

## 1 Introduction

The following introduction to Electromagnetically Induced Transparency (EIT) is based on Stephen E. Harris' article in *PHYSICS TODAY*, 1997. [Link](#), and various other sources including this well-written [thesis](#) by Furman of Reed College, this [paper](#), and of course *Principles of Laser Spectroscopy and Quantum Optics* by Paul Berman.

In shortest possible terms, EIT is a technique which renders an otherwise opaque atomic medium transparent with electromagnetic radiation. The medium is typically a [three-level](#) atomic system, with specific requirements: two of the possible three transitions must be dipole-allowed (so transition rules satisfied) and one not dipole-allowed. The spectrum of the medium without (blue) and with (red) EIT is shown below:

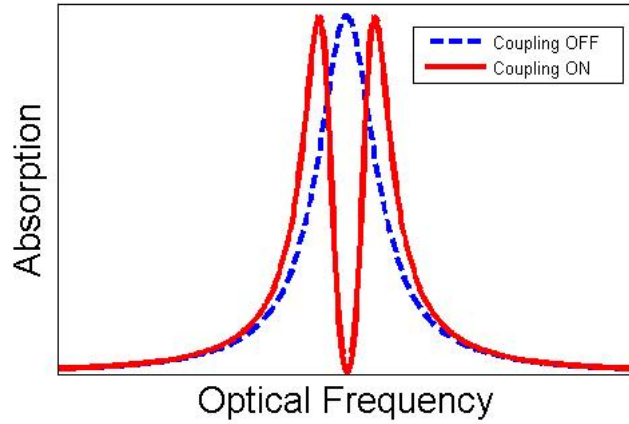


Figure 1

Notice how there is no absorption at resonance frequency with EIT. Typically, in the presence of a near-resonance field, a two-level atom with ground state  $|1\rangle$  and excited state  $|2\rangle$  will interact with the field, resulting in a non-zero probability amplitude of the excited state,  $|P_2|^2 = \langle 2|2\rangle > 0$ . If  $|P_2(\delta\omega)|^2$  is the population of the excited state as a function of the detuning, then it follows the blue, Lorentzian line in the above figure. In EIT where there are two radiation fields, though, the energy levels of a three-level atom are altered. This in turn creates a window of frequencies at which the medium is transparent.

## 2 Derivation Overview

The following derivation is (heavily) inspired by Furman's derivation and the Heidelberg paper, but roughly condensed and sprinkled with a bit of my own narratives and insights here and there.

Let's consider a  $\Lambda$  configuration:

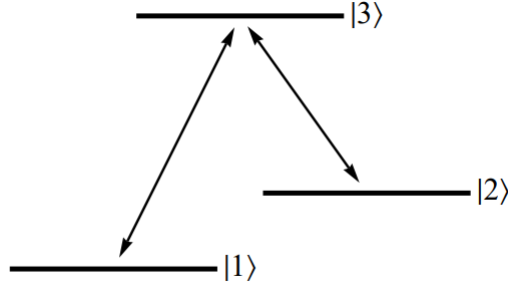


Figure 2

Let the energy of each state  $|n\rangle$ ,  $n = \{1, 2, 3\}$  be

$$E_n = \langle n | \hat{H} | n \rangle = \hbar\omega_n. \quad (1)$$

where  $\hat{H}$  is the neutral atom Hamiltonian. Assume that the transition  $|1\rangle \rightarrow |2\rangle$  is forbidden (just as shown in Figure 2). Since  $E_n$  and  $|n\rangle$  are the eigenvalues and eigenstates of  $\hat{H}$ , respectively, we let the bare-atom Hamiltonian  $\hat{H}_0$  be:

$$\hat{H}_0 = \hbar \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}. \quad (2)$$

We can do this because the eigenstates  $|n\rangle$  form an orthonormal basis. Next, let the applied fields be

$$\begin{aligned} \vec{E} &= \vec{E}_p \cos(\omega_p t - \vec{k}_p \cdot \vec{r}) + \vec{E}_c \cos(\omega_c t - \vec{k}_c \cdot \vec{r}) \\ &\approx \vec{E}_p \cos(\omega_p t) + \vec{E}_c \cos(\omega_c t) \\ &= \frac{\vec{E}_p}{2} (e^{i\omega_p t} - e^{-i\omega_p t}) + \frac{\vec{E}_c}{2} (e^{i\omega_c t} + e^{-i\omega_c t}). \end{aligned} \quad (3)$$

where  $\omega_p \approx \omega_{13} = \omega_3 - \omega_1$ , and  $\omega_c \approx \omega_{23} = \omega_3 - \omega_2$ . The subscripts  $p$  and  $c$  means **probe** and **coupling**, respectively. We should also denote the relevant detuning  $\delta_p = \omega_{13} - \omega_p$  and  $\delta_c = \omega_{12} - \omega_c$ . It makes sense to label our subscripts this way, because in the end we are interested in the probability amplitude of  $|3\rangle$  as a function of the detuning  $\delta_p$  of the probe beam from (bare atom) resonance.

