

Matrices in Quantum Computing

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Matrix Analysis

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CLAS, May 2, 2019

Presentation layout

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Qubits & Quantum Gates

Qubit: A quantum system with measurable eigenstates $|0\rangle$ and $|1\rangle$,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow \text{like a Classical Bit.}$$

But before measurement,

$$\text{Wavefunction : } |\psi\rangle = a|0\rangle + b|1\rangle \in \mathbb{C}^2, \quad |a|^2 + |b|^2 = 1.$$

Probabilistic:

$$P(|\psi\rangle \rightarrow |0\rangle) = |a|^2 \quad P(|\psi\rangle \rightarrow |1\rangle) = |b|^2.$$

Quantum gate: unitary transformation on $|\psi\rangle$ of one or many qubits.

Multiple Qubits

How to express two qubits, $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$ as one *composite* state?

$$|\psi_1\psi_2\rangle \stackrel{?}{\sim} |\psi_1\rangle, |\psi_2\rangle$$

What if there are more than two $|\psi_i\rangle$'s $\in \mathbf{V}_i$'s

$$|\psi_1\psi_2 \dots \psi_n\rangle \stackrel{?}{\sim} |\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_n\rangle?$$

Questions:

- Is there a vector space that contains $|\psi_1\psi_2 \dots \psi_n\rangle$?
- What is the vector space containing $|\psi_1\psi_2 \dots \psi_n\rangle$?
- How does $|\psi_1\psi_2 \dots \psi_n\rangle$ change w.r.t $\mathcal{A}_1 |\psi_1\rangle$ where $\mathcal{A}_1 \in \mathfrak{L}(\mathbf{V})$?
- What about for $\mathcal{A}_1 |\psi_1\rangle, \dots, \mathcal{A}_n |\psi_n\rangle$, where $\mathcal{A}_i \in \mathfrak{L}(\mathbf{V})$?

Tensor Product

Postulate (QM): [NC02]

The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

For $|\psi_1\rangle \in \mathbf{V}_1, \dots, |\psi_n\rangle \in \mathbf{V}_n$,

$$|\psi_1 \dots \psi_n\rangle \in \mathbf{V}_1 \otimes \dots \otimes \mathbf{V}_n,$$

where the joint state $|\psi_1 \dots \psi_n\rangle$ is given by

$$|\psi_1 \dots \psi_n\rangle = |\psi_1\rangle |\psi_2\rangle \dots |\psi_n\rangle = |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle.$$

$|\psi_1 \dots \psi_n\rangle$ is an *elementary tensor* in $\mathbf{V}_1 \otimes \dots \otimes \mathbf{V}_n$.

Not all $|\phi\rangle \in \mathbf{V}_1 \otimes \dots \otimes \mathbf{V}_n$ are elementary.

Tensor Product: Definition

What is this “ \otimes ” object?

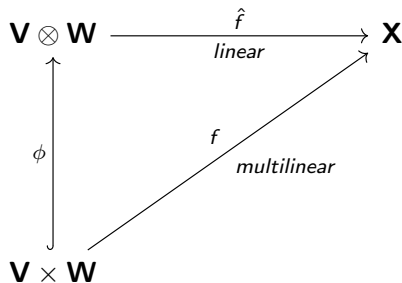
Definition [Kam]

The *tensor product* of \mathbf{V} and \mathbf{W} is a vector space $\mathbf{V} \otimes \mathbf{W}$ with the *bilinear map* $\phi : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{V} \otimes \mathbf{W}$, such that for every vector space \mathbf{X} and every bilinear map $f : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{X}$, there exists a *unique linear map* $\hat{f} : \mathbf{V} \otimes \mathbf{W} \longrightarrow \mathbf{X}$ such that $f = \hat{f} \circ \phi$.

In other words...

Giving the $\hat{f} : \mathbf{V} \otimes \mathbf{W} \xrightarrow{\text{linear}} \mathbf{X}$ is the same as giving $f : \mathbf{V} \times \mathbf{W} \xrightarrow{\text{bilinear}} \mathbf{X}$.

