Linear response theory and applications in many-body quantum physics

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In this paper, we...

I. INTRODUCTION

Talk about linear response theory. Significant because from it we can calculate how complex systems, in particular system of interacting quantum gases, respond to external perturbations which may or may not be time-dependent. What else can we get out of linear response theory? What is linear response theory useful? Because its results are things we can measure in the lab: correlation functions. Will talk about the relationship between measurements and correlations. Will talk about how from linear response theory one obtains sum rules which are related to correlation functions – which again are things one can measure in the lab...

The object of study in linear response theory is the linear response function. In the simplest case, the response function is simply a proportionality constant χ relating the response of a system $x(t)$ due to a perturbation $h(t)$ at that instant t. For linear, time-invariant systems with memory, the response $x(t)$ gets contributions due to past perturbations $h(t')$ and is given by the convolution of the response function with the perturbation:

$$x(t) = \int_{-\infty}^{\infty} \chi(t-t')h(t')\,dt'. \tag{1}$$

For nonlinear systems, $x(t)$ contains higher orders of the perturbation. This leads to the full Volterra expansion. citation here.

In analysis of classical systems, linear response theory is standard: χ solves the equation $L\chi(t-t') = \delta(t-t')$ where L is the linear differential operator associated with the system. It is identically the Green's function associated with L and characterizes the system's response to an external impulse. As Section blah will show, the response function often contains rich physics. For the case of the damped harmonic oscillator, we can fully characterize its resonance behavior and energy dissipation based solely on the (complex) response function.

For quantum systems, an analogous framework exists...

This paper is structured as follows:

say more about what happens in the quantum case. Talk about this thing called the Kubo formula and dissipation-fluctuation theorem. Talk about the Kramer-Kronig relation. Talk about this thing called the structure factor.

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Talk about magnetic susceptibility, which is how much the magnetization change as a function of the applied/external magnetic field. Of course, talk about the trivial example of the damped (driven) harmonic oscillator. s

A non-trivial example is how does one study, say, sound in a BEC or a Fermi gas? Typically the setup is such that one has the Hamiltonian governing the dynamics of some quantum gas and then turn on perturbation: like shaking the trapping potential or stirring the BEC with a laser or something like that. How do we predict the response of the gas under such perturbations, as a function of parameters of the perturbation and as a function of other state variables?

II. CLASSICAL LINEAR RESPONSE

A. Causality

The response function tells us how a system responds *as a consequence* of some perturbation, which means that it respects causality, i.e., $\chi(t) = 0$ for all $t < t_0 = 0$ when the perturbation is turned on. Now consider the Fourier expansion of $\chi(t)$:

$$\chi(t) = \int_{-\infty}^{\infty} e^{-i\omega t}\tilde{\chi}(\omega)\,d\omega.$$

For $t < 0$, we can perform the integral by closing the contour in the upper half-plane. Since the answer is zero, it must be that $\chi(\omega)$ is analytic in the upper half-plane. In other words,

$$\text{Causality} \implies \chi(\omega) \text{ analytic for } \operatorname{Im}(\chi) > 0.$$

Let $\chi'(\omega) = \operatorname{Re}\chi(\omega)$ and $\chi''(\omega) = \operatorname{Im}\chi(\omega)$. $\chi'(\omega)$ is called the *reactive part* of the response, and $\chi''(\omega)$ the *dissipative part*. From the time-invariance of $\chi(t)$, one can show that $\chi'(\omega)$ and $\chi''(\omega)$ are even and odd functions, respectively.

If $\chi(\omega)$ decays faster than $1/|\omega|$ as $\omega \to \infty$ in addition to being analytic in the upper half-plane, then $\chi'(\omega)$ and $\chi''(\omega)$ are related via the Kramer-Kronig relations. In particular, one can reconstruct the full complex response function knowing only its imaginary part:

$$\chi(\omega) = \int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\operatorname{Im}\chi'(\omega')}{\omega' - \omega - i\epsilon}.$$

In the following two examples from classical mechanics, we will see that the imaginary part of the response function describes the dissipative behavior of the system. It turns out that a similar result holds in quantum mechanics, with a bonus that is the fluctuation-dissipation theorem.

