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Cartesian Product v.s. Tensor pdt

Why is $\mathbb{C}^3 \otimes \mathbb{C}^2$ different from $\mathbb{C}^3 \times \mathbb{C}^2$?

Consider bases $\left\{ \underset{t_1}{\begin{pmatrix}1\\0\\0\end{pmatrix}}, \underset{t_2}{\begin{pmatrix}0\\1\\0\end{pmatrix}}, \underset{t_3}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \right\}, \left\{ \underset{s_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{s_2}{\begin{pmatrix}0\\1\end{pmatrix}} \right\}$

Then a basis for

$\mathbb{C}^3 \otimes \mathbb{C}^2$ is $\underbrace{\{t_i \otimes s_a\}_{i=1, a=1}^{3 \quad 2}}_{6 \text{ dimensional.}}$

$\mathbb{C}^3 \times \mathbb{C}^2 \quad \underbrace{\{(e_i, 0), (0, s_a)\}_{i=1, a=1}^{3 \quad 2}}_{5 \text{ dimensional.}}$

In general, w/ $W = \mathrm{Span}\{w_i\}_{i=1}^m$, $V = \mathrm{Span}\{v_a\}_{a=1}^n$

$\Rightarrow$ Basis of $W \otimes V$ is $\underbrace{\{w_i \otimes v_a\}}_{m \cdot n \text{ dimensional}}$

Basis of $W \times V$ is $\underbrace{\{(w_i, 0), (0, v_a)\}}_{m + n \text{ dimensional}}$

Note that $w_1 \otimes v_1 + w_1 \otimes v_2 = w_1 \otimes (v_1 + v_2)$

$w_1 \otimes v_1 + w_2 \otimes v_1 = (w_1 + w_2) \otimes v_1$

But in general

$w_1 \otimes v_1 + w_2 \otimes v_2 \neq \tilde{w} \otimes \tilde{v}$

for any choice $\tilde{w} \in W$, $\tilde{v} \in V$

Roughly speaking this is the phenomenon of "quantum entanglement"

Tensor products for composite QM Sys

How do we represent tensor product vectors?

Ket notation

$\mathbb{C}^3 \otimes \mathbb{C}^2$: $|1\rangle|1\rangle \quad |1\rangle|2\rangle \quad |2\rangle|1\rangle \quad |2\rangle|2\rangle \quad |3\rangle|1\rangle \quad |3\rangle|2\rangle$

$\rightarrow \begin{pmatrix}1\\0\\0\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\1\\0\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\1\\0\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\0\\1\\0\end{pmatrix} \begin{pmatrix}0\\0\\0\\0\\0\\1\end{pmatrix}$

In general $\Rightarrow V = \sum_{ij} a_{ij} |i\rangle|j\rangle$

$= (a_{11}, a_{12}, a_{21}, a_{22}, a_{31}, a_{32})^T$

So the k^{th} element of V is $a_{(\mathrm{ceil}(\frac{k}{2}))(\mathrm{remainder}(\frac{k}{2})+1)}$

Tensor pdt of ops

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix} \otimes \begin{pmatrix}e & f\\ g & h\end{pmatrix} = \left(\begin{array}{c|c} a\begin{pmatrix}e & f\\ g & h\end{pmatrix} & b\begin{pmatrix}e & f\\ g & h\end{pmatrix} \\ \hline c\begin{pmatrix}e & f\\ g & h\end{pmatrix} & d\begin{pmatrix}e & f\\ g & h\end{pmatrix}\end{array}\right)$$

$\leftarrow$ 4x4 matrix!

General Case

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & a_{2n} \\ \vdots & & & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} = n \times n \text{ matrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & & & b_{2m} \\ \vdots & & & \vdots \\ b_{m1} & \cdots & & b_{mm} \end{pmatrix} = m \times m \text{ matrix}$$

$(A \otimes B) = M \rightarrow$ an $mn \times mn$ matrix

Exercise: What is M_{xy} in terms of the a_{ij} and $b_{\alpha\beta}$?

Notation: $i, j \in \{1, \ldots, n\}$, $\alpha, \beta \in \{1, \ldots, m\}$

$x, y \in \{1, \ldots, mn\}$

A:

$$M_{xy} = a_{(\mathrm{Quotient}(\frac{x-1}{m})+1)(\mathrm{Quotient}(\frac{y-1}{m})+1)} \cdot b_{(\mathrm{Mod}(\frac{x-1}{m})+1)(\mathrm{Mod}(\frac{y-1}{m})+1)}$$