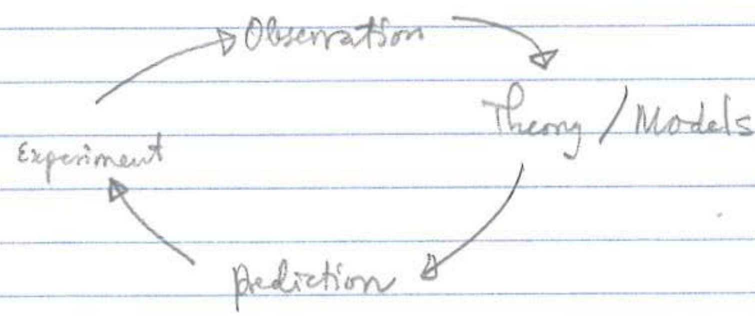
PH311: Classical Mechanics

Feb 8, 2018

The Scientific Method



basis for all scientific disciplines. However, it was first developed for the science of mechanics

Why study classical mechanics?

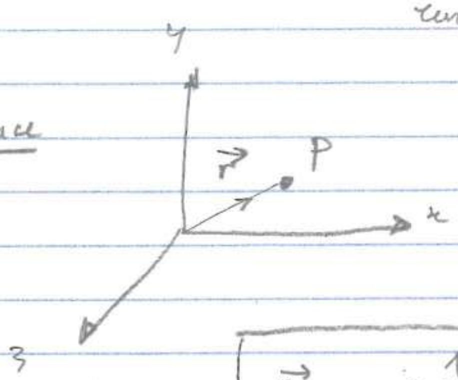
- first well-developed branch of science
- Creativity — brings together diverse phenomena
- Impacts on other fields (philosophy, religion ...)
- Conservation laws
- Lagrangian / Hamiltonian $\Longrightarrow$ Quantum Mechanics
- Build other physical theory (statistical mech / astro)

$\boxed{\text{Read Taylor's 1.1 — 1.3}}$

Chapter 1

Review: Newton's 3 laws of motion depends on 4 underlying concepts: Space, Time, Mass, Force

1) Space



$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{x} + y\hat{y} + z\hat{z}$$

let $\hat{e}$ = unit vector

$$\boxed{\vec{r} = r_1\hat{e}_1 + r_2\hat{e}_2 + r_3\hat{e}_3 + \dots = \sum_{i=1}^{3\dots} r_i\hat{e}_i}$$

Review of vector calculus $\vec{r} = (r_1, r_2, r_3)$

$\vec{s} = (s_1, s_2, s_3)$

Dot product $\boxed{\vec{r}\cdot\vec{s} = r_1s_1 + r_2s_2 + r_3s_3 = |\vec{r}||\vec{s}|\cos\theta = \sum_{n=1}^{3} r_n s_n}$

<u>Cross product</u> $\vec{r} \times \vec{s} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{bmatrix}$

(+) $p_x = r_2 s_3 - s_2 r_3$

(−) $p_y = s_1 r_3 - r_1 s_3$

(+) $p_z = r_1 s_2 - s_1 r_2$

<u>Differentiation</u> $\vec{r}(t)$: pos

$\frac{d\vec{r}(t)}{dt}$: velocity

$\frac{d^2\vec{r}(t)}{dt^2}$: acceleration

$\frac{d}{dt}(\vec{f}+\vec{r}) = \frac{d\vec{f}}{dt} + \frac{d\vec{r}}{dt}$

$\frac{d}{dt}(fr) = r\frac{df}{dt} + f\frac{dr}{dt}$

#

Feb 9, 2018

2) <u>Time</u>

In the domain of Newtonian mechanics → $v \ll c$

→ differences among measured times are entirely negligible.

→ Single, universal time.

3) <u>Reference frames</u>

A set of 3-D, mutually ⊥ coordinate axes

★ <u>Inertial ref frames / system</u>

→ coordinate axes where $a = 0$

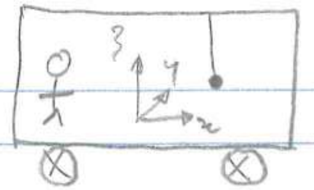
→ ~~All~~ physical laws are only true in an inertial reference frame.

★ <u>Accelerating / Non inertial / Rotating Reference frames</u>

→ Newton's laws do not hold here.

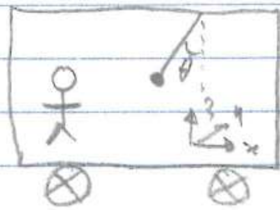
$\rightarrow v = \text{const}$

Example



inertial (can't fall if moving at v)

$\rightarrow a$



non-inertial

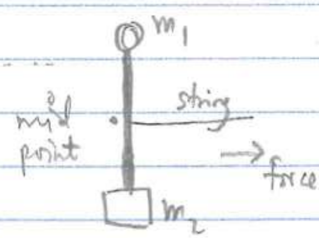
4) Mass & Force

↳ Mass characterizes the object's INERTIA ↗ tendency of the object to remain in the same state of motion (rest/const velocity)

INERTIA is quantifiable with mass

☆ The practical way to know the mass of objects is to weigh them in same location

☆ Scientific way → $\boxed{\text{Inertial balance}}$



if $m_1 = m_2 \Rightarrow$ rod will ~~xxxx~~ accelerate w/o rotating

if $m_1 \neq m_2 \Rightarrow$ rod will rotate.

5 ↳ Force

Something which changes the state of motion of object

Newton's law

$$\left(\frac{\text{Cause}}{\text{Force}}\right) \longrightarrow \left(\frac{\text{Effect}}{\text{acceleration}}\right)$$

Direction of $\vec{F}$ = direction of $\vec{a}$

if $\vec{F} = \vec{0} \Rightarrow \vec{a} = 0$

Unit of force (N) = $kg\, m/s^2$

$\boxed{\text{Newton's first law}}$ → { Every body continues in its state of rest or uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it }

$\boxed{\text{Newton's second law}}$

→ { The change of motion is proportional to the net force impressed and changes is made in the direction of the line in which that force is impressed }

Linear momentum : a quantity of motion

→ $\vec{p} = m\vec{v}$ → change of motion = change of mom.

→ $\frac{d\vec{p}}{dt}$

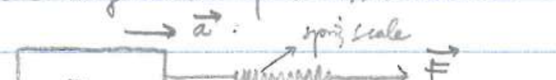
$$\boxed{\vec{F} = \frac{d\vec{p}}{dt}} \to \vec{F} = \left(\frac{dm}{dt}\vec{v}\right)^{0} + \frac{d\vec{v}}{dt}m = m\vec{a}$$

← desired.

most fundamental

~~Newton's third law~~

Operational definition of a force scale.



1) Let $m = 1\,kg$
2) Apply F such that $\dot{v} = 1\,m/s^2$
3) $|\vec{F}| = 1N$
4) Apply same force on different masses

$$F = m_1 a_1 = m_2 a_2$$

$$\to \boxed{\frac{m_1}{m_2} = \frac{a_2}{a_1}}$$

5) Apply different forces to the same mass

$$m = \frac{F_1}{a_1} = \frac{F_2}{a_2}$$

2 different kinds of mass

- Inertial mass

$$\boxed{\vec{F} = m\vec{a}} \to m_I$$

- Gravitational mass

Newton's laws of gravitation

m_1 → ← m_2

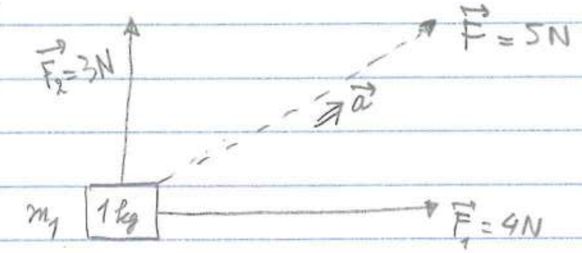
$\underbrace{\bullet \qquad\qquad \bullet}_{r}$

$$\boxed{F_{21} = F_{12} = \frac{G m_1 m_2}{r^2}}$$ ← defines gravitational mass

$$m_1, m_2 = m_G$$

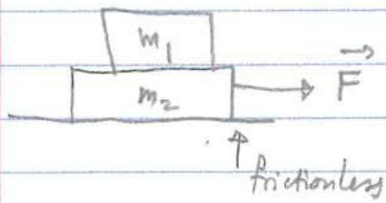
Experimentally, for the same object $\frac{m_G}{m_I} = 1 \pm (10^{-12})$

Multiple forces or multiple masses



$$\boxed{\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = \sum_{i=1}^{N} \vec{F}_i = m\vec{a}}$$

↳ Forces add like vectors.

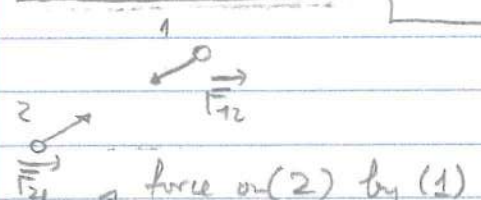


$$\boxed{\vec{F} = M\vec{a} = (m_1 + m_2)\vec{a}}$$

↳ Masses add as scalars

Feb 12, 2017 $\boxed{\text{Pset \# 1} \rightarrow \text{due Feb 19}}$

Newton's 3rd law $\boxed{\text{To every action there is an equal \& opposite reaction}}$

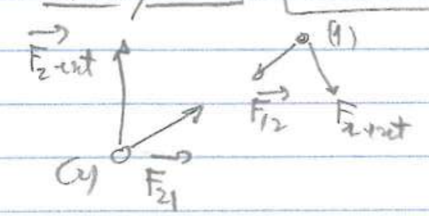


If obj 1 exerts a force $\vec{F}_{21}$ on obj 2

↳ then obj 2 exerts a reaction force $\vec{F}_{12}$ on obj 1

$$\boxed{\vec{F}_{21} = -\vec{F}_{12}}$$

Consequences $\boxed{\text{3rd law is related to the law of conservation of momentum}}$

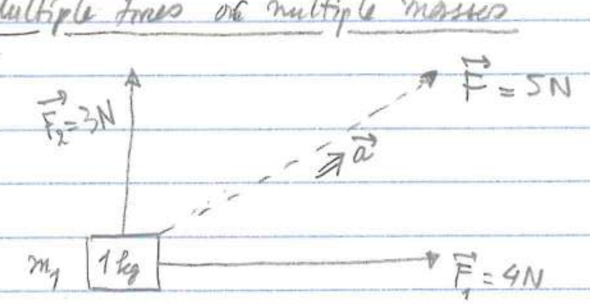


$$\left\{ \begin{aligned} \vec{F}_2 &= \vec{F}_{21} + \vec{F}_{2\text{-ext}} \\ \vec{F}_1 &= \vec{F}_{12} + \vec{F}_{1\text{-ext}} \end{aligned} \right\}$$

From Newton's II law: $\left\{ \begin{aligned} \vec{F}_2 &= \dot{\vec{p}}_2 \\ \vec{F}_1 &= \dot{\vec{p}}_1 \end{aligned} \right.$

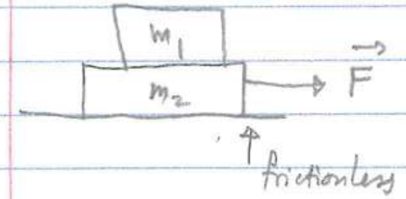
Total momenta: $\vec{P} = \vec{P}_1 + \vec{P}_2 \Rightarrow \boxed{\dot{\vec{P}} = \dot{\vec{P}}_1 + \dot{\vec{P}}_2}$

Multiple forces or multiple masses



$$\boxed{\vec{F}_{net} = \vec{F_1} + \vec{F_2} = \sum_{i=1}^{N} \vec{F_i} = m\vec{a}}$$

Forces add like vectors.

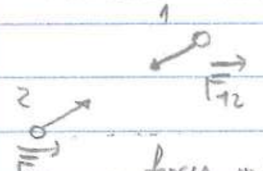


$$\boxed{\vec{F} = M\vec{a} = (m_1 + m_2)\vec{a}}$$

Masses add as scalars

Feb 12, 2017 | Pset #1 → due Feb 19 |

Newton's 3rd law | To every action there is an equal & opposite reaction |



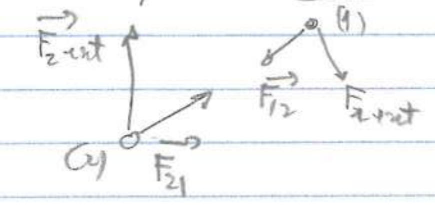
force on (2) by (1)

If obj 1 exerts a force $\vec{F_{21}}$ on obj 2

then obj 2 exerts a reaction force $\vec{F_{12}}$ on obj 1

$$\boxed{\vec{F_{21}} = -\vec{F_{12}}}$$

Consequences | 3^{rd} law is related to the law of conservation of momentum |



$$\begin{cases} \vec{F_2} = \vec{F_{21}} + \vec{F}_{2\,ext} \\ \vec{F_1} = \vec{F_{12}} + \vec{F}_{1\,ext} \end{cases}$$

From Newton's II law: $\begin{cases} \vec{F_2} = \dot{\vec{p_2}} \\ \vec{F_1} = \dot{\vec{p_1}} \end{cases}$

Total momenta: $\vec{P} = \vec{p_1} + \vec{p_2} \Rightarrow \boxed{\dot{\vec{P}} = \dot{\vec{p_1}} + \dot{\vec{p_2}}}$

$\rightarrow \dot{\vec{P}} = (\vec{F}_{12} + \vec{F}_{1ext}) + (\vec{F}_{21} + F_{2ext}) = \vec{F}_{1ext} + \vec{F}_{2ext} = \vec{F}^{ext}_{tot}$

$\rightarrow \boxed{\dot{\vec{P}} = \vec{F}_{ext\ total}} \rightarrow$ if $\vec{F}_{ext\ total} = 0$, then $\dot{\vec{p}} = 0$

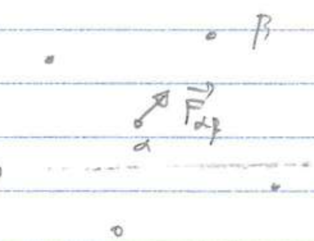
$\rightarrow$ <u>Momentum is conserved.</u>

<u>Principle of conservation of momentum</u> $\rightarrow$ in the absence of external forces $\rightarrow$ momentum is conserved

$\boxed{\text{Multiple-particle system}}$ System of N particles (α or β)

Mass of particle $\alpha \rightarrow m_\alpha$

momenta $\rightarrow \vec{p}_\alpha$ Particle α $\rightarrow$ can feel force from $(N-1)$ other particle forces $(\vec{F}_{\alpha\beta})$



Net external force $= F^{ext}_\alpha$

Net force on α $= \vec{F}_\alpha = \sum_{\substack{\beta \\ \beta \neq \alpha}} \vec{F}_{\alpha\beta} + \vec{F}_{\alpha\,ext} = \vec{F}_\alpha = \dot{\vec{P}}_\alpha$

Total momentum of N-particle sys

$$\vec{P} = \sum_\alpha \vec{P}_\alpha \qquad (\alpha = 1, \ldots, N)$$

$$\dot{\vec{P}} = \sum_\alpha \dot{\vec{P}}_\alpha$$

$$\rightarrow \dot{\vec{P}} = \underbrace{\sum_\alpha \sum_\beta \vec{F}_{\alpha\beta}}_{0} + \sum_\alpha \vec{F}_\alpha^{ext} \Rightarrow \boxed{\dot{\vec{P}} = \sum_\alpha \vec{F}_\alpha^{ext}} = \vec{F}^{ext}_{tot}$$

$$\parallel \sum_\alpha \sum_\beta (\vec{F}_{\alpha\beta} + \vec{F}_{\beta\alpha}) = 0 \quad (\text{for } \beta > \alpha)$$

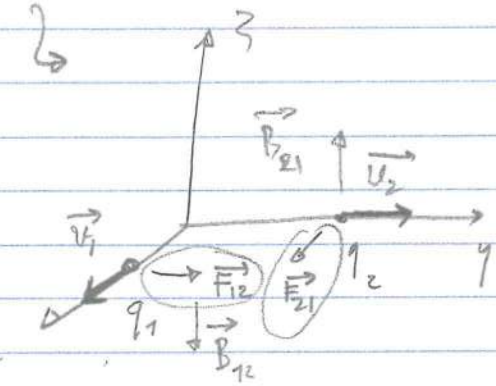
$\rightarrow$ $\boxed{\text{For a multi-particle system, the rate of change of momentum } \dot{\vec{P}} \text{ is the total external force on the system}}$

& If no ext force $\rightarrow$ momentum is <u>conserved</u>

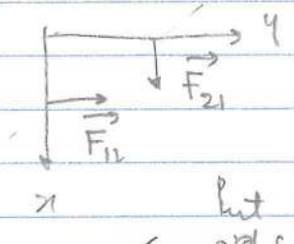
Validity of Newton's third law → Relativity → time not absolute

→ Newton's third law does not hold true

Counter Ex to N's III



→ Newton's third law doesn't hold here (?)



but with Coulomb's force

↳ 3rd law conserved

→ $\boxed{\text{HOLD}}$

$\boxed{\text{Newton's 2nd law in Cartesian}}$

↳ $\boxed{\vec{F} = m\ddot{\vec{r}}}$

$$\left\{ \begin{array}{l} \vec{F} = F_x \hat{x} + F_y \hat{y} + F_z \hat{z} \\ \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \quad \rightarrow \quad \dot{\vec{r}} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z} \\ \qquad\qquad\qquad\qquad\qquad \hookrightarrow \ddot{\vec{r}} = \ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z} \end{array} \right\}$$

$$\Rightarrow \left\{ \begin{array}{l} F_x = m\ddot{x} \\ F_y = m\ddot{y} \\ F_z = m\ddot{z} \end{array} \right.$$

Example Constant Force, solve II by integration

$$F = ma = \text{const} \rightarrow \int a = \int_0 \frac{dv}{dt} \rightarrow \cancel{F = m\int_0^t}$$

$$\rightarrow \boxed{v - v_0 = at}$$

$$v = \frac{dx}{dt} \Rightarrow \int_0^t v\,dt = \int_{x_0}^{x} dx \rightarrow \cancel{x - x_0 = vt}$$

$(v + at)$

$$\rightarrow \boxed{x - x_0 = v_0 t + \frac{1}{2}at^2}$$

$$a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}\cdot\frac{dx}{dt} = v\frac{dv}{dx} = a$$

$$\rightarrow \int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a\,dx \rightarrow \boxed{v^2 - v_0^2 = 2a(x - x_0)}$$

Feb 13, 2017

Time dependent, 2-variable force

time dependent $a(t) = \frac{F(t)}{m} = \frac{d}{dt}v(t) \rightarrow \boxed{v(t) = \int_0^{t_f} \frac{F(t)}{m}\,dt + v_0}$

Example $a(t) = kt^{1/2}$

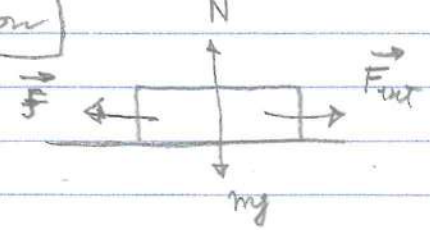
$$v - v_0 = \int_0^t kt^{1/2}\,dt = \frac{2}{3}kt^{3/2}\Big|_0^t = \boxed{\frac{2}{3}kt^{3/2} = v - v_0}$$

$$\Rightarrow \boxed{v(t) = \frac{2}{3}kt^{3/2} + v_0}$$

$$v(t) = \frac{dx}{dt} \rightarrow \int_{x_0}^{x} dx = \int_0^t \left(\frac{2}{3}kt^{3/2} + v_0\right) dt$$

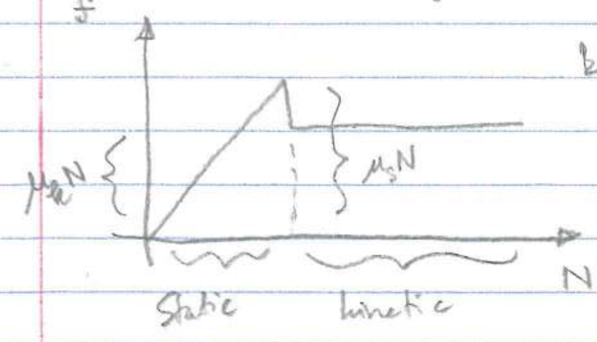
$$x - x_0 = \frac{4}{15}kt^{5/2} + v_0 t \rightarrow \boxed{x(t) = x_0 + \frac{4}{15}kt^{5/2} + v_0 t}$$

Friction



$\boxed{|\vec f| \neq \mu|\vec N|}$

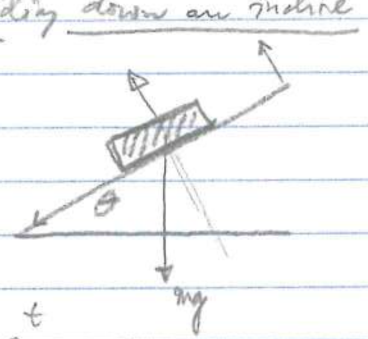
$\left\{\begin{array}{l}\text{static}\\ \text{kinetic}\end{array}\right.$ friction



both $\mu_k N$ & $\mu_s N$ are independent of contact area.

μ_k independent of velocity

Blocks sliding down an incline



Mass m accelerating from rest down an incline, μ_k is at θ w/ the horizontal.

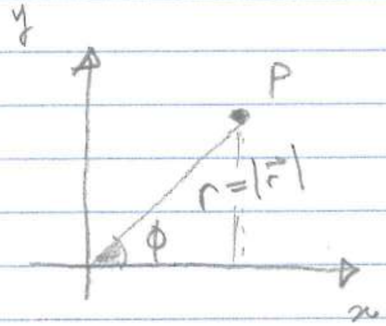
$$\begin{cases} x: \; mg\sin\theta - \mu_k mg\cos\theta = m\dfrac{dv}{dt} \\ y: \; N - mg\cos\theta = 0 \end{cases}$$

$$\int_0^v dv = \int_0^t g(\sin\theta - \mu\cos\theta)\,dt$$

$$v = gt(\sin\theta - \mu\cos\theta) = \frac{dx}{dt}$$

$$\Rightarrow \int_{x_0}^{x} dx = \frac{1}{2}gt^2(\sin\theta - \mu\cos\theta) = x - x_0$$

$$\boxed{x(t) = x_0 + \frac{1}{2}gt^2(\sin\theta - \mu\cos\theta)}$$

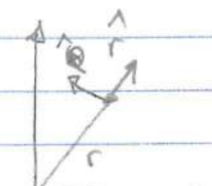


2-D Polar coordinates

(r, ϕ)

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases} \quad r = \sqrt{x^2+y^2}, \; \tan\phi = y/x$$

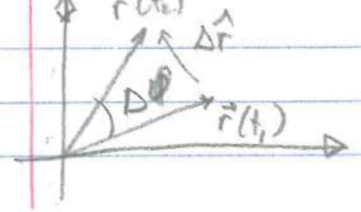
What is $\hat{r}$, $\hat{\phi}$?



$\vec{r}$ $\quad$ $\hat{r}$: unit vector pointing the direction we move when r increases & ϕ stays fixed.

$\hat{\phi}$: ϕ increases, r fixed.

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} \Rightarrow \boxed{\vec{r} = |\vec{r}|\,\hat{r}} \quad \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$



position of particle at time $t_1 \Rightarrow t_2$, where $t_2 = t_1 + \Delta t$

For $\Delta t \to 0$ $\quad \Delta\hat{r} \approx \Delta\phi\,\hat{\phi} \rightarrow \boxed{\Delta\hat{r} \approx \dot{\phi}\,dt\,\hat{\phi}}$

$$\rightarrow \boxed{\frac{d\hat{r}}{dt} = \dot{\varphi}\hat{\varphi}}$$

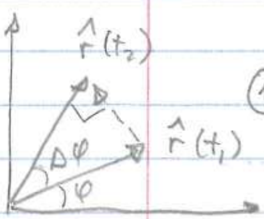
$$\vec{v} = \dot{\vec{r}} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} = \boxed{\dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} = v}$$

Feb 14, 2018 Reading notes:

Proof of $\frac{d\hat{r}}{dt} = \dot{\varphi}\hat{\varphi}$ and more

So, to start $\vec{r} = r(t)\hat{r} \Rightarrow \boxed{\vec{v} = \frac{d}{dt}\vec{r} = \frac{dr(t)}{dt}\hat{r} + r(t)\frac{d\hat{r}}{dt}}$

What is $\frac{d\hat{r}}{dt}$?



direction

(1) $\frac{d\hat{r}}{dt} = \lim_{t\to 0}\frac{\Delta\hat{r}}{\Delta t} = \lim_{t\to 0}\frac{\overset{1}{\cancel{|\hat{r}(t)|}}\Delta\varphi}{\Delta t}\cdot\hat{\varphi} = \overset{1}{\cancel{|\hat{r}(t)|}}\cdot\frac{d\varphi}{dt}\hat{\varphi} = \boxed{\dot{\varphi}(t)\hat{\varphi}}$

Correct

For small $\Delta\varphi \Rightarrow |\Delta\hat{r}| \approx |\hat{r}(t)\cdot\sin(\Delta\varphi)| \approx |\Delta\varphi\cdot\hat{r}(t)|$

(2) Alternative proof $\hat{r}(t) = \sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$ $\boxed{\text{fix to } \hat{r}(t) = \cos\varphi\hat{x} + \sin\varphi\hat{y}}$

$$\frac{d\hat{r}}{dt} = \cos\varphi\,\dot{\varphi}\,\hat{x} + \sin\varphi\,\dot{\varphi}\,\hat{y} = \dot{\varphi}(t)\underbrace{(\cos\varphi\,\hat{x} - \sin\varphi\,\hat{y})}_{\hat{\varphi}} = \boxed{\dot{\varphi}\hat{\varphi}}$$

Recall

$\hat{r} = \sin\varphi\,\hat{x} + \cos\varphi\,\hat{y}$

$\hat{\varphi} = \cos\varphi\,\hat{x} - \sin\varphi\,\hat{y}$

radial angular

So $\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi}} = v$

What is $\boxed{\frac{d\hat{\varphi}}{dt}? = -\dot{\varphi}\hat{r}}$

So what is $\vec{a}$? $\vec{a} = \dot{v}$

$$a = \dot{v} = \frac{dv}{dt} = (\ddot{r}\hat{r} + \dot{r}\dot{\varphi}\hat{\varphi}) + (\dot{r}\dot{\varphi}\hat{\varphi} + r(\dot{\varphi}\hat{\varphi})^{\cdot})$$

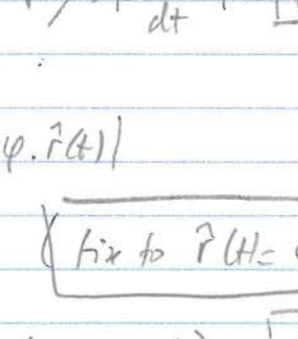
$$= (\ddot{r}\hat{r} + \dot{r}\dot{\varphi}\hat{\varphi}) + (\dot{r}\dot{\varphi}\hat{\varphi}) + r\left(\ddot{\varphi}\hat{\varphi} + \dot{\varphi}(-\dot{\varphi}\hat{r})\right)$$

$$\rightarrow \boxed{a = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi}}$$

Proof $\hat{\varphi} = \cos\varphi\,\hat{x} - \sin\varphi\,\hat{y}$

$\frac{d\hat{\varphi}}{dt} = -\sin\varphi\,\dot{\varphi}\,\hat{x} - \cos\varphi\,\dot{\varphi}\,\hat{y}$

$= \boxed{-\dot{\varphi}\hat{r}}$



$\frac{d\hat{\varphi}}{dt} = \frac{\Delta\hat{\varphi}}{\Delta t} = -\frac{\cancel{|\hat{\varphi}|}\Delta\varphi}{\Delta t}\hat{r} = -\dot{\varphi}\hat{r}$

$\hat{\varphi} = \cos\varphi\,\hat{x} - \sin\varphi\,\hat{y}$

$\boxed{\dot{\hat{\varphi}} = -\dot{\varphi}\,\hat{r}}$

$\boxed{\dot{\hat{\varphi}} = -\dot{\varphi}\,\hat{r}}$

Feb 15, 2018

$$\boxed{\vec{v} = \dot{\vec{r}} = \underbrace{\dot{r}\hat{r}}_{\text{radial}} + \underbrace{r\dot{\varphi}\hat{\varphi}}_{\text{angular}}} \rightarrow \begin{cases} v_r = \dot{r} & \text{(radial velocity)} \\ v_\varphi = r\omega & \text{(angular velocity)} \end{cases}$$

$$a = \dot{\vec{v}} = \left(\ddot{r}\hat{r} + \dot{r}\dot{\varphi}\hat{\varphi}\right) + \dot{r}\dot{\varphi}\hat{\varphi} + r\left(\ddot{\varphi}\hat{\varphi} + \dot{\varphi}(-\dot{\varphi})\hat{r}\right)$$

$$\boxed{\vec{a} = \left(\ddot{r} - r\dot{\varphi}^2\right)\hat{r} + \left(2\dot{r}\dot{\varphi} + r\ddot{\varphi}\right)\hat{\varphi}}$$

For circular motion: $\vec{a} = \left(-r\dot{\varphi}^2\right)\hat{r} + \left(\overset{x}{\dot{r}}\ddot{\varphi}\right)\hat{\varphi}$ $\quad \begin{pmatrix} r = \text{const} \\ \dot{r} = \ddot{r} = \vec{0} \end{pmatrix}$

$$= \boxed{\left(-r\omega^2\right)\hat{r} + (r\alpha)\hat{\varphi}}$$

angular velocity $\quad$ angular acceleration

where (only true if $r = \text{const}$)

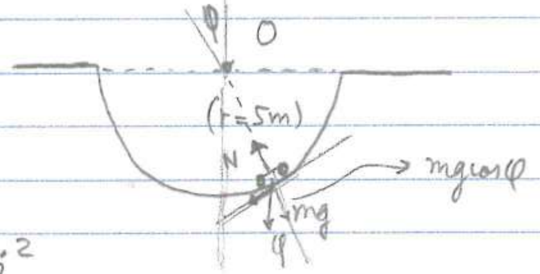
$-r\omega^2 \rightarrow$ centripetal acceleration

$(r\alpha) \rightarrow$ tangential acceleration

Newton's 2^{nd} law

$$\begin{cases} \vec{F}_r = m\left(\ddot{r} - r\dot{\varphi}^2\right) \\ \vec{F}_\varphi = m\left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right) \end{cases}$$

Example of oscillating skateboard



Question: Discuss motion

$$\vec{F}_r = m\left(\cancel{\ddot{r}} - r\dot{\varphi}^2\right) = -mR\dot{\varphi}^2$$

$$\vec{F}_\varphi = m\left(r\ddot{\varphi} + \cancel{2\dot{r}\dot{\varphi}}\right) = mR\ddot{\varphi}$$

$F_r = mg\cos\varphi - N$

$F_\varphi = -mg\sin\varphi = mR\ddot{\varphi} \Rightarrow \boxed{\ddot{\varphi} = -\frac{g}{R}\sin\varphi}$

If $\varphi \ll$, then $\boxed{\ddot{\varphi} = \frac{-g}{R}\varphi}$ $\rightarrow$ Define $\omega = \sqrt{\frac{g}{R}}$

$\rightarrow \boxed{\ddot{\varphi} = -\omega^2\varphi}$

$\rightarrow$ Solution $\boxed{\varphi(t) = A\sin(\omega t) + B\cos(\omega t)}$

at $t=0$, $\varphi = \varphi_0 = B$

at $t=0$, $\dot{\varphi} = \omega A$

Released from rest $\dot{\varphi} = 0 \Rightarrow A = 0$
(also $t=0$)

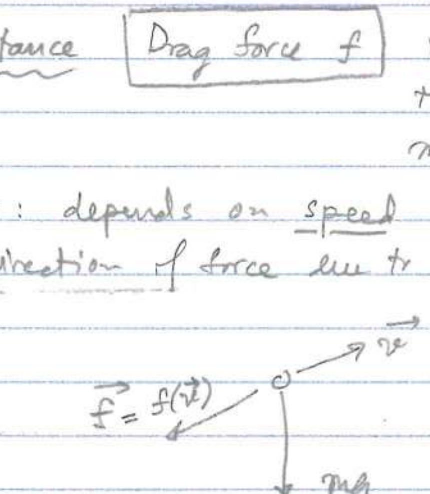
$$\Rightarrow \boxed{\varphi(t) = \varphi_0 \cos(\omega t)}$$

indeed $\boxed{\ddot{\varphi} = -\varphi_0 \omega^2 \cos(\omega t) = -\omega^2 \varphi}$

CHAPTER 2 — Projectiles - Charged Particles

Air resistance $\boxed{\text{Drag force } f}$ resistive force of any medium (air) through which the object (projectile) is moving

→ f : depends on speed of the object
→ direction of force due to motion through air opposes to velocity $\vec{v}$



Drag force as $\boxed{\vec{f} = -f(v)\,\vec{v}}$

Feb 16, 2018 Practice problem # 3

$$\begin{cases} r = be^{kt} \\ \theta = ct = \varphi \end{cases}$$

$$\vec{v} = \dot{r}\hat{r} + r\dot{\varphi}\hat{\varphi} = (kbe^{kt})\hat{r} + (be^{kt})c\hat{\varphi} = be^{kt}(k\hat{r} + c\hat{\varphi})$$

$$\vec{a} = (\ddot{r} - r\dot{\varphi}^2)\hat{r} + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{\varphi}$$

$$= (k^2be^{kt} - be^{kt}c^2)\hat{r} + (\underbrace{be^{kt}\cdot 0}_{0} + 2kbe^{kt}c)\hat{\varphi}$$

$$= (k^2be^{kt} - be^{kt}c^2)\hat{r} + 2(kbe^{kt}c)\hat{\varphi}$$

$$= be^{kt}\left[(k^2 - c^2)\hat{r} + 2kc\hat{\varphi}\right]$$

Find $\cos(\gamma) = \dfrac{\vec{v}\cdot\vec{a}}{|\vec{v}||\vec{a}|} = \dfrac{(k^3 - c^2k) + (2c^2k)}{\sqrt{k^2+c^2}\sqrt{(k^2-c^2)^2 + (2kc)^2}}$ = const → $\underline{\gamma \text{ const}}$

$= \dfrac{k}{\sqrt{k^2+c^2}}$ (no t dependence)

Feb 19, 2018

Air resistance (recall)

$\vec{v}$

$\vec{f}$ o m

$m\vec{g}$

$$\boxed{|\vec{f}| = -f(v)\,\hat{v}} \qquad \hat{v} = \frac{\vec{v}}{|\vec{v}|}$$

$$\boxed{f(v) = bv + cv^2 = f_{linear} + f_{quadr}}$$

1. f_{linear} arises from the viscous drag of the medium

$$\left\{ f_{linear} \propto \begin{array}{l} \text{1) viscosity of medium} \\ \text{2) linear size of projectile} \end{array} \right\}$$

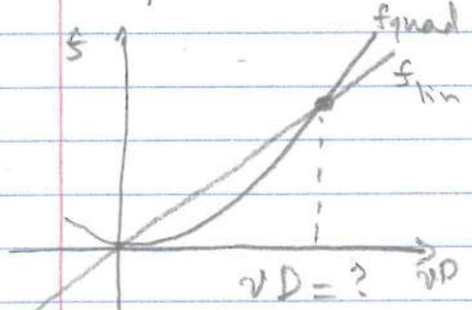
2. f_{quad} arises from the projectile's having to accelerate the mass of air around it.

$$\left\{ f_{quad} \propto \begin{array}{l} \text{1) density of the medium} \\ \text{2) cross-sectional area of projectile.} \end{array} \right\}$$

$\boxed{\text{For a spherical projectile}}$ $\left\{ \begin{array}{l} b = \beta D \leftarrow \text{diameter.} \\ c = \gamma D^2 \end{array} \right.$

STP in air $\left\{ \begin{array}{l} \beta = 1.6 \times 10^{-4}\ Ns/m^2 \\ \gamma = 0.25\ Ns^2/m^4 \end{array} \right.$

Compare two terms f_{linear} & f_{quad} in air @ STP for a spherical proj



$$\frac{cv^2}{bv} = \frac{\gamma D^2 v^2}{\beta D v} = \frac{\gamma D}{\beta} v = \left(1.6 \times 10^3 \frac{s}{m^2}\right) Dv$$

In a given problem, we can neglect f_{lin} / f_{quad} ...

