Solid State Theory II

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Spring 2022

Course material online. Lecture notes from previous years available from Open Courseware

Course description: 2 lectures per week (Tuesday, Thursday 1-2:30, 4-159) by Leonid Levitov

Recitations (Friday 1-2, 4-153) by Cristian Zanoci (czanoci@mit.edu)

Make-up lectures by LL, Friday 1-2:30, 4-153

Homeworks: 6 total, posted bi-weekly, due Mondays

Term paper: due April 29; presentations in the final week: May 5, 10 (tentative dates)

Grade: 70% homework, 30% term paper

Questions: email LL or CZ with subject 8.512

Course topics (tentative)

Circle 3 most interesting topics, cross 3 least interesting topics, indicate other fun topics that you'd like to cover in 8.512 and submit on canvas

- Bose–Einstein Condensation and Superfluidity. Quasiparticles, collective excitations, topological excitations 2 weeks
- Superconductivity: Basic Phenomena and Phenomenological Theories 1 week
- Microscopic Theory of Superconductivity: BCS theory, Bogoliubov-deGennes equation, topological superconductivity, unconventional and high-Tc superconductivity 2 weeks
- Magnetism: Magnetically Ordered States and Spin-Wave Excitations, The Hubbard Model and Mott Insulators 2 weeks
- Many-body theory (second quantization, canonical transformation, quasiparticles) 1 week
- Linear response in many-body systems (fluctuation-dissipation theorem, sum rules, optical conductivity, neutron and x-ray scattering, collective modes) 2 weeks

- Interacting electrons in metals. Static and dynamic screening. Landau's Fermi liquid theory, quasiparticles, collective modes 2 weeks
- Quantum-coherent transport: nonlocal transport of electron waves, Landauer formula and conductance quantization, scattering and universal conductance fluctuations, fluctuations and noise in mesoscopic systems 1 week
- Anderson localization: classical and semiclassical diffusion, quantum corrections to diffusion, weak localization in 2D, strong localization in 1D, localization and metal–insulator transition (Thouless picture and scaling theory) 2 weeks
- Disordered systems 2: transport at finite temperature (mobility gap and activated transport, variable-range hopping), many-body localization, spin glasses 2 weeks
- Transport in superconductors (Josephson effect, Andreev scattering, superconducting circuits, qubits) 2 weeks
- Exotic superconductivity (topological SC, Majorana states, braiding and quantum information applications) 2 weeks

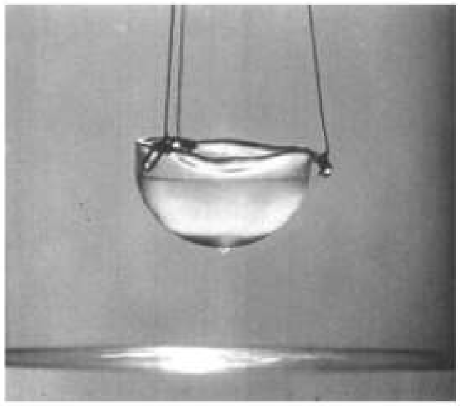
Your name, year and interests:

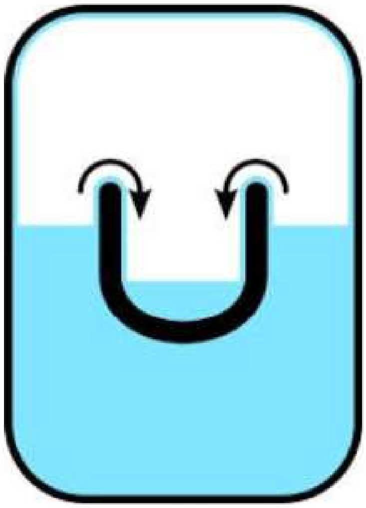
Suggest topics:

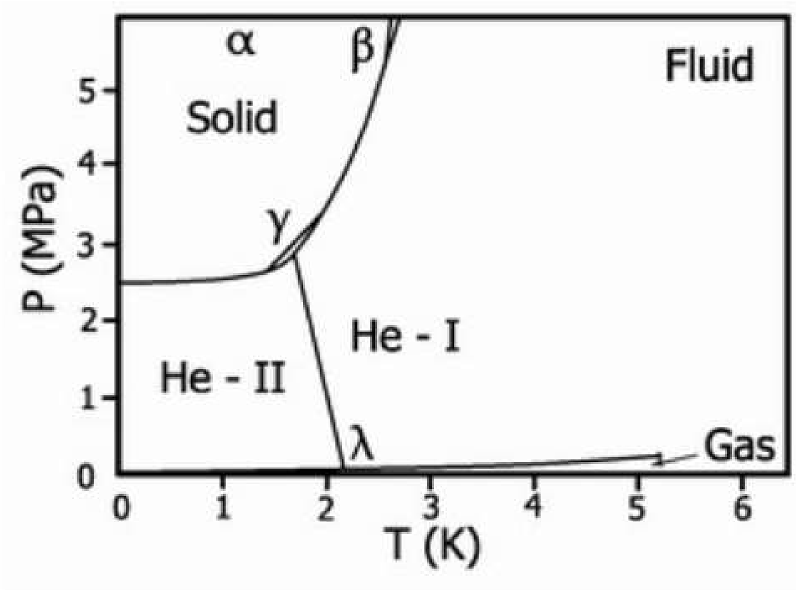
All things being equal, would you rather prefer fewer topics covered at a greater depth or more topics covered at a fleeting pace:

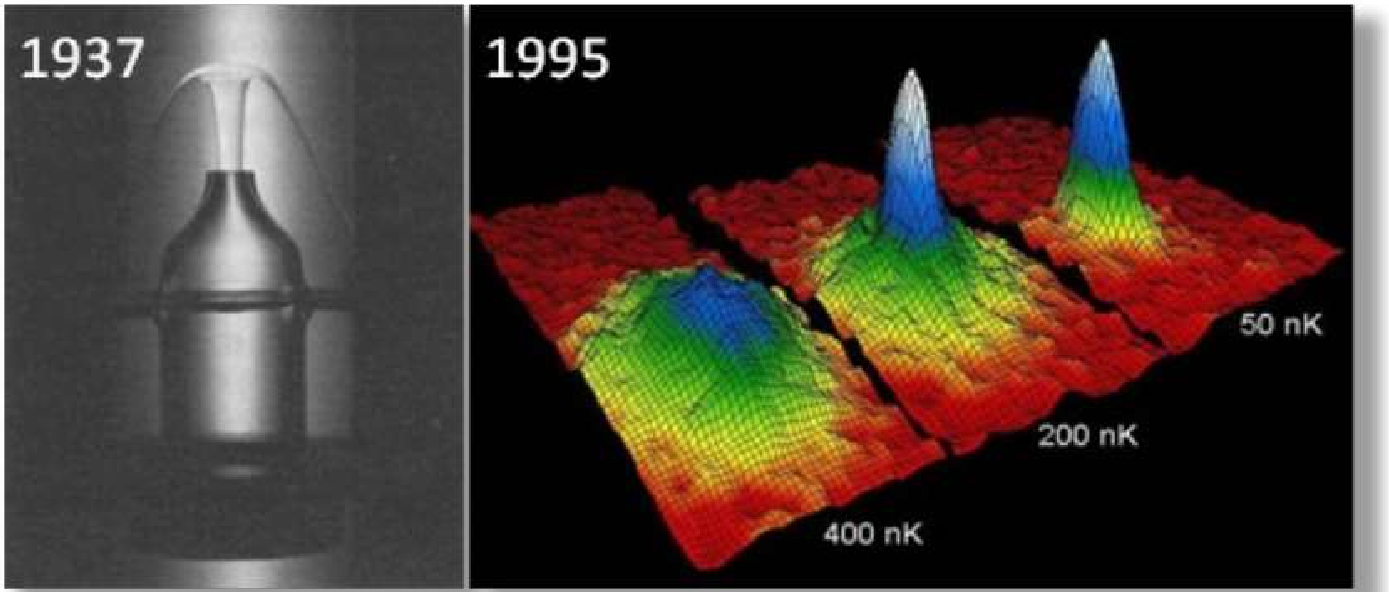
Please tell us about anything else that would be useful to mention:

