Thurs., Sept. 2

Quantum measurements

Why is it that $[\mathcal{O}(x), \mathcal{O}'(y)] = 0$ implies no information is being passed along?

Let $\mathcal{O}$ be some quantum operator that is observable. Let $\mathcal{O}'$ be some other quantum observable operator. Let the eigenstates of $\mathcal{O}$ be

$$\mathcal{O}|\psi_n\rangle = \lambda_n |\psi_n\rangle \qquad n = 1, 2, \cdots N$$

and let P_n be the associated spectrum projection operator

$$P_n = |\psi_n\rangle\langle\psi_n|$$

(we take $\langle\psi_n|\psi_n\rangle = 1$)

Similarly let $\mathcal{O}'|\psi_n'\rangle = \lambda_n' |\psi_n\rangle \qquad n = 1, 2, \cdots, N'$

$$P_n' = |\psi_n'\rangle\langle\psi_n'|$$

Since these are projection operators

$$P_n^2 = P_n \qquad \sum_{n=1}^{N} P_n = 1$$

$$P_n'^2 = P_n' \qquad \sum_{n=1}^{N'} P_n' = 1$$

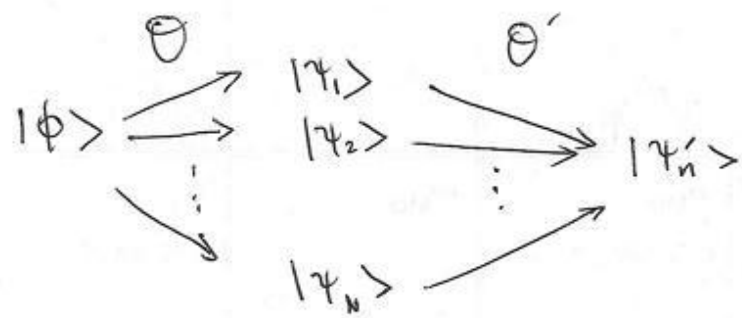
The act of a quantum measurement collapses the wavefunction to one of eigenstates. Let $|\phi\rangle$ be the initial quantum state. We take $\langle\phi|\phi\rangle = 1$. Then the act of measurement means

$$|\phi\rangle \longrightarrow \begin{cases} |\psi_1\rangle & \text{with probability } |\langle\psi_1|\phi\rangle|^2 \\ |\psi_2\rangle & \text{with probability } |\langle\psi_2|\phi\rangle|^2 \\ \vdots & \text{or} \\ |\psi_N\rangle & \text{with probability } |\langle\psi_N|\phi\rangle|^2 \end{cases}$$

<u>Case I</u>: Suppose $|\phi\rangle$ is the initial state and we measure $\mathcal{O}'$. The probability we get λ_n' is

$$|\langle \psi_n' | \phi \rangle|^2 = \langle \phi | P_n' | \phi \rangle$$

Case II: Suppose $|\phi\rangle$ is the initial state. We first measure Θ and then measure Θ'. Let's calculate the probability we get λ_n'.



The total probability is

$$|\langle \psi_1 | \phi \rangle|^2 \, |\langle \psi_n' | \psi_1 \rangle|^2 + |\langle \psi_2 | \phi \rangle|^2 \, |\langle \psi_n' | \psi_2 \rangle|^2 + \cdots$$

We can rewrite this as

$$\langle\phi|\psi_1\rangle\langle\psi_1|\psi_n'\rangle\langle\psi_n'|\psi_1\rangle\langle\psi_1|\phi\rangle$$
$$+\langle\phi|\psi_2\rangle\langle\psi_2|\psi_n'\rangle\langle\psi_n'|\psi_2\rangle\langle\psi_2|\phi\rangle$$
$$+ \quad \vdots$$

We can write this more simply as

$$\langle\phi|P_1 P_n' P_1|\phi\rangle + \langle\phi|P_2 P_n' P_2|\phi\rangle + \cdots$$

If Θ and Θ' commute then they can be simultaneously diagonalized. In that case P_n and P_n' are also made diagonal, which shows that they must commute. So we have

$$\langle\phi|P_n' P_1^2|\phi\rangle + \langle\phi|P_n' P_2^2|\phi\rangle + \cdots$$

$$= \langle\phi|P_n' P_1|\phi\rangle + \langle\phi|P_n' P_2|\phi\rangle + \cdots$$

$$= \langle\phi|P_n'(P_1 + P_2 + \cdots + P_N)|\phi\rangle$$

$$= \langle\phi|P_n'|\phi\rangle$$

We have shown that measuring Θ has no effect on the statistics of the $\mathcal{O}'$ measurement if the operators commute.

Consider the function $\frac{1}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})}$

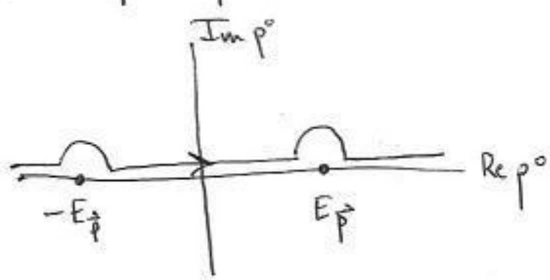
It has poles at $p^0 = E_{\vec{p}}$ and $p^0 = -E_{\vec{p}}$.

Suppose we fourier transform back to a function of time

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dp^0 \, e^{-ip^0 t}}{(p^0 - E_{\vec{p}})(p^0 + E_{\vec{p}})}$$

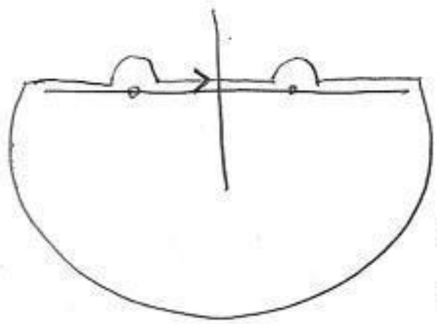
We need to define the contour... how we avoid the poles.

Suppose we take a contour passing above the poles in the complex plane.



Notice that the $e^{-ip^0 t}$ means that for $t > 0$ as $p^0 \to -i\infty$ we have $e^{-ip^0 t} \to e^{-\infty} \to 0$.

So we continue the contour in the lower half plane for $t > 0$.



$t > 0$

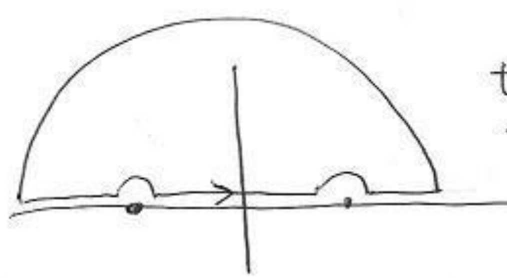
$e^{-ip^0 t} \to 0$ as $p^0 \to -i\infty$

residue of pole at $-E_{\vec{p}}$ ... at $E_{\vec{p}}$

In this case we get $f(t) = (-2\pi i) \times \left[\frac{e^{+iE_{\vec{p}}t}}{2\pi(-2E_{\vec{p}})} + \frac{e^{-iE_{\vec{p}}t}}{2\pi(2E_{\vec{p}})}\right]$

$$= -\frac{i}{2E_{\vec{p}}}\left(e^{-iE_{\vec{p}}t} - e^{iE_{\vec{p}}t}\right) \quad (\text{for } t > 0)$$

If $t < 0$ then we must continue the contour in the upper half plane



$t < 0$

$e^{-ip^0 t} \to 0$ as $p^0 \to +i\infty$

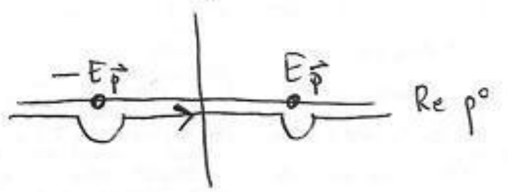
So for $t<0$ $\quad f(t)=0$. $\qquad \theta(x)=\begin{cases}1 & x\geq 0\\ 0 & x<0\end{cases}$

Putting all together, $f(t) = \theta(t)\times\frac{(-i)}{2E_{\vec{p}}}\left(e^{-iE_{\vec{p}}t}-e^{iE_{\vec{p}}t}\right)$

if we go above the two poles at $\pm E_{\vec{p}}$.

This is called a retarded Green's function or forward propagating Green's function since the signal is nonzero for $t>0$.

If we instead we go below both poles

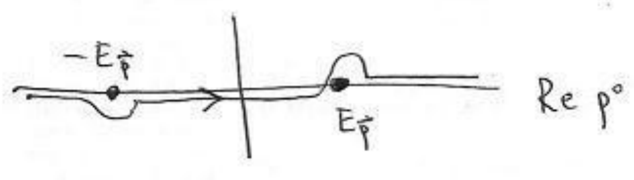


then we get

$$f(t) = \theta(-t)\frac{(+i)}{2E_{\vec{p}}}\left(e^{-iE_{\vec{p}}t}-e^{iE_{\vec{p}}t}\right)$$

This an advanced or backward propagating Green's functions since the signal is nonzero for $t<0$.

Suppose we go below the $-E_{\vec{p}}$ pole (hence backward propagating) and above the $E_{\vec{p}}$ pole (hence forward propagating).



We then have $\quad f(t) = \theta(t)(-i)\dfrac{e^{-iE_{\vec{p}}t}}{2E_{\vec{p}}} + \theta(-t)(-i)\dfrac{e^{iE_{\vec{p}}t}}{2E_{\vec{p}}}$

This is a time-ordered Green's function. The terminology will be clear shortly...

Let us to the two-point function for free fields

$$D(x-y) \equiv \langle 0 |\, \phi(x)\, \phi(y)\, | 0 \rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{2E_{\vec{p}}}\, e^{-ip\cdot(x-y)} \Big|_{(p^0 = E_{\vec{p}})}$$

Suppose now I define a time-ordered product of fields...

$$T\{\phi(x)\,\phi(y)\} \equiv \phi(x)\,\phi(y)\,\theta(x^0-y^0) + \phi(y)\,\phi(x)\,\theta(y^0-x^0)$$

[mnemonic: latest on the left]

Then $\langle 0|T\{\phi(x)\,\phi(y)\}|0\rangle$

$$= D(x-y)\,\theta(x^0-y^0) + D(y-x)\,\theta(y^0-x^0)$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{+i\vec{p}\cdot(\vec{x}-\vec{y})} \underbrace{\left[\theta(x^0-y^0)\,\frac{e^{-iE_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}} + \theta(y^0-x^0)\,\frac{e^{iE_{\vec{p}}(x^0-y^0)}}{2E_{\vec{p}}}\right]}_{\substack{i \text{ times our time-ordered Green's function} \\ \int \frac{dp^0}{2\pi} \frac{i e^{-ip^0(x^0-y^0)}}{(p^0-E_{\vec{p}})(p^0+E_{\vec{p}})}}}$$

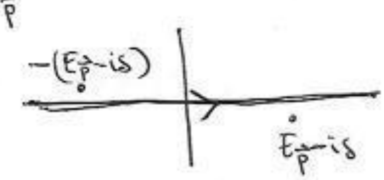
$$= \int \frac{dp}{(2\pi)^4}\, \frac{i e^{-ip\cdot(x-y)}}{(p^0)^2 - E_{\vec{p}}^2} = \int \frac{d^4p}{(2\pi)^4}\, \frac{i\, e^{-ip\cdot(x-y)}}{p^2-m^2}$$

where we go below the pole at $p^0 = -E_{\vec{p}}$
... above the pole at $p^0 = +E_{\vec{p}}$.

The prescription for the contour can be rewritten as limit...

$$\frac{1}{(p^0-E_{\vec{p}})(p^0+E_{\vec{p}})} = \lim_{\delta\to 0^+} \frac{1}{(p^0-(E_{\vec{p}}-i\delta))(p^0+(E_{\vec{p}}-i\delta))}$$

below $-E_{\vec{p}}$
above $E_{\vec{p}}$



Since $(p^0-(E_{\vec{p}}-i\delta))(p^0+(E_{\vec{p}}-i\delta)) = (p^0)^2-(E_{\vec{p}}-i\delta)^2$

$$= (p^0)^2 - E_{\vec{p}}^2 + 2i\delta E_{\vec{p}} \cancel{+\delta^2}$$

$$= p^2 - m^2 + 2i\delta E_{\vec{p}}$$

Since $E_{\vec{p}} > 0$ we can write as

$$\lim_{\varepsilon\to 0^+} \frac{1}{p^2-m^2+i\varepsilon}$$

This is how it is usually written.

So we have

$$\langle 0 | T\{\phi(x)\,\phi(y)\} | 0 \rangle$$

$$= D(x-y)\,\Theta(x^0-y^0) + D(y-x)\,\Theta(y^0-x^0)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon}\, e^{-ip\cdot(x-y)} \quad \text{as} \quad \varepsilon \to 0^+$$

This is called the Feynman propagator and written as $D_F(x-y)$.

Note that $(\partial_\mu \partial^\mu + m^2)\, D_F(x) = -i\,\delta^{(4)}(x)$.

$$\left[\text{recall } \delta^{(4)}(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot x}\right]$$

Lorentz Invariance

Consider an arbitrary Lorentz transformation

$$x^\mu \to x'^\mu = \underbrace{\Lambda^\mu{}_\nu}_{4\times 4} x^\nu$$

This induces a transformation

$$\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)$$

Why Λ^{-1} and not Λ? Because we require

$$\phi'(x') = \phi(x)$$
$$\Rightarrow \phi'(x) = \phi(\Lambda^{-1}x)$$

How about $\partial_\mu \phi$?

$$\begin{aligned}\partial_\mu \phi(x) &\to \partial_\mu\left[\phi(\Lambda^{-1}x)\right] \\ &= \partial_\mu (\Lambda^{-1}x)^\nu (\partial_\nu \phi)(\Lambda^{-1}x) \\ &= (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x)\end{aligned}$$

This is how a lower index object transforms.

upper index : $x^\mu \to \Lambda^\mu{}_\nu \, x^\nu$

$$\underset{4\times4}{\left[\ \Lambda\ \right]} \left[x\right] = \left[x'\right]$$

lower index: $x_\mu \to x_\nu \left(\Lambda^{-1}\right)^\nu{}_\mu$

$$\left[\ x\ \right] \underset{4\times4}{\left[\ \Lambda^{-1}\ \right]} = \left[\ x'\ \right]$$