The normalization of $|\vec{p}\rangle_{NR}$ is not invariant under Lorentz boosts. Let us consider a particle with momentum $\vec{p} = (p_x, p_y, p_z)$ and energy E. Suppose there is another inertial frame whose axes are moving at velocity

$$\vec{v} = (0, 0, -\beta)$$

units where $c=1$

relative to the original frame.

In the new frame the momentum of the particle is

$$\vec{p}' = (p_x', p_y', p_z')$$

where

$$p_x' = p_x$$
$$p_y' = p_y$$
$$p_z' = \frac{1}{\sqrt{1-\beta^2}}(p_z + \beta E)$$

and the energy is

$$E' = \frac{1}{\sqrt{1-\beta^2}}(E + \beta p_z)$$

Now let us consider

$$\delta^{(3)}(\vec{p}-\vec{q}) = \delta(p_x-q_x)\,\delta(p_y-q_y)\,\delta(p_z-q_z)$$

For a function $f(x)$ with a simple zero at $x = x_0$, we have

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\,\delta(x-x_0)$$

[e.g., $\delta(2x) = \frac{1}{2}\delta(x)$, $\delta(2x-14) = \frac{1}{2}\delta(x-7)$]

we have $\delta(p_z'-q_z') = \dfrac{1}{\left|\dfrac{dp_z'}{dp_z}\right|}\cdot\delta(p_z-q_z)$

$$= \frac{1}{\frac{1}{\sqrt{1-\beta^2}}\left(1+\beta\frac{dE}{dp_z}\right)}\,\delta(p_z-q_z)$$

$$\frac{dE}{dp_z} = \frac{d\left(\sqrt{p_x^2+p_y^2+p_z^2+m^2}\right)}{dp_z}$$

$$= \frac{2p_z}{2\sqrt{p_x^2+p_y^2+p_z^2+m^2}} = \frac{p_z}{E}$$

$$= \frac{1}{\frac{1}{\sqrt{1-\beta^2}}\left(1+\beta\frac{p_z}{E}\right)}\,\delta(p_z-q_z) = \frac{E}{\frac{1}{\sqrt{1-\beta^2}}\left(E+\beta p_z\right)}\,\delta(p_z-q_z)$$

$$= \frac{E}{E'}\,\delta(p_z-q_z)$$

So $\quad \delta^{(3)}(\vec{p}-\vec{q}) \neq \delta^{(3)}(\vec{p}\,'-\vec{q}\,')$ but instead

$$E\,\delta^{(3)}(\vec{p}-\vec{q}) = E'\,\delta^{(3)}(\vec{p}\,'-\vec{q}\,')$$

So we define the relativistically normalized states

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\;|\vec{p}\rangle_{NR} \quad \text{where } E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2}$$

$$\langle\vec{q}|\vec{p}\rangle = 2E_{\vec{p}}\,(2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{q})$$

[The factor of 2 is for later convenience.]

We <u>don't</u> change the normalization of $a_{\vec{p}}, a^\dagger_{\vec{p}}$

$$|\vec{p}\rangle_{NR} = a^\dagger_{\vec{p}}|0\rangle$$

$$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\;a^\dagger_{\vec{p}}|0\rangle$$

$$[a_{\vec{p}}, a^\dagger_{\vec{p}\,'}] = (2\pi)^3\,\delta^{(3)}(\vec{p}-\vec{p}\,')$$

The corresponding completeness relation for these states is

$$1 = \int \frac{d^3\vec{p}}{(2\pi)^3}\,|\vec{p}\rangle\frac{1}{2E_{\vec{p}}}\langle\vec{p}|$$

Up to now we have considered $\phi(\vec{x})$ and $\Pi(\vec{x})$, which are analogs of q and p in the time-independent Schrödinger equation. We now consider time dependence.

In the "Schrödinger" picture, $U(t) = e^{-iHt}$ is the time evolution operator and

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$$

$$\langle\psi(t)| = \langle\psi(0)|\,U^{\dagger}(t)$$

In the "Heisenberg" picture, operators depend on time rather than quantum states,

$$\Theta(t) = U^{\dagger}(t)\,\Theta(0)\,U(t)$$

The two pictures give the same observable matrix elements...

$$\overset{\text{Heisenberg}}{\langle\psi_1|\,\Theta(t)\,|\psi_2\rangle} = \overset{\text{Schrödinger}}{\langle\psi_1(t)|\,\Theta\,|\psi_2(t)\rangle}$$

time independent $\qquad\qquad\qquad$ time independent

We work mostly in the Heisenberg picture.

Let $\phi(x) = \phi(\vec{x}, t)$

$\pi(x) = \pi(\vec{x}, t)$

where $\phi(\vec{x}, 0)$ and $\pi(\vec{x}, 0)$ correspond with the independent fields $\phi(\vec{x})$ and $\pi(\vec{x})$ we discussed previously.

