Name: **Huan Q. Bui**
Course: **8.370 - QC**
Problem set: **#9**
Due: Wednesday, Dec 7, 2022
Collaborators/References: Piazza

1. The SWAP test

The SWAP test tests whether two pure quantum states $|\phi\rangle$ and $|\psi\rangle$ are the same. Before the measurement in the first qubit is made, the circuit does the following (ignoring normalization):

$$\begin{aligned}|+\rangle|\psi\rangle|\phi\rangle &\to |0\rangle|\psi\rangle|\phi\rangle + |1\rangle|\phi\rangle|\psi\rangle \\ &\to |+\rangle|\psi\rangle|\phi\rangle + |-\rangle|\phi\rangle|\psi\rangle\end{aligned}$$

(a) If $|\phi\rangle = |\psi\rangle$, then the state of the circuit before the measurement is

$$(|+\rangle + |-\rangle)|\psi\rangle|\psi\rangle = |0\rangle|\psi\rangle|\psi\rangle .$$

So the probability that we observe $|0\rangle$ in the first wire is $\boxed{1}$.

(b) The state of the circuit before the measurement is

$$\frac{1}{2}|0\rangle(|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle) + \frac{1}{2}|1\rangle(|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle) .$$

The probability that we observe $|0\rangle$ in the first wire is

$$\frac{1}{4}(\langle\psi|\langle\phi| + \langle\phi|\langle\psi|)(|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle) = \frac{1}{4}(1+1) = \boxed{\frac{1}{2}}$$

(c) Suppose we apply the SWAP test with the inputs being two identical density matrices:

$$\rho_1 = \rho_2 = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| .$$

We can do this problem probabilistically. The initial states of the circuit and associated probabilities are:

$$\begin{aligned}\Pr(|+\rangle|0\rangle|0\rangle) &= p^2 \\ \Pr(|+\rangle|0\rangle|1\rangle) &= p(1-p) \\ \Pr(|+\rangle|1\rangle|0\rangle) &= (1-p)p \\ \Pr(|+\rangle|1\rangle|1\rangle) &= (1-p)(1-p)\end{aligned}$$

From the previous parts, the probability that we observe $|0\rangle$ on the top wire is

$$p^2 + (1-p)(1-p) + \frac{1}{2}[p(1-p) + (1-p)p] = \boxed{1 - p + p^2}.$$

2. kl-qubit code

The generalization of the 9-qubit code to a kl-qubit code has the codewords:

$$|0\rangle_L = \frac{1}{2^{l/2}}(|\underbrace{000\ldots0}_{k}\rangle + |\underbrace{111\ldots1}_{k}\rangle)^{\otimes l}$$

$$|1\rangle_L = \frac{1}{2^{l/2}}(|\underbrace{000\ldots0}_{k}\rangle - |\underbrace{111\ldots1}_{k}\rangle)^{\otimes l}$$

Here k, l are odd numbers. In the known case where $k = l = 3$, we know that the code can correct 1 bit error and 1 phase error. To correct the bit error we need to measure 2 syndrome bits. To correct the phase error we also need to measure 2 syndrome bits.

3.

(a)

(b)

4.

(a)

(b)

(c)

(d)