PY 711 Fall 2010
Homework 12: Due Tuesday, November 30

1. (15 points) Let ϕ be a massless real scalar field that interacts with the electron field through a Yukawa interaction,

$$\mathcal{L} = \mathcal{L}_{\mathrm{QED}}(A_\mu, \psi, \bar{\psi}) + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - g\bar{\psi}\psi\phi. \tag{1}$$

Here $\mathcal{L}_{\mathrm{QED}}(A_\mu, \psi, \bar{\psi})$ is the usual Lagrange density for quantum electrodynamics involving electrons and photons. Take the mass of the electron to be zero. Consider unpolarized scattering of an incoming electron and photon that produces an outgoing electron and massless scalar particle,

$$e^- + \gamma \to e^- + \phi. \tag{2}$$

To lowest non-vanishing order in perturbation theory, determine the differential cross section $\frac{d\sigma}{d\Omega}$ in the center-of-mass frame. Let $\vec{p}$ be the incoming momentum of the electron and $\vec{p}'$ be the outgoing momentum of the electron. Determine the differential cross section $\frac{d\sigma}{d\Omega}$ as a function of $|\vec{p}|$, and θ, the angle between $\vec{p}$ and $\vec{p}'$. Simplify your final result as much as possible.

Comments: This process of producing new particles by photon scattering is called photoproduction. In the real world there is no known scalar field that couples to the electron, but this toy scattering calculation is similar to actual photoproduction of neutral pions from protons.

KELLY PATTON
HW #12

$\frac{4.5}{15}$

1. LET ϕ BE A MASSLESS REAL SCALAR FIELD THAT INTERACTS WITH THE ELECTRON FIELD THROUGH A YUKAWA INTERACTION,

$$\mathcal{L} = \mathcal{L}_{QED}(A_\mu, \psi, \bar{\psi}) + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - g\bar{\psi}\psi\phi$$

HERE $\mathcal{L}_{QED}(A_\mu, \psi, \bar{\psi})$ IS THE USUAL LAGRANGE DENSITY FOR QUANTUM ELECTRODYNAMICS INVOLVING ELECTRONS AND PHOTONS. TAKE THE MASS OF THE ELECTRON TO BE ZERO. CONSIDER UNPOLARIZED SCATTERING OF AN INCOMING ELECTRON AND PHOTON THAT PRODUCES AN OUTGOING ELECTRON AND MASSLESS SCALAR PARTICLE,

$$e^- + \gamma \to e^- + \phi.$$

TO LOWEST NON-VANISHING ORDER IN PERTURBATION THEORY, DETERMINE THE DIFFERENTIAL CROSS SECTION $\frac{d\sigma}{d\Omega}$ IN THE CENTER-OF-MASS FRAME. LET $\vec{p}$ BE THE INCOMING MOMENTUM OF THE ELECTRON AND $\vec{p}\,'$ BE THE OUTGOING MOMENTUM OF THE ELECTRON. DETERMINE THE DIFFERENTIAL CROSS SECTION $\frac{d\sigma}{d\Omega}$ AS A FUNCTION OF $|\vec{p}|$, AND θ, THE ANGLE BETWEEN $\vec{p}$ AND $\vec{p}\,'$. SIMPLIFY YOUR FINAL RESULT AS MUCH AS POSSIBLE.

Two diagrams

✓

Where $p_1 = (p, \vec{p})$ $\qquad p_3 = (p, -\vec{p}\,')$

$p_2 = (p, -\vec{p})$ $\qquad p_4 = (p, \vec{p}\,')$

Since everything is massless and we're in center of mass, all particles have energy $p = |\vec{p}| = |\vec{p}\,'|$.

For now, I will use p_1, p_2, p_3, p_4 and will replace with definitions later in the calculation.

1 CONTINUED

$$i\mathcal{M}_1 = \left(\bar{u}^r(p_4)(-ie\gamma^\mu)\epsilon_\mu(p_2)\right)\left(\frac{i(\not{p}_1 - \not{p}_3)}{(p_1-p_3)^2 + i\epsilon}\right)\left(-ig\, u^s(p_1)\right)$$

From the Dirac equation

$$(\not{p}_1 + \cancel{m}^{\,0})\, u^s(p_1) = 0 \quad \Rightarrow \quad \not{p}_1 u^s(p_1) = 0$$

