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I was going to send you an email with this document and the new images attached, but it (as usual) got too long. So I decided to put everything into this document instead.

0. The $\phi : \mathbb{Z}^2 \to \mathbb{C}$ that is used in this document is given by:

$$\begin{cases} \dfrac{301}{384} - \dfrac{7i}{48}, & (x,y) = (0,0) \\ \dfrac{7}{96} + \dfrac{i}{24}, & (x,y) = (-1,0) \\ \dfrac{3}{32} + \dfrac{i}{24}, & (x,y) = (1,0) \\ -\dfrac{1}{48}, & (x,y) = (2,0) \\ -\dfrac{1}{48}, & (x,y) = (-2,0) \\ \dfrac{7}{96} + \dfrac{i}{24}, & (x,y) = (0,1) \\ \dfrac{7}{96} + \dfrac{i}{24}, & (x,y) = (0,-1) \\ -\dfrac{7}{192} - \dfrac{i}{96}, & (x,y) = (0,2) \\ -\dfrac{7}{192} - \dfrac{i}{96}, & (x,y) = (0,-2) \\ \dfrac{1}{96}, & (x,y) = (0,3) \\ \dfrac{1}{96}, & (x,y) = (0,-3) \\ -\dfrac{1}{768}, & (x,y) = (0,4) \\ -\dfrac{1}{768}, & (x,y) = (0,-4) \\ \dfrac{1}{192}, & (x,y) = (-1,-1) \\ \dfrac{1}{192}, & (x,y) = (-1,1) \\ -\dfrac{1}{192}, & (x,y) = (1,1) \\ -\dfrac{1}{192}, & (x,y) = (1,-1) \end{cases} \tag{1}$$

1.

And so the FT of ϕ, or $\hat{\phi}$, is the following:

$$\boxed{\hat{\phi}(\xi_1,\xi_2)=\frac{1}{3}\left(3-\frac{i}{2}\sin^2(\xi_1/2)-\sin^4(\xi_1/2)-\frac{i}{2}\sin^4(\xi_2/2)-\sin^8(\xi_2/2)-\frac{i}{16}(\sin\xi_1\cos\xi_2-\sin\xi_1)\right)} \tag{2}$$

Taylor expanding this around $(0,0)$ gives

$$\left(1-\frac{i\xi_2^4}{96}+\frac{i\xi_2^6}{576}+\mathcal{O}\left(\xi_2^7\right)\right)+\xi_1\left(\frac{i\xi_2^2}{96}-\frac{i\xi_2^4}{1152}+\frac{i\xi_2^6}{34560}+\mathcal{O}\left(\xi_2^7\right)\right)-\frac{i\xi_1^2}{24}$$
$$+\xi_1^3\left(-\frac{i\xi_2^2}{576}+\frac{i\xi_2^4}{6912}-\frac{i\xi_2^6}{207360}+\mathcal{O}\left(\xi_2^7\right)\right)-\left(\frac{1}{48}-\frac{i}{288}\right)\xi_1^4+\mathcal{O}\left(\xi_1^5\right). \tag{3}$$

I have checked that $\hat{\phi}(0,0)=1$. With this, Taylor-expanding

$$\log\left(\frac{\hat{\phi}((\xi_1,\xi_2)+(0,0))}{\hat{\phi}(0,0)}\right) \tag{4}$$

gives

$$\left(-\frac{i\xi_2^4}{96}+\mathcal{O}\left(\xi_2^5\right)\right)+\xi_1\left(\frac{i\xi_2^2}{96}-\frac{i\xi_2^4}{1152}+\mathcal{O}\left(\xi_2^5\right)\right)+\xi_1^2\left(-\frac{i}{24}+\frac{\xi_2^4}{2048}+\mathcal{O}\left(\xi_2^5\right)\right)+\mathcal{O}\left(\xi_1^3\right). \tag{5}$$