Tensor Product: Construction



Let v_1, \dots, v_n be a basis for \mathbf{V} and w_1, \dots, w_m be a basis for \mathbf{W} ,

- For $i \in [1, n], j \in [1, m]$, $\{v_i \otimes w_j\}$ is a basis of $\mathbf{V} \otimes \mathbf{W}$:

$$v \otimes w = \sum_i^n \alpha_i v_i \otimes \sum_j^m \beta_j w_j = \sum_{i,j}^{n,m} \alpha_i \beta_j (v_i \otimes w_j)$$

- $\dim(\mathbf{V} \otimes \mathbf{W}) = \dim(\mathbf{V}) \dim(\mathbf{W}) = nm$.

Tensor Product

Let $\mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$, where $\mathcal{A} \in \mathcal{L}(\mathbf{V})$, and $\mathcal{B} \in \mathcal{L}(\mathbf{W})$.

$$(\mathcal{A} \otimes \mathcal{B})(v \otimes w) \stackrel{?}{\sim} \mathcal{F}(v, w) \stackrel{\Delta}{=} \mathcal{A}(v) \otimes \mathcal{B}(w).$$

One way to see this...

A commutative diagram illustrating the relationship between the tensor product and the bilinear map \mathcal{F} . The diagram consists of two nodes: $\mathbf{V} \otimes \mathbf{W}$ at the top and $\mathbf{V} \times \mathbf{W}$ at the bottom. A vertical arrow labeled ϕ points from $\mathbf{V} \times \mathbf{W}$ to $\mathbf{V} \otimes \mathbf{W}$. A diagonal arrow labeled \mathcal{F} points from $\mathbf{V} \times \mathbf{W}$ to $\mathbf{V} \otimes \mathbf{W}$. A horizontal arrow labeled $\mathcal{A} \otimes \mathcal{B}$ points from $\mathbf{V} \otimes \mathbf{W}$ to $\mathbf{V} \otimes \mathbf{W}$.

By uniqueness,

$$(\mathcal{A} \otimes \mathcal{B}) \circ \phi = \mathcal{F} \iff \boxed{(\mathcal{A} \otimes \mathcal{B})(v \otimes w) = \mathcal{A}(v) \otimes \mathcal{B}(w)}$$

Tensor Product to Kronecker Product

Let Γ be a basis for $\mathbf{V} \otimes \mathbf{W}$, and $\{\cdot\}_\Gamma = \mathcal{A}_\Gamma^{-1}$ is the coordinatization from $\mathbf{V} \otimes \mathbf{W}$ to \mathbb{C}^{nm} , where $n = \dim(\mathbf{V})$, $m = \dim(\mathbf{W})$.

$$\begin{array}{ccc}
 \mathbf{V} \otimes \mathbf{W} & \xrightarrow[\text{linear}]{\mathcal{L} \otimes \mathcal{M}} & \mathbf{V} \otimes \mathbf{W} \\
 \downarrow \{\cdot\}_\Gamma & & \uparrow \mathcal{A}_\Gamma \\
 \mathbb{C}^{nm} & \xrightarrow[\text{linear}]{\{\mathcal{L} \otimes \mathcal{M}\}_\Gamma \leftarrow \Gamma} & \mathbb{C}^{nm}
 \end{array}$$

Kronecker Product

$$[\mathcal{L} \otimes \mathcal{M}]_{\Gamma \leftarrow \Gamma} = [\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma}.$$

If

$$[\mathcal{L}]_{\Gamma \leftarrow \Gamma} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \quad \text{and} \quad [\mathcal{M}]_{\Gamma \leftarrow \Gamma} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

then the *Kronecker product* $[\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma}$ is defined as

$$\begin{aligned} [\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma} &= \begin{bmatrix} l_{11}\mathcal{M} & l_{12}\mathcal{M} \\ l_{21}\mathcal{M} & l_{22}\mathcal{M} \end{bmatrix} \\ &= \begin{bmatrix} l_{11} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{12} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \\ l_{21} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{22} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Kronecker Products

Doesn't care where scalar goes...

$$(\alpha \mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\alpha \mathcal{B}) = \alpha(\mathcal{A} \otimes \mathcal{B})$$

Associative:

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$$

Left-distributive:

$$\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = \mathcal{A} \otimes \mathcal{B} + \mathcal{A} \otimes \mathcal{C}$$

Right-distributive:

$$(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes \mathcal{C} + \mathcal{B} \otimes \mathcal{C}$$

