1. Consider the state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ on two qubits. What are the states that you get when you apply $\sigma_x \otimes I$, $\sigma_y \otimes I$, and $\sigma_z \otimes I$ to this state? Show that they are all orthogonal.

   Solution: Here we have

$$\begin{aligned}
|\psi_1\rangle &:= (\sigma_x \otimes I)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle) \\
|\psi_2\rangle &:= (\sigma_y \otimes I)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{i}{\sqrt{2}}(|11\rangle + |00\rangle) \\
|\psi_3\rangle &:= (\sigma_z \otimes I)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle).
\end{aligned}$$

   Since $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ are orthonormal states, we have

$$\begin{aligned}
\langle\psi_1|\psi_2\rangle &= \frac{i}{2}\langle 11|11\rangle - \frac{i}{2}\langle 00|00\rangle = 0 \\
\langle\psi_1|\psi_3\rangle &= 0, \langle\psi_2|\psi_3\rangle = 0.
\end{aligned}$$

2. I'd like you to work out an example of an observable with repeated eigenvalues, but I don't want the computations to be too painful, so I'm giving you a hint.

   I don't think I said very much about this in class, so as a reminder, if an observable has repeated eigenvalues, the state being measured is projected onto the subspace corresponding to the eigenvalue.

   Consider the observable

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}.$$

   Hint: Three eigenvectors of M (not normalized) are

$$\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

   (a) What are the eigenvalues and the corresponding projection matrices for this observable?
   Solution: The eigenvector

$$|v_1\rangle = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

   has eigenvalue 4, because

$$M|v_1\rangle = \begin{pmatrix} 2 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix} = 4|v_1\rangle$$

   Similarly, one can compute that the other two eigenvectors both have eigenvalues -2.

To compute the projection matrix, we must use normalized eigenvectors $|w_1\rangle$, $|w_2\rangle$, $|w_3\rangle$. Because $|\,|v_1\rangle\,| = \sqrt{6}$, $|\,|v_2\rangle\,| = \sqrt{3}$, and $|\,|v_3\rangle\,| = \sqrt{2}$, we have

$$|w_1\rangle = \frac{1}{\sqrt{6}}\,|v_1\rangle\,. \quad |w_2\rangle = \frac{1}{\sqrt{3}}\,|v_2\rangle\,, \quad \text{and} \quad |w_3\rangle = \frac{1}{\sqrt{2}}\,|v_3\rangle\,.$$

The projector onto the outcome 4 is $\Pi_4 = |w_1\rangle\langle w_1|$, and the projector onto the outcome -2 is $\Pi_{-2} = |w_2\rangle\langle w_2| + |w_3\rangle\langle w_3|$. Computing these,

$$\Pi_4 = |w_1\rangle\langle w_1| = \frac{1}{6}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}(2,1,1) = \frac{1}{6}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Similarly,

$$\begin{aligned}
\Pi_{-2} &= |w_2\rangle\langle w_2| + |w_3\rangle\langle w_3| \\
&= \frac{1}{3}\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}(1,-1,-1) + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}(0,1,-1) \\
&= \frac{1}{3}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 2 & -2 & -2 \\ -2 & 5 & -1 \\ -2 & -1 & 5 \end{pmatrix}
\end{aligned}$$

To check that we've performed this calculation correctly, we can see whether $4\Pi_4 - 2\Pi_{-2} = M$. (This also gives us another way to compute Π_{-2}.)

(b) If the qutrit

$$\frac{2}{3}\,|0\rangle + \frac{2}{3}\,|1\rangle - \frac{1}{3}\,|2\rangle$$

is measured using this observable, what are the possible outcomes, with what probabilities do you observe them, and what are the resulting quantum states?

Solution: If we call the above qutrit $|\psi\rangle$, the probabilities are $|\Pi_4\,|\psi\rangle\,|^2$ and $|\Pi_{-2}\,|\psi\rangle\,|^2$. We can compute that

$$\begin{aligned}
\Pi_4\,|\psi\rangle &= \frac{1}{18}(10\,|0\rangle + 5\,|1\rangle + 5\,|2\rangle) \\
\Pi_{-2}\,|\psi\rangle &= \frac{1}{18}(2\,|0\rangle + 7\,|1\rangle - 11\,|2\rangle)
\end{aligned}$$

