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Course: **8.321 - Quantum Theory I**
Problem set: **#4**

1. Particle in a box. It is well known that the solution to the SE

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x) = E\psi(x)$$

where

$$V(x) = \begin{cases} 0, & 0 < x < a \\ \infty, & \text{else} \end{cases}$$

is normalized standing waves:

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin(k_n x) = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi n x}{a}\right)$$

where k_n are the wavenumbers and $E_n = n^2\pi^2\hbar^2/2ma^2$. One can solve this quickly by noting that $\psi = 0$ outside of $[0, a]$, and the solution inside $[0, a]$ must be a sinusoid. Because $\psi(0) = 0$, the only satisfactory solution is $\psi(x) \sim \sin(k_n x)$. To satisfy the boundary conditions $\psi(0) = \psi(a) = 0$, the wavenumbers $k_n = n\pi/a$. Plugging this back into the SE, we find that $E_n = n^2\pi^2\hbar^2/2ma^2$. Finally, we integrate $\int_0^a \sin^2(k_n x)\,dx$ to find the normalization factor $N = \sqrt{2/a}$, so that $\psi_n(x) = \sqrt{2/a}\sin(k_n x)$.

We now wish to calculate the uncertainty product for the ground state and first excited states. For the ground state, the wavefunction is $\psi_1 = \sqrt{2/a}\sin(\pi x/a)$.

$$\langle x\rangle_1 = \int_0^a x\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\right]^2 dx = \frac{a}{2} \quad \text{(as expected by symmetry)}$$

$$\langle x^2\rangle_1 = \int_0^a x^2\left[\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\right]^2 dx = a^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right).$$

So,

$$\langle(\Delta x)^2\rangle_1 = \langle x^2\rangle_1 - \langle x\rangle_1^2 = \frac{a^2}{12} - \frac{a^2}{2\pi^2}.$$

Next, we find the moments of the momentum by using $\hat{p} = -i\hbar\partial_x$:

$$\langle p\rangle_1 = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)[-i\hbar\partial_x]\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) dx = 0 \quad \text{(as expected by symmetry)}$$

$$\langle p^2\rangle_1 = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)[-i\hbar]^2\,\partial_x^2\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right) dx = \frac{\hbar^2\pi^2}{a}.$$

So,

$$\langle(\Delta p)^2\rangle_1 = \langle p^2\rangle_1 - \langle p\rangle_1^2 = \frac{\hbar^2\pi^2}{2}.$$

With these, we find

$$\boxed{\langle(\Delta x)^2\rangle_1\langle(\Delta p)^2\rangle_1 = \frac{\pi^2 - 6}{12}\hbar^2 > \frac{\hbar^2}{4}}$$

Next, we do the same for the first excited state, whose wavefunction is $\psi_2(x) = \sqrt{2/a}\,\sin(2\pi x/a)$.

$$\langle x\rangle_2 = \int_0^a x\left[\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)\right]^2 dx = \frac{a}{2} \quad \text{(as expected by symmetry)}$$
$$\langle x^2\rangle_2 = \int_0^a x^2\left[\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)\right]^2 dx = a^2\left(\frac{1}{3} - \frac{1}{8\pi^2}\right).$$

So,

$$\langle(\Delta x)^2\rangle_2 = \langle x^2\rangle_2 - \langle x\rangle_2^2 = \frac{a^2}{12} - \frac{a^2}{8\pi^2}.$$

Next, we find the moments of the momentum by using $\hat{p} = -i\hbar\partial_x$:

$$\langle p\rangle_2 = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)[-i\hbar\partial_x]\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right) dx = 0 \quad \text{(as expected by symmetry)}$$
$$\langle p^2\rangle_2 = \int_0^a \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right)[-i\hbar]^2\,\partial_x^2\sqrt{\frac{2}{a}}\sin\left(\frac{2\pi x}{a}\right) dx = \frac{4\hbar^2\pi^2}{a}.$$

So,

$$\langle(\Delta p)^2\rangle_2 = \langle p^2\rangle_2 - \langle p\rangle_2^2 = \frac{4\hbar^2\pi^2}{2}.$$

With these, we find

$$\boxed{\langle(\Delta x)^2\rangle_2\langle(\Delta p)^2\rangle_2 = \frac{2\pi^2 - 3}{6}\hbar^2 > \frac{\hbar^2}{4}}$$

Mathematica code:

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(* GROUND STATE *)

In[36]:= x11 =
Integrate[x*(Sqrt[2/a]*Sin[(Pi/a)*x])^2, {x, 0, a}] // FullSimplify

Out[36]= a/2

In[37]:= x12 =
Integrate[x^2*(Sqrt[2/a]*Sin[(Pi/a)*x])^2, {x, 0, a}] // Expand

Out[37]= a^2/3 - a^2/(2 \[Pi]^2)

In[40]:= x12 - x11^2

Out[40]= a^2/12 - a^2/(2 \[Pi]^2)

