Optical Bloch Equations

Atom in presence of monochromatic light.

Difficulty: "infinitely long" coherence time (memory).
Master equation (Markov type) doesn't work.
$G(z)$ Resolvent approach does not work: too many important intermediate states

$\rightarrow$ Describe incident field as external classical field.

$$H = H_A + H_R - \vec{d} \cdot \left(\vec{E}_e(\vec{0},t) + \vec{E}_\perp(\vec{0}) \right)$$

$$\vec{E}_e(\vec{0},t) = \mathcal{E}_0 \cos(\omega_L t)$$

Without vacuum: $i\hbar \dot{\sigma} = \left[H_A - \vec{d} \cdot \mathcal{E}_0 \cos(\omega_L t), \sigma \right]$

Assume independent rates of variation

$$\Rightarrow \quad \dot{\sigma}_{bb} = i\Omega_1 \cos(\omega_L t) (\sigma_{ba} - \sigma_{ab}) - \Gamma \sigma_{bb}$$

$$\dot{\sigma}_{aa} = -i\Omega_1 \cos(\omega_L t) (\sigma_{ba} - \sigma_{ab}) + \Gamma \sigma_{bb}$$

$$\dot{\sigma}_{ab} = i\omega_0 \sigma_{ab} - i\Omega_1 \cos(\omega_L t) (\sigma_{bb} - \sigma_{aa}) - \frac{\Gamma}{2} \sigma_{ab}$$

$$\dot{\sigma}_{ba} = -i\omega_0 \sigma_{ba} + i\Omega_1 \cos(\omega_L t) (\sigma_{bb} - \sigma_{aa}) - \frac{\Gamma}{2} \sigma_{ba}$$

with $\hbar \Omega_1 = - \vec{d}_{ab} \cdot \mathcal{E}_0$ Rabi frequency

$\vec{d}_{ab} = \langle a | \vec{d} | b \rangle = \langle b | \vec{d} | a \rangle$ (assume $\vec{d}$ real)

approach valid if $\Omega_1 \ll \frac{1}{\tau_c} \approx \omega_0$.

Rotating wave approximation

$$\vec{d} = \vec{d}_{ab}(|b\rangle\langle a| + |a\rangle\langle b|) = \vec{d}_+ + \vec{d}_-$$

$$\vec{d}_\pm = \vec{d}_{ab} S_\pm$$

$$S_+ = |b\rangle\langle a|$$

$$S_- = |a\rangle\langle b|$$

resonant!

$|a\rangle \to |b\rangle$ absorption $|b\rangle \to |a\rangle$ emission

$$-\vec{d}\cdot\mathcal{E}_0\cos(\omega_L t) = \frac{1}{2}\hbar\Omega_1\Big[S_+ e^{-i\omega_L t} + S_- e^{i\omega_L t}$$

$$+ S_- e^{-i\omega_L t} + S_+ e^{i\omega_L t}\Big]$$

$|b\rangle \to |a\rangle$ absorption $|a\rangle \to |b\rangle$ emission

off-resonant $\to$ neglect!

Now: Go to "rotating frame"

$$\hat{\sigma}_{ba} = \sigma_{ba} e^{i\omega_L t} \qquad \hat{\sigma}_{aa} = \sigma_{aa}$$

$$\hat{\sigma}_{ab} = \sigma_{ab} e^{-i\omega_L t} \qquad \hat{\sigma}_{bb} = \sigma_{bb}$$

$$\Rightarrow \quad \frac{d}{dt}\hat{\sigma}_{bb} = i\frac{\Omega_1}{2}(\hat{\sigma}_{ba} - \hat{\sigma}_{ab}) - \Gamma\hat{\sigma}_{bb}$$

$$\frac{d}{dt}\hat{\sigma}_{aa} = -i\frac{\Omega_1}{2}(\hat{\sigma}_{ba} - \hat{\sigma}_{ab}) + \Gamma\hat{\sigma}_{bb}$$

$$\frac{d}{dt}\hat{\sigma}_{ab} = -i\delta_L\hat{\sigma}_{ab} - i\frac{\Omega_1}{2}(\hat{\sigma}_{bb} - \hat{\sigma}_{aa}) - \frac{\Gamma}{2}\hat{\sigma}_{ab}$$

