8.422 AMO II Lecture Notes

Chapter 5

Optical Bloch Equations

Until now, our discussion has primarily been about *closed* quantum systems, which evolve unitarily. Specifically, we considered single and multiple component systems, such as atoms and photons, but we kept every part, and never threw anything away.

However, in real systems, we often want to be able to disregard certain microscopic dynamics, or do not have access to certain parts of a system. For example, we may want to cool an atom, but will not keep track of the microscopic state of the cooling laser beam after it has interacted with the atom. Or an atom may interact with the vacuum, emitting a photon which we do not track.

The study of such topics is the subject of open quantum system dynamics. This chapter begins our investigation of open quantum systems, with the goal of developing a fully quantum-mechanical model of an atom undergoing spontaneous emission, while interacting with a classical electromagnetic field.

5.1 Derivation

The *master equation* is an equation of motion for a density matrix describing an open quantum system, much like the Schrodinger equation describes the evolution of a closed quantum system. This section provides a derivation of the master equation for a spontaneously emitting atom, driven by a classical field, which is known as the Optical Bloch Equation.

5.1.1 Classical model of atom and field

A good starting point, to appreciate the problem of open quantum systems, is the classical model for a two-level atom coupled to a black-body electromagnetic field, the Einstein rate equations

$$\frac{dN_g}{dt} = A_{eg}N_e + u(\omega_{eg})\left[B_{eg}N_e - B_{ge}N_g\right] \tag{5.1.1}$$

$$\frac{dN_e}{dt} = -A_{eg}N_e + u(\omega_{eg})\left[B_{ge}N_g - B_{eg}N_e\right], \tag{5.1.2}$$

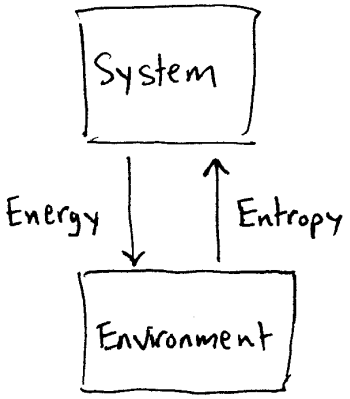
where A_{eg} is the spontaneous emission rate, B_{ge} and B_{eg} are stimulated emission rates, and $u(\omega_{eg})$ is the field energy density at atomic frequency ω_{eg}, for levels denoted by $|e\rangle$ and $|g\rangle$.

Is there a straightforward quantum analogue of this? We might be tempted to simply add a damping term to the Schrödinger equation, like

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle - i\Gamma|\psi\rangle\,, \tag{5.1.3}$$

but this is not physically allowed by quantum mechanics!

How, then, can we construct a fully quantum-mechanical description of open system dynamics? The key concept is that we must properly account for *noise*:



From classical thermodynamics, we know that any time there is energy transfer from a system to the environment, there is entropy exchanged back from the environment to the system. This is a simple illustration of the very basic *fluctuation–dissipation* principle: there can be no relaxation without con-commitment noise! To study quantum open system, we must model the appropriate quantum noise contribution which goes along with relaxation.

5.1.2 Density matrices and closed system dynamics

The main tool we shall use to model open quantum systems is the density matrix representation for quantum states, so it is helpful to begin with a review of density matrices and how they evolve under Hamiltonian dynamics.

Recall that a density matrix ρ for a pure state $|\psi\rangle$ is the matrix $|\psi\rangle\langle\psi|$. Thus, for example

$$|0\rangle \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{5.1.4}$$

$$|1\rangle \to \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.1.5}$$

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \to \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \tag{5.1.6}$$

Density matrices may also represent *statistical mixtures* of pure states; this state

$$\frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.1.7}$$

can be interpreted as a 50/50 mixture of $|0\rangle$ and $|1\rangle$. However, one must be careful, because there are infinitely many ways to *unravel* a density matrix into statistical mixtures of pure states. For example,

$$\rho = \frac{1}{4}\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.1.8}$$

is

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|\,, \tag{5.1.9}$$

but it is also

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|\,, \tag{5.1.10}$$

where

$$|a\rangle = \frac{\sqrt{3}|0\rangle + |1\rangle}{2} \tag{5.1.11}$$

$$|b\rangle = \frac{\sqrt{3}|0\rangle - |1\rangle}{2}\,. \tag{5.1.12}$$

In general, a density matrix ρ may always be written as a statistical mixture of pure states,

$$\rho = \sum_k p_k |\psi_k\rangle\,, \tag{5.1.13}$$

where p_k are probabilities, such that $\sum_k p_k = 1$.

A matrix ρ is a valid density matrix if and only if the eigenvalues of ρ are non-negative, and sum to one, such that $\mathrm{Tr}(\rho) = 1$. ρ represents a pure state if and only if $\mathrm{Tr}(\rho^2) = 1$.

How does a density matrix evolve in a closed system? From the Schrodinger equation

$$i\hbar\partial_t|\psi\rangle = H|\psi\rangle\,, \tag{5.1.14}$$

it follows that a pure state density matrix $\rho = |\psi\rangle\langle\psi|$ evolves as

$$\dot{\rho} = |\dot{\psi}\rangle\langle\psi| + |\psi\rangle\langle\dot{\psi}| \tag{5.1.15}$$

$$= -\frac{i}{\hbar}[H, \rho]\,. \tag{5.1.16}$$

For example, if ρ is a two-level atom evolving under the classical field Jaynes-Cummings Hamiltonian

$$H = \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \Omega(|g\rangle\langle e| + |e\rangle\langle g|)\,, \tag{5.1.17}$$

which we may express using Pauli matrices as

$$H = \frac{\hbar\omega_0}{2} Z + \Omega X \,, \tag{5.1.18}$$

then for

$$\rho = \begin{bmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{bmatrix} \,, \tag{5.1.19}$$

the equation of motion for ρ is

$$\dot{\rho} = \begin{bmatrix} i\Omega(\rho_{eg} - \rho_{ge}) & i\omega_0\rho_{ge} - i\Omega(\rho_{ee} - \rho_{gg}) \\ -i\omega_0\rho_{eg} + i\Omega(\rho_{ee} - \rho_{gg}) & i\Omega(\rho_{eg} - \rho_{ge}) \end{bmatrix} \,. \tag{5.1.20}$$

We can recognize this as a rotation of the Bloch sphere about the axis defined by

$$\boldsymbol{n} = \frac{\hbar\omega_0}{2}\hat{z} + \Omega\hat{x} \,, \tag{5.1.21}$$

by using the Bloch sphere representation for a density matrix,

$$\rho = \frac{I + \boldsymbol{r}\cdot\sigma}{2} \,. \tag{5.1.22}$$

5.1.3 Density matrices and open system dynamics: approach

We can build a mathematical model for quantum open system dynamics, based on four basic ideas:

1. **Density matrix evolution**

   Instead of pure states (eg $|\psi\rangle$), we describe the system state using a density matrix ρ. The equation of motion for ρ is

   $$\dot{\rho} = \mathcal{L}[\rho] \,, \tag{5.1.23}$$

   where $\mathcal{L}$ is known as the Liouvillian operator (or a "superoperator"). For example, for Hamiltonian evolution, we have:

   $$i\hbar\partial_t|\psi\rangle = H|\psi\rangle \tag{5.1.24}$$

   which, for $\rho = |\psi\rangle\langle\psi|$ gives

   $$\begin{aligned} \partial_t\rho &= |\dot{\psi}\rangle\langle\psi| + |\psi\rangle\langle\dot{\psi}| && (5.1.25)\\ &= -\frac{i}{\hbar}H|\psi\rangle\langle\psi| + |\psi\rangle\langle\psi|\frac{i}{\hbar}H && (5.1.26)\\ &= -\frac{i}{\hbar}[H,\rho] \,. && (5.1.27) \end{aligned}$$

   This is unitary evolution, but in general, the differential equation for ρ can describe *non-unitary* evolution. Such differential equations are known as "master equations." They are nontrivial to construct, because they must restrict ρ to be a legitimate density matrix at all times.

