8.09 & 8.309 Classical Mechanics III, Fall 2019
FINAL
Friday December 20, 1:30pm-4:30pm
You have 180 minutes.

Answer all problems in the white books provided. Write YOUR NAME on EACH book you use.

There are six problems, totaling 150 points. You should do all six. The problems are worth 12, 27, 26, 27, 34, and 24 points. You may do the problems in any order.

None of the problems requires extensive algebra. If you find yourself lost in a calculational thicket, stop and think.

No books, notes, or calculators allowed.

Some potentially useful information

- Euler-Lagrange equations for generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j} \;, \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot q_j}$$

constraints: holonomic $f_\alpha(q,t) = 0$ or semiholonomic $g_\beta = \sum_j a_{\beta j}(q,t)\dot q_j + a_{\beta t}(q,t) = 0$

- Generalized forces: $d/dt(\partial L/\partial \dot q_j) - \partial L/\partial q_j = R_j$

Friction forces: $\vec f_i = -h(v_i)\vec v_i/v_i$, $\vec v_i = \dot{\vec r}_i$ gives $R_j = -\partial \mathcal{F}/\partial \dot q_j$, $\mathcal{F} = \sum_i \int_0^{v_i} dv_i'\, h(v_i')$

- Hamilton's equations for canonical variables (q_j, p_j): $\quad \dot q_j = \frac{\partial H}{\partial p_j}$, $\quad \dot p_j = -\frac{\partial H}{\partial q_j}$

- Hamiltonian for a Lagrangian quadratic in velocities

$$L = L_0(q,t) + \dot{\vec q}^{\,T}\cdot\vec a + \frac{1}{2}\dot{\vec q}^{\,T}\cdot\hat T\cdot\dot{\vec q} \quad \Rightarrow \quad H = \frac{1}{2}(\vec p - \vec a)^T\cdot\hat T^{-1}\cdot(\vec p - \vec a) - L_0(q,t)$$

- The Moment of Inertia Tensor and its relations:

$$I_{ab} = \int dV\, \rho(\vec r)[\vec r^{\,2}\delta_{ab} - r_a r_b] \qquad \text{or} \qquad I^{ab} = \sum_i m_i[\delta^{ab}\vec r_i^{\,2} - r_i^a r_i^b]$$

$$I_{ab}^{(Q)} = M(\delta_{ab}\vec R^2 - R_a R_b) + I_{ab}^{(\mathrm{CM})} \;, \qquad \hat I' = \hat U\,\hat I\,\hat U^T$$

- Euler's Equations:

$$\begin{aligned} I_1\dot\omega_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1 \\ I_2\dot\omega_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2 \\ I_3\dot\omega_3 - (I_1 - I_2)\omega_1\omega_2 &= \tau_3 \end{aligned}$$

- Vibrations: $L = \frac{1}{2}\dot{\vec\eta}^{\,T}\cdot\hat T\cdot\dot{\vec\eta} - \frac{1}{2}\vec\eta^{\,T}\cdot\hat V\cdot\vec\eta$ has Normal modes $\vec\eta^{\,(k)} = \vec a^{\,(k)}\exp(-i\omega^{(k)}t)$

$$\det(\hat V - \omega^2\hat T) = 0 \;, \qquad \left(\hat V - [\omega^{(k)}]^2\,\hat T\right)\cdot\vec a^{\,(k)} = 0 \;, \qquad \vec\eta = \mathrm{Re}\sum_k C_k\,\vec\eta^{\,(k)}$$

- Generating functions for Canonical Transformations: $K = H + \partial F_i/\partial t$ and

$$F_1(q,Q,t): \quad p_i = \frac{\partial F_1}{\partial q_i} \;, \; P_i = -\frac{\partial F_1}{\partial Q_i} \;, \qquad F_2(q,P,t): \quad p_i = \frac{\partial F_2}{\partial q_i} \;, \; Q_i = \frac{\partial F_2}{\partial P_i}$$

- Poisson Brackets: $\quad [u,v]_{q,p} = \sum_j \left[\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j}\right]$, $\quad \frac{du}{dt} = [u,H] + \frac{\partial u}{\partial t}$

- Relations for Hamilton's Principle function, $S = S(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n,t)$

$$K = 0 \;, \qquad P_i = \alpha_i \;, \qquad Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i} \;, \qquad p_i = \frac{\partial S}{\partial q_i}$$

- Relations for Hamilton's Characteristic function, $W = W(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n)$

$$K = H = \alpha_1 \;, \qquad P_i = \alpha_i \;, \qquad \beta_1 + t = \frac{\partial W}{\partial \alpha_1} \;, \qquad \beta_{i>1} = \frac{\partial W}{\partial \alpha_i} \;, \qquad p_i = \frac{\partial W}{\partial q_i}$$

- Action Angle Variables: $\quad J = \oint p\, dq$, $\quad w = \frac{\partial W(q,J)}{\partial J}$, $\quad \dot w = \frac{\partial H(J)}{\partial J} = \nu(J)$

- Time Dependent Perturbation Theory for $H_0 + \Delta H$. Solve $H_0(p,q)$ with the Hamilton-Jacobi method to obtain constant canonical variables (β,α) where $[\beta,\alpha]=1$. Then
$$\dot{\alpha}^{(n)} = -\frac{\partial \Delta H}{\partial \beta}\bigg|_{n-1}, \qquad \dot{\beta}^{(n)} = \frac{\partial \Delta H}{\partial \alpha}\bigg|_{n-1}$$

- Fluid volume and continuity equations $\quad \dfrac{dV}{dt} = \displaystyle\int dV\, \vec{\nabla}\cdot\vec{v}\,, \qquad \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}\,) = 0$

- Euler equation ($\nu = 0$) or Navier-Stokes equation ($\nu = \eta/\rho \neq 0$), with gravity:
$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\,\vec{v} + \frac{1}{\rho}\vec{\nabla}p - \nu\nabla^2\vec{v} = \frac{\vec{f}}{\rho} = \vec{g}$$

- For direction i the force/unit area on a surface $= -\hat{n}_i\, p + \hat{n}_k \sigma'_{ki}$

- Ideal fluid has $ds/dt = 0$ so $p = p(\rho, s)$. Viscous fluid has $ds/dt \propto \sigma'_{ik}\ \partial v_i/\partial x_k$.

- Bernoulli's equation for a steady incompressible ideal fluid in gravity $\vec{g} = -g\hat{z}$:
$$\frac{\vec{v}^{\,2}}{2} + gz + \frac{p}{\rho} = \text{constant}$$

- Irrotational incompressible ideal fluid flow (potential flow): $\vec{v} = \nabla\phi$, $\nabla^2\phi = 0$

- Sound waves: $\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right)\{p', \rho', \vec{v}\,\} = 0$. Mach number $M = v_0/c_s$.