The probability of seeing these projections are thus the squares of the length of these vectors, or $\frac{150}{324}$ and $\frac{174}{324}$. The resulting states, conditional on the measurement outcomes, are the normalizations of the above vectors, namely,

$$\frac{1}{\sqrt{150}}\Big(10\,|0\rangle + 5\,|1\rangle + 5\,|2\rangle\Big) \text{ and } \frac{1}{\sqrt{174}}\Big(2\,|0\rangle + 7\,|1\rangle - 11\,|2\rangle\Big).$$

3. A spin-1 particle has three quantum states. We will take the basis of the quantum state space to be the values of the spin along the z-axis; the three basis states are $|1\rangle$, $|0\rangle$ and $|-1\rangle$. The observable corresponding the the spin along the z-axis in this basis is then

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In this basis, the observables for the spin along the x- and y-axes are:

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Two observables can be measured simultaneously if the matrices corresponding to them commute.

(a) Show that the matrices J_x and J_z do not commute, and thus cannot be measured simultaneously.
Solution: This is just matrix multiplication.

$$J_x J_z = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$J_z J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

(b) Show that the observables J_x^2, J_y^2, and J_z^2 all commute, Find the three simultaneous eigenvectors and their associated eigenvalues. What is the observable $J^2 = J_x^2 + J_y^2 + J_z^2$?
Solution: Here we have

$$J_x^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$J_y^2 = \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$J_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

One can show that these commute using matrix multiplication, but there might be an easier way. Notice that if we define

$$|\alpha\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad |\beta\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad |\gamma\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

then

$$
\begin{aligned}
J_x^2 &= |\alpha\rangle\langle\alpha| + |\beta\rangle\langle\beta| \,. \\
J_y^2 &= |\beta\rangle\langle\beta| + |\gamma\rangle\langle\gamma| \,, \\
J_z^2 &= |\alpha\rangle\langle\alpha| + |\gamma\rangle\langle\gamma| \,,
\end{aligned}
$$

and the fact that $|\alpha\rangle$, $|\beta\rangle$, $|\gamma\rangle$ are orthogonal implies that J_x^2, J_y^2, and J_z^2 all commute. Note that they are also their simultaneous eigenvalues. Finally, $J_x^2 + J_y^2 + J_z^2 = 2I$.

4. In this problem, we will derive the matrix J_x in problem (2). Suppose we have two qubits A and B. The observable giving the spin in the z direction is

$$\frac{1}{2}\left(\sigma_z^A \otimes I^B + I^A \otimes \sigma_z^B\right).$$

Similarly, the observable giving the spin in the x direction is

$$\frac{1}{2}\left(\sigma_x^A \otimes I^B + I^A \otimes \sigma_x^B\right).$$

There is a 3-dimensional subspace of the 4-dimensional state space of two qubits which corresponds to the state space of a spin-1 particle. This is the subspace orthogonal to the state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. (The state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ has spin 0 when measured along any axis). Use these facts to find the matrix J_x of problem (2).

Solution: We have

$$\tfrac{1}{2}(\sigma_z \otimes I + I \otimes \sigma_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\tfrac{1}{2}(\sigma_x \otimes I + I \otimes \sigma_x) = \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

We'd like to eliminate the coordinate $\frac{1}{\sqrt{2}}\,|01\rangle - |10\rangle$. To do this, we can apply the change-of-basis matrix

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to put the state $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$ in the second coordinate and the state the state $\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ in the third coordinate. Doing this, we get

$$\tfrac{1}{2}B(\sigma_z \otimes I + I \otimes \sigma_z)B^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\frac{1}{2}B(\sigma_x \otimes I + I \otimes \sigma_x)B^\dagger = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Eliminating the third row and column (corresponding to the state $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ gives the matrices J_z and J_x as given in problem 2.

5. **Generalized Measurements**

In this problem, you will derive an example of a non-von Neumann measurement (which you will implement using a sequence of unitary transformations and von Neumann measurements).

Suppose you are given one of the three states:

$$|0\rangle\,, \qquad -\frac{1}{2}\,|0\rangle + \frac{\sqrt{3}}{2}\,|1\rangle\,, \qquad -\frac{1}{2}\,|0\rangle - \frac{\sqrt{3}}{2}\,|1\rangle\,,$$

with equal probabilities. Your task is to identify the state while minimizing the probability that you get it wrong.

(a) Suppose you measure the state using a basis $\{|v\rangle\,, |\bar{v}\rangle\}$ where $|v\rangle$ and $|\bar{v}\rangle$ are orthonormal quantum states. Show that the probability you get it correct is strictly less than $\frac{2}{3}$. (There's a simple argument for this that barely uses any calculation, although if you can't find this argument feel free to use a more calculationally intensive one.)