2. **Partial trace**

   We are interested in the state of the system alone, and want to disregard the state of the environment. If ρ_{total} is the state of the whole system + environment, then the state of the system alone is

$$\rho_{sys} = \text{Tr}_{env}\left[\rho_{\text{total}}\right] \tag{5.1.28}$$

$$= \sum_{\text{env}} \langle \text{env}|\rho_{\text{total}}|\text{env}\rangle \,. \tag{5.1.29}$$

3. **Assumptions about the environment**

   The environment is also known as a "bath" or a "reservoir" (cf API). We model it as being an ensemble of oscillators, of a variety of frequencies, which are weakly coupled to the system. It has several important properties:

   - Large and unchanging – Born approximation
   - Short correlation time τ_c – Markov approximation

4. **Two (very different) timescales**

   There are two important timescales in this model:

   - T_{relax}: A slow evolution of the system
   - τ_c: The fast fluctuations of the environment

   We will build equations of motion which have a timescale Δt, chosen such that $\tau_c \ll \Delta t \ll T_{relax}$.

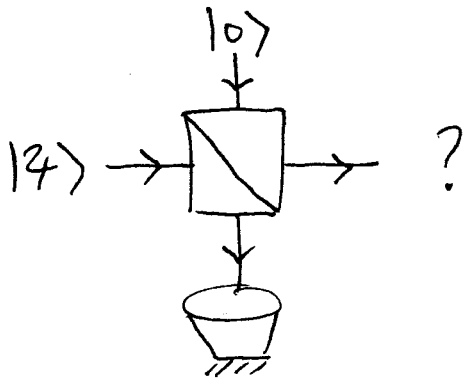
Our goal is to construct a model dynamical equation of motion for ρ_{sys} of the form

$$\frac{\Delta \rho_{sys}}{\Delta t} = M \rho_{sys} \,, \tag{5.1.30}$$

where M is time independent. This is known as a "coarse grained" evolution equation. It is desirable to obtain M for a variety of scenarios, including interactions where the system + environment are atom + light, light + light, and atom + motion, for example. Below, we construct a master equation for the atom + vacuum using two different approaches.

5.1.4 Beamsplitter model of the master equation

The physical intuition behind the master equation we desire to construct can be captured with a simple example, which builds on the beamsplitter we studied in previous lectures. Consider a single photon state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ (for simplicity, let the coefficients be real-valued) entering a beamsplitter of angle θ:



A vacuum state $|0\rangle$ is input to the other port, whose output we discard. Let us consider the photon as being our system, and the other (initially vacuum) mode as being our environment. What is the quantum state of the undiscarded output? Naively, we might argue that a single photon is discarded into the environment with probability $p_1 = \beta^2 \sin^2\theta$, so that we might expect the output to be $|0\rangle$ with probability p_1, and $|\psi\rangle$ with probability $1 - p_1$. However, that (semi-)classical argument is incorrect.

The output state $|\phi\rangle$ of the system + environment is

$$|\phi\rangle = e^{i\theta(a^\dagger b + b^\dagger a)} \Big[|\psi\rangle \otimes |0\rangle\Big] \tag{5.1.31}$$

$$= \alpha|00\rangle + \beta\Big[\cos\theta|10\rangle + \sin\theta|01\rangle\Big] \tag{5.1.32}$$

$$= \Big[\alpha|0\rangle + \beta\cos\theta|1\rangle\Big] \otimes |0\rangle + \Big[\beta\sin\theta|0\rangle\Big] \otimes |1\rangle\,. \tag{5.1.33}$$

Thus, the correct result is that the output is $|\psi_0\rangle = (\alpha|0\rangle + \beta\cos\theta|1\rangle)/\sqrt{\alpha^2 + \beta^2\cos^2\theta}$, with probability $1 - p_1$, and $|\psi_1\rangle = |0\rangle$ with probability p_1.

These states can conveniently be written as density matrices. The input state is

$$\rho_{in} = |\psi\rangle\langle\psi| = \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix}, \tag{5.1.34}$$

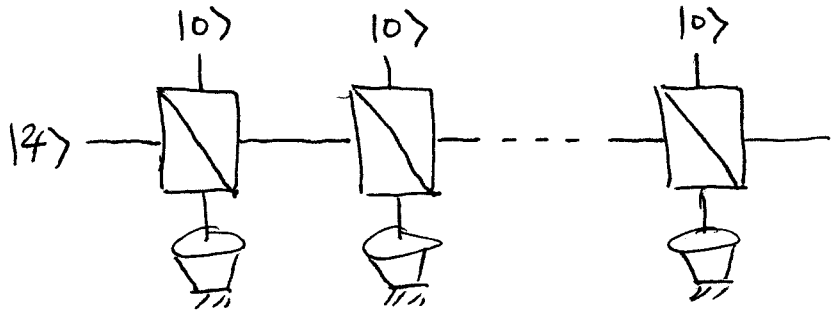
and the output state is

$$\rho_{out} = p_1|\psi_1\rangle\langle\psi_1| + p_0|\psi_0\rangle\langle\psi_0| = \begin{bmatrix} \alpha^2 + \beta^2\sin^2\theta & \alpha\beta\cos\theta \\ \alpha\beta\cos\theta & \beta^2\cos^2\theta \end{bmatrix}. \tag{5.1.35}$$

Note that ρ_{out} cannot be written as $|\chi\rangle\langle\chi|$ for any pure state $|\chi\rangle$, because it is not pure (it is a statistical mixture). The change in ρ is

$$\Delta\rho = \rho_{out} - \rho_{in} = \begin{bmatrix} -\beta^2(\cos 2\theta - 1)/2 & \alpha\beta(\cos\theta - 1) \\ \alpha\beta(\cos\theta - 1) & \beta^2(\cos 2\theta - 1)/2 \end{bmatrix}. \tag{5.1.36}$$

Now imagine that we send the single photon state through *many* beamsplitters in a sequence, each with some small tap angle $\theta = \sqrt{\Gamma\Delta t/2}$:



We make two assumptions: the environment modes always begin in the vacuum $|0\rangle$ (this is the Born approximation), and the environment is completely different and uncorrelated between scattering events (this is the Markov approximation).

This allows us to write a coarse-grained differential equation for the photon state

$$\frac{\Delta\rho}{\Delta t} \approx \begin{bmatrix} \dot\rho_{00} & \dot\rho_{01} \\ \dot\rho_{10} & \dot\rho_{11} \end{bmatrix} = -\Gamma \begin{bmatrix} -\beta^2 & \alpha\beta/2 \\ \alpha\beta/2 & \beta^2 \end{bmatrix} . \tag{5.1.37}$$

Expressed as differential equations for each of the independent matrix elements, we get (using Eq.(5.1.34) for $\rho(t=0)$)

$$\frac{d}{dt}\rho_{00} = +\Gamma\rho_{11} \tag{5.1.38}$$

$$\frac{d}{dt}\rho_{11} = -\Gamma\rho_{11} \tag{5.1.39}$$

for the diagonal elements. These describe the evolutions of the probabilities of finding the photon in the $|0\rangle$ and $|1\rangle$ states, and are analogous to the Einstein rate equations. And for the off-diagonal elements we get

$$\frac{d}{dt}\rho_{01} = -\frac{\Gamma}{2}\rho_{01} \tag{5.1.40}$$

$$\frac{d}{dt}\rho_{10} = -\frac{\Gamma}{2}\rho_{10} \tag{5.1.41}$$

which show the decay of the *quantum coherence* of the state.

The form of these differential equations, which are master equations, is very general, and almost exactly the same result is obtained for a two-level atom interacting with the vacuum. In that situation, the solution differs essentially only in that the coherences evolve as

$$\frac{d}{dt}\rho_{01} = -\left(i\Delta + \frac{\Gamma}{2}\right)\rho_{01} , \tag{5.1.42}$$

where Δ is a frequency shift of the system known as the "Lamb shift," which is due to virtual excitations to higher atomic levels. In the atomic master equation, Γ is the spontaneous emission rate, given by Fermi's golden rule

$$\Gamma = \frac{2\pi}{\hbar}\sum_{\kappa\epsilon} |\langle g;\kappa\epsilon|V|e;0\rangle|^2 \delta(\hbar\omega - \hbar\omega_{ge}) , \tag{5.1.43}$$

as derived in API.

