Quick way to get solution to Dirac equation...

In the Weyl representation we have

$$\begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix}\begin{pmatrix} u(p) \end{pmatrix} = 0$$

Recall $\sigma^\mu = (1, \vec{\sigma})$

$\bar{\sigma}^\mu = (1, -\vec{\sigma})$

Note that $p\cdot\sigma$ and $p\cdot\bar{\sigma}$ commute:

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0 - \vec{p}\cdot\vec{\sigma})(p^0 + \vec{p}\cdot\vec{\sigma})$$
$$= (p^0 + \vec{p}\cdot\vec{\sigma})(p^0 - \vec{p}\cdot\vec{\sigma})$$
$$= (p\cdot\bar{\sigma})(p\cdot\sigma)$$

Also $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p\cdot\bar{\sigma})(p\cdot\sigma) = (p^0)^2 - \vec{p}^2 = p^2$

$$= m^2$$

So we can write $m = \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}$, $p\cdot\sigma = \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma}$,

$p\cdot\bar{\sigma} = \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}}$

$$\begin{pmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar{\sigma}}\sqrt{p\cdot\bar{\sigma}} & -\sqrt{p\cdot\sigma}\cdot\sqrt{p\cdot\bar{\sigma}} \end{pmatrix}\begin{pmatrix} ? \\ ? \end{pmatrix} = 0$$

Solution is clearly

$$\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix} \qquad \text{for any } \xi$$

which is what we had before

For negative frequency solutions... $\psi(x) = v(p)\, e^{+ip\cdot x}$

$$\begin{pmatrix} -m & -p\cdot\sigma \\ -p\cdot\bar\sigma & -m \end{pmatrix} \begin{pmatrix} v(p) \end{pmatrix} = 0$$

The quick way gives

$$\begin{pmatrix} -\sqrt{p\cdot\sigma}\,\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\sigma}\,\sqrt{p\cdot\sigma} \\ -\sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\bar\sigma} & -\sqrt{p\cdot\sigma}\,\sqrt{p\cdot\bar\sigma} \end{pmatrix} \begin{pmatrix} v(p) \end{pmatrix} = 0$$

So $$\begin{pmatrix} v(p) \end{pmatrix} = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$ actually standard notation is $\eta$ instead of $\xi$ for $v(p)$

We note that

$$u^\dagger u = \left( \xi^\dagger \sqrt{p\cdot\sigma} \quad \xi^\dagger \sqrt{p\cdot\bar\sigma} \right) \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger (p\cdot\sigma + p\cdot\bar{\sigma})\xi$$

$$= \xi^\dagger (E_{\vec{p}} + E_{\vec{p}})\xi = 2E_{\vec{p}} \overbrace{\xi^\dagger \xi}^{=1}$$

We will choose an orthonormal basis

$$\xi^1, \xi^2$$

(typical choice will be $\xi^1 = \begin{pmatrix}1\\0\end{pmatrix}$, $\xi^2 = \begin{pmatrix}0\\1\end{pmatrix}$)

Then $\quad u^{r\dagger}(p)\, u^s(p) = 2E_{\vec{p}}\, \delta^{rs}$

Similarly $\quad v^{r\dagger}(p)\, v^s(p) = 2E_{\vec{p}}\, \delta^{rs}$

We can now calculate

$$\bar{u}^r(p)\, u^s(p) = u^{r\dagger}(p)\, \gamma^0 u^s(p)$$

$$= \left(\xi^{r\dagger}\sqrt{p\cdot\sigma} \;\; \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix}$$

$$= \xi^{r\dagger}\xi^s \cdot 2m = 2m\,\delta^{rs}$$

$\boxed{\text{Dirac adjoint} \quad \bar{\psi} = \psi^\dagger \gamma^0}$

Similarly

$$\bar{v}^r(p)\, v^s(p) = -2m\,\delta^{rs}$$

We can also check

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$$

orthogonality with respect to Dirac adjoint

## Spin sums

These results will be useful later when summing over all spin-$\frac{1}{2}$ polarizations.

Since $\xi^s$ are an orthonormal basis

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So
$$\sum_{s=1,2} u^s(p)\, \bar{u}^s(p) = \sum_{s=1,2} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \begin{pmatrix} \xi^{s\dagger}\sqrt{p\cdot\sigma} & \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} p\cdot\sigma & m \\ m & p\cdot\bar{\sigma} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$$= p\cdot\gamma + m$$

Similarly
$$\sum_{s=1,2} v^s(p)\, \bar{v}^s(p) = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = p\cdot\gamma - m$$

Feynman "Slash" notation

$$\not{p} \equiv p_\mu \gamma^\mu = p \cdot \gamma$$

## Dirac Matrices and Bilinears

Consider $\bar{\psi}\,\Gamma\,\psi$ where $\Gamma$ is $4\times 4$

16 possible terms

Start with powers of $\gamma$'s...

$$1,\ \gamma^\mu,\ \gamma^\mu\gamma^\nu,\ \gamma^\mu\gamma^\nu\gamma^\rho,$$

The symmetric combination $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and so it suffices to consider only completely antisymmetric products.

$$\underset{\substack{\#=1\\ \text{such}\\ \text{term}}}{1},\quad \underset{\substack{\#=4\\ \text{such}\\ \text{terms}}}{\gamma^\mu},\quad \underset{\substack{\#=\binom{4}{2}=6\\ \text{such}\\ \text{terms}}}{\gamma^{[\mu}\gamma^{\nu]}} \equiv \frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu),$$

$$\underset{\#=\binom{4}{3}=4 \text{ such terms}}{\gamma^{[\mu}\gamma^\nu\gamma^{\rho]}},\quad \underset{\#=\binom{4}{4}=1 \text{ such term}}{\gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}}$$

This adds up to 16.

