Classical Mechanics III (8.09)
Assignment 1: Solutions

September 14, 2021

1. Two Particles in a Gravitational Field [12 points]

(a) [4 points] Let the Cartesian coordinates of the two particles be $\vec{r_1} = (x_1, y_1, z_1)^T$ and $\vec{r_2} = (x_2, y_2, z_2)^T$. We will also define the relative displacement

$$\vec{r} = \vec{r_1} - \vec{r_2},$$

the total mass

$$M = m_1 + m_2,$$

center of mass coordinates

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2},$$

and the *reduced mass*

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Then the kinetic energy is

$$\begin{aligned} T &= \frac{m_1}{2}\dot{\vec{r}}_1^2 + \frac{m_2}{2}\dot{\vec{r}}_2^2 \\ &= \frac{M}{2}\dot{\vec{R}}^2 + \frac{m_1}{2}(\dot{\vec{r}}_1 - \dot{\vec{R}})^2 + \frac{m_2}{2}(\dot{\vec{r}}_2 - \dot{\vec{R}})^2. \end{aligned}$$

A quick calculation shows that

$$\vec{r}_1 - \vec{R} = \frac{m_2}{m_1 + m_2}\vec{r}, \quad \vec{r}_2 - \vec{R} = -\frac{m_1}{m_1 + m_2}\vec{r},$$

and hence

$$\begin{aligned}
T &= \frac{M}{2}\dot{\vec{R}}^2 + \frac{m_1 m_2^2}{2(m_1+m_2)^2}\dot{\vec{r}}^2 + \frac{m_1^2 m_2}{2(m_1+m_2)^2}\dot{\vec{r}}^2 \\
&= \frac{M}{2}\dot{\vec{R}}^2 + \frac{\mu}{2}\dot{\vec{r}}^2.
\end{aligned}$$

(If a student uses Goldstein (1.31) without going through the calculation above themselves, then this is also okay.) The potential energy is given by the external gravity acting on the total mass M at the CM coordinate, and the gravitational attraction of the two masses:

$$V = -MgX - \frac{Gm_1 m_2}{r},$$

where $\vec{R} = (X, Y, Z)^T$. Therefore the Lagrangian $L = T - V$ is

$$L = \Big(\frac{M}{2}\dot{\vec{R}}^2 + MgX\Big) + \Big(\frac{\mu}{2}\dot{\vec{r}}^2 + \frac{Gm_1 m_2}{r}\Big).$$

Thus the Lagrangian splits into two parts, one involving only the center of mass coordinates, while the other involving only the relative coordinates $\vec{r}$. This is to be expected, as the motion of this system (without constraints) can be split into the motion of the center of mass with external forces, and the motion of the particles relative to the center of mass with internal forces.

(b) [4 points] The internal part of the above Lagrangian depending on $\vec{r}$ and $\dot{\vec{r}}$ is

$$L^{int}(\vec{r}, \dot{\vec{r}}) = \frac{\mu}{2}\dot{\vec{r}}^2 + \frac{Gm_1 m_2}{r}.$$

Let's use spherical coordinates (r, θ, ϕ). We can write $\vec{r} = r\hat{r}$. The time derivative of $\hat{r}$ is

$$\begin{aligned}
\frac{d\hat{r}}{dt} &= \frac{\partial \hat{r}}{\partial \theta}\dot{\theta} + \frac{\partial \hat{r}}{\partial \phi}\dot{\phi} \\
&= \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}
\end{aligned}$$

where we used $\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$ and $\frac{\partial \hat{r}}{\partial \phi} = \sin\theta\hat{\phi}$. Therefore

$$\dot{\vec{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi}.$$

(It is fine if you just wrote this result down without derivation.) This gives

$$L^{int} = \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) + \frac{Gm_1 m_2}{r}.$$

Let's now compute the Euler-Lagrange equations:

