09/19/2010

Introduction

Non-relativistic quantum mechanics, rotational invariance simplifies scattering problems

$|\vec{k}| = |\vec{k}'| = k$ (condition for elastic scattering)

Central potential $\quad V(r) \quad (r = |\vec{r}|)$

$$\Rightarrow \psi(\vec{r}) = \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming}} + \underbrace{f(\vec{k},\vec{k}')\frac{e^{ikr}}{r}}_{\text{outgoing}}$$

Scattering amplitude

$$f(\vec{k},\vec{k}') = \sum_{l} f_l(k)\, P_l(\cos(\theta))$$

So why does relativity make things more complicated?

→ Larger symmetry group - Lorentz invariance

⇒ Particles can appear out of the vacuum.

p = proton $\qquad \bar{p}$ = antiproton

$$p + p \rightarrow p + p + p + \bar{p}$$

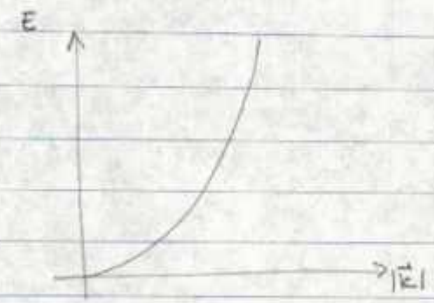
In nonrelativistic physics, the Hamiltonian is block diagonal in particle number

$$H = \begin{pmatrix} \text{0-body} & 0 & 0 & 0 \\ 0 & \text{1-body} & 0 & 0 \\ 0 & 0 & \text{2-body} & 0 \end{pmatrix}$$

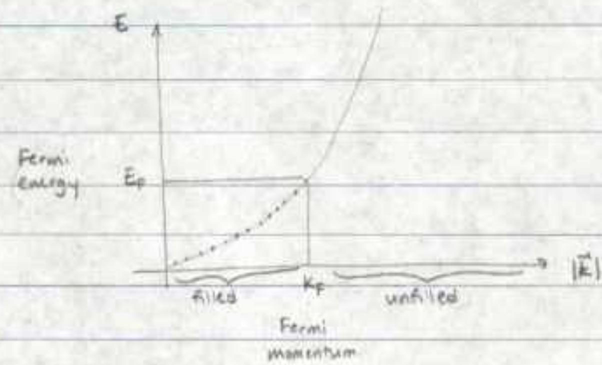
A similar thing happens in nonrelativistic many body systems

Consider a nonrelativistic electron. The kinetic energy as a function of momentum is

$$E = \frac{|\vec{k}|^2}{2m}$$



Neglect interactions for the moment. Suppose we have lots of electrons



For $|\vec{k}| > k_F$, we can add an extra electron. This is called a "particle" or "quasi-particle" excitation.

For $|\vec{k}| < k_F$, we can remove an electron. This is called a "hole" or "quasihole" excitation

Nonrelativistic scattering :

particle + particle → particle + particle + particle + hole

Relativistic Conventions + Notation

"Natural" units ⇒ $\hbar = c = 1$ (later, $k_B = 1$)

[length] = [time]

[energy] = [momentum] = [mass]

Everything can be written as MeV or kg

Metric tensor $g_{\mu\nu}$

Greek indices $\mu = \underbrace{0}_{\text{time}}, \underbrace{1, 2, 3}_{\text{space}}$ time and space

Regular indices $i = 1, 2, 3$ space only

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$g_{00} = +1$ $g_{ii} = -1$ all others $= 0$

Four vectors

$$x^\mu = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$$

For two four vectors, a^μ and b^μ, we can define a Lorentz scalar product (contraction)

$$a \cdot b = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = a^\mu b^\nu g_{\mu\nu}$$

This quantity $(a \cdot b)$ is Lorentz invariant.

Lower index object

$$x_\mu = g_{\mu\nu} x^\nu = g_{\mu 0} x^0 + g_{\mu 1} x^1 + g_{\mu 2} x^2 + g_{\mu 3} x^3$$

$$x_\mu = (x^0, -\vec{x})$$

Note: $a \cdot b = a^\mu b^\nu g_{\mu\nu} = a^\mu b_\mu = a_\mu b^\mu$

Energy-momentum four vector

$$p^\mu = (\underset{\uparrow \text{ energy}}{p^0}, \underset{\uparrow \text{ spatial momentum}}{\vec{p}})$$

$$p \cdot p = (p^0)^2 - \vec{p} \cdot \vec{p} = E^2 - \vec{p}^2 = m^2$$

Short hand: $a^2 = a \cdot a \Rightarrow p^2 = m^2$

Classical Field Theory

Action

$$S = \int dt\, L = \int \underbrace{\mathcal{L}\, d^3x}_{=L}\, dt = \int \mathcal{L}\, d^4x$$

L - Lagrangian

$\mathcal{L}$ - Lagrange density

Let ϕ be a real-valued function of spacetime

$\mathcal{L}$ depends on ϕ and derivatives of ϕ $(\partial_\mu \phi)$

Initial configuration $\phi(t_i, \vec{x}) = f_{initial}(\vec{x})$

Final configuration $\phi(t_f, \vec{x}) = f_{final}(\vec{x})$

We get ϕ at intermediate times by extremizing S.

$$\phi = \phi_{stationary} + \delta\phi$$

$\phi \to \phi + \delta\phi$ must give $\delta S = 0$

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right) \qquad \left(\delta(\partial_\mu \phi) = \partial_\mu (\delta\phi) \right)$$

$$= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) - \left(\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi \right)$$

$$= \int d^4x \left(\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right) \delta\phi \right) + \cancel{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \Big|_{initial}^{final}}^{\to 0} = 0$$

$$\Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0}$$ Euler-Lagrange equations.

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Euler-Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$$

Ex. $\mathcal{L} = \phi^2$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 2\phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$$

$\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = ?$$

$$\mathcal{L} = \partial_\mu \phi_1 \, \partial^\mu \phi_2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \bigg|_{\phi_1 = \phi_2 = \phi}$$

$$= \partial^\mu \phi_2 + \partial^\mu \phi_1 \bigg|_{\phi_1 = \phi_2 = \phi}$$

$$= 2 \partial^\mu \phi$$

Klein-Gordon Field (Real)

Let $\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \qquad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

Euler-Lagrange $\rightarrow$ $\quad -m^2 \phi = \partial_\mu (\partial^\mu \phi)$

$$\boxed{(\partial_\mu \partial^\mu + m^2) \phi = 0}$$ Klein-Gordon Equation

$$\partial_\mu \partial^\mu = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2$$

$$\phi(x) = e^{-ip\cdot x}$$

$$\partial_\mu \partial^\mu \left(e^{-ip\cdot x}\right) = (-ip_\mu)(-ip^\mu)\, e^{-ip\cdot x}$$

$$= -p^2 e^{-ip\cdot x}$$

$$(-p^2 + m^2) = 0 \quad \Rightarrow \quad p^2 = m^2$$

Relativistic particle with mass m

Noether's Theorem

For every continuous symmetry, there exists a conserved current j^μ ($\mu = 0,1,2,3$), which implies a local conservation law.

A conserved current is an object j^μ such that

$$\partial_\mu j^\mu = 0$$

$$\partial_0 j^0 + \partial_1 j^1 + \partial_2 j^2 + \partial_3 j^3 = 0$$

$$\partial_0 = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial t}$$

$$\partial_1 = \frac{\partial}{\partial x^1}$$

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right\rangle = \langle \partial_1, \partial_2, \partial_3 \rangle$$

$$\partial_0 j^0 + \vec{\nabla}\cdot\vec{j} = 0$$

$$\boxed{\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j}}$$

Charge Q in some spatial region

$$Q = \int j^0 \, d^3x$$

If j^μ is a conserved current, then

$$\frac{dQ}{dt} = \int \frac{\partial j^0}{\partial t} \, d^3x = -\int \vec{\nabla} \cdot \vec{j} \, d^3x$$

$$= -\oint \vec{j} \cdot d^2s \quad \text{(surface integral - Gauss' law)}$$

Local conservation law

Assume there exists a symmetry group that leaves $\mathcal{L}$ the same.

Consider an infinitesimal change

$$\phi(x) \to \phi'(x) = \phi(x) + \underset{\substack{\uparrow \\ \text{infinitesimal} \\ \text{parameter}}}{\alpha} \cdot \Delta\phi(x)$$

If $\mathcal{L}$ is invariant, then

$$\mathcal{L} \to \mathcal{L} + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi} (\alpha \cdot \Delta \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi)}_{=0}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} (\alpha \Delta \phi) = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi)$$

Euler-Lagrange $\to$ $\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) (\alpha \Delta \phi) = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi)$$

$$\partial_\mu \left(\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) (\alpha \Delta \phi) \right) = 0$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi \quad \to \quad \text{conserved current (density)}$$

Example : Massless real Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$$

Symmetry $\rightarrow$ $\phi(x) \rightarrow \phi(x) + \alpha$ (α - constant)

$\Delta\phi(x) = 1$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi = \partial^\mu \phi$$

Check $\rightarrow$ $\partial_\mu j^\mu = 0$

$$\partial_\mu \partial^\mu \phi = 0 \ ?$$

Klein-Gordon $(\partial_\mu \partial^\mu + m^2)\phi = 0$

$m = 0$ $\partial_\mu \partial^\mu \phi = 0$ ✓

Complex Klein-Gordon Field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

ϕ is a complex field

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$\phi^* \phi = \frac{1}{2}\phi_1^2 + \frac{1}{2}\phi_2^2$$

$$(\partial_\mu \phi^*)(\partial^\mu \phi) = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2)$$

This is just 2 real Klein-Gordon fields with the same mass m

Lagrange density has a symmetry:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Infinitesimal $\rightarrow$ $\begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 - \alpha\phi_2 \\ \phi_2 + \alpha\phi_1 \end{pmatrix}$$

$$\Delta\phi_1 = -\phi_2$$
$$\Delta\phi_2 = \phi_1$$

Conserved current $\rightarrow$ $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} \Delta\phi_1 + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} \Delta\phi_2$

$$j^\mu = (\partial^\mu \phi_1)(-\phi_2) + (\partial^\mu \phi_2)(\phi_1)$$

$$= -\phi_2 \partial^\mu \phi_1 + \phi_1 \partial^\mu \phi_2$$

Check $\partial_\mu j^\mu = 0$

$$\partial_\mu j^\mu = -(\partial_\mu \phi_2)(\partial^\mu \phi_1) - \phi_2(\partial_\mu \partial^\mu \phi_1) + (\partial_\mu \phi_1)(\partial^\mu \phi_2) + \phi_1(\partial_\mu \partial^\mu \phi_2)$$

$$= -\phi_2(-m^2\phi_1) + \phi_1(-m^2\phi_2)$$

$$\partial_\mu j^\mu = 0 \quad \checkmark$$

Or write

$$j^\mu = i\phi\partial^\mu\phi^* - i\phi^*\partial^\mu\phi$$

We can get this faster.

Shortcut: Think of ϕ and ϕ^* as independent fields

$$\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \phi^* = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

$$\phi \to \phi' = e^{+i\alpha}\phi$$
$$\phi^* \to \phi^{*\prime} = e^{-i\alpha}\phi^*$$

Infinitesimal $\quad \phi \to \phi + i\alpha\phi \qquad \Delta\phi = i\phi$

$\qquad \phi^* \to \phi^* - i\alpha\phi^* \qquad \Delta\phi^* = -i\phi^*$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)}\Delta\phi^*$$

$$= (\partial^\mu \phi^*)(i\phi) + (\partial^\mu \phi)(-i\phi^*)$$

$$j^\mu = i\phi(\partial^\mu \phi^*) - i\phi^*(\partial^\mu \phi) \checkmark$$

Not necessary that $\mathcal{L}(x)$ is invariant. As long as

$$S = \int d^4x \, \mathcal{L}(x)$$

is the same, the equations of motion are the same.

$$\phi \to \phi + \alpha\Delta\phi$$

$$\mathcal{L} \to \mathcal{L} + \alpha\Delta\mathcal{L}$$

Suppose $\quad \Delta\mathcal{L} = \partial_\mu I^\mu$

$$S \to S + \alpha\int \Delta\mathcal{L}\, d^4x = S + \alpha\underbrace{\int \partial_\mu I^\mu \, d^4x}_{=0}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) = \Delta \mathcal{L}$$

In our case $\quad \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) = \partial_\mu I^\mu$

Define $\quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - I^\mu$

$\partial_\mu j^\mu = 0 \quad$ - conserved current.

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Noether's Theorem

$$\phi \to \phi + \alpha \Delta \phi$$

$$\mathcal{L} \to \mathcal{L} + \alpha \Delta \mathcal{L} \qquad \Delta \mathcal{L} = \partial_\mu I^\mu$$

$$\to \quad j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - I^\mu$$

$$\partial_\mu j^\mu = 0$$

Example: Spacetime translation

$$\phi(x) \to \phi'(x) = \phi(x+a)$$

a^μ spacetime independent four-vector $(\mu = 0, 1, 2, 3)$

μ - type of translation ($\mu = 0$ - time, $\mu = 1, 2, 3$ - space)

Infinitesimal a^μ

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x)$$

For translations in the μ direction

$$\mathcal{L} \to \mathcal{L} + \alpha \partial_\nu I^\nu \qquad (\alpha = a^\mu)$$

$$I^\nu = \delta^\nu_\mu \mathcal{L}$$

$$a^\mu \partial_\nu I^\nu = a^\mu \partial_\nu (\delta^\nu_\mu \mathcal{L}) = a^\mu \partial_\mu \mathcal{L}$$

For time translations $(\mu=0)$

$$I^\nu = (\mathcal{L}, 0, 0, 0)$$

For spatial translations $(\mu=1)$

$$I^\nu = (0, \mathcal{L}, 0, 0)$$

Similarly for $\mu = 2, 3$

Now look at j^ν

$$j^\nu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \Delta\phi - I^\nu$$

$$= \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L}$$

Make μ explicit

$$T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L}$$

$$T^{\nu\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial^\mu \phi - g^{\nu\mu} \mathcal{L} \qquad (g^{\mu\alpha}\delta^\nu_\alpha = g^{\nu\mu})$$

$T^{\nu\mu}$ is the energy-momentum tensor

For j^ν, $\nu = 0 \rightarrow$ charge density, $\nu = 1,2,3 \rightarrow$ current density

Consider time translation $(\mu=0)$ and charge density $(\nu=0)$

Hamiltonian $H = \int T^{00} \, d^3x$

Consider spatial translation $(\mu=1,2,3)$ and charge density $(\nu=0)$

Momentum (physical) $P^i = \int T^{0i} \, d^3x$

Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L}$$

$$= \frac{1}{2}(\partial_0 \phi)(\partial^0 \phi) - \frac{1}{2}(\partial_i \phi)(\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

$$\therefore H = \int T^{00} d^3x = \int \left(\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 \right) d^3x$$

Note: All terms are positive (sum of squares of things)
$\rightarrow$ can't fall to arbitrarily negative energy

Quantization

$$[q,p] = qp - pq = i$$

Harmonic oscillator

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2$$

Ladder operators

annihilation $\quad a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$

creation $\quad a^+ = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$

$$a^+ a = \frac{1}{\omega} H - \frac{1}{2}$$

$$H = \omega\left(a^+ a + \frac{1}{2}\right)$$

Ground state $|0\rangle$

Satisfies $a|0\rangle = 0$

Build the other states using powers of a^+

$$|n\rangle = \frac{1}{\sqrt{n!}} \underbrace{a^+a^+ \cdots a^+}_{n} |0\rangle$$

$$[a^+a, a] = -a$$

$$[a^+a, a^+] = +a^+$$

$$[H, a] = -\omega a \qquad [H, a^+] = +\omega a^+$$

a lowers the energy by ω

a^+ raises the energy by ω

$$Ha|n\rangle = aH|n\rangle - \omega a|n\rangle$$
$$= aE_n|n\rangle - \omega a|n\rangle = (E_n - \omega)\, a|n\rangle$$

$$Ha^+|n\rangle = (E_n + \omega)\, a^+|n\rangle$$

$$H|0\rangle = \tfrac{1}{2}\omega|0\rangle \rightarrow E_0 = \tfrac{1}{2}\omega$$

$$E_n = (n + \tfrac{1}{2})\,\omega$$

Quantization of Free Field Theory

Ground state consists of vacuum $|0\rangle$ (no particles)

Next there are one-particle states with momentum $\vec{p}$.

Label these states as $|\vec{p}\rangle_{NR}$ (non-relativistic normalization)

$$_{NR}\langle\vec{p}\,|\,\vec{p}\,'\rangle_{NR} = (2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')$$

In this course, we use the following conventions

$$\tilde{f}(p) = \int_{-\infty}^{\infty} dx\, e^{ipx}\, f(x)$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, e^{-ipx}\, \tilde{f}(p)$$

* For each dp you have a $\frac{1}{2\pi}$.

This results in a (2π) for each $\delta(p-p')$

Example: $\frac{d^3p}{(2\pi)^3}$ or $\frac{d^4p}{(2\pi)^4}$

$(2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')$ or $(2\pi)^4\,\delta^4(p-p')$

The energy for each particle with momentum $\vec{p}$ is

$$\omega_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$$

(Later we will use $E_{\vec{p}}$ for this)

For each $\vec{p}$, we have a harmonic oscillator

$$a_{\vec{p}},\ a^\dagger_{\vec{p}} \rightarrow \quad a_{\vec{p}}|0\rangle = 0 \qquad a^\dagger_{\vec{p}}|0\rangle = |\vec{p}\rangle_{NR}$$

The harmonic oscillators for each $\vec{p}$ do not interfere with each other.

$$[a_{\vec{p}}, a_{\vec{p}'}] = 0 = [a^{\dagger}_{\vec{p}}, a^{\dagger}_{\vec{p}'}]$$

$$[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] \propto \delta^3(\vec{p}-\vec{p}\,')$$

$$\underbrace{[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}]}_{\text{just a number, not an operator}} = \langle 0| [a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] |0\rangle \qquad (\langle 0|0\rangle = 1)$$

$$= \langle 0| (a_{\vec{p}} a^{\dagger}_{\vec{p}'} - \cancelto{0}{a^{\dagger}_{\vec{p}'} a_{\vec{p}}}) |0\rangle$$

$$= {}_{NR}\langle \vec{p} | \vec{p}\,' \rangle_{NR}$$

$$[a_{\vec{p}}, a^{\dagger}_{\vec{p}'}] = (2\pi)^3 \, \delta^3(\vec{p}-\vec{p}\,')$$

What is the Hamiltonian for free field theory?

Single harmonic oscillator $\rightarrow$ $H = \omega(a^{\dagger}a + \frac{1}{2})$

$$H = \omega(a^{\dagger}a + \tfrac{1}{2}[a, a^{\dagger}])$$

$$H = \tfrac{1}{2}\omega(a^{\dagger}a + a a^{\dagger})$$

Free field theory $\rightarrow$

$$H = \int \frac{d^3p}{(2\pi)^3} \, \omega_{\vec{p}} \left(a^{\dagger}_{\vec{p}} a_{\vec{p}} + \underbrace{\tfrac{1}{2}[a_{\vec{p}}, a^{\dagger}_{\vec{p}}]}_{\text{strange}} \right) \quad \tfrac{1}{2}(2\pi)^3 \delta^3(\vec{0})$$

Correct ladder structure:

$$[H, a_{\vec{p}}] = -\omega a_{\vec{p}} \qquad [H, a^{\dagger}_{\vec{p}}] = +\omega a^{\dagger}_{\vec{p}}$$

The "strange" term is a divergence in the energy in the ground state.

→ Energy differences between ground state and excited states are still finite.

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Quantization of Free Field

When we integrate $\frac{d^3p}{(2\pi)^3} a^\dagger_{\vec{p}} a_{\vec{p}}$, we get the number of particles with momentum $\vec{p}$

Physical momentum operator $\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$

Harmonic oscillator

$$q = \frac{1}{\sqrt{2m\omega}}(a + a^\dagger) \qquad p = -i\sqrt{\frac{m\omega}{2}}(a - a^\dagger)$$

Absorb $\sqrt{m}$ into the definitions of q and p

$$q = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \qquad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$$

$\phi_{\vec{p}}$ analog of q (for every $\vec{p}$)

$\pi_{\vec{p}}$ analog of p

$$\phi_{\vec{p}} = \frac{1}{\sqrt{2\omega_{\vec{p}}}}(a_{\vec{p}} + a^\dagger_{\vec{p}})$$

$$\pi_{\vec{p}} = -i\sqrt{\frac{\omega_{\vec{p}}}{2}}(a_{\vec{p}} - a^\dagger_{\vec{p}})$$

Convenient to work in position space $\rightarrow$ Fourier transform

Spatial coordinate $\vec{x}$

Should we use $e^{-i\vec{p}\cdot\vec{x}}$ or $e^{+i\vec{p}\cdot\vec{x}}$?

Actually, one of each.

$$\langle 0|\phi(\vec{x})|\vec{p}\rangle_{NR} \propto e^{+i\vec{p}\cdot\vec{x}}$$ (from 1 particle to no particles)

$$\text{like} \quad \langle \vec{x}|\vec{p}\rangle \propto e^{+i\vec{p}\cdot\vec{x}}$$

$${}_{NR}\langle\vec{p}|\phi(\vec{x})|0\rangle \propto e^{-i\vec{p}\cdot\vec{x}}$$ (from no particles to 1 particle)

$$\text{like} \quad \langle\vec{p}|\vec{x}\rangle \propto e^{-i\vec{p}\cdot\vec{x}}$$

$a_{\vec{p}}$ goes with $e^{+i\vec{p}\cdot\vec{x}}$

$a_{\vec{p}}^{\dagger}$ goes with $e^{-i\vec{p}\cdot\vec{x}}$

This gives

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}}\right)$$

$$\pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \left(-i\sqrt{\frac{\omega_{\vec{p}}}{2}}\right)\left(a_{\vec{p}} e^{+i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^{\dagger} e^{-i\vec{p}\cdot\vec{x}}\right)$$

We can show

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 \quad \text{for any } \vec{x} \text{ and } \vec{y}$$

$$[\pi(\vec{x}), \pi(\vec{y})] = 0$$

$$[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^3(\vec{x}-\vec{y})$$

From last time, Hamiltonian for free field

$$H = \int d^3x \underbrace{\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - g^{00}\mathcal{L}\right)}_{=\ \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \frac{1}{2}m^2\phi^2}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi$$

In order to quantize, we define the canonical conjugate

$$\pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi(\vec{x}))} = \partial^0 \phi(\vec{x})$$

$$\left(\text{just like} \quad p = \frac{\partial L}{\partial \dot{q}}\right)$$

We can write the Hamiltonian as

$$H = \int d^3x \, \mathcal{H} = \int d^3x \left(\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \tfrac{1}{2}m^2\phi^2 \right)$$

$$p^i = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} \partial^i \phi - \cancel{g^{0i}\mathcal{L}}^{\,0} \right)$$

$$= \int d^3x \left(\pi \, \partial^i \phi \right)$$

$$\vec{P} = -\int d^3x \left(\pi \, \vec{\nabla} \phi \right)$$

Check: $H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \tfrac{1}{2}[a_{\vec{p}}, a^\dagger_{\vec{p}}] \right)$ (in book)

$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \, a^\dagger_{\vec{p}} a_{\vec{p}}$ (homework)

Lorentz Transformations and New Normalization

$|\vec{p}\rangle_{NR}$ is not invariant under Lorentz boosts.

Let's consider a particle with momentum $\vec{p} = (p_x, p_y, p_z)$ and energy E. Suppose there is another inertial frame whose axes are moving at velocity $\vec{v} = (0, 0, -\beta)$ relative to the original frame.

In the new frame the momentum is

$$\vec{p}\,' = (p_x', p_y', p_z')$$

where $p_x' = p_x$

$p_y' = p_y$

$p_z' = \frac{1}{\sqrt{1-\beta^2}} (p_z + \beta E)$

and the energy is

$$E' = \frac{1}{\sqrt{1-\beta^2}} (E + \beta p_z)$$

Before $_{NR}\langle \vec{p} | \vec{q} \rangle_{NR} = (2\pi)^3 \delta^3(\vec{p} - \vec{q})$

$$\delta^3(\vec{p} - \vec{q}) = \delta(p_x - q_x)\, \delta(p_y - q_y)\, \delta(p_z - q_z)$$

For a function $f(x)$ with a simple zero at $x = x_0$, we have

$$\delta(f(x)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

$$\delta(2x) = \frac{1}{2} \delta(x)$$

$$\delta(2x - 14) = \frac{1}{2} \delta(x - 7)$$

$$\delta(p_z' - q_z') = \frac{1}{\left|\frac{dp_z'}{dp_z}\right|} \delta(p_z - q_z)$$

$$= \frac{1}{\left(1/\sqrt{1-\beta^2}\right)\left(1 + \beta \frac{dE}{dp_z}\right)} \delta(p_z - q_z)$$

$$\frac{dE}{dp_z} = \frac{d}{dp_z}\left(\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}\right)$$

$$= \frac{1}{2\sqrt{p_x^2 + p_y^2 + p_z^2 + m^2}} (2p_z) = \frac{p_z}{E}$$

$$\delta(p_z' - q_z') = \frac{\sqrt{1-\beta^2}}{\left(1 + \beta \frac{p_z}{E}\right)} \delta(p_z - q_z)$$

$$= \frac{E}{\left(\frac{1}{\sqrt{1-\beta^2}}\right)(E + \beta p_z)} \delta(p_z - q_z)$$

$$\boxed{\delta(p_z' - q_z') = \frac{E}{E'} \delta(p_z - q_z)}$$

$$\delta^3(\vec{p}-\vec{q}) \neq \delta^3(\vec{p}'-\vec{q}')$$

$$E\,\delta^3(\vec{p}-\vec{q}) = E'\,\delta^3(\vec{p}'-\vec{q}')$$

Relativistically normalized states

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\;|\vec{p}\rangle_{NR} \qquad \left(E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}\right)$$

$$\langle\vec{q}\,|\vec{p}\rangle = 2E_{\vec{p}}\,(2\pi)^3\,\delta^3(\vec{p}-\vec{q})$$

We don't change definitions for $a_{\vec{p}}$ and $a_{\vec{p}}^{\dagger}$

$$a_{\vec{p}}^{\dagger}|0\rangle = |\vec{p}\rangle_{NR}$$

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\;a_{\vec{p}}^{\dagger}|0\rangle$$

$$[a_{\vec{p}}, a_{\vec{p}'}^{\dagger}] = (2\pi)^3\,\delta^3(\vec{p}-\vec{p}')$$

Completeness relation

$$1 = \int\frac{d^3p}{(2\pi)^3}\,|\vec{p}\rangle\,\frac{1}{2E_{\vec{p}}}\,\langle\vec{p}| + \dots$$

Time Dependence

In the Schrödinger picture $\quad U(t) = e^{-iHt} \quad$ is the time evolution operator, states evolve in time.

