

Hele-Shaw Flow & Saffman-Taylor Instability

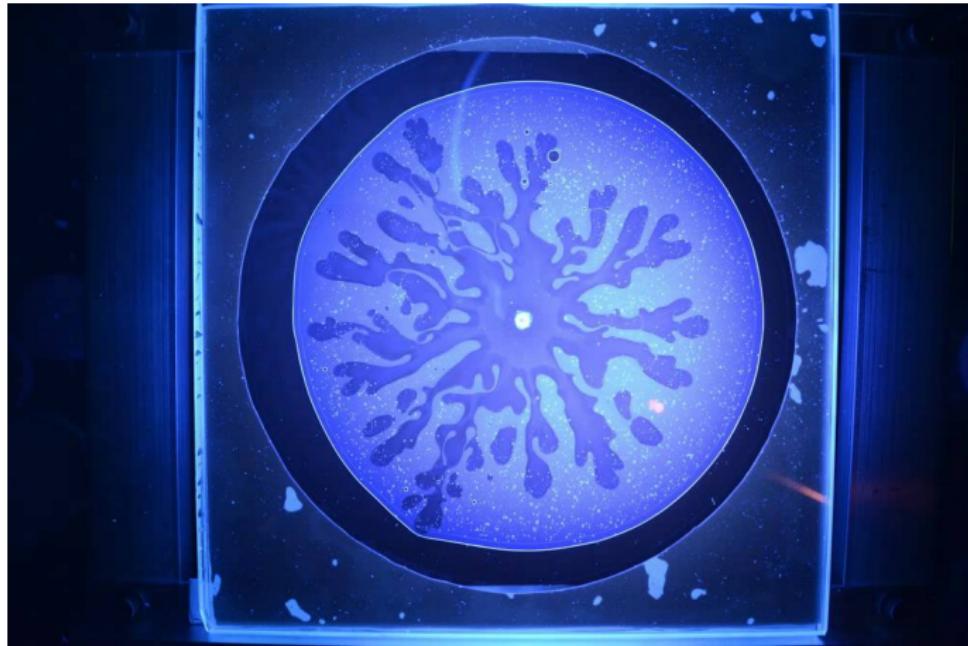
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PH333: Experimental Soft-Matter Physics

Professor Jonathan McCoy

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Hele-Shaw flow: A virtual demo ([link](#))



Hele-Shaw flow



Figure: (1) Hele-Shaw (1854–1941), British engineer [3], (2) Hele-Shaw apparatus

- ♣ Hele-Shaw invented the apparatus to study how water flows around a ship's hull.[2] Think 19th century “wind” tunnels!

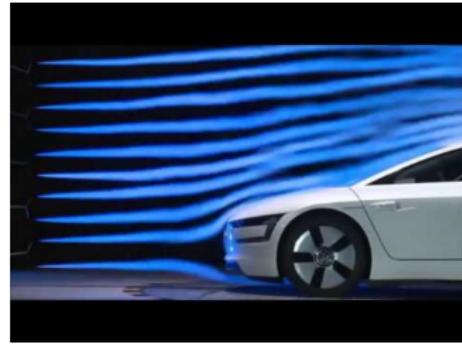
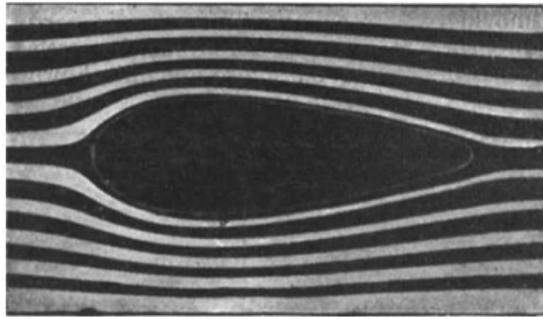


Figure: (1) HS's set up [3], (2) Car in a wind tunnel

Hele-Shaw flow



Figure: (1) Saffman (1931-2008), (2) Taylor (1886-1975)

- ♣ Saffman-Taylor in 1958 used Hele-Shaw's device to investigate fingering instability in various branching phenomena. This is now known as "Saffman-Taylor instability." [2]

Hele-Shaw flow: The Experiment

Experimental setup: (Source: [Karno Widjaja's lab at CMU](#))

