8.512 Recitation 2

- Pset 1 due 02/14 at 5 pm, on Canvas
- OH 2–3 pm in 8-308

Today: 1) Bogoliubov transform for squeezed states
2) Gross-Pitaevskii equation
3) Vortex sol. to GP equation

Ref: Pitaevskii & Stringari Ch. 5
"BEC and Superfluidity"

1) • Recall the Harmonic Oscillator

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m \omega^2 \hat{x}^2$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a^\dagger + a) \qquad [\hat{x}, \hat{p}] = i\hbar$$

$$\Downarrow$$

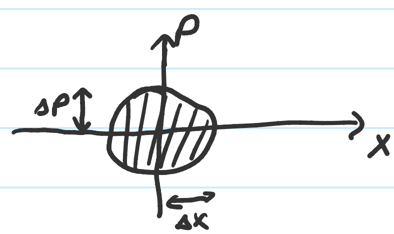
$$\hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\,(a^\dagger - a) \qquad [a, a^\dagger] = 1 \qquad H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

• Uncertainty relation in ground state $|0\rangle$:

$$\Delta X = \sqrt{\langle \hat{x}^2\rangle - \langle \hat{x}\rangle^2} = \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\underbrace{\langle (a^\dagger + a)^2\rangle}_{=\langle a a^\dagger\rangle = 1} - \underbrace{\langle a^\dagger + a\rangle^2}_{=0}} = \sqrt{\frac{\hbar}{2m\omega}}$$

$$\Delta P = \ldots = \sqrt{\frac{\hbar m\omega}{2}}$$

$\Rightarrow$ Saturates $\quad \Delta X \Delta P = \frac{\hbar}{2}$



• Define squeezing operator $\quad S(z) = e^{\frac{1}{2}(z^* a^2 - z (a^\dagger)^2)}$
where $\quad z = r e^{i\theta} \in \mathbb{C}$

Squeezed state $|z\rangle = S(z)|0\rangle$

Operator is unitary $S^\dagger(z) = S^{-1}(z) = S(-z)$
Its action on $a, a^\dagger$ is

$$S^\dagger(z)\, a\, S(z) \equiv b$$
$$S^\dagger(z)\, a^\dagger\, S(z) \equiv b^\dagger$$

- Recall the BCH formula:

$$e^{\hat{A}}\hat{B}e^{-\hat{A}} = \hat{B} + [\hat{A},\hat{B}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{B}]] + \dots$$

and plug in $\hat{A} = \frac{1}{2}(z(a^\dagger)^2 - z^* a^2)$
$\hat{B} = a$

Easy to check that $[(a^\dagger)^2, a] = -2a^\dagger$
$[a^2, a^\dagger] = -2a$

$$\Rightarrow [\hat{A}, a] = -z a^\dagger \quad , \quad [\hat{A}, a^\dagger] = -z^* a$$

$$\underbrace{[\hat{A},[\hat{A},\dots[\hat{A},}_{n}\hat{B}]\dots] = \begin{cases} |z|^n a & \text{if } n\text{-even} \\ -z|z|^{n-1}a^\dagger & \text{if } n\text{-odd} \end{cases}$$

$$\Rightarrow b = e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[\hat{A},\dots[\hat{A},}_{n}\hat{B}]\dots]$$

$$= \sum_{n\text{-even}} \frac{|z|^n}{n!} a - \sum_{n\text{-odd}} \frac{z|z|^{n-1}}{n!} a^\dagger$$

$$= a\sum_{k=0}^{\infty} \frac{r^{2k}}{(2k)!} - a^\dagger e^{i\theta} \sum_{k=0}^{\infty} \frac{r^{2k+1}}{(2k+1)!}$$

$$= a\cosh r - a^\dagger e^{i\theta}\sinh r$$

Similarly $b^\dagger = a^\dagger \cosh r - a e^{-i\theta} \sinh r$

- This defines a Bogoliubov transformation

$$\begin{pmatrix} b \\ b^\dagger \end{pmatrix} = \begin{pmatrix} u & v \\ v^* & u^* \end{pmatrix} \begin{pmatrix} a \\ a^\dagger \end{pmatrix}$$

with $u = \cosh r$, $v = -e^{i\theta} \sinh r$

$|u|^2 - |v|^2 = 1$

- With these new operators, uncertainty relation becomes

$$x' = \sqrt{\frac{\hbar}{2m\omega}} \, (b^\dagger + b)$$

$$p' = i \sqrt{\frac{\hbar m\omega}{2}} \, (b^\dagger - b)$$

$$\Delta x' = \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\langle (a\cosh r - a^\dagger e^{i\theta} \sinh r + a^\dagger \cosh r - a\, e^{-i\theta} \sinh r)^2 \rangle - \underbrace{\langle b^\dagger + b \rangle^2}_{=0}}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\cosh^2(r) + \sinh^2(r) - 2\sinh(r)\cosh(r)\cos(\theta)}$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \cdot \sqrt{\cosh(2r) - \sinh(2r)\cos(\theta)}$$

