PY 711 Fall 2010
Homework 2: Due Tuesday, September 7

1. In class we defined for a free real scalar field,

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} \right), \tag{1}$$

$$\pi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}} \right). \tag{2}$$

(a) (5 points) By explicit calculation show that $[\phi(\vec{x}), \phi(\vec{y})] = 0$ for any $\vec{x}$ and $\vec{y}$.

(b) (5 points) By explicit calculation show that $[\pi(\vec{x}), \pi(\vec{y})] = 0$ for any $\vec{x}$ and $\vec{y}$.

(c) (5 points) The momentum operator $\vec{P}$ is the Noether charge associated with spatial translations. It terms of ϕ and π it has the form

$$\vec{P} = -\int d^3\vec{x}\ \pi(\vec{x}) \vec{\nabla} \phi(\vec{x}). \tag{3}$$

Using Eq. (1), (2), and parity invariance, show that $\vec{P}$ can be written as

$$\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} a_{\vec{p}}^{\dagger} a_{\vec{p}}. \tag{4}$$

1. IN CLASS WE DEFINED FOR A FREE REAL SCALAR FIELD,

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^{+}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^{+}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right).$$

a. BY EXPLICIT CALCULATION SHOW THAT $[\phi(\vec{x}), \phi(\vec{y})] = 0$ FOR ANY $\vec{x}$ AND $\vec{y}$.

I'm going to use the rearrangement in Eq 2.27 and 2.28.

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} + a^{+}_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}}$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}} - a^{+}_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}}$$

$$[\phi(\vec{x}), \phi(\vec{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \left(\left[(a_{\vec{p}} + a^{+}_{-\vec{p}})\, e^{i\vec{p}\cdot\vec{x}}, (a_{\vec{q}} + a^{+}_{-\vec{q}})\, e^{i\vec{q}\cdot\vec{y}} \right] \right)$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} e^{i(\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} \Big(\xcancel{[a_{\vec{p}}, a_{\vec{q}}]}^{\,0} + [a^{+}_{\vec{p}}, a_{\vec{q}}]$$

$$+ [a_{\vec{p}}, a^{+}_{-\vec{q}}] + \xcancel{[a^{+}_{-\vec{p}}, a^{+}_{-\vec{q}}]}^{\,0} \Big)$$

✓

Recall $\quad [a_{\vec{p}}, a^{+}_{\vec{q}}] = (2\pi)^3\, \delta^3(\vec{p} - \vec{q})$

$$[\phi(\vec{x}), \phi(\vec{y})] = \int \frac{d^3p\, d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} e^{i(\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} \underbrace{\left(-\delta^3(\vec{q} + \vec{p}) + \delta^3(\vec{p} + \vec{q}) \right)}_{=0}$$

✓

$$\Rightarrow \boxed{[\phi(\vec{x}), \phi(\vec{y})] = 0}$$

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b. BY EXPLICIT CALCULATION SHOW THAT $[\pi(\vec{x}), \pi(\vec{y})] = 0$ FOR ANY $\vec{x}$ AND $\vec{y}$.

$$[\pi(\vec{x}), \pi(\vec{y})] = -\int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{\sqrt{E_{\vec{p}} E_{\vec{q}}}}{2} \left(\left[(a_{\vec{p}} - a^{\dagger}_{-\vec{p}}) e^{i\vec{p}\cdot\vec{x}}, (a_{\vec{q}} - a^{\dagger}_{-\vec{q}}) e^{i\vec{q}\cdot\vec{y}} \right] \right)$$

$$= -\int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{\sqrt{E_{\vec{p}} E_{\vec{q}}}}{2} e^{i(\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} \Big(\cancelto{0}{[a_{\vec{p}}, a_{\vec{q}}]} - [a_{\vec{p}}, a^{\dagger}_{-\vec{q}}]$$

$$- [a^{\dagger}_{-\vec{p}}, a_{\vec{q}}] + \cancelto{0}{[a^{\dagger}_{-\vec{p}}, a^{\dagger}_{-\vec{q}}]} \Big)$$

$$= -\int \frac{d^3p\, d^3q}{(2\pi)^3} \frac{\sqrt{E_{\vec{p}} E_{\vec{q}}}}{2} e^{i(\vec{p}\cdot\vec{x} + \vec{q}\cdot\vec{y})} \underbrace{\left(-\delta^3(\vec{p} + \vec{q}) - (-\delta^3(\vec{p} + \vec{q})) \right)}_{=0}$$

