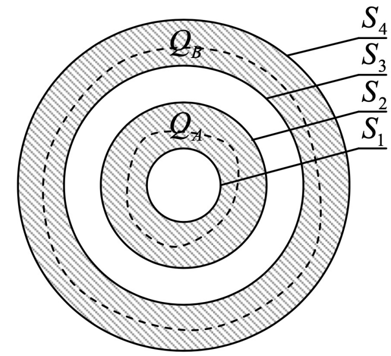
Electromagnetic Theory
Problem Set #2 (8.311)

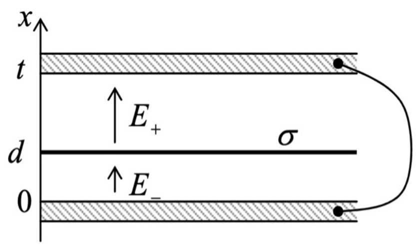
[Due Wed. Feb 16, 2022]

Problem 2.2. Electric charges Q_A and Q_B have been put on two metallic, concentric spherical shells – see the figure on the right. What is the full charge of each of the surfaces S_1-S_4?

(20 Pts)



Problem 2.6. A wide, thin plane film, carrying a uniform electric charge density σ, is placed inside a similarly wide plane capacitor, whose plates are connected with a wire (see the figure on the right) and were initially electroneutral. Neglecting the fringe effects, calculate the surface charges of the plates, and the net force exerted upon the film (per unit area).



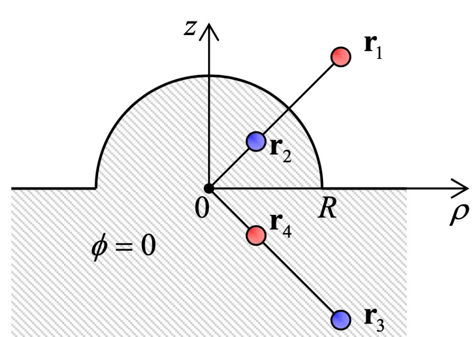
(30 pts)

Problem 2.29. Use the image charge method to calculate the full surface charges induced in the plates of a very broad, externally-unbiased plane capacitor of thickness D by a point charge q separated from one of the electrodes by distance d.

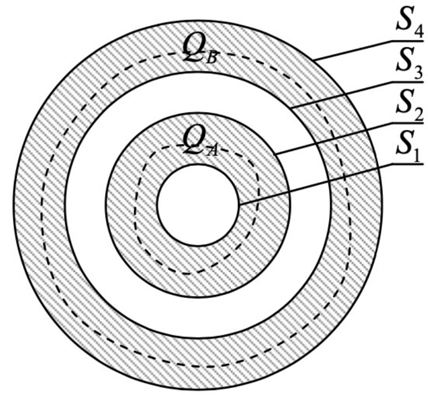
(50 pts)

Problem 2.32. Use the method of images to find the Green's function of the system shown in the figure on the right, where the bulge on the conducting plane has the shape of a semi-sphere of radius R.

(50 pts)



Problem 2.2. Electric charges Q_A and Q_B have been put on two metallic, concentric spherical shells – see the figure on the right. What is the full charge of each of the surfaces S_1-S_4?



Solution: Let us start with applying the Gauss law (1.16),

$$\int_S E_n d^2r = \frac{Q}{\varepsilon_0} \qquad (*)$$

(where Q is the full charge within a closed surface S), to any volume located inside the inner sphere and containing the surface S_1 inside it – see the inner dashed line in the figure on the right. As was discussed in Sec. 2.1 of the lecture notes, in statics, the electric field inside any metal should be zero, so that the left-hand side of Eq. (*) for the surface vanishes, indicating that in the full charge of surface S_1 equals zero: $Q_1 = 0$. Hence the full charge Q_A of the first sphere should sit on its external surface: $Q_2 = Q_A$.

