

Sep 28, 2020

## Dirac Matrices & Dirac Field Bilinears

Oct 2, 2020 Recall that  $\bar{\psi}\psi$  is Lorentz scalar...

Recall that  $\bar{\psi}\gamma^\mu\psi$  is also a 4-vector.

⇒  $\boxed{?}$  Consider  $\bar{\psi}\tilde{\Gamma}\psi$ , where  $\tilde{\Gamma}$  is any  $4 \times 4$   
 → can we decompose  $\tilde{\Gamma}$  into terms that have  
 definite transformation properties under the Lorentz  
 group?

↳  $\tilde{\Gamma}$  can be written as combo of 16-element basis  
 defined by

$$\left. \begin{array}{lll}
 1: & \mathbb{1} & \rightarrow 1 \\
 4: & \gamma^\mu & \rightarrow 4C2 \\
 6: & \gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \equiv \gamma^{\mu\nu\rho\sigma} & \rightarrow 4C3 \\
 4: & \gamma^{\mu\nu\rho} = \gamma^{\mu\rho}\gamma^\nu & \rightarrow 4C2 \\
 1: & \gamma^{\mu\nu\rho\sigma} = \gamma^{\mu\rho}\gamma^\nu\gamma^\sigma & \rightarrow 4C2
 \end{array} \right\}$$

16 total.

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisymmetric tensor of successively higher ranks

→ Introduction

$$\bar{\gamma}^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

$$\begin{aligned}
 0123 &\rightarrow 1 \\
 7023 &\rightarrow -1
 \end{aligned}$$

↳ totally  
anti-symmetric

$$\text{Note that } \rightarrow \boxed{(\gamma^5)^2 = 1}$$

$$\rightarrow (\gamma^5)^+ = -i(\gamma^2)^+ - i(\gamma^0)^+$$

$$= +i\gamma^2\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

also

$$\{\gamma^5, \gamma^m\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^m + i\gamma^m\gamma^0\gamma^1\gamma^2\gamma^3$$

(-1)

$$= 0$$

and thus

$$[\gamma^5, \gamma^{uv}] = [\gamma^5, \frac{i}{4}\{\gamma^u, \gamma^v\}] = 0$$

$\Rightarrow$  Eigenstates of  $\gamma^5$  with different eigenvalues don't mix under Lorentz transform.

$\rightarrow$  In basis, can write

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \text{for } \Psi_L \text{ (left-hd)}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \text{for } \Psi_R \text{ (right-hd)}$$

$\rightarrow$  a Dirac spinor with only L/R component is an eigenstate of  $\gamma^5$  with eigenvalue  $(-1)/(1)$ .

With  $\gamma^5$ , can rewrite the table of  $4 \times 4$  matrices as

$\gamma^m$	scalar	1
$\gamma^m\gamma^5$	vector	4
$\gamma^5$	tensor	6
$\gamma^5$	pseudo vector	4
$\gamma^5$	pseudo scalar	7
		16

pseudo-vector/scalar is due to the fact that they transform like vector/scalar, BUT with an additional under Lorentz transf  $\rightarrow$  in charge under parity-transf.

Ex Parity transf:  $\vec{x} \rightarrow -\vec{x}$

$$\hookrightarrow (x^0, x^i) \rightarrow (x^0, -x^i)$$

If instead  $(x^0, \vec{x}) \rightarrow -(\vec{x}, x^0) = (-x^0, \vec{x})$   
under parity, we call this a pseudo-vector

$\rightarrow$  pseudo vector/scalar flips sign under parity transf.

$\rightarrow$  From vector + pseudo-vector we can form 2 currents out of Dirac field bilinears -

$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x) \rightarrow$ vector current
$j^{\mu 5}(x) = \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x) \rightarrow$ pseudo vector current

Assume that  $\psi$  satisfies Dirac eqn..  $\bar{\psi} = \psi^\dagger \gamma^0$

$$\rightarrow i \not{D} \psi = m \psi \quad \rightarrow i \not{D} \bar{\psi} = m \bar{\psi} \quad (\text{Given } \not{D} = \not{\partial}^0,$$

$\rightarrow$  compute div of these currents -

$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi}) \not{\partial}^\mu \psi + \bar{\psi} \not{\partial}^\mu (\partial_\mu \psi)$$

$$= i m \bar{\psi} \psi + \bar{\psi} (-i m \psi) = 0$$

$$\rightarrow \boxed{\partial_\mu j^\mu = 0}$$

$\rightarrow j^m$  is always conserved if  $\psi(x)$  satisfies  
Dirac eqn

$\rightarrow$  It is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{i\alpha} \psi(x)$$

Similarity

$$\begin{aligned}\partial_m j^{ms} &= (\partial_m \bar{\psi}) \gamma^m \gamma^5 \psi + \cancel{\bar{\psi} \gamma^m \gamma^5 \partial_m \psi} \\ &= (\partial_m \bar{\psi}) \gamma^m \gamma^5 \psi + (-1) \bar{\psi} \gamma^5 \gamma^m \cancel{\partial_m \psi} \\ &= \text{im } \bar{\psi} \gamma^5 \psi + (-1)(-i) m \bar{\psi} \gamma^5 \psi\end{aligned}$$

$\rightarrow \boxed{\partial_m j^{ms} = 2 \text{im } \bar{\psi} \gamma^5 \psi} \rightsquigarrow$  axial vector current

$\rightarrow$  if  $m=0$  then  $\partial_m j^{ms}$  is conserved.

$\rightarrow$  When  $m=0$ ,  $j^m$  is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{i\alpha \gamma^5} \psi(x)$$

(we worry about the rest of this section in ~~Wojciech~~ Pashkin's ...)

