VII.E The Degenerate Fermi Gas

At zero temperature, the fermi occupation number,

$$\langle n_{\vec{k}} \rangle_- = \frac{1}{e^{\beta\left(\mathcal{E}(k)-\mu\right)}+1}, \tag{VII.40}$$

is one for $\mathcal{E}(\vec{k}) < \mu$, and zero otherwise. The limiting value of μ at zero temperature is called the *fermi energy*, $\mathcal{E}_F$, and all one-particle states of energy less than $\mathcal{E}_F$ are occupied, forming a *fermi sea*. For the ideal gas with $\mathcal{E}(\vec{k}) = \hbar^2 k^2/(2m)$, there is a corresponding *fermi wavenumber* k_F, calculated from

$$N = \sum_{|\vec{k}| \leq k_F} (2s+1) = gV \int^{k<k_F} \frac{d^3\vec{k}}{(2\pi)^3} = g\frac{V}{6\pi^2}k_F^3. \tag{VII.41}$$

In terms of the density $n = N/V$,

$$k_F = \left(\frac{6\pi^2 n}{g}\right)^{1/3}, \qquad \Longrightarrow \qquad \mathcal{E}_F(n) = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}. \tag{VII.42}$$

Note that while in a classical treatment the ideal gas has a large density of states at $T=0$ (from $\Omega_{\text{Classical}} = V^N/N!$), the quantum fermi gas has a unique ground state with $\Omega = 1$. Once the one-particle momenta are specified (all $\vec{k}$ for $|\vec{k}| < k_F$), there is only one anti-symmetrized state, as constructed in eq.(VII.7).

To see how the fermi sea is modified at small temperatures, we need the behavior of $f_m^-(z)$ for large z which, after integration by parts, is

$$f_m^-(z) = \frac{1}{m!}\int_0^\infty dx\, x^m \frac{d}{dx}\left(\frac{-1}{z^{-1}e^x+1}\right).$$

Since the fermi occupation number changes abruptly from one to zero, its derivative in the above equation is sharply peaked. We can expand around this peak by setting $x = \ln z + t$, and extending the range of integration to $-\infty < t < +\infty$, as

$$\begin{aligned} f_m^-(z) &\approx \frac{1}{m!}\int_{-\infty}^{\infty} dt\ (\ln z + t)^m \frac{d}{dt}\left(\frac{-1}{e^t+1}\right) \\ &= \frac{1}{m!}\int_{-\infty}^{\infty} dt \sum_{\alpha=0}^{\infty}\binom{m}{\alpha} t^\alpha (\ln z)^{m-\alpha}\frac{d}{dt}\left(\frac{-1}{e^t+1}\right) \\ &= \frac{(\ln z)^m}{m!}\sum_{\alpha=0}^{\infty}\frac{m!}{\alpha!(m-\alpha)!}(\ln z)^{-\alpha}\int_{-\infty}^{\infty} dt\ t^\alpha \frac{d}{dt}\left(\frac{-1}{e^t+1}\right). \end{aligned} \tag{VII.43}$$

Using the (anti-) symmetry of the integrand under $t \to -t$, and un-doing the integration by parts yields,

$$\frac{1}{\alpha!}\int_{-\infty}^{\infty} dt\; t^{\alpha}\frac{d}{dt}\left(\frac{-1}{e^t+1}\right) = \begin{cases} 0 & \text{for } \alpha \text{ odd,} \\ \dfrac{2}{(\alpha-1)!}\displaystyle\int_0^{\infty} dt\,\frac{t^{\alpha-1}}{e^t+1} = 2f_\alpha^-(1) & \text{for } \alpha \text{ even.} \end{cases}$$

Inserting the above into eq.(VII.43), and using tabulated values for the integrals $f_\alpha^-(1)$, leads to the *Sommerfeld expansion*,

$$\begin{aligned} \lim_{z\to\infty} f_m^-(z) = & \frac{(\ln z)^m}{m!}\sum_{\alpha=0}^{\text{even}} 2f_\alpha^-(1)\;\frac{m!}{(m-\alpha)!}\;(\ln z)^{-\alpha} \\ = & \frac{(\ln z)^m}{m!}\left[1+\frac{\pi^2}{6}\;\frac{m(m-1)}{(\ln z)^2}+\frac{7\pi^4}{360}\;\frac{m(m-1)(m-2)(m-3)}{(\ln z)^4}+\cdots\right]. \end{aligned} \tag{VII.44}$$