Striking macroscopic quantum phenomena

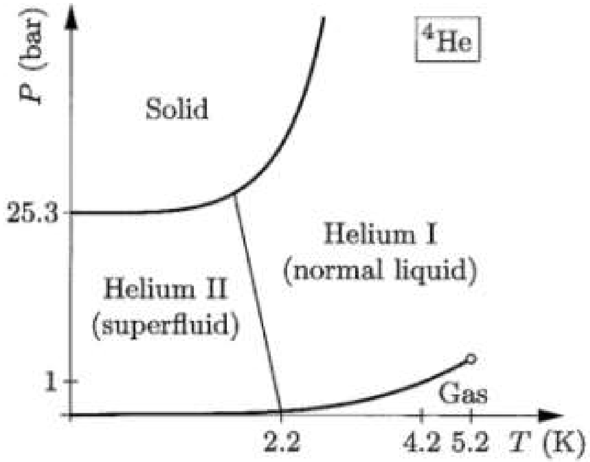


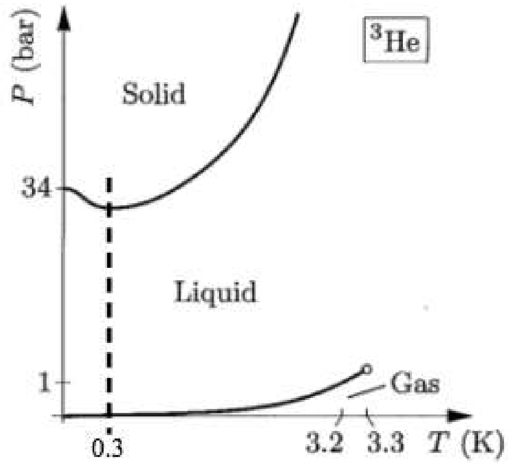


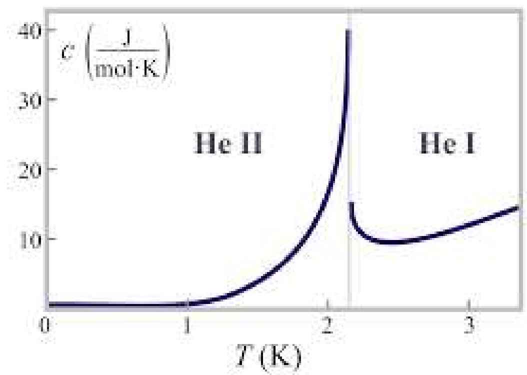




Quantum statistics. Two quantum liquids: 4He and 3He







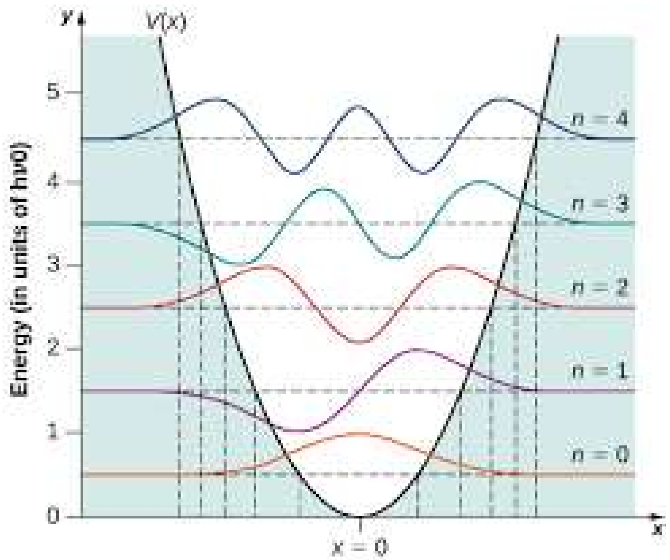
Schematic phase diagrams for 4He (bosons) and 3He (fermions). Near-identical interactions, but different statistics. 4He is a superfluid below 2K; 3He is a degenerate Fermi gas below 0.5K. Right panel: specific heat of 4He near the normal/super transition (λ point): a diverging non-analytic behavior but no jump in entropy (zero latent heat), a type-II phase transition. How to explain?

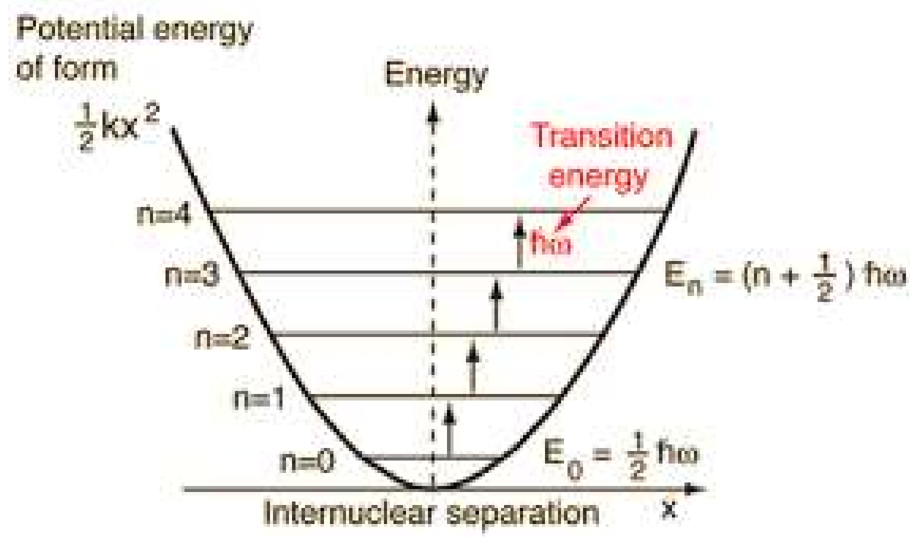
Liquid Helium II the superfluid
an overview of experiment (Caltech 2020)
highlights (BBC documentary)

Discussion

How to describe many identical particles. Harmonic oscillator representation

Quantum particle in a harmonic potential: $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$, $p = -i\hbar\partial_x$.
Energy spectrum: $E_n = \hbar\omega(n + 1/2)$, $\omega = \sqrt{k/m}$.





The key idea: view different energy excitations as independent particles in the same quantum state

Consider photons

Q: Is it a coincidence that quantum harmonic oscillators and photons have energy quantized as $E = hf$?

* Think of EM radiation by an oscillator
* Think of EM field normal modes

Describe many identical particles. Harmonic oscillator

Quantum particle in a harmonic potential: $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$, $p = -i\hbar\partial_x$.
Energy spectrum: $E_n = \hbar\omega(n + 1/2)$, $\omega = \sqrt{k/m}$.
Ladder operators (creation and annihilation of excitations):

$$b = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{i}{m\omega}p\right), \quad b^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(x - \frac{i}{m\omega}p\right)$$

The energy eigenstates $|n\rangle$, when operated on by these ladder operators, give

$$b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad b|n\rangle = \sqrt{n}|n-1\rangle$$

Define the number operator $N = b^\dagger b$, which is related to the Hamiltonian as

$$N|n\rangle = n|n\rangle, \quad H = \hbar\omega\left(N + \frac{1}{2}\right)$$

The quantity N can be thought of as a number of some indistinguishable particles.
This approach is generalizable to more complicated problems, providing a relation to quantum fields. A basic fact with far reaching implications.

