

# Advanced Classical Mechanics

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Classical Mechanics III  
MIT 8.09 & 8.309

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# Chapter 1

## A Review of Analytical Mechanics

### 1.1 Introduction

These lecture notes cover the third course in Classical Mechanics, taught at MIT by Professor Stewart to advanced undergraduates (course 8.09) as well as to graduate students (course 8.309). In the prerequisite classical mechanics II course the students are taught both Lagrangian and Hamiltonian dynamics, including Kepler bound motion and central force scattering, and the basic ideas of canonical transformations. This course briefly reviews the needed concepts, but assumes some familiarity with these ideas. References used for this course include

- Goldstein, Poole & Safko, *Classical Mechanics*, 3rd edition.
- Landau and Lifshitz vol.6, *Fluid Mechanics*. Symon, *Mechanics* for reading material on non-viscous fluids.
- Strogatz, *Nonlinear Dynamics and Chaos*.
- Review: Landau & Lifshitz vol.1, *Mechanics*. (Typically used for the prerequisite Classical Mechanics II course and hence useful here for review)

### 1.2 Lagrangian & Hamiltonian Mechanics

#### Newtonian Mechanics

In Newtonian mechanics, the dynamics of a system of  $N$  particles are determined by solving for their coordinate trajectories as a function of time. This can be done through the usual vector spatial coordinates  $\mathbf{r}_i(t)$  for  $i \in \{1, \dots, N\}$ , or with generalized coordinates  $q_i(t)$  for  $i \in \{1, \dots, 3N\}$  in 3-dimensional space. These generalized coordinates could be angles, distances between objects, etc.

## CHAPTER 1. A REVIEW OF ANALYTICAL MECHANICS

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Velocities will be represented through  $\mathbf{v}_i \equiv \dot{\mathbf{r}}_i$  for spatial coordinates, or through  $\dot{q}_i$  for generalized coordinates. Note that dots above a symbol will always denote the total time derivative  $\frac{d}{dt}$ . Momenta are likewise either Newtonian momenta  $\mathbf{p}_i = m_i \mathbf{v}_i$  or generalized momenta  $p_i$ .

For a fixed set of masses  $m_i$  Newton's 2<sup>nd</sup> law can be expressed in 2 equivalent ways:

1. It can be expressed as  $N$  second-order equations  $\mathbf{F}_i = \frac{d}{dt} (m_i \dot{\mathbf{r}}_i)$  with  $2N$  boundary conditions given in  $\mathbf{r}_i(0)$  and  $\dot{\mathbf{r}}_i(0)$ . The problem then becomes one of determining the  $N$  vector variables  $\mathbf{r}_i(t)$ .
2. It can also be expressed as an equivalent set of  $2N$  1<sup>st</sup> order equations  $\mathbf{F}_i = \dot{\mathbf{p}}_i$  &  $\mathbf{p}_i/m_i = \dot{\mathbf{r}}_i$  with  $2N$  boundary conditions given in  $\mathbf{r}_i(0)$  and  $\mathbf{p}_i(0)$ . The problem then becomes one of determining the  $2N$  vector variables  $\mathbf{r}_i(t)$  and  $\mathbf{p}_i(t)$ .

Note that  $\mathbf{F} = m\mathbf{a}$  holds in *inertial frames*. Inertial frames describe time and space homogeneously (invariant to displacements), isotropically (invariant to rotations), and in a time independent manner. These are frames where the motion of a particle not subject to any force is in a straight line with constant velocity. (The converse, that straight line motion with constant velocity implies there are no forces present, is not true. For example, there may be forces that are not visible for motion in a particular direction, like motion parallel to a constant magnetic field.) Noninertial frames also generically have “fictitious forces”, such as the centrifugal and Coriolis effects. Inertial frames also play a key role in special relativity. In general relativity the concept of inertial frames is replaced by that of geodesic motion.

The existence of an inertial frame is a useful approximation for working out the dynamics of particles, and non-inertial terms can often be included as perturbative corrections. Examples of approximate inertial frames are that of a fixed Earth, or better yet, of fixed distant stars. We can still test for how noninertial we are by looking for fictitious forces that (a) may point back to an origin with no source for the force or (b) behave in a non-standard fashion in different frames (i.e. they transform in an unusual manner when going between different frames).

We will use primes to denote coordinate transformations. If  $\mathbf{r}$  is measured in an inertial frame  $S$ , and  $\mathbf{r}'$  is measured in frame  $S'$  whose relation to  $S$  is given by a transformation  $\mathbf{r}' = \mathbf{f}(\mathbf{r}, t)$ , then  $S'$  is inertial iff  $\ddot{\mathbf{r}} = 0 \leftrightarrow \ddot{\mathbf{r}}' = 0$ . This is solved by the Galilean transformations,

$$\begin{aligned}\mathbf{r}' &= \mathbf{r} + \mathbf{v}_0 t, \\ t' &= t,\end{aligned}$$

with constant  $\mathbf{v}_0$ . This transformation preserves the inertiality of frames, with  $\mathbf{F} = m\ddot{\mathbf{r}}$  equivalent to  $\mathbf{F}' = m\ddot{\mathbf{r}}'$ . Galilean transformations are the non-relativistic limit,  $v \ll c$ , of Lorentz transformations which preserve inertial frames in special relativity. A few examples related to the concepts of inertial frames are:

1. In a rotating frame, the transformation is given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If  $\theta = \omega t$  for some constant  $\omega$ , then  $\ddot{\mathbf{r}} = 0$  does not imply  $\{\ddot{x}', \ddot{y}'\} = 0$ , so the primed frame is noninertial.

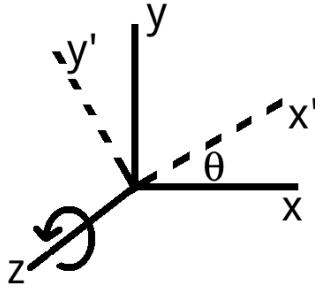


Figure 1.1: Frame rotated by an angle  $\theta$

2. In polar coordinates,  $\mathbf{r} = r\hat{r}$ , and the unit vectors are time dependent,

$$\frac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}, \quad \frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r} \quad (1.1)$$

and thus

$$\ddot{\mathbf{r}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\left(\ddot{\theta}\hat{\theta} - \dot{\theta}^2\hat{r}\right). \quad (1.2)$$

Even if  $\ddot{\mathbf{r}} = 0$  we can still have  $\ddot{r} \neq 0$  and  $\ddot{\theta} \neq 0$ , and we can not in general form a simple Newtonian force law equation  $m\ddot{q} = F_q$  for each of these coordinates. This is different than the first example, since here we are picking coordinates rather than changing the reference frame, so to remind ourselves about their behavior we will call these “non-inertial coordinates” (which we may for example decide to use in an inertial frame). In general, curvilinear coordinates are non-inertial.

## Lagrangian Mechanics

In Lagrangian mechanics, the key function is the Lagrangian

$$L = L(q, \dot{q}, t). \quad (1.3)$$

Here,  $q = (q_1, \dots, q_N)$  and likewise  $\dot{q} = (\dot{q}_1, \dots, \dot{q}_N)$ . We are now letting  $N$  denote the number of scalar (rather than vector) variables, and will often use the short form to denote

dependence on these variables, as in Eq. (1.3). Typically we can write  $L = T - V$  where  $T$  is the kinetic energy and  $V$  is the potential energy. In the simplest cases,  $T = T(\dot{q})$  and  $V = V(q)$ , but we also allow the more general possibility that  $T = T(q, \dot{q}, t)$  and  $V = V(q, \dot{q}, t)$ . It turns out, as we will discuss later, that even this generalization does not describe all possible classical mechanics problems.

The solution to a given mechanical problem is obtained by solving a set of  $N$  second-order differential equations known as *Euler-Lagrange equations of motion*,

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \quad (1.4)$$

These equations involve  $\ddot{q}_i$ , and reproduce the Newtonian equations  $\mathbf{F} = m\mathbf{a}$ . The principle of stationary action (Hamilton's principle),

$$\delta S = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0, \quad (1.5)$$

is the starting point for deriving the Euler-Lagrange equations. Although you have covered the Calculus of Variations in an earlier course on Classical Mechanics, we will review the main ideas in Section 1.5.

There are several advantages to working with the Lagrangian formulation, including

1. It is easier to work with the scalars  $T$  and  $V$  rather than vectors like  $\mathbf{F}$ .
2. The same formula in equation (1.4) holds true regardless of the choice of coordinates.

To demonstrate this, let us consider new coordinates

$$Q_i = Q_i(q_1, \dots, q_N, t). \quad (1.6)$$

This particular sort of transformation is called a *point transformation*. Defining the new Lagrangian by

$$L' = L'(Q, \dot{Q}, t) = L(q, \dot{q}, t), \quad (1.7)$$

we claim that the equations of motion are simply

$$\frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{Q}_i} \right) - \frac{\partial L'}{\partial Q_i} = 0. \quad (1.8)$$

**Proof:** (for  $N = 1$ , since the generalization is straightforward)

Given  $L'(Q, \dot{Q}, t) = L(q, \dot{q}, t)$  with  $Q = Q(q, t)$  then

$$\dot{Q} = \frac{d}{dt} Q(q, t) = \frac{\partial Q}{\partial q} \dot{q} + \frac{\partial Q}{\partial t}. \quad (1.9)$$

Therefore

$$\frac{\partial \dot{Q}}{\partial \dot{q}} = \frac{\partial Q}{\partial q}, \quad (1.10)$$

a result that we will use again in the future. Then

$$\begin{aligned} \frac{\partial L}{\partial q} &= \frac{\partial L'}{\partial q} = \frac{\partial L'}{\partial Q} \frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}} \frac{\partial \dot{Q}}{\partial q}, \\ \frac{\partial L}{\partial \dot{q}} &= \frac{\partial L'}{\partial \dot{q}} = \frac{\partial L'}{\partial \dot{Q}} \frac{\partial \dot{Q}}{\partial \dot{q}} = \frac{\partial L'}{\partial \dot{Q}} \frac{\partial Q}{\partial q}. \end{aligned} \quad (1.11)$$

Since  $\frac{\partial Q}{\partial q} = 0$  there is no term  $\frac{\partial L'}{\partial Q} \frac{\partial Q}{\partial \dot{q}}$  in the last line.

Plugging these results into  $0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q}$  gives

$$\begin{aligned} 0 &= \left[ \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{Q}} \right) \frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}} \frac{d}{dt} \left( \frac{\partial Q}{\partial q} \right) \right] - \left[ \frac{\partial L'}{\partial Q} \frac{\partial Q}{\partial q} + \frac{\partial L'}{\partial \dot{Q}} \frac{\partial \dot{Q}}{\partial q} \right] \\ &= \left[ \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{Q}} \right) - \frac{\partial L'}{\partial Q} \right] \frac{\partial Q}{\partial q}. \end{aligned} \quad (1.12)$$

Here we have used  $\frac{d}{dt} \frac{\partial Q}{\partial q} = (\dot{q} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}) \frac{\partial Q}{\partial q} = \frac{\partial}{\partial q} (\dot{q} \frac{\partial}{\partial q} + \frac{\partial}{\partial t}) Q = \frac{\partial \dot{Q}}{\partial q}$  so that the second and fourth terms in the first line of Eq. (1.12) cancel. Finally for non-trivial transformation where  $\frac{\partial Q}{\partial q} \neq 0$  we have, as expected,

$$0 = \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{Q}} \right) - \frac{\partial L'}{\partial Q}. \quad (1.13)$$

Note two things:

- This implies we can freely use the Euler-Lagrange equations for noninertial coordinates.
  - We can formulate  $L$  in whatever coordinates are easiest, and then change to convenient variables that better describe the symmetry of a system (for example, Cartesian to spherical).
3. Continuing our list of advantages for using  $L$ , we note that it is also easy to incorporate *constraints*. Examples include a mass constrained to a surface or a disk rolling without slipping. Often when using  $L$  we can avoid discussing forces of constraint (for example, the force normal to the surface).

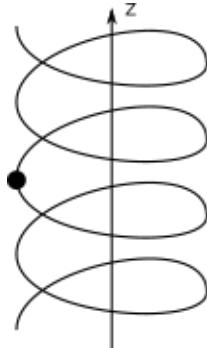


Figure 1.2: Bead on a helix

Lets discuss the last point in more detail (we will also continue to discuss it in the next section). The method for many problems with constraints is to simply make a good choice for the generalized coordinates to use for the Lagrangian, picking  $N - k$  independent variables  $q_i$  for a system with  $k$  constraints.

**Example:** For a bead on a helix as in Fig. 1.2 we only need one variable,  $q_1 = z$ .

**Example:** A mass  $m_2$  attached by a massless pendulum to a horizontally sliding mass  $m_1$  as in Fig. 1.3, can be described with two variables  $q_1 = x$  and  $q_2 = \theta$ .

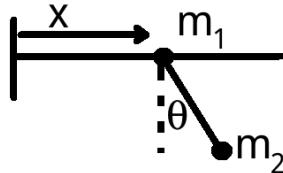


Figure 1.3: Pendulum of mass  $m_2$  hanging on a rigid bar of length  $\ell$  whose support  $m_1$  is a frictionless horizontally sliding bead

**Example:** As an example using non-inertial coordinates consider a potential  $V = V(r, \theta)$  in polar coordinates for a fixed mass  $m$  at position  $\mathbf{r} = r\hat{r}$ . Since  $\dot{\mathbf{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$  we have  $T = \frac{m}{2}\dot{\mathbf{r}}^2 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$ , giving

$$L = \frac{m}{2} \left( \dot{r}^2 + r^2\dot{\theta}^2 \right) - V(r, \theta). \quad (1.14)$$

For  $r$  the Euler-Lagrange equation is

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \frac{d}{dt} (m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r}. \quad (1.15)$$

This gives

$$m\ddot{r} - mr\dot{\theta}^2 = -\frac{\partial V}{\partial r} = F_r, \quad (1.16)$$

from which we see that  $F_r \neq m\ddot{r}$ . For  $\theta$  the Euler-Lagrange equation is

$$0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \frac{d}{dt} \left( mr^2\dot{\theta} \right) + \frac{\partial V}{\partial \theta}. \quad (1.17)$$

This gives

$$\frac{d}{dt} \left( mr^2\dot{\theta} \right) = -\frac{\partial V}{\partial \theta} = F_\theta, \quad (1.18)$$

which is equivalent to the relation between angular momentum and torque perpendicular to the plane,  $\dot{L}_z = F_\theta = \tau_z$ . (Recall  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$  and  $\tau = \mathbf{r} \times \mathbf{F}$ .)

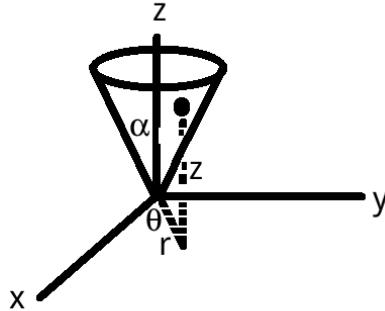


Figure 1.4: Particle on the inside of a cone

**Example:** Let us consider a particle rolling due to gravity in a frictionless cone, shown in Fig. 1.4, whose opening angle  $\alpha$  defines an equation for points on the cone  $\tan(\alpha) = \sqrt{x^2 + y^2}/z$ . There are 4 steps which we can take to solve this problem (which are more general than this example):

1. Formulate  $T$  and  $V$  by  $N = 3$  generalized coordinates. Here it is most convenient to choose cylindrical coordinates denoted  $(r, \theta, z)$ , so that  $T = \frac{m}{2} (\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2)$  and  $V = mgz$ .
2. Reduce the problem to  $N - k = 2$  independent coordinates and determine the new Lagrangian  $L = T - V$ . In this case we eliminate  $z = r \cot(\alpha)$  and  $\dot{z} = \dot{r} \cot(\alpha)$ , so

$$L = \frac{m}{2} \left[ (1 + \cot^2 \alpha) \dot{r}^2 + r^2 \dot{\theta}^2 \right] - mgr \cot \alpha. \quad (1.19)$$

3. Find the Euler-Lagrange equations. For  $r$ ,  $0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r}$ , which here is

$$0 = \frac{d}{dt} [m (1 + \cot^2 \alpha) \dot{r}] - mr\dot{\theta}^2 + mg \cot \alpha \quad (1.20)$$

giving

$$(1 + \cot^2 \alpha) \ddot{r} - r\dot{\theta}^2 + g \cot \alpha = 0. \quad (1.21)$$

For  $\theta$  we have  $0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta}$ , so

$$0 = \frac{d}{dt} \left( mr^2 \dot{\theta} \right) - 0, \quad (1.22)$$

giving

$$(2\dot{r}\dot{\theta} + r\ddot{\theta})r = 0. \quad (1.23)$$

4. Solve the system analytically or numerically, for example using Mathematica. Or we might be only interested in determining certain properties or characteristics of the motion without a full solution (we will discuss this last point in detail in Ch.7).

## Hamiltonian Mechanics

In Hamiltonian mechanics, the canonical momenta  $p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$  are promoted to coordinates on equal footing with the generalized coordinates  $q_i$ . The coordinates  $(q, p)$  are *canonical variables*, and the space of canonical variables is known as *phase space*.

The Euler-Lagrange equations say  $\dot{p}_i = \frac{\partial L}{\partial q_i}$ . These need not equal the kinematic momenta  $m_i \dot{q}_i$  if  $V = V(q, \dot{q})$ . Performing the Legendre transformation

$$H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t) \quad (1.24)$$

(where for this equation, and henceforth, repeated indices will imply a sum unless otherwise specified) yields the *Hamilton equations of motion*

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \quad (1.25)$$

which are  $2N$  1<sup>st</sup> order equations. We also have the result that

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (1.26)$$

**Proof:** (for  $N = 1$ ) Consider how the left and right hand side of the Legendre transformation behave under differential changes to the variables,

$$\begin{aligned} dH &= \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt \\ &= pd\dot{q} + \dot{q}dp - \cancel{\frac{\partial L}{\partial q} dq} - \cancel{\frac{\partial L}{\partial \dot{q}} d\dot{q}} - \frac{\partial L}{\partial t} dt = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t} dt. \end{aligned} \quad (1.27)$$

In the last line we used the definition  $p = \frac{\partial L}{\partial \dot{q}}$  and the E-L equation,  $\dot{p} = \frac{\partial L}{\partial q}$ . Since we are free to independently vary  $dq$ ,  $dp$ , and  $dt$  this implies (generalizing now to  $N$  variables):

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i, \quad \frac{\partial H}{\partial p_i} = \dot{q}_i, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (1.28)$$

The first two results are Hamilton's equations. We can interpret these two Hamilton equations as follows:

- $\dot{q}_i = \frac{\partial H(q,p,t)}{\partial p_i} = \dot{q}_i(q, p, t)$  is an inversion of  $p_i = \frac{\partial L(q,\dot{q},t)}{\partial \dot{q}_i} = p_i(q, \dot{q}, t)$ .
- $\dot{p}_i = -\frac{\partial H}{\partial q_i}$  provides the Newtonian dynamics.

However, these two equation have an equal footing in Hamiltonian mechanics, since the coordinates and momenta are treated on a common ground. We can use  $p_i = \frac{\partial L}{\partial \dot{q}_i}$  to construct  $H$  from  $L$  and then forget about  $L$ .

As an example of the manner in which we will usually consider transformations between Lagrangians and Hamiltonians, consider again the variables relevant for the particle on a cone from Fig. 1.4:

$$\begin{array}{ccc} L(r, \theta, z, \dot{r}, \dot{\theta}, \dot{z}) & \xrightarrow{z=r \cot \alpha} & \text{new } L(r, \theta, \dot{r}, \dot{\theta}) \\ \Downarrow & & \Downarrow \\ H(r, \theta, z, p_r, p_\theta, p_z) & \xrightarrow{\text{not here}} & H(r, \theta, p_r, p_\theta) \end{array} \longrightarrow \begin{array}{c} \text{Euler-Lagrange Eqtns.} \\ \Downarrow \\ \text{Hamilton Eqtns.} \end{array} \quad (1.29)$$

Here we consider transforming between  $L$  and  $H$  either before or after removing the redundant coordinate  $z$ , but in this course we will only consider constraints imposed on Lagrangians and not in the Hamiltonian formalism (the step indicated by  $\Rightarrow$ ). For the curious, the topic of imposing constraints on Hamiltonians, including even more general constraints than those we will consider, is covered well in Dirac's little book "Lectures on Quantum Mechanics". Although Hamiltonian and Lagrangian mechanics provide equivalent formalisms, there is often an advantage to using one or the other. In the case of Hamiltonian mechanics potential advantages include the language of phase space with Liouville's Theorem, Poisson Brackets and the connection to quantum mechanics, as well as the Hamilton-Jacobi transformation theory (all to be covered later on).

**Special case:** Let us consider a special case that is sufficient to imply that the Hamiltonian is equal to the energy,  $H = E \equiv T + V$ . If we only have quadratic dependence on velocities in the kinetic energy,  $T = \frac{1}{2}T_{jk}(q)\dot{q}_j\dot{q}_k$ , and  $V = V(q)$  with  $L = T - V$ , then

$$\dot{q}_i p_i = \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \frac{1}{2}\dot{q}_i T_{ik}\dot{q}_k + \frac{1}{2}\dot{q}_j T_{ji}\dot{q}_i = 2T. \quad (1.30)$$

Hence,

$$H = \dot{q}_i p_i - L = T + V = E \quad (1.31)$$

which is just the energy.

**Another Special case:** Consider a class of Lagrangians given as

$$L(q, \dot{q}, t) = L_0 + a_j \dot{q}_j + \frac{1}{2} \dot{q}_j T_{jk} \dot{q}_k \quad (1.32)$$

where  $L_0 = L_0(q, t)$ ,  $a_j = a_j(q, t)$ , and  $T_{jk} = T_{kj} = T_{jk}(q, t)$ . We can write this in shorthand as

$$L = L_0 + \vec{a} \cdot \dot{\vec{q}} + \frac{1}{2} \dot{\vec{q}} \cdot \hat{T} \cdot \dot{\vec{q}}. \quad (1.33)$$

Here the generalized coordinates, momenta, and coefficients have been collapsed into vectors, like  $\vec{q}$  (rather than the boldface that we reserve for Cartesian vectors), and dot products of vectors from the left imply transposition of that vector. Note that  $\vec{q}$  is an unusual vector, since its components can have different dimensions, eg.  $\vec{q} = (x, \theta)$ , but nevertheless this notation is useful. To find  $H$ ,

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = a_j + T_{jk} \dot{q}_k, \quad (1.34)$$

meaning  $\vec{p} = \vec{a} + \hat{T} \cdot \dot{\vec{q}}$ . Inverting this gives  $\dot{\vec{q}} = \hat{T}^{-1} \cdot (\vec{p} - \vec{a})$ , where  $\hat{T}^{-1}$  will exist because of the positive-definite nature of kinetic energy, which implies that  $\hat{T}$  is a positive definite matrix. Thus,  $H = \dot{\vec{q}} \cdot \vec{p} - L$  yields

$$H = \frac{1}{2} (\vec{p} - \vec{a}) \cdot \hat{T}^{-1} \cdot (\vec{p} - \vec{a}) - L_0(q, t) \quad (1.35)$$

as the Hamiltonian. So for *any* Lagrangian in the form of Eq. (1.32), we can find  $\hat{T}^{-1}$  and write down the Hamiltonian as in Eq. (1.35) immediately.

**Example:** let us consider  $L = \frac{1}{2} m \mathbf{v}^2 - e\phi + e\mathbf{A} \cdot \mathbf{v}$ , where  $e$  is the electric charge and SI units are used. In Eq. (1.32), because the coordinates are Cartesian,  $\mathbf{a} = e\mathbf{A}$ ,  $\hat{T} = m\mathbb{1}$ , and  $L_0 = -e\phi$ , so

$$H = \frac{1}{2m} (\mathbf{p} - e\mathbf{A})^2 + e\phi. \quad (1.36)$$

As you have presumably seen in an earlier course, this Hamiltonian does indeed reproduce the Lorentz force equation  $e(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = m\dot{\mathbf{v}}$ .

**A more detailed Example.** Find  $L$  and  $H$  for the frictionless pendulum shown in Fig. 1.3. This system has two constraints, that  $m_1$  is restricted to lie on the x-axis sliding without friction, and that the rod between  $m_1$  and  $m_2$  is rigid, giving

$$y_1 = 0, \quad (y_1 - y_2)^2 + (x_1 - x_2)^2 = \ell^2. \quad (1.37)$$

Prior to imposing any constraints the Lagrangian is

$$L = T - V = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2) - m_2gy_2 - m_1gy_1. \quad (1.38)$$

Lets choose to use  $x \equiv x_1$  and the angle  $\theta$  as the independent coordinates after imposing the constraints in Eq. (1.37). This allows us to eliminate  $y_1 = 0$ ,  $x_2 = x + \ell \sin \theta$  and  $y_2 = -\ell \cos \theta$ , together with  $\dot{x}_2 = \dot{x} + \ell \cos \theta \dot{\theta}$ ,  $\dot{y}_2 = \ell \sin \theta \dot{\theta}$ ,  $\dot{x}_1 = \dot{x}$ . The Lagrangian with constraints imposed is

$$L = \frac{m_1}{2}\dot{x}^2 + \frac{m_2}{2}(\dot{x}^2 + 2\ell \cos \theta \dot{x}\dot{\theta} + \ell^2 \cos^2 \theta \dot{\theta}^2 + \ell^2 \sin^2 \theta \dot{\theta}^2) + m_2g\ell \cos \theta. \quad (1.39)$$

Next we determine the Hamiltonian. First we find

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + m_2(\dot{x} + \ell \cos \theta \dot{\theta}) = (m_1 + m_2)\dot{x} + m_2\ell \cos \theta \dot{\theta}, \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m_2\ell \cos \theta \dot{x} + m_2\ell^2 \dot{\theta}. \end{aligned} \quad (1.40)$$

Note that  $p_x$  is not simply proportional to  $\dot{x}$  here (actually  $p_x$  is the center-of-mass momentum). Writing  $\begin{pmatrix} p_x \\ p_\theta \end{pmatrix} = \hat{T} \cdot \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix}$  gives

$$\hat{T} = \begin{pmatrix} m_1 + m_2 & m_2\ell \cos \theta \\ m_2\ell \cos \theta & m_2\ell^2 \end{pmatrix}, \quad (1.41)$$

with  $L = \frac{1}{2}(\dot{x} \dot{\theta}) \cdot \hat{T} \cdot \begin{pmatrix} \dot{x} \\ \dot{\theta} \end{pmatrix} + L_0$  where  $L_0 = m_2g\ell \cos \theta$ . Computing

$$\hat{T}^{-1} = \frac{1}{m_1m_2\ell^2 + m_2\ell^2 \sin^2 \theta} \begin{pmatrix} m_2\ell^2 & -m_2\ell \cos \theta \\ -m_2\ell \cos \theta & m_1 + m_2 \end{pmatrix}, \quad (1.42)$$

we can simply apply Eq. (1.35) to find the corresponding Hamiltonian

$$\begin{aligned} H &= \frac{1}{2}(p_x p_\theta) \cdot \hat{T}^{-1} \cdot \begin{pmatrix} p_x \\ p_\theta \end{pmatrix} - m_2g\ell \cos \theta \\ &= \frac{1}{2m_2\ell^2(m_1 + m_2 \sin^2 \theta)} [m_2\ell^2 p_x^2 + (m_1 + m_2)p_\theta^2 - 2m_2\ell \cos \theta p_x p_\theta] - m_2g\ell \cos \theta. \end{aligned} \quad (1.43)$$

Lets compute the Hamilton equations of motion for this system. First for  $(x, p_x)$  we find

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m_1 + m_2 \sin^2 \theta} - \frac{\cos \theta p_\theta}{\ell(m_1 + m_2 \sin^2 \theta)}, \\ \dot{p}_x &= -\frac{\partial H}{\partial x} = 0. \end{aligned} \quad (1.44)$$

As we might expect, the CM momentum is time independent. Next for  $(\theta, p_\theta)$ :

$$\begin{aligned}\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{1}{m_2 \ell^2 (m_1 + m_2 \sin^2 \theta)} \left[ (m_1 + m_2)p_\theta - m_2 \ell \cos \theta p_x \right], \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{\sin \theta \cos \theta}{\ell^2 (m_1 + m_2 \sin^2 \theta)^2} \left[ m_2 \ell^2 p_x^2 + (m_1 + m_2)p_\theta^2 - 2m_2 \ell \cos \theta p_x p_\theta \right] \\ &\quad - m_2 g \ell \sin \theta - \frac{\sin \theta p_x p_\theta}{\ell (m_1 + m_2 \sin \theta)}.\end{aligned}\tag{1.45}$$

These non-linear coupled equations are quite complicated, but could be solved in mathematica or another numerical package. To test our results for these equations of motion analytically, we can take the small angle limit, approximating  $\sin \theta \simeq \theta$ ,  $\cos \theta \simeq 1$  to obtain

$$\begin{aligned}\dot{x} &= \frac{p_x}{m_1} - \frac{p_\theta}{\ell m_1}, \quad \dot{p}_x = 0, \quad \dot{\theta} = \frac{1}{m_1 m_2 \ell^2} \left[ (m_1 + m_2)p_\theta - m_2 \ell p_x \right], \\ \dot{p}_\theta &= \frac{\theta}{\ell^2 m_1^2} \left[ m_2 \ell^2 p_x^2 + (m_1 + m_2)p_\theta^2 - 2m_2 \ell \cos \theta p_x p_\theta \right] - \frac{\theta p_x p_\theta}{\ell m_1} - m_2 g \ell \theta.\end{aligned}\tag{1.46}$$

To simplify it further we can work in the CM frame, thus setting  $p_x = 0$ , and linearize the equations by noting that  $p_\theta \sim \dot{\theta}$  should be small for  $\theta$  to remain small, and hence  $\theta p_\theta^2$  is a higher order term. For the non-trivial equations this leaves

$$\dot{x} = -\frac{p_\theta}{\ell m_1}, \quad \dot{\theta} = \frac{p_\theta}{\mu \ell^2}, \quad \dot{p}_\theta = -m_2 g \ell \theta,\tag{1.47}$$

where  $\mu = m_1 m_2 / (m_1 + m_2)$  is the reduced mass for the two-body system. Thus  $\ddot{\theta} = \dot{p}_\theta / (\mu \ell^2) = -\frac{m_2 g}{\mu \ell} \theta$  as expected for simple harmonic motion.

### 1.3 Symmetry and Conservation Laws

A *cyclic coordinate* is one which does not appear in the Lagrangian, or equivalently in the Hamiltonian. Because  $H(q, p, t) = \dot{q}_i p_i - L(q, \dot{q}, t)$ , if  $q_j$  is absent in  $L$  for some particular  $j$ , it will be absent in  $H$  as well. The absence of that  $q_j$  corresponds with a symmetry in the dynamics.

In this context, *Noether's theorem* means that a symmetry implies a cyclic coordinate, which in turn produces a conservation law. If  $q_j$  is a cyclic coordinate for some  $j$ , then we can change that coordinate without changing the dynamics given by the Lagrangian or Hamiltonian, and hence there is a symmetry. Furthermore the corresponding canonical momentum  $p_j$  is conserved, meaning it is a constant through time.

The proof is simple. If  $\frac{\partial L}{\partial q_j} = 0$  then  $\dot{p}_j = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0$ , or even more simply,  $\frac{\partial H}{\partial q_j} = 0$  is equivalent to  $\dot{p}_j = 0$ , so  $p_j$  is a constant in time.

Special cases and examples of this abound. Lets consider a few important ones:

1. Consider a system of  $N$  particles where no external or internal force acts on the center of mass (CM) coordinate  $\mathbf{R} = \frac{1}{M} m_i \mathbf{r}_i$ , where the total mass  $M = \sum_i m_i$ . Then the CM momentum  $\mathbf{P}$  is conserved. This is because

$$\mathbf{F}_{\mathbf{R}} = -\nabla_{\mathbf{R}} V = 0 \quad (1.48)$$

so  $V$  is independent of  $\mathbf{R}$ . Meanwhile,  $T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2$ , which when using coordinates relative to the center of mass,  $\mathbf{r}'_i \equiv \mathbf{r}_i - \mathbf{R}$ , becomes

$$T = \frac{1}{2} \left( \sum_i m_i \right) \dot{\mathbf{R}}^2 + \dot{\mathbf{R}} \cdot \frac{d}{dt} \left( \sum_i m_i \mathbf{r}'_i \right) + \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}'^2_i = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}'^2_i. \quad (1.49)$$

Note that  $\sum_i m_i \mathbf{r}'_i = 0$  from the definitions of  $M$ ,  $\mathbf{R}$ , and  $\mathbf{r}'_i$ , so  $T$  splits into two terms, one for the CM motion and one for relative motion. We also observe that  $T$  is independent of  $\mathbf{R}$ . This means that  $\mathbf{R}$  is cyclic for the full Lagrangian  $L$ , so  $\mathbf{P} = M \dot{\mathbf{R}}$  is a conserved quantity. In our study of rigid bodies we will also need the forms of  $M$  and  $\mathbf{R}$  for a continuous body with mass distribution  $\rho(\mathbf{r})$ , which for a three dimensional body are  $M = \int d^3r \rho(\mathbf{r})$  and  $\mathbf{R} = \frac{1}{M} \int d^3r \rho(\mathbf{r}) \mathbf{r}$ .

Note that  $\dot{\mathbf{P}} = 0$  is satisfied by having no total external force, so  $\mathbf{F}^{\text{ext}} = \sum_i \mathbf{F}_i^{\text{ext}} = 0$ , and by the internal forces obeying Newton's 3<sup>rd</sup> law  $\mathbf{F}_{i \rightarrow j} = -\mathbf{F}_{j \rightarrow i}$ . Hence,

$$M \ddot{\mathbf{R}} = \sum_i \mathbf{F}_i^{\text{ext}} + \sum_{i,j} \mathbf{F}_{i \rightarrow j} = 0. \quad (1.50)$$

2. Let us consider a system that is invariant with respect to rotations of angle  $\phi$  about a symmetry axis. This has a conserved angular momentum. If we pick  $\phi$  as a generalized coordinate, then  $L = T - V$  is independent of  $\phi$ , so  $\dot{p}_\phi = \frac{\partial L}{\partial \dot{\phi}} = 0$  meaning  $p_\phi$  is constant. In particular, for a system where  $V$  is independent of the angular velocity  $\dot{\phi}$  we have

$$p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\phi}} = \sum_i m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \phi}. \quad (1.51)$$

Simplifying further using the results in Fig. 2.2 yields

$$p_\phi = \sum_i m_i \mathbf{v}_i \cdot (\hat{n} \times \mathbf{r}_i) = \hat{n} \cdot \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i = \hat{n} \cdot \mathbf{L}_{\text{total}}. \quad (1.52)$$

Note that  $\mathbf{L}$  about the CM is conserved for systems with no external torque,  $\boldsymbol{\tau}^{\text{ext}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = 0$  and internal forces that are all central. Defining  $\mathbf{r}_{ij} \equiv \mathbf{r}_i - \mathbf{r}_j$  and its magnitude appropriately, this means  $V_{ij} = V_{ij}(r_{ij})$ . This implies that  $\mathbf{F}_{ji} = -\nabla_i V_{ij}$  (no sum on the repeated index) is parallel to  $\mathbf{r}_{ij}$ . Hence,

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ji}. \quad (1.53)$$

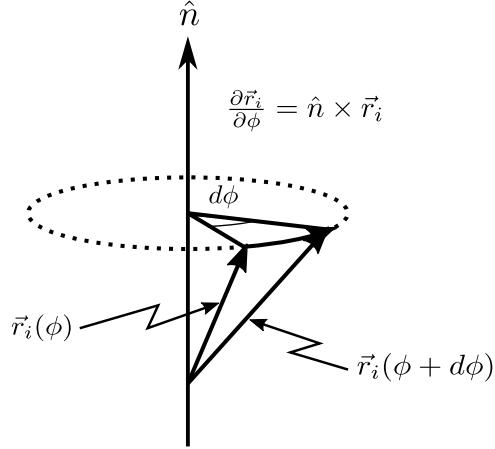


Figure 1.5: Rotation about a symmetry axis

However,  $\sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = 0$ , so

$$\frac{d\mathbf{L}}{dt} = \sum_{i < j} \mathbf{r}_{ij} \times \mathbf{F}_{ji} = 0. \quad (1.54)$$

3. One can also consider a scaling transformation. Suppose that under the transformation  $\mathbf{r}_i \rightarrow \lambda \mathbf{r}_i$  the potential is homogeneous and transforms as  $V \rightarrow \lambda^k V$  for some constant  $k$ . Letting  $T$  be quadratic in  $\dot{\mathbf{r}}_i$  and taking time to transform as  $t \rightarrow \lambda^{1-k/2} t$  then gives  $\dot{\mathbf{r}}_i \rightarrow \lambda^{k/2} \dot{\mathbf{r}}_i$ . So by construction  $T \rightarrow \lambda^k T$  also, and thus the full Lagrangian  $L \rightarrow \lambda^k L$ . This overall factor does not change the Euler-Lagrange equations, and hence the transformation is a symmetry of the dynamics, only changing the overall scale or units of the coordinate and time variables, but not their dynamical relationship. This can be applied for several well known potentials:
  - a)  $k = 2$  for a harmonic oscillator. In this case the scaling for time is given by  $1 - k/2 = 0$ , so it does not change with  $\lambda$ . The amplitude for the coordinates scales with  $\lambda$ . Thus, the frequency of the oscillator, which is a time variable, is independent of the amplitude.
  - b)  $k = -1$  for the Coulomb potential. Here  $1 - k/2 = 3/2$  so there is a more intricate relation between coordinates and time. This power is consistent with the behavior of bound state orbits, where the period of the orbit  $T$  obeys  $T^2 \propto a^3$ , for  $a$  the semi-major axis distance (Kepler's 3rd law).
  - c)  $k = 1$  for a uniform gravitational field. Here  $1 - k/2 = 1/2$  so for a freely falling object, the time of free fall goes as  $\sqrt{h}$  where  $h$  is the distance fallen.
4. Consider the Lagrangian for a charge in the presence of scalar ( $\phi$ ) and vector ( $\mathbf{A}$ ) electromagnetic fields,  $L = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\phi + e\mathbf{A} \cdot \dot{\mathbf{r}}$ . As a concrete example, let us take  $\phi$  and  $\mathbf{A}$  to be independent of the Cartesian coordinate  $x$ . The canonical momentum is

$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = m\dot{\mathbf{r}} + e\mathbf{A}$ , which is notably different from the kinetic momentum. Then  $x$  being cyclic means the canonical momentum  $p_x$  is conserved.

5. Let us consider the conservation of energy and the relationship between energy and the Hamiltonian. Applying the time derivative gives  $\dot{H} = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} + \frac{\partial H}{\partial t}$ . However,  $\dot{q} = \frac{\partial H}{\partial p}$  and  $\dot{p} = -\frac{\partial H}{\partial q}$ . Thus

$$\dot{H} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (1.55)$$

There are two things to consider.

- If  $H$  (or  $L$ ) has no explicit time dependence, then  $H = \dot{q}_i p_i - L$  is conserved.
- Energy is conserved if  $\dot{E} = 0$ , where energy is defined by  $E = T + V$ .

If  $H = E$  then the two points are equivalent, but otherwise either of the two could be true while the other is false.

**Example:** Let us consider a system which provides an example where  $H = E$  but energy is not conserved, and where  $H \neq E$  but  $H$  is conserved. The two situations will be obtained from the same example by exploiting a coordinate choice. Consider a system consisting of a mass  $m$  attached by a spring, with spring constant  $k$ , to a cart moving at a constant speed  $v_0$  in one dimension, as shown in Fig. 1.6. Let us call  $x$

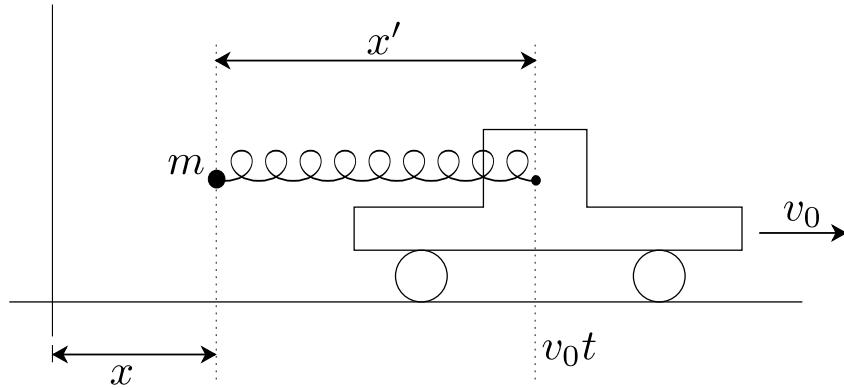


Figure 1.6: Mass attached by a spring to a moving cart

the displacement of  $m$  from the fixed wall and  $x'$  is its displacement from the center of the moving cart. Using  $x$ ,

$$L(x, \dot{x}) = T - V = \frac{m}{2}\dot{x}^2 - \frac{k}{2}(x - v_0 t)^2, \quad (1.56)$$

where the kinetic term is quadratic in  $\dot{x}$  and the potential term is independent of  $\dot{x}$ . This means that  $H$  falls in the special case considered in Eq. (1.31) so

$$H = E = T + V = \frac{p^2}{2m} + \frac{k}{2} (x - v_0 t)^2, \quad (1.57)$$

However  $\frac{\partial H}{\partial t} \neq 0$  so the energy is not conserved. (Of course the full energy would be conserved, but we have not accounted for the energy needed to pull the cart at a constant velocity, treating that instead as external to our system. That is what led to the time dependent  $H$ .)

If we instead choose to use the coordinate  $x' = x - v_0 t$ , then

$$L'(x', \dot{x}') = \frac{m}{2} \dot{x}'^2 + mv_0 x' + \frac{m}{2} v_0^2 - \frac{k}{2} x'^2. \quad (1.58)$$

Note that  $p' = m\dot{x}' + mv_0 = m\dot{x} = p$ . This Lagrangian fits the general form in equation (1.32) with  $a = mv_0$  and  $L_0 = mv_0^2/2 - kx'^2/2$ . So

$$H'(x', p') = \dot{x}' p' - L' = \frac{1}{2m} (p' - mv_0)^2 + \frac{k}{2} x'^2 - \frac{m}{2} v_0^2, \quad (1.59)$$

Here the last terms is a constant shift. The first and second terms in this expression for  $H'$  look kind of like the energy that we would calculate if we were sitting on the cart and did not know it was moving, which is not the same as the energy above. Hence,  $H' \neq E$ , but  $\dot{H}' = 0$  because  $\frac{\partial H'}{\partial t} = 0$ , so  $H'$  is conserved.

## 1.4 Constraints and Friction Forces

So far, we've considered constraints to a surface or curve that are relationships between coordinates. These fall in the category of *holonomic constraints*. Such constraints take the form

$$f(q_1, \dots, q_N, t) = 0 \quad (1.60)$$

where explicit time dependence is allowed as a possibility. An example of holonomic constrain is mass in a cone (Figure 1.4), where the constrain is  $z - r \cot \alpha = 0$ . Constraints that violate the form in Eq. (1.60) are *non-holonomic constraints*.

- An example of a non-holonomic constraint is a mass on the surface of a sphere. The constraint here is an *inequality*  $r^2 - a^2 \geq 0$  where  $r$  is the radial coordinate and  $a$  is the radius of the sphere.
- Another example of a non-holonomic constraint is an object rolling on a surface without slipping. The point of contact is stationary, so the constraint is actually on the *velocities*.

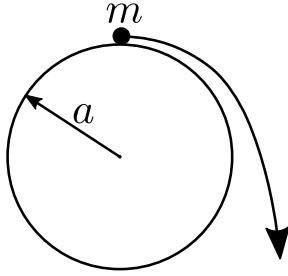


Figure 1.7: Mass on a sphere

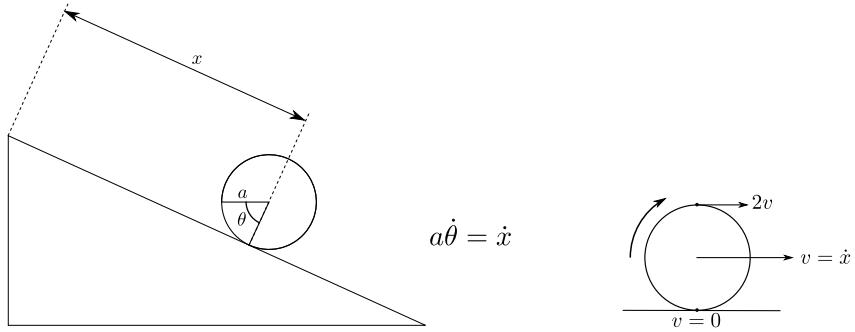


Figure 1.8: Disk rolling down an incline without slipping

A simple example is a disk of radius  $a$  rolling down an inclined plane without slipping, as shown in Fig. 1.8. Here the condition on velocities,  $a\dot{\theta} = \dot{x}$  is simple enough that it can be integrated into a holonomic constraint.

As a more sophisticated example, consider a vertical disk of radius  $a$  rolling on a horizontal plane, as shown in Fig. 1.9. The coordinates are  $(x, y, \theta, \phi)$ , where  $(x, y)$  is the point of contact,  $\phi$  is the rotation angle about its own axis, and  $\theta$  is the angle of orientation along the  $xy$ -plane. We will assume that the flat edge of the disk always remains parallel to  $z$ , so the disk never tips over. The no-slip condition is  $v = a\dot{\phi}$  where  $\mathbf{v}$  is the velocity of the center of the disk, and  $v = |\mathbf{v}|$ . This means  $\dot{x} = v \sin(\theta) = a \sin(\theta)\dot{\phi}$  and  $\dot{y} = -v \cos(\theta) = -a \cos(\theta)\dot{\phi}$ , or in differential notation,  $dx - a \sin(\theta)d\phi = 0$  and  $dy + a \cos(\theta)d\phi = 0$ .

In general, constraints of the form

$$\sum_j a_j(q) dq_j + a_t(q) dt = 0 \quad (1.61)$$

are not holonomic. We will call this a *semi-holonomic constraint*.

Let us consider the special case of a holonomic constraint in differential form,  $f(q_1, \dots, q_{3N}, t) = 0$ . This means

$$df = \sum_j \frac{\partial f}{\partial q_j} dq_j + \frac{\partial f}{\partial t} dt = 0, \quad (1.62)$$

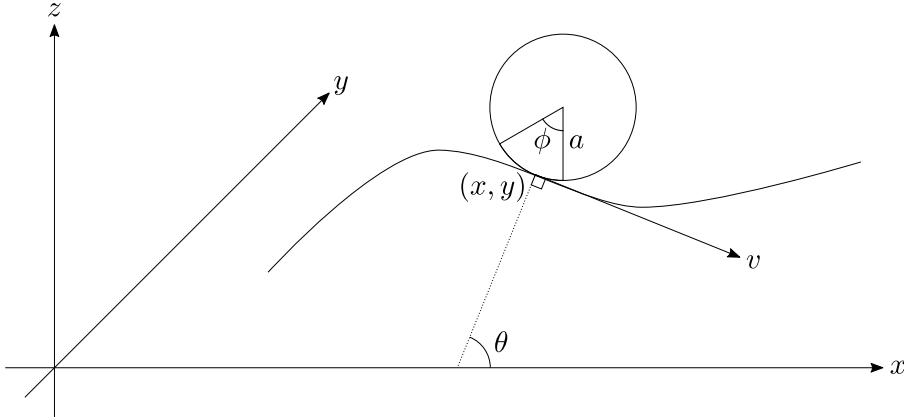


Figure 1.9: Vertical rolling disk on a two dimensional plane

so  $a_j = \frac{\partial f}{\partial q_j}$  and  $a_t = \frac{\partial f}{\partial t}$ . The properties of mixed partial derivatives means

$$\frac{\partial a_j}{\partial q_i} = \frac{\partial a_i}{\partial q_j}, \quad \frac{\partial a_t}{\partial q_i} = \frac{\partial a_i}{\partial t}. \quad (1.63)$$

These conditions imply that a seemingly semi-holonomic constraint is in fact holonomic. (In math we would say that we have an exact differential form  $df$  for the holonomic case, but the differential form in Eq.(1.61) need not always be exact.)

**Example:** To demonstrate that not all semiholonomic constraints are secretly holonomic, consider the constraint in the example of the vertical disk. Here there is no function  $h(x, y, \theta, \phi)$  that we can multiply the constraint  $df = 0$  by to make it holonomic. For the vertical disk from before, we could try  $(dx - a \sin(\theta) d\phi)h = 0$  with  $a_x = h$ ,  $a_\phi = -a \sin(\theta)h$ ,  $a_\theta = 0$ , and  $a_y = 0$ , all for some function  $h$ . As we must have  $\frac{\partial a_\phi}{\partial \theta} = \frac{\partial a_\theta}{\partial \phi}$ , then  $0 = -a \cos(\theta) - a \sin(\theta) \frac{\partial h}{\partial \theta}$ , so  $h = \frac{k}{\sin(\theta)}$ . That said,  $\frac{\partial a_x}{\partial \theta} = \frac{\partial a_\theta}{\partial x}$  gives  $\frac{\partial h}{\partial \theta} = 0$ , yielding a contradiction for any non-trivial  $h$  with  $k \neq 0$ .

If the rolling is instead constrained to a line rather than a plane, then the constraint is holonomic. Take as an example  $\theta = \frac{\pi}{2}$  for rolling along  $\hat{x}$ , then  $\dot{x} = a\dot{\phi}$  and  $\dot{y} = 0$ . Integrating we have  $x = a\varphi + x_0$ ,  $y = y_0$ , and  $\theta = \frac{\pi}{2}$ , which together form a set of holonomic constraints.

A useful concept for discussing constraints is that of the *virtual displacement*  $\delta \mathbf{r}_i$  of particle  $i$ . There are a few properties to be noted of  $\delta \mathbf{r}_i$ .

- It is infinitesimal.
- It is consistent with the constraints.
- It is carried out at a fixed time (so time dependent constraints do not change its form).

**Example:** let us consider a bead constrained to a moving wire. The wire is oriented along

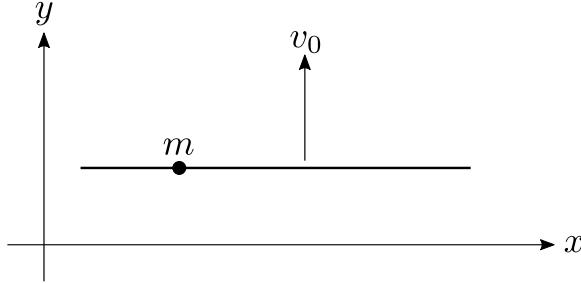


Figure 1.10: Bead on a moving wire

the  $x$ -axis and is moving with coordinate  $y = v_0 t$ . Here the virtual displacement of the bead  $\delta \mathbf{r}$  is always parallel to  $\hat{x}$  (since it is determined at a fixed time), whereas the real displacement  $d\mathbf{r}$  has a component along  $\hat{y}$  in a time interval  $dt$ .

For a large number of constraints, the constraint force  $\mathbf{Z}_i$  is perpendicular to  $\delta \mathbf{r}_i$ , meaning  $\mathbf{Z}_i \cdot \delta \mathbf{r}_i = 0$ , so the “virtual work” (in analogy to work  $W = \int \mathbf{F} \cdot d\mathbf{r}$ ) of a constraint force vanishes. More generally, there is no *net* work from constraints, so  $\sum_i \mathbf{Z}_i \cdot \delta \mathbf{r}_i = 0$  (which holds for the actions of surfaces, rolling constraints, and similar things). The Newtonian equation of motion is  $\dot{\mathbf{p}}_i = \mathbf{F}_i + \mathbf{Z}_i$ , where  $\mathbf{F}_i$  encapsulates other forces. Vanishing virtual work gives

$$\sum_i (\dot{\mathbf{p}}_i - \mathbf{F}_i) \cdot \delta \mathbf{r}_i = 0 \quad (1.64)$$

which is the *D'Alembert principle*. This could be taken as the starting principle for classical mechanics instead of the Hamilton principle of stationary action.

Of course Eq.(1.64) is not fully satisfactory since we are now used to the idea of working with generalized coordinates rather than the cartesian vector coordinates used there. So lets transform to generalized coordinates through  $\mathbf{r}_i = \mathbf{r}_i(q, t)$ , so  $\delta \mathbf{r}_i = \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$ , where again we sum over repeated indicies (like  $j$  here). This means

$$\mathbf{F}_i \cdot \delta \mathbf{r}_i = \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \equiv Q_j \delta q_j \quad (1.65)$$

where we have defined *generalized forces*

$$Q_j \equiv \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \quad (1.66)$$

We can also transform the  $\dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i$  term using our earlier point transformation results as well as the fact that  $\frac{d}{dt} \left( \frac{\partial \mathbf{r}_i}{\partial q_j} \right) = \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t} + \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k} \dot{q}_k = \frac{\partial \mathbf{v}_i}{\partial q_j}$ . Writing out the index sums explicitly,

this gives

$$\begin{aligned}
 \sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i &= \sum_{i,j} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \\
 &= \sum_{i,j} \left( \frac{d}{dt} \left( m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left( \frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right) \delta q_j \\
 &= \sum_{i,j} \left( \frac{d}{dt} \left( m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} \right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j} \right) \delta q_j \\
 &= \sum_j \left( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} \right) \delta q_j
 \end{aligned} \tag{1.67}$$

for  $T = \frac{1}{2} \sum_i m_i \mathbf{v}_i^2$ . Together with the D'Alembert principle, we obtain the final result

$$\sum_j \left( \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j \right) \delta q_j = 0. \tag{1.68}$$

We will see momentarily that this result is somewhat more general than the Euler-Lagrange equations, containing them as a special case.

### Holonomic Constraints:

We will start by considering systems with only holonomic constraints, postponing other types of constraints to the next section. Here we can find the independent coordinates  $q_j$  with  $j = 1, \dots, N - k$  that satisfy the  $k$  constraints. This implies that the generalized virtual displacements  $\delta q_j$  are independent, so that their coefficients in Eq. (1.68) must vanish,

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} - Q_j = 0, \quad \text{for } j = 1, \dots, N - k. \tag{1.69}$$

There are several special cases of this result, which we derived from the d'Alembert principle.

1. For a conservative force  $\mathbf{F}_i = -\nabla_i V$ , then

$$Q_j = -(\nabla_i V) \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \tag{1.70}$$

where we assume that the potential can be expressed in the generalized coordinates as  $V = V(q, t)$ . Then using  $L = T - V$ , we see that Eq. (1.69) simply reproduces the Euler-Lagrange equations  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$ .

2. If  $Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt} \left( \frac{\partial V}{\partial \dot{q}_j} \right)$  for  $V = V(q, \dot{q}, t)$ , which is the case for velocity dependent forces derivable from a potential (like the electromagnetic Lorentz force), then the Euler-Lagrange equations  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$  are again reproduced.

3. If  $Q_j$  has forces obtainable from a potential as in case 2, as well as generalized forces  $R_j$  that cannot, then

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = R_j \quad (1.71)$$

is the generalization of the Euler-Lagrange equations with non-conservative generalized forces.

We will continue to use the notation  $R_j$  for generalized forces that can not be obtained from a potential. An important example of such nonconservative forces  $R_j$  is given by *friction*.

- Static friction is  $F_s \leq F_s^{\max} = \mu_s F_N$  for a normal force  $F_N$ .
- Sliding friction is  $\mathbf{F} = -\mu F_N \frac{\mathbf{v}}{v}$ , so this is a constant force that is always opposite the direction of motion (but vanishes when there is no motion).
- Rolling friction is  $\mathbf{F} = -\mu_R F_N \frac{\mathbf{v}}{v}$ .
- Fluid friction at a low velocity is  $\mathbf{F} = -bv \frac{\mathbf{v}}{v} = -b\mathbf{v}$ .

A general form for a friction force is  $\mathbf{F}_i = -h_i(v_i) \frac{\mathbf{v}_i}{v_i}$  (where as a reminder there is no implicit sum on  $i$  here since we specified  $i$  on the left-hand-side). For this form

$$R_j = - \sum_i h_i \frac{\mathbf{v}_i}{v_i} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = - \sum_i h_i \frac{\mathbf{v}_i}{v_i} \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}. \quad (1.72)$$

Simplifying further gives

$$\begin{aligned} R_j &= - \sum_i \frac{h_i}{2v_i} \frac{\partial}{\partial \dot{q}_j} (\mathbf{v}_i^2) = - \sum_i h_i \frac{\partial v_i}{\partial \dot{q}_j} = - \sum_i \frac{\partial v_i}{\partial \dot{q}_j} \frac{\partial}{\partial v_i} \int_0^{v_i} dv'_i h_i(v'_i) = - \frac{\partial}{\partial \dot{q}_j} \sum_i \int_0^{v_i} dv'_i h_i(v'_i) \\ &= - \frac{\partial \mathcal{F}}{\partial \dot{q}_j} \end{aligned} \quad (1.73)$$

where

$$\mathcal{F} = \sum_i \int_0^{v_i} dv'_i h_i(v'_i) \quad (1.74)$$

is the “dissipation function”. This is a scalar function like  $L$  so it is relatively easy to work with.

**Example:** Consider a sphere of radius  $a$  and mass  $m$  falling in a viscous fluid. Then  $T = \frac{1}{2}m'y^2$  where  $m' < m$  accounts for the mass of displaced fluid (recall Archimedes principle that the buoyant force on a body is equal to the weight of fluid the body displaces). Also

$V = m'gy$ , and  $L = T - V$ . Here  $h \propto \dot{y}$ , so  $\mathcal{F} = 3\pi\eta a \dot{y}^2$ , where by the constant of proportionality is determined by the constant  $\eta$ , which is the viscosity. From this,  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = -\frac{\partial \mathcal{F}}{\partial \dot{y}}$  gives the equation of motion  $m'\ddot{y} + m'g = -6\pi\eta a \dot{y}$ . The friction force  $6\pi\eta a \dot{y}$  is known as Stokes Law. (We will derive this equation for the friction force from first principles later on, in our discussion of fluids.) This differential equation can be solved by adding a particular solution  $y_p(t)$  to a solution of the homogeneous equation  $m'\ddot{y}_H + 6\pi\eta a \dot{y}_H = 0$ . For the time derivatives the results are  $\dot{y}_p = -m'g/(6\pi\eta a)$  and  $\dot{y}_H = A \exp(-6\pi\eta a t/m')$ , where the constant  $A$  must be determined by an initial condition. The result  $\dot{y} = \dot{y}_H + \dot{y}_p$  can be integrated in time once more to obtain the full solution  $y(t)$  for the motion.

**Example:** if we add sliding friction to the case of two masses on a plane connected by a spring (considered on problem set #1), then  $h_i = \mu_f m_i g$  for some friction coefficient  $\mu_f$ , and

$$\mathcal{F} = \mu_f g(m_1 v_1 + m_2 v_2) = \mu_f g \left( m_1 \sqrt{\dot{x}_1^2 + \dot{y}_1^2} + m_2 \sqrt{\dot{x}_2^2 + \dot{y}_2^2} \right). \quad (1.75)$$

If we switch to a suitable set of generalized coordinates  $q_j$  that simplify the equations of motion without friction, and then compute the generalized friction forces  $R_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$ , we can get the equations of motion including friction. Further details of how this friction complicates the equations of motion were provided by an example done in lecture.

## 1.5 Calculus of Variations & Lagrange Multipliers

### Calculus of Variations

In the calculus of variations, we wish to find a set of functions  $y_i(s)$  between  $s_1$  and  $s_2$  that extremize the following functional (a function of functions),

$$J[y_i] = \int_{s_1}^{s_2} ds f(y_1(s), \dots, y_n(s), \dot{y}_1(s), \dots, \dot{y}_n(s), s), \quad (1.76)$$

where for this general discussion only we let  $\dot{y}_i \equiv \frac{dy_i}{ds}$  rather than  $\frac{d}{dt}$ . To consider the action of the functional under a variation we consider  $y'_i(s) = y_i(s) + \eta_i(s)$  where  $\eta_i(s_1) = \eta_i(s_2) = 0$ , meaning that while the two endpoints are fixed during the variation  $\delta y_i = \eta_i$ , the path in between is varied. Expanding the variation of the functional integral  $\delta J = J[y'_i] - J[y_i] = 0$  to 1<sup>st</sup> order in  $\delta y_i$  we have

$$0 = \delta J = \int_{s_1}^{s_2} ds \sum_i \left[ \delta y_i \frac{\partial f}{\partial y_i} + \delta \dot{y}_i \frac{\partial f}{\partial \dot{y}_i} \right]. \quad (1.77)$$

Using integration by parts on the second term, and the vanishing of the variation at the endpoints to remove the surface term,  $\delta J$  vanishes when

$$\int_{s_1}^{s_2} \sum_i \left[ \frac{\partial f}{\partial y_i} - \frac{d}{ds} \left( \frac{\partial f}{\partial \dot{y}_i} \right) \right] \delta y_i(s) ds = 0. \quad (1.78)$$

If we identify a set of independent variations  $\delta y_j$  (for example, after imposing holonomic constraints), this can only occur if

$$\frac{\partial f}{\partial y_j} - \frac{d}{ds} \left( \frac{\partial f}{\partial \dot{y}_j} \right) = 0. \quad (1.79)$$

The scope of this calculus of variation result for extremizing the integral over  $f$  is more general than its application to classical mechanics.

**Example:** Hamilton's principle states that motion  $q_i(t)$  extremizes the action, so in this case  $s = t$ ,  $y_i = q_i$ ,  $f = L$ , and  $J = S$ . Demanding  $\delta S = 0$  then yields the Euler-Lagrange equations of motion from Eq. (1.79).

**Example:** As an example outside of classical mechanics, consider showing that the shortest distance between points on a sphere of radius  $a$  are great circles. This can be seen by minimizing the distance  $J = \int_{s_1}^{s_2} ds$  where for a spherical surface,

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{a^2(d\theta)^2 + a^2 \sin^2(\theta)(d\phi)^2} \quad (1.80)$$

since  $dr = 0$ . Taking  $s = \theta$  and  $y = \phi$ , then

$$ds = a \sqrt{1 + \sin^2(\theta) \left( \frac{d\phi}{d\theta} \right)^2} d\theta, \quad (1.81)$$

so  $f = \sqrt{1 + \sin^2(\theta)\dot{\phi}^2}$ . The solution for the minimal path is given by solving  $\frac{d}{d\theta} \left( \frac{\partial f}{\partial \dot{\phi}} \right) - \frac{\partial f}{\partial \phi} = 0$ . After some algebra these are indeed found to be great circles, described by  $\sin(\phi - \alpha) = \beta \cot(\theta)$  where  $\alpha, \beta$  are constants.

**Example:** Hamilton's principle can also be used to yield the Hamilton equations of motion, by considering the variation of a path in phase space. In this case

$$\delta J[q, p] = \delta \int_{t_1}^{t_2} dt \left[ p_i \dot{q}_i - H(q, p, t) \right] = 0 \quad (1.82)$$

must be solved with fixed endpoints:  $\delta q_i(t_1) = \delta q_i(t_2) = 0$  and  $\delta p_i(t_1) = \delta p_i(t_2) = 0$ . Here, the role of  $y_i$  is played by the  $2N$  variables  $(q_1, \dots, q_N, p_1, \dots, p_N)$ . As  $f = p_i \dot{q}_i - H$ , then

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{q}_i} \right) - \frac{\partial f}{\partial q_i} &= 0 & \implies \dot{p}_i &= -\frac{\partial H}{\partial q_i}, \\ \frac{d}{dt} \left( \frac{\partial f}{\partial \dot{p}_i} \right) - \frac{\partial f}{\partial p_i} &= 0 & \implies \dot{q}_i &= \frac{\partial H}{\partial p_i}, \end{aligned} \quad (1.83)$$

giving Hamilton's equations as expected. Note that because  $f$  is independent of  $\dot{p}_i$ , the term  $(\partial f / \partial \dot{p}_i) \delta \dot{p}_i = 0$ , and it would seem that we do not really need the condition that  $\delta p_i(t_1) = \delta p_i(t_2) = 0$  to remove the surface term. However, these conditions on the variations  $\delta p_i$  are actually *required* in order to put  $q_i$  and  $p_i$  on the same footing, including making transformation that mix up these variables (we will exploit this in detail later on when discussing canonical transformations).

It is interesting and useful to note that D'Alembert's principle

$$\left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} - R_j \right) \delta q_j = 0 \quad (1.84)$$

is a “differential” version of the equations that encode the classical dynamics, while Hamilton's principle

$$\delta J = \int_{t_1}^{t_2} dt \left( \frac{\partial L}{\partial q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) \right) \delta q_j = 0 \quad (1.85)$$

(for  $R_j = 0$  where all forces come from a potential) is an integrated version.

## Method of Lagrange Multipliers

Next we will consider the method of *Lagrange multipliers*. For simplicity we will assume there are no generalized forces outside the potential,  $R_j = 0$ , until further notice. (It is straightforward to add back the  $R_j$  to the final equations derived below.) The method of Lagrange multipliers will be useful for two situations that we will encounter:

1. When we actually want to study the forces of constraint that are holonomic.
2. When we have semi-holonomic constraints.

Let us consider  $k$  constraints for  $n$  coordinates, with  $\alpha \in \{1, \dots, k\}$  being the index running over the constraints. These holonomic or semi-holonomic constraints take the form

$$g_\alpha(q, \dot{q}, t) = a_{j\alpha}(q, t)\dot{q}_j + a_{t\alpha}(q, t) = 0 \quad (1.86)$$

where again repeated indices are summed. Thus,  $g_\alpha dt = a_{j\alpha} dq_j + a_{t\alpha} dt = 0$ . For a virtual displacement  $\delta q_j$  we have  $dt = 0$ , so

$$\sum_{j=1}^n a_{j\alpha} \delta q_j = 0, \quad (1.87)$$

which gives us  $k$  equations constraining the virtual displacements. For each equation we can multiply by a function  $\lambda_\alpha(t)$  known as *Lagrange multipliers*, and sum over  $\alpha$ , and the combination will still be zero. Adding this zero to D'Alembert's principle (taking  $R_j = 0$ ) yields

$$\left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} - \lambda_\alpha a_{j\alpha} \right] \delta q_j = 0, \quad (1.88)$$

where the sums implicitly run over both  $\alpha$  and  $j$ . It is clear that the Lagrange multiplier term is zero if we sum over  $j$  first, but now we want to consider summing first over  $\alpha$  for a fixed  $j$ . Our goal is make the term in square brackets zero. Only  $n - k$  of the virtual displacements  $\delta q_j$  are independent, so for these values of  $j$  the square brackets must vanish. For the remaining  $k$  values of  $j$  we can simply choose the  $k$  Lagrange multipliers  $\lambda_\alpha$  to force the  $k$  square bracketed equations to be satisfied. This is known as the method of Lagrange multipliers. Thus all square bracketed terms are zero, and we have the generalization of the Euler-Lagrange equations which includes terms for the constraints:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = \lambda_\alpha a_{j\alpha}. \quad (1.89)$$

This is  $n$  equations, for the  $n$  possible values of  $j$ , and on the right-hand-side we sum over  $\alpha$  for each one of these equations. The sum  $\lambda_\alpha a_{j\alpha}$  can be interpreted as a generalized constraint force  $Q_j$ . The Lagrange multipliers  $\lambda_\alpha$  and generalized coordinates  $q_j$  together form  $n + k$  parameters, and equation (1.89) in conjunction with  $g_\alpha = 0$  for each  $\alpha$  from (1.86) together form  $n + k$  equations to be solved.