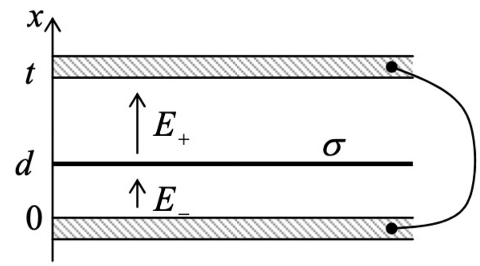
Now let us consider a similar Gaussian volume with the boundary inside the second metallic sphere, containing the surfaces S_1, S_2, and S_3. Similarly, the left-hand-part of Eq. (*) equal zero as well, and the Gauss law gives

$$0 = \frac{Q_1 + Q_2 + Q_3}{\varepsilon_0}.$$

With our prior results, this means that $Q_3 = -Q_A$. Hence the charge of the last, external surface S_4 of the same sphere is

$$Q_4 = Q_B - Q_3 = Q_B - (-Q_A) = Q_A + Q_B.$$

Problem 2.6. A wide, thin plane film, carrying a uniform electric charge density σ, is placed inside a similarly wide plane capacitor, whose plates are connected with a wire (see the figure on the right) and were initially electroneutral. Neglecting the fringe effects, calculate the surface charges of the plates, and the net force exerted upon the film (per unit area).



Solution: Let us pursue the most methodical (but not the shortest) way to solve the problem: solving the Poisson equation (1.41) for the electrostatic potential ϕ. Since, due to the problem's symmetry, ϕ may depend on just one Cartesian coordinate (say, x), normal to the surfaces' and the film's planes, the equation is reduced to

$$\frac{d^2\phi}{dx^2} = -\frac{\sigma}{\varepsilon_0}\delta(x-d). \qquad (*)$$

Here the origin of x is taken at the bottom plate's surface, so that at the top plate's surface, $x = d$ – see the figure above. Since the plates are connected by a conducting wire, in statics their potentials should be equal. Taking this value for zero, we may write the boundary conditions for the function $\phi(x)$ as

$$\phi(0) = \phi(d) = 0. \qquad (**)$$

The solution of this boundary problem is straightforward. Indeed, according to Eq. (*), in each of the two gaps between the film and the plates, $d^2\phi/d^2x = 0$, so that ϕ has to be a linear function of x:

$$\phi(x) = \begin{cases} c_- - E_- x, & \text{at } 0 < x < d, \\ c_+ - E_+ x, & \text{at } d < x < t, \end{cases}$$

where $E_\pm$ are the corresponding values of the electric field, $E = -d\phi/dx$ (see the figure above).Integrating Eq. (*) over an infinitesimal interval $[d-0, d+0]$, we get

$$\left.\frac{d\phi}{dx}\right|_{x=d+0} - \left.\frac{d\phi}{dx}\right|_{x=d-0} \equiv -E_+ + E_- = -\frac{\sigma}{\varepsilon_0}.$$

The boundary conditions (**) give two more equations for the constants $c_\pm$ and $E_\pm$:

$$c_- = 0, \quad c_+ - E_+ t = 0.$$

Solving this simple system of four equations, we get[26]

$$E_- = -\frac{\sigma}{\varepsilon_0}\frac{t-d}{t}, \qquad E_+ = \frac{\sigma}{\varepsilon_0}\frac{d}{t}, \qquad c_- = 0, \qquad c_+ = \frac{\sigma}{\varepsilon_0}d. \qquad (****)$$

Due to the fundamental relation (2.3), the first two of these formulas immediately yield the surface charges of capacitor's plates:

$$\sigma_{\text{bottom plate}} = \varepsilon_0 E_- = -\sigma\frac{t-d}{t}, \qquad \sigma_{\text{top plate}} = -\varepsilon_0 E_+ = -\sigma\frac{d}{t}.$$

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[26] Alternatively (and easier), these relations for the fields $E_\pm$ may be obtained by applying the Gauss theorem to three pillboxes with various positions of their lids – see the derivation of Eq. (1.23) in the lecture notes. This simple exercise is highly recommended to the reader.