Bose-Einstein condensation

Start with free bosons, add interactions later
introduce second quantization, the long-range order
Free bosons in a box: single-particle orbitals $\frac{1}{\sqrt{V}}e^{i\boldsymbol{kr}}$ with periodic boundary conditions. The grand-canonical Hamiltonian

$$H_G = \sum_{\boldsymbol{k}} \left(\frac{\hbar^2 k^2}{2m} - \mu \right) n_{\boldsymbol{k}}, \quad n_{\boldsymbol{k}} = 0, 1, 2...$$

The chemical potential μ negative at $T > T_c$ (non-BEC), and zero in the BEC state, at $T < T_c$.
Macroscopically large number of particles populating the $\boldsymbol{k} = 0$ state at $T < T_c$.
Since $n_k < \infty$, positive μ are not allowed for free bosons (unlike fermions).

Creation and annihilation operators:

Define ladder operators for harmonic oscillators associated with single-particle orbitals:

$$[b_k, b^\dagger_{k'}] = \delta_{k,k'}, \quad [b_k, b_{k'}] = [b^\dagger_k, b^\dagger_{k'}] = 0.$$

Occupancies are then represented as $n_k = b^\dagger_k b_k$.
Interpretation: individual bosons populating each orbital $\frac{1}{\sqrt{V}} e^{i\boldsymbol{kr}}$ are viewed as different energy excitations of quantum oscillators associated with the orbitals. Similar to how photons are introduced as excitations of quantized EM field oscillators, $E_n = h\nu n$.
A useful bookkeeping tool, also a powerful formalism...

As a simple illustration of the second quantization approach, let's derive the Bose-Einstein statistics for many identical bosons $N(\varepsilon) = \frac{1}{e^{\beta\varepsilon}-1}$ from a single QM harmonic oscillator at a finite temperature.

$$H = \hbar\omega b^\dagger b, \quad \varepsilon_n = \hbar\omega n, \quad |n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle$$

Maxwell distribution for thermal excitations of different levels gives probabilities $p_n = p_0 e^{-\beta\hbar\omega n}$, $\beta = 1/T$. For the thermal average of the number of excitations n this gives (here $x = e^{-\beta\hbar\omega} < 1$):

$$\langle n\rangle = \frac{\sum_{n\geq 0} np_n}{\sum_{n\geq 0} p_n} = \frac{x + 2x^2 + 3x^3 + ...}{1 + x + x^2 + ...} = \frac{x}{1-x}$$

This is nothing but $N(\varepsilon) = \frac{1}{e^{\beta\varepsilon}-1}$ with $\varepsilon = \hbar\omega$.
QED

The ground states at zero temperature

Since $\mu = 0$, the $\boldsymbol{k} = 0$ state can be populated with any number of particles without changing system energy. Massive degeneracy of the ground states:

$$|N\rangle = \frac{1}{\sqrt{N!}}(b_0^\dagger)^N|0\rangle, \quad N = 0, 1, 2\ldots$$

these states are orthogonal and form a complete set.
Another convenient basis are coherent states

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha b_0^\dagger}|0\rangle = \sum_{n\geq 0} \frac{\alpha^N}{\sqrt{N!}}(b_0^\dagger)^N|0\rangle$$

with complex-valued α. Superpositions of number states, non-orthogonal and an overcomplete set.

An interesting property: $b_0|\alpha\rangle = \alpha|\alpha\rangle$ (eigenstates of the ladder operators but not of partilcle number!)

Coherent states as BEC ground states

As we will see, the BEC ground states will resemble the coherent states with suitably chosen parameter α values. The modulus $|\alpha|$ will be fixed by the average particle number, however the phase will be undetermined and completely arbitrary.
Phase ambiguity is related to the U(1) symmetry:

$$b_k \to e^{i\theta} b_k, \quad b_k^\dagger \to e^{-i\theta} b_k^\dagger$$

which leaves the occupation numbers $n_k = b_k^\dagger b_k$ invariant. The ground states, however, aren't invariant. They change as $|\alpha\rangle \to |e^{i\theta}\alpha\rangle$.
Therefore, U(1) is a symmetry of the Hamiltonian but not of the ground states. It is a classic example of spontaneously broken symmetry (in a Mexican hat potential). A link to type-II transition in 4He.

Off-diagonal long-range order (ODLRO)

Consider a local bosonic field $\psi(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \frac{1}{\sqrt{V}} e^{i\boldsymbol{k}\boldsymbol{r}} b_k$.

Analyze a pair correlation function $\langle\langle\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}')\rangle\rangle$ ($\langle\langle ... \rangle\rangle$ is a QM and thermal average). Expressing it through average occupancies $\bar{n}_k = \langle\langle b_k^\dagger b_k \rangle\rangle$ gives

$$\frac{1}{V}\sum_{k,k'} e^{-i\boldsymbol{k}\boldsymbol{r}} e^{i\boldsymbol{k}'\boldsymbol{r}'} \langle\langle b_k^\dagger b_{k'} \rangle\rangle = \sum_k e^{i\boldsymbol{k}(\boldsymbol{r}'-\boldsymbol{r})} \frac{\bar{n}_k}{V}$$

For $T = 0$, have $\bar{n}_k = N\delta_{k,0}$ and, therefore, $\langle\langle\psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r}')\rangle\rangle = N/V$ which is independent of point separation $\boldsymbol{r} - \boldsymbol{r}'$, indicating a long-range order which survives for all temperatures $0 \le T < T_c$.

This resembles the behavior of pair spin correlations in Stat Mech, which are short-ranged at $T > T_c$ but cease to decay at $T < T_c$.

Watch video (active links)

Liquid Helium II the superfluid (by Alfred Leitner)

* highlights
* part 1 Introduction and equipment 1:30-3:30
* part 2 The transition to the superfluid state 2:00-6:30
* part 3 The superfluid has zero viscosity 0:00-3:00
* part 4 The fountain effect
* a summary of experimental setup (Caltech 2020)

* Emergent states of matter (by Nigel Goldenfeld)

Detour: Bose-Einstein condensation. Ideal Bose gas

Particle density from Bose-Einstein statistics

$$\frac{N}{V} = \frac{1}{V}\sum_{\mathrm{k}} \underbrace{\frac{1}{e^{\beta(\varepsilon(\mathrm{k})-\mu)}-1}}_{\text{occupation number}} = \frac{1}{(2\pi)^3}\int 4\pi k^2 dk \frac{1}{z^{-1}e^{\beta\varepsilon(\mathrm{k})}-1}$$

$z = e^{\beta\mu}$ with $\beta = \frac{1}{k_B T}$. Possible z values?
Recall: need $\mu < 0$ ($z \leq 1$) for occupation numbers to be positive.
Integrate taking $\varepsilon(\mathrm{k}) = \frac{\hbar^2 \mathrm{k}^2}{2m}$ as a new variable.
$y = \beta\varepsilon(k) \Rightarrow k = \sqrt{\frac{2my}{\beta\hbar^2}}$. Find