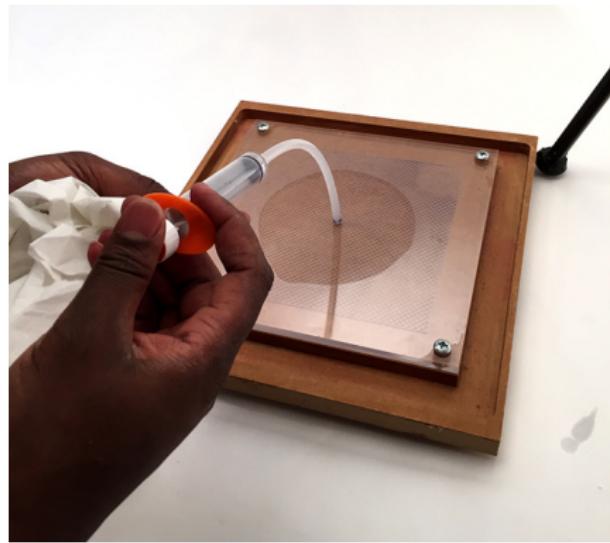


Figure: Before



Figure: After

Hele-Shaw flow: The Physics

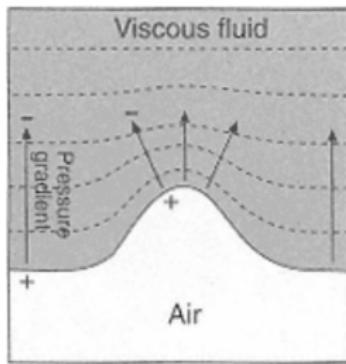
Governing equation [1]:

$$\text{Flow: } \vec{u}(x, y) = -\frac{b^2}{12\mu} \nabla \vec{p}(x, y) \quad \& \quad \text{Conservation law: } \nabla \cdot \vec{u} = 0$$

where

- $\vec{u}(x, y)$: average 2D velocity of the fluid between the plates
- $\vec{p}(x, y)$: the pressure field
- b : the (small) gap between the two plates
- μ : viscosity of the fluid

⇒ Can be derived from the Navier-Stokes equation!
⇒ Can be derived from Newton's 2nd law of motion!



$$\vec{u}(x, y) = -\frac{b^2}{12\mu} \nabla p(x, y)$$

- Flow velocity is negatively proportional to gradient of pressure field
- Saffman & Taylor \implies the gradient in pressure around a bulge at the air/oil interface gets steeper as the bulge gets sharper.
- Self-amplifying process: a small initial bulge begins to move faster than the interface to either side
 \implies Saffman-Taylor instability

Saffman-Taylor Instability: The Forming of Instabilities



Figure: [Link](#)

At first the fluid will appear to grow as a circle until it reaches a 'critical wavelength' where instabilities form on its boundary in the form of fingers.

These instabilities come from disturbances in the liquid such as:

- Pressure differences of the fluid
- Irregularities in the container
- Particulate matter in the fluid

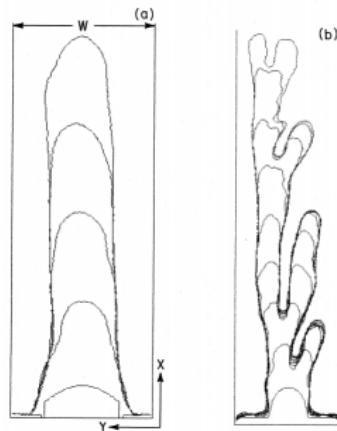
Instabilities on the boundary form into finger-like patterns.

Finger Size and Pattern Ornateness

Characteristic length scale related to the average width of the fingers. [3]

The number and width of “fingers” relates to surface tension and increased pressure means decreased surface tension.

The splitting of the fingers is due to low surface tension.



$$d_0 = \frac{\pi^3}{3} \frac{b^2}{W^2} \frac{T}{\mu U}$$

Figure: [1], [2] Surface tension d_0 is inversely proportional to the fluid velocity U , thus for slower velocity and lower pressure (a) we see less tip-splitting and for higher velocity and higher pressure (b) we see defined tip-splitting emerge.

Saffman-Taylor Instability:

Surface tension tends to want to decrease surface area.

Saffman-Taylor instability wants to increase the critical length on all scales.

