Oct, 19, 2020

→ well... Each H_I has 4 ϕ_I's : $\phi(z)\phi(z)\phi(z)\phi(z)$,

→ interchanging the contraction "ends" will give the same amplitude...

→ So for each H_I we expect a factor of 4!

→ cancels out the $\frac{1}{4!}$ in $\frac{\lambda}{4!}\phi^4$.

- In a diagram with more there are power of H_I

We can exchange all the contractions ends of one H_I with contraction ends of the other H_I.

→ Since we integrate over all z_1, z_2 ⇒ these give the same amplitude.

→ For a diagram with n "internal vertices",

(i.e. # of H_I's), we get a factor of $n!$ This cancels the $\frac{1}{n!}$ factor from Taylor series expansion.

of $\exp\{-i\int H_I(t)\,dt\}$.

→ ~~[illegible]~~ is

For example... consider the λ^3 term:

$$\langle 0|\phi(x)\phi(y)\,\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,\phi\phi\phi\phi\int d^4w\,\phi\phi\phi\phi\int d^4u\,\phi\phi\phi\phi$$

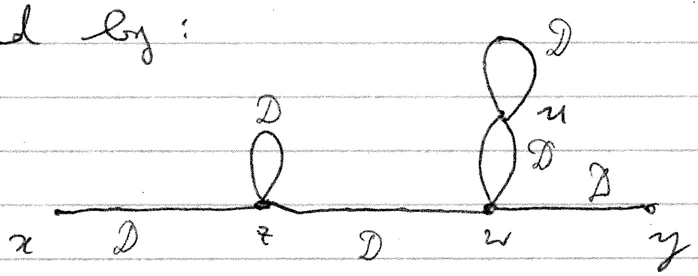
$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,d^4w\,d^4u\,D_F(x-z)\,D_F(z-z)\,D_F(z-w)$$
$$D_F(w-y)\,D_F^2(w-u)\,D_F(u-u)$$

The number of contractions that give this same expression is

$$3! \times (4.3) \times (4.3.2.1) \times (4.3) \times (1/2)$$

- $3!$: interchange vertices
- (4.3): placement of contractions into z vertex
- $(4.3.2.1)$: placement of contraction into w vertex
- (4.3): " … for u vertex
- $(1/2)$: interchange of w-u interaction

Represented by:



Now… there is a subtlety that goes will all of this, and that's symmetry factors.

Symmetry factors

→ Best to consider the simplest diagram with the most general problem.

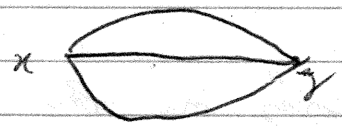
→ consider $\frac{\lambda}{3!}\phi^3$ theory ...

At 2nd order in λ, $\langle 0|T\left\{\exp\int_{-\infty}^{\infty} H_I(t)\,dt\right\}|0\rangle$

gives sth like

$$\int \langle 0|\phi(x)\phi(x)\phi(x)\,\phi(y)\phi(y)\phi(y)|0\rangle\, d^4x\, d^4y$$

↳ Feynman diagram is



$$D_F(x-y)\,D_F(x-y)\,D_F(x-y).$$

Naively we expect $2!$ from interchanging x-y.

and $3! \times 3! = 36$ from interchanging the ϕ's at x and ϕ's at y.

→ Expect 72. But are actually only 6:

$$\phi_x^1 \;\; \phi_y^1 \;\; \rightarrow \;\; \left\{ \begin{array}{l} \phi_x\phi_x\phi_x \;\; \phi_y\phi_y\phi_y \\ \phi_x\phi_x\phi_x \;\; \phi_y\phi_y\phi_y \end{array} \right.$$

Similarly for $\phi_x^1 - \phi_y^2$ $(\times 2)$

$\phi_x^1 - \phi_y^3$ $(\times 2)$

} total = 6.

→ we have overshot by 12, because

$$\cancel{\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y = \phi_x\phi_x\phi_x\phi_y\phi_y\phi_y}$$

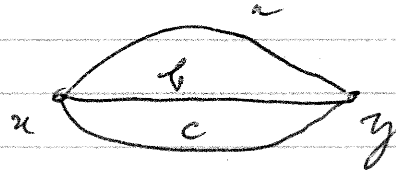
~~also interchanging x-y~~

(1) Simultaneously swap 1^{st} & 2^{nd} x
& 1^{st} & 2^{nd} y

$$\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y = \phi_x\phi_x\phi_x\phi_y\phi_y\phi_y$$

(2) exchanging all x's & y's doesn't do anything either...

We can see what's going on by labelling the vertices & propagators...



$a \leftrightarrow b$, $b \leftrightarrow c$, $c \leftrightarrow a$ don't change the diagram

$x \leftrightarrow y$ also doesn't change the diagram

⇒ diagram has a permutation symmetry

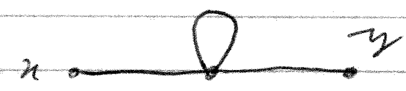
This has $2! \times 3! = 12$ elements; which is the factor we overshot with...

This is called the symmetry factor S.

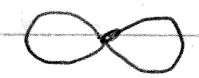
The number of diagrams is then

$$\frac{1}{s}(3!)(3!)(2!) = 6.$$

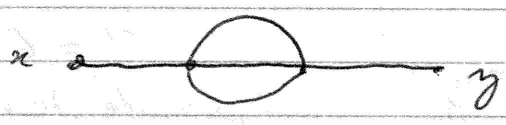
<u>Some examples</u>



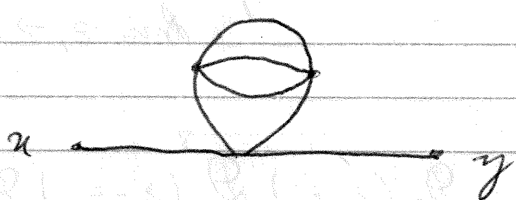
$s = 2 \qquad (x \leftrightarrow y).$



$s = 2 \cdot 2 \cdot 2 = 8$



$s = 3! = 6$



$s = 3! \cdot 2 = 12$

Now, we are ready to state the Feynman rules for position space...

Recall that this rule lets us find

$$\langle 0 | T\left\{ \phi(x)\phi(y) \exp\left\{ -i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\,dt \right\} \right\} |0\rangle$$

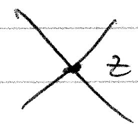
$$= \left(\begin{array}{l} \text{sum of all possible diagrams with} \\ \text{two external points} \end{array} \right)$$

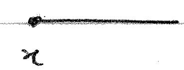
where each diagram is built out of
- propagators
- vertices
- external pts

...