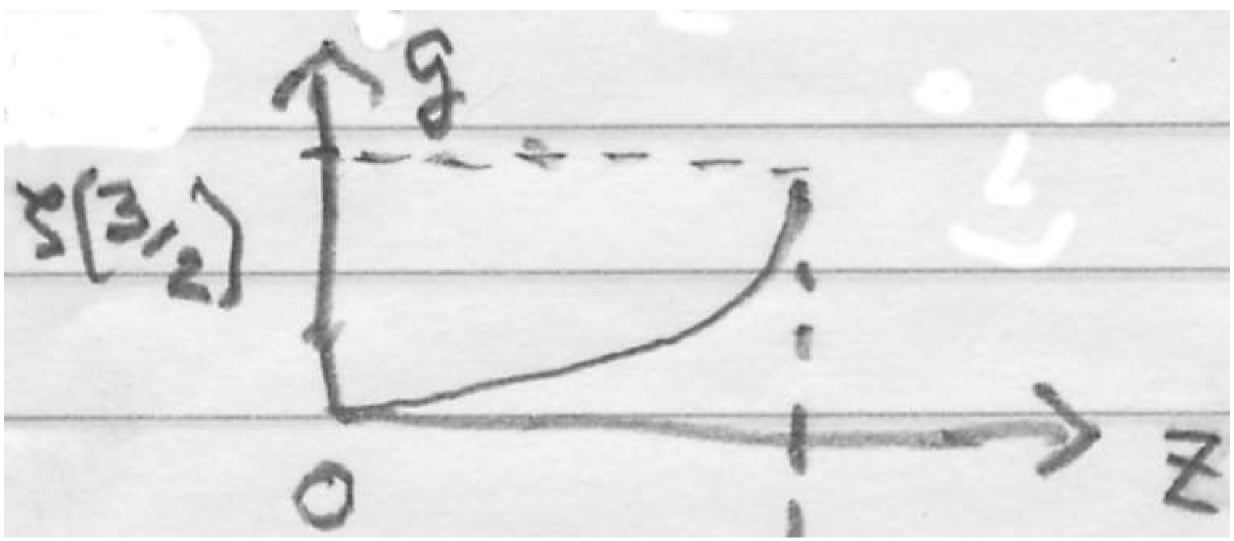
$$\frac{N}{V} = \left(\frac{2m}{\beta\hbar^2}\right)^{3/2} \frac{4\pi}{(2\pi)^3}\frac{1}{2}\int_0^\infty dy \frac{\sqrt{y}}{z^{-1}e^y - 1}$$

$$\frac{N}{V} = \frac{1}{\lambda^3} g_{3/2}(z), \quad \lambda = \left(\frac{\hbar^2}{2\pi m k_B T}\right)^{1/2} \quad \text{thermal wavelength}$$

$$g_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \frac{\sqrt{y}}{z^{-1} e^y - 1} = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \dots$$

Features: $g_{3/2}(z)$ is monotonically increasing. As $z \to 1$ $g_{3/2}$ remains finite:

$g_{3/2}(1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots = \zeta(3/2) \approx 2.61$ where $\zeta(z)$ is Riemann ζ-function:



Conclude:

$$n = \frac{N}{V} = \frac{g_{3/2}(z)}{\lambda^3} \leq \frac{g(1)}{\lambda^3} \approx \frac{2.61}{\lambda^3} = 2.61\left(\frac{2\pi m k_B T}{\hbar^2}\right)^{3/2}$$

We encounter a contradiction (!) Indeed, as n remains fixed, the inequality is violated as temperature decreases below the critical value

$$T_c \approx \left(\frac{n}{2.61}\right)^{2/3} \frac{\hbar^2}{2\pi m k_B}.$$

Solution to the paradox?

We therefore have to consider a finite density of particles in the ground state. We treat $k = 0$ and $k \neq 0$ separately:

$$\frac{1}{V}\sum n(\varepsilon(k)) = \underset{\substack{\uparrow \\ \text{ground state occupation} \\ \text{finite density}}}{\frac{n(0)}{V}} + \frac{1}{(2\pi)^3}\int_0^\infty dk 4\pi k^2 n(\varepsilon(k))$$

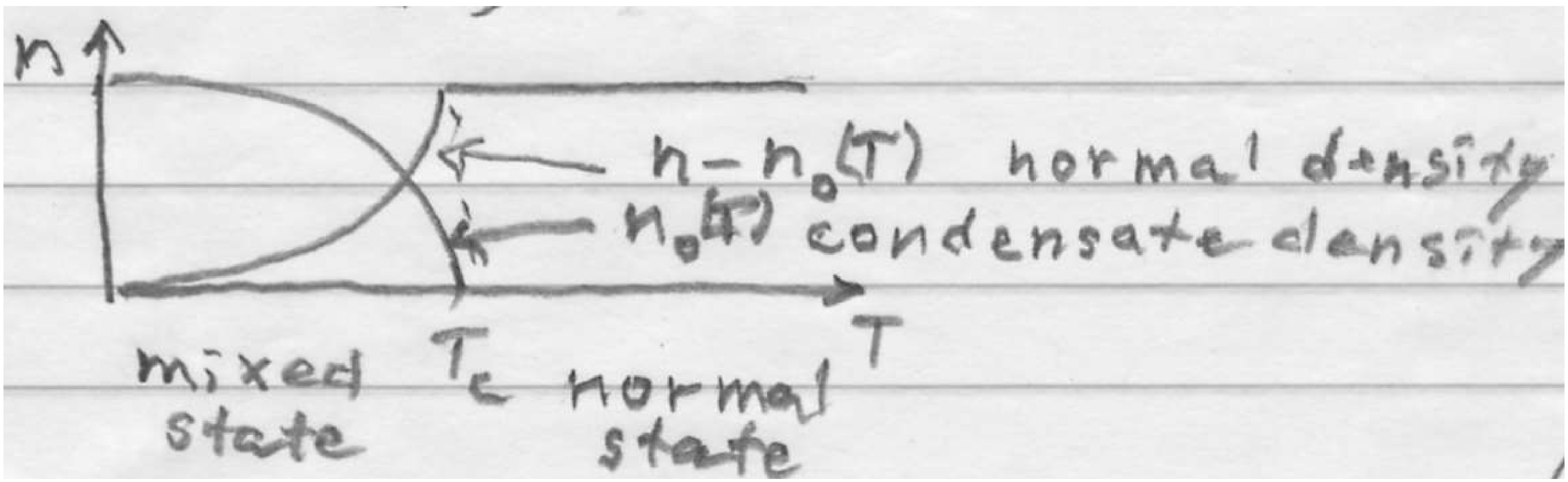
Hence $n(0) \sim V$. Then we get

$$n = \frac{N}{V} = \frac{n(0)}{V} + \frac{g_{3/2}(z)}{\lambda^3} \Rightarrow n = n_0 + \frac{g_{3/2}(z)}{\lambda^3}$$

with n_0 the density of bosons in the ground state.

For $T > T_c$ can always choose z such that $n_0 = 0$ and $n = g_{3/2}(z)/\lambda^3$.

At $T < T_c$ it is necessary to have $n_0 > 0$, hence $\mu = 0$ $(z = 1)$. As $T \to T_c$ from above, $z(T) \to 1$. Particle density components, normal and BEC:



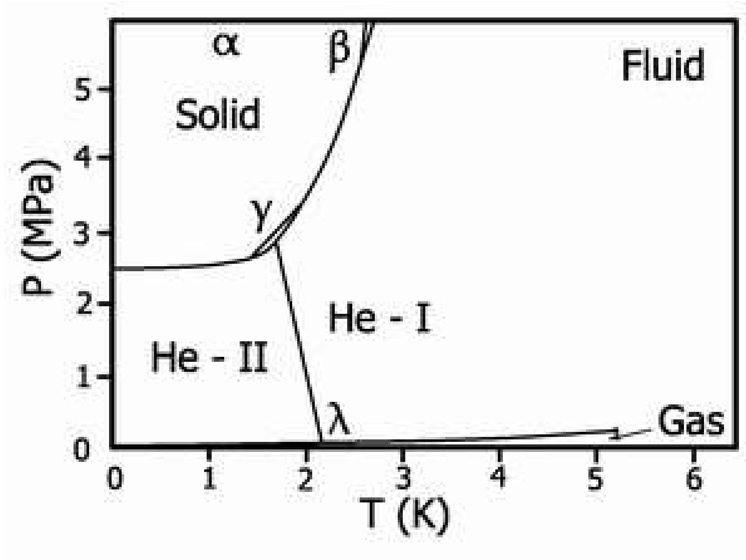
One can analyze pressure, specific heat, equation of state, and so on

1. The normal and BEC compnts coexist at $T < T_c$
2. Suggests a ‘two-fluid’ model for mixed region.

Is it applicable to real-life interacting bosons?