- Momentum conservation: $\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot\hat{T} = \vec{f}$ where the energy momentum tensor is $T_{ki} = v_k v_i\,\rho + \delta_{ki}\,p - \sigma'_{ki}$. For $\vec{\nabla}\cdot\vec{v} = 0$ the viscous stress tensor $\sigma'_{ki} = \eta\big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\big)$.

- Reynolds Number: $R = uL/\nu$

- Bifurcations at $\mu = 0$. In 1-dim: "saddle-node" $\dot{x} = \mu + x^2$, "transcritical" $\dot{x} = x(\mu - x)$, "supercritical pitchfork" $\dot{x} = \mu x - x^3$, "subcritical pitchfork" $\dot{x} = \mu x + x^3$. In 2-dim: "supercritical Hopf" $\dot{r} = r(\mu - r^2)$, "subcritical Hopf" $\dot{r} = r(\mu + r^2)$.

- Linearization for 2-dim fixed points: $\begin{pmatrix}\dot{u}\\ \dot{v}\end{pmatrix} = M\begin{pmatrix}u\\ v\end{pmatrix}, \quad \begin{pmatrix}u\\ v\end{pmatrix} = \vec{a}\,e^{\lambda t}, \quad M\vec{a} = \lambda\,\vec{a}$
$$\lambda_{\pm} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \tau = \mathrm{tr}\, M\,, \qquad \Delta = \det M$$

- 2-dim conserved system $\dot{x} = f_x(x,y)$, $\dot{y} = f_y(x,y)$ with $\vec{\nabla}\cdot\vec{f} = 0$, has conserved $H(x,y) = \int^y dy' f_x(x,y') - \int^x dx' f_y(x',y)$.

- 1-dim map $x_{n+1} = f(x_n)$. Its fixed points satisfy $x^* = f(x^*)$. Here x^* is stable for $|f'(x^*)| < 1$ and unstable for $|f'(x^*)| > 1$.

- Fractal dimension: $\quad d_F = \lim\limits_{a\to 0} \dfrac{\ln N(a)}{\ln(a_0/a)}$

Fundamental Poisson brackets → $[Q_i, Q_j] = 0$ $[P_i, P_j] = 0$ $[Q_i, P_j] = \delta_{ij}$

1. **Short answer problems** [12 points]

These problems require no algebra or a little algebra. Your answers should be short.

(a) [3 points] Suppose (q_1, p_1) and (q_2, p_2) are pairs of canonical variables. What equations could you check to ensure that new variables defined by the transformations $Q_i = Q_i(q, p)$ and $P_i = P_i(q, p)$ for $i = 1, 2$ are canonical?

ex: $G = p$

(b) [2 points] Give an example of a generating function G for an infinitesimal canonical transformation, and explain what transformation it generates.

(c) [4 points] What is the Kaplan-Yorke relation between fractal dimensions and Liapunov exponents? Describe the variables appearing in any formula you write.

(d) [3 points] What are two characteristics of turbulence in a fluid that are not shared by a steady laminar fluid flow?

scale invariance, energy dissipation

velocity is time dependent

better, broad answer not great, but ok.

2. **Two Unrelated Problems** [27 points]

(a) [15 points] Consider the Hamiltonian for the Kepler problem which describes motion in a plane with polar coordinates (r, ψ):

$$H = \frac{1}{2\mu}\left(p_r^2 + \frac{p_\psi^2}{r^2}\right) - \frac{k}{r}.$$

→ written that ψ cyclic

ok

Here μ is the reduced mass and $k > 0$ determines the strength of the gravitational potential. Using the Hamilton-Jacobi method, find Hamilton's Characteristic function. Then use this result to derive two equations, one for the radial motion in the form $t = t(r)$, and one for the orbital equation in the form $\psi = \psi(r)$. In all cases your results can be left in the form of indefinite integrals.

$\vec{\nabla} \cdot \vec{v} = 0$

$\vec{\nabla} \times \vec{v} = 0$

(b) [12 points] Consider an irrotational incompressible ideal fluid. To describe the flow of this fluid we consider the following function in cylindrical coordinates (r,θ,z)

$$\phi(r, \theta, z) = c\, r^a \cos(b\,\theta + d) .$$

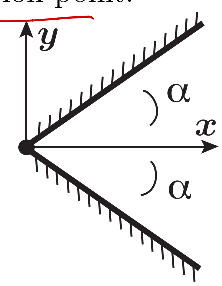
→ potential

In this expression a, b, c, d are four constants.

i. [7 points] Find any relations these four constants must satisfy in order for ϕ to be a valid velocity potential. Also find any constraints on the four constants that are obtained by demanding that the origin is a stagnation point.

$\boxed{v(0) = 0}$

ii. [5 points] This velocity potential can be used to describe the flow near two walls that meet at an angle 2α as shown. Find the additional constraints on the constants that are required to describe this geometry (a single valid answer suffices).



Note that in cylindrical coordinates (r, θ, z):

$$\vec{\nabla} = \hat{r}\,\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\,\frac{\partial}{\partial z}\,, \qquad \nabla^2 = \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}\right)$$

(continue)

(1b) Ex... look at $F_2(q, P) = qP$

$$p = \frac{\partial F_2}{\partial q} + \varepsilon \frac{\partial G}{\partial q} = \underline{P} + \varepsilon \, \partial G / \partial q$$

$$Q = \frac{\partial F_2}{\partial \underline{P}} + \varepsilon \frac{\partial G}{\partial \underline{P}} = q + \varepsilon \, \partial G / \partial \underline{P}$$

make $G = p$ then it generates translation

$\varepsilon = \cancel{d}x \ldots$

$G = L$ then it generates rotation

$\varepsilon = \partial\theta$

$G = \mathcal{H}$ then it generates time evolution

$\varepsilon = \partial t$

(1c) Kaplan Yorke relation for fractal dim

$$K = \lim_{a \to 0} \frac{\ln(N)}{\ln(a_0/a)} \sim 1 + \frac{\lambda_1}{|\lambda_2|}$$

$e^{\lambda_1 t}$ $e^{\lambda_2 t}$

or sth like this

Lyapunov exp... $\delta \sim e^{\lambda t}$

$\uparrow$

sensitivity to initial condition...