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## QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\psi} (\gamma \not{d} - m) \psi = \bar{\psi} (\gamma^0 \partial_0 + \vec{\gamma} \cdot \vec{\partial}) \psi - m \bar{\psi} \psi .$$

→ Canonical momentum conjugate to  $\psi$  is

$$\frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = \gamma^0 \bar{\psi} \gamma^0 = \gamma^0 \bar{\psi} \gamma^0 = i \psi^+ .$$

→ Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} \partial_0 \psi - \mathcal{L} = i \bar{\psi} \gamma^0 \partial_0 \psi - i \bar{\psi} \gamma^0 \partial_0 \psi \\ &\quad - i \bar{\psi} \vec{\gamma} \cdot \vec{\partial} \psi + m \bar{\psi} \psi \\ &= -i \bar{\psi} \vec{\gamma} \cdot \vec{\partial} \psi + m \bar{\psi} \psi \end{aligned}$$

Thus,

$$\boxed{\mathcal{H} = \int \mathcal{H} d^3x = \int d^3x \bar{\psi} (-i \vec{\gamma} \cdot \vec{\partial} + m) \psi}$$

→ now let's figure out the commutators to make this a quantum field theory...

→ DO NOT QUANTIZE THE DIRAC FIELD

This won't work!

Guess  $\left[ \psi_a(\vec{x}), i\psi_b^+(\vec{y}) \right] = i\delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}$

↑ spin ↑  
components  $(a, b = 1, 2, 3, 4)$

i.e.

$$\left[ \psi_a(\vec{x}), \psi_b^+(\vec{y}) \right] = \delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}.$$

In matrix notation ...

$$\left[ \psi_a(\vec{x}), \psi_b^+(\vec{y}) \right] = \mathbf{1}_{4 \times 4} \delta^{(3)}(\vec{x}-\vec{y})$$

↓ ↓  
[ : ] [ --- ]

Also guess  $\left[ \psi_a(\vec{x}), \psi_b(\vec{y}) \right] = 0$

$$\left[ \psi_a^+(\vec{x}), \psi_b^+(\vec{y}) \right] = 0$$

No heat

$$\left[ \psi(\vec{x}), \psi(\vec{y}) \right] = \left[ \psi(\vec{x}), \psi^+(\vec{y}) \right] \gamma^0$$

$$= \left[ \psi(\vec{x}), \psi^+(\vec{y}) \right] \gamma^0 = \gamma^0 \delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote -

(real) field  $\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \vec{a}_p + \vec{a}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}$ . (FT)

For complex field  $\rightarrow$  we get

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \vec{a}_p + \vec{b}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}.$$

In the case of Dirac field, need spin degrees of freedom.

Try --

$$\Psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{+i\vec{p} \cdot \vec{x}}$$

↑  
Spin degrees of freedom

Former components:  $\Psi(\vec{x}) = u(p) e^{i\vec{p} \cdot \vec{x}}$

$$2 \quad \Psi^+(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{-i\vec{p} \cdot \vec{x}}$$

Recall about  $u, v$  also solves Dirac eqn in the reverse  
heat (in momentum space --)

$$p^m \delta_m u^r(p) = mu^r(p) \quad p^m \delta_m v^r(p) = -mv^r(p)$$

We can by the commutators --

$$[\hat{a}_p^r, \hat{a}_{p'}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{b}_p^r, \hat{b}_{p'}^{s\dagger}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{a}_p^r, \hat{b}_{p'}^{s\dagger}] = 0$$

The rest are all zero --

We find heat  $\rightarrow$  as desired --

$$[\Psi_a(\vec{x}), \Psi_b(\vec{y})] = 0 = [\Psi_a^+(\vec{x}), \Psi_b^+(\vec{y})]$$

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We also find that

$$\{\Psi_a(\vec{x}), \Psi_b^+(\vec{y})\} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian ...

$$H = \int d^3x \left[ -i\vec{\nabla} \cdot \vec{\psi} + m\vec{\psi}\vec{\psi} \right]$$

$$= \int d^3x \left\{ \psi^0 \underbrace{\left[ -i\vec{\nabla} \cdot \vec{\psi} + m \right]}_{\text{just const}} \psi \right\}$$

Now, with  $\vec{p}^\mu \partial_\mu u^r(p) = mu^r(p)$

$$\rightarrow (\vec{p} \cdot \vec{\nabla} + m) u^r(p) = \vec{p}^\mu \delta^\mu_r u^r(p) = E_p \delta^\mu_r u^r(p)$$

$$\text{Similarly, } \text{sic } \vec{p}^\mu \partial_\mu v^r(p) = -mv^r(p)$$

$$(\vec{p} \cdot \vec{\nabla} + m) v^r(p) = -E_p \delta^\mu_r v^r(p).$$

So ...

$$\begin{aligned} \rightarrow [-i\vec{\nabla} \cdot \vec{\psi} + m] \psi &= [-i\vec{\nabla} \cdot \vec{\psi} + m] \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_p^r u_p^r + b_p^r v_p^r] e^{ip \cdot \vec{x}} \\ &= \psi^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ E_p a_p^r u_p^r(p) - E_p b_p^r v_p^r(p) \right\} e^{ip \cdot \vec{x}} \end{aligned}$$

So ...

$$H = \int d^3x \left\{ \psi^+ \psi^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \dots \right\} e^{ip \cdot \vec{x}} \right\}$$

play in ...

$$\rightarrow H = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} E_p \left\{ a_p^r a_p^r - b_{+p}^r b_{+p}^{r+} \right\}$$

$\downarrow$   
 $b_{+p}^{r+} b_{+p}^r + \text{const}$

!

→ By creating more and more particles with  $b_{+p}^r$ , we can lower the energy indefinitely

→ This is bad...

→ So we should use Fermi-Dirac statistics instead → anti-commutators instead of commutators...

