Name: **Huan Q. Bui**
Course: **8.309 - Classical Mechanics III**
Problem set: **#2**

1. Spherical Pendulum with Friction

(a) In this problem, we will ignore the effect of buoyancy and assume that the effective mass m' is equal to m, the real mass of the pendulum blob. Stokes law states that $\vec{F}_{\text{friction}} = -6\pi\eta R\vec{v} \equiv -b\vec{v}$, where $b > 0$ is a constant which depends on the radius R of the mass and the viscosity μ of the fluid. Putting this force in the form $F_i = -h_i(v_i)\vec{v}_i/v_i$, we find that $h_i = bv_i$. Thus, the dissipation function is

$$\mathcal{F} = \sum_{i=1}^{3} \int_0^{v_i} h_i(v_i')\, dv_i' = \frac{1}{2}b(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

We choose the spherical coordinates such that the origin is the pivot of the pendulum and $\theta = \pi$ when the pendulum is in its equilibrium position: $(x, y, z) = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)$, we find

$$\mathcal{F} = \frac{1}{2}b\left[\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right] = \frac{b}{2}\left[\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right] = \frac{bl^2}{2}(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

where we have used $r = l$ which is fixed.

(b) The Lagrangian is

$$\begin{aligned}\mathcal{L} &= T - U \\ &= \frac{1}{2}m\dot{\vec{r}}^2 - mgr\cos\theta \\ &= \frac{m}{2}\left[\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)\right] - mgr\cos\theta \\ &= \frac{ml^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) - mgl\cos\theta\end{aligned}$$

where we have used $r = l$ which is fixed. We find two equations of motion:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) - \frac{\partial\mathcal{L}}{\partial\theta} = \frac{-\partial\mathcal{F}}{\partial\dot{\theta}} &\implies l^2m\ddot{\theta} - glm\sin\theta + l^2m\cos\theta\sin\theta\dot{\phi}^2 = -bl^2\dot{\theta} \\ &\implies \boxed{\ddot{\theta} = -\frac{b\dot{\theta}}{m} + \frac{\sin\theta}{l}\left[g + l\cos\theta\dot{\phi}^2\right]}\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = \frac{-\partial\mathcal{F}}{\partial\dot{\phi}} &\implies l^2m\sin\theta\left(2\dot{\theta}\cos\theta\dot{\phi} + \sin\theta\ddot{\phi}\right) = -bl^2\sin^2\theta\dot{\phi} \\ &\implies \boxed{\ddot{\phi} = -\frac{\dot{\phi}(b + 2m\cot\theta\dot{\theta})}{m}}\end{aligned}$$

(c) Mathematica code:

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(*Problem 1*)

In[1]:= L = (m*l^2/2)*(\[Theta]'[t]^2 + Sin[\[Theta][t]]^2*\[Phi]'[t]^2) -
m*g*l*Cos[\[Theta][t]]

Out[1]= -g l m Cos[\[Theta][t]] +
1/2 l^2 m (Derivative[1][\[Theta]][t]^2 +
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Sin[\[Theta][t]]^2 Derivative[1][\[Phi]][t]^2)

In[2]:= F = (b*l^2/2)*(\[Theta]'[t]^2 + Sin[\[Theta][t]]^2*\[Phi]'[t]^2)

Out[2]= 1/2 b l^2 (Derivative[1][\[Theta]][t]^2 +
Sin[\[Theta][t]]^2 Derivative[1][\[Phi]][t]^2)

In[13]:= (*Theta equation*)

In[12]:= Solve[FullSimplify[
D[D[L, \[Theta]'[t]], t] -
D[L, \[Theta][t]] == -D[F, \[Theta]'[t]]], \[Theta]''[
t]] // FullSimplify

Out[12]= {{(\[Theta]^\[Prime]\[Prime])[
t] -> -((b Derivative[1][\[Theta]][t])/m) + (
Sin[\[Theta][t]] (g +
l Cos[\[Theta][t]] Derivative[1][\[Phi]][t]^2))/l}}

In[5]:= D[D[L, \[Theta]'[t]], t]

Out[5]= l^2 m (\[Theta]^\[Prime]\[Prime])[t]

In[8]:= D[L, \[Theta][t]]

