PH312: Physics of Fluids (Prof. McCoy) – Reflection
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1.

(a) The solution to the PDE

$$\partial_t u = \nu \partial_y^2 u$$

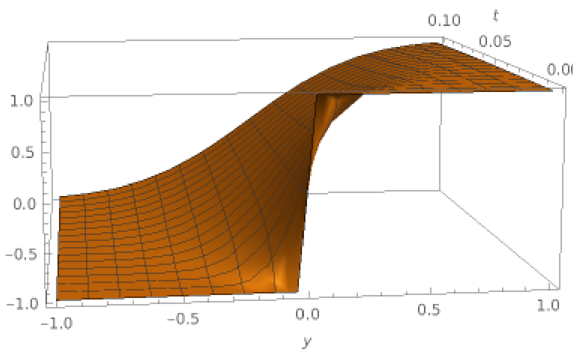
with initial condition

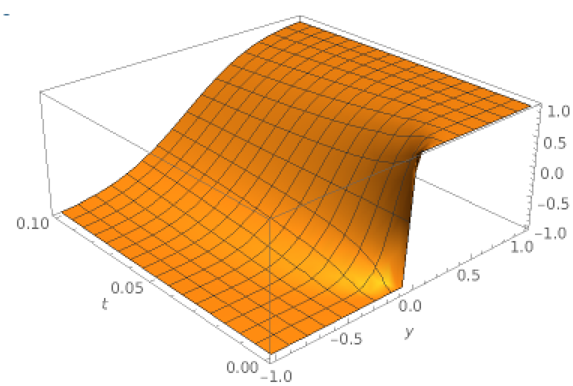
$$u(y,0) = \begin{cases} +U, & y > 0 \\ -U, & y < 0 \end{cases}$$

is given by

$$u(\eta) = U \operatorname{erf}(\eta), \qquad \eta = y/\sqrt{4\nu t}.$$

Let $U = 1, \nu = 1$, we find the following space-time plots of the solution: This makes sense: At





$t = 0$ there is a discontinuity due to the initial condition. But the solution (flow field) becomes smoother as t increases.

Mathematica code:

```
eta[y\_,t\_] = := y/Sqrt[4t];
Plot3D[Erf[eta[y,t]], {y,-1,1},{t,0,1}, AxesLabel -> Automatic]
```

(b) To see how $u(\eta)$ is obtained, we use the heat kernel G. For each y, t, we set $\alpha(y') = (y-y')/\sqrt{4\nu t}$. This means that

$$\int_0^\infty dy' \cdots \to -\int_{y/\sqrt{4\nu t}}^{-\infty} d\alpha \cdots = +\int_{-\infty}^{y/\sqrt{4\nu t}} d\alpha \ldots$$

since y' carries a minus sign in α.

$$
\begin{aligned}
u(y,t) &= G * u(y,0) \\
&= \frac{1}{\sqrt{4\nu t}} \cdot \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{(y-y')^2}{4\nu t}\right) \cdot u(y',0)\, dy' \\
&= \frac{U}{\sqrt{4\nu t}} \cdot \frac{1}{\sqrt{\pi}} \left[\int_{0}^{\infty} \exp\left(-\frac{(y-y')^2}{4\nu t}\right) dy' - \int_{-\infty}^{0} \exp\left(-\frac{(y-y')^2}{4\nu t}\right) dy'\right] \\
&= \frac{U}{\sqrt{\pi}} \left[\int_{-\infty}^{y/\sqrt{4\pi t}} \exp(-\alpha^2)\, d\alpha - \int_{y/\sqrt{4\pi t}}^{\infty} \exp(-\alpha^2)\, d\alpha\right] \\
&= \frac{U}{\sqrt{\pi}} \left[2 \int_{0}^{y/\sqrt{4\nu t}} \exp(-\alpha^2)\, d\alpha\right] \equiv \frac{2U}{\sqrt{\pi}} \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right),
\end{aligned}
$$

where we have used the symmetry of the Gaussian to cancel the integrals $-\int_0^\infty$ and $\int_{-\infty}^0$ and keep 2 terms of $\int_0^{y/\sqrt{4\nu t}}$ and the definition of the error function.

