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Course: **8.321 - Quantum Theory I**
Problem set: **#5**

1. Coherent states

(a)

$$\left|\phi\right\rangle = e^{\phi a^\dagger}\left|0\right\rangle = \sum_{n=0}^{\infty}\frac{\phi^n (a^\dagger)^n}{n!}\left|0\right\rangle = \sum_{n=0}^{\infty}\frac{\phi^n}{n!}\sqrt{n!}\left|n\right\rangle = \sum_{n=0}^{\infty}\frac{\phi^n}{\sqrt{n!}}\left|n\right\rangle.$$

(b)

$$a\left|\phi\right\rangle = \sum_{n=0}^{\infty}\frac{\phi^n}{\sqrt{n!}}a\left|n\right\rangle = \sum_{n=0}^{\infty}\frac{\phi^n}{\sqrt{n!}}\sqrt{n}\left|n-1\right\rangle = \phi\sum_{n-1=0}^{\infty}\frac{\phi^{n-1}}{\sqrt{(n-1)!}}\left|n-1\right\rangle = \phi\left|\phi\right\rangle.$$

(c)

$$\left\langle\phi\middle|\phi'\right\rangle = \sum_{m=0}^{\infty}\frac{(\phi^*)^m}{\sqrt{m!}}\sum_{n=0}^{\infty}\frac{\phi'^n}{\sqrt{n!}}\left\langle m\middle|n\right\rangle = \sum_{n=0}^{\infty}\frac{(\phi^*\phi')^n}{n!} = e^{\phi^*\phi'}.$$

(d)

$$\left\langle\phi\right| : A(a^\dagger, a) : \left|\phi'\right\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C(m,n)\left\langle\phi\right|(a^\dagger)^m a^n\left|\phi'\right\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C(m,n)(\phi^*)^m\phi'^n\left\langle\phi\middle|\phi'\right\rangle = e^{\phi^*\phi'}A(\phi^*,\phi')$$

(e)

$$\frac{1}{2\pi i}\int d\phi^* d\phi e^{-\phi^*\phi}\left|\phi\right\rangle\!\left\langle\phi\right| = \frac{1}{2\pi i}\sum_{n,m}^{\infty}\frac{\left|m\right\rangle\left\langle n\right|}{\sqrt{m!n!}}\int d\phi^* d\phi(\phi^*)^n\phi^m e^{-\phi^*\phi}$$

In polar coordinates, $\phi = re^{i\theta}$, and $\int d\phi^* d\phi = 2i\int r\,drd\theta$. With this,

$$\begin{aligned}
\frac{1}{2\pi i}\sum_{n,m}^{\infty}\frac{\left|m\right\rangle\left\langle n\right|}{\sqrt{m!n!}}\int d\phi^* d\phi e^{-\phi^*\phi}\left|\phi\right\rangle\!\left\langle\phi\right| &= \frac{2i}{2\pi i}\sum_{n,m}^{\infty}\frac{\left|m\right\rangle\left\langle n\right|}{\sqrt{m!n!}}\int_0^{2\pi}d\theta e^{i(m-n)\theta}\int_0^{\infty}dr r^{m+n+1}e^{-r^2}\\
&= \frac{2i}{2\pi i}\sum_{n,m}^{\infty}\frac{\left|m\right\rangle\left\langle n\right|}{\sqrt{m!n!}}2\pi\delta_{mn}\frac{1}{2}\Gamma\left(\frac{2+m+n}{2}\right)\\
&= \frac{2i}{2i}\sum_{n=0}^{\infty}\frac{\left|n\right\rangle\left\langle n\right|}{n!}\Gamma(n+1)\\
&= \frac{2i}{2i}\sum_{n=0}^{\infty}\frac{\left|n\right\rangle\left\langle n\right|}{n!}n!\\
&= \mathbb{I}.
\end{aligned}$$

2. Squeezed states

(a) When $\beta = 0$ we have

$$\begin{aligned}
\left\langle\alpha,0,\gamma\middle|\alpha,0,\gamma\right\rangle &= e^{\alpha^*\alpha}\left\langle 0\right|\left(e^{\gamma(a^\dagger)^2}\right)^\dagger e^{\gamma(a^\dagger)^2}\left|0\right\rangle\\
&= e^{\alpha^*\alpha}\left\langle 0\right|e^{\gamma^* a^2}e^{\gamma(a^\dagger)^2}\left|0\right\rangle
\end{aligned}$$

Let's calculate $e^{\gamma(a^\dagger)^2}|0\rangle$:

$$\begin{aligned}
e^{\gamma(a^\dagger)^2}|0\rangle &= \sum_{n=0}^{\infty}\frac{\gamma^n (a^\dagger)^n (a^\dagger)^n}{n!}|0\rangle \\
&= \sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}(a^\dagger)^n|n\rangle \\
&= \sum_{n=0}^{\infty}\frac{\gamma^n}{\sqrt{n!}}\sqrt{\frac{(2n)!}{n!}}|2n\rangle \\
&\sum_{n=0}^{\infty}\frac{\gamma^n}{n!}\sqrt{(2n)!}\,|2n\rangle\,.
\end{aligned}$$

With this,

$$\langle\alpha,0,\beta|\alpha,0,\beta\rangle = e^{\alpha^*\alpha}\sum_{n,m}^{\infty}\frac{\gamma^n\gamma^m}{n!m!}\sqrt{(2n)!(2m)!}\delta_{mn} = e^{\alpha^*\alpha}\sum_{n=0}^{\infty}\frac{\gamma^{2n}}{(n!)^2}(2n)!$$

In order for this norm to converge, the series satisfies the ratio test:

$$1 > e^{|\alpha|^2}\lim_{n\to\infty}\frac{\gamma^{2(n+1)}(2(n+1))!/((n+1)!)^2}{\gamma^{2n}(2n)!/(n!)^2} = \lim_{n\to\infty}e^{|\alpha|^2}\gamma^2\frac{(2n+1)(2n+2)}{(n+1)(n+1)} = 4e^{|\alpha|^2}\gamma^2 \implies \boxed{e^{|\alpha|^2}\gamma^2 < 1/4}$$

Extend this result for $\beta \neq 0$?