<u>Ex</u>: We have a baseball & some drop of liquid.
Assess the relative importance of linear & quadratic & linear drag for a baseball: $D = 7\,cm$, $v = 5\,m/s$ (1)
raindrop: $D = 1\,mm$, $v = 0.6\,m/s$ (2)
drop of oil: $D = 1.5\,\mu m$, $v = 5\times10^{-5}\,m/s$ (3)

<u>Baseball</u>: $\frac{f_q}{f_l} = \left(1.6\times10^{3}\,\frac{s}{m^2}\right)(0.07\,m)(5\,m/s) = 560 \to f_q \gg f_{lin}$

<u>Rain</u> $\frac{f_q}{f_l} = \left(1.6\times10^{3}\,\frac{s}{m^2}\right)(1\times10^{-3}\,m)(0.6\,m/s) = 0.96 \to f_q \approx f_{lin}$

<u>Oil</u> $\frac{f_q}{f_l} = \left(1.6\times10^{3}\,\frac{s}{m^2}\right)(1.5\times10^{-6}\,m)(5\times10^{-5}\,m/s) = 1.2\times10^{-7} \to f_{lin} \gg f_q$

$\boxed{\text{Reynolds Number}}$

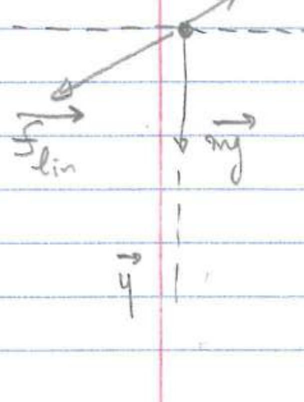
Linear drag ~ viscosity (η)
Quadratic drag ~ density (ρ)

$$\Rightarrow \frac{f_{quad}}{f_{linear}} \sim \boxed{R = \frac{Dv\rho}{\eta}}$$ ← dimensionless number...

If R is large, f_{quad} dominates,
" small, f_{linear} dominates.

//

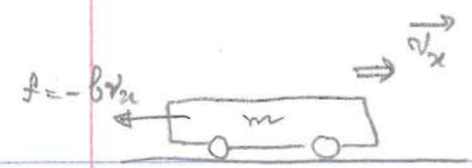
$\boxed{\text{LINEAR DRAG}}$ → $\boxed{R \ll 1}$ and $\boxed{\vec{f} = -bv\hat{v}}$



Newton's 2nd law:

$$\vec{F} = \vec{F}_W + \vec{f} = m\ddot{\vec{r}}$$

$$\left\{\begin{array}{ll} x: \; m\dot{v}_x = -bv_x & (1) \\ y: \; m\dot{v}_y = mg - bv_y & (2) \end{array}\right.$$

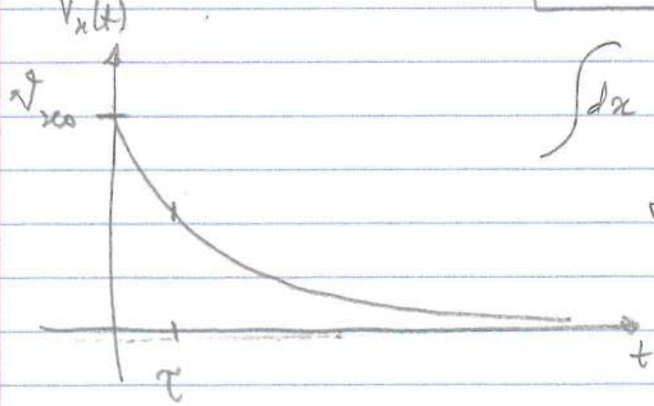


$$(1): \; m\dot{v}_x = -bv_x$$

Eqn (1) $\Rightarrow \int_{v_{x0}}^{v_x(t)} \frac{1}{v_x}\dot{v}_x\,dt = \int \frac{-b}{m}\,dt \rightarrow \ln\left(\frac{v_x(t)}{v_{x0}}\right) = \frac{-bt}{m}$

$$\Rightarrow \boxed{v_x(t) = v_{x0}\, e^{-bt/m}}$$

Define $\tau = \frac{m}{b} \rightarrow \boxed{v_x(t) = v_{x0}\, e^{-t/\tau}}$



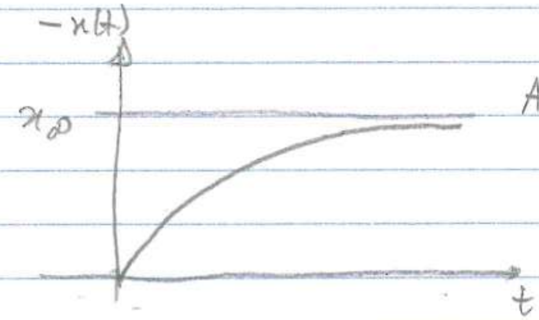
$$\int dx = \int v_{x0}\, e^{-t/\tau}\, dt$$

$$\rightarrow x(t) = v_{x0}(-\tau)\left(e^{-t/\tau} - 1\right)$$

$$\boxed{x(t) = -v_{x0}\tau\left(1 - e^{-t/\tau}\right)}$$

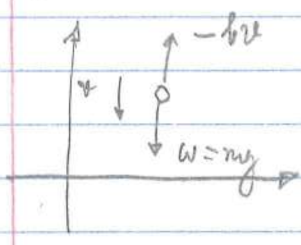
Define $v_{x0}\tau = x_\infty$

$$\Rightarrow \boxed{x(t) = -x_\infty\left(1 - e^{-t/\tau}\right)}$$



As cart slows down, $x(t) \rightarrow x_\infty$ asymptotically.

Feb 20, 2018 | Vertical motion with linear drag |



linear air resistance

$$m\ddot{y} = mg - bv_y = m\dot{v}_y$$

terminal velocity

v small $\rightarrow mg > bv_y$. If $mg = bv_y \Rightarrow \boxed{v_t = \frac{mg}{b}}$

Terminal velocity for small liq drop

1. Find V_{ter} of tiny droplet in Millikan & small drop of mist.

$$F = \beta D v \quad , \quad b = \beta D = 2\beta r$$

$$v_T = \frac{mg}{b} = \frac{\left(\frac{4}{3}\pi r^3\right)\rho g}{2\beta r} = \frac{2}{3}\pi \frac{\rho g}{\beta} \cdot r^2$$

$\left\{ \begin{array}{ll} \text{oil drop} : V_T = 6.06 \times 10^{-5} \text{ m/s} \\ \text{Mist} \quad V_T = 1.30 \text{ m/s} \end{array} \right\}$ ← lol... who cares...

$$m\dot{v}_y = mg - bv_y$$

$$\boxed{m\dot{v}_y = -b(v_y - v_{ter})} \rightarrow \boxed{v(t) = v_{ter}\left(1 - e^{-\frac{bt}{m}}\right)}$$

for $V_0 = 0$

Define $\tau = \frac{m}{b}$

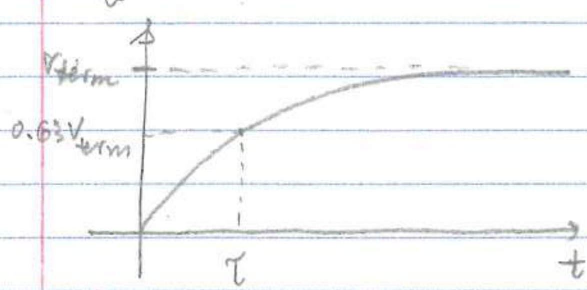
$$\Rightarrow \boxed{v_y = v_{term}\left(1 - e^{-t/\tau}\right) + v_0 e^{-t/\tau}}$$

If $t \to \infty \rightarrow v_y \rightarrow v_{term}$

Special case $v_0 = 0 \Rightarrow \boxed{v_y = \left(1 - e^{-t/\tau}\right) v_{term}}$

$$t \to \infty \rightarrow v = v_{term}$$

$$t \to \tau \Rightarrow v = v_{term}\left(1 - \frac{1}{e}\right) = 0.63\, v_{term}$$



t	% Vterm
0	0
τ	63%
2τ	86%
3τ	95%

$$t \to 3\tau \rightarrow v = (v_{term})(0.95) \approx v_{term}$$

Example Find τ for oil drop & mist.

$$\tau = \frac{m}{b} = \frac{m}{\beta D} = \frac{\left(\frac{4}{3}\pi r^3\right)\rho}{\beta D} = \frac{\left(\frac{\pi D^3}{6}\right)\rho}{\beta D} = \boxed{\frac{\pi \rho}{6\beta} D^2}$$

$$\boxed{v_{term} = \frac{mg}{b} = g\tau} \rightarrow \begin{cases} \text{oil drop: } 6.2\times10^{-6}\,s = \tau \\ \text{mist drop: } 0.13\,s = \tau \end{cases}$$

↳ After falling just 18 μs, the oil drop acquires 95% of terminal speed
" 0.389 s a mist " 95%

general → $$\boxed{v_y(t) = v_{term} + (v_{y0} - v_{term})e^{-t/\tau}}$$

Assume, $t=0$, $y=0$ → what is $y(t)$?

$$\int_0^y dy = \int_0^t v_{term} + (v_{y0} - v_{term})e^{-t/\tau}\,dt$$

$$\rightarrow y(t) = v_T t - (\tau)(v_{y0} - v_{Term})(e^{-t/\tau} - 1)$$

$$\hookrightarrow \boxed{y(t) = v_T t + (\tau)(v_{y0} - v_{term})(1 - e^{-t/\tau})}$$

Trajectory & Range in a Linear Medium

From E.O.M of projectile in both x & y direction

$$\rightarrow \begin{cases} x(t) = -v_{x0}\tau(1 - e^{-t/\tau}) & (1) \\ y(t) = -v_T t + (\tau)(v_{y0} + v_{term})(1 - e^{-t/\tau}) \end{cases}$$

Switch signs of v_{term}

↳ Find y as a function of x & (1) → $t = -\tau \ln\left(1 - \frac{x}{v_{0x}\tau}\right)$

$$\Rightarrow \quad y = (v_{y0} + v_{term})\,\tau\left(\frac{x}{v_{0x}\tau}\right) + v_{term}\cdot\tau \ln\left(1 - \frac{x}{v_{0x}\tau}\right)$$

$$\boxed{y(x) = \left(\frac{v_{y0} + v_{term}}{v_{0x}}\right) x + v_{term}\cdot\tau \ln\left(1 - \frac{x}{v_{0x}\tau}\right)}$$

Feb 22, 2018

<u>Horizontal Range</u> Range is the value of x when $y = 0$

$$\boxed{0 = \left(\frac{v_{y0} + v_{term}}{v_{0x}}\right)(R) + v_T\cdot\tau \ln\left(1 - \frac{R}{v_{0x}T}\right)}$$ transcendental equation.

$\Rightarrow$ <u>Approximation</u> let air resistance small $\Rightarrow v_{term},\ \tau \to$ large

$$\ln(1-a) \simeq -\left(a + \frac{a^2}{2} + \frac{a^3}{3} + \dots\right)$$

$$\boxed{\frac{v_{y0} + v_{term}}{v_{x0}} R - v_{term}\tau\left[\frac{R}{v_{0x}\tau} + \frac{1}{2}\left(\frac{R}{v_0\tau}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{0x}\tau}\right)^3 + \dots\right] = 0}$$

$$\boxed{\frac{v_{y0}}{v_{x0}} - \frac{v_{term}}{2}\frac{R}{v_{0x}^2\tau} - \frac{v_{term}}{3}\frac{R^2}{v_{x0}^3\tau^2} = 0}$$

$$\tau = \frac{v_{ter}}{g} \Rightarrow \boxed{\frac{v_{0y}}{v_{0x}} - \frac{gR}{2v_{0x}^2} - \frac{gR^2}{3v_{0x}^3\tau} = 0}$$

$$\boxed{\frac{gR}{2v_{0x}^2} = \frac{v_{y0}}{v_{x0}} - \frac{gR^2}{3v_{0x}^2\tau}}$$

$$\boxed{R = \frac{2v_{0x}v_{y0}}{g} - \frac{2}{3v_{x0}\tau}R^2}$$ small.

$$\boxed{R_{vac} \approx \frac{2v_{0x}v_{0y}}{g}} = R_{vacuum}$$

$mg - bv = 0$

$v = \frac{mg}{b} = \frac{2}{3\pi}\frac{\rho g}{\beta} r^2$

$3\pi\eta$

To make approximation better...

$$\boxed{\begin{aligned} R &\approx R_{vac} - \frac{2}{3 v_{0x} \tau}(R_{vac})^2 \\ &= R_{vac}\left(1 - \frac{4}{3}\frac{v_{y0}}{v_{term}}\right) \end{aligned}}$$

where $R_{vac} \approx \frac{2 v_{x0} v_{y0}}{g}$

$\left(\frac{v_{y0}}{v_{term}}\right)$ — correction depends only on this ratio

Example tiny metal pellet $D = 0.2\,mm$. $\vec{v} = 1\,m/s$ @ $45°$
Find R assuming it is gold ($\rho = 16\,g/cm^3$), if it is Al ($\rho = 2.7\,g/cm^3$)

$R_{vac} = \frac{2 v_{0x} v_{0y}}{g} = 0.102\,m$

$v_{ter} = \frac{mg}{b} = \frac{2}{3}\pi\frac{\rho g}{\beta} r^2 = \begin{cases} 20.52\,m/s \text{ for Au} \\ 3.46\,m/s \text{ for Al} \end{cases}$

$(\beta = 1.6\times10^{-4}\,Ns/m^2)$

$\frac{4}{3}\frac{v_{y0}}{v_{term}} = \frac{4}{3}\cdot\frac{\sqrt{2}/2}{20.52} = 0.0459 \ll 1$ (Au)

$\frac{4}{3}\frac{v_{y0}}{v_{term}} = \frac{4}{3}\cdot\frac{\sqrt{2}/2}{3.46} = 0.27$ not negligible (Al)

$\boxed{\text{Quadratic air resistance}}$

Linear : $f(v) = -bv\hat{v}$ (small objects)

Quadratic : $\boxed{f(v) = -cv^2\hat{v}}$

$\frac{m\,d\vec{v}}{dt} = m\vec{g} + \vec{f}$

but in general → this is a nonlinear DE.

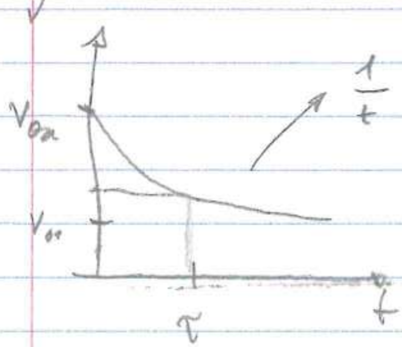
Horizontal direction

$$\vec{F} = m\ddot{x} = -cv^2 = m\frac{dv}{dt}$$

$$\Rightarrow \int_{v_{0x}}^{v} \frac{1}{v^2}\,dv = \int_0^t \frac{-c}{m}\,dt \Rightarrow \boxed{\frac{1}{v_{0x}} - \frac{1}{v} = \frac{-ct}{m}}$$

for quad drag

$$\Rightarrow \boxed{v_x(t) = \frac{v_{0x}}{1+v_{0x}ct/m} = \frac{v_{0x}}{1+t/\tau}} \quad \text{where } \boxed{\tau = \frac{v_{0x}c}{m}}$$



$\frac{1}{t}$, not e^{-t}

goes down much more slowly than linear drag
($1/t$) (exp)

$$\boxed{x(t) = x_0 + \int_0^t \frac{v_{0x}}{1+t/\tau}\,dt = x_0 + v_{0x}\tau \ln(1+t/\tau)}$$

Vertical motion

$$\boxed{m\dot{v}_y = mg - cv^2} \qquad V_{term} = \sqrt{\frac{mg}{c}}, \quad c = \frac{mg}{v_{term}^2}$$

$$m\dot{v}_y = mg - \frac{mg}{v_{term}^2}\cdot v^2 \Rightarrow \boxed{\dot{v}_y = g\left(1 - \frac{v^2}{v_{term}^2}\right)}$$

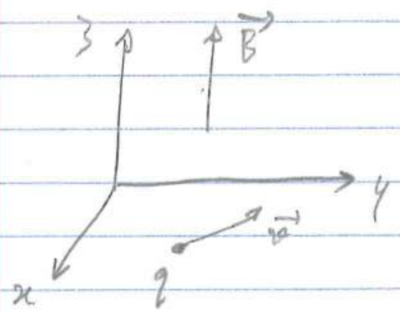
$$\int \frac{dv_y}{dt} = \int g\left(1 - \frac{v^2}{v_{term}^2}\right) dt \Rightarrow \boxed{\int_{v_0=0}^{v} \frac{1}{1-\frac{v^2}{v_{term}^2}}\,dv_y = \int_0^t g\,dt}$$

0

$\to v(t) = v_{term} \cdot \tanh\left(\frac{gt}{v_{term}}\right)$

$$\boxed{y = \int v(t)\,dt = \frac{(v_{term})^2}{g}\ln\left(\cosh\left(\frac{gt}{v_{term}}\right)\right)}$$

<u>Motion of charge in uniform magnetic field</u>



<u>Net force</u> $\vec{F} = q\,\vec{v}\times\vec{B}$

$$m\dot{\vec{v}} = q\,\vec{v}\times\vec{B} = q\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}\times\begin{bmatrix} 0 \\ 0 \\ B \end{bmatrix} = q\left(v_y B,\ -v_x B,\ 0\right)$$

$$\Rightarrow \begin{Bmatrix} m\dot{v}_x = q v_y B \\ m\dot{v}_y = -q v_x B \\ m\dot{v}_z = 0 \end{Bmatrix} \Rightarrow \begin{cases} \\ \\ v_z = \text{const} \Rightarrow \text{particle in } x, y \text{ plane} \end{cases}$$

<u>Define</u> $\omega = \frac{qB}{m} \Rightarrow \begin{Bmatrix} \dot{v}_x = \omega v_y \\ \dot{v}_y = -\omega v_x \end{Bmatrix}$ — coupled differential equation.

<u>Define</u> $\boxed{\eta = v_x + i v_y}$

$$\Rightarrow \dot{\eta} = \dot{v}_x + i\dot{v}_y = \omega v_y - i\omega v_x = \boxed{-i\omega\overbrace{(v_x + i v_y)}^{\eta}}$$

$$\to \boxed{\dot{\eta} = -i\omega\eta}$$

$$\to \boxed{\eta(t) = A e^{-i\omega t}}$$

$$\Rightarrow x + iy = \int \eta(t)\,dt$$

$$x + iy = \frac{iA}{\omega}e^{-i\omega t} + \overbrace{(X + iY)}^{\text{const}}$$

redefine coordinate such that $X + iY = 0$

$\Rightarrow$ $\boxed{x + iy = Ce^{-i\omega t}}$, where $C = \frac{iA}{\omega}$

if there's E in z-direction $\rightarrow$ $\boxed{F = q(-v_y B, -v_x B, E)}$

Feb 26, 2018 — **Momentum & Angular Momentum**

↳ Conservation of momentum

Consider N particles $\alpha = 1, \ldots, N$

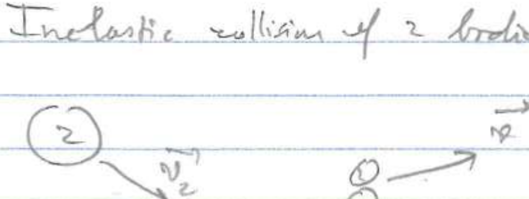
- If the internal forces obey Newton's III law, $\Rightarrow$ cancel out
- System's total $P = p_1 + p_2 + \ldots + p_N$

↳ $\boxed{\dot{p} = F_{external}}$ ← Conservation of linear momentum.

If the system is isolated, then the system's total momentum
$p = \sum m_\alpha v_\alpha$ = constant

Special case $N = 1$ → all forces are external CoM reduces to Momentum of a single particle, is constant, Newton's I law.
→ Trivial for $N=1$. Non-trivial for $N \geq 2$

Exp Inelastic collision of 2 bodies (m_1, m_2 & $\vec{v}_1$, $\vec{v}_2$)



Assume $\vec{F}_{ext} = \vec{0}$ during the brief moment of collision.

Find $\vec{v}$ just after the collision.

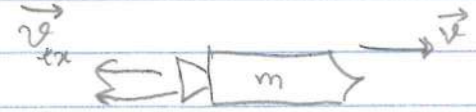
$$m_1\vec{v}_1 + m_2\vec{v}_2 = (m_1 + m_2)\vec{v}$$

$$\vec{v} = \frac{m_1}{m_1+m_2}\vec{v}_1 + \frac{m_2}{m_1+m_2}\vec{v}_2$$

Special case : $\vec{v}_2 = \vec{0}$

↳ $\boxed{\vec{v} = \frac{m_1}{m_1+m_2}\vec{v}_1}$

Rockets Principle of momentum conservation $\Rightarrow$ rocket propulsion



v_{ex} relative to rocket

$v_{exhaust} \rightarrow$ relative to the rocket. Rocket's mass m is steadily decreasing,

For rocket

At t ___ $P = mv \leftarrow$ of rocket

At $t+\Delta t$ $P(t+dt) = (m+dm)(v+dv)$

dm negative

The fuel ejected in time dt, has a mass $(-dm)$ & velocity $(v - v_{ex})$ (relative to ground)

$\hookrightarrow$ total momentum $= P(t+dt) = (m+dm)(v+dv) - dm(v - v_{ex})$

$$\Rightarrow \boxed{P(t+dt) = mv + mdv + (dm)v_{ex}}$$

$\hookrightarrow$ $dm\,dv$ very small

$$\boxed{\begin{aligned} dP &= P(t+dt) - P(t) \\ &= mdv + v_{ex}dm \end{aligned}}$$

(dm negative)

Assume there's no external force $\Rightarrow$ $0 = mdv + v_{ex}dm$

$$\Rightarrow \boxed{m\frac{dv}{dt} = -v_{ex}\frac{dm}{dt}} \Rightarrow \boxed{m\dot{v} = -v_{ex}\dot{m}}$$

where $\dot{m}$ the rate at which the rocket is ejecting its mass

We call: $\boxed{-v_{ex}\dot{m}} \rightarrow$ **THRUST**

(1) $\boxed{dv = -v_{ex}\dfrac{dm}{m}}$ Assume that v_{ex} is constant

$$\hookrightarrow \; v - v_0 = v_{ex}\ln\left(\frac{m_0}{m_f}\right)$$

where v_0 = initial velocity
m_0 = initial mass
(include fuel + pay load)

<u>Ratio</u> $\frac{m_0}{m_f}$ largest if all the burned

if the original mass is 90% fuel $\rightarrow$ $\frac{m_0}{m_f} = 10$ in the end

$\boxed{\ln(10) = 2.3}$

The speed gained by the rocket $(v - v_0)$ cannot be more than (2.3) times v_{ex}.

Center of Mass

Consider a group of N particles m_α & $\vec{r}_\alpha$ from origin-O

Define C.O.M of the system:

$$\boxed{\vec{R} = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_\alpha}$$ where M is the total mass of the system.

$r_\alpha = \begin{pmatrix} x_\alpha \\ y_\alpha \\ z_\alpha \end{pmatrix}$ and $\vec{R} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ $\Rightarrow$ $RX = \frac{1}{M}\sum_{\alpha=1}^{N} m_\alpha \vec{r}_{\alpha x}$ and so on...

<u>Example</u> 2 particles $\boxed{\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}}$

Rewriting the total momentum $\vec{P}$ of the N-particle system

$$\boxed{P = \sum_\alpha p_\alpha = \sum_\alpha m_\alpha \dot{\vec{r}}_\alpha = M\dot{R}}$$

total momentum of system is the product of M and rate of change of position of the C.O.M

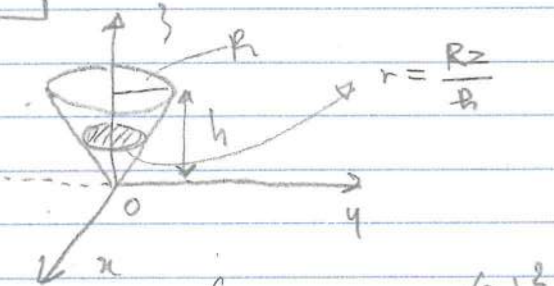
$$\boxed{F_{ext} = \dot{P} = M\ddot{R}}$$

 25

If mass is distributed continuously

$$\boxed{R = \frac{1}{M}\int \underset{\downarrow\, r(m)}{r\,dm} = \frac{1}{M}\int \rho\, r\, dV} \qquad dm = \rho\, dV$$

Feb 27, 2018 — **C.O.M of a solid cone**



- Due to symmetry, COM on z axis
- let height COM $= z$

$$z = \frac{1}{M}\int \rho\, r\, dV = \frac{\rho}{M}\int z\,dx\,dy\,dz \qquad \int dx\,dy = \pi r^2 = \pi\left(\frac{Rz}{h}\right)^2$$

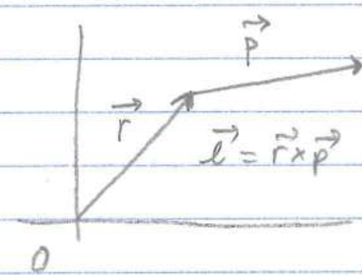
$$= \frac{\rho}{M}\pi\int\left(\frac{Rz}{h}\right)^2 z\,dz$$

$$z = \frac{\rho\pi\cdot R^2}{M\cdot h^2}\int_0^h z^3\,dz \Rightarrow \boxed{z = \frac{1}{4}\frac{\rho\pi R^2}{M h^2}h^4} = \boxed{\frac{1}{4}\frac{\rho\pi R^2}{M}h^2}$$

$$\boxed{M = \frac{1}{3}\rho\pi R^2 h} \rightarrow z = \frac{3}{4}h$$

ANGULAR MOMENTUM

For a single particle : $\boxed{\vec{\ell} = \vec{r}\times\vec{p}}$

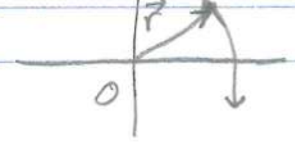


$$\dot{\ell} = \frac{d}{dt}(\vec{r}\times\vec{p}) = (\dot{\vec{r}}\times\vec{p}) + (\vec{r}\times\dot{\vec{p}})$$

$$= \underbrace{(\dot{\vec{r}}\times m\dot{\vec{r}})}_{0} + (\vec{r}\times\vec{F}) \Rightarrow \boxed{\dot{\vec{\ell}} = \vec{r}\times\vec{F} = \vec{\tau}}$$

$\boxed{\dot{\ell} = \tau}$ ← rotational analog of newton's II law $\boxed{\dot{p} = F}$

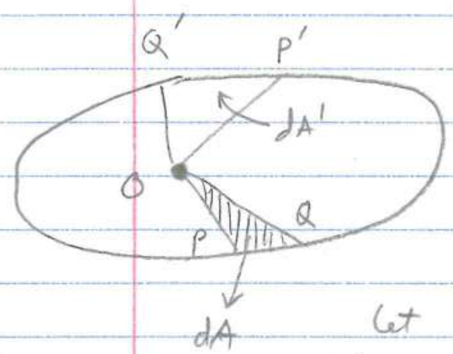
Ex



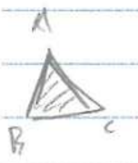
planet orbiting the Sun, $F = \frac{GmM}{r^2}$

$\vec{r} \times \vec{F} = \vec{0} \rightarrow$ planet's angular momentum is constant

$\boxed{\text{Kepler's } 2^{nd} \text{ law}}$ $\left\{ \begin{array}{l} \text{As each planet move around the Sun, a line drawn from} \\ \text{the planet to the Sun sweeps out equal areas in} \\ \text{equal times.} \end{array} \right\}$



$QOP \Rightarrow dA$ in $dt' = dt \Rightarrow dA = dA'$
$Q'OP' \rightarrow dA'$



Let $\vec{OP} = \vec{r}$, then $\vec{PQ} = d\vec{r} = \vec{v}dt$ &

$\hookrightarrow$ Area of $OPQ = \left[\frac{1}{2} \left| \vec{OP} \times \vec{PQ} \right| \right] \rightarrow dA = \frac{1}{2} \left| \vec{r} \times \vec{v}\, dt \right|$

$A = \frac{1}{2} \left\{ \vec{AB} \times \vec{AC} \right\}$

$\hookrightarrow \boxed{\frac{dA}{dt} = \frac{1}{2m} \left| \vec{r} \times \vec{p} \right| = \frac{1}{2m} \vec{\ell}} \rightarrow$ constant

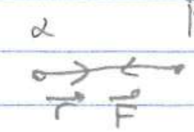
So $\frac{dA}{dt}$ always constant!

