Tychonoff Spaces

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Chapter 1

Preliminaries

A mathematical text is seldom read linearly, and understanding the material in its first chapter is not a prerequisite to reading later chapters. On the other hand, there are a great many books whose preliminary chapter is worth reading even though there is not enough time to wade through the rest of the book. This chapter is written in the spirit that it will be here when the reader is ready to read it.

1.1 Set theory

One defines an equivalence relation on a family of sets by declaring $A \approx B$ when there exists a bijection $f : A \to B$. We might denote the equivalence class containing the empty set by 0, then inductively think of the natural number n as the equivalence class containing $\{0, 1, 2, \ldots, n-1\}$. A general equivalence class is called a **cardinal**, and the ones just constructed are referred to as the finite cardinals. It is sometimes difficult to establish equality of two cardinals because finding bijections can be difficult. There is a powerful theorem that lets us deduce the existence of a bijection from A to B if we can demonstrate the existence of two injections, one from A into B and the other from B into A. Let us write $A \leq B$ if there exists an injection from A into B. The aforementioned theorem is called the **Schroeder-Bernstein Theorem**.

Theorem 1 *If A and B are two sets such that $A \leq B$ and $B \leq A$, then $A \approx B$.*

Proof. Assume that $A \leq B$ and $B \leq A$, so with no loss of generality we may assume $A \subseteq B$ and $f : B \to A$ is an injective function. For each

$x \in B \setminus A$ let $\mathcal{O}_x$ denote the orbit

$$\mathcal{O}_x \equiv \{ f^i(x) \ : i = 1, 2, \dots \},$$

where f^0 is the identity function and $f^i = f \circ f^{i-1}$ for $i \in \mathbb{N}$. As a consequence of the injectivity of f, we have that each orbit $\mathcal{O}_x$ is infinite, and that A is a disjoint union of these orbits together with the subset of A consisting of those points that lie on no orbit. Define $g : A \to B$ by

$$g(a) = \begin{cases} a & \text{if } a \text{ lies on no orbit} \\ f^{i-1}(x) & \text{if } a = f^i(x) \text{ is in } \mathcal{O}_x \end{cases}$$

Verify that g is a bijection, from which we deduce $A \approx B$.

$\square$

A proper subset of a finite parent set can not have the same cardinality as its parent, but proper subsets of infinite sets often do have the same cardinality as their parents. In fact, any infinite subset E of $\mathbb{N}$ $\mathbb{N}$ has the same cardinality as $\mathbb{N}$. A bijection may be described inductively by letting e_1 be the smallest element of E, and having specified $e_1, \dots, e_{n-1}$, we let e_n be the smallest element of $E \setminus \{e_1, \dots, e_{n-1}\}$. In this way we see that the cardinality of the natural numbers is the smallest cardinal of an infinite set. The sets with this cardinality are termed the **denumerable** sets, and a set is called **countable** if it is either denumerable or finite.

The natural numbers may be partitioned into an infinite family of disjoint infinite sets. One way to do this is to let E_p denote all the powers of the prime number p, and let E denote those natural numbers that are not a power of a prime, from which we see $\mathbb{N} = (\cup_p E_p) \cup E$. This observation is the root of the assertion that a countable union of countable sets is countable. In particular we see that the integers are countable, as the union of the natural numbers with their negatives (plus zero), and if we let $\mathbb{Q}_r$ denote the subset of rational numbers of the form p/r with p and integer, then $\mathbb{Q}_r$ is in natural correspondence with the integers, and so $\mathbb{Q}$ is a countable union of the countable sets $\mathbb{Q}_r$ $(r \in \mathbb{N})$, so it is countable itself.

The **power set** of a set A is defined to be the set of all subsets of A. If f is any function mapping A into its power set, then the set $E = \{ a \in A \ : a \notin f(a) \}$ can not be in the range of f, as a simple proof by contradiction reveals. If we let $P(A)$ denote the power set of A, then we obtain an infinite chain of distinct infinite cardinals

$$\mathbb{N}, P(\mathbb{N}), P(P(\mathbb{N})), \dots,$$

and a consideration of how the real numbers are constructed reveals that the cardinality of $\mathbb{R}$ equals that of $P(\mathbb{N})$, both examples of what we call **uncountable** sets. Whether there exists a set whose cardinality lies strictly between $\mathbb{N}$ and $P(\mathbb{N})$ is the famous **continuum hypothesis**, which is known to be independent of the standard operating axioms we use for the bulk of our mathematics.

Among our standard operating axioms is the **axiom of choice**, which says that an infinite product of nonempty sets is nonempty. It requires some effort to see how we might generalize the idea of a product of two sets to a product of arbitrarily many, and what we arrive at is the following. Let $\mathcal{I}$ denote an arbitrary index set, and for each $i \in \mathcal{I}$ assume that $\mathcal{X}_i$ is a nonempty set. Then the **product** of the indexed family $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ is the set of functions with domain I such that $f(i) \in \mathcal{X}_i$ for all $i \in \mathcal{I}$. Such functions are called **choice functions**, and the axiom of choice simply asserts the existence of at least one choice function. We denote this product by $\Pi_{i\in\mathcal{I}} X_i$, and we often think of the choice functions as generalized "tuples", and write (x_i) to indicate the choice function whose value at i is x_i, just as we might denote a sequence.

The axiom of choice seems innocuous enough when phrased as it is above, but the axiom leads to strange and mysterious consequences. One equivalent formulation of the axiom asserts that every set can be well ordered, which enables proofs by **transfinite induction**, a technique that mimics a usual induction proof except that the natural numbers are replaced by an arbitrary (potentially very large) well ordered set. Another equivalent formulation, called **Zorn's lemma**, is phrased in the language of partially ordered sets. It says that a nonempty partially ordered set, in which every chain has an upper bound, must contain a maximal element. A consequence of this is the **ultrafilter property**. A filter $\mathcal{F}$ in a set $\mathcal{X}$ is a family of nonempty subsets of $\mathcal{X}$ for which $F_1, F_2 \in \mathcal{F}$ implies $F_1 \cap F_2 \in \mathcal{F}$ (filters are closed under finite intersection), and if $F \in \mathcal{F}$ and $F \subseteq G$ then $G \in \mathcal{F}$ (filters are closed under supersets). Two filters may be related by set inclusion, which makes the collection of filters a partially ordered set. One can check that a chain of filters may be unioned to obtain a filter containing each member of the chain. Zorn's lemma then asserts the existence of maximal filters (filters that are not properly contained in any other filter), and such a maximal filter is called an **ultrafilter**. The ultrafilter property asserts that every filter is contained in an ultrafilter.

Exercises

1. Assume that $f : \mathcal{X} \to \mathcal{Y}$ is a function. Prove that f is injective if and only if there exists a surjection $g : \mathcal{Y} \to \mathcal{X}$ with $g \circ f$ the identity function on $\mathcal{X}$.
2. Prove that $f : \mathcal{X} \to \mathcal{Y}$ is surjective if and only if there exists an injection $g : \mathcal{Y} \to \mathcal{X}$ with $f \circ g$ the identity function on $\mathcal{Y}$.
3. Assume that $\mathcal{X}$ is a finite set. Prove that $f : \mathcal{X} \to \mathcal{X}$ is injective if and only if it is surjective.
4. Show that any two nonempty open intervals of real numbers have the same cardinality.
5. If (a, b) is not empty, show that it has the same cardinality as $[a, b]$.
6. Mimic the discussion on page 7 to prove that Zorn's lemma implies every filter is contained in an ultrafilter.
7. A **well ordered** set is a totally ordered set in which every subset has a least element. Use induction to prove that the natural numbers is well ordered.
8. Indicate how to endow the integers with a well ordering. Do the same thing for the rational numbers.
9. Assume $\mathcal{F}$ is a filter in $\mathcal{X}$ and $E \subset \mathcal{X}$. Prove that $\mathcal{F} \cup \{E\}$ is contained in a filter if and only if $E \cap F \neq \emptyset$ for every $F \in \mathcal{F}$.
10. Assume that $\mathcal{F}$ is a filter in a set $\mathcal{X}$. Prove that $\mathcal{F}$ is an ultrafilter if and only if for every subset $E \subset \mathcal{X}$ either $E \in \mathcal{F}$ or $\mathcal{X} \setminus E \in \mathcal{F}$. (We denote the **complement** of E in $\mathcal{X}$ by $\mathcal{X} \setminus E$ and it is defined as the set $\mathcal{X} \setminus E \equiv \{x \in \mathcal{X} \ : x \notin E\}$.)

1.2 Linear Algebra

A **vector space** is a formal abstraction of Euclidean spaces. For those who are familiar with the terminology of abstract algebra, a vector space $\mathcal{V}$ over a field $\mathbb{F}$ is an abelian group $\mathcal{V}$ together with a unital embedding of the field $\mathbb{F}$ into the ring of homomorphisms $\hom(\mathcal{V})$. We identify $\mathbb{F}$ with the image of this embedding and refer to the elements there as **scalars**, or as **scalar transformations**. The **linear transformations** are the homomorphisms that commute with the scalars, and linear algebra is the study of the structure of the linear transformations.

Just as every vector in $\mathbb{R}^3$ is uniquely expressible as $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, one can use the axiom of choice to prove that every vector space contains a subset $\mathcal{B}$ for which every vector is uniquely expressible as

$$\beta_1 b_1 + \ldots + \beta_n b_n$$

with $b_1, \ldots, b_n \in B$. The set B is called a **basis** of $\mathcal{V}$, and it is far from unique. One thinks of a choice of a particular basis as a coordinatization of $\mathcal{V}$, and the scalars $\beta_1, \ldots, \beta_n$ are thought of as the coordinates of the vector $\sum \beta_i b_i$. One may prove that any two bases of $\mathcal{V}$ must have the same cardinality, and this cardinal is called the **dimension** of $\mathcal{V}$.

Every linear transformation is completely determined by its behavior on a basis, and conversely, starting with a basis $\mathcal{B}$, one can construct a linear transformation by specifying where the basis elements are sent. In other words, any function $f : \mathcal{B} \to \mathcal{V}$ gives rise to a linear transformation $T : \mathcal{V} \to \mathcal{V}$, which is defined by

$$T(\sum \beta_i b_i) = \sum \beta_i f(b_i).$$

In case $\mathcal{B}$ has only two elements, one expresses the two outputs of f in two columns, each column containing the two coordinates of the output basis vector. In this way we visualize the transformations as matrices of scalars. Matrix addition and multiplication are defined in such a way as to correspond to the addition and composition of the associated transformations. There is then a one–to–one correspondence between the set of functions mapping an n element basis $\mathcal{B}$ into $\mathcal{V}$ and the set of all $n \times n$ scalar matrices. When this correspondence is viewed as a mapping between the set of linear transformations on $\mathcal{V}$ and the set of all $n \times n$ scalar matrices, it is a ring isomorphism.

As we change from a basis $\mathcal{B}$ to another basis $\mathcal{C}$, the isomorphism between the set of linear transformations and the $n \times n$ matrices changes, so a given transformation T is associated with two possibly different matrices A_T and B_T. In this case we say that A_T is **similar** to B_T, and we can prove the existence of an invertible matrix S, the **change of basis matrix**, such that

$$A_T S = S B_T.$$

We also say that two transformations T_1 and T_2 are similar when there exists an invertible transformation S with $T_1 S = S T_2$, and this happens exactly when there are two bases so that the matrix of T_1 relative to one of the bases is the same as the matrix of T_2 relative to the other basis. The essence of the so called **canonical forms** is to determine a natural common matrix form that every transformation may obtain, and be able to determine whether two transformations are similar by checking whether their associated canonical forms are the same.

We will briefly describe two canonical forms, the **Jordan form** and the **rational form**. The Jordan form is valid for transformations over complex fields, where the fundamental theorem of algebra is employed. It says that given any transformation T one can find a basis $\mathcal{B}$ that can be partitioned into blocks $\mathcal{B} = \mathcal{B}_1 \cup \ldots \cup \mathcal{B}_k$ so that the matrix of T relative to this basis looks like

$$\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_k \end{bmatrix}$$

where all the entries in the matrix are zero except along the main diagonal containing the submatrices A_i, and each submatrix A_i represents the matrix of a transformation acting on the span of the subbasis $\mathcal{B}_i$, and that submatrix has the special Jordan form

$$\begin{bmatrix} \alpha & 1 & & \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ & & & \alpha \end{bmatrix}$$

called a **Jordan block**. The block has the single scalar α running down the main diagonal, it has 1 running down the diagonal just above the main diagonal, and it has 0 elsewhere. One says that T has been put into block diagonal form, with each block a Jordan block. The rational form is described similarly, but instead of Jordan blocks appearing on the diagonal, one finds the **rational blocks** which look like

$$\begin{bmatrix} \alpha_1 & 1 & & \\ \alpha_2 & & \ddots & \\ \vdots & & & 1 \\ \alpha_t & & & \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} & & & \alpha_1 \\ 1 & & & \alpha_2 \\ & \ddots & & \vdots \\ & & 1 & \alpha_t \end{bmatrix},$$

depending on the author's preference.

Exercises

1. Find somewhere the formal axioms defining a vector space, then prove that the abstract algebraic description given on page 8 is equivalent to them.

2. Assume that $\mathcal{V}$ is a real vector space, $T : \mathcal{V} \to \mathcal{V}$ is a group homomorphism, and $\iota : \mathbb{R} \to \text{Hom}\ (\mathcal{V})$ is the unital ring homomorphism of the real field into the homomorphisms of $\mathcal{V}$ defined by $\iota_r(\mathbf{v}) = r\mathbf{v}$. Prove that T satisfies

$$T(\sum r_i \mathbf{v}_i) = \sum r_i T(\mathbf{v}_i)$$

   if and only if T commutes with every homomorphism ι_r.

3. Use Zorn's lemma to prove that every vector space has a basis.

4. A vector $\mathbf{e}$ is a **cyclic vector** for a transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ if $\{\mathbf{e}, T(\mathbf{e}), \ldots, T^{n-1}(\mathbf{e})\}$ is a basis of $\mathbb{R}^n$. The algebra $\mathcal{A}_T$ generated by the transformation T is the set of all polynomials in T, i.e. it is

$$\left\{ \alpha_0 + \alpha_1 T + \alpha_2 t^2 + \ldots + \alpha_n T^n \ \ : n \in \mathbb{Z},\ n \geq 0,\ \alpha_i \in \mathbb{C} \right\}.$$

   Assuming that $\mathbf{e}$ is a cyclic vector for T, prove that the map from $\mathcal{A}_T$ to $\mathbb{C}^n$ defined by $A \mapsto A(\mathbf{e})$ is a bijection. Use this to find a basis of $\mathcal{A}_T$.

5. The transpose of a matrix A is the one whose i^{th} column is the i^{th} row A, in other words, if $A = (a_{ij})$ then the transpose is (a_{ji}). Use the Jordan form to prove that a complex square matrix is similar to its transpose.

6. Prove that any square matrix is similar to its transpose.

1.3 Rings and Fields

Chapter 2

Metric and Norm Spaces

The first attempt to isolate the concept of continuity resulted in the formalism of a metric space, introduced by M. Frechet in 1906. The intuition is encoded by a **distance function** that specifies the distance between any two points in a set. The distance formulas we meet early in our mathematical lives, such as $|x - y|$ and

$$\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

are concrete examples.

2.1 Formal definitions

Let $\mathcal{X}$ denote any set. A **metric** on $\mathcal{X}$ is a function (sometimes called a **distance function**) $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ that satisfies, for all $x_1, x_2, x_3 \in X$

1. $d(x_1, x_2) \geq 0$ and $d(x_1, x_2) = 0$ if and only if $x_1 = x_2$.
2. $d(x_1, x_2) = d(x_2, x_1)$ (the function d is symmetric).
3. $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$ (d satisfies the triangle inequality).

Often a metric arises from objects that carry quite a lot of structure in addition to the distance function. If $\mathcal{V}$ is a real or complex vector space, then a **norm** on $\mathcal{V}$ is a function $\eta : \mathcal{V} \to \mathbb{R}$ that satisfies, for all $v, v_1, v_2 \in \mathcal{V}$,

1. $\eta(v) \geq 0$ and $\eta(v) = 0$ if and only if $v = 0$ (nonnegative).
2. for all scalars α, we have $\eta(\alpha v) = |\alpha| \eta(v)$ (positive homogenous).
3. $\eta(v_1 + v_2) \leq \eta(v_1) + \eta(v_2)$ (triangle inequality).

The function $|\alpha|$ is the usual absolute value of a real number in case the scalar is real, and it is the complex absolute value

$$|\alpha| = \sqrt{(\text{Re }\alpha)^2 + (\text{Im }\alpha)^2}$$

when the scalar is complex. The real or complex absolute value is itself a norm, and we will often mimic its notation by writing

$$||v|| \equiv \eta(v)$$

when dealing with a general norm. A norm on $\mathcal{V}$ induces a metric on $\mathcal{V}$ by defining $d(v_1, v_2) = ||v_1 - v_2||$.

Examples

1. On the set $\mathbb{R}$ of real numbers let $d(s,t) = |s - t|$. More generally, on the set of vectors in Euclidean space $\mathbb{R}^n$ one has the distance formula

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}.$$

   This metric is induced by the norm $||\mathbf{u}||^2 = \sum_i u_i^2$.

2. On the set of complex numbers $\mathbb{C}$ we also let $d(z, w) = |z - w|$, and on $\mathbb{C}^n$ we obtain a metric from the norm

$$||\mathbf{z}||^2 = \sum_i |z_i|^2.$$

   The norm on $\mathbb{C}$ can be identified with the norm described in Example 1 on $\mathbb{R}^2$, and the norm on $\mathbb{C}^n$ can be identified with the norm on $\mathbb{R}^{2n}$. We call these the real and complex **Euclidean norms**.

3. A **Hilbert space** is an infinite dimensional version of Euclidean space. It may be described as the set of all square summable sequences of scalars, with the same norm as above. We may speak of either a real or complex Hilbert space, depending on whether the scalars are real or complex.

4. A large supply of metric spaces are available by specifying any particular subset of a normed space. This supply of examples is universal, in the sense that every metric space can be seen to be isometrically isomorphic to a subset of a normed space.

The strongest notion of equivalence for metric spaces is that of isometric isomorphism. Two metric spaces are **isometrically isomorphic** if there is a distance preserving bijection between them. Given any set $\mathcal{X}$, we let $\mathcal{B}(\mathcal{X})$

denote the set of real (or complex) valued bounded functions with domain $\mathcal{X}$, and for $f \in \mathcal{B}(\mathcal{X})$, we define a norm

$$||f||_\infty = \sup_{x \in \mathcal{X}} |f(x)|,$$

so $||f||_\infty$ being finite is exactly what we mean by saying f is bounded.

