Single photons

From Atomic Physics Wiki

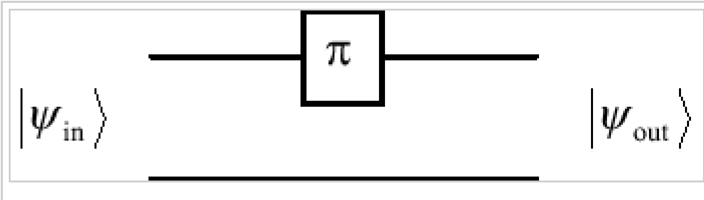
A single photon, mathematically represented by the number eigenstate $|1\rangle$, physically describes the electromagnetic field corresponding to the lowest nonzero energy eigenstate of a single mode cavity. $|0\rangle$ is the vacuuum state. A great deal of physics can be understood by considering what happens to just the $|1\rangle$ and $|0\rangle$ states, through a variety of optical components. This section uses such an approach to explore three of the most basic components -- two linear components: phase shifters, beam splitters, and one nonlinear component: Kerr cross-phase modulation. These are the building blocks of linear and non-linear interferometers; their physical behavior provides helpful intuition for quantum behavior.

▼ Category:8.422
▶ Category:Quantum Light
▶ Category:Optical Bloch Equations
▶ Category:Bose-Einstein condensates
▶ Category:Ion traps and quantum information
▶ Category:Light forces
▶ Category:Photon-atom interactions
Introduction to Atomic Physics
Quantum light: states and dynamics
Optical Bloch equations
Photon-atom interactions
Frontiers in Atomic Physics
Ion traps and quantum information
Quantum gases

Contents

Beamsplitters and phase shifters

The number state $|1\rangle$ evolves through propagation in free space to become $e^{i\omega t}|1\rangle$ after time t. In a medium with a different index of refraction, however, light propagates at a different velocity, giving, for example, $e^{i\omega' t}|1\rangle$. Such a phase difference is only physically meaningful, however, when compared with a reference. Let us therefore introduce a pair of modes, each with either 0 or 1 photons, depicted as two lines, and using a box to indicate a segment through which one mode propagates at a different velocity. For example:



depicts two modes, in which the top photon has its phase shifted by π relative to the bottom one. It is clear that any relative phase shift can be imparted between the two modes, by an appropriate experimental setup; experimentally, this can be accomplished with different thicknesses of glass, or by lengthening one path versus the other.

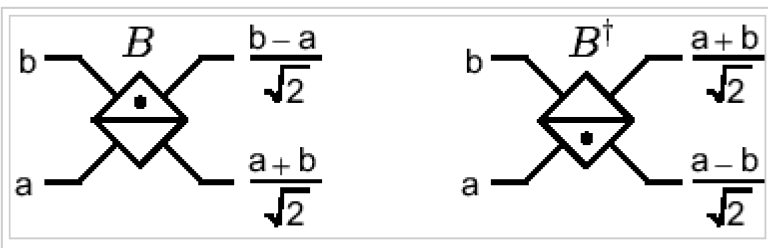
Two modes of light can be mixed by a beamsplitter (as we have previously seen). A beamsplitter B acts with Hamiltonian

$$H_{bs} = i\theta\left(ab^\dagger - a^\dagger b\right) ,$$

on the two modes, with corresponding operators a and b. Transformation of light through this beamsplitter is given by $e^{iH\theta}$, where θ is the angle of the beamsplitter, giving the unitary operation

$$B = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right] .$$

We may depict this as:



Note the use of a $\cdot$ to distinguish the ports; this is needed because we have adopted a phase convention for the beamsplitter which obviates the need to keep track of an extra factor of i. In the Heisenberg picture, B transforms a and b as

$$BaB^\dagger = a\cos\theta + b\sin\theta \quad \text{and} \quad BbB^\dagger = -a\sin\theta + b\cos\theta .$$

This can be verified using the the Baker--Campbell--Hausdorf formula,

$$e^{\lambda G} A e^{-\lambda G} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n ,$$

where λ is a complex number, A, G, and C_n are operators, and C_n is defined recursively as the sequence of commutators $C_0 = A, C_1 = [G, C_0], C_2 = [G, C_1], C_3 = [G, C_2], \ldots, C_n = [G, C_{n-1}]$.