$$\frac{d}{dt}\hat{\sigma}_{ba} = i\delta_L\hat{\sigma}_{ba} + i\frac{\Omega_1}{2}(\hat{\sigma}_{bb} - \hat{\sigma}_{aa}) - \frac{\Gamma}{2}\hat{\sigma}_{ba}$$

$$\delta_L = \omega_L - \omega_0. \qquad \text{Note: } \frac{d}{dt}(\hat{\sigma}_{aa} + \hat{\sigma}_{bb}) = 0$$

Other forms of the OBE:

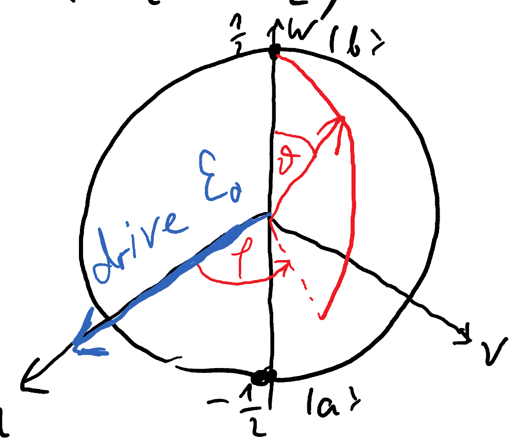
$$\hat{S}_+ = e^{-i\omega_L t} S_+ = e^{-i\omega_L t} |b\rangle\langle a|$$
$$\hat{S}_- = e^{i\omega_L t} S_- = e^{i\omega_L t} |a\rangle\langle b|$$
$$\hat{S}_z = \frac{1}{2}\left(|b\rangle\langle b| - |a\rangle\langle a|\right)$$

$$\langle \hat{S}_+ \rangle = Tr\left(\sigma S_+ e^{-i\omega_L t}\right) = \sigma_{ab} e^{-i\omega_L t} = \hat{\sigma}_{ab}$$
$$\langle \hat{S}_- \rangle = Tr\left(\sigma S_- e^{i\omega_L t}\right) = \sigma_{ba} e^{i\omega_L t} = \hat{\sigma}_{ba}$$
$$\langle \hat{S}_z \rangle = Tr\left(\sigma \tfrac{1}{2}(|b\rangle\langle b| - |a\rangle\langle a|\right) = \frac{1}{2}(\hat{\sigma}_{bb} - \hat{\sigma}_{aa})$$

$$\rightarrow \langle \dot{\hat{S}}_+ \rangle = -\left(i\delta_L + \frac{\Gamma}{2}\right)\langle \hat{S}_+ \rangle - i\Omega_1 \langle S_z \rangle$$
$$\langle \dot{\hat{S}}_- \rangle = -\left(-i\delta_L + \frac{\Gamma}{2}\right)\langle \hat{S}_- \rangle + i\Omega_1 \langle S_z \rangle$$
$$\langle \dot{\hat{S}}_z \rangle = \frac{i\Omega_1}{2}\left[\langle \hat{S}_- \rangle - \langle \hat{S}_+ \rangle\right] - \Gamma\left(\langle S_z \rangle + \frac{1}{2}\right)$$

Introduce Bloch vector:

$$u = \frac{1}{2}(\hat{\sigma}_{ab} + \hat{\sigma}_{ba})$$
$$v = \frac{1}{2i}(\hat{\sigma}_{ab} - \hat{\sigma}_{ba})$$
$$w = \frac{1}{2}(\hat{\sigma}_{bb} - \hat{\sigma}_{aa})$$



$$\boxed{\begin{aligned} \dot{u} &= \delta_L v - \frac{\Gamma}{2} u \\ \dot{v} &= -\delta_L u - \Omega_1 w - \frac{\Gamma}{2} v \\ \dot{w} &= \Omega_1 v - \Gamma w - \frac{\Gamma}{2} \end{aligned}}$$

$$\langle \vec{d} \rangle = 2 d_{ab} \left(u \cos(\omega_L t) - v \sin(\omega_L t)\right)$$

drive acts as "magnetic field" on our "spin" $\vec{S}$

$\rightarrow$ No precession if $\vec{S}$ is along driving field

Geometric interpretation in terms of a fictitious spin $\frac{1}{2}$

Every two-level system is equivalent to fictitious spin $\frac{1}{2}$:

$$|+\rangle \leftrightarrow |a\rangle$$
$$|-\rangle \leftrightarrow |b\rangle$$

Operators in the basis $\{|+\rangle, |-\rangle\}$:

