8.(3)09 Final Exam Solutions 2018

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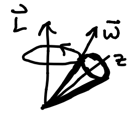
1. (a) 4 The top can undergo <u>rotation</u> & <u>precession</u>

$\boxed{\text{or}}$ Euler angles $\dot{\psi}$ = constant & $\dot{\phi}$ = constant with $\dot{\theta} = 0$

$\boxed{\text{or}}$  body cone rolls on space fixed cone of $\vec{\omega}$

(b) 3 The Origin used to calculate $\hat{I}$ is fixed.

(c) 3 General relativity causes secular change to perihelion of a Kepler orbit

$\boxed{\text{or}}$ θ^4 corrections to simple harmonic pendulum cause secular change to frequency

$\boxed{\text{or}}$ Relativistic corr to simple harmonic osc. cause secular change to frequency

(d) 4 Fixed points $x^* = r x^*(1-x^*)$

$\Rightarrow$ $x^* = 0$ any r <u>and</u> $x^* = 1 - 1/r$ for $r \geq 1$.

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2. 22

$$E = H = \frac{p^2}{2m} + V(x) \quad , \quad E > 0 \, , \qquad x_L \leq x \leq x_R$$

(a) 10

1. • $E > 2\lambda b$, $p \neq 0$ not periodic

3. • $\lambda b < E < 2\lambda b$ periodic with $E = -2\lambda(x_L + b) = 2\lambda(x_R - b)$

so $x_L = -b - \frac{E}{2\lambda}$, $x_R = +b + E/2\lambda$

3. • $0 < E < \lambda b$ & $x(t=0) > 0$ right well

$E = \lambda(b - x_L) = 2\lambda(x_R - b) \Rightarrow x_L = b - \frac{E}{\lambda}$, $x_R = b + \frac{E}{2\lambda}$

3. • $0 < E < \lambda b$ & $x(t=0) < 0$ left well

$E = -2\lambda(x_L + b) = \lambda(b + x_R) \Rightarrow x_L = -b - \frac{E}{2\lambda}$, $x_R = -b + \frac{E}{\lambda}$

2. (b) (10) $\quad 0 < E < \lambda b \qquad E = \frac{p^2}{2m} + V(x), \quad p = \pm\sqrt{2m}\,(E - V(x))^{1/2}$

$$J = 2\int_{x_L}^{x_R} dx\,\sqrt{2m}\,\sqrt{E - V(x)} = 2\sqrt{2m}\int_{x_L}^{b} dx\,\sqrt{E - \lambda(b-x)} + 2\sqrt{2m}\int_{b}^{x_R} dx\,\sqrt{E - 2\lambda(x-b)}$$

same for $p>0$ & $p<0$

$$= 2\sqrt{2m}\left[\frac{2}{3\lambda}\left(E - \lambda b + \lambda x\right)^{3/2}\Big|_{x_L}^{b} + \left(\frac{-2}{6\lambda}\right)\left(E + 2\lambda b - 2\lambda x\right)^{3/2}\Big|_{b}^{x_R}\right]$$

$$= \frac{2\sqrt{2m}}{\lambda}\,\frac{2}{3}\left[E^{3/2} - 0 - \frac{1}{2}\left(0 - E^{3/2}\right)\right]$$

$$J = \frac{2\sqrt{2m}}{\lambda}E^{3/2}, \qquad H = E = \left(\frac{\lambda J}{2\sqrt{2m}}\right)^{2/3}$$

$$\nu = \frac{\partial H}{\partial J} = \frac{2}{3}\left(\frac{\lambda}{2\sqrt{2m}}\right)^{2/3} J^{-1/3} = \frac{2}{3}\left(\frac{\lambda}{2\sqrt{2m}}\right)^{2/3}\left(\frac{2\sqrt{2m}}{\lambda}\right)^{-1/3} E^{-1/2}$$

$$\nu = \frac{\lambda}{3\sqrt{2mE}} \quad\Rightarrow\quad \boxed{\tau = \frac{1}{\nu} = \frac{3\sqrt{2mE}}{\lambda}}$$