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L^{int}}{\partial \dot r} - \frac{\partial L^{int}}{\partial r} &= \mu\ddot r - \mu r\dot\theta^2 - \mu r\sin^2\theta\dot\phi^2 + \frac{Gm_1m_2}{r^2} = 0\,,\\
\frac{d}{dt}\frac{\partial L^{int}}{\partial \dot\theta} - \frac{\partial L^{int}}{\partial \theta} &= \frac{d}{dt}(\mu r^2\dot\theta) - \mu r^2\sin\theta\cos\theta\dot\phi^2 = \mu r^2\ddot\theta + 2\mu r\dot r\dot\theta - \mu r^2\sin\theta\cos\theta\dot\phi^2 = 0\\
\frac{d}{dt}\frac{\partial L^{int}}{\partial \dot\phi} - \frac{\partial L^{int}}{\partial \phi} &= \frac{d}{dt}(\mu r^2\sin^2\theta\dot\phi) = 0.
\end{aligned}$$

(c) [4 points] We have

$$\begin{aligned}
p_r &= \frac{\partial L^{int}}{\partial \dot r} = \mu\dot r\\
p_\theta &= \frac{\partial L^{int}}{\partial \dot\theta} = \mu r^2\dot\theta\\
p_\phi &= \frac{\partial L^{int}}{\partial \dot\phi} = \mu r^2\sin^2\theta\dot\phi
\end{aligned}$$

and hence

$$\begin{aligned}
H^{int} &= p_r\dot r + p_\theta\dot\theta + p_\phi\dot\phi - L^{int}\\
&= \frac{p_r^2}{\mu} + \frac{p_\theta^2}{\mu r^2} + \frac{p_\phi^2}{\mu r^2\sin^2\theta} - \left(\frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{p_\phi^2}{2\mu r^2\sin^2\theta} + \frac{Gm_1m_2}{r}\right)\\
&= \frac{p_r^2}{2\mu} + \frac{p_\theta^2}{2\mu r^2} + \frac{p_\phi^2}{2\mu r^2\sin^2\theta} - \frac{Gm_1m_2}{r}.
\end{aligned}$$

(One could have obtained the latter more quickly by using $H = T + V$; we could also have used the form $L = L_0(q) + \dot{\vec q}^T\cdot\hat T\cdot\dot{\vec q}$ which implies $H = \vec p^T\cdot\hat T^{-1}\cdot\vec p - L_0(q)$ using a result from lecture.) The Hamilton's equations of motions are

$$\dot r = \frac{\partial H^{int}}{\partial p_r} = \frac{p_r}{\mu}$$

$$\dot\theta = \frac{\partial H^{int}}{\partial p_\theta} = \frac{p_\theta}{\mu r^2}$$

$$\dot\phi = \frac{\partial H^{int}}{\partial p_\phi} = \frac{p_\phi}{\mu r^2\sin^2\theta}$$

$$\dot p_r = -\frac{\partial H^{int}}{\partial r} = \frac{p_\theta^2}{\mu r^3} + \frac{p_\phi^2}{\mu r^3\sin^2\theta} - \frac{Gm_1m_2}{r^2}$$

$$\dot p_\theta = -\frac{\partial H^{int}}{\partial \theta} = \frac{p_\phi^2\cos\theta}{\mu r^2\sin^3\theta}$$

$$\dot p_\phi = -\frac{\partial H^{int}}{\partial \phi} = 0.$$

2. Double Pendulum in a Plane with Gravity [20 points]

(a) [10 points] To find the kinetic and potential energies of the system, the most straightforward method is to go back to Cartesian coordinates. Call (x_1, y_1) the coordinates of m_1, and (x_2, y_2) the coordinates of m_2 (and assume $+x$ points to the right and $+y$ upwards). We have

$$x_1 = l_1 \sin\theta_1 \quad , \quad y_1 = -l_1 \cos\theta_1$$
$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \quad , \quad y_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2$$

and

$$\dot{x}_1 = l_1 \cos\theta_1 \dot{\theta}_1 \quad , \quad \dot{y}_1 = l_1 \sin\theta_1 \dot{\theta}_1$$
$$\dot{x}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2 \quad , \quad \dot{y}_1 = l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2$$

and so the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 (\cos^2\theta_1 + \sin^2\theta_1) + \frac{1}{2} m_2 [(l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2)^2 + (l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2)^2] \\ &= \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] \end{aligned}$$

where we used the identity $\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$. Continuing, the potential energy is

$$V = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

and so

$$L = T - V = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] + (m_1 + m_2) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2.$$

