Spring, 2021

Physics 312: Physics of Fluids
Assignment #8 (Solutions)

Background Reading

Friday, Apr. 2: None!

Monday, Apr. 5: Tritton 6.5, 6.6,
Kundu & Cohen 5.1, 5.2, 5.5

Wednesday, Apr. 7: Tritton 10.1 - 10.3,
Kundu & Cohen 5.4, 6.1, 6.2

Informal Written Reflection

Due: Thursday, April 8 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, April 9 (in class)

1. Download and read through Ed Purcell's famous 1977 paper on low Reynolds swimming (originally given, amazingly enough, as a lecture at a symposium honoring quantum electrodynamics master Victor Weisskopf!). Here's the reference:

   "Life at low Reynolds number" (reprinted: E. M. Purcell, *American Journal of Physics* **45**, 3 (1977))

Write a short summary of the major highlights of this article, focusing on his *scallop theorem* and the peculiarities of low Reynolds number swimming mechanisms. Feel free to supplement your understanding with additional online research, but be careful of your sources and don't overdo it. Have fun with this!

2. The strength of a vortex tube is defined by the *circulation* Γ around a closed curve C that encircles the tube

$$\Gamma = \oint_C \mathbf{u} \cdot \mathrm{d}\mathbf{s}$$

or, equivalently, the vorticity flux through any surface A enclosed by this curve,

$$\Gamma = \int_A \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{A}.$$

Show that the strength of a vortex tube, defined in this way, does not depend on the choice of C or A.

(Hint: Choose two different curves, C_1 and C_2, which do not intersect and think of a vector calculus expression that relates the circulations associated with these curves. If it helps, you might think about the analogy between vortex tubes and streamtubes. . .)

Solution:

Choose two nointersecting surfaces, A_1 and A_2, which are bounded by C_1 and C_2. These surfaces, together with that of the vortex tube itself, form a boundary B that completely enclose a tube-shaped volume V. The total vorticity flux through B is

$$\int_B \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{A} = \int_{A_2} \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{A} - \int_{A_1} \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{A} \; = \; \Gamma_2 - \Gamma_1,$$

since there is no flux through the sides of the tube (drawing a good picture will help you understand the minus sign). Since the vorticity field has zero divergence everywhere, we must have

$$\int_B \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{A} = \int_V (\nabla \cdot \omega) \, \mathrm{d}V = 0.$$

Thus, $\Gamma_1 = \Gamma_2$, which tells us that the strength of a vortex tube does not depend on which curve (or area) is used to compute it. Note, of course, this assumes that C encircles the tube exactly once!

3. Consider axisymmetric flow emanating from a point source of strength Q (measured in units of m^3/s). Argue that the velocity components in spherical coordinates are

$$u_r = \frac{Q}{4\pi r^2}, \quad u_\theta = 0$$

and that the streamfunction for this flow has the form $\psi = \psi(\theta)$. Finally, show that

$$\psi = -\frac{Q}{4\pi}\cos\theta$$

and that this flow is irrotational.

(Hint: Start with the continuity equation in spherical coordinates...)

Solution:

For flow emanating from a point source, the spherical angles θ and ϕ are indistinguishable and, therefore, we expect a purely radial velocity field. The continuity equation for a purely radial flow,

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = 0,$$

then demands an inverse square solution, $u_r = C/r^2$. Integrating over all angles to get the volume flow rate through a spherical shell should give us our known strength Q (and this condition sets the value of the unknown constant C):

$$Q = 4\pi r^2 u_r = 4\pi C \quad \Rightarrow \quad u_r = \frac{Q}{4\pi r^2}.$$

Next, since the flow is axisymmetric, we know the streamfunction in spherical coordinates is related to the velocity components through

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

Clearly, u_θ doesn't vanish unless ψ depends only on θ. Integrating the u_r expression gives us $\psi(\theta) = -C\cos\theta$. For an axisymmetric flow, there is at most one nonzero component of vorticity:

$$\omega_\phi = \frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial}{\partial \theta}u_r\right]$$

For the purely radial flow considered here, this expression vanishes.

4. In this problem, we consider another two examples of irrotational flow (both of which are fairly simple)...

   (a) Show that
   $$\psi = \frac{1}{2}Ur^2\sin^2\theta$$
   is the streamfunction for *uniform* flow in the x-direction (a flow that is clearly both irrotational and inviscid).

   (b) A source-sink pair with a vanishingly small separation produces a streamfunction,
   $$\psi = -\frac{m}{r}\sin^2\theta,$$
   known as an axisymmetric *doublet*. Using your favorite plotting program, such as Wolfram Alpha, plot the streamlines of this flow. What does this flow pattern remind you of?

   (Hint: You may need to write out u_r and u_θ to work out the directions of flow along each streamline...)

   (c) For irrotational flow problems, *velocity potentials* provide an alternative to streamfunctions. Show that
   $$\phi = Ur\cos\theta$$
   is the potential function for the uniform flow considered in part (a) above and that
   $$\phi = \frac{m}{r^2}\cos\theta$$
   is the potential function for the doublet considered in part (b).

Solution:

(a) Plugging this streamfunction into our spherical coordinate expressions for the velocity components, we get

$$u_r = U\cos\theta, \quad u_\theta = -U\sin\theta.$$

It's not hard to show, using a little trig, that $u_r\,\hat{\mathbf{r}} + u_\theta\,\hat{\boldsymbol{\theta}} = U\hat{\mathbf{x}}$.

