# Classical Mechanics III (8.(3)09)

# Assignment 10: Solutions

November 17, 2021

## 1. Classifying Fixed Points [12 points]

Setting the derivatives to be zero at the fixed points, we have $x^*(4 + y^* - x^{*2}) = y^*(x^* - 1) = 0$, which gives the solutions

$$(x^*, y^*) = (0, 0),\ (2, 0),\ (-2, 0),\ (1, -3).$$

Let's linearize around the fixed points (in the following we will ignore all quadratic terms or above):
(i) $(x^*, y^*) = (0, 0)$. Then

$$\begin{aligned} \dot{x} &= 4x \\ \dot{y} &= -y \end{aligned} \qquad\qquad \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

This is clearly a saddle point, where the eigenvectors are just $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, for the repelled and attracted modes, respectively.
(ii) $(x^*, y^*) = (2, 0)$. Defining $u = x - x^* = x - 2$ and $v = y - y^* = y$, we have

$$\begin{aligned} \dot{u} &= (u + 2)(4 + v - (u + 2)^2) = -8u + 2v \\ \dot{v} &= v((u + 2) - 1) = v \end{aligned}$$

(ignoring second order terms), or

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -8 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

We can read off the eigenvalues from the diagonal (since the matrix is upper-triangular); they are $-8$ and $+1$. Hence the fixed point is a saddle point. (The relevant directions, or (unnormalized) eigenvectors, are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 9 \end{pmatrix}$, for the attracted and repelled solutions, respectively.)

(iii) $(x^*, y^*) = (-2, 0)$. Defining $u = x - x^* = x + 2$ and $v = y - y^* = y$, we have

$$\begin{aligned} \dot{u} &= (u-2)(4+v-(u-2)^2) = -8u + 2v \\ \dot{v} &= v((u-2)-1) = -3v \end{aligned}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -8 & -2 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Again we can read off the eigenvalues $-8$ and $-3$, so this is a stable node (with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$).

(iv) $(x^*, y^*) = (1, -3)$. Defining $u = x - x^* = x - 1$ and $v = y - y^* = y + 3$, we have

$$\begin{aligned} \dot{u} &= (u+1)(4+(v-3)-(u+1)^2) = -2u + v \\ \dot{v} &= u(v-3) = -3u \end{aligned}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

The trace of the matrix is $-2$, while the determinant is 3. Hence the characteristic equation is $\lambda^2 + 2\lambda + 3 = 0$ with roots $-1 \pm \sqrt{-2}$, and the fixed point is an attractor (with spiral trajectories).

## 2. Chaos in an Undamped Nonlinear Oscillator [12 points]

See the Mathematica notebook posted with the solutions.

## 3. Bead on a Rotating Hoop [20 points]

(a) [3 points] Dividing by $ma\omega_0^2$, we have

$$\frac{\ddot{\theta}}{\omega_0^2} = -\frac{\beta}{ma\omega_0}\frac{\dot{\theta}}{\omega_0} + \sin\theta\left(\cos\theta - \frac{g}{a\omega_0^2}\right)$$

Now defining $b = \frac{\beta}{ma\omega_0} \geq 0$, $\frac{1}{\gamma} = \frac{g}{a\omega_0^2} > 0$, and rescaling to $t' = \omega_0 t$ (so $\frac{d\theta}{dt'} = \frac{1}{\omega_0}\frac{d\theta}{dt}$ and $\frac{d^2\theta}{dt'^2} = \frac{1}{\omega_0^2}\frac{d^2\theta}{dt^2}$, we have, setting $\omega = \dot{\theta}$,

$$\dot{\theta} = \omega$$
$$\dot{\omega} = -b\omega + \sin\theta\left(\cos\theta - \frac{1}{\gamma}\right)$$

where the dots now indicate differentiation with respect to the new time $t'$. This system of equations is invariant under the transformation $\theta \to -\theta$ and $\omega \to -\omega$.

(b) [9 points] Any fixed point must satisfy $\omega^* = 0$, so $b\omega^* = 0$ whether $b = 0$ or not; the fixed points are the same in the two cases. For the fixed point either $\sin\theta^* = 0$ or $\cos\theta^* - \frac{1}{\gamma} = 0$ (the latter case can happen only if $\gamma \geq 1$). We can thus break into three cases: $\theta^* = n\pi$ for even $n$, $\theta^* = n\pi$ for odd $n$, and $\theta^* = 2\pi n \pm \cos^{-1}(\frac{1}{\gamma})$ for any integer $n$.

We'll start with the case $b > 0$, and do the analysis case-by-case (since the $b = 0$ case will then make use of the same calculations):

- $(\theta^*, \omega^*) = (n\pi, 0)$ for even $n$ ($b > 0$). Without loss of generality set $\theta^* = 0$. Then expanding to first order we have $\dot{\theta} = \omega$ and $\dot{\omega} = -b\omega + (1 - \frac{1}{\gamma})\theta$, or

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 - \frac{1}{\gamma} & -b \end{pmatrix} \begin{pmatrix} \theta \\ \omega \end{pmatrix}$$

The trace is $\tau = -b < 0$, and the determinant is $\Delta = \frac{1}{\gamma} - 1$. Thus the eigenvalues are $\lambda_\pm = [\tau \pm \sqrt{\tau^2 - 4\Delta}]/2 = [-b \pm \sqrt{b^2 - 4(\frac{1}{\gamma} - 1)}]/2$. Therefore:

1. $\gamma > 1$. Then $\Delta < 0$, $\lambda_+ > 0 > \lambda_-$, and the fixed point is a saddle point.
2. $\gamma = 1$. Then $\Delta = 0$, so $\lambda_+ = 0 > \lambda_-$. This is a boundary case and the stability cannot be determined without further analysis. (In the linearized case we would have a non-isolated fixed point.)
3. $\gamma < 1$. Then $\Delta > 0$, and we need to break into further cases depending on the sign of the term in the square root (the discriminant) :
   - $b^2 - 4(\frac{1}{\gamma} - 1) > 0$. Then we have two negative real eigenvalue, and the fixed point is a stable node.
   - $b^2 - 4(\frac{1}{\gamma} - 1) = 0$. There is then one single real eigenvalue $\lambda = -\frac{b}{2}$. In this case

$$\begin{pmatrix} \dot{\theta} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{b^2}{4} & b \end{pmatrix} \begin{pmatrix} \theta \\ \omega \end{pmatrix} = -\frac{b}{2} \begin{pmatrix} \theta \\ \omega \end{pmatrix}$$

     and it can be checked that there is only one independent eigenvector $\begin{pmatrix} 1 \\ b/2 \end{pmatrix}$. In the linearized case this is therefore a degenerate node, but in the general case all we can conclude is that the fixed point is stable (since $\tau < 0$).
   - $b^2 - 4(\frac{1}{\gamma} - 1) < 0$. The eigenvalues are complex conjugates with negative real part, and so we have a stable spiral.

- $(\theta^*, \omega^*) = (n\pi, 0)$ for odd $n$ ($b > 0$). Set $\theta' = \theta - n\pi$, and expanding to first order we have $\sin\theta = \sin(n\pi + \theta') = -\theta'$ and $\cos\theta' = -1$. Hence $\dot{\theta} = \omega$ and $\dot{\omega} = -b\omega + (1 - \frac{1}{\gamma})\theta'$, or

$$\begin{pmatrix} \dot{\theta}' \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 + \frac{1}{\gamma} & -b \end{pmatrix} \begin{pmatrix} \theta' \\ \omega \end{pmatrix}.$$

The trace is $\tau = -b < 0$, and the determinant is $\Delta = -(1 + \frac{1}{\gamma}) < 0$. Thus the eigenvalues are $\lambda_\pm = [\tau \pm \sqrt{\tau^2 - 4\Delta}]/2 = [-b \pm \sqrt{b^2 + 4(\frac{1}{\gamma} + 1)}]/2$. Regardless of the values of $b$ and $\gamma$, we always have $\lambda_+ > 0 > \lambda_-$, and the fixed point is a saddle point.

- $(\theta^*, \omega^*) = (2\pi n \pm \cos^{-1}\frac{1}{\gamma}, 0)$ ($b > 0$, $\gamma > 1$; the $\gamma = 1$ case reduces to the first case). Without loss of generality set $\theta^* = \pm\cos^{-1}\frac{1}{\gamma}$. Then $\cos\theta^* = \frac{1}{\gamma}$, and $\sin\theta^* = \pm\frac{\sqrt{\gamma^2-1}}{\gamma}$. Setting

$\theta' = \theta - \theta^*$, we have to first order

$$\begin{aligned}\sin\theta &= \sin(\theta^* + \theta') = \sin\theta^*\cos\theta' + \cos\theta^*\sin\theta' \\ &\approx \sin\theta^* + \theta'\cos\theta^*\end{aligned}$$

$$\begin{aligned}\cos\theta &= \cos(\theta^* + \theta') = \cos\theta^*\cos\theta' - \sin\theta^*\sin\theta' \\ &\approx \cos\theta^* - \theta'\sin\theta^*\end{aligned}$$

(this can also be obtained by just Taylor expanding around $\theta^*$), and hence

$$\begin{aligned}\dot{\theta}' &= \omega \\ \dot{\omega} &= -b\omega + (\sin\theta^* + \theta'\cos\theta^*)\left(\cos\theta^* - \theta'\sin\theta^* - \frac{1}{\gamma}\right) \\ &= -b\omega + \left(\pm\frac{\sqrt{\gamma^2-1}}{\gamma} + \frac{1}{\gamma}\theta'\right)\left(\frac{1}{\gamma} \mp \frac{\sqrt{\gamma^2-1}}{\gamma}\theta' - \frac{1}{\gamma}\right) \\ &= -b\omega - \left(1 - \frac{1}{\gamma^2}\right)\theta'\end{aligned}$$

$$\begin{pmatrix}\dot{\theta}' \\ \dot{\omega}\end{pmatrix} = \begin{pmatrix}0 & 1 \\ \frac{1}{\gamma^2} - 1 & -b\end{pmatrix}\begin{pmatrix}\theta' \\ \omega\end{pmatrix}.$$

The trace is $\tau = -b$, and the determinant is $\Delta = 1 - \frac{1}{\gamma^2} > 0$. The eigenvalues are $\lambda_\pm = [\tau \pm \sqrt{\tau^2 - 4\Delta}]/2 = \left[-b \pm \sqrt{b^2 - 4\left(1 - \frac{1}{\gamma^2}\right)}\right]/2$. To determine the stability, we look at the discriminant $\tau^2 - 4\Delta$:

1. If $\tau^2 - 4\Delta = b^2 - 4\left(1 - \frac{1}{\gamma^2}\right) > 0$, both eigenvalues are real and negative; we have a stable node.
2. If $\tau^2 - 4\Delta = b^2 - 4\left(1 - \frac{1}{\gamma^2}\right) = 0$, we have a boundary case, and all we can conclude is that the fixed point is stable (since $\tau < 0$). (In the linear case, following the argument as in the case $\theta^* = 0$ we can conclude this is a degenerate node.)
3. If $\tau^2 - 4\Delta = b^2 - 4\left(1 - \frac{1}{\gamma^2}\right) < 0$, the eigenvalues are complex conjugates with a negative real part, and we have a stable spiral.

Now let us deal with the case $b = 0$. Plugging this into our analysis above, we get instead that

- $(\theta^*, \omega^*) = (n\pi, 0)$ for even $n$ ($b = 0$). Then $\tau = -b = 0$.

  1. $\gamma > 1$. Then $\Delta < 0$, $\lambda_+ > 0 > \lambda_-$, and the fixed point is a saddle point.
  2. $\gamma = 1$. Then $\Delta = 0$, so $\lambda_+ = 0 > \lambda_-$. This is a boundary case and the stability cannot be determined without further analysis. (In the linearized case we would have a non-isolated fixed point.)
  3. $\gamma < 1$. Then $\Delta > 0$, so $\tau^2 - 4\Delta < 0$, and the two eigenvalues $\lambda_\pm = \pm i\sqrt{\Delta}$ are pure imaginary complex conjugates. Thus the fixed point is a center. (Actually all we can

conclude at this point is that this is a center for the linearized system; however in part (c) it will be shown that the system is conservative for $b = 0$, and hence the fixed point is a center as well for the nonlinear system.)