With $\vec{\xi}\equiv(\xi_1,\xi_2)$, we read off $iP(\vec{\xi})$:

$$\boxed{\Pi(\vec{\xi})=iP(\vec{\xi})=-\frac{i\xi_2^4}{96}+\frac{i\xi_1\xi_2^2}{96}-\frac{i\xi_1^2}{24}} \tag{6}$$

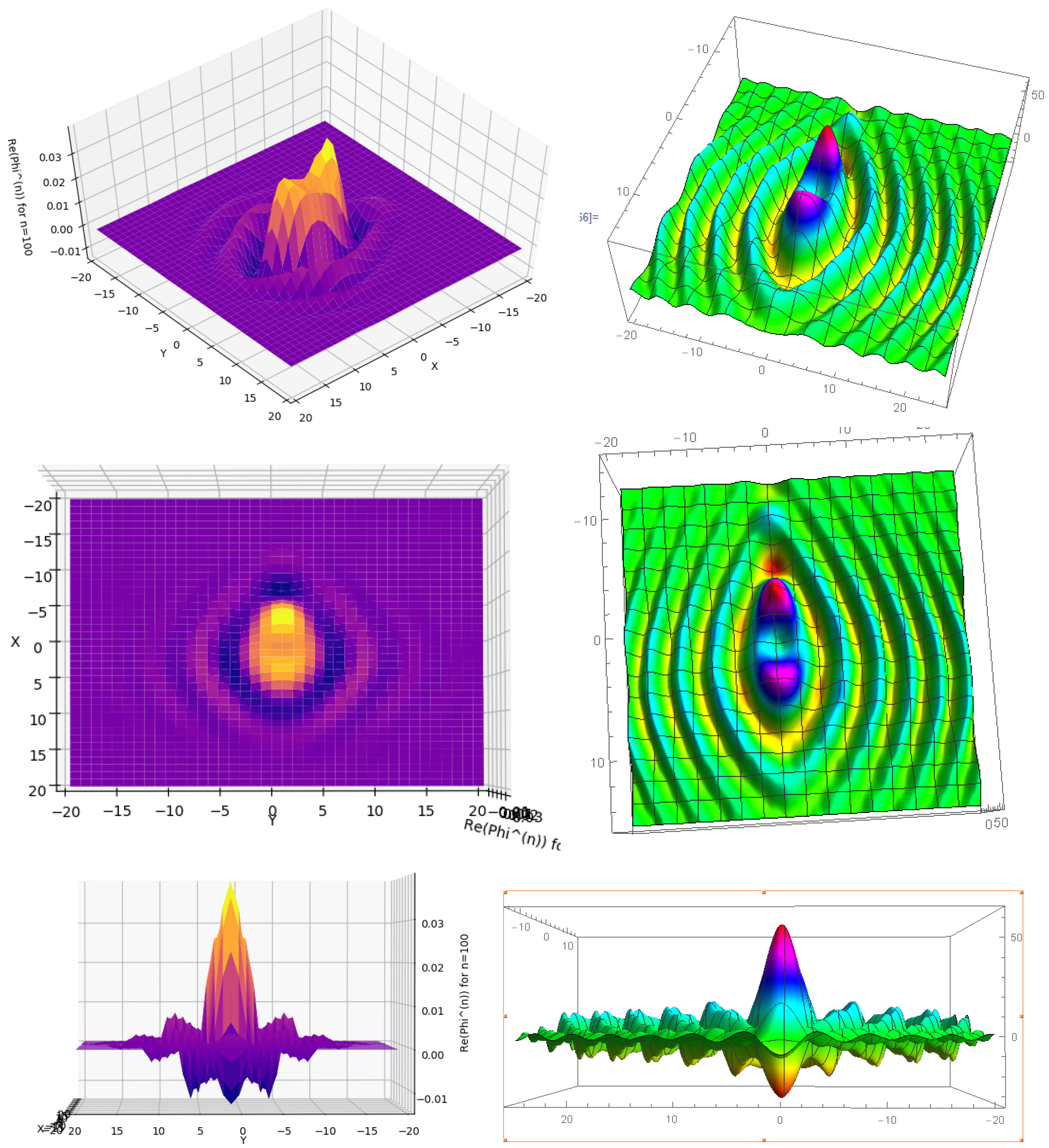
Once cos and sin in $\hat{\phi}$ have been replaced by $e^{i\cdots}$ we can write $\hat{\phi}$ as