Out[8]= g l m Sin[\[Theta][t]] +
l^2 m Cos[\[Theta][t]] Sin[\[Theta][t]] Derivative[1][\[Phi]][t]^2

In[9]:= -D[F, \[Theta]'[t]]

Out[9]= -b l^2 Derivative[1][\[Theta]][t]
(*Phi equation*)

In[20]:= Solve[FullSimplify[
D[D[L, \[Phi]'[t]], t] -
D[L, \[Phi][t]] == -D[F, \[Phi]'[t]]], \[Phi]''[
t]] // FullSimplify

Out[20]= {{(\[Phi]^\[Prime]\[Prime])[
t] -> -(((b +
2 m Cot[\[Theta][t]] Derivative[1][\[Theta]][t]) Derivative[
1][\[Phi]][t])/m)}}

In[19]:= D[D[L, \[Phi]'[t]], t] // FullSimplify // TeXForm

Out[19]//TeXForm=
l^2 m \sin (\theta (t)) \left(2 \theta '(t) \cos (\theta (t)) \phi '(t)+\sin (\theta
(t)) \phi ''(t)\right)

In[16]:= D[L, \[Phi][t]]

Out[16]= 0

In[17]:= -D[F, \[Phi]'[t]]

Out[17]= -b l^2 Sin[\[Theta][t]]^2 Derivative[1][\[Phi]][t]
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2. Bead Spiraling on a Helix

(a) The Lagrangian only has the kinetic term. In cylindrical coordinates, we have $(x, y, z) = (r\cos\theta, r\sin\theta, z) = (bz\cos(az), bz\sin(az), z)$

$$\begin{aligned}\mathcal{L} &= \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)\\ &= \frac{m}{2}(b^2\dot{z}^2 + a^2b^2z^2\dot{z}^2 + \dot{z}^2)\\ &= \frac{m}{2}(1 + b^2 + a^2b^2z^2)\dot{z}^2.\end{aligned}$$

From the Euler-Lagrange equation for z we find

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{z}}\right) = \frac{\partial\mathcal{L}}{\partial z} \implies m\ddot{z}(1+b^2+a^2b^2z^2) + 2ma^2b^2z\dot{z}^2 = ma^2b^2z\dot{z}^2$$
$$\implies \ddot{z} = \frac{-a^2b^2z\dot{z}^2}{1+b^2+a^2b^2z^2}$$

Following the hint, we divide both sides by $\dot{z}$ and integrate:

$$\int_0^t \frac{\ddot{z}}{\dot{z}}\,dt' = \int_{z(0)}^{z(t)} \frac{-a^2b^2z}{1+b^2+a^2b^2z^2}\,dz \implies \ln\left(\frac{\dot{z}(t)}{v_0}\right) = \ln\frac{\sqrt{1+b^2+a^2b^2h^2}}{\sqrt{1+b^2+a^2b^2z^2}}.$$

Thus, we find $\dot{z}$ as a function of a, b, m, z, v_0, h:

$$\boxed{\dot{z} = v_0\sqrt{\frac{1+b^2+a^2b^2h^2}{1+b^2+a^2b^2z^2}}}$$

There are two ways to find the constraint torque Z_θ. In the first way, we simply compute the term $(d/dt)(\partial\mathcal{L}/\partial\dot{\theta})$ and plug in what we know about $z, \dot{z}, \ddot{z}$:

$$Z_\theta = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = mb^2a\frac{d}{dt}\left(z^2\dot{z}\right)$$
$$= \boxed{mab^2v_0^2(1+b^2+a^2b^2h^2)\frac{z(2+2b^2+a^2b^2z^2)}{(1+b^2+a^2b^2z^2)^2}}$$