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

$$\langle\psi(t)| = \langle\psi(0)|\,U^{\dagger}(t) \qquad \left(U^{\dagger}(t) = e^{+iHt}\right)$$

In the Heisenberg picture, operators evolve in time

$$\mathcal{O}(t) = U^\dagger(t)\,\mathcal{O}(0)\,U(t)$$

$$\underbrace{\langle \psi_1 | \mathcal{O}(t) | \psi_2 \rangle}_{\text{Heisenberg}} = \underbrace{\langle \psi_1(t) | \mathcal{O} | \psi_2(t) \rangle}_{\text{Schrödinger}}$$

- We will work mainly in the Heisenberg picture

$$\phi(x) = \phi(\vec{x}, t)$$

$$\pi(x) = \pi(\vec{x}, t)$$

$\phi(\vec{x}, 0)$ and $\pi(\vec{x}, 0)$ corresponds with $\phi(\vec{x})$ and $\pi(\vec{x})$ from before.

$$\phi(x) = e^{+iHt}\,\phi(\vec{x}, 0)\,e^{-iHt}$$

$$\pi(x) = e^{iHt}\,\pi(\vec{x}, 0)\,e^{-iHt}$$

$$\partial_t \phi(x) = iH\phi - \phi iH = -i[\phi(x), H]$$

$$i\frac{\partial}{\partial t}\phi(x) = [\phi(x), H]$$

04/02/2010

Time Evolution

$$\mathcal{O}(\vec{x},t) = e^{iHt}\,\mathcal{O}(\vec{x},0)\,e^{-iHt}$$

$$i\frac{\partial}{\partial t}\mathcal{O}(x) = [\mathcal{O}(x), H]$$

This is true for any operator $\mathcal{O}(x)$ (combination of a and a^+)

We know

$$[H, a_{\vec{p}}] = -E_{\vec{p}}\,a_{\vec{p}}$$

$$H a_{\vec{p}} = a_{\vec{p}} H - a_{\vec{p}} E_{\vec{p}} = a_{\vec{p}}(H - E_{\vec{p}})$$

More generally

$$H^n a_{\vec{p}} = H^{n-1} H a_{\vec{p}} = H^{n-1} a_{\vec{p}}(H - E_{\vec{p}})$$

Can prove by induction

$$H^n a_{\vec{p}} = a_{\vec{p}}(H - E_{\vec{p}})^n$$

Consider $e^{iHt} a_{\vec{p}}$

$$e^{iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H^n$$

$$e^{iHt} a_{\vec{p}} = \sum \frac{(it)^n}{n!} H^n a_{\vec{p}}$$

$$= \sum \frac{(it)^n}{n!} a_{\vec{p}}(H - E_{\vec{p}})^n$$

$$= a_{\vec{p}} \sum \frac{(it)^n}{n!} (H - E_{\vec{p}})^n$$

$$e^{iHt} a_{\vec{p}} = a_{\vec{p}}\, e^{i(H - E_{\vec{p}})t}$$

$$e^{iHt}\, a_{\vec p}\, e^{-iHt} = a_{\vec p}\, e^{i(H - E_{\vec p})t}\, e^{-iHt}$$

$$= a_{\vec p}\, e^{-iE_{\vec p} t}$$

Can do the same for $a^\dagger_{\vec p}$. Or take Hermitian conjugate

$$e^{iHt}\, a^\dagger_{\vec p}\, e^{-iHt} = a^\dagger_{\vec p}\, e^{+iE_{\vec p} t}$$

Recall

$$\phi(\vec x, 0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p}\, e^{i\vec p\cdot\vec x} + a^\dagger_{\vec p}\, e^{-i\vec p\cdot\vec x} \right)$$

$$\phi(\vec x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p}\, e^{-iE_{\vec p}t}\, e^{i\vec p\cdot\vec x} + a^\dagger_{\vec p}\, e^{iE_{\vec p}t}\, e^{-i\vec p\cdot\vec x} \right)$$

$$p\cdot x\big|_{p^0 = E_{\vec p}} = p^0 x^0 - \vec p\cdot\vec x\big|_{p^0 = E_{\vec p}} = E_{\vec p}\, t - \vec p\cdot\vec x$$

$$\phi(\vec x, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(a_{\vec p}\, e^{-ip\cdot x} + a^\dagger_{\vec p}\, e^{+ip\cdot x} \right)\Big|_{p^0 = E_{\vec p}}$$

$$\pi(\vec x, t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec p}}{2}} \left(a_{\vec p}\, e^{-ip\cdot x} - a^\dagger_{\vec p}\, e^{+ip\cdot x} \right)\Big|_{p^0 = E_{\vec p}}$$

Note that $\pi(\vec x, t) = \frac{\partial}{\partial t}\phi(\vec x, t)$

$\phi(x)$ is a Heisenberg field. Consider

$$D(x - y) = \langle 0|\, \underset{\substack{\uparrow \\ a_{\vec p},\, a^\dagger_{\vec p} \\ \text{gives zero} \\ 0 = \langle 0| a^\dagger_{\vec p}}}{\phi(x)}\; \underset{\substack{\uparrow \\ a_{\vec p'},\, a^\dagger_{\vec p'} \\ \text{gives zero} \\ a_{\vec p'}|0\rangle = 0}}{\phi(y)}\, |0\rangle$$

So $$D(x-y) = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_{\vec{p}}}} \frac{1}{\sqrt{2E_{\vec{p}'}}} \langle 0| e^{-ip\cdot x} e^{+ip'\cdot y} a_{\vec{p}} a^\dagger_{\vec{p}'} |0\rangle \Big|_{p^0=E_{\vec{p}},\, p^{0\prime}=E_{\vec{p}'}}$$

$$\langle 0| a_{\vec{p}} a^\dagger_{\vec{p}'} |0\rangle = {}_{NR}\langle \vec{p}|\vec{p}'\rangle_{NR} = (2\pi)^3 \delta^3(\vec{p}-\vec{p}')$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0=E_{\vec{p}}}$$

Consider the case $x^0-y^0=t$ and $\vec{x}=\vec{y}$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-i(p^0(x^0-y^0) - \vec{p}\cdot(\vec{x}-\vec{y}))} \Big|_{p^0=E_{\vec{p}}}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-iE_{\vec{p}}t} \qquad (E_{\vec{p}} = \sqrt{\vec{p}^2+m^2})$$

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t} \qquad p=|\vec{p}|$$

As $t\to\infty$, the integrand oscillates rapidly. Dominated by the region of integration where the oscillation is smallest $\to p\approx 0$

$D(x-y)$ for large t is $\propto e^{-imt}$

Now consider the case $x^0=y^0$ and $\vec{x}-\vec{y}=\vec{r}$, $r=|\vec{r}|$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}}$$

$$\int_0^{2\pi} d\phi \int_0^\pi \sin(\theta)\, d\theta = \int_0^{2\pi} d\phi \int_{-1}^1 d\cos(\theta)$$

$$= \int_0^\infty \int_0^{2\pi} \int_{-1}^1 d\cos(\theta)\, d\phi\, dp\, p^2 \frac{1}{(2\pi)^3} \frac{1}{2\sqrt{p^2+m^2}} e^{ipr\cos(\theta)}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dp\, p \frac{1}{\sqrt{p^2+m^2}} \frac{1}{2ir} \left(e^{ipr} - e^{-ipr}\right)$$

Find as $r\to\infty$

$$D(x-y) \propto \frac{e^{-mr}}{r} \qquad \text{non zero!}$$

Causality?

$x, y \qquad (x-y)^\mu (x-y)_\mu = (x-y)^2 < 0$ spacelike separation

They cannot influence each other (causally disconnected)

$D(x-y) \neq 0 \quad$ for $(x-y)^2 < 0$

This does not violate causality because no signal can be transmitted. No information is exchanged.

Suppose you have a local measurement at x. $\theta(x)$
Suppose a local measurement at y $\theta'(y)$

So long as the operators commute $[\theta(x), \theta'(y)] = 0$, then the two measurements don't affect each other.

To check causality, is $[\theta(x), \theta'(y)] = 0$? for spacelike separation $(x-y)^2 < 0$.

$$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}} E_{\vec{p}'}}} \Big(e^{-ip\cdot x} \underbrace{[a_{\vec{p}}, a_{\vec{p}'}^\dagger]}_{(2\pi)^3 \delta(\vec{p}-\vec{p}')} e^{ip'\cdot y} + e^{ip\cdot x} \underbrace{[a_{\vec{p}}^\dagger, a_{\vec{p}'}]}_{-(2\pi)^3 \delta^3(\vec{p}-\vec{p}')} e^{-ip'\cdot y} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{-ip\cdot(y-x)} \right)$$

$$= D(x-y) - D(y-x)$$

$\langle \vec{p} | \vec{q} \rangle = (2\pi)^3 2E_{\vec{p}} \delta^3(\vec{p} - \vec{q})$ Lorentz invariant

$$\frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} = \frac{d^3p'}{(2\pi)^3 2E_{\vec{p}'}}$$ also Lorentz invariant.

$D(x-y)$ is Lorentz invariant (in new frame $D(x'-y')$)

$$D(x'-y') = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x'-y')}$$

$$= \int \frac{d^3p''}{(2\pi)^3} \frac{1}{2E_{\vec{p}''}} e^{-ip''\cdot(x-y)}$$

$\vec{p}''$ is $\vec{p}$ viewed in the original frame

$$= D(x-y)$$

Suppose $(x-y)^2 < 0$ spacelike separated

There exists some frame where they are at the same time

$(\vec{x}, t)$ $(\vec{y}, t)$

$$(x-y) = (\vec{x} - \vec{y}, 0)$$
$$(y-x) = (\vec{y} - \vec{x}, 0)$$

Rotate into each other?

$$D(x-y) = D(y-x) \quad \text{for } (x-y)^2 < 0$$

09/07/2010 Two Point Correlation Function

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

$$[\phi(x), \phi(y)] = D(x-y) - D(y-x)$$

For $(x-y)^2 < 0$ spacelike seperated, can find a frame where they are the same time. Then rotate about midpoint

$$[\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 < 0$$

Why does $[\theta(x), \theta'(y)] = 0$ imply no information is passed along? → See notes online.

Green's Functions

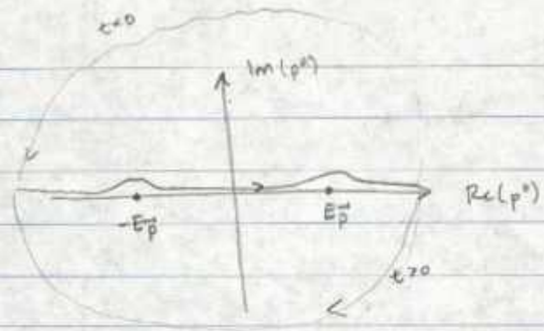
Consider the function $\frac{1}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})}$ $\quad (E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2})$

Poles at $p^0 = E_{\vec{p}}$ and $p^0 = -E_{\vec{p}}$

Fourier transform back to a function of time.

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dp^0}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})} e^{-ip^0 t}$$

We need to define a contour of integration



Contour passes above both poles.

$$t>0: \quad e^{-ip^0 t} \quad \text{as} \quad p^0 \to -i\infty, \; e^{-ip^0 t} \to e^{-\infty} \to 0$$

close below

$$\text{For } t>0, \quad f(t) = (-2\pi i)\left(\underbrace{\frac{e^{iE_{\vec p}t}}{2\pi(-2E_{\vec p})}}_{\text{residue of pole at } -E_{\vec p}} + \underbrace{\frac{e^{-iE_{\vec p}t}}{2\pi(2E_{\vec p})}}_{\text{residue of pole at } E_{\vec p}}\right)$$

$$f(t) = \frac{-i}{2E_{\vec p}}\left(e^{-iE_{\vec p}t} - e^{iE_{\vec p}t}\right)$$

$$t<0: \quad \text{Close above} \quad e^{-ip^0 t} \to 0 \quad \text{as} \quad p^0 \to i\infty$$

$$f(t) = 0 \quad \text{(don't enclose either pole)}$$

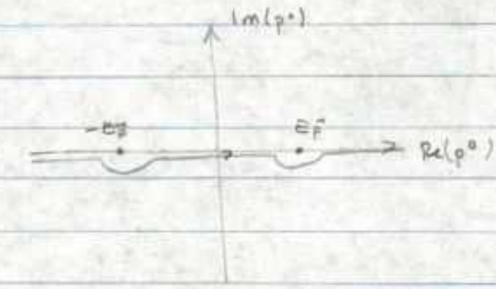
All together

$$f(t) = \Theta(t)\left(\frac{-i}{2E_{\vec p}}\right)\left(e^{-iE_{\vec p}t} - e^{iE_{\vec p}t}\right)$$

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t<0 \end{cases}$$

Retarded or <u>Forward Propagating</u> Green's function

Suppose we go below both poles



In this case

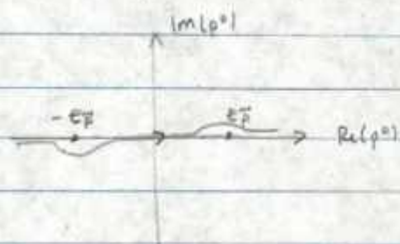
$t > 0$: close below $\quad f(t) = 0$

$t < 0$: close above

$$f(t) = \Theta(-t)\left(\frac{i}{2E_{\vec{p}}}\right)\left(e^{-iE_{\vec{p}}t} - e^{iE_{\vec{p}}t}\right)$$

Advanced or Backward Propagating Green's Function

Suppose we go below pole at $-E_{\vec{p}}$ and above pole at $+E_{\vec{p}}$



$-E_{\vec{p}}$ - backward propagating

$+E_{\vec{p}}$ - forward propagating

$$f(t) = \Theta(t)\left(\frac{-i}{2E_{\vec{p}}}\right)e^{-iE_{\vec{p}}t} + \Theta(-t)\left(\frac{-i}{2E_{\vec{p}}}\right)e^{+iE_{\vec{p}}t}$$

This is a time-ordered Green's function.

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \qquad p^0 = E_{\vec{p}}$$

Define a time-ordered product

$$T\{\phi(x)\phi(y)\} \equiv \phi(x)\phi(y)\,\Theta(x^0-y^0) + \phi(y)\phi(x)\,\Theta(y^0-x^0)$$

"latest on left"

$$\langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle = D(x-y)\,\Theta(x^0-y^0) + D(y-x)\,\Theta(y^0-x^0)$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \Big(\Theta(x^0-y^0) \frac{e^{-iE_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} + \Theta(y^0-x^0) \frac{e^{iE_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \underbrace{\left(i f(x^0-y^0)\right)}_{= \int \frac{dp^0}{2\pi} i \frac{e^{ip^0(x^0-y^0)}}{(p^0-E_{\vec{p}})(p^0+E_{\vec{p}})}}$$

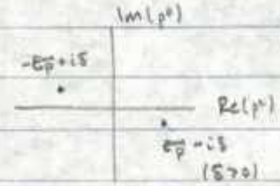
$$= \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{(p^0)^2 - (E_{\vec{p}})^2}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot(x-y)}}{p^2 - m^2}$$

We go below the pole at $p^0 = -E_{\vec{p}}$,

above the pole at $p^0 = +E_{\vec{p}}$

Let's rewrite this...



Move poles off the real axis and take limit as $\delta \to 0$

$$\lim_{\delta\to 0^+} \frac{1}{(p^0-(E_{\vec p}-i\delta))(p^0+(E_{\vec p}-i\delta))}$$

$$(p^0-(E_{\vec p}-i\delta))(p^0+(E_{\vec p}-i\delta)) = (p^0)^2 - (E_{\vec p}-i\delta)^2$$

$$= (p^0)^2 - E_{\vec p}^2 + \underbrace{2i\delta E_{\vec p}}_{i\epsilon\ (\epsilon>0)} + \cancel{\delta^2}^{\to 0}$$

$$\lim_{\epsilon\to 0^+} \frac{1}{p^2-m^2+i\epsilon}$$

Summarize:

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = D(x-y)\,\Theta(x^0-y^0) + D(y-x)\,\Theta(y^0-x^0)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)} \quad \left(\lim_{\epsilon\to 0^+}\right)$$

Feynman propagator $D_F(x-y)$

$$(\partial_\mu\partial^\mu + m^2) D_F(x) = -i\delta^4(x) \qquad \left(\delta^4(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x}\right)$$

Lorentz Transformations

$$x^\mu \longrightarrow x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{4\times4 \text{ matrix}} x^\nu$$

$$\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)$$

Why Λ^{-1} not Λ? Because we require

$$\phi'(x') = \phi(x)$$

$$\phi'(x) = \phi(\Lambda^{-1}x)$$

What about $\partial_\mu \phi(x)$?

Under transformation $\partial_\mu(\phi(\Lambda^{-1}x))$

$$\partial_\mu(\phi(\Lambda^{-1}x)) = \left(\partial_\mu(\Lambda^{-1}x)^\nu\right)(\partial_\nu\phi)\Big|_{\Lambda^{-1}x}$$

$$= (\Lambda^{-1})^\nu{}_\mu(\partial_\nu\phi)\Big|_{\Lambda^{-1}x}$$

Upper index object → $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu$ $\quad \underset{4\times4}{[\Lambda]}[x] = [x']$

Lower index object → $x_\mu \rightarrow x_\nu(\Lambda^{-1})^\nu{}_\mu$ $\quad [x]\underset{4\times4}{[\Lambda^{-1}]} = [x']$

$$x_\mu x^\mu \rightarrow \left(x_\alpha(\Lambda^{-1})^\alpha{}_\mu\right)\left(\Lambda^\mu{}_\beta x^\beta\right)$$

$$= x_\alpha \delta^\alpha{}_\beta x^\beta$$

$$= x_\beta x^\beta \quad \text{Lorentz invariant.}$$

Lorentz group is the group of transformations that leave $g_{\mu\nu}$ the same.

In all inertial frames, $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$g_{\mu\nu}$ has two lower index components

$$g_{\mu\nu} \rightarrow (\Lambda^{-1})^{\alpha}{}_{\mu} (\Lambda^{-1})^{\beta}{}_{\nu} \, g_{\alpha\beta}$$

$$\underset{4\times4}{[g]} \rightarrow \underset{4\times4}{[\Lambda^{-1}]^T} \underset{4\times4}{[g]} \underset{4\times4}{[\Lambda^{-1}]}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = (\Lambda^{-1})^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} (\Lambda^{-1})$$

09/09/2010

Lorentz Transformations

Groups

Set of elements $g_1, g_2, \ldots \in G$

1. Closed under multiplication $g_i \cdot g_j \in G$

2. Exists an identity element $e \in G$ $e \cdot g = g$ $g \cdot e = g$

3. Exists an inverse element g^{-1} for each $g \in G$ $g \cdot g^{-1} = e$ $g^{-1} \cdot g = e$

4. Associativity $(g_i \cdot g_j) \cdot g_k = g_i \cdot (g_j \cdot g_k)$

Group of real $N \times N$ matrices M

$$M^T M = I_{N \times N} = 1$$ Orthogonal group of $N \times N$ matrices $O(N)$

$$v^T v = |v|^2 \rightarrow (Mv)^T (Mv) = v^T M^T M v = v^T v = |v|^2$$

Typically use $\underbrace{\text{special}}_{\det(M)=1}$ orthogonal group of $N \times N$ matrices

Group of rotations in three dimensional Euclidean space $SO(3)$

Our group of Lorentz transformations is similar to $SO(4)$

$$M^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$ Lorentz group $(SO(3,1))$

$$M^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ $SO(4)$

SO(3,1) includes three dimensional rotations

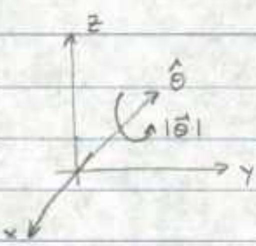
From quantum mechanics: Spin s representation $s = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$

dimension of representation $= 2s+1$

Spin $\frac{1}{2}$ representation

$\vec{\sigma}$ Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



$$U(\vec{\theta}) = \exp[-i\vec{\theta}\cdot\vec{\sigma}/2]$$

Lie group: Elements form a differentiable manifold.
Group operation is smooth.

$$U(\vec{\theta}) = \exp[-i\,\underbrace{\vec{\theta}\cdot\vec{\sigma}/2}_{\text{spin } 1/2 \text{ representation of the Lie algebra.}}]$$

In the case for arbitrary spin representation

$$U(\vec{\theta}) = \exp[-i\vec{\theta}\cdot\vec{J}]$$

$$\vec{J} = (J^1, J^2, J^3)$$

$$[J^j, J^k] = i\sum_{l}\epsilon^{jkl}J^l$$

Check it works for spin 1/2

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i \sum_\ell \epsilon^{jk\ell} \sigma^\ell/2$$

Spinless particle $\psi(\vec{x})$ decomposed into orbital angular momentum states

$J = 0, 1, 2, \ldots$

(no intrinsic spin $\rightarrow$ $J = L$)

$$\vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$$

$$J^j = i \sum_\ell \epsilon^{jk\ell} x^k \nabla^\ell \qquad \left(\nabla^\ell = -\nabla_\ell = -\frac{\partial}{\partial x^\ell}\right)$$

In 3D $\rightarrow$ 3 rotations $\rightarrow$ 3 pairs of axes

Rotation $x^1 + x^2 \rightarrow J^3$

$x^2 + x^3 \rightarrow J^1$

$x^3 + x^1 \rightarrow J^2$

In 4D $\rightarrow$ 6 rotations $\rightarrow$ 6 pairs of axes

Two index object

$$J^{k\ell} = i(x^k \nabla^\ell - x^\ell \nabla^k) \quad \text{rotate between } x^k + x^\ell$$

Guess $\rightarrow$ We know the answer for $SO(4)$. For $SO(3,1)$, guess the answer is the same with upper index objects

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

Spatial rotations $\rightarrow$ $J^{jk} = i(x^j \partial^k - x^k \partial^j)$

Lorentz boosts along x^j axis $\rightarrow$ $J^{0j} = i(x^0 \partial^j - x^j \partial^0)$

From homework 3, problem 2

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} + g^{\mu\sigma} J^{\nu\rho} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} \right)$$

Lorentz algebra

3 rotations
$$J^{12} = -J^{21}$$
$$J^{23} = -J^{32}$$
$$J^{31} = -J^{13}$$

3 boosts
$$J^{01} = -J^{10}$$
$$J^{02} = -J^{20}$$
$$J^{03} = -J^{30}$$

Elements of the Lorentz group

$$U(\omega_{\mu\nu}) = \exp\left[\frac{-i}{2} \omega_{\mu\nu} J^{\mu\nu} \right]$$

$$\omega_{\mu\nu} = -\omega_{\nu\mu}$$

$\frac{1}{2}$ to take into account two copies $(\omega_{10} J^{10} = \omega_{01} J^{01})$

Four-vector representation

Case I: $\omega_{12} = \theta = -\omega_{21}$

$$\left[V^\mu \right] \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \left[V^\mu \right]$$

Infinitesimal θ: $\left[V^\mu \right] \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \left[V^\mu \right]$

$$(V^1)' = V^1 - \theta V^2$$
$$(V^2)' = V^2 + \theta V^1$$

Change is $-i\theta[J^{12}v]^\mu = (0, -\theta v^2, \theta v^1, 0)$

Write as $-i[J^{12}v]^\mu = g^{\mu 1}v^2 - g^{\mu 2}v^1$

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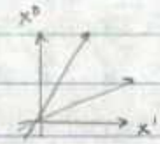
Lorentz Transformations

Case II : $\omega_{01} = -\omega_{10} = \eta$



$$[V^\mu] \longrightarrow \begin{pmatrix} \cosh(\eta) & \sinh(\eta) & 0 & 0 \\ \sinh(\eta) & \cosh(\eta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [V^\mu]$$

Infinitesimal η $\begin{pmatrix} 1 & \eta & 0 & 0 \\ \eta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$$(V^0)' = V^0 + \eta V^1$$
$$(V^1)' = V^1 + \eta V^0$$

$$-i [J^{01} V]^\mu = g^{\mu 0} V^1 - g^{\mu 1} V^0$$

$$[J^{\mu\nu}_{(4)}]^\alpha{}_\beta = i (g^{\mu\alpha} \delta^\nu{}_\beta - g^{\nu\alpha} \delta^\mu{}_\beta)$$

Dirac Equation

Define $\{A, B\} = AB + BA$ anticommutator

$n \times n$ Matrices $\mu = 0, 1, 2, 3$

γ^μ (Dirac matrices) such that

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \quad \text{(Dirac algebra)}$$

then $S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$

satisfy the Lorentz algebra

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\mu\sigma} S^{\nu\rho} + g^{\nu\rho} S^{\mu\sigma} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho}\right)$$

Look at spatial components.

Try 2×2 matrices → $\gamma^j = i\sigma^j$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\sigma^j)^2 = 1$$

$$\begin{aligned} \sigma^1\sigma^2 &= -\sigma^2\sigma^1 = i\sigma^3 \\ \sigma^2\sigma^3 &= -\sigma^3\sigma^2 = i\sigma^1 \qquad \Rightarrow \quad \{\sigma^j, \sigma^k\} = 2\delta^{jk} \\ \sigma^3\sigma^1 &= -\sigma^1\sigma^3 = i\sigma^2 \qquad\quad\;\; [\sigma^j, \sigma^k] = 2i\epsilon^{jkl}\sigma^l \end{aligned}$$

$$\{\gamma^j, \gamma^k\} = \{i\sigma^j, i\sigma^k\} = -\{\sigma^j, \sigma^k\} = -2\delta^{jk} = -2g^{jk}$$

$$S^{jk} = \frac{i}{4}[\gamma^j, \gamma^k] = \frac{i}{4}[i\sigma^j, i\sigma^k] = \frac{-i}{4}[\sigma^j, \sigma^k]$$

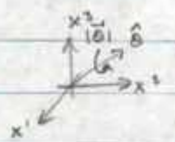
$$S^{jk} = \frac{1}{2}\epsilon^{jkl}\sigma^l$$

Spin ½ : $J^3 = S^{12} = \frac{1}{2}\sigma^3$

$J^2 = S^{31} = \frac{1}{2}\sigma^2$

$J^1 = S^{23} = \frac{1}{2}\sigma^1$

Rotation by $\vec{\theta}$



$$\exp\left[-i\,\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right]$$

"Weyl" or "chiral" Representation

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \left(\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right)$$

$$\gamma^i = \left(\begin{array}{c|c} 0 & \sigma^i \\ \hline -\sigma^i & 0 \end{array}\right)$$

for example $\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$

Boost in j-direction

$$S^{0j} = \frac{i}{4}[\gamma^0, \gamma^j] = \frac{i}{4}(\gamma^0\gamma^j - \gamma^j\gamma^0)$$

$$\gamma^0\gamma^j = \begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} \qquad \gamma^j\gamma^0 = \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$$

$$S^{0j} = \frac{i}{4}\begin{pmatrix} -2\sigma^j & 0 \\ 0 & 2\sigma^j \end{pmatrix} = \frac{-i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$$

Rotation i-j axes:

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{i}{4}(\gamma^i\gamma^j - \gamma^j\gamma^i)$$

$$\gamma^i\gamma^j = \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} \qquad \gamma^j\gamma^i = \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{i}{4}\begin{pmatrix} [\sigma^i, \sigma^j] & 0 \\ 0 & [\sigma^j, \sigma^i] \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \epsilon^{ijk}\sigma^k & 0 \\ 0 & \epsilon^{ijk}\sigma^k \end{pmatrix}$$

Four component objects these matrices transform are Dirac (bi)spinors.

S^{ij} rotations are Hermitian

S^{0i} boosts are antiHermitian

Dirac bispinor $\rightarrow$ 4 component column vector $\begin{bmatrix}\cdot\\\cdot\\\cdot\\\cdot\end{bmatrix}$

$$[\cdot\ \cdot\ \cdot\ \cdot]\ \Lambda^{-1}_{1/2}\begin{pmatrix}\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot\end{pmatrix}\Lambda_{1/2}\begin{bmatrix}\cdot\\\cdot\\\cdot\\\cdot\end{bmatrix}$$

$\Lambda_{\frac{1}{2}}$ spin $\frac{1}{2}$ representation

What is $\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2}$?

$$\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2} \approx \left(1+\tfrac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)\gamma^\mu\left(1-\tfrac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$$\approx \gamma^\mu - \tfrac{i}{2}\left[\gamma^\mu, \omega_{\alpha\beta}S^{\alpha\beta}\right] \qquad (\omega_{\alpha\beta}\ \text{small})$$

$$= \gamma^\mu - \tfrac{i}{2}\omega_{\alpha\beta}\left[\gamma^\mu, S^{\alpha\beta}\right]$$

$$= \gamma^\mu - \tfrac{i}{2}\omega_{\alpha\beta}\left[\gamma^\mu, \tfrac{i}{4}[\gamma^\alpha,\gamma^\beta]\right]$$

$$= \gamma^\mu - \tfrac{i}{2}\omega_{\alpha\beta}\tfrac{i}{4}\left(\underbrace{\gamma^\mu\gamma^\alpha}_{-\gamma^\alpha\gamma^\mu+2g^{\alpha\mu}}\gamma^\beta - \gamma^\mu\gamma^\beta\gamma^\alpha - \gamma^\alpha\underbrace{\gamma^\beta\gamma^\mu}_{-\gamma^\mu\gamma^\beta+2g^{\beta\mu}} + \gamma^\beta\gamma^\alpha\gamma^\mu\right)$$

$$[\gamma^\mu, S^{\alpha\beta}] = \left[J^{\alpha\beta}_{(4)}\right]^\mu{}_\nu\gamma^\nu = i\left(g^{\alpha\mu}\delta^\beta{}_\nu - g^{\beta\mu}\delta^\alpha{}_\nu\right)\gamma^\nu$$

$\rightarrow$ γ^μ transforms like a 4-vector

$$\gamma^\mu \rightarrow \gamma^\mu - \tfrac{i}{2}\omega_{\alpha\beta}\left[J^{\alpha\beta}_{(4)}\right]^\mu{}_\nu\gamma^\nu$$

$$\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{1/2} = (\Lambda_4)^\mu{}_\nu\gamma^\nu$$

Dirac equation

$$\boxed{(i\gamma^\mu \partial_\mu - m)\,\psi(x) = 0}$$

This is Lorentz invariant. Let's check

$$\psi(x) \to \Lambda_{\frac{1}{2}}\, \psi(\Lambda_4^{-1} x)$$

$$i\gamma^\mu \partial_\mu \psi(x) \to i\gamma^\mu \Lambda_{\frac{1}{2}} \partial_\mu (\psi(\Lambda_4^{-1} x))$$

$$= \Lambda_{\frac{1}{2}} \Lambda_{\frac{1}{2}}^{-1}\, i\gamma^\mu \Lambda_{\frac{1}{2}}\, \partial_\mu (\psi(\Lambda_4^{-1} x))$$

$$= \Lambda_{\frac{1}{2}}\, i\, (\Lambda_4)^\mu{}_\nu\, \gamma^\nu\, \partial_\mu (\psi(\Lambda_4^{-1} x))$$

$$= \Lambda_{\frac{1}{2}}\, i\, (\Lambda_4)^\mu{}_\nu\, \gamma^\nu\, (\Lambda_4^{-1})^\alpha{}_\mu\, \partial_\alpha \psi \big|_{\Lambda_4^{-1} x}$$

$$= \Lambda_{\frac{1}{2}}\, i\, \delta^\alpha{}_\nu\, \gamma^\nu \partial_\alpha \psi \big|_{\Lambda_4^{-1} x}$$

$$i\gamma^\mu \partial_\mu \psi(x) \to \Lambda_{\frac{1}{2}}\, i\gamma^\mu \partial_\mu \psi \big|_{\Lambda_4^{-1} x}$$

Transforms the same way as $\psi(x)$ ($\Lambda_{\frac{1}{2}}$ on far left)

So if we have $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$, in the new frame

$$\Lambda_{\frac{1}{2}} (i\gamma^\mu \partial_\mu - m)\psi \big|_{\Lambda_4^{-1} x} = 0$$

Multiply Dirac Equation by $(-i\gamma^\mu\partial_\mu - m)$ on the left

$$(-i\gamma^\mu\partial_\mu - m)(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

$$(\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi(x) = 0$$

$$\left(\tfrac{1}{2}\underbrace{\{\gamma^\mu,\gamma^\nu\}}_{2g^{\mu\nu}}\partial_\mu\partial_\nu + m^2\right)\psi(x) = 0$$

$$(\partial_\mu\partial^\mu + m^2)\psi(x) = 0 \rightarrow \text{Klein-Gordon equation}$$

Solution of Dirac equation is a solution of Klein-Gordon equation.