$$\frac{1}{192}e^{i\xi_2-i\xi_1}-\frac{1}{192}e^{i\xi_1+i\xi_2}+\frac{1}{192}e^{-i\xi_1-i\xi_2}-\frac{1}{192}e^{i\xi_1-i\xi_2}+\left(\frac{3}{32}+\frac{i}{24}\right)e^{i\xi_1}$$
$$+\left(\frac{7}{96}+\frac{i}{24}\right)e^{-i\xi_1}-\frac{1}{48}e^{-2i\xi_1}-\frac{1}{48}e^{2i\xi_1}+\left(\frac{7}{96}+\frac{i}{24}\right)e^{-i\xi_2}-\frac{1}{768}e^{-4i\xi_2}+\frac{1}{96}e^{-3i\xi_2}$$
$$-\left(\frac{7}{192}+\frac{i}{96}\right)e^{-2i\xi_2}+\left(\frac{7}{96}+\frac{i}{24}\right)e^{i\xi_2}-\left(\frac{7}{192}+\frac{i}{96}\right)e^{2i\xi_2}+\frac{1}{96}e^{3i\xi_2}-\frac{1}{768}e^{4i\xi_2}+\left(\frac{301}{384}-\frac{7i}{48}\right) \tag{7}$$

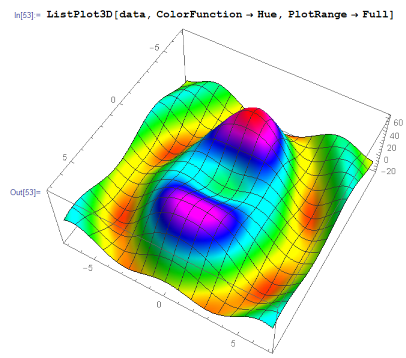
from which we can read off the values of $\phi:\mathbb{Z}^2\to\mathbb{C}$.

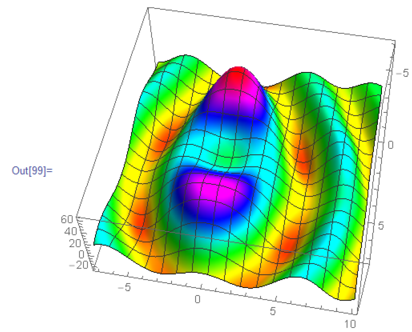
2. I rescaled the x, y in the previous $H(x, y)$ with $t = 100$ and got good correspondence after some hours of numerical integration. (t here appears in H_p^t and t^E in the paper). I wish I had thought about the possibility that the stretchingcould be so extreme that the peaks appear rotated earlier...

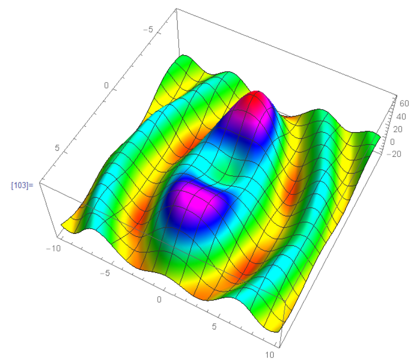
Here are the images. Purple-ly plots are convolution powers with $n = 100$. Green-ish plots are the approximated attractor $H_P^t(x, y)$ with $t = 100$. (I don't know what the exact correspondence between t and n is for now, but I think setting them equal is an o.k. starting point.)



