THE QUANTUM ISING CHAIN FOR BEGINNERS

Sep 28, 2020

1 The Jordan - Wigner transformation

There are techniques to deal with large assemblies of bosons & fermions. But not with spin systems.

→ need a way to "map" the hard problem to easy!

→ Consider single spin-$\frac{1}{2}$ ⇒ 3 components of spin operator:

- σ^x, σ^y, σ^z. Hilbert space is $\{|\uparrow\rangle, |\downarrow\rangle\}$

- Eigenstates: $\begin{cases} \sigma^z|\uparrow\rangle = |\uparrow\rangle \\ \sigma^z|\downarrow\rangle = -|\downarrow\rangle \end{cases}$

- Commutation relation (from angular momentum $J^i \leftrightarrow \sigma^i$)

index

$$\boxed{[\sigma_j^i, \sigma_{j'}^{i'}] = 0}, \quad \boxed{[\sigma_j^x, \sigma_j^y] = 2i\sigma_j^z}$$

site

same site, obey usual comm. relation (can be written with ε^{ijk} ...)

cyclic

- Define $\sigma^\pm = \dfrac{\sigma_j^x \pm i\sigma_j^y}{2}$

gives $\boxed{\sigma^+|\downarrow\rangle = |\uparrow\rangle, \quad \sigma^-|\uparrow\rangle = |\downarrow\rangle}$

& $\boxed{\{\sigma_j^+, \sigma_j^-\} = \mathbb{1}}$ → anticommutator, typical of rules for fermions

→ Should we describe spins w/ bosons or fermions?

→ Let's start w/ bosons... (hard...)

Suppose have single boson $\hat{b}^+$ with associated vacuum state $|0\rangle$ s.t. $\hat{b}|0\rangle = |0\rangle$

then because $[\hat{b}, \hat{b}^+] = 1$, can have

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^+)^n|0\rangle \qquad n = 0, 1, \dots, \infty$$

→ Now, since we want only spin-$\frac{1}{2}$ ⇒ truncate Hilbert space

so that $(\hat{b}^+)^2|0\rangle = 0$ ⇒ get sth like Hilbert space of single spin-$\frac{1}{2}$.

(!) this kind of truncation is called "hard-core boson"

Now, how do we relate $\hat{b}^+$, $\hat{b}$, to the Pauli matrices?

→ Observe that if we identify $\begin{cases} |0\rangle \leftrightarrow |\uparrow\rangle \\ |1\rangle \leftrightarrow |\downarrow\rangle \end{cases}$

then $\hat{b}^+|0\rangle = |1\rangle \longleftrightarrow |\downarrow\rangle = \hat{\sigma}^-|\uparrow\rangle$
and so on...

so
$$\boxed{\begin{cases} \hat{\sigma}_j^+ \equiv \hat{b}_j \\ \hat{\sigma}_j^- \equiv \hat{b}_j^+ \\ \hat{\sigma}_j^z \equiv 1 - 2\hat{b}_j^+\hat{b}_j \end{cases} \rightarrow \begin{cases} \hat{\sigma}_j^x = \hat{b}_j^+ + \hat{b}_j \\ \hat{\sigma}_j^y = i(\hat{b}_j^+ - \hat{b}_j) \\ \hat{\sigma}_j^z = 1 - 2\hat{b}_j^+\hat{b}_j \end{cases}}$$

• Note that $[\hat{b}_j, \hat{b}_i] = 0 = [\hat{b}_j^\dagger, \hat{b}_i^\dagger]$ $(j \neq i)$

(like how $\vec{\sigma}$ commutes @ different sites)

→ But $\hat{b}_j^\dagger, \hat{b}_j$ are not ordinary bosonic operators.

• Also ~~Note~~ that b/c of the truncation, $(\hat{b}_j^\dagger)^2|0\rangle = 0$, and that

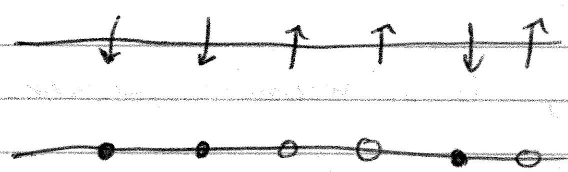
$$\{\hat{b}_j, \hat{b}_j^\dagger\} = \mathbb{1}.$$

⇒ $\boxed{\text{At most one boson is allowed at one site}}$

Now, if we pay close attention... the hard-core boson representation is calling out "spinless fermions" $\hat{c}_j^\dagger$

↳ Why? b/c the absence of double occupancy is actually enforced by the Pauli Exclusion Princip

& that the anti-commutation rule comes for free!



h.c. bosons

spinless fermions

$$= \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_5^\dagger |0\rangle = \hat{c}_1^\dagger \hat{c}_2^\dagger \hat{c}_5^\dagger |0\rangle$$

There's a difficulty, however. The mapping $\hat{\sigma}_j^- \to \hat{b}_j^\dagger$ can be done in any dimension.

But writing $\hat{b}_j^\dagger$ in terms of $\hat{c}_j^\dagger$ is only useful in 1D.

b/c there's a natural ordering of the sites!

What is this mapping $\hat{b}_j \to \hat{c}_j$?

$\to$ the Jordan - Wigner transformation!

\# operator

$$\boxed{\hat{b}_j = \hat{K}_j \hat{c}_j = \hat{c}_j \hat{K}_j \quad \text{where} \quad \hat{K}_j = e^{i\pi \sum_{j'=1}^{j-1} \hat{n}_{j'}} = \prod_{j'=1}^{j-1} (1 - 2\hat{n}_{j'})}$$

where we've introduced the "non-local" string operator $\hat{K}_j$

$\to$ now, $\hat{K}_j$ is just a sign! $\hat{K}_j = \pm 1$.

$\to$ Intuitively, $\boxed{\hat{K}_j \text{ counts the parity of \# of fermions } \underline{\text{before}} \text{ site } j.}$

Now, $\hat{K}_j = e^{i\pi \sum_{j'=1}^{j-1} \hat{n}_{j'}}$, b/c $\hat{K}_j = \pm 1$

$\to$ $\boxed{\hat{K}_j = \hat{K}_j^{\dagger} = \hat{K}_j^{-1}, \text{ and } \hat{K}_j^{2} = 1.}$

<u>Why?</u> $\hat{n}_j = \hat{c}_j^{\dagger} \hat{c}_j$ is a number operator.

$\to$ we will now show that <u>if</u> we take c_j to be the fermionic operators with the anti-comm. rel.

$$\boxed{\{\hat{c}_j, \hat{c}_{j'}^{\dagger}\} = \delta_{j,j'} \quad \& \quad \{\hat{c}_j, \hat{c}_{j'}\} = \{\hat{c}_j^{\dagger}, \hat{c}_{j'}^{\dagger}\} = 0}$$

then the expected properties of the $\hat{b}_j, \hat{b}_j^{\dagger}$'s will follow...

Same-site property: (anti-commutation relation)

$$\{ \hat{b}_j , \hat{b}_j^+ \} = 1$$

$$\{ \hat{b}_j , \hat{b}_j \} = \{ \hat{b}_j^+ , \hat{b}_j^+ \} = 0$$

Different-site property: (commutation relation)

$$[\hat{b}_j , \hat{b}_{j'}^+] = 0 \qquad [\hat{b}_j^+ , \hat{b}_{j'}^+] = 0$$

$$[\hat{b}_j , \hat{b}_{j'}] = 0$$

@ different sites, always commute.

i.e. that $\hat{b}_j^{(+)}$'s are hard-core bosons.

→ To show the same-site property, just use the fact that

$$\hat{b}_j^+ \hat{b}_j = \underbrace{(\hat{c}_j^+ \hat{K}_j^+)(\hat{K}_j}_{\hat{K}_j^2} \hat{c}_j) = \hat{c}_j^+ (1) \hat{c}_j = \hat{c}_j^+ \hat{c}_j$$

⇒ $\hat{b}_j$'s follow the same anti-comm. relation as $\hat{c}_j$

Similarly $\hat{b}_j \hat{b}_j^+ = \hat{c}_j \hat{c}_j^+$.

Now, to show the different-site property ... ~~use the~~

↳ Consider $[\hat{b}_{j_1} , \hat{b}_{j_2}]$, assuming $j_2 > j_1$.

By the JW transform, we find that

$$\boxed{\hat{b}^\dagger_{j_2}\hat{b}_{j_1}} = \hat{c}^\dagger_{j_2}\hat{K}^\dagger_{j_2}\hat{K}_{j_1}\hat{c}_{j_1}$$

$$= \hat{c}^\dagger_{j_2}\left\{e^{-i\pi\sum_{j'=1}^{j_2-1}\hat{n}_{j'}}\right\}\left\{e^{i\pi\sum_{j'=1}^{j_1-1}\hat{n}_{j'}}\right\}\hat{c}_{j_1}$$

$$= \hat{c}_{j_2}\, e^{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}}\,\hat{c}_{j_1}$$

$$= e^{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}}\,\hat{c}^\dagger_{j_2}\hat{c}_{j_1} \qquad K_{j_1} \leftrightarrow c_{j_2}$$

$$= -\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}_{j_1}\hat{c}^\dagger_{j_2} \qquad \leftarrow \text{by anti comm. relation}$$

$$= \boxed{+\hat{c}_{j_1}\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}^\dagger_{j_2}}$$

where the last eq comes from the fact that

$$-\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}_{j_1}\hat{c}^\dagger_{j_2}$$

annihilates site $j_1 \Rightarrow \hat{n}_{j_1} = 0$

whereas

$\hat{c}_{j_1}\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}^\dagger_{j_2}$ has $\hat{n}_{j_1} = 1$ since there is no $\hat{c}_{j_1}$ present.

→ differ by (−)

Similarly, can show that

$$\boxed{\hat{b}_{j_1}\hat{b}^\dagger_{j_2} = \hat{c}_{j_1}\exp\left\{-i\pi\sum_{j'=j_1}^{j_2-1}\hat{n}_{j'}\right\}\hat{c}^\dagger_{j_2}}$$

With these, ... can check that $\boxed{\{\hat{b}_{j_1}, \hat{b}^{\dagger}_{j_2}\} = 0}$

$\rightarrow$ all other different-site relations are proven similarly.

<u>Facts</u>

$$\prod_{j'=1}^{j-1}(1-2\hat{n}_{j'}) \prod_{j'=1}^{j}(1-2\hat{n}_{j'}) = 1-2\hat{n}_j$$

since $(1-2\hat{n}_{j'})^2 = 1$ & terms with different j''s commute.

<u>note</u> true b/c $\hat{n}_j$ can only be 0 or 1.

With this relation, we get ...

