Physics 8.321, Fall 2021
Homework #1

Due **Wednesday, September 22** by 8:00 PM.

The operator measuring the spin of a spin-1/2 particle along the axis parallel to a general unit vector $\widehat{\mathbf{n}}$ is given by

$$S_{\mathbf{n}} = \mathbf{S} \cdot \widehat{\mathbf{n}}$$

where $S_i = \sigma_i \, \hbar/2$ for $i = 1, 2, 3$, and

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These operators are used in problems 1-5.

You may find it helpful to use the result mentioned in class that when an operator O is measured and the (normalized) initial state is the ket/column vector $|i\rangle$, the probability that the final state is $|f\rangle$ is just $|\langle f|i\rangle|^2$, where $\langle f|$ is the bra/row vector (*dual vector*) formed by the adjoint/transpose conjugate of $|f\rangle$, when $|f\rangle$ is a (normalized) eigenstate of O, and there are no eigenvalue degeneracies. (This is just a convenient way of picking out the coefficient α of $|f\rangle$ when writing $|i\rangle$ in a basis of eigenstates of O.)

1. (a) Measurement of an electron's spin along the z-axis (S_z) using a Stern-Gerlach apparatus gives the eigenvalue $\hbar/2$. What is the probability that a subsequent measurement of the spin in the direction $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ yields $\hbar/2$?

(b) Measurement of an electron's spin along the axis $\widehat{\mathbf{n}}$ gives the eigenvalue $\hbar/2$. What is the probability that a subsequent measurement of the spin along the z-axis yields $\hbar/2$?

Answer:

As above, the probability can always be expressed as $|\langle f|i\rangle|^2$, where $|i\rangle$ is the initial state, and $|f\rangle$ the final state, where the final state is an eigenstate of the operator being measured. In this case the operator is

$$\begin{aligned} S_{\mathbf{n}} &= \mathbf{S} \cdot \widehat{\mathbf{n}} \\ &= \frac{\hbar}{2}\left(\sin\theta\cos\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\ &= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \end{aligned}$$

with eigenvalues and eigenvectors:

$$\frac{\hbar}{2}, \qquad \chi_+ = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix}$$

$$-\frac{\hbar}{2}, \qquad \chi_- = \begin{pmatrix} \sin\frac{\theta}{2} e^{-i\frac{\phi}{2}} \\ -\cos\frac{\theta}{2} e^{i\frac{\phi}{2}} \end{pmatrix}$$

The answers to the questions raised are

(a) $|i\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|f\rangle = \chi_+$, prob $= \cos^2 \frac{\theta}{2}$.

(b) $|i\rangle = \chi_+$, $|f\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, prob $= \cos^2 \frac{\theta}{2}$.

The general form of these answers should not be surprising and might be anticipated without doing any algebra: $\cos^2 f(\theta)$. The subtlety here is that $f(\theta)$ is $\frac{\theta}{2}$, rather than simply θ; this is closely related to the fact that, for electrons, the 360° rotated state differs from the original wave function by a minus sign.

2. The *expectation value* of an operator O in a state $|s\rangle$ is $\langle O\rangle = \langle s|O|s\rangle$. If $|\lambda_i\rangle$ is a basis of (normalized) eigenvectors of O with eigenvalues λ_i, then if $|s\rangle = \sum_i c_i|\lambda_i\rangle$ then $\langle O\rangle = \sum_i |c_i|^2\lambda_i$, i.e. the probabilistically weighted average of the measured values.

Show that it is impossible for an electron to be in a state such that

$$\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = 0\,.$$

Answer:

Assume the electron is in the state $|\alpha\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, where $|a|^2 + |b|^2 = 1$. Then

$$S_x|\alpha\rangle = \frac{\hbar}{2}\begin{pmatrix} b \\ a \end{pmatrix} \Rightarrow a^*b + b^*a = 0 \tag{1}$$

$$S_y|\alpha\rangle = i\frac{\hbar}{2}\begin{pmatrix} -b \\ a \end{pmatrix} \Rightarrow -a^*b + b^*a = 0 \tag{2}$$

$$S_z|\alpha\rangle = \frac{\hbar}{2}\begin{pmatrix} a \\ -b \end{pmatrix} \Rightarrow |a|^2 - |b|^2 = 0 \tag{3}$$

The only solution for (1), (2), and (3) is $a = b = 0$, which is not a physical state.