Next derive the equations of motion. For θ_1 we have

$$\begin{aligned} 0 &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} \\ &= \frac{d}{dt}\left[m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right] + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 \end{aligned}$$

$$\Rightarrow (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \left[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right] + (m_1 + m_2) g l_1 \sin\theta_1 = 0.$$

and for θ_2 we have

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = \frac{d}{dt}\left[m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right] + m_2 g l_2 \sin\theta_2$$

$$\Rightarrow m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \left[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\right] + m_2 g l_2 \sin\theta_2 = 0.$$

(b) [10 points] In this part we take $m_1 = m_2 = m$. Note that the Lagrangian in (a) has the form $L = L_0(q,t) + \frac{1}{2}\dot{\vec{q}}^T \cdot T \cdot \dot{\vec{q}}$, where $q = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}$, and the matrix

$$T = \begin{pmatrix} 2ml_1^2 & ml_1 l_2 \cos(\theta_1 - \theta_2) \\ ml_1 l_2 \cos(\theta_1 - \theta_2) & ml_2^2 \end{pmatrix}.$$

Therefore we can use the general transformation worked out in lecture for an L of this form into H, which will yield $H = \frac{1}{2}(\vec{p} - \vec{a})^T \cdot T^{-1} \cdot (\vec{p} - \vec{a}) - L_0$ for the case $\vec{a} = 0$. We calculate

$$T^{-1} = \frac{1}{ml_1^2 l_2^2 (1 + \sin^2(\theta_1 - \theta_2))} \begin{pmatrix} l_2^2 & -l_1 l_2 \cos(\theta_1 - \theta_2) \\ -l_1 l_2 \cos(\theta_1 - \theta_2) & 2l_1^2 \end{pmatrix}$$

where the prefactor is the determinant and the matrix is the matrix of cofactors. Thus we get

$$H = \frac{l_2^2 p_{\theta_1}^2 + 2l_1^2 p_{\theta_2}^2 - 2l_1 l_2 \cos(\theta_1 - \theta_2) p_{\theta_1} p_{\theta_2}}{2ml_1^2 l_2^2 (1 + \sin^2(\theta_1 - \theta_2))} - 2mgl_1 \cos\theta_1 - mgl_2 \cos\theta_2.$$

Lets let the numerator of the momentum dependent term be $X = ml_2^2 p_{\theta_1}^2 + 2ml_1^2 p_{\theta_2}^2 - 2ml_1 l_2 \cos(\theta_1 - \theta_2) p_{\theta_1} p_{\theta_2}$. Calculating the Hamilton equations of motion and simplifying we find

$$\dot{\theta}_1 = \frac{\partial H}{\partial p_{\theta_1}} = \frac{l_2 p_{\theta_1} - l_1 p_{\theta_2} \cos(\theta_1 - \theta_2)}{ml_1^2 l_2 \left[1 + \sin^2(\theta_1 - \theta_2)\right]}$$

$$\dot{\theta}_2 = \frac{\partial H}{\partial p_{\theta_2}} = \frac{2l_1 p_{\theta_2} - l_2 p_{\theta_1} \cos(\theta_1 - \theta_2)}{ml_1 l_2^2 [1 + \sin^2(\theta_1 - \theta_2)]}$$

$$\begin{aligned} \dot{p}_{\theta_1} &= -\frac{\partial H}{\partial \theta_1} = \frac{2X \sin(\theta_1 - \theta_2)\cos(\theta_1 - \theta_2)}{2m^2 l_1^2 l_2^2 [1 + \sin^2(\theta_1 - \theta_2)]^2} - \frac{p_{\theta_1} p_{\theta_2} \sin(\theta_1 - \theta_2)}{l_1 l_2 m [1 + \sin^2(\theta_1 - \theta_2)]} - 2mgl_1 \sin\theta_1 \\ &= \frac{[l_2 \cos(\theta_1 - \theta_2) p_{\theta_1} - 2l_1 p_{\theta_2}][l_2 p_{\theta_1} - l_1 p_{\theta_2} \cos(\theta_1 - \theta_2)] \sin(\theta_1 - \theta_2)}{ml_1^2 l_2^2 [1 + \sin^2(\theta_1 - \theta_2)]^2} \\ &\quad -2mgl_1 \sin\theta_1 \end{aligned}$$