The time dependence is given by

$$\phi(x) = \phi(\vec{x}, t) = e^{iHt} \phi(\vec{x}, 0) e^{-iHt}$$

$$\pi(x) = \pi(\vec{x}, t) = e^{iHt} \pi(\vec{x}, 0) e^{-iHt}$$

We note that

$$\frac{\partial}{\partial t} \phi(x) = \frac{\partial}{\partial t}\left[e^{iHt} \phi(\vec{x}, 0) e^{-iHt}\right]$$

$$= iH\left[e^{iHt} \phi(\vec{x}, 0) e^{-iHt}\right] - \left[e^{iHt} \phi(\vec{x}, 0) e^{-iHt}\right] iH$$

$$= -i\left[\phi(x), H\right]$$

So $i \frac{\partial}{\partial t} \phi(x) = \left[\phi(x), H\right]$.

Actually, $\underline{\underline{\text{any}}}$ operator $\mathcal{O}(x)$ satisfies

$$i\frac{\partial}{\partial t}\mathcal{O}(x) = [\mathcal{O}(x), H].$$

Here is the argument... Previously we noted that

$$[H, a_{\vec{p}}] = -E_{\vec{p}}\, a_{\vec{p}}$$ (we used $\omega_{\vec{p}}$ notation back then)

So $$H a_{\vec{p}} = a_{\vec{p}} H - a_{\vec{p}} E_{\vec{p}}$$
$$= a_{\vec{p}}(H - E_{\vec{p}})$$

More generally $$H^n a_{\vec{p}} = H^{n-1} H a_{\vec{p}}$$
$$= H^{n-1} a_{\vec{p}}(H - E_{\vec{p}})$$

By induction we can show

$$H^n = a_{\vec{p}}(H - E_{\vec{p}})^n$$

Now consider $e^{iHt} a_{\vec{p}}$. We can write

$$e^{iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H^n$$

So we then have

$$e^{iHt} a_{\vec{p}} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \underbrace{H^n a_{\vec{p}}}_{= a_{\vec{p}}(H - E_{\vec{p}})^n}$$

$$= \sum_{n=0}^{\infty} a_{\vec{p}} \frac{(it)^n}{n!} (H - E_{\vec{p}})^n$$

$$= a_{\vec{p}}\, e^{i(H - E_{\vec{p}})t}$$

So $e^{iHt} a_{\vec{p}} e^{-iHt} = e^{-iE_{\vec{p}}t} a_{\vec{p}}$.

Taking the Hermitian conjugate gives

$$e^{iHt} a^{\dagger}_{\vec{p}} e^{-iHt} = e^{+iE_{\vec{p}}t} a^{\dagger}_{\vec{p}}$$

So $\displaystyle \phi(\vec{x}, t) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + a^{\dagger}_{\vec{p}} e^{+ip\cdot x} \right)\Big|_{p^0 = E_{\vec{p}}}$

$e^{-iE_{\vec{p}}t} e^{+i\vec{p}\cdot\vec{x}}$ $\qquad$ $e^{+iE_{\vec{p}}t} e^{-i\vec{p}\cdot\vec{x}}$

$\left[\text{Note } p\cdot x\Big|_{p^0 = E_{\vec{p}}} = p^0 x^0 - \vec{p}\cdot\vec{x}\Big|_{p^0 = E_{\vec{p}}} = E_{\vec{p}} t - \vec{p}\cdot\vec{x} \right]$

Also $\Pi(\vec{x},t) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{-ip\cdot x} - a^\dagger_{\vec{p}}\, e^{+ip\cdot x} \right) \Big|_{p^0 = E_{\vec{p}}}$

Note that $\Pi(\vec{x},t) = \frac{\partial}{\partial t} \phi(\vec{x},t)$.

Sometimes we are lazy about writing $p^0 = E_{\vec{p}}$ explicitly ...

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{+ip\cdot x} \right)$$

Let us now consider the vacuum expectation value of the product of two Heisenberg fields

$$D(x-y) = \langle 0 | \phi(x)\, \phi(y) | 0 \rangle$$

The product $\phi(x)\,\phi(y)$ will contain terms like

$$a_{\vec{p}}\, a_{\vec{p}\,'}, \quad a_{\vec{p}}\, a^\dagger_{\vec{p}\,'}, \quad a^\dagger_{\vec{p}}\, a_{\vec{p}\,'}, \quad a^\dagger_{\vec{p}}\, a^\dagger_{\vec{p}\,'}$$

Since $a_{\vec{p}\,'} | 0 \rangle = 0$ and $\langle 0 | a^\dagger_{\vec{p}} = 0$,

we need only look at the $a_{\vec{p}}\, a^\dagger_{\vec{p}\,'}$ term

So $$D(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\frac{d^3\vec{p}\,'}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}\,'}}} \langle 0| e^{-ip\cdot x} a_{\vec{p}} a^\dagger_{\vec{p}\,'} e^{ip'\cdot y}|0\rangle$$

$$(p^0 = E_{\vec{p}},\ p^{0\prime} = E_{\vec{p}\,'})$$

Since $$\langle 0| a_{\vec{p}} a^\dagger_{\vec{p}\,'} |0\rangle = {}_{NR}\langle \vec{p}|\vec{p}\,'\rangle_{NR}$$

$$= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,'),$$

$$D(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \qquad (p^0 = E_{\vec{p}}).$$

