Homework 1 6.6410/8.370/18.435/2.111 Due 9/21

1. We start with some properties of unitary matrices that will turn out to be very useful.

   (a) Consider the d-dimensional quantum space with basis is $|\,1\rangle, |\,2\rangle, \ldots, |\,d\rangle$. Suppose you have a different orthonormal basis $|\,v_1\rangle, |\,v_2\rangle, \ldots, |\,v_d\rangle$. for it. Show that there is a unitary transformation U such that $U\,|\,j\rangle = |\,v_j\rangle$. For this problem, you should assume that the definition of a unitary matrix is a matrix for which $UU^\dagger = I$, or equivalently, $U^\dagger U = I$.

   If you don't recall from linear algebra what an orthonormal basis is, it's one that satisfies $\langle v_i | v_j \rangle = \delta_{i,j}$, where δ is the Kronecker delta function.

   (b) Suppose $|\,v_1\rangle, |\,v_2\rangle, \ldots, |\,v_d\rangle$ are an orthonormal basis of a d-dimensional quantum state space. Show that

$$\sum_{i=1}^{d} |\,v_i\rangle\langle v_i\,| = I.$$

2. In this problem, we will see the relation between the angle between two quantum states and the angle between the associated points of the Bloch sphere.

   (a) Recall from lecture that the point $p_i = (x_i, y_i, z_i)$ on the Bloch sphere associated the quantum state of a qubit $|\,v_i\rangle$ satisfies

$$x_i\sigma_x + y_i\sigma_y + z_i\sigma_z = |\,v_i\rangle\langle v_i\,| - |\,\bar{v}_i\rangle\langle \bar{v}_i\,|\,, \tag{1}$$

   where $|\,\bar{v}_i\rangle$ is a state orthogonal to $|\,v_i\rangle$. (There are many orthogonal states $|\,\bar{v}_i\rangle$, but they all give the same value of $|\,\bar{v}_i\rangle\langle \bar{v}_i\,|$.) Find an expression for $|\,v_i\rangle\langle v_i\,|$ in terms of x_i, y_i, z_i, the three Pauli matrices and the 2×2 identity matrix.

   (b) Use this formula and the half-angle formula from trigonometry to find a relation between $\arccos |\langle v_1 | v_2 \rangle|$ and $\arccos p_1 \cdot p_2$. You may need the fact that

$$|\langle v_1|v_2\rangle|^2 = \langle v_1|v_2\rangle\langle v_2|v_1\rangle = \mathrm{Tr}\left(|\,v_1\rangle\langle v_1\,|\;|\,v_2\rangle\langle v_2\,|\right),$$

   which can be proved using the cyclic property of trace, $\mathrm{Tr}\,ABC = \mathrm{Tr}\,CBA$.

3. Suppose a qubit is in the state

$$|\,\psi\rangle = \frac{1}{\sqrt{3}}\,|\,0\rangle + \frac{1+i}{\sqrt{3}}\,|\,1\rangle$$

   If a von Neumann measurement is applied using the basis

$$\left\{ \tfrac{1}{\sqrt{2}}(|\,0\rangle + i\,|\,1\rangle), \tfrac{1}{\sqrt{2}}(|\,0\rangle - i\,|\,1\rangle) \right\},$$

   what are the probabilities of the various outcomes? Remember that when you go from the ket to the bra, you have to take the complex conjugate.

4. A qutrit is a three-state quantum system.

   (a) Show that

   $$\begin{aligned} |\, a\rangle &= \frac{1}{2}\,|\, 0\rangle + \frac{1}{\sqrt{2}}\,|\, 1\rangle + \frac{1}{2}\,|\, 2\rangle\,, \\ |\, b\rangle &= \frac{1}{2}\,|\, 0\rangle - \frac{1}{\sqrt{2}}\,|\, 1\rangle + \frac{1}{2}\,|\, 2\rangle\,, \\ |\, c\rangle &= \frac{1}{\sqrt{2}}\,|\, 0\rangle - \frac{1}{\sqrt{2}}\,|\, 2\rangle\,, \end{aligned}$$

   is an orthonormal basis for a qutrit.

   (b) Suppose a qutrit is in the state $\frac{1}{\sqrt{3}}(|\, 0\rangle + |\, 1\rangle - |\, 2\rangle)$ and is measured using the von Neumann measurement associated with the basis in part (a). What are the probabilities of the various outcomes?

5. Imagine perfect polarizing filters that let all horizontally polarized photons through and filter out all vertically polarized photons. If we put two such polarizing filters on top of each other, and rotate the top one by $90°$, no light will get through.

   (a) Now, suppose we put a polarizing filter between them at an angle of $45°$. What fraction of the photons make it through, on average?

   (b) Now, put two filters between them, one rotated by $30°$ and the other by $60°$. What order should you put them in to make the most light come through, and what fraction of the photons will make it through?

6. Suppose somebody gives me a qubit which is equally likely to be $|\, 0\rangle$, $|\, 1\rangle$, $|+\rangle$, and $|-\rangle$, and challenges me to make a copy of it. There is a no-cloning theorem in quantum mechanics that says I cannot succeed 100% of the time.

   (a) One thing I could do is measure it in the $\{|\, 0\rangle\,, |\, 1\rangle\}$ basis, and make two copies of the resulting state. I do this, and hand the challenger both qubits. They measure them in either the $\{|\, 0\rangle\,, |\, 1\rangle\}$ basis, or the $\{|+\rangle\,, |-\rangle\}$ basis, depending on which basis their original qubit was in. I succeed only if both qubits are measured to be equal to the challenger's original qubit. What is the probability I pass the test?

   (b) I could try to do better by measuring in an intermediate basis, say

   $$\{\cos(\theta)\,|\, 0\rangle + \sin(\theta)\,|\, 1\rangle\,, -\sin(\theta)\,|\, 0\rangle + \cos(\theta)\,|\, 1\rangle\}$$

   What is the probability that I succeed if I use this basis (as a function of θ)?

   (c) How should I pick θ in part b to maximize my probability of success? Does it matter?