MA434, Spring 2020 — Final Exam Solutions

Not really a complete set of solutions, sorry. On the other hand, I do give at least a sketch of a solution for all the problems that were attempted by someone.

1. [10 points] Show that the affine variety in $\mathbb{A}^2$ defined by $xy = 1$ is not isomorphic to $\mathbb{A}^1$.

The most common problem with the solutions I saw was taking the obvious map and showing that it is not an isomorphism. But that's not enough. After all, the map $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$ that sends t to t^2 is not an isomorphism, but $\mathbb{A}^1$ is certainly isomorphic to itself.

The way to show two things are not isomorphic is to find a property that distinguishes them. In this case, the easiest thing is to use the coordinate rings.

We showed that the variety in $\mathbb{A}^2$ defined by $xy = 1$ is isomorphic to the standard open subset of $\mathbb{A}^1$ defined by $x \neq 0$. So its coordinate ring is $k[x, x^{-1}]$, which is not isomorphic to $k[x]$, the coordinate ring of $\mathbb{A}^1$. Since isomorphic varieties have isomorphic coordinate rings, the varieties cannot be isomorphic.

An easy way to see the two rings are not isomorphic is to consider a k-algebra homomorphism $f : k[x, x^{-1}] \longrightarrow k[x]$. Since x is invertible in $k[x, x^{-1}]$, its image $f(x)$ must be an invertible element of $k[x]$, hence must be a nonzero constant. It follows that the image of f is contained in k, so it cannot be surjective.

2. [20 points] Given two affine varieties V and W and a polynomial map $f : V \longrightarrow W$, we get a ring homomorphism $f^* : k[W] \longrightarrow k[V]$. When is f^* injective?

This is basically in the book, in section 4.10. I think several of you found it difficult to juggle the two categories in play: f is a map between affine varieties, f^* is a ring homomorphism, so there are two distinct worlds to play in.

Given a nonzero $u \in \ker(f^*)$ we have $u \in k[W]$ and $f^*(u) = 0$. Since $f^*(u) = u \circ f$, we are saying that this is the zero *function* in $k[V]$. That means $u(f(x)) = 0$ for any $x \in V$. That happens if and only if the image of f is contained in the closed subset $V(u) \neq W$ (since we know $V(u) = W$ if and only if $u = 0$).

So f^* is injective if and only if the image of f is not contained in any closed proper subset of W, i.e., if the image of f is dense in W. We called such f *dominant* maps.

3. [20 points] Consider the polynomial map $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ defined by $\varphi(t) = (t^2, t^5)$ (one of the maps considered in problem 4.3). Let C_0 be the image of φ. We showed in that problem that there is a rational map $\pi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ so that $\pi|_{C_0}$ is the inverse of φ.

a. Show that C_0 is an irreducible algebraic subset of $\mathbb{A}^2$, i.e., an affine variety.

   If we set $f(x, y) = x^5 - y^2$ then $f(t^2, t^5) = 0$, so that $\varphi(\mathbb{A}^1) = C_0 \subset V(f)$. Conversely, if $x^5 = y^2$ with $x \neq 0$, let $t = \pi(x, y) = y/x^2$. It's easy to check that $(x, y) = \varphi(t)$. For example, $t^2 = (y/x^2)^2 = y^2/x^4 = x^5/x^4 = x$. It remains to check points with $x = 0$, but the only such point in $V(f)$ is $(0, 0) = \varphi(0)$. So every point in $V(f)$ belongs to the image, and we have shown $C_0 = V(f)$.

   It remains to show C_0 is irreducible. One could just ask Sage whether the ideal generated by $x^5 - y^2$ is prime, which amounts to whether the polynomial $x^5 - y^2$ is irreducible. But here's a cheaper way: notice that since the image of φ is all of C_0 it follows from problem 1 that $\varphi* : k[C_0] \longrightarrow k[\mathbb{A}^1] = k[t]$ is injective, hence $k[C_0]$ is isomorphic to a subring of a domain, hence is a domain.

b. Find a rational map $\tilde{\varphi} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ that extends φ.

   Easy: $\tilde{\varphi}([t_o, t_1]) = [t_0^5, t_0^3 t_1^2, t_1^5]$. That gives $\tilde{\varphi}([1 : t]) = [1 : t^2 : t^5]$, so it agrees with φ on the standard $\mathbb{A}^1$ given by $t_0 \neq 0$.

c. Let C be the projective completion C_0, so that C is a projective variety. Show that C is the image of $\tilde{\varphi}$.

