

Midterm Quiz

1. Atomic structure (8 points)

In Dirac theory, which two states of hydrogen in the $n=2$ manifold are degenerate?

Discuss which of the following effects lifts the degeneracy, and why? The Lamb shift, the Darwin term, the size of the proton, the mass of the proton, spin-orbit coupling.

The Lamb shift in the $n=2$ state is about 1 GHz. What is your estimate for the Lamb shift in the $n=1$ state?

How can you experimentally distinguish the Lamb shift from effects of the finite proton radius? Keep your answer short.

2. Lande g factor (12 points)

The spin S of an atom couples with the orbital angular momentum L to the total angular momentum J .

a. What is the g-factor g (or the magnetic moment μ) in the following cases: (4 points)

- i. $S=0$, L arbitrary
- ii. $L=0$, S arbitrary
- iii. $L=S$

(Use $g_e=2$ for the electron spin).

b. Find the magnetic moment μ (or the g factor) of the atom as a function of S , L , J . (8 points)

3. Concept questions about Stark and Zeeman effects (17 points)

a) DC Stark effect in a high B field ---- does it matter whether the E field is parallel or perpendicular to B? **(9 points)** For this question, address the following steps:

Assume a two-level atom without spin, ground state is an s state, excited state is p. Show the relevant states at zero and strong B field.

Identify which states are coupled by an E field parallel or perpendicular to the B field.

What is the leading order Zeeman shift of those states (try to get the correct sign)?

Write down the leading order result of the Stark effect for parallel and perpendicular E fields.

Compare the results for parallel and perpendicular E fields for zero and high magnetic fields.

At high magnetic fields, what other effect should be included?

b) For spinless positronium, how does the Zeeman splitting of the p levels differ from the case above? (2 points)

- c) For Rydberg states with zero angular momentum (s states) and large principal quantum number n , how does the electric polarizability scale with n ? Consider the dominant coupling to be to the p state with the same n . (3 points)
- d) Provide an estimated expression for the electric field strength above which you expect a linear Stark effect. How does it scale with n ? (3 points)

o Perturbation theory

$$|n\rangle = |n^{(0)}\rangle + \sum_{k \neq n} \frac{\langle k^{(0)} | V | n^{(0)} \rangle}{E_n^{(0)} - E_k^{(0)}} |k^{(0)}\rangle$$

$$E_n^{(0)} = E_n^{(0)} + \langle n^{(0)} | V | n^{(0)} \rangle + \sum_{k \neq n} \frac{|\langle k^{(0)} | V | n^{(0)} \rangle|^2}{E_n^{(0)} - E_k^{(0)}} \quad \text{2nd order}$$

4. Photon number and saturation (15 points)

- a) For a single mode cavity with mode volume V , what is photon number n necessary to saturate (i.e. realize a saturation parameter of one) an atomic transition with dipole matrix element d , and transition frequency ω ? Express your answer for n in terms of a ratio of two characteristic frequencies or rates. You can neglect factors of 2 or other numerical factors. (3 points)
- b) How does the photon number scale with d , V , ω ? (2 points)
- c) Is it possible to saturate the transition with a single photon? Why or why not? (2 points)
- d) (now don't neglect numerical factors) In a black body cavity of volume V , what average occupation number (near resonance) do you need to saturate the atomic transition? What is temperature of the black body radiation? (3 points)
- e) Give the expression for the spectral density of modes. What is the number of modes of the black body radiation which can resonantly excite the atom? How does this number scale with volume V , d and ω ? (3 points)
- f) Compare the result for the single mode cavity to the black body cavity and explain the difference. (2 points)

①

Dirac theory $P_{1/2}$ and $S_{1/2}$ are degenerate

Lamb shift lifts degeneracy:
 deviation from Coulomb potential
 affect S states

Size of proton: same

Dirac term, spin-orbit coupling: included in
 Dirac equation, these terms (together with
 relativistic kinetic energy) lead to
 the degeneracy

$$\text{Lamb shift} \propto |\psi(0)|^2 \propto n^{-3}$$

$$n=2 \quad 1 \text{ GHz} \Rightarrow n=1 \quad \underline{8 \text{ GHz}}$$

Finite proton radius leads to shift of S states
 as the Lamb shift. Both scale as n^{-3} .
 Can be distinguished by studying isotopes
 (H , D , positronium, muonic hydrogen)

(2) a) Lande g factor

(2)

i. $S=0, L$
 $g = g_L = 1$

ii. $S, L=0$ $g = g_S = 2$

iii. $L=S$

$$g = \frac{g_S + g_L}{2} = 1.5$$

(Follows from symmetry
 \vec{S} and \vec{L} and \vec{J} form isosceles triangle

b) $g_J = \frac{\vec{L} \cdot \vec{J}}{|\vec{J}|^2} + \frac{2 \vec{S} \cdot \vec{J}}{|\vec{J}|^2}$ $\vec{J} = \vec{S} + \vec{L}$