Superfluid ^{4}He

^{4}He P-T phase diagram



1. *liquid/gas and liquid/solid, first order, latent heat*
2. *normal/super, second order, no latent heat*
3. ^{4}*He mass* $m \approx 6.65 \times 10^{-24}$ *g, specific volume* $v = 27.6\ cm^3/mole$ *gives* $T_c^{BEC} \approx 3.13^{o}K$ *vs* $T_\lambda \approx 2.18^{o}K$. **This is pretty close, so is 4He a BEC?**

Critical superfluid velocity (Landau's argument)

Superfluid of mass M, flowing with velocity v. Can friction arise due to generation of excitations with momentum k and energy $\varepsilon(\mathrm{k})$?
Energetic instability (Galilei transformation).
Suppose one excitation is produced, $\mathrm{v} \to \mathrm{v} - \Delta\mathrm{v}$.

$$M\mathrm{v} = M\mathrm{v} - M\Delta\mathrm{v} + \hbar\mathrm{k}, \quad \text{conservation of momentum},$$

$$\frac{M\mathrm{v}^2}{2} = \frac{M(\mathrm{v} - \Delta\mathrm{v})^2}{2} + \varepsilon(\mathrm{k}), \quad \text{conservation of energy}.$$

In the large-M limit the values k, v, and $\varepsilon(k)$ obey

$$\hbar\mathrm{k} \cdot \mathrm{v} = \varepsilon(\mathrm{k})$$

The LHS attains max value for $\mathrm{k} \parallel \mathrm{v}$. Can this criterion be satisfied?

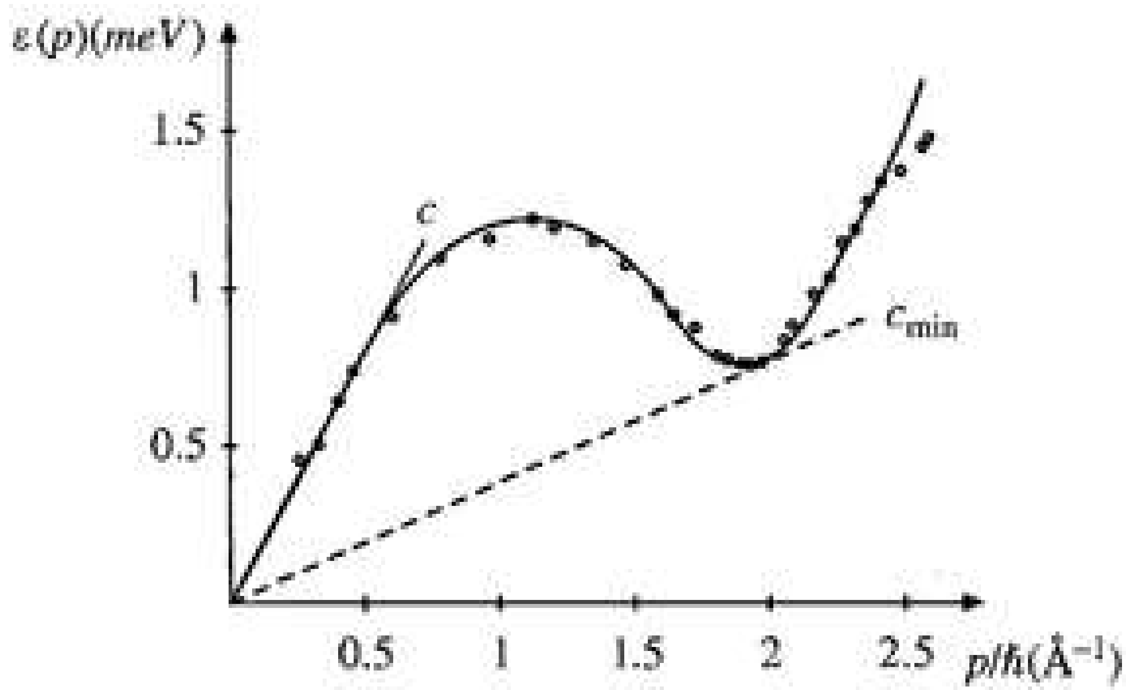
If $\hbar k v < \varepsilon(k)$ this dissipation process is forbidden.
Landau argument suggests critical velocity value for the onset of dissipation

$$v_c = \min\left(\frac{\varepsilon(k)}{\hbar k}\right)$$

For noninteracting bosons:
$\varepsilon(k) = \frac{\hbar^2 k^2}{2m} \;\Rightarrow\; v_c = 0.$
Therefore, BEC for a noninteracting Bose gas cannot explain superfluidity. Interactions are essential. Once included, enable superfluidity: yield $v_c > 0$ and stabilize the superfluid state.

Dispersion of excitations in 4He (from neutron scattering)

The critical velocity value v_c is finite, controlled by "rotons"



credit: J. E. Annett, Superconductivity, Superfluids, and Condensates, Oxford University Press, Oxford, 2004.

Theory: Y. Pomeau and S. Rica, Model of superflow with rotons, Phys. Rev. Lett. 71, 247 (1993) (active link)
Gravity-capillary waves on water surface (ripples); dispersion with a "roton bump" $\omega^2 = g|k| + \frac{\sigma}{\rho}|k|^3$

Experiment: T. Burghelea and V. Steinberg, Onset of Wave Drag Due to Generation of Capillary-Gravity Waves by a Moving Object as a Critical Phenomenon, Phys. Rev. Lett. 86, 2557 (2001) (active link)

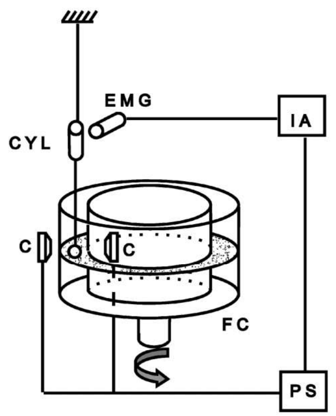


FIG. 1. Experimental setup: FC—channel with a fluid; CYL—cylinder to measure a deviation from an initial position by eddy current gauge (EMG); IA—amplifier; PS—power amplifier; C—coils of electromagnets.

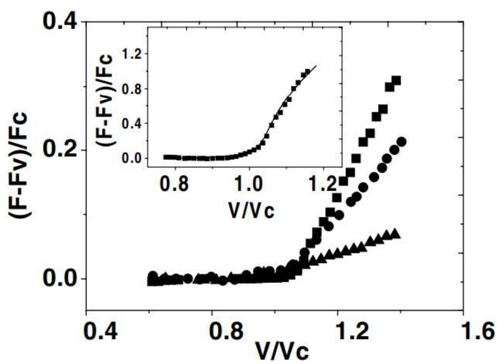


FIG. 4. The reduced drag force vs the reduced velocity for three fluids and a 3.14 mm ball: squares—DC200/50 cS; circles—glycerol water 30 cS; triangles—glycerol water 46 cS. Inset: the same for water, solid curve is the fit by the Ginzburg-Landau equation with a field (see text).

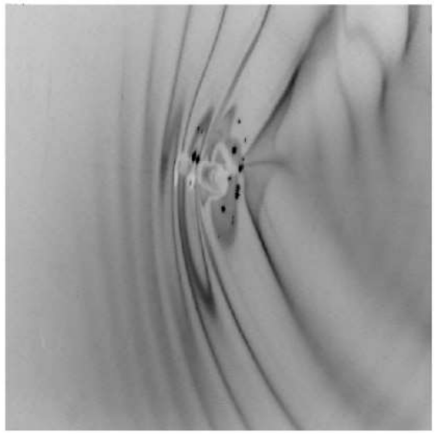


FIG. 6. Image of the surface waves on water.

Theory of collective excitations in a weakly interacting Bose gas (Bogoliubov)

- Predicts quasiparticles with dispersion $\varepsilon(p) = \sqrt{(p^2/2m)^2 + s^2p^2}$.
- Behaves as acoustic sound at long wavelengths, and as free particles at short wavelengths.
- Landau criterion fulfilled. Expect superfluidity.
- So: BEC of noninteracting bosons is not a superfluid but becomes one in the presence of interactions.
- Repulsive interactions, no matter how weak, stabilize the superfluid behavior.

