3} \delta^{(3)}(p-p')$$

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip\cdot(x-y)} \qquad \square.$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$...

(1) Suppose that $x-y = (t, 0, 0, 0)$, then

$$D(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \, \frac{p^2}{2\sqrt{p^2+m^2}} \, e^{-i\sqrt{p^2+m^2}\,t}$$

(timelike)

$$= \frac{1}{4\pi^2} \int_m^\infty dE \, \sqrt{E^2-m^2} \, e^{-iEt}$$

$$\underset{t\to\infty}{=} e^{-imt} \; \longrightarrow \; \text{dominated by region where } p \approx 0.$$

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

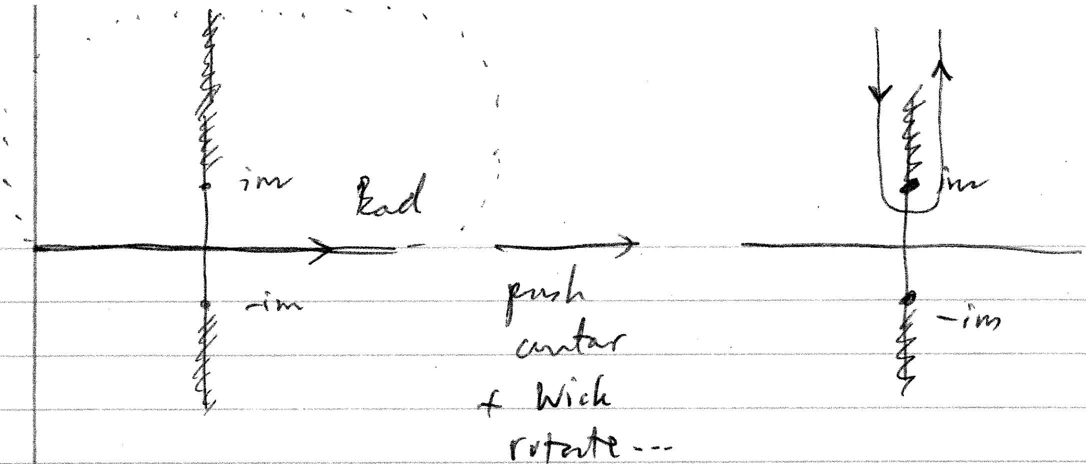
$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \, \frac{p^2}{2E_p} \, \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \, \frac{p e^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ $\pm im$ (singularity)...

→ must change contours... ≈ Wick rotate



$\rho = -ip$ (Wick rotate)

To get
$$D(x-y) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \, \frac{\rho e^{-i p r}}{\sqrt{\rho^2 - m^2}}$$

$$\Rightarrow \boxed{D(x-y) \underset{r\to\infty}{=} \frac{e^{-mr}}{r}} \longrightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $D(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^\mu (x-y)_\mu = -(\vec{x}-\vec{y})^2 < 0$ is spacelike, cannot have causality between x & y.

$D(x-y) \neq 0 \Rightarrow$??? paradox?

→ __NO!__ To discuss causality, we should ask not whether particles can propagate over spacelike intervals ...

... but whether a "<u>measurement</u>" ~~at~~ at one point can affect a measurement at another point whose separation from the first is spacelike –

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement @ x, call this $\Phi(x)$. & a local measurement @ y, called $\Phi'(y)$

So long as $[\Phi(x), \Phi'(y)] = 0$, the 2 measurements don't affect one another.

~ measure the field ϕ @ x & @ y,

If $[\phi(x), \phi(y)] = 0$ ~~when~~ $(x-y)^2 < 0$ then we're good

→ ~~$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x}[a_p, a_{p'}^\dagger] e^{ip'\cdot y} + e^{ip\cdot x}[a_p^\dagger, a_{p'}] e^{-ip'\cdot y} \right\}$~~

→ $$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \times \left\{ \left[(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_{p'} e^{-ip'\cdot y} + a_{p'}^\dagger e^{ip'\cdot y}) \right] \right.$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6\, 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x} \underbrace{[a_p, a_{p'}^\dagger]}_{(2\pi)^3 \delta^3(p-p')} e^{ip'\cdot y} + e^{ip\cdot x} \underbrace{[a_p^\dagger, a_{p'}]}_{-(2\pi)^3 \delta^3(p-p')} e^{-ip'\cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p}\right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

So $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$.

Since $D(y-x)$ is Lorentz invariant, can take

$y-x \to x-y$. (possible b/c $(x-y)^2 < 0$)

Note that when $(x-y)^2 > 0 \to$ there's no continuous transf that takes $y-x \to x-y$

$\to$ so this is only possible because $(x-y)^2 < 0$ (spacelike).