3. A beam produced by a Stern-Gerlach filter contains electrons that are all in the same spin state, which can be written as

$$|\alpha\rangle = s_+|+\rangle + s_-|-\rangle$$

where $|+\rangle, |-\rangle$ are eigenstates of S_z with eigenvalues $\pm\hbar/2$.

Part of the beam is passed through an analyzer oriented in the z direction, giving

$$\langle S_z\rangle = 0\,.$$

The other part of the beam is passed through an analyzer oriented in the x direction, giving

$$\langle S_x\rangle = \hbar/4\,.$$

(a) Calculate $\langle S_y\rangle$.

(b) What are the possible directions along which the original Stern-Gerlach filter may have been oriented?

Answer: Similar to Prob(2), we have

$$|s_+|^2 + |s_-|^2 = 1 \tag{4}$$

$$|s_+|^2 - |s_-|^2 = 0 \tag{5}$$

$$\frac{\hbar}{2}\left(s_+^* s_- + s_-^* s_+\right) = \frac{\hbar}{4} \tag{6}$$

from (4), (5) $\Rightarrow s_+ = \frac{1}{\sqrt{2}}e^{i\phi_+}$, and $s_- = \frac{1}{\sqrt{2}}e^{i\phi_-}$. Substitute into (6) $\Rightarrow \cos(\phi_- - \phi_+) = 1/2 \Rightarrow \sin(\phi_- - \phi_+) = \pm\sqrt{3}/2$.

(a)

$$\begin{aligned}
\langle S_y \rangle &= i\frac{\hbar}{2}\left(-s_+^* s_- + s_-^* s_+\right) \\
&= \left(i\frac{\hbar}{2}(\frac{-1}{2})(e^{i(\phi_- - \phi_+)} - e^{-i(\phi_- - \phi_+)})\right) \\
&= \pm\frac{\sqrt{3}\hbar}{4}
\end{aligned}$$

(b)

$$\begin{pmatrix} s_+ \\ s_- \end{pmatrix} = e^{i\frac{\phi_+ + \phi_-}{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}e^{-i\frac{\phi_- - \phi_+}{2}} \\ \frac{1}{\sqrt{2}}e^{+i\frac{\phi_- - \phi_+}{2}} \end{pmatrix} \quad \text{compared to } \chi_+ = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\frac{\phi}{2}} \\ \sin\frac{\theta}{2}e^{i\frac{\phi}{2}} \end{pmatrix}$$

$$\Rightarrow \theta = \frac{\pi}{2}, \quad \phi = \pm\frac{\pi}{3}$$

You get the same result if you compare to χ_-.

4. [Sakurai and Napolitano Problem 1.19 (page 62); typo there corrected]

(a) Compute

$$\langle (\Delta S_x)^2 \rangle \cong \langle S_x^2 \rangle - \langle S_x \rangle^2 \,,$$

where the expectation value is taken for the S_z+ state. Using this result, check the generalized uncertainty relation

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle \geq \frac{1}{4} |\langle [A, B] \rangle|^2 \,,$$

with $A \to S_x, B \to S_y$, and where $[A, B] = AB - BA$.

(b) Check the uncertainty relation with $A \to S_x, B \to S_y$ for the S_x+ state.

Answer:

(a)

$$\langle S_x^2 \rangle = \left(\frac{\hbar}{2} \right)^2, \quad \langle S_x \rangle = 0 \Rightarrow \langle (\Delta S_x)^2 \rangle = \left(\frac{\hbar}{2} \right)^2.$$

also

$$\langle S_y^2 \rangle = \left(\frac{\hbar}{2} \right)^2, \quad \langle S_y \rangle = 0 \Rightarrow \langle (\Delta S_y)^2 \rangle = \left(\frac{\hbar}{2} \right)^2.$$

and

$$[S_x, S_y] = i\hbar S_z \Rightarrow \frac{1}{4} |i\hbar \langle S_z \rangle|^2 = \frac{1}{4} \hbar^2 \left(\frac{\hbar}{2} \right)^2$$

So the equality holds for $|+\rangle$ state ($\left(\frac{\hbar}{2}\right)^2 = \left(\frac{\hbar}{2}\right)^2$ in this case).