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$S_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \frac{1}{2} \times \text{Pauli matrices}$$
$$S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$H_A = \hbar\omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \frac{\hbar\omega_0}{2}\mathbb{1} + \hbar\omega_0 S_z$$

$$-\hat{\vec{d}} \cdot \vec{\mathcal{E}}_0 \cos(\omega_L t) = \hbar\Omega_1 \cos(\omega_L t)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= 2\hbar\Omega_1 \cos(\omega_L t)\, S_x$$

$\Rightarrow$ H_A and $-\hat{\vec{d}} \cdot \vec{\mathcal{E}}_0 \cos(\omega_L t)$ describe interaction of fictitious spin with magnetic fields $\vec{B}_0$ and $2\vec{B}_1 \cos(\omega_L t)$, parallel to Oz and Ox, resp.,

Larmor frequencies ω_0 and $2\Omega_1 \cos(\omega_L t)$

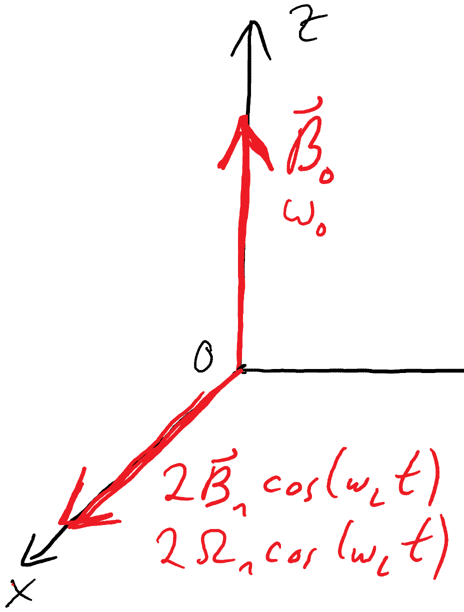
$$2\vec{B}_1 \cos(\omega_L t) = \vec{B}_1 e^{i\omega_L t} + \vec{B}_1 e^{-i\omega_L t}$$

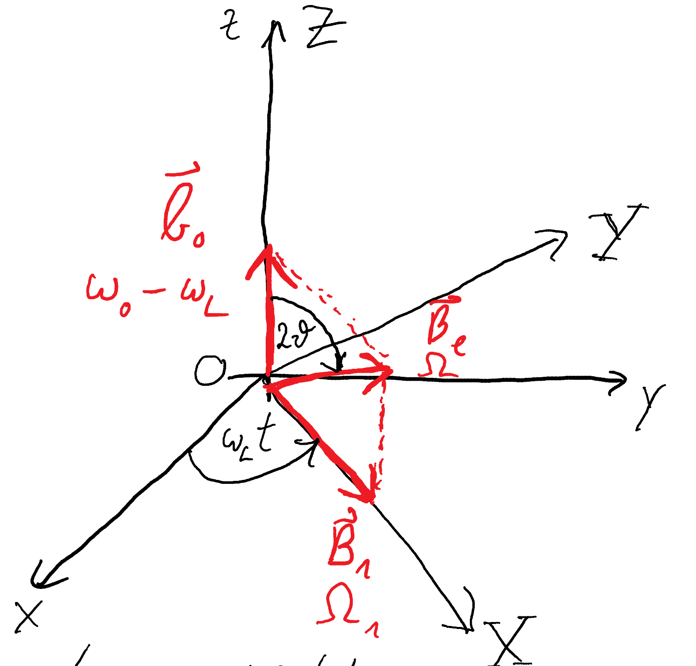
$\hat{=}$ two fields, amplitude B_1, rotating clockwise + anti-clockwise in the plane xOy at frequency ω_L.