Feynman Rules for ϕ^4 theory

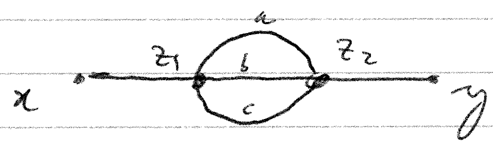
1. For each propagator  $= \mathcal{D}_F(x-y)$

2. For each vertex  $= (-i\lambda)\int d^4z$

3. For each external point  $= 1$

4. Divide by symmetry factor.

no $\frac{1}{4!}$ here

Ex 

$S = 3! = 6$
(permute the 3 propagators from $z_1 \to z_2$)

Amplitude:

$$\left(\frac{-i\lambda}{1}\right)^2 \cdot \frac{1}{6} \cdot \int d^4z_1\, d^4z_2\; \mathcal{D}_F(x-z_1)\, \mathcal{D}_F^3(z_1-z_2)\, \mathcal{D}_F(z_2-y)$$

Interpretation

- Each of the vertex factor $(-i\lambda)$ is the amplitude for the emission and/or absorption of particles at a vertex.

- The integral $\int d^4z$ instructs us to sum over all points where this process can occur.

$\rightarrow$ this is the principle of superposition!

- $\int d^4z$ is addition of amplitudes

$\hookrightarrow$ Feynman rules tell us to multiply the amplitudes for each independent part of the process.

Now, in most calculations, it is simpler to express the Feynman rules in terms of momentum

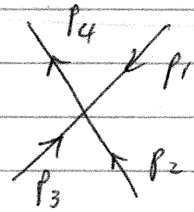
$\longrightarrow$ We want <u>momentum-space Feynman diagrams</u>

To do this, we write $D_F(x-y)$ in Fourier space

$$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\epsilon} e^{-ip\cdot(x-y)}}$$

Now, we present this in a diagram by assigning a 4-momentum p to each propagator...

When 4 lines meet at a vertex, we get



$$\longrightarrow \int d^4z\, e^{-ip_1 z} e^{-ip_2 z} e^{-ip_3 z} e^{+ip_4 z}$$

$$= (2\pi)^4 \delta^{(4)}(p_1+p_2+p_3-p_4)$$

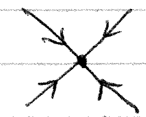
i.e. <u>momentum is conserved</u> at each vertex.

The delta functions can be used to perform integrals from the propagators...

$\longrightarrow$ From here we get momentum space Feynman rules.

Momentum-space Feynman rules

1. Each propagator $\xrightarrow[p]{\hspace{2em}} \; = \dfrac{i}{p^2-m^2+i\varepsilon}$

2. Each vertex  $= -i\lambda$

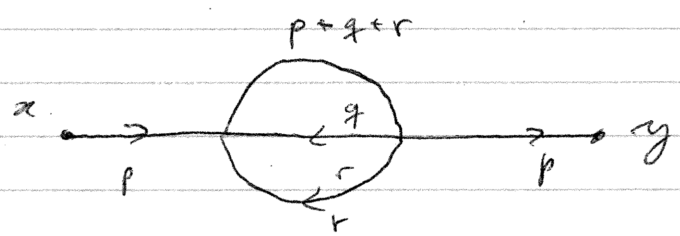
3. Each external point $\overset{\cdot}{x}\;\underset{p}{\longleftarrow} \; = e^{-ip\cdot x}$

4. Impose momentum conservation at each vertex

5. Integrate over each ~~a~~ undetermined momentum
$$\int \frac{d^4p}{(2\pi)^4}$$

6. Divide by symmetry factor.

Ex



$$= (-i\lambda)^2 \, \frac{1}{6} \cdot \int \left\{ \left(\frac{i}{p^2-m^2+i\varepsilon} \right)^2 \frac{i}{(q+p+r)^2-m^2+i\varepsilon} \; \frac{i}{q^2-m^2+i\varepsilon} \right.$$

$$\left. \cdot \frac{i}{r^2-m^2+i\varepsilon} \right\} \frac{d^4q}{(2\pi)^4} \; \frac{d^4r}{(2\pi)^4} \,.$$

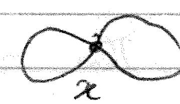
There is one subtlety, however...

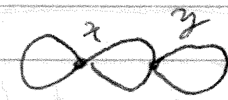
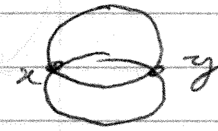
Consider diagrams without external vertices.

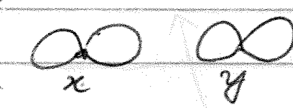
↳ these are diagrams from the form:

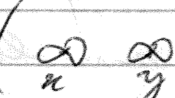
$$\langle 0| \, T\left\{ \exp\left(-i \int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt \right) \right\} |0\rangle$$

→ these are called "vacuum diagrams".

At order λ we have ... 

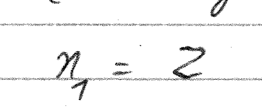
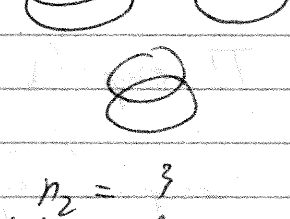
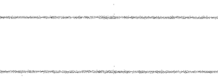
λ^2  and 

and () → disconnected diagram.

⇒ there is $S = 2$ for ()

In general, a vacuum diagram has connected subdiagrams V_i which appears n_i times

<u>Ex</u> V_1 V_2 V_3

<u># copies</u> → $n_1 = 2$ $n_2 = 3$ $n_3 = 1$

The amplitude for the total diagram is the product:

$$\boxed{\prod_i \left(\frac{1}{n_i!} V_i^{n_i} \right)}$$

Now, ~~So~~, the sum over all ~~[illegible]~~ diagrams can be written as

$$\boxed{\sum_{\substack{\text{all possible}\\\text{connected}\\\text{pieces}}} \sum_{\substack{\text{all}\\\{n_i\}}} \left(\begin{array}{c}\text{value of}\\\text{connected}\\\text{piece}\end{array}\right) \times \left\{ \prod_i \frac{1}{n_i!} (V_i)^{n_i} \right\}}$$

The sum of the connected pieces factors out, giving

$$\boxed{= \left(\sum \text{connected}\right) \times \sum_{\substack{\text{all}\\\{n_i\}}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)}$$

sum all the values of the connected pieces

Now...

$$\sum_{\{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)$$

$$= \prod_i \sum_{\{n_i\}} \frac{1}{n_i!} (V_i)^{n_i}$$

$$= \prod_i \exp(V_i)$$

$$= \exp\left(\sum_i V_i \right)$$

$$\rightarrow \boxed{\sum \text{all diagrams} = \sum \text{connected} \times \exp\left(\sum \text{disconnected}\right)}$$

Now, recall that the sum of all vacuum diagrams is just going to be

$$\exp\left(\sum_i V_i\right).$$

$$\Rightarrow \langle 0| T\left\{ \exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt\right)\right\}|0\rangle = \exp\left(\sum_i V_i\right)$$

and as we have argued...

$$\langle 0| T\left\{ \phi_I(x)\phi(y) \exp\left\{-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt\right\}\right\}|0\rangle$$

$$= \left(\sum \text{connected}\right) \times \exp\left(\sum_i V_i\right).$$

$\Rightarrow$ And so we have that

$$\langle \Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim \frac{\langle 0| T\left(\phi(x)\phi(y)\left\{\exp\int iH(t)dt\right)|0\rangle\right.}{\langle 0| T\left(\exp\left(-i\int H(t)dt\right)\right)}$$

$= \sum'$ all connected diagrams with 2 external pts.