Theorem 2 *Every metric space (X, d) is isometrically isomorphic to a subset of $\mathcal{B}(\mathcal{X})$.*

Proof. Fix an element $x_0 \in \mathcal{X}$ and for each $x \in \mathcal{X}$ define a real valued function f_x by

$$f_x(x') = d(x', x) - d(x', x_0).$$

Verify that f_x is a bounded function, and

$$||f_{x_1} - f_{x_2}||_\infty = \sup_{x'} |d(x', x_1) - d(x', x_2)| = d(x_1, x_2).$$

□

Exercises

1. Let d denote the Euclidean metric on $\mathbb{R}^3$. Prove that d actually is a metric.
2. Let $\mathcal{X} = \{a, b, c\}$, and suppose d is a symmetric function with $d(a, b) = 1$, $d(b, c) = 1$, $d(a, c) = \sqrt{2}$, and $d(a, a) = d(b, b) = d(c, c) = 0$. Show that d is a metric, and find a subset of $\mathcal{B}(\mathcal{X})$ that is isometric to $(\mathcal{X}, d)$.
3. If d is obtained from a norm via $d(s, t) = ||s - t||$, prove that d is a metric.
4. On $\mathbb{R}^2$ define a function $|| \cdot ||_3 : \mathbb{R}^2 \to \mathbb{R}$ by $||(x, y)||_3 = (|x|^3 + |y|^3)^{\frac{1}{3}}$. Prove this is a norm.
5. Provide the details in the proof of Theorem 2.
6. Assume that $\rho : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ satisfies the first two conditions for a metric, but does not satisfy the triangle inequality. Define a function d by

   $$d(x, y) = \inf \{ \textstyle\sum_{i=1}^n \rho(x_i, x_{i-1}) \ : \ \{x_0, x_1, \ldots, x_n\} \subset \mathcal{X}, \ x_0 = x, \ x_n = y \}.$$

   Show that d is a metric.

7. Assume T is an $n \times n$ matrix (real or complex), and view T as a transformation of Euclidean space in the usual way, by writing the Euclidean vector $\mathbf{x}$ as a column, so the transformed vector is obtained by multiplying T on the left of the column vector, which is written $T\mathbf{x}$. Define $||T||$ to be

$$||T|| = \sup \{ \, ||T\mathbf{x}|| \ : ||\mathbf{x}|| \leq 1 \, \},$$

where the norms on the right hand side of the equality are the Euclidean norms. Prove that this defines a norm on the set of matrices. (It is called the **operator norm**.)

8. Assume that $||\cdot||$ denotes the operator norm and T is a diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$. Show that $||T|| = \sup_i |\alpha_i|$.

9. Given two vectors $\mathbf{x}$ and $\mathbf{y}$ in Euclidean space, let $\mathbf{x} \otimes \mathbf{y}$ denote the rank one matrix $\mathbf{x}\mathbf{y}^*$, where $\mathbf{y}^*$ denotes the transpose (or conjugate transpose in the complex case) of the column vector $\mathbf{y}$. Show that every square matrix can be written as a finite sum of these rank one matrices, and then define

$$||T||_1 = \inf \{ \, \textstyle\sum_i ||x_i|| \, ||y_i|| \ : \ T = \sum_i x_i \otimes y_i \, \}.$$

(Once again, the Euclidean norm appears on the right hand side of the equality.) Show that $||\cdot||_1$ is a norm on the set of $n \times n$ matrices, and

$$||T|| \leq ||T||_1$$

for all T. (This is called the **trace norm** of a matrix.)

10. If T is a diagonal matrix with diagonal entries $\alpha_1, \ldots, \alpha_n$, show that

$$||T||_1 = \sum_i |\alpha_i|.$$

11. Assume T is a square matrix with entries (t_{ij}), and define $||T||_2$ by

$$||T||_2 = \sqrt{\sum_{ij} |t_{ij}|^2}.$$

Show that $||T||_2$ is a norm, and

$$||T|| \leq ||T||_2 \leq ||T||_1$$

for all T. (This is called the **Hilbert-Schmidt norm** of a matrix.)

12. Assume that T is an $n \times n$ matrix, and assume $\alpha_1, \ldots, \alpha_n$ are the singular values of T, i.e. they are the eigenvalues, including multiplicity, of the matrix $(T^*T)^{\frac{1}{2}}$. Prove that $||T|| = \sup_i |\alpha_i|$, $||T||_1 = \sum_i |\alpha_i|$, and $||T||_2 = \sqrt{\sum_i |\alpha_i|^2}$.

2.2 Continuity

If $\mathcal{X}$ is the set of real numbers with the metric $d(s,t) = |s-t|$, then the definition found in a Calculus text of a **continuous function** will read: for every number $s \in \mathcal{X}$ and every $\epsilon > 0$, there exists $\delta > 0$ so that $d(s,t) < \delta$ implies $d(f(s), f(t)) < \epsilon$. We take this to be the definition of a continuous function f between two arbitrary metric spaces $\mathcal{X}$ and $\mathcal{Y}$. The intuition being encoded is that a continuous function should take elements that are close to one another in the domain, to nearby elements in the codomain. The notion of distance in the codomain might be different than the one in the domain, so we will use different symbols to differentiate between the two metrics.

We seek to find other carriers of this notion of continuity and are led to consider the sets

$$B_d(x,r) = \{\, x' \in \mathcal{X} \ : d(x,x') < r \,\}$$

as basic building blocks. We call this set the d**-ball** centered at x of radius r. We use these sets to build **open sets** be declaring a subset $O \subseteq \mathcal{X}$ to be open if it can be obtained as a union of d-balls: in other words, the set O is open if and only if for every $x \in O$ there exists $\epsilon > 0$ such that

$$B_d(x,\epsilon) \subseteq O.$$

Theorem 3 *A function $f : (X,d) \to (Y,d')$ is continuous if and only if inverse images of open sets in Y are open in X.*

Proof. Assume that f is continuous and assume that O is open in Y. Let $x \in f^{-1}(O)$ be given. Then $f(x) \in O$ so there exists an $\epsilon > 0$ with

$$B_{d'}(f(x),\epsilon) \subseteq O.$$

The definition of continuity gives us a $\delta > 0$ such that

$$f(B_d(x,\delta)) \subseteq B_{d'}(f(x),\epsilon) \subseteq O,$$

from which it follows that $B_d(x,\delta) \subseteq f^{-1}(O)$, so $f^{-1}(O)$ is open.

Conversely, assume inverse images of open sets are open, and assume $x \in \mathcal{X}$ and $\epsilon > 0$ are given. Verify that $B_{d'}(f(x),\epsilon)$ is an open subset of Y, so that $f^{-1}(B_{d'}(f(x),\epsilon))$ is open in $\mathcal{X}$. Since $x \in f^{-1}(B_{d'}(f(x),\epsilon))$ we get $\delta > 0$ with

$$B_d(x,\delta) \subseteq f^{-1}(B_{d'}(f(x),\epsilon)),$$

so $d(x,z) < \delta$ implies $d'(f(x),f(z)) < \epsilon$.

□

Sequences provide another carrier for the concept of continuity in metric spaces. Given a metric space $(\mathcal{X},d)$, a **sequence** in $\mathcal{X}$ is a function whose domain is the natural numbers and whose range is a subset of $\mathcal{X}$. We will denote a sequence by (x_i), where x_i is intended to mean the value of the function at the natural number i. We say that this sequence is **eventually** in the set O if there exists $N \in \mathbb{N}$ with $x_i \in O$ for all $i \geq N$. We might describe this by saying O contains a tail of the sequence (x_i). A sequence (x_i) **converges** to x means that every d-ball $B(x,\epsilon)$ contains a tail of (x_i); in other words, for every ϵ there exists $N \in \mathbb{N}$ such that $d(x,x_i) < \epsilon$ for every $i \geq N$. Notice that this is exactly the same definition found in a calculus book when $d(s,t) = |s-t|$. When (x_i) converges to x we write $x_i \to x$.

Theorem 4 *A function $f:(X,d) \to (Y,d')$ is continuous if and only if convergence $x_i \to x$ in the domain implies convergence $f(x_i) \to f(x)$ in the codomain.*

Proof. Assume f is continuous and $x_i \to x$. Towards proving that $f(x_i) \to f(x)$, let $\epsilon > 0$ be given. The definition of continuity then gives us $\delta > 0$, and the definition of of sequential convergence gives us $N \in \mathbb{N}$ so that $d(x,x_i) < \delta$, and hence $d'(f(x),f(x_i)) < \epsilon$, when $i \geq N$.

Conversely, if f is not continuous at x, then there exists an $\epsilon > 0$ so that for each natural number n there exists x_n with $d(x,x_n) < \frac{1}{n}$ but $d'(f(x),f(x_n)) \geq \epsilon$. Apparently we have $x_i \to x$, but $f(x_i)$ does not converge to $f(x)$.

□

Examples

1. The continuous functions from calculus are exactly the continuous functions mapping the metric space $(\mathbb{R},d)$ into itself, when $d(s,t) = |s-t|$.

2. The continuous functions from one Euclidean space to another are the ones that are continuous relative to the Euclidean metrics

$$d(\mathbf{s}, \mathbf{t}) = \sqrt{\sum_i (s_i - t_i)^2}.$$

3. The continuous functions mapping the complex numbers into itself are the ones continuous relative to the distance function $d(z, w) = |z - w|$. This is a particular case of the previous example. (The two dimensional Euclidean space case.)

4. On any set S it is possible to define the **discrete** metric d that assigns a distance 1 between any pair of distinct points in the set (and self-distances are defined to be 0). Every function mapping (S, d) to itself is continuous relative to this silly metric.

5. Let f and g be two continuous functions with domain $[0, 1]$, and define the distance between f and g by

$$d(f, g) = ||f - g|| = \sup_{x \in [0,1]} |f(x) - g(x)|.$$

   The extreme value theorem implies that $d(f, g)$ is finite. Convergence of a sequence f_i to a function f is called **uniform convergence**, and a basic result is that integration, when view as a function with this metric space as its domain, is a continuous operator.

Exercises

1. Prove that a finite intersection of open sets is open, and an arbitrary union of open sets is open.

2. A subset $F \subseteq \mathcal{X}$ of a metric space is **closed** if $\mathcal{X} \setminus F$ is an open subset. A common mistake by beginners is to prove a set closed by showing it is not open. Give an example that shows the fallacy of this reasoning.

3. Recall that a **sequence in** F is a sequence (x_i) with $x_i \in F$ for all $i \in \mathbb{N}$. Prove that the following statement is equivalent to the subset F being closed: if (x_i) is a sequence in F that converges to x, then $x \in F$.

4. Prove that a finite union of closed sets is closed, and an arbitrary intersection of closed sets is closed.

5. Prove that every set E gives rise to two sets $\bar{E}$ and E^o such that $E^o \subseteq E \subseteq \bar{E}$, where $\bar{E}$ is the smallest closed set containing E, and E^o is the largest open set contained in E. Give examples to show that there might not exist a smallest open set containing E, and there might not exists a largest closed set contained in E. (The set $\bar{E}$ is called the **closure** of E and E^o is called the **interior** of E.)

6. A point x is called an **interior point** of a subset $E \subseteq \mathcal{X}$ if there exists $\epsilon > 0$ so that $B(x, \epsilon) \subseteq E$. Prove that E^o is exactly the set of interior points of E.

7. Given any subset E of a metric space, let Bd E denote the set of x with the property that every ball containing x intersects both E and $\mathcal{X} \setminus E$. (This set is called the **boundary** of E.) Prove that E^o, Bd E, and $(\mathcal{X} \setminus E)^o$ form a partition of $\mathcal{X}$. (The latter set is often called the **exterior** of E.)

8. Give an example of a subset of $\mathbb{R}$ whose interior is empty but whose boundary is all of $\mathbb{R}$.

9. If E is a subset of a metric space $(\mathcal{X}, d)$ and $x \in \mathcal{X}$, we will say that x is a **limit point** of E when, for every $\epsilon > 0$, the ball $B(x, \epsilon)$ contains an element of E other than x. If x is a limit point of E, prove that the set $B(x, \epsilon) \cap E$ is infinite.

10. Let E' denote the set of limit points of E. Prove that the closure of E equals $E \cup E'$.

11. Prove that a function mapping one metric space to another is continuous if and only if inverse images of closed sets are closed.

12. Assume $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$ and there exists a positive real number K for which $d_2(f(x), f(y)) \leq K d_1(x, y)$ for all $x, y \in \mathcal{X}_1$. Prove that f is continuous. (Such functions are called **Lipschitz functions**.)

13. Assume $x \in \mathcal{X}$ and d is a metric on $\mathcal{X}$. We define a family of sets $\mathcal{F}_x$ by $H \in \mathcal{F}_x$ if and only if there exists ϵ such that $B_d(x, \epsilon) \subseteq H$. Prove that $\mathcal{F}_x$ is a filter. (It is called the **neighborhood filter** of x.)

14. We will write $\mathcal{F} \to x$, and say that the **filter** $\mathcal{F}$ **converges to** x, when $\mathcal{F}_x \subseteq \mathcal{F}$. Given a filter $\mathcal{F}$ in the domain of a function f, we denote by $f(\mathcal{F})$ the family of sets defined as follows: $H \in f(\mathcal{F})$ if and only if H contains a subset of the form $f(F)$ with $F \in \mathcal{F}$. Prove that $f(\mathcal{F})$ is a filter, and a mapping f between metric spaces is continuous if and only if filter convergence $\mathcal{F} \to x$ in the domain of f implies $f(\mathcal{F}) \to f(x)$.

15. Prove that $f(\mathcal{F})$ is an ultrafilter when $\mathcal{F}$ is.

16. Let $\mathcal{V}$ denote an n-dimensional real or complex Euclidean space, and think of the $n \times n$ matrix T as a transformation of $\mathcal{V}$. Prove this transformation is continuous.

17. Assume that d_1 and d_2 are two metrics on the set $\mathcal{X}$, and let $\mathcal{O}_i$ denote the collection of open sets relative to $(\mathcal{X}, d_i)$ $(i = 1, 2)$. In case $\mathcal{O}_1 \subseteq \mathcal{O}_2$, we say that the topology of $(\mathcal{X}, d_1)$ is **weaker** than $(\mathcal{X}, d_2)$, and refer to the topology of $(\mathcal{X}, d_2)$ as **stronger** than $(\mathcal{X}, d_1)$. Prove that $(\mathcal{X}, d_1)$ is weaker than $(\mathcal{X}, d_2)$ if and only if the identity map on $\mathcal{X}$, viewed as a mapping from $(\mathcal{X}, d_2)$ to $(\mathcal{X}, d_1)$, is continuous.

18. A mapping $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$ is called a **homeomorphism** if it is a bijection and both f and f^{-1} are continuous. Prove that if f is a homeomorphism, then the inverse image of f, viewed as a mapping from the open subsets of $(\mathcal{X}_2, d_2)$ into the open subsets of $(\mathcal{X}_1, d_1)$, is a bijection. Determine whether the converse is true or false.

19. Give an example of two metric spaces $(\mathcal{X}_1, d_1)$ and $(\mathcal{X}_2, d_2)$ and a continuous bijection $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$ that is not a homeomorphism.

20. Assume $f : \mathbb{R} \to \mathbb{R}$ is a continuous bijection. Prove that f is a homeomorphism.

21. An **embedding** $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$ is an injective mapping for which $f : (\mathcal{X}_1, d_1) \to f((\mathcal{X}_1, d_1))$ is a homeomorphism. Prove that there is an embedding of $\mathbb{R}$ into $[0, 1]$.

22. Prove that there is an embedding of $\mathbb{R}$ into the unit circle
$$\left\{ (x, y) \ : \ x^2 + y^2 = 1 \right\}.$$

2.3 Compactness

A close look at the proof of the extreme value theorem from calculus reveals a property that all closed intervals have, and this property of closed intervals is the essence of that theorems proof. We call that property compactness, and it generalizes to arbitrary metric spaces and beyond. It is hard to say exactly what the intuition underlying compactness should be, because compact spaces are a diverse breed. In practice, however, we exploit a type of finiteness imposed on compact spaces by their very definition, and that enables us to say quite a lot about them.

We will begin by giving a definition of compact spaces in terms of closed sets. A family of sets $\mathcal{F}$ is said to have the **finite intersection property** if the intersection of finitely many of the sets in $\mathcal{F}$ is always nonempty. A metric space $(\mathcal{X}, d)$ is **compact** if every family of closes sets $\mathcal{F}$ with the finite intersection property has a common element in *all* the $F \in \mathcal{F}$. In other words, every family with the property $F_1 \cap \ldots \cap F_n \neq \emptyset$ for any finite subset $\{F_1, \ldots, F_n\} \subseteq \mathcal{F}$, actually has the propery $\cap_{F \in \mathcal{F}} F \neq \emptyset$.

Using set complementation and DeMorgan's law, a characterization of compactness emerges in terms of open sets. Given a metric space $(\mathcal{X}, d)$, a **cover** of $\mathcal{X}$ is a family of sets that contain $\mathcal{X}$ in its union. An **open cover** is a family of open sets that contain $\mathcal{X}$ in its union. If $\mathcal{C}$ is an open cover, then it is itself a set (whose elements are open sets), and a subset $\mathcal{C}_o$ of $\mathcal{C}$ is called a subcover if $\mathcal{X} \subseteq \cup_{G \in \mathcal{C}_o} G$, i.e. if the subset is itself a cover of $\mathcal{X}$.

Corresponding to a given positive number r one has a prototypical cover of a metric space, given by

$$\mathcal{C}_r = \{ B_d(x, r) \ : x \in \mathcal{X} \}.$$

One then proves that a metric space is compact if and only if every open cover has a finite subcover, i.e. every open cover contains a *finite* subset that still covers $\mathcal{X}$. We call a metric space **totally bounded** in case every cover of the form $\mathcal{C}_r$ contains a finite subcover. Compact implies totally bounded, but the converse is not true. The converse becomes true if we add another property to total boundedness, a property called **completeness**. A sequence (x_i) in a metric space is a **Cauchy** sequence if, given any $\epsilon > 0$, the sequence is eventually in a single d-ball $B(x, \epsilon)$ for some $x \in \mathcal{X}$. A metric space is **complete** if every Cauchy sequence converges. In preparation for our next theorem, recall that a **subsequence** of a sequence (x_i) is obtained from an increasing function $n : \mathbb{N} \to \mathbb{N}$, the subsequence being (x_{n_i}). We say that the sequence (x_i) is **frequently** in the set O if

$$\{ i \ : x_i \in O \}$$

is an infinite set.

