Local Limit Theorems for Convolution Powers of Complex Functions on $\mathbb{Z}^d$

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November 27, 2021

Abstract

Denote by $\ell^1(\mathbb{Z}^d)$ the set of functions $\phi : \mathbb{Z}^d \to \mathbb{C}$ for which

$$\|\phi\|_1 := \sum_{x\in\mathbb{Z}^d} |\phi(x)| < \infty.$$

For a fixed $\phi \in \ell^1(\mathbb{Z}^d)$, we define the convolution powers $\phi^{(n)} \in \ell^1(\mathbb{Z}^d)$ of ϕ iteratively by putting $\phi^{(1)} = \phi$ and, for each integer $n \geq 2$,

$$\phi^{(n)}(x) = \sum_{y\in\mathbb{Z}^d} \phi^{(n-1)}(x-y)\phi(y)$$

for $x \in \mathbb{Z}^d$. Motivated by its central importance in random walk theory [13, 21] and its applications to data smoothing algorithms [8, 19] and numerical solutions to partial differential equations [4, 24, 25], we are interested in the asymptotic behavior of $\phi^{(n)}(x)$ as $n \to \infty$. Following the articles [5], [15], and [16], our goal in this article is to broaden the class of functions $\phi \in \ell^1(\mathbb{Z})$ for which it is possible to obtain "simple" pointwise descriptions of $\phi^{(n)}(x)$, for sufficiently large n, in the form of local limit theorems. To better understand this goal, let us first discuss some background whose origins are rooted in probability.

1 Introduction

Denote by $\ell^1(\mathbb{Z}^d)$ the set of functions $\phi : \mathbb{Z}^d \to \mathbb{C}$ for which

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In the case that ϕ is a probability distribution on $\mathbb{Z}^d$, i.e., $\phi \geq 0$ and $\|\phi\|_1 = \sum_{x\in\mathbb{Z}^d} \phi(x) = 1$, there is a natural Markov process on $\mathbb{Z}^d$ whose nth-step transition kernels are given by $k_n(x,y) = \phi^{(n)}(y-x)$; we call this process the **random walk on $\mathbb{Z}^d$ driven by ϕ**. In particular, $\phi^{(n)}(x) = k_n(0,x)$ represents the probability that the "random walker" starting at the origin will be at position x after n steps. The study of random walks on $\mathbb{Z}^d$ has a long and storied history and we encourage the reader to take a look at the wonderful book of F. Spitzer [21] for an account (see also [13]). In the case that the random walk is aperiodic, irreducible and of finite range, the classical local (central) limit theorem states that

$$\phi^{(n)}(x) = n^{-d/2} G_\phi\left(n^{-1/2}(x - n\alpha_\phi)\right) + o(n^{-d/2}) \tag{1}$$

uniformly for $x \in \mathbb{Z}^d$ where $\alpha_\phi \in \mathbb{R}^d$ denotes the mean of ϕ and G_ϕ is the generalized Gaussian given by

$$G_\phi(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(C_\phi)}} \exp\left(-\frac{x \cdot C_\phi^{-1} x}{2}\right) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-P_\phi(\xi)} e^{-ix\cdot\xi}\, d\xi$$

where $\cdot$ denotes the dot product, C_ϕ is the symmetric and positive-definite covariance matrix of ϕ, and $P_\phi(\xi) = \xi \cdot (C_\phi \xi)$ is its associated positive-definite homogeneous second-order polynomial [13, 16, 21]. Under the additional assumption that ϕ is symmetric, $\alpha_\phi = 0$ and so (1) yields the two-sided estimate

$$C_1 n^{-d/2} \leq \phi^{(n)}(0) = k_n(x,x) \leq C_2 n^{-d/2}$$

for all $n \in \mathbb{N}_+ := \{1,2,3,\dots\}$ and $x \in \mathbb{Z}^d$; here, C_1 and C_2 are positive constants. This so-called on-diagonal estimate describes the return probabilities of the random walk and was used by G. Pólya to establish the dichotomy of recurrence/transience of simple random walk [14]. The hypotheses that a probability distribution ϕ is symmetric, aperiodic, irreducible, and of finite range can be weakened significantly and still a local limit theorem will hold. For example, if one assumes only that a probability distribution ϕ has finite second moments and is genuinely d-dimensional[1], then

$$\phi^{(n)}(x) = n^{-d/2}\Theta(n,x) G_\phi\left(n^{-1/2}(x - n\alpha_\phi)\right) + o(n^{-d/2}) \tag{2}$$

uniformly for $x \in \mathbb{Z}^d$ where $\Theta(n,x)$ is a "support function" which characterizes the periodicity of the random walk. For a proof of the local limit theorem (2), we refer the reader to Subsection 7.6 of [16]. Also, there is a rich theory for improving the error $o(n^{-d/2})$ in (2) which is nicely presented in [13].

In taking our discussion beyond the realm of probability, it is helpful to introduce some basic objects that play a role in the theory. For a given $\phi \in \ell^1(\mathbb{Z}^d)$, we define its Fourier transform $\widehat{\phi}$ by the trigonometric series

$$\widehat{\phi}(\xi) = \sum_{x\in\mathbb{Z}^d} \phi(x) e^{ix\cdot\xi}$$

which is everywhere uniformly convergent and has $|\widehat{\phi}(\xi)| \leq \|\phi\|_1$ for all $\xi \in \mathbb{R}^d$. As in [3] and [16], we shall focus on the subspace $\mathcal{S}_d$ of $\ell^1(\mathbb{Z}^d)$ consisting of those $\phi \in \ell^1(\mathbb{Z}^d)$ for which

$$\|x^\beta \phi\|_1 = \sum_{x\in\mathbb{Z}^d} \left|x^\beta \phi(x)\right| < \infty$$

[1] In the language of F. Spitzer, ϕ is said to be **genuinely d-dimensional** if it is not supported in any affine hyperplane of $\mathbb{Z}^d$ [21].

for each $\beta = (\beta_1, \beta_2, \dots, \beta_d) \in \mathbb{N}^d$ where $x^\beta := (x_1)^{\beta_1}(x_2)^{\beta_2}\cdots(x_d)^{\beta_d}$ for $x = (x_1, x_2, \dots, x_d) \in \mathbb{Z}^d$. We observe that $\mathcal{S}_d$ contains all finitely supported functions in $\phi \in \ell^1(\mathbb{Z}^d)$. It is easy to see that, for each $\phi \in \mathcal{S}_d$, $\widehat{\phi} \in C^\infty(\mathbb{R}^d)$ and, in fact, $\widehat{\phi}$ is analytic whenever ϕ is finitely supported. As discussed in [3–5, 15, 16, 24], the asymptotic behavior of $\phi^{(n)}$ is characterized by the local behavior of $\widehat{\phi}$ near points in $\mathbb{T}^d := (-\pi, \pi]^d$ at which $|\widehat{\phi}|$ is maximized. The fact is seen evident through the Fourier transform identity

$$\phi^{(n)}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \tag{3}$$

which holds for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. For simplicity, we shall assume that $\phi \in \mathcal{S}_d$ is normalized so that $\sup_\xi |\widehat{\phi}(\xi)| = 1$ and with this we define

$$\Omega(\phi) = \left\{\xi \in \mathbb{T}^d : \left|\widehat{\phi}(\xi)\right| = 1\right\}. \tag{4}$$

Remark 1. In the case that $\phi \geq 0$ is a probability distribution, the normalization $\sup|\widehat{\phi}| = 1$ is automatic. In this case, recognizing $\mathbb{T}^d$ as the d-dimensional torus group, $\Omega(\phi)$ is a subgroup of $\mathbb{T}^d$ and the support function $\Theta(n, x)$ appearing in (2) can be expressed in terms of the elements $\xi \in \Omega(\phi)$ [16]. The additional hypotheses that the random walk driven by ϕ is aperiodic and irreducible are equivalent to the hypothesis that $\Omega(\phi) = \{0\}$ and, in this case, $\Theta(n, x) \equiv 1$ making (1) a special case of (2).

For each $\xi_0 \in \Omega(\phi)$, consider $\Gamma_{\xi_0} : \mathcal{U} \to \mathbb{C}$ defined by

$$\Gamma_{\xi_0}(\xi) = \log\left(\frac{\widehat{\phi}(\xi + \xi_0)}{\widehat{\phi}(\xi_0)}\right) \tag{5}$$

where log is the principal branch of the logarithm and $\mathcal{U} \subseteq \mathbb{R}^d$ is an open convex neighborhood of 0 which is small enough to ensure that log is continuous on $\{\widehat{\phi}(\xi + \xi_0)/\widehat{\phi}(\xi_0)\}$ as ξ varies in $\mathcal{U}$. Our supposition that $\phi \in \mathcal{S}_d$ guarantees that $\Gamma_{\xi_0} \in C^\infty(\mathcal{U})$ and so it makes sense to consider its Taylor expansion at 0. It is the nature of these Taylor expansions (and hence the nature of Γ_{ξ_0}) that determines the asymptotic nature of $\phi^{(n)}$.

The vast majority of existing theory on the asymptotic behavior of convolution powers pertains to the 1-dimensional setting, i.e., $d = 1$. In fact, determining the asymptotic behavior of $\phi^{(n)}$ where ϕ is finitely supported on $\mathbb{Z}$ and takes on only real values is known as de Forest's problem and dates back to its initial investigation by Erastus L. de Forest in the nineteenth century driven by de Forest's interest in data smoothing [5, 23]. This study was continued by I. J. Schoenberg and T. N. E. Greville [8, 19], both of whom proved local limit theorems under hypotheses (stronger than) described below. Tied to advancements in scientific computing, the general problem of understanding the asymptotic behavior of the convolution powers of a complex function on $\mathbb{Z}$ was reinvigorated in the second half of the twentieth century by its appearance in numerical solution algorithms for partial differential equations; we encourage the reader to look at the excellent survey of V. Thomée [25] for an account of this story (see also Subsection 5.2 of [5]). From the perspective of numerical PDEs, one is often concerned with the so-called max-norm stability[2] of operators $L_\phi : L^\infty(\mathbb{R}) \to L^\infty(\mathbb{R})$ defined by $L_\phi f(x) = \sum_y \phi(y) f(x + hy)$ for $f \in L^\infty(\mathbb{R})$ where $\phi \in \ell^1(\mathbb{Z})$ and h is a fixed positive parameter. In 1965, V. Thomée characterized

[2]We say that L_ϕ **stable in the maximum norm** if, for some $C > 0$, $\|L_\phi^n f\|_{L^\infty(\mathbb{R})} \leq C\|f\|_{L^\infty(\mathbb{R})}$ for all $n \in \mathbb{N}_+$ and $f \in L^\infty(\mathbb{R})$. Upon noting that $L_\phi^n = L_{\phi^{(n)}}$, it is easily seen that max-norm stability is equivalent to the property that $\sup_n \|\phi^{(n)}\|_1 < \infty$. Recognizing $\ell^1(\mathbb{Z}^d)$ as a Banach algebra equipped with the convolution product, this property is referred to as power boundedness [6, 20].

max-norm stability for operators L_ϕ associated to finitely supported functions $\phi \in \ell^1(\mathbb{Z})$ [24, Theorem 1]. Though this characterization is not directly relevant to our present goals, in the course of his proof, Thomée introduced the following definition which was found essential to the theory in [15].

Definition 1.1. *Let $\phi : \mathbb{Z} \to \mathbb{C}$ be such that $\sup_\xi |\widehat{\phi}| = 1$ and let $\xi_0 \in \Omega(\phi)$.*

1. *We say that ξ_0 is of Type 1 (or Type γ) of order m for $\widehat{\phi}$ if there is an even natural number $m = m_{\xi_0} \geq 2$, a real number α_{ξ_0}, and a complex number β_{ξ_0} with $\operatorname{Re}(\beta_{\xi_0}) > 0$ for which*
$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0}\xi - \beta_{\xi_0}\xi^{m_{\xi_0}} + o\left(\xi^{m_{\xi_0}}\right)$$
as $\xi \to 0$.
2. *We say that ξ_0 is of Type 2 (or Type β) of order m for $\widehat{\phi}$ if there is a natural number $m = m_{\xi_0} \in \{2, 3, \dots,\}$, a real number α_{ξ_0}, a real-valued polynomial $q_{\xi_0}(\xi)$ with $\beta_{\xi_0} = iq_{\xi_0}(0) \neq 0$, an even number $k_{\xi_0} > m_{\xi_0}$, and a positive number γ_{ξ_0} for which*
$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0}\xi - iq_{\xi_0}(\xi)\xi^{m_{\xi_0}} - \gamma_{\xi_0}\xi^{k_{\xi_0}} + o\left(\xi^{k_{\xi_0}}\right)$$
as $\xi \to 0$.

The distinction made by the definition above essentially concerns the nature of the lowest order (non-linear) polynomial appearing in the Taylor expansion for Γ_{ξ_0}. If the polynomial has a coefficient β_{ξ_0} with strictly positive real part, ξ_0 is of Type 1 for $\widehat{\phi}$ and, if the coefficient $\beta_{\xi_0} = ip_{\xi_0}(0)$ is non-zero and purely imaginary, ξ_0 is of Type 2 for $\widehat{\phi}$. As observed by Thomée (see Section 3 of [24]) , for a finitely supported $\phi : \mathbb{Z} \to \mathbb{C}$ which is supported on more than one point in $\mathbb{Z}$ and has $\sup_\xi |\widehat{\phi}| = 1$, $\Omega(\phi)$ is necessarily a finite set and every element $\xi \in \Omega(\phi)$ is a point of Type 1 or Type 2 for $\widehat{\phi}$. As a consequence and in view of (3), understanding the asymptotic behavior of $\phi^{(n)}(x)$ reduces to the problem of analyzing the nature of the contributions from points of Type 1 and Type 2 to $\widehat{\phi}(\xi)^n$ which we explain as follows.

For illustrative purposes, let's assume that $\Omega(\phi)$ consists only of a single element, i.e., $\Omega(\phi) = \{\xi_0\}$. If ξ_0 is a point of Type 1 for $\widehat{\phi}$ with associated even integer $m = m_{\xi_0}$, real number α_{ξ_0} and complex number $\beta = \beta_{\xi_0}$ with $\operatorname{Re}(\beta) > 0$, Theorem 2.3 of [5] guarantees that

$$\phi^{(n)}(x) = \widehat{\phi}(\xi_0)^n e^{-ix\xi_0} H^n_{m,\beta}(x - n\alpha) + o(n^{-1/m}) \tag{6}$$

uniformly for $x \in \mathbb{Z}$ where $H^{(\cdot)}_{m,\beta}(\cdot)$ is defined by

$$H^t_{m,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-t\beta\xi^m} e^{-ix\xi}\, d\xi. \tag{7}$$

$t > 0$ and $x \in \mathbb{R}$. Because $\operatorname{Re}(\beta) > 0$, the integral defining $H_{m,\beta}$ converges absolutely and uniformly for $x \in \mathbb{R}$. In fact, $H^t_{m,\beta}(\cdot)$ is a Schwartz function (for each $t > 0$) and is a fundamental solution to the heat-type equation

$$\frac{\partial u}{\partial t} + i^m\beta\frac{\partial^m u}{\partial x^m} = 0.$$

The function $H_{m,\beta}$ also enjoys the property

$$H^t_{m,\beta}(x) = t^{-1/m} H^1_{m,\beta}(t^{-1/m}x) \tag{8}$$

for all $t > 0$ and $x \in \mathbb{R}$ and so (6) can be equivalently stated as

$$\phi^{(n)}(x) = n^{-1/m}\widehat{\phi}(\xi_0)^n e^{ix\xi_0} H^1_{m,\beta}(n^{-1/m}(x - n\alpha)) + o(n^{-1/m})$$

uniformly for $x \in \mathbb{Z}$.

Place Example 1 Here

In the case that $\Omega(\phi) = \{\xi_0\}$ and ξ_0 is a point of Type 2 for $\widehat{\phi}$ with integer $m = m_{\xi_0}$, real number $\alpha = \alpha_{\xi_0}$ and purely imaginary $\beta = \beta_{\xi_0} = iq_{\xi_0}(0)$, Theorem 1.3 of [15] guarantees that, for any compact set K,

$$\phi^{(n)}(x) = \widehat{\phi}(\xi_0)^n e^{-ix\xi_0} H^n_{m,\beta}(x - n\alpha) + o(n^{-1/m}) \tag{9}$$

uniformly for $x \in n\alpha + n^{1/m}K$ where $H^{(\cdot)}_{m,\beta}(\cdot)$ is given by

$$H^t_{m,\beta}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-t\beta\xi^m} e^{-ix\xi}\, d\xi = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itq\xi^m} e^{-ix\xi}\, d\xi \tag{10}$$

for $t > 0$ and $x \in \mathbb{R}$ where $\beta = iq$ for the real non-zero $q = q_{\xi_0}(0)$. In contrast to (7), the integrand $e^{-itq\xi^m}e^{-ix\xi}$ is not absolutely integrable and so we interpret the integral in (10) as a sum of the improper Riemann integrals

$$\int_{-\infty}^0 e^{-itq\xi^m} e^{-ix\xi}\, d\xi \qquad \text{and} \qquad \int_0^\infty e^{-itq\xi^m} e^{-ix\xi}\, d\xi$$

which converge on account of the highly oscillatory nature of the integrand as $\xi \to \pm\infty$. Despite the differing interpretations of the integrals in (7) and (10), the function $H_{m,\beta} = H_{m,iq}$ is smooth and also satisfies the property (8). We remark that, when $m = 2$, $H^t_{2,\beta} = H^t_{2,iq}$ is the heat kernel evaluated at imaginary time $\tau = itq$, i.e.,

$$H^t_{2,iq}(x) = \frac{1}{\sqrt{4\pi iqt}} \exp\left(-\frac{x^2}{4iqt}\right)$$

for $t > 0$ and $x \in \mathbb{R}^d$ and, when $m = 3$, $H^t_{3,\beta} = H^t_{3,iq}$ is a scaled Airy function.

Put Example 2 here

Remark 2. In comparing the local limit theorems (6) and (9), we observe that (9) is guaranteed to hold uniformly for $x \in n\alpha + n^{1/m}K$ where K is a compact set whereas (6) is valid uniformly for $x \in \mathbb{Z}$. This limitation of (9) is seen to be necessary in light of the fact that $\phi^{(n)}$ is necessarily finitely supported whenever ϕ is and (in the case that $m = 2$) the heat kernel $H^n_{2,iq}(y)$ has constant modulus $(2\pi qt)^{-1/2}$ for all $y \in \mathbb{R}$. An analogous limitation will also play a role in our d-dimensional theory. We remark that, when $m > 2$, the local limit theorem (9) does hold uniformly for $x \in \mathbb{Z}$ (see Theorem 1.2 of [15]). this needs to be rewritten and reference the preceding example

Beyond the case that $\Omega(\phi) = \{\xi_0\}$, i.e., in cases where $|\widehat{\phi}(\xi)|$ is maximized at more than one point, local limit theorems for $\phi^{(n)}$ involve sums of the so-called attractors $H_{m,\beta}$ of the forms (7) and (10) according to whether the points $\xi \in \Omega(\phi)$ are of Type 1 or Type 2 for $\widehat{\phi}$. For example, if $\Omega(\phi) = \{\xi_1, \xi_2, \dots, \xi_l\}$ and, for each $k = 1, 2, \dots, l$, ξ_k is a point of Type 1 for $\widehat{\phi}$ with $m_k = m_{\xi_k}$, $\alpha_k = \alpha_{\xi_k}$, and $\beta_k = \beta_{\xi_k}$, Theorem 2.3 of [5] (see also Theorem 1.2 of [15] and Theorem 1.5 of [16]) guarantees that

$$\phi^{(n)}(x) = \sum_{j=1}^{A} \widehat{\phi}(\xi_{k_j})^n e^{-ix\xi_{k_j}} H^n_{m,\beta_{k_j}}(x - n\alpha_{k_j}) + o(n^{-1/m}) \tag{11}$$

uniformly for $x \in \mathbb{Z}$ where $m = \max_{k=1,2,\dots,l} m_k$ and the points $\xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_A} \in \Omega(\phi)$ are those for which $m_{k_j} = m$ for all $j = 1, 2, \dots, A$. In fact, according to Theorem 1.2 of [16], (11) also holds in the case that the points $\xi_1,\xi_2,\dots,\xi_l\in\Omega(\phi)$ are of Type 1 or Type 2 (which is always true when ϕ is finitely supported) as long as $m>2$. Thanks to Thomée's key observations, proving a local limit theorem robust enough to handle the entire class of finitely supported functions $\phi:\mathbb{Z}\to\mathbb{C}$ required the consideration that $\Omega(\phi)$ can consist of more than one point *and* that its element(s) can be of both Type 1 and Type 2 for $\widehat{\phi}$. The results of I. J. Schoenberg [19], T. N. E. Greville [8], and P. Diaconis and L. Saloff-Coste [5] treated this theory for (proper) subsets of these possibilities (see also the work of K. Hochberg [11] who proved a generalized central limit theorem for the convolution powers of a signed Borel measure satisfying analogous hypotheses). Though we will not state it here due to its complexity, Theorem 1.3 of [15] (a local limit theorem) provides a complete description of the asymptotic behaviors of $\phi^{(n)}$ when ϕ is any (normalized) finitely supported complex-valued function on $\mathbb{Z}$. When applied to real-valued and finitely supported functions on $\mathbb{Z}$, it represents a full solution to de Forest's problem.

