Physics 8.321, Fall 2021
Homework #5

Due **Friday, November 5** by 8:00 PM.

1. Define the *coherent state* $|\phi\rangle = e^{\phi a^\dagger}|0\rangle$, where ϕ is a complex number, $a^\dagger$ is the creation operator for a harmonic oscillator, and $|0\rangle$ is the oscillator ground state. Show that $|\phi\rangle$ has the following properties:

(a) $|\phi\rangle = \sum \frac{\phi^n}{\sqrt{n!}}|n\rangle$

(b) $a|\phi\rangle = \phi|\phi\rangle$

(c) $\langle\phi|\phi'\rangle = e^{\phi^*\phi'}$

(d) $\langle\phi| : A(a^\dagger, a) : |\phi'\rangle = e^{\phi^*\phi'} A(\phi^*, \phi')$,
where $: A(a^\dagger, a) :$ is "normal ordered" so that all creation operators $a^\dagger$ are to the left of all annihilation operators a. You may assume that the function $A(x, y)$ can be expressed as a power series in the arguments x, y (don't worry about convergence)

(e) $\int \frac{d\phi^* d\phi}{2\pi i} e^{-\phi^*\phi}|\phi\rangle\langle\phi| = \mathbb{1}$. (completeness for coherent states)

2. Define a *squeezed state* to be a state of the form

$$|\alpha, \beta, \gamma\rangle = e^{\alpha + \beta a^\dagger + \gamma(a^\dagger)^2}|0\rangle \tag{1}$$

in the single harmonic oscillator Hilbert space

(a) Compute the norm $\langle\alpha, \beta, \gamma|\alpha, \beta, \gamma\rangle$ in the special case $\beta = 0$. What is the condition needed for this norm to be finite? Extra credit: can you generalize your result to $\beta \neq 0$?

(b) Show that the position basis state $|x'\rangle$ can be written in the form (1), and find the associated values $\alpha(x'), \beta(x'), \gamma(x')$. Does your expression for $|x'\rangle$ give a state of finite norm in the Hilbert space?

3. For each part of this problem you are asked to find an approximation to the energies of one or more of the lowest-lying quantum states for a particular potential. You may use any approximation technique you wish to determine the energy eigenvalues. You may use a computer if you wish, or you can work by hand. (Note that a good way to check your answers yourself is to try using several different methods!) Please include a sketch or graph of the eigenfunctions in each case. In all parts you should use units with $\hbar = m = 1$.

(a) Find the ground state and first excited state energies for a particle in the 1D potential

$$V(x) = \frac{1}{4}x^4 .$$

(b) Find the ground state and first excited state energies for a particle in the 1D potential

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{24}x^4 .$$

(c) Find the ground state energy for a pair of particles in the harmonic oscillator potential $V(x) = x^2/2$. The interaction energy between the particles is given by $-\sqrt{2}|x - y|$, where x, y are the positions of the two particles. You may assume that these particles are fermions, so that $\psi(x, y) = -\psi(y, x)$. Note that the Hamiltonian for this system is equivalent to that of a single particle moving in two dimensions x, y in the potential

$$W(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2 - \sqrt{2}|x - y| \, .$$

(d) (Extra credit, optional): Find the ground state energy for a particle in the 2D potential

$$V(x, y) = \frac{1}{4}x^4 + \frac{1}{6}y^6 + 2xy \, .$$