1 Introduction

The following introduction to Electromagnetically Induced Transparency (EIT) is based on Stephen E. Harris' article in *PHYSICS TODAY*, 1997. Link, and various other sources including this well-written thesis by Furman of Reed College, this paper, and of course *Principles of Laser Spectroscopy and Quantum Optics* by Paul Berman.

In shortest possible terms, EIT is a technique which renders an otherwise opaque atomic medium transparent with electromagnetic radiation. The medium is typically a three-level atomic system, with specific requirements: two of the possible three transitions must be dipole-allowed (so transition rules satisfied) and one not dipole-allowed. The spectrum of the medium without (blue) and with (red) EIT is shown below:

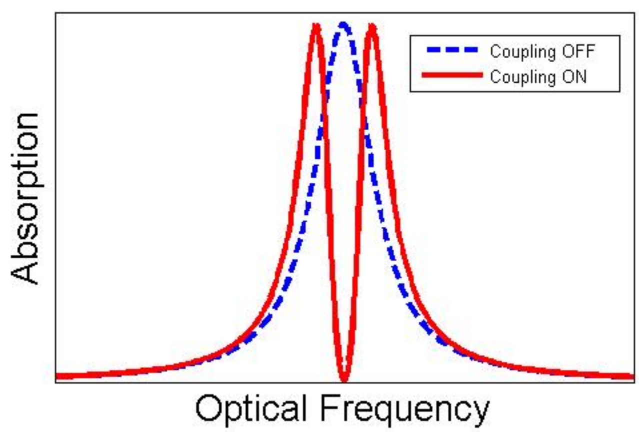

Figure 1

Notice how there is no absorption at resonance frequency with EIT. Typically, in the presence of a near-resonance field, a two-level atom with ground state $|1\rangle$ and excited state $|2\rangle$ will interact with the field, resulting in a nonzero probability amplitude of the excited state, $|P_2|^2 = \langle 2|2\rangle > 0$. If $|P_2(\delta\omega)|^2$ is the population of the excited state as a function of the detuning, then it follows the blue, Lorentzian line in the above figure. In EIT where there are two radiation fields, though, the energy levels of a three-level atom are altered. This in turn creates a window of frequencies at which the medium is transparent.

2 Derivation: Static Dark State

The following derivation is (heavily) inspired by Furman's derivation and the Heidelberg paper, but roughly condensed and sprinkled with a bit of my own narratives and insights here and there.

Let's consider a Λ configuration:

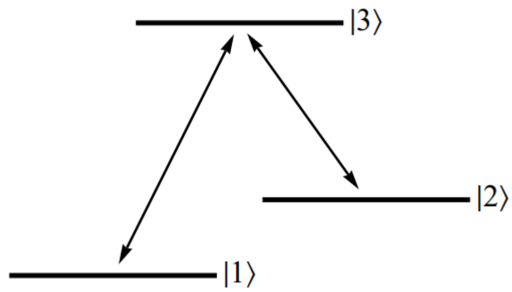

Figure 2

Let the energy of each state $|n\rangle$, $n = \{1, 2, 3\}$ be

$$E_n = \langle i| \hat{H} |n\rangle = \hbar\omega_n. \tag{1}$$

where $\hat{H}$ is the neutral atom Hamiltonian. Assume that the transition $|1\rangle \to |2\rangle$ is forbidden (just as shown in Figure 2). Since E_n and $|n\rangle$ are the eigenvalues and eigenstates of $\hat{H}$, respectively, we let the bare-atom Hamiltonian $\hat{H}_0$ be:

$$\hat{H}_0 = \hbar \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}. \tag{2}$$

We can do this because the eigenstates $|n\rangle$ form an orthonormal basis. Next, let the applied fields be

$$\begin{aligned} \vec{E} &= \vec{E}_p \cos\left(\omega_p t - \vec{k_p} \cdot \vec{r}\right) + \vec{E}_c \cos\left(\omega_c t - \vec{k_c} \cdot \vec{r}\right) \\ &\approx \vec{E}_p \cos(\omega_p t) + \vec{E}_c \cos(\omega_c t) \\ &= \frac{\vec{E}_p}{2}\left(e^{i\omega_p t} - e^{-i\omega_p t}\right) + \frac{\vec{E}_c}{2}\left(e^{i\omega_c t} + e^{i\omega_c t}\right). \end{aligned} \tag{3}$$

where $\omega_p \approx \omega_{13} = \omega_3 - \omega_1$, and $\omega_c \approx \omega_{23} = \omega_3 - \omega_2$. The subscripts p and c means **probe** and **coupling**, respectively. We should also denote the relevant detuning $\delta_p = \omega_{13} - \omega_p$ and $\delta_c = \omega_{12} - \omega_c$. It makes sense to label our subscripts this way, because in the end we are interested in the probability amplitude of $|3\rangle$ as a function of the detuning δ_p of the probe beam from (bare atom) resonance.