Alternatively, we can first find L_z, the z-component of the angular momentum, then find its rate of change: $Z_\theta = dL_z/dt$. The angular momentum vector is $\vec{L} = (L_x, L_y, L_z) = \vec{r}\times m\vec{v} = m(x,y,z)\times(\dot{x},\dot{y},\dot{z})$. Using Mathematica, we find that

$$L_z = mab^2v_0z^2\sqrt{\frac{1+b^2+a^2b^2h^2}{1+b^2+a^2b^2z^2}}.$$

Since $\ddot{z}$ and $\dot{z}$ can ultimately be expressed in terms of z, we can find the desired constraint torque:

$$Z_\theta = \frac{d}{dt}L_z$$
$$= \frac{mab^2v_0z\dot{z}\left(a^2b^2z^2+2b^2+2\right)}{b^2\left(a^2h^2+1\right)+1}\left(\frac{a^2b^2h^2+b^2+1}{a^2b^2z^2+b^2+1}\right)^{3/2}$$
$$= \boxed{mab^2v_0^2(1+b^2+a^2b^2h^2)\frac{z(2+2b^2+a^2b^2z^2)}{(1+b^2+a^2b^2z^2)^2}}$$

(b) The particle only has kinetic energy:

$$E = \frac{m}{2}(1+b^2+a^2b^2z^2)\dot{z}^2.$$

Rearranging the relation between $\dot{z}$ and z gives

$$(1+b^2+a^2b^2z^2)\dot{z}^2 = (1+b^2+a^2b^2h^2)v_0^2 = \text{constant}.$$

Therefore,

$$E = \frac{m}{2}(1+b^2+a^2b^2h^2)v_0^2$$

which is constant.

On the other hand, the rate of change of L_z in z is

$$\frac{d}{dz}L_z \propto \frac{d}{dz}\left(\frac{z^2}{\sqrt{C+z^2}}\right) = \frac{z^3+2Cz}{(C+z^2)^{3/2}}, \quad C \text{ is a positive constant.}$$

When $|z|$ increases, it is clear that L_z increases, as desired.

(c) Mathematica code:

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In[1]:= (*Problem 2*)

In[2]:= (*Lagrangian*)

In[3]:= L = (m/2)*(D[x[t], t]^2 + D[y[t], t]^2 + D[z[t], t]^2)

Out[3]= 1/2 m (Derivative[1][x][t]^2 + Derivative[1][y][t]^2 +
Derivative[1][z][t]^2)

In[4]:= cyl = {x[t] -> b*z[t]*Cos[a*z[t]],
y[t] -> b*z[t]*Sin[a*z[t]] , z[t] -> z[t],
D[x[t], t] -> D[b*z[t]*Cos[a*z[t]], t],
D[y[t], t] -> D[b*z[t]*Sin[a*z[t]], t] };

In[5]:= L = L /. cyl // FullSimplify

Out[5]= 1/2 m (1 + b^2 + a^2 b^2 z[t]^2) Derivative[1][z][t]^2

(*Euler-Lagrange Equation for z*)

In[6]:= FullSimplify[D[D[L, z'[t]], t] == D[L, z[t]]]

Out[6]= a^2 b^2 m z[t] Derivative[1][z][t]^2 +
m (1 + b^2 + a^2 b^2 z[t]^2) (z^\[Prime]\[Prime])[t] == 0

In[7]:= Solve[FullSimplify[D[D[L, z'[t]], t] == D[L, z[t]]], z''[t]]

Out[7]= {{(z^\[Prime]\[Prime])[t] -> -((
a^2 b^2 z[t] Derivative[1][z][t]^2)/(1 + b^2 + a^2 b^2 z[t]^2))}}

In[8]:= Integrate[-((a^2 b^2 z[t])/(1 + b^2 + a^2 b^2 z[t]^2)), z[t]]

Out[8]= -(1/2) Log[1 + b^2 + a^2 b^2 z[t]^2]

In[9]:= zPrime[t] =
v0*Sqrt[(1 + b^2 + a^2*b^2*h^2)/(1 + b^2 + a^2*b^2*z[t]^2)]

Out[9]= v0 Sqrt[(1 + b^2 + a^2 b^2 h^2)/(1 + b^2 + a^2 b^2 z[t]^2)]

In[10]:= (*Torque in two ways*)

In[41]:= q =
m*b^2*a*D[z[t]^2*z'[t],
t] /. {z''[t] -> -((a^2 b^2 z[t] Derivative[1][z][t]^2)/(
1 + b^2 + a^2 b^2 z[t]^2)),
z'[t] ->
v0 Sqrt[(1 + b^2 + a^2 b^2 h^2)/(1 + b^2 + a^2 b^2 z[t]^2)]} //
FullSimplify