Solution: There are a few ways to approach this problem. The most direct one is as follows.

Enumerate the three possible states as $|\phi_1\rangle\,, |\phi_2\rangle\,, |\phi_3\rangle$. Suppose the probability that these states measure to $|v\rangle$ is p_1, p_2, p_3 respectively. Further suppose that our probabilistic strategy, when the measurement outcome is $|v\rangle$, guesses $|\phi_i\rangle$ with probability a_i, and when the measurement outcome is $|\bar{v}\rangle$, guesses $|\phi_i\rangle$ with probability b_i. Then our probability of success is

$$\begin{aligned} &\frac{1}{3}[p_1 a_1 + (1-p_1)b_1 + p_2 a_2 + (1-p_2)b_2 + p_3 a_3 + (1-p_3)b_3] \\ &\qquad = \frac{1}{3}[b_1 + b_2 + b_3 + p_1(a_1 - b_1) + p_2(a_2 - b_2) + p_3(a_3 - b_3)]. \end{aligned}$$

Note that $b_1 + b_2 + b_3 = 1$, therefore it suffices for us to show that

$$p_1(a_1 - b_1) + p_2(a_2 - b_2) + p_3(a_3 - b_3) < 1.$$

Note that $(a_1 - b_1) + (a_2 - b_2) + (a_3 - b_3) = 0$, $|a_i - b_i| \leq 1$, and $p_i \leq 1$ for all i. Therefore the only case where $p_1(a_1-b_1)+p_2(a_2-b_2)+p_3(a_3-b_3) = 1$ is when for some i, $p_i = 1$ and $a_i - b_i = 1$, and $p_j = 0$ for all $j \neq i$. Translating back to the measurement language, this means $|v\rangle = |\phi_i\rangle$. However, since these three states are not orthogonal, we can't have $p_j = 0$ for all $j \neq i$. Therefore our success probability is strictly less than $2/3$.

Now, let's try to do better. Suppose you take the first qubit and tensor it with a second qubit inthe state $|0\rangle$.

(b) Find α and β such that the following quantum states form an orthonormal basis:

$$\left\{ \begin{array}{cc} |11\rangle\,, & -\frac{1}{2}\alpha\,|00\rangle + \frac{\sqrt{3}}{2}\alpha\,|10\rangle + \beta\,|01\rangle\,, \\ \alpha\,|00\rangle + \beta\,|01\rangle\,, & -\frac{1}{2}\alpha\,|00\rangle - \frac{\sqrt{3}}{2}\alpha\,|10\rangle + \beta\,|01\rangle\,. \end{array} \right\}$$

Solution: Enumerate the states, from left to right, from top to bottom as $|\psi_1\rangle\,, |\psi_2\rangle\,, |\psi_3\rangle\,, |\psi_4\rangle$. Obviously $|\psi_1\rangle$ is orthogonal to the other three states. We have

$$\langle\psi_2|\psi_3\rangle = -\frac{1}{2}\alpha^2 + \beta^2 = 0 \Rightarrow \beta^2 = \frac{1}{2}\alpha^2$$
$$\langle\psi_3|\psi_3\rangle = \alpha^2 + \beta^2 = 1 \Rightarrow \alpha^2 = \frac{2}{3}.$$

Therefore we choose $\alpha = \sqrt{\frac{2}{3}}$, $\beta = \sqrt{\frac{1}{3}}$, and we see that the states are indeed orthonormal.

(c) Suppose you use the measurement corresponding to the orthonormal basis above to try to identify the state. What is the probability that you succeed?

Solution: Following our notation from the previous part, we note that $|00\rangle$ measures to $|\psi_3\rangle$ with probability $2/3$, $-\frac{1}{2}\,|00\rangle + \frac{\sqrt{3}}{2}\,|10\rangle$ measures to $|\psi_2\rangle$ with probability $2/3$, and $-\frac{1}{2}\,|00\rangle - \frac{\sqrt{3}}{2}\,|10\rangle$ measures to $|\psi_4\rangle$ with probability $2/3$. Therefore, we can simply preform our POVM measurements, and guess $|00\rangle$ if the measurement outcome is 3, guess $-\frac{1}{2}\,|00\rangle + \frac{\sqrt{3}}{2}\,|10\rangle$ is the outcome is 2, and guess $-\frac{1}{2}\,|00\rangle - \frac{\sqrt{3}}{2}\,|10\rangle$ if the outcome is 4. This strategy has success probability $2/3$.