- $\hat{b}_j^{\dagger}\hat{b}_j = \hat{c}_j^{\dagger}\hat{c}_j$
- $\hat{b}_j^{\dagger}\hat{b}^{\dagger}_{j+1} = \hat{c}_j^{\dagger}(1-2\hat{n}_j)c^{\dagger}_{j+1} = \hat{c}_j^{\dagger}\hat{c}^{\dagger}_{j+1}$
- $\hat{b}_j^{\dagger}\hat{b}_{j+1} = \hat{c}_j^{\dagger}(1-2\hat{n}_j)\hat{c}_{j+1} = \hat{c}_j^{\dagger}\hat{c}_{j+1}$
- $\hat{b}_j\hat{b}_{j+1} = \hat{c}_j(1-2\hat{n}_j)\hat{c}_{j+1} = \hat{c}_j[1-2(1-\hat{c}_j\hat{c}_j^{\dagger})]\hat{c}_{j+1}$

  $= -\hat{c}_j\hat{c}_{j+1}$
- $\hat{b}_j\hat{b}^{\dagger}_{j+1} = \hat{c}_j(1-2\hat{n}_j)\hat{c}^{\dagger}_{j+1} = \hat{c}_j[1-2(1-\hat{c}_j\hat{c}_j^{\dagger})]\hat{c}^{\dagger}_{j+1}$

  $= -\hat{c}_j\hat{c}^{\dagger}_{j+1}$

To summarize, the JW transformation is given by

$$\begin{cases} \hat{\sigma}_j^x = \hat{K}_j (\hat{c}_j^{\dagger} + \hat{c}_j) = \hat{b}_j^{\dagger} + \hat{b}_j \\ \hat{\sigma}_j^y = \hat{K}_j\, i (\hat{c}_j^{\dagger} - \hat{c}_j) = i(\hat{b}_j^{\dagger} - \hat{b}_j) \\ \hat{\sigma}_j^z = 1 - 2\hat{n}_j = 1 - 2\hat{c}_j^{\dagger} \hat{c}_j = 1 - 2 \hat{b}_j^{\dagger} \hat{b}_j \end{cases}$$

where

$$\hat{K}_j = \prod_{j'=1}^{j-1} (1 - 2\hat{n}_{j'})$$

Under this map, spin operators become local ferm. op.

$$\hat{\sigma}_j^z = 1 - 2\hat{n}_j = (\hat{c}_j^{\dagger} + \hat{c}_j)(\hat{c}_j^{\dagger} - \hat{c}_j)$$

$$\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x = \left[\hat{c}_j^{\dagger} \hat{c}_{j+1}^{\dagger} + \hat{c}_j^{\dagger} \hat{c}_{j+1} + h.c. \right]$$

$$\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y = -\left[\hat{c}_j^{\dagger} \hat{c}_{j+1}^{\dagger} - \hat{c}_j^{\dagger} \hat{c}_{j+1} + h.c. \right]$$

<u>Note</u> a longitudinal field term involving a single $\hat{\sigma}_j^x$ <u>cannot</u> be translated into a simple <u>local</u> fermionic operator.

Oct 6, 2020

<u>Note</u> Boundary conditions are very important.

Often assumes periodic boundary conditions → (PBC)

i.e. model is defined on a <u>ring geometry</u>

where we understand that $\hat{\sigma}_0^{\alpha} \equiv \hat{\sigma}_L^{\alpha}$, $\hat{\sigma}_{L+1}^{\alpha} = \hat{\sigma}_1^{\alpha}$

This implies the same PBC for hard-core bosons

Hence, e.g. $\hat{b}_L^\dagger \hat{b}_{L+1} \equiv \hat{b}_L^\dagger \hat{b}_1$.

Oct 7, 2020 → But things can go wrong for fermions when we look at

$$\hat{b}_L^\dagger \hat{b}_1 = e^{i\pi \sum_{j=1}^{L-1} \hat{n}_j} \hat{c}_L^\dagger \hat{c}_1$$ → $\mathbb{1}$ due to $\hat{c}_L^\dagger$

$$= e^{i\pi \sum_{j=1}^{L-1} \hat{n}_j + \hat{n}_L} \cdot (-1)\, \hat{c}_L^\dagger \hat{c}_1$$

$$= -e^{i\pi \sum_{j=1}^{L} \hat{n}_j} \hat{c}_L^\dagger \hat{c}_1$$

$$\Rightarrow \boxed{\hat{b}_L^\dagger \hat{b}_1 = -e^{i\pi \hat{N}} \hat{c}_L^\dagger \hat{c}_1}$$ "fermion parity"

where $\hat{N} = \sum_{j=1}^{L} \hat{n}_j = \sum_{j=1}^{L} \hat{c}_j^\dagger \hat{c}_j = \sum_{j=1}^{L} \hat{b}_j^\dagger \hat{b}_j$

is the total # of particles.

Similarly, one finds that

$$\hat{b}_L^\dagger \hat{b}_1^\dagger = e^{i\pi \sum_{j=1}^{L-1} \hat{n}_j} \hat{c}_L^\dagger \hat{c}_1^\dagger$$

$$\Rightarrow \boxed{\hat{b}_L^\dagger \hat{b}_1^\dagger = -e^{i\pi \hat{N}} \hat{c}_L^\dagger \hat{c}_1^\dagger}$$

→ This shows that boundary conditions are affected by the <u>fermion parity</u>:

$$e^{i\pi \hat{N}} = (-1)^{\hat{N}}$$

(ABC)

In particular, PBC ⟶ anti-periodic when $\hat{N}$ is <u>even</u>
when PBC is open (OBC) ⇒ no problem.

TFIM

(2) $\boxed{\text{The transverse-field Ising model : fermionic formulation}}$

$\boxed{\text{Info}}$ There is a class of 1D spin systems in which a ~~f~~- fermionic re-formulation can be useful.

↳ most noteworthy is the XXZ chain (Heisenberg)

$$\hat{H}_{XXZ} = \sum_j \left(J_j^{\perp} \left(\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) + J_j^{zz} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z \right) - \sum_j h_j \hat{\sigma}_j^z$$

The corresponding fermionic formulation reads

$$\hat{H}_{XXZ} \rightarrow \sum_j \left(2J_j^{\perp} \left\{ \hat{c}_j^{\dagger} \hat{c}_{j+1} + h.c. \right\} + J_j^{zz} (2\hat{n}_j - 1)(2\hat{n}_{j+1} - 1) \right) + \sum_j h_j (2\hat{n}_j - 1)$$

single-site

→ shows that the fermions interact at <u>nearest-neighbors</u>, due to the J_j^{zz} term.

—— // ——

→ now look at 1D models where the fermionic Hamiltonian can be diagonalized exactly since it is quadratic in the fermions...

→ e.g. XY model & TFIM

After a rotation in spin space, can write the Hamiltonian $\longrightarrow$ (allowing for non-uniform, possibly random couplings) as follows:

$$\boxed{\hat{H} = -\sum_{j=1}^{L}\left(J_j^x \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + J_j^y \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y\right) - \sum_{j=1}^{L} h_j \hat{\sigma}_j^z}$$

can be chosen to be iid from $\mathcal{U}(0,1)$

For system size $L<\infty$, with OBC, then the sum only actually runs from $1 \longrightarrow L-1$, with $J_L^{x,y}=0$

But if we have PBC $\Rightarrow$ Sum runs from $1 \to L$, and we assume that

$$\hat{\sigma}_{L+1}^{\alpha} = \hat{\sigma}_1^{\alpha}.$$

$\rightarrow$ When $J_j^y = 0$, we have the TFIM:

$$\hat{H}_{IS} = -\sum_{j=1}^{L} J_j^x \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x - \sum_{j=1}^{L} h_j \hat{\sigma}_j^z.$$

$\rightarrow$ When $J_j^y = J_j^x$, we have the isotropic XY model

$$\hat{H}_{XY\,iso} = -\sum_{j=1}^{L} J_j^{x,y}\left\{\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y\right\} - \sum_{j=1}^{L} h_j \hat{\sigma}_j^z$$

Now, let's write $\hat{H}$ in terms of hard-core bosons.

$$\boxed{\hat{H} \rightarrow -\sum_{j=1}^{L}\left(J_j^+ \hat{b}_j^\dagger \hat{b}_{j+1} + J_j^- \hat{b}_j^\dagger \hat{b}_{j+1}^\dagger + h.c.\right) + \sum_{j=1}^{L} h_j\left(2\hat{n}_j - 1\right)}$$

where $J_j^{\pm} = J_j^x \pm J_j^y$

Next, we switch from hard core bosons to spinless fermions.

Now, since

$$\hat{b}_j^{+}\hat{b}_{j+1} = \hat{c}_j^{+}\hat{c}_{j+1}$$

$$\hat{b}_j^{+}\hat{b}_{j+1}^{+} = \hat{c}_j^{+}\hat{c}_{j+1}$$

$$\hat{b}_j^{+}\hat{b}_j = \hat{n}_j = \hat{n}_j = \hat{c}_j^{+}\hat{c}_j$$

none depends on the string operator $\hat{K}_j$

→ the Hamiltonian in the fermionic picture is identical.

<u>Remark</u>: { In the fermionic context the pair creation & annihilation terms are characteristic of the BCS theory of superconductivity }

⟹ <u>tricky point</u> is just the boundary conditions

→ If we use <u>OBC</u> → first sum runs over $1 \to L-1$, and there is no term involving the site $L+1$

$$\to \hat{H}_{OBC} = -\sum_{j=1}^{L-1}\left(J_j^{+}\hat{c}_j^{+}\hat{c}_{j+1} + J_j^{-}\hat{c}_j^{+}\hat{c}_{j+1}^{+} + h.c.\right) + \sum_{j=1}^{L} h_j\,(2\hat{n}_j - 1)$$

notice the range of the summations

If we use PBC, terms like $\hat{b}^{+}\hat{b}$ can close @ $L+1$:

$$\hat{b}_L^{+}\hat{b}_{L+1} \equiv \hat{b}_L^{+}\hat{b}_1 = -(-1)^{\hat{N}}\hat{c}_L^{+}\hat{c}_1$$

$$\hat{b}_L^{+}\hat{b}_{L+1}^{+} \equiv \hat{b}_L^{+}\hat{b}_1^{+} = -(-1)^{\hat{N}}\hat{c}_L^{+}\hat{c}_1^{+}$$

→ just as we showed before...

$$\Rightarrow \boxed{\hat{H}_{PBC} = \hat{H}_{OBC} + (-1)^{\hat{N}}\left\{J_L^{+}\hat{c}_L^{+}\hat{c}_1 + J_L^{-}\hat{c}_L^{+}\hat{c}_1^{+} + h.c.\right\}}$$

Info Notice that the number of fermions $\hat{N}$ is not conserved by the Hamiltonian in the PBC:

$$\hat{N} = \sum_{j=1}^{L} \hat{c}_j^{\dagger} \hat{c}_j \;\not\longleftrightarrow\; \hat{\mathcal{H}}_{PBC} \quad \text{(can check this)}$$

<u>But</u> the fermionic parity $(-1)^{\hat{N}} = e^{i\pi\hat{N}}$ is a constant of motion since

$$e^{i\pi\hat{N}} = \pm 1 \longleftrightarrow \hat{\mathcal{H}}_{PBC} \checkmark$$

→ From the fermionic perspective, it is as if we have anti-boundary condition (ABC)

<u>where</u> $\left\{ \hat{c}_{L+1} = -\hat{c}_1 \right.$ if $\hat{N}$ even

or <u>PBC</u> if $\left\{ \hat{c}_{L+1} = \hat{c}_1 \right.$ if $\hat{N}$ odd

→ This symmetry can also be seen from the nearest-neighbor $\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x$ & $\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y$

→ these can only flip <u>pairs</u> of spins.

→ <u>Parity</u> of the overall magnetization along z is unchanged.

$$\boxed{\hat{\mathcal{P}} = \prod_{j=1}^{L} \hat{\sigma}_j^z = \prod_{j=1}^{L} (1 - 2\hat{n}_j)}$$ → parity operator.

→ <u>Hermitian</u>

there is a $\mathbb{Z}_2$-symmetry which the system breaks in the ordered ferromagnetic phase

<u>Remark:</u> $\hat{\mathcal{P}}$ flips all the $\hat{\sigma}_j^x$ & $\hat{\sigma}_j^y$:

i.e. $$\boxed{\hat{\mathcal{P}} \hat{\sigma}_j^{x,y} \hat{\mathcal{P}} = -\hat{\sigma}_j^{x,y}}$$ (parity transform)

→ $\hat{\mathcal{P}} \hat{\mathcal{H}} \hat{\mathcal{P}} = \hat{\mathcal{H}}$.

