Today's plan

Lectures

- What is a symmetry?
- A crash course in group theory

Activities

- GeoGebra
- Brainstorming
- Breakout rooms

Note: I will be writing on top of these slides. I'll send you a link to the blank slides for now and upload the pdf with my written notes later today.

How to interact during the lectures

- This class is a safe (virtual) place. Questions are always welcome, no matter how trivial you may think they are.
- You can ask questions at any time during the lecture. You have a few options:
  - "Raise your hand" through Zoom and ask in person
  - Use the Zoom chat
  - Ask on sli.do (#W613)
- I will often ask you questions during the lectures. I **do not know** how well this is going to work online, but I'll still do it.

Socrative

To answer polls on socrative, go to `socrative.com` and login as a sudent with the room name **SYMMETRIES**.

What is a symmetry?

Groups and subgroups

Definition (group)

A group is a set G together with an operation $* : G \times G \to G$ satisfying the following properties:

- there is a special element $e \in G$, called the *identity*, such that

$$g * e = e * g = g, \quad \forall g \in G$$

- each element of G has an *inverse*, that is for each $g \in G$ there is an element $g^{-1} \in G$ such that

$$g^{-1} * g = g * g^{-1} = e$$

- the operation $*$ is *associative*, that is

$$a * (b * c) = (a * b) * c, \quad \forall a, b, c \in G.$$

Additionally, we say that the group G is *abelian* or *commutative* if

$$a * b = b * a, \quad \forall a, b \in G.$$

Definition (subgroup)

Let G be a group with operation $*$. A subgroup of G is a subset $H \subseteq G$ that contains the identity element and is closed under the operation $*$ and under inversion, that is

- $e \in H$
- $a * b \in H$ for all $a, b \in H$
- $a \in H \implies a^{-1} \in H$

The notation $H \leq G$ is commonly used to indicate that H is a subgroup of G.

Exercise

Prove the following consequences of the definition of a group:

1. the identity element is unique (if two elements satisfy the identity property, they are necessarily equal)
2. for each $g \in G$ the inverse g^{-1} is unique
3. $(g^{-1})^{-1} = g$
4. $(gh)^{-1} = h^{-1}g^{-1}$

Homomorphisms and isomorphisms

Definition (group homomorphism)

A group homomorphism is a map $\varphi : G \to H$ between two groups G and H such that

$$\varphi(a *_G b) = \varphi(a) *_H \varphi(b), \quad \forall a, b \in G.$$

Exercise

Prove that if $\varphi : G \to H$ is a group homomorphism, then

1. $\varphi(e_G) = e_H$ (Hint: look at $\varphi(e_G e_G)$)
2. $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.

Definition (kernel)

The kernel of a group homomorphism $\varphi : G \to H$ is the set

$$\ker \varphi = \{g \in G \mid \varphi(g) = e_H\}$$

of all the elements of G that are sent to the identity in H.

Proposition

A group homomorphism $\varphi : G \to H$ is injective if and only if its kernel is trivial, that is

$$\ker \varphi = \{e_G\}.$$

Definition (isomorphism)

Two groups G and H are *isomorphic* (denoted by $G \cong H$) if there exists an invertible group homomorphism $\varphi : G \to H$. Such a map is called an *isomorphism* between G and H.

Exercise

Prove that $\mathbb{Z}_2$ is isomorphic to the subgroup $\{\mathbb{I}_n, -\mathbb{I}_n\} \leq \mathrm{GL}(n, \mathbb{C})$. While you are at it, prove that the latter is indeed a subgroup!

Exercise

I'll do you one better: prove that *any* group with only two elements is isomorphic to $\mathbb{Z}_2$.

Quotient group and isomorphism theorem

Definition (normal subgroup)

Let G be a group. A subgroup $N \leq G$ is called *normal* if

$$gng^{-1} \in N, \quad \forall g \in G, \quad \forall n \in N.$$

The notation $N \trianglelefteq G$ is commonly used to indicate that N is a normal subgroup of G.

Definition (quotient group)

Let N be a normal subgroup of a group G. We can define an equivalence relation on G as

$$g \sim h \iff h^{-1}g \in N,$$

with equivalence classes

$$[g] = \{h \in G \,|\, h^{-1}g \in N\}.$$

The quotient group G/N (pronounced "G mod N") is the set of equivalence classes

$$G/N = \{[g] \,|\, g \in G\}$$

which is made into a group by defining

$$[g][h] = [gh], \quad [g]^{-1} = [g^{-1}], \quad e_{G/N} = [e_G].$$

Exercise

Show that the $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ is a normal subgroup of $(\mathbb{Z}, +)$ and that $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Theorem (first isomorphism theorem)

Let $\varphi : G \to H$ *be a group homomorphism. Then:*

- $\operatorname{Im} \varphi$ *is a subgroup of* H
- $\ker \varphi$ *is a normal subgroup of* G
- $\operatorname{Im} \varphi$ *is isomorphic to the quotient group* $G / \ker \varphi$

Exercise

Prove the first two points of the isomorphism theorem.