There are two important cases to be considered.

1. In the holonomic case,  $f_\alpha(q, t) = 0$ . Here,  $g_\alpha = \dot{f}_\alpha = \frac{\partial f_\alpha}{\partial q_j} \dot{q}_j + \frac{\partial f_\alpha}{\partial t}$ , so  $a_{j\alpha} = \frac{\partial f_\alpha}{\partial q_j}$ . This gives

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = \sum_{\alpha=1}^k \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j} \quad (1.90)$$

for holonomic constraints. The same result can be derived from a generalized Hamilton's principle

$$J[q_j, \lambda_\alpha] = \int_{t_1}^{t_2} (L + \lambda_\alpha f_\alpha) dt \quad (1.91)$$

by demanding that  $\delta J = 0$ . It is convenient to think of  $-\lambda_\alpha f_\alpha$  as an extra potential energy that we add into  $L$  so that a particle does work if it leaves the surface defined by  $f_\alpha = 0$ . Recall that given this potential, the Force <sub>$q$</sub>  =  $-\nabla_q(-\lambda_\alpha f_\alpha) = \lambda_\alpha \nabla_q f_\alpha$ , where the derivative  $\nabla_q f_\alpha$  gives a vector that is normal to the constraint surface of constant  $f_\alpha = 0$ . This agrees with the form of our generalized force above.

2. In the semi-holonomic case, we just have  $g_\alpha = a_{j\alpha}(q, t)\dot{q}_j + a_{t\alpha}(q, t) = 0$ , with  $a_{j\alpha} = \frac{\partial g_\alpha}{\partial \dot{q}_j}$ . This gives

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = \sum_{\alpha=1}^k \lambda_\alpha \frac{\partial g_\alpha}{\partial \dot{q}_j} \quad (1.92)$$

for semi-holonomic constraints. This result cannot be derived from Hamilton's principle in general. This justifies the time we spent discussing d'Alembert's principle, which we have used to obtain (1.92). Recall that static friction imposes a no-slip constraint in the form of our equation  $g_\alpha = 0$ . For  $g \propto \dot{q}$ , the form  $\frac{\partial g}{\partial \dot{q}}$ , is consistent with the form of generalized force we derived from our dissipation function,  $\frac{\partial \mathcal{F}}{\partial \dot{q}}$  in our discussion of friction.

We end this chapter with several examples of the use of Lagrange multipliers.

**Example:** Consider a particle of mass  $m$  at rest on the top of a sphere of radius  $a$ , as shown above in Fig. 1.7. The particle is given an infinitesimal displacement  $\theta = \theta_0$  so that it slides down. At what angle does it leave the sphere?

We use the coordinates  $(r, \theta, \phi)$  but set  $\phi = 0$  by symmetry as it is not important. The constraint  $r \geq a$  is non-holonomic, but while the particle is in contact with the sphere the constraint  $f = r - a = 0$  is holonomic. To answer this question we will look for the point where the constraint force vanishes. Here  $T = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 \right)$  and  $V = mgz = mgr \cos(\theta)$  so that  $L = T - V$ , then  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = \lambda \frac{\partial f}{\partial r}$  gives

$$m\ddot{r} - mr\dot{\theta}^2 + mg \cos(\theta) = \lambda, \quad (1.93)$$

while  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial f}{\partial \theta} = 0$  gives

$$\frac{d}{dt} \left( mr^2 \dot{\theta} \right) - mgr \sin(\theta) = 0. \quad (1.94)$$

This in conjunction with  $r = a$  gives 3 equations for the 3 variables  $(r, \theta, \lambda)$ . At this point we can now consider solving the equations together. The constraint equation gives  $\dot{r} = 0$  so  $\ddot{r} = 0$ . This means

$$ma^2 \ddot{\theta} = mga \sin(\theta), \quad -ma\dot{\theta}^2 + mg \cos(\theta) = \lambda.$$

Multiply the first of these by  $\dot{\theta}$  and integrate over time, knowing that  $\dot{\theta} = 0$  when  $\theta = 0$ , gives  $\dot{\theta}^2 = \frac{2g}{a} (1 - \cos(\theta))$ . Plugging this into the second equation we find that

$$\lambda = mg(3 \cos(\theta) - 2) \quad (1.95)$$

is the radial constraint force. The mass leaves the sphere when  $\lambda = 0$  which is when  $\cos(\theta) = \frac{2}{3}$  (so  $\theta \approx 48^\circ$ ).

What if we instead imposed the constraint  $f' = r^2 - a^2 = 0$ ? If we call its Lagrange multiplier  $\lambda'$  we would get  $\lambda' \frac{\partial f'}{\partial r} = 2a\lambda'$  when  $r = a$ , so  $2a\lambda' = \lambda$  is the constraint force from before. The meaning of  $\lambda'$  is different, and it has different units, but we still have the same constraint force.

What are the equations of motion for  $\theta > \arccos(\frac{2}{3})$ ? Now we no longer have the constraint so

$$m\ddot{r} - mr\dot{\theta}^2 + mg \cos(\theta) = 0 \quad \text{and} \quad \frac{d}{dt} \left( mr^2\dot{\theta} \right) - mgr \sin(\theta) = 0.$$

The initial conditions are  $r_1 = a$ ,  $\theta_1 = \arccos(\frac{2}{3})$ ,  $\dot{r}_1 = 0$ , and  $\dot{\theta}_1^2 = \frac{2g}{3a}$  from before. Simpler coordinates are  $x = r \sin(\theta)$  and  $z = r \cos(\theta)$ , giving

$$L = \frac{m}{2} (\dot{x}^2 + \dot{z}^2) - mgz, \quad (1.96)$$

so  $\ddot{x} = 0$  and  $\ddot{z} = -g$  with initial conditions  $z_1 = \frac{2a}{3}$ ,  $x_1 = \frac{\sqrt{5}a}{3}$ , and the initial velocities  $v_{z1} = \dot{z}(t_1) = -a \sin \theta_1 \dot{\theta}_1 = -\frac{\sqrt{5}a}{3} \left(\frac{2g}{3a}\right)^{1/2}$  and  $v_{x1} = \dot{x}(t_1) = a \cos \theta_1 \dot{\theta}_1 = \frac{2a}{3} \left(\frac{2g}{3a}\right)^{1/2}$ . Here  $t_1$  is the time when the mass leaves the sphere. This means

$$x(t) = v_{x1}(t - t_1) + x_1, \quad (1.97)$$

$$z(t) = -\frac{g}{2} (t - t_1)^2 + v_{z1}(t - t_1) + z_1. \quad (1.98)$$

The time  $t_1$  can be found from

$$\dot{\theta}^2 = \frac{2g}{a} (1 - \cos(\theta)) = \frac{4g}{a} \sin^2 \left( \frac{\theta}{2} \right), \quad (1.99)$$

so  $t_1 = \sqrt{\frac{a}{4g}} \int_{\theta_0}^{\arccos(\frac{2}{3})} \frac{d\theta}{\sin(\frac{\theta}{2})}$  where  $\theta_0$  is the small initial angular displacement from the top of the sphere. Note that the calculation of  $t_1$  is the only place where we see that it was important to have a small  $\theta_0$  different from zero.

**Example:** Consider a hoop of radius  $a$  and mass  $m$  rolling down an inclined plane of angle  $\phi$  without slipping as shown in Fig. 1.11, where we define the  $\hat{x}$  direction as being parallel to the ramp as shown. What is the friction force of constraint, and how does the acceleration compare to the case where the hoop is sliding rather than rolling?

The no-slip constraint means  $a\dot{\theta} = \dot{x}$ , so  $h = a\dot{\theta} - \dot{x} = 0$ , which can be made holonomic but which we will treat as semi-holonomic. Here the kinetic energy has contributions from the center mass motion of the hoop and its rotation,  $T = T_{\text{CM}} + T_{\text{rotation}} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2$ , where the hoops moment of inertia is  $I_{\text{hoop}} = ma^2$ . (We will discuss moments of inertia in detail in chapter 2.) Meanwhile,  $V = mg(l - x) \sin(\phi)$  where we are free to pick the length  $l$  where  $V(x = l) = 0$ , such as at the bottom of the ramp. This means

$$L = T - V = \frac{m}{2} \dot{x}^2 + \frac{ma^2}{2} \dot{\theta}^2 + mg(x - l) \sin(\phi). \quad (1.100)$$

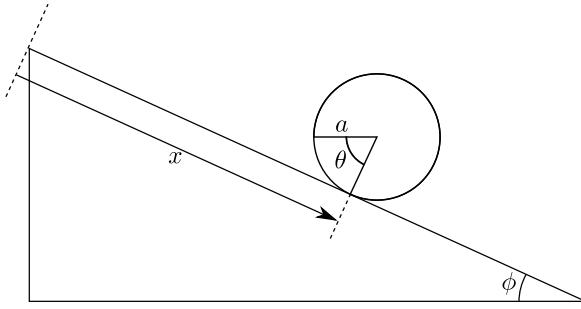


Figure 1.11: Hoop rolling on inclined plane

The equations of motion from  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = \lambda \frac{\partial h}{\partial \dot{x}}$  and  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = \lambda \frac{\partial h}{\partial \dot{\theta}}$  are

$$m\ddot{x} - mg \sin(\phi) = -\lambda \quad \text{and} \quad ma^2\ddot{\theta} = \lambda a, \quad (1.101)$$

along with  $\dot{x} = a\dot{\theta}$ . Taking a time derivative of the constraint gives  $\ddot{x} = a\ddot{\theta}$ , so  $m\ddot{x} = \lambda$ , and  $\ddot{x} = \frac{g}{2} \sin(\phi)$ . This is one-half of the acceleration of a sliding mass. Plugging this back in we find that

$$\lambda = \frac{1}{2}mg \sin(\phi) \quad (1.102)$$

is the friction force in the  $-\hat{x}$  direction for the no-sliding constraint, and also  $\ddot{\theta} = \frac{g}{2a} \sin(\phi)$ .

**Example:** Consider a wedge of mass  $m_2$  and angle  $\alpha$  resting on ice and moving without friction. Let us also consider a mass  $m_1$  sliding without friction on the wedge and try to find the equations of motion and constraint forces. The constraints are that  $y_2 = 0$  so the

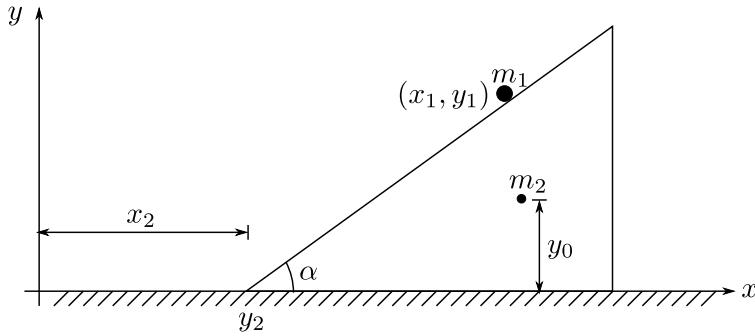


Figure 1.12: Wedge sliding on ice

wedge is always sitting on ice, and  $\frac{y_1 - y_2}{x_1 - x_2} = \tan(\alpha)$  so the point mass is always sitting on the wedge. We will ignore the constraint force for no rotation of the wedge, and aim to find the constraint forces for these two constraints, and the equations for the motion.

The kinetic energy is simply  $T = \frac{m_1}{2}(\dot{x}_1^2 + \dot{y}_1^2) + \frac{m_2}{2}(\dot{x}_2^2 + \dot{y}_2^2)$ , while the potential energy is  $V = m_1gy_1 + m_2g(y_2 + y_0)$ , where  $y_0$  is the height of the center of mass of the wedge. Then  $L = T - V$ , with the constraints  $f_1 = (y_1 - y_2) - (x_1 - x_2)\tan(\alpha) = 0$  and  $f_2 = y_2 = 0$ . The equations of motion from the Euler-Lagrange equations with these holonomic constraints are

$$\begin{aligned}\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} &= \lambda_1 \frac{\partial f_1}{\partial x_1} + \lambda_2 \frac{\partial f_2}{\partial x_1} \quad \Rightarrow \quad m_1\ddot{x}_1 = -\lambda_1 \tan(\alpha), \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_1} - \frac{\partial L}{\partial y_1} &= \lambda_1 \frac{\partial f_1}{\partial y_1} + \lambda_2 \frac{\partial f_2}{\partial y_1} \quad \Rightarrow \quad m_1\ddot{y}_1 + m_1g = \lambda_1, \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} &= \lambda_1 \frac{\partial f_1}{\partial x_2} + \lambda_2 \frac{\partial f_2}{\partial x_2} \quad \Rightarrow \quad m_2\ddot{x}_2 = \lambda_1 \tan(\alpha), \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{y}_2} - \frac{\partial L}{\partial y_2} &= \lambda_1 \frac{\partial f_1}{\partial y_2} + \lambda_2 \frac{\partial f_2}{\partial y_2} \quad \Rightarrow \quad m_2\ddot{y}_2 + m_2g = -\lambda_1 + \lambda_2,\end{aligned}\tag{1.103}$$

which in conjunction with  $y_1 - y_2 = (x_1 - x_2)\tan(\alpha)$  and  $y_2 = 0$  is six equations for six variables. We number them (1) to (6) and now consider putting the equations together. Equation (6) gives  $\ddot{y}_2 = 0$  so (4) gives  $m_2g = \lambda_2 - \lambda_1$  where  $\lambda_2$  is the force of the ice on the wedge and  $\lambda_1$  is the *vertical* force (component) of the wedge on the point mass. Adding (1) and (3) gives  $m_1\ddot{x}_1 + m_2\ddot{x}_2 = 0$  meaning that the CM of  $m_1$  and  $m_2$  has no overall force acting on it.

Additionally, as (5) implies  $\ddot{y}_1 = (\ddot{x}_1 - \ddot{x}_2)\tan(\alpha)$ , then using (1), (2), and (3) we find the constant force

$$\lambda_1 = \frac{g}{\frac{1}{m_1 \cos^2(\alpha)} + \frac{\tan^2(\alpha)}{m_2}}.\tag{1.104}$$

With this result in hand we can use it in (1), (2), and (3) to solve for the trajectories. Since

$$\begin{aligned}\ddot{x}_2 &= \frac{\tan(\alpha)}{m_2} \lambda_1, \\ \ddot{x}_1 &= -\frac{\tan(\alpha)}{m_1} \lambda_1, \\ \ddot{y}_1 &= \frac{\lambda_1}{m_1} - g,\end{aligned}\tag{1.105}$$

the accelerations are constant. As a check on our results, if  $m_2 \rightarrow \infty$ , then  $\ddot{x}_2 = 0$  so indeed the wedge is fixed; and for this case,  $\ddot{x}_1 = -g \sin(\alpha) \cos(\alpha)$  and  $\ddot{y}_1 = -g \sin^2(\alpha)$  which both vanish as  $\alpha \rightarrow 0$  as expected (since in that limit the wedge disappears, flattening onto the icy floor below it).

# Chapter 2

## Rigid Body Dynamics

### 2.1 Coordinates of a Rigid Body

A set of  $N$  particles forms a *rigid body* if the distance between any 2 particles is fixed:

$$r_{ij} \equiv |\mathbf{r}_i - \mathbf{r}_j| = c_{ij} = \text{constant}. \quad (2.1)$$

Given these constraints, how many generalized coordinates are there?

If we know 3 non-collinear points in the body, the remaining points are fully determined by triangulation. The first point has 3 coordinates for translation in 3 dimensions. The second point has 2 coordinates for spherical rotation about the first point, as  $r_{12}$  is fixed. The third point has one coordinate for circular rotation about the axis of  $\mathbf{r}_{12}$ , as  $r_{13}$  and  $r_{23}$  are fixed. Hence, there are *6 independent coordinates*, as represented in Fig. 2.1. The result that there are six continuous coordinates is independent of  $N$ , so it also applies to a continuous body (in the limit of  $N \rightarrow \infty$ ).

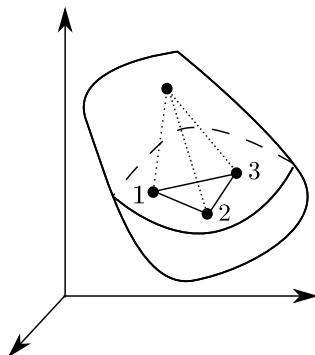


Figure 2.1: Three non-collinear points can be fully determined by using 6 continuous coordinates. Since the distances between any two other points are fixed in the rigid body, any other point of the body is fully determined by the distance to these 3 points (and a discrete orientation which is not a dynamical variable).

The translations of the body require three spatial coordinates. These translations can be taken from any fixed point in the body. Typically the fixed point is the center of mass (CM), defined as:

$$\mathbf{R} = \frac{1}{M} \sum_i m_i \mathbf{r}_i, \quad (2.2)$$

where  $m_i$  is the mass of the  $i$ -th particle and  $\mathbf{r}_i$  the position of that particle with respect to a fixed origin and set of axes (which will notationally be unprimed) as in Fig. 2.2. In the case of a continuous body, this definition generalizes as:

$$\mathbf{R} = \frac{1}{M} \int_{\mathcal{V}} \mathbf{r} \rho(\mathbf{r}) d\mathcal{V}, \quad (2.3)$$

where  $\rho(\mathbf{r})$  is the mass density at position  $\mathbf{r}$  and we integrate over the volume  $\mathcal{V}$ .

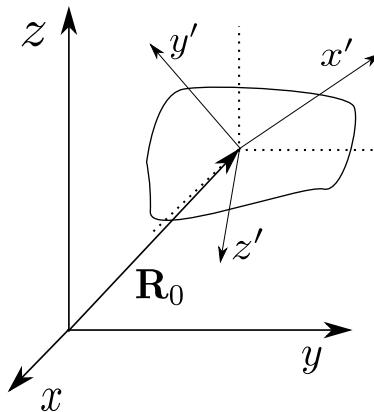


Figure 2.2: The three translational coordinates correspond to the position of the Center of Mass, and the three rotational coordinates correspond to the three angles necessary to define the orientation of the axis fixed with the body.

Rotations of the body are taken by fixing axes with respect to the body (we will denote these body fixed axes with primes) and describing their orientation with respect to the unprimed axes by 3 angles  $(\phi, \theta, \psi)$ .

A particularly useful choice of angles are called *Euler angles*. The angle  $\phi$  is taken as a rotation about the  $z$ -axis, forming new  $\tilde{x}$ - and  $\tilde{y}$ -axes while leaving the  $z$ -axis unchanged, as shown in Fig. 2.3. The angle  $\theta$  is then taken as a rotation about the  $\tilde{x}$ -axis, forming new  $\tilde{y}'$ - and  $\tilde{z}'$ -axes while leaving the  $\tilde{x}$ -axis unchanged, as shown in Fig. 2.4. Finally, the angle  $\psi$  is taken as a rotation about the  $\tilde{z}'$ -axis, forming new  $x'$ - and  $y'$ -axes while leaving the  $\tilde{z}'$ -axis unchanged, as shown in Fig. 2.5. (The  $\tilde{x}$ -axis is called the line of nodes, as it is the intersection of the  $xy$ - and  $x'y'$ -planes.)

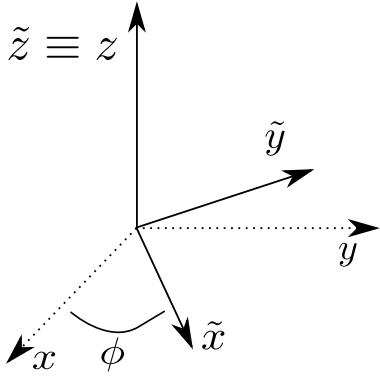


Figure 2.3: First rotation is by  $\phi$  around the original  $z$  axis.

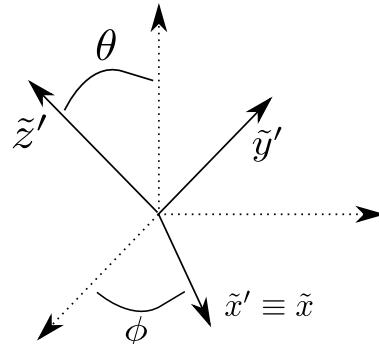


Figure 2.4: Second rotation is by  $\theta$  around the intermediate  $\tilde{x}$  axis.

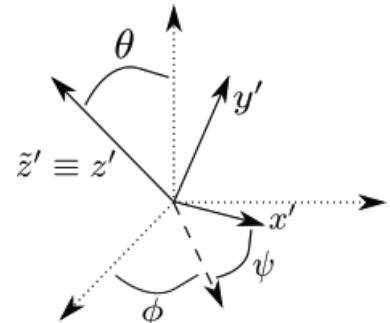


Figure 2.5: Final rotation is by  $\psi$  around the final  $z'$  axis.

Rotations can be described by  $3 \times 3$  matrices  $U$ . This means each rotation step can be described as a matrix multiplication. Where  $\mathbf{r} = (x, y, z)$ , then

$$\tilde{\mathbf{r}} = U_\phi \mathbf{r} = \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \quad (2.4)$$

Similar transformations can be written for the other terms:

$$\tilde{\mathbf{r}}' = U_\theta \tilde{\mathbf{r}} \quad , \quad \mathbf{r}' = U_\psi \tilde{\mathbf{r}}' = U_\psi U_\theta \tilde{\mathbf{r}} = U_\psi U_\theta U_\phi \mathbf{r}.$$

Defining the total transformation as  $U$ , it can be written as:

$$U \equiv U_\psi U_\theta U_\phi \Rightarrow \mathbf{r}' = U \mathbf{r}. \quad (2.5)$$

Care is required with the order of the terms since the matrices don't commute. Writing  $U$  out explicitly:

$$U = \begin{bmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & \sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (2.6)$$

All rotation matrices, including  $U_\phi$ ,  $U_\theta$ ,  $U_\psi$ , and  $U$  are *orthogonal*. Orthogonal matrices  $W$  satisfy

$$W^\top W = WW^\top = \mathbb{1} \Leftrightarrow W^\top = W^{-1}, \quad (2.7)$$

where  $\mathbb{1}$  refers to the identity matrix and  $\top$  to the transpose. This ensures that the length of a vector is invariant under rotations:

$$\mathbf{r}'^2 = \mathbf{r}^\top (W^\top W) \mathbf{r} = \mathbf{r}^2. \quad (2.8)$$

Orthogonal matrices  $W$  have 9 entries but need to fulfill 6 conditions from orthogonality, leaving only 3 free parameters, corresponding to the 3 angles necessary to determine the rotation.

We can also view  $\mathbf{r}' = U\mathbf{r}$  as a transformation from the vector  $\mathbf{r}$  to the vector  $\mathbf{r}'$  in the same coordinate system. This is an active transformation, as opposed to the previous perspective which was a passive transformation.

Finally, note that inversions like

$$U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{or} \quad U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad (2.9)$$

are not rotations. These have  $\det(U) = -1$ , so they can be forbidden by demanding that  $\det(U) = 1$ . All orthogonal matrices have  $\det(W) = \pm 1$  because  $\det(W^\top W) = (\det(W))^2 = 1$ . In the language of group theory, the restriction to  $\det(W) = 1$  gives the special orthogonal group  $SO(3)$  as opposed to simply  $O(3)$ , the orthogonal group. We disregard the  $\det(U) = -1$  subset of transformations because it is impossible for the system to undergo these transformations without the distance between the particles jumping abruptly in the process (they are not continuously connected to the identity transformation), so they are not allowed mechanical motions of our rigid body.

Intuitively, we could rotate the coordinates  $(x, y, z)$  directly into the coordinates  $(x', y', z')$  by picking the right axis of rotation. In fact, the *Euler theorem* states that a general displacement of a rigid body with one point fixed is equivalent to a single rotation about *some axis* that runs through the fixed point. This theorem will be true if a general rotation  $U$  leaves some axis fixed, which is satisfied by

$$U\mathbf{r} = \mathbf{r} \quad (2.10)$$

for any point  $\mathbf{r}$  on this axis. This is an eigenvalue equation for  $U$  with eigenvalue 1. To better understand this, we need to use the linear algebra concepts of eigenvalue equations, which we now review.

Although the notion of an eigenvalue equation generally holds for linear operators, for now the discussion will be restricted to orthogonal rotation matrices  $U$ . The eigenvalue equation is

$$U\xi = \lambda\xi, \quad (2.11)$$

where  $\xi$  is an eigenvector and  $\lambda$  is the associated eigenvalue. Rewriting this as

$$(U - \lambda\mathbb{1})\xi = 0 \quad (2.12)$$

requires that  $\det(U - \lambda\mathbb{1}) = 0$ , so that  $U - \lambda\mathbb{1}$  is not invertible and the solution can be non-trivial,  $\xi \neq 0$ .  $\det(U - \lambda\mathbb{1}) = 0$  is a cubic equation in  $\lambda$ , which has 3 solutions, which are the eigenvalues  $\lambda_\alpha$  for  $\alpha \in \{1, 2, 3\}$ . The associated eigenvectors are  $\xi^{(\alpha)}$  and satisfy

$$U\xi^{(\alpha)} = \lambda_\alpha\xi^{(\alpha)}, \quad (2.13)$$

where no implicit sum over repeated indices is taken. Forming a matrix from the resulting eigenvectors as columns:

$$X = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \xi^{(1)} & \xi^{(2)} & \xi^{(3)} \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \quad (2.14)$$

then we can rewrite Eq.(2.13) as

$$UX = X \cdot \text{diag}(\lambda_1, \lambda_2, \lambda_3) \Rightarrow X^{-1}UX = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \quad (2.15)$$

This means  $X$  diagonalizes  $U$ . Since  $U$  is orthogonal, the matrix  $X$  is unitary ( $X^\dagger X = XX^\dagger = \mathbb{1}$ ). Note that  $^\top$  indicates transposition whereas  $^\dagger$  indicates Hermitian conjugation (i.e. complex conjugation  $^*$  combined with transposition  $^\top$ ).

Next we note that since  $\det(U) = 1$ , then  $\lambda_1\lambda_2\lambda_3 = 1$ . Additionally,  $|\lambda_\alpha|^2 = 1$  for each  $\alpha$  because:

$$U\xi = \lambda\xi \Rightarrow \xi^\dagger U^\top = \lambda^* \xi^\dagger \Rightarrow \xi^\dagger \xi = \xi^\dagger U^\top U \xi = |\lambda|^2 \xi^\dagger \xi. \quad (2.16)$$

Finally, if  $\lambda$  is an eigenvalue, then so is  $\lambda^*$ , since we can take the complex conjugate of the original equation:

$$U\xi = \lambda\xi \Rightarrow U\xi^* = \lambda^* \xi^* \quad (2.17)$$

where  $\xi^*$  is still a column vector but with its elements undergoing complex conjugation with respect to  $\xi$ . Without loss of generality, let us say for a rotation matrix  $U$  that  $\lambda_2 = \lambda_3^*$ . Then  $1 = \lambda_1|\lambda_2|^2 = \lambda_1$ , so one of the eigenvalues is 1, giving Eq.(2.10), and thus proving Euler's Theorem. The associated eigenvector  $\xi^{(1)}$  to the eigenvalue  $\lambda_1 = 1$  is the rotation axis, and if  $\lambda_2 = \lambda_3^* = e^{i\Phi}$  then  $\Phi$  is the rotation angle about that axis.

In fact, we can make good use of our analysis of Euler's theorem. Together the rotation axis and rotation angle can be used to define the instantaneous *angular velocity*  $\omega(t)$  such that:

$$|\omega| = \frac{d\Phi}{dt} \quad \text{and} \quad \omega \parallel \xi^{(1)}. \quad (2.18)$$

The angular velocity will play an important role in our discussion of time dependence with rotating coordinates in the next section. If we consider several consecutive displacements of the rigid body, then each can have its own axis  $\xi^{(1)}$  and its own  $\dot{\Phi}$ , so  $\omega$  changes at each instance of time, and hence  $\omega = \omega(t)$  (for the entire rigid body).

## 2.2 Time Evolution with Rotating Coordinates

Lets use unprimed axes  $(x, y, z)$  for the fixed (inertial) axes, with fixed basis vectors  $\mathbf{e}_i$ . We will also use primed axes  $(x', y', z')$  for the body axes with basis vectors  $\mathbf{e}'_i$ .

If we consider *any* vector then it can be decomposed with either set of basis vectors:

$$\mathbf{b} = \sum_i b_i \mathbf{e}_i = \sum_i b'_i \mathbf{e}'_i. \quad (2.19)$$

For fixed axes basis vectors by definition  $\dot{\mathbf{e}}_i = 0$ , while for those in the body frame,

$$\dot{\mathbf{e}}'_i = \boldsymbol{\omega}(t) \times \mathbf{e}'_i \quad (2.20)$$

meaning vectors of fixed length undergo a rotation at a time  $t$ . The derivation of this result is shown in the figure on the right, by considering the change to the vector after an infinitesimal time interval  $dt$ .

Summing over repeated indices, this means:

$$\dot{\mathbf{b}} = \dot{b}_i \mathbf{e}_i = \dot{b}'_i \mathbf{e}'_i + \boldsymbol{\omega}(t) \times (b'_i \mathbf{e}'_i) = \dot{b}'_i \mathbf{e}'_i + \boldsymbol{\omega}(t) \times \mathbf{b}$$

Defining  $\frac{d}{dt}$  as the time evolution in the fixed (F) frame and  $\frac{d_R}{dt}$  the time evolution in the rotating/body (R) frame, then vectors evolve in time according to

$$\frac{d\mathbf{b}}{dt} = \frac{d_R \mathbf{b}}{dt} + \boldsymbol{\omega} \times \mathbf{b}. \quad (2.21)$$

As a mnemonic we have the operation “ $(d/dt) = d_R/dt + \boldsymbol{\omega} \times$ ” which can act on any vector.

Let us apply this to the position  $\mathbf{r}$  of a particle of mass  $m$ , which gives

$$\frac{d\mathbf{r}}{dt} = \frac{d_R \mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r} \Leftrightarrow \mathbf{v}_F = \mathbf{v}_R + \boldsymbol{\omega} \times \mathbf{r}. \quad (2.22)$$

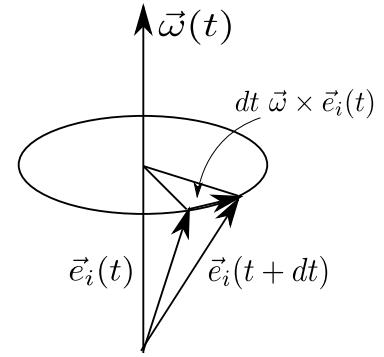
Taking another time derivative gives us the analog for acceleration:

$$\begin{aligned} \frac{\mathbf{F}}{m} &= \frac{d\mathbf{v}_F}{dt} = \frac{d_R \mathbf{v}_F}{dt} + \boldsymbol{\omega} \times \mathbf{v}_F \\ &= \frac{d_R \mathbf{v}_R}{dt} + \frac{d_R \boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \frac{d_R \mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{v}_R + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}). \end{aligned} \quad (2.23)$$

As  $\frac{d_R \mathbf{r}}{dt} = \mathbf{v}_R$  is the velocity within the body frame and  $\frac{d_R \mathbf{v}_R}{dt} = \mathbf{a}_R$  is the acceleration within the body frame, and  $d_R \boldsymbol{\omega}/dt = d\boldsymbol{\omega}/dt = \dot{\boldsymbol{\omega}}$  then

$$m\mathbf{a}_R = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times \mathbf{v}_R - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad (2.24)$$

gives the acceleration in the body frame with respect to the forces that seem to be present in that frame. The terms  $-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$  and  $-2m\boldsymbol{\omega} \times \mathbf{v}_R$  are, respectively, the centrifugal



and Coriolis fictitious forces respectively, while the last term  $-m \frac{d_{\text{R}}\omega}{dt} \times \mathbf{r}$  is a fictitious force that arises from non-uniform rotational motion, so that there is angular acceleration within the body frame. The same result could also have been obtained with the Euler-Lagrange equations for  $L$  in the rotating coordinates:

$$L = \frac{m}{2}(\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r})^2 - V, \quad (2.25)$$

and you will explore this on a problem set.

Note that the centrifugal term is radially outward and perpendicular to the rotation axis. To see this, decompose  $\mathbf{r}$  into components parallel and perpendicular to  $\boldsymbol{\omega}$ ,  $\mathbf{r} = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp}$ , then  $\boldsymbol{\omega} \times \mathbf{r} = \boldsymbol{\omega} \times \mathbf{r}_{\perp}$ , so  $-\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{\perp}) = w^2 \mathbf{r}_{\perp}$ . This term is present for any rotating body. On the other hand, the Coriolis force is nonzero when there is a nonzero velocity in the rotating/body frame:  $\mathbf{v}_R \neq 0$ .

**Example:** Consider the impact of the Coriolis force on projectile motion on the rotating Earth, where the angular velocity is  $\omega_{\text{Earth}} = \frac{2\pi}{24 \times 3600 \text{ s}} \approx 7.3 \times 10^{-5} \text{ s}^{-1}$ . We work out the cross-product  $-\boldsymbol{\omega} \times \mathbf{v}_r$  as shown in Fig. 2.6 for a particle in the northern hemisphere, where  $\boldsymbol{\omega}$  points to the north pole. Thus a projectile in the northern/southern hemisphere would be perturbed to the right/left relative to its velocity direction  $\mathbf{v}_r$ .

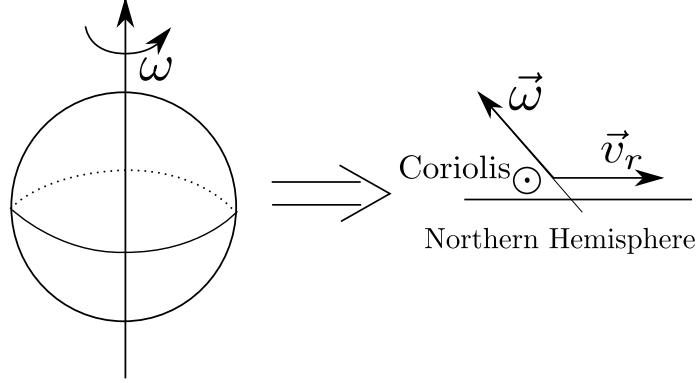
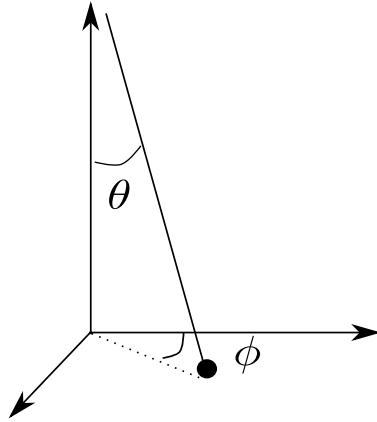
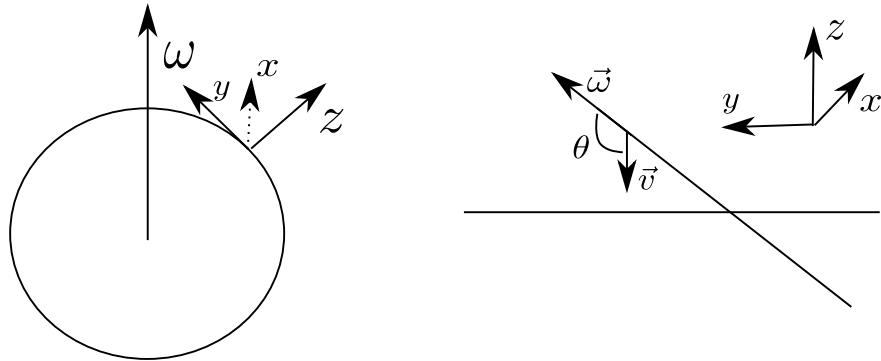


Figure 2.6: For a projectile, in the Northern Hemisphere, the Coriolis pushes it to its right, relative to its direction of motion.

**Exercise:** Consider a Foucault pendulum which hangs from a rigid rod, but is free to oscillate in two angular directions, as shown in Fig. 2.2. For  $\theta \ll 1$  and working to first order in the small  $\boldsymbol{\omega}$ , the result derived from the Coriolis force gives  $\dot{\phi} \approx \omega_{\text{Earth}} \sin(\lambda)$ . Here  $\lambda$  is the latitude angle measured from equator. The precession is clockwise in the northern hemisphere, and is maximized at the north pole where  $\lambda = 90^\circ$ .



**Example:** Consider the Coriolis deflection of a freely falling body on Earth in the northern hemisphere. We use the coordinate system shown below, where  $z$  is perpendicular to the surface of the earth and  $y$  is parallel to the earth's surface and points towards the north pole.



Working to first order in the small  $\omega$  gives us

$$m\mathbf{a}_R = m\dot{\mathbf{v}}_R = -mg\hat{z} - 2m\boldsymbol{\omega} \times \mathbf{v}, \quad (2.26)$$

where the centrifugal terms of order  $O(\omega^2)$  are dropped. As an initial condition we take  $\mathbf{v}(t=0) = v_0\hat{z}$ . The term  $-\boldsymbol{\omega} \times \mathbf{v}$  points along  $\hat{x}$ , so:

$$\ddot{z} = -g + O(\omega^2) \Rightarrow v_z = v_0 - gt \quad (2.27)$$

Moreover implementing the boundary condition that  $\dot{x}(t=0) = 0$ :

$$\ddot{x} = -2(\boldsymbol{\omega} \times \mathbf{v})_x = -2\omega \sin(\theta)v_z(t) \Rightarrow \dot{x} = -2\omega \sin(\theta) \left( v_0 t - \frac{g}{2} t^2 \right). \quad (2.28)$$

Taking also  $x(t=0) = 0$ , and integrating one further time, the motion in the  $x$  direction is:

$$x(t) = -2\omega \sin(\theta) \left( \frac{v_0}{2} t^2 - \frac{g}{6} t^3 \right). \quad (2.29)$$

Lets consider this effect for a couple simple cases. If the mass  $m$  is dropped from a height  $z(t = 0) = h_{\max}$  with zero velocity,  $v_0 = 0$ , then:

$$z = h_{\max} - \frac{g}{2}t^2 \quad (2.30)$$

and the mass reaches the floor at time

$$t_1 = \sqrt{\frac{2h_{\max}}{g}}. \quad (2.31)$$

From Eq.(2.28) we see that  $\dot{x} > 0$  for all  $t$ , and that:

$$x(t = t_1) = \frac{2\sqrt{2}\omega \sin(\theta)h_{\max}^{3/2}}{3\sqrt{g}} > 0. \quad (2.32)$$

However, if the mass  $m$  is thrown up with an initial  $\dot{z}(t = 0) = v_0 > 0$  from the ground ( $z = 0$ ), then :

$$z(t) = v_0 t - \frac{g}{2}t^2 > 0. \quad (2.33)$$

Here the particle rises to a maximum height  $z = v_0^2/(2g)$  at time  $t = v_0/g$ , and then falls back to earth. Using Eq.(2.28) we see that  $\dot{x} < 0$  for all  $t$ . If  $t_1$  is the time it reaches the ground again ( $t_1 = \frac{2v_0}{g}$ ), then:

$$x(t = t_1) = -\frac{4\omega \sin(\theta)v_0^3}{3g^2} < 0. \quad (2.34)$$

## 2.3 Kinetic Energy, Angular Momentum, and the Moment of Inertia Tensor for Rigid Bodies

Returning to rigid bodies, consider one built out of  $N$  fixed particles. The kinetic energy is best expressed using CM coordinates, where  $\mathbf{R}$  is the CM and we here take  $\mathbf{r}_i$  to be the *displacement* of particle  $i$  relative to the CM. Once again making sums over repeated subscripts as implicit, the kinetic energy ( $T$ ) of the system is given by:

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum m_i \dot{\mathbf{r}}_i^2. \quad (2.35)$$

As the body is rigid, then points cannot translate relative to the body but can only rotate so that  $\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i$ . The rotational kinetic energy is then

$$T_R = \frac{1}{2}\sum m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2}\sum m_i (\boldsymbol{\omega} \times \mathbf{r}_i)^2 = \frac{1}{2}\sum m_i [\boldsymbol{\omega}^2 \mathbf{r}_i^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_i)^2]. \quad (2.36)$$

Labeling Cartesian indices with  $a$  and  $b$  to reserve  $i$  and  $j$  for particle indices, then we can write out this result making the indices all explicit as

$$T_R = \frac{1}{2} \sum_{i,a,b} m_i (\delta_{ab} \mathbf{r}_i^2 - r_{ia} r_{ib}) \omega_a \omega_b. \quad (2.37)$$

It is convenient to separate out the parts in this formula that depend on the shape and distributions of masses in the body by defining the *moment of inertia tensor*  $\hat{I}$  for the discrete body as

$$\hat{I}_{ab} \equiv \sum_i m_i (\delta_{ab} \mathbf{r}_i^2 - r_{ia} r_{ib}). \quad (2.38)$$

The analog for a continuous body of mass density  $\rho(\mathbf{r})$  is:

$$\hat{I}_{ab} \equiv \int_V (\mathbf{r}^2 \delta_{ab} - r_a r_b) \rho(\mathbf{r}) dV. \quad (2.39)$$

In terms of the moment of inertia tensor, the kinetic energy from rotation can now be written as:

$$T_R = \frac{1}{2} \sum_{a,b} \hat{I}_{ab} \omega_a \omega_b = \frac{1}{2} \boldsymbol{\omega} \cdot \hat{I} \cdot \boldsymbol{\omega}, \quad (2.40)$$

where in the last step we adopt a convenient matrix multiplication notation.

The moment of inertia tensor can be written with its components as a matrix in the form

$$\hat{I} = \sum_i m_i \begin{bmatrix} y_i^2 + z_i^2 & -x_i y_i & -x_i z_i \\ -x_i y_i & x_i^2 + z_i^2 & -y_i z_i \\ -x_i z_i & -y_i z_i & x_i^2 + y_i^2 \end{bmatrix}, \quad (2.41)$$

where the diagonal terms are the “moments of inertia” and the off-diagonal terms are the “products of inertia”. Note also that  $\hat{I}$  is symmetric in any basis, so  $\hat{I}_{ab} = \hat{I}_{ba}$ .

**Special case:** if the rotation happens about only one axis which can be defined as the  $z$ -axis for convenience so that  $\boldsymbol{\omega} = (0, 0, \omega)$ , then  $T_R = \frac{1}{2} \hat{I}_{zz} \omega^2$  which reproduces the simpler and more familiar scalar form of the moment of inertia.

Lets now let  $\mathbf{r}_i$  be measured from a stationary point in the rigid body, which need not necessarily be the CM. The *angular momentum* can be calculated about this fixed point. Since  $\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$ , we can write the angular momentum as:

$$\mathbf{L} = m_i \mathbf{r}_i \times \mathbf{v}_i = m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = m_i [\mathbf{r}_i^2 \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_i) \mathbf{r}_i]. \quad (2.42)$$

Writing out the components

$$L_a = \sum_i m_i (\mathbf{r}_i^2 \omega_a - (\boldsymbol{\omega} \cdot \mathbf{r}_i) r_{ia}) = \sum_{i,b} \omega_b m_i (\delta_{ab} \mathbf{r}_i^2 - r_{ia} r_{ib}) = \sum_b \hat{I}_{ab} \omega_b, \quad (2.43)$$

which translates to the matrix equation:

$$\mathbf{L} = \hat{\mathbf{I}} \cdot \boldsymbol{\omega}. \quad (2.44)$$

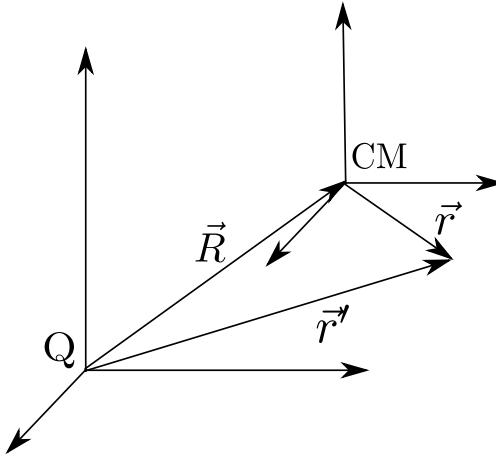
This allows us to write the corresponding rotational kinetic energy about the fixed point as:

$$T_R = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}. \quad (2.45)$$

Note that in general,  $\mathbf{L}$  is *not* parallel to  $\boldsymbol{\omega}$ . We will see an explicit example of this below. Also note that the formula that we used for  $\hat{\mathbf{I}}$  in this case is the same as we gave above. We use these formulas whether or not  $\mathbf{r}_i$  is taken with respect to the CM. (Do note that if the body is translating, such that no point in the body is fixed, then the CM point has a special role since the translational and rotational kinetic energy can be separated when using this point, and not in general other points.)

It is useful to pause to see what precisely the calculation of  $\hat{\mathbf{I}}$  depends on. Since it involves components of the vectors  $r_i$  it depends on *the choice of the origin* for the rotation. Furthermore the entries of the matrix  $\hat{\mathbf{I}}_{ab}$  obviously depend on the *orientation of the axes* used to define the components labeled by  $a$  and  $b$ . Given this, it is natural to ask whether given the result for  $\hat{\mathbf{I}}_{ab}$  with one choice of axes and orientation, whether we can determine an  $\hat{\mathbf{I}}'_{a'b'}$  for a different origin and axes orientation. This is always possible with the help of a couple of theorems.

**The parallel axis theorem:** Given  $\hat{\mathbf{I}}^{\text{CM}}$  about the CM, it is simple to find  $\hat{\mathbf{I}}^Q$  about a different point Q with the *same* orientation for the axes. Referring to the figure below,



we define  $\mathbf{r}'_i$  as the coordinate of a particle  $i$  in the rigid body with respect to point Q and  $\mathbf{r}_i$  to be the coordinate of that particle with respect to the CM, so that:

$$\mathbf{r}'_i = \mathbf{R} + \mathbf{r}_i. \quad (2.46)$$

By definition of the CM:

$$\sum_i m_i \mathbf{r}_i = 0 \quad \text{and we let} \quad M = \sum_i m_i. \quad (2.47)$$

The tensor of inertia around the new axis is then:

$$\begin{aligned} \hat{I}_{ab}^Q &= m_i (\delta_{ab} \mathbf{r}'_i^2 - r'_{ia} r'_{ib}) \\ &= m_i \left( \delta_{ab} (\mathbf{r}_i^2 + 2\mathbf{r}_i \cdot \mathbf{R} + \mathbf{R}^2) - \mathbf{r}_{ia} \mathbf{r}_{ib} - \mathbf{r}_{ia} \mathbf{R}_b - \mathbf{R}_a \mathbf{r}_{ib} - \mathbf{R}_a \mathbf{R}_b \right), \end{aligned} \quad (2.48)$$

where the cross terms involving a single  $\mathbf{r}_i$  or single component  $\mathbf{r}_{ia}$  sum up to zero by Eq.(2.47). The terms quadratic in  $\mathbf{r}$  are recognized as giving the moment of inertia tensor about the CM. This gives the *parallel axis theorem* for translating the origin:

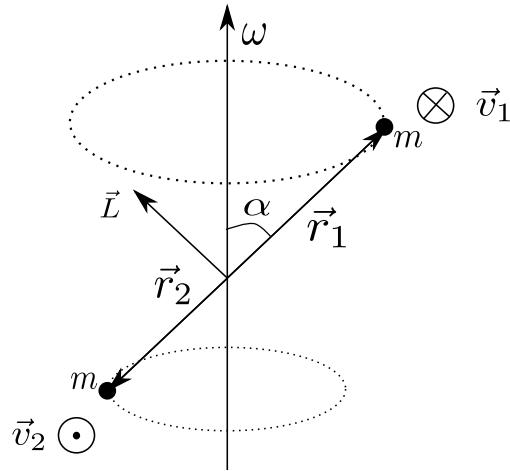
$$\hat{I}_{ab}^Q = M(\delta_{ab} \mathbf{R}^2 - \mathbf{R}_a \mathbf{R}_b) + \hat{I}_{ab}^{\text{CM}}, \quad (2.49)$$

If we wish to carry out a translation between  $P$  and  $Q$ , neither of which is the CM, then we can simply use this formula twice. Another formula can be obtained by projecting the parallel axis onto a specific axis  $\hat{n}$  where  $\hat{n}^2 = 1$  (giving a result that may be familiar from an earlier classical mechanics course):

$$\begin{aligned} \hat{n} \cdot \hat{I}^Q \cdot \hat{n} &= M(\mathbf{R}^2 - (\hat{n} \cdot \mathbf{R})^2) + \hat{n} \cdot \hat{I}^{\text{CM}} \cdot \hat{n} = M\mathbf{R}^2[1 - \cos^2(\theta)] + \hat{n} \cdot \hat{I}^{\text{CM}} \cdot \hat{n} \\ &= M\mathbf{R}^2 \sin^2(\theta) + \hat{n} \cdot \hat{I}^{\text{CM}} \cdot \hat{n} \end{aligned} \quad (2.50)$$

where  $\hat{n} \cdot \mathbf{R} \equiv R \cos(\theta)$ .

**Example:** Lets consider an example of the calculation of  $\hat{I}$  for a situation where  $\mathbf{L}$  is not parallel to  $\boldsymbol{\omega}$ . Consider a dumbbell made of 2 identical point masses  $m$  attached by a massless rigid rod (but with different separations  $r_1$  and  $r_2$  from the axis of rotation), spinning so that  $\boldsymbol{\omega} = \omega \hat{z}$  and so that the rod makes an angle  $\alpha$  with the axis of rotation, as shown



We define body axes where the masses lie in the  $yz$ -plane. Here,

$$\mathbf{r}_1 = (0, r_1 \sin \alpha, r_1 \cos \alpha) \text{ and } \mathbf{r}_2 = (0, -r_2 \sin \alpha, -r_2 \cos \alpha). \quad (2.51)$$

Then using the definition of the moment inertia tensor:

$$\begin{aligned} I_{zz} &= m(x_1^2 + y_1^2) + m(x_2^2 + y_2^2) = m(r_1^2 + r_2^2) \sin^2 \alpha \\ I_{xx} &= m(y_1^2 + z_1^2) + m(y_2^2 + z_2^2) = m(r_1^2 + r_2^2) \\ I_{yy} &= m(x_1^2 + z_1^2) + m(x_2^2 + z_2^2) = m(r_1^2 + r_2^2) \cos^2 \alpha \\ I_{xy} &= I_{yx} = m(-x_1 y_1 - x_2 y_2) = 0 \\ I_{xz} &= I_{zx} = m(-x_1 z_1 - x_2 z_2) = 0 \\ I_{yz} &= I_{zy} = m(-y_1 z_1 - y_2 z_2) = -m(r_1^2 + r_2^2) \sin \alpha \cos \alpha \end{aligned} \quad (2.52)$$

Plugging these into  $\mathbf{L} = \hat{I} \cdot \boldsymbol{\omega}$ , recalling that only  $\omega_z$  is non-zero, this gives

$$\mathbf{L} = (0, I_{yz}\omega, I_{zz}\omega). \quad (2.53)$$

Thus in this example  $\mathbf{L}$  is not parallel to  $\boldsymbol{\omega}$ .

Next, instead of translating the axes in a parallel manner, let us keep the origin fixed and rotate the axes according to an orthogonal rotation matrix  $U$  satisfying  $U^\top U = UU^\top = 1$ . Vectors are rotated as

$$\mathbf{L}' = U\mathbf{L} \quad , \quad \boldsymbol{\omega}' = U\boldsymbol{\omega} \quad \text{and therefore} \quad \boldsymbol{\omega} = U^\top \boldsymbol{\omega}'. \quad (2.54)$$

Putting these together

$$\mathbf{L}' = U\hat{I} \cdot \boldsymbol{\omega} = (U\hat{I}U^\top) \cdot \boldsymbol{\omega}' \quad \Rightarrow \quad \hat{I}' = U\hat{I}U^\top, \quad (2.55)$$

where  $\hat{I}'$  is the new moment of inertia tensor. (The fact that it transforms this way *defines* it as a tensor.) This allows us to calculate the new moment of inertia tensor after a rotation.

For a real symmetric tensor  $\hat{I}$ , there always exists a rotation from an orthogonal matrix  $U$  that diagonalizes  $\hat{I}$  giving a diagonal matrix  $\hat{I}'$ :

$$\hat{I}_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \quad (2.56)$$

The entries of the diagonal moment of inertia tensor,  $I_\alpha$ , are real and positive. This is just a special case of saying a Hermitian matrix can always be diagonalized by a unitary transformation (which is often derived in a Quantum Mechanics course as part of showing that a Hermitian matrix has real eigenvalues and orthogonal eigenvectors). The positivity

## CHAPTER 2. RIGID BODY DYNAMICS

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of diagonal matrix follows immediately from the definition of the moment of inertia tensor for the situation with zero off-diagonal terms.

The axes that make  $\hat{I}$  diagonal are called the *principal axes* and the components  $I_\alpha$  are the *principal moments of inertia*. We find them by solving the eigenvalue problem

$$\hat{I} \cdot \xi = \lambda \xi, \quad (2.57)$$

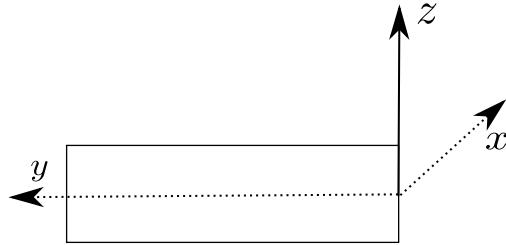
where the 3 eigenvalues  $\lambda$  give the principal moments of inertia  $I_\alpha$ , and are obtained from solving  $\det(\hat{I} - \lambda \mathbf{1}) = 0$ . The corresponding 3 real *orthogonal* eigenvectors  $\xi^{(\alpha)}$  are the principal axes. Here  $U^\top = [\xi^{(1)} \ \xi^{(2)} \ \xi^{(3)}]$ , where the eigenvectors vectors fill out the columns. Then, without summing over repeated indices:

$$L_\alpha = I_\alpha \omega_\alpha \quad \text{and} \quad T = \frac{1}{2} \sum_\alpha I_\alpha \omega_\alpha^2, \quad (2.58)$$

where  $L_\alpha$  and  $\omega_\alpha$  are the components of  $\mathbf{L}$  and  $\boldsymbol{\omega}$ , respectively, evaluated along the principal axes.

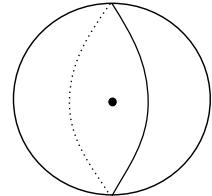
To summarize, for any choice of origin for any rigid body, there is a choice of axes that diagonalizes  $\hat{I}$ . For  $T$  to separate into translational and rotational parts, we must pick the origin to be the CM. Often, the principal axes can be identified by a symmetry of the body.

**Example:** for a thin rectangle lying in the  $yz$ -plane with one edge coinciding with the  $z$ -axis, and the origin chosen as shown below, then  $I_{yz} = 0$  as the body is symmetric under  $z \leftrightarrow -z$ , while  $I_{xz} = I_{xy} = 0$  as the body lies entirely within  $x = 0$ . Hence these are principal axes.

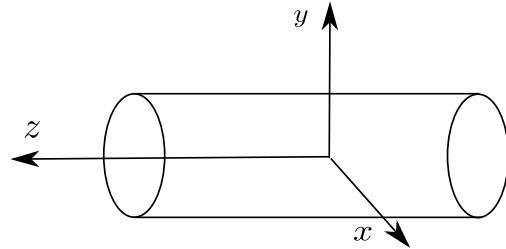


Sometimes, symmetry allows multiple choices for the principal axes.

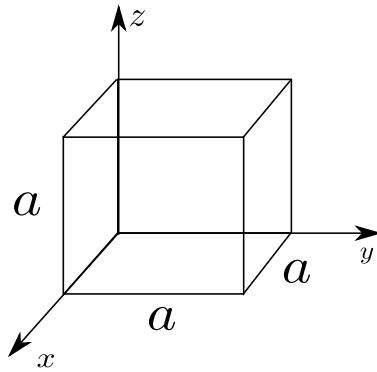
**Example:** for a sphere, any orthogonal axes through the origin are principal axes.



**Example:** for a cylinder whose central axis is aligned along the  $z$ -axis, because of rotational symmetry any choice of the  $x$ - and  $y$ -axes gives principal axes.



**Example:** Lets consider an example where the principal axes may not be apparent, which we can solve through the eigenvalue problem. Consider a uniform cube with sides of length  $a$ , mass  $m$ , and having the origin at one corner, as shown below.



By symmetry we have

$$\begin{aligned} I_{xx} = I_{yy} = I_{zz} &= \frac{m}{a^3} \int_0^a \int_0^a \int_0^a (x^2 + y^2) dx dy dz = \frac{2}{3} ma^2, \\ I_{xy} = I_{yz} = I_{xz} &= \frac{m}{a^3} \int_0^a \int_0^a \int_0^a -xz dx dy dz = -\frac{1}{4} ma^2. \end{aligned} \quad (2.59)$$

Thus the matrix is

$$\hat{I} = ma^2 \begin{pmatrix} +\frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & +\frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & +\frac{2}{3} \end{pmatrix}. \quad (2.60)$$

The principal moments of inertia are found from

$$\det(\hat{I} - \lambda \mathbb{1}) = \left( \frac{11}{12} ma^2 - \lambda \right)^2 \left( \frac{1}{6} ma^2 - \lambda \right) = 0. \quad (2.61)$$

This gives  $I_1 = \lambda_1 = \frac{1}{6} ma^2$ . Solving

$$(\hat{I} - \lambda_1 \mathbb{1}) \boldsymbol{\xi}^{(1)} = 0 \quad \text{we find} \quad \boldsymbol{\xi}^{(1)} = (1, 1, 1). \quad (2.62)$$

The remaining eigenvalues are degenerate:

$$I_2 = I_3 = \lambda_2 = \lambda_3 = \frac{11}{12}ma^2 \quad (2.63)$$

so there is some freedom in determining the corresponding principal axes from  $(\hat{I} - \lambda_2 \mathbb{1})\xi^{(2,3)} = 0$ , though they still should be orthogonal to each other (and  $\xi^{(1)}$ ). One example of a solution is:

$$\xi^{(2)} = (1, -1, 0) \quad \text{and} \quad \xi^{(3)} = (1, 1, -2) \quad (2.64)$$

Using these principal axes and the same origin, the moment of inertia tensor becomes

$$\hat{I}_D = \frac{ma^2}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{11}{2} & 0 \\ 0 & 0 & \frac{11}{2} \end{pmatrix}. \quad (2.65)$$

In contrast, if we had chosen the origin as the center of the cube, then one choice for the principal axes would have the same orientation, but with  $\hat{I}_{CM} = \frac{1}{6}ma^2\mathbb{1}$ . This result could be obtained from Eq. (2.65) using the parallel axis theorem. (Note that for a moment of inertia tensor  $\propto \mathbb{1}$  that any set of orthogonal axes are principal axes due to the symmetry.)

## 2.4 Euler Equations

Consider the rotational motion of a rigid body about a fixed point (which could be the CM but could also be another point). We aim to describe the motion of this rigid body by exploiting properties of the body frame. To simplify things as much as possible, for this fixed point, we choose the principal axes fixed in the body frame indexed by  $\alpha \in \{1, 2, 3\}$ . Using the relation between time derivatives in the inertial and rotating frames, the torque is then given by:

$$\tau = \frac{d\mathbf{L}}{dt} = \frac{d_R\mathbf{L}}{dt} + \boldsymbol{\omega} \times \mathbf{L} \quad (2.66)$$

where  $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ . For example:

$$\tau_1 = \frac{d_R L_1}{dt} + \omega_2 L_3 - \omega_3 L_2. \quad (2.67)$$

Not summing over repeated indices and using the formula for angular momentum along the principal axes gives  $L_\alpha = I_\alpha \omega_\alpha$ . Since we have fixed moments of inertia within the body we have  $d_R I_\alpha / dt = 0$ . Note that  $d\boldsymbol{\omega}/dt = d_R \boldsymbol{\omega}/dt + \boldsymbol{\omega} \times \boldsymbol{\omega} = d_R \boldsymbol{\omega}/dt$ , so its rotating and inertial time derivatives are the same, and we can write  $\dot{\omega}_\alpha$  without possible cause of confusion. Thus  $d_R L_\alpha / dt = I_\alpha \dot{\omega}_\alpha$ . This yields *the Euler equations*:

$$\begin{aligned} I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 &= \tau_1 \\ I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_3 \omega_1 &= \tau_2 \\ I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 &= \tau_3 \end{aligned} \quad (2.68)$$

where in all of these  $\boldsymbol{\omega}$  and  $\boldsymbol{\tau}$  are calculated in the rotating/body frame. This can also be written as

$$\tau_\alpha = I_\alpha \dot{\omega}_\alpha + \epsilon_{\alpha l k} \omega_l \omega_k I_k , \quad (2.69)$$

with  $\alpha$  fixed but a sum implied over the repeated  $l$  and  $k$  indices. Here  $\epsilon_{abc}$  is the fully antisymmetric Levi-Civita symbol.

Solving these equations gives  $\omega_\alpha(t)$ . Since the result is expressed in the body frame, rather than the inertial frame of the observer, this solution for  $\boldsymbol{\omega}(t)$  may not always make the physical motion transparent. To fix this we can connect our solution to the Euler angles using the relations

$$\begin{aligned} \omega_1 &= \omega_{x'} = \dot{\phi} \sin \theta \sin \psi + \dot{\theta} \cos \psi , \\ \omega_2 &= \omega_{y'} = \dot{\phi} \sin \theta \cos \psi - \dot{\theta} \sin \psi , \\ \omega_3 &= \omega_{z'} = \dot{\phi} \cos \theta + \dot{\psi} . \end{aligned} \quad (2.70)$$

These results should be derived as exercise for the student.

**Example:** let us consider the stability of rigid-body free rotations ( $\boldsymbol{\tau} = 0$ ). Is a rotation  $\boldsymbol{\omega} = \omega_1 \mathbf{e}_1$  about the principal axis  $\mathbf{e}_1$  stable?

Perturbations can be expressed by taking  $\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \kappa_2 \mathbf{e}_2 + \kappa_3 \mathbf{e}_3$ , where  $\kappa_2$  and  $\kappa_3$  are small and can be treated to 1<sup>st</sup> order. The Euler equations are:

$$\dot{\omega}_1 = \frac{(I_2 - I_3)}{I_1} \kappa_2 \kappa_3 = O(\kappa^2) \approx 0 , \quad (2.71)$$

so  $\omega_1$  is constant at this order, and

$$\dot{\kappa}_2 = \frac{(I_3 - I_1)}{I_2} \omega_1 \kappa_3 \quad \text{and} \quad \dot{\kappa}_3 = \frac{(I_1 - I_2)}{I_3} \omega_1 \kappa_2 . \quad (2.72)$$

Combining these two equations yields

$$\ddot{\kappa}_2 = \left[ \frac{(I_3 - I_1)(I_1 - I_2)\omega_1^2}{I_2 I_3} \right] \kappa_2 , \quad (2.73)$$

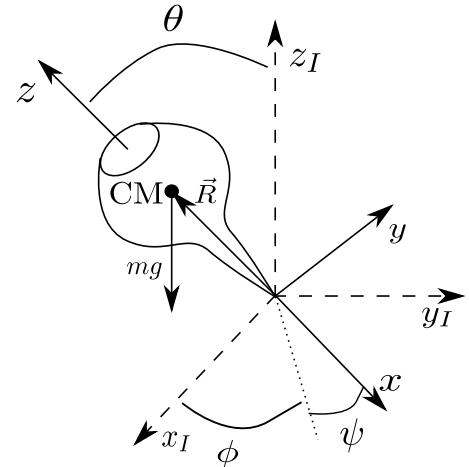
and the same equation for  $\kappa_3$ . The term in the square bracket is constant, and is either negative  $= -w^2$  with an oscillating solution  $\kappa_2 \propto \cos(wt + \phi)$ , or is positive  $= \alpha^2$  with exponential solutions  $\kappa_2 \propto ae^{\alpha t} + be^{-\alpha t}$ . If  $I_1 < I_{2,3}$  or  $I_{2,3} < I_1$  then the constant prefactor is negative, yielding stable oscillatory solutions. If instead  $I_2 < I_1 < I_3$  or  $I_3 < I_1 < I_2$  then the constant prefactor is positive, yielding an unstable exponentially growing component to their solution! This behavior can be demonstrated by spinning almost any object that has three distinct principal moments of inertia. A smart phone or tennis racket are good examples.

## 2.5 Symmetric Top with One Point Fixed

This section is devoted to a detailed analysis of a particular example that appears in many situations, the symmetric top with one point fixed, acted upon by a constant force.

Labeling the body axes as  $(x, y, z)$  and the fixed axes as  $(x_I, y_I, z_I)$ , as depicted in the right, symmetry implies that  $I_1 = I_2$ , and we will assume that  $I_{1,2} \neq I_3$ . The Euler angles are as usual  $(\phi, \theta, \psi)$ . From the figure we see that  $\psi$  is the rotation rate of the top about the (body)  $z$ -axis,  $\dot{\phi}$  is the precession rate about the  $z_I$  fixed inertial axis, and  $\dot{\theta}$  is the nutation rate by which the top may move away or towards the  $z_I$  axis. The Euler equations in this case are

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1, \\ I_1\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2, \\ I_3\dot{\omega}_3 &= 0 = \tau_3. \end{aligned} \quad (2.74)$$



Since the CM coordinate  $\mathbf{R}$  is aligned along the  $z$ -axis there is no torque along  $z$ ,  $\tau_3 = 0$ , leading to a constant  $\omega_3$ .

There are two main cases that we will consider.

**Case:  $\tau = 0$  and  $\dot{\theta} = 0$**

The first case we will consider is when  $\tau = 0$  (so there is no gravity) and  $\dot{\theta} = 0$  (so there is no nutation). Then

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} = 0 \quad \Rightarrow \quad \mathbf{L} = \text{constant} \quad (2.75)$$

Let us define the constant:

$$\Omega \equiv \frac{I_3 - I_1}{I_1}\omega_3. \quad (2.76)$$

Then the Euler equations for this situation reduce to:

$$\dot{\omega}_1 + \Omega\omega_2 = 0 \quad \text{and} \quad \dot{\omega}_2 - \Omega\omega_1 = 0. \quad (2.77)$$

The simplest solution correspond to  $\omega_1(t) = \omega_2(t) = 0$ , where we just have a rotation about the  $z$ -axis. Here:

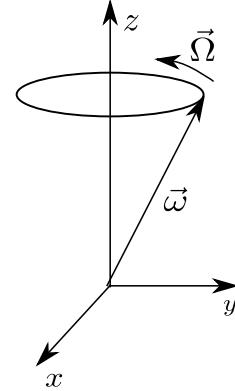
$$\begin{aligned} \mathbf{L} &= L_3 \mathbf{e}_3 \quad \text{where} \quad L_3 = I_3\omega_3 \\ \omega_1 = \omega_2 &= 0 \quad \Rightarrow \quad \dot{\theta} = \dot{\phi} = 0 \text{ and } \dot{\psi} = \omega_3. \end{aligned} \quad (2.78)$$

In this case  $\mathbf{L} \parallel \boldsymbol{\omega}$ . A more general situation is when  $\mathbf{L}$  and  $\boldsymbol{\omega}$  are not necessarily parallel, and  $\omega_1$  and  $\omega_2$  do not vanish. In this case Eq. (2.77) is solved by:

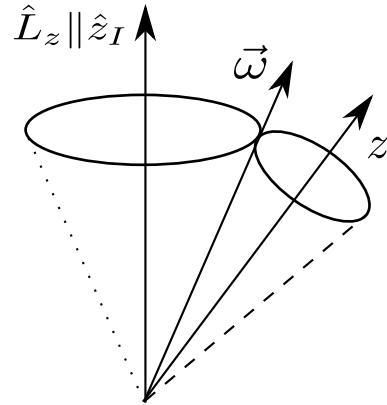
$$\omega_1 = C \sin(\Omega t + D) \quad \text{and} \quad \omega_2 = -C \cos(\Omega t + D). \quad (2.79)$$

The simple case corresponds to  $C = 0$ , so now we take  $C > 0$  (since a sign can be accounted for by the constant phase  $D$ ). This solution means  $\boldsymbol{\omega}$  precesses about the body  $z$ -axis at the rate  $\Omega$ , as pictured on the right. Since  $\omega_1^2 + \omega_2^2$  is constant, the full  $\omega = |\boldsymbol{\omega}|$  is constant, and is given by  $\omega^2 = C^2 + \omega_3^2$ .

The total energy here is just rotational kinetic energy  $T_R = \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{L}$  which is constant too. Thus  $\boldsymbol{\omega}$  also precesses about  $\mathbf{L}$ .



We can picture this motion by thinking about a body cone that rolls around a cone in the fixed coordinate system, where in the case pictured with a larger cone about  $\mathbf{L}$  we have  $I_1 = I_2 > I_3$ .



To obtain more explicit results for the motion we can relate Eq.(2.79) to Euler angles. Since  $\dot{\theta} = 0$ , we take  $\theta = \theta_0$  to be constant. The other Euler angles come from:

$$\boldsymbol{\omega} = \begin{bmatrix} C \sin(\Omega t + D) \\ -C \cos(\Omega t + D) \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \sin(\theta_0) \sin(\psi) \dot{\phi} \\ \sin(\theta_0) \cos(\psi) \dot{\phi} \\ \cos(\theta_0) \dot{\phi} + \dot{\psi} \end{bmatrix}. \quad (2.80)$$

Adding the squares of the 1<sup>st</sup> and 2<sup>nd</sup> components gives

$$C^2 = \sin^2(\theta_0) \dot{\phi}^2. \quad (2.81)$$

To be definite, take the positive square root of this equation to give

$$\dot{\phi} = \frac{C}{\sin(\theta_0)} \Rightarrow \phi = \frac{C}{\sin(\theta_0)} t + \phi_0. \quad (2.82)$$

The first two equations in Eq. (2.80) are then fully solved by taking  $\psi = \pi - \Omega t - D$ , so we find that both  $\phi$  and  $\psi$  have linear dependence on time. Finally the third equation gives a relation between various constants

$$\omega_3 = C \cot(\theta_0) - \Omega. \quad (2.83)$$

Thus, we see that the solution has  $\dot{\phi}$  and  $\dot{\psi}$  are constants with  $\dot{\theta} = 0$ . If we had picked the opposite sign when solving Eq. (2.81) then we would have found similar results:

$$\dot{\phi} = -\frac{C}{\sin(\theta_0)} \Rightarrow \psi = -\Omega t - D \quad \text{and} \quad \omega_3 = -C \cot(\theta_0) - \Omega. \quad (2.84)$$

### Case: $\tau \neq 0$ and $\dot{\theta} \neq 0$

Now we consider the general case where  $\tau \neq 0$  and  $\dot{\theta} \neq 0$ . It is now more convenient to use the Lagrangian than the Euler equations directly. Since  $I_1 = I_2$ , using

$$T = \frac{1}{2} (I_1(\omega_1^2 + \omega_2^2) + I_3\omega_3^2) \quad \text{and} \quad \boldsymbol{\omega} = \begin{bmatrix} \sin(\theta) \sin(\psi) \dot{\phi} + \cos(\psi) \dot{\theta} \\ \sin(\theta) \cos(\psi) \dot{\phi} - \sin(\psi) \dot{\theta} \\ \cos(\theta) \dot{\phi} + \dot{\psi} \end{bmatrix}, \quad (2.85)$$

gives us the kinetic energy

$$T = \frac{I_1}{2} (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) + \frac{I_3}{2} (\dot{\psi} + \cos \theta \dot{\phi})^2. \quad (2.86)$$

Moreover,  $V = mgR \cos(\theta)$ , so in the Lagrangian  $L = T - V$  the variables  $\phi$  and  $\psi$  are cyclic. This means that the momenta

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = [I_1 \sin^2(\theta) + I_3 \cos^2(\theta)] \dot{\phi} + I_3 \cos(\theta) \dot{\psi} \quad (2.87)$$

$$p_\psi = \frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \cos(\theta)\dot{\phi}) = I_3\omega_3 \quad (2.88)$$

are conserved (constant). Here  $p_\psi$  is same as the angular momentum  $L_3$  discussed in the case above. The torque is along the line of nodes, and  $p_\phi$  and  $p_\psi$  correspond to two projections of  $\mathbf{L}$  that are perpendicular to this torque (i.e. along  $\hat{z}_I$  and  $\hat{z}$ ). Additionally, the energy is given by

$$E = T + V = \frac{I_1}{2} (\dot{\theta}^2 + \sin^2(\theta) \dot{\phi}^2) + \frac{I_3}{2} (\dot{\psi} + \cos(\theta) \dot{\phi})^2 + mgR \cos(\theta) \quad (2.89)$$

and is also conserved. Solving the momentum equations, Eq. (2.87), for  $\dot{\phi}$  and  $\dot{\psi}$  gives

$$\begin{aligned}\dot{\phi} &= \frac{p_\phi - p_\psi \cos(\theta)}{I_1 \sin^2(\theta)} \\ \dot{\psi} &= \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi \cos(\theta)) \cos(\theta)}{I_1 \sin^2(\theta)}.\end{aligned}\tag{2.90}$$

Note that once we have a solution for  $\theta(t)$  that these two equations then allow us to immediately obtain solutions for  $\phi(t)$  and  $\psi(t)$  by integration. Eq. (2.90) can be plugged into the energy formula to give

$$E = \frac{I_1}{2} \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos(\theta))^2}{2I_1 \sin^2(\theta)} + \frac{p_\psi^2}{2I_3} + mgR \cos(\theta),\tag{2.91}$$

which is a (nonlinear) differential equation for  $\theta$ , since all other quantities that appear are simply constants. To simplify this result take  $u = \cos(\theta)$  so that:

$$1 - u^2 = \sin^2(\theta), \quad \dot{u} = -\sin(\theta)\dot{\theta}, \quad \dot{\theta}^2 = \frac{\dot{u}^2}{1 - u^2}.\tag{2.92}$$

Putting all this together gives:

$$\frac{\dot{u}^2}{2} = \left( \frac{2EI_3 - p_\psi^2}{2I_1 I_3} - \frac{mgR}{I_1} u \right) (1 - u^2) - \frac{1}{2} \left( \frac{p_\phi - p_\psi u}{I_1} \right)^2 \equiv -V_{\text{eff}}(u),\tag{2.93}$$

which is a cubic polynomial that we've defined to be the effective potential  $V_{\text{eff}}(u)$ . The solution to this from

$$dt = \pm \frac{du}{\sqrt{-2V_{\text{eff}}(u)}}\tag{2.94}$$

yields a complicated elliptic function, from which it is hard to get intuition for the motion.

Instead, we can look at the form of  $V_{\text{eff}}(u)$ , because

$$\frac{1}{2} \dot{u}^2 + V_{\text{eff}}(u) = 0\tag{2.95}$$

is the equation for the energy of a particle of unit mass  $m = 1$ , kinetic energy  $\dot{u}^2/2$ , a potential  $V_{\text{eff}}(u)$ , and with vanishing total energy. The cubic equation will have in general three roots where  $V_{\text{eff}}(u) = 0$ . Since the kinetic energy is always positive or zero, the potential energy must be negative or zero in the physical region, and hence the particle can not pass through any of the roots. The roots therefore serve as turning points. Furthermore, physical solutions are bounded by  $-1 \leq (u = \cos \theta) \leq 1$ . While the precise values for the roots will depend on the initial conditions or values of  $E$ ,  $p_\psi$ , and  $p_\phi$ , we can still describe the solutions in a fairly generic manner.

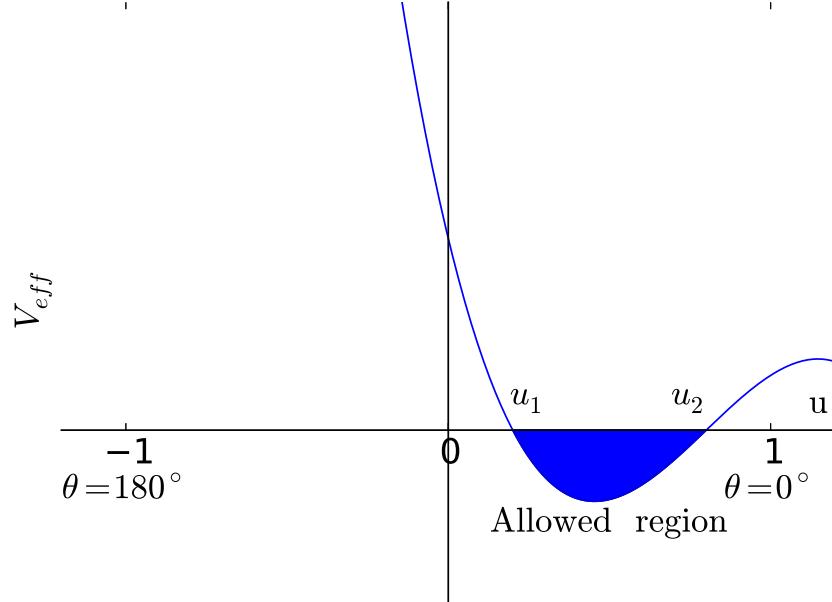


Figure 2.7: Allowed region for solutions for the top's nutation angle  $\theta$  that solve Eq. (2.95).

Consider two roots  $u_1$  and  $u_2$  (corresponding respectively to some angles  $\theta_1$  and  $\theta_2$  as  $u = \cos(\theta)$ ) satisfying  $V_{\text{eff}}(u_1) = V_{\text{eff}}(u_2) = 0$ , where  $V_{\text{eff}}(u) < 0$  for  $u_1 < u < u_2$ ; as shown in Fig. 2.7. We see that  $u_1$  and  $u_2$  correspond to the turning points of the motion. The region  $u_1 < u < u_2$  corresponds to the region where the motion of our top lives and gives rise to a periodic nutation, where the solution bounces between the two turning points. Depending on the precise value of the various constants that appear in this  $V_{\text{eff}}$  this gives rise to different qualitative motions, with examples shown in Figs. 2.8–2.11. Recalling that  $\dot{\phi} = (p_\phi - p_\psi u)/[I_1(1 - u^2)]$ , we see that the possible signs for  $\dot{\phi}$  will depend on  $p_\phi$  and  $p_\psi$ . In Fig. 2.8 the top nutates between  $\theta_1$  and  $\theta_2$  while always precessing in the same direction with  $\dot{\phi} > 0$ , whereas in Fig. 2.9 the precession is also in the backward direction,  $\dot{\phi} < 0$ , for part of the range of motion. In Fig. 2.10 the top has  $\dot{\phi} = 0$  at  $\theta_2$ , before falling back down in the potential and gaining  $\dot{\phi} > 0$  again. This figure also captures the case where we let go of a top at  $\theta = \theta_2 \geq 0$  that initially has  $\dot{\psi} > 0$  but  $\dot{\phi} = 0$ . Finally in Fig. 2.11 we have the situation where there is no nutation oscillation because the two angles coincide,  $\theta_1 = \theta_2$ .

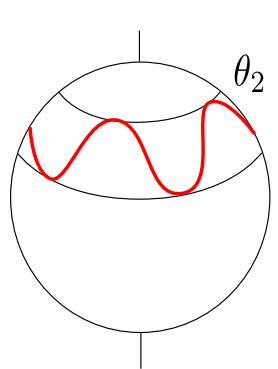


Figure 2.8:  $\dot{\phi} > 0$

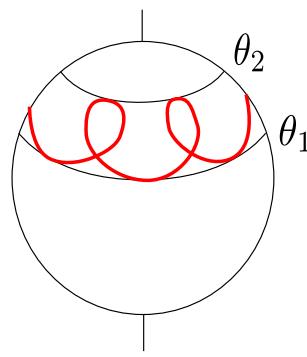


Figure 2.9:  $\dot{\phi}$  has both signs

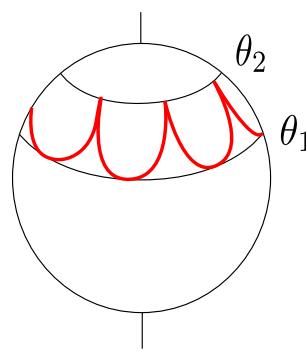


Figure 2.10: at  $\theta_2$  we have  $\dot{\phi} = 0, \dot{\theta} = 0$

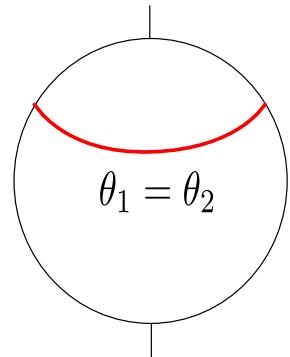


Figure 2.11: No nutation

# Chapter 3

## Vibrations & Oscillations

The topic of vibrations and oscillations is typically discussed in some detail in a course on waves (at MIT this is 8.03). Our goal for this chapter is to revisit aspects of oscillation phenomena using generalized coordinates. Many equations of motion we have encountered have been nonlinear. Here, we will expand about a minimum of the potential  $V(q_1, \dots, q_n)$ , yielding linear equations.

Let us take  $q_i = q_{0i} + \eta_i$ , where  $\vec{q}_0$  minimizes  $V(q)$ , and expand in the  $\eta_i$ . Henceforth and until further notice, repeated indices will implicitly be summed over. Then

$$V(q_1, \dots, q_n) = V(q_{01}, \dots, q_{0n}) + \frac{\partial V}{\partial q_i} \Big|_0 \eta_i + \frac{1}{2} \frac{\partial^2 V}{\partial q_i \partial q_j} \Big|_0 \eta_i \eta_j + \dots, \quad (3.1)$$

where  $|_0$  means “evaluate the quantity at  $\vec{q}_0$ ”. We already know that  $\frac{\partial V}{\partial q_i} \Big|_0 = 0$  as by definition  $\vec{q}_0$  minimizes  $V(q)$ . As a matter of convention, we choose  $V(q_0) = 0$ , since this just corresponds to picking the convention for the zero of the Energy. Finally, we define the constants  $V_{ij} \equiv \frac{\partial^2 V}{\partial q_i \partial q_j} \Big|_0$ . There is no time dependence in the definition of our generalized coordinates, so the kinetic energy is

$$T = \frac{1}{2} m_{ij}(q_1, \dots, q_n) \dot{q}_i \dot{q}_j = \frac{1}{2} m_{ij}(q_{01}, \dots, q_{0n}) \dot{\eta}_i \dot{\eta}_j + \mathcal{O}(\eta \dot{\eta}^2), \quad (3.2)$$

where  $m_{ij}(q_{01}, \dots, q_{0n}) \equiv T_{ij}$  are constants, and terms of  $\mathcal{O}(\eta \dot{\eta}^2)$  and beyond are neglected. Thus, the Lagrangian to quadratic order in the  $\eta_i$ s is

$$L = \frac{1}{2} (T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j) . \quad (3.3)$$

From this, the equations of motion are

$$T_{ij} \ddot{\eta}_j + V_{ij} \eta_j = 0 \quad (3.4)$$

to the same order. These are coupled linear equations of motion.

### 3.1 Simultaneous Diagonalization of $\hat{T}$ and $\hat{V}$

To solve Eq. (3.4) lets try

$$\eta_i = a_i e^{-i\omega t} \quad (3.5)$$

where  $a_i \in \mathbb{C}$  for all  $i \in \{1, \dots, n\}$ , and the frequency  $\omega$  is the same along all directions in the generalized coordinate space. Notationally,  $i$  and  $j$  will denote coordinate indices, while  $i = +\sqrt{-1}$  is the imaginary unit. This gives

$$V_{ij}a_j = \omega^2 T_{ij}a_j \quad (3.6)$$

which can be rewritten in matrix form as

$$\hat{V} \cdot \vec{a} = \lambda \hat{T} \cdot \vec{a} \quad (3.7)$$

with  $\lambda = \omega^2$ . This looks like an eigenvalue equation except that when we act with the linear operator  $\hat{V}$  on  $\vec{a}$  we get back  $\hat{T} \cdot \vec{a}$  instead of just the eigenvector  $\vec{a}$ . This can be rewritten as

$$(\hat{V} - \lambda \hat{T}) \cdot \vec{a} = 0 \quad (3.8)$$

where  $\hat{V}$  and  $\hat{T}$  are *real* and *symmetric*  $n \times n$  matrices. In order to have a non-trivial solution of this equation we need

$$\det(\hat{V} - \lambda \hat{T}) = 0 \quad (3.9)$$

which is an  $n^{\text{th}}$  order polynomial equation with  $n$  solutions eigenvalues  $\lambda_\alpha$  with  $\alpha \in \{1, \dots, n\}$ . The solutions of  $(\hat{V} - \lambda_\alpha \hat{T}) \cdot \vec{a}^{(\alpha)} = 0$  are the eigenvectors  $\vec{a}^{(\alpha)}$ . This means

$$\hat{V} \cdot \vec{a}^{(\alpha)} = \lambda_\alpha \hat{T} \cdot \vec{a}^{(\alpha)}, \quad (3.10)$$

and the solutions are much like a standard eigenvalue problem. Here and henceforth, there will be no implicit sum over repeated eigenvalue indices  $\alpha$  (so any sums that are needed will be made explicit), but we will retain implicit sums over repeated coordinate indices  $i$  &  $j$ .

There are two cases we will consider.

1) Let us start by considering the case when  $\hat{T}$  is diagonal. In particular, let us consider the even easier case proportional to the unit matrix, where  $T_{ij} = m\delta_{ij}$ . This means

$$m\ddot{\eta}_i + V_{ij}\eta_j = 0. \quad (3.11)$$

Here we have the standard eigenvalue equation

$$\hat{V} \cdot \vec{a}^{(\alpha)} = m\lambda_\alpha \vec{a}^{(\alpha)}. \quad (3.12)$$

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### CHAPTER 3. VIBRATIONS & OSCILLATIONS

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The eigenvalues  $\lambda_\alpha$  are real and nonnegative as  $\lambda_\alpha = \omega_\alpha^2$ ; the quantities  $\omega_\alpha$  are the *normal mode frequencies*. The eigenvectors  $\vec{a}^{(\alpha)}$  are orthogonal, and we can choose their normalization so that

$$m \vec{a}^{(\beta)\dagger} \cdot \vec{a}^{(\alpha)} = \delta_{\beta\alpha} \quad (\text{or } \vec{a}^{(\beta)\dagger} \cdot \vec{a}^{(\alpha)} = \delta_{\beta\alpha}). \quad (3.13)$$

This implies that

$$\lambda_\alpha = \vec{a}^{(\alpha)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)} \quad (\text{or } m\lambda_\alpha = \vec{a}^{(\alpha)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)}). \quad (3.14)$$

The time-dependent eigenvectors are then

$$\vec{\eta}^{(\alpha)} = \vec{a}^{(\alpha)} e^{-i\omega_\alpha t}, \quad \text{or} \quad \eta_i^{(\alpha)} = a_i^{(\alpha)} e^{-i\omega_\alpha t}. \quad (3.15)$$

These are the normal mode solutions for the  $n$  coordinates labeled by  $i$ , and there are  $n$  such solutions labeled by  $\alpha$ . The general solution of a linear equation is a superposition of the independent normal mode solutions:

$$\vec{\eta} = \sum_{\alpha} C_{\alpha} \vec{\eta}^{(\alpha)} \quad (3.16)$$

where  $C_{\alpha} \in \mathbb{C}$  are fixed by initial conditions. To find real coordinate solutions, we take the real parts of these equations.

Lets prove the statements made above. Again, there will be no implicit sum over the eigenvalue index  $\alpha$ . Dotting in  $\vec{a}^{(\beta)\dagger}$  into Eq. (3.12) gives

$$\vec{a}^{(\beta)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)} = m\lambda_\alpha \vec{a}^{(\beta)\dagger} \cdot \vec{a}^{(\alpha)}, \quad (3.17)$$

taking the Hermitian conjugate of both sides, noting that  $\hat{V}^\dagger = \hat{V}$ , and then swapping  $\alpha \leftrightarrow \beta$  gives  $\vec{a}^{(\beta)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)} = m\lambda_\beta^* \vec{a}^{(\beta)\dagger} \cdot \vec{a}^{(\alpha)}$ . Taking the difference of these results gives

$$(\lambda_\alpha - \lambda_\beta^*) \vec{a}^{(\beta)\dagger} \cdot \vec{a}^{(\alpha)} = 0, \quad (3.18)$$

and if  $\alpha = \beta$  then  $(\lambda_\alpha - \lambda_\alpha^*) \vec{a}^{(\alpha)\dagger} \cdot \vec{a}^{(\alpha)} = 0$  implies the eigenvalues are real  $\lambda_\alpha \in \mathbb{R}$ . For  $\lambda_\alpha \neq \lambda_\beta$ , Eq. (3.18) then implies  $\vec{a}^{(\beta)\dagger} \cdot \vec{a}^{(\alpha)} = 0$  so the eigenvectors are orthogonal. If by chance  $\lambda_\alpha = \lambda_\beta$  for some  $\alpha \neq \beta$  then we can always simply choose the corresponding eigenvectors to be orthogonal. By convention, we then normalize the eigenvectors so that they satisfy Eq. (3.13). Finally, if  $\alpha = \beta$  then Eq. (3.17) now gives  $\lambda_\alpha = \vec{a}^{(\alpha)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)}$ . The statement that we are at a local minimum of the multivariable potential and not a saddle point or a maximum implies then that  $\lambda_\alpha \geq 0$  (we have positive second derivatives in each eigenvector direction).

2) Let us now consider when  $\hat{T}$  is not diagonal and summarize which parts of the result are the same and where there are differences. Here we have  $(\hat{V} - \lambda\hat{T}) \cdot \vec{a} = 0$ . Again, the eigenvalues  $\lambda_\alpha$  are real and nonnegative, with  $\lambda_\alpha = \omega_\alpha^2$ . Now, however,

$$\vec{a}^{(\beta)\dagger} \cdot \hat{T} \cdot \vec{a}^{(\alpha)} = 0, \quad (3.19)$$

for  $\alpha \neq \beta$ , and we can replace the old normalization condition by a new one stating that

$$\vec{a}^{(\beta)\dagger} \cdot \hat{T} \cdot \vec{a}^{(\alpha)} = \delta_{\alpha\beta}, \quad (3.20)$$

which up to an overall prefactor reduces to the old orthonormality condition when  $\hat{T} = m\hat{1}$ . Here again,

$$\lambda_\alpha = \vec{a}^{(\alpha)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)}, \quad (3.21)$$

and the  $\alpha^{\text{th}}$  normal mode solution is

$$\vec{\eta}^{(\alpha)} = \vec{a}^{(\alpha)} e^{-i\omega_\alpha t}. \quad (3.22)$$

The general solution is again the superposition

$$\vec{\eta} = \sum_{\alpha} C_{\alpha} \vec{\eta}^{(\alpha)}, \quad (3.23)$$

with the complex coefficients  $C_{\alpha}$  fixed by the initial conditions (and a real part taken to get real coordinates).

Lets repeat the steps of our proof for this case. Dotting  $\vec{a}^{(\beta)\dagger}$  into Eq. (3.10) gives

$$\vec{a}^{(\beta)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)} = \lambda_{\alpha} \vec{a}^{(\beta)\dagger} \cdot \hat{T} \cdot \vec{a}^{(\alpha)}. \quad (3.24)$$

Taking the Hermitian conjugate and swapping  $\alpha$  and  $\beta$  yields  $\vec{a}^{(\beta)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)} = \lambda_{\beta}^* \vec{a}^{(\beta)\dagger} \cdot \hat{T} \cdot \vec{a}^{(\alpha)}$ . Subtracting the two results this gives

$$(\lambda_{\alpha} - \lambda_{\beta}^*) \vec{a}^{(\beta)\dagger} \cdot \hat{T} \cdot \vec{a}^{(\alpha)} = 0. \quad (3.25)$$

If we consider  $\alpha = \beta$  then  $(\lambda_{\alpha} - \lambda_{\alpha}^*) \vec{a}^{(\alpha)\dagger} \cdot \hat{T} \cdot \vec{a}^{(\alpha)} = 0$  implies  $\lambda_{\alpha} \in \mathbb{R}$  since  $\hat{T} = \hat{T}^* = \hat{T}^T$  and physically we know that the kinetic energy  $T = \dot{\vec{\eta}} \cdot \hat{T} \cdot \dot{\vec{\eta}} > 0$  for any  $\dot{\vec{\eta}} \neq 0$ . For  $\lambda_{\alpha} \neq \lambda_{\beta}$ , then the condition instead implies  $\vec{a}^{(\beta)\dagger} \cdot \hat{T} \cdot \vec{a}^{(\alpha)} = 0$  so the eigenvectors are orthogonal (with respect to an inner product defined with  $\hat{T}$ ). If by chance  $\lambda_{\alpha} = \lambda_{\beta}$  for some  $\alpha \neq \beta$  then we can simply choose the corresponding eigenvectors to be orthogonal in this sense. By convention, we normalize the eigenvectors so that they will be orthonormal as in Eq. (3.20). Finally, if  $\alpha = \beta$  then  $\lambda_{\alpha} = \vec{a}^{(\alpha)\dagger} \cdot \hat{V} \cdot \vec{a}^{(\alpha)}$ , which is positive, so  $\lambda_{\alpha} > 0$  also. The statement that we are at a local minimum of the potential and not a saddle point or a maximum implies then that  $\lambda_{\alpha} \geq 0$ .

## 3.2 Vibrations and Oscillations with Normal Coordinates

Given these results, it is natural to ask whether a different set of generalized coordinates might be better for studying motion about the minimum?

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### CHAPTER 3. VIBRATIONS & OSCILLATIONS

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We form the matrix  $A$  by placing the eigenvectors in columns

$$A = [\vec{a}^{(1)} \quad \vec{a}^{(2)} \quad \dots \quad \vec{a}^{(n)}] \quad (3.26)$$

and construct a diagonal eigenvalue matrix  $\hat{\lambda} = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ . The matrix  $A$  can be ensured to be real because each  $\vec{a}^{(\alpha)}$  only has at most an overall phase<sup>1</sup>, and these can be removed by putting them into the coefficients  $C_\alpha$ . The matrix  $A$  simultaneously diagonalizes  $\hat{T}$  and  $\hat{V}$  since

$$A^\top \hat{T} A = \mathbb{1} \quad \text{and} \quad A^\top \hat{V} A = \hat{\lambda}. \quad (3.27)$$

We choose new *normal coordinates*  $\vec{\xi}$  by letting

$$\vec{\eta} = A \vec{\xi} \quad \text{and} \quad \dot{\vec{\eta}} = A \dot{\vec{\xi}} \quad (3.28)$$

so that the Lagrangian

$$\begin{aligned} L &= \frac{1}{2} \dot{\vec{\eta}} \cdot \hat{T} \cdot \dot{\vec{\eta}} - \frac{1}{2} \vec{\eta} \cdot \hat{V} \cdot \vec{\eta} \\ &= \frac{1}{2} \dot{\vec{\xi}} \cdot (A^\top \hat{T} A) \cdot \dot{\vec{\xi}} - \frac{1}{2} \vec{\xi} \cdot (A^\top \hat{V} A) \cdot \vec{\xi} \\ &= \frac{1}{2} \sum_{\alpha} (\dot{\xi}_{\alpha}^2 - \omega_{\alpha}^2 \xi_{\alpha}^2). \end{aligned} \quad (3.29)$$

This gives the simple equations of motion for each  $\alpha$ :

$$\ddot{\xi}_{\alpha} + \omega_{\alpha}^2 \xi_{\alpha} = 0. \quad (3.30)$$

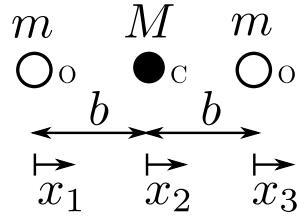
Thus, each normal coordinate describes the oscillations of the system with normal mode frequency  $\omega_{\alpha}$ .

**Example:** Let us consider the triatomic molecule CO<sub>2</sub> shown in Figure 3.1. We can picture it as a carbon atom of mass  $M$  in the middle of two oxygen atoms each of mass  $m$ . For the three particles there are 9 coordinates given by  $\mathbf{r}_1$ ,  $\mathbf{r}_2$ , and  $\mathbf{r}_3$ . Six of these coordinates correspond to translations and rotations of the mass system treated as a rigid body. This leaves 3 coordinates that correspond to internal motions of the system. To model the potential we connect each oxygen atom to the carbon atom with a spring of constant  $k$  and relaxed length  $b$ . This does not add any cost to relative motion of the atoms with fixed spring length, which we will address below by adding another potential term in order to favor the linear configuration.

It is straightforward to guess what the normal modes could be:

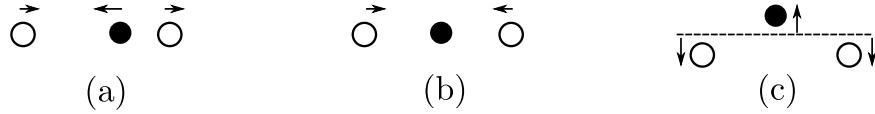
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<sup>1</sup>Why is it just an overall phase? The equation  $\hat{V} \cdot \vec{a}^{(\alpha)} = \lambda_{\alpha} \hat{T} \cdot \vec{a}^{(\alpha)}$  alone does not fix the normalization of  $\vec{a}^{(\alpha)}$ . Let us say we pick  $a_i^{(\alpha)} \in \mathbb{R}$  for some  $i$ . Then  $V_{kj} a_j^{(\alpha)} = \lambda_{\alpha} K_{kj} a_j^{(\alpha)}$  is a set of equations with all real coefficients and one real term in the sums. Hence the solutions  $a_j^{(\alpha)} / a_i^{(\alpha)} \in \mathbb{R}$  for all  $j \in \{1, \dots, n\}$ , implying that at most there is an overall phase in  $\vec{a}^{(\alpha)}$ .


 Figure 3.1: The  $\text{CO}_2$  molecule.

- The oxygen atoms moving in the same direction along the line and the carbon atom moving in the opposite direction. This is a longitudinal oscillation.
- The oxygen atoms opposing each other along the line while the carbon atom remains at rest. This is a longitudinal oscillation.
- The oxygen atoms move in the same direction perpendicular to the line and the carbon atom moving in the opposite direction. This is a transverse oscillation.

These three normal modes are shown in Figure 3.2.


 Figure 3.2: The three Normal Modes of the  $\text{CO}_2$  molecule

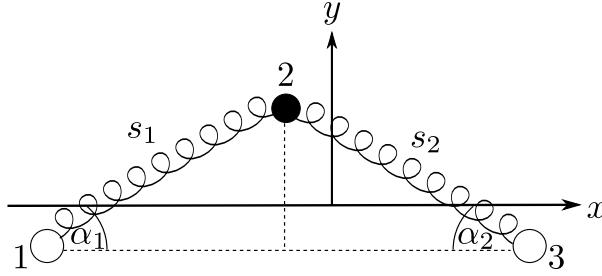
We pick the body frame axes as follows:

- the three particles are in the  $xy$ -plane fixing 3 coordinates  $z_i = 0$  for  $i \in \{1, 2, 3\}$ ,
- the origin is the CM so  $m(x_1 + x_3) + Mx_2 = m(y_1 + y_3) + My_2 = 0$ , which fixes two more coordinates,
- the axes are oriented so that  $y_1 = y_3$ , which fixes one coordinate.

Defining the mass ratio as  $\rho \equiv \frac{m}{M}$ , then  $x_2 = -\rho(x_1 + x_3)$  and  $y_2 = -2\rho y_1$  can be eliminated. Altogether this fixes 6 coordinates, leaving the coordinates  $(x_1, x_3, y_1)$ . This setup is shown in Figure 3.3.

For the potential we take

$$V = \frac{k}{2} (s_1 - b)^2 + \frac{k}{2} (s_2 - b)^2 + \frac{\lambda b^2}{2} (\alpha_1^2 + \alpha_2^2). \quad (3.31)$$


 Figure 3.3: The orientation of the  $\text{CO}_2$  molecule on  $xy$  plane

The first two terms are the springs discussed previously, and the last two provide a quadratic energy cost to the springs rotating away from the linear configuration, with strength given by  $\lambda$ . The spring lengths are

$$s_1 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{[x_1 + \rho(x_1 + x_3)]^2 + (1 + 2\rho)^2 y_1^2} \quad (3.32)$$

$$s_2 = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} = \sqrt{[x_3 + \rho(x_1 + x_3)]^2 + (1 + 2\rho)^2 y_1^2},$$

and the two angles are

$$\alpha_1 = \tan^{-1} \left( \frac{y_3 - y_2}{x_3 - x_2} \right) = \tan^{-1} \left[ \frac{(1 + 2\rho)y_1}{(1 + \rho)x_1 + x_3} \right], \quad (3.33)$$

$$\alpha_2 = \tan^{-1} \left( \frac{y_2 - y_1}{x_2 - x_1} \right) = \tan^{-1} \left[ \frac{(1 + 2\rho)y_1}{(1 + \rho)x_3 + x_1} \right].$$

These results give  $V = V(x_1, y_1, x_3)$ .

For the kinetic energy we have

$$T = \frac{m}{2} (\dot{x}_1^2 + \dot{x}_3^2) + \frac{M}{2} \dot{x}_2^2 + \frac{m}{2} (\dot{y}_1^2 + \dot{y}_3^2) + \frac{M}{2} \dot{y}_1^2, \quad (3.34)$$

which after eliminating coordinates becomes

$$T = \frac{m}{2} (1 + \rho) (\dot{x}_1^2 + \dot{x}_3^2) + m\rho\dot{x}_1\dot{x}_3 + m(1 + 2\rho)\dot{y}_1^2. \quad (3.35)$$

Equilibrium comes from taking  $y_1 = 0$ ,  $x_3 = -x_1 = b$ , which implies  $\alpha_1 = \alpha_2 = 0$ ,  $s_1 = s_2 = b$ , and  $V = 0$ . We define coordinates for expanding about this potential minimum as  $\eta_1 = x_1 + b$ ,  $\eta_3 = x_3 - b$ , and  $\eta_2 = y_1$ . Then as  $V(-b, 0, b) = 0$  in equilibrium and  $\frac{\partial V}{\partial \eta_i} \Big|_0 = 0$  then  $V = \frac{1}{2}V_{ij}\eta_i\eta_j + \dots$  where

$$V_{ij} = \frac{\partial^2 V}{\partial \eta_i \partial \eta_j} \Big|_0 = \begin{bmatrix} k(1 + 2\rho + 2\rho^2) & 0 & 2k\rho(1 + \rho) \\ 0 & 2\lambda(1 + 2\rho)^2 & 0 \\ 2k\rho(1 + \rho) & 0 & k(1 + 2\rho + 2\rho^2) \end{bmatrix} \quad (3.36)$$

for this system. Additionally,

$$T_{ij} = \begin{bmatrix} m(1+\rho) & 0 & m\rho \\ 0 & 2m(1+2\rho) & 0 \\ m\rho & 0 & m(1+\rho) \end{bmatrix} \quad (3.37)$$

for this system. Since there are no off-diagonal terms in the 2nd row or 2nd column in either  $\hat{V}$  or  $\hat{T}$ , the transverse and the longitudinal modes decouple. For the transverse mode, we are left with

$$\ddot{y}_1 + \frac{2\lambda(1+2\rho)^2}{2m(1+2\rho)} y_1 = 0, \quad (3.38)$$

which is a simple harmonic oscillator. For the longitudinal modes, we have  $\vec{\eta} = (\eta_1, \eta_3)$ . The frequencies come from

$$\det \begin{bmatrix} k(1+2\rho+2\rho^2) - \lambda m(1+\rho) & 2k\rho(1+\rho) - \lambda m\rho \\ 2k\rho(1+\rho) - \lambda m\rho & k(1+2\rho+2\rho^2) - \lambda m(1+\rho) \end{bmatrix} = 0 \quad (3.39)$$

The solutions give the normal mode frequencies

$$\lambda_1 = \omega_1^2 = \frac{k}{m}, \quad \lambda_2 = \omega_2^2 = \frac{k}{m}(1+2\rho), \quad (3.40)$$

with associated eigenvectors

$$\vec{a}^{(1)} = \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{a}^{(2)} = \frac{1}{\sqrt{2m(1+2\rho)}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad (3.41)$$

which were chosen to satisfy  $\vec{a}^{(\alpha)} \cdot \hat{T} \cdot \vec{a}^{(\beta)} = \delta_{\alpha\beta}$ . Thus, the normal coordinates for the longitudinal modes are  $\xi_1 \propto x_1 - x_3$  and  $\xi_2 \propto x_1 + x_3$ . Oscillations in these coordinates correspond to the normal mode motions in Fig. 3.2(b) and Fig. 3.2(a) respectively.

# Chapter 4

## Canonical Transformations, Hamilton-Jacobi Equations, and Action-Angle Variables

We've made good use of the Lagrangian formalism. Here we'll study dynamics with the Hamiltonian formalism. Problems can be greatly simplified by a good choice of generalized coordinates. How far can we push this?

**Example:** Let us imagine that we find coordinates  $q_i$  that are all cyclic. Then  $\dot{p}_i = 0$ , so  $p_i = \alpha_i$  are all constant. If  $H$  is conserved, then:

$$H = H(\alpha_1, \dots, \alpha_n) \quad (4.1)$$

is also constant in time. In such a case the remaining equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial \alpha_i} = \omega_i(\alpha) \quad \Rightarrow \quad q_i = \omega_i t + \delta_i \quad (4.2)$$

All coordinates are linear in time and the motion becomes very simple.

We might imagine searching for a variable transformation to make as many coordinates as possible cyclic. Before proceeding along this path, we must see what transformations are allowed.

### 4.1 Generating Functions for Canonical Transformations

Recall the Euler-Lagrange equations are invariant when:

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- a point transformation occurs  $Q = Q(q, t)$  with  $L[q, t] = L'[Q, t]$ ;
- a total derivative is summed to the Lagrangian  $L' = L + \frac{dF[q, t]}{dt}$ .

For  $H$  we consider point transformations in phase space:

$$Q_i = Q_i(q, p, t) \quad \text{and} \quad P_i = P_i(q, p, t), \quad (4.3)$$

where the Hamilton's equations for the evolution of the canonical variables  $(q, p)$  are satisfied:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \quad (4.4)$$

Generally, not all transformations preserve the equations of motion. However, the transformation is *canonical* if there exists a new Hamiltonian:

$$K = K(Q, P, t), \quad (4.5)$$

where

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad \text{and} \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}. \quad (4.6)$$

For notational purposes let repeated indices be summed over implicitly. Hamilton's principle can be written as:

$$\delta \int_{t_1}^{t_2} (p_i \dot{q}_i - H(q, p, t)) dt = 0, \quad (4.7)$$

or in the new Hamiltonian as:

$$\delta \int_{t_1}^{t_2} (P_i \dot{Q}_i - K(Q, P, t)) dt = 0. \quad (4.8)$$

For the Eq.(4.7) to imply Eq.(4.8), then we need:

$$\lambda(p_i \dot{q}_i - H) = P_i \dot{Q}_i - K + \dot{F}. \quad (4.9)$$

Since  $\dot{F}$  is a total derivative and the ends of the path are fixed:

$$\delta q|_{t_1}^{t_2} = 0 \quad \text{and} \quad \delta p|_{t_1}^{t_2} = 0 \quad \Rightarrow \quad \delta F|_{t_1}^{t_2} = 0 \quad (4.10)$$

There are a few things to be said about transformations and  $\lambda$ .

- If  $\lambda = 1$  then the transformation is *canonical*, which is what we will study.
- If  $\lambda \neq 1$  then the transformation is *extended canonical*, and the results from  $\lambda = 1$  can be recovered by rescaling  $q$  and  $p$  appropriately.

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- If  $Q_i = Q_i(q, p)$  and  $P_i = P_i(q, p)$  without explicit dependence on time, then the transformation is *restricted canonical*.

We will always take transformations  $Q_i = Q_i(q, p, t)$  and  $P_i = P_i(q, p, t)$  to be invertible in any of the canonical variables. If  $F$  depends on a mix of old and new phase space variables, it is called a *generating function* of the canonical transformation. There are four important cases of this.

**1.** Let us take

$$F = F_1(q, Q, t) \quad (4.11)$$

where the old coordinates  $q_i$  and the new coordinates  $Q_i$  are independent. Then:

$$p_i \dot{q}_i - H = P_i \dot{Q}_i - K + \dot{F}_1 = P_i \dot{Q}_i - K + \frac{\partial F_1}{\partial t} + \frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i \quad (4.12)$$

from this we see that  $P_i \dot{Q}_i$  cancels and equating the terms with a  $\dot{q}_i$ , a  $\dot{Q}_i$  and the remaining terms gives:

$$p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i} \quad \text{and} \quad K = H + \frac{\partial F_1}{\partial t}, \quad (4.13)$$

which gives us formula for a transformation:

$$p_i = p_i(q, Q, t) \quad \text{and} \quad P_i = P_i(q, Q, t) \quad (4.14)$$

and connects  $K$  to an initial  $H$ .

**Example:** if

$$F_1 = -\frac{Q}{q}, \quad (4.15)$$

then:

$$p = \frac{\partial F_1}{\partial q} = \frac{Q}{q^2} \quad \text{and} \quad P = -\frac{\partial F_1}{\partial Q} = \frac{1}{q}. \quad (4.16)$$

Writing the new coordinates as function of the old ones yields

$$Q = pq^2 \quad \text{and} \quad P = \frac{1}{q} \quad (4.17)$$

**Example:** Given the transformations

$$Q = \ln\left(\frac{p}{q}\right) \quad \text{and} \quad P = -\left(\frac{q^2}{2} + 1\right) \frac{p}{q}, \quad (4.18)$$

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we can prove they are canonical by finding a corresponding generating function. We know:

$$\frac{\partial F_1}{\partial q} = p = qe^Q, \quad (4.19)$$

which gives us

$$F_1 = \int qe^Q dq + g(Q) = \frac{q^2}{2} e^Q + g(Q), \quad (4.20)$$

and

$$\begin{aligned} P &= -\frac{\partial F_1}{\partial Q} = -\frac{q^2}{2} e^Q - \frac{dg}{dQ} = -\left(\frac{q^2}{2} + 1\right) \frac{p}{q} = -\left(\frac{q^2}{2} + 1\right) e^Q \\ \Rightarrow g(Q) &= e^Q. \end{aligned} \quad (4.21)$$

Thus  $F_1$  is given by:

$$F_1 = \left(\frac{q^2}{2} + 1\right) e^Q. \quad (4.22)$$

**2.** Let:

$$F = F_2(q, P, t) - Q_i P_i \quad (4.23)$$

where we wish to treat the old coordinates  $q_i$  and new momenta  $P_i$  as independent variables. Then:

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \dot{F}_2 - \dot{Q}_i P_i - Q_i \dot{P}_i = -Q_i \dot{P}_i - K + \frac{\partial F_2}{\partial t} + \frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial P_i} \dot{P}_i \quad (4.24)$$

This corresponds to

$$p_i = \frac{\partial F_2}{\partial q_i} \quad ; \quad Q_i = \frac{\partial F_2}{\partial P_i} \quad \text{and} \quad K = H + \frac{\partial F_2}{\partial t}. \quad (4.25)$$

**3.** We could also take

$$F = F_3(p, Q, t) + q_i p_i \quad (4.26)$$

with the new coordinates  $Q_i$  and the old momenta  $p_i$  as independent variables.

**4.** Finally we could take

$$F = F_4(p, P, t) + q_i p_i - Q_i P_i \quad (4.27)$$

with the old momenta  $p_i$  and new momenta  $P_i$  as independent variables.

This can be summarized in the table below.

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Function	Transformations	Simplest case	
$F_1(q, Q, t)$	$p_i = \frac{\partial F_1}{\partial q_i}, P_i = -\frac{\partial F_1}{\partial Q_i}$	$F_1 = q_i Q_i$	$Q_i = p_i, P_i = -q_i$
$F_2(q, P, t)$	$p_i = \frac{\partial F_2}{\partial q_i}, Q_i = \frac{\partial F_2}{\partial P_i}$	$F_2 = q_i P_i$	$Q_i = q_i, P_i = p_i$
$F_3(p, Q, t)$	$q_i = -\frac{\partial F_3}{\partial p_i}, P_i = -\frac{\partial F_3}{\partial Q_i}$	$F_3 = p_i Q_i$	$Q_i = -q_i, P_i = -p_i$
$F_4(p, P, t)$	$q_i = -\frac{\partial F_4}{\partial p_i}, Q_i = \frac{\partial F_4}{\partial P_i}$	$F_4 = p_i P_i$	$Q_i = p_i, P_i = -q_i$

The simplest case of the 2<sup>nd</sup> ( $F_2$ ) transformation is just an identity transformation. For any of these  $F_i$  cases we also have:

$$K = H + \frac{\partial F_i}{\partial t}. \quad (4.28)$$

If  $F_i$  is independent of time then this implies

$$K = H \quad (4.29)$$

Mixed cases may also occur when more than two old canonical coordinates are present. (In this chapter we will be using Einstein's repeated index notation for implicit summation, unless otherwise stated.)

**Example:** consider

$$F_2 = f_i(q, t)P_i \quad (4.30)$$

for some functions  $f_i$  where  $i \in \{1, \dots, n\}$ . Then

$$Q_i = \frac{\partial F_2}{\partial P_i} = f_i(q, t) \quad (4.31)$$

is a coordinate point transformation. It is canonical with

$$p_i = \frac{\partial f_i}{\partial q_j} P_j, \quad (4.32)$$

which can be inverted to get  $P_j = P_j(q, p, t)$ .

**Example:** Consider the harmonic oscillator:

$$H = \frac{p^2}{2m} + \frac{kq^2}{2} \quad \text{where } k = m\omega^2 \quad (4.33)$$

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Let us try the transformation:

$$\begin{aligned} p &= \alpha\sqrt{2mP} \cos(Q) \\ q &= \frac{\alpha}{m\omega}\sqrt{2mP} \sin(Q) \end{aligned} \quad (4.34)$$

for  $\alpha$  constant. Then:

$$K = H = P\alpha^2 (\cos^2(Q) + \sin^2(Q)) = P\alpha^2, \quad (4.35)$$

so the new momentum

$$P = \frac{E}{\alpha^2} \quad (4.36)$$

is just proportional to the energy, while  $Q$  is a cyclic variable.

Is this transformation canonical? We can find a generating function  $F = F_1(q, Q)$  by dividing the old variables:

$$\frac{p}{q} = m\omega \cot(Q). \quad (4.37)$$

This gives us:

$$p = \frac{\partial F_1}{\partial q} \Rightarrow F_1 = \int p(q, Q) dq + g(Q) = \frac{1}{2}m\omega q^2 \cot(Q) + g(Q) \quad (4.38)$$

$$P = -\frac{\partial F_1}{\partial Q} = \frac{m\omega q^2}{2 \sin^2(Q)} - \frac{dg}{dQ}$$

Setting:

$$\frac{dg}{dQ} = 0 \Rightarrow q^2 = \frac{2P}{m\omega} \sin^2(Q), \quad (4.39)$$

which tells us the transformation is canonical if  $\alpha = \sqrt{\omega}$ . This means:

$$P = \frac{E}{\omega} \quad (4.40)$$

By Hamilton's equations Eq.(4.4):

$$\dot{Q} = \frac{\partial K}{\partial P} = \omega \Rightarrow Q = \omega t + \delta. \quad (4.41)$$

Putting this altogether, this gives the familiar results:

$$\begin{aligned} q &= \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \delta) \\ p &= \sqrt{2mE} \cos(\omega t + \delta). \end{aligned} \quad (4.42)$$

Lets record for future use our final canonical transformation here:

$$q = \sqrt{\frac{2P}{m\omega}} \sin(Q), \quad p = \sqrt{2m\omega P} \cos(Q).$$

So far, a transformation  $Q = Q(q, p, t)$  and  $P = P(q, p, t)$  is canonical if we can find a generating function  $F$ . This involves integration, which could be complicated, so it would be nice to have a test that only involves differentiation. There is one!

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## 4.2 Poisson Brackets and the Symplectic Condition

In Classical Mechanics II (8.223) the *Poisson bracket* of the quantities  $u$  and  $v$  was defined as

$$\{u, v\}_{q,p} = \sum_i \left( \frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial v}{\partial q_i} \frac{\partial u}{\partial p_i} \right) \quad (4.43)$$

It is easy to check that the following fundamental Poisson bracket relations are satisfied:

$$\{q_i, q_j\}_{q,p} = \{p_i, p_j\}_{q,p} = 0 \quad \text{and} \quad \{q_i, p_j\}_{q,p} = \delta_{ij}. \quad (4.44)$$

There are a few other properties of note. These include:

$$\{u, u\} = 0, \quad (4.45)$$

$$\{u, v\} = -\{v, u\}, \quad (4.46)$$

$$\{au + bv, w\} = a\{u, w\} + b\{v, w\}, \quad (4.47)$$

$$\{uv, w\} = u\{v, w\} + \{u, w\}v, \quad (4.48)$$

$$\{u, \{v, w\}\} + \{v, \{w, u\}\} + \{w, \{u, v\}\} = 0, \quad (4.49)$$

for  $a, b$  constants. Eq.(4.49) is the Jacobi identity.

The above looks a lot like the commutators of operators in quantum mechanics, such as:

$$[\hat{x}, \hat{p}] = i\hbar \quad (4.50)$$

Indeed, quantizing a classical theory by replacing Poisson brackets with commutators through:

$$[u, v] = i\hbar\{u, v\} \quad (4.51)$$

is a popular approach (first studied by Dirac). It is also the root of the name “canonical quantization”. (Note that Eq.(4.48) was written in a manner to match the analogous formula in quantum mechanics where the operator ordering is important, just in case its familiar. Here we can multiply functions in either order.)

Now we can state the desired criteria that only involves derivatives.

**Theorem:** A transformation  $Q_j = Q_j(q, p, t)$  and  $P_j = P_j(q, p, t)$  is canonical if and only if:

$$\{Q_i, Q_j\}_{q,p} = \{P_i, P_j\}_{q,p} = 0 \quad \text{and} \quad \{Q_i, P_j\}_{q,p} = \delta_{ij}. \quad (4.52)$$

To prove it, we’ll need some more notation. Let’s get serious about treating  $q_i$  and  $p_i$  on an equal footing together, defining the following two quantities:

$$\vec{\eta} = \begin{pmatrix} q_1 \\ \vdots \\ q_n \\ p_1 \\ \vdots \\ p_n \end{pmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0_{n \times n} & \mathbb{1}_{n \times n} \\ -\mathbb{1}_{n \times n} & 0_{n \times n} \end{bmatrix} \quad (4.53)$$

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where  $0_{n \times n}$  is the  $n \times n$  zero matrix,  $\mathbb{1}_{n \times n}$  is the  $n \times n$  identity matrix. The following properties of  $J$  will be useful:

$$J^\top = -J \quad , \quad J^2 = -\mathbb{1}_{2n \times 2n} \quad \text{and} \quad J^\top J = JJ^\top = \mathbb{1}_{2n \times 2n}. \quad (4.54)$$

We also note that  $\det(J) = 1$ .

With this notation Hamilton's equations, Eq.(4.4), can be rewritten as:

$$\dot{\vec{\eta}} = J \frac{\partial H}{\partial \vec{\eta}} \quad \text{or} \quad \dot{\vec{\eta}} = J \nabla_{\vec{\eta}} H. \quad (4.55)$$

The notation  $\nabla_{\vec{\eta}} H$  better emphasizes that this quantity is a vector, but we will stick to using the first notation for this vector,  $\partial H / \partial \vec{\eta}$ , below.

Although the Theorem is true for time dependent transformations, lets carry out the proof for the simpler case of time independent transformations  $Q_i = Q_i(q, p)$  and  $P_i = P_i(q, p)$ . This implies  $K = H$ . Let us define:

$$\vec{\xi} = \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \\ P_1 \\ \vdots \\ P_n \end{pmatrix} \quad (4.56)$$

which is a function of the original coordinates, so we can write:

$$\vec{\xi} = \vec{\xi}(\vec{\eta}) \quad (4.57)$$

Now consider the time derivative of  $\vec{\xi}$ :

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial \eta_j} \dot{\eta}_j \quad \Leftrightarrow \quad \dot{\vec{\xi}} = M \dot{\vec{\eta}} \quad \text{where} \quad M_{ij} = \frac{\partial \xi_i}{\partial \eta_j}. \quad (4.58)$$

Here  $M$  corresponds to the *Jacobian* of the transformation.

From the Hamilton's equations, we know that

$$\dot{\vec{\eta}} = J \frac{\partial H}{\partial \vec{\eta}}. \quad (4.59)$$

We want to show that :

$$\dot{\vec{\xi}} = J \frac{\partial H}{\partial \vec{\xi}} \quad \text{for} \quad \vec{\xi} = \vec{\xi}(\vec{\eta}) \text{ a canonical transformation.} \quad (4.60)$$

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Let us now consider:

$$\dot{\xi}_i = \frac{\partial \xi_i}{\partial \eta_j} \dot{\eta}_j = \frac{\partial \xi_i}{\partial \eta_j} J_{jk} \frac{\partial H}{\partial \eta_k} = \frac{\partial \xi_i}{\partial \eta_j} J_{jk} \frac{\partial \xi_l}{\partial \eta_k} \frac{\partial H}{\partial \xi_l} \Leftrightarrow \dot{\vec{\xi}} = M J M^\top \frac{\partial H}{\partial \vec{\xi}} \quad (4.61)$$

for any  $H$ . Then

$$\vec{\xi} = \vec{\xi}(\vec{\eta}) \text{ is a canonical transformation iff } M J M^\top = J \quad (4.62)$$

is satisfied. This is known as the “symplectic condition”. Moreover, since

$$M J = J(M^\top)^{-1} \quad \text{and} \quad J^2 = -\mathbb{1}, \quad (4.63)$$

we can write:

$$J(M J) J = -J M = J(J M^{\top-1}) J = -M^{\top-1} J \Rightarrow J M = M^{\top-1} J. \quad (4.64)$$

Thus we see the equivalence

$$M J M^\top = J \Leftrightarrow M^\top J M = J. \quad (4.65)$$

Now consider Poisson brackets in this matrix notation:

$$\{u, v\}_{q,p} = \{u, v\}_{\vec{\eta}} = \left( \frac{\partial u}{\partial \vec{\eta}} \right)^\top J \frac{\partial v}{\partial \vec{\eta}} \quad (4.66)$$

and the fundamental Poisson brackets are:

$$\{\eta_i, \eta_j\}_{\vec{\eta}} = J_{ij} \quad (4.67)$$

Then we can calculate the Poisson brackets that appeared in the theorem we are aiming to prove as

$$\{\xi_i, \xi_j\}_{\vec{\eta}} = \left( \frac{\partial \xi_i}{\partial \vec{\eta}} \right)^\top J \frac{\partial \xi_j}{\partial \vec{\eta}} = (M^\top J M)_{ij} \quad (4.68)$$

This last equation is the same as Eq.(4.65). The new variables satisfy the Poisson bracket relationships Eq.(4.67):

$$\{\xi_i, \xi_j\}_{\vec{\eta}} = J_{ij} \quad (4.69)$$

*if and only if*

$$M^\top J M = J \quad (4.70)$$

which using Eqs. (4.62) and (4.65) is true if and only if  $\vec{\xi} = \vec{\xi}(\vec{\eta})$  is canonical. This completes the proof.

There are two facts that arise from this.

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- Poisson brackets are canonical invariants

$$\{u, v\}_{\vec{\eta}} = \{u, v\}_{\vec{\xi}} = \{u, v\}. \quad (4.71)$$

This is true because:

$$\{u, v\}_{\vec{\eta}} = \left( \frac{\partial u}{\partial \vec{\eta}} \right)^T J \frac{\partial v}{\partial \vec{\eta}} = \left( M^T \frac{\partial u}{\partial \vec{\xi}} \right)^T J \left( M^T \frac{\partial v}{\partial \vec{\xi}} \right) \quad (4.72)$$

$$= \left( \frac{\partial u}{\partial \vec{\xi}} \right)^T M J M^T \frac{\partial v}{\partial \vec{\xi}} = \left( \frac{\partial u}{\partial \vec{\xi}} \right)^T J \frac{\partial v}{\partial \vec{\xi}} = \{u, v\}_{\vec{\xi}} \quad (4.73)$$

- Phase space volume elements are preserved by canonical transformations, as discussed in 8.223. Phase space volume is given by:

$$V_{\vec{\xi}} = \prod_i dQ_i dP_i = |\det(M)| \prod_j dq_j dp_j = |\det(M)| V_{\vec{\eta}}. \quad (4.74)$$

Since we have

$$\det(M^T J M) = \det(J) = (\det(M))^2 \det(J) \Rightarrow |\det(M)| = 1, \quad (4.75)$$

and the two phase space volumes are the same.

### 4.3 Equations of Motion & Conservation Theorems

Let us consider a function:

$$u = u(q, p, t) \quad (4.76)$$

Then:

$$\dot{u} = \frac{\partial u}{\partial q_i} \dot{q}_i + \frac{\partial u}{\partial p_i} \dot{p}_i + \frac{\partial u}{\partial t} = \frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}, \quad (4.77)$$

which can be written more concisely as

$$\dot{u} = \{u, H\} + \frac{\partial u}{\partial t} \quad (4.78)$$

for any canonical variables  $(q, p)$  and corresponding Hamiltonian  $H$ . Performing canonical quantization on this yields the Heisenberg equation of time evolution in quantum mechanics. There are a few easy cases to check.

- If  $u = q_i$  then:

$$\dot{q}_i = \{q_i, H\} + \frac{\partial q_i}{\partial t} = \frac{\partial H}{\partial p_i} \quad (4.79)$$

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- If  $u = p_i$  then:

$$\dot{p}_i = \{p_i, H\} + \frac{\partial p_i}{\partial t} = -\frac{\partial H}{\partial q_i} \quad (4.80)$$

Together the above two cases yield Hamilton's equations of motion.

- Also, if  $u = H$  then:

$$\dot{H} = \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} \quad (4.81)$$

Next consider what we can say if a quantity  $u$  is conserved. Here:

$$\dot{u} = 0 = \{u, H\} + \frac{\partial u}{\partial t}. \quad (4.82)$$

As a corollary, if

$$\frac{\partial u}{\partial t} = 0, \quad (4.83)$$

then

$$\{u, H\} = 0 \Rightarrow u \text{ is conserved.} \quad (4.84)$$

(In quantum mechanics this is the analog of saying that  $u$  is conserved if  $u$  commutes with  $H$ .)

Another fact, is that if  $u$  and  $v$  are conserved then so is  $\{u, v\}$ . This could potentially provide a way to compute a new constant of motion. To prove it, first consider the special case where:

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0 \quad (4.85)$$

then using the Jacobi identity we have:

$$\{H, \{u, v\}\} = -\{u, \{v, H\}\} - \{v, \{H, u\}\} = -\{u, 0\} - \{v, 0\} = 0 \quad (4.86)$$

For the most general case we proceed in a similar manner:

$$\begin{aligned} \{\{u, v\}, H\} &= \{u, \{v, H\}\} + \{v, \{H, u\}\} = -\left\{u, \frac{\partial v}{\partial t}\right\} + \left\{v, \frac{\partial u}{\partial t}\right\} \\ &= -\frac{\partial}{\partial t}\{u, v\} \quad \Rightarrow \quad \frac{d}{dt}\{u, v\} = 0 \end{aligned} \quad (4.87)$$

### Infinitesimal Canonical Transformations

Let us now consider the generating function:

$$F_2(q, P, t) = q_i P_i + \epsilon G(q, P, t), \quad (4.88)$$

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where  $F_2 = q_i P_i$  is an identity transformation, and  $|\epsilon| \ll 1$  is infinitesimal. The function  $G(q, P, t)$  is known as the generating function of an infinitesimal canonical transformation. Using the properties of an  $F_2$  generating function we have:

$$p_j = \frac{\partial F_2}{\partial q_j} = P_j + \epsilon \frac{\partial G}{\partial q_j} \quad \Rightarrow \quad \delta p_j = P_j - p_j = -\epsilon \frac{\partial G}{\partial q_j} \quad (4.89)$$

giving the infinitesimal transformation in the momentum. Likewise:

$$Q_j = \frac{\partial F_2}{\partial P_j} = q_j + \epsilon \frac{\partial G}{\partial P_j} \quad (4.90)$$

Since  $P_j = p_j + O(\epsilon)$  and  $\epsilon$  is infinitesimal we can replace  $\partial G(q, P, t)/\partial P_j = \partial G(q, p, t)/\partial p_j + O(\epsilon)$ . Therefore we have:

$$Q_j = q_j + \epsilon \frac{\partial G}{\partial p_j} + O(\epsilon^2) \quad \Rightarrow \quad \delta q_j = Q_j - q_j = \epsilon \frac{\partial G}{\partial p_j} \quad (4.91)$$

where now we note that we can consider  $G = G(q, p, t)$ , a function of  $q$  and  $p$ , to this order. Returning to the combined notation of  $\vec{\eta}^\top = (q_1, \dots, q_n, p_1, \dots, p_n)$ , Eq.(4.89) and Eq.(4.90) can be consisely written as the following Poisson bracket:

$$\delta \vec{\eta} = \epsilon \{ \vec{\eta}, G \} \quad (4.92)$$

**Example:** if  $G = p_i$  then  $\delta p_i = 0$  and  $\delta q_j = \epsilon \delta_{ij}$ , which is why *momentum is the generator of spatial translations*.

**Example:** if  $G$  is the  $z$  component of the angular momentum:

$$G = L_z = \sum_i (x_i p_{iy} - y_i p_{ix}) \quad \text{and} \quad \epsilon = \delta \theta \quad (4.93)$$

then the infinitesimal change correponds to a rotation

$$\delta x_i = -y_i \delta \theta \quad , \quad \delta y_i = x_i \delta \theta \quad , \quad \delta z_i = 0 \quad (4.94)$$

$$\delta p_{ix} = -p_{iy} \delta \theta \quad , \quad \delta p_{iy} = p_{ix} \delta \theta \quad , \quad \delta p_{iz} = 0 \quad (4.95)$$

which is why *angular momentum is the generator of rotations*.

**Important Example:** if  $G = H$  and  $\epsilon = dt$  then

$$\epsilon \{ \vec{\eta}, G \} = \{ \vec{\eta}, H \} dt = \dot{\vec{\eta}} dt = d\vec{\eta}$$

On the left hand side we have the change to the phase space coordinates due to our transformation. On the right hand side we have the physical increment to the phase space variables

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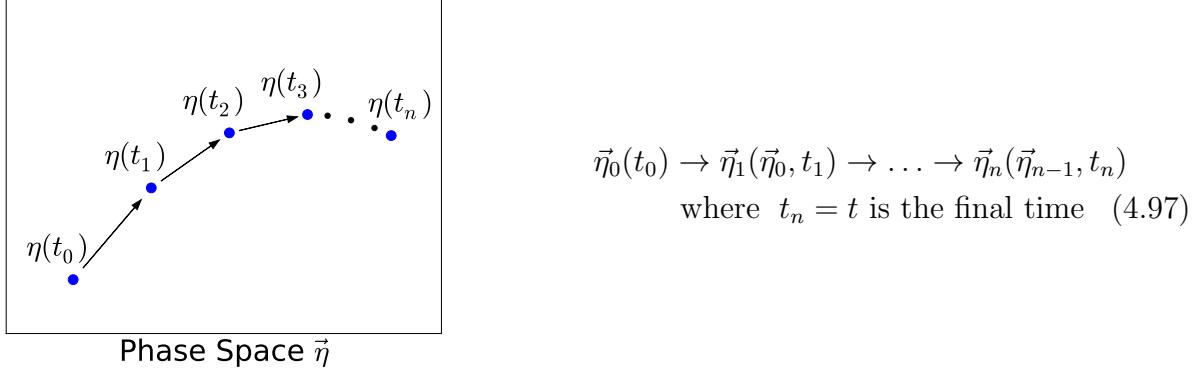
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that takes place in time  $dt$ . The fact that these are equivalent tells us that *the Hamiltonian is the generator of time evolution*. The infinitesimal transformation generated by the Hamiltonian gives the same function as solving for the physical motion.

Rather than trying to think of this as a passive coordinate change  $(q, p) \rightarrow (Q, P)$ , it is useful if we can take an *active view* of the infinitesimal canonical transformation generated by  $H$ . Let the time  $t$  be a parameter for the family of transformations with  $\epsilon = dt$ : the initial conditions are:

$$\vec{\eta}_0(t_0) = \vec{\eta}_0 \quad (4.96)$$

The result is a series of transformations of  $\vec{\eta}$  that move us in a fixed set of phase space coordinates from one point to another:



All together, combining an infinite number of infinitesimal transformations allows us to make a finite transformation, resulting in:

$$\vec{\eta} = \vec{\eta}(\vec{\eta}_0, t) \quad \text{or} \quad \vec{\eta}_0 = \vec{\eta}_0(\vec{\eta}, t) \quad (4.98)$$

This is a canonical transformation that yields a solution for the motion!

How could we directly find this transformation, without resorting to stringing together infinitesimal transformations? We can simply look for a canonical transformation with new coordinates  $Q_i$  and new momenta  $P_i$  that are all constants, implying an equation of the type:

$$\vec{\eta}_0 = \vec{\eta}_0(\vec{\eta}, t) \quad (4.99)$$

Inverting this then gives the solution for the motion.

This logic can be used to extend our proof of the Theorem in Section 4.2 to fully account for time dependent transformations. (see Goldstein). Using  $K = H + \epsilon \partial G / \partial t$ , Goldstein also describes in some detail how the change to the Hamiltonian  $\Delta H$  under an active infinitesimal canonical transformation satisfies:

$$\Delta H = -\epsilon \{G, H\} - \epsilon \frac{\partial G}{\partial t} = -\epsilon \dot{G} \quad (4.100)$$

This says “the constants of motion are generating functions  $G$  of the infinitesimal canonical transformation that leave  $H$  invariant”; that is,  $\dot{G} = 0$  if and only if  $\Delta H = 0$  under the transformation. Thus a conservation law exists if and only if there is a symmetry present.

## 4.4 Hamilton-Jacobi Equation

Let us take the suggestion from the end of the previous section seriously and look for new canonical variables that are all cyclic, such that:

$$\dot{Q}_i = \dot{P}_i = 0 \quad \Rightarrow \quad (Q, P) \text{ are all constants.} \quad (4.101)$$

If the new Hamiltonian  $K$  is independent of  $(Q, P)$  then:

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = 0 \quad \text{and} \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i} = 0. \quad (4.102)$$

We could look for a constant  $K$ , but it is simplest to simply look for  $K = 0$ .

Using a generating function  $F = F_2(q, P, t)$ , then we need

$$K = H(q, p, t) + \frac{\partial F_2}{\partial t} = 0. \quad (4.103)$$

Because  $p_i = \frac{\partial F_2}{\partial q_i}$ , then we can rewrite this condition as

$$H\left(q_1, \dots, q_n, \frac{\partial F_2}{\partial q_1}, \dots, \frac{\partial F_2}{\partial q_n}, t\right) + \frac{\partial F_2}{\partial t} = 0, \quad (4.104)$$

which is the time dependent *Hamilton-Jacobi equation* (henceforth abbreviated as the H-J equation). This is a 1<sup>st</sup> order partial differential equation in  $n + 1$  variables  $(q_1, \dots, q_n, t)$  for  $F_2$ . The solution for  $F_2$  has  $n + 1$  independent constants of integration. One of these constants is trivial ( $F_2 \rightarrow F_2 + C$  for a pure constant  $C$ ), so we'll ignore this one. Hence, suppose the solution is:

$$F_2 \equiv S = S(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t), \quad (4.105)$$

where  $S$  is called Hamilton's principal function and each  $\alpha_i$  is an independent constant. We can pick our new momenta to be the constants of integration  $P_i = \alpha_i$  for  $i \in \{1, \dots, n\}$  (so that  $\dot{P}_i = 0$ ), thus specifying  $F_2 = F_2(q, P, t)$  as desired. Then, using again the property of an  $F_2$  generating function (and  $K = 0$ ), we have that the new constant variables are:

$$P_i \equiv \alpha_i \quad \text{and} \quad Q_i \equiv \beta_i = \frac{\partial S(q, \alpha, t)}{\partial \alpha_i}. \quad (4.106)$$

We introduce the notation  $\beta_i$  to emphasize that these are constants.

From these results we can obtain a solution for the motion as follows. From the invertibility of our transformations we have:

$$\begin{aligned} \beta_i(q, \alpha, t) &= \frac{\partial S}{\partial \alpha_i} \quad \Rightarrow \quad q_i = q_i(\alpha, \beta, t), \\ p_i(q, \alpha, t) &= \frac{\partial S}{\partial q_i} \quad \Rightarrow \quad p_i = p_i(q, \alpha, t) = p_i(q(\alpha, \beta, t), \alpha, t) = p_i(\alpha, \beta, t). \end{aligned} \quad (4.107)$$

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(Note that function argument notation has been abused slightly here since  $p_i(q, \alpha, t)$  and  $p_i(\alpha, \beta, t)$  are technically different functions of their three arguments. Since we are always sticking explicit variables into the slots this should not cause confusion.) If desired, we can also swap our  $2n$  constants  $\alpha_i$  and  $\beta_i$  for  $2n$  initial conditions  $q_{i0}$  and  $p_{i0}$ , to obtain a solution for the initial value problem. We obtain one set of constants in terms of the other set by solving the  $2n$  equations obtained from the above results at  $t = t_0$ :

$$q_{i0} = q_i(\alpha, \beta, t_0), \quad p_{i0} = p_i(\alpha, \beta, t_0). \quad (4.108)$$

Thus we see that Hamilton's principal function  $S$  is the generator of canonical transformations of constant  $(Q, P)$ , and provides a method of obtaining solutions to classical mechanics problems by way of finding a transformation.

There are a few comments to be made about this.

1. The choice of constants  $\alpha_i$  is somewhat arbitrary, as any other independent choice  $\gamma_i = \gamma_i(\alpha)$  is equally good. Thus, when solving the H-J equation, we introduce the constants  $\alpha_i$  in whatever way is most convenient.
2. What is  $S$ ? We know that:

$$\dot{S} = \frac{\partial S}{\partial q_i} \dot{q}_i + \frac{\partial S}{\partial P_i} \dot{P}_i + \frac{\partial S}{\partial t}, \quad (4.109)$$

but we also know that:

$$\frac{\partial S}{\partial q_i} = p_i \quad , \quad \dot{P}_i = 0 \quad \text{and} \quad \frac{\partial S}{\partial t} = -H \quad (4.110)$$

Putting Eq.(4.109) and Eq.(4.110) together we have:

$$\dot{S} = p_i \dot{q}_i - H = L \quad \Rightarrow \quad S = \int L dt \quad (4.111)$$

Thus  $S$  is the classical action which is an indefinite integral over time of the Lagrangian (so it is no coincidence that the same symbol is used).

3. The H-J equation is also the semiclassical limit of the quantum mechanical Schrödinger equation (0'th order term in the WKB approximation). To see this consider the Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + V(q) \right) \psi, \quad (4.112)$$

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with the wavefunction  $\psi = \exp(iS/\hbar)$ . At this point we are just making a change of variable, without loss of generality, and  $S(q, t)$  is complex. Plugging it in, and canceling an exponential after taking the derivative, we find

$$-\frac{\partial S}{\partial t} = -\frac{i\hbar}{2m} \frac{\partial^2 S}{\partial q^2} + \frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 + V(q). \quad (4.113)$$

This equation is just another way of writing the Schrödinger equation, to solve for a complex  $S$  instead of  $\psi$ . If we now take  $\hbar \rightarrow 0$  then we find that the imaginary term goes away leaving

$$0 = \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial q} \right)^2 + V(q) = \frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right), \quad (4.114)$$

which is the Hamilton-Jacobi equation for  $S$  with a standard  $p^2/2m$  kinetic term in  $H$ .

Having set things up, it is always good for us to test a new formalism on an example where we know the solution.

**Example:** let us consider the harmonic oscillator Eq.(4.33):

$$H = \frac{1}{2m} (p^2 + (m\omega q)^2) = E \quad (4.115)$$

Here we will look for one constant  $P = \alpha$  and one constant  $Q = \beta$ . The H-J equation says

$$\frac{1}{2m} \left( \left( \frac{\partial S}{\partial q} \right)^2 + (m\omega q)^2 \right) + \frac{\partial S}{\partial t} = 0. \quad (4.116)$$

In solving this, we note that the dependence of  $S$  on  $q$  and  $t$  is *separable*

$$S(q, \alpha, t) = W(q, \alpha) + g(\alpha, t), \quad (4.117)$$

which gives:

$$\frac{1}{2m} \left( \left( \frac{\partial W}{\partial q} \right)^2 + (m\omega q)^2 \right) = -\frac{\partial g}{\partial t} = \alpha. \quad (4.118)$$

Since the left side is independent of  $t$  and the right hand side is independent of  $q$ , then the result must be equal to a separation constant  $\alpha$  that is independent of  $q$  and  $t$ . We will choose our new  $P = \alpha$ . Now we have

$$\frac{\partial g}{\partial t} = -\alpha \Rightarrow g = -\alpha t \quad (4.119)$$

where we have avoided the addition of a further additive constant (since our convention was to always drop an additive constant when determining  $S$ ). To identify what  $\alpha$  is note that

$$H = -\frac{\partial S}{\partial t} = -\frac{\partial g}{\partial t} = \alpha, \quad (4.120)$$

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which corresponds to the constant energy,

$$\alpha = E. \quad (4.121)$$

The other equation we have to solve is

$$\frac{1}{2m} \left( \left( \frac{\partial W}{\partial q} \right)^2 + (m\omega q)^2 \right) = \alpha, \quad (4.122)$$

so rearranging and integrating this gives the indefinite integral

$$W = \pm \int \sqrt{2m\alpha - (m\omega q)^2} dq, \quad (4.123)$$

which we will leave unintegrated until we must do so. The full solution is then given by:

$$S = -\alpha t \pm \int \sqrt{2m\alpha - (m\omega q)^2} dq. \quad (4.124)$$

With this result for Hamilton's Principal function in hand we can now solve for the equations of motion. The equations of motion come from (we now do the integral, after taking the partial derivative):

$$\beta = \frac{\partial S}{\partial \alpha} = -t \pm m \int \frac{dq}{\sqrt{2m\alpha - (m\omega q)^2}} \Rightarrow t + \beta = \pm \frac{1}{\omega} \arcsin \left( \sqrt{\frac{m\omega^2}{2\alpha}} q \right). \quad (4.125)$$

Inverting gives:

$$q = \pm \sqrt{\frac{2\alpha}{m\omega^2}} \sin(\omega(t + \beta)), \quad (4.126)$$

so  $\beta$  is related to the phase. Next we consider  $p$  and use this result to obtain:

$$p = \frac{\partial S}{\partial q} = \pm \sqrt{2m\alpha - (m\omega q)^2} = \pm \sqrt{2m\alpha} \cos(\omega(t + \beta)) \quad (4.127)$$

These results are as expected. We can trade  $(\alpha, \beta)$  for the initial conditions  $(q_0, p_0)$  at  $t = 0$ . The choice of phase (from shifting  $\beta$  so that  $\omega\beta \rightarrow \omega\beta + \pi$ ) allows taking the positive sign of each square root in the solutions above.

Separation of variables is the main technique to solve the H-J equation. In particular, for a time independent  $H$  where

$$\dot{H} = \frac{\partial H}{\partial t} = 0 \quad (4.128)$$

we can always separate time by taking:

$$S(q, \alpha, t) = W(q, \alpha) - \alpha_1 t, \quad (4.129)$$

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where  $\alpha_1$  has been chosen as the separation constant, then plugging this into the time dependent H-J equation yields (just as in our Harmonic Oscillator example):

$$H\left(q_1, \dots, q_n, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_n}\right) = \alpha_1. \quad (4.130)$$

This result is referred to as the *time independent Hamilton-Jacobi equation*. Since  $\dot{H} = 0$ ,  $H$  is conserved, and equal to a constant  $\alpha_1$ . If  $H = E$  then this constant is energy,  $\alpha_1 = E$ . The function  $W$  is called Hamilton's characteristic function.

The idea is now to solve the time independent H-J equation for  $W = W(q, \alpha)$  where  $P = \alpha$  still. If we follow the setup from our time dependent solution above then the equations of motion are obtained from the following prescription for identifying variables:

$$\begin{aligned} p_i &= \frac{\partial W}{\partial q_i} \quad \text{for } i \in \{1, \dots, n\}, \\ Q_1 &= \beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial W}{\partial \alpha_1} - t, \\ Q_j &= \beta_j = \frac{\partial W}{\partial \alpha_j} \quad \text{for } j \in \{2, \dots, n\} \text{ for } n > 1. \end{aligned} \quad (4.131)$$

Here all the  $Q_i$  are constants.

There is an alternative to the above setup, which allows us to not refer to the time dependent solution. The alternative is to consider  $W = F_2(q, P)$  as the generating function, instead of  $S$  and only demand that all the new momenta  $P_i$  are constants with  $P_1 = \alpha_1 = H$  for a time independent Hamiltonian  $H$ . At the start of chapter 4 we saw that this less restrictive scenario would lead to  $Q$ s that could have a linear time dependence, which is still pretty simple.

This is almost identical to the above setup but we rename and reshuffle a few things. The following three equations are the same as before:

$$p_i = \frac{\partial W}{\partial q_i}, \quad P_i = \alpha_i \quad \text{and} \quad H\left(q, \frac{\partial W}{\partial q}\right) = \alpha_1 \quad (4.132)$$

However, now we have a non-zero  $K$  and different equation for  $Q_1$ :

$$K = H = \alpha_1 \quad \text{and} \quad Q_i = \frac{\partial W}{\partial \alpha_i} \text{ for all } i \in \{1, \dots, n\}. \quad (4.133)$$

This means:

$$\dot{Q}_1 = \frac{\partial K}{\partial \alpha_1} = 1 \quad \Rightarrow \quad Q_1 = t + \beta_1 = \frac{\partial W}{\partial \alpha_1} \quad (4.134)$$

which is Eq. (4.131) but rearranged from the perspective of  $Q_1$ . For  $j > 1$ , the equations are the same as before Eq.(4.131):

$$\dot{Q}_j = \frac{\partial K}{\partial \alpha_j} = 0 \quad \Rightarrow \quad Q_j = \beta_j = \frac{\partial W}{\partial \alpha_j} \quad (4.135)$$

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In this language we do not need to mention  $S$  only  $W$ . There are a few comments to be made:

1. Again, the choice of  $\alpha$  is arbitrary, and  $\alpha_i = \alpha_i(\gamma)$  is fine. If we do replace  $\alpha_1 = \alpha_1(\gamma)$  then  $\dot{Q}_i = \frac{\partial K}{\partial \gamma_i} = v_i$  is a constant so that (potentially) all of the  $Q_i$  become linear in time:

$$Q_i = v_i t + \beta_i \text{ for all } i \in \{1, \dots, n\} \quad (4.136)$$

2. What is  $W$ ? We know that:

$$W = \frac{\partial W}{\partial q_i} \dot{q}_i = p_i \dot{q}_i \Rightarrow W = \int p_i \dot{q}_i dt = \int p_i dq_i, \quad (4.137)$$

which is a different sort of “action”.

3. The time independent H-J equation has some similarity to the time-independent Schrödinger energy eigenvalue equation (both involve  $H$  and a constant  $E$ , but the former is a non-linear equation for  $W$ , while the latter is a linear equation for the wavefunction  $\psi$ ).

The H-J method is most useful when there is a separation of variables in  $H$ .

**Example:** if

$$H = h_1(q_1, q_2, p_1, p_2) + h_2(q_1, q_2, p_1, p_2)f(q_3, p_3) = \alpha_1, \quad (4.138)$$

so that  $q_3$  is separable, then

$$f(q_3, p_3) = \frac{\alpha_1 - h_1}{h_2} \quad (4.139)$$

is a constant because the right hand side is independent of  $q_3$  and  $p_3$ . Thus we assign

$$f(q_3, p_3) = \alpha_2 \quad (4.140)$$

for convenience. We can then write:

$$W = W'(q_1, q_2, \alpha) + W_3(q_3, \alpha) \Rightarrow f\left(q_3, \frac{\partial W_3}{\partial q_3}\right) = \alpha_2 \quad \text{and} \quad (4.141)$$

$$h_1\left(q_1, q_2, \frac{\partial W'}{\partial q_1}, \frac{\partial W'}{\partial q_2}\right) + \alpha_2 h_2\left(q_1, q_2, \frac{\partial W'}{\partial q_1}, \frac{\partial W'}{\partial q_2}\right) = \alpha_1 \quad (4.142)$$

Here,  $q_1$  and  $q_2$  may or may not be separable.

If all variables are separable then we use the solution:

$$W = \sum_i W_i(q_i, \alpha) \quad (4.143)$$

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We can simply try a solution of this form to test for separability.

Note that cyclic coordinates are always separable.

**Proof:** let us say that  $q_1$  is cyclic. Then

$$p_1 \equiv \gamma \quad \text{and} \quad H\left(q_2, \dots, q_n, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \dots, \frac{\partial W}{\partial q_n}\right) = \alpha_1, \quad (4.144)$$

where  $\gamma$  is constant. Let us now write

$$W(q, \alpha) = W_1(q_1, \alpha) + W'(q_2, \dots, q_n, \alpha). \quad (4.145)$$

This gives us:

$$p_1 = \frac{\partial W_1}{\partial q_1} = \gamma \Rightarrow W_1 = \gamma q_1. \quad (4.146)$$

Which gives us:

$$W(q, \alpha) = \gamma q_1 + W'(q_2, \dots, q_n, \alpha) \quad (4.147)$$

This procedure can be repeated for any remaining cyclic variables.

Note that the choice of variables is often important in finding a result that separates. A problem with spherical symmetry may separate in spherical coordinates but not Cartesian coordinates.

## 4.5 Kepler Problem

As an extended example, let us consider the Kepler problem of two masses  $m_1$  and  $m_2$  in a central potential (with the CM coordinate  $\mathbf{R} = 0$ ). The Lagrangian is:

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 - V(r) \quad \text{where} \quad \frac{1}{m} \equiv \frac{1}{m_1} + \frac{1}{m_2}, \quad (4.148)$$

and here  $m$  is the reduced mass. Any  $V(r)$  conserves  $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ , so the motion of  $\mathbf{r}$  and  $\mathbf{p}$  is in a plane perpendicular to  $\mathbf{L}$ . The coordinates in the plane can be taken as  $(r, \psi)$ , so:

$$L = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\psi}^2 \right) - V(r), \quad (4.149)$$

with  $\psi$  being cyclic, which implies:

$$p_\psi = mr^2\dot{\psi} \text{ is a constant.} \quad (4.150)$$

In fact  $p_\psi = |\mathbf{L}| \equiv \ell$ . Notationally, we use  $\ell$  for the magnitude of the angular momentum  $\mathbf{L}$  to distinguish it from the Lagrangian  $L$ .

The energy is then:

$$E = \frac{m}{2}\dot{r}^2 + \frac{\ell^2}{2mr^2} + V(r), \quad (4.151)$$

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which is constant, and this can be rewritten as:

$$E = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r) \quad \text{where} \quad V_{\text{eff}}(r) = V(r) + \frac{\ell^2}{2mr^2}, \quad (4.152)$$

where  $V_{\text{eff}}$  is the effective potential, as plotted below for the gravitational potential.

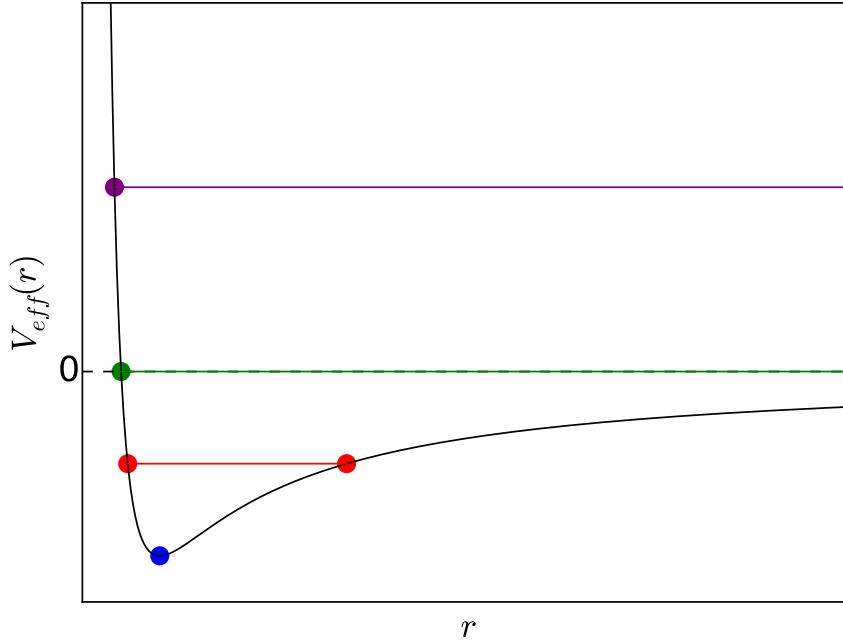


Figure 4.1: Plot of the effective potential  $V_{\text{eff}}$  along with the different qualitative orbits allowed in a gravity-like potential. The points correspond to turning points of the orbit.

Writing the E-L equation for  $\dot{r} = dr/dt = \dots$  and then solving for it as  $dt = dr/(\dots)$ , and integrating yields

$$t = t(r) = \int_{r_0}^r \frac{dr'}{\sqrt{\frac{2}{m} (E - V(r') - \frac{\ell^2}{2mr'^2})}} \quad (4.153)$$

as an implicit solution to the radial motion.

The orbital motion comes as  $r = r(\psi)$  or  $\psi = \psi(r)$  by using Eq.(4.150) and substituting, in Eq.(4.153). We have  $\dot{\psi} = d\psi/dt = \ell/(mr^2)$ , so we can use this to replace  $dt$  by  $d\psi$  in  $dt = dr/(\dots)$  to get an equation of the form  $d\psi = dr/(\dots)$ . The result is given by

$$\psi - \psi_0 = \ell \int_{r_0}^r \frac{dr'}{r'^2 \sqrt{2m (E - V(r') - \frac{\ell^2}{2mr'^2})}} \quad (4.154)$$

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In the particular case of  $V(r) = -\frac{k}{r}$ , the solution of the orbital equation is:

$$\frac{1}{r(\psi)} = \frac{mk}{\ell^2} (1 + \varepsilon \cos(\psi - \psi')) \quad (4.155)$$

where the eccentricity  $\varepsilon$  is given by:

$$\varepsilon \equiv \sqrt{1 + \frac{2E\ell^2}{mk^2}} \quad (4.156)$$

Below are plotted the different qualitative orbits for the gravitic potential, with different  $\varepsilon$  or  $E$  (circular, elliptical, parabolic, and hyperbolic respectively).

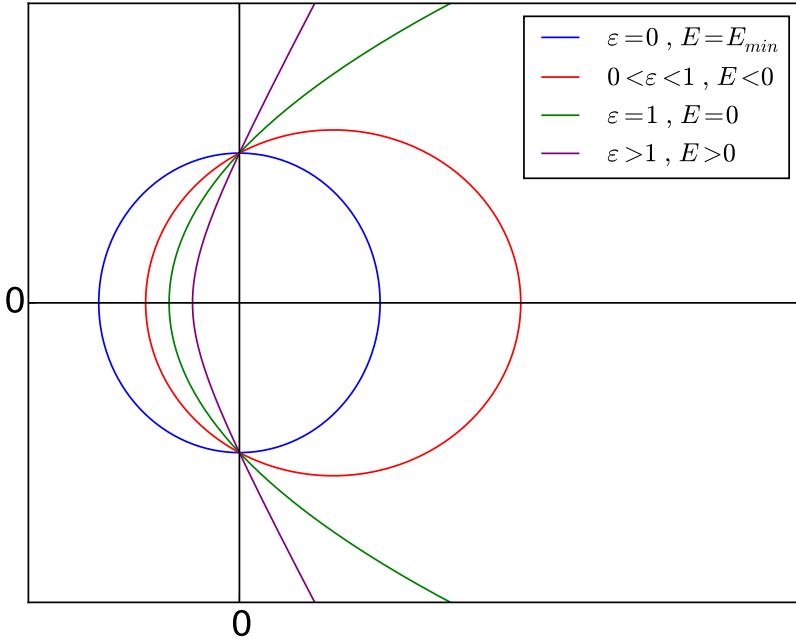


Figure 4.2: Different Orbits for the gravity-like potential. The orbits' colors match those of Fig.(4.1). The unbounded orbits occur for  $E \geq 0$ . The different curves correspond to the different possible conic sections.

Consider solving this problem instead by the H-J method. Lets start by considering as the variables  $(r, \psi)$  so that we assume that the motion of the orbit is in a plane. Here

$$H = \frac{1}{2m} \left( p_r^2 + \frac{p_\psi^2}{r^2} \right) + V(r) = \alpha_1 = E . \quad (4.157)$$

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As  $\psi$  is cyclic, then  $p_\psi \equiv \alpha_\psi$  is constant. Using:

$$W = W_1(r) + \alpha_\psi \psi, \quad (4.158)$$

then the time independent H-J equation is:

$$\frac{1}{2m} \left( \left( \frac{\partial W_1}{\partial r} \right)^2 + \frac{\alpha_\psi^2}{r^2} \right) + V(r) = \alpha_1. \quad (4.159)$$

This is simplified to

$$\frac{\partial W_1}{\partial r} = \sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\psi^2}{r^2}} \quad (4.160)$$

and solved by

$$W = \alpha_\psi \psi + \int \sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\psi^2}{r^2}} dr. \quad (4.161)$$

The transformation equations are:

$$\begin{aligned} t + \beta_1 &= \frac{\partial W}{\partial \alpha_1} = m \int \frac{dr}{\sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\psi^2}{r^2}}}, \\ \beta_2 &= \frac{\partial W}{\partial \alpha_\psi} = \psi - \alpha_\psi \int \frac{dr}{r^2 \sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\psi^2}{r^2}}}. \end{aligned} \quad (4.162)$$

Thus we immediately get the radial equation  $t = t(r)$  and orbital equation  $\psi = \psi(r)$  from this, with  $\alpha_\psi = \ell$  and  $\alpha_1 = E$ , showing that the constants are physically relevant parameters.

Let's solve this problem again, but suppose the motion is in 3 dimensions (as if we did not know the plane of the orbit). Using spherical coordinates  $(r, \theta, \varphi)$  this corresponds to

$$H = \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2 \sin^2 \theta} \right) + V(r) = \alpha_1. \quad (4.163)$$

Lets try a separable solution

$$W = W_r(r) + W_\theta(\theta) + W_\varphi(\varphi). \quad (4.164)$$

Since  $\varphi$  is cyclic we know it is separable and that:

$$W_\varphi(\varphi) = \alpha_\varphi \varphi. \quad (4.165)$$

Together, this leaves;

$$\left( \frac{\partial W_r}{\partial r} \right)^2 + \frac{1}{r^2} \left( \left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{\alpha_\varphi^2}{\sin^2(\theta)} \right) + 2mV(r) = 2m\alpha_1. \quad (4.166)$$

Because the term:

$$\left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{\alpha_\varphi^2}{\sin^2(\theta)} \quad (4.167)$$

only depends on  $\theta$  while the rest of the equation depends on  $r$ , it must be a constant so we can say:

$$\left( \frac{\partial W_\theta}{\partial \theta} \right)^2 + \frac{\alpha_\varphi^2}{\sin^2(\theta)} \equiv \alpha_\theta^2 \quad (4.168)$$

and the separation works. This then gives:

$$\left( \frac{\partial W_r}{\partial r} \right)^2 + \frac{\alpha_\theta^2}{r^2} = 2m(\alpha_1 - V(r)), \quad (4.169)$$

which is the same equation we considered before when assuming the motion was in a plane, with  $\alpha_1 = E$  and  $\alpha_\theta = \ell$ . Eq. (4.168) says that

$$p_\theta^2 + \frac{p_\varphi^2}{\sin^2(\theta)} = \ell^2. \quad (4.170)$$

Here  $p_\varphi$  is the constant angular momentum about the  $\hat{z}$  axis.

## 4.6 Action-Angle Variables

For many problems, we may not be able to solve analytically for the exact motion or for orbital equations, but we can still characterize the motion. For *periodic* systems we can find the frequency by exploiting *action-angle variables*.

The simplest case is for a single dimension of canonical coordinates  $(q, p)$ . If  $H(q, p) = \alpha_1$  then  $p = p(q, \alpha_1)$ . There are two types of periodic motion to consider.

1. Libration (oscillation) is characterized by a closed phase space orbit, so that  $q$  and  $p$  evolve periodically with the same frequency.

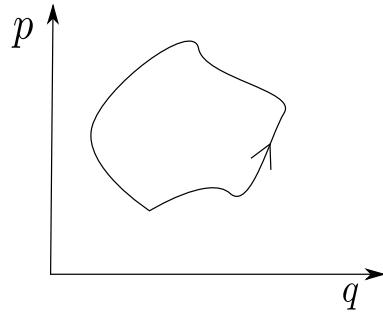


Figure 4.3: Phase space orbit of a libration (oscillation). The trajectory closes on itself, the state returns to the same position after some time  $\tau$ .

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2. Rotation is characterized by an open phase space path, so  $p$  is periodic while  $q$  evolves without bound.

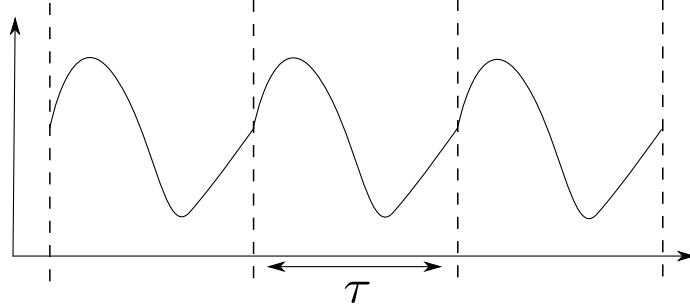


Figure 4.4: Phase space orbit of a rotation. Although the orbit is not closed, each period the evolution of the system is the same, leading to a orbit that repeats itself with a translation.

**Example:** a pendulum of length  $a$  may be characterized by canonical coordinates  $(\theta, p_\theta)$ , where:

$$E = H = \frac{p_\theta^2}{2ma^2} - mga \cos \theta \quad (4.171)$$

This means:

$$p_\theta = \pm \sqrt{2ma^2(E + mga \cos \theta)} \quad (4.172)$$

must be real. A rotation occurs when  $E > mga$ , and oscillations occur when  $E < mga$ . The critical point in between (when the pendulum just makes it to the top) is when  $E = mga$  exactly, and is depicted by a dashed line in the figure below.

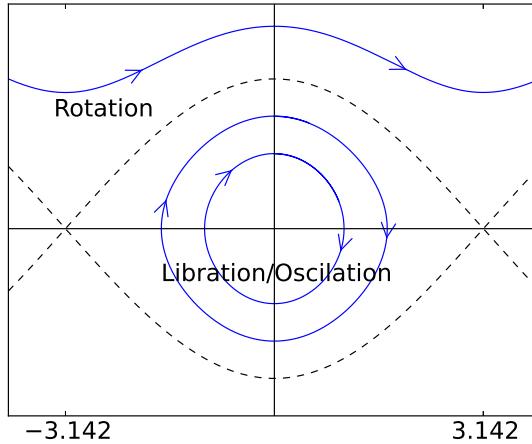


Figure 4.5: The pendulum exhibits both librations and rotations depending on the initial conditions.

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For either type of periodic motion, it is useful to replace  $P = \alpha_1$  by a different (constant) choice called the *action variable*

$$J = \oint p dq, \quad (4.173)$$

where  $\oint$  refers to a definite integral over one period in phase space. To see that  $J$  is constant, recall that  $p = p(q, \alpha_1)$ , so plugging this into the definite integral we are left with  $J = J(\alpha_1)$ . Also, we have the inverse,  $\alpha_1 = H = H(J)$ , and can rewrite Hamilton's characteristic function in terms of  $J$  by  $W = W(q, \alpha_1) = W(q, H(J)) = W(q, J)$  (where again the argument notation is abused slightly).

The coordinate conjugate to  $J$  is the *angle variable*

$$\omega = \frac{\partial W}{\partial J} \quad (4.174)$$

(where  $\omega$  is not meant to imply an angular velocity). This means

$$\dot{\omega} = \frac{\partial H(J)}{\partial J} = \nu(J) \text{ is a constant.} \quad (4.175)$$

As a result the angle variable has linear time dependence,

$$\omega = \nu t + \beta, \quad (4.176)$$

for some initial condition  $\beta$ . Dimensionally,  $J$  has units of angular momentum, while  $\omega$  has no dimensions (like an angle or a phase).

To see why it is useful to use the canonical variables  $(\omega, J)$ , let us consider the change in  $\omega$  when  $q$  goes through a complete cycle.

$$\Delta\omega = \oint \frac{\partial \omega}{\partial q} dq = \oint \frac{\partial^2 W}{\partial q \partial J} dq = \frac{\partial}{\partial J} \oint \frac{\partial W}{\partial q} dq = \frac{\partial}{\partial J} \oint p dq = 1 \quad (4.177)$$

where in the last equality we used the definition of  $J$  in Eq.(4.173). Also, we have  $\Delta\omega = \nu\tau$  where  $\tau$  is the period. Thus

$$\nu = \frac{1}{\tau} \quad (4.178)$$

is the *frequency* of periodic motion. If we find  $H = H(J)$  then

$$\nu = \frac{\partial H(J)}{\partial J} \quad (4.179)$$

immediately gives the frequency  $\nu = \nu(J)$  for the system. Often, we then  $J = J(E)$  to get  $\nu = \nu(E)$  the frequency at a given energy. This is a very efficient way of finding the frequency of the motion without solving for extraneous information.

**Example:** let us consider a pendulum with action-angle variables. We define:

$$\tilde{E} = \frac{E}{mga} \quad (4.180)$$

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so that  $\tilde{E} > 1$  corresponds to rotation and  $\tilde{E} < 1$  corresponds to oscillation. This means

$$p_\theta = \pm \sqrt{2m^2ga^3} \sqrt{\tilde{E} + \cos \theta}. \quad (4.181)$$

For  $\tilde{E} > 1$ :

$$J = \sqrt{2m^2ga^3} \int_{-\pi}^{\pi} d\theta \sqrt{\tilde{E} + \cos \theta}, \quad (4.182)$$

which is an elliptic integral. For  $\tilde{E} < 1$ :

$$\begin{aligned} J &= \sqrt{2m^2ga^3} \int_{-\theta_0}^{+\theta_0} d\theta \sqrt{\tilde{E} + \cos \theta} + \sqrt{2m^2ga^3} \int_{\theta_0}^{-\theta_0} d\theta \left[ -\sqrt{\tilde{E} + \cos \theta} \right] \\ &= 4\sqrt{2m^2ga^3} \int_0^{\theta_0} d\theta \sqrt{\tilde{E} + \cos \theta}, \end{aligned} \quad (4.183)$$

as the contributions from the four intervals that the pendulum swings through in one period are all equivalent. Here  $\theta_0$  is the turning point of the oscillation, and  $\tilde{E} = -\cos(\theta_0)$ .

From this:

$$\nu = \frac{\partial E}{\partial J} = \left( \frac{\partial J}{\partial E} \right)^{-1} \quad (4.184)$$

which we can solve graphically by making a plot of  $J$  vs  $E$ , then  $dJ/dE$  versus  $E$ , and finally the inverse  $\nu = dE/dJ$  versus  $E$ .

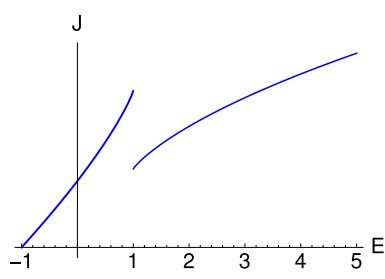


Figure 4.6: Plot of  $J(E)$  versus  $\tilde{E}$ . The discontinuity corresponds to the transition from Oscillation to Rotation.

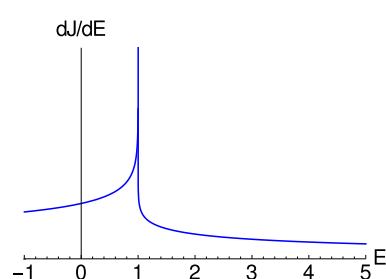


Figure 4.7: Plot of  $\frac{dJ}{dE}$ . The discontinuity is logarithmic divergent so it is integrable.

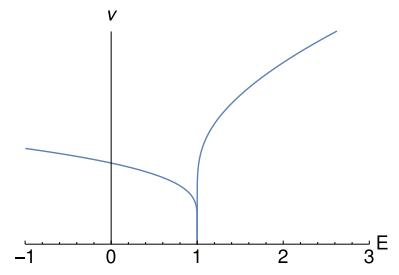


Figure 4.8: Plot of the frequency of oscillation  $\nu(E)$  versus  $\tilde{E}$ . As  $\tilde{E} \rightarrow -1$  we approach the small amplitude limit, where  $\nu = (2\pi)^{-1} \sqrt{g/a}$ .

**Example:** let us consider the limit  $|\theta| \ll 1$  of a pendulum, so:

$$H = \frac{p_\theta^2}{2ma^2} + \frac{mga}{2}\theta^2 - mga \quad (4.185)$$

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We can actually consider this in the context of a general harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{m\Omega^2}{2}x^2, \quad (4.186)$$

where:

$$\Omega = \sqrt{\frac{g}{a}} \quad , \quad x = a\theta \quad \text{and} \quad p = \frac{p_\theta}{a} \quad (4.187)$$

Notationally,  $\Omega$  is used for the harmonic oscillator frequency to distinguish from the transformed angle variable  $\omega$ . We then have:

$$J = \oint p dq = \oint \pm \sqrt{2mE - m^2\Omega^2x^2} dx \quad (4.188)$$

Note that the coordinate does not need to be an angle, as may be the case for general  $x$ . This gives:

$$J = 4\sqrt{2mE} \int_0^{x_0} \sqrt{1 - \frac{x^2}{x_0^2}} dx \quad \text{where} \quad x_0 \equiv \frac{\sqrt{2mE}}{m\Omega}. \quad (4.189)$$

Solving the integral yields

$$J = \pi\sqrt{2mE}x_0 = \frac{2\pi m E}{m\Omega} = \frac{2\pi E}{\Omega}, \quad (4.190)$$

which gives us

$$\frac{\partial E}{\partial J} = \frac{\Omega}{2\pi}, \quad (4.191)$$

the expected cyclic frequency for the harmonic oscillator.

Multiple Variables: We can treat multiple variables  $(q_1, \dots, q_n, p_1, \dots, p_n)$  with the action-angle formalism if *each* pair  $(q_i, p_i)$  has an oscillatory or rotating orbit. Lets also assume that the H-J equation is completely separable into:

$$W = \sum_j W_j(q_j, \alpha). \quad (4.192)$$

Here we have

$$p_i = \frac{\partial W_i}{\partial q_i} = p_i(q_i, \alpha_1, \dots, \alpha_n) \Rightarrow J_i = \oint p_i dq_i = J_i(\alpha_1, \dots, \alpha_n) \quad (4.193)$$

where repeated indices do not correspond to implicit sums here. This implies that the inverse will be  $\alpha_j = \alpha_j(J_1, \dots, J_n)$  and thus  $\alpha_1 = H = H(J_1, \dots, J_n)$ . Likewise:

$$\omega_i = \frac{\partial W}{\partial J_i} = \sum_j \frac{\partial W_j}{\partial J_i} = \omega_i(q_1, \dots, q_n, J_1, \dots, J_n). \quad (4.194)$$

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Just as in the one dimensional case the time derivative of the angle variables is a constant

$$\dot{\omega}_i = \frac{\partial H}{\partial J_i} = \nu_i(J_1, \dots, J_n) \quad (4.195)$$

which are the frequencies describing motion in this “multi-periodic” system. Due to the presence of multiple frequencies, the motion through the whole  $2n$ -dimensional phase space need not be periodic in time.

**Example:** in the 2-dimensional harmonic oscillator:

$$x = A \cos(2\pi\nu_1 t) \text{ and } y = B \cos(2\pi\nu_2 t) \quad (4.196)$$

$$p_x = m\dot{x} \text{ and } p_y = m\dot{y} \quad (4.197)$$

The overall motion is not periodic in time unless  $\frac{\nu_1}{\nu_2}$  is a rational number.

**Kepler Problem Example:**

Let us do a more extended and detailed example. Returning to the Kepler problem:

$$V(r) = -\frac{k}{r} \quad (4.198)$$

with its separable  $W$ :

$$W = W_r(r, \alpha) + W_\theta(\theta, \alpha) + W_\varphi(\varphi, \alpha). \quad (4.199)$$

If we take  $E < 0$ , we have oscillation in  $r$  and  $\theta$ , along with a rotation in  $\varphi$ . In particular from solving our earlier differential equations for  $W_\theta$  and  $W_r$ , we have

$$\begin{aligned} W_\varphi &= \alpha_\varphi \varphi \\ W_\theta &= \pm \int d\theta \sqrt{\alpha_\theta^2 - \frac{\alpha_\varphi^2}{\sin^2(\theta)}} \\ W_r &= \pm \int dr \sqrt{2m(\alpha_1 - V(r)) - \frac{\alpha_\theta^2}{r^2}} \end{aligned}$$

Here we have

$$J_\varphi = \oint p_\varphi d\varphi = \oint \frac{\partial W}{\partial \varphi} d\varphi = \oint \alpha_\varphi d\varphi \quad (4.200)$$

For the cyclic variable  $\varphi$ , we still call the constant  $p_\varphi$  periodic and will take the period to be  $2\pi$  (arbitrarily since any period would work), which corresponds to particle returning to the original point in space. Thus

$$J_\varphi = 2\pi\alpha_\varphi, \quad (4.201)$$

where  $\alpha_\varphi$  is the angular momentum about  $\hat{z}$ .

Continuing, in a similar manner we have

$$J_\theta = \oint p_\theta d\theta = \oint \frac{\partial W}{\partial \theta} d\theta = \oint \pm \sqrt{\alpha_\theta^2 - \frac{\alpha_\varphi^2}{\sin^2(\theta)}} d\theta \quad (4.202)$$

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Let us call:

$$\cos(\gamma) \equiv \frac{\alpha_\varphi}{\alpha_\theta} \quad (4.203)$$

the angular momentum fraction. Then:

$$J_\theta = \alpha_\theta \oint \pm \sqrt{1 - \frac{\cos^2(\gamma)}{\sin^2(\theta)}} d\theta \quad (4.204)$$

If we let  $\sin(\theta_0) = \cos(\gamma)$ , then  $p_\theta = 0$  at the turning points,  $\theta \in \{\theta_0, \pi - \theta_0\}$ , as expected.

Here one oscillator goes from  $\pi - \theta_0 \rightarrow \theta_0$  when  $p_\theta > 0$ , and in reverse for  $p_\theta < 0$ . Moreover,  $\sin(\theta)^{-2}$  is even about  $\theta = \frac{\pi}{2}$ . This gives

$$J_\theta = 4\alpha_\theta \int_{\frac{\pi}{2}}^{\theta_0} \sqrt{1 - \frac{\cos^2(\gamma)}{\sin^2(\theta)}} d\theta. \quad (4.205)$$

Making two more substitutions

$$\cos(\theta) \equiv \sin(\gamma) \sin(\psi), \text{ and then } u \equiv \tan(\psi), \quad (4.206)$$

after some work the expression becomes

$$\begin{aligned} J_\theta &= 4\alpha_\theta \int_0^\infty \left( \frac{1}{1+u^2} - \frac{\cos^2(\gamma)}{1+u^2 \cos^2(\gamma)} \right) du = 2\pi\alpha_\theta(1 - \cos(\gamma)) \\ &= 2\pi(\alpha_\theta - \alpha_\varphi), \end{aligned} \quad (4.207)$$

which gives

$$J_\theta + J_\varphi = 2\pi\alpha_\theta. \quad (4.208)$$

Finally we can consider

$$J_r = \oint \sqrt{2mE - 2mV(r) - \frac{(J_\theta + J_\varphi)^2}{4\pi^2 r^2}} dr \quad (4.209)$$

We can immediately make some observations. We observe that  $J_r = J_r(E, J_\theta + J_\varphi)$  is a function of two variables for any  $V = V(r)$ , and thus if we invert  $E = E(J_r, J_\theta + J_\varphi)$ . This implies:

$$\frac{\partial E}{\partial J_\theta} = \frac{\partial E}{\partial J_\varphi} \Rightarrow \nu_\theta = \nu_\varphi \quad (4.210)$$

The two frequencies are degenerate for any  $V = V(r)$ .

For the  $V(r) = -kr^{-1}$  potential, the integration can be performed (for example, by contour integration) to give (for  $E < 0$ )

$$J_r = -(J_\theta + J_\varphi) + \pi k \sqrt{\frac{2m}{-E}} \Rightarrow J_r + J_\theta + J_\varphi = \pi k \sqrt{\frac{2m}{-E}}. \quad (4.211)$$

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This means:

$$E = -\frac{2\pi^2 k^2 m}{(J_r + J_\theta + J_\varphi)^2} \Rightarrow \nu_\theta = \nu_\varphi = \nu_r \quad (4.212)$$

In particular:

$$\nu_r = \frac{\partial E}{\partial J_r} = 4\pi^2 k^2 (J_r + J_\theta + J_\varphi)^{-3} = \frac{1}{\pi k} \sqrt{\frac{-2E^3}{m}} \quad (4.213)$$

which is the correct orbital frequency in a bound Kepler orbit.

Using the relations between  $\{\alpha_1 = E, \alpha_\theta, \alpha_\varphi\}$  and  $\{J_r, J_\theta, J_\varphi\}$ , we can also get Hamilton's characteristic function for this system as

$$\begin{aligned} W &= W_\varphi + W_\theta + W_r \\ &= \frac{\varphi J_\varphi}{2\pi} \pm \int \sqrt{(J_\theta + J_\varphi)^2 - \frac{J_\varphi^2}{\sin^2(\theta)}} \frac{d\theta}{2\pi} \pm \int \sqrt{\frac{-(2\pi mk)^2}{(J_r + J_\theta + J_\varphi)^2} + \frac{2mk}{r} - \frac{(J_\theta + J_\varphi)^2}{(2\pi r)^2}} dr. \end{aligned}$$

This then gives the angle variables:

$$\begin{aligned} \omega_r &= \frac{\partial W}{\partial J_r} = \omega_r(r, J_r + J_\theta + J_\varphi, J_\theta + J_\varphi) \\ \omega_\theta &= \frac{\partial W}{\partial J_\theta} = \omega_\theta(r, \theta, J_r + J_\theta + J_\varphi, J_\theta + J_\varphi, J_\varphi) \\ \omega_\varphi &= \frac{\partial W}{\partial J_\varphi} = \omega_\varphi(r, \theta, \varphi, J_r + J_\theta + J_\varphi, J_\theta + J_\varphi, J_\varphi) \end{aligned} \quad (4.214)$$

where  $\dot{\omega}_r = \nu_r$ ,  $\dot{\omega}_\theta = \nu_\theta$ , and  $\dot{\omega}_\varphi = \nu_\varphi$ . Of course, in this case,  $\nu_r = \nu_\theta = \nu_\varphi$ .

At this point we can identify five constants for the Kepler problem

$$\begin{aligned} J_1 &= J_\varphi \\ J_2 &= J_\theta + J_\varphi \\ J_3 &= J_r + J_\theta + J_\varphi \\ \omega_1 &= \omega_\varphi - \omega_\theta \\ \omega_2 &= \omega_\theta - \omega_r. \end{aligned} \quad (4.215)$$

(These 5 constants could also be identified from the angular momentum  $\vec{L}$ , energy  $E$ , and Laplace-Runge-Lenz vector  $\vec{A}$ .) What are they? There are two constants specifying the plane of the orbit (the  $x'y'$ -plane), which are the inclination angle  $i$  and the longitude of the ascending node  $\Omega$ . There are three constants specifying the form of the ellipse, which are the semi-major axis  $a$  (giving the size), the eccentricity  $\varepsilon$  (giving the shape), and the angle  $\omega$  (giving the orientation within the plane). These are all shown in Fig. 4.9.

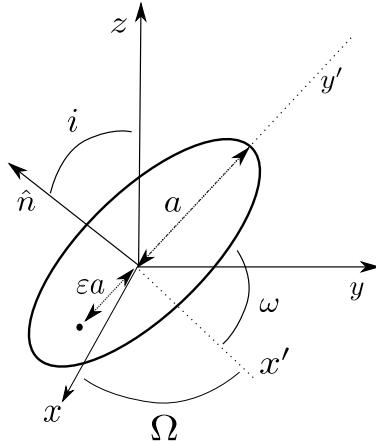


Figure 4.9: Picture of an orbit in 3d and the five parameters necessary to fully specify it. The angles  $i$ ,  $\Omega$  and  $\omega$  provide the orientation in space while  $a$  and  $\varepsilon$  provide the size and shape of the conic section.

It can be shown that the relations between these constants and the ones above are

$$\begin{aligned} \cos(i) &= \frac{J_1}{J_2} & a &= -\frac{k}{2E} = \frac{J_3^2}{4\pi^2 mk} & \epsilon &= \sqrt{1 - \left(\frac{J_2}{J_3}\right)^2} \\ \Omega &= 2\pi\omega_1 & \omega &= 2\pi\omega_2 & & \end{aligned} \quad (4.216)$$

providing a fairly simple physical interpretations to  $(J_1, J_2, J_3, \omega_1, \omega_2)$ . Also recall that  $J_2 = 2\pi\alpha_\theta = 2\pi\ell$ .

When the orbit is perturbed by additional forces (like a moon, another planet, general relativistic corrections, and so on), these action-angle variables provide a natural way to describe the modified orbit. We will soon see that they become functions of time. For example, from the above list of constants, the perturbed  $\omega = \omega(t)$  is the precession of the perihelion of an orbit. We will learn how to derive this time dependence next, in our chapter on Perturbation Theory.

# Chapter 5

## Perturbation Theory

In this chapter we will discuss time dependent perturbation theory in classical mechanics. Many problems we have encountered yield equations of motion that cannot be solved analytically. Here, we will consider cases where the problem we want to solve with Hamiltonian  $H(q, p, t)$  is “close” to a problem with Hamiltonian  $H_0(q, p, t)$  for which we know the exact solution. We say

$$H(q, p, t) = H_0(q, p, t) + \Delta H(q, p, t), \quad (5.1)$$

where  $\Delta H$  is small. The general idea is to expand variables

$$z(t) = z_0(t) + \epsilon z_1(t) + \epsilon^2 z_2(t) + \dots, \quad (5.2)$$

for  $z \in \{q, p\}$  and use the expanded equations of motion to determine the series

$$z_0(t) \rightarrow z^{(1)}(t) \rightarrow z^{(2)}(t) \rightarrow \dots, \quad \text{where} \quad z^{(k)}(t) = \sum_{j=0}^k \epsilon^j z_j(t). \quad (5.3)$$

We can do this with any of our methods for solving problems in classical mechanics, including the Euler-Lagrange equations, Hamilton equations, Poisson bracket equations, or Hamilton-Jacobi equations. Since there are some practical benefits, our focus will be on doing this for the Hamilton-Jacobi equations, but let's first start with an example where we carry out an expansion for the Hamilton equations.

**Example** Consider  $H_0 = \frac{p^2}{2m}$  the free Hamiltonian, and  $\Delta H = \frac{m\omega^2}{2}x^2$ . Here  $\omega$  is an oscillator frequency. The full Hamiltonian  $H = H_0 + \Delta H$  in this case is just a Harmonic oscillator where we already know the solution, so we have the opportunity to see how this solution is built up perturbatively. Without any approximation, the Hamilton equations are

$$\dot{x} = \frac{p}{m}, \quad \dot{p} = -m\omega^2 x. \quad (5.4)$$

To carry out perturbation theory we are going to count  $w^2$  as  $\mathcal{O}(\epsilon)$  and then at each order we balance the number of  $\epsilon$ s on each side of the equations of motion. For  $H_0$ , we have

$\dot{p}_0 = 0$  so the momentum  $p_0$  is a constant which we fix as the initial condition value. We also have  $\dot{x}_0 = \frac{p_0}{m}$  (since  $w^2$  does not appear we have not dropped anything in this equation). Integrating we get  $x_0 = \frac{p_0}{m}t$ , where we have taken the initial condition  $x(t = 0) = 0$  for simplicity.

Having setup the 0'th order solution, lets now consider determining the solution at 1'st order. At first order the RHS of the equations of motion should be  $\mathcal{O}(\epsilon)$ . Therefore

$$\begin{aligned}\dot{p}^{(1)} &= -m\omega^2 x^{(0)} = -m\omega^2 \frac{p_0}{m} t = -\omega^2 t p_0, \\ p^{(1)}(t) &= p_0 - \frac{1}{2} p_0 \omega^2 t^2.\end{aligned}\tag{5.5}$$

For the other equation of motion at this order we then have

$$\begin{aligned}\dot{x}^{(1)} &= \frac{p^{(1)}}{m} = \frac{p_0}{m} - \frac{p_0 \omega^2 t^2}{2m}, \\ x^{(1)}(t) &= \frac{p_0}{m} t - \frac{p_0 \omega^2 t^3}{6m}.\end{aligned}\tag{5.6}$$

These are precisely the 1<sup>st</sup> order terms in the full solution

$$p(t) = p_0 \cos(\omega t), \quad x(t) = \frac{p_0}{m\omega} \sin(\omega t).\tag{5.7}$$

## 5.1 Time Dependent Perturbation Theory for the Hamilton-Jacobi Equations

From here on we will focus on using H-J methods. If  $H = H_0 + \Delta H$ , then the solution for  $H_0$  has a principal function  $S(q, \alpha, t)$  that is the generating function that makes a canonical transformation  $(q, p) \rightarrow (\alpha, \beta)$ , so that

$$H_0\left(q, \frac{\partial S}{\partial q}, t\right) + \frac{\partial S}{\partial t} = 0.\tag{5.8}$$

For the dynamics generated by  $H_0$  the variables  $(\alpha, \beta)$  are constants. However, the resulting canonical transformation provides a new set of variables that is valid for use with any Hamiltonian, they are just particularly simple variables for  $H_0$ . Therefore, for  $H$ , we can still use the canonical transformation generated by  $S$ , but now the new variables

$$P_i = \alpha_i = \alpha_i(p, q), \quad Q_i = \beta_i = \beta_i(p, q),\tag{5.9}$$

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will no longer be constant in time. The new Hamiltonian is

$$K = H_0 + \Delta H + \frac{\partial S}{\partial t} = \Delta H = \Delta H(\alpha, \beta, t) \quad (5.10)$$

The new Hamilton equations  $\dot{Q}_i = \frac{\partial K}{\partial P_i}$  and  $\dot{P}_i = -\frac{\partial K}{\partial Q_i}$  now yield exact equations of motion for these variables

$$\dot{\alpha}_i = -\frac{\partial \Delta H}{\partial \beta_i}, \quad \dot{\beta}_i = \frac{\partial \Delta H}{\partial \alpha_i}. \quad (5.11)$$

The idea of perturbation theory is to solve these equations with an expansion. Since here the small  $\Delta H \sim \epsilon$  appears on the RHS of both equations, we will always use lower order solutions on the RHS to obtain the higher order results on the LHS. Thus we use  $\alpha^{(0)}$  and  $\beta^{(0)}$  to get the first order  $\alpha^{(1)}$  and  $\beta^{(1)}$ :

$$\begin{aligned} \dot{\alpha}_i^{(1)} &= -\left. \frac{\partial \Delta H}{\partial \beta_i} \right|_{\substack{\alpha_i=\alpha_i^{(0)} \\ \beta_i=\beta_i^{(0)}}} = -\left. \frac{\partial \Delta H}{\partial \beta_i} \right|_0, \\ \dot{\beta}_i^{(1)} &= \left. \frac{\partial \Delta H}{\partial \alpha_i} \right|_{\substack{\alpha_i=\alpha_i^{(0)} \\ \beta_i=\beta_i^{(0)}}} = \left. \frac{\partial \Delta H}{\partial \alpha_i} \right|_0, \end{aligned} \quad (5.12)$$

where the  $|_0$  is a shorthand notation. We then use  $\alpha^{(1)}$  and  $\beta^{(1)}$  to get  $\alpha^{(2)}$  and  $\beta^{(2)}$ , and so on. At  $n^{\text{th}}$  order we have

$$\begin{aligned} \dot{\alpha}_i^{(n)} &= -\left. \frac{\partial \Delta H}{\partial \beta_i} \right|_{\substack{\alpha_i=\alpha_i^{(n-1)} \\ \beta_i=\beta_i^{(n-1)}}} = -\left. \frac{\partial \Delta H}{\partial \beta_i} \right|_{n-1}, \\ \dot{\beta}_i^{(n)} &= \left. \frac{\partial \Delta H}{\partial \alpha_i} \right|_{\substack{\alpha_i=\alpha_i^{(n-1)} \\ \beta_i=\beta_i^{(n-1)}}} = \left. \frac{\partial \Delta H}{\partial \alpha_i} \right|_{n-1}. \end{aligned} \quad (5.13)$$

**Example** Lets once again consider  $H_0 = \frac{p^2}{2m}$  and  $\Delta H = \frac{m\omega^2}{2}x^2$ . For  $H_0$ , the H-J equation is  $\frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + \frac{\partial S}{\partial t} = 0$ . As  $x$  is cyclic, the solution is  $S = \alpha x - \frac{\alpha^2}{2m}t$ . Here,

$$P = \alpha, \quad Q = \beta = \frac{\partial S}{\partial \alpha} = x - \frac{\alpha}{m}t, \quad (5.14)$$

giving the exact transformation equations

$$x = \frac{\alpha}{m}t + \beta, \quad p = \frac{\partial S}{\partial x} = \alpha.$$

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For simplicity, we can take the initial constants as  $\alpha^{(0)} = \alpha_0$  and  $\beta^{(0)} = \beta_0 = 0$ . In terms of the new variables our perturbing Hamiltonian is  $\Delta H = \frac{m\omega^2}{2} \left( \frac{\alpha}{m}t + \beta \right)^2$ , so prior to expanding the full equations of motion are

$$\begin{aligned}\dot{\alpha} &= -\frac{\partial \Delta H}{\partial \beta} = -m\omega^2 \left( \frac{\alpha}{m}t + \beta \right), \\ \dot{\beta} &= \frac{\partial \Delta H}{\partial \alpha} = \omega^2 t \left( \frac{\alpha}{m}t + \beta \right).\end{aligned}\tag{5.15}$$

Plugging in 0<sup>th</sup> order solutions on the RHS, to 1<sup>st</sup> order we have

$$\begin{aligned}\dot{\alpha}^{(1)} &= -\omega^2 \alpha_0 t \quad \Rightarrow \quad \alpha^{(1)}(t) = \alpha_0 - \frac{1}{2}\omega^2 \alpha_0 t^2, \\ \dot{\beta}^{(1)} &= \frac{\omega^2}{m} \alpha_0 t^2 \quad \Rightarrow \quad \beta^{(1)}(t) = \frac{\omega^2 \alpha_0 t^3}{3m}.\end{aligned}\tag{5.16}$$

If we change back to our original variables with the inverse transformation (which we may wish to do at any point) this gives

$$p^{(1)} = \alpha^{(1)} = \alpha_0 - \frac{1}{2}\omega^2 \alpha_0 t^2,\tag{5.17}$$

and

$$x^{(1)}(t) = \frac{\alpha^{(1)}(t)}{m}t + \beta^{(1)}(t) = \frac{\alpha_0}{m}t - \frac{\omega^2 \alpha_0}{m} \frac{t^3}{2} + \frac{\omega^2 \alpha_0}{m} \frac{t^3}{3} = \frac{\alpha_0}{m}t - \frac{\omega^2 \alpha_0}{m} \frac{t^3}{3!},\tag{5.18}$$

which are the same results we previously obtained by solving Hamilton's equations perturbatively.

## 5.2 Periodic and Secular Perturbations to Finite Angle Pendulum

**Example** Let us consider a case where we do not have a simple solution. Consider a pendulum, with

$$H = \frac{p^2}{2ma^2} - mga \cos(\theta)\tag{5.19}$$

with  $\theta \ll 1$ . Expanding the cosine term we have

$$H = -mga + \frac{p^2}{2ma^2} + \frac{mga}{2}\theta^2 + \frac{mga}{2}\theta^2 \left( -\frac{\theta^2}{12} + \frac{\theta^4}{360} + \dots \right).\tag{5.20}$$

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In this case, the first term is a constant that will not play a role in our equations of motion, so we can identify  $H_0 = \frac{p^2}{2ma^2} + \frac{mga}{2}\theta^2$ . If we are only interested in applying first order perturbation theory we can simply take  $\Delta H = -\frac{mga}{24}\theta^4$  and drop terms of  $\mathcal{O}(\theta^6)$  and higher. The Hamiltonian  $H_0$  is just a harmonic oscillator with ‘moment of inertia  $I = ma^2$  and frequency  $\Omega^2 = \frac{g}{a}$ . Again we use  $\Omega$  here for angular frequency of the  $H_0$  harmonic oscillator, to avoid confusion with the angle variable  $\omega$ .

The action-angle variables for  $H_0$  are

$$\alpha = H_0 = \frac{\Omega}{2\pi} J, \quad \omega = \nu t + \beta \quad (5.21)$$

where  $\nu = \frac{\Omega}{2\pi}$ ,  $J$  is the action variable, and  $\omega$  is the angle variable. This gives

$$\begin{aligned} \theta &= \sqrt{\frac{2\alpha}{I\Omega^2}} \sin(\Omega t + \delta) = \sqrt{\frac{J}{\pi I\Omega}} \sin [2\pi(\nu t + \beta)], \\ p &= \sqrt{2I\alpha} \cos(\Omega t + \delta) = \sqrt{\frac{IJ\Omega}{\pi}} \cos [2\pi(\nu t + \beta)]. \end{aligned} \quad (5.22)$$

Since  $\omega$  and  $\beta$  are linearly related, the fact that  $(J, \omega)$  are canonical implies that  $(J, \beta)$  are also a canonical pair. So we are free to take  $(J, \beta)$  as our new canonical variables when using the transformation in Eq. (5.22).

If we use  $(J, \beta)$  as the new variables, with  $J^{(0)} = J_0$  and  $\beta^{(0)} = \beta_0$  as given constants fixed by the initial conditions, then in terms of the new variables

$$\Delta H = -\frac{mga}{24}\theta^4 = -\frac{J^2}{24\pi^2 I} \sin^4(2\pi(\nu t + \beta)). \quad (5.23)$$

Expanding by using the 0<sup>th</sup> order solution gives

$$\begin{aligned} \dot{\beta}^{(1)} &= \left. \frac{\partial \Delta H}{\partial J} \right|_0 = -\frac{J_0}{12\pi^2 I} \sin^4(2\pi(\nu t + \beta_0)), \\ j^{(1)} &= -\left. \frac{\partial \Delta H}{\partial \beta} \right|_0 = \frac{J_0^2}{3\pi I} \sin^3(2\pi(\nu t + \beta_0)) \cos(2\pi(\nu t + \beta_0)). \end{aligned} \quad (5.24)$$

These results can be integrated to give  $\beta^{(1)} = \beta^{(1)}(J_0, \beta_0, \nu, t)$  and  $J^{(1)} = J^{(1)}(J_0, \beta_0, \nu, t)$ . Before we consider computing these functions, lets pause to characterize two types of solution that occur in a more general context than simply this example.

Often we can characterize the nature of the perturbative solution without requiring a full study of the analytic form of a solution. A common situation where this is the case is when  $H_0$  exhibits *periodic orbits* (as in the harmonic oscillator) with some frequency  $\nu$ . In this case a relevant question is the following: what cumulative effect does the small perturbation have after going through one or more periods  $T = \frac{1}{\nu}$ ? There are two possibilities:

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- The perturbation itself could be *periodic*, where the parameter returns to its initial value. Here the perturbed trajectory looks much like the unperturbed one.
- Alternatively, we could have a net increment in the parameter after each orbit, called a *secular* change. After many periods, the parameter will be quite different from its value in  $H_0$ .

**Example** Returning to our pendulum from before, the interesting quantity to study is the average over one period of the time rate of change of the variable,

$$\overline{\dot{J}^{(1)}} = \frac{1}{T} \int_0^T \dot{J}^{(1)}(t) dt = \frac{J^{(1)}(T) - J^{(1)}(0)}{T}, \quad (5.25)$$

since this tells us how much the variable changes over one period. For our example  $\overline{\dot{J}^{(1)}} = 0$  because  $\int_0^{2\pi} \sin^3(\theta) \cos(\theta) d\theta = 0$ , and therefore the perturbation to  $J$  is periodic. Actually, from integrating Eq. (5.24) we have

$$J^{(1)}(t) = J_0 + \frac{J_0^2}{24\pi^2 I \nu} \sin^4(2\pi(\nu t + \beta_0)). \quad (5.26)$$

Note from Eq. (5.22) that  $J$  determines the amplitude for  $\theta(t)$  and  $p(t)$ . A comparison between the trajectory with  $J_0$  and with  $J^{(1)}(t)$  is made in Fig. 5.1, where for this figure we set  $\beta_0 = 0$ .

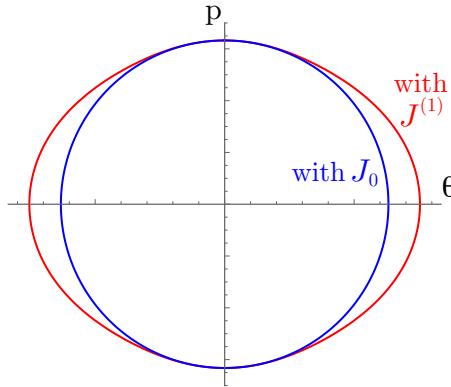


Figure 5.1: Comparison of the pendulum's periodic phase space trajectory using  $J_0$  and  $J^{(1)}(t)$ .

In contrast, using  $\int_0^{2\pi} \sin^4(\theta) \frac{d\theta}{2\pi} = \frac{3}{8}$ , we find

$$\overline{\dot{\beta}^{(1)}} = \frac{1}{T} \int_0^T \dot{\beta}^{(1)}(t) dt = \frac{\beta^{(1)}(T) - \beta^{(1)}(0)}{T} = -\frac{J_0}{32\pi^2 I}, \quad (5.27)$$

which means  $\beta$  experiences a secular change. After many periods ( $t \gg T$ ) the change continues to build up, and we have on average that  $\beta^{(1)}(t) \approx \overline{\dot{\beta}^{(1)}} t + \beta_0$ . (If we look at the exact solution for  $\beta^{(1)}(t)$  then it has precisely this linear term in  $t$ , plus terms that are periodic over the period  $\tau$ , and that is what we mean by the  $\approx$  here.) Looking back at how the  $\beta(t)$  dependence appears in  $\theta = \theta(J, \beta, t)$  and  $p = p(J, \beta, t)$  from Eq. (5.22), we see that on average the 1<sup>st</sup> order perturbation simply shifts the frequency to  $\nu' = \nu + \overline{\dot{\beta}^{(1)}}$ .

Recall that we determined the full frequency  $\nu_{\text{full}}(E)$  numerically as an example in our study of action-angle variables, which is shown below in Figure 5.2. Recalling that  $J_0 =$

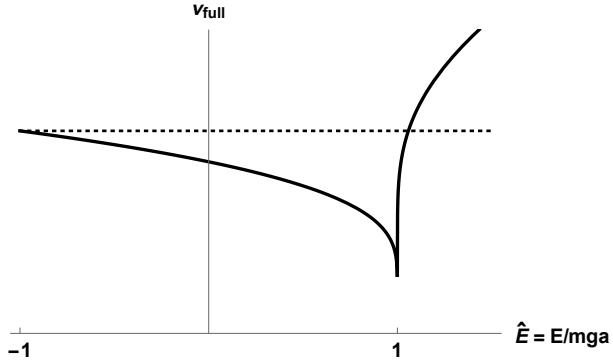


Figure 5.2: The full frequency  $\nu_{\text{full}}$  vs.  $\hat{E}$

$\frac{H_0}{\nu} = \frac{E+mga}{\nu}$ , we can write our perturbative shift to the frequency as a function of energy

$$\nu' - \nu = \overline{\dot{\beta}^{(1)}} = -\frac{(E + mga)}{32\pi^2 ma^2 \nu}. \quad (5.28)$$

This is the first order correction to  $\nu_{\text{full}}(E)$  when it is expanded about the simple harmonic oscillator minimum at  $E = -mga$ , which in Fig. 5.2 gives the negative linear correction to the frequency that occurs just above  $\hat{E} = E/(mga) = -1$ .