- $(\theta^*, \omega^*) = (n\pi, 0)$ for odd $n$ ($b = 0$). The fixed point is a saddle point.
- $(\theta^*, \omega^*) = (2\pi n \pm \cos^{-1}\frac{1}{\gamma}, 0)$ ($b = 0$, $\gamma > 1$). The trace is $\tau = -b = 0$, and the determinant is $\Delta = 1 - \frac{1}{\gamma^2} > 0$. The eigenvalues are $\lambda_\pm = [\tau \pm \sqrt{\tau^2 - 4\Delta}]/2 = \pm i\sqrt{\Delta}$, so the fixed point is a center. (The case $\gamma = 1$, $\Delta = 0$ is covered above in $(\theta^*, \omega^*) = (n\pi, 0)$ for even $n$; it is a boundary case.)

(c) [5 points] Setting $\vec{f} = (f_\theta(\theta, \omega), f_\omega(\theta, \omega))$, where $f_\theta = \dot{\theta} = \omega$ and $f_\omega = \dot{\omega} = \sin\theta\left(\cos\theta - \frac{1}{\gamma}\right)$, we see immediately that $\vec{\nabla} \cdot \vec{f} = 0$: the system is conservative (phase volumes remain constant). For 2D, this means that we can find a conserved energy $E$ by setting

$$\begin{aligned} H(\theta, \omega) &= \int^\omega f_\theta(\theta, \omega')d\omega' - \int^\theta f_\omega(\theta', \omega)d\theta' \\ &= \frac{\omega^2}{2} + \cos\theta\left(\frac{\cos\theta}{2} - \frac{1}{\gamma}\right). \end{aligned}$$

We can also go back to the physical system and look at its Hamiltonian and energy (Pset #1 with $\theta \to \theta - \pi$ to obtain our convention for the angle here) which are:

$$\begin{aligned} H &= \frac{m}{2}(a^2\dot{\theta}^2 - a^2\omega_0^2 \sin^2\theta) - mga\cos\theta \\ E = T + V &= \frac{m}{2}(a^2\dot{\theta}^2 + a^2\omega_0^2 \sin^2\theta) - mga\cos\theta. \end{aligned}$$

This $H$ agrees with our $H(\theta, \omega)$ above after using $\sin^2\theta = 1 - \cos^2\theta$ and ignoring the difference in multiplicative and additive constants. In $E$ the first two terms are the kinetic energy, and the last term is the gravitational potential energy. So $H$ is not the energy because of the difference in sign in front of the second term. Physically the rotating hoop is doing work on the bead through its constraint force which is not part of $H$. Our $H(\theta, \omega)$ is equal to the Hamiltonian which is conserved, but is not equal to the non-conserved energy. The plots of constant $H$ are given in the Mathematica notebook.

(d) [3 points] For $b = 1$ and $\gamma = 2$, if we follow our analysis we can conclude that the only stable fixed points are $(\theta^*, \omega^*) = (2\pi n \pm \cos^{-1}\frac{1}{\gamma}, 0)$, which are stable spirals. The plots are given in the Mathematica notebook.

## 4. Bifurcation of a Limit Cycle and Fixed Point [12 points]

(a) [4 points] The first order equations are

$$\begin{aligned} \dot{x} &= \omega \\ \dot{\omega} &= -a\omega(x^2 + \omega^2 - 1) - x \end{aligned}$$

Proving that it has a circular limit cycle can be done by converting into polar coordinates (as in part (c)), but in this case we can also just notice that if $x^2+\omega^2=1$ then this reduces to $\dot{x}=\omega$ and $\dot{\omega}=-x$, which has solution $x=A\sin t$ and $\omega=A\cos t$ for some $A$ (and so $x^2+\omega^2=A^2$). Hence $x^2+\omega^2=1$ is a closed trajectory, with amplitude 1 and period $2\pi$.

For the closed trajectory to be a limit cycle, it also needs to be isolated. We will show below in (c) that it is isolated for $a \neq 0$, but for $a=0$ there is no limit cycle and we simply have a trajectory about a center.

