COMPLEX ANALYSIS

- A Quick Guide -

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Preface

Greetings,

Complex Analysis: A Quick Guide to is compiled based on my MA352: Complex Analysis notes with professor Evan Randles. This guide is almost entirely based on *Complex Variables and Applications, Eighth edition* by Churchill and Brown.

Enjoy!

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Chapter 1

COMPLEX NUMBERS

1.1 Sums and Products

Let $z \in \mathbb{C}$, it is customary to write

$$z = x + iy = (x, y) \tag{1.1}$$

where

$$x = \mathrm{Re}(z) \in \mathbb{R} \qquad y = \mathrm{Im}(z) \in \mathbb{R}. \tag{1.2}$$

For $z_1, z_2 \in \mathbb{C}$,

$$z_1 = z_2 \iff \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \wedge \mathrm{Im}(z_1) = \mathrm{Im}(z_2). \tag{1.3}$$

Addition works as we expect

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \tag{1.4}$$

So does multiplication

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y - 1y_2, y_1 x_2 + x_1 y_2). \tag{1.5}$$

Of course,

$$i^2 = -1 = (-1, 0). \tag{1.6}$$

1.2 Basic Algebraic Properties

It is easy to see that complex number multiplication and addition are both commutative and associative:

$$z_1 + z_2 = z_2 + z_1, \qquad z_1 z_2 = z_2 z_1 \tag{1.7}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1 z_2) z_3 = z_1 (z_2 z_3). \tag{1.8}$$

The additive identity is $0 = (0, 0)$. The multiplicative identity is $1 = (1, 0)$. For $z = (x, y) \in \mathbb{C}$, the additive inverse is

$$-z = (-x, -y). \tag{1.9}$$

For any nonzero complex number $z = (x, y)$, there exists an multiplicative inverse z^{-1} such that $zz^{-1} = z^{-1}z = 1$. We can find that

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right). \tag{1.10}$$

The existence of the multiplicative inverse allows us to show that if a product of two complex numbers is zero, then at least one of them is zero. And of course, if two complex numbers are nonzero, then so is their product.

Subtraction and division are defined in terms of addition and multiplication. For $z_1 = (x_1, y_2)$ and $z_2 = (x_2, y_2) \neq 0$,

$$z_1 - z_2 = (x_1 - x_2, y_1 - y_2) \tag{1.11}$$

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = (x_1, y_2) \left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2} \right) = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right). \tag{1.12}$$

This formula can be difficult to remember, so here's way to obtain it:

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}. \tag{1.13}$$

1.3 Further Properties

By the distributive law, we can show that

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2) z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3}. \tag{1.14}$$

Beside some other expected properties involving quotients that follow, we also have the binomial formula. If z_1, z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2 n - k \qquad (n = 1, 2, \dots) \tag{1.15}$$

1.4 Vectors and Moduli

It is natural to associate $z = (x, y)$ to a point of a plane with coordinates (x, y). The modulus of z is defined as

$$|z| = \sqrt{x^2 + y^2}. \tag{1.16}$$

The distance between two points z_1, z_2 is the same as the modulus of $z_1 - z_2$:

$$|z_2 - z_1| = \sqrt{(x_1 - x^2)^2 + (y_1 + y_2)^2}. \tag{1.17}$$

It is easy to see that

$$|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 \tag{1.18}$$

so that

$$\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z| \tag{1.19}$$
$$\mathrm{Im}(z) \leq |\mathrm{Im}(z)| \leq |z|. \tag{1.20}$$

Next, we have the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.21}$$

An immediate consequence of this inequality is another inequality:

$$|z_1 + z_2| \geq ||z_1| - |z_2||. \tag{1.22}$$

To prove this, we simply write $|z_1| = |(z_1 + z_2) - z_2|$. The triangle inequality takes care of the rest of the proof.

In summary, we have

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|. \tag{1.23}$$

The triangle inequality can be generalized by induction to sums involving any *finite* number of terms:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|. \tag{1.24}$$

1.5 Complex Conjugates

For $z = (x, y) \in \mathbb{C}$, the complex conjugate of z, denoted $\bar{z}$, is

$$\bar{z} = (x, -y). \tag{1.25}$$

We note

$$\bar{\bar{z}} = z, \qquad |\bar{z}| = |z|. \tag{1.26}$$

We can show that

$$\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2} \tag{1.27}$$
$$\overline{z_1 z_2} = \bar{z_1}\bar{z_2} \tag{1.28}$$
$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}, \qquad (z_2 \neq 0) \tag{1.29}$$
$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \tag{1.30}$$
$$\mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \tag{1.31}$$
$$z\bar{z} = |z|^2 \tag{1.32}$$
$$|z_1 z_2| = |z_1||z_2| \tag{1.33}$$
$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad (z_2 \neq 0). \tag{1.34}$$

1.6 Exponential Form

For any nonzero complex number $z = (x, y)$, the polar form is

$$z = x + iy = r\cos\theta + ir\sin\theta, \tag{1.35}$$

where $r = |z| \geq 0$. Note that for $z = 0$, the angle θ is not defined. Each value of θ is called an argument of z, denoted $\arg(z)$. However, because $\arg(z)$ is "multiple-valued," we define the *principal value* of $\arg(z)$, $\mathrm{Arg}(z)$ as

$$\arg(z) = \mathrm{Arg}(z) + 2n\pi, \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{1.36}$$

Note that when z is a negative real number, $\mathrm{Arg}(z) = \pi$, not $-\pi$.