$$\Delta p' = \dots = \sqrt{\frac{\hbar m\omega}{2}} \cdot \sqrt{\cosh(2r) + \sinh(2r)\cos(\theta)}$$

$$\Rightarrow \Delta x' \Delta p' = \frac{\hbar}{2} \sqrt{\cosh^2(2r) - \sinh^2(2r)\cos^2(\theta)}$$

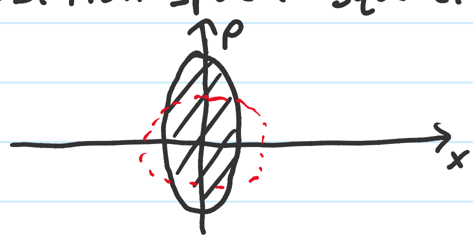
- $\theta = 0$

$$\Delta x' = \sqrt{\frac{\hbar}{2m\omega}} \cdot e^{-r}$$

$$\Delta p' = \sqrt{\frac{\hbar m\omega}{2}} \cdot e^{r}$$

$$\Delta x' \Delta p' = \frac{\hbar}{2}$$

position space squeezing

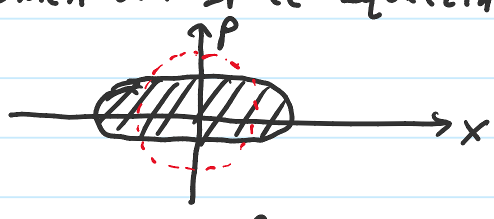


- $\theta = \pi$

$$\Delta x' = \sqrt{\frac{\hbar}{2m\omega}} \cdot e^{r}$$

$$\Delta p' = \sqrt{\frac{\hbar m\omega}{2}} \cdot e^{-r}$$

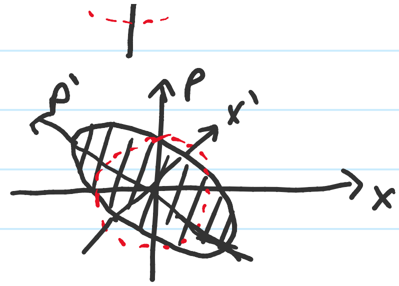
momentum space squeezing



- $\theta = \frac{\pi}{4}$ more complicated

  need to rotate

  $p' \to e^{-i\theta/2} p'$, $\quad x' \to e^{-i\theta/2} x'$



- Set $\theta = 0$. Intuition behind squeezing op?

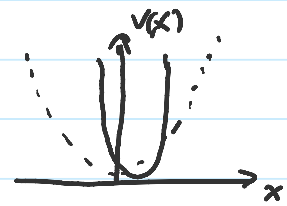
Equivalent to time evolution under Hamiltonian

$$e^{\frac{r}{2}(a^2 - (a^\dagger)^2)} = e^{-i\frac{Ht}{\hbar}} \quad \Rightarrow \quad H = i\frac{r\hbar}{t}\left(a^2 - (a^\dagger)^2\right)$$

$$H \sim (a^2 + h.c.)$$

two-photon process

Such a Hamiltonian term can be engineered by applying an electric field and a potential that restricts the H.O. to a finite region



$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2 - e E_0 (\alpha \hat{X} - \beta \hat{x}^2)$$

displacement   barrier

$\hat{x}^2$ will generate terms of the form $a^2 + (a^\dagger)^2$

2) Weakly interacting Bose gas:

$$H = \int d^3x \left[\psi^\dagger \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu\right)\psi + \frac{g}{2}\psi^\dagger\psi^\dagger\psi\psi \right]$$

Heisenberg eq. $\partial_t \psi = i[H, \psi]$ yields

$$-i\hbar\partial_t \psi(x,t) = \frac{\hbar^2}{2m}\nabla^2\psi(x,t) + \mu\psi(x,t) - g\,\psi^\dagger(x,t)\psi^2(x,t)$$

Define the classical field associated w/ operator $\psi(x,t)$

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$$\psi|\varphi\rangle = \varphi|\varphi\rangle$$
$$\langle\psi(x,t)\rangle = \varphi(x,t)$$