Consider the case $x^0 - y^0 = t$ and $\vec{x} = \vec{y}$.

$$D(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}} e^{-iE_{\vec{p}}t} = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t}$$

As $t \to \infty$ the integrand oscillates rapidly. It is therefore dominated by the point where $\frac{d}{dp}\left(\sqrt{p^2+m^2}\right) = 0$, which $p = 0$. So as $t \to \infty$,

$D(x-y)$ looks like $\sim e^{-imt}$ for $\vec{x} = \vec{y}$.
This makes sense, we produce an excitation with energy m (particle at rest).

Now consider the case $x^0 = y^0$, $\vec{x} - \vec{y} = \vec{r}$, $r = |\vec{r}|$

$$D(x-y) = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{i\vec{p}\cdot\vec{r}}$$

We write the angular integration as

$$\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta = \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta$$

$$= \int_0^{\infty} \int_0^{2\pi} \int_{-1}^{1} \frac{dp \, d\phi \, d\cos\theta}{(2\pi)^3 2\sqrt{p^2+m^2}} p^2 e^{ipr\cos\theta}$$

$$= \frac{1}{(2\pi)^2} \int_0^{\infty} \frac{dp \, p}{\sqrt{p^2+m^2}} \frac{e^{ipr} - e^{-ipr}}{2ir}$$

If you perform this integral (using Mathematica or some complex analysis tricks), you find that as $r \to \infty$,

$$D(x-y) \sim \frac{e^{-mr}}{r}$$

In particular it is nonzero. But since $x^0 = y^0$, doesn't this imply instantaneous signals and violate causality? $(x-y)^2 = -r^2 < 0$. Spacelike separation should be causally disconnected.

Answer: No. Because $D(x-y) \neq 0$ for $(x-y)^2 < 0$ does <u>not</u> imply <u>information</u> is travelling faster than light.

Imagine some local measurement at x represented by $\mathcal{O}(x)$ and some local measurement at y represented by $\mathcal{O}'(y)$. So long as the two operators commute

$$[\mathcal{O}(x), \mathcal{O}'(y)] = 0$$

for $(x-y)^2 < 0$, then the two measurements do not affect each other at spacelike separation. We discuss this more later.

The question then is whether for $(x-y)^2 < 0$, the commutator vanishes,

$$[\phi(x), \phi(y)] = 0 ?$$

Let's check ...

$$[\phi(x), \phi(y)] = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \frac{d^3\vec{p}\,'}{(2\pi)^3 \sqrt{2E_{\vec{p}'}}}$$

$$\times \Big\{ e^{-ip\cdot x} \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}'}]}_{(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,')} e^{+ip'\cdot y} + e^{+ip\cdot x} \underbrace{[a^\dagger_{\vec{p}}, a_{\vec{p}'}]}_{-(2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,')} e^{-ip'\cdot y} \Big\}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \left[e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right]$$

$$= D(x-y) - D(y-x)$$

The $\frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}$ is a Lorentz invariant measure. It goes with our Lorentz invariant normalization

$$\langle \vec{p} \,|\, \vec{q} \rangle = (2\pi)^3\, 2E_{\vec{p}}\, \delta^{(3)}(\vec{p}-\vec{q})$$

(this is why we put the factor of 2 with $E_{\vec{p}}$)

Notice also that $D(x-y)$ is Lorentz invariant...

$$\underbrace{D(x'-y')}_{\text{primed inertial frame}} = \int \frac{d^3\vec{p}}{(2\pi)^3\, 2E_{\vec{p}}}\, e^{-ip\cdot(x'-y')}$$

$$= \int \frac{d^3\vec{p}''}{(2\pi)^3\, 2E_{\vec{p}''}}\, e^{-ip''\cdot(x-y)}$$

($\vec{p}''$ is momentum $\vec{p}$ as viewed in the original inertial frame)

$$= D(x-y)$$

Now suppose $(x-y)^2 < 0$ so the separation is spacelike. Claim: There exists a Lorentz transformation which takes $x-y \to -(x-y)$.

Proof: Let us choose a primed frame where $x'-y' = (0, \vec{x}'-\vec{y}')$. Now do a rotation which reverses the direction of $\vec{x}' - \vec{y}'$. The result of this rotation in the original frame takes $x - y \to -(x-y)$.

So for $(x-y)^2 < 0$, $D(x-y) = D(y-x)$ and thus $[\phi(x), \phi(y)] = 0$.

since it is Lorentz invariant