Hele-Shaw Flow & Saffman-Taylor Instability

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PH333: Experimental Soft-Matter Physics

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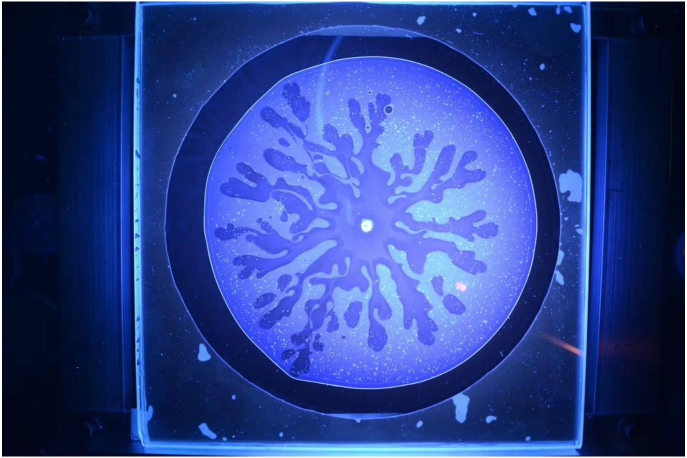
September, 2020

Hele-Shaw flow: A virtual demo (link)



Hele-Shaw flow

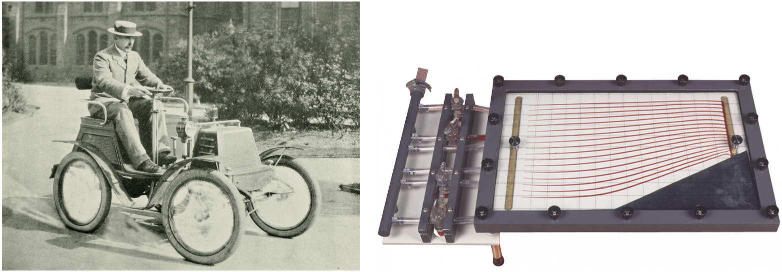


Figure: (1) Hele-Shaw (1854–1941), British engineer [3], (2) Hele-Shaw apparatus

♣ Hele-Shaw invented the apparatus to study how water flows around a ship's hull.[2] Think 19th century "wind" tunnels!

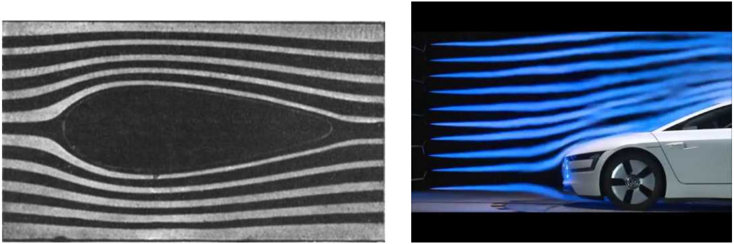


Figure: (1) HS's set up [3], (2) Car in a wind tunnel

Hele-Shaw flow

Figure: (1) Saffman (1931-2008), (2) Taylor (1886-1975)

♣ Saffman-Taylor in 1958 used Hele-Shaw's device to investigate fingering instability in various branching phenomena. This is now known as "Saffman-Taylor instability." [2]

Hele-Shaw flow: The Experiment

Experimental setup: (Source: Karno Widjaja's lab at CMU)

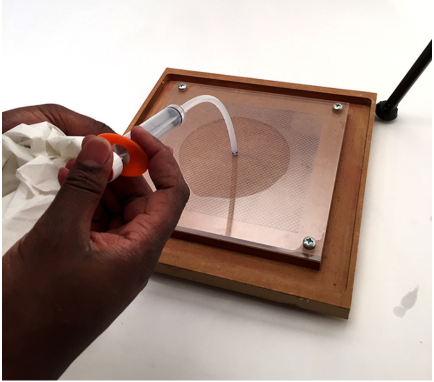


Figure: Before

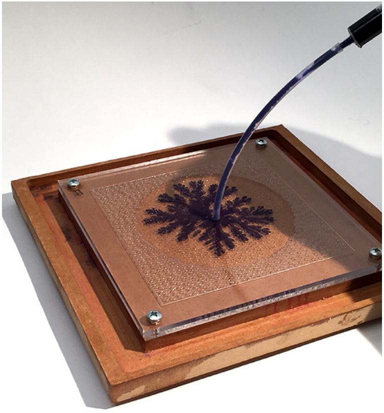


Figure: After

Hele-Shaw flow: The Physics

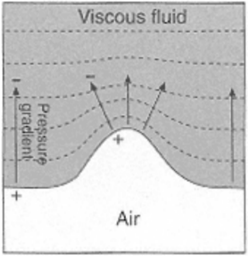
Governing equation [1]:

$$\boxed{\text{Flow: } \vec{u}(x,y) = -\frac{b^2}{12\mu}\nabla \vec{p}(x,y) \quad \& \quad \text{Conservation law: } \nabla \cdot \vec{u} = 0}$$

where

- $\vec{u}(x,y)$: average 2D velocity of the fluid between the plates
- $\vec{p}(x,y)$: the pressure field
- b: the (small) gap between the two plates
- μ: viscosity of the fluid

$\Longrightarrow$ Can be derived from the Navier-Stokes equation!
$\Longrightarrow$ Can be derived from Newton's 2nd law of motion!



$$\vec{u}(x,y) = -\frac{b^2}{12\mu}\nabla\vec{p}(x,y)$$

- Flow velocity is negatively proportional to gradient of pressure field $\implies$ Think ball on a hill
- Saffman & Taylor $\implies$ the gradient in pressure around a bulge at the air/oil interface gets steeper as the bulge gets sharper.
- Self-amplifying process: a small initial bulge begins to move faster than the interface to either side
  $\implies$ Saffman-Taylor instability

Saffman-Taylor Instability

Figure: Link

At first the fluid will appear to grow as a circle until it reaches a critical size where instabilities form on its boundary. These instabilities come from disturbances in the liquid such as:

- Pressure differences of the fluid
- Irregularities in the container
- Particulate matter in the fluid

Instabilities on the boundary form into finger-like patterns.

