Problem Set 11- Optional

Due (to get extra credit): Monday 5pm, May 2nd, via Canvas upload or in envelope outside 26-255
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1 Scattering of light by a trapped atom

This problem will investigate the matrix element involved in scattering of light by a trapped atom. The light field propagating in the x-direction has a phase factor e^{ikx}. In the quantum description, the operator e^{ikx} generates a translation in momentum space by $\hbar k$ - this is the origin of photon recoil. If the wavefunction for the initial and final motional state of the atom are $\psi_i(x)$ and $\psi_f(x)$, the matrix element will be $\langle \psi_f | e^{ikx} | \psi_i \rangle = \int \mathrm{d}x\, \psi_f^*(x) e^{ikx} \psi_i(x)$.

a) Show that $\langle \psi_f | e^{ikx} | \psi_i \rangle = \langle \phi_f(p+k) | \phi_i(p) \rangle = \int \frac{\mathrm{d}p}{2\pi} \phi_f^*(p+k) \phi_i(p)$, where $\phi_{i,f}(p) = \int \frac{\mathrm{d}p}{2\pi} e^{-ipx} \psi_{i,f}(x)$ is the Fourier transform of $\psi_{i,f}(x)$.

For scattering of free particles, this means that the final x-momentum is $\hbar k$ larger than the initial one, so the atom received a kick. Let's now consider a harmonically trapped atom (mass M, frequency ω). We assume for simplicity that the harmonic trap is the same in the initial and the final electronic state (that's surely quite a simplifying assumption, we could certainly be more general). We need to evaluate the matrix element

$$\left\langle m | e^{ikx} | n \right\rangle$$

where n and m are the quantum numbers in the initial and final state of the scattering process. We know that $\psi_n(x) = \langle x | n \rangle = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} e^{-M\omega x^2/2\hbar} H_n\left(\sqrt{\frac{M\omega}{\hbar}} x\right)$, with $H_n(x)$ the Hermite polynomials. We could use the fact that $\phi_n(p)$ must have the exact same form, just replacing x by $p/M\omega$, thanks to the symmetry of the harmonic oscillator Hamiltonian, and then evaluate the matrix element in momentum representation: It will be the overlap of the "kicked" wavefunction $\phi_m(p+k)$ with the original $\phi_n(p)$. But to obtain an analytical result more easily, we can instead use raising and lowering operators a, $a^\dagger$ for the atomic motion (not to be confused with photon raising / lowering operators), writing

$$x = x_0(a + a^\dagger)$$
$$p = ix_0 M\omega(a^\dagger + a)$$

with $x_0 = \sqrt{\frac{\hbar}{2M\omega}}$ a measure of the harmonic oscillator ground state size.

b) "Disentangle" $e^{ikx} = e^{ikx_0(a+a^\dagger)}$ into a product of $e^{ikx_0 a^\dagger}$ times $e^{ikx_0 a}$ and an extra factor.
Hint: Make use of $e^{A+B} = e^A e^B e^{-[A,B]/2}$, valid for operators A and B that commute with $[A, B]$.

c) Find $a^j |n\rangle$ and therefore $e^{ikx_0 a} |n\rangle = \sum_j \frac{(ikx_0)^j a^j}{j!} |n\rangle$.

d) Show that the matrix element is given by

$$\left\langle m|e^{ikx}|n\right\rangle = \exp\left(-\frac{1}{2}(kx_0)^2\right)\sqrt{\frac{n_<!}{n_>!}}\,(ikx_0)^{\Delta n} L_{n_<}^{\Delta n}\left[(kx_0)^2\right]$$

where $n_<$ and $n_>$ is the smaller and larger of m and n, respectively, $\Delta n = |m - n|$, and the generalized Laguerre polynomials are defined as

$$L_n^\alpha(X) = \sum_{m=0}^{n} (-1)^m \left(\begin{array}{c} n+\alpha \\ n-m \end{array}\right) \frac{X^m}{m!}$$

e) For large quantum numbers $m, n \gg 1$, we should retrieve the classical result from lecture. For this, use the asymptotic behavior of the generalized Laguerre polynomials, for fixed α and large $n \gg 1$:

$$L_n^\alpha(X) \to \left(\frac{n}{X}\right)^{\alpha/2} e^{X/2} J_\alpha(2\sqrt{nX})$$

where $J_\alpha(x)$ is the Bessel function, and relate the (large) energy $\sim n\hbar\omega$ of the quantum h.o. to the energy of a classical harmonic oscillation with amplitude x_{cl}. You should find

$$\left|\left\langle m|e^{ikx}|n\right\rangle\right|^2 = J_{\Delta n}^2(\beta)$$

with $\beta = kx_{\mathrm{cl}}$, which is the result found in lecture for the classical case, where the oscillating atom experiences a phase-modulated electric field.