With the perturbation from the radiation, the new Hamiltonian of the atom is:

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{4}$$

where, with $\hat{\rho} \equiv qd$ being the dipole moment operator:

$$\hat{H}_{1,ij} = -E\hat{\rho}_{ij}, \tag{5}$$

where E is the field strength. We will see how E relates to the Rabi rate Ω later. Now, a state does not dipole-interact with itself, hence $\rho_{ii} = 0$. In addition, since the transition $|1\rangle \to |2\rangle$ is forbidden, $\rho_{12} = \rho 21 = 0$. It follows that:

$$\begin{aligned}\hat{H} &= \hat{H}_0 + \hat{H}_1 \\ &= \hbar \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} - E \begin{pmatrix} 0 & 0 & \rho_{13} \\ 0 & 0 & \rho 23 \\ \rho_{31} & \rho_{32} & 0 \end{pmatrix}.\end{aligned} \tag{6}$$

Now, what we have been working with so far are the time-independent eigenstates $|n\rangle$. In the following steps we shall bring in the time-dependent parts. To do this, we invoke the unitary matrix $U(t)$, which transforms $|n\rangle$ into full time-dependent wavefunctions:

$$U(t) = e^{iH_0 t/\hbar} \begin{pmatrix} e^{i\omega_1 t} & 0 & 0 \\ 0 & e^{i\omega_2 t} & 0 \\ 0 & 0 & e^{i\omega_3 t} \end{pmatrix}. \tag{7}$$

Obviously, if we apply $\hat{U}(t)$ to $\hat{H}_0$ there wouldn't be anything interesting, since $\hat{U}(t) = d/dt\hat{H}_0$. But we can apply $\hat{U}(t)$ to $\hat{H}_1$. The change of basis rule gives

$$\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t) = -E \begin{pmatrix} 0 & 0 & \rho_{13}e^{-i\omega_{13}t} \\ 0 & 0 & \rho_{23}e^{-i\omega_{23}t} \\ \rho_{31}e^{i\omega_{13}t} & \rho_{32}e^{i\omega_{32}t} & 0 \end{pmatrix}. \tag{8}$$

Multiplying E into $\hat{H}_1$ and applying rotating wave approximation to result gives the non-zero matrix elements:

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{13} = -\frac{1}{2}E_p\rho_{13}e^{i(\omega_p - \omega_{13})t} \tag{9}$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{23} = -\frac{1}{2}E_c\rho_{23}e^{i(\omega_c - \omega_{23})t} \tag{10}$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{31} = -\frac{1}{2}E_p\rho_{31}e^{i(\omega_p - \omega_{31})t} \tag{11}$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{32} = -\frac{1}{2}E_c\rho_{32}e^{i(\omega_c - \omega_{32})t} \tag{12}$$

Transforming $\hat{H}_1$ back to the vector space of $|n\rangle$ gives:

$$\begin{aligned}\hat{H}_1 &= \hat{U}^\dagger(t)\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t)\right)\hat{U}(t) \\ &= -\frac{1}{2}\begin{pmatrix} 0 & 0 & E_p\rho_{13}e^{i\omega_p t} \\ 0 & 0 & E_c\rho_{23}e^{i\omega_c t} \\ E_p\rho_{31}e^{-i\omega_p t} & E_c\rho_{32}e^{-i\omega_c t} & 0 \end{pmatrix}.\end{aligned} \tag{13}$$

Now, to put the dipole moment in terms of the Rabi frequency Ω:

$$\Omega_p = \frac{E_p|\rho_{13}|}{\hbar} = \frac{E_p\rho_{13}e^{-i\phi_p}}{\hbar} \tag{14}$$

$$\Omega_c = \frac{E_c|\rho_{23}|}{\hbar} = \frac{E_c\rho_{23}e^{-i\phi_c}}{\hbar} \tag{15}$$

So the total Hamiltonian is

$$\hat{H} = \frac{\hbar}{2}\begin{pmatrix} 2\omega_1 & 0 & -\Omega_p e^{i(\omega_p t+\phi_p)} \\ 0 & 2\omega_2 & -\Omega_c e^{i(\omega_c t+\phi_c)} \\ -\Omega_p e^{-i(\omega_p t+\phi_p)} & -\Omega_c e^{-i(\omega_c t+\phi_c)} & 2\omega_3 \end{pmatrix} \tag{16}$$