Since it follows from $[a, a^\dagger] = 1$ and $[b, b^\dagger] = 1$ that $[G, a] = b$ and $[G, b] = -a$, for $G \equiv ab^\dagger - a^\dagger b$, we obtain for the expansion of $BaB^\dagger$ the series coefficients $C_0 = a$, $C_1 = [G, a] = b$, $C_2 = [G, C_1] = -a$, $C_3 = [G, C_2] = -[G, C_0] = -b$, which in general are

$$
\begin{aligned}
C_{n\,\text{even}} &= i^n a \\
C_{n\,\text{odd}} &= -i^{n+1} b\,.
\end{aligned}
$$

From this, our desired result follows straightforwardly:

$$
\begin{aligned}
BaB^\dagger &= e^{\theta G} a e^{-\theta G} \\
&= \sum_{n=0}^{\infty} \frac{\theta^n}{n!} C_n \\
&= \sum_{n\,\text{even}} \frac{(i\theta)^n}{n!} a - i \sum_{n\,\text{odd}} \frac{(i\theta)^n}{n!} b \\
&= a\cos\theta + b\sin\theta\,.
\end{aligned}
$$

The transform $BbB^\dagger$ is trivially found by swapping a and b in the above solution.

Beamsplitters with single photon inputs

How does B act on a single photon input? On input $|10\rangle$, letting the modes be a on the right, and b on the left, we get

$$
\begin{aligned}
B|10\rangle &= Bb^\dagger|00\rangle = Bb^\dagger B^\dagger B|00\rangle \\
&= (-a^\dagger \sin\theta + b^\dagger \cos\theta)|00\rangle \\
&= -\sin\theta|01\rangle + \cos\theta|10\rangle\,.
\end{aligned}
$$

Similarly, $B|01\rangle = \cos\theta|01\rangle + \sin\theta|10\rangle$. This indicates that $\theta = \pi/4$ corresponds to a $50/50$ beamsplitter. Note that B does not destroy any photons; it can only move them between the two modes. Mathamatically, this arises from the fact that it commutes with the total photon number operator, $a^\dagger a + b^\dagger b$.

If both input modes contain photons, the output state does not have as simple a form as above. In particular, we find that

$$
\begin{aligned}
B|11\rangle &= Ba^\dagger B^\dagger Bb^\dagger B^\dagger B|00\rangle \\
&= (a^\dagger \cos\theta + b^\dagger \sin\theta)(-a^\dagger \sin\theta + b^\dagger \cos\theta)|00\rangle \\
&= (-{a^\dagger}^2 \cos\theta\sin\theta + a^\dagger b^\dagger \cos^2\theta - b^\dagger a^\dagger \sin^2\theta + {b^\dagger}^2 \sin\theta\cos\theta)|00\rangle \\
&= -\sqrt{2}\cos\theta\sin\theta|02\rangle + (\cos^2\theta - \sin^2\theta)|11\rangle + \sqrt{2}\sin\theta\cos\theta|20\rangle\,,
\end{aligned}
$$

so it is possible for the output to be found to have both photons in one mode. Since we'd like to avoid such cases in this section, let us restrict our attention here to the case when inputs to beampsplitters have a total of one photon at most.

Dual-rail photon states and two-level systems

Omitting the vacuum state $|00\rangle$, the two-mode state space we shall consider thus has a basis state spanned by $|01\rangle$, and $|10\rangle$. We call this the dual-rail photon state space (this Hilbert space behaves much like a a two-level system). Note that an arbitrary state in this space as $|\psi\rangle = \alpha|01\rangle + \beta|10\rangle$, where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$.

From Bloch's Theorem, it follows that any dual-rail photon state can be generated using phase shifters and beamsplitters. Specifically, if we write $|\psi\rangle$ as a two-component vector, then the action of B is

$$B(\theta)\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}.$$

Similarly, the action of a phase shifter P of phase ϕ is

$$P(\phi)\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=e^{-i\theta/2}\begin{bmatrix}e^{i\theta/2} & 0\\ 0 & e^{-i\theta/2}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix},$$

where the overall phase $e^{i\theta/2}$ is irrelevant and can be dropped in the following.

Arbitrary single qubit operations with phase shifters and beam splitters

Employing Pauli matrices

Let us define the Pauli matrices as

$$\begin{aligned} X &= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\ Y &= \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\\ Z &= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \end{aligned}$$

Phase shifter and Beamsplitter operators

In terms of these, we find that the phase shifter and beamsplitter operators may be expressed as

$$\begin{aligned} B(\theta) &= \exp[i\theta Y]\\ P(\phi) &= \exp[i\phi Z/2]. \end{aligned}$$

Rotations of a two-level quantum system

These are rotations of a two-level system, about the axes $\hat{y}$ and $\hat{z}$, by angles 2θ and ϕ, respectively. The standard rotation operator definitions are

$$
\begin{aligned}
R_x(\theta) &\equiv e^{-i\theta X/2} = \cos\tfrac{\theta}{2} I - i\sin\tfrac{\theta}{2} X = \begin{bmatrix} \cos\tfrac{\theta}{2} & -i\sin\tfrac{\theta}{2} \\ -i\sin\tfrac{\theta}{2} & \cos\tfrac{\theta}{2} \end{bmatrix} \\
R_y(\theta) &\equiv e^{-i\theta Y/2} = \cos\tfrac{\theta}{2} I - i\sin\tfrac{\theta}{2} Y = \begin{bmatrix} \cos\tfrac{\theta}{2} & -\sin\tfrac{\theta}{2} \\ \sin\tfrac{\theta}{2} & \cos\tfrac{\theta}{2} \end{bmatrix} \\
R_z(\theta) &\equiv e^{-i\theta Z/2} = \cos\tfrac{\theta}{2} I - i\sin\tfrac{\theta}{2} Z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}.
\end{aligned}
$$

Bloch's Theorem

In terms of these, Bloch's Theorem states the following:

> Theorem: (Z-Y decomposition of rotations)
>
> Suppose U is a unitary operation on a two-dimensional Hilbert space. Then there exist real numbers α, β, γ and δ such that
>
> $$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta).$$

Proof:~ Since U is unitary, the rows and columns of U are orthonormal, from which it follows that there exist real numbers α, β, γ,and δ such that

$$
U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\tfrac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\tfrac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\tfrac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\tfrac{\gamma}{2} \end{bmatrix}.
$$

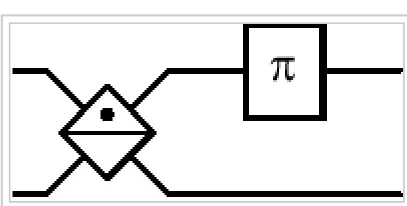
Equation (\ref{eqtn:alg:qubit_decomp}) now follows immediately from the definition of the rotation matrices and matrix multiplication.

What we have just done, expressed in modern language, is to introduce an optical quantum bit, a "qubit," and showed that arbitrary single qubit operations ("gates") can be performed using phase-shifters and beamsplitters.

For example, one widely useful single-qubit transform is the Hadamard gate,

$$
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.
$$

This operation can be performed by doing:

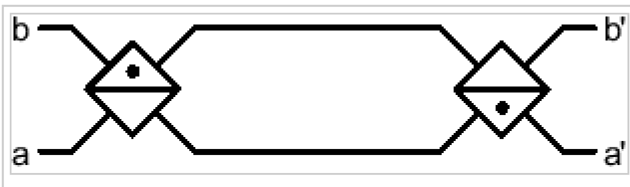


where the beamsplitter has $\theta = \pi/4$. From inspection, it is easy to verify that it transforms $|01\rangle \rightarrow (|01\rangle + |10\rangle)/\sqrt{2}$ and $|10\rangle \rightarrow (|01\rangle - |10\rangle)/\sqrt{2}$ up to an overall phase, as desired. Up to a phase shift, a $50/50$ beamsplitter can thus be thought of as being a Hadamarad gate, and vice-versa.

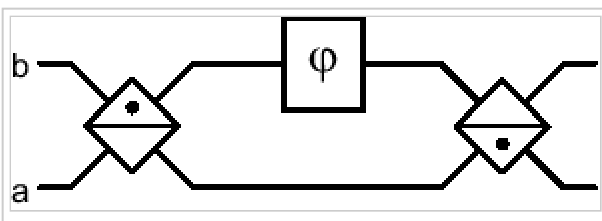
Mach-Zehnder interferometer

The reason we have introduced the dual-rail photon representation of a qubit is because this will allow us to clarify the universality of certain quantum optical ideas, namely interference and interferometers, which will be ubiquitous through our treatment of atoms and quantum information.

Let us begin by developing a model for the Mach-Zehnder interferometer, which is constructed from two beamsplitters. Recall that two beamsplitters B and $B^\dagger$, configured as



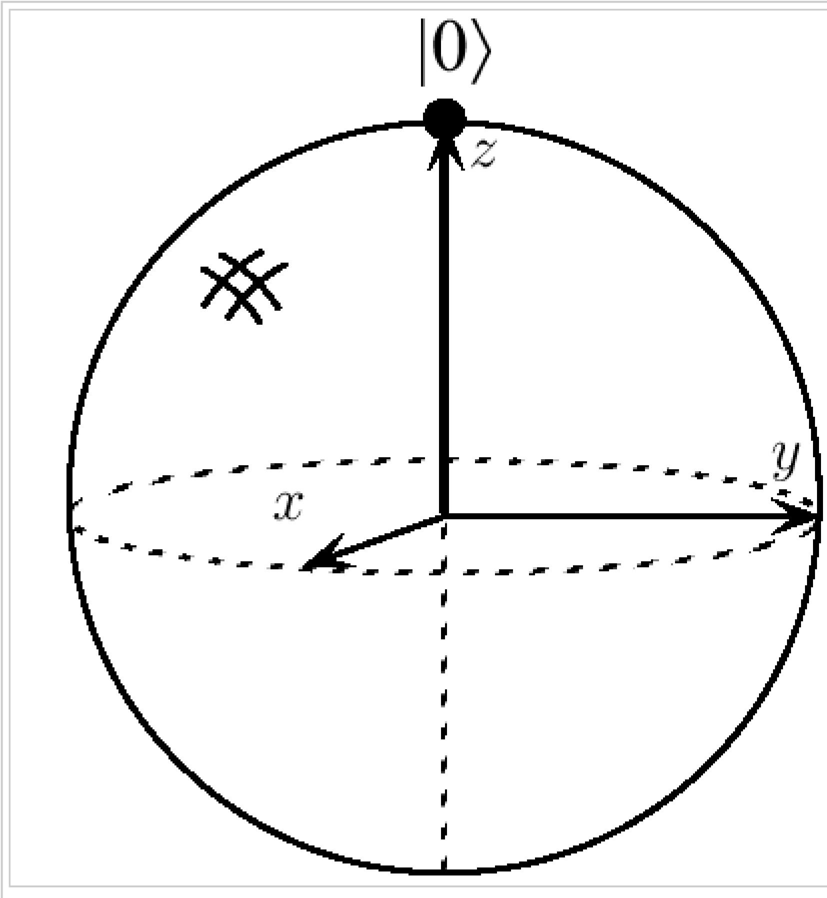
naturally leave the output identical to the input, as $B^\dagger B = I$. If a phase shifter $P(\phi)$ is placed inbetween two $50/50$ beamsplitters,



then the input is transformed by

$$\begin{aligned} B^\dagger(\pi/4)P(\phi)B(\pi/4) &= \exp[-i\pi Y/4]\exp[i\phi Z/2]\exp[i\pi Y/4] \\ &= R_y(\pi/2)R_z(-\phi)R_y(-\pi/2)\,. \end{aligned}$$

It is convenient to visualize this sequence of three rotations on the Bloch sphere:



The first $R_y(-\pi/2)$ rotates $\hat{z}$ into $\hat{x}$, and $-\hat{x}$ into $\hat{z}$. The system is then rotated around $\hat{z}$ by angle ϕ. Then the last $R_y(\pi/2)$ rotates $\hat{z}$ back to $-\hat{x}$. The overall sequence is thus a rotation by ϕ about $-\hat{x}$:

$$B^\dagger(\pi/4)P(\phi)B(\pi/4) \quad = \quad R_x(-\phi)\,.$$

If the input is $|01\rangle$, then the output will thus be $\cos(\phi/2)|01\rangle - i\sin(\phi/2)|10\rangle$, so the photon is found in mode a with probability $\cos^2(\phi/2)$, and in mode b with probability $\sin^2(\phi/2)$. This is exactly what a classical interferometer should do. Two important limits are that when $\phi = 0$ (the interferometer is "balanced"), the input is unchanged, and when $\phi = \pi$, the two modes are swapped.

Nonlinear Mach-Zehnder interferometer

The two components we have studied so far, phase shifters and beamsplitters, are linear optics elements. Such elements have an electric polarization which is linear with the applied electric field, $\vec{P} = \epsilon_0 \chi \vec{E}$. Nonlinear optical elements, have $\vec{P} = \epsilon_0 (\chi^{(1)}\vec{E} + \chi^{(2)}\vec{E}^2 + \chi^{(3)}\vec{E}^3 + \cdots)$. Previously, we have seen that an optical parametric oscillator (with $\chi^{(2)} \neq 0$) can be used for creating quantum states such as squeezed light. What do nonlinear optical elements do to single photons?

Optical Kerr media

Consider a material with $\chi \sim \chi^{(3)} \neq 0$, which we may model as having the Hamiltonian

$$H_{xpm} = -\chi a^\dagger a b^\dagger b\,,$$

where a and b describe two modes propagating through the medium. For a crystal of length L we obtain the unitary transform

$$K = e^{i\chi L a^\dagger a b^\dagger b}\,.$$

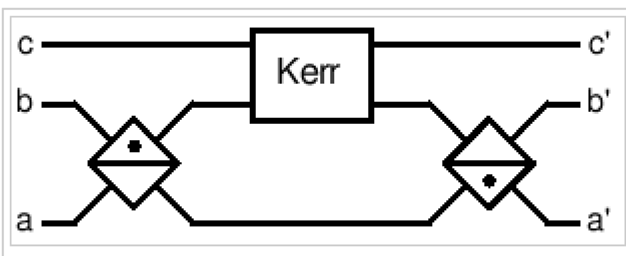
Here, χ parametrizes the third order nonlinear susceptibility coefficient. We will refer to H_{xpm} as the Kerr cross-phase modulation Hamiltonian, and the nonlinear crystal as being a Kerr medium.

Interferometer with Kerr medium inside

Interesting non-classical behavior can be obtained using interferometers constructed with Kerr media used as nonlinear phase shifters. For single photon states, we find that

$$\begin{aligned} K|00\rangle &= |00\rangle \\ K|01\rangle &= |01\rangle \\ K|10\rangle &= |10\rangle \\ K|11\rangle &= e^{i\chi L}|11\rangle\,. \end{aligned}$$

Let us take $\chi L = \pi$, such that $K|11\rangle = -|11\rangle$. Suppose we now place the Kerr medium inside a Mach-Zehnder interferometer in this manner:



\noindent Intuitively, we expect that when no photons are input into c, then the Mach-Zehnder interferometer is balanced, leading to $a' = a$ and $b' = b$. But when a photon is input into c, if the cross-phase modulation due to K is sufficiently large ($\sim \pi$), then the inputs are swapprd, producing $a' = b$ and $b' = a$.

Mathematically, we may write the transform performed by this nonlinear Mach-Zehnder interferometer as the unitary transform $U = B^\dagger K B$, where B is a 50/50 beamsplitter, K is the Kerr cross phase modulation operator $K = e^{i\xi\, b^\dagger b\, c^\dagger c}$, and $\xi = \chi L$ is the product of the coupling constant and the interaction distance. The transform simplifies to give

$$\begin{aligned} U &= \exp\left[i\xi c^\dagger c\left(\frac{b^\dagger - a^\dagger}{\sqrt{2}}\right)\left(\frac{b-a}{\sqrt{2}}\right)\right] \\ &= e^{i\frac{\pi}{2}b^\dagger b}\, e^{\frac{\xi}{2}c^\dagger c(a^\dagger b - b^\dagger a)}\, e^{-i\frac{\pi}{2}b^\dagger b}\, e^{i\frac{\xi}{2}a^\dagger a\, c^\dagger c}\, e^{i\frac{\xi}{2}b^\dagger b\, c^\dagger c}\,. \end{aligned}$$

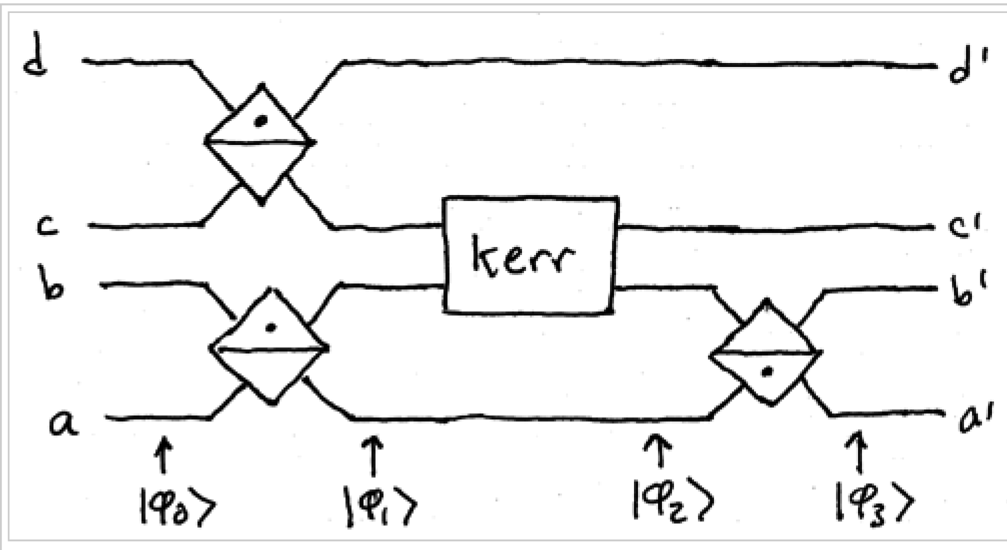
The first and third exponentials are constant phase shifts, and the last two phase shifts come from cross phase modulation. All those effects are not fundamental, and can be compensated for. The interesting term is the second exponential, which we define as

$$F(\xi) = \exp\left[\frac{\xi}{2}c^\dagger c\,(a^\dagger b - b^\dagger a)\right]\,.$$

For $\xi = \pi$, when no photons are input at c, then $a' = a$ and $b' = b$, but when a single photon is input at c, then $a' = b$ and $b' = a$, as we expected. We may also interpret $F(\chi)$ as being like a controlled-beamsplitter operator, where the rotation angle is $\xi c^\dagger c$.

Squeezing with nonlinear interferometer

How does this nonlinear interferometer produce non-classical behavior? Well, one thing it can be used for is to create a state very much like two-mode squeezed light, as we now show in the limit of single photons. Consider this setup, with two dual-rail qubits, and one Kerr medium:



This has two Mach-Zehnder interferometers coupled with a Kerr medium, which we shall take to have $\chi L = \pi$. If the input state is $|\phi_0\rangle = |0101\rangle$, using mode labeling $|dcba\rangle$, then the state after the first two 50/50 beamsplitters is

$$\begin{aligned} |\phi_1\rangle &= (|01\rangle + |10\rangle)(|01\rangle + |10\rangle) \\ &= |0101\rangle + |0110\rangle + |1001\rangle + |1010\rangle \,, \end{aligned}$$

up to a normalization factor which we shall suppress for clarity. The Kerr medium takes $K|11\rangle \to -|11\rangle$ and leaves all other basis states unchanged. Thus,

$$|\phi_2\rangle = |0101\rangle - |0110\rangle + |1001\rangle + |1010\rangle \,.$$

Finally, the output state, given by applying $B^\dagger$ to modes a and b, is

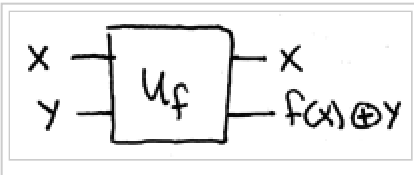
$$|\phi_3\rangle = \frac{|1001\rangle + |0110\rangle}{\sqrt{2}} \,.$$

Compare this with the two-mode infinitely squeezed state $\sum_n f(n)|n\rangle_1|n\rangle_2$ which we used in the discussion of teleportation. This state has exactly the same feature that when mode a has a single photon, mode d does also, and vice versa. The same is true also for modes b and c. This state has an extra spatial correlation that the two-mode infinitely squeezed state did not. But is is not hard to imagine that they have similar properties. In the section on Entangled Photons we show that both are entangled quantum states, which have correlations beyond what is possible with classical states.

Deutsch-Jozsa algorithm

Nonlinear Mach-Zender interferometers are also useful for implementing and understanding simple quantum algorithms. One of the most elementary of these is known as the Deutsch-Jozsa algorithm, which solves the following problem.

Suppose you are given the following box, which accepts two inputs x and y, and produces two outputs, x and $y \oplus f(x)$, where $\oplus$ denotes addition modulo two:



Each signal is a single bit, and the box is promised to implement one of four functions, computing either f_0, f_1, f_2, or f_3:

x	$f_0(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$
0	0	0	1	1
1	0	1	1	0

Call f_0 and f_2 the even functions, and f_1 and f_3 the odd functions. How many queries to the box must you perform to determine whether it is implementing an even or odd function?

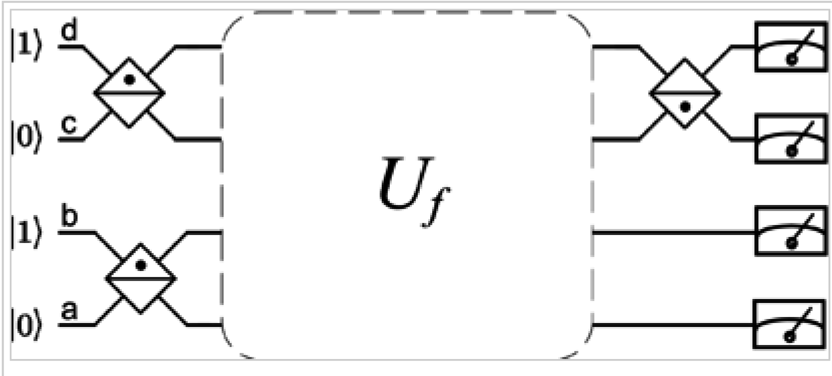
If 0 and 1 are the only two values you can input for x and y, then at least two queries to the box are needed to answer this question. This can be seen by direct examination of the full input-output table:

y	x	$f_0(x) \oplus y$	$f_1(x) \oplus y$	$f_2(x) \oplus y$	$f_3(x) \oplus y$
0	0	0	0	1	1
0	1	0	1	1	0
1	0	1	1	0	0
1	1	1	0	0	1

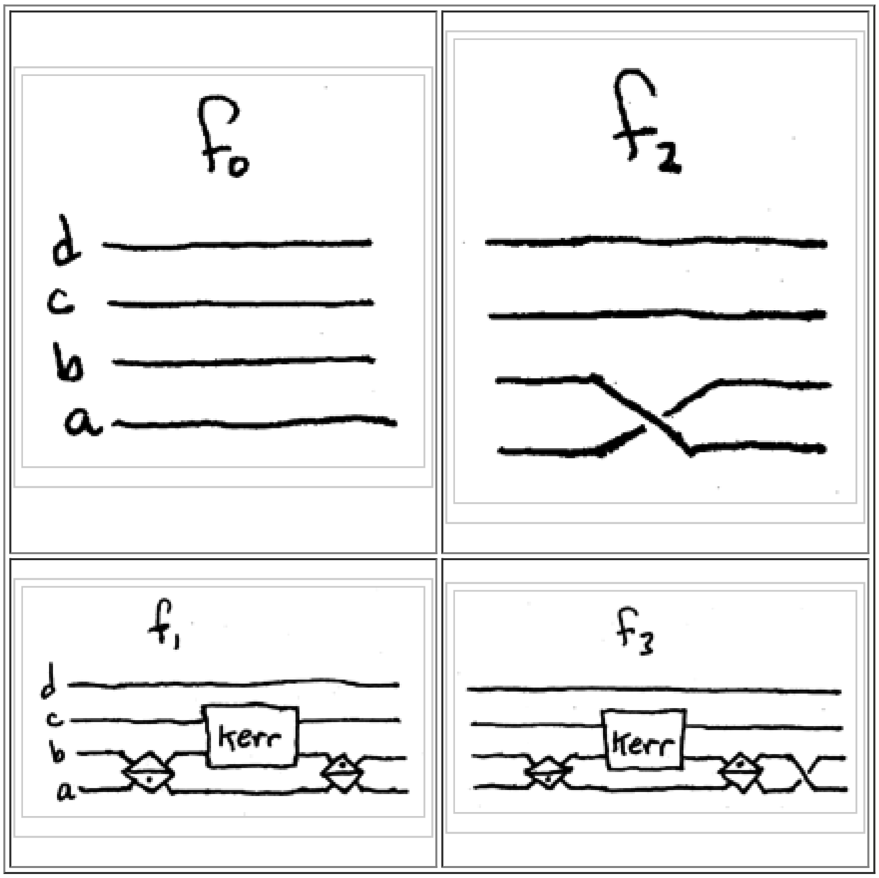
and by observing that (1) changing y gives no additional information about whether U_f implements an even or odd function, and (2) for any single input value of x, there are both even and odd functions which give the same output. Indeed, whether the function is even or odd is given by $f(0) \oplus f(1)$, and this expression clearly needs two evaluations of f to be computed, in general.