Theorem 5 *A metric space $(\mathcal{X}, d)$ is totally bounded if and only if every sequence has a Cauchy subsequence.*

Proof. Assume that $(\mathcal{X}, d)$ is a totally bounded metric space and assume (x_i) is an arbitrary sequence in $\mathcal{X}$. Our plan is to construct a sequence of subsequences in such a way that, when this sequence of sequences is written as a 2×2 infinite grid, the sequence extracted along its main diagonal is our desired Cauchy subsequence. (This demonstrates an important and standard technique in the manipulation of sequences.) We will continue to denote the open cover of d-balls of radius r by $\mathcal{C}_r$. By total boundedness, there is a finite subcover of $\mathcal{C}_1$, and the sequence (x_i) must eventually be in one the finitely many balls. In this way we obtain our first subsequence $(x_{n_i^1})$ so that $x_{n_i^1}$ lies in a single ball of radius 1 for all $i \in \mathbb{N}$. We proceed inductively: assuming that we have constructed $k-1$ subsequences, we produce the k^{th} subsequence by repeating the reasoning above with the cover $\mathcal{C}_{\frac{1}{k}}$ and the sequence $(x_{n_i^{k-1}})$, resulting in a subsequence of $(x_{n_i^{k-1}})$, which we denote $(x_{n_i^k})$, all of whose terms lie in a single ball of radius $1/k$. We emphasize that each of these sequences is not only a subsequence of the original, but also a subsequence of each of its predecessors. The reader should verify that diagonal sequence $(x^k_{n_k})$ is a Cauchy sequence.

Conversely, assume that $(\mathcal{X}, d)$ is not totally bounded, so for some $\epsilon > 0$ there does not exists finitely many balls of radius ϵ that covers $\mathcal{X}$. Pick x_1 in $\mathcal{X}$ arbitrarily, then pick an x_2 that does not lie in $B(x_1, \epsilon)$. Continuing inductively, assuming we have constructed $x_1, \ldots, x_{k-1}$, pick x_k outside of $B(x_1, \epsilon) \cup \ldots \cup B(x_{k-1}, \epsilon)$. The result is a sequence for which $d(x_i, x_j) \geq \epsilon$ for all $i \neq j$, so this sequence can not have a Cauchy subsequence.

□

If E is a subset of a metric space $(\mathcal{X}, d)$, we define the **diameter** of E to be

$$\text{diam } E \equiv \sup \{ d(x, y) \ : x, y \in E \}.$$

We will say that a metric space $(\mathcal{X}, d)$ is **sequentially compact** if every sequence in $\mathcal{X}$ has a convergent subsequence (to a limit in $\mathcal{X}$).

Lemma 6 *If $(\mathcal{X}, d)$ is sequentially compact, and if $\mathcal{C}$ is an open cover of $\mathcal{X}$, then there exists a number δ such that diam $E \leq \delta$ implies there exists $G \in \mathcal{C}$ such that $E \subseteq G$.*

(Such a number δ is called a **Lebesgue number** of the cover $\mathcal{C}$.)

Proof. We prove this assertion by contrapositive, so assume no such number δ exists. Thus for each δ of the form $\frac{1}{n}$ we obtain a subset E_n, with diam $E \leq \frac{1}{n}$, and such that E_n is not contained in any single element of $\mathcal{C}$. For each $n \in \mathbb{N}$ let x_n be an element of E_n. This is our sequence that contains no convergent subsequence, which we prove by contradiction. If x_n were to have a convergent subsequence, we could find $G \in \mathcal{C}$ containing the subsequential limit x, and we could find a natural number n with

$$B(x, \frac{2}{n}) \subseteq G.$$

We then could find a term x_k of the subsequence, beyond the n^{th} term and inside the ball $B(x, \frac{1}{n})$, from which the triangle inequality shows that

$$E_k \subseteq B(x, \frac{2}{n}) \subseteq G.$$

That E_k lies in a single element of the cover contradicts how E_k was chosen, so (x_n) has no convergent subsequence.

□

Theorem 7 *For a metric space $(\mathcal{X}, d)$, the following are equivalent:*

1. $(\mathcal{X}, d)$ *is compact.*
2. $(\mathcal{X}, d)$ *is complete and totally bounded.*
3. $(\mathcal{X}, d)$ *is sequentially compact.*

Proof. Assume $(\mathcal{X}, d)$ is compact and let a Cauchy sequence (x_i) be given. Let F_n denote the closure of the set $\{ x_i \ : i \geq n \}$ and let $\mathcal{F}$ denote the collection of these closed sets. Then $\mathcal{F}$ has the finite intersection property, so there exists $x \in F_n$ for all n. Verify that $x_i \to x$. It follows that compactness implies completeness, and it trivially implies total boundedness, so we have established the first implication.

Now assume that our metric space is complete and totally bounded, and let (x_i) be any sequence in $\mathcal{X}$. By theorem 5, (x_i) has a Cauchy subsequence which, by completeness, must converge. Thus the second assertion implies the third.

Assume that every sequence has a convergent subsequence. Let $\mathcal{C}$ be an open cover of $\mathcal{X}$. Let δ be a Lebesgue number for this cover, given by lemma 6. Every sequence having a convergent subsequence implies every sequence has a Cauchy subsequence, so theorem 5 tells us $(\mathcal{X}, d)$ is totally bounded. Thus we obtain a finite subcover of $\mathcal{C}_\delta$,

$$B(x_1, \delta), \ldots, B(x_n),$$

and we then can find $G_i \in \mathcal{C}$ with $B(x_i, \delta) \subseteq G_i$, and $\{G_1, \ldots, G_n\}$ is evidently a finite subcover of $\mathcal{C}$.

□

Examples

Exercises

1. Assume $(\mathcal{X}, d)$ this a metric space and prove the following: every open cover of $\mathcal{X}$ has a finite subcover if and only if every family of closed sets with the finite intersection property has nonempty intersection.
2. Prove that a continuous image of a compact metric space is compact.

3. Prove that a continuous real valued function with a compact domain attains a maximum value and a minimum value.

4. Assume that $\mathcal{Y}$ is a subset of $(\mathcal{X}, d)$, and $\mathcal{Y}$ is compact with the metric d. Prove that Y is a closed subset of $\mathcal{X}$.

5. Prove that a closed subset of a compact metric space is compact.

6. A metric space $(\mathcal{X}, d)$ is **bounded** if there exists one positive real number R such that $d(x, y) \leq R$ for all $x, y \in \mathcal{X}$. Prove that compact metric spaces are bounded. Conclude that a compact subset of a metric space must be closed and bounded.

7. Prove that a closed and bounded subset of the real numbers is compact. (This is called the **Heine-Borel theorem**.)

8. Give an example of a closed and bounded subset of a metric space that is not compact.

9. Give an example of a closed subset $E \subseteq (\mathcal{X}_1, d_1)$ and an embedding

$$f : E \to (\mathcal{X}_2, d_2)$$

for which $f(E)$ is not closed in $(\mathcal{X}_2, d_2)$. Consequently, we say that the property of being "closed" is a **relative property**. Show that the property of being "compact" is not a relative property: i.e. prove that if $E \subseteq (\mathcal{X}_1, d_1)$ and $f : E \to (\mathcal{X}_2, d_2)$ is an embedding, and if E is compact, then $f(E)$ is compact.

10. Given a sequence (x_i), one constructs the family $\mathcal{F}$ that consists of all sets that contain a tail of the sequence. (A **tail** of a sequence is a set of the form $\{ x_i \ : i \geq n \}$ for some $n \in \mathbb{N}$.) Prove that $\mathcal{F}$ is a filter, and $\mathcal{F} \to x$ if and only if $x_i \to x$.

11. If $\mathcal{F}$ is a filter in a metric space $(\mathcal{X}, d)$, we say that $\mathcal{F}$ is a **Cauchy filter** if $\mathcal{F}$ contains balls of all radii; i.e. if $\epsilon > 0$ is given, then there exists $x \in \mathcal{X}$ such that $B(x, \epsilon) \in \mathcal{F}$. Prove that, if $\mathcal{F}$ is a Cauchy filter in a compact space $(\mathcal{X}, d)$, then $\mathcal{F}$ converges.

12. Prove that a metric space $(\mathcal{X}, d)$ is totally bounded if and only if every filter is contained in a Cauchy filter.

13. Prove that a metric space is complete if and only if every Cauchy filter converges.

14. Prove that a metric space is compact if and only if every ultrafilter converges. (When passing from metric spaces to the arbitrary topological spaces introduced in the next section, the equivalent conditions presented in Theorem 7 are no longer equivalent! The equivalence of the first two conditions can be recovered within the setting of uniform spaces, as we will see in the next chapter, but the third condition becomes a new property, and gives rise to

the definition of a *sequentially compact* space. We wish to note that the ultrafilter statement in this problem remains equivalent to compactness in general topological spaces.)

15. Show that a closed subset of a complete metric space is complete.

16. Recall that $\mathcal{B}(\mathcal{X})$ denotes the set of all bounded complex valued functions on $\mathcal{X}$ with the norm $||f||_\infty = \sup\{\,|f(x)| \ : x \in \mathcal{X}\ \}$. Prove that $\mathcal{B}(\mathcal{X})$ is a complete metric space.

17. If $(\mathcal{X}, d)$ is a metric space, we let $\mathcal{C}^*(\mathcal{X})$ denote the set of all bounded continuous complex valued functions. Show that $\mathcal{C}^*(\mathcal{X})$ is a closed subset of $\mathcal{B}(\mathcal{X})$, and conclude that $\mathcal{C}^*(\mathcal{X})$ is complete.

18. If $E \subseteq (\mathcal{X}, d)$, we say that E is **dense** in $\mathcal{X}$ if $\mathcal{X}$ is the closure of E. Use the preceding exercises and Theorem 2 to prove that every metric space can be isometrically embedded as a dense subset of a complete metric space.

19. Assume that $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$. We say that f is **uniformly continuous** if, for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon$. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ that is not uniformly continuous.

20. Prove that the uniformly continuous image of a Cauchy sequence is a Cauchy sequence.

21. Assume that $E \subseteq (\mathcal{X}_1, d_1)$ is dense in $\mathcal{X}_1$, $f : E \to (\mathcal{X}_2, d_2)$ is uniformly continuous, and $(\mathcal{X}_2, d_2)$ is complete. Prove that f extends to a continuous function on $\mathcal{X}_1$.

22. Assume that a metric space (E, ρ) embeds isometrically as a dense subset of $(\mathcal{X}_1, d_1)$, and it also embeds isometrically as a dense subset of $(\mathcal{X}_2, d_2)$. Prove that $(\mathcal{X}_1, d_1)$ is isometric to $(\mathcal{X}_2, d_2)$. It follows that the space obtained in Exercise 18 is unique (up to isometric isomorphism), and it is called the **metric completion** of (E, ρ).

2.4 Weak Topologies and Products

A product of metric spaces gives rise to a **Tychonoff space**, an object that need not be a metric space, but arises frequently in mathematics, and carries a lot of desirable topological properties. The standard way of describing product spaces is to specify their subbasic open sets, and an effective strategy in dealing with a product space is by manipulation of its subbasic open sets. Recall from Exercise 2.2.1 that the collection $\mathcal{O}$ of open sets in a metric space satisfy

1. If $G_i \in \mathcal{O}$ with $i \in \mathcal{I}$, and $\mathcal{I}$ is a finite index set, then $\cap_{i\in\mathcal{I}} G_i \in \mathcal{O}$.

2. If $G_i \in \mathcal{O}$, $i \in \mathcal{I}$, with no restrictions on $\mathcal{I}$, then $\cup_{i \in \mathcal{I}} G_i \in \mathcal{O}$.

We define a **topology** on a set $\mathcal{X}$ to be any family of subsets $\mathcal{O}$ that satisfy these two conditions, and we refer to the pair $(\mathcal{X}, \mathcal{O})$ as a **topological space**. We allow the possibility that $\mathcal{I} = \emptyset$, and we will consider $\emptyset$ to be finite, so the first condition implies that $\mathcal{X} \in \mathcal{O}$ and the second condition implies $\emptyset \in \mathcal{O}$. The elements of $\mathcal{O}$ are called the **open sets** in this topology. A subset $\mathcal{B} \subseteq \mathcal{O}$ is called a **base** for the topology, and its elements are called **basic open sets**, if every element of $\mathcal{O}$ can be obtained as a union of elements in $\mathcal{B}$. A subset $\mathcal{S} \subseteq \mathcal{O}$ is called a **subbase** for the topology if the family of sets obtained from $\mathcal{S}$ by taking all possible finite intersections (of elements in $\mathcal{S}$) forms a base for the topology. A prototype of a base is the set of all d-balls in a metric space $(\mathcal{X}, d)$. In the set of real numbers with its usual metric $d(s,t) = |s - t|$, the d-balls are the open intervals, so the set of all open intervals forms a base for the topology, and a subbase is seen to be the collection

$$\{ (-\infty, x) \ : \ x \in \mathbb{R} \} \cup \{ (x, \infty) \ : \ x \in \mathbb{R} \}.$$

When any of the equivalent conditions in the following lemma hold, we will say that f is a **continuous** function. The proof of the lemma is one of the exercises.

Lemma 8 *If f is a mapping between two topological spaces, then the following are equivalent:*

1. *Inverse images of open sets are open.*

2. *Inverse images of basic open sets are open.*

3. *Inverse images of subbasic open sets are open.*

Given any collection of sets $\mathcal{S}$, we could intersect all the topologies that contain $\mathcal{S}$ and we would be left with the smallest, or **weakest**, topology containing $\mathcal{S}$. This topology can also be viewed as generated by the collection $\mathcal{S}$, by first taking all finite intersections of the elements in $\mathcal{S}$, giving a set $\mathcal{B}$, then taking all possible unions of the elements in $\mathcal{B}$ resulting in a topology $\mathcal{O}$, which must be contained in any other topology containing $\mathcal{S}$. The original family $\mathcal{S}$ is then a subbase for $\mathcal{O}$, and $\mathcal{B}$ is a base for $\mathcal{O}$. We say that $\mathcal{O}$ is the **weak topology** generated by $\mathcal{S}$. We might also indicate that we are building this topology by *declaring* $\mathcal{S}$ to be its subbase.

If $\mathcal{X}$ is a set and $f_i : \mathcal{X} \to \mathcal{Y}_i$ $(i \in \mathcal{I})$ is a family of functions mapping $\mathcal{X}$ into topological spaces $\mathcal{Y}_i$, then there is a weakest topology on $\mathcal{X}$ that makes all the functions f_i continuous, namely the topology with the subbase

$$\left\{ f_i^{-1}(G) \ : i \in \mathcal{I},\ G \text{ open in } \mathcal{Y}_i \right\}.$$

This is called the **weak topology** induced by the functions f_i. The product topology on a product of metric spaces, or more generally, on a product of topological spaces, is a special case of this weak topology, the case being the weakest topology that making all the coordinate projections continuous. Recall that the product space $\Pi_{i\in\mathcal{I}}X_i$ consists of the family of choice functions

$$\mathbf{x} : \mathcal{I} \to \cup_{i\in\mathcal{I}}X_i$$

with $\mathbf{x}(i) \in \mathcal{X}_i$ for all $i \in \mathcal{I}$. The i^{th} coordinate projection p_i is the function $p_i(\mathbf{x}) = \mathbf{x}(i)$. Consequently, a subbase for the product topology consists of the sets

$$\left\{ p_i^{-1}(G) \ : i \in \mathcal{I},\ G \text{ open in } \mathcal{X}_i \right\}.$$

If the index set is $\{1, 2, 3\}$, then we see that these subbasic open sets look like either $G \times \mathcal{X}_2 \times \mathcal{X}_3$, $\mathcal{X}_1 \times G \times \mathcal{X}_3$, or $\mathcal{X}_1 \times \mathcal{X}_2 \times G$. In the general case, we think of the subbasic open subsets of the product space as the same product, except a single coordinate space has been replaced with an open subset of that coordinate space.

Assume that $\mathcal{X}$ is a set and $f_i : \mathcal{X} \to \mathcal{Y}_i$ $(i \in \mathcal{I})$ is a family of functions mapping $\mathcal{X}$ into topological spaces $\mathcal{Y}_i$. Then we can form the product $\Pi_i\mathcal{Y}_i$, and we obtain the family of coordinate functions $p_i : \Pi_i\mathcal{Y}_i \to \mathcal{Y}_i$. The product is *universal* in that, whenever there is set, such as $\mathcal{X}$, and a family of functions mapping that set into each $\mathcal{Y}_i$, such as the functions f_i, then this family must factor through the product space. Diagramatically speaking, we are saying that there exists a mapping Γ so that

$$\begin{tikzcd}
\mathcal{X} \arrow[r, dotted, "\Gamma"] \arrow[dr, "f_i"'] & \Pi_i\mathcal{Y}_i \arrow[d, "p_i"] \\
& \mathcal{Y}_i
\end{tikzcd}$$

is a commutative diagram. The mapping Γ is defined to force the commutativity, so $\Gamma(x)$ is the element of the product space whose value at i is $f_i(x)$. Bringing the topology back into the discussion, we note that the weak topology on $\mathcal{X}$ and the product topology on $\Pi_i\mathcal{Y}_i$ are exactly what is needed to make Γ continuous. Better still, as long as Γ is injective, then it will also be an embedding. This is the content of the **topological embedding theorem**.

Theorem 9 *With the setup as above, if for every pair of distinct points $x_1, x_2 \in \mathcal{X}$, there exists f_i such that $f_i(x_1) \neq f_i(x_2)$, then the function Γ is an embedding of $\mathcal{X}$ into the product space $\Pi_i \mathcal{Y}_i$.*

The condition in the theorem stating that two distinct points in $\mathcal{X}$ are not collapsed by all the functions f_i is what we mean when we say the family of functions **separates the points** of $\mathcal{X}$, and sometimes the family is referred to as a **separating family** of functions. The image of $\mathcal{X}$ in the product space acquires what is called a **subspace topology**, where the open sets in the image are the intersection of the image with open sets in the product space.

Proof. To see the continuity of Γ, take the inverse image of a subbasic open set $p_i^{-1}(G)$ in the product space, then verify that

$$\Gamma^{-1}(p_i^{-1}(G)) = f_i^{-1}(G),$$

hence this set is open relative to the weak topology. The fact that the family f_i separates the points of $\mathcal{X}$ says exactly that Γ is injective. We need only check that Γ^{-1} is a continuous mapping from the image space $\Gamma(\mathcal{X})$ back to $\mathcal{X}$, so take a subbasic open subset $f_i^{-1}(G)$ of $\mathcal{X}$ and verify that the image of this set under Γ is exactly the intersection of $p_i^{-1}(G)$ with $\Gamma(X)$, and hence open in the subspace topology on $\Gamma(X)$.

$\square$

A topological space is **compact** if every open cover has a finite subcover, or equivalently, every family of closes sets with the finite intersection property has nonempty intersection. When dealing with normed spaces, compactness abounds as long as the underlying vector space is finite dimensional, but compactness is generally lost when the vector space is infinite dimensional. In particular, the closed unit ball, which is compact in finite dimensions, is not compact in infinite dimensions because it fails to be totally bounded. Compactness may be recovered by introducing a second topology, in addition to the one that arises from the norm, in which the closed unit ball is compact. This second topology is a weak topology induced by a family of linear functions, and the resulting compactness is a consequence of the embedding theorem and the fundamental **Tychonoff product theorem**.