   The homogenization of $f(x, y)$ is $F(x_0, x_1, x_2] = x_1^5 - x_0^3 x_2^2$, so $C = V(F)$. Let's show this is the image of $\tilde{\varphi}$. We already know that

   $$\tilde{\varphi}(\mathbb{A}^1) = \varphi(\mathbb{A}^1) = C_0 \subset C,$$

   and there is only one more point on $\mathbb{P}^1$, the point at infinity. But $\tilde{\varphi}([0, 1]) = [0, 0, 1]$ and clearly $F(0, 0, 1) = 0$. So the image is contained in C.

   We already know that every point in C_0 is in the image, so it remains to check the points in C that are not in the standard affine $\mathbb{A}^2$. Those look like

$[0 : x_1 : x_2]$. Since $F(0, x_1, x_2) = x_1^5$, the only point of that form in C is $[0 : 0 : 1]$, which we have already checked is in the image. So $C = \tilde{\varphi}(\mathbb{P}^1)$.

d. Find a rational map $\tilde{\pi} : C \dashrightarrow \mathbb{P}^1$ that extends $\pi|_{C_0}$.

We know that $\pi([1 : \frac{x_1}{x_0} : \frac{x_2}{x_0}])$ is equal to

$$\left[1 : \frac{(x_2/x_0)}{(x_1/x_0)^2}\right] = \left[1 : \frac{x_2 x_0}{x_1^2}\right],$$

so we can set

$$\tilde{\pi}([x_0, x_1, x_2]) = [x_1^2 : x_2 x_0].$$

Let's check that this is an inverse. For example,

$$\tilde{\pi}\tilde{\varphi}([t_0 : t_1]) = \tilde{\pi}([t_0^5 : t_0^3 t_1^2 : t_1^5]) = [t_0^6 t_1^4 : t_0^5 t_1^5] = [t_0 : t_1],$$

as expected, and

$$\tilde{\varphi}\tilde{\pi}([x_0 : x_1 : x_2]) = \tilde{\varphi}([x_1^2 : x_2 x_0]) = [x_1^{10} : x_1^6 x_2^2 x_0^2 : x_2^5 x_0^5],$$

and since $x_1^5 = x_0^3 x_2^2$ this is equal to

$$[x_0^6 x_2^4 : x_1 x_0^5 x_2^4 : x_0^5 x_2^5] = [x_0 : x_1 : x_2].$$

e. Determine the domains of $\tilde{\pi}$ and $\tilde{\varphi}$.

The formulas are

$$\tilde{\varphi}([t_0 : t_1] = [t_0^5 : t_0^3 t_1^2 : t_1^5]$$

and

$$\tilde{\pi}([x_0, x_1, x_2]) = [x_1^2 : x_2 x_0].$$

There are no denominators, so we just need to check for common zeros.

In the first formula, $t_0^5 = 0$ says $t_0 = 0$ and $t_1^5 = 0$ says $t_1 = 0$, which cannot both happen at once. So the domain of $\tilde{\varphi}$ is all of $\mathbb{P}^1$.

In the second formula, $x_1^2 = 0$ forces $x_1 = 0$ and $x_0 x_2 = 0$ means one of the two is zero. So the only possible problem points are $[0 : 0 : 1]$ and $[1 : 0 : 0]$, and both points are on C. So the domain of $\tilde{\pi}$ is

$$C - \{[0 : 0 : 1], [1 : 0 : 0]\}.$$

So the rational map $\tilde{\varphi} : \mathbb{P}^1 \dashrightarrow C$ is birational but not an isomorphism.

4. [20 **points**] Suppose that f is a rational function on $\mathbb{P}^1$.

a. Show that if f is regular at every point of $\mathbb{P}^1$ then it is constant. (Hint: consider the two affine pieces $\mathbb{A}^1_{(0)}$ and $\mathbb{A}^1_{(1)}$.)

Suppose $f([u : v]) = g(u, v)/h(u, v)$ with g and h homogeneous of the same degree. Let f_0 be the restriction of f to $\mathbb{A}^1_{(0)}$ and let f_1 be the restriction of f to $\mathbb{A}^1_{(1)}$.

Since f_0 is regular on the affine variety $\mathbb{A}^1_{(0)}$, we know, by Theorem 4.8 (II), that $f_0 \in k[v]$. So $g(u, v)/h(u, v) = a(v) \in k[v]$. For the same reasons, $g(u, v)/h(u, v) = b(u) \in k[u]$. But the only way a polynomial in u can be equal to a polynomial in v is if both are constants. Hence $f \in k$.

b. Show that there are no non-constant morphisms $\mathbb{P}^1 \longrightarrow \mathbb{A}^m$.

Such a morphism would have to be an m-tuple $(f_1, f_2, \dots, f_m)$ of rational functions that are regular on all of $\mathbb{P}^1$. By part (a), the f_i are all constant and so the image is a point.

