o5. Since a sandwich by itself is pretty boring, students from the school in Problem 3 are offered a choice of a drink (from among five different kinds), a sandwich, and a fruit (from among four different kinds). In how many ways may a student make a choice of the three items now?

Solution: $5 \cdot 15 \cdot 4 = 300$. Why do we multiply? Multiplying five by 15 is equivalent to adding 15, the number of sandwiches, once for each drink, giving us 75 combinations of drink and sandwich. For each such pair we have 4 choices of fruit, and we can either think of adding 75 fours or adding four 75s to get three hundred. Thus we multiply because multiplication is repeated addition. ∎

+13. Let us now return to Problem 7 and justify—or perhaps finish—our answer to the question about the number of functions from a three-element set to a 12-element set.

(a) How can you justify your answer in Problem 7 to the question "How many functions are there from a three element set (say $[3] = \{1, 2, 3\}$) to a twelve element set (say $[12]$)? "

Solution: For a function f, we can decide on $f(1)$ in twelve ways, then, given the decision we make for $f(1)$, we have 12 ways to decide on $f(2)$, and given the decisions we have made for $f(1)$ and $f(2)$, we have 12 ways to decide on $f(3)$. Therefore by the general product principle, there are $12^3 = 1728$ functions from $[3]$ to $[12]$, or from any three element set to any twelve element set. ∎

(b) Based on the examples you've seen so far, make a conjecture about how many functions there are from the set

$$[m] = \{1, 2, 3, \ldots, m\}$$

to $[n] = \{1, 2, 3, \ldots, n\}$ and prove it.

Solution: n^m. We can think of choosing a function f as making a sequence of m decisions, namely deciding on $f(1), f(2), \ldots, f(m)$. We have n choices for $f(1)$. Given the choices we have made for $f(1)$ through $f(i-1)$, we have n choices for $f(i)$. Thus by the general product principle we have a product of m terms each equal to n, which is n^m, as the number of ways to choose f. ∎

o19. Assuming $k \leq n$, in how many ways can we pass out k distinct pieces of fruit to n children if each child may get at most one? What is the number if $k > n$? Assume for both questions that we pass out all the fruit.

Solution: There are n choices for the child to whom the first piece of fruit goes, then $n-1$ choices for the second, and, in general, $n-i+1$ choices for the ith piece of fruit. By the general product principle, this gives us $\prod_{i=1}^{k} n-i+1$ ways to pass out the fruit. The number of ways to pass out the fruit is zero if $k > n$, because the problem says each child has to get at most one piece of fruit, and that all the fruit must be passed out. This is impossible if $k > n$, so there are zero ways to pass out the fruit. It is a nice coincidence that our formula for the first question gives 0 if $k > n$. ∎

•26. Digraphs of functions help us visualize the ideas of one-to-one functions and onto functions.

(a) What does the digraph of a one-to-one function (injection) from a finite set X to a finite set Y look like? (Look for a test somewhat similar to the one we described for when a digraph is the digraph of a function.)

Solution: A function from X to Y is one-to-one if, in its digraph, at most one arrow goes into each vertex representing a member of Y. (For a digraph to be the digraph of a function from X to Y, one and only one arrow must come out of each vertex representing a member of X.) ▮

(b) What does the digraph of an onto function look like?

Solution: A function is onto if, in its digraph, at least one arrow goes into each vertex representing a member of Y. (For a digraph to be the digraph of a function from X to Y, one and only one arrow must come out of each vertex representing a member of X.) ▮

(c) What does the digraph of a one-to-one and onto function from a finite set S to a set T look like?

Solution: A function from X to Y is one-to one and onto if, in its digraph, exactly one arrow goes into each vertex representing a member of Y. (For a digraph to be the digraph of a function from X to Y, one and only one arrow must come out of each vertex representing a member of X.) ▮

28. The *binary* representation of a number m is a list, or string, $a_1a_2\dots a_k$ of zeros and ones such that $m = a_1 2^{k-1} + a_2 2^{k-2} + \cdots + a_k 2^0$. Describe a bijection between the binary representations of the integers between 0 and $2^n - 1$ and the subsets of an n-element set. What does this tell you about the number of subsets of the n-element set $[n]$?

Solution: The sequence $a_1a_2\dots a_k$ corresponds to the set of i such that $a_i = 1$. This is a bijection because each sequence gives a subset of $[n]$, and each subset of $[n]$ is the set of places where exactly one sequence has its ones. Since there are 2^n integers which are between 0 and $2^n - 1$, and they correspond to sequences of length n (notice, we have another bijection, the one between a number and its binary representation), there are 2^n subsets of an n-element set. ▮

Supp ➔1. Remember that we can write n as a sum of n ones. How many plus signs do we use? In how many ways may we write n as a sum of a list of k positive numbers? Such a list is called a *composition* of n into k parts.

Solution: We use $n-1$ plus signs. Write down such a sum and choose $k-1$ of the plus signs. Then each string of ones and plusses between two chosen plus signs, before the first chosen plus sign or after the last chosen one corresponds to a part of a composition of n. Thus the number of compositions of n with k parts is the number of ways to choose the $k-1$ places, which is $\binom{n-1}{k-1}$. ▮