Lemma 10 *A space is compact if and only if every ultrafilter converges.*

Proof. Assume $\mathcal{F}$ is an ultrafilter in a compact space $\mathcal{X}$. The the family of closed set contained in $\mathcal{F}$ has the finite intersection property, so there is an element x that lies in every closed set contained in $\mathcal{F}$. If G is an open neighborhood of x, and K is its complement, then either G or K is in $\mathcal{F}$, since it is an ultrafilter. But x lies in every closed set in $\mathcal{F}$, but it does not lie in K, so G must be in $\mathcal{F}$, hence $\mathcal{F} \to x$.

Conversely, let $\mathcal{F}_0$ be a family of closed sets with the finite intersection property. Let $\mathcal{F}$ consist of $\mathcal{F}_0$ together with all sets that contain a member of $\mathcal{F}_0$. Then $\mathcal{F}$ is a filter, which is contained in an ultrafilter $\mathcal{F}'$. Since, by hypothesis, $\mathcal{F}'$ converges to some element x, we conclude that x lies in every element of $\mathcal{F}_0$, so $\mathcal{X}$ is compact.

□

Theorem 11 (Tychonoff) *A product of compact spaces is compact.*

Proof. Assume that $\mathcal{F}$ is an ultrafilter in the product $\Pi_i \mathcal{X}_i$, with each $\mathcal{X}_i$ compact. Then for each coordinate projection p_i, the ultrafilter $p_i(\mathcal{F})$ (Exercise 2.2.15) converges to a point x_i in $\mathcal{X}_i$, by the compactness of $\mathcal{X}_i$. Given any subbasic open set $S = p_k^{-1}(G)$ in the product space that contains the choice function $\mathbf{x} = (x_i)$, the set $p_k(S) = G$ is an open set containing the point x_k, and as such, it must by in $p_k(\mathcal{F})$, which implies $p_k(E) \subseteq p_k(S)$, for some $E \in \mathcal{F}$. Taking inverse images then reveals that $E \subseteq S$, so $S \in \mathcal{F}$. It follows that $\mathcal{F}$ contains every neighborhood of $\mathbf{x}$ and hence $\mathcal{F} \to \mathbf{x}$, and the product is compact. (Compare this proof to Exercises 8 and 12.)

□

Examples

1. In $\mathbb{R}^2$, one subbasic open set is $(0,1) \times \mathbb{R}$, which looks like a vertical strip, and another subbasic open set is $\mathbb{R} \times (0,1)$, which looks like a horizontal strip. The intersection of these two strips gives us an open rectangle $(0,1) \times (0,1)$, which is a basic open set.
2. One way to think of the set of all sequences of real numbers is to realize that it is the product space
$$\Pi_{i \in \mathbb{N}} \mathcal{X}_i$$
with each $\mathcal{X}_i = \mathbb{R}$.
3. The set of all real functions can be realized as the product $\Pi_{r \in \mathbb{R}} \mathcal{X}_r$ with $\mathcal{X}_r = \mathbb{R}$ for all $r \in \mathbb{R}$. The product topology is the **topology of pointwise convergence**: a sequence f_i converges to f in the product topology if and only if $f_i(x) \to f(x)$ for all $x \in \mathbb{R}$.

4. Assume $\mathcal{H}$ denotes **Hilbert space**, the set of (real or complex) sequences $\mathbf{x} = (x_i)$ with $\sum_i |x_i|^2 < \infty$, with the norm $||\mathbf{x}|| = \sqrt{\sum_i |x_i|^2}$ and the inner product $< \mathbf{x}, \mathbf{y} >= \sum_i x_i \bar{y}_i$. The unit ball is not totally bounded, which can be seen by considering the distance between the standard basis vectors (the sequences with zeros everywhere except in one coordinate, where a one appears). However, the weak topology induced by the functionals $f_{\mathbf{y}}$, where $f_{\mathbf{y}}(\mathbf{x}) = < \mathbf{x}, \mathbf{y} >$, gives a second topology on $\mathcal{H}$ in which the unit ball is compact.

Exercises

1. Describe the set of choice functions in the following specific instances:
   (a) $\mathcal{I}$ is the set $\{1, 2\}$ and $\mathcal{X}_i = \mathbb{R}$ for all $i \in \mathcal{I}$.
   (b) $\mathcal{I}$ is the set $\{1, 2, 3\}$ and $\mathcal{X}_1 = \mathbb{R}$, $\mathcal{X}_2 = [0, 1)$, and $\mathcal{X}_3 = \mathbb{R}$.
   (c) $\mathcal{I}$ is the set of natural numbers, and $\mathcal{X}_i = \{0, 1\}$ for all $i \in \mathcal{I}$.
   (d) $\mathcal{I}$ is the set of real numbers, and $\mathcal{X}_i = \mathbb{R}$ for all $i \in \mathcal{I}$.
2. Prove Lemma 8.
3. Prove that the intersection of a family of topologies is a topology.
4. In $\mathbb{R}^2$, prove that every open disc is a union of open rectangles, and prove that every open rectangle is a union of open discs. Conclude that the open sets generated from the Euclidean metric are the same as the open sets in the product topology.
5. In an arbitrary topological space $\mathcal{X}$, we say that a sequence (x_i) converges to x (and write $x_i \to x$) if, for every open set G containing x, the sequence (x_i) is eventually in G. Prove that $x_i \to x$ if and only if, for every subbasic open set S, (x_i) is eventually in S.
6. In arbitrary topological spaces, the neighborhood filter $\mathcal{F}_x$ of a point x is defined to be the collection of all subsets that contain an open set containing x, and we again define $\mathcal{F} \to x$ to mean $\mathcal{F}_x \subseteq \mathcal{F}$. Prove that $\mathcal{F} \to x$ if and only if every subbasic open set containing x is in $\mathcal{F}$.
7. Assume $\mathcal{X}$ is given the weak topology induced by the functions $f_i : \mathcal{X} \to \mathcal{Y}_i$ ($i \in \mathcal{I}$), and assume the f maps a topological space $\mathcal{Y}$ into $\mathcal{X}$. Prove that f is continuous if and only if $f_i \circ f$ is continuous for all $i \in \mathcal{I}$.
8. Assume $\mathcal{X}$ is given the weak topology induced by the functions $f_i : \mathcal{X} \to \mathcal{Y}_i$ ($i \in \mathcal{I}$), and assume $\mathcal{F}$ is a filter in $\mathcal{X}$. Prove that $\mathcal{F} \to x$ if and only if $f_i(\mathcal{F}) \to f_i(x)$ for all $i \in \mathcal{I}$.
9. Dual to the concept of a weak topology induced by a family of functions mapping out of a set is the concept of a strong topology induced by a family of functions mapping into a set. Let $\mathcal{X}$ denote a set, and assume $f_i : \mathcal{Y}_i \to \mathcal{X}$

is a family of functions mapping topological spaces $\mathcal{Y}_i$ into $\mathcal{X}$. The **strong topology** induced by this family of functions is the collection of subsets G of $\mathcal{X}$ with the property that $f_i{-}1(G)$ is open in $\mathcal{Y}_i$ for all i in the indexing set. Prove that this is a topology.

10. Assume that $\mathcal{X}$ has the strong topology induced by a family of functions f_i, and g maps $\mathcal{X}$ into a topological space $\mathcal{Z}$. Prove that g is continuous if and only if $g \circ f$ is continuous for all indices i.

11. Given a topological space $\mathcal{X}$, an equivalence relation $\sim$ on $\mathcal{X}$ induces a quotient map
$$q : \mathcal{X} \to \mathcal{X}/\sim,$$
where $\mathcal{X}/\sim$ denotes the set of equivalence classes, and q maps an element x to the equivalence class containing x. The **quotient** topology on $\mathcal{X}/\sim$ is the strong topology induced by this single function q. Prove that the open subsets G of $\mathcal{X}/\sim$ are exactly those subsets of the quotient for which
$$\cup_{q(x)\in G} \; q(x)$$
is an open subset of $\mathcal{X}$.

12. Assume that $S = p_k^{-1}(G)$ is a subbasic open set in a product space $\Pi_i \mathcal{X}_i$. Prove that $S = p_k^{-1}(p_k(S))$, and if $p_k(E) \subseteq p_k(S)$, then $E \subseteq S$.

13. Prove that a topological space is compact if and only if every open covering by basic open sets has a finite subcover.

14. Prove that a topological space is compact if and only if every open covering by subbasic open sets has a finite subcover. (This requires the axiom of choice.)

15. Use the preceding exercise to give another proof of Tychonoff's theorem.

16. A motto to remember is, "in a metric space, sequences suffice". What this means is that sequences are enough to detect all topological properties such as continuity (Theorem 4), compactness (Theorem 7), closure (Exercise 2.2.3), et cetera. The real reason to know the motto is to be reminded that *sequences no longer suffice* when passing to products of metric spaces! Consider the set of all functions mapping the real numbers into itself, viewed with the product topology (Example 3). A subset of this space is the set of *finitely non-zero functions*, i.e. the set of functions f for which $\{ x \in \mathbb{R} \ : \ f(x) \neq 0 \}$ is a finite set. Prove that a limit of a sequence in E can be non-zero on, at most, a countable subset of $\mathbb{R}$, while the closure of E is everything. Conclude that no metric gives the topology of pointwise convergence. (Sequences are inadequate in some topologies, but filters are not. They provide a tool that replaces sequences in these spaces.)

17. Prove that the unit ball of Hilbert space is not compact relative to the norm topology.

18. Let $(\mathbf{e}_i)$ denote the standard basis vectors in Hilbert space. Prove that $\mathbf{e}_i \to \mathbf{0}$ relative to the weak topology described in Example 4.

19. With the notation of the preceding problem, let E be the subset of $\mathcal{H}$ given by
$$E = \{\mathbf{e}_1, \sqrt{2}\mathbf{e}_2, \sqrt{3}\mathbf{e}_3, \ldots\}.$$
Prove that the vector $\mathbf{0}$ is in the closure of E, but no sequence in E converges to $\mathbf{0}$.

20. A topological space $\mathcal{X}$ is **locally compact** if there exists a base $\mathcal{B}$ for which every element of $\mathcal{B}$ has compact closure. Give an example of a locally compact space that is not compact.

21. Give an example of a space that is not locally compact.

2.5 Tychonoff topology

A **Tychonoff topology** can be thought of as a generalization of a metric space, one whose open sets are determined not by a single metric, but by a family of metric–like functions called **pseudometrics**. A pseudometric is defined by the same axioms as a metric, except that we allow $d(x, y) = 0$ without insisting that x equals y. We will require, however, that there are enough pseudometrics in the family so that they separate the points of $\mathcal{X}$, i.e. given $x, y \in \mathcal{X}$ one can find a pseudometric d in the family with $d(x, y) \neq 0$. A **Tychonoff space** is a topological space whose topology is the weak topology generated by the d-balls of a separating family of pseudometrics. In this case, we will say that the family of pseudometrics **generates** the topology on $\mathcal{X}$.

An arbitrary product of metric spaces $(\mathcal{X}_i, d_i)$ is easily seen to be a Tychonoff space, since the product topology is generated by the family of pseudometrics
$$\mathbf{d}_i(\mathbf{x}, \mathbf{y}) = d_i(x_i, y_i),$$
where $\mathbf{d}_i$ is the pseudometric on the product that uses the metric d_i on $\mathcal{X}_i$ to measure the distance between the i^{th} coordinates of $\mathbf{x}$ and $\mathbf{y}$. Conversely, if $\mathcal{X}$ is a Tychonoff space whose topology is generated by a family of pseudometrics $\mathcal{G}$, then the topology on $\mathcal{X}$ is the weak topology induced by the family of functions
$$f_x^d(y) = d(x, y) \quad (x \in \mathcal{X}, d \in \mathcal{G}),$$
so by the embedding theorem, $\mathcal{X}$ embeds into a product of many copies of $\mathbb{R}$, one copy for each pair (x, d), with $x \in \mathcal{X}, d \in \mathcal{G}$. Thus, every Tychonoff space is homeomorphic to a subspace of a product of metric spaces.

If $\mathcal{X}$ is a topological space and d is a pseudometric on $\mathcal{X}$, we will say that d is a **continuous pseudometric** if every d-ball is an open subset of $\mathcal{X}$. For a given Tychonoff space, there may be many generating families of pseudometrics, but there is a natural largest generating family, namely, the set of all continuous pseudometrics. The metric spaces are precisely the Tychonoff spaces that have a generating family consisting of a singleton. We will call a topological space **metrizable** if the open sets are generated by a single metric. A basic technique is the following method of reducing a countable generating family to a singleton.

Theorem 12 *A Tychonoff space is metrizable if and only if there exists a countable generating set of pseudometrics.*

Proof. Assume a topology on $\mathcal{X}$ is generated by the pseudometrics $d_1, d_2, \ldots$. Notice that we can truncate the distances so that none of the metrics attains a value greater than 1, without disturbing the weak topology generated by the d-balls; in other words, there is no loss of generality in assuming that $d_i(x, y) \leq 1$ for all $x, y \in \mathcal{X}$. Then

$$d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(x, y)}{2^i}$$

is a metric on $\mathcal{X}$ that generates the same topology.

□

A metric is to a norm what a pseudometric is to a seminorm: specifically, a **seminorm** on a (real or complex) vector space is a function that satisfies all the properties of a norm, except we do not require $\mathbf{x} = 0$ when $||\mathbf{x}|| = 0$. The analogue of a Tychonoff space is what people call a **locally convex space**, which is a vector space with a separating family of seminorms. It is fair to say that most of the Tychonoff spaces that arise in analysis are, in fact, locally convex spaces, and we have already met an important one in Example 2.4.4. Interesting locally convex spaces are an infinite dimensional phenomenon. In finite dimensions, we will soon see that the topologies arising from any two locally convex structures are the same.

Let $\mathcal{V}$ denote either $\mathbb{R}^n$ or $\mathbb{C}^n$, and let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denote the standard basis. The coordinate pseudometrics on $\mathcal{V}$ are thus obtained from the seminorms

$$||\sum_i \alpha_i \mathbf{e}_i||_k = |\alpha_k|,$$

and consequently, these n seminorms form a generating family for the product topology on $\mathcal{V}$. As in Theorem 12, the sum of these seminorms is a single generator for the product topology, and it is the norm $||\sum_i \alpha_i \mathbf{e}_i|| = \sum_i |\alpha_i|$.

Theorem 13 *The topology on $\mathcal{V}$ induced by any seminorm is weaker than the product topology. The topology induced by any separating family of seminorms (i.e. any locally convex topology) is the product topology.*

Before we begin the proof, we need to emphasize that the statement above is about the *finite dimensionality of* $\mathcal{V}$, so when we move to infinite dimensions the theorem is no longer valid. The locally convex topologies of interest that arise in infinite dimensional settings tend to be strictly weaker than a related norm topology, as is the case in Example 2.4.4.

Proof. Let $||\cdot||$ denote the norm described in the paragraph preceding the statement of our theorem, so that

$$||\sum_i \alpha_i \mathbf{e}_i|| = \sum_i |\alpha_i|,$$

and assume $|||\cdot|||$ is any seminorm on $\mathcal{V}$. The triangle inequality gives us

$$|||\sum_i \alpha_i \mathbf{e}_i||| \le \sum_i |\alpha_i| |||\mathbf{e}_i||| \le \sum_i |\alpha_i| K = K||\sum_i \alpha_i \mathbf{e}_i||,$$

where K is the largest of $|||\mathbf{e}_i|||$ $(i = 1, \ldots, n)$. We conclude that the identity function, viewed as a mapping from $(\mathcal{V}, ||\cdot||)$ to $(\mathcal{V}, |||\cdot|||)$, is continuous (in fact Lipschitz, see Exercise 2.2.12), so that the pseudometric induced by $|||\cdot|||$ is continuous relative to the product topology, which verifies the first assertion of the theorem.

Assume now that $\mathcal{G}$ is a separating family of seminorms on $\mathcal{V}$. It follows from the previous paragraph that the topology generated by $\mathcal{G}$ is weaker than the product topology, so we are left to verify that this topology is also stronger than the product topology. Starting with a non-zero vector $\mathbf{v}_1$ in $\mathcal{V}$, we choose a seminorm $||\cdot||_1$ in $\mathcal{G}$ so that $||\mathbf{v}_1||_1 \neq 0$. We then pick a non-zero element $\mathbf{v}_2$ in the "kernel" of $||\cdot||_1$, i.e. in the set

$$\mathcal{M}_1 = \{ \mathbf{v} \ : \ ||\mathbf{v}||_1 = 0 \},$$

and select an element $||\cdot||_2$ in $\mathcal{G}$ so that $||\mathbf{v}_2||_2 \neq 0$. We continue inductively, picking non-zero $\mathbf{v}_i$ in the set

$$\mathcal{M}_{i-1} = \{ \mathbf{v} \ : \ ||\mathbf{v}||_j = 0 \text{ for all } j = 1, \ldots, i-1 \}$$

and selecting $||\cdot||_i$ in $\mathcal{G}$ so that $||\mathbf{v}_i||_i \neq 0$, repeating the process until $\mathcal{M}_i = \{\mathbf{0}\}$. At each step, the set $\mathcal{M}_k$ is a proper subset of $\mathcal{M}_{k-1}$, and being subspaces, the dimension decreases. It follows that the process must terminate in at least n steps, resulting in finitely many elements of $\mathcal{G}$ whose sum, which we denote $|||\cdot|||$, is a norm on $\mathcal{V}$.

The closed set $\{\sum_i \alpha_i \mathbf{e}_i \ : \ \sum_i |\alpha_i| = 1\}$ is compact in the product topology, and $|||\cdot|||$, being continuous relative to the product topology, attains a minimum value on this set. It follows that

$$0 < r \leq |||\frac{\mathbf{x}}{||\mathbf{x}||}|||,$$

for all $\mathbf{x} \in \mathcal{V}$, and $r||\mathbf{x}|| \leq |||\mathbf{x}|||$, showing that the identity is continuous as a mapping from $(\mathcal{V}, |||\cdot|||)$ to $(\mathcal{V}, ||\cdot||)$. Thus the topology generated by $|||\cdot|||$ is stronger than the product topology.

