

**Partial Differential Equations  
&  
The Euler-Lagrange Equations**

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**Topic:** PDEs and Euler-Lagrange Equations.

**Plan:** I will first discuss two motivating examples of optimization problems that use calculus of variations: showing that a straight line gives the shortest distance between two points (in Euclidean geometry), and the famous Brachistochrone problem. These examples will lead up to the idea of a common **functional** of the form

$$J[\phi_i] = \int_a^b \mathcal{L}(t, \phi_i, \dot{\phi}_i) dt$$

and how the **Euler-Lagrange equations**

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \frac{\partial}{\partial t} \left( \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \right),$$

where  $\mathcal{L}$  is called the **Lagrangian**, arise when finding the function  $\bar{\phi}_i$  that minimizes  $J[\phi_i]$ . I will then show we can:

1. Solve PDE's by formulating them as minimization problems. For example, the solution to the Dirichlet problem

$$(*) \begin{cases} \nabla^2 u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

minimizes

$$J[\phi] = \frac{1}{2} \int_{\Omega} |\nabla \phi|^2 dx$$

where

$$\mathcal{L} = \frac{1}{2} |\nabla \phi|^2.$$

2. Recognize the Euler-Lagrange equations as PDE's whose solutions minimize  $J[\phi_i]$ . I will discuss *Hamilton's Principle of Least Action* and how "equations on motion" including Newton's second law of motion, the wave equation, etc. can be derived from this principle.