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 Course: **8.421 - AMO I**
 Problem set: **#7**
 Due: Friday, April 1, 2022.

1. Spherical Harmony. We want to evaluate matrix elements

$$\langle J' m'_J | Y_{LM} | J m_J \rangle = \int d\Omega Y_{J' m'_J}^* Y_{LM} Y_{J m_J}.$$

To do this, we consider two particles with angular momenta j_1 and j_2 . The total angular momentum is $J = j_1 + j_2$. We can go between the coupled and uncoupled basis via

$$\begin{aligned} |(j_1 j_2) JM\rangle &= \sum_{m_1, m_2} |j_1 m_1\rangle |j_2 m_2\rangle \langle j_1 m_1 j_2 m_2 | JM\rangle \\ |j_1 m_1\rangle |j_2 m_2\rangle &= \sum_{J, M} |(j_1 j_2) JM\rangle \langle JM | j_1 m_1 j_2 m_2 \rangle. \end{aligned}$$

The sum over M has only one nonzero term $M = m_1 + m_2$, and $|j_1 - j_2| < J < j_1 + j_2$. We also have the wavefunction of each particle at polar angle $\Omega_i = (\theta_i, \phi_i)$ is

$$\langle \Omega_i | j_i m_i \rangle = Y_{j_i m_i}(\Omega_i).$$

For the state of definite total angular momentum, we have

$$\Phi_{JM}(\Omega_1, \Omega_2) = \langle \Omega_1, \Omega_2 | (j_1 j_2) JM \rangle.$$

Now consider the function

$$F_{JM}(\Omega) \equiv \langle \Omega, \Omega | (j_1 j_2) JM \rangle$$

where $\Omega_1 = \Omega_2 = \Omega$. This is a wavefunction of an effective particle with angular momentum quantum numbers J, M . Indeed, it inherits its eigenvalues J^2 and J_z from $\Phi_{JM}(\Omega_1, \Omega_2)$. We conclude that $F_{JM}(\Omega)$ must be proportional to the spherical harmonic $Y_{JM}(\Omega)$. Let us call

$$F_{JM}(\Omega) = A_{(j_1 j_2)J} Y_{JM}(\Omega).$$

The factor $A_{(j_1 j_2)J}$ cannot depend on M as F_{JM} must behave exactly like Y_{JM} , in particular when acted upon by J_{\pm} which changes M . From here we have that

$$A_{(j_1 j_2)J} Y_{JM}(\Omega) = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle Y_{j_1 m_1}(\Omega) Y_{j_2 m_2}(\Omega).$$

(a) To find $A_{(j_1 j_2)J}$ we consider the special case where $\Omega = (\theta = 0, \phi)$. In this case, we have that

$$Y_{j_i m_i}(\Omega) = Y_{j_i m_i}(\theta = 0, \phi) = \sqrt{\frac{2j_i + 1}{4\pi}} \delta_{m_i 0}.$$

From the equation above we find that

$$A_{(j_1 j_2)J} \sqrt{\frac{2J + 1}{4\pi}} \delta_{M 0} = \sum_{m_1, m_2} \langle j_1 m_1 j_2 m_2 | JM \rangle \sqrt{\frac{2j_1 + 1}{4\pi}} \delta_{m_1 0} \sqrt{\frac{2j_2 + 1}{4\pi}} \delta_{m_2 0}.$$

This equation is nontrivial if $M = m_1 = m_2 = 0$, in which case we can solve for $A_{(j_1 j_2)J}$:

$$A_{(j_1, j_2)J} = \sqrt{\frac{(2j_1 + 1)(2j_2 + 1)}{4\pi(2J + 1)}} \langle j_1 0 j_2 0 | J 0 \rangle$$

(b) By applying $\langle \Omega, \Omega |$ to the LHS of

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{J,M} |(j_1 j_2) JM\rangle \langle JM | j_1 m_1 j_2 m_2 \rangle$$

we find that

$$\begin{aligned} \boxed{Y_{j_1 m_1}(\Omega) Y_{j_2 m_2}(\Omega)} &= \sum_{J,M} F_{JM}(\Omega) \langle JM | j_1 m_1 j_2 m_2 \rangle \\ &= \sum_{J,M} A_{(j_1 j_2)J} Y_{JM}(\Omega) \langle JM | j_1 m_1 j_2 m_2 \rangle \\ &= \boxed{\sum_{J,M} \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2J+1)}} \langle j_1 0 j_2 0 | J 0 \rangle \langle JM | j_1 m_1 j_2 m_2 \rangle Y_{JM}(\Omega)} \end{aligned}$$

(c) It remains to find the matrix element given at the top. To do this, we simply plug things in and use orthonormality of spherical harmonics:

$$\begin{aligned} \boxed{\langle j_3 m_3 | Y_{j_2 m_2} | j_1 m_1 \rangle} &= \int d\Omega Y_{j_3 m_3}^*(\Omega) Y_{j_2 m_2}(\Omega) Y_{j_1 m_1}(\Omega) \\ &= \int d\Omega Y_{j_3 m_3}^*(\Omega) \sum_{J,M} \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2J+1)}} \langle j_1 0 j_2 0 | J 0 \rangle \langle j_3 m_3 | j_1 m_1 j_2 m_2 \rangle Y_{JM}(\Omega) \\ &= \sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j_3+1)}} \langle j_1 0 j_2 0 | j_3 0 \rangle \langle j_3 m_3 | j_1 m_1 j_2 m_2 \rangle \underbrace{\int d\Omega Y_{j_3 m_3}^*(\Omega) Y_{j_3 m_3}(\Omega)}_1 \\ &= \boxed{\sqrt{\frac{(2j_1+1)(2j_2+1)}{4\pi(2j_3+1)}} \langle j_1 0 j_2 0 | j_3 0 \rangle \langle j_3 m_3 | j_1 m_1 j_2 m_2 \rangle} \end{aligned}$$