□

Exercises

1. Let $\mathcal{X}$ be a topological space. Prove that if d is a continuous pseudometric, then the sets $\{ y \in \mathcal{X} \ : \ d(x,y) > \delta \}$ are open, where $x \in \mathcal{X}$ and $\delta \in \mathbb{R}$.
2. Let $\mathcal{X}$ be a topological space. Prove that d is a continuous pseudometric on $\mathcal{X}$ if and only if the function f_x^d is continuous for every $x \in \mathcal{X}$. (See page 33 for the definition of f_x^d.)
3. Let $\mathcal{X}$ be a Tychonoff space whose topology is generated by the family of pseudometrics $\mathcal{G}$. Prove that the topology on $\mathcal{X}$ is the same as the weak topology induced by the family of functions f_x^d, where $x \in \mathcal{X}$ and $d \in \mathcal{G}$.
4. Assume $\mathcal{X}$ is a Tychonoff space with generating family $\mathcal{G}$. If E is a subset of $\mathcal{X}$, let $\mathcal{G}_E$ denote the set of restrictions of elements of $\mathcal{G}$ to E. Prove that the resulting Tychonoff Topology on E generated by the family $\mathcal{G}_E$ is the same as the topological **subspace topology** that E inherits from the topology on $\mathcal{X}$ (see the definition of subspace topology on page 29).
5. Give an example of a continuous pseudometric on $(0,1)$ that is not the restriction of a continuous pseudometric on $\mathbb{R}$ to $(0,1)$.
6. Prove that a bounded continuous pseudometric on $(0,1)$ is the restriction of a continuous pseudometric on $\mathbb{R}$ to $(0,1)$. (?CHECK?)
7. If d_1 and d_2 are continuous relative to a topology on $\mathcal{X}$, prove that $d_1 + d_2$ is continuous also.
8. Assume that the topology on $\mathcal{X}$ is generated by the family of pseudometrics $\mathcal{G}$, and let $\mathcal{G}'$ be the set of all finite sums of elements of $\mathcal{G}$. Show that the set of d-balls with $d \in \mathcal{G}'$ forms a base for the topology.

9. Two pseudometrics are **topologically equivalent** if they give rise to the same open sets. Prove that two pseudometrics are topologically equivalent if and only if each is continuous relative to the topology generated by the other.

10. Assume d is a pseudometric on a set $\mathcal{X}$ and $d(x, y) = 0$ for some $x, y \in \mathcal{X}$. Prove that
$$d(x, z) = d(y, z)$$
for all $z \in \mathcal{X}$.

11. Assume d is a pseudometric on $\mathcal{X}$, and define a relation by $x \sim y$ if and only if $d(x, y) = 0$. Verify that this defines an equivalence relation on $\mathcal{X}$, and show that the quotient topology on the quotient space is metrizable.

12. A topological space $\mathcal{X}$ is called **Hausdorff** if every pair of distinct points in $\mathcal{X}$ are contained in disjoint open subsets of $\mathcal{X}$. Prove that every Tychonoff space is Hausdorff.

13. A topological space $\mathcal{X}$ is **normal** if every pair of disjoint closed sets are contained in a pair of disjoint open sets. Prove that every metric space is normal.

14. A topological space is **completely regular** if every pair consisting of a closed set and a point not in that set can be separated with a continuous function. Prove that every Tychonoff space is completely regular.

15. Prove that every completely regular Hausdorff space is a Tychonoff space.

16. Prove that an arbitrary product of Tychonoff spaces is a Tychonoff space.

17. A **quasi-pseudometric** is a function that satisfies the axioms of a pseudometric, except we drop the requirement that $d(x, y) = d(y, x)$. Prove that every topological space is generated by a family of quasi-pseudometrics.

18. A real (or complex) valued function f defined on a topological space $\mathcal{X}$ is said to have **compact support** if there exists a compact subset K of $\mathcal{X}$ for which $f(x) = 0$ whenever $x \in \mathcal{X} \setminus K$. If $\mathcal{X}$ is a locally compact Tychonoff space, prove that the topology on $\mathcal{X}$ coincides with the weak topology generated by the family of continuous functions with compact support.

19. If $\mathcal{X}$ is a locally compact Tychonoff space, prove that there exists a family of d-balls that form a base for the topology, such that, every member of this family has compact closure.

Chapter 3

Uniformity

Topological spaces are the carriers for the concepts of limits and continuity. Certain natural concepts arising from metrics, such as completeness, total boundedness, and uniform continuity, require additional structure beyond the topological, and a uniform space is what results when we augment topology to include this extra structure.

Assume that $f : (\mathcal{X}_1, d_1) \to (\mathcal{X}_2, d_2)$ is a function between metric spaces, and recall the definition of uniform continuity given in Exercise 2.3.19: f is **uniformly continuous** if, for all $\epsilon > 0$ there exists $\delta > 0$ such that $d_1(x, y) < \delta$ implies $d_2(f(x), f(y)) < \epsilon$. The difference between uniform continuity and plain continuity is that the δ depends only on ϵ in uniform continuity, so one δ works everywhere throughout $\mathcal{X}_1$, whereas in plain continuity we are allowed to use different δ's at different points $x \in \mathcal{X}_1$. There is an "inverse image" interpretation of uniform continuity that parallels the one for continuity, and leads nicely towards the definition of an abstract uniform space.

On a general metric space $(\mathcal{X}, d)$, consider the open covers $\mathcal{C}_\epsilon$ given by

$$\mathcal{C}_\epsilon = \{\, B(x, \epsilon) \ : x \in \mathcal{X} \,\},$$

which consists of sets, all with *uniform size*, covering $\mathcal{X}$. Consequently, we name such covers **basic uniform covers**, and think of them as uniform analogues of basic open sets. If we take the inverse image of such a cover, i.e. we take the inverse image of every set in the cover, we obtain a family which we denote $f^{-1}(\mathcal{C}_\epsilon)$,

$$f^{-1}(\mathcal{C}_\epsilon) = \left\{\, f^{-1}(B(x, \epsilon)) \ : x \in \mathcal{X}_2 \,\right\},$$

which, if f is continuous, is an open cover of $\mathcal{X}_1$. By looking closely at this cover when f is uniformly continuous, we arrive at a pertinent definition, which follows. We say that a cover $\mathcal{C}_1$ **refines** a second cover $\mathcal{C}_2$ if $G \in \mathcal{C}_1$ implies the existence of $H \in \mathcal{C}_2$ with $G \subseteq H$. (In this scenario we also say that $\mathcal{C}_2$ is refined by $\mathcal{C}_1$.) We define a cover $\mathcal{C}$ to be a **uniform cover** if it is refined by a basic uniform cover. The uniform covers are then to uniform spaces what open sets are to topological spaces, and a function f is uniformly continuous if and only if inverse images of basic uniform covers are uniform covers (Exercise 3.1.1).

3.1 Uniform Spaces and their Mappings

The entire theory of uniform spaces can be developed using uniform covers as the primitive object, but we intend to use them in an auxiliary role. We take as the primitive object a family of pseudometrics $\mathcal{U}$ on a set $\mathcal{X}$ that satisfy two axioms:

1. If $d_1, d_2 \in \mathcal{U}$ then $d_1 + d_2 \in \mathcal{U}$, and
2. If d is a pseudometric on $\mathcal{X}$ with the property that, for every $\epsilon > 0$, there exists $\delta > 0$ and $d' \in \mathcal{U}$ so that $d'(x, y) < \delta$ implies $d(x, y) < \epsilon$, then we require $d \in \mathcal{U}$.

Such a pair $(\mathcal{X},\mathcal{U})$ will be called a **uniform space**, and we will assume that the pairs we call uniform spaces are Hausdorff, i.e. that the associated topology is a Tychonoff space. It is natural to think of the axioms defining a uniform space as analogues to their topological counterparts, namely, the first is an analogue to the axiom that finite intersections of open sets are open, and the second is an analogue to the axiom that arbitrary unions of open sets are open.

The second axiom for a uniform space seems mysterious at first, so a comment or two to dispel the mystery is in order. We may restate the axiom in terms of the d-basic uniform covers

$$\mathcal{C}^d_\epsilon = \{\, B_d(x, \epsilon) \ : \ x \in \mathcal{X} \,\},$$

and that restatement is, if d is a pseudometric so that every d-basic uniform cover is refined by some basic uniform cover corresponding to an element of $\mathcal{U}$, then d should be in $\mathcal{U}$. Given any collection of pseudometrics $\mathcal{G}$, we will say that a pseudometric d **lies below** $\mathcal{G}$ if every d-basic uniform cover is refined by one corresponding to an element of $\mathcal{G}$, in which language the second axiom becomes, if d lies below $\mathcal{U}$, then d is in $\mathcal{U}$.

We now mimic the concepts of subbase and base for a topological space, defining a subset $\mathcal{B} \subseteq \mathcal{U}$ to be a **base for the uniformity** if $\mathcal{U}$ is exactly the set of all pseudometrics that lie below $\mathcal{B}$. We say that $\mathcal{S}$ is a **subbase for the uniformity** if the set of all finite sums of pseudometrics from $\mathcal{S}$ forms a base for the uniformity. Starting with any separating family of pseudometrics $\mathcal{S}$, one can generate a uniformity by first taking all possible finite sums of elements of $\mathcal{S}$, resulting in the larger family $\mathcal{B}$, then taking all pseudometrics that lie below $\mathcal{B}$.

We could have phrased the first axiom in terms of the supremum $d_1 \vee d_2$ of two pseudometrics, i.e. in terms of the pseudometric

$$(d_1 \vee d_2)(x, y) = \max\{d_1(x, y), d_2(x, y)\}.$$

If the first axiom is replaced with the condition

1. If $d_1, d_2 \in \mathcal{U}$ then $d_1 \vee d_2 \in \mathcal{U}$,

and the second axiom is unchanged, then one has an equivalent definition for a uniform space. We will feel free to use whichever is more convenient at the time it is used.

Given a function $f : (\mathcal{X}_1, \mathcal{U}_1) \to (\mathcal{X}_2, \mathcal{U}_2)$ between uniform spaces, and given $d \in \mathcal{U}_2$, we will let $d \circ \mathbf{f}$ denote the pseudometric

$$(d \circ \mathbf{f})(x, y) = d(f(x), f(y)).$$

We will say that f is **uniformly continuous** if the equivalent conditions in the following theorem are true.

Theorem 14 *For every $d \in \mathcal{U}_2$ we have $d \circ \mathbf{f} \in \mathcal{U}_1$ if and only if inverse images of uniform covers are uniform covers.*

Proof. Assume that $d \in \mathcal{U}_2$ implies $d \circ \mathbf{f} \in \mathcal{U}_1$, and let $\mathcal{C}$ be a uniform cover of $\mathcal{X}_2$. Then there exists $\epsilon > 0$ and $d \in \mathcal{U}_2$ such that $\mathcal{C}^d_\epsilon$ refines $\mathcal{C}$. One may now see that $\mathcal{C}^{d\circ\mathbf{f}}_\epsilon$ refines $f^{-1}(\mathcal{C})$: let $B_{d\circ\mathbf{f}}(x, \epsilon)$ be given. Then $B_d(f(x), \epsilon)$ must be contained in some element $H \in \mathcal{C}$, and thus

$$\mathcal{C}^{d\circ\mathbf{f}}_\epsilon \subseteq f^{-1}(B_d(f(x), \epsilon)) \subseteq f^{-1}(H).$$

It follows that $f^{-1}(\mathcal{C})$ is a uniform cover.

Now assume that inverse images of uniform covers are uniform, and let $d \in \mathcal{U}_2$ be given. We intend to prove that $d \circ \mathbf{f}$ lies below $\mathcal{U}_1$, so towards this end, let ϵ be given. Thus $f^{-1}(\mathcal{C}^d_\epsilon)$ is a uniform cover of $\mathcal{X}_1$, and $\mathcal{C}^{d'}_\delta$ refines $f^{-1}(\mathcal{C}^d_\epsilon)$ for some $d' \in \mathcal{U}_1$ and $\delta > 0$. If $d'(x, y) < \delta$, then because $B_{d'}(x, \delta) \subseteq f^{-1}(B_d(z, \epsilon))$ for some $z \in \mathcal{X}_2$, we get

$$(d \circ \mathbf{f})(x, y) = d(f(x), f(y)) < 2\epsilon,$$

which is close enough.

□

Examples

1. Let $(\mathcal{X}, d)$ be a metric space. Then the singleton $\{d\}$ is a base for a uniformity $\mathcal{U}_d$ that consists of all pseudometrics lying below $\{d\}$.

2. The set of all continuous pseudometrics on a Tychonoff space $\mathcal{X}$ forms a uniformity on $\mathcal{X}$ called the **fine uniformity** on $\mathcal{X}$.

3. On the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ we consider two topologically equivalent metrics, the usual metric d on $\mathbb{R}$ restricted to $(-\frac{\pi}{2}, \frac{\pi}{2})$, and the metric

$$d'(x,y) = |\tan(x) - \tan(y)|.$$

The sequence $\{\frac{\pi}{2} - \frac{1}{n}\}$ is a Cauchy sequence relative to d, but not relative to d'. The interval is complete metric space relative to d', but not relative to d.

Exercises

1. Using the definition of uniform continuity given in Exercise 2.3.19, and again on page 39, prove that a function between metric spaces is uniformly continuous if and only if inverse images of basic uniform covers are uniform covers.

2. Prove that uniform continuity implies continuity.

3. Assume $(\mathcal{X},\mathcal{U})$ is a uniform space, $E \subseteq \mathcal{X}$, and $\mathcal{B}_E$ is the the set of restrictions of elements of $\mathcal{U}$ to E. Prove that $\mathcal{B}_E$ is a base for a uniformity on E. The corresponding uniformity $\mathcal{U}_E$ on E is called the **subspace uniformity** on E.

4. If $(\mathcal{X},\mathcal{U})$ is a uniform space, prove that the set of restrictions $\mathcal{B}_E$ of elements of $\mathcal{U}$ to a subset E need not be a uniformity.

5. Prove that inverse images of basic uniform covers are uniform covers if and only if inverse images of uniform covers are uniform covers.

6. With $f : (\mathcal{X}_1,\mathcal{U}_1) \to (\mathcal{X}_2,\mathcal{U}_2)$, and if $\mathcal{S}$ and $\mathcal{B}$ constitute a subbase and a base (respectively) of $\mathcal{U}_2$, prove that the following are equivalent:

   (a) if $d \in \mathcal{U}_2$ then $d \circ \mathbf{f} \in \mathcal{U}_1$.

   (b) if $d \in \mathcal{B}$ then $d \circ \mathbf{f} \in \mathcal{U}_1$.

   (c) if $d \in \mathcal{S}$ then $d \circ \mathbf{f} \in \mathcal{U}_1$.

7. On page 41 it is asserted that the first axiom for a uniform space can be replaced with the assertion that the supremum of two elements of the uniformity is again in the uniformity, without changing the logical content of the definition. Prove the equivalence of the two definitions.

8. Modify the proof of Theorem 14 to arrive at $(d \circ \mathbf{f})(x, y) = d(f(x), f(y)) < \epsilon$, instead of $(d \circ \mathbf{f})(x, y) = d(f(x), f(y)) < 2\epsilon$.
9. Prove that the two metrics given in Example 3 are topologically equivalent.
10. Prove that the sequence given in Example 3 is not d'-Cauchy.
11. Show that the interval in Example 3 is a complete metric space relative to d', but not relative to d.
12. Prove that the fine uniformity on the interval $(\frac{-\pi}{2}, \frac{\pi}{2})$ is not the same as the uniformity obtained from the standard metric.
13. If d is an element of a uniformity $\mathcal{U}$, prove that the truncation

$$(d \wedge 1)(x, y) = \begin{cases} d(x, y) & \text{if } d(x, y) \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

   is also in $\mathcal{U}$.
14. If $\{d_i\}$ is a sequence in $\mathcal{U}$, prove that there exists a single element d of $\mathcal{U}$ so that the set of pseudometrics that lie below $\{d_i\}_{i=1}^{\infty}$ is the same as the set of pseudometrics below d.
15. Prove the analogue of Theorem 12 for uniform spaces.

3.2 Weak and Product Uniformity

Assume that $\{f_i\}$ is a family of functions mapping a set $\mathcal{X}$ into uniform spaces $(\mathcal{X}_i, \mathcal{U}_i)$. The **weak uniformity** on $\mathcal{X}$ generated by the family $\{f_i\}$ is the one generated by the family of pseudometrics $\{d \circ \mathbf{f}_i\}$, where i varies in the index set and d varies in $\mathcal{U}_i$. As such, it is the weakest uniformity on $\mathcal{X}$ making all of the functions f_i uniformly continuous. The **product uniformity** on $\Pi_i \mathcal{X}_i$ is the weak uniformity induced by the family of coordinate projections. A **uniform isomorphism** is a uniformly continuous bijection between uniform spaces with a uniformly continuous inverse, and a **uniform embedding** is a uniformly continuous injection that is a uniform isomorphism onto its range. The **uniform embedding theorem** proceeds just like its topological counterpart.

Theorem 15 *If $(\mathcal{X}, \mathcal{U})$ is the weak uniformity induced by the family*

$$f_i : \mathcal{X} \to \mathcal{X}_i,$$

then the corresponding mapping of $\mathcal{X}$ into the product $\Pi_i \mathcal{X}_i$ is a uniform embedding, as long as the family $\{f_i\}$ separates the points of $\mathcal{X}$.

Proof. Let ι denote the natural mapping of $\mathcal{X}$ into the product, and let p_i denote the i^{th} coordinate projection, so that $p_i \circ \iota = f_i$. Take any subbasic pseudometric on the product space, which is of the form $d \circ \mathbf{p}_i$, for some index i and $d \in \mathcal{U}_i$. It follows that

$$(d \circ \mathbf{p}_i) \circ \iota = d \circ (\mathbf{p}_i \circ \iota) = d \circ \mathbf{f}_i,$$

which shows that ι is uniformly continuous. To see that ι^{-1} is uniformly continuous, take a subbasic pseudometric of the weak uniformity $d \circ \mathbf{f}_i$, and note that

$$(d \circ \mathbf{f}_i) \circ \iota^{-1} = d \circ (\mathbf{f}_i \circ \iota^{-1}) = d \circ \mathbf{p}_i.$$

□

Assume that $(\mathcal{X},\mathcal{U})$ is any uniform space, and index the elements of $\mathcal{U}$ like so $\mathcal{U} = \{d_i\}_{i \in \mathcal{I}}$. Each space $(\mathcal{X}, d_i)$ is a pseudometric space, and, just as in Theorem 2, it maps isometrically into the normed space $\mathcal{B}(\mathcal{X})$. Let us denote the isometry by $f_i : \mathcal{X} \to \mathcal{B}(\mathcal{X})$. We then have a family of functions mapping $\mathcal{X}$ into the uniform space $\mathcal{B}(\mathcal{X})$. If the metric induced by the norm on $\mathcal{B}(\mathcal{X})$ is denoted with d, then f_i, being an isometry, satisfies $d_i = d \circ \mathbf{f}_i$, which shows that the weak uniformity induced on $\mathcal{X}$ by the family $\{f_i\}$ is stronger than $\mathcal{U}$. On the other hand, every other pseudometric in the uniformity of $\mathcal{B}(\mathcal{X})$ is one that lies below the singleton $\{d\}$, and pulling this back via f_i will result in a pseudometric lying below d_i, which will be in $\mathcal{U}$. It follows that the uniformity $\mathcal{U}$ equals the weak uniformity generated by the family $\{f_i\}$. The uniform embedding theorem then says that, as long as the topology associated with $\mathcal{U}$ is Tychonoff, the uniform space $(\mathcal{X},\mathcal{U})$ is uniformly isomorphic to a subspace of a product of many copies of $\mathcal{B}(\mathcal{X})$, one factor for each element in the uniformity. We record this observation for future reference.

Theorem 16 *Every Tychonoff uniform space $(\mathcal{X},\mathcal{U})$ is uniformly isomorphic to a subspace of a (possibly large) product of the normed space $\mathcal{B}(\mathcal{X})$ with itself.*

Examples

1. If $\mathcal{V}$ is a locally convex space induced by a separating family of seminorms $\mathcal{G}$, then the set of all continuous seminorms $\mathcal{G}'$ on $\mathcal{V}$ induces the same locally convex structure on $\mathcal{V}$, and $\mathcal{G}'$ is a base for the associated uniformity.

Exercises

1. Using the definition of uniform continuity given in Exercise 2.3.19, and again on page 39, prove that a function between metric spaces is uniformly continuous if and only if inverse images of basic uniform covers are uniform covers.

2. Prove that inverse images of basic uniform covers are uniform covers if and only if inverse images of uniform covers are uniform covers.

3. Let $\mathcal{X}$ be a Tychonoff space, and recall that d is a continuous pseudo-metric when every d-ball is open. Prove that, if d is continuous, then

$$d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

is a continuous function, when $\mathcal{X} \times \mathcal{X}$ is given the product topology.

Solution: Assume that the net (x_i, y_i) converges to (x, y) in the product topology, so $x_i \to x$ and $y_i \to y$. Since all d-balls are open, then $d(x_i, x) \to 0$ and $d(y_i, y) \to 0$. The triangle inequality gives us

$$d(x_i, y_i) \leq d(x_i, x) + d(x, y) + d(y_i, y)$$

and

$$d(x, y) \leq d(x_i, x) + d(x_i, y_i) + d(y_i, y),$$

from which it follows that

$$|d(x_i, y_i) - d(x, y)| \leq d(x_i, x) + d(y_i, y) \to 0.$$

4. Give an example of a function

$$d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$$

for which every set of the form $\{\, y \ : d(x, y) < \epsilon \,\}$ is open, but d is not continuous when $\mathcal{X} \times \mathcal{X}$ is given the product topology.

Solution: Let $\mathcal{X} = \mathbb{R}$ and let d be the function with $d(0, 0) = 0$ and

$$d(x, y) = \frac{xy}{x^2 + y^2}$$

for $(x, y) \neq (0, 0)$. Since for each $x \in \mathbb{R}$ the function $g_x(y) = d(x, y)$ is continuous, we have

$$\{\, y \ : d(x, y) < \epsilon \,\} = g_x^{-1}(-\infty, \epsilon)$$

open, but d is not continuous at $(0, 0)$, since when $x_i \to 0$ we have

$$d(x_i, x_i) \to \frac{1}{2} \neq d(0, 0).$$

5. When $(\mathcal{X},\mathcal{U})$ is a uniform space with $d \in \mathcal{U}$, then $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ is uniformly continuous, when $\mathcal{X} \times \mathcal{X}$ is given the product uniformity, and $\mathbb{R}$ has its usual uniform structure (the one induced by the usual metric $|r - s|$). Prove this statement and its converse.

   Solution: Assume that $d \in \mathcal{U}$, and we set out to prove that $\tau \circ \mathbf{d}$ is in the product uniformity, where τ denotes the standard metric on $\mathbb{R}$. Let ϵ be given. We have

$$\begin{aligned} |d(x_1, y_1) - d(x_2, y_2)| &\leq |d(x_1, y_1) - d(x_1, y_2)| + |d(x_1, y_2) - d(x_2, y_2)| \\ &\leq d(y_1, y_2) + d(x_1, x_2) \end{aligned}$$

   so $|d(x_1, y_1) - d(x_2, y_2)| < \epsilon$ as soon as both $d \circ \mathbf{p_1}$ and $d \circ \mathbf{p_2}$ are less than $\epsilon/2$ (where p_i denotes the two projection functions on $\mathcal{X} \times \mathcal{X}$). Conversely, if d is uniformly continuous, then there are elements $d_i \in \mathcal{U}$ and $\rho_i \in \mathcal{U}$, and a $\delta > 0$, so that $d_i(x_1, x_2) < \delta$ and $\rho_i(y_1, y_2) < \delta$ forces

$$|d(x_1, y_1) - d(x_2, y_2)| < \epsilon.$$

   Setting $y_1 = y_2 = x_2$ in the above equation gives

$$d(x_1, x_2) < \epsilon$$

   when $d_i(x_1, x_2) < \delta$, so $d \in \mathcal{U}$.

3.3 Complete Uniform Spaces

A filter $\mathcal{F}$ in a uniform space $(\mathcal{X},\mathcal{U})$ is **Cauchy** if, given any $\epsilon > 0$ and any $d \in \mathcal{U}$, there exists $x \in \mathcal{X}$ with $B_d(x, \epsilon) \in \mathcal{F}$. A uniform space is **complete** if every Cauchy filter converges.

The definition of a complete metric space was given in terms of sequences, but in the generality of a uniform space the sequential definition is inadequate. Indeed, the set E constructed in Exercise 2.4.19 is an example of a uniform space that is not complete by the definition in the previous paragraph, but every Cauchy sequence in E does converge, i.e. E is *sequentially complete*. There is another object, in addition to filters, which can fully carry topological and uniform properties in our current generality, and these objects have the advantage that they can be manipulated almost exactly like sequences. They were originally called **generalized sequences**, but today they are more commonly referred to as **nets**. A sequence is a function whose domain is the well ordered set $\mathbb{N}$. A **net** is a function whose domain is a partially ordered set $\mathcal{D}$ that is **directed**: if $i, j \in \mathcal{D}$, then there exists $k \in \mathcal{D}$ such that $i \leq k$ and $j \leq k$. We call such a domain a **directed set**.

In a metric space, imagine the set $\mathcal{D}$ of all balls with a common center x. This is a directed set whose order is set inclusion *directed downwards*, so in this ordering $B_1 \leq B_2$ means $B_2 \subseteq B_1$. To describe local topological properties at x, we can make do with the subset

$$\left\{ B(x, \tfrac{1}{n}) \;:\; n \in \mathbb{N} \right\},$$

which, as a directed set (directed downwards), is isomorphic to the natural numbers $\mathbb{N}$, which is why we need only deal with sequences in the case of metric spaces. This subset is what is called a **local base** at x, meaning that, given any open set G containing x, there exists an element B of the local base with $B \subseteq G$. As long as x has a countable local base, then local topological properties can be encoded via sequences, and countable local bases are what we lose when passing to large product spaces.

When describing local convergence properties at a point x, in the situation where sequences are inadequate, what is often taken as the domain $\mathcal{D}$ of our net is the neighborhood filter at x, which is a directed set (with downward direction). The peripheral definitions for nets, such as convergence, a net being eventually or frequently in a set, a Cauchy net, et cetera, are all almost exactly the same as the corresponding peripheral definitions for sequences, and the techniques of manipulating nets offer few surprises to those comfortable handling sequences. Thus, if E is a set with x in the closure of E, then every neighborhood G of x, i.e. every element of the neighborhood filter $\mathcal{D}$, must intersect E, so there exists $x_G \in E \cap G$. The function taking G to x_G is a net that satisfies: for every open set H containing x, the net x_G is eventually in H, and consequently, the net $\{x_G\}$ converges to x. This is half of the argument that x is in the closure of E if and only if there exists a net in E that converges to x.

Here is the definition for a **Cauchy net**: $\{x_i\}$ $(i \in \mathcal{D})$ is a Cauchy net in a uniform space $(\mathcal{X},\mathcal{U})$ if, for every $\epsilon > 0$ and every $d \in \mathcal{U}$, there exists $k \in \mathcal{D}$ so that $d(x_i, x_j) < \epsilon$ for all $i \geq k$ and $j \geq k$. The following proof is virtually its metric space counterpart, with the word "sequence" replaced with "net" everywhere it appears.

Lemma 17 *The uniformly continuous image of a Cauchy net is Cauchy.*

Proof. Assume $\{x_i\}$ $(i \in \mathcal{D})$ is a Cauchy net in the domain of a uniformly continuous function $f : (\mathcal{X}_1,\mathcal{U}_1) \to (\mathcal{X}_2,\mathcal{U}_2)$. In order to prove $\{f(x_i)\}$ is a Cauchy net, assume $\epsilon > 0$ and $d \in \mathcal{U}_2$ are given. Uniform continuity tells us that $d \circ \mathbf{f} \in \mathcal{U}_1$, and the fact that $\{x_i\}$ is Cauchy, paired with the $\epsilon > 0$

and $d \circ \mathbf{f} \in \mathcal{U}_1$, gives us $k \in \mathcal{D}$ so that $(d \circ \mathbf{f})(x_i, x_j) < \epsilon$ for all $i \geq k$ and $j \geq k$. But this says $d(f(x_i), f(x_j)) < \epsilon$ when $i, j \geq k$, which means $\{f(x_i)\}$ is Cauchy.

□

Theorem 18 *Assume that* $(\mathcal{X}_1, \mathcal{U}_1)$ *is a uniform space,* E *is a dense subset of* $\mathcal{X}_1$*, and* $f : E \to \mathcal{X}_2$ *is a uniformly continuous function into a complete uniform space* $(\mathcal{X}_2, \mathcal{U}_2)$*. Then* f *extends to a uniformly continuous function on* $\mathcal{X}_1$*.*

Proof. Let $x \in \mathcal{X}_1$ be given, and let $\mathcal{F}_x$ denote its neighborhood filter. The density of E implies that

$$\mathcal{F}^x = \{\, E \cap F \;:\; F \in \mathcal{F}_x \,\}$$

is a filter in E, which is easily seen to be Cauchy. Just as in Lemma 17 (Exercise 9), we have $f(\mathcal{F}^x)$ is Cauchy in $\mathcal{X}_2$, and hence converges. Denote the limit of this filter by $f(x)$. The continuity of f on E ensures that this is a continuous extension (Exercise 10).

It remains to prove that this extension is uniformly continuous, i.e. that $d \circ \mathbf{f}$ is in $\mathcal{U}_1$. We intend to prove that $d \circ \mathbf{f}$ lies below $\mathcal{U}_1$, so let an $\epsilon > 0$ be given. Recall that the subspace uniformity on E consists of all pseudometrics that lie below the restrictions of elements of $\mathcal{U}_1$ to E, so the hypothesis of uniform continuity implies $d \circ \mathbf{f}$ lies below that set of restrictions, i.e there exists $d' \in \mathcal{U}_1$ and $\gamma > 0$ such that

$$d'(u, v) < \gamma \text{ implies } d(f(u), f(v)) < \frac{\epsilon}{3},$$

for all $u, v \in E$. Now let $\delta = \frac{\gamma}{3}$, let $x, y \in \mathcal{X}_1$ be given, and prepare to prove that

$$d'(x, y) < \delta \text{ implies } d(f(x), f(y)) < \epsilon.$$

Assume $d'(x, y) < \delta$, and note that the filter $f(\mathcal{F}^x)$ must contain the ball $B_d(f(x), \frac{\epsilon}{3})$. It follows that there exists $\delta_x > 0$, with $\delta_x < \delta$, and

$$f(E \cap B'_d(x, \delta_x)) \subseteq B_d(f(x), \frac{\epsilon}{3}).$$

Select $u_x \in E \cap B'_d(x, \delta_x)$, and repeat the previous sentence with y in place of x, obtaining $u_y \in E \cap B'_d(y, \delta_y)$. It follows that

$$d'(u_x, u_y) \leq d'(u_x, x) + d'(x, y) + d'(y, u_y) < \gamma,$$

and

$$d(f(x), f(y)) \leq d(f(x), f(u_x)) + d(f(u_x), f(u_y)) + d(f(u_y, y) < \epsilon.$$

□

The product topology may be called the **topology of coordinatewise convergence**, since a filter $\mathcal{F}$ in the product space converges to x if and only if each coordinate filter $p_i(\mathcal{F})$ converges to the corresponding coordinate $p_i(x)$ of x, which is the essential content of Exercise 2.4.8. The corresponding statement in terms of nets is, a net $\{x_\alpha\}$ in a product space converges to x if and only if each coordinate net $\{p_i(x_\alpha)\}$ converges to $p_i(x)$ (Exercise 22). Coupled with the net characterization of continuity, that a function f is continuous if and only if $x_\alpha \to x$ in the domain of f implies $f(x_\alpha) \to f(x)$ (Exercise 4), the proof of the following theorem becomes very easy.

Theorem 19 *The arbitrary product of complete uniform spaces is complete.*

Proof. Assume $(\mathcal{X}_i, \mathcal{U}_i)$ is a family of complete uniform spaces, and $\{x_\alpha\}$ is a Cauchy net in $\Pi_{i \in \mathcal{I}} \mathcal{X}_i$. The product uniformity forces each coordinate projection to be uniformly continuous, so each coordinate net $\{p_i(x_\alpha)\}$ is Cauchy in $(\mathcal{X}_i, \mathcal{U}_i)$ (Lemma 17), and hence converges in that coordinate space by the completeness assumption. Since each coordinate net converges, we conclude the net $\{x_\alpha\}$ itself converges in $\Pi_{i \in \mathcal{I}} \mathcal{X}_i$.

□

By a **completion** of a uniform space $(\mathcal{X}, \mathcal{U})$, we mean a complete uniform space $(\mathcal{X}^c, \mathcal{U}^c)$ and a uniform embedding

$$\iota : \mathcal{X} \to \mathcal{X}^c$$

whose range is dense. Given two completions, $(\mathcal{X}_1^c, \mathcal{U}_1^c)$ and $(\mathcal{X}_2^c, \mathcal{U}_2^c)$, of $(\mathcal{X}, \mathcal{U})$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "\iota_1"] \arrow[dr, "\iota_2"'] & \mathcal{X}_1^c \arrow[d, "\Gamma"] \\
 & \mathcal{X}_2^c
\end{tikzcd}$$

where Γ is the uniformly continuous extension obtained from Theorem 18 and the mapping

$$\iota_2 \circ \iota_1^{-1} : \iota_1(\mathcal{X}) \to \mathcal{X}_2^c.$$

A moments thought convinces us that Γ^{-1} is obtained in the same way from $\iota_1 \circ \iota_2^{-1}$, so Γ is a uniform isomorphism. It follows that any two completions of $(\mathcal{X},\mathcal{U})$ are uniformly isomorphic, which is why folks refer to any such completion as *the* completion of $(\mathcal{X},\mathcal{U})$.

Theorem 20 *Every uniform space has a completion.*

Proof. Theorem 16 told us that every uniform space $(\mathcal{X},\mathcal{U})$ embeds uniformly into a product of normed spaces, each factor of the form $\mathcal{B}(\mathcal{X})$. We intend to prove that $\mathcal{B}(\mathcal{X})$ is complete. Theorem 19 will then imply that the product is complete, and since closed subspaces of complete spaces are complete (Exercise 1), we may obtain our completion by taking the closure of the image of $(\mathcal{X},\mathcal{U})$ under this embedding.

Assume that $\{f_i\}$ is a Cauchy sequence in $\mathcal{B}(\mathcal{X})$. It follows that $\{f_i\}$ is a bounded sequence, so that for some $M \in \mathbb{R}$, $||f_i|| \leq M$ for all $i \in \mathbb{N}$. The inequality

$$|f_i(x) - f_j(x)| \leq ||f_i - f_j||$$

implies that, for every $x \in \mathcal{X}$, the sequence $\{f_i(x)\}$ is Cauchy in $\mathbb{C}$, and hence converges by the completeness of $\mathbb{C}$. Denote its limit by $f(x)$. It follows that

$$|f_i(x)| \leq M$$

for all $x \in \mathcal{X}$ and $i \in \mathbb{N}$, so taking the limit as $i \to \infty$, we see that f is bounded.

It remains to prove that $||f_i - f|| \to 0$, so let an $\epsilon > 0$ be given. The Cauchy assumption gives us $k \in \mathbb{N}$ so that $||f_i - f_j|| < \frac{\epsilon}{2}$ whenever $i, j \geq k$. Now let $x \in \mathcal{X}$ be given and choose any $i \geq k$. It follows that

$$|f_i(x) - f(x)| \leq |f_i(x) - f_j(x)| + |f_j(x) - f(x)|,$$

and taking j large enough so that $j \geq k$ and $|f_j(x) - f(x)| \leq \frac{\epsilon}{2}$, we have

$$|f_i(x) - f(x)| \leq \epsilon,$$

independent of x. It follows that $||f_i - f|| \leq \epsilon$.

$\square$

Exercises

1. Prove that a closed subset of a complete uniform space is complete.
2. If $\mathcal{F}$ is a Cauchy filter and $\mathcal{F} \subseteq \mathcal{F}'$, prove that $\mathcal{F}'$ is a Cauchy.
3. Prove that in a Hausdorff space, limits of a net are unique.
4. Prove that a function f is continuous if and only if convergence of a net $x_\alpha \to x$ in the domain of f implies $f(x_\alpha) \to f(x)$.
5. An element x of a Tychonoff space is a **cluster point** of a net $\{x_i\}$ if the net is frequently in every neighborhood of x. Prove that a Cauchy net converges to any of its cluster points.
6. Give an example of a net with infinitely many cluster points.
7. Any filter $\mathcal{F}$ is a directed set, and if, for $F \in \mathcal{F}$ we choose $x_F \in F$, we obtain a **net based on the filter** (there are many of them). Prove that the filter converges to x if and only if the net $\{x_F\}_{F\in\mathcal{F}}$ converges to x.
8. If $\{x_F\}_{F\in\mathcal{F}}$ is a net based on the filter $\mathcal{F}$, prove that $\{x_F\}_{F\in\mathcal{F}}$ is a Cauchy net if and only if $\mathcal{F}$ is a Cauchy filter.
9. If f is uniformly continuous, and $\mathcal{F}$ is a Cauchy filter in the domain of f, prove that $f(\mathcal{F})$ is a Cauchy filter.
10. Prove that the extension obtained in the first paragraph of the proof of Theorem 18 is continuous (do so directly, not by deducing it from the uniform continuity obtained in the second paragraph of the proof).
11. In a compact uniform space, prove that a Cauchy ultrafilter converges.
12. If $\{x_i\}_{i\in\mathcal{I}}$ is a net in $\mathcal{X}$, then the collection of all sets F for which F contains a tail of $\{x_i\}_{i\in\mathcal{I}}$ forms a filter in $\mathcal{X}$ called the **filter of tails** of $\{x_i\}_{i\in\mathcal{I}}$. Prove that a net converges to x if and only if the corresponding filter of tails converges to x.
13. Prove that a net is Cauchy if and only if the corresponding filter of tails is Cauchy.
14. Assume that $\{x_i\}_{i\in\mathcal{I}}$ is a net in $\mathcal{X}$ for which the corresponding filter of tails $\mathcal{F}$ is an ultrafilter. Prove that, for every subset $E \subseteq \mathcal{X}$, $\{x_i\}_{i\in\mathcal{I}}$ is either eventually in E or it is eventually in the complement of E. A net with this property is called an **ultranet**.
15. Assume that $\mathcal{F}$ is is an ultrafilter and $\{x_F\}_{F\in\mathcal{F}}$ is a net based on $\mathcal{F}$. Prove that $\{x_F\}_{F\in\mathcal{F}}$ is an ultranet.
16. If $\mathcal{D}$ and $\mathcal{D}'$ are directed sets, we call a function $k : \mathcal{D}' \to \mathcal{D}$ and increasing cofinal function if $k_i < k_j$ whenever $i < j$, and for all $k_0 \in \mathcal{D}$ there exists $i \in \mathcal{D}'$ with $k_0 < k_i$. A **subnet** of a net $x : \mathcal{D} \to \mathcal{X}$ is a composition

$$\mathcal{D}' \xrightarrow{k} \mathcal{D} \xrightarrow{x} \mathcal{X},$$

with k and increasing cofinal function. Such a subnet will be denoted $\{x_{k_i}\}_{i\in\mathcal{D}'}$. Prove that any subnet of a convergent net converges.