09/16/2010

Dirac Equation

Weyl representation

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Block diagonal → Reducible representation

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

ψ_L - left-handed Weyl spinor

ψ_R - right-handed Weyl spinor

Infinitesimal rotation and boost $(\vec{\theta},\ \tanh(|\vec{\beta}|) = |\vec{v}|)$

$$\psi_L \to \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L$$

$$\psi_R \to \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_R$$

The transformation of ψ_R is equivalent to the transformation of ψ_L^*

$$\psi_L^* \to \left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

$$\sigma^2\vec{\sigma}^* = -\vec{\sigma}\sigma^2$$

$$\sigma^2\psi_L^* \to \sigma^2\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

$$= \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*$$

Recall

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\begin{pmatrix} -m & i\partial_0 + i\vec{\sigma}\cdot\vec{\nabla} \\ i\partial_0 - i\vec{\sigma}\cdot\vec{\nabla} & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

If $m = 0$, there is no coupling of ψ_L and ψ_R

$$i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0$$

Weyl equations

$$i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0$$

(non-interacting, massless, spin $\frac{1}{2}$ particle)

For later convenience, define

$$\sigma^\mu = (1, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

Dirac equation

$$\begin{pmatrix} -m & i\sigma^\mu\partial_\mu \\ i\bar{\sigma}^\mu\partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Weyl equations $(m=0)$

$$i\bar{\sigma}\cdot\partial\psi_L = 0$$

$$i\sigma\cdot\partial\psi_R = 0$$

Plane wave solutions of Dirac equation

$$\psi(x) = u(p)\, e^{-ip\cdot x} \qquad (p^0 = E_{\vec{p}} \text{ to satisfy Klein-Gordon})$$

$$e^{-i\omega t} \quad \text{positive frequency solutions} \quad (\omega > 0)$$

$$(i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$$

$$(\gamma^\mu p_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$$

$$(\gamma^\mu p_\mu - m)\, u(p) = 0$$

Let's assume $m \neq 0$. Go to rest frame of particle

$$p^0 = m \qquad \vec{p} = 0$$

$$(m\gamma^0 - m)\, u(p) = 0$$

$$m \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p) = 0$$

$$u(p) = \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix} \quad \text{for any } a \text{ and } b$$

$$\xi = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$u(p) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$S^{12} = S_z = \frac{1}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{gives spin-z} = +\frac{1}{2}$$

$$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{gives spin-z} = -\frac{1}{2}$$

Move out of rest frame ...

$$p^{\mu} = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Boost in the z-direction $\quad \vec{v} = v\hat{z}$

Rapidity η: $\quad \tanh(\eta) = |\vec{v}|$

$$p^{\mu} = \begin{pmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} m\cosh(\eta) \\ 0 \\ 0 \\ m\sinh(\eta) \end{pmatrix}$$

$$E = m\cosh(\eta) \qquad p^3 = m\sinh(\eta)$$

$$S^{03} = \frac{-i}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$$

Transformation matrix

$$M = \exp\left[-i\,\eta\, S^{03}\right]$$

$$= \exp\left[\frac{-\eta}{2} \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]$$

$$= \exp\left[\frac{-\eta}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \begin{pmatrix} \exp[-\eta/2] & 0 & 0 & 0 \\ 0 & \exp[\eta/2] & 0 & 0 \\ 0 & 0 & \exp[\eta/2] & 0 \\ 0 & 0 & 0 & \exp[-\eta/2] \end{pmatrix}$$

$\sigma^3 \cdot \sigma^3 = 1$

$$\exp[a \cdot \sigma^3] = 1 + a\sigma^3 + \frac{a^2}{2!} \cdot 1 + \frac{a^3}{3!}\sigma^3 + \frac{a^4}{4!} \cdot 1 + \cdots$$

even terms $\rightarrow$ $\cosh(a)$

odd terms $\rightarrow$ $\sinh(a)\,\sigma^3$ (true even for not σ^3)

$$M = \begin{pmatrix} \cosh(\eta/2) - \sinh(\frac{\eta}{2})\sigma^3 & 0 \\ 0 & \cosh(\frac{\eta}{2}) + \sinh(\frac{\eta}{2})\,\sigma^3 \end{pmatrix}$$

Now transform $u(p) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$

$$M\left(\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}\right) = \sqrt{m}\begin{pmatrix} (\cosh(\frac{\eta}{2}) - \sinh(\frac{\eta}{2})\,\sigma^3)\,\xi \\ (\cosh(\frac{\eta}{2}) + \sinh(\frac{\eta}{2})\,\sigma^3)\,\xi \end{pmatrix}$$

$$\left(\cosh(\tfrac{\eta}{2}) - \sinh(\tfrac{\eta}{2})\sigma^3\right)^2 = \underbrace{\cosh^2(\tfrac{\eta}{2}) + \sinh^2(\tfrac{\eta}{2})}_{\cosh(\eta)} - \underbrace{2\cosh(\tfrac{\eta}{2})\sinh(\tfrac{\eta}{2})}_{\sinh(\eta)}\sigma^3$$

$$= \cosh(\eta) - \sinh(\eta)\,\sigma^3$$ (positive eigenvalues $\cosh(\eta) > \sinh(\eta)$)

$$= \frac{E}{m} - \frac{p^3}{m}\sigma^3$$

$$= \frac{1}{m}\, p \cdot \sigma \qquad (\sigma^\mu = (1, \vec{\sigma}))$$

Define $\sqrt{p \cdot \sigma} = \sqrt{m}\left(\cosh(\frac{\eta}{2}) - \sinh(\frac{\eta}{2})\,\sigma^3\right)$

$$\sqrt{p \cdot \bar{\sigma}} = \sqrt{m}\left(\cosh(\tfrac{\eta}{2}) + \sinh(\tfrac{\eta}{2})\,\sigma^3\right)$$

$$\Rightarrow u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma}\,\xi \\ \sqrt{p \cdot \bar{\sigma}}\,\xi \end{pmatrix}$$ (correct for arbitrary p)

Useful facts . . .

$$(\vec{p}\cdot\vec{\sigma})(\vec{p}\cdot\vec{\sigma}) = (p^1)^2 + (p^2)^2 + (p^3)^2 = \vec{p}^{\,2}$$

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0 - \vec{p}\cdot\vec{\sigma})(p^0 + \vec{p}\cdot\vec{\sigma})$$

$$= (p^0)^2 - \vec{p}^{\,2}$$

$$= p^2 \quad (= m^2 \text{ if satisfies Klein-Gordon})$$

09/21/2010

Solutions to Dirac Equation

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix} \qquad \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For our example

$$p^\mu = (E, 0, 0, p^3)$$

$$\sigma^\mu = (1, \vec{\sigma}) \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

$$p\cdot\sigma = \begin{pmatrix} E-p^3 & 0 \\ 0 & E+p^3 \end{pmatrix}$$

$$\sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}$$

$$\sqrt{p\cdot\bar{\sigma}} = \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}$$

Spin-z $= +\frac{1}{2}$

$$u(p) = \begin{pmatrix} \sqrt{E-p^3} \\ 0 \\ \sqrt{E+p^3} \\ 0 \end{pmatrix}$$

Spin-z $= -\frac{1}{2}$

$$u(p) = \begin{pmatrix} 0 \\ \sqrt{E+p^3} \\ 0 \\ \sqrt{E-p^3} \end{pmatrix}$$

Massless limit $\rightarrow$ $E = p^3$ $(c=1)$

$$S = +\tfrac{1}{2} \quad u(p) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2E} \\ 0 \end{pmatrix} \qquad S = -\tfrac{1}{2} \quad u(p) = \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix}$$

Helicity - spin in direction of motion

$$h = \hat{p} \cdot \vec{S} \qquad \vec{S} = (S^1, S^2, S^3) = (S^{23}, S^{31}, S^{12})$$

$$h = \frac{1}{2}\hat{p} \cdot \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$h = +\frac{1}{2} \rightarrow$ right handed

$h = -\frac{1}{2} \rightarrow$ left handed

left-handed

Suppose $m \neq 0$ (does not move at speed of light)

We can pass the particle

right-handed

$\Rightarrow$ Helicity is frame dependent

However, if the particle is massless, then h is a good quantum number for everyone.

Remember, for massless case $\rightarrow$ Weyl equations

$$i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 = i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla})\psi_L = 0$$

$$i\sigma^\mu \partial_\mu \psi_R = 0 = i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla})\psi_R = 0$$

Plane wave solutions

$$\partial_0 e^{-ip\cdot x} = (-iE)\, e^{-ip\cdot x}$$

$$\vec{\nabla} e^{-ip\cdot x} = (i\vec{p})\, e^{-ip\cdot x}$$

Massless $\Rightarrow$ $\vec{p} = |\vec{p}|\,\hat{p} = E\hat{p}$

$$(E + E\hat{p}\cdot\vec{\sigma})\,\psi_L = 0$$

$$(E - E\hat{p}\cdot\vec{\sigma})\,\psi_R = 0$$

$$\Rightarrow E(1 + 2h)\,\psi_L = 0 \longrightarrow h = -\tfrac{1}{2}$$

$$E(1 - 2h)\,\psi_R = 0 \longrightarrow h = +\tfrac{1}{2}$$

Quick Way to Solve Dirac Equation

$$(p\cdot\gamma - m)\,u(p) = 0$$

Weyl representation $\rightarrow$ $\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} u(p) = 0$

Recall $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p^2 = (p\cdot\bar{\sigma})(p\cdot\sigma) = m^2$

$$m = \sqrt{m^2} = \sqrt{p\cdot\sigma}\,\sqrt{p\cdot\bar{\sigma}}$$

$$p\cdot\sigma = \sqrt{p\cdot\sigma}\,\sqrt{p\cdot\sigma} \qquad p\cdot\bar{\sigma} = \sqrt{p\cdot\bar{\sigma}}\,\sqrt{p\cdot\bar{\sigma}}$$

Now, Dirac equation

$$\begin{pmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} \end{pmatrix} u(p) = 0$$

$A = \sqrt{p\cdot\sigma} \quad B = \sqrt{p\cdot\bar\sigma}$

$$\begin{pmatrix} -AB & AA \\ BB & -AB \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Solution: $u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$ for any ξ (we derived this before)

What about $e^{+ip\cdot x}$? $(p^0 = E_{\vec p} > 0)$

Negative frequency solutions $v(p)$

$$\begin{pmatrix} -m & -p\cdot\sigma \\ -p\cdot\bar\sigma & -m \end{pmatrix} v(p) = 0$$

$$\begin{pmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ -\sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} \end{pmatrix} v(p) = 0$$

Solution: $v(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$ for any ξ (call it η later)

Some identities...

$$u^\dagger(p)\,u(p) = \left(\xi^\dagger\sqrt{p\cdot\sigma} \quad \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}\sqrt{p\cdot\sigma}\;\xi \\ \sqrt{p\cdot\bar\sigma}\;\xi\end{pmatrix}$$

$$= \xi^\dagger(p\cdot\sigma)\xi + \xi^\dagger(p\cdot\bar\sigma)\xi$$

$$= \xi^\dagger\underbrace{(p\cdot\sigma + p\cdot\bar\sigma)}_{2p^0 = 2E_{\vec p}}\xi \qquad (\xi^\dagger\xi = 1)$$

$$u^\dagger u = 2E_{\vec p}$$

$$\xi^1 = \begin{pmatrix}1\\0\end{pmatrix} \qquad \xi^2 = \begin{pmatrix}0\\1\end{pmatrix} \qquad \left(u^s = \begin{pmatrix}\sqrt{p\cdot\sigma}\;\xi^s \\ \sqrt{p\cdot\bar\sigma}\;\xi^s\end{pmatrix}\right)$$

$$u^{r\dagger}u^s = 2E_{\vec p}\,\xi^{r\dagger}\xi^s = 2E_{\vec p}\,\delta^{rs}$$

$r=1,2$
$s=1,2$

Similarly for $v(p)$ →

$$v^{r\dagger}(p)\,v^s(p) = 2E_{\vec p}\,\delta^{rs}$$

Dirac Adjoint

$$\psi = [\psi] \qquad \psi^\dagger = [\psi^\dagger] \qquad \bar\Psi = \psi^\dagger\gamma^0 = [\psi^\dagger]\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$\bar u^r(p)\,u^s(p) = u^{r\dagger}(p)\gamma^0 u^s(p)$$

$$= \left(\xi^{r\dagger}\sqrt{p\cdot\sigma} \quad \xi^{r\dagger}\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\;\xi^s \\ \sqrt{p\cdot\bar\sigma}\;\xi^s\end{pmatrix}$$

$$= 2\,\xi^{r\dagger}\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma}\,\xi^s$$

$$\bar u^r u^s = 2m\,\delta^{rs}$$

$$\boxed{\bar{v}^r(p)\,v^s(p) = -2m\,\delta^{rs} \qquad \bar{u}^r(p)\,u^s(p) = 2m\,\delta^{rs}}$$

$$\bar{u}^r(p)\,v^s(p) = \xi^{r\dagger}\left(-\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} + \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\sigma}\right)\xi = 0$$

$$\bar{v}^r(p)\,u^s(p) = 0$$

Spin Sums

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_{s=1,2} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\sigma} & \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} p\cdot\sigma & m \\ m & p\cdot\bar{\sigma} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$$\boxed{\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = p\cdot\gamma + m}$$

$$\boxed{\sum_{s=1,2} v^s(p)\,\bar{v}^s(p) = p\cdot\gamma - m}$$

Feynman "Slash" Notation

$$\not{p} = \gamma^\mu p_\mu = \gamma_\mu p^\mu = p\cdot\gamma$$

$\bar{\psi}\Gamma\psi$ - $\Gamma = 4\times4$ matrix (16 entries)

	total number	
1	1	
γ^μ	4	
$\gamma^\mu\gamma^\nu$	6	(only consider $[\gamma^\mu,\gamma^\nu]$ since $\{\gamma^\mu,\gamma^\nu\}=2g^{\mu\nu}$)
$\gamma^\mu\gamma^\nu\gamma^\rho$	4	(completely antisymmetric)
$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$	1	
	16	

Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\gamma^5 = -\frac{i}{4!}\epsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

09/23/2010

More on γ Matrices

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$\gamma^{0\dagger} = \gamma^0 \qquad \gamma^{1\dagger} = -\gamma^1 \qquad \gamma^{2\dagger} = -\gamma^2 \qquad \gamma^{3\dagger} = -\gamma^3$$

$$\gamma^{5\dagger} = -i\,\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} = i\gamma^3\gamma^2\gamma^1\gamma^0$$

$$= -i\gamma^3\gamma^2\gamma^0\gamma^1$$

$$\vdots$$

$$= i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

$$\{\gamma^5, \gamma^\mu\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + \underbrace{i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3}_{(-1)^3(+1)}$$

$$= 0$$

$$[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]] = 0$$

$\rightarrow$ Eigenstates of γ^5 don't mix under Lorentz transformations.

In the Weyl representation

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix}$$

-1 for left-handed

$+1$ for right-handed

Parity $\rightarrow$ $\vec{x} \rightarrow -\vec{x}$

For four-vector $(x^0, \vec{x}) \rightarrow (x^0, -\vec{x})$

Pseudovector/axial vector $(x^0, \vec{x}) \rightarrow (-x^0, \vec{x})$

Scalar invariant under parity (no sign flip)

Pseudoscalar Invariant under rotations/boosts, flips sign under parity

16 possible 4x4 matrices

1	Scalar	1
γ^μ	Vector	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$	Tensor	6
$\gamma^\mu \gamma^5$	Axial vector	4
γ^5	Pseudoscalar	1
		16

Vector Current

$$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

Axial vector current

$$j^{\mu 5}(x) = \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x)$$

Defined $\bar{\psi} = \psi^\dagger \gamma^0$

For 4x4 matrix $\bar{M} = \gamma^0 M^\dagger \gamma^0$

$$\overline{M_1 M_2} = \gamma^0 (M_1 M_2)^\dagger \gamma^0 = \gamma^0 M_2^\dagger M_1^\dagger \gamma^0$$

$$= \gamma^0 M_2^\dagger \gamma^0 \gamma^0 M_1^\dagger \gamma^0$$

$$= \bar{M}_2 \bar{M}_1$$

$$\bar{\gamma}^\mu = \gamma^\mu \quad \text{(self-bar)}$$

Suppose $\psi(x)$ satisfies the free Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) = 0$$

$$i\gamma^\mu \partial_\mu \psi(x) = m\psi(x)$$

Take Dirac adjoint of this equation

$$-i(\partial_\mu \bar{\psi}(x))\gamma^\mu = m\bar{\psi}(x)$$

Consider $\partial_\mu j^\mu(x)$

$$\partial_\mu j^\mu(x) = \underbrace{(\partial_\mu \bar{\psi}(x))\gamma^\mu}_{im\bar{\psi}(x)} \psi(x) + \bar{\psi}(x) \underbrace{\gamma^\mu \partial_\mu \psi(x)}_{-im\psi(x)}$$

$$= (im\bar{\psi}(x))\gamma^\mu \psi(x) + \bar{\psi}(x)\gamma^\mu(-im\psi(x))$$

$$\partial_\mu j^\mu = 0$$

$\Rightarrow$ $j^\mu(x)$ is a conserved current associated with the symmetry $\psi \to e^{i\alpha}\psi$

Consider $\partial_\mu j^{\mu 5}(x)$

$$\partial_\mu j^{\mu 5}(x) = (\partial_\mu \bar{\psi}(x))\gamma^\mu\gamma^5\psi(x) + \bar{\psi}(x)\gamma^\mu\gamma^5\partial_\mu\psi(x)$$

$$= (im\bar{\psi}(x))\gamma^5\psi(x) - \bar{\psi}(x)\gamma^5\gamma^\mu\partial_\mu\psi(x)$$

$$= im\bar{\psi}(x)\gamma^5\psi(x) + \bar{\psi}\gamma^5(im\psi(x))$$

$$= 2im\,\bar{\psi}(x)\gamma^5\psi(x)$$

$j^{\mu 5}(x)$ is conserved if $m=0$.

If particle is massless, this is a conserved current associated with $\psi \to e^{i\alpha\gamma^5}\psi$

(rotates left-handed one direction, right-handed the other → axial rotation)

<u>Quantization of Dirac Field</u>

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

Canonical variable to ψ

$$\frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = i\bar{\psi}\gamma^0 = i(\psi^\dagger\gamma^0)\gamma^0 = i\psi^\dagger$$

$$\mathcal{H} = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)}(\partial_0\psi) - \mathcal{L}$$

$$= i\psi^\dagger(\partial_0\psi) - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$= i\bar{\psi}\gamma^0\partial_0\psi - \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

$$= i\bar{\psi}\gamma^0\partial_0\psi - (i\bar{\psi}\gamma^0\partial_0\psi + i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi - m\bar{\psi}\psi)$$

$$\mathcal{H} = -i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi$$

$$H = \int d^3x\, \mathcal{H}$$

$$H = \int d^3x \left(\bar{\psi} (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi \right)$$

First try (this won't work), guess

$$[\, \psi_a(\vec{x}),\, i\psi_b^\dagger(\vec{y}) \,] = i\,\delta^3(\vec{x}-\vec{y})\,\delta_{ab} \qquad (a,b = 1,2,3,4)$$

$$[\, \psi_a(\vec{x}),\, \psi_b^\dagger(\vec{y}) \,] = \delta_{ab}\,\delta^3(\vec{x}-\vec{y})$$

$$[\, \psi_a(\vec{x}),\, \psi_b(\vec{y}) \,] = 0$$

$$[\, \psi_a^\dagger(\vec{x}),\, \psi_b^\dagger(\vec{y}) \,] = 0$$

$$[\, \psi_a(\vec{x}),\, \bar{\psi}_b(\vec{x}) \,] = \gamma^0_{ab}\,\delta^3(\vec{x}-\vec{y})$$

Free real scalar,

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} + a^\dagger_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}} \qquad (\phi^\dagger(\vec{x}) = \phi(\vec{x}))$$

Complex scalar field

$$\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} + b^\dagger_{-\vec{p}} \right) e^{i\vec{p}\cdot\vec{x}}$$

For Dirac field

$$\psi(\vec{x}) = \underbrace{\sum_{r=1,2}}_{\text{sum over spin}} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a^r_{\vec{p}}\, u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}}\, v^r(-\vec{p}) \right) e^{i\vec{p}\cdot\vec{x}}$$

$$\psi^\dagger(\vec{x}) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a^{r\dagger}_{\vec{p}}\, u^{r\dagger}(\vec{p}) + b^r_{-\vec{p}}\, v^{r\dagger}(-\vec{p}) \right) e^{-i\vec{p}\cdot\vec{x}}$$

Try commutators

$$[a^r_{\vec p}, a^s_{\vec p'}] = 0 \qquad [b^r_{\vec p}, b^s_{\vec p'}] = 0$$

$$[a^r_{\vec p}, a^{s\dagger}_{\vec p'}] = (2\pi)^3 \delta^{rs} \delta^3(\vec p - \vec p')$$

$$[b^r_{\vec p}, b^{s\dagger}_{\vec p'}] = -(2\pi)^3 \delta^{rs} \delta^3(\vec p - \vec p')$$

→ All commutators of $\psi, \psi^\dagger$ work with these.

Back to Hamiltonian

$$H = \int d^3x \left(\psi^\dagger \gamma^0 (-i\vec\gamma \cdot \vec\nabla + m) \psi \right)$$

Put in definitions for $\psi, \psi^\dagger$

Recall $\quad (p^\mu \gamma_\mu) u^r(\vec p) = m u^r(p) \qquad (p^\mu \gamma_\mu = p^0\gamma^0 - \vec p \cdot \vec\gamma)$

$$(\vec p \cdot \vec\gamma + m) u^r(\vec p) = p^0 \gamma^0 u^r(\vec p) = E_{\vec p} \gamma^0 u^r(\vec p)$$

$$(p^0\gamma^0 \underset{\substack{\uparrow \\ \text{sign flip} \\ \text{because of } -\vec p}}{+} \vec p \cdot \vec\gamma) v^r(-\vec p) = -m v^r(-\vec p)$$

$$(\vec p \cdot \vec\gamma + m) v^r(-\vec p) = -p^0\gamma^0 v^r(-\vec p) = -E_{\vec p} \gamma^0 v^r(-\vec p)$$

$$\Rightarrow (-i\vec\gamma \cdot \vec\nabla + m)\psi(\vec x) = \gamma^0 \sum_r \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \left(E_{\vec p} a^r_{\vec p} u^r(\vec p) - E_{\vec p} b^{r\dagger}_{-\vec p} v^r(-\vec p) \right) e^{i\vec p \cdot \vec x}$$

$$H = \sum_r \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \left(a^{r\dagger}_{\vec p} a^r_{\vec p} - b^r_{-\vec p} b^{r\dagger}_{-\vec p} \right) \qquad \text{(some steps skipped)}$$

$$= \sum_r \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \Big(a^{r\dagger}_{\vec p} a^r_{\vec p} \underset{\substack{\uparrow \\ \text{energy is unbounded below.}}}{-} b^{r\dagger}_{-\vec p} b^r_{-\vec p} + \text{const.} \Big) \qquad \text{(use commutator)}$$

→ Replace commutators with anticommutators.

09/28/2010

Quantization of Dirac Field

Try anticommutators

$$\{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})$$

$$\{b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})$$

All other anticommutators are zero.

Define

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right) e^{i\vec{p}\cdot\vec{x}}$$

Will find that

$$\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta^3(\vec{x}-\vec{y})\, \delta_{ab}$$

$$\{\psi_a(\vec{x}), \psi_b(\vec{y})\} = 0 \qquad \{\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})\} = 0$$

Following derivation from last class

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_{\vec{p}} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r - \underbrace{b_{-\vec{p}}^r b_{-\vec{p}}^{r\dagger}}_{= b_{-\vec{p}}^{r\dagger} b_{-\vec{p}}^r + \text{const}} \right)$$

(from anticommutator)

$$H = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_{\vec{p}} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r + \text{const} \right)$$

Energy is bounded below.

In homework, find

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r \right)$$

Usually, we'll write

$$\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \left(a^r_{\vec{p}} u^r(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + b^{r\dagger}_{\vec{p}} v^r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)$$

Heisenberg field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=1,2} \left(a^r_{\vec{p}} u^r(\vec{p}) e^{-ip\cdot x} + b^{r\dagger}_{\vec{p}} v^r(\vec{p}) e^{ip\cdot x} \right)\Big|_{p^0 = E_{\vec{p}}}$$

ψ contains the annihilation of a particle, and creation of antiparticle

$a \to$ annihilates particles

$b \to$ annihilates antiparticles

$a^\dagger \to$ creates particles

$b^\dagger \to$ creates antiparticles

Define vacuum $|0\rangle$

$$a^s_{\vec{p}} |0\rangle = 0 \qquad b^s_{\vec{p}} |0\rangle = 0$$

One particle states: (relativistic normalization)

$$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\, a^{s\dagger}_{\vec{p}} |0\rangle$$

One antiparticle states:

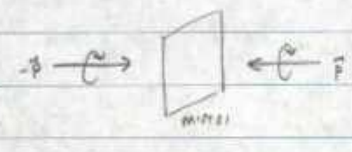
$$|\vec{p}, s\rangle = \sqrt{2E_{\vec{p}}}\, b^{s\dagger}_{\vec{p}} |0\rangle$$

(antiparticle)
(no new notation)

Discrete Symmetries: Parity, Time Reversal, Charge Conjugation

Parity (P) flips the directions of spatial vectors

$$P : (t, \vec{x}) \to (t, -\vec{x})$$



Handedness will change (because momentum flips), spin remains same.

