1. Consider a quantum gate on two qubits that applies a σ_x to the second qubit if the first qubit is $|0\rangle$ and a σ_z to the second qubit if the first qubit is $|1\rangle$.

   (a) Write down a 4×4 matrix for the action of this gate.

   (b) Show how to build a quantum circuit for this gate using CNOT gates and one-qubit gates.

2. Suppose that we have three parties, Alice, Bob, and Charlie. Alice and Bob share a pair of qubits in the Bell state

$$\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)$$

   Bob and Charlie also share a pair of qubits in that Bell state. (So between all three of them, they hold two Bell states consisting of four qubits total.)

   Alice, Bob, and Charlie would like to perform some operations so that Alice and Charlie end up with a pair of qubits in this Bell state. Can they? If so, how? (They are allowed to measure any of their original qubits that they want.)

3. Alice has two classical bits, a_1 and a_2. Bob has two classical bits, b_1 and b_2. They would like to send Charlie two classical bits which are functions of their bits, namely $c_1 = a_1 \oplus b_1$ and $c_2 = a_2 \oplus b_2$, where $\oplus$ is the XOR operation. However, they have only very limited resources to do this. Alice and Charlie share a pair of entangled bits in the state

$$\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)$$

   They are allowed to send one qubit from Alice to Bob, and one qubit from Bob to Charlie (and none of them are allowed to send any classical bits). Can they accomplish their task? If so, how?

4. Recall that on an earlier homework, we considered a scenario where a challenger gave you an unknown quantum state which was either $|0\rangle$, $|+\rangle$, $|1\rangle$, $|-\rangle$, and challenged you to clone it. You give them back two qubits, and they measure both of them to check whether they were both the original state. You succeed if both the measurement results agree with the original state. There was a strategy that succeeded with probability $\frac{5}{8}$. In this problem, you will analyze a batter strategy, that succeeds with a larger probability.

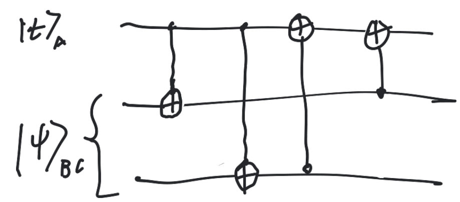
   Suppose we have a mapping B that embeds a single qubit into a subspace of a three-qubit system that behaves as follows (and takes superpositions of the inputs to superpositions of the outputs, as you would expect):

$$B\,|0\rangle = \frac{\sqrt{2}}{\sqrt{3}}\,|000\rangle + \frac{1}{\sqrt{6}}\,|011\rangle + \frac{1}{\sqrt{6}}\,|101\rangle$$
$$B\,|1\rangle = \frac{\sqrt{2}}{\sqrt{3}}\,|111\rangle + \frac{1}{\sqrt{6}}\,|100\rangle + \frac{1}{\sqrt{6}}\,|010\rangle$$

You take the qubit they give you and apply B, then give them the first two qubits of the result. What is the probability that you pass the chalenger's test?

In fact, this is the optimal cloning strategy ... you cannot succeed with a higher probability.

5. You can implement the strategy in problem 2 by taking the qubit $|t\rangle$ they gave you, appending some two-qubit state $|\psi\rangle$, and applying the following circuit with four CNOT gates:



Find the state $|\psi\rangle$ and show that this circuit gives the desired results.