Out[41]= (a b^2 m (2 (1 + b^2 (1 + a^2 h^2)) v0^2 z[t] -
a^2 b^2 z[t]^3 Derivative[1][z][t]^2))/(1 + b^2 + a^2 b^2 z[t]^2)

In[45]:= q /. {z'[t] ->
v0 Sqrt[(1 + b^2 + a^2 b^2 h^2)/(
1 + b^2 + a^2 b^2 z[t]^2)]} // Simplify

Out[45]= (a b^2 (1 + b^2 (1 + a^2 h^2)) m v0^2 z[
t] (2 + 2 b^2 + a^2 b^2 z[t]^2))/(1 + b^2 + a^2 b^2 z[t]^2)^2

In[42]:= AngMom =
FullSimplify[
m*Cross[{x[t], y[t], z[t]}, {D[x[t], t], D[y[t], t],
D[z[t], t] }] /. cyl /. {z'[t] -> zPrime[t]}]

Out[42]= {-a b m v0 Cos[a z[t]] z[t]^2 Sqrt[(1 + b^2 + a^2 b^2 h^2)/(
1 + b^2 + a^2 b^2 z[t]^2)], -a b m v0 Sin[a z[t]] z[t]^2 Sqrt[(
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1 + b^2 + a^2 b^2 h^2)/(1 + b^2 + a^2 b^2 z[t]^2)],
a b^2 m v0 z[t]^2 Sqrt[(1 + b^2 + a^2 b^2 h^2)/(
1 + b^2 + a^2 b^2 z[t]^2)]}

In[49]:= DAngMomZ = D[AngMom[[3]], t] // FullSimplify

Out[49]= (a b^2 m v0 z[t] ((1 + b^2 + a^2 b^2 h^2)/(
1 + b^2 + a^2 b^2 z[t]^2))^(
3/2) (2 + 2 b^2 + a^2 b^2 z[t]^2) Derivative[1][z][t])/(1 +
b^2 (1 + a^2 h^2))

In[51]:= DAngMomZ /. {z'[t] ->
v0 Sqrt[(1 + b^2 + a^2 b^2 h^2)/(
1 + b^2 + a^2 b^2 z[t]^2)]} // FullSimplify

Out[51]= (a b^2 (1 + b^2 (1 + a^2 h^2)) m v0^2 z[
t] (2 + 2 b^2 + a^2 b^2 z[t]^2))/(1 + b^2 + a^2 b^2 z[t]^2)^2

In[53]:= D[z^2/Sqrt[C + z^2], z] // FullSimplify

Out[53]= (2 C z + z^3)/(C + z^2)^(3/2)
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3. Hoop Rolling on a Cylinder

We will assume that the hoop falls to the right. The coordinates for this problem are (ρ, α, β) where $\rho = r + R$, α is the angle which $\vec{\rho}$ makes with the horizontal direction (measured counterclockwise from the horizontal), and β is the angle $\vec{r}$ makes with the vertical, measured clockwise from the vertical.

The Lagrangian is

$$\begin{aligned}\mathcal{L} &= T_{\text{trans}} + T_{\text{rot}} - U \\ &= \frac{m}{2}\left(\dot{\rho}^2 + \rho^2\dot{\alpha}^2\right) + \frac{1}{2}mr^2\dot{\beta}^2 - mg\rho\sin\alpha.\end{aligned}$$

The constraint equations are

$$f = \rho - (r + R) = 0 \qquad \text{and} \qquad g = \dot{\beta} - \frac{(r+R)}{r}\dot{\alpha} = 0.$$

The first equation comes from the fact that the hoop is always in contact with the larger cylidner before leaving it. The second equation is due to the no-slip condition: the distance that the center of the hoop travels is the same as the distance travelled by any point on its body. We notice that f_1 is a holonomic constraint, while g_2 is semi-holonomic.