$$
\begin{aligned}
\dot{p}_{\theta_2} &= -\frac{\partial H}{\partial \theta_2} = \frac{-2X\sin(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{2m^2 l_1^2 l_2^2[1+\sin^2(\theta_1-\theta_2)]^2} + \frac{p_{\theta_1}p_{\theta_2}\sin(\theta_1-\theta_2)}{ml_1l_2[1+\sin^2(\theta_1-\theta_2)]} - mgl_2\sin\theta_1 \\
&= \frac{[l_2\cos(\theta_1-\theta_2)p_{\theta_1} - 2l_1p_{\theta_2}][l_2p_{\theta_1} - l_1p_{\theta_2}\cos(\theta_1-\theta_2)]\sin(\theta_2-\theta_1)}{ml_1^2l_2^2[1+\sin^2(\theta_1-\theta_2)]^2} \\
&\quad -mgl_2\sin\theta_2.
\end{aligned}
$$

Note that the first term in H was symmetric in $\theta_1 \leftrightarrow \theta_2$, so it is no surprise that the first terms in $\dot{p}_{\theta_1}$ and $\dot{p}_{\theta_2}$ are related by $\theta_1 \leftrightarrow \theta_2$. This complete simplification for the $\dot{p}_{\theta_1}$ and $\dot{p}_{\theta_2}$ results is not needed for full marks. Note that in a problem like this, that use of a computer algebra package like mathematica to double check your algebra is encouraged!

3. Point Mass on a Hoop [12 points]

Start with spherical coordinates (r,θ,ϕ) centered on the center of the hoop. Here θ is the angle that the point mass makes with the upward vertical z-axis, and the constraints are $\dot{\phi}=\omega$, and $r=a$. In terms of these coordinates we have $\dot{r}=0$,

$$T = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta) = \frac{m}{2}(a^2\dot{\theta}^2 + a^2\omega^2\sin^2\theta),$$

and the potential is $V = mga\cos\theta$ when $V=0$ is chosen for the $\theta=\pi/2$ plane. Thus

$$L = \frac{ma^2}{2}(\dot{\theta}^2 + \omega^2\sin^2\theta) - mga\cos\theta.$$

The Lagrange equation of motion for our single generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = ma^2\ddot{\theta} - ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta = 0.$$

The Lagrangian is independent of time, so the constant of motion in this case is the Hamiltonian function

$$H = \dot{\theta}\frac{\partial L}{\partial\dot{\theta}} - L = \frac{m}{2}(a^2\dot{\theta}^2 - a^2\omega^2\sin^2\theta) + mga\cos\theta.$$

(Note that this differs from the energy

$$E = T + V = \frac{m}{2}(a^2\dot{\theta}^2 + a^2\omega^2\sin^2\theta) + mga\cos\theta\,,$$

because of the applied force that keeps the hoop spinning with $\dot{\phi}=\omega$.)

For a stationary solution $\theta=\theta_0$ we have $\dot{\theta}=0$, and the equation of motion gives

$$\sin\theta_0(a\omega^2\cos\theta_0 + g) = 0.$$

For $\sin\theta_0 \neq 0$ i.e. the particle is neither on the top (an unstable solution) nor on the bottom (the trivial solution), this gives $\cos\theta_0 = -g/(a\omega^2)$.

This implies that $\cos\theta_0$ is negative so the particle is on the bottom half of the hoop. Since the range of $\cos\theta_0$ is $[-1, 1]$, we can impose $-g/(a\omega^2) = \cos\theta_0 > -1$ which implies that a nontrivial stationary solution always exists for $g/(a\omega^2) < 1$, which is $\omega^2 > g/a$. Thus we require

$$\omega > \omega_0 \equiv \sqrt{g/a}$$

As we increase ω we have that the angle θ_0 where the mass is stationary travels up the hoop, from the bottom $\theta_0 = \pi$ towards $\theta_0 = \pi/2$.