If $\omega_L = \omega_0$, counter-clockwise component accompanies the spin in its Larmor precession around $\vec{B}_0$.

can act efficiently on it. Other component rotates too rapidly (at $-2\omega_L$) to have appreciable effect.

Rotating wave approximation $\hat{=}$ retain only component rotating in the <u>same</u> direction as the spin.





Magnetic Fields acting on Fictitious Spin

Lab frame

Rotating frame OXYZ (neglecting counter-rotating field)

In rotating frame rotating about z with frequency ω_L, the component rotating with the spin is stationary. Field along z is reduced from $\vec{B}_0$ to $\vec{b}_0$ as Larmor precession is reduced from ω_0 to $\omega_0 - \omega_L$.

$\Rightarrow$ Spin experiences effective field

$$\vec{B}_e = \vec{b}_0 + \vec{B}_1,$$

precession frequency $\Omega = \sqrt{\Omega_1^2 + \delta_L^2}$.

At resonance $\delta_L = 0$, spin precesses about $\vec{B}_1$ at frequency $\Omega_1 \Rightarrow$ Rabi oscillation between $|a\rangle$ and $|b\rangle$.

Note $\quad \mathcal{S}_\pm = \mathcal{S}_x \pm i\mathcal{S}_y$, so

$$u = \frac{1}{2}\left(\hat{\sigma}_{ab} + \hat{\sigma}_{ba}\right)$$
$$= \frac{1}{2}\operatorname{Tr}\left(\sigma\left[(\mathcal{S}_x + i\mathcal{S}_y)e^{-i\omega_L t} + (\mathcal{S}_x - i\mathcal{S}_y)e^{i\omega_L t}\right]\right)$$
$$= \langle \mathcal{S}_x \rangle \cos(\omega_L t) + \langle \mathcal{S}_y \rangle \sin(\omega_L t)$$
$$= \langle \vec{\mathcal{S}} \rangle \cdot \vec{e}_X$$

Same way: $\quad v = \langle \mathcal{S}_Y \rangle \quad , \quad w = \langle \mathcal{S}_Z \rangle$

$$\Rightarrow \quad \frac{d}{dt}\langle \mathcal{S}_X \rangle = \delta_L \langle \mathcal{S}_Y \rangle - \frac{\Gamma}{2}\langle \mathcal{S}_X \rangle$$
$$\frac{d}{dt}\langle \mathcal{S}_Y \rangle = -\delta_L \langle \mathcal{S}_Y \rangle - \Omega_1 \langle \mathcal{S}_Z \rangle - \frac{\Gamma}{2}\langle \mathcal{S}_Y \rangle$$
$$\frac{d}{dt}\langle \mathcal{S}_Z \rangle = \Omega_1 \langle \mathcal{S}_Y \rangle - \Gamma \langle \mathcal{S}_Z \rangle - \frac{\Gamma}{2} .$$

$\hat{=}$ Bloch equations for magnetic resonance with $T_1 = \frac{1}{\Gamma}$, $T_2 = \frac{2}{\Gamma}$ for spontaneous emission

Note: evolutions of populations and coherences coupled

$\Rightarrow$ No simple interpretation in terms of transition rates.
(In some cases still possible, i.e. when coherences evolve much faster than populations).

Solutions of the OBE

atomic observables from u, v, w:

mean dipole $\langle \vec{d} \rangle$ and population difference

Can also analyze radiative forces exerted by light on atom

1. Internal degrees of freedom:

OBE - linear diff. equations with constant coefficients

$\Rightarrow$ solutions = superpositions of exponentials $e^{-r_i t}$

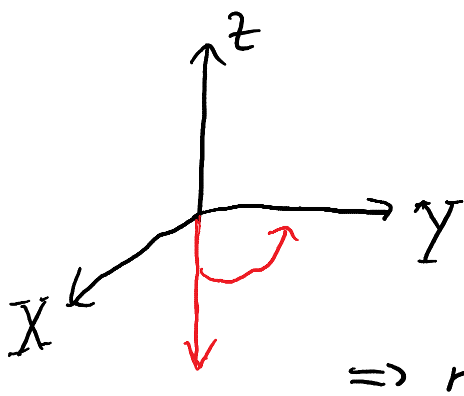
$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -\frac{\Gamma}{2} & \delta_L & 0 \\ -\delta_L & -\frac{\Gamma}{2} & -\Omega_1 \\ 0 & \Omega_1 & -\Gamma \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\frac{\Gamma}{2} \end{pmatrix}$$