We now move beyond one dimension and consider a general $\phi\in S_d\subseteq\ell^1(\mathbb{Z}^d)$ for which $\sup_\xi|\widehat{\phi}(\xi)|=1$. In light of the natural complexity of $\mathbb{R}^d$, a categorization of the possible behaviors of Γ_{ξ_0} along the lines of Definition 1.1, even for finitely supported functions, appears to be a very difficult task (if it is possible at all). Despite this drawback, we still seek a reasonable generalization of Definition 1.1 which will capture much of the behaviors commonly seen for $\phi\in\mathcal{S}_d$. To produce such a generalization, we must first consider what will replace the monomials ξ^m appearing as the lowest order (non-linear) terms in the expansion for Γ_{ξ_0} in Definition 1.1.

First, let E be a (linear) endomorphism of $\mathbb{R}^d$ (written $E\in\operatorname{End}(\mathbb{R}^d)$) and define

$$T_t=t^E=\exp(\log(t)E)=\sum_{k=0}^{\infty}\frac{(\ln(t))^k}{k!}E^k$$

for each $t>0$. The map $t\mapsto T_t=t^E$ is a Lie group homomorphism from the positive real numbers (under multiplication) to the general linear group, $\mathrm{Gl}(\mathbb{R}^d)$, and its image is a one-parameter subgroup of $\mathrm{Gl}(\mathbb{R}^d)$ which we denote by $\{T_t\}$ or $\{t^E\}$. This one parameter group $\{t^E\}$ is said to be **generated by** E and E is said to be **the generator of** $\{t^E\}$. In fact, all such one-parameter subgroups $\{T_t\}$ of $\mathrm{Gl}(\mathbb{R}^d)$ are of this form. In Subsection A.1 of the Appendix, we have collected some useful facts about these one-parameter groups. For a function $P:\mathbb{R}^d\to\mathbb{C}$, we say that P **is homogeneous with respect to** $\{t^E\}$ if

$$P(t^E\xi)=tP(\xi)\tag{12}$$

for all $t>0$ and $\xi\in\mathbb{R}^d$. By an abuse of language, we also say that P **is homogeneous with respect to** E. The set of all $E\in\operatorname{End}(\mathbb{R}^d)$ for which (12) is satisfied is said to be the **exponent set of** P and will be denoted by $\operatorname{Exp}(P)$. When P is real valued, we say that P is **positive-definite** provided that $P\geq 0$ and $P(\xi)=0$ only when $\xi=0$. Also, the set $S_P=\{\eta\in\mathbb{R}^d:P(\eta)=1\}$ is called the **unital level set of** P.

Definition 1.2. *Let $P:\mathbb{R}^d\to\mathbb{R}$ be continuous and positive-definite. If $\operatorname{Exp}(P)$ is non-empty and S_P is compact, we say that P is a positive homogeneous function.*

Example 1. *Given any $\nu>0$, consider the function $\mathbb{R}^d\ni\xi\mapsto|\xi|^\nu$ where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$. It is not hard to see that $|\xi|^\nu$ is continuous, positive-definite and $S_{|\cdot|^\nu}$ is precisely the unit sphere $\mathbb{S}\subseteq\mathbb{R}^d$, which is compact. Consider $E_{1/\nu}:=(1/\nu)I$ where $I\in\mathrm{Gl}(\mathbb{R}^d)$ is the identity transformation. Observe that*

$$|t^{E_{1/\nu}}\xi|^\nu=|t^{1/\nu}\xi|^\nu=t|\xi|^\nu$$

for all $t > 0$ and $\xi \in \mathbb{R}^d$. Thus $E_{1/\nu} \in \mathrm{Exp}(|\cdot|^\nu)$ and so $\xi \mapsto |\xi|^\nu$ is a positive homogeneous function. We remark that

$$\mathrm{Exp}(|\cdot|^\nu) = E_{1/\nu} + \mathfrak{o}(d)$$

where $\mathfrak{o}(d)$ is the Lie algebra of the orthogonal group $\mathrm{O}(\mathbb{R}^d)$ and is characterized by the set of skew symmetric matrices. Correspondingly, $\mathrm{End}(|\cdot|^\nu)$ is not a singleton when $d > 1$. In the context of one dimension, our argument above ensures that, for each even natural number $m \in \mathbb{N}_+$, the monomial ξ^m is positive homogeneous. Also, for each $n \in \mathbb{N}_+$, $\mathbb{R} \ni \xi \mapsto |\xi^n|$ is positive homogeneous. $\triangle$

We refer the reader to the recent article [3] which develops a theory of positive homogeneous functions and presents many examples and pictures. Before we introduce a central class of positive homogeneous functions that will be of interest for us, it is helpful to state the following characterization of positive-homogeneous functions which appears as Proposition 1.1. of [3] and is proved in Section 2 therein.

Proposition 1.3. *Let $P : \mathbb{R}^d \to \mathbb{R}$ be continuous, positive-definite, and have non-empty exponent set $\mathrm{Exp}(P)$. Then the following are equivalent.*

1. *P is positive homogeneous.*
2. *There exists $M > 0$ for which $P(\xi) > 1$ for all $|\xi| \geq M$.*
3. *For each $E \in \mathrm{Exp}(P)$, $\{t^E\}$ is **contracting** in the sense that*

   $$\lim_{t \to 0} \|t^E\| = 0$$

   where $\|\cdot\|$ is the operator norm on $\mathrm{End}(\mathbb{R}^d)$.
4. *There exists $E \in \mathrm{Exp}(P)$ for which $\{t^E\}$ is contracting.*
5. *We have $\lim_{\xi \to \infty} P(\xi) = \infty$.*

Armed with this proposition, we now introduce the class of semi-elliptic polynomials.

Example 2. *Let $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}_+^d$ be a d-tuple of positive integers and, for a multindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$, define*

$$|\alpha : \mathbf{m}| = \sum_{k=1}^{d} \frac{\alpha_k}{m_k}.$$

In the language of L. Hörmander [12], a polynomial $P : \mathbb{R}^d \to \mathbb{C}$ is said to be semi-elliptic if it vanishes only at the origin and can be written in the form

$$P(\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha \xi^\alpha \tag{13}$$

for some $\mathbf{m} \in \mathbb{N}_+^d$ where $\{a_\alpha\} \subseteq \mathbb{C}$ and $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_d^{\alpha_d}$ for each multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$ and $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$. Given such a polynomial $P(\xi)$, we consider $E = E_{1/\mathbf{m}} \in \mathrm{End}(\mathbb{R}^d)$ with standard matrix representation $\mathrm{diag}(m_1^{-1}, m_2^{-1}, \dots, m_d^{-1})$. We observe that

$$P(t^E\xi) = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha (t^{1/m_1}\xi_1)^{\alpha_1}(t^{1/m_2}\xi_2)^{\alpha_2}\cdots(t^{1/m_d}\xi_d)^{\alpha_d} = \sum_{|\alpha:\mathbf{m}|=1} a_\alpha t^{|\alpha:\mathbf{m}|}\xi^\alpha = tP(\xi)$$

for all $t > 0$ and $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$. Thus, $E \in \mathrm{Exp}(P)$. It is clear that $\{t^E\}$ is contracting and so, by virtue of Proposition 1.3, we obtain the following result:

If a semi-elliptic polynomial of the form (13) is positive-definite, it is positive homogeneous.

For two concrete examples, consider the polynomials P_1 and P_2 on $\mathbb{R}^2$ given by

$$P_1(\xi_1,\xi_2)=\xi_1^2+\xi_1\xi_2^2+\xi_2^4 \qquad \text{and} \qquad P_2(\xi_1,\xi_2)=\xi_1^6+\xi_2^4.$$

It is easy to see that both polynomials are of the form (13) *with* $\mathbf{m}=(2,4)$ *for* P_1 *and* $\mathbf{m}=(6,4)$ *for* P_2*. A routine verification shows that both are, in fact, positive-definite and therefore positive homogeneous. As illustrated in these examples, a real-valued polynomial of the form* (13) *can only be positive-definite (and hence positive homogeneous) provided that* $\mathbf{m}$ *is a d-tuple of positive even integers, i.e.,* $\mathbf{m}=2\mathbf{n}$*, for* $\mathbf{n}\in\mathbb{N}_+^d$*.* $\triangle$

Remark 3. In [16], a positive-homogeneous polynomial is a complex-valued polynomial P for which $R(\xi)=\mathrm{Re}(P(\xi))$ is positive-definite and $\mathrm{Exp}(P)$ contains an endomorphism whose spectrum is real. By virtue of Proposition 2.2 of [16], every such positive-homogeneous polynomial P is semi-elliptic in some coordinate system and from this it follows that R is a positive homogeneous function in the sense of the present article.

Let P be a positive homogeneous function and consider the set $\mathrm{Sym}(P)$ consisting of those $O\in\mathrm{End}(\mathbb{R}^d)$ for which

$$P(O\xi)=P(\xi)$$

for all $\xi\in\mathbb{R}^d$. In the case that P is the Euclidean norm $\mathbb{R}^d\ni\xi\mapsto|\xi|$, it is easy to see that $\mathrm{Sym}(|\cdot|)$ coincides with the orthogonal group, $\mathrm{O}(\mathbb{R}^d)$. As the following proposition shows, this is archetypal of the situation in general. We note that this proposition appears in Section 2.1 of [3]; we give a proof here for completeness.

Proposition 1.4. *For a positive homogeneous function* $P:\mathbb{R}^d\to\mathbb{R}$*,* $\mathrm{Sym}(P)$ *is a compact subgroup of the general linear group,* $\mathrm{Gl}(\mathbb{R}^d)$*.*

Proof. In view of the fact that P is positive-definite, it is straightforward to see that $\mathrm{Sym}(P)$ is a subgroup of $\mathrm{Gl}(\mathbb{R}^d)$. We shall prove that $\mathrm{Sym}(P)$ is compact by showing it is both closed and bounded. To see that $\mathrm{Sym}(P)$ is closed, let $\{O_n\}\subseteq\mathrm{Sym}(P)$ be a sequence which converges to $O\in\mathrm{Gl}(\mathbb{R}^d)$. By virtue of the continuity of P, for each $\xi\in\mathbb{R}^d$, we have

$$P(O\xi)=\lim_{n\to\infty}P(O_n\xi)=\lim_{n\to\infty}P(\xi)=P(\xi)$$

which proves that $O\in\mathrm{Sym}(P)$ and therefore $\mathrm{Sym}(P)$ is closed. To see that $\mathrm{Sym}(P)$ is bounded, we assume, to reach a contradiction, that there exists a sequence $\{O_n\}\subseteq\mathrm{Sym}(P)$ and a sequence $\{\xi_n\}$ of elements on the unit sphere $\mathbb{S}$ of $\mathbb{R}^d$ for which $\lim_{n\to\infty}|O_n\xi_n|=\infty$. By virtue of Item 5 of Proposition 1.3, we have

$$\lim_{n\to\infty}P(\xi_n)=\lim_{n\to\infty}P(O_n\xi_n)=\infty$$

but this is impossible for we know that P is continuous on $\mathbb{R}^d$ and therefore bounded on $\mathbb{S}$. $\square$

In view of the preceding proposition, we shall refer to $\mathrm{Sym}(P)$ as the **symmetric group of P.** The fact that $\mathrm{Sym}(P)$ is compact allows us to establish the following important invariant of $\mathrm{Exp}(P)$.

Proposition 1.5. *Let P be a positive homogeneous function and let $E_1,E_2\in\mathrm{Exp}(P)$. Then*

$$\operatorname{tr}E_1=\operatorname{tr}E_2>0.$$

Proof. Suppose that, for $E\in\mathrm{End}(\mathbb{R}^d)$, $\{t^E\}$ is contracting. Then, by virtue of the continuity of the determinant map and the fact that $\det(t^E)=t^{\operatorname{tr}E}$, we find that

$$\lim_{t\to0}t^{\operatorname{tr}E}=\lim_{t\to0}\det(t^E)=0$$

and so $\operatorname{tr} E > 0$. Thus, in view of Proposition 1.3, every element of $\operatorname{Exp}(P)$ has positive trace. It remains to show that $\operatorname{tr} E_1 = \operatorname{tr} E_2$ for all $E_1, E_2 \in \operatorname{Exp}(P)$. To this end, let E_1 and E_2 be two such elements and, for $t > 0$, set $O_t = t^{E_1} t^{-E_2} = t^{E_1}(1/t)^{E_2}$. We observe that

$$P(O_t\xi) = P(t^{E_1}t^{-E_2}\xi) = tP((1/t)^{E_2}\xi) = t(1/t)P(\xi) = P(\xi)$$

for all $\xi \in \mathbb{R}^d$. Consequently, $O_t \in \operatorname{Sym}(P)$ for each $t > 0$. Because $\operatorname{Sym}(P)$ is a compact group by virtue of Proposition 1.4, $\det(O_t) = \pm 1$ and consequently,

$$1 = |\det(O_t)| = \left|\det(t^{E_1})\det(t^{-E_2})\right| = \left|t^{\operatorname{tr} E_1} t^{-\operatorname{tr} E_2}\right| = t^{\operatorname{tr} E_1 - \operatorname{tr} E_2}$$

for all $t > 0$ and therefore $\operatorname{tr} E_1 = \operatorname{tr} E_2$. $\square$

By virtue of the above proposition, given a positive homogeneous function P, we define **the homogeneous order of** P to be the unique positive number μ_P for which

$$\mu_P = \operatorname{tr} E$$

for all $E \in \operatorname{Exp}(P)$. Henceforth, when we speak of a positive homogeneous function, we take along with it its exponent set $\operatorname{Exp}(P)$, its symmetric group $\operatorname{Sym}(P)$, its unital level set S_P, and its homogeneous order μ_P.

Example 3. *In Example 1, we considered the positive homogeneous function $\xi \mapsto |\xi|^\nu$ where $\nu > 0$. In this case, $S_{|\cdot|^\nu}$ coincides with the unit sphere $\mathbb{S} \subseteq \mathbb{R}^d$, as we mentioned previously,* $\operatorname{Sym}(|\cdot|^\nu)$ *is precisely the orthogonal group $O(\mathbb{R}^d)$ and, because $E = I/\nu \in \operatorname{Exp}(|\cdot|^\nu)$, we have $\mu_{|\cdot|^\nu} = d/\nu$. In particular, when $d = 1$ and $\nu = m$ for some even natural number m, we find that $\xi \mapsto \xi^m$ is positive homogeneous with $\mu_{\xi^m} = 1/m$. This one-dimensional case is noteworthy because, in addition to the fact that ξ^m appears as the dominant term in the Type 1 in Definition 1.1, the error $o(n^{-1/m})$ in* (6) *(and the so-called on-diagonal scaling $t^{-1/m}$ of $H^t_{m,\beta}$ in* (8)*) can be expressed in terms of the homogeneous order $\mu_{\xi^m} = 1/m$. A similar observation can be made for points of Type 2 where m is not necessarily even. As we will see, these observations are emblematic of the situation in $\mathbb{Z}^d$ for $d > 1$ (see also Theorem 1.4. of [16]).*

In reference to Example 2, for the positive homogeneous semi-elliptic polynomials P of the form (13) *(with $\mathbf{m} = (m_1, m_2, \dots, m_d) \in \mathbb{N}^d_+$), the unital level set S_P and symmetric group* $\operatorname{Sym}(P)$ *will vary from example to example. It is straightforward to see that the homogeneous order of such a semi-elliptic polynomial P is given by*

$$\mu_P = |\mathbf{1} : \mathbf{m}| = \frac{1}{m_1} + \frac{1}{m_2} + \cdots + \frac{1}{m_d}.$$

$\triangle$

We are almost ready to present our generalization of Definition 1.1 in the context of $\mathbb{Z}^d$. It remains to remains to develop an appropriate concept of $o(\xi^m)$ where ξ^m is replaced by a positive-homogeneous function. To this end, we present the following definition taken from [3].

Definition 1.6. *Let $\widetilde{P}$ be a complex-valued function which is defined and continuous on an open neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^d$. Also, let $E \in \operatorname{End}(\mathbb{R}^d)$ be such that $\{t^E\}$ is contracting.*

1. *We say that $\widetilde{P}$ is **subhomogeneous with respect to** E if, for each $\epsilon > 0$ and compact set $K \subseteq \mathbb{R}^d$, there exists $\tau > 0$ for which*

$$\left|\widetilde{P}(t^E\xi)\right| \le \epsilon t$$

for all $0 < t < \tau$ and $\xi \in K$.

2. *Given $l \in \mathbb{N}_+$, we say that $\widetilde{P}$ is* ***strongly subhomogeneous with respect to*** *E* ***of order*** *l if, $\widetilde{P} \in C^l(\mathcal{U})$ and, for each $\epsilon > 0$ and compact set $K \subseteq \mathbb{R}^d$, there exists $\tau > 0$ such that*

$$\left| t^k \partial_t^k \widetilde{P}(t^E \xi) \right| \leq \epsilon t$$

   for all $k \in \{0, 1, 2, \ldots, l\}$, $0 < t < \tau$, and $\xi \in K$.

When E is understood (and fixed), we will say that $\widetilde{P}$ is subhomogeneous if it is subhomogeneous with respect to E. Also, we will say that $\widetilde{P}$ is l-strongly subhomogeneous if it is strongly subhomogeneous with respect to E of order l.

From the definition, it is apparent that all strongly subhomogenous functions are subhomogenous and it is reasonable to interpret subhomogeneous as strongly subhomogeneous of order 0. So that the inequalities in the definition makes sense, we remark that the supposition that $\{t^E\}$ is contracting guarantees that, for each compact set K, there is $\tau_0 > 0$ for which $t^E \xi \in \mathcal{U}$ for all $0 < t < \tau_0$ and $\xi \in K$ (see Proposition A.6 of [3]). In Section 4, we introduce a large class smooth functions which, for a given E, are strongly subhomogeneous of all orders. The following proposition connects the concept of subhomogeneity with that of a function being $o(P(\xi))$ as $\xi \to 0$ for some positive homogeneous function P. Its proof appears in Subsection A.2 of the Appendix.

Proposition 1.7. *Let P be a positive homogeneous function and $\widetilde{P}$ be a complex-valued function which is continuous on a neighborhood 0 of $\mathbb{R}^d$. The following are equivalent:*

1. *$\widetilde{P}(\xi) = o(P(\xi))$ as $\xi \to 0$.*
2. *For every $E \in \operatorname{Exp}(P)$, $\widetilde{P}$ is subhomogeneous with respect to E.*
3. *There exists $E \in \operatorname{Exp}(P)$ for which $\widetilde{P}$ is subhomogeneous with respect to E.*

We are now ready to introduce the d-dimensional generalization of Definition 1.1 considered in this article. We recall that, because $\widehat{\phi}$ is smooth for $\phi \in S_d$, $\Gamma_{\xi_0} \in C^\infty(\mathcal{U})$ and so we can use Taylor's theorem to approximate Γ_{ξ_0} near 0. Presicsely, we write

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - i\left(Q_{\xi_0}(\xi) + \widetilde{Q}_{\xi_0}(\xi)\right) - \left(R_{\xi_0}(\xi) + \widetilde{R}_{\xi_0}(\xi)\right) \tag{14}$$

where $\alpha_{\xi_0} \in \mathbb{R}^d$, Q_{ξ_0} and R_{ξ_0} are real-valued polynomials which vanish at 0 and contain no linear terms, and $\widetilde{Q}_{\xi_0}$ and $\widetilde{R}_{\xi_0}$ are real-valued smooth functions on $\mathcal{U}$ which vanish at 0. The fact that this expansion contains no real linear part is seen necessary because $|\widehat{\phi}|$ has a local maximum at ξ_0.

