Due: **Monday April 18 by 5pm**
Reading: Linear response theory is covered well in various textbooks and online course notes. For instance, see David Tong's notes and 8.512 class notes (active links)

Linear response theory: Kubo theorem, susceptibility, collective modes

1. [**Polarizability of a harmonic oscillator from Kubo formula**]
Consider charge on a spring, described by the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 - exE(t).$$

(a) [10 pts] Treating charge motion as classical, show that a time-dependent field oscillating at a frequency ω, $E(t) \sim \mathrm{Re}\, E_0 e^{-i\omega t}$, induces an oscillating electric polarization of amplitude $d_\omega = ex_\omega = \frac{e^2}{m(\omega_0^2 - (\omega + i\eta)^2)} E_0$, where η is a damping parameter.
(b) [10 pts] Now consider a quantum oscillator. Derive the susceptibility from Kubo formula

$$\chi_{\mathrm{Kubo}}(t - t') = -\frac{i}{\hbar} \langle G | [\widehat{d}(t), \widehat{d}(t')] | G \rangle$$

where $\widehat{d}(t) = ex(t)$ is the operator of electric dipole and $|G\rangle$ is the oscillator ground state.
Hint: Use the Heisenberg equations of motion for position and momentum $\widehat{x}(t) = \widehat{x}(t') \cos\omega_0(t - t') + \frac{\widehat{p}(t')}{m\omega_0} \sin\omega_0(t - t')$, $\widehat{p}(t) = p(t') \cos\omega_0(t - t') - m\omega_0 \widehat{x}(t') \sin\omega_0(t - t')$.

2. [**Static polarization function of a one-dimensional free-electron gas**]
Consider a degenerate Fermi gas in $D = 1$ in the presence of a weak scalar potential coupled to fermion density through

$$H' = \int dx' \rho(x') \phi(x')$$

As discussed in class, the polarization function describing the linear response $\langle \delta\rho(q) \rangle = \Pi(q)\phi(q)$ is given by the expression

$$\Pi(q) = 2 \sum_k \frac{f_k - f_{k+q}}{\epsilon_k - \epsilon_{k+q}}$$

where $f_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$ is the Fermi function, $\sum_k ... = \int_{-\infty}^{\infty} ... \frac{dk}{2\pi}$ and $\epsilon_k = \frac{\hbar^2 k^2}{2m}$.
(a) [10 pts] Find $\Pi(q)$ at zero temperature. To facilitate integration, it is convenient to shift the integration variable k by $q/2$ and rewrite the expression for $\Pi(q)$ as

$$\Pi(q) = \int_{-\infty}^{\infty} \frac{dk}{\pi} \frac{f_{k-q/2} - f_{k+q/2}}{\epsilon_{k-q/2} - \epsilon_{k+q/2}}, \quad f_k = \begin{cases} 1, & |k| < k_F \\ 0, & |k| > k_F \end{cases}. \tag{1}$$

Evaluate the integral over k taking into account that $\epsilon_{k-q/2} - \epsilon_{k+q/2} = -(\hbar^2 q/m)k$.
(b) [10 pts] Plot the result for $\Pi(q)$ vs. q. Comment on the behavior at small q, large q and $q \sim 2k_F$.

3. [**Friedel oscillations in one dimension**]
As we saw in class, the response of a Fermi system to an external periodic potential $U(\mathbf{x}) \sim \cos \mathbf{kx}$ changes abruptly when the wavenumber k exceeds $2k_F$, where k_F is the Fermi momentum. The system is relatively well compressible at k smaller than $2k_F$ and is poorly compressible at k greater than $2k_F$. Furthermore, the polarization function of a Fermi gas, $\Pi(\mathbf{k})$, has a nonanalytic behavior at $|\mathbf{k}| = 2k_F$. The rigidity of a Fermi system manifests itself most clearly in spatial Friedel oscillations arising in the presence of a

localized perturbation such as a potential of a defect or a boundary. Friedel oscillations feature spatial periodicity of $\lambda_F/2$ and an amplitude decaying with distance as a power law.

Here we investigate Friedel oscillations of a one-dimensional Fermi gas near a wall. Consider electrons moving freely in 1D in the region $x > 0$. Electronic states can be written as superpositions of an incident and reflected plane wave, $\phi_k(x) = e^{ikx} - e^{-ikx}$, $k > 0$, with a minus sign mandated by the hard-wall boundary condition at $x = 0$.

(a) [10 pts] Consider the particle density operator $n(x) = \psi^\dagger(x)\psi(x)$ where

$$\psi(x) = \sum_{k>0} \phi_k(x) c_k, \quad \psi^\dagger(x) = \sum_{k>0} \phi_k^*(x) c_k^\dagger,$$

where c_k and $c_k^\dagger$ are Fermi operators, $[c_k, c_q^\dagger]_+ = \delta_{kq}$, $[c_k, c_q]_+ = 0$, and $\sum_{k>0} \ldots = \int_0^\infty \ldots \frac{dk}{2\pi}$ [we suppressed electron spin for conciseness].

