

QUANTUM FIELD THEORY

Sep 13, 2020

Before. These notes come from Prof. Paltin's QFT notes, Peskin-Schroeder QFT, and "Quantum Field Theory & Condensed Matter" - Shankar.

Enjoy!

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Conventions

$$t = c = 1$$

$$[\text{length}] = [\text{time}] = [\text{energy}] = [\text{mass}]$$

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1 - 1 - 1 - 1)$$

$$p^2 = p_\mu p^\mu = E^2 - |\vec{p}|^2 = mc^2 = m$$

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, \vec{x})$$

$$x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

$$\epsilon^{0123} = +1 ; \quad \epsilon_{0123} = -1$$

$$\epsilon^{1230} = -1$$

$$E = i \frac{\partial}{\partial x^0}, \quad \vec{p} \rightarrow \vec{\nabla}$$

$$p^\mu = i \partial^\mu$$

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- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx} \theta(x)$

- n -dimensional Dirac δ -fn:

$$\int d^n x \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \tilde{f}(k)$$

$$\tilde{f}(k) = \int d^4 x e^{ik \cdot x} f(x)$$

- $\int d^4 x e^{ik \cdot x} = (2\pi)^4 \delta^{(4)}(k)$

- EM $\Phi = \frac{Q}{4\pi r} \leftarrow$ Coulomb potential

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- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi hc} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\vec{E}, \vec{A}), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of classical Field Theory

- Lagrangian Field Theory:

$$S = \int \underline{L} dt = \int \underline{L}(\phi, \partial_\mu \phi) d^4x \quad \left(\underline{L} = L d^4x \right)$$

Principle of least action:

$$0 = \delta S \quad (\text{Lagrangian})$$

$$(\text{E-L eqn}) = \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi + \frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta \phi \right] \right\}$$

$$\Rightarrow 0 = \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right] - \frac{\partial \underline{L}}{\partial \phi}$$

FTC \rightarrow term vanishes
@ Boundary

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Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_m \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \right\}$$

Ex $\mathcal{L} = \phi^2 : \quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi ; \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 0 \Rightarrow 2\phi = 0$

$$\begin{aligned} \mathcal{L} &= (\partial_m \phi) (\partial^m \phi) \\ \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} &= 0 ; \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 2\partial^m \phi \end{aligned} \quad \left. \right\} \Rightarrow \partial^m \phi = 0,$$

Ex Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2} (\partial_m \phi) (\partial^m \phi) - \frac{1}{2} m^2 \phi^2$$

$$\therefore \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = \partial^m \phi .$$

relativistic particle
of mass m .

$$\mathcal{E} - L \Rightarrow -m^2 \phi = \partial_m \partial^m \phi = \square \phi$$

$$\rightarrow \boxed{(\square + m^2) \phi = 0}$$

(Klein - Gordon Eqn.)

$$\text{Ex } \phi = e^{-ip \cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$$

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Noether's Thm

Thm

For every continuous symmetry, there exists a conserved current j^μ which implies a local conservation law

$$\partial_\nu j^\nu = -\vec{\nabla} \cdot \vec{j} \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

Ex Conservation of charge -- (gauss' law)

$$\begin{aligned} Q &= \int j^0 d^3x \\ \Rightarrow \frac{\partial Q}{\partial t} &= \int \frac{d j^0}{dt} d^3x = - \int \vec{\nabla} \cdot \vec{j} d^3x \\ &= - \oint \vec{j} d^2s \end{aligned}$$

Idea Consider continuous transf. \rightarrow infinitesimally

$$(\star) \quad \phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

↑
Small

(\star) is a symmetry if EOM invariant under (\star).

$\Rightarrow S$ is invariant.

$\Rightarrow L$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$,
for some J^μ .

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Let us compare this expectation for ΔL to the result obtained by varying the fields ...

$$\begin{aligned}\alpha \Delta L &= \frac{\partial L}{\partial \phi} (\alpha \Delta \phi) + \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \partial_\mu (\alpha \Delta \phi) \\ &= \alpha \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right) + \alpha \underbrace{\left(\frac{\partial L}{\partial \phi} - \partial_\mu \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \right\} \right) \Delta \phi}_{0}\end{aligned}$$

$$\Rightarrow \Delta L = \partial_\mu \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right\}$$

So $\frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi$ is the desired J^μ .

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi - J^\mu$$

Ex massless KG field

$$L = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)$$

Consider transformation:

$$\phi(x) \rightarrow \phi(x) + \alpha, \quad \Delta \phi = 1$$

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi = \partial^\mu \phi.$$

Check $\partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = \square \phi = 0$ since
 $(m^2 + \nabla^2) \phi = 0 \Rightarrow m = 0$ \uparrow

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Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^+) (\partial^\mu \phi) - m^2 \phi^+ \phi.$$

again, EOM \Rightarrow

$$(m^2 + \Box) \phi = 0.$$

Symmetry: $\phi \rightarrow e^{i\alpha} \phi$.

For infinitesimal transf we have:

$$\begin{aligned}\alpha D\phi &= i\alpha \phi && \text{(Taylor expand)} \\ \alpha D\phi^+ &= -i\alpha \phi^+\end{aligned}$$

One can check that

$$j^\mu = i [(\partial^\mu \phi^+) \phi - \phi^+ (\partial^\mu \phi)]$$

\Rightarrow the conserved current.

[Ex]

Space-time translation:

Consider inf. translation:

$$x^\mu \rightarrow x^\mu - a^\mu$$

\hookrightarrow in field transforms this becomes:

$$\phi(x) \rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

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Lagrangian is a scalar \Rightarrow must transform the same way:

$$L \rightarrow L + a^{\mu} \partial_{\mu} L = L + a^{\nu} \partial_{\mu} (s_{\nu}^{\mu} L)$$

Compare this to the eqn:

$$L \rightarrow L + a \partial_{\mu} J^{\mu}$$

we have

$$J^{\mu} = \cancel{s}_{\nu}^{\mu} L$$

\Rightarrow apply this, we find:

$$J^{\mu} = \frac{\partial L}{\partial (\partial_{\mu} \phi)} (\partial_{\mu} \phi) - s_{\nu}^{\mu} L$$

value μ explicit...

$$\boxed{T_{\mu}^{\nu} = \frac{\partial L}{\partial (\partial_{\nu} \phi)} \partial_{\mu} \phi - \delta_{\mu}^{\nu} L}$$

\hookrightarrow STRESS-ENERGY TENSOR, (or Energy-momentum tensor)

Conserved charge \Rightarrow the Hamiltonian

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad (\text{time-translation})$$

$$\text{momentum} \rightarrow p^i = \int T^{0i} d^3x = - \int \mathcal{H} \partial_i \phi d^3x \quad (\text{spatial translation})$$

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Ex Klein-Gordon field again

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.$$

$$T^{00} = \frac{\partial F}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} = \dots$$

$$= \frac{1}{2} (\partial_0 \phi) (\partial^0 \phi) - \frac{1}{2} (\partial_i \phi) (\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

so

$$H = \int T^{00} d^3x = \int \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} d^3x$$

(note: all terms are positive ... (sum of squares))

→ can't fall into arbitrary negative energy

THE KLEIN-GORDON FIELD as HARMONIC OSCILLATOR

promote: ϕ, π to operators \Rightarrow impose suitable commutation relations

Reall ...

$$[q_i, p_j] = i \hbar \delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0.$$

$$\text{Harmonic oscillator: } H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2$$

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Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}} \left(g\sqrt{m\omega} + ip \frac{1}{\sqrt{m\omega}} \right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}} \left(g\sqrt{m\omega} - ip \frac{1}{\sqrt{m\omega}} \right)$

- $a^\dagger a = \frac{1}{\omega} H - \frac{1}{2}$ ($\Rightarrow H = \omega(a^\dagger a + \frac{1}{2})$)



operator...

- $|0\rangle, a|0\rangle = 0.$

$$|n\rangle = \frac{1}{\sqrt{n!}} \underbrace{a^\dagger \dots a^\dagger}_{n} |0\rangle$$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -\omega a ; [H, a^\dagger] = \omega a^\dagger$

a lowers by ω

a^\dagger raises by ω

- $H a^\dagger |n\rangle = (E_n + \omega) a^\dagger |n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

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For continuous system ... commutation relations become ..

$$[\phi(x), \pi(y)] = i\delta^{(1)}(x-y)$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0.$$

Next, the Hamiltonian is now also an operator...
To find $\text{spec}(H)$, Fourier transf $\phi(x)$

$$\rightarrow \phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \phi(\vec{p}, t)$$

Recall KG eqn: $(m^2 + D)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|\vec{p}|^2 + m^2) \right\} \phi(\vec{p}, t) = 0$$

\rightarrow This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}$$

$$\text{Now, } H_{SHO} = \frac{1}{2} \vec{p}^2 + \frac{1}{2} \omega^2 \phi^2 \quad (m=1)$$

\rightarrow know spectrum! $(n + \frac{1}{2})\omega$.

$$\phi = \frac{1}{\sqrt{2\omega}} (at + a) ; \vec{p} = -i\sqrt{\frac{\omega}{2}} (a - at)$$

$$[a, a^\dagger] = 1.$$

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Since it's more convenient to work in position space

$$\boxed{\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p e^{ip \cdot x} + a_p^\dagger e^{-ip \cdot x})}$$

$$\boxed{\pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p e^{ip \cdot x} - a_p^\dagger e^{-ip \cdot x})}$$

Note

$$\left\{ \begin{array}{l} a_p \text{ goes with } e^{+ip \cdot x} \\ a_p^\dagger \text{ goes with } e^{-ip \cdot x}. \end{array} \right.$$