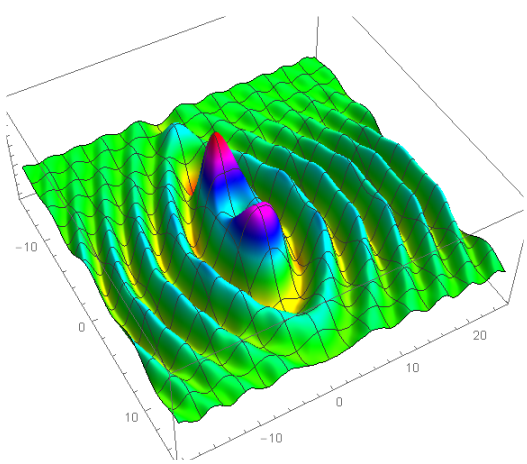
3. I'm still running the calculations for $H_P^t(x,y)$ again with $t = 100$. The output looks good. I'm integrating in "batches" and will aggregate the data as I go along. This should give us a lot of data for future stretching/contracting/scaling at different values of t. Below is the first few batches (around the origin). The peaks have the correct orientation, and are very much like the convolution powers we've been generating!

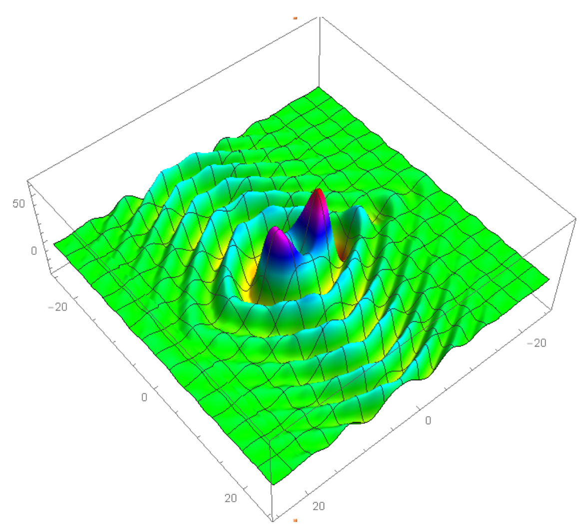






This is what I have as of March 9 and March 21, 2020:





There are about 2×10^5 data points in the second figure.

