

12

1. (a) Poisson Brackets $[Q_1, Q_2] = 0$
 (b) $[P_1, P_2] = 0$
 $[Q_i, P_j] = \delta_{ij}$

(b) Take $G = P_z$. This generates translation in z direction

(c) **or** $G = L_z$ " " rotation about $\frac{z}{2}$

or $G = H$ " " evolution in time

(d) Kaplan-Yorke : $d_F = 1 + \frac{\lambda_1}{|\lambda_2|}$ for $\lambda_2 < 0 < \lambda_1$
 $(\lambda_3 = 0)$

d_F = fractal dimension, λ_i describe contraction / expansion of phase space trajectories that start out differing by δ_{00}

$$\delta_i \approx \delta_{00} e^{\lambda_i t} \quad \text{for } i=1,2 \text{ directions}$$

- (d) Two of : • chaotic • larger dissipation of energy
 • vortices at all scales
 • effectively transports energy (or momentum or temperature)
 • reduces drag on boundary layer of an object

2. 27

(a) $H = \frac{1}{2\mu} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) - \frac{k}{r} = \alpha_1$ constant

+ cyclic : $p_\theta = \alpha_4 = \text{constant}$, $\omega = \omega_1(r) + \alpha_4 \varphi$

H.T eqtn: $\frac{1}{2\mu} \left[\left(\frac{d\omega_1}{dr} \right)^2 + \frac{\alpha_4^2}{r^2} \right] - \frac{k}{r} = \alpha_1$

$$\frac{d\omega_1}{dr} = \left[2\mu \left(\alpha_1 + \frac{k}{r} \right) - \frac{\alpha_4^2}{r^2} \right]^{\frac{1}{2}} \Rightarrow \boxed{\omega = \alpha_4 \varphi + \int dr \left[2\mu \left(\alpha_1 + \frac{k}{r} \right) - \frac{\alpha_4^2}{r^2} \right]^{\frac{1}{2}}}$$

Transf'n Eqtns:

$$t + p_1 = \frac{d\omega}{2\alpha_1} = \int \frac{\mu dr}{\left[2\mu \left(\alpha_1 + \frac{k}{r} \right) - \frac{\alpha_4^2}{r^2} \right]^{\frac{1}{2}}} = t(r) \quad \begin{matrix} 3. \\ \text{radial motion} \end{matrix}$$

$$p_2 = \frac{d\omega}{2\alpha_4} = \varphi - \int \frac{\alpha_4 dr}{r^2 \left[2\mu \left(\alpha_1 + \frac{k}{r} \right) - \frac{\alpha_4^2}{r^2} \right]^{\frac{1}{2}}} = \varphi(r) \quad \begin{matrix} 3. \\ \text{orbital equation} \end{matrix}$$

2.(b) $\phi(r, \theta, z) = c r^a \cos(b\theta + d)$

(12)

$$\text{Need } \nabla^2 \phi = 0 = \left(\frac{1}{r} \frac{\partial^2}{\partial r^2} \underbrace{r^2 \frac{\partial^2}{\partial r^2}}_{\alpha^2} r^a \right) c \cos(b\theta + d) + r^{a-2} c \frac{\partial^2}{\partial \theta^2} \cos(b\theta + d)$$

$$= a^2 (r^{a-2} c \cos(b\theta + d)) - b^2 (r^{a-2} c \cos(b\theta + d))$$

$\therefore \boxed{a^2 = b^2}$

3.

Origin $r=0$ stagnation point. $\vec{v}(r=0) = 0$

$$\vec{v} = \nabla \phi = \hat{r} \frac{\partial \phi}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial \phi}{\partial \theta} = \hat{r} c a r^{a-1} \cos(b\theta + d) - \hat{\theta} c r^{a-1} b \sin(b\theta + d)$$

$\therefore \boxed{a > 1}$ so $a = b > 1$

4.

Require $\hat{\theta} \cdot \vec{v} = 0$ for $\theta = \alpha$ and $\theta = -\alpha$

$$\sin(b\alpha + d) = 0$$

$$\sin(-b\alpha + d) = 0$$

\therefore take $b\alpha + d = \pi$ $\Rightarrow \boxed{d = \frac{\pi}{2}}$

$$-b\alpha + d = 0 \quad \Rightarrow \quad \boxed{b = \frac{\pi}{2\alpha}}$$

5.