$\boxed{\text{Angular Momentum for Several particles}}$ N particles.. labelled α

$\hookrightarrow \boxed{\vec{\ell_\alpha} = \vec{r_\alpha} \times \vec{p_\alpha}} \rightarrow \vec{L} = \sum_{\alpha=1}^{N} \vec{\ell_\alpha}$

So $\boxed{\dot{\vec{L}} = \sum_{\alpha=1}^{N} \dot{\vec{\ell_\alpha}} = \sum_{\alpha}^{N} \vec{r_\alpha} \times \vec{F_\alpha}}$

Now, net force on particle $\boxed{\alpha = F_\alpha = \sum_{\beta \neq \alpha} F_{\alpha} F_{\beta} + F_\alpha^{ext}}$



$\hookrightarrow \dot{\vec{L}} = \underbrace{\left(\sum_{\alpha} \sum_{\beta \neq \alpha} \vec{r_\alpha} \times \vec{F_{\alpha\beta}} \right)}_{\parallel} + \left(\sum_{\alpha} \vec{r_\alpha} \times \vec{F_\alpha}^{ext} \right)$

$\sum_{\alpha} \sum_{\beta > \alpha} \left(\vec{r_\alpha} \times \vec{F_{\alpha\beta}} + \vec{r_\beta} \times \vec{F_{\beta\alpha}} \right)$, with $\vec{F_{\alpha\beta}} = -\vec{F_{\beta\alpha}}$

$\underline{1^{st} \text{ term}} = \sum_{\alpha} \sum_{\beta > \alpha} \left(\vec{r_\alpha} - \vec{r_\beta} \right) \times \vec{F_{\alpha\beta}} = \sum_{\alpha} \sum_{\beta > \alpha} \underbrace{\vec{r_{\alpha\beta}} \times \vec{F_{\alpha\beta}}}_{\vec{0}} = \vec{0}$

$$\boxed{\dot{\vec{L}} = \sum_{\alpha}^{N} \vec{r}_\alpha \times \vec{F}_\alpha^{ext} = \vec{\tau}^{ext}_{net}}$$

when $\vec{\tau}^{ext}_{net} = \vec{0}$, the system has constant $\vec{L}$

Moment of Inertia

$\boxed{I = \sum m_\alpha R_\alpha^2}$ → {distance from m_α to the axis of rotation}

$\boxed{\vec{L} = I\vec{\omega}}$ —— angular velocity of rotation.

Uniform disk (M,R) → $I = \frac{1}{2}MR^2$

Uniform sphere → $I = \frac{2}{5}MR^2$

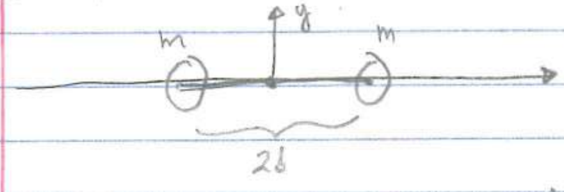
Angular Momentum about CM $\quad \boxed{\dot{L} = \tau^{ext}}$

↳ derivation was based on assumption of inertial frame. Same result also holds if $\vec{L}$, τ are measured about the COM, even if in a non-inertial frame

i.e. $\frac{d}{dt}L = \tau^{ext}$

if $\tau^{ext} = 0$, $\vec{L}$ about CM is conserved.

March 1, 2018 **Sliding a Spinning Dumbbell**



At $t=0$, left mass is given a sharp tap, in the y-direction with $\vec{F}$, lasting for Δt. Describe the subsequent motion.

$\tau = I\alpha =$

$F = ma$

$\dot{p} = F^{ext} \Rightarrow p = F^{ext}\Delta t = M\dot{R} \Rightarrow v_{CM} = \dot{R} = \frac{F\Delta t}{2m}$

$\Gamma^{ext} = F.b$ about CM ↳ $\dot{\vec{L}} = \Gamma^{ext}$ $\quad \boxed{\vec{L} = Fb\Delta t}$

$L = I\omega \quad , \quad I = (2m)b^2$

$\hookrightarrow (2mb^2)\,\omega = Fb\Delta t \rightarrow \boxed{\omega = \frac{F\Delta t}{2mb}}$ angular velocity. clockwise

Speed : $\begin{cases} v_l = v_{CM} + \omega b = \frac{F\Delta t}{m} \\ v_r = v_{CM} - \omega r = 0 \end{cases}$ (instantaneously after F)

Chapter 4 $\boxed{\text{CONSERVATION OF ENERGY}}$

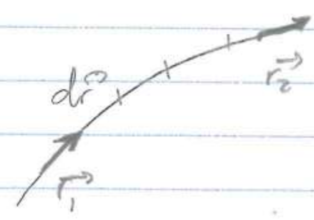
(A) kinetic energy - work : $T = \frac{1}{2}mv^2 \rightarrow \boxed{\frac{dT}{dt} = m\dot{\vec{v}}\vec{v}}$ $\vec{F}$

Newton's law $\boxed{\frac{dT}{dt} = \vec{F}\cdot\vec{v}}$ $\rightsquigarrow \boxed{dT = \vec{F}\cdot d\vec{r}} \longrightarrow$ work done by the force F with displacement dr.

(B) Work - kinetic energy theorem

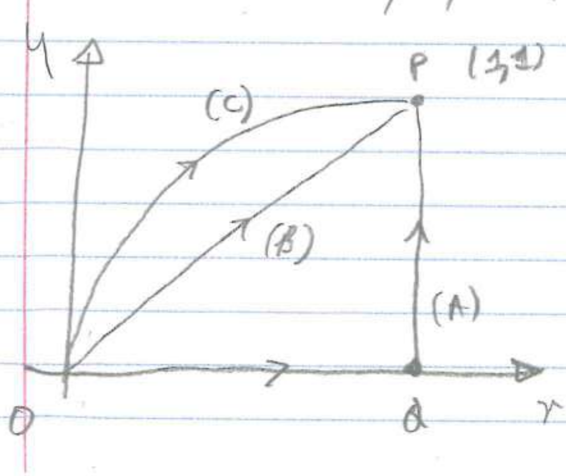
The change in the particle's T between 2 neighboring points on its path is equal to the work done by the net force.

$\boxed{\Delta T = \int \vec{F} d\vec{r}} \longrightarrow$ line integral



path dependent.

Example $\rightsquigarrow$ evaluate line integral for work done by $\vec{F} = (y, 2x)$ going from origin O to P(1,1), along 3 different paths.



Path A

$$W_a = \int \vec{F} d\vec{r} = \int_O^Q \vec{F}\cdot d\vec{r} + \int_Q^P \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \underbrace{F_x(x,0)}_{y} dx + \int_0^1 \underbrace{F_y(1,y)}_{2x = 2} dy$$

$$= 0 + 2 = 2$$

Path b $W_b = \int_b \vec{F} \cdot d\vec{r} = \int_1^2 (F_x\, dx + F_y\, dy)$

$= \int_0^1 (x + 2x)\, dx = 1.5$

Path c

$\vec{F} = (y, 2x)$
$\Rightarrow y = \sin\theta = F_x$
$2x = 1 - \cos\theta = F_y$

point Q

$\vec{r} = (x, y) = (1-\cos\theta, \sin\theta) \quad 0 \le \theta \le \pi/2$

$d\vec{r} = (dx, dy) = (\sin\theta, \cos\theta)\, d\theta$

$W_c = \int_c \vec{F} \cdot d\vec{r} = \int_c (F_x dx + F_y dy) = \int_0^{\pi/2} (\sin^2\theta + 2(1-\cos\theta)\cos\theta)\, d\theta$

$= \boxed{2 - \frac{\pi}{4} = 1.21}$ (?)

Since $W_a \ne W_b \ne W_c$, path dependent.

$\boxed{\text{Potential Energy}} \rightarrow$ Conservative Force

Mar 5, 2018

2 conditions to be conservative Force:

(1) F depends only on position (r) of the object on which it acts. (cannot depend on velocity / time)

Example $\vec{F}(\vec{r}) = \frac{GMm}{r^2}\hat{r}$

(2) Path independence. Work $W(1 \to 2)$ is the same for all paths between $1 \to 2$.

Ex $W = -mgh$ (height between 1, 2)

If all forces on an obj are conservative $\rightarrow$ we can define potential energy $U(r)$

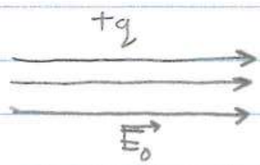
$\rightarrow$ Total mechanical energy $\boxed{E = T + U(r)}$ constant

E is conserved

$$\boxed{U(r) = -W(\vec{r}_0 - \vec{r}) = -\int_{r_0}^{r} \vec{F}(\vec{r}')\,d\vec{r}'}$$

$\hookrightarrow$ r_0 is a reference point at which $U = 0$

Exp potential energy of a charge in a uniform $\vec{E}$ field.



$\vec{F} = q\vec{E}_0$. Show $\vec{F}$ is conservative. find $U(r)$

$$W(1 \mapsto 2) = +\int_1^2 \vec{F}\cdot d\vec{r} = +\int_1^2 q\vec{E}_0\cdot d\vec{r} = +\int_1^2 qE_0\,\hat{x}\,d\vec{r}$$

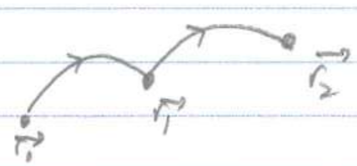
$$= +\int_1^2 qE_0\,dx = \boxed{qE_0\,\Delta x}$$

To define potential energy $U(r) \to$ pick a reference point $r_0 @ U = 0$.

$$\boxed{U(\vec{r}) = -W(\vec{r}_0 - \vec{r})} \rightarrow \boxed{U = -qE_0 x}$$

Mechanical Energy

Single force



$\vec{r}_1, \vec{r}_2$ any 2 points. $\vec{r}_0$ is the reference point where $U = 0$

$$W(\vec{r}_0 \to \vec{r}_2) = W(\vec{r}_0 \to \vec{r}_1) + W(\vec{r}_1 \to \vec{r}_2)$$

$$\hookrightarrow W(\vec{r}_1 \to \vec{r}_2) = W(\vec{r}_0 - \vec{r}_2) - W(\vec{r}_0 \to \vec{r}_1)$$

$$= -U(\vec{r}_2) + U(\vec{r}_1)$$

$$= -(U(\vec{r}_2) - U(\vec{r}_1))$$

$$W(\vec{r}_1 \to \vec{r}_2) = -\Delta U$$

Work-KE theorem: $W(\vec{r}_1 \to \vec{r}_2) = \Delta T$

$$\boxed{\Delta T + \Delta U = 0}$$

So $\Delta(T+U) = 0 \to \Delta E = 0 \to \boxed{E \text{ constant}} \leftrightarrow \vec{F}$ is conservative

For several forces — Particle is subject to several conservative forces.
→ will the result still be valid?

Ex: mass suspended from a ceiling by a spring.

Two forces: → gravity (U_{grav})
→ spring force (U_{spring})

Use work = KE theorem
$$\Delta T = W_{gravity} + W_{spring} = -(\Delta U_{grav} + \Delta U_{spring})$$
→ get $\Delta(T + U_{grav} + U_{spring}) = \Delta E = 0$
→ total mech. energy conserved.

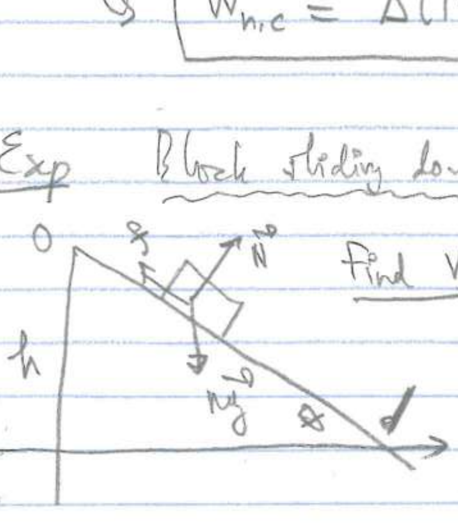
$\boxed{\text{Generalize}}$ If all n forces $\vec{f_i}$ ($i = 1, 2, 3, \dots n$) acting on a particle are conservative, each with its corresponding potential $E = U_i(r)$ then the total mechanical energy, then
$$[E = T + U = \text{constant in time}]$$

$\boxed{\text{Non-conservative force}}$ → $F_{net} = F_{conservative} + F_{n.c}$

work KE. th → $\Delta T = W = W_{con} + W_{nc} = -\Delta U + W_{n.c}$

→ $\boxed{W_{n.c} = \Delta(T+U) = \Delta E}$ → work done by n.c force = change in total energy

Exp Block sliding down an incline.



Find V_f

Weight. $U = m.g.y$
Normal. $\perp \vec{v}$ does no work.
Friction, $f = N\mu = \mu mg\cos\theta$

$d\sin\theta$

$$W_{friction} = -f.d = -\mu mg d\cos\theta = \frac{1}{2}mv^2 - mgh$$

$$- \mu mg \cos\theta \, d$$

So $mg\, d \sin\theta - f_r d = \frac{1}{2} m v^2$

$$\boxed{v = \sqrt{\frac{2dg(\sin\theta - \mu\cos\theta)}{1}}}$$

March 6, 2018

Force as gradient of potential energy

corresponding potential energy

Particle is acted on by conservative force $\vec{F}(\vec{r}) \sim U(\vec{r})$

Suppose work done: $\vec{r} \to \vec{r} + d\vec{r}$

$$\boxed{W(\vec{r} \to \vec{r} + d\vec{r}) = \vec{F}(\vec{r}) \cdot d\vec{r} = F_x dx + F_y dy + F_z dz} \quad (A)$$

$$W(\vec{r} \to \vec{r} + d\vec{r}) = -dU = -\left[U(\vec{r} + d\vec{r}) - U(\vec{r})\right] =$$
$$= -\left[U(x+dx, y+dy, z+dz) - U(x,y,z)\right]$$

1D $df = f(x+dx) - f(x) = \frac{df}{dx} dx$

$$\boxed{dU = \frac{\partial U}{\partial x} dx + \frac{\partial U}{\partial y} dy + \frac{\partial U}{\partial z} dz = -W(\vec{r} \to d\vec{r} + \vec{r})} \quad (B)$$

$$\boxed{dU = \vec{\nabla} U \cdot d\vec{r}}$$

So (A) (B) $\Longrightarrow$ $F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}$

So

$$\boxed{\vec{F} = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z}}$$

$$\boxed{\vec{F} = -\nabla U}$$ force is derivable from a potential energy.

Example $U = Axy^2 + B\sin(Cz)$ (A, B, C constants)

Find f.

$$\vec{f} = -\nabla U = -\frac{\partial U}{\partial x}\hat{x} - \frac{\partial U}{\partial y}\hat{y} - \frac{\partial U}{\partial z}\hat{z} = -Ay^2\hat{x} - 2Axy\hat{y} - BC\cos(Cz)\hat{z}$$

$$\vec{f} = \begin{pmatrix} -Ay^2 \\ -2xyA \\ -BC\cos(Cz) \end{pmatrix}$$

2nd condition for conservative force: Work $\int_1^2 \vec{F}\, d\vec{r}$ → independent of path.

→ Simpler equivalent test. Test the curl of $\vec{F}$.

→ Work independent of path $\Longleftrightarrow$ $\boxed{\vec{\nabla}\times\vec{F} = 0}$

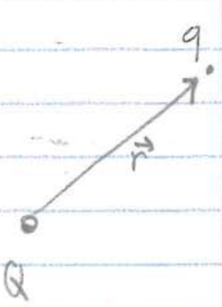
Cross-product $\vec{A}$ & $\vec{B}$

Vector	$\hat{x}$	$\hat{y}$	$\hat{z}$
$\vec{A}$	A_x	A_y	A_z
$\vec{B}$	B_x	B_y	B_z
$\vec{A}\times\vec{B}$	$A_yB_z - B_yA_z$	$-A_xB_z + A_zB_x$	$A_xB_y - B_xA_y$

What is $\vec{\nabla}\times\vec{F}$?

$$\vec{\nabla}\times\vec{F} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{x} - \left(\frac{\partial F_z}{\partial x} - \frac{\partial F_x}{\partial z}\right)\hat{y} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{z}$$

Example: Coulomb force. Is the Coulomb force conservative?



Consider $\vec{F}$ on small charge q due to a fixed charge Q at $(0,0)$. Show that it is conservative. Find corresponding potential energy.

$$\boxed{F = k\frac{Qq}{r^2}\hat{r} = \frac{\gamma}{r^3}\vec{r}}$$

What is $(\vec{\nabla}\times\vec{F})_x$?

So $\vec{\nabla}\times\vec{F} = \vec{0}$ → $\vec{F}$ conservative

$$(\vec{\nabla}\times\vec{F})_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\gamma z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{\gamma y}{r^3}\right)$$

$r = \sqrt{x^2+y^2+z^2} \rightarrow \left(\frac{\partial r}{\partial y} = \frac{y}{r}\right)$ → $\left(\frac{\partial r}{\partial z} = \frac{z}{r}\right)$

and $\frac{\partial}{\partial r} r^{-3} = (-3)\, r^{-4}$

$$(\vec{\nabla}\times\vec{F})_x = \gamma z\left(\frac{\partial}{\partial r}r^{-3}\cdot\frac{\partial r}{\partial y}\right) - \gamma y\left(\frac{\partial}{\partial r}r^{-3}\frac{\partial r}{\partial z}\right)$$

$$= \gamma z\left(\frac{-3}{r^4}\cdot\frac{y}{r}\right) - \gamma y\left(\frac{-3}{r^4}\cdot\frac{z}{r}\right)$$

$$= \frac{-3yz\gamma}{r^5} + \frac{3\gamma yz}{r^5} = 0$$

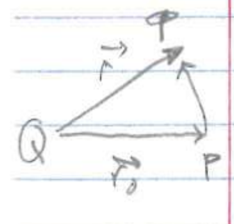
Similarly, for $y, z \Rightarrow 0$

⇒ F satisfies both 1st, 2nd condition ⇒ F conservative.

$\boxed{\text{What is } U\,?}$

$$U = -\int_{r_0}^{r} F(\vec r\,')\, d\vec r\,' \qquad \left(r_0 \to \text{reference point where } U = 0 \right)$$

← independent of path.
→ can choose any path.



Choose a path that goes radially outwards to point P then around a circle to $\vec r$.

In QP: $\vec F(\vec r\,')$ in $d\vec r\,'$ in the same direction → $\vec F(\vec r\,') \cdot d\vec r\,' = \frac{\gamma}{r^2}\, dr$

In Pq: $\vec F(\vec r\,') \perp d\vec r\,'$ → 0 work done.

$$\boxed{U(\vec r) = -\int_{r_0}^{r} \frac{\gamma}{r^2}\, dr' = \left(\frac{\gamma}{r} - \frac{\gamma}{r_0}\right)}$$

choose $r_0 = \infty$ ⇒ $\boxed{U(\vec r) = \frac{kqQ}{r}}$

$U(\vec r)$ only depends on the magnitude of $\vec r$, not the direction

check $(\nabla U)_x = \frac{\partial U}{\partial x} = \left(\frac{\partial U}{\partial r}\right)\left(\frac{\partial r}{\partial x}\right) = \frac{-kqQ}{r^2}\left(\frac{x}{r}\right) = \frac{-kqQ}{r^3}\,x$

So $\boxed{\nabla U_x = -f_x}$ same goes for y, z

$\boxed{\nabla U = -f}$

March 8, 2018

$\boxed{\text{Time-dependent potential energy}}$

$\vec{F}(\vec{r},t) \longrightarrow$ not conservative (fails the first spatial condition).

We can still define a potential energy: $U(\vec{r},t)$ such that $\vec{F} = -\vec{\nabla} U$

However, the total mechanical energy is no longer conserved...

Exp

Charge on sphere slowly leaking away!

$$\boxed{\vec{F}(\vec{r},t) = \frac{1}{4\pi\epsilon_0} \cdot \frac{kq\,Q(t)}{r^2}\hat{r}}$$

So, force is time-dependent even if $\vec{r}$ = constant...

If $\vec{r}$ constant, the spatial dependence of the force is the same as for time-independent Coulomb force.

$$\vec{\nabla} \times \vec{F} = 0$$

$\Rightarrow \int \vec{F} d\vec{r}$ path independent.

Define $U(\vec{r},t) = -\int_{r_0}^{r} \vec{F}(\vec{r}\,',t)\,d\vec{r}\,'$

But Note $E = T + U$ not conserved... E changes as the particle move on this path...

Consider 2 points on the particle's path. @ t & $t+dt$.

Calculate ΔT

$\hookrightarrow dT = \frac{dT}{dt}dt = (m\dot{\vec{v}}\cdot\vec{v})dt = (\vec{F}\cdot d\vec{r})$

$U(\vec{r},t) = U(x,y,z,t)$. $dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz + \frac{\partial U}{\partial t}dt$

$= \vec{\nabla} U \cdot d\vec{r} + \frac{\partial U}{\partial t}dt$

Add U & T

$$\boxed{dU = \underbrace{-\vec{F}\cdot d\vec{r}}_{-dT} + \frac{\partial U}{\partial t}dt}$$

$$\boxed{d(U+T) = \frac{\partial U}{\partial t}dt}$$

So E only conserved

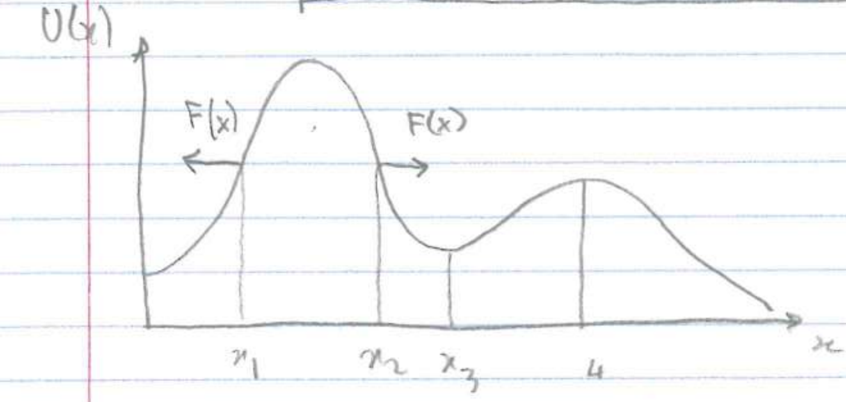
$\Leftrightarrow \frac{\partial U}{\partial t} = 0 \Leftrightarrow U$ independent of time.

$\boxed{\text{1-D systems}}$ $W(x_1 - x_2) = \int_{x_1}^{x_2} F(x)\,dx$

Object is constrained to move only in the x-axis.

<u>Potential energy</u>: $U(x)$ — function of only 1-dependent variable

$$\boxed{U(x) = -\int_{x_0}^{x} F(x')dx' \quad , \quad F(x) = \frac{-dU}{dx}}$$



The force $F_x = \frac{-dU}{dx}$ tends to push the object "downhill" @ x_1, x_2.

@ x_3 & x_4 → $\frac{dU}{dx} = 0 = F$

→ <u>object in equilibrium</u> @ x_3 : $\frac{d^2U}{dx^2} > 0 \Rightarrow U(x)$ is minimum
⇒ STABLE EQ

@ x_4 $\frac{d^2U}{dx^2} < 0 \Rightarrow U(x)$ is maximum
⇒ UNSTABLE EQ.

Solution of motion for 1-D system

$$\boxed{T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2 = E - U(x)}$$

So $\dot{x} = \pm\sqrt{\frac{2}{m}(E - U(x))}$

$$\int dt = \int \frac{\pm 1}{\sqrt{2/m(E-U(x))}}\,dx \Rightarrow \boxed{t_f - t_i = \int_{x_i}^{x_f} \frac{dx}{\dot{x}}}$$

$$\Rightarrow \boxed{\Delta t = \sqrt{\frac{m}{2}} \int \frac{dx'}{\sqrt{E - U(x)}}}$$

<u>Example</u> → free fall Drop a stone from top. Use conservation of E to find the stone's position x $(x=0)$
(Neglect air resistance)

Only force is gravity $\rightarrow$ $U(x) = -mgx$ $\Big\} \rightarrow$ $E = 0$
Since the stone is at rest @ $x = 0$

$$\dot{x} = \sqrt{\frac{2}{m}}\sqrt{E - U(x)} = \sqrt{\frac{2}{m}}\sqrt{(mgx)} = \sqrt{2gx}$$

$$t = \int_0^x \frac{dx'}{\sqrt{2gx'}} = \frac{1}{\sqrt{2g}}\int_0^x \frac{1}{\sqrt{x'}}\,dx = \frac{1}{\sqrt{2g}}\left[2\sqrt{x}\right]\Big|_0^x = \sqrt{\frac{2x}{g}}$$

So $t^2 = \frac{2x}{g} \rightarrow \boxed{x = \frac{1}{2}gt^2}$

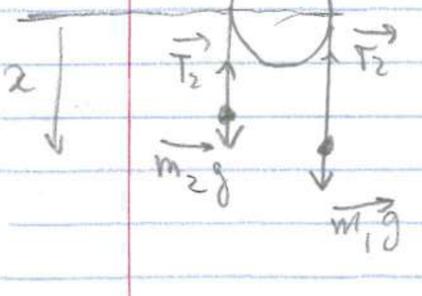
$\boxed{\text{Curvilinear 1-D system}}$

Position of the particle $\rightarrow$ distance "s" measured along the wire

$$\left\{ \begin{array}{cc} T = \frac{1}{2}m\dot{s}^2 & \vec{F}_{tangent} = m\ddot{s} = -\frac{dU}{ds} \\ & E = T + U(s) = \text{constant} \end{array} \right\}$$

$\boxed{\text{Everything holds}}$ $\rightarrow$ look at example of a cube on cylinder

Atwood's machine



2 masses suspended from a string that goes over a pulley.

$\Delta T_1 + \Delta U_1 = W_1^{ten}$ $\rightarrow$ work done by tension force on m_1 & m_2

$\Delta T_2 + \Delta U_2 = W_2^{ten}$

Since $W_1^{ten} = -W_2^{ten} \rightarrow \boxed{\Delta T_1 + \Delta U_1 + \Delta T_2 + \Delta U_2 = 0}$ $\rightarrow$ $\sum E$ constant.

Mar 12, 2018

$\boxed{\text{Central forces}}$ Force directed towards or away from a fixed "force center"

⊛ If the force center is the origin

$$\boxed{\vec{F}(\vec{r}) = f(\vec{r}) \cdot \hat{r}}$$

Ex Coulomb force. force on q, due to Q @ origin

$$\boxed{\vec{F}(r) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Qq}{r^2}\,\hat{r} \quad , \quad f(\vec{r}) = \frac{k_e q Q}{r^2}}$$

Coulomb force has 2 additional properties:

(1) Coulomb force is CONSERVATIVE

(2) Spherically symmetric (or rotationally invariant).

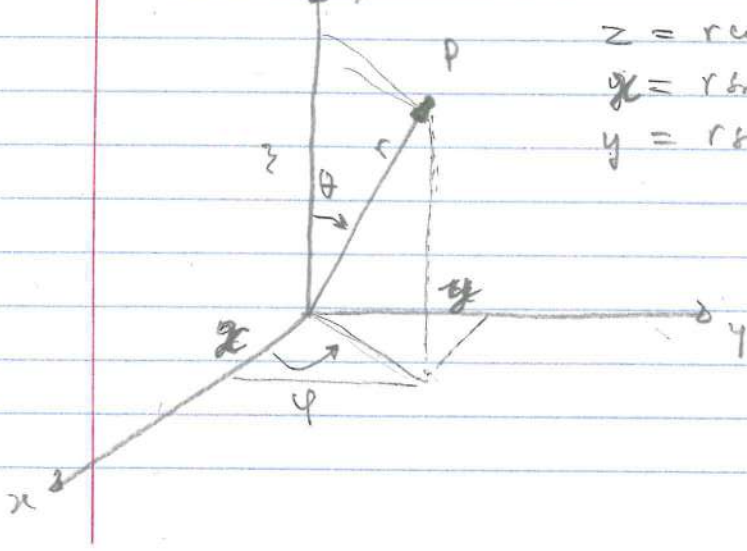
$f(\vec{r}) = f(|\vec{r}|)$ → only depends on the magnitude of $\vec{r}$ & not direction

magnitude of F.

Note A central force that is conservative ⟺ it is spherically symmetric

If $f(\vec{r})$ is symmetric spherically → $f(\vec{r}) = f(|\vec{r}|) = f(r)$

Now, $\boxed{\text{Spherical Polar coordinate}}$



$z = r\cos\theta$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial \varphi} = 0$$

{ Since f only depends on r }

$\hat{r} = \sin\theta\cos\varphi\,\hat{i} + \sin\theta\sin\varphi\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{\theta} = \cos\theta\cos\varphi\,\hat{i} + \cos\theta\sin\varphi\,\hat{j} + (-\sin\theta)\,\hat{k}$

$\hat{\varphi} = -\sin\varphi\,\hat{i} + \cos\varphi\,\hat{j}$

$\hat{r}, \hat{\theta}, \hat{\varphi}$ are orthogonal $\Rightarrow$ for any 2 vectors: $\vec{a}, \vec{b}$

$$\boxed{\vec{a} \cdot \vec{b} = a_r b_r + a_\theta b_\theta + a_\varphi b_\varphi}$$

Gradient for Spherical Polar Coordinate

$$\vec{\nabla} f = \hat{x}\frac{\partial f}{\partial x} + \hat{y}\frac{\partial f}{\partial y} + \hat{z}\frac{\partial f}{\partial z} \qquad \text{[cartesian]}$$

<u>Recall</u> $df = \vec{\nabla} f \cdot d\vec{r}$

where $d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}$ [spherical]

$$df = (\vec{\nabla} f)_r\,dr + (\vec{\nabla} f)_\theta\,r\,d\theta + (\vec{\nabla} f)_\varphi\,r\sin\theta\,d\varphi$$
$$= \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial \theta}d\theta + \frac{\partial f}{\partial \varphi}d\varphi$$

So
$$\boxed{\vec{\nabla} f = \underbrace{\hat{r}\frac{\partial f}{\partial r}}_{(\vec{\nabla} f)_r} + \underbrace{\hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta}}_{(\vec{\nabla} f)_\theta} + \underbrace{\hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}}_{(\vec{\nabla} f)_\varphi}}$$

<u>Central Force</u> Conservation & Spherically symmetric

$\hookrightarrow$ Assume central force <u>is conservative</u>, Test if it is spherically sym

If $\vec{F}(\vec{r})$ is conservative $\Rightarrow$ $\vec{F}(\vec{r}) = -\vec{\nabla} U$

$$= -\hat{r}\frac{\partial U}{\partial r} - \hat{\theta}\frac{1}{r}\frac{\partial U}{\partial \theta} - \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial U}{\partial \varphi}$$

but since $\frac{\partial U}{\partial \theta} = \frac{\partial U}{\partial \varphi} = \emptyset$ (since $\vec{F}(\vec{r})$ only depends on r)

$\hookrightarrow$ $\vec{F}(\vec{r}) = -\hat{r}\frac{\partial U}{\partial r}$ $\Rightarrow$ U is ~~sphericallly~~ symmetric

<u>So F is spherically symmetric</u>

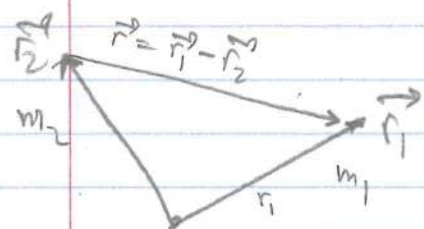
Energy of interactions for 2 particles

Suppose 2 particles interact via forces $\vec{F}_{12}$ (on 1, by 2), $\vec{F}_{ext} = \vec{0}$

$\vec{F}_{21}$ (on 2, by 1)

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}$$

Example Isolated binary star



The only 2 forces are gravitational attraction

$$\vec{F}_{12} = -\frac{Gm_1m_2}{r^2}\hat{r}$$

$$= -\frac{Gm_1m_2}{r^3}\vec{r}$$

Force only depends on the combination $(\vec{r}_1 - \vec{r}_2)$

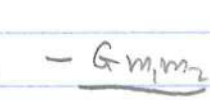
$$= -\frac{Gm_1m_2}{r^3}(\vec{r}_1 - \vec{r}_2)$$

→ Translationally invariant

$$= -\frac{Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2)$$

→ If system is translated w/o changing the relative position → interparticle forces remain the same.