Requirement.

$$\left\{ a_p^r, a_q^{s+} \right\} = \left\{ b_{+p}^r, b_{+q}^{s+} \right\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q})$$

↑  
 no longer harmonic!      ↗ all other  
 anti-commutators  
 are zero...

When this is true, we find that

$$\left\{ \psi_a(\vec{x}), \psi_b^+(\vec{y}) \right\} = S^{(3)}(\vec{x} - \vec{y}) \delta_{ab}$$

$$\left\{ \psi_a(\vec{x}), \psi_b(\vec{y}) \right\} = \left\{ \psi_a^+(\vec{x}), \psi_b^+(\vec{y}) \right\} = 0$$

where we're using

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[ a_p^r u_r(\vec{p}) + b_{-p}^{r+} v_r(-\vec{p}) \right] e^{i\vec{p} \cdot \vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p (\hat{a}_p^{rt} \hat{a}_p^r - \hat{b}_{-p}^r \hat{b}_{-p}^{rt}) - \hat{b}_{-p}^{rt} \hat{b}_p^r + \text{const}$$

$$\Rightarrow \mathcal{H} = \boxed{\int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ \hat{a}_p^{rt} \hat{a}_p^r + \hat{b}_{-p}^r \hat{b}_{-p}^{rt} \right\}}$$

now good, b/c  $E$  is bold below...

→ also can compute

$$\tilde{P} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} \tilde{p} (\hat{a}_p^{rt} \hat{a}_p^r + \hat{b}_{-p}^{rt} \hat{b}_{-p}^r)$$

To avoid sign confusion, we will usually write

$$\Psi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left( \hat{a}_p^r u_r^r(\vec{p}) e^{+i\vec{p} \cdot \vec{x}} + \hat{b}_{-p}^{rt} v_r^r(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} \right)$$

As a Heisenberg field,

$$\Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left( \hat{a}_p^r u_r^r(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} + \hat{b}_{-p}^{rt} v_r^r(\vec{p}) e^{+i\vec{p} \cdot \vec{x}} \right)$$

where:

- |   |                     |                              |
|---|---------------------|------------------------------|
| { | $\hat{a}_p^r$       | : annihilates particles      |
|   | $\hat{a}_p^{rt}$    | : creates particles          |
|   | $\hat{b}_p^r$       | : annihilates anti-particles |
|   | $\hat{b}_{-p}^{rt}$ | : creates anti-particles.    |

Vacuum state as  $|0\rangle$  where

$$\begin{cases} \hat{a}_p^\dagger |0\rangle = 0 \\ \hat{b}_p^\dagger |0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ conserved norm:

$$|\vec{p}, s\rangle = \sqrt{2E_p} \hat{a}_p^\dagger |0\rangle$$

so that

$$|\vec{p}, s\rangle |\vec{q}, r\rangle = \sqrt{2E_p} \sqrt{2E_q} \delta^{(3)}(\vec{p} - \vec{q}) \delta^{rs}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lorentz}} \psi'(x) = \gamma \frac{1}{\sqrt{2}} \psi(\gamma^{-1} x)$$

recall that with  $\omega_{12} = -\omega_{21} = \theta$

$$\begin{cases} \omega_{12} = -\omega_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \end{cases} \Rightarrow \exp\left\{-i\omega_{\mu\nu} \gamma^\nu \frac{\gamma^\mu}{2}\right\} = 1 - i\vec{\theta} \cdot \vec{\gamma}$$

$$= 1 - i\vec{\theta} \cdot \vec{\gamma}$$

$$\rightarrow \text{and } \psi(\gamma^{-1} x) \approx [1 - \vec{\theta} \cdot \vec{\gamma}] \psi(x)$$

$$\vec{\gamma} = \vec{x} \times (-i\vec{\nabla})$$

so we'd  $\psi \rightarrow \psi + S\psi$  where

$$S\psi = \psi' - \psi = \left(\frac{i}{2}\vec{\theta} \cdot \vec{\gamma}\right)\psi(x) - \vec{\theta} \cdot (\vec{x} \times \vec{\nabla})\psi(x)$$

By Noether's Thm,

$$\vec{J}_{\text{total}} \text{ (total spin)} = \int \frac{d^3x}{2} \left[ \bar{\psi}^\dagger (-i\vec{\gamma} \cdot \vec{\nabla}) \psi + \frac{1}{2} \bar{\psi}^\dagger \vec{\Sigma} \psi \right].$$

~~to~~

We won't worry about the rest of this section about propagators

$\rightarrow$  we'll come back to them later when looking at Feynman diagrams.

~~to~~

### DISCRETE SYMMETRIES OF THE DIRAC THEORY

Basically, we have

Parity — Time reversal

Charge conjugation

~~to~~

Recall that we before, we looked at implementation of continuous Lorentz transform -

$\rightarrow$  found that  $\pm 1 \in$  Lorentz group

$\exists U(1)$  unitary for which

$$U(1) \psi(x) \bar{U}(1) = \Lambda \frac{1}{2}' \psi(\Lambda x).$$

$\rightarrow$  Now, we'll look about discrete symmetries on the Dirac field.

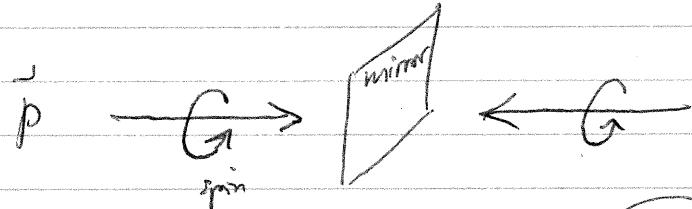
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant:

→ e.g. { time-reversal },  
{ parity }.

[Parity] (P) : flips direction of spatial vectors

$$P: (t, \vec{x}) \rightarrow (t, -\vec{x})$$

↔ mirror sym → change the handedness.



→ Note momentum flip sign, but spin is unchanged.