What happens to creation/annihilation operators?

$$P^{\dagger} a^{s}_{\vec{p}} P = a^{s}_{-\vec{p}} \qquad (P^{\dagger}P = 1 = PP^{\dagger},\ P \text{ is unitary})$$

$$P^{\dagger} b^{s}_{\vec{p}} P = b^{s}_{-\vec{p}}$$

$$P^{\dagger} a^{s\dagger}_{\vec{p}} P = a^{s\dagger}_{-\vec{p}} \qquad P^{\dagger} b^{s\dagger}_{\vec{p}} P = b^{s\dagger}_{-\vec{p}}$$

We could have

$$P^{\dagger} a^{s}_{\vec{p}} P = \eta_a a^{s}_{-\vec{p}} \qquad P^{\dagger} b^{s}_{\vec{p}} P = \eta_b b^{s}_{-\vec{p}}$$

η_a, η_b are phases

$$|\eta_a|^2 = 1 \qquad |\eta_b|^2 = 1$$

$$P^{\dagger}\psi(x)P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(\eta_a a^{s}_{-\vec{p}} u^{s}(\vec{p}) e^{-ip\cdot x} + \eta_b^{*} b^{s\dagger}_{-\vec{p}} v^{s}(\vec{p}) e^{ip\cdot x} \right)\Big|_{p^0 = E_{\vec{p}}}$$

Define $\tilde{p} = (E_{\vec{p}}, -\vec{p})$, $\tilde{x} = (t, -\vec{x})$ $(\tilde{p}\cdot\tilde{x} = p\cdot x)$

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \qquad \sigma^\mu = (1, \vec{\sigma}) \qquad \xi^{1,2} = \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$$

$$\bar{\sigma}^\mu = (1, -\vec{\sigma})$$

$$= \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi^s \\ \sqrt{\tilde{p}\cdot\sigma}\,\xi^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u^s(-\vec{p}) = \gamma^0 u^s(-\vec{p})$$

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi^s \\ -\sqrt{\tilde{p}\cdot\sigma}\,\xi^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} v^s(-\vec{p}) = -\gamma^0 v^s(-\vec{p})$$

$$P^\dagger \psi(x) P = \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(\eta_a a^s_{-\vec{p}} u^s(-\vec{p}) e^{-i\tilde{p}\cdot\tilde{x}} - \eta_b^* b^{s\dagger}_{-\vec{p}} v^s(-\vec{p}) e^{i\tilde{p}\cdot\tilde{x}} \right)$$

Let $\eta_a = 1$ (arbitrary, never measurable)

If $\eta_b^* = -1$ $(\eta_b = -1)$, then (relative minus sign between particles/antiparticles)

$$P^\dagger \psi(x) P = \gamma^0 \psi(\tilde{x})$$

Pions $\pi^+ = (u\bar{d})$

Parity $= -1$ → one particle + one antiparticle

What about $\bar{\Psi}(x)$?

$$P^\dagger \bar{\Psi}(x) P = P^\dagger \psi^\dagger(x) \gamma^0 P = P^\dagger \psi^\dagger(x) P \gamma^0$$

$$= (P^\dagger \psi(x) P)^\dagger \gamma^0$$

$$= (\gamma^0 \psi(\tilde{x}))^\dagger \gamma^0$$

$$= \psi^\dagger(\tilde{x}) \gamma^0 \gamma^0 \qquad (\gamma^{0\dagger} = \gamma^0)$$

$$P^\dagger \bar{\Psi}(x) P = \bar{\Psi}(\tilde{x}) \gamma^0$$

$$P^\dagger (\bar{\Psi}(x) \psi(x)) P = P^\dagger \bar{\Psi}(x) P P^\dagger \psi(x) P$$

$$= \bar{\Psi}(\tilde{x}) \gamma^0 \gamma^0 \psi(\tilde{x}) = \bar{\Psi}(\tilde{x}) \psi(\tilde{x})$$

So

$\bar{\Psi}\psi$	scalar
$i\bar{\Psi}\gamma^5\psi$	pseudoscalar
$\bar{\Psi}\gamma^\mu\psi$	vector ($\mu=0 \to$ same, $\mu=1,2,3 \to$ flips sign)
$\bar{\Psi}\gamma^\mu\gamma^5\psi$	axial vector ($\mu=0 \to$ flips sign, $\mu=1,2,3 \to$ same)

Time reversal

Suppose we have some unitary operator T $(T^\dagger T = 1)$

Want $[T, H] = 0$ (simultaneous eigenstates)

$$T^\dagger e^{-iHt} T = e^{-iHt} \qquad \underline{\text{NOT}} \text{ time reversal!}$$

Guess instead $\{T, H\} = 0$

$$T^\dagger H T = -H$$

Any operator with this property has symmetric eigenvalues about $0 \to$ unbounded above $\Rightarrow$ unbounded below

Weird...

Assume time reversal is antilinear (conjugate linear)

$$T^\dagger c T = c^* \qquad (c \text{ - complex number})$$

$$[T, H] = 0 \qquad T^\dagger T = T T^\dagger = 1$$

Then $T^\dagger e^{-iHt} T = e^{+iHt}$

Time is imaginary!

09/30/2010

Time Reversal

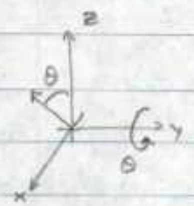
Time reversal is like watching a movie backwards

Idea: $T^{+}\psi(\underset{(t,\vec{x})}{x})T = \text{Matrix } \psi(\underset{(-t,\vec{x})}{x_T})$

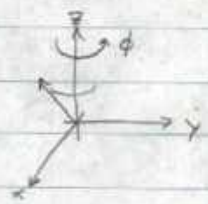
Reversing spins $\quad \xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$ spin up z-axis

$\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$ spin down z-axis

(θ, ϕ)



θ about y-axis



ϕ about z-axis

Transformation $\quad M(\theta,\phi) = \exp\left[-i\phi\frac{\sigma^3}{2}\right]\exp\left[-i\theta\frac{\sigma^2}{2}\right]$

$$\exp\left[-i\theta\frac{\sigma^2}{2}\right] = 1 - i\frac{\theta}{2}\sigma^2 + \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{i}{3!}\left(\frac{\theta}{2}\right)^3\sigma^3$$

odd terms $\to -i\sin\left(\frac{\theta}{2}\right)\sigma^2$

even terms $\to \cos\left(\frac{\theta}{2}\right)$

$$\exp\left[-i\theta\frac{\sigma^2}{2}\right] = \cos\left(\frac{\theta}{2}\right) - i\sin\left(\frac{\theta}{2}\right)\sigma^2 = \begin{pmatrix}\cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right)\\ \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right)\end{pmatrix}$$

$$\exp\left[-i\phi\frac{\sigma^3}{2}\right] = \cos\left(\frac{\phi}{2}\right) - i\sin\left(\frac{\phi}{2}\right)\sigma^3 = \begin{pmatrix}\exp[-i\phi/2] & 0\\ 0 & \exp[i\phi/2]\end{pmatrix}$$

$$M(\theta,\phi) = \begin{pmatrix} e^{-i\phi/2}\cos(\frac{\theta}{2}) & -e^{-i\phi/2}\sin(\frac{\theta}{2}) \\ e^{i\phi/2}\sin(\frac{\theta}{2}) & e^{i\phi/2}\cos(\frac{\theta}{2}) \end{pmatrix}$$

Spin up along (θ,ϕ) direction

$$\xi^1 = M(\theta,\phi)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\phi/2}\cos(\theta/2) \\ e^{i\phi/2}\sin(\theta/2)\end{pmatrix}$$

Spin down along (θ,ϕ) direction

$$\xi^2 = M(\theta,\phi)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} -e^{-i\phi/2}\sin(\frac{\theta}{2}) \\ e^{i\phi/2}\cos(\theta/2)\end{pmatrix}$$

Trivia: 2π rotation for a spinor gives an overall minus sign.

Set $\theta = 2\pi$, $\phi = 0$

$$\Rightarrow \quad \xi^1 = M(2\pi,0)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$
$$\xi^2 = M(2\pi,0)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

Set $\theta = 0$, $\phi = 2\pi$

$$\Rightarrow \quad \xi^1 = M(0,2\pi)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$
$$\xi^2 = M(0,2\pi)\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

For spin flip $\rightarrow$ rotate $\theta+\pi$ about y-axis, then ϕ about z-axis

$$\xi^1(\theta+\pi,\phi) = \begin{pmatrix} e^{-i\phi/2}\cos(\frac{\theta+\pi}{2}) \\ e^{i\phi/2}\sin(\frac{\theta+\pi}{2})\end{pmatrix} = \begin{pmatrix} -e^{-i\phi/2}\sin(\theta/2) \\ e^{i\phi/2}\cos(\theta/2)\end{pmatrix}$$

$$\xi^2(\theta+\pi,\phi) = \begin{pmatrix} -e^{-i\phi/2}\sin(\frac{\theta+\pi}{2}) \\ e^{i\phi/2}\cos(\frac{\theta+\pi}{2})\end{pmatrix} = \begin{pmatrix} -e^{-i\phi/2}\cos(\theta/2) \\ -e^{i\phi/2}\sin(\theta/2)\end{pmatrix}$$

$$\xi^s = (\xi^1, \xi^2) \rightarrow \xi^{-s} = (\xi^2, -\xi^1)$$

spin-reversed spinor

$$\xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{1*} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\phi/2}\cos(\theta/2) \\ e^{-i\phi/2}\sin(\theta/2) \end{pmatrix}$$

$$\xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{2*} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -e^{i\phi/2}\sin(\theta/2) \\ e^{i\phi/2}\cos(\theta/2) \end{pmatrix}$$

$$\xi^{-s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{s*} = -i\sigma^2 \xi^{s*}$$

$$u^{-s}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\ \xi^{-s} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\ \xi^{-s} \end{pmatrix} \qquad (\tilde{p} = (E_{\vec{p}}, -\vec{p}))$$

$$= \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\ (-i\sigma^2)\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\ (-i\sigma^2)\xi^{s*} \end{pmatrix}$$

$$\sigma^2 \vec{\sigma} = -\vec{\sigma}^* \sigma^2$$

$$\sqrt{\tilde{p}\cdot\sigma} = \sqrt{E_{\vec{p}} + \vec{p}\cdot\vec{\sigma}}$$

$$\sqrt{\tilde{p}\cdot\sigma}\ \sigma^2 = \sigma^2\sqrt{E_{\vec{p}} - \vec{p}\cdot\vec{\sigma}^*} = \sigma^2\sqrt{p\cdot\sigma^*}$$

$$u^{-s}(-\vec{p}) = -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma^*}\ \xi^{s*} \\ \sqrt{p\cdot\bar{\sigma}^*}\ \xi^{s*} \end{pmatrix}$$

$$= -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}(u^s(\vec{p}))^*$$

$$= -\gamma^1\gamma^3 (u^s(\vec{p}))^*$$

$$v^{-s}(-\vec{p}) = -\gamma^1\gamma^3 (v^s(\vec{p}))^*$$

So,

$$T^\dagger a^s_{\vec{p}} T = a^{-s}_{-\vec{p}}$$

$$T^\dagger b^s_{\vec{p}} T = b^{-s}_{-\vec{p}}$$

$$a^{-s}_{-\vec{p}} = (a^2_{-\vec{p}}, -a^1_{-\vec{p}}) \qquad b^{-s}_{-\vec{p}} = (b^2_{-\vec{p}}, -b^1_{-\vec{p}})$$

$$T^\dagger \psi(x) T = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(T^\dagger \left(a^s_{\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{ip\cdot x} \right) T \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^{-s}_{-\vec{p}} (u^s(\vec{p}))^* e^{ip\cdot x} + b^{-s\dagger}_{-\vec{p}} (v^s(\vec{p}))^* e^{-ip\cdot x} \right)$$

$$-i \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \rightarrow (\gamma^1\gamma^3)(-\gamma^1\gamma^3) = 1$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_s \left(a^{-s}_{-\vec{p}} (\gamma^1\gamma^3 u^{-s}(-\vec{p})) e^{ip\cdot x} + b^{-s\dagger}_{-\vec{p}} (\gamma^1\gamma^3 v^{-s}(-\vec{p})) e^{-ip\cdot x} \right)$$

$$e^{iE_{\vec{p}}t} e^{-i\vec{p}\cdot\vec{x}}, \text{ let } \vec{p}' = -\vec{p}, t \rightarrow -t \Rightarrow e^{-iE_{\vec{p}}t} e^{+i\vec{p}'\cdot\vec{x}}$$

$$\boxed{T^\dagger \psi(x) T = \gamma^1\gamma^3 \psi(x_T)} \qquad (x_T = (-t, \vec{x}))$$

$$T^\dagger \bar{\psi}(x) T = T^\dagger (\psi^\dagger(x)\gamma^0) T = T^\dagger \psi^\dagger(x) T \gamma^0 \qquad (\gamma^0 \text{ is real})$$

$$= (T^\dagger \psi(x) T)^\dagger \gamma^0$$

$$= (\gamma^1\gamma^3 \psi(x_T))^\dagger \gamma^0$$

$$= \psi^\dagger(x_T) \gamma^{3\dagger}\gamma^{1\dagger}\gamma^0 \qquad (\gamma^{1\dagger} = -\gamma^1, \gamma^{3\dagger} = -\gamma^3)$$

$$= \psi^\dagger(x_T) \gamma^3\gamma^1\gamma^0$$

$$= \psi^\dagger(x_T) \gamma^0\gamma^3\gamma^1$$

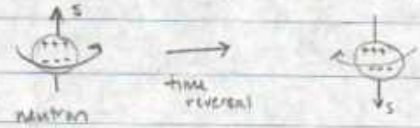
$$\boxed{T^\dagger \bar{\psi}(x) T = -\bar{\psi}(x_T) \gamma^1\gamma^3}$$

$$T^\dagger \bar{\psi}(x)\psi(x) T = (-\bar{\psi}(x_T)\gamma^1\gamma^3)(\gamma^1\gamma^3\psi(x_T)) = \bar{\psi}(x_T)\psi(x_T)$$
$$((-\gamma^1\gamma^3)(\gamma^1\gamma^3) = 1)$$

See notes for other transformations

Current work in our department ... Electric dipole moment of neutron.

→ Would mean T invariance violation



Charge Conjugation

Interchanges particles and antiparticles.

- Spin and momentum are left alone.

$$C^\dagger a^s_{\vec{p}} C = b^s_{\vec{p}} \qquad C^\dagger b^s_{\vec{p}} C = a^s_{\vec{p}}$$

$$C^\dagger \psi(x) C \stackrel{?}{=} \text{Matrix } \psi(x)$$

Won't be able to find a matrix to do this.

Instead, look for

$$C^\dagger \psi(x) C = \text{Matrix } \psi^*(x)$$

Note: C is a linear operator even though the equation
$$C^\dagger \psi(x) C = \text{Matrix } \psi^*(x)$$
looks conjugate linear.

$$C^\dagger i\psi C = i \text{Matrix } \psi^*$$
$$T^\dagger i\psi T = -i \text{Matrix } \psi$$ T is a conjugate linear operator, even though $T^\dagger \psi T = \text{Matrix } \psi$ looks linear.

10/05/2010

Charge Conjugation

We want to connect ψ^* with $C^+\psi C$

$$v^{s*}(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\ \xi^{-s} \end{pmatrix}^*$$

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\ \xi^{-s} \end{pmatrix}$$

$$\xi^{-s} = -i\sigma^2 \xi^{s*}$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\ (-i\sigma^2 \xi^{s*}) \\ -\sqrt{p\cdot\bar\sigma}\ (-i\sigma^2 \xi^{s*}) \end{pmatrix}^*$$

$$-i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \to (-i\sigma^2)^* = -i\sigma^2$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma^*}\ (-i\sigma^2)\ \xi^s \\ -\sqrt{p\cdot\bar\sigma^*}\ (-i\sigma^2)\ \xi^s \end{pmatrix}$$

$$\vec\sigma^*\sigma^2 = -\sigma^2\vec\sigma$$

$$\sigma^{\mu *}\sigma^2 = \sigma^2\bar\sigma^\mu$$

$$= \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma}\ \xi^s \\ i\sigma^2\sqrt{p\cdot\sigma}\ \xi^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} u^s(p) \qquad u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar\sigma}\ \xi^s \end{pmatrix}$$

$$\boxed{v^{s*}(p) = -i\gamma^2 u^s(p)}$$

$$-i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \to (-i\gamma^2)^* = -i\gamma^2$$

$$v^s(p) = -i\gamma^2\ u^{s*}(p)$$

$$\boxed{-i\gamma^2 v^s(p) = u^{s*}(p)} \qquad (-i\gamma^2)(-i\gamma^2) = 1$$

$$C^\dagger \psi(x) C = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \sum_s \left(b^s_{\vec p} u^s(p) e^{-ip\cdot x} + a^{s\dagger}_{\vec p} v^s(p) e^{ip\cdot x} \right)$$

$$u^s(p) = -i\gamma^2 v^{s*}(p)$$
$$v^s(p) = -i\gamma^2 u^{s*}(p)$$

$$= -i\gamma^2 \psi^*(x)$$

$$C^\dagger \psi(x) C = -i\gamma^2 \left(\psi^\dagger(x) \right)^T \qquad (\psi^\dagger(x) = (\psi^*(x))^T)$$

$$= -i\gamma^2 \left(\bar\psi(x)\gamma^0 \right)^T \qquad (\gamma^2)^T = \gamma^2$$

$$= -i \left(\bar\psi(x)\gamma^0\gamma^2 \right)^T$$

$$C^\dagger \bar\psi(x) C = C^\dagger (\psi^\dagger(x)) C \gamma^0$$

$$= (C^\dagger \psi(x) C)^\dagger \gamma^0$$

$$= (-i\gamma^2 \psi^*(x))^\dagger \gamma^0 \qquad \gamma^{2\dagger} = -\gamma^2$$

$$= i\,\psi^T(x)\,\gamma^2\gamma^0 \qquad \gamma^{0T} = \gamma^0$$

$$= -i \left(\gamma^0\gamma^2\psi(x) \right)^T$$

Bilinears

$$C^\dagger \bar\psi \psi C = (C^\dagger \bar\psi C)(C^\dagger \psi C)$$

$$= \left(-i(\gamma^0\gamma^2\psi(x))^T\right)\left(-i(\bar\psi(x)\gamma^0\gamma^2)^T\right)$$

$$= -\left((-i\bar\psi\gamma^0\gamma^2)(-i\gamma^0\gamma^2\psi) \right)^T \qquad \text{(negative because of anticommutator)}$$

components of $\bar\psi$ and ψ anticommute

$$\{\psi_a(x), \bar\psi_b(y)\} = \gamma^0_{ab}\, \delta^3(\vec x - \vec y)$$

$$= \bar\psi\psi \quad \text{(homework)}$$

Define $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

looks like $S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] = \frac{\sigma^{\mu\nu}}{2}$

	$\bar\Psi\psi$	$i\bar\Psi\gamma^5\gamma$	$\bar\Psi\gamma^\mu\psi$	$\bar\Psi\gamma^\mu\gamma^5\psi$	$\bar\Psi\sigma^{\mu\nu}\psi$	γ^μ
P	+1	−1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	+1	−1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	+1	+1	−1	+1	−1	+1
CPT	+1	+1	−1	−1	+1	−1

$$(-1)^\mu = \begin{cases} +1 & \mu = 0 \\ -1 & \mu = 1,2,3 \end{cases}$$

Under CPT $\to (-1)^{\#\text{ indices}}$

$$\bar\Psi\psi \to 0 \quad \bar\Psi\gamma^5\psi \to 0 \quad \bar\Psi\sigma^{\mu\nu}\psi \to 2 \quad \Rightarrow +1$$
$$\bar\Psi\gamma^\mu\psi \to 1 \quad \bar\Psi\gamma^\mu\gamma^5\psi \to 1 \quad \gamma^\mu \to 1 \quad \Rightarrow -1$$

Invariance under CPT is required for any Lorentz invariant local Hermitian operator.

Correlation Functions of Dirac Fields

$$\langle 0|\, \psi_a(x)\, \bar\psi_b(y) |\, 0\rangle$$

only a contributes (annihilate particle) — only $a^\dagger$ contributes (create particle)

$$\langle 0|\, \psi_a(x)\, \bar\psi_b(y) |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} \underbrace{\sum_s u^s_a(p)\, \bar u^s_b(p)}_{(\not p + m)_{ab}} e^{-ip\cdot x}\, e^{ip\cdot y}$$

$$= (i\not\partial_x + m)_{ab} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip\cdot(x-y)}}_{D(x-y)}$$

$$= (i\not\partial_x + m)_{ab}\, D(x-y)$$

Similarly

$$\langle 0|\, \underset{b}{\bar\psi_b(y)}\, \underset{b^\dagger}{\psi_a(x)} |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} \underbrace{\sum_s v^s_a(p)\, \bar v^s_b(p)}_{(\not p - m)_{ab}} e^{ip\cdot(x-y)}$$

$$= (-i\not\partial_x - m)_{ab} \underbrace{\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip\cdot(y-x)}}_{D(y-x)}$$

$$= -(i\not\partial_x + m)_{ab}\, D(y-x)$$

Feynman propagator

$$S_F^{ab}(x-y) = \begin{cases} \langle 0|\psi_a(x)\, \bar\psi_b(y)|0\rangle & x^0 > y^0 \\ -\langle 0|\bar\psi_b(y)\, \psi_a(x)|0\rangle & x^0 < y^0 \end{cases}$$

$$\equiv \underbrace{\langle 0|\, T\{\psi_a(x)\, \bar\psi_b(y)\}\,|0\rangle}_{\text{time ordered product}}$$

$$T\{\psi_a(x)\bar\psi_b(y)\} = \psi_a(x)\bar\psi_b(y)\, \Theta(x^0-y^0) - \bar\psi_b(y)\psi_a(x)\, \Theta(y^0-x^0)$$

Adding Interactions

To preserve causality, we consider only local interactions. For example,

$$H_{int} = \int d^3x \, \mathcal{H}_{int}(\phi(\vec{x})) = -\int d^3x \, \mathcal{L}_{int}(\phi(\vec{x}))$$

products of fields at the same point.

Common example in particle and condensed matter...

$$\mathcal{L}_{int} = \frac{-\lambda}{4!} \phi^4(x)$$

$$\mathcal{L}(x) = \frac{1}{2}(\partial_\mu \phi(x))(\partial^\mu \phi(x)) - \frac{1}{2} m^2 \phi^2(x) - \underbrace{\frac{\lambda}{4!} \phi^4(x)}_{\text{does not involve } \partial_0 \phi}$$

$$\Pi(x) = \partial_0 \phi(x) \qquad \text{(just like free field theory)}$$

In general, adding interactions produces divergences

High momentum divergences ("Ultraviolet")

10/12/2010

Adding Interactions

$$\mathcal{H}_{int} = \frac{\lambda}{4!}\phi^4$$

Effective interactions at low energies have mass dimension ≥ 0

→ Renormalizable interactions

$\lambda = 1\ \text{MeV}^{-3}$ Λ - high momentum cutoff scale

$\Lambda \sim 1000$ MeV → perturbation theory diverges

In 3+1 dimensions

$$S = \int d^4x\, \mathcal{L} = \int d^4x \left(\tfrac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)\right)$$

S has mass dimension 0.

d^4x has -4

$\partial_\mu, \partial^\mu$ each have $+1$

→ ϕ must have mass dimension $+1$

$$\mathcal{L}_{int} = -\mathcal{H}_{int} = \frac{-\lambda}{4!}\phi^4 \quad \rightarrow \lambda = (\text{MeV})^0$$

Perturbative expansion

$$H = H_0 + H_{int}$$

H_0 - free field (Klein-Gordon)

H_{int} - for ex $\int d^3x \frac{\lambda}{4!}\phi^4$

Generate a power series in λ

Commutation relations etc. at $t = t_0$

$$\phi(t_0, \vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^{+}_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$e^{-iE_{\vec{p}}t_0}$ absorbed into $a_{\vec{p}}$

Heisenberg field

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\, \phi(t_0, \vec{x})\, e^{-iH(t-t_0)}$$

If we temporarily shut off interaction

$$= e^{iH_0(t-t_0)}\, \phi(t_0, \vec{x})\, e^{-iH_0(t-t_0)}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^{+}_{\vec{p}}\, e^{+ip\cdot x} \right) \Big|_{\substack{x^0 = t - t_0 \\ p^0 = E_{\vec{p}}}}$$

$$= \phi_I(t, \vec{x}) \quad \text{interaction picture field}$$

Back to Heisenberg field

$$\phi(t, \vec{x}) = e^{iH(t-t_0)}\, \phi(t_0, \vec{x})\, e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH(t-t_0)}\, e^{-iH_0(t-t_0)}}_{U^{+}(t, t_0)}\, \phi_I(t, \vec{x})\, \underbrace{e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}}_{U(t, t_0)}$$

$$= U^{+}(t, t_0)\, \phi_I(t, \vec{x})\, U(t, t_0)$$

$$U(t, t_0) = e^{iH_0(t-t_0)}\, e^{-iH(t-t_0)}$$

Can think of $U(t, t_0)$ as time evolution of the quantum states in the interaction picture

$$i\partial_t U(t,t_0) = e^{iH_0(t-t_0)}(-H_0 + H)\, e^{-iH(t-t_0)} \qquad ([H_0, H] \neq 0)$$

$$= e^{iH_0(t-t_0)}(H_{int})\, e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}_{H_{int}(\phi_I(t,\vec{x}))} \underbrace{e^{iH_0(t-t_0)} e^{-iH(t-t_0)}}_{U(t,t_0)}$$

$$= H_{int}(\phi_I(t,\vec{x}))\, U(t,t_0)$$

$$\equiv H_I(t)\, U(t,t_0)$$

Dysons formula

$$U(t,t_0) = T \exp\left(-i\int_{t_0}^{t} dt'\, H_I(t')\right) \qquad \text{(time ordered)}$$

As a power series in H_I

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt'\, H_I(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2\, T(H_I(t_1)\, H_I(t_2)) + \ldots$$

General definition for $U(t,t')$

$$U(t,t') = T \exp\left(-i\int_{t'}^{t} dt''\, H_I(t'')\right) \qquad (\text{for } t \geq t')$$

$$i\partial_t U(t,t') = H_I(t)\, U(t,t')$$

$$i\partial_{t'} U(t,t') = -U(t,t')\, H_I(t')$$

Easy to check that this works

$$U(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)}$$

$$U^\dagger(t,t') = U^{-1}(t,t') = U(t',t)$$

$$U^\dagger(t,t') = e^{iH_0(t'-t_0)} e^{iH(t-t')} e^{-iH_0(t-t_0)}$$

$$U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$$

Let $|0\rangle$ be the ground state of H_0

Let us zero out H_0 by adding constant so that $H_0|0\rangle = 0$

Let $|\Omega\rangle$ be the ground state of H, the full Hamiltonian

Let $|n\rangle$ label all energy states of H

Let the corresponding energies be E_n

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nt}|n\rangle\langle n|0\rangle$$

Consider limit as $T \to \infty$

Actually, consider $T \to \infty(1-i\epsilon)$ (make time slightly imaginary)

Assume there is a gap in energy between the vacuum and the next energy levels.

As $T \to (1-i\epsilon)\infty$,

$$e^{-iHT}|0\rangle \to e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$$

(higher energy states increasingly dampened)

$$|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty}\left(\left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle\right)$$

Also

$$|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \left(\left(e^{-iE_0(T+t_0)} \langle\Omega|0\rangle\right)^{-1} e^{\overbrace{-iH(T+t_0)}^{-iH(t_0-(-T))}} \underset{e^{-iH_0(-T-t_0)}}{\uparrow} |0\rangle \right) \qquad (H_0|0\rangle = 0)$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(\left(e^{-iE_0(t_0-(-T))} \langle\Omega|0\rangle\right)^{-1} U(t_0,-T)\,|0\rangle \right)$$

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \left(\left(e^{-iE_0(T-t_0)} \langle 0|\Omega\rangle\right)^{-1} \langle 0| \underset{e^{+iH_0(T-t_0)}}{\uparrow} e^{-iH(T-t_0)} \right)$$

$$= \lim_{T\to\infty(1-i\epsilon)} \left(\left(e^{-iE_0(T-t_0)} \langle 0|\Omega\rangle\right)^{-1} \langle 0|U(T,t_0) \right)$$

Now, consider

$$\langle\Omega|\,\phi(x)\,\phi(y)\,|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \Big(\left(e^{-iE_0(T-t_0)} \langle 0|\Omega\rangle\right)^{-1} \left(e^{-iE_0(t_0-(-T))} \langle\Omega|0\rangle\right)^{-1}$$

$$\times \langle 0|\,U(T,t_0)\,\phi(x)\,\phi(y)\,U(t_0,-T)\,|0\rangle \Big)$$

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Continue from last time ...

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty}\Big(\big(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\big)^{-1}\big(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\big)^{-1}$$

$$\times \langle 0|U(T,t_0)\,\phi(x)\phi(y)\,U(t_0,-T)|0\rangle\Big)$$

$$U(t,t') = e^{iH_0(t-t_0)}e^{-iH(t-t')}e^{-iH_0(t'-t_0)} = T\left(\exp\left(-i\int_{t'}^{t}dt''\,H_I(t'')\right)\right) \quad t>t'$$

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty}\Big(\big(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\big)^{-1}\big(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\big)^{-1}$$

$$\times \langle 0|U(T,t_0)U^\dagger(x^0,t_0)\,\phi_I(x)\,U(x^0,t_0)U^\dagger(y^0,t_0)\,\phi_I(y)\,U(y^0,t_0)U(t_0,-T)|0\rangle$$

$$= \lim_{T\to(1-i\epsilon)\infty}\Big(\big(|\langle 0|\Omega\rangle|^2 e^{-iE_0 2T}\big)^{-1}$$

$$\times \langle 0|U(T,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-T)|0\rangle\Big)$$

$\langle\Omega|\Omega\rangle = 1$ (normalized)

$$= \lim_{T\to(1-i\epsilon)\infty}\Big(\big(|\langle 0|\Omega\rangle|^2 e^{-iE_0 2T}\big)^{-1}\langle 0|U(T,-T)|0\rangle\Big)$$

$$\Rightarrow \langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty}\left(\frac{\langle 0|U(T,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}\right)$$

$(x^0 \geq y^0)$

$$= \lim_{T\to(1-i\epsilon)\infty}\left(\frac{\langle 0|T\left(\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right)|0\rangle}{\langle 0|T\left(\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right)|0\rangle}\right)$$

In general,

$$\langle\Omega|T(\phi(x)\phi(y))|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty}\left(\frac{\langle 0|T\left(\phi_I(x)\phi_I(y)\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right)|0\rangle}{\langle 0|T\left(\exp\left[-i\int_{-T}^{T}dt\,H_I(t)\right]\right)|0\rangle}\right)$$

What is $\langle 0| T(\phi_I(x)\,\phi_I(y))|0\rangle$?

$$\phi_I(x) = \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\, a_{\vec p}\, e^{-ip\cdot x}}_{\phi_I^+(x)} + \underbrace{\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}\, a_{\vec p}^+\, e^{ip\cdot x}}_{\phi_I^-(x)}$$

"positive" frequency ($e^{-iE_{\vec p}t}$) only annihilation operators

"negative" frequency ($e^{+iE_{\vec p}t}$) only creation operators

$$\phi_I^+(x)|0\rangle = 0 \qquad \langle 0|\phi_I^-(x) = 0$$

Take $x^0 \geq y^0$ for now

$$T(\phi_I(x)\,\phi_I(y)) = \phi_I^+(x)\,\phi_I^+(y) + \phi_I^-(x)\,\phi_I^+(y) + \underbrace{\phi_I^+(x)\,\phi_I^-(y)}_{\phi_I^-(y)\,\phi_I^+(x) + [\phi_I^+(x),\,\phi_I^-(y)]} + \phi_I^-(x)\,\phi_I^-(y)$$

Move annihilations to right and creations to left.

