Noether's theorem for a scalar field in flat space

quasi-symmetry version (read the regular one first!)

Def: The smooth one-parameter subgroup of transformations $\tilde{x}^\mu = X^\mu_\varepsilon(x)$, $\tilde{\phi}(\tilde{x}) = F_\varepsilon(\phi(x))$ is an infinitesimal quasi-symmetry of the action $S[\phi] = \int_{U \subseteq \mathbb{R}^n} d^n x\, \mathcal{L}(\phi(x), \partial\phi(x), x)$ if for all $\phi(x)$ there is a vector field $\Lambda^\mu(x)$ on U such that

$$\delta S = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{S}[\tilde{\phi}] = \int_U \partial_\mu \Lambda^\mu(x)\, d^n x$$

where Λ^μ is a vector field on U. Note that the integral gives a boundary term (by Stokes' theorem).

Theorem: if $\tilde{x}^\mu = X^\mu_\varepsilon(x)$, $\tilde{\phi}(\tilde{x}) = F_\varepsilon(\phi(x))$ is an infinitesimal quasi-symmetry of $S[\phi] = \int_{U \subseteq \mathbb{R}^n} d^n x\, \mathcal{L}(\phi(x), \partial\phi(x), x)$ for all (nice) $U \subseteq \mathbb{R}^n$ → integration over U should be defined

(where Λ^μ does not depend on U) → but generally depends on ϕ

then $\partial_\mu \left[\mathcal{L}\, \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta\phi - \phi_{,\nu}\, \delta x^\nu) - \Lambda^\mu \right] = 0$ when ϕ satisfies the Euler-Lagrange eqs. $\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} = 0$. $\left(\delta x^\mu = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X^\mu_\varepsilon(x),\ \delta\phi = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F_\varepsilon(\phi) \right)$

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proof: Almost exactly the same as the regular case. When we get to the step

$$\delta S = \int_U d^n x\, \partial_\mu \left[\mathcal{L}\, \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta\phi - \phi_{,\nu}\, \delta x^\nu) \right]$$

we find that

$$\int_U d^n x\, \partial_\mu \left[\mathcal{L}\, \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta\phi - \phi_{,\nu}\, \delta x^\nu) - \Lambda^\mu \right] = 0 \qquad \forall U$$

$$\Rightarrow \partial_\mu \left[\mathcal{L}\, \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta\phi - \phi_{,\nu}\, \delta x^\nu) - \Lambda^\mu \right] = 0$$