9 Easy to show that

$$\left\{ \begin{array}{l} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}). \end{array} \right.$$

* Can re-arrange...

$$\left\{ \begin{array}{l} \phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p + a_{-p}^\dagger) e^{ip \cdot x} \\ \pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{-ip \cdot x}. \end{array} \right.$$

→ set commutation relation between a_p :

$$\boxed{[a_p; a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')}$$

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Now, can check that

$$\begin{aligned} [\phi(x), \pi(x')] &= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{w_p}{w_{p'}}} x \cdot e^{ip \cdot x} (p \cdot x + p' \cdot x') \\ &\quad \left([a_{-p}^\dagger, a_p] - [a_p, a_{-p}^\dagger] \right) \\ &= i \delta^{(3)}(x - x') \quad \checkmark \end{aligned}$$

• Now, can express Hamiltonian in terms of ladder ops

recall that

KG field, but
done

$$H = \int d^3 x \left\{ \frac{\partial f}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \partial^0 f \right\}$$

$$= \int d^3 x \left\{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial f}{\partial (\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial f}{\partial \dot{\phi}} \right)$$

so ...

$$H = \int d^3 x \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi) \cdot (\nabla \phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

$$\text{with } \pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{ip \cdot x}$$

we get

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p + a_{-p}^\dagger) e^{-ip \cdot x}$$

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$$H = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(p+p') \cdot x} \left\{ -\frac{\sqrt{w_p w_{p'}}}{4} (a_p - a_{-p}^\dagger) (a_{p'} + a_{-p'}^\dagger) \right. \\ \left. + \frac{-p \cdot p' + m^2}{4\sqrt{w_p w_{p'}}} (a_p + a_{-p}^\dagger) (a_{p'} + a_{-p'}^\dagger) \right\}$$

results in $C(p-p')$
 $\Rightarrow p = p'$

Some $S^{(3)}$
 will appear...

$$= \int \frac{d^3p}{(2\pi)^3} w_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

Σ

$$H = \int \frac{d^3p}{(2\pi)^3} w_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

→ can evaluate commutators...

$$[H, a_p^\dagger] = w_p a_p^\dagger; [H, a_p] = -w_p a_p$$

With H , can find momentum operator...

kG field \rightarrow form $p^i = \int d^3x T^{0i} = - \int \nabla_i \phi d^3x$, we set

$$\tilde{P} = - \int d^3x \nabla_i \phi(x) \\ = \int \frac{d^3p}{(2\pi)^3} \tilde{p} a_p^\dagger a_p$$

$E_p \xrightarrow{0}$
 \parallel

a_p^\dagger creates momentum \tilde{p} & energy $w_p = \sqrt{|\tilde{p}|^2 + m^2}$.

Excitation: $a_p^\dagger a_q^\dagger \dots |0\rangle$ = "particles".

↳ such excitation at p is a particle.

\Rightarrow set particle statistics --

Consider 2-particle state $a_p^+ a_q^+ |0\rangle$.

Since $[a_p^+, a_q^+] = 0$, we have

$$a_p^+ a_q^+ |0\rangle = a_q^+ a_p^+ |0\rangle$$

\Rightarrow Klein Gordon particles follow Bose-Einstein state.

* Normalization $\langle 0|0 \rangle = 1$.

$$\langle p | \propto a_p^+ |0\rangle$$

This $\rightarrow \langle q | p \rangle = (2\pi)^3 \delta^{(3)}(q-p) \rightarrow$ NOT Lorentz inv

PF Under a Lorentz boost $\rightarrow p'_3 = \gamma(p_3 + \beta E)$

using the identity $E' = \gamma(E + \beta p_3)$

$$\rightarrow \delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(n - n_0)$$

$$\text{we can write: } \delta^{(3)}(p-q) = \delta^6(p'-q') \cdot \left(\frac{dp'_3}{dp_3} \right)$$

$$\begin{aligned} & \underbrace{\delta(p_1-q_1)}_{\text{same}} \underbrace{\delta(p_2-q_2)}_{\text{boosted}} \underbrace{\delta(p_3-q_3)}_{\text{boosted}} \\ &= \delta^{(3)}(p'-q') \cdot \gamma \left(1 + \beta \frac{dE}{dp_3} \right) \\ &= \delta^{(3)}(p'-q') \frac{\gamma}{E} (E + \beta p_3) \\ &= \delta^{(3)}(p'-q') \left(\frac{E'}{E} \right) \end{aligned}$$

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$$\text{So } \boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \left(\frac{E'}{E}\right)} \rightarrow 1 \Leftrightarrow E = E_p = E'$$

For normalization to work \rightarrow use E_p , not E .

\rightarrow define: $|p\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle$

to find $\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$

Completeness relation ...

1 particle $\rightarrow \boxed{1 = \int \frac{d^3 p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$

RS Interpret $\phi(x)|0\rangle \dots$ we know that a_p^\dagger creates momentum p energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle , \quad a_p^\dagger |0\rangle = \frac{1}{\sqrt{2E_p}} |p\rangle$$

Pf. By defn ...

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ipx} + a_p^\dagger e^{-ipx})$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle \quad \square$$

$\rightarrow \phi(x)|0\rangle$ is a lin. superposition of single-particle states

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Hint here: well-defn momentum.

When nonrelativistic $\rightarrow E_p \approx \text{constant}!$

\Rightarrow $\phi(x)$ acting on the vacuum, "creates a particle at position x ".

\hookrightarrow Confirm this by computing -

$$\langle 0 | \phi(x) | p \rangle = \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p^\dagger e^{ip \cdot x} + a_p^- e^{-ip \cdot x}) \sqrt{2E_p} a_p | p \rangle$$

$$\boxed{\langle 0 | \phi(x) | p \rangle = e^{ip \cdot x}}$$

\hookrightarrow Interpretation: position-space representation of the single-particle wf_n of the state $|p\rangle$, just like

$$\boxed{\langle n | p \rangle \propto e^{ip \cdot x}} \text{ in QM!}$$

$\langle 0 | \phi(x) | \sim x | \dots$ (don't take this literally, ofc).

Note Hw1, Hw2 are copy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time \rightarrow we quantized KG field in the Schrödinger picture.

\rightarrow Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$U(t) = e^{-iHt}$ is the time evolution.

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \xrightarrow{\text{state evolves in time}}$$

\rightarrow In the Heisenberg picture, ... Operators evolve in time

$$\theta(t) = U^\dagger(t) \theta(0) U(t).$$

to treat

$$\langle \psi_1 | \theta(t) | \psi_2 \rangle = \langle \psi_1(t) | \theta(t) | \psi_2(t) \rangle$$

\downarrow

Heisenberg

\downarrow

Schrödinger.

\rightarrow make the operators ϕ, π time-dependent like this

$$\phi(x) \rightarrow \phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$$

$$\pi(x) \rightarrow \pi(x, t) = e^{iHt} \pi(x) e^{-iHt}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t} \theta = [\theta, H]$

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which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$\frac{i}{\partial t} \phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\} \right]$$

$$(\phi \leftrightarrow \phi) \Rightarrow \int d^3x' \left(i\delta^{(3)}(x-x') \pi'(x,t) \right)$$

\rightarrow only continual term is 1^{st} .

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t} \phi(x,t) = \pi(x,t)}$$

and

$$\frac{i}{\partial t} \pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\pi)^2 + \frac{1}{2}m^2\phi^2 \right\} \right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x') (-\nabla^2 + m^2) \phi(x',t) \right)$$

(integrate by parts here)

$$= -i(-\nabla^2 + m^2) \phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \pi(x,t) = (m^2 - \nabla^2) \phi(x,t)}$$

Combining these 2 results we get ...

$$\boxed{\frac{\partial^2}{\partial t^2} \phi(x,t) = (\nabla^2 - m^2) \phi(x,t)}$$

\hookrightarrow rearranging this gives

$$\boxed{(\nabla^2 + m^2) \phi(x,t) = 0} \rightarrow \text{just the KG eqn...}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

Recall: $H_{\text{ap}} = a_p^{\dagger} (H - E_p) \rightarrow$ from comm. rule -

\Rightarrow (proof by induction)

$$H^n a_p = a_p (H - E_p)^n$$

Similarly,

$$H^n a_p^{\dagger} = a_p^{\dagger} (H + E_p)^n$$

\rightarrow So, we have

$$e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t} \rightarrow \text{from above...}$$

and

$$e^{iHt} a_p^{\dagger} e^{-iHt} = a_p^{\dagger} e^{+iE_p t}$$

\rightarrow Now -- we want to write $\phi(x, t)$ in terms of these operators. (since $\phi(x)$ is a comb of a & a^{\dagger})

$\pi(x)$
we know that $\phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before ...

$$\phi(x) = \phi(x, 0) = \int \frac{dp}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ip \cdot x} + a_p^{\dagger} e^{-ip \cdot x})$$

substitute this into $\phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$ we find

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$$\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-ip \cdot x} + a_p^+ e^{ip \cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow p \cdot x = E_p \cdot x^0 - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p} = E_p t - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p}$$

So,

$$\boxed{\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p} \cdot \vec{x}} + a_p^+ e^{+i\vec{p} \cdot \vec{x}} \right\} \Big|_{p^0 = E_p}}$$

Similarly, we can find

$$\boxed{\Pi(x, t) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} (a_p e^{-i\vec{p} \cdot \vec{x}} - a_p^+ e^{+i\vec{p} \cdot \vec{x}}) \Big|_{p^0 = E_p}}$$

from defn or from $\Pi(x, t) = \frac{\partial}{\partial t} \phi(x, t)$.

Note we can also do everything, but starting from P and not H . But we won't worry about that.