$[\phi_I^+(x), \phi_I^-(y)]$ is just a number

$[\phi_I^+(x), \phi_I^-(y)] = \langle 0|[\phi_I^+(x), \phi_I^-(y)]|0\rangle$ (because it's a number)

$$= \langle 0|\phi_I^+(x)\,\phi_I^-(y)|0\rangle$$

$$= \langle 0|\phi_I(x)\,\phi_I(y)|0\rangle$$

$$T(\phi_I(x)\,\phi_I(y)) = \phi_I^+(x)\,\phi_I^+(y) + \phi_I^-(x)\,\phi_I^+(y)$$

$$+ \phi_I^-(y)\,\phi_I^+(x) + \langle 0|\phi_I(x)\,\phi_I(y)|0\rangle + \phi_I^-(x)\,\phi_I^-(y)$$

All annihilation operators on the right, all creation operators on the left

Normal ordering

Consider function of creation and annihilation operators $f(a, a^{\dagger})$

$N(f(a,a^{\dagger}))$ - rearrange the string of as and $a^{\dagger}$s so that a on the right, $a^{\dagger}$ on left

$$N(a^{\dagger}_{\vec{p}}\, a_{\vec{q}}) = a^{\dagger}_{\vec{p}}\, a_{\vec{q}}$$

$$N(a_{\vec{q}}\, a^{\dagger}_{\vec{p}}) = a^{\dagger}_{\vec{p}}\, a_{\vec{q}}$$

$$N(a_{\vec{p}}\, a_{\vec{q}}\, a^{\dagger}_{\vec{r}}) = a^{\dagger}_{\vec{r}}\, \underbrace{a_{\vec{p}}\, a_{\vec{q}}}_{\text{order does not matter}}$$

Normal ordering is not a mathematical operation.

→ Just a lexicographic convention

$$A = B \not\Rightarrow N(A) = N(B)$$

$$N([a_{\vec{p}}, a^{\dagger}_{\vec{p}'}]) = N(a_{\vec{p}}\, a^{\dagger}_{\vec{p}'} - a^{\dagger}_{\vec{p}'}\, a_{\vec{p}})$$
$$= a^{\dagger}_{\vec{p}'}\, a_{\vec{p}} - a^{\dagger}_{\vec{p}'}\, a_{\vec{p}} = 0$$

$$N((2\pi)^3 \delta^3(\vec{p}-\vec{p}')) = (2\pi)^3 \delta^3(\vec{p}-\vec{p}')$$

In the literature you see $:f(a,a^{\dagger}):$

Back to problem. Consider general x^0, y^0.

$$T(\phi_I(x)\, \phi_I(y)) = N(\phi_I(x)\, \phi_I(y)) + \underbrace{\begin{cases} [\phi_I^{+}(x), \phi_I^{-}(y)] & \text{if } x^0 \geq y^0 \\ [\phi_I^{+}(y), \phi_I^{-}(x)] & \text{if } x^0 \leq y^0 \end{cases}}$$

$$= \contraction{}{\phi}{{}_I(x)\,}{\phi}\phi_I(x)\,\phi_I(y) = \contraction{}{\phi}{{}_I(y)\,}{\phi}\phi_I(y)\,\phi_I(x)$$

contraction of $\phi_I(x)$ and $\phi_I(y)$

What is the contraction?

$$\overline{\phi_I(x)\,\phi_I(y)} = \begin{cases} \langle 0|\,\phi_I^+(x)\,\phi_I^-(y)\,|0\rangle & x^0 \geq y^0 \\ \langle 0|\,\phi_I^+(y)\,\phi_I^-(x)\,|0\rangle & x^0 \leq y^0 \end{cases}$$

$$= \langle 0|\,T\,(\phi_I(x)\,\phi_I(y))\,|0\rangle$$

$$= D_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\epsilon}\,e^{-ip\cdot(x-y)}$$

Wick's Theorem

Claim: $T(\phi(x_1)\cdots\phi(x_N)) = N(\phi(x_1)\cdots\phi(x_N) + \text{all possible contractions})$

(drop I subscript)
(these are $\phi_I(x_1)\cdots\phi_I(x_N)$)

Ex. $T(\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)) = T(\phi_1\,\phi_2\,\phi_3\,\phi_4)$

$$\begin{aligned} = N(&\phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 \\ &+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 \\ &+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 \\ &+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 \\ &+ \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4\,) \end{aligned}$$

Proof by induction

We have shown this for $n=2$.

Assume it's true for $n-1$.

Let us define $W(\phi_1,\ldots,\phi_N) \equiv N(\phi_1\cdots\phi_N + \text{all possible contractions})$

We want to show $W(\phi_1,\ldots,\phi_N) = T(\phi_1\cdots\phi_N)$.

Without loss of generality, we consider the case

$$x_1^0 \geq x_2^0 \geq \cdots \geq x_N^0 \quad \text{(already ordered)}$$

$$T(\phi_1 \cdots \phi_N) = \phi_1 \, T(\phi_2 \cdots \phi_N)$$

$$= \phi_1 \, W(\phi_2, \ldots, \phi_N)$$

$$= (\phi_1^+ + \phi_1^-) \, W(\phi_2, \ldots, \phi_N)$$

$$= \underbrace{\phi_1^- W(\phi_2, \ldots, \phi_N) + W(\phi_2, \ldots, \phi_N) \, \phi_1^+}_{X} + \underbrace{[\phi_1^+, W(\phi_2, \ldots, \phi_N)]}_{Y}$$

Everything is normal ordered.

In Y, ϕ_1 is always contracted with something

X contains all contractions of $\phi_1, \ldots, \phi_N$ which does not contract ϕ_1

Y, is all contractions of $\phi_1, \ldots, \phi_N$ which does contract ϕ_1

$$\Rightarrow \quad T(\phi_1 \ldots \phi_N) = W(\phi_1, \ldots, \phi_N)$$

By induction, it holds for all N.

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Wick's Theorem

$$T(\phi_1 \phi_2 \ldots \phi_N) = N\left(\phi_1 \cdots \phi_N + \text{all possible contractions}\right)$$

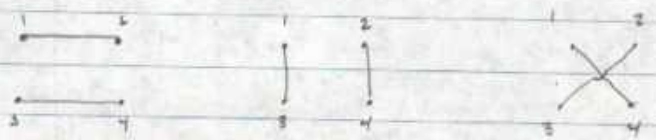
$$= N\left(\exp\left(\tfrac{1}{2}\sum_{i\neq j} \phi_i\phi_j \tfrac{\partial}{\partial\phi_i}\tfrac{\partial}{\partial\phi_j}\right) \phi_1\phi_2\ldots\phi_N\right)$$

$$T(\phi_1\phi_2\phi_3\phi_4) = N(\phi_1\phi_2\phi_3\phi_4 + \text{all possible contractions})$$

$$\langle 0| T(\phi_1\phi_2\phi_3\phi_4)|0\rangle = \text{only fully contracted terms}$$

$$= \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$

Feynman Diagrams



$\phi_1\phi_2\phi_3\phi_4$ $\quad$ $\phi_1\phi_2\phi_3\phi_4$ $\quad$ $\phi_1\phi_2\phi_3\phi_4$

Consider $\langle 0| T\left(\phi(x)\phi(y) \exp\left[-i\int_{-\infty}^{\infty} dt\, H_I(t)\right]\right)|0\rangle$ $\quad$ ($\frac{\lambda}{4!}\phi^4$ interaction)

Power series in λ

Lowest order λ^0: $\langle 0|T(\phi(x)\phi(y))|0\rangle =$  $= D_F(x-y)$

First order λ^1: $\langle 0| T\left(\phi(x)\phi(y)(-i)\int d^4z\, \frac{\lambda}{4!}\phi^4(z)\right)|0\rangle$

$$= \frac{-i\lambda}{4!}\int d^4z\, \langle 0| T(\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z))|0\rangle$$

$$= \frac{-i\lambda}{4!}\int d^4z \Big(\phi(x)\phi(y) \big(\phi(z)\phi(z)\phi(z)\phi(z)$$
$$+ \phi(z)\phi(z)\phi(z)\phi(z)$$
$$+ \phi(z)\phi(z)\phi(z)\phi(z) \big)$$
$$+ \left(\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z) + \cdots\right)\Big)$$ 12 such terms

We can write as:

$$\propto \int d^4z\, D_F(x-y)\, D_F(z-z)\, D_F(z-z) + \propto \int d^4z\, D_F(x-z)\, D_F(z-z)\, D_F(z-y)$$

3 such terms — 12 such terms

x, y are external vertices

z is internal vertex

How do we count combinatorial factors?

Each H_I has 4 ϕs: $\phi(z)\phi(z)\phi(z)\phi(z)$

$$\cdots \underbrace{\phi(z)\,\phi(z)}_{\substack{\text{exchange}\\ \text{contraction}\\ \text{ends}}}\phi(z)\,\phi(z) \cdots = \cdots \phi(z)\,\phi(z)\,\phi(z)\,\phi(z)\cdots$$

$$= \cdots \phi(z)\,\phi(z)\,\phi(z)\,\phi(z) \cdots$$

4! different ways to exchange contraction ends

→ cancels the $\frac{1}{4!}$ in the interaction $\frac{\lambda}{4!}$

For a diagram with more than 1 power of H_I, we can exchange all contraction ends of one H_I with all contraction ends of another H_I

$$\cdots \phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)\,\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2) \cdots$$

$$= \int d^4z_1\, d^4z_2\;\; \phi(z_1)\phi(z_1)\phi(z_1)\phi(z_1)\,\phi(z_2)\phi(z_2)\phi(z_2)\phi(z_2)$$

For a diagram with n powers of H_I (n internal vertices), we get a factor of $n!$. This $n!$ cancels the $\frac{1}{n!}$ in the power series expansion of the exponential $\exp\left[-i\int dt\, H_I(t)\right]$

Small subtlety ... Symmetry factors

Let's consider $\frac{\lambda}{3!}\phi^3$ theory to make the argument easier to follow.

At second order in λ, consider $\langle 0| T\left(\exp\left[-i\int H_I(t)\,dt\right]\right)|0\rangle$

 $\propto \int d^4x\, d^4y\, D_F(x-y)\, D_F(x-y)\, D_F(x-y)$

Naively, we expect a $2!$ from exchanging the internal vertices x and y. and expect $3! \times 3!$ from exchanging the ϕs at x and the ϕs at y.

$$2! \times 3! \times 3! = 2 \cdot 6 \cdot 6 = 72$$

$\phi_x \phi_x \phi_x \phi_y \phi_y \phi_y$ − first ϕ_x with first ϕ_y $\left(\begin{array}{l} \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \\ \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \end{array}\right.$

first ϕ_x with second ϕ_y $\left(\begin{array}{l} \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \\ \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \end{array}\right.$

first ϕ_x with third ϕ_y $\left(\begin{array}{l} \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \\ \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \end{array}\right.$

There are only 6. We overshot by a factor of $\frac{72}{6} = 12$.

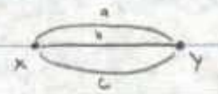
Why? Let's start with $\phi_x \phi_x \phi_x \phi_y \phi_y \phi_y$

Consider exchanging first ϕ_x and second ϕ_x and simultaneously the first ϕ_y and second ϕ_y.

$\phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \longrightarrow \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y$ same!

Also interchanging all ϕ_x and all ϕ_y give the same contraction pattern

$\phi_x \phi_x \phi_x \phi_y \phi_y \phi_y \longrightarrow \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y$



Label internal vertices and propagators.

Consider exchanging $a \longleftrightarrow b$.

topologically identical.

Similarly, exchange $x \leftrightarrow y$

topologically identical

Symmetry group : $3!$ from exchanging set $\{a,b,c\}$
$2!$ from exchanging $\{x,y\}$

$3! \times 2! = 6 \cdot 2 = 12$ (our factor of overcounting)

S = symmetry factor = number of elements in the symmetry group

$$\frac{3! \times 3! \times 2!}{3! \times 2!} = 6$$

Feynman Rules in Position Space

For each propagator

$$x \bullet\!\!\!-\!\!\!-\!\!\!-\!\!\!\bullet\, y = D_F(x-y)$$

For each internal vertex

$$\times_z = (-i\lambda) \int d^4 z \quad (\text{no } \tfrac{1}{4!} \text{ here})$$

Divide by symmetry factor S.

Example:

$$S = 3! = 6 \quad (\text{exchanging } \{z_1, z_2\} \text{ not identical})$$

$$\text{Amplitude} = \frac{(-i\lambda)^2}{6} \int d^4z_1\, d^4z_2\, D_F(x-z_1) \big(D_F(z_1-z_2)\big)^3 D_F(z_2-y)$$

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Note on Symmetry Factors

symmetry factor of 2

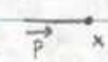
(for loops with no directionality)

Feynman Rules in Momentum Space

For each propagator

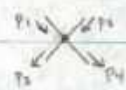
$= \dfrac{i}{p^2 - m^2 + i\epsilon}$

For each external vertex



$= e^{-ip\cdot x}$

For each internal vertex



$= -i\lambda$

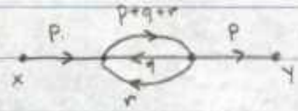
Momentum conservation $(2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$

Integrate over momenta that are unconstrained

$$\int \frac{d^4p}{(2\pi)^4}$$

Divide by the total symmetry factor S.

Example



$$= (-i\lambda)^2 \; \frac{1}{6}\int \frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\frac{d^4r}{(2\pi)^4}\Bigg(e^{-ip\cdot y}\, e^{ip\cdot x} \left(\frac{i}{p^2-m^2+i\epsilon}\right)^2$$

$$\times \left(\frac{i}{(p+q+r)^2-m^2+i\epsilon}\right)\left(\frac{i}{q^2-m^2+i\epsilon}\right)\left(\frac{i}{r^2-m^2+i\epsilon}\right)\Bigg)$$

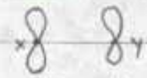
Consider diagrams with only internal vertices (vacuum diagrams)

$$\langle 0|\, T\left(\exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,dt\right]\right)|0\rangle$$

At order λ^1 :

$(S=4)$

At order λ^2 :

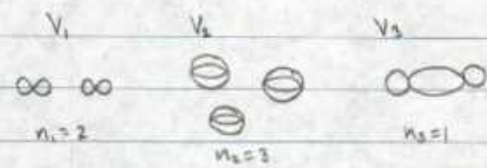


$(S=4\cdot 4\cdot 2)$

disconnected

Amplitude for disconnected diagram is product of the amplitudes for connected pieces

Consider a general vacuum diagram with connected subdiagrams V_i each appearing n_i times.



We will use V_i to represent the diagram and the amplitude.

The amplitude for the total diagram is

$$\frac{V_1^{2}}{2!} \cdot \frac{V_2^{3}}{3!} \cdot \frac{V_3^{1}}{1!} = \prod_i \left(\frac{V_i^{n_i}}{n_i!} \right)$$

What is the sum of all vacuum diagrams?

$$\left(1 + \frac{(V_1)^1}{1!} + \frac{(V_1)^2}{2!} + \frac{(V_1)^3}{3!} + \cdots\right)\left(1 + \frac{(V_2)^1}{1!} + \frac{(V_2)^2}{2!} + \frac{(V_2)^3}{3!} + \cdots\right) \cdots$$

Each monomial in this product will be

$$\frac{(V_1)^{n_1}}{n_1!} \times \frac{(V_2)^{n_2}}{n_2!} \times \cdots$$

We can write this product as

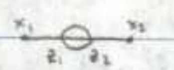
$$\prod_i \left(\sum_{n_i=0}^{\infty} \left(\frac{(V_i)^{n_i}}{n_i!} \right) \right) = \prod_i \exp\left[V_i\right] = \exp\left[\sum_i V_i\right]$$

Sum of all vacuum diagrams = exp [Sum of connected vacuum diagrams]

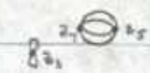
If we have n external vertices

$$\langle 0 | T\left(\phi(x_1) \cdots \phi(x_n) \exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,dt\right]\right) | 0 \rangle$$

Example : $n=2$



connected diagram (all internal vertices connected to an external vertex)



disconnected

$$\langle 0 | T\left(\phi(x_1) \cdots \phi(x_n) \exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,dt\right]\right) | 0 \rangle$$

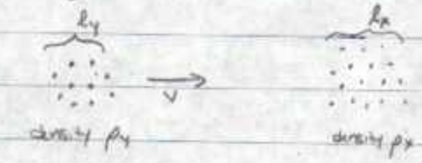
= (Sum of connected diagrams) × (Sum of all vacuum diagrams)

Recall $\langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle = \dfrac{\langle 0|T(\phi(x_1)\cdots\phi(x_n)\exp[-i\int_{-\infty}^{\infty}H_I(t)dt])|0\rangle}{\langle 0|T(\exp[-i\int_{-\infty}^{\infty}H_I(t)dt])|0\rangle}$

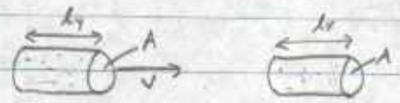
= sum of connected diagrams with external vertices $x_1, x_2, \ldots, x_n$

Cross Sections and the Scattering Matrix (S Matrix)

Consider a target of particles of type X at rest and incoming particles of type Y moving with speed v, towards the target.



ρ_X, ρ_Y are densities observed in the lab (target at rest)



$$N_X = \rho_X l_X A$$
$$N_Y = \rho_Y l_Y A$$

Total number of scatterings is proportional to $N_X \cdot N_Y$

Let the total number of scatterings be $N_X \cdot N_Y \cdot \frac{\sigma}{A}$

$\frac{\sigma}{A}$ = probability that a particular X particle and a particular Y particle scatter.

σ is the cross section (total cross section)

When $N_x = 1$ (1 target particle)

$$\text{total number of scatterings} = N_y \frac{\sigma}{A} = \rho_y l_y \sigma$$

$$\text{So} \quad \sigma = \frac{\text{total \# scatterings}}{\rho_y l_y}$$

Consider what happens in time duration Δt

$$\sigma = \frac{\Delta \text{\# scatterings}/\Delta t}{\rho_y \underbrace{\Delta l_y/\Delta t}_{v}} = \frac{\text{\# scatterings per unit time}}{\underbrace{\rho_y v}_{\text{particle flux}}}$$

Differential cross section is the fraction of σ in which final particles lie in some window of momenta

$$\frac{d\sigma}{\underbrace{d^3p_1 \cdots d^3p_n}_{\text{final particle momenta}}}$$

Simplest case $\to$ 2 final particles

two spatial momenta $\quad d^3p_1\, d^3p_2 \quad$ (6 parameters)

four-momentum conservation $\quad$ 4 constraints

$6-4 = 2$ free parameters

We can take these two free parameters to be angles θ, ϕ

$$\to \frac{d\sigma}{d\Omega}(\theta, \phi)$$

Peskin + Schroeder use wave packets. We consider instead a periodic box with length L on all sides.

10/20/2010

Cross Sections and the Scattering Matrix

Consider space is a periodic cube of length L.

Spatial momentum modes are discrete $\quad \vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z) \quad n_{x,y,z}$ - integers

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \cdot V \qquad V \text{ - volume of cube } (L^3)$$

Check consistency with infinite volume limit.

$$\delta_{\vec{k},\vec{k}'} \cdot V = \int_0^L\int_0^L\int_0^L dx_1 dx_2 dx_3 \, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$\text{As } V \to \infty \qquad = \iiint d^3x \; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} = (2\pi)^3 \delta^3(\vec{k}-\vec{k}')$$

The field $\phi(x)$ in our periodic box

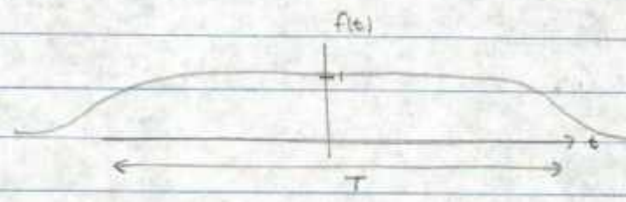
$$\phi(x) = \sum_{\vec{k}} \left(\frac{2\pi}{L}\right)^3 \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} e^{+ik\cdot x}\right)$$

$$\phi(x) = \sum_{\vec{k}} \frac{1}{V} \frac{1}{\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} e^{+ik\cdot x}\right)$$

At some early time, $\to$ free field theory

At some late time $\to$ free field theory

Replace $H_I(t)$ with $H_I(t) f(t)$



$$\int_{-\infty}^{\infty} f(t)\,dt = T \qquad \int_{-\infty}^{\infty} (f(t))^2 dt = T$$

Scattering matrix

$$S = \underset{\substack{\uparrow \\ \text{time ordering}}}{T}\left(\exp\left[-i \int_{-\infty}^{\infty} dt \; H_I(t) f(t) \right] \right)$$

$$\langle \underset{\substack{\uparrow \\ \text{free particle state}}}{\text{final}} | \, S \, | \underset{\substack{\uparrow \\ \text{free particle state}}}{\text{initial}} \rangle$$

$|\text{initial}\rangle \rightarrow$ momenta $\vec{k}_i^I$, energies E_i^I (I - initial, i - particle index)

$|\text{final}\rangle \rightarrow$ momenta $\vec{k}_i^F$, energies E_i^F (F - final)

Consider the nontrivial part of the S matrix

$$\langle \text{final} | \, S - 1 \, | \text{initial} \rangle$$

$$= i \mathcal{M} (2\pi)^4 \, \delta(E^F_{tot} - E^I_{tot}) \, \delta^3(\vec{k}^F_{tot} - \vec{k}^I_{tot}) \qquad \text{(infinite volume)}$$

$\mathcal{M}$ is a function of momenta (Feynman amplitude)

Return to our periodic cube. Finite V and $f(t)$

$$\langle \text{final} | S - 1 | \text{initial} \rangle = i \mathcal{M} \left(\int_{-\infty}^{\infty} f(t) \, e^{i(E^F_{tot} - E^I_{tot}) t} \, dt \right) \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V$$

$$|\langle \text{final} | S - 1 | \text{initial} \rangle|^2 = |\mathcal{M}|^2 \, \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \, V^2 \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) f(t') \, e^{i(E^F_{tot} - E^I_{tot})(t - t')} \, dt \, dt'}_{\substack{\text{as } T \to \infty, \text{ this is proportional to } \delta(E^F_{tot} - E^I_{tot}) \\ \text{What is constant of proportionality?}}}$$

Integrate with respect to E^F_{tot}.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dE^F_{tot}\; e^{i(E^F_{tot}-E^I_{tot})(t-t')} f(t)f(t')\,dt\,dt'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (2\pi)\,\delta(t-t')\, f(t)f(t')\,dt\,dt'$$

$$= (2\pi)\int_{-\infty}^{\infty} (f(t))^2\,dt \;=\; 2\pi T$$

$$|\langle \text{final}|S-1|\text{initial}\rangle|^2 = |\mathcal{M}|^2\, \delta_{\vec{K}^F_{tot},\vec{K}^I_{tot}}\, V^2\, (2\pi)T\,\delta(E^F_{tot}-E^I_{tot})$$

What are the normalizations of the final and initial states?

Relativistically normalized states

$$\langle \text{initial}|\text{initial}\rangle = \prod_i (2E^I_i \cdot V)$$

$$\langle \text{final}|\text{final}\rangle = \prod_i (2E^F_i V)$$

$$\frac{\text{probability}}{\text{time}} = \frac{1}{T}\,\frac{|\langle \text{final}|S-1|\text{initial}\rangle|^2}{\langle \text{final}|\text{final}\rangle\langle \text{initial}|\text{initial}\rangle}$$

$$= \frac{|\mathcal{M}|^2 (2\pi)\,\delta(E^F_{tot}-E^I_{tot})\,\delta_{\vec{K}^F_{tot},\vec{K}^I_{tot}}\,V^2}{\prod_i (2E^F_i V)\,\prod_i (2E^I_i V)}$$

As $V\to\infty$, $\quad \delta_{\vec{K}^F_{tot},\vec{K}^I_{tot}}\cdot V \to (2\pi)^3\,\delta^3(\vec{K}^F_{tot}-\vec{K}^I_{tot})$

$$\sum_{\vec{K}^F_1,\dots,\vec{K}^F_{n_F}} \left(\frac{1}{2E^F_1 V}\right)\cdots\left(\frac{1}{2E^F_{n_F} V}\right) \to \int\frac{d^3k^F_1}{(2\pi)^3(2E^F_1)}\;\cdots\;\int\frac{d^3K^F_{n_F}}{(2\pi)^3(2E^F_{n_F})}$$

n_F = # particles in final state.

Let $n_I = 1$ (one particle initial state)

Total decay rate (probability per unit time for decay)

$$\Gamma = \int d\Gamma$$

$$d\Gamma = \frac{1}{2E^I} \left(\prod_{i=1}^{n_F} \frac{d^3k_i^F}{(2\pi)^3 (2E_i^F)} \right) |\mathcal{M}|^2 (2\pi)^4 \delta^4 (k_{tot}^F - k_{tot}^I)$$

Let $n_I = 2$ (two particle initial state)

Cross section is given by

σ = probability / (time · flux density)

flux density = (relative speed between incoming beam and target)(density of particles in beam)

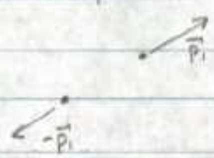
We have normalized the probability in our periodic cube

→ 1 incoming particle → density $= \frac{1}{V}$

$$\text{flux density} = \frac{|\vec{v}_A - \vec{v}_B|}{V} \qquad \vec{v}_A, \vec{v}_B \text{ - velocities in lab frame}$$

$$d\sigma = \underbrace{\left(\prod_i \frac{d^3k_i^F}{(2\pi)^3 (2E_i^F)} \right) (2\pi)^4 \delta^4 (k_{tot}^F - k_{tot}^I)}_{\equiv d\Pi_{n_F} \text{ - phase space measure of integration}} \frac{|\mathcal{M}|^2}{2E_A 2E_B |\vec{v}_A - \vec{v}_B|}$$

Consider $n_f = 2$ (two final particles) in center of mass frame.



$E_{cm} = E_{tot}^{I}$

$$\int d\Pi_2 = \int \frac{dp_1 \; p_1^2 \, d\Omega}{(2\pi)^3 (2E_1)(2E_2)} (2\pi)\, \delta(E_1 + E_2 - E_{cm}) \qquad p_1 = |\vec{p}_1|$$

$$E_1 = \sqrt{p_1^2 + m_1^2}$$
$$E_2 = \sqrt{p_1^2 + m_2^2}$$

$$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^3} \int_0^\infty dp_1 \, \frac{p_1^2 \, \delta\left(\sqrt{p_1^2+m_1^2} + \sqrt{p_1^2+m_2^2} - E_{cm}\right)}{\sqrt{p_1^2+m_1^2}\,\sqrt{p_1^2+m_2^2}}$$

$$\delta(f(x)) = \frac{\delta(x - x_0)}{f'(x_0)}$$

$$\frac{dE_1}{dp_1} = \frac{p_1}{E_1} \qquad \frac{dE_2}{dp_1} = \frac{p_1}{E_2}$$

$$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^2} \frac{p_1^2}{E_1 E_2 \left(\frac{p}{E_1} + \frac{p}{E_2}\right)} \Bigg|_{p_1 \text{ chosen so } E_1 + E_2 = E_{cm}}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1^2}{p_1(E_1 + E_2)}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_{cm}}$$

10/28/2010

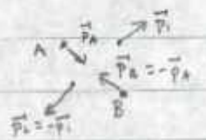
Cross Sections

From last time: Final state 2 particles in center of mass frame

$$d\Pi_2 = \frac{d\Omega}{16\pi^2}\frac{p_1}{E_{cm}}$$ —

Now consider two particles → two particles

$$A + B \to \#1 + \#2 \qquad \text{(center of mass)}$$



$$d\sigma = d\Pi_2 \frac{|\mathcal{M}|^2}{(2E_A)(2E_B)|\vec{v}_A - \vec{v}_B|}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{p_{final}\,|\mathcal{M}|^2}{(2E_A)(2E_B)|\vec{v}_A - \vec{v}_B|}\frac{1}{16\pi^2 E_{cm}}$$

$$E_{cm} = E_A + E_B$$

T matrix → $S = 1 + iT$

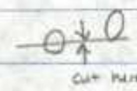
Claim: $\langle \vec{p}_1^{\,F}, \dots, \vec{p}_{n_F}^{\,F} | iT | \vec{p}_A, \vec{p}_B \rangle$

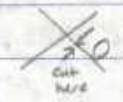
$$= \left({}_{free}\langle \vec{p}_1^{\,F}, \dots, \vec{p}_{n_F}^{\,F} | T\left(\exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,dt\right]\right) | \vec{p}_A, \vec{p}_B \rangle_{free} \right)_*$$

$*$ - only connected diagrams and amputated diagrams.