(c) (2) $\quad$ dimensions $\quad [mE] = kg^2\, m^2/s^2, \quad [\lambda] = \left[\frac{E}{m}\right] = \frac{kg\, m}{s^2}$

$$[\tau] = s = \left(\cancel{kg}\,\cancel{m}/s^2\right)^{1/2}\frac{s^2}{\cancel{kg}\cdot\cancel{m}} = s \quad ✓$$

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3. (30)

(a) (11) $\quad$ steady $\frac{\partial \vec{v}}{\partial t} = 0, \quad \vec{f} = 0 \quad \therefore \quad \underline{(\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p = \nu\nabla^2\vec{v}} \quad (*)$

$$\vec{v} = \hat{\phi}\, v_\phi(r), \qquad p = p(r)$$

$$\vec{\nabla}p(r) = \hat{r}\,\frac{\partial}{\partial r}p(r)$$

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = \left(v_\phi \frac{1}{r}\frac{\partial}{\partial\phi}\right)\left(\hat{\phi}\, v_\phi(r)\right) = -\frac{v_\phi^2}{r}\hat{r}$$

$$\nabla^2\vec{v} = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} + \cancel{\frac{\partial^2}{\partial z^2}}\right)\left(\hat{\phi}\, v_\phi(r)\right), \qquad \frac{\partial^2}{\partial\phi^2}\hat{\phi} = \frac{\partial}{\partial\phi}(-\hat{r}) = -\hat{\phi}$$

$$= \hat{\phi}\left(v_\phi''(r) + \frac{1}{r}v_\phi'(r)\right) - \frac{1}{r^2}v_\phi\,\hat{\phi}$$

$\circledast \cdot \hat{r}$: $\boxed{\dfrac{1}{\rho}\dfrac{\partial p(r)}{\partial r} = \dfrac{v_\phi^2(r)}{r}}$

$\circledast \cdot \hat{\phi}$: $\boxed{\dfrac{\partial^2 v_\phi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial v_\phi(r)}{\partial r} - \dfrac{v_\phi}{r^2} = 0}$

(b) 4

$v_\phi = ar + \dfrac{b}{r}$ $\qquad \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial r} = \dfrac{a}{r} - b/r^3$

$\dfrac{\partial^2 v_\phi}{\partial r^2} = \dfrac{2b}{r^3}$

$-\dfrac{v_\phi}{r^2} = -\dfrac{a}{r} - b/r^3$

sum $= 0$ ✓

(c) 8 boundary conditions

$r = R_2 \qquad v_\phi(r=R_2) = 0 = aR_2 + \dfrac{b}{R_2} \rightarrow \quad 0 = a + \dfrac{b}{R_2^2}$

$r = R_1 \qquad v_\phi(r=R_1) = \Omega R_1 = aR_1 + \dfrac{b}{R_1} \rightarrow \quad \Omega = a + \dfrac{b}{R_1^2}$

subtract $\quad \Omega = b\left(\dfrac{1}{R_1^2} - \dfrac{1}{R_2^2}\right) = b\left(\dfrac{R_2^2 - R_1^2}{R_1^2 R_2^2}\right) \rightarrow \boxed{b = +\dfrac{\Omega R_1^2 R_2^2}{R_2^2 - R_1^2}}$

$\boxed{a = -\dfrac{b}{R_2^2} = \dfrac{-\Omega R_1^2}{R_2^2 - R_1^2}}$ $\qquad v_\phi(r) = \dfrac{\Omega R_1^2}{R_2^2 - R_1^2}\left(-r + \dfrac{R_2^2}{r}\right)$

(d) 7 Friction Force $= -\sigma'_{r\phi}$ for $\hat{n} = -\hat{r}$ at $r = R_2$

& force in $\hat{\phi}$ direction

$= -\eta\left[\{(\hat{r}\cdot\vec{\nabla})\vec{v}\}\cdot\hat{\phi} + \{(\hat{\phi}\cdot\vec{\nabla})\vec{v}\}\cdot\hat{r}\right]$

