

Matrix Theory in a 2-Qubit Entangler

Huan Q. Bui

Matrix Analysis

Professor Leo Livshits

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Presentation layout

- 1 Qubits & Quantum Gates
- 2 Some Matrix Theory
- 3 Simulation on IBM-Q
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Qubits & Quantum Gates

Qubit: A quantum system with two measurable physical states $|0\rangle$ and $|1\rangle$,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Before measurement,

$$|\psi\rangle = a|0\rangle + b|1\rangle \in \mathbb{C}^2, \quad |a|^2 + |b|^2 = 1.$$

Physically,

$$P(|\psi\rangle \rightarrow |0\rangle) = |a|^2 \quad P(|\psi\rangle \rightarrow |1\rangle) = |b|^2.$$

Quantum gate: a unitary transformation on $|\psi\rangle$.

Qubits & Quantum Gates

Multiple Qubits: States of k qubits is a vector in $\otimes^k \mathbb{C}^2$ with basis vectors

$$|x_1 \dots x_k\rangle = |x_1\rangle \otimes \dots \otimes |x_k\rangle, \quad x_i \in \{0, 1\}.$$

“ \otimes ”: Kronecker product. If $\mathcal{A} \in \mathbb{M}_{m \times n}$ and $\mathcal{B} \in \mathbb{M}_{p \times q}$, then

$$\mathcal{A} \otimes \mathcal{B} = \begin{bmatrix} a_{11}\mathcal{B} & \dots & a_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathcal{B} & \dots & a_{mn}\mathcal{B} \end{bmatrix}.$$

Kronecker Products

Example: Representing the classical numbers “1” and “0” with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$0_2 \equiv |00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = [0 \ 0 \ 1 \ 0]^T, |11\rangle = [0 \ 0 \ 0 \ 1]^T.$$

In fact, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis for $\otimes^2 \mathbb{C}^2$, the 2-qubit system.

Kronecker Products

Doesn't care where scalar goes...

$$(\alpha \mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\alpha \mathcal{B}) = \alpha(\mathcal{A} \otimes \mathcal{B})$$

Associative:

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$$

Left-distributive:

$$\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = \mathcal{A} \otimes \mathcal{B} + \mathcal{A} \otimes \mathcal{C}$$

Right-distributive:

$$(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes \mathcal{C} + \mathcal{B} \otimes \mathcal{C}$$

Quantum Gates

Quantum Gates: Represented by unitary matrices \rightarrow Reversible. Act on spaces of one or many qubits. Example:

$$\text{Hadamard} : H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad CNOT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Measurements: Irreversible \rightarrow Not quantum gates.

Multi-qubit systems

- Representing a multi-qubit state as many single-qubit states?
- Representing a multi-qubit gate as many single-qubit gates?

What do we need to entangle two qubits?

- Tensor products
- Hadamard gate
- CNOT gate
- Measure

Tensor Products

The *tensor product* of $\mathbf{V} = \mathbb{C}^{\Sigma_1}$ and $\mathbf{W} = \mathbb{C}^{\Sigma_2}$ is

$$\mathbf{V} \otimes \mathbf{W} = \mathbb{C}^{\Sigma_1 \times \Sigma_2}.$$

Elementary tensors span $\mathbf{V} \otimes \mathbf{W}$. For $|v\rangle \in \mathbf{V}$ and $|w\rangle \in \mathbf{W}$,

$$|v\rangle \otimes |w\rangle \equiv |v\rangle |w\rangle \equiv |vw\rangle \in \mathbf{V} \otimes \mathbf{W}.$$

Example: Representing the classical number “1” with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Tensor Products (cont.)

$\text{span}(|00\rangle, |01\rangle, |10\rangle, |11\rangle) = \mathbf{V} \otimes \mathbf{W}$, where

$$|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T, |10\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^T, |11\rangle = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T.$$

Linear independence $\rightarrow (|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ form a computational basis.

A *generic state*: For $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$,

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.$$

Tensor Products (cont.)

