8.09 & 8.309 Classical Mechanics III, Fall 2021
FINAL
Friday December 17, 9:00am-12:00pm
You have 180 minutes.

Answer all problems in the white books provided. Write YOUR NAME on EACH book you use.

There are six problems, totaling 150 points. You should do all six. The problems are worth 13, 16, 30, 26, 35 and 30 points. You may do the problems in any order.

None of the problems requires extensive algebra. If you find yourself lost in a calculational thicket, stop and think.

No books, notes, or calculators allowed.

Some potentially useful information

- Euler-Lagrange equations for generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j} \;, \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot{q}_j}$$

constraints: holonomic $f_\alpha(q,t) = 0$ or semiholonomic $g_\beta = \sum_j a_{\beta j}(q,t)\dot{q}_j + a_{\beta t}(q,t) = 0$

- Generalized forces: $d/dt(\partial L/\partial \dot{q}_j) - \partial L/\partial q_j = R_j$

Friction forces: $\vec{f}_i = -h(v_i)\vec{v}_i/v_i$, $\vec{v}_i = \dot{\vec{r}}_i$ gives $R_j = -\partial \mathcal{F}/\partial \dot{q}_j$, $\mathcal{F} = \sum_i \int_0^{v_i} dv_i' \, h(v_i')$

- Hamilton's equations for canonical variables (q_j, p_j): $\quad \dot{q}_j = \dfrac{\partial H}{\partial p_j}$, $\quad \dot{p}_j = -\dfrac{\partial H}{\partial q_j}$

- Hamiltonian for a Lagrangian quadratic in velocities

$$L = L_0(q,t) + \dot{\vec{q}}^T \cdot \vec{a} + \frac{1}{2}\dot{\vec{q}}^T \cdot \hat{T} \cdot \dot{\vec{q}} \quad \Rightarrow \quad H = \frac{1}{2}(\vec{p} - \vec{a})^T \cdot \hat{T}^{-1} \cdot (\vec{p} - \vec{a}) - L_0(q,t)$$

- The Moment of Inertia Tensor and its relations:

$$I_{ab} = \int dV \, \rho(\vec{r})[\vec{r}^{\,2}\delta_{ab} - r_a r_b] \qquad \text{or} \qquad I^{ab} = \sum_i m_i[\delta^{ab}\vec{r}_i^{\,2} - r_i^a r_i^b]$$

$$I_{ab}^{(Q)} = M(\delta_{ab}\,\vec{R}^2 - R_a R_b) + I_{ab}^{(\mathrm{CM})} \;, \qquad \hat{I}' = \hat{U}\,\hat{I}\,\hat{U}^T$$

- Euler's Equations:

$$I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = \tau_1$$
$$I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 = \tau_2$$
$$I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 = \tau_3$$

- Vibrations: $L = \frac{1}{2}\dot{\vec{\eta}}^T \cdot \hat{T} \cdot \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T \cdot \hat{V} \cdot \vec{\eta}$ has Normal modes $\vec{\eta}^{(k)} = \vec{a}^{(k)} \exp(-i\omega^{(k)} t)$

$$\det(\hat{V} - \omega^2 \hat{T}) = 0 \;, \qquad \big(\hat{V} - [\omega^{(k)}]^2\,\hat{T}\big) \cdot \vec{a}^{(k)} = 0 \;, \qquad \vec{\eta} = \mathrm{Re}\sum_k C_k\,\vec{\eta}^{(k)}$$

- Generating functions for Canonical Transformations: $K = H + \partial F_i/\partial t$ and

$$F_1(q,Q,t): \quad p_i = \frac{\partial F_1}{\partial q_i} \;, \; P_i = -\frac{\partial F_1}{\partial Q_i} \;, \qquad F_2(q,P,t): \quad p_i = \frac{\partial F_2}{\partial q_i} \;, \; Q_i = \frac{\partial F_2}{\partial P_i}$$

- Poisson Brackets: $\quad [u,v]_{q,p} = \displaystyle\sum_j \left[\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j}\right]$, $\quad \dfrac{du}{dt} = [u,H] + \dfrac{\partial u}{\partial t}$

- Relations for Hamilton's Principle function, $S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n, t)$

$$K = 0 \;, \qquad P_i = \alpha_i \;, \qquad Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i} \;, \qquad p_i = \frac{\partial S}{\partial q_i}$$