(b) We claim that

$$\boxed{|x'\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x'^2\right)\exp\left(\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger - \frac{1}{2}(a^\dagger)^2\right)|0\rangle}$$

from which we read off the coefficients:

$$\gamma = -\frac{1}{2}, \qquad \beta = \sqrt{\frac{2m\omega}{\hbar}}x', \qquad \alpha = -\frac{m\omega}{2\hbar}x'^2 + \frac{1}{4}\ln\left(\frac{m\omega}{\pi\hbar}\right).$$

Now we prove that the boxed equation is true. To this end, we check that the normalization is correct and that the the equation $\hat{x}|x'\rangle = x'|x'\rangle$ is satisfied.

$$\begin{aligned}
\hat{x}|x'\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)|x'\rangle \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x'^2\right)\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\exp\left(\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger - \frac{1}{2}(a^\dagger)^2\right)|0\rangle \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x'^2\right)\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger)\exp\left(-\frac{1}{2}(a^\dagger)^2\right)\exp\left(\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger\right)|0\rangle
\end{aligned}$$

since things commute. This is rather complicated to deal with. However, we may insert the identity operator I defined by

$$I = \exp\left(-\frac{1}{2}(a^\dagger)^2\right)\exp\left(\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger\right)\exp\left(-\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger\right)\exp\left(\frac{1}{2}(a^\dagger)^2\right)$$

to the left and observe that

$$\begin{aligned}
\exp\left(\frac{1}{2}(a^\dagger)^2\right)(a+a^\dagger)\exp\left(-\frac{1}{2}(a^\dagger)^2\right) &= \exp\left(\frac{1}{2}(a^\dagger)^2\right) a \exp\left(-\frac{1}{2}(a^\dagger)^2\right) + a^\dagger \\
&= a + \frac{1}{2}[a^\dagger a^\dagger, a] + a^\dagger \\
&= a + \frac{1}{2}(a^\dagger[a^\dagger, a] + [a^\dagger, a]a^\dagger) + a^\dagger \\
&= a - a^\dagger + a^\dagger \\
&= a,
\end{aligned}$$

where we have used the identity for $e^A B e^{-A}$ from Pset 1 and the fact that $a^\dagger$ commute with itself. Next, we find (using the same identity)

$$\begin{aligned}
\exp\left(-\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger\right) a \exp\left(\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger\right) &= a - \sqrt{\frac{2m\omega}{\hbar}}x'[a^\dagger, a] \\
&= a + \sqrt{\frac{2m\omega}{\hbar}}x'.
\end{aligned}$$

Since $a|0\rangle = 0$, we have

$$\begin{aligned}
\hat{x}|x'\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x'^2\right) \cancel{\sqrt{\frac{\hbar}{2m\omega}}} \exp\left(-\frac{1}{2}(a^\dagger)^2\right) \exp\left(\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger\right) \cancel{\sqrt{\frac{2m\omega}{\hbar}}}\, x'|0\rangle \\
&= x'\left\{\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x'^2\right) \exp\left(\sqrt{\frac{2m\omega}{\hbar}}x'a^\dagger - \frac{1}{2}(a^\dagger)^2\right)|0\rangle\right\} \\
&= x'|x'\rangle \qquad \checkmark
\end{aligned}$$

The normalization is obtained by finding $\langle 0|x'\rangle$. Suppose that it is N, then

$$\langle 0|x'\rangle = N\langle 0|\exp\left(\sqrt{\frac{2m\omega}{\hbar}}xa^\dagger - \frac{1}{2}(a^\dagger)^2\right)|0\rangle = N\langle 0|0\rangle = N \implies N = \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega}{2\hbar}x'^2\right).$$

With this we're done.

To see if $\langle x'|x'\rangle$ is bounded or not, we may look at $e^{|\alpha|^2}\gamma^2$ for this case. Notice that $e^{|\alpha|^2} \geq 1$ for all α, and so the norm is finite only if $\gamma^2 < 1/4$. However, in this case we have $\gamma = -1/2 \implies \gamma^2 = 1/4$. We therefore conclude that $\langle x'|x'\rangle$ is infinite, as expected.

3. Low-lying states

(a) Ground and first excited energy for particle in the potential:

$$V(x) = \frac{1}{4}x^4$$

(b) Ground and first excited energy for particle in the potential:

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{24}x^4$$

(c) Ground state energy for particle in the potential:

$$W(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \sqrt{2}|x-y|$$

(d) Ground state energy for particle in the potential:

$$V(x,y) = \frac{1}{4}x^4 + \frac{1}{6}y^6 + 2xy$$