⇒ So the Hamiltonian (in the spin pic) is invariant

Now, let us focus on diagonalizing the Hamiltonian:

→ Define projectors on the subspaces with even & odd # of parti

$$\boxed{\begin{aligned} \hat{P}_{even} &= \frac{1}{2}\left(\mathbb{1} + e^{i\pi\hat{N}}\right) = \hat{P}_0 \\ & \& \\ \hat{P}_{odd} &= \frac{1}{2}\left(\mathbb{1} - e^{i\pi\hat{N}}\right) = \hat{P}_1 \end{aligned}}$$

$(-1)^{\hat{N}}$

$(-1)^{\hat{N}}$

With these, can define 2 fermionic Ham's acting on the 2^{L-1}-dim even/odd parity subspaces of the full Hilbert space:

$$\boxed{\hat{H}_0 = \hat{P}_0 \hat{H}_{PBC} \hat{P}_0 \quad \& \quad \hat{H}_1 = \hat{P}_1 \hat{H}_{PBC} \hat{P}_1}$$

so that

$$\hat{H}_{PBC} = \underbrace{\begin{pmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_1 \end{pmatrix}}_{2^{L-1}} \rightarrow 2^{L-1}$$

Now, observe that if we write a 2^L fermionic Hamilt. of the form

$$\hat{\mathbb{H}}_{p=0,1} = -\sum_{j=1}^{L-1} \left(J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1} + h.c.\right)$$
$$+ (-1)^p \left(J_L^+ \hat{c}_L^\dagger \hat{c}_1^\dagger + J_L^- \hat{c}_L^\dagger \hat{c}_1 + h.c.\right)$$
$$+ \sum_{j=1}^{L} h_j \left(2\hat{n}_j - 1\right)$$

then we have that:

→ $\boxed{\text{When } p=1}$

$$\left\{ \hat{H}_{p=1} = -\sum_{j=1}^{L} \left(J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1} + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger \right) + \sum_{j=1}^{L} h_j (2\hat{n}_j - 1) \right\}$$
$$+h.c.$$

$=$ a <u>legitimate PBC-fermionic Hamiltonian.</u>

$(\hat{c}_{L+1} = \hat{c}_1)$

→ $\boxed{\text{When } p=0}$

$$\left\{ \hat{H}_{p=0} = -\sum_{j=1}^{L} \left(J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1} + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger \right) + \left(J_L^+ \hat{c}_L^\dagger \hat{c}_1 + J_L^- \hat{c}_L^\dagger \hat{c}_1^\dagger \right) \right.$$
$$+h.c. \qquad\qquad +h.c.$$
$$\left. + \sum_{j=1}^{L} h_j (2\hat{n}_j - 1) \right\}$$

$=$ a ABC - fermionic Hamiltonian, where we identify $\hat{c}_{L+1} = -\hat{c}_1$.

→ <u>But</u>, since $p = 0, 1 \neq \hat{N}$ in general since $\hat{N} = \sum_{j=1}^{L} \hat{c}_j^\dagger \hat{c}_j$

→ $H_{p=0,1}$ are not exactly the PBC-spin Hamiltonian form.

However, it is true that

$$\left\{ \begin{aligned} &\hat{H}_0 = \hat{P}_0 \hat{H}_{PBC} \hat{P}_0 = \hat{P}_0 \hat{H}_0 \hat{P}_0 = \hat{H}_0 \hat{P}_0 \hat{P}_0 = \hat{H}_0 \hat{P}_0 . \\ \text{and } &\hat{H}_1 = \hat{P}_1 H_{PBC} \hat{P}_1 = \hat{P}_1 \hat{H}_1 \hat{P}_1 = \hat{H}_1 \hat{P}_1 \hat{P}_1 = \hat{H}_1 \hat{P}_1 \end{aligned} \right\}$$

- <u>sanity check:</u>

$$\hat{P}_0 \hat{P}_0 = \frac{1}{4} \left(\mathbb{1} + e^{i\pi \hat{N}} \right) \left(1 + e^{i\pi \hat{N}} \right) = \frac{1}{4} \left(\underbrace{\mathbb{1} + e^{i\pi 2\hat{N}}}_{2\cdot\mathbb{1}} + 2 e^{i\pi \hat{N}} \right)$$

$$= \frac{1}{2} \left(\mathbb{1} + e^{i\pi \hat{N}} \right) = \hat{P}_0$$

→ similarly for $\hat{P}_1$.

Further, show that $\vec{P}_0 \leftrightarrow \vec{\mathbb{H}}_0$:

This is easy b/c $\vec{\mathbb{H}}_{p=0,1}$ conserves fermionic parity ($e^{i\pi\vec{N}}$), just like $\overset{\hookrightarrow}{H}_{PBC}$.

$\rightarrow \vec{\mathbb{H}}_{0,1} \leftrightarrow \vec{P}_{0,1}$.

$\triangle\!!$ $\hat{H}_{0,1} \neq \hat{\mathbb{H}}_{0,1}$ - $\vec{H}_{0,1}$ acts on 2^{L-1}-dim vector space,

$\hat{\mathbb{H}}_{0,1}$ acts on 2^L-dim vector space.

$\vec{H}_{0,1}$ are blocks with 2^{L-1} eigenvalues.

$\hat{\mathbb{H}}_{0,1}$ live in the full Hilbert space.

<u>Note</u> In the OBC case, since we don't have to worry about what happens when $j = L+1$,

there is no distinction between $\vec{\mathbb{H}}_0$ & $\vec{\mathbb{H}}_1$.

$\rightarrow \vec{\mathbb{H}}_0 = \vec{\mathbb{H}}_1 = \mathbb{H}$.

In the OBC case, can just set $\vec{H}_{OBC} = \vec{\mathbb{H}}$ & work with a single fermionic Hamiltonian.

---

3 The Uniform Ising model

To put the formalism into context, recall how the numerics show that some states of the ~~uniform~~ (non) uniform Ising model can ~~be~~ obey $\prod X = 1$ or $\prod X = -1$,

↳ This is exactly what we mean by partitioning $\hat{\mathcal{H}}$ into $\hat{\mathcal{H}}_0$ & $\hat{\mathcal{H}}_1$ where $\hat{\mathcal{H}}_0$ ~~deals with~~ states with $\prod X = 1$ & $\hat{\mathcal{H}}_1$ with $\prod X = -1$.

→ In any case, let's look at the uniform case where

$J_j^x = J^x$; $J_j^y = J^y$; $h_j = h$ → uniform!

→ Customary to parameterize
$$\begin{cases} J^x = J(1+\chi)/2 \\ J^y = J(1+\chi)/2 \end{cases}$$

So that
$$J^{\pm} = J^x \pm J^y = J \text{ and } \chi J, \text{ respectively.}$$

With these, the Ham looks like...

$$\boxed{\begin{aligned}\hat{H}_{OBC} = &-J\sum_{j=1}^{L-1}\left(\hat{c}_j^{\dagger}\hat{c}_{j+1} + \chi\hat{c}_j^{\dagger}\hat{c}_{j+1}^{\dagger} + h.c.\right) \\ &+ h\sum_{j=1}^{L}\left(2\hat{c}_j^{\dagger}\hat{c}_j - 1\right)\end{aligned}}$$

(remember that OBC is "nice")

and

$$\boxed{\hat{H}_{PBC} = \hat{H}_{OBC} + (-1)^{\hat{N}} J\left(\hat{c}_L^{\dagger}\hat{c}_1 + \chi\hat{c}_L^{\dagger}\hat{c}_1^{\dagger} + h.c.\right)}$$

Now, let us assume that L is even. This is not a big restriction & is useful.

(!) Should not confound L even/odd with even/odd parity of states!

↳ Recall that in the spin-PBC, if

$$\begin{cases} \hat{N} \text{ odd} \Rightarrow \hat{c}_{L+1} \equiv \hat{c}_1 , \\ \hat{N} \text{ even} \Rightarrow \hat{c}_{L+1} \equiv -\hat{c}_1 . \end{cases}$$

The Hamiltonian conserves $e^{i\pi\hat{N}}$ → need to consider ~~the~~ both cases where $\hat{N}$ = odd/even when diagonalizing $\hat{H}$.

→ need to introduce $\hat{H}_0$ & $\hat{H}_1$:

$$\boxed{\hat{H}_{p=0,1} = -J\sum_{j=1}^{L} \left(\hat{c}_j^\dagger \hat{c}_{j+1} + x\, \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + h.c. \right) + h \sum_{j=1}^{L} (2\hat{n}_j - 1)}$$

compact way to write dependence on p

where $p = 0$ goes with even parity

$p = 1$ " odd.

and ~~such that~~ we're assuming that

$$\boxed{\hat{c}_{L+1} \equiv (-1)^{p+1} \hat{c}_1}$$

→ This $\hat{H}_{p=0,1}$ might look "wrong" but it isn't!

Now, to proceed, we will look at fermionic ops in "momentum" space:

$$\left\{ \begin{array}{ll} \hat{c}_k = \dfrac{e^{-i\phi}}{\sqrt{L}} \displaystyle\sum_{j=1}^{L} e^{-ikj} \, \hat{c}_j & \text{(FT)} \\ \hat{c}_j = \dfrac{e^{i\phi}}{\sqrt{L}} \displaystyle\sum_{k} e^{+ijk} \, \hat{c}_k & \text{(inv FT)} \end{array} \right\}$$

- We'll use $e^{i\phi}$ to correct phase later... (only useful math term)
- k depends on p!

$\rightarrow$ For $p = 1$, $\hat{c}_{L+1} = \hat{c}_1$

$$\Rightarrow \hat{c}_{L+1} = \frac{e^{i\phi}}{\sqrt{L}} \sum_k e^{i(L+1)k} \, \hat{c}_k$$

$$\parallel$$

$$\hat{c}_1 = \frac{e^{i\phi}}{\sqrt{L}} \sum_k e^{ik} \, \hat{c}_k.$$

$$\rightarrow e^{ikL} = 1$$

standard PBC choices for k

$$\rightarrow \boxed{p = 1 \Rightarrow \mathcal{K}_{p=1} = \left\{ k = \frac{2\pi n}{L} \,;\, n = -\frac{L}{2}+1, \ldots, 0, \ldots \frac{L}{2} \right\}}$$

Similarly,

$$\boxed{p = 0 \Rightarrow \mathcal{K}_{p=0} = \left\{ k = \pm \frac{(2n-1)\pi}{L} \,;\, n = 1, \ldots, \frac{L}{2} \right\}}$$

Now, let us try to express $\hat{H}_{p=0,1}$ in terms of the $\hat{c}_k$'s

$\hookrightarrow$ $\hat{H}_{p=0,1}$ in "momentum" space...

$$\hat{H}_{p=0,1} = -J\sum_{j=1}^{L}\left(\hat{c}_j^\dagger \hat{c}_{j+1} + x\,\hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + h.c.\right) + h\sum_{j=1}^{L}\left(2\hat{n}_j - 1\right)$$

where remember that $\hat{c}_{L+1} = (-1)^{p+1}\hat{c}_1$.

