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Course: **8.422 - AMO II**
Problem set: **#3**
Due: Friday, Mar 3, 2022
References:

1. Squeezing Hamiltonian.

The dimensionless squeezing Hamiltonian is

$$\mathcal{H} = \frac{\hbar\omega}{2}(\tilde{p}^2 - \tilde{x}^2).$$

With $p = (\tilde{p} - \tilde{x})/\sqrt{2}$ and $x = (\tilde{p} + \tilde{x})/\sqrt{2}$, we have $[x, p] = \tilde{x}, \tilde{p} = i$ and the Hamiltonian becomes

$$\mathcal{H} = \hbar\omega x p$$

plus an offset which we ignore.

1. Consider a wavefunction $\psi_0(x)$ at $t = 0$. From Schrödinger's equation we find

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = \hbar\omega x p \psi(x,t) = -i\hbar\omega x\frac{\partial}{\partial x}\psi(x,t).$$

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2. Disentangling the Squeezing Operator.

In this problem we show that

$$e^{\frac{r}{2}(a^{\dagger 2} - a^2)}\left|0\right\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}(\tanh r)^n\left|2n\right\rangle.$$

To do this, we follow the parts below.

(a) We first calculate the following commutators:

$$\begin{aligned}
[a^2, a^{\dagger 2}] &= aaa^\dagger a^\dagger - a^\dagger a^\dagger aa \\
&= a(1 + a^\dagger a)a^\dagger - a^\dagger(aa^\dagger - 1)a \\
&= 1 + a^\dagger a + (1 + a^\dagger a)(1 + a^\dagger a) - a^\dagger aa^\dagger a + a^\dagger a \\
&= 4a^\dagger a + 2.
\end{aligned}$$

$$\begin{aligned}
[a^2, a^\dagger a] &= aaa^\dagger a - a^\dagger aaa \\
&= a(1 + a^\dagger a)a - (aa^\dagger - 1)aa \\
&= 2a^2.
\end{aligned}$$

$$\begin{aligned}[a^{\dagger 2}, a^\dagger a] &= a^\dagger a^\dagger a^\dagger a - a^\dagger a a^\dagger a^\dagger \\ &= a^\dagger a^\dagger (a a^\dagger - 1) - a^\dagger (1 + a^\dagger a) a^\dagger \\ &= -2 a^{\dagger 2}.\end{aligned}$$

From these, we conclude that the Lie algebra of operators $\{a^2, a^{\dagger 2}, a^\dagger a + 1/2\}$ is closed under commutation. This means we must be able to write

$$e^{\frac{r}{2}(a^{\dagger 2} - a^2)} = e^{\frac{u}{2} a^{\dagger 2}} e^{t(^\dagger a + 1/2)} e^{\frac{v}{2} a^2}.$$

Our job now is to find the numbers u, t, v, which are functions of r. To do this, we find any other Lie algebra whose three operators obey the same commutation relations which allows us to more easily find u, t, v. It turns out that Pauli matrices work.

(b) Consider the replacement:

$$a^\dagger a + \frac{1}{2} \to \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$a^2 \to -\sigma_- = -\sigma_x + i\sigma_y = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$

$$a^{\dagger 2} \to \sigma_+ = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Let us check that the commutation relations above still hold.

$$[a^2, a^{\dagger 2}] \to [-\sigma_-, \sigma_+] = [-\sigma_x, i\sigma_y] + [i\sigma_y, \sigma_x] = 4\sigma_z \leftarrow 4a^\dagger a + \frac{1}{2} \quad \checkmark$$

$$[a^2, a^\dagger a] \to [-\sigma_-, \sigma_z - 1/2] = [-\sigma_-, \sigma_z] = 2i\sigma_y - 2\sigma_x = 2(-\sigma_-) \leftarrow 2a^2 \quad \checkmark$$

$$[a^{\dagger 2}, a^\dagger a] = [\sigma_+, \sigma_z - 1/2] = [\sigma_+, \sigma_z] = -2i\sigma_y - 2\sigma_x = -2\sigma_+ \leftarrow -2a^{\dagger 2} \quad \checkmark$$

(c) With

$$\frac{r}{2}\left(a^{\dagger 2} - a^2\right) = \frac{r}{2}(\sigma_+ + \sigma_-) = r\sigma_x = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix},$$

we find

$$e^{\frac{r}{2}\left(a^{\dagger 2} - a^2\right)} = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}.$$