- Show that the expectation value of $n(x)$ in the ground state is given by

$$\langle n(x) \rangle = \sum_{k>0} |\phi_k(x)|^2 f_k, \quad f_k = \frac{1}{e^{\beta(\epsilon_k - \mu)} + 1}$$

  where the Fermi function f_k describes occupancy of different states.
- For zero temperature, taking $f_k = \theta(k_F - k)$, find the density $\langle n(x) \rangle$ and show that it exhibits Friedel oscillations vs. distance from the wall.
- Relate the period of these oscillations to k_F. Sketch $\langle n(x) \rangle$ vs. x and comment on the behavior at $x \ll k_F^{-1}$ and $x \gg k_F^{-1}$.

(b) [10 pts] Consider Friedel oscillations at a finite temperature $T > 0$ which is small compared to the Fermi energy, $T \ll \mu$. Argue that the oscillations in $\langle n(x) \rangle$ decay as a power law at short distances and exponentially at large distances. Estimate by order of magnitude the crossover lengthscale—known as "thermal length"—as a function of T.

4. [**Plasma oscillations in a two-dimensional electron gas**]

(a) [10 pts] Consider a gas of electrons confined to a plane (say, $z = 0$). The electrons are freely moving in the plane and interact through the Coulomb potential $V(\mathbf{x} - \mathbf{x}') = \frac{e^2}{|\mathbf{x} - \mathbf{x}'|\kappa}$ (here κ is the dielectric constant). As discussed in class, plasmon oscillations in this case can be described using the polarization function $\Pi(q, \omega)$ evaluated for high frequencies $\omega \gg v_F q$ and the dielectric function $\epsilon(q, \omega) = 1 - \frac{2\pi e^2}{q\kappa}\Pi(q, \omega)$, where $\frac{2\pi e^2}{q\kappa}$ is derived as a 2D Fourier transform of the 3D Coulomb potential: $\frac{2\pi e^2}{q} = \int d^2 r e^{-i\mathbf{qr}} \frac{e^2}{r}$.

Here we develop a different approach (hydrodynamic, or mean-field) in which the charge fluid movement is described by a local 'center of mass' velocity field $\mathbf{v}(\mathbf{r}, t)$ arising in response to the electric field produced by distant charges, represented as an oscillating charge density $\delta\rho(\mathbf{r}, t)$. The dynamics can be obtained from the action $S = \int dt L = \int dt\, [E_{\rm kin} - E_{\rm pot}]$,

$$L = E_{\rm kin} - E_{\rm pot} = \int d^2 r \frac{m\mathbf{v}^2}{2} n - \frac{1}{2} \int\int d^2 r d^2 r' \frac{\delta\rho(\mathbf{r})\delta\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|\kappa}$$

where m is the free-particle band mass (which in 8.511 we called the effective mass), n is the electron number density. The flow velocity and the charge density $\delta\rho$ are not independent variables since they are constrained by the continuity equation $\partial_t \rho + \nabla \mathbf{j} = 0$ with $\mathbf{j} = en\mathbf{v}$ the electric current density.

- Consider the Fourier representation $\mathbf{v}(\mathbf{r},t)=\sum_q e^{i\mathbf{qr}}\mathbf{v}_q(t)$, $\delta\rho(\mathbf{r},t)=\sum_q e^{i\mathbf{qr}}\delta\rho_q(t)$ and, substituting it in the Lagrangian, derive an expression

$$S=\int dtL=\int dt\sum_q\left(\frac{mn}{2}\mathbf{v}_{-q}(t)\mathbf{v}_q(t)-\frac{\pi}{q\kappa}\delta\rho_{-q}(t)\delta\rho_q(t)\right)$$

where the harmonics with different $\mathbf{q}$ are treated as independent quantities.
- Apply the variational principle $\frac{\delta S}{\delta(\delta\rho)}=0$ taking into account the relation between the density and velocity harmonics $\partial_t\delta\rho_q+en(i\mathbf{q}\cdot\mathbf{v}_q)=0$ that reflects the continuity equation constraint.
- Derive the dispersion relation for charge oscillations $\omega^2(\mathbf{q})=B|\mathbf{q}|$ and find the coefficient B.

(b) [10 pts] Here we analyze the peculiar $q^{1/2}$ dispersion of the modes found in part a).