17. Prove that any subnet of a Cauchy net is Cauchy.
18. Assume $\{x_i\}$ is a net with cluster point x. Prove that there exists a subnet of $\{x_i\}$ that converges to x.
19. If $\{x_i\}$ is a Cauchy net with a convergent subnet, prove that $\{x_i\}$ converges.
20. Prove that the filter $\mathcal{F}^x$ in the proof of Theorem 18 is Cauchy.
21. Prove that the extension in Theorem 18 is unique.
22. Prove that a net $\{x_\alpha\}_{\alpha\in\mathcal{D}}$ in a product space $\Pi_{i\in\mathcal{I}}\mathcal{X}_i$ converges to x if and only if $p_i(x_\alpha) \to p_i(x)$ for all $i \in \mathcal{I}$.

3.4 Totally Bounded Uniform Spaces

Recall that a pseudometric space $(\mathcal{X}, d)$ is totally bounded if each basic uniform cover $\mathcal{C}_\epsilon$ has a finite subcover. We say that a uniform space $(\mathcal{X},\mathcal{U})$ is **totally bounded** if, for every $d \in \mathcal{U}$, the pseudometric space $(\mathcal{X}, d)$ is totally bounded, i.e., if for each $d \in \mathcal{U}$ and every $\epsilon > 0$, there are finitely many elements $x_i \in \mathcal{X}$ $(i = 1, \ldots, n)$ such that the balls $B_d(x_i, \epsilon)$ cover $\mathcal{X}$.

Theorem 21 *A uniform space $(\mathcal{X},\mathcal{U})$ is totally bounded if and only if every filter in $\mathcal{X}$ is contained in a Cauchy filter.*

Proof. Assume that $(\mathcal{X},\mathcal{U})$ is totally bounded and let an arbitrary filter $\mathcal{F}$ in $\mathcal{X}$ be given. The ultrafilter property says that $\mathcal{F}$ is contained in some ultrafilter $\mathcal{F}'$, which we intend to prove is a Cauchy filter. Towards this end, let $\epsilon > 0$ and $d \in \mathcal{U}$ be given. The total boundedness says that $\mathcal{X}$ is covered by a finite collection $\{B_d(x_1, \epsilon), \ldots, B_d(x_n, \epsilon)\}$, and we assert that at least one of these balls must lie in $\mathcal{F}'$. If, by way of contradiction, this were not true, then, being an ultrafilter, all of the complements would lie in $\mathcal{F}'$, as would the intersection of these complements. But the intersection of the complements is empty (as the union of those balls is all of $\mathcal{X}$). Since $\mathcal{F}'$ can not contain the empty set, we have our contradiction.

Let $d \in \mathcal{U}$ and, proceeding contrapositively, assume $(\mathcal{X}, d)$ is not totally bounded. Thus, for some $\epsilon > 0$, the basic cover $\mathcal{C}^d_\epsilon$ has no finite subcover. In particular, with $x_1 \in \mathcal{X}$, $B_d(x_1, \epsilon)$ fails to cover $\mathcal{X}$, and there exists x_2 outside this ball. Similarly, $B_d(x_1, \epsilon) \cup B_d(x_2, \epsilon)$ is not all of $\mathcal{X}$, and there exists x_3 outside this union. Continuing inductively, we obtain a sequence $x_1, x_2, x_3, \ldots$ for which $d(x_i, x_j) \geq \epsilon$ for all $i \neq j$. Let $\mathcal{F}$ denote the filter of tails of this sequence. If $\mathcal{F}'$ is any filter containing $\mathcal{F}$, and $E \in \mathcal{F}'$, then E intersects every tail of $\{x_i\}$, so that this sequence is frequently in E. It follow that $\mathcal{F}'$ contains no d-ball of radius $\epsilon/2$, since such a ball can contain at most one element of the sequence. Thus, $\mathcal{F}'$ is not Cauchy.

□

Let $\mathcal{U}_{\mathcal{X}}$ denote the family of uniformities on a set $\mathcal{X}$. This family is naturally partitioned into blocks, in which, two uniformities lie in the same block exactly when they give rise to the same Tychonoff topology. Thus the blocks are in one–to–one correspondence with the various Tychonoff topologies on $\mathcal{X}$. Each such block contains a maximal element, namely, the uniformity consisting of all pseudometrics continuous relative to that corresponding Tychonoff topology. This is called the **fine uniformity** on that topology. A block contains a minimal uniformity if and only if the corresponding topology is **locally compact**. In this case, we will soon see that this minimal uniformity is generated by the family of pseudometrics

$$d_f(x, y) = |f(x) - f(y)|,$$

as f varies over all the continuous functions with **compact support**.

Given any uniformity, one can show that the set of all totally bounded pseudometrics contained therein is itself a uniformity (Exercise 2), so that every uniformity may be associated with a weaker uniformity which is totally bounded. (Among all totally bounded uniformities that are weaker than our given uniformity, the associated one is strongest). Imagine now a partitioning of each block, so that two uniformities find themselves in the same subblock exactly when they contain the same subset of totally bounded pseudometrics, i.e., their associated totally bounded uniformities coincide. Each subblock contains a minimal element, namely, the common associated totally bounded uniformity, and this is the unique totally bounded uniformity in that subblock. Thus there is a one–to–one correspondence between these subblocks and the set of all totally bounded uniformities on $\mathcal{X}$. Two uniformities lying in the same subblock are said to be **proximinal**, and the subblock is called a **proximity class**.

Theorem 22 *Two uniformities $\mathcal{U}_1$ and $\mathcal{U}_2$ are proximinal if and only if a bounded real (or complex) valued function f on $\mathcal{X}$ is uniformly continuous relative to $\mathcal{U}_1$ exactly when it is uniformly continuous relative to $\mathcal{U}_2$. In this case, the unique totally bounded uniformity in the proximity class is the one generated by the family*

$$d_f(x, y) = |f(x) - f(y)|,$$

as f varies over all such uniformly continuous functions.

Proof. Assume that $(\mathcal{X},\mathcal{U})$ is a uniform space, and let $\mathcal{U}'$ denote the subset of $\mathcal{U}$ consisting of those d for which $(\mathcal{X}, d)$ is totally bounded. For each $d \in \mathcal{U}'$ and $x \in \mathcal{X}$, the functions

$$f_x^d(y) = d(x, y)$$

are bounded and uniformly continuous (in fact, they are Lipschitz). We will prove that this family of pseudometrics

$$d_{f_x^d}(y, z) = |f_x^d(y) - f_x^d(z)| = |d(x, y) - d(x, z)|$$

generates $\mathcal{U}'$, so towards this end, let $d \in \mathcal{U}'$ and $\epsilon > 0$ be given. Being totally bounded, there exists $x_1, \ldots, x_n \in \mathcal{X}$ for which the balls $B_\rho(x_i, \epsilon/3)$ cover $\mathcal{X}$. For each $i \in \{1, \ldots, n\}$, let us avoid excessive subscripts by writing $f_i(y) = d(y, x_i)$ and

$$d_i(x, y) = |f_i(x) - f_i(y)| = |d(x, x_i) - d(y, x_i)|.$$

Let $\delta = \epsilon/3$, and let $x, y \in \mathcal{X}$ be given so that $\vee_{i=1}^n d_i(x, y) < \delta$, i.e.

$$|d(x, x_i) - d(y, x_i)| < \epsilon/3$$

for each i. Choose k so that $x \in B_d(x_k, \epsilon/3)$, which, combined with the inequality above, gives

$$d(y, x_k) < \epsilon/3 + d(x, x_k) < 2\epsilon/3,$$

and

$$d(x, y) \leq d(x, x_k) + d(y, x_k) < \epsilon.$$

It follows that d lies below the family $\{d_{f_x^d}\}$, so it belongs to the uniformity generated by this family, i.e. $\mathcal{U}'$ is contained in the uniformity generated by this family. As each pseudometric $d_{f_x^d}$ is a totally bounded element of $\mathcal{U}$, the opposite inclusion holds, so $\mathcal{U}'$ is the uniformity generated by the family $\{d_{f_x^d}\}$. This proves the last assertion of the theorem, and the equivalence in the first assertion is a consequence of this.

$\square$

Exercises

1. Prove that a uniform space is totally bounded if and only if every net has a Cauchy subnet.
2. Let $(\mathcal{X},\mathcal{U})$ be a uniform space, and let $\mathcal{U}'$ denote the set of totally bounded uniformities contained in $\mathcal{U}$. Prove that $\mathcal{U}'$ is a uniformity on $\mathcal{X}$.
3. Prove that a subspace of a totally bounded uniform space is totally bounded.
4. Assume $(\mathcal{X},\mathcal{U})$ is a uniform space and $\mathcal{Y}$ is a dense subspace of $\mathcal{X}$ that is totally bounded relative to its subspace uniformity. Prove that $(\mathcal{X},\mathcal{U})$ is totally bounded.
5. A **proximity** on a set $\mathcal{X}$ is a relation σ on the power set of $\mathcal{X}$ that satisfies, for all subsets A, B, and C, of $\mathcal{X}$,
   - If $A \sigma B$ then both A and B are non-empty.
   - If $A \cap B$ is non-empty, then $A \sigma B$.
   - If $A \sigma B$, then $B \sigma A$.
   - $A \sigma (B \cup C)$ if and only if $A \sigma B$ or $A \sigma C$.
   - If $A \not\sigma B$, then $A \not\sigma C$ and $B \not\sigma (\mathcal{X} \setminus C)$ for some C.
   - If $\{x\} \sigma \{y\}$, then $x = y$.

   When reading "$A \sigma B$", we say "A is close to B".

   If $(\mathcal{X},\mathcal{U})$ is a uniform space, define $A \sigma B$ to mean that A and B can not be separated with a uniformly continuous real function. Equivalently, we mean $A \not\sigma B$ when there exists a uniformly continuous real function that is 0 on A and 1 on B. Prove that this defines a proximity on $\mathcal{X}$. (It turns out that *every* proximity arises this way. Assume this fact in the following exercises on proximity.)
6. If $(\mathcal{X},\mathcal{U}_1)$ is proximinal to $(\mathcal{X},\mathcal{U}_2)$, show that the proximities (as defined in the previous exercise) on $\mathcal{X}$ are the same.
7. Show that the proximities on $\mathcal{X}$ are in one–to–one correspondence with the proximity classes.

3.5 Compactification

We have just seen that the proximity classes over a Tychonoff space are in one–to–one correspondence with the totally bounded uniformities on that Tychonoff space. We will soon see that these classes are also in bijective correspondence with the **compactifications** of the Tychonoff space. A compactification of a Tychonoff space $\mathcal{X}$ consists of a compact Hausdorff space $\alpha\mathcal{X}$ and a topological embedding

$$\mathcal{X} \stackrel{\iota_\alpha}{\to} \alpha\mathcal{X}$$

whose range is dense in $\alpha\mathcal{X}$. Two compactifications, that are homeomorphic via a homeomorphism that acts as the identity map on the embedded images of $\mathcal{X}$, are identified, so it is, in fact, an equivalence class that corresponds to a totally bounded uniformity on $\mathcal{X}$. Two compactifications are called **equivalent compactifications** when they are identified by such a homeomorphism.

The pertinent facts, which are forthcoming, are that any compact Hausdorff space carries a unique uniformity, which happens to be totally bounded. A compactification of $\mathcal{X}$ then gives rise to a totally bounded uniformity on $\mathcal{X}$ by restriction. In this way, we obtain a mapping from the compactifications of $\mathcal{X}$ into the totally bounded uniformities. We construct an inverse for this mapping, which simply takes a totally bounded uniform structure on $\mathcal{X}$ to its completion, and these two mappings constitute the bijective correspondence.

Theorem 23 *A uniform space is compact if and only if it is complete and totally bounded.*

Proof. If $(\mathcal{X},\mathcal{U})$ is compact, then it is clearly totally bounded. If $\mathcal{F}$ is Cauchy, then by Exercise 2.3.11 it converges, so our uniform space is complete.

Conversely, assume that $(\mathcal{X},\mathcal{U})$ is complete and totally bounded, and assume $\mathcal{F}$ is an ultrafilter in $\mathcal{X}$. Since $(\mathcal{X},\mathcal{U})$ is totally bounded $\mathcal{F}$ is contained in a Cauchy filter (Theorem 21), and being an ultrafilter, we conclude $\mathcal{F}$ is itself Cauchy. By completeness, $\mathcal{F}$ converges, and $(\mathcal{X},\mathcal{U})$ is compact.

□

Theorem 24 *A compact Tychonoff space $\mathcal{X}$ carries a unique uniformity.*

Proof. Assume that $\mathcal{X}$ is compact and $\mathcal{U}$ is any uniformity on $\mathcal{X}$. We will show that $\mathcal{U}$ contains all continuous semi-metrics, and hence there is a unique uniformity on $\mathcal{X}$, the fine uniformity. Let ρ be any continuous semi-metric. Let $\epsilon > 0$ be given. Then $B_\rho(x, \frac{\epsilon}{3})$ covers $\mathcal{X}$ so there is a finite subcover

$$\left\{ B_\rho(x_i, \tfrac{\epsilon}{3}) \ : 1 \le i \le n \right\}.$$

For each $i \in \{1, \ldots, n\}$ let K_i be the closure of $B_\rho(x_i, \frac{\epsilon}{3})$ and let G_i equal $B_\rho(x_i, \frac{\epsilon}{2})$, so K_i is compact and $K_i \subseteq G_i$. For each $x \in G_i$ we have $\rho_x \in \mathcal{G}$ and $\delta_x > 0$ such that

$$B_{\rho_x}(x, \delta_x) \subseteq G_i,$$

and the sets $B_{\rho_x}(x, \frac{\delta_x}{2})$ form a cover of K_i, from which we extract a finite subcover with corresponding center points $x_{i1}, x_{i2}, \ldots, x_{in_i}$ and radii $\frac{\delta_{i1}}{2}, \frac{\delta_{i2}}{2}, \ldots, \frac{\delta_{in_i}}{2}$. Put $\delta = min\{\delta_{ij}/2 \ : 1 \leq i \leq n, 1 \leq j \leq n_i\ \}$ and assume that

$$\vee \rho_{x_{ij}}(y, z) < \delta.$$

Find i with $y \in B_\rho(x_i, \frac{\epsilon}{3})$, then find $j \in \{1, \ldots, n_i\}$ with $y \in B_{\rho_{x_{ij}}}(x_{ij}, \frac{\delta_{ij}}{2})$. It follows that

$$\rho_{x_{ij}}(x_{ij}, z) \leq \rho_{x_{ij}}(x_{ij}, y) + \rho_{x_{ij}}(y, z) < \delta_{ij},$$

so that both y and z are in

$$B_{\rho_{x_{ij}}}(x_{ij}, \delta_{ij}) \subseteq B_\rho(x_i, \frac{\epsilon}{2}),$$

so $\rho(y, z) < \epsilon$.

□

When proving, in a Calculus course, that a continuous function on a closed interval $[a, b]$ is Riemann integrable, we usually employ a result that says a continuous function whose domain is $[a, b]$ must be uniformly continuous. This follows from the fact that the compact space $[a, b]$ carries a unique uniformity.

Corollary 25 *If f is a continuous mapping of a compact Hausdorff space into a uniform space, then f is uniformly continuous.*

Proof. For any pseudometic d in the uniformity of the codomain, the continuity of f implies $d \circ \mathbf{f}$ is a continuous pseudometric on the domain, and hence an element of the unique uniformity on the domain (which consists of all continuous pseudometrics on the domain).

□

It is worth noticing that Theorem 23 gives a second proof of Tychonoff's theorem.

Corollary 26 *A product of compact Tychonoff spaces is compact.*

Proof. Each factor space carries a unique uniformity which is complete and totally bounded. Since products of complete spaces are complete, and products of totally bounded spaces are totally bounded, we conclude that the product space is complete and totally bounded, i.e. compact.

□

Theorem 27 *There is a bijective correspondence between the (equivalence classes of) compactifications of a Tychonoff space $\mathcal{X}$ and the totally bounded uniform structures on $\mathcal{X}$.*

Proof. Let Γ denote the mapping that takes a compactification of $\mathcal{X}$ to the uniform structure on $\mathcal{X}$ that it inherits as a subspace of the given compact space. Thus, if $\alpha\mathcal{X}$ is a compactification, with topological embedding ι_α, then $\Gamma(\alpha\mathcal{X})$ is the uniformity on $\mathcal{X}$ that makes ι_α a uniform embedding. It follows from Exercise 3.4.3 that this is a totally bounded uniformity. Let Δ be the mapping that takes a totally bounded uniform space $(\mathcal{X},\mathcal{U})$ to its completion. Since the embedding of a uniform space into its completion is a uniform embedding, it follows from Exercise 3.4.4 that $\Delta(\mathcal{X},\mathcal{U})$ is both complete and totally bounded, hence compact. Thus $\Delta(\mathcal{X},\mathcal{U})$ is a compactification.