Let us define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

We can write $\gamma^5 = -\frac{i}{4!}\varepsilon^{\mu\nu\rho\sigma}\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$

$(\varepsilon^{0123} = 1, \varepsilon^{1023} = -1, \cdots)$
totally antisymmetric

Note that

$$(\gamma^5)^\dagger = -i(\gamma^3)^\dagger(\gamma^2)^\dagger(\gamma^1)^\dagger(\gamma^0)^\dagger$$
$$= i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5$$

Also $$\{\gamma^5, \gamma^\mu\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\underbrace{\gamma^0\gamma^1\gamma^2\gamma^3}_{\text{overall } (-1)}$$
$$= 0$$

Also $$[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}[\gamma^\mu, \gamma^\nu]]$$
$$= 0$$

So eigenstates of $\gamma^5$ with different eigenvalues don't mix under Lorentz transformations. In the Weyl representation

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$-1$ for $\psi_L$ left-handed
$+1$ for $\psi_R$ right-handed

Parity : $\vec{x} \to -\vec{x}$

... A Lorentz vector $(x^0, \vec{x}) \to (x^0, -\vec{x})$

If instead $(x^0, \vec{x}) \to -(x^0, -\vec{x}) = (-x^0, \vec{x})$ under parity, we call this a pseudo- or axial-vector

A scalar is invariant under parity.
A pseudoscalar flips sign under parity.

| | | # of such matrices |
|---|---|---|
| $\mathbb{1}$ | scalar | 1 |
| $\gamma^\mu$ | vector | 4 |
| $\sigma^{\mu\nu} \equiv \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ | tensor | 6 |
| $\gamma^\mu \gamma^5$ | axial vector | 4 |
| $\gamma^5$ | pseudoscalar | 1 |

We will discuss parity in more detail later

Vector current:

$$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x)$$

Axial vector current:

$$j^{\mu 5}(x) = \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x)$$

Note on Dirac adjoints...

We defined $\bar{\psi} = \psi^\dagger \gamma^0$.

It is useful to define $\bar{M}$ for matrix $M$

$$\bar{M} = \gamma^0 M^\dagger \gamma^0 \qquad \left(\begin{array}{l}\text{we discussed} \\ \text{this earlier}\end{array}\right)$$

We can take the adjoint of products

$$\overline{M_1 M_2} = \gamma^0 M_2^\dagger M_1^\dagger \gamma^0 = \gamma^0 M_2^\dagger \overbrace{\gamma^0 \gamma^0}^{\gamma^0\gamma^0 = 1} M_1^\dagger \gamma^0$$

$$= \bar{M}_2 \bar{M}_1$$

So it behaves just like a Hermitian conjugate

Note that $\bar{\gamma}^\mu = \gamma^0 \gamma^{\mu\dagger} \gamma^0 = \gamma^\mu$.

If $\psi(x)$ satisfies the Dirac equation then

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 \Rightarrow i\gamma^\mu \partial_\mu \psi = m\psi$$

and the Dirac adjoint gives...

$$-i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0 \Rightarrow -i\partial_\mu \bar{\psi}\gamma^\mu = m\bar{\psi}$$

So
$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi})\gamma^\mu \psi + \bar{\psi}\gamma^\mu(\partial_\mu \psi)$$
$$= im\bar{\psi}\psi - im\bar{\psi}\psi = 0$$

$j^\mu$ is a <u>conserved</u> current

It is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha}\psi(x)$$

Similarly

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar{\psi})\gamma^\mu \gamma^5 \psi + \bar{\psi}\gamma^\mu\gamma^5 \partial_\mu \psi$$
$$= (\partial_\mu \bar{\psi})\gamma^\mu \gamma^5 \psi - \bar{\psi}\gamma^5\gamma^\mu \partial_\mu \psi$$
$$= 2im\bar{\psi}\gamma^5\psi$$

So $j^{\mu 5}$ is conserved if $m=0$.

When $m=0$, $j^{\mu 5}$ is the Noether current for the symmetry $\psi(x) \to e^{i\alpha\gamma^5}\psi(x)$

## Quantization of Dirac field

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

The canonical momentum conjugate to $\psi$ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger$$

The Hamiltonian is

$$H = \int d^3\vec{x}\, \left(i\bar{\psi}\gamma^0\partial_0\psi - i\bar{\psi}\gamma^0\partial_0\psi - i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi\right)$$

$$= \int d^3\vec{x}\, \left(-i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi\right)$$

You may have seen this notation before...

$$\vec{\alpha} = \gamma^0\vec{\gamma}\,, \quad \beta = \gamma^0$$

in which case

$$H = \int d^3\vec{x}\, \psi^\dagger(-i\vec{\alpha}\cdot\vec{\nabla} + m\beta)\psi$$