More generally...

$$\boxed{\frac{\langle 0| T\left\{\phi(x_1)\ldots\phi(x_n)\exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)dt\right)\right\}|0\rangle}{\langle 0| T\left\{\exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)dt\right)\right\}|0\rangle} = \sum \text{connected diagrams with end points } x_1 \ldots x_n}$$

Cross section & the S-matrix

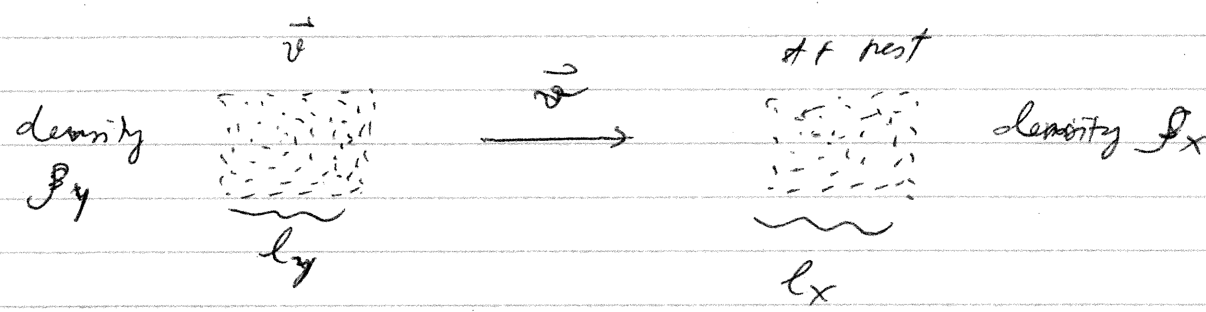
Now that we have a formula for computing the n-point correlation function...

→ Next task is to compute quantities that can be measured

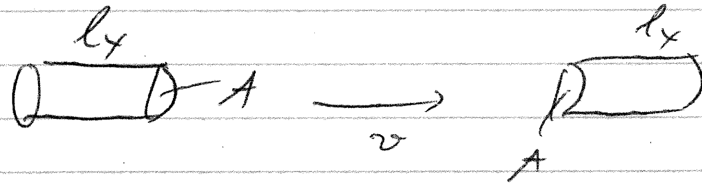
↳ cross section & decay rates.

The cross section:

Consider collision of 2 beams of particles with relatively well-defined momenta.



{ l_y, l_x are observed from rest.
{ ρ_y, ρ_x are densities at rest. Let A be the cross sectional area of overlap...



Total # of particles... $N_x = \rho_x l_x A$

$N_y = \rho_y l_y A$

→ Total # of scatterings is proportional to $N_x N_y$..

Let total number of scatterings be

$$N_X \cdot N_Y \left(\frac{\sigma}{A}\right)$$

probability one particular X particle & Y collide.

Call σ the effective area or "cross section" of the scattering process.

Let $N_X = 1$, then

$$\boxed{\sigma = \frac{\text{total \# scatterings}}{\rho_Y \cdot l_Y}}$$

For small time interval...

$$\sigma = \frac{\text{total \# scatterings}/\Delta T}{\rho_Y \cdot (l_Y/\Delta T)}$$

scattering rate

particle flux.

The differential cross section is the portion of σ in which the final particle momenta lie inside some window of momenta.

↳ write this as

$$\frac{d\sigma}{d^3\vec{p}_1 \cdots d^3\vec{p}_n}, \text{ so-that}$$

final particle momenta.

$$\int \frac{d\sigma}{d^3\vec{p}_1 \cdots d^3\vec{p}_n} \, d^3\vec{p}_1 \cdots d^3\vec{p}_n = \int d\sigma = \sigma .$$

Now, if there are only 2 final particles then there are only two free parameters ...

<u>Why?</u> two spatial momenta → 6 params

4-momentum conservation → 4 ~~params~~ constraints

⇒ can take these two params to be orient. angles θ & ϕ

→ Then we can measure $\boxed{\frac{d\sigma}{d\Omega}(\theta,\phi)}$

where $d\Omega$ is the solid angle differential

$$\boxed{d\Omega = d\cos\theta \, d\phi}$$

"Differential cross section" refers to $\frac{d\sigma}{d\Omega}$

Let's look at an example ... Consider a periodic box with length L on all sides.

Spatial momentum modes are now discrete:

$$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z) . \qquad n_i \in \mathbb{Z} .$$

there canon. relation:

$$[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} \, V \longrightarrow V = L^3 \text{ (volume)}$$

as $V \to \infty$

$$\delta_{\vec{k},\vec{k}'} \cdot V = \int_0^L\!\!\int_0^L\!\!\int_0^L dx_1\, dx_2\, dx_3 \; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$\to \int_0^\infty\!\!\int_0^\infty\!\!\int_0^\infty dx_1\, dx_2\, dx_3 \; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$= (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}').$$

In this box ...

$$\phi(x) = \sum_{\vec{k}} \frac{(2\pi/L)^3}{(2\pi)^3\sqrt{2E_{\vec{k}}}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} e^{+ik\cdot x} \right)$$

$$= \sum_{k} \frac{1}{\sqrt{2E_k}} \left(a_{\vec{k}} e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} e^{+ik\cdot x} \right)$$

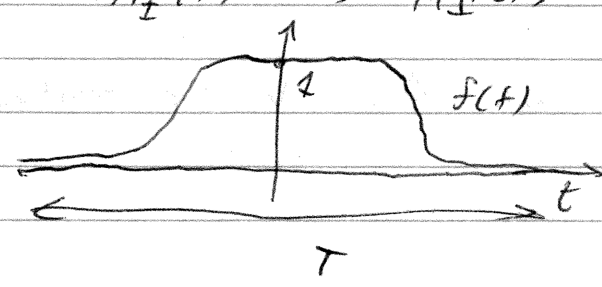
Oct 25, 2020

$\boxed{\text{S-Matrix}}$

Now imagine starting with free field theory ...

at some early time then switching on the interaction slowly and then slowly switching off the interactions ...