$$\mathcal{H} = \frac{1}{2\mu}\left(p_r^2 + \frac{p_\psi^2}{r^2}\right) - \frac{k}{r} = E$$

$$\boxed{W = a_\psi \psi + W_r}$$ since ψ cyclic

$\Downarrow$

$a_\psi = p_\psi$ = constant

$$\mathcal{H} = \frac{1}{2\mu}\left(\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{a_\psi^2}{r^2}\right) \cancel{- \frac{k}{r}} = E + \frac{k}{r}$$

$$\left(\frac{\partial W_r}{\partial r}\right)^2 = 2\mu\left(E + \frac{k}{r}\right) - \frac{a_\psi^2}{r^2}$$

$$\frac{\partial W_r}{\partial r} = \pm\sqrt{2\mu\left(E + \frac{k}{r}\right) - \frac{a_\psi^2}{r^2}}$$

$$\boxed{W_r = \int \pm\sqrt{2\mu\left(E + \frac{k}{r}\right) - \frac{a_\psi^2}{r^2}}\, dr}$$

$$\boxed{\beta_1 + t = \frac{\partial W}{\partial E}} \to \text{get } t \text{ eqn}$$

$$\boxed{\beta_\varphi = \frac{\partial W}{\partial \alpha_\varphi}} = \cdots \text{ something independent of } \varphi$$

constant

$\Downarrow$

eqn for $\varphi = \varphi(r)$

$$\boxed{W = \alpha_\varphi \varphi + W_r}$$

$$\boxed{\phi = c r^a \cos(b\theta + d)}$$

$$\vec{\nabla}\phi = \hat{r}\frac{\partial}{\partial r}\phi + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\phi + \cancel{\hat{z}\frac{\partial}{\partial z}} \quad 0$$

$$= a\hat{r}\left[c r^{a-1}\cos(b\theta + d)\right]$$

$$+ \hat{\theta}\left[c r^{a-1}(-\sin(b\theta + d)) \cdot b\right] = \vec{v}$$

<u>So</u> $\vec{v} = c a r^{a-1}\cos(b\theta + d)\,\hat{r}$

$$+ c r^{a-1}(-\sin(b\theta + d))\,\hat{\theta}$$

$v(0) = 0 \Rightarrow$ so $\boxed{a > 1}$

<u>Now</u> $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}^2\phi = \left[\frac{1}{r}\left(\partial_r(r\partial_r)\right) + \frac{1}{r^2}\partial_\theta^2\right]\phi$

$$= \frac{1}{r}\partial_r\left[a c r^a \cos(b\theta + d)\right]$$

$$+ (-)c r^{a-2}\cos(b\theta + d)\,b^2 = 0$$

$$a^2 \cancel{c} \cancel{r^{a-2}} \cos(\cancel{b}\theta + d)$$

$$- \cancel{c} \cancel{r^{a-2}} b^2 \cos(\cancel{b}\theta - d) = 0$$

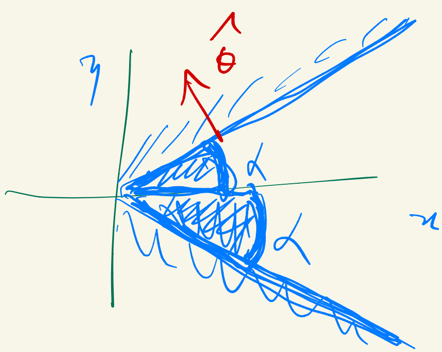
$\forall\ \theta, r \ldots$

$$(a^2 - b^2)\cos(b\theta - d) = 0 \quad \forall\theta$$

$$a^2 = b^2 \Rightarrow \boxed{b = \pm a}$$

Finally $\tilde{\nabla} \times \tilde{v} = 0$ well this is always true... right? since $\tilde{v} = \tilde{\nabla} \cdot \phi$.

Now, flow near 2 walls.



Boundary condition...

$$\boxed{\vec{v}_{surface} \cdot \hat{n}\Big|_{wall} = 0}$$

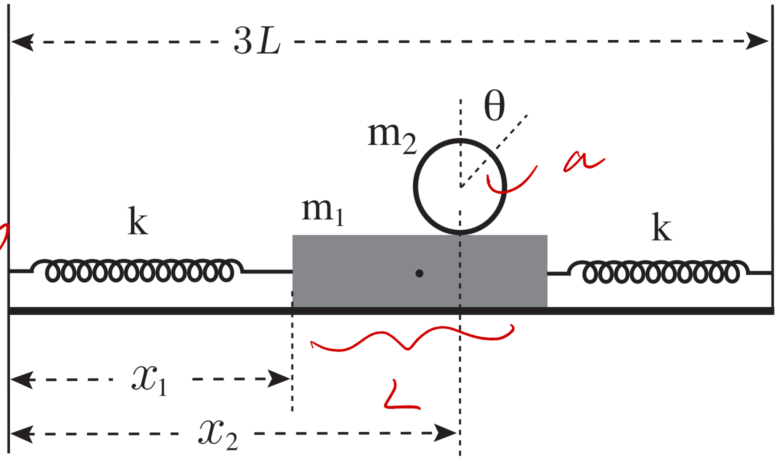
$$\boxed{\vec{v} \cdot \hat{\theta} = 0}$$

This means that

$$\boxed{V_\theta \Big|_{\pm a} = 0} \Rightarrow \text{get something from here} \cdots$$

3. Dancing Hoop on a Block [26 points]

no gravity
pulley, and
nothing moving
in ↕



A hoop of radius a and mass m_2 rolls without slipping on a block (the hoop's moment of inertia is $m_2 a^2$). The block has length L, mass m_1, and slides on the floor without friction under the influence of two springs with spring constants k and zero relaxed length. As shown, the springs are attached to the ends of the block and to two walls that are $3L$ apart. We will assume that the hoop does not fall off the block. The system can be described by 3 variables (x_1, x_2, θ) plus a Lagrange multiplier for the no slip constraint.

(a) [3 points] What is the constraint for no slipping for the hoop?

(b) [5 points] Without imposing the constraint, what is the Lagrangian for this system?

(c) [6 points] Including the Lagrange multiplier, what are the equations of motion?

(d) [6 points] Find an expression for the force of constraint in terms of the variables. Your result should not involve time derivatives.

(e) [6 points] What is the frequency of oscillation for the block? for the hoop?