So $D(y-x) = D(x-y)$

So, $[\phi(x), \phi(y)] = 0$ ✓

The Klein-Gordon Propagator

Let's look at $[\phi(x), \phi(y)]$ in more details ...

$[\phi(x), \phi(y)]$ is just a number

$\Rightarrow$ can write $[\phi(x), \phi(y)] \equiv \langle 0| [\phi(x), \phi(y)] |0\rangle$

$$\to \langle 0|[\phi(x), \phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right\}$$

(assuming $x^0 > y^0$)

Poles @ $p^2 = m^2$ $\Updownarrow$ $p^0 = \pm E_p$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_p} - \frac{1}{2E_p} e^{-ip(x-y)} \Big|_{p^0 = -E_p} \right.$$

The Klein-Gordon Propagator

→ Before looking at this... need to look at Green's Function & Contour Integrals...

→ Consider function

$$\frac{1}{(p^0_0 - E_p)(p^0_0 + E_p)} \qquad E_p = \sqrt{|p|^2 + m^2}.$$

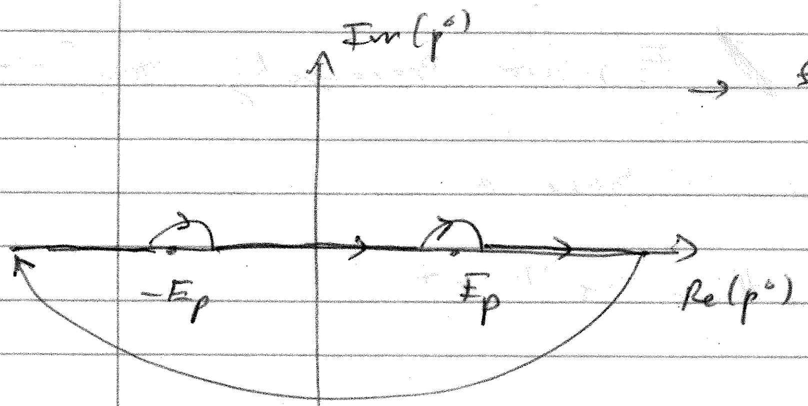
→ Poles at $p^0_0 = \pm E_p$.

Look at FT:

$$\boxed{f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}}$$

→ How to integrate this?

If $t > 0$ →



↑ close contour...
but clockwise → reverse sign

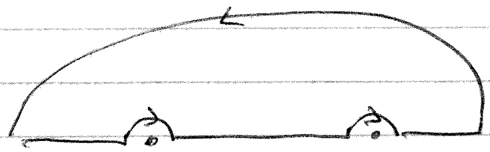
$$\rightarrow f(t) = (-2\pi i) \sum \text{Res}(f)$$

$$= (-2\pi i) \left\{ \frac{e^{iE_p t}}{2\pi(-2E_p)} + \frac{e^{-iE_p t}}{2\pi(2E_p)} \right\}$$

→

$$\boxed{f(t) = \frac{-i}{2E_p} \left(e^{+iE_p t} - e^{-iE_p t} \right)} \qquad (t > 0)$$

If $t<0$ close contour above poles



$\rightarrow \quad f(t) = 0.$

$\rightarrow$ So, altogether, we have...

$$f(t) = \int \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}$$

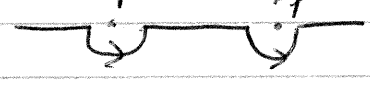
$$= \Theta(t) \left(\frac{-i}{2E_p}\right)\left(e^{-iE_p t} - e^{+iE_p t}\right)$$

where $\Theta(t)$ is the Heaviside Step fn...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

$\rightarrow$ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



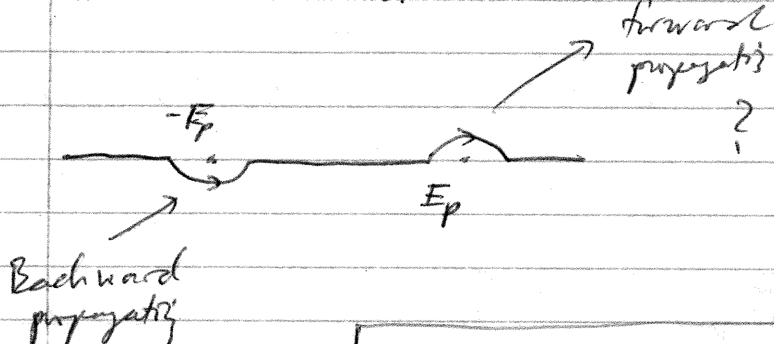
, then we'll get

$t > 0 \rightarrow f(t) = 0$

$t < 0 \rightarrow f(t) = \Theta(-t) \frac{i}{2E_p}\left(e^{-iE_p t} - e^{iE_p t}\right)$

$\rightarrow$ Advanced / Backward Propagating Green's fn.