"Energetic Cost" [3]

Infinitesimally small surface tension allows for fractal Patterns

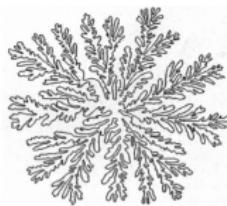


Fig. 5.15

Viscous fingering has a characteristic length scale, which determines the minimum width of the branches. So the fingers are fatter than the fine filaments of DLA clusters. (Image: Yves Couder, Ecole Normale Supérieure, Paris.)

Surface tension & Saffman-Taylor instability compete \implies why we get fingers of certain thicknesses (fingers that are too fat/too thin are prohibited)

Not all fingers are the same

When the tip of a finger is disturbed, regularity might appear.

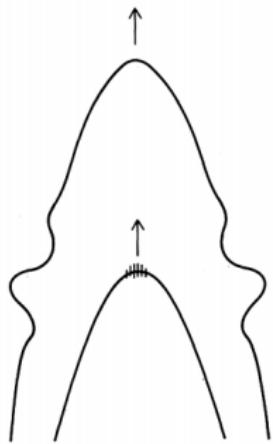
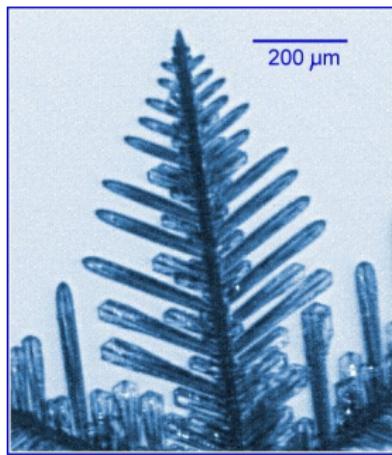
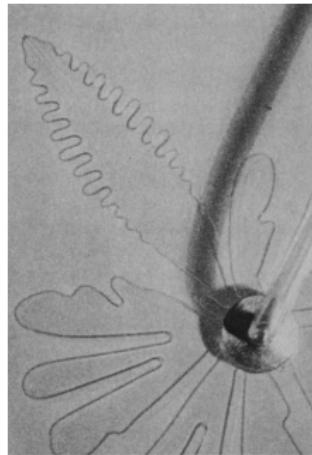


Figure: (1) Dendrite-like finger due to a Couder's bubble [5]; (2) Dendrite on a snowflake ([source](#)); (3) Formation mechanism [5]

Simulation: Harder than it seems

Computer simulations of Hele-Shaw fingers is highly non-trivial, requiring complex-analytical methods [4]:

- Treat \mathbb{R}^2 as \mathbb{C} and use conformal mapping
- Cauchy-Green coordinates (cf. CIF) to evolve the boundary

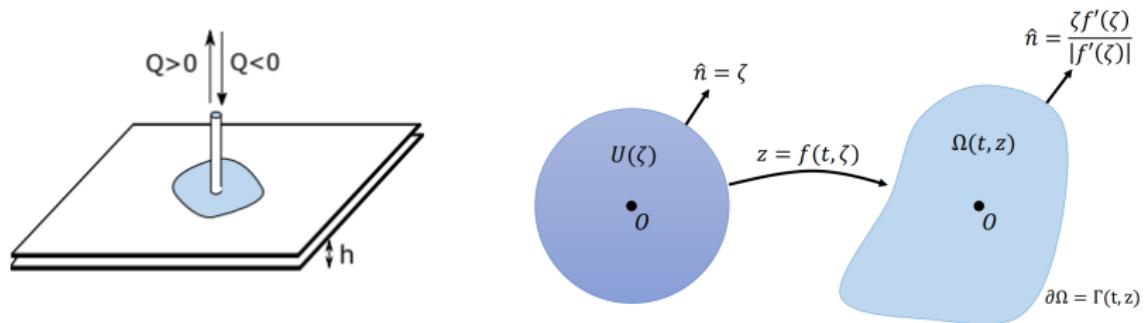


Figure: [4] Initial setup & Conformal mapping

Simulation: Results



Figure: Computer-generated Hele-Shaw flow by [4]

- ♣ Watch a computer-generated Hele-Shaw flow [now](#).

Real-world occurrences



Figure: Fingering pattern in Mud!

Real-world occurrences



Photographie : Pierre Thomas

Figure: Finger pattern in Rocks!

References

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