Definition 1.8. *Let $\phi \in \mathcal{S}_d$ with $\sup_\xi |\widehat{\phi}(\xi)| = 1$ and, given $\xi_0 \in \Omega(\phi)$, consider the expansion (14) above.*

1. *We say that ξ_0 is of* ***positive homogeneous type*** *for $\widehat{\phi}$ if R_{ξ_0} is positive homogeneous and, there exists $E \in \operatorname{Exp}(R_{\xi_0})$ for which Q_{ξ_0} is homogeneous with respect to E and both $\widetilde{R}_{\xi_0}$ and $\widetilde{Q}_{\xi_0}$ are subhomogeneous with respect to E. In this case, we will write $\mu_{\xi_0} = \mu_{R_{\xi_0}}$.*

2. *We say that ξ_0 is of* ***imaginary homogeneous type*** *for $\widehat{\phi}$ if $|Q_{\xi_0}|$ and R_{ξ_0} are both positive homogeneous and, there exists $E \in \operatorname{Exp}(|Q_{\xi_0}|)$ and $k > 1$ for which R_{ξ_0} is homogeneous with respect to E/k, $\widetilde{Q}_{\xi_0}$ is strongly subhomogeneous with respect to E of order 2, and $\widetilde{R}_{\xi_0}$ is strongly subhomogeneous with respect to E/k of order 1. In this case, we write $\mu_{\xi_0} = \mu_{|Q_{\xi_0}|}$.*

In either case, μ_{ξ_0} is said to be the homogeneous order associated to ξ_0 and $\alpha_{\xi_0} \in \mathbb{R}^d$ is said to be the drift associated to ξ_0.

On account of the simplicity of positive homogeneous functions in one dimension, it is straightforward to verify that the notions of positive homogeneous type and imaginary homogeneous type coincide with Thomée's notions of Type 1 and Type 2 presented in Definition 1.1, respectively, when $d = 1$. In Definition 1.3 of [16], a point $\xi_0 \in \Omega(\phi)$ is said to be of positive-homogeneous type for $\widehat{\phi}$ provided that

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - P_{\xi_0}(\xi) - \widetilde{P}_{\xi_0}(\xi) \tag{15}$$

for $\xi \in \mathcal{U}$ where P_{ξ_0} is a positive-homogeneous polynomial (in the sense of Remark 3) and $\widetilde{P}_{\xi_0}(\xi) = o(R_{\xi_0}(\xi))$ as $\xi \to 0$ where $R_{\xi_0} = \operatorname{Re} P_{\xi_0}$. To see this in the context of Definition 1.8, let us write $P_{\xi_0}(\xi) = R_{\xi_0}(\xi) + iQ_{\xi_0}(\xi)$ and $\widetilde{P}_{\xi_0}(\xi) = \widetilde{R}_{\xi_0}(\xi) + i\widetilde{Q}_{\xi_0}(\xi)$ and, in this case, (14) coincides with (15). If ξ_0 is of positive homogeneous type for $\widehat{\phi}$ in the sense of the definition above, it follows that P_{ξ_0} is a complex-valued polynomial which is homogeneous with respect to E (and so $\operatorname{Exp}(P_{\xi_0})$ contains $E \in \operatorname{End}(\mathbb{R}^d)$ for which $\{r^E\}$ is contracting) and $R_{\xi_0} = \operatorname{Re} P_{\xi_0}$ is positive-definite. In view of Remark 3, this is consistent with (and perhaps generalizes) the assumption that P_{ξ_0} is a positive-homogeneous polynomial in [16]. Further, the assumption that $\widetilde{Q}_{\xi_0}$ and $\widetilde{R}_{\xi_0}$ are subhomogeneous with respect to E guarantees that $\widetilde{P}_{\xi_0}(\xi) = o(R_{\xi_0}(\xi))$ as $\xi \to 0$ by virtue of Proposition 1.7. With these two observations, we see that our definition of a point ξ_0 being of positive homogeneous type, which is stated in terms of subhomogeneity, is consistent with that of [16].

Remark 4. In the case that ξ_0 is of imaginary homogeneous type, the assumption that $|Q_{\xi_0}|$ is positive-homogeneous guarantees that, for each $E \in \operatorname{Exp}(|Q_{\xi_0}|)$, Q_{ξ_0} is homogeneous with respect to E. In fact, by virtue of Lemma A.4, $\operatorname{Exp}(|Q_{\xi_0}|) = \operatorname{Exp}(Q_{\xi_0})$. We shall use this fact many time throughout this article. In particular, when $\xi_0 \in \Omega(\phi)$ is of imaginary homogeneous type for $\widehat{\phi}$, we may always choose $E \in \operatorname{Exp}(Q_{\xi_0})$ which has $\operatorname{tr} E = \mu_{|Q_{\xi_0}|} = \mu_{\xi_0}$.

Our hypotheses for local limit theorems in this article will be stated under the assumption that, given $\phi \in \mathcal{S}_d$ with $\sup_\xi |\widehat{\phi}| = 1$, every point $\xi_0 \in \Omega(\phi)$ is of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$. In either case, because R_{ξ_0} is positive-definite and $\widetilde{R}_{\xi_0}(\xi) = o(R_{\xi_0}(\xi))$ as $\xi \to 0$ (by virtue of Proposition 1.7), ξ_0 is necessarily an isolated point of $\mathbb{T}^d$ and so it follows that $\Omega(\phi)$ is a finite set. Under these hypotheses, we define **the homogeneous order of** ϕ to be the positive number

$$\mu_\phi = \min_{\xi \in \Omega(\phi)} \mu_\xi > 0. \tag{16}$$

Of course, in the case that $\Omega(\phi) = \{\xi_0\}$, $\mu_\phi = \mu_{\xi_0}$.

As we did in one dimension, to state our local limit theorems, it is necessary to introduce the attractors appearing therein. In the case that $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\widehat{\phi}$ with associated R_{ξ_0} and Q_{ξ_0}, we consider the polynomial

$$P_{\xi_0}(\xi) := R_{\xi_0}(\xi) + iQ_{\xi_0}(\xi)$$

and define $H^{(\cdot)}_{P_{\xi_0}}(\cdot) : (0,\infty) \times \mathbb{R}^d \to \mathbb{C}$ by

$$H^t_{P_{\xi_0}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-tP_{\xi_0}(\xi)} e^{-ix\cdot\xi} \, d\xi \tag{17}$$

for $t > 0$ and $x \in \mathbb{R}^d$. Like its one-dimensional analogue (7), the integral above converges absolutely and uniformly for $x \in \mathbb{R}^d$. In fact, as we will show in Section 2, $x \mapsto H^t_{P_{\xi_0}}(x)$ is a Schwartz function for each $t>0$ and, for each $E\in\operatorname{Exp}(P_{\xi_0})=\operatorname{Exp}(R_{\xi_0})\cap\operatorname{Exp}(Q_{\xi_0})$,

$$H_{P_{\xi_0}}^t(x)=t^{-\mu_{\xi_0}}H_{P_{\xi_0}}^1(t^{-E^*}x) \tag{18}$$

for $t>0$ and $x\in\mathbb{R}^d$ where E^* is the adjoint of E; this scaling property generalizes (8). The article [16] studies extensively the properties of the attractors $H_{P_{\xi_0}}$ and their role as fundamental solutions to higher-order heat-type equations, i.e., generalized heat kernels. In particular, [16] establishes Gaussian-type estimates for $H_{P_{\xi_0}}$ and its derivatives; these estimates are given in terms of the Legendre-Fenchel transform of R_{ξ_0}. For more on this perspective, we encourage the reader to see the articles [17] and [18], both of which focus on related variable-coefficient heat-type equations and their associated heat kernel estimates. Let us assume now that $\xi_0\in\Omega(\phi)$ is of imaginary homogeneous type for $\widehat{\phi}$ with associated Q_{ξ_0}. In this case, taking our cues from (10) and (17), we expect that, at least formally, the associated attractor will be given by

$$H_{iQ_{\xi_0}}^t(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}e^{-tiQ_{\xi_0}(\xi)}e^{-ix\cdot\xi}\,d\xi \tag{19}$$

for $t>0$ and $x\in\mathbb{R}^d$. Unlike its positive homogeneous counterpart, the convergence of the above integral is a delicate matter. Since $iQ_{\xi_0}(\xi)$ is purely imaginary, this integral is oscillatory in nature and does not converge in the sense of Lebesgue (for any values of $t>0$ and $x\in\mathbb{R}^d$). Also, because there is no canonical notion of improper Riemann integral in $\mathbb{R}^d$ for $d>1$, extending (10) and proving an associated generalization of (9) are not straightforward tasks. In some sense, the major hurdle faced in this article is establishing the convergence of oscillatory integrals of the form (19) in an appropriate sense. We will interpret the integral in (19) as a *renormalized integral* in the sense of C. Bär [1] (see Section 2) and, using the generalized polar-coordinate integration formula of [3], we prove that the integral converges for all $t>0$ and $x\in\mathbb{R}^d$ provided that $\mu_{\xi_0}<1$; this is Theorem 2.4. The theorem also shows that $x\mapsto H_{iQ_{\xi_0}}^t(x)$ is continuous for each $t>0$ and, for each $E\in\operatorname{Exp}(Q_{\xi_0})=\operatorname{Exp}(|Q|_{\xi_0})$,

$$H_{iQ_{\xi_0}}^t(x)=t^{-\mu_{\xi_0}}H_{iQ_{\xi_0}}^1(t^{-E^*}x)$$

for $t>0$ and $x\in\mathbb{R}^d$ where E^* is the adjoint of E. Though we suspect the attractors $H_{iQ_{\xi_0}}^t(x)$ are smooth, this remains an open question. With the attractors $H_{P_{\xi_0}}$ and $H_{iQ_{\xi_0}}$ in hand, we are ready to state our first local limit theorem; this result is new and extends the local limit theorems (6) and (9).

Theorem 1.9. *Let $\phi\in\mathcal{S}_d$ be such that $\sup|\widehat{\phi}|=1$. Suppose that $\Omega(\phi)=\{\xi_0\}$ and that ξ_0 is of positive homogeneous type or imaginary homogeneous type for $\widehat{\phi}$ with drift α_{ξ_0} and homogeneous order μ_{ξ_0}. In either case, note that $\mu_\phi=\mu_{\xi_0}$. When ξ_0 is of imaginary homogeneous type for $\widehat{\phi}$, we assume additionally that $\mu_\phi=\mu_{\xi_0}<1$.*

1. *In the case that ξ_0 is of positive homogeneous type for $\widehat{\phi}$, let $H_{P_{\xi_0}}$ be as given in the previous paragraph. Then,*

$$\phi^{(n)}(x)=\widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0}H_{P_{\xi_0}}^n(x-n\alpha_{\xi_0})+o(n^{-\mu_\phi})$$

   uniformly for $x\in\mathbb{Z}^d$.

2. *In the case that ξ_0 is of imaginary homogeneous type for $\widehat{\phi}$, let $H_{iQ_{\xi_0}}$ be as discussed above (and defined precisely in Section 2). Then, for each compact set $K\subseteq\mathbb{R}^d$ and $E\in\operatorname{Exp}(Q_{\xi_0})=\operatorname{Exp}(|Q_{\xi_0}|)$,*

$$\phi^{(n)}(x)=\widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0}H_{iQ_{\xi_0}}^n(x-n\alpha_{\xi_0})+o(n^{-\mu_\phi})$$

   uniformly for $x\in\left(n\alpha_{\xi_0}+n^{E^*}(K)\right)\bigcap\mathbb{Z}^d$ where E^* is the adjoint of E.

Include a Huan Example here.

For simplicity of presentation, we have stated Theorem 1.9 with the assumption that $\Omega(\phi)$ is a singleton. In Section 3, we state and prove a general theorem which allows for $\Omega(\phi)$ to consist of more than one point; this is Theorem 3.8. Our theorem recaptures Theorem 1.6 of [16], a result that assumes that every point of $\Omega(\phi)$ is of positive homogeneous type for $\widehat{\phi}$. What is new in Theorem 3.8 (and Theorem 1.9) is the consideration of points of imaginary homogeneous type for $\widehat{\phi}$ and this consideration, in its essence, is the major task treated in this article. While our results are not able to treat the full generality in which all points of $\Omega(\phi)$ are either of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$ (with no other restrictions), our results partially extend the local limit theorems of [15] and do so in the spirit of that article.

Notation: In this article, we denote by $\mathcal{M}_d$ the σ-algebra of Lebesgue measurable set of $\mathbb{R}^d$ and, for $A \in \mathcal{M}_d$, χ_A is the characteristic function of A. We denote by $m = m_d$ the Lebesgue measure and we write $dm = m(d\xi) = d\xi$. For $X \in \mathcal{M}_d$ and $1 \leq p \leq \infty$, $L^p(X) = L^p(X, m)$ will denote the usual Lebesgue space equipped with its norm $\|\cdot\|_{L^p(X)}$. When $k \in \mathbb{N}$ and $X \subseteq \mathbb{R}^d$ is non-empty and open (or, generally, when the following concept makes sense), we will denote by $C^k(X)$ the set of k-times continuously differentiable complex-valued functions on X. Of course, $C^0(X)$ and $C^\infty(X)$ are, respectively, the set of continuous and smooth functions on X. For $0 \leq k \leq \infty$, the notation $C^k(X;\mathbb{R})$ will denote the subclass of function $C^k(X)$ which are real valued. The set of locally integrable functions on $\mathbb{R}^d$ will be denoted by $L^1_{\text{loc}}(\mathbb{R}^d)$; this is set of complex-valued measurable functions f for which, given any compact set $K \subseteq \mathbb{R}^d$, $\xi \mapsto f(\xi)\chi_K(\xi)$ belongs to $L^p(\mathbb{R}^d)$. Finally, $\mathcal{S}(\mathbb{R}^d)$ will denote the Schwartz class of complex-valued functions on $\mathbb{R}^d$.

2 The Attractors

In this section, we study the attractors (17) and (19) discussed in the introduction. Our first task is to establish some basic facts about the attractor (17) corresponding to a point ξ_0 of positive homogeneous type. To this end, we present the following proposition which ensures, in particular, that (17) is smooth and exhibits the scaling property (18). The proposition is stated in the context of positive homogeneous functions (that is, it does not assume that $P = R + iQ$ is a polynomial), however, thanks to the theory developed in [3], working in this level of generality is natural. We remark that a similar statement can be found as Proposition 2.6 of [16].

Proposition 2.1. *Let $R : \mathbb{R}^d \to \mathbb{R}$ be a positive homogeneous function and let $Q : \mathbb{R}^d \to \mathbb{R}$ be continuous and such that $\operatorname{Exp}(R) \cap \operatorname{Exp}(Q)$ is non-empty. Set $P(\xi) = R(\xi) + iQ(\xi)$ for $\xi \in \mathbb{R}^d$, observe that $\operatorname{Exp}(P) = \operatorname{Exp}(R) \cap \operatorname{Exp}(Q)$, and set $\mu_P = \mu_R$. Then $e^{-tP(\xi)} \in L^1(\mathbb{R}^d)$ for each $t > 0$ and we may therefore define*

$$H_P^t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-tP(\xi)} e^{-ix\cdot\xi}\, d\xi \tag{20}$$

for $t > 0$ and $x \in \mathbb{R}^d$. There holds the following:

1. *As a function on $(0,\infty) \times \mathbb{R}^d$, $H_P = H_P^{(\cdot)}(\cdot)$ is smooth. If, additionally, R and Q are polynomials, then, for each $t > 0$, $H_P^t \in \mathcal{S}(\mathbb{R}^d)$.*
2. *For any $E \in \operatorname{Exp}(P)$,*
$$H_P^t(x) = t^{-\mu_P} H_P^1(t^{-E^*} x)$$
for $t > 0$ and $x \in \mathbb{R}^d$.

3. *We have*

$$\left|H_P^t(x)\right| \le t^{-\mu_P} H_R(0) = \frac{t^{-\mu_P}}{(2\pi)^d} m(B_R)\Gamma(\mu_P+1)$$

for all $x \in \mathbb{R}^d$ *and* $t > 0$*; here,* $B_R = \{\xi \in \mathbb{R}^d : R(\xi) < 1\}$ *(and has finite measure by virtue of Proposition 1.3),* $\Gamma(\cdot)$ *is Euler's Gamma function, and* H_R *is defined in precisely the same way at* H_P*; they coincide, of course, when* $Q \equiv 0$*.*

Proof. We first claim to that, for each $k \in \mathbb{N}$ and multiindex $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$,

$$\sup_{\xi\in\mathbb{R}^d} \left|\xi^\alpha P(\xi)^k e^{-tP(\xi)}\right| < \infty. \tag{21}$$

for each $t > 0$; here, $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}\cdots\xi_d^{\alpha_d}$ for $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{R}^d$. From this claim, the absolute integrability of $e^{-tP(\xi)}$ follows immediately. Also, standard arguments (e.g., those which show that the Fourier transform exchanges decay at infinity for smoothness) show that, by virtue of (21), one may differentiate (in both t and x) through the integral sign in (20) as many times as one likes. In this way, (21) ensures that $H_P \in C^\infty((0,\infty)\times\mathbb{R}^d)$. To prove (21), in view of the fact that $\xi \mapsto \xi^\alpha P(\xi)^k e^{-tP(\xi)}$ is continuous and $\mathrm{Re}(P) = R$, it suffices to prove that

$$\sup_{\xi\in\mathbb{R}^d/B_R} \left|\xi^\alpha P(\xi)^k e^{-tR(\xi)}\right| < \infty$$

where, as given in the statement of the proposition, $B_R = \{\xi \in \mathbb{R}^d : R(\xi) < 1\}$. For $E \in \mathrm{Exp}(P) \subseteq \mathrm{Exp}(R)$, the fact that $\{r^E\}$ is contracting guarantees that $\mathbb{R}^d \setminus B_R = \{r^E\eta : r \ge 1, \eta \in S_R\}$ where S_R is the unital level set of R. From this it follows that

$$\sup_{\xi\in\mathbb{R}^d\setminus B_R} \left|\xi^\alpha P(\xi)^k e^{-tR(\xi)}\right| = \sup_{r\ge1,\eta\in S_R} \left|(r^E\eta)^\alpha P(r^E\eta)^k e^{-tR(r^E\eta)}\right| = \sup_{r\ge1,\eta\in S_R} \left|(r^E\eta)^\alpha r^k P(\eta)^k e^{-tr}\right|.$$

Now, for $r \ge 1$, $\|r^E\| \le r^{\|E\|}$ and therefore $\left|(r^E\eta)^\alpha\right| \le \left|r^E\eta\right|^{|\alpha|} \le |\eta|^{|\alpha|} r^{|\alpha|\|E\|}$ where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$. Given that P is continuous, S_R is compact, and $t > 0$, we conclude that

$$\sup_{\xi\in\mathbb{R}^d\setminus B_R} \left|\xi^\alpha P(\xi)^k e^{-tP(\xi)}\right| \le \sup_{r\ge1,\eta\in S_R} |\eta|^{|\alpha|} |P(\eta)|^k r^{\|E\||\alpha|+k} e^{-tr} < \infty,$$

as was asserted.

In the case that R and Q are polynomials, for each pair of multiindices $\alpha, \beta \in \mathbb{N}^d$,

$$\xi^\alpha D^\beta \left(e^{-tP(\xi)}\right) = U_{\alpha,\beta}(t,\xi) e^{-tP(\xi)}$$

where $U_{\alpha,\beta}(t,\xi)$ is a polynomial in t and ξ; here $D^\beta = (i\partial_{\xi_1})^{\beta_1}(i\partial_{\xi_2})^{\beta_2}\cdots(i\partial_{\xi_d})^{\beta_d}$. By a similar argument to that given above, we find that

$$\sup_{\xi\in\mathbb{R}^d\setminus B_R} \left|\xi^\alpha D^\beta \left(e^{-tP(\xi)}\right)\right| = \sup_{r\ge1,\eta\in S_R} \left|U(t, r^E\eta) e^{-tr}\right| < \infty$$

for each pair of multiindices α and β and therefore $\xi \mapsto e^{-tP(\xi)} \in \mathcal{S}(\mathbb{R}^d)$. Because the Fourier transform is an automorphism of $\mathcal{S}(\mathbb{R}^d)$, it follows that $H_P^t \in \mathcal{S}(\mathbb{R}^d)$ for each $t > 0$ and this concludes the proof of Item 1.