### 5.3 Perihelion Precession from Perturbing a Kepler Orbit

**Kepler Example:** Consider a central force perturbation

$$H = \underbrace{\frac{1}{2m} \left( p_r^2 + \frac{p_\psi^2}{r^2} \right) - \frac{k}{r}}_{H_0} - \underbrace{\frac{h}{r^n}}_{\Delta H} \quad (5.29)$$

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where the coupling parameter  $h$  is small and  $n \geq 2$ . The action-angle variables for  $H_0$  are

$$\begin{aligned} J_1 &= J_\phi & \omega_1 &= \omega_\phi - \omega_\theta \\ J_2 &= J_\theta + J_\phi & \omega_2 &= \omega_\theta - \omega_r \\ J_3 &= J_r + J_\theta + J_\phi & \omega_3 &= \omega_r \end{aligned}$$

where only  $\dot{\omega}_3 = \nu_r \neq 0$ , and all the others are constant in time. The pairs  $(\omega_i, J_i)$  for  $i \in \{1, 2, 3\}$  are all canonically conjugate. One way to see this is to note that we can implement a change of variables from the canonical pairs  $\{(\omega_r, J_r), (\omega_\theta, J_\theta), (\omega_\phi, J_\phi)\}$  that we considered earlier, to these variables by using the generating function

$$F_2 = (\omega_\phi - \omega_\theta)J_1 + (\omega_\theta - \omega_r)J_2 + \omega_r J_3. \quad (5.30)$$

Let us study the perihelion precession, with the parameter  $\omega = 2\pi\omega_2$  determining the perihelion angle. Some examples of precession are shown in Fig. 5.3, where in the case of a planet, the precession is like that of Fig. 5.3(b) with the sun at the focus of the ellipse. We

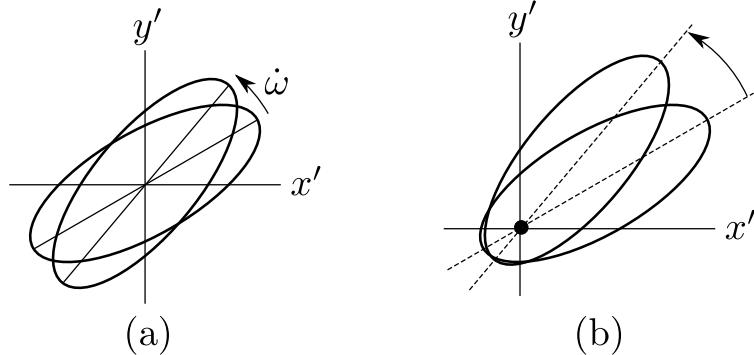


Figure 5.3: Precession of the perihelion from the point of view of coordinates centered on (a) the center of the ellipse, and (b) the focus of the ellipse. The latter is relevant for a planet like mercury orbiting the sun (shown with exaggerated eccentricity here).

know that

$$\dot{\omega}_2 = \frac{\partial \Delta H}{\partial J_2}, \quad J_2 = J_\theta + J_\phi = 2\pi\alpha_\theta = 2\pi\ell \quad (5.31)$$

where  $\ell = |\mathbf{L}|$  is the magnitude of the angular momentum. From the equation of motion  $\dot{\omega}_2 = \partial \Delta H / \partial J_2$  we therefore have

$$\dot{\omega} = 2\pi \frac{\partial \Delta H}{\partial J_2} = \frac{\partial \Delta H}{\partial \ell}, \quad (5.32)$$

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and perturbatively,  $\dot{\omega}^{(1)} = \frac{\partial \Delta H}{\partial \ell}|_0$ . We can average over the orbit to find the secular change:

$$\overline{\dot{\omega}^{(1)}} = \frac{1}{T} \int_0^T \frac{\partial \Delta H}{\partial \ell} \Big|_0 dt. \quad (5.33)$$

Recall for the Kepler problem that  $E^{(0)} = -(2\pi^2 k^2 m)/J_3^2$ , and that at 0<sup>th</sup> order the period is

$$T = \frac{1}{\nu_r} = \pi k \sqrt{\frac{m}{-2E^{(0)3}}} = T(J_3). \quad (5.34)$$

Therefore  $\frac{\partial}{\partial \ell} = 2\pi \frac{\partial}{\partial J_2}$  does not act on  $T = T(J_3)$ , so we can pull the partial derivative outside the integral,

$$\overline{\dot{\omega}^{(1)}} = \frac{\partial}{\partial \ell} \left( \frac{1}{T} \int_0^T \Delta H|_0 dt \right) = \frac{\partial}{\partial \ell} \overline{\Delta H|_0}. \quad (5.35)$$

Thus we must calculate the average of  $\Delta H$  over one period,

$$\overline{\Delta H|_0} = -\frac{h}{T} \int_0^T \frac{dt}{r^n(t)}. \quad (5.36)$$

Using  $\ell = mr^2 \dot{\psi}$  so that  $dt = \frac{mr^2}{\ell} d\psi$  to switch variables from  $t$  to  $\psi$ , and then using the orbital equation for  $r(\psi)$  we have

$$\overline{\Delta H|_0} = -\frac{hm}{\ell T} \int_0^{2\pi} \frac{d\psi}{r^{n-2}(\psi)} = -\frac{hm}{\ell T} \left( \frac{mk}{\ell^2} \right)^{n-2} \int_0^{2\pi} [1 + \epsilon \cos(\psi - \psi')]^{n-2} d\psi, \quad (5.37)$$

where the eccentricity  $\epsilon = \sqrt{1 + \frac{2E\ell^2}{mk^2}}$  also depends on  $\ell$ . There are two simple cases where we can perform this integral:

- If  $n = 2$ , then  $\overline{\Delta H} = -\frac{2\pi hm}{\ell T}$ , so  $\overline{\dot{\omega}^{(1)}} = \frac{2\pi hm}{\ell^2 T}$ .
- If  $n = 3$ , then  $\overline{\Delta H} = -\frac{hkm^2}{\ell^3 T} \int_0^{2\pi} d\psi (1 + \epsilon \cos(\psi - \psi'))$ , where the cos term vanishes upon integration, so  $\overline{\dot{\omega}^{(1)}} = \frac{\partial \overline{\Delta H|_0}}{\partial \ell} = \frac{6\pi m^2 hk}{\ell^4 T}$ .

The latter type of potential ( $n = 3$ ) is induced by corrections from general relativity to the Newtonian potential. The Schwarzschild metric is

$$ds^2 = -c^2(d\tau)^2 = \left(1 - \frac{r_s}{r}\right) c^2(dt)^2 - \frac{(dr)^2}{1 - \frac{r_s}{r}} - r^2(d\theta)^2 - r^2 \sin^2(\theta)(d\phi)^2 \quad (5.38)$$

for  $r_s = \frac{2GM}{c^2}$  where  $M$  is the central mass (say of the sun),  $G$  is Newton's gravitational constant, and  $c$  is the speed of light. The geodesic equation for radial motion is given by

$$E = \frac{m}{2} \left( \frac{dr}{d\tau} \right)^2 + V_{\text{eff}}(r) \quad (5.39)$$

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where  $V_{\text{eff}}(r) = -\frac{mc^2 r_s}{2r} + \frac{\ell^2}{2mr^2} - \frac{r_s \ell^2}{2mr^3}$ . (More background details on the Schwarzschild metric and the derivation of this geodesic equation are discussed below in the Side Note on page 103.) Defining  $k = GMm$ , then the effective potential can be rewritten as

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2mr^2} - \frac{k\ell^2}{c^2 m^2 r^3}, \quad (5.40)$$

from which we can identify  $h = \frac{k\ell^2}{c^2 m^2}$ . Note that  $h$  must be treated as a constant independent of the canonical variable  $\ell$  for the purpose of the above perturbative analysis (we simply substitute this value for  $h$  at the end).

Plugging in the  $h$  for the general relativistic correction we have

$$\overline{\dot{\omega}^{(1)}} = \frac{6\pi m^2 h k}{\ell^4 T} = \frac{6\pi}{T} \frac{k^2}{\ell^2 c^2} = \frac{6\pi}{T} \frac{k}{mc^2} \frac{1}{a(1-\varepsilon^2)}, \quad (5.41)$$

where in the last equality we used  $\ell^2 = mka(1-\varepsilon^2)$ , which follows from Eq. (4.216). For Mercury, the period  $T = 0.2409T_{\text{Earth}}$ , eccentricity  $\varepsilon = 0.2056$ , and semi-major axis  $a = 5.79 \times 10^7$  km, while  $\frac{k}{mc^2} = \frac{GM_{\text{sun}}}{c^2} = 1.4766$  km, so we get a precession rate of

$$\overline{\dot{\omega}^{(1)}} = 2.0836 \times 10^{-6} \text{ radians/year} = 42.98 \text{ arcseconds/century} \quad (5.42)$$

from general relativity. (An arcsecond is  $1/3600$ 'th of a degree.) After removing other contributions, such as a shift of 531.54 arcseconds/century from perturbations by other planets, the data on mercury's orbit shows a shift of 43.1 arcseconds/century (excellent agreement!). This was historically one of the first tests of general relativity, and still remains an important one.

We could also consider perturbations involving momentum variables rather than coordinates, and things work out in a very similar manner. To see this lets do one example.

**Example** Consider the relativistic correction to harmonic oscillator where the relativistic energy

$$E = \sqrt{c^4 m^2 + c^2 p^2} = mc^2 + \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + \dots \quad (5.43)$$

In this case to analyze the first order perturbative correction we take

$$H = \underbrace{\frac{1}{2m} (p^2 + m^2 \Omega^2 q^2)}_{H_0} - \underbrace{\frac{p^4}{8c^2 m^3}}_{\Delta H}. \quad (5.44)$$

From  $H_0$ , the variables have a canonical transformation from the H-J analysis that gives

$$q = \sqrt{\frac{J}{\pi m \Omega}} \sin(2\pi(\nu t + \beta)), \quad p = \sqrt{\frac{J m \Omega}{\pi}} \cos(2\pi(\nu t + \beta)). \quad (5.45)$$

This gives

$$\Delta H = -\frac{J^2 \Omega^2}{8\pi^2 c^2 m} \cos^4 [2\pi(\nu t + \beta)]. \quad (5.46)$$

Since  $\dot{J}^{(1)}$  is odd over one period, it turns out that  $J$  is periodic once again

$$\dot{J}^{(1)} = -\frac{\partial \Delta H}{\partial \beta} \Big|_0 \Rightarrow \overline{\dot{J}^{(1)}} = 0. \quad (5.47)$$

Meanwhile, the change for  $\dot{\beta}^{(1)}$  is secular,

$$\dot{\beta}^{(1)} = \frac{\partial \Delta H}{\partial J} \Big|_0 = -\frac{J_0 \Omega^2}{4\pi^2 m c^2} \cos^4 [2\pi(\nu t + \beta)] \Rightarrow \overline{\dot{\beta}^{(1)}} = -\frac{3J_0 \Omega^2}{32\pi^2 m c^2}. \quad (5.48)$$

Thus,  $\overline{\dot{\beta}^{(1)}}$  from the relativistic correction  $\Delta H$  is again a negative shift to the frequency of the oscillator.

This ends our discussion of perturbation theory in classical mechanics.

### Side Note: The Schwarzschild Geodesic from Action Minimization

The Schwarzschild metric is given by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \left(1 - \frac{r_s}{r}\right) c^2 t^2 - \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2, \quad (5.49)$$

where  $r_s = \frac{2GM}{c^2}$  is the Schwarzschild radius. The geodesic orbit for a test particle is a curve which minimizes proper distance with this metric. In this case, we have

$$0 = \delta s = \delta \int ds = \delta \int \left( g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right)^{\frac{1}{2}} d\tau \quad (5.50)$$

where  $\tau$  is the proper time and  $ds^2 = c^2 d\tau^2$ . (One method of determining the geodesic equations is to use  $0 = \frac{d^2 x^\lambda}{d\tau^2} + \Gamma_{\mu\nu}^\lambda \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$  with the Christoffel symbols  $\Gamma_{\mu\nu}^\lambda$  determined from the metric, but we will follow a different approach.)

The minimization in Eq. (5.50) is equivalent to applying the minimal action principal for the Lagrangian

$$\begin{aligned} L &= \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \\ &= \frac{m}{2} \left[ \left(1 - \frac{r_s}{r}\right) c^2 \left(\frac{dt}{d\tau}\right)^2 - \frac{1}{\left(1 - \frac{r_s}{r}\right)} \left(\frac{dr}{d\tau}\right)^2 - r^2 \left(\frac{d\theta}{d\tau}\right)^2 - r^2 \sin^2 \theta \left(\frac{d\phi}{d\tau}\right)^2 \right] \end{aligned} \quad (5.51)$$

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since the presence of the extra square root  $(\dots)^{\frac{1}{2}}$  does not matter for this minimization. Here we have the generalized coordinates  $x^\mu = (t, r, \theta, \phi)$  which are to be considered as functions of the proper time variable  $\tau$ . Also, the mass  $m$  is a test mass (which also gives us the proper units).

Because  $t$  and  $\phi$  are cyclic variables in  $L$ , we have

$$\begin{aligned} p_t &= \frac{\partial L}{\partial \dot{t}} = m \left(1 - \frac{r_s}{r}\right) c^2 \frac{dt}{d\tau} = E^{\text{tot}} && \text{energy} \\ p_\phi &= -\frac{\partial L}{\partial \dot{\phi}} = mr^2 \sin^2 \theta \frac{d\phi}{d\tau} = \ell && \text{angular momentum} \end{aligned} \quad (5.52)$$

Using the E-L equation on  $\theta$  we obtain

$$\begin{aligned} 0 &= \frac{d}{d\tau} \frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} \\ &= \frac{d}{d\tau} \left( mr^2 \frac{d\theta}{d\tau} \right) - mr^2 \sin \theta \cos \theta \dot{\phi}^2 \end{aligned} \quad (5.53)$$

where planar motion with  $\theta = \frac{\pi}{2}$  is a solution that suffices for our purposes. Now

$$\frac{ds^2}{d\tau^2} = c^2 = \left(1 - \frac{r_s}{r}\right) c^2 \frac{E_{\text{tot}}^2}{m^2 \left(1 - \frac{r_s}{r}\right)^2 c^4} - \frac{1}{\left(1 - \frac{r_s}{r}\right)} \left(\frac{dr}{d\tau}\right)^2 - r^2 \frac{\ell^2}{m^2 r^4} \quad (5.54)$$

gives a radial equation. Separating out the rest mass and expanding  $E = E^{\text{tot}} - mc^2 \ll mc^2$  we have:

$$E^{\text{tot}} = mc^2 + E \Rightarrow (E^{\text{tot}})^2 \approx m^2 c^4 + 2mc^2 E \quad (5.55)$$

Putting this together we have

$$\underbrace{\frac{m}{2} \left(c^2 + \frac{l^2}{m^2 r^2}\right) \left(1 - \frac{r_s}{r}\right) - \frac{m}{2} c^2}_{V_{\text{eff}}(r)} + \frac{m}{2} \left(\frac{dr}{d\tau}\right)^2 = E \quad (5.56)$$

Note that the rest mass  $mc^2/2$  terms cancel. Therefore the effective potential is

$$\begin{aligned} V_{\text{eff}}(r) &= -\frac{r_s mc^2}{2r} + \frac{l^2}{2mr^2} - \frac{r_s l^2}{2mr^3} \\ &= -\frac{k}{r} + \frac{l^2}{2mr^2} - \frac{kl^2}{m^2 c^2 r^3} \end{aligned}$$

where  $r_s = \frac{2GM}{c^2}$  and  $k = GMm$ , so  $mr_s c^2 = 2k$ . This is the result that was quoted above in Eq. (5.40).

# Chapter 6

## Fluid Mechanics

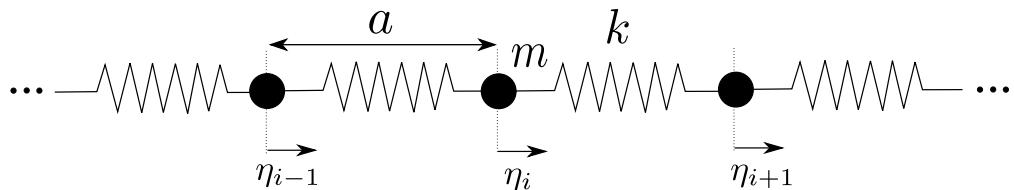
So far, our examples of mechanical systems have all been discrete, with some number of masses acted upon by forces. Even our study of rigid bodies (with a continuous mass distribution) treated those bodies as single objects. In this chapter, we will treat an important continuous system, which is that of fluids.

*Fluids* include both liquids and gases. (It also includes plasmas, but for them a proper treatment requires the inclusion of electromagnetic effects, which will not be discussed here.) For our purposes, a fluid is a material which can be treated as *continuous*, which has the ability to *flow*, and which has very little resistance to deformation (that is, it has only a small support for shear stress, which refers to forces parallel to an applied area). Applications include meteorology, oceanography, astrophysics, biophysics, condensed matter physics, geophysics, medicine, aerodynamics, plumbing, cosmology, heavy-ion collisions, and so on.

The treatment of fluids is an example of classical field theory, with continuous field variables as the generalized coordinates, as opposed to the discrete set of variables  $q_i$  that we have treated so far. Therefore the first step we have to take is understanding how to make the transition from discrete to continuum.

### 6.1 Transitioning from Discrete Particles to the Continuum

Rather than starting with fluids, let's instead consider a somewhat simpler system, that of an infinite one dimensional chain of masses  $m$  connected by springs with spring constant  $k$ , which we take as an approximation for an infinite continuous elastic one-dimensional rod.



If the equilibrium separation of masses is  $a$  and the distance the  $i$ 'th mass is translated from equilibrium is  $\eta_i$ , then

$$V = \frac{k}{2} \sum_i (\eta_{i+1} - \eta_i)^2 \quad T = \frac{m}{2} \sum_i \dot{\eta}_i^2, \quad (6.1)$$

where  $V$  is the potential energy from the springs, and  $T$  is the kinetic energy. It is convenient to write the Lagrangian as

$$L = T - V = \frac{1}{2} \sum_i a \left( \frac{m}{a} \dot{\eta}_i^2 - ka \left( \frac{\eta_{i+1} - \eta_i}{a} \right)^2 \right), \quad (6.2)$$

and the corresponding equations of motion obtained from the Euler-Lagrange equations as

$$\frac{m}{a} \ddot{\eta}_i - ka \left( \frac{\eta_{i+1} - \eta_i}{a^2} \right) + ka \left( \frac{\eta_i - \eta_{i-1}}{a^2} \right) = 0. \quad (6.3)$$

Technically both the Lagrangian and the equations of motion are independent of  $a$ , but we have introduced factors of  $a$  to facilitate taking the continuous limit  $a \rightarrow 0$ . In this limit the masses become progressively more densely distributed along the line. The important question when taking this limit is which quantities do we hold fixed.

Lets define  $\mu \equiv \frac{m}{a}$  as the mass density and  $Y = ka$  as the Young's modulus. Here  $Y$  is equivalent to the spring constant for a continuous rod. (For a rod, the force  $F = Y\xi$  where  $\xi$  is the longitudinal extension per unit length, or in other words, the strain.) We intend to hold  $\mu$  and  $Y$  fixed when taking the continuous limit.

The key change in the continuous limit is that the discrete position index  $i$  becomes a continuous position label  $x$ , so instead of  $\eta_i = \eta_i(t)$ , now  $\eta_x = \eta_x(t)$ , or with more conventional notation,  $\eta = \eta(x, t)$ . This also means that

$$\frac{\eta(x+a, t) - \eta(x, t)}{a} \rightarrow \frac{\partial \eta}{\partial x}, \quad (6.4)$$

$$\frac{1}{a} \left( \frac{\eta(x+a, t) - \eta(x, t)}{a} - \frac{\eta(x, t) - \eta(x-a, t)}{a} \right) \rightarrow \frac{\partial^2 \eta}{\partial x^2}, \quad (6.5)$$

$$\sum_i a \rightarrow \int dx. \quad (6.6)$$

Using these results in Eq. (6.2) gives  $L = \int \mathcal{L} dx$  where

$$\mathcal{L} = \frac{1}{2} \left( \mu \left( \frac{\partial \eta(x, t)}{\partial t} \right)^2 - Y \left( \frac{\partial^2 \eta(x, t)}{\partial x^2} \right)^2 \right) \quad (6.7)$$

is the *Lagrangian density*. Likewise, using them in Eq. (6.3) gives the equations of motion

$$\mu \frac{\partial^2 \eta}{\partial t^2} - Y \frac{\partial^2 \eta}{\partial x^2} = 0, \quad (6.8)$$

which we recognize as the wave equation.

The results for the Lagrange density and Euler Lagrange equations are of course not independent. We can also use

$$\mathcal{L} = \mathcal{L}\left(\eta, \frac{\partial\eta}{\partial x}, \frac{\partial\eta}{\partial t}, x, t\right) \quad (6.9)$$

with Hamilton's principle,

$$\delta S = \delta \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \mathcal{L} dx dt = 0. \quad (6.10)$$

to formulate the dynamics, and thus derive the Euler-Lagrange equations. Because  $\eta = \eta(x, t)$  has two parameters, if we follow the standard procedure of varying the path  $\eta$  takes between the two endpoints, we get variations from the dependence of the Lagrange density on its first three arguments. Integrating by parts in each of  $t$  and  $x$ , and setting the surface terms to zero, then yields

$$\frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial (\frac{\partial \eta}{\partial t})} \right) + \frac{\partial}{\partial x} \left( \frac{\partial \mathcal{L}}{\partial (\frac{\partial \eta}{\partial x})} \right) - \frac{\partial \mathcal{L}}{\partial \eta} = 0 \quad (6.11)$$

as the continuum Euler-Lagrange equation. Recall that for  $N$  particles we expect  $N$  E-L equations for the time dependence, but here we have just one equation. However actually by that counting, this result corresponds to an infinite number of equations, one for each value of  $x$ . From this point of view, the derivatives with respect to  $x$  are what couples these equations together.

**Example** For the Lagrangian density of the elastic rod,

$$\frac{\partial \mathcal{L}}{\partial (\frac{\partial \eta}{\partial t})} = \mu \frac{\partial \eta}{\partial t}, \quad \frac{\partial \mathcal{L}}{\partial (\frac{\partial \eta}{\partial x})} = -Y \frac{\partial \eta}{\partial x}, \quad \frac{\partial \mathcal{L}}{\partial \eta} = 0. \quad (6.12)$$

Putting these results together in Eq. (6.11) gives the wave equation  $\mu \frac{\partial^2 \eta}{\partial t^2} - Y \frac{\partial^2 \eta}{\partial x^2} = 0$  as anticipated.

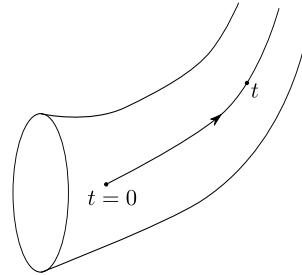
In our above analysis,  $\eta = \eta(x, t)$  is a continuum generalized coordinate called a *classical field*. Here  $t$  is a parameter and  $x$  is a continuous label as well.

Although we have been talking about one dimension so far, it is easy to generalize the above discussion to a higher number of dimensions. For example, in three dimensions we simply have dependence on three continuous label parameters,  $\eta = \eta(x, y, z, t)$  or  $\eta = \eta(\mathbf{r}, t)$ . The field  $\eta(\mathbf{r}, t)$  is called a scalar field because the output is a single number. With multiple dimensions we also have vector fields  $\eta(\mathbf{r}, t)$ , where the output is a vector. An example of vector fields that you are familiar with are the electromagnetic fields  $\mathbf{E}(\mathbf{r}, t)$  and  $\mathbf{B}(\mathbf{r}, t)$ . In fact, classical fields of exactly this sort are also the starting point for formulating quantum field theory. One formulates a classical Lagrangian density  $\mathcal{L}$  (that is most often Lorentz invariant) which depends on fields like the electromagnetic scalar and vector potentials  $\phi(\mathbf{r}, t)$  and  $\mathbf{A}(\mathbf{r}, t)$ . Then one quantizes these fields.

Our description of fluids will make use of classical field variables in 3-dimensions without considering quantization.

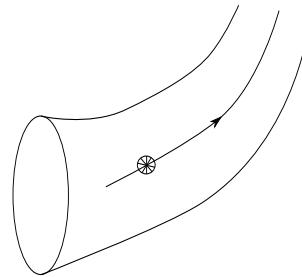
## Eulerian & Lagrangian Variables

Let us consider fluid flowing in a tube. A natural set of variables would be to take some initial time  $t = 0$  and label all fluid elements by their coordinates at this time:  $\mathbf{r}_0$ . We will often refer to fluid element, which is a point in the fluid, or sometimes an infinitesimally small region of the fluid. The motion of the fluid could then be described by  $\mathbf{r} = \mathbf{r}(\mathbf{r}_0, t)$ , which determines the subsequent coordinates of the fluid element labeled by  $\mathbf{r}_0$  at the later time  $t$ .



If we continue in this direction it leads to the Lagrangian formulation of fluid dynamics. The advantage is that the usual laws of classical mechanics apply to fluid particles. The disadvantage is that it is more work to make the connection to measurable quantities that characterize the fluid.

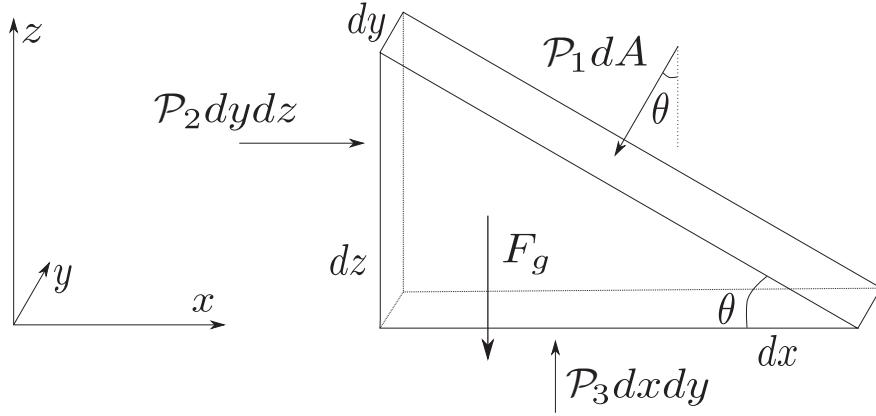
Instead, we will explore an alternate formulation. Pretend we sit at a fixed point  $\mathbf{r}$  in the fluid and ask what happens there as a function of the time  $t$ . We can think of it like placing a measuring device at  $\mathbf{r}$  and measuring, at any time  $t$ , quantities like the fluid velocity  $\mathbf{v}$ , density  $\rho$ , or pressure  $\mathcal{P}$ . In fact, these 5 variables (as density and pressure are scalars, while velocity is a vector) are enough to describe the state of the moving fluid. This is the Eulerian formulation of fluid dynamics. Additionally,  $\rho$  and  $\mathcal{P}$  are enough to determine all thermodynamic quantities (assuming we know the equation of state).



**Example** for an ideal gas at temperature  $T$ , we have  $\mathcal{P}V = nRT$ , where  $V$  is volume,  $n$  is the number of moles,  $R$  is constant, and  $T$  is temperature. Dividing by  $V$  we have that pressure is simply proportional to density,  $\mathcal{P} = \rho R'T$  for a rescaled gas constant  $R'$ .

Pressure is *isotropic* in a fluid. This means it is the same from all directions, so there is only 1 number for each  $(\mathbf{r}, t)$ ; thus, pressure is a scalar field,  $\mathcal{P}(\mathbf{r}, t)$ .

To prove this consider the infinitesimal wedge-shaped fluid element below, which we take to be at rest in a gravitational field. Recall that pressure is force per unit area,  $\mathcal{P} = \hat{\mathbf{n}} \cdot \mathbf{F}/A$ , where  $\hat{\mathbf{n}}$  is a unit vector perpendicular to the area  $A$  that the force  $\mathbf{F}$  is acting on. In the figure we label three pressures  $\mathcal{P}_1$ ,  $\mathcal{P}_2$ ,  $\mathcal{P}_3$  acting on three of the sides, that have corresponding force components in the  $x$  and  $z$  directions. (There is also pressure on the other two faces with forces in the  $y$  direction, but the above will suffice to argue why the pressure is isotropic.)



Next we balance forces for the wedge at rest. Gravity pulls down, and the volume of the wedge is  $dxdydz/2$  so the force of gravity is

$$F_g = \rho g \frac{dxdydz}{2}. \quad (6.13)$$

Also by simple trigonometry the area  $dA$  of the slanted face can be written in two different ways

$$dA = dy \frac{dz}{\sin(\theta)} = dy \frac{dx}{\cos(\theta)}.$$

Balancing forces in the  $x$  and  $z$  directions then means

$$\begin{aligned} 0 &= dF_x = P_2 dy dz - P_1 \sin(\theta) dA = (P_2 - P_1) dy dz, \\ 0 &= dF_z = P_3 dx dy - P_1 \cos(\theta) dA - \frac{1}{2} \rho g dxdydz. \end{aligned} \quad (6.14)$$

The first equation implies  $P_1 = P_2$ . In the second equation we can pull out a common  $dxdy$  to give

$$P_3 = P_1 + \frac{1}{2} \rho g dz, \quad (6.15)$$

then as the infinitesimal distance  $dz \rightarrow 0$  we have

$$P_1 = P_3. \quad (6.16)$$

Thus, pressure is the same in all directions. Even if the fluid is moving or even accelerating we would come to the same conclusion. For example, if we had to balance the force against acceleration this would lead to adding term

$$\rho \mathbf{a} dxdydz \quad (6.17)$$

for acceleration  $\mathbf{a}$ , which again drops out for an infinitesimal fluid element just like the gravitational force did.

#### Time Derivatives:

The total time derivative  $\frac{d}{dt}$  tells us the rate at which a quantity changes as we move with a fluid element. The partial time derivative  $\frac{\partial}{\partial t}$  tells us the rate of change of a quantity at a fixed position  $\mathbf{r}$ . We can work out a relation between them.

#### Example For $\mathcal{P}$

$$\begin{aligned}\frac{d\mathcal{P}}{dt} &= \frac{\partial\mathcal{P}}{\partial t} + \frac{\partial\mathcal{P}}{\partial x}\dot{x} + \frac{\partial\mathcal{P}}{\partial y}\dot{y} + \frac{\partial\mathcal{P}}{\partial z}\dot{z} \\ &= \frac{\partial\mathcal{P}}{\partial t} + \mathbf{v} \cdot \nabla \mathcal{P}.\end{aligned}\quad (6.18)$$

In general, the time derivative acts as

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \quad (6.19)$$

on any fluid field (scalar, vector, or tensor) that is a function of  $(x, y, z, t)$ .

## 6.2 Fluid Equations of Motion

### 6.2.1 Continuity Equations

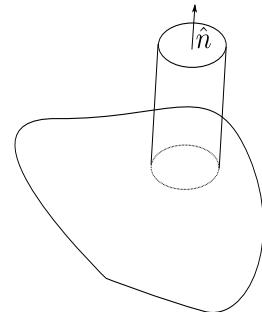
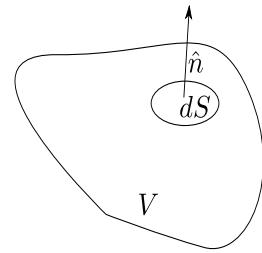
Let us consider a volume  $V$  of fluid. Then from Gauss' divergence theorem (which you may recall from electromagnetism, but which is a general result of vector calculus for any vector field):

$$\int_V dV \nabla \cdot \mathbf{v} = \int_{\partial V} d\mathcal{S} \hat{n} \cdot \mathbf{v} = \int_{\partial V} d\mathcal{S} \cdot \mathbf{v}, \quad (6.20)$$

where  $\partial V$  is the closed area that bounds the volume  $V$ ,  $\hat{n}$  is unit vector orthogonal to the surface element  $d\mathcal{S}$ , and  $d\mathcal{S} \equiv dS\hat{n}$ .

Lets ask: As the fluid moves, how does  $V$  change?

The quantity  $\hat{n} \cdot \mathbf{v}$  is the outward velocity of the surface  $d\mathcal{S}$ , so  $\hat{n} \cdot \mathbf{v} d\mathcal{S} dt$  is the volume added in a time interval  $dt$ , as illustrated on the right.



This means that the change in volume can be determined by adding up all the changes from integrating over the entire surface

$$\frac{dV}{dt} = \int_{\partial V} \mathbf{v} \cdot d\mathcal{S} = \int_V \nabla \cdot \mathbf{v} dV. \quad (6.21)$$

This result becomes even simpler if we consider an infinitesimal volume  $\delta V$  over which  $\nabla \cdot \mathbf{v}$  does not vary, then we can pull  $\nabla \cdot \mathbf{v}$  outside the integral to obtain simply

$$\frac{d\delta V}{dt} = \delta V \nabla \cdot \mathbf{v}. \quad (6.22)$$

Thus the divergence of the velocity,  $\nabla \cdot \mathbf{v}$ , controls how the fluid volume expands with time. If  $\nabla \cdot \mathbf{v} = 0$  everywhere then we say the fluid is *incompressible* because for every volume element  $\frac{dV}{dt} = 0$ .

Even if the volume changes, the mass of the fluid element will not,

$$\frac{d\delta m}{dt} = \frac{d}{dt} (\rho \delta V) = 0. \quad (6.23)$$

This implies that

$$0 = \delta V \frac{d\rho}{dt} + \rho \frac{d\delta V}{dt} = \delta V \left( \frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} \right), \quad (6.24)$$

so

$$\frac{d\rho}{dt} + \rho \nabla \cdot \mathbf{v} = 0. \quad (6.25)$$

Expanding out the time derivative into partial derivatives this yields

$$\frac{\partial \rho}{\partial t} + \mathbf{v} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{v} = 0, \quad (6.26)$$

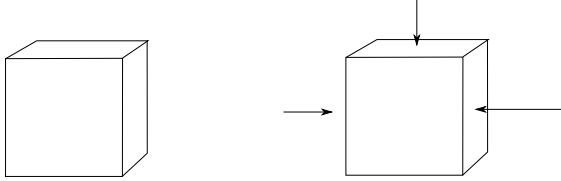
and simplifying the result gives

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \quad (6.27)$$

This is an important result the *continuity equation* for mass of the fluid, which is a partial differential equation in the fluid variables.

Here  $\rho \mathbf{v}$  is the mass density flux (the flow of mass density), and this equation says that nowhere is the matter making up the fluid created or destroyed. If the density changes, then there must be an incoming or outgoing flux. This is easier to see if we use the divergence theorem for the vector  $(\rho \mathbf{v})$  to write this result in integral form,

$$\frac{\partial}{\partial t} \int_V dV \rho = - \int_{\partial V} dS \hat{n} \cdot (\rho \mathbf{v}). \quad (6.28)$$



Mass increase equals the inflowing mass

Here  $\frac{\partial}{\partial t} \int_V dV \rho$  is the increase of mass in the volume  $V$ , while  $\int_{\partial V} dS \hat{n} \cdot (\rho \mathbf{v})$  is the outflow of mass through the surface surrounding this volume (which becomes an inflow with the minus sign).

We've talked about  $\nabla \cdot \mathbf{v}$  thus far, so it's natural to ask: is there a physical interpretation to  $\nabla \times \mathbf{v}$ ? There is indeed.

The quantity  $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$  is the *vorticity*. If the velocity is constant or uniform in a single direction  $\vec{v} = v_x(x)\hat{x}$ , then it's obvious that  $\boldsymbol{\Omega} = 0$ . To consider what it measures we can use Stoke's theorem for the velocity vector field,

$$\int_S (\nabla \times \mathbf{v}) \cdot \hat{n} dS = \oint_{\partial S} \mathbf{v} \cdot d\mathbf{l}, \quad (6.29)$$

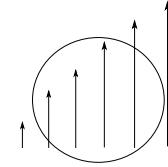
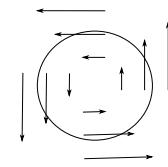
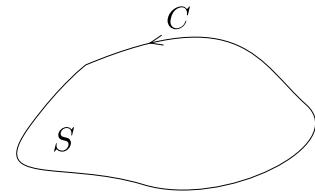
where  $S$  is now an open surface and  $\partial S$  is its closed boundary curve. We can use this result to determine when  $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$  is non-zero.

Consider a rotating fluid and the circular curve shown. When we integrate the velocity along this circular curve the terms always add up, so  $\boldsymbol{\Omega} \neq 0$ . It thus provides a measure of the rate of rotation of the fluid.

If we consider flow in one direction with a velocity gradient, again  $\boldsymbol{\Omega} \neq 0$ . Here a larger positive contribution to the integral is obtained from the right side of the circle, relative to the negative contribution from the left side.

If  $\nabla \times \mathbf{v} = 0$  everywhere in a moving fluid, we say the flow is *irrotational*.

We can determine more precisely what  $\Omega$  is as follows. Consider a rotating coordinate



system with constant angular velocity  $\boldsymbol{\omega}$ , so  $\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}$ . Then

$$\begin{aligned}\nabla \times \mathbf{v} &= \nabla \times \mathbf{v}' + \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \nabla \times \mathbf{v}' + \boldsymbol{\omega} \nabla \cdot \mathbf{r} - \boldsymbol{\omega} \cdot \nabla \mathbf{r} \\ &= \nabla \times \mathbf{v}' + 2\boldsymbol{\omega}.\end{aligned}$$

Now  $\boldsymbol{\omega}$  was constant, but we have not yet specified its value. If we pick  $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \mathbf{v}$  at a point  $\mathbf{r}$ , then  $\nabla \times \mathbf{v}' = 0$  and the fluid is irrotational at  $\mathbf{r}$  in the rotating frame. This argument can be repeated for other points  $\mathbf{r}$ . Considering this from the point of view of the original frame we thus see that  $\Omega/2$  is the angular velocity of the fluid at position  $\mathbf{r}$ .

### 6.2.2 Ideal Fluid: Euler's Equation and Entropy Conservation

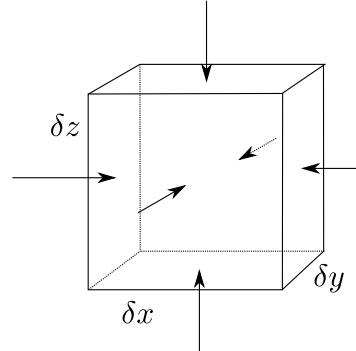
Let us consider an *ideal fluid* which has no energy dissipation due to internal friction (meaning no viscosity) and no heat exchange between different parts of the fluid (meaning no thermal conductivity).

The force on a fluid element  $\delta V$  from pressure in the  $\hat{x}$  direction is

$$\begin{aligned}\delta F_x &= F_x(x) - F_x(x + \delta x) = \\ &= \frac{\mathcal{P}(x) - \mathcal{P}(x + \delta x)}{\delta x} \delta y \delta z \delta x = -\frac{\partial \mathcal{P}}{\partial x} \delta V.\end{aligned}\quad (6.30)$$

More generally accounting for all directions we have

$$\delta \mathbf{F} = -\nabla \mathcal{P} \delta V. \quad (6.31)$$



From external forces  $\delta \mathbf{F}$  it is useful to define the force density  $\mathbf{f}$  by  $\delta \mathbf{F} = \mathbf{f} \delta V$ . This means that Newton's law,  $m\mathbf{a} = \mathbf{F}$  becomes

$$\rho \delta V \frac{d\mathbf{v}}{dt} = (-\nabla \mathcal{P} + \mathbf{f}) \delta V. \quad (6.32)$$

Writing out the total time derivatives this becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\nabla \mathcal{P}}{\rho} = \frac{\mathbf{f}}{\rho} \quad (6.33)$$

which is the Euler equation for fluid dynamics. There are two special cases which are useful to consider.

- Often  $\mathbf{f}$  can be derived from a potential:  $\mathbf{f} = -\rho \nabla \Phi$ . Note that  $\Phi$  here is a potential energy per unit mass. For example, with gravity,  $\Phi = gz$ , so  $\mathbf{f} = -\rho g \hat{z}$ . In general, then

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\nabla \mathcal{P}}{\rho} + \nabla \Phi = 0 \quad (6.34)$$

is a rewriting of the Euler equation.

2. We can use  $\mathbf{v} \times \boldsymbol{\Omega} = \mathbf{v} \times (\nabla \times \mathbf{v}) = \nabla \left( \frac{\mathbf{v}^2}{2} \right) - \mathbf{v} \cdot \nabla \mathbf{v}$ . This gives

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{\mathbf{v}^2}{2} + \Phi \right) + \frac{\nabla \mathcal{P}}{\rho} - \mathbf{v} \times \boldsymbol{\Omega} = 0 \quad (6.35)$$

as another rewriting of the Euler equation. For constant  $\rho$ , we can take the curl and use the fact that  $\nabla \times (\nabla h) = 0$  for any  $h$  to obtain

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = \nabla \times (\mathbf{v} \times \boldsymbol{\Omega}). \quad (6.36)$$

We will have occasion to use this result later on.

To solve the Euler and continuity partial differential equations we need boundary conditions. The boundary conditions for ideal fluids are simply that they cannot penetrate solid surfaces, so if a surface with normal vector  $\hat{n}$  is stationary then

$$\mathbf{v} \cdot \hat{n} \Big|_{\text{surface}} = 0. \quad (6.37)$$

If the surface does move with velocity  $\mathbf{v}_{\text{surface}}$  then

$$\mathbf{v} \cdot \hat{n} \Big|_{\text{surface}} = \mathbf{v}_{\text{surface}} \cdot \hat{n}. \quad (6.38)$$

So far we have four equations (continuity for the scalar density  $\rho$ , and the Euler equation for the vector velocity  $\mathbf{v}$ ) for five unknowns. For an ideal fluid the 5<sup>th</sup> equation,

$$\frac{dS}{dt} = 0, \quad (6.39)$$

is the statement that the entropy  $S$  is conserved, so there is no heat exchange. Effectively, this provides a relationship between pressure and density through  $\mathcal{P} = \mathcal{P}(\rho, S)$ . A simple example is an ideal gas at constant temperature, where  $\mathcal{P} = \rho R' T$ .

### 6.2.3 Conservation of Momentum and Energy

Due to the term  $(\mathbf{v} \cdot \nabla) \mathbf{v}$  which has a  $v_i v_j$ , the Euler equation is nonlinear. For simpler situations it is therefore very useful to consider conservation laws.

Let us start by considering *Momentum Conservation*. The quantity  $\rho \mathbf{v}$  is the flux of mass density, which is also the density of momentum (in direct analogy to  $\mathbf{p} = m \mathbf{v}$ ). Consider  $\frac{\partial}{\partial t} (\rho v_i) = \frac{\partial \rho}{\partial t} v_i + \rho \frac{\partial v_i}{\partial t}$ . Using the continuity and Euler equations to replace these two partial derivatives, and once again implicitly summing over repeated indices until further notice, this becomes

$$\frac{\partial}{\partial t} (\rho v_i) = -v_i \frac{\partial}{\partial x_j} (\rho v_j) + \rho \left( -v_j \frac{\partial v_i}{\partial x_j} - \frac{1}{\rho} \frac{\partial \mathcal{P}}{\partial x_i} \right) + f_i. \quad (6.40)$$

This is rearranged to give

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial \mathcal{P}}{\partial x_i} + \frac{\partial}{\partial x_j}(\rho v_i v_j) = f_i. \quad (6.41)$$

We define the *stress tensor* for an ideal fluid as

$$T_{ij} = \mathcal{P}\delta_{ij} + \rho v_i v_j \quad (6.42)$$

which gives the momentum flux density (which is to say, the density of momentum in the direction  $\mathbf{e}_i$  flowing in the direction of  $\mathbf{e}_j$ ). Note that  $T_{ij}$  is symmetric. Then, in vector form, the equation above becomes

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot \mathbb{T} = \mathbf{f}. \quad (6.43)$$

This describes the conservation of linear momentum density with sources given by the external force densities  $\mathbf{f}$ . Comparing to the continuity equation where the density  $\rho$  is a scalar so its flux  $\rho \mathbf{v}$  is a vector, here the momentum density  $\rho \mathbf{v}$  is a vector so its flux  $\mathbb{T}$  is a tensor.

Next consider *Conservation of Energy*. Conservation of energy can be said to arise from the Euler equation  $d\mathbf{v}/dt + \nabla \mathcal{P}/\rho + \nabla \Phi = 0$ . Note that here we are switching back to the total time derivative since this is more useful for our discussion of energy. For a volume element  $\delta V$ , we take the inner product of the Euler equation with  $\rho \mathbf{v} \delta V$  to obtain

$$\delta V \rho \mathbf{v} \cdot \frac{d\mathbf{v}}{dt} + \delta V \mathbf{v} \cdot \nabla \mathcal{P} + \delta V \rho \mathbf{v} \cdot \nabla \Phi = 0. \quad (6.44)$$

Given that  $\frac{d}{dt}(\rho \delta V) = 0$  by the conservation of mass we can move this combination inside of total time derivatives. Also recall that  $\mathbf{v} \cdot \nabla = d/dt - \partial/\partial t$ . Using these two facts we obtain

$$\frac{d}{dt} \left( \frac{1}{2} \rho \mathbf{v}^2 \delta V \right) + (\mathbf{v} \cdot \nabla \mathcal{P}) \delta V + \frac{d}{dt}(\rho \Phi \delta V) - \rho \frac{\partial \Phi}{\partial t} \delta V = 0. \quad (6.45)$$

Next we try to convert the second term to time derivatives. Consider using

$$\frac{d}{dt}(\mathcal{P} \delta V) = \frac{\partial \mathcal{P}}{\partial t} \delta V + (\mathbf{v} \cdot \nabla \mathcal{P}) \delta V + \mathcal{P}(\nabla \cdot \mathbf{v}) \delta V, \quad (6.46)$$

where we recalled that  $d\delta V/dt = (\nabla \cdot \mathbf{v}) \delta V$ . Using this to eliminate  $(\mathbf{v} \cdot \nabla \mathcal{P}) \delta V$  gives

$$\frac{d}{dt} \left( \frac{1}{2} \rho \mathbf{v}^2 \delta V + \rho \Phi \delta V + \mathcal{P} \delta V \right) = \left( \frac{\partial \mathcal{P}}{\partial t} + \rho \frac{\partial \Phi}{\partial t} \right) \delta V + \mathcal{P} \nabla \cdot \mathbf{v} \delta V, \quad (6.47)$$

where  $\frac{1}{2} \rho \mathbf{v}^2 \delta V$  is the kinetic energy,  $\rho \Phi \delta V$  is the external potential energy, and  $\mathcal{P} \delta V$  is the internal potential energy due to pressure. The terms with partial time derivatives act like sources. Unfortunately there is still a term without a total or partial time derivative, however

this term is easier to interpret. It is related to the work  $W_u$  done by  $\delta V$  when it expands and exerts pressure on the surrounding fluid. In particular

$$\frac{dW_u}{dt} = \mathcal{P} \frac{d(\delta V)}{dt} = \mathcal{P} \nabla \cdot \mathbf{v} \delta V \equiv -\frac{d}{dt}(U \delta m) \quad (6.48)$$

where in the last step we have defined the work as a negative potential energy  $U$  (per unit mass). If the equation of state is given, this  $U$  can be calculated either as an integral in density or pressure,

$$U = \int d(\delta V) \frac{\mathcal{P}}{\delta m} = \int_{\rho_0}^{\rho} d\rho' \frac{\mathcal{P}(\rho')}{\rho'^2} = \int_{\mathcal{P}_0}^{\mathcal{P}} \mathcal{P}' \left( \frac{1}{\rho^2} \frac{d\rho}{d\mathcal{P}'} \right) (\mathcal{P}') d\mathcal{P}' \quad (6.49)$$

where we used the fixed  $\delta m = \delta V \rho$  to switch variables between  $\delta V$  and  $\rho$ . Using Eq. (6.48) means we can now write everything in terms of time derivatives,

$$\frac{d}{dt} \left( \left( \frac{1}{2} \rho \mathbf{v}^2 + \rho \Phi + \mathcal{P} + \rho U \right) \delta V \right) = \left( \frac{\partial \mathcal{P}}{\partial t} + \rho \frac{\partial \Phi}{\partial t} \right) \delta V \quad (6.50)$$

This is the equation for *energy conservation* in an ideal fluid. If  $\mathcal{P}$  and  $\Phi$  are not explicitly dependent on time at any point in space (which is often the case), then any fluid element has a constant total energy as it moves along (recall that this is the meaning of  $d/dt$ ). For applications to fluids it is more convenient to divide this result by  $\delta m = \rho \delta V$  to give

$$\frac{d}{dt} \left( \frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} + U \right) = \frac{1}{\rho} \frac{\partial \mathcal{P}}{\partial t} + \frac{\partial \Phi}{\partial t}. \quad (6.51)$$

This is *Bernoulli's equation*.

### 6.3 Static Fluids & Steady Flows

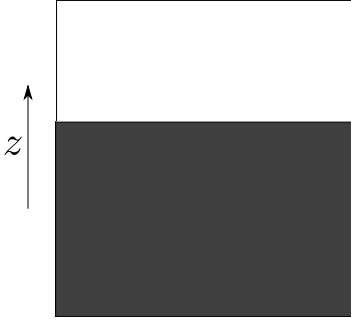
Having derived the equations of motion and conservation laws for ideal fluids, let us now consider some important special cases.

#### Static Fluids

Static fluids have  $\mathbf{v} = 0$  everywhere, so the fluid is at rest (implying mechanical equilibrium). Continuity then says  $\frac{\partial \rho}{\partial t} = 0$ , so  $\rho$  and  $\mathcal{P}$  are independent of time. If  $\Phi = gz$  for gravity, the Euler equation says

$$\frac{\nabla \mathcal{P}}{\rho} = -g \hat{z}, \quad (6.52)$$

so  $\mathcal{P}$  and  $\rho$  can each only be functions of  $z$ , while in fact  $\frac{1}{\rho} \frac{\partial \mathcal{P}}{\partial z}$  is independent of  $z$ .



**Example** if the density  $\rho$  is constant, then if  $\Phi = gz$ , then

$$\mathcal{P}(z) = \mathcal{P}_0 - \rho g z, \quad (6.53)$$

so the pressure decreases with height.

**Example** let us pretend the atmosphere is an ideal gas at a uniform constant temperature  $T$ . As  $\mathcal{P} = \rho R' T$ , then

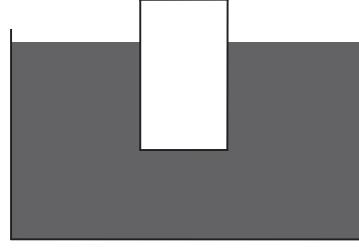
$$\frac{\partial \mathcal{P}}{\partial z} = -\frac{g}{R' T} \mathcal{P}, \quad (6.54)$$

so

$$\mathcal{P}(z) = \mathcal{P}_0 e^{-\frac{gz}{R' T}}, \quad (6.55)$$

and the pressure falls exponentially.

**Example** the Archimedes principle says that the pressure balances the weight of any displaced fluid. This follows from our first example above. The pressure on the bottom of the object displacing the fluid is  $\rho g z$ , where  $z$  is the distance from the surface. The force is pressure times area, so that is the volume displaced times  $\rho g$ , or the mass displaced times  $g$ .



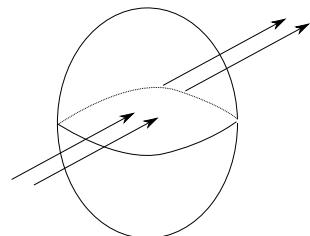
Note that if the temperature is not uniform, then the mechanical equilibrium is not stable, as the temperature gradients result in *convection* currents which mix the fluid. Therefore we have used the fact that we are discussing an ideal fluid.

### Steady Flows

Steady flows are ones in which  $\frac{\partial \rho}{\partial t} = 0$ ,  $\frac{\partial \mathcal{P}}{\partial t} = 0$ , and  $\frac{\partial \mathbf{v}}{\partial t} = 0$  at every position  $\mathbf{r}$ .

In this case, the continuity equation becomes  $\nabla \cdot (\rho \mathbf{v}) = 0$ , so what flows in must flow out. This is most easily implemented by using the integral form, where for any closed surface  $S$  we have

$$\int_S d\mathbf{S} \cdot (\rho \mathbf{v}) = 0. \quad (6.56)$$



For a steady flow the Bernoulli equation becomes the statement that

$$\frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} + U = B \quad (6.57)$$

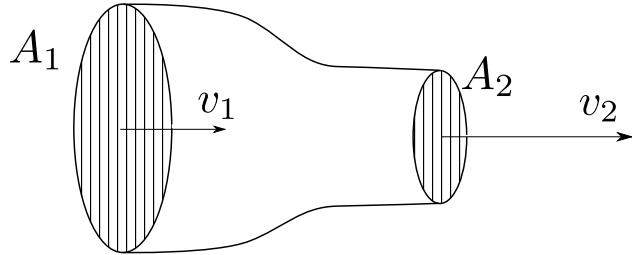
where  $B$  is a constant along the paths of fluid elements. Most often we will consider gravity where  $\Phi = gz$ .

A *steady incompressible fluid* is one where  $\nabla \cdot \mathbf{v} = 0$  and  $U$  is constant. Continuity now says  $\nabla \rho = 0$  so  $\rho$  is constant as well. Moreover, the Bernoulli equation now says

$$\frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} = B' \quad (6.58)$$

is also constant.

**Example** for a horizontal pipe filled with an incompressible fluid (which is approximately true of water at room temperature) of constant density  $\rho$ .



Lets consider the two ends to be at approximately at the same height so we can drop the term  $\Phi = gz$ . At the hatched areas shown we know that the flow must be tangential to the edge of the pipe, so a valid solution is to simply consider the velocities to be uniform and tangential to the enclosing pipe across each of these areas. Using Bernoulli this implies that

$$\frac{\rho v_1^2}{2} + \mathcal{P}_1 = \frac{\rho v_2^2}{2} + \mathcal{P}_2. \quad (6.59)$$

Furthermore, the continuity equation for the enclosed areas shown (some sides being those of the pipe) implies that the flux in at one end must equal the flux out at the other  $A_1 v_1 = A_2 v_2$ . Together this gives

$$\mathcal{P}_1 = \mathcal{P}_2 + \frac{1}{2} \rho v_2^2 \left( 1 - \left( \frac{A_2}{A_1} \right)^2 \right). \quad (6.60)$$

Since  $A_1 > A_2$  we have  $v_2 > v_1$ , and this implies  $\mathcal{P}_1 > \mathcal{P}_2$ .

**Example** let us consider a water tank filled to a height  $z_1$  with a hole at height  $z_2 < z_1$  that produces a jet of water.

Let us assume  $v_1 \approx 0$  (so the tank is much larger than the hole). Then

$$\frac{1}{2}\rho v_2^2 + \mathcal{P}_2 + \rho g z_2 = \mathcal{P}_1 + \rho g z_1. \quad (6.61)$$

Additionally,  $\mathcal{P}_2 = \mathcal{P}_1 = \mathcal{P}_{\text{atmosphere}}$ , so the pressure terms cancel out, and we can solve for the  $v_2$  velocity to give

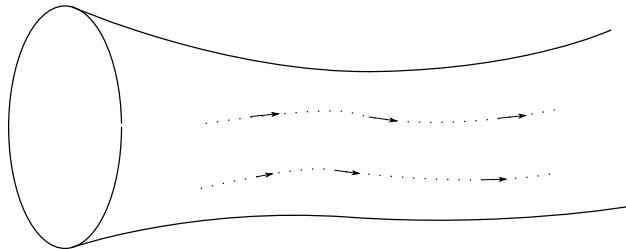
$$v_2 = \sqrt{2g(z_1 - z_2)}. \quad (6.62)$$

This is the same velocity as that for any mass falling from rest through a height  $z_1 - z_2$ . Of course a key difference for the jet of water is that this velocity is horizontal rather than vertical.

Lets discuss two common ways to picture flows. One is through *stream lines*, which are lines that are everywhere tangent to the instantaneous velocity, meaning

$$\frac{d\mathbf{x}(s)}{ds} \times \mathbf{v} = 0 \quad (6.63)$$

for some parameter  $s$  that determines the distance along the streamline. These lines are drawn at some fixed time and never cross since there is a unique velocity at every point.



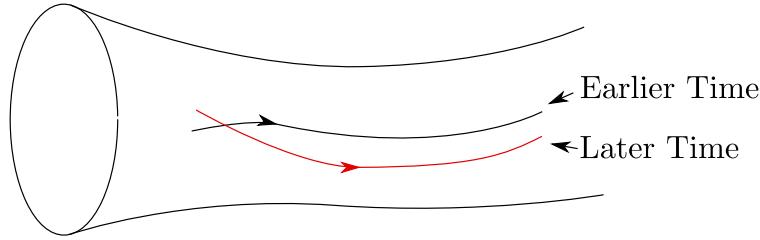
Writing out the cross product we find  $0 = d\mathbf{x}/ds \times \mathbf{v} = \hat{z}(v_y \frac{dx}{ds} - v_x \frac{dy}{ds}) + \dots$  implying that we have

$$\frac{dy}{dx} = \frac{v_y}{v_x}, \quad \frac{dz}{dx} = \frac{v_z}{v_x}, \quad \left( \text{or } \frac{dz}{dy} = \frac{v_z}{v_y} \right). \quad (6.64)$$

When we use the equations in this form we would need to be able to switch from  $s$  to the variable  $x$  to uniquely parameterize the curve.

Another method of picturing the flow is through *flow lines*, which are paths that are followed by fluid elements, meaning

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}(\mathbf{x}, t). \quad (6.65)$$

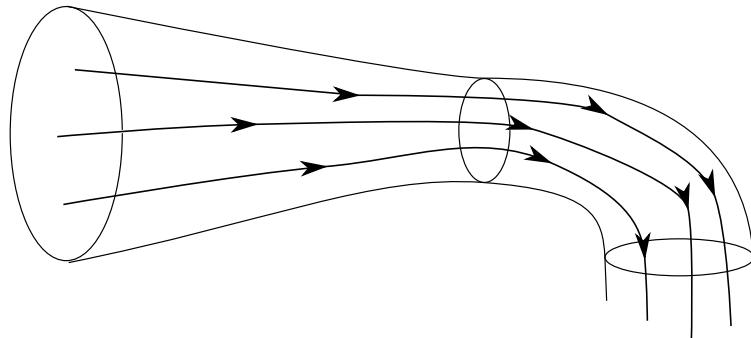


Since flow lines are time dependent, they can in general cross since the path a fluid element takes through a point may differ at a later time.

For a steady flow, the lines are time independent, and stream lines and flow lines are identical. To prove this we consider the flow line equations  $dx/dt = v_x$ ,  $dy/dt = v_y$ , and note that since the velocities are time independent that we can eliminate time through the ratio  $dy/dx = v_y/v_x$  (and similar for other directions), which is the equation for the stream lines. To go the opposite direction we simply pick  $s = t$  to parameterize the streamline, and note that  $\mathbf{v} \times \mathbf{v} = 0$ . Furthermore, for a steady fluid we have Bernoulli's law

$$\frac{\mathbf{v}^2}{2} + \Phi + \frac{\mathcal{P}}{\rho} + U = B \quad (6.66)$$

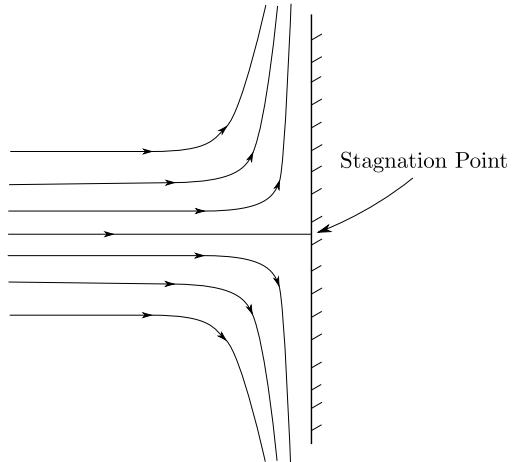
which we now understand is constant along stream lines.



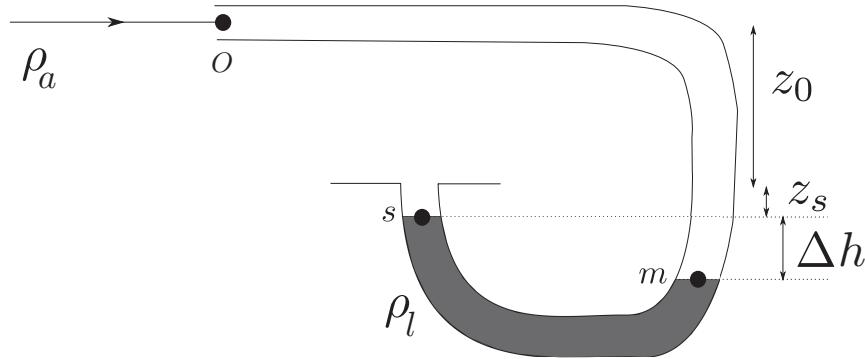
### Stagnation Points and a Pitot Tube

Consider an ideal incompressible fluid in a steady flow which flows in an almost uniform manner, and then hits a wall.

Since the velocity must be tangential at the surface, the flow can either curve left or right, and it is often the case that parts of the flow will go in both directions. In this case there is a stagnation point  $o$ , where  $v_o = 0$ . If we consider the streamline that hits the stagnation point then  $\mathcal{P}_o = \mathcal{P}_\infty + \frac{1}{2}\rho v_\infty^2$ , where  $v_\infty$  and  $\mathcal{P}_\infty$  are, respectively, the velocity and pressure infinitely far away. Thus the largest pressure in the entire flow occurs at the stagnation point where there is only pressure and no kinetic energy.



Now let us consider a Pitot tube, which is a device used to measure velocity (for example on airplanes).



If the density is  $\rho_a$  in the air and  $\rho_l$  in the liquid, we can write down the Bernoulli equations obtained by comparing the air and liquid flows at  $\infty$ , the stagnation point  $o$ , at the point  $s$  near the surface (where the air velocity is the same as at  $\infty$  and liquid is at rest), and at the point  $m$  at the top of the liquid inside the column. This gives:

$$\begin{aligned} \mathcal{P}_o &= \mathcal{P}_\infty + \frac{1}{2}\rho_a v_\infty^2 \\ \mathcal{P}_s &= \mathcal{P}_\infty + \frac{1}{2}\rho_a g(z_o + z_s) \\ \mathcal{P}_m &= \mathcal{P}_o + \rho_a g(z_0 + z_s + \Delta h) \\ &= \mathcal{P}_s + \rho_l g \Delta h \end{aligned}$$

Subtracting the 2<sup>nd</sup> equation from the 1<sup>st</sup>, subtracting the 4<sup>th</sup> equation from the 3<sup>rd</sup>, and then adding these two results cancels all the pressure terms, and leaves

$$\frac{1}{2}\rho_a v_\infty^2 = g(\rho_l - \rho_a)\Delta h. \quad (6.67)$$

This can be rearranged to write

$$v_\infty^2 = 2g \left( \frac{\rho_l}{\rho_a} - 1 \right) \Delta h \approx 2g \frac{\rho_l}{\rho_a} \Delta h \quad (6.68)$$

allowing us to determine the velocity of the air  $v_\infty$  in terms of the known ratio of densities  $\frac{\rho_l}{\rho_a} \gg 1$  and simply the measured height between the liquid on each side,  $\Delta h$ .

## 6.4 Potential Flow

When a flow is everywhere both irrotational and incompressible it is known as *potential flow*. Such flows may be steady or not steady. Since  $\boldsymbol{\Omega} = \nabla \times \mathbf{v} = 0$  the velocity field is conservative. This means there exists a *velocity potential*

$$\phi(\mathbf{r}) = \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{v}(\mathbf{r}', t) \cdot d\mathbf{r}' \quad (6.69)$$

which depends only on the endpoints of the integration, and not the path taken between them, such that

$$\mathbf{v} = \nabla \phi \quad (6.70)$$

Since  $\nabla \cdot \mathbf{v} = 0$  for an incompressible fluid, then  $\phi$  must solve Laplace's equation

$$\nabla^2 \phi = 0. \quad (6.71)$$

Solving for the scalar  $\phi$  (with suitable boundary conditions), then immediately gives  $\mathbf{v}$ .

We can then use the Euler equation to immediately get the pressure. For  $\boldsymbol{\Omega} = 0$  one form of the Euler equation was

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( \frac{\mathbf{v}^2}{2} + \Phi \right) + \frac{\nabla P}{\rho} = 0. \quad (6.72)$$

If  $\rho$  is constant, then

$$\nabla \left( \frac{\partial \phi}{\partial t} + \frac{\mathbf{v}^2}{2} + \Phi + \frac{P}{\rho} \right) = 0, \quad (6.73)$$

so

$$\frac{\partial \phi}{\partial t} + \frac{\mathbf{v}^2}{2} + \Phi + \frac{P}{\rho} = b(t) \quad (6.74)$$

for some function  $b$ . For each  $t$  we can pick the zero of  $\phi$  so that  $b(t)$  is constant. (This is equivalent to shifting  $\phi \rightarrow \phi + \int^t b(t') dt'$ , where adding this constant that is independent of  $\mathbf{x}$  gives a solution that is equally valid.) The remaining constant  $b$  be fixed by a boundary condition on the pressure. Thus the full pressure as a function of  $\mathbf{x}$  and  $t$  is determined by

$$\frac{P}{\rho} = -\frac{\partial \phi}{\partial t} - \frac{\mathbf{v}^2}{2} - \Phi + b, \quad (6.75)$$

where in principal the first three terms on the right hand side carry both spatial and time dependence. Often we are interested in a steady flow, in which case the term  $\partial\phi/\partial t = 0$ .

**Example** if  $\phi = v_0x$ , then  $\mathbf{v} = v_0\hat{x}$ , which is a specific case of a uniform flow.

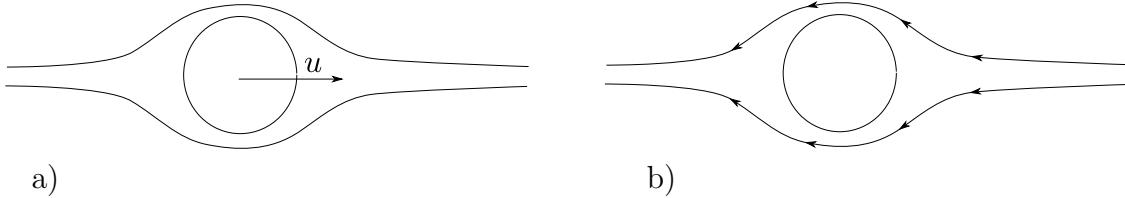
**Example** if  $\phi = A \ln(r)$  in 2 dimensions, then for all  $r > 0$

$$\mathbf{v} = \frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\hat{\theta} = \frac{A}{r}\hat{r}, \quad (6.76)$$

which is a point source. An analogous point source in 3 dimensions would arise from  $\phi = -\frac{A}{r}$ , which gives  $\mathbf{v} = \frac{A}{r^2}\hat{r}$  for all  $r > 0$ . (At  $r = 0$  there would be a delta function source, so we do not satisfy Laplace's equation at this point.)