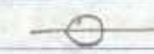
Amputated or one particle irreducible means it cannot be disconnected by cutting one internal line.

Not amputated →





Amputated →



$$|\vec{p}_1, \dots, \vec{p}_n\rangle \propto \lim_{T\to\infty(1-i\epsilon)} e^{-iHT} |\vec{p}_1, \dots, \vec{p}_n\rangle_{free}$$

$$\phi_I^+(x)|\vec{p}\rangle_{free} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}} e^{-ik\cdot x} \left(\sqrt{2E_{\vec{p}}}\, a_{\vec{p}}^{\dagger}|0\rangle\right)$$

$$= e^{-ip\cdot x}|0\rangle$$

$$\overbrace{\phi_I(x)|\vec{p}}\rangle_{free} = e^{-ip\cdot x}$$

$${}_{free}\langle\vec{p}|\phi_I^-(x) = \langle 0| e^{ip\cdot x}$$

$${}_{free}\langle\overbrace{\vec{p}|\phi_I(x)} = e^{ip\cdot x}$$

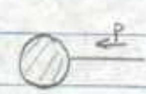
Feynman Rules with External Particles (Position Space)

For each propagator

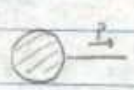
$D_F(x-y)$

For each vertex 

$(-i\lambda)\int d^4z$

For each external line 

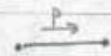
$e^{-ip\cdot x}$



$e^{ip\cdot x}$

Divide by symmetry factor S

Feynman Rules with External Particles (Momentum Space)

For each propagator  $\dfrac{i}{p^2-m^2+i\epsilon}$

For each vertex  $-i\lambda$ and momentum conservation

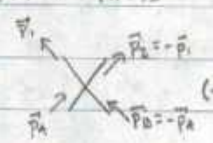
Integrate over all unconstrained momenta

For external line  no extra factor $(\times 1)$

Divide by symmetry factor S

Two particle → two particle in $\frac{\lambda}{4!}\phi^4$ (lowest nontrivial order)

$\langle \vec{p}_1, \vec{p}_2 | iT | \vec{p}_A, \vec{p}_B \rangle$



$(-i\lambda) = i\mathcal{M}$ Feynman amplitude

In center of mass frame

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{p_{final}\,|\mathcal{M}|^2}{(2E_A)(2E_B)|\vec{v}_A-\vec{v}_B|}\frac{1}{16\pi^2 E_{CM}}$$

Only one type of particle → $|\vec{p}_1| = |\vec{p}_A| = p_{final} = p$

$$E_{CM} = E_A + E_B = 2\sqrt{\vec{p}^2+m^2}$$

$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2p}{E_A} \quad (\vec{p}_A = \gamma m \vec{v}_A)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2 p}{16\pi^2\left(\frac{2p}{E_A}\right)(2E_A)(2E_A)(2E_A)}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\lambda^2}{64\pi^2 E_{CM}^2}$$

What is the total cross section?

Careful! Particles are indistinguishable → factor of $\frac{1}{2}$ to avoid double counting.

$$\sigma_{tot} = 4\pi \times \frac{1}{2} \times \frac{\lambda^2}{64\pi^2 E_{cm}^2}$$

$$\sigma_{tot} = \frac{\lambda^2}{32\pi E_{cm}^2}$$

Feynman Rules for Fermions

Recall $\quad T(\psi_a(x)\bar{\psi}_b(y)) = \begin{cases} \psi_a(x)\bar{\psi}_b(y) & x^0 \geq y^0 \\ -\bar{\psi}_b(y)\psi_a(x) & y^0 \geq x^0 \end{cases}$

The Feynman propagator:

$$[S_F(x-y)]_{ab} = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)_{ab}}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}$$

$$= \langle 0| T(\psi_a(x)\bar{\psi}_b(y))|0\rangle$$

$$T(\psi_1\psi_2\psi_3\psi_4) = \begin{cases} \psi_1\psi_2\psi_3\psi_4 & x_1^0 \geq x_2^0 \geq x_3^0 \geq x_4^0 \\ -\psi_2\psi_1\psi_3\psi_4 & x_2^0 \geq x_1^0 \geq x_3^0 \geq x_4^0 \\ -\psi_3\psi_2\psi_1\psi_4 & x_3^0 \geq x_2^0 \geq x_1^0 \geq x_4^0 \\ \vdots & \vdots \end{cases}$$

→ $\times(-1)$ for odd permutation

$\times(+1)$ for even permutation

Normal ordering

$$N(a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3} a^\dagger_{\vec{p}_4}) = (-1)^3 a^\dagger_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3}$$

$\times(-1)$ for odd permutation

$\times(-1)$ for even permutation

$$T(\psi_a(x)\bar{\psi}_b(y)) = N(\psi_a(x)\bar{\psi}_b(y)) + \overbrace{\psi_a(x)\bar{\psi}_b(y)}$$

$$\overbrace{\psi_a(x)\bar{\psi}_b(y)} = \langle 0|T(\psi_a(x)\bar{\psi}_b(y))|0\rangle$$

$$= \begin{cases} \{\psi_a^+(x), \bar{\psi}_b^-(y)\} & x^0 \geq y^0 \\ -\{\bar{\psi}_b^+(y), \psi_a^-(x)\} & y^0 \geq x^0 \end{cases}$$

$$\overbrace{\bar{\psi}_b(y)\psi_a(x)} = \langle 0|T(\bar{\psi}_b(y)\psi_a(x))|0\rangle = -\overbrace{\psi_a(x)\bar{\psi}_b(y)}$$

$$\overbrace{\psi_a(x)\psi_b(y)} = 0 \qquad \overbrace{\bar{\psi}_a(x)\bar{\psi}_b(y)} = 0$$

Wick's theorem holds for fermions.

$$T(\psi_1\bar{\psi}_2\psi_3\cdots) = N(\psi_1\bar{\psi}_2\psi_3\cdots + \text{all possible contractions})$$

(Same type of induction proof as for bosons)

$$N(\psi_1\psi_2\bar{\psi}_3\bar{\psi}_4) = -\overbrace{\psi_1\bar{\psi}_3}\,N(\psi_2\bar{\psi}_4) \qquad (-\text{ because } \bar{\psi}_3 \text{ must hop over } \psi_2)$$

Helpful fact: For any fully contracted quantity, count the number of times the contraction lines cross each other and this tells you whether it's even or odd.

Ex $-$ $\psi_1\psi_2\psi_3\bar{\psi}_4\bar{\psi}_5\bar{\psi}_6$ $-$ even (0 crossings)

$\psi_1\psi_2\psi_3\bar{\psi}_4\bar{\psi}_5\bar{\psi}_6$ $-$ odd (3 crossings)

Yukawa Theory

$$H = \underset{\text{Dirac}}{H_{\text{free}}(\psi, \bar{\psi})} + \underset{\text{Klein-Gordon}}{H_{\text{free}}(\phi)} + g \int d^3x \, \bar{\psi} \psi \phi$$

Let $|\vec{p}, s\rangle$ be a fermion with momentum $\vec{p}$, spin s

$$\psi_I^+(x) |\vec{p}, s\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}} \sum_{s'} a_{\vec{p}'}^{s'} u^{s'}(p') e^{-ip' \cdot x} \left(\sqrt{2E_{\vec{p}}} \, a_{\vec{p}}^{s} |0\rangle \right)$$

$$= e^{-ip \cdot x} u^s(p) |0\rangle$$

$$\psi_I(x) |\vec{p}, s\rangle = e^{-ip \cdot x} u^s(p)$$

Antifermion with momentum $\vec{k}$ and spin s

$$\bar{\psi}_I(x) |\vec{k}, s\rangle = e^{-ik \cdot x} \bar{v}^s(k)$$

$$\langle \vec{p}, s | \bar{\psi}_I(x) = e^{ip \cdot x} \bar{u}^s(p)$$

$$\langle \vec{k}, s | \psi_I(x) = e^{ik \cdot x} v^s(k)$$

11/02/2010

Feynman Rules in Momentum Space for Yukawa Theory

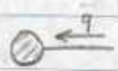
$\phi\phi$ (contracted)  $\dfrac{i}{q^2 - m_\phi^2 + i\epsilon}$

$\psi\bar{\psi}$ (contracted)  $\dfrac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$

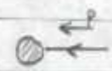

This arrow denotes the flow of particle number

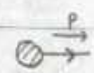
$\leftarrow$ this is the flow convention of momentum

 $= -ig$

$\phi|\vec{q}\rangle$ =  $= 1$

$\langle\vec{q}|\phi$ =  $= 1$

$\psi|\vec{p},s\rangle$ (fermion) =  $= u^s(p)$

$\langle\vec{p},s|\bar{\psi}$ (fermion) =  $= \bar{u}^s(p)$

$\bar{\psi}|\vec{k},s\rangle$ (antifermion) =  $= \bar{v}^s(k)$

$\langle\vec{k},s|\psi$ (antifermion) =  $= v^s(k)$

Note for external particles $\rightarrow$ The particle number arrow points the same as momentum for fermions, opposite for antifermions.

Integrate over unconstrained momenta.

Divide by symmetry factor S.

Example: Two fermions $\vec{p}, \vec{k}$ $\longrightarrow$ Two fermions $\vec{p}', \vec{k}'$

(don't worry about spin in this example)

Momentum conservation $\longrightarrow$ $q = p - p'$

Incoming state $|\vec{p}, \vec{k}\rangle = \sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{k}}}\, a^\dagger_{\vec{p}} a^\dagger_{\vec{k}} |0\rangle$

Let $\langle \vec{k}, \vec{p}|$ be the corresponding bra for $|\vec{p}, \vec{k}\rangle$ ket

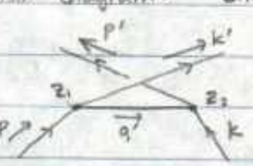
$$\langle \vec{k}, \vec{p}| = \langle 0| \sqrt{2E_{\vec{k}}}\sqrt{2E_{\vec{p}}}\, a_{\vec{k}} a_{\vec{p}}$$

$$\int d^4z_1\, d^4z_2 \, \langle \vec{k}', \vec{p}' | \bar{\psi}_1 \psi_1 \phi_1 \; \bar{\psi}_2 \psi_2 \phi_2 | \vec{p}, \vec{k}\rangle$$

(2 crossings $\rightarrow \times (+1)$)

$$= (-ig)^2 \frac{i}{q^2 - m_\phi^2 + i\epsilon} \left(\bar{u}(p') u(p)\right)\left(\bar{u}(k') u(k)\right) \qquad (q = p - p')$$

There's another diagram! Exchange diagram



$q' = p - k'$

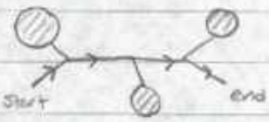
$$\int d^4z_1\, d^4z_2 \, \langle \vec{k}', \vec{p}' | \bar{\psi}_1 \psi_1 \phi_1 \; \bar{\psi}_2 \psi_2 \phi_2 | \vec{p}, \vec{k}\rangle$$

(3 crossings $\rightarrow \times(-1)$) Sign difference!

$$= -(ig)^2 \frac{i}{q'^2 - m_\phi^2 + i\epsilon} \left(\bar{u}(k') u(p)\right)\left(\bar{u}(p') u(k)\right) \qquad (q' = p - k')$$

Quick Tips on Feynman Rules

For every fermion line that doesn't close into a loop, follow the particle number arrow to the end.



Mathematical operations go from right to left.

In English, we write from left to right.

→ Start at end of particle number arrow

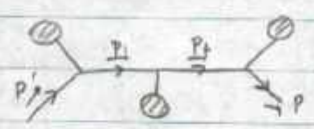
If the end is an outgoing fermion, then write $\bar{u}(p)$

If the end is an incoming antifermion, then write $\bar{v}(p)$

$\bar{u}(p)$

$\bar{v}(p)$

Then write down the propagators you get by flowing backwards against the particle number arrow.



$$\bar{u}(p)\ \frac{i(\not{p}_2+m)}{p_2^2-m^2+i\epsilon}\ \frac{i(\not{p}_1+m)}{p_1^2-m^2+i\epsilon}\cdots$$

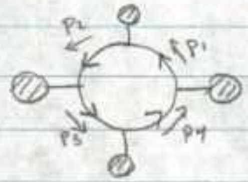
Last step, If the start is an incoming fermion, then write $u(p')$

If the start is an outgoing antifermion, then write $v(p')$

$u(p')$

$v(p')$

If the fermion line makes a loop



Start anywhere and go backwards while writing propagators

$$(-1)\,\mathrm{Tr}\left(\frac{i(\not p_4+m)}{p_4^2-m^2+i\epsilon}\;\frac{i(\not p_3+m)}{p_3^2-m^2+i\epsilon}\;\frac{i(\not p_2+m)}{p_2^2-m^2+i\epsilon}\;\frac{i(\not p_1+m)}{p_1^2-m^2+i\epsilon}\right)$$

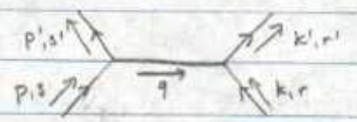
for every closed fermion loop you get a (-1)

$$\underbrace{\bar\psi_1\psi_1}_{\text{propagator}}\;\bar\psi_2\psi_2\;\bar\psi_3\psi_3\;\bar\psi_4\psi_4$$

$$(\bar\psi_1)_a(\psi_4)_b = -(\psi_4)_b(\bar\psi_1)_a$$

Yukawa Potential

Nonrelativistic scattering of two different fermions (just to keep things simple) with the same mass and same Yukawa coupling to a scalar particle



$q = p - p'$

$$i\mathcal{M} = (-ig)^2\,\frac{i}{q^2-m_\phi^2+i\epsilon}\,\left(\bar u^{s'}(p')u^s(p)\right)\left(\bar u^{r'}(k')u^r(k)\right)$$

$(q=p-p')$

Nonrelativistic limit $\rightarrow$

$$p=(E,\vec p\,) = (m,\vec p\,) + \mathcal{O}\left(\frac{\vec p^{\,2}}{m^2}\right)$$
$$k=(m,\vec k\,) + \mathcal{O}\left(\frac{\vec k^{\,2}}{m^2}\right)$$

same for p' and k'

$$q^2 = (p-p')^2 = -(\vec{p}-\vec{p}')^2 + \cdots$$

$$u^s(p) = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} + \mathcal{O}\left(\frac{|\vec{p}|}{m}\right)$$

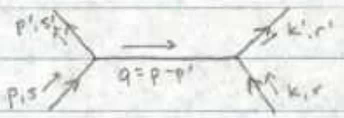
$$\bar{u}^{s'}(p')\,u^s(p) = \sqrt{m}\begin{pmatrix} \xi^{s'T} & \xi^{s'T} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}\sqrt{m}$$

$$= 2m\,\delta^{ss'}$$

$$i\mathcal{M} = ig^2 \frac{1}{(\vec{p}-\vec{p}')^2 + m_\phi^2}\,(2m\,\delta^{ss'})\,(2m\,\delta^{rr'})$$

11/04/2010

Yukawa Potential



$$iM = (-ig)^2 \frac{i}{(p-p')^2 - m_\phi^2 + i\epsilon} \left(\bar{u}^{s'}(p') u^s(p)\right)\left(\bar{u}^{r'}(k') u^r(k)\right)$$

Non-relativistic limit

$$\bar{u}^{s'}(p') u^s(p) \approx 2m\,\delta^{ss'} \qquad \text{(preserves spin)}$$

$$iM = ig^2 \frac{1}{(\vec{p}-\vec{p}\,')^2 + m_\phi^2} (2m\,\delta^{ss'})(2m\,\delta^{rr'})$$

In the Born approximation, the nonrelativistic scattering amplitude is:

$${}_{NR}\langle \vec{p}\,' | iT | \vec{p} \rangle_{NR} = -i\,\tilde{V}(\vec{q})\,(2\pi)\,\delta(E_{\vec{p}'} - E_{\vec{p}})$$

$\vec{q} = \vec{p} - \vec{p}\,'$ $\quad\tilde{V}$ - Fourier transform of potential

$$\Rightarrow \tilde{V}(\vec{q}) = \frac{-g^2}{\vec{q}^{\,2} + m_\phi^2} \qquad \text{(2m factors from relativistic normalization)}$$

$$\Rightarrow V(\vec{x}) = \int \frac{d^3q}{(2\pi)^3} \frac{-g^2}{\vec{q}^{\,2} + m_\phi^2} e^{i\vec{q}\cdot\vec{x}}$$

$$= \frac{-g^2}{8\pi^3} \int_0^\infty \int_{-1}^{1} \int_0^{2\pi} d\phi\, d\cos(\theta)\, dq\, q^2\, e^{iqr\cos(\theta)} \frac{1}{q^2 + m_\phi^2}$$

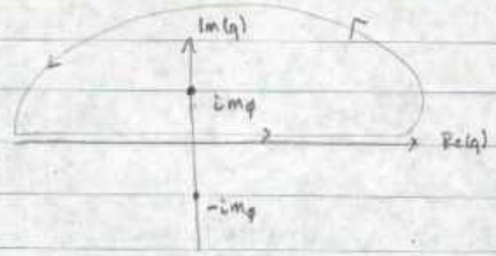
$$r = |\vec{x}|$$

$$= \frac{-g^2}{4\pi^2} \int_0^\infty dq\, q^2 \left(\frac{1}{iqr}\right)\left(e^{iqr} - e^{-iqr}\right) \frac{1}{q^2 + m_\phi^2}$$

Let $q \to -q$ for second exponential term

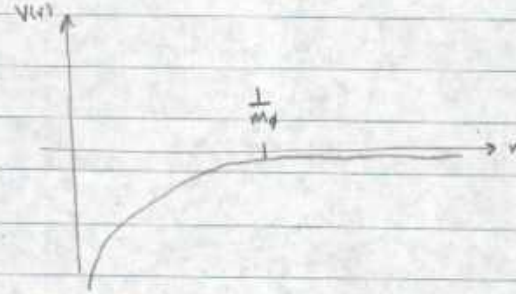
$$= \frac{-g^2}{i4\pi^2 r} \int_{-\infty}^{\infty} dq\, \frac{e^{iqr}\, q}{q^2 + m_\phi^2}$$

Use complex analysis (Cauchy's Theorem)



$$V(\vec{x}) = (2\pi i)\left(\frac{-g^2}{i4\pi^2 r}\right)\left(\frac{1}{2}e^{-m_\phi r}\right)$$

$$\boxed{V(\vec{x}) = \frac{-g^2}{4\pi}\frac{e^{-m_\phi r}}{r}}$$ Yukawa Potential



Now to Quantum Electrodynamics . . .

Quantum Electrodynamics (QED)

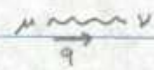
Quantum field $A_\mu(x)$ photon field

$$H_{int} = e \int d^3x \; \bar{\psi} \gamma^\mu \psi A_\mu$$

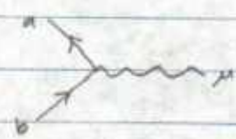
The photon is a spin-1 particle

Polarization vector $\epsilon^\mu(p)$

Feynman Rules in Momentum Space

Photon propagator 

$\dfrac{-i g_{\mu\nu}}{q^2 + i\epsilon}$

Vertex 

$-ie(\gamma^\mu)_{ab}$

a,b - spinor index

External photons

$\overbrace{A_\mu |p,\epsilon}\rangle =$ 

$\epsilon_\mu(p)$

$\langle \overbrace{p,\epsilon| A_\mu} =$ 

$\epsilon^*_\mu(p)$

Lorentz gauge $\rightarrow$ $\quad \partial_\mu A^\mu = 0$

$$\partial_\mu F^{\mu\nu} = 0$$

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$$

In Lorentz gauge $\rightarrow$ $\quad \partial_\mu \partial^\mu A^\nu = 0$

Klein-Gordon field with massless particles

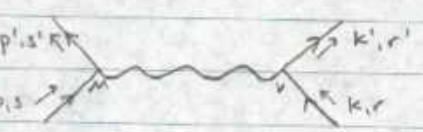
$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \sum_{r=0}^{3} \left(a_{\vec{p}}^r \epsilon_\mu^r(p)\, e^{-ip\cdot x} + a_{\vec{p}}^{r\dagger} \epsilon_\mu^{r*}(p)\, e^{ip\cdot x} \right)$$

where $p^0 = E_{\vec{p}} = |\vec{p}|$

Only two physical polarizations.

Coulomb Potential

Consider nonrelativistic scattering of two different fermions with the same mass and same charge.



$$i\mathcal{M} = (-ie)^2 \left(\bar{u}^{s'}(p')\, \gamma^\mu\, u^s(p)\right)\left(\bar{u}^{r'}(k')\, \gamma^\nu\, u^r(k)\right) \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

$$\bar{u}^{s'}(p')\, \gamma^i\, u^s(p) = \begin{pmatrix} \xi^{s'\dagger} & \xi^{s'\dagger} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} m$$

$i = 1,2,3$ (nonrelativistic limit)

$$= 0$$

$$\bar{u}^{s'}(p')\gamma^0 u^s(p) = (\xi^{s'\dagger}\ \xi^{s'T}) \underbrace{\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}}_{=1} \begin{pmatrix}\xi^s\\ \xi^s\end{pmatrix} m$$

$$= 2m\,\delta^{ss'}$$

Almost like the Yukawa model

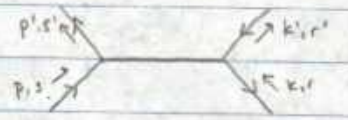
$$iM = \frac{ie^2}{-(\vec{p}-\vec{p}')^2}(2m)^2 g_{00}\,\delta^{s's}\delta^{r'r}$$

$$\boxed{V(r) = \frac{e^2}{4\pi}\frac{1}{r} = \frac{\alpha}{r}}$$ $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$ fine structure constant

Fermion - fermion is repulsive

<u>Fermion - Antifermion Interactions</u> (1. Yukawa, 2. QED)

Yukawa



$$|\vec{p},s;\vec{k},r\rangle = \sqrt{2E_{\vec p}}\sqrt{2E_{\vec k}}\, a_{\vec p}^{s\dagger} b_{\vec k}^{r\dagger}|0\rangle$$

$$\langle\vec{k},r;\vec{p},s| = \langle 0|\, b_{\vec k}^{r} a_{\vec p}^{s}\sqrt{2E_{\vec k}}\sqrt{2E_{\vec p}}$$

$$\langle\vec{k}',r';\vec{p}',s'|\,\bar\psi\psi\phi\ \ \bar\psi\psi\phi\,|\vec{p},s;\vec{k},r\rangle$$ 3 crossings $\to \times(-1)$

For fermion $\bar{u}^{s'}(p')u^s(p) = 2m\,\delta^{ss'}$

For antifermion $\bar{v}^r(k)\,v^{r'}(k')$

$$v^{r'}(k') = \sqrt{m}\begin{pmatrix}\xi^{r'}\\ -\xi^{r'}\end{pmatrix}$$

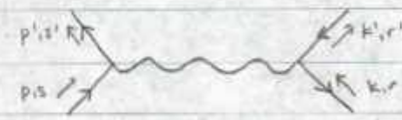
$$\bar{v}^r(k)\,v^{r'}(k') = (\xi^{r\dagger}\ \ -\xi^{r\dagger})\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\xi^{r'}\\ -\xi^{r'}\end{pmatrix} m$$ (spin flip)

$$= -2m\,\delta^{rr'}$$

Two minus signs $\rightarrow$ $V_{f\bar{f}}^{\text{Yukawa}}(r) = V_{ff}^{\text{Yukawa}}(r)$

Attractive also

QED



Overall minus sign from anticommuting (3 crossings like before).

$$\bar{u}^{s'}(p)\,\gamma^{\mu}\,u^{s}(p) \xrightarrow[\substack{\text{only} \\ \mu=0 \\ \text{nonzero}}]{} \bar{u}^{s'}(p')\,\gamma^{0}\,u^{s}(p) \simeq 2m\,\delta^{ss'}$$

$$\bar{v}^{r}(k)\,\gamma^{0}\,v^{r'}(k') = \left(\xi^{-r\dagger} \;\; -\xi^{-r\dagger}\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}\xi^{-r'}\\-\xi^{-r'}\end{pmatrix} m$$

$$= +\,2m\,\delta^{rr'}$$

One minus sign $\rightarrow$ $V_{f\bar{f}}^{\text{Coulomb}}(r) = -V_{ff}^{\text{Coulomb}}(r)$

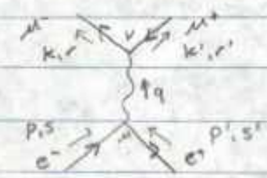
Attractive

Exchange Particle	ff or $\bar{f}\bar{f}$	$f\bar{f}$
scalar (Yukawa)	attractive	attractive
vector (spin-1, QED)	repulsive	attractive
tensor (spin-2, graviton)	attractive	attractive

11/09/2010

Electron-Positron Scattering

$$e^+ + e^- \to \mu^+ + \mu^-$$



$q = p + p'$

$$i\mathcal{M} = (-ie)^2 \left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma^\nu v^{r'}(k')\right)\left(\frac{-ig_{\mu\nu}}{q^2+i\epsilon}\right)$$

$$= \frac{ie^2}{(p+p')^2}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)$$

We actually need $|\mathcal{M}|^2$.

Consider $\bar{v}\gamma^\mu u$.

$$(\bar{v}\gamma^\mu u)^* = (\bar{v}\gamma^\mu u)^\dagger \qquad (\text{it's a } 1\times1 \text{ matrix}, \dagger = *)$$

$$= \overline{(\bar{v}\gamma^\mu u)}$$

$$(\bar{\gamma}^\mu = \gamma^\mu)$$

$$= \bar{u}\gamma^\mu v$$

$$|\mathcal{M}|^2 = \frac{e^4}{((p+p')^2)^2}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)$$

$$\times\left(\bar{u}^s(p)\gamma^\nu v^{s'}(p')\right)\left(\bar{v}^{r'}(k')\gamma_\nu u^r(k)\right)$$

$$= \frac{e^4}{((p+p')^2)^2}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^s(p)\gamma^\nu v^{s'}(p')\right)$$

$$\times\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)\left(\bar{v}^{r'}(k')\gamma_\nu u^r(k)\right)$$

We're going to consider unpolarized scattering → equal probability for all initial spins.

→ Sum over final spins, average over initial spins.

$$\frac{1}{4}\sum_{r,r',s,s'} |\mathcal{M}|^2 \qquad \text{(4 initial spin states)}$$

Consider A (n×m matrix), B (m×n matrix)

$$Tr(AB) = \sum_{i,j} A_{ij} B_{ji} = \sum_{i,j} B_{ij} A_{ji} = Tr(BA)$$

$$|\mathcal{M}|^2 = \frac{e^4}{((p+p')^2)^2} Tr\left(\gamma^\mu u^s(p)\,\bar{u}^s(p)\,\gamma^\nu\, v^{s'}(p')\,\bar{v}^{s'}(p')\right)$$

$$\times\, Tr\left(\gamma_\mu v^{r'}(k')\,\bar{v}^{r'}(k')\,\gamma_\nu\, u^r(k)\,\bar{u}^r(k)\right)$$

Recall $\quad \sum_s u^s(p)\,\bar{u}^s(p) = \not{p} + m$

$$\sum_s v^s(p)\,\bar{v}^s(p) = \not{p} - m$$

$$\frac{1}{4}\sum_{r,r',s,s'} |\mathcal{M}|^2 = \frac{e^4}{4((p+p')^2)^2} Tr\left(\gamma^\mu(\not{p}+m_e)\gamma^\nu(\not{p}'-m_e)\right) Tr\left(\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu(\not{k}+m_\mu)\right)$$

Now let's play with Dirac matrices to simplify...