Causality Note that causality is broken when without the presence of fields.

→ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture --

-- the amplitude for a particle to propagate from $y \rightarrow x$ is given by

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

"Two-point correlation function"

Let $\boxed{D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle}$

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \langle 0 | \text{in terms of } a_p^+ a_q^- | 0 \rangle$$

$$= \langle 0 | a_p^+ a_q^- | 0 \rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)}$$

More explicitly --

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$\begin{cases} \textcircled{2} p' = \vec{p} \\ p'_0 = E_{p'} \end{cases}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | e^{-ipx} e^{ip'y} a_p^+ a_{p'}^- | 0 \rangle$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{1}{\sqrt{2E_p}} \right) \left(\frac{1}{\sqrt{2E_{p'}}} \right) e^{-ip \cdot x - ip' \cdot y} (2\pi)^3 \delta^{(3)}(p-p')$$

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip \cdot (x-y)} \quad \square$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$.

(1) Suppose that $x-y = (t, \vec{v}, 0, 0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t}$$

$$(\text{timelike}) = \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2} e^{-iEt}$$

$$= e^{-imt} \xrightarrow{\text{dominated by region above}} \text{dominated by region above}$$

$t - i\omega$

$p \approx 0 -$

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

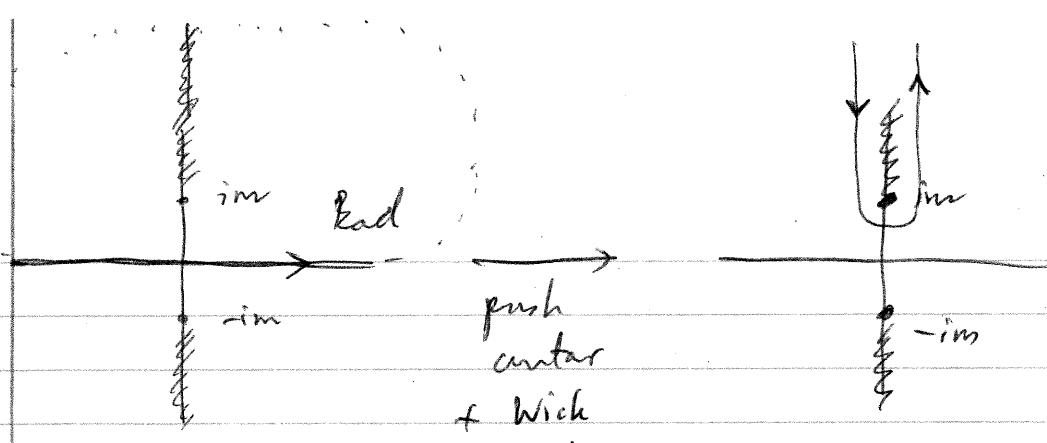
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2Ep} e^{i\vec{p} \cdot \vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2Ep} \frac{e^{ipr} - e^{-ipr}}{i pr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ tim (singularity) ...

→ must change contour... \rightarrow which rotate



push
cancel
+ Wick
rotate ...

$$p = -i\bar{p}$$

To get

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty dp \frac{pe^{-ipr}}{\sqrt{p^2 - m^2}} \quad (\text{Wick rotate})$$

$$\Rightarrow \boxed{\mathcal{D}(x-y) = \frac{e^{-mr}}{r}} \rightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^m(x-y)_n = -(\vec{x}-\vec{y})^2 \delta_{mn}$
is spacelike, cannot have causality between $x-y$.

$\mathcal{D}(x-y) \neq 0 \Rightarrow ???$ paradox?

\rightarrow No! To discuss causality, we should ask not whether particles can propagate over spacelike intervals ...

... but whether a "measurement" at one point can affect a measurement at another point where separation from the first is spacelike -

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement $\phi(x)$, call this $\phi(x)$. or a local measurement $\phi(y)$, called $\phi(y)$

So long as $[\phi(x), \phi(y)] = 0$, the 2 measurements don't affect one another.

→ measure the field $\phi @ x + @ y$,

If $[\phi(x), \phi(y)] = 0$ when $(x-y)^2 < 0$ then we've good

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} \right. \\ \left. + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\}$$

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \times \left\{ (a_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}), \right. \\ \left. (a_p^\dagger e^{-ip \cdot y} + a_p e^{ip \cdot y}) \right]$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} \right. \\ \left. + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\} \\ (2\pi)^3 \delta^3(p - p') \quad -(2\pi)^3 \delta^3(p - p')$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2E_p} \right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

$$\text{So } [\phi(x), \phi(y)] = D(x-y) - D(y-x).$$

Since $D(y-x)$ is Lorentz invariant, can take

$$y-x \rightarrow x-y. \text{ (possible b/c } (x-y)^2 < 0)$$

Note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

\rightarrow so this is why possible because $(x-y)^2 < 0$
(negative).

$$\text{So } D(y-x) = D(x-y)$$

$$\text{So, } [\phi(x), \phi(y)] = 0 \quad \checkmark$$

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~~The Klein-Gordon Propagator~~

Let's look at $[\phi(x), \phi(y)]$ in more details..

$[\phi(x), \phi(y)]$ is just a number

~~can write~~ $[\phi(x), \phi(y)] \equiv \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$

$$\rightarrow \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right\}$$

$$\Delta \text{ Poles}$$

$$E_p^2 = m^2$$

$$p^0 = \pm E_p$$

(assuming $x^0 > y^0$)

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} \Big|_{p^0 = E_p} - \frac{1}{2E_p} e^{-ip \cdot (x-y)} \Big|_{p^0 = -E_p}$$

$= -E_0$

The Klein-Gordon Propagator

→ Before looking at this -- need to look at Green's Function & Contour Integrals --

→ Consider function

$$\frac{1}{(p_0^0 - E_p)(p_0^0 + E_p)} \quad E_p = \sqrt{|p|^2 + m^2}.$$

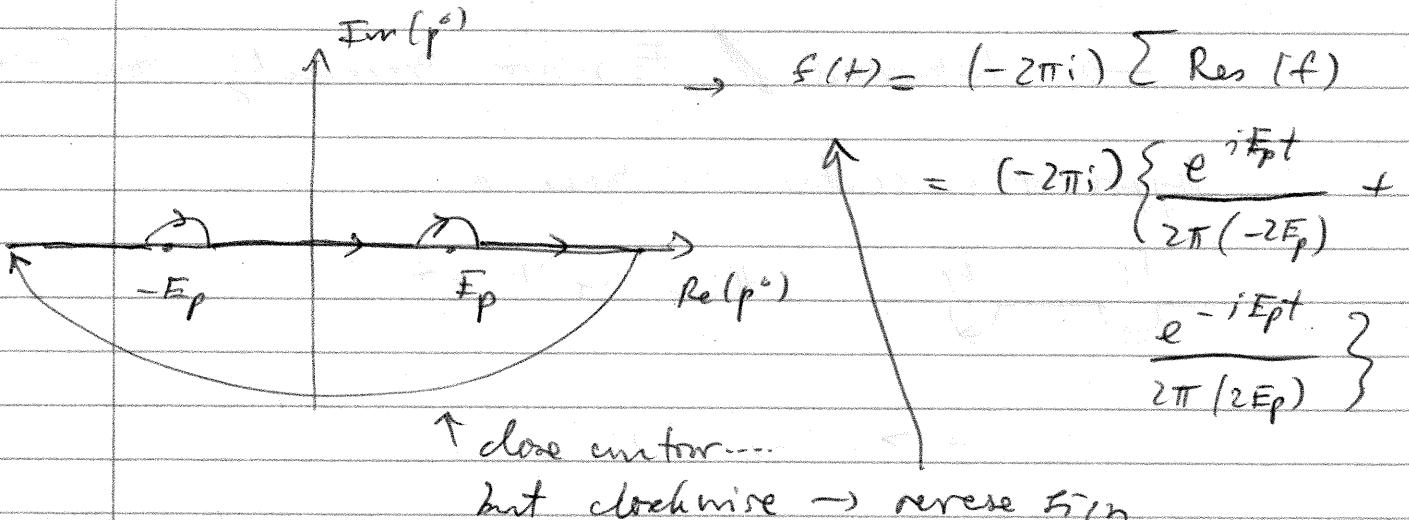
→ Poles at $p_0^0 = \pm E_p$.

Lab at FT:

$$f(t) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{e^{-ip^0 t}}{(p^0 - E_p)(p^0 + E_p)}$$

→ How to integrate this?

If $t > 0 \rightarrow$ ~~crosses poles~~



$$\rightarrow f(t) = \frac{-i}{2E_p} (e^{+iE_p t} - e^{-iE_p t}) \quad (t > 0)$$

If $t < 0$ close contours above poles



$$\rightarrow f(t) = 0.$$

→ So, altogether, we have ...

$$\begin{aligned} f(t) &= \int \frac{1}{2\pi} \frac{dp^0}{(p_0^0 E_p)(p^0 + E_p)} e^{-ip^0 t} \\ &= \Theta(t) \left(\frac{-i}{2E_p} \right) (e^{-iE_p t} - e^{+iE_p t}) \end{aligned}$$

where $\Theta(t)$ is the Heaviside Step fn ...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

→ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



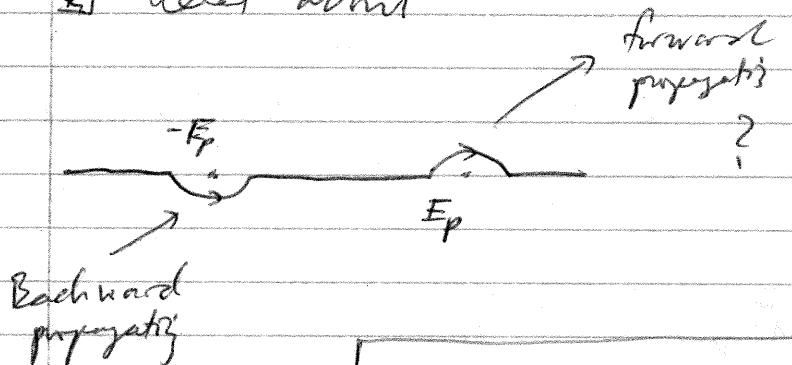
then we'll get

$$t > 0 \rightarrow f(t) = 0$$

$$t < 0 \rightarrow f(t) = \Theta(-t) \frac{i}{2E_p} (e^{-iE_p t} - e^{+iE_p t})$$

→ Advanced / Backward Propagating Green's fn.