$= -\eta\left[\left\{\dfrac{\partial}{\partial r} v_\phi(r)\hat{\phi}\right\}\cdot\hat{\phi} + \left\{\dfrac{1}{r}\dfrac{\partial}{\partial\phi}\left(v_\phi(r)\hat{\phi}\right)\right\}\cdot\hat{r}\right]$ $\qquad$ use $\dfrac{\partial}{\partial\phi}\hat{\phi} = -\hat{r}$

$= -\eta\left[\underbrace{\dfrac{\partial v_\phi}{\partial r}}_{a - \frac{b}{r^2}} - \underbrace{\dfrac{1}{r}v_\phi}_{a + \frac{b}{r^2}}\right] = -\eta\,\dfrac{(-2b)}{R_2^2} = \boxed{2\eta\,\dfrac{\Omega R_1^2}{R_2^2 - R_1^2}}$, $r = R_2$

4. $\boxed{36}$

$$\dot{x} = (y-1)(x^2-3y-4)$$
$$\dot{y} = y(x-1)$$

(a) (14)

$\dot{y}=0 \Rightarrow y=0$ or $x=1$

$\Downarrow$ $\qquad\qquad$ $\Downarrow$

$\dot{x}=0 \qquad x=\pm 2 \qquad\qquad y=1$ or $-3y-3=0$ ie $y=-1$

So fixed points : $(2,0)$, $(-2,0)$, $(1,1)$, $(1,-1)$ 4.

$\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = M\begin{pmatrix}u\\ v\end{pmatrix}$ expanding about each fixed point 1.

$(2,0)$ $\quad x=2+u,\ y=v$

$\dot{u} = (\cancel{v}-1)(\cancel{4}+4u+\cancel{u^2}-3v-\cancel{4}) = -4u+3v+\ldots$

$\dot{v} = v(1+u) = v+\ldots$

2. $M = \begin{pmatrix}-4 & 3\\ 0 & 1\end{pmatrix}$ $\quad \tau=-3$, $\Delta=-4$ $\Rightarrow$ saddle node

$(-2,0)$ $\quad x=-2+u,\ y=v$

$\dot{u} = (v-1)(\cancel{4}-4u+\cancel{u^2}-3v-\cancel{4}) = 4u+3v+\ldots$

$\dot{v} = v(-3+u) = -3v+\ldots$

2. $M = \begin{pmatrix}4 & 3\\ 0 & -3\end{pmatrix}$ $\quad \tau=1$, $\Delta=-12$ $\Rightarrow$ saddle node

$(1,1)$ $\quad x=1+u,\ y=1+v$

$\dot{u} = v(1+2\cancel{u}+\cancel{u^2}-3-3\cancel{v}-4) = -6v+\ldots$

$\dot{v} = (v+1)(u) = +u+\ldots$

2. $M = \begin{pmatrix}0 & -6\\ +1 & 0\end{pmatrix}$ $\quad \tau=0$, $\Delta=+6$ $\Rightarrow$ center

$(1,-1)$ $\quad x=1+u,\ y=-1+v$

$\dot{u} = (-2+\cancel{v})(\cancel{1}+2u+\cancel{u^2}+\cancel{3}-3v-\cancel{4}) = -4u+6v+\ldots$

$\dot{v} = (-1+\cancel{v})u = -u$

3. $M = \begin{pmatrix}-4 & +6\\ -1 & 0\end{pmatrix}$ $\quad \tau=-4$, $\Delta=+6$ $\quad \tau^2-4\Delta = 16-24=-8<0$

$\Rightarrow$ stable spiral

(b) 10

$\underline{(2,0)}$ $\quad \tau^2 - 4\Delta = 9 + 16 = 25 \qquad \lambda_\pm = -\frac{3}{2} \pm \frac{5}{2} \qquad \begin{matrix} \lambda_+ = 1 \\ \lambda_- = -4 \end{matrix} \qquad M = \begin{pmatrix} -4 & 3 \\ 0 & 1 \end{pmatrix}$

$\lambda_+ : \quad M - \lambda_+ \mathbb{1} = \begin{pmatrix} -4-1 & 3 \\ 0 & 1-1 \end{pmatrix} = \begin{pmatrix} -5 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{a}_+ = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

3.