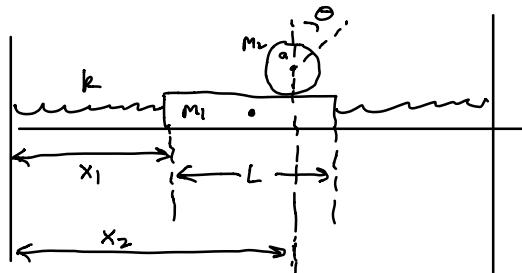
or $b\alpha + d = 0$ $\Rightarrow \boxed{d = \frac{\pi}{2}}$

$$-b\alpha + d = \pi \quad \Rightarrow \quad \boxed{b = -\frac{\pi}{2\alpha}}$$

or $\pi \rightarrow -\pi$ in one of above solutions is also OK

or $b\alpha + d = n\pi$ $d = \frac{(n+m)\pi}{2}$, $b = \frac{(n-m)\pi}{2\alpha}$ (general solution)

3. [26]



$$I = m_2 a^2 \text{ for hoop}$$

③ (a) constraint $\dot{x}_2 - \dot{x}_1 = a\dot{\theta}$ $(x_2 - x_1 = a\theta + k)$
 $g = \dot{x}_2 - \dot{x}_1 - a\dot{\theta} = 0$ also OK

⑤ (b) $T = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2 + \frac{m_2 a^2}{2} \dot{\theta}^2$ 3. length of z^{th} spring
 $V = \frac{k}{2} x_1^2 + \frac{k}{2} (2L - x_1)^2$ 2. $= 3L - L - x_1$

$$L = T - V = \frac{m_1}{2} \dot{x}_1^2 + \frac{m_2}{2} \dot{x}_2^2 + \frac{m_2 a^2}{2} \dot{\theta}^2 - \frac{k}{2} x_1^2 - \frac{k}{2} (2L - x_1)^2$$

③ c. ⑥
 $x_1 // \frac{d}{dt} \frac{2L}{2\dot{x}_1} - \frac{2L}{2x_1} = \lambda \frac{2\ddot{x}_1}{2\dot{x}_1} \Rightarrow m_1 \ddot{x}_1 + kx_1 + k(x_1 - zL) = -\lambda$
 $m_1 \ddot{x}_1 + 2k(x_1 - L) = -\lambda$ 2.
 $x_2 // \frac{d}{dt} \frac{2L}{2\dot{x}_2} - \frac{2L}{2x_2} = \lambda \frac{2\ddot{x}_2}{2\dot{x}_2} \Rightarrow m_2 \ddot{x}_2 = \lambda$ 2.
 $\theta // \frac{d}{dt} \frac{2L}{2\dot{\theta}} - \frac{2L}{2\theta} = \lambda \frac{2\ddot{\theta}}{2\dot{\theta}} \Rightarrow m_2 a^2 \ddot{\theta} = -a\lambda \therefore m_2 a \ddot{\theta} = -\lambda$ 2.

④ solve with $g = 0 = \ddot{x}_2 - \dot{x}_1 - a\dot{\theta} \Rightarrow \ddot{x}_2 = \ddot{x}_1 + a\ddot{\theta}$ 2.

⑥ $\Rightarrow \frac{\lambda}{m_2} = \left(\frac{-\lambda - 2k(x_1 - L)}{m_1} \right) + \left(\frac{-\lambda}{m_2} \right)$
 $\frac{2\lambda}{m_2} + \frac{\lambda}{m_1} = -\frac{2k(x_1 - L)}{m_1} = \lambda \left(\frac{2m_1 + m_2}{m_1 m_2} \right)$

$\therefore \boxed{\lambda = \frac{-2k m_2}{(2m_1 + m_2)} (x_1 - L)}$ 4.

⑦ block $\ddot{x}_1 = -\frac{2k}{m_1} (x_1 - L) - \frac{\lambda}{m_1} = -\frac{2k}{m_1} (x_1 - L) \left[1 - \frac{m_2}{2m_1 + m_2} \right]$

⑧ $= -\frac{4k}{2m_1 + m_2} (x_1 - L)$ Harmonic osc in $x'_1 = x_1 - L$

freq. $\boxed{\omega = \sqrt{\frac{4k}{2m_1 + m_2}}}$ 4.

hoop $\ddot{x}_2 + a\ddot{\theta} = 0 \quad \& \quad \ddot{x}_2 = \ddot{x}_1 + a\ddot{\theta}$

so $2\ddot{x}_2 = \ddot{x}_1 = -2a\ddot{\theta}$ hoop will have 2.
same oscillation frequency as block

4. 27

-4-

(a) Ideal fluid. $z=0$ at 3.