Recall $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ (Weyl representation)

$$\boxed{Tr(1_{4\times4}) = 4}$$

$$\boxed{Tr(\gamma^\mu) = 0}$$

$$\gamma^5\gamma^5 = 1 \qquad \{\gamma^\mu, \gamma^5\} = 0$$

$$Tr(\gamma^\mu) = Tr(\gamma^5\gamma^5\gamma^\mu) = Tr(\gamma^5\gamma^\mu\gamma^5) = -Tr(\gamma^5\gamma^5\gamma^\mu) = -Tr(\gamma^\mu) = 0$$

$$Tr(\gamma^\mu\gamma^\nu) = \frac{1}{2} Tr(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)$$

$$= \frac{1}{2}\cdot 2g^{\mu\nu}\, Tr(1_{4\times4}) = \boxed{4g^{\mu\nu} = Tr(\gamma^\mu\gamma^\nu)}$$

$$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = Tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma)$$

$$= Tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma)$$

$$= Tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho - \gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu)$$

$$2Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = Tr(2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho)$$

$$\boxed{Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}}$$

$$\boxed{Tr(\gamma^5) = Tr(i\gamma^0\gamma^1\gamma^2\gamma^3) = 0}$$

$Tr(1_{4\times4}) = 4$

$Tr(\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{odd}) = 0$

$Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$

$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$

$Tr(\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{odd}\gamma^5) = 0$

$Tr(\gamma^\mu\gamma^\nu\gamma^5) = 0$

$Tr(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\underbrace{\epsilon^{\mu\nu\rho\sigma}}_{\text{completely antisymmetric}}$

$\epsilon^{0123} = +1$

$Tr(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}) = Tr(\underbrace{\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}}_{\text{reverse order}})$ (n - even)

Contraction Identities

$\gamma_\mu\gamma^\mu = \gamma^\mu\gamma^\nu g_{\mu\nu} = \frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)g_{\mu\nu} = \frac{1}{2}(2g^{\mu\nu})g_{\mu\nu} = 4$

$\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(2g^{\nu\mu} - \gamma^\mu\gamma^\nu) = 2\gamma^\nu - 4\gamma^\nu = -2\gamma^\nu$

$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$

$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$

Back to our scattering problem

$$\mathrm{Tr}\left[\gamma^\mu (\not{p} + m_e)\gamma^\nu (\not{p}' - m_e)\right] = \mathrm{Tr}\left[\gamma^\mu \not{p}\gamma^\nu \not{p}'\right]$$

$$- m_e^2\, \mathrm{Tr}\left[\gamma^\mu \gamma^\nu\right]$$

(all other terms = 0 because odd number of γs)

$$= \mathrm{Tr}\left[\gamma^\mu p_\alpha \gamma^\alpha \gamma^\nu p'_\beta \gamma^\beta\right] - 4 m_e^2 g^{\mu\nu}$$

$$= p_\alpha p'_\beta \left(4\left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\nu} g^{\alpha\beta} + g^{\mu\beta} g^{\alpha\nu}\right)\right)$$

$$- 4 m_e^2 g^{\mu\nu}$$

$$= 4\left(p^\mu p'^\nu - g^{\mu\nu} p\cdot p' + p^\nu p'^\mu - m_e^2 g^{\mu\nu}\right)$$

$$= 4\left(p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}\left(p\cdot p' + m_e^2\right)\right)$$

$$\mathrm{Tr}\left[\gamma_\mu (\not{k}' - m_\mu)\gamma_\nu (\not{k} + m_\mu)\right] = 4\left(k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}\left(k'\cdot k + m_\mu^2\right)\right)$$

$$\frac{1}{4}\sum_{r,r',s,s'} |\mathcal{M}|^2 = \frac{4e^4}{((p+p')^2)^2}\Big(2(p\cdot k')(p'\cdot k) + 2(p\cdot k)(p'\cdot k') - 2(p\cdot p')(k\cdot k' + m_\mu^2)$$

$$- 2(k\cdot k')(p\cdot p' + m_e^2)$$

$$+ 4(k'\cdot k + m_\mu^2)(p\cdot p' + m_e^2)\Big)$$

11/11/2010

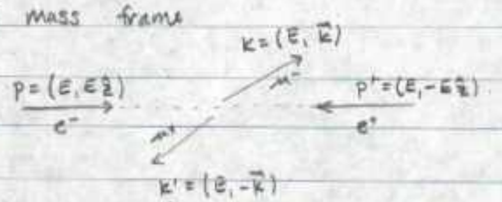
Electron- Positron Scattering

$$\frac{1}{4}\sum_{s,s',r,r'} |\mathcal{M}|^2 = \frac{4e^4}{(q^2)^2}\Big(2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) - 2(k'\cdot k)(p\cdot p' + m_e^2)$$
$$- 2(p\cdot p')(k'\cdot k + m_\mu^2) + 4(k'\cdot k + m_\mu^2)(p\cdot p' + m_e^2)\Big)$$

Assume sufficiently high energies so that m_e is neglible $(m_e \approx 0)$

$$\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{4e^4}{(q^2)^2}\Big(2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) + 2m_\mu^2 (p\cdot p')\Big)$$

Center of mass frame



$$|\vec{k}| = \sqrt{E^2 - m_\mu^2}$$

$$\vec{k}\cdot\hat{z} = |\vec{k}|\cos(\theta)$$

$$q^2 = (p+p')^2 = (2E)^2 = 4E^2$$

$$p\cdot p' = E^2 + E^2 = 2E^2$$

$$p\cdot k = E^2 - E|\vec{k}|\cos(\theta) \qquad p\cdot k' = E^2 + E|\vec{k}|\cos(\theta)$$

$$p'\cdot k' = E^2 - E|\vec{k}|\cos(\theta) \qquad p'\cdot k = E^2 + E|\vec{k}|\cos(\theta)$$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{8e^4}{(4E^2)^2}\Big((E^2 - E|\vec{k}|\cos(\theta))^2 + (E^2 + E|\vec{k}|\cos(\theta))^2 + m_\mu^2(2E^2)\Big)$$

$$= \frac{e^4}{2E^4}\Big(E^2(E - |\vec{k}|\cos(\theta))^2 + E^2(E + |\vec{k}|\cos(\theta))^2 + 2m_\mu^2 E^2\Big)$$

$$= \frac{e^4}{E^2}\Big((E^2 + |\vec{k}|^2\cos^2(\theta)) + m_\mu^2\Big)$$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = e^4\left(1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right)$$

Two body final state cross section in center of mass frame

$$\left(\frac{d\sigma}{d\Omega}\right) = \frac{1}{(2E_A)(2E_B)|\vec{v}_A-\vec{v}_B|}\,|\vec{k}|\,\frac{1}{16\pi^2 E_{cm}}\,\frac{1}{4}\sum_{spins}|\mathcal{M}|^2$$

$$|\vec{v}_A-\vec{v}_B| = 2 \qquad \text{(both } e^- \text{ and } e^+ \text{ moving at speed of light)}$$

$$E_A = E = E_B \qquad E_{cm} = 2E$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2E)(2E)(2)}\sqrt{E^2-m_\mu^2}\,\frac{1}{16\pi^2(2E)}\left(e^4\left(1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right)\right)$$

$$= \frac{e^4\sqrt{1-m_\mu^2/E^2}}{256\pi^2 E^2}\left(1+\frac{m_\mu^2}{E^2}+\left(1-\frac{m_\mu^2}{E^2}\right)\cos^2(\theta)\right)$$

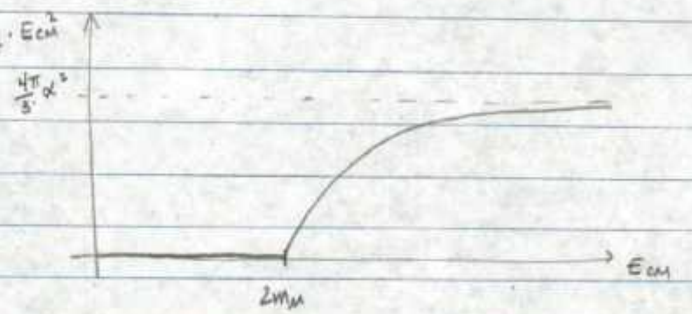
Integrate over solid angle to find total cross section $\left(\int_0^{2\pi}\int_{-1}^{1} d\cos(\theta)\, d\phi\right)$

$$\sigma_{tot} = \frac{e^4\, 2\pi}{256\pi^2 E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(2\left(1+\frac{m_\mu^2}{E^2}\right)+\frac{2}{3}\left(1-\frac{m_\mu^2}{E^2}\right)\right)$$

$$\sigma_{tot} = \frac{e^4}{64\pi E^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(\frac{4}{3}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)\right)$$

$$\alpha = \frac{e^2}{4\pi} \rightarrow \alpha^2 = \frac{e^4}{16\pi^2} \qquad E_{cm} = 2E \rightarrow E_{cm}^2 = 4E^2$$

$$\sigma_{tot} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E^2}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)$$



As $E_{cm}\rightarrow\infty$, $\sigma_{tot}\rightarrow\frac{4\pi\alpha^2}{3E_{cm}^2}$, $\frac{d\sigma}{d\Omega}\rightarrow\frac{\alpha^2}{4E_{cm}^2}(1+\cos^2(\theta))$

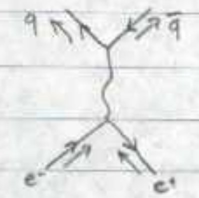
Quarks are fermions which couple to the photon, but they also have strong interactions → Quantum Chromodynamics

At long distances (low energies), interactions are strong.

At short distances (high energies), interactions are weak.

Gross, Politzer, Wilczek - Nobel Prize 2004 → Asymptotic freedom

At high energies, you produce a quark q and antiquark $\bar{q}$ pair



So we expect

$$\sigma(e^-e^+ \to \text{any hadrons}) \approx \sum_{i \in \{\text{all quarks}\}} Q_i^2 \, \sigma(e^-e^+ \to \mu^-\mu^+)$$

Q_i = electric charge of quark i

$$R = \frac{\sigma(e^-e^+ \to \text{any hadrons})}{\sigma(e^-e^+ \to \mu^-\mu^+)}$$

Below ~ 4 GeV , 3 flavors of light quarks

u $Q = +2/3$

d $Q = -1/3$ $\qquad \Sigma Q_i^2 = \frac{4}{9} \neq 2$

s $Q = -1/3$

3 colors for each flavor

$$R = \underset{\substack{\uparrow \\ \text{color}}}{3} \left(\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{-1}{3}\right)^2 + \left(\tfrac{-1}{3}\right)^2 \right) = 2$$

Between 4 GeV and 10 GeV, fourth flavor

c $Q = +2/3$

$$R = 3\left(\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{-1}{3}\right)^2 + \left(\tfrac{-1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 \right) = 3\tfrac{1}{3}$$

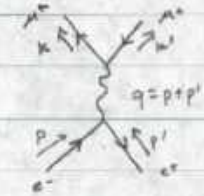
Above 10 GeV, fifth flavor

b $Q = -1/3$

$$R = 3\left(\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{-1}{3}\right)^2 + \left(\tfrac{-1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(\tfrac{-1}{3}\right)^2 \right) = 3\tfrac{2}{3}$$

Top quark has mass ~ 180 GeV and unstable (decays to W bosons)

Crossing Symmetry



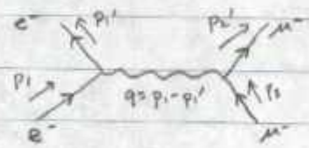
$e^+e^- \to \mu^+\mu^-$ $\quad m_e = 0$

$$i\mathcal{M} = \frac{ie^2}{q^2}\left(\bar v(p')\gamma^\mu u(p)\right)\left(\bar u(k)\gamma_\mu v(k')\right)$$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4(q^2)^2}\,Tr\left[\not p'\gamma^\mu \not p\,\gamma^\nu\right] Tr\left[(\not k + m_\mu)\gamma_\mu(\not k' - m_\mu)\gamma_\nu\right]$$

$$= \frac{8e^4}{(q^2)^2}\left((p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\right)$$

$e^-\mu^- \to e^-\mu^-$ $\quad m_e = 0$



$$i\mathcal{M} = \frac{ie^2}{(q^2)}\left(\bar u(p_1')\gamma^\mu u(p_1)\right)\left(\bar u(p_2')\gamma_\mu u(p_2)\right)$$

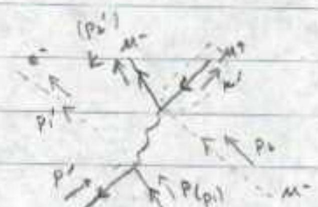
$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{e^4}{4(q^2)^2}\,Tr\left[\not p_1'\gamma^\mu \not p_1\gamma^\nu\right] Tr\left[(\not p_2' + m_\mu)\gamma_\mu(\not p_2 + m_\mu)\gamma_\nu\right]$$

Compare to previous

$p \leftrightarrow p_1 \qquad k \leftrightarrow p_2'$

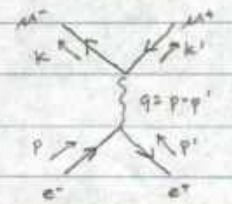
$p' \leftrightarrow -p_1' \qquad k' \leftrightarrow -p_2$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{8e^4}{(q^2)^2}\left((p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_\mu^2(p_1\cdot p_1')\right)$$



14/10/2010

Crossing Symmetry



$$\frac{1}{4}\sum_{spins} |M|^2 = \frac{8e^4}{(q^2)^2}\Big((p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2 (p\cdot p')\Big)$$

$$\frac{1}{4}\sum_{spins} |M|^2 = \frac{8e^4}{(q^2)^2}\Big((p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_\mu^2 (p_1\cdot p_1')\Big)$$

$p \leftrightarrow p_1 \qquad k \leftrightarrow p_2'$

$p' \leftrightarrow -p_1' \qquad k' \leftrightarrow -p_2$

$e^+ + e^- \to \mu^+ + \mu^-$

$e^- + \mu^- \to e^- + \mu^-$

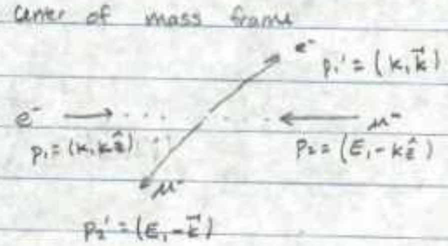
$e^+ + \mu^+ \to e^+ + \mu^+$

$e^+ + \mu^- \to e^+ + \mu^-$

$\vdots$

Look at $e^- + \mu^- \to e^- + \mu^-$ in more detail

center of mass frame



$E^2 = k^2 + m_\mu^2$

$p_1 \cdot p_2 = Ek + k^2 = p_1' \cdot p_2'$

$p_1' \cdot p_2 = Ek + k^2\cos(\theta) = p_1 \cdot p_2'$

$p_1 \cdot p_1' = k^2 - k^2\cos(\theta)$

$q = p_1 - p_1' \to q^2 = \overbrace{p_1^2}^{0} - \overbrace{p_1'^2}^{0} - 2p_1\cdot p_1' = -2(k^2 - k^2\cos(\theta))$

$(m_e = 0)$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{4k^4(1-\cos(\theta))^2}\left((kE + k^2\cos(\theta))^2 + (kE+k^2)^2 - m_\mu^2(k^2 - k^2\cos(\theta)) \right)$$

$$= \frac{8e^4}{4k^4(1-\cos(\theta))^2}\left(k^2(E+k\cos(\theta))^2 + k^2(E+k)^2 - m_\mu^2 k^2(1-\cos(\theta)) \right)$$

$$= \frac{2e^4}{k^2(1-\cos(\theta))^2}\left((E+k)^2 + (E+k\cos(\theta))^2 - m_\mu^2(1-\cos(\theta)) \right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{k}{(2E_A)(2E_B)|\vec{v}_A - \vec{v}_B|}\frac{1}{16\pi^2 E_{CM}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$E_{CM} = E + k$$

$$E_A = k \qquad E_B = E$$

$$(v_A)_z = 1 \qquad (v_B)_z = -\frac{k}{E}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{2k^2(E+k)^2(1-\cos(\theta))^2}\left((E+k)^2 + (E+k\cos(\theta))^2 - m_\mu^2(1-\cos(\theta)) \right)$$

As $\theta \to 0$, $\left(\frac{d\sigma}{d\Omega}\right)_{CM} \propto \frac{1}{\theta^4}$ (infinite range interaction (Coulomb))

General Crossing Symmetry

Scalar particle ϕ

$$\mathcal{M}(\phi(p) + X \to Y) = \mathcal{M}(X \to Y + \bar{\phi}(-p))$$

just flip sign of p and incoming $\leftrightarrow$ outgoing

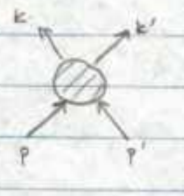
Fermions (spinors) then worry about minus signs from contraction intersections and

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m \longrightarrow \sum_s v^s(-p)\bar{v}^s(-p) = -\not{p} - m$$

Mandelstam Variables

Convenient for showing crossing symmetries

Consider two body → two body scattering



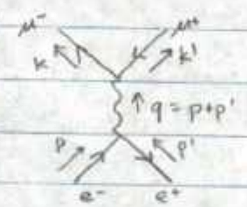
$$s = (p+p')^2 = (k+k')^2$$

$$t = (p-k)^2 = (p'-k')^2 \quad \text{(momentum transfer)}$$

$$u = (p-k')^2 = (p'-k)^2 \quad \text{(exchange momentum transfer)}$$

Ambiguity in labelling k + k'

→ If one incoming and one outgoing are the same type of particle then these are p + k.



Set $m_e = 0$ and $m_\mu = 0$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{(q^2)^2}\Big((p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k)\Big)$$

$$s = (p+p')^2 = q^2$$

$$t = (p-k)^2 = p^2 + k^2 - 2p\cdot k = -2p\cdot k$$

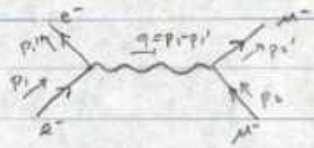
$$= (p'-k')^2 = -2p'\cdot k'$$

$$u = (p-k')^2 = p^2 + k'^2 - 2p\cdot k' = -2p\cdot k'$$

$$= (p'-k) = -2p'\cdot k$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{s^2}\Big(\left(-\tfrac{t}{2}\right)\left(-\tfrac{t}{2}\right) + \left(-\tfrac{u}{2}\right)\left(-\tfrac{u}{2}\right)\Big)$$

$$= \frac{2e^4}{s^2}\left(t^2 + u^2\right)$$



Set $m_e = 0$ and $m_\mu = 0$

$$\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{8e^4}{(q^2)^2}\Big((p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') \Big)$$

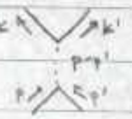
$$s = (p_1+p_2)^2 = +2p_1\cdot p_2$$
$$= (p_1'+p_2')^2 = +2p_1'\cdot p_2'$$
$$t = (p_1 - p_1')^2 = q^2$$
$$u = (p_1 - p_2')^2 = -2p_1\cdot p_2'$$
$$= (p_2 - p_1')^2 = -2p_1'\cdot p_2$$

$$\frac{1}{4}\sum_{spins} |\mathcal{M}|^2 = \frac{8e^4}{t^2}\Big(\left(\frac{-u}{2}\right)\left(\frac{-u}{2}\right) + \left(\frac{s}{2}\right)\left(\frac{s}{2}\right) \Big)$$

$$= \frac{2e^4}{t^2}\left(u^2 + s^2 \right)$$

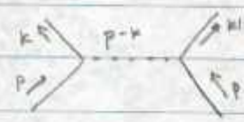
Compare these two diagrams → exchange s and t

s-channel diagram →



$\mathcal{M} \propto \frac{1}{s-m^2}$ s-channel pole

t-channel diagram →



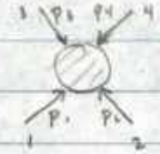
$\mathcal{M} \propto \frac{1}{t-m^2}$ t-channel pole

u-channel diagram →

$\mathcal{M} \propto \frac{1}{u-m^2}$ u-channel pole

Claim: $s + t + u = \sum_{i=1}^{4} m_i^2$ (masses of each incoming or outgoing particle)

Proof:



3,4 outgoing – physical momenta are $-p_3, -p_4$

$$p_1 + p_2 + p_3 + p_4 = 0$$

$$2(s+t+u) = \overbrace{(p_1+p_2)^2 + (p_3+p_4)^2}^{2s} + \overbrace{(p_1+p_3)^2 + (p_2+p_4)^2}^{2t}$$

$$+ \underbrace{(p_1+p_4)^2 + (p_2+p_3)^2}_{2u}$$

$$= 3(p_1^2 + p_2^2 + p_3^2 + p_4^2) + 2(p_1 \cdot p_2 + p_1 \cdot p_3 + p_1 \cdot p_4 + p_2 \cdot p_3 + p_2 \cdot p_4 + p_3 \cdot p_4)$$

$$= 3 \sum_i p_i^2 + 2 \sum_{i<j} p_i \cdot p_j$$

$$\sum_i p_i = 0 \Rightarrow \left(\sum_i p_i\right)^2 = 0 = \sum_i p_i^2 + 2 \sum_{i<j} p_i \cdot p_j$$

$$2(s+t+u) = 2 \sum_i p_i^2 + \left(\sum_i p_i^2 + 2 \sum_{i<j} p_i \, p_j\right)$$

$$= 2 \sum_i p_i^2$$

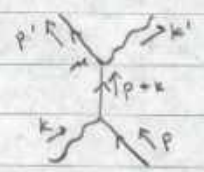
$$s + t + u = \sum_i p_i^2 = \sum_i m_i^2 \quad //$$

Compton Scattering

In 1923, Compton looked at the scattering of X-rays and γ-rays off electrons.

$$e^- + \gamma \to e^- + \gamma$$

(II)



(I)

For photons $\epsilon_\nu(k)$ and $\dot{\epsilon}_\nu(k')$ are polarization vectors

(I) $$\bar{u}(p')(-ie\gamma^\mu)\epsilon^*_\mu(k')\frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\epsilon}(-ie\gamma^\nu)\epsilon_\nu(k)u(p)$$

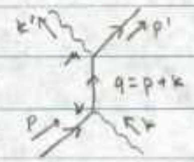
(II) $$+\bar{u}(p')(-ie\gamma^\nu)\epsilon_\nu(k)\frac{i(\not{p}-\not{k}'+m)}{(p-k)^2-m^2+i\epsilon}(-ie\gamma^\mu)\epsilon^*_\mu(k')u(p)$$

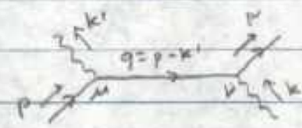
$$=(-ie^2)\,\epsilon^*_\mu(k')\,\epsilon_\nu(k)\,\bar{u}(p')\left(\frac{\gamma^\mu(\not{p}+\not{k}+m)\gamma^\nu}{(p+k)^2-m^2+i\epsilon}+\frac{\gamma^\nu(\not{p}-\not{k}'+m)\gamma^\mu}{(p-k)^2-m^2+i\epsilon}\right)\bar{u}(p)$$

11/18/2010

Compton Scattering

$e^- + \gamma \to e^- + \gamma$





$\epsilon_\nu(k)$ and $\epsilon_\nu(k')$ are polarization vectors

$$i\mathcal{M} = -ie^2\, \epsilon_\mu^*(k')\, \epsilon_\nu(k)\, \bar{u}(p') \left(\frac{\gamma^\mu(\not p + \not k + m)\gamma^\nu}{(p+k)^2 - m^2 + i\epsilon} + \frac{\gamma^\nu(\not p - \not k' + m)\gamma^\mu}{(p-k')^2 - m^2 + i\epsilon} \right) u(p)$$

$$k^2 = 0 \qquad (k')^2 = 0$$
$$p^2 = m^2 \qquad (p')^2 = m^2$$

$$(p+k)^2 = p^2 + k^2 + 2p\cdot k = m^2 + 2p\cdot k$$

$$(p-k')^2 = p^2 + k'^2 - 2p\cdot k' = m^2 - 2p\cdot k'$$

$$(\not p + m)\gamma^\nu u(p) = \gamma^\nu \underbrace{(-\not p + m)u(p)}_{=0} + 2p^\nu u(p) = 2p^\nu u(p)$$

$$\{\not p, \gamma^\nu\} = p_\mu \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} p_\mu = 2p^\nu$$

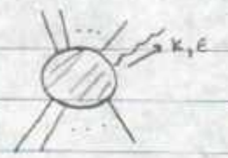
$$i\mathcal{M} = -ie^2\, \epsilon_\mu^*(k')\, \epsilon_\nu(k)\, \bar{u}(p') \left(\frac{2\gamma^\mu p^\nu + \gamma^\mu \not k \gamma^\nu}{2p\cdot k} + \frac{-2\gamma^\nu p^\mu + \gamma^\nu \not k' \gamma^\mu}{+2p\cdot k'} \right) u(p)$$

Photon Polarizations and Ward Identity

Interaction in QED has the form

$$e\int d^4x\, j^\mu(x) A_\mu(x) \qquad j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$$

electromagnetic current density



1 outgoing photon

$$i\mathcal{M} = i\,\epsilon^*_\mu(k)\,\mathcal{M}^\mu(k)$$

$$\mathcal{M}^\mu(k) \propto \int d^4x\, e^{ik\cdot x} \langle f | j^\mu(x) | i\rangle$$

final state without external photon

Since $j^\mu(x)$ is a conserved current density, $\partial_\mu j^\mu(x) = 0$

$$k_\mu \mathcal{M}^\mu(k) \propto \int d^4x\, e^{ik\cdot x}\, k_\mu \langle f | j^\mu(x) | i\rangle$$

$$= -i\int d^4x\, \partial_\mu e^{ik\cdot x} \langle f | j^\mu(x) | i\rangle$$

integrate by parts

$$= i\int d^4x\, e^{ik\cdot x} \langle f | \partial_\mu j^\mu(x) | i\rangle = 0$$

If we replace the $\epsilon_\mu(k)$ by k_μ (the four-momentum of the photon) the amplitude vanishes

$$k_\mu \mathcal{M}^\mu(k) = 0 \quad \text{(for any incoming or outgoing photon)}$$

Ward Identity

What can we say about the polarization sum?

$$\sum_{\substack{\text{polarization} \\ i}} \epsilon_i^{*\mu}(k)\, \epsilon_i^{\nu}(k)$$

Example: $k \rightarrow$ photon four-momentum

$$k^\mu = (k, 0, 0, k) \qquad \text{(pointing in z-direction)}$$

Two physical polarizations

$$\epsilon_1 = (0,1,0,0) \qquad \epsilon_2 = (0,0,1,0)$$

$$\sum \epsilon_i^{*\mu}(k)\, \epsilon_i^{\nu}(k) = \begin{array}{c|c} & \xrightarrow{\nu} \\ \big\downarrow \mu & \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \qquad \text{Ugly!}$$

Try something more symmetrical.

We are always calculating something of the form

$$\sum_i |\epsilon_{i\mu}^{*}(k)\, \mathcal{M}^\mu(k)|^2$$

$$= \sum_i \epsilon_i^{*\mu}(k)\, \epsilon_i^{\nu}(k)\, \mathcal{M}_\mu(k)\, \mathcal{M}_\nu^{*}(k)$$

$$= |\mathcal{M}_1(k)|^2 + |\mathcal{M}_2(k)|^2$$

From the Ward Identity, $k_\mu \mathcal{M}^\mu(k) = 0$

$$\mathcal{M}^0(k) - \mathcal{M}^3(k) = 0$$

$$\mathcal{M}^0(k) = \mathcal{M}^3(k)$$

So we could take

$$\sum_i \epsilon_i^{*\mu}(k)\, \epsilon_i^{\nu}(k) = -g^{\mu\nu}$$

Not actually equal to this, but it works

Then we get

$$\sum_i \epsilon_i^{*\mu}(k)\, \epsilon_i^{\nu}(k)\, \mathcal{M}_\mu(k)\, \mathcal{M}_\nu^{*}(k) = -\mathcal{M}_\mu(k)\, \mathcal{M}^{*\mu}(k)$$

$$= \cancel{-|\mathcal{M}_0|^2} + |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + \cancel{|\mathcal{M}_3|^2}$$

$$= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 \checkmark$$

Back to Compton Scattering

$$i\mathcal{M} = -ie^2\, \epsilon_\mu^{*}(k')\, \epsilon_\nu(k)\, \bar{u}(p') \left(\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'} \right) u(p)$$

Initial spins $\rightarrow$ (2 electron spins) $\times$ (2 photon polarizations) $= 4$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4}\, g_{\mu\rho}\, g_{\nu\sigma}$$

$$\times \mathrm{Tr}\left[(\not{p}'+m) \left(\frac{\gamma^\mu \not{k} \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'} \right) (\not{p}+m) \left(\frac{\gamma^\sigma \not{k} \gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{\gamma^\rho \not{k}' \gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'} \right) \right]$$

Look at just one term.

$$\mathrm{Tr}\left[\not{p}' \gamma^\mu \not{k} \gamma^\nu \not{p} \gamma_\nu \not{k} \gamma_\mu \right]$$

$$\gamma^\nu \not{p} \gamma_\nu = -2 \not{p}$$

$$= -2\,\mathrm{Tr}\left[\not{p}' \gamma^\mu \not{k} \not{p} \not{k} \gamma_\mu \right]$$

$$\gamma^\mu \not{a} \not{b} \not{c} \gamma_\mu = -2 \not{c} \not{b} \not{a}$$

$$= 4\,\mathrm{Tr}\left[\not{p}' \not{k} \not{p} \not{k} \right]$$

$$= 4 \cdot 4 \left((p'\cdot k)(p\cdot k) - (p'\cdot p)\cancel{(k\cdot k)}^{0} + (p'\cdot k)(p\cdot k) \right)$$

$$= 32 (p'\cdot k)(p \cdot k)$$

One of the terms is $2e^4 \frac{p'\cdot k}{p\cdot k}$

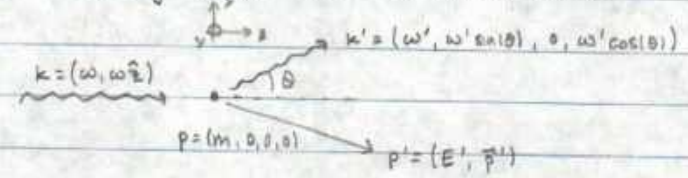
(other terms similarly)

Result ...