$\lambda_- : \quad M - \lambda_- \mathbb{1} = \begin{pmatrix} 0 & 3 \\ 0 & 5 \end{pmatrix} \Rightarrow \vec{a}_- = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

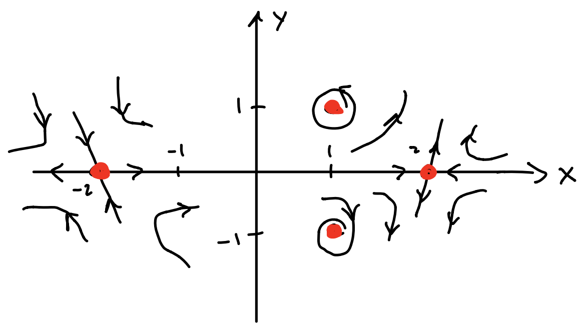
$\underline{(-2,0)}$ $\quad \tau^2 - 4\Delta = 1 + 48 = 49 \qquad \lambda_\pm = \frac{1}{2} \pm \frac{7}{2}, \quad \begin{matrix} \lambda_+ = 4, \\ \lambda_- = -3 \end{matrix} \qquad M = \begin{pmatrix} 4 & 3 \\ 0 & -3 \end{pmatrix}$

$\lambda_+ : \quad M - \lambda_+ \mathbb{1} = \begin{pmatrix} 0 & 3 \\ 0 & -7 \end{pmatrix} \Rightarrow \vec{a}_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

3.

$\lambda_- : \quad M - \lambda_- \mathbb{1} = \begin{pmatrix} +7 & 3 \\ 0 & 0 \end{pmatrix} \Rightarrow \vec{a}_- = \begin{pmatrix} 3 \\ -7 \end{pmatrix}$

4



(c) 7

$$\dot{r} = r(r^2 - a)(r-1) = r(r - \sqrt{a})(r + \sqrt{a})(r-1)$$

$\dot{\theta} = 1 \qquad\qquad$ if $a > 0$ $\qquad\qquad r \geq 0$

$\underline{r=0}$ fixed point $\qquad \dot{r} = ar + \ldots \qquad a < 0$ stable

2 $\qquad\qquad\qquad\qquad\qquad\qquad a > 0$ unstable

$\underline{r=1}$ limit cycle $\qquad \dot{r} = (1-a)(r-1) \qquad a < 1$ unstable

2 $\qquad\qquad\qquad\qquad\qquad\qquad a > 1$ stable

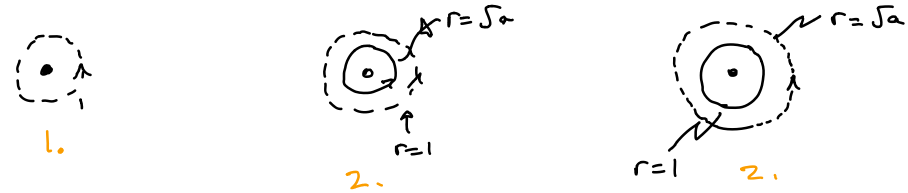
$\underline{a > 0,\ r = \sqrt{a}}$ limit cycle $\qquad \dot{r} = 2a(\sqrt{a} - 1)(r - \sqrt{a}) \qquad 0 < a < 1$ stable

2 $\qquad\qquad\qquad\qquad\qquad\qquad a > 1$ unstable

1. Bifurcations at $a = 0$ and $a = 1$

(d) 5

$a < 0$ $\qquad$ $0 < a < 1$ $\qquad$ $a > 1$



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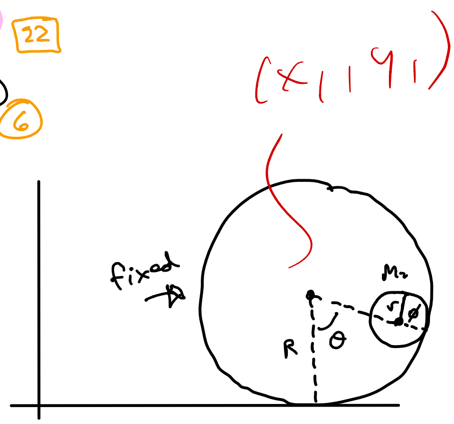
5. $\boxed{22}$