**Example** Consider  $\phi = \frac{\Gamma}{2\pi}\theta$  in polar coordinates in the 2 dimensional plane. Then  $v_r = \frac{\partial\phi}{\partial r} = 0$  and  $v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$  for all  $r > 0$ . This corresponds to a potential vortex about the point  $r = 0$ .

**Example** Consider a sphere of radius  $R$  moving with constant velocity  $\mathbf{u}$  through an incompressible ideal fluid and find its velocity by solving for its velocity potential. Equivalently we can consider the problem of finding the velocity when a sphere is held at rest and the fluid flows in from far away with uniform velocity  $-\mathbf{u}$  at infinity. The first situation is shown in figure a) and the second as figure b).



We use coordinates centered on the sphere, and define the axes so that  $\mathbf{u} = u\hat{x}$ . The problem is spherically symmetric other than the directionality from  $\mathbf{u}$ , so it's natural to expect  $\mathbf{v} \propto \mathbf{u}$ . Since  $\nabla\phi = \mathbf{v}$  this means that we expect  $\phi \propto \mathbf{u}$ . (Effectively the boundary condition is linear in  $\mathbf{u}$  and the equation  $\nabla^2\phi = 0$  is linear. We could also explicitly demonstrate the proportionality  $\mathbf{v} \propto \mathbf{u}$  using dimensional analysis, as we will discuss in more detail later in this chapter.)

It is actually easier to consider the sphere being at rest with the fluid moving past it as in b), so let's start with this case. Since  $\nabla^2\phi = 0$  is linear, we can solve using superposition. The velocity potential

$$\phi(\mathbf{x}) = -ux + \phi'(\mathbf{x}) \quad (6.77)$$

has a term  $-ux$  giving the correct uniform flow far away from the sphere. Therefore, with  $r$  the distance from the center of the sphere, we have

$$\lim_{r \rightarrow \infty} \phi'(\mathbf{x}) = 0. \quad (6.78)$$

Another way to see this is that taking the gradient of Eq. (6.77) gives

$$\mathbf{v} = -u\hat{x} + \mathbf{v}', \quad (6.79)$$

which is simply the translation between the velocity field  $\mathbf{v}$  for b) and the velocity field  $\mathbf{v}'$  for a). For the situation a) we would anticipate Eq. (6.78) as the correct boundary condition, since the fluid is at rest at infinity when it is infinitely far away from the disturbance caused by dragging the sphere. We can look for a solution for  $\phi'$ .

As  $\nabla^2\phi' = 0$  with  $\lim_{r \rightarrow \infty}\phi' = 0$ , one option could be  $\phi' = \frac{1}{r}$  in 3 dimensions, but this would give a point source solution with velocity moving radially outward from our sphere and hence make it impossible to satisfy the appropriate boundary condition on the sphere (it's also not  $\propto \mathbf{u}$ ). Instead, let us consider a dipole source

$$\phi' = A\mathbf{u} \cdot \nabla \left( \frac{1}{r} \right) \quad (6.80)$$

for some constant  $A$ . This proposal is linear in  $\mathbf{u}$ , and satisfies  $\nabla^2\phi' = 0$  for  $r > 0$  since the derivatives commute:  $\nabla^2\phi' = A(\mathbf{u} \cdot \nabla)\nabla^2(1/r) = 0$ . It remains to compute  $\phi'$  and  $\mathbf{v}$  explicitly and check that we can satisfy the boundary conditions (and proper dimensions) with this solution. Since

$$\nabla \left( \frac{1}{r} \right) = -\frac{\mathbf{r}}{r^3}, \quad (6.81)$$

we have

$$\phi' = -\frac{A\mathbf{u} \cdot \mathbf{r}}{r^3}. \quad (6.82)$$

Moreover,

$$\nabla(\mathbf{u} \cdot \mathbf{r}) = \mathbf{u}, \quad \nabla r^{-n} = -nr^{-(n+2)}\mathbf{r}, \quad (6.83)$$

so the solution for  $\mathbf{v}'$  is

$$\mathbf{v}' = \nabla\phi' = \frac{A}{r^3} \left( -\mathbf{u} + \frac{3(\mathbf{u} \cdot \mathbf{r})\mathbf{r}}{r^2} \right) \quad (6.84)$$

and we then also have obtained  $\mathbf{v} = -\mathbf{u} + \mathbf{v}'$ . The boundary condition on the surface of the sphere (which is  $r = R$  for case b) where the sphere is at rest) is  $\mathbf{v} \cdot \hat{r} = 0$ . This means

$$0 = -\mathbf{u} \cdot \hat{r} - \frac{A\mathbf{u} \cdot \hat{r}}{R^3} + \frac{3A\mathbf{u} \cdot \hat{r}}{R^3}, \quad (6.85)$$

which has the solution  $A = \frac{R^3}{2}$ . Thus,

$$\mathbf{v} = -\mathbf{u} + \frac{R^3}{2r^3} \left( \frac{3(\mathbf{u} \cdot \mathbf{r})\mathbf{r}}{r^2} - \mathbf{u} \right). \quad (6.86)$$

This solution has the right dimensions and satisfies the boundary conditions on the sphere and at infinity.

For steady flow, we can then use Bernoulli's equation to get the pressure on the sphere, constant =  $(\mathcal{P} + \frac{1}{2}\rho v^2)_{r=R} = (\mathcal{P} + \frac{1}{2}\rho v^2)_{r=\infty}$ . Squaring our result for the velocity on the sphere, setting  $\mathbf{u} \cdot \mathbf{r} = ur \cos \theta$ , and simplifying we find

$$\mathcal{P} = \mathcal{P}_\infty + \frac{\rho \mathbf{u}^2}{8} (9 \cos^2 \theta - 5). \quad (6.87)$$

This result for the pressure says that it is the same on the front and back of the sphere, since its unchanged by taking  $\theta \rightarrow \pi/2 - \theta$ . This is quite counterintuitive, since we expect a force on the sphere in b) that would try to push it downstream. This actually results from our approximation that the fluid is ideal (viscosity can not be neglected when trying to answer questions near surfaces).

Another possibility is that our approximation of potential flow is suspect. To explore this, lets ask how common is potential flow? Consider

$$\frac{d}{dt} (\nabla \times \mathbf{v}) = \nabla \times \frac{d\mathbf{v}}{dt} = \nabla \times \left( \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right), \quad (6.88)$$

which expands into

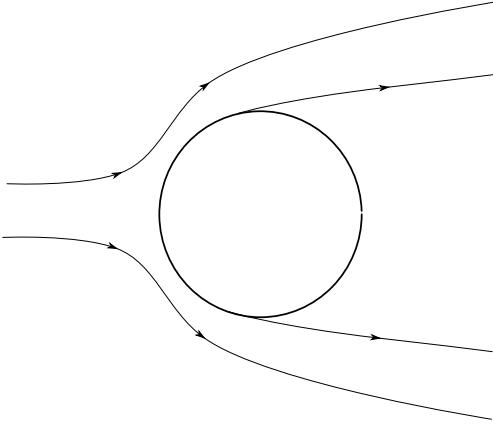
$$\frac{\partial}{\partial t} (\nabla \times \mathbf{v}) + \nabla \times \left( \nabla \left( \frac{\mathbf{v}^2}{2} \right) - \mathbf{v} \times (\nabla \times \mathbf{v}) \right) = \frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\Omega}) = 0 \quad (6.89)$$

where in the last step we used our result in Eq. (6.36) that we derived from the Euler equation. Thus,  $\nabla \times \mathbf{v}$  is conserved along flow lines. If we consider a steady flow which starts out as uniform infinitely far away, then there is no vorticity at infinity, and

$$\nabla \times \mathbf{v} = 0 \quad (6.90)$$

on every stream line, and remains that way for the entire flow. Thus we have a potential flow with no vorticity.

If we wanted to get around the counterintuitive behavior we have found in our ideal fluid solution, but stick with the ideal fluid framework, then we would have to allow for the existence of discontinuous solutions. For an ideal fluid flowing past a sphere, we could propose that stream lines exist that start tangential to the spherical surface, and hence satisfy the boundary conditions, and can have  $\nabla \times \mathbf{v} \neq 0$  since they are not connected to infinity. Behind the sphere we could then imagine having fluid at rest.



The correct treatment of boundary layers near objects and of wakes, does not require discontinuous solutions of this sort, but instead simply requires the inclusion of viscosity, which we will turn to shortly, after treating one final important example from ideal fluids.

## 6.5 Sound Waves

In this section we will explore an example where  $\nabla \cdot \mathbf{v} \neq 0$  plays an important role. To set things up, consider a compressible fluid at rest with pressure  $\mathcal{P}_0$  and density  $\rho_0$  in equilibrium with an external force density  $\mathbf{f}_0$ . If  $\mathcal{P}_0$  and  $\rho_0$  are constant and uniform, then

$$\frac{\nabla \mathcal{P}_0}{\rho_0} = \frac{\mathbf{f}_0}{\rho_0} \quad (6.91)$$

from the Euler equation. Now lets add disturbances  $\mathcal{P}'$  and  $\rho'$  to this system

$$\mathcal{P} = \mathcal{P}_0 + \mathcal{P}', \quad (6.92)$$

$$\rho = \rho_0 + \rho', \quad (6.93)$$

with  $\mathcal{P}' \ll \mathcal{P}_0$  and  $\rho' \ll \rho_0$ . These disturbances will induce a velocity field as well,  $\mathbf{v}(\mathbf{r}, t)$ , which we will also assume is small, so that perturbation theory applies. We will therefore drop terms that are second order or higher in any of  $\{\mathcal{P}', \rho', \mathbf{v}\}$ . Using perturbation theory on the Euler equation, that is

$$0 = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho (\mathbf{v} \cdot \nabla) \mathbf{v} + \nabla p - \mathbf{f}_0 = \nabla \mathcal{P}_0 - \mathbf{f}_0 + \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \nabla \mathcal{P}' + \dots \quad (6.94)$$

where we have dropped terms  $\rho \mathbf{v} \cdot \nabla \mathbf{v} = \mathcal{O}(\rho_0 \mathbf{v}^2)$  and  $\mathcal{O}(\rho' \mathbf{v})$ . Using  $\mathbf{f}_0 = \nabla \mathcal{P}_0$  then gives

$$\frac{\partial \mathbf{v}}{\partial t} = - \frac{\nabla \mathcal{P}'}{\rho_0} \quad (6.95)$$

as the Euler equation to 1<sup>st</sup> order in perturbations. To the same order, continuity says

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) = -\rho_0 \nabla \cdot \mathbf{v}. \quad (6.96)$$

where we have again dropped second order terms. Finally the appropriate thermodynamic equation of state is

$$\rho' = \frac{\rho_0}{B} \mathcal{P}' \quad (6.97)$$

where  $B$  is a constant known as the bulk modulus. The bulk modulus describes a substance's resistance to compression, and this formula arises from  $B = \rho \frac{\partial P}{\partial \rho} \approx \rho_0 \frac{P'}{\rho'}$ . Using this result we can eliminate density  $\rho'$  to get a second equation involving only the pressure and velocity disturbances,

$$\frac{\partial \mathcal{P}'}{\partial t} = \frac{B}{\rho_0} \frac{\partial \rho'}{\partial t} = -B \nabla \cdot \mathbf{v} \quad (6.98)$$

Combining Eq. (6.95) and Eq. (6.98) we can derive a differential equation for the pressure disturbance

$$\frac{\partial^2 \mathcal{P}'}{\partial t^2} = -B \nabla \cdot \frac{\partial \mathbf{v}}{\partial t} = \frac{B}{\rho_0} \nabla^2 \mathcal{P}', \quad (6.99)$$

which can be written more simply as

$$\frac{\partial^2 \mathcal{P}'}{\partial t^2} - c_s^2 \nabla^2 \mathcal{P}' = 0 \quad (6.100)$$

which is a wave equation for  $\mathcal{P}'$ , whose solutions move at a velocity  $c_s = \sqrt{\frac{B}{\rho_0}}$  which is known as the *speed of sound*. Due to the simple proportionality from the equation of state we also immediately know that

$$\frac{\partial^2 \rho'}{\partial t^2} - c_s^2 \nabla^2 \rho' = 0 \quad (6.101)$$

so the same wave equation also holds for the density.

It remains to derive a differential equation for the velocity. Taking the curl of Eq. (6.95) yields

$$\frac{\partial}{\partial t} (\nabla \times \mathbf{v}) = -\frac{1}{\rho_0} \nabla \times \nabla \mathcal{P}' = 0. \quad (6.102)$$

This means  $\nabla \times \mathbf{v}$  does not explicitly depend on time, so we can conveniently take  $\nabla \times \mathbf{v} = 0$  initially everywhere, and hence for all times. Using Eq. (6.95) and Eq. (6.98) now gives

$$\begin{aligned} \frac{\partial^2 \mathbf{v}}{\partial t^2} &= -\frac{1}{\rho_0} \nabla \left( \frac{\partial \mathcal{P}'}{\partial t} \right) = \frac{B}{\rho_0} \nabla (\nabla \cdot \mathbf{v}) \\ &= \frac{B}{\rho_0} \left( \nabla^2 \mathbf{v} - \nabla \times (\nabla \times \mathbf{v}) \right) \\ &= \frac{B}{\rho_0} \nabla^2 \mathbf{v}, \end{aligned} \quad (6.103)$$

since  $(\nabla \times \mathbf{v}) = 0$ . This yields

$$\frac{\partial^2 \mathbf{v}}{\partial t^2} - c_s^2 \nabla^2 \mathbf{v} = 0 \quad (6.104)$$

which means that the velocity of the fluid disturbance also satisfies the same wave equation. The solutions are thus *sound waves* in pressure, density, and velocity, with speed  $c_s$ . An example of a solution is a plane wave (here written for pressure) which looks like

$$\mathcal{P}' = \mathcal{P}'(\mathbf{r} \cdot \hat{n} - c_{st}t) \quad (6.105)$$

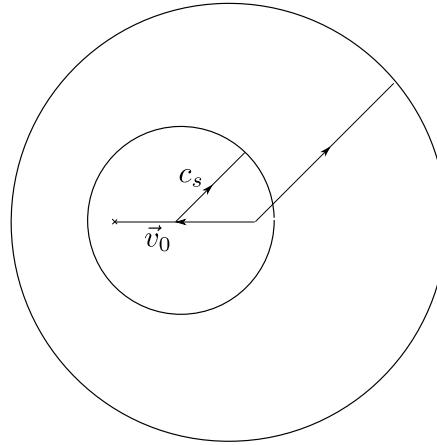
when traveling in the direction of  $\hat{n}$ .

Note that  $\nabla \cdot \mathbf{v} \neq 0$  was important for this derivation. It is reasonable to ask if there is a way that we can determine when the approximation  $\nabla \cdot \mathbf{v} = 0$  may be justified. For a flow with characteristic velocity  $v_0$  this can be done by defining the *Mach number*

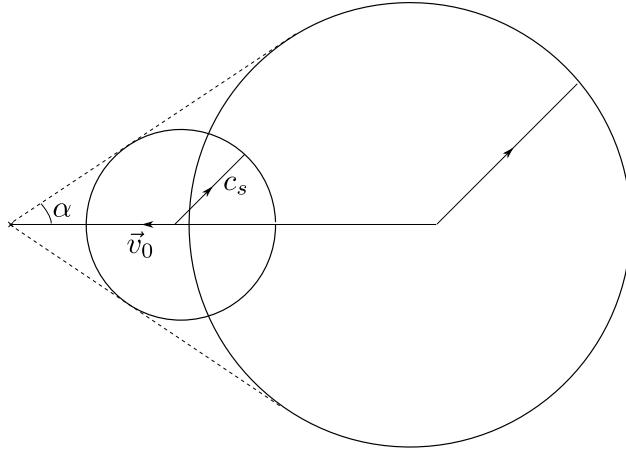
$$M = \frac{v_0}{c_s}, \quad (6.106)$$

since the scaling of terms involving  $\nabla \cdot \mathbf{v}$  will be determined by this ratio. If  $M \ll 1$  then we can treat flow as approximately incompressible, with  $\nabla \cdot \mathbf{v} = 0$ .

Considering flows with large values of  $M$  leads to the concept of shock waves. Consider a flow with initial velocity  $v_0$  in which there is a disturbance. If  $M < 1$  then the flow is said to be subsonic, and the perturbation spreads everywhere, because the speed of the perturbation is larger than that of the flow.



On the other hand if  $M > 1$ , then the disturbance is swept downstream to the right by the flow, and actually propagates downstream within a cone of angle  $\gamma$  defined by  $\sin(\gamma) = c_s/v_0 = \frac{1}{M}$ , as shown below.



If we consider a supersonic plane, then we should view this picture the other way around, where the fluid is static and the disturbance (plane) moves through it, traveling to the left at faster than the speed of sound. This causes a sonic boom, which is the air pressure wave given by the dashed lines trailing the plane, which moves at speed  $c_s$ . (Another example is thunder, where the rapid increase in temperature of plasma of ions causes rapid air expansion, making a shockwave.)

## 6.6 Viscous Fluid Equations

Internal friction occurs in a fluid when neighboring fluid particles move with different velocities, which means that  $\frac{\partial v_i}{\partial x_j} \neq 0$ . Viscous stresses oppose such relative motion. Adding these friction terms changes some of our fluid equations. In particular, the continuity equation remains unchanged, while the Euler equations along with the conservation laws for momentum, energy, and entropy must be modified.

To consider this friction we will work to first order in the  $\frac{\partial v_i}{\partial x_j}$  partial derivatives, treating these as the most important terms. These derivatives can be arranged into

$$\sigma_{ij} = \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right) + \zeta \delta_{ij} \nabla \cdot \mathbf{v} \quad (6.107)$$

to define the viscous stress tensor with elements  $\sigma_{ij}$ . The constant coefficients of the two terms are the shear viscosity  $\eta$  and the bulk viscosity  $\zeta$ , where  $\eta > 0$  &  $\zeta > 0$ . The form of the viscous stress tensor  $\sigma_{ij}$  is dictated by the fact that it must vanish for constant  $\mathbf{v}$  and for uniform rotation  $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$  where there is no friction. Writing out  $\mathbf{v} = \omega_x(y\hat{z} - z\hat{y}) + \dots$  we see that  $\nabla \cdot \mathbf{v} = 0$  and  $\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} = 0$ , etc., for the uniform rotation, dictating the symmetric form of the terms in  $\sigma_{ij}$ . The remaining organizational choice is to let  $\eta$  multiply a traceless tensor.

Momentum conservation still comes from

$$\frac{\partial}{\partial t} (\rho \mathbf{v}) + \nabla \cdot \mathbb{T} = \mathbf{f} \quad (6.108)$$

where now we include a friction term in the stress tensor to account for the viscous transfer of momentum. Thus

$$T_{ij} = \mathcal{P}\delta_{ij} + \rho v_i v_j - \sigma_{ij} \quad (6.109)$$

is the new total stress tensor.

A simple rule for incorporating  $\sigma_{ij}$  is simply to replace  $\mathcal{P}\delta_{ij} \rightarrow \mathcal{P}\delta_{ij} - \sigma_{ij}$ . With this we can add friction to the Euler equation. In particular we have

$$(\nabla\mathcal{P})_i = \frac{\partial}{\partial x_k} \delta_{ki} \mathcal{P} \rightarrow \frac{\partial}{\partial x_k} (\delta_{ki} \mathcal{P} - \sigma_{ki}), \quad (6.110)$$

where we can compute that

$$\begin{aligned} \frac{\partial}{\partial x_k} \sigma_{ki} &= \eta \left( \frac{\partial^2 v_i}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_i} \nabla \cdot \mathbf{v} - \frac{2}{3} \frac{\partial}{\partial x_i} \nabla \cdot \mathbf{v} \right) + \zeta \frac{\partial}{\partial x_i} \nabla \cdot \mathbf{v} \\ &= \eta \nabla^2 v_i + \left( \zeta + \frac{\eta}{3} \right) \frac{\partial}{\partial x_i} \nabla \cdot \mathbf{v}. \end{aligned} \quad (6.111)$$

Plugging this into the Euler equation yields

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + \frac{\nabla \mathcal{P}}{\rho} - \frac{\eta}{\rho} \nabla^2 \mathbf{v} - \frac{1}{\rho} \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \cdot \mathbf{v}) = \frac{\mathbf{f}}{\rho} \quad (6.112)$$

which is the *Navier-Stokes equation*.

A common case we will study is when  $\mathbf{f} = 0$  &  $\nabla \cdot \mathbf{v} = 0$ , which reduces the Navier-Stokes equation to

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla \mathcal{P}}{\rho} + \nu \nabla^2 \mathbf{v} \quad (6.113)$$

where

$$\nu \equiv \frac{\eta}{\rho} \quad (6.114)$$

is the *kinematic viscosity*, and the bulk viscosity term has dropped out. The dimensions of the kinematic viscosity are  $[\nu] = m^2/s$ , which is simpler than  $[\eta] = kg/(ms)$ .

Other useful equations can be derived for the situation where  $\nabla \cdot \mathbf{v} = 0$  &  $\rho$  is constant. Taking the divergence of the Navier-Stokes equation removes both the  $\partial \mathbf{v} / \partial t$  term, and the  $\nu \nabla^2 \mathbf{v}$  term, leaving  $\rho \nabla \cdot [(\mathbf{v} \cdot \nabla) \mathbf{v}] = -\nabla^2 \mathcal{P}$ . Writing this out in components we find

$$\nabla^2 \mathcal{P} = -\rho \frac{\partial}{\partial x_j} v_i \frac{\partial}{\partial x_i} v_j = -\rho \frac{\partial v_i}{\partial x_j} \frac{\partial v_j}{\partial x_i}, \quad (6.115)$$

since  $\partial v_j / \partial x_j = \nabla \cdot \mathbf{v} = 0$ . This equation can be used to compute the pressure if the velocity is determined, since it simply acts like a source term. Taking the curl of the Navier-Stokes equation, and recalling that  $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$  we find

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} - \nabla \times (\mathbf{v} \times \boldsymbol{\Omega}) = \nu \nabla^2 \boldsymbol{\Omega}, \quad (6.116)$$

where the algebra to arrive at the terms on the LHS was consider earlier in our discussion of the Euler equation, and the new pieces is the term on the RHS.

In the presence of viscosity the boundary conditions change from what we had previously. Molecular forces between the viscous fluid & any surface mean that the fluid adheres to the surface, and hence that the velocity of the fluid and surface must be the same for both the tangential and longitudinal components. Therefore the boundary condition for a moving wall is  $\mathbf{v} = \mathbf{v}_{\text{wall}}$ , which also covers the case  $\mathbf{v} = 0$  for a wall at rest.

Another important concept is the force exerted by the fluid on a surface. This has a contribution both from the pressure as well as from the friction. The force per unit area  $\mathcal{F}_i$  is given by

$$\mathcal{F}_i = -n_j(\mathcal{P}\delta_{ji} - \sigma_{ji}) = -\mathcal{P}n_i + \sigma_{ij}n_j, \quad (6.117)$$

where  $\hat{n}$  is the normal vector pointing out of the surface, and the first term is the pressure acting along this normal vector, while the second is the friction that has tangential components.

Starting with the Navier-Stokes equation we can also derive a modified form for energy conservation. Rather than carrying out this derivation explicitly, we will just examine the final result in integral form, which is a bit more intuitive:

$$\frac{\partial}{\partial t} \int_V \left( \frac{1}{2} \rho \mathbf{v}^2 \right) dV = - \oint_{\partial V} \left( \rho \left( \frac{\mathbf{v}^2}{2} + \frac{\mathcal{P}}{\rho} \right) v_i - v_j \sigma_{ji} \right) dS_i - \int_V \sigma_{ij} \frac{\partial v_i}{\partial x_j} dV. \quad (6.118)$$

Here the term on the LHS is the change of the kinetic energy in the volume  $V$  with time. The first integral on the RHS is the energy flux through the closed bounding surface  $\partial V$ , and the second integral is the decrease in energy that is caused by dissipation. To see this even more clearly we can consider integrating over the whole fluid with  $\mathbf{v} = 0$  at  $\infty$  (or on  $\partial V$ ). This removes the flux term and leaves

$$\frac{\partial E}{\partial t} = - \int_V \eta \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \frac{\partial v_i}{\partial x_j} dV = -\frac{\eta}{2} \int_V \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)^2 dV < 0. \quad (6.119)$$

where we can check the second equality by squaring and manipulating the summed over dummy indices  $i$  and  $j$ . Thus we see that friction causes energy to dissipate just as we would expect (and this also justifies our sign choice of  $\eta > 0$ ).

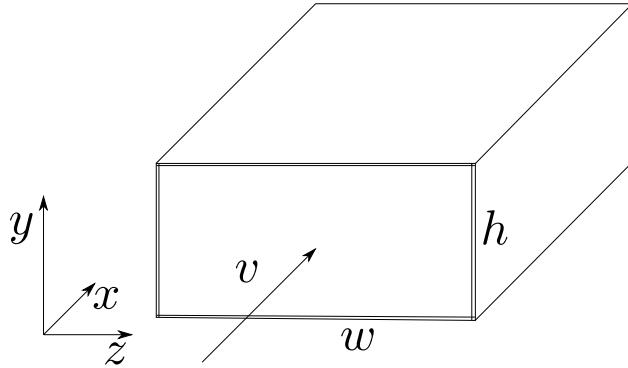
Entropy conservation is modified at temperature  $T$  to  $\rho T \dot{S} = \sigma_{ij} \frac{\partial v_i}{\partial x_j}$ , where the left-hand side of the equation is the heat gain per unit volume, and the right-hand side of the equation is the energy dissipated by viscosity. (If we allow thermal conduction (from temperature gradients) in the fluid, then there is another term on the right-hand side that appears as  $\nabla \cdot (\kappa \nabla T)$  for conductivity  $\kappa$ .)

## 6.7 Viscous Flows in Pipes and Reynolds Number

We start our investigation of fluid flow with viscosity, by studying steady flows in pipes, typically caused by a pressure gradient. This type of steady viscous flow is also called Poiseuille flow, after Jean Poiseuille who first studied them experimentally. We take the flow to be incompressible,  $\nabla \cdot \mathbf{v} = 0$ , and hence the continuity equation implies that  $\rho$  is constant (just as it did in the ideal fluid case). Thus the Navier-Stokes equation for such flows reduces to

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla \mathcal{P}}{\rho} + \nu \nabla^2 \mathbf{v} \quad (6.120)$$

**Example** Lets start by considering flow in a long rectangular duct, aligned with the  $x$ -axis, with height  $h$  and width  $w$ . We also take it be a thin duct with  $h \ll w$ , and with a pressure gradient along the pipe in the  $x$ -direction.



This can be approximated by infinite parallel plates, taking  $w \rightarrow \infty$  and holding  $h$  fixed. Here

$$\mathbf{v} = v_x(y) \hat{x}, \quad (6.121)$$

since there is no dependence on  $x$  or  $z$  by the translational symmetry in the plane, and we equally well can not develop velocity components  $v_y$  or  $v_z$  due to this symmetry. This implies that the term  $\mathbf{v} \cdot \nabla \mathbf{v} = v_x \frac{\partial \mathbf{v}}{\partial x} = 0$ . Taking the inner product of Eq.(6.120) with  $\hat{y}$  and  $\hat{z}$  then removes all the terms that depends on the velocity (which is only in the  $\hat{x}$  direction), giving

$$\frac{\partial \mathcal{P}}{\partial y} = \frac{\partial \mathcal{P}}{\partial z} = 0. \quad (6.122)$$

Thus the pressure  $\mathcal{P} = \mathcal{P}(x)$  and can have a gradient only in the  $\hat{x}$  direction. Taking the inner product of Eq.(6.120) with  $\hat{x}$  gives

$$\frac{\partial \mathcal{P}}{\partial x} = \rho \nu \nabla^2 v_x = \eta \frac{\partial^2 v_x}{\partial y^2} = k, \quad (6.123)$$

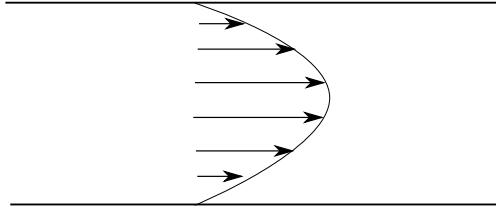
where we have introduced a constant  $k$ . Since  $\frac{\partial P}{\partial x}$  only depends on  $x$ , while  $\eta \frac{\partial^2 v_x}{\partial y^2}$  only depends on  $y$ , they must both be equal to a constant. Let us say  $\frac{\partial P}{\partial x} = k < 0$ , so that the pressure drops as we move in the  $\hat{x}$  direction. (In the Navier-Stokes equation, this pressure drop balances the viscous stress term.) Then integrating the equation for  $v_x(y)$  gives

$$v_x(y) = \frac{k}{2\eta}y^2 + ay + b, \quad (6.124)$$

where we have introduced two integration constants  $a$  and  $b$ . To solve for  $a$  and  $b$  we impose the boundary conditions that the velocity must vanish at  $y = 0$  and  $y = h$ , giving

$$v_x(y) = \frac{(-k)}{2\eta}y(h - y). \quad (6.125)$$

Recalling that  $k < 0$  we see that  $v_x(y) > 0$ , flowing down the pipe. (If we had reversed  $k$  the flow would be in the opposite direction.) The velocity field we have derived flows in the pipe with a parabolic profile with its maximum in the center:

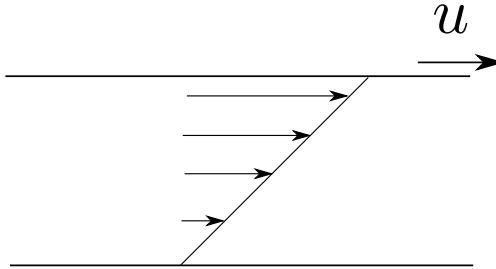


Lets also calculate the friction force per unit area that the fluid exerts on the pipe wall. The bottom plate at  $y = 0$  has a unit vector  $\hat{n} = \hat{y}$ , so from Eq. (6.117) the force along  $\hat{x}$  is

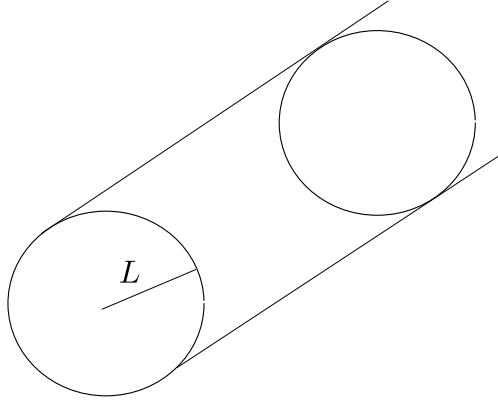
$$\mathcal{F}_x = \sigma_{yx} = \eta \frac{\partial v_x}{\partial y} \Big|_{y=0} = -\frac{hk}{2} > 0. \quad (6.126)$$

Intuitively this is the direction we expect, the fluid tries to drag the pipe along with it.

**Example** Lets now consider the same example of fluid between infinite parallel plates, but now with no pressure gradient. Instead we take the top plate to move with velocity  $\mathbf{u} = u\hat{x}$ . Here  $P = P_0$  is constant and uniform with  $k = 0$ , so the second derivative of  $v_x(y)$  is zero,  $\nabla^2 v_x = 0$ , and the solution for  $v_x(y)$  can at most be linear. The solution for this case is  $v_x(y) = \frac{uy}{h}$ , which satisfies the boundary conditions  $v_x(0) = 0$  and  $v_x(h) = u$ . Thus the fluids velocity field is linear for this case:



**Example** Next consider a long cylindrical pipe of radius  $L$ , oriented along  $\hat{x}$ , again with a pressure gradient along the pipe. We will approximate the pipe as being infinitely long so there is a translational symmetry along  $x$ .



Due to the translational symmetry we know that  $\mathbf{v} = \mathbf{v}(y, z)$ . To fully exploit the consequences of the symmetry it is useful to use cylindrical coordinates  $(x, r, \theta)$  so we can also easily impose the rotational symmetry about  $\hat{x}$  to conclude  $\mathbf{v} = \mathbf{v}(r)$ . The fact that there is  $r$  dependence makes sense since we know that  $\mathbf{v}$  must vanish at the edge of the pipe,  $r = L$ , but we do not want it to vanish everywhere. Continuity and symmetry also imply that the velocity is only in the  $\hat{x}$  direction, so in fact

$$\mathbf{v} = v_x(r)\hat{x}. \quad (6.127)$$

For example, consider an annulus shaped closed surface formed by the region between two cylinders concentric with the pipe. The flow into and out of this surface must be balanced by continuity. The flow on the ends of the surface automatically balance each other since  $\mathbf{v}$  is independent of  $x$ . But since  $\mathbf{v}$  is  $r$  dependent, the only way the flow through the circular sides can balance each other is if there is no flow in the  $\hat{r}$  direction.

With this setup we can again confirm that  $(\mathbf{v} \cdot \nabla)\mathbf{v} = v_x(r)\frac{\partial}{\partial x}v_x(r)\hat{x} = 0$ , leaving  $\nabla P = \eta\nabla^2\mathbf{v}$  from the Navier-Stokes equation. Taking the inner product with  $\hat{y}$  and  $\hat{z}$  we see that  $\frac{\partial P}{\partial y} = \frac{\partial P}{\partial z} = 0$ , so  $P = P(x)$ . Taking the inner product with  $\hat{x}$  gives

$$\frac{\partial P(x)}{\partial x} = \eta\nabla^2v_x(r) = k \quad (6.128)$$

where since  $\frac{\partial P}{\partial x}$  depends only on  $x$  while  $\eta\nabla^2v_x(r)$  is independent of  $x$ , the two must be equal to a constant  $k$ . Again we choose  $k < 0$  to have a pressure gradient that pushes the fluid down the pipe in the  $\hat{x}$  direction ( $k > 0$  would simply reverse the flow). For the velocity this gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_x}{\partial r}\right) = \frac{k}{\eta}, \quad (6.129)$$

and integrating this gives

$$v_x(r) = \frac{k}{4\eta} r^2 + a \ln\left(\frac{r}{r_0}\right) + b \quad (6.130)$$

for some constants  $a$  and  $b$ . (The constant  $r_0$  is introduced to make the equation dimensionfully correct, but is not independent, since any change to  $r_0$  can be compensated by a change to  $b$ ). Since  $v_x(r)$  has to be finite at  $r = 0$  we must have  $a = 0$  (if the geometry excluded the region at the middle). The condition  $v_x = 0$  at  $r = L$  fixes  $b$  so that

$$v_x(r) = \frac{(-k)}{4\eta} (L^2 - r^2) . \quad (6.131)$$

Lets calculate the discharge rate of fluid in such a pipe, as mass per unit time. This is given by

$$\text{discharge rate} = \rho \int_0^L 2\pi r v_x(r) dr = -\frac{\pi k L^4}{8\nu} > 0 . \quad (6.132)$$

Note that this rate is quite sensitive to the radius, being proportional to  $L^4$ . This is why you don't want narrow blood vessels.

### Reynolds Number and Similarity

How do we quantify the importance of viscosity? Let us consider a flow and identify a characteristic length  $L$  (examples of which might be the radius of a pipe or the size of a sphere inserted in a flow) and a characteristic speed  $u$  (examples of which might be the fluid velocity in the middle of a pipe or the fluid velocity upstream from flow around a sphere). The quantity

$$R = \frac{uL}{\nu} = \frac{\rho u L}{\eta} \quad (6.133)$$

is the dimensionless *Reynolds number*. A large  $R$  means viscosity is not as important compared to the characteristic  $u$  and  $L$  of the system, while small  $R$  means that viscosity plays an important role. It is important to note that viscosity is always important near a surface where it modifies the boundary condition on  $\mathbf{v}$ . (The above examples of Poiseuille flow have  $R \lesssim 2000$ .)

The introduction of  $R$  leads nicely into a discussion of the utility of exploiting dimensional analysis in fluid dynamics problems. This is best explained by way of examples.

**Example** Consider a steady incompressible flow in a system that is characterized by a single length scale  $L$  and initial velocity  $u$ . What could the possible functional form be for the velocity and pressure in such a system? Considering the coordinates and fluid field variables we can only have

$$\frac{\mathbf{v}}{u} = \mathbf{h}\left(\frac{\mathbf{r}}{L}, R\right), \quad \frac{\mathcal{P}}{\rho u^2} = g\left(\frac{\mathbf{r}}{L}, R\right) . \quad (6.134)$$

Here  $\mathbf{h}$  and  $g$  are a dimensionless vector and scalar function respectively. The ratio  $\mathbf{r}/L$  is dimensionless, as is  $R$ , and the dimensions of  $\mathbf{u}$  and  $\mathcal{P}$  are compensated by  $u$  and  $\rho u^2$  respectively. Note that if we consider flows that have the same  $R$ , then the properties of those flows are related by changes to the scales associated with  $\mathbf{v}$ ,  $\mathbf{r}$ , or  $\mathcal{P}$ . Such flows are called *similar*.

**Example** Consider a viscous flow past a sphere of radius  $a$  with initial velocity given by  $\lim_{x \rightarrow -\infty} \mathbf{v} = u\hat{x}$ . Here the Reynolds number is  $R = \frac{ua}{\nu}$ . If we double  $\nu$  and double  $u$  then  $R$  is unchanged. Due to the relations in Eq. (6.134) we thus can predict that we will have the exact same form for the solutions with  $\mathbf{v}$  twice as large as before, and  $\mathcal{P}$  being four times as large as before.

Note that in general other dimensionless ratios, like the ratio of two length scales, or the Mach number  $M = \frac{u}{c_s}$  could also appear. (For  $M \ll 1$  we treat the fluid as incompressible and neglect  $M$  for the dimensional analysis.) To determine how many independent dimensionless ratios can be formed for the analysis of a general dimensional analysis problem, one counts the number of variables and subtracts the number of unrelated types of dimensions that appear in these variables. For most fluid problems this will mean subtracting three for kg, meters, and seconds.

## 6.8 Viscous Flow Past a Sphere (Stokes Flow)

Lets consider a steady, incompressible, viscous fluid with small  $R$  flowing past a sphere of radius  $a$ . The fluid has velocity  $\mathbf{u}$  when it is far away from the sphere. This is the same problem we treated for an ideal fluid with potential flow, but now we want to consider the impact of viscosity, and resolve the puzzle we found in our solution for an ideal fluid. To make the problem solvable we work in the limit  $R \ll 1$ .

Here the Navier-Stokes equation is

$$\mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla \mathcal{P}}{\rho} + \frac{\eta}{\rho} \nabla^2 \mathbf{v}. \quad (6.135)$$

Lets determine the relative importance of the two velocity terms in the  $R \ll 1$  limit. Using  $a$  to characterize spatial variations, and  $u$  to characterize velocity we find

$$\mathbf{v} \cdot \nabla \mathbf{v} = \mathcal{O}\left(\frac{u^2}{a}\right), \quad \frac{\eta}{\rho} \nabla^2 \mathbf{v} = \mathcal{O}\left(\frac{\eta u}{\rho a^2}\right) = \mathcal{O}\left(\frac{u^2}{a R}\right). \quad (6.136)$$

Therefore the viscosity term, which is enhanced by a factor of  $1/R$ , dominates. Neglecting the  $\mathbf{v} \cdot \nabla \mathbf{v}$  term, the Navier-Stokes equation reduces to

$$\nabla \mathcal{P} = \eta \nabla^2 \mathbf{v}. \quad (6.137)$$

Note that when written in terms of the shear viscosity  $\eta$ , that the density  $\rho$  has dropped out of this equation, and hence the constant  $\rho$  will not play a part in the solution. Taking the

divergence of this equation we find  $\nabla^2 \mathcal{P} = \eta \nabla^2 \nabla \cdot \mathbf{v} = 0$ , so the pressure satisfies Laplace's equation. Using dimensional analysis for the pressure we expect a solution of the form

$$\mathcal{P} = \frac{\eta \mathbf{u} \cdot \mathbf{r}}{a^2} g\left(\frac{\mathbf{r}}{a}\right), \quad (6.138)$$

where the dimensional analysis requirement of having only a single factor of the velocity  $\mathbf{u}$  and a scalar result, leads to including the factor of  $\mathbf{u} \cdot \mathbf{r}$ . Note that we have not included  $R$  as a possible argument since we are expanding with  $R \ll 1$ . Due to the fact that  $\mathcal{P}$  must satisfy Laplace's equation and is proportional to  $\mathbf{u} \cdot \mathbf{r}$ , we can immediately recognize the dipole solution (which we met earlier for the potential  $\phi$  in the ideal fluid case):

$$\mathcal{P} = \mathcal{P}_0 + k \frac{\eta a \mathbf{u} \cdot \mathbf{r}}{r^3}. \quad (6.139)$$

Here we have included an overall constant pressure  $\mathcal{P}_0$  to satisfy any boundary condition on the pressure at  $r \rightarrow \infty$ , and a dimensionless constant  $k$  which is still to be determined. Note that with  $\mathbf{u} \cdot \mathbf{r} = ur \cos \theta$  that the pressure is no longer symmetric on the front and back of the sphere (with respect to the incoming flow), thus qualitatively differing from the ideal fluid case.

Next we note that the vorticity of the fluid satisfies

$$\begin{aligned} \nabla \times \boldsymbol{\Omega} &= \nabla \times (\nabla \times \mathbf{v}) = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2 \mathbf{v} = -\nabla^2 \mathbf{v} = -\frac{1}{\eta} \nabla \mathcal{P} \\ &= -\frac{ka}{r^3} \left( \mathbf{u} - \frac{3(\mathbf{u} \cdot \mathbf{r})\mathbf{r}}{r^2} \right). \end{aligned} \quad (6.140)$$

Enforcing that  $\boldsymbol{\Omega}$  vanishes as  $r \rightarrow \infty$ , the solution for this equation is

$$\boldsymbol{\Omega} = ka \frac{\mathbf{u} \times \mathbf{r}}{r^3}. \quad (6.141)$$

Thus we see that unlike the ideal fluid case, there is now a non-zero vorticity in the fluid here.

Next we turn to determining the velocity, which can only depend on  $\mathbf{v} = \mathbf{v}(\mathbf{r}, \mathbf{u}, a)$ . Again we do not include  $R$  as an argument since we are expanding for  $R \ll 1$ . By dimensional analysis the velocity must be linear in  $\mathbf{u}$  so the most general possible solution takes the form

$$\mathbf{v} = \mathbf{u} f\left(\frac{r}{a}\right) + \frac{\mathbf{r}(\mathbf{u} \cdot \mathbf{r})}{a^2} g\left(\frac{r}{a}\right), \quad (6.142)$$

where the functions  $f$  and  $g$  are dimensionless. The gradient of one of these dimensionless functions gives  $\nabla f(r/a) = (\mathbf{r}/(ar))f'(r/a)$ . Computing the divergence of the velocity with this form we have

$$0 = \nabla \cdot \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{r}}{ar} f'\left(\frac{r}{a}\right) + \frac{(\mathbf{u} \cdot \mathbf{r})r}{a^3} g'\left(\frac{r}{a}\right) + \frac{4(\mathbf{u} \cdot \mathbf{r})}{a^2} g\left(\frac{r}{a}\right). \quad (6.143)$$

Therefore we find that the scalar functions must satisfy

$$f'\left(\frac{r}{a}\right) = -\frac{r^2}{a^2}g'\left(\frac{r}{a}\right) - \frac{4r}{a}g\left(\frac{r}{a}\right). \quad (6.144)$$

Next we equate the Laplacian of the velocity and gradient of the pressure, which can be simplified to give

$$\begin{aligned} \nabla^2 \mathbf{v} &= \left(\nabla^2 f + \frac{2g}{a^2}\right) \mathbf{u} + \left(\nabla^2 g + \frac{4g'}{ar}\right) \frac{(\mathbf{u} \cdot \mathbf{r}) \mathbf{r}}{a^2} \\ &= \frac{1}{\eta} \nabla \mathcal{P} = \frac{ka}{r^3} \mathbf{u} + \left(-\frac{3ka^3}{r^5}\right) \frac{(\mathbf{u} \cdot \mathbf{r}) \mathbf{r}}{a^2}. \end{aligned} \quad (6.145)$$

Note that here  $\nabla^2 g(r/a) = (1/r^2)(d/dr)r^2(d/dr)g(r/a)$ . Equating the coefficients of the two structures we find

$$\begin{aligned} \nabla^2 g + \frac{4g'}{ar} &= \frac{g''}{a^2} + \frac{6g'}{ar} = -\frac{3ka^3}{r^5}, \\ \frac{f''}{a^2} + \frac{2f'}{ar} + \frac{2g}{a^2} &= \frac{ka}{r^3}. \end{aligned} \quad (6.146)$$

To solve the equation for  $g$  we try a polynomial solution of the form  $g(x) = C_n x^n$  giving

$$C_n [n(n-1) + 6n] x^{n-2} = -3kx^{-5}. \quad (6.147)$$

Here  $n = -3$  is a particular solution to the full inhomogeneous equation with  $C_{-3} = k/2$ . Also  $n = 0$  and  $n = -5$  are homogeneous solutions where the LHS vanishes, and the corresponding coefficients  $C_0$  and  $C_5$  must be fixed by boundary conditions. Looking back at our starting point in Eq. (6.142) we see that the full set of boundary conditions are

$$\begin{aligned} \mathbf{v} = \mathbf{u} \text{ at } r = \infty : \quad &\lim_{r \rightarrow \infty} r^2 g(r/a) \rightarrow 0, \quad \lim_{r \rightarrow \infty} f(r/a) \rightarrow 1, \\ \mathbf{v} = 0 \text{ at } r = a : \quad &g(1) = 0, \quad f(1) = 0, \end{aligned} \quad (6.148)$$

This fixes  $C_0 = 0$  and  $C_{-5} = -k/2$  so that

$$g(r/a) = \frac{ka^3}{2r^3} \left(1 - \frac{a^2}{r^2}\right). \quad (6.149)$$

Using Eq. (6.144) and integrating once we find that

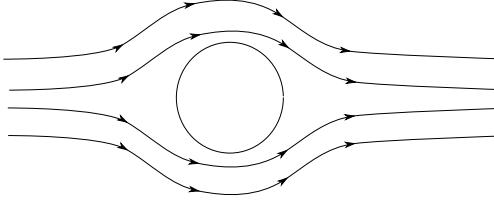
$$f(r/a) = 1 + \frac{ka}{2r} + \frac{ka^3}{6r^3}, \quad (6.150)$$

where we have set the integration constant to one to satisfy the boundary condition at  $r = \infty$ . The final boundary condition,  $f(1) = 1$  then requires us to take the constant  $k = -3/2$ .

Note that this fixes the constant  $k$  that appeared in the vorticity  $\boldsymbol{\Omega}$  and in the pressure  $\mathcal{P}$ . All together we have that the final solution for the velocity is

$$\mathbf{v} = \mathbf{u} \left( 1 - \frac{3a}{4r} - \frac{a^3}{4r^3} \right) - \frac{3a^3}{4r^3} \frac{\mathbf{r}(\mathbf{u} \cdot \mathbf{r})}{a^2} \left( 1 - \frac{a^2}{r^2} \right). \quad (6.151)$$

The flow looks like:



Next we turn to determining the drag force on the sphere. In general the drag force on an object in the direction  $j$  is given by an integral of the force per unit area over the surface,

$$\mathbf{F}_{Dj} = \int_{\partial V} d\mathcal{S}_i (\mathcal{P}\delta_{ij} - \sigma_{ij}). \quad (6.152)$$

Lets take the inflowing velocity to be in the  $\hat{x}$  direction,  $\mathbf{v}(r \rightarrow \infty) = u\hat{x}$ . Then  $\mathbf{F}_D \cdot \hat{x}$  will be the drag force on our sphere in the direction of the bulk fluid flow. With spherical coordinates  $(r, \theta, \phi)$  where  $\theta$  is the polar angle, we have  $d\mathbf{S} = \hat{r}a^2 d\cos\theta d\phi$  with  $r = a$ , as well as  $\hat{r} \cdot \hat{x} = \cos\theta$  and  $\hat{\theta} \cdot \hat{x} = -\sin\theta$ . Thus

$$\hat{x} \cdot \mathbf{F}_D = a^2 \int d\cos\theta d\phi \left( -\mathcal{P} \cos\theta + \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta \right). \quad (6.153)$$

Computing the needed components on the sphere  $r = a$  we find

$$\begin{aligned} \sigma_{rr} &= 2\eta \frac{\partial v_r}{\partial r} \Big|_{r=a} = 0, & \sigma_{r\theta} &= \eta \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right)_{r=a} = -\left( \frac{3\eta}{2a} \right) u \sin\theta, \\ \mathcal{P} &= \mathcal{P}_0 - \left( \frac{3\eta}{2a} \right) u \cos\theta. \end{aligned} \quad (6.154)$$

Thus the drag force on the sphere is

$$\hat{x} \cdot \mathbf{F}_D = \left( \frac{3\eta u}{2a} \right) \int d\cos\theta d\phi a^2 = (6\pi\eta a)u, \quad (6.155)$$

which is the famous Stoke's formula for the viscous (friction) drag force on a sphere. (Note that we could have obtained the factor of  $\eta au$  by dimensional analysis.)

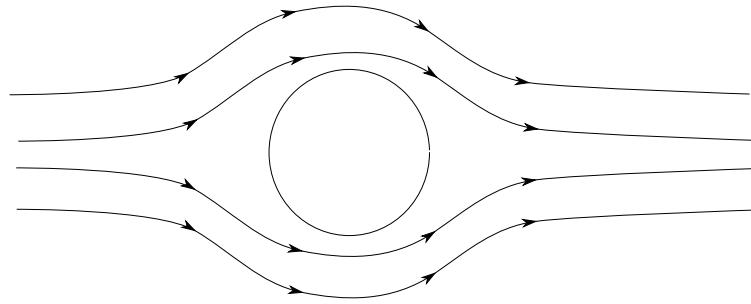
In addition to drag forces like this, that point in the direction of the fluid flow, objects may also experience *lift forces* that are tangential to direction of the fluid flow. Such forces

occur for wing-shaped objects and are important for many physical phenomena, including lift on airplanes.

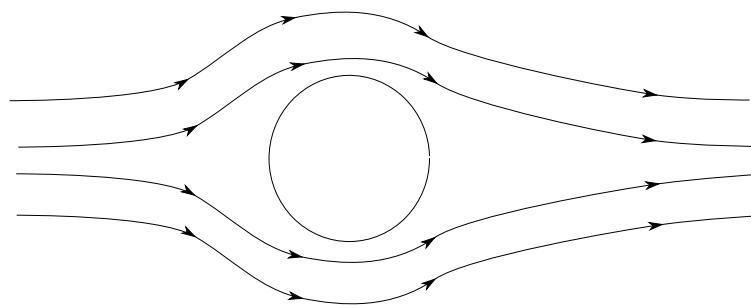
### Dynamic Vortices and Turbulence

For our flow about the fixed sphere, let's consider what happens as we increase  $R$ . From our analysis above it is clear that at some point the non-linear  $\mathbf{v} \cdot \nabla \mathbf{v}$  term we dropped will become important. The  $\partial \mathbf{v} / \partial t$  will also become important, with flows that are more dynamical, changing with time. Let's consider how the flow appears for various values of  $R$ :

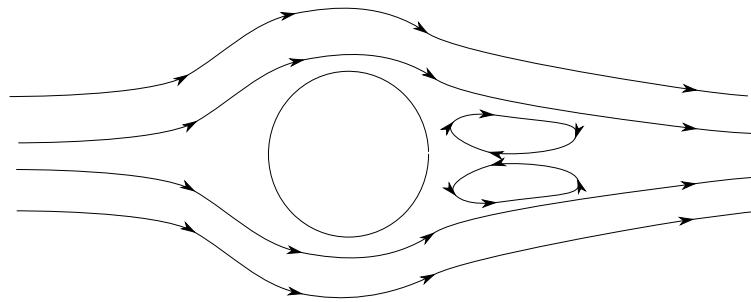
- For  $R \ll 1$ , the flow is symmetric and is (somewhat counter intuitively) qualitatively like the case of  $\eta \rightarrow 0$ . This is also called “Stokes flow” or a “laminar flow”.



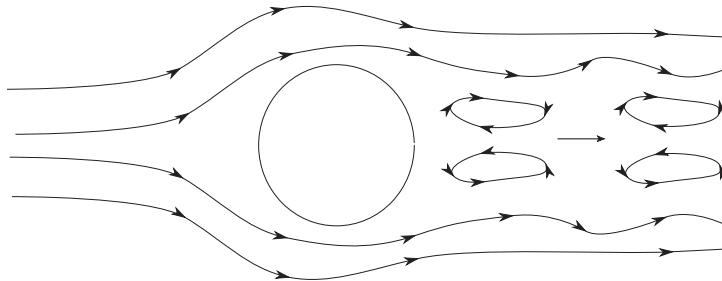
- For  $R \approx 1$ , the flow is still like Stokes flow, but the stream lines are no longer as symmetric, with a more clear wake developing behind the sphere.



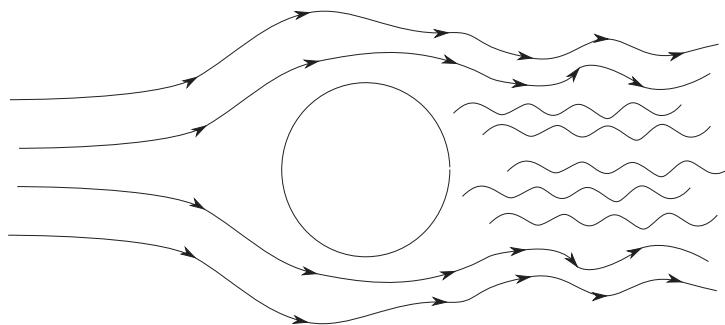
- For  $10 \lesssim R \lesssim 100$ , detached vortices called *eddies* form behind the sphere, though the flow is still steady. Note that directly behind the sphere between the vortices that the fluid is now flowing in the opposite direction to the asymptotic inflow  $\mathbf{u}$ . As  $R$  increases, the flow becomes less steady, with the time dependence emerging through oscillations of the vortices.



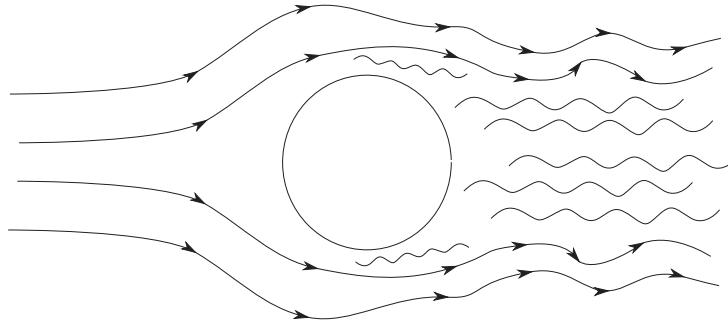
- For  $150 \lesssim R \lesssim 1000$ , vortices start to be cyclically shed and drift downstream in a *wake* behind the sphere. This time dependent solution appears like it has interaction between the eddies, where one pair pushes the next downstream.



- For  $10^3 \lesssim R \lesssim 2 \times 10^5$ , the wake becomes highly irregular, exhibiting a phenomena known as turbulence which we will discuss in more detail below. Here there are unsteady, interacting vortices at all length scales.



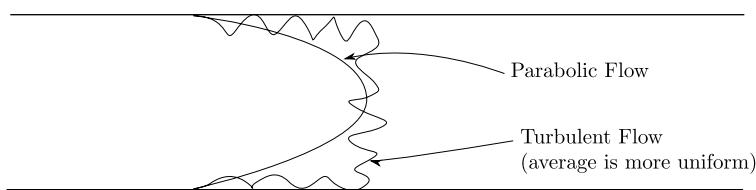
- For  $R \gtrsim 2 \times 10^5$ , the turbulent wake narrows and the boundary layer around the sphere is no longer laminar, also becoming turbulent.



*Turbulence* is characterized by a flow that: i) is highly irregular (technically chaotic) in  $\mathbf{x}$  and/or  $t$ , ii) has nonzero vorticity, iii) has much greater dissipation of energy compared to more uniform laminar viscous flows, and iv) has eddies and vortices forming over many length scales with energy that in a three dimensional flow cascades from the largest eddies down to the smallest eddies where it dissipates into heat due to viscous friction. Turbulent mixing is a very effective mechanism of transport for energy, momentum, temperature, and so on. Examples of turbulence include many familiar phenomena: the circulation of air in the atmosphere, the water flow from a faucet which becomes white at a certain point when the flow rate is increased, water in rapids, dust kicked up by wind, water beside a ship moving in an otherwise smooth lake, clear air turbulence causing a drop in lift for airplanes, and so on.

In the last value of  $R$  discussed for flow around a sphere, a turbulent boundary layer appeared. This causes an abrupt drop in drag in the flow over objects, and is a very useful phenomena. In particular, by introducing imperfections we can cause this turbulent boundary layer to form at smaller values  $R$ , meaning smaller velocities. This is why golf balls have dimples and baseballs and tennis balls have visible seams.

We also get turbulence in flow through pipes at large  $R$ . The viscous flow in pipes we previously considered were laminar flow at smaller  $R$  values and had velocity distributions that were parabolic, while in contrast a turbulent flow will be non-uniform at small scales, but when averaged causes a more uniform flow down the pipe at larger length scales.



In general the chaotic and irregular nature of turbulence makes it difficult to treat with analytic methods, and a complete description of turbulence remains an unsolved problem.

# Chapter 7

## Chaos and Non-Linear Dynamics

By a deterministic systems of equations, we mean equations that given some initial conditions have a unique solution, like those of classical mechanics. In a deterministic system we will define *chaos* as aperiodic long-term behavior that exhibits sensitive dependence on initial conditions.

- Here “aperiodic behavior” means that phase space trajectories do not converge to a point or a periodic orbit, they are irregular and undergo topological mixing (discussed below).
- By “sensitive to initial conditions” we mean that trajectories that start nearby initially, separate exponentially fast. Defining  $\delta(t)$  as the difference between points on two such trajectories at time  $t$ , then this means that  $|\delta(t)| \propto \delta_0 e^{\lambda t}$  for some  $\lambda > 0$ , as depicted in Fig. 7.1.

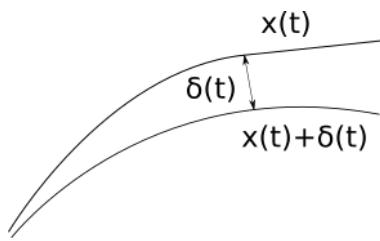


Figure 7.1: The difference in initial condition leads to different orbits. Their difference is given by  $\delta(t)$ , which grows exponentially with time.

This means that even though they are deterministic, chaotic systems are most often not predictable. In particular, there will always be a small difference  $\delta_0$  between the true and measured initial conditions for the system (from statistical or systematic measurement error), which grows exponentially to yield inaccurate predictions for predictions far enough in the future.

The sensitivity to initial conditions is important to chaos but does not itself differentiate from simple exponential growth, so the aperiodic behavior is also important. In the definition of this somewhat undescriptive phrase we include that the system should undergo *Topological Mixing*. This means that any points starting in a region (open set) of the phase space will evolve to overlap any other region of the phase space, so chaotic systems tend to explore a larger variety of regions of the phase space.<sup>1</sup>

## 7.1 Introduction to Chaos

We will now explore some properties of non-linear dynamical systems, including methods to characterize solutions, and the study of solutions with chaotic behavior.

### 7.1.1 Evolution of the system by first order differential equations

The dynamical system can be defined by a system of first order differential equations:

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, \dots, x_n) \\ \dot{x}_2 &= f_2(x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n)\end{aligned}\tag{7.1}$$

where the quantities  $x_i$ , for  $i \in \{1, \dots, n\}$ , are any variables that evolve in time, which could be coordinates, velocities, momenta, or other quantities. For our applications in this chapter we will often assume that the  $x_i$  equations are also chosen to be dimensionless, and the procedure for this type of conversion will be discussed further below.

**Example:** the Hamilton equations of motion are 1<sup>st</sup> order equations in the canonical variables, so they are an example of equation of the form in Eq. (7.1) with an even number of  $x_i$  variables.

#### Deterministic evolution from the existence and uniqueness theorem

Assume that we have a set of differential equations in the form in Eq. (7.1), which we can write in a shorthand as

$$\dot{\vec{x}} = \vec{f}(\vec{x}),\tag{7.2}$$

and that  $f_j$  and  $\frac{\partial f_j}{\partial x_i}$  (for  $i, j \in \{1, \dots, n\}$ ) are continuous in a connected region  $\mathbb{D} \in \mathbb{R}^n$ . Then if we have an initial condition  $\vec{x}(t=0) = \vec{x}_0 \in \mathbb{D}$ , then the theorem states that there exists a *unique* solution  $\vec{x} = \vec{x}(t)$  on some interval  $(-\tau, \tau)$  about  $t = 0$ . Time evolution in such a system is therefore deterministic from this existence and uniqueness theorem.

---

<sup>1</sup>For a dissipative chaotic system there are further restrictions on the choice of the open sets in this definition of topological mixing since it is otherwise obvious that we could pick a region that the system will not return to.

For this chapter the damped nonlinear forced oscillator will be a good example on which to base our discussion. In the case of a pendulum with damping and a periodic driving force, its evolution is given by the equation of motion:

$$ml^2\ddot{\theta} + ml^2\gamma\dot{\theta} + mgl \sin(\theta) = A \cos(\omega_D t), \quad (7.3)$$

where  $\ell$  is the length of the pendulum,  $\theta$  is the oscillator angle,  $\gamma$  is the damping coefficient, and  $A \cos(\omega_D t)$  is the driving force with an amplitude  $A$  and constant frequency  $\omega_D$ . It is useful to turn this into a dimensionless equation. First we divide by  $mgl$  to make the third term dimensionless, defining

$$a \equiv \frac{A}{mgl}, \quad (7.4)$$

to give a dimensionless amplitude for the forcing term. This leaves

$$\frac{l}{g}\ddot{\theta} + \frac{\gamma l}{g}\dot{\theta} + \sin\theta = a \cos(\omega_D t). \quad (7.5)$$

Next to make the first term dimensionless we rescale the time derivatives so that they involve a dimensionless time  $t'$ , and change to a dimensionless frequency  $\omega'_D$  for the forcing term via

$$t' \equiv \sqrt{\frac{g}{l}}t, \quad \omega'_D \equiv \sqrt{\frac{l}{g}}\omega_D, \quad \dot{u} \equiv \frac{du}{dt} \Rightarrow \dot{u} \equiv \frac{du}{dt'} . \quad (7.6)$$

As indicated we also now let dots indicate derivatives with respect to the dimensionless time. Finally we define

$$\frac{1}{q} \equiv \sqrt{\frac{l}{g}}\gamma, \quad (7.7)$$

where  $q$  is the dimensionless quality factor for the damping term.

Dropping all the added primes for simplicity, our final differential equation is now fully dimensionless:

$$\ddot{\theta} + \frac{1}{q}\dot{\theta} + \sin(\theta) = a \cos(\omega_D t) \quad (7.8)$$

Here  $a$ ,  $q$ , and  $\omega_D$  are all dimensionless constants. We can convert this into 1<sup>st</sup> order form by defining  $\varphi \equiv \omega_D t$  to get rid of the explicit time dependence in the forcing term, and  $\dot{\theta} \equiv \omega$  to eliminate the double time derivatives. This gives the system of three equations that are in the form in Eq. (7.1) with  $\vec{x} = (\theta, \omega, \varphi)$ :

$$\begin{aligned} \dot{\theta} &= \omega, \\ \dot{\omega} &= -\frac{1}{q}\omega - \sin(\theta) + a \cos(\varphi), \\ \dot{\varphi} &= \omega_D. \end{aligned} \quad (7.9)$$

### 7.1.2 Evolution of Phase Space

#### Phase space trajectories never cross

From the uniqueness theorem, phase space trajectories never cross. To prove this, note that any point  $\vec{x}(t)$  on a trajectory could be used as an initial condition for a new trajectory. Since a point can only be part of one single trajectory, no crossings can occur.

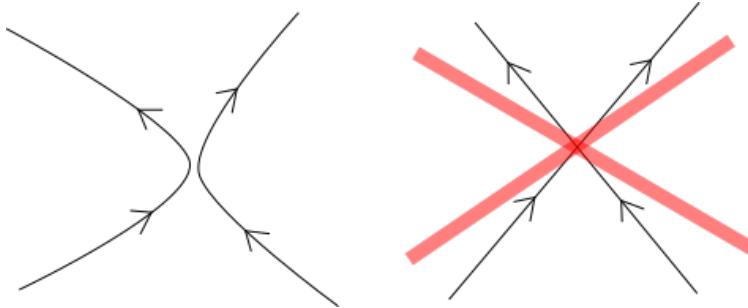


Figure 7.2: By the uniqueness theorem, no two trajectories can cross, only come arbitrarily close.

#### Evolution of phase space volume

The phase space volume is given by:

$$\mathcal{V} = \int_{\mathcal{V}} \prod_{j=1}^n dx_j \quad (7.10)$$

Recall that for Hamiltonian systems, canonical transformations do not change volume elements. If we take this transformation to be a solution for the motion (via the H-J equation), then it is clear that the motion generated by a Hamiltonian preserves the volume, so  $\dot{\mathcal{V}} = 0$ .

What happens with damping/friction (which is not in our Hamiltonian formalism)? To determine the answer we can exploit an analogy with our results for changes in volume for fluids:

$$\begin{aligned} \dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}) &\Leftrightarrow \dot{\vec{x}} = \vec{f}(\vec{x}), \\ \dot{V} = \int dV \nabla \cdot \mathbf{v} &\Rightarrow \dot{\mathcal{V}} = \int d\mathcal{V} \nabla \cdot \vec{f}. \end{aligned} \quad (7.11)$$

where in the context of a general nonlinear system,  $\nabla$  refers to derivatives with respect to  $\vec{x}$ . Thus we see that  $\nabla \cdot \vec{f}$  determines the change to a volume of our phase space variables. For this reason we define  $\nabla \cdot \vec{f} = 0$  as a *conservative system* (whether or not a general Hamiltonian exists), while  $\nabla \cdot \vec{f} < 0$  is a *dissipative system* where the phase space volume shrinks.

For our damped nonlinear driven oscillator example we have:

$$\nabla \cdot \vec{f} = \frac{\partial \omega}{\partial \theta} + \frac{\partial}{\partial \omega} \left( -\frac{1}{q} \omega - \sin(\theta) + a \cos(\varphi) \right) + \frac{\partial \omega_D}{\partial \varphi} = -\frac{1}{q} < 0, \quad (7.12)$$

as expected for a dissipative system.

