Practice Solutions 2.111/8.370/18.435

1. Suppose we have two qubits. We can apply a measurement on the first qubit using the basis

$$\left\{ \frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle , \quad \frac{-1}{\sqrt{5}} |0\rangle + \frac{2}{\sqrt{5}} |1\rangle \right\}$$

In the projection-matrix description of a von Neumann measurement on the two-qubit system, what two 4×4 projection matrices does this correspond to?

Solution:

The two 2×2 projection matrix for the first qubit are

$$\left(\frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle \right) \left(\frac{2}{\sqrt{5}} \langle 0| + \frac{1}{\sqrt{5}} \langle 1| \right) = \frac{1}{5} \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

and

$$\left(-\frac{1}{\sqrt{5}} |0\rangle + \frac{2}{\sqrt{5}} |1\rangle \right) \left(-\frac{1}{\sqrt{5}} \langle 0| + \frac{2}{\sqrt{5}} \langle 1| \right) = \frac{1}{5} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

We are applying the identity measurement (i.e., not doing anything) on the second qubit, so to get the 4×4 projection matrices, we take the projection matrices on the first qubit and tensor them with the identity. Doing this, we get:

$$\frac{1}{5} \begin{pmatrix} 4 & 0 & 2 & 0 \\ 0 & 4 & 0 & 2 \\ 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \frac{1}{5} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ -2 & 0 & 4 & 0 \\ 0 & -2 & 0 & 4 \end{pmatrix}.$$

2. Suppose we have two qubits on system AB in the state

$$\frac{1}{3} |01\rangle_{AB} + \frac{2}{3} |10\rangle_{AB} + \frac{2}{3} |11\rangle_{AB}$$

We now apply a measurement on the first qubit, A, using the basis

$$\left\{ \frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle , \quad \frac{-1}{\sqrt{5}} |0\rangle + \frac{2}{\sqrt{5}} |1\rangle \right\}$$

What is the probability of getting the first element of this basis, and if we do, what is the resulting state of qubit B?

Solution:

We take the inner product of the first basis element and the joint quantum state:

$$\begin{aligned} \left(\frac{2}{\sqrt{5}} {}_A\langle 0| + \frac{1}{\sqrt{5}} {}_A\langle 1| \right) \left(\frac{1}{3} |01\rangle_{AB} + \frac{2}{3} |10\rangle_{AB} + \frac{2}{3} |11\rangle_{AB} \right) &= \frac{2}{3\sqrt{5}} |1\rangle_B + \frac{2}{3\sqrt{5}} |0\rangle_B + \frac{2}{3\sqrt{5}} |1\rangle_B \\ &= \frac{2}{3\sqrt{5}} \left(|0\rangle_B + 2 |1\rangle_B \right) \end{aligned}$$

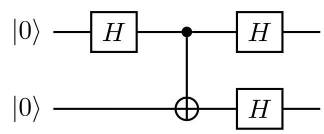
The probability of seeing this outcome is just the square of the length of this state, which is

$$\frac{4}{45}(1+2^2) = \frac{4}{9}.$$

To find the state of B's system after the measurement, you need to normalize the state above. You can either divide it by the square root of the probability of seeing it or just directly normalize the state $|0\rangle + 2\,|1\rangle$. Either way, you get

$$\frac{1}{\sqrt{5}}\big(\,|0\rangle_B + 2\,|1\rangle_B\,\big)$$

3. Suppose we have the quantum circuit:



What is the state of the two qubits at the output?

Solution: After the Hadamard gate, the system is in the state $\frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)$.
After the CNOT, it is is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.
After the next two Hadamard gates, it is in the state $\frac{1}{\sqrt{2}}(|{++}\rangle + |{--}\rangle)$.
Expanding this gives $\frac{1}{\sqrt{8}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle + |00\rangle - |01\rangle - |10\rangle + |11\rangle)$,
which is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

4. (a) What is the density matrix ρ representing an equal mixture (probability $\frac{1}{2}$ each) of the quantum states:

$$\frac{1}{\sqrt{3}}\,(|00\rangle + |01\rangle + |10\rangle) \quad \text{and} \quad \frac{1}{\sqrt{3}}\,(|00\rangle - |01\rangle + |10\rangle)$$

Solution: The density matrix is

$$\frac{1}{2}\left(\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) What is $\mathrm{Tr}_A\rho$, where ρ is as in part (a)?
Solution:

$$\mathrm{Tr}_A\rho = \frac{1}{3}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

(c) What is $\mathrm{Tr}_B\rho$?