Need to develop many-body techniques

Second quantization for bosons

BEC model for weakly interacting bosons

Two main contributions to the Hamiltonian of an interacting Bose gas: the single particle kinetic energy H_{KE} and the inter-particle interaction H_{int}.

Kinetic energy: Using boson creation and annihilation operators for plane-wave states in a box of size L, have

$$H_{\mathrm{KE}} = \sum_{\mathrm{k}} \frac{\hbar^2 \mathrm{k}^2}{2m} c_{\mathrm{k}}^{\dagger} c_{\mathrm{k}}.$$

For periodic boundary conditions the wavectors are $\boldsymbol{k} = \left(\frac{2\pi}{L} n_1, \frac{2\pi}{L} n_2, \frac{2\pi}{L} n_3\right)$, $n_i = 0, \pm 1, \pm 2 \ldots$
This is totally general though eventually we'll be interested in large L

Second-quantized Hamiltonian

Short-range repulsive interactions $V(\boldsymbol{r}-\boldsymbol{r}')$ are represented in first-quantized form by

$$H_{\text{int}} = \frac{g}{2}\sum_{i\neq j} V(\boldsymbol{r}_i-\boldsymbol{r}_j).$$

The interaction Hamiltonian, second-quantized, is

$$H_{\text{int}} = \frac{1}{2L^3}\sum_{\mathrm{kpq}} c^\dagger_{\mathrm{k+p}} c^\dagger_{\mathrm{q-p}} c_{\mathrm{q}} c_{\mathrm{k}} \tilde{V}(p), \quad \tilde{V}(p) = \int d^3 r e^{-ipr} V(r).$$

The full Hamiltonian then reads $H = H_{\text{KE}} + H_{\text{int}}$. From now on focus on a contact interaction parameterized by coupling g: $V(\boldsymbol{r}-\boldsymbol{r}') = g\delta(\boldsymbol{r}-\boldsymbol{r}')$. In this case

$$H_{\text{int}} = \frac{g}{2L^3}\sum_{\mathrm{kpq}} c^\dagger_{\mathrm{k+p}} c^\dagger_{\mathrm{q-p}} c_{\mathrm{q}} c_{\mathrm{k}}.$$

Approximate diagonalization

Try approximate H by a quadratic Hamiltonian.
The approach to take is suggested by recalling the ground state of the non-interacting Bose gas, in which all particles occupy the $\mathrm{k} = 0$ state.
Suppose that the occupation of this orbital remains macroscopic for small g, so that the ground state expectation value $\langle c_0^\dagger c_0 \rangle$ takes a value N_0 which is of the same order as N, the total number of particles.
In this case we can approximate the operators $c_0^\dagger$ and c_0 by the c-number $\sqrt{N_0}$ and expand H in decreasing powers of N_0. We find

$$H_{\mathrm{int}} = \frac{gN_0^2}{2L^3} + \frac{gN_0}{2L^3} \sum_{\mathrm{k} \ ne0} \left[2c_{\mathrm{k}}^\dagger c_{\mathrm{k}} + 2c_{-\mathrm{k}}^\dagger c_{-\mathrm{k}} + c_{\mathrm{k}}^\dagger c_{-\mathrm{k}}^\dagger + c_{\mathrm{k}} c_{-\mathrm{k}} \right] + \mathcal{O}\left([N_0]^0 \right)$$

- At this stage N_0 is unknown, but we can write an operator expression for it, as

$$N_0 = N - \sum_{\mathrm{k}\neq 0} c^\dagger_{\mathrm{k}} c_{\mathrm{k}}.$$

- It is also useful to introduce notation for the average number density $\rho = N/L^3$.
- Substituting for N_0 gives $2c^\dagger_{\mathrm{k}} c_{\mathrm{k}} + 2c^\dagger_{-\mathrm{k}} c_{-\mathrm{k}} \to c^\dagger_{\mathrm{k}} c_{\mathrm{k}} + c^\dagger_{-\mathrm{k}} c_{-\mathrm{k}}$, and we obtain

$$H_{\mathrm{int}} = \frac{g\rho}{2} N + \frac{g\rho}{2} \sum_{\mathrm{k}\neq 0} \left[c^\dagger_{\mathrm{k}} c_{\mathrm{k}} + c^\dagger_{-\mathrm{k}} c_{-\mathrm{k}} + c^\dagger_{\mathrm{k}} c^\dagger_{-\mathrm{k}} + c_{\mathrm{k}} c_{-\mathrm{k}} \right] + \mathcal{O}\left([N_0]^0\right)$$

- Therefore, the full Hamiltonian is

$$H = \frac{g\rho}{2} N + \frac{1}{2} \sum_{\mathrm{k}\neq 0} \left[E(k) \left(c^\dagger_{\mathrm{k}} c_{\mathrm{k}} + c^\dagger_{-\mathrm{k}} c_{-\mathrm{k}} \right) + g\rho \left(c^\dagger_{\mathrm{k}} c^\dagger_{-\mathrm{k}} + c_{\mathrm{k}} c_{-\mathrm{k}} \right) \right] + \dots$$

  with $E(k) = \frac{\hbar^2 k^2}{2m} + g\rho$.
- At this order we have a quadratic Hamiltonian, which we can diagonalize using the Bogoliubov transformation for a pair of bosons as before.

Bogoliubov quasiparticles

$$\varepsilon(k) = \left[\left(\frac{\hbar^2 k^2}{2m} + g\rho\right)^2 - (g\rho)^2\right]^{1/2}$$

At large k ($\hbar^2 k^2/2m \gg g\rho$), this reduces to the dispersion relation for free particles, but in the opposite limit it has the form

$$\varepsilon(k) \simeq \hbar v k \quad \text{with } v = \sqrt{\frac{g\rho}{m}}.$$

Obtain a critical velocity for superfluid flow, which is proportional to the interaction strength g. Thus interactions can lead to a behavior quite different from that in a non-interacting system.

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Quantum systems of many identical particles

Two approaches:

- Second quantization: 1-particle QM with multiple occupancies of eigenstates $\rightarrow$ Quantum Field
- Field quantization: Classical field with normal modes turned QM oscillators $\rightarrow$ Quantum Field

Comparison:

- different viewpoints: particles (1) vs. fields (2)
- Equivalent results
- Bosons & Fermions
- "2nd quantization" vs. "1st quantization": historic, the notation of QFT

The canonical field quantization approach

Recipe for quantizing fields:

- Determine the classical normal modes. If the equations are nonlinear, this may be difficult. Linearize the equations if necessary. The nonlinear terms can be included later as perturbations.
- Quantize the normal modes as simple harmonic oscillators.
- Classical fields become field-operators obeying free-field commutation relations
- From the distribution of the quantum states, predict thermodynamic quantities, correlation functions, etc.