5. [20 **points**] Below are three formulas that possibly define rational maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Decide whether the formulas do define rational maps. If they do, determine $\mathrm{dom}(f)$ and decide whether f is birational.

Two mistakes were very popular. First, people forgot that points in projective space have multiple representations, because everything is up to scaling. This is crucial when we are looking for regular points: all we need is a representation that is defined at the point, so we can't just looko at whatever we were given.

Second, people forgot that the definition of a rational function requires a quotient of homogeneous polynomials *of the same degree*. So x or $1/x$ are *not* rational functions.

a. $f([x : y : z]) = [1/x : 1/y : 1/z]$.

Since $f([x : y : x]) = [yz : xz : xy]$ is given by a triple of homogeneous polynomials of degree two, it is a rational function. We can also give the proper representations explicitly:

$$f([x:y:z]) = \left[1 : \frac{x}{y} : \frac{z}{z}\right] = \left[\frac{y}{x} : 1 : \frac{y}{z}\right] = \left[\frac{z}{x} : \frac{z}{y} : 1\right].$$

It's clear that f is its own inverse on the open set $xyz \neq 0$, hence it is a birational equivalence $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. We get $yz = xz = xy = 0$ if and only if two of the variables are 0, so the domain is

$$\mathbb{P}^2 - \{[1:0:0], [0:1:0], [0:0:1]\}.$$

It is fun to notice that the three "bad" points are exactly the image of the three lines $x = 0$, $y = 0$, $z = 0$.

b. $f([x:y:z]) = [x:y:1]$.

This is not well defined, since

$$f([tx:ty:tz]) = [tx:ty:1] \neq [x:y:1] = f([x:y:z])$$

unless $x = y = 0$.

c. $f([x:y:z]) = [\frac{x^3+y^3}{z^3} : \frac{y^2}{z^2} : 1]$.

This one is clearly a rational map, since each function is a ratio of homogeneous polynomials of the same degree. I like to rewrite as

$$f([x:y:z]) = [x^3 + y^3 : y^2z : z^3].$$

We get $z^3 = 0$ only if $z = 0$, which also makes $y^2z = 0$. So if $y^3 = -x^3$ and $z = 0$ we get a point that is not in the domain. If $y^3 = -x^3$ then $y = \omega x$ where $\omega^3 = -1$. One solution is $\omega = -1$ but since we are working over $\mathbb{C}$ there are two others: call them ω_1 and ω_2. So any point of the form $[x : \omega x : 0]$ with $x \neq 0$ is not in the domain. There are three such points: $[1:-1:0]$, $[1, \omega_1 : 0]$ and $[1 : \omega_2 : 0]$.

The easiest way to see that f is not birational is probably to notice that in the open dense set $z \neq 0$ the map is never injective, since for any nonzero $\gamma \in \mathbb{C}$ there are three choices of z such that $z^3 = \gamma$, and for any choice of z we can then solve $y^2z = \beta$ and $x^3 + y^3 = \alpha$. So there are at least three $[x : y : z]$ mapping to $[\alpha : \beta : \gamma]$. A birational equivalence can be restricted to an isomorphism between dense open sets, but any such set would intersect $z \neq 0$. So f is not birational.

Notice that it is not enough to find two points that map to the same point in the image, because the points you found might belong to the closed subset on which f is not a bijection.

6. [10 points] Prove statements (i), (ii), (iii), (iv) from Example I from section 5.7 of *Undergraduate Algebraic Geometry*.

Nathaniel gave a complete proof in the forum for that class, so it was mostly of matter of writing up his argument.

7. [20 points] Consider the rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$f([u : v : w]) = [uw : w^2 : vw].$$

Describe the domain and the image of f.

I'm pretty sure this one didn't come out as the professor intended. Notice that we can rewrite f as $f([u : v : w]) = [u : w : v]$, at which point it is clear that it is in fact an isomorphism $\mathbb{P}^2 \longrightarrow \mathbb{P}^2$.

A lot of people forgot that they could move to an equivalent representation and so missed that this is actually a bijection.

8. [25 points] Solve problem 5.2 in *Undergraduate Algebraic Geometry*.

No one attempted this one, alas.

9. [25 points] Solve problem 5.11 in *Undergraduate Algebraic Geometry*.

No one attempted this one either.

10. [25 points] Recall that a hyperfurface is a variety in $\mathbb{P}^n$ defined by a single polynomial (equivalently, by a principal ideal). Show that every variety is birational to a hypersurface. (Hint: Use the Noether Normalization Theorem.)

Basically section 5.10 of *UAG*, which I hoped people would write up in more detail. Let V be a variety. The basic steps are:

a. If V is affine, leave it alone. If V is projective, it has a dense open subset that is an affine variety and is birational to it (the identity map gives an isomorphism with a dense open subset). So we can *assume V is affine*.

b. Let $A = k[V]$. Since this is a finitely generated k-algebra, we can use Noether Normalization to find $y_1, y_2, \ldots, y_m \in k[V]$ such that $y_1, y_2, \ldots, y_m$ are algebraically independent and $k[V]$ is a finite $k[y_1, y_2, \ldots, y_m]$-algebra.

c. $k[y_1, y_2, \ldots, y_m]$ is just a ring of polynomials, hence we can think of it as $k[\mathbb{A}^m]$. The inclusion $k[y_1, y_2, \ldots, y_m] \hookrightarrow k[V]$ translates to a dominant map $V \longrightarrow \mathbb{A}^m$.

d. (Here's the step many folks missed.) Since $k[V]$ is a domain and finite over $k[y_1, y_2, \ldots, y_m]$, we can pass to fields of fractions $k(y_1, y_2, \ldots, y_m) \subset k(V)$. The primitive element theorem in 3.17 now says that $k(V)$ is generated by taking one more element: $k(V) = k(y_1, y_2, \ldots, y_m, y_{m+1})$.

e. Since y_{m+1} is algebraic over $k(y_1, y_2, \ldots, y_m)$, it satisfies a single equation with coefficients in $k[y_1, y_2, \ldots, y_m]$. This equation defines a hypersurface in $\mathbb{A}^{m+1}$.

f. The function field of that hypersurface is clearly $k(V)$, so it is birational to V.

11. [30 points] Given an invertible matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with complex coefficients, define a function $f_A : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ by

$$f_A([u : v]) = [au + bv : cu + dv].$$

a. Show that f_A is a morphism.

The smart way to write f_A is with matrix multiplication. If we write the pairs $[u : v]$ as column vectors, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} au + bv \\ cu + dv \end{bmatrix}.$$

We know f_A is a rational function because both components are homogeneous of degree one. The domain is all pairs such that the answer is not the zero vector. But A is invertible, so only the zero vector maps to the zero vector. Thus, the domain is all of $\mathbb{P}^1$, i.e., f_A is a morphism.

b. How does f_{AB} relate to f_A and f_B?

The formula with matrix multiplication makes it clear that $f_{AB} = f_A f_B$.

c. Show that f_A has an inverse morphism, so that f_A defines an automorphism of $\mathbb{P}^1_C$.

Notice that if $A = I$ is the identity then f_A is the identity as well. From the previous part, it follows that if $B = A^{-1}$ we have $f_B f_A = f_A f_B = f_I$, so $f_{A^{-1}} = (f_A)^{-1}$.

d. If we identify $\mathbb{C}$ with the standard $\mathbb{A}^1 \subset \mathbb{P}^1$ defined by $v \neq 0$, show that the restriction of f_A to $\mathbb{C}$ is a rational function, and find its formula.

It is the function $f_A(z) = \dfrac{az+b}{cz+d}$. This is only a rational function, since $z = -d/c$ is not in the domain and a/c is not in the image. Functions like this are known as *Möbius transformations*.

One way to think of the previous problem is to remember that $\mathbb{P}^1$ can be gotten from $\mathbb{C}^2$ by deleting the origin and then modding out by equivalence. An invertible matrix A gives an automorphism of $\mathbb{C}^2$ that sends $(0,0)$ (and only that point) to $(0,0)$, so we can delete that point and get a function $\mathbb{C}^2 = \{(0,0)\} \longrightarrow \mathbb{C}^2 - \{(0,0)\}$. Since it is linear, $A(tb) = t(Ab)$, so it respects the equivalence relation, and therefore induces an automorphism $\mathbb{P}^1 \longrightarrow \mathbb{P}^1$. That's our f_A.

12. [30 points] Show that any automorphism of $\mathbb{P}^1_{\mathbb{C}}$ is of the form f_A as in the previous problem.

Using the last paragraph in the last solution, what we are trying to prove is that automorphisms of $\mathbb{P}^1$ (in the world of algebraic geometry) come from automorphisms of $\mathbb{C}^2$ (in the world of linear algebra). This is not obvious!

Suppose we have a rational function $f : \mathbb{P}^1 - \to \mathbb{P}^1$. Then we can write $f([u : v]) = [A(u, v) : B(u : v)]$ where A and B are homogeneous polynomials of the same degree.

We need to prove that if f is an isomorphism then the degree must be one. We can try to do this by brute force, but it is messy. The cleanest way, I think, is to use the theorem in section 4.11: if f is an automorphism, then f^* is a field isomorphism $k(t) \longrightarrow k(t)$. The theorem then follows by showing that every field isomorphism of that kind is a Möbius transformation.