Classical Mechanics III (8.09 & 8.309) Fall 2021
Assignment 5

Massachusetts Institute of Technology
Physics Department
Mon. October 11, 2021

Due Mon. October 18, 2021
6:00pm

Announcements

This week we continue to study Canonical Transformations.

- On this problem set, both **8.09 students** and **8.309 students** should do the same problems, 1 through 6.

Reading Assignment

- The reading for Canonical Transformations is **Goldstein** Ch.9 sections 9.1-9.7. (We will not discuss active infinitesimal canonical transformations with the same level of detail that Goldstein does in 9.6, but it is still good reading.)
- The reading on the Hamilton-Jacobi equations and Action-Angle Variables is **Goldstein** Ch.10 sections 10.1-10.6, and 10.8. There is one problem here. We will cover more examples of this material on problem set #6.

Problem Set 5

On this problem set there are 6 problems. Some are short. 5 problems involve canonical transformations, Poisson brackets, and conserved quantities, and in a sixth problem you apply the Hamilton-Jacobi method to a simple example. (Problem 6 does not illustrate the power of the method, and will actually be harder than the classic solution, but it does allow you to practice in a situation where you know the answer.)

1. **Canonical Transformations** [12 points, everyone]

   In this problem we get some practice with canonical transformations from (q, p) to (Q, P). We will also look at generating functions $F(q, p, Q, P, t)$, following the notation in Goldstein for $F_1(q, Q, t)$, $F_2(q, P, t)$, $F_3(p, Q, t)$, and $F_4(p, P, t)$.

   (a) [2 points] Determine two possible generating functions for $Q_i = q_i$ and $P_i = p_i$.

   (b) [2 points] Find a generating function $F_1(q, Q, t)$ for: $Q = p/t$ and $P = -qt$.

   (c) [4 points] For which parameters k, ℓ, m, n is there a generating function $F_1(q, Q)$ for: $Q = q^k p^\ell$ and $P = q^m p^n$?

   (d) [4 points] For a particle with charge q and mass m moving in an electromagnetic field the Hamiltonian is given by

   $$H = \frac{1}{2m}(\vec{p} - q\vec{A})^2 + q\phi \tag{1}$$

   where $\vec{A} = \vec{A}(\vec{x}, t)$ and $\phi = \phi(\vec{x}, t)$ are the vector and scalar potentials. Here $\{x_i, p_j\}$ are canonical coordinates and momenta.

   Under a *gauge transformation* of the electromagnetic field:

   $$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla} f(\vec{x}, t)\,, \qquad \phi \to \phi' = \phi - \frac{\partial f(\vec{x}, t)}{\partial t}\,,$$

   while $\vec{p} - q\vec{A}$ is unchanged. Show that this is a canonical transformation for the coordinates and momenta of a charged particle, and determine a generating function $F_2(\vec{x}, \vec{P}, t)$ for this transformation.

2. **Harmonic Oscillator** [7 points, everyone] (Related to Goldstein Ch.9 #24)

   (a) [2 points] For constant a and canonical variables $\{q, p\}$, show that the transformation

   $$Q = p + iaq\,, \qquad P = \frac{p - iaq}{2ia}$$

   is canonical by using the theorem that allows you to check this by using Poisson brackets.

(b) [5 points] With a suitable choice for a, obtain a new Hamiltonian for the linear harmonic oscillator problem $K = K(Q, P)$. Solve the equations of motion with K to find $Q(t)$, $P(t)$, and then find $q(t)$ and $p(t)$.

3. **Poisson Brackets and Conserved Quantities** [4 points, everyone]

A system of two degrees of freedom is described by the Hamiltonian

$$H = q_1 p_1 - q_2 p_2 + a q_1^2 + b q_2^2 \,,$$

with constants a and b. Show that $u_1 = (p_1 + a q_1)/q_2$ and $u_2 = q_1 q_2$ are constants of the motion.

4. **Angular Momentum and the Laplace-Runge-Lenz vector** [13 points, everyone]

Consider the angular momentum $\vec{L} = \vec{x} \times \vec{p}$ for canonical variables $\{x_i, p_j\}$ in 3-dimensions. The components can be written as $L_i = \epsilon_{ijk} x_j p_k$ with an implicit sum on the repeated indices j and k. Here ϵ_{ijk} is the Levi-Civita tensor

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk = 123 \text{ or a cyclic combination of this} \\ -1 & \text{if } ijk = 321 \text{ or a cyclic combination of this} \\ 0 & \text{otherwise} \end{cases}$$

Often ϵ_{ijk} is handy when we are considering cross-products: $\vec{c} = \vec{a} \times \vec{b}$ is equivalent to $c_i = \epsilon_{ijk} a_j b_k$. Some properties you may find useful are: $\epsilon_{ijk} = \epsilon_{jki}$, $\epsilon_{jik} = -\epsilon_{ijk}$, and $\sum_k \epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$.

(a) [4 points] As warm up, calculate the Poisson brackets $[x_i, L_j]$, $[p_i, L_j]$, $[L_i, L_j]$, and $[L_i, \vec{L}^2]$.

Now consider two particles attracted to each other by a central potential $V(r) = -k/r$, where $r = |\vec{r}|$ is the distance between them. Taking the origin at the CM, the Hamiltonian for this system is $H = \vec{p}^2/(2\mu) - k/r$ where μ is the reduced mass and the r_i and p_j are canonical variables. The angular momentum, $\vec{L} = \vec{r} \times \vec{p}$, is conserved so you may assume that $[L_i, H] = 0$ (some of you may recall proving this in 8.223).

(b) [7 points] Show that the Laplace-Runge-Lenz vector, $\vec{A} = \vec{p} \times \vec{L} - \mu\, k\, \vec{r}/r$, is conserved.

Recall that the conservation of $\vec{L}$ implies that the motion of the particles in this central force take place in a plane that is perpendicular to $\vec{L}$. The set of $H, \vec{L}, \vec{A}$ gives 7 constants of motion, but for two particles there are at most 6 constants from integrating the equations of motion. Furthermore, at least one constant must refer to an initial time, and none of $H, \vec{L}, \vec{A}$ do so. Hence there must be at least two relations between these constants. It is easy to see that $\vec{L} \cdot \vec{A} = 0$ provides one relation.

(c) [2 points] Show that the other relation is $\vec{A}^2 = \mu^2 k^2 + 2\mu H\, \vec{L}^2$.

[Read Goldstein section 3.9 to see how $\vec{A}$ can be used to very easily find the orbital equation $r = r(\theta)$ for motion in the plane.]

5. **An Exponential Potential** [13 points, everyone]

   A particle with mass $m = 1/2$ is moving along the x-axis inside a potential $V(x) = \exp(x)$, so its Hamiltonian is $H = p^2 + e^x$. You may assume $p > 0$.

   (a) [6 points] Determine a generating function $F_2(x, P)$ that yields a new Hamiltonian $K = P^2$. (Feel free to check your results with mathematica.)

   (b) [3 points] What are the transformation equations $P = P(x, p)$ and $Q = Q(x, p)$?

   (c) [4 points] Determine $x(t)$ and $p(t)$.

   Question [not for points]: How would your analysis change if $p < 0$?

6. **Projectile with Hamilton-Jacobi** [11 points, everyone] (Goldstein Ch.10 #17)

   Solve the problem of the motion of a point projectile of mass m in a vertical plane using the Hamilton-Jacobi method. Find both the equation of the trajectory and the dependence of the coordinates on time. Assume that the projectile is fired off at time $t = 0$ from the origin with the velocity v_0, making an angle θ with the horizontal.