Also

$$(p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1\cdot p_3 = -2p_1\cdot p_3 \quad \text{(massless particles)}$$

$$i\mathcal{M}_1 = \frac{-ieg}{2p_1\cdot p_3}\, \epsilon_\mu(p_2)\, \bar{u}^r(p_4)\, \gamma^\mu \not{p}_3\, u^s(p_1)$$

$$i\mathcal{M}_2 = \left(-ig\, \bar{u}^r(p_4)\right)\left(\frac{i(\not{p}_1 + \not{p}_2)}{(p_1+p_2)^2 + i\epsilon}\right)\left(\epsilon_\mu(p_2)(-ie\gamma^\mu)\, u^s(p_1)\right)$$

As in class

$$\not{p}_1 \gamma^\mu u^s(p_1) = 2p_1^\mu u^s(p_1) - \gamma^\mu \cancel{\not{p}_1 u^s(p_1)}^{\,0}$$

And

$$(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1\cdot p_2 = 2p_1\cdot p_2$$

$$i\mathcal{M}_2 = \frac{-ieg}{2p_1\cdot p_2}\, \epsilon_\mu(p_2)\, \bar{u}^r(p_4)\left(2p_1^\mu + \not{p}_2\gamma^\mu\right) u^s(p_1)$$

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$$

$$= -\frac{eg}{2}\, \epsilon_\mu(p_2)\bar{u}^r(p_4)\left(\frac{\gamma^\mu \not{p}_3}{p_1\cdot p_3} + \frac{2p_1^\mu + \not{p}_2\gamma^\mu}{p_1\cdot p_2}\right) u^s(p_1)$$

1 CONTINUED

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{-(eg)^2}{\cancel{4}\;16}\, g_{\mu\nu}\, \mathrm{Tr}\left[\not p_4 \left(\frac{\gamma^\mu \not p_3}{p_1\cdot p_3} + \frac{2p_1^\mu + \not p_2\gamma^\mu}{p_1\cdot p_2}\right)\not p_1 \left(\frac{\not p_3\gamma^\nu}{p_1\cdot p_3} + \frac{2p_1^\nu + \gamma^\nu \not p_2}{p_1\cdot p_2}\right)\right]$$

$$= \frac{-(eg)^2}{\cancel{4}\;16}\Bigg(\frac{1}{(p_1\cdot p_3)^2}\overbrace{\mathrm{Tr}\left[\not p_4 \gamma^\mu \not p_3 \not p_1 \not p_3 \gamma_\mu\right]}^{\mathrm{I}}$$

$$+ \frac{1}{(p_1\cdot p_3)(p_1\cdot p_2)}\Big(2\overbrace{\mathrm{Tr}\left[\not p_4\, p_1^\mu \not p_1 \not p_3 \gamma_\mu\right]}^{\mathrm{II}} + \overbrace{\mathrm{Tr}\left[\not p_4 \not p_2 \gamma^\mu \not p_1 \not p_3 \gamma_\mu\right]}^{\mathrm{III}}$$

$$+ 2\underbrace{\mathrm{Tr}\left[\not p_4 \gamma^\mu \not p_3 \not p_1\, p_{1\mu}\right]}_{\mathrm{IV}} + \underbrace{\mathrm{Tr}\left[\not p_4 \gamma^\mu \not p_3 \not p_1 \gamma_\mu \not p_2\right]}_{\mathrm{V}}\Big)$$

$$+ \frac{1}{(p_1\cdot p_2)^2}\Big(4\overbrace{\mathrm{Tr}\left[\not p_4\, p_1^\mu \not p_1\, p_{1\mu}\right]}^{\mathrm{VI}}$$

$$+ 2\overbrace{\mathrm{Tr}\left[\not p_4\, p_1^\mu \not p_1 \gamma_\mu \not p_2\right]}^{\mathrm{VII}}$$

$$+ 2\overbrace{\mathrm{Tr}\left[\not p_4 \not p_2 \gamma^\mu \not p_1\, p_{1\mu}\right]}^{\mathrm{VIII}}$$

$$+ \underbrace{\mathrm{Tr}\left[\not p_4 \not p_2 \gamma^\mu \not p_1 \gamma_\mu \not p_2\right]}_{\mathrm{IX}}\Big)\Bigg)$$

