Physics 8.321, Fall 2020
Final Exam

You have **3 hours** to complete the exam and a grace period of an additional 20 minutes to get your solutions uploaded to canvas. You may use your books and notes including the notes on canvas from the course, you may freely use any results derived in homework assignments this semester, and you may use symbolic manipulation tools like mathematica and matlab, but you may not consult other online resources, and you may not communicate with other people in any way while doing the final. You also may not communicate any information about the exam to anyone after you have completed it until the exam period is over at the end of the day on 12/15/20.

Note: you are expected to upload your completed exam to the canvas website immediately after completing the exam and within 200 minutes or less of downloading it. You have a few extra minutes in case of technical complications. The system will log your download and upload times. If for some reason you have difficulty uploading your exam after completion, please email it immediately to one of the course staff.

1. Particle in a linear symmetric potential (40 points)

Consider a particle moving in one dimension in the potential $V = A|x|$, where A is a constant. You may fix $m = \hbar = 1$.

(a) Estimate the ground state energy using the variational principle for the family of Gaussian wavefunctions $e^{-ax^2/2}$ parameterized by a, suitably normalized.

(b) Estimate the ground state energy using the WKB approximation and compare with your result from part (a).

Answer:

(a) Our trial wavefunction is $\psi = ce^{-ax^2/2}$, with a a free parameter, $c = (a/\pi)^{1/4}$. Integrating we get

$$\langle H\rangle_a = 2\int_0^\infty \psi^2(Ax + a/2 - a^2x^2/2) = \frac{A}{\sqrt{a\pi}} + a/4.$$

This is minimized at $a = a^* = (4A^2/\pi)^{1/3}$, giving

$$\langle H\rangle_{a^*} = \frac{3A^{2/3}}{2(2\pi)^{1/3}} \cong 0.8128A^{2/3}.$$

(b) This is a special case of the potential considered in homework for 6, problem 4c, with parameters $k = A, \alpha = 1$, for which we estimate a ground state energy

$$E_0 \cong (\frac{\pi/2}{4\sqrt{2\pi}/3\sqrt{\pi}})^{2/3}A^{2/3} \cong 0.8853A^{2/3}.$$

The scaling agrees and the coefficients are close!

2. Rotating wave trick for a time-dependent Hamiltonian (30 points)

Consider the system with *time-dependent* Hamiltonian

$$H(t) = \frac{\hbar}{2}\begin{bmatrix} \omega & \Omega e^{-i\nu t} \\ \Omega e^{i\nu t} & -\omega \end{bmatrix} \quad \text{where} \quad \omega, \Omega, \nu \in \mathbb{R}. \tag{1}$$

(a) Consider the unitary operator

$$U_R(t) = \begin{bmatrix} e^{iat} & 0 \\ 0 & e^{-iat} \end{bmatrix}, \tag{2}$$

with $a \in \mathbb{R}$. Show that $U_R(t)H(t)U_R^\dagger(t)$ is time independent for suitable choice of a. Find a. Henceforth, we are going to use this a in $U_R(t)$.

(b) Let $|\psi(t)\rangle$ be the solution of the time-dependent Schrodinger equation for the Hamiltonian $H(t)$ above. That is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{3}$$

Show that the state $U_R(t)|\psi(t)\rangle$ solves a Schrodinger equation with a *time-independent* Hamiltonian H_{eff}. Find H_{eff}.

(c) Use the results above to write a formal expression for $|\psi(t)\rangle$ for given initial state $|\psi(0)\rangle$ and in terms of H_{eff} and $U_R(t)$. You don't have to evaluate any matrix exponential explicitly.

(d) Now consider the situation $\omega = \nu$, and let the initial state be

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4}$$

Find the time-evolved state $|\psi(t)\rangle$. Are there time(s) $t = t_c$ for which collapsing to the initial state after measurement of the operator

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5}$$

is certain? If so, when does that happen (i.e. find t_c)?

