Final Review 8.09

Pert. Theory

Fluids

Nonlinear dynamics & Chaos

Perturbation Theory

Idea: $H = H_0 + \Delta H$, treat ΔH by expanding eqtns of motion in a series

With Hamilton-Jacobi: solve $H_0(p,q)$ by H-J method finding canonical variables (α,β) that are constant

$p = p(\alpha,\beta)$, $q = q(\alpha,\beta)$.

$K = \Delta H$ $\quad \dot{\alpha}_i = -\frac{\partial \Delta H}{\partial \beta_i} \xrightarrow{\text{approx}} \dot{\alpha}_i^{(n)} = -\frac{\partial \Delta H}{\partial \beta_i}\Big|_{n-1}$

$\dot{\beta}_i = +\frac{\partial \Delta H}{\partial \alpha_i} \rightarrow \dot{\beta}_i^{(n)} = \frac{\partial \Delta H}{\partial \alpha_i}\Big|_{n-1}$

If H_0 has periodic orbits, perturbations could be <u>periodic</u> (return to initial value) or <u>secular</u> (net change which builds up)

eg. Pendulum $(p,\theta) \rightarrow (J,\beta)$ for $H_0 = \frac{p^2}{2I} + \frac{I\omega^2}{2}\theta^2$

$\theta = \sqrt{\frac{J}{\pi I \omega}} \sin[2\pi(\nu t + \beta)]$ $\quad \Delta H \propto \theta^4$

$p = \sqrt{\frac{IJ\omega}{\pi}} \cos[2\pi(\nu t + \beta)]$ $\quad$ find $\overline{\dot{J}^{(1)}} = \frac{1}{\tau}\int_0^\tau dt\, \dot{J}^{(1)}(t) = 0$

$\overline{\dot{\beta}^{(1)}} = -J^{(0)}/32\pi^2 I$

eg. Kepler $H_0 = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) - \frac{k}{r}$, $\Delta H = -\frac{h}{r^n}$ $\quad n \geq 2$

J_ϕ, J_θ, J_r, $\underbrace{w_\phi, w_\theta, w_r}_{\dot{w}_i = \nu_r}$ $\qquad w_2 = w_\theta - w_r = \frac{\omega}{2\pi}$ ← ellipse orientation

$\dot{w}_2 = 0$

$J_2 = J_\theta + J_\phi = 2\pi l$

$\dot{w}^{(1)} = \frac{\partial \Delta H}{\partial l}\Big|_0$ $\quad \left[\ \overline{\dot{w}^{(1)}} = \frac{\partial}{\partial l}\frac{(-h)}{\tau}\int_0^\tau \frac{dt}{[r(t)]^n} = \cdots\ \right]$ precession of perihelion of mercury

Fluid Mechanics

Transition to Continuous System $\quad i \to$ continuous label x

$$\eta_i(t) \to \eta(x,t) \quad \text{field}$$

eg springs in 1-dim $\quad L = \int dx\, \mathcal{L}\left(\eta, \frac{d\eta}{dx}, \frac{d\eta}{dt}, x, t\right) = \int \frac{dx}{2}\left[\mu\left(\frac{d\eta}{dt}\right)^2 - Y\left(\frac{d\eta}{dx}\right)^2\right]$

equation of motion $\quad \mu \frac{d^2\eta}{dt^2} - Y\frac{d^2\eta}{dx^2} = 0 \quad$ (wave eqtn)

Fluid Variables 5

velocity $\vec{v}(x,y,z,t)$ $\quad$ measurements at fixed point (x,y,z)

density $\rho(x,y,z,t)$ $\quad$ in fluid at time t

pressure $p(x,y,z,t)$ $\quad$ "Eulerian"