What about



$$\rightarrow \boxed{f(t) = \Theta(+)(-\frac{i}{2E_p})e^{-iE_pt} + \Theta(-)(-\frac{i}{2E_p})e^{+iE_pt}}$$

Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$

Consider this quantity...

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left[e^{-ip(x-y)} - e^{ip(x-y)} \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip(x-y)} + \frac{1}{-2E_p} e^{-ip(x-y)} \right\} \\ &\quad \uparrow \quad \downarrow \\ &\quad \text{pole} \quad \text{pole @} \\ &\quad @ p_0 = E_p \quad p_0 = -E_p \end{aligned}$$

$$\begin{aligned} \text{(4) integral} \rightarrow &= \int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-i}{p^2 - m^2} e^{-ip(x-y)} \\ &\quad \underbrace{\qquad\qquad\qquad}_{f(+)} \end{aligned}$$

$f(+)$ before, where

$$(p^0 - E_p)(p^0 + E_p) = p^{0^2} - |p|^2 - m^2 = p^2 - m^2$$

Now, call ...

$$D_R(x-y) = \theta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle$$

Then

$$\begin{aligned}
 \rightarrow (\square + m^2) D_R(x-y) &= \square D_R(x-y) + m^2 D_R(x-y) \\
 &= (\square \theta(x^0 - y^0)) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + 2(\partial_\mu \theta(x^0 - y^0)) \partial^\mu \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + \theta(x^0 - y^0) (\square + m^2) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &= -\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \quad \text{cancel} \\
 &\quad + 2\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle + 0 \\
 &= \delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \quad \text{milds} \\
 &= -i \delta^{(4)}(x-y) \quad \text{renormalization} \\
 &\quad \downarrow \quad \text{(easy)} \\
 &\quad -i \delta^{(3)}(x-y)
 \end{aligned}$$

So

$$(\square + m^2) D_R(x-y) = -i \delta^{(4)}(x-y)$$

$\rightarrow D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0 @ x^0 < y^0$

$\Rightarrow D_R(x-y) \equiv \text{"Retarded" Green's fn}$

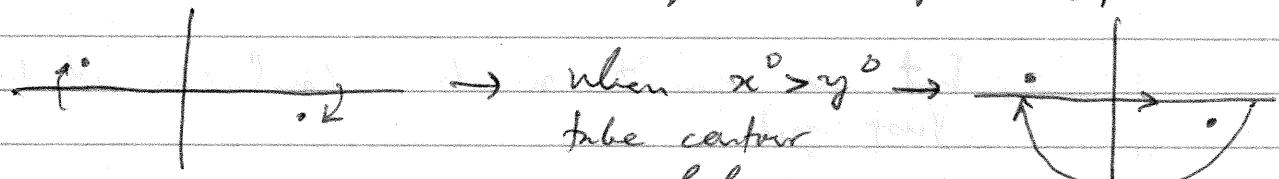
Now ... As we have seen, there are many ways to take the contour ...



→ Use the Feynman prescription instead

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

Convenient! B/c now poles are $p^0 = \pm(E_p - i\epsilon)$



when $x^0 < y^0 \rightarrow$
take contour above.

→ get same expression
but with $x \leftrightarrow y$.

So,

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \Theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \Theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle$$

$$= \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

"time-ordering" symbol -

Time-ordering symbol \Rightarrow instructs us to place the operators & heat follows in order with the latest to the left.

\rightarrow apply $(D + m^2)$ to last line, set D_F is Green's fn of Klein-Gordon Operator,

$$() \quad \overbrace{\hspace{10em}}^{\text{---}}$$

$D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator--

\hookrightarrow propagation amplitude

\rightarrow But we can't much calculation at this point just yet.

\rightarrow B/c we've only looked at the free K-G theory

\rightarrow Field eqn in this case is linear : there are no interactions--

\rightarrow this theory is too simple to make any predictions--

\rightarrow need perturbation --

One kind of interaction it's can also be solved



Particle Creation by a classical Source

Consider the source $j(x)$

Result... free field: $(D + m^2)\phi = 0$

→ now... $(D + m^2)\phi = j(x)$ Field ϕ is
 ↗ space time.

$j(x)$ is nonzero only for a finite time interval

The associated lagrangian is

$$L = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is
 enough to solve

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip_i x} + a_p^+ e^{ip_i x})$$

With a source...

$$\phi(x) = \phi_0(x) + i \int d^3y D_R(x-y)j(y)$$

We won't worry about this for now...

Some problems & Insights

(1) Classical EM (no sources) follow from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \epsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \quad \boxed{(\vec{\nabla} \cdot \vec{E} = 0) \quad (\nu = 0)}$$

\downarrow

$$\boxed{-\partial_t \vec{E} + \vec{\nabla} \times \vec{E} = 0} \quad (\nu = i)$$

(2) Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi^- - m^2 \phi^+ \phi^- \right)$$

Derive E-L eqn:

$$\boxed{i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ = 0}$$

$$\boxed{-i\partial_t \phi^- - \frac{1}{2m} \nabla^2 \phi^- = 0}$$

Now... write $\phi \rightarrow e^{-i\theta} \phi$, $\phi^+ \rightarrow e^{i\theta} \phi^+$

$$\begin{aligned} &\sim \phi - i\theta \phi \\ &\rightarrow \Delta \phi \sim -i\theta \end{aligned}$$

$$\begin{aligned} &\sim \phi^+ + i\theta \phi^+ \\ &\Delta \phi^+ \sim i\theta \end{aligned}$$

So that

$$j^{\mu} = \frac{\partial f}{\partial (\partial_{\mu} \phi)} \Delta \phi + \frac{\partial f}{\partial (\partial_{\mu} \phi^+)} \Delta \phi^+$$

↑
conserved current -

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial f}{\partial (\partial_x \phi)} \rightarrow \dots \text{conjugate\dots}$$

→ can get Hamiltonian → there's a formula in book,
but we worry abt this.

3) If we take $(x-y)^2 = -r^2 \rightarrow$ can implicitly write

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2 \rightarrow D(x-y)$ can be written in terms of Bessel Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation...

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

→ What happens to $\phi(x)$ under Λ ?

We require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ What about $\partial_\mu \phi(x)$?

Under transform -- $\partial_\mu \phi(x) \rightarrow \partial_\mu (\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu (\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\rho_\mu (\partial_\rho \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho_\mu (\Lambda^{-1})^\sigma_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu \phi(x))^2 \rightarrow g^{\mu\nu} (\partial_\mu \phi'(x)) (\partial_\nu \phi'(x))$$

$$= g^{\mu\nu} [\partial_\mu (\phi(\Lambda^{-1}x))] [\partial_\nu (\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu} \{ (\Lambda^{-1})^\rho_\mu \partial_\rho \phi \} \{ (\Lambda^{-1})^\sigma_\nu \partial_\sigma \phi \} (\Lambda^{-1}x)$$

$$= (\partial_\mu \phi)^2 (\Lambda^{-1}x) \leftarrow = g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi (\Lambda^{-1}x) \rightarrow \text{evaluated}$$

(37)

Exercise

$$\phi(x) \rightarrow \phi(\tilde{x})$$

$$\partial_\mu \phi(x) \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)(\tilde{x})$$

$$(\partial_\mu \phi(x))^2 \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)^2 (\tilde{x})^\nu$$

So it is clear that

$$L \rightarrow L(\tilde{x})$$

↑

Lagrangian is Lorentz-invariant.

→ The action $S = \int d^4x L$ is also Lorentz inv.

→ also clear that EOM is also Lorentz inv.

$$\begin{aligned} (\square + m^2) \phi(x) &= (\tilde{x})^\mu \partial_\mu (\tilde{x})^\nu \partial_\nu + m^2 \phi(\tilde{x}) \\ &= (\partial^\mu \partial_\mu + m^2) \phi(\tilde{x}) \\ &= 0 \quad \checkmark \end{aligned}$$

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Lep 10, 2020

→ How do we find Lorentz-invariant theories, in general?

→ For simplicity, restrict attention to lin. transf

→ $\phi_a = \phi \in \mathbb{C}^n$, → matrix giving Lorentz transf in A .

$$\rightarrow \boxed{\Phi_a(x) \rightarrow M_{ab}(A) \Phi_b(\tilde{x})}$$

$n \times n$

The

→ most general nonlinear draft can be built
out of linear ones \Rightarrow suffices to consider M
only.

↳ for short, write $\phi \mapsto M(\alpha) \phi$.

→ What are the possible allowed $M(\alpha)$?

◻ $\{M(\alpha)\}$ form a group $M(\alpha') M(\alpha) = M(\alpha')$
 $\curvearrowright \alpha'' \alpha = \alpha'$

→ the correspondence between $\alpha \in M$ must be
preserved under multiplication.