Let λ_1, λ_2 be Lagrange multipliers associated with our two constraints. The method of Lagrange multiplier states that

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\rho}}\right) - \frac{\partial \mathcal{L}}{\partial \rho} = \lambda_1\frac{\partial f}{\partial \rho} + \lambda_2\frac{\partial g}{\partial \dot{\rho}} &\implies m\ddot{\rho} - m\rho\dot{\alpha}^2 + mg\sin\alpha = \lambda_1 \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\alpha}}\right) - \frac{\partial \mathcal{L}}{\partial \alpha} = \lambda_1\frac{\partial f}{\partial \alpha} + \lambda_2\frac{\partial g}{\partial \dot{\alpha}} &\implies m\rho^2\ddot{\alpha} + 2m\rho\dot{\rho}\dot{\alpha} + m\rho g\cos\alpha = -\lambda_2\frac{r+R}{r} \\
\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\beta}}\right) - \frac{\partial \mathcal{L}}{\partial \beta} = \lambda_1\frac{\partial f}{\partial \beta} + \lambda_2\frac{\partial g}{\partial \dot{\beta}} &\implies mr^2\ddot{\beta} = \lambda_2
\end{aligned}$$

To find at which point the hoop leaves the cylinder, we need to solve the equation $\lambda_1 = 0$, since λ_1 represents the constraint that the hoop stays on the cylinder (i.e., that the center-to-center distance is exactly $\rho = r + R$). To this end, we want to find $\dot{\alpha}$ in terms of α. Plugging in the constraints equations into these equations of motion we find

$$\begin{aligned} mg\sin\alpha - m(r+R)\dot\alpha^2 &= \lambda_1 \\ m(r+R)g\cos\alpha + m(r+R)^2\ddot\alpha &= -\lambda_2\frac{r+R}{r} \\ mr^2\left(\frac{r+R}{r}\right)\ddot\alpha &= \lambda_2 \end{aligned}$$

Combining the α and β equations we find

$$m(r+R)g\cos\alpha + m(r+R)^2\ddot\alpha = mr^2\left(\frac{r+R}{r}\right)^2\ddot\alpha \implies \ddot\alpha = \frac{-g}{2(r+R)}\cos\alpha.$$

Multiplying both sides of the above differential equation for α by $\dot\alpha$ and integrating we find

$$\int_0^t \ddot\alpha\dot\alpha\, dt' = \frac{-g}{2(r+R)}\int\cos\alpha\, d\alpha \implies \dot\alpha^2 = \frac{g}{r+R}\left(1-\sin\alpha\right).$$

Here we have used the initial conditions $\dot\alpha(0) = 0$ (the hoop starts from rest) and $\alpha(0) = \pi/2$. Now, it remains to solve the equation $\lambda_1 = 0$ for α:

$$\begin{aligned} 0 &= \lambda_1 \\ &= mg\sin\alpha - m(r+R)\dot\alpha^2 \\ &= mg\sin\alpha - m(r+R)\frac{g}{r+R}(1-\sin\alpha) \\ &= mg(1-2\sin\alpha) \implies \boxed{\alpha = \pi/6} \end{aligned}$$

So, the hoop leaves the cylidner when $\vec\rho$ is 30° from the horizontal.

4. An Infinite Rope

Assume that the rope, in equilibrium, forms some curve C. We will solve this problem using Lagrangian mechanics. The Lagrangian for the rope only has the potential term (since it is static): $\mathcal{L} = -U$. Moreover, since U cannot be velocity-dependent (again, the rope is static), we must have

$$\cancel{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q}\right)} - \frac{\partial\mathcal{L}}{\partial q} = 0 \implies \frac{\partial V}{\partial q} = 0.$$

So, the problem comes down to finding C for which the potential energy U is minimized. Consider an infinitesimal element $d\vec{l}$ of the curve C, we have

$$dl = \sqrt{dx^2 + dy^2} = \sqrt{1+x'^2}\, dy.$$

Since the rope has uniform linear mass density – let us call it λ – we have $dm = \lambda\, dl$. The potential energy associated with this infinitesimal element of the rope is therefore

$$dU = gy\, dm = \lambda gy\, dl = \lambda gy\sqrt{1+x'^2}\, dy.$$

Let the rope be between points (1) and (2), our goal is to minimize the total potential energy

$$U = \int dU = \int_{(1)}^{(2)} \lambda gy\sqrt{1+x'^2}\, dy.$$

We see that this quantity is proportional to

$$2\pi \int_{(1)}^{(2)} y\sqrt{1+x'^2}\,dy$$

which is the area of the surface generated by revolving the rope around the x-axis. Thus, the problem of finding the curve which the rope assumed between the two pulleys is equivalent to minimizing the area of this surface.