(a) $\boxed{a\dot{\theta}} = v_{block/hoop} = \boxed{\dot{x}_2 - \dot{x}_1}$

why? if $\dot{x}_1 = 0$ then just $\dot{x}_2$

if $\dot{x}_2 = 0$ then just $\dot{x}_1$, but $(-)$

(continue)

$$L = T_{hoop} + T_{block} - V$$

$$= T_{h,trans} + T_{h,rot} + T_{block} - V$$

$$= \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 a^2 \dot{\theta}^2$$

$$- \frac{1}{2} k x_1^2 - \frac{1}{2} k (3L - L - x_1)^2$$

$$= \boxed{\frac{1}{2} m_2 \dot{x}_2^2} + \boxed{\frac{1}{2} m_1 \dot{x}_1^2} + \boxed{\frac{1}{2} m_2 a^2 \dot{\theta}^2}$$

$$- \frac{1}{2} k x_1^2 - \frac{1}{2} k (2L - x_1)^2$$

From here get EOMs without Lagrange multiplier...

But including Lagrange multiplier...

The constraint is semi holonomic (only one)

$$g = a\dot{\theta} - (\dot{x}_2 - \dot{x}_1) = 0$$

↳ E-L eqn ...

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} - \frac{\partial \mathcal{L}}{\partial x_1} = \lambda \frac{\partial g}{\partial \dot{x}_1} \quad (1)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} - \frac{\partial \mathcal{L}}{\partial x_2} = \lambda \frac{\partial g}{\partial \dot{x}_2} \quad (2)$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = \lambda \, \partial g / \partial \dot{\theta}_2 \quad (3)$$

From here do the algebra ...

$$\boxed{\begin{aligned} &(1) \quad m_1\ddot{x}_1 + \underbrace{kx_1 + k(x_1 - 2L)}_{2k(x_1 - L)} = \lambda \\ &(2) \quad m_2\ddot{x}_2 = -\lambda \\ &(3) \quad m_2 a^2 \ddot{\theta} = a\lambda \end{aligned}}$$

<u>what is λ ?</u> well ...

$$m_2 a\ddot{\theta} = \lambda$$

And $$a\ddot{\theta} = \ddot{x}_2 - \ddot{x}_1$$

$$\ddot{x}_2 = \frac{-\lambda}{m_2} \quad ; \quad \ddot{x}_1 = \frac{\lambda - 2k(x_1 - L)}{m_1}$$

$$\ddot{x}_2 - \ddot{x}_1 = \frac{-\lambda}{m_2} - \frac{\lambda - 2k(x_1 - L)}{m_1}$$

$$\Rightarrow m_2\left[\frac{-\lambda}{m_2} - \frac{\lambda - 2k(x_1 - L)}{m_1}\right] = \lambda$$

$$\cancel{-\lambda} - \frac{m_2}{m_1}\left[\lambda - 2k(x_1 - L)\right] = 2\lambda$$

$$+\frac{m_2}{m_1} 2k(x_1 - L) = 2\lambda + \frac{m_2}{m_1}\lambda$$

$$= \left(2 + \frac{m_2}{m_1}\right)\lambda$$

$$\lambda = \frac{+m_2/m_1\,(2k)(x_1 - L)}{\left(2 + m_2/m_1\right)}$$

Simplify from here...

Frequency of oscillation for the blocks...

Look at eqn for x_1...

$$\ddot{x}_1 = \frac{1}{m_1}\Big(\lambda - 2k\underbrace{(x_1 - L)}\Big)$$

$$\lambda = \frac{\cancel{-\frac{m_2}{2m_1}} + 2k(x_1 - L)}{\left(2 + m_2/m_1\right)\left(\frac{m_1}{m_2}\right)}$$

$$= \frac{+2k(x_1 - L)}{\frac{2m_1}{m_2} + 1}$$

$$= \frac{+2km_2(x_1 - L)}{2m_1 + m_2}$$

$$= \boxed{\frac{+2km_2}{2m_1 + m_2}(x_1 - L)}$$

do change of variables...

So... $$\ddot{x}_1 = \underbrace{\frac{1}{m_1}\left[\frac{+2km_2}{2m_1 + m_2} - 2k\right]}_{-\omega^2}(x_1 - L)$$

$$\ddot{x}_1 = \ddot{(x_1 - L)}$$

$$-\omega^2 = \frac{1}{m_1}(2k)\left(\frac{m_2}{2m_1+m_2} - 1\right)$$

$$= \frac{2k}{m_1}\,\frac{\cancel{m_2} - 2m_1 - \cancel{m_2}}{2m_1+m_2}$$

$$= \frac{-\cancel{2}4k}{m_1+m_2}$$

$$\boxed{\omega = \sqrt{\frac{4k}{m_1+m_2}}}$$

solve for x_1

plug back in here.

For the hang,

$$\ddot{x}_2 = -\frac{\lambda}{m_2}$$

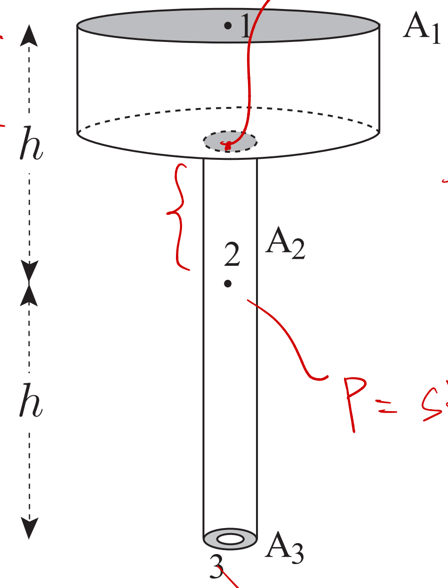
$$\ddot{\theta} = \frac{\lambda}{m_2 a}$$

$$\boxed{\lambda = \frac{2km_2}{2m_1+m_2}(x_1 - L)}$$

$p = g\rho\ell + 1\,atm$

4. **Fluid in a Funnel with and without Viscosity** [27 points]

Consider the steady state flow of incompressible water in a set of cylindrical pipes. The fluid has a constant density ρ and is acted upon by gravity. At 1 the fluid has velocity $v_1 \simeq 0$ and pressure $p_1 = p_{atm}$. The cross sectional area A_1 is much much larger than the area A_2. The second cylinder ends with a funnel that has a hole at the bottom of area $A_3 = A_2/4$. Point 3 is at the center of this hole and the pressure there is $p_3 = p_{atm}$.



ℓ

$P = \rho g h$

$p = 1\,atm$

For the first part lets consider the fluid to be ideal.

(a) [7 points] Consider the three points labeled 1, 2, 3 in the figure. Determine the velocity and pressure at 2 and velocity at 3 in terms of the height h and other parameters.

(b) [2 points] Does the pressure increase or decrease as the fluid goes down the cylinder with area A_2?