Special case Let $\vec{r}_2 = \vec{0}$.



→ $\vec{F}_{12} = \vec{F}_{12}(\vec{r}_{12})$

if $\vec{F}_{12}$ conservative → $\vec{\nabla}_1 \times \vec{F}_{12} = 0$

where $\vec{\nabla}_1 = \hat{x}\frac{\partial}{\partial x_1} + \hat{y}\frac{\partial}{\partial y_1} + \hat{z}\frac{\partial}{\partial z_1}$ → with respect to particle (2) → $(x, y, z$ of (1))

$$\vec{F}_{12} = -\vec{\nabla}_1 U(\vec{r}_{12}) = -\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)$$

From Newton's III law → $\vec{F}_{12} = -\vec{F}_{21}$ → $\boxed{\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2) = -\vec{\nabla}_2 U(\vec{r}_1 - \vec{r}_2)}$

For multiparticle system $\Rightarrow$ force on (1) $= -\nabla_1 U$
force on (2) $= -\nabla_2 U$
↳ Single position energy U, from which we can derive both forces.

Conservation of energy for 2-particle sys

$$\begin{cases} dT_1 = \text{work on (1)} = d\vec{r}_1 \cdot \vec{F}_{12} \\ dT_2 = \text{work on (2)} = d\vec{r}_2 \cdot \vec{F}_{21} \end{cases}$$

Total chge in KE $= dT_1 + dT_2 = d\vec{r}_1 \cdot \vec{F}_{12} + d\vec{r}_2 \cdot \vec{F}_{21}$
$= d\vec{r}_1 \cdot \vec{F}_{12} - d\vec{r}_2 \cdot \vec{F}_{12}$
$= \vec{F}_{12} \, (d\vec{r}_1 - d\vec{r}_2)$

where $\vec{F}_{12} = -\vec{\nabla}_1 U(\vec{r}_1 - \vec{r}_2)$

$\Rightarrow dT = dT_1 + dT_2 = \left[-\vec{\nabla} U(\vec{r}_1 - \vec{r}_2) \right] \cdot d\,(\vec{r}_1 - \vec{r}_2)$

$= -\nabla U(\vec{r}) \cdot d\vec{r} = -dU$

↳ $\vec{\nabla} U = \frac{\partial U_x}{\partial x}\hat{x} + \frac{\partial U_y}{\partial r}\hat{y} + \frac{\partial U}{\partial y}\hat{z} = \frac{dU}{dr}$

$\boxed{\text{So } dT = -dU}$

↳ $d(T+E) = 0 \rightarrow \boxed{dE = 0}$

Total energy is conserved.

Mar 14, 2018 $\boxed{\text{Elastic Collision}}$

{ Collision between 2 particles via a conservative force $\rightarrow 0$ as separation $\vec{r}_1 \rightarrow \vec{r}_2$ increases }

The potential energy $U = U(\vec{r}_1 - \vec{r}_2) \rightarrow \text{const} = 0$ (can be taken to be 0)

$T + U = \text{constant}$ - when particles are far apart, $\boxed{T_{in} = T_{final}}$

Consider elastic collision 2 particles $m = m_1 = m_2$

Prove if m_2 initially at rest, then $(\vec{r}_2, \vec{r}_1) = 90°$

$\frac{1}{2} m \dot{r}_1^2 = \frac{1}{2} m_2 \dot{r}_2^2 \Rightarrow \quad m\vec{v}_1 = m\vec{v}_1{}' + m\vec{v}_2{}'$

$\Sigma_p = m\dot{r}_1 = \quad$ also $\quad \vec{v}_1^{\,2} = \vec{v}_1{}' + \vec{v}_2{}'$

$\Rightarrow \vec{v}_1{}' \vec{v}_2{}' = 0$

$\downarrow \boxed{\vec{v}_1{}' \perp \vec{v}_2{}'}$

Energy of multi-particle system 2 particles → N particles

N = 4

4 particles, interacting, subject to $\vec{F}_{ext}$

For each pair of particles, $\alpha\beta$, there is an action-reaction pair of forces $\vec{F}_{\alpha\beta}$ & $\vec{F}_{\beta\alpha}$ + each particle α – subject to ext force $\hookrightarrow \vec{F}_\alpha^{ext}$

Total KE $\searrow$ $\boxed{T = T_1 + T_2 + T_3 + T_4 \quad \text{where } T_\alpha = \frac{1}{2} m_\alpha v_\alpha^2}$

Potential E ⤷ take 2 particles, look @ pair. For forces $(\vec{F}_{43}$ & $\vec{F}_{34})$ (conservative) then

$$U_{34} = -U_{43} = U_{34}(\vec{r}_3 - \vec{r}_4)$$

with $F_{34} = -\vec{\nabla}_3 U_{34}$

$F_{43} = -\vec{\nabla}_4 U_{34}$

6 pairs of particles. $U_{12}, U_{13}, U_{14}, U_{23}, U_{24}, U_{34}$,

External forces $\vec{F}_\alpha^{ext} \longrightarrow \vec{r}_\alpha$

$\vec{F}_1^{ext}$ depends only on position $\vec{r}_1$, but not on $\vec{r}_2, \vec{r}_3, \vec{r}_4$

$\vec{F}_1^{ext}$ → a force on a single particle

→ conservative → pot^n energy $\boxed{\vec{F}_\alpha^{ext} = -\nabla_\alpha U_\alpha^{ext}(\vec{r}_\alpha)}$

Total potential E

$$U = U^{int} + U^{ext} = \left(U_{12} + U_{13} + U_{14} + U_{23} + U_{24} + U_{34}\right) + \left(U_1^{ext} + U_2^{ext} + U_3^{ext} + U_4^{ext}\right)$$

Force on α = $-\nabla U$ with respect to the coordinate of that particle.

Force on 1 $= -\nabla_1 U = -\nabla_1 \left(U_{12} + U_{13} + U_{14} + U_1^{ext}\right)$

Force on 1 $= \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14} + \vec{F}_1^{ext}$

To generalize

$\boxed{\text{Net force on any particle } \alpha \text{ is } = -\nabla_\alpha U}$ $E = T + U$ (conserved)

Total potential E

$$\boxed{U = U^{int} + U^{ext} = \sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta} + \sum_\alpha U_\alpha^{ext}}$$

Example → Rigid body made out of N atoms.

Potential Energy of the internal forces

$$\boxed{U^{int} = \sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta}\left(\vec{r}_\alpha - \vec{r}_\beta\right)}$$

$$= \sum_\alpha \sum_{\beta > \alpha} U_{\alpha\beta}\left(|\vec{r}_\alpha - \vec{r}_\beta|\right)$$

since Interatomic forces are central

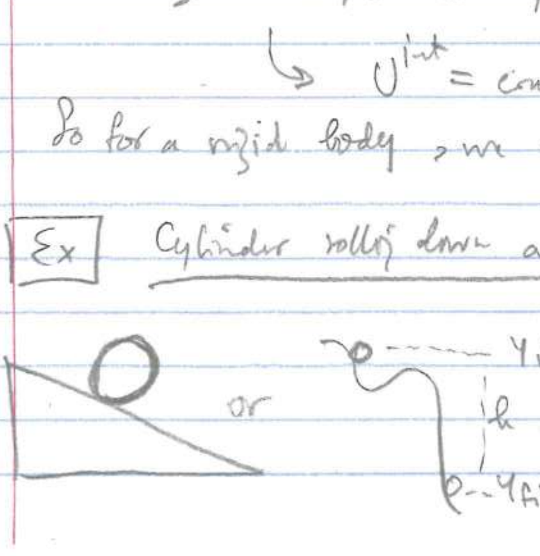
↳ $U_{\alpha\beta}$ only depends on magnitude

For a rigid body, $\vec{r}_\alpha - \vec{r}_\beta$ is constant $\forall\, \alpha, \beta$.

↳ $U^{int} = \text{const} \rightarrow$ take it to be 0

So for a rigid body, we only take into account U^{ext} & KE

$\boxed{\text{Ex}}$ Cylinder rolling down an incline



A uniform, rigid cylinder, of radius R, rolls w/o slipping down the hill. Use E conservation to find its speed when it reaches a vertical height (h) below y-initial.

↳ f, N don't do any work

↳ External force = gravity = conservative force.

$U^{ext} = Mgy$ ← height of the COM from a reference level.

$$\boxed{KE = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2} \qquad I = \frac{1}{2}MR^2, \ \omega = \frac{v}{R}$$

$$= \frac{1}{2}Mv^2 + \frac{1}{4}MR^2\omega^2$$

$$KE = \frac{3}{4}Mv^2$$

$$\Delta KE = \frac{3}{4}Mv^2 = MgY \Rightarrow v = \sqrt{\frac{4MgY}{3M}} = \sqrt{\frac{4}{3}gh}$$

CHAPTER 5: OSCILLATION

↳ System is displaced from stable equilibrium → OSC

Hooke's Law → $F_x(x) = -kx$ (restoring force, eq. stable)

{force constant} {displacement}

$$\boxed{PE = \frac{1}{2}kx^2 = U_x}$$

Nov 15, 2018

HARMONIC OSCILLATOR

Conservative 1-D system with coordinate x & potential energy $U(x)$

stable equilibrium @ $x = x_0 = 0$

$U(x)$ in the vicinity of $x_0 = 0$

Expand $U(x)$ in taylor series

$$\boxed{U(x) = U(0) + U'(0)\,x + U''(0)\,\frac{x^2}{2!} + \cdots}$$

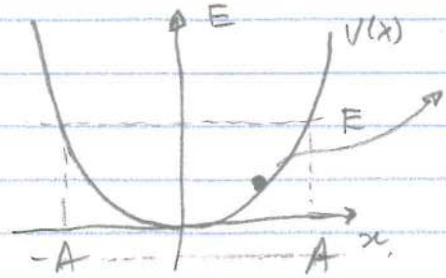
$$U(x) = \underbrace{U(0)}_{0} + \underbrace{U'(0)x}_{0} + \frac{1}{2!}U''(0)x^2 + \dots = \frac{1}{2!}U''(0)x^2$$

Let $U''(0) = k \Rightarrow \boxed{U(x) = \frac{1}{2}kx^2}$

For ideal harmonic oscillator $\longrightarrow$ $F = -kx$ (Hooke's law)

$U_x = \frac{1}{2}kx^2$

$\left\{\begin{array}{l} \text{periodic motion} \\ \text{~~period~~ – independent of amplitude of motion – period} \\ \text{amplitude = constant (no damping)} \end{array}\right\}$

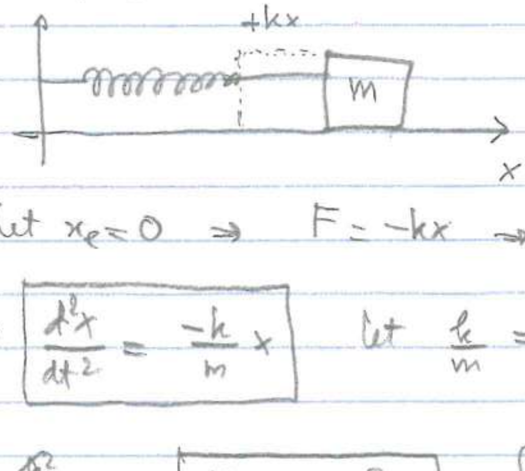


If $E > 0$, particle is trapped $\rightarrow$ oscillate.

$\rightsquigarrow$ turning point $= \pm A$

$\boxed{\text{Single harmonic motion}}$ Diff eqn of motion

Mass m displaced from position of stable equilibrium. Mass m attached to a spring... frictionless



$\boxed{F = -k(x - x_e)}$ k: N/m

$\hookrightarrow U = +\frac{1}{2}k(x - x_e)^2$

Let $x_e = 0 \Rightarrow F = -kx \Rightarrow m\frac{d^2x}{dt^2} = -kx$

$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \frac{-k}{m}x}$ let $\frac{k}{m} = \omega_0^2 \Rightarrow \omega_0$ = angular frequency.

$\underline{\text{Or}}$: $\boxed{\ddot{x} = -\omega_0^2 x}$ $\rightarrow \left\{ \begin{array}{c} 2^{nd}\text{-order, linear, homogenous} \\ \text{diff eqn.} \end{array} \right\}$

Review Homogeneous, 2^{nd} order different equation.

↳ linear, generalised 2^{nd} order diff

↳ $$\boxed{a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = F(t)}$$

$$\left\{ \begin{array}{l} a_2, a_1, a_0 \text{ are constants} \rightarrow \text{linear} \\ 2^{nd} \text{ order} \Rightarrow \frac{d^2x}{dt^2} \rightarrow 2^{nd} \text{ derivative} \\ \text{Homogeneous } F(t) = 0 \end{array} \right\}$$

Okay... $a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = 0$

trial solution $x = Ae^{pt}$

↳ $\frac{dx}{dt} = pAe^{pt}$, $\frac{d^2x}{dt^2} = p^2 Ae^{pt}$

$\Rightarrow a_2 p^2 Ae^{pt} + a_1 p Ae^{pt} + a_0 Ae^{pt} = 0$

$\Rightarrow \boxed{(a_2 p^2 + a_1 p + a_0) = 0} \Rightarrow p = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_2}$ $\left\{ \begin{array}{l} p_+ \\ p_- \end{array} \right\}$

↳ General solution $\boxed{x(t) = A_+ e^{p_+ t} + A_- e^{p_- t}}$

Solution for SHM ↳ General soln ~~to~~ $\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$

↳ $\boxed{\frac{d^2x}{dt^2} + \omega_0^2 x = 0}$

$a_2 = 1$

$a_1 = 0$

$a_0 = \omega_0^2$

$p_+ = \frac{-i\,\omega_0}{2} \cdot 2 = \boxed{+i\omega_0}$

$p_- = \boxed{+i\omega_0}$

General sol $x(t) = A_+ e^{p_+ t} + A_- e^{p_- t} =$

$$\boxed{x(t) = A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t}}$$

↳ superposition principle

Alternative form $\rightarrow$ $x(t) = A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t}$

Use Euler's identity $e^{\pm i\theta} = \cos\theta \pm i\sin\theta$

$$\rightarrow x(t) = A_+ \left(\cos(\omega_0 t) + i\sin(\omega_0 t)\right) + A_- \left(\cos(\omega_0 t) - i\sin(\omega_0 t)\right)$$
$$= (A_+ + A_-)\cos(\omega_0 t) + i(A_+ - A_-)\sin(\omega_0 t)$$

Let $\left.\begin{array}{l} A_+ + A_- = A_c \\ i(A_+ - A_-) = A_s \end{array}\right\} \rightarrow \boxed{x(t) = A_c\cos(\omega_0 t) + A_s\sin(\omega_0 t)}$

This is SHM

The arbitrary constants

$\rightarrow \left.\begin{array}{l} A_+ \text{ or } A_- \\ A_c \text{ or } A_s \end{array}\right\} \rightarrow$ set by initial condition!

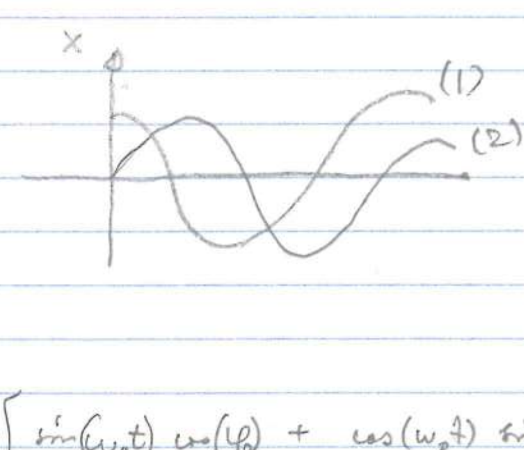
Suppose $t=0$, $\xrightarrow{v=0}$ $x(0) = A_c = x_0$ ← initial position

$$v = \frac{dx}{dt} = -\omega_0 A_c \sin(\omega_0 t) + \omega_0 A_s \cos(\omega_0 t)$$

$$@\, t=0 \rightarrow v = \omega_0 A_s = 0 \rightarrow A_s = 0$$

$\rightarrow \boxed{x(t) = x_0\cos(\omega_0 t)}$ (1) ← Using this boundary condition!

or $\boxed{x(t) = \frac{v_0}{\omega_0}\sin(\omega_0 t)}$ (2)



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Nov 19, 2019

Phase-shifted Cosine sol^n

$$x(t) = A\sin(\omega_0 t + \varphi_0) = A\left[\sin(\omega_0 t)\cos(\varphi_0) + \cos(\omega_0 t)\sin(\varphi_0)\right]$$

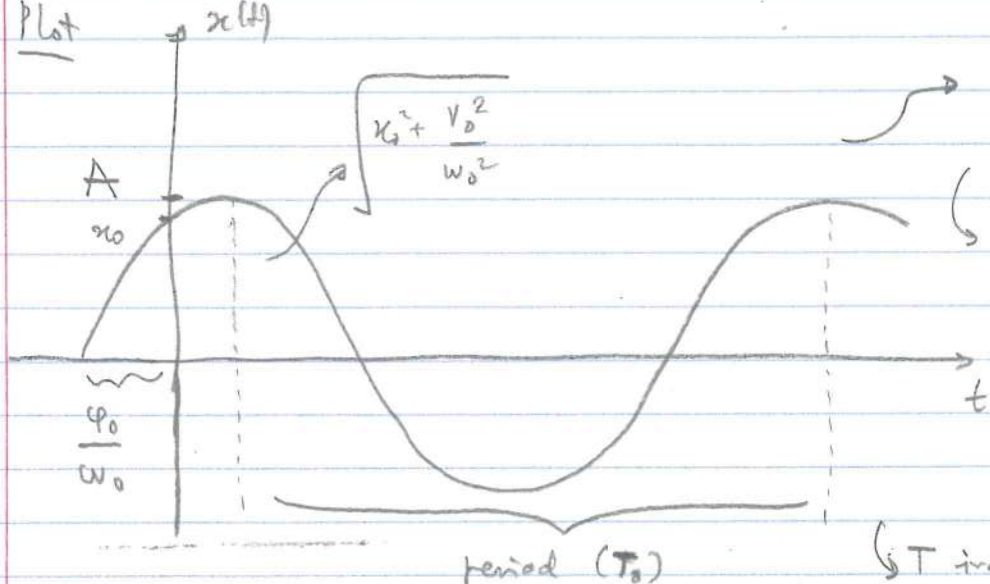
So $= \left[A\sin(\varphi_0)\right]\cos\omega_0 t + \left[A\cos(\varphi_0)\right]\sin(\omega_0 t)$

$$\boxed{x(t) = x_0\cos(\omega_0 t) + \left(\frac{v_0}{\omega_0}\right)\sin(\omega_0 t)}$$

So $A = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$ → so $\boxed{\tan \varphi_0 = \frac{x_0 \omega_0}{v_0}}$

and $\boxed{x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}} \cdot \sin(\omega_0 t + \varphi_0)}$

Plot



→ Periodic $\boxed{x(t+T) = x(t)}$

↳ $A \sin(\omega_0(t+T) + \varphi_0) = A\sin(\omega_0 t + \varphi_0)$

↳ $\omega_0 T = 2\pi$

↳ $\boxed{T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}}$

↳ T independent of A

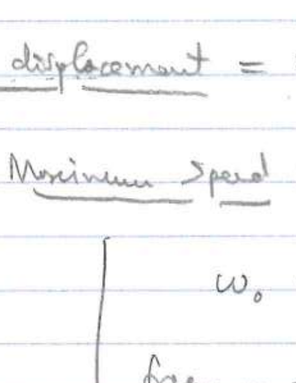
Max displacement $= A =$ amplitude $= \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}$

Maximum Speed $= v_{max} = \omega_0 A = (v_0^2 + \omega_0^2 x_0^2)^{1/2}$

ω_0 is the natural freq of osc → rad/s (angular)

frequency → $f_0 = \frac{\omega_0}{2\pi}$, $T = \frac{1}{f_0} = \frac{2\pi}{\omega_0}$

Ex Bottle in Bucket



Bottle is floating upright in a bucket of H_2O. In eq, it's submerged to a depth d_0 below the surface of water. Show if it's pushed down to a depth d and released

↳ it'll execute SHM

Find ω_0.

2 forces $mg\downarrow$, $\rho g A d\uparrow$ @ E_q: $mg = \rho g A d_0$

Suppose submerged $\rightarrow d = d_0 + x$

$\hookrightarrow mg - \rho g A(d_0+x) = m\ddot{x}$

$\hookrightarrow m\ddot{x} = mg - \rho g A(d_0+x) = \cancel{mg} - \cancel{\rho g A d_0} - \rho g A x$

$\rightarrow m\ddot{x} = -\rho g A x \rightarrow \boxed{\ddot{x} = -\frac{g}{d_0}x}$ (SHM)

Freq $= \omega_0 = \sqrt{\frac{g}{d_0}}$ $\hookrightarrow \tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{d_0}{g}} = 2\pi\sqrt{\frac{0.02m}{9.8m/s^2}} = 0.95$

Energy of SHO

$\hookrightarrow x(t) = A\sin(\omega_0 t + \varphi_0)$

$$E = T + V = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}m\left(\omega_0 A\cos(\omega_0 t+\varphi_0)\right)^2 + \frac{1}{2}k\left(A\sin(\omega_0 t+\varphi_0)\right)^2$$

$$= \frac{1}{2}m\omega_0^2A^2\cos^2(\omega_0 t+\varphi_0) + \frac{1}{2}kA^2\sin^2(\omega_0 t+\varphi_0)$$

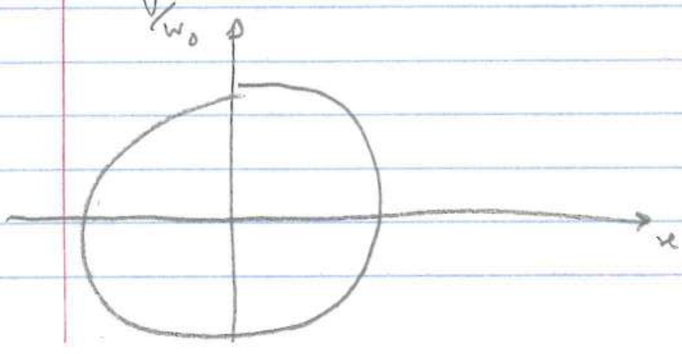
$$= \frac{1}{2}m\frac{k}{m}A^2\cos^2(\omega_0 t+\varphi_0) + \frac{1}{2}kA^2\sin^2(\omega_0 t+\varphi_0)$$

$$= \boxed{\frac{1}{2}kA^2}$$

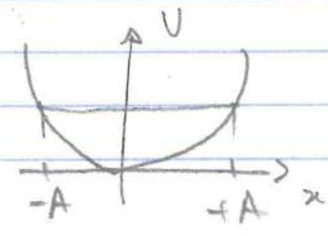
E is constant $\rightarrow$ Spring force is conservative.

$\hookrightarrow E = \frac{1}{2}kA^2 = \frac{1}{2}m\omega_0^2A^2 = \frac{1}{2}mv_{max}^2 = T_{max}$

$$E = \frac{1}{2}m\omega_0^2\left(\frac{v_0^2}{\omega_0^2} + x^2\right)$$



and



<u>2D oscillator</u> → Isotropic → $F = -k\mathbf{r} \rightarrow \begin{cases} F_x = -kx \\ F_y = -ky \end{cases}$

↳ Anisotropic ↳ $F = \begin{cases} -k_x x \\ -k_y y \end{cases}$ $k_x \neq k_y \rightarrow$ | richer motion |

$\ddot{\vec{r}} = \frac{\vec{F}}{m} \Rightarrow \ddot{x} = -\omega_0^2 x \;,\; \ddot{y} = -\omega_0^2 y$ (Isotropic ...)

$\sqrt{\frac{k}{m}}$

<u>Solution</u> $x(t) = A_x \sin(\omega_0 t + \varphi_{0x})$ (4 unknowns, ...)

$y(t) = A_y \sin(\omega_0 t + \varphi_{0y})$ $(A_x, A_y, \varphi_x, \varphi_y)$

~ determined by initial condition

Redefine t $x(t) = A_x \sin(\omega_0 t)$

$y(t) = A_y \sin(\omega_0 t + \varphi)$ where $\varphi = \varphi_x - \varphi_y$

relative phase ...

different φ →

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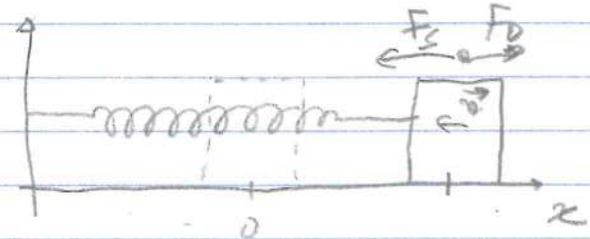
$\boxed{\text{Damped harmonic oscillator}}$ → "Damped" – Resistive force taken into account.

<u>Ex</u> mass + spring on horizontal surface or oil bath → linear drag



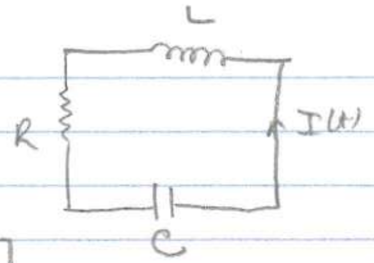
$$\boxed{m\frac{d^2x}{dt^2} = -kx - c\frac{dx}{dt}}$$

↳ $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$ let $\omega_0^2 = \frac{k}{m}$, $2\beta = \frac{c}{m}$

natural freq damping constant

↳ $\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0}$

Recall LRC circuit



$I(t) = \dot{q}(t)$

Inductor $L\dot{I} = L\ddot{q}$

Resistor $IR = R\dot{q}$

Cap $\frac{q}{C}$

$$\boxed{L\ddot{q} + R\dot{q} + \frac{q}{C} = 0} \leftarrow \text{same form}$$

$L \leftrightarrow m$

$\frac{R}{L} \leftrightarrow 2\beta$

$\frac{1}{CL} \leftrightarrow k/m$

Trial solution

$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x \longrightarrow x = Ae^{pt}$ auxiliary eqn

$\ddot{x} = p^2 A e^{pt} \rightarrow p^2 + 2\beta p + \omega_0^2 = 0$

$\dot{x} = pAe^{pt}$

$p = \frac{-2\beta \pm \sqrt{4\beta^2 - 4\omega_0^2}}{2}$

$p = \frac{-\beta \pm \sqrt{\beta^2 - \omega_0^2}}{1}$

So $\boxed{p = -\beta \pm \sqrt{\beta^2 - \omega_0^2}} = -\beta \pm q$

Define $q = +\left(\beta^2 - \omega_0^2\right)^{1/2}$

$= \left(-(\omega_0^2 - \beta^2)\right)^{1/2}$

$= \left[-\omega_0^2\left(1 - \beta^2/\omega_0^2\right)\right]^{1/2} = i\omega_0\left(1 - \beta^2/\omega_0^2\right)^{1/2} = i\omega_1$

where $\omega_1 = \left(1 - \frac{\beta^2}{\omega_0^2}\right)^{1/2}\omega_0$

Gen solution $\boxed{\text{For } \beta > \omega_0} \rightarrow x(t) = A_+ e^{(-\beta+q)t} + A_- e^{(-\beta-q)t}$

$\hookrightarrow \boxed{x(t) = e^{-\beta t}\left(A_+ e^{\sqrt{\beta^2-\omega_0^2}\, t} + A_- e^{-\sqrt{\beta^2-\omega_0^2}\, t}\right)}$

(a) Undamped → no damping → $\beta = 0$ → $x(t) = A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t}$

SHO

(b) Under-damped (weak damping) β is small

$\hookrightarrow \boxed{\beta < \omega_0}$ $\boxed{x(t) = A_+ e^{-\beta t} e^{i\omega_1 t} + A_- e^{-\beta t} e^{-i\omega_1 t}}$

$$= e^{-\beta t}\left[A_+ e^{i\omega_1 t} + A_- e^{-i\omega_1 t}\right]$$

↳ rewrite

$$\boxed{x(t) = e^{-\beta t}\left[A_c \cos(\omega_1 t) + A_s \sin(\omega_1 t)\right]}$$

Boundary conditions

$$\left.\begin{aligned} \dot{x}(0) &= v_0 \\ x(0) &= x_0 \end{aligned}\right\}$$

Under-~~Endamped~~damped

$$\left\{\begin{aligned} &\cancel{x(0) = A_c \cos(\omega_1 \cdot 0) + A_s \sin(\omega_1 0) = A_c = x_0} \\ &\cancel{\dot{x}(0) = -\beta\left[A_c\right.} \end{aligned}\right.$$

$$\left\{\begin{aligned} x(0) &= A_c = x_0 \\ \dot{x}(t) &= -\beta e^{-\beta t}\left[A_c \cos(\omega_1 t) + A_s \sin(\omega_1 t)\right] + e^{-\beta t}\left[-A_c \omega_1 \sin(\omega_1 t) + \omega_1 A_s \cos(\omega_1 t)\right] \end{aligned}\right.$$

$$\dot{x}(0) = -\beta\left(A_c \cos(0)\right) + \omega_1 A_s \cos(\omega_1 0)$$

$$= \boxed{-\beta A_c + \omega_1 A_s = V_0}$$

$$x(0) = \boxed{A_c = x_0}$$

$$\rightarrow V_0 = -\beta x_0 + \omega_1 A_s$$

$$\Rightarrow \boxed{A_s = \frac{V_0 + \beta x_0}{\omega_1}}$$

So $\boxed{A_c = x_0}$ and $\boxed{A_s = \frac{1}{\omega_1}\left(v_0 + \beta x_0\right)}$

Full Under-damped s/n

$$\boxed{x(t) = e^{-\beta t}\left[x_0 \cos(\omega_1 t) + \frac{1}{\omega_1}(v_0 + \beta x_0) \sin(\omega_1 t)\right]}$$

↳ rewrite Let $x_0 = A' A_C$, $\frac{1}{\omega_1}(v_0 + \beta x_0) = A' A_S$ where $A_c^2 + A_s^2 = A'^2$

(53)

$$\boxed{x(t) = e^{-\beta t} A' \sin(\omega_1 t + \varphi_0')}$$ where $A' = \sqrt{A_c^2 + A_s^2}$

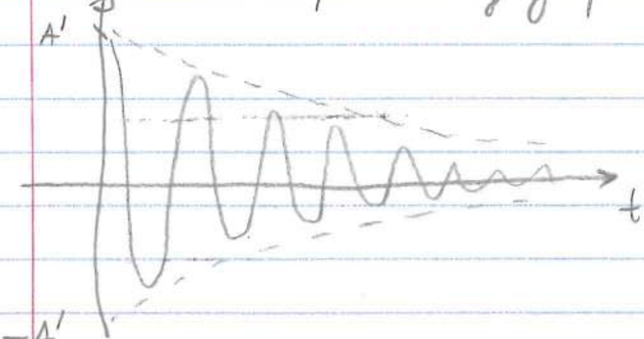
and $\varphi_0' = \tan^{-1}\left(\dfrac{A_c}{A_s}\right)$ $\qquad = \sqrt{x_0^2 + \dfrac{1}{\omega_1^2}(v_0 + \beta x_0)^2}$

$$= \tan^{-1}\left(\frac{x_0 \omega_1}{v_0 + \beta x_0}\right)$$

Note A_c plays the role of sin( ) & A_s plays cos( )

Note solution describes SHM w/ freq ω_1 with exp decreasing amplitude $A' e^{-\beta t}$

β = decaying parameter.