2. Dipole Operator. A symmetric top molecule has a Hamiltonian $\mathcal{H} = B\mathbf{J}^2$, with B the rotational constant. The dipole moment operator is $\hat{\mathbf{d}} = d\hat{\mathbf{r}}$, with d the value of the “permanent dipole moment” (in the molecular frame).

(a) We will prove the spherical tensor decomposition:

$$\sum_m C_{1m}^* \hat{\mathbf{e}}_m = \sum_m C_{1m} \hat{\mathbf{e}}_m = \hat{\mathbf{r}}$$

where $C_{1m}(\theta, \phi) = \sqrt{4\pi/3} Y_{1m}(\theta, \phi)$,

$$\hat{\mathbf{e}}_{\pm} = \mp \frac{\hat{\mathbf{e}}_x \pm i\hat{\mathbf{e}}_y}{\sqrt{2}} \quad \hat{\mathbf{e}}_0 = \hat{\mathbf{e}}_z$$

To this end, we simply write everything out explicitly. We will show that the left-most term is equal to $\hat{\mathbf{r}}$. Once done, the other equality follows immediately from the fact that $\hat{\mathbf{r}}$ is real (and therefore the second term is equal to the (conjugate of) the first term).

$$\begin{aligned} &C_{1-}^* \hat{\mathbf{e}}_- + C_{10}^* \hat{\mathbf{e}}_0 + C_{1+}^* \hat{\mathbf{e}}_+ \\ &= \frac{1}{2} e^{+i\phi} \sqrt{\frac{3}{2\pi}} \sqrt{\frac{4\pi}{3}} \sin \theta \frac{\hat{\mathbf{e}}_x - i\hat{\mathbf{e}}_y}{\sqrt{2}} + \frac{1}{2} \sqrt{\frac{3}{\pi}} \sqrt{\frac{4\pi}{3}} \cos \theta \hat{\mathbf{e}}_z + \frac{1}{2} e^{-i\phi} \sqrt{\frac{3}{2\pi}} \sqrt{\frac{4\pi}{3}} \sin \theta \frac{\hat{\mathbf{e}}_x + i\hat{\mathbf{e}}_y}{\sqrt{2}} \\ &= \sin \theta \cos \phi \hat{\mathbf{e}}_x + \sin \theta \sin \phi \hat{\mathbf{e}}_y + \cos \theta \hat{\mathbf{e}}_z \\ &= \hat{\mathbf{r}}. \quad \checkmark \end{aligned}$$

(b) Now we will show that

$$\hat{e}_m^* \cdot \hat{e}_n = \sum_p \delta_{mp} \delta_{np} = \delta_{mn}.$$

It suffices to demonstrate the following cases:

$$\hat{e}_+^* \cdot \hat{e}_- = -\frac{\hat{e}_x - i\hat{e}_y}{\sqrt{2}} \cdot \frac{\hat{e}_x - i\hat{e}_y}{\sqrt{2}} = 0 \iff \hat{e}_-^* \cdot \hat{e}_+ = 0$$

and

$$\hat{e}_\pm^* \cdot \hat{e}_\pm = \frac{\hat{e}_x \mp i\hat{e}_y}{\sqrt{2}} \cdot \frac{\hat{e}_x \mp i\hat{e}_y}{\sqrt{2}} = \frac{2}{2} = 1.$$

With these we are done.

(c) Suppose we have two unit vectors \hat{r} and \hat{r}' pointing in the direction of solid angle (θ, ϕ) and (θ', ϕ') . Let us call Θ the angle between the vectors, then we have

$$\begin{aligned} \cos \Theta &= \hat{r} \cdot \hat{r}' \\ &= \sum_{m,n} C_{1m}(\theta, \phi) C_{1n}^*(\theta', \phi') \hat{e}_m^* \cdot \hat{e}_n \\ &= \sum_{m,n} C_{1m}(\theta, \phi) C_{1n}^*(\theta', \phi') \delta_{mn} \\ &= \sum_m C_{1m}(\theta, \phi) C_{1m}^*(\theta', \phi') \\ &= \cos \theta \cos \theta' + \frac{1}{2} e^{-i\phi - i\phi'} \sin \theta \sin \theta' + \frac{1}{2} e^{i\phi + i\phi'} \sin \theta \sin \theta' \\ &= \cos \theta \cos \theta' + \cos(\phi - \phi') \sin \theta \sin \theta', \end{aligned}$$

as expected from standard geometry. A generalization of this result (for which $l = 1$) is

$$P_l(\cos \Theta) = \sum_m C_{lm}^*(\theta, \phi) C_{lm}(\theta', \phi')$$

where

$$C_{lm}(\theta, \phi) = \sqrt{\frac{4\pi}{2l+1}} Y_{lm}(\theta, \phi).$$

The proof is done by setting one of the unit vectors the z-axis, and the angles simplify.

(d) The electric field can be written

$$\begin{aligned} \mathbf{E} &= E_z \hat{e}_z + E_x \hat{e}_x + E_y \hat{e}_y \\ &= E_0 \hat{e}_0 + E_+ \hat{e}_+ + E_- \hat{e}_- \\ &= \sum_m E_m^* \hat{e}_m = \sum_m E_m \hat{e}_m^* \end{aligned}$$

where E_0, E_\pm defined in terms of $E_{x,y,z}$ in a similar way as the \hat{e}_m 's are defined in terms of $\hat{e}_{x,y,z}$. The dipole operator may be decomposed into spherical harmonics as

$$\begin{aligned} -\hat{d} \cdot \mathbf{E} &= -d \hat{r} \cdot \mathbf{E} \\ &= -d \sum_{m,n} C_{1m}^* E_n \hat{e}_m \cdot \hat{e}_n^* = -d \sum_{m,n} C_{1m} E_n^* \hat{e}_m^* \cdot \hat{e}_n \\ &= -d \sum_m C_{1m}^* E_m = -d \sum_m C_{1m} E_m^*. \end{aligned}$$

- (e) (Extra credit) Take $\mathbf{E} = E\hat{e}_z$. The matrix elements of the Hamiltonian $\mathcal{H} = B\mathbf{J}^2 - \hat{\mathbf{d}} \cdot \mathbf{E}$ in the $\{|Jm_J\rangle\}$ basis are given by

$$\begin{aligned}
\langle J'm_J'|\mathcal{H}|Jm_J\rangle &= BJ(J+1)\delta_{JJ'}\delta_{m_J'm_J} - dE\langle J'm_J'|C_{10}|Jm_J\rangle \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_J'm_J} - dE\sqrt{\frac{4\pi}{3}} \int d\Omega \underbrace{Y_{J'm_J'}^* Y_{10} Y_{Jm_J}} \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_J'm_J} - dE\sqrt{\frac{(2J+1)(2+1)}{3(2J'+1)}} \langle (J,0)(1,0)|(J',0)\rangle \langle J'm_J'|(Jm_J)(1,0)\rangle \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_J'm_J} - dE\sqrt{\frac{(2J+1)(2+1)}{3(2J'+1)}} \langle (J,0)(1,0)|(J',0)\rangle \langle J'm_J'|(Jm_J)(1,0)\rangle.
\end{aligned}$$

where we have used the fact that $C_{10} = C_{10}^*$ and remove the conjugation symbol. To get the matrix elements in the second term, we must use Wigner's 3-j symbols:

$$\langle j_1 m_1 j_2 m_2 | JM \rangle = (-1)^{-j_1+j_2-M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix}$$

which with we write the Hamiltonian matrix elements as

$$\begin{aligned}
\langle J'm_J'|\mathcal{H}|Jm_J\rangle &= BJ(J+1)\delta_{JJ'}\delta_{m_J'm_J} \\
&\quad - dE\sqrt{\frac{(2J+1)(2+1)}{3(2J'+1)}} (-1)^{-J+1} (-1)^{-J+1-m_J'} \sqrt{2J'+1} \sqrt{2J'+1} \begin{pmatrix} J & 1 & J' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J & 1 & J' \\ m_J & 0 & -m_J' \end{pmatrix} \\
&= BJ(J+1)\delta_{JJ'}\delta_{m_J'm_J} - dE(-1)^{-m_J'} \sqrt{\frac{(2J+1)(2+1)(2J'+1)}{3}} \begin{pmatrix} J & 1 & J' \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} J & 1 & J' \\ m_J & 0 & -m_J' \end{pmatrix}.
\end{aligned}$$

Using MATLAB, we can generate this matrix and diagonalize to find the eigenstates and their energies. Since it is convenient, I actually generated the Hamiltonian and carried out exact diagonalization in MATLAB but then plotted the probabilities in using `SphericalPlot3D[]` in Mathematica. Perhaps the grader will tell me that this solution is *cursed*. Figure 1 shows the first six energy levels up to an electric field $E \approx 10B/d$. For this calculation, I have picked $J_{\max} = 10$.

Using Mathematica, we plot $|\langle \theta, \phi | \Psi \rangle|^2$ of the lowest state for $dE/B = 0, 1, 10$. To do this, I have used MATLAB to find the lowest energy eigenstate for each value of dE/B . Then, I express these states in terms of spherical harmonics by identifying each entry of the state vector with the correct $|Jm_J\rangle$ state in the basis. For this part of the problem, I have used $J_{\max} = 4$. See Figures 2a, 2b, 3 for the results.

MATLAB code for calculating:

```

clear all
close all

J = 10;
strength = 0:0.05:10;
size = (J+1)^2;
H = zeros(size,size);

% creat a basis for the Hamiltonian
basis = [];

for j = 0:1:J
    for mj = -j:1:j
        basis = [basis; [j mj]];
    end
end

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```

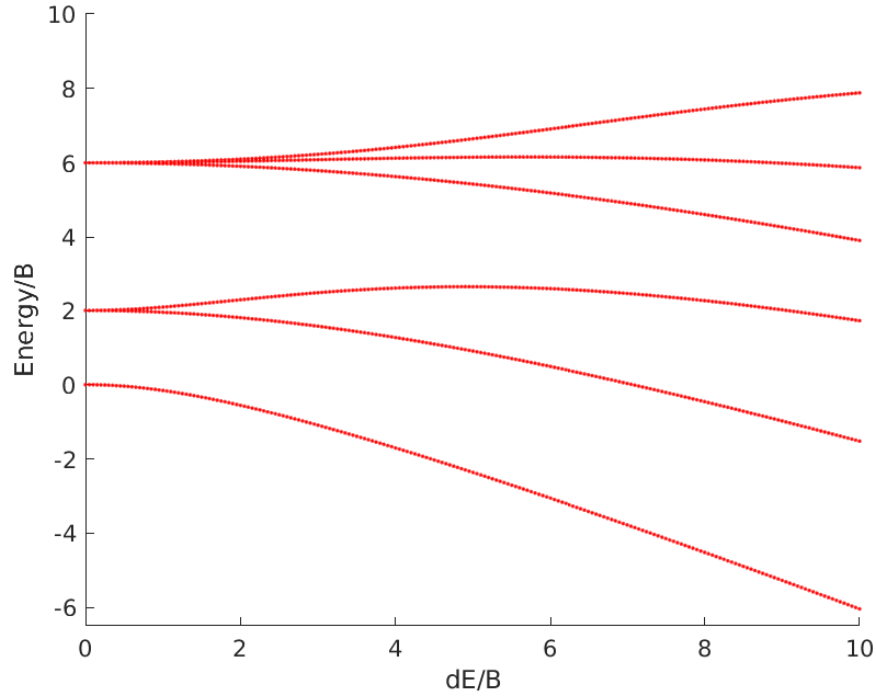
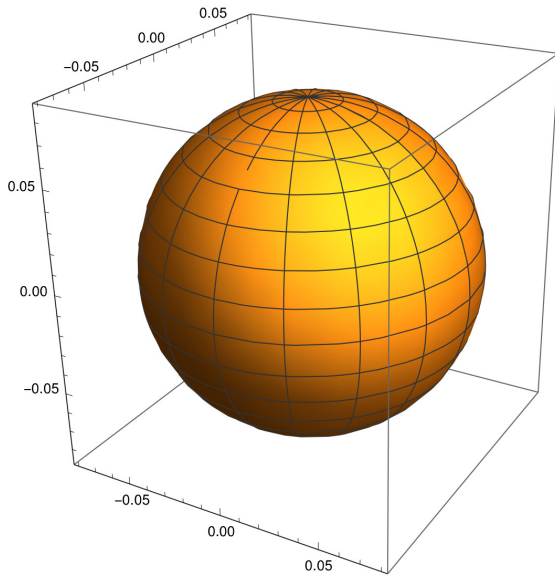
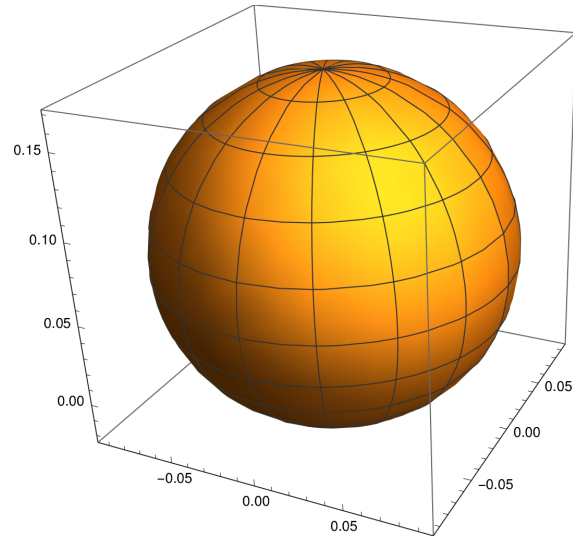


Figure 1: Energies of the first six energy levels p to an electric field $E \approx 10B/d$.



(a) $|\langle \theta, \phi | \Psi \rangle|^2$ for $dE/B = 0$.



(b) $|\langle \theta, \phi | \Psi \rangle|^2$ for $dE/B = 1$.

```
%% first plot energies as a fn of dE/B

stark = figure(1);
for a = strength % loop over field strengths
% create the Hamiltonian, element-by-element
for r = 1:size
j = basis(r,1);
mj = basis(r,2);
```

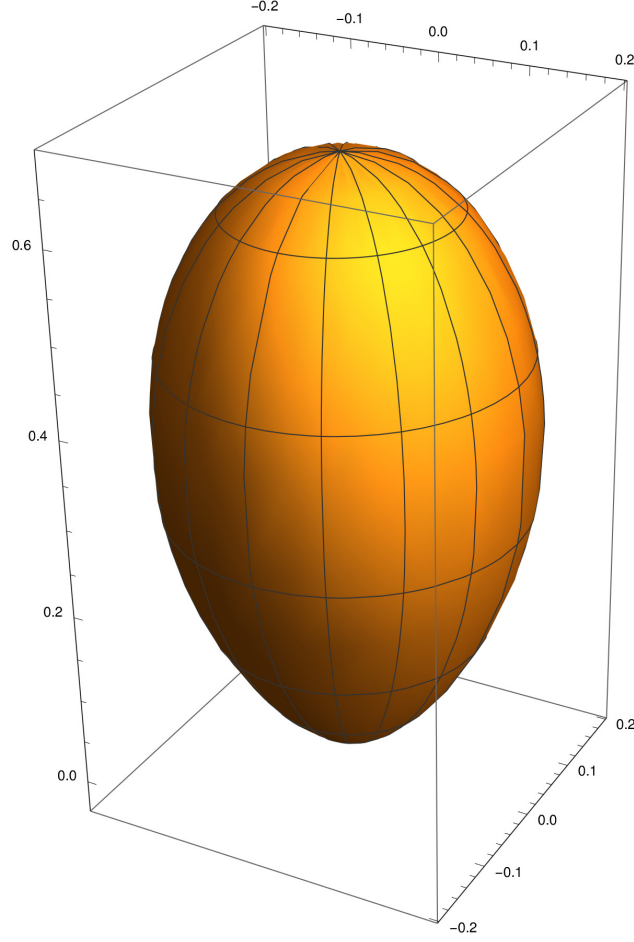


Figure 3: $|\langle \theta, \phi | \Psi \rangle|^2$ for $dE/B = 10$.

```

for c = 1:size
    jj = basis(c,1);
    mjj = basis(c,2);

    H(r,c) = j*(j+1)*(j==jj)*(mj==mjj)...
    -a*(-1)^(-mjj)*sqrt((2*j+1)*(2*j+1)*(2*j+1)/3)...
    *Wigner3j([j,1,jj],[0,0,0])*Wigner3j([j,1,jj],[mj,0,-mjj]);
end
end
% diag n plot eigenvalues associated with field strength a = dE/B
energies = eig(H);
hold on
plot(a*ones(size), energies, '.', 'Color', 'red', 'MarkerSize',4);
end

% disp(H)

% plot includes up to 6 lowest energies only

hold off
%title('Energy vs dE/B')
ylim([-6.5 10])
xlabel('dE/B')
ylabel('Energy/B')

```

```

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```

```

% now find the lowest state and associated energy for various a = dE/B
% then take result over to mathematica to plot wavefunction^2

strength = [0 1 10];
wavefunction = 0;

for a = strength % loop over field strengths
% create the Hamiltonian, element-by-element
for r = 1:size
j = basis(r,1);
mj = basis(r,2);
for c = 1:size
jj = basis(c,1);
mjj = basis(c,2);

H(r,c) = j*(j+1)*(j==jj)*(mj==mjj)...
-a*(-1)^(-mjj)*sqrt((2*j+1)*(2*j+1)*(2*jj+1)/3)...
*Wigner3j([j,1,jj],[0,0,0])*Wigner3j([j,1,jj],[mj,0,-mjj]);
end
end
% diag n plot eigenvalues associated with field strength a = dE/B
[state,energy] = eigs(H,1,'SA');
disp('Ground state energy:')
disp(energy)
disp('Ground state:')
disp(state)

end

```

Mathematica code for plotting:

```

(*dE/B=0 ----> |0,0> state*)

Energy1 = 0;

State1 = SphericalHarmonicY[0, 0, \[Theta], \[Phi]];

SphericalPlot3D[
Conjugate[State1]*State1, {\[Theta], 0, Pi}, {\[Phi], 0, 2 Pi},
PlotRange -> All]

(*dE/B=1 ----> get superposition state*)

Jmax2 = 4;

Base2 = Flatten[
Table[SphericalHarmonicY[J, mJ, \[Theta], \[Phi]], {J, 0,
Jmax2}, {mJ, -J, J}], 1];

Energy2 = -0.1577;

State2 = {-0.9644, 0.0000, -0.2634, -0.0000, -0.0000,
0.0000, -0.0222, -0.0000, -0.0000, 0.0000, 0.0000,
0.0000, -0.0009, -0.0000, 0.0000, 0.0000, 0.0000, -0.0000, -0.0000,
0.0000, -0.0000, -0.0000, 0.0000, -0.0000, 0.0000};

State2 = State2/Norm[State2];

wfn2 = Dot[Base2, State2]

-0.27206 - 0.128702 Cos[\[Theta]] -
0.0070019 (-1 + 3 Cos[\[Theta]]^2) -
0.000335869 (-3 Cos[\[Theta]] + 5 Cos[\[Theta]]^3)

SphericalPlot3D[
Conjugate[wfn2]*wfn2, {\[Theta], 0, Pi}, {\[Phi], 0, 2 Pi},
AspectRatio -> Full]

(*dE/B=10 ----> get superposition state*)

Jmax3 = 4;

Base3 = Flatten[
Table[SphericalHarmonicY[J, mJ, \[Theta], \[Phi]], {J, 0,
Jmax2}, {mJ, -J, J}], 1];

Energy3 = -6.0448;

```

```

State3 = {-0.6477, 0.0000, -0.6782, -0.0000,
0.0000, -0.0000, -0.3323, -0.0000, 0.0000, -0.0000, 0.0000,
0.0000, -0.0987, -0.0000, -0.0000, 0.0000, 0.0000, -0.0000,
0.0000, -0.0191, -0.0000, 0.0000, -0.0000, 0.0000};

State3 = State3/Norm[State3];

wfn3 = Dot[Base3, State3]

-0.182713 - 0.33137 Cos[Theta] -
0.104805 (-1 + 3 Cos[Theta]^2) -
0.0368325 (-3 Cos[Theta] + 5 Cos[Theta]^3) -
0.0020205 (3 - 30 Cos[Theta]^2 + 35 Cos[Theta]^4)

SphericalPlot3D[
Conjugate[wfn3]*wfn3, {Theta, 0, Pi}, {Phi, 0, 2 Pi},
PlotRange -> All, AspectRatio -> Full]

```

3. The Stark Effect in Hydrogen.

- (a) **Stark quenching of the 2S state.** In hydrogen, the 2S state is metastable. In the absence of external electric fields, its lifetime is 1/8 seconds. When an external electric field is applied, the 2S becomes mixed with the 2P state, which is strongly coupled to the ground state. The 2P state lifetime is only 1.6 ns. Depending on the strength of the electric field, the lifetime of the 2S state can be shortened by many orders of magnitude. This process is known as “quenching.” To see how this works, we will look at how the amplitude $a(t)$ of $|a\rangle$ (2S state) evolves over time in the presence of a DC Stark perturbation with matrix element $\hbar V = \langle b | e \mathbf{E} \cdot \mathbf{r} | a \rangle$ where $|b\rangle$ stands for the 2P state.

Assuming that the atom is initially in the 2S state, i.e., $a(0) = 1, b(0) = 0$. Working in the interaction picture, we can derive the following differential equations for $a(t)$ and $b(t)$:

$$\begin{aligned}
 i\dot{a} &= V^* e^{i\omega_0 t} b - i \frac{\Gamma_a}{2} a \\
 i\dot{b} &= V e^{-i\omega_0 t} a - i \frac{\Gamma_b}{2} b
 \end{aligned}$$

where $\Gamma_a = 8 \text{ s}^{-1}$ and $\Gamma_b = 6.3 \times 10^8 \text{ s}^{-1}$. Here, $\hbar\omega_0$ is the energy difference $E_a - E_b$. To solve these equations, we make the following ansatz

$$\begin{aligned}
 a(t) &= a_1 e^{-\mu_1 t} + a_2 e^{-\mu_2 t} \\
 b(t) &= b_1 e^{-(\mu_1 + i\omega_0)t} + b_2 e^{-(\mu_2 + i\omega_0)t}
 \end{aligned}$$

where a_1, a_2, b_1, b_2 are constants. Applying the initial condition $a(0) = 1, b(0) = 0$ we find that

$$a_1 + a_2 = 1 \quad b_1 + b_2 = 0.$$

With this, we may write our ansatz as

$$\begin{aligned}
 a(t) &= a_1 e^{-\mu_1 t} + (1 - a_1) e^{-\mu_2 t} \\
 b(t) &= b_1 e^{-(\mu_1 + i\omega_0)t} - b_1 e^{-(\mu_2 + i\omega_0)t}
 \end{aligned}$$

From this point, we may proceed using Mathematica. Plugging this ansatz into the system of differential equations above and set $t = 0$, we can solve for μ_1 and μ_2 in terms of a_1, b_1 . The result is

$$\mu_1 = \frac{\Gamma_a}{2} - \frac{i(a_1 - 1)V}{b_1} \quad \mu_2 = \frac{\Gamma_a}{2} - \frac{ia_1 V}{b_1}$$

It remains to find a_1, b_1 . To this end, we pick $t = 1/V$. By writing μ_1, μ_2 in terms of a_1, b_1 , we can solve for a_1, b_1 . The result is

$$a_1 = \frac{1}{2} + \frac{i(\Gamma_a - \Gamma_b) - 2\omega_0}{2\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)}}$$

$$b_1 = -\frac{2V}{\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)}}$$

Note that there is another solution (a_1, b_1) , but it doesn't matter all that much what we pick to compute μ_1, μ_2 since the result is the same. So we will pick the solution above. We should simplify this even more by writing the denominator as

$$\begin{aligned}\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)} &= \sqrt{16V^2 - ((\Gamma_a - \Gamma_b) + 2i\omega_0)^2} \\ &= \sqrt{16V^2 - (\Gamma_a - \Gamma_b)^2 + 4\omega_0^2 - 4i\omega_0(\Gamma_a - \Gamma_b)}.\end{aligned}$$

Now recall that for $x, y \in \mathbb{R}$,

$$\sqrt{x + iy} = \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} + i \operatorname{sgn}(y) \sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}.$$

Let $x = 16V^2 - (\Gamma_a - \Gamma_b)^2 + 4\omega_0^2$ and $y = -4\omega_0(\Gamma_a - \Gamma_b) > 0$, then we have

$$\begin{aligned}\sqrt{-(\Gamma_a - \Gamma_b - 4V + 2i\omega_0)(\Gamma_a - \Gamma_b + 4V + 2i\omega_0)} &= \sqrt{x + iy} \\ &= \sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} + i \sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}}.\end{aligned}$$

From here we can calculate μ_1, μ_2 :

$$\begin{aligned}\mu_1 &= \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{i}{4}\sqrt{x + iy} - \frac{i\omega_0}{2} \\ \mu_2 &= \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} - \frac{i}{4}\sqrt{x + iy} - \frac{i\omega_0}{2}\end{aligned}$$

From here, we can find the real and imaginary parts of μ_1, μ_2 :

$$\begin{aligned}\operatorname{Re}(\mu_1) &= \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} - \frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} \\ \operatorname{Im}(\mu_1) &= \frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} - \frac{\omega_0}{2} \\ \operatorname{Re}(\mu_2) &= \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} - x}{2}} \\ \operatorname{Im}(\mu_2) &= -\frac{1}{4}\sqrt{\frac{\sqrt{x^2 + y^2} + x}{2}} - \frac{\omega_0}{2}\end{aligned}$$

Here, the real parts of μ_1, μ_2 give the decay rate of the 2S state, and the imaginary parts tell us the level shifts. With these expressions (for $\mu_1, \mu_2, a_1, a_2, b_1, b_2$) in terms of the known constants, we can write down the full solution for $a(t)$. While it is possible, I won't do that here since this is simply plugging things into the ansatz.

In the small V limit, we may Taylor expand these expressions about $V = 0$ to second order in V to find

$$\begin{aligned}\text{Re}(\mu_1) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{\Gamma_a}{4} - \frac{\Gamma_b}{4} - \frac{2(\Gamma_b - \Gamma_a)}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2} V^2 \\ \text{Im}(\mu_1) &\approx \frac{\omega_0}{2} + \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2} V^2 - \frac{\omega_0}{2} = \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2} V^2 \\ \text{Re}(\mu_2) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} - \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} + \frac{2(\Gamma_b - \Gamma_a)}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2} V^2 \\ \text{Im}(\mu_2) &\approx -\frac{\omega_0}{2} - \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2} V^2 - \frac{\omega_0}{2} = -\omega_0 + \frac{4\omega_0}{(\Gamma_b - \Gamma_a)^2 + 4\omega_0^2} V^2.\end{aligned}$$

Now let us make various assumptions to simplify things. Let us assume that $\Gamma_a \ll \Gamma_b$ and $\Gamma_a \ll \mu_1, \mu_2$. From these, we have

$$\begin{aligned}\text{Re}(\mu_1) &\approx -\frac{\Gamma_b/2}{(\Gamma_b/2)^2 + \omega_0^2} V^2 \\ \text{Im}(\mu_1) &\approx \frac{\omega_0}{(\Gamma_b/2)^2 + \omega_0^2} V^2 \\ \text{Re}(\mu_2) &\approx \frac{\Gamma_b}{2} + \frac{(\Gamma_b/2)}{(\Gamma_b/2)^2 + \omega_0^2} V^2 \\ \text{Im}(\mu_2) &\approx -\omega_0 + \frac{\omega_0}{(\Gamma_b/2)^2 + \omega_0^2} V^2\end{aligned}$$

On the other hand, we have in the large V limit, $x \rightarrow 16V^2$ and $x \gg y$, and so

$$\begin{aligned}\text{Re}(\mu_1) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} \approx \frac{\Gamma_b}{4} \\ \text{Im}(\mu_1) &\approx V - \frac{\omega_0}{2} \\ \text{Re}(\mu_2) &\approx \frac{\Gamma_a}{4} + \frac{\Gamma_b}{4} \approx \frac{\Gamma_b}{4} \\ \text{Im}(\mu_2) &\approx -V - \frac{\omega_0}{2}\end{aligned}$$

We see that these results agree well with perturbation theory results for Stark shift energies.

- (b) **Effect of the Lamb shift on quenching.** Here we want to find the electric field in V/cm for which the 2S state lifetime is equal to 1 μs for two cases:

- **Case 1:** Weak coupling limit ($V^2 \ll \omega_0^2$) assuming that ω_0 is much smaller than the actual 2P linewidth Γ_b , so $\omega_0 \ll \Gamma_b$. First, we can identify from the solution above that the decay rate of the 2S state in the weak field limit is given by

$$\gamma_{2S} = 2 \times \frac{\Gamma_b/2}{(\Gamma_b/2)^2 + \omega_0^2} V^2 \rightarrow \frac{4V^2}{\Gamma_b}$$

where the factor of 2 comes from the fact that $|a(t)|^2$ and $|b(t)|^2$ are the relevant quantities for population. The lifetime is therefore

$$\tau_{2S} = \frac{\Gamma_b}{4V^2}.$$

Since $V = 3ea_0\mathcal{E}/\hbar$, we can set τ_{2S} to 1 μs to find that, the desired electric field strength is

$$\mathcal{E} = \sqrt{\frac{\Gamma_b}{4(1\mu\text{s})(3ea_0/\hbar)^2}} \approx \boxed{0.5203 \text{ V/cm}}$$

- **Case 2:** Weak field limit as above but with $\omega_0 = \omega_{\text{Lamb}} = 2\pi \times 1057.864$ MHz. In this case, we find that

$$\tau_{2S} = \frac{(\Gamma_b/2)^2 + \omega_0^2}{\Gamma_b V^2} \Rightarrow \mathcal{E} = \sqrt{\frac{(\Gamma_b/2)^2 + \omega_0^2}{\Gamma_b (1 \mu\text{s}) (3a_0 e/\hbar)^2}} \approx \boxed{1.82 \text{ V/cm}}$$

Including the Lamb shift increases the electric field required by a factor of ~ 3.5 on this time scale.