☒ What about



$$\longrightarrow \quad \boxed{f(t) = \Theta(t)\left(\frac{-i}{2E_p}\right)e^{-iE_pt} + \Theta(-t)\left(\frac{-i}{2E_p}\right)e^{+iE_pt}}$$

↑ Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$ ...

Consider this quantity...

$$\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle = \int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{ \underset{\substack{\uparrow \\ \text{pole} \\ @\, p_0 = E_p}}{\frac{1}{2E_p}e^{-ip(x-y)}} + \underset{\substack{\downarrow \\ \text{pole @} \\ p_0 = -E_p}}{\frac{1}{-2E_p}e^{-ip(x-y)}} \right\}$$

4D integral →

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3} \underbrace{\int \frac{dp^0}{2\pi i}\frac{-1}{p^2-m^2}e^{-ip(x-y)}}_{f(t) \text{ before, where}}$$

$$(p^0 - E_p)(p^0 + E_p) = p^{0\,2} - |p|^2 - m^2 = p^2 - m^2.$$

Now, call ...

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle}$$

Then

$$\rightarrow (\Box+m^2)D_R(x-y) = \Box D_R(x-y) + m^2 D_R(x-y)$$

$$= (\Box\theta(x^0-y^0))\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ 2(\partial_\mu\theta(x^0-y^0))\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ \theta(x^0-y^0)(\Box+m^2)\underbrace{\langle 0|[\phi(x),\phi(y)]|0\rangle}_{\cancel{\text{constant}}}$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$+ 2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0$$

$$= \delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

needs verification (easy)

$$= -i\delta^{(4)}(x-y)$$

$$-i\delta^{(3)}(x-y)$$

so

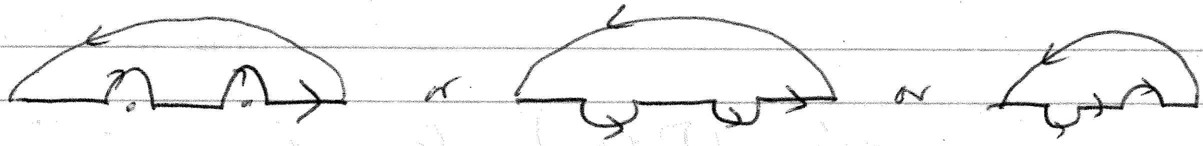
$$\boxed{(\Box+m^2)D_R(x-y) = -i\delta^{(4)}(x-y)}$$

$\longrightarrow$ $D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0$ @ $x^0 < y^0$

$\Rightarrow$ $D_R(x-y) \equiv$ "Retarded" Green's fn

Now ... As we have seen, there are many ways to take the contour ...

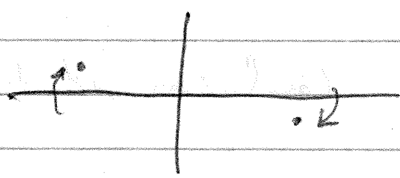


or ... or ... etc

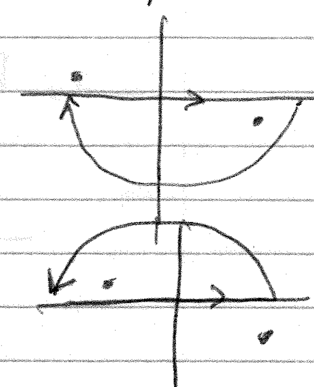
$\rightarrow$ Use the <u>Feynman prescription</u> instead

$$\boxed{D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}}$$

$\rightarrow$ Convenient! B/c now poles are $p^0 = \pm(E_p - i\varepsilon)$



$\rightarrow$ when $x^0 > y^0$ $\rightarrow$ take contour below



when $x^0 < y^0$ $\rightarrow$ take contour above.

$\rightarrow$ get same expression but with $x \leftrightarrow y$.

So,

$$\boxed{\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases} \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &\equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \end{aligned}}$$

"time-ordering" symbol

<u>Time-ordering symbol</u> $\Rightarrow$ instructs us to place the operators that follows in order with the latest to the left -

→ apply $(\Box + m^2)$ to last line, get D_F is Green's fn of Klein-Gordon Operator,

→ $D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator...

→ propagation amplitude

→ But we can't much calculation at this point just yet.

→ B/c We've only looked at the <u>free</u> KG theory

→ field eqn in this case is linear : there are <u>no interactions</u>...