(b) Using $S_x +$ state $= \frac{1}{\sqrt{2}} (|+\rangle + |-\rangle)$, we get

$$\langle S_x^2 \rangle = \left(\frac{\hbar}{2} \right)^2, \quad \langle S_x \rangle = \frac{\hbar}{2} \Rightarrow \langle (\Delta S_x)^2 \rangle = 0. \quad \text{There is no uncertainty here.}$$

and

$$\langle S_z \rangle = 0$$

So the equality holds here. ($0 = 0$ in this case).

5. [Sakurai and Napolitano Problem 1.20 (page 62)]

Find the (normalized) linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle \,.$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

Answer: Note for *any* normalized state (linear combination of $|+\rangle$ and $|-\rangle$), it is always true that $\langle S_x^2 \rangle = \langle S_y^2 \rangle = \left(\frac{\hbar}{2}\right)^2$. Also, $|+\rangle$ or $|-\rangle$ (or their rays) has $\langle S_x \rangle = \langle S_y \rangle = 0$, so this maximizes $\langle (\Delta S_x)^2 \rangle$ and $\langle (\Delta S_y)^2 \rangle$. What we are looking for therefore is nothing but $|+\rangle$ or $|-\rangle$ (or their rays), and is what we did in Prob 4(a).

6. Prove that the equation $AB - BA = \mathbb{1}$ cannot be satisfied by any finite-dimensional matrices A, B.

Answer: Proof by contradiction. Assume

$$AB - BA = \mathbb{1}$$

Take the trace on both sides, Tr $(AB-BA) = \text{Tr}\,(AB) - \text{Tr}\,(BA) = 0$, while Tr $\mathbb{1} =$ dimension $\neq 0$.

This means that for finite-dimensional matrices, $AB - BA$ is equal to a traceless matrix, and cannot be the identity.

7. (a) Consider two operators A, B that do not necessarily commute. Show that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\}$$

where

$$A^0\{B\} = B, \quad A^1\{B\} = [A, B], \quad A^2\{B\} = [A, [A, B]], \text{ etc.}$$

Hint: treat $e^A = 1 + A + A^2/2 + \cdots$ as a formal power series.

(b) Let $A(x)$ be an operator that depends on a continuous parameter x. Derive the following identity

$$e^{-iA(x)} \frac{d}{dx} e^{iA(x)} = i \sum_{n=0}^{\infty} \frac{(-i)^n}{(n+1)!} A^n\{\frac{dA}{dx}\}\,.$$

Answer:

(a) We introduce a dummy variable, a c-number, t, and let

$$F(t) \equiv e^{At} B e^{-At}$$
$$G(t) \equiv \textstyle\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n\{B\}$$

then we can check that

$$\tfrac{dF}{dt} = [A, F(t)]$$

$$\tfrac{dG}{dt} = [A, G(t)]$$

and $F(t=0) = G(t=0) = B$ (note we use $A^n\{B\} = A(A^{n-1}\{B\}) - (A^{n-1}\{B\})A$, from the definition). So F and G satisfy the same equation of motion (a 1-order ordinary differential equation) with the same initial condition, and we conclude $F(t) = G(t), \forall\, t$.

If we want to *derive,* rather than prove, then we can integrate $F(t)$ from the equation of motion:

$$F(t) = \int_0^t dt\,[A, F(t)] + F(t=0)$$

and proceed by expanding $F(t)$ in powers of t on both sides with some operator-valued coefficients O_n,

$$\begin{aligned} O_0 + O_1 t + ... + O_n t^n + ... &= \int_0^t dt\, [A, O_0 + O_1 t + ... + O_n t^n + ...] + F(t=0) \\ \Rightarrow O_0 + O_1 t + ... + O_n t^n + ... &= B + [A, O_0]t + [A, O_1]t^2/2 + ... + [A, O_{n-1}]t^n/n! \end{aligned} \tag{7}$$