(7) continuity

$$\oint dV \nabla \cdot (\rho \vec{v}) = \rho \oint dS \hat{n} \cdot \vec{v} = 0$$

$$\text{so } A_2 v_2 = A_3 v_3$$

$$v_2 = \frac{A_3}{A_2} v_3 = \frac{v_3}{4} \quad z,$$

Bernoulli

$$\cancel{\frac{v_1^2}{2}} + z h g + \frac{p_{\text{atm}}}{\rho} = \frac{v_2^2}{2} + h g + \frac{p_2}{\rho} = \frac{v_3^2}{2} + 0 + \frac{p_{\text{atm}}}{\rho}$$

$$\text{so } v_3^2 = 4gh$$

$$\boxed{v_3 = 2\sqrt{gh}}$$

$$v_2 = v_3/4 = \frac{\sqrt{gh}}{2}$$

3.

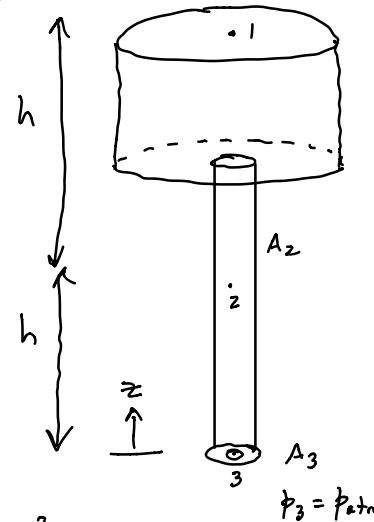
$$\text{and } \frac{p_2}{\rho} = \frac{p_{\text{atm}}}{\rho} + \frac{v_3^2 - v_2^2}{2} - hg$$

$$= \frac{p_{\text{atm}}}{\rho} + \frac{4gh - \frac{1}{4}gh}{2} - hg, \quad 2 - \frac{1}{8} - 1 = \frac{7}{8}$$

$$\boxed{p_2 = p_{\text{atm}} + \frac{7}{8} \rho hg}$$

z.

$$v_1 \approx 0, \quad p_1 = p_{\text{atm}}, \quad A_1$$



$$p_3 = p_{\text{atm}}$$

- (b) As z decreases in cylinder 2 we have some v_2 (by continuity) but zg gravity term decreases
so pressure must increase

$$\textcircled{C} \text{ (ii)} \quad \vec{v} = v_z(r) \hat{z}$$

$$\sigma = n/e$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} + \underbrace{\frac{1}{e} \vec{\nabla} \phi}_{\text{0}} - \sigma \nabla^2 \vec{v} = \vec{g} = -g \hat{z}$$

$$\underbrace{v_z}_{\text{0}} \underbrace{\frac{\partial}{\partial z} v_z(r)}_{\text{0}} \quad \frac{1}{e} \vec{\nabla} \phi - \sigma \nabla^2 v_z(r) \hat{z} = -g \hat{z}$$

so $\vec{\nabla} \phi$ must be in \hat{z} $\therefore \phi = \phi(z)$

$$\underbrace{\frac{1}{e} \frac{\partial}{\partial z} \phi(z)}_{\text{function of } z} = \underbrace{\sigma \nabla^2 v_z(r) - g}_{\text{function of } r} = \text{constant} = K \quad \text{2.}$$

$$\therefore \frac{d\phi(z)}{dz} = eK \quad \boxed{\phi(z) = eKz + \phi_0} \quad \text{2.}$$

$$\therefore \sigma \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} v_z(r) = g + K$$

$$\frac{\partial}{\partial r} r \frac{\partial}{\partial r} v_z(r) = \frac{g+K}{\sigma} r$$

$$r \frac{\partial}{\partial r} v_z(r) = \left(\frac{g+K}{\sigma}\right) \frac{r^2}{2} + a$$

$$\frac{\partial}{\partial r} v_z(r) = \left(\frac{g+K}{\sigma}\right) \frac{r}{2} + \frac{a}{r}$$

$$v_z(r) = \left(\frac{g+K}{\sigma}\right) \frac{r^2}{4} + a \ln r + b \quad \text{3.}$$