$$= \frac{L^2 + 2S^2 + 3\vec{L} \cdot \vec{S}}{|\vec{J}|^2}$$

$$\vec{L} \cdot \vec{S} = \frac{1}{2} (J^2 - L^2 - S^2)$$

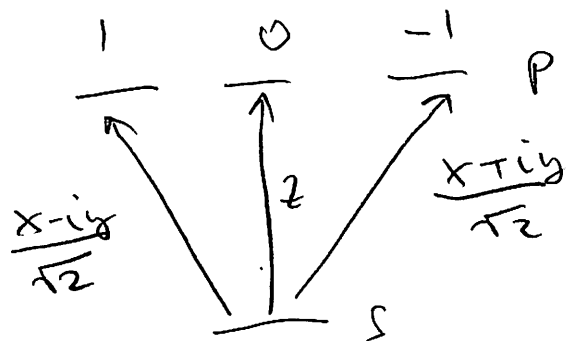
$$\Rightarrow g_J = \frac{1}{|\vec{J}|^2} \left(L^2 + 2S^2 + \frac{3}{2} (J^2 - L^2 - S^2) \right)$$

$$= 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}$$

3.) Stark and Zeeman effects

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a) Zero Field



parallel fields: $z, 0 \rightarrow 0$

perpendicular fields $x, y, 0 \rightarrow \pm 1$

linear Zeeman shift $\Delta = 1 \mu_B \cdot B$

$m = +1, \mu_z < 0$ (neg. charge of e^-)

$\Rightarrow -\vec{\mu} \cdot \vec{B}$ is positive \Rightarrow up shift



Stark effect for s state:

2nd order pert. theory

$$\Delta E_s = \sum_m \frac{|\langle P, m | \vec{E} \cdot \vec{r} | S \rangle|^2}{E_s - E_{P, m}} \quad M^2 \quad V = \vec{E} \cdot \vec{r}$$

$$\Delta E_{s, 11} = e^2 E^2 \frac{|\langle P, 0 | z | S \rangle|^2}{E_s - E_{P, 0}} = \Delta E_{s, 0} = - \frac{M^2 e^2 E^2}{4 \omega_{sp}}$$

independent of B field

\uparrow
zero B field
Stark shift

$$\Delta E_{s,\perp} = \Delta E_{s,0} \quad \text{at zero B field}$$

Finite B Field

$$\Delta E_{s,\perp} = -\mu_B^2 e^2 E^2 \frac{1}{2} \left(\frac{1}{\hbar \omega_{sp} + \Delta} + \frac{1}{\hbar \omega_{sp} - \Delta} \right)$$

$$\epsilon = \frac{\Delta}{\hbar \omega_{sp}} = \frac{\mu_B B}{\hbar \omega_{sp}}$$

$$\frac{1}{\hbar \omega_{sp} + \Delta} = \frac{1}{\hbar \omega_{sp}} \left(1 - \epsilon + \epsilon^2 \right) \quad \text{linear term cancels}$$

$$\Delta E_{s,\perp} = \Delta E_{s,0} (1 + \epsilon^2) > \Delta E_{s,||}$$

At high field, we also have a quadratic Zeeman shift (diamagnetic A^2 term).

b) spinless positronium

p state does not have a linear Zeeman term

e^+ , e^- have opposite contributions to $\vec{\mu}$

$$\text{or } \mathcal{H} = \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + \frac{(\mathbf{p} + e\mathbf{A})^2}{2m} \quad \text{has no linear term in } A$$

c) matrix element $\propto n^2$ (\approx size of orbit)

$$\left. \begin{array}{l} \text{S state } E_n = -\frac{Ryd}{(n-\delta_s)^2} \\ \text{P state } E_p = -\frac{Ryd}{(n-\delta_p)^2} \end{array} \right\} \text{Diff } \propto \frac{\delta_s - \delta_p}{n^3}$$

$$\text{Polarizability } \propto \frac{\mu^2}{\Delta E} \propto n^7$$

(5)

d) Linear Zeeman effect breaks down when

$$E_c M \sim \Delta E$$

$$E_c a_0 n^2 \sim \frac{Ryd}{n^3} (\delta_s - \delta_p) \quad Ryd \sim \frac{e^2}{a_0}$$

$$E \sim \frac{e}{a_0^2} \frac{(\delta_s - \delta_p)}{n^5} \propto n^{-5}$$

$\underbrace{\hspace{1cm}}$
atomic unit of E Field

4) a) single photon Rabi Freq ω_1

$$\hbar \omega_1 = E_1^2 V$$

$$E_1 = \sqrt{\frac{\hbar \omega_1}{V}} \quad E \text{ Field}$$

$$\omega_1 = d E_1 = d \sqrt{\frac{\hbar \omega_1}{V}} \quad d = \text{dipole matrix element}$$

Saturation:

$$\omega_{\text{Rabi}} = \omega_1 \sqrt{n} \approx \Gamma \quad \Rightarrow n = \left(\frac{\Gamma}{\omega_1} \right)^2$$

\uparrow
photons

$$b) \quad \Gamma \propto d^2, \quad \omega_1 \propto d \quad \Rightarrow n \propto d^2$$

$$n \propto V$$

$$\Gamma \propto \omega_1^3, \quad \omega_1 \propto \sqrt{V} \quad \Rightarrow n \propto V^5$$

c) Single photon can saturate for small volume V

d/

Rate of $(n+1)$

$$n = \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$\Rightarrow kT = h\nu / \ln 3$$

e)

$$= \underbrace{\frac{V_0 I}{\pi^2} \frac{\omega^2}{c^3}}_{\text{Energy density}} d\omega$$

$S(\omega)$ Spectral density

of modes which resonantly excite atom

$$\approx \rho(w_0) \Gamma = \frac{Vol}{\pi^2} \frac{w^2}{c^3} \Gamma$$

$\propto Vol, d^2, n^5$

f) In both cases is the number of photons per mode times the number of resonant modes the same