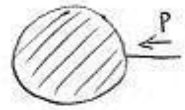
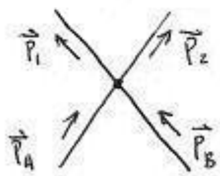
For each internal vertex  $-i\lambda$ and momentum conservation

For each external line 

no extra factor (i.e, 1)

Integrate over all unconstrained momenta and divide by symmetry factor S.

Example $\langle \vec{p}_1, \vec{p}_2 | iT | \vec{p}_A, \vec{p}_B \rangle$ at lowest order



Feynman amplitude
$i\mathcal{M} = -i\lambda$

So $$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\vec{p}^{\,final}|\,|\mathcal{M}|^2}{2E_A\,2E_B\,|\vec{v}_A - \vec{v}_B|\;16\pi^2 E_{cm}}$$

Let $p = |\vec{p}^{\,final}| = |\vec{p}_A| = |\vec{p}_B|$ all same since masses are all the same

$$E_{cm} = 2E_A = 2E_B = 2\sqrt{p^2+m^2}$$

$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2p}{E_A}$$

So $\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^2 \, p}{\frac{2p}{E_A}(2E_A)(2E_B)16\pi^2 E_{cm}} = \frac{\lambda^2}{64\pi^2 E_{cm}^2}$

This is spherical symmetric, and so

$$\sigma_{tot} = 4\pi \frac{\lambda^2}{64\pi^2 E_{cm}^2} \cdot \left(\frac{1}{2}\right)$$

tricky... particles in final state are identical and so we need $\frac{1}{2}$ factor

$$= \frac{\lambda^2}{32\pi^2 E_{cm}^2}$$

Feynman rules for fermions

Recall that $T\{\psi_a(x)\bar{\psi}_b(y)\} = \begin{cases} \psi_a(x)\bar{\psi}_b(y) & \text{for } x^0 > y^0 \\ -\bar{\psi}_b(y)\psi_a(x) & \text{for } x^0 < y^0 \end{cases}$

The Feynman propagator is

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$$

$$= \langle 0| T\{\psi(x)\bar{\psi}(y)\}|0\rangle$$

The generalization of T for more than two fermion fields...

$$T\{\psi_1\psi_2\psi_3\psi_4\} = \begin{cases} \psi_1\psi_2\psi_3\psi_4 & \text{if } x_1^0 > x_2^0 > x_3^0 > x_4^0 \\ -\psi_2\psi_1\psi_3\psi_4 & \text{if } x_2^0 > x_1^0 > x_3^0 > x_4^0 \\ -\psi_3\psi_2\psi_1\psi_4 & \text{if } x_3^0 > x_2^0 > x_1^0 > x_4^0 \\ \vdots & \vdots \end{cases}$$

$\times(-1)$ if odd permutation of fields

$\times 1$ if even permutation of fields

Similarly we define normal ordering...

$$N\left\{ a_{\vec{p}_1}\, a_{\vec{p}_2}\, a_{\vec{p}_3}\, a^\dagger_{\vec{p}_4} \right\} = (-1)^3\, a^\dagger_{\vec{p}_4}\, a_{\vec{p}_1}\, a_{\vec{p}_2}\, a_{\vec{p}_3}$$

$\times(-1)$ if odd permutation of fields
$\times\, 1$ if even permutation of fields

Just as in the bosonic case,

$$T\left\{ \psi_a(x)\, \bar{\psi}_b(y) \right\} = N\left\{ \psi_a(x)\, \bar{\psi}_b(y) \right\} + \contraction{}{\psi}{_a(x)}{\bar{\psi}}\psi_a(x)\, \bar{\psi}_b(y)$$

where
$$\contraction{}{\psi}{_a(x)}{\bar{\psi}}\psi_a(x)\, \bar{\psi}_b(y) = \begin{cases} \left\{ \psi^+_a(x), \bar{\psi}^-_b(y) \right\} & \text{for } x^0 > y^0 \\ -\left\{ \bar{\psi}^+_b(y), \psi^-_a(x) \right\} & \text{for } y^0 > x^0 \end{cases}$$
$$= \langle 0 | T\left\{ \psi_a(x)\, \bar{\psi}_b(y) \right\} | 0 \rangle$$
$$= S_{F,ab}(x-y) = -\contraction{}{\bar{\psi}}{_b(y)}{\psi}\bar{\psi}_b(y)\, \psi_a(x)$$