5.1.5 Full derivation – walk-through

We now turn to a full derivation of the general master equation. Following the notation used in API, Chapter 4, let A denote the system, and R the environment (known as the reservoir in API). The full Hamiltonian is

$$H = H_A + H_R + V\,, \tag{5.1.44}$$

where V is the system-reservoir interaction potential. In the interaction picture defined by H_A and H_R, the equation of motion for the full system + reservoir density matrix is

$$i\hbar\dot{\rho} = [V, \rho]\,. \tag{5.1.45}$$

Integrating this once gives

$$i\hbar\rho(t) = \int_0^t [V(t'), \rho(t')]\, dt'\,. \tag{5.1.46}$$

Substituting this back into Eq.(5.1.45) gives

$$\dot{\rho} = -\frac{1}{\hbar^2}\int_0^t [V(t), [V(t'), \rho(t')]]\, dt'\,. \tag{5.1.47}$$

If we assume that the system and reservoir are initially uncorrelated, and make the approximation that the reservoir stays unchanged (the Born approximation), then

$$\rho(t) \approx \rho_A(t) \otimes \rho_R(0) = \rho_A(t) \otimes \rho_R \tag{5.1.48}$$

This gives us our starting point for a general master equation:

$$\dot{\rho}_A = -\frac{1}{\hbar^2}\mathrm{Tr}_R \int_0^t [V(t), [V(t'), \rho_A(t') \otimes \rho_R]]\, dt'\,. \tag{5.1.49}$$

An example is helpful in seeing how this equation works. Generally, we will take system + reservoir interactions of the form $V = -AR$, where A acts only on the system, and R acts only on the reservoir. Specifically, let the system be a two-level atom, and the environment be a single electromagnetic mode initially in the vacuum state $|0\rangle$. The atom interacts with the usual dipole interaction,

$$V = \frac{\hbar\Omega}{2}(b^\dagger|g\rangle\langle e| + b|e\rangle\langle g|)\,, \tag{5.1.50}$$

which is conveniently written using Pauli raising and lowering operators

$$\sigma_\pm = \frac{\sigma_x \pm i\sigma_y}{2} \tag{5.1.51}$$

where

$$\sigma_- = |g\rangle\langle e| \qquad \text{and} \qquad \sigma_+ = |e\rangle\langle g|\,. \tag{5.1.52}$$

Insert this now into Eq.(5.1.49), but disregard the integral over time (this

lets us see what the essential dynamics are, at the expense of not obtaining the correct specific rates). The relevant commutators are (writing $V = \frac{\hbar\Omega}{2}\hat{V}$)

$$\left[\hat{V}, \rho_A \otimes |0\rangle\langle 0|\right] = (\sigma_- \rho_A) \otimes |1\rangle\langle 0| - (\rho_A \sigma_+) \otimes |0\rangle\langle 1| \tag{5.1.53}$$

and

$$\left[\hat{V}, \left[\hat{V}, \rho_A \otimes |0\rangle\langle 0|\right]\right] = (\sigma_+\sigma_-\rho_A)\otimes|0\rangle\langle 0| - 2(\sigma_-\rho_A\sigma_+)\otimes|1\rangle\langle 1| + (\rho_A\sigma_+\sigma_-)|0\rangle\langle 0| + \text{other}\,, \tag{5.1.54}$$

where the "other" terms are not diagonal in the electromagnetic mode states, and thus disappear in the partial trace over the reservoir. We find, finally:

$$\dot{\rho}_A = -\frac{\Gamma}{2}\left[\sigma_+\sigma_-\rho_A - 2\sigma_-\rho_A\sigma_+ + \rho_A\sigma_+\sigma_-\right]. \tag{5.1.55}$$

This is the master equation for a two-level atom dipole coupled to a single electromagnetic mode initially in the vacuum state. It is written in a form known as the "Lindblad" form, which is very common. In atomic physics, you will often see master equations like this. The Lindblad form has the special property that it ensures ρ is a legitimate density matrix at all times; not only does $\mathrm{Tr}(\rho) = 1$ always, but also, its eigenvalues remain non-negative. And more importantly, the map from $\rho(t)$ to $\rho(t')$ is *completely positive*, meaning that if the map operates on just part of a larger system, the state of the larger system remains described by a valid, positive density matrix.

Using the definitions for $\sigma_\pm$, if we express ρ_A as

$$\rho_A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{5.1.56}$$

then we find

$$\dot{\rho}_A = -\frac{\Gamma}{2}\begin{bmatrix} 2a & b \\ c & -2a \end{bmatrix}, \tag{5.1.57}$$

which is identical to the master equation we constructed for the beamsplitter example, Eq.(5.1.37), up to a relabeling of $|g\rangle$ and $|e\rangle$.

As shown by this example, the physical picture behind the master equation is not so complicated, even though the mathematics (used in all its glory) can be overwhelming.

The equation of motion for $\dot{\rho}$ we have obtained is very close to the classical Einstein equations we began with, as we can see by writing out equations of motion for the individual components of ρ. Explicitly, and including Hamiltonian evolution under the classical field Jaynes-Cummings interaction, we find

$$\dot{\rho_{ee}} = i\Omega(\rho_{eg} - \rho_{ge}) - \Gamma\rho_{ee} \tag{5.1.58}$$

$$\dot{\rho_{ge}} = i\omega_0\rho_{ge} - i\Omega(\rho_{ee} - \rho_{gg}) - \frac{\Gamma}{2}\rho_{ge}\,, \tag{5.1.59}$$

where the other two components are given by $\rho_{gg} = 1 - \rho ee$ and $\rho_{eg} = \rho^*_{ge}$. These differential equations are known as the *Optical Bloch Equations*, and we will base

a great deal of our study of atoms and light forces on this quantum description of open system dynamics of a spontaneously emitting atom driven by a classical field.

5.2 Steady-state solutions

The optical Bloch equations provide a time-dependent quantum description of a spontaneously emitting atom driven by a classical electromagnetic field. Considerable insight into the physical processes involved can be gained by studying these equations in the transient excitation limit, as well as the steady-state limit, as we see in this section. We begin by considering the coherent part of the evolution, then extend this to re-visit the Bloch sphere picture of the optical Bloch equations, which provides useful visualizations of transient responses and steady state solutions.

5.2.1 Eigenstates of the Jaynes-Cummings Hamiltonian

The optical Bloch equations are

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{\Gamma}{2}\left[\sigma_+\sigma_-\rho - 2\sigma_-\rho\sigma_+ + \rho\sigma_+\sigma_-\right], \tag{5.2.1}$$

where the Hamiltonian is

$$H = \frac{\hbar\omega_0}{2}(|e\rangle\langle e| - |g\rangle\langle g|) + \frac{\hbar\Omega}{2}(|g\rangle\langle e| + |e\rangle\langle g|). \tag{5.2.2}$$

Useful limits to this equation of motion can be obtained, for example, by solving for ρ in the steady state, when $\dot{\rho} = 0$. Both the Hamiltonian part of this equation, and the damping part, are important to consider, and we begin here by reviewing the coherent evolution under H.