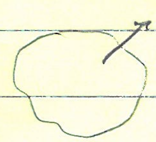
("Lagrangian" approach would have $x(\vec{x}_0,t)$)

pressure isotropic, same in all direction (scalar field)

$$d/dt = \frac{\partial}{\partial t} + \vec{v}\cdot\vec{\nabla} \quad \text{on fields}$$

change as we move with fluid $\quad$ change at fixed point (x,y,z) in fluid

Continuity Equation



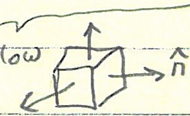
$$\frac{dV}{dt} = \int_V dV\, \vec{\nabla}\cdot\vec{v}, \qquad \vec{\nabla}\cdot\vec{v} = 0 \text{ for } \underline{\text{incompressible}} \text{ fluid}$$

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 \quad \text{mass conservation}$$

<u>equation of continuity</u>

for both ideal & viscous fluids

$$\underbrace{\frac{\partial}{\partial t}\int_{\text{fixed } V} \rho\, dV}_{\text{increase in mass}} = -\int_V dV\, \vec{\nabla}\cdot(\rho\vec{v}) = \underbrace{-\int_S dS\, \hat{n}\cdot(\rho\vec{v})}_{\text{outflow}}$$

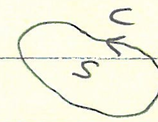


for steady flow: what enters a given V must also exit

Vorticity $\quad \vec{\Omega} = \vec{\nabla} \times \vec{v}$ $\quad$ measure angular velocity of fluid at (x,y,z) by $\frac{\vec{\Omega}}{2}$

$\vec{\nabla} \times \vec{v} = 0$ is irrotational flow

$$\int_S \hat{n} \cdot (\vec{\nabla} \times \vec{v})\, dS = \oint_{C = \partial S} \vec{v} \cdot d\vec{l}$$



Ideal Fluids

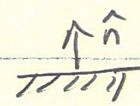
- Euler Equation for ideal fluid (viscosity $= 0$)

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla} p = \frac{\vec{f}}{\rho} = \vec{g}$$

gravity

boundary condition $\vec{v} \cdot \hat{n} = 0$ $\quad \perp$ to surface

or $\vec{v} \cdot \hat{n} = v_{surface}$



- $\frac{ds}{dt} = 0$ $\quad$ no heat exchange

- Conservation of Momentum $\quad \frac{\partial}{\partial t}(\rho \vec{v}) + \vec{\nabla} \cdot \hat{T} = \vec{f}$

stress-tensor for ideal fluid $\quad T_{ki} = \delta_{ki}\, p + v_k v_i \rho$

- Energy Conservation / Bernoulli $\quad \Phi = gz$ $\quad$ gravity

$$\frac{d}{dt}\left(\frac{\vec{v}^2}{2} + \Phi + \frac{p}{\rho} + u\right) = \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{\partial \Phi}{\partial t}$$

u is work from expansion of δV $\quad p\, \vec{\nabla} \cdot \vec{v}\, \delta V = -\frac{d}{dt}(u\, \delta m)$

Applications to Static Fluids $\quad \vec{v} = 0$

ρ, p independent of time

Euler: $\quad \frac{1}{\rho}\vec{\nabla} p = \frac{\vec{f}}{\rho} = -g\hat{z}$, $\quad p = p(z)$

pressure changes with height

Application to Steady Flows $\quad \frac{\partial}{\partial t}(\vec{v}, \rho, p) = 0$

$$\oint_{\substack{\text{closed}\\\text{surface}}} d\vec{S} \cdot (\rho \vec{v}) = 0 \quad , \quad \frac{\vec{v}^2}{2} + \Phi + \frac{p}{\rho} + u = \text{constant} = B$$

incompressible $\quad \vec{\nabla} \cdot \vec{v} = 0 , \quad u = \text{constant}$