I $\quad \mathrm{Tr}\left[\not p_4 \gamma^\mu \not p_3 \not p_1 \not p_3 \gamma_\mu\right] = -2\,\mathrm{Tr}\left[\not p_4 \not p_3 \not p_1 \not p_3\right]$

$$= -8\left((p_4\cdot p_3)(p_1\cdot p_3) - (p_4\cdot p_1)\cancelto{0}{(p_3\cdot p_3)} + (p_4\cdot p_3)(p_1\cdot p_3)\right)$$

$$= -16\,(p_4\cdot p_3)(p_1\cdot p_3)$$

II $\quad \mathrm{Tr}\left[\not p_4\, p_1^\mu \not p_1 \not p_3 \gamma_\mu\right] = \mathrm{Tr}\left[\not p_4 \not p_1 \not p_3 \not p_1\right]$

$$= 4\left((p_4\cdot p_1)(p_3\cdot p_1) - (p_4\cdot p_3)\cancelto{0}{(p_1\cdot p_1)} + (p_4\cdot p_1)(p_1\cdot p_3)\right)$$

$$= 8\,(p_4\cdot p_1)(p_3\cdot p_1)$$

III $\quad \mathrm{Tr}\left[\not p_4 \not p_2 \gamma^\mu \not p_1 \not p_3 \gamma_\mu\right] = 4\, p_1\cdot p_3\, \mathrm{Tr}\left[\not p_4 \not p_2\right]$

$$= 4(p_1\cdot p_3)(4\, p_4\cdot p_2)$$

$$= 16\,(p_1\cdot p_3)(p_4\cdot p_2)$$

1 CONTINUED

IV $\quad Tr[\not{p}_4 \gamma^\mu \not{p}_3 \not{p}_1 p_{1\mu}] = Tr[\not{p}_4 \not{p}_1 \not{p}_3 \not{p}_1]$

$$= 4\left((p_4\cdot p_1)(p_3\cdot p_1) - (p_4\cdot p_3)\cancel{(p_1\cdot p_1)}^{0} + (p_4\cdot p_1)(p_3\cdot p_1)\right)$$

$$= 8(p_4\cdot p_1)(p_3\cdot p_1)$$

V $\quad Tr[\not{p}_4 \gamma^\mu \not{p}_3 \not{p}_1 \gamma_\mu \not{p}_2] = 4(p_3\cdot p_1)\,Tr[\not{p}_4 \not{p}_2]$

$$= 4(p_3\cdot p_1)(4(p_4\cdot p_2))$$

$$= 16(p_3\cdot p_1)(p_4\cdot p_2)$$

VI $\quad Tr[\not{p}_4 p_1^\mu \not{p}_1 p_{1\mu}] = Tr[\not{p}_4 \not{p}_1 (p_1\cdot p_1)]$

$$= 0$$

✓ good work

VII $\quad Tr[\not{p}_4 p_1^\mu \not{p}_1 \gamma_\mu \not{p}_2] = Tr[\not{p}_4 \not{p}_1 \not{p}_1 \not{p}_2]$

$$= 4\left((p_4\cdot p_1)(p_1\cdot p_2) - (p_4\cdot p_1)(p_1\cdot p_2) + (p_4\cdot p_2)\cancel{(p_1\cdot p_1)}^{0}\right)$$

$$= 0$$

VIII $\quad Tr[\not{p}_4 \not{p}_2 \gamma^\mu \not{p}_1 p_{1\mu}] = Tr[\not{p}_4 \not{p}_2 \not{p}_1 \not{p}_1]$

$$= 4\left((p_4\cdot p_2)\cancel{(p_1\cdot p_1)}^{0} - (p_4\cdot p_1)(p_2\cdot p_1) + (p_4\cdot p_1)(p_2\cdot p_1)\right)$$

$$= 0$$

IX $\quad Tr[\not{p}_4 \not{p}_2 \gamma^\mu \not{p}_1 \gamma_\mu \not{p}_2] = -2\,Tr[\not{p}_4 \not{p}_2 \not{p}_1 \not{p}_2]$

$$= -8\left((p_4\cdot p_2)(p_1\cdot p_2) - (p_4\cdot p_1)\cancel{(p_2\cdot p_2)}^{0} + (p_4\cdot p_2)(p_1\cdot p_2)\right)$$

$$= -16(p_4\cdot p_2)(p_1\cdot p_2)$$

$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{-(eg)^2}{4}\left(\frac{-16\,p_4\cdot p_3}{p_1\cdot p_3} + \frac{16\,p_4\cdot p_1}{p_1\cdot p_2} + \frac{16\,p_4\cdot p_2}{p_1\cdot p_2} + \frac{16\,p_4\cdot p_1}{p_1\cdot p_2} + \frac{16\,p_4\cdot p_2}{p_1\cdot p_2} - \frac{16\,p_4\cdot p_2}{p_1\cdot p_2}\right)$$

$$= -4(eg)^2\left(-\frac{p_4\cdot p_3}{p_1\cdot p_3} + \frac{2\,p_4\cdot p_1}{p_1\cdot p_2} + \frac{p_4\cdot p_2}{p_1\cdot p_2}\right)$$

1 CONTINUED

Now go back to the definitions of p_1, p_2, p_3, p_4

$$p_1 \cdot p_2 = p^2 + p^2 = 2p^2$$

$$p_1 \cdot p_3 = p^2 + \vec{p} \cdot \vec{p}\,' = p^2 (1 + \cos(\theta))$$

$$p_1 \cdot p_4 = p^2 - \vec{p} \cdot \vec{p}\,' = p^2 (1 - \cos(\theta))$$

$$p_2 \cdot p_4 = p^2 - \vec{p} \cdot \vec{p}\,' = p^2 (1 + \cos(\theta))$$

$$p_3 \cdot p_4 = p^2 + p^2 = 2p^2$$

$$\frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2 = -4 (eg)^2 \left(\frac{-2p^2}{p^2(1+\cos(\theta))} + \frac{2p^2(1-\cos(\theta))}{2p^2} + \frac{p^2(1+\cos(\theta))}{2p^2} \right)$$

$$= -4(eg)^2 \left(\frac{-2}{1+\cos(\theta)} + (1 - \cos(\theta)) + \frac{1}{2}(1 + \cos(\theta)) \right)$$

$$= -4(eg)^2 \left(\frac{-2}{1+\cos(\theta)} + \frac{3}{2} - \frac{1}{2}\cos(\theta) \right)$$

$$= (eg)^2 \left(\frac{8}{1+\cos(\theta)} - 6 + 2\cos(\theta) \right)$$

We know from class and previous assignments

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\vec{p}\,'|}{(2E_A)(2E_B)|\vec{v}_A - \vec{v}_B|} \frac{1}{16\pi^2 E_{CM}} \frac{1}{4} \sum_{\text{spins}} |\mathcal{M}|^2$$

Since these are massless particles

$$|\vec{v}_A - \vec{v}_B| = 2 \qquad E_A = E_B = p \qquad E_{CM} = 2p \qquad |\vec{p}\,'| = p$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{p}{4p^2 (2) 16\pi^2 (2p)} (eg)^2 \left(\frac{8}{1+\cos(\theta)} - 6 + 2\cos(\theta) \right)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{(eg)^2}{128\pi^2 p^2} \left(\frac{4}{1+\cos(\theta)} - 3 + \cos(\theta) \right)$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{(eg)^2}{\cancel{128}\,\pi^2 p^2} \left(\frac{(\cos(\theta) - 1)^2}{(1 + \cos(\theta))} \right)}$$

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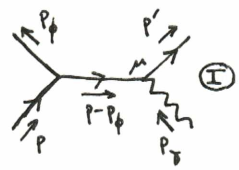
Solutions #12 PY 711

1. $\mathcal{L} = \mathcal{L}_{QED} + \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - g\bar{\psi}\psi\phi$

$$e^- + \gamma \rightarrow e^- + \phi$$



$|\vec{p}| = |\vec{p}\,'|$

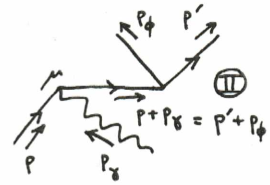
$p = (|\vec{p}|, \vec{p})$

$p' = (|\vec{p}|, \vec{p}\,')$

$p_\gamma = (|\vec{p}|, -\vec{p})$

$p_\phi = (|\vec{p}|, -\vec{p}\,')$

$\hat{p}\cdot\hat{p}' = \cos\theta$



$$i\mathcal{M}_I = (-ig)\epsilon_\mu(p_\gamma)\bar{u}(p')(-ie\gamma^\mu)\, i(\not{p} - \not{p}_\phi)u(p)\frac{1}{\underbrace{(p-p_\phi)^2}_{= -2p\cdot p_\phi} + i\epsilon}$$

$$i\mathcal{M}_{II} = (-ig)\epsilon_\mu(p_\gamma)\bar{u}(p')\, i(\not{p}' + \not{p}_\phi)(-ie\gamma^\mu)u(p)\frac{1}{\underbrace{(p'+p_\phi)^2}_{= 2p'\cdot p_\phi} + i\epsilon}$$