$\{1\}$ Lorentz group $\longleftrightarrow \{M(\alpha)\} \rightarrow$ n-dim
representation of the
Lorentz group

↳ [?] What are the finite-dim matrix reps
of the Lorentz group?

Ex in $\mathfrak{so}(4)$... spin $\frac{1}{2} \rightarrow \{M\}$ are the 2×2 unitary
matrices with determinant 1

$$\rightarrow \boxed{U = e^{-i\vec{\theta}^i \vec{\sigma}^i/2}} \rightarrow \{\theta^i \vec{\sigma}^i/2\}$$

$$\begin{pmatrix} \vec{\theta} \\ \vec{\sigma} \end{pmatrix}$$

3 arbitrary parameters
& Pauli matrices.

$$\{U(\vec{\theta}) : e^{-i\vec{\theta}^i \vec{\sigma}^i/2}\}$$

→ In the case for arbitrary spin representations...

$$U(\vec{\theta}) = e^{-i \vec{\theta} \cdot \vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i \sum_l \epsilon^{ijk} J^l$$

→ Check that this works for spin $\frac{1}{2}$:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2} \right] = i \sum_l \epsilon^{jkl} \frac{\sigma^l}{2} \quad \checkmark$$

→ for spinless particles... $\psi(\vec{x})$ can be decomposed into orbital angular momentum states. $J=0, 1, 2, \dots$
(no intrinsic spin $\Rightarrow J=L$)

$$\bullet \quad \vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i \vec{\nabla})$$

$$\bullet \quad J^j = i \sum_l \epsilon^{jkl} x^k \nabla^l$$

$$\bullet \quad \nabla^l = -\partial_x^l = -\frac{\partial}{\partial x^l}$$

But the cross product is special to 3D case.

→ write operators in antisymmetric tensor...

$$J^{ij} = -i(x^i \partial^j - x^j \partial^i) \quad \rightarrow \text{represents free cross product.}$$

$$\text{so that } J^3 = J^{12}, \text{ etc.}$$

→ generate to 4D: → 6 operators that generate 3 boosts, 3 rotations,

$$J^{\mu\nu} = +i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad \text{of the Lorentz group.}$$

$\{ \rightarrow$ Spatial Rotations: $J^{ik} = i(x^i \partial^k - x^k \partial^i)$

\rightarrow Lorentz boosts along x^j axis: $J^{ij} = i(x^i \partial^j - x^j \partial^i)$

\rightarrow Now, want to get commutation rules.

\rightarrow compute the commutators of differential ops

to get

$$[J^{mn}, J^{pq}] = i(g^{rp}J^{m\sigma} - g^{n\sigma}J^{m\rho} - g^{n\sigma}J^{mp} + g^{m\sigma}J^{np})$$

$$\left. \begin{array}{l} \text{Ex 1 rotations: } J^{12} = -J^{21} \\ J^{23} = -J^{32} \\ J^{13} = -J^{31} \end{array} \right\} \Rightarrow 6 \text{ total}$$

$$\left. \begin{array}{l} \text{3 boost: } J^{01} = -J^{10} \\ J^{02} = -J^{20} \\ J^{03} = -J^{30} \end{array} \right\} \text{6 Lorentz metrics...}$$

Ex Consider the 4×4 matrix $(J^{mn})_{\alpha\beta}$ where m, n label which of the 6 matrices, while α, β label the component/matrix element.

$$(J^{mn})_{\alpha\beta} = i(\delta_\alpha^m \delta_\beta^n - \delta_\alpha^n \delta_\beta^m)$$

\hookrightarrow can verify that $(J^{mn})_{\alpha\beta}$ satisfies the comm. relation...

\rightarrow These are matrices that act on ordinary Lorentz 4-vectors...

to see this...

→ Look at elements of the Lorentz group

$$U(w_{\mu\nu}) = \exp \left[-i \frac{\omega_{\mu\nu}}{2} J^{\mu\nu} \right]$$

infinitesimally $\rightarrow \sim I + \frac{-i}{2} w_{\mu\nu} J^{\mu\nu}$

$$\sim \delta_p^\alpha + \frac{-i}{2} w_{\mu\nu} (J^{\mu\nu})_p^\alpha$$

So, infinitesimally...

$$V^\alpha \rightarrow \delta_p^\alpha + \frac{-i}{2} w_{\mu\nu} (J^{\mu\nu})_p^\alpha V^\beta$$

$w_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles.

$V_\alpha, V_\beta \rightarrow 4$ -vectors..

Ex 1 When $w_{12} = -w_{21} = \theta$, $w_{\mu\nu} = 0$ else, we get

$$[V^\mu] \rightarrow \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [V^\mu]$$

→ Infinitesimal ROTATION on xy plane.

Ex 2 when $w_{01} = -w_{10} = \beta \Rightarrow$ get
 $w_{\mu\nu} = 0$ else

$$[V^\mu] \rightarrow \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} [V^\mu] \rightarrow \boxed{\text{BOOST along } x}$$

THE DIRAC EQUATION

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to develop formalism for finding all other ~~formalisms~~ representations...
(problem 3.1)

→ focus on spin $\frac{1}{2}$ systems...

→ In this case, use Dirac's trick due to -

Suppose we had a set of 4 $n \times n$ matrices γ^{μ} satisfying:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\mathbf{I}$$

Then we could write down an n -dim representation of the Lorentz algebra...

$$S^{\mu\nu} = \frac{i}{4} [\gamma^{\mu}, \gamma^{\nu}]$$

These matrices satisfy the commutation relation...

$$\rightarrow [S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

→ Verify that this trick works in 3D Euclidean space

in which case, $\gamma^0 = \gamma^5$ → $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \sum_k i \epsilon^{ijk} \sigma^k = J^i$$

Which is what we saw before as angular momentum.

$$\left\{ J^1 = S^{12} = \frac{1}{2} \sigma^3 \right\}$$

$$\left\{ J^2 = S^{31} = \frac{1}{2} \sigma^2 \right\}$$

$$\left\{ J^3 = S^{23} = \frac{1}{2} \sigma^1 \right\}$$

→ now, want S^{mn} for 4D Minkowski space...

→ Matrices γ^m must be at least 4×4 .

→ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2 \times 2} \\ \mathbb{I}_{2 \times 2} & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

"Weyl" / "Chiral" representations.

→ In this case, the boost + rotation generators are ..

Boots
in

$$S^{0i} = \frac{i}{4} [\gamma^0, \gamma^i] = -\frac{i}{2} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Rotations
in

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \sum_k \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \sum_k \frac{1}{2} \epsilon^{ijk} \sum_l \sigma^l$$

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at all transformations under which the ~~law~~ ^{Action's} ~~gives~~ ~~it~~ is invariant

→ In particular, we want \mathcal{S} to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\left\{ \begin{array}{l} \phi(x) \rightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x^\mu) \rightarrow \phi(\Lambda^{\mu\nu} x^\nu). \end{array} \right. \rightarrow \begin{array}{l} \text{check that} \\ \mathcal{S} \text{ is invariant} \end{array}$$

→ But this is very simple ... ⇒ There are many more transformations that leave \mathcal{S} Lorentz invariant.

→ How do we find all of them?

→ For simplicity, we'll just restrict ourselves to linear combinations of transformations

→ Look at transformations of the form

$$\phi_a(x) \rightarrow \sum_b M_{ab}(\Lambda) \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \rightarrow M(\Lambda) \phi}$$

These matrices M must be "nice" in the sense that M must obey...

Ex

$$\boxed{\phi \rightarrow M(\Lambda') M(\Lambda) \phi = M(\Lambda' \Lambda) \phi}$$

This says that $\{M\}$ (the collection of M 's) must be a representation of the Lorentz group.

What?? So, recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

 \rightarrow

$$\boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

of a group to

A representation Π is a function π satisfying the property

$$\pi(g_1) \pi(g_2) = \pi(g_1 g_2)$$

 \uparrow \uparrow \uparrow g_1 $\in G$ $\in G$

With this, it is clear that

$$\boxed{\{\Lambda\} \text{ Lorentz group} \Rightarrow \{M\} \text{ is a representation of } \{\Lambda\}}$$

So... what are these M ?

 \rightarrow Ex

Rotation group for spin $1/2$ particles

For spin - $\frac{1}{2}$, the most important nontrivial representation is the 2D representation:

→ These are unitary matrices with $\det = 1$
 (2×2)

$$\Rightarrow \text{In general: } U = e^{-i \vec{\sigma} \cdot \vec{\theta}/2}$$

$\vec{\sigma}$ → Pauli matrices
 $\vec{\theta}$ → angle.

For infinitesimal rotations, we can write

$$U = I - i \frac{\vec{\sigma}}{\hbar} \cdot \vec{\theta} = I - \vec{\tau} \cdot \vec{\theta}$$

{U} form a Lie-algebra of the L-group.

$\vec{\tau}$ here are the "generators" of the Lie algebra

when {U} is a representation of the rotational group, we identify

$$\vec{\tau} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{\tau}$ is the quantum angular momentum operator

→ satisfies the commutation relation

$$[\vec{\tau}^i, \vec{\tau}^j] = i \epsilon^{ijk} \vec{\tau}^k$$

like the generators of $SO(3)$, namely the Pauli matrices -

→ finite rotations are formed by matrix exp.

$$R = \exp \left[-i \theta^i \hat{J}^i \right]$$

\longleftarrow \rightarrow θ^i

Sep 27, 2020

back to present problem...

to get generator of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

$$(3D) \quad \vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\sigma})$$

$$(4D) \quad \boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})}$$

→ any matrices that are to represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu)}$$

→ by symmetry, μ, ν take label which of the six matrices we want;

→ α, β label components.

The Dirac Eqn.