- With $\hat{c}_j$ written in $\hat{c}_k$'s, ~~[illegible]~~ we need some useful identities first before writing $\hat{H}_p$ in k:

① $$\frac{1}{L}\sum_{j=1}^{L} e^{-i(k-k')j} = \delta_{kk'}$$

② $$\sum_k 2\cos(k)\,\hat{c}_k^\dagger \hat{c}_k = \sum_k \cos(k)\left(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k}\hat{c}_{-k}^\dagger\right)$$

where we used the anti-comm relation and

③ $$\sum_k \cos(k) = 0 \text{ , and}$$

④ $$\sum_k \left(2\hat{c}_k^\dagger \hat{c}_k - 1\right) = \sum_k\left(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k}\hat{c}_{-k}^\dagger\right)$$

→ With these, we find that:

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$$\hat{H}_p = -J\sum_{j=1}^{L}\left(\hat{c}_j^\dagger \hat{c}_{j+1} + x\,\hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + h.c.\right) + h\sum_{j=1}^{L}\left(2\hat{n}_j - 1\right)$$

where $\hat{c}_{L+1} = (-1)^{p+1}\hat{c}_1$

in terms of $\hat{c}_k, \hat{c}_k^\dagger$ →

$$= \boxed{-J\sum_{k}^{K_p}\left[2\cos k\,\hat{c}_k^\dagger \hat{c}_k + x\left(e^{-2i\phi} e^{ik}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger + h.c.\right)\right] + h\sum_k^{K_p}\left(2\hat{c}_k^\dagger \hat{c}_k - 1\right)}$$

Let's verify this ...

$$\hat{c}_j^{\dagger}\hat{c}_{j+1} = \left(\frac{e^{-i\phi}}{\sqrt{L}}\sum_k e^{-ijk}\hat{c}_k^{\dagger}\right)\left(\frac{e^{i\phi}}{\sqrt{L}}\sum_{k'} e^{+i(j+1)k'}\hat{c}_{k'}\right)$$

$$= \frac{1}{L}\sum_{k,k'} e^{-i(kj-k'(j+1))}\hat{c}_k^{\dagger}\hat{c}_{k'}$$

$$= \frac{1}{L}\sum_{k,k'} e^{-i(k-k')j} e^{ik'}\hat{c}_k^{\dagger}\hat{c}_{k'}$$

so $$\sum_{j=1}^{L}\hat{c}_j^{\dagger}\hat{c}_{j+1} = \frac{1}{L}\sum_{j=1}^{L}\sum_{k,k'} e^{-i(k-k')j} e^{ik'}\hat{c}_k^{\dagger}\hat{c}_{k'}$$

$$= \frac{1}{L}\sum_{k,k'}\sum_{j=1}^{L} e^{-i(k-k')j} e^{ik'}\hat{c}_k^{\dagger}\hat{c}_{k'}$$

now look at $k \in K_{p=0}$ & $K_{p=1}$ $\to 2\cos k$. $$= \sum_{k,k'}\delta_{kk'} e^{ik'}\hat{c}_k^{\dagger}\hat{c}_{k'} \rightsquigarrow \text{not quite ..}$$

since $p=0,1$ gives different answers ..

Rather ...

$$\sum_{j=1}^{L}\hat{c}_j^{\dagger}\hat{c}_{j+1} = \sum_{j=1}^{L-1}\hat{c}_j^{\dagger}\hat{c}_{j+1} + (-1)^{p+1}\hat{c}_L^{\dagger}\hat{c}_1$$

$$= \frac{1}{L}\sum_{k,k'}\sum_{j=1}^{L-1} e^{-i(k-k')j} e^{ik'}\hat{c}_k^{\dagger}\hat{c}_k + \cancel{(-1)^{p+1}}$$

$$+ \frac{(-1)^{p+1}}{L}\sum_{k,k'} e^{-i(k-k')L} e^{ik'}\hat{c}_k^{\dagger}\hat{c}_{k'}$$

$$e^{-iLk} e^{iLk'} e^{ik'}\hat{c}_k^{\dagger}\hat{c}_{k'}$$

some trig identity comes in here, Basically, when $k \in K_{p=0}$ or $K_{p=1}$

$$= \dots$$

$$= \sum_{k}^{K_p} 2\cos(k)\, \hat{c}_k^{\dagger}\hat{c}_{k'}$$ ✓

2cos k works ✓

That's good enough of a check -- we'll just take the answer as it is ---

→ Now, notice that the coupling of $-k$ & k in the anomalous pair-creation term:

$$\left[e^{-2i\phi} e^{ik} \hat{c}_k^{\dagger} \hat{c}_{-k}^{\dagger} \right]$$

with the exceptions for $p=1$ (PBC), $k=0, \pi$.

Recall that $K_{p=1} = \left\{ k = \frac{2\pi n}{L} , n = -\frac{L}{2}+1, \ldots, 0, \ldots \frac{L}{2} \right\}$

when $k = 0, \pi$, $\underline{e^{ik} = \pm 1} \rightarrow$ no separate $-k$ partner since h.c. cancels ---

⇒ Useful to manipulate the (normal) number-conserving terms to rewrite the Hamiltonian ...

We need the fact that

$$\sum_k 2\cos(k)\, \hat{c}_k^{\dagger} \hat{c}_k = \sum_k \cos(k)\, 2\hat{c}_k^{\dagger} \hat{c}_k$$

anti-comm relation

$$= \sum_k \cos(k) \left[\hat{c}_k^{\dagger} \hat{c}_k - \hat{c}_{-k}^{*} \hat{c}_{-k}^{\dagger} \right]$$

and by defn

to write the Ham as:

$$\boxed{\hat{H}_p = \sum_p^{K_p} \left\{ (h - J\cos k)(\hat{c}_k^{\dagger} \hat{c}_k - \hat{c}_{-k}^{*} \hat{c}_{-k}^{\dagger}) - \chi J (e^{-2i\phi} e^{ik} \hat{c}_k^{\dagger} \hat{c}_{-k}^{\dagger} + h.c.) \right\}}$$

Might be a typo in the document.

$$= -2J\hat{n}_0 + 2J\hat{n}_\pi + h(2\hat{n}_0 - 1) + h(2\hat{n}_\pi - 1)$$

$$= -2J(\hat{n}_0 - \hat{n}_\pi) + 2h(\hat{n}_0 + \hat{n}_\pi - 1)$$

The remaining $p=1$ terms and all terms for $p=0$, come into pairs $(k,-k)$.

→ Define the positive k values as follows ...

leaving out $k = 0,\pi$

$$\boxed{\begin{aligned} K^+_{p=1} &= \left\{ k = \frac{2n\pi}{L}, \quad n = 1, 2, \ldots, \frac{L}{2} - 1 \right\} \\ K^+_{p=0} &= \left\{ k = \frac{(2n-1)\pi}{L}, \quad n = 1, 2, \ldots, \frac{L}{2} \right\} \end{aligned}}$$

With this, we can write the Hamiltonian as ...

$$\boxed{\hat{H}_0 = \sum_{k \in K^+_{p=0}} \hat{H}_k \quad \& \quad \hat{H}_1 = \sum_{k \in K^+_{p=1}} \hat{H}_k + \hat{H}_{k=0,\pi}}$$

OBC PBC

where

factor of 2 due to the $(-k,k)$ symmetry ...

$$\hat{H}_k = 2(h - J\cos k)(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k}^\dagger \hat{c}_{-k}) - 2xJ(\sin k)(ie^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger - ie^{2i\phi}\hat{c}_{-k}\hat{c}_k)$$

each is 4×4.

and

$$\hat{H}_{k=0,\pi} = -2J(\hat{n}_0 - \hat{n}_\pi) + 2h(\hat{n}_0 + \hat{n}_\pi - 1)$$

this is a "4×4" matrix

where we have used the fact that

$$\sum_k \left(2\hat{c}_k^\dagger \hat{c}_k - 1\right) = \sum_k \left(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k} \hat{c}_{-k}^\dagger\right)$$

to manipulate the transverse field term & write

$$\hat{H}_p = -J \sum_k^{K_p} \left(2\cos k\, \hat{c}_k^\dagger \hat{c}_k\right) + h \sum_k^{K_p} \left(2\hat{c}_k^\dagger \hat{c}_k - 1\right) - J \sum_k^{K_p} \chi \left(e^{-2i\phi} e^{ik} \hat{c}_k^\dagger \hat{c}_{-k}^\dagger + h.c.\right).$$

$$= \sum_k^{K_p} \Big\{ (h - J\cos k)\left(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k} \hat{c}_{-k}^\dagger\right) - \chi J \left(e^{-2i\phi} e^{ik} \hat{c}_k^\dagger \hat{c}_{-k}^\dagger + h.c.\right) \Big\}.$$

now, notice that when $p=1$ & $k=0$ or π, we can take out the two terms with these indices and write

$$\hat{H}_{k=0,\pi} = -2J\left(\underbrace{\hat{n}_0}_{\hat{c}_0^\dagger \hat{c}_0} - \underbrace{\hat{n}_\pi}_{\hat{c}_\pi^\dagger \hat{c}_\pi}\right) + 2h\left(\underbrace{\hat{n}_0}_{\hat{c}_0^\dagger \hat{c}_0} + \underbrace{\hat{n}_\pi}_{\hat{c}_\pi^\dagger \hat{c}_\pi} - 1\right)$$

where does this come from? well... look at terms with $p=1$, $k=0, \pi$:

$$\cancel{(-J+h)\left(\hat{c}_0^\dagger \hat{c}_0 - \hat{c}_{-0} \hat{c}_{-0}^\dagger\right) + (h+J)\left(\hat{c}_\pi^\dagger \hat{c}_\pi - \hat{c}_{-\pi} \hat{c}_{-\pi}^\dagger\right)}$$

$$-J\left(2\hat{c}_0^\dagger \hat{c}_0\right) - J(-2)\left(\hat{c}_\pi^\dagger \hat{c}_\pi\right) + h\left(2\hat{c}_0^\dagger \hat{c}_0 - 1\right) + h\left(2\hat{c}_\pi^\dagger \hat{c}_\pi - 1\right)$$

$$-J\chi\{\underbrace{e^{-i\phi} \hat{c}_0^\dagger \hat{c}_0^\dagger + h.c.}_{0}\} - J\chi\{\underbrace{e^{-i\phi} \hat{c}_\pi^\dagger \hat{c}_\pi^\dagger (-1) + h.c.}_{0}\}$$

Now, a closer look at the Hamiltonian $\hat{H}_k$:

$$\hat{H}_k = 2(h - J\cos k)\left(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k}\hat{c}_{-k}^\dagger\right)$$
$$- 2i\chi J(\sin k)\left\{ e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger - e^{2i\phi}\hat{c}_{-k}\hat{c}_k \right\}$$

Look at the collection (basis): $\boxed{\left\{ \hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle, |0\rangle, \hat{c}_k^\dagger|0\rangle, \hat{c}_{-k}^\dagger|0\rangle \right\}}$

① $\hat{H}_k \underbrace{\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle} = 2(h - J\cos k)\left(\hat{c}_k^\dagger \hat{c}_k - \cancel{\hat{c}_{-k}\hat{c}_{-k}^\dagger}\right)\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle$
$$- J2i\chi\sin k\left\{ \cancel{e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger} - e^{+2i\phi}\hat{c}_{-k}\hat{c}_k \right\}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle$$
$$= 2(h - J\cos k)\underbrace{\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle} + 2i\chi J(\sin k)e^{i2\phi}\underbrace{|0\rangle}$$

②
$$\hat{H}_k|0\rangle = 2(h - J\cos k)\left(\cancel{\hat{c}_k^\dagger \hat{c}_k} - \hat{c}_{-k}\hat{c}_{-k}^\dagger\right)|0\rangle$$
$$- J2i\chi(\sin k)\left\{ e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger - \cancel{e^{2i\phi}\hat{c}_{-k}\hat{c}_k} \right\}|0\rangle$$
$$= -2(h - J\cos k)\underbrace{\cancel{\hat{c}_{-k}\hat{c}_{-k}^\dagger}}_{1}|0\rangle - 2i\chi J\sin k\, e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle$$

③ $\hat{H}_k \hat{c}_k^\dagger|0\rangle = \cancel{2(h - J\cos k)}\ 0$

④ $\hat{H}_k \hat{c}_{-k}^\dagger|0\rangle = 0$

So we see that in the subspace $\left\{ \overset{A}{\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle}, \overset{B}{|0\rangle}, \overset{C}{\hat{c}_k^\dagger|0\rangle}, \overset{D}{\hat{c}_{-k}^\dagger|0\rangle} \right\}$,

$$\hat{H}_k = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array}\begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \left(\begin{array}{cc|cc} 2(h - J\cos k) & -2i\chi J\sin k\, e^{-2i\phi} & 0 & 0 \\ 2i\chi J\sin k\, e^{2i\phi} & -2(h - J\cos k) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Check of dimensions

- Recall that both $\hat{H}_{p=0,1}$ have 2^L eigenvalues.