Interpretation of β:

$1/\beta$ → time in which the amplitude for $A' e^{-\beta t}$ falls to $1/e$ of its initial value.

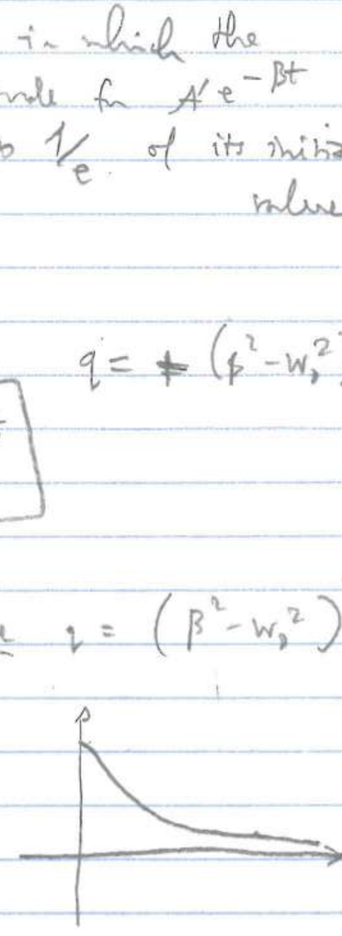
$\boxed{\text{STRONG DAMPING CASE}}$ $(\beta > \omega_0)$

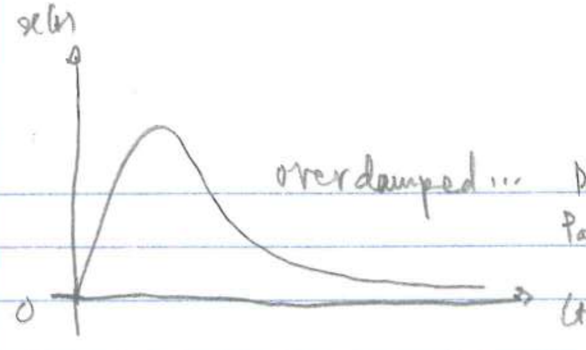
↳ ω_1 is imaginary → $q = i\omega_1$ is real $\qquad q = \pm(\beta^2 - \omega_0^2)^{1/2}$

$$\boxed{x(t) = A_+ e^{-(\beta - \sqrt{\beta^2 - \omega_0^2})t} + A_- e^{-(\beta + \sqrt{\beta^2 - \omega_0^2})t}}$$

↳ $\boxed{x(t) = e^{-\beta t}\underbrace{\left[A_+ e^{qt} + A_- e^{-qt}\right]}_{\text{no } i \,\to\, \text{no oscillation}}}$ where $q = (\beta^2 - \omega_0^2)^{1/2}$

Two real exp → both decay over time (t) } → Complete no oscillation





overdamped... Decay Param $\boxed{\beta - \sqrt{\beta^2 - w_0^2}} = \beta - q$

$\boxed{\text{CRITICAL DAMPING}}$ → boundary between underdamped & overdamped

when $\boxed{\beta = w_0}$ → $w_1 = q = 0$

$$\boxed{x(t) = e^{-\beta t}\left(A_+ + A_-\right) = A e^{-\beta t}}$$

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Note only 1 solution, since there's only 1 root in the auxiliary eqn for $\beta = w_0$

Our guess solution $x = e^{\beta t}$ failed.

This happens because the two solutions of the auxillary eqn coincide, when $\beta = w_0$ → our trial solution fails! $x(t) = e^{\beta t}$

To find 2^{nd} solution, check $\boxed{x(t) = te^{-\beta t}}$ → satisfies!

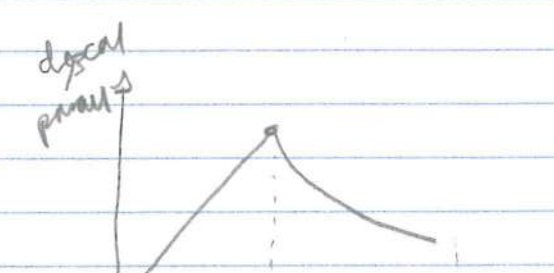
E.o.M $\boxed{\ddot{x}(t) + 2\beta\dot{x} + w_0^2 x = 0}$ → 2^{nd} order ODE → needs 2 solutions

General solution $\boxed{x(t) = c_1 e^{-\beta t} + c_2 \cdot t \cdot e^{-\beta t} = (c_1 + c_2 t)e^{-\beta t}}$

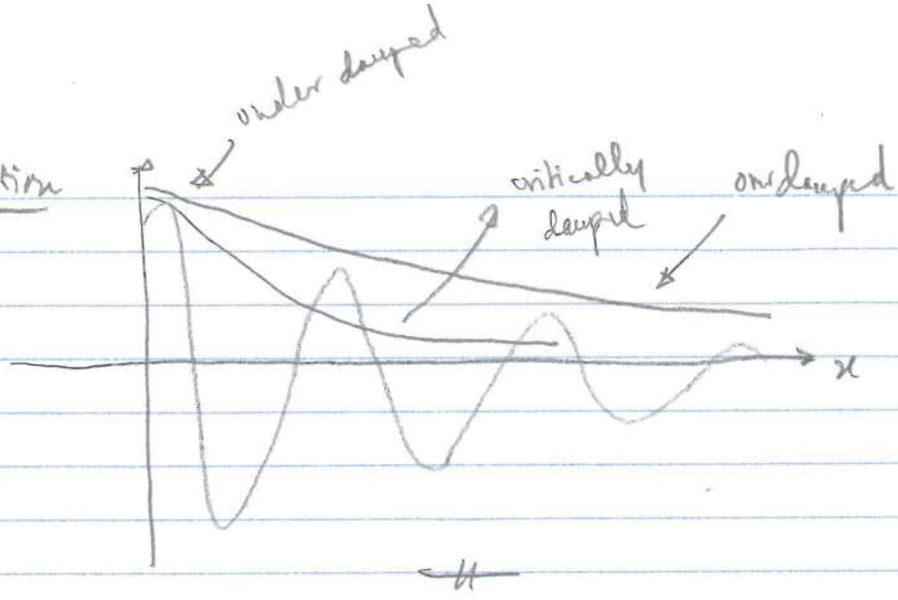
Both terms decay at the same rate.

Compare various type of damping

	decay param
Undamped	0
underdamped	$\beta < w_0$
overdamped	$\beta > w_0$, $\beta - \sqrt{\beta^2 - w_0^2}$
critically damped	$\beta = w_0$

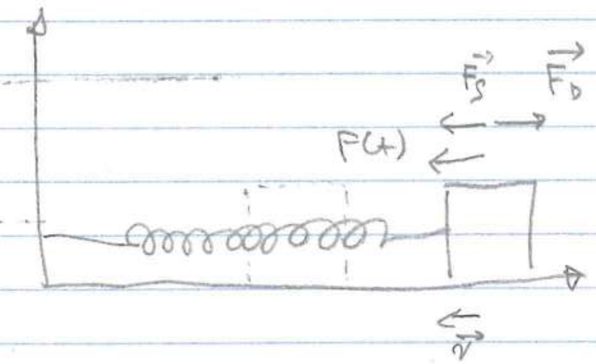


Motion



$\boxed{\text{Driven and damped SHM}}$ → any natural oscillator with damping forces come to a rest, So to continue the osc, we need another external driving force to maintain motion.

Ex mass on spring in oil bath with another time-dependent driving force



$$\vec{F} = \vec{F_s} + \vec{F_D} + \vec{F}(t)$$

external driving force

So $\boxed{m\ddot{x} = -kx - c\dot{x} + F(t)}$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \rightarrow \boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t)}$$

inhomogeneous 2^{nd} order ODE

Review $a_2\ddot{x} + a_1\dot{x} + a_0 x = F(t)$

Eqn has 2^{sol}: $x(t) = x_t + x_s$ → time-independent? / → time-dependent.

$$\left.\begin{aligned}\dot{x}(t) &= \dot{x}_t + \dot{x}_s\\ \ddot{x}(t) &= \ddot{x}_t + \ddot{x}_s\end{aligned}\right\} \rightarrow \text{plug in} \quad a_2(\ddot{x}_t + \ddot{x}_s) + a_1(\dot{x}_t + \dot{x}_s) + a_0(x_t + x_s)$$

→ Can write

$$a_2\ddot{x}_t + a_1\dot{x}_t + a_0 x_t = 0 \rightsquigarrow \underbrace{x_t = e^{-\beta t}\left(A_+e^{i\omega_0 t} + A_-e^{-i\omega_0 t}\right)}$$

↓ Transient sol $x(t) \to 0$ as $t \to \infty$

particular eqn ← $\boxed{a_2\ddot{x}_s + a_1\dot{x}_s + a_0 x_s = F(t)}$

Solution x_s is called the "STEADY STATE SOLUTION"
↳ generally does not die away!

$\boxed{\text{Solution to } F(t) = \cos(\ )}$

↳ let $f(t) = \frac{F(t)}{m} = f_0 \cos(\omega t)$ → ω: driving angular frequency, ω_0 (different from the natural freq)

So $\boxed{\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f_0 \cos(\omega t)}$

Guess solution → $x(t) = A\cos(\omega t - \varphi)$, A = amplitude, φ = phase shift

$\dot{x}(t) = -A\omega \sin(\omega t - \varphi)$
$\ddot{x}(t) = -A\omega^2 \cos(\omega t - \varphi)$

So $\boxed{-A\omega^2 \cos(\omega t - \varphi) + (-2\beta) A\omega \sin(\omega t - \varphi) + \omega_0^2 A \cos(\omega t - \varphi) = f_0 \cos(\omega t)}$

$\cos(\omega t - \varphi) = \cos(\omega t)\cos(\varphi) + \sin(\omega t)\sin(\varphi)$
$\sin(\omega t - \varphi) = \sin(\omega t)\cos(\varphi) - \cos(\omega t)\sin(\varphi)$

↳ $-A\omega^2\left[\cos(\omega t)\cos(\varphi) + \sin(\omega t)\sin(\varphi)\right] - 2\beta A\omega\left[\sin(\omega t)\cos(\varphi) - \cos(\omega t)\sin(\varphi)\right]$
$+ \omega_0^2 A\left[\cos(\omega t)\cos(\varphi) + \sin(\omega t)\sin(\varphi)\right] = f_0 \cos(\omega t)$

$\Rightarrow \sin(\omega t)\left[-A\omega^2 \sin(\varphi) - 2\beta A\omega \cos(\varphi) + \omega_0^2 A \sin(\varphi)\right]$
$+ \cos(\omega t)\left[-A\omega^2 \cos(\varphi) + 2\beta A\omega \sin(\varphi) + \omega_0^2 A \cos(\varphi)\right] = f_0 \cos(\omega t)$

$$\Rightarrow \begin{cases} A\cos(\omega t)\left[(-\omega^2 + \omega_0^2)\cos\varphi + 2\beta\omega \sin\varphi\right] = f_0 \cos(\omega t) \\ \left[(-\omega^2 + \omega_0^2)\sin\varphi - 2\beta\omega\cos(\varphi)\right]A = 0 \end{cases}$$

$$\Rightarrow \begin{cases} A\left[(-\omega^2 + \omega_0^2)\cos\varphi + 2\beta\omega\sin\varphi\right] = f_0 \\ (-\omega^2 + \omega_0^2)\sin\varphi - 2\beta\omega\cos(\varphi) = 0 \Rightarrow \boxed{\tan(\varphi) = \frac{2\beta\omega}{\omega_0^2 - \omega^2}} \end{cases}$$

<u>Since</u> $\tan\varphi = \dfrac{2\beta\omega}{\omega_0^2-\omega^2}$

$\hookrightarrow$ $\sin\varphi = \dfrac{2\beta\omega}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2}}$ and $\cos\varphi = \dfrac{\cancel{2\beta\omega}\ \omega_0^2-\omega^2}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2}}$

<u>Plug in...</u>

$$\hookrightarrow (\omega_0^2-\omega^2)\frac{(\omega_0^2-\omega^2)}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2}} - (2\beta\omega)\frac{(2\beta\omega)}{\left[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right]^{1/2}} = \frac{f_0}{A}$$

$$\hookrightarrow \frac{\left((\omega_0^2-\omega^2)^2+4\beta^2\omega^2\right)}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}} = \frac{f_0}{A} \rightarrow \boxed{A(\omega) = \frac{f_0}{\sqrt{(\omega_0^2-\omega^2)^2+4\beta^2\omega^2}}}$$

<u>with</u> $\varphi = \tan^{-1}\left(\dfrac{2\beta\omega}{\omega_0^2-\omega^2}\right)$

transient soln

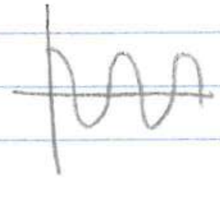
<u>full soln</u>
$$\boxed{x(t) = e^{-\beta t}\left(A_+e^{i\omega_0 t}+A_-e^{-i\omega_0 t}\right) + \underbrace{A(\omega)\cos(\omega t-\varphi)}_{\text{steady state}}}$$

<u>Can also use</u>

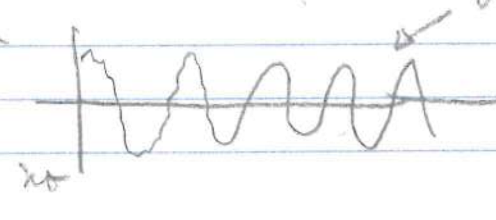
$$\hookrightarrow \boxed{x(t) = A\cos(\omega t-\varphi) + e^{-\beta t}\left[B_1\cos(\omega_1 t)+B_2\sin(\omega_1 t)\right]}$$

<u>where</u> $\omega_1 = \sqrt{\omega_0^2-\beta^2}$

<u>if</u> $f = \ldots$



then



like $f(t)$

April 2, 2018 driving force $f(t) = f_0 \cos(\omega t)$

other than transient motion, which dies down quickly, the system responses to oscillate sinusoidally, @ the same freq ω

$$x(t) = A\cos(\omega t - \delta)$$

A = amplitude, δ = phase shift

Today ↳ $\boxed{\text{Resonance}}$

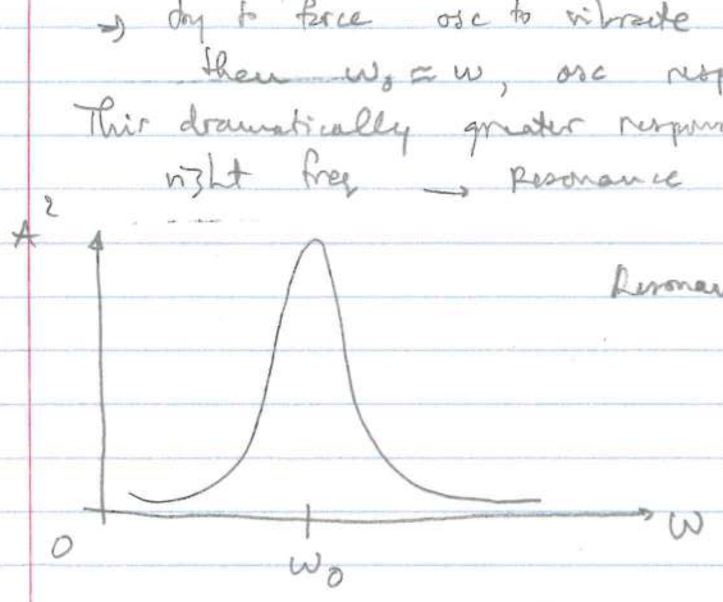
(a) Amplitude resonance A depends on driving freq ω

↳ ω at which A = max, osc vibrates @ natural freq

⇒ try to force osc to vibrate at ω

then $\omega_0 \approx \omega$, osc responds well.

This dramatically greater response of an osc when driven @ right freq → Resonance



Resonance occurs for $\omega_0 = \omega$

Recall $A(\omega) = \dfrac{F_0/m}{\sqrt{(\omega^2-\omega_0^2)^2 + (2\beta\omega)^2}} = \dfrac{f_0}{\left((\omega_0^2-\omega^2)^2 + (2\beta\omega)^2\right)^{1/2}}$

$$\frac{dA}{d\omega} = f_0 \cdot \left(-\frac{1}{2}\right)\left[(\omega^2-\omega_0^2)^2 + (2\beta\omega)^2\right]^{-\frac{3}{2}} \cdot \left(2(\omega^2-\omega_0^2)2\omega + 4\beta^2 \cdot 2\omega\right)$$

↳ So $\dfrac{dA}{d\omega} = 0 \Rightarrow \boxed{\omega = 0}$ or $2\omega^2 - 2\omega_0^2 = -4\beta^2 \Rightarrow \boxed{\omega = \sqrt{-2\beta^2 + \omega_0^2}}$

not natural freq

So $\boxed{\omega = 0}$ or $\boxed{\omega_r = (\omega_0^2 - 2\beta^2)^{1/2}}$ ← resonance freq → obtain max amplitude

$\boxed{\text{Velocity resonance}}$ → at which ω do we get maximum velocity?

where $A = A(\omega)$

$$x(t) = A\cos(\omega t - \delta) \rightarrow \dot{x}(t) = -A\omega \sin(\omega t - \delta)$$

<u>Define</u> $\boxed{v(\omega) = \omega A(\omega)}$ ← velocity amplitude

Max v occurs when $\frac{dv}{d\omega} = 0 = \dot{\omega} A + \omega \dot{A}$

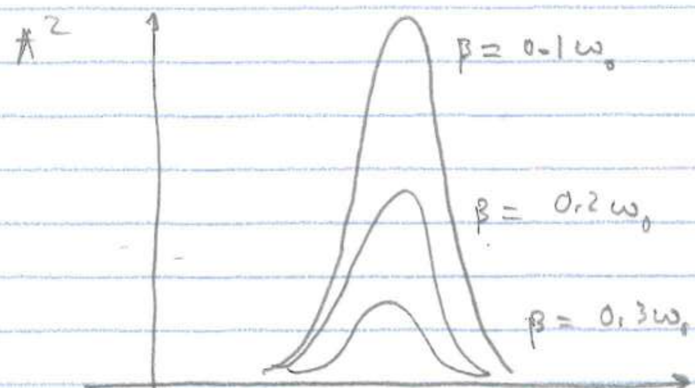
<u>So</u> $A + \omega . \frac{dA}{d\omega} = 0 \Leftrightarrow$

$$\Leftrightarrow \frac{F_0}{m}\left((\omega^2-\omega_0^2)^2 + (2\beta\omega)^2\right)^{-1/2} + \omega . \left(\frac{-1}{2}\right)\left(\frac{F_0}{m}\right) \frac{2(\omega^2-\omega_0^2)2\omega + 4\beta^2 2\omega}{\left((\omega^2-\omega_0^2)^2 + (2\beta\omega^2)^2\right)^{3/2}}$$

$$\Leftrightarrow \left[(\omega^2-\omega_0^2)^2 + (2\beta\omega)^2\right] - 2\omega^2(\omega^2-\omega_0^2) - 4\beta^2\omega^2 = 0$$

$\Rightarrow \boxed{\omega = \omega_0}$ → obtain max vel of system.

<u>Width of resonance, Q - factor</u>



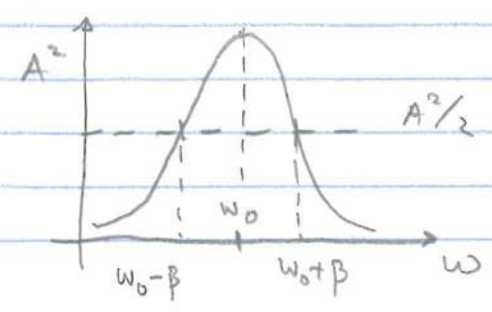
3 different damping const

↳ $\beta \uparrow$, resonance peak gets higher & sharper.

↳ <u>width</u> FWHM

↳ interval between 2 pts where $A^2 = \frac{1}{2}$ max height

<u>FWHM</u>



<u>FWHM</u> $\boxed{\Delta\omega = (\omega_0+\beta) - (\omega_0 - \beta) = 2\beta}$

↳ <u>So</u> HWHM → $\Delta\omega = \beta$

$\boxed{Q\text{-factor}} \rightarrow \boxed{\dfrac{\omega_0}{2\beta}}$ Narrow/sharp peak → needs large Q
→ β small

- Typical pendulum (grandfather clock)
  ↳ $Q = 100$
- Watch → $Q = 10{,}000$

#

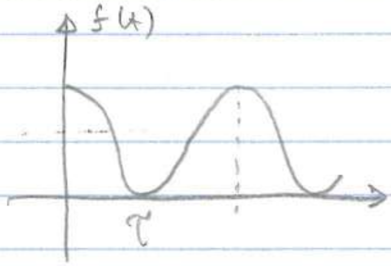
FOURIER SERIES

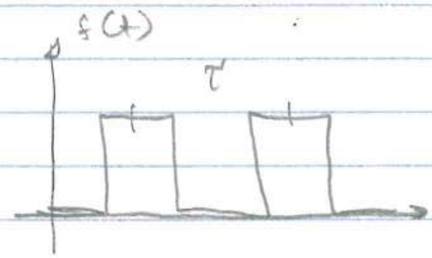
→ any periodic function (or driving force) can be built-up from sinusoidal forces using Fourier series

↳ Consider $f(t)$ → periodic with period τ

So $f(t+\tau) = f(t)$

Ex





Fourier: "Every τ-periodic fn can be written as a linear combo of sines & cosines ($\cos(n\omega t)$ & $\sin(n\omega t)$, $n = 0, 1, 2, \ldots$, $\omega = \frac{2\pi}{\tau}$)

Ex $$\boxed{f(t) = \sum_{n=0}^{\infty}\left[a_n \cos(n\omega t) + b_n \sin(n\omega t)\right]}$$ fourier series

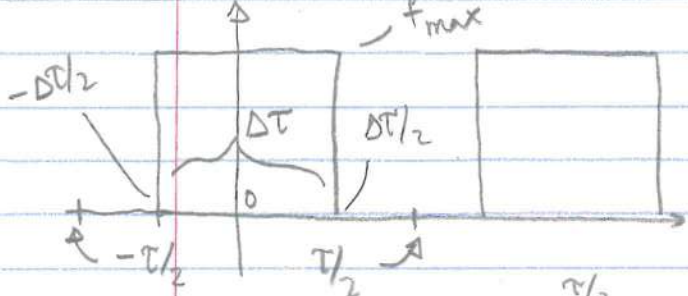
where $$\boxed{a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)\,dt} \quad (n \geq 1)$$

when $n = 0$, $b_0 = 0$

$$\boxed{a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt}$$

$$\boxed{b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)\,dt} \quad (n \geq 1)$$

Ex For rectangular pulse $\quad a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt$



So $\quad a_0 = \frac{1}{\tau}\int_{-\Delta\tau/2}^{\Delta\tau/2} f_{max}\,dt = \boxed{\frac{f_{max}\,\Delta\tau}{\tau} = a_0}$

Now $\quad a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\cos(n\omega t)\,dt = \frac{2f_{max}}{\tau}\int_{-\Delta\tau/2}^{\Delta\tau/2}\cos(n\omega t)\,dt \quad (n \geq 1)$

April 3, 2018 $\quad$ and $\quad b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)\,dt \quad (n \geq 1)$

Fourier series for rectangular pulse in terms of τ, pulse height (f_{max}), and pulse width ($\Delta\tau$). ~~Get~~ Plot $f(t)$ for the sum of the first 3, 11 terms of the series.

$$f(t) = \begin{cases} f_{max} & \text{if} \\ 0 & \text{if} \end{cases}$$

$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt = \frac{1}{\tau}\int_{-\Delta\tau/2}^{\Delta\tau/2} f_{max}\,dt = \frac{1}{\tau} f_{max}\left(\frac{\Delta\tau}{2} - \left(-\frac{\Delta\tau}{2}\right)\right) = \boxed{\frac{f_{max}\,\Delta\tau}{\tau}}$$

$$a_n = \frac{2}{\tau}\int_{-\Delta\tau/2}^{+\Delta\tau/2} f_{max}\cdot\cos(n\omega t)\,dt = \frac{2f_{max}}{\tau}\int_{-\Delta\tau/2}^{\Delta\tau/2}\cos(n\omega t)\,dt \qquad \text{Note } \omega = \frac{2\pi}{\tau}$$

$$= 2\left(\frac{2f_{max}}{\tau}\right)\int_0^{\Delta\tau/2}\cos\left(\frac{2\pi}{\tau}nt\right)dt$$

$$= \frac{4f_{max}}{\tau}\left(\frac{\tau}{2\pi n}\right)\sin\left(\frac{2\pi}{\tau}nt\right)\Bigg|_0^{\Delta\tau/2} = \frac{4f_{max}}{\tau}\left(\frac{\tau}{2\pi n}\right)\cdot\sin\left(\frac{\pi n\Delta\tau}{\tau}\right)$$

So $\boxed{a_n = \frac{2f_{max}}{\pi n}\sin\left(\frac{\pi n\,\Delta\tau}{\tau}\right)}$

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odd fn

$$\cdot\ b_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\sin(n\omega t)\,dt = \frac{2f_{max}}{\tau}\int_{-\Delta\tau/2}^{\Delta\tau/2} \sin(n\omega t)\,dt = 0$$

So $\boxed{b_n = 0}$

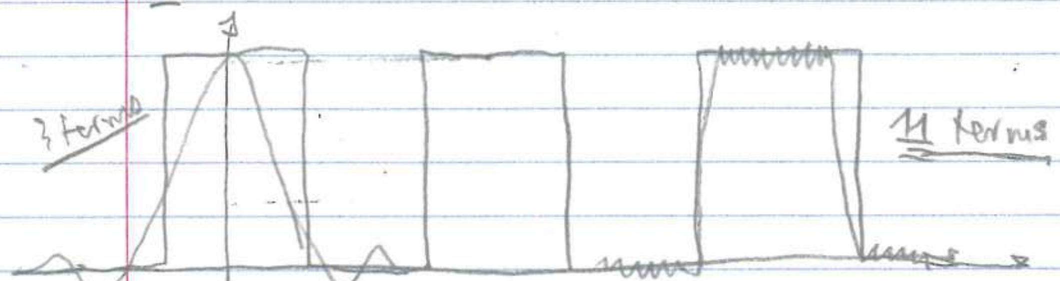
$$\cdot\ f(t) = \sum_{n=1}^{\infty}\left[a_n\cos(n\omega t) + \cancel{b_n}^{\,0}\sin(n\omega t)\right] + a_0$$

$$= \sum_{n=1}^{\infty}\left[\frac{2f_{max}}{\pi n}\sin\left(\frac{\pi n\Delta\tau}{\tau}\right)\cdot\cos(n\omega t)\right] + \left(\frac{f_{max}\Delta\tau}{\tau}\right)$$

$$= \left(\frac{f_{max}\Delta\tau}{\tau}\right) + \sum_{n=1}^{\infty}\left[\frac{2f_{max}}{\pi n}\sin\left(\frac{\pi n\Delta\tau}{\tau}\right)\cdot\cos\left(\frac{n2\pi}{\tau}t\right)\right]$$

For $n = 11$, $f(t) = 0.25 + 0.45\cos(2\pi t) + 0.32\cos(4\pi t) +$
$0.15\cos(6\pi t) - 0.05\cos(10\pi t) - 0.11\cos(12\pi t) + \ldots$

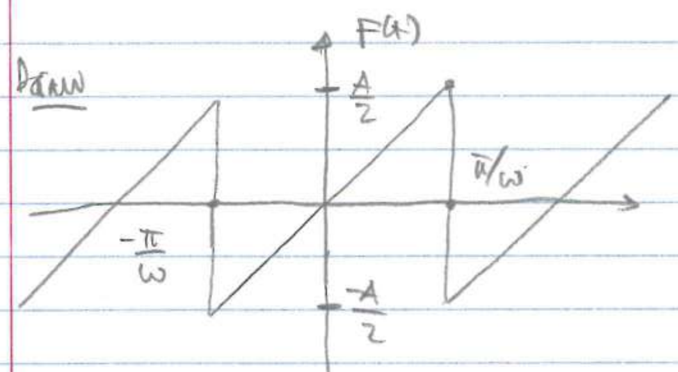
So



practice

$\hookrightarrow$ find Fourier series $F(t) = \frac{At}{T} = \frac{\omega A}{2\pi}t$, $\frac{-\pi}{\omega} < t \leq \frac{\pi}{\omega}$

Draw



$$a_0 = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} f(t)\,dt = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\frac{\omega A}{2\pi}t\,dt$$

$$= \frac{1}{\tau}\int_{-\tau/2}^{\tau/2}\left(\frac{\omega A}{2\pi}\right)t\,dt$$

$$= \left(\frac{1}{\tau}\right)\left(\frac{\omega A}{2\pi}\right)\frac{1}{2}t^2\Big|_{-\tau/2}^{\tau/2} = 0$$

So $\boxed{a_0 = 0}$

odd

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(n\omega t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} \frac{\omega A}{2\pi} t \cdot \cos(n\omega t)\,dt = 0$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} \frac{\omega A}{2\pi}\sin(n\omega t)\,dt = \frac{2}{T}\int_{-T/2}^{T/2} \frac{\omega A}{2\pi} t \cdot \sin(n\omega t)\,dt = \frac{2}{T}\frac{\omega A}{2\pi}\int_{-T/2}^{T/2} t\sin(n\omega t)\,dt$$

$$= \frac{\omega A}{2\pi}\frac{4}{T}\int_{-T/2}^{T/2} t\sin(n\omega t)\,dt = \left(\frac{\omega A}{2\pi}\frac{4}{T}\right)\left[-t\cos(n\omega t)\frac{1}{n\omega}\Big|_0^{T/2} - \int_0^{T/2} -\frac{1}{n\omega}\cos(n\omega t)\,dt\right]$$

take $\sin(t)dt = dv \rightarrow v = \frac{-1}{n\omega}\cos(n\omega t)$

$t = u \rightarrow du = dt$

$$= \frac{2\omega A}{\pi T}\left[-\frac{T}{2}\cos\left(n\omega\frac{T}{2}\right) + \frac{1}{n\omega}\frac{1}{n\omega}\sin(n\omega t)\Big|_0^{T/2}\right]$$

$$= \frac{2\omega A}{\pi T}\left(-\frac{T}{2}\cos\left(n\omega\frac{T}{2}\right) + \frac{1}{n^2\omega^2}\sin\left(n\omega\frac{T}{2}\right)\right)$$

where $\omega = \frac{2\pi}{T}$ so simplify...

$$b_n = \frac{A}{2\pi^2 n}\cdot(-2\pi n)\cos(\pi n) \Rightarrow b_n = \frac{-A}{\pi^2 n^2}\pi n\cos(\pi n)$$

so $\boxed{b_n = \frac{A}{n\pi}(-1)^{n+1}}$

so

$$\boxed{\begin{aligned} F(t) &= \sum_{n=0}^{\infty}\left[\frac{A}{\pi n}(-1)^{n+1}\cdot\sin(n\omega t)\right] \\ &= \frac{A}{\pi}\left(\frac{\sin\omega t}{1} - \frac{\sin 2\omega t}{2} + \frac{\sin 3\omega t}{3} - \frac{\sin 4\omega t}{4} + \dots\right)\end{aligned}}$$

RMS displacement & Parseval's Theorem

Root mean square (RMS) displacement

$$\boxed{x_{rms} = \sqrt{\langle x^2 \rangle}}$$ where time avg $\langle x^2 \rangle = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} x^2\, dt$

To evaluate this, use Fourier series, use

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega t - \delta_n)$$

So

$$\langle x^2 \rangle = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} \sum_n \sum_m A_n \cos(n\omega t - \delta_n)\, A_m \cos(m\omega t - \delta_m)\, dt$$

where $\int_{-\tau/2}^{\tau/2} \cos(n\omega t - \delta_n)\cos(m\omega t - \delta_m)\, dt = \begin{cases} \tau & m = n = 0 \\ \tau/2 & m = n \neq 0 \\ 0 & m \neq n \end{cases}$

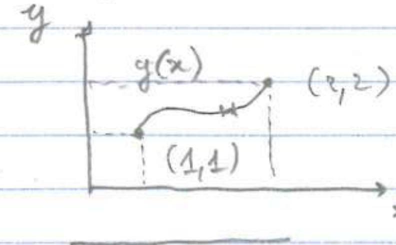
$$\boxed{\langle x^2 \rangle = A_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} A_n^2}$$

April 5, 2018

LAGRANGIAN EQUATION

Calculus of Variation — Variation Principle

Exp Shortest path between 2 points.



