Some potentially useful information

- Euler-Lagrange equations for generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j} \;, \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot{q}_j}$$

constraints: holonomic $f_\alpha(q,t) = 0$ or semiholonomic $g_\beta = \sum_j a_{\beta j}(q,t)\dot{q}_j + a_{\beta t}(q,t) = 0$

- Generalized forces: $d/dt(\partial L/\partial \dot{q}_j) - \partial L/\partial q_j = R_j$

Friction forces: $\vec{f}_i = -h(v_i)\vec{v}_i/v_i$, $\vec{v}_i = \dot{\vec{r}}_i$ gives $R_j = -\partial \mathcal{F}/\partial \dot{q}_j$, $\mathcal{F} = \sum_i \int_0^{v_i} dv_i' \, h(v_i')$

- Hamilton's equations for canonical variables (q_j, p_j): $\quad \dot{q}_j = \dfrac{\partial H}{\partial p_j}$, $\quad \dot{p}_j = -\dfrac{\partial H}{\partial q_j}$

- Hamiltonian for a Lagrangian quadratic in velocities

$$L = L_0(q,t) + \dot{\vec{q}}^T \cdot \vec{a} + \frac{1}{2}\dot{\vec{q}}^T \cdot \hat{T} \cdot \dot{\vec{q}} \quad \Rightarrow \quad H = \frac{1}{2}(\vec{p} - \vec{a})^T \cdot \hat{T}^{-1} \cdot (\vec{p} - \vec{a}) - L_0(q,t)$$

- The Moment of Inertia Tensor and its relations:

$$I_{ab} = \int dV \, \rho(\vec{r})[\vec{r}^{\,2}\delta_{ab} - r_a r_b] \qquad \text{or} \qquad I^{ab} = \sum_i m_i [\delta^{ab}\vec{r}_i^{\,2} - r_i^a r_i^b]$$

$$I_{ab}^{(Q)} = M(\delta_{ab}\,\vec{R}^2 - R_a R_b) + I_{ab}^{(\mathrm{CM})} \;, \qquad \hat{I}' = \hat{U}\,\hat{I}\,\hat{U}^T$$

- Euler's Equations:

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= \tau_3 \end{aligned}$$

- Vibrations: $L = \frac{1}{2}\dot{\vec{\eta}}^T \cdot \hat{T} \cdot \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T \cdot \hat{V} \cdot \vec{\eta}$ has Normal modes $\vec{\eta}^{(k)} = \vec{a}^{(k)} \exp(-i\omega^{(k)}t)$

$$\det(\hat{V} - \omega^2\hat{T}) = 0 \;, \qquad \left(\hat{V} - [\omega^{(k)}]^2\,\hat{T}\right) \cdot \vec{a}^{(k)} = 0 \;, \qquad \vec{\eta} = \mathrm{Re}\sum_k C_k\,\vec{\eta}^{(k)}$$

- Generating functions for Canonical Transformations: $K = H + \partial F_i/\partial t$ and

$$F_1(q,Q,t): \quad p_i = \frac{\partial F_1}{\partial q_i} \;,\; P_i = -\frac{\partial F_1}{\partial Q_i} \;, \qquad F_2(q,P,t): \quad p_i = \frac{\partial F_2}{\partial q_i} \;,\; Q_i = \frac{\partial F_2}{\partial P_i}$$

- Poisson Brackets: $\quad [u,v]_{q,p} = \displaystyle\sum_j \left[\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j}\right]$, $\quad \dfrac{du}{dt} = [u,H] + \dfrac{\partial u}{\partial t}$

- Relations for Hamilton's Principle function, $S = S(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n, t)$

$$K = 0 \;, \qquad P_i = \alpha_i \;, \qquad Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i} \;, \qquad p_i = \frac{\partial S}{\partial q_i}$$

- Relations for Hamilton's Characteristic function, $W = W(q_1, \ldots, q_n; \alpha_1, \ldots, \alpha_n)$

$$K = H = \alpha_1 \;, \qquad P_i = \alpha_i \;, \qquad \beta_1 + t = \frac{\partial W}{\partial \alpha_1} \;, \qquad \beta_{i>1} = \frac{\partial W}{\partial \alpha_i} \;, \qquad p_i = \frac{\partial W}{\partial q_i}$$