  Since there are $\frac{L}{2}$ such terms for H_k, we get a dimension of

$$\underbrace{4}_{\substack{(4\times 4)\\ H_k}}{}^{\overbrace{L/2}^{(L/2)\ H_k\text{'s}}} = 2^L \checkmark$$

$\hookrightarrow$ each H_k has 4 eigs.

- notice that $\hat{H}_{k=0,\pi}$, also works in a 4-dim subspace:

$$\{ |0\rangle,\ \hat{c}_0^\dagger \hat{c}_\pi^\dagger |0\rangle,\ \hat{c}_0^\dagger |0\rangle,\ \hat{c}_\pi^\dagger |0\rangle \}.$$

  and there are $\frac{L}{2}-1$ wave vectors in K_1^+

  $\Rightarrow$ again a total dimension for $\hat{H}_1$ of

$$\underbrace{4^{L/2-1}}_{\substack{\left(\frac{L}{2}-1\right)\text{ of } H_k \\ \text{where } k\in K^+_{p=1}}} \cdot \underbrace{4}_{\substack{\text{just}\\ \hat{H}_{k=0,\pi}}} = 2^L \checkmark$$

- Finally, recall that the correct eigs are obtained from the block Hamiltonians $\hat{H}_{p=0,1} = P\hat{H}P$ which have 2^{L-1} eigs, those with even $(p=0)$ & odd $(p=1)$ fermion parity.

→ Now, we want to further simplify things...

Look at

$$H_k = \begin{pmatrix} 2(h - J\cos k) & -2i\chi J \sin k\, e^{-2i\phi} & 0 & 0 \\ 2i\chi J \sin k\, e^{2i\phi} & -2(h - J\cos k) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$$

in the basis $\{ \vec{c}_k^{\,\dagger} \vec{c}_{-k}^{\,\dagger} |0\rangle,\ |0\rangle,\ \vec{c}_k^{\,\dagger}|0\rangle,\ \vec{c}_{-k}^{\,\dagger}|0\rangle \}$.

→ We want to isolate just the nontrivial block of this matrix, so that

$$\boxed{H_k = \begin{pmatrix} \mathcal{H}_k & 0 \\ 0 & 0 \end{pmatrix}_{4\times 4} \quad \text{where } \mathcal{H}_k \text{ is } 2\times 2.}$$

⇒ Need a Bogoliubov transformation.

To this end, define a 2-component spinor:

$$\boxed{\vec{\Psi}_k = \begin{pmatrix} \vec{c}_k \\ \vec{c}_{-k}^{\,\dagger} \end{pmatrix} \quad \& \quad \vec{\Psi}_k^{\,\dagger} = \left(\vec{c}_k^{\,\dagger},\ \vec{c}_{-k} \right)}$$

with the anti-commutation relation...

$$\boxed{\{ \vec{\Psi}_{k,\alpha},\ \vec{\Psi}^{\,\dagger}_{k',\alpha'} \} = \delta_{\alpha,\alpha'}\, \delta_{k,k'}}$$

(α denotes the component of $\vec{\Psi}_k$: $\alpha = 1, 2$)

With this, can rewrite H_k (4×4) as

$$\hat{\mathcal{H}}_k = \sum_{\alpha,\alpha'} \underbrace{\hat{\vec{\Psi}}^\dagger_{k\alpha}}_{\substack{4\times 1 \\ \text{or } (2)\times 2}} \underbrace{(\mathcal{H}_k)_{\alpha\alpha'}}_{(2\times 2)\text{ in }(2\times 2)} \hat{\vec{\Psi}}_{k\alpha'}$$

$$= \left(\hat{\vec{c}}_k^{\,\dagger}, \hat{\vec{c}}_{-k}\right) \underbrace{\begin{pmatrix} 2(h - J\cos k) & -2i\gamma J \sin k\, e^{-2i\phi} \\ 2i\gamma J \sin k\, e^{2i\phi} & -2(h - J\cos k) \end{pmatrix}}_{(\mathcal{H}_k)_{2\times 2}} \begin{pmatrix} \hat{\vec{c}}_k \\ \hat{\vec{c}}_{-k}^{\,\dagger} \end{pmatrix}$$

So

$$\boxed{\hat{\mathcal{H}}_k = \sum_{\alpha,\alpha'} \hat{\Psi}_{k\alpha} (\mathcal{H}_k)_{\alpha\alpha'} \hat{\Psi}_{k\alpha'}}$$

$$\left(\boxed{2}, \boxed{2}\right) \begin{pmatrix} \boxed{2}' & \boxed{2} \\ \boxed{2} & \boxed{2} \end{pmatrix} \begin{pmatrix} \boxed{2} \\ \boxed{2} \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{2} & 0 \\ 0 & 0 \end{pmatrix}$$

Now, look at this new matrix

$$\boxed{\mathcal{H}_k = \begin{pmatrix} 2(h - J\cos k) & -2i\gamma J \sin k\, e^{-2i\phi} \\ 2i\gamma J \sin k\, e^{2i\phi} & -2(h - J\cos k) \end{pmatrix}}$$

This can be expressed in terms of new pseudo-spin Pauli matrices $\hat{\tau}^{x,y,z}$ as

$$\boxed{\mathcal{H}_k = \mathbf{R}_k \cdot \vec{\tau}}$$

where

$$\mathbf{R}_k = 2\left(-\chi J \sin 2\phi \sin k,\ \chi J \cos 2\phi \sin k,\ (h - J\cos k)\right)^T$$

"effective magnetic field".

Let's verify this ...

$$\mathbf{R}_k \cdot \vec{\tau} = 2\begin{pmatrix} -\chi J \sin 2\phi \sin k \\ \chi J \cos 2\phi \sin k \\ h - J\cos k \end{pmatrix} \cdot \hat{\tau} \qquad \begin{pmatrix} \hat{\sigma}^x \\ \hat{\sigma}^y \\ \hat{\sigma}^z \end{pmatrix}$$

$$= 2(-\chi J \sin 2\phi \sin k)\,\hat{\sigma}^x + 2\chi J \cos 2\phi \sin k\,\hat{\sigma}^y + (h - J\cos k)\,\hat{\sigma}^z$$

$$= 2\begin{pmatrix} 0 & -\chi J \sin 2\phi \sin k \\ -\chi J \sin 2\phi \sin k & 0 \end{pmatrix} + 2\begin{pmatrix} h - J\cos k & 0 \\ 0 & -h + J\cos k \end{pmatrix}$$

$$+ 2\begin{pmatrix} 0 & -i\chi J \cos 2\phi \sin k \\ +i\chi J \cos 2\phi \sin k & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} h - J\cos k & -i\chi J \sin k\, e^{-2i\phi} \\ +i\chi J \sin k\, e^{+2i\phi} & -h + J\cos k \end{pmatrix}$$

✓

Info We can now see the role of the arbitrary phase ... ϕ

For $\phi = 0$, $\mathbf{R}_k$ lives in the y-z plane

$\phi = \pi/4$, $\mathbf{R}_k$ lives in the x-z plane

$\Rightarrow \mathcal{H}_k$ is real .

Now, one can diagonalize $\mathcal{H}_k$ and find the eigs:

$$\boxed{\varepsilon_{k\pm} = \pm \varepsilon_k} \quad \text{with}$$

$$\varepsilon_k = |R_k| = 2J\sqrt{\left(\cos k - \frac{h}{J}\right)^2 + \chi^2 \sin^2 k}$$

<u>Now, fix $\phi = 0$.</u> to start

$$\mathbf{R}_k = \left(0,\ 2\chi J \sin k,\ h - J\cos k\right)^T \equiv \left(0, y_k, z_k\right)^T$$

→ For the positive energy eigenvectors ... we have ...

$$\mathcal{H}_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} 0 \\ y_k \\ z_k \end{pmatrix} \cdot \begin{pmatrix} \hat{\sigma}^x \\ \hat{\sigma}^y \\ \hat{\sigma}^z \end{pmatrix} = y_k \hat{\sigma}^y + z_k \hat{\sigma}^z = \varepsilon_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

$$\rightarrow \boxed{\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{\sqrt{2\varepsilon_k(\varepsilon_k + z_k)}} \begin{pmatrix} \varepsilon_k + z_k \\ i y_k \end{pmatrix}}$$

more explicitly ...

$$\begin{cases} z_k u_k - i y_k v_k = \varepsilon_k u_k \\ i y_k u_k - z_k v_k = \varepsilon_k v_k \end{cases}$$

gives

eigv 1

$$\boxed{\begin{pmatrix} u_{k+} \\ v_{k+} \end{pmatrix} \equiv \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{\sqrt{2\varepsilon_k(\varepsilon_k + z_k)}} \begin{pmatrix} \varepsilon_k + z_k \\ i y_k \end{pmatrix}}$$

For the negative energy eigenvectors ... $\varepsilon_{k-} = -\varepsilon_k$, we have

$$\begin{cases} z_k(-v_k^*) - i y_k u_k^* = -\varepsilon_k(-v_k^*) = \varepsilon_{k-}(-v_k^*) \\ i y_k(-v_k^*) - z_k u_k^* = -\varepsilon_k(u_k^*) = \varepsilon_{k-}(-u_k^*) \end{cases}$$

to get

eigv 2

$$\boxed{\begin{pmatrix} u_{k-} \\ v_{k-} \end{pmatrix} \equiv \begin{pmatrix} -v_k^* \\ u_k^* \end{pmatrix} = \frac{1}{\sqrt{2\varepsilon_k(\varepsilon_k + z_k)}} \begin{pmatrix} i y_k \\ \varepsilon_k + z_k \end{pmatrix}}$$

So, the unitary U_k that diagonalizes $\mathcal{H}_k$ is

$$U_k = \begin{pmatrix} u_k & -v_k^* \\ v_k & u_k^* \end{pmatrix}$$

$\hookrightarrow$ $$\boxed{U_k^\dagger \mathcal{H}_k U_k = \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix}}$$

Define a new fermion two-component operator:

$$\boxed{\hat{\Phi}_k = \mathcal{U}_k^\dagger \hat{\Psi}_k = \begin{pmatrix} u_k^* & +v_k^* \\ -v_k & u_k^* \end{pmatrix}\begin{pmatrix} \hat{c}_k \\ \hat{c}_{-k}^\dagger \end{pmatrix} \equiv \begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}_{-k}^\dagger \end{pmatrix}}$$

where

$$\left\{ \begin{array}{l} \hat{\gamma}_k = u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^\dagger \\ \hat{\gamma}_{-k}^\dagger = -v_k \hat{c}_k + u_k^* \hat{c}_{-k}^\dagger \end{array} \right\}$$

~~Verify~~

~~claim: $\hat{\gamma}_k$ is indeed a fermion~~

~~Proof~~ $\cancel{\{\hat{\gamma}_k, \hat{\gamma}_k^\dagger\} = \{u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^{*}, -v_k \hat{c}_k + u_k \hat{c}_{-k}^\dagger\}}$

$\cancel{= |u_k|^2 \{\hat{c}_k, \hat{c}_{+k}^\dagger\} + |v_k|^2 \{\hat{c}_{-k}^\dagger, \hat{c}_{-k}\}}$

Why can we define $\hat{\gamma}_k$ this way?

