# A. Applying Hadamard gates to general states.

Background: In almost all “real” quantum computing problems we want to use some form of “quantum parallelism” where we evaluate some function f(x) on all states simultaneously.

Goal: To recap how using a n-Qbit Hadamard creates a superposition and its properties..

# 1. Single-Qbit Hadamard

## a. Creating a superposition

$$\mathbf{H}|0\rangle = \left\{ \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right\} = |+\rangle$$

$$\mathbf{H}|1\rangle = \left\{ \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right\} = |-\rangle$$

$$\mathbf{H}|x\rangle = \left\{ \frac{|0\rangle + (-1)^x|1\rangle}{\sqrt{2}} \right\}$$

$$= \frac{1}{\sqrt{2}} \sum_{0 \le y < 2} (-1)^{x \cdot y} |y\rangle$$

## b. The Hadamard is its own inverse

$$\mathbf{H}|+\rangle = |0\rangle$$

$$\mathbf{H}|-\rangle = |1\rangle$$

$$\mathbf{H}\frac{1}{\sqrt{2}}\{|0\rangle + (-1)^x|1\rangle\} = |x\rangle$$

# 2. Two-Qbit Hadamard

## a. Quantum parallelism

$$\begin{aligned}
\mathbf{H}^{\otimes 2}|0\rangle_2 &= \mathbf{H}|0\rangle\mathbf{H}|0\rangle \\
&= \left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right\}\left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right\} \\
&= \left\{\frac{1}{\sqrt{2}}\sum_{0\le y_1<2}|y_1\rangle\right\}\left\{\frac{1}{\sqrt{2}}\sum_{0\le y_0<2}|y_0\rangle\right\} \\
&= \frac{1}{2}\left\{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle\right\}
\end{aligned}$$

$$\mathbf{H}^{\otimes 2}|0\rangle_2 = \frac{1}{2}\sum_{0\le y<2^2}|y\rangle_2$$

$$|y\rangle_2 = |y_1\rangle|y_0\rangle$$

## b. Two-Qbit Hadamard on a general state

$$|x\rangle_2 = |x_1\rangle|x_0\rangle$$
$$|y\rangle_2 = |y_1\rangle|y_0\rangle$$

$$\begin{aligned}
\mathbf{H}^{\otimes 2}|x\rangle_2 &= \mathbf{H}|x_1\rangle\mathbf{H}|x_0\rangle \\
&= \left\{\frac{|0\rangle + (-1)^{x_1}|1\rangle}{\sqrt{2}}\right\}\left\{\frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}\right\} \\
&= \left\{\frac{1}{\sqrt{2}}\sum_{0\le y_1<2}(-1)^{x_1 y_1}|y_1\rangle\right\}\left\{\frac{1}{\sqrt{2}}\sum_{0\le y_0<2}(-1)^{x_0 y_0}|y_0\rangle\right\} \\
&= \frac{1}{2}\left\{|0\rangle|0\rangle + (-1)^{x_0}|0\rangle|1\rangle + (-1)^{x_1}|1\rangle|0\rangle + (-1)^{x_1+x_0}|1\rangle|1\rangle\right\}
\end{aligned}$$

$$\mathbf{H}^{\otimes 2}|x\rangle_2 = \frac{1}{2}\sum_{0\le y<2^2}(-1)^{x\cdot y}|y\rangle_2$$

$$\begin{aligned}
(-1)^{x\cdot y} &= (-1)^{x_1 y_1 + x_0 y_0} \\
&= (-1)^{x_1 y_1 \bigoplus x_0 y_0}
\end{aligned}$$

# c. Inverting a two-Qbit Hadamard

$$
\mathbf{H}^{\otimes 2}\left\{\frac{|0\rangle + (-1)^{x_1}|1\rangle}{\sqrt{2}}\right\}\left\{\frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}\right\}
$$

$$
= |x_1\rangle|x_0\rangle
$$

$$
= |x\rangle_2
$$

# 3. n-Qbit Hadamard

## a. Quantum parallelism

$$\mathbf{H}^{\otimes n}|0\rangle_n =$$

$$= \left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right\}\left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right\}\cdots\left\{\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right\}$$

$$= \left\{\frac{1}{\sqrt{2}}\sum_{0\le y_{n-1}<2}|y_{n-1}\rangle\right\}\cdots\left\{\frac{1}{\sqrt{2}}\sum_{0\le y_1<2}|y_1\rangle\right\}\left\{\frac{1}{\sqrt{2}}\sum_{0\le y_0<2}|y_0\rangle\right\}$$

$$\mathbf{H}^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{2^n}}\sum_{0\le y<2^n}|y\rangle_n$$

$$|y\rangle_n = |y_{n-1}\rangle|y_{n-2}\rangle\cdots|y_1\rangle|y_0\rangle$$

## b. n-Qbit Hadamard on a general state

$$|x\rangle_n = |x_{n-1}\rangle|x_{n-2}\rangle\dots|x_1\rangle|x_0\rangle$$

$$\begin{aligned}\mathbf{H}^{\otimes n}|x\rangle_n &= \mathbf{H}|x_{n-1}\rangle\dots\mathbf{H}|x_1\rangle\mathbf{H}|x_0\rangle\\ &= \left\{\frac{|0\rangle+(-1)^{x_{n-1}}|1\rangle}{\sqrt{2}}\right\}\dots\left\{\frac{|0\rangle+(-1)^{x_1}|1\rangle}{\sqrt{2}}\right\}\left\{\frac{|0\rangle+(-1)^{x_0}|1\rangle}{\sqrt{2}}\right\}\\ &= \left\{\frac{1}{\sqrt{2}}\sum_{0\le y_{n-1}<2}(-1)^{x_{n-1}y_{n-1}}|y_{n-1}\rangle\right\}\dots\left\{\frac{1}{\sqrt{2}}\sum_{0\le y_1<2}(-1)^{x_1y_1}|y_1\rangle\right\}\left\{\frac{1}{\sqrt{2}}\sum_{0\le y_0<2}(-1)^{x_0y_0}|y_0\rangle\right\}\end{aligned}$$

$$\mathbf{H}^{\otimes n}|x\rangle_n = \frac{1}{\sqrt{2^n}}\sum_{0\le y<2^n}(-1)^{x\cdot y}|y\rangle_n$$

$$|y\rangle_n = |y_{n-1}\rangle|y_{n-2}\rangle\dots|y_1\rangle|y_0\rangle$$

$$\begin{aligned}(-1)^{x\cdot y} &= (-1)^{x_{n-1}y_{n-1}+\dots+x_1y_1+x_0y_0}\\ &= (-1)^{x_{n-1}y_{n-1}\oplus\dots\oplus x_1y_1\oplus x_0y_0}\end{aligned}$$

# c. Inverting an n-Qbit Hadamard

$$\mathbf{H}^{\otimes n}\left\{\frac{|0\rangle + (-1)^{x_{n-1}}|1\rangle}{\sqrt{2}}\right\}\cdots\left\{\frac{|0\rangle + (-1)^{x_1}|1\rangle}{\sqrt{2}}\right\}\left\{\frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}\right\}$$

$$= \mathbf{H}^{\otimes n}\left\{\frac{1}{\sqrt{2^n}}\sum_{0\le y<2^n}(-1)^{x\cdot y}|y\rangle_n\right\}$$

$$= |x_{n-1}\rangle\ldots|x_1\rangle|x_0\rangle$$

$$= |x\rangle_n$$