We then inductively find that

$$O_0 = B, \quad O_1 = [A, B], \quad O_n = \frac{1}{n!} A^n\{B\}. \tag{8}$$

Alternative Solution: Alternatively we can compute the series directly and match the terms on the LHS to those on the RHS for all n. (This is the formal version of "matching the terms".) Notice that the RHS series has exactly n As appearing in the n^{th} term, so we will group the terms on the LHS in the same way.

$$\begin{aligned} e^A B e^{-A} &= \sum_{n_1=0}^{\infty} \frac{A^{n_1}}{(n_1)!} \cdot B \cdot \sum_{n_2=0}^{\infty} \frac{(-A)^{n_2}}{(n_2)!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{A^k B (-A)^{n-k}}{k!(n-k)!}. \end{aligned} \tag{9}$$

The proof will be done if we can show that for any $n \in \mathbb{N}$ we have

$$\sum_{k=0}^{n} \frac{A^k B (-A)^{n-k}}{k!(n-k)!} = \frac{A^n\{B\}}{n!}. \tag{10}$$

We will show this by induction. First we show that it is true for $n = 0$. The LHS is

$$\frac{A^0 B (-A)^{0-0}}{0!(0-0)!} = B, \tag{11}$$

and the LHS is

$$\frac{A^0\{B\}}{0!} = B. \tag{12}$$

So the formula we need is true for $n = 0$. We now assume that it is true up to n and show that this implies it is also true for $n + 1$. We start from the RHS

$$\begin{aligned} \frac{A^{n+1}\{B\}}{(n+1)!} &= \frac{1}{(n+1)!} \left(A \cdot A^n\{B\} - A^n\{B\} \cdot A \right) \\ &= \left(A \cdot \sum_{k=0}^{n} \frac{A^k B (-A)^{n-k}}{k!(n-k)!} - \sum_{k=0}^{n} \frac{A^k B (-A)^{n-k}}{k!(n-k)!} \cdot A \right) \cdot \frac{1}{n+1} \\ &= \sum_{k=0}^{n+1} \frac{A^k B (-A)^{n+1-k}}{k!(n+1-k)!}, \end{aligned} \tag{13}$$

finishing the proof. Note that one obtains the final line by adding the coefficients of the terms in the second line with k As in front:

$$\frac{1}{n+1}\left(\frac{(-1)^{n-k+1}}{(k-1)!(n-k+1)!}-\frac{(-1)^{n-k}}{(k)!(n-k)!}\right)=(-1)^{n+1-k}\frac{1}{n+1}\cdot\frac{n+1}{k!(n+1-k)!}=\frac{(-1)^{n+1-k}}{k!(n+1-k)!} \tag{14}$$

(b) Like in part (a), we can introduce a function

$$F(t)=e^{-iA(x)t}\frac{\partial}{\partial x}e^{iA(x)t} \tag{15}$$

Taking a derivative with respect to t, we find

$$\frac{\partial F(t)}{\partial t}=[-iA(x),F(t)]+i\frac{dA}{dx} \tag{16}$$

Now if we again expand

$$F(t)=O_0+O_1t+...+O_nt^n+..., \tag{17}$$

we can formally integrate to get

$$O_0+O_1t+...+O_nt^n+...=F(t=0)+\int_{t=0}^{t}dt[-iA(x),O_0+O_1t+...+O_nt^n+...]+i\int_{t=0}^{t}dt\frac{dA(x)}{dx} \tag{18}$$

Since $F(t=0)=0$, we find inductively that

$$O_0=0,\quad O_1=i\frac{\partial A(x)}{\partial x},\quad O_2=\frac{1}{2}[-iA(x),i\frac{\partial A(x)}{\partial x}],\quad O_n=\frac{1}{n!}(-i)^{n-1}iA^{n-1}\{\frac{dA(x)}{dx}\} \tag{19}$$

Hence,

$$e^{-iA(x)t}\frac{\partial}{\partial x}e^{iA(x)t}=i\sum_{n=1}^{\infty}\frac{(-i)^{n-1}}{n!}A^{n-1}\{\frac{dA}{dx}\}=i\sum_{n=0}^{\infty}\frac{(-i)^n}{(n+1)!}A^n\{\frac{dA}{dx}\} \tag{20}$$