Problem Set 10

Due: Friday 5pm, April 22nd, via Canvas upload or in envelope outside 26-255
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1 Optical Bloch Equations with Spontaneous Emission

Consider a two level system driven with Rabi frequency ω_R with damping rate Γ. We denote the ground state and the excited state of the atom as $|a\rangle$ and $|b\rangle$. In this problem, we compute the population fraction in the excited state $|b\rangle$ at the limit $t \to \infty$.

a) Let us begin by guessing the population in the excited state $|b\rangle$ in the limit $t \to \infty$ at the large detuning $|\delta| \gg \Gamma, \omega_R$. We estimate $\rho_{bb}(t \to \infty)$ by two different approaches.

i. For $\Gamma = 0$ (without spontaneous emission), what is the excited state fraction, $\rho_{bb}(t)$, given by the solution for undamped Rabi oscillations? What do you expect will happen if a weak damping term is added to account for spontaneous emission? Guess the result for ρ_{bb} in the limit $t \to \infty$ by assuming that the oscillatory term will damp out to the average value.

Solution

The Hamiltonian describing the system is

$$H = \frac{\hbar}{2}\begin{pmatrix} -\delta & \omega_R \\ \omega_R & \delta \end{pmatrix}$$

and the initial state is $|\psi(t=0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. After time t this state becomes

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{\omega_{R,\text{eff}}}{2}t\right) + i\frac{\delta}{\omega_{R,\text{eff}}}\sin\left(\frac{\omega_{R,\text{eff}}}{2}t\right) \\ -i\frac{\omega_R}{\omega_{R,\text{eff}}}\sin\left(\frac{\omega_{R,\text{eff}}}{2}t\right) \end{pmatrix}$$

where the effective Rabi frequency is defined as $\omega_{R,\text{eff}} = \sqrt{\omega_R^2 + \delta^2}$. The excited state population is $\rho_{bb}(t) = \frac{\omega_R^2}{\omega_{R,\text{eff}}^2}\sin^2\left(\frac{\omega_{R,\text{eff}}}{2}t\right)$.

It can be expected that a weak damping will not change the population on average but will make the oscillations in the populations smear out to their average values. One reasoning is that due to damping the initial doesn't matter after long time so the phase cannot be well defined. Another reasoning is that a small decay smears out the exact energy of the excited state which in turn smears out the precise value of the detuning δ which slightly smears out $\omega_{R,\text{eff}}$ and the oscillation phase will not be well defined after long time.

Substituting $1/2$ instead of the $\sin^2$ term gives the average excited state occupancy after long time $\bar{\rho}_{bb} = \frac{\omega_R^2}{2\omega_{R,\text{eff}}^2} = \frac{\omega_R^2}{2(\omega_R^2+\delta^2)}$.

ii. Compare your guess with the result obtained for ρ_{bb} in the lowest order perturbation theory, that is exactly how we obtained the AC Stark shift. Is it the same or not?

For this, assume that the two states, $|a, 1\text{photons}\rangle$ and $|b, 0\text{photons}\rangle$, are coupled by $H_{int} = -\boldsymbol{d} \cdot \boldsymbol{E}$ with $\boldsymbol{E} = i\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{\epsilon}(a - a^\dagger)$. Identify the Rabi frequency as $\hbar\omega_R = 2\sqrt{\frac{2\pi\hbar\omega}{V}}\hat{\epsilon} \cdot \boldsymbol{d}_{ab}\sqrt{n}$.

Solution

In first order perturbation theory the mixing amplitude of the excited state into to lowest energy state is $\langle b|H_{\rm int}|a\rangle / (E_b - E_a) = \omega_R/(2\delta)$. That gives a probability of being in the excited state of $\rho_{bb} = \frac{\omega_R^2}{4\delta^2}$ which is a factor of 2 different from the previous result in the case of large detunings.

b) In order to consider the effect of spontaneous emission properly, we need to consider the time-evolution of the density matrix for the system: $\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$.

The density matrix ρ consists of two parts: the population fractions (ρ_{aa} and ρ_{bb}) and the coherence of the system (ρ_{ab} and ρ_{ba}). Here, let us denote the damping rate for the population fraction (ρ_{aa} and ρ_{bb}) as Γ_1 and the damping rate for the coherence (ρ_{ab} and ρ_{ba}) as Γ_2. Then, the evolution of the system, including spontaneous emission, can be completely determined by the following equation of motion for the density matrix:

$$\dot{\rho} = \frac{1}{i\hbar}[H, \rho] + \begin{pmatrix} \Gamma_1\rho_{bb} & -\Gamma_2\rho_{ab} \\ -\Gamma_2\rho_{ba} & -\Gamma_1\rho_{bb} \end{pmatrix}.$$

where $H = \frac{\hbar}{2}\begin{pmatrix} -\omega_0 & \omega_R e^{i\omega t} \\ \omega_R e^{-i\omega t} & \omega_0 \end{pmatrix}$ and $\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix}$ with $\rho_{ab} = \rho_{ba}^*$ and normalization condition $\rho_{aa} + \rho_{bb} = 1$.

The above equations of motion for the density matrix are called the *optical Bloch equations*. Here we only obtain the steady state solution without solving the optical Bloch equations directly.

i. By making the the substitutions $\hat{\rho}_{ab} = \rho_{ab}e^{-i\omega t}$ and $\hat{\rho}_{ba} = \rho_{ba}e^{i\omega t}$, obtain the following equations of motion for each element in the density matrix:

$$\begin{aligned}
\dot{\rho}_{aa} &= i\tfrac{\omega_R}{2}(\hat{\rho}_{ab} - \hat{\rho}_{ba}) + \Gamma_1\rho_{bb} \\
\dot{\rho}_{bb} &= -i\tfrac{\omega_R}{2}(\hat{\rho}_{ab} - \hat{\rho}_{ba}) - \Gamma_1\rho_{bb} \\
\dot{\hat{\rho}}_{ab} &= (-i\delta - \Gamma_2)\hat{\rho}_{ab} + i\tfrac{\omega_R}{2}(\rho_{aa} - \rho_{bb}) \\
\dot{\hat{\rho}}_{ba} &= (i\delta - \Gamma_2)\hat{\rho}_{ba} - i\tfrac{\omega_R}{2}(\rho_{aa} - \rho_{bb})
\end{aligned}$$

Solution
This can be either done using pen and paper or a computer (Mathematica) but either way one eventually gets the above answer.

ii. Show that the steady state solution for arbitrary δ, Γ_1, Γ_2, and ω_R is:

$$\rho_{bb} = \frac{\omega_R^2}{2} \frac{\frac{\Gamma_2}{\Gamma_1}}{\delta^2 + \Gamma_2^2 + \frac{\Gamma_2}{\Gamma_1}\omega_R^2}$$

Solution
By setting the left hand sides of the above equations to zero and using that $\rho_{aa} + \rho_{bb} = 1$ one can find the above steady state solution for ρ_{bb}.

c) In part b), we denoted the damping rate for the population fraction as Γ_1 and the damping rate for the coherence as Γ_2. Accordingly, the result we obtained depends on both Γ_1 and Γ_2. Now we need to represent Γ_1 and Γ_2 in terms of the spontaneous emission rate Γ.

i. Consider the case where there is no driving force ($H = 0$). Then the density matrix ρ evolves as follows:

$$\dot{\rho} = \begin{pmatrix} \Gamma_1 \rho_{bb} & -\Gamma_2 \rho_{ab} \\ -\Gamma_2 \rho_{ba} & -\Gamma_1 \rho_{bb} \end{pmatrix}.$$

Solve for the density matrix $\rho(t)$ at time t. Use $\rho_{aa}(0)$, $\rho_{ab}(0)$, $\rho_{ba}(0)$ and $\rho_{bb}(0)$ as initial conditions.

Solution
In this case the density matrix simply decays exponentially giving the solution

$$\rho(t) = \begin{pmatrix} \rho_{aa}(0) + \rho_{bb}(0)\left(1 - \exp\left(-\Gamma_1 t\right)\right) & \rho_{ab}(0)\exp\left(-\Gamma_2 t\right) \\ \rho_{ba}(0)\exp\left(-\Gamma_2 t\right) & \rho_{bb}(0)\exp\left(-\Gamma_1 t\right) \end{pmatrix}$$

ii. Let us suppose that the atom starts out in a superposition state

$$|\psi\rangle = (\alpha_a(0)|a\rangle + \alpha_b(0)|b\rangle) \otimes |0\rangle \tag{1}$$

where $\alpha_a(0)|a\rangle + \alpha_b(0)|b\rangle$ is the atomic state and $|0\rangle$ represents the vacuum. At time t, it will be in a state

$$|\psi\rangle = \alpha_a(t)|a\rangle \otimes |0\rangle + \alpha_b(t)|b\rangle \otimes |0\rangle + \sum_k c_k(t)|a\rangle \otimes |1_k\rangle$$

where $|n_k\rangle$ is a n-photon state in mode k.

Represent the density matrix $\rho(t)$ in terms of $\alpha_a(t)$, $\alpha_b(t)$ and $c_k(t)$. By comparing this with the density matrix $\rho(t)$ obtained in *i*, show that

$$\Gamma_1 = \Gamma \qquad \Gamma_2 = \frac{1}{2}\Gamma \tag{2}$$

when there is no driving force. Explain why the off-diagonal element decay at half rate of the excited population.

Solution

The full density matrix is

$$\begin{aligned}
|\psi\rangle\langle\psi| =& |\alpha_a|^2 |a0\rangle\langle a0| + |\alpha_b|^2 |b0\rangle\langle b0| + \sum_k |c_k|^2 |a1_k\rangle\langle a1_k| \\
&+ \alpha_a\alpha_b^* |a0\rangle\langle b0| + \alpha_a^*\alpha_b |b0\rangle\langle a0| \\
&+ \left(\sum_k c_k^*(\alpha_a |a\rangle + \alpha_b |b\rangle)\langle a| \otimes |0\rangle\langle 1_k| + h.c.\right) \\
&+ \sum_{k,k'\neq k} c_k c_{k'}^* |a1_k\rangle\langle a1_{k'}|
\end{aligned}$$

where h.c. means Hermitian conjugate. When taking the trace over the photon states the last two lines vanish due to the orthogonality of the photon modes and we get the reduced density matrix

$$\rho = \begin{pmatrix} |\alpha_a|^2 + \sum_k |c_k|^2 & \alpha_a\alpha_b^* \\ \alpha_a^*\alpha_b & |\alpha_b|^2 \end{pmatrix}.$$

For a decaying system the excited state population ($\rho_{bb} = |\alpha_b|^2$) decays as $\exp(-\Gamma t)$ and one can identify $\Gamma_1 = \Gamma$. From that we also know that α_b decays as $\exp(-\Gamma t/2)$. Since the decay increases the c_k amplitudes the α_a amplitude stays constant. Therefore the off diagonal matrix elements decay as $\exp(-\Gamma t/2)$ and we can identify $\Gamma_2 = \Gamma/2$.

d) In fact, the relations $\Gamma_1 = \Gamma$ and $\Gamma_2 = \frac{1}{2}\Gamma$ we have obtained in c) still hold in the presence of the driving force.

i. By using this, rewrite the steady state solution for ρ_{bb}. Also, represent your result in terms of the saturation parameter $s = 2\omega_R^2/\Gamma^2$ which is obtained in the last problem set. This is an important result, that one can also derive from Fermi's Golden Rule.

Solution

Making the substitutions one arrives to

$$\rho_{bb} = \frac{s/2}{1 + s + \left(\frac{\delta}{\Gamma/2}\right)^2}.$$

For small intensities the excited state population increases linearly and it saturates at 1/2 for large driving intensities. With detuning the lineshape is Lorentzian.

ii. Finally, obtain the population fraction at the large detuning limit: $|\delta| \gg \Gamma, \omega_R$. Compare your result with your guess in a).

Solution

For large detuning the excited state fraction is

$$\rho_{bb} = \frac{\omega_R^2}{4\delta^2}$$

which is the result obtained in part *a) ii* and it is a factor of 2 different from part *a) i*.

2 Intensity Distribution Due to Spontaneous Emission

An atom of total angular momentum J has a spontaneous radiation rate A. It radiates to a lower level with angular momentum $J' = J - 1$. The problem is to find the rates for the various allowed transitions, i.e. the fraction of the radiation that goes into each of the possible transitions $(J, m) \longrightarrow (J', m')$. The rates can be found by applying the following considerations:

- The sum of the rates out of each state (J, m) must equal A.
- The sum of the rates into each state (J', m') must equal $A \times \frac{2J+1}{2J'+1}$.
- An unpolarized mixture of radiators in level J must emit equal intensities of light with each of the three polarization components.
- The rate for a transition $(J, m) \longrightarrow (J', m')$ must be the same as for $(J, -m) \longrightarrow (J', -m')$.

For $J = 2$, $J' = 1$, designate the transitions by letters as follows:

$$\begin{aligned}
a&: \quad m = 2 \longrightarrow m' = 1 \\
b&: \quad m = 1 \longrightarrow m' = 1 \\
c&: \quad m = 0 \longrightarrow m' = 1 \\
d&: \quad m = 1 \longrightarrow m' = 0 \\
e&: \quad m = 0 \longrightarrow m' = 0
\end{aligned}$$

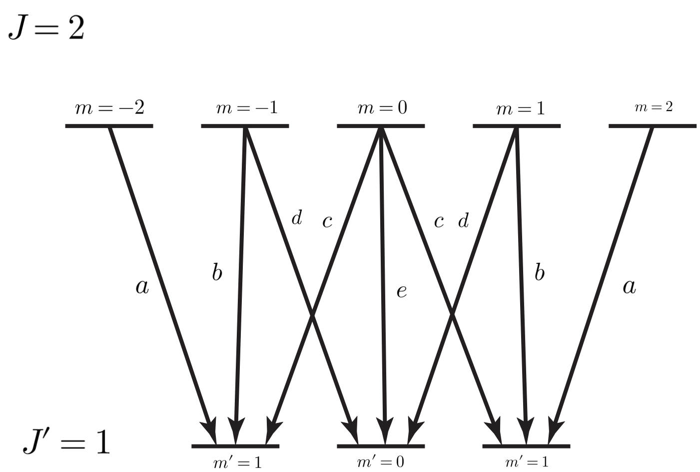

Figure 1: **The energy level diagram.** The 5 excited states can decay to the 3 ground states with the indicated transition strengths.

1. Find the rates for a though e, and present your results on a figure.

   Solution
   The transitions can be seen on Fig. 1. The probability to emit a photon from any excited state is A, therefore $A = a = b + d = 2c + e$. The probability of arriving into any ground state is $5A/3$ so $5A/3 = a + b + c = 2d + e$. The probability of having a transition of $\Delta m = \pm 1$ is the same as that of $\Delta m = 0$ as the emitted photons are unpolarized so $a + d + c = 2b + e$. Solving the set of linear equations yields

$$a = A$$
$$b = A/2$$
$$c = A/6$$
$$d = A/2$$
$$e = 2A/3.$$

2. Find the rates for a though e, using the Wigner-Eckart theorem (see Note 1. below). Clebsch-Gordan coefficients can either be worked out from first principles (manageable for this problem), or taken from a table in one of the quantum mechanics or spectroscopy texts.

   Solution
   To find the relative transition rates we just need to find the Clebsch-Gordan coefficients. The transition rates are proportional to the coefficients squared. In

this problem we have $J' = 1, L = 1, J = 2$. We know that $|J=2, m_J=2\rangle = |J'=1, m'_J=1\rangle\,|L=1, m_L=1\rangle$ so this Clebsch-Gordan coefficient is 1 so that transition strength has relative value $a = 1$. To find the other Clebsch-Gordan coefficients, apply the angular momentum lowering operator $J_- = J'_- + L_-$ to that state. This gives

$$
\begin{aligned}
|J=2, m_J=2\rangle &= \left|J'=1, m'_J=1\right\rangle |L=1, m_L=1\rangle \\
|J=2, m_J=1\rangle &= \frac{1}{\sqrt{2}}\left|J'=1, m'_J=0\right\rangle |L=1, m_L=1\rangle \\
&\quad + \frac{1}{\sqrt{2}}\left|J'=1, m'_J=1\right\rangle |L=1, m_L=0\rangle \\
|J=2, m_J=0\rangle &= \frac{1}{\sqrt{6}}\left|J'=1, m'_J=-1\right\rangle |L=1, m_L=1\rangle \\
&\quad + \frac{2}{\sqrt{6}}\left|J'=1, m'_J=0\right\rangle |L=1, m_L=0\rangle \\
&\quad + \frac{1}{\sqrt{6}}\left|J'=1, m'_J=1\right\rangle |L=1, m_L=-1\rangle \\
|J=2, m_J=-1\rangle &= \frac{1}{\sqrt{2}}\left|J'=1, m'_J=-1\right\rangle |L=1, m_L=0\rangle \\
&\quad + \frac{1}{\sqrt{2}}\left|J'=1, m'_J=0\right\rangle |L=1, m_L=-1\rangle \\
|J=2, m_J=-2\rangle &= \left|J'=1, m'_J=-1\right\rangle |L=1, m_L=-1\rangle\,.
\end{aligned}
$$

These coefficients squared give the relevant transition rates between the appropriate J, m_J and J', m'_J states. By comparison it can be seen that we get the same results.

Notes:

1. The Wigner-Eckart Theorem is

$$\langle J m_J \alpha | T_{Lm} | J' m_{J'} \alpha' \rangle = c(J'LJ; m_{J'} m m_J)\langle J\alpha \parallel T_L \parallel J'\alpha'\rangle$$

α and α' are the other quantum numbers not related to angular momentum. $\langle J\alpha \parallel T_L \parallel J'\alpha'\rangle$ is a quantity which is independent of $m_J, m_{J'}$ and m. The prefactor $c(J'LJ; m_{J'} m m_J)$, which is also often written as $\langle J'L; m_{J'} m | J m_J\rangle$, is the Clebsch-Gordan coefficient for adding two angular momenta J' and L with z-components $m_{J'}$ and m, to get a resultant angular momentum J with z-component m_J.

2. The transition rates calculated here are important in experiments involving laser excitation. Because emission and absorption rates are proportional, the distribution of emission rates yields the relative strengths of the transitions, i.e. their relative rates of excitation.