MA439: Functional Analysis
Tychonoff Spaces: Exercises 1-6 on p.36, Ben Mathes

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Exercise 1 (Ex 1, p.36). *Let $\mathcal{X}$ be a topological space. Prove that if d is a continuous pseudometric, then the sets $\{y \in \mathcal{X} : d(x,y) > \delta\}$ are open, where $x \in \mathcal{X}$ and $\delta \in \mathbb{R}$.*

Proof. Let $O = \{y \in \mathcal{X} : d(x,y) > \delta\}$. We want to show that each $y \in O$ is an interior point of O. Let $y \in O$ be given, then $d(x,y) > \delta$. This means that $d(x,y) \geq \delta + \epsilon$ for some $\epsilon > 0$. d is a continuous pseudometric, so every d-ball is an open subset of $\mathcal{X}$. In particular, $B_d(y, \epsilon/2)$ is an open subset of $\mathcal{X}$. By the triangle inequality, for any $z \in B_d(y, \epsilon/2)$, $z \in O$. Thus, $B_d(y, \epsilon/2) \subseteq O$. So, O is open as desired. $\square$

Exercise 2 (Ex 2, p.36). *Let $\mathcal{X}$ be a topological space. Prove that d is a continuous pseudometric on $\mathcal{X}$ if and only if the function $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$.*

Proof. ($\implies$) Suppose that d is a continuous pseudometric on $\mathcal{X}$. Let $\epsilon > 0$ and $x \in \mathcal{X}$. f_x^d is continuous at $y \in \mathcal{X}$ if and only if for every $\epsilon > 0 \exists f(y) \in G \subseteq \mathcal{X}$ open for which $\left|f_x^d(y) - f_x^d(y')\right| < \epsilon$ whenever $y' \in G$. Note that $\left|f_x^d(y) - f_x^d(y')\right| = |d(x,y) - d(x,y')| \leq d(y,y')$. So, we just take $G = B_d(y, \epsilon)$.

($\impliedby$) Let d be a pseudometric and suppose that $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$. We want to show that every d-ball is open in $\mathcal{X}$. To this end, let $x \in \mathcal{X}$ and $\delta > 0$ be given and consider $B_d(x, \delta) = \{y \in \mathcal{X} : d(x,y) < \delta\} = \{y \in \mathcal{X} : f_x^d(y) < \delta\} = \{y \in \mathcal{X} : f_x^d(y) \in (-\delta, \delta)\}$ which is open by continuity of f_x^d. So we're done. $\square$

Exercise 3 (Ex 3, p.36). *Let $\mathcal{X}$ be a Tychonoff space whose topology is generated by the family of pseudometrics $\mathcal{G}$. Prove that the topology on $\mathcal{X}$ is the same as the weak topology induced by the family of functions f_x^d where $x \in \mathcal{X}$, $d \in \mathcal{G}$.*[1]

Proof. One inclusion is trivial. It remains to show the other inclusion. Tychonoff: for every closed set $F \subseteq \mathcal{X}$ and every $x \in F$, there exists a continuous function $f : \mathcal{X} \to \mathbb{R}$ for which $f[F] = \{0\}$ and $f(x) = 1$. From $\mathcal{G}$, use balls as a subbase and build the topology from those balls. Alternatively, we can build the functions $\{f_x^d : x \in \mathcal{X}, d \in \mathcal{G}\}$ and build the (open-ball) topology by taking inverse images of open sets. From the previous exercise, we have that weak topology $\implies$ f_x^d are all continuous, which implies that all balls are open relative to the weak topology, which implies that the new (open ball) topology is contained in the weak topology. $\square$

Exercise 4 (Ex 4, p.36). *Assume $\mathcal{X}$ is a Tychonoff space with generating family $\mathcal{G}$. If E is a subset of $\mathcal{X}$, let $\mathcal{G}_E$ denote the set of restrictions of elements of $\mathcal{G}$ to E. Prove that the resulting Tychonoff Topology on E generated by the family $\mathcal{G}_E$ is the same as the topological* ***subspace topology*** *that E inherits from the topology on $\mathcal{X}$.*

Proof. get base from finite intersection of balls. G open iff for every $x \in G$ there exist finitely many $d_1, \ldots, d_k \in \mathcal{G}$ and $\epsilon_1, \ldots, \epsilon_k > 0$ such that $\cap_{i=1}^k B_{d_i}(x, \epsilon_i) \subseteq G$. $\square$

[1] completely regular $\equiv$ Tychonoff

Exercise 5 (Ex 5, p.36)**.** *Give an example of a continuous pseudometric on* $(0, 1)$ *that is not the restriction of a continuous pseudometric on* $\mathbb{R}$ *to* $(0, 1)$.

Proof. blah $\square$

Exercise 6 (Ex 6, p.36)**.** *Prove that a bounded continuous pseudometric on* $(0, 1)$ *is the restriction of a continuous pseudometric on* $\mathbb{R}$ *to* $(0, 1)$. *(?CHECK?)*

Proof. blah $\square$