B. Example 1: Damped driven harmonic oscillator

Consider the damped driven harmonic oscillator. Fourier-transforming the equation of motion and using (1) gives the response function in frequency domain:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = h(t) \implies \tilde{\chi}(\omega) = (\omega_0^2 - \omega^2 - i\gamma\omega)^{-1}.$$

The static response function is $\tilde{\chi}(\omega = 0) = 1/\omega_0^2$, giving $x = h/\omega_0^2$ as expected. For a drive with frequency ω, the response function has poles $\omega_\star$ given by

$$\omega_* = -i\gamma/2 \pm \sqrt{\omega_0^2 - \gamma^2/4}.$$

When the oscillator is underdamped ($\gamma/2 < \omega_0$), the poles have both real and imaginary parts and are in the lower half-plane. When the oscillator is overdamped ($\gamma/2 > \omega_0$), the poles are on the negative imaginary axis. In both cases, $\tilde{\chi}(\omega)$ is analytic in the upper half-plane, consistent with causality.

Suppose the drive is $h(t) = h_0 \operatorname{Re}(e^{-i\omega t})$, then the average power dissipated by the system[1] is

$$\langle P \rangle = 2h_0^2 \omega \operatorname{Im}\chi(\omega) = 2h_0^2 \gamma\omega^2 [(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2]^{-1}.$$

which has a Lorentzian line shape with FWHM γ near resonance. Note that the first equality holds in general and depends only on the imaginary part of the response function.

C. Example 2: Hydrodynamics

In this example, the dynamical variables (mass density ρ and current $\vec{J}$) are functions of both space and time. Consider a simple model of a fluid with diffusion:

$$\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0 \implies \partial_t \rho - D\nabla^2 \rho = -\vec{\nabla} \cdot \vec{F}.$$

where the current is due to the diffusion process (with diffusion coefficient D) and a driving force $\vec{F}$: $\vec{J} = -D\vec{\nabla}\rho + \vec{F}$. To understand how the density and current responds to $\vec{F}$, we introduce their respective linear response functions $\chi_{\rho J}(t'-t, x'-x)$ and $\chi_{JJ}(t'-t, x'-x)$[2]. Note that by writing the arguments as $t'-t$ and $x'-x$, we are assuming that the response functions are invariant under both time and spatial translations.

$$\rho(t,x) = \int \chi_{\rho J}(t'-t, x'-x) F(t', x')\, dt'\, dx'$$
$$J(t,x) = \int \chi_{JJ}(t'-t, x'-x) F(t', x')\, dt'\, dx'$$

In frequency and momentum space, we have

$$\tilde{\rho}(\omega,k) = \tilde{\chi}_{\rho J}(\omega,k)\tilde{F}(\omega,k) \implies \tilde{\chi}_{\rho J}(\omega,k) = \frac{ik}{-i\omega + Dk^2}$$
$$\tilde{J}(\omega,k) = \tilde{\chi}_{JJ}(\omega,k)\tilde{F}(\omega,k) \implies \chi_{JJ}(\omega,k) = \frac{-i\omega}{i\omega + Dk^2}.$$

III. QUANTUM LINEAR RESPONSE

A. The Kubo formula

B. Structure factor

C. Fluctuation-Dissipation Theorem

IV. FROM LINEAR RESPONSE FUNCTION TO MEASUREMENTS

A. From linear response functions to correlation functions

B. From correlation functions to measurements

V. APPLICATIONS

VI. CONCLUSIONS

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[1] which we find by calculating the average power absorbed by the system, since the system is in steady state.

[2] The subscript J denotes the fact that $\vec{F}$ acts on the current $\vec{J}$