4. Spring System on a Plane [10 points]

(a) [2 points] In Cartesian coordinates we simply have

$$L = \frac{m_1}{2}|\dot{\vec{r}}_1|^2 + \frac{m_2}{2}|\dot{\vec{r}}_2|^2 - \frac{k}{2}(|\vec{r}_1 - \vec{r}_2| - b)^2.$$

Note the presence of b here, the relaxed length of the spring.
(b) [5 points] Let's pick our four independent coordinates to be the Cartesian coordinates for the center of mass, the distance between the two particles, and the angle that the spring makes with respect to a fixed axis:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}, \qquad \vec{r} = \vec{r}_1 - \vec{r}_2 = (r\cos\theta, r\sin\theta)$$

Here our independent coordinates are (R_x, R_y, r, θ). Defining the total mass $M = m_1 + m_2$ and the reduced mass $\mu = \frac{m_1 m_2}{m_1+m_2}$, we have (see example Goldstein pg.71)

$$\begin{aligned} L &= \frac{M}{2}|\dot{\vec{R}}|^2 + \frac{\mu}{2}|\dot{\vec{r}}|^2 - \frac{k}{2}(r-b)^2 \\ &= \frac{M}{2}(\dot{R}_x^2 + \dot{R}_y^2) + \frac{\mu}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{2}(r-b)^2. \end{aligned}$$

We can now read off the equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{R}}}\right) - \frac{\partial L}{\partial \vec{R}} = \frac{d}{dt}(M\dot{\vec{R}}) = M\ddot{\vec{R}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \mu\ddot{r} - \mu r\dot{\theta}^2 + k(r-b) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt}(\mu r^2\dot{\theta}) = 0$$

(c) [3 points] From the above we see that three of our four coordinates are cyclic: R_x, R_y, and θ. The corresponding conserved generalized momenta are the center-of-mass momenta $P_x = M\dot{R}_x$ and $P_y = M\dot{R}_y$, and the angular momentum $p_\theta = \mu r^2\dot{\theta}$.

For a solution that rotates but does not oscillate, we have that r is constant, and then p_θ conservation implies that $\dot{\theta}$ is constant. The E-L equation for r then gives $\dot{\theta}^2 = \frac{k}{\mu}(1 - b/r)$, which has a solution with $r > 0$ as long as $\dot{\theta}^2 < k/\mu$. (In contrast, the angular momentum $p_\theta = \mu r^2\dot{\theta}$ can be as large as we like, since it increases with increasing r.) Here $\frac{k}{\mu}(1 - b/r)$ is a strictly increasing function as r increases. Thus r always increases as the rate of rotation $|\dot{\theta}|^2$ increases. A plot of $\dot{\theta}^2$ versus r also suffices.

5. Jerky Mechanics [6 points, 8.09 ONLY]

We can write the action as $S = \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, t)dt$. The action needs to be stationary under the infinitesimal variation $q_i' = q_i + \eta_i$, where the variations and their time derivatives vanish at the fixed endpoints, $\eta_i(t_1) = \eta_i(t_2) = 0$ and $\dot{\eta}_i(t_1) = \dot{\eta}_i(t_2) = 0$. Thus following the derivation of the E-L equations but with an extra partial derivative we have

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} L(q', \dot{q}', \ddot{q}', t)dt - \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, t)dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\eta_i + \frac{\partial L}{\partial \dot{q}_i}\dot{\eta}_i + \frac{\partial L}{\partial \ddot{q}_i}\ddot{\eta}_i \right) dt \\
&= \left[\frac{\partial L}{\partial \dot{q}_i}\eta_i + \frac{\partial L}{\partial \ddot{q}_i}\dot{\eta}_i \right]_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\eta_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\dot{\eta}_i \right) dt \\
&= \left[-\left(\frac{d}{dt}\frac{\partial L}{\partial \ddot{q}_i} \right)\eta_i \right]_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i}\eta_i - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)\eta_i + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right)\eta_i \right) dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) \right) \eta_i dt = 0
\end{aligned}
$$

For the last two terms on the second line we integrated by parts to get the third line. Then the last term on the third line was integrated once more by parts to get the fourth line. The various surface terms evaluated at $t = t_1$ and $t = t_2$ are zero because of our restrictions on $\eta_i(t)$. Since $\delta S = 0$ for arbitrary infinitesimal variations η_i, we obtain the desired result

$$
\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) = 0
$$

6. Routhian Mechanics [6 points, 8.309 ONLY]

The Routhian, R, is defined as

$$R(q_1, \ldots, q_n, p_1, \ldots, p_s, \dot{q}_{s+1}, \ldots, \dot{q}_n, t) = \left(\sum_{k=1}^{s} p_k \dot{q}_k\right) - L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)\,,$$