If quantum superpositions are allowed as inputs, but the outputs are simply measured in the usual "computational" basis, then the problem still takes two queries to be solved.

However, if quantum superpositions are allowed as inputs, and outputs can also be intefered, then only one query is needed. This is done using the following procedure. Let us use dual-rail photon qubits, and choose $|01\rangle$ to represent 0, and $|10\rangle$ to represent 1. The optical setup implementing the quantum algorithm to solve the Deutsch-Jozsa problem is:



The key to understanding how this works is to explicitly write down what is inside the U_f box for the four possible functions. These are



Note how f_0 is trivial, since $f_0(x) \oplus y = y$; also straighforward is $f_2(x) \oplus y = 1 \oplus y$, since this is just an inversion of y, that is accomplished by swapping modes a and b. The two odd functions involve an interaction between modes ba and dc, because $f_1(x) \oplus y = x \oplus y$, and $f_3(x) \oplus y = 1 \oplus x \oplus y$. These two are implemented with nonlinear Mach-Zehnder interferometers, which cause modes a and b to be swapped if mode c has a photon, or left alone if mode c has no photon.

Inserting these into the algorithm, we find that if the input state is $|1010\rangle$ (designating modes as $|dcba\rangle$), the outputs are

function	output state
f_0	$\|10\rangle(\|01\rangle + \|10\rangle)$
f_1	$\|01\rangle(\|01\rangle + \|10\rangle$

f_2	$\|10\rangle(\|10\rangle - \|01\rangle)$
f_3	$\|01\rangle(\|10\rangle - \|01\rangle$

When the function is f_0 or f_2, the ba modes completely decouple from the cd modes, so the output is trivially obtained.

Thus, for those two cases, modes dc end up in $|10\rangle$, so a photon is found in mode d. When the function is f_1, then the two initial beamsplitters on ba cancel, leaving a photon in mode b. This photon then causes the nonlinear Mach-Zehnder interferometer in modes dc to flip the photons between those modes. A similar thing happens for function f_3, leaving modes dc in state $|01\rangle$, so a photon is found in mode c. The measurement of whether a photon ends up in mode c or in mode d thus determines whether the function is even or odd.

The main insight given by this example, which generalizes to more complex quantum algorithms, is that phases and interference are central to their operation. Another important insight is that quantum algorithms are somewhat of like a kind of spectroscopy: just as the standard Mach-Zehnder interferometer may be used to measure the index of refraction of an unknown crystal, nonlinear, coupled Mach-Zehnder interferometers can be used to measure periods of certain functions. Indeed, it is through period measurement that Shor's quantum factoring algorithm works.

Another important insight gained by this example is that quantum algorithms are likely complex and difficult to implement, if they require a multitude of coupled interferometers. This is because well balanced, stable interferometers are experimentally challenging to realize. Nonlinear optical Kerr media that have no loss, and can impart a π cross phase modulation between single photons, are also rather exotic.

Finally, it is worthwhile considering exactly what we used which was quantum-mechanical in implementing the Deutsch-Jozsa algorithm. Would this implementation have worked with coherent states, instead of single photons?

References

Category: Quantum Light