- Relations for Hamilton's Characteristic function, $W = W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n)$

$$K = H = \alpha_1 \;, \qquad P_i = \alpha_i \;, \qquad \beta_1 + t = \frac{\partial W}{\partial \alpha_1} \;, \qquad \beta_{i>1} = \frac{\partial W}{\partial \alpha_i} \;, \qquad p_i = \frac{\partial W}{\partial q_i}$$

- Action Angle Variables: $\quad J = \displaystyle\oint p\,dq$, $\quad w = \dfrac{\partial W(q,J)}{\partial J}$, $\quad \dot{w} = \dfrac{\partial H(J)}{\partial J} = \nu(J)$

- Time Dependent Perturbation Theory for $H_0 + \Delta H$. Solve $H_0(p,q)$ with the Hamilton-Jacobi method to obtain constant canonical variables (β, α) where $[\beta, \alpha] = 1$. Then

$$\dot{\alpha}^{(n)} = -\frac{\partial \Delta H}{\partial \beta}\bigg|_{n-1}, \qquad \qquad \dot{\beta}^{(n)} = \frac{\partial \Delta H}{\partial \alpha}\bigg|_{n-1}$$

- Fluid volume and continuity equations $\quad \dfrac{dV}{dt} = \displaystyle\int dV\, \vec{\nabla} \cdot \vec{v}\,, \qquad \dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}\,) = 0$

- Euler equation ($\nu = 0$) or Navier-Stokes equation ($\nu = \eta/\rho \neq 0$), with gravity:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\, \vec{v} + \frac{1}{\rho} \vec{\nabla} p - \nu \nabla^2 \vec{v} = \frac{\vec{f}}{\rho} = \vec{g}$$

- For direction i the force/unit area on a surface $= -\hat{n}_i\, p + \hat{n}_k \sigma'_{ki}$

- Ideal fluid has $ds/dt = 0$ so $p = p(\rho, s)$. Viscous fluid has $ds/dt \propto \sigma'_{ik}\, \partial v_i / \partial x_k$.

- Bernoulli's equation for a steady incompressible ideal fluid in gravity $\vec{g} = -g\hat{z}$:

$$\frac{\vec{v}^{\,2}}{2} + gz + \frac{p}{\rho} = \text{constant}$$

- Irrotational incompressible ideal fluid flow (potential flow): $\vec{v} = \nabla \phi$, $\nabla^2 \phi = 0$

- Sound waves: $\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right)\{p', \rho', \vec{v}\,\} = 0$. Mach number $M = v_0/c_s$.

- Momentum conservation: $\frac{\partial}{\partial t}(\rho \vec{v}) + \vec{\nabla} \cdot \hat{T} = \vec{f}$ where the energy momentum tensor is $T_{ki} = v_k v_i\, \rho + \delta_{ki}\, p - \sigma'_{ki}$. For $\vec{\nabla} \cdot \vec{v} = 0$ the viscous stress tensor $\sigma'_{ki} = \eta\big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\big)$.

- Reynolds Number: $R = uL/\nu$

- Bifurcations at $\mu = 0$. In 1-dim: "saddle-node" $\dot{x} = \mu + x^2$, "transcritical" $\dot{x} = x(\mu - x)$, "supercritical pitchfork" $\dot{x} = \mu x - x^3$, "subcritical pitchfork" $\dot{x} = \mu x + x^3$. In 2-dim: "supercritical Hopf" $\dot{r} = r(\mu - r^2)$, "subcritical Hopf" $\dot{r} = r(\mu + r^2)$.

- Linearization for 2-dim fixed points: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = \vec{a}\, e^{\lambda t}, \quad M\vec{a} = \lambda\, \vec{a}$

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}\,, \qquad \tau = \mathrm{tr}\, M\,, \qquad \Delta = \det M$$

- 2-dim conserved system $\dot{x} = f_x(x,y)$, $\dot{y} = f_y(x,y)$ with $\vec{\nabla} \cdot \vec{f} = 0$, has conserved $H(x,y) = \int^y dy' f_x(x,y') - \int^x dx' f_y(x',y)$.

- 1-dim map $x_{n+1} = f(x_n)$. Its fixed points satisfy $x^* = f(x^*)$. Here x^* is stable for $|f'(x^*)| < 1$ and unstable for $|f'(x^*)| > 1$.

- Fractal dimension: $\qquad d_F = \lim\limits_{a \to 0} \dfrac{\ln N(a)}{\ln(a_0/a)}$

1. **Short answer problems** [13 points]

   These problems require no algebra or a little algebra. Your answers should be short.

   (a) [2 points] For a transcritical bifurcation in one dimension, draw a bifurcation diagram.

