1 Moment of Inertial of an Off-Balance, Non-Uniform Cylinder

(a) $I_{ij} = \int dm \left(\delta_{ij} \vec{r}^2 - r_i r_j\right)$

$I_{x'y'} = I_{x'z'} = I_{y'z'} = 0$

$I_{z'z'} = \int_{-b/2}^{b/2} dz \int_0^a dr (2\pi r) r^2 \rho_0 (r/a)^2 = \frac{\pi \rho_0 a^4 b}{3} = \lambda_3$

$$
\begin{aligned}
I_{x'x'} = I_{y'y'} &= \int_{-b/2}^{b/2} dz \int_0^a r\,dr \int_0^{2\pi} d\theta \left(r^2 \sin^2\theta + z^2\right) \rho_0 (r/a)^2 \\
&= \int_{-b/2}^{b/2} dz \int_0^a dr \left(\pi r^2 + 2\pi z^2\right) r^3 \rho_0 / a^2 \\
&= \int_0^a dr\, \pi \left(b r^2 + \frac{1}{6} b^3\right) r^3 \rho_0 / a^2 \\
&= \frac{\pi \rho_0}{24} a^2 b \left(b^2 + 4a^2\right) = \lambda_1
\end{aligned}
$$

(b) $\alpha = \arctan \frac{a}{b/2}$, $\sin\alpha = \frac{2a}{\sqrt{4a^2+b^2}}$, $\cos\alpha = \frac{b}{\sqrt{4a^2+b^2}}$

For any vector $\vec{v} = v_x \hat{x} + v_y \hat{y} + v_z \hat{z}$
$= v_x' \hat{x}' + v_y' \hat{y}' + v_z' \hat{z}'$,

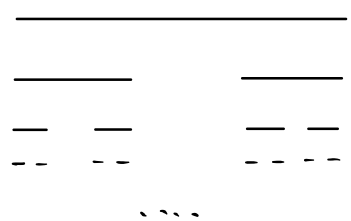
$$
v_{i'} = \begin{bmatrix} v_x' \\ v_y' \\ v_z' \end{bmatrix} = R_{i'i} v_i = \begin{bmatrix} \cos\alpha & & -\sin\alpha \\ & 1 & \\ \sin\alpha & & \cos\alpha \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
$$

$$
\begin{aligned}
I_{ij} = (R^T)_{ii'} I'_{i'j'} R_{j'j} &= \begin{bmatrix} c & & s \\ & 1 & \\ -s & & c \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_3 \end{bmatrix} \begin{bmatrix} c & & -s \\ & 1 & \\ s & & c \end{bmatrix} \\
&= \begin{bmatrix} c\lambda_1 & & s\lambda_3 \\ & \lambda_1 & \\ -s\lambda_1 & & c\lambda_3 \end{bmatrix} \begin{bmatrix} c & & -s \\ & 1 & \\ s & & c \end{bmatrix} \\
&= \begin{bmatrix} c^2\lambda_1 + s^2\lambda_3 & & -sc(\lambda_1 - \lambda_3) \\ & \lambda_1 & \\ -sc(\lambda_1 - \lambda_3) & & s^2\lambda_1 + c^2\lambda_3 \end{bmatrix}
\end{aligned}
$$

(c)
$$
\begin{aligned}
T &= \frac{1}{2} \omega^2 I_{zz} \\
&= \frac{1}{2} \omega^2 \left(s^2 \lambda_1 + c^2 \lambda_3\right) \\
&= \frac{1}{2} \omega^2 \frac{1}{4a^2+b^2} \left[4a^2 (4a^2+b^2) + b^2 \cdot 8a^2\right] \frac{\pi \rho_0 a^2 b}{24} \\
&= \frac{\omega^2 \pi \rho_0 a^4 b \left(4a^2 + 3b^2\right)}{12 \left(4a^2+b^2\right)}
\end{aligned}
$$

2. Moment of Inertia of Fractals

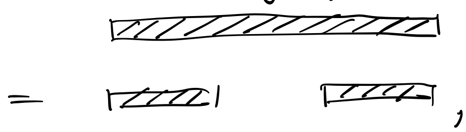
(a)



after ∞ times

We denote the final object as , with mass m and lenth ℓ

This object consists of two small objects,



Scale like ML^2

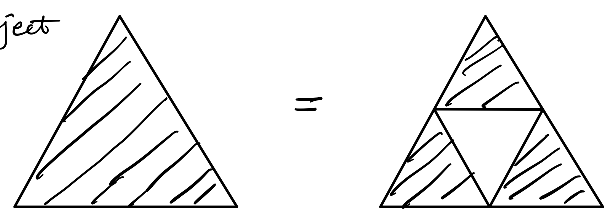
each of which has mass $m/2$ and lenth $\ell/3$, inertial $I_1 = I_2 = \frac{1}{2}\left(\frac{1}{3}\right)^2 I$

By parrellel axis theorem,

$$I = I_1 + I_2 + 2\,\frac{m}{2}\left(\frac{\ell}{3}\right)^2$$

$$\Rightarrow \quad I = \frac{1}{8} m\ell^2$$

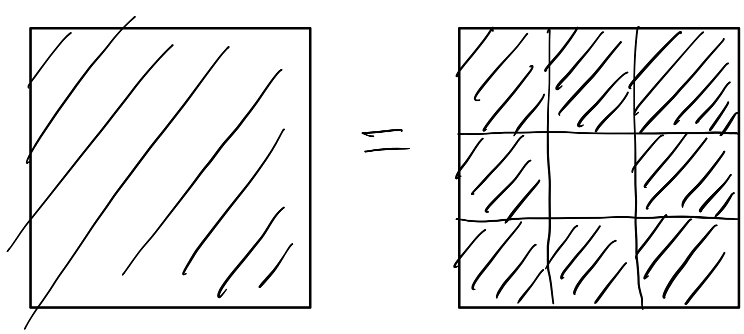
(b) The final object



$$I = 3\cdot\frac{1}{3}\left(\frac{1}{2}\right)^2 I + 3\cdot\frac{m}{3}\left(\frac{\sqrt{3}}{6}\ell\right)^2$$

$$\Rightarrow I = \frac{1}{9} m\ell^2$$

(c)



$$I = 8 \cdot \frac{1}{8}\left(\frac{1}{3}\right)^2 I + 4\,\frac{m}{8}\left(\frac{\ell}{3}\right)^2 + 4\,\frac{m}{8}\left(\frac{\sqrt{2}\ell}{3}\right)^2$$

$$\Rightarrow I = \frac{3}{16} m\ell^2$$

(d) (a) $3^d = 2$ $\quad d = \log_3 2 \approx 0.63$

(b) $2^d = 3$ $\quad d = \log_2 3 \approx 1.58$

(c) $3^d = 8$ $\quad d = \log_3 8 \approx 1.89$