Mathematica code:

```
In[7]:= MatrixExp[r*PauliMatrix[1]] // FullSimplify
Out[7]= {{Cosh[r], Sinh[r]}, {Sinh[r], Cosh[r]}}
```

(d) Similarly, we have

$$U = e^{\frac{u}{2} a^{\dagger 2}} = \exp\left[\frac{u}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right] = \exp\left[\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

$$T = e^{t\left(a^\dagger a + 1/2\right)} = e^{t\sigma_z} = \exp\left[\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}\right] = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$V = e^{\frac{v}{2}a^2} = \exp\left[\frac{v}{2}\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}\right] = \exp\left[\begin{pmatrix} 0 & 0 \\ -v & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}.$$

With these,

$$UTV = \begin{pmatrix} e^t - e^{-t}uv & e^{-t}u \\ -e^{-t}v & e^{-t} \end{pmatrix}.$$

Mathematica code:

```
In[14]:= U = MatrixExp[{{0, u}, {0, 0}}]
Out[14]= {{1, u}, {0, 1}}

In[15]:= T = MatrixExp[t*PauliMatrix[3]]
Out[15]= {{E^t, 0}, {0, E^-t}}

In[16]:= V = MatrixExp[{{0, 0}, {-v, 0}}]
Out[16]= {{1, 0}, {-v, 1}}

In[18]:= U . T . V // FullSimplify
Out[18]= {{E^t - E^-t u v, E^-t u}, {-E^-t v, E^-t}}
```

(e) Comparing the results of Parts (c) and (d) we easily find that

$$t = -\ln\cosh r, \qquad u = -v = e^t \sinh r = \tanh r.$$

so we have

$$e^{\frac{r}{2}\left(a^{\dagger 2} - a^2\right)} = e^{\frac{\tanh r}{2}a^{\dagger 2}} e^{-\ln\cosh r\left(a^\dagger a + 1/2\right)} e^{-\frac{\tanh r}{2}a^2} = \frac{1}{\sqrt{\cosh r}} e^{\frac{\tanh r}{2}a^{\dagger 2}} e^{-\ln\cosh r(a^\dagger a)} e^{-\frac{\tanh r}{2}a^2}$$

(f) Applying the operator above to $|0\rangle$, we realize that the two right most operators act on $|0\rangle$ as the identity, so we end up with

$$\begin{aligned} e^{\frac{r}{2}\left(a^{\dagger 2} - a^2\right)} |0\rangle &= \frac{1}{\sqrt{\cosh r}} e^{\frac{\tanh r}{2}a^{\dagger 2}} |0\rangle \\ &= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{2^n n!} a^{\dagger 2n} |0\rangle \\ &= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh r)^n |2n\rangle, \end{aligned}$$

as desired.

3. Generation of Squeezed States by Two-Photon Interactions.

Consider a mode $(\vec{k}, \vec{\varepsilon})$ with wavevector $\vec{k}$ and polarization $\vec{\varepsilon}$ of the EM field with frequency ω whose Hamiltonian is given by

$$H = \hbar\omega a^\dagger a + i\hbar\Lambda\left[(a^\dagger)^2 e^{-2i\omega t} - a^2 e^{2i\omega t}\right].$$

The first term is the energy of the mode of the free field. The second term describes a two-photon interaction process such as parametric amplification (a classical wave of frequency 2ω generating two photons with frequency ω). Λ is a real quantity characterizing the strength of the interaction. In this problem, we will show that this Hamiltonian produces squeezed vacuum and explore how it acts on coherent states.

(a) The equation of motion for $a(t)$ in the Heisenberg picture is

$$\frac{d}{dt}a_H = \frac{i}{\hbar}[H_H, a_H],$$

where

???

Now we compute the commutators:

$$[a^\dagger a, a] = [a, a^\dagger a] = -[a, a^\dagger]a - a^\dagger[a, a] = -a.$$

$$[(a^\dagger)^2, a] = -[a, (a^\dagger)^2] = -[a, a^\dagger]a^\dagger - a^\dagger[a, a^\dagger] = -2a^\dagger$$

$$[a^2, a] = 0$$

From these, we find the equations of motion for a and $a^\dagger$

$$\begin{aligned}\dot{a} &= \frac{i}{\hbar}\left(-\hbar\omega a - 2i\hbar\Lambda a^\dagger e^{-2i\omega t}\right) = -i\omega a + 2\Lambda a^\dagger e^{-2i\omega t}\\ \dot{a}^\dagger &= i\omega a^\dagger + 2\Lambda a e^{2i\omega t}.\end{aligned}$$

In matrix form,

(b)

(c)

(d)

(e)