- Action Angle Variables: $\quad J = \displaystyle\oint p\,dq$, $\quad w = \dfrac{\partial W(q,J)}{\partial J}$, $\quad \dot{w} = \dfrac{\partial H(J)}{\partial J} = \nu(J)$

will be (β, J)

- Time Dependent Perturbation Theory for $H_0 + \Delta H$. Solve $H_0(p,q)$ with the Hamilton-Jacobi method to obtain constant canonical variables (β, α) where $[\beta, \alpha] = 1$. Then

$$\dot\alpha^{(n)} = -\frac{\partial \Delta H}{\partial \beta}\bigg|_{n-1}, \qquad \dot\beta^{(n)} = \frac{\partial \Delta H}{\partial \alpha}\bigg|_{n-1}$$

- Fluid volume and continuity equations $\quad \dfrac{dV}{dt} = \displaystyle\int dV\, \vec\nabla\cdot\vec v\,, \qquad \dfrac{\partial \rho}{\partial t} + \vec\nabla\cdot(\rho\vec v\,) = 0$

- Euler equation ($\nu = 0$) or Navier-Stokes equation ($\nu = \eta/\rho \neq 0$), with gravity:

$$\cancel{\frac{\partial \vec v}{\partial t}} + (\vec v\cdot\vec\nabla)\,\vec v + \frac{1}{\rho}\vec\nabla p - \nu\cancel{\nabla^2}\vec v = \frac{\vec f}{\rho} = \vec g$$

- For direction i the force/unit area on a surface $= -\hat n_i\, p + \hat n_k \sigma'_{ki}$

- Ideal fluid has $ds/dt = 0$ so $p = p(\rho, s)$. Viscous fluid has $ds/dt \propto \sigma'_{ik}\, \partial v_i/\partial x_k$.

- Bernoulli's equation for a steady incompressible ideal fluid in gravity $\vec g = -g\hat z$:

$$\frac{\vec v^{\,2}}{2} + gz + \frac{p}{\rho} = \text{constant}$$

- Irrotational incompressible ideal fluid flow (potential flow): $\vec v = \nabla\phi$, $\nabla^2\phi = 0$

- Sound waves: $\left(\frac{\partial^2}{\partial t^2} - c_s^2\nabla^2\right)\{p', \rho', \vec v\,\} = 0$. Mach number $M = v_0/c_s$.

- Momentum conservation: $\frac{\partial}{\partial t}(\rho\vec v) + \vec\nabla\cdot\hat T = \vec f$ where the energy momentum tensor is $T_{ki} = v_k v_i\,\rho + \delta_{ki}\,p - \sigma'_{ki}$. For $\vec\nabla\cdot\vec v = 0$ the viscous stress tensor $\sigma'_{ki} = \eta\big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\big)$.

- Reynolds Number: $R = uL/\nu$

- Bifurcations at $\mu = 0$. In 1-dim: "saddle-node" $\dot x = \mu + x^2$, "transcritical" $\dot x = x(\mu - x)$, "supercritical pitchfork" $\dot x = \mu x - x^3$, "subcritical pitchfork" $\dot x = \mu x + x^3$. In 2-dim: "supercritical Hopf" $\dot r = r(\mu - r^2)$, "subcritical Hopf" $\dot r = r(\mu + r^2)$.

- Linearization for 2-dim fixed points: $\begin{pmatrix}\dot u\\ \dot v\end{pmatrix} = M\begin{pmatrix}u\\ v\end{pmatrix}, \quad \begin{pmatrix}u\\ v\end{pmatrix} = \vec a\, e^{\lambda t}, \quad M\vec a = \lambda\,\vec a$

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}\,, \qquad \tau = \mathrm{tr}\, M\,, \qquad \Delta = \det M$$

- 2-dim conserved system $\dot x = f_x(x,y)$, $\dot y = f_y(x,y)$ with $\vec\nabla\cdot\vec f = 0$, has conserved $H(x,y) = \int^y dy' f_x(x,y') - \int^x dx' f_y(x',y)$.

- 1-dim map $x_{n+1} = f(x_n)$. Its fixed points satisfy $x^* = f(x^*)$. Here x^* is stable for $|f'(x^*)| < 1$ and unstable for $|f'(x^*)| > 1$.

- Fractal dimension: $\quad d_F = \displaystyle\lim_{a\to 0} \frac{\ln N(a)}{\ln(a_0/a)}$