(a) 6



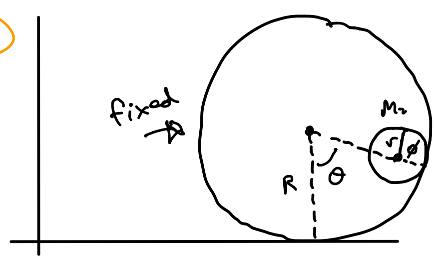
$$V = m_2 g y_2 = m_2 g [\overset{\text{drop}}{R} - (R-r)\cos\theta] \quad 2.$$

$$T = \frac{m_2 r^2 \dot{\phi}^2}{2} + \frac{m_2}{2}(R-r)^2\dot{\theta}^2 \quad 4.$$

variables: θ, ϕ

$$L = T - V = \frac{m_2 r^2 \dot{\phi}^2}{2} + \frac{m_2}{2}(R-r)^2\dot{\theta}^2 + m_2 g (R-r)\cos\theta$$

(b) 6



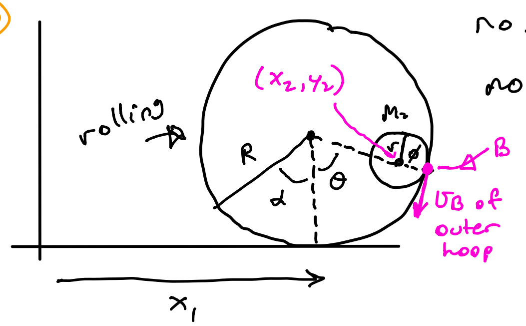
no slip: $v = \underline{r\dot{\phi} = (R-r)\dot{\theta}}$ $\quad$ 2.

without constraint $\to$ same as in (a) $\quad$ 2.

with constraint $\to$ just θ $\quad$ 2.

$$L = m_2 (R-r)^2 \dot{\theta}^2 + m_2 g (R-r)\cos\theta$$

(c) 10



no slip outer: $\dot{x}_1 = R\dot{\alpha}$ $\quad$ 2.

no slip inner: $r\dot{\phi} = (R-r)\dot{\theta} + v_B$

$$v_B = R\dot{\alpha} - \dot{x}_1\cos\theta = R\dot{\alpha}(1-\cos\theta)$$

$$r\dot{\phi} = (R-r)\dot{\theta} + R\dot{\alpha}(1-\cos\theta) \quad 2.$$

(go easy on grading here!)

$$y_2 = R - (R-r)\cos\theta \qquad x_2 = x_1 + (R-r)\sin\theta$$

$$\dot{y}_2 = (R-r)\sin\theta\,\dot{\theta} \qquad \dot{x}_2 = \dot{x}_1 + (R-r)\cos\theta\,\dot{\theta}$$

$\downarrow$ 6. rest

(see next page)

translate of big loop rotate of big loop

prior to rolling constraints:

$$T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_1 R^2}{2}\dot{\alpha}^2 + \frac{m_2}{2} r^2 \dot{\phi}^2 + \frac{m_2}{2}\left((R-r)^2 \sin^2\theta\, \dot{\theta}^2 + \left(\dot{x}_1 + (R-r)\cos\theta\, \dot{\theta}\right)^2\right)$$

after rolling constraints $\Rightarrow$ (α, θ) variables

$$T = m_1 R^2 \dot{\alpha}^2 + \frac{m_2}{2}\left[(R-r)\dot{\theta} + R\dot{\alpha}(1-\cos\theta)\right]^2 + \frac{m_2}{2}\left[(R-r)^2\sin^2\theta\,\dot{\theta}^2 + \left\{R\dot{\alpha} + (R-r)\cos\theta\,\dot{\theta}\right\}^2\right]$$

$$V = -m_2 g (R-r)\cos\theta$$

$$L = T - V$$

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5. 26

$$H = \frac{1}{2m}\left(\vec{p} - e\vec{A}(\vec{r})\right)^2 = \frac{1}{2m}\left(p_j - eA_j(\vec{r})\right)\left(p_j - eA_j(\vec{r})\right)$$