Reference: An excellent read for this problem is the paper *Laser Cooling of Atoms* by Wineland and Itano, PRA 20, 1521 (1979), discussing laser cooling in free space as well as for confined particles. Using the above result, they derive that one could cool typical atoms to a temperature of hundreds of Nanokelvin. Wineland won the Nobel prize 2012 (jointly with Serge Haroche) not directly for laser cooling (that prize was given out in 1997 to only a subset of those involved in its invention, to Bill Phillips, Claude Cohen-Tannoudji and Steven Chu, see e.g. Bill Phillips Nobel prize lecture, https://www.nobelprize.org/prizes/physics/1997/phillips/lecture/), but "for ground-breaking experimental methods that enable measuring and manipulation of individual quantum systems“.

2 Saturation Spectroscopy

This problem guides you through the concepts of saturation spectroscopy. This is one of the techniques to perform Doppler-free spectroscopy, i.e. to extract a narrow line (with the natural linewidth) in a gas with a broad velocity distribution. It nicely illustrates the combination of homogenous and inhomogeneous line broadening. Saturation spectroscopy is frequently used to lock lasers to atomic lines. You should not get into nasty integrals for this problem. The drawn lineshapes should clearly show the basic features, but don't have to be exact.

a)**Homogeneous broadening**
Consider a dilute gas of density n composed of atoms with resonant frequency ω_0 and linewidth Γ. The gas is exposed to monochromatic light of frequency $\omega_L = \omega_0 + \delta$ and intensity $I = s\, I_{SAT}$ where I_{SAT} is the saturation intensity. Let us ignore the effects of the motion of the atoms, i.e. consider temperature $T = 0$. What are the densities of atoms in the ground state n_1 and the excited state n_2, including the effect of saturation? What is the cross-section for absorption? The gas is in a box of length L along the direction of the incoming light. What fraction of the light is absorbed? (This is a small fraction, so don't worry about the effect of light attenuation on the saturation of the sample).

b)**The Bennet hole**
Now, let's endow these atoms with a mass m and a temperature T. Let the incoming light have a wavevector $\mathbf{k_L}$ along the z-axis. What is the population density distribution in the ground state $n_1(v_z)$ as a function of v_z, the component of velocity in the z-direction? You should find that the light "burns a hole" (known as the Bennet hole) in the distribution of absorbers. What is the position of the hole? How do the width and depth (relative to the population for $s = 0$) vary with saturation parameter s?

c)**Inhomogeneous broadening**
Consider that we sweep the frequency of the incident laser ω_L and measure the (small) absorption of the beam. Determine the fraction of the light absorbed as a function of s and δ and compare with its value at $s = 0$ (you don't need to solve the integral). For high temperatures ($k_L \bar{v_z} \gg \Gamma$), what is the width of the absorption line? Does saturation affect the width?

d)**Saturation spectroscopy**
To actually get some benefit from saturating the gas, we introduce a second laser beam.

i) We can add a weak (counter-propagating) probe beam at frequency ω_p with wavevector k_p. What is the absorption of this beam, including the effects of the saturating beam (k_L, ω_L)? Again, just write the integral, and take the length of the box along

$\mathbf{k_p}$ to be L. Draw the absorption line shape, identifying the position and width of its features.

ii) We can also just retroreflect the original beam. Draw the population distribution $n_1(v_z)$ for $\omega_L \neq \omega_0$ and $\omega_L = \omega_0$. Identifying the depth of the Bennet hole(s), draw the lineshape of absorption of the retroreflected beam (i.e. we scan ω_L). What is the width of the central feature (at $\delta = 0$)?