i.e. $H_I(t) \longrightarrow H_I(t)\, f(t)$ where $f(t)$ looks like



such that

Such that $\int_{-\infty}^{\infty} f(t)\,dt = T$, $\int_{-\infty}^{\infty} (f(t))^2\,dt = T$

Let $S = T \exp\left\{ -i \int_{-\infty}^{\infty} dt\; H_I(t) f(t) \right\}$

Define the S matrix as

$$\langle \text{final} | S | \text{initial} \rangle$$

where $|\text{initial}\rangle$ is a free particle state with momenta $\vec{k}_i^{\,I}$ & energies E_i^{I}

and $|\text{final}\rangle$ is a free particle state with momenta $\vec{k}_f^{\,F}$ + energies E_i^{F}

Now look at $S - I$:

$$\langle \text{final} | S - I | \text{initial} \rangle$$

- as $T \to \infty$, $V \to \infty$ we can write this amplitude as

$$\langle \text{final} | S - I | \text{initial} \rangle = i \cdot \mathcal{M} \cdot (2\pi)^4\, \delta(E^F_{tot} - E^I_{tot})\, \delta^{(3)}(\vec{k}^F_{tot} - \vec{k}^I_{tot})$$

fn of the momenta

- For finite T and finite V we instead have

$\langle \text{final} | S - I | \text{initial} \rangle$

$$= i\mathcal{M} \int_{-\infty}^{\infty} f(t)\, e^{i(E^F_{tot} - E^I_{tot})t} \cdot \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V$$

So,

$$|\langle \text{final} | S - I | \text{initial} \rangle|^2 = |\mathcal{M}|^2 \cdot \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V^2$$

$$\cdot \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) f(t')\, e^{i(E^F_{tot} - E^I_{tot})(t'-t)}\, dt'\, dt}$$

as $T \to \infty$, this is some constant times $\delta(E^F_{tot} - E^I_{tot})$

What is this constant?
To get this → integrate w.r.t E^F_{tot}..

$$\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left\{ \int_{-\infty}^{\infty} dE^F_{tot}\, e^{i(E^F_{tot} - E^I_{tot})(t'-t)} \right\}}_{2\pi\delta(t-t')} f(t) f(t')\, dt\, dt'$$

$$= 2\pi \int_{-\infty}^{\infty} f^2(t)\, dt = 2\pi \cdot T.$$

So the constant is $2\pi \cdot T$ and so

$$|\langle \text{final} | S - I | \text{initial} \rangle|^2 = |\mathcal{M}|^2 (2\pi)\, \delta(E^F_{tot} - E^I_{tot})\, \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}}\, V^2 T$$

Now, we have been using relativistic normalizations

$$\begin{cases} \langle \text{initial} | \text{initial} \rangle = \prod_i (2E_i^I \cdot V) \quad \text{becomes } (2\pi)^3 \delta^{(3)}(0) \text{ as } V \to \infty \\ \\ \langle \text{final} | \text{initial} \rangle = \prod_i (2E_i^F \cdot V) \end{cases}$$

To get transition probability per unit time ...

$$\frac{\text{probability}}{\text{time}} = \frac{1}{T} \frac{|\langle \text{final} | S - I | \text{initial} \rangle|^2}{\langle \text{final} | \text{final} \rangle \langle \text{initial} | \text{initial} \rangle}$$

$$= \frac{|\mathcal{M}|^2 (2\pi) \delta(E^F_{tot} - E^I_{tot}) \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V^2}{\prod_i (2E_i^F \cdot V) \prod_i (2E_i^I \cdot V)}$$

As $V \to \infty$, $\delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V \Rightarrow (2\pi)^3 \delta^{(3)}(\vec{k}^F_{tot} - \vec{k}^I_{tot})$

If we sum over final states in some window then we have

$$\underbrace{\sum_{\vec{k}_1^F, \ldots \vec{k}_{n_F}^F} \frac{1}{(2E_1^F \cdot V)} \cdots \frac{1}{(2E_{n_F}^F \cdot V)}} \frac{|\mathcal{M}|^2 (2\pi)^4 \delta(k^F_{tot} - k^F_{tot}) \cdot V}{(2E_1^I \cdot V) \cdots (2E_{n_I}^I \cdot V)}$$

As $V \to \infty$, $\dfrac{d^3 \vec{k}_1^F}{(2\pi)^3 2E_1^F} \cdots \dfrac{d^3 k_{n_F}^F}{(2\pi)^3 2E_{n_F}^F}$ $\begin{cases} n_I = \#\text{ initial particles} \\ n_F = \#\text{ final particles} \end{cases}$

Consider single particle decay... ($n_I = 1$)

The total decay rate is $\Gamma = \int d\Gamma$ where

$$d\Gamma = \frac{1}{2E^I}\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3\, 2E_i^F}\right) |M|^2 (2\pi)^4 \delta^{(4)}(k_{tot}^F - k_{tot}^I)$$

For a 2-particle initial state, the cross section is given by

$$\sigma = \frac{\text{probability}}{\text{time} \cdot \text{flux density}}$$

flux density = relative velocity between beam and target × density of incoming beam in lab frame.

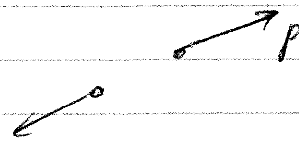
We have normalized probability for one incoming beam particle → density = $1/V$, and flux...

$$\text{flux} = \frac{|\vec{v}_A - \vec{v}_B|}{V}$$

$\vec{v}_A, \vec{v}_B$ = velocity of particles in lab frame.

So
$$d\sigma = \underbrace{\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3\, 2E_i^F}\right)(2\pi)^4 \delta^{(4)}(k_{tot}^F - k_{tot}^I)}_{\text{call this } d\Pi_{n_F}} \frac{|M|^2}{2E_A E_B |\vec{v}_A - \vec{v}_B|}$$

Now consider special case: 2 final particles ($n_f = 2$) in com frame



then $n_f = 2$ $\quad p_1 = |\vec{p}_1|$

$$\rightarrow \int d\Pi_2 = \int \frac{dp_1\, p_1^2\, d\Omega}{(2\pi)^3\, 2E_1\, 2E_2} (2\pi)\, \delta(E_{com} - E_1 - E_2)$$

where final particle energies are
$$\begin{cases} E_1 = \sqrt{|\vec{p}_1|^2 + m_1^2} \\ E_2 = \sqrt{|\vec{p}_2|^2 + m_2^2} \end{cases}$$

So

$$\cancel{= \int d\Pi_2 = \int d\Omega \frac{p_1^2}{16\pi^2 E_1 E_2}}$$

$$\rightarrow = \int d\Pi_2 = \int \frac{d\Omega}{16\pi^2} \int_0^\infty \frac{p_1^2\, \delta\left(-E_{com} + \sqrt{|\vec{p}_1|^2 + m_1^2} + \sqrt{|\vec{p}_2|^2 + m_2^2}\right)}{\sqrt{|\vec{p}_1|^2 + m_1^2} \cdot \sqrt{|\vec{p}_2|^2 + m_2^2}}\, dp_1$$

Recall that $\delta(f(x)) = \dfrac{\delta(x - x_0)}{|f'(x_0)|}$

$$\frac{dE_1}{dp_1} = \frac{d\sqrt{|\vec{p}_1|^2 + m_1^2}}{dp_1} = \frac{1}{2} \frac{2p_1}{\sqrt{|\vec{p}_1|^2 + m_1^2}} = \frac{p_1}{E_1}$$

likewise $\dfrac{dE_2}{dp_2} = \ldots = \dfrac{p_2}{E_2}$.