In[41]:= p11 =
Integrate[(Sqrt[2/a]*Sin[(Pi/a)*x])*(I*h)*
D[(Sqrt[2/a]*Sin[(Pi/a)*x]), x], {x, 0, a}] // FullSimplify

Out[41]= 0

In[42]:= p12 =
Integrate[(Sqrt[2/a]*Sin[(Pi/a)*x])*(I*h)^2*
D[D[(Sqrt[2/a]*Sin[(Pi/a)*x]), x], x], {x, 0, a}] // FullSimplify

Out[42]= (h^2 \[Pi]^2)/a^2

In[44]:= (x12 - x11^2)*(p12 - p11^2) // FullSimplify

Out[44]= 1/12 h^2 (-6 + \[Pi]^2)
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(* FIRST EXCITED STATE *)
In[46]:= x21 =
Integrate[x*(Sqrt[2/a]*Sin[(2*Pi/a)*x])^2, {x, 0, a}] // FullSimplify

Out[46]= a/2

In[47]:= x22 =
Integrate[x^2*(Sqrt[2/a]*Sin[(2*Pi/a)*x])^2, {x, 0, a}] // Expand

Out[47]= a^2/3 - a^2/(8 \[Pi]^2)

In[48]:= x22 - x21^2

Out[48]= a^2/12 - a^2/(8 \[Pi]^2)

In[49]:= p21 =
Integrate[(Sqrt[2/a]*Sin[(2*Pi/a)*x])*(I*h)*
D[(Sqrt[2/a]*Sin[(2*Pi/a)*x]), x], {x, 0, a}] // FullSimplify

Out[49]= 0

In[50]:= p22 =
Integrate[(Sqrt[2/a]*Sin[(2*Pi/a)*x])*(I*h)^2*
D[D[(Sqrt[2/a]*Sin[(2*Pi/a)*x]), x], x], {x, 0, a}] // FullSimplify

Out[50]= (4 h^2 \[Pi]^2)/a^2

In[52]:= (x22 - x21^2)*(p22 - p21^2) // FullSimplify

Out[52]= 1/6 h^2 (-3 + 2 \[Pi]^2)
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2. Balancing an Ice Pick. Let us model the ice pick of length L and mass m as an inverted pendulum initially in the classical equilibrium position $\theta(0) = 0, \dot{\theta}(0) = 0$ (the angle θ is measured from the vertical). Quantum mechanically, we have

$$\Delta x \Delta p \geq \frac{\hbar}{2} \implies L\Delta\theta(0) mL\Delta\dot{\theta}(0) \geq \frac{\hbar^2}{2}.$$

Let us assume for simplicity that the Heisenberg uncertainty is saturated, so that

$$\Delta\theta(0)\Delta\dot{\theta}(0) = \frac{\hbar}{2mL^2}.$$

To proceed, let us assume that the uncertainty in the initial conditions is also on the same order as the values of the initial conditions themselves, i.e., $\Delta\theta(0) \sim \theta(0)$ and $\Delta\dot{\theta}(0) \sim \dot{\theta}(0)$ so that

$$\theta(0)\dot{\theta}(0) = \frac{\hbar^2}{2mL^2}.$$

Now, by Newton's second law of motion and the small angle approximation, we have a simple equation of motion

$$\ddot{\theta} = \frac{g}{L}\theta \implies \theta(t) = A\exp(\omega t) + B\exp(-\omega t)$$

where $\omega = \sqrt{g/L}$. We're interested in the exponentiall growing solution, so we'll set $B = 0$. Plugging in the initial conditions, we find that $\theta(t) = \theta(0)\exp\left(t\sqrt{g/L}\right)$ and therefore we may set $\dot{\theta}(0) = \theta(0)\sqrt{g/L}$. With the Heisenberg uncertainty conditions, we find that

$$\theta(0) \sim \sqrt{\frac{\sqrt{L/g}\hbar^2}{2mL^2}} \qquad \dot{\theta}(0) = \sqrt{\frac{\hbar^2}{2mL^2\sqrt{L/g}}}.$$

With these the solution is

$$\theta(t) = \sqrt{\frac{\sqrt{L/g}\hbar^2}{2mL^2}} \exp\left(\sqrt{\frac{g}{L}}t\right).$$

Inverting this and solve for t we find

$$t(\theta) = \sqrt{\frac{L}{g}} \ln \left\{ \theta \sqrt{\frac{2mL^2}{\hbar^2 \sqrt{L/g}}} \right\}.$$

We will now assume the dimension and weight of the ice pick to be $L = 0.1$m, $m = 0.1$kg. Assume also that the angle at which we define the ice pick as "falling" to be $\theta \sim 1$. Putting these numbers into the formula above we find that the length of time that this ice pick can be balanced on a point is

$$t \approx 3.91\text{s} \approx 4\text{s}$$

So, the answer is **"in a few seconds."**

3.

(a)

$$\sum_{a'} |\langle a| x |a'\rangle|^2 (E_{a'} - E_a) = \sum_{a'} \langle a| x |a'\rangle \langle a'| x |a\rangle (E_{a'} - E_a)$$
$$=$$

(b)

(c)

4.