H arises from the Jaynes-Cummings interaction we have previously considered in the context of cavity QED, describing a single two-level atom interacting with a single mode of the electromagnetic field:

$$H_{JC} = \frac{\hbar\omega_0}{2}\left[|e\rangle\langle e| - |g\rangle\langle g|\right] + \hbar\omega a^\dagger a + \frac{\hbar\Omega}{2}\left[|g\rangle\langle e| + |e\rangle\langle g|\right]\left[a + a^\dagger\right]. \tag{5.2.3}$$

The last term in this expression is H_I, the dipole interaction between atom and field. By defining $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$, we may write this interaction as

$$H_I = \frac{\hbar\Omega}{2}\left[\sigma_+ + \sigma_-\right]\left[a + a^\dagger\right]. \tag{5.2.4}$$

In the frame of reference of the atom and field, recall that

$$\sigma_+ a^\dagger \to e^{i(\omega+\omega_0)t}\sigma_+ a^\dagger \tag{5.2.5}$$
$$\sigma_- a \to e^{-i(\omega+\omega_0)t}\sigma_- a \tag{5.2.6}$$
$$\sigma_+ a \to e^{i(\omega_0-\omega)t}\sigma_+ a \tag{5.2.7}$$
$$\sigma_- a^\dagger \to e^{-i(\omega_0-\omega_0)t}\sigma_- a^\dagger\,. \tag{5.2.8}$$

When near resonance, $\omega \approx \omega_0$, and because the $\sigma_+ a^\dagger$ and $\sigma_- a$ terms oscillate at nearly twice the frequency of ω, those terms can be dropped. Doing so is known as the *rotating wave approximation*, and it gives us a simplified interaaction Hamiltonian

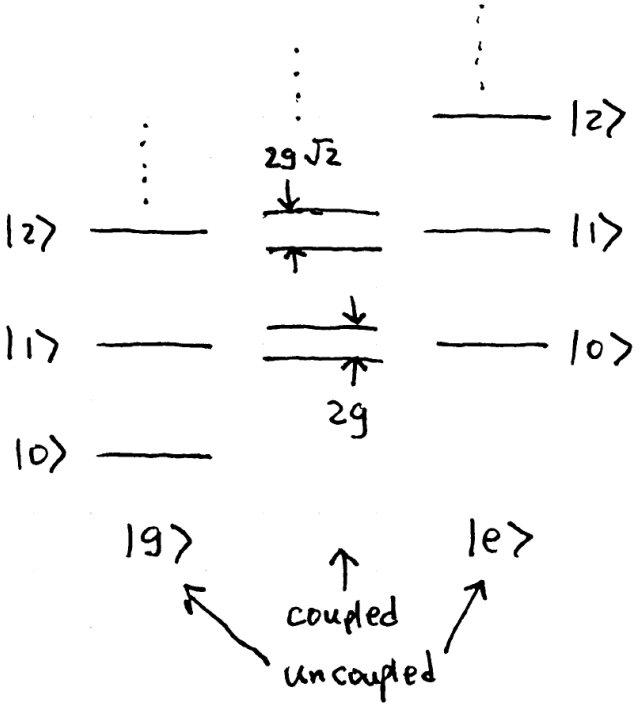
$$H_I = \frac{\hbar\Omega}{2}\left[a^\dagger\sigma_+ + a\sigma_-\right]\,. \tag{5.2.9}$$

Under this approximation, it is useful to note that this interaction merely exchanges one quantum of excitation from atom to field, and back, so that the total number of excitations $N = a^\dagger a + |e\rangle\langle e|$ is a constant of the motion. We may thus write the total Hamiltonian, in the rotating wave approximation, as

$$H = \hbar\omega N + \delta\sigma_z + \frac{\hbar\Omega}{2}(a^\dagger\sigma_- + a\sigma_+)\,, \tag{5.2.10}$$

where we have defined $\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$, and $\delta = \omega_0 - \omega$. Below, we may use $g = \hbar\Omega/2$ to simplify writing.

What are the eigenstates of this Hamiltonian? It describes a two-level system coupled to a simple harmonic oscillator; when uncoupled, if $\delta = 0$, then the eigenstates are simply those of $|e\rangle$, $|g\rangle$ and $|n\rangle$, as shown here:



When coupled, degenerate energy levels split, with harmonic oscillator levels $|n\rangle$ and $|n+1\rangle$ splitting into two energy levels separated by $2g\sqrt{n+1}$. Since the

coupling only pairs levels separated by one quantum of excitation, it is straightforward to show that the eigenstates of the Jaynes-Cummings Hamiltonian fall into well defined pairs of states, which we may label as $|\pm, n\rangle$; these are

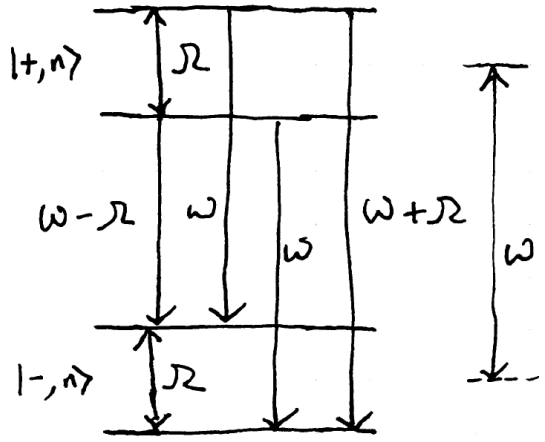
$$|\pm, n\rangle = \frac{1}{\sqrt{2}} \left[|e, n\rangle \pm |g, n+1\rangle \right] , \tag{5.2.11}$$

and they have energies

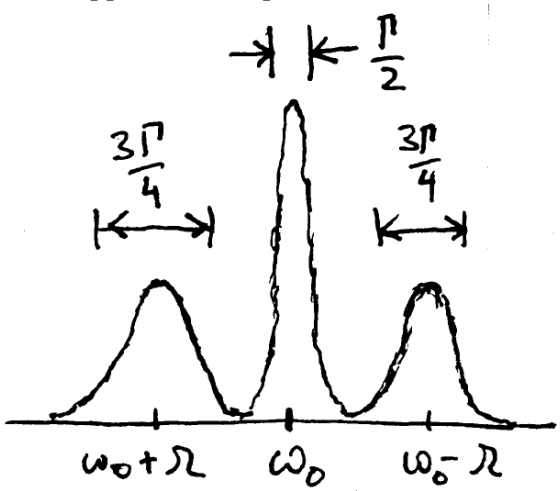
$$E_{\pm,n} = \hbar\omega(n+1) \pm g\sqrt{n+1} \, . \tag{5.2.12}$$

When $\delta \neq 0$, similar physics result, but with slightly more complicated expressions describing the eigenstates, as we shall see when we later return to the "dressed states" picture.

An atom strongly coupled to a single mode electromagnetic field, or an atom driven strongly by a single mode field, will thus have an emission spectrum described by the coupled energy level diagram:



where, to good approximation, the energy level differences are ω and $\omega \pm \Omega$. These three lines which appear in the spectrum are known as the Mollow triplet:



The Mollow triplet is experimentally observed in a wide variety of systems. However, while our energy eigenstate analysis has predicted the number and

frequencies of the emission lines, it fails to explain a key characteristic: the widths are not the same. If the central peak at ω_0 has width $\Gamma/2$, the two sidebands each have a width of $3\Gamma/4$. To explain this, we need the optical Bloch equations.

5.2.2 Rotating frame of reference for atom + field

A simplification worth using in the study of the optical Bloch equations is a transformation into the rotating frame of the light field. The Hamiltonian for the atom + classical field may be written in general as

$$H = \frac{\omega_0}{2} Z + g(X \cos\omega t + Y \sin\omega t)\,, \tag{5.2.13}$$

where g parameterizes the strength field, ω_0 is the atomic transition frequency, and X, Y, Z are the Pauli matrices as usual. Define $|\phi(t)\rangle = e^{i\omega t Z/2}|\chi(t)\rangle$, such that the Schrödinger equation

$$i\partial_t|\chi(t)\rangle = H|\chi(t)\rangle \tag{5.2.14}$$

can be re-expressed as

$$i\partial_t|\phi(t)\rangle = \left[e^{i\omega Z t/2} H e^{-i\omega Z t/2} - \frac{\omega}{2} Z\right]|\phi(t)\rangle\,. \tag{5.2.15}$$

Since

$$e^{i\omega Z t/2} X e^{-i\omega Z t/2} = (X\cos\omega t - Y \sin\omega t)\,, \tag{5.2.16}$$

Eq.(8.4.10) simplifies to become

$$i\partial_t|\phi(t)\rangle = \left[\frac{\omega_0 - \omega}{2} Z + gX\right]|\phi(t)\rangle\,, \tag{5.2.17}$$

where the terms on the right multiplying the state can be identified as the effective 'rotating frame' Hamiltonian. The solution to this equation is

$$|\phi(t)\rangle = e^{i\left[\frac{\omega_0-\omega}{2} Z + gX\right]t}|\phi(0)\rangle\,. \tag{5.2.18}$$

The concept of *resonance* arises from the behavior of this time evolution, which can be understood as being a single qubit rotation about the axis

$$\hat{n} = \frac{\hat{z} + \frac{2g}{\omega_0-\omega}\,\hat{x}}{\sqrt{1 + \left(\frac{2g}{\omega_0-\omega}\right)^2}} \tag{5.2.19}$$

by an angle

$$|\boldsymbol{n}| = t\sqrt{\left(\frac{\omega_0-\omega}{2}\right)^2 + g^2}\,. \tag{5.2.20}$$

When ω is far from ω_0, the qubit is negligibly affected by the laser field; the axis of its rotation is nearly parallel with $\hat{z}$, and its time evolution is nearly exactly that

of the free atom Hamiltonian. On the other hand, when $\omega_0 \approx \omega$, the free atom contribution becomes negligible, and a small laser field can cause large changes in the state, corresponding to rotations about the $\hat{x}$ axis. The enormous effect a small field can have on the atom, when tuned to the appropriate frequency, is responsible for the concept of atomic 'resonance,' as well as nuclear magnetic resonance.