• nothing special happens at $r=0$ (non-singular) so $a=0$

$$\bullet v_z(r=R) = 0 \quad \therefore \left(\frac{g+K}{\sigma}\right) \frac{R^2}{4} + b = 0$$

$$\boxed{b = -\left(\frac{g+K}{\sigma}\right) \frac{R^2}{4}} \quad \text{2.}$$

$$\boxed{v_z(r) = \frac{(g+K)}{4\sigma} (r^2 - R^2)}$$

$$\text{where } \sigma = \frac{n}{e}$$

(d) friction force/unit area = $(-\hat{r})_k \sigma_{ki} (\hat{z})_i$ -6-

(4) $= -\sigma'_{rz} = -\eta \frac{\partial v_z}{\partial r}$ friction force in + \hat{z}

$= -\frac{\rho(g+k)}{4} \frac{\partial r}{\partial r} \Big|_{r=R} = -\frac{\rho(g+k)R}{2}$

$\boxed{\vec{F}_{\text{friction}} = -\frac{\rho(g+k)R}{2} \hat{z}}$

- (e)
- Energy goes into heating fluid through friction force
 - or does work on walls if we don't hold them fixed
 - or there is an increase in pressure potential energy as fluid drops
- 2 of these

5. [34] $\ddot{x} + a \dot{x} (x^2 + \dot{x}^2 - 1) + x = 0$

(a) Let $\omega = \dot{x}$ then $\dot{\omega} = \ddot{x}$

(2) $\dot{\omega} = -a\omega(x^2 + \omega^2 - 1) - x$

(b) Fixed pts $\omega^* = 0$

(10) $x^* = 0$ only

Linear Analysis

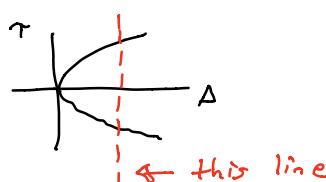
$\dot{x} = \omega$

$\dot{\omega} = a\omega - x$

$$\frac{d}{dt} \begin{pmatrix} x \\ \omega \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}}_{M} \begin{pmatrix} x \\ \omega \end{pmatrix}$$

$\Delta = \det M = +1$

$\tau = \text{tr } M = a$



$$\sqrt{\tau^2 - \Delta} = \sqrt{a^2 - 4}$$

$$\lambda_{\pm} = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

1. $a = 0$

center

$\dot{x} = \omega$

$\dot{\omega} = -x$

1.

$0 < a < 2$

unstable spiral

λ_{\pm} complex

2. $a = 2$ $\lambda_{\pm} = 1$ one eigenvalue

$$M = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} \vec{a} = \vec{a}, \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \vec{a} = 0$$

$\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ only 1-eigenvector \therefore unstable degenerate node

1. $a > 2$ λ_{\pm} real distinct unstable node

1. $-2 < a < 0$ stable spiral

1. $a = -2$ $\lambda_{\pm} = -1$ $\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \vec{a} = -\vec{a}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \vec{a} = 0$

$\therefore \vec{a} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ 1 eigenvector stable degenerate node

1. $a < -2$ stable node

⑥ limit cycle $x^2 + \omega^2 = 1$ $\dot{x} = \omega$ $x = \sin t$
 ⑦ $\dot{\omega} = -x$ $\omega = \cos t$

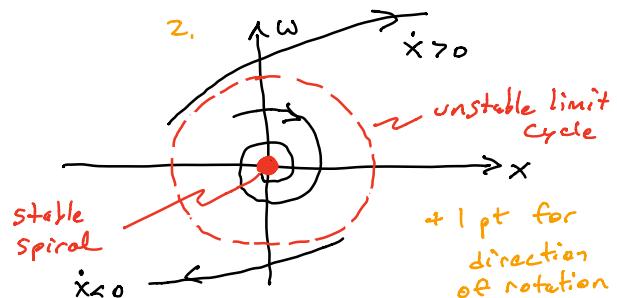
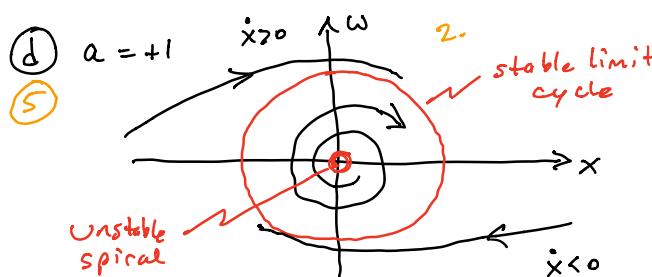
radial coords: $x = r \sin \theta$ $\omega = r \cos \theta$ $r^2 = x^2 + \omega^2$

$$r \dot{r} = x \dot{x} + \omega \dot{\omega} = x \omega + (-x \omega) - a \omega^2 (r^2 - 1)$$

$$= -a r^2 \cos^2 \theta (r^2 - 1) \quad 3.$$

if $a > 0$ $r > 1$ $\dot{r} < 0$
 $r < 1$ $\dot{r} > 0$ \therefore stable limit cycle 2.
 ($\$$ isolated)

if $a < 0$ $r > 1$ $\dot{r} > 0$
 $r < 1$ $\dot{r} < 0$ \therefore unstable limit cycle 2.