(As a sanity check, $\sigma_{\text{bottom plate}} + \sigma_{\text{top plate}} + \sigma = 0$, as it should be for the initially electroneutral system.)

Next, plugging Eqs. (****) for the fields $E_\pm$ into Eq. (1.65) of the lecture notes, we may calculate the potential energy of the electric field in the system (per unit area):

$$\frac{U}{A} = \frac{\varepsilon_0}{2}\left[E_+^2(t-d) + E_-^2 d\right] = \frac{\sigma^2}{2\varepsilon_0}\left[\left(\frac{d}{t}\right)^2(t-d) + \left(\frac{t-d}{t}\right)^2 d\right] \equiv \frac{\sigma^2}{2\varepsilon_0}\frac{(t-d)d}{t}.$$

Now the force exerted on the film (also per unit area) may be found just as the (minus) gradient of its potential energy – see, e.g., Eq. (1.32). In our geometry, the only nonvanishing component of the gradient is vertical, so that $\mathbf{F} = F_x\mathbf{n}_x$, with[27]

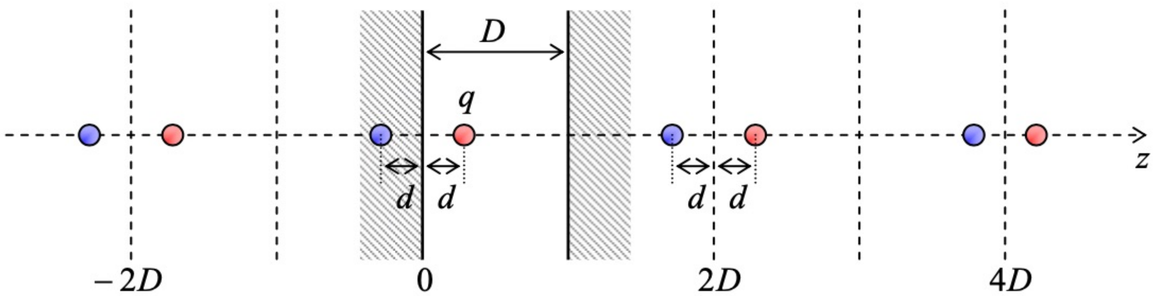
$$\frac{F_x}{A} = -\frac{\partial(U/A)}{\partial d} = \frac{\sigma^2}{2\varepsilon_0}\frac{t-2d}{t}.$$

This result shows that the film is attracted to the nearest plate, so that at the middle between the plates, at $d = t/2$, the net force vanishes, i.e. the film is in equilibrium. Since at this point (and actually at any position d)

$$\frac{\partial^2 U}{\partial d^2} = -\frac{\sigma^2}{\varepsilon_0 t} < 0,$$

this equilibrium is unstable.[28]

Solution: The system of image charges that satisfies the corresponding Poisson equation and the boundary conditions ($\phi(0) = \phi(D) = 0$) for this problem has already been discussed in Sec. 2.6 of the lecture notes – see Fig. 2.28c, reproduced below in an extended version. Here the red balls denote point charges $+q$ (including the real one), while the blue ones, charges $-q$.



As a result, the total charge induced in any electrode surface (say, the one located at $z = 0$) may be calculated as was discussed in Sec. 2.9:

$$Q = \int_{z=0} \sigma d^2r = -\varepsilon_0 \int_{z=0} \frac{\partial \phi}{\partial z}\Big|_{z=0} d^2r = -\varepsilon_0 \int_{z=0} \frac{\partial}{\partial z} \sum_j \phi_j \Big|_{z=0} d^2r = -\varepsilon_0 \int_0^\infty \frac{\partial}{\partial z} \sum_{j,\pm} \frac{\pm q}{4\pi\varepsilon_0 \left[\rho^2 + z_j^2\right]^{1/2}}\Big|_{z=0} 2\pi\rho d\rho, \quad (*)$$

where ρ is the distance of from the z-axis, and z_j is the position of an image belonging to the j^{th} pair of adjacent charges, centered at $2Dj$. According to Eq. (2.194) (but also as evident from the figure above),

$$z_j = 2Dj \pm d\ .$$

The differentiation and integration in Eq. (*) may be swapped with the summation, making two first of these operations easy; however, the final analytical summation is not too pleasing.