$\quad \nabla \rho = 0 \quad \rho = \text{constant}$

Here B is constant along streamlines (everywhere tangent to $\vec{v}$) which are equal to flow lines (path followed by fluid particles)

eg. pipe flow, changing area

water tanks

pitot tube & stagnation pressure

Irrotational Incompressible (Potential) Flow

$\vec{\nabla} \times \vec{v} = 0 \quad$ so $\quad \vec{v} = \vec{\nabla} \phi$

$\vec{\nabla} \cdot \vec{v} = 0 \quad$ so $\quad \nabla^2 \phi = 0$

ϕ ← potential for velocity field

eg. $\phi = v_0 x \quad$ uniform flow

$\phi = A \ln r \quad$ in 2-dim, pt source $\qquad \phi = \frac{\Gamma}{2\pi} \theta \quad$ 2-dim potential vortex

$\phi = A/r \quad$ in 3-dim, pt source

$\phi = A \vec{u} \cdot \vec{\nabla} \frac{1}{r} \quad$ dipole flow in 3-dim (used for ideal fluid flow around a sphere)

$\phi = A \vec{u} \cdot \vec{\nabla} \ln r \quad$ flow around cylinder

- $\frac{d}{dt} \vec{\nabla} \times \vec{v} = 0$, conserved along flow lines for ideal fluid

Sound Waves $\quad \left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right) \{p', \rho', \vec{v}\} = 0$

$c_s = \sqrt{\frac{B}{\rho_0}}$ speed of sound, $\quad M = \frac{v_0}{c_s}$ Mach number

can treat as incompressible if $M \ll 1$

Travelling wave solution: $\quad p' = f(\hat{n}\cdot\vec{r} - c_s t) \quad$ moves in $\hat{n}$

Viscous Fluids

viscous stress tensor

$$\sigma'_{ik} = \underbrace{\eta}_{\text{shear viscosity}} \left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\vec{\nabla}\cdot\vec{v}\right) + \underbrace{\zeta}_{\text{bulk viscosity}}\, \delta_{ik}\vec{\nabla}\cdot\vec{v}$$

modified equations

- Momentum conservation $\quad T_{ki} = v_k v_i \rho + \delta_{ki} p - \sigma'_{ki}$

$$\frac{\partial}{\partial t}(\rho\vec{v}) + \vec{\nabla}\cdot\hat{T} = \vec{f}$$

- Navier-Stokes (incompressible $\vec{\nabla}\cdot\vec{v} = 0$, with gravity)

$$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p \underbrace{- \nu\nabla^2\vec{v}}_{\text{new}} = \frac{\vec{f}}{\rho}$$

$\nu = \frac{\eta}{\rho}$ is kinematic viscosity $\quad [\nu] = m^2/s$

boundary condition: $\quad \vec{v} = 0$ at walls

or $\vec{v} = \vec{v}_{wall}$

Force on surface: $\quad -\hat{n}_i p + \hat{n}_k \sigma'_{ki}$

Energy: viscosity dissipates energy

causes heat gain $\left[\rho T \frac{ds}{dt} = \sigma'_{ik} \frac{\partial v_i}{\partial x_k}\right]$

Application

- Steady flows in pipes $\rightarrow$ most often quadratic profiles eg. $v_x \propto (L^2 - r^2)$ in circular pipe

Reynolds Number $R = \frac{uL}{\nu}$ $\quad u, L$ characteristic velocity, length

$\rightarrow$ dimensional analysis

- Flows with small R, $F_D = (6\pi\eta a)u$ for drag on sphere $\quad$ onset of detached vortices, turbulence with increasing R

$\boxed{\text{Chaos \& Non-linear Dynamics}}$ $\quad \dot{\vec{x}} = \vec{f}(\vec{x})$ 1st order form

- existence & uniqueness, trajectories do not cross
- **chaos** is aperiodic (irregular) behavior with sensitive dependence to initial conditions