What are the representations of the Lorentz group?
especially for spin- $\frac{1}{2}$?

Dirac's trick: if we have a set of $4 \times n \times n$ matrices γ^μ which satisfies:

Dirac algebra

$$\rightarrow \boxed{\{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\gamma^{\mu\nu} \star I_{n \times n}}$$

Then the n -dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4} \{ \gamma^\mu, \gamma^\nu \}}$$

\rightarrow In other words, $S^{\mu\nu}$ satisfies:-

$$[\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = i(g^{\mu\rho} S^{\nu\sigma} - g^{\nu\rho} S^{\mu\sigma} - g^{\mu\sigma} S^{\nu\rho} + g^{\nu\sigma} S^{\mu\rho})$$

* Note that this trick works also in any dim.

e.g. take $\gamma^0 = i\sigma^3$ so that $\{ \gamma^i, \gamma^j \} = -2\delta^{ij}$

$$\Rightarrow \boxed{S^{ij} = \frac{1}{2} \epsilon^{ijk} S^k} \rightarrow \text{just as before.}$$

2D representation of the rotation group.

$$\text{Spin } \frac{1}{2}: J^1 = J^{12} = \frac{1}{2}\sigma^3; J^2 = \frac{1}{2}\sigma^2 = S^{21}; J^3 = S^{23} = \frac{1}{2}\sigma^1$$

One such representation for the Dirac algebra is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4 \times 4}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4 \times 4}$$

Weyl / chiral representation.

get

Boosts $S^{0j} = \frac{i}{4} [\gamma^0, \gamma^j] = \frac{-i}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} 0 & \sigma^k \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \epsilon^{ijk} \epsilon^{lk}$$

Hermitian Def'n

not ~~rotation~~
but Ψ is also

classical

field, not a
wfn

All 4-component field Ψ that transforms under
boosts + rotations according to is called
a Dirac spinor

S^{ij} are Hermitian

S^{0i} are anti-Hermitian

Since b/c Ψ is a classical field, not a wfn.

Now, what is the field eqn for ψ ?

→ try $(\square + m^2)\psi = 0 \leftarrow \text{KG field eqn.}$

But this obviously works because the representations are block-diagonal...

→ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~at~~ at transformation of δ matrices

In an expression we can think of...

$$[\dots] \Delta_{\frac{1}{2}} [\dots]_{4 \times 4} \Delta_{\frac{1}{2}} [\dots] \xrightarrow{\frac{1}{2} \text{ for spin } \frac{1}{2}}$$

where $\Delta_{\frac{1}{2}} = \exp \left\{ \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right\}$

$$\simeq 1 - \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}$$

$$\Rightarrow [\gamma^1] \rightarrow [\Delta_{\frac{1}{2}}] [\gamma^1] [\Delta_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right) \gamma^1 \left(1 - \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right)$$

$$= \dots \quad (\text{some terms of higher order cancelled...})$$

$$= \gamma^1 - \frac{i}{2} \omega_{\alpha\beta} \underbrace{[\gamma^1, S^{\alpha\beta}]}_{?}$$

above a quick computation shows that

$$[\gamma^\mu, \gamma^\nu] = (\gamma^{\mu\nu})_\nu \gamma^\nu$$

where

$$\gamma^{\mu\nu} = i(g^{\mu\nu}\gamma_5 - g^{\nu\mu}\gamma_5)$$

So ...

$$\boxed{\gamma^\mu \rightarrow \gamma^\mu - \frac{i}{2} \omega_{\alpha\beta} (\gamma^{\alpha\beta})_\nu \gamma^\nu = \Gamma_{\frac{1}{2}} \gamma^\mu \Gamma_{\frac{1}{2}}}$$

$\rightarrow \gamma^\mu$ transforms like 4-vectors ... !

$\Rightarrow \gamma^\mu$ are invariant under simultaneous rotations of
their vectors & spinor indices.

I can treat " μ " or γ^μ as a vector index!

\rightarrow can dot γ^μ into ∂_μ to form a Lorentz-

inv. differential operator ...

Dine eqn

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

check that this is Lorentz-inv:

Lit $\psi(x) \rightarrow \Gamma_{\frac{1}{2}} \psi(\Gamma' x)$ then

$$i\gamma^\mu \partial_\mu \psi \rightarrow (i\gamma^\mu \Gamma_{\frac{1}{2}}) \partial_\mu (\psi(\Gamma' x))$$

$$= i\Gamma_{\frac{1}{2}} (\Gamma_{\frac{1}{2}} \gamma^\mu \Gamma_{\frac{1}{2}}) \cdot (\Gamma')^\mu_\mu (\partial_\mu \psi)(\Gamma' x)$$

some Lorentz transform

$$\begin{aligned}
 &= i \Delta_{\frac{1}{2}} (\Delta)^{\mu}_{\nu} \gamma^{\nu} \cdot (\Delta)_{\mu}^{\alpha} (\partial_2 \psi)(\Delta' x) \\
 &= i \Delta_{\frac{1}{2}} \gamma^{\mu} \underbrace{(\Delta)^{\mu}_{\nu} (\Delta)_{\nu}^{\alpha}}_{\delta^{\alpha}_{\nu}} (\partial_2 \psi)(\Delta' x) \\
 &= i \Delta_{\frac{1}{2}} \gamma^{\mu} \partial_{\mu} \psi(\Delta' x)
 \end{aligned}$$

$$\Rightarrow i \gamma^{\mu} \partial_{\mu} \psi(x) \rightarrow \Delta_{\frac{1}{2}} i \gamma^{\mu} \psi(\Delta' x)$$

→ transforms the same way as $\psi(\Delta' x)$

Cleaner way:

$$\begin{aligned}
 \text{Let } [i \gamma^{\mu} \partial_{\mu} - m] \psi(x) &\rightarrow [\overbrace{i \gamma^{\mu} (\Delta')_{\mu}^{\nu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \psi(\Delta' x)] \Delta_{\frac{1}{2}} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \Delta_{\frac{1}{2}}^{-1} [\overbrace{i \gamma^{\mu} (\Delta')_{\mu}^{\nu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \psi(\Delta' x)] \Delta_{\frac{1}{2}} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \left\{ i \gamma^{\mu} \overbrace{(\Delta')_{\mu}^{\nu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \right\} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \left\{ i \gamma^{\nu} \partial_{\nu} - m \right\} \psi(\Delta' x) \\
 &= 0 \quad \checkmark
 \end{aligned}$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i \gamma^{\mu} \partial_{\mu} - m) \psi$$

$$\begin{aligned}
 \rightarrow 0 &= (-i \gamma^{\mu} \partial_{\mu} - m) (+i \gamma^{\nu} \partial_{\nu} - m) \psi \\
 &= (\gamma^{\mu} \gamma^{\nu} \partial_{\nu} \partial_{\mu} + m^2) \psi = ...
 \end{aligned}$$

$$\begin{aligned}
 &= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2) \psi \\
 &= \left[\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2 \right] \psi \\
 &= \left[\frac{1}{2} \{ \gamma^\mu \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right] \psi \quad \xrightarrow{\text{LG eqn.}} \\
 &= (g^{\mu\nu} \partial_\mu \partial_\nu + m^2) \psi = (D + m^2) \psi = 0
 \end{aligned}$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger \psi$ doesn't work b/c under a boost,

$$\psi^\dagger \frac{1}{\sqrt{2}} \Delta_{\frac{1}{2}} \psi \neq \psi^\dagger \psi \text{ since } \frac{1}{\sqrt{2}} = \exp \left\{ -i \omega \gamma^\mu S^\mu \right\}$$

not unitary ... since not all $S^{\mu\nu}$ are Herms.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, set

$$\bar{\psi} \rightarrow \bar{\psi} \frac{1}{\sqrt{2}} \gamma^0 \simeq \bar{\psi} \left(1 + i \frac{\omega}{2} \gamma_\mu (S^{\mu\nu})^\dagger \right) \gamma^\nu$$

when ~~assume~~ $\omega \neq 0 \Rightarrow \gamma \neq 0$, $(S^{\mu\nu})^\dagger = (S^{\mu\nu})$

$$\therefore (S^{\mu\nu} \leftrightarrow \gamma^0)$$

When $\mu=0$ or $\nu=0$, $(S^{\mu\nu})^+ = -S_{\mu\nu}^\mu$

$S^{\mu\nu}$ anti-commutes w/ γ^0 .

$$\rightarrow \bar{\psi} \rightarrow \psi^+ \left(1 + \frac{i}{2} \gamma_\mu \nu (S^{\mu\nu})^+ \right) \gamma^0$$

$$= \underbrace{\psi^+}_{\gamma^0} \gamma^0 \left(1 + \frac{i}{2} \gamma_\mu \nu S^{\mu\nu} \right)$$

$$= \bar{\psi} \left(1 + \frac{i}{2} \gamma_\mu \nu S^{\mu\nu} \right) = \bar{\psi} \gamma_1^{-1} \text{ as desired.}$$

$$\rightarrow \boxed{\bar{\psi} \rightarrow \bar{\psi} \gamma_1^{-1}}$$

and so $\boxed{\bar{\psi} \psi = \psi^+ \gamma^0 \psi}$ is a Lorentz scalar.

Similarly, can show that

$$\boxed{\bar{\psi} \gamma^\mu \psi}$$
 is a Lorentz vector.

\rightarrow the correct Lorentz-invariant Dirac Lagrangian is

$$\boxed{L_{\text{Dirac}} = \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi}$$

{-L eqn for $\bar{\psi}$ gives $(\gamma^\mu \partial_\mu - m) \psi = 0$

{-L eqn for ψ gives $-i \partial_\mu \bar{\psi} \gamma^\mu - m \bar{\psi} = 0$

WEYL SPINOR

Recall that

$$\begin{aligned} S^{0j} &= \frac{-i}{2} \begin{pmatrix} \sigma^i & \alpha \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & \alpha \\ 0 & \sigma^k \end{pmatrix} \end{aligned}$$

Since block-diagonal \Rightarrow Dirac representation of the Lorentz group is reducible.

\rightarrow Can form 2-D representations by considering each block separately.

$$\rightarrow \text{write } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{\text{left-handed Weyl spinors}}$$

Under infinitesimal boost $\vec{\beta}$ + rotation $\vec{\theta}$, these transform as

$$\begin{aligned} \psi_L &\rightarrow \left(1 - i \vec{\theta} \cdot \vec{\sigma}/2 - i \vec{\beta} \cdot \vec{\sigma}/2 \right) \psi_L \\ \psi_R &\rightarrow \left(1 - i \vec{\theta} \cdot \vec{\sigma}/2 + i \vec{\beta} \cdot \vec{\sigma}/2 \right) \psi_R \end{aligned}$$

Recall that $(\tanh(\vec{\beta}) = \frac{1+i}{i})$.

\rightarrow Transform of ψ_R is equiv to trans of ψ_L^\dagger

By writing that

$$\psi_L^* \rightarrow \left(1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}^*}{2} - \vec{\beta} \cdot \frac{\vec{\sigma}^*}{2} \right) \psi_L^*$$

noting that $\vec{\sigma}^2 \vec{\sigma}^* = -\vec{\sigma} \vec{\sigma}^*$ ($\vec{\sigma}^2 = \vec{\sigma} \cdot \vec{\sigma}$)

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we find.

$$\underbrace{\vec{\sigma}^2 \psi_L^*}_{\psi_L^*} \rightarrow \vec{\sigma}^2 \left[1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}}{2} - \vec{\beta} \cdot \frac{\vec{\sigma}^*}{2} \right] \psi_L^*$$

$$= \underbrace{\left[1 - i\vec{\sigma} \cdot \frac{\vec{\sigma}}{2} + \vec{\beta} \cdot \frac{\vec{\sigma}}{2} \right]}_{\text{like } \psi_R \text{ transform.}} \psi_L^*$$

$\underline{\text{So }} \vec{\sigma}^2 \psi_L^* \text{ transform like } \psi_R \dots$

With $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, the Dirac eqn has form.

$$(i\vec{\sigma}^m \partial_m - m) \Psi = 0 \Leftrightarrow \begin{pmatrix} -m & i(\vec{\sigma}_0 + \vec{\sigma} \cdot \vec{\beta}) \\ i(\vec{\sigma}_0 - \vec{\sigma} \cdot \vec{\beta}) & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

When $m=0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left\{ \begin{array}{l} i(\vec{\sigma}_0 - \vec{\sigma} \cdot \vec{\beta}) \psi_L = 0 \\ i(\vec{\sigma}_0 + \vec{\sigma} \cdot \vec{\beta}) \psi_R = 0 \end{array} \right\} \rightarrow \underline{\text{Welfl eqns.}}$$

\rightarrow important for neutrinos & weak force studies..

For convenience let us define -

$$\sigma^{\mu} = (1, \vec{\sigma}) ; \quad \bar{\sigma}^{\mu} = (1, -\vec{\sigma})$$

So that $\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$ $\sigma^{\mu} = (1, \vec{\sigma}, \vec{\sigma}^2, \vec{\sigma}^3)$

With this, can simply rotation. Dirac eqn becomes -

$$\begin{pmatrix} -m & i\vec{\sigma} \cdot \vec{\alpha} \\ i\vec{\sigma} \cdot \vec{\alpha} & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$i(\vec{\alpha} + \vec{\sigma} \cdot \vec{\nabla}) \psi_L = 0$$

$$i(\vec{\alpha} - \vec{\sigma} \cdot \vec{\nabla}) \psi_R = 0$$

∴ the Weyl eqns become :

$$(i\vec{\sigma} \cdot \vec{\alpha}) \psi_L = 0$$

$$(i\vec{\sigma} \cdot \vec{\alpha}) \psi_R = 0$$

A hint

$$p^* = \sqrt{p^2 + m^2} = E_p$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p) e^{-ip \cdot x} , \quad p^2 = m^2$$

Look only solutions with positive frequency ... that is
 $E_p = p^0 > 0 \dots$

Ψ solves Dirac eqn $\rightarrow (i\gamma^\mu \partial_\mu - m)\Psi = 0$

$$\rightarrow (i\gamma^\mu \partial_\mu - m) u(p) e^{-ip \cdot x} = 0$$

$$\rightarrow \boxed{(i\gamma^\mu p_\mu - m) u(p) = 0}$$

Get rest frame $\Rightarrow p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $\Lambda_{\frac{1}{2}}$.

In rest frame, we have

$$(i\gamma^\mu p_\mu - m) u(p) \rightarrow (m\gamma^0 - m) u(p_0) = m(\gamma^0 - 1) u(p_0) = 0$$

$$\Rightarrow m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$$

$$\rightarrow \boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \xrightarrow{\text{two-component spinor}}}$$

just a factor ξ with norm. constraint.

$$\xi^\dagger \xi = 1,$$

$\cancel{\alpha}$

What are those ξ ?

Look at rotation generators ...

$$\boxed{s^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

$$\text{In particular, } S^2 = S'^2 = \frac{1}{2} \begin{pmatrix} 6^2 & 0 \\ 0 & 0^2 \end{pmatrix}$$

$$\text{So if } \left\{ \begin{array}{l} S = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ S = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$$

$$\text{Now, we're in rest frame, so } p' = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, boost to frame where particle has velocity ...

$$\vec{v} = v \cdot \hat{z} \cdot \circ \quad \text{Let } \tanh(\eta) = \frac{v}{c}.$$

↗ "rapidity"

$$\text{Then } \begin{pmatrix} E \\ p^3 \end{pmatrix} = p' = \begin{pmatrix} \cosh \eta & 0 & 0 & \sinh \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{minh})$$

(infinitesimal $\frac{1}{2}$)

$\frac{1}{2} \rightarrow$ just the Lorentz transform.

$$\rightarrow \text{In this frame, } \left\{ \begin{array}{l} E = m \cosh \eta \\ p^3 = m \sinh \eta \end{array} \right.$$

Now, apply the same boost to $\alpha(p)$...

$$\begin{aligned} \alpha(p) &= \frac{1}{2} \sqrt{m} \begin{pmatrix} s \\ s \end{pmatrix} \rightarrow \left(\frac{1}{2} \right) = \exp \left(\frac{-i}{2} \omega_{\mu\nu} S^{\mu\nu} \right) \\ &= \exp \left(\frac{-i}{2} \gamma \begin{pmatrix} 0^3 & 0 \\ 0 & -0^3 \end{pmatrix} \right) \sqrt{m} \begin{pmatrix} s \\ s \end{pmatrix} \\ &\quad \text{~} \uparrow i \cdot (0^3 - s) \end{aligned}$$

So, infinitesimally -

$$\exp \left\{ \frac{-i}{2} \eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right\} \approx \begin{pmatrix} \cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3 & 0 \\ 0 & \cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3 \end{pmatrix}$$

So Rest

$$u(p) \approx \sqrt{m} \begin{pmatrix} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) \xi \\ (\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) \xi \end{pmatrix} \quad \text{---}$$

Simplify ... note that

$$\begin{aligned} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3)^2 &= \dots \\ &= \cosh \eta - \sinh \eta \sigma^3 \\ &= \frac{E}{m} - \frac{P^3}{m} \sigma^3 = \frac{p \cdot \sigma}{m} \\ &= \frac{p^{\mu} \sigma^{\mu}}{m} \quad \text{where } \sigma^{\mu} = (1, \vec{\sigma}) \end{aligned}$$

So ... $\{ (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \sigma}{m}}$

and $(\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \bar{\sigma}}{m}}$

So -
$$\boxed{u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}} \rightarrow \text{current = valid for any arbitrary direction of } p.$$

Fact
$$\{(p \cdot \sigma)(p \cdot \bar{\sigma}) = (p^0)^2 - \vec{p}^2 = p^2 = m^2\}$$

(61)

Now, back to example

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow p \cdot \sigma = \dots = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$$

$$\sqrt{p \cdot \sigma} = \begin{pmatrix} \sqrt{E - p^3} & 0 \\ 0 & \sqrt{E + p^3} \end{pmatrix}$$

and

$$\sqrt{p \cdot \bar{\sigma}} = \begin{pmatrix} \sqrt{E + p^3} & 0 \\ 0 & \sqrt{E - p^3} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E - p^3} (1) \\ \sqrt{E + p^3} (0) \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then (spin $-\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E + p^3} (0) \\ \sqrt{E - p^3} (1) \end{pmatrix}$$

In the massless limit, $E \rightarrow p^3$ ($E^2 = \sqrt{mc^2 + (p^3)^2}$)

$$\Rightarrow \boxed{u(p) = \begin{pmatrix} (0) \\ \sqrt{2E} (1) \end{pmatrix} \text{ spin } \frac{1}{2}}$$

$$\boxed{u(p) = \begin{pmatrix} \sqrt{2E} (0) \\ (0) \end{pmatrix} \text{ spin } -\frac{1}{2}}$$

These states: $u(p) = \sqrt{2E} \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$, $u(p) = \sqrt{2E} \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$ are eigenstates of the helicity operator

$$\boxed{h = \vec{p} \cdot \vec{S} = \sum_i \frac{1}{2} p_i^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}} = \frac{1}{2} \vec{p} \cdot \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

