1. Consider the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$ on two qubits. What are the states that you get when you apply $\sigma_x \otimes I$, $\sigma_y \otimes I$, and $\sigma_z \otimes I$ to this state? Show that they are all orthogonal.

2. I'd like you to work out an example of an observable with repeated eigenvalues, but I don't want the computations to be too painful, so I'm giving you a hint.

   I don't think I said very much about this in class, so as a reminder, if an observable has repeated eigenvalues, the state being measured is projected onto the subspace corresponding to the eigenvalue.

   Consider the observable

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

   Hint: Three eigenvectors of M (not normalized) are

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

   (a) What are the eigenvalues and the corresponding projection matrices for this observable?

   (b) If the qutrit

$$\frac{2}{3}\,|\,0\rangle + \frac{2}{3}\,|\,1\rangle - \frac{1}{3}\,|\,2\rangle$$

   is measured using this observable, what are the possible outcomes, with what probabilities do you observe them, and what are the resulting quantum states?

3. A spin-1 particle has three quantum states. We will take the basis of the quantum state space to be the values of the spin along the z-axis; the three basis states are $|\,1\rangle$, $|\,0\rangle$ and $|\,-1\rangle$. The observable corresponding the the spin along the z-axis in this basis is then

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   In this basis, the observables for the spin along the x- and y-axes are:

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Two observables can be measured simultaneously if the matrices corresponding to them commute.

(a) Show that the matrices J_x and J_z do not commute, and thus cannot be measured simultaneously.

(b) Show that the observables J_x^2, J_y^2, and J_z^2 all commute, Find the three simultaneous eigenvectors and their associated eigenvalues. What is the observable $J^2 = J_x^2 + J_y^2 + J_z^2$?

4. In this problem, we will derive the matrix J_x in problem (2). Suppose we have two qubits A and B. The observable giving the spin in the z direction is

$$\frac{1}{2}\left(\sigma_z^A \otimes I^B + I^A \otimes \sigma_z^B\right).$$

Similarly, the observable giving the spin in the x direction is

$$\frac{1}{2}\left(\sigma_x^A \otimes I^B + I^A \otimes \sigma_x^B\right).$$

There is a 3-dimensional subspace of the 4-dimensional state space of two qubits which corresponds to the state space of a spin-1 particle. This is the subspace orthogonal to the state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$. (The state $\frac{1}{\sqrt{2}}(|\,01\rangle - |\,10\rangle)$ has spin 0 when measured along any axis). Use these facts to find the matrix J_x of problem (2).

5. **Generalized Measurements**

In this problem, you will derive an example of a non-von Neumann measurement (which you will implement using a sequence of unitary transformations and von Neumann measurements).

Suppose you are given one of the three states:

$$|\,0\rangle\,, \qquad -\frac{1}{2}\,|\,0\rangle + \frac{\sqrt{3}}{2}\,|\,1\rangle\,, \qquad -\frac{1}{2}\,|\,0\rangle - \frac{\sqrt{3}}{2}\,|\,1\rangle\,,$$

with equal probabilities. Your task is to identify the state while minimizing the probability that you get it wrong.

(a) Suppose you measure the state using a basis $\{|\,v\rangle\,, |\,\bar{v}\rangle\}$ where $|\,v\rangle$ and $|\,\bar{v}\rangle$ are orthonormal quantum states. Show that the probability you get it correct is strictly less than $\frac{2}{3}$. (There's a simple argument for this that barely uses any calculation, although if you can't find this argument feel free to use a more calculationally intensive one.)

Now, let's try to do better. Suppose you take the first qubit and tensor it with a second qubit in the state $|\,0\rangle$.

(b) Find α and β such that the following quantum states form an orthonormal basis:

$$\left\{ \begin{array}{cc} |\,11\rangle\,, & -\frac{1}{2}\alpha\,|\,00\rangle + \frac{\sqrt{3}}{2}\alpha\,|\,10\rangle + \beta\,|\,01\rangle\,, \\ \alpha\,|\,00\rangle + \beta\,|\,01\rangle\,, & -\frac{1}{2}\alpha\,|\,00\rangle - \frac{\sqrt{3}}{2}\alpha\,|\,10\rangle + \beta\,|\,01\rangle\,. \end{array} \right\}$$

(c) Suppose you use the measurement corresponding to the orthonormal basis above to try to identify the state. What is the probability that you succeed?