Answer:

(a) Matrix multiplication simply gives

$$\begin{aligned} U_R(t)H(t)U_R^\dagger(t) &= \frac{\hbar}{2}\begin{bmatrix} e^{iat} & 0 \\ 0 & e^{-iat} \end{bmatrix}\begin{bmatrix} \omega & \Omega e^{-i\nu t} \\ \Omega e^{i\nu t} & \omega \end{bmatrix}\begin{bmatrix} e^{-iat} & 0 \\ 0 & e^{iat} \end{bmatrix} \\ &= \frac{\hbar}{2}\begin{bmatrix} \omega & \Omega e^{i(2a-\nu)t} \\ \Omega e^{-i(2a-\nu)t} & -\omega \end{bmatrix}. \end{aligned} \tag{6}$$

From this it is apparent that $U_R(t)H(t)U_R^\dagger(t)$ time-independent if and only if

$$a = \frac{\nu}{2}. \tag{7}$$

In this case we have

$$U_R(t) = \begin{bmatrix} e^{i\frac{\nu}{2}t} & 0 \\ 0 & e^{-i\frac{\nu}{2}t} \end{bmatrix} = \exp\left(i\frac{\nu}{2}\sigma_z t\right), \tag{8}$$

where σ_z is the Pauli matrix for the z-direction, and

$$U_R(t)H(t)U_R^\dagger(t) = \frac{\hbar}{2}\begin{bmatrix} \omega & \Omega \\ \Omega & -\omega \end{bmatrix}. \tag{9}$$

Henceforth we are going to plug $a = \nu/2$, as the question instructs us to do. Therefore we are going to use the matrices above for $U_R(t)$ and $U_R(t)H(t)U_R^\dagger(t)$.

(b) Just apply $i\hbar\frac{\partial}{\partial t}$ to $U_R(t)|\psi(t)\rangle$ and observe,

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left[U_R(t)|\psi(t)\rangle\right] &= i\hbar\frac{\partial U_R(t)}{\partial t}|\psi(t)\rangle + i\hbar U_R(t)\frac{\partial}{\partial t}|\psi(t)\rangle \\ &= i\hbar\left(i\frac{\nu}{2}\sigma_z\right)U_R(t)|\psi(t)\rangle + U_R(t)H(t)|\psi(t)\rangle \\ &= -\frac{\hbar\nu}{2}\sigma_z U_R(t)|\psi(t)\rangle + U_R(t)H(t)U_R^\dagger(t)U_R(t)|\psi(t)\rangle \\ &= \left[-\frac{\hbar\nu}{2}\sigma_z + U_R(t)H(t)U_R^\dagger(t)\right]U_R(t)|\psi(t)\rangle. \end{aligned} \tag{10}$$

Note that in the second line we used the derivative of the matrix $U_R(t)$ from part above and the fact that the state $|\psi(t)\rangle$ satisfies the time-dependent Schrodinger equation; and in the third line we inserted the identity $U_R^\dagger(t)U_R(t) = 1$ appropriately. We see from the last line that the state $U_R(t)|\psi(t)\rangle$ solves the Schrodinger equation

$$i\hbar\frac{\partial}{\partial t}\left[U_R(t)|\psi(t)\rangle\right] = H_{eff}U_R(t)|\psi(t)\rangle, \tag{11}$$

with a *time-independent* Hamiltonian

$$H_{eff} = -\frac{\hbar\nu}{2}\sigma_z + U_R(t)H(t)U_R^\dagger(t) = \frac{\hbar}{2}\begin{bmatrix} \omega-\nu & \Omega \\ \Omega & -(\omega-\nu) \end{bmatrix}, \tag{12}$$

which we have denoted as H_{eff}.