Tensor Products: more basic properties

Other properties:

- Bilinear: linear in both arguments.
- Associative
- Distributive
- Not commutative
- $(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$.
- $\text{Tr}(\mathcal{A} \otimes \mathcal{B}) = \text{Tr}(\mathcal{A}) \cdot \text{Tr}(\mathcal{B})$.
- $\det(\mathcal{A} \otimes \mathcal{B}) = (\det(\mathcal{A}))^m \cdot \det(\mathcal{B})^n$, where $m = \text{size}(\mathcal{A})$, $n = \text{size}(\mathcal{B})$.

Entangling 2 qubits

- Entanglement, intuitively (or not)
- Entanglement, mathematically.
- Recipe for a 2-qubit entangler.
- Running on IBM-Q.

Composite State as a Kronecker Product

Example: Representing the classical numbers “1” and “0” with two qubits:

$$\begin{aligned}1_2 \equiv |01\rangle &= |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\0_2 \equiv |00\rangle &= |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\|10\rangle &= [0 \ 0 \ 1 \ 0]^T, |11\rangle = [0 \ 0 \ 0 \ 1]^T.\end{aligned}$$

In fact, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis for $\otimes^2 \mathbb{C}^2$, the 2-qubit system.

Entanglement

Not every $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|c\rangle, |d\rangle \in \mathbb{C}^2$ such that

$$\begin{aligned} |c\rangle \otimes |d\rangle = |\beta_{00}\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \\ &= \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \rightarrow \textbf{Entangled} \end{aligned}$$

Examples: Bell states, also entangled [CMTH]

$$\begin{aligned} |\beta_{10}\rangle &= \frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle \\ |\beta_{01}\rangle &= \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle \end{aligned}$$

“Entangled” operators

For operators: $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W}), \mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}|v\rangle) \otimes (\mathcal{B}|w\rangle).$$

Not all $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ can be written as $\mathcal{A} \otimes \mathcal{B}, \mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$.

Example:

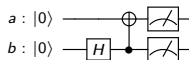
$$CNOT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What do we need to entangle two qubits?

- Hadamard gate
- CNOT gate
- Measure

2-Qubit Entanglement Circuit

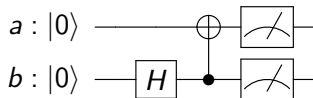
[EF04]



$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \left\{ \begin{array}{l} |00\rangle \rightarrow |00\rangle \\ |10\rangle \rightarrow |10\rangle \\ |01\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |01\rangle \end{array} \right.$$

Entanglement (cont.)



$$\begin{aligned} C_b(I \otimes H) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_a \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b \right) &= C_b \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_a \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b \right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |1\rangle \\ &\rightarrow \textbf{Entangled} \end{aligned}$$

Entanglement (cont.)

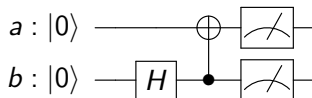
Notice:

$$\begin{aligned}(I|0\rangle) \otimes (H_b|0\rangle) &= (I \otimes H_b)(|0\rangle \otimes |0\rangle) \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \mathcal{O} \\ \mathcal{O} & \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \left[\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \right]^T &= \left[\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \right]^T\end{aligned}$$

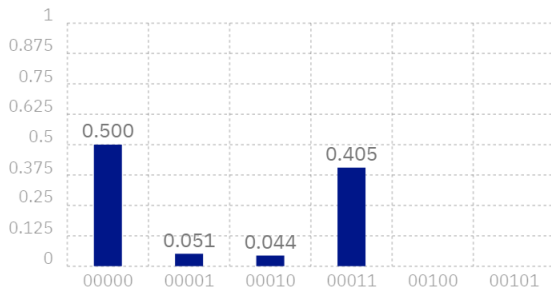
→ Possible to write H as $I \otimes H_b$. Not possible for $CNOT_b$.

Simulation on IBM-Q

Entanglement circuit, revisited








Quantum State: Computation Basis



Recap

- Representing multi-qubit systems with tensor products.
- Representing multi-qubit operators with tensor products.
- Entanglement, mathematically.
- 2-qubit entangler, mathematically.
- Entanglement on IBM-Q.

References

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