[Time-reversal]

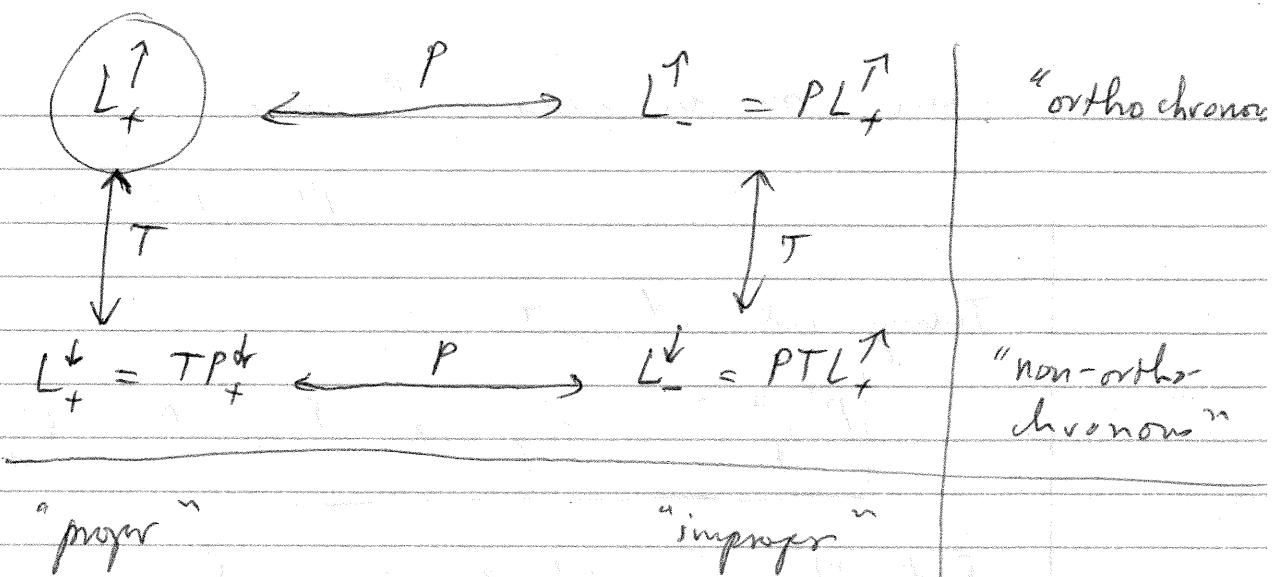
$$T: (t, \vec{x}) \rightarrow (-t, \vec{x})$$

P,T don't belong to the "proper" Lorentz group  $L_+$

→ the full Lorentz group breaks into 4 disjoint subsets ...

(L)

(03)



charge conjugation  $\rightarrow$  intercharge particles & anti-particles.

$\hookrightarrow$  non-space-time.

Let's look at Parity.

Note that because  $P: (t, \vec{x}) \rightarrow (t, -\vec{x})$

$\rightarrow$  momentum flips sign

but not spin!  $\rightarrow$  what is  $P$ ? As an operator?

$$\overrightarrow{\alpha} \rightarrow \cdots \overset{P}{\rightarrow} \leftarrow \overleftarrow{\beta}$$

As an operator on creation/annihilation ops, we want

$$P^\dagger \hat{a}_p^s P = \hat{a}_p^s \quad \& \quad P^\dagger \hat{b}_p^s P = \hat{b}_{-p}^s$$

where, as discussed,  $P$  must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, set

$$\boxed{P^\dagger a_{\vec{p}}^s P = a_{-\vec{p}}^{s\dagger} \quad P^\dagger b_{\vec{p}}^s P = b_{-\vec{p}}^{s\dagger}}$$

But there might be too restrictive --- we can get better constraints by requiring that:

$$\boxed{P^\dagger a_{\vec{p}}^s P = \eta_a a_{-\vec{p}}^{s\dagger} \quad P^\dagger b_{\vec{p}}^s P = \eta_b b_{-\vec{p}}^{s\dagger}}$$

as long as  $\eta_a = (\eta_b) = 1$  are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases  $\eta_a, \eta_b$  will cancel:

$$\left. \begin{aligned} P^\dagger a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s P &= a_{-\vec{p}}^{s\dagger} a_{-\vec{p}}^s \\ P^\dagger b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s P &= b_{-\vec{p}}^{s\dagger} b_{-\vec{p}}^s \end{aligned} \right\}$$

With this, let's ~~try to~~ implement parity condition on  $\psi(x)$

$$\rightarrow P^\dagger \psi P = ? \quad \left( \begin{array}{l} \text{to find out what these} \\ \eta_a + \eta_b \text{ must be...} \end{array} \right)$$

$$P^t \chi(x) P = \int \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2\tilde{p}}} \sum_{s=1,2} (\gamma_a^s a_{-\vec{p}}^s u^s(p) e^{-i\tilde{p} \cdot \vec{x}} + \gamma_b^s b_{-\vec{p}}^s v^s(\vec{p}) e^{i\tilde{p} \cdot \vec{x}})$$

Define  $\begin{cases} \tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}) \end{cases}$

Note that

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad \text{where } \sigma = (1, \vec{\sigma}) \quad \bar{\sigma} = (11, -\vec{\sigma})$$

$$\begin{aligned} &= \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} u^s(-\tilde{p}) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u^s(-\tilde{p}) \end{aligned}$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\tilde{p})}$$

and

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \dots = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} v^s(-\tilde{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(-\tilde{p})}$$

With these, we find that

$$\tilde{p} \cdot \tilde{x} = p \cdot x$$

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$$P^+ \bar{\psi}(x) P = 8^0 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left( \gamma_a \frac{a^s}{-p} u^s(-p) e^{-ip \cdot \tilde{x}} + \gamma_b^* \frac{b^s}{-p} v^s(-p) e^{ip \cdot \tilde{x}} \right)$$

Now, notice that if  $\gamma_a = \gamma_b^*$  then it's "nice":

$$( \gamma_a = \gamma_b^* ) \Rightarrow P \bar{\psi}(x) P = \gamma_a 8^0 \bar{\psi}(\tilde{x}) \quad \rightarrow P_{\text{transf}} \text{ in final form}$$

$\rightarrow$  sufficient to choose  $\gamma_a = 1 = -\gamma_b^*$

relative sign between fermions - antifermions --

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Now, useful to know how various Dirac field bilinears transform under parity ...