$$\frac{d\mathcal{V}}{dt} = \int d\mathcal{V}\, \vec{\nabla}\cdot\vec{f}$$

phase space

$\vec{\nabla}\cdot\vec{f} = 0$ conservative

$\vec{\nabla}\cdot\vec{f} < 0$ dissipative

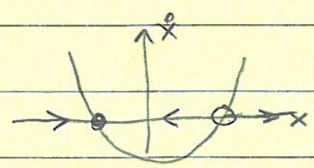
eg. damped nonlinear forced oscillator

Fixed Points $\quad \vec{f}(\vec{x}^*) = 0$ for $\vec{x}^*$ = fixed point

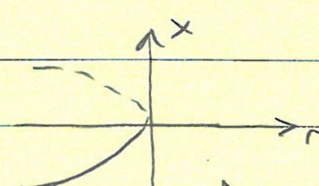
Bifurcations

"saddle-node" $\quad \dot{x} = r + x^2$

normal form



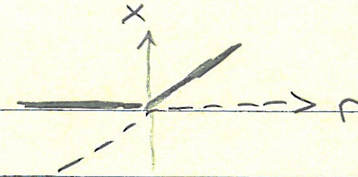
$r < 0$



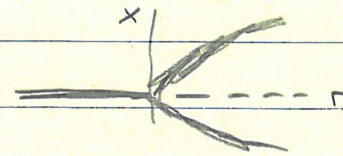
bifurcation diagram.

fixed points annihilate

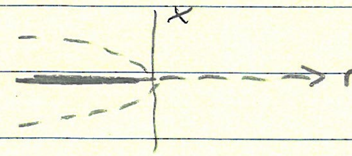
"transcritical" $\dot{x} = x(r-x)$



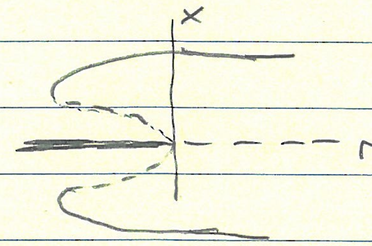
"supercritical pitchfork" $\dot{x} = rx - x^3$



"subcritical pitchfork" $\dot{x} = rx + x^3$



$\dot{x} = rx + x^3 - x^5$



hysteris

2-dim Fixed Points & Stability

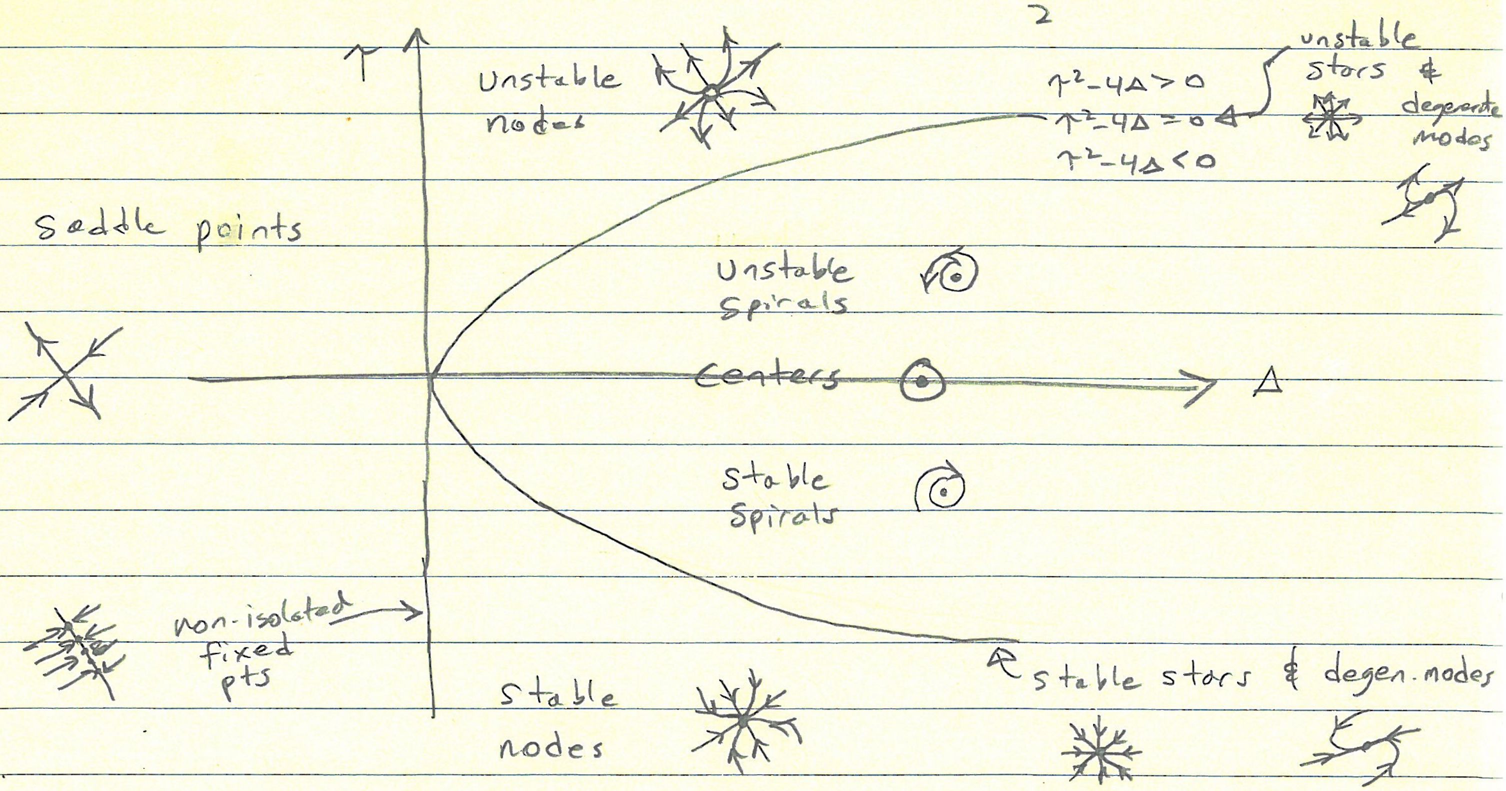
$\dot{x} = f(x,y)$ expand in $u = x - x^*$