A simpler way to calculate Q is to apply the Gauss law to the system of charges (including the actual charge q) shown in the figure above. Let us first consider the set of *only the "positive"*[57] *charges* ($+q$) inside a round cylinder with its axis coinciding with axis z, a very large radius $R >> D$, and the following special choice of position of the lids (both parallel to capacitor planes): one just to the right of the surface of our current interest, i.e. at $z = +0$, while the other lid located exactly in the middle between the actual charge and the first "positive" charge image on the right of it, i.e. at

$$z = L_+ \equiv \frac{d + (2D + d)}{2} \equiv D + d\ .$$

Due to the condition $R >> D$, the electric field E_+ of the "positive" charges at the lateral (curved) wall of the Gaussian cylinder is virtually uniform, normal to the z-axis, and is the same as of a continuous charge line with uniform linear density $\lambda_+ = q/2D$. Hence, according to the (very easy) solution of Problem 1.1, $E_+ = \lambda_+/2\pi\varepsilon_0$, so that its flux through the wall is

$$\int_{\rho=R>>D} (E_+)_n d^2r = \frac{\lambda_+ L_+}{\varepsilon_0} \equiv \frac{q}{2D\varepsilon_0}(D + d).$$

[57] I am taking this word in quotes because our calculation is correct for any sign of the actual point charge q.

The electric flux through the cylinder's lids should be calculated more carefully because some parts of them are at distances of the order of $d \sim D$ from the nearest charges. Fortunately, with our (smart :-) choice of lid positions, the flux through the lid located at $z = L_+$ equals zero due to the symmetry, while, according to Eq. (2.3), the flux through the lid with $z = +0$ is proportional to the surface charge Q_+ of the electrode, induced by the "positive" charges:

$$\int_{z=0} (E_+)_n d^2r = -\frac{Q_+}{\varepsilon_0},$$

where the minus sign is due to the fact that we are calculating the flux *out of* the considered cylinder. Now taking into account that inside this cylinder we have just one "positive" charge q, the Gauss law for it reads

$$\oint (E_+)_n d^2r \equiv \frac{q}{2D\varepsilon_0}(D+d) - \frac{Q_+}{\varepsilon_0} = \frac{q}{\varepsilon_0},$$

giving

$$Q_+ = -\frac{q}{2}\left(1 - \frac{d}{D}\right).$$

Now we may repeat this calculation for the system of "negative" charges ($-q$), with the replacement of L_+ with the exact middle between two such charges, for example

$$L_- \equiv \frac{-d + (2D - d)}{2} \equiv D - d,$$

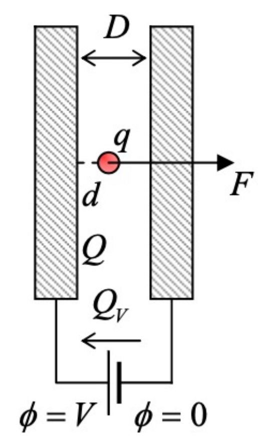
and of λ_+ with $\lambda_- = -q/2D$, and taking into account that there are no "negative" charges inside this new cylinder, with $+0 \leq z \leq L_-$. The result is similar:

$$Q_- = -\frac{q}{2}\left(1 - \frac{d}{D}\right),$$

so that the total surface charge $Q = Q_+ + Q_-$ of the surface is[58]

$$Q = -q\left(1 - \frac{d}{D}\right). \qquad (**)$$

Note that this result may be obtained more easily by the following simple reasoning. Let us assume that our plane capacitor, with the point charge inside, is connected to a battery fixing the voltage V across the plates, independent of the charge position – see the figure on the right. Then the uniform external electric field $E = V/D$ creates a position-independent force $F = qE = qV/D$, applied to the charge and directed perpendicular to electrode surfaces. On a small displacement Δd, this force performs work $\Delta\mathscr{W} = F\Delta d = qV\Delta d/D$. On the other hand ("from the point of view of the battery"), the same work $\Delta\mathscr{W}$ has to be equal to $V\Delta Q_V$, where ΔQ_V is the charge that has flown through the battery as a result of this displacement. But this charge has nowhere to go rather than to change of the surface charge of the