5.2.3 Bloch vector evolution

We have previously seen that an arbitrary qubit state $|\psi\rangle = \cos\theta|0\rangle + e^{i\phi}\sin\phi|1\rangle$ can be represented as being a point on a unit sphere, located at (θ, ϕ) in polar coordinates. Similarly, a density matrix ρ may be depicted as being a point *inside* or on the unit sphere, using

$$\rho = \frac{I + \boldsymbol{r}\cdot\sigma}{2}\,, \tag{5.2.21}$$

where $\boldsymbol{r}$ is the Bloch vector representation of ρ. Explicitly, if we let

$$\rho = \begin{bmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{bmatrix}\,, \tag{5.2.22}$$

then

$$r_x = \rho_{ge} + \rho_{eg} \tag{5.2.23}$$
$$r_y = (\rho_{ge} - \rho_{eg})/i \tag{5.2.24}$$
$$r_z = \rho_{gg} - \rho_{ee}\,. \tag{5.2.25}$$

Visulization of the evolution of a density matrix under the optical Bloch equations is thus helped by rewriting them in terms of a differential equation for $\boldsymbol{r}$. A convenient starting point for this is the optical Bloch equation

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] - \frac{\Gamma}{2}\left[\sigma_+\sigma_-\rho - 2\sigma_-\rho\sigma_+ + \rho\sigma_+\sigma_-\right]\,, \tag{5.2.26}$$

using the rotating frame Hamiltonian (suppressing $\hbar$)

$$H = \frac{\delta}{2}Z + gX\,. \tag{5.2.27}$$

This gives us the equations of motion

$$\dot{r}_x = \delta\, r_y - \frac{\Gamma}{2}r_x \tag{5.2.28}$$
$$\dot{r}_y = -\delta\, r_x - g\, r_z - \frac{\Gamma}{2}r_y \tag{5.2.29}$$
$$\dot{r}_z = g\, r_y - \Gamma(r_z - 1)\,. \tag{5.2.30}$$

Note how these equations of motion provide a simple set of flows on the Bloch sphere: the δ terms correspond to a rotation in the $\hat{x} - \hat{y}$ plane, g corresponds to a rotation in the $\hat{y} - \hat{z}$ plane, and Γ drives a relaxation process which shrinks

$\hat{x}$ and $\hat{y}$ components of the Bloch vector, while moving the $\hat{z}$ component toward $r_z = 1$.

Physically, what is the meaning of r_x, r_y, and r_z? r_z is manifestly the population difference between the excited and ground states. The other two components may be interpreted by recognizing that the average dipole moment of the atom is

$$\begin{aligned} \langle d \rangle &= d_{ge} \mathrm{Tr}(\rho(|e\rangle\langle g| + |g\rangle\langle e|)) && (5.2.31) \\ &= d_{ge}(\rho_{ge} e^{i\omega t} + \rho_{eg} e^{-i\omega t}) && (5.2.32) \\ &= d_{ge}(r_x \cos\omega t + r_y \sin\omega t)\,. && (5.2.33) \end{aligned}$$

Thus, r_x and r_y correspond to the phase components of the atomic dipole moment which are in-phase and in quadrature with the incident electromagnetic field.

5.2.4 Transient repsonse of the optical Bloch equations

The optical Bloch equations allow us to study the internal state of the atom as it changes due to the external driving field, and due to spontaneous emission.

Starting from the time-independent form of the equations,

$$\begin{aligned} \dot{r}_x &= \delta\, r_y - \frac{\Gamma}{2} r_x && (5.2.34) \\ \dot{r}_y &= -\delta\, r_x - g\, r_z - \frac{\Gamma}{2} r_y && (5.2.35) \\ \dot{r}_z &= g\, r_y - \Gamma(r_z - 1)\,, && (5.2.36) \end{aligned}$$

we may note that when $g \to 0$ and at resonance, $\delta = 0$, the Bloch vector exhibits pure damping behavior, towards $r_z = 1$, and $r_x = r_y = 0$.

When $g \gg \Gamma$, Rabi oscillations occur, represnted by rapid rotations of the Bloch vector about $\hat{x}$. Since the relaxation along $\hat{y}$ occurs at rate $\Gamma/2$, and the relaxation about $\hat{z}$ occurs at rate Γ, we might expect that the average relaxation rate of the rotating components under such a strong driving field would be $(\Gamma + \Gamma/2)/2 = 3\Gamma/4$. The remaining component r_x does not rotate, because it sits along $\hat{x}$, the axis of rotation. Thus, it relaxes with rate $\Gamma/2$. Computation of the eigenvalues of the equations of motion verify this qualitative picture, and show that for $g \gg \Gamma$, and $\delta = 0$, the eigenvalues of motion are $\pm ig + 3\Gamma/4$ and $\Gamma/2$. These correspond to a main peak at ω_0 with width $\Gamma/2$, and two sidebands at $\omega_0 \pm \Omega$, with widths $3\Gamma/4$, thus explaining the widths of the observed Mollow triplet lines.

5.2.5 Steady-state solution of the optical Bloch equations

The steady state solution of the optical Bloch equations are found by setting all the time derivatives to zero, giving a set of three simultaneous equations,

$$0 = \delta\, r_y - \frac{\Gamma}{2} r_x \tag{5.2.37}$$

$$0 = -\delta\, r_x - g\, r_z - \frac{\Gamma}{2} r_y \tag{5.2.38}$$

$$0 = g\, r_y - \Gamma(r_z - 1)\,. \tag{5.2.39}$$

The solutions are (up to overall minus signs which can be absorbed into definitions):

$$r_x = (g\delta)\frac{1}{\delta^2 + g^2/2 + (\Gamma/2)^2} \tag{5.2.40}$$

$$r_y = \left(\frac{g\Gamma}{2}\right)\frac{1}{\delta^2 + g^2/2 + (\Gamma/2)^2} \tag{5.2.41}$$

$$r_z = \left(\delta^2 + \frac{\Gamma^2}{4}\right)\frac{1}{\delta^2 + g^2/2 + (\Gamma/2)^2} \tag{5.2.42}$$

Physically, these are Lorentzians; the r_y solution (the component in quadrature with the dipole) corresponds to an absorption curve with half-width

$$\sqrt{\frac{\Gamma^2}{4} + \frac{g^2}{2}}\,, \tag{5.2.43}$$

and the r_x solution (the component in-phase with the dipole) corresponds to a dispersion curve. And under a strong driving field, as $g \to \infty$, $r_z \to 0$, indicating that the populations in the excited and ground states are equalizing. The steady-state population in the excited state is

$$\rho_{ee} = \frac{1 - r_z}{2} = \frac{g^2/4}{\delta^2 + g^2/2 + \Gamma^2/4}\,, \tag{5.2.44}$$

an important result that will later be used in studying light forces.

These solutions can be re-expressed in a simplified manner by defining the saturation parameter

$$s = \frac{g^2/2}{\delta^2 + \Gamma^2/4}\,, \tag{5.2.45}$$

in terms of which we find

$$r_x = \frac{2\delta}{g}\frac{s}{1+s} \tag{5.2.46}$$

$$r_y = \frac{\Gamma}{g}\frac{s}{1+s} \tag{5.2.47}$$

$$r_z = \frac{1}{1+s}\,. \tag{5.2.48}$$

As $s \to \infty$, the atomic transitions become *saturated*, and the linewidth of the

transition broadens from its natural value Γ, becoming $\Gamma' = \Gamma\sqrt{1+s}$ on resonance, at $\delta = 0$.