→ this theory is too simple to make any predictions...

→ need perturbation ....

One kind of interaction that ~~can~~ can als be solved ...

→

Particle Creation by a classical Source

Consider the source $j(x)$

Recall... free field: $(\Box + m^2)\phi = 0$

→ now... $(\Box + m^2)\phi = j(x)$ fixed fn in space time.

$j(x)$ is nonzero only for a finite time interval

The associated Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is easiest to solve...

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)$$

With a source...

$$\phi(x) = \phi_0(x) + i\int d^4y \, D_R(x-y) j(y)$$

⋮

We won't worry about this for now...

Some problems & Insights

① **Classical EM (no sources)** follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

ⓐ Identify $\begin{cases} E^i = -F^{0i} \\ \varepsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} ; \quad \vec{\nabla} \cdot \vec{E} = 0 \quad (\nu = 0)$$

$$-\partial_t \mathcal{E} + \vec{\nabla} \times \vec{E} = 0 \quad (\nu = i)$$

② **Complex scalar field**

$$S = \int d^4x \left(\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \right)$$

Derive E-L eqn:

$$\boxed{\begin{aligned} & i\partial_t \phi^\dagger - \frac{1}{2m} \nabla^2 \phi^\dagger = 0 \\ & -i\partial_t \phi - \frac{1}{2m} \nabla^2 \phi = 0 \end{aligned}}$$

now... write $\phi \Rightarrow e^{-i\theta}\phi$, $\phi^\dagger \to e^{i\theta}\phi^\dagger$

$\sim \phi - i\theta\phi$ $\qquad \sim \phi^\dagger + i\theta\phi^\dagger$

$\to \Delta\phi \sim -i\theta$ $\qquad \Delta\phi^\dagger \sim i\phi^\dagger$

So that

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\Delta\phi^\dagger$$

conserved current.

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \to \ \ldots \text{compute} \ldots$$

→ can get Hamiltonian... there's a Ham. formula in book, but we won't worry abt this.

$$\mathcal{H} = \int d^3x \left(\pi^\dagger \pi + (\nabla\phi)^\dagger(\nabla\phi) + m^2\phi^\dagger\phi\right)$$

(3) If we take $(x-y)^2 = -r^2$ → can implicitly write

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2$ → $D(x-y)$ can be written ~~as~~ in terms of Bessels Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation... $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu}$

→ what happens to $\phi(x)$ under Λ?

we require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \longrightarrow \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform ... $\partial_\mu \phi(x) \longrightarrow \partial_\mu(\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu(\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\nu{}_\mu \, (\partial_\nu\phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu \, g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu \phi(x))^2 \longrightarrow g^{\mu\nu}(\partial_\mu \phi'(x))(\partial_\nu \phi'(x))$$

$$= g^{\mu\nu}\left[\partial_\mu(\phi(\Lambda^{-1}x))\right]\left[\partial_\nu(\phi(\Lambda^{-1}x))\right]$$

$$= g^{\mu\nu}\{(\Lambda^{-1})^\rho{}_\mu \partial_\rho \phi\}\{(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \phi\}(\Lambda^{-1}x)$$

$$= (\partial\phi)^2(\Lambda^{-1}x) \longleftarrow = g^{\rho\sigma}\partial_\rho\phi\, \partial_\sigma\phi\,(\Lambda^{-1}x) \text{ evaluated}$$

Recall

$$\boxed{\begin{aligned} \phi(x) &\longrightarrow \phi(\Lambda^{-1}x) \\ \partial_\mu \phi(x) &\longrightarrow (\Lambda^{-1})^{\nu}{}_{\mu} (\partial_\nu \phi)(\Lambda^{-1}x) \\ (\partial_\mu \phi(x))^2 &\longrightarrow (\partial_\mu \phi)^2(\Lambda^{-1}x) \end{aligned}}$$

So it is clear that

$$\boxed{\mathcal{L} \longrightarrow \mathcal{L}(\Lambda^{-1}x)}$$

Lagrangian is Lorentz-invariant.

$\Rightarrow$ The action $S = \int d^4x\, \mathcal{L}$ is also Lorentz inv.

$\rightarrow$ also clear that EOM is also Lorentz inv.

$$(\Box + m^2)\,\phi'(x) = \left((\Lambda^{-1})^{\mu}{}_{\nu}\, \partial_\mu (\Lambda^{-1})^{\sigma\mu} \partial_\sigma + m^2\right) \phi(\Lambda^{-1}x)$$

$$= (\partial^\sigma \partial_\sigma + m^2)\, \phi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Sep ~~[illegible]~~, 2020