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Matrix manipulations

Group Exercise

i) Find eigenvalues of $A = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}$, (λ_1, λ_2)

ii) Find corresponding eigenvectors $(\vec{v}_1, \vec{v}_2)$

iii) Define $U \equiv (\vec{v}_1 | \vec{v}_2)$

↳ compute U^{-1}

iv) Compute $U \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} U^{-1}$

v) Compute $U^{-1} A^2 U$, $U^{-1} A^3 U$

A: i) $\begin{vmatrix} -\lambda & p \\ q & -\lambda \end{vmatrix} = \lambda^2 - pq = 0 \Rightarrow \lambda_1 = \sqrt{pq}, \lambda_2 = -\sqrt{pq}$

ii) $\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix} = \pm\sqrt{pq} \begin{pmatrix} x_\pm \\ y_\pm \end{pmatrix} \Rightarrow p y_\pm = \pm\sqrt{pq}\, x_\pm$

or $y_\pm = \pm\sqrt{\frac{q}{p}}\, x_\pm$

Normalize: $|x_\pm|^2 (1 + \frac{q}{p}) = 1 \Rightarrow |x_\pm| = \sqrt{\frac{p}{p+q}}$

Set overall phase $= 0 \Rightarrow \vec{v}_\pm = \frac{1}{\sqrt{p+q}} \begin{pmatrix} \sqrt{p} \\ \pm\sqrt{q} \end{pmatrix}$

iii) $$U = \frac{1}{\sqrt{p+q}} \begin{pmatrix} \sqrt{p} & \sqrt{p} \\ \sqrt{q} & -\sqrt{q} \end{pmatrix}$$

$$\det(U) = \frac{-\sqrt{pq} - \sqrt{pq}}{p+q} = \frac{-2\sqrt{pq}}{p+q}$$

$$\Rightarrow U^{-1} = \frac{-p-q}{2\sqrt{pq}} \cdot \frac{1}{\sqrt{p+q}} \begin{pmatrix} -\sqrt{q} & -\sqrt{p} \\ -\sqrt{q} & \sqrt{p} \end{pmatrix} = \frac{\sqrt{p+q}}{2} \begin{pmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{q}} \\ \frac{1}{\sqrt{p}} & -\frac{1}{\sqrt{q}} \end{pmatrix}$$

iv) $$\frac{1}{\sqrt{p+q}} \begin{pmatrix} \sqrt{p} & \sqrt{p} \\ \sqrt{q} & -\sqrt{q} \end{pmatrix} \begin{pmatrix} \sqrt{pq} & 0 \\ 0 & -\sqrt{pq} \end{pmatrix} \frac{\sqrt{p+q}}{2} \begin{pmatrix} \frac{1}{\sqrt{p}} & \frac{1}{\sqrt{q}} \\ \frac{1}{\sqrt{p}} & -\frac{1}{\sqrt{q}} \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \sqrt{p} & \sqrt{p} \\ \sqrt{q} & -\sqrt{q} \end{pmatrix} \begin{pmatrix} \sqrt{q} & \sqrt{p} \\ -\sqrt{q} & \sqrt{p} \end{pmatrix} = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}.$$

$$\text{So } A = U \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{\text{call this } \Sigma_\lambda} U^{-1} \ !$$

v) • $U^{-1} A^2 U = U^{-1} (U \Sigma_\lambda U^{-1})^2 U = U^{-1} (U \Sigma_\lambda^2 U^{-1}) U$

$$= \Sigma_\lambda^2 = \begin{pmatrix} pq & 0 \\ 0 & pq \end{pmatrix}$$

• $U^{-1} A^3 U = \Sigma_\lambda^3 = \begin{pmatrix} (pq)^{3/2} & 0 \\ 0 & -(pq)^{3/2} \end{pmatrix}$

So we see that computing powers of A can be easily done by using the diagonalized form:

$$A^n = U \Sigma_\lambda^n U^{-1}$$

What does this mean for matrix exp?

$$\exp(A) \equiv \sum_{n=0}^{\infty} \frac{A^n}{n!} = \sum_{n=0}^{\infty} \frac{U \Sigma_\lambda^n U^{-1}}{n!}$$

$$= U \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} \sqrt{pq}^{\,n} & 0 \\ 0 & (-\sqrt{pq})^n \end{pmatrix} \cdot U^{-1}$$

$$= U \begin{pmatrix} e^{\sqrt{pq}} & 0 \\ 0 & e^{-\sqrt{pq}} \end{pmatrix} U^{-1}$$

General theory

Diagonalizeable matrix $A \Leftrightarrow \exists P, P^{-1}$ s.t. $P^{-1}AP$ is diagonal

$P =$ has eigenvectors as columns

$P^{-1}AP$ has eigenvalues of A on diagonal

useful since diagonal matrices are easy to work with!

Manipulating Sums (apply to prob 7)

$$\sum_{n_1=0}^{\infty} C(n_1) A^{n_1} \cdot B \cdot \sum_{n_2=0}^{\infty} D(n_2) A^{n_2}$$

Count # of A's: $n_1 + n_2 = N \in \mathbb{N}$

How to get total of N A's?

→ if have k A's on left

⇒ must have $N-k$ A's on right

So we can equivalently write the sum as

$$\sum_{N=0}^{\infty} \sum_{k=0}^{N} C(k) D(N-k)\, A^k B A^{N-k}$$

Proofs by induction

E.g. want to show that, for any $N \in \mathbb{N}$

$F(N) = g(N)$ ← two different looking functions of N

Maybe easy to show $F(0) = