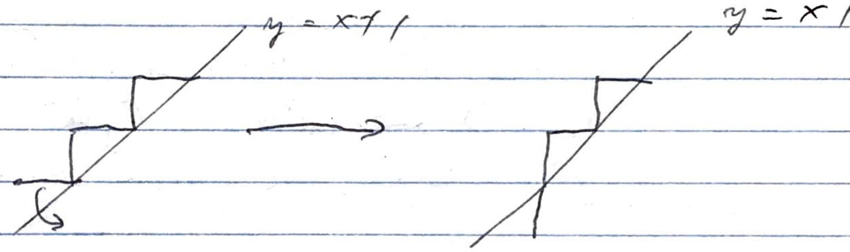
Huan Rui (1)

MA355: Due March 5, 2021

(51) (a) There paths ~~that~~ go outside the triangle, so there's always a point on the path that's above the line $y=x$. So the path touches/crosses $y=x+1$.

(b) Every path from $(-1,1) \to (n,n)$ can be bijectively mapped to a path from $(0,0) \to (n,n)$ that touches/crosses $y=x+1$ by reflecting the part of the path before the 1st time it touches $y=x+1$ around $y=x+1$.

E.g.



This is a bijection simply because the reflection is bijective...

(c) $\binom{n+n}{n} - \binom{n+n}{n-1} = \boxed{\binom{2n}{n} - \binom{2n}{n-1}}$

\# total   \# cross/touch

(52) $\frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!} = \binom{2n}{n}\left[1 - \frac{n}{n+1}\right] = \boxed{\frac{1}{n+1}\binom{2n}{n}}$

(a) Because those k steps don't end below the y-coord of the first point, so still a Dyck path.

(b) $\boxed{0}$ It's easy then to see that if the last point by y-coord of 1, then we have a Dyck path from $(0,0) \to (2n,0) \Rightarrow$ Catalan path.

(c) $\boxed{n+1}$ since i goes from $0 \to n$.

$\boxed{B_0} \to$ set of lattice paths w/ 0 upsteps following the last abs minimum $\to \boxed{\{\ \}}$

(d) Each Catalan path has exactly $\boxed{n}$ upsteps.

(e) Fix i. Notice that when $FUB \to BUF$, this BUF path must attain absolute minimum at B-U. This is because no absolute minimum can be created after B-U by F. (otherwise F will have to cross the x-axis in FUB) $\Rightarrow$ Since F has $i-1$ upsteps, the # of upsteps following the abs minimum BUF is $i-1+1 = i$

$\Rightarrow$ BUF belongs to B_i.

Now this is a bijection because we can simply reverse this process and obtain FUB again: (by locating the absolute minimum, etc.).

$\to$ $|B_i| = |\text{Catalan Paths}|$ ~~for i [1, n+1]~~ $(i = 1, 2, .., n)$

(f) By quotient principle:

size of each block

~~thus~~ $\frac{1}{n+1}\binom{2n}{n} = |B_i| = \#$ catalan paths.

\# blocks B_i

\# ~~blocks of total #~~ ~~of Catalan paths~~

\# total number of possible diagonal paths

55 $\displaystyle\sum_{i=1}^{10}\binom{10}{i}3^i = \sum_{i=0}^{10}\binom{10}{i}3^i - 1$

$\displaystyle = (1+3)^{10} - 1 = \boxed{4^{10}-1}$

56 $\displaystyle\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots\pm\binom{n}{n}$

$\displaystyle = \sum_{i=0}^{n}\binom{n}{i}(-1)^i = (1-1)^n = \boxed{0}$

$\boxed{\text{Sup \#5}}$ This is the same problem as counting Catalan paths (up/down).

Let (be $\rightarrow$
) be $\uparrow$

The total we want is just the total # paths from $(0,0) \rightarrow (n,n)$ without crossing $y=x$.

$\displaystyle \rightarrow \# = \boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$

58 $\boxed{\text{TRUE}}$

When n even, we can do the following...

sets of size $2k$ is (even-sized)

$\displaystyle\sum_{k=0}^{n/2}\binom{n}{2k} = \binom{n}{0}+\binom{n}{2}+\cdots+\binom{n}{n}$

According to (56)

$\displaystyle \rightarrow \sum_{k=0}^{n/2}\binom{n}{2k} = \binom{n}{1}+\binom{n}{3}+\cdots+\binom{n}{n-1} = \sum_{k=0}^{n/2}\binom{n}{2k+1},$

which # odd-sized sets. So we're done. $\square$

(59) Claim: This is $\boxed{n 2^{n-1}}$

$\hookrightarrow \frac{d}{dx} \cancel{x} (1+x)^{n}$

$= n(1+x)^{n-1} = \frac{d}{dx} \sum_{i=0}^{n} x^i \binom{n}{i}$

$= \sum_{i=0}^{n} i x \binom{n}{i}$

Now, set $x=1$, then $n \cdot 2^{n-1} = \sum_{i=0}^{n} i \binom{n}{i}$ ✓

(63) ~~By induction.. The base case is the regular "special" pigeon hole principle. So assume that~~

By contradiction... Assume that all blocks have fewer than $k+1$ elements, then

$\#$ elements ~~per~~ in box $i < k+1$

$\Rightarrow \#$ elements ~~per~~ in box $i \leq k$

$\Rightarrow$ Total $= \sum_{i=1}^{n} \#$ elements in box $i \leq nk$.

But the total is $> nk$. So this is a contradiction.

Supp #8

$\binom{n}{k}\binom{n-k}{m} =$ choose k from n
then choose m from left over: $n-k$.
~~[illegible]~~

(1) which is the same as choosing m from n, then k from the rest...

(2) $\binom{n}{k}\binom{n-k}{m} = \frac{n!}{k!\cancel{(n-k)!}} \frac{\cancel{(n-k)!}}{m!(n-k-m)!} = \frac{n!\,(n-m)!}{k!m!(n-m)!(n-m-k)!}$

$= \binom{n}{m}\binom{n-m}{k}$ ✓