- resonance $\omega_L = \omega_0$ $(\delta_L = 0)$, $\Omega_1 \to 0$

$\Rightarrow$ two $r_{1,2} = \frac{\Gamma}{2}$, $r_3 = \Gamma$

$\Rightarrow$ purely damped, transient, no oscillation

- resonance, but $\Omega_1 \gg \Gamma$: Rabi oscillation



along OY : damping $\frac{\Gamma}{2}$

along OZ : damping Γ

average over $\frac{1}{\Omega_1}$: $\frac{3}{4}\Gamma$

$\Rightarrow r_{1,2} = \pm i\Omega_1 + \frac{3}{4}\Gamma$;

$r_3 = \frac{\Gamma}{2}$ motion along X (pure damping)

Recall $\langle \vec{d} \rangle = 2\,\vec{d}_{ab}\,(u\cos(\omega_L t) - v\sin(\omega_L t))$

in phase   in quadrature
components
with respect to driving field
$\mathcal{E}_0 \cos(\omega_L t)$

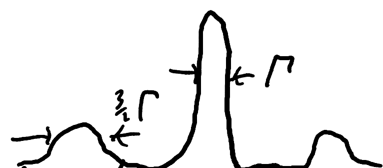
Oscillating dipole → emitted (scattered light)

spectrum of Bloch vector → spectrum of emitted light.

For $\delta_L = 0$, $\Omega_1 \gg \Gamma$ Rabi oscillation

3 peaks at ω_L, $\omega_L \pm \Omega_1$

widths Γ $\frac{3}{2}\Gamma$



Mollow triplet

• case $|\delta_L| \gg |\Omega_1|$:

effective field $\vec{B}_e$ is almost $\parallel Oz$.

Motion of spin in XOY is precession at δ_L, damped at $\frac{\Gamma}{2}$, while motion along Oz is purely damped at rate Γ.

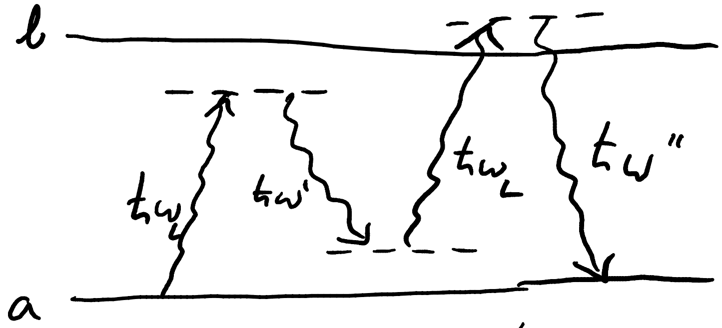
$$r_{1/2} = \pm i\delta_L + \frac{\Gamma}{2} \; ; \quad r_3 = \Gamma$$

Spectrum: 3 peaks at ω_L, $\omega_L \pm \delta_L = \omega_0, 2\omega_L - \omega_0$

widths 2Γ Γ

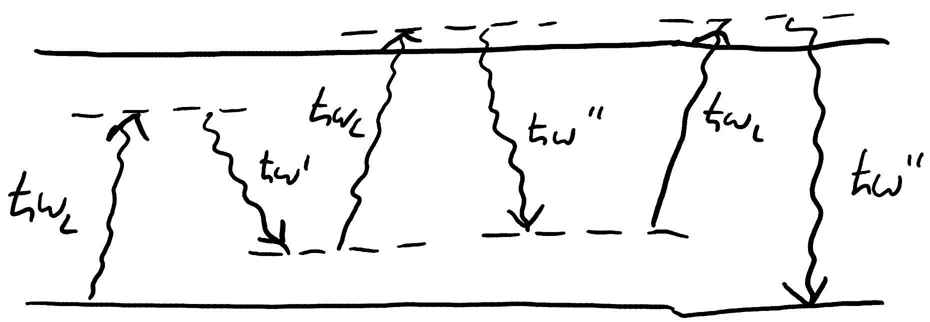
(+ coherent δ-peak at ω_L whose weight $\to 0$ as $\Omega_1 \gg \Gamma$)

Interpretation:
(possible for $|\Delta| \gg |\Omega_R|$)



→ explains sidebands at ω_0 and $2\omega_L - \omega_0$ of width Γ (process above resonant if intermediate state is within Γ of b).