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4 \left[\frac{p'\cdot k}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)\right]$$

(check primes)

Compton scattering in rest frame of initial electron



Solve for ω'

$$p'^2 = m^2$$

$$p'^2 = (p+k-k')^2 = \underbrace{p^2}_{m^2} + \cancel{k^2}^{0} + \cancel{k'^2}^{0} + 2p\cdot k - 2p\cdot k' - 2k\cdot k' = m^2$$

$$= m^2 + 2p\cdot(k-k') - 2k\cdot k' = m^2$$

$$2m(\omega-\omega') - 2(\omega\omega' - \omega\omega'\cos(\theta)) = 0$$

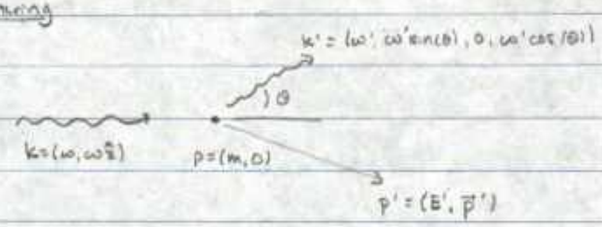
$$2m(\omega-\omega') - 2\omega\omega'(1-\cos(\theta)) = 0$$

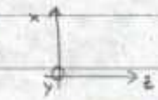
$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos(\theta))$$

$$\Rightarrow \quad \omega' = \frac{1}{\frac{1}{m}(1-\cos(\theta)) + \frac{1}{\omega}} = \frac{\omega}{1+\frac{\omega}{m}(1-\cos(\theta))}$$

11/23/2010

Compton Scattering





$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = 2e^4\left(\frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k}{p \cdot k'} + 2m^2\left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right) + m^4\left(\frac{1}{p \cdot k} - \frac{1}{p \cdot k'} \right)\right)$$

$$\omega' = \frac{\omega}{1 + \frac{\omega}{m}(1-\cos(\theta))}$$

In rest frame of initial electron

$$\int \frac{d^3k'}{(2\pi)^3 2\omega'} \frac{d^3p'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4(k' + p' - k - p)$$

Use spatial δ^3 to kill d^3p' $\rightarrow$ $\vec{p}\,' = \vec{k} + \vec{p} - \vec{k}'$

$$= \int \frac{d^3k'}{(2\pi)^3 2\omega'} \frac{1}{2E'} 2\pi\, \delta(\omega' + E' - \omega - m)$$

$$E' = \sqrt{(\vec{k} + \vec{p} - \vec{k}')^2 + m^2}$$

$$= \sqrt{(\vec{k} - \vec{k}')^2 + m^2}$$

$$(\vec{k} - \vec{k}')^2 = \vec{k}^2 + \vec{k}'^2 - 2\vec{k} \cdot \vec{k}' = \omega^2 + \omega'^2 - 2\omega\omega'\cos(\theta)$$

$$E' = \sqrt{\omega^2 + \omega'^2 - 2\omega\omega'\cos(\theta) + m^2}$$

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|} \quad \text{at simple zero } x = x_0$$

$$= \int \frac{\omega'^2 d\omega' d\Omega}{(2\pi)^3 4\omega' E'} 2\pi \frac{\delta(\omega' - \omega/(1 + \frac{\omega}{m}(1-\cos(\theta))))}{\left| 1 + \frac{1}{E'}(\omega' - \omega\cos(\theta)) \right|}$$

$$= \int \frac{d\cos(\theta)\,\omega'}{8\pi\,\left|1+\frac{\omega'-\omega\cos(\theta)}{E'}\right|E'} \qquad \text{(did } d\omega' \text{ integral with } \delta\text{)}$$

$$= \frac{1}{8\pi}\int d\cos(\theta)\,\frac{\omega'}{|\underbrace{E'+\omega'}_{\substack{\text{total energy}\\ m+\omega}}-\omega\cos(\theta)|}$$

$$= \frac{\omega'}{8\pi}\int d\cos(\theta)\,\frac{1}{m+\omega(1-\cos(\theta))}$$

$|\vec{v}_A-\vec{v}_B| = 1$ (photon at speed of light, e^- at rest)

$$\frac{d\sigma}{d\cos(\theta)} = \frac{1}{2\omega}\,\frac{1}{2m}\,\frac{\omega'}{8\pi}\underbrace{\frac{1}{m+\omega(1-\cos(\theta))}}_{\frac{\omega'}{m\omega}}\,\frac{1}{4}\sum_{\text{spins}}|m|^2$$

$$\frac{d\sigma}{d\cos(\theta)} = \frac{1}{32\pi}\left(\frac{\omega'}{m\omega}\right)^2\left(2e^4\left(\frac{p\cdot k}{p\cdot k'}+\frac{p\cdot k}{p\cdot k'}+2m^2\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)+m^4\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)^2\right)\right)$$

$p\cdot k = m\omega$

$p\cdot k' = m\omega'$

etc.

$$\frac{d\sigma}{d\cos(\theta)} = \frac{1}{32\pi}\left(\frac{\omega'}{m\omega}\right)^2\left(2e^4\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}+2m\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)+m^2\left(\frac{1}{\omega}-\frac{1}{\omega'}\right)^2\right)\right)$$

Now use $\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos(\theta))}$

$$m\left(\frac{1}{\omega}-\frac{1}{\omega'}\right) = m\left(\frac{1}{\omega}-\frac{1+\frac{\omega}{m}(1-\cos(\theta))}{\omega}\right)$$

$$= -(1-\cos(\theta))$$

$$\frac{d\sigma}{d\cos(\theta)} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-2(1-\cos(\theta))+(1-\cos(\theta))^2\right)$$

$$\boxed{\frac{d\sigma}{d\cos(\theta)} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left(\frac{\omega'}{\omega}+\frac{\omega}{\omega'}-\sin^2(\theta)\right)}$$ Klein-Nishina Formula

As $\omega \to 0$ . . .

$$\frac{\omega'}{\omega} \to 1 \quad , \quad \frac{d\sigma}{d\cos(\theta)} \to \frac{\pi \alpha^2}{m^2}\left(1 + \cos^2(\theta)\right)$$

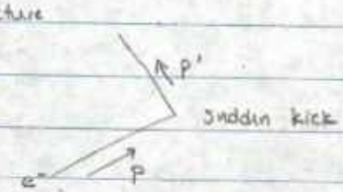
$$\sigma_{tot} \to \frac{8\pi}{3}\frac{\alpha^2}{m^2}$$

These are the Thompson cross section limits for classical E+M scattering off an electron.

Soft Brehmsstrahlung

Low frequency radiation when an electron undergoes sudden acceleration

Classical picture



We get radiation from Maxwell's equations once we know the current density.

For a particle at rest at the origin $j^\mu(x) = ?$

$$j^\mu(x) = e\,(\text{particle density}, \vec{0})^\mu$$

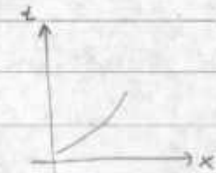
$$= e\,(1,0,0,0)^\mu\, \delta^3(\vec{x})$$

$$= \int dt'\, (1,0,0,0)^\mu\, e\, \delta^4(x - y(t'))$$

$$y(t') = (t', 0, 0, 0)$$

For a moving particle, generalize

Proper time



$$d\tau = \sqrt{(dt)^2 - (dx)^2}$$

Particle world line $y^\mu(\tau)$

$$\frac{dy^\mu(\tau)}{d\tau} = u^\mu \quad \text{(four-velocity)}$$

$$j^\mu(x) = e \int d\tau \, \frac{dy^\mu}{d\tau} \, \delta^4(x - y(\tau))$$

Pick time t. Find τ such that $y^0(\tau) = x^0 = t$.

$\frac{dy^\mu}{d\tau}$ is the four-velocity of the particle at time t.

Is this a conserved current?

$$\partial_\mu j^\mu(x) = 0$$

For any well behaved function at infinity

$$f(x) \to 0 \text{ as } x^\mu \to \infty$$

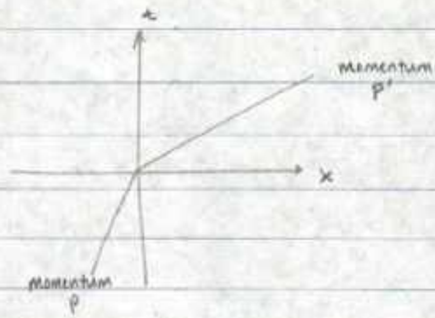
$$\int d^4x \, f(x) \, \partial_\mu j^\mu(x) = \int d^4x \, f(x) \, e \int d\tau \, \frac{dy^\mu}{d\tau} \, \partial_\mu^x (\delta^4(x - y(\tau)))$$

$$= -e \int d\tau \, \frac{dy^\mu}{d\tau} \, \partial_\mu f(x) \Big|_{x = y(\tau)}$$

$$= -e \int d\tau \, \frac{df(y^\mu(\tau))}{d\tau}$$

$$= -e f(y^\mu(\tau)) \Big|_{\tau = -\infty}^{\tau = +\infty}$$

$$= 0$$



$$y^\mu(\tau) = \begin{cases} \frac{p^\mu}{m}\tau & \tau < 0 \\ \frac{p'^\mu}{m}\tau & \tau > 0 \end{cases}$$

$$j^\mu(x) = e\int_0^\infty d\tau \left(\frac{p'^\mu}{m}\right)\delta^4(x - \tfrac{p'}{m}\tau) + e\int_{-\infty}^0 d\tau \left(\frac{p^\mu}{m}\right)\delta^4(x - \tfrac{p}{m}\tau)$$

Make use of Maxwell's equations.

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu$$

Pick Lorentz gauge $\rightarrow$ $\partial_\mu A^\mu = 0$

$$\partial_\mu \partial^\mu A^\nu = j^\nu$$

In momentum space

$$\tilde{A}^\mu(k) = \frac{-1}{k^2}\tilde{j}^\mu(k)$$

$$\tilde{j}^\mu(k) = \int d^4x\, e^{ik\cdot x} j^\mu(x)$$

$$= e\int_0^\infty d\tau \frac{p'^\mu}{m} \exp\left[i\left(\frac{k\cdot p'}{m} + i\epsilon\right)\tau\right] \qquad (\epsilon \to 0^+)$$

$$+ e\int_{-\infty}^0 d\tau \frac{p^\mu}{m} \exp\left[i\left(\frac{k\cdot p}{m} - i\epsilon\right)\tau\right]$$

$$= ie\left(\frac{p'^\mu}{k\cdot p' + i\epsilon}\right) - ie\left(\frac{p^\mu}{k\cdot p - i\epsilon}\right)$$

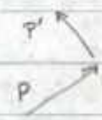
$$\tilde{A}^\mu(k) = \frac{-ie}{k^2}\left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon}\right)$$

Back to position space

$$A^{\mu}(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \frac{(-ie)}{k^2} \left(\frac{p'^{\mu}}{k\cdot p' + i\epsilon} - \frac{p^{\mu}}{k\cdot p - i\epsilon} \right)$$

11/30/2010

Soft Brehmsstrahlung

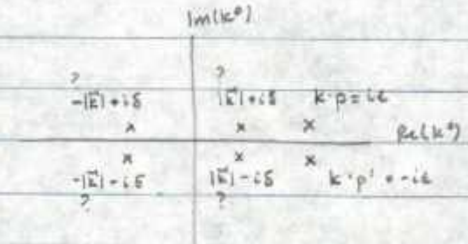


at $t=0$, $\vec{x}=0 \rightarrow$ kick

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \frac{(-ie)}{k^2} \left(\frac{p'^\mu}{k\cdot p' + i\epsilon} - \frac{p^\mu}{k\cdot p - i\epsilon} \right)$$

$$k^2 = (k^0)^2 - \vec{k}^2$$

Integral over k^0 $\qquad$ $k^0 = \pm|\vec{k}|$



Consider $t = x^0 < 0 \rightarrow$ Close above

Answer should not involve p' (kick hasn't happened yet)

$\Rightarrow$ Poles at $-|\vec{k}| - i\delta$, $|\vec{k}| - i\delta$

$$A^\mu(x) = \int \frac{d^3k}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}} e^{-i(\vec{k}\cdot\vec{p}/p^0)t} (2\pi i) \frac{ie}{k^2} \frac{p^\mu}{p^0}$$

$$k\cdot p = i\epsilon \rightarrow k^0 p^0 \approx \vec{k}\cdot\vec{p} \rightarrow k^0 = \frac{\vec{k}\cdot\vec{p}}{k^0}$$

In the rest frame of p

$$p^\mu = (m; 0, 0, 0)$$

$$A^\mu(x) = e \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot\vec{x}} (1,0,0,0)^\mu}{|\vec{k}|^2}$$

Static Coulomb potential

What about $t = x^0 > 0$?

Close below. Three poles

Residue at $k\cdot p' = -i\epsilon$ is completely analagous and gives the Coulomb field due to the momentum p' electron.

$$A^\mu_{rad}(x) = \int \frac{d^3k}{(2\pi)^3}\left(\frac{-e}{2|\vec{k}|} e^{i\vec{k}\cdot\vec{x}} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)\Big|_{k^0=|\vec{k}|} e^{-i|\vec{k}|t} \right.$$

$$\left. + \frac{e}{2|\vec{k}|} e^{i\vec{k}\cdot\vec{x}} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)\Big|_{k^0=-|\vec{k}|} e^{i|\vec{k}|t} \right)$$

Second term $\rightarrow$ $\vec{k} \rightarrow -\vec{k}$

$$\int \frac{d^3k}{(2\pi)^3} \frac{-e}{2|\vec{k}|} e^{-i\vec{k}\cdot\vec{x}} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)\Big|_{k^0=|\vec{k}|} e^{i|\vec{k}|t}$$

$=$ complex conjugate of first term

$$A^\mu_{rad}(x) = \mathrm{Re}\left[\int \frac{d^3k}{(2\pi)^3} a^\mu(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} e^{-i|\vec{k}|t} \right]$$

$$a^\mu(\vec{k}) = \frac{-e}{|\vec{k}|}\left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p}\right)\Big|_{k^0=|\vec{k}|}$$

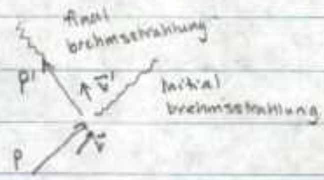
Choose a frame where the initial and final energies of the electron are the same.

$$p^0 = p'^0 = E$$

$$k^\mu = (|\vec{k}|, \vec{k}) \qquad p^\mu = (E, E\vec{v}) \qquad p'^\mu = (E, E\vec{v}')$$

$$\frac{1}{k\cdot p'} = \frac{1}{|\vec{k}|E(1-\hat{k}\cdot\vec{v}')} \qquad \frac{1}{k\cdot p} = \frac{1}{|\vec{k}|E(1-\hat{k}\cdot\vec{v})}$$

When $\hat{k}$ points in the direction of $\vec{v}'$ or $\vec{v}$, there is relative maximum.



Electron Vertex Function

amputated diagrams  $= -ie\,\Gamma^\mu(p,p')$

If we have a classical E+M field

$$\Delta H_{int} = \int d^3x \; e\, A^{cl}_\mu(x)\, j^\mu(x)$$

$$j^\mu(x) = \bar\psi(x)\,\gamma^\mu\,\psi(x)$$

$$\text{Scattering amplitude} = -ie\,\bar u(p')\,\Gamma^\mu(p',p)\,u(p)\,\tilde A^{cl}_\mu(p'-p)$$

$$\Gamma^\mu(p,p') = \gamma^\mu + \mathcal{O}(e^2)$$

function of p, p', gamma matrices, e, m

γ^μ is a Lorentz vector

$$\Gamma^\mu(p,p') = \gamma^\mu A + (p'^\mu + p^\mu)B + (p'^\mu - p^\mu)C + \cancel{\gamma^\mu\gamma^5}\,D \quad \text{parity is wrong}$$

A, B, C, D are functions of p, p' with no uncontracted indices

$\not{p}\, u(p) = m\, u(p) \qquad \bar{u}(p')\, \not{p}' = \bar{u}(p')\, m$

Say $B = \not{p}\, \not{p}'$. We can anticommute until we get $\not{p}'$ on left and $\not{p}$ on right. Then all slashes will be gone using the Dirac equation

A, B, C are functions of

$$p \cdot p = m^2$$
$$p' \cdot p' = m^2$$
$$p \cdot p'$$
$$q^2 = (p' - p) = p'^2 + p^2 - 2p' \cdot p$$

They are functions of q^2.

Word - Takahashi Identity

$$q_\mu \bar{u}(p')\, \Gamma^\mu (p, p')\, u(p) = 0$$

$$q_\mu \bar{u}(p')\, \gamma^\mu u(p) = \bar{u}(p')\, (\not{p}' - \not{p})\, u(p) = \bar{u}(p')(m - m)\, u(p) = 0$$

$$q_\mu \bar{u}(p')\, (\gamma^\mu A + (p'^\mu + p^\mu) B + (p'^\mu - p^\mu) C)\, u(p)$$

$$= \bar{u}(p')\, \left(\cancelto{0}{(\not{p}' - \not{p})} A + \cancelto{0}{((p')^2 - p^2)} B + q^2 C \right) u(p)$$

$\Rightarrow \quad C = 0 \quad$ for Word Identity to hold.

Gordon Identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

Proof:

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\frac{\not{p}'}{m}\gamma^\mu u(p) = \bar{u}(p')\gamma^\mu\frac{\not{p}}{m}u(p)$$

$$= \frac{1}{2}\bar{u}(p')\left(\frac{\not{p}'}{m}\gamma^\mu + \gamma^\mu\frac{\not{p}}{m}\right)u(p)$$

$$= \frac{1}{4m}\bar{u}(p')\left(\{\not{p}', \gamma^\mu\} + [\not{p}', \gamma^\mu]\right)u(p)$$

$$+ \frac{1}{4m}\bar{u}(p')\left(\{\gamma^\mu, \not{p}\} + [\gamma^\mu, \not{p}]\right)u(p)$$

$$= \frac{1}{4m}\bar{u}(p')\left(2p'^\mu + p'_\nu[\gamma^\nu, \gamma^\mu]\right)u(p)$$

$$+ \frac{1}{4m}\bar{u}(p')\left(2p^\mu + p_\nu[\gamma^\mu, \gamma^\nu]\right)u(p)$$

$$= \bar{u}(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{1}{4m}\left(-p'_\nu[\gamma^\mu, \gamma^\nu] + p_\nu[\gamma^\mu, \gamma^\nu]\right)\right)u(p)$$

$$= \bar{u}(p')\left(\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p)$$

We use the Gordon identity to replace the $(p'^\mu + p^\mu)B$

$$\Gamma^\mu(p, p') = \gamma^\mu \underbrace{F_1(q^2)}_{\text{Dirac Form Factor}} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\underbrace{F_2(q^2)}_{\text{Pauli Form Factor}}$$

At lowest order, $\Gamma^\mu(p,p') = \gamma^\mu$

So $F_1(q^2) = 1$, $F_2(q^2) = 0$

Classical field

$$A_\mu^{cl}(x) = (\phi(\vec{x}), 0, 0, 0) \quad \text{time independent electrostatic potential}$$

$$\tilde{A}_\mu^{cl}(q) = (2\pi)\,\delta(q^0)\,(\tilde{\phi}(\vec{q}), 0, 0, 0)$$

$$\text{Scattering amplitude} = -ie\,\bar{u}(p')\,\Gamma^\mu(p,p')\,u(p)\,\tilde{A}_\mu^{cl}(q)$$

$$= -ie\,\bar{u}(p')\,\Gamma^0(p,p')\,u(p)\,\tilde{\phi}(\vec{q})\,(2\pi\delta(q^0))$$

Assume that $\tilde{\phi}(\vec{q})$ is peaked at $\vec{q} = 0$, Then we can approximately write

$$\bar{u}(p')\,\Gamma^0(p,p')\,u(p)\,\tilde{\phi}(\vec{q}) = \bar{u}(p')\,\gamma^0 F_1(0)\,u(p)\,\tilde{\phi}(\vec{q})$$

$$\approx u^\dagger(p')\,u(p)\,F_1(0)\,\tilde{\phi}(\vec{q})$$

In nonrelativistic limit

$$u^\dagger(p')\,u(p) \approx 2m\,\xi'^\dagger\xi$$

$$\text{Nonrelativistic scattering amplitude} = -ie\,F_1(0)\,\tilde{\phi}(\vec{q})\,(2m\,\xi'^\dagger\xi)\,(2\pi\,\delta(q^0))$$

$eF_1(0)$ - physically measured charged $\Rightarrow$ $F_1(0) = 1$.

17/02/2010

Electron Vertex Function

$$\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m} F_2(q^2)$$

amputated diagrams

$$= -ie\Gamma^\mu(p',p)$$

For static potential $\phi(\vec{x})$ (nearly uniform in space)

$$i\mathcal{M} = -ie\, F_1(0)\, \tilde{\phi}(\vec{q})\, (2m\, \xi'^\dagger \xi)$$

Scattering due to an $eF_1(0)\phi(\vec{x})$ potential

$$\Rightarrow F_1(0) = 1.$$

Magnetic field $\rightarrow$ $A^0_{cl} = 0$ $\quad \vec{A}_{cl}(\vec{x})$

$$i\mathcal{M} = \sum_j ie\, \bar{u}(p')\left(\gamma^j F_1 + \frac{i\sigma^{j\nu}q_\nu}{2m} F_2\right) u(p)\, \tilde{A}^j_{cl}(\vec{q})$$

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} = \begin{pmatrix} \sqrt{m-\vec{p}\cdot\vec{\sigma}}\,\xi \\ \sqrt{m+\vec{p}\cdot\vec{\sigma}}\,\xi \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \\ \left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \end{pmatrix}$$ (nonrelativistic limit)

$$\bar{u}(p') = \sqrt{m}\left(\xi'^\dagger\left(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right) \quad \xi'^\dagger\left(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\right)\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\gamma^0}$$

F_1 term $\rightarrow$

$$\bar{u}(p')\gamma^i u(p) = m\left(\xi'^T\left(1-\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right) \quad \xi'^T\left(1+\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\right)\right)\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix}\begin{pmatrix}\left(1-\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi\\\left(1+\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi\end{pmatrix}$$

$$= 2m\,\xi'^T\left(\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\sigma^i + \sigma^i\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi$$

$$\sigma^i\sigma^j = \frac{1}{2}\left[\sigma^i,\sigma^j\right] + \frac{1}{2}\left\{\sigma^i,\sigma^j\right\}$$

$$= i\epsilon^{ijk}\sigma^k + \delta^{ij}$$

$$(\vec{p}'\cdot\vec{\sigma})\sigma^i = p'^j\sigma^j\sigma^i = p'^i + i\epsilon^{jik}p'^j\sigma^k$$

$$= p'^i - i\epsilon^{ijk}p'^j\sigma^k$$

$$\sigma^i(\vec{p}\cdot\vec{\sigma}) = \sigma^i p^j\sigma^j = p^i + i\epsilon^{ijk}p^j\sigma^k$$

$$\bar{u}(p')\gamma^i u(p) = 2m\,\xi'^\dagger\left(\frac{p'^i+p^i}{2m} - \underbrace{\frac{i\epsilon^{ijk}q^j\sigma^k}{2m}}_{\text{this will give a } (\nabla\times\vec{A})\cdot\vec{\sigma}}\right)\xi$$

F_2 term $\rightarrow$

$$\bar{u}(p')\left(\frac{i}{2m}\sigma^{i\nu}q_\nu\right)u(p)$$

$$\sigma^{ij} = \frac{i}{2}\left[\gamma^i,\gamma^j\right] = \frac{i}{2}\left[\begin{pmatrix}0&\sigma^i\\-\sigma^i&0\end{pmatrix},\begin{pmatrix}0&\sigma^j\\-\sigma^j&0\end{pmatrix}\right]$$

$$= \frac{-i}{2}\begin{pmatrix}[\sigma^i,\sigma^j]&0\\0&[\sigma^i,\sigma^j]\end{pmatrix}$$

$$= \epsilon^{ijk}\begin{pmatrix}\sigma^k&0\\0&\sigma^k\end{pmatrix}$$

$$\bar{u}(p')\left(\frac{i}{2m}\sigma^{i\nu}q_\nu\right)u(p) = 2m\,\xi'^{\dagger}\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\right)\xi$$

index raised gives extra minus sign.

$$i\mathcal{M} = \vec{p}\cdot\vec{A}\text{ term} + 2m\,ie\,\xi'^{\dagger}\left(\frac{-i}{2m}\epsilon^{ijk}q^j\sigma^k\left(F_1(q^2)+F_2(q^2)\right)\right)\xi\,\tilde{A}^i_{cl}(\vec{q})$$

$$F_1(q^2) \approx F_1(0) = 1$$

$$F_2(q^2) \approx F_2(0)$$

Weak magnetic field scattering due to a potential

$$V(\vec{x}) = -\underbrace{\langle\vec{\mu}\rangle}_{\text{magnetic moment}}\cdot\vec{B}(\vec{x}) \qquad (\vec{B}(\vec{x}) = \nabla\times\vec{A}(\vec{x}))$$

$$\langle\vec{\mu}\rangle = \frac{e}{m}\left(1+F_2(0)\right)\xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi$$

Classical electrodynamics $\rightarrow$ $\langle\vec{\mu}\rangle = g\left(\frac{e}{2m}\right)\vec{S}$

g is called Landé g-factor (gyromagnetic ratio)

If we neglect things at order e^2 $\rightarrow$ $g = 2 + \mathcal{O}(\alpha)$

Loop Diagrams (Preview for next semester)

$$\int \frac{d^4k}{(2\pi)^4} \left(\frac{-i g_{\nu\rho}}{(p-k)^2+i\epsilon} \, \bar{u}(p') (-ie\gamma^\nu) \left(\frac{i(\not{k}'+m)}{k'^2-m^2+i\epsilon} \right) (-ie\gamma^\mu) \left(\frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} \right) (-ie\gamma^\rho) u(p) \right)$$

Looking at just denominator . . .

$$\int \frac{d^4k}{(2\pi)^4} \frac{1}{(p-k)^2+i\epsilon} \frac{1}{k'^2-m^2+i\epsilon} \frac{1}{k^2-m^2+i\epsilon}$$

How do you do this?

Feynman's Trick for Combining Denominators

$$\frac{1}{A\cdot B} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}$$

Proof:

$A = B$: $\displaystyle\int_0^1 dx \frac{1}{(xA+(1-x)B)^2} = \int_0^1 dx \frac{1}{A^2} = \frac{1}{A^2} = \frac{1}{A\cdot B}$ ✓

$A \neq B$: $\displaystyle\int_0^1 dx \frac{1}{(xA+(1-x)B)^2} = \frac{-1}{(xA+(1-x)B)} \frac{1}{(A-B)} \Bigg|_0^1$

$$= \frac{-1}{(A-B)} \left(\frac{1}{A} - \frac{1}{B} \right)$$

$$= \frac{-1}{(A-B)} \left(\frac{-(A-B)}{AB} \right) = \frac{1}{A\cdot B} \checkmark$$

Generalizations

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \frac{\delta(\sum x_i - 1)\,(n-1)!}{(x_1 A_1 + x_2 A_2 + \cdots + x_n A_n)^n}$$

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^1 dx_1 \cdots dx_n \, \frac{\delta(\sum x_i - 1) \prod x_i^{m_i - 1}\, \Gamma(\sum m_i)}{(\sum x_i A_i)^{\sum m_i} (\Gamma(m_1) \cdots \Gamma(m_n))}$$