(b) [4 points] The only fixed point is $(x,\omega)=(0,0)$. Only keeping linear terms around this fixed point, we have

$$\begin{pmatrix} \dot{x} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} x \\ \omega \end{pmatrix}$$

with trace $\tau=a$ and determinant $\Delta=1$. The eigenvalues are $\lambda_\pm=(\tau\pm\sqrt{\tau^2-4\Delta})/2=(a\pm\sqrt{a^2-4})/2$. We break into cases:

1. $a>2$: $\tau>0$ and $\tau^2-4\Delta>0$, so this is an unstable node (two negative eigenvalues).

2. $a=2$: $\tau>0$ and $\tau^2-4\Delta=0$, so this node is an unstable boundary case (for the linear case this is a degenerate stable node, since it only has one independent eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$).

3. $0<a<2$: $\tau>0$ and $\tau^2-4\Delta<0$, so this is an unstable spiral (two complex eigenvalues with negative real part).

4. $a=0$: $\tau=0$ and $\tau^2-4\Delta<0$, so this is a boundary case (for the linear case it's a center, since there are two purely imaginary eigenvalues). We'll see in part (c) that this is a center in the general case too.

5. $-2<a<0$: $\tau<0$ and $\tau^2-4\Delta<0$, so this is a stable spiral (two complex eigenvalues with positive real part).

6. $a=-2$: $\tau<0$ and $\tau^2-4\Delta=0$, so this is a stable boundary case (for the linear case this is a degenerate stable node).

7. $a<-2$: $\tau<0$ and $\tau^2-4\Delta>0$, so this is a stable node (two positive eigenvalues).

The fixed point at the origin is stable for $a<0$ and unstable for $a>0$. We can thus guess that for $a>0$ a trajectory that starts from $r=0$ will tend to the limit cycle at $r=1$, and hence the limit cycle at $r=1$ is stable; conversely, for $a<0$ a trajectory that starts at the limit cycle at $r=1$ could tend to the fixed point at $r=0$, and so the limit cycle is unstable. (This is merely a hypothesis at this point, but we'll prove it in the next part.)

(c) [4 points] We make the change of variables $x=r\cos\theta$, $\omega=r\sin\theta$; or $r=\sqrt{x^2+\omega^2}$, $\tan\theta=\frac{\omega}{x}$. So

$$\begin{aligned} \dot{r} &= \frac{1}{r}\frac{d}{dt}(x^2+\omega^2) = \frac{x\dot{x}+\omega\dot{\omega}}{r} \\ &= -\frac{a\omega^2(x^2+\omega^2-1)}{r} \\ &= -ar(r^2-1)\sin^2\theta. \end{aligned}$$

Then:

1. $a = 0$: $\dot{r} = 0$, and hence in this case all trajectories are circles. (Hence the center is a center.) The closed trajectory $r = 1$ is not a limit cycle in this case (nearby trajectories aren't attracted or repelled). [No points off if this case is not discussed.]

2. $a > 0 : \dot{r} < 0$ for $r > 1$ and $\dot{r} > 0$ for $r < 1$, so the limit cycle is stable (and the fixed point is unstable).

3. $a < 0 : \dot{r} > 0$ for $r > 1$ and $\dot{r} < 0$ for $r < 1$, so the limit cycle is unstable (and the fixed point is stable).

(Note that for $a \neq 0$ trajectories near the circle $r = 1$ are either all attracted or all repelled, so $r = 1$ is a limit cycle.) There is a bifurcation at $a = 0$ where the limit cycle and fixed point swap stabilities. We could reasonably call this a transcritical bifurcation of a limit cycle and a fixed point.

## 5. Chaos in Maps [15 points]

See the Mathematica notebook except for part (c):
(c) [3 points] The two values of $x$ for the attractor are in a period-2 orbit, and hence if we take $x = x_0 = x_2$ then

$$\begin{aligned} x &= x_2 = r x_1 (1 - x_1) \\ &= r^2 x (1 - x)[1 - r x (1 - x)] \end{aligned}$$

If we divide out by $x$ (assuming $x \neq 0$, since $x = 0$ isn't actually an attractor) we get

$$r^3 x^2 (2 - x) - r^2 (r + 1) x + (r^2 - 1) = 0$$

as desired.