Now, since we want our final result in terms of the frequencies only, we shall express $\hat{H}$ in a new basis that is time and phase independent. Let a unitary matrix $\tilde{U}(t)$ be given such that it satisfies the required change of basis. It follows that the eigenstates transform as

$$|\tilde{n}\rangle = \tilde{U}(t)\,|n'\rangle\,. \tag{17}$$

Let the new Hamiltonian in this new basis be $\tilde{H}$. The new eigenstate also has to solve the Schrödinger equation in the new basis, so

$$\begin{aligned}
\tilde{H}\,|\tilde{n}\rangle &= i\hbar\frac{\partial}{\partial t}\,|\tilde{n}\rangle \\
&= i\hbar\frac{\partial}{\partial t}\left(\tilde{U}\,|n'\rangle\right) \\
&= i\hbar\left(\frac{\partial\tilde{U}}{\partial t}\,|n'\rangle + \tilde{U}\frac{\partial}{\partial t}\,|n'\rangle\right) \\
&= i\hbar\left(\frac{\partial\tilde{U}}{\partial t}\,|n'\rangle + \frac{1}{i\hbar}\tilde{U}\hat{H}\,|n'\rangle\right) \\
&= \left(i\hbar\frac{\partial\tilde{U}}{\partial t}\tilde{U}^\dagger + \tilde{U}\hat{H}\tilde{U}^\dagger\right)\tilde{U}\,|n'\rangle \\
&= \left(i\hbar\frac{\partial\tilde{U}}{\partial t}\tilde{U}^\dagger + \tilde{U}\hat{H}\tilde{U}^\dagger\right)|\tilde{n}\rangle\,.
\end{aligned} \tag{18}$$

So $\tilde{H}$ can be readily computed:

$$\begin{aligned}
\tilde{H} &= \left(i\hbar\frac{\partial\tilde{U}}{\partial t}\tilde{U}^\dagger + \tilde{U}\hat{H}\tilde{U}^\dagger\right) \\
&= \frac{\hbar}{2}\begin{pmatrix} 2(\omega_1+\omega_p) & 0 & \Omega_p \\ 0 & 2(\omega_2+\omega_c) & -\Omega_c \\ -\Omega_p & -\Omega_c & 2\omega_3 \end{pmatrix}
\end{aligned} \tag{19}$$

Note that we can add $\hbar(\omega_1 + \omega_p)\hat{I}$ to $\tilde{H}$ without changing the physical interpretation. In fact, the form of $\tilde{H}$ now is a result of our definition of the ω's. Now, let us define the detunings $\delta_c = \omega_{23} - \omega_c = \omega_3 - \omega_2 - \omega_c$ and $\delta_p = \omega_{13} - \omega_p = \omega_3 - \omega_1 - \omega_p$. So, the "better" Hamiltonian is

$$\begin{aligned}\tilde{H}' &= \tilde{H} - \hbar(\omega_1 + \omega_p)\hat{I} \\ &= \hbar \begin{pmatrix} 0 & 0 & -\Omega_p \\ 0 & 2(\delta_p - \delta_c) & -\Omega_c \\ -\Omega_p & -\Omega_c & 2\delta_p \end{pmatrix}\end{aligned} \tag{20}$$

Assuming $\delta_p \approx \delta_c$, the eigenvalues of $\tilde{H}'$ are:

$$\begin{aligned} E^+ &= \hbar\omega^+ = \frac{\hbar}{2}\left(\delta_p + \sqrt{\delta_p^2 + \Omega_p^2 + \Omega_c^2}\right) \\ E^- &= \hbar\omega^- = \frac{\hbar}{2}\left(\delta_p - \sqrt{\delta_p^2 + \Omega_p^2 + \Omega_c^2}\right) \\ E^0 &= \hbar\omega = 0. \end{aligned} \tag{21}$$

And the eigenstates are:

$$\begin{aligned} \left|a^0\right\rangle &= \frac{\Omega_c \left|1\right\rangle - \Omega_c \left|2\right\rangle}{\sqrt{\Omega_p^2 + \Omega_c^2}} = \cos\theta \left|1\right\rangle - \sin\theta \left|2\right\rangle \\ \left|a^-\right\rangle &= -\frac{\Omega_p \left|1\right\rangle + \Omega_c \left|2\right\rangle}{\delta_p - \sqrt{\delta_p^2 + \Omega_p^2 + \Omega_c^2}} + \left|3\right\rangle = \sin\theta\cos\phi \left|1\right\rangle + \cos\theta\cos\phi \left|2\right\rangle - \sin\phi \left|3\right\rangle \\ \left|a^+\right\rangle &= +\frac{\Omega_p \left|1\right\rangle + \Omega_c \left|2\right\rangle}{\delta_p + \sqrt{\delta_p^2 + \Omega_p^2 + \Omega_c^2}} - \left|3\right\rangle = \sin\theta\cos\phi \left|1\right\rangle + \cos\theta\cos\phi \left|2\right\rangle + \cos\phi \left|3\right\rangle \end{aligned} \tag{22}$$

where

$$\tan\theta = \frac{\Omega_p}{\Omega_c} \tag{23}$$

$$\tan\phi = \frac{\sqrt{\Omega_p^2 + \Omega_c^2}}{\delta_p + \sqrt{\delta_p^2 + \Omega_p^2 + \Omega_c^2}}. \tag{24}$$