Throughout the remainder of the proof, we shall write $\mu = \mu_P$ and note that $\mu = \mu_R = \operatorname{tr} E$ for any $E \in \operatorname{Exp}(R)$. To see Item 2, observe that, for any $E \in \operatorname{Exp}(P) = \operatorname{Exp}(R) \cap \operatorname{Exp}(Q)$, $t > 0$, and $x \in \mathbb{R}^d$,

$$H_P^t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-tP(\xi)} e^{-ix\cdot\xi}\, d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-P(t^E\xi)} e^{-ix\cdot\xi}\, d\xi.$$

Upon making the change of variables $\xi = t^{-E}\zeta$, for any $t > 0$ and $x \in \mathbb{R}^d$, we have

$$\begin{aligned} H_P^t(x) &= \frac{1}{(2\pi)} \int_{\mathbb{R}^d} e^{-P(\zeta)} e^{-ix\cdot(t^{-E}\zeta)} \det(t^{-E})\, d\zeta \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-P(\zeta)} e^{-i(t^{-E^*}x)\cdot\zeta} t^{-\mu}\, d\zeta \\ &= t^{-\mu} H_P^1(t^{-E^*}x) \end{aligned}$$

where we have used the fact that $\det(t^{-E}) = t^{-\operatorname{tr} E} = t^{-\mu}$ and $(t^{-E})^* = t^{-E^*}$. This proves Item 2.

To prove Item 3, we first appeal the the previous item to see that, for any $t > 0$ and $x \in \mathbb{R}^d$,

$$\left|H_P^t(x)\right| = t^{-\mu}\left|H_P^1(t^{-E^*}x)\right| \le \frac{t^{-\mu}}{(2\pi)^d} \int_{\mathbb{R}^d} \left|e^{-P(\xi)} e^{-i(t^{-E^*}x)\cdot\xi}\right| d\xi = \frac{t^{-\mu}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-R(\xi)}\, d\xi;$$

Thus,

$$\left|H_P^t(x)\right| \le \frac{t^{-\mu}}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-R(\xi)}\, d\xi = t^{-\mu} H_R^1(0)$$

for all $t > 0$ and $x \in \mathbb{R}^d$. It remains to compute $H_R^1(0)$. To this end, we appeal to the polar coordinate integration formula (Theorem 1.3) of [3]. For the positive homogeneous function R, the theorem gives us a Radon measure σ_R on S_R for which, given any $f \in L^1(\mathbb{R}^d)$ and $E \in \operatorname{Exp}(R)$,

$$\int_{\mathbb{R}^d} f(\xi)\, d\xi = \int_{S_R} \int_0^\infty f(r^E\eta) r^{\mu-1}\, dr \sigma_R(d\eta).$$

Applying this to $f(\xi) = e^{-R(\xi)}$, we have

$$\begin{aligned} H_R^1(0) &= \frac{1}{(2\pi)^d} \int_{S_R} \int_0^\infty e^{-R(r^E\eta)} r^{\mu-1}\, dr \sigma_R(d\eta) \\ &= \frac{1}{(2\pi)^d} \int_{S_R} \int_0^\infty e^{-r} r^{\mu-1}\, dr\, \sigma_R(d\eta) \\ &= \frac{\sigma_R(S_R)}{(2\pi)^d} \int_0^\infty e^{-r} r^{\mu-1}\, dr \\ &= \frac{\mu \cdot m(B_R)}{(2\pi)^d} \Gamma(\mu) \\ &= \frac{\Gamma(\mu+1)}{(2\pi)^d} m(B_R) \end{aligned}$$

where we have noted that $\sigma_R(S_R) = \mu \cdot m(B_R)$ by virtue of Item 3 of Theorem 1.3 of [3], recognized the integral representation of the Gamma function, and made use of the fundamental identity, $\mu\Gamma(\mu) = \Gamma(\mu + 1)$. $\square$

We now turn our attention to the attractors corresponding to points ξ_0 of imaginary homogeneous type. As we discussed in the introduction, we must first discuss a theory for the convergence of oscillatory integrals given formally by (19). The integral we introduce below can be seen as a special case the renormalized integral of [1]. While still having its limitations, this renormalized integral is broad enough to handle a large variety of possible Qs and sufficiently robust to make tractable our analysis.

Given $X \in \mathcal{M}_d$, a family of measurable set $\mathcal{A} = \{\mathcal{O}_\tau \subseteq X : \tau > 0\} \subseteq \mathcal{M}_d$ is said to be an **approximating family of** X if $\mathcal{A}$ is nested increasing and covers X, i.e., $\mathcal{O}_{\tau_1} \subseteq \mathcal{O}_{\tau_2}$ whenever $\tau_1 \leq \tau_2$, and $X = \cup_{\tau>0}\mathcal{O}_\tau$.

Definition 2.2. *Let $X \in \mathcal{M}_d$ and suppose that $\mathcal{A} = \{\mathcal{O}_\tau : \tau > 0\}$ is an approximating family of X.*

1. *Given a measurable function $f : X \to \mathbb{C}$, we say that the renormalized integral of f over X associated to $\mathcal{A}$ converges if the limit*

$$\lim_{\tau\to\infty} \int_{\mathcal{O}_\tau} f\, dm \tag{22}$$

exists. In this case, the renormalized integral of f over X associated to $\mathcal{A}$ is defined to be the value of the limit (22) and denoted by

$$\fint_X f\, d_{\mathcal{A}} \qquad or \qquad \fint_X f(\xi)\, d_{\mathcal{A}}\xi.$$

When the context is clear, we will omit the vocabulary "over X associated to $\mathcal{A}$".

2. *Let Y be a non-empty set and, for a function $g = g_{(\cdot)}(\cdot) : Y \times X \to \mathbb{C}$, and suppose that $\xi \mapsto g_y(\xi)$ is measurable for each $y \in Y$. If, the renormalized integral $\fint_X g_y\, d_{\mathcal{A}} = \fint_X g_y(\xi)\, d_{\mathcal{A}}\xi$ converges for each $y \in Y$ and, for all $\epsilon > 0$, there exists $\tau_0 > 0$ for which*

$$\left| \fint g_y\, d_{\mathcal{A}} - \int_{\mathcal{O}_\tau} g_y\, dm \right| < \epsilon$$

for all $y \in Y$ and $\tau \geq \tau_0$, we say that the renormalized integral of g_y converges uniformly on Y. We also say that the renormalized integral of g_y converges uniformly for $y \in Y$.

As mentioned above, the above definition is a special case[3] of the renormalized integral of [1], a notion which itself recaptures several instances of Cauchy's principal value. Some simple observations following Definition 2.2 are presented in the following examples.

Example 4. *In one dimension, the improper integral*

$$\int_0^\infty f(\xi)\, d\xi = \lim_{s\to\infty} \int_0^s f(\xi)\, d\xi,$$

coincides with the renormalized integral of f over $\mathbb{R}$ and associated to the family $\mathcal{A} = \{[0, s), s > 0\}$. As noted in the introduction (c.f., the discussion surrounding (10)), this integral is used to define the attractors in [15]. △

[3] Looking at Definition 2.2 from the perspective of [1], the relevant family of measure spaces is $\Omega = \{\mathcal{O}_\tau, d\xi|_{\mathcal{O}_\tau}\}_{\tau>0}$ which is indexed by the directed set of positive real numbers with its usual ordering.

Example 5. *Given $X \in \mathcal{M}_d$ and $f \in L^1(X)$, the Lebesgue integral of f over X coincides with the renormalized integral of f over X associated to any (and so every) approximating family $\mathcal{A}$ of X. In other words, given any $f \in L^1(X)$ and approximating family $\mathcal{A}$ of X, the renormalized integral of f over X associated to $\mathcal{A}$ converges and*

$$\fint_X f(\xi)\, d_{\mathcal{A}}\xi = \int_X f(\xi)\, d\xi.$$

In fact, the absolute Lebesgue integrability of a measurable function f can be characterized by the convergence of the renormalized integral of $|f(\xi)|$ over X associated any (and every) approximating family $\mathcal{A}$ of X. These assertions can be seen straightforwardly as consequences of the monotone and dominated convergence theorems.

As an application, let $P = R + iQ$ be as in the statement of Proposition 2.1 and consider the approximating family $\mathcal{A} = \{\mathcal{O}_\tau : \tau > 0\}$ of $\mathbb{R}^d$ where, for each $\tau > 0$, $\mathcal{O}_\tau = \{\xi \in \mathbb{R}^d : R(\xi) < \tau\}$. Then the attractor H_P^t of Proposition 2.1 is equivalently given by

$$H_P^t(x) = \frac{1}{(2\pi)^d} \fint_{\mathbb{R}^d} e^{-tP(\xi)} e^{-ix\cdot\xi}\, d_{\mathcal{A}}\xi = \lim_{\tau\to\infty} \frac{1}{(2\pi)^d} \int_{\mathcal{O}_\tau} e^{-tP(\xi)} e^{-ix\cdot\xi}\, d\xi$$

for $t > 0$ and $x \in \mathbb{R}^d$. It is easy to see that, for each $t_0 > 0$, this renormalized integral converges uniformly for $(t,x) \in [t_0,\infty) \times \mathbb{R}^d$. △

As [1] points out, the renormalized integral is linear, monotonic, and satisfies the triangle inequality. Unsurprisingly however, the renormalized integral does not satisfy basic limit theorems of measure theory, including Fatou's lemma and the monotone and dominated convergence theorems. The following proposition is a characterization of uniform convergence of renormalized integrals in terms of a Cauchy condition. Its proof is straightforward and omitted.

Proposition 2.3. *Let $\mathcal{A}$ be an approximating family of $X \in \mathcal{M}_d$ and let Y be a non-empty set. Suppose that $g = g_{(\cdot)}(\cdot) : Y \times X \to \mathbb{C}$ is such that $\xi \mapsto g_y(\xi)$ is Lebesgue measurable for each $y \in Y$. Then renormalized integral of g_y converges uniformly on Y if and only if, the following Cauchy condition is satisfied: For each $\epsilon > 0$, there exists $\tau_0 > 0$ such that*

$$\left| \int_{\mathcal{O}_{\tau_2} \setminus \mathcal{O}_{\tau_1}} g_y\, dm \right| < \epsilon$$

for all $y \in Y$ and $\tau_0 \le \tau_1 \le \tau_2$.

Armed with the renormalized integral as a basic tool, we turn our focus to the attractors given formally by (19). In what follows, we let $Q : \mathbb{R}^d \to \mathbb{R}$ be a continuous function for which $|Q|$ is positive homogeneous with homogeneous order $\mu = \mu_Q = \mu_{|Q|}$ and exponent set $\operatorname{Exp}(Q) = \operatorname{Exp}(|Q|)$ (c.f., Remark 4). For each $\tau > 0$, define $\mathcal{O}_\tau = \{\xi \in \mathbb{R}^d : |Q(\xi)| < \tau\}$ and observe that $\mathcal{A} = \{\mathcal{O}_\tau : \tau > 0\}$ is an approximating family of $\mathbb{R}^d$. The following theorem, stated in terms of the renormalized integral, introduces the functions which will appear as attractors in local limit theorems corresponding to points of imaginary homogeneous type.

Theorem 2.4. *Let $Q : \mathbb{R}^d \to \mathbb{R}$ be continuous and such that $|Q|$ is positive homogeneous with homogeneous order $\mu = \mu_{|Q|} > 0$. Also, let $\mathcal{A}$ be as above. If $\mu < 1$, then the following statements hold.*

1. *For any $t > 0$ and $x \in \mathbb{R}^d$, the renormalized integral of $\xi \mapsto e^{-itQ(\xi) - ix\cdot\xi}$ over $\mathbb{R}^d$ associated to the family $\mathcal{A}$ converges and we set*

$$H_{iQ}^t(x) = \frac{1}{(2\pi)^d} \fint_{\mathbb{R}^d} e^{-itQ(\xi) - ix\cdot\xi}\, d_{\mathcal{A}}\xi. \tag{23}$$

Further, $H_{iQ}^t(x)$ is continuous on its domain, $(0,\infty)\times\mathbb{R}^d$.

2. *For each $t_0>0$ and compact set $K\subseteq\mathbb{R}^d$, the renormalized integral in (23) converges uniformly for $t\geq t_0$ and $x\in K$.*
3. *For any $E\in\operatorname{Exp}(Q)$,*
$$H_{iQ}^t(x)=\frac{1}{t^\mu}H_{iQ}^1\left(t^{-E^*}x\right) \tag{24}$$
for every $t>0$ and $x\in\mathbb{R}^d$.

To see that the hypothesis $\mu<1$ cannot be weakened, in general, consider $Q:\mathbb{R}\to\mathbb{R}$ defined by $Q(\xi)=-\xi$ for $\xi\in\mathbb{R}$. We see immediately that $|Q(\xi)|=|\xi|$ is positive homogeneous with $\mu=1$. In this case, for each $\tau>0$,
$$\frac{1}{2\pi}\int_{\mathcal{O}_\tau}e^{-itQ(\xi)-ix\xi}\,d\xi=\frac{1}{2\pi}\int_{-\tau}^{\tau}e^{i(t-x)\xi}\,d\xi=\begin{cases}\frac{\sin((t-x)\tau)}{\pi(t-x)} & t\neq x\\ \frac{\tau}{\pi} & t=x\end{cases}$$
for $t>0$ and $x\in\mathbb{R}$. Consequently, for no values of $t>0$ and $x\in\mathbb{R}$ does the associated renormalized integral converge. From the perspective of distribution theory, this comes as no surprise for the (formal) integral
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i(t-x)\xi}\,d\xi$$
is a representation of the Dirac distribution, $\delta(t-x)$.

As we will see below, an important part of the theorem's proof makes use of the generalized polar-coordinate integration formula developed in [3]. Specifically, Theorem 1.3 of [3] hands us a Radon measure $\sigma=\sigma_{|Q|}$ on the unital level set $S=S_{|Q|}=\{\xi\in\mathbb{R}^d:|Q(\xi)|=1\}$ for which
$$\int_{\mathbb{R}^d}f(\xi)\,d\xi=\int_S\int_0^\infty f(r^E\eta)r^{\mu-1}\,dr\,\sigma(d\eta)$$
for any $f\in L^1(\mathbb{R}^d)$ and $E\in\operatorname{Exp}(Q)=\operatorname{Exp}(|Q|)$. We note that, for $E\in\operatorname{Exp}(Q)=\operatorname{Exp}(|Q|)$, $\left|Q(r^E\eta)\right|=r|Q(\eta)|=r$ for all $\eta\in S$ and $r>0$ and so it follows that, for each $\tau>0$,
$$\chi_{\mathcal{O}_\tau}(r^E\eta)=\chi_{[0,\tau)}(r)$$
for all $\eta\in S$ and $r>0$. We note that, by virtue of the continuity of Q and Proposition 1.3, each $\mathcal{O}_\tau$ is relatively compact with compact closure $\overline{\mathcal{O}_\tau}=\{\xi\in\mathbb{R}^d:|Q(\xi)|\leq\tau\}$. With these observations in mind, we find that, for any $f\in L^1_{\text{loc}}(\mathbb{R}^d)$, $E\in\operatorname{Exp}(Q)=\operatorname{Exp}(|Q|)$ and $\tau>0$,
$$\int_{\mathcal{O}_\tau}f(\xi)\,d\xi=\int_S\int_0^\tau f(r^E\eta)r^{\mu-1}\,dr\,\sigma(d\eta). \tag{25}$$
When applied to $f(\xi)=e^{-itQ(\xi)-ix\cdot\xi}$, we obtain
$$\int_{\mathcal{O}_\tau}e^{-itQ(\xi)-ix\cdot\xi}\,d\xi=\int_S\int_0^\tau e^{-itrQ(\eta)-ix\cdot(r^E\eta)}r^{\mu-1}\,dr\sigma(d\eta)$$
for $x\in\mathbb{R}^d$ and $\tau>0$. As we will see, the advantage of this representation is that it allows us to handle the oscillatory behavior of the integrand by studying its oscillations in the "radial" direction r. In this way, we reduce the question of convergence to the analysis of an oscillatory integral in one dimension. For this reason, we will make use of the famous Van der Corupt lemma which we state here fore completeness (for a proof, see [22], [15], or [2]).

Proposition 2.5 (Van der Corput Lemma). *Given a compact interval $I=[a,b]$, let $f\in C^2(I)$ and $g\in C^1(I)$ be real valued and denote by f' and g' their first derivatives, respectively. If f' is monotonic on I and, for $\lambda>0$, $|f'(\theta)|\geq\lambda$ for all $\theta\in I$, then*

$$\left|\int_a^b e^{-if(\theta)}g(\theta)\,d\theta\right|\leq 4\frac{\|g\|_{L^\infty(I)}+\|g'\|_{L^1(I)}}{\lambda}.$$

Proof of Theorem 2.4. We first prove Item 2. Let $\epsilon>0$, $t_0>0$ and $K\subseteq\mathbb{R}^d$ be a fixed compact set. By virtue of Proposition 2.3, it suffices to show that there is a constant $\tau_0>0$ for which

$$\left|\int_{\mathcal{O}_{\tau_2}\setminus\mathcal{O}_{\tau_1}}\exp(-itQ(\xi)-ix\cdot\xi)\,d\xi\right|<\epsilon$$

for all $t\geq t_0$, $x\in K$ and $\tau_2\geq\tau_1\geq\tau_0$. Fix $E\in\operatorname{Exp}(Q)=\operatorname{Exp}(|Q|)$ and observe that, by virtue of (25), for $0\leq\tau_1\leq\tau_2$, $t>0$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}\int_{\mathcal{O}_{\tau_2}\setminus\mathcal{O}_{\tau_1}}\exp(-itQ(\xi)-ix\cdot\xi)\,d\xi &= \int_S\int_{\tau_1}^{\tau_2}\exp\left(-itQ(r^E\eta)-ix\cdot(r^E\eta)\right)r^{\mu-1}\,dr\,\sigma_P(d\eta)\\ &= \int_S\int_{\tau_1}^{\tau_2}\exp\left(-itrQ(\eta)-ix\cdot(r^E\eta)\right)r^{\mu-1}\,dr\,\sigma_P(d\eta)\\ &= \int_S I_{\tau_1,\tau_2,t,x}(\eta)\sigma_P(d\eta)\end{aligned}$$

where

$$I_{\tau_1,\tau_2,t,x}(\eta)=\int_{\tau_1}^{\tau_2}\exp\left(-itrQ(\eta)-ix\cdot(r^E\eta)\right)r^{\mu-1}\,dr$$

for each $\eta\in S$. Upon making the change of variables $r=\theta^{1/\mu}$ and noting that $d\theta=r^{\mu-1}\,dr$, we have

$$I_{\tau_1,\tau_2,t,x}(\eta)=\int_{\tau_1^\mu}^{\tau_2^\mu}e^{if_{t,x,\eta}(\theta)}\,d\theta$$

where

$$f_{t,x,\eta}(\theta)=-\left(tQ(\eta)\theta^{1/\mu}+x\cdot(\theta^F\eta)\right),$$

$F=E/\mu$ and we have used the fact that $d\theta=r^{\mu-1}\,dr$. For simplicity, we write $f=f_{t,x,\eta}$ and observe that

$$\partial_\theta f(\theta)=-\left(\frac{t}{\mu}Q(\eta)\theta^{1/\mu-1}+x\cdot\left(\theta^{F-I}F\eta\right)\right)\tag{26}$$

and

$$\theta^2\partial_\theta^2 f(\theta)=-\left(\frac{t}{\mu}\left(\frac{1}{\mu}-1\right)Q(\eta)\theta^{1/\mu}+x\cdot(\theta^F(F-I)F\eta\right)\tag{27}$$

for all $t>0$, $x\in\mathbb{R}^d$, $\eta\in S$, and $\theta>0$. Using the estimates

$$\left|x\cdot(\theta^F(F-I)F\eta)\right|\leq|x||(F-I)F\eta|\|\theta^F\|\quad\text{ and }\quad\left|x\cdot(\theta^{F-I}F\eta)\right|\leq|x||F\eta|\|\theta^F\|\theta^{-1},$$

the compactness of K and S, and our hypothesis that $\mu<1$, an appeal to Corollary A.3 (with $\alpha=\mu$) hands us $\tau_0>0$ for which

$$\left|x\cdot(\theta^F(F-I)F\eta)\right|\leq\frac{t_0}{2\mu}\left(\frac{1}{\mu}-1\right)\theta^{1/\mu}$$

and

$$\left|x \cdot (\theta^{F-I} F\eta)\right| \leq \frac{t_0}{2\mu}\theta^{1/\mu-1}$$

for all $x \in K$, $\eta \in S$ and $\theta \geq \tau_0^{\mu}$. Using these inequalities and the fact that $|Q(\eta)| = 1$ for all $\eta \in S$, from (26) and (27) it follows that, for all $t \geq t_0$, $x \in K$, $\eta \in S$ and $\theta \geq \tau_0^{\mu}$,

$$\left|\theta^2 \partial_\theta^2 f(\theta)\right| \geq \frac{t}{2\mu}\left(\frac{1}{\mu} - 1\right)\theta^{1/\mu} > 0$$

and

$$|\partial_\theta f(\theta)| \geq \frac{t}{2\mu}\theta^{1/\mu-1};$$

in particular, $\theta \mapsto \partial_\theta f(\theta)$ is monotonic on $[\tau_0^{\mu}, \infty)$. By further enlarging τ_0 so that $\theta^{1/\mu-1} \geq 16\mu\, \sigma(S)/(\epsilon t_0)$ for all $\theta \geq \tau_0^{\mu}$, if necessary, we have

$$|\partial_\theta f(\theta)| \geq \frac{t}{2\mu}\theta^{1/\mu-1} \geq 8\frac{\sigma(S)}{\epsilon}\frac{t}{t_0} \geq 8\frac{\sigma(S)}{\epsilon}$$

for all $t \geq t_0$, $x \in K$, $\eta \in S$ and $\theta \geq \tau_0^{\mu}$. By an appeal to Proposition 2.5 (with $f(\theta)$ as above and $g(\theta) \equiv 1$), we obtain

$$|I_{\tau_1,\tau_2,t,x}(\eta)| \leq \frac{4(1+0)}{8\sigma(S)/\epsilon} = \frac{1}{2}\frac{\epsilon}{\sigma(S)}$$

for all $t \geq t_0$, $x \in K$, $\eta \in S$ and $\tau_2 \geq \tau_1 \geq \tau_0$. Consequently,

$$\left|\int_{\mathcal{O}_{\tau_2}\setminus\mathcal{O}_{\tau_1}} \exp(-itQ(\xi) - ix\cdot\xi)\, d\xi\right| \leq \int_S |I_{\tau_1,\tau_2,t,x}(\eta)|\, \sigma(d\eta) \leq \frac{1}{2}\int_S \frac{\epsilon}{\sigma(S)}\, \sigma(d\eta) < \epsilon$$

for all $t \geq t_0$, $x \in K$ and $\tau_2 \geq \tau_1 \geq \tau_0$, as was asserted.