In the degenerate limit, the density and chemical potential are related by

$$\frac{n\lambda^3}{g} = f_{3/2}^-(z) = \frac{(\ln z)^{3/2}}{(3/2)!}\left[1+\frac{\pi^2}{6}\;\frac{3}{2}\;\frac{1}{2}\;(\ln z)^{-2}+\cdots\right] \gg 1. \tag{VII.45}$$

The lowest order result reproduces the expression in eq.(VII.41) for the fermi energy,

$$\lim_{T\to 0}\ln z = \left[\frac{3}{4\sqrt{\pi}}\;\frac{n\lambda^3}{g}\right]^{2/3} = \frac{\beta\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} = \beta\mathcal{E}_F.$$

Inserting the zero temperature limit into eq.(VII.45) gives the first order correction,

$$\ln z = \beta\mathcal{E}_F\left[1+\frac{\pi^2}{8}\;\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\cdots\right]^{-2/3} = \beta\mathcal{E}_F\left[1-\frac{\pi^2}{12}\;\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\cdots\right]. \tag{VII.46}$$

The appropriate dimensionless expansion parameter is $(k_BT/\mathcal{E}_F)$. Note that the fermion chemical potential $\mu = k_BT\ln z$, is positive at low temperatures, and negative at high temperatures (from eq.(VII.38)). It changes sign at a temperature proportional to $\mathcal{E}_F/k_B$.

The low temperature expansion for the pressure is

$$\begin{aligned} \beta P = & \frac{g}{\lambda^3}f_{5/2}^-(z) = \frac{g}{\lambda^3}\frac{(\ln z)^{5/2}}{(5/2)!}\left[1+\frac{\pi^2}{6}\;\frac{5}{2}\;\frac{3}{2}\;(\ln z)^{-2}+\cdots\right] \\ = & \frac{g}{\lambda^3}\frac{8(\beta\mathcal{E}_F)^{5/2}}{15\sqrt{\pi}}\left[1-\frac{5}{2}\;\frac{\pi^2}{12}\;\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\cdots\right]\left[1+\frac{5\pi^2}{8}\;\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\cdots\right] \\ = & P_F\left[1+\frac{5}{12}\pi^2\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\cdots\right], \end{aligned} \tag{VII.47}$$

where $P_F = (2/5)n\mathcal{E}_F$ if the *fermi pressure*. Unlike its classical counterpart, the fermi gas at zero temperature has finite pressure and internal energy.

The low temperature expansion for the internal energy is obtained easily from eq.(VII.47) using

$$\frac{E}{V} = \frac{3}{2}P = \frac{3}{5}nk_BT_F\left[1 + \frac{5}{12}\pi^2\left(\frac{T}{T_F}\right)^2 + \cdots\right], \tag{VII.48}$$

where we have introduced the *fermi temperature* $T_F = \mathcal{E}_F/k_B$. Eq.(VII.48) leads to a low temperature heat capacity,

$$C_V = \frac{dE}{dT} = \frac{\pi^2}{2}Nk_B\left(\frac{T}{T_F}\right) + \mathcal{O}\left(\frac{T}{T_F}\right)^2. \tag{VII.49}$$

The linear vanishing of the heat capacity as $T \to 0$ is a general feature of a fermi gas, valid in all dimensions. It has the following simple physical interpretation: The probability of occupying single-particle states, eq.(VII.40), is very close to a step function at small temperatures. Only particles within a distance of approximately k_BT of the fermi energy can be thermally excited. This represents only a small fraction T/T_F, of the total number of electrons. Each excited particle gains an energy of the order of k_BT, leading to a change in the internal energy of approximately $k_BTN(T/T_F)$. Hence the heat capacity is given by $C_V = dE/dT \sim Nk_BT/T_F$. This conclusion is also valid for an interacting fermi gas. The fact that only a small number, $N(T/T_F)$, of fermions are excited at small temperatures accounts for many interesting properties of fermi gases. For example, the magnetic susceptibility of a classical gas of N non-interacting particles of magnetic moment μ_B follows the Curie law, $\chi \propto N\mu_B^2/(k_BT)$. Since quantum mechanically, only a fraction of spins contributes at low temperatures, the low temperature susceptibility saturates to a (Pauli) value of $\chi \propto N\mu_B^2/(k_BT_F)$. (See review problems for the details of this calculation.)