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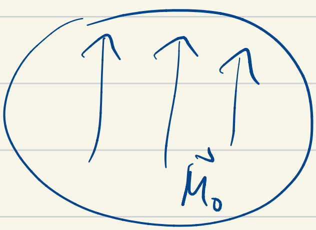
8.311 EM theory

Pset 6 due April 13, 2022

(7) $\vec{B}$ around spherical magnet with uniform magnetization $\vec{M}_0 = \text{const}$?



There's no stand-alone current sources, so

$$\vec{H} = -\vec{\nabla}\Phi_m$$

scalar field

where $\nabla^2 \phi_m = 0$.

In spherical coordinates...

$$\phi_m(r,\theta) = \sum_{\ell=0}^{\infty} \left(a_\ell r^\ell + \frac{b_\ell}{r^{\ell+1}}\right) P_\ell(\cos\theta)$$

in general.

ϕ is axially symmetric

Want field @ ∞ $= 0$ $\Rightarrow$ set $a_0 = 0$

Also $\vec{\nabla} \cdot \vec{\mathcal{H}} = 0 \Rightarrow \partial_r \mathcal{H} = 0$ @ ∞

$\Rightarrow$ Want $b_0 = 0$ as well

$\Rightarrow$ Now set

$$\boxed{\begin{aligned} &\underline{r \geq R} \quad \phi_m(r,\theta) = \sum_{\ell=1}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \\ &\underline{r \leq R} \quad \phi_m(r,\theta) = \sum_{\ell=1}^{\infty} a_\ell r^\ell P_\ell(\cos\theta) \end{aligned}}$$

To get a_ℓ, b_ℓ ... need boundary conditions...

@ R ... $\mathcal{H}_{\text{tangential}}$ constant, so

$$\boxed{\sum_{\ell=1}^{\infty} a_\ell R^\ell P_\ell(\cos\theta) = \sum_{\ell=1}^{\infty} b_\ell \frac{1}{R^{\ell+1}} P_\ell(\cos\theta)}$$

Also need B_{normal} constant across boundary.

We have that

inside

$$B_n = \mu_0(\mathcal{H}_n + M_n)$$

$$= \mu_0 \left\{ \frac{-\partial}{\partial r}\phi_m + M_0 \cos\theta \right\}$$

So

$$\boxed{\mu_0 \left\{ -\sum_{\ell=1}^{\infty} \ell a_\ell R^{\ell-1} P_\ell(\cos\theta) + M_0 \cos\theta \right\} = \mu_0 \sum_{\ell=1}^{\infty} (\ell+1) \frac{b_\ell}{R^{\ell+2}} P_\ell(\cos\theta)}$$

$P_1(\cos\theta)$

From the two boxed equations and the fact that $P_\ell(\cos\theta)$ are orthogonal functions ... we have that

$$\begin{cases} a_\ell R^\ell = \dfrac{b_\ell}{R^{\ell+1}} \\ -\ell a_\ell R^{\ell-1} = (\ell+1) \dfrac{b_\ell}{R^{\ell+2}} \end{cases} \quad \forall \ell \neq 1$$

$$\Rightarrow \begin{cases} R^{2\ell+1} = b_\ell / a_\ell \\ R^{2\ell+1} = -\dfrac{(\ell+1)}{\ell} b_\ell / a_\ell \end{cases}$$

$\Rightarrow$ may set b_ℓ, a_ℓ to zero ... and solve only for $\ell = 1$

For $\ell = 1$ ...

$$\begin{cases} -a_1 \cos\theta + M_0 \cos\theta = 2 \dfrac{b_1}{R^3} \cos\theta \\[2ex] a_1 R \cos\theta = \dfrac{b_1}{R^2} \cos\theta \end{cases}$$

$$\rightarrow \boxed{a_1 = \frac{M_0}{3} \qquad b_1 = \frac{M_0 R^3}{3}}$$

So

$$\boxed{\begin{aligned} \phi_{m,in} &= \frac{M_0}{3} r \cos\theta \\ \phi_{m,out} &= \frac{M_0 R^3}{3r^2} \cos\theta \end{aligned}}$$

With this can find the field outside the magnet ...

$$\vec{\mathcal{H}}_{out} = -\vec{\nabla}\phi_m = \frac{2M_0R^3}{3r^3}\cos\theta\,\hat{r} + \frac{M_0R^3\sin\theta}{3r^3}\,\hat{\theta}$$

$$\boxed{\begin{aligned}\vec{B}_{out} &= \mu_0\vec{\mathcal{H}}_{out} \\ &= \mu_0\left\{\frac{2M_0R^3}{3r^3}\cos\theta\,\hat{r} + \frac{M_0R^3}{3r^3}\sin\theta\,\hat{\theta}\right\}\end{aligned}}$$

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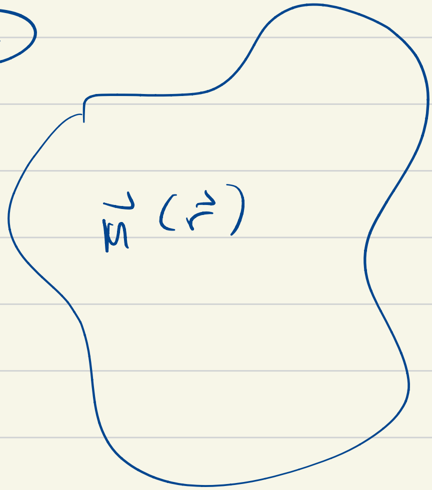


What is $\vec{B}$ / $\vec{\mathcal{H}}$ induced?
and its potential?