$\dot{y} = g(x,y)$ $\quad v = y - y^*$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix}$$

$\tau = \mathrm{tr}\, M$ eigenvalues $M\vec{a} = \lambda \vec{a}$ from $\begin{pmatrix} u \\ v \end{pmatrix} = \vec{a} e^{\lambda t}$

$\Delta = \det M$ $\qquad \lambda_\pm = \dfrac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$



linearized analysis suffices for cases not on a "borderline"

Systems with conserved $E(\vec{x})$

- no attracting fixed points
- stable centers for isolated $\vec{x}^*$ at minimum of E

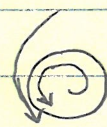
2-dim $\quad H(x,y) = \int^y dy' f_x(x,y') - \int^x dx' f_y(x',y)$

$\frac{dH}{dt} = 0 \iff \vec{\nabla}\cdot\vec{f} = 0$, critical points of $H(x,y)$ $\iff$ fixed points $\vec{f}(\vec{x}^*)=0$

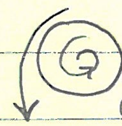
Also $\dot{x} = \mu\, \partial H/\partial y$, $\dot{y} = -\mu\, \partial H/\partial x$ Hamilton type equations

eg. any equation $\ddot{x} = -V'(x)$

Limit Cycles



stable



unstable

half-stable

eg $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$

fixed points for radius coordinate are limit cycles in x,y

Poincaré-Bendixson $\boxed{\text{2-dim}}$ trajectory C that is confined inside bounded region R either goes to fixed pt or limit cycle (no chaotic motion with strange attractor)

2-dim Bifurcations

saddle points, transcritical, pitchforks involve "collision" of fixed points as we change r (captured by 1-dim subspace)

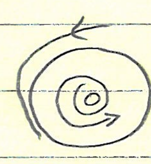
Hopf Bifurcation: $\quad \dot{r} = \mu r - r^3$ (supercritical) like pitchfork but now