Let $i = 1, \ldots, n$ and $k = 1, \ldots, s$ and $j = s+1, \ldots, n$. Starting with L, we know that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

Using the Legendre transform with the differential dR, sum on repeated indices in the ranges above:

$$\begin{aligned} dR &= \frac{\partial R}{\partial \dot{q}_i} dq_i + \frac{\partial R}{\partial p_k} dp_k + \frac{\partial R}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial R}{\partial t} dt \\ &= dp_k \dot{q}_k + d\dot{q}_k p_k - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i - \frac{\partial L}{\partial t} dt \end{aligned}$$

We can conclude that $\frac{\partial R}{\partial t} = -\frac{\partial L}{\partial t}$ from equating the coefficients of dt. For $k = 1, \ldots, s$:

$$\frac{\partial R}{\partial q_k} dq_k + \frac{\partial R}{\partial p_k} dp_k = dp_k \dot{q}_k + d\dot{q}_k p_k - \frac{\partial L}{\partial q_k} dq_k - \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k$$

Since $d\dot{q}_k p_k = \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k$, those two terms cancel. We can then equate coefficients of the independent variable differentials dq_k and dp_k, giving

$$\frac{\partial R}{\partial q_k} = -\frac{\partial L}{\partial q_k} \text{ and } \dot{q}_k = \frac{\partial R}{\partial p_k}.$$

We are now able to get Hamilton type equations for the q_k and p_k variables:

$$\begin{aligned} \dot{q}_k &= \frac{\partial R}{\partial p_k} \\ \dot{p}_k &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k} = -\frac{\partial R}{\partial q_k} \end{aligned}$$

For $j = s+1, \ldots, n$:

$$\frac{\partial R}{\partial q_j} dq_j + \frac{\partial R}{\partial \dot{q}_j} d\dot{q}_j = -\frac{\partial L}{\partial q_j} dq_j - \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j$$

Similar to before, we can equate coefficients of the independent variables dq_j and $d\dot{q}_j$ to get

$$\frac{\partial R}{\partial q_j} = -\frac{\partial L}{\partial q_j}, \qquad \frac{\partial R}{\partial \dot{q}_j} = -\frac{\partial L}{\partial \dot{q}_j}$$

which leads to the Euler-Lagrange style equations

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_j}\right) - \frac{\partial R}{\partial q_j} = 0.$$

Since the first s variables (q_k, p_k) obey Hamilton type equations and the last $(n-s)$ variables $(q_j, \dot{q}_j)$ obey Euler-Lagrange equations, the Routhian is between the H&L formalisms.

Note - we could have also derived the last two displayed equations directly from R without considering dR, and solutions of this nature are also fine.

7.* Equivalent Lagrangians [NOT FOR POINTS]

Given a Lagrangian $L(q, \dot{q}, t)$ obeying the Euler-Lagrange equation $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$, we need to show that $L' = L + \frac{d}{dt}F(q,t)$ also obeys the E-L equation. This can be verified by direct calculation:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) - \frac{\partial L'}{\partial q} = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q}\right] + \left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\frac{dF}{dt}\right) - \frac{\partial}{\partial q}\frac{dF}{dt}\right]$$

and the first term is zero by the E-L equation for L, so we only need to check the second term is also zero. We have

$$\frac{\partial}{\partial \dot{q}}\frac{dF}{dt} = \frac{\partial}{\partial \dot{q}}\left(\frac{\partial F}{\partial t} + \dot{q}\frac{\partial F}{\partial q}\right) = \frac{\partial F}{\partial q}$$

and also that

$$\begin{aligned} \frac{d}{dt}\frac{\partial F}{\partial q} &= \dot{q}\frac{\partial}{\partial q}\frac{\partial F}{\partial q} + \frac{\partial}{\partial t}\frac{\partial F}{\partial q} \qquad \text{(chain rule)} \\ &= \frac{\partial}{\partial q}\left(\dot{q}\frac{\partial F}{\partial q} + \frac{\partial F}{\partial t}\right) = \frac{\partial}{\partial q}\frac{dF}{dt}. \end{aligned}$$

Therefore $\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\frac{dF}{dt}\right) - \frac{\partial}{\partial q}\frac{dF}{dt} = 0$, as desired.