1 Ramsey Fringes Overview:

Following a double pulse, the population of the excited state is:

$$\boxed{P_2 = 4\sin^2\theta \sin^2\frac{\Omega'\tau}{2}\left\{\cos\frac{\Omega'\tau}{2}\cos\frac{\Delta_0 T}{2} - \cos\theta\sin\frac{\Omega'\tau}{2}\sin\frac{\Delta_0 T}{2}\right\}}$$

under the assumption that initially,

$$\begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The final state vector is:

$$\begin{pmatrix} C_1(2\tau + T) \\ C_2(2\tau + T) \end{pmatrix} = \rho_2 D \rho_1 \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix}$$

where $|C_1|^2 + |C_2|^2 = 1$ for all value of time, and ρ_1 and ρ_2 are propagators associated with Pulse 1 and Pulse 2 (both with width τ), respectively. D is a propagator associated with the field-free evolution of duration T.

Specifically, in the interaction representation:

$$\rho_1 = e^{-i\overline{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau - \phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} \\ ie^{i\left(\frac{\Delta_0}{2}\tau + \phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}$$

$$\rho_2 = e^{-i\overline{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau + \phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{-i\Delta_0(\tau+T)} \\ ie^{i\left(\frac{\Delta_0}{2}\tau + \phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{i\Delta_0(\tau+T)} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that D, in the interaction representation, is the identity matrix. This is different from Ramsey's original approach in which the state vector does evolve and change during the delay time T. The angle θ is defined as:

$$\sin\theta = \frac{{\Omega_0}^*}{\Omega'}$$

and

$$\cos\theta = \frac{\Delta_0 + \Delta_d}{\Omega'}$$

where ${\Omega_0}^*$ is the complex conjugate of the Rabi rate, and Ω' can be defined as the "effective Rabi rate."

$$\Omega' = \sqrt{|{\Omega_0}^*| + (\Delta_0 + \Delta_d)^2}$$

2 Detailed Derivation for P_f

In the interaction representation,

$$\boxed{i\begin{pmatrix}\dot{a}_i(t)\\ \dot{a}_f(t)\end{pmatrix} = \begin{pmatrix}\Delta_i & -\frac{\Omega_0^*}{2}e^{-i\Delta_0 t}\\ -\frac{\Omega_0}{2}e^{i\Delta_0 t} & \Delta_f\end{pmatrix}\begin{pmatrix}a_i(t)\\ a_f(t)\end{pmatrix}} \tag{1}$$

where Δ_i is the ac Stark shift in the $|i\rangle$ state and Δ_f is the ac Stark shift in the $|f\rangle$ state. We first solve for a_i:

$$i\ddot{a}_i = \Delta_i\dot{a}_i - \frac{\Omega_0^*}{2}e^{-i\Delta_0 t}\dot{a}_f + \frac{i\Delta_0}{2}\Omega_0^* e^{-i\Delta_0 t}a_f, \tag{2}$$

where

$$a_f = \left(\frac{\Omega_0^*}{2}e^{-\Delta_0 t}\right)^{-1}(\Delta_i a_i - i\dot{a}_i) = \frac{2}{\Omega_0^*}e^{i\Delta_0 t}(\Delta_i a_i - i\dot{a}_i). \tag{3}$$