$\Rightarrow$ GP equation for order parameter

$$-i\hbar\,\partial_t\varphi(x,t) = \frac{\hbar^2}{2m}\nabla^2\varphi(x,t) + \mu\,\varphi(x,t) - g\,|\varphi(x,t)|^2\varphi(x,t) \quad (*)$$

- In BEC, local particle density is $\rho(x,t) = |\varphi(x,t)|^2$

Multiply $(*)$ by φ^* and subtract complex conjugate

$$\Rightarrow \frac{d}{dt}\rho(x,t) + \nabla\cdot\vec{j}(x,t) = 0 \quad \text{(continuity eq.} \Rightarrow \text{particle conservation)}$$

where $j = -\frac{i\hbar}{2m}(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*)$ is the particle current density

- We can write $\varphi(x,t) = \sqrt{\rho(x,t)}\,e^{i\theta(x,t)}$

$$\Rightarrow \vec{j} = \rho\frac{\hbar}{m}\vec{\nabla}\theta$$

$\Rightarrow$ superfluid velocity $\boxed{V_s = \frac{\hbar}{m}|\nabla\theta|}$ see HW

$(*)$ Becomes $$\hbar\frac{d\theta}{dt} + \frac{1}{2}mv_s^2 - \mu + g\rho - \frac{\hbar^2}{2m\sqrt{\rho}}\nabla^2\rho = 0 \quad (**)$$

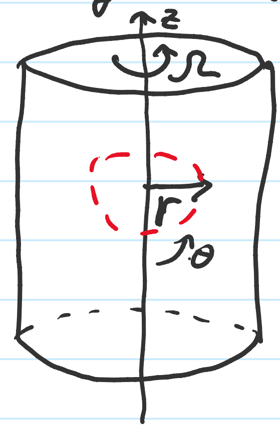
Heisenberg uncertainty between particle # and phase

$(**)$ admits stationary solution (ground state)

$$\left.\begin{aligned}\rho(x,t) &= \rho_0\\ \theta(x,t) &= \theta_0\end{aligned}\right\} \Rightarrow \varphi_0 = \sqrt{\frac{\mu}{g}}\,e^{i\theta_0}$$

$(\mu \geq 0)$

3) Consider now a superfluid inside a cylinder (r, θ, z)



Look for radially-symmetric stationary sol.

$$\varphi(r) = \sqrt{\rho(r)}\, e^{is\theta}$$

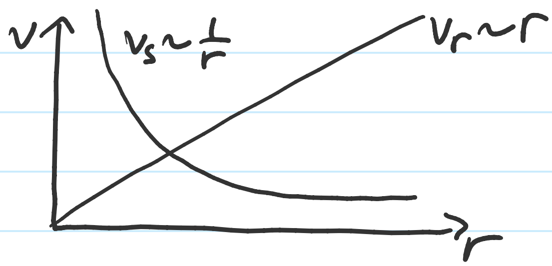
φ is single-valued $\Rightarrow$ $s \in \mathbb{Z}$

- Superfluid is irrotational $\quad \vec{\nabla} \times \vec{v}_s = \frac{\hbar}{m} \vec{\nabla} \times (\vec{\nabla}\theta) = 0$

$$v_s = \frac{\hbar}{m} \cdot \left| \frac{1}{r} \frac{d}{d\theta} (is\theta) \right| = \frac{\hbar s}{mr}$$

- on the other hand, for a rigid system

$$\vec{v}_r = \vec{\Omega} \times \vec{r}$$
$$\vec{\nabla} \times \vec{v}_r = 2\vec{\Omega} \neq 0$$
$$v_r = \Omega r$$



- Onsager-Feynman quantization condition for velocity:

$$\oint \vec{v}_s \cdot d\vec{l} = \oint v_s \cdot r\, d\theta = \frac{\hbar s}{m} \oint d\theta = 2\pi s \frac{\hbar}{m}$$

(**) becomes $\quad -\frac{\hbar^2}{2m\sqrt{\rho}} \left(\frac{1}{r} \frac{d}{dr} \left(r \frac{d}{dr} \sqrt{\rho(r)} \right) \right) + \frac{\hbar^2 s^2}{2mr^2} - \mu + g\rho(r) = 0$

- Further let $\rho(r) = \overset{\sqrt{\frac{\mu}{g}}}{\rho_0} f(\eta)$ where $\eta = r/\xi$

(healing length) $\quad \xi \equiv \frac{\hbar}{\sqrt{2mg\rho_0}}$

$f(\infty) = 1$

$$\Rightarrow \quad \frac{1}{\eta f}\left(\frac{d}{d\eta}\left(\eta \frac{df}{d\eta}\right)\right) + 1 - \frac{s^2}{\eta^2} - f^2 = 0$$

Solve numerically for different s: