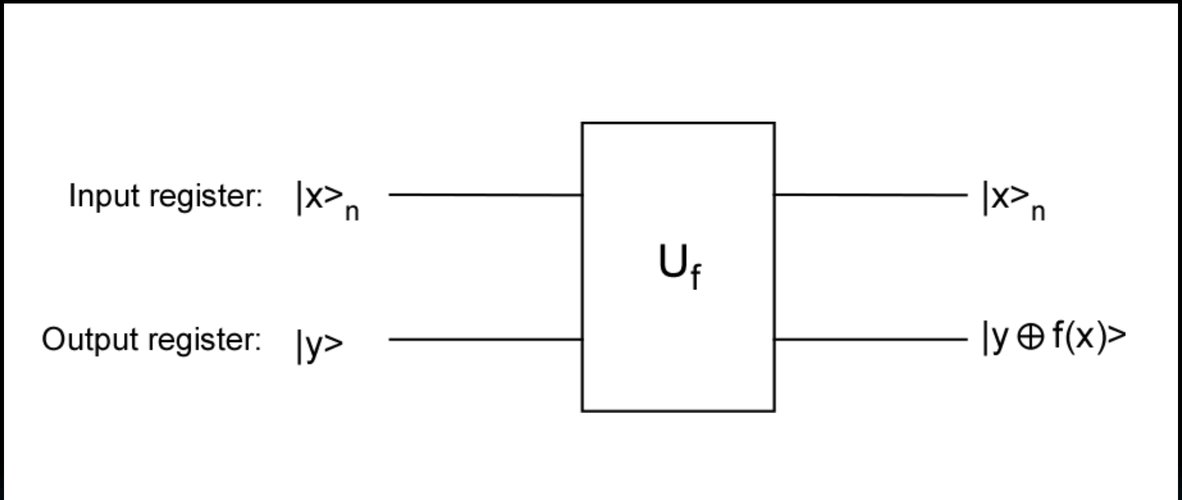
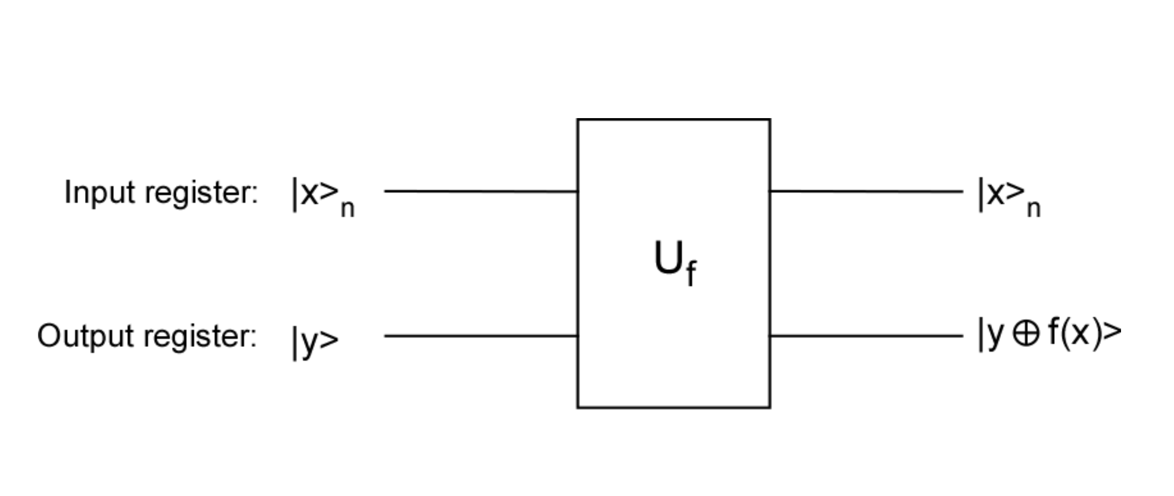
C. The Deutsch-Jozsa problem

Background: The Deutsch-Jozsa problem and the Bernstein-Vazirani try to uncover the behavior of an “oracle.”



Goal: To show how we can uncover the behavior of an oracle using quantum parallelism and phase kickback.

1. Deutsch-Jozsa and Bernstein-Vazirani



$$f : \{0,1\}^n \to \{0,1\}$$

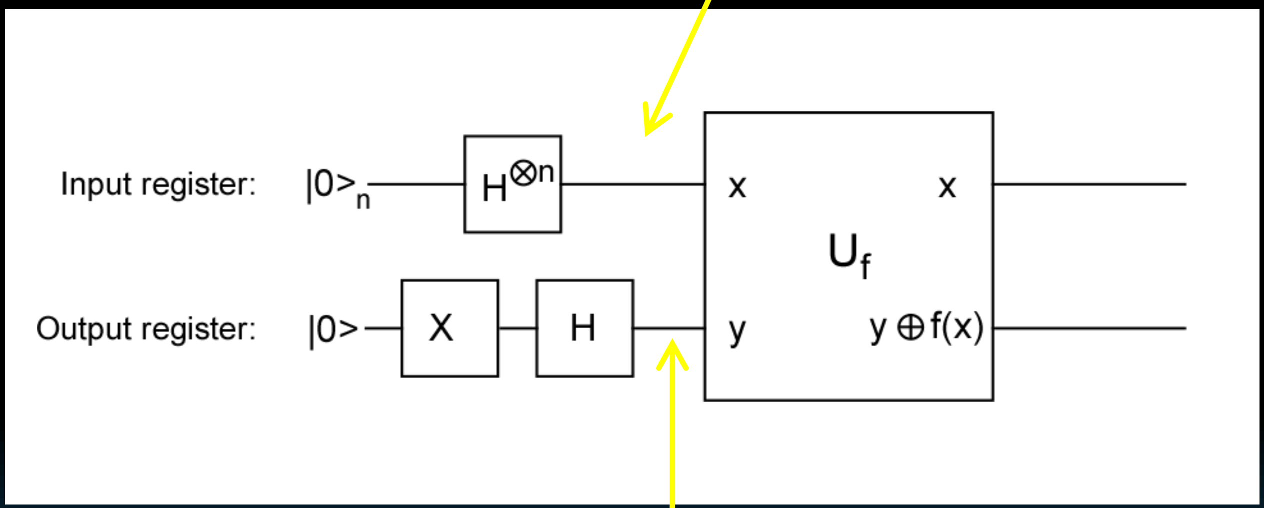
Deutsch-Jozsa: $f(x)$ is either constant or balanced. Which is it?

Bernstein-Vazirani: $f(x) = a \cdot x$ for some a. What is a?

2. Solving the Deutsch-Jozsa Problem

a. Quantum Parallelism and phase kickback

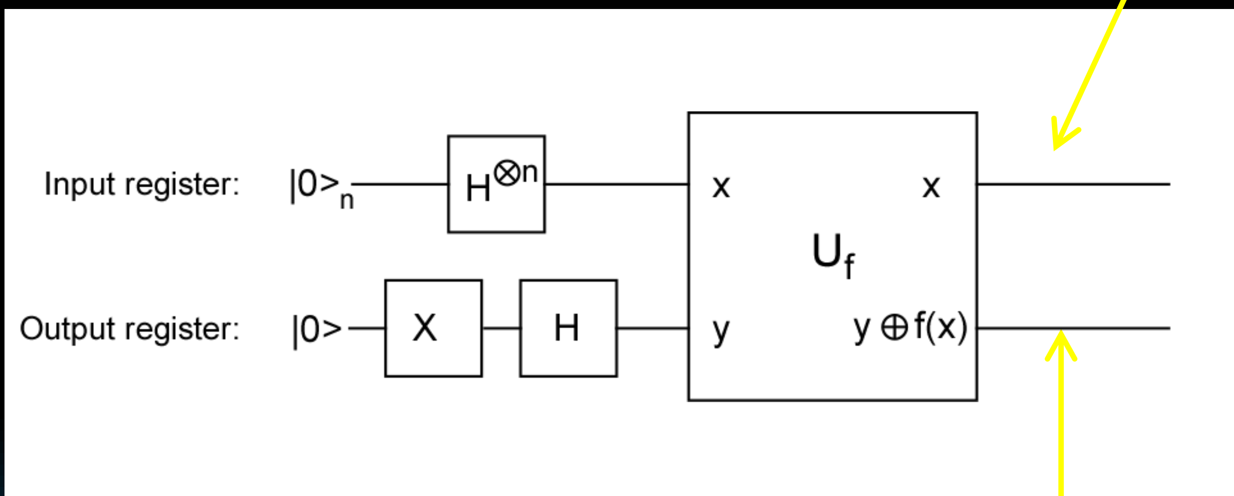
$$|\psi_i\rangle = \mathbf{H}^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{2^n}} \sum_{0\leq x<2^n} |x\rangle_n$$



$$|\psi_o\rangle = \mathbf{HX}|0\rangle = \mathbf{H}|1\rangle = |-\rangle$$

b. After the oracle

$$|\psi_i\rangle = \frac{1}{\sqrt{2^n}} \sum_{0 \le x < 2^n} (-1)^{f(x)} |x\rangle_n$$

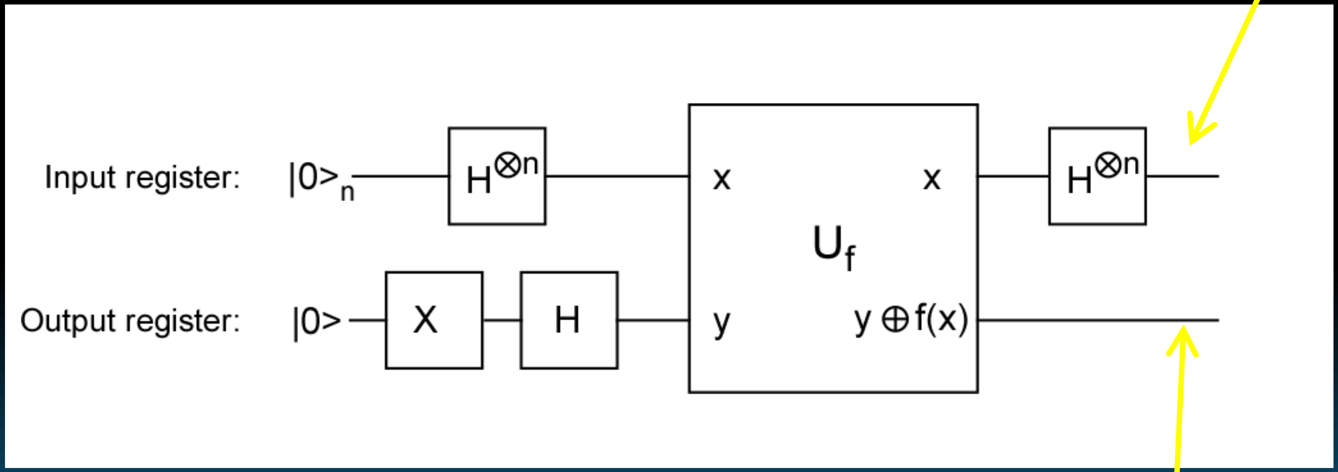


$$|\psi_o\rangle = |-\rangle$$

c. Manipulating the output to get an answer!

$$|\psi_i\rangle = \frac{1}{\sqrt{2^n}} \sum_{0\le x<2^n} (-1)^{f(x)} \mathbf{H}^{\otimes n} |x\rangle_n$$

$$= \frac{1}{\sqrt{2^n}} \sum_{0\le x<2^n} (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{0\le y<2^n} (-1)^{x\cdot y} |y\rangle_n$$



$$|\psi_o\rangle = |-\rangle$$

d. Thinking about the input register.

$$|\psi_i\rangle = \frac{1}{2^n} \sum_{0\le x<2^n} (-1)^{f(x)} \sum_{0\le y<2^n} (-1)^{x\cdot y} |y\rangle_n$$

$$= \frac{1}{2^n} \sum_{0\le y<2^n} \sum_{0\le x<2^n} (-1)^{f(x)} (-1)^{x\cdot y} |y\rangle_n$$

What is the coefficient for the state $|y\rangle_n = |0\rangle_n$?

$$\alpha_0 = \frac{1}{2^n} \sum_{0\le x<2^n} (-1)^{f(x)} \overbrace{(-1)^{x\cdot 0}}^{=+1}$$

$$= \frac{1}{2^n} \sum_{0\le x<2^n} (-1)^{f(x)}$$

For a constant function f(x) = c:

$$\alpha_0 = \frac{1}{2^n} \sum_{0 \le x < 2^n} \overbrace{(-1)^c}^{2^n \text{ terms, all of value } +1 \text{ or } -1}$$

$$= \pm 1 \quad \Longrightarrow \quad P_0 = |\alpha_0|^2 = 1.$$

For a balanced function $f_b(x)$:

$$\alpha_0 = \frac{1}{2^n} \sum_{0 \le x < 2^n} \overbrace{(-1)^{f_b(x)}}^{2^n \text{ terms: half } +1, \text{ and half } -1}$$

$$= 0 \quad \Longrightarrow \quad P_0 = |\alpha_0|^2 = 0.$$