The convergence statement of Item 1 follows immediately from Item 2. Since the approximants

$$(t,x) \mapsto \frac{1}{(2\pi)^d}\int_{\mathcal{O}_\tau} e^{-itQ(\xi)-ix\cdot\xi}\, d\xi$$

of $H_{iQ}^t(x)$ are all continuous and, by virtue of Item 2, converge uniformly on all compact subsets of $(0,\infty)\times\mathbb{R}^d$, it follows that $H_{iQ}^t(x)$ is continuous on its domain. This completes the proof of Item 1.

Finally, for an arbitrary $E \in \operatorname{Exp}(Q) = \operatorname{Exp}(|Q|)$, observe that, for each $t > 0$, $x \in \mathbb{R}^d$ and $\tau > 0$,

$$\begin{aligned}
\int_{\mathcal{O}_\tau} e^{-itQ(\xi)-ix\cdot\xi}\, d\xi &= \int_{\mathcal{O}_\tau} e^{-iQ(t^E\xi)-ix\cdot\xi}\, d\xi \\
&= \int_{t^E(\mathcal{O}_\tau)} e^{iQ(\zeta)-ix\cdot t^{-E}\zeta} \det(t^{-E})\, d\zeta \\
&= t^{-\mu}\int_{\mathcal{O}_{t\tau}} e^{iQ(\zeta)-i(t^{-E^*}x)\cdot\zeta}\, d\zeta
\end{aligned} \tag{28}$$

where we have made the change of variables $\zeta = t^E\xi$ and noted that $\det(t^{-E}) = t^{-\operatorname{tr} E} = t^{-\mu}$ and

$$t^E(\mathcal{O}_\tau) = \{t^E\xi : |Q(\xi)| < \tau\} = \{t^E\xi : \left|Q(t^E\xi)\right| = t|Q(\xi)| < t\tau\} = \{\zeta : |Q(\zeta)| < t\tau\} = \mathcal{O}_{t\tau}.$$

Consequently

$$\begin{aligned}H_{iQ}^t(x) &= \frac{1}{(2\pi)^d}\lim_{\tau\to\infty}\int_{\mathcal{O}_\tau} e^{-itQ(\xi)-ix\cdot\xi}\,d\xi\\ &= \frac{t^{-\mu}}{(2\pi)^d}\lim_{\tau\to\infty}\int_{\mathcal{O}_{t\tau}} e^{-iQ(\zeta)-i(t^{-E^*}x)\cdot\zeta}\,d\zeta\\ &= t^{-\mu}H_{iQ}^1(t^{-E^*}x)\end{aligned}$$

for each $t>0$ and $x\in\mathbb{R}^d$; this is precisely the assertion made in Item 3. $\square$

3 Local limit Theorems

In this section, we establish local limit theorems for a class of complex-valued functions on $\mathbb{Z}^d$. To this end, we shall fix $\phi\in\mathcal{S}_d\subseteq\ell^1(\mathbb{Z}^d)$ which is suitably normalized so that $\sup_\xi|\widehat{\phi}|=1$ and satisfies various hypotheses concerning the nature of the points in $\Omega(\phi)=\{\xi\in\mathbb{T}^d:|\widehat{\phi}(\xi)|=1\}$ described below. All of our results include the assumption that every point of $\Omega(\phi)$ is either of positive homogeneous type or imaginary homogeneous type for $\widehat{\phi}$. Our main result, Theorem 3.8, allows for $\Omega(\phi)$, under certain conditions, to contain a mixture of points positive homogeneous and imaginary homogeneous type for $\widehat{\phi}$. This theorem partially extends Theorem 1.3 of [15] to the d-dimensional setting and it extends Theorem 1.6 of [16] to include points of imaginary homogeneous type. As we will see, Theorem 1.9 follows immediately from Theorem 3.8.

In view of (3), our analysis will done by studying $\widehat{\phi}(\xi)$ locally at each point $\xi_0\in\Omega(\phi)\subseteq\mathbb{T}^d$. For these local studies, it helpful (though not essential) for the points $\xi_0\in\Omega(\phi)$ to live in the interior of $\mathbb{T}^d$, which is not always the case (e.g., simple random walk on $\mathbb{Z}^d$). As we previously discussed, our hypothesis that all points of $\Omega(\phi)$ are either of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$ ensures that $\Omega(\phi)$ is a finite set and so it follows that, for some $\xi_\phi\in\mathbb{R}^d$,

$$\mathbb{T}^d_\phi:=\mathbb{T}^d+\xi_\phi$$

contains $\Omega(\phi)$ in its interior (as a subset of $\mathbb{R}^d$). Using this representation of the torus, we have

$$\phi^{(n)}(x)=\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d_\phi}\widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi \tag{29}$$

for all $x\in\mathbb{R}^d$ and $n\in\mathbb{N}_+$. The following lemma addresses the contribution from points $\xi_0\in\Omega(\phi)$ which are of positive homogeneous type for $\widehat{\phi}$. Though this lemma and its proof can be found, essentially, as Lemma 4.3 in [16], the proof we give here is simplified and allows us to highlight the ingredients which carry over easily to the imaginary homogeneous setting and those which do not.

Lemma 3.1. *Let $\xi_0\in\Omega(\phi)$ be of positive homogeneous type for $\widehat{\phi}$ with associated $\mu=\mu_{\xi_0}$, $\alpha=\alpha_{\xi_0}$ and $P=R+iQ$ where $R=R_{\xi_0}$ and $Q=Q_{\xi_0}$. Then, for any $\epsilon>0$, there exists an open neighborhood of $\mathcal{U}_{\xi_0}\subseteq\mathbb{T}^d_\phi$ of ξ_0, which can be taken as small as desired, and a natural number N for which*

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_{\xi_0}}\widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi-\widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0}H_P^n(x-n\alpha)\right|<\epsilon n^{-\mu}$$

for all $n\geq N$ and $x\in\mathbb{Z}^d$.

Proof. Let $\epsilon > 0$ be fixed and, for each $\tau > 0$, define $\mathcal{O}_\tau = \{\xi \in \mathbb{R}^d : R(\xi) < \tau\}$. Upon writing $\widetilde{R} = \widetilde{R}_{\xi_0}$, $\widetilde{Q} = \widetilde{Q}_{\xi_0}$ and $\widetilde{P} = \widetilde{R} + i\widetilde{Q}$, we have

$$\Gamma(\xi) := \Gamma_{\xi_0}(\xi) = i\alpha \cdot \xi - P(\xi) - \widetilde{P}(\xi)$$

on some convex neighborhood $\mathcal{U}$ of 0. With the help of Proposition 1.7, choose $\delta > 0$ for which $\mathcal{O}_\delta \subseteq \mathcal{U}$, $\mathcal{U}_{\xi_0} := \mathcal{O}_\delta + \xi_0$ is as small as desired and (minimally) ensures that $\mathcal{U}_{\xi_0} \subseteq \mathbb{T}^d_\phi$, and

$$\left|\widetilde{R}(\xi)\right| < R(\xi)/2 \tag{30}$$

whenever $\xi \in \mathcal{O}_\delta$. With this neighborhood $\mathcal{U}_{\xi_0}$ of ξ_0, we define

$$\mathcal{E} = \mathcal{E}_{n,x} = n^\mu \left| \frac{1}{(2\pi)^d} \int_{\mathcal{U}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0} H_P^n(x - n\alpha) \right|$$

for $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. It is clear that, to prove the lemma, we must find $N \in \mathbb{N}_+$ for which $\mathcal{E} < \epsilon$ for all $x \in \mathbb{Z}^d$ and $n \geq N$. To this end, we first make the change of vaiables $\xi \mapsto \xi + \xi_0$ to see that

$$\begin{aligned}
\int_{\mathcal{U}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi &= \int_{\mathcal{O}_\delta} \widehat{\phi}(\xi + \xi_0)^n e^{-ix\cdot(\xi+\xi_0)}\, d\xi \\
&= \int_{\mathcal{O}_\delta} \widehat{\phi}(\xi_0)^n e^{n\Gamma(\xi)} e^{-ix\cdot\xi_0} e^{-ix\cdot\xi}\, d\xi \\
&= \widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0} \int_{\mathcal{O}_\delta} e^{-n(P(\xi)+\widetilde{P}(\xi))} e^{-i(x-n\alpha)\cdot\xi}\, d\xi
\end{aligned} \tag{31}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. Consequently,

$$\mathcal{E} = n^\mu \left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_\delta} e^{-nP(\xi) - n\widetilde{P}(\xi)} e^{-i(x-n\alpha)\cdot\xi}\, d\xi - H_P^n(x - n\alpha) \right|$$

for $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. Now, let $E \in \operatorname{Exp}(R) \cap \operatorname{Exp}(Q) = \operatorname{Exp}(P)$ be as given by Definition 1.8 and observe that

$$H_P^n(x - n\alpha) = n^{-\mu} H_P^1(y)$$

where

$$y = y_{n,x} = n^{-E^*}(x - n\alpha)$$

by virtue of Proposition 2.1. With this $E \in \operatorname{Exp}(P)$, the change change of variables $\xi \mapsto n^{-E}\xi$ yields

$$\int_{\mathcal{O}_\delta} e^{-nP(\xi) - n\widetilde{P}(\xi)} e^{-i(x-n\alpha)\cdot\xi}\, d\xi = n^{-\mu} \int_{\mathcal{O}_{n\delta}} e^{-P(\xi) - n\widetilde{P}(n^{-E}\xi)} e^{-iy\cdot\xi}\, d\xi$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$ where we have made use of the fact that $n^E(\mathcal{O}_\delta) = \mathcal{O}_{n\delta}$. Consequently,

$$\mathcal{E} = \left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{n\delta}} e^{-P(\xi) - n\widetilde{P}(n^{-E}\xi)} e^{-iy\cdot\xi}\, d\xi - H_P^1(y) \right|$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$ where $y = n^{-E^*}(x - n\alpha)$. In view of Proposition 2.1 (applied to the positive homogeneous function R), let $\tau_0 > 0$ be such that

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d \setminus \mathcal{O}_{\tau_0}} e^{-R(\xi)}\, d\xi < \epsilon/2.$$

Thus, for any $\tau \geq \tau_0$,

$$\begin{aligned}
\left|\frac{1}{(2\pi)^d}\int_{\mathcal{O}_\tau} e^{-P(\xi)}e^{-iy\cdot\xi}\,d\xi - H_P^1(y)\right| &= \left|\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d\setminus\mathcal{O}_\tau} e^{-P(\xi)}e^{-iy\cdot\xi}\,d\xi\right| \\
&\leq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d\setminus\mathcal{O}_{\tau_0}} e^{-R(\xi)}\,d\xi \\
&< \epsilon/2 \qquad (32)
\end{aligned}$$

for all $y \in \mathbb{R}^d$. Observe that

$$\begin{aligned}
&\left|\frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)-n\widetilde{P}(n^{-E}\xi)}e^{-iy\cdot\xi}\,d\xi - \frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)}e^{-iy\cdot\xi}\,d\xi\right| \\
&\qquad = \frac{1}{(2\pi)^d}\left|\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)}\left(e^{-n\widetilde{P}(n^{-E}\xi)}-1\right)e^{-iy\cdot\xi}\,d\xi\right| \\
&\qquad \leq \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-R(\xi)}\left|e^{-n\widetilde{P}(n^{-E}\xi)}-1\right|\chi_{\mathcal{O}_{n\delta}}(\xi)\,d\xi \qquad (33)
\end{aligned}$$

for all $y \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$; here, for a measurable set A of $\mathbb{R}^d$, χ_A denotes the characteristic function of A. Upon noting that $n^{-E}\xi \in \mathcal{O}_\delta$ whenever $\xi \in \mathcal{O}_{n\delta}$, we see that

$$\left|e^{-n\widetilde{P}(n^{-E}\xi)}-1\right| \leq e^{-\widetilde{R}(n^{-E}\xi)}+1 \leq e^{nR(n^{-E}\xi)/2}+1 = e^{R(\xi)/2}+1 \leq 2e^{R(\xi)/2}$$

whenever $n^{-E}\xi \in \mathcal{O}_{n\delta}$ where we have used (30). Consequently,

$$e^{-R(\xi)}\left|e^{-n\widetilde{P}(n^{-E}\xi)}-1\right|\chi_{\mathcal{O}_{n\delta}}(\xi) \leq 2e^{-R(\xi)/2}$$

for all $\xi \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$ and we note that $\xi \mapsto 2e^{-R(\xi)/2} \in L^1(\mathbb{R}^d)$ thanks to Propsotion 2.1. Also, by virtue of the subhomogeneity of $\widetilde{R}$ and $\widetilde{Q}$ with respect to E, we have

$$\lim_{n\to\infty} n\widetilde{P}(n^{-E}\xi) = \lim_{n\to\infty} n\widetilde{R}(n^{-E}\xi) + i\lim_{n\to\infty} n\widetilde{Q}(n^{-E}\xi) = 0$$

for each $\xi \in \mathcal{O}_{n\delta}$ and so, for every $\xi \in \mathbb{R}^d$,

$$\lim_{n\to\infty} e^{-R(\xi)}\left|e^{-n\widetilde{P}(n^{-E}\xi)}-1\right|\chi_{\mathcal{O}_{n\delta}}(\xi) = 0.$$

We may therefore appeal to the dominated convergence theorem to produce a natural number $N \geq \tau_0/\delta$ so that, in view of (33),

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)-n\widetilde{P}(n^{-E}\xi)}e^{-iy\cdot\xi}\,d\xi - \frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)}e^{-iy\cdot\xi}\,d\xi\right| < \epsilon/2 \qquad (34)$$

for all $y \in \mathbb{R}^d$ and $n \geq N$. Upon noting that $n\delta \geq \tau_0$ whenever $n \geq N$, the estimates (32), (33), and (34) together yield

$$\begin{aligned}
\mathcal{E} \leq &\left|\frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)-n\widetilde{P}(n^{-E}\xi)}e^{-iy\cdot\xi}\,d\xi - \frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)}e^{-iy\cdot\xi}\,d\xi\right| \\
&+\left|\frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}} e^{-P(\xi)}e^{-iy\cdot\xi}\,d\xi - H_P^1(y)\right| < \epsilon
\end{aligned}$$

for all $x \in \mathbb{Z}^d$ and $n \geq N$, as desired. □

From the preceding lemma and in view of Item 3 of Proposition 2.1, we immediately obtain the following useful corollary. The corollary appears as Lemma 3.3 of [3] and Lemma 4.4 of [16].

Lemma 3.2. *Let $\xi_0 \in \Omega(\phi)$ be of positive homogeneous type for $\widehat{\phi}$ with associated homogeneous order $\mu_{\xi_0} > 0$. Then there exists an open neighborhood $\mathcal{U}_{\xi_0} \subseteq \mathbb{T}^d_\phi$ of ξ_0, which can be taken as small as desired, and a constant C_{ξ_0} for which*

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi\right| \leq C_{\xi_0} n^{-\mu_{\xi_0}}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$.

We shall now focus on the case in which $\xi_0 \in \Omega(\phi)$ is of imaginary-homogeneous type for $\widehat{\phi}$. In view of Definition 1.8, there is an open neighborhood $\mathcal{U}$ of 0 on which

$$\Gamma(\xi) = i\alpha\cdot\xi - i\left(Q(\xi) + \widetilde{Q}(\xi)\right) - \left(R(\xi) + \widetilde{R}(\xi)\right)$$

where $|Q|$ and R are both positive-homogeneous and there exists $E \in \operatorname{Exp}(Q) = \operatorname{Exp}(|Q|)$ and $\kappa > 1$ for which R is homogeneous with respect to E/κ, $\widetilde{Q}$ is strongly subhomogeneous with respect to E of order 2 and $\widetilde{R}$ is strongly subhomogeneous with respect to E/κ of order 1. Before we are able to prove a result analogous to Lemma 3.1, we shall first treat three preliminary lemmas, all of which are aimed at handling the oscillatory integrals which arise in this imaginary-homogeneous setting. For the reader's convenience, we remark that Lemmas 3.3 and 3.4 are established using basic properties of homogeneous and subhomogeneous functions. Lemma 3.5 is our central oscillatory integral estimate and its proof makes use of the generalized polar-coordinate integration formula of [3] (Theorem 1.3 of [3]) to decompose an oscillatory integral over $\mathbb{R}^d$ into a "radial" integral and a surface integral. The radial integral, which captures the primary oscillatory behavior, is then estimated by an application of the Van der Corput lemma (Proposition 2.5) using Lemma 3.3 to handle phase and Lemma 3.4 to handle the amplitude. We then present Lemma 3.6, our analogue of Lemma 3.1 for the imaginary-homogeneous setting; its proof makes use of Theorem 2.4 and Lemma 3.5 (and does not rely directly on Lemmas 3.3 or 3.4).

Lemma 3.3. *Let $Q : \mathbb{R}^d \to \mathbb{R}$ be a continuous function for which $|Q|$ is positive-homogeneous with unital level set S and homogeneous order μ. Also, given an open neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^d$, let $\widetilde{Q}$ be a real-valued function which is twice continuously differentiable on $\mathcal{U}$, i.e., $Q \in C^2(\mathcal{U};\mathbb{R})$. Given $E \in \operatorname{Exp}(Q) = \operatorname{Exp}(|Q|)$, define $F = E/\mu$ and*

$$f_{n,y,\eta}(\theta) = Q(\theta^F\eta) + n\widetilde{Q}(n^{-E}\theta^F\eta) + y\cdot(\theta^F\eta)$$

for $n \in \mathbb{N}_+$, $y \in \mathbb{R}^d$, $\eta \in S$ and $\theta > 0$ for which $n^{-E}\theta^F\eta \in \mathcal{U}$. If $\mu < 1$ and $\widetilde{Q}$ is strongly subhomogeneous with respect to E of order 2, then, for each compact set $K \subseteq \mathbb{R}^d$, there exist $\delta > 0$ and $\theta_0 \geq 1$ such that, for any natural number n such that $\theta_0 \leq (n\delta)^\mu$, $\partial_\theta f_{n,y,\eta}(\theta)$ is monotonic on $[\theta_0,(n\delta)^\mu]$ and

$$|\partial_\theta f_{n,y,\eta}(\theta)| \geq \frac{1}{2\mu}\theta^{1/\mu-1}$$

for all $\theta_0 \leq \theta \leq (n\delta)^\mu$, $y \in K$ and $\eta \in S$.