From Eq. (1), we also have

$$\begin{aligned}\dot{a}_f &= i^{-1}\left(-\frac{\Omega_0}{2}e^{i\Delta_0 t}a_i + \Delta_f a_f\right)\\ &= i^{-1}\left[-\frac{\Omega_0}{2}e^{i\Delta_0 t}a_i + \Delta_f(\Delta_i a_i - i\dot{a}_i)\left(\frac{2}{\Omega_0^*}e^{i\Delta_0 t}\right)\right]\\ &= \frac{2i}{\Omega_0^*}e^{i\Delta_0 t}\left(\frac{|\Omega_0|^2}{4}a_i - \Delta_f\Delta_i a_i + i\Delta_f\dot{a}_i\right).\end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned}\ddot{a}_i &= -i\Delta_i a_i + \frac{i\Omega_0^*}{2}e^{-i\Delta_0 t}\left[\frac{2i}{\Omega_0^*}e^{i\Delta_0 t}\left(\frac{|\Omega_0|^2}{4}a_i - \Delta_f\Delta_i a_i + i\Delta_f\dot{a}_i\right)\right]\\ &+ \frac{\Delta_0}{2}\Omega^* e^{-i\Delta_0 t}\frac{2}{\Omega_0^*}e^{i\Delta_0 t}(\Delta_i a_i - i\dot{a}_i)\\ &= -i\Delta_i\dot{a}_i - \frac{|\Omega_0|^2}{4}a_i + \Delta_f\Delta_i a_i - i\Delta_f\dot{a}_i + \Delta_0\Delta_i a_i - i\Delta_0\dot{a}_i\\ &= -i(\Delta_0 + \Delta_i + \Delta_f)\dot{a}_i - \left[\frac{|\Omega_0|^2}{4} - \Delta_i(\Delta_f + \Delta_0)\right]a_i\\ &= -i(\Delta_0 + \Delta_i + \Delta_f)\dot{a}_i - \frac{1}{4}\left(|\Omega_0|^2 - 4\Delta_i\Delta_f - 4\Delta_i\Delta_0\right)a_i.\end{aligned} \tag{5}$$

We obtain the first second-order homogeneous differential equation:

$$\ddot{a}_i + i(\Delta_0 + \Delta_i + \Delta_f)\dot{a}_i + \left[\frac{|\Omega_0|^2}{4} - \Delta_i(\Delta_f + \Delta_0)\right]a_i = 0. \tag{6}$$

Let a guess solution be $a_i(t) = a_0 e^{i\omega t}$. The characteristic equation is:

$$\begin{aligned}-\omega^2 + i(\Delta_0 + \Delta_i + \Delta_f)(i\omega) + \frac{1}{4}(|\Omega_0|^2 - 4\Delta_0\Delta_i - 4\Delta_i\Delta_f) = 0\\ -\omega^2 - (\Delta_0 + \Delta_i + \Delta_f)\omega + \frac{1}{4}(|\Omega_0|^2 - 4\Delta_0\Delta_i - 4\Delta_i\Delta_f) = 0.\end{aligned} \tag{7}$$

Solving the quadratic equation (7) and obtain w:

$$\begin{aligned}\omega &= -\frac{\Delta_0+\Delta_f+\Delta_i}{2} \pm \frac{1}{2}\sqrt{(\Delta_0+\Delta_f+\Delta_i)^2 + |\Omega_0|^2 - 4\Delta_0\Delta_i - 4\Delta_i\Delta_f} \\ &= -\frac{\Delta_0+\Delta_f+\Delta_i}{2} \\ &\pm \frac{1}{2}\sqrt{(\Delta_0+\Delta_f+\Delta_i)^2 + \Delta_0^2 + |\Omega_0|^2 + (\Delta_i-\Delta_f)^2 - 2\Delta_0(\Delta_i-\Delta_f)}\end{aligned} \tag{8}$$

Let $\bar{\Delta} = (\Delta_i+\Delta_f)/2$ and $\Delta_d = \Delta_f - \Delta_i$, this gives

$$\boxed{\omega = -\frac{\Delta_0}{2} - \bar{\Delta} \pm \frac{1}{2}\sqrt{|\Omega_0|^2 + (\Delta_0+\Delta_d)^2}} \tag{9}$$

Next, let the "effective Rabi rate" be Ω', defined as

$$\Omega' = \sqrt{|\Omega_0|^2 + (\Delta_0+\Delta_d)^2}. \tag{10}$$

The general solution to eq. (7) is:

$$\begin{aligned}a_i &= a_+ e^{i\omega_+ t} + a_- e^{i\omega_- t} \\ &= e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t}\left(a_+ e^{i\frac{\Omega'}{2}t} + a_- e^{-i\frac{\Omega'}{2}t}\right)\end{aligned} \tag{11}$$