[58] With this result for Q on hand, the surface charge Q' of the counter-electrode (located at $z = D$) may be most simply calculated by applying the Gauss law to a similar cylinder, but with its lids inside the opposite capacitor's plates, so that the electric field on them vanishes. This yields $Q + Q' + q = 0$, finally giving $Q' = -qd/D$.

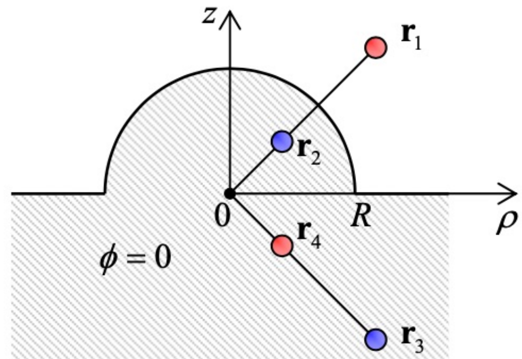
capacitor plate: $\Delta Q_V = \Delta Q$. Comparing these two expressions for $\Delta \mathscr{W}$, we get $\Delta Q = q(\Delta d/D)$, i.e. $Q = qd/D$ + const. Since the surface charge Q should tend to $-q$ when the point charge approaches the surface ($d \to 0$), and hence has to be fully compensated by the closest surface charge, the constant in the last expression has to equal $-q$, so that we return to Eq. (**).

Finally, one more way[59] to obtain the same result is to use the reciprocity theorem whose proof was the task of Problem 1.17. Let us take our problem's situation (with grounded capacitor electrodes, and the point charge q between them) for the charge and potential distributions number 1, and those in the same capacitor but voltage-biased as shown in the last figure above, but *without* the charge q, for distributions number 2. Then the theorem reads

$$QV + qV\left(1 - \frac{d}{D}\right) = 0,$$

immediately yielding Eq. (**) again. (An additional task for the reader: explain why the last two derivations of the result are not entirely independent.)

Problem 2.32. Use the method of images to find the Green's function of the system shown in the figure on the right, where the bulge on the conducting plane has the shape of a semi-sphere of radius R.



Solution: Let a (real) charge q_1 be at point $\mathbf{r}_1$ in the free-space part of the system. Then, according to Eq. (2.198) of the lecture notes, all the boundary conditions may be satisfied using the three charge images shown in the figure on the right, with

$$q_2 = -q_1 \frac{R}{r_1}, \quad \mathbf{r}_2 = \left(\frac{R}{r_1}\right)^2 \mathbf{r}_1; \qquad q_3 = -q_1, \quad \boldsymbol{\rho}_3 = \boldsymbol{\rho}_1, \quad z_3 = -z_1;$$

$$q_4 = -q_3 \frac{R}{r_3}, \quad \mathbf{r}_4 = \left(\frac{R}{r_3}\right)^2 \mathbf{r}_3;$$

where $\boldsymbol{\rho}_j$ are the horizontal components of the radius vectors $\mathbf{r}_j$. As a result, the Green's function may be represented simply as

$$G(\mathbf{r}, \mathbf{r}_1) = \sum_{j=1}^{4} \frac{q_j / q_1}{\left|\mathbf{r} - \mathbf{r}_j\right|}.$$

However, when expressed explicitly in the apparently most suitable, cylindrical coordinates, in which $r_1 = (\rho_1^2 + z_1^2)^{1/2}$, this compact expression becomes somewhat bulky, because each denominator looks like

$$\left|\mathbf{r} - \mathbf{r}_j\right| = \left[\rho^2 + \rho_j^2 - 2\rho\rho_j \cos(\varphi - \varphi_j) + (z - z_j)^2\right]^{1/2}.$$