Matrices in Quantum Computing

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Matrix Analysis

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Presentation layout

Qubits & Quantum Gates

Qubit: A quantum system with measurable eigenstates $|0\rangle$ and $|1\rangle$,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \to \text{like a Classical Bit.}$$

But before measurement,

$$\text{Wavefunction} : |\psi\rangle = a\,|0\rangle + b\,|1\rangle \in \mathbb{C}^2, \quad |a|^2 + |b|^2 = 1.$$

Probabilistic:

$$P(|\psi\rangle \to |0\rangle) = |a|^2 \quad P(|\psi\rangle \to |1\rangle) = |b|^2.$$

Quantum gate: unitary transformation on $|\psi\rangle$ of one of many qubits.

Multiple Qubits

How to express the state of two qubits, $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$?

$$|\psi_1\psi_2\rangle \stackrel{?}{\sim} |\psi_1\rangle, |\psi_2\rangle$$

More than two, $|\psi_i\rangle \in \mathbf{V}_i$?

$$|\psi_1\psi_2\ldots\psi_n\rangle \stackrel{?}{\sim} |\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$$

Questions:

- What is the vector space containing $|\psi_1\psi_2\ldots\psi_n\rangle$?
- How does $|\psi_1\psi_2\ldots\psi_n\rangle$ change w.r.t $\mathcal{A}_1|\psi_1\rangle$ where $\mathcal{A}_1 \in \mathfrak{L}(\mathbf{V})$?
- What about for $\mathcal{A}_1|\psi_1\rangle, \ldots \mathcal{A}_n|\psi_n\rangle$, where $\mathcal{A}_i \in \mathfrak{L}(\mathbf{V})$?

Tensor Product

Postulate: [Mike & Ike] The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

For $|\psi_1\rangle \in \mathbf{V}_1, \ldots, |\psi_n\rangle \in \mathbf{V}_n$,

$$|\psi_1 \ldots \psi_n\rangle \in \mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n.$$

The joint state $|\psi_1 \ldots \psi_n\rangle$ is given by

$$|\psi_1 \ldots \psi_n\rangle = |\psi_1\rangle |\psi_2\rangle \ldots |\psi_n\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle .$$

$|\psi_1 \ldots \psi_n\rangle$ is an *elementary tensor* in $\mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n$.

Not all $|\phi\rangle \in \mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n$ are elementary.

Tensor Product

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\hat{f}", "\textit{linear}"'] & & \mathbb{C} \\
& & \\
\mathbf{V} \times \mathbf{W} \arrow[uu, "\otimes"] \arrow[uurr, "f", "\textit{multilinear}"'] & &
\end{tikzcd}
$$

Tensor Product

Let $\dim(\mathbf{V}) = n, \dim(\mathbf{W}) = m$.

- Dimensions multiply:

$$\dim(\mathbf{V} \otimes \mathbf{W}) = \dim(\mathbf{V}) \otimes \dim(\mathbf{W}).$$

- For $v \in \mathbf{V} \otimes \mathbf{W}$,

$$v = \sum_{i,j}^{n,m} a_{ij} \left| v_i \right\rangle \left| w_j \right\rangle .$$

- $\left| v_1 \right\rangle, \ldots, \left| v_n \right\rangle$ & $\left| w_1 \right\rangle, \ldots, \left| w_m \right\rangle$ form orthonormal bases for $\mathbf{V}$ & $\mathbf{W}$, then $\left| v_i \right\rangle \otimes \left| w_j \right\rangle$ form a basis for $\mathbf{V} \otimes \mathbf{W}$.

Kronecker Product