So,

$$\boxed{a_i = e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t}\left(A\cos\frac{\Omega' t}{2} + B\sin\frac{\Omega' t}{2}\right)} \tag{12}$$

Next, we solve for a_f. From eq. (3):

$$\begin{aligned}a_f &= \frac{2}{\Omega_0^*} e^{i\Delta_0 t}\left(\Delta_i a_i - i\dot{a}_i\right) \\ &= \frac{2}{\Omega_0^*} e^{i\Delta_0 t}\left[\Delta_i e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t}\left(A\cos\frac{\Omega' t}{2} + B\sin\frac{\Omega' t}{2}\right) - i\dot{a}_i\right] \\ &= \frac{2}{\Omega_0^*} e^{i\frac{\Delta_0}{2}t}\left[\Delta_i e^{-i\bar{\Delta}t}\left(A\cos\frac{\Omega' t}{2} + B\sin\frac{\Omega' t}{2}\right) - i\dot{a}_i\right]\end{aligned} \tag{13}$$

where

$$\begin{aligned}-i\dot{a}_i &= (-i)^2\left(\bar{\Delta} + \frac{\Delta_0}{2}\right) e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t}\left[\left(A\cos\frac{\Omega' t}{2} + B\sin\frac{\Omega' t}{2}\right)\right. \\ &\quad \left. + - i\frac{\Omega'}{2}\left(-A\sin\frac{\Omega' t}{2} + B\cos\frac{\Omega' t}{2}\right)\right] \\ &= e^{-i\bar{\Delta}t} e^{-i\frac{\Delta_0}{2}t}\left[-\left(\bar{\Delta} + \frac{\Delta_0}{2}\right)\left(A\cos\frac{\Omega' t}{2} + B\sin\frac{\Omega' t}{2}\right)\right. \\ &\quad \left. + i\frac{\Omega'}{2}\left(A\sin\frac{\Omega' t}{2} - B\cos\frac{\Omega' t}{2}\right)\right].\end{aligned} \tag{14}$$

Assume that at $t = 0$, $A = a_i(0)$ and

$$B = i\frac{\Omega_0^*}{\Omega'}a_f(0) + i\frac{\Delta_d + \Delta_0}{\Omega'}a_i(0). \tag{15}$$

So, from Eq. (12):

$$\boxed{a_i(t) = e^{-i\left(\bar{\Delta}+\frac{\Delta_0}{2}\right)t}\left\{a_i(0)\left[\cos\frac{\Omega' t}{2} + i\frac{\Delta_0 + \Delta_d}{\Omega'}\sin\frac{\Omega' t}{2}\right] + a_f(0)\frac{i\Omega_0^*}{\Omega'}\sin\frac{\Omega' t}{2}\right\}} \tag{16}$$

From Eq. (13) and (14), we obtain an expression for $a_f(t)$:

$$\begin{aligned}
a_f(t) = &\frac{2}{\Omega_0^*}e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{\Delta_i\left(a_i(0)\cos\frac{\Omega' t}{2} + \left(\frac{i\Omega_0^*}{\Omega'}a_f(0) + i\frac{\Delta_d + \Delta_0}{\Omega'}a_i(0)\right)\sin\frac{\Omega' t}{2}\right)\right.\\
&-\left(\bar{\Delta} + \frac{\Delta_0}{2}\left(a_i\cos\frac{\Omega' t}{2} + \left(\frac{i\Omega_0^*}{\Omega'}a_f(0) + \frac{i}{2}\frac{\Delta_d + \Delta_0}{\Omega'}a_i(0)\right)\sin\frac{\Omega' t}{2}\right)\right)\\
&\left. i\frac{\Omega'}{2}\left(a_i\sin\frac{\Omega' t}{2} - \left(\frac{i\Omega_0^*}{\Omega'}a_f(0) + \frac{i}{2}\frac{\Delta_d + \Delta_0}{\Omega'}a_i(0)\right)\cos\frac{\Omega' t}{2}\right)\right\}\\
= &\frac{2}{\Omega_0^*}e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{\left(\Delta_i - \bar{\Delta} - \frac{\Delta_0}{2}\right)\left(A\cos\frac{\Omega' t}{2} + B\sin\frac{\Omega' t}{2}\right)\right.\\
&\left.\frac{i\Omega'}{2}\left(A\sin\frac{\Omega' t}{2} - B\cos\frac{\Omega' t}{2}\right)\right\}.
\end{aligned} \tag{17}$$