Mathematica code:

```
(*Problem 3: Stark stuff*)

In[1]:= a[t_] := a1*Exp[-m1*t] + (1 - a1)*Exp[-m2*t];
In[2]:= b[t_] := b1*Exp[-(m1 + I*w0)*t] - b1*Exp[-(m2 + I*w0)*t];

In[3]:= FullSimplify[
I*a'[t] == V*Exp[I*w0*t]*b[t] - I*Ga/2*a[t]] /. {t -> 0}

Out[3]= I (a1 (Ga - 2 m1) + 2 I b1 V) -
I ((-1 + a1) (Ga - 2 m2) + 2 I b1 V) == 0

In[4]:= FullSimplify[
I*b'[t] == V*Exp[-I*w0*t]*a[t] - I*Gb/2*b[t]] /. {t -> 0}

Out[4]= I b1 (Gb - 2 m1) + 2 (-1 + a1) V - 2 a1 V -
I b1 (Gb - 2 (m2 + I w0)) + 2 b1 w0 == 0

(*Solve for m1, m2*)

In[5]:= Solve[{I (a1 (Ga - 2 m1) + 2 I b1 V) -
I ((-1 + a1) (Ga - 2 m2) + 2 I b1 V) == 0,
I b1 (Gb - 2 m1) + 2 (-1 + a1) V - 2 a1 V -
I b1 (Gb - 2 (m2 + I w0)) + 2 b1 w0 == 0}, {m1,
m2}] // FullSimplify

Out[5]= {{m1 -> Ga/2 - (I (-1 + a1) V)/b1, m2 -> Ga/2 - (I a1 V)/b1}}

(*Pick t=1/V, solve for a1, b1*)

In[6]:= eqn1 =
I*a'[t] == V*Exp[I*w0*t]*b[t] - I*Ga/2*a[t] /. {t -> 1/V,
m1 -> Ga/2 - (I (-1 + a1) V)/b1, m2 -> Ga/2 - (I a1 V)/b1} //
FullSimplify

Out[6]= (((-1 + a1) a1 + b1^2) E^((I (-1 + a1))/b1 - Ga/(
2 V)) (-1 + E^(I/b1)) V)/b1 == 0

In[8]:= eqn2 =
I*b'[t] == V*Exp[-I*w0*t]*a[t] - I*Gb/2*b[t] /. {t -> 1/V,
m1 -> Ga/2 - (I (-1 + a1) V)/b1, m2 -> Ga/2 - (I a1 V)/b1} //
FullSimplify

Out[8]= E^((I (-1 + a1))/b1 - (Ga + 2 I w0)/(
2 V)) (-1 + E^(I/b1)) (2 (-1 + 2 a1) V +
I b1 (Ga - Gb + 2 I w0)) == 0

(*Solve for a1, b1*)
In[9]:= Solve[{eqn1, eqn2}, {a1, b1}] // FullSimplify

Out[9]= {{a1 -> (
I Ga - I Gb +
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] -
2 w0)/(2 Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V +
2 I w0))]),
b1 -> -((2 V)/
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V +
2 I w0))])}, {a1 -> (-I Ga + I Gb +
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] +
2 w0)/(2 Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V +
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2 I w0))]],
b1 -> (2 V)/
Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))]]}

In[135]:= (*Find mu1*)

In[30]:= Ga/2 - (I (-1 + a1) V)/b1 // Expand

Out[30]= Ga/4 + Gb/4 -
1/4 I Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] - (
I w0)/2

In[13]:= (*Find mu2*)

In[31]:= Ga/2 - (I a1 V)/b1 // Expand

Out[31]= Ga/4 + Gb/4 +
1/4 I Sqrt[-((Ga - Gb - 4 V + 2 I w0) (Ga - Gb + 4 V + 2 I w0))] - (
I w0)/2

(*Weak field limit --> get V^2 scaling*)

In[26]:= Series[
Sqrt[(Sqrt[x^2 + y^2] - x)/2], {V, 0, 2}] // FullSimplify

Out[26]= SeriesData[V, 0, {
2^Rational[-1, 2] (
Gab^2 - 4 w0^2 + ((Gab^2 + 4 w0^2)^2)^Rational[1, 2])^Rational[
1, 2], 0, (-4)
2^Rational[1, 2] ((Gab^2 + 4 w0^2)^2)^Rational[-1, 2] (
Gab^2 - 4 w0^2 + ((Gab^2 + 4 w0^2)^2)^Rational[1, 2])^Rational[
1, 2]}, 0, 3, 1]

In[146]:=
Series[Sqrt[(Sqrt[x^2 + y^2] + x)/2], {V, 0, 2}] // FullSimplify

Out[146]= SeriesData[V, 0, {
2^Rational[-1,
2] (-Gab^2 + 4 w0^2 + ((Gab^2 + 4 w0^2)^2)^Rational[
1, 2])^Rational[1, 2],
0, (4 ((Gab^2 + 4 w0^2)^2)^Rational[-1, 2]) ((-2)
Gab^2 + 8 w0^2 + 2 ((Gab^2 + 4 w0^2)^2)^Rational[
1, 2])^Rational[1, 2]}, 0, 3, 1]

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