(c) Now we can write the formal expression for $|\psi(t)\rangle$ for a given initial state $|\psi(0)\rangle$ and in terms of H_{eff} and $U_R(t)$. From the formal solution of the Schrodinger equation with a time-independent Hamiltonian we have,

$$U_R(t)|\psi(t)\rangle = \exp\left[-\frac{iH_{eff}t}{\hbar}\right]U_R(0)|\psi(0)\rangle = \exp\left[-\frac{iH_{eff}t}{\hbar}\right]|\psi(0)\rangle, \tag{13}$$

using $U_R(0) = 1$. Note that no time ordering is necessary since H_{eff} is time-independent. From here then it is easy to see

$$\begin{aligned} |\psi(t)\rangle &= U_R(t)^\dagger \exp\left[-\frac{iH_{eff}t}{\hbar}\right]|\psi(0)\rangle \\ &= \begin{bmatrix} e^{-i\frac{\nu}{2}t} & 0 \\ 0 & e^{i\frac{\nu}{2}t} \end{bmatrix}\exp\left(-\frac{i}{2}\begin{bmatrix} \omega-\nu & \Omega \\ \Omega & -(\omega-\nu) \end{bmatrix}t\right)|\psi(0)\rangle. \end{aligned} \tag{14}$$

One can evaluate this exponential further but it wasn't necessary. This is our formal solution, and you have just solved Schrodinger equation with a time-dependent Hamiltonian! The trick that introduces $U_R(t)$ and relating the original Schrodinger equation to one with a time-independent Hamiltonian as above is called the *rotating wave* trick. (The name is justified because we roughly considered a frame rotating with the Hamiltonian in time first, then went back to the original frame.)

(d) Now consider the situation $\omega = \nu$. First note that H_{eff} takes the form

$$H_{eff} = \frac{\hbar}{2}\begin{bmatrix} 0 & \Omega \\ \Omega & 0 \end{bmatrix} = \frac{\hbar\Omega}{2}\sigma_x, \tag{15}$$

where σ_x is the Pauli matrix for the x-direction. Note that the initial state given in the problem

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{16}$$

is the eigenstate of σ_x with eigenvalue 1. From this fact we easily see that the time evolved state $|\psi(t)\rangle$ is, from the formal solution above,

$$\begin{aligned} |\psi(t)\rangle &= U_R(t)^\dagger \exp\left[-\frac{i\Omega t}{2}\right]|\psi(0)\rangle \\ &= \frac{e^{-\frac{i\Omega t}{2}}}{\sqrt{2}}\begin{bmatrix} e^{-i\frac{\nu}{2}t} & 0 \\ 0 & e^{i\frac{\nu}{2}t} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{e^{-\frac{i\Omega t}{2}}}{\sqrt{2}}\begin{bmatrix} e^{-i\frac{\nu}{2}t} \\ e^{i\frac{\nu}{2}t} \end{bmatrix}. \end{aligned} \tag{17}$$

Clearly at the times

$$\nu t_c = 2\pi n \implies t_c = \frac{2\pi n}{\nu}, \quad n \in \mathbb{Z}_{\geq 0}, \tag{18}$$

time evolved state differs from the initial state by an unimportant global phase and collapsing to the initial state after the measurement of σ_x is certain by the initial state being an eigenstate of σ_x. This happens at times $t = t_c$ given above.

3. Quantum Rigid Rotor (30 points)

Consider a rigid rotor immersed in a magnetic field, with Hamiltonian

$$H = \frac{\mathbf{L}^2}{2I} + \omega_0 L_z. \tag{19}$$

Suppose the system is in an initial state $|\psi_0\rangle$ such that

$$\langle\theta, \phi|\psi_0\rangle = \sqrt{3/4\pi}\sin\theta\sin\phi. \tag{20}$$

(a) What values of L_z will be obtained if a measurement is carried out in the initial state, and with what probability do these values occur?

(b) What is $\langle\theta, \phi|\psi(t)\rangle$?

(c) What are $\langle L_x(t)\rangle, \langle L_y(t)\rangle, \langle L_y(t)^2\rangle, \langle \Delta L_y(t)\rangle$ in this state?