Now, notice that

$$\Delta_i - \bar{\Delta} = \Delta_i - \frac{\Delta_i + \Delta_f}{2} = -\frac{\Delta_d}{2}. \tag{18}$$

So,

$$\begin{aligned}
a_f(t) &= e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{A\left(-\frac{\Delta_d + \Delta_0}{\Omega_0^*}\cos\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\sin\frac{\Omega' t}{2}\right)\right.\\
&\quad\left.-B\left(\frac{\Delta_d + \Delta_0}{\Omega_0^*}\sin\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\cos\frac{\Omega' t}{2}\right)\right\}\\
&= e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{a_i(0)\left(-\frac{\Delta_d + \Delta_0}{\Omega_0^*}\cos\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\sin\frac{\Omega' t}{2}\right)\right.\\
&\quad\left.-\left(i\frac{\Omega_0^*}{\Omega'}a_f(0) + i\frac{\Delta_d + \Delta_0}{\Omega'}a_i(0)\right)\left(\frac{\Delta_d + \Delta_0}{\Omega_0^*}\sin\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\cos\frac{\Omega' t}{2}\right)\right\}\\
&= e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{a_i\sin\frac{\Omega' t}{2}\left(\frac{i\Omega'}{\Omega_0^*\Omega'} - \frac{i(\Delta_d + \Delta_0)^2}{\Omega_0^*\Omega'}\right)\right.\\
&\quad\left. a_f(0)\left(\cos\frac{\Omega' t}{2} - i\frac{\Delta_d + \Delta_0}{\Omega'}\sin\frac{\Omega' t}{2}\right)\right\}.
\end{aligned} \tag{19}$$

Next, note that

$$\Omega'^2 = (\Delta_0 + \Delta_d)^2 + |\Omega_0|^2 = (\Delta_0 + \Delta_d)^2 + \Omega_0\Omega_0^*. \tag{20}$$

So

$$\boxed{a_f(t) = e^{-i\bar{\Delta}t} e^{i\frac{\Delta_0}{2}t} \left\{ a_i(0) \frac{i\Omega_0}{\Omega'} \sin\frac{\Omega' t}{2} + a_f(0) \left(\cos\frac{\Omega' t}{2} - i\frac{\Delta_d + \Delta_0}{\Omega'} \sin\frac{\Omega' t}{2} \right) \right\}}. \tag{21}$$

Finally, let us put everything together in matrix form:

$$\begin{pmatrix} a_i(t) \\ a_f(t) \end{pmatrix} = \mathcal{M} \begin{pmatrix} a_i(0) \\ a_f(0) \end{pmatrix}, \tag{22}$$

where $\mathcal{M}$ is the matrix

$$\boxed{e^{-i\bar{\Delta}t} \begin{pmatrix} e^{-i\frac{\Delta_0}{2}t} \left(\cos\frac{\Omega' t}{2} + i\frac{\Delta_d+\Delta_0}{\Omega'} \sin\frac{\Omega' t}{2} \right) & e^{-i\frac{\Delta_0}{2}t} \frac{i\Omega_0^*}{\Omega'} \sin\frac{\Omega' t}{2} \\ e^{i\frac{\Delta_0}{2}t} \frac{i\Omega_0}{\Omega'} \sin\frac{\Omega' t}{2} & e^{i\frac{\Delta_0}{2}t} \left(\cos\frac{\Omega' t}{2} - i\frac{\Delta_d+\Delta_0}{\Omega'} \sin\frac{\Omega' t}{2} \right) \end{pmatrix}} \tag{23}$$