$\psi^+(x), \bar{\psi}^+(x)$ is the positive frequency part of $\psi(x), \bar{\psi}(x)$
... ie, the part with annihilation operators

$\psi^-(x), \bar{\psi}^-(x)$ is the negative frequency part of $\psi(x), \bar{\psi}(x)$
... ie, the part with creation operators

Note: $\contraction{}{\psi}{_a(x)}{\psi}\psi_a(x)\, \psi_b(y) = 0 = \contraction{}{\bar{\psi}}{_a(x)}{\bar{\psi}}\bar{\psi}_a(x)\, \bar{\psi}_b(y)$

Just as we proved Wick's theorem for bosons, we can show the same for fermions.

$$T\{\psi_1 \bar{\psi}_2 \psi_3 \cdots\} = N\{\psi_1 \bar{\psi}_2 \psi_3 \cdots + \text{all possible contractions}\}$$

We note that an expression such as

$$N\{\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4\} = -\psi_1 \bar{\psi}_3 \, N\{\psi_2 \bar{\psi}_4\}$$

gets a minus sign since the $\bar{\psi}_3$ must hop over the ψ_2.

Helpful fact: for any fully contracted quantity count the number of times the contraction lines criss-cross and this tells you if it is an odd or even permutation...

$\psi_1 \psi_2 \psi_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6 \sim$ even (0 criss-crosses)

$\psi_1 \psi_2 \psi_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6 \times(-1)$ odd (1 criss-cross)

$\psi_1 \psi_2 \psi_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6 \times(-1)$ odd (3 criss-crosses)

$\psi_1 \psi_2 \psi_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6 \times(-1)$ odd (3 criss-crosses)

We consider the simplest possible interacting theory with fermions...

<u>Yukawa theory</u>

$$H = H_{Dirac}(\psi, \bar{\psi}) + H_{Klein\text{-}Gordon}(\phi) + g\int d^3\vec{x}\ \bar{\psi}\psi\phi$$

Let $|\vec{p}, s\rangle$ be a fermion state with momentum $\vec{p}$ and spin s.

$$\psi_I(x)\,|\vec{p}, s\rangle = \int \frac{d^3\vec{p}'}{(2\pi)^3\sqrt{2E_{\vec{p}'}}} \sum_{s'} a^{s'}_{\vec{p}'}\, u^{s'}(p')\, e^{-ip'\cdot x}\left(\sqrt{2E_{\vec{p}}}\, a^{s\dagger}_{\vec{p}}\,|0\rangle\right)$$

$$= e^{-ip\cdot x}\, u^s(p)\,|0\rangle$$

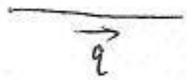
Let us define $\quad \overbrace{\psi_I(x)\,|\vec{p}, s\rangle} = e^{-ip\cdot x}\, u^s(p)$

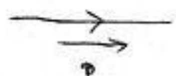
Similarly ... $\quad \overbrace{\bar{\psi}_I(x)\,|\underset{\substack{\uparrow \\ \text{antifermion} \\ \text{state}}}{\vec{K}}, s\rangle} = e^{-ik\cdot x}\, \bar{v}^s(K)$

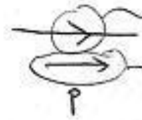
$$\overbrace{\langle \underset{\substack{\uparrow \\ \text{fermion}}}{\vec{p}}, s\,|\,\bar{\psi}_I(x)} = e^{+ip\cdot x}\, \bar{u}^s(p)$$

$$\overbrace{\langle \underset{\substack{\uparrow \\ \text{antifermion}}}{\vec{K}}, s\,|\,\psi_I(x)} = e^{+ik\cdot x}\, v^s(K)$$

Feynman rules (momentum space)

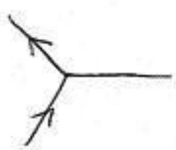
$\overbrace{\phi\,\phi}$  $= \dfrac{i}{q^2 - m_\phi^2 + i\varepsilon}$

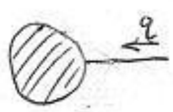
$\overbrace{\psi\,\bar{\psi}}$  $= \dfrac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}$

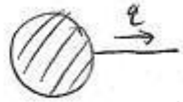
note  this arrow shows flow of particle number

this shows flow of momentum

Peskin + Schroeder always orients both in the same direction when possible for internal lines.