Permanent magnet ... no stand alone current source ...

$$\vec{\nabla} \cdot \vec{B} = 0 = \mu_0 \nabla \cdot (\vec{\mathcal{H}} + \vec{M})$$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{\mathcal{H}} = - \vec{\nabla} \cdot \vec{M}$$

Integrate this to find

$$\vec{\mathcal{H}}(\vec{r}) = \frac{1}{4\pi} \int \underline{- \vec{\nabla}' \cdot \vec{M}(\vec{r}')} \, \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \, d^3 \vec{r}' \quad \checkmark$$

some scalar field

So ... magnetic field outside is

$$\boxed{\begin{aligned}\vec{B}_{out}(\vec{r}) &= \mu_0 \vec{\mathcal{H}}(\vec{r}) \\ &= \frac{\mu_0}{4\pi}\int_V \left(-\vec{\nabla}_{r'}\cdot\vec{M}(\vec{r}\,')\right)\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'\end{aligned}}$$

whereas $\vec{B}$ inside is

$$\boxed{\begin{aligned}\vec{B}_{in}(\vec{r}) &= \mu_0\left(\vec{\mathcal{H}}(\vec{r}) + \vec{M}(\vec{r})\right) \\ &= \mu_0 \vec{M}(\vec{r}) + \frac{\mu_0}{4\pi}\int \left(-\vec{\nabla}_{r'}\cdot\vec{M}(\vec{r}\,')\right)\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'\end{aligned}}$$

The potential is gotten by analogy to electrostatics ...

$$\boxed{\phi_m = \frac{1}{4\pi}\int_V \left(-\vec{\nabla}'\cdot\vec{M}(\vec{r}\,')\right)\frac{1}{|\vec{r}-\vec{r}\,'|}\, d^3\vec{r}\,'}$$

When $\vec{M}$ is constant inside the volume and zero on the outside, then

$\vec{\nabla} \cdot \vec{M} = 0$ everywhere inside the volume

and $\vec{\nabla} \cdot \vec{M} \neq 0$ ... $\frac{\partial M_n}{\partial n} \neq 0$

the surface charge

When this is the case, then we replace

$-\vec{\nabla} \cdot \vec{M}$ with $M_n \big|_{\text{surface}}$ and change the

volume integral into surface integral

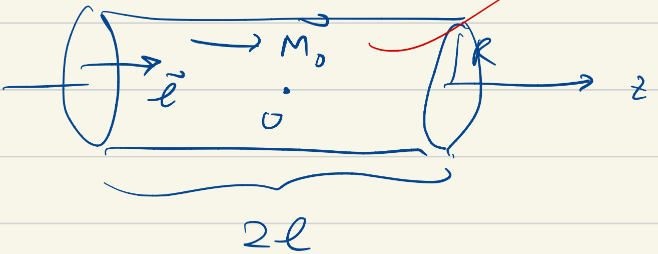
$$\vec{\mathcal{H}}(\vec{r}) \sim \frac{1}{4\pi} \int_{\partial V} M_n \Big|_{\partial V} \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \, d^2\vec{r}\,'$$

and the potential is modified accordingly –

$$\boxed{\phi_m \sim \frac{1}{4\pi} \int_{\partial V} M_n\Big|_{\partial V} \frac{1}{|\vec{r}-\vec{r}\,'|} \, d^2\vec{r}\,'}$$

③ Distribution of $\vec{\mathcal{H}}$ along axis of permanent magnet...

$\vec{M}_0$ constant.



Since $\vec{M}_0$ constant, we have that

$$\vec{\mathcal{H}}(\vec{r}) = \frac{1}{4\pi}\int_{\partial V} M_n\Big|_{\partial V} \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\, d^3\vec{r}\,'.$$

- Now along axis $(x, y) = (0, 0)$ ... $\vec{\mathcal{H}}$ points in $\hat{z}$ only.

- Also since $\vec{M}_0 \parallel \hat{z}$ ... $M_n \neq 0$ only at the end caps ...

So in particular, $M_n\Big|_+ = M_0$

$M_n\Big|_- = -M_0$

With this... Write $\vec{\mathcal{H}} = \mathcal{H}_z \vec{z}$ and

$$\mathcal{H}_z = \frac{M_0}{4\pi}\int_+ \frac{z-z'}{|\vec{\sigma}-\vec{r}'|^3}\,d^2\sigma$$

$$-\frac{M_0}{4\pi}\int_- \frac{z-z'}{|\vec{\sigma}-\vec{r}'|^3}\,d^2\sigma$$

$$= \frac{M_0}{4\pi}\left\{\int_0^R \frac{(z-\ell)\,r\,d\varphi\,dr}{[(z-\ell)^2+r^2]^{3/2}}\right.$$

$$\left. - \int_0^R \frac{(z+\ell)\,r\,d\varphi\,dr}{[(z+\ell)^2+r^2]^{3/2}}\right\}$$

$=$ ... Mathematica ... this is similar to a problem we did before ...

$$\boxed{\begin{aligned} H_z = \frac{M_0}{2} \Bigg\{ & -\frac{z-l}{\left((z-l)^2+R^2\right)^{1/2}} + \operatorname{sign}(z-l) \\ & + \frac{z+l}{\left((z+l)^2+R^2\right)^{1/2}} - \operatorname{sign}(z+l) \Bigg\} \end{aligned}}$$