So $$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^2} \frac{p_1^2}{E_1 E_2 \left(\frac{p_1}{E_1} + \frac{p_2}{E_2}\right)} \Bigg|_{p_1 \text{ s.t. } E_{com} = E_1 + E_2}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_1 + E_2} = \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_{com}}$$

So for 2 particles $\longrightarrow$ 2 particles ...

$$\left(\frac{d\sigma}{d\Omega}\right)_{com} = \frac{|\vec{p}^{\,final}|\cdot|\mathcal{M}|^2}{2E_A E_B\,|\vec{v}_A-\vec{v}_B|\cdot 16\pi^2 E_{com}} \qquad (E_{com} = E_A + E_B)$$

Now, it is conventional to define the T-matrix as

$$\boxed{S = 1 + iT}$$

Claim

$$\langle \vec{p}_1^{\,F},\dots,\vec{p}_{n_F}^{\,F}|\, iT\,|\vec{p}_A,\vec{p}_B\rangle = \left({}_{final}\langle \vec{p}_1^{\,F},\dots\vec{p}_{n_F}^{\,F}|\,T\exp\left[i\int_{-\infty}^{\infty}dt\,H_I(t)\right]|\vec{p}_A,\vec{p}_B\rangle_{free}\right)_{☆}$$

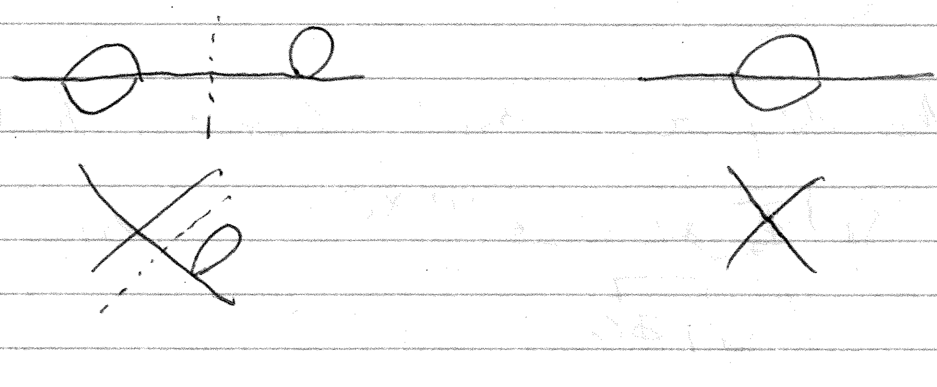
where ☆ = connected diagrams only
+ "amputated" diagrams only

"amputated" $\equiv$ diagrams can't be broken into disconnected pieces by cutting one internal line

(i.e. 1 particle irreducible)

Ex

(not amputated) (amputated)



The claim won't be proven now, but the idea is similar as before...

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0 T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

and we would like something similar...

$$|\vec{p}_1 \dots \vec{p}_{n_F}\rangle \propto \lim_{T\to\infty(1-i\varepsilon)} e^{-iHT}|\vec{p}_1, \dots, \vec{p}_n\rangle_{free}$$

For now, we'll just take this claim as true...

→ Note that

$$\phi_I^+(x)|\vec{p}\rangle_{free} = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}}\, a_{\vec{k}}\, e^{-ik\cdot x} \underbrace{\sqrt{2E_p}}_{\text{relativistic normalization}}\, a^{\dagger}_{\vec{p}}|0\rangle$$

$$= e^{-ip\cdot x}|0\rangle$$

We can think of taking the commutator of $\phi_I^+(x)$ with the $a_p^\dagger$ from $|p\rangle_{free}$

→ suggests the notation

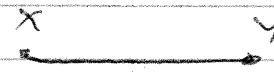
$$\phi_I^{+}(x)\,|\vec{p}\rangle_{free} = e^{-ip\cdot x}$$

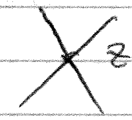
Now drop the "free" subscript → finally...

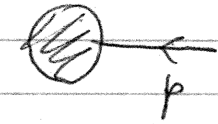
$\langle\vec{p}|\Phi_I^-(x) = e^{+ip\cdot x}\langle 0|$ → define

$$\langle\vec{p}|\Phi(x) = e^{ip\cdot x}$$

→ Set Feynman Rules in position space with external lines

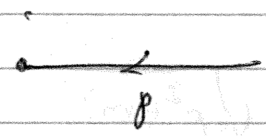
Propagator :  $D_F(x-y)$

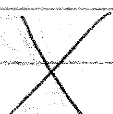
Internal vertex :  $(-i\lambda)\int d^4 z$

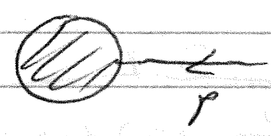
Each external line  $e^{-ip\cdot x}$

Divide by symmetry factor S.

Feynman Rules in momentum space with external line

Propagator :  $\frac{i}{p^2-m^2+i\varepsilon}$

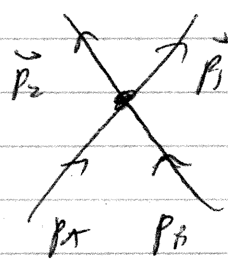
Internal vertex :  $-i\lambda$ & momentum conservation

External line  no extra factor (just 1)

Integrate over all unconstrained momenta & divide by S.

Ex $\langle \vec{p}_1, \vec{p}_2 | iT | \vec{p}_A, \vec{p}_B \rangle$ at lowest order...

well...



Feynman amplitude:

$$i\mathcal{M} = -i\lambda$$

So
$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\vec{p}^{\,final}|\,|\mathcal{M}|^2}{2E_A E_B |\vec{v}_A - \vec{v}_B|\, 16\pi^2 E_{cm}}$$

Let $p = |\vec{p}^{\,final}| = |\vec{p}_A| = |\vec{p}_B|$ all same since masses are all same

$$E_{cm} = 2E_A = 2\sqrt{p^2+m^2}$$

$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2p}{E_A}.$$

So
$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^2 p}{\frac{2p}{E_A}(2E_A)(2E_B)16\pi^2 E_{cm}} = \frac{\lambda^2}{64\pi^2 E_{cm}^2}$$

This is spherically symmetric, so

$$\sigma_{tot} = (4\pi)\frac{\lambda^2}{64\pi^2 E_{cm}^2}\cdot\frac{1}{2}$$ → particles in final state are identical so need a 1/2 factor...