   (b) [4 points] What performance advantage is obtained by having dimples on a golf ball? What do they do that causes this?

   (c) [2 points] Give an example of an equation that exhibits hysteresis when a parameter in your equation is changed.

   (d) [5 points] When considering perturbations, explain the meaning of a "periodic change" and a "secular change". You may either explain this in words or give an example.

2. **Two Unrelated Problems** [16 points]

   (a) [8 points] **Dimensional Analysis**

   A river flows at velocity v past a circular pylon of radius r, and sheds vortices at a frequency f. Here f is also a function of the water density ρ, shear viscosity η, and the acceleration due to gravity g. Use dimensional analysis to write f in terms of a dimensionless function F, with as many dimensionless arguments as you can construct. What is the Reynolds number R for this system? We build a model in the lab to test this system, where the pylon is $\frac{1}{10}$th the size, and we adjust the velocity and density of the fluid to keep the arguments of F fixed. What frequency of vortex shedding do we expect for the model system relative to the frequency of vortex shedding in the river?

   (b) [8 points] **Period of the $|x|$ Potential**

   A particle of mass m exhibits periodic motion in one dimension under the influence of a potential $V(x) = k|x|$ where $k > 0$ is a constant. Using action-angle variables, find the period of the motion as a function of the particle's energy.

(continue)

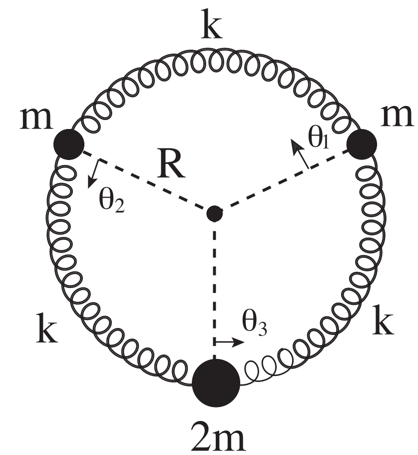
3. **Vibrating Masses on a Circular Spring** [30 points]

Consider three point masses that are connected by springs, all of which are constrained to lie on a circle of radius R. Two of the point masses have mass m, and the third has mass $2m$, while all three springs have spring constant k. In the equilibrium positions shown the springs all have equal length when the system is at rest. For all problems below, assume that motion of the masses away from these points stretch the springs by only a small amount.

(a) [8 points] What is the Lagrangian of this system using the angular variables θ_1, θ_2, and θ_3?

(b) [8 points] Determine a matrix eigenvalue equation that could be used to find the normal mode frequencies and eigenvectors for the motion.

(c) [14 points] Either solve the equation from (c) or apply physical intuition to find the three eigenvectors and eigenvalues. Make a sketch of the normal mode motion for each case.



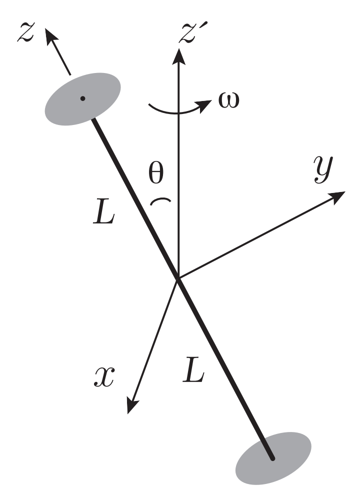
4. **Rotating Dumbbell** [26 points]

Consider two thin uniform disks, each with radius R and mass M, which are attached to a massless rod of length $2L$ to form a dumbbell. The dumbbell is fixed at the origin at its center-of-mass, and the angle θ shown in the figure is fixed. The dumbbell rotates with angular velocity ω about the z' axis as shown in the figure.

(a) [12 points] What is the moment of inertia tensor for one disk about its center of mass with principle axes? What is the moment of inertia tensor for the two disks on the dumbbell with respect to the (xyz) body axes?

(b) [7 points] Take the z' axis to be in the y-z plane. What is the angular velocity $\vec{\omega}$ in the body system (xyz)? Find the angular momentum $\vec{L}$ in the body system.

(c) [7 points] What instantaneous torque $\vec{\tau}$ in the body frame (xyz) is required to obtain a constant angular acceleration α for rotation about the z' axis, so that $\omega = \alpha t$?