Not every $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|c\rangle, |d\rangle \in \mathbb{C}^2$ such that

$$\begin{aligned} |c\rangle \otimes |d\rangle = |\beta_{00}\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \\ &= \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \end{aligned}$$

Examples: Bell states

$$\begin{aligned} |\beta_{10}\rangle &= \frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle \\ |\beta_{01}\rangle &= \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle \end{aligned}$$

Tensor Products (cont.)

For operators: $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W}), \mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}|v\rangle) \otimes (\mathcal{B}|w\rangle).$$

Not all $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ can be written as $\mathcal{A} \otimes \mathcal{B}, \mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$.

Example:

$$CNOT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$SWAP \neq S_1 \otimes S_2$$

Consider the 2-qubit *SWAP* map:

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W}).$$

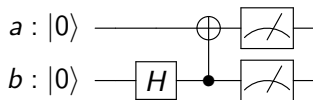
Observe:

$$SWAP(|0\rangle \otimes |1\rangle) = |1\rangle \otimes |0\rangle.$$

Suppose for $S_1 \in \mathcal{L}(\mathbf{V})$, $S_2 \in \mathcal{L}(\mathbf{W})$

$$SWAP = S_1 \otimes S_2$$

Example: 2-Qubit Entanglement Circuit



$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{cases} |00\rangle \rightarrow |00\rangle \\ |10\rangle \rightarrow |10\rangle \\ |01\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |01\rangle \end{cases}$$

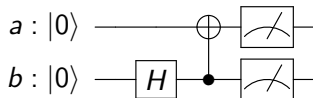
Example: Entanglement (cont.)

Notice:

$$\begin{aligned}(I|0\rangle) \otimes (H_b|0\rangle) &= (I \otimes H_b)(|0\rangle \otimes |0\rangle) \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \mathcal{O} \\ \mathcal{O} & \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \left[\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \right]^T &= \left[\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \right]^T\end{aligned}$$

→ Possible to write H as $I \otimes H_b$. Not possible for $CNOT_b$.

Example: Entanglement (cont.)



$$\begin{aligned} C_b(I \otimes H) \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}_a \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b \right) &= C_b \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_a \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b \right) \\ &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |1\rangle \\ &\rightarrow \textbf{Entangled} \end{aligned}$$

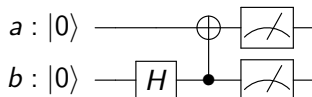
Tensor Products (cont.)

Other properties:

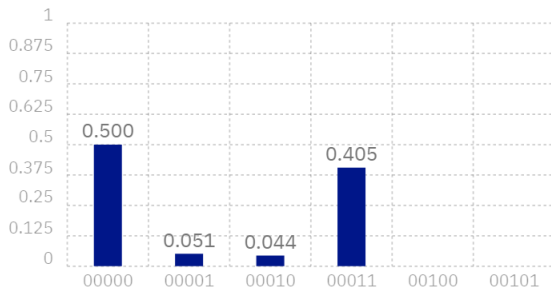
- Bilinear: linear in both arguments.
- Associative
- Distributive
- Not commutative
- $(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$.
- $\text{Tr}(\mathcal{A} \otimes \mathcal{B}) = \text{Tr}(\mathcal{A}) \cdot \text{Tr}(\mathcal{B})$.
- $\det(\mathcal{A} \otimes \mathcal{B}) = (\det(\mathcal{A}))^m \cdot \det(\mathcal{B})^n$, where $m = \text{size}(\mathcal{A})$, $n = \text{size}(\mathcal{B})$.

Simulation on IBM-Q

Entanglement circuit, revisited










Quantum State: Computation Basis



- Representing multi-qubit systems with tensor products.
- Representing multi-qubit operators with tensor products.
- Not all multi-qubit states are a tensor product of 1-qubit states.
- Not all multi-qubit gates are a tensor product of 1-qubit gates.
- Entanglement on IBM-Q.

References

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