(5e) $x_{n+1} = r x_n - x_n^3 = f(x_n)$, $f'(x) = r - 3x^2$. -8-

(10) $|f'(x^*)| < 1$ stable, $|f'(x^*)| > 1$ unstable

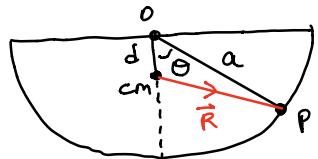
fixed points $x^* = r x^* - x^{*3}$ $\therefore (r-1)x^* = x^{*3}$

so $x^* = 0$ $f'(0) = r$ so $0 < r < 1$ stable
 $1 < r < \frac{3}{2}$ unstable 4.

$x^* = +\sqrt{r-1}$ only if $r > 1$ $f'(x^*) = r - 3(r-1) = 3-2r$
 $x^* = -\sqrt{r-1}$ for both
so stable $1 < r < \frac{3}{2}$ 4.

(6) 24

(a)
(10)



Use II axes theorem for I_{zz}' 's
 $\vec{R} = \hat{x} a \sin \theta + \hat{y} (d - a \cos \theta)$ 2.

$$I_{zz}' = I_{zz}^{cm} + M(\delta_{zz} \vec{R}^2 - R_z R_z) \quad 2.$$

$$I_{zz}^{cm} = I_{zz} + M d^2$$

$$I_{zz}^{cm} = \frac{M}{\frac{1}{2}\pi a^2} \int_0^a dr r \int_0^{2\pi} d\theta (x^2 + y^2) = \frac{2M}{\pi a^2} \pi \frac{a^4}{4} = \frac{Ma^2}{2}$$

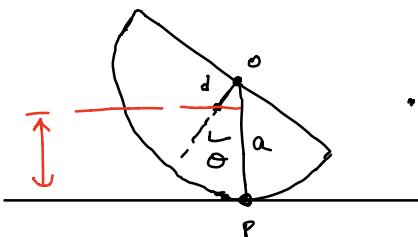
$$I_{zz}^{cm} = M \left(\frac{a^2}{2} - d^2 \right) \quad 3.$$

$$\begin{aligned} I_{zz}' &= M \left(\frac{a^2}{2} - d^2 \right) + M \left(a^2 \sin^2 \theta + (d - a \cos \theta)^2 \right) \\ &= M \frac{a^2}{2} - \cancel{Md^2} + M a^2 + \cancel{Md^2} - 2M ad \cos \theta \\ &= M \left(\frac{3}{2} a^2 - 2 ad \cos \theta \right) \quad 3. \end{aligned}$$

← don't have to simplify

(b) $L = T - V$

(8) height
 $= a - d \cos \theta$



$\therefore V = mg(a - d \cos \theta)$ 4.

T: contact point P is instantaneously at rest

rotates about \hat{z}' axis through P, angular velocity $\omega = \dot{\theta} \hat{z}'$

$$T = \frac{m}{2} \dot{\theta}^2 \left[\frac{3}{2} a^2 - 2ad \cos\theta \right] \quad 4.$$

(C) At $t=0 \quad \theta=0 \quad \& \quad \dot{\theta} = \frac{v_0}{a}$

(6) Use energy conservation $E = T + V$

at $t=0 \quad E = \frac{m}{2} \frac{v_0^2}{a^2} \left[\frac{3}{2} a^2 - 2ad \right] + mg(a-d)$

at $\theta = \frac{\pi}{2}$ need $\dot{\theta} = 0$ to not tip over

$$E = 0 + m g a$$

$$\therefore \frac{m}{2} v_0^2 \left(\frac{3}{2} a^2 - 2ad \right) + m g a - m g d = m g a \quad 4.$$

so $v_0 = \left[\frac{2gd}{\frac{3}{2}a - 2d} \right]^{\frac{1}{2}} = \left[\frac{4gd}{3 - 4\frac{d}{a}} \right]^{\frac{1}{2}}$

The END