For the remaining parts we will include viscosity $\eta \neq 0$ and only consider the fluid flow in the 2nd cylinder with area A_2. Let $A_2 = \pi R^2$ where R is the radius. We will also assume h is very large and concentrate on the flow away from the transition regions at the ends. Assume the flow is steady and not turbulent.

(c) [11 points] Assume that $\boxed{\vec{v} = v_z(r)\hat{z}}$. Solve the Navier-Stokes equation in the 2nd cylinder. Impose suitable boundary conditions and determine $v_z(r)$. Also determine a formula for the pressure in terms of unknown constants. (The cylinder is treated as infinite here so there is no need to consider boundary conditions at its ends.)

(d) [4 points] What is the friction force per unit area on the walls of this 2nd cylinder?

(e) [3 points] Since $\vec{v}$ is independent of z there is no change to the kinetic energy as the fluid falls in cylinder 2, but the potential energy from gravity is decreasing. Identify in words two places where this energy is going.

dissipation due to viscosity & friction.

Note: In cylindrical coordinates (r, θ, z) the Laplacian is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

(continue)

Bernoulli's eqn and gravity ...

$$\boxed{\frac{v^2}{2} + gz + \frac{P}{\rho}} = \text{constant} = C$$

① $\frac{0}{2} + \boxed{g(2h) + \frac{1\,atm}{\rho}} = C$

② $\frac{V_2^2}{2} + gh + \frac{P_2}{\rho} = C$

③ $\frac{V_3^2}{2} + \underbrace{g \cdot 0}_{0} + \frac{P_{atm}}{\rho} = C$

$$\Rightarrow \quad \frac{V_3^2}{2} = \cancel{-\frac{1\,atm}{\rho}} + \cancel{\frac{1\,atm}{\rho}} + \boxed{g\,2h}$$

$$\Rightarrow \quad \boxed{V_3 = 2\sqrt{gh}}$$

also $A_2 V_2 = A_3 V_3 \rightarrow$ $V_2 = \frac{A_3}{A_2} 2\sqrt{gh}$

(incompressible)

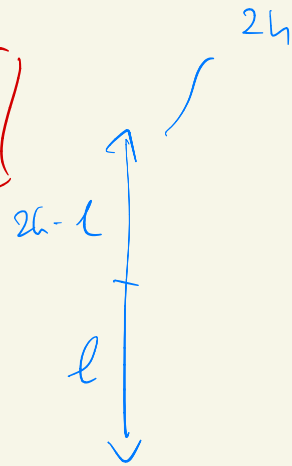
$$= \boxed{\frac{\sqrt{gh}}{2}} \rightarrow \text{find } \boxed{P_2}$$

$$P_2 = \rho \left[C - gh - v_2^2/2 \right]$$

$$= \rho \left[\frac{1\,\text{atm}}{\rho} + \underbrace{2gh - gh}_{gh} - v_2^2/2 \right]$$

$$= 1\,\text{atm} + \rho g h - \rho/2 \cdot \frac{1}{4} g h$$

$$P_2 = \boxed{1\,\text{atm} + \frac{7}{8}\rho g h}$$



Can repeat calculation, with

$$P_2 = \rho \left[\frac{1\,\text{atm}}{\rho} + 2gh - gl - v_2^2/2 \right]$$

$$= 1\,\text{atm} + 2gh\rho - \rho g l - \rho\, v_2^2/2$$

$$= 1\,\text{atm} + 2gh\rho - \rho g l - \rho/2 \cdot \frac{1}{4} g h$$

$$\boxed{l \downarrow \;\Rightarrow\; P_2 \uparrow}$$

$$\vec{v} = v_z(r)\,\hat{z}$$

$\boxed{NS}$

$$(\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p - \nu\nabla^2\vec{v} = \vec{g}$$

$$\underbrace{\begin{pmatrix} 0 \\ 0 \\ v_z(r) \end{pmatrix}}_{\hat{z}} \cdot \underbrace{\begin{pmatrix} \cdots \\ \cdots \\ \partial_z \end{pmatrix}}_{\hat{z}} = v_z(r)\,\partial_z$$

$$\partial_z \hat{z} = 0$$

$$v_z(r)\,\partial_z\left[v_z(r)\,\hat{z}\right]$$

$$= 0$$

$$\nabla^2\vec{v} = \left[\frac{1}{r}\partial_r(r\partial_r) + \cancelto{0}{\frac{1}{r^2}\partial_\theta^2} + \cancelto{0}{\partial_z^2}\right] v_z(r)\,\hat{z}$$

$$= \left\{\frac{1}{r}\partial_r\,(r\partial_r)\,v_z(r)\right\}\hat{z}$$

$$= \hat{z}\cdot\left(\frac{1}{r}\right)\left[\partial_r\left\{r\,\partial_r\,v_z(r)\right\}\right]$$

$$= \hat{z} \cdot \frac{1}{r}\left[\partial_r \left[r\, v_z'(r) \right]\right]$$

$$= \hat{z} \cdot \frac{1}{r}\left\{ v_z'(r) + r\, v_z''(r) \right\}$$

$$= \hat{z} \left\{ \frac{1}{r} v_z'(r) + v_z''(r) \right\}$$

$$\vec{\nabla} \cdot \underline{P} = \partial_z P \, \hat{z} \qquad P = P(z)$$

$$\left(\frac{1}{\rho} \partial_z \underset{\underset{P(z)}{\parallel}}{P} - \nu \left\{ \frac{1}{r} v_z'(r) + v_z''(r) \right\} = g \right) \hat{z}$$

$$\frac{1}{\rho} \partial_z P(z) = \nu \left\{ \frac{1}{r} v_z'(r) + v_z''(r) \right\}$$

$$- g \qquad = \text{constant} = K$$

$$(K+g)$$

$$\frac{1}{\rho} \partial_z P(z) = K + g \rightarrow P'(z) = \rho (K + g)$$

$$\boxed{P(z) = P_0 + \rho \cancel{g} z}$$

$P(z) = P_0 + \rho(g+K)z$

$\downarrow$

1 atm

Some unknown constant

what is K? Don't know... but ok

$$\frac{1}{r} v_z' + v_z'' = K/\nu$$

$$\boxed{V_z(r) = \frac{K}{\cancel{2}4\nu}(r^2 - R^2)}$$

$$v_z = A\ln(r) + (ar^2 + br + c)$$

$$v_z' = \frac{A}{r} + (2ar + b)$$

$$v_z'' = -\frac{A}{r^2} + (2a)$$

So $\boxed{V_z(r) = \frac{K}{\cancel{2}4\nu} r^2 + c}$

$r = R \to v = 0$

$$\frac{A}{r^2} + \frac{-A}{r^2} = K/\nu$$

$$C = \frac{-KR^2}{2\nu}$$

$$+ 2a + \frac{b}{r} + 2a = K/\nu$$

$b = 0$; $\cancel{2}4a = K/\nu$; $A \in R$

Friction is

$$\sigma'_{ki} = \eta\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)$$

Friction is only in $\hat{z}$ ... look at

$$\sigma'_{kz} = \eta\left(\frac{\partial v_z}{\partial x_k}\right) + \cancel{\frac{\partial v_k}{\partial z}} \rightarrow 0$$

since no z dependence.