Saffman-Taylor Instability

talk about characteristic wavelengths/regularity/the effect of surface tension etc (in the paragraph...)

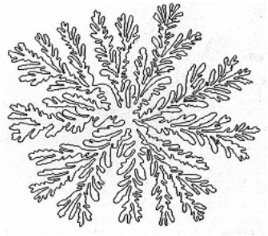
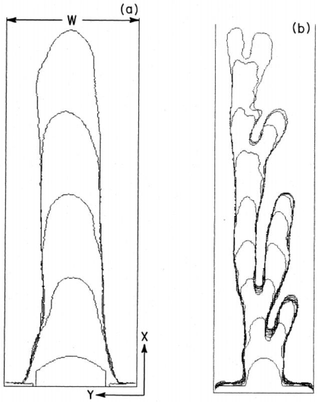


Fig. 5.15
Viscous fingering has a characteristic length scale, which determines the minimum width of the branches. So the fingers are fatter than the fine filaments of DLA clusters. (Image: Yves Couder, Ecole Normale Supérieure, Paris.)

Surface tension & Saffman-Taylor instability compete $\Longrightarrow$ why we get fingers of certain thicknesses (fingers that are too fat/too thin are prohibited)

Saffman-Taylor Instability

The number of "fingers" is ideally proportional to the square root of the radial velocity. The splitting of the fingers is due to low surface tension.



$$\boxed{d_0 = \frac{\pi^3}{3}\frac{b^2}{W^2}\frac{T}{\mu U}}$$

Figure: [1], [2] Surface tension d_0 is inversely proportional to the fluid velocity U, thus for slower velocity (a) we see less tip-splitting and for higher velocity (b) we see defined tip-splitting emerge.

Not all fingers are the same

When the tip of a finger is disturbed, a different pattern might appear.

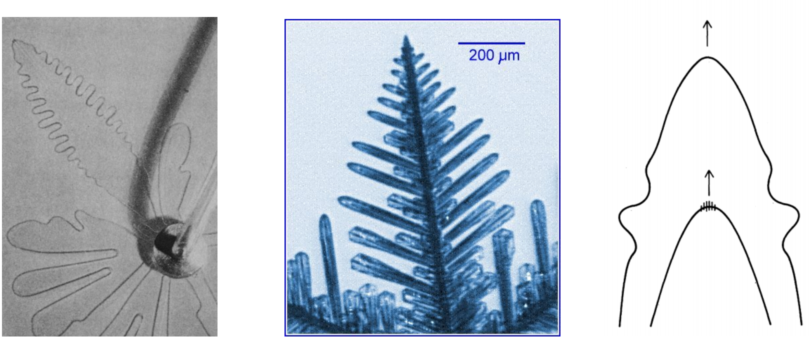


Figure: (1) Dendrite-like finger due to a Couder's bubble [5]; (2) Dendrite on a snowflake (source); (3) Formation mechanism [5]

Simulation: Harder than it seems

Computer simulations of Hele-Shaw fingers is highly non-trivial, requiring complex-analytical methods [4]:

- Approximate boundary by a polygonal
- Treat $\mathbb{R}^2$ as $\mathbb{C}$ and use conformal mapping
- Cauchy-Green coordinates (cf. CIF) to evolve the boundary

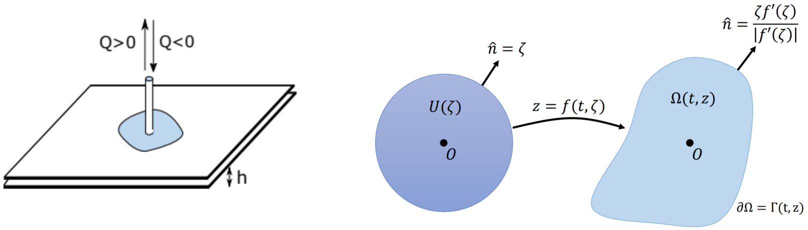


Figure: [4] Initial setup & Conformal mapping

Simulation: Results

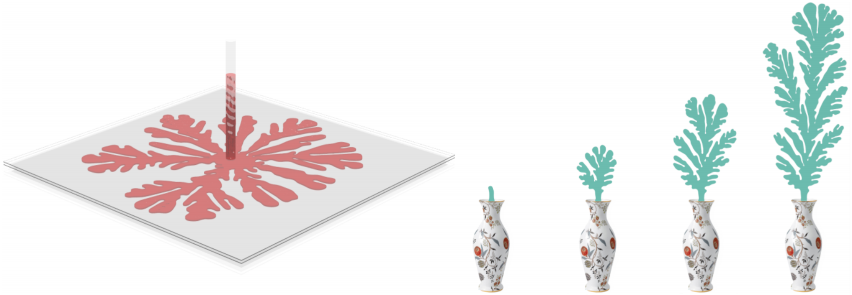


Figure: Computer-generated Hele-Shaw flow by [4]

♣ Watch a computer-generated Hele-Shaw flow now.

References

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