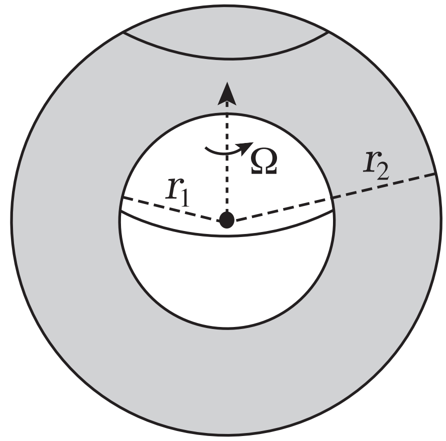


Note: $\int_0^{2\pi} d\theta \cos^2(\theta) = \pi$.

(continue)

5. **Viscous Fluid between Two Spheres** [35 points]

Consider two co-centric spheres with radii r_1 and r_2 where $r_2 > r_1$. The outer sphere is fixed, while the inner sphere rotates with a constant angular velocity $\vec{\Omega} = \Omega\,\hat{z}$. An incompressible viscous fluid with constant shear viscosity η and density ρ is placed between the two spheres. Assume that the flow is steady and non-turbulent, and do not include gravity in this problem.
[Note that if you get stuck somewhere you can still complete later parts.]



(a) [4 points] Using $\vec{\Omega}$, what is the velocity for a general point $\vec{r}_1$ on the surface of the inner sphere? Write your answer using spherical coordinates (r, θ, ϕ).

(b) [4 points] Based on symmetry and continuity, what variables from the spherical coordinates should the velocity $\vec{v}$ and pressure p of the fluid depend on? You should simply state the result, a proof is not required. [Hint: the velocity $\vec{v}$ should be solely in the direction identified in part (a).]

(c) [2 points] Provide an estimate for the Reynold's number R for this fluid.

For $R \ll 1$ the Navier-Stokes equation becomes $\vec{\nabla} p = \eta \nabla^2 \vec{v}$. Use this result below.

(d) [8 points] Prove that the pressure must be constant.

(e) [10 points] For the velocity consider the ansatz $\vec{v} = \vec{\Omega} \times [\vec{\nabla} f(r)]$ and show it points in the desired direction. Then use the Navier-Stokes equation to find a differential equation for the function $f(r)$. Finally, verify that

$$f(r) = Ar^2 + B/r$$

is a solution of your differential equation with constants A and B.
Hint: Recall that $\vec{\nabla}$ and ∇^2 commute.

(f) [7 points] Using the boundary conditions for $\vec{v}$ on the surface of the two spheres, solve for A and B. To check your result, consider $\vec{v}$ in the limit $r_2 \to \infty$ where the outer sphere is removed. Does your solution in this limit behave as expected?

The following spherical coordinate identities might be useful:

$$\vec{\nabla} = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\,, \qquad \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\,,$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\,, \quad \text{and non-zero derivatives:}$$

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}\,, \quad \frac{\partial \hat{r}}{\partial \phi} = \sin\theta\,\hat{\phi}\,, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}\,, \quad \frac{\partial \hat{\theta}}{\partial \phi} = \cos\theta\,\hat{\phi}\,, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\cos\theta\,\hat{\theta} - \sin\theta\,\hat{r}\,.$$

(continue)

6. **Nonlinear Attractions** [30 points]

In this problem you will find fixed points and/or limit cycles for three examples.

(a) [16 points] Consider the system of equations

$$\dot{x} = x\,y - 1$$
$$\dot{y} = y(1 - y^2)$$

Find the two fixed points. Using a linear analysis classify each fixed point (as a center, saddle node, unstable node, stable spiral, unstable spiral, etc). What are the corresponding eigenvalues and eigenvectors? Plot them in the x-y plane and draw trajectories near these two fixed points.

(b) [6 points] In polar coordinates r and θ consider the equations

$$\dot{r} = \mu\,r - r^3$$
$$\dot{\theta} = w + b\,r^2$$

for constant parameters μ, $w > 0$, and $b > 0$. Consider both the cases $\mu < 0$ and $\mu > 0$ and identify all fixed points and limit cycles, as well as stating their stability. For each case draw a picture in the x-y plane for these attractors, and draw nearby trajectories. If there is a bifurcation in μ state what type it is.

(c) [8 points] Consider the Tent Map with a parameter r and values $0 \leq x_n \leq 1$ defined by

$$x_{n+1} = \begin{cases} r x_n & 0 \leq x_n \leq 1/2 \\ r(1 - x_n) & 1/2 \leq x_n \leq 1 \end{cases}$$

Find all fixed points and determine their stability for the case $0 < r < 1$. Then repeat for the case $1 < r < 2$.

(the end)