$$\Rightarrow \boxed{[\pi(\vec{x}), \pi(\vec{y})] = 0}$$

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C. THE MOMENTUM OPERATOR $\vec{P}$ IS THE NOETHER CHARGE ASSOCIATED WITH SPATIAL TRANSLATIONS. IN TERMS OF ϕ AND π IT HAS THE FORM

$$\vec{P} = -\int d^3x\, \pi(\vec{x})\, \vec{\nabla}\phi(\vec{x}).$$

USING EQ (1), (2) AND PARITY INVARIANCE, SHOW THAT $\vec{P}$ CAN BE WRITTEN AS

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$$

$$\vec{P} = -\int d^3x\, \pi(\vec{x})\, \vec{\nabla}\phi(\vec{x})$$

$$= -\int d^3x \int \frac{d^3q\, d^3p}{(2\pi)^6}\, \frac{1}{2}\sqrt{\frac{E_{\vec{q}}}{E_{\vec{p}}}}\,(-i)(i\vec{p})\left(a_{\vec{q}}\, e^{i\vec{q}\cdot\vec{x}} - a^\dagger_{\vec{q}}\, e^{-i\vec{q}\cdot\vec{x}}\right) * \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$= -\int d^3x \int \frac{d^3q\, d^3p}{(2\pi)^6}\, \frac{1}{2}\sqrt{\frac{E_{\vec{q}}}{E_{\vec{p}}}}\,\vec{p}\,\Big(a_{\vec{q}} a_{\vec{p}}\, e^{i(\vec{p}+\vec{q})\cdot\vec{x}} - a_{\vec{q}} a^\dagger_{\vec{p}}\, e^{i(\vec{q}-\vec{p})\cdot\vec{x}} - a^\dagger_{\vec{q}} a_{\vec{p}}\, e^{i(\vec{p}-\vec{q})\cdot\vec{x}} + a^\dagger_{\vec{q}} a^\dagger_{\vec{p}}\, e^{i(-\vec{p}-\vec{q})\cdot\vec{x}}\Big)$$

✓

By definition, $\int d^3x\, e^{i(\vec{p}-\vec{q})\cdot\vec{x}} = (2\pi)^3\, \delta(\vec{p}-\vec{q})$.

$$\vec{P} = -\int \frac{d^3q\, d^3p}{(2\pi)^3}\, \frac{1}{2}\sqrt{\frac{E_{\vec{q}}}{E_{\vec{p}}}}\,\vec{p}\,\Big(a_{\vec{q}} a_{\vec{p}}\, \delta^3(\vec{p}+\vec{q}) - a_{\vec{q}} a^\dagger_{\vec{p}}\, \delta^3(\vec{q}-\vec{p}) - a^\dagger_{\vec{q}} a_{\vec{p}}\, \delta^3(\vec{p}-\vec{q}) + a^\dagger_{\vec{q}} a^\dagger_{\vec{p}}\, \delta^3(-\vec{p}-\vec{q})\Big)$$

$$\sqrt{\frac{E_{\vec{p}}}{E_{\vec{p}}}} = \sqrt{\frac{E_{-\vec{p}}}{E_{\vec{p}}}} = 1$$

c. CONTINUED

$$\vec{P} = -\int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \vec{p} \left(a_{-\vec{p}} a_{\vec{p}} - a_{\vec{p}} a^{\dagger}_{\vec{p}} - a^{\dagger}_{\vec{p}} a_{\vec{p}} + a^{\dagger}_{-\vec{p}} a^{\dagger}_{\vec{p}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \vec{p} \left(\underbrace{a_{\vec{p}} a^{\dagger}_{\vec{p}}}_{a^{\dagger}_{\vec{p}} a_{\vec{p}} + [a_{\vec{p}}, a^{\dagger}_{\vec{p}}]} + a^{\dagger}_{\vec{p}} a_{\vec{p}} \right) - \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \vec{p} \left(a_{-\vec{p}} a_{\vec{p}} + a^{\dagger}_{-\vec{p}} a^{\dagger}_{\vec{p}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \vec{p} \left(2 a^{\dagger}_{\vec{p}} a_{\vec{p}} \right) + \cancel{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \vec{p} \underbrace{[a_{\vec{p}}, a^{\dagger}_{\vec{p}}]}_{(2\pi)^3 \delta^3(0)}} \rightarrow 0$$ ✓

$$- \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \vec{p} \left(a_{-\vec{p}} a_{\vec{p}} + a^{\dagger}_{-\vec{p}} a^{\dagger}_{-\vec{p}} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} \right) - \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \vec{p} \left(a_{-\vec{p}} a_{\vec{p}} + a^{\dagger}_{-\vec{p}} a^{\dagger}_{\vec{p}} \right)}_{=0}$$ ✓

Since we're integrating over all space and we have terms with $+\vec{p}$ and $-\vec{p}$, the integration goes to zero and this term disappears.

