Today's plan

Lectures

- What is a symmetry?
- A crash course in group theory

Activities

- GeoGebra
- Brainstorming
- Breakout rooms

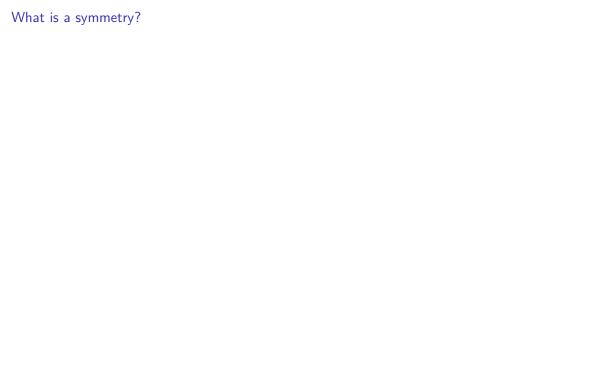
Note: I will be writing on top of these slides. I'll send you a link to the blank slides for now and upload the pdf with my written notes later today.

How to interact during the lectures

- This class is a safe (virtual) place. Questions are always welcome, no matter how trivial you may think they are.
- You can ask questions at any time during the lecture. You have a few options:
 - "Raise your hand" through Zoom and ask in person
 - Use the Zoom chat
 - Ask on sli.do (#W613)
- I will often ask you questions during the lectures. I do not know how well this is going to work online, but I'll still do it.

Socrative

To answer polls on socrative, go to socrative.com and login as a sudent with the room name **SYMMETRIES**.



Groups and subgroups

Definition (group)

A group is a set G together with an operation $*: G \times G \rightarrow G$ satisfying the following properties:

- there is a special element $e \in G$, called the *identity*,
 - such that

$$g * e = e * g = g, \quad \forall g \in G$$

 each element of G has an inverse, that is for each $g \in G$ there is an element $g^{-1} \in G$ such that

$$g^{-1} * g = g * g^{-1} = e$$

• the operation * is associative, that is

$$a*(b*c) = (a*b)*c, \forall a,b,c \in G.$$

Additionally, we say that the group G is abelian or commutative if

$$a * b = b * a$$
, $\forall a, b \in G$.

Definition (subgroup)

Let G be a group with operation *. A subgroup of G is a subset $H \subseteq G$ that contains the identity element and is closed under the operation * and under inversion,

that is
$$e \in H$$

3. $(g^{-1})^{-1} = g$ 4. $(gh)^{-1} = h^{-1}g^{-1}$

■
$$a*b \in H$$
 for all $a, b \in H$

H is a subgroup of G.

Exercise

 \bullet $a \in H \implies a^{-1} \in H$ The notation $H \leq G$ is commonly used to indicate that

Homomorphisms and isomorphisms

Definition (group homomorphism)

A group homomorphism is a map $\varphi: G \to H$ between two groups G and H such that

$$\varphi(a*_G b) = \varphi(a)*_H \varphi(b), \quad \forall a, b \in G.$$

Exercise

Prove that if $\varphi: G \to H$ is a group homomorphism, then

- 1. $\varphi(e_G) = e_H$ (Hint: look at $\varphi(e_G e_G)$)
- 2. $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.

Definition (kernel)

The kernel of a group homomorphism $\varphi:G\to H$ is the set $\ker \varphi=\{g\in G\,|\, \varphi(g)=e_H\}$

of all the elements of G that are sent to the identity in H.

Proposition

A group homomorphism $\varphi: G \to H$ is injective if and only if its kernel is trivial, that is

 $\ker \varphi = \{e_G\}.$

Definition (isomorphism)

Two groups G and H are isomorphic (denoted by $G \cong H$) if there exists an invertible group homomorphism $\varphi: G \to H$. Such a map is called an isomorphism between G and H.

Exercise

Prove that \mathbb{Z}_2 is isomorphic to the subgroup $\{\mathbb{I}_n, -\mathbb{I}_n\} \leq \operatorname{GL}(n, \mathbb{C})$. While you are at it, prove that the latter is indeed a subgroup!

Exercise

I'll do you one better: prove that any group with only two elements is isomorphic to \mathbb{Z}_2 .

Quotient group and isomorphism theorem

Definition (normal subgroup)

Let G be a group. A subgroup $N \leq G$ is called *normal* if

$$gng^{-1} \in N$$
, $\forall g \in G$, $\forall n \in N$.

The notation $N \subseteq G$ is commonly used to indicate that N is a normal subgroup of G.

Definition (quotient group)

Let N be a normal subgroup of a group G. We can define an equivalence relation on G as

$$g \sim h \iff h^{-1}g \in N$$
,

with equivalence classes

$$[g] = \{ h \in G \mid h^{-1}g \in N \}.$$

The quotient group G/N (pronounced " $G \mod N$ ") is the set of equivalence classes

$$G/N = \{[g] \mid g \in G\}$$

which is made into a group by defining

$$[g][h] = [gh], \quad [g]^{-1} = [g^{-1}], \quad e_{G/N} = [e_G].$$

Exercise

Show that the $2\mathbb{Z}=\{2n\,|\,n\in\mathbb{Z}\}$ is a normal subgroup of $(\mathbb{Z},+)$ and that $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}.$

Theorem (first isomorphism theorem)

Let $\varphi: G \to H$ be a group homomorphism. Then:

- $\operatorname{Im} \varphi$ is a subgroup of H
- lacksquare ker arphi is a normal subgroup of G
- $\operatorname{Im} \varphi$ is isomorphic to the quotient group $\operatorname{G}/\ker \varphi$

Exercise

Prove the first two points of the isomorphism theorem.