Back to minimizing U. Let $x(y,0)$ denote the desired curve and introduce a perturbation $\eta(y)$ for which $\eta(y_1) = \eta(y_2) = 0$. A perturbed curve may take the form $x(y,\alpha) = y(x,0) + \alpha\eta(y)$ where α is some parameter. Let $f = y\sqrt{1+x'^2}$, we see that U is minimized whenever

$$0 = \left(\frac{dU}{d\alpha}\right)\bigg|_{\alpha=0} = \int_{y_2}^{y_1} \left(\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'}\right)\frac{\partial x}{\partial \alpha}\,dy\bigg|_{\alpha=0},$$

which we obtained by first differentiating under the integral sign and using the vanishing boundary conditions for η plus integration by parts (This is standard calculus of variations – the full step-by-step derivation is given in Chapter 2 of Goldstein's). In any case, we find that f must satisfy the Euler-Lagrange equation:

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0.$$

With $f = y\sqrt{1+x'^2}$, we find that $\partial f/\partial x = 0$ and therefore

$$\frac{d}{dy}\frac{\partial f}{\partial x'} = \frac{d}{dt}\left[\frac{yx'}{\sqrt{1+x'^2}}\right] = 0 \implies \frac{yx'}{\sqrt{1+x'^2}} = C = \text{constant}.$$

Squaring, moving all x's to one side and y's to the other, and finally integrating we find

$$y^2x'^2 = C^2(1+x'^2) \implies x'^2(y^2-C^2) = C^2 \implies \int dx = \int \frac{C\,dy}{\sqrt{y^2-C^2}} \implies x = C \operatorname{arc\,cosh}\frac{y}{C} + B$$

where B is some constant and antiderivative of the integrand is well-known and can be found in various tables. Inverting this gives

$$\boxed{y = C\cosh\frac{x-B}{C}}$$

Finally, we solve for B, C using the fact that the curve must go through (x_1, y_1) and (x_2, y_2):

$$\boxed{y_1 = C\cosh\frac{x_1-B}{C} \qquad \text{and} \qquad y_2 = C\cosh\frac{x_2-B}{C}}$$

Remark: The form $f = y\sqrt{1+x'^2}$ may feel awkward on first glance, but it turns out that the choice $dl = \sqrt{dx^2+dy^2} = \sqrt{1+x'^2}\,dy$ over $dl = \sqrt{dx^2+dy^2} = \sqrt{1+y'^2}\,dx$ is actually deliberate to make the calculations easier. In the latter form, derivatives get a bit uglier when we plug f into the Euler-Lagrange equation, as we will see in the alternative approach below.

Another way to do this problem reflects what happens when the latter form of dl is used. The Lagrangian is once again just $\mathcal{L} = -U$ where

$$U = \int_{x_1}^{x_2} \lambda g y\sqrt{1+\dot{y}^2}\,dx$$

(which is the same but written slightly differently then before). Calculus of variations implies

$$-\frac{d}{dx}\left(\frac{\partial U}{\partial \dot{y}}\right) + \frac{\partial U}{\partial y} = \Lambda\frac{\partial f_1}{\partial y} \implies -\frac{d}{dx}\left[\frac{y\dot{y}}{\sqrt{1+\dot{y}^2}}\right] + \sqrt{1+\dot{y}^2} = 0.$$

Doing the x derivative is a little involved, but it is doable. After simplying we find the following differential equation

$$y\ddot{y} - \dot{y}^2 - 1 = 0.$$

Unlike the previous problem where we can find a formula for y in a straightforward manner, this differential equation requires special techniques to solve. Even though we won't solve this equation explicitly, we will check that the solution we found before for y indeed solves this equation. Suppose

$$y = C\cosh\frac{x-B}{C}$$

then

$$\dot{y} = -\sinh\frac{x-B}{C} \implies \ddot{y} = \frac{1}{C}\cosh\frac{x-B}{C}.$$

So

$$y\ddot{y} - \dot{y}^2 y - 1 = \cosh^2\frac{x-B}{C} - \sinh^2\frac{x-B}{C} - 1 = 0,$$

as desired.