Recall ... 5 of them:

$$\bar{\psi} \psi, \bar{\psi} \gamma^\mu \psi, ; \bar{\psi} [\gamma^\mu, \gamma^\nu] \psi$$

$$\bar{\psi} \gamma^\mu \gamma^\nu \psi, ; \bar{\psi} \gamma^\nu \psi.$$

$\rightarrow$  find these, first compute:  $P \bar{\psi}(x) P$  --

$$P^+ \bar{\psi}(x) P = P^+ \bar{\psi}^+(x) \gamma^0 P \stackrel{\curvearrowright}{=} (P^+ \bar{\psi} P)^+ \gamma^0 \quad (\gamma^0 = \gamma^0)$$

$$\rightarrow = \gamma_a^* (\gamma^0 \bar{\psi}(\tilde{x}))^+ \gamma^0 = \gamma_a^* \bar{\psi}^+(\tilde{x}) \gamma^0 \gamma^0$$

$$\rightarrow \boxed{P^+ \bar{\psi} P = \gamma_a^* \bar{\psi}(\tilde{x}) \gamma^0}$$

With this --

$$\begin{aligned}
 p^\dagger \bar{\psi} \psi p &= \underbrace{p^\dagger \bar{\psi}(x) p}_{(x)(x)} \underbrace{p^\dagger \psi(x) p}_{\text{II}} \\
 &= \gamma_a^\dagger \bar{\psi}(\tilde{x}) \gamma^0 \gamma_a \gamma^0 \psi(\tilde{x}) \\
 &= |\gamma_a|^2 \bar{\psi}(\tilde{x}) \psi(\tilde{x})
 \end{aligned}$$

scalar

$$p^\dagger \bar{\psi} \psi p(x) = \bar{\psi} \psi(\tilde{x}). \quad (\text{scalar})$$

scalar.

can also show --

$$\begin{aligned}
 p^\dagger \bar{\psi}(x) \underbrace{\gamma^\mu}_{\text{(vector)} \atop \text{fields}} \psi p &= \bar{\psi} \gamma^0 \gamma^\mu \gamma^0 \psi(\tilde{x}) \\
 &= \begin{cases} + \bar{\psi} \gamma^\mu \psi(\tilde{x}) & \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi(\tilde{x}) & \mu = 1, 2, 3 \end{cases}
 \end{aligned}$$

$$p^\dagger (i \cancel{\partial} \gamma^5 \psi) p = i \bar{\psi}(\tilde{x}) \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) = -i \cancel{\partial} \gamma^5 \psi(\tilde{x})$$

↑  
pseudo  
scalar  
(-)

~~$$\begin{aligned}
 &\cancel{\partial} \gamma^5 \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) \\
 &+ \cancel{\partial} \gamma^5 \gamma^0 \gamma^5 \psi(\tilde{x}) \quad \mu = 0 \\
 &- \cancel{\partial} \gamma^5 \gamma^0 \gamma^5 \psi(\tilde{x}) \quad \mu = 1, 2, 3
 \end{aligned}$$~~

$$p^\dagger \bar{\psi} \gamma^\mu \gamma^5 \psi p = \bar{\psi}(\tilde{x}) \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(\tilde{x})$$

↑  
pseudo  
vector.  
(-)

~~$$\begin{aligned}
 &- \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) \quad \mu = 0 \\
 &+ \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) \quad \mu = 1, 2, 3
 \end{aligned}$$~~

Note The relative sign:  $-\gamma_a = \gamma_b^*$  is important.

for the relationship between fermion - anti - fermi

Consider ~~and~~ fermion - anti fermion state...

$$\begin{aligned}
 & a_p^{st} b_q^{st} |0\rangle \xrightarrow{P} P(a_p^{st} b_q^{st} |0\rangle) \\
 &= P^+ (a_p^{st} b_q^{st}) P |0\rangle \\
 &= \underbrace{P^+ a_p^{st} P P^+ b_q^{st} P}_{\gamma_a} |0\rangle \\
 &= (\gamma_a) a_{-p}^{st} \gamma_b b_{-q}^{st} |0\rangle \\
 &= -(\gamma_b \gamma_b^*) a_{-p}^{st} b_{-q}^{st} |0\rangle \\
 &= -a_{-p}^{st} b_{-q}^{st} |0\rangle
 \end{aligned}$$

→ a state containing a fermion-antifermion pair gets an  $(-1)$  under parity transformation.

extra

—

### [TIME REVERSAL].

if  $T$  is unitary  $\Rightarrow [T, H] = 0$

$$\rightarrow T^+ e^{iHt} T = e^{iHt + T^+ T} = e^{iHt}$$

→ no good...

What if  $T^+ T = -H$ ? or  $[T, H] = 0$ ?

But this  $\Rightarrow$  no good either since implies that  $H$  is unbounded ...

$\rightarrow$  Assume this ...

"Time-reversal is conjugate-linear/anti-linear"

Assume:

$T$  is unitary

$$T^* T = c^* \quad (c \in \mathbb{C})$$

$$[T, H] = 0$$

With those

$$T^* e^{-iHt} T = e^{-iHt} \quad \checkmark$$

$\rightarrow$  Time-reversal:

momentum

$\downarrow$

spin

are reversed

$\rightarrow$  like watching a movie played back-wards

$$G \xrightarrow{\quad} T \xrightarrow{\quad} \leftarrow \int$$

Flipping momentum is easy.