Say $\hat{\gamma}_k = u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^\dagger$

$\rightarrow \hat{\gamma}_k^\dagger = u_k \hat{c}_k^\dagger + v_k \hat{c}_{-k}$

$\rightarrow \hat{\gamma}_{-k}^\dagger = u_{-k} \hat{c}_{-k}^\dagger + v_{-k} \hat{c}_{+k}$

but $u_{-k} = u_k$ & $v_{-k} = -v_k$ (eigvalues)

$\Rightarrow \hat{\gamma}_{-k}^\dagger = u_k \hat{c}_{-k}^\dagger - v_k \hat{c}_k$ ✓

So the defn above makes sense!

Claim $\boxed{\hat{\gamma}_k \text{ is a fermion}}$

Pf $\{\hat{\gamma}_k, \hat{\gamma}_k^\dagger\} = \{ u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^\dagger , u_k \hat{c}_k^\dagger + v_k \hat{c}_{-k} \}$

$$= |u_k|^2 \underbrace{\{\hat{c}_k, \hat{c}_k^\dagger\}}_{1} + |v_k|^2 \underbrace{\{\hat{c}_{-k}^\dagger, \hat{c}_{-k}\}}_{1}$$

$$= |u_k|^2 + |v_k|^2 = 1 .$$

$\square$.

With this new definition, we can write $\hat{H}_k$ (4x4) as

$$\hat{H}_k = \hat{\Psi}_k^\dagger H_k \hat{\Psi}_k$$

$$= \underbrace{\hat{\Psi}_k^\dagger U_k}_{(\hat{\gamma}_k^\dagger, \hat{\gamma}_{-k})} U_k^\dagger H_k U_k \underbrace{U_k^\dagger \hat{\Psi}_k}_{\begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}_{-k}^\dagger \end{pmatrix}}$$

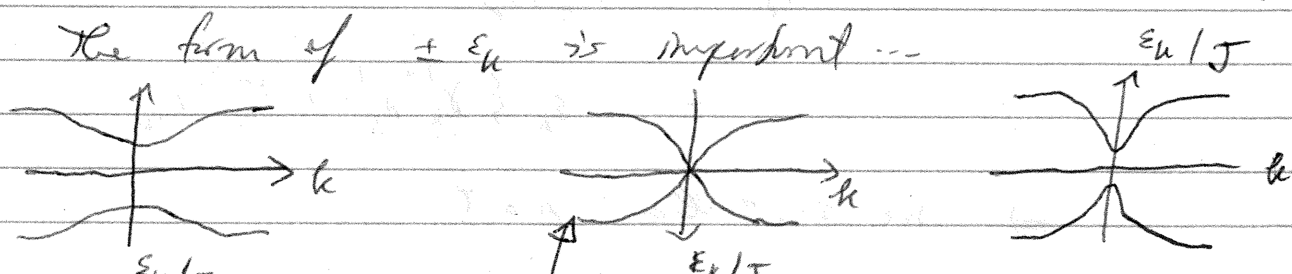
$$= \hat{\Phi}_k^\dagger \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix} \hat{\Phi}_k = \varepsilon_k \hat{\Phi}_k^\dagger \hat{\sigma}^z \hat{\Phi}_k$$

$$= \varepsilon_k \left(\hat{\gamma}_k^\dagger \hat{\gamma}_k - \hat{\gamma}_{-k} \hat{\gamma}_{-k}^\dagger \right) \longrightarrow \text{anti-comm relation,}$$

$$= \varepsilon_k \left(\hat{\gamma}_k^\dagger \hat{\gamma}_k + \hat{\gamma}_{-k}^\dagger \hat{\gamma}_{-k} - 1 \right)$$

$$\Rightarrow \boxed{\hat{H}_k = \varepsilon_k \left(\hat{\gamma}_k^\dagger \hat{\gamma}_k + \hat{\gamma}_{-k}^\dagger \hat{\gamma}_{-k} - 1 \right)}$$

The form of $\pm \varepsilon_k$ is important ...



(ferromagnetic) (critical) (paramagnetic)

GAPLESS

- Critical point: Gapless linear spectrum.
- Ferro vs Paramagnetic: indistinguishable, but topology is distinctly different

$\longrightarrow$ we'll see this later.

Oct 20, 2020

3.1. Ground state & Excited states of the Ising model

Recall that with $\hat{\vec{\Phi}}_k^{\dagger} = \hat{\vec{\Psi}}_k^{\dagger} U_k = (\hat{\gamma}_k^{\dagger}, \hat{\gamma}_{-k})$ we have

$$\hat{H}_k = \vec{\Psi}_k^{\dagger} U_k U_k^{\dagger} \mathcal{H}_k U_k U_k^{\dagger} \vec{\Psi}_k$$

$$= \hat{\vec{\Phi}}_k^{\dagger} \begin{pmatrix} \varepsilon_k & \\ & -\varepsilon_k \end{pmatrix} \hat{\vec{\Phi}}_k = (\varepsilon_k) \{ \hat{\gamma}_k^{\dagger} \hat{\gamma}_k - \hat{\gamma}_{-k} \hat{\gamma}_{-k}^{\dagger} \}$$

$$= (\varepsilon_k) \{ \hat{\gamma}_k^{\dagger} \hat{\gamma}_k + \hat{\gamma}_{-k}^{\dagger} \hat{\gamma}_{-k} - \mathbb{1} \}.$$

From here, we notice that if $|\phi_g\rangle$ denotes the ground state of $\hat{\mathcal{H}}_k$ then

$$\hat{\mathcal{H}}_k |\phi_g\rangle = -\varepsilon_k |\phi_g\rangle$$

$$= -\varepsilon_k \{ \hat{\gamma}_k^{\dagger} \hat{\gamma}_k + \hat{\gamma}_{-k}^{\dagger} \hat{\gamma}_{-k} - \mathbb{1} \} |\phi_g\rangle$$

$\Rightarrow$ we must have that

$$(\hat{\gamma}_k^{\dagger} \hat{\gamma}_k + \hat{\gamma}_{-k}^{\dagger} \hat{\gamma}_{-k}) |\phi_g\rangle = 0$$

This occurs if $|\phi_\gamma\rangle$ annihilates the $\gamma_k^\dagger$'s for all k. (positive or negative)

→ These are called the "Bogoliubov vacuum".

$$\boxed{\gamma_k^\dagger |\phi_\gamma\rangle = 0 \qquad \forall k}$$

Even | # of particles
Odd | ↗

In general, one can define 2 such states, one in the $p=0$ (~~ABC~~) and one in the $p=1$ (PBC) sector (odd)

However, the "winner" between the two is the actual global ground state is the one in the $p=0$ (even) sector

→ Energy of this gnd state is simply

$$\boxed{E_0^{ABC} = -\sum_{k>0}^{ABC} \varepsilon_k}$$

Explicitly, the ground state is given by

$$\boxed{|\phi_\gamma\rangle^{ABC} \propto \prod_{k>0}^{ABC} \gamma_{-k}\gamma_k |0\rangle}$$

s.t. $\gamma_k^\dagger|0\rangle = 0$

where $|0\rangle$ is the vacuum for the original fermions

$$c_k^\dagger |0\rangle = 0.$$

Now, let's expand this...

by defn of $\hat{\gamma}_k$

$$\prod_{k>0} \hat{\gamma}_{-k} \hat{\gamma}_k |0\rangle = \prod_{k>0} \left\{ u^*_{-k} \hat{c}_{-k} + v^*_{-k} \hat{c}^\dagger_k \right\} \left\{ \cancel{u^*_k \hat{c}_k} + v^*_k \hat{c}^\dagger_{-k} \right\} |0\rangle$$

c-no.

$$= \prod_{k>0} \left\{ u^*_{-k} \hat{c}_{-k} + v^*_{-k} \hat{c}^\dagger_k \right\} \left\{ v^*_k \hat{c}^\dagger_{-k} |0\rangle \right\}$$

$$= \prod_{k>0} v^*_k \Big(\underbrace{\hat{c}_{-k} \hat{c}^\dagger_{-k}}_{\mathbb{1}} u^*_{-k} + v^*_{-k} \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \Big) |0\rangle$$

$$= \prod_{k>0} v^*_k \left(u^*_{-k} + v^*_{-k} \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right) |0\rangle$$

$$= \prod_{k>0} v^*_k \left\{ u^*_k - v^*_k \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right\} |0\rangle$$

where we have used $\begin{cases} u_k = u_{-k} \\ v_k = -v_{-k} . \end{cases}$

With this, we find $|\phi_\gamma\rangle$ in terms of $\hat{c}_k$ & coefs ... after normalising ...

$$\boxed{\begin{aligned} |\phi_\gamma\rangle^{ABC} &= \prod_{k>0}^{ABC} \left(u^*_k - v^*_k \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right) |0\rangle \\ &= \prod_{k>0}^{ABC} \left(u_k + v_k \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right) |0\rangle \end{aligned}}$$

note that we can $u^*_k \to u_k$ by choosing a phase in which

u_k is real & v_k is purely imaginary.

This is a $\boxed{\text{BCS form}}$

What about the PBC sector state? I.e. what about the ground state of the odd-fermion sector?

↳ Since a BCS-paired state is always fermion-even, the unpaired $\hat{H}_{k=0,\pi}$ must contribute <u>exactly</u> <u>one</u> fermion in the ground state.

Suppose $h>0$, the gnd state has $\begin{cases} \hat{n}_{k=0} \to 1 \\ \hat{n}_{k=\pi} \to 0 \end{cases}$ (since gnd state)

↳ ~~Recall that~~ and so...

~~$\hat{H}_k = 2(h - J\cos k)(\hat{c}_k^{\dagger}\hat{c}_k - \hat{c}_{-k}\hat{c}_{-k}^{\dagger}) - 2J\sin k\,(ie^{-2i\phi}\hat{c}_k^{\dagger}\hat{c}_{-k}^{\dagger} + ie^{2i\phi}\hat{c}_{-k}\hat{c}_k)$~~

~~when $k = 0,\pi \Rightarrow \hat{H}_k = 2(h \mp J)(\ldots)$~~

~~→ also, we have that the gnd state energy has an extra term~~

~~$\boxed{\delta E_{0,\pi} = \min(\hat{H}_{0,\pi}) = -2J}$~~

Now, recall that

$$\hat{H}_{k=0,\pi} = -2J(\overset{1}{\hat{n}_0} - \overset{0}{\hat{n}_\pi}) + 2h(\overset{1}{\hat{n}_0} + \overset{0}{\hat{n}_\pi} - 1)$$

So, the ~~fact~~ we get ~~$\hat{H}_{k=0}$~~ a term of the form $\boxed{\delta E_{0,\pi} = \min(\hat{H}_{0,\pi}) = -2J}$

single fermion

The PBC ground state is therefore

$$\boxed{\begin{aligned} |\phi_\gamma\rangle^{PBC} &= \hat{c}^\dagger_{k=0} \prod_{\pi > k > 0}^{PBC} \left(u_k^* - v_k^* \hat{c}_k^\dagger \hat{c}_{-k}^\dagger\right)|0\rangle \\ &= \hat{\gamma}_0 \prod_{0<k<\pi}^{PBC} \left(u_k^* - v_k^* \hat{c}_k^\dagger \hat{c}_{-k}^\dagger\right)|0\rangle \end{aligned}}$$

where we defined $\hat{\gamma}_0 = \hat{c}^\dagger_{k=0}$; $\hat{\gamma}_\pi = \hat{c}_{k=\pi}$

for the unpaired states.