(b) There is a very strong formal resemblance between the axisymmetric doublet of strength m,

$$u_r = -\frac{2m}{r^3}\cos\theta, \quad u_\theta = -\frac{m}{r^3}\sin\theta,$$

and the electric field of a pure dipole of strength p,

$$E_r = \frac{2p}{4\pi\epsilon_0 r^3}\cos\theta, \quad E_\theta = \frac{p}{4\pi\epsilon_0 r^3}\sin\theta.$$

Note that choosing a dipole strength of $p = -4\pi\epsilon_0\, m$ gives exactly the same field as our doublet... This should give you a pretty good idea what the doublet flow pattern looks like! Remember, however, that we've been using the x-axis as our symmetry axis for spherical coordinates where, in an electrodynamics course, one typically uses the z-axis instead.

(c) For $\phi = Ur\cos\theta$, the definition $\mathbf{u} = \nabla\phi$ gives us the same answers we found using a streamfunction in part (a) (look up the components of the gradient vector in spherical coordinates):

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta, \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta.$$

Likewise, for $\phi = (m/r^2)\cos\theta$, we get the same answers we found in part (b):

$$u_r = \frac{\partial\phi}{\partial r} = -\frac{2m}{r^3}\cos\theta, \quad u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{m}{r^3}\sin\theta.$$

5. In our discussion of reference frames in fluid mechanics, we compared the flow observed by passengers on a ship to the flow observed by a stationary observer watching the ship pass by. In this problem, we derive the observed patterns (shown in Kundu and Cohen, Figures 3.7 and 3.8) by considering inviscid flow around a sphere...

   (a) Show that irrotational flow around a sphere or radius a can be represented as a superposition of a uniform flow and an axisymmetric doublet,

   $$\psi = \frac{1}{2}Ur^2 \sin^2\theta - \frac{m}{r}\sin^2\theta,$$

   What does the doublet strength m have to be to satisfy the appropriate boundary condition at $r = a$? Is there slip?

   (Hint: Use the streamfunction relations to show that the vorticity component ω_ϕ vanishes (this proves that the flow is, indeed, irrotational) and think carefully the behavior of the velocity components at $r = a$... Note also that the uniform flow arises from the motion of the boat relative to the water and, thus, a stationary observer sees only the doublet!)

   (b) As with viscous flow over a sphere, we can now derive the pressure field from the velocity field. Use Euler's equation, in spherical coordinates, to show that

   $$p - p_\infty = \frac{1}{2}\rho U^2\Big(1 - \frac{9}{4}\sin^2\theta\Big).$$

   What is the pressure drag implied by this pressure field?

   (Hint: You'll need to check the appendices to make sure you catch all the terms arising from the use of spherical coordinates, but remember that the viscous terms all vanish (and need not be included). This simplifies your calculation quite a bit...)

Solution:

(a) What would the flow look like if there were no sphere? Our implicit assumption is that the flow would be uniform and, thus, the flow around a sphere should look like uniform flow. The given streamfunction

$$\psi = \frac{1}{2}Ur^2 \sin^2\theta - \frac{m}{r}\sin^2\theta,$$

clearly satisfies this far-field condition, since the effect of the doublet term falls off with increasing r. The other boundary condition, zero radial flow at the surface of the sphere, can be saisfied by picking the doublet strength m carefully:

$$u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \big(U - \frac{2m}{r^3}\big)\cos\theta$$

vanishes at $r = a$ if $2m = Ua^3$. In this case,

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -U\big(1 + \frac{a^3}{2r^3}\big)\sin\theta$$

Thus, as expected for irrotational flow (which usually assumes inviscid flow), there *is* slip at the sphere's surface.

(b) From Kundu and Cohen's appendices, we find the following expression for Euler's equation (keeping only those terms present for an inviscid, axisymmetric flow):

$$u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Plugging in our results from (a), the terms on the left become

$$\begin{aligned} u_r\frac{\partial u_r}{\partial r} &= U^2\big(\frac{3a^3}{r^4} - \frac{3a^6}{r^7}\big)\cos^2\theta \\ \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} &= U^2\big(\frac{1}{r} - \frac{a^3}{2r^4} - \frac{a^6}{2r^7}\big)\sin^2\theta \\ -\frac{u_\theta^2}{r} &= U^2\big(\frac{1}{r} - \frac{a^3}{2r^4} - \frac{a^6}{2r^7}\big)\sin^2\theta. \end{aligned}$$

Thus, combining expressions, we find

$$U^2\big(\frac{3a^3}{r^4} - \frac{3a^6}{r^7}\big)\cos^2\theta - U^2\big(\frac{3a^3}{2r^4} + \frac{3a^6}{4r^7}\big)\sin^2\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

Integrating both sides over radius and evaluating at $r = a$ gives us the pressure on the surface of the sphere:

$$\begin{aligned} p - p_\infty &= \frac{1}{2}\rho U^2\cos^2\theta - \frac{5}{8}\rho U^2\sin^2\theta \\ &= \frac{1}{2}\rho U^2\big(1 - \frac{9}{4}\sin^2\theta\big). \end{aligned}$$

One can tell by inspection, or by carrying a simple integration similar to what we did for creeping viscous flow over a sphere, that this expression implies *zero* pressure drag.