Notice that the transition probability of $\left|a^0\right\rangle \to \left|3\right\rangle$ is 0, because:

$$\left\langle a^0\right| \hat{H} \left|3\right\rangle = 0. \tag{25}$$

We can also see this because $\left|a^0\right\rangle$ is not expressed in terms of $\left|3\right\rangle$, i.e., it does have $\left|3\right\rangle$ component. We call $\left|a^0\right\rangle$ the *dark state* because the transition to $\left|3\right\rangle$ is not possible.

Now, in the case where $\Omega_p \ll \Omega_c$, i.e., the probe beam is much weaker than the coupling beam, we get

$$\sin\theta \to 0, \cos\theta \to 1, \left|a^0\right\rangle \to |1\rangle\,, \tag{26}$$

which means that $|1\rangle$ becomes a dark state, hence this opens a window of frequency where $|1\rangle$ becomes transparent to previously resonance frequencies. This is exactly electromagnetically induced transparency.

3 Derivation: Dynamic EIT

We have derived and explained how $|1\rangle$ becomes a dark state when $\Omega_p \ll \Omega_c$. Now, in order to derive the absorption profile, we need a dynamic description of the three-level system where we also take into account spontaneous emission. We do this using the *density matrix formalism.* Let r be the density operator, defined as:

$$r = \sum_n P_n |n\rangle \langle n|\,, \tag{27}$$

where P_i is the probability of finding the atom in state i. Note that the definition is simply describing a statistical mixture of the n states. If given an operator $\mathcal{O}$, the expectation value of the measurement of a statistical mixture of states is given by:

$$\begin{aligned}
\langle\mathcal{O}\rangle &= \sum_n P_n \langle n|\mathcal{O}|n\rangle \\
&= \sum_n P_n \operatorname{tr}(|n\rangle\langle n|\mathcal{O}) \\
&= \sum_n \operatorname{tr}(P_n |n\rangle\langle n|\mathcal{O}) \\
&= \operatorname{tr}\left(\sum_n P_n |n\rangle\langle n|\mathcal{O}\right) \\
&= \operatorname{tr}(r\mathcal{O}).
\end{aligned} \tag{28}$$

Another way to derive the above equality is given in Furman's thesis:

$$\begin{aligned}
\langle\mathcal{O}\rangle &= \sum_n P_n \langle n|\mathcal{O}|n\rangle \\
&= \sum_{m,n} P_n \langle n|\mathcal{O}|m\rangle \langle m|n\rangle \\
&= \sum_{m,n} \langle m| P_n |n\rangle \langle n|\mathcal{O}|m\rangle \\
&= \sum_m \langle m| r\mathcal{O} |m\rangle \\
&= \operatorname{tr}(r\mathcal{O}).
\end{aligned} \tag{29}$$

Note that for pure states:

$$\langle \mathcal{O} \rangle = \langle n | \mathcal{O} | n \rangle . \tag{30}$$

To relate r to the Hamiltonian $\hat{H}$, we invoke the von Neumann equation. The derivation of the von Neumann equation begins with the Schrödinger equation and its adjoint:

$$\frac{d}{dt} |n\rangle = -\frac{i}{\hbar} \hat{H} |n\rangle \tag{31}$$

$$\frac{d}{dt} \langle n| = \frac{i}{\hbar} \langle n| \hat{H}. \tag{32}$$

To get the von Neumann equation, we take the time derivative of r.

$$\begin{aligned}
\dot{r} &= \frac{d}{dt} \sum_n P_n |n\rangle \langle n| \\
&= \sum_n P_n \left(|\dot{n} \langle n| + |n\rangle \langle \dot{n}| \rangle \right) \\
&= -\frac{i}{\hbar} \sum_n P_n \left(\hat{H} |n\rangle \langle n| - |n\rangle \langle n| \hat{H} \right) \\
&= -\frac{i}{\hbar} (\hat{H} r - r \hat{H}) \\
&= -\frac{i}{\hbar} [\hat{H}, r].
\end{aligned} \tag{33}$$

Note that r is an operator, so $\dot{r}$ is expressed in terms of a commutator.