For the special case of  $q \rightarrow \infty$  and  $a = 0$  (undamped and undriven system), then:

$$\dot{\theta} = \omega, \quad \text{and} \quad \dot{\omega} = -\sin(\theta). \quad (7.13)$$

The corresponding trajectories in phase space are illustrated below:

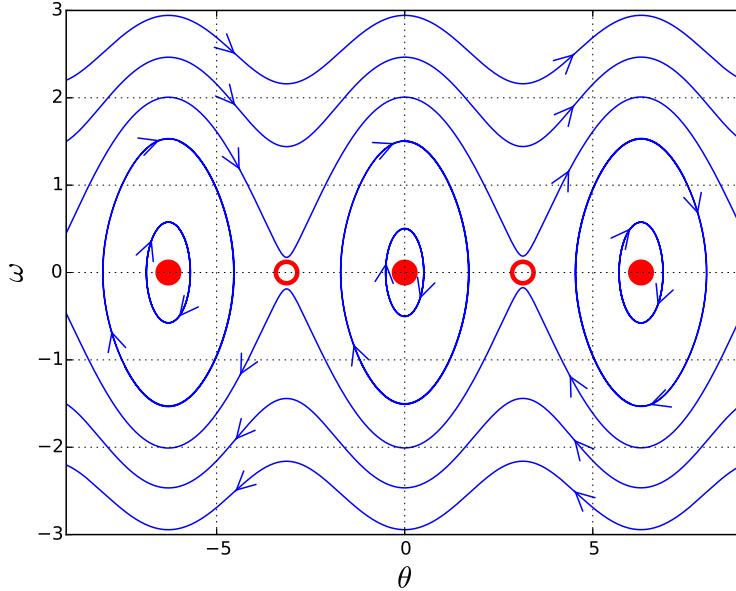


Figure 7.3: Phase space picture of the undamped, unforced oscillator. Filled circles are the stable fixed points and empty circles are the saddle points which are fixed points that are unstable in one direction and stable in another.

### 7.1.3 Fixed Points

Of particular interest in a system are its *fixed points*,  $\vec{x}^*$ , defined as the locations where

$$\vec{f}(\vec{x}^*) = 0. \quad (7.14)$$

At these points the state of the system is constant throughout time. Depending on the behavior of the trajectories nearby the fixed point they can be characterized as:

- *Stable* - nearby trajectories approach the stable point

- *Unstable* - nearby trajectories move away from the fixed point
- *Saddle Point* - in different directions trajectories can either approach or move away

For the undriven, undamped oscillator (Eq.(7.13)), the system has fixed points for  $\omega = 0$  and  $\theta = n\pi$  for any integer  $n$ . For this pendulum, the fixed point at the bottom  $\theta = 2\pi n$  is stable, while the fixed point at the top is unstable  $\theta = \pi(2n+1)$ , as shown in Fig. 7.3. Note that this fixed point at the top is not a crossing trajectory because we can only get to this point if  $E = 0$  exactly, and in that case the trajectory would stop at this fixed point. Any small perturbation knocks it off the unstable point at the top and determines which way it goes.

If there is dissipation, then all trajectories in the neighborhood of a stable fixed point converge upon it, so this region is called the *basin of attraction* and the fixed point is an *attractor*; energy dissipates as motion decays to the attractor. In our example it occurs if  $q$  is finite, and the basins of attraction in this case are diagonal strips in phase space. The result for two trajectories in phase space are shown below.

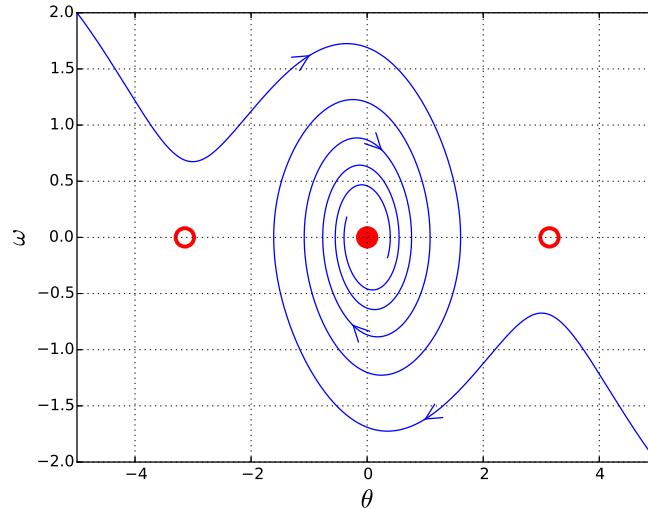


Figure 7.4: With finite damping ( $q = 5$ ) in our oscillator example the trajectories converge to the stable fixed points of the system with spiraling motion.

#### Conditions for chaotic behavior

In general, two necessary conditions for chaos are:

- The equations of motion must be nonlinear. (For linear systems we already know the solutions, which are exponential or oscillating and hence not chaotic.)

- There must be at least 3 variables, so if  $i \in \{1, \dots, n\}$ , then  $n \geq 3$ . (We will see why this is necessary later.)

In our non-linear damped oscillator example, now including a non-zero forcing term gives rise to a wider range of qualitative behaviors. In particular for certain values of  $(a, q, \omega_D)$  the system can be chaotic.

If we start instead with the linearized version of the forced damped oscillator then we have:

$$\dot{\omega} = -\frac{1}{q}\omega - \theta + a \cos(\varphi) \quad (7.15)$$

For this case the solution, which is non-chaotic, are well known and often studied in elementary courses in classical mechanics or waves. The general solutions come in three cases, underdamped ( $q > 1/2$ ), critically damped ( $q = 1/2$ ), or overdamped ( $q < 1/2$ ). For example the general underdamped solution is given by:

$$\theta(t) = Be^{-\frac{t}{2q}} \cos\left(t\sqrt{1 - \frac{1}{4q^2}} + \varphi_0\right) + \frac{a}{\omega_D \sqrt{q^{-2} + (\omega_D^{-1} - \omega_D)^2}} \cos(\omega_D t - \delta), \quad (7.16)$$

where  $\tan(\delta) = \omega_D/(q - q\omega_D^2)$ , and  $B$  and  $\varphi_0$  are constants that are determined by the initial conditions. The first term in Eq. (7.16) is the transient that decays away exponentially, whereas the second term describes the steady state forced motion (whose amplitude exhibits resonant behavior at  $\omega_D = 1$ ).

A projection of the trajectories into the 2-dimensional  $\theta$ - $\omega$  plane, as shown in Fig. 7.5 shows that they converge onto ellipses after many cycles. This does not break the uniqueness theorem since  $\varphi = \omega_D t$  is increasing, so the trajectory never crosses itself when all three variables are plotted. If restrict  $\varphi \in [0, 2\pi]$  then the trajectory converges to a closed orbit. Note that the nonlinear forcing term  $\cos(\varphi)$  is important to ensure that this closed orbit is an isolated stable endpoint for the motion.

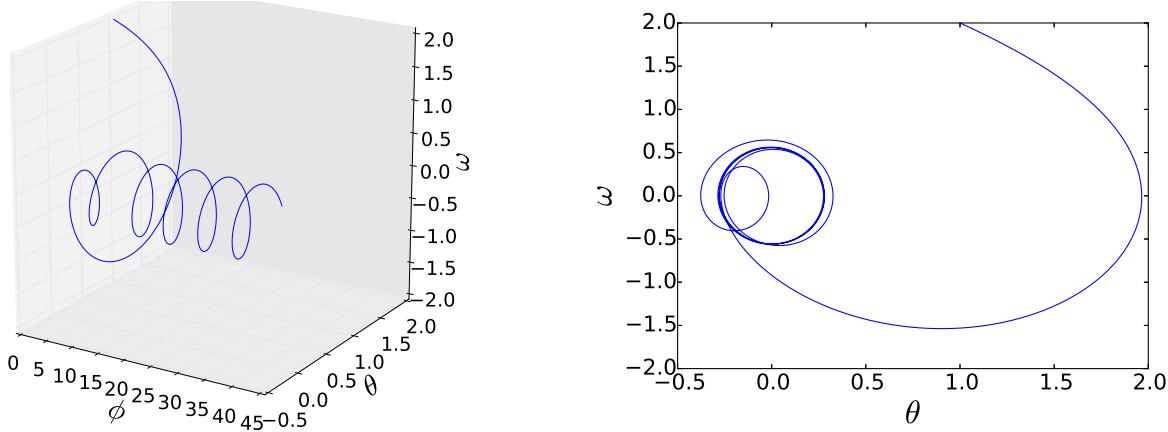


Figure 7.5: Although the  $\theta - \omega$  projection of the system shows crossings, which would seem to violate the uniqueness theorem, plotting the 3 variables we see that no crossing occurs, and uniqueness is preserved. In the projection plot we also clearly see the system evolving to a closed orbit.

An attractor that is a closed orbit rather than a single point is called a *limit cycle*. We will give a more careful definition of a limit cycle shortly.

#### 7.1.4 Picturing Trajectories in Phase Space

##### 2-dim projections

To solve the full nonlinear damped forced oscillator, described by the solution to Eq. (7.9), we use a computer. Note that we can examine chaos and sensitivity to initial conditions on a computer since the various phenomena, including the exponential growth of differences due to chosen initial conditions, occur much before the differences due to truncation errors associated with machine precision take over. In order to give a taste of what chaos looks like, we will first simply examine some of the results especially as applied to the nonlinear oscillator.

One way to see chaos is to do a projection of trajectories in the full  $n$ -dimensional space of variables to a lower dimension (usually down to 2 dimensions so we can plot results in a plane). For the nonlinear oscillator, this is typically the  $\theta - \omega$  plane where we project away  $\varphi$  (as in the right most images of Fig. 7.5). For chaotic motion this projection yields a two dimensional picture which in general gets quite messy, with the trajectory filling out a large area of the plane.

##### Poincaré Section (Poincaré Map)

To simplify things further we can use a Poincaré section (also called a Poincaré map). Here we sample the trajectory periodically in  $\varphi$  say when  $\varphi = 2\pi n$  which is the periodicity

of  $\cos \varphi$  for our example, and plot only these values  $(\theta_n, \omega_n)$  in the  $\theta$ - $\omega$  plane. The results we track are then much like what we would observe by looking at the system with a stroboscope. For creating the Poincaré section of any such system, we wait until the transients die out.

For the nonlinear oscillator, this might be at  $\varphi = 2\pi n$  for integer  $n$ , yielding a sampling frequency of exactly  $\omega_D$ , so the map is a plot of only these values  $(\theta_n, \omega_n)$ . For example, we could take  $\omega_D = \frac{2}{3}$  and  $q = 2$  while varying  $a$  as in Fig. 7.6; where we have waited for 30 cycles to ensure that the transients have died out.

In figure Fig. 7.6 we show both 2-dimensional phase portraits and Poincaré maps for various values of  $a$ . As  $a$  increases the plots show singly periodic long term behavior ( $a = 0.9$ ), to doubly periodic ( $a = 1.07$ ), to chaotic ( $a = 1.19$ ), and finally to periodic again occurring amidst neighboring chaos ( $a = 1.35$ ).

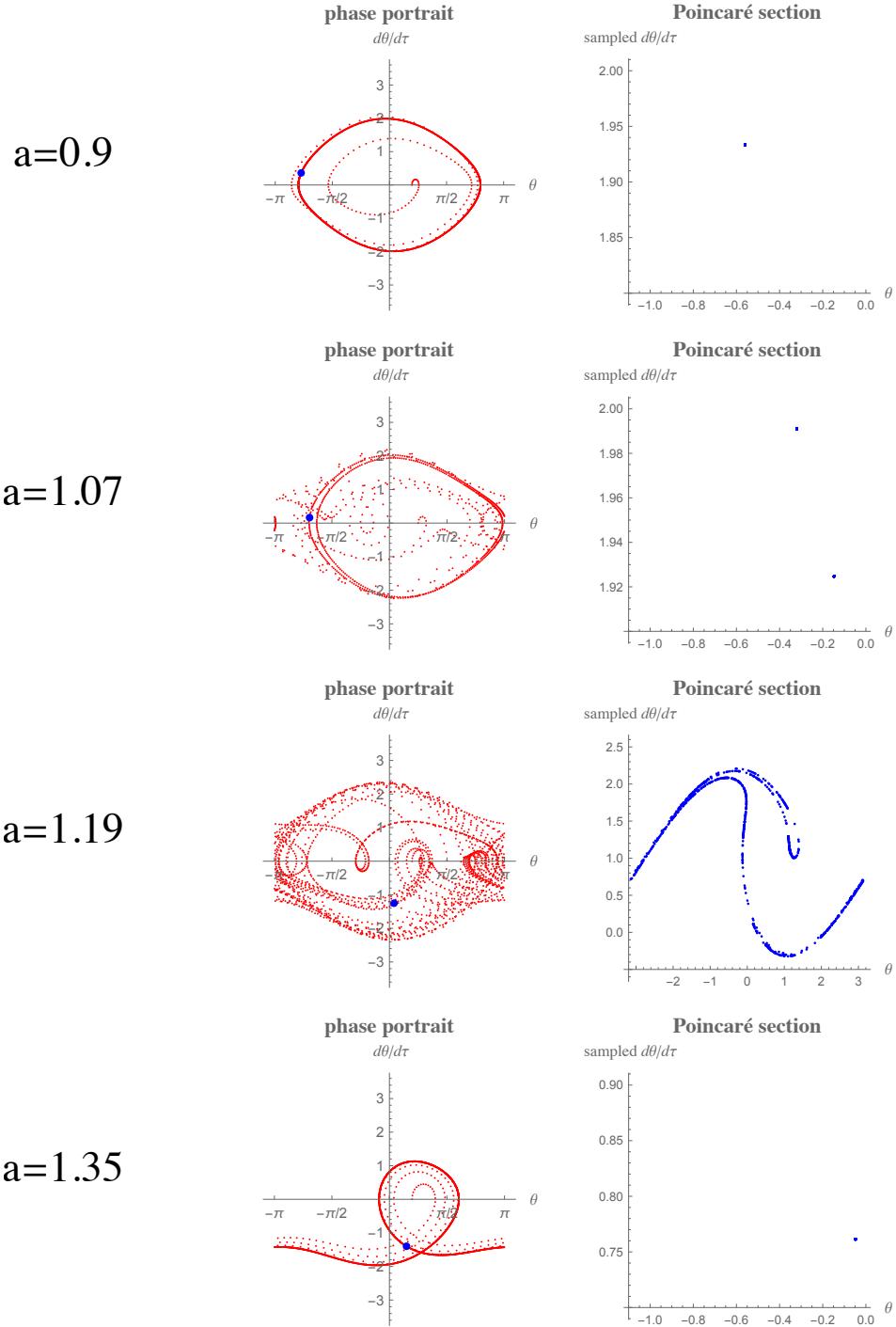


Figure 7.6: Phase portraits and Poincaré sections for the nonlinear driven damped oscillator with  $\omega_D = 2/3$ ,  $q = 2$ , and various values of  $a$ . The plots show singly periodic, doubly periodic, chaotic, and singly periodic behavior respectively. (Plots generated with the Mathematica demonstration package, Chaotic Motion of a Damped Driven Pendulum, by Nasser Abbasi.)

### Bifurcation Map

Yet another way is through a bifurcation diagram, which takes the Poincaré map results but plots one of the variables against one of the parameters of the system. This plot allows us to see the transitions between different behaviors, in particular a change in the fixed points of the system. For the nonlinear oscillator, this could be a plot of  $\omega$  against  $a$ , as shown in Fig. 7.7.

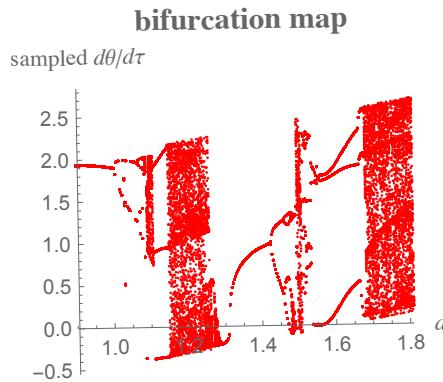


Figure 7.7: For the Driven damped nonlinear oscillator, plot of  $\omega = \dot{\theta}$  values obtained from the Poincaré map as a function of  $a$  with  $Q = 2$  and  $\omega_D = 2/3$  fixed. This bifurcation plot show the qualitative transitions of the system, such as where period doubling/bifurcation occurs, and where chaos starts. (Plot generated with the Mathematica demonstration package, Chaotic Motion of a Damped Driven Pendulum, by Nasser Abbasi.)

There are a few notable features in this bifurcation plot which we summarize in the following table:

$a$	Features
1.0	only a single $\omega$
1.07	two values of $\omega$ from the same initial conditions ( <i>period doubling</i> )
1.15-1.28	mostly chaos (some periodic windows)
1.35	periodic again

Other parameter choices also lead to qualitatively similar bifurcation plots, with quantitatively different windows of periodic behavior and chaos. We can also obtain bifurcation plots which exhibit both periodic and chaotic windows by plotting  $\omega$  against other parameters of the system, such as  $\omega_D$ .

## 7.2 Bifurcations

In our analysis of the nonlinear damped forced oscillator, we took snapshots  $(\theta_n, \omega_n)$  at  $\varphi = 2\pi n$  for integer  $n$  to form the Poincaré map. When we changed the driving amplitude  $a$ , there were qualitative changes to the  $(\theta, \omega)$  projected trajectories (which are also generally called phase portraits) captured by the Poincaré map results. In particular, we observed period doubling at certain values of  $a$ ; period doubling is a particular example of a bifurcation (Fig.(7.7)).

A simple example of an abrupt change is when the existence/type of fixed points changes with the system's parameters (or limit cycles, attractors, or so on) abruptly changes. These changes are generally known as *bifurcations*. Since bifurcations already occur in 1-dimensional systems, so we will start by studying these systems. We will later on find out that many examples of bifurcations in higher-dimensions are simple generalizations of the 1D case.

For a 1-dimensional system we study the equation:

$$\dot{x} = f(x) \quad (7.17)$$

Trajectories in 1 dimension are pretty simple, we either have flow to a finite fixed point  $x \rightarrow x^*$  or a divergence to  $x \rightarrow \pm\infty$ .

**Example:** The system  $\dot{x} = x^2 - 1$ , pictured in Fig. 7.8, has a stable fixed point at  $x^* = -1$  and an unstable fixed point at  $x^* = 1$ . For one dimension the motion is simple enough that we can determine whether fixed points are stable or unstable simply from this picture. Imagine a particle moving on the  $x$ -axis. For  $x < -1$  the red curve of  $x^2 - 1$  is above the  $x$ -axis, so  $\dot{x} > 0$  and the particle moves to the right, as indicated by the blue arrow. For  $-1 < x < 1$  the red curve is below,  $\dot{x} < 0$ , and the particle moves to the left. For  $x > 1$  the curve is again above,  $\dot{x} > 0$  and the particle moves to the right. The left point is stable since the particle always moves towards it, while the right point is unstable and the particle moves away from it.

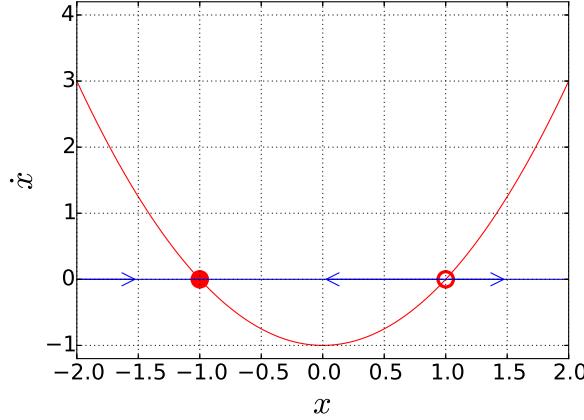


Figure 7.8: In this system there are two fixed points, one stable (represented by a full circle) and one unstable (represented by the empty circle)

Stability can also be determined by linearizing about a fixed point. Although this is overkill for one variable, the general method remains applicable for analyzing situations with more variables, so it's useful to discuss it here. Using  $x = x^* + \eta$  and expanding to  $\mathcal{O}(\eta)$ , then  $\dot{\eta} = \dot{x} = x^2 - 1 \approx 2x^*\eta$ , so for  $x^* = -1$ , then  $\dot{\eta} = -2\eta$  which decays according to  $\eta \propto e^{-2t}$  making the fixed point stable, while for  $x^* = 1$ , then  $\dot{\eta} = 2\eta$  which grows according to  $\eta \propto e^{2t}$  and the fixed point is unstable.

To find the stability of fixed points in multiple dimensions, we would similarly set  $\vec{x} = \vec{x}^* + \vec{\eta}$  and expand, giving a linearized system of equations after dropping  $\mathcal{O}(\eta^2)$  terms:

$$\dot{\vec{\eta}} = M\vec{\eta} \quad (7.18)$$

Here  $M$  is a  $n \times n$  matrix, whose eigenvalues and eigenvectors give us the solutions near the fixed point, of the form  $\vec{\eta} = \vec{a}e^{\lambda t}$ . We will come back later on to discuss higher dimensional fixed points in much more detail.

First we will categorize several types of bifurcations in one dimension, by considering the equation

$$\dot{x} = f(x, r), \quad (7.19)$$

where  $r$  is a parameter that we vary. The fixed points  $x^*$  of  $f(x, r)$  are functions of  $r$ , and drawing them in the  $r - x$ -plane gives a *bifurcation diagram*.

### 7.2.1 Saddle-Node Bifurcation

A saddle-node bifurcation is the basic mechanism by which fixed points are created and destroyed. As we vary  $r$  two fixed points can either appear or disappear (one stable and one unstable).

**Example:** Consider the equation

$$\dot{x} = r + x^2, \quad (7.20)$$

which exhibits a saddle-node bifurcation at  $r = 0$ . The two fixed points disappear as we increase  $r$  from negative to positive values, as shown in the images below.

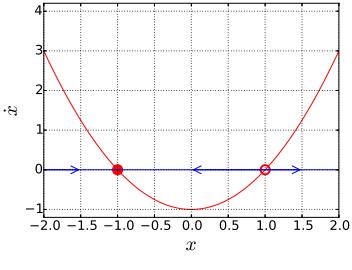


Figure 7.9: Two fixed points, one stable and one unstable exist for  $r < 0$

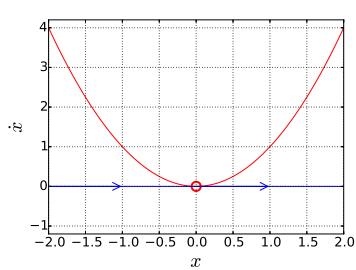


Figure 7.10: A single semi-stable point exists for  $r = 0$

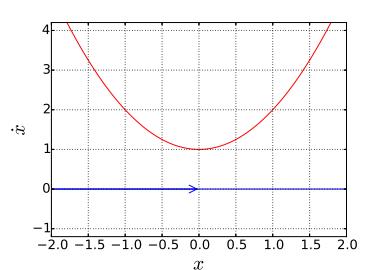


Figure 7.11: No fixed points occur for  $r > 0$

This saddle-node bifurcation transition can be best pictured by the bifurcation diagram in Fig. 7.12 below, where the full lines correspond to the stable fixed points and the dashed lines the unstable ones.

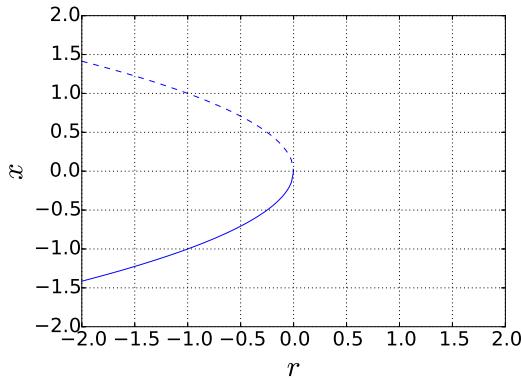


Figure 7.12: Bifurcation diagram for the system  $\dot{x} = r + x^2$

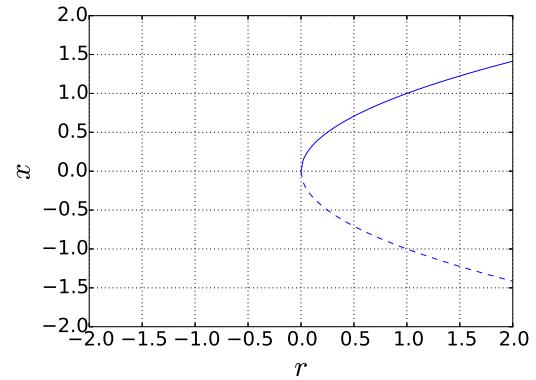


Figure 7.13: Bifurcation diagram for the system  $\dot{x} = r - x^2$

For the analogous equation  $\dot{x} = r - x^2$  we can obtain the results by interchanging  $x \rightarrow -x$  and  $r \rightarrow -r$ . This gives the bifurcation diagram shown in Fig. 7.13.

**Example:** Some flow equations like

$$\dot{x} = r - x - e^{-x} = f(x, r) \quad (7.21)$$

are hard to solve analytically for the fixed points, which are given by the transcendental equation

$$r - x^* = e^{-x^*} \quad (7.22)$$

Here a graphical approach suffices, where we separately plot  $r - x$  and  $e^{-x}$  and look for intersections of the curves to provide the position of the fixed points, as displayed in Fig. 7.14.

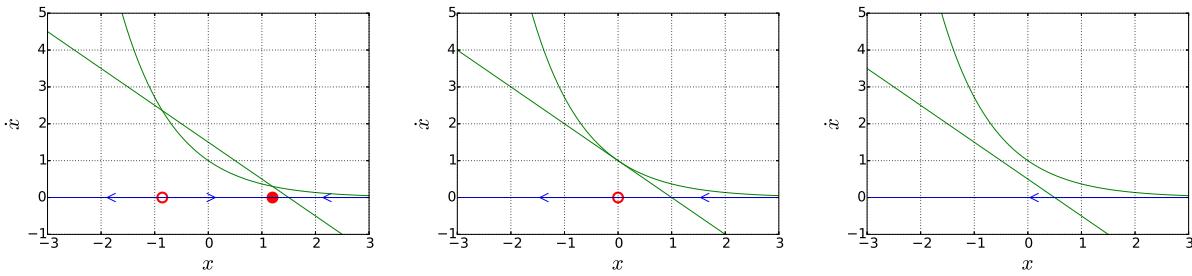


Figure 7.14: Fixed points of the system correspond to the intersections of the curves  $e^{-x}$  and  $r - x$  for  $r = 1.5$ ,  $r = 1.0$ ,  $r = 0.5$  respectively. As  $r$  is varied the position of the fixed points varies and a Saddle-Node Bifurcation occurs.

Examining which curve is larger also determines the direction of the one-dimensional flow, and hence the stability of the fixed points.

Here the bifurcation occurs at  $r = r_C$ , when the two curves are tangential and hence only touch once:

$$\left. \frac{\partial f}{\partial x} \right|_{x=x^*, r=r_C} = 0 \quad (7.23)$$

This gives  $-1 = -\exp(-x^*)$  so  $x^*(r_C) = 0$ . Plugging  $x^* = 0$  into Eq. (7.22) we find that  $r_C = 1$ .

By a simple generalization, we can argue that the quadratic examples  $\dot{x} = r \pm x^2$  are representative of all saddle-node bifurcations. Taylor expanding  $f(x, r)$  near the bifurcation point and fixed point we have

$$\begin{aligned} \dot{x} &= f(x, r) = f(x^*, r_C) + (x - x^*) \left. \frac{\partial f}{\partial x} \right|_{x^*, r_C} + (r - r_C) \left. \frac{\partial f}{\partial r} \right|_{x^*, r_C} + \frac{(x - x^*)^2}{2} \left. \frac{\partial^2 f}{\partial x^2} \right|_{x^*, r_C} + \dots \\ &= a(r - r_C) + b(x - x^*)^2 + \dots, \end{aligned} \quad (7.24)$$

where we have kept the first non-trivial dependence on  $r$  and  $x$  (noting that the partial derivatives are simply some constants  $a$  and  $b$ ), and two terms have vanished due to the

fixed point and by the tangential bifurcation conditions:

$$f(x^*; r_C) = 0 \quad \text{and} \quad \left. \frac{\partial f}{\partial x} \right|_{x=x^*, r=r_C} = 0. \quad (7.25)$$

Thus  $\dot{x} = r \pm x^2$  is the normal form of a saddle-node bifurcation. This can be determined explicitly from Eq. (7.24) by making the change of variable  $r' = a(r - r_C)$  and  $x' = \sqrt{|b|}(x - x^*)$  to obtain  $\dot{x}' = r' \pm x'^2$  where the  $\pm$  sign is determined by the sign of  $b$ .

### 7.2.2 Transcritical Bifurcation

In a transcritical bifurcation a fixed point exists for all values of the parameter, but changes its stability as the parameter is varied.

**Example:** Consider the equation

$$\dot{x} = x(r - x). \quad (7.26)$$

Here there are fixed points at  $x^* = 0$  and  $x^* = r$ . These fixed points change their stability at  $r = 0$  but never disappears as illustrated graphically in Fig. 7.15.

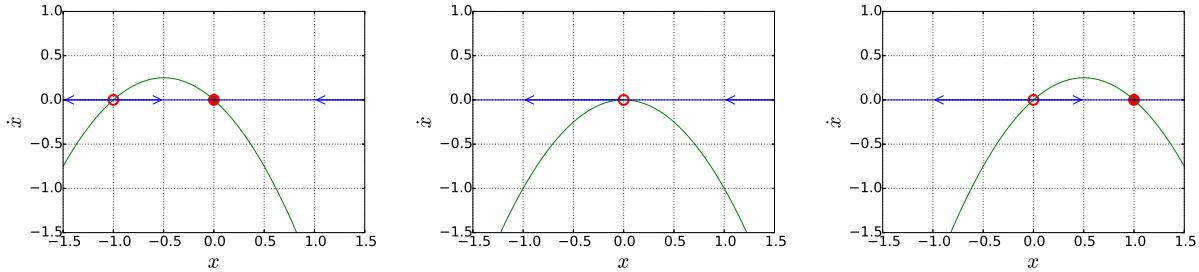


Figure 7.15: Analysis of  $\dot{x} = x(r - x)$  for  $r = -1$ ,  $r = 0$  and  $r = 1$  respectively. As  $r$  changes the same type of fixed points remain, but the stability of the fixed points is swapped.

This gives us the following bifurcation diagram:

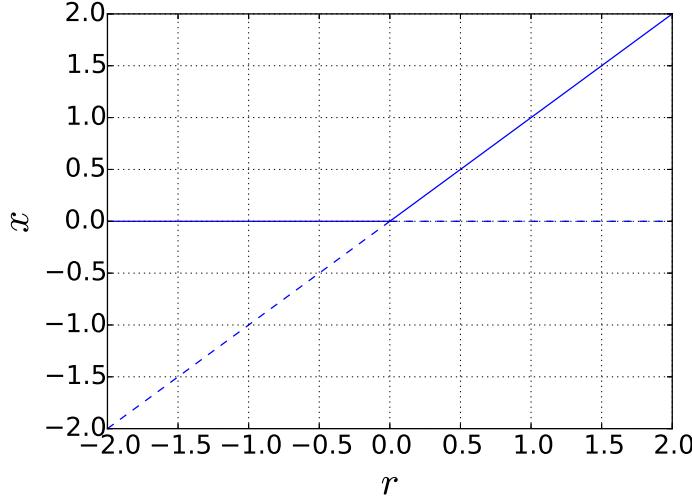


Figure 7.16: Bifurcation diagram for the system  $\dot{x} = x(r - x)$  which plots the position of the fixed points, with a full and dashed line for the stable and unstable points respectively. Here the transcritical bifurcation at  $r = 0$  becomes clear.

In fact, the equation “ $\dot{x} = x(r - x)$ ” is the normal form of a transcritical bifurcation obtained by expanding in a Taylor series near  $x = x^*$  and  $r = r_C$ .

**Example:** Lets consider an example with physical content, namely a model for the threshold behavior of a laser. This can be modeled as:

$$\begin{aligned}\dot{n} &= GnN - Kn, \\ \dot{N} &= -GnN - fN + p.\end{aligned}\tag{7.27}$$

where the variables are  $N$  the number of excited atoms and  $n$  the number of laser photons. The constant parameters include,  $f$  for the term governing the spontaneous emission decay rate,  $G$  for the stimulated emission gain coefficient,  $K$  as the photon loss rate, and  $p$  as the pump strength. Since there are two equations this is in general a two dimensional system (which we will discuss how to analyze shortly). Here to make the equation one dimensional we will assume rapid relaxation so that  $\dot{N} \approx 0$ , this allows us to solve for  $N(t)$  from the second equation in Eq. (7.27) to give

$$N(t) = \frac{p}{Gn(t) + f}.\tag{7.28}$$

Plugging this back into the first equation in Eq. (7.27) then gives

$$\dot{n} = \frac{n}{Gn + f} [pG - K(Gn + f)] \approx n(r - x) + O(n^3)\tag{7.29}$$

Expanding this result near  $n = 0$  we find

$$\dot{n} = n(r - bn) + O(n^3) \quad (7.30)$$

where the constant coefficients are

$$r \equiv \frac{pG}{f} - K, \quad b \equiv \frac{G^2 p}{f^2}. \quad (7.31)$$

Only  $n > 0$  makes sense, so the critical parameter value is when  $r = 0$ , or when the pump strength  $p = \frac{Kf}{G}$ . For larger values of  $p$  this lamp turns into a laser, with a fixed point at a non-zero  $n = r$  as illustrated in Figs. 7.17 and 7.18. The fixed point for  $p > \frac{Kf}{G}$  indicates the coherent laser action.

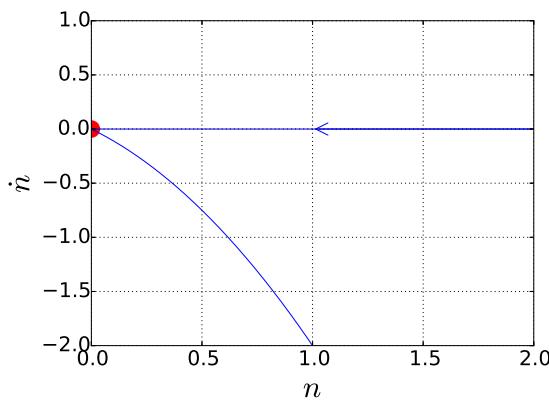


Figure 7.17: When  $p < Kf/G$ , the only stable point is when there are no photons.

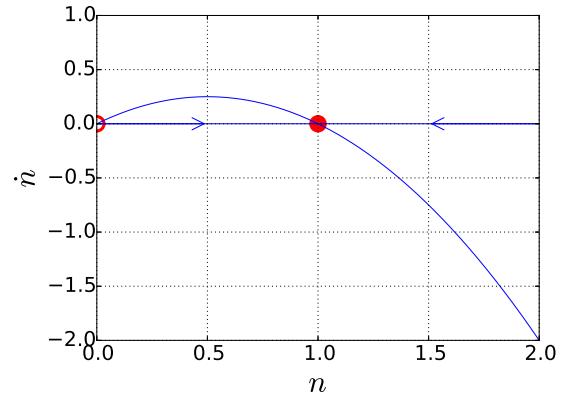


Figure 7.18: When  $p > Kf/G$ , the stable point of the system is with a non-zero number of laser photons.

### 7.2.3 Supercritical Pitchfork Bifurcation

A supercritical pitchfork bifurcation is a type of bifurcation common in problems with symmetries such that fixed points appear or disappear in pairs. In particular, as the parameter is varied, one fixed point is always present but changes from being stable to being unstable at the same place where two stable fixed points appear.

**Example:** The normal form for this type of bifurcation is

$$\dot{x} = rx - x^3. \quad (7.32)$$

This equation is invariant under  $x \leftrightarrow -x$ , so the fixed point  $x^* = 0$  is always present. On the other hand, the fixed points at  $x^* = \pm\sqrt{r}$  only appear when  $r$  crosses from negative to positive values. The different cases that occur as we change  $r$  are plotted in Fig. 7.19.

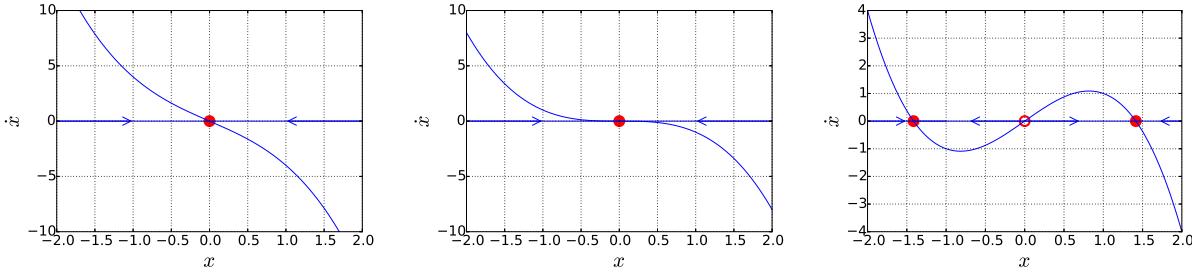


Figure 7.19: Plots of  $\dot{x} = rx - x^3$  for  $r = -2, 0, +2$  respectively. When  $r$  becomes positive the fixed point at  $x = 0$  loses its stability, and two new stable fixed points appear in the system.

This gives rise to the following bifurcation diagram:

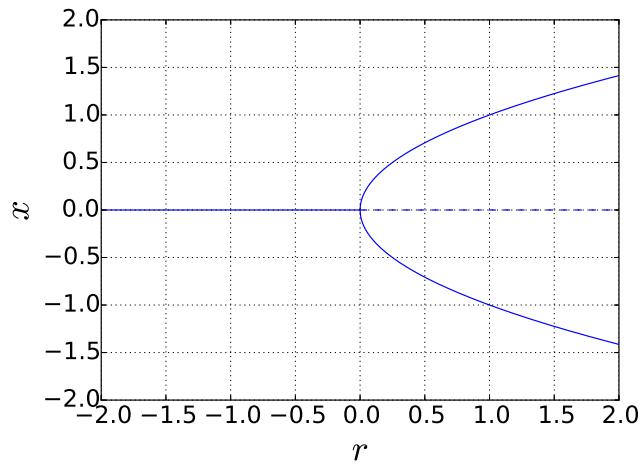


Figure 7.20: Diagram for the supercritical pitchfork bifurcation. The stability of the fixed point at  $x = 0$  changes while two new stable points appear.

#### 7.2.4 Subcritical pitchfork bifurcation

A subcritical pitchfork bifurcation essentially the opposite of a supercritical pitchfork bifurcation in that if the parameter is varied, one fixed point that is always present changes from unstable to stable, while two unstable fixed points appear.

**Example** As an example consider the normal form

$$\dot{x} = rx + x^3, \quad (7.33)$$

which again has a  $x \rightarrow -x$  symmetry. Here the cubic term is destabilizing, so this exhibits a subcritical pitchfork bifurcation as depicted in Fig. 7.21.

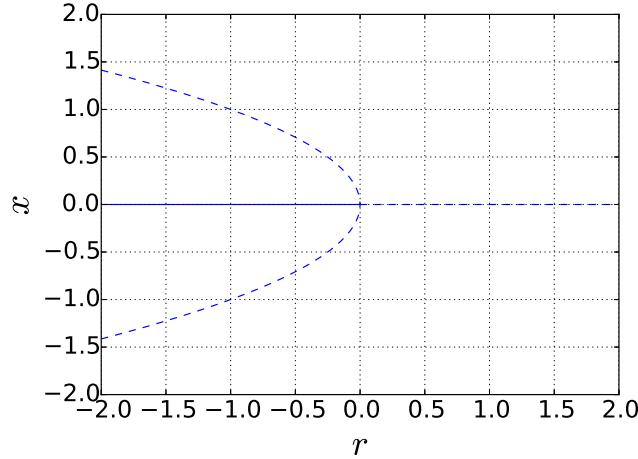


Figure 7.21: Bifurcation Diagram for the Subcritical Bifurcation  $\dot{x} = rx + x^3$ . Here there is a change in the stability of the fixed point at  $x = 0$  and the appearance of two new fixed points at  $x = \pm\sqrt{-r}$  as  $r$  becomes negative.

It is interesting to consider what happens if we add a higher order stabilizing term, such as in the equation

$$\dot{x} = rx + x^3 - x^5. \quad (7.34)$$

This equation supports five real solutions for a finite range of  $r$  values. This system supports hysteresis as we increase and decrease  $r$  as illustrated in Fig. 7.22. We can imagine a path where we start with a particle at  $x = 0$  and  $r = -0.2$  and then slowly increase  $r$ . When we get to  $r = 0$  the  $x = 0$  fixed point becomes unstable and a small perturbation will push the particle to another branch, such as that at  $x > 0$ . Increasing  $r$  further the particle travels up this branch. If we then start to decrease  $r$ , the particle will travel back down this same branch, and continue on it even below  $r = 0$ , and thus not following the same path. Then suddenly at the critical  $r_C < 0$  where there is a saddle-node bifurcation, the particle will again lose its stability and will jump back down to  $x = 0$ , after which the process can be repeated.

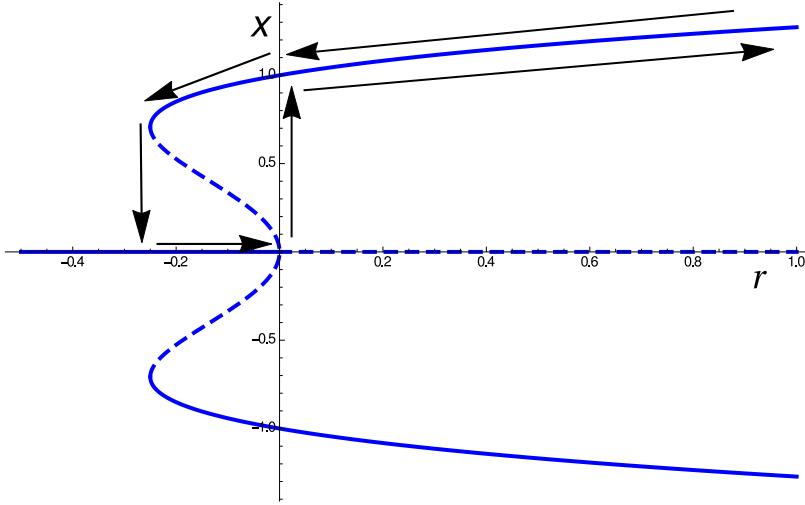


Figure 7.22: Bifurcation Diagram of the system  $\dot{x} = rx + x^3 - x^5$ . The arrows show the motion of the system as we increase and decrease  $r$ ; it undergoes hysteresis.

**Example:** Lets consider a bead on a rotating hoop with friction, described by the equation of motion:

$$ma\ddot{\theta} + b\dot{\theta} = m \sin \theta (a\omega^2 \cos \theta - g). \quad (7.35)$$

Here  $\theta$  is the angle of the bead of mass  $m$  from the bottom of the hoop,  $a$  is the radius of the hoop,  $\omega$  is the constant angular velocity for the rotation of the hoop (about an axis through the center of the hoop and two points on the hoop), and  $g$  is the coefficient of gravity. Once again to turn this into a one-dimensional problem we consider the overdamped solution. Overdamping means we can take  $ma\ddot{\theta} \rightarrow 0$ . The fixed points are then  $\theta^* = 0$  which changes from being stable (when  $a\omega^2 < g$ ) to being unstable (when  $a\omega^2 > g$ ), while  $\theta^* = \pi$  is always present and unstable. Additionally, the stable fixed points  $\theta^* = \pm \arccos(\frac{g}{a\omega^2})$  appear when  $a\omega^2 > g$ . This corresponds to a supercritical pitchfork bifurcation. The system's bifurcation diagram is shown in Fig. 7.23.

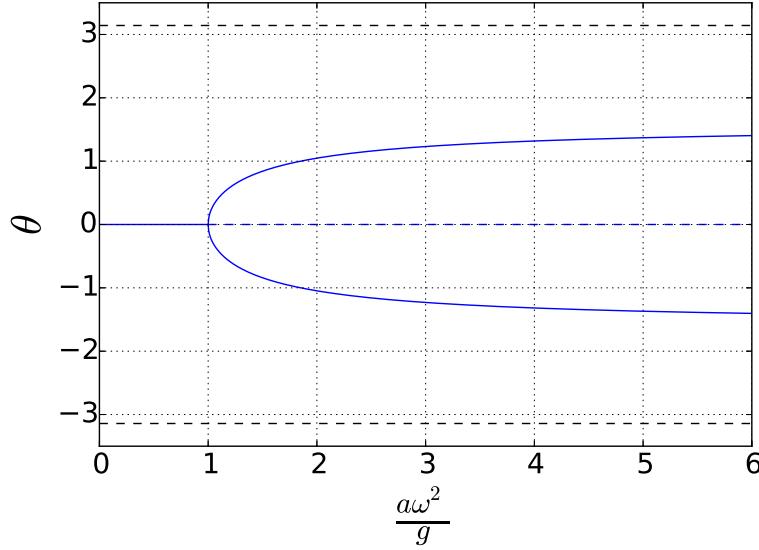


Figure 7.23: When  $\omega^2 > g/a$  the bottom of the loop becomes an unstable fixed point and two new stable fixed points arise that move away from the bottom as the rotation speed is increased.

**Example:** As another example, consider an overdamped pendulum driven by a constant torque described by:

$$\dot{\theta} = \tau - b \sin(\theta) , \quad (7.36)$$

where  $\tau > 0$ ,  $b > 0$ , and  $\theta \in [-\pi, \pi]$ . For  $b > \tau$ , the gravity beats the torque and there is one stable and one unstable fixed points as shown in Fig. 7.24. For  $b < \tau$ , there are no fixed points as shown in Fig. 7.25, and here the torque wins resulting in a rotating solution. Even when  $b < \tau$ , there is a remnant of the influence of the fixed point in the slowing down of the pendulum as it goes through the “bottleneck” to overcome gravity. Combined this is thus a saddle-node bifurcation at  $\tau = b$  as shown in Fig. 7.26.

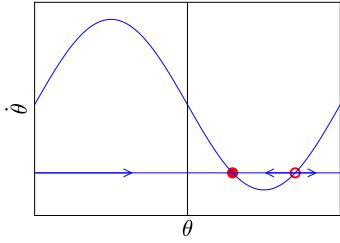


Figure 7.24:  $\dot{\theta}$  as a function of  $\theta$  when  $\tau < b$ . Gravity dominates torque and there is a stable fixed point.

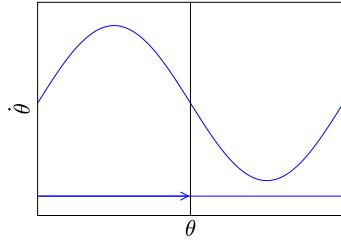


Figure 7.25:  $\dot{\theta}$  as a function of  $\theta$  when  $\tau > b$ . Torque dominates gravity so there are no fixed points.

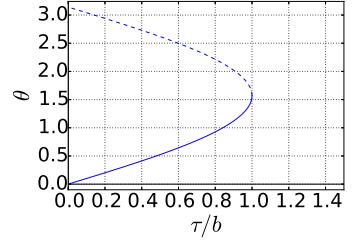


Figure 7.26: Bifurcation plot of the system as a function of the applied torque ( $\tau/b$ ).

## 7.3 Fixed Points in Two-Dimensional Systems

### 7.3.1 Motion Near a Fixed Point

#### General Categorization

In 2-dimensions, to analyze the trajectories near a fixed point  $\vec{x}^* = (x^*, y^*)$ , we can again linearize the equations of the system. Therefore, we'll start by analyzing a general 2-dimensional *linear* system with  $\vec{x}^* = 0$ . This can be written as

$$\begin{aligned}\dot{x} &= ax + by, \\ \dot{y} &= cx + dy,\end{aligned}\tag{7.37}$$

or

$$\dot{\vec{x}} = M\vec{x} \quad \text{where} \quad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\tag{7.38}$$

and the matrix of coefficients here has no restrictions.

**Example:** Let us consider a system of equations that consists of two independent 1-dimensional flows,

$$\dot{x} = ax, \quad \dot{y} = -y.\tag{7.39}$$

We have the two independent solutions:

$$x(t) = x_0 e^{at} \quad \text{and} \quad y(t) = y_0 e^{-t}\tag{7.40}$$

The parameter regions  $a < -1$ ,  $a = -1$ , and  $-1 < a < 0$  all produce a stable and attracting fixed point  $\vec{x}^* = 0$  in qualitatively different ways, because the decay rate of  $x(t)$  is

either larger, equal, or smaller than that of  $y(t)$ . This is shown in Figs. 7.27, 7.28, and 7.29. If  $a = 0$ , then  $\vec{x}^* = 0$  is no longer isolated as there is a line of fixed points at  $y = 0$  and for all values of  $x$ , see Fig. 7.30. If  $a > 0$ , then  $\vec{x}^* = 0$  is a saddle point (with the  $y$ -axis being the “stable manifold” and the  $x$ -axis being the “unstable manifold”), see Fig. 7.31.

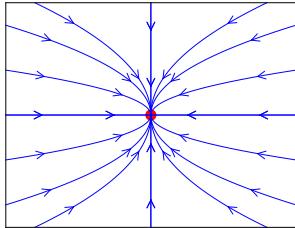


Figure 7.27: Stable Node  
 $a < -1$

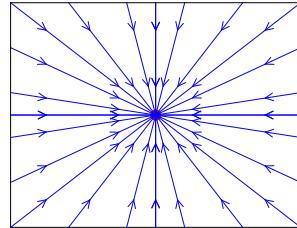


Figure 7.28: Stable Node  
 $a = -1$

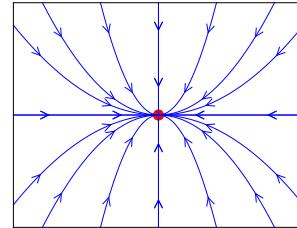


Figure 7.29: Stable Node  
 $-1 < a < 0$

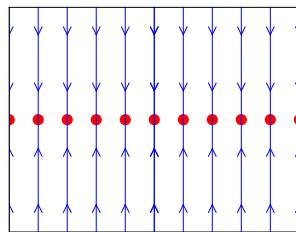


Figure 7.30: Non-isolated fixed points  
 $a = 0$

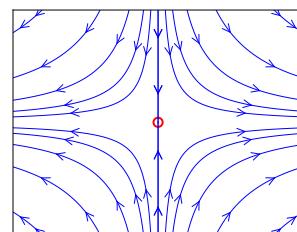


Figure 7.31: Saddle Point  
 $a > 0$

In general in two dimensions there are more possibilities for the motion than in one-dimension and we should be more careful about our definition for when a fixed point is stable. For a fixed point  $\vec{x}^*$  we will say that

- it is *attracting* if all trajectories starting in its neighborhood approach it as  $t \rightarrow \infty$ ,
- it is *Lyapunov stable* if all trajectories starting in its neighborhood remain in that neighborhood for all time,
- it is *stable* if it is both attracting and Lyapunov stable.

Picking one fixed point from Fig. 7.30, most trajectories will be Lyapunov stable but not attracting. If a fixed point allows a trajectory to wander away but eventually return, then it may also be attracting but not Lyapunov stable.

In the general case defined in Eq.(7.38), we need to solve the eigenvalue problem for the linear system, and thus find the eigenvalues and eigenvectors of  $M$ . Here we have

$$\dot{\vec{a}} = M\vec{a} = \lambda\vec{a} \Rightarrow \vec{a}(t) = \vec{a}_0 e^{\lambda t} \quad (7.41)$$

Therefore, as usual, we set  $\det(M - \lambda\mathbb{1}) = 0$  where  $\mathbb{1}$  is the identity matrix of the same dimension as  $M$ . From this, defining

$$\Delta \equiv \det(M) = ad - bc, \quad \tau \equiv \text{tr}(M) = a + d, \quad (7.42)$$

then the eigenvalues are given by

$$\lambda_{\pm} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}. \quad (7.43)$$

The corresponding eigenvectors are then  $\vec{a}_{\pm}$ , and for a generic  $M$  they will not be orthogonal. Assuming that two different eigenvectors exist a general solution is by linearity given by

$$\vec{x}(t) = \text{Re} \left[ C_+ \vec{a}_+ e^{\lambda_+ t} + C_- \vec{a}_- e^{\lambda_- t} \right] \quad (7.44)$$

assuming for the moment that  $\lambda_+ \neq \lambda_-$  and taking the real part at the end if needed. There are three main cases to consider.

1. Real eigenvalues  $\lambda_+, \lambda_- \in \mathbb{R}$  with  $\lambda_+ \neq \lambda_-$ . This is like the system in Eq.(7.40), but with the  $x$  and  $y$  axes replaced by the directions defined by  $\vec{a}_+$  and  $\vec{a}_-$ .

**Example:** Consider for example a solution where  $\vec{a}_+ = (1, 1)$  and  $\vec{a}_- = (1, -4)$ , ignoring normalization. If  $\lambda_- < 0 < \lambda_+$ , then growth occurs along  $\vec{a}_+$  and decay occurs along  $\vec{a}_-$ , so  $\vec{x}^* = 0$  is a saddle point, as drawn in Fig. 7.32

If instead  $\lambda_- < \lambda_+ < 0$ , then decay occurs slower with  $\lambda_+$  so it occurs first onto  $\vec{a}_+$ , making  $\vec{x}^* = 0$  a stable node, as drawn in Fig. 7.33

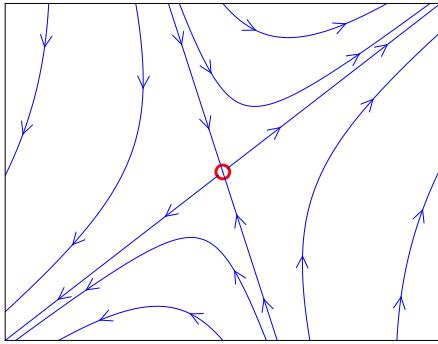


Figure 7.32: Saddle Point with  $\vec{a}_+ = (1, 1)$  and  $\vec{a}_- = (1, -4)$

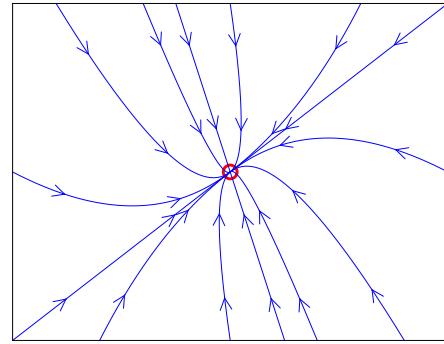


Figure 7.33: Stable Node with  $\vec{a}_+ = (1, 1)$  and  $\vec{a}_- = (1, -4)$

2. Let us now consider when  $\lambda_+ = \lambda_- = \lambda \in \mathbb{R}$ . In this situation there can either be two independent eigenvectors or only one. Two independent eigenvectors can only occur if

$$M = \lambda \mathbb{1}, \quad (7.45)$$

in which case the fixed point is called a *star*, and is shown in Fig. 7.34.

If instead there is only one independent eigenvector, then the fixed point is called a *degenerate node*. An example of this is

$$M = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix},$$

where the eigenvalue is  $\lambda$  and which has  $\vec{a} = (1, 0)$  as its only independent eigenvector. Here the phase space portrait is as given in Fig. 7.35, where the trajectory decays first onto the eigenvalue direction and then down onto the fixed point.

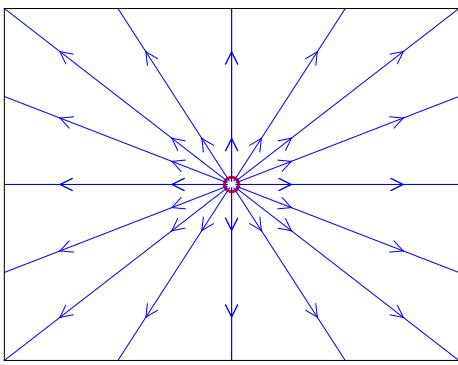


Figure 7.34: Unstable star with  $\lambda > 0$ .

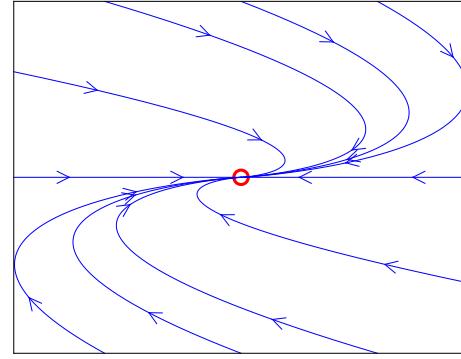


Figure 7.35: Degenerate Node with  $\lambda = -1$  and  $b = 2$ .

3. The final case to consider is when the eigenvalues have complex parts,  $\lambda_{\pm} = \alpha \pm i\omega$  for  $\omega \neq 0$ . If  $\alpha < 0$ , the fixed point is a *stable spiral* where the trajectories spiral into it, as in Fig. 7.36. If  $\alpha = 0$ , the fixed point is a *center*, with neighboring trajectories being closed orbits around it, as in Fig. 7.37. If  $\alpha > 0$ , the fixed point is an *unstable spiral* where trajectories spiral out from it, as in Fig. 7.38.

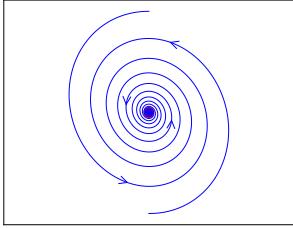


Figure 7.36: Stable spiral with  $\alpha < 0$

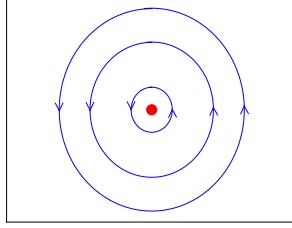


Figure 7.37: Trajectories about a center fixed point

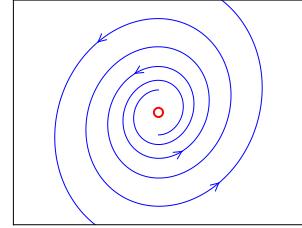


Figure 7.38: Unstable Spiral with  $\alpha > 0$

As a summary if  $\Delta < 0$  then the fixed points are saddle points, while if  $\Delta = 0$  then the fixed points are not isolated but form a continuous line of fixed points. If  $\Delta > 0$ , then there are a number of possibilities:

- $\tau < -2\sqrt{\Delta}$  produces stable nodes;
- $\tau > 2\sqrt{\Delta}$  produces unstable nodes;

- $-2\sqrt{\Delta} < \tau < 0$  produces stable spirals;
- $\tau = 0$  produces centers;
- $0 < \tau < 2\sqrt{\Delta}$  produces unstable spirals;
- $\tau = \pm 2\sqrt{\Delta}$  produces stars or degenerate nodes.

Note that all unstable fixed points have  $\tau > 0$ , while all stable fixed points have  $\tau < 0$ ; this is true even for stars and degenerate nodes. This information can be summarized by the following diagram:

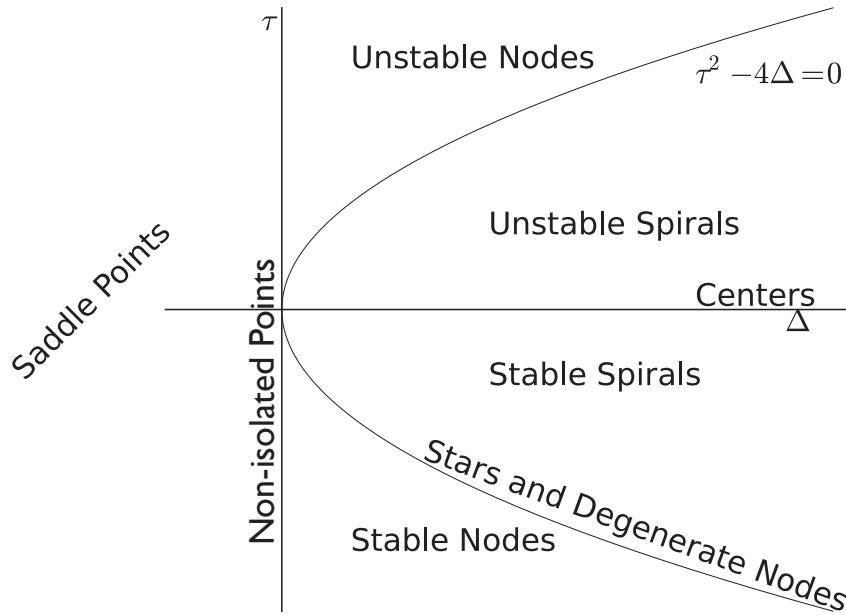


Figure 7.39: Diagram determining the type of fixed point given the determinant  $\Delta$  and trace  $\tau$  of the linearized system.

This linearized analysis yields the correct classification for saddle points, stable/unstable nodes, and stable/unstable spirals, but not necessarily for the borderline cases that occur on a line rather than in an area of the  $\Delta$ - $\tau$  plane (centers, stars, degenerate nodes, or non-isolated fixed points). Nonlinear terms can tip a borderline case to a nearby case in the  $\Delta$ - $\tau$  plane. This implies nonlinear terms may only affect the stability of centers.

#### Analysis of a General 2-Dimensional System

Consider a general 2-dimensional system:

$$\dot{\vec{x}} = \vec{f}(\vec{x}) = (f_x(x, y), f_y(x, y)), \quad (7.46)$$

which may have several fixed points  $(x^*, y^*)$ . We can analyze their types by linearizing about each one, defining  $u = x - x^*$  and  $v = y - y^*$  and expanding about  $(u, v) = (0, 0)$ . Defining  $\vec{u} = (u, v)$ , then this expansion yields

$$\dot{\vec{u}} = M\vec{u} \quad \text{where} \quad M_{ij} = \left. \frac{\partial f_i}{\partial x_j} \right|_{\vec{x}=\vec{x}^*} \quad (7.47)$$

This is the same as a Taylor series about  $\vec{x} = \vec{x}^*$ , where  $\vec{f}(\vec{x}^*) = 0$ .

**Example:** Lets consider a population growth model where rabbits ( $x$ ) compete with sheep ( $y$ ). With one species, the model might look like  $\dot{x} = x(1 - x)$ , where for small  $x$  there is population growth, but above  $x > 1$  food resources become scarce and the population shrinks. For two species there can be coupling between the equations, so we could consider

$$\begin{aligned} \dot{x} &= x(3 - x - 2y), \\ \dot{y} &= y(2 - y - x). \end{aligned} \quad (7.48)$$

which is called the Lotka-Volterra model. Here the parameters have been chosen to model the fact that rabbits produce faster ( $3 > 2$  in the linear terms) and sheep compete better for resources ( $2 > 1$  in the quadratic cross terms). To determine how solutions to these equations behave we can analyze the structure of the fixed points.

The fixed points for this system are:

$$\vec{x} \in \{(0, 0), (0, 2), (3, 0), (1, 1)\} \quad (7.49)$$

For each one we carry out a linear analysis:

- $\vec{x}^* = (0, 0)$  simply gives  $\dot{x} = 3x$  and  $\dot{y} = 2y$ , so it is an unstable node.
- $\vec{x}^* = (0, 2)$ . Here we define  $u = x$  and  $v = y - 2$  and the linear equations become  $\dot{u} = -u$  and  $\dot{v} = -2u - 2v$ . Taking the trace and determinant we find  $\tau = -3$  and  $\Delta = 2$  giving  $\lambda_+ = -1$  and  $\lambda_- = -2$ . This is a stable node.
- $\vec{x}^* = (2, 0)$  gives  $\lambda_+ = -1$  and  $\lambda_- = -3$ , making it too a stable node.
- $\vec{x}^* = (1, 1)$  gives  $\lambda_{\pm} = -1 \pm \sqrt{2}$ , making it a saddle point.

From knowing the behavior of trajectories near these fixed points we can complete the picture for an approximate behavior of the entire system, as shown in Fig. 7.40. A diagonal line passing through the unstable node and saddle point divides the basins of attraction for the fixed points where the sheep win  $(0, 2)$  or where the rabbits win  $(3, 0)$ .

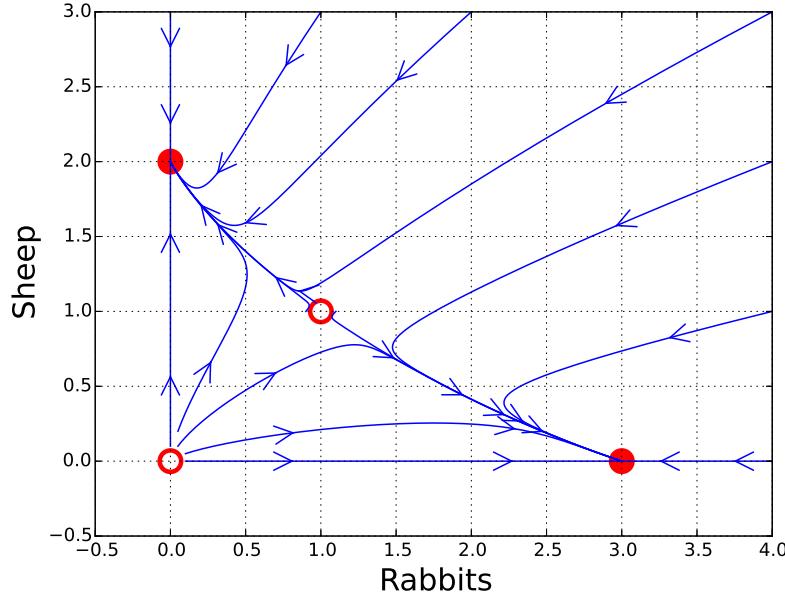


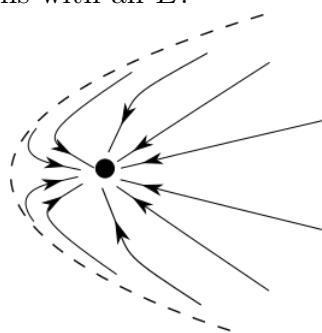
Figure 7.40: Picture of the behavior of trajectories for the population growth model in Eq. (7.48).

### 7.3.2 Systems with a conserved $E(\vec{x})$

The mechanical systems of equations that we are most familiar with are those where the energy is constant along trajectories. We will generalize this slightly and say systems with *any* function  $E = E(\vec{x})$  that is conserved (so  $\dot{E} = 0$ ) are *conservative systems with  $E$* . To rule out taking a trivial constant value of  $E$  (which would work for any system), we demand that  $E(\vec{x})$  must not be constant on any closed region in the space of  $\vec{x}$ . Note that  $\dot{E} = 0$  is generally not equivalent to  $\nabla \cdot \vec{f} = 0$ , and hence we do not simply call these conservative systems.

Several results follow from considering conservative systems with an  $E$ :

- Conservative systems with  $E$  do not have attracting fixed points. We can prove this by contradiction, by imagining that such a point did exist. Since all the points in the basin of attraction of that point must go to this single fixed point, they must all share the same value of  $E$ , which contradicts  $E$  not being constant within a closed region.