Using $\not{p}\,u(p) = 0$, $\bar{u}(p')\not{p}' = 0$, we have

$$i\mathcal{M} = i\mathcal{M}_I + i\mathcal{M}_{II} = (-ieg)\,\epsilon_\mu(p_\gamma)\bar{u}(p')\left[\frac{\gamma^\mu\not{p}_\phi}{2p\cdot p_\phi} + \frac{\not{p}_\phi\gamma^\mu}{2p'\cdot p_\phi}\right]u(p)$$

Using $\sum_{\text{spins}} \epsilon_\mu(p_\gamma)\epsilon^*_\nu(p_\gamma) = -g_{\mu\nu}$ and $\sum_{\text{spins}} u(p)\bar{u}(p) = \not{p}$,

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = -\frac{e^2g^2}{4}\,\mathrm{Tr}\left\{\not{p}'\left[\frac{\gamma^\mu\not{p}_\phi}{2p\cdot p_\phi} + \frac{\not{p}_\phi\gamma^\mu}{2p'\cdot p_\phi}\right]\not{p}\left[\frac{\not{p}_\phi\gamma_\mu}{2p\cdot p_\phi} + \frac{\gamma_\mu\not{p}_\phi}{2p'\cdot p_\phi}\right]\right\}$$

From the contraction identities: $\gamma^\mu\gamma^\alpha\gamma_\mu = -2\gamma^\alpha$,

$$\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4g^{\alpha\beta}, \quad \gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\delta\gamma_\mu = -2\gamma^\delta\gamma^\beta\gamma^\alpha,$$

we have

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = -\frac{e^2g^2}{4}\mathrm{Tr}\left[\not{p}'\not{p}_\phi\not{p}\not{p}_\phi\right]\frac{-2}{4(p\cdot p_\phi)^2} - \frac{e^2g^2}{4}\mathrm{Tr}\left[\not{p}'\not{p}_\phi\right]\frac{4p\cdot p_\phi}{4(p\cdot p_\phi)(p'\cdot p_\phi)}$$

$$-\frac{e^2g^2}{4}\mathrm{Tr}\left[\not{p}'\not{p}_\phi\right]\frac{4p\cdot p_\phi}{4(p\cdot p_\phi)(p'\cdot p_\phi)} - \frac{e^2g^2}{4}\mathrm{Tr}\left[\not{p}'\not{p}_\phi\not{p}\not{p}_\phi\right]\frac{-2}{4(p'\cdot p_\phi)^2}$$

So now

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^2g^2}{2}\,\frac{2(p'\cdot p_\phi)(p\cdot p_\phi)}{(p\cdot p_\phi)^2} - e^2g^2\,\frac{(p'\cdot p_\phi)(p\cdot p_\phi)}{(p\cdot p_\phi)(p'\cdot p_\phi)}$$

$$- e^2g^2\,\frac{(p'\cdot p_\phi)(p\cdot p_\phi)}{(p\cdot p_\phi)(p'\cdot p_\phi)} + \frac{e^2g^2}{2}\,\frac{2(p'\cdot p_\phi)(p\cdot p_\phi)}{(p'\cdot p_\phi)^2}$$

$$= e^2g^2\left[\frac{p'\cdot p_\phi}{p\cdot p_\phi} - 2 + \frac{p\cdot p_\phi}{p'\cdot p_\phi}\right]$$

Since $p\cdot p_\phi = |\vec{p}|^2(1+\cos\theta)$ and $p'\cdot p_\phi = 2|\vec{p}|^2$, we have

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = e^2g^2\left[\frac{2}{1+\cos\theta} - 2 + \frac{1+\cos\theta}{2}\right) = e^2g^2\,\frac{(1-\cos\theta)^2}{2(1+\cos\theta)}$$

The differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{(2\underbrace{E_A}_{|\vec{p}|})(2\underbrace{E_B}_{|\vec{p}|})\underbrace{|\vec{v}_A-\vec{v}_B|}_{2}}\;\frac{|\vec{p}|}{16\pi^2\underbrace{E_{cm}}_{2|\vec{p}|}} \times \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2$$

$$= \frac{e^2g^2}{512\pi^2|\vec{p}|^2}\;\frac{(1-\cos\theta)^2}{1+\cos\theta}$$