→ $$\boxed{\sigma_{tot} = \frac{\lambda^2}{32\pi^2 E_{cm}^2}}$$ → our first QFT cross section!

Oct 26
2020

Feynman Rules for Fermions

So far we've discussed only the ϕ^4 theory...

→ need to generalize results to theories containing fermions.

→ need to generalize defn. of time-ordering & normal ordering symbols to include fermions...

Recall...

$$T\{\psi_a(x)\,\overline{\psi}_b(y)\} = \begin{cases} \psi_a(x)\,\overline{\psi}_b(y) & x^0 > y^0 \\ -\overline{\psi}_b(y)\,\psi_a(x) & x^0 < y^0 \end{cases}$$

The Feynman propagator, under this defn is

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\,\frac{i(\not p + m)}{p^2 - m^2 + i\varepsilon}\,e^{-ip\cdot(x-y)}$$

$$= \langle 0|\,T\{\psi_a(x)\,\overline{\psi}_b(y)\}\,|0\rangle$$

(Recall that $\not p = \gamma^\mu p_\mu$

Generalize of T to more than two fermion fields...

$$T\{\psi_1\psi_2\psi_3\psi_4\} = \begin{cases} \psi_1\psi_2\psi_3\psi_4 & \text{if } x_1^0 > x_2^0 > x_3^0 > x_4^0 \\ -\psi_2\psi_1\psi_3\psi_4 & \text{if } x_2^0 > x_1^0 > x_3^0 > x_4^0 \\ -\psi_3\psi_2\psi_1\psi_4 & \text{if } x_3^0 > x_2^0 > x_1^0 > x_4^0 \\ \vdots & \end{cases}$$

Rule: $\times(-1)$ if odd # permutations.

$\times(+1)$ if even # permutations.

Similarly, for normal ordering symbol...

$$N\left\{ a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3} a^{\dagger}_{\vec{p}_4} \right\} = (-1)^3\, a^{\dagger}_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3}$$

$\times(-1)$ if odd permutation of fields

$\times(+1)$ if even

With these, can generalize to get Wick's theorem...

1st case: 2 Dirac fields $T\{ \psi_a(x)\, \overline{\psi}_b(y) \}$

$$T\{ \psi_a(x)\, \overline{\psi}_b(y) \} = N\{ \psi_a(x)\, \overline{\psi}_b(y) \} + \contraction{}{\psi}{{}_a(x)}{\overline{\psi}} \psi_a(x)\, \overline{\psi}_b(y)$$

where $\contraction{}{\psi}{{}_a(x)}{\overline{\psi}} \psi_a(x)\, \overline{\psi}_b(y) \equiv \begin{cases} \{ \psi_a^+(x), \overline{\psi}_b^-(y) \} & x^0 > y^0 \\ -\{ \overline{\psi}_b^-(y), \psi_a^+(x) \} & x^0 < y^0 \end{cases}$

$$= \langle 0 |\, T\{ \psi_a(x)\, \overline{\psi}_b(y) \} \,| 0 \rangle$$

$$= S_F(x-y)$$

$$= - \contraction{}{\overline{\psi}}{{}_b(y)}{\psi} \overline{\psi}_b(y)\, \psi_a(x)$$

where recall that

$\psi^+, \bar\psi^+$ are the positive frequency part of $\psi, \bar\psi$

$\rightarrow$ i.e. part with annihilation operators.

$\psi^-, \bar\psi^-$ ... " negative " creation operators.

Also ~~the~~ note that:

$$\boxed{\psi_a(x)\psi_b(y) = \bar\psi_a(x)\bar\psi_b(y) = 0}$$

$\rightarrow$ Just as we proved Wick's thm for bosons, we can show the same for fermions

$$T\{\psi_1\bar\psi_2\psi_3 \cdots\} = N\left[\psi_1\bar\psi_2\psi_3 \cdots + \text{all possible contractions}\right]$$

where we note that an expression such as

$$N[\psi_1\psi_2\bar\psi_3\bar\psi_4] = -\psi_1\bar\psi_3 N[\psi_2\bar\psi_4]$$

gets a minus sign since the $\bar\psi_3$ must leap over the ψ_2

<u>Helpful hint</u> for any <u>fully</u> contracted quantity, count the number of times the contraction lines criss-cross tells you if the # of perm. is odd/even...

<u>Ex</u> $\psi_1\psi_2\psi_3\bar\psi_4\bar\psi_5\bar\psi_6 \rightarrow$ even

$\psi_1\psi_2\psi_3\bar\psi_4\bar\psi_5\bar\psi_6 \rightarrow$ odd.

Yukawa theory

↳ Next we consider the simplest theory with fermions.

$$\boxed{H_{Yukawa} = H_{Dirac} + H_{Klein-Gordon} + \int d^3x \, g \bar{\Psi}\Psi\phi}$$

↑ simplified model of QED. We will carefully work out the rules of calculations for Yukawa theory before going to QED.

We will consider two-particle scattering reaction:

$$\text{fermion } (p) + \text{fermion } (k) \longrightarrow \text{fermion } (p') + \text{fermion } (k')$$

The leading contribution comes from the H_I^2 term of the S-matrix:

$${}_0\langle p', k'| \, T\left\{ \frac{1}{2!}(-ig)\int d^4x \, \bar{\Psi}_I \Psi_I \phi_I \, (-ig)\int d^4y \, \bar{\Psi}_I \Psi_I \phi_I |p, k\rangle_0 \right.$$

Now use Wick's theorem to reduce this to N-product of contractions → then act on uncontracted fields

Represent this as the contraction:

$$\Psi_I(x)|\vec{p}, s\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}} \sum_{s'} a_{\vec{p}'}^{s'} u^{s'}(p') e^{-ip'\cdot x} \underbrace{\sqrt{2E_{\vec{p}}}\, a_{\vec{p}}^{\dagger} |0\rangle}_{|p,s\rangle}$$

$$= \underline{e^{-ip\cdot x} u^s(p)|0\rangle}$$

↑ fermion state with momentum $\vec{p}$, spin s

Define:

So
$$\boxed{\psi_I(x)\,|\vec{p},s\rangle = e^{-ip\cdot x}\,u^s(p)\,\cancel{|0\rangle}}$$

Similarly,
$$\bar{\psi}_I(x)\,|k,s\rangle = e^{-ik\cdot x}\,\bar{v}^s(k)\,\cancel{|0\rangle}$$
$$\langle \vec{p},s|\,\bar{\psi}_I(x) = e^{+ip\cdot x}\,\bar{u}^s(p)$$
$$\langle k,s|\,\psi_I(x) = e^{+ik\cdot x}\,v^s(k)$$

So, typically, a contribution to the matrix element is the contraction ...

$$\langle \vec{p}\,', \vec{k}\,'|\,\frac{1}{2!}(-ig)\int d^4x\,\bar{\psi}\psi\phi\,(-ig)\int d^4y\,\bar{\psi}\psi\phi\,|\vec{p}, \vec{k}\rangle$$

Up to a $(-)$ sign, the value of this quantity is

$$J = (-ig)^2\int \frac{d^4q}{(2\pi)^4}\,\frac{i}{q^2-m_\phi^2}\,(2\pi)^4\,\delta^{(4)}(p'-p+q)$$
$$\times\,(2\pi)^4\,\delta^{(4)}(k'-k-q)\,\bar{u}(p')\,u(p)\,\bar{u}(k')\,u(k)$$

$\phi-\phi$

$\vec{p}\,'\leftrightarrow\bar{\psi}$ ψ-p $\vec{k}\,'$-$\bar{\psi}$ ψ-k

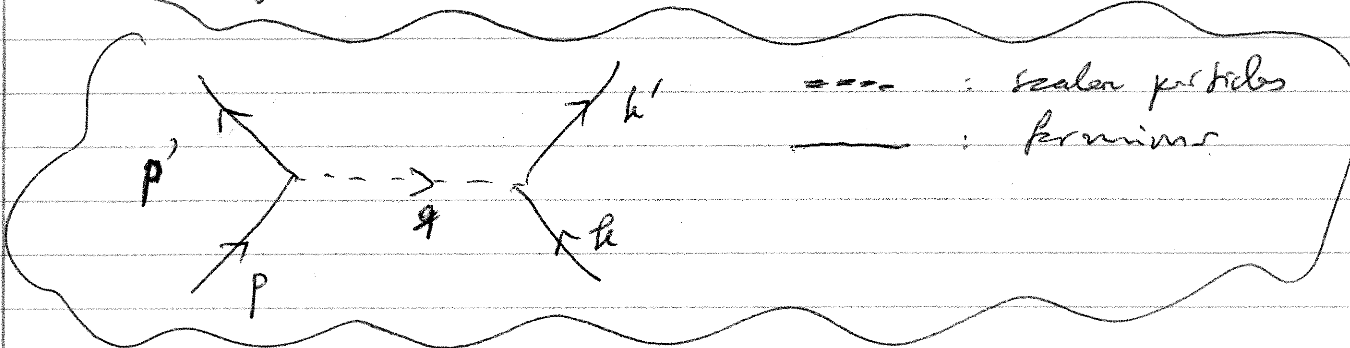
int. over all intermediate momenta.

where $q = p - p' = k' - k$.

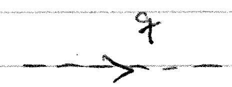
$\rightarrow$ Upon using the δ functions, $J = iM(2\pi)^4\,\delta^{(4)}(\Sigma p)$

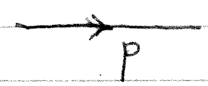
where
$$iM = \frac{-ig^2}{q^2-m_\phi^2}\,\bar{u}(p')\,u(p)\,\bar{u}(k')\,u(k)$$

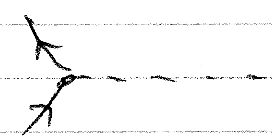
The Feynman diagram for this is ...

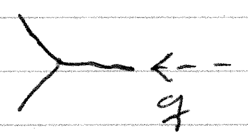


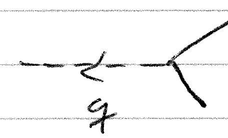
Feynman rules for fermions in momentum space

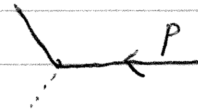
① Propagators: $\phi(x)\phi(y)$ =  $= \dfrac{i}{q^2 - m_\phi^2 + i\varepsilon}$

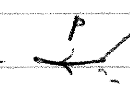
$\psi(x)\bar{\psi}(y)$ =  $= \dfrac{i(\not{p}+m)}{p^2 - m^2 + i\varepsilon}$

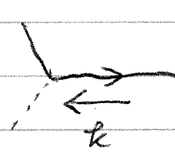
② Vertices  $= (-ig)$

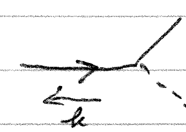
③ External leg contractions: ⓐ $\phi|q\rangle$ =  $= 1$

ⓑ $\langle q|\phi$ =  $= 1$

ⓒ $\psi|\vec{p}, s\rangle$ =  $= u^s(p)$ (fermions)

ⓓ $\langle\vec{p}, s|\bar{\psi}$ =  $= \bar{u}^s(p)$

ⓔ $\bar{\psi}|k, s\rangle$ =  $= \bar{v}^s(k)$ (antifermion)

ⓕ $\langle k, s|\psi$ = 

④ Mom. consv @ each vertex

⑤ Integrate over intermediate mom.!

⑥ Figure out sign of diagram

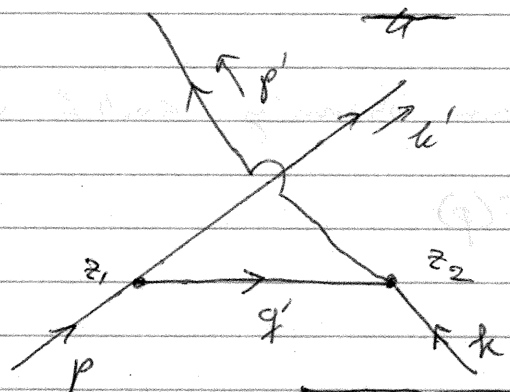
Note { initial states have momentum pointing in
final ________ out

external particle / antiparticle $\Rightarrow$ $\rightarrow$ / $\leftarrow$

same direction   opposite direction

Back to example: there are 2 criss-cross, so $(+1)$.

Ex



$$\underline{M} = \int d^4 z_1\, d^4 z_2 \,\langle k', p'|\; \bar{\Psi}_1 \Psi_1 \phi_1 \;\bar{\Psi}_2 \Psi_2 \phi_2 \,|\vec{p}, \vec{k}\rangle$$

criss-cross = 3 $\Rightarrow$ (-1)

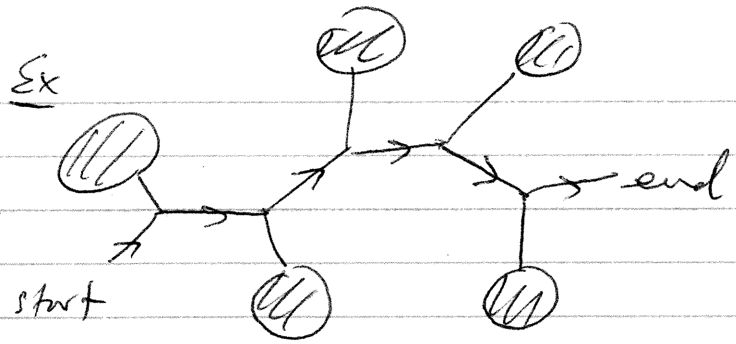
$$\rightarrow \underline{M} = (-ig)^2 (-1) \cdot \frac{i}{q'^2 - m_\phi^2 + i\varepsilon} \big(\bar{u}(k')\, u(p)\big)\big(\bar{u}(p)\, u(k)\big)$$

2 vertices   3 c-c ϕ-ϕ $\langle k'|\bar{\Psi}$ $\Psi|p\rangle$ $\langle p'|\bar{\Psi}$ $\Psi|k\rangle$

where $\underline{q' = p - k'}$.

{{{ Tips for each fermion line that doesn't close into loop, follow the particle number arrow to the end

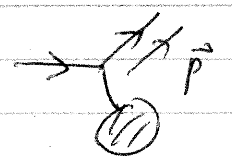
Ex



(note that

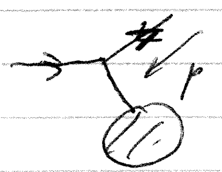
)

If the end is an outgoing fermion write down as



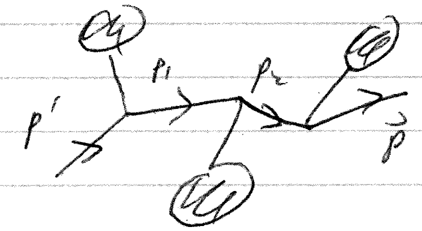
$\bar{u}(p)$

If the end is an incoming antifermion ---



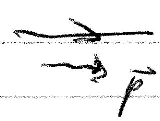
$\bar{v}(p)$

Write down fermion propagators you encounter as you follow the particle number arrow backwards



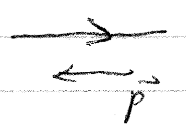
$$\bar{u}(p)\,\frac{i(\not{p}_2+m)}{p_2^2-m^2+i\varepsilon}\cdot\frac{i(\not{p}_3+m)}{p_3^2-m^2+i\varepsilon}\cdots$$

Note if



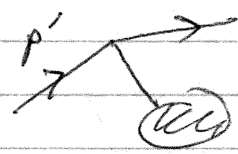
$$,\quad \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$$

if



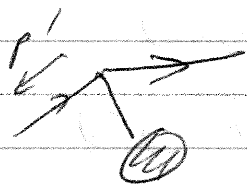
$$,\quad \frac{i(-\not{p}+m)}{p^2-m^2+i\varepsilon}$$

If the start is an incoming fermion...



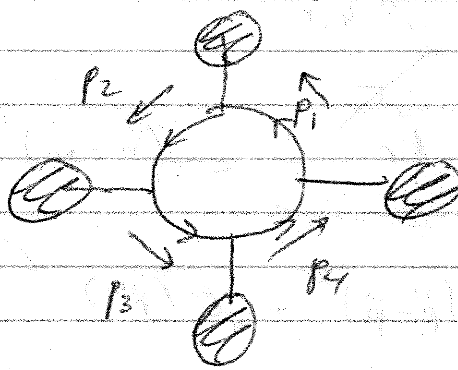
$u(p')$

If ... incoming antifermion



$v(p')$

If the fermion lines form a closed loop



$$= \bar\psi\psi\,\bar\psi\psi\,\bar\psi\psi\,\bar\psi\psi$$

$$= (-1)\,\mathrm{tr}\left\{\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\right\}$$

$$= (-1)\,\mathrm{tr}\left\{S_F\, S_F\, S_F\, S_F\right\}$$

→ a closed loop always gives a factor of (-1) & the **trace** of a product of Dirac matrices.

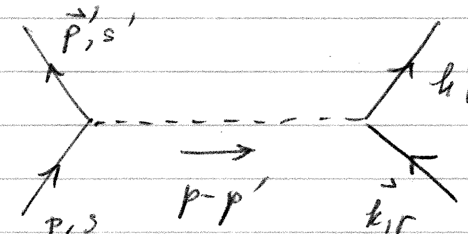
→ "trace" because we sum over the spinor indices.

The Yukawa Potential

Consider non-relativistic scattering of 2 different fermions → interact via exchange of a scalar particle.

Ignore $\mathcal{O}(\vec{p}^{\,2}/m^2)$ corrections, momenta are

$$\begin{cases} p = (m, \vec{p}) \quad , \quad k = (m, \vec{k}) \\ p' = (m, \vec{p}\,') \quad , \quad k' = (m, \vec{k}') \end{cases}$$



$= (m-m)^2 - (\vec{p}\,' - \vec{p})^2 + \mathcal{O}(\cdots)$

$$(p'-p)^2 = -|\vec{p}\,' - \vec{p}|^2 + \mathcal{O}(\vec{p}^{\,4})$$

$$u^s(p) = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} \quad , \text{ etc.}$$

where $\xi^{s'\dagger}\xi^s = \delta_{ss'}$

<u>Spinor products</u>:
$$\begin{cases} \bar{u}^{s'}(p')u^s(p) = 2m\,\xi^{s'\dagger}\xi^s = 2m\,\delta^{ss'} \\ \bar{u}^{r'}(k')u^r(k) = 2m\,\xi^{r'\dagger}\xi^r = 2m\,\delta^{rr'} \end{cases}$$

→ the spin of each particle is conserved.

⇒ amplitude: $i\mathcal{M} = \dfrac{ig^2}{(p-p')^2 + m_\phi^2}\left(\bar{u}^{s'}(p')u^s(p)\right)\left(\bar{u}^{r'}(k')u^r(k)\right)$

$$\boxed{i\mathcal{M} = \frac{ig^2}{|\vec{p}\,' - \vec{p}|^2 + m_\phi^2}\, 2m\,\delta^{rr'}\, 2m\,\delta^{ss'}}$$

Okay so $\quad iM = \dfrac{ig^2}{(\vec{p}-\vec{p}\,')^2 + m_\phi^2} \quad 2m\,\delta^{ss'}\,2m\,\delta^{rr'}$

Compared with the Born approximation to the scattering amplitude in nonrelativistic ~~scattering~~ ~~interaction~~ ~~...~~ QM, in terms of the potential $V(x)$:

$$\langle p' | iT | p \rangle = -i\,\tilde{V}(\vec{q})\,(2\pi)\,\delta(E_{\vec{p}\,'} - E_{\vec{p}})$$

where $\vec{q} = \vec{p}\,' - \vec{p}$.

where does this come from?

(cf. Griffiths Chapter 11 on the Born approximation)