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} \right)}$$

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PY711 Solutions #2

1. We write

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \{a_{\vec{p}} + a^\dagger_{-\vec{p}}\} e^{i\vec{p}\cdot\vec{x}}$$

$$\pi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \{a_{\vec{p}} - a^\dagger_{-\vec{p}}\} e^{i\vec{p}\cdot\vec{x}}$$

a) The terms $[a_{\vec{p}}, a_{\vec{p}'}]$ and $[a^\dagger_{-\vec{p}}, a^\dagger_{-\vec{p}'}]$ vanish and so

$$[\phi(\vec{x}), \phi(\vec{y})] = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{d^3\vec{p}'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}}$$
$$\times \left\{ [a_{\vec{p}}, a^\dagger_{-\vec{p}'}] + [a^\dagger_{-\vec{p}}, a_{\vec{p}'}] \right\} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{p}'\cdot\vec{y}}$$

Since $[a_{\vec{p}}, a^\dagger_{-\vec{p}'}] = (2\pi)^3 \delta^{(3)}(\vec{p}+\vec{p}') = -[a^\dagger_{-\vec{p}}, a_{\vec{p}'}]$, we find $[\phi(\vec{x}), \phi(\vec{y})] = 0$.

b) In this case we get

$$[\pi(\vec{x}), \pi(\vec{y})] = -\int \frac{d^3\vec{p}}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}}}{2}} \frac{d^3\vec{p}'}{(2\pi)^3} \sqrt{\frac{E_{\vec{p}'}}{2}}$$
$$\times \left\{ -[a_{\vec{p}}, a^\dagger_{-\vec{p}'}] - [a^\dagger_{-\vec{p}}, a_{\vec{p}'}] \right\} e^{i\vec{p}\cdot\vec{x}} e^{i\vec{p}'\cdot\vec{y}}$$

In the same manner we conclude that $[\pi(\vec{x}), \pi(\vec{y})] = 0$.

c) For the momentum operator,

$$\vec{P} = -\int d^3\vec{x}\; \pi(\vec{x})\,\vec{\nabla}\phi(\vec{x})$$

$$= -\int \frac{d^3\vec{p}}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\,\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}\,'}}}\left\{(a_{\vec{p}}-a^\dagger_{-\vec{p}})\,\vec{p}\,'(a_{\vec{p}\,'}+a^\dagger_{-\vec{p}\,'})\right\}\int d^3\vec{x}\; e^{i\vec{p}\cdot\vec{x}}e^{i\vec{p}\,'\cdot\vec{x}}$$

using the fact that $\vec{\nabla}e^{i\vec{p}\,'\cdot\vec{x}} = i\vec{p}\,'e^{i\vec{p}\,'\cdot\vec{x}}$

Since $\int d^3\vec{x}\; e^{i\vec{p}\cdot\vec{x}}e^{i\vec{p}\,'\cdot\vec{x}} \to (2\pi)^3\delta^{(3)}(\vec{p}+\vec{p}\,')$,

$$\vec{P} = \int\frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2}\left\{(a_{\vec{p}}-a^\dagger_{-\vec{p}})\,\vec{p}\,(a_{-\vec{p}}+a^\dagger_{\vec{p}})\right\}$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2}\left\{\vec{p}\,(a_{\vec{p}}a_{-\vec{p}} - a^\dagger_{-\vec{p}}a_{-\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}} - a^\dagger_{-\vec{p}}a^\dagger_{\vec{p}})\right\}$$

By parity symmetry $(\vec{p}\to-\vec{p})$

$$\vec{P} = \int\frac{d^3\vec{p}}{(2\pi)^3}\,\frac{1}{2}\,\vec{p}\left\{a^\dagger_{\vec{p}}a_{\vec{p}} + a_{\vec{p}}a^\dagger_{\vec{p}}\right\}$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\,\vec{p}\left\{a^\dagger_{\vec{p}}a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}},a^\dagger_{\vec{p}}]\right\}$$

integrates to zero due to parity $(\vec{p}\to-\vec{p})$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3}\,\vec{p}\,a^\dagger_{\vec{p}}a_{\vec{p}} \quad \text{as desired}$$