With the perturbation from the radiation, the new Hamiltonian of the atom is:

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad (4)$$

where, with  $\hat{\rho} \equiv qd$  being the dipole moment operator:

$$\hat{H}_{1,ij} = -E\hat{\rho}_{ij}, \quad (5)$$

where  $E$  is the field strength. We will see how  $E$  relates to the Rabi rate  $\Omega$  later. Now, a state does not dipole-interact with itself, hence  $\rho_{ii} = 0$ . In addition, since the transition  $|1\rangle \rightarrow |2\rangle$  is forbidden,  $\rho_{12} = \rho_{21} = 0$ . It follows that:

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_1 \\ &= \hbar \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} - E \begin{pmatrix} 0 & 0 & \rho_{13} \\ 0 & 0 & \rho_{23} \\ \rho_{31} & \rho_{32} & 0 \end{pmatrix}. \end{aligned} \quad (6)$$

Now, what we have been working with so far are the time-independent eigenstates  $|n\rangle$ . In the following steps we shall bring in the time-dependent parts. To do this, we invoke the unitary matrix  $U(t)$ , which transforms  $|n\rangle$  into full time-dependent wavefunctions:

$$U(t) = e^{iH_0 t/\hbar} \begin{pmatrix} e^{i\omega_1 t} & 0 & 0 \\ 0 & e^{i\omega_2 t} & 0 \\ 0 & 0 & e^{i\omega_3 t} \end{pmatrix}. \quad (7)$$

Obviously, if we apply  $\hat{U}(t)$  to  $\hat{H}_0$  there wouldn't be anything interesting, since  $\hat{U}(t) = d/dt \hat{H}_0$ . But we can apply  $\hat{U}(t)$  to  $\hat{H}_1$ . The change of basis rule gives

$$\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t) = -E \begin{pmatrix} 0 & 0 & \rho_{13}e^{-i\omega_{13}t} \\ 0 & 0 & \rho_{23}e^{-i\omega_{23}t} \\ \rho_{31}e^{i\omega_{13}t} & \rho_{32}e^{i\omega_{32}t} & 0 \end{pmatrix}. \quad (8)$$

Multiplying  $E$  into  $\hat{H}_1$  and applying rotating wave approximation to result gives the non-zero matrix elements:

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t)\right)_{13} = -\frac{1}{2}E_p\rho_{13}e^{i(\omega_p-\omega_{13})t} \quad (9)$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t)\right)_{23} = -\frac{1}{2}E_c\rho_{23}e^{i(\omega_c-\omega_{23})t} \quad (10)$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t)\right)_{31} = -\frac{1}{2}E_p\rho_{31}e^{i(\omega_p-\omega_{31})t} \quad (11)$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t)\right)_{32} = -\frac{1}{2}E_c\rho_{32}e^{i(\omega_c-\omega_{32})t} \quad (12)$$

Transforming  $\hat{H}_1$  back to the vector space of  $|n\rangle$  gives:

$$\begin{aligned} \hat{H}_1 &= \hat{U}^\dagger(t) \left( \hat{U}(t)\hat{H}_1\hat{U}^\dagger(t) \right) \hat{U}(t) \\ &= -\frac{1}{2} \begin{pmatrix} 0 & 0 & E_p\rho_{13}e^{i\omega_p t} \\ 0 & 0 & E_c\rho_{23}e^{i\omega_c t} \\ E_p\rho_{31}e^{-i\omega_p t} & E_c\rho_{32}e^{-i\omega_c t} & 0 \end{pmatrix}. \end{aligned} \quad (13)$$

Now, to put the dipole moment in terms of the Rabi frequency  $\Omega$ :

$$\Omega_p = \frac{E_p |\rho_{13}|}{\hbar} = \frac{E_p \rho_{13} e^{-i\phi_p}}{\hbar} \quad (14)$$

$$\Omega_c = \frac{E_c |\rho_{23}|}{\hbar} = \frac{E_c \rho_{23} e^{-i\phi_c}}{\hbar} \quad (15)$$