What abt flipping spinor? We know that

In some basis --

$$\xi(\uparrow) = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let  $\xi^s = (\xi(\uparrow), \xi(\downarrow))$  for  $s=1, 2$  & define

reversed  
spin

$$\xi^{-s} = -i\sigma^2 (\xi^s)^{\dagger}$$

→ This is the flipped spinor

It is clear that

$$\begin{aligned} \xi^{-s} &= -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow), \xi(\downarrow))^{\dagger} \\ &= (\xi(\downarrow), -\xi(\uparrow))^{\dagger} \end{aligned}$$

where  $\xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{\dagger \dagger}$

$$\xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{\dagger \dagger}$$

→ This is convenient since our time reversal op. involves complex conjugation --

→ Can show:  $\tilde{m}^s(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{s\dagger} \\ \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{s\dagger} \end{pmatrix}$

So if we use the identity ...

$$\{\sqrt{\tilde{p} \cdot \sigma} \gamma^2 = \sigma^2 \sqrt{\tilde{p} \cdot \sigma^2}\}$$

(prove using  $\sigma^2 \bar{\sigma}^2 = -\bar{\sigma}^2 \sigma^2$ )

then we get

$$\begin{aligned} u^{-s}(\tilde{p}) &= \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} (-i\sigma^2) \xi^{s\pm} \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} (-i\sigma^2) \xi^{s\mp} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2) \sqrt{\tilde{p} \cdot \sigma^2} \xi^{s\pm} \\ (-i\sigma^2) \sqrt{\tilde{p} \cdot \bar{\sigma}^2} \xi^{s\mp} \end{pmatrix} \\ &= (-i) \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} [u^s(p)]^\star = -\gamma^1 \gamma^3 [u^s(p)]^\star \end{aligned}$$

$\uparrow$   
 $\sigma^2 = \sigma^2$

element-wise  
cmplx conjugation

similarly,

$$v^{-s}(\tilde{p}) = -\gamma^1 \gamma^3 [\vartheta^s(p)]^\star$$

in this relation,  $v^{-s}$  contains

$$\xi^{(-s)} = -\xi^s$$

a  $360^\circ$  flip  
introduces  
a  $(-)$  sign.

~~introduces~~  
~~effect~~

Now we can define time reversal operation on the creation - annihilation operators ---

here  $\rightarrow$   $T^+ a_p^s T = \bar{a}_{-\vec{p}}^{-s}$  &  $T^+ b_p^s T = \bar{b}_{-\vec{p}}^{-s}$

↑ flip  $\vec{p}$   
↓ flip momentum

can't  
switch here ---

where  $\left\{ \begin{array}{l} \bar{a}_{-\vec{p}}^{-s} = (a_{-\vec{p}}^\downarrow, -a_{-\vec{p}}^\uparrow) \\ \bar{b}_{-\vec{p}}^{-s} = (b_{-\vec{p}}^\downarrow, -b_{-\vec{p}}^\uparrow) \end{array} \right.$

we now just like what we  
define  $\left\{ \begin{array}{l} \bar{a}_{-\vec{p}}^{-s} = (a_{-\vec{p}}^\downarrow, -a_{-\vec{p}}^\uparrow) \\ \bar{b}_{-\vec{p}}^{-s} = (b_{-\vec{p}}^\downarrow, -b_{-\vec{p}}^\uparrow) \end{array} \right.$  did with  
 $\zeta^s = (s(\downarrow), -s(\uparrow))$

if  $\left\{ \begin{array}{l} a_p^s = (a_p^\uparrow, a_p^\downarrow) \\ b_p^s = (b_p^\uparrow, b_p^\downarrow) \end{array} \right.$  analogous to what  
we did before ---

With this, let's evaluate  $T^\dagger \psi(x) T$ :

$$\begin{aligned} T^\dagger \psi(x) T &= \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1,2} T^+ (a_p^s u_s^s(p) e^{-ip \cdot x} + b_p^{s+} v_s^s(p) e^{+ip \cdot x}) T \\ &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^2 \left\{ \bar{a}_{-\vec{p}}^{-s} [u_s^s(p)]^* e^{-ip \cdot x} \right. \\ &\quad \left. + \bar{b}_{-\vec{p}}^{-s} [v_s^s(p)]^* e^{-ip \cdot x} \right\} \end{aligned}$$

where under  $T$ ,  $= \gamma^1 \gamma^2 \psi(x_T)$ ,  $x_T = (-t, \vec{x})$

$a_p^s \xrightarrow{T} \bar{a}_{-\vec{p}}^{-s}$

$\psi(x_T) = \bar{\psi}(-t, \vec{x})$

$\rightarrow \bar{\psi}(-t, \vec{x}) = \psi(t, \vec{x})^*$

$\bullet T^\dagger e^{-ip \cdot x} T = \mathbb{1} e^{+ip \cdot x}; T^\dagger u_p^s T = [u_p^s]^*$

note sign here  
choose ↑  
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Becare  $\{u^s(p)\}^* = \gamma_1 \gamma_3 u^{-s}(\tilde{p})$ , we have

$$\begin{aligned} T^+ \psi(x) T &= \gamma' \gamma^3 \int \frac{d^2 \tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^2 \left\{ a_{\tilde{p}}^{-s} u^{-s}(\tilde{p}) e^{i\tilde{p}(t_1, \tilde{x})} \right. \\ &\quad \left. + b_{\tilde{p}}^{-s} v^{-s}(\tilde{p}) e^{-i\tilde{p}(t_1, \tilde{x})} \right\} \\ &= \gamma' \gamma^3 \psi(-t, x) \\ &= -\tilde{\rho}(-t, \tilde{x}), \end{aligned}$$

$$\Rightarrow \boxed{T^+ \psi(x, t) T = \gamma' \gamma^3 \psi(x, -t)}$$

Next, can check the action of  $T$  on bilinears...