Answer:

(a) Note that

$$\langle \theta, \phi | \psi_0 \rangle = -\frac{1}{\sqrt{2}i}(Y_{1,1}(\theta,\phi) + Y_{1,-1}(\theta,\phi)) \tag{21}$$

where

$$Y_{l,m}(\theta,\phi) \equiv \langle \theta, \phi | l, m \rangle \tag{22}$$

are the spherical harmonics. So

$$|\psi_0\rangle = -\frac{1}{\sqrt{2}i}|1,1\rangle - \frac{1}{\sqrt{2}i}|1,-1\rangle . \tag{23}$$

In this state, the two possible values of L_z are $+1$ and -1, and both are obtained with equal probability.

(b) Since $|l,m\rangle$ is an eigenstate of the Hamiltonian with eigenvalue

$$E_{l,m} = \frac{\hbar^2 l(l+1)}{2I} + \omega_0 \hbar m, \tag{24}$$

the time-evolved state is given by

$$\langle \theta, \phi | \psi(t) \rangle = -\frac{1}{\sqrt{2}i}\alpha_+(t)\, Y_{1,1}(\theta,\phi) - \frac{1}{\sqrt{2}i}\alpha_-(t)\, Y_{1,-1}(\theta,\phi) \tag{25}$$

where

$$\alpha_\pm(t) = e^{-i(\hbar/I \pm \omega_0)t} \tag{26}$$

(c) L_x can be written in the θ, ϕ basis as

$$L_x = \frac{\hbar}{i}(-\sin\phi\,\frac{\partial}{\partial\theta} - \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi}) \tag{27}$$

Hence,

$$\begin{aligned}
\langle L_x(t)\rangle &= \frac{1}{2}\frac{3}{4\pi}\int d\theta \sin\theta\; d\phi\;\; \sin\theta(\alpha_+^* e^{-i\phi} + \alpha_-^* e^{i\phi}) \\
&\frac{\hbar}{i}(-\sin\phi\,\frac{\partial}{\partial\theta} - \cot\theta\,\cos\phi\,\frac{\partial}{\partial\phi})((\alpha_+ e^{i\phi} + \alpha_- e^{-i\phi})\sin\theta) \\
&= (-\frac{\hbar}{i})\frac{3}{4\pi}\int d\theta \sin^2\theta\cos\theta \int d\phi\;\; \sin\phi(\alpha_+^* e^{-i\phi} + \alpha_-^* e^{i\phi})(\alpha_+ e^{i\phi} + \alpha_- e^{-i\phi}) \\
&+(-\hbar)\frac{3}{4\pi}\int d\theta \sin^2\theta\cos\theta \int d\phi\;\; \cos\phi(\alpha_+^* e^{-i\phi} + \alpha_-^* e^{i\phi})(\alpha_+ e^{i\phi} - \alpha_- e^{-i\phi}) \\
&= 0
\end{aligned} \tag{28}$$

as in both cases, the integral over θ evaluates to zero.

We can similarly use the expression for L_y in the θ, ϕ basis,

$$L_y = \frac{\hbar}{i}\left(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}\right) \tag{29}$$

to find

$$\langle L_y\rangle = 0, \quad \langle L_y^2\rangle = \hbar^2\sin^2(\omega_0 t), \quad \langle\Delta L_y\rangle = \hbar\sin(\omega_0 t) \tag{30}$$

An alternative method for the computation is by noticing that we can equivalently use the spin-1 matrices

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix} \quad L_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix} \tag{31}$$

and $|\psi(t)\rangle$ expressed in the $|1,m\rangle$ basis, $|\psi(t)\rangle = -\frac{1}{\sqrt{2}i}\begin{pmatrix}\alpha_+\\0\\\alpha_-\end{pmatrix}$, to find the expectation values:

$$\langle L_x(t)\rangle = \frac{\hbar}{2\sqrt{2}}\begin{pmatrix}\alpha_+^* & 0 & \alpha_-^*\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\alpha_+\\0\\\alpha_-\end{pmatrix} = 0 \tag{32}$$

$$\langle L_y(t)\rangle = \frac{\hbar}{2\sqrt{2}i}\begin{pmatrix}\alpha_+^* & 0 & \alpha_-^*\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix}\alpha_+\\0\\\alpha_-\end{pmatrix} = 0 \tag{33}$$

$$\langle L_y(t)^2\rangle = -\frac{\hbar^2}{4}\begin{pmatrix}\alpha_+^* & 0 & \alpha_-^*\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}\begin{pmatrix}\alpha_+\\0\\\alpha_-\end{pmatrix} = \frac{1}{4}|\alpha_+ - \alpha_-|^2\hbar^2 = \hbar^2\sin^2(\omega_0 t) \tag{34}$$

$$\langle\Delta L_y\rangle = \hbar\sin(\omega_0 t) \tag{35}$$

4. Interacting spins in a magnetic field (30 points)

Consider the Hamiltonian

$$H = H_0 + BH_1 = -J\ \mathbf{S}^{(1)}\cdot\mathbf{S}^{(2)} - BS_z^{(1)}\,, \tag{36}$$

with $J > 0$.

(a) Solve for the spectrum exactly when $B = 0$, including multiplicities, when $S^{(1)}, S^{(2)}$ are each systems of general spin j_1, j_2.

(b) Find the corrections to the energy eigenvalues using perturbation theory at leading order in small B in the case $j_1 = j_2 = 1/2$.

(c) Solve exactly in the case $j_1 = j_2 = 1/2$ and check your result from part (b).

Answer:

(a) Suppose the two particles have eigenvalues $\hbar^2 j_1(j_1+1)$, $\hbar^2 j_2(j_2+1)$ for J_1^2 and J_2^2. Recall we can write

$$\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} = \frac{1}{2}((\mathbf{S}^{(1)} + \mathbf{S}^{(2)})^2 - (\mathbf{S}^{(1)})^2 - (\mathbf{S}^{(2)})^2) \tag{37}$$

The eigenstates are thus given by the eigenstates $|j, m\rangle$ for $(\mathbf{S}_1 + \mathbf{S}_2)^2$, with eigenvalues given by

$$\mathbf{S}_1 \cdot \mathbf{S}_2 \, |j, m\rangle = \frac{\hbar^2}{2}(j(j+1) - j_1(j_1+1) - j_2(j_2+1)) \, |j, m\rangle \tag{38}$$

For a given j_1 and j_2, j can take all possible integer values from $j_1 + j_2$ to $|j_1 - j_2$. For each value of j, m can take $2j+1$ values from j to $-j$. We can see that (38) is independent of m, so all $2j+1$ states $\mathbf{j}, \mathbf{m}$ for a given j are all degenerate.

(b) In the case $j_1 = j_2 = 1/2$, recall that the $|j, m\rangle$ states can be written in the $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$ basis (which we abbreviate below as $|m_1\rangle \otimes |m_2\rangle$) as

$$\begin{aligned} |1,1\rangle &= |1/2\rangle \otimes |1/2\rangle \\ |1,0\rangle &= \frac{1}{\sqrt{2}}(|-1/2\rangle \otimes |1/2\rangle + |1/2\rangle \otimes |-1/2\rangle) \\ |1,-1\rangle &= |-1/2\rangle \otimes |-1/2\rangle \end{aligned} \tag{39}$$

and

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|-1/2\rangle \otimes |1/2\rangle - |1/2\rangle \otimes |-1/2\rangle) \tag{40}$$

The shift in the energy of the $|0,0\rangle$ state can be found using first-order non-degenerate perturbation theory, to be

$$\Delta E_{0,0} = -B \, \langle 0,0|S_z^{(1)}|0,0\rangle = 0 \tag{41}$$

To find the shift in the remaining states, we need to find the matrix $-B \, \langle 1,m|S_z^{(1)}|1,m'\rangle$:

$$\hbar \begin{pmatrix} -B/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B/2 \end{pmatrix} \tag{42}$$

Since this matrix is already diagonal in the $|1,m\rangle$ basis, we simply find

$$\begin{aligned} \Delta E_{1,1} &= -\frac{\hbar B}{2} \\ \Delta E_{1,0} &= 0 \\ \Delta E_{1,-1} &= \frac{\hbar B}{2} \end{aligned} \tag{43}$$

so that overall, the energies are now given by

$$\begin{aligned} E_{1,1} &= -J\frac{\hbar^2}{4} - \frac{\hbar B}{2} \\ E_{1,0} &= -J\frac{\hbar^2}{4} \\ E_{1,-1} &= -J\frac{\hbar^2}{4} + \frac{\hbar B}{2} \\ E_{0,0} &= J\frac{3\hbar^2}{4} \end{aligned} \tag{44}$$

(c) In this case, we need to write the full Hamiltonian in either the $|m_1\rangle \otimes |m_2\rangle$ basis, or in the $|1,m\rangle, |0,0\rangle$ basis. Since we have already found some of the matrix elements in the latter basis, let us write the full Hamiltonian in this basis. The additional non-zero matrix elements besides the ones we have already found are

$$\langle 1,0|H|0,0\rangle = \langle 0,0|H|1,0\rangle = -\frac{\hbar B}{2}. \tag{45}$$

$$H = \begin{pmatrix} -J\hbar^2/4 - \hbar B/2 & 0 & 0 & 0 \\ 0 & -J\hbar^2/4 & 0 & -\hbar B/2 \\ 0 & 0 & -J\hbar^2/4 + \hbar B/2 & 0 \\ 0 & -\hbar B/2 & 0 & 3J\hbar^2/4 \end{pmatrix} \tag{46}$$

On diagonalizing this matrix, we find that the energies are given by

$$\begin{aligned} E_{1,1} &= -J\frac{\hbar^2}{4} - \frac{\hbar B}{2} \\ E_{1,0} &= -J\frac{\hbar^2}{4} + J\frac{\hbar^2}{2}\sqrt{1+(B/\hbar J)^2} \\ E_{1,-1} &= -J\frac{\hbar^2}{4} + \frac{\hbar B}{2} \\ E_{0,0} &= -J\frac{\hbar^2}{4} - J\frac{\hbar^2}{2}\sqrt{1+(B/\hbar J)^2} \end{aligned} \tag{47}$$

By Taylor-expanding the square root, we can check that the answers at first order agree with those obtained from perturbation theory above.

5. Three entangled quantum spins (30 points)

(a) Consider a system of three spin-1/2 particles in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle). \tag{48}$$

Compute the entanglement entropy for a subsystem containing one or two of the spins. (Recall that the entanglement entropy of a subsystem is the entropy of the density matrix of the subsystem after tracing out the degrees of freedom of the complement of the subsystem, and quantifies the randomness/uncertainty in the subsystem taken in isolation.)

(b) Now consider a system of three spin-1/2 particles in the state

$$|\psi\rangle = \frac{1}{2}(|++\rangle + |+-+\rangle - |-+-\rangle - |--\rangle)\,. \tag{49}$$

Again, compute the entanglement entropy for a subsystem containing each subset of one or two of the spins. Explain your answer in terms of how the density matrix for the different subsystems is described by mixed or pure states. [Hint: Try factorizing the state!]