Hilbert space for N identical particles (B or F)

Fock space: $F_N = V_0 \oplus V_1 \oplus V_2 \oplus \ldots = \oplus_{n=0}^{\infty} SV^{\otimes n}$

V_0 vacuum, V_1 one-particle states, V_2 two-particle states (symmetric for B, antisymmetric for F), etc

Hamiltonian in F_N

$$H = -\sum_{i=1\ldots N} -\frac{\hbar^2}{2m}\nabla_i^2 + V(\boldsymbol{r}_1 \ldots \boldsymbol{r}_N)$$

Eigenfunctions $H\psi_n(\boldsymbol{r}_1 \ldots \boldsymbol{r}_N) = E_n\psi_n(\boldsymbol{r}_1 \ldots \boldsymbol{r}_N)$ symmetric for B, antisymmetric for F

An equivalent quantum field picture (justify later)

Introduce field operators:

$$\psi(\mathrm{r}) = \sum_i \varphi_i(\mathrm{r}) c_i, \quad \psi^\dagger(\mathrm{r}) = \sum_i \varphi_i^*(\mathrm{r}) c_i^\dagger.$$

$\langle \varphi_i | \varphi_j \rangle = \delta_{ij}$, $[c_i, c_j^\dagger]_\pm = \delta_{ij}$.
The operator $\psi(\boldsymbol{r})$ annihilates particle at $\boldsymbol{r}$, $\psi^\dagger(\boldsymbol{r})$ creates particle at $\boldsymbol{r}$

$$[\psi(\boldsymbol{r}), \psi^\dagger(\boldsymbol{r}')]_\pm = \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}'), \quad [\psi(\boldsymbol{r}), \psi(\boldsymbol{r}')]_\pm = 0$$

notation: $[A, B]_\pm = AB \pm BA$

An equivalent quantum field picture (justify later)

One-particle operators:

$$A = \sum_{i=1}^{N} A(\mathrm{r}_i) \; \rightarrow A = \sum_{\mathrm{pq}} A_{\mathrm{pq}} c_{\mathrm{p}}^{\dagger} c_{\mathrm{q}}$$

with $A_{\mathrm{pq}} = \int d^3 r \varphi_{\mathrm{p}}^{*}(\mathrm{r}) A \varphi_{\mathrm{q}}(\mathrm{r})$. Here $A(\boldsymbol{r})$, say, a 1-particle kinetic or potential energy operator:

$$A(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\nabla^2, \quad A(\boldsymbol{r}) = U(\mathrm{r}), \text{ etc}$$

Two(three)-particle operators are constructed in a similar manner. The field operators appear to be basis-dependent. We'll show later that they are not.

An equivalent quantum field picture (justify later)

1-particle/many-particle correspondence:

$$\sum_i f(\boldsymbol{r}_i) \to \int d^3r \psi^\dagger(\boldsymbol{r}) f(\boldsymbol{r}) \psi(\boldsymbol{r})$$

2-particle/many-particle correspondence:

$$\sum_{ij} g(\boldsymbol{r}_i, \boldsymbol{r}_j) \to \frac{1}{2} \int d^3r_1 d^3r_2 \psi^\dagger(\boldsymbol{r}_1) \psi^\dagger(\boldsymbol{r}_2) g(\boldsymbol{r}_1, \boldsymbol{r}_2) \psi(\boldsymbol{r}_2) \psi(\boldsymbol{r}_1)$$

and so on. For $V(\boldsymbol{r}_1 ... \boldsymbol{r}_N) = \sum_{ij} V(\boldsymbol{r}_i, \boldsymbol{r}_j)$ arrive at

$$H = \int d^3r \psi^\dagger(\boldsymbol{r}) \frac{p^2}{2m} \psi(\boldsymbol{r}) + \frac{1}{2} \int d^3r d^3r' \psi^\dagger(\boldsymbol{r}) \psi^\dagger(\boldsymbol{r}') V(\boldsymbol{r}, \boldsymbol{r}') \psi(\boldsymbol{r}') \psi(\boldsymbol{r})$$

Properties:

- Define particle # operator $N = \int d^3 \psi^\dagger(\boldsymbol{r})\psi(\boldsymbol{r})$;
  N obeys:

  $$[N, H] = 0,\ [\psi(\boldsymbol{r}), N] = \psi(\boldsymbol{r})],\ [\psi^\dagger(\boldsymbol{r}), N] = -\psi^\dagger(\boldsymbol{r})$$

  same for B and F!
  Interpretation: action of ψ ($\psi^\dagger$) on eigenstate of N is to decrease (increase) eigenvalue by 1
- The vacuum state $\psi(\boldsymbol{r})|0\rangle = 0$ (for any $\boldsymbol{r}$) where $|0\rangle$ is a nonzero vector and 0 is the null vector
- Eigenstates of N: $\psi^\dagger(\boldsymbol{r}_1)\psi^\dagger(\boldsymbol{r}_2)...\psi^\dagger(\boldsymbol{r}_m)|0\rangle$ with the eigenvalue $N = m$

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Topics in Bose-Einstein condensation

- Is superfluid 4He a BEC?
- Mixing creation and annihilation operators: quasiparticles from Bogoliubov transform
- BEC: microscopic approach, order parameter, long-range order, broken U(1) symmetry
- Phase rigidity and dissipationless flow
- Quantized vorticity; vortices in a rotating superfluid; vortex rings
- Superfuid hydrodynamics at $T > 0$: the two-fluid model, the fountain effect (HW1)
- Excitations in 4He: phonons and rotons (HW1)
- Time-dependent Gross-Pitaevskii eqn: collective modes, modulus and phase (HW1)

Weakly interacting Bose gas (recap)

Recall the 2nd quantized Hamiltonian

$$H = \int d^3x \left[\psi^\dagger \left(-\frac{\hbar^2}{2m} \nabla^2 - \mu \right) \psi + \frac{g}{2} \psi^\dagger \psi^\dagger \psi \psi \right]$$

with bosonic quantum fields $\psi(x)$, $\psi^\dagger(x)$ describing identical bosons, obeying $[\psi(x), \psi^\dagger(y)] = \delta^3(x-y)$, $[\psi(x), \psi(y)] = 0 = [\psi^\dagger(x), \psi^\dagger(y)]$. Contact interaction $V(x-y) = g\delta^3(x-y)$ corresponds to a pseudopotential for low-energy scattering with $g = \frac{4\pi a}{m}$ with a the s-wave scattering length.

H is invariant under global gauge transformation, $\psi \to e^{i\alpha}\psi$, $\psi^\dagger \to e^{-i\alpha}\psi^\dagger$ (a continuous symmetry). The U(1) symmetry which is spontaneously broken in the ground state.

Incorporate condensate in 2nd quantization

* We wrote fields $\psi(x)$, $\psi^\dagger(x)$ in terms of an orthornomal set of single-particle basis states, $\psi(x) = \sum_j u_j(x) a_j$, $\psi^\dagger(x) = \sum_j u_j^*(x) a_j^\dagger$, $\langle u_i | u_j \rangle = \delta_{ij}$ (for the time being, $u_i(x) = \frac{1}{\sqrt{V}} e^{i\boldsymbol{kx}}$).

* We found it useful to decompose fields into the condensate and out-of-condensate parts, $\psi = \psi_0 + \varphi$, where $\psi_0 = u_0(x) a_0$, $\varphi = \sum_{j \neq 0} u_j(x) a_j$.

* Since $a_0 \sim \sqrt{N}$, where N is the total particle number, we can replace a_0 by a c-number.

The meaning of c-numbers:

* A displaced state for the simple harmonic oscillator described by $a_0 = \tilde{a}_0 + \sqrt{V}\varphi$, $a_0^\dagger = \tilde{a}_0^\dagger + \sqrt{V}\varphi^*$,
* For particle number N the shift parameter is $\varphi = \sqrt{N/V}$,
* Shifted state $\tilde{a}_0|\varphi\rangle = 0$ is $|\varphi\rangle = e^{\sqrt{V}(\varphi a_0^\dagger - \varphi^* a_0)}|0\rangle = e^{-\frac{V}{2}|\varphi|^2} \sum_{n=0}^{\infty} \frac{(\sqrt{V}\varphi)^n}{\sqrt{n!}}|n\rangle$.
* Poisson distribution:
the mean = the variance = $V|\varphi|^2$.
* NB: fluctuations growing with system size! (after interactions are added, number fluctuations persist although are somewhat reduced)

Plugging $\psi = \psi_0 + \varphi$ in H and expanding in φ:

$$H = \int d^3x \left[\psi_0^* H_0 \psi_0 + \frac{g}{2}(\psi_0^*\psi_0)^2 \right.$$
$$\left. + \varphi^\dagger H_0 \varphi + g|\psi_0|^2 \varphi^\dagger \varphi + \frac{g}{2}\varphi^\dagger\varphi^\dagger\psi_0\psi_0 + \frac{g}{2}\psi_0^*\psi_0^*\varphi\varphi \right]$$