$$\begin{aligned} T^+ \bar{\psi} T &= T^+ \psi^+ \gamma^0 T = T^+ \psi^+ T \gamma^0 \xrightarrow{\text{real}} \\ &= (\gamma' \gamma^3 \psi(x_T))^+ \gamma^0 = \psi^+(x_T) \gamma^3 \gamma^1 \gamma^0 \\ &= \psi^+(x_T) \gamma^3 \gamma^1 \gamma^0 \quad \begin{matrix} \uparrow \\ -\gamma^3 \end{matrix} \quad \begin{matrix} \uparrow \\ -\gamma^1 \end{matrix} \\ &= +\psi^+(x_T) \gamma^0 \gamma^3 \gamma^1 \\ &\Rightarrow \boxed{T^+ \bar{\psi} T = -\bar{\psi}(x_T) \gamma^1 \gamma^3} \end{aligned}$$

with this, can compute the rest---

Scalar  $\boxed{T \bar{\psi} \psi T = \bar{\psi} (-\gamma' \gamma^3) \underbrace{(\gamma' \gamma^3)}_{11} \psi(x_T) = \bar{\psi}(x_T) \psi(x_T)}$

Pseudoscalar  $\rightarrow$  set (-)  $\rightarrow$  "pseudo"

$$\boxed{T^+ \bar{\psi} \gamma^5 \psi T = -i \bar{\psi} (-\gamma' \gamma^3) (\gamma' \gamma^3) \psi(x_T) = -i \bar{\psi}(x_T) \gamma^5 \psi(x_T)}$$

Vector

$$\boxed{T^+ \bar{\psi} \gamma^\mu \psi T = \bar{\psi} (-\gamma^1 \gamma^3) (\gamma^\mu)^T (\gamma^1 \gamma^3) \psi}$$

(x)

$$= \begin{cases} + \bar{\psi} \gamma^\mu \psi (x_T) & \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi (x_T) & \mu = 1, 2, 3 \end{cases}$$

This makes sense... Recall that  $\bar{\psi} \gamma^0 \psi$  is the charge density

↳  $\bar{\psi} \gamma^0 \psi$  should be the same under T -

as we saw:  $T^+ \bar{\psi} \gamma^0 \psi T = \bar{\psi} \gamma^0 \psi$ .

but current density (time-dy) must reverse sign

$$\rightarrow T^+ \bar{\psi} \gamma^5 \psi T = - \bar{\psi} \gamma^5 \psi \quad \checkmark.$$

→

Charge Conjugation - Matter-anti-matter flip

(s) anti-particles  $\rightarrow$  particles are swapped.

{ spin + momentum are the same.

Let  $\left\{ \begin{array}{l} C^\dagger a_p^+ C = b_p^- \\ C^\dagger b_p^- C = a_p^+ \end{array} \right\} \rightarrow$  ignore phases...

How should C act on  $\psi(x)$ ?

First, look at relation ...

$$(v^s(p))^{\pm} = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}} & (-i\gamma^2) \xi^{s\pm} \\ \sqrt{p\cdot\bar{\sigma}} & (-i\gamma^2) \xi^{s\pm} \end{pmatrix}^{\pm} = \begin{pmatrix} -i\gamma^2 \sqrt{p\cdot\bar{\sigma}} \xi^{s\pm} \\ i\gamma^2 \sqrt{p\cdot\bar{\sigma}} \xi^{s\pm} \end{pmatrix}^{\pm}$$

$$= \begin{pmatrix} 0 & -i\gamma^2 \\ -i\gamma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}} \xi^s \\ \sqrt{p\cdot\bar{\sigma}} \xi^s \end{pmatrix} = \cancel{\text{both}}$$

→ set

$$\boxed{u^s(p) = -i\gamma^2 (v^s(p))^{\pm}}$$

$$\boxed{v^s(p) = -i\gamma^2 (u^s(p))^{\pm}}$$

$$\rightarrow C^+ \psi(x) C = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^2 \left\{ -i\gamma^2 b_p^s (v^s(p))^* e^{-ip \cdot x} - i\gamma^2 a_p^{s\pm} (u^s(p))^{\pm} e^{ip \cdot x} \right\}$$

$$= -i\gamma^2 \psi^*(x) = -i\gamma^2 (\psi^+)^T = -i(\bar{\psi} \gamma^0 \gamma^2)^T$$

$$\Rightarrow \boxed{C^+ \psi(x) C = -i(\bar{\psi} \gamma^0 \gamma^2)^T} \rightarrow C \text{ is a unitary op.}$$

On bilinear ... first, find  $\bar{\psi} = (\psi^+)^+ \gamma^0 = \psi^0$

$$\boxed{C^+ \bar{\psi} \psi^0 C = C^+ \psi^+ \gamma^0 C = \underbrace{C^+ \psi^+}_{\psi^0} \gamma^0 = -i \psi^T \gamma^0 \gamma^0}$$

$$= (-i \gamma^2 \psi)^T \gamma^0 = (-i \gamma^0 \gamma^2 \psi)^T$$

Next ...