5.3 Unraveling quantum open system dynamics

Open quantum system dynamics are traditionally studied with the master equation, a differential equation for the density matrix describing the system. An alternative, and equivalent approach, utilizing pure-state wavefunctions and stochastic evolution, is presented in this section. This approach, known variously as the "quantum monte-carlo wavefunction" technique, or the method of quantum jumps or quantum trajectories, is based on the fact that density matrices can be represented as stochastic combinations of pure states. We apply this technique to modeling an atom coupled to a vacuum, and show how the monte-carlo approach generates average evolution equivalent to the optical Bloch equations. The approach offers new ways to physically interpret open quantum system evolution, but must be used with caution, since there are an infinite number of equivalent interpretations, many of which may appear to conflict with each other.

5.3.1 Quantum jumps

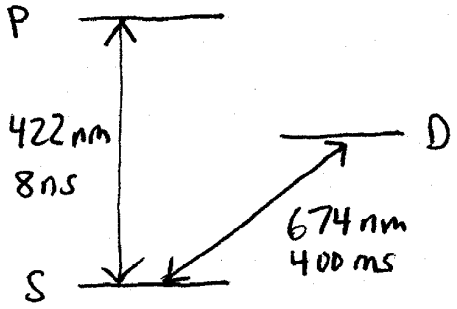
Much of traditional atomic physics has focused on ensembles of many atoms; correspondingly, the density matrix formalism we have employed in studying an atom coupled to the vacuum provides an *ensemble* description of quantum dynamics. In fact, the concept of ensembles is vital to the foundations of quantum mechanics; only ensemble averages can be measured, and one might even say that in a sense, only ensemble averages are "real." Schrdinger, in 1952, wrote (and we quote here from Gerry and Knight, "Introductory Quantum Optics," Cambridge University Press, 2005):

> *. . . we never experiment with just one electron or atom or (small) molecule. In thought experiments, we sometimes assume that we do; this invariably entails ridiculous consequences. . . In the first place, it is fair to state that we are not experimenting with single particles, any more than we can raise ichthysauria in the zoo.*

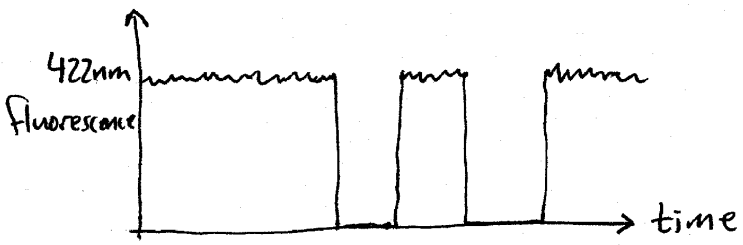
Despite this rather extreme opinion, modern atomic physics experiments are now broadly moving in the direction of coherently addressing and manipulating single atoms and single molecules in the laboratory. Experiments have been successful with both (hot and cold) neutral atoms and ions, and with a variety of single (hot) molecules. These date back to Dehmelt's 1975 proposal to trap single ions to provide an atomic time standard.

Such experiments with single quantum systems motivate a description of open quantum system dynamics in terms of statistical averages over single realizations, as illustrated by the following experiment. Consider a single three-level

ion, such as strontium, with a 8 ns lifetime 422 nm S-P transition, and a $\sim$ 400ms lifetime 674 nm S-D transition:



Experimental observation of the blue fluorescence from this atom, when strongly excited at 422 nm, and weakly excited at 674 nm, shows a very interesting signature: quantum jumps. When the atom is in the metastable D state, the 422 nm fluorescence ceases, because no photons can be scattered from the strong transition until it transitions back to the ground S state. A typical observation record might look like this:



Naturally, the *average* fluorescence rate will be consistent with that expected by the relative strengths of the red and blue excitations, but this would be just a constant value. Such quantum jumps, however, are ubiquitous in experiments involving single atoms and other single quanta, and it would be nice to have a theoretical prescription for describing such observations. In addition, it would be nice to have available possible *interpretations* of what how the individual systems might be evolving to give some observed ensemble behavior, such as an exponential decay.

5.3.2 Quantum Monte-Carlo wavefunction technique for atom + vacuum

Consider a single two-level atom evolving under the Hamiltonian H_0, while also undergoing spontaneous emission. This is the same scenario we studied to formulate the optical Bloch equations. It can be modeled using the quantum Monte-Carlo wavefunction (QMCWF) technique, using the following steps.

Let us assume the atom begins in some state $|\psi\rangle$, and let the atomic levels be $|e\rangle$ and $|g\rangle$. Let Γ be the decay rate of the excited state, as determined, for example, from Fermi's golden rule. The QMCWF technique introduces a new *non-Hermitian* "Hamiltonian"

$$H = H_0 - i\frac{\Gamma}{2}|e\rangle\langle e|\,, \tag{5.3.1}$$

which governs evolution of the state, according to the following. For each differential time step dt, the state $|\psi\rangle$ evolves as:

- **1.** Compute $dp = \Gamma dt |\langle e|\psi\rangle|^2$
- **2.** Let $0 \leq \epsilon \leq 1$ be a uniformly distributed random number
- **3.** If $\epsilon < dp$ then $|\psi\rangle \leftarrow |g\rangle$
- **4.** If $\epsilon \geq dp$ then $|\psi\rangle \leftarrow \left[e^{-iHdt}/\sqrt{1-dp}\right]|\psi\rangle$
- **5.** Go to **1.**

Physically, dp is the probability of the atom jumping from $|e\rangle$ to $|g\rangle$, that is, spontaneously emitting a photon, during the time interval dt. Note that typically $dp \ll 1$, and

$$dp = 1 - \langle\psi|e^{iH^\dagger dt}e^{-iHdt}|\psi\rangle\,, \tag{5.3.2}$$

so $1 - dp$ is the normalization of the state after dt of evolution under the non-Hermitian Hamiltonian H. Essentially, the QMCWF scheme models an imaginary observer watching for a spontaneously emitted photon coming from the atom. If a photon is emitted, then the atom transitions immediately into $|g\rangle$; this is the rule given in step 3, and such a transition is known as a *quantum jump*.

Otherwise, if no photon is emitted, the state changes nevertheless, albeit by only a small amount. Because the observer saw no photon, and thus the probability of the atom being in $|e\rangle$ is slightly diminished. Specifically, if $H_0 = 0$, and $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle$, then according to the rule in step 4,

$$|\psi\rangle \leftarrow \frac{\alpha|g\rangle + \beta e^{-\Gamma dt/2}|e\rangle}{\sqrt{|\alpha|^2 + |\beta|^2 e^{-\Gamma dt}}}\,. \tag{5.3.3}$$

This is exactly the same physics we have seen earlier, in the beamspliter model of the optical Bloch equations.

Equivalence of QMCWF and the optical Bloch equations

The equivalence of QMCWF and the optical Bloch equations (OBE) is shown by demonstrating that the evolution of the density matrix

$$\rho(t) = \overline{|\psi(t)\rangle\langle\psi(t)|} \tag{5.3.4}$$

satisfies the OBE, where the average (denoted by the overline) is taken over instances of running the QMCWF procedure. This follows from computing the

density matrix for the state after one QMCWF procedure step:

$$\rho(t+dt) = dp\,|g\rangle\langle g| + (1-dp)\frac{e^{-iHdt}|\psi\rangle\langle\psi|e^{+iH^\dagger dt}}{1-dp} \tag{5.3.5}$$

$$\approx dp\,|g\rangle\langle g| + (1-iHdt)\rho(t)(1+iH^\dagger dt) \tag{5.3.6}$$

$$\approx dp\,|g\rangle\langle g| + \rho(t) - i(H\rho(t) - \rho(t)H^\dagger)dt \tag{5.3.7}$$

$$= \Gamma dt\langle e|\rho(t)|e\rangle\,|g\rangle\langle g| + \rho(t) - i(H\rho(t) - \rho(t)H^\dagger)dt \tag{5.3.8}$$

$$= \Gamma dt|g\rangle\langle e|\rho(t)|e\rangle\langle g| + \rho(t) - i\,[H_0,\rho(t)] - \frac{\Gamma}{2}\Big(|e\rangle\langle e|\rho(t) + \rho(t)|e\rangle\langle e|\Big), \tag{5.3.9}$$

Writing this as a coarse-grained differential equation, taking the limit of small dt, we find

$$\frac{d}{dt}\rho(t) \approx \frac{\rho(t+dt)-\rho(t)}{dt} \tag{5.3.10}$$

$$= -i\,[H_0,\rho(t)] - \frac{\Gamma}{2}\Big(|e\rangle\langle e|\rho(t) + \rho(t)|e\rangle\langle e|\Big) + \Gamma dt|g\rangle\langle e|\rho(t)|e\rangle\langle g|. \tag{5.3.11}$$

This is the optical Bloch equation.

5.3.3 Quantum Monte-Carlo wavefunction technique: general case

The method of the quantum Monte-Carlo wavefunction technique can be applied not just to the atom + vacuum scenario, but also to model any open quantum system dynamics. The relaxation part of a master equation may be described by a Lindblad operator

$$\mathcal{L}(\rho) = -\frac{1}{2}\sum_k C_k^\dagger C_k\rho + \rho C_k^\dagger C_k + \sum_k C_k\rho C_k^\dagger\,, \tag{5.3.12}$$

where C_k are known as "quantum jump" operators. The corresponding QMCWF procedure employs the non-Hermitian Hamiltonian

$$H = H_0 - \frac{i\hbar}{2}\sum_k C_k^\dagger C_k\,, \tag{5.3.13}$$

and the steps

- **1.** Compute $dp = \sum_k dp_k$, and $dp_k = dt\langle\psi|C_k^\dagger C_k|\psi\rangle$
- **2.** Let $0 \le \epsilon \le 1$ be a uniformly distributed random number
- **3.** If $\epsilon < dp$ then $|\psi\rangle \leftarrow C_k|\psi\rangle/\sqrt{dp_k/dt}$, with k randomly chosen with probability dp_k/dp.
- **4.** If $\epsilon \ge dp$ then $|\psi\rangle \leftarrow \left[e^{-iHdt}/\sqrt{1-dp}\right]|\psi\rangle$
- **5.** Go to **1.**

There are some computational advantages to the QMCWF approach, over numerical solution of the differential equations normally obtained with master

equations. In particular, an N dimensional Hilbert space described by a density matrix requires $O(N^2)$ variables, whereas a pure-state wavefunction requires only $O(N)$ variables. Of course, calculating expectation values means that the stochastic evolution of the QMCWF technique must be repeated $\sim N$ times, so the overall effect is a tradeoff between storage space and computational time. However, the QMCWF technique is immediately parallelizable, and often (but not always) desired observables converge quickly.

For more about this, see, for example, the nice article by Molmer, Castin, and Dalibard, *Monte Carlo wave-function method in quantum optics*, J. Opt. Soc. Am. B, v10, p524, 1993.

5.3.4 Example: phase and amplitude damping

Spontaneous emission, which is mainly what we have studied so far, with the OBE and QMCWF, is described by the density matrix evolution

$$\rho(t=0) = \begin{bmatrix} 1-c & b^* \\ b & c \end{bmatrix} \longrightarrow \rho(t) = \begin{bmatrix} 1-ce^{-\Gamma t} & b^* e^{-\Gamma t/2} \\ be^{-\Gamma t/2} & ce^{-\Gamma t} \end{bmatrix}. \tag{5.3.14}$$

Here, Γ is the spontaneous emission rate of the atom, parameterizing the loss of energy from the atom (the decay of the diagonal elements of ρ to $|g\rangle\langle g|$), and the simultaneous loss of phase coherence of the atom (the decay of the off-diagonal elements of ρ to zero). The quantum jump operator for spontaneous emission, in a two-level atom, is $C = \sqrt{\Gamma}\sigma^-$.

More generally, however, we can have two parameters which describe separate decay of the diagonal and off-diagonal elements,

$$\rho(t) = \begin{bmatrix} 1-ce^{-t/T_1} & b^* e^{-t/T_2} \\ be^{-t/T_2} & ce^{-t/T_1} \end{bmatrix}, \tag{5.3.15}$$

where T_1 is the energy loss rate, and T_2 is the rate of loss of quantum coherence, also known as the phase damping rate. Both of these effects are known as decoherence.

In particular, we define decoherence as *any process which can turn pure states into statistical mixtures* (mixed states). A density matrix ρ is *pure* if either $\mathrm{Tr}(\rho^2) = 1$, or $\rho = |\psi\rangle\langle\psi|$ for some state $|\psi\rangle$, or the entropy $S(\rho) = -\mathrm{Tr}(\rho \log \rho) = 0$. Otherwise, it is mixed.

Phase damping, which gives rise to T_2, is in a sense the most "quantum" kind of noise; it comes along with spontaneous emission, but can also exist by itself. Historically, the term "decoherence" has sometimes been identified exclusively with phase damping, because of its important role in the emergence of classical behavior from quantum systems, but today, decoherence is used as a much more general term, because there are so many ways to loose coherence from a quantum system other than just phase damping.

5.3.5 Open system dynamics have infinitely many equivalent unravelings

The various physical origins of phase damping are interesting to consider, because they can teach us an important fact about models of decoherence for single quantum systems: just as ρ can be *unraveled* in infinitely many ways as $\sum_k p_k|\psi_k\rangle\langle\psi_k|$, decoherence *processes* may also be unraveled in an infinite number of ways, each equally equivalent and equally physically meaningful.

We consider three different physical models of phase damping, on a two-level atom.

Random phase noise

A two-level atom of frequency ω_0 excited by far off-resonance light, $\omega \gg \omega_0$, experiences an AC stark shift of amount proportional to the light intensity. If this intensity fluctuates, then the atom's phase is randomly modulated, causing the evolution

$$|e\rangle \to e^{i\theta}|e\rangle\,, \tag{5.3.16}$$

where θ is the phase imparted by the AC stark shift. Suppose θ is modeled as a Gaussian distributed random variable, with mean zero and variance $2\lambda t$, such that

$$\text{prob}(\theta) = \frac{1}{\sqrt{4\pi\lambda t}} e^{-\theta^2/4\lambda t}\,. \tag{5.3.17}$$

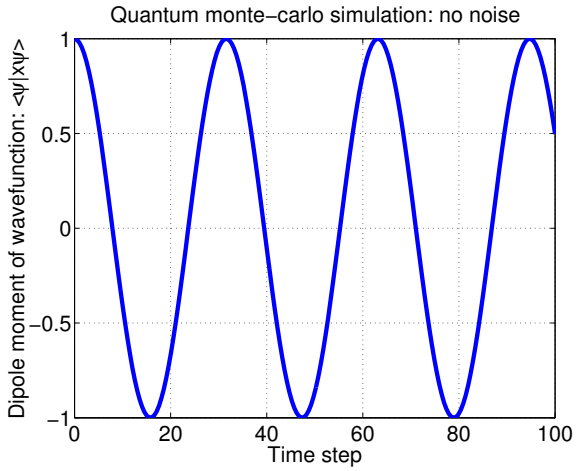
If the initial state of the atom is $\psi\rangle = a|g\rangle + b|e\rangle$, then after time t, it evolves into the average state described by the density matrix

$$\rho(t) = \int_{-\infty}^{+\infty} \begin{bmatrix} |a|^2 & ab^* e^{i\theta} \\ a^* b e^{-i\theta} & |b|^2 \end{bmatrix} \text{prob}(\theta)\, d\theta\,, \tag{5.3.18}$$

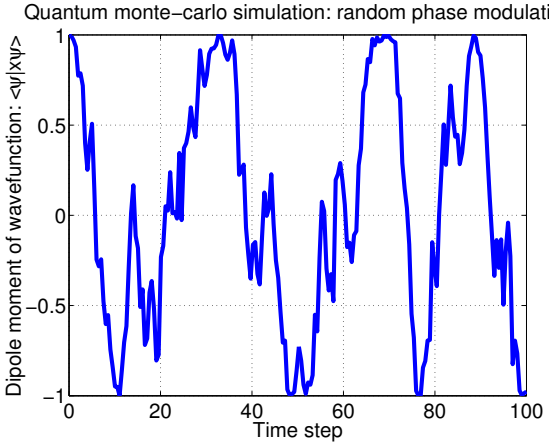
which is found to be

$$\rho(t) = \begin{bmatrix} |a|^2 & ab^* e^{-\lambda t} \\ a^* b e^{-\lambda t} & |b|^2 \end{bmatrix}\,. \tag{5.3.19}$$