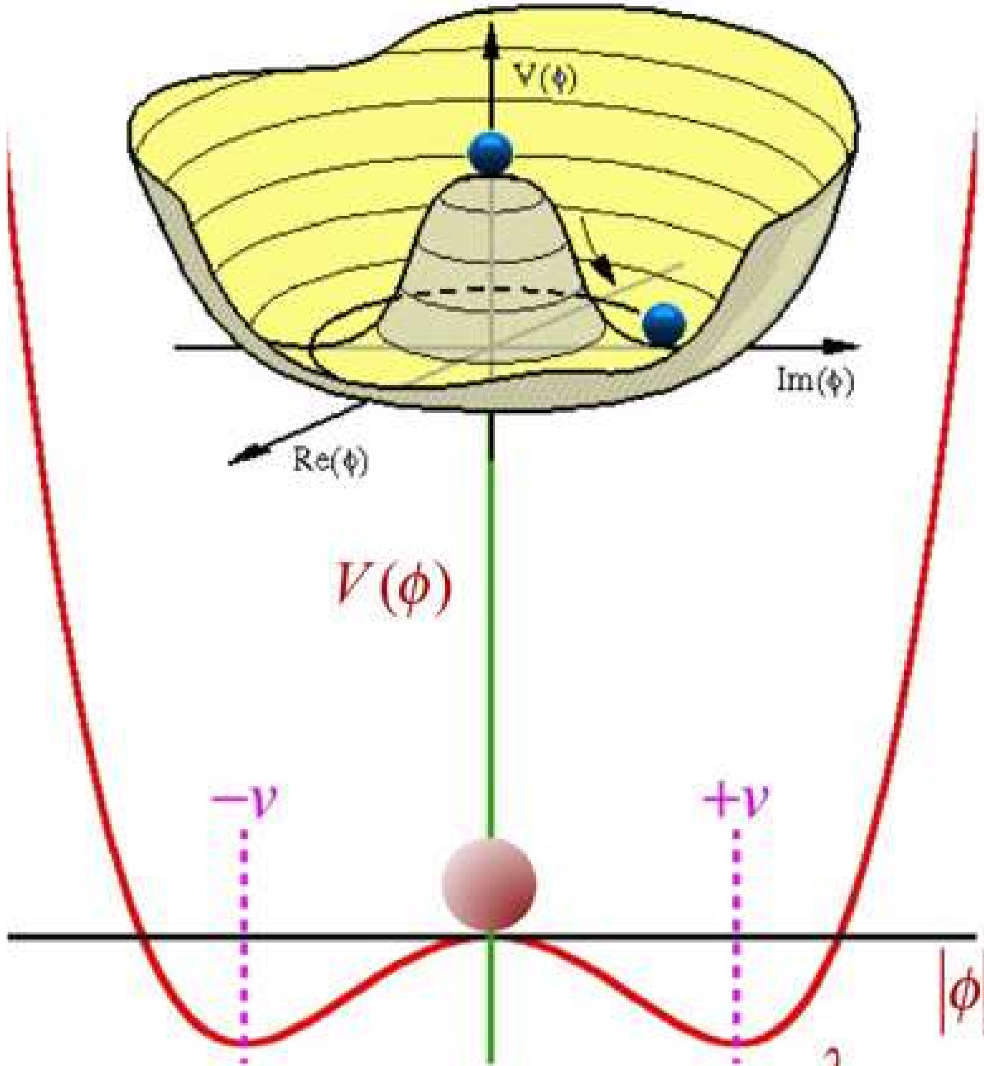
where $H_0 = -\frac{\hbar^2}{2m}\nabla^2 - \mu$. NB: first-order terms vanish, whereas cubic and quartic terms are small. Expression in line 1 is a c-number (an innocent energy shift, ignore it for now), the rest is a quadratic Hamiltonian which can be diagonalized by a Bogoliubov transformation (see last lecture), giving qusiparticles with a new dispersion relation $\varepsilon(k) = \sqrt{(\varepsilon_0(k) + gn_0)^2 - g^2 n_0^2}$, $n_0 = |\psi_0|^2$.

is understood most easily from the c-number part. Adding boundary terms, rewrite H:

$$H[\psi_0] = \int d^3x \left[\frac{\hbar^2}{2m}\nabla\psi_0^*\nabla\psi_0 + V(\psi_0)\right],$$

$$V(\psi_0) = -\mu|\psi_0|^2 + \frac{g}{2}|\psi_0|^4$$

For $\mu > 0$, $V(\psi_0)$ is a mexican hat potential invariant under the gauge transformation $\psi_0 \to e^{i\alpha}\psi_0$. The ground states form a degenerate manifold parameterized as $\psi_0 = \sqrt{\mu/g}e^{i\theta}$. Gauge transformation is a symmetry of $H[\psi_0]$, but maps different ground states onto one another.



$$V(\phi) = \tfrac{1}{2}\mu^2\phi^\dagger\phi + \tfrac{1}{4}\lambda(\phi^\dagger\phi)^2$$

Groundstate at $|\phi_0| = \sqrt{\dfrac{-\mu^2}{\lambda}} \equiv \upsilon$

$$|\phi| = \sqrt{\phi^\dagger\phi} = \sqrt{\phi^{+\dagger}\phi^+ + \phi^{0\dagger}\phi^0}$$

$$V(\phi_0) = -\frac{\lambda}{4}\upsilon^4$$

Discussion:

- QM uncertainty relation for particle $\#$ and phase, $\delta N \delta\theta \geq \frac{1}{2}$, implies that a well defined θ must go together with strong fluctuations in particle number (consistent with the behavior of the variance $V|\psi_0|^2$, see above)
- Strong particle $\#$ fluctuations arise microscopically because the Bogoliubov hamiltonian is particle-nonconserving
- Phase rigidity: the degeneracy in θ global rather than local. Long-range correlations maintained microscopically by the gradient term $-\psi_0^* \nabla^2 \psi_0$
- Intuition: phase rigidity arises due to zero-point particle $\#$ fluctuations

Analyze BEC with a fixed particle number (say N_0 particles in a trap). System state is an eigenstate of the particle # operator, $N|\varphi_0\rangle = N_0|\varphi_0\rangle$. From the comm relation $[N, \psi(x)] = \psi(x)$ can show

$$e^{i\alpha N}\psi(x)e^{-i\alpha N} = e^{i\alpha}\psi(x)$$

using $e^B A e^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \ldots$ (Baker-Hausdorff).
Combined with $e^{i\alpha N}|\varphi_0\rangle = e^{i\alpha N_0}|\varphi_0\rangle$ it gives

$$\langle\varphi_0|\psi(x)|\varphi_0\rangle = \langle\varphi_0|e^{i\alpha N}\psi(x)e^{-i\alpha N}|\varphi_0\rangle = e^{i\alpha}\langle\varphi_0|\psi(x)|\varphi_0\rangle$$

Therefore $\langle\varphi_0|\psi(x)|\varphi_0\rangle = 0$. Meaning: global phase undefined (and, on general QM grounds, not measurable) despite local phase rigidity and long-range phase correlations.

Symmetry breaking summary

- H invariant under symmetry operation but ground state is not: symmetry spontaneously broken
- Degenerate g.s. family: infinite degeneracy for continuous symmetry; the actual g.s. is one of those, ergodicty failure!
- Nontrivial topology (Mexican hat!)
- Low-energy states $|\varphi(x)\rangle$, slowly varying $\varphi(x)$
- Excitation energy vanishing for slowly varying $\varphi(x)$ — a manfestation of a general theorem
- Goldstone-Nambu modes (discuss later)
- Hydrodynamics (discuss later)