Solution:

$$\mathrm{Tr}_B\rho = \frac{1}{3}\left(\begin{array}{cc} \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} & \mathrm{Tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{array}\right) = \frac{1}{3}\mathrm{Tr}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

5. Show that for any density matrix ρ

$$\tfrac{1}{4}\left(\rho + \sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y + \sigma_z\rho\sigma_z\right) = \tfrac{1}{2}I\,.$$

Solution: Let $\rho' = \frac{1}{2}(\rho + \sigma_z\rho\sigma_z)$, and let $\rho'' = \frac{1}{2}(\sigma_x\rho'\sigma_x)$. The density matrix we are looking for is ρ''. Why? Because $\sigma_x\sigma_z\rho\sigma_z\sigma_x = \sigma_y\rho\sigma_y$. (This follows from $\sigma_z\sigma_x = i\sigma_y$.)

Now, if $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\sigma_z\rho\sigma_z = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$, and $\rho' = \frac{1}{2}(\rho + \sigma_z\rho\sigma_z) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Similarly, $\sigma_x\rho'\sigma_x = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$, and $\frac{1}{2}(\rho + \sigma_z\rho\sigma_z) = \frac{1}{2}I$, because $a + d = \mathrm{Tr}\rho = 1$.

6. What is the value of: $\langle 0001111100|\, H^{\otimes 10}\, |1111110000\rangle$?

Justify your answer.

Solution: Recall that

$$H^{\otimes n}\,|j\rangle = \frac{1}{2^{n/2}}\sum_{k=0}^{2^n-1}(-1)^{j\cdot k}\,|k\rangle\,.$$

Thus, because the inner product of the two bit strings is 3, the amplitude on the $|0001111100\rangle$ term of $H^{\otimes 10}\,|111110000\rangle$ is

$$\frac{1}{2^5}(-1)^3 = \frac{-1}{32}.$$

7. Suppose we have two quantum circuits, C_1 and C_2, made up of unitary gates. If you input the state $|00000000\rangle$ into C_1, the output state is $\sum_{j=0}^{255}\alpha_j\,|j\rangle$. If you input the state $|00000001\rangle$ into C_2, the output state is $\sum_{k=0}^{255}\beta_k\,|k\rangle$. Now, suppose you make up a new quantum circuit C_3 by taking circuit C_1 and appending the conjugate transpose of the gates of circuit C_2 in reverse order. If you input $|00000000\rangle$ into C_3, and measure the output of C_3, what is the probability that you see $|00000001\rangle$?

Solution: We are asked to find the probability of observing $|00000001\rangle$ if we input $|00000000\rangle$ into C_3. This is the square of

$$\langle 00000001|\, V_1^\dagger V_2^\dagger V_3^\dagger \ldots V_n^\dagger U_m \ldots U_3 U_2 U_1\, |00000000\rangle\,.$$

However, we can divide this quantity as

$$\Big(\langle 00000001 |\, V_1^\dagger V_2^\dagger V_3^\dagger \dots V_n^\dagger \Big) \Big(U_m \dots U_3 U_2 U_1 \, |00000000\rangle \Big) ,$$

and using what we know about C_1 and C_2, we see that the answer is

$$\left| \Big(\sum_{k=0}^{255} \beta_k^* \langle k| \Big) \Big(\sum_{j=0}^{255} \alpha_j \, |j\rangle \Big) \right|^2 = \left| \sum_{j=0}^{255} \beta_j^* \alpha_j \right|^2$$

8. Give a unitary transformation on one qubit that takes $|0\rangle$ to $\frac{1}{\sqrt{2}}\big(|0\rangle + i\,|1\rangle \big)$ and takes $\frac{1}{\sqrt{2}}\big(|0\rangle + i\,|1\rangle \big)$ to $|0\rangle$.

