•125. A *multiset* chosen from a set S may be thought of as a subset with repeated elements allowed. To determine a multiset we must say how many times (including, perhaps, zero) each member of S appears in the multiset. The number of times an element appears is called its *multiplicity*. For example if we choose three identical red marbles, six identical blue marbles and four identical green marbles, from a bag of red, blue, green, white and yellow marbles then the multiplicity of a red marble in our multiset is three, while the multiplicity of a yellow marble is zero. The size of a multiset is sum of the multiplicities of its elements. For example if we choose three identical red marbles, six identical blue marbles and four identical green marbles, then the size of our multiset of marbles is 13. What is the number of multisets of size k that can be chosen from an n-element set?

Solution: There is a bijection between arrangements of identical books on n shelves and multisets chosen from an n-element set: the multiplicity of element i is the number of books on shelf i. Thus we have $\binom{n+k-1}{k}$ ways to choose a k-element multiset from an n-element set by Problem 124. ▮

→126. Your answer in the previous problem should be expressible as a binomial coefficient. Since a binomial coefficient counts subsets, find a bijection between subsets of something and multisets chosen from a set S.

Solution: We will show a bijection between ways of choosing $n-1$ things out of $n+k-1$ things and multisets. Namely, take $n+k-1$ objects and line them up in a row. Choose $n-1$ of them. Now let the multiplicity of element 1 of our multiset be the number of objects before the first object we chose. If $1 < i < n$, let the multiplicity of element i of our multiset be the number of objects between the $i-1$th object we choose and the ith object we choose. Let the multiplicity of the nth element of our multiset be the number of objects after the last one we choose. Another way to say the essentially same thing is to make a list of $n+k-1$ blank spaces. We choose k of them in which we put ones and $n-1$ of them in which we put plus signs. Then the multiplicity of element 1 is the number of ones before the first plus sign, the multiplicity of element n is the number of ones after the last plus sign and if $1 < i < n$, the multiplicity of element i is the number of ones between the $(i-1)$th plus sign and the ith plus sign. Notice that we could have two plus signs in a row if some element has multiplicity 0. ▮

127. How many solutions are there in nonnegative integers to the equation $x_1 + x_2 + \cdots + x_m = r$, where m and r are constants?

Solution: We can think of x_i as the multiplicity of element i of a multiset chosen from among m things. The total number of elements of the multiset will be r. Thus we have $\binom{m+r-1}{r}$ solutions. ▮

➔131. Your answer in Problem 130 can be expressed as a binomial coefficient. This means it should be possible to interpret a composition as a subset of some set. Find a bijection between compositions of k into n parts and certain subsets of some set. Explain explicitly how to get the composition from the subset and the subset from the composition.

Solution: If we line up k identical books, there are $k-1$ places in between two books. If we choose $n-1$ of these places and slip dividers into those places, then we have a first clump of books, a second clump of books, and so on. The ith element of our list is the number of books in the ith clump. Clearly using books is irrelevant; we could line up any k identical objects and make the same argument. Our bijection is between compositions and $n-1$-element subsets of the set of $k-1$ spaces between our objects. ∎

·132. Explain the connection between compositions of k into n parts and the problem of distributing k identical objects to n recipients so that each recipient gets at least one.

Solution: Since the recipients are distinct, we can think of them as a first recipient, a second, and so on. Given a composition of k into n parts, let the ith element of the list be the number of objects given to recipient number i. ∎

136. Extend Stirling's triangle enough to allow you to answer the following question and answer it. (Don't fill in the rows all the way; the work becomes quite tedious if you do. Only fill in what you need to answer this question.) A caterer is preparing three bag lunches for hikers. The caterer has nine different sandwiches. In how many ways can these nine sandwiches be distributed into three identical lunch bags so that each bag gets at least one?

Solution: We need $S(9,3)$. Thus we need to extend our table for four more rows, but only out to the column labeled 3. These rows are 6,0,1,31,90; 7,0,1,63,301, 8,0,1,127,966; 9,0,1,255,3025. Thus there are 3025 ways to distribute the sandwiches into the lunch bags. If you work backwards from $S(9,3)$, you will see we don't need the first thee entries of row 9, the first two entries of row 8 and the first entry of row 7 (which is zero anyhow). ∎

137. The question in Problem 136 naturally suggests a more realistic question; in how many ways may the caterer distribute the nine sandwiches into three identical bags so that each bag gets exactly three? Answer this question.

Solution: $\binom{9}{3}\binom{6}{3}\binom{3}{3}/3!$. First we choose three sandwiches for bag 1, then three for bag 2, and put the remainder in bag 3. However, it doesn't matter which bags the sandwiches are in so we have counted each partition 3! times. ∎

8. Using multisets:
Part(a) is just $\binom{12}{3}$.

Part (b) is the number of multisets of size 3 chosen from a set of 12, i.e. $\binom{12+3-1}{12-1} = \binom{14}{11} = \frac{14!}{11!\,3!} = 364$.