Answer:

(a) The density operator for the full system is

$$\rho = \frac{1}{2}\left(|++\rangle\langle+++| - |++\rangle\langle---| - |--\rangle\langle+++| + |--\rangle\langle---|\right)$$

If we trace out two of the spins (without loss of generality trace out the second and third spins), then the reduced density matrix in the $\pm$ basis is

$$\rho_1 = \mathrm{Tr}_{23}\,\rho = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \tag{50}$$

which we recognize as an equal mixture of pure spin out and down states; entropy is (in units of (k_B))

$$S_1 = -\mathrm{Tr}\;\rho \ln \rho = \ln 2.$$

Since entanglement entropy satisfies $S_A = S_B$ when a system is broken into two parts A, B, we also have for the entanglement entropy of any pair $S_{12} = \ln 2$, etc.; this can also be confirmed directly by noting that the density matrix ρ_{12} has nonzero elements that form a block of the form (50) in the basis elements $|++\rangle, |--\rangle$.

(b) We can simplify our understanding by factorizing the state in the form

$$|\psi\rangle = \frac{1}{2}(|+_1+_3\rangle - |-_1-_3\rangle)(|+_2\rangle + |-_2\rangle)\,.$$

Thus, the second particle is in a pure state and the first and third are entangled. It follows immediately that

$$S_2 = S_{13} = 0\,,$$

since each of these subsystems is in a pure state. If we split the two entangled particles, the analysis is the same as the previous part and we have (in units of k_B)

$$S_1 = S_3 = S_{23} = S_{12} = \ln 2\,.$$

6. Simple Harmonic Oscillator & Magnetic Field in 2D (40 points)

Consider a spinless particle of mass m and charge q moving in two dimensions, in the $xy-$plane. The particle feels a uniform magnetic field $\mathbf{B} = B\widehat{z}$ and is also subject to a harmonic potential that is independent of x

$$V(x,y) = \frac{1}{2}m\omega^2 y^2. \tag{51}$$

Find the energy eigenvalues and corresponding energy eigenfunctions. You can express your answer in terms of ψ_n, $n = 0, 1, 2, \ldots$, which are the energy eigenfunctions corresponding to the one-dimensional simple harmonic oscillator. That is you don't have to derive the full explicit expression for all states, but other than that the answer should depend on the given parameters, fundamental constants, and/or quantum numbers needed to label the state. Check that your results for energy eigenvalues make sense in the limit of vanishing magnetic field $B \to 0$, and separately for the case without harmonic oscillator potential, i.e. $\omega \to 0$.

Answer: Hamiltonian describing this situation is

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{q}{c}\mathbf{A}\right)^2 + \frac{1}{2}m\omega^2 y^2. \tag{52}$$

Now we have to make a gauge choice in order to progress further and it is crucial to choose the most convenient gauge, otherwise the problem becomes intractable. It makes sense to make the gauge choice so that there is a translational symmetry in x (i.e. the case where no explicit factors of x appears in the Hamiltonian). This is provided by Landau gauge where we pick

$$\mathbf{A} = (-By, 0, 0) \implies \mathbf{B} = \nabla \times \mathbf{A} = (0, 0, B). \tag{53}$$

Now writing $\mathbf{p} = -i\hbar\nabla$ and ignoring $z-$direction since we take the system in xy-plane, the Hamiltonian in the position space takes the form

$$H = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x} + \frac{qBy}{c}\right)^2 + \frac{1}{2}m\omega^2 y^2. \tag{54}$$

As we can see above, this is translation-invariant in the $x-$direction now. So let our ansatz for the energy eigenfunction with the energy eigenvalue E be

$$\Psi(x, y) = e^{ikx}\phi(y), \quad k \in \mathbb{R}, \tag{55}$$

for which $\psi(y)$ satisfies,

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial y^2} + \frac{1}{2m}\left(\hbar k + \frac{qBy}{c}\right)^2 + \frac{1}{2}m\omega^2 y^2\right]\phi(y) = E\phi(y). \tag{56}$$

Now notice the equation above is quadratic in y, which means that if we complete the square appropriately then we may relate it to the usual one-dimensional simple harmonic oscillator's energy eigenvalue equation. In order to do that, first define the *cyclotron frequency*

$$\omega_c \equiv \frac{qB}{mc}, \tag{57}$$

as usual and then notice that we can complete the terms to squares as follows:

$$\begin{aligned}
\frac{1}{2m}\left(\hbar k + \frac{qBy}{c}\right)^2 + \frac{1}{2}m\omega^2 y^2 &= \frac{\hbar^2 k^2}{2m} + \hbar k\omega_c y + \frac{1}{2}m(\omega^2 + \omega_c^2)y^2 \qquad (58)\\
&= \frac{\hbar^2 k^2}{2m} + \hbar k\omega_c y + \frac{1}{2}m(\omega^2 + \omega_c^2)y^2 + \frac{\hbar^2 k^2}{2m}\frac{\omega_c^2}{\omega^2 + \omega_c^2} - \frac{\hbar^2 k^2}{2m}\frac{\omega_c^2}{\omega^2 + \omega_c^2}\\
&= \frac{1}{2}m(\omega^2 + \omega_c^2)\left(y + \frac{\hbar k\omega_c}{m(\omega^2 + \omega_c^2)}\right)^2 + \frac{\hbar^2 k^2}{2m}\left(\frac{\omega^2}{\omega^2 + \omega_c^2}\right).
\end{aligned}$$

Defining the variable

$$y' = y + \frac{\hbar k \omega_c}{m(\omega^2 + \omega_c^2)}, \tag{59}$$

this leads to following equation for $\phi(y)$

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial y'^2} + \frac{1}{2} m(\omega^2 + \omega_c^2) y'^2 \right] \phi(y) = \left[E - \frac{\hbar^2 k^2}{2m} \left(\frac{\omega^2}{\omega^2 + \omega_c^2} \right) \right] \phi(y). \tag{60}$$

Indeed, this shows $\phi(y)$ is related to the eigenfunctions of one-dimensional simple harmonic oscillator with the frequency $\sqrt{\omega^2 + \omega_c^2}$. As a result, the energy eigenfunctions for our system can be written as, with ψ_n, $n = 0, 1, 2, \ldots$ are energy eigenfunctions corresponding to one-dimensional simple harmonic oscillator,

$$\Psi_{k,n}(x, y) = e^{ikx} \psi_n \left(y + \frac{\hbar k \omega_c}{m(\omega^2 + \omega_c^2)} \right), \tag{61}$$

and their associated energy eigenvalues are

$$E_{k,n} = \hbar \sqrt{\omega^2 + \omega_c^2} \left(n + \frac{1}{2} \right) + \frac{\hbar^2 k^2}{2m} \left(\frac{\omega^2}{\omega^2 + \omega_c^2} \right). \tag{62}$$

Recall the quantum numbers $k \in \mathbb{R}$ and $n \in \mathbb{Z}_{\geq 0}$ from above.

Clearly when $B \to 0$ we have $\omega_c \to 0$ and in this limit energy eigenfunctions takes the form

$$\Psi_{k,n}(x, y) = e^{ikx} \psi_n (y), \tag{63}$$

and they have the associated eigenvalues

$$E_{k,n} = \hbar \omega \left(n + \frac{1}{2} \right) + \frac{\hbar^2 k^2}{2m}. \tag{64}$$

This is precisely the result one should expect, since in this situation x and y directions are decoupled and contribution to energies are sum of a simple harmonic oscillator and a free particle.

On the other hand when we have $\omega \to 0$, we find

$$\Psi_{k,n}(x, y) = e^{ikx} \psi_n \left(y + \frac{\hbar k}{m \omega_c} \right), \tag{65}$$

and their associated energy eigenvalues becomes

$$E_n = \hbar \omega_c \left(n + \frac{1}{2} \right). \tag{66}$$

Again this is precisely the expected energies from our discussion of Landau levels. Note that k dependence drops out in this situation and there is a huge degeneracy for each levels!