The corresponding energy is

$$\boxed{E_0^{PBC} = -2J - \sum_{0<k<\pi}^{PBC} \varepsilon_k} \quad \text{(as expected)}$$

What happens in the thermodynamic limit $L \to \infty$??

→ We would expect that the energy per site

$e_0 = \frac{E_0}{L}$, the gnd state energy tends to an integral...

Expect: $$e_0 = -\lim_{L\to\infty} \frac{1}{L} \sum_{k>0}^{ABC} \varepsilon_k = -\int_0^\pi \frac{dk}{2\pi} \varepsilon_k$$

But ...

There is some subtlety when we treat the boundary points at 0, π

notice that for E_0^{ABC} involves $\frac{L}{2}$ k-points in $(0,\pi)$

but E_0^{PBC} involves $\frac{L}{2}-1$ in $(0,\pi)$

+ an extra term $-2J$

It turns out that the energy splitting

$$\boxed{\Delta E_0 = E_0^{PBC} - E_0^{ABC}} \quad \text{when } -J<h<J$$

(ferro. ordered)

decays exponentially fast when $L \to \infty$

$\Rightarrow$ the two sectors ABC, PBC provide the required double degeneracy of the ferromag phase so long as $|h|<J$.

When $h=0$ for instance (easy to see...)

$\Rightarrow$ On the contrary, ΔE_0 is finite in the quantum disordered region $|h|>J$

$$\rightarrow \boxed{\Delta E_0 = 2(|h|-J)}$$

and goes to zero as a power law $\sim \frac{\pi}{2L}$

at the critical points $h_c = \pm J$

(figures)

What about excited states?

Let's look at excited states in the $p=0$ (even) sector...

→ Consider the state $\hat{\gamma}^{\dagger}_{k_1} |\phi_\gamma\rangle^{ABC}$.

Have that

$$\hat{\gamma}^{\dagger}_{k_1} |\phi_\gamma\rangle^{ABC} = \hat{\gamma}^{\dagger}_{k_1} \prod_{k>0}^{ABC} \left(u_k^* - v_k^* \hat{c}^{\dagger}_k \hat{c}^{\dagger}_{-k} \right) |0\rangle$$

$$= \left(-v_{-k_1} \hat{c}_{-k_1} + u_{-k_1} \hat{c}^{\dagger}_{k_1} \right) \prod_{k>0}^{ABC} \left(u_k^* - v_k^* \hat{c}^{\dagger}_k \hat{c}^{\dagger}_{-k} \right) |0\rangle$$

→ $$\boxed{\hat{\gamma}^{\dagger}_{k_1} |\phi_\gamma\rangle^{ABC} = \hat{c}^{\dagger}_{k_1} \prod_{\substack{k>0 \\ k \neq |k_1|}}^{ABC} \left(u_k^* - v_k^* \hat{c}^{\dagger}_k \hat{c}^{\dagger}_{-k} \right) |0\rangle}$$

⇒ $\hat{\gamma}^{\dagger}_{k_1}$ transforms the Cooper pair at momentum $(|k_1|, -|k_1|)$ into an unpaired fermion in the state

$$\hat{c}^{\dagger}_{k_1} |0\rangle :$$

$\hat{\gamma}^{\dagger}_{k_1}$ ⇒ this cost an extra energy $+\varepsilon_{k_1}$ over the gnd state.

→ There's a problem with parity here because a single unpaired fermion changes the overall fermion parity.

→ Lowest allowed states must involve 2 creation ops: $\hat{\gamma}^{\dagger}_{k_1}$, $\hat{\gamma}^{\dagger}_{k_2}$, with $k_1 \neq |k_2|$;

$$\rightarrow \boxed{\hat{\gamma}^{\dagger}_{k_1} \hat{\gamma}^{\dagger}_{k_2} |\phi_\gamma\rangle^{ABC} = \hat{c}^{\dagger}_{k_1} \hat{c}^{\dagger}_{k_2} \prod^{ABC}_{\substack{k>0 \\ k \neq |k_1|, |k_2|}} \left(u_k^* - v_k^* \hat{c}^{\dagger}_k \hat{c}^{\dagger}_{-k} \right) |0\rangle}$$

The energy of such an excitation is $\boxed{E_0^{ABC} + \varepsilon_1 + \varepsilon_2}$.

<u>Fun fact</u> : if we consider the special case $\hat{\gamma}^{\dagger}_{k_1} \hat{\gamma}^{\dagger}_{-k_1}$ we find that

$$\hat{\gamma}^{\dagger}_{k_1} \hat{\gamma}^{\dagger}_{-k_1} |\phi_\gamma\rangle^{ABC} = \left(v_{k_1} + u_{k_1} \hat{c}^{\dagger}_{k_1} \hat{c}^{\dagger}_{-k_1} \right) \prod^{ABC}_{\substack{k>0 \\ k \neq |k_1|}} \left(u_k^* - v_k^* \hat{c}^{\dagger}_k \hat{c}^{\dagger}_{-k} \right) |0\rangle$$

$\Rightarrow$ $\hat{\gamma}^{\dagger}_{k_1} \hat{\gamma}^{\dagger}_{-k_1}$ transforms the Cooper pair at momentum $(|k_1|, -|k_1|)$ into the corresponding anti-bonding pair:

↳ Explicitly ...

$$\underbrace{\left(u_{k_1}^* - v_{k_1}^* \hat{c}^{\dagger}_{k_1} \hat{c}^{\dagger}_{-k_1} \right) |0\rangle}_{\substack{\text{Cooper pair} \\ @\ (|k_1|, -|k_1|)}} \xrightarrow{\hat{\gamma}^{\dagger}_{k_1} \hat{\gamma}^{\dagger}_{-k_1}} \underbrace{\left(v_{k_1} + u_{k_1} \hat{c}^{\dagger}_{k_1} \hat{c}^{\dagger}_{-k_1} \right) |0\rangle}_{\substack{\text{anti-bonding} \\ \text{pair}}}$$

this can be checked by substituting in the defn of $\hat{\gamma}^{\dagger}_{\pm k_1}$.

$\Rightarrow$ this costs an energy of $2\varepsilon_{k_1}$.

(42)

- From here, we can construct all excited states for the even ($p=0$) sector... by applying an even number of $\hat{\gamma}_k^{\dagger}$ to $|\phi_\gamma\rangle^{ABC}$

  where each $\hat{\gamma}_k^{\dagger}$ costing an energy ε_k (makes sense...)

→ In the occupation number representation (Fock) we have...

$$\boxed{\begin{aligned} |\Psi_{\{n_k\}}\rangle &= \prod_k^{ABC} \left(\hat{\gamma}_k^{\dagger}\right)^{n_k} |\phi_\gamma\rangle^{ABC} \quad \text{with } n_k = 0,1, \ \sum_k^{ABC} n_k \text{ even.} \\ E_{\{n_k\}} &= E_0^{ABC} + \sum_k^{ABC} n_k \varepsilon_k \end{aligned}}$$

↳ Note that there are a total of 2^{L-1} such states, as required.

In the $p=1$ sector, we must be careful... We should still apply an even number of $\hat{\gamma}_k^{\dagger}$ to the gnd state $|\phi_\gamma\rangle^{PBC}$ (to preserve fermion parity), including in the choice ~~of~~ unpaired ops $\hat{\gamma}_0^{\dagger}$, amounting to removing the fermion from the $k=0$ state, and $\hat{\gamma}_\pi^{\dagger}$ amounting to creating a fermion in the $k=\pi$ state.

—

Next, we look at how we can relate all this back to the spin representation...

→ $\boxed{\text{Relationship with the spin representation}}$

→ here we relate the spectrum in the fermionic representation & the corresponding physics in the original spin representation.

→ here let us fix $\chi = 1$.

→ look at classical Ising model...

$$\hat{H}_{cl} = -J \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x$$

→ there are 2 degenerate gnd states...

$$|++\ldots+\rangle \quad \& \quad |--\ldots-\rangle$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(1, \pm 1)^T$ denote the 2 eigenstates of σ^x with eigv ± 1.

Recall parity op: $\hat{P} = \prod_{j=1}^{L} \sigma_j^z$; and that $\sigma^z|\pm\rangle = |\mp\rangle$

we have... $\begin{cases} \hat{P}|++\ldots+\rangle = |--\ldots-\rangle \\ \hat{P}|--\ldots-\rangle = |++\ldots+\rangle \end{cases}$

⇒ 2 eig states of $\hat{P}$ must be

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|++\ldots+\rangle \pm |--\ldots-\rangle\right) \rightarrow \hat{P}|\Psi_\pm\rangle = \mp|\Psi_\pm\rangle$$

These two opposite parity states $|\psi_\pm\rangle$ must be rep. by 2 fermionic gnd states belonging to the ABC/PBC sectors, which are degenerate when $h=0$.

→ Now, let $h \neq 0$ but $|h| \ll J$. Let us consider ...

$$\hat{\mathcal{H}}_{OBC} = -J \sum_{j=1}^{L-1} \hat{\sigma}^x_j \hat{\sigma}^x_{j+1} - h \sum_{j=1}^{L} \hat{\sigma}^z_j .$$

At lowest-perturbative order in $|h|/J$, the 2 lowest-energy states have the same form: $\prod |+\rangle$, $\prod |-\rangle$

or $|\psi_+\rangle$, $|\psi_-\rangle$.

→ to get higher excitations, consider the domain-wall configs of the form...

$$|l\rangle = |\underbrace{-\,-\,- \dots -}_{1 \to l \text{ sites}}\, + + \dots + \rangle , \quad l = 1 \dots L-1$$

For $h=0$, all $|l\rangle$'s are degenerate, and separated from the 2 gnd states by a gap $(2J)$.

~~→ can study the effect of small transverse field by perturbation theory...~~

~~The Hamiltonian restricted to the $L-1$ (dim) subspace of the domain-wall excitations has the form~~

$$\cancel{\hat{H}_{eff} = 2J \sum_{l=1}^{L-1} |l\rangle\langle l| - h \sum_{l=1}^{L-2} \left(|l\rangle\langle l+1| + h.c. \right)}$$

let's estimate the separation between the two gnd states originating from the $h=0$ doublet when $h>0$.

$h=0 \Rightarrow \prod|+\rangle$ & $\prod|-\rangle$ are degenerate.

& this doublet is separated from other states by $2J$.

Now... $\prod|+\rangle$ & $\prod|-\rangle$ are coupled only at order L in perturbation theory...

↳ b/c we need to flip L spins using $\hat{\sigma}_j^z$ to couple one to another.

→ Expect their splitting to be

$$\Delta E_L \sim \left(h/J\right)^L \rightarrow \text{exponentially small in } L \text{ for small } |h|.$$

⇒ the resulting eigenstates $|\Psi_\pm(h)\rangle$, even and odd approach the 2 eigenstates $|\Psi_\pm\rangle$

where again... $|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(\prod|+\rangle \pm \prod|-\rangle\right)$.

for $h \to 0$.

⇒ $\boxed{\text{In the thermodynamic limit, break } \mathbb{Z}_2 \text{ symmetry.}}$

At any finite size we have the symmetry-preserving gnd states $|\Psi_\pm(h)\rangle \to |\Psi_\pm\rangle$ as $h\to 0$.