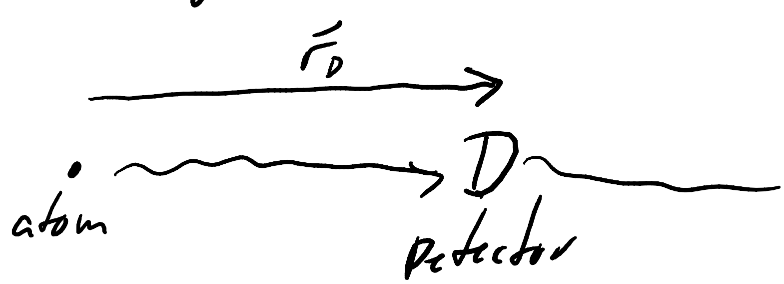
central peak of width 2Γ:



For interpretation of resonant case: Dressed atom!

Properties of light emitted by atom API p. 379

Photodetection signals



$$E(\vec{r}_D, t) = \eta\, d\left(t - \frac{r_D}{c}\right)$$ η proportionality const.

$$E^{\pm}(\vec{r}_D, t) = \eta\, e^{\mp i\omega_L\left(t - \frac{r_D}{c}\right)} S_{\pm}\left(t - \frac{r_D}{c}\right)$$

$$\left(\text{since } d = d_{ab}\left(S_+ e^{-i\omega_L t} + S_- e^{i\omega_L t}\right)\right)$$

total average intensity:

$$\langle I(t)\rangle = \langle E^{(-)}(\vec{r}_D, t)\, E^{(+)}(\vec{r}_D, t)\rangle$$

spectral density of radiation

$$\mathcal{J}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{-i\omega\tau} \langle E^{(-)}(\vec{r}_D, t+\tau)\, E^{(+)}(\vec{r}_D, t)\rangle$$

$$\Rightarrow \langle I(t)\rangle = \eta^2 \left\langle S_+\left(t - \frac{r_D}{c}\right) S_-\left(t - \frac{r_D}{c}\right)\right\rangle$$

$$\mathcal{J}(\omega) = \frac{\eta^2}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i(\omega_L - \omega)\tau} \left\langle S_+\left(t + \tau - \frac{r_D}{c}\right) S_-\left(t - \frac{r_D}{c}\right)\right\rangle$$

Total intensity:

$$\langle I(t) \rangle = \eta^2 \sigma_{bb}\left(t - \frac{r_D}{c}\right)$$

Split $S_\pm$ into mean + fluctuations

$$S_\pm\left(t - \frac{r_D}{c}\right) = \langle S_\pm\left(t - \frac{r_D}{c}\right)\rangle + \delta S_\pm\left(t - \frac{r_D}{c}\right)$$

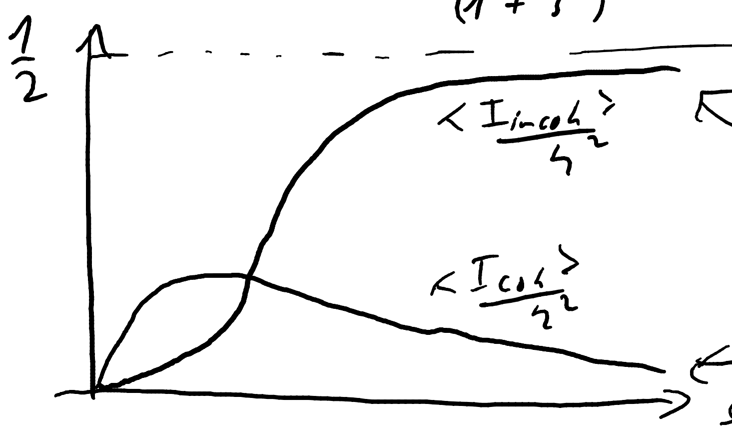
$$\langle \delta S_\pm\left(t - \frac{r_D}{c}\right) \rangle = 0.$$