Friction is

$$V_z = \frac{K}{4\nu}(r^2 - R^2)$$

$$\Downarrow$$

$$v_z' = \frac{K}{2\nu} r$$

$$\hat{n}_k \cdot \sigma'_{ki} = \hat{n}_r \cdot \sigma'_{rz}$$

$$= \hat{r} \cdot \sigma'_{rz}$$

So look at

$$\eta\left(\frac{\partial v_z}{\partial r}\right)\Bigg|_R = \boxed{\frac{\eta K}{2\nu} \cdot R} \rightarrow \frac{\rho K}{2} R$$

$$\nu = \eta/\rho$$

5. **Nonlinear Attractions** [34 points]

Consider the differential equation

$$\ddot{x} + a\,\dot{x}\,(x^2 + \dot{x}^2 - 1) + x = 0$$

where a is a real constant.

(a) [2 points] Introduce a variable w to write the equation as two first order equations.

(b) [10 points] For all values of a find all fixed points, and then classify them with a linear analysis (as a center, saddle node, unstable node, stable spiral, unstable spiral, stable star, stable degenerate node, etc). Only find the corresponding eigenvalues and eigenvectors if its necessary to use them to make a decision.

(c) [7 points] Show that for any $a \neq 0$ the system has a circular limit cycle, and determine its stability.

(d) [5 points] For both $a = +1$ and $a = -1$ sketch trajectories to exhibit the behavior of the system near the fixed points and the limit cycle for various initial conditions.

okay,

This final question is independent of those above. This is not hard...

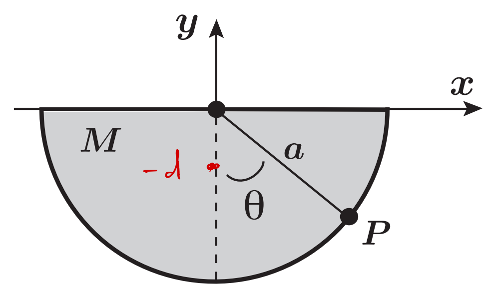
(e) [10 points] Consider the Cubic Map with a parameter r and values $-1 \leq x_n \leq 1$ defined by

$$x_{n+1} = r x_n - x_n^3$$

Find all fixed points and determine their stability for $0 < r < 3/2$.

6. **A Rotating Half-Disk** [24 points]

Consider a uniform thin half-circular disk of radius a and mass M as shown. Here the z-axis is coming out of the page. Let the center of mass of the disk be at $\vec{R} = -d\,\hat{y}$. (You do not need to calculate d.)
The disk is acted upon by gravity g.



(a) [10 points] What is the moment of inertia for rotations about an axis parallel to the z axis which passes through the point P on the edge of the disk? Write your answer using a, M, d, and θ.

(b) [8 points] Now take the half disk to carry out simple oscillations on a horizontal table, balancing on its rounded edge without slipping. Use your result from (a) and find its Lagrangian in terms of θ, $\dot{\theta}$, and constants.

(c) [6 points] Suppose that at $t = 0$ the disk is sitting with $\theta = 0$, and someone pushes down on the upper-right corner of the half-disk so that it has an instantaneous velocity $-v_0\hat{y}$. Find the maximum value of v_0 such that the half-disk does not flip over onto its flat side.

(the end)

$$\ddot{x} - a\dot{x}\,(x^2 + \dot{x}^2 - 1) + x = 0$$

$$w = \dot{x}$$

$$\rightarrow \begin{cases} \dot{w} = -a w\,(x^2 + w^2 - 1) - x = 0 \\ \dot{x} = w \end{cases}$$

$$\boxed{\begin{cases} \dot{x} = w \\ \dot{w} = -a w\,(x^2 + w^2 - 1) - x \end{cases}}$$

Now, fixed point

$$\begin{cases} 0 = w \\ 0 = -a w\,(x^2 + w^2 - 1) - x \end{cases}$$

$$\boxed{w = 0\,; \quad x = 0} \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & +a \end{pmatrix}$$

Linear...

$$\begin{cases} w \simeq w^* + \varepsilon_w = \varepsilon_w \\ x \simeq x^* + \varepsilon_x = \varepsilon_x \end{cases}$$

$$\begin{cases} \dot{\varepsilon}_x = \varepsilon_w = \boxed{\varepsilon_w} \\ \dot{\varepsilon}_w = -a\varepsilon_w\,(\cancelto{0}{\varepsilon_x^2} + \cancelto{0}{\varepsilon_w^2} - 1) - \varepsilon_x \simeq \boxed{-a\varepsilon_w + \varepsilon_x} \end{cases}$$

$$M = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$$

Eigv... $\det = 1$

$$\begin{pmatrix} 0-\lambda & 1 \\ -1 & a-\lambda \end{pmatrix}$$

$$\det \quad (a-\lambda)(-\lambda) + 1 = 0$$

$$\lambda^2 - \lambda a + 1 = 0$$

$$\lambda_\pm = \frac{a \pm \sqrt{a^2 - 4}}{2}$$

$$\boxed{\lambda_\pm = \frac{a}{2} \pm \frac{1}{2}\sqrt{a^2-4}}$$

Basically done.

(b) $a \neq 0$

go to polar coord, ...