- From our experience in expanding about the minima of potentials, we also expect to find stable centers for conservative systems with  $E$ . This result is achieved by the following theorem which we have essentially proven in our analysis in the chapter on vibrations:

For  $\dot{\vec{x}} = \vec{f}(\vec{x})$ , where  $\frac{\partial f_i}{\partial x_j}$  is continuous for all  $i, j$ , if  $E = E(\vec{x})$  is conserved with an isolated fixed point  $\vec{x}^*$  at the minimum of  $E$ , then all trajectories sufficiently close are centers.

- In 2 dimensions the  $\nabla \cdot \vec{f} = 0$  definition of conservative is equivalent to having a conserved  $E = E(\vec{x})$  along the systems trajectories.

Knowing that  $\dot{\vec{x}} = \vec{f}(\vec{x})$  and  $\nabla \cdot \vec{f} = 0$ , then let us define:

$$\begin{aligned} H(\vec{x}) &= \int^y f_x(x, y') dy' - \int^x f_y(x', y) dx' \\ \Rightarrow \frac{\partial H}{\partial y} &= f_x(x, y) - \int^x \frac{\partial f_y(x', y)}{\partial y} dx' = f_x + \int^x \frac{\partial f_x(x', y)}{\partial x'} dx' \\ \Rightarrow \frac{\partial H}{\partial x} &= -f_y(x, y) + \int^y \frac{\partial f_x(x, y')}{\partial x} dy' = -f_y - \int^y \frac{\partial f_y(x, y')}{\partial y'} dy' \end{aligned} \quad (7.50)$$

Then  $\frac{\partial H}{\partial y} \in \{f_x, 2f_x\}$  and  $\frac{\partial H}{\partial x} \in \{-f_y, -2f_y\}$ . The first case of each occurs if  $f_x = f_x(y)$  and  $f_y = f_y(x)$ , respectively. Thus  $\dot{x} = \mu \frac{\partial H}{\partial y}$  and  $\dot{y} = -\mu \frac{\partial H}{\partial x}$  for  $\mu \in \{1, 2\}$ . After a trivial rescaling, these are the Hamilton equations for a conserved Hamiltonian  $H(\vec{x})$  (independent of  $t$ ) which serves here as our function  $E(\vec{x})$ . Additionally, from the relations the critical points  $\vec{x}^*$  of  $H$  where  $\nabla H|_{\vec{x}=\vec{x}^*} = 0$  are identical to the fixed points where  $\vec{f}(\vec{x}^*) = 0$ .

**Example:** Consider the one-dimensional classical mechanics motion given by:

$$\ddot{x} = ax - x^2 \equiv -U'(x) \quad (7.51)$$

with  $a > 0$ . We first turn this into one-dimensional form by writing

$$\dot{x} = y = f_x, \quad \dot{y} = ax - x^2 = f_y. \Rightarrow \quad (7.52)$$

Since here  $f_x$  is independent of  $x$ , and  $f_y$  is independent of  $y$ , we obviously have  $\nabla \cdot \vec{f} = 0$ . We can define a conserved scalar quantity from  $f_x$  and  $f_y$  using Eq. (7.50) to give

$$H = \frac{y^2}{2} - \frac{ax^2}{2} + \frac{x^3}{3} = K(y) + U(x) \quad (7.53)$$

where we have Hamilton's equations

$$\dot{x} = \frac{\partial H}{\partial y} = \frac{\partial K}{\partial y} \quad \text{and} \quad \dot{y} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x}. \quad (7.54)$$

For this system the fixed points occur at  $\vec{x}^* = (0, 0)$  and  $(a, 0)$ , which are also the extremal points of  $H$ . For  $\vec{x}^* = (0, 0)$  we have

$$\frac{\partial^2 H}{\partial x^2} = -a < 0 \quad , \quad \frac{\partial^2 H}{\partial y^2} = 1 > 0 \quad , \text{ and} \quad \frac{\partial^2 H}{\partial x \partial y} = 0 , \quad (7.55)$$

so the fixed point is a saddle point. For  $\vec{x}^* = (a, 0)$  we have

$$\frac{\partial^2 H}{\partial x^2} = a > 0 \quad , \quad \frac{\partial^2 H}{\partial y^2} = 1 > 0 \quad , \text{ and} \quad \frac{\partial^2 H}{\partial x \partial y} = 0 , \quad (7.56)$$

so the fixed point is a center. These fixed points and some representative trajectories are illustrated in Fig. 7.41. Here the bound trajectories have  $H < 0$ , while the unbound trajectories have  $H > 0$ . The dividing case with energy  $H = 0$  is the trajectory is that would stop at the saddle point  $(0, 0)$ .

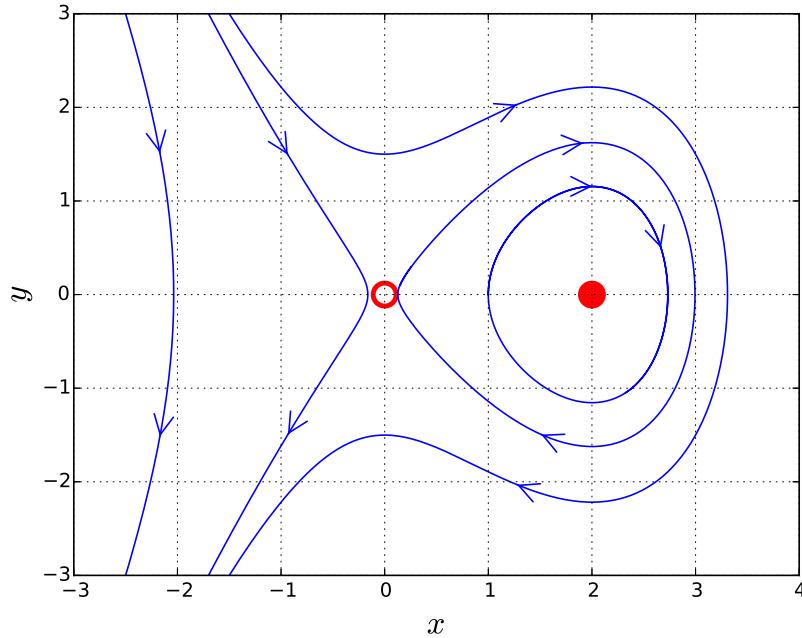


Figure 7.41: Phase space picture of the system  $\ddot{x} = ax - x^2$  with  $a = 2$ .

## 7.4 Limit Cycles and Bifurcations

In two dimensions, we can have a new type of 2-dimensional attractor called a *limit cycle*, which is an isolated closed trajectory. For stable limit cycles trajectories nearby converge to it as in Fig. 7.42, while for unstable limit cycles the nearby trajectories diverge from it as in

Fig. 7.43. We could also imagine a semi-stable limit cycles, where the trajectories converge or diverge on opposite sides of the cycle (an example is shown in Fig. 7.44).

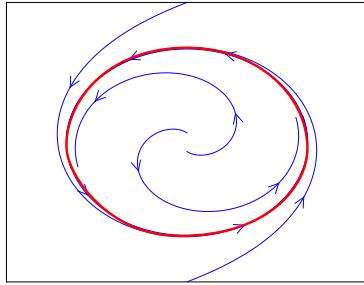


Figure 7.42: Stable Limit Cycle

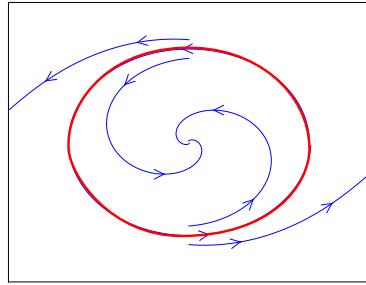


Figure 7.43: Unstable limit Cycle

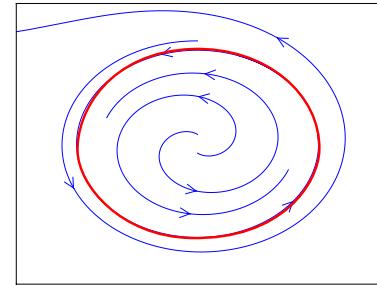


Figure 7.44: Semi-stable Limit Cycle

Note that a limit cycle is not like a center trajectory about a fixed point, because a limit cycle is isolated from other closed trajectories, whereas around centers nearby trajectories are also closed.

**Example:** Lets consider a system of equations written with polar coordinates,  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$  so that

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1, \quad (7.57)$$

with  $r \geq 0$ . Here the circle  $r^* = 1$  corresponds to a stable limit cycle, as in Fig. 7.42. Since only the radial coordinate matters for the stability of the limit cycle we can picture this in one dimension, as in Fig. 7.45.

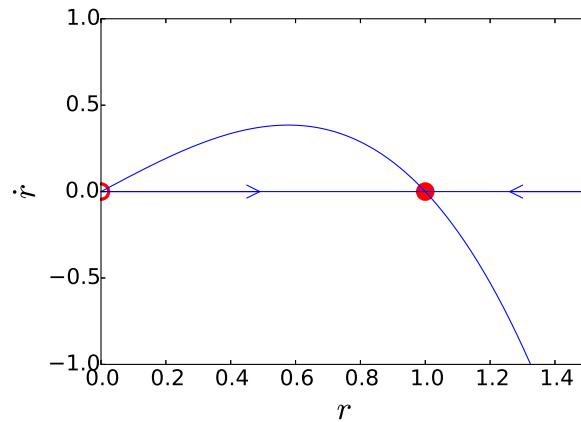


Figure 7.45: Behavior of the radial component of the system. The stable point is at  $r = 1$ , meaning the system has a stable limit cycle of radius  $r = 1$ .

**Example:** Lets consider the van der Pol oscillator (first studied in 1927 in electric circuits and found to exhibit chaotic “noise” when driven)

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \Rightarrow \begin{cases} \dot{x} = \omega \\ \dot{\omega} = \mu(1 - x^2)\omega - x \end{cases} \quad (7.58)$$

If  $x^2 > 1$  then the term involving  $\mu$  gives (nonlinear) positive damping, while if  $x^2 < 1$  then the term involving  $\mu$  gives (nonlinear) negative damping, which is growth. For different  $\mu$  values the phase portrait is depicted in the figures below.

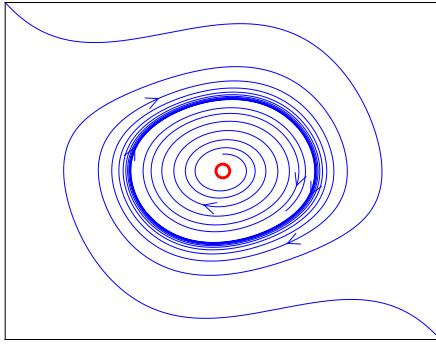


Figure 7.46: Van Der Pol Oscillator with  $\mu = 0.1$

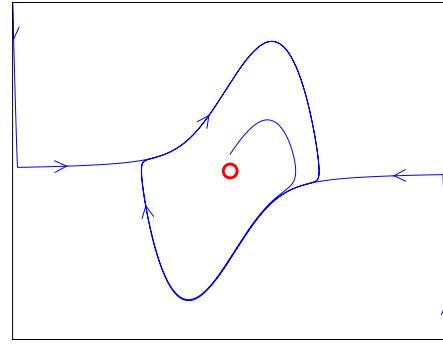


Figure 7.47: Van Der Pol Oscillator with  $\mu = 2$

There are several known methods for ruling out limit cycles, but we will instead focus on a method for showing they exist.

### 7.4.1 Poincaré-Bendixson Theorem

Take a 2-dimensional system  $\dot{x} = f_x(x, y)$  and  $\dot{y} = f_y(x, y)$  with continuous and differentiable  $\vec{f}$ . Let  $\mathbb{D}$  be a closed, bounded region. Suppose there exists a trajectory  $C$  confined inside  $\mathbb{D}$  for all times  $t \geq 0$ , then  $C$  either goes to a fixed point or a limit cycle as  $t \rightarrow \infty$ .

The proof requires the use of some topology, so we won’t study it. To understand how we can use this theorem, let us suppose we have determined that there are no fixed points in a closed, bounded region  $\mathbb{D}$ , and at the boundary’s surface the  $\dot{x}$  points “inward” to trap the trajectory in  $\mathbb{D}$ . An example of this situation is shown in Fig.(7.48). Then due to the theorem we must have a limit cycle in this region. Intuitively, the trajectory  $C$  wanders around  $\mathbb{D}$ , but it cannot self intersect and it eventually runs out of room to wander. Therefore, it must converge to a fixed point or a limit cycle. This implies that there is *no chaos in 2 dimensions*.

In 3 or more dimensions, trajectories have “more room” to wander and can do so forever, allowing for chaos!

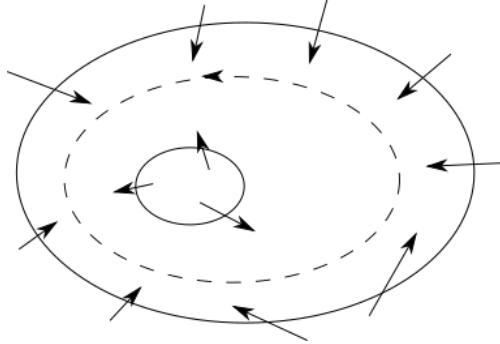


Figure 7.48: If at the boundary, the flow of a two-dimensional system pushes it into a region where there are no fixed points, then the system has a stable limit cycle in that region.

**Example:** Lets consider whether a limit cycle exists for

$$\begin{cases} \dot{x} = x - y - x(x^2 + 5y^2) \\ \dot{y} = x + y - y(x^2 + y^2) \end{cases} \quad (7.59)$$

Using polar coordinates

$$r\dot{r} = x\dot{x} + y\dot{y} \Rightarrow r\dot{r} = r^2(1 - r^2 - r^2 \sin^2(2\theta)) \quad (7.60)$$

In particular,  $1 - r^2 - r^2 \sin^2(2\theta) > 0$  for  $r < 1/\sqrt{2}$ , while  $1 - r^2 - r^2 \sin^2(2\theta) < 0$  for  $r > 1$ . Since there are no fixed points for  $1/\sqrt{2} < r < 1$  there must be a limit cycle.

### 7.4.2 Fixed Point Bifurcations Revisited and Hopf Bifurcations

We can revisit bifurcations by adding a varying parameter to the discussion of fixed points and limit cycles. In particular, we now include limit cycles popping in or out of existence in the range of things that can occur if we change a parameter.

Saddle-node, transcritical, and pitchfork bifurcations for fixed points can still occur here.

**Example:** As a simple example consider a system of uncoupled equations

$$\dot{x} = \mu - x^2, \quad \dot{y} = -y. \quad (7.61)$$

which has a saddle-node bifurcation at  $\mu = 0$ , as shown in the phase portraits below.

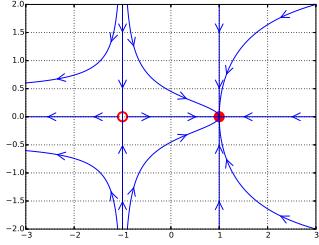


Figure 7.49: System with  $\mu = 1$  with two fixed points

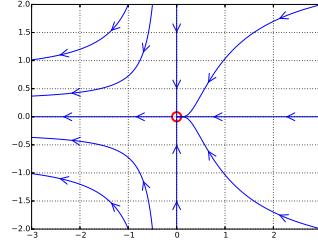


Figure 7.50: System with  $\mu = 0$  and one fixed point

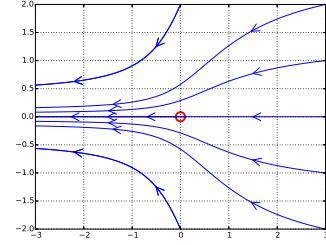


Figure 7.51: System with  $\mu = -1$  and no fixed points

More generally, we can think about determining the fixed points by drawing the curves  $\dot{x} = f_x(x, y) = 0$  and  $\dot{y} = f_y(x, y) = 0$  separately. Fixed points require both equalities to be satisfied, so we look for crossings of these two curves. Varying a parameter of the system then leads the fixed points to slide into one another, which corresponds to a 1-dimensional motion. This is why our study of the various types of bifurcation of fixed points in one-dimension (saddle-node, transcritical, supercritical and subcritical pitchforks) immediately carry over to bifurcation of fixed points in higher dimensional equations.

**Example:** consider the system of equations

$$\dot{x} = \mu x + y + \sin(x), \quad \dot{y} = x - y. \quad (7.62)$$

Note that these equations have a symmetry under  $x \rightarrow -x$  and  $y \rightarrow -y$ . This always has  $\vec{x}^* = (0, 0)$  as a fixed point. Linearizing for this fixed point yields  $\tau = \mu$  and  $\Delta = -(\mu + 2)$ . So the fixed point is stable if  $\mu < -2$  or a saddle point if  $\mu > -2$ .

Due to the symmetry we might expect a pitchfork bifurcation. If so, then near  $\mu = -2$ , there should be two more fixed points. We would need  $x = y$ , so expanding and solving we write

$$\dot{x} = (\mu + 1)x + x - \frac{x^3}{6} + \dots = 0. \quad (7.63)$$

Since we are studying points near  $x \simeq 0$ , but with  $\mu \simeq -2$  the term with  $x^3$  can be equally important, whereas the higher terms are subleading. This yields a solution where  $x^* = y^* = \pm\sqrt{6(\mu + 2)}$  for  $\mu > -2$ , implying that there is a supercritical pitchfork bifurcation. This occurs when  $\Delta = \lambda_+ \lambda_- = 0$ , which actually means  $\lambda_+ = 0$  first. As we vary  $\mu$  here the eigenvalue crosses from negative to positive values and the stability changes.

### Hopf Bifurcations

A *Hopf bifurcation* occurs when a spiral trajectory changes stability when a parameter is varied, and this stability change is accompanied by the creation or destruction of limit

cycles. A Hopf bifurcation is like a pitchfork bifurcation except that the limit cycle replaces the “fork” in the pitchfork. Both supercritical and subcritical Hopf bifurcations exist in analogy to pitchfork bifurcations. Here the transition of the eigenvalues of the linearized system is different, with the real part of both eigenvalues switching sign simultaneously, as pictured below:

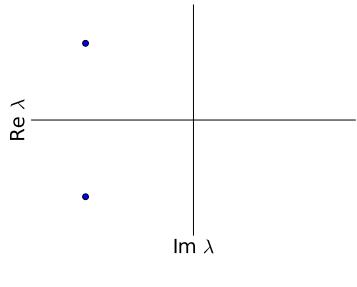


Figure 7.52:  $\text{Re}(\lambda) < 0$   
which gives us a stable spiral

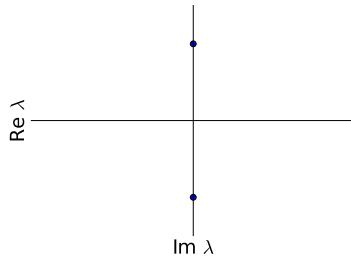


Figure 7.53:  $\text{Re}(\lambda) = 0$

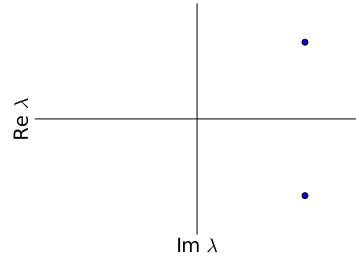


Figure 7.54:  $\text{Re}(\lambda) > 0$   
which gives us an unstable spiral

**Example:** Consider in polar coordinates the system

$$\dot{r} = \mu r - r^3, \quad \dot{\theta} = \omega + br^2. \quad (7.64)$$

It has a stable spiral into  $r^* = 0$  for  $\mu < 0$  and no limit cycles. For  $\mu > 0$ , then  $r^* = \sqrt{\mu}$  is a stable limit cycle, while the spiral from  $r^* = 0$  becomes unstable. Thus,  $\mu = 0$  is a supercritical Hopf bifurcation.

If we look at the eigenvalues of the linearized system at  $r = 0$  by setting  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ , then  $\dot{x} \approx \mu x - \omega y$  and  $\dot{y} \approx \omega x + \mu y$ , so  $\lambda_{\pm} = \mu \pm i\omega$  which indeed hits  $\text{Re}(\lambda) = 0$  when  $\mu = 0$  as expected. The flows for this Hopf bifurcation are depicted below.

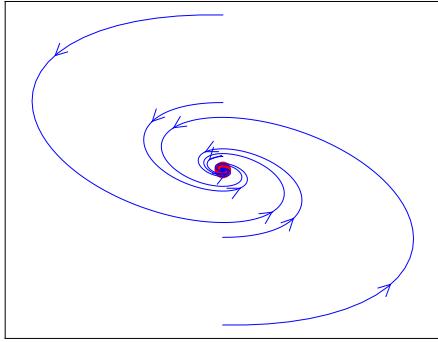


Figure 7.55: System for  $\mu = -0.5$ ,  $b = 2$  and  $\omega = 1$

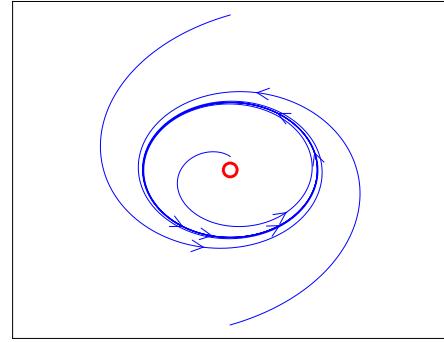


Figure 7.56: System for  $\mu = 1$ ,  $b = 2$  and  $\omega = 1$

**Example:** Consider the following system of equations in polar coordinates:

$$\dot{r} = \mu r + r^3 - r^5, \quad \dot{\theta} = \omega + br^2, \quad (7.65)$$

which has a subcritical Hopf bifurcation at  $\mu = 0$ .

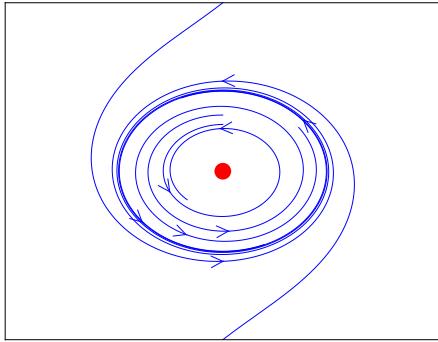


Figure 7.57: System for  $\mu = -0.2$ ,  $b = 2$  and  $\omega = 1$ . One of the inner orbits converges to the center while the other converges to the outer limit cycle, there is an unstable limit cycle between the two.

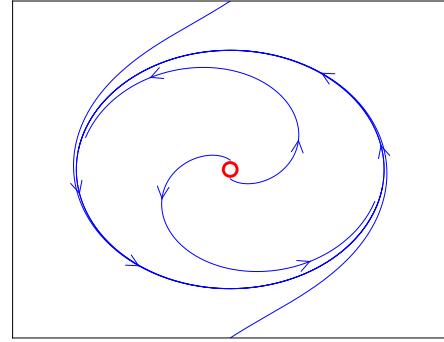


Figure 7.58: System for  $\mu = 1$ ,  $b = 2$  and  $\omega = 1$ . There is no longer an unstable limit cycle in the inner region of the phase space.

**Example:** As a physics example with a limit cycle, let's consider a damped pendulum driven

by a constant torque whose equation of motion is

$$\ddot{\theta} + \frac{1}{q}\dot{\theta} + \sin(\theta) = \tau \quad \Rightarrow \quad \begin{cases} \dot{\theta} = \omega \\ \dot{\omega} = -\frac{1}{q}\omega - \sin(\theta) + \tau \end{cases} \quad (7.66)$$

For  $\tau \leq 1$ , the fixed points are  $\omega^* = 0$  and  $\sin(\theta^*) = \tau$ , for which there are two solutions given by the solutions to  $\theta^* = \arcsin(\tau)$ . The graphical solution for the fixed points is shown below where we compare  $\sin \theta$  to the constant  $\tau$  and observe where they cross. One fixed point is stable and the other is a saddle point.<sup>2</sup>

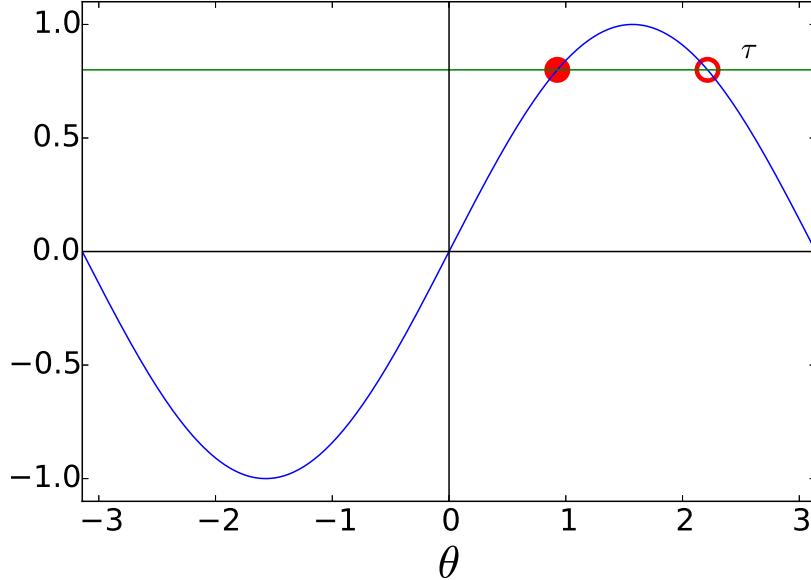


Figure 7.59: Graphical determination of the  $\theta$  value of the fixed points. We see that they cannot occur if  $\tau > 1$ .

What if  $\tau > 1$ ? It turns out that there is a unique stable limit cycle attractor. Consider

$$\dot{\omega} = -\frac{1}{q}[\omega - q(\tau - \sin \theta)] \quad (7.67)$$

For  $\tau > 1$  there are no fixed points, however for very negative  $\omega$ , then  $\dot{\omega} > 0$  and for very positive  $\omega$ ,  $\dot{\omega} < 0$ . There is thus a trapping region where the system has no fixed points, which by the Poincaré-Bendixson theorem implies the existence of a limit cycle. This limit

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<sup>2</sup>See also our earlier analysis of the overdamped oscillator in Eq. (7.35), which used a slightly different definition for the constants ( $q\tau \rightarrow \tau$  and  $q \rightarrow b$ ).

cycle corresponding to rotations of the pendulum over the top. The motion of two trajectories with the same initial conditions, but with  $\tau < 1$  and  $\tau > 1$ , are shown in Fig. 7.60.

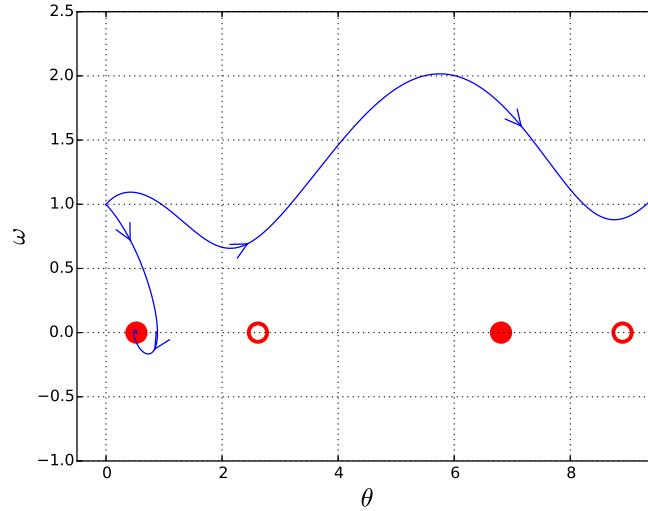


Figure 7.60: Two trajectories shown from the same initial condition, one with  $\tau < 1$  (where the trajectory converges to stable point), and one with  $\tau > 1$  (where the trajectory continues indefinitely).

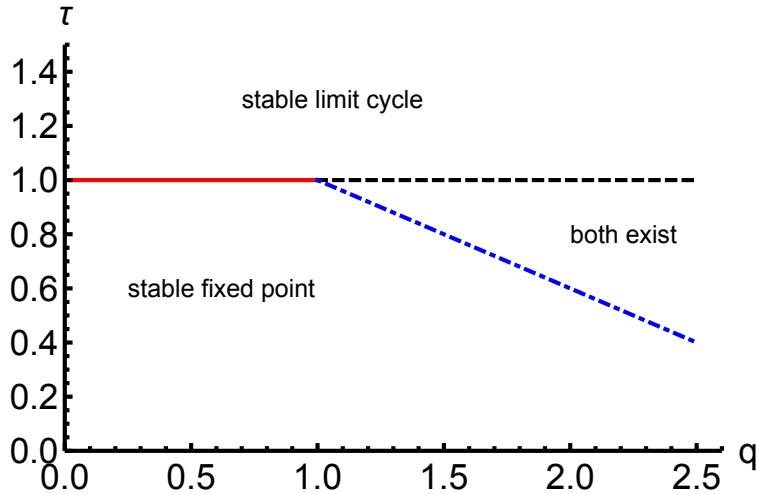


Figure 7.61: Stable attractors and bifurcation transitions for a pendulum with a constant applied torque.

In fact for  $q > 1$  the limit cycle also exists for a range of values  $\tau_c < \tau < 1$ . Since both the fixed points and limit cycle exist for these parameter values the endpoint of the motion

depends on the choice of initial conditions. Here  $\tau_c$  is a constant that depends on  $q$ , and as  $q \rightarrow 1$  then  $\tau_c \rightarrow 1$ . For  $q \leq 1$  the limit cycle only exists for  $\tau \geq 1$ . The boundaries between these regions are sketched in Fig. 7.61. For the transition between the cycle and fixed points shown as a solid (red) line, the saddle and stable node fixed points are born on the cycle which then disappears (called an  $\infty$ -period bifurcation). The transition across the dashed (black) line is a saddle node bifurcation where the two fixed points are born, but the saddle persists. Finally, for the transition across the dot-dashed (blue) line the saddle point collides with and destabilizes the cycle, so that it seeks to exist in the region to the left (this is called a homoclinic bifurcation). Although we have not tried to classify the full range of possible bifurcations for systems involving a limit cycle, this example has illustrated a few of the possibilities.

## 7.5 Chaos in Maps

In nonlinear systems with 2 variables, we have obtained a qualitative analytic understanding of the motion by analyzing fixed points and limit cycles. The analysis of 2 variables includes the possible motion for a 1-dimensional particle with two phase space variables. There is no chaos with 2 variables. We could study chaos with 3 variables, but is there a simpler way?

Recall that chaos in the Poincaré map of the damped driven nonlinear oscillator could be found from

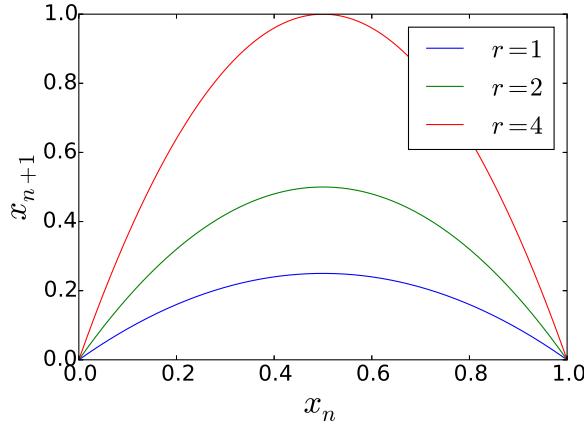
$$\begin{cases} \theta_{N+1} = f_1(\theta_N, \omega_N) \\ \omega_{N+1} = f_2(\theta_N, \omega_N) \end{cases} \quad (7.68)$$

which are 2 discrete variables. Here we set  $\varphi = 2\pi N$  to be discrete with  $N$  an integer. Uniqueness for the 3 continuous variable solution implies the existence of  $f_1$  and  $f_2$ . In fact, for general systems, we can go a step further. Chaos already exists in 1-dimensional maps  $x_{N+1} = f(x_N)$  for a nonlinear function  $f$ .

The example we will be using to illustrate chaos in maps is the logistic map:

$$x_{n+1} = f(x_n) = rx_n(1 - x_n) \quad (7.69)$$

which has a parameter  $r$ . If we take  $0 < r \leq 4$ , then the  $\{x_n\}$  are bounded by  $0 \leq x \leq 1$ , since the maximum is  $f(1/2) = r/4$ . We can visualize this solution by a plot in the  $x_n$ - $x_{n+1}$  plane:



The fixed points of a general map satisfy:

$$x^* = f(x^*) \quad (7.70)$$

which is slightly different from nonlinear differential equations, as these are now iterated difference equations. For our logistic map example this gives

$$x^* = rx^*(1 - x^*) \Rightarrow \begin{cases} x^* = 0 & \text{for all } r \\ x^* = 1 - \frac{1}{r} & \text{for } r > 1 \end{cases} \quad (7.71)$$

The stability of a fixed point can be found by checking a small perturbation

$$x_N = x^* + \eta_N \Rightarrow x_{N+1} = x^* + \eta_{N+1} = f(x^* + \eta_N) = f(x^*) + \left. \frac{df}{dx} \right|_{x=x^*} \eta_N + O(\eta^2) \quad (7.72)$$

to obtain

$$\eta_{N+1} = \left. \frac{df}{dx} \right|_{x=x^*} \eta_N \quad (7.73)$$

Therefore if

- $\left| \frac{df}{dx} \right|_{x=x^*} < 1 \Rightarrow \lim_{N \rightarrow \infty} \eta_N = 0$ :  $x^*$  is stable.
- $\left| \frac{df}{dx} \right|_{x=x^*} > 1 \Rightarrow \lim_{N \rightarrow \infty} \eta_N \rightarrow \infty$ :  $x^*$  is unstable.
- $\left| \frac{df}{dx} \right|_{x=x^*} = 1$ , then  $x^*$  is marginal (requiring an expansion beyond linear analysis).

For the logistic map, Eq. (7.69) we have:

$$\frac{df}{dx} = r - 2rx \Rightarrow \begin{cases} \left. \frac{df}{dx} \right|_{x=0} = r & x^* = 0 \\ \left. \frac{df}{dx} \right|_{x=x^*} = 2 - r & x^* = 1 - \frac{1}{r} \end{cases} \quad (7.74)$$

The first case is stable if  $r < 1$ , and the second is stable if  $1 < r < 3$  and unstable otherwise, which we show graphically in Fig. 7.62. Thus we find that

$$\lim_{n \rightarrow \infty} x_n = \begin{cases} 0 & r < 1 \\ 1 - \frac{1}{r} & 1 < r < 3 \end{cases} . \quad (7.75)$$

For  $r > 3$ , the  $\lim_{N \rightarrow \infty} x_N$  is not well-defined as a single number given by a fixed point. So what happens?

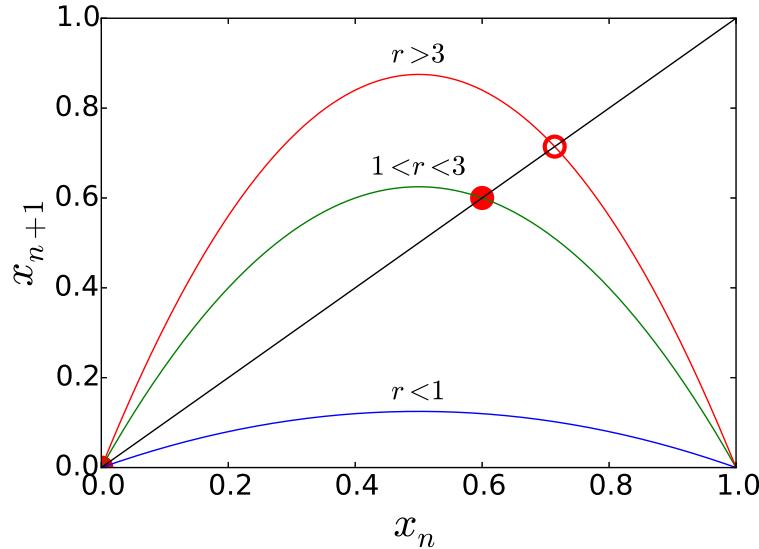


Figure 7.62: Stability of fixed points of the Logistic Map.

To find out, consider two iterations of the map:

$$x_{n+2} = f(f(x_n)) \quad (7.76)$$

which makes  $x_{n+2}$  a 4<sup>th</sup> order polynomial of  $x_n$  as shown in Fig. 7.63 on the right. Here there are three fixed points of the double iterated map, two are stable (which we call  $p$  and  $q$ ) and one is unstable. Furthermore we find that  $p = f(q)$  and  $q = f(p)$ , so the  $n \rightarrow \infty$  state of the

Logistic map is an oscillating 2-cycle as shown in Fig. 7.63 on the left. Thus the (discrete) period has doubled and we call this a pitchfork bifurcation of the map at  $r = 3$ .

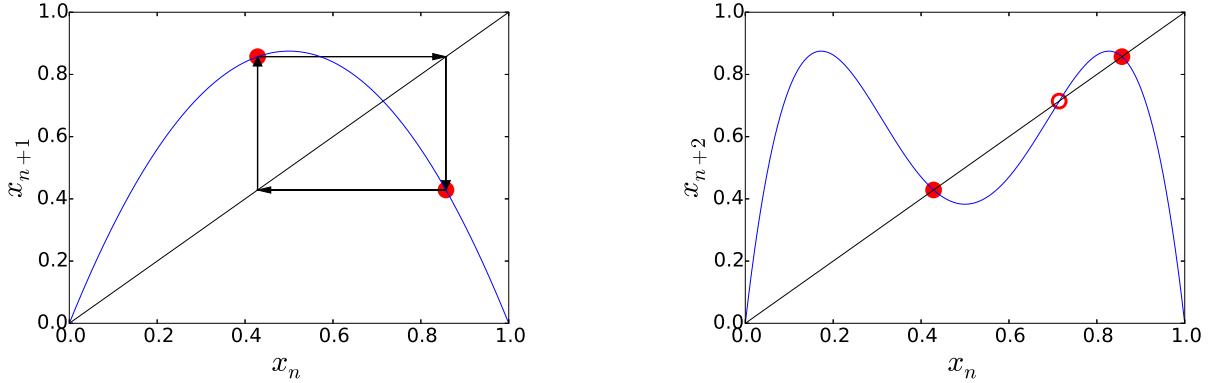


Figure 7.63: The logistic map for  $r > 3$  has fixed points for the double iterated mapping, which are a two-cycle for the original map.

If we analyze the stability of  $p$  and  $q$ , we find that more bifurcations occur for higher values of  $r$ . Since

$$\frac{d}{dx} (f(f(x)))|_{x=x^*} = \frac{df(p)}{dp} \frac{df(q)}{dq} \quad (7.77)$$

for  $x^* = p$  or  $x^* = q$ , this implies that  $p$  and  $q$  lose their stability simultaneously when  $\left| \frac{df(p)}{dp} \frac{df(q)}{dq} \right| > 1$ . At this point the 2-cycle bifurcates into a 4-cycle. This pattern of period doubling continues,  $2 \rightarrow 4 \rightarrow 8 \rightarrow 16 \rightarrow 32 \rightarrow \dots$ , until  $r = 3.5699456\dots$ . Beyond that point the map becomes chaotic. This behavior is shown in Fig. 7.64.

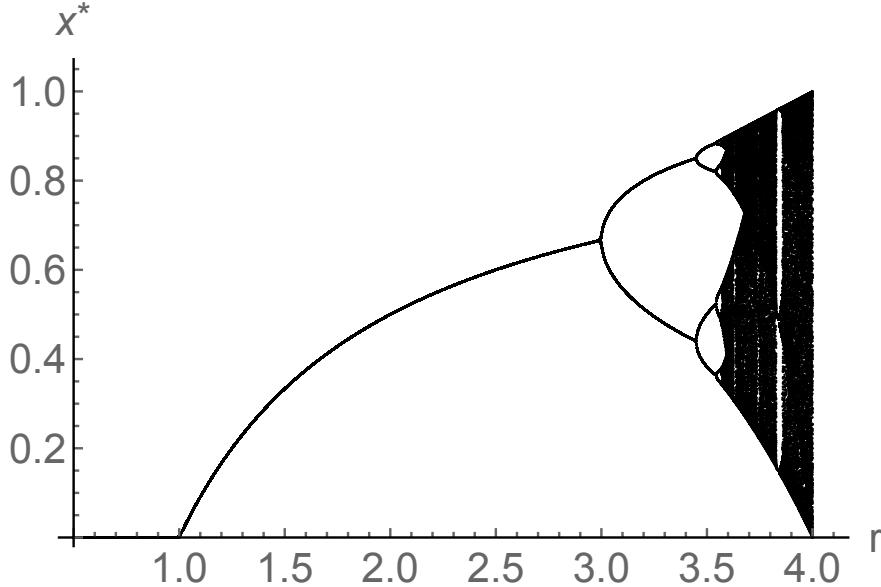


Figure 7.64: Bifurcation plot for the Logistic Map, showing points that are part of the non-transient motion as a function of  $r$ . Below  $r = 3$  there is a single fixed point. The first two bifurcations occur for  $r = 3$  and  $r = 1 + \sqrt{6}$ . Each new bifurcation is closer to the previous, until we reach the chaotic regime. In the middle of the chaotic region there are non-chaotic regions, such as the one near  $r = 3.84$  visible as a white stripe.

This is called a *period doubling road to chaos* and is one common mechanism by which chaos emerges. Indeed, this phenomena also occurs in the nonlinear damped driven oscillator. But how do we know that it is chaos?

If chaos occurs in a map, then we should have sensitivity to initial conditions. Examine

$$\begin{aligned} x_0 &\rightarrow x_1 \rightarrow x_2 \rightarrow \dots \\ x_0 + \delta_0 &\rightarrow x_1 + \delta_1 \rightarrow x_2 + \delta_2 \rightarrow \dots \end{aligned}$$

where  $\delta_n$  is the separation between two initially neighboring trajectories after  $n$  iterations. As such we expect  $\lim_{n \gg 1} |\delta_n| \approx |\delta_0| e^{\lambda n}$ ; there should be exponential separation with  $\lambda > 0$  for chaos to occur, where  $\lambda$  is called the *Lyapunov exponent*.

For maps we can derive a formula for  $\lambda$  as follows. We know that:

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{f^n(x_0 + \delta_0) - f^n(x_0)}{\delta_0} \right| \quad (7.78)$$

where  $f^n(x) = \underbrace{f(f(f(\dots x)))}_{n \text{ times}}$ . Assuming that  $\delta_0$  is very small this implies

$$\lambda \simeq \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \frac{d}{dx_0} f^n(x_0) \right| = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left| \prod_{j=0}^{n-1} \frac{df(x_j)}{dx_j} \right|, \quad (7.79)$$

where the  $x_j$  are the points along the map trajectory so far. This gives

$$\lambda = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \ln \left| \frac{df(x_j)}{dx_j} \right| \quad (7.80)$$

as a formula we can use to compute the Lyapunov exponent by keeping track of this sum as we increase  $n$ . The result is shown in Fig. 7.65. In period doubling regions  $\lambda < 0$ , while in chaotic regions  $\lambda > 0$ . There may also be periodic windows with chaos on either side. For the logistic map, the largest such window is the 3-cycle near  $r \approx 3.83$ . These windows are also clearly visible in the bifurcation diagram.

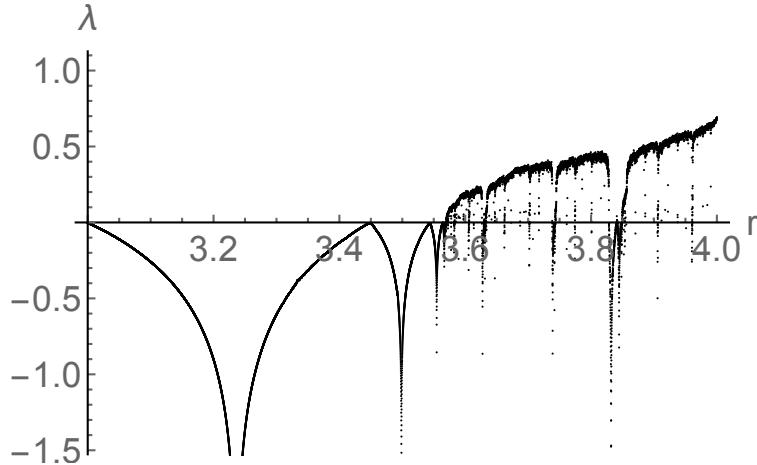


Figure 7.65: Value of the Lyapunov Exponent as a function of  $r$ . The chaotic regimes correspond to  $\lambda > 0$ . (Finite sampling leads to the discrete points.)

You may have noticed that period doubling occurs after progressively shorter intervals as the parameter  $r$  is increased in the case of the logistic map. In fact, for a wide class of maps (and nonlinear differential equations), this speed-up is characterized by a *universal* number. For a parameter  $r$ , denoting  $r_\alpha$  as the value where the  $\alpha^{\text{th}}$  period doubling occurs, then

$$\delta_F = \lim_{\alpha \rightarrow \infty} \frac{r_\alpha - r_{\alpha-1}}{r_{\alpha+1} - r_\alpha} \approx 4.669201 \quad (7.81)$$

is the *Feigenbaum number*. For the logistic map, it is easy to check that we are already pretty close to this number for small  $\alpha$ . Given this, we can estimate where chaos starts as follows:

$$\delta_1 = r_2 - r_1 = \sqrt{6} - 2 \quad \text{and} \quad \delta_n = r_{n+1} - r_n = \frac{\delta_{n-1}}{\delta_F} = \dots = \frac{\delta_1}{\delta_F^{n-1}}, \quad (7.82)$$

so our estimate for where chaos starts is

$$r = \lim_{n \rightarrow \infty} r_n = 3 + \sum_{n=1}^{\infty} \delta_n \simeq 3 + \delta_1 \sum_{n=1}^{\infty} \delta_F^{-(n-1)} = 3 + \frac{\delta_1}{1 - \frac{1}{\delta_F}} = 3.572 \quad (7.83)$$

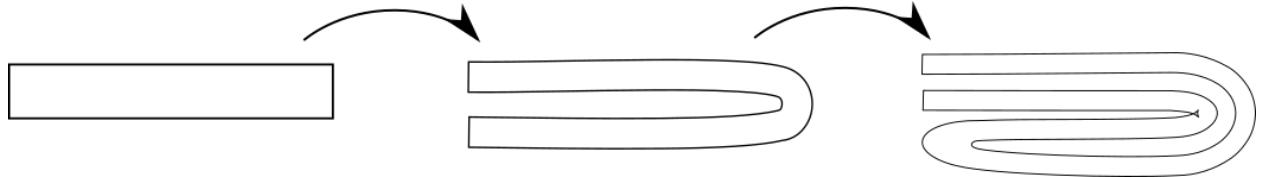
which is fairly close to the real value of  $r = 3.5699456$ .

You may be wondering how trajectories can diverge exponentially (initially) while still remaining bounded. The mechanism is by *stretching and folding* of trajectories. Think of a drop of food coloring on cookie dough which you then fold and knead. To see this more explicitly, lets once again returning to the logistic map in Eq.(7.69), but now we take  $r \simeq 4$ . Then:

$$x_n \in \left(0, \frac{1}{2}\right) \text{ is sent to } x_{n+1} \in (0, 1) \quad (7.84)$$

$$x_n \in \left(\frac{1}{2}, 1\right) \text{ is sent to } x_{n+1} \in (1, 0) \quad (7.85)$$

so the two resulting intervals are the same however in opposite directions. Together the original  $(0, 1)$  interval is both stretched and bent. We then repeat this, with the phase space structure getting progressively more complicated as depicted below:



Finally, there is a self-similarity property of the bifurcation diagram for the logistic map. When we zoom in on regions of smaller scales of  $r$ , we see the same picture again, including the periodic windows, chaotic regions, and period doubling. This is a property of fractals that we'll see shortly.

## 7.6 Chaos in Differential Equations, Strange Attractors, and Fractals

### 7.6.1 The Lorenz Equations

The prototypical example of chaos in differential equations consists of the *Lorenz equations*

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases} \quad (7.86)$$

where the 3 parameters  $\sigma$ ,  $r$ , and  $b$  are all positive. Note the symmetry under  $x \rightarrow -x$  and  $y \rightarrow -y$ . Lorenz discovered chaotic behavior in his study of atmospheric modeling, which he showed also appeared in these simpler three equations. This serves as a simplified model of a fluid in a convection roll, with  $x$  being the average velocity in the loop,  $y$  being the temperature difference between the flow on the two halves of the roll, and  $z$  being the temperature difference between the inside and outside of the roll. One can think of these equations as an approximation arising from the full Navier-Stokes and heat transfer equations.

The fixed points are  $x^* = y^* = z^* = 0$  which is stable for  $r < 1$  or a saddle point for  $r > 1$ , and  $x^* = y^* = \pm\sqrt{b(r-1)}$  and  $z^* = r-1$  which only exist for  $r > 1$ . At  $r = 1$  there is a supercritical pitchfork bifurcation of the fixed point. With some work, we can show that the  $r > 1$  stable fixed points only remain stable up to  $r = r_H$ , and are unstable beyond that. At this point  $r = r_H$  the stable fixed point prongs each become unstable under subcritical Hopf bifurcations, which involve a collision with an unstable limit cycle that shrinks onto each of the fixed points. This is shown in Fig. 7.66.

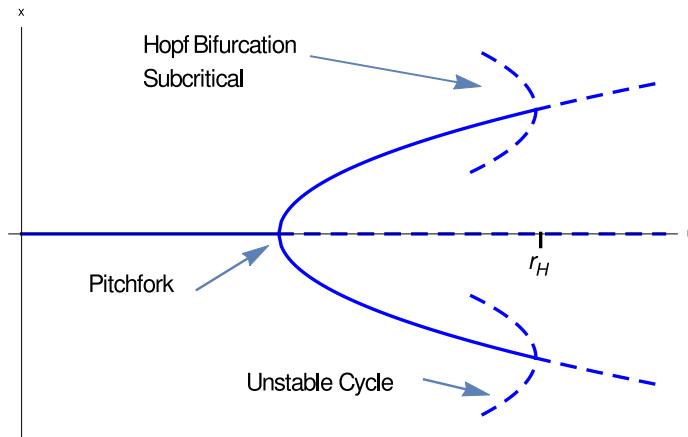


Figure 7.66: Bifurcation Diagram of the  $x$  coordinate of the Lorenz system

We also know that in this system the phase space volumes contract  $\nabla \cdot \vec{f} = -(\sigma+b+1) < 0$  (so the system is dissipative). It can also be shown that trajectories are (eventually) bounded by a sphere  $x^2 + y^2 + (z - r - \sigma)^2 = \text{constant}$ .

In the Lorenz system, for  $r < r_H$  the trajectories converge on a stable fixed point. What happens for  $r > r_H$ ? The trajectories are bounded and the phase space volume shrinks, but there are no stable fixed points or stable limit cycles to serve as attractors. Instead, we have chaos with a *strange attractor*, which is depicted in Fig. 7.67.

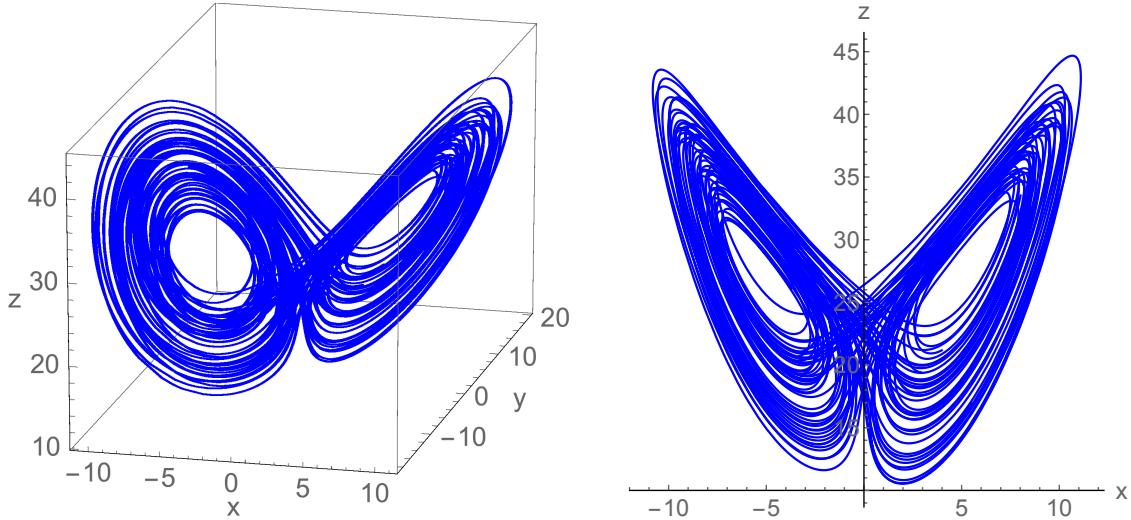


Figure 7.67: Strange attractor in the chaotic regime of the Lorenz equations, shown in the 3-dimensional space as well as for the  $x-z$  projection.

In a strange attractor, the trajectories still never cross (in the 3 dimensions), and the attractor trajectory exhibits *exponential* sensitivity to initial conditions. It also has zero volume consistent with  $\dot{V} = \int \nabla \cdot \vec{f} dV$ , but interestingly, it has infinite surface area! There are infinitely many surfaces traced out by cycles near the fixed points, so the attractor is a *fractal*. For the Lorenz system, surfaces are different after each pass from  $x > 0$  to  $x < 0$  and vice versa; this attractor is a fractal with dimension  $2 < D < 3$ . In fact  $D \simeq 2.05$  in this case.

How can we have exponential divergence of trajectories while the phase space volume shrinks? For the Lorenz system, we have 3 variables, so there are 3 directions in which trajectories can converge or diverge. (In general, these directions are more complicated than simply fixed Cartesian axes. We must find the principal axes at each time.) Thus there are in principle 3 exponents governing the trajectories:

$$\delta_j = \delta_{j0} e^{\lambda_j t} \text{ for } j \in \{1, 2, 3\} \quad \Rightarrow \quad V(t) \approx V_0 e^{(\lambda_1 + \lambda_2 + \lambda_3)t} \quad (7.87)$$

For a case where  $\nabla \cdot \vec{f}$  is constant this means that:

$$\dot{V} = (\lambda_1 + \lambda_2 + \lambda_3)V(t) = \int \nabla \cdot \vec{f} dV = \nabla \cdot \vec{f} V(t) < 0. \quad (7.88)$$

Note that  $\nabla \cdot \vec{f} = -(\sigma + b + 1)$  is constant for the Lorenz system. This means  $\lambda_1 + \lambda_2 + \lambda_3 = \nabla \cdot \vec{f} < 0$ , and the system is dissipative if the sum of the exponents is negative, indicating that the volume shrinks overall. But exponential sensitivity to initial conditions only requires  $\lambda_j > 0$  for at least one value of  $j$ . Here, the *Lyapunov exponent* is defined as  $\lambda \equiv \max(\lambda_1, \lambda_2, \lambda_3)$ .

For the nonlinear damped driven oscillator, we also have  $\lambda_1 + \lambda_2 + \lambda_3 = \nabla \cdot \vec{f} = -\frac{1}{q} < 0$ . Here, things are even simpler because  $\varphi = \omega_D t$  has  $\lambda_3 = 0$ , so  $\lambda_1 + \lambda_2 = -\frac{1}{q}$ . For the undamped case ( $q \rightarrow \infty$ ), we can still have chaos with  $\lambda_1 = -\lambda_2 > 0$ . Thus we note that chaos can occur in both conservative and dissipative systems.

We can think of an area in phase space as it gets stretched and contracted as pictured below. Here it is stretched by the exponent  $\lambda_1 > 0$  and contracted by  $\lambda_2 < 0$ . If trajectories are bounded then it must also get folded.

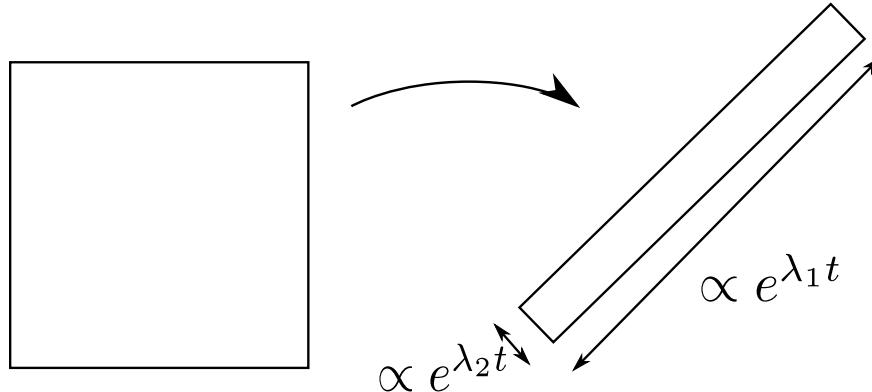


Figure 7.68: The action of the system leads to stretching and rotation of phase space volume.

### 7.6.2 Fractals and the Connection to Lyapunov Exponents

Fractals are characterized by a nontrivial structure at arbitrarily small length scales. In particular, they are self-similar (in that they contain copies of themselves at arbitrarily small scales).

**Example:** The Cantor set fractal is created by iteratively removing the middle  $\frac{1}{3}$  of a line segment. If  $S_0$  is the line segment  $0 \leq x \leq 1$ , then  $S_1$  is formed by removing the middle  $\frac{1}{3}$  of  $S_0$ ,  $S_2$  is formed by removing the middle  $\frac{1}{3}$  of each piece of  $S_1$ , and so on until the true Cantor set emerges as  $S_\infty$ . This is pictured in Fig. 7.69. Here the number of separate pieces grows infinitely large (and is in fact non-denumerable), while the total length of the pieces

tends to zero. (This is the one dimensional analog of area  $\rightarrow \infty$  with volume  $\rightarrow 0$  for the Lorentz equations strange attractor.) The Cantor set also has the self-similar property.



Figure 7.69: Illustration of the iterative procedure that produces the Cantor Set.

How do we define a non-integer dimension for fractals? Let us consider covering a line of length  $a_0$  with segments of length  $a$ . We would need  $N(a) = \frac{a_0}{a}$  segments. For a square of side length  $a_0$  covered by smaller squares of side length  $a$ , we would need  $N(a) = \left(\frac{a_0}{a}\right)^2$  squares. In general, for a  $D$ -dimensional hypercube of side length  $a_0$  covered by  $D$ -dimensional hypercubes of side length  $a$ , we would need  $N(a) = \left(\frac{a_0}{a}\right)^D$  such hypercubes for integer  $D$ . This can be generalized beyond integers to

$$d_F = \lim_{a \rightarrow 0} \frac{\ln(N(a))}{\ln\left(\frac{a_0}{a}\right)} \quad (7.89)$$

which is the *Hausdorff dimension* (also called the capacity dimension or the fractal dimension).

**Example:** in the Cantor set, after  $n$  steps, the number of segments is:

$$N(a) = 2^n \quad (7.90)$$

while the length of each segment goes as:

$$a_n = \frac{a_0}{3^n} \quad (7.91)$$

Thus the fractal dimension is given by:

$$d_F = \lim_{n \rightarrow \infty} \frac{\ln(2^n)}{\ln(3^n)} = \frac{\ln(2)}{\ln(3)} \simeq 0.6309 \quad (7.92)$$

indicating that it is less than a line with  $d_F = 1$  but more than a point with  $d_F = 0$ .

In general, fractal dimensions are not integers and are usually irrational.

**Example:** The Koch curve is like the Cantor set, except that instead of deleting the middle  $\frac{1}{3}$  of every segment, we replace it by an equilateral triangle on the other two sides, so segments are overall added rather than removed. The Koch curve corresponds to one of the sides of the Koch Snowflake depicted below in Fig. 7.70. In this case:

$$N(a) = 4^n \quad \text{and} \quad a_n = \frac{a_0}{3^n} \quad \Rightarrow \quad d_F = \frac{\ln(4)}{\ln(3)} \simeq 1.262 \quad (7.93)$$

which satisfies  $1 < d_F < 2$ . This means the Koch curve has infinite length (1-dimensional volume) but zero area (2-dimensional volume).

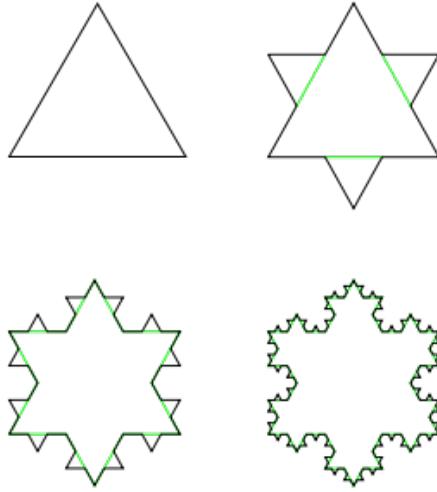


Figure 7.70: The Koch Curve corresponds to starting with just one of the 3 sides of the triangle used to generate the Koch Snowflake shown here.

We can connect the notion of a fractal dimension to Lyapunov exponents which govern the behavior of physical trajectories. For simplicity, let us consider an example with  $\lambda_1 > 0$ ,  $\lambda_2 < 0$ , and  $\lambda_3 = 0$ . The area of a square of phase space points evolves as:

$$A(t=0) = a_0^2 \rightarrow A_0(t) = a_0^2 e^{(\lambda_1 + \lambda_2)t} \quad (7.94)$$

while the squares covering it have area  $A(t) = a_0^2 e^{2\lambda_2 t}$ , see Fig. 7.71. Therefore

$$N(t) = \frac{A_0(t)}{A(t)} = e^{(\lambda_1 - \lambda_2)t} \quad (7.95)$$

This gives rise to a fractal dimension of:

$$d_F = 1 + \frac{\lambda_1}{|\lambda_2|} \quad (7.96)$$

which is the *Kaplan-Yorke relation*. A fixed point attractor has  $d_F = 0$ , and a limit cycle attractor has  $d_F = 1$ . By contrast, a strange attractor generally has a non-integer  $d_F$ , and this dimension is related to the sensitivity to initial conditions (given by  $\lambda_1$ ) as well as to the contraction of phase space (given by  $\lambda_2$ ).

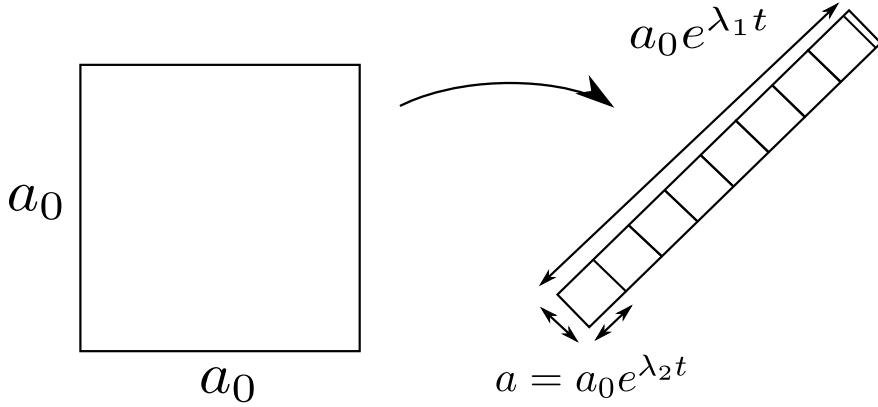


Figure 7.71: As the system evolves the phase space volume changes, so our tiling volume changes as well.

### 7.6.3 Chaos in Fluids

Chaos can occur in fluids as well. If we take  $\nabla \cdot \mathbf{v} = 0$  and  $\rho$  to be constant and uniform, the Navier-Stokes equation says:

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{v} - \frac{\nabla \mathcal{P}}{\rho} + \nu \nabla^2 \mathbf{v} \quad (7.97)$$

and this should be used in conjunction with the heat transfer equation. In the language we have been using in this chapter, the velocity field  $\mathbf{v}(\mathbf{x}, t)$  corresponds to a continuum of variables (each labeled by  $\mathbf{x}$ ). One can also think of the terms involving  $\nabla \mathbf{v}$  as couplings between these variables, like finite differences, for example:

$$\frac{\partial v_x}{\partial x} \approx \frac{v_x(x + \epsilon) - v_x(x - \epsilon)}{2\epsilon} \quad (7.98)$$

In some cases (as in convection rolls per the Lorenz equations), we can have aperiodic time dependence but spatial regularity in  $\mathbf{x}$ . Here, many of the ideas that we have studied (like, for example, the period doubling road to chaos) apply. In other cases, the spatial structure in  $\mathbf{x}$  also becomes irregular. The regularity (or lack thereof) can also depend on initial conditions. This happens, for example, in fat convection rolls in shallow fluids. Essentially there could be multiple attractors present. For the case with irregularity in  $\mathbf{x}$ , the dimensionality of the attractor is proportional to the size of the system, which is very large! Here it makes more sense to speak of a “dimension density”.

Strong turbulence in a fluid falls in the category of being irregular in  $\mathbf{x}$  with no characteristic size for features. This is certainly more advanced than our examples, and indeed a full formalism for turbulence remains to be invented. One thing we can do to characterize strong turbulence is apply dimensional analysis.

There are several scaling laws for turbulence in 3 dimensions. Recall that vortices (eddies) appear at all length scales  $\lambda$  and are efficient at transferring energy. Let us define  $L$  as the size of the fluid container,  $\lambda_0$  as the scale where dissipation is important (for Reynolds number  $R \approx 1$ ),  $\epsilon$  as the mean energy transfer per unit time per unit mass, and  $v_\lambda$  as the velocity variation at length scale  $\lambda$ . Note that the dimensions  $[\nu] = m^2/s$  and  $[\epsilon] = (kgm^2/s^2)(1/kg) = m^2/s^3$ . There are three scales to consider.

1. At  $\lambda \approx L$ , there can be no dependence on  $\nu$ , so  $\epsilon \propto \frac{v_L^3}{L}$ . (This is the scale with the most kinetic energy and the largest energy.)
2. At  $\lambda_0 \ll \lambda \ll L$ , there can still be no  $\nu$ , so here  $\epsilon \propto \frac{v_\lambda^3}{\lambda}$ . Note that this is independent of the properties  $\rho$ ,  $\nu$  and the scale  $L$  of the fluid!
3. At  $\lambda \approx \lambda_0$ , because  $R = \frac{v_0 \lambda_0}{\nu} \approx 1$ , then  $v_0 \approx \frac{\nu}{\lambda_0}$ . This is where the energy dissipation occurs. Here we only have  $\nu$  and  $\lambda_0$  present, so  $\epsilon \propto \frac{\nu^3}{\lambda_0^4}$ .

Rather than using  $\lambda$  and  $v_\lambda$ , the universal result for the case  $\lambda_0 \ll \lambda \ll L$  is often written in terms of the wavenumber  $k \propto \frac{1}{\lambda}$  and kinetic energy per unit mass per unit wave number,  $E(k)$ . The kinetic energy per unit mass can be written as  $E(k) dk$ . Here  $E(k)$  behaves as a rescaled version of the energy with slightly different dimensions,  $[E(k)] = m^3/s^2$ . Analyzing its dimensions in relation to  $\epsilon$  and  $k$  we note that  $m^3/s^2 = (m^2/s^3)^{2/3}(1/m)^{-5/3}$  which yields

$$E(k) \sim \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}} \quad (7.99)$$

This is the famous Kolmogorov scaling law for strong turbulence. It provides a mechanism by which we can make measurements and probe a universal property of turbulence in many systems.

The End.