In steady state, $\langle S_+(t)\rangle$ and $\langle S_+(t) S_-(t)\rangle$ don't depend on time.

$$\Rightarrow \langle I \rangle = \underset{\substack{\uparrow \\ \text{mean dipole} \Rightarrow \text{coherent} \\ \langle I_{coh}\rangle}}{\eta^2 \langle S_+ \rangle\langle S_- \rangle} + \underset{\substack{\uparrow \\ \text{incoherent} = \text{fluctuations} \\ \langle I_{incoh}\rangle \text{ of dipole.}}}{\eta^2 \langle \delta S_+ \delta S_- \rangle}$$

$$\frac{1}{\eta^2}\langle I_{coh}\rangle = |\langle S_+\rangle|^2 = |u_{st} + i v_{st}|^2 = \frac{1}{2}\frac{s}{(1+s)^2}$$

$$\frac{1}{\eta^2}\langle I_{incoh}\rangle = \langle S_+ S_-\rangle - |\langle S_+\rangle|^2$$
$$= \sigma_{bb}^{st} - |u_{st} + i v_{st}|^2$$
$$= \frac{1}{2}\frac{s^2}{(1+s)^2}$$



saturates at $\frac{1}{2}$
cannot emit more than $\frac{\Gamma}{2}$ photons/unit time

linear with s for small s
$\rightarrow 0$ for large s, $\langle d \rangle \rightarrow 0$!

Spectral distribution

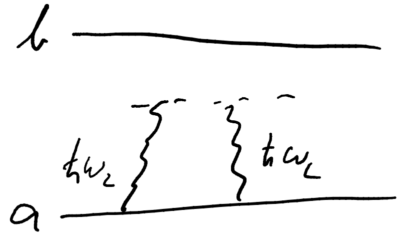
$$\mathcal{I}(\omega) = \mathcal{I}_{coh}(\omega) + \mathcal{I}_{incol.}(\omega)$$

$$\mathcal{I}_{coh}(\omega) = \frac{\eta^2}{2\pi} \int_{-\infty}^{\infty} d\tau \, e^{i(\omega_L - \omega)\tau} \left| \langle S_+ \rangle \right|^2$$

$$\mathcal{I}_{incoh.}(\omega) = \frac{\eta^2}{2\pi} \int_{-\infty}^{\infty} d\tau \, e^{i(\omega_L - \omega)\tau} \langle \delta S_+(t+\tau) \, \delta S_-(t) \rangle$$

$$\mathcal{I}_{coh}(\omega) = \langle I_{coh} \rangle \, \delta(\omega - \omega_L)$$

$\rightarrow$ monochromatic. mean dipole oscillating in the forced regime at frequency ω_L will emit radiation at ω_L

low intensity: diagram 

Inelastic spectrum

$$\mathcal{I}_{incol}(\omega) = \frac{\eta^2}{2\pi} 2 \mathrm{Re} \int_0^{\infty} d\tau \, e^{i(\omega_L - \omega)\tau} \langle \delta S_+(\tau) \, \delta S_-(0) \rangle$$

Quantum regression theorem (Complement A_{V} API):
$\langle \delta S_+(\tau) \, \delta S_-(0) \rangle$ obeys same equation as $\langle \delta S_+(t) \rangle$

$$(\tau > 0) \quad \frac{d}{d\tau} \langle \delta S_+(\tau) \delta S_-(0) \rangle = \sum_{q'} B_{qq'} \langle \delta S_{q'}(\tau) \, \delta S_-(0) \rangle$$

Bloch matrix we had before

$\Rightarrow \langle \delta S_+(\tau) \delta S_-(0) \rangle$ is a superposition of $e^{-r_\lambda t}$ with 3 r_λ eigenvalues of $B_{qq'}$, as before.