4. Here's the Python code I use to calculate the convolution powers

```
import numpy as np
import matplotlib.pyplot as plt
from scipy import signal
from matplotlib import cm, colors
from mpl_toolkits.mplot3d.axes3d import Axes3D
import operator
import time
from numpy import unravel_index


def fast_convolve(n_times, support_bound, drift):
    Phi = np.zeros(shape=(9,9),dtype=np.complex_)

    Phi[ 0+9//2][ 0+9//2]  = complex(301/384,-7/48)
    Phi[ 0+9//2][-1+9//2]  = complex(7/96,1/24)
    Phi[ 0+9//2][ 1+9//2]  = complex(3/32,1/24)
    Phi[ 0+9//2][ 2+9//2]  = -1/48
    Phi[ 0+9//2][-2+9//2]  = -1/48
    Phi[ 1+9//2][ 0+9//2]  = complex(7/96,1/24)
    Phi[-1+9//2][ 0+9//2]  = complex(7/96,1/24)
    Phi[ 2+9//2][ 0+9//2]  = -complex(7/192,1/96)
    Phi[-2+9//2][ 0+9//2]  = -complex(7/192,1/96)
    Phi[ 3+9//2][ 0+9//2]  = 1/96
    Phi[-3+9//2][ 0+9//2]  = 1/96
    Phi[ 4+9//2][ 0+9//2]  = -1/768
    Phi[-4+9//2][ 0+9//2]  = -1/768
    Phi[-1+9//2][-1+9//2]  =  1/192
    Phi[ 1+9//2][-1+9//2]  =  1/192
    Phi[ 1+9//2][ 1+9//2]  = -1/192
    Phi[-1+9//2][ 1+9//2]  = -1/192

    conv_power = np.copy(Phi)
    offset = np.array([0,0])

    i=0
    if drift:
        while i < n_times:
            i += 1
            init_vec = unravel_index(np.absolute(conv_power).argmax(), np.absolute(conv_power).shape)
            conv_power = signal.convolve2d(Phi, conv_power, 'full')
            after_vec = unravel_index(np.absolute(conv_power).argmax(), np.absolute(conv_power).shape)
            offset += np.subtract(init_vec , after_vec)

            dim_f = np.shape(conv_power)

            if dim_f[0] > support_bound or dim_f[0] > support_bound:
                conv_power = crop(conv_power, support_bound)
    else:
        while i < n_times:
            i += 1
            conv_power = signal.convolve2d(Phi, conv_power, 'full')
            dim_f = np.shape(conv_power)

            if dim_f[0] > support_bound or dim_f[0] > support_bound:
                conv_power = cropND(conv_power, support_bound)
return conv_power

def cropND(img, sup_bd):
    if sup_bd < np.shape(img)[0] and sup_bd < np.shape(img)[1]:
    dim = np.shape(img)
    return img[(dim[0]//2)-sup_bd//2:(dim[0]//2)+sup_bd//2,
        (dim[1]//2)-sup_bd//2:(dim[1]//2)+sup_bd//2]

def crop(img, sup_bd):
    center = unravel_index(np.absolute(img).argmax(), np.absolute(img).shape)
    return img[center[0]-sup_bd//2:center[0]+sup_bd//2,
        center[1]-sup_bd//2:center[1]+sup_bd//2]

if __name__ == '__main__':

while True:

    n_times = int(input('Convolve how many times? '))
    support_bound = int(input('NxN suppport bound, N = '))
    drift_ans = str(input('Expect asymetric drift? [y/n]: '))
    print('Calculating...')
    start = time.time()

    if drift_ans == 'y':
```

```
        drift = True
    elif drift_ans == 'n':
        drift = False
    else:
        print('WARNING: Write "y" for YES and "n" for NO.')
        print('-----------------------------------------')
        print('\n')
        continue

    data = np.real(fast_convolve(n_times, support_bound, drift))
    dim = np.shape(data)
    x = range((-dim[0]//2)+1,(dim[0]//2)+1)
    y = range((-dim[1]//2)+1,(dim[1]//2)+1)

    hf = plt.figure()
    ha = hf.add_subplot(projection='3d')
    ha.set_xlim(-np.shape(data)[0]//2, np.shape(data)[0]//2)
    ha.set_ylim(-np.shape(data)[0]//2, np.shape(data)[0]//2)

    drift = False # I'm setting this for now for testing
    if drift:
        ha.set_xlabel('\n \n X \n \n DRIFTING CONVOLUTION POWERS!')
        ha.set_ylabel('\n \n Y \n \n DRIFTING CONVOLUTION POWERS!')
        ha.set_zlabel(' \n \n Re(Phi^(n)) for n='+str(n_times))
    else:
        ha.set_xlabel('X')
        ha.set_ylabel('Y')
        ha.set_zlabel(' \n \n Re(Phi^(n)) for n='+str(n_times))

    X, Y = np.meshgrid(x, y)
    surf = ha.plot_surface(X, Y, data , rstride=1, cstride=1, cmap='plasma', edgecolor='none', linewidth=0.2)

    end = time.time()
    print('Time elapsed (s): ', end - start)

    plt.show()
    print('-------------------------------------')
```

5. Here's the Mathematica code that I use to approximate and plot the attractor:

```
H[i_, j_] := NIntegrate[Cos[(-i*x/(100^(1/2)) - j*y/(100^(1/4)) - y^4/96 + y^2*x/96 - x^2/24)],
 {x, -11, 11}, {y, -11, 11}, PrecisionGoal -> 4,
Method -> "OscillatorySelection"]
data = Flatten[
Table[{i, j, H[i, j]}, {i, -7, 7, 0.1}, {j, 7, 10, 0.1}], 1];
Export["ConvolutionPowers/data5.csv", data, "CSV"]


ListPlot3D[Import["ConvolutionPowers/data5.csv"], ColorFunction -> Hue, PlotRange -> Full]
```

The output looks something like