Proof. Let K be a compact subset of $\mathbb{R}^d$. Given that $\widetilde{Q}$ is strongly subhomogeneous with respect to E of order 2, let $\delta > 0$ be such that

$$\left|\partial_r\widetilde{Q}(r^E\eta)\right| \leq \frac{1}{4} \tag{35}$$

and

$$\left|r\partial_r^2\widetilde{Q}(r^E\eta)\right| \le \frac{1-\mu}{4} = \frac{\mu}{4}\left(\frac{1}{\mu}-1\right) \tag{36}$$

for all $0 < r < \delta$ and $\eta \in S$. Using the estimates

$$\left|y\cdot(\theta^{F-I}F\eta)\right| \le |y||F\eta|\|\theta^F\|\theta^{-1} \quad \text{and} \quad \left|y\cdot(\theta^F(F-I)F\eta)\right| \le |y||(F-I)F\eta|\|\theta^F\|,$$

the compactness of K and S, and the hypothesis that $\mu < 1$, an application of Corollary A.3 (with $\alpha = \mu$) hands us $\theta_0 \ge 1$ for which

$$\left|y\cdot(\theta^{F-I}F\eta)\right| \le \frac{1}{4\mu}\theta^{1/\mu-1} \tag{37}$$

and

$$\left|y\cdot(\theta^F(F-I)F\eta)\right| \le \frac{1}{4\mu}\left(\frac{1}{\mu}-1\right)\theta^{1/\mu} \tag{38}$$

for all $\theta \ge \theta_0$, $y \in K$ and $\eta \in S$. With these estimates in hand, let us now write $f = f_{n,y,\eta}$ and observe that, when $r = r_n(\theta) = \theta^{1/\mu}/n$, $r^E = n^{-E}\theta^F$ and so

$$f(\theta) = \theta^{1/\mu}Q(\eta) + n\widetilde{Q}(r^E\eta) + y\cdot(\theta^F\eta).$$

We have

$$\begin{aligned}
\partial_\theta f(\theta) &= \frac{1}{\mu}\theta^{1/\mu-1}Q(\eta) + n\partial_r\widetilde{Q}(r^E\eta)\frac{\partial r}{\partial\theta} + \partial_\theta\left(y\cdot(\theta^F\eta)\right) \\
&= \frac{1}{\mu}\theta^{1/\mu-1}Q(\eta) + n\partial_r\widetilde{Q}(r^E\eta)\frac{1}{\mu}\frac{\theta^{1/\mu-1}}{n} + y\cdot(\theta^{F-I}F\eta) \\
&= \frac{1}{\mu}\theta^{1/\mu-1}\left(Q(\eta) - \partial_r\widetilde{Q}(r^E\eta)\right) + y\cdot(\theta^{F-I}F\eta)
\end{aligned} \tag{39}$$

for all $n \in \mathbb{N}_+$, $y \in K$, $\eta \in S$, and $\theta > 0$ for which $r^E\eta = n^{-E}\theta^F\eta \in \mathcal{U}$. By virtue of (35) and the fact that $|Q(\eta)| = 1$ for all $\eta \in S$, we see that $r^E\eta \in \mathcal{U}$ and

$$\left|Q(\eta) - \partial_r\widetilde{Q}(r^E\eta)\right| \ge \frac{3}{4} \tag{40}$$

whenever $\eta \in S$ and $0 < \theta < (n\delta)^\mu$. Combining the estimates (37), (39) and (40), guarantees that

$$|\partial_\theta f(\theta)| \ge \frac{3}{4\mu}\theta^{1/\mu-1} - \frac{1}{4\mu}\theta^{1/\mu-1} = \frac{1}{2\mu}\theta^{1/\mu-1}$$

for all $n \in \mathbb{N}_+$, $y \in K$, $\eta \in S$ and $\theta_0 \le \theta \le (n\delta)^\mu$.

It remains to show that $\partial_\theta f(\theta)$ is monotonic. Making use of (36) and (38), analogous reasoning shows that

$$\left|\theta^2\partial_\theta^2 f(\theta)\right| \ge \frac{1}{4\mu}\left(\frac{1}{\mu}-1\right)\theta^{1/\mu} > 0$$

for all $n \in \mathbb{N}_+$, $y \in K$, $\eta \in S$ and $\theta_0 \le \theta \le (n\delta)^\mu$. In particular, $\partial_\theta^2 f(\theta)$ is non-vanishing on $[\theta_0, (n\delta)^\mu]$ and so $\partial_\theta f(\theta)$ is monotonic. □

Lemma 3.4. *Given a compact set $S \subseteq \mathbb{R}^d$ which does not contain 0 and a positive homogeneous function R, let m and M be positive constants for which*

$$m \leq R(\eta) \leq M$$

for all $\eta \in S$. Also, let $\mathcal{U} \subseteq \mathbb{R}^d$ be an open neighborhood of 0 and let $\widetilde{R}$ be a real-valued function which is once continuously differentiable on $\mathcal{U}$. Given $E \in \operatorname{End}(\mathbb{R}^d)$ for which $\{r^E\}$ is a contracting group, define $F := E/\mu$ where $\mu = \operatorname{tr} E$ and

$$g_{n,\eta}(\theta) = \exp\left(-nR(n^{-E}\theta^F\eta) - n\widetilde{R}(n^{-E}\theta^F\eta)\right)$$

for $n \in \mathbb{N}_+$, $\eta \in S$ and $\theta > 0$ for which $n^{-E}\theta^F\eta \in \mathcal{U}$. If, for some $\kappa > 0$, R is homogeneous with respect to E/κ and $\widetilde{R}$ is strongly subhomogeneous with respect to E/κ of order 1, then, for any $\beta > 1$, there exists a $\delta > 0$ for which

$$\|g_{n,\eta}\|_{L^\infty[\theta_1,\theta_2]} + \|\partial_\theta g_{n,\eta}\|_{L^1[\theta_1,\theta_2]} \leq 1 + \frac{\beta M}{m}$$

for all $n \in \mathbb{N}_+$, $\eta \in S$ and $0 < \theta_1 \leq \theta_2 \leq (n\delta)^\mu$.

Proof. Define

$$h_{n,\eta}(r) = \exp\left(-nR(r^E\eta) - n\widetilde{R}(r^E\eta)\right)$$

for $n \in \mathbb{N}_+$, $\eta \in S$ and $r > 0$ for which $r^E\eta \in \mathcal{U}$. Observe that, for $r = r_n(\theta) = \theta^{1/\mu}/n$, we have $r^E = n^{-E}\theta^F$ and

$$g_{n,\eta}(\theta) = h_{n,\eta}(r_n(\theta))$$

for all $n \in \mathbb{N}_+$, $\eta \in S$ and $\theta > 0$ for which $r^E\eta = n^{-E}\theta^F\eta \in \mathcal{U}$. Let us fix $\beta > 1$ and let $0 < \epsilon < 1$ be such that

$$\beta = \frac{1+\epsilon}{1-\epsilon}.$$

In view of our supposition $\widetilde{R}$ is strongly subhomogeneous with respect to E/κ, there exists $\delta_1 > 0$ for which

$$\left|\widetilde{R}(r^E\eta)\right| \leq \epsilon m r^\kappa$$

for all $0 < r < \delta_1$ and $\eta \in S$. Further, by virtue of the strong subhomogeneity of $\widetilde{R}$ with respect to E/κ, we can choose $\delta_2 > 0$ for which

$$\left|\partial_r \widetilde{R}(r^E\eta)\right| = \left|\frac{\partial \widetilde{R}((r^\kappa)^{E/\kappa}\eta)}{\partial(r^\kappa)}\right|\left|\frac{\partial(r^\kappa)}{\partial r}\right| < \epsilon M\kappa r^{\kappa-1}$$

for all $0 < r < \delta_2$. For $\delta := \min\{\delta_1, \delta_2\}$, the preceding estimates guarantee that

$$R(r^E\eta) + \widetilde{R}(r^E\eta) \geq r^\kappa R(\eta) - \epsilon m r^\kappa \geq m(1-\epsilon)r^\kappa \tag{41}$$

and

$$\left|\partial_r\left(R(r^E\eta) + \widetilde{R}(r^E\eta)\right)\right| \leq \kappa r^{\kappa-1}R(\eta) + \epsilon M\kappa r^{\kappa-1} \leq M(1+\epsilon)\kappa r^{\kappa-1} \tag{42}$$

for all $0 < r < \delta$ and $\eta \in S$. Thus, for all $n \in \mathbb{N}_+$, $\eta \in S$, and $0 < \theta_1 \leq \theta_2 \leq (\delta n)^\mu$ (equivalently, $0 < \rho_1 \leq \rho_2 \leq \delta$ where $\rho_j := r_n(\theta_j) = \theta_j^{1/\mu}/n$ for $j = 1, 2$),

$$\|g_{n,\eta}\|_{L^\infty[\theta_1,\theta_2]} = \sup_{\rho_1 \leq r \leq \rho_2}\left(e^{-n(R(r^E\eta)+\widetilde{R}(r^E\eta))}\right) \leq \sup_{0<r\leq\delta}\left(e^{-n(1-\epsilon)mr^\kappa}\right) \leq 1$$

by virtue of (41) and the fact that $\epsilon < 1$. Appealing to both (41) and (42), we have

$$\begin{aligned}
\|\partial_\theta g_{n,\eta}\|_{L^1[\theta_1,\theta_2]} &= \int_{\theta_1}^{\theta_2} |\partial_\theta g_{n,\eta}(\theta)|\,d\theta \\
&= \int_{\theta_1}^{\theta_2} |\partial_\theta h_{n,\eta}(r_n(\theta))|\,d\theta \\
&= \int_{\rho_1}^{\rho_2} |\partial_r h_{n,\eta}(r)|\,dr \\
&= \int_{\rho_1}^{\rho_2} n\Big|\partial_r\left(R(r^E\eta)+\widetilde{R}(r^E\eta)\right)\Big| e^{-n\left(R(r^E\eta)+\widetilde{R}(r^E\eta)\right)}\,dr \\
&\leq \int_{\rho_1}^{\rho_2} nM(1+\epsilon)\kappa r^{\kappa-1}e^{-n(1-\epsilon)mr^\kappa}\,dr \\
&\leq \frac{(1+\epsilon)M}{(1-\epsilon)m}\int_0^\infty e^{-u}\,du = \frac{\beta M}{m}
\end{aligned}$$

for all $n \in \mathbb{N}_+$, $\eta \in S$ and $0 < \theta_1 \leq \theta_2 \leq (\delta n)^\mu$. With this, our desired estimate follows without trouble. $\square$

Lemma 3.5. *Let $Q : \mathbb{R}^d \to \mathbb{R}$ be a continuous function for which $|Q|$ is positive-homogeneous with unital level set S and homogeneous order μ. Let $R : \mathbb{R}^d \to \mathbb{R}$ be a positive homogeneous function and, given an open neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^d$, let $\widetilde{Q} \in C^2(\mathcal{U};\mathbb{R})$ and $\widetilde{R} \in C^1(\mathcal{U};\mathbb{R})$. Suppose that, for some $E \in \operatorname{Exp}(Q) = \operatorname{Exp}(|Q|)$, $\widetilde{Q}$ is strongly subhomogeneous with respect to E of order 1 and, for some $\kappa > 0$, R is homogeneous with respect to E/κ and $\widetilde{R}$ is strongly subhomogeneous with respect to E/κ of order 2. Define*

$$\Psi_{n,y}(\xi) = nQ(n^{-E}\xi) + n\widetilde{Q}(n^{-E}\xi) + y\cdot\xi = Q(\xi) + n\widetilde{Q}(n^{-E}\xi) + y\cdot\xi \tag{43}$$

and

$$A_n(\xi) = \exp\Big(-n\left(R(n^{-E}\xi)+\widetilde{R}(n^{-E}\xi)\right)\Big) = \exp\Big(-n^{1-\kappa}R(\xi) - n\widetilde{R}(n^{-E}\xi)\Big) \tag{44}$$

for $n \in \mathbb{N}_+$, $y \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$ for which $n^{-E}\xi \in \mathcal{U}$. If $\mu < 1$, then, for each $\epsilon > 0$ and compact set $K \subseteq \mathbb{R}^d$, there is a $\delta > 0$ and $\tau_0 \geq 1$ for which

$$\left|\int_{\mathcal{O}_{n\delta}\setminus\mathcal{O}_\tau} e^{-i\Psi_{n,y}(\xi)}A_n(\xi)\,d\xi\right| \leq \epsilon$$

for all $y \in K$, $n \in \mathbb{N}_+$, and $\tau > 0$ for which $\tau_0 \leq \tau \leq n\delta$.

Proof. Let $\epsilon > 0$ and $K \subseteq \mathbb{R}^d$ be a compact set. Though we will further restrict the size of δ, for the moment, let $\delta > 0$ be small enough that $n^{-E}\xi \in \mathcal{U}$ for all $n \in \mathbb{N}_+$ and $\xi \in \mathcal{O}_\delta$. Just as we did leading up to the proof of Theorem 2.4, we shall denote by S the unital level set of $|Q|$ and, by virtue of Theorem 1.3 of [3], let σ be the Radon measure on S for which (25) holds. If $\tau \leq n\delta$,

$$\begin{aligned}
\int_{\mathcal{O}_{n\delta}\setminus\mathcal{O}_\tau} e^{-i\Psi_{n,y}(\xi)}A_n(\xi)\,d\xi &= \int_S\int_\tau^{n\delta} e^{-i\Psi_{n,y}(r^E\eta)}A_n(r^E\eta)r^{\mu-1}\,dr\sigma(d\eta) \\
&= \int_S I_{\tau,n,y}(\eta)\,\sigma(d\eta)
\end{aligned}$$

where

$$I_{\tau,n,y}(\eta)=\int_{\tau}^{n\delta}e^{-i\Psi_{n,y}(r^E\eta)}A_n(r^E\eta)r^{\mu-1}\,dr.$$

As in the proof of Theorem 2.4, we make the change of variables $r\mapsto\theta^{1/\mu}$ to observe that

$$I_{\tau,n,y}(\eta)=\int_{\tau^\mu}^{(n\delta)^\mu}e^{-i\Psi_{n,y}(\theta^F\eta)}A_n(\theta^F)\,d\theta=\int_{\tau^\mu}^{(n\delta)^\mu}e^{-if_{n,y,\eta}(\theta)}g_{n,\eta}(\theta)\,d\theta$$

where, for $F=E/\mu$,

$$f_{n,y,\eta}(\theta)=\Psi_{n,y}(\theta^F\eta)=Q(\theta^F\eta)+n\widetilde{Q}(n^{-E}\theta^F\eta)+y\cdot(\theta^F\eta),$$

and

$$g_{n,\eta}(\theta)=A_n(\theta^F\eta)=\exp\left(-n\left(R(n^{-E}\theta^F\eta)+\widetilde{R}(n^{-E}\theta^F\eta)\right)\right)$$

in the notation of the preceding two lemmas (with $E_=E$). Our immediate goal is to give a uniform estimate for $|I_{\tau,n,y}(\eta)|$ and we shall do this using the Van der Corput lemma. To this end, we first make an appeal to Lemma 3.3 to obtain $\delta_1>0$ and $\theta_0\geq 1$ be such that $\partial_\theta f_{n,y,\eta}(\theta)$ is monotonic on $[\theta_0,(n\delta)^\mu]$ and

$$|\partial_\theta f_{n,y,\eta}(\theta)|\geq\frac{1}{2\mu}\theta^{1/\mu-1}$$

for all $n\in\mathbb{N}_+$, $y\in K$, $\eta\in S$ and $\theta>0$ such that $\theta_0\leq\theta\leq(n\delta_1)^\mu$. Also, by an appeal to Lemma 3.4, choose $\delta_2>0$ for which

$$\|g_{n,\eta}\|_{L^\infty[\theta_1,\theta_2]}+\|\partial_\theta g_{n,\eta}\|_{L^1[\theta_1,\theta_2]}\leq 1+\frac{2M}{m}$$

for all $n\in\mathbb{N}_+$, $\eta\in S$ and $0<\theta_1\leq\theta_2\leq(n\delta_2)^\mu$ where $m=\inf_{\eta\in S}R(\eta)$ and $M=\sup_{\eta\in S}R(\eta)$, both of which are necessarily finite and positive because R is positive-homogeneous. Upon setting $\delta=\min\{\delta_1,\delta_2\}$ and selecting τ_0 such that $\tau_0\geq\theta_0^{1/\mu}$ and

$$\tau_0^{\mu-1}\leq\frac{\epsilon}{16\mu(1+2M/m)\sigma(S)},$$

we appeal to the Van der Corput lemma, Proposition 2.5, to see that

$$\begin{aligned}|I_{\tau,n,y}(\eta)| &= \left|\int_{\tau^\mu}^{(n\delta)^\mu}e^{-if_{n,y,\eta}(\theta)}g_{n,\eta}(\theta)\,d\theta\right|\\ &\leq 4\frac{\|g_{n,y,\eta}\|_{L^\infty[\tau^\mu,(n\delta)^\mu]}+\|\partial_\theta g_{n,y,\eta}\|_{L^\infty[\tau^\mu,(n\delta)^\mu]}}{\inf_{\theta\in[\tau^\mu,(n\delta)^\mu]}\theta^{1/\mu-1}/2\mu}\\ &\leq 4\left(1+\frac{2M}{m}\right)\frac{2\mu}{\tau^{1-\mu}}\\ &\leq 8\mu\left(1+\frac{2M}{m}\right)\tau_0^{\mu-1}\\ &\leq \frac{\epsilon}{2\sigma(S)}\end{aligned}$$

for all $n\in\mathbb{N}_+$, $y\in K$, $\eta\in S$, and $\tau>0$ for which $\tau_0\leq\tau\leq n\delta$. Thus,

$$\left|\int_{\mathcal{O}_{n\delta}\setminus\mathcal{O}_\tau}e^{-i\Psi_{n,y}(\xi)}A_n(\xi)\,d\xi\right|\leq\int_S|I_{\tau,n,y}(\eta)|\sigma(d\eta)\leq\int_S\frac{\epsilon}{2\sigma(S)}\sigma(d\eta)<\epsilon$$

for all $n\in\mathbb{N}_+$, $y\in K$, and $\tau>0$ for which $\tau_0\leq\tau\leq n\delta$. $\square$

With the preceding lemma in hand, we are now in a position to prove a limit statement analogous to Lemma 3.1 in the case that $\xi_0 \in \Omega(\phi)$ is a point of imaginary-homogeneous type for $\widehat{\phi}$.

Lemma 3.6. *Let $\xi_0 \in \Omega(\phi)$ be of imaginary homogeneous type for $\widehat{\phi}$ with associated $\alpha = \alpha_{\xi_0}$, $Q = Q_{\xi_0}$, and $\mu = \mu_{\xi_0}$. If $\mu < 1$, then, for any compact set $K \subseteq \mathbb{R}^d$ and $\epsilon > 0$, there exists an open neighborhood $\mathcal{U}_{\xi_0} \subseteq \mathbb{T}_\phi$ of ξ_0, which can be taken as small as desired, and a natural number N for which*

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{U}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0} H^n_{iQ}(x - n\alpha) \right| < \epsilon n^{-\mu}$$

whenever $n \geq N$ and $x \in \mathbb{Z}^d$ is such that $(x - n\alpha) \in n^{E^*}(K)$ for $E \in \operatorname{Exp}(Q) = \operatorname{Exp}(|Q|)$.