2 given points (x_1, y_1), (x_2, y_2) & path $y = y(x)$. Find the path s.t. the length is shortest length.

$ds = \sqrt{dx^2 + dy^2}$ = length of short segment of the path.

$\frac{dy}{dx}dx = y'(x)\,dx$ so $\boxed{ds = \sqrt{1 + y'(x)^2}\; dx}$

So the length of the path between $x_1 \to x_2$

$$= \boxed{L = \int_{x_1}^{x_2} \sqrt{1+(y'(x))^2}\, dx}$$

Problem boils down to finding a $y(x)$ such that L is minimum

Ex <u>Fermat's principle</u> Find the path that light follows between 2 points.

Suppose, the time for light to travel a short distance $ds = \left(\frac{ds}{v}\right)$ where

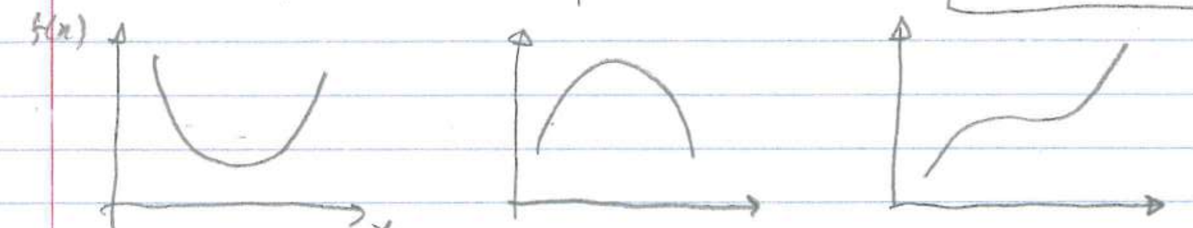
v = speed of light in the medium, with refractive index $= n \Rightarrow v = \frac{c}{n}$

$\hookrightarrow$ The Fermat's principle says that the correct path between 1 & 2 is the path for which the <u>time of travel is minimum</u>

$$\hookrightarrow \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 n\, ds \quad . \text{ In general, } n = n(x,y)$$

$$\hookrightarrow \boxed{T = \int_1^2 dt = \frac{1}{c}\int_{x_1}^{x_2} n(x,y)\sqrt{1+y'(x)^2}\, dx} \text{ is minimum}$$

Necessary condition for max/min of an $f(x) \Rightarrow \boxed{\frac{df}{dx} = 0 \Rightarrow x_0 = \text{stationary point}}$
but insufficient $\Rightarrow$ 3 possibilities.

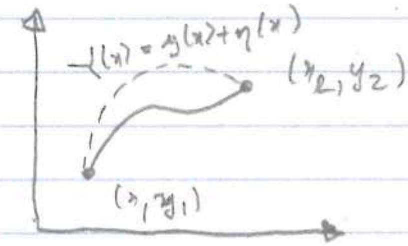


$\frac{d^2f}{dx^2} > 0 \Rightarrow$ <u>Min</u> $\qquad \frac{d^2f}{dx^2} < 0 \Rightarrow$ max $\qquad \frac{d^2f}{dx^2} = 0 \Rightarrow$ don't know

The EULER – LAGRANGE Equation

$$\boxed{S = \int_{x_1}^{x_2} f\left[y(x), y'(x), x\right] dx} \quad \text{— (A)}$$

$y(x)$ = unknown curve joining (x_1, y_1) & (x_2, y_2)



Now, imagine an alternative path

↳ $Y(x) = y(x) + \eta(x)$

and

$$Y(x_1) = y(x_1) = y_1 \ , \ Y(x_2) = y(x_2) = y_2 \quad \text{(B)}$$

Goal

↳ find S min } → Suppose S in (A) is evaluated for $y = y(x)$ is less than for any neighboring curve

→ $Y(x) = y(x) + \eta(x)$

Since $Y(x)$ must satisfy (B) → $\eta(x)$ must satisfy $\eta(x_1) = \eta(x_2) = 0$ and the integral S taken over $Y(x)$ is larger than the "right" path

"wrong curve"

↳ To express this requirement ⟶ introduce param α

↳ redefine $Y(x)$ as $\boxed{Y(x) = y(x) + \alpha \eta(x)}$

The integral S taken along $Y(x)$ – the wrong path – depends on α

↳ $S(\alpha)$ → problem is reduced to $S(\alpha)$ ~~→ find~~ has a min at a specified point → $\frac{dS}{d\alpha} = 0$

$$\boxed{S(\alpha) = \int_{x_1}^{x_2} f(Y, Y', x)\, dx = \int_{x_1}^{x_2} f\left[y + \alpha\eta, y' + \alpha\eta', x\right] dx}$$

$$\frac{\partial f}{\partial \alpha} = \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \Rightarrow \frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial \alpha} dx$$

So $$\boxed{\frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} \right) dx = 0} \rightarrow \text{C}$$

2^{nd} term $$\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'} dx = \underbrace{\left(\eta(x) \frac{\partial f}{\partial y'} \right)_{x_1}^{x_2}}_{0} - \int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx$$

So $$\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'} dx = -\int_{x_1}^{x_2} \eta(x) \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) dx$$

So in C

$$\boxed{\frac{\partial S}{\partial \alpha} = \int_{x_1}^{x_2} \eta(x) \left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) \right] dx = 0}$$ of the form $\int \eta(x) g(x) dx = 0$

Since $\frac{\partial S}{\partial \alpha} = 0 \ \forall \ \eta(x)$, $\boxed{g(x) = 0}$

$$\rightarrow \boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0} \rightarrow \text{Euler - lagrange equation}$$

Example shortest path between 2 points $\rightarrow L = \int_{x_1}^{x_2} \underbrace{\sqrt{1 + y'(x)^2}}_{f(\,)} dx$

So $f = f(y, y', x) = (1 + y'^2)^{1/2}$

$$\frac{\partial f}{\partial y} = 0; \quad \frac{\partial f}{\partial y'} = \frac{1}{2}(1 + y'^2)^{-1/2} \cdot 2y' = y'(1 + y'^2)^{-1/2}$$

plug in $\rightarrow 0 - \frac{d}{dx}\left[y'(1+y'^2)^{-1/2} \right] = 0 \rightarrow y'(1+y'^2)^{-1/2} = \text{constant} = \tilde{C}$

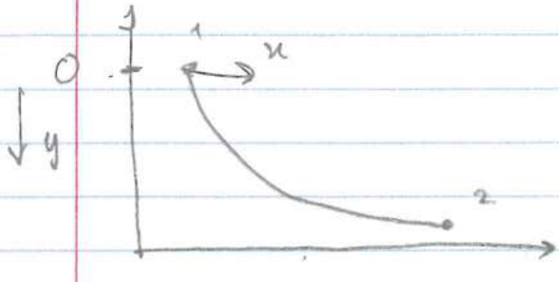
$\rightarrow y'^2 = \tilde{C}^2 \cdot (1 + y'^2)$

$$\hookrightarrow y''(1+y'^2)^{1/2} + y'\left(-\frac{1}{2}\right)(1+y'^2)^{-3/2}(2y') = 0$$

$\rightarrow y'^2 = \text{const}$

$\rightarrow \boxed{y' = m} \rightarrow \boxed{y = Cx + b}$

Ex $\boxed{\text{Brachistochrone}}$ Given 2 points 1 and 2 with 1 higher above the ground → find in what shape should we build a track for a frictionless rollercoaster s.t a car released from (1) reaches (2) in the shortest possible time.



Fermat's principle

$$T = \int dt = \int \frac{ds}{v}$$

where $v = \sqrt{2gy}$ → y no longer independent

So $T = \cancel{\int \frac{1}{\sqrt{2gy}} \sqrt{1+y'^2}\, dx}$ where $\frac{dy}{dx} = y'$

Unknown path is $x = x(y)$

So $$T = \int \frac{1}{\sqrt{2gy}} \sqrt{1+x'^2}\, dy \quad , \quad x' = \frac{dx}{dy} = x'(y)$$

$$= \frac{1}{\sqrt{2g}} \int_{y_1}^{y_2} \underbrace{\frac{\sqrt{1+x'^2}}{\sqrt{y}}}_{f}\, dy$$ → role of x & y has been changed

So $$f[x, x', y] = \frac{1}{\sqrt{y}} \cdot \sqrt{1+x'^2}$$

Euler-Lagrange Eq. of new form $\frac{\partial f}{\partial x} = \frac{d}{dy} \frac{\partial f}{\partial x'}$

So $$\frac{\partial f}{\partial x} = 0 = \frac{d}{dy} \frac{\partial f}{\partial x'} = \frac{d}{dy}\left[\frac{1}{2} \frac{2x'}{(x'^2+1)^{1/2}\sqrt{y}}\right] = 0$$

So $$\frac{x'}{(x'^2+1)^{3/2}\sqrt{y}} = \text{const} = \frac{1}{2a} \Rightarrow (2a x'^2) = (y + y x'^2)$$

any const

So $x'^2(2a-y) = y$

So $x' = \sqrt{\dfrac{y}{2a-y}}$

So $x = \displaystyle\int \sqrt{\frac{y}{2a-y}}\, dy$

Use parametric solution $\rightarrow$ $y = a(1-\cos\theta)$

$$x = \int \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}\; a\sin\theta\, d\theta$$

$$= a\int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\, \sin\theta\, d\theta$$

$$= a\int \sqrt{\frac{(1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)}}\, \sin\theta\, d\theta$$

$$= a\int (1-\cos\theta)\, d\theta$$

$$\boxed{x = a(\theta - \sin\theta) + \text{const}}$$

initially, $x = y = 0 \Rightarrow 0 = \text{const}$

so $\left\{\begin{array}{l} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta) \end{array}\right\} \rightarrow$ Cycloid

April 9, 2018 → Euler-Lagrange Eqn: $\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0}$ → 2 vars: indep (x) & dep (y)

$S(\alpha) \to \frac{dS}{d\alpha} = 0$

$(= y(x) + \alpha\eta(x))$

Maximum - Minimum vs Stationary

{ The E-L eqn guarantees only to give a path for which the original integral is stationary (max/min/neither) }

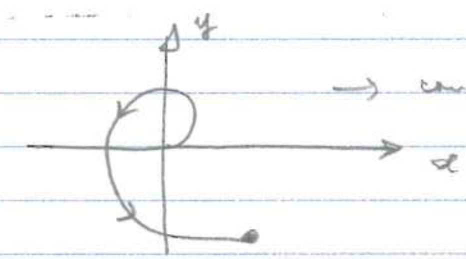
In some ex, obvious for distance between 2 pts on a plane
→ straight lines give minimum distance.

In other cases → Brachistochrone problem → NOT so obvious.

$\boxed{\text{Extend Euler-Lagrange Eqn for more than 2 variables}}$

→ Usually, we only have 1 independent variable, and several dependent variables.

Ex
→ Shortest distance between 2 points. → $y = y(x)$



→ can't be written as $y = y(x)$
or $x = x(y)$

→ parametric form

To find the shortest path among all possible paths.
→ write down paths as
$$\left\{ \begin{array}{l} x = x(u) \\ y = y(u) \end{array} \right\}$$
where u is a parameter (convenient variable in which the curve can be parameterized).

The length of a small segment of path

$$ds = \sqrt{\frac{dx}{du}^2 + \frac{dy}{du}^2}\,du = \sqrt{(x'(u))^2 + (y'(u))^2}\,du$$

Total path length $L = \int_{u_1}^{u_2} \sqrt{x'(u)^2 + y'(u)^2}\, du = \int ds$

Our job is to find 2 functions $x(u)$, $y(u)$ for which L stationary

Integral $\boxed{S = \int_{u_1}^{u_2} f[x(u), y(u), x'(u), y'(u), u]\, du}$

2 fixed point $\to$ $[x(u_1), y(u_1)]$ and $[x(u_2), y(u_2)]$

Goal: Find $[x(u), y(u)]$ such that L stationary.

↳ With 2 dependent variables, we get 2 Euler-Lagrange eqns

↳ Procedure is similar to the 1-var case

Let correct path $= x = x(u)$, $y = y(u)$

2 perturbed path

$$\left.\begin{aligned} x &= x(u) + \varepsilon \xi(u) \\ y &= y(u) + \gamma \eta(u) \end{aligned}\right\}$$

So

$S(\varepsilon, \gamma)$ has to be such that $\frac{\partial S}{\partial \varepsilon} = 0$ and $\frac{\partial S}{\partial \gamma} = 0$

These 2 conditions are equiv. to 2 E-L eqns, which are:

$$\boxed{\frac{\partial f}{\partial y} = \frac{d}{du}\frac{\partial f}{\partial y'}} \quad \text{and} \quad \boxed{\frac{\partial f}{\partial x} = \frac{d}{du}\frac{\partial f}{\partial x'}}$$

Example Shortest dist between 2 points in this generalized case

$$f[x, x', y, y', u] = \sqrt{x'^2 + y'^2}$$

$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \Rightarrow$ So $\left\{ \frac{d}{du}\frac{\partial f}{\partial x'} = \frac{d}{du}\frac{\partial f}{\partial y'} = 0 \right.$

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2+y'^2}} = \text{constant} \quad \text{and} \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2+y'^2}} = \text{constant}$$

So $\dfrac{x'}{\sqrt{x'^2+y'^2}} = c_1$ and $\dfrac{y'}{\sqrt{x'^2+y'^2}} = c_2$

So $\boxed{\dfrac{x'}{y'} = \dfrac{c_1}{c_2}}$

So $y' = \dfrac{c_2}{c_1} x' \rightarrow \dfrac{dy}{du} = \dfrac{c_2}{c_1}\dfrac{dx}{du} \rightarrow \int dy = \int \dfrac{c_1}{c_2}\,dx = \int m\,dx$

So $\boxed{y = mx + c}$ → required path is a straight line.

Generalized Euler-Lagrange Equation to Arbitrary # of dep. vars

→ Suppose independent variable in Lagrangian mechanics is time (t)

& dependent variables are the coordinates that specify the position (or the configuration) of the system:

$$(q_1, q_2, q_3 \ldots q_n) \qquad (\# \; n \text{ depends on nature of system})$$

- For N particles in 3D, then $n = 3N$, and coordinates $q_1 \ldots q_n$ → cartesian coordinates $(x_1, y_1, z_1), \ldots (x_N, y_N, z_N)$

We think of the n-generalized coordinates as defining 1 point in an n-dimensional configuration space

S → called the "action integral" in Lagrangian mech

And the integral = called the Lagrangian ($\mathcal{L}$)

And $\boxed{\mathcal{L} = \mathcal{L}(q_1, \dot{q}_1, \ldots\ldots, q_n, \dot{q}_n, t)}$

<u>Action integral</u>

$$\boxed{S = \int_{t_1}^{t_2} \mathcal{L}\left(q_1, \dot{q}_1, \dots q_n, \dot{q}_n, t\right) dt}$$

(least action) to be stationary

We get n - Euler - Lagrange equations

$$\left\{ \begin{aligned} \frac{\partial \mathcal{L}}{\partial q_1} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} \\ \frac{\partial \mathcal{L}}{\partial q_2} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2} \\ \frac{\partial \mathcal{L}}{\partial q_3} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3} \\ &\vdots \\ \frac{\partial \mathcal{L}}{\partial q_n} &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_n} \end{aligned} \right\}$$

CHAPTER 7: LAGRANGIAN MECHANICS

Before, Ex Lagrange's eqn for unconstrained motion.

- Consider a particle in 3D, unconstrained, subject to a conservative force

The particle's kinetic energy $= \frac{1}{2}mv^2$

$$\left.\begin{aligned} &\hookrightarrow KE = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) \\ &\underline{\text{where}}\ PE = U(\vec{r}) = U(x,y,z) \end{aligned}\right\}$$

<u>Define</u> $\boxed{\text{the Lagrangian}} \rightarrow \mathcal{L} = T - U = \mathcal{L}(x,y,z,\dot{x},\dot{y},\dot{z},t)$

$$\left\{ \frac{\partial \mathcal{L}}{\partial x} = -\frac{\partial U}{\partial x} = +F_x \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m\dot{x} = p_x \right\}$$

<u>Note</u> - $F_x = \dot{p}_x \Rightarrow \boxed{\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}}$

April 10, 2011 <u>Recall</u> $\boxed{\mathcal{L} = T - U}$ + $\boxed{\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}}}$

Action integral

$$S(\alpha) = \int \mathcal{L}\, dt$$ is stationary for the path followed by the particle.

Hamilton's Principle

The path that a particle follows between 2 points (1) + (2) in a given time interval t_1 to t_2, is such that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\, dt$$ is stationary when taking along the actual path.

Lagrange's equations hold true in any coordinate system
- Cartesian $\vec{r} = (x, y, z)$
- Spherical Polar (r, θ, φ)
- Cylindrical (ρ, φ, z)

or any set of "generalized coordinates" $\longrightarrow$ (orthogonal)

$q_1, q_2, q_3 \ldots$ where each position $\vec{r}$ specifies a unique value of $(q_1, q_2, q_3) \rightarrow q_i = q_i(\vec{r})$

$\vec{r} = \vec{r}(q_1, q_2, q_3)$

$$\boxed{\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^{\,2} - U(\vec{r})}$$

Action Int

$$S = \int_{t_1}^{t_2} \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)\, dt$$

<u>So</u> $\mathcal{L} = \mathcal{L}(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3, t)$

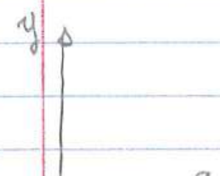
The action integral is stationary for the correct path in the new coordinate system. The correct path must also satisfy the 3 E-L equations:

$$\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}, \quad \frac{\partial \mathcal{L}}{\partial q_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_2}, \quad \frac{\partial \mathcal{L}}{\partial q_3} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_3}$$

Example

→ take 1 particle in 2-D, cartesian coordinates.

Write down the Lagrangian equation for cartesian for a particle moving in a conservative force field in 2D, & show that they imply Newton's 2nd law.



$$\mathcal{L} = \frac{1}{2}m\dot{\vec{r}}^2 - U(\vec{r}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 - \underbrace{U(r)}_{U(x,y)}$$

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Leftrightarrow -\frac{\partial U}{\partial x} = \frac{d}{dt}\left[m\dot{x}\right] \Rightarrow \boxed{F_x = m\ddot{x}}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \Leftrightarrow -\frac{\partial U}{\partial y} = \frac{d}{dt}(m\dot{y}) \Rightarrow \boxed{F_y = m\ddot{y}}$$

Observe —

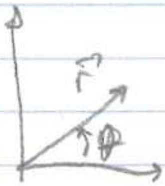
$\frac{\partial \mathcal{L}}{\partial x}$ gives force, $\frac{\partial \mathcal{L}}{\partial \dot{x}}$ gives momentum,

Generalize $\Longrightarrow$ $\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = i^{th} \text{ component of generalized force}}$

$\boxed{\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = i^{th} \text{ component of generalized momentum}}$

Each Lagrangian eqn

↳ $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \Rightarrow \begin{pmatrix}\text{generalized}\\ \text{force}\end{pmatrix} = \begin{pmatrix}\text{rate of}\\ \text{change of}\end{pmatrix}\begin{pmatrix}\text{generalized}\\ \text{momentum}\end{pmatrix}$



Ex Find Lagrangian for the same system, but in polar coordinates

↳ $\mathcal{L} = T - U = \frac{1}{2}m\dot{\vec r}^2 - U(r,\theta)$

$$\vec v = \frac{d\vec r}{dt} = \frac{d}{dt}(r\hat r) = \dot r\hat r + r\frac{d\theta}{dt}\hat\theta = \dot r\hat r + r\dot\theta\hat\theta = v_r\hat r + v_\theta\hat\theta$$

$$T = \frac{1}{2}m\dot{\vec r}^2 = \frac{1}{2}m\left[v_r\hat r + v_\theta\hat\theta\right]^2 = \frac{1}{2}mv_r^2 + \frac{1}{2}mv_\theta^2$$

$$\rightarrow \mathcal{L} = \mathcal{L}(r,\theta,\dot r,\dot\theta,t) = \frac{1}{2}m\dot r^2 + \frac{1}{2}mr^2\dot\theta^2 - U(r,\theta)$$

2 Lagrange eqn: • $\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot r}$

↳ $mr\dot\theta^2 - \frac{\partial U}{\partial r} = \frac{d}{dt}(m\dot r) = m\ddot r$ so $\boxed{mr\dot\theta^2 - \frac{\partial U}{\partial r} = m\ddot r}$

so $-\frac{\partial U}{\partial r} = \boxed{F_r = m(\ddot r - r\dot\theta^2) = m\vec a_r}$

centripetal acc. (radial of Newton's 2nd law)

• $\frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} \rightarrow -\frac{\partial U}{\partial\theta} = \frac{d}{dt}(mr^2\dot\theta)$

↳ $-\frac{\partial U}{\partial\theta} = ?$ need to know gradient of U in polar

$-\frac{\partial U}{\partial\theta} = \left[2mr\dot\theta\dot r + mr^2\ddot\theta\right]$

$\vec\nabla U = \frac{\partial U}{\partial r}\hat r + \frac{1}{r}\frac{\partial U}{\partial\theta}\hat\theta$ (prove $dU = \vec\nabla U\cdot d\vec r$)

so $F_\theta = -\frac{1}{r}\frac{\partial U}{\partial\theta}$

$-\frac{\partial U}{\partial\theta} = rF_\varphi$

so $\boxed{F_\varphi = -\frac{1}{r}\frac{\partial U}{\partial\theta}}$ (verified)

so $-\frac{\partial U}{\partial\theta} = rF_\varphi$ ⟶ torque Γ on the particle about origin

so $\left\{\begin{array}{l} \frac{\partial\mathcal{L}}{\partial\theta} = \text{change in angular momentum} = \text{torque} \\ \frac{\partial\mathcal{L}}{\partial\dot\theta} = \text{angular momentum.} \end{array}\right\}$

no constrained x, y, z

$\boxed{\text{Lagrange's equation for a system of N unconstrained particles}}$

April 12, 2018

Let $N = 2$

$$\mathcal{L} = T - U$$

$$\mathcal{L}(\vec{r_1}, \vec{r_2}, \dot{\vec{r_1}}, \dot{\vec{r_2}}, t) = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - U(\vec{r_1}, \vec{r_2})$$

$$\vec{F_1} = -\vec{\nabla}_1 U$$
$$\vec{F_2} = -\vec{\nabla}_2 U$$

Bey Newton's 2nd law can be applied to each particle

$$\left.\begin{matrix} \vec{F_{1x}} = \dot{p}_{1x} & \text{and} & \vec{F_{2x}} = \dot{p}_{2x} \\ \vec{F_{1y}} = \dot{p}_{1y} & & \vec{F_{2y}} = \dot{p}_{2y} \\ \vec{F_{1z}} = \dot{p}_{1z} & & \vec{F_{2z}} = \dot{p}_{2z} \end{matrix}\right\} \text{correspond to 6 E–L equation}$$

$$\left.\begin{matrix} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} \\ \vdots \\ \frac{\partial \mathcal{L}}{\partial z_2} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{z}_2} \end{matrix}\right\} \text{The action integral } S = \int_{t_1}^{t_2} \mathcal{L}\,dt \text{ is stationary}$$

If I changed coordinates $\frac{\partial \mathcal{L}}{\partial q_1} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_1}$

$$\vdots$$

$$\frac{\partial \mathcal{L}}{\partial q_6} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_6}$$

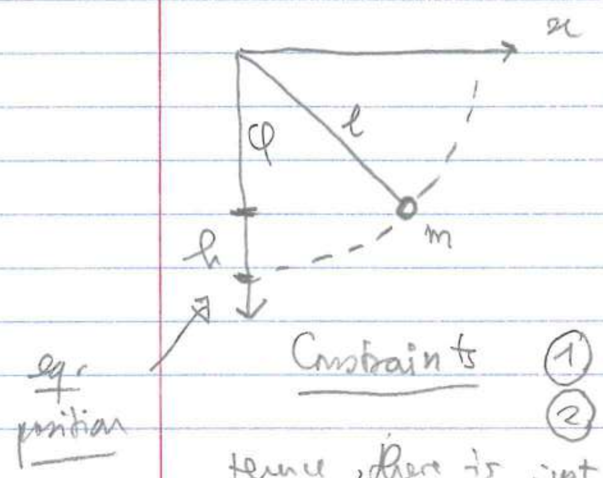
In place of 6 coordinates of $\vec{r_1}, \vec{r_2}$, we can also use 3 coordinates of center of mass

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$ and 3 corresponding relative position $\vec{r} = \vec{r_1} - \vec{r_2}$

For N number of particles (unconstrained) particles, there are 3N Lagrange's equation $\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ $(i = 1, 2, \ldots 3N)$

~~Consta~~

Constrained System → Example: plane pendulum



A mass m, fixed to a massless rod, which is pivoted at O and is free to swing without friction in the x-y plane

(constrained to x-y plane)

Constraints ① planar motion ⇒ $z = 0$
② Length of the rod cannot change : $l = \sqrt{x^2+y^2}$

Hence, there is just **1** degree of freedom for this particle

↳ Should use only 1 independent coordinate to solve the motion

→ $\boxed{\text{Use the angle } \varphi}$ → angle between pendulum & its eqn position

$$\begin{cases} T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\varphi}^2 \\ U = mgh = mgl(1-\cos\varphi) \end{cases}$$

(h: height above eq position

So $\boxed{\mathcal{L} = T - U = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl(1-\cos\varphi)}$

So $\mathcal{L} = \mathcal{L}(\varphi, \dot{\varphi}, t)$

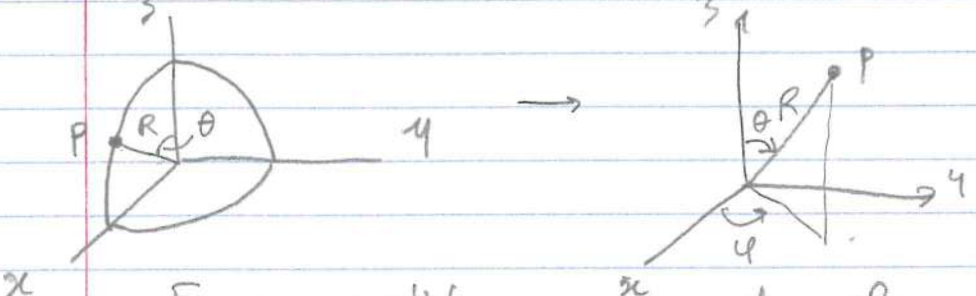
E-L eqn : $\dfrac{\partial\mathcal{L}}{\partial\varphi} = \dfrac{d}{dt}\dfrac{\partial\mathcal{L}}{\partial\dot{\varphi}}$

$$\boxed{\underbrace{-mgl\sin\varphi}_{\tau} = \frac{d}{dt}\left(ml^2\dot{\varphi}\right) = \underbrace{\overbrace{ml^2}^{I}\,\overbrace{\ddot{\varphi}}^{\alpha}}_{\dot{L}}}$$

So (torque by gravity)

$\boxed{\tau = I\alpha}$

Example 2 Motion on the surface of a sphere



For any particle on surface of sphere $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Constraints $\longrightarrow$ $\|\vec{r}\| = R = \sqrt{x^2+y^2+z^2}$

3 coordinates − 1 constraints = 2 ⇒ needs 2 coordinate to solve (independent)

$$\Rightarrow \left\{ \begin{array}{l} x = R\sin\theta\cos\varphi \\ y = R\sin\theta\sin\varphi \\ z = R\cos\theta \end{array} \right\} \quad \theta, \varphi \text{ independent}$$

$\hookrightarrow$ $\boxed{\text{Define Degree of Freedom}}$

Suppose a collection of N-points like objects

- To describe each particle ⇒ need 3 coordinates (e.g: x, y, z)
- To describe N particles → need 3N coordinates $(x_1, y_1, \dots z_N)$
  $\hookrightarrow (x_i, y_i, z_i)\quad i \to N$

Constraints : maybe motion of collection of particles is constrained in some way

Exp $\hookrightarrow$ distance between particle j & k is fixed (ℓ)

i.e $(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 = \ell^2$

Suppose there are "m" such equations of constraints

$\hookrightarrow$ need $n = \boxed{3N - m}$ E-L eqns to describe the system!

$\boxed{\text{Lagrange's eqn under transformation of coordinates}}$

Consider a system of N particles, subjected to m # of constraints

↳ $\boxed{n = 3N - m}$ generalized coordinates are needed to describe system

↳ $\mathcal{L}_i = T_i - V_i = \mathcal{L}(x_{\alpha,i}, \dot{x}_{\alpha,i}, t)$ where $\alpha = (1,2,3)$

$i = 1, \dots, n$

$x_1 = x$

$x_2 = y$

$x_3 = z$

$\boxed{\text{So}}$

$$\boxed{\frac{\partial \mathcal{L}}{\partial x_{\alpha,i}} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{\alpha,i}}\right)}$$

→ there are 3N of for all $x_{\alpha,i}$ must be subjected to constraints

Say $x_{\alpha,i} = x_{\alpha i}(q_k, t)$

<u>Under transform of coordinates</u>

$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k}$ <u>and</u> k from 1 to n

there are only 3N − m of these

constraints are built-in)

$\boxed{\text{Ignorable coordinates}}$ → $\boxed{\text{\underline{if} } \mathcal{L} \text{ independent of } q_k \text{, then } q_k \text{ ignorable}}$

→ q_k = general coordinate, so $\dot{q}_k$ = generalized velocity

and so $\frac{\partial \mathcal{L}}{\partial \dot{q}_n}$ = generalized momentum p_x

Ex in 1D → $\mathcal{L} = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} = mv = p_x$

So if $\frac{\partial \mathcal{L}}{\partial q_k} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \Rightarrow \boxed{\frac{d}{dt}p_k = \frac{\partial \mathcal{L}}{\partial q_k}}$

If $\frac{\partial \mathcal{L}}{\partial q_k} = 0$, then

$\dot{p}_k = 0$

$\Rightarrow p_k$ = constant

<u>Hence</u> if $\mathcal{L}$ ind of $q_k \Rightarrow q_k$ ignorable

Application of Lagrange's equation

April 16, 2018

Problem solving

(1) Pick a convenient set of generalized coordinates ($q_1, q_2, \ldots$ to q_n) where n = # of degrees of freedom

(2) Express KE in terms of q_k's ... $\dot{q}_k$'s ...

(3) Express PE

(4) Form Lagrange function: $\mathcal{L} = T - V$

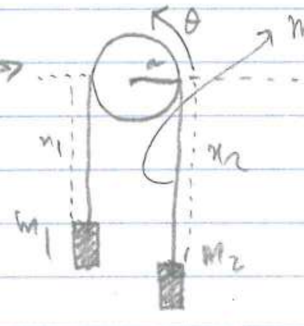
(5) Form for each generalized coordinate, a E-L eqn

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k}$$

(6) Solve for q_k's.

Example

(1) Atwood machine



massless string ...

$I_{pulley} = I$

→ **3 objects** need ~~3~~ 3 generalized coordinates

$\boxed{x_1,\ x_2 \text{ and } \theta}$

But note

Constraints (1) length of string = $\boxed{L = x_1 + x_2 + \pi a}$

(2) String does not slip relative to pulley → Speed = $\boxed{\dot{x}_1 = a\dot{\theta}}$

So we only need 1 independent coordinate

$$\left\{ \begin{aligned} T &= \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}I\dot{\theta}^2 \\ V &= -mgx_1 - mgx_2 \end{aligned} \right\}$$

$\boxed{\text{Simplify}}$ → Let x = independent coordinate $= x_1$

Then $x_2 = l - \pi a - x$, $\dot{x}_2 = -\dot{x}$

$\dot{x}_1 = \dot{x}$ and $\dot{\theta} = \dfrac{\dot{x}}{a}$

So $KE = \frac{1}{2}(m_1+m_2)\dot{x}^2 + \frac{1}{2}\frac{I}{a^2}\dot{x}^2 = \frac{1}{2}\left(m_1+m_2+\frac{I}{a^2}\right)\dot{x}^2$

$PE = \cancel{-mgx + mgx = 0} = -mgx - m_{2}g(l-\pi a - x)$

$PE = m_2 g(\pi a - l) - g(m_1 - m_2)x$

So $\mathcal{L} = T - V$

$= \frac{1}{2}\left(m_1+m_2+\frac{I}{a^2}\right)\dot{x}^2 - mg(\pi a - l) - g(m_2-m_1)x$

$-g(m_2-m_1) = \frac{\partial \mathcal{L}}{\partial x}$, $\frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(m_1+m_2+\frac{I}{a^2}\right)\dot{x}$, $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(m_1+m_2+\frac{I}{a^2}\right)\ddot{x}$

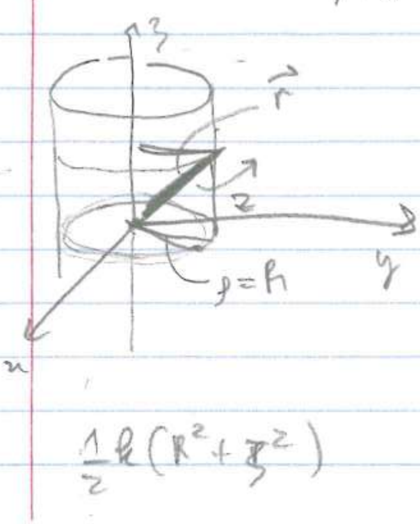
$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \Rightarrow g(-m_2+m_1) = \left(m_1+m_2+\frac{I}{a^2}\right)\ddot{x}$

So $\boxed{\ddot{x} = \dfrac{g(-m_2+m_1)}{m_2+m_1+I/a^2}}$

$\boxed{\text{Example}}$

A particle confined to move in a cylinder Particle of mass m, constrained to move on a frictionless cylinder, (R), given by $\rho = R$ (in cylindrical polar coordinates) (ρ, φ, z)

$\vec{r} = R\hat{e}_\rho + z\hat{z}$



Since $\rho = R$ → then 2 coordinates

↳ $\vec{r} = \varphi\hat{\varphi} + z\hat{z}$

$v_\varphi = R\dot{\varphi}$
$v_z = \dot{z}$

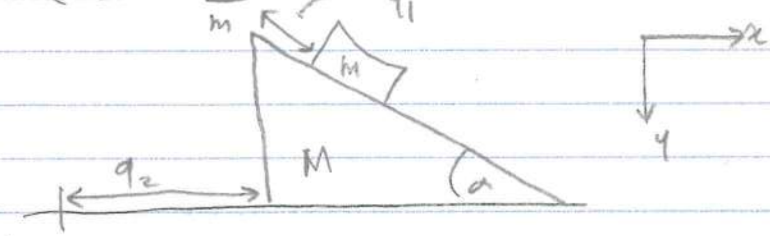
$KE = \frac{1}{2}mv^2 = \frac{1}{2}m\left(R^2\dot{\varphi}^2 + \dot{z}^2\right)$

$\frac{1}{2}k(R^2+z^2) = PE = \frac{1}{2}kr^2$, where $r = \|\vec{r}\| = \sqrt{R^2+z^2}$

$$\mathcal{L} = T - V = \frac{1}{2}m\left(R^2\dot{\varphi}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mR^2\dot{\varphi}, & \dfrac{\partial \mathcal{L}}{\partial \varphi} = 0, \quad \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{\varphi}} = mR^2\ddot{\varphi} \\ \dfrac{\partial \mathcal{L}}{\partial \dot{z}} = m\dot{z}, & \dfrac{\partial \mathcal{L}}{\partial z} = -kz, \quad \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{z}} = m\ddot{z} \end{cases}$$

So $\begin{cases} mR^2\ddot{\varphi} = 0 \\ m\ddot{z} = -kz \end{cases}$ so $\begin{cases} \ddot{\varphi} = 0 \\ \ddot{z} = -\dfrac{kz}{m} \end{cases}$ → angular momentum conserved



Example 3

↳ Block sliding on a wedge
wedge can also slide on table
with negligible friction. → Suppose the block is released from the top, with both initially at rest

If the wedge has sloping face $= L$, how long does it take the block to reach the bottom?

System has 2 degrees of freedom: q_1 & q_2

KE $\frac{1}{2}M\dot{q}_2^2$ for wedge → Note block's velocity relative to the wedge, but the wedge slides

For block $= \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2$

$$v = (v_x, v_y) = \left(\dot{q}_1\cos\alpha + \dot{q}_2,\ \dot{q}_1\sin\alpha\right)$$

So $T = \dfrac{1}{2}M\dot{q}_2^2 + \dfrac{1}{2}m\left[\dot{q}_1^2 + \dot{q}_2^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right]$

$= \dfrac{1}{2}(M+m)\dot{q}_2^2 + \dfrac{1}{2}m\dot{q}_1^2 + \dfrac{1}{2}m\dot{q}_2^2 + \dot{q}_1\dot{q}_2\cos\alpha$

and

April 17, 2018 $PE = -mgy = -mg\,q_1\sin\alpha$

$$\mathcal{L} = T - V = \frac{1}{2}(M+m)\dot{q}_2^2 + \frac{1}{2}m\left(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2\cos\alpha\right) + mgq_1\sin\alpha$$

$$\frac{\partial\mathcal{L}}{\partial q_1} = mg\sin\alpha\ ,\quad \frac{\partial\mathcal{L}}{\partial\dot{q}_1} = m\dot{q}_1 + m\dot{q}_2\cos\alpha\ ,\quad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_1}\right) = m\ddot{q}_1 + m\cos\alpha\,\ddot{q}_2$$

$$\frac{\partial\mathcal{L}}{\partial q_2} = 0\ ,\quad \frac{\partial\mathcal{L}}{\partial\dot{q}_2} = (M+m)\dot{q}_2 + m\cos\alpha\,\dot{q}_1\ ,\quad \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_2}\right) = (M+m)\ddot{q}_2 + m\cos\ddot{q}_1$$

So
$$\begin{cases} mg\sin\alpha = m(\ddot{q}_1 + \ddot{q}_2\cos\alpha) \\ (m+M)\ddot{q}_2 + m\cos\ddot{q}_1 = 0 \end{cases} \Rightarrow \begin{cases} g\sin\alpha = \ddot{q}_1 + \ddot{q}_2\cos\alpha \\ (m+M)\ddot{q}_2 + m\cos\alpha\,\ddot{q}_1 = 0 \end{cases}$$

$$\ddot{q}_1 = g\sin\alpha - \ddot{q}_2\cos\alpha = g\sin\alpha + \frac{m\cos^2\alpha}{m+M}\ddot{q}_1 \rightarrow \left(1 - \frac{m\cos^2\alpha}{m+M}\right)\ddot{q}_1 = g\sin\alpha$$

So
$$\ddot{q}_1 = \left[\frac{1}{g\sin\alpha}\left(1 - \frac{m\cos^2\alpha}{m+M}\right)\right]^{-1}$$

$$\ddot{q}_1 = g\sin\alpha\left(1 - \frac{m\cos^2\alpha}{m+M}\right)^{-1}$$

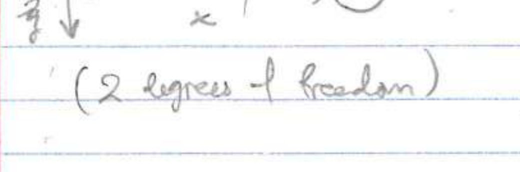
And
$$\ddot{q}_2 = \frac{-m\cos\ddot{q}_1}{m+M} = \frac{-m}{m+M}\cos\alpha\cdot g\sin\alpha\left(1 - \frac{m\cos\alpha}{m+M}\right)^{-1}$$

$$= \frac{-m}{m+M}\,g\cos\alpha\sin\alpha\,\frac{m+M}{m+M-m\cos\alpha}$$

$$= \frac{-mg\cos\alpha\sin\alpha}{m+M-m\cos\alpha}$$

//

Example 4 Driven pendulum



$$\begin{cases} T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \\ \quad = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[(\dot{X} + l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2\right] \\ V = 0 + (-mgl\cos\theta) \end{cases}$$

(2 degrees of freedom)

$$\begin{cases} x = X + l\sin\theta \Rightarrow \dot{x} = \dot{X} + l\cos\theta\,\dot{\theta} \\ y = -l\cos\theta \Rightarrow \dot{y} = +l\sin\theta\,\dot{\theta} \end{cases}$$

So $\mathcal{L} = T - V = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[(\dot{x} + l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2\right] + mgl\cos\theta$

$\frac{\partial\mathcal{L}}{\partial x} = 0$, $\frac{\partial\mathcal{L}}{\partial\dot{x}} = M\dot{x} + m(\dot{x} + l\cos\theta\,\dot{\theta})$, $\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta$

$$\frac{\partial\mathcal{L}}{\partial\theta} = m(\dot{x} + l\cos\theta\,\dot{\theta})\left[-l\dot{\theta}\sin\theta\right] + \frac{m}{2}2l\sin\theta\,\dot{\theta}\cos\theta - mgl\sin\theta$$

$$= m(\dot{x} + l\cos\theta\,\dot{\theta})(-l\dot{\theta})\sin\theta + ml^2\dot{\theta}^2\sin\theta\cos\theta - mgl\sin\theta$$

$$= -ml\dot{x}\dot{\theta}\sin\theta - mgl\sin\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m(\dot{x} + l\cos\theta\,\dot{\theta})(l\cos\theta) + ml^2\sin\theta\,\dot{\theta} = ml\dot{x}\cos\theta + ml^2\dot{\theta}$$

$\cancel{\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = m(\ddot{x}}$

So

"x" $\Rightarrow (M+m)\ddot{x} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta = 0 \to \boxed{(M+m)\ddot{x} = -ml\frac{\partial}{\partial t}(\dot{\theta}\cos\theta)}$

"θ" $\Rightarrow \frac{\partial\mathcal{L}}{\partial\theta} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}} \Rightarrow ml^2\ddot{\theta} = -ml(\ddot{x}\cos\theta + g\sin\theta)$

$$\to \boxed{\ddot{\theta} = -\frac{1}{l}(\ddot{x}\cos\theta + g\sin\theta)}$$

Assume that $\theta \ll 1$

so $\sin\theta \approx \theta$, and $\cos\theta \approx 1$

↳ find $x(t)$, let $x(t) = x_0\cos(\omega_0 t) \Rightarrow \ddot{x} = -\omega_0^2 x_0\cos(\omega_0 t)$

Then "θ" $\Rightarrow \ddot{\theta} = -\frac{1}{l}\left[-\omega_0^2 x_0\cos(\omega_0 t)\cdot 1 + g\theta\right]$

$$\hookrightarrow \ddot{\theta} = \frac{-\omega_0^2 x_0\cos(\omega_0 t)}{l} - \frac{g}{l}\theta$$

$$\hookrightarrow \boxed{\ddot{\theta} + \frac{g}{l}\theta = \frac{\omega_0^2}{l}x_0\cos(\omega_0 t)}$$

→ driven, undamped harmonic osc

Let $\ddot{x} = a = \text{constant}$ $\qquad \omega = \sqrt{\frac{g}{\ell}}$

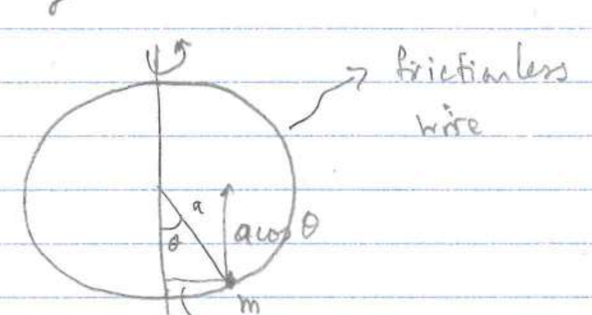
$\hookrightarrow \ddot{\theta} + \frac{g}{\ell}\theta = \frac{a}{\ell} = \frac{a}{g}\omega^2$

$\hookrightarrow \boxed{\theta(t) = \frac{a}{g} + A\cos(\omega t + \alpha)}$

osc about an angle $\theta_0 = \frac{a}{g}$

April 19, 2018 Going over exam 2

April 20, 2018 Bead on a rotating loop



The bead of mass m is threaded on a frictionless circular hoop of radius a rotating with ω

- Find any equilibrium position for the bead.

$KE = \frac{1}{2}m\underbrace{a^2\dot{\theta}^2}_{a\dot{\theta} = \omega_0 r^2} + \frac{1}{2}m\left[a\sin\theta\,\omega\right]^2 = \frac{1}{2}ma^2\underbrace{\left[\dot{\theta}^2 + \omega^2\sin^2\theta\right]}_{\text{due to spinning of the hoop}}$

$V = mga(1-\cos\theta)$

So $\mathcal{L} = T - V = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) - mga(1-\cos\theta)$

$\frac{\partial \mathcal{L}}{\partial\theta} = \frac{1}{2}ma^2\omega^2 \cdot 2\sin\theta\cos\theta + -mga\sin\theta$

$= ma^2\omega^2\sin\theta\cos\theta - mga\sin\theta$

$\frac{\partial\mathcal{L}}{\partial\dot{\theta}} = \frac{1}{2}ma^2 \cdot 2\dot{\theta} \Rightarrow \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\theta}}\right) = ma^2\ddot{\theta}$

So $\cancel{m}a^2\ddot{\theta} = \cancel{m}a^2\omega^2\sin\theta\cos\theta - \cancel{m}ga\sin\theta$

So $\boxed{\ddot{\theta} = \omega^2\sin\theta\cos\theta - \frac{g}{a}\sin\theta = \sin\theta\left(\omega^2\cos\theta - \frac{g}{a}\right)}$

$\ddot{\theta} = \sin\theta\left(\omega^2\cos\theta - \frac{g}{a}\right)$ can't be solved analytically.

Equilibrium position $\ddot{\theta} = 0$. $\boxed{\sin\theta = 0}$, $\theta = 0$, π ...

(stable) (unstable)

or $\boxed{\omega^2\cos\theta = \frac{g}{a}}$, $\theta = \cos^{-1}\left(\frac{g}{a\omega^2}\right)$

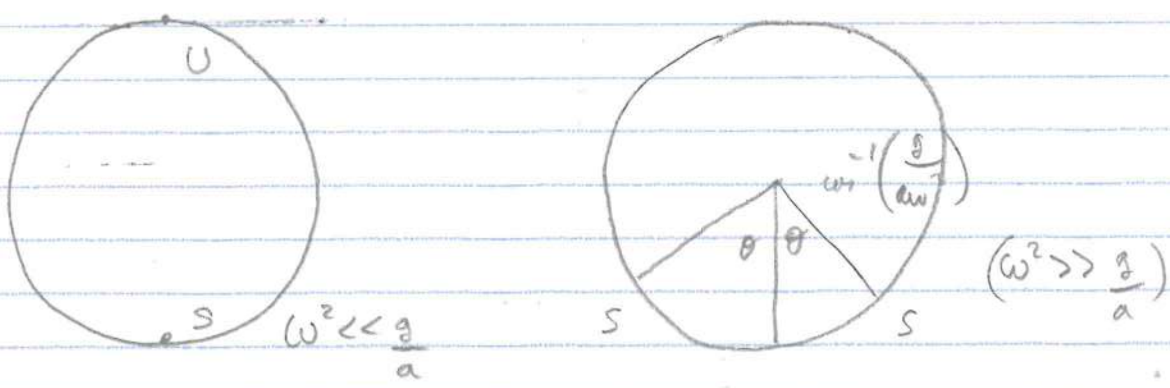
(stable)

Approx ① If $\omega^2 \ll \frac{g}{a}$, θ small

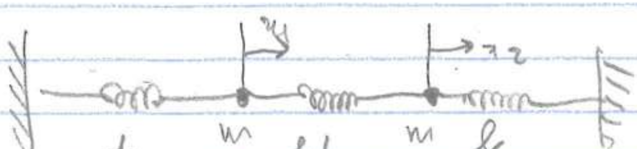
$\ddot{\theta} = -\frac{g}{a}\theta \Rightarrow$ pendulum (around $\theta = 0$) (stable)

② If $\omega^2 \gg g/a$, θ small

$\ddot{\theta} \simeq \omega^2\theta \Rightarrow$ "repelled from $\theta = 0$" (unstable)



⑥ Small oscillations Two coupled oscillators



What are x_1 & x_2?

$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2)$ $V = \frac{1}{2}kx_1^2 + \frac{1}{2}k'(x_2 - x_1)^2 + \frac{1}{2}kx_2^2$

$\mathcal{L} = T - V = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 - \frac{1}{2}k'(x_2 - x_1)^2$

$\frac{\partial\mathcal{L}}{\partial x_1} = -\frac{1}{2}kx_1 \cdot 2 - \frac{1}{2}k'(-2x_2 + 2x_1)$, $\frac{\partial\mathcal{L}}{\partial x_2} = -\frac{1}{2}kx_2 \cdot 2 - \frac{1}{2}k'(-2x_1 + 2x_2)$

$$\frac{\partial L}{\partial \dot{x}_1} = m\dot{x}_1 \;,\; \frac{\partial L}{\partial \dot{x}_2} = m\dot{x}_2 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m\ddot{x}_1 \;,\; \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = m\ddot{x}_2$$

(Hint : Eigenvector)

So
$$\begin{cases} m\ddot{x}_1 = -kx_1 - k'x_1 + k'x_2 \\ m\ddot{x}_2 = -kx_2 - k'x_2 + k'x_1 \end{cases} \rightarrow \begin{cases} \ddot{x}_1 = -\omega^2 x_1 + \omega'^2 x_2 & (1) \\ \ddot{x}_2 = -\omega^2 x_2 + \omega'^2 x_1 & (2) \end{cases}$$

where $\omega^2 = \frac{k+k'}{m}$, $\omega'^2 = \frac{k'}{m}$

(1) + (2)

$$\hookrightarrow m(\ddot{x}_1 + \ddot{x}_2) + (k+k')(x_1+x_2) - k'(x_1+x_2) = 0$$

$$(\ddot{x}_1 + \ddot{x}_2) + \frac{k}{m}(x_1+x_2) = 0 \qquad \left(\omega_- = \sqrt{\frac{k}{m}}\right)$$

So $\omega_-^2 \rightarrow (\ddot{x}_1 + \ddot{x}_2) + \omega_-^2(x_1+x_2) = 0$

$$\boxed{\ddot{X} = -\omega_-^2 X} \Rightarrow \boxed{X = x_1 + x_2 = A\cos(\omega_- t + \varphi)}$$

(1) − (2)

$$m(\ddot{x}_1 - \ddot{x}_2) + (k+k')(x_1-x_2) - k'(x_2-x_1) = 0$$

$$\Rightarrow \underbrace{(\ddot{x}_1 - \ddot{x}_2)} + \underbrace{\left(\frac{k+2k'}{m}\right)}_{\omega_+^2}(x_1-x_2) = 0$$

$$\boxed{\ddot{Y} = -\omega_+^2 Y} \qquad \left(\omega_+ = \sqrt{\frac{k+2k'}{m}}\right)$$

April 23, 2018

Normal mode

$\hookrightarrow$ let $x_1 = A_1\cos\omega t$, $x_2 = A_2\cos\omega t$ $\Rightarrow \begin{cases} \ddot{x}_1 = -\omega^2 A_1 \cos\omega t \\ \ddot{x}_2 = -\omega^2 A_2\cos\omega t \end{cases}$

So $\left[\left(-m\omega^2 + (k+k')\right)A_1 - k'A_2\right]\cos\omega t = 0$

and $\left[\left(-m\omega^2 + (k+k')\right)A_2 - k'A_1\right]\cos\omega t = 0$

Can be written as a matrix eqn

$$\rightarrow \overbrace{\begin{pmatrix} -m\omega^2+(k+k') & -k' \\ -k' & -m\omega^2+(k+k') \end{pmatrix}}^{M} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$$

<u>Solutions</u>: (i) $A_1 = A_2 = 0$ $\rightarrow (M - 0 I_2)$

(ii) $\det M = 0 = \left[\left(-m\omega^2 + (k+k')\right)^2 - (-k')^2 \right] = 0$

<u>then</u> $-m\omega^2 + (k+k') = \pm k'$

$$\omega^2 = \frac{k + (k' \pm k')}{m}$$

<u>So</u> $\boxed{\omega_- = \sqrt{\frac{k}{m}} \quad \text{or} \quad \omega_+ = \sqrt{\frac{k+2k'}{m}}}$ normal modes

<u>So</u>

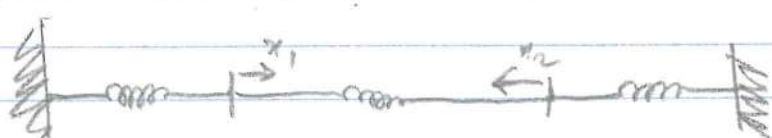
$$\begin{cases} x_1 = A_1 \cos(\omega_+ t) \\ x_2 = A_2 \cos(\omega_+ t) \end{cases} \quad \text{or} \quad \begin{cases} x_1 = B_1 \cos(\omega_- t) \\ x_2 = B_2 \cos(\omega_- t) \end{cases}$$

<u>Pluging in</u> (ω_+) $\left[-m\omega_+^2 + (k+k')\right] A_1 - k' A_2 = 0$

<u>So</u> $\left[-m\,\frac{k+2k'}{m} + k + k'\right] A_1 - k' A_2 = 0$

<u>So</u> $\boxed{A_1 = -A_2} = A$

<u>So</u> $\boxed{x_1 = A\cos(\omega_+ t) \quad \text{and} \quad x_2 = -A\cos(\omega_+ t)}$



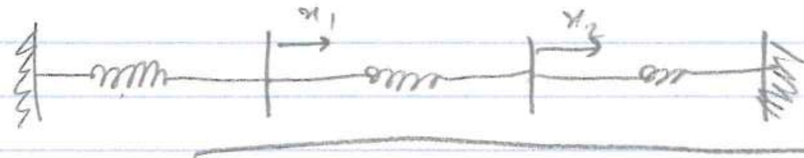
asymmetric mode

Plugging in $\omega_- = \sqrt{\frac{k}{m}}$

so $\left[-m\omega^2 + (k+k')\right]B_1 - k'B_2 = 0$

so $\boxed{B_1 = B_2} = B$

get $\boxed{x_1 = B\cos(\omega_- t) \text{ and } x_2 = B\cos(\omega_- t)}$



Symmetric mode

In general

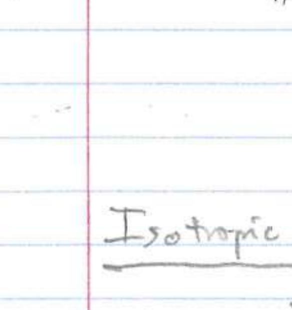
$$\left\{\begin{aligned} x_1 &= A\cos(\omega_+ t + \varphi) + B\cos(\omega_- t + \theta) \\ x_2 &= -A\cos(\omega_+ t + \varphi) + B\cos(\omega_- t + \theta) \end{aligned}\right\}$$

and

A, B, φ, θ can be solved with initial conditions

Central Force

Central force always act towards / away from one point in space



use spherical polar coordinates → central force $\vec{F}(\vec{r}) = F(r, \theta, \varphi)\hat{r}$

$\boxed{\text{Conserves angular momentum}}$

"Isotropic" central force $\vec{F}(\vec{r}) = F(r)\hat{r}$ (spherically sym.)

$\boxed{\text{Conserves angular momentum + energy}}$

Isotropic central forces

$\vec{\nabla} \times \vec{F} = \vec{0}$ → can define potential energy function

$U(\vec{r}) = -\int_{r} \vec{F}(\vec{r}') \cdot d\vec{r}'$

And $\boxed{\vec{F}(\vec{r}) = -\vec{\nabla} U(r)}$

Examples (a) Gravity $\vec{F}(r) = -\frac{GMm}{r^2}\hat{r} = -\frac{GMm}{r^3}\vec{r}$

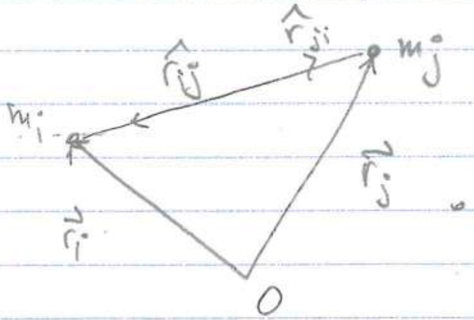
(b) Coulomb force $\vec{F}(r) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^3}\vec{r}$

(c) Interatomic force / Molecular force $\Big\}$ $\vec{F}(\vec{r}) = -\frac{V_0}{a}\left[\left(\frac{a}{r}\right)^7 - 2\left(\frac{a}{r}\right)^{13}\right]\hat{r}$ (VdW)

(d) Nuclear Force (Yukawa force) $\rightarrow$ $\vec{F}(\vec{r}) = aV_0\left(\frac{1 + r/a}{r^2}\right)e^{-r/a}\hat{r}$

$\boxed{\text{Force between any 2 point masses}}$

Newton's law of gravitation: Every pair of point masses in the universe attract one another.



$\vec{r}_{ji} = \vec{r}_j - \vec{r}_i$ and $\hat{r}_{ij} = -\hat{r}_{ji}$

- Force on m_j due to m_i:

$$\vec{F}_{ji} = -\frac{Gm_j m_i}{r_{ji}^2}\hat{r}_{ji}$$

where $\vec{F}_{ij} = -\vec{F}_{ji}$

- Force on m_i due to m_j

$G = 6.67\times10^{-11}\ Nm^2/kg^2$

$$\vec{F}_{ij} = \frac{-Gm_j m_i}{r_{ij}^2}\hat{r}_{ij} = -\vec{F}_{ji}$$

$U = -\int \vec{F}(\vec{r})\cdot d\vec{r}$ potential energy function

$= \frac{-GMm}{|\vec{r}_i - \vec{r}_j|}$ where $\vec{r} = \vec{r}_i - \vec{r}_j$

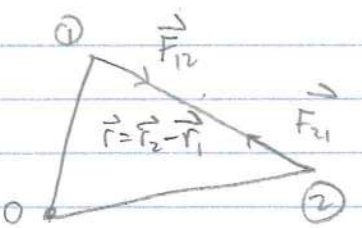
$\rightarrow$ U only depends on $\|\vec{r}_i - \vec{r}_j\|$

Want to find the possible motion of 2 bodies with

Lagrangian $\mathcal{L} = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - U(r)$

April 24, 2018 $\boxed{\text{Center of Mass and Relative Coordinates ; Reduced Mass}}$

↳ System of 2 masses that exert forces on each other → Internal forces... CENTRAL. But no other external force...



$\hat{r} = \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}$ $\qquad \vec{F}_{12} = -\vec{F}_{21} = -F(r)\hat{r}$

In Cartesian $\rightarrow \begin{Bmatrix} \vec{r}_1 = x_1\hat{i} + y_1\hat{j} + z_1\hat{k} \\ \vec{r}_2 = x_2\hat{i} + y_2\hat{j} + z_2\hat{k} \end{Bmatrix}$

$F(r) = F(x_1, y_1, z_1, x_2, y_2, z_2)$

Newton's 2nd Law $\longrightarrow$ $m_1\ddot{\vec{r}}_1 = \vec{F}_{12} = -F(r)\hat{r}$

$m_2\ddot{\vec{r}}_2 = \vec{F}_{21} = F(r)\hat{r}$

$$\boxed{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2 = 0}$$

Def COM $\quad \vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \Rightarrow \ddot{\vec{R}} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1 + m_2}$

$(m_1 + m_2)\ddot{\vec{R}} = \vec{0}$

$\searrow \boxed{\vec{R}(t) = \vec{R}_0 + \vec{V}_0 t} \rightarrow \vec{R}$ moves at constant v_0

The CM of 2 particles lies on the line joining them.

$M = m_1 + m_2 \rightarrow \boxed{\vec{P} = M\dot{\vec{R}}}$ → Same as if total mass M were concentrated @ center of mass

$\ddot{\vec{R}} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1 + m_2}$, $\vec{r} = \vec{r}_2 - \vec{r}_1 \Rightarrow \ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = \frac{F(r)}{m_2}\hat{r} - \left(\frac{-F(r)\hat{r}}{m_1}\right)$

$\ddot{\vec{r}} = \left(\frac{1}{m_2} + \frac{1}{m_1}\right)F(r)\hat{r}$

μ = reduced mass of system

So $\left(\frac{m_1 m_2}{m_1 + m_2}\right)\ddot{\vec{r}} = F(r)\hat{r} \Rightarrow \mu\ddot{\vec{r}} = F(r)\hat{r}$

<u>To express KE</u> $\Rightarrow$ $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$ and $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$

<u>KE</u> $\frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2}$

$$= \frac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \frac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2$$

$$= \frac{1}{2}m_1\dot{\vec{R}}^2 + \cancel{\frac{1}{2}m_1\cdot 2\dot{\vec{R}}\frac{m_2}{M}\dot{\vec{r}}} + \frac{1}{2}m_1\left(\frac{m_2}{M}\dot{\vec{r}}\right)^2$$

$$+ \frac{1}{2}m_2\dot{\vec{R}}^2 - \cancel{\frac{1}{2}m_2\, 2\dot{\vec{R}}\frac{m_1}{M}\dot{\vec{r}}} + \frac{1}{2}m_2\left(\frac{m_1}{M}\dot{\vec{r}}\right)^2$$

$$= \frac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{M}\right)\dot{\vec{r}}^2\cdot\left(\frac{m_2+m_1}{M}\right)$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\left(\frac{m_1 m_2}{M}\right)\dot{\vec{r}}^2\cdot\underbrace{\left(\frac{m_2+m_1}{M}\right)}_{1}$$

$$\boxed{KE = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2}$$

$|\vec{r}_2 - \vec{r}_1|$

$$\mathcal{L} = T - U = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

$$= \frac{1}{2}M\dot{\vec{R}}^2 + \left(\frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)\right)$$

$$\boxed{\mathcal{L} = \mathcal{L}_{CM} + \mathcal{L}_{rel}}$$

we have split the Lagrangian into 2 seperate pieces ...

only involve $\dot{\vec{R}}$ only involve $\dot{\vec{r}}$

<u>The "R" equation</u>

$$\frac{\partial\mathcal{L}}{\partial R} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{R}} \Rightarrow \boxed{M\ddot{\vec{R}} = 0}$$

direct consequence of conservation of total momentum

If $\mathcal{L}$ is independent of R $\to$ R is called the "ignorable" coordinate

<u>The "r" eqn</u>

$$\frac{\partial\mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{r}} \Rightarrow \frac{-dU(r)}{dr} = \mu\ddot{\vec{r}} = -\vec{\nabla}U = F(r)$$

April 26, 2018 $\boxed{\text{The CM reference frame}}$

CM frame $\rightarrow$ $\begin{cases} \text{As } \dot{R} = \text{constant, we choose an inertial reference frame where the COM is} \\ \text{at the origin (at rest) \& total momentum } = 0 \end{cases}$

if $\dot{R} = 0$, the CM part of $\mathcal{L}$ is 0 & $\mathcal{L} = \mathcal{L}_{rel}$

In this frame, $\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu\dot{r}^2 - U(r)$ (1-body problem)

Both particles have equal & opposite momentum.