When $\{ h = \frac{1}{2} \Rightarrow \text{call Right-handed}$

$\{ h = -\frac{1}{2} \Rightarrow \text{call Left-handed}$

Note: Dirac helicity is frame-dependent... (for massive particle). — since can boost so that momentum is in the opposite direction,

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\left\{ \begin{array}{l} i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) \psi_L = i(\vec{\sigma} \cdot \vec{\partial}) \psi_L = 0 \\ i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \psi_R = i(\vec{\sigma} \cdot \vec{\partial}) \psi_R = 0 \end{array} \right.$$

Plug $\psi = u(p) e^{-ip \cdot x} \sim$, $\partial_0 \rightarrow -iE$

$$\vec{\nabla} \rightarrow i\vec{p}$$

↓, with $m=0$, $\tilde{p} = E\vec{p}$.

$$\Rightarrow h = \frac{-1}{2}$$

$$\Rightarrow \text{get } \left\{ (E + E\vec{p} \cdot \vec{\sigma}) \psi_L = 0 \Rightarrow (E)(1+2h) \psi_L = 0 \right.$$

$$\left. (E - E\vec{p} \cdot \vec{\sigma}) \psi_R = 0 \Rightarrow (E)(1-2h) \psi_R = 0 \right. \Rightarrow h = \frac{1}{2}$$

$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}$, as expected

#

Recap... $\psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 > 0) \rightarrow \text{positive frequency}$

 $\Rightarrow u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix} \rightarrow \text{spinor.}$

when $\psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 < 0) \rightarrow \text{negative frequency}$

$$\Rightarrow (u(p)) = \dots \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ -\sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix}$$

#

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = (\xi^+ \sqrt{p \cdot \sigma} \xi^+ \sqrt{p \cdot \bar{\sigma}}) \cdot \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix}$$

$$= \xi^+ \left[(p \cdot \sigma) + (p \cdot \bar{\sigma}) \right] \xi$$

$$\Rightarrow u^\dagger u = 2E_p \xi^+ \xi \quad \xrightarrow{\text{depends on } p!}$$

\sim ~~also~~ $u^\dagger u$ is not a Lorentz-inv scalar.
just like $\psi^\dagger \psi$.

\Rightarrow to make one such Lorentz-inv scalar, define

$$\bar{u}(p) = u^\dagger(p) \gamma^0$$



$$\bar{u}u = 2m \xi^+ \xi \quad \begin{matrix} \text{Lorentz-inv} \\ (\text{indep of } \vec{p}) \end{matrix}$$

$$\text{L}, \text{ wish after } \bar{u}n = u^r \gamma^0 n = 2m \xi^+ \xi^- = 2m$$

→ convenient to choose ONB spinors, ξ^1, ξ^2 .

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ \sqrt{p \cdot \bar{\sigma}} & \xi^s \end{pmatrix} \quad s=1,2}$$

Normalize:

$$\boxed{\bar{u}^r(p) u^s(p) = 2m \delta^{rs} \Leftrightarrow u^{r\dagger}(p) u^s(p) = 2E_p \delta^{rs}}$$

For the negative-freq solns, we get

$$\boxed{\bar{v}^r(p) v^s(p) = -2m \delta^{rs} \Leftrightarrow v^{r\dagger}(p) v^s(p) = +2E_p \delta^{rs}}$$

and

v, u are orthogonal to each other...

$$\boxed{\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0}$$

†

Finally, talk about spin sumrs

→ useful when evaluating Feynman diagrams.

→ when we need to sum all spin- $\frac{1}{2}$ polarizations

Since $\{\xi^s\}$ form an ONB,

$$\sum_{s=1,2} \xi^s \xi^{s*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\begin{aligned} \sum_{s=1,2} n^s(p) \bar{n}^s(p) &= \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \sigma} \xi^s} \right) \cdot \left(\xi^{s*} \sqrt{p \cdot \sigma}, \xi^{s*} \sqrt{p \cdot \sigma} \right) \\ &= \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \sigma} \xi^s} \right) \cdot \left(\xi^{s*} \sqrt{p \cdot \sigma}, \xi^{s*} \sqrt{p \cdot \sigma} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \xrightarrow{\text{"completeness"}} &= \begin{pmatrix} p \cdot \sigma & m \\ m & p \cdot \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} &= \sqrt{(p \cdot \sigma + p \cdot \sigma - p \cdot \sigma + p \cdot \sigma) m m} \\ &= \sqrt{(p \cdot \sigma)(p \cdot \sigma)} = \sqrt{((p^0, \vec{p}) \cdot (1, -\vec{\sigma})) ((p^0, \vec{p}) \cdot (1, \vec{\sigma}^2))} \\ &= \sqrt{(p^0)^2 - p^2} = m. \end{aligned}$$

$\frac{p}{\cancel{p}}$

$$\boxed{\sum_{s=1,2} n^s(p) \bar{n}^s(p) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix} = p \cdot \gamma + m I} \quad \begin{array}{l} \text{Feyn-} \\ \text{man's} \\ \text{slash} \\ \text{notation} \end{array}$$

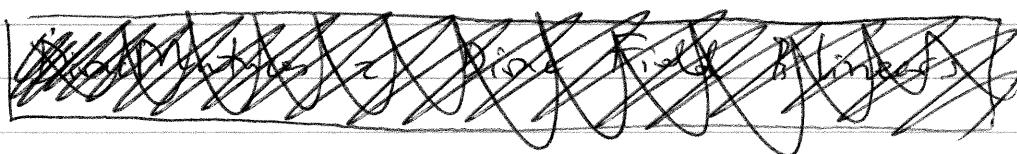
$\frac{p}{\cancel{p}}$

$$\boxed{\sum_{s=1,2} n^s(p) \bar{n}^s(p) = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \sigma & -m \end{pmatrix} = p \cdot \gamma - m I}$$

→ The combos $\partial \cdot p$ occur so often that Feynman introduced the notation:

$$\not{p} = \partial^\mu p_\mu = p_\mu \partial^\mu$$

#

Exercise

Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L .
The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

m = Majorana mass.

- (a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.
- (b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

a) In general, infinitesimal Lorentz transform on $\Psi_L(x)$ has the form

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} + \vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

→ Rotation has the form:

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

$$\Rightarrow \sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L(x) \quad (\text{notes})$$

Lorentz transformed:

$$\Psi_L(x) \rightarrow \Lambda_{\frac{1}{2}} \Psi_L(\Lambda^{-1}x)$$

$$\partial_\mu \Psi_L(x) \rightarrow (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x)$$

→ put these together ...

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(\Lambda^{-1}x)$$

$$\sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(\Lambda^{-1}x)$$

$$\Rightarrow -im \sigma^2 \Psi_L^*(x) \rightarrow -im \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(\Lambda^{-1}x)$$

$$\text{Next, } i\vec{\sigma} \cdot \partial \Psi_L(x) = i\vec{\sigma}^\mu \partial_\mu \Psi_L(x)$$

$$\Rightarrow i\vec{\sigma}^\mu (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x) \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right)$$

$$= i\vec{\sigma}^\mu \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x)$$

we find: multiply:

$$1 = \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \left(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \quad (\text{rot} \times \text{inv-rot})$$

$$\Rightarrow \rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\mu (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \\ \times (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \boxed{?} (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

Want is $\boxed{?}$

$$\rightarrow \boxed{?} = (1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\mu (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})$$

$$\approx \bar{\sigma}^\mu + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} \bar{\sigma}^\mu - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} \bar{\sigma}^\mu$$

$$= \bar{\sigma}^\mu - \frac{i}{2} \vec{\theta} [\bar{\sigma}^\mu, \frac{\vec{\sigma}}{2}]$$

\downarrow
 \downarrow can show want

$$= \bar{\sigma}^\mu - i\vec{\theta} [J_\mu^{\alpha\beta}] \bar{\sigma}^\nu$$

\downarrow

$$i(g^{\mu\nu} \delta_\nu^\alpha - g^{\mu\nu} \delta_\nu^\alpha)$$

$$\Rightarrow \boxed{?} = (\Delta_4)^\mu_\nu \bar{\sigma}^\nu \rightarrow \bar{\sigma}^\mu transforms like 4-vector$$

$$\Rightarrow i\vec{\theta} \cdot \partial \Psi_L(x) \Rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \Delta_\nu^\mu \bar{\sigma}^\nu (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \delta_\nu^\alpha \bar{\sigma}^\nu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\nu \partial_\nu \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma} \cdot \partial \Psi_L(\Delta' x)$$

✓

$$\Rightarrow i\bar{\sigma} \cdot \partial \psi_c(x) - im \bar{\sigma}^2 \psi_c^*(x) = 0$$

\rightarrow leads to infinitesimal solution ...

$$(1 - i\tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \underbrace{\{ i\bar{\sigma} \cdot \partial \psi_c(\bar{x}) - im \bar{\sigma}^2 \psi_c^*(\bar{x}) \}}_{=0} = 0$$

\Rightarrow done! So Majorana eqn is invariant under infinitesimal rotations.

\rightarrow

① Bosons (proceed in a similar way ...)

Key

$$(1 - \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \tilde{\beta} \cdot \frac{\vec{\sigma}}{2})$$

$$= \bar{\sigma}^M - \frac{1}{2} \tilde{\beta} \{ \bar{\sigma}^M, \bar{\sigma} \}$$

$$\Rightarrow \bar{\sigma}^M \rightarrow \bar{\sigma}^M - i\tilde{\beta} [\bar{\sigma}^M]_r \bar{\sigma}^r$$

$$\Rightarrow (1 - \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) = (1)_r \bar{\sigma}^r \text{ as before}$$

so plug this back into the original eqn.

$\cancel{*}$