$\dot{\theta} = \ldots$ has cycles for $r \neq 0$

$\mu < 0$

stable $r=0$ spiral fixed pt

$\mu > 0$



unstable $r=0$ spiral fixed pt

stable $r=\sqrt{\mu}$ limit cycle

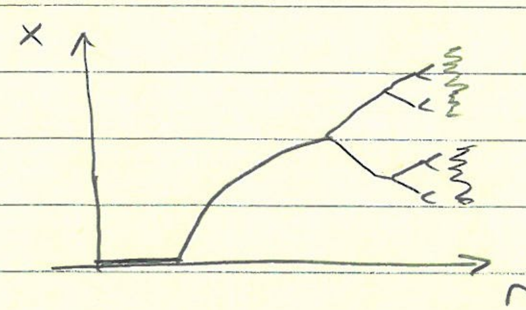
Chaos in Maps

Logistic Map: $x_{n+1} = r x_n (1 - x_n)$

fixed pts $\quad x^* = f(x^*)$

stability $\quad |f'(x^*)| > 1 \quad$ unstable

$\qquad\qquad |f'(x^*)| < 1 \quad$ stable



period doubling route to chaos

Lyapunov exponent

$x_0 \to \quad \cdots \to x_n \to$

$x_0 + \delta_0 \to \quad \to x_n + \delta_n \to$

$|\delta_n| = |\delta_0| \, e^{n\lambda}$

$\lambda < 0$ in period doubling regium

$\lambda > 0$ for chaos, sensitive exponentially to initial cond.

Feigenbaum number

spacings $r_k - r_{k-1}$ between period doublings are characterized by universal # $\quad \delta = \lim_{n\to\infty} \dfrac{r_n - r_{n-1}}{r_{n+1} - r_n}$

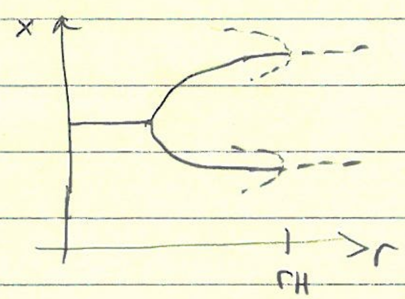
Stretchin & Folding

Lorenz Equations

$\dot{x} = \sigma (y - x)$

$\dot{y} = rx - y - xz$

$\dot{z} = -bz + xy$



- dissipative,
- trajectories bounded
- $r > r_H$, chaos with "strange attractor"
- itself has exp. sensitivity to initial conditions
- fractal dimension (often)

Lyapunov Exponents $\quad \lambda_1 + \lambda_2 + \lambda_3 = \vec{\nabla} \cdot \vec{f} < 0$

$V(t) = V_0 \, e^{(\lambda_1 + \lambda_2 + \lambda_3)t}$

$\uparrow$ $\lambda_1 > 0$ chaotic

phase space gets stretched, contracted & folded

Fractals

$$d_F = \lim_{a \to 0} \frac{\log N(a)}{\log(a_0/a)}$$

$N(a)$: # objects length a that are needed

$\log(a_0/a)$: size a of objects relative

$$\log\left(\frac{a_0}{a}\right) = \underbrace{\log(a_0)}_{\text{choice here doesn't matter}} - \log(a)$$

eg. Cantor Set $\quad 0 < d_F < 1$

Koch curve $\quad 1 < d_F < 2$

Also saw $\quad d_F = 1 - \frac{\lambda_1}{\lambda_2}$ for case with $\lambda_1 > 0,\ \lambda_2 < 0$, $\lambda_3 = 0$

Kolmogorov scaling for turbulence (3-d fluid)

$$\lambda_0 \ll \lambda \ll L$$

λ_0: scale where energy dissipates

λ: vortices at scale λ

L: box size

$R = \frac{v_0 \lambda_0}{\nu} \sim 1$

mean energy transfer / unit time / unit mass $= \epsilon \sim \frac{v_\lambda^3}{\lambda}$

$\Updownarrow$

Kinetic energy / unit mass / unit wave # k $= E(k) \sim \epsilon^{2/3} k^{-5/3}$

$$\lambda = \frac{1}{k}$$