<u>Conservation of angular momentum</u>

$$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2 = m_1 \vec{r}_1 \times \dot{\vec{r}}_1 + m_2 \vec{r}_2 \times \dot{\vec{r}}_2$$

<u>In CM frame</u>

$$\vec{r}_1 = \frac{m_2}{M}\vec{r}, \quad \vec{r}_2 = -\frac{m_1}{M}\vec{r}$$

$$L = \frac{m_1 m_2}{M}\vec{r} \times \frac{m_2}{M}\dot{\vec{r}} + \frac{m_2 m_1}{M}\vec{r} \times \frac{m_1}{M}\dot{\vec{r}}$$

$$= \frac{m_1 m_2}{M^2}\left[m_2 \vec{r} \times \dot{\vec{r}} + m_1 \vec{r} \times \dot{\vec{r}}\right] = \frac{m_1 m_2}{M}\left(\vec{r} \times \dot{\vec{r}}\right) = \boxed{\mu \vec{r} \times \dot{\vec{r}} = \vec{L}}$$

So $\boxed{\vec{L} = \vec{r} \times \mu \dot{\vec{r}}}$ $\rightarrow$ total angular momentum in CM frame is exactly the same as angular momentum of a single particle with mass μ & position $\vec{r}$

As the angular momentum is conserved

$\vec{r} \times \dot{\vec{r}} = \text{constant}$

$\hookrightarrow$ $\vec{r}$ & $\dot{\vec{r}}$ in a fixed plane

$\rightarrow$ $\boxed{\text{2D problem}}$

$\Rightarrow$ choose polar coordinate r, φ

$$\boxed{\mathcal{L} = \mathcal{L}_{rel} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r)}$$

z-component of angular momentum

<u>φ eqn</u>

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0 = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(\mu r^2 \dot{\varphi}\right) = \mu r^2 \dot{\varphi} \quad \text{so } \dot{\varphi} = \text{constant}$$

$$\boxed{\dot{\varphi} = \frac{l_z}{\mu r^2} = \frac{l}{\mu r^2}}$$

E-L eqn with respect to r

Newton's 2nd law

$$\frac{\partial \mathcal{L}}{\partial r} = -\frac{d}{dr}U(r) + \mu\dot{\varphi}^2 r = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{d}{dt}\left(\mu\dot{r}\right) = \boxed{\mu\ddot{r} = -\frac{d}{dr}U(r) + \mu\dot{\varphi}^2 r}$$

particle in 1D with $\vec{r}$, μ

fictitious centrifugal force

$$F_{cf} = \mu\dot{\varphi}^2 r = \mu\left(\frac{l}{\mu r^2}\right)^2 r = \boxed{\frac{l^2}{\mu r^3} = -\frac{d}{dr}(U_{cf})}$$

So $\boxed{U_{cf} = \left(\frac{l^2}{2\mu r^2}\right)}$ Note $\vec{r}$ is the relative position of m_1 & m_2

centrifugal potential energy, given by $\frac{l^2}{2\mu r^2}$

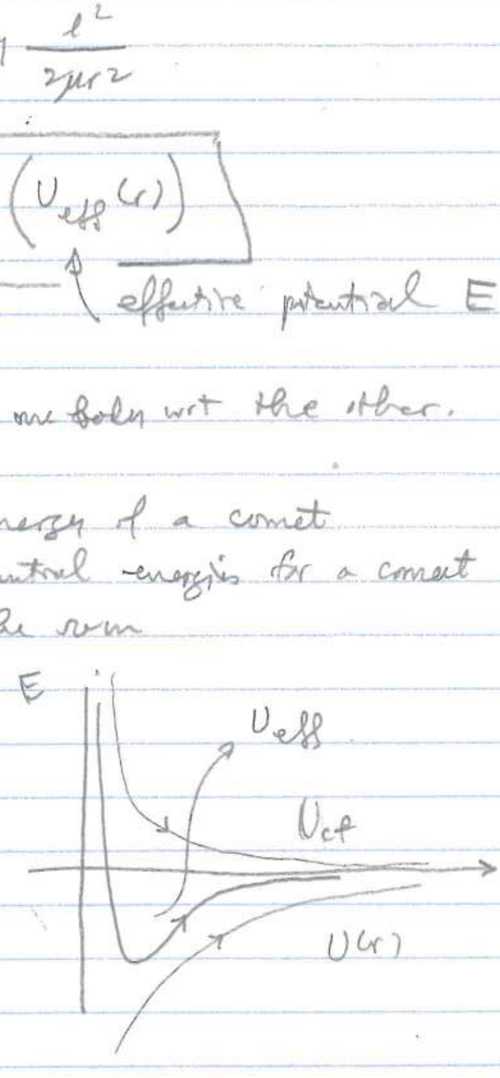
Rewrite the radial eqn as

$$\boxed{\mu\ddot{r} = -\frac{d}{dr}\left[U(r) + U_{cf}(r)\right] = -\frac{d}{dr}\left(U_{eff}(r)\right)}$$

effective potential E

where $\boxed{U_{eff} = U(r) + \frac{l^2}{2\mu r^2}}$ → one body wrt the other.

Example Kepler problem: effective potential energy of a comet
Write down the actual & eff potential energies for a comet moving in the grav. field of the sun
Sketch the pot wrt r.



$$U(r) = -\frac{GMm}{r} \quad \text{and} \quad U_{cf} = \frac{l^2}{2\mu r^2}$$

$$U_{eff} = -\frac{GMm}{r} + \frac{l^2}{2\mu r^2}$$

$r \to 0$ $\quad \ddot{r} = -\frac{d}{dr}U > 0$ → away from sun

$r \to \infty$ $\quad \ddot{r} = -\frac{d}{dr}U < 0$ → towards sun

Conservation of Energy

$$\hookrightarrow \left[\mu\ddot{r} = -\frac{d}{dr}U(r)\right] \times \dot{r} \Rightarrow \mu\dot{r}\ddot{r} = -\frac{d}{dr}U(r)\cdot\frac{dr}{dt}$$

so $$\frac{d}{dt}\left(\frac{1}{2}\mu\dot{r}^2\right) = -\frac{d}{dt}U_{eff}(r)$$

so

$$\frac{d}{dt}\left[\frac{1}{2}\mu\dot{r}^2 + U(r)\right] = 0 \quad \text{so} \quad \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r) = \text{constant}$$

so

$$\boxed{\frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{l^2}{2\mu r^2} = \frac{1}{2}\mu\dot{r}^2 + U(r) + \frac{(\mu r^2\dot{\varphi})^2}{2\mu r^2} = E}$$

constant !!!

$\frac{1}{2}\mu r^2\dot{\varphi}^2$

radial pot. angular

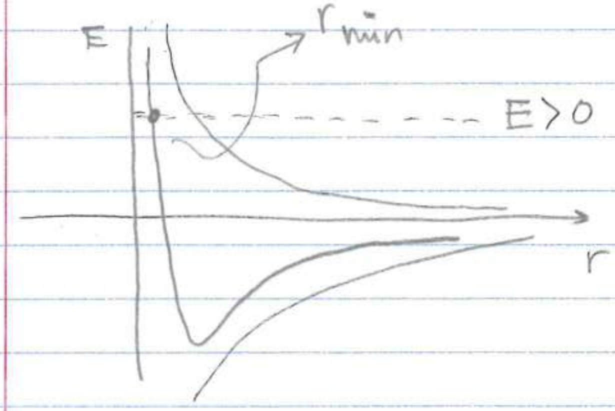
Simplify $\quad \overbrace{\hphantom{xxxxxx}}^{U_{eff}}$

$$\boxed{E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\varphi}^2 + U(r) = \text{constant}}$$

Consider total energy for the Comet, Find max & min distance of the comet from the Sun for $E>0$, $E<0$

$E = \frac{1}{2}\mu\dot{r}^2 + U_{eff}$. Since $\frac{1}{2}\mu\dot{r}^2 \geq 0$, the comet's motion is governed by U_{eff}.

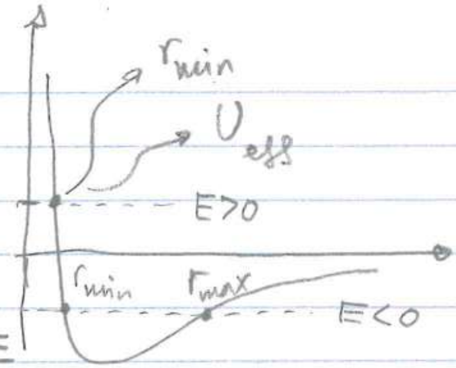
<u>If $E>0$</u>



Comet with $E>0$ can't move anywhere inside U_{eff}.

$\rightarrow$ Turning point

$$\boxed{U_{eff}(r_{min}) = E}$$



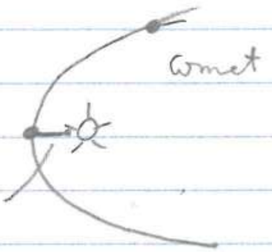
Conservation of Energy (for Comet)

April 27, 2018

$E>0$

As the comet cannot move anywhere inside the turning point r_{min}

Given by the conditions $U_{eff}(r_{min}) = E$

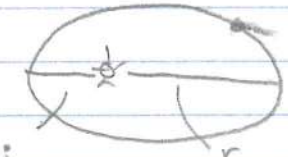


r_{min}

If the comet is initially moving towards the sun, it'll continue to move until r_{min} $(\dot{r}=0)$, then it moves outward to ∞
$\rightarrow$ Unbounded orbit

$E<0$ $\rightarrow$ the line $E<0$ meets the curve at 2 turning points r_{min} & r_{max}

$\hookrightarrow$ Comet trapped between 2 points $\Rightarrow$



Bounded orbit

If the comet is moving away from the sun $(\dot{r}>0)$, it continues to do so until r_{max} $(\dot{r}=0)$, moves inward until reaches r_{min} $(\dot{r}=0)$ $\rightarrow$ Comet oscillates in a orbit between r_{min} & r_{max} $\rightarrow$ Bounded orbit

$\hookrightarrow$ If $E = U_{eff}$, then $r_{min} = r_{max}$ $\Rightarrow$ comet trapped in a circular orbit

April 30, 2018 $\rightarrow$ $\boxed{\text{Orbit equation}}$

Recall $\mu\ddot{r} = \frac{-d}{dr}U_{eff}(r) = \frac{-d}{dr}\left[U(r) + \frac{l^2}{2\mu r^2}\right]$ (angular momentum: l)

So $\mu\ddot{r} = \frac{-d}{dr}U(r) + \frac{l^2}{\mu r^3}$

So $\boxed{\mu\ddot{r} = F(r) + \frac{l^2}{\mu r^3}}$ where $F(r) = \frac{-d}{dr}U(r)$ & $\underbrace{\frac{l^2}{\mu r^3}}_{\text{centrifugal force}}$

taylor class

Goal Find r as a function of $\varphi(\theta) \rightarrow r(\varphi)$

$\boxed{\text{Change of variables}}$ $u = \frac{1}{r} \Rightarrow r = \frac{1}{u}$ $\quad dr = (-1)\frac{1}{u^2}du$

(1) $F(r) = F(u)$ and (2) $\frac{\ell^2}{\mu r^3} = \frac{\ell^2 u^3}{\mu}$

(3) $\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d}{dt}\left(\frac{dr}{du}\frac{du}{d\varphi}\frac{d\varphi}{dt}\right)$

$= \frac{d}{dt}\left(\frac{-1}{u^2}\frac{du}{d\varphi}\frac{d\varphi}{dt}\right) = \frac{d}{dt}\left(\frac{-1}{u^2}\frac{du}{d\varphi}\dot{\varphi}\right) = \frac{d}{dt}\left(\frac{-\dot{\varphi}}{u^2}\frac{du}{d\varphi}\right)$

Now

$\ell = \mu r^2\dot{\varphi} = \frac{\mu\dot{\varphi}}{u^2} \Rightarrow \boxed{\frac{\dot{\varphi}}{u^2} = \frac{\ell}{\mu}}$

So $\frac{d^2r}{dt^2} = \frac{d}{dt}\left(\frac{-\ell}{\mu}\frac{du}{d\varphi}\right) = \frac{-\ell}{\mu}\frac{d}{dt}\left(\frac{du}{d\varphi}\right)$

$= \frac{-\ell}{\mu}\frac{d\varphi}{dt}\frac{d}{d\varphi}\left(\frac{du}{d\varphi}\right)$

$= \frac{-\ell}{\mu}\dot{\varphi}\left(\frac{d^2u}{d\varphi^2}\right) = \frac{-\ell}{\mu}\frac{\ell u^2}{\mu}\left(\frac{d^2u}{d\varphi^2}\right)$

So $\frac{d^2r}{dt^2} = \frac{-\ell^2u^2}{\mu^2}\left(\frac{d^2u}{d\varphi^2}\right)$

Therefore

$$\boxed{\frac{-\ell^2u^2}{\mu}\left(\frac{d^2u}{d\varphi^2}\right) = F(u) + \frac{\ell^2u^3}{\mu}}$$

So $F(u) = \frac{-\ell^2u^2}{\mu}\left(\frac{d^2u}{d\varphi^2} + u\right)$

or $\boxed{\frac{d^2u}{d^2\varphi} + u = \frac{-\mu}{\ell^2u^2}F(u)}$ $\rightarrow$ orbit equation ...

Example Radial eqn for a free particle

(Solve the orbit eqn for a free particle & confirm that the resulting orbit is a straight line.

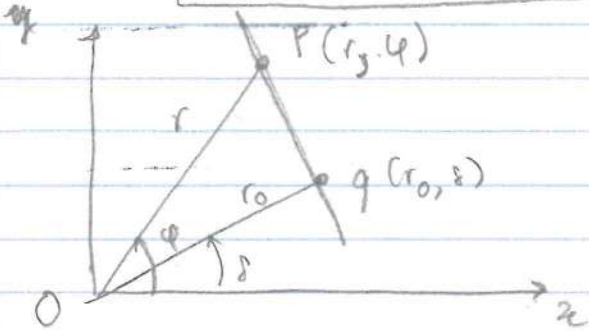
↳ From orbit eqn → $F(u) = 0 \Rightarrow \dfrac{d^2u}{d\varphi^2} + u = 0$

or $u''(\varphi) = -u(\varphi)$

So $u(\varphi) = A\cos(\varphi - \delta)$ A, δ constant

↳ $r(\varphi) = \dfrac{1}{u(\varphi)} = \dfrac{1}{A\cos(\varphi-\delta)} = \dfrac{r_0}{\cos(\varphi-\delta)}$ $\left(A = \dfrac{1}{r_0}\right)$

So $\boxed{r_0 = r(\varphi)\cos(\varphi - \delta)}$ → eqn of straight line in polar coordinates...



$P(r, \varphi)$, $Q(r_0, \delta)$

P lies on the line only if

$$\boxed{r(\varphi) = \frac{r_0}{\cos(\varphi - \delta)}}$$

(Example 2 Find possible orbits of a comet subject to inverse square force → Kepler's orbit)

$F = -\dfrac{GMm}{r^2}$, $F = \dfrac{+1\, q_1 q_2}{4\pi\varepsilon_0 r^2}$

So $\boxed{F(r) = \dfrac{-\gamma}{r^2} = -\gamma u^2}$ (γ = positive force constant)

↳ $\dfrac{d^2u}{d\varphi^2} + u = \dfrac{-\mu}{\ell^2 u^2}(-\gamma u^2) = \dfrac{\mu\gamma}{\ell^2}$

So $\boxed{u''(\varphi) = -u(\varphi) + \frac{\gamma\mu}{l^2}}$

To solve ... let $w = u(\varphi) - \frac{\gamma\mu}{l^2}$

$\Rightarrow \quad w'' = u''(\varphi) = -w(\varphi)$

So $\boxed{w(\varphi) = A\cos(\varphi - \delta)}$ A - positive constant, δ can be taken to zero for a suitable choice of dir φ

So $\boxed{w(\varphi) = A\cos(\varphi)}$

So $u(\varphi) = A\cos(\varphi) + \frac{\gamma\mu}{l^2} = \boxed{\frac{\gamma\mu}{l^2}\left(1 + \epsilon\cos\varphi\right) = u(\varphi)}$

where $\epsilon = \text{const} = \frac{Al^2}{\mu\gamma}$ = eccentricity.

Since $u(\varphi) = \frac{1}{r(\varphi)}$ ... introduce $C = \frac{l^2}{\gamma\mu}$

$\rightarrow \frac{1}{r(\varphi)} = \frac{1}{C}\left(1 + \epsilon\cos\varphi\right)$

So $\boxed{r(\varphi) = \frac{C}{1 + \epsilon\cos\varphi}}$ $\rightarrow$ orbit due to inverse square law force ...

The behavior of the orbit is determined by $\epsilon < 1$ or $\epsilon \geq 1$

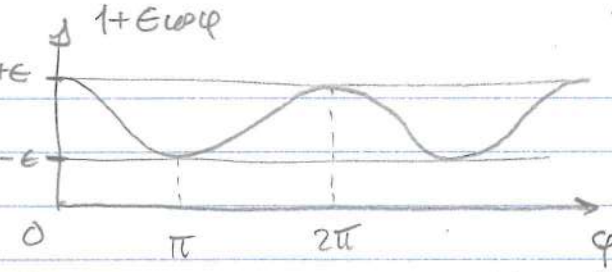
If $\epsilon = \frac{Al^2}{\gamma\mu} < 1$, then the denominator never vanishes $\Rightarrow r(\varphi)$ bounded

If $\epsilon \geq 1$, then denominator vanishes at some angle $\Rightarrow r(\varphi) \rightarrow \infty$ at some angle...

If $\epsilon = 1 \Rightarrow$ is boundary between unbounded & bounded orbit

Note

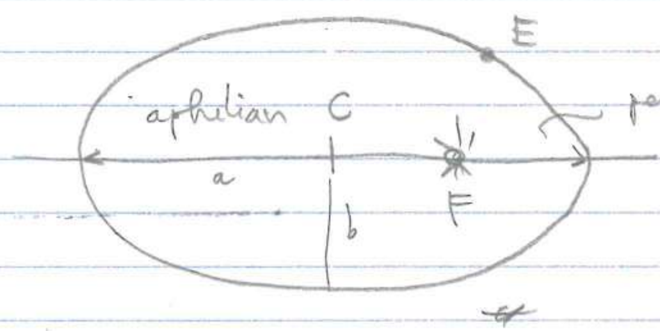
For $\epsilon < 1$



The denominator $1+\epsilon\cos\varphi$ oscillates between $1+\epsilon$ & $1-\epsilon$
also periodic $\rightarrow 2\pi$

$$\underbrace{\boxed{r_{min} = \frac{C}{1+\epsilon}}}_{\text{perihelion } \varphi=0}, \quad \boxed{r_{max} = \frac{C}{1-\epsilon}}$$

perihelion $\varphi = 0$ aphelion $\varphi = \pi$



a: semimajor axis
b: semiminor axis

May 4, 2018

Rewrite $R(\theta) = \dfrac{C}{1+\epsilon\cos\theta}$ in Cartesian coordinates - cast it in form of ellipse

$$\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{where } a = \frac{C}{1-\epsilon^2},\ b = \frac{C}{\sqrt{1-\epsilon^2}},\ d = a\epsilon$$

$x = r\cos\varphi$ & $C = r(1+\epsilon\cos\theta)$ $\qquad C = \dfrac{l^2}{\gamma\mu}$, $\gamma = GMm$

$C = r + x\epsilon$

$\Rightarrow r^2 = (C - x\epsilon)^2$

$\Rightarrow r^2 = c^2 - 2xc\epsilon + (x\epsilon)^2$

$$\Rightarrow x^2 + \frac{2c\epsilon}{1-\epsilon^2}x + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}$$

$$d = \frac{c\epsilon}{1-\epsilon^2} \rightarrow x^2 + 2dx + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}$$

$$(x+d)^2 + \frac{y^2}{1-\varepsilon^2} = \frac{c^2}{1-\varepsilon^2} + d^2 = \frac{c^2}{1-\varepsilon^2}\left(1+\frac{\varepsilon^2}{1-\varepsilon^2}\right) = \left(\frac{c^2}{1-\varepsilon^2}\right)^2 = a^2$$

So

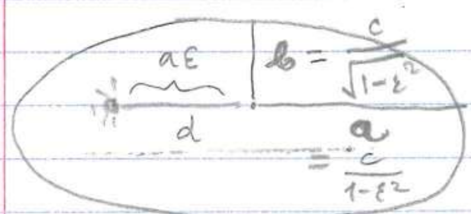
$$\frac{(x+d)^2}{a^2} + \frac{y^2}{a^2(1-\varepsilon^2)} = 1 \Rightarrow \boxed{\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1}$$

Let $b^2 = a^2(1-\varepsilon^2) = \left(\frac{c^2}{1-\varepsilon^2}\right)^2(1-\varepsilon^2) = \frac{c^2}{1-\varepsilon^2}$ (✓)

Note $\boxed{\frac{b}{a} = \sqrt{1-\varepsilon^2}}$ → ratio of major to minor axes.

And

ε = eccentricity. If $\varepsilon = 0$ → circle.
If $\varepsilon = 1$ → elongated



Position of the sun : $d = a\varepsilon$

↳ Exactly the distance from the center to either focus of the ellipse

→ Orbit of each planet is an ellipse with the sun located at one of the focal points — KEPLER'S FIRST LAW

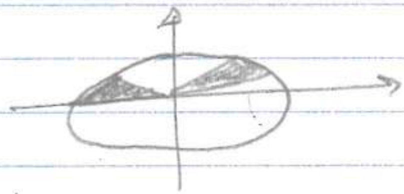
Halley's Comet → follows a very eccentric orbit $\varepsilon = 0.967$
At closest approach (perihelion), the comet is
$1AU = 1.5\times10^8$ km   0.59 AU from the sun.
What is the comet's greatest distance from the Sun?

$$r_{min} = \frac{c}{1+\varepsilon} \rightarrow c = r_{min}(1+\varepsilon)$$

$$r_{max} = \frac{c}{1-\varepsilon} = \frac{r_{min}(1+\varepsilon)}{1-\varepsilon} = \frac{0.59(1.967)}{0.033} = 35.2\,AU$$

KEPLER'S SECOND LAW (law of equal areas)

↳ A line drawn between the sun - planet sweeps out equal areas in equal time



$$\boxed{dA = \frac{1}{2} r^2 d\theta = \frac{1}{2}(r)(r d\theta)}$$

Or $r^2\dot{\theta} = \text{const}$

or $mr^2\dot{\theta} = \text{const}$ or $|\vec{L}|$ is contant (conserved)

$\Rightarrow$ gravity is a central force.

KEPLER'S THIRD LAW — Harmonic law

↳ The square of the period of a planet is directly portional to the cube of the semi major axis of the orbit

$$\frac{a^3}{T^2} = \text{constant}$$

Ex Mercury : $T = 87.97$ days, $a = 0.579 \times 10^8$ km $\Big\}$ $\frac{a^3}{T^2} = 2.51 \times 10^{19}$ km^3/d^2

Earth : $T = 365.3$ year, $a = 1.5 \times 10^8$ km $\Big\}$ $\frac{a^3}{T^2} = 2.51 \times 10^{19}$ km^3/d^2

Derivation $\frac{dA}{dt} = \frac{\ell}{2\mu}$ Total area = πab

$$T = \frac{A}{dA/dt} = \frac{\pi ab}{\ell/2\mu} = \frac{2\pi ab\mu}{\ell} \rightarrow T^2 = \frac{4\pi^2 a^2 b^2 \mu^2}{\ell^2}$$

Note $b^2 = \frac{c^2}{1-\varepsilon^2} = a^2(1-\varepsilon^2) = a^2\left(\frac{c}{1-\varepsilon^2}\right)(1-\varepsilon^2) = a^2 c$

So $T^2 = \frac{4\pi^2 a^3 c\mu^2}{\ell^2}$ Note $c = \frac{\ell^2}{\gamma\mu}$ $\rightarrow$ $\boxed{T^2 = \frac{4\pi^2 a^3 \mu}{\gamma}}$

$$T = \sqrt{\frac{4\pi^2 a^3}{G(M+m)}}$$ $\to$ total mass

$$T^2 = \frac{4\pi^2 a^3 \mu}{\gamma} \quad \text{where } F = \frac{-\gamma}{r^2} \quad \gamma = Gm_1 m_2 \simeq G\mu M$$

So $\boxed{T^2 = \frac{4\pi^2 a^3}{GM_s} \to T^2 \propto a^3}$ Kepler's 3rd law

Exp Period of a low-orbit Earth satellite

$$T^2 = \frac{4\pi^2}{GM_{earth}} \cdot r_{earth}^3 \quad \text{where} \quad g = \frac{GM}{r^2}$$

$$= \frac{4\pi^2}{g} r_{earth} = 25.78 \times 10^6 \, s^2$$

So

$$T = 5077 s \Rightarrow \simeq 85 \text{ mins} \ldots$$

Relation between Energy and eccentricity

$$@ \, r_{min} \Rightarrow E = U_{effective} \, @ \, r_{min} = \frac{-\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$

$$\Rightarrow \boxed{E = \frac{1}{2r_{min}}\left(\frac{\ell^2}{\mu r_{min}} - 2\gamma\right)}$$

$$r_{min} = \frac{c}{1+\varepsilon}, \quad c = \frac{\ell^2}{\gamma\mu}, \quad r_{min} = \frac{\ell^2}{\gamma\mu(1+\varepsilon)}$$

So $$\frac{\ell^2}{\mu r_{min}} = \gamma(1+\varepsilon)$$

$$E = \frac{\gamma\mu(1+\varepsilon)}{2\ell^2}\left[\gamma(1+\varepsilon) - 2\gamma\right] = \frac{\gamma\mu(1+\varepsilon)}{2\ell^2}\left(\gamma(\varepsilon - 1)\right)$$

$$\boxed{E = \frac{\gamma^2\mu(\varepsilon^2 - 1)}{2\ell^2}}$$

For $E < 0 \Rightarrow \varepsilon < 1 \to$ bounded

$E > 0 \Rightarrow \varepsilon > 1 \to$ unbounded..

May 9, 2018

The unbounded Kepler orbit

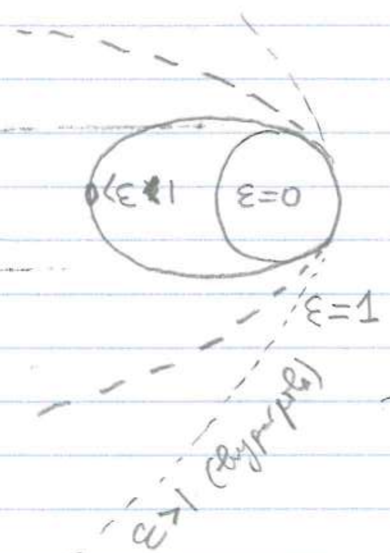
For general Kepler orbit $r(\varphi) = \dfrac{C}{1+\varepsilon\cos\varphi}$

$\varepsilon < 1 \sim E < 0$ (bounded orbit)
$\varepsilon \geq 1 \sim E \geq 0$ (unbounded orbit)

For $\varepsilon \geq 1$, $1+\varepsilon\cos\varphi = 0$ when $\varphi = \pm\pi \to r(\varphi) \to \pm\infty$

Case $\varepsilon = 1$. $r = \dfrac{C}{1+\cos\varphi} \to r(1+\cos\varphi) = C \Rightarrow r + x = C$

So $r = C - x$
But $r^2 = x^2 + y^2 = C^2 + x^2 - 2xC \Rightarrow \boxed{y^2 = C^2 - 2xC} \Rightarrow$ parabola



$\longrightarrow \dfrac{(x-f)^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Summary $r(\varphi) = \dfrac{C}{1+\varepsilon\cos\varphi}$, $E = \dfrac{\gamma^2\mu}{2\ell^2}(\varepsilon^2 - 1)$

ε	E	orbit
$\varepsilon = 0$	<0	circular
$0<\varepsilon<1$	<0	ellipse
$\varepsilon = 1$	$=0$	parabola
$\varepsilon > 1$	>0	hyperbola

where $C = \dfrac{\ell^2}{\gamma\mu}$

$\gamma = G m_1 m_2$

Change of orbit Suppose orbit has bounded, elliptical orbit

$$r(\varphi) = \frac{c}{1+\varepsilon\cos(\varphi-\delta)}$$

Initial orbit → E_1, l_1, orbital parameters c_1 ε_1 δ_1

To change orbits, it fires rockets vigorously for a short time → Impulse → suppose Impulse @ angle φ_0 → gives instantaneous change in velocity.

From change of velocity → E_2, l_2, c_2, ε_2, δ_2 ...

$$\boxed{\frac{c_1}{1+\varepsilon_1\cos(\varphi_0-\delta_1)} = \frac{c_2}{1+\varepsilon_2(\cos(\varphi_0-\delta_2))}}$$

Special cases of eqn

1) Satellite is firing rockets in tangential direction (forward/backwards) @ perigee of initial orbit

Choose axis, $\varphi = 0$, $\varphi_0 = 0$, $\delta_1 = 0$. Also, rockets are in tangential direction, so velocity just after the firing is still in the same direction

So $\boxed{\frac{c_1}{1+\varepsilon_1} = \frac{c_2}{1+\varepsilon_2}}$

Let $\boxed{\lambda = \frac{v_2}{v_1}}$ → thrust factor

- $\lambda > 1$ ⇒ forward
- $0 < \lambda < 1$ ⇒ backward
- $\lambda < 0$ ⇒ reversal of direction

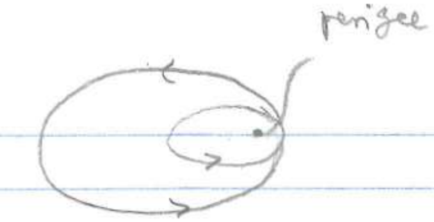
At perigee $l \propto v$

$$l = \mu r v \Rightarrow \boxed{l_2 = \lambda l_1} \rightarrow \boxed{c_2 = \lambda^2 c_1}$$

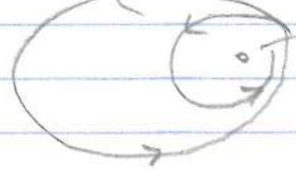
Because $c = \frac{l^2}{\mu \gamma}$

$$\frac{c_1}{c_2} = \frac{1}{\lambda^2} = \frac{1+\varepsilon_1}{1+\varepsilon_2} \rightarrow \frac{1+\varepsilon_2}{1+\varepsilon_1} = \lambda^2 \rightarrow \boxed{\varepsilon_2 = \lambda^2\varepsilon_1 + \lambda^2 - 1}$$

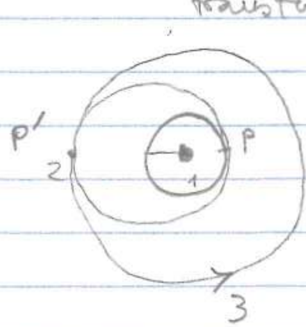
$\boxed{\text{If } \lambda > 1 \text{, so } \varepsilon_2 > \varepsilon_1}$



$\boxed{\text{If } \lambda < 1, \quad \varepsilon_2 < \varepsilon_1}$



Example A satellite in circular orbit (R_1) wants to transfer to a different orbit $2R_1$. The satellite uses 2 successive boosts. The 1^{st} boost $\rightarrow$ into elliptical transfer orbit 2. Secondly, when it reaches desired radius $(2R_1)$ @ P' $\Rightarrow$ it boosts to desired circular orbit (3)



By what factor must it increase speed in each of these boosts? What are the required thrusts? By what factor does the satellite speed increases as a result of the whole maneuver?

Circular orbit $\varepsilon_1 = 0 \Rightarrow c_1 = R_1$
Final orbit $\varepsilon_3 = 0 \Rightarrow c_3 = R_3 = 2R_1$
Transfer orbit $\varepsilon_2 \neq 0 \rightarrow c_2 = \lambda^2 R_1$, $\varepsilon_2 = \lambda^2 - 1$ $\Big\}$ $\lambda \rightarrow$ thrust after first boost @ P

By the time satellite reaches P' (apogee of transfer orbit)

$$R_3 = \frac{c_2}{1-\varepsilon_2} = \frac{\lambda^2 R_1}{1-\lambda^2+1} = \frac{\lambda^2 R_1}{2-\lambda^2} = 2R_1$$

$$\text{So} \quad \lambda^2 = 2(2-\lambda^2) \Rightarrow \lambda^2 = \frac{4}{3} \Rightarrow \lambda = \sqrt{\frac{4}{3}} \quad (\lambda > 0)$$

{ So satellite must boost its speed by 15% to move to TRANSFER orbit }

Boost factor @ $P' = \lambda'$

Second orbit @ P' $\quad \dfrac{c_2}{1-\varepsilon_2} = \dfrac{c_3}{1} \Rightarrow c_3 = \lambda'^2 c_2$

$c_3 = \lambda'^2 c_2 \rightarrow \dfrac{c_2}{1-\varepsilon_2} = \lambda'^2 c_2$ <u>so</u> $\lambda'^2 = \dfrac{1}{1-\varepsilon_2}$

<u>So</u> $\lambda'^2 = \dfrac{1}{1-\lambda^2+1} = \dfrac{1}{2-\lambda^2}$ <u>or</u> $\boxed{\lambda' = 1.22}$

Boost by 22% to move from transfer to final orbit.