PH312: Physics of Fluids (Prof. McCoy)
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Mar 18 2021

1. From Leibniz's Theorem, for a scalar field $F = F(\mathbf{x}, t)$, the time derivative of integrals such as

$$\frac{d}{dt}\int_{V(t)} F\,dV = \int_{V(t)} \frac{\partial F}{\partial t}\,dV + \int_{A(t)} \mathbf{dA}\cdot\mathbf{u}_A F$$

can be put into the material derivative language by replacing the d/dt by D/Dt, $V(t)$ by $\mathcal{V}$, and $\mathbf{u}_A$ by $\mathbf{u}$, the velocity field:

$$\frac{D}{Dt}\int_{\mathcal{V}} F(\mathbf{x}, t)\,d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial F}{\partial t}\,d\mathcal{V} + \int_{A} \mathbf{dA}\cdot\mathbf{u}F.$$

Conservation of mass requires that the mass of any given volume $\mathcal{V}$ of fluid as it flows remain constant. This means that the time material derivative of the mass of $\mathcal{V}$ is zero, i.e.,

$$0 = \frac{D}{Dt}\int_{\mathcal{V}} \rho(\mathbf{x}, t)\,d\mathcal{V} = \int_{\mathcal{V}} \frac{\partial \rho}{\partial t}\,d\mathcal{V} + \int_{A} \mathbf{dA}\cdot\mathbf{u}\rho,$$

where $F = \rho = \rho(\mathbf{x}, t)$ is the (scalar) density field. Apply Gauss' theorem to the RHS:

$$0 = \int_{\mathcal{V}} \dot{\rho}\,d\mathcal{V} + \int_{A} \mathbf{dA}\cdot\mathbf{u}\rho = \int_{\mathcal{V}} \dot{\rho}\,d\mathcal{V} + \int_{\mathcal{V}} \boldsymbol{\nabla}\cdot\left(\rho\mathbf{u}\right)d\mathcal{V}.$$

Since the integral is linear and the material volume $\mathcal{V}$ is arbitrary, we obtain the *continuity equation* as desired:

$$\dot{\rho} + \boldsymbol{\nabla}\cdot\left(\rho\mathbf{u}\right) = \dot{\rho} + \frac{\partial}{\partial x_i}(\rho u_i) = 0.$$

2. Fixed volume derivation of momentum conservation:

(a) The ith component of momentum and force on a fixed volume V of fluid are given by

$$M_i = \int_V \rho u_i\,dV \quad \text{and} \quad F_i = \int_V \left[\rho g_i + \frac{\partial}{\partial x_j}\tau_{ij}\right] dV,$$

respectively. The *momentum principle* states that

$$F_i = \frac{d}{dt}M_i + \int_A \rho u_i u_j n_j\,dA.$$

The second term on the RHS can be interpreted as the rate of outflux of i-momentum: The term $\rho(\mathbf{u}\cdot\mathbf{dA})$ is the mass outflux rate (units: [mass]/[time]) through an area element $\mathbf{dA}$ on ∂V. And so when we multiply this the velocity component u_i, we have [mass outflux rate] $\times$ [i-velocity] = [i-momentum outflux rate].

(b) By the momentum principle we have:

$$\int_V \left[\rho g_i + \frac{\partial}{\partial x_j}\tau_{ij}\right] dV = \frac{d}{dt}\int_V \rho u_i \, dV + \int_A \rho u_i u_j n_j \, dA.$$

Next, we apply Gauss' theorem to turn the surface integral on the RHS to a volume integral, then move the t-derivative inside the second integral (which is allowed by the fixed volume assumption). One these are done, we rearrange to find:

$$\int_V \left\{\left[\rho g_i + \frac{\partial}{\partial x_j}\tau_{ij}\right] - \frac{d}{dt}(\rho u_i) - \frac{\partial}{\partial x_j}(\rho u_i u_j)\right\} dV = 0.$$

Since V is arbitrary, the integrand must vanish, i.e.,

$$\begin{aligned}
\rho g_i + \frac{\partial}{\partial x_j}\tau_{ij} &= \frac{d}{dt}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_i u_j) \\
&= \dot{\rho} u_i + \rho \frac{\partial}{\partial t} u_i + u_i \frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial u_i}{\partial x_j} \\
&= u_i \underbrace{\left[\dot{\rho} + \frac{\partial}{\partial x_j}(\rho u_j)\right]}_{=0 \text{ by continuity eqn.}} + \rho \underbrace{\left(\frac{\partial}{\partial t} u_i + u_j \frac{\partial u_i}{\partial x_j}\right)}_{\equiv Du_i/Dt} \\
&= \rho \frac{D}{Dt} u_i,
\end{aligned}$$

[1] And so we have just derived Newton's law from the fixed-volume perspective:

$$\rho \frac{D}{Dt} u_i = \rho g_i + \frac{\partial}{\partial x_j}\tau_{ij}.$$

3. 1D Diffusion. The (heat) kernel $G(x,t)$ given by

$$G(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{x^2}{4kt}\right)$$

is a Gaussian whose maximum $1/\sqrt{4\pi k t}$ is attained at $x = 0$. As t increases, this maximum decreases monotonically, and the "width" of $G(x,t)$, which is proportional to $\sqrt{t}$, increases. Thus, as t increases, $G(x,t)$ decays and spreads out in space. Further, since

$$\int_{\mathbb{R}} G(x,t)\, dx = \int_{\mathbb{R}} \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}\, dx = 1,$$

the area under the curve of $G(x,t)$ in space is constant, which means that the kernel "conserves" the total "amount" of heat. Since the solution to the 1D heat equation is the convolution of the initial data $\phi(x,0)$ with $G(x,t)$, as $t > 0$, $\phi(x,t)$ also decays, spreads out (hence "diffuse"), and "looks" more and more like $G(x,t)$, a Gaussian.[2]

[1]Recall that the material derivative of a field v along the flow field $\mathbf{u}$ is given by $Dv/Dt \equiv \partial v/\partial t + \mathbf{u} \cdot \nabla v$

[2]There are precise mathematical statements to characterize our qualitative interpretations, but we won't worry about them here.