Proof of Noether's theorem

$\tilde t = T_\varepsilon(t) \qquad \tilde q(\tilde t) = Q_\varepsilon(q(t)) = Q_\varepsilon(q(T_{-\varepsilon}(\tilde t)))$

with T_ε and Q_ε smooth one-parameter subgroups ($T_0 = id \quad T_{-\varepsilon} = T_\varepsilon^{-1} \quad T_{\varepsilon+\varepsilon'} = T_\varepsilon \circ T_{\varepsilon'}$)

→ i.e. $\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon$ makes sense

$$S[q] = \int_a^b L(q(t), \dot q(t), t)$$

$$\tilde S[\tilde q] = \int_{T_\varepsilon(a)}^{T_\varepsilon(b)} L\Big(\tilde q(\tilde t), \overset{\frac{d\tilde q}{d\tilde t}}{\dot{\tilde q}}(\tilde t), \tilde t\Big)\, d\tilde t = \int_a^b \frac{dT_\varepsilon}{dt} L\big(\tilde q(T_\varepsilon(t)), \dot{\tilde q}(T_\varepsilon(t)), T_\varepsilon(t)\big)\, dt$$

suppose that $\tilde S[\tilde q] = S[q]$ for all paths (even those that are not solutions of E-L eqs.) and for all choices of a, b.

Then $\delta S := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde S[\tilde q] = 0$

Some notation:

- $\delta t := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} T_\varepsilon$ (function of t)
- $\delta q := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} Q_\varepsilon$ (function of q)

} generators of T_ε and Q_ε (see Lie algebras later)

Let's apply $D := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$ everywhere we can!

- $D\frac{dT_\varepsilon}{dt} = \frac{d}{dt} DT_\varepsilon = \frac{d}{dt}\delta t$
- $D\,\tilde q(\tilde t) = D\,Q_\varepsilon(q(t)) = \delta q(q(t))$
- $\frac{d\tilde q}{d\tilde t}(\tilde t) = \frac{d}{d\tilde t} Q_\varepsilon(q(t)) = \frac{\partial Q_\varepsilon}{\partial q}(q(t))\,\dot q(t)\,\underbrace{\frac{dt}{d\tilde t}(\tilde t)}_{= \frac{1}{\frac{d\tilde t}{dt}(t)} = \frac{1}{\frac{dT_\varepsilon}{dt}(t)}}$

$$\Rightarrow \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \dot{\tilde q}(\tilde t) = \frac{\partial}{\partial q} DQ_\varepsilon(q(t))\,\dot q(t)\,\frac{1}{\underbrace{\frac{dT_0}{dt}}_{=1}} + \underbrace{\frac{\partial}{\partial q} Q_0(q(t))}_{=1}\,\dot q(t)\,\underbrace{D\frac{1}{\frac{dT_\varepsilon}{dt}}}_{-\frac{d}{dt}\delta t}$$

$$\Rightarrow D\,\dot{\tilde q}(\tilde t) = \frac{\partial}{\partial q}\delta q\,\dot q(t) - \dot q(t)\frac{d}{dt}\delta t = \frac{d}{dt}\delta q - \dot q(t)\frac{d}{dt}\delta t$$

Now suppose that q satisfies $\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0$

$$0 = D\tilde S[\tilde q] = \int_a^b dt\, D\left[\frac{dT_\varepsilon}{dt} L(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t)\right]$$

$$= \int_a^b dt \left\{ \frac{d\delta t}{dt} L(q,\dot q,t) + \underbrace{\frac{dT_0}{dt}}_{=1} DL(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t)\right\}$$

$$DL(\tilde q(\tilde t), \dot{\tilde q}(\tilde t), \tilde t) = \frac{\partial L}{\partial t} D\tilde t + \frac{\partial L}{\partial q} D\tilde q(\tilde t) + \frac{\partial L}{\partial \dot q} D\dot{\tilde q}(\tilde t)$$

$$= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot q}\left(\frac{d}{dt}\delta q - \dot q \frac{d}{dt}\delta t\right)$$

$$= \frac{\partial L}{\partial t}\delta t + \frac{\partial L}{\partial q}\delta q + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\delta q\right) - \delta q \frac{d}{dt}\frac{\partial L}{\partial \dot q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\, \delta t\right) + \delta t \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\dot q\right)$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \frac{\partial L}{\partial t}\delta t + \delta t\left(\frac{d}{dt}\frac{\partial L}{\partial \dot q}\right)\dot q + \delta t \frac{\partial L}{\partial \dot q}\ddot q$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \delta t\left[\frac{\partial L}{\partial t} + \frac{\partial L}{\partial q}\dot q + \frac{\partial L}{\partial \dot q}\ddot q\right]$$

$$= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] + \delta t \frac{d}{dt} L(q(t),\dot q(t),t)$$

$$\Rightarrow \quad 0 = \int_a^b \left\{ L(q(t),\dot q(t),t)\frac{d}{dt}\delta t + \delta t \frac{d}{dt}L(q(t),\dot q(t),t) + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right]\right\} dt$$

$$= \int_a^b \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] dt$$

$$\Rightarrow \quad \frac{d}{dt}\left[L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t)\right] \quad \text{or} \quad L\,\delta t + \frac{\partial L}{\partial \dot q}(\delta q - \dot q\,\delta t) \ \ \underline{\text{conserved}}$$

<u>Note</u>: for this to work we need to check that $\tilde S[\tilde q] = S[q]$ or $D\tilde S[\tilde q] = 0$ <u>whithout</u> using E-L eqs!