↳ there are superpositions of macroscopically ordered states
$$|\pm\rangle_h = \frac{1}{\sqrt{2}}\left(|\Psi_+(h)\rangle \pm |\Psi_-(h)\rangle\right).$$

→ There can be explicit symmetry breaking in the subspace generated by $|\Psi_\pm(h)\rangle$ <u>only</u> in the thermodynamic limit in which

↳ the 2 macrostates are degenerate ⇒ the slightest perturbation selects one of the two macroscopically ordered superpositions $|\pm\rangle_h$.

~~Symmetry~~

Nov 11, 2020

$\boxed{\text{Nambu formalism for the general case}}$

Summary of what we've seen so far..

In the ordered case, $\mathcal{H}$ can be diagonalized by

1) Fourier transform : reducing problem to a collection of (2×2) pseudo spin 1/2 problems
2) Bogoliubov transform ✓

→ In the disordered case, we do kind of the same thing, but we won't be able to reduce to 2×2 problems in a simple way.

Instead, we'll need the Nambu formalism.

↳

Define a column vector $\hat{\Psi}$ and its Hermitian conjugate row vector $\hat{\Psi}^\dagger$, each of length $2L$ by

$$\hat{\Psi} = \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_L \\ \hat{c}_1^\dagger \\ \vdots \\ \hat{c}_L^\dagger \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{c}} \\ \hat{\mathbf{c}}^\dagger \end{pmatrix}$$

$$\hat{\Psi}^\dagger = \left(\hat{\mathbf{c}}^\dagger \;\; \hat{\mathbf{c}} \right)$$

OR

$$\hat{\Psi}_j = \hat{c}_j \; ; \; \hat{\Psi}_{j+L} = \hat{c}_j^\dagger$$
$$\hat{\Psi}_j^\dagger = \hat{c}_j^\dagger \; ; \; \hat{\Psi}_{j+L}^\dagger = \hat{c}_j \qquad j \leq L$$

where the $\mathbf{c}_j$'s are from the Jordan-Wigner trans f.

Warning: $\hat{\Psi}_j$ satisfies the standard fermionic anti-commutation rule

$$\{ \hat{\Psi}_j , \hat{\Psi}_{j'}^\dagger \} = \delta_{j,j'} \quad \forall j \leq 2L$$

But note that

$$\{ \hat{\Psi}_j , \hat{\Psi}_{j+L} \} = 1 \quad \forall j \leq L$$

$\hookrightarrow$ we'll worry abt this later

Next, we introduce the SWAP matrix ($2L \times 2L$):

$$\$ = \begin{pmatrix} 0_{L\times L} & \mathbb{1}_{L\times L} \\ \mathbb{1}_{L\times L} & 0_{L\times L} \end{pmatrix}$$

Next, consider a general fermionic quadratic form

$$\hat{H} = \hat{\Psi}^\dagger \mathbb{H} \hat{\Psi} = (\hat{c}^\dagger \; \hat{c}) \begin{pmatrix} A & B^* \\ -B^* & -A \end{pmatrix} \begin{pmatrix} \hat{c} \\ \hat{c}^\dagger \end{pmatrix}$$

- $\mathbb{H}$ is Hermitean
- A ~~is~~ ~~are~~ also Hermitean ($A = A^\dagger$)
- B is anti-symmetric. ($B = -B^T$)

Note

(particle-hole symmetry : $\mathbb{H}\$ = -\$\mathbb{H}^*$)

we won't worry about this now...

Now, let's look ~~also~~ back at the Ising case:

$$\hat{\mathbb{H}}_{p=0,1} = -\sum_{j=1}^{L} (J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1} + h.c.) + \sum_{j=1}^{L} h_j (\hat{c}_j \hat{c}_j^\dagger - \hat{c}_j^\dagger \hat{c}_j)$$

with boundary condition

$$\hat{c}_{L+1} = (-1)^{p+1} \hat{c}_1$$

Note that J_j, h_j are real.

Okay now that we have $\mathbb{H}_{p=0}$ $(2L\times 2L)$ and $\mathbb{H}_{p=1}$ $(2L\times 2L)$ for each ~~symmetry~~ parity sector ... $p=0,1$

$\downarrow$ since each $\hat{c}_j$ is 2×2

→ The corresponding $2L\times 2L$ matrices $\mathbb{H}_p$ are ~~un~~ real & symmetric.

→ A is real & symmetric, B is real & antisymmetric

So,

$$\mathbb{H} = \begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \xrightarrow{\text{Ising}} \mathbb{H}_p = \begin{pmatrix} A & B \\ \underbrace{-B}_{B^T} & \underbrace{-A}_{-A^T} \end{pmatrix}$$

The structure of A, B are given by

$$\boxed{\begin{aligned} &\begin{cases} A_{j,j} = h_j \\ A_{j,j+1} = A_{j+1,j} = -\dfrac{J_j^+}{2} = \dfrac{-J_j}{2} \end{cases} \\ &\text{and} \qquad\qquad \underline{\text{else} = 0} \\ &\begin{cases} B_{j,j} = 0 \\ B_{j,j+1} = -B_{j+1,j} = -\dfrac{J_j^-}{2} = -\dfrac{\chi J_j}{2} \end{cases} \end{aligned}}$$

where $J_j^x = J_j(1+\chi)/2$; $J_j^y = J_j(1-\chi)/2$

In the PBC-spin case, we get additional matrix elements

$$A_{L,1} = A_{1,L} = (-1)^p \frac{J_L^+}{2} = (-1)^p \frac{J_L}{2}$$
$$B_{L,1} = -B_{1,L} = (-1)^p \frac{J_L^-}{2} = (-1)^p \frac{x J_L}{2}$$

both depending on the fermion parity p.

The OBC case is recovered by setting $J_L = 0$, which makes $H_1 = H_0$. $\rightsquigarrow$ note that there are no longer the $H_{p=0,1}$ as we've seen before...

$\boxed{\text{Now, let us diagonalize } \vec{H}}$

<u>The Bogoliubov - de Gennes Eqn</u>

Consider the eigv problem

$$H \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix} = \begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix} = \varepsilon_\mu \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix}$$

where u, v are L-dimensional

μ : index referring to the μ-th eigv.

This gives the Bogoliubov - de Gennes eqn

$$\begin{cases} A u_\mu + B v_\mu = \varepsilon_\mu u_\mu \\ -B^* u_\mu - A^* v_\mu = \varepsilon_\mu v_\mu \end{cases}$$

easy to show that if $(u_\mu, v_\mu)^T$ is eigv with eigv ε_μ

then $(v_\mu^* \; u_\mu^*)^T$ is eigv with eigv $-\varepsilon_\mu$.

In Ising case, A, B are real, so solutions ~~are~~ can be taken to be real...

→ We can organize the eigv in a unitary (or orth. if solutions are real) $2L \times 2L$ matrix:

$$\mathbb{U} = \left(\begin{array}{c|c} u_1 \cdots u_L & v_1^* \cdots v_L^* \\ \hline v_1 \cdots v_L & u_1^* \cdots u_L^* \end{array} \right) = \begin{pmatrix} \mathcal{U} , & \mathcal{V}^* \\ \mathcal{V} & \mathcal{U}^* \end{pmatrix}$$

$$(u_\mu = \{ u_1, u_2, \ldots u_L \})$$

where $\mathcal{U}, \mathcal{V}$ are $L \times L$ matrices. With this, we find that $\mathbb{U}$ diagonalizes $\mathbb{H}$.

$$\boxed{\mathbb{U}^\dagger \mathbb{H} \mathbb{U} = \left(\begin{array}{cccc|cccc} \varepsilon_1 & & & & & & & \\ & \varepsilon_2 & & & & & \mathbb{O} & \\ & & \ddots & & & & & \\ & & & \varepsilon_L & & & & \\ \hline & & & & -\varepsilon_1 & & & \\ & \mathbb{O} & & & & \ddots & & \\ & & & & & & & -\varepsilon_L \end{array} \right)}$$

let's call this $\mathbb{E}_{diag}$.

If we define new Number fermion operators $\hat{\Phi}, \hat{\Phi}^\dagger$ s.t.

$$\hat{\Psi} = \mathbb{U} \hat{\Phi}$$

then
$$\hat{H} = \hat{\Psi}^\dagger \mathbb{H} \hat{\Psi} = \hat{\Phi}^\dagger \mathbb{U}^\dagger \mathbb{H} \mathbb{U} \hat{\Phi} = \hat{\Phi}^\dagger \mathbb{E}_{diag} \hat{\Phi}$$

In this case,

$$\hat{\vec{\Phi}} = \begin{pmatrix} \hat{\gamma} \\ \hat{\gamma}^{+} \end{pmatrix} = \mathbb{U}^{+}\hat{\Psi} = \begin{pmatrix} U^{+} & V^{+} \\ V^{T} & U^{T} \end{pmatrix} \begin{pmatrix} \hat{c} \\ \hat{c}^{+} \end{pmatrix}$$

More explicitly ...

$$\boxed{\begin{cases} \hat{\gamma}_{\mu} = \sum_{j=1}^{L} \left(U^{*}_{j\mu}\, \hat{c}_j + V^{*}_{j\mu}\, \hat{c}_j^{+} \right) \\ \hat{\gamma}^{+}_{\mu} = \sum_{j=1}^{L} \left(V_{j\mu}\, \hat{c}_j + U_{j\mu}\, \hat{c}_j^{+} \right) \end{cases}}$$

(One can check that $\hat{\vec{\Phi}}, \hat{\Phi}$ are indeed fermion operators ...)

this can be inverted ... $(\hat{\Psi} = \mathbb{U}\hat{\Phi})$

$$\boxed{\begin{cases} \hat{c}_j = \sum_{\mu} \left(U_{j\mu}\, \hat{\gamma}_{\mu} + V^{*}_{j\mu}\, \hat{\gamma}^{+}_{\mu} \right) \\ \hat{c}_j^{+} = \sum_{\mu} \left(V_{j\mu}\, \hat{\gamma}_{\mu} + U^{*}_{j\mu}\, \hat{\gamma}^{+}_{\mu} \right) \end{cases}}$$

So, in terms of $\hat{\gamma}_{\mu}$, $\hat{H}$ reads

$$\hat{H} = \sum_{\mu=1}^{L} \left(\varepsilon_{\mu}\, \hat{\gamma}^{+}_{\mu}\hat{\gamma}_{\mu} - \varepsilon_{\mu}\, \hat{\gamma}_{\mu}\hat{\gamma}^{+}_{\mu} \right)$$

$$= \sum_{\mu=1}^{L} 2\varepsilon_{\mu} \left(\hat{\gamma}^{+}_{\mu}\hat{\gamma}_{\mu} - \frac{1}{2} \right)$$

The gnd state is that annihilated by all $\hat{\gamma}_\mu$...

$$\hat{\gamma}_\mu |\phi_0\rangle = 0 \;\; \forall_\mu \iff \hat{H}|\phi_0\rangle = E_0 |\phi_0\rangle \text{ with } E_0 = -\sum_{\mu=1}^{L} \varepsilon_\mu\,,$$

The 2^L Fock states can be expressed as

$$\begin{cases} |\Psi_{\{n_\mu\}}\rangle = \prod_{\mu=1}^{L} (\hat{\gamma}_\mu^\dagger)^{n_\mu} |\phi_0\rangle & n_\mu = \{0,1\} \\ E_{\{n_\mu\}} = E_0 + 2\sum_\mu n_\mu \varepsilon_\mu \end{cases}$$

<u>BCS - form of the ground state</u>