$\dot{x} = w$

$x^2 + w^2 = r^2$

$x = r\cos\phi$

$w = r\sin\phi$

$r\dot{r} = x\dot{x} + w\dot{w}$

$\dot{x} = \dot{r}\cos\phi - r\sin\phi\,\dot{\phi}$

$r\dot{r} = x\dot{x} + w\dot{w}$

$= (r\cos\phi)(r\sin\phi)$

$+ (r\sin\phi)\left[-a\,r\sin\phi\,(r^2-1) - r\cos\phi\right]$

$= r^2\cos\phi\sin\phi$

$\boxed{r=1} \Rightarrow \dot{r} = 0$

$\Downarrow$

$+ r^2\sin^2\phi\,(-a)(r^2-1)$

limit cycle circular of radius 1

$- r^2\sin\phi\cos\phi$

$r\dot{r} = r^2\sin^2\phi\,(-a)(r^2-1)$

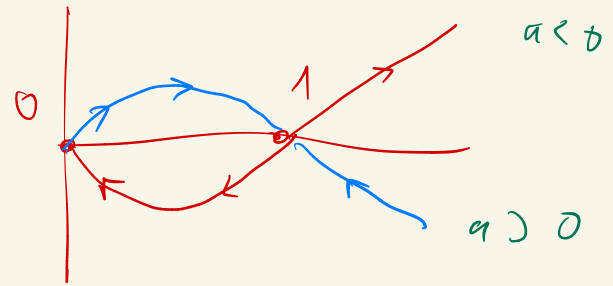
Stability?

$$r\dot{r} = r^2 \sin^2\phi\,(-a)(r^2-1)$$

$$\dot{r} = \underbrace{r \sin^2\phi}_{>0}\,(-a)\,(r-1)\underbrace{(r+1)}_{>0}$$

suppose $a > 0$

$$\dot{r} \sim r\,(r-1)\,(-a)$$



$a>0 \Rightarrow$ stable $r=1$, unstable fixed pt.

$a<0 \Rightarrow$ unstable $r=1$

$$x_{n+1} = r x_n - x_n^3$$

<u>Fixed point</u>

$$x^* = r x^* - x^{*3}$$

$$\boxed{x^* = 0}$$

or

$$1 = r - x^{*2}$$

$$x^{*2} = r - 1$$

$$\boxed{x^* = \pm\sqrt{1-r}}$$

stable iff $|r| < 1$

un

$$x_n = r^n \varepsilon$$
$$= r^n x_0$$

Stability ...

at $x^* = 0$ look at $x_0 = \varepsilon$

$$x_1 = r\varepsilon + \varepsilon^3 \sim r\varepsilon$$

$$x_2 = r^2\varepsilon$$

$\boxed{x^* = 0}$ stable for $0 < r < 1$

unstable for $1 < r < \frac{3}{2}$

$$\boxed{x^* = \pm\sqrt{r-1}}$$

only valid if $r > 1$

$x \simeq \pm\sqrt{r \cancel{+} 1} + \varepsilon$

stable ~~unstable~~ limit

$0 < 1 < r < \frac{3}{2}$

$f(x_n) = r x_n \cancel{+} x_n^3$

$|f'(x)| = r \cancel{+} 3x_n^2$

$-2r + 3$

$\sim r \cancel{+} 3(r-1)$

$= |4r - 3|$

not valid

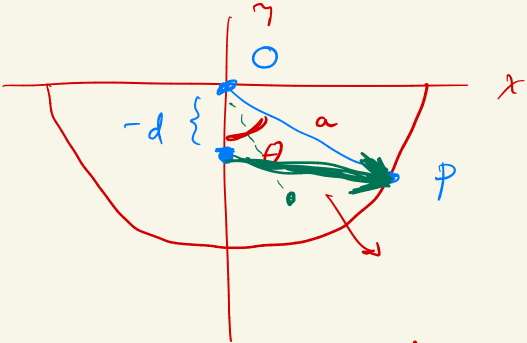
anyway, do algebra

$|4r-3| = 4r - 3 < 1$

$\Leftrightarrow r < 4/4 = 1$

$4r - 3 > 1 \Leftrightarrow r > 4/4 = 1$

... finish this.



$$\begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}\begin{pmatrix} w_x \\ w_y \\ w_z \end{pmatrix}$$

$$\frac{M}{\pi a^2/2} = \frac{2M}{\cancel{\pi} a^2}$$

(a) Calculate $I_0 \xrightarrow{} I_d \xrightarrow{} I_P$

$dx\,dy$

$$I_{ab} = \int dV \rho \left(r^2 \delta_{ab} - \cancel{r_a r_b}^{\,0}\right)$$

I_{zz} $\quad dx\,dy$

I_{zx}

only care about I_{zz}

$$I^0_{zz} = \int dA\, \rho \left(r^2 - \cancel{xy}^{\,0}\right)$$

$dx\,dy$

$$= \int r\,dr\,d\theta \left(r^2 - \cancel{r^2 \sin\theta\cos\theta}\right)$$

$$= \rho \int_0^a r^3\,dr \int_0^\pi d\theta - \cancel{\rho \int_0^a r^3\,dr \int_0^\pi \sin\theta\cos\theta\,d\theta}$$

$$= \left(\frac{\rho a^4}{4}\pi\right) - \cancel{\left(\frac{a^4}{4}\pi\right)} \cdot 0 = \frac{\rho a^4 \cancel{\pi}}{4} = \boxed{\frac{Ma^2}{2}}$$

$\cos\theta = u$

$du = -\sin\theta\, d\theta$

$$-\int_1^{-1} u\,du = \int_{-1}^{1} u\,du$$

$$I^{O}_{zz} = \frac{Ma^2}{2}$$

Parallel axis thm

$$I^{cm}_{zz} = ?$$

$$\boxed{I^{o}_{zz} = I^{cm}_{zz} + Md^2}$$

$$I^{0}_{zz} = \underbrace{M\left(\delta_{zz}\vec{R}^2 - \overbrace{R_a R_b}^{r_z = 0}\right)}_{Md^2} + I^{cm}_{zz}$$

$$I^{0}_{zz} = Md^2 + I^{cm}_{zz}$$

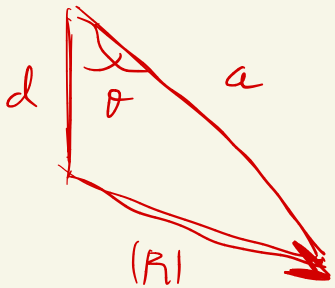
$$\to I^{cM}_{zz} = \frac{Ma^2}{2} - Md^2$$

<u>Next,</u>

$$I^{P}_{zz} = I^{cM}_{zz} + M\left(\delta_{zz}\vec{R}^2 - \cancel{R_a R_b}^{0}\right)$$

$$= I^{cM}_{zz} + M$$

What is $|\vec{R}|$?