The polar form can also be re-written in a different way using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.37}$$

With this,

$$z = re^{i\theta} = |z|e^{i\theta}. \tag{1.38}$$

1.7 Products and Powers in Exponential Forms

With a simple trigonometry identity, we can show that

$$e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}. \tag{1.39}$$

So,

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}. \tag{1.40}$$

Similarly,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}. \tag{1.41}$$

It is then easy to see that for $z \neq 0$,

$$z^{-1} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}. \tag{1.42}$$

And of course, we can see that

$$z^n = r^n e^{in\theta}, \qquad (n = 0, \pm 1, \pm 2, \dots). \tag{1.43}$$

This can be verified by induction.

1.8 Arguments of Products and Quotients

For $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$,

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}. \tag{1.44}$$

So,

$$\arg(z_1 + z_2) = \arg(z_1) + \arg(z_2). \tag{1.45}$$

We also have

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2). \tag{1.46}$$

1.9 Roots of Complex Numbers

Let $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$. By imagining z_2 and z_2 as points on a circle, we can see that

$$z_1 = z_2 \iff r_1 = r_2 \wedge \theta_1 = \theta_2 + 2n\pi \tag{1.47}$$

where n is some integer $n = 0, \pm 1, \pm 2, \dots$

Next, it is useful to find roots of any nonzero complex number $z_0 = r_0 e^{i\theta_0}$. Well, suppose

$$r^n e^{in\theta} = r_0 e^{i\theta_0}, \tag{1.48}$$

where $n = 2, 3, \dots$ Then

$$r^n = r_0, \qquad n\theta = \theta_0 + 2k\pi. \tag{1.49}$$

So,

$$r = \sqrt[n]{r_0} \tag{1.50}$$

$$\theta = \frac{\theta_0}{n} + \frac{2k\pi}{n}. \tag{1.51}$$

So, the complex numbers

$$z = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right], \tag{1.52}$$

where all the distinct roots are obtained when $k = 0, 1, 2, \ldots, n-1$. These distinct roots are called c_k:

$$c_k = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \tag{1.53}$$

where $k = 0, 1, 2, \ldots, n-1$.

$z_0^{1/n}$ denotes the *set* of nth roots of z_0, while $\sqrt[n]{z_0}$ is reserved for the one positive root. If θ_0 happens to be the principal argument of z_0 then c_0 is called the *principal root*. This makes sense because when z_0 is a positive real number r_0 its principal root is nothing but $\sqrt[n]{r_0}$.

More compactly, we can also write

$$c_k = c_0\omega_n^k \tag{1.54}$$

where of course

$$c_0 = \sqrt[n]{r_0} \exp\left(i\frac{\theta_0}{n}\right) \tag{1.55}$$

and

$$\omega_k = \exp\left(\frac{2k\pi}{n}\right). \tag{1.56}$$

A much more convenient way to remember all this is to first write

$$z_0 = r_0 e^{i(\theta_0 + 2k\pi)}. \tag{1.57}$$

Then from there, using the laws of exponential multiplication, we can get the *set* of roots of z_0

$$z_0^{1/n} = \left[r_0 e^{i(\theta_0+2k\pi)}\right]^{1/n} = \sqrt[n]{r_0} \exp\left[i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)\right] \tag{1.58}$$

where $k = 0, 1, 2, \ldots n-1$.

1.10 Regions in the Complex Plane

An ϵ *neighborhood*

$$|z - z_0| < \epsilon \tag{1.59}$$

of a given point z_0 contains all points z lying inside but NOT on a circle centered at z_0 with a specified radius ϵ.

Occasionally it's convenient to speak of a *deleted neighborhood* (or punctured disk)

$$0 < |z - z_0| < \epsilon \tag{1.60}$$

consisting of all points z in an $e\epsilon$ neighborhood of z_0 except for z_0 itself.

A point z_0 is an *interior point* of a set S when there is some neighborhood of z_0 that contains only points of S.

A point z_0 is an *exterior point* of a set S when there is some neighborhood of z_0 that contains no points of S.

If z_0 is neither of these then it is called a *boundary point* of S. In this case, any neighborhood of z_0 contains at least one point in S and at least one point not in S.

The *totality* of all boundary points is called the *boundary* of S.

A set is *open* if it contains none of its boundary points. In fact, a set is open $\iff$ each of its points is an interior point.

A set is *closed* if it contains all of its boundary points.

The *closure* of a set S is the closed set consisting of all points in S together with the boundary of S.

Some sets are neither open nor closed. For example, the punctured disk $0 < |z - z_0| \leq 1$ is neither an open nor a closed set because (1) it contains at least one boundary point and (2) the boundary point z_0 is NOT contained in the set.

An open set S is *connected* if each pair of points in it can be joined by a *polygonal line*, consisting of a finite number of line segments joined end to end, that lies entirely in S.

A nonempty open set that is connected is called a *domain*. Note: any neighborhood is a domain.

A *region* is a domain together with some/none/all of its boundary points.

A set is *bounded* if every point in it lies inside some circle. Otherwise, it is *unbounded*.

A point z_0 is said to be an *accumulation point* of a set S if each deleted neighborhood of z_0 contains at least one point in S. Note: if a set is closed, then it contains each of its accumulation points.

A set is closed $\iff$ it contains all of its accumulation points.

A point z_0 is NOT an accumulation point of S when there exists a deleted neighborhood which contains no points in S.

Chapter 2

Analytic Functions

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