$$C^+ \bar{\psi} \psi C = (-i(\bar{\psi} \gamma^0 \gamma^2 \psi)^T) (-i(\bar{\psi} \gamma^0 \gamma^2 \psi)^T) = \dots =$$

$$= -[(-i \bar{\psi} \gamma^0 \gamma^2)(-i \bar{\psi} \gamma^0 \gamma^2)]^T = +\bar{\psi} \gamma^0 \gamma^2 \gamma^0 \gamma^2 \psi$$

$$= +\bar{\psi} \gamma^0 \gamma^2 \gamma^2 \gamma^0 \psi = +\bar{\psi} \psi$$

(P)

$$\text{So } \boxed{C^\dagger \bar{\gamma}^4 C = \bar{\gamma}^\dagger \gamma} \rightarrow \text{reduces}$$

vector

$$\boxed{C_i^\dagger \bar{\gamma}^i \gamma^i C = i (-i \gamma^0 \gamma^2 \gamma)^T \gamma^i (-i \bar{\gamma}^0 \bar{\gamma}^2 \bar{\gamma})^T = i \bar{\gamma}^i \gamma^i}$$

pseudo-scalar

$$\boxed{C^\dagger \bar{\gamma}^m \gamma^m C = - \bar{\gamma}^m \gamma^m}$$

pseudo scalar

$$\boxed{C^\dagger \bar{\gamma}^m \gamma^m \gamma^i C = + \bar{\gamma}^m \gamma^m \gamma^i}$$

(I'll skip the derivations... to save time)

### Summary

	$\bar{\gamma} \gamma$	$i \bar{\gamma} \gamma^5 \gamma$	$\bar{\gamma} \gamma^m \gamma$	$\bar{\gamma} \gamma^m \gamma^5 \gamma$	$\bar{\gamma} \gamma^m \gamma^5 \gamma$	$\partial_\mu$
P	+1	-1	$(-1)^m$	$-(-1)^m$	$(-1)^m (-1)^v$	$(-1)^m$
T	+1	-1	$(-1)^m$	$(-1)^m$	$-(-1)^m (-1)^v$	$-(-1)^m$
C	+1	+1	-1	+1	-1	+1
CPT	+1	+1	-1	-1	+1	-1

Notice that

$$L = \bar{\gamma} (i \gamma^\mu \partial_\mu - m) \gamma \text{ is invariant under } C, P, T \text{ separately}$$

→ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

Problem 5

↳ (to be continued...)

Invariance under CPT is required for any Lorentz invariant local Hermitian op.

## Correlation functions for Dirac fields

$\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle \rightarrow$  Dirac propagation amplitudes  
 ↓      ↑  
 only "a"      only "a"  
 term contributes      term contributes

Recall -

$$\rightarrow \bar{\psi}_A(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ a_A^S u_A^S(p) e^{-ip \cdot x} + b_A^{S+} v_A^S(p) e^{-ip \cdot x} \right\}$$

$$\rightarrow \bar{\psi}_B(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ b_B^S \bar{v}_B^S(p) e^{-ip \cdot x} + a_B^{S+} \bar{u}_B^S(p) e^{ip \cdot x} \right\}$$

where  $\{a_A^S, a_B^{S+}\} = \{b_A^S, b_B^{S+}\} = (2\pi)^3 \delta^{(3)}(p-q)/8$

$$\rightarrow \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_S u_A^S(p) \bar{v}_B^S(p)}_{AB} e^{-ip(x-y)}$$

$$= (i\gamma_x - m) \underbrace{\int \frac{d^3 p}{(2\pi)^3 / 2E_p}}_{AB} e^{-ip(x-y)}$$

$$(p+m)_{AB}$$

$$\boxed{\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = (i\gamma_x + m)_{AB} D(x-y)}$$

$$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \underbrace{\bar{\psi}_A^s(p) \psi_B^s(p)}_{(\phi-m)_{AB}} e^{-ip(x-y)}$$

↑                      ↑  
 6 terms              6 terms  
 contribute            contribute

$$= \int \frac{d^3 p}{(2\pi)^3} \left( \frac{1}{2E_p} \right) (\phi-m)_{AB} e^{-ip(x-y)}$$

$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = - (i\partial_x + m)_{AB} \delta(y-x)$

### Feynman Propagator

$$S_f^{AB}(x-y) = \begin{cases} \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle & \text{for } x^0 > y^0 \\ -\langle 0 | \bar{\psi}_B(y) \bar{\psi}_A(x) | 0 \rangle & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0 | T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \} | 0 \rangle$$

↑  
--- time-ordering ---

where  $T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \}$

$$= \theta(x^0 - y^0) \bar{\psi}_A(x) \bar{\psi}_B(y)$$

$$- \theta(y^0 - x^0) \bar{\psi}_B(y) \bar{\psi}_A(x)$$

minus sign for Fermions

Let's do the calculations.

$$\langle 0 | \psi_A(x) \bar{\psi}_B(y) | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | \left\{ \sum_s a_{A\vec{p}}^s \bar{u}_A^s(p) e^{-ip \cdot x} + b_{A\vec{p}}^{s+} \bar{u}_A^{s+}(p) e^{ip \cdot x} \right\}$$

$$\times \left\{ \sum_s b_{B\vec{p}}^{s+} u_B^s(p') e^{-ip' \cdot y} + a_{B\vec{p}}^{s+} u_B^{s+}(p') e^{ip' \cdot y} \right\} | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \langle 0 | \sum_s a_{A\vec{p}}^s \bar{u}_A^s(p) \sum_s a_{B\vec{p}}^{s+} \bar{u}_B^{s+}(p') e^{i(p \cdot x - p' \cdot y)} | 0 \rangle$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \sum_s \langle 0 | \bar{u}_A^s(p) \bar{u}_B^{s+}(p') | 0 \rangle (2\pi)^3 \delta^{(3)}(p - p') e^{i(p \cdot x - p' \cdot y)}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \bar{u}_A^s(p) \bar{u}_B^{s+}(p) e^{-ip(x-y)}$$

$$(p+m)_{AB} = (\gamma^\mu p_\mu + m)_{AB} \quad \begin{matrix} \text{(spin sum)} \\ \text{relations} \end{matrix}$$

$$= (i\cancel{x} + m)_{AB} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}. \checkmark$$

Similarly, we can get the other relation too..

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