So the total Hamiltonian is

$$\hat{H} = \frac{\hbar}{2} \begin{pmatrix} 2\omega_1 & 0 & -\Omega_p e^{i(\omega_p t + \phi_p)} \\ 0 & 2\omega_2 & -\Omega_c e^{i(\omega_c t + \phi_c)} \\ -\Omega_p e^{-i(\omega_p t + \phi_p)} & -\Omega_c e^{-i(\omega_c t + \phi_c)} & 2\omega_3 \end{pmatrix} \quad (16)$$

Now, since we want our final result in terms of the frequencies only, we shall express  $\hat{H}$  in a new basis that is time and phase independent. Let a unitary matrix  $\tilde{U}(t)$  be given such that it satisfies the required change of basis. It follows that the eigenstates transform as

$$|\tilde{n}\rangle = \tilde{U}(t) |n'\rangle. \quad (17)$$

Let the new Hamiltonian in this new basis be  $\tilde{H}$ . The new eigenstate also has to solve the Schrödinger equation in the new basis, so

$$\begin{aligned} \tilde{H} |\tilde{n}\rangle &= i\hbar \frac{\partial}{\partial t} |\tilde{n}\rangle \\ &= i\hbar \frac{\partial}{\partial t} (\tilde{U} |n'\rangle) \\ &= i\hbar \left( \frac{\partial \tilde{U}}{\partial t} |n'\rangle + \tilde{U} \frac{\partial}{\partial t} |n'\rangle \right) \\ &= i\hbar \left( \frac{\partial \tilde{U}}{\partial t} |n'\rangle + \frac{1}{i\hbar} \tilde{U} \hat{H} |n'\rangle \right) \\ &= \left( i\hbar \frac{\partial \tilde{U}}{\partial t} \tilde{U}^\dagger + \tilde{U} \hat{H} \tilde{U}^\dagger \right) \tilde{U} |n'\rangle \\ &= \left( i\hbar \frac{\partial \tilde{U}}{\partial t} \tilde{U}^\dagger + \tilde{U} \hat{H} \tilde{U}^\dagger \right) |\tilde{n}\rangle. \end{aligned} \quad (18)$$

So  $\tilde{H}$  can be readily computed:

$$\begin{aligned} \tilde{H} &= \left( i\hbar \frac{\partial \tilde{U}}{\partial t} \tilde{U}^\dagger + \tilde{U} \hat{H} \tilde{U}^\dagger \right) \\ &= \frac{\hbar}{2} \begin{pmatrix} 2(\omega_1 + \omega_p) & 0 & \Omega_p \\ 0 & 2(\omega_2 + \omega_c) & -\Omega_c \\ -\Omega_p & -\Omega_c & 2\omega_3 \end{pmatrix} \end{aligned} \quad (19)$$

Note that we can add  $\hbar(\omega_1 + \omega_p)\hat{I}$  to  $\tilde{H}$  without changing the physical interpretation. In fact, the form of  $\tilde{H}$  now is a result of our definition of the  $\omega$ 's. Now, let us define the detunings  $\delta_c = \omega_{23} - \omega_c = \omega_3 - \omega_2 - \omega_c$  and  $\delta_p = \omega_{13} - \omega_p = \omega_3 - \omega_1 - \omega_p$ . So, the “better” Hamiltonian is

$$\begin{aligned}\tilde{H}' &= \tilde{H} - \hbar(\omega_1 + \omega_p)\hat{I} \\ &= \hbar \begin{pmatrix} 0 & 0 & -\Omega_p \\ 0 & 2(\delta_p - \delta_c) & -\Omega_c \\ -\Omega_p & -\Omega_c & 2\delta_p \end{pmatrix}\end{aligned}\quad (20)$$

The eigenvalues of  $\tilde{H}'$  are:

$$\begin{aligned}E^+ &= \hbar\omega^+ = \frac{\hbar}{2} \left( \delta_p + \sqrt{\delta_p^2 + \Omega_p^2 + \Omega_c^2} \right) \\ E^- &= \hbar\omega^- = \frac{\hbar}{2} \left( \delta_p - \sqrt{\delta_p^2 + \Omega_p^2 + \Omega_c^2} \right) \\ E^0 &= \hbar\omega = 0.\end{aligned}\quad (21)$$