Proof. Let $\epsilon > 0$ and $K \subseteq \mathbb{R}^d$ be a compact set. Set $\mathcal{U}_{\xi_0} = \xi_0 + \mathcal{O}_\delta$ where $\delta > 0$ is yet to be specified but small enough to ensure that $\mathcal{O}_\delta \subseteq \mathcal{U}$ and $\mathcal{U}_{\xi_0} \subseteq \mathbb{T}^d_\phi$ is as small as desired. For $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$, define

$$\mathcal{E} = n^\mu \left| \frac{1}{(2\pi)^d} \int_{\mathcal{U}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0} H^n_{iQ}(x - n\alpha) \right|.$$

Also, let $E \in \operatorname{Exp}(|Q|) = \operatorname{Exp}(Q)$ be that which appears in Definition 1.8 and, for $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$, set

$$y = y_{n,x} = n^{-E^*}(x - n\alpha).$$

We observe that

$$\begin{aligned}
\mathcal{E} &= \left| \frac{n^\mu}{(2\pi)^d} \int_{\mathcal{O}_\delta} \widehat{\phi}^n(\xi + \xi_0) e^{-ix\cdot(\xi+\xi_0)}\, d\xi - \widehat{\phi}^n(\xi_0) e^{-ix\cdot\xi_0} H^n_{iQ}(n^{E^*} y) \right| \\
&= \left| \frac{n^\mu}{(2\pi)^d} \int_{\mathcal{O}_\delta} \widehat{\phi}^n(\xi_0) e^{-ix\cdot\xi_0} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi - \widehat{\phi}^n(\xi_0) e^{-ix\cdot\xi_0} H^n_{iQ}(n^{E^*} y) \right| \\
&= \left| \frac{n^\mu}{(2\pi)^d} \int_{\mathcal{O}_\delta} e^{n\Gamma(\xi)} e^{-ix\cdot\xi}\, d\xi - n^\mu H^n_{iQ}(n^{E^*} y) \right| \\
&= \left| \frac{n^\mu}{(2\pi)^d} \int_{\mathcal{O}_\delta} e^{n(\Gamma(\xi) - i\alpha\cdot\xi)} e^{-i(n^{E^*} y)\cdot\xi}\, d\xi - n^\mu H^n_{iQ}(n^{E^*} y) \right|
\end{aligned}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$ where we have made an analogous computation to (31) and written $\Gamma = \Gamma_{\xi_0}$. In view of (14), making the change of variables $\xi \mapsto n^{-E}\xi$ yields

$$\begin{aligned}
n^\mu \int_{\mathcal{O}_\delta} e^{n(\Gamma(\xi) - i\alpha\cdot\xi)} e^{-i(n^{E^*} y)\cdot\xi}\, d\xi &= n^\mu \int_{\mathcal{O}_\delta} \exp\left(-in\left(Q(\xi) + \widetilde{Q}(\xi)\right) - n\left(R(\xi) + \widetilde{R}(\xi)\right)\right) e^{-iy\cdot(n^E\xi)}\, d\xi \\
&= \int_{\mathcal{O}_{n\delta}} e^{-i\Psi_{n,y}(\xi)} A_n(\xi)\, d\xi
\end{aligned}$$

where $\Psi_{n,y}(\xi)$ and $A_n(\xi)$ are those defined in (43) and (44), respectively, and we have noted that $n^E(\mathcal{O}_\delta) = \mathcal{O}_{n\delta}$. Also, by virtue of (24), we have $n^\mu H^n_{iQ}(n^{E^*} y) = H^1_{iQ}(y)$. Consequently,

$$\mathcal{E} = \left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{n\delta}} e^{-i\Psi_{n,y}(\xi)} A_n(\xi)\, d\xi - H^1_{iQ}(y) \right|$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$.

Given that $\mathrm{Sym}(|Q|)$ is compact by virtue of Proposition 1.4, let K' be a compact subset of $\mathbb{R}^d$ for which $O^*(K) \subseteq K'$ for all $O \in \mathrm{Sym}(|Q|)$. One can take, for example, K' to be the closed ball of radius

$$M = \left(\sup_{x\in K} |x|\right)\left(\sup_{O\in \mathrm{Sym}(|Q|)} \|O^*\|\right) = \left(\sup_{x\in K} |x|\right)\left(\sup_{O\in \mathrm{Sym}(|Q|)} \|O\|\right) < \infty.$$

Suppose that, for some $E' \in \mathrm{Exp}(Q) = \mathrm{Exp}(|Q|)$ (which is possibly different from E), $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$, $(x - n\alpha) \in n^{E'^*}(K)$, then

$$y = n^{-E^*}(x - n\alpha) \subseteq n^{-E^*} n^{E'^*}(K) = (n^{E'} n^{-E})^*(K) \subseteq K'$$

by virtue of Proposition A.1 and the simple fact that $n^{E'} n^{-E} \in \mathrm{Sym}(|Q|)$. With this observation in mind, to prove the lemma, it suffices to find a $\delta > 0$ and an $N \in \mathbb{N}_+$ for which $\mathcal{E} < \epsilon$ for all $n \geq N$ and $y \in K'$.

By virtue of Lemma 3.5, let us now (and finally) fix $\delta > 0$ and $\tau_0 >$ for which

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{O}_{n\delta}\setminus\mathcal{O}_\tau} e^{-i\Psi_{n,y}(\xi)} A_n(\xi)\, d\xi\right| < \epsilon/3$$

for all $y \in K'$, $n \in \mathbb{N}_+$, and $\tau > 0$ for which $\tau_0 \leq \tau \leq n\delta$. By an appeal to Theorem 2.4, let $\tau_1 \geq \tau_0$ be such that

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{O}_{\tau_1}} e^{-iQ(\xi) - iy\cdot\xi}\, d\xi - H^1_{iQ}(y)\right| < \epsilon/3$$

for all $y \in K'$. Let us now observe that

$$\begin{aligned}\int_{\mathcal{O}_{\tau_1}} \left|e^{-i\Psi_{n,y}(\xi)} A_n(\xi) - e^{-iQ(\xi) - iy\cdot\xi}\right| d\xi &= \int_{\mathcal{O}_{\tau_1}} \left|e^{-n\left(i\widetilde{Q}(n^{-E}\xi) + R(n^{-E}\xi) + \widetilde{R}(n^{-E}\xi)\right)} - 1\right| d\xi\\ &= \int_{\mathcal{O}_{\tau_1}} \left|e^{-n\widetilde{P}(n^{-E}\xi)} - 1\right| d\xi\end{aligned}$$

where

$$\widetilde{P}(\zeta) := i\widetilde{Q}(\zeta) + R(\zeta) + \widetilde{R}(\zeta)$$

for $\zeta \in \mathbb{R}^d$. By our supposition that $\widetilde{Q}$ is subhomogeneous with respect to E, R is homogeneous with respect to E/κ with $\kappa > 1$ and $\widetilde{R}$ is subhomogeneous with respect to E/κ, from Proposition 1.7 it follows that $\widetilde{P}(\zeta) = o(|Q(\zeta)|)$ as $\zeta \to 0$. Thus, given that $\{n^E\}$ is a contracting group and $\mathcal{O}_{\tau_1}$ is relatively compact,

$$n\widetilde{P}(n^{-E}\xi) = \frac{\widetilde{P}(n^{-E}\xi)}{|Q(n^{-E}\xi)|}|Q(\xi)| \to 0 \qquad \text{as} \qquad n \to \infty$$

uniformly for $\xi \in \mathcal{O}_{\tau_1}$. Thus, there is a natural number $N \geq \tau_1/\delta$ for which

$$\frac{1}{(2\pi)^d}\int_{\mathcal{O}_{\tau_1}} \left|e^{-i\Psi_{n,y}(\xi)} A_n(\xi) - e^{-iQ(\xi) - iy\cdot\xi}\right| d\xi < \frac{\epsilon}{3}$$

for all $n \geq N$ and $y \in K'$. With these estimates, we observe that, for $y \in K'$ and $n \geq N$, $\tau_1 < N\delta \leq n\delta$

and

$$
\begin{aligned}
\mathcal{E} &= \Bigg| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{n\delta}\setminus\mathcal{O}_{\tau_1}} e^{-i\Psi_{n,y}(\xi)} A_n(\xi)\, d\xi \\
&\qquad + \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\tau_1}} e^{-i\Psi_{n,y}(\xi)} A_n(\xi)\, d\xi - \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\tau_1}} e^{-iQ(\xi)-iy\cdot\xi}\, d\xi \\
&\qquad\qquad + \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\tau_1}} e^{-iQ(\xi)-iy\cdot\xi}\, d\xi - H^1_{iQ}(y) \Bigg| \\
&\leq \left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{n\delta}\setminus\mathcal{O}_{\tau_1}} e^{-i\Psi_{n,y}(\xi)} A_n(\xi)\, d\xi \right| \\
&\qquad + \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\tau_1}} \left| e^{-i\Psi_{n,y}(\xi)} A_n(\xi) - e^{-iQ(\xi)-iy\cdot\xi} \right| d\xi \\
&\qquad\qquad + \left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\tau_1}} e^{-iQ(\xi)-iy\cdot\xi}\, d\xi - H^1_{iQ}(y) \right| \\
&< \epsilon,
\end{aligned}
$$

as desired. □

Corollary 3.7. *Let $\xi_0 \in \Omega(\phi)$ be of imaginary homogeneous type for $\widehat{\phi}$ with associated drift $\alpha_{\xi_0} \in \mathbb{R}^d$, homogeneous order $0 < \mu_{\xi_0}$, and polynomial Q_{ξ_0}. If $\mu_{\xi_0} < 1$, then, for any compact set $K \subseteq \mathbb{R}^d$, there exists an open neighborhood $\mathcal{U}_{\xi_0} \subseteq \mathbb{T}^d_\phi$ of ξ_0, which can be taken as small as desired, and a constant C_{ξ_0} for which*

$$
\left| \frac{1}{(2\pi)^d} \int_{\mathcal{U}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \right| \leq C_{\xi_0} n^{-\mu_{\xi_0}}
$$

whenever $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$ is such that $(x - n\alpha_{\xi_0}) \in n^{E^*}(K)$ for $E \in \operatorname{Exp}(Q_{\xi_0}) = \operatorname{Exp}(|Q_{\xi_0}|)$.

Proof. By virtue of the continuity of $y \mapsto H^1_{iQ_{\xi_0}}(y)$ ensured by Theorem 2.4, we note that

$$
M := \sup_{y\in K} \left| H_{iQ_{\xi_0}}(y) \right| < \infty
$$

and so, thanks to Item 3 of Theorem 2.4, it follows that

$$
\left| H^n_{iQ_{\xi_0}}(x - n\alpha_{\xi_0}) \right| = n^{-\mu_{\xi_0}} \left| H^1_{iQ_{\xi_0}}\left(n^{-E^*}(x - n\alpha_{\xi_0}) \right) \right| \leq n^{-\mu_{\xi_0}} M
$$

whenever $x \in n\alpha_{\xi_0} + n^{E^*}(K)$ for $E \in \operatorname{Exp}(Q_{\xi_0})$. Consequently, an application of Lemma 3.6 and the triangle inequality yield $C_{\xi_0} > 0$ for which the desired estimate holds for n sufficiently large. By modifying C_{ξ_0}, if necessary, we obtain the desired estimate for all $n \in \mathbb{N}_+$. □

Theorem 3.8. *Let $\phi \in \mathcal{S}_d$ be such that $\sup_\xi |\widehat{\phi}(\xi)| = 1$ and suppose that $\Omega(\phi)$ consists only of points of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$ and let $\mu_\phi > 0$ be the homogeneous order of ϕ defined by (16). Denote by $\Omega_p(\phi)$ and $\Omega_i(\phi)$ those point of $\Omega(\phi)$ of positive homogeneous type and imaginary homogeneous type for $\widehat{\phi}$, respectively. If $\Omega_i(\phi)$ is non-empty, assume additionally that*

(i) $\mu_\xi < 1$ for every $\xi \in \Omega_i(\phi)$,

(ii) there exists $\xi \in \Omega_i(\phi)$ for which $\mu_\xi = \mu_\phi$,

and

(iii) there exists $\alpha \in \mathbb{R}^d$ for which $\alpha_\xi = \alpha$ for all $\xi \in \Omega_i(\phi)$.

Then, there holds the following.

Case 1 In the case that $\Omega(\phi) = \Omega_p(\phi)$, i.e., every point of $\Omega(\phi)$ is of positive homogeneous type for $\widehat{\phi}$, let $\{\xi_1, \xi_2, \ldots, \xi_A\}$ be the set of points in $\Omega(\phi)$ for which $\mu_{\xi_k} = \mu_\phi$ and, for each $k = 1, 2, \ldots, A$, set $\alpha_k = \alpha_{\xi_k}$ and $P_k = P_{\xi_k} = R_{\xi_k} + iQ_{\xi_k}$ in view of Definition 1.8 and let H_{P_k} be the corresponding attractor defined in Proposition 2.1. Then,

$$\phi^{(n)}(x) = \sum_{k=1}^{A} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H_{P_k}^n(x - n\alpha_k) + o(n^{-\mu_\phi})$$

uniformly for $x \in \mathbb{Z}^d$.

Case 2 Alternatively, in the case that $\Omega_i(\phi)$ is nonempty, we have the following:

(a) If $\Omega(\phi) = \Omega_i(\phi)$ or $\mu_\xi > \mu_\phi$ for all $\xi \in \Omega_p(\phi)$, let $\{\xi_1, \xi_2, \ldots, \xi_A\}$ be the set of points in $\Omega(\phi)$ for which $\mu_{\xi_k} = \mu_\phi$. Necessarily, $\{\xi_1, \xi_2, \ldots, \xi_A\} \subseteq \Omega_i(\phi)$ and so, for each $k = 1, 2, \ldots, A$, let $Q_k = Q_{\xi_k}$ in view of Definition 1.8 and let H_{iQ_k} be the corresponding attractor defined in Theorem 2.4. Then, for each compact set $K \subseteq \mathbb{R}^d$, there exists a nested increasing sequence of compact sets $\{K_n\}$ all containing K and whose union is $\mathbb{R}^d$ for which

$$\phi^{(n)}(x) = \sum_{k=1}^{A} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H_{iQ_k}^n(x - n\alpha) + o(n^{-\mu_\phi})$$

uniformly for $x \in (n\alpha + K_n) \bigcap \mathbb{Z}^d$.

(b) If, otherwise, $\Omega_i(\phi)$ and $\Omega_p(\phi)$ both contain points of order μ_ϕ, we shall denote by $\{\xi_1, \xi_2, \ldots, \xi_{A_i}\}$ those points of $\Omega_i(\xi)$ for which $\mu_{\xi_k} = \mu_\phi$ for $k = 1, 2, \ldots, A_i$. Also, denote by $\{\zeta_1, \zeta_2, \ldots, \zeta_{A_p}\}$ those points of $\Omega_p(\xi)$ for which $\mu_{\zeta_j} = \mu_\phi$ for $j = 1, 2, \ldots A_p$. For each $k = 1, 2 \ldots, A_i$, let $Q_k = Q_{\xi_k}$ be that given in Definition 1.8 and let H_{iQ_k} be the associated attractor given in Theorem 2.4. For each $j = 1, 2 \ldots, A_p$, let $\alpha_j = \alpha_{\zeta_j} \in \mathbb{R}^d$ and $P_j = P_{\zeta_j} = R_{\zeta_j} + iQ_{\zeta_j}$ be those given by Definition 1.8 and let H_{P_j} be the associated attractor given in Propsition 2.1. Then, for each compact set $K \subseteq \mathbb{R}^d$, there exists a nested increasing sequence of compact sets $\{K_n\}$ all containing K and whose union is $\mathbb{R}^d$ for which

$$\phi^{(n)}(x) = \sum_{k=1}^{A_i} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H_{iQ_k}^n(x - n\alpha) + \sum_{j=1}^{A_p} \widehat{\phi}(\zeta_j)^n e^{-ix\cdot\zeta_j} H_{P_j}^n(x - n\alpha_j) + o(n^{-\mu_\phi})$$

uniformly for $x \in (n\alpha + K_n) \bigcap \mathbb{Z}^d$.

Proof. Throughout the proof, we will write $\mu = \mu_\phi$.

Case 1. Let $\epsilon > 0$. In view of our supposition that every point of $\Omega(\phi)$ is of positive homogeneous type for $\widehat{\phi}$, we write

$$\Omega(\phi) = \{\xi_1, \xi_2, \ldots, \xi_A, \ldots, \xi_B\}$$

where $\mu_{\xi_k} = \mu$ for $k = 1, 2, \ldots A$ and, if $A < k \leq B$, $\mu_{\xi_k} > \mu_\phi$. For $k = 1, 2, \ldots, A$, let $\alpha_k = \alpha_{\xi_k} \in \mathbb{R}^d$ and $P_k = P_{\xi_k} = R_{\xi_k} + iQ_{\xi_k}$ be those associated to ξ_k in view of Definition 1.8. By making simultaneous appeals to Lemma 3.1 and Corollary 3.2, we may choose a disjoint collection of open sets $\mathcal{U}_1, \mathcal{U}_2, \dots, \mathcal{U}_B \subseteq \mathbb{T}_\phi^d$, each containing $\xi_k \in \Omega(\phi)$ for the appropriate $k = 1, 2 \dots, B$, and a natural number N_1 for which, if $k = 1, 2, \dots, A$,

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H_{P_k}^n(x - n\alpha_k) \right| < \frac{\epsilon n^{-\mu}}{2B} \tag{45}$$

for all $n \geq N_1$, $x \in \mathbb{Z}^d$ and, if $A < k \leq B$,

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \right| < C_k n^{-\mu_{\xi_k}}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$; here, $C_k > 0$. We remark that, for $A < k \leq B$, $n^{-\mu_{\xi_k}} = o(n^{-\mu})$ as $n \to \infty$ because $\mu < \mu_{\xi_k}$. Thus, there exists a natural number N_2 for which

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \right| < \frac{\epsilon n^{-\mu}}{2B} \tag{46}$$

for $n \geq N_2$, $x \in \mathbb{Z}^d$ and $A < k \leq B$. For the compact set

$$G = \mathbb{T}_\phi^d \setminus \left(\bigcup_{k=1}^{B} \mathcal{U}_k \right),$$

we observe that, for each $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$,

$$\left| \frac{1}{(2\pi)^d} \int_G \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \right| \leq \frac{1}{(2\pi)^d} \int_G |\widehat{\phi}(\xi)|^n\, d\xi \leq s^n$$

where

$$s := \sup_{\xi \in G} |\widehat{\phi}(\xi)| < 1$$

because $\Omega(\phi) \subseteq \bigcup_{k=1}^B \mathcal{U}_k$. Upon noting that $s^n = o(n^{-\mu})$ as $n \to \infty$, we can therefore select $N_3 \in \mathbb{N}_+$ for which

$$\left| \frac{1}{(2\pi)^d} \int_G \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi \right| \leq s^n < \frac{\epsilon n^{-\mu}}{2} \tag{47}$$

for all $n \geq N_3$ and $x \in \mathbb{Z}^d$. By virtue of the identity (29), observe that

$$\begin{aligned}
&\left|\phi^{(n)}(x) - \sum_{k=1}^{A} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H^n_{P_k}(x - n\alpha_k)\right| \\
&= \left|\frac{1}{(2\pi)^d}\int_{\mathbb{T}^d_\phi} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \sum_{k=1}^{A} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H^n_{P_k}(x - n\alpha_k)\right| \\
&= \left|\sum_{k=1}^{B} \frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi + \frac{1}{(2\pi)^d}\int_{G} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \sum_{k=1}^{A} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H^n_{P_k}(x - n\alpha_k)\right| \\
&= \left|\sum_{k=1}^{A} \left(\frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H^n_{P_k}(x - n\alpha_k)\right)\right. \\
&\qquad\qquad \left. + \sum_{k=A+1}^{B} \frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi + \frac{1}{(2\pi)^d}\int_{G} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi\right| \\
&\leq \sum_{k=1}^{A} \left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi - \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H^n_{P_k}(x - n\alpha_k)\right| \\
&\qquad\qquad \sum_{k=A+1}^{B} \left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi\right| + \left|\frac{1}{(2\pi)^d}\int_{G} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\, d\xi\right| \qquad (48)
\end{aligned}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. Making use of the estimates (45), (46), and (47), the preceding inequality guarantees that

$$\left|\phi^{(n)}(x) - \sum_{k=1}^{A} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H^n_{P_k}(x - n\alpha_k)\right| < B\left(\frac{\epsilon n^{-\mu}}{2B}\right) + \frac{\epsilon n^{-\mu}}{2} = \epsilon n^{-\mu}$$

whenever $n \geq N = \max\{N_1, N_2, N_3\}$ and $x \in \mathbb{Z}^d$, as desired. //

Case 2a. Let $\epsilon > 0$ and $K \subseteq \mathbb{R}^d$ be compact. In view of our supposition, we can write

$$\Omega(\phi) = \{\xi_1, \xi_2, \dots, \xi_A, \cdots, \xi_B\}$$

where, for each $k = 1, 2, \dots, A$, ξ_k is a point of imaginary homogeneous type for $\widehat{\phi}$ with homogeneous order $\mu_{\xi_k} = \mu < 1$ and, for $A < k \leq B$, ξ_k is a point of positive homogeneous type or imaginary homogeneous type for $\widehat{\phi}$ with homogeneous order $\mu < \mu_{\xi_k}$; in the case that ξ_k is of imaginary homogeneous type, we also know that $\mu < \mu_{\xi_k} < 1$.