- Show that strong $q^{1/2}$ dispersion translates into propagation velocity that can be much greater than the Fermi velocity of the 2D electrons. Find the group velocity $u_g=\partial\omega/\partial q$ in units of v_F and plot it as a function of the mode frequency. For simplicity, assume the dielectric constant value $\kappa=1$.
- Consider the ratio $E_{\text{kin}}/E_{\text{pot}}$. Since both E_{kin} and E_{pot} are quadratic in the osillation amplitude, the ratio $E_{\text{kin}}/E_{\text{pot}}$ takes an amplitude-independent universal value. Find the value $E_{\text{kin}}/E_{\text{pot}}$ and interpret your result.

5. The Anderson-Higgs Mechanism [20 pts]

If the vacuum state of the system is non-invariant under a continuous symmetry of system Hamiltonian, we normally expect to find massless particles (gapless modes) generated by the Nambu-Goldstone mechanism. However, as people realized in the early 60's, the situation is quite different when you have both gauge symmetry and spontaneous symmetry breaking. In this case, the Goldstone massless mode can combine with the massless gauge field modes to produce a physical massive field: Massless photon eats the Goldstone boson and becomes massive.

This important idea was first developed by Anderson in the context of superconductivity (worked out in the summer of 1962, paper submitted to Physical Review that November, and appeared in the April 1963 issue of the journal, in the particle physics section). All that was missing was an explicit relativistic example to supplement the non-relativistic superconductivity one. This was provided by several authors in 1964, with Higgs giving the first explicit relativistic model. Higgs was also the first to explicitly discuss the existence in models like his of a massive mode, of the sort that we now call a "Higgs particle." In essence, Higgs constructed the first relativistic model that provided a realization of the Anderson mechanism.

Here we discuss a simple example that illustrates the Anderson-Higgs phenomenon, using as a model the hydrodynamic action for the phase mode of a superfluid condensate coupled to electromagnetic fields by a minimal coupling mandated by the gauge symmetry. As discussed in class (see class notes), the action is the difference of the kinetic and potential energy terms, $S=\int dtL(\theta,\mathbf{A},\phi)=\int dt\,[E_{\text{kin}}-E_{\text{pot}}]$

$$L(\theta,A,\phi)=\int d^3x\left[\frac{\hbar^2 n_s}{2ms^2}\left(\partial_t\theta+\frac{2e}{\hbar}\phi\right)^2-\frac{\hbar^2 n_s}{2m}\left(\nabla\theta-\frac{2e}{\hbar c}\mathbf{A}\right)^2+\frac{\mathbf{E}^2}{8\pi}-\frac{\mathbf{H}^2}{8\pi}\right]$$

with the phase mode velocity s being a function of microscopic interactions. The EM fields are the usual functions of the EM potentials $\mathbf{E}=-\frac{1}{c}\partial_t\mathbf{A}-\nabla\phi$, $\mathbf{H}=\nabla\times\mathbf{A}$.

The dynamics governed by this action can be analyzed by using the gauge symmetry to absorb θ into $\mathbf{A}$ and ϕ. After fixing the gauge by a gauge transformation

$$\mathbf{A} \to \mathbf{A} + c\nabla\chi, \quad \phi \to \phi - \partial_t\chi$$

with $\chi = \frac{\hbar}{2e}\theta$, the Lagrangian depends only on $\mathbf{A}$ and ϕ:

$$L = \int d^3x \left[\frac{4e^2 n_s}{2m} \left(\frac{1}{s^2}\phi^2 - \frac{1}{c^2}\mathbf{A}^2 \right) + \frac{\mathbf{E}^2}{8\pi} - \frac{\mathbf{H}^2}{8\pi} \right].$$

Our goal will be to show that all modes described by this Lagrangian, three in total, are gapped.

- As a first step, use the Fourier representation

$$\mathbf{A}(\mathbf{r}, t) = \sum_q e^{i\mathbf{qr}} \mathbf{A}_q(t), \quad \phi(\mathbf{r}, t) = \sum_q e^{i\mathbf{qr}} \phi_q(t)$$

  to write the action as a sum of the contributions due to different harmonics $\mathbf{A}_q$, ϕ_q.
- Next, decompose $\mathbf{A}_q$ into a sum of the longitudinal and transverse parts, $\mathbf{A}'_q \parallel \mathbf{q}$, $\mathbf{A}''_q \perp \mathbf{q}$. Show that the longitudinal contribution, which includes $\mathbf{A}'_q$ and ϕ_q, decouples from the contribution of the transverse part $\mathbf{A}''_q$.
- Lastly, derive mode dispersion for each part and show that the dispersion is gapped,

$$\omega(\mathbf{q} \to 0) = \omega_0 > 0.$$

  Compare the gap values ω_0 to the plasma frequency $\omega_p = \left(\frac{4\pi n e^2}{m}\right)^{1/2}$.