If the atomic Hamiltonian is $H_0 = \Omega|e\rangle\langle e|$, and the atom's initial state is $|\psi\rangle = (|g\rangle + |e\rangle)/\sqrt{2}$, then the dipole moment of the atom shows a simple Rabi oscillation:



When random phase noise is imposed on the atom, then its dipole moment decays with time. The evolution of a *single* atom, according to the "trajectory" described by the random walk of Eq.(5.3.16), is a noisy Rabi oscillation:



Elastic collisions

Another physical origin for phase damping is elastic collisions. Assume the two-level atom bounces along a waveguide, interacting with the walls without loosing kinetic energy, but changing its trajectory slightly at each bounce, in a manner depending on the state of the atom. This can be modeled by a Hamiltonian interacting the atom with a single mode environment,

$$H_{SE} = |e\rangle\langle e| \otimes \left[\gamma|0\rangle\langle 1| + \gamma^*|1\rangle\langle 0|\right] , \tag{5.3.20}$$

with coupling constant γ. During a small differential timestep dt, an initial atomic state $a|g\rangle + b|e\rangle$ coupled to an environment $|0\rangle$ evolves to become

$$(a|g\rangle + b|e\rangle) \otimes |0\rangle \rightarrow a|g\rangle|0\rangle + b|e\rangle(\cos\theta|0\rangle + \sin\theta|1\rangle) \tag{5.3.21}$$

$$= \left[a|g\rangle + b\cos\theta|e\rangle\right]|0\rangle + \left[b\sin\theta|e\rangle\right]|1\rangle , \tag{5.3.22}$$

where $e^{-\lambda dt} = \cos\theta$. This expression is very similar to that obtained for the gedankenexperiment used in the QMCWF model of spontaneous emission; the difference is that when a photon is observed in the environment, the atom does not collapse into $|g\rangle$, but rather, into $|e\rangle$. In other words, it does not loose energy; it only looses *information* about what state it was in, prior to the collapse.

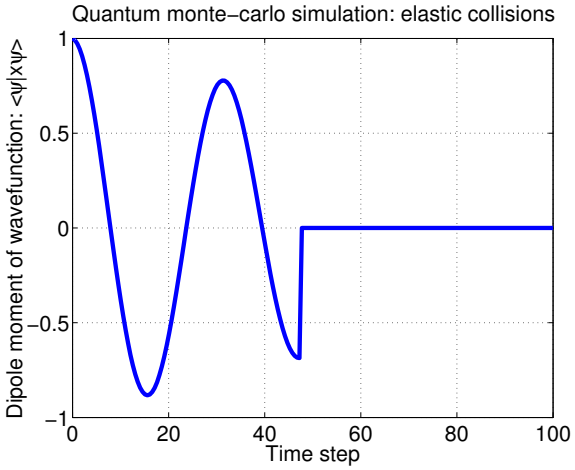
Just as in the proof of the equivalence of QMCWF to the OBE, we can compute the density matrix evolution which this model gives rise to, by writing down an expression for $\rho(t)$, based on Eq.(5.3.22),

$$\rho(t) = \left[a|g\rangle + b\cos\theta|e\rangle\right]\left[\langle g|a^* + \langle e|b^*\cos\theta\right] + |b|^2\sin^2\theta|e\rangle\langle e| \tag{5.3.23}$$

$$= \begin{bmatrix} |a|^2 & ab^*e^{-\lambda t} \\ a^*be^{-\lambda t} & |b|^2 \end{bmatrix} . \tag{5.3.24}$$

Note that this is exactly the same evolution as we obtained for the random phase noise model, Eq.(5.3.19).

Despite the density matrix evolution being identical to that of the random phase model, the elastic collision model implies a different single-particle evolution trajectory. In contrast to the noisy Rabi oscillations previously seen, for the elastic collisions, the atomic state initially decays, then *jumps* into the $|e\rangle$ state at some random time; this is illustrated by this sample trajectory:



Phase flips

A third physical model for phase damping is based on quantum jumps. The Lindblad operator for phase damping is evidently

$$\mathcal{L}(\rho) = -\frac{1}{T_2}\left[|e\rangle\langle e|\rho|g\rangle\langle g| + |g\rangle\langle g|\rho|e\rangle\langle e|\right] . \tag{5.3.25}$$

Equivalently, it may be rewritten in standard form as

$$\mathcal{L}(\rho) = -\frac{1}{2}\left[C^\dagger C\rho + \rho C^\dagger C\right] + C\rho C^\dagger , \tag{5.3.26}$$

where

$$C = \frac{1}{\sqrt{2T_2}}\left[|g\rangle\langle g| - |e\rangle\langle e|\right] . \tag{5.3.27}$$

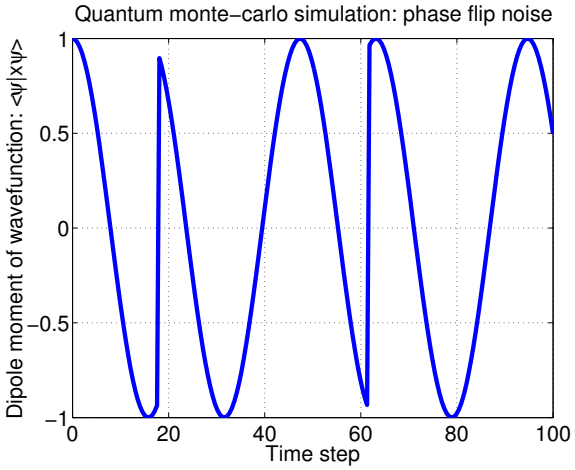
Note that $C^\dagger C$ is proportional to the identity, and thus $H = H_0$, such that no evolution occurs due to this relaxation process except when a quantum jump occurs. Moreover, the effect of a quantum jump is to flip the phase of the atom by -1, changing $|e\rangle \to -|e\rangle$.

If such a flip happens with probability $(1 - e^{-\lambda t})/2$ at each moment in time, then the density matrix for this evolution is thus

$$\begin{aligned}\rho(t) &= \frac{1+e^{-\lambda t}}{2}\left[a|g\rangle + b|e\rangle\right]\left[\langle g|a^* + \langle e|b^*\right] + \frac{1-e^{-\lambda t}}{2}\left[a|g\rangle - b|e\rangle\right]\left[\langle g|a^* - \langle e|b^*\right] \\ &= \begin{bmatrix} |a|^2 & ab^* e^{-\lambda t} \\ a^* b e^{-\lambda t} & |b|^2 \end{bmatrix} .\end{aligned} \tag{5.3.28, 5.3.29}$$

This is again the same density matrix dynamics as previously obtained for the random phase noise and elastic collision models. However, the trajectories of individual evolutions is different; at each moment in time, the two-level atom

either remains *completely unchanged*, or its excited state flips sign, inverting its dipole moment:



Discussion

We have seen three models of phase damping, all of which produce the same density matrix evolution, but each of which has very different microscopic trajectories for individually evolving systems.

Which is correct?

The answer is that all of them are correct, and yet none are. Any of the three can be used for physical intuition and interpretation, but only as long as the only conclusions drawn depend on statistical averages. In fact, in the absence of control over the environment, *no experiment can distinguish between phase damping processes* described by these three models, even in principle. This strong statement arises from the fact that there are an infinite number of ways a (mixed) density matrix can be written as a statistical mixture of pure states; correspondingly, there are an infinite number of "unravelings" of density matrix time evolutions, into statistical evolutions of pure state wavefunctions.

The freedom of *interpretation* which arises in studying decoherence processes arises from a unitary degree of freedom. For example, in the QMCWF model, a gedankenexperiment is performed, in which the atom is allowed to decay and the emitted photon is captured. The evolution of the atom cannot depend on what measurement basis is used for the photodetection. This basis choice is a unitary transform which can be chosen arbitrarily in the gedankenexperiment, and different choices lead to the different unravelings of the master equation into trajectories.