Exercises for Lecture 2
Foundations of Stat Mech

May 29, 2020

1 Time avetrages by Hamiltonian evolution

Be $\mathcal{U}(X) = e^{iHt} X e^{-iHt} \equiv X_t$ the time evolution generated by a non degenerate Hamiltonian $H = \sum_n E_n P_n$ and $\phi_t = \mathcal{U}\phi = e^{1iHt}\phi e^{iHt}$ the time evolved state. Let p_n be the probability of being in the state P_n, namely $p_n = \mathrm{tr}(\phi P_n)$.

1.1 Time averaged state

Let ϕ be a pure state with amplitudes $|\phi\rangle = \sum_n a_n |E_n\rangle$. Show that

$$\overline{\phi} = \sum_n P_n \phi P_n = \sum_n p_n P_n = \sum_n |a_n|^2 P_n = D_H \phi \tag{1}$$

and compute its purity.

1.2 Average Loschmidt Echo

Define the Loschmidt echo as $\mathcal{L}_t = |\langle\phi|\,\phi_t\rangle|^2$. Show that $\overline{\mathcal{L}_t} = \mathrm{Tr}\overline{\phi}^2$.

1.3 Typicality in time (Challenge)

Be H a non-degenerate Hamiltonian with also non-degenerate gaps. Define $\omega = D_H \phi$ Show that

$$\overline{D(\phi_{St}, \omega_S)} \leq \frac{1}{2}\sqrt{d_S^2 \mathrm{tr}\omega^2} \tag{2}$$

2 Averages in $\mathcal{H}$

2.1 Haar-average Loschmidt Echo

Show that

$$\langle\overline{\mathcal{L}_t}\rangle_\phi = \frac{2}{d+1} \tag{3}$$

2.2 Markov inequality

Use Markov inequality to upper bound

$$\langle \Pr\left(|A_t - \overline{A}| > \epsilon\right)\rangle_\phi \tag{4}$$

2.3 Haar-average purity

Show that

$$\langle \mathrm{tr}\phi_A^2\rangle_\phi = \frac{d_A + d_B}{d_A d_B + 1} \tag{5}$$

2.4 Typicality of canonical state (challenge problem)

Define $\omega = D_H\phi$. Be $d(\cdot,\cdot)$ the trace distance. Show that

$$\langle d\left(\omega_S, \frac{1}{d_S}\right)\rangle_\phi \leq \frac{1}{2}\sqrt{\frac{1}{d_B}} \tag{6}$$