(c) We have $\omega = \nabla \times u$. Since the curl can go through derivatives, the PDE $\partial_t u = \nu \partial_y^2 u$ can be transformed to $\partial_t \nabla \times u = \nu \partial_y^2 \nabla \times u$, which gives $\partial_t \omega = \nu \partial_y^2 \omega$. By the geometry of the problem, ω can be treated as a scalar field, which is the only nonzero component of the vorticity (more formally denoted by $\vec{\omega}$). With the expression for $u(\eta)$, we write

$$
\omega(y,t) = -\partial_y u(y,t) = -\frac{\partial}{\partial y} \frac{U}{\sqrt{\pi}} \left[2 \int_0^{y/\sqrt{4\nu t}} \exp(-\alpha^2)\, d\alpha\right].
$$

By Leibniz's rule for integrals we find

$$
\omega(y,t) = \frac{-2U}{\sqrt{\pi}} \exp\left(-\frac{y^2}{4\nu t}\right) \frac{d}{dy} \frac{y}{\sqrt{4\nu t}} - 0 + \cancel{\int_0^{y/\sqrt{4\nu t}} \frac{\partial}{\partial y} \exp(-\alpha^2)\, d\alpha}.
$$

So,

$$
\omega(y,t) = \frac{-U}{\sqrt{\pi \nu t}} \exp\left(-\frac{y^2}{4\nu t}\right).
$$

This is a negative Gaussian whose width is proportional to $\sqrt{t}$ and for any $t > 0$ decreases as $|y|$ increases. At small t, the vorticity ω is highly concentrated near the origin. As t increases, ω away *diffuses* towards infinity. At any $t > 0$, the total vorticity is $\int_{\mathbb{R}} \omega\, dy$. Since ω is a scaled Gaussian, $\int_{\mathbb{R}} \omega\, dy$ is a constant depending only on U (since ν can be absorbed into t, the analogue of the standard deviation). We therefore conclude that the total amount of vorticity is constant.

(d) Starting with $\omega = -2U\delta(y)$ at $t = 0$, we find

$$
\begin{aligned}
\omega(y,t) &= -2U \cdot G * \delta(y) = -2U \cdot G(y,t) \\
&= \frac{-2U}{\sqrt{4\pi\nu t}} \exp\left(-\frac{y^2}{4\nu t}\right) = \frac{-U}{\sqrt{\pi \nu t}} \exp\left(-\frac{y^2}{4\nu t}\right),
\end{aligned}
$$

since convolving G with the delta function is an evaluation of G at $y' = 0$.

2.

(a) To derive the stream function K&C 9.64, we start by taking the curl on both sides of $\nabla p = \mu \nabla^2 \mathbf{u}$. This gives $0 = \nabla^2 \omega$, since ∇p is conservative. Next, since the nonzero components of the flow field $\mathbf{u}$ are u_r and u_θ, there is only one nonzero component of ω, which is the axial ω_φ (right-hand rule). The spherical curl gives us ω_φ in terms of u_r, u_θ.

Next, since we're in axisymmetric flow, a stream function ψ can be defined such that $\mathbf{u} = -\nabla\varphi \times \nabla\psi$, which allows us to write u_r and u_θ in terms of derivatives ψ. With this, we can write ω_φ in terms of derivatives of ψ. From $\nabla^2\omega = 0$, we obtain (9.64).