If we start with a totally bounded uniform space $(\mathcal{X},d)$ and form its completion $\Delta(\mathcal{X},\mathcal{U})$, then $(\Gamma\circ\Delta)(\mathcal{X},\mathcal{U})$ is defined to be the uniformity on $\mathcal{X}$ making the embedding a uniform embedding. Since the embedding of a space into its completion is a uniform embedding, we conclude that $(\mathcal{X},\mathcal{U}) = (\Gamma\circ\Delta)(\mathcal{X},\mathcal{U})$, so that $\Gamma\circ\Delta$ is the identity. If we start with a compactification $\alpha\mathcal{X}$ of $\mathcal{X}$, then $\alpha\mathcal{X}$ is a completion of any of its dense subspaces, and in particular it is a completion of $\Gamma(\alpha\mathcal{X})$. Since any two completions are uniformly isomorphic (page 49), we conclude that $(\Delta\circ\Gamma)(\alpha\mathcal{X})$ is homeomorphic to $\alpha\mathcal{X}$, and $\Delta\circ\Gamma$ is the identity map on the corresponding equivalence classes of compactifications.

□

There is the natural order of set inclusion defined on the collection of totally bounded uniformities, which, when transported to the compactifications using the bijection above, gives a corresponding relation on the compactifications. If $\alpha_1\mathcal{X}$ and $\alpha_2\mathcal{X}$ are two compactifications of $\mathcal{X}$, then $\alpha_1\mathcal{X} \leq \alpha_2\mathcal{X}$ means that there exists a continuous surjection $\tau : \alpha_2\mathcal{X} \to \alpha_1\mathcal{X}$

that makes the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\alpha_2"] \arrow[dr, "\alpha_1"'] & \alpha_2\mathcal{X} \arrow[d, "\tau"] \\ & \alpha_1\mathcal{X} \end{tikzcd}$$

commute. With this order structure, the mappings Γ and Δ above become order isomorphisms.

Examples

1. The unit circle in the complex plane is a compactification of $\mathbb{R}$. Indeed, the interval $(0, 2\pi)$ is homeomorphic to $\mathbb{R}$, and the mapping $f(t) = e^{it}$ is a dense embedding of this open interval into the unit circle. This is called the **one point compactification** of $\mathbb{R}$.

2. With $\alpha_1(t) = e^{it}$, the one point compactification of $\mathcal{X} = (0, 2\pi)$ is the unit circle, and a second compactification (which we might call the *two point compactification*) is the closed interval $[0, 2\pi]$, with $\alpha_2(t) = t$ (the inclusion mapping). The continuous surjection $\tau(t) = e^{it}$ of the two point compactification onto the one point compactification makes a commutative diagram $\tau \circ \alpha_2 = \alpha_1$, so as one would expect, $\alpha_1\mathcal{X} \leq \alpha_2\mathcal{X}$.

3. Let $\mathcal{X}$ be any Tychonoff space, and let $\mathcal{U}$ consist of all continuous pseudometrics d such that $(\mathcal{X}, d)$ is totally bounded. The compactification corresponding to $(\mathcal{X},\mathcal{U})$ is called the **Stone-Cech** compactification of $\mathcal{X}$, and it is traditionally denoted $\beta\mathcal{X}$. It is characterized by the property that every bounded continuous function

$$f : \mathcal{X} \to \mathbb{R}$$

   has a continuous extension to $\beta\mathcal{X}$. This follows from the fact that such a function is uniformly continuous if and only if d_f is in the uniformity, together with the fact that uniformly continuous functions extend to their closures.

Exercises

1. Prove that

3.1.15

Chapter 4

Function Algebras

4.1 Entire functions

Let $\mathcal{P}(\mathbb{C})$ denote the set of all complex polynomials,

$$\mathcal{P}(\mathbb{C}) \equiv \{\, a_0 + \ldots + a_n x^n \;\; : a_i \in \mathbb{C}, (0 \leq i \leq n), \text{ and } n \in \mathbb{N} \,\}$$

and, given a compact subset $K \subset \mathbb{C}$, define a corresponding seminorm $||\cdot||_K$ on $\mathcal{P}(\mathbb{C})$ by

$$||p||_K = \sup_{z \in K} |p(z)|.$$

The resulting family of seminorms $\mathcal{G}$ turns $\mathcal{P}(\mathbb{C})$ into a locally convex space. Let us look at the completion $\mathcal{E}$ of this space, the **algebra of entire functions**.

If $K_n \subset \mathbb{C}$ denotes the closed disc centered at the origin of radius n, then the countable family $\{K_n\}$ corresponds to a countable family of seminorms, which are easily seen to constitute a base for the uniformity. It follows that the uniformity of $(\mathcal{P}(\mathbb{C}), \mathcal{G})$ is singly generated (Exercise 3.1.15), so that sequences suffice when discussing convergence. A specific element of $\mathcal{E}$ may then be indicated by specifying a Cauchy sequence of complex polynomials $\{p_k\}$, and if z is a complex number, then $\{p_k(z)\}$ must be a Cauchy sequence of complex numbers, which converges to a number $f(z)$ by completeness of $\mathbb{C}$. In this way, every element of $\mathcal{E}$ is naturally associated with a function $f : \mathbb{C} \to \mathbb{C}$. A moments thought should convince us that each such function f is continuous, so we may think of $\mathcal{E}$ a set lying between $\mathcal{P}(\mathbb{C})$ and the algebra of all continuous complex functions. The functions in $\mathcal{E}$ are called **entire functions**.

Exercises

1. If $\{p_k\}$ is a Cauchy sequence in $\mathcal{P}(\mathbb{C})$, and $z \in \mathbb{C}$, prove that $\{p_k(z)\}$ is a Cauchy sequence of complex numbers.
2. If $\{p_k\}$ and $\{q_k\}$ are two equivalent Cauchy sequences in $\mathcal{P}(\mathbb{C})$ (so that they converge to the same element in the completion), prove that
$$\lim_{k\to\infty} p_k(z) = \lim_{k\to\infty} q_k(z)$$
for any $z \in \mathbb{C}$. What does this tell us about the function f we attached to the Cauchy sequence $\{p_k\}$?
3. Exhibit a metric on $\mathcal{P}(\mathbb{C})$ that gives the same uniformity as the family of seminorms $\mathcal{G}$.
4. Show that there is no norm on $\mathcal{P}(\mathbb{C})$ that gives the same uniformity as $\mathcal{G}$.
5. Give a direct proof that each function in $\mathcal{E}$ is continuous.

4.2 Continuous Functions

Assume that $\mathcal{X}$ is a compact Tychonoff space, and let $C(\mathcal{X})$ denote the set of all continuous complex valued functions on $\mathcal{X}$. The norm
$$||f|| = \sup_{x\in\mathcal{X}} |f(x)|$$
turns $C(\mathcal{X})$ into a complete normed space, i.e. a **Banach space**, and with pointwise addition and multiplication, $C(\mathcal{X})$ is an algebra over $\mathbb{C}$. The inequality
$$||fg|| \leq ||f||\;||g||$$
holds for all $f, g \in C(\mathcal{X})$, which makes $C(X)$ a **Banach algebra**. Finally, there is the involution
$$f^*(x) = \overline{f(x)}$$
on $C(X)$, which is a pointwise complex conjugation, and this involution satisfies
$$||f^*f|| = ||f||^2,$$
which, in addition to all the properties already stated, makes $C(X)$ a **C*-algebra**.

As it happens, all topological information carried by a compact space is recoverable from the C*-algebra $C(\mathcal{X})$, which allows us to investigate topological properties using C*-algebraic tools, and it allows us to **quantize** topology by seeking a theory that describes all C*-algebras, among whom the spaces $C(\mathcal{X})$ are the commutative examples.

The topological properties of $\mathcal{X}$ are encoded in what we will call the **character space** of $C(\mathcal{X})$. The **characters** of $C(\mathcal{X})$ are defined to be the nonzero **multiplicative linear functionals** defined on $C(\mathcal{X})$, i.e. the surjective linear mappings $\phi : C(\mathcal{X}) \to \mathbb{C}$ that satisfy $\phi(fg) = \phi(f)\phi(g)$ for all $f, g \in C(\mathcal{X})$, and the character space is the set of all characters. Among the characters are the **point evaluations**, denoted $\hat{t}$, which are the functionals that take an element f to $f(t)$, i.e. $\hat{t}(f) = f(t)$, and there is one such point evaluation $\hat{t}$ for each element $t \in \mathcal{X}$. The fact that addition and multiplication of functions in $C(\mathcal{X})$ is defined pointwise translates into the statement that each point evaluation is a multiplicative functional. It is sometimes useful to employ the notation of **dual pairing**

$$[f, t]$$

to denote the number $f(t) = \hat{t}(f)$. As such, we may think of f as being a function of t, we may think of t as being a function of f, and we might as well think of f as being a function of $\hat{t}$, since in the dual pairing viewpoint there is no difference between t and $\hat{t}$. The set of characters is topologized by giving it the weak topology induced by the family of functions $f \in C(\mathcal{X})$.

Theorem 28 *Every character on $C(\mathcal{X})$ is given by a point evaluation, and the mapping of $\mathcal{X}$ onto the character space is a homeomorphism.*

Proof. Assume that ϕ is a character on $C(\mathcal{X})$, let

$$\mathcal{M} = \ker \phi = \{ f \ : \phi(f) = 0 \},$$

and let $Z_f = \{ x \ : f(x) = 0 \}$ denote the **zero set** of f. The set $\mathcal{M}$ is an ideal in $C(\mathcal{X})$, so it can not contain invertible elements, which implies that for $f \in \mathcal{M}$ we must have $Z_f \neq \emptyset$. Again, because $\mathcal{M}$ is an ideal, when $f_1, f_2 \in \mathcal{M}$ we have $f_1\bar{f}_1 + f_2\bar{f}_2 \in \mathcal{M}$, and

$$Z_{f_1\bar{f}_1 + f_2\bar{f}_2} = Z_{f_1} \cap Z_{f_2},$$

which shows that the family of all Z_f with $f \in \mathcal{M}$ is a family of closed sets with the finite intersection property. Being in a compact space, we conclude that

$$\cap_{f \in \mathcal{M}} Z_f \neq \emptyset.$$

Choose t in this intersection, so $\phi(f) = 0$ implies $\hat{t}(f) = 0$. The multiplicativity of ϕ, and the fact that $\phi \neq 0$, implies $\phi(I) = 1$, where I denotes the multiplicative identity of $C(\mathcal{X})$. Thus for any $f \in C(\mathcal{X})$, we have $\phi(f - \phi(f)I) = 0$, so $\hat{t}(f - \phi(f)I) = 0$, and $\hat{t}(f) = \phi(f)$.

$\square$

Chapter 5

C^*-Algebra

Chapter 6

Scratch Work 1

Linear algebra over complex vector spaces is especially beautiful, as a consequence of the **fundamental theorem of algebra**. This theorem says that every non-constant polynomial with complex coefficients has a complex root. More precisely, if

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

is a polynomial, with $n \geq 0$ and $a_i \in \mathbb{C}$ $(1 \leq i \leq n)$, then there exists $w \in \mathbb{C}$ such that

$$p(w) = a_0 + a_1 w + \ldots + a_n w^n = 0.$$

The number w is called a **root** of the polynomial p. By division, we obtain $p(z) = (z - w)q(z)$, repeat the procedure on q, and continue inductively, arriving at

$$p(z) = a(z - w_1) \cdots (z - w_k),$$

with $w_1 \ldots w_k$ denoting all the roots of p, (roots repeated according to multiplicity). One says that the polynomial p has **split**.

In terms of a linear transformation T on a finite dimensional complex vector space, it implies a factorization

$$p(T) = a(T - \alpha_1) \cdots (T - \alpha_k).$$

In finite dimensions, it is possible to find an **annihilating polynomial** p for T, i.e. a polynomial such that $p(T) = 0$, and then the splitting of that polynomial implies the existence of an eigenvector for T. We intend to broadly generalize this important phenomenon, but before we do so, we record a proof of the fundamental theorem.

Our crucial ingredient of the fundamental theorem's proof is **maximum modulus principle** for polynomials. It says that, for non–constant polynomials p, local maxima do not exist in the complex plane, and that local minima can occur only at roots of p.

Theorem 29 *Assume that p is a non–constant complex polynomial, and assume $z_0 \in \mathbb{C}$ and $\epsilon > 0$. Then there exists $w \in B(z_0, \epsilon)$ such that*

$$|p(z_0)| < |p(w)|.$$

Moreover, if $|p(z_0)| \neq 0$, then there exists a second $w \in B(z_0, \epsilon)$ with

$$|p(w)| < |p(z_0)|.$$

Before moving to the proof of this theorem, let us derive the **fundamental theorem of algebra** from it.

Theorem 30 *Every non–constant complex polynomial p has a root.*

Proof. Assume that

$$p(z) = a_0 + a_1 z + \ldots + a_n z^n$$

is a polynomial with $a_i \in \mathbb{C}$ and $a_n \neq 0$. Since

$$\lim_{z\to\infty} |p(z)| = \lim_{z\to\infty} |z^n||a_n|||f(z)|$$

with $\lim_{z\to\infty} |f(z)| = 1$, it follows that

$$\lim_{z\to\infty} |p(z)| = \infty,$$

so eventually $|p(z)| \geq 1$, say if z is outside of $B(0, R)$. It follows that the minimum of $|p(z)|$ on the compact closure of $B(0, R)$ exists (Exercise 2.3.3), and must lie within $B(0, R)$. If that minimum occurs at $z_0 \in B(0, R)$, then necessarily $p(z_0) = 0$, because otherwise we could find $\epsilon > 0$ with

$$B(z_0, \epsilon) \subseteq B(0, R),$$

and Theorem 29 then tells us z_0 can not be a minimum on $B(z_0, \epsilon)$, contradicting that it was chosen as a minimum on $B(0, R)$.

□

The proof of the maximum modulus principle is simplified by a few reductions to special cases, which lose no generality. The first reduction is to realize that if the theorem holds for all polynomials in the case when $z = 0$, then it holds in full generality. To see this, write p as a polynomial centered at z_0, so that

$$p(z_0) = a_0 + a_1(z - z_0) + \ldots + a_n(z - z_0)^n.$$

If $q(z) = a_0 + a_1 z + \ldots + a_n z^n$, then the conclusion of the theorem holding for q on $B(0, \epsilon)$ implies that the conclusion holds for p on $B(z_0, \epsilon)$. We therefore assume, with no loss of generality, that $z_0 = 0$, and we are left to prove the existence of $w \in B(0, \epsilon)$ with

$$|p(0)| < |p(w)|,$$

and if $p(0) \neq 0$, we prove the existence of the second $w \in B(0, \epsilon)$ with

$$|p(w)| < |p(0)|.$$

The second reduction is that we might as well assume $p(0) = 1$. This is because the conclusion is trivial when $p(0) = 0$, since the non-constant p can have at most finitely many roots, and $B(0, \epsilon)$ being an infinite set, ensures some element w in $B(0, \epsilon)$ has $p(w) \neq 0$. When $a = p(0) \neq 0$, the conclusion will be true if the conclusion holds for $q(z) = \frac{p(z)}{a}$, and q is a polynomial with $q(0) = 1$.

Assume $p(z) = 1 + a_1 z + \ldots + a_n z^n$, and write this as $p(z) = 1 + q(z)$. We will prove the existence of $w \in B(0, \epsilon)$ with $|p(w)| < 1$, and leave the easier second statement as an exercise. In order to get $|1 + q(w)| < 1$, we must find w that puts $q(w)$ into the unit ball centered at -1, i.e. into $B(-1, 1)$. We do this by proving that every ball centered at the origin contains an element z such that $q(z)$ lies in the left cone defined as the intersection of the left half plane with those points between the lines $y = x$ and $y = -x$. Since $q(z) \to 0$ as $z \to 0$, we are able to find a w with $q(w)$ in this region and with length small enough to land in $B(-1, 1)$ (length less than $\sqrt{2}$ will do: see Figure 6.1). Assume, by way of contradiction, that no element of $B(0, \delta)$ maps into the left cone. The polynomial $q = a_1 z + \ldots + a_n z^n$ might have $a_1 = 0$, but there is a coefficient $a_j \neq 0$ with smallest index j, and writing $a = a_j$ for this coefficient we have

$$\lim_{z \to 0} \frac{q(z)}{z^j} = a \neq 0,$$

or equivalently, $\frac{q(z)}{a\,z^j} \to 1$ as $z \to 0$. Choose a sequence $\{z_n\}$ in $B(0, \delta)$ converging to 0 such that az_n^j is a negative real number for all n. Then, $q(z)$ never in the left cone implies that $\frac{q(z_n)}{a\,z_n^j}$ is never in the right cone, defined as the region in the right half plane between the lines $y = x$ and $y = -x$ (between the dotted lines in Figure 6.1). This contradicts $\frac{q(z_n)}{a\,z_n^j} \to 1$ as $n \to \infty$.

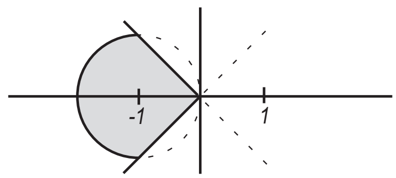

Figure 6.1: Finding w with $q(w)$ in the shaded region.

Exercises

1. Assume T is a linear transformation of a finite dimensional vector space $\mathcal{V}$, and assume $\mathbf{v} \in \mathcal{V}$. Prove that there exists a polynomial p such that $p(T)\mathbf{v} = \mathbf{0}$.

2. Assume T as in the previous exercise, and assume that

$$\mathbf{e}_1, \ldots, \mathbf{e}_n$$

is a basis of $\mathcal{V}$. Prove that there is a polynomial p so that $p(\mathbf{e}_i) = \mathbf{0}$ for all $1 \le i \le n$, and conclude that p **annihilates** T, i.e. that $p(T) = 0$.

3. If q is a polynomial with $q(0) = 0$ and $\epsilon > 0$, prove there exists $w \in B(0, \epsilon)$ with $q(w)$ in the right half plane. Conclude that there exists w in $B(0, \epsilon)$ with $|p(0)| < |p(w)|$, when $p(z) = 1 + q(z)$.

Chapter 7

Scratch Work 2

Suppose $A \subseteq B$ are unital commutative C*-algebras. Assume τ is a pure state of A, and a_o is an invertible elements of B not in A. Then τ extends to a pure state on the C*-algebra generated by A and a_o.

Proof. Generate the multiplicative semigroup $\{ a_o^n b \ : b \in A, n \in \mathbb{Z} \}$ and extend τ by declaring $\tau(a_o) = 1$. Check that it is well defined, then extend to the linear span of that semigroup. This is a dense subalgebra of the C*-algebra generated by A and a_o.

□

Theorem 31 *Pure states extend in unital commutative C*-algebras.*

Proof. Use Zorn's lemma and the preceding.

□

Corollary 32 *If $A \subseteq B$ are unital commutative C*-algebras, the restriction map, that takes a pure state of B to its restriction on A, is a surjection.*

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