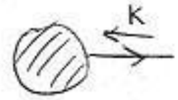
 $= -ig$

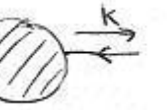
$\overbrace{\phi\,|\vec{q}\rangle}$ =  $= 1$

$\overbrace{\langle\vec{q}|\,\phi}$ =  $= 1$

$\overbrace{\psi\,|\vec{p},s\rangle}$ (fermion) =  $= u^s(p)$

$\overbrace{\langle\vec{p},s|\,\bar{\psi}}$ (fermion) =  $= \bar{u}^s(p)$

$$\bar{\psi}\,|\vec{k},s\rangle = $$  $$= \bar{v}^s(k)$$
antifermion

$$\langle\vec{k},s|\,\psi = $$  $$= v^s(k)$$
antifermion

Note: initial states have momentum pointing inwards
final states have momentum pointing outwards

external particle/antiparticle $\Leftrightarrow$ $\rightrightarrows$ / $\rightleftarrows$
same direction / opposite direction

Integrate over all unconstrained momenta.

Divide by symmetry factors, S.

Since all three lines coming out of a given vertex are different, S is very often 1.

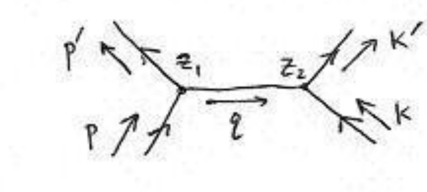
One notable exception are vacuum diagrams...

$S = 2$

An added complication is keeping track

of the overall sign. Count the number of criss-crosses of $\overbracket{\psi\bar{\psi}}$ lines. If you encounter a $\overbracket{\bar{\psi}\psi}$, this also gets a minus sign, since $\overbracket{\bar{\psi}\psi} = -\overbracket{\psi\bar{\psi}}$.

Example



(don't worry about spin polarizations in this example)

Let the incoming state be $|\vec{p},\vec{k}\rangle$, where $|\vec{p},\vec{k}\rangle = \sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{k}}}\, a^{\dagger}_{\vec{p}} a^{\dagger}_{\vec{k}} |0\rangle$.

Note how the order of $a^{\dagger}_{\vec{p}}$ + $a^{\dagger}_{\vec{k}}$ could cause confusion with minus signs.

Let $\langle\vec{k},\vec{p}|$ be the corresponding "bra" with this "ket" (or dual vector). Note how we reversed the order of $\vec{k}$ + $\vec{p}$ (P + S do not use this reordering). It is convenient since it reminds us that $\langle\vec{k},\vec{p}| = \langle 0| a_{\vec{k}} a_{\vec{p}} \sqrt{2E_{\vec{k}}}\sqrt{2E_{\vec{p}}}$

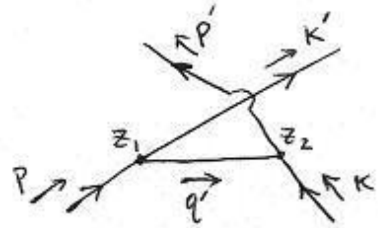
$$\int d^4z_1\, d^4z_2 \quad \langle \vec{k}', \vec{p}' |\, \bar\psi_1 \psi_1 \phi_1 \bar\psi_2 \psi_2 \phi_2 \,| \vec{p}, \vec{k} \rangle$$

[2 criss-crossing intersections so $\times 1$]

$$= (-ig)^2 \frac{i}{q^2 - m_\phi^2 + i\varepsilon} \left(\bar u(p')\, u(p)\right)\left(\bar u(k')\, u(k)\right)$$

where $q = p - p'$

We can compare this to



$$\int d^4z_1\, d^4z_2 \; \langle \vec{k}', \vec{p}' |\, \bar\psi_1 \psi_1 \phi_1 \bar\psi_2 \psi_2 \phi_2 \,| \vec{p}, \vec{k} \rangle$$

[3 criss-crosses so $\times(-1)$]

$$= (-ig)^2 \times (-1) \times \frac{i}{q'^2 - m_\phi^2 + i\varepsilon} \left(\bar u(k')\, u(p)\right)\left(\bar u(p')\, u(k)\right)$$

where $q' = p - k'$

Tips! For each fermion line that doesn't close into a loop, follow the particle number arrow to the end.