(b) With

$$\psi(r,\theta) = U r^2 \sin^2\theta \left[\frac{1}{2} - \frac{3a}{2r} + \frac{a^3}{4r^3}\right],$$

we find

$$\begin{aligned}
u_r &= \frac{1}{r^2 \sin\theta} \partial_\theta \psi \\
&= \frac{1}{r^2 \sin\theta} \partial_\theta \left\{ U r^2 \sin^2\theta \left[\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right]\right\} \\
&= \frac{2Ur^2 \sin\theta\cos\theta}{r^2 \sin\theta} \left[\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right] \\
&= U\cos\theta \left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right]
\end{aligned}$$

and

$$\begin{aligned}
u_\theta &= -\frac{1}{r\sin\theta} \partial_r \psi \\
&= -\frac{1}{r\sin\theta} \partial_r \left\{ U r^2 \sin^2\theta \left[\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right]\right\} \\
&= -\frac{U\sin^2\theta}{r\sin\theta} \left[r - \frac{3ar}{4r} - \frac{ra^3}{4r^3}\right] \\
&= -U\sin\theta \left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3}\right].
\end{aligned}$$

(c) We have $\nabla p = \mu \nabla^2 \mathbf{u}$, so we need to take the spherical laplacian of $\mathbf{u}$. While one can try to do this by hand using the formula of laplacian for a vector in the Appendix of K&C (which can be quite tedious), we can also do this in Mathematica:

```
Ur[r_,\[Theta]_,\[Phi]_]:=U*Cos[\[Theta]]*(1-3a/(2r)+a^3/(2r^3) );
U\[Theta][r_,\[Theta]_,\[Phi]_]:= -U*Sin[\[Theta]]*(1-3a/(4r)-a^3/(4r^3));
U\[Phi][r_,\[Theta]_,\[Phi]_] :=0;

Laplacian[{Ur[r,\[Theta],\[Phi]], U\[Theta][r,\[Theta]
,\[Phi]], U\[Phi][r,\[Theta],\[Phi]]}, {r, \[Theta], \[Phi] },
 "Spherical"] //Expand//MatrixForm
```

to find

$$\nabla p = \nabla^2 \mathbf{u} = \begin{pmatrix} \frac{3aU\cos\theta}{r^3} & \frac{3aU\sin\theta}{2r^3} & 0 \end{pmatrix}^\top .$$

Taking the spherical gradient of p, we find the following equations:

$$\partial_r p = \frac{3aU\cos\theta}{r^3} \quad \text{and} \quad \frac{1}{r}\partial_\theta p = \frac{3aU\sin\theta}{2r^3}.$$

Using the method of inspection [1] and setting the integration constant to be p_∞, we get the following expression for p:

$$p = -\mu U \cos\theta \frac{3a}{2r^2}.$$

We remark that this expression is analogous to the electric potential due to a small dipole in electrostatics: $V = kp\cos\theta/r^2$ where $\vec{p} = q\vec{d}$ is the electric dipole moment.

(d) Now we calculate the stress components:

$$\sigma_{rr} = 2\mu\partial_r u_r = 2\mu U\cos\theta\left[\frac{3a}{2r^2} - \frac{3a^3}{2r^4}\right]$$

and

$$\sigma_{r\theta} = \mu\left[r\partial_r(u_\theta/r) + (1/r)\partial_\theta u_r\right] = -\frac{3\mu U a^3}{2r^4}\sin\theta.$$

With these, we can compute the component of the drag force per unit area in the direction of the uniform stream:

$$\begin{aligned}
[-p\cos\theta + \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta]_{r=a} &= \frac{3a\mu U\cos^2\theta}{2a^2} + \cancel{2\mu U\cos^2\theta\left[\frac{3a}{2a^2} - \frac{3a^3}{2a^4}\right]} + \frac{3\mu U a^3\sin^2\theta}{2a^4} \\
&= \cos^2\theta\frac{3\mu U}{2a} + \sin^2\theta\frac{3\mu U}{2a} \\
&= \frac{3\mu U}{2a}.
\end{aligned}$$

Integrating this (a constant) over the surface of the sphere is just multiply it by the surface area $4\pi a^2$ of the sphere, so the total force is

$$F = \frac{3\mu U}{2a} \times 4\pi a^2 = 6\pi\mu a U.$$

We have just derived Stokes' law of resistance!

[1] more commonly known as "equation staring"