5. A Falling Ladder for 8.09 students

6. A Falling Ladder for 8.309 students

(a) The CM is the center of the ladder. Assuming uniform linear mass density, i.e., $dm = \lambda\, dl = (M/L)\, dl$, the moment of inertia of the ladder for rotations about this point is

$$I = \int_{-L/2}^{L/2} l^2\, dm = \frac{M}{L}\int_{-L/2}^{L/2} l^2\, dl = \frac{M}{3L}\left(\frac{L^3}{8}+\frac{L^3}{8}\right) = \boxed{\frac{1}{12}ML^2}$$

The Lagrangian is

$$\boxed{\mathcal{L} = \frac{m}{2}(\dot{x}^2+\dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 - mgy}$$

There are two constraints:

$$\boxed{f_1 = x - \frac{L}{2}\cos\theta = 0 \qquad \text{and} \qquad f_2 = y - \frac{L}{2}\sin\theta = 0}$$

(b) Let λ_1, λ_2 be Lagrange multipliers, we find the following equations:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) - \frac{\partial \mathcal{L}}{\partial x} = \lambda_1\frac{\partial f_1}{\partial x} + \lambda_2\frac{\partial f_2}{\partial x} \implies M\ddot{x} = \lambda_1$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial \mathcal{L}}{\partial y} = \lambda_1\frac{\partial f_1}{\partial y} + \lambda_2\frac{\partial f_2}{\partial y} \implies M\ddot{y} + Mg = \lambda_2$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = \lambda_1\frac{\partial f_1}{\partial \theta} + \lambda_2\frac{\partial f_2}{\partial \theta} \implies I\ddot{\theta} = \lambda_1\frac{L}{2}\sin\theta - \lambda_2\frac{L}{2}\cos\theta.$$

From the two constraints, we can compute $\ddot{x}$ and $\ddot{y}$ in order to solve for λ_1, λ_2:

$$\lambda_1 = M\ddot{x} = \frac{-ML}{2}\left[\cos\theta\dot{\theta}^2 + \sin\theta\ddot{\theta}\right] \quad \text{and} \quad \lambda_2 = Mg + \frac{ML}{2}\left[-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\right].$$

Plugging these back in and replacing $I = ML^2/12$, we find the equations of motion for the system:

$$\boxed{\ddot{x} = -\frac{L}{2}\left[\cos\theta\dot{\theta}^2 + \sin\theta\ddot{\theta}\right]}$$

$$\boxed{\ddot{y} = \frac{L}{2}\left[-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\right]}$$

$$\boxed{\ddot{\theta} = -\frac{3g}{2L}\cos\theta}$$

(c) To solve for θ, we will find the formula for $t = t(\theta)$. The first step is the old trick where one multiplies both sides of the θ equation by $\dot{\theta}$ and integrate:

$$\ddot{\theta}\dot{\theta} = -\frac{3g}{2L}\cos\theta\dot{\theta} \implies \dot{\theta}^2 = \frac{3g}{L}(\sin\theta_0 - \sin\theta),$$

where we have used $\dot{\theta}(0) = 0$. Since θ decreases in time, we find that

$$t = \int dt = \sqrt{\frac{L}{3g}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{\sin\theta' - \sin\theta_0}} \implies \boxed{t(\theta) = \sqrt{\frac{L}{3g}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{\sin\theta' - \sin\theta_0}}}$$

(d) The ladder leaves the wall when the normal force due to the wall vanishes. In this case, we will have $\lambda_1 = 0$, which implies that

$$\cos\theta\dot{\theta}^2 = -\sin\theta\ddot{\theta} \implies \cos\theta\frac{3g}{L}(\sin\theta_0 - \sin\theta) = \sin\theta\frac{3g}{2L}\cos\theta \implies \sin\theta = \frac{2}{3}\sin\theta_0.$$