Cos thm

$$|R|^2 = d^2 + a^2 - 2\cos\theta\,|d||a|$$

$$= d^2 + a^2 - 2ad\cos\theta$$

So

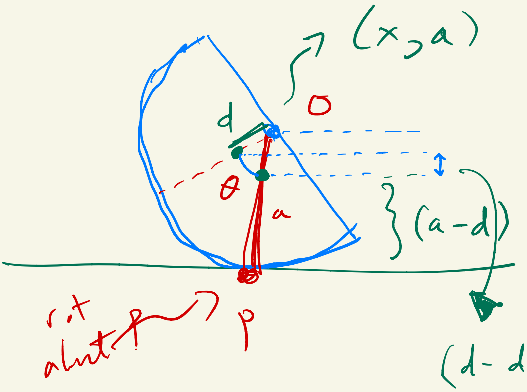
$$I^P_{zz} = \frac{Ma^2}{2} - \cancel{Md^2} + M\left[\cancel{d^2} + a^2 - 2ad\cos\theta\right]$$

$$= \frac{Ma^2}{2} + Ma^2 - 2Mad\cos\theta$$

$$= +\frac{3Ma^2}{2} - 2Mad\cos\theta$$

$$= \frac{Ma}{2}\left(3a - 4d\cos\theta\right)$$

rolling without slip

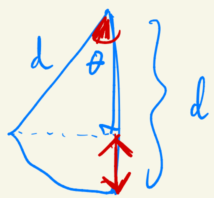


$$\boxed{a\dot{\theta} = \dot{x}} \quad (\text{in } x)$$

take V, at

$$y = d - d\cos\theta$$

$$V = -Mgd\cos\theta$$



$$\mathcal{L} = T_{trans} + T_{rot} - V$$

$$= T_{trans} + T_{rot} + Mgd\cos\theta$$

$$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}I^P_{zz}\,\dot{\theta}^2 + Mgd\cos\theta$$

$$\boxed{\frac{Ma}{2}(3a - 4d\cos\theta)}$$

$$\boxed{\text{min when } \theta = 0}$$

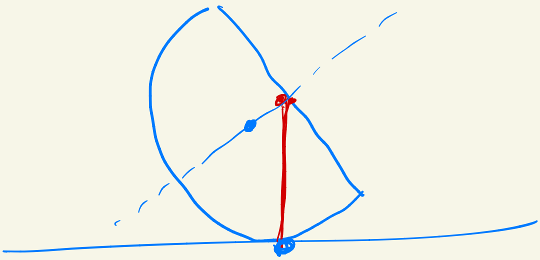
$$\dot{x} = a\dot{\theta}$$

$$\boxed{\mathcal{L} = \cancel{\frac{1}{2}Ma^2\dot{\theta}^2} + \frac{1}{2}[\cdots]\dot{\theta}^2 + Mgd\cos\theta}$$

can take $y = (a-d) - d\cos\theta + d = \boxed{a - d\cos\theta}$

more realistic but ok.

Disk does not flip if ... at turning pt.



instantaneous velocity

$V = -v_0 \hat{y}$

This translate to

$$\boxed{\dot{\theta}\Big|_0 = \frac{-v_0}{a}}$$

apparently don't have this

$$\mathcal{L} = \cancel{\frac{1}{2} M a^2 \dot{\theta}^2} + \frac{1}{2} \frac{Ma}{2} (3a - 4d\cos\theta)\dot{\theta}^2 + Mgd\cos\theta$$

initially, energy is all rotational ...

$$\theta\Big|_0 = 0 \; ; \; \dot{\theta}\Big|_0 = \frac{v_0}{a}$$

conserved.

$$\Rightarrow \boxed{\mathcal{E} = \frac{1}{2} M \cancel{a}^2 \frac{v_0^2}{\cancel{a}^2} + \frac{1}{2} \frac{Ma}{2} (3a - 4d) \frac{v_0^2}{a^2} - Mgd}$$

at the end -- all potential, no translate, no rolling ...

$$\mathcal{E}_f = -Mgd\cos\theta = 0$$

$$\theta = \pi/2$$

$$\frac{1}{2} M v_0^2 + \frac{1}{4} Ma(3a - 4d) \frac{v_0^2}{a^2} = Mgd$$

$$\Big[\underbrace{\frac{1}{2} M + \frac{1}{4} \frac{M}{a}(3a - 4d)}_{} \Big] v_0^2 = \cancel{M} gd$$

$$\left(\frac{2}{4} \cancel{M} + \frac{3}{4} \cancel{M} - \frac{\cancel{M} d}{a} \right)$$

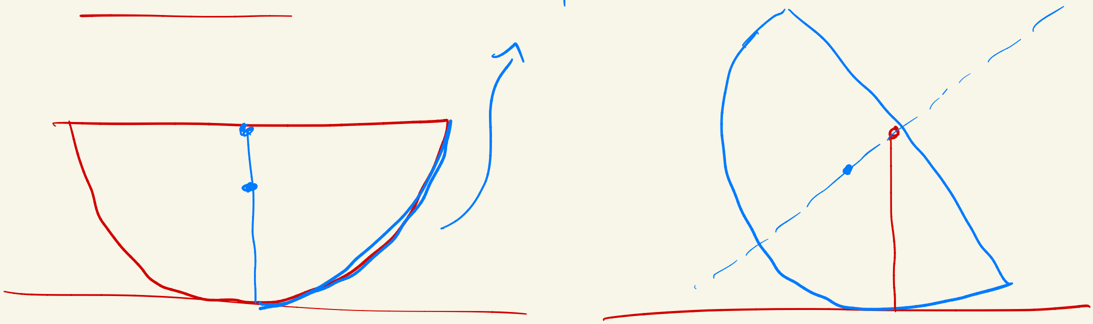
$$v_0 = 2\sqrt{\frac{gda}{5a - 4d}}$$

$$\left(\frac{5}{4} - \frac{d}{a} \right) v_0^2 = gd$$

$$v_0^2 = \frac{gd}{5/4 - d/a} = \frac{4gda}{5a - 4d}$$

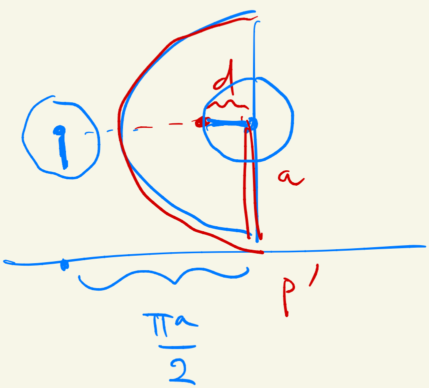
But wait

$$\frac{2\pi a}{4} = \frac{\pi a}{2}$$



Does the CM move?

well... look at $\theta = 0$ & $\theta = \pi/2$



center moved by $\frac{\pi a}{2}$ in x

$d < a$

cm position of dish changed as well...

$x_d = 0$

$x_d' = \frac{\pi a}{2} - d > 0$