   Solution:

   A unitary transformation that takes $|0\rangle$ to $\frac{1}{\sqrt{2}}\big(|0\rangle + i\,|1\rangle \big)$ will have the first column determined:

   $$\begin{pmatrix} \frac{1}{\sqrt{2}} & ? \\ \frac{i}{\sqrt{2}} & ? \end{pmatrix}.$$

   (Although because global phases don't affect the state, you could multiply this by any unit complex number.) Now, because it takes $\frac{1}{\sqrt{2}}\big(|0\rangle + i\,|1\rangle \big)$ to $|0\rangle$, we can see what the second column must be:

   $$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

   (again, up to a unit complex phase.) We can check that this is indeed unitary, and that's easy: each column is clearly length 1, and the inner product of the two columns is

   $$\frac{1}{2}(\langle 0| - i\,\langle 1|)(-i\,|0\rangle - |1\rangle) = 0.$$

9. Suppose that we have four parties, Alice, Bob, Cathy, and David. Alice and Cathy share a pair of qubits in one of the four Bell states

   $$\frac{1}{\sqrt{2}}\big(|01\rangle \pm |10\rangle \big), \quad \frac{1}{\sqrt{2}}\big(|00\rangle \pm |11\rangle \big),$$

   but they don't know what state it's in. Bob and David share a pair of qubits in the same Bell state. Suppose further that Alice and Bob share a qubit in the state $\frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle \big)$. Show that there is a protocol that lets Cathy and David end up sharing a pair of qubits in the state $\frac{1}{\sqrt{2}}\big(|01\rangle - |10\rangle \big)$.

   Justify your answer.

Solution: The idea is for Alice to teleport her state to Cathy, and Bob to teleport his state to David. Even though the participants don't know the state that Alice shares with Cathy, the fact that Bob shares the same state with David means that the errors this causes in the teleportation cancel each other out, and Cathy and David end up sharing a pair of qubits in the state $\frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big)$.

Alice and Cathy share a Bell state, which can be written as

$$\sigma_1^C \,|\psi_{EPR}\rangle_{AC}\,,$$

where σ_1 is either one of the three Pauli matrices or the identity. The superscript C represents that it is applied to Cathy's qubit [note that this really should be written $\mathrm{id}^B \otimes \sigma_1^C$, but we are leaving out implied identity matrices, as this notation gets cumbersome very quickly]. Alice and Cathy don't know what σ_1 is, but they know that it is the same as the σ_1 in the state Bob and David share, which is

$$\sigma_1^D \,|\psi_{EPR}\rangle_{BD}\,.$$

Now, if Alice uses

$$\sigma_1^C \,|\psi_{EPR}\rangle_{AC}$$

to teleport her qubit of $|\psi_{EPR}\rangle_{AB}$ to Cathy, what happens is that Cathy and Bob now hold $\sigma_1^C\sigma_2^C \,|\psi_{EPR}\rangle_{CB}$, where Cathy knows what σ_2^C is (because this depends on the results of Alice's measurement) but not σ_1. Now, Bob uses

$$\sigma_1^D \,|\psi_{EPR}\rangle_{BD}$$

to teleport his qubit of $\sigma_1^C\sigma_2^C \,|\psi_{EPR}\rangle_{CB}$ to David. Now, Cathy and David share

$$\sigma_1^C\sigma_2^C \otimes \sigma_1^D\sigma_3^D \,|\psi_{EPR}\rangle_{CD} = \pm\sigma_2^C\sigma_1^C \otimes \sigma_3^D\sigma_1^D \,|\psi_{EPR}\rangle_{CD}\,,$$

where we can interchange the two pairs of Pauli matrices because any two Pauli matrices either commute or anticommute. But since Cathy and David know σ_2 and σ_3, they can undo them, leaving

$$\pm\sigma_1^C \otimes \sigma_1^D \,|\psi_{EPR}\rangle_{CD}\,.$$

The ± 1 phase factor does not change the quantum state, and since the state $|\psi_{EPR}\rangle_{CD}$ is invariant when the same basis transformation is applied to both of its qubits, Cathy and David now share

$$\pm\,|\psi_{EPR}\rangle_{CD}\,,$$

which is what we wanted.