We now construct the sequence of compact sets $\{K_n\}$. For each $k = 1, 2, \dots, B$ for which ξ_k is of imaginary homogeneous type for $\widehat{\phi}$ with associated $Q_k = Q_{\xi_k}$, let $E_k \in \mathrm{Exp}(Q_k) = \mathrm{Exp}(|Q_k|)$. Define $\psi_k : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ by

$$\psi_k(s,x) = \begin{cases} s^{E_k^*}x & s > 0 \\ 0 & s = 0 \end{cases}$$

for $s \geq 0$ and $x \in \mathbb{R}^d$. Upon noting that $\{t^{E_k^*}\}$ is a contracting group (this follows from Proposition 1.3 and the fact that $(t^{E_k})^* = t^{E_k^*}$), it is straightforward to verify that ψ_k is continuous. Let us fix a closed ball $\overline{B} = \overline{B_R(0)}$ containing K and observe that

$$J_k := \psi_k([0,1] \times \overline{B}) = \bigcup_{0<r\leq 1} r^{E_k^*}(\overline{B})$$

which is necessarily compact by virtue of the continuity of ψ_k. Thanks to the above identity, we see that, for any $t \geq 1$, we have

$$J_k \subseteq \bigcup_{0<s\leq t} s^{E_k^*}(\overline{B}) = \bigcup_{0<r\leq 1} (t\cdot r)^{E_k^*}(\overline{B}) = t^{E_k^*}\left(\bigcup_{0<r\leq 1} r^{E_k^*}(\overline{B})\right) = t^{E_k^*}(J_k).$$

Consequently, for any natural numbers $n \leq m$, we have

$$n^{E_k^*}(J_k) \subseteq m^{E_k^*}(J_k).$$

With the above observations in mind, and in view of the fact that $\{t^{E_k^*}\}$ is contracting, it follows that

$$J_k^n := n^{E_k^*}(J_k)$$

is a nested increasing sequence of compact sets all containing K and whose union is $\mathbb{R}^d$. Finally, for each $n \in \mathbb{N}_+$, we define

$$K_n = \bigcap_{\substack{k=1,2,\dots,B\\ \xi_k\in\Omega_i(\phi)}} J_k^n = \bigcap_{\substack{k=1,2,\dots,B\\ \xi_k\in\Omega_i(\phi)}} n^{E_k^*}(J_k)$$

From our construction of J_k^n, it is straightforward to see that $\{K_n\}$ is a nested increasing sequence of compact sets all containing K and whose union is $\mathbb{R}^d$.

We now make simultaneous appeals to Lemma 3.6, Corollary 3.2, and Corollary 3.7 to produce a collection of disjoint open sets $\mathcal{U}_1, \mathcal{U}_2, \dots \mathcal{U}_B \subseteq \mathbb{T}_\phi^d$, each containing $\xi_k \in \Omega(\phi)$ for the appropriate $k = 1, 2, \dots B$, and a natural number N_1 for which the following estimates hold. For $k = 1, 2, \dots, A$,

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi - \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H_{iQ_k}^n(x-n\alpha)\right| < \frac{\epsilon n^{-\mu}}{2B} \tag{49}$$

whenever $n \geq N_1$, $x \in \mathbb{Z}^d$, and $(x - n\alpha) \in K_n \subseteq n^{E_k^*}(J_k)$. If $A < k \leq B$ and $\xi_k \in \Omega_p(\phi)$,

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi\right| < C_k n^{-\mu_{\xi_k}}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. If $A < k \leq B$ and $\xi_k \in \Omega_i(\phi)$,

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi\right| < C_k n^{-\mu_{\xi_k}}$$

whenever $n \in \mathbb{N}_+$, $x \in \mathbb{Z}^d$ and $(x - n\alpha) \in K_n \subseteq n^{E_k^*}(J_k)$. Upon noting that $n^{-\mu_{\xi_k}} = o(n^{-\mu})$ as $n \to \infty$ whenever $A < k \leq B$, we may further choose a natural number N_2 for which

$$\left|\frac{1}{(2\pi)^d}\int_{\mathcal{U}_k} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi\right| < \frac{\epsilon n^{-\mu}}{2B} \tag{50}$$

whenever $n \geq N_1$, $x \in \mathbb{Z}^d$ and $(x - n\alpha) \in K_n$. Finally, as in Case 1, we select a natural number N_3 for which

$$\left|\frac{1}{(2\pi)^d}\int_{G} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi}\,d\xi\right| < \frac{\epsilon n^{-\mu}}{2} \tag{51}$$

for all $n \geq N_3$ and $x \in \mathbb{Z}^d$ where $G = \mathbb{T}^d_\phi \setminus \bigcup_{k=1}^B \mathcal{U}_k$. By an analogous argument to that given in Case 1, we combine the estimates (29), (49), (50), and (51) to find that

$$\left|\phi^{(n)}(x) - \sum_{k=1}^{A} \widehat{\phi}(\xi_k)^n e^{-ix\cdot\xi_k} H^n_{iQ_k}(x - n\alpha)\right| < \epsilon n^{-\mu}$$

for all $n \geq N = \max\{N_1, N_2, N_3\}$ and $x \in \mathbb{Z}^d$ for which $(x - n\alpha) \in K_n$. //

Case 2b. The proof of this final case is almost identical to the proof of Case 2a. The only difference is that, in addition to appealing to Lemma 3.6 to handle those points $\xi_k \in \Omega_i(\phi)$ for which $\mu_{\xi_k} = \mu$, one also appeals to Lemma 3.1 to obtain local limits for those points $\eta_j \in \Omega_p(\phi)$ for which $\mu_{\eta_j} = \mu$. We leave this argument to the interested and committed reader. //

□

Proof of Theorem 1.9. We assume that $\Omega(\phi) = \{\xi_0\}$ where ξ_0 is of positive homogeneous type for $\widehat{\phi}$ or ξ_0 is of imaginary homogeneous type for $\widehat{\phi}$ with $\mu_\phi = \mu_{\xi_0} < 1$. In the positive-homogeneous case, the stated local limit theorem follows directly from Case 1 of Theorem 3.8 with $A = 1$. In the imaginary-homogeneous case, the hypotheses (i)-(iii) of Theorem 3.8 are met automatically and we find ourselves in Case 2a with $A = 1$ because $\Omega(\phi) = \Omega_i(\phi) = \{\xi_0\}$. Let us select a compact set $K \subseteq \mathbb{R}^d$ and, using the argument given in the proof of Lemma 3.6, choose a compact set $K' \subseteq \mathbb{R}^d$ for which $O^*(K) \subseteq K'$ for all $O \in \mathrm{Sym}(|Q_{\xi_0}|)$. With K' in hand, an appeal to Theorem 3.8 guarantees that

$$\phi^{(n)}(x) = \widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0} H^n_{iQ_{\xi_0}}(x - n\alpha_{\xi_0}) + o(n^{-\mu_\phi})$$

uniformly for $x \in (n\alpha_{\xi_0} + K_n) \cap \mathbb{Z}^d$ where K_n is a sequence of compact sets containing K' and whose union is $\mathbb{R}^d$. Upon careful study of the construction of the sets K_n in the proof of Theorem 3.8, we see that, for a chosen $E_0 \in \mathrm{Exp}(|Q_{\xi_0}|)$,

$$n^{E_0^*}(K') \subseteq n^{E_0^*}(\overline{B}) \subseteq n^{E_0}(J_0) = K_n$$

for each $n \in \mathbb{N}_+$ where $\overline{B} = \overline{B_R(0)}$ is a closed ball containing K' and $J_0 = \bigcup_{0<r\leq 1} r^{E_0^*}(\overline{B})$. Consequently, for any $E \in \mathrm{Exp}(Q_{\xi_0}) = \mathrm{Exp}(|Q_{\xi_0}|)$ and $n \in \mathbb{N}_+$, we have

$$n^{E^*}(K) = n^{E_0^*}(n^E n^{-E_0})^*(K) \subseteq n^{E_0^*}(K') \subseteq K_n$$

because $n^E n^{-E_0} \in \mathrm{Sym}(|Q_{\xi_0}|)$. Thus, for any $E \in \mathrm{Exp}(Q_{\xi_0}) = \mathrm{Exp}(|Q_{\xi_0}|)$, we have

$$\phi^{(n)}(x) = \widehat{\phi}(\xi_0)^n e^{-ix\cdot\xi_0} H^n_{iQ_{\xi_0}}(x - n\alpha_{\xi_0}) + o(n^{-\mu_\phi})$$

uniformly for $x \in (n\alpha_{\xi_0} + n^{E^*}(K)) \cap \mathbb{Z}^d$. □

4 Examples

Some examples

A Appendix

A.1 Contracting Groups

As discussed in the introduction, we shall denote by $\mathrm{End}(\mathbb{R}^d)$ the ring of (linear) endomorphisms of $\mathbb{R}^d$ which we take to be equipped with the operator norm $\|\cdot\|$ (inherited from the usual Euclidean norm $|\cdot|$ on $\mathbb{R}^d$). For a given $A \in \mathrm{End}(\mathbb{R}^d)$, we shall denote by $\det(A)$, $\operatorname{tr} A$ and A^*, its determinant, trace, and adjoint/transpose, respectively. The associated general linear group will be denoted by $\mathrm{Gl}(\mathbb{R}^d)$ and its identity element by I. Given $E \in \mathrm{End}(\mathbb{R}^d)$, we define

$$T_r = r^E = \exp(\ln(r)E) = \sum_{k=0}^{\infty} \frac{(\ln(r))^k}{k!} E^k$$

for $r > 0$. The following amasses some basic facts about $T_r = r^E$; proofs can be found in the references [7, 9, 10].

Proposition A.1. *For $E, F \in \mathrm{End}(\mathbb{R}^d)$ and $A \in \mathrm{Gl}(\mathbb{R}^d)$, the following properties hold:*

1. *For every $r > 0$, $r^E \in \mathrm{Gl}(\mathbb{R}^d)$.*
2. *For every $r > 0$, $(r^E)^* = r^{E^*}$.*
3. *For every $r > 0$, $\det(r^E) = r^{\operatorname{tr} E}$.*
4. *For every $r > 0$, $A^{-1} r^E A = r^{A^{-1}EA}$.*
5. *For every $r \geq 1$, $\|r^E\| \leq r^{\|E\|}$.*
6. *If $EF = FE$, then $r^E r^F = r^{E+F}$ for every $r > 0$.*

Finally, the map $(0,\infty) \ni r \mapsto r^E \in \mathrm{Gl}(\mathbb{R}^d)$ is a Lie group homomorphism. In particular, it is continuous and satisfies:

1. *$1^E = I$*
2. *For each $r > 0$, $r^{-E} = (1/r)^E = (r^E)^{-1}$.*
3. *For each $r, s > 0$, $r^E s^E = (ts)^E$.*

As guaranteed by the preceding proposition, for each $E \in \mathrm{End}(\mathbb{R})$, $\{r^E\}$ is a subgroup of $\mathrm{Gl}(\mathbb{R}^d)$ which we commonly refer to as a one-parameter group. As stated in the introduction, the one-parameter group $\{r^E\}$ is said to be **contracting** if

$$\lim_{r\to 0} \|r^E\| = 0.$$

The contracting property for $\{r^E\}$ is easily seen to be equivalent to Lyapunov stability of the (additive) one-parameter group $\mathbb{R} \ni t \mapsto e^{tE}$ (see [7]). We refer the reader to Appendix A of [3] which contains many results on one-parameter contracting groups, many of which are used in this article. In particular, Proposition A.2 of [3] guarantees that $\{r^E\}$ is contracting if and only if, for each $\xi \in \mathbb{R}^d$,

$$\lim_{r\to 0} |r^E \xi| = 0.$$

Perhaps unsurprisingly, this result is a consequence of the Banach-Steinhaus theorem. the preceding sentence can probably be omitted. This break is an excellent place to put additional information that we don't want to prove here. For the remainder of this subsection, we focus on two results concerning the large-r behavior of one-parameter contracting groups, neither of which can be found in [3]. By definition, contracting groups $\{r^E\}$ enjoy the property that their norms are well controlled as $r \to 0$. On the other hand, Item 5 of Proposition A.1 informs the large-r behavior of $\|r^E\|$ based on the norm $\|E\|$. The following lemma gives us an estimate for $\|r^E\|$ using E's trace.

Lemma A.2. *Let $E \in \mathrm{End}(\mathbb{R}^d)$ be such that $\{r^E\}$ is a contracting group and suppose that $\operatorname{tr} E < 1$. Then, for any $\epsilon > 0$ there is an $r_0 \geq 1$ for which*

$$\|r^E\| \leq \epsilon r$$

for all $r \geq r_0$. In other words, $\|r^E\| = o(r)$ as $r \to \infty$.

Proof. We write $\mathrm{Spec}(E) = \{\lambda_1, \lambda_2, \dots, \lambda_d\}$. Our assumption that $\{r^E\}$ is contracting guarantees that $\mathrm{Re}(\lambda_k) > 0$ for all $k = 1, 2, \dots, d$ by virtue of the classical Lyapunov theorem (see, e.g., Theorem 2.10 of [7]). Set $\rho = \max_{k=1,2,\dots,d} \mathrm{Re}(\lambda_k)$ and observe that

$$0 < \rho \leq \sum_{k=1}^{d} \mathrm{Re}(\lambda_k) = \mathrm{Re}\left(\sum_{k=1}^{d} \lambda_k\right) = \mathrm{Re}(\operatorname{tr} E) = \operatorname{tr} E < 1.$$

Now, for $A \in \mathrm{End}(\mathbb{C}^d)$, we denote by $\|A\|'$ its operator norm. It is easy to see that the, the inclusion map $\iota : (\mathrm{End}(\mathbb{R}^d), \|\cdot\|) \to (\mathrm{End}(\mathbb{C}^d), \|\cdot\|')$ is a contraction, i.e, for all $A \in \mathrm{End}(\mathbb{R}^d)$, $\|A\| \leq \|A\|'$. Viewing E as an element of $\mathrm{End}(\mathbb{C}^d)$ and making use of the Jordan-Chevelley decomposition, we write $E = D + N$ where $D \in \mathrm{End}(\mathbb{C}^d)$ is diagonalizable with $\mathrm{Spec}(D) = \mathrm{Spec}(E)$, $N \in \mathrm{End}(\mathbb{C}^d)$ is nilpotent and $DN = ND$. Because D is diagonalizable, there is a constant $M \geq 1$ for which

$$\|r^D\|' = \|\exp(\ln(r)D)\|' \leq M \max_{\lambda \in \mathrm{Spec}(D)} \left| e^{\ln(r)\lambda} \right| \leq M \max_{\lambda \in \mathrm{Spec}(D)} e^{\ln(r)\,\mathrm{Re}(\lambda)} = M r^\rho$$

for $r \geq 1$ where we have used the fact that $\mathrm{Spec}(E) = \mathrm{Spec}(D)$. Thus, by virtue of the fact that N and D commute and N is nilpotent, we have

$$\|r^E\| \leq \|r^E\|' = \|r^N r^D\|' \leq M \|r^N\|' r^\rho \leq P(\ln(r)\|N\|') r^\rho$$

for $r \geq 1$ where P is a polynomial. In view of the logarithm's slow growth and the fact that $\rho \leq \operatorname{tr} E < 1$, the preceding inequality guarntees that, for any $\rho < \omega < 1$, there is an $M' \geq 1$ for which

$$\|r^E\| \leq M' r^\omega = \left(M' r^{\omega - 1}\right) r$$

for all $r \geq 1$. With this, the desired estimate follows immediately. $\square$

The following corollary follows immediately from the lemma above.

Corollary A.3. *Let $E \in \mathrm{End}(\mathbb{R}^d)$ and, for $\alpha > 0$, define $F = E/\alpha$. If $\{r^E\}$ is a contracting group, then $\{\theta^F\}_{\theta > 0}$ is a contracting group. Further, if $\operatorname{tr} E < 1$, then, for any $\epsilon > 0$, there is a $\theta_0 \geq 1$ for which*

$$\|\theta^F\| \leq \epsilon \theta^{1/\alpha}$$

for all $\theta \geq \theta_0$.

A.2 Homogeneous and Subhomogeneous functions

As mentioned in the introduction, the article [3] develops the theory of positive homogeneous functions and related subhomogeneous functions. This development is more complete than that treated (or needed) in the present article and, for this reason, we have often appealed directly to the results of [3]. In this short appendix, we present two results concerning positive homogeneous and subhomogeneous functions which are of particular interest for us.

Lemma A.4. *Let $Q : \mathbb{R}^d \to \mathbb{R}$ be continuous and suppose that $|Q|$ is positive-homogeneous. Then* $\operatorname{Exp}(Q) = \operatorname{Exp}(|Q|)$.

Proof. If $d = 1$, it is easy to see that Q is necessarily of the form

$$Q(\xi) = \begin{cases} Q(1)|\xi|^\alpha & \xi \geq 0 \\ Q(-1)|\xi|^\alpha & \xi < 0 \end{cases}$$

for $\xi \in \mathbb{R}$ where $\alpha > 0$. In this case, it is easy to see that $\operatorname{Exp}(Q) = \operatorname{Exp}(|Q|) = \{E_{1/\alpha}\}$ where $E_{1/\alpha} \in \operatorname{End}(\mathbb{R})$ is the transformation taking ξ to ξ/α. For $d > 1$, given that Q is non-vanishing on the connected set $\mathbb{R}^d \setminus \{0\}$, Q must be single signed, i.e., given any fixed non-zero $\xi_0 \in \mathbb{R}^d$, $Q(\xi) = \operatorname{sgn}(Q(\xi_0))|Q(\xi)|$ for all $\xi \in \mathbb{R}^d$ and from this it follows immediately that Q and $|Q|$ share the same exponent set. ☐

Proof of Proposition 1.7. In what follows, P is a positive homogeneous function and $\widetilde{P}$ is a continuous complex-valued function defined on an open set $\mathcal{U}$ of $\mathbb{R}^d$.

1 $\Longrightarrow$ 2. Let $E \in \operatorname{Exp}(P)$ and fix $\epsilon > 0$ and a compact set $K \subseteq \mathbb{R}^d$. By our supposition, let $\mathcal{O} \subseteq \mathcal{U}$ be a neighborhood of 0 for which

$$\left|\widetilde{P}(\zeta)\right| \leq \frac{\epsilon}{M+1} P(\zeta)$$

for all $\zeta \in \mathcal{O}$ where

$$M = \sup_{\xi \in K} P(\xi).$$

Since $\{t^E\}$ is contracting thanks to Proposition 1.3, there exists $\tau > 0$ for which $t^E\xi \in \mathcal{O}$ for all $0 < t < \tau$ and $\xi \in K$ (Proposition A.6 of [3]). Consequently, for any $\xi \in K$ and $0 < t < \tau$,

$$\left|\widetilde{P}(t^E\xi)\right| \leq \frac{\epsilon}{M+1} P(t^E\xi) = \epsilon t \frac{P(\xi)}{M+1} \leq \epsilon t \frac{M}{M+1} < \epsilon t.$$

//

2 $\Longrightarrow$ 3. This is immediate. //

3 $\Longrightarrow$ 1. Let $\epsilon > 0$. Because the unital level set S of P is compact (Proposition 1.3), our supposition guarantees $\tau > 0$ for which

$$\left|\widetilde{P}(t^E\eta)\right| \leq \epsilon t$$

for all $0 < t < \tau$ and $\eta \in S$. For the open set $\mathcal{O}_\tau = \{\zeta \in \mathbb{R}^d : P(\zeta) < \tau\}$, we claim that

$$\mathcal{O}_\tau \setminus \{0\} = \{t^E\eta : 0 < t < \tau, \eta \in S\}.$$

To see this, first suppose that $\zeta \in \mathcal{O}_\tau \setminus \{0\}$ or, equivalently, $0 < P(\zeta) < \tau$. Then, for $t = P(\zeta) \in (0,\tau)$, observe that $\eta := t^{-E}\zeta \in S$ because $P(\eta) = P(t^{-E}\zeta) = P(\zeta)/t = 1$. Consequently, $\zeta = t^E\eta$ for $\eta \in S$ and $0 < t < \tau$ and so $\mathcal{O}_\tau \setminus \{0\} \subseteq \{t^E\eta : 0 < t < \tau, \eta \in S\}$. Of course, for $0 < t < \tau$ and $\eta \in S$, $P(t^E\eta) = tP(\eta) = t \in (0,\tau)$ and so we have justified our claim.

With this identification, we observe that

$$\left|\widetilde{P}(\zeta)\right| \leq \epsilon t = \epsilon P(\zeta).$$

for each $\zeta = t^E\eta \in \mathcal{O} \setminus \{0\}$. By the continuity of $\widetilde{P}$, it immediately follows that $\widetilde{P}(0) = 0 \leq \epsilon P(0)$. Thus, we have found an open neighborhood $\mathcal{O} = \mathcal{O}_\tau$ of 0 for which

$$\left|\widetilde{P}(\zeta)\right| \leq \epsilon P(\zeta)$$

for all $\zeta \in \mathcal{O}$ which is precisely the statement that $\widetilde{P}(\xi) = o(P(\xi))$ as $\xi \to 0$. //

□

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