(a) 4

$$\dot{r}_i = \frac{\partial H}{\partial p_i} = \frac{1}{m}\left(p_i - eA_i(\vec{r})\right) = v_i \qquad 2.$$

$$\dot{p}_i = -\frac{\partial H}{\partial r_i} = +\frac{1}{m}\left(p_j - eA_j\right)\left(-e\frac{\partial A_j}{\partial r_i}\right) \qquad 2.$$

(b) 8

$$[v_i, r_j] = \frac{1}{m}[p_i - eA_i, r_j] = \frac{1}{m}[p_i, r_j] = \frac{1}{m}(0 - \delta_{ij}) = -\frac{1}{m}\delta_{ij}$$

$$\boxed{a = -\frac{1}{m}} \qquad 3.$$

$$[v_i, v_j] = \frac{1}{m^2}[p_i - eA_i, p_j - eA_j] = -\frac{e}{m^2}[p_i, A_j] - \frac{e}{m^2}[A_i, p_j]$$

$$= -\frac{e}{m^2}\left(0 - \frac{\partial p_i}{\partial p_k}\frac{\partial A_j}{\partial r_k}\right) - \frac{e}{m^2}\left(\frac{\partial A_i}{\partial r_k}\frac{\partial p_j}{\partial p_k} - 0\right)$$

$$= \frac{e}{m^2}\left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)$$

$$\epsilon_{ijk} B_k = \epsilon_{ijk}\underbrace{\epsilon_{k\ell m}}_{\epsilon_{\ell m k}}\frac{\partial}{\partial r_\ell} A_m = \left(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}\right)\frac{\partial}{\partial r_\ell}A_m$$

$$= \left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)$$

$$[v_i, v_j] = \frac{e}{m^2}\epsilon_{ijk}B_k \qquad \boxed{b = \frac{e}{m^2}} \qquad 5.$$

(c) 10

$\vec B = g\frac{\hat r}{r^2} = g\frac{\vec r}{r^3}$, $J_j = m\,\epsilon_{jkl}\, r_k v_l - \frac{g e r_j}{r}$

$H = \frac{m}{2}\vec v^{\,2} = \frac{m}{2} v_i v_i$

$$\dot J_j = [J_j, H] = m v_i [J_j, v_i]$$

$$= \underbrace{m^2 v_i\, \epsilon_{jkl} \underbrace{[r_k, v_i]}_{-a\,\delta_{ik}} v_l}_{0} + m^2 v_i\, \epsilon_{jkl}\, r_k \underbrace{[v_l, v_i]}_{\substack{b\,\epsilon_{lim} B_m \\ = b\,\epsilon_{iml} B_m}} - m v_i \frac{ge}{r} \overbrace{[r_j, v_i]}^{-a\,\delta_{ij}} - m v_i\, g e\, r_j \underbrace{[1/r, v_i]}_{(-r_i/m r^3)}$$

$$= b m^2 v_i r_k \left(\frac{g r_m}{r^3}\right)\left(\delta_{ij}\delta_{km} - \delta_{jm}\delta_{ki}\right) + a m g e \frac{v_j}{r} + g e \frac{\vec r\cdot\vec v\, r_j}{r^3}$$

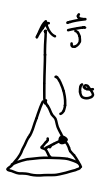
$$= \underbrace{b m^2}_{e} g \left(\frac{v_j}{r} - \frac{r_j\, \vec r\cdot\vec v}{r^3}\right) + \underbrace{a m}_{-1} g e \frac{v_j}{r} + g e \frac{\vec r\cdot\vec v\, r_j}{r^3}$$

$$= \frac{v_j}{r}(eg - eg) + \frac{r_j\, \vec r\cdot\vec v}{r^3}(-eg + eg) = 0$$

(d) 4

$\hat r\cdot\vec J = -eg$ $\quad$ $\vec J$ is vector that is constant w.r.t. time

$\cos\theta = \frac{\hat r\cdot\vec J}{|\vec J|} = \frac{-eg}{|\vec J|}$ constant in time



particle travels on cone of constant angle θ

$\cos\theta < 0$ so $\frac{\pi}{2} < \theta < \pi$

The END