Thus, the ladder leaves the wall when

$$\boxed{\theta = \arcsin\left(\frac{2}{3}\sin\theta_0\right)}$$

At this point, the height of the upper end of the ladder is

$$h = 2y = 2\frac{L}{2}\sin\theta = \boxed{\frac{2L}{3}\sin\theta_0}$$

(e) After the ladder loses contact with the wall, the equations of motion are obtained by simply setting $\lambda_1 = 0$ to the previous equations of motion. For the θ equation, we replace I by $ML^2/12$ and set $\lambda_1 = 0$. After simplifying the expressions in Mathematica we find

$$\boxed{\ddot{x} = 0}$$

$$\boxed{\ddot{y} = \frac{L}{2}\left[-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\right]}$$

$$\boxed{\ddot{\theta} = \frac{3\cos\theta\left(-2g/L + \sin\theta\dot{\theta}^2\right)}{3\cos^2\theta + 1}}$$

(f) Assume that the ladder does not bounce back when $\theta = 0$. We wish to show that the normal force on the ladder due to the floor, λ_2, never vanishes (whether the ladder is in contact with the wall or not).

Consider the case where the ladder is still in contact with the wall. From Part (b), we have

$$\lambda_2 = Mg + \frac{ML}{2}\left[-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}\right]$$

Plugging in known results for $\dot{\theta}$ and $\ddot{\theta}$ in terms of constants such as g, L, and θ_0:

$$\ddot{\theta} = -\frac{3g}{2L}\cos\theta \quad \text{and} \quad \dot{\theta}^2 = \frac{3g}{L}(\sin\theta_0 - \sin\theta)$$

we find

$$\lambda_2 = \frac{3gM}{2}\left[\frac{5}{3} - \frac{1}{2}\sin^2\theta - \sin\theta_0\sin\theta\right].$$

We notice that this is independent of L. Moreover, since $\theta \leq \theta \in [0, \pi/2]$, we have

$$\lambda_2 \geq \frac{3gM}{2}\left[\frac{5}{3} - \frac{3}{2}\sin^2\theta_0\right] > 0.$$

for all $\theta_0 \in [0, \pi/2]$. Therefore, λ_2 never vanishses, and the ladder never leaves the floor when it is in contact with the wall.

Next, we consider the case where the ladder is no longer in contact with the wall, i.e., λ_1.

$$\lambda_2 = -\frac{2I\ddot{\theta}}{L\cos\theta} = -\frac{ML}{2}\frac{-2g/L + \sin\theta\dot{\theta}^2}{3\cos^2\theta + 1},$$

where the second equality is obtained by plugging in the equation of motion for θ. We want to do the same thing as previously, that is to express $\dot{\theta}$ and $\ddot{\theta}$ in terms of the known constants and θ so that the final expression for λ_2 is only dependent on θ and the constants. From the equation above, it sufffices to find $\dot{\theta}$. To this end, we make sure of energy conservation. Consider two points in the dynamics, first at $\theta = \theta_c = \arcsin((2/3) * \sin(\theta_0))$ where θ_c is the angle the ladder makes with the floor when it leaves the wall, and $\theta(t)$ at some time t, after the ladder has left the wall. Conservation of energy states that

$$\frac{M}{2}\dot{x}^2 + \frac{M}{2}\dot{y}^2 - mgy + \frac{1}{2}\frac{1}{12}ML^2\dot{\theta}^2 = \frac{M}{2}\dot{x}^2 + \frac{M}{2}\dot{y}(\theta_c)^2 - mgy(\theta_c) + \frac{1}{2}\frac{1}{12}ML^2\dot{\theta}^2(\theta_c)$$

At θ_c, many of the above quantities are known:

$$blah$$

Moreover, by definition, we can write the left-hand side entirely in terms of $\dot{\theta}$ and θ:

$$blah$$

Putting everything together and solving for $\dot{\theta}$ gives

$$\dot{\theta}^2 = \frac{4g\left(9\sin\theta - \sin^3\theta_0 - 3\sin\theta_0\right)}{3L\left(3\cos^2\theta + 1\right)}.$$

Plugging this into the expression for λ_2 we find

$$\lambda_2 = \frac{gM\left[2\sin\theta\left(\sin^3\theta_0 + 3\sin\theta_0 - 9\sin\theta\right) + 9\cos^2\theta + 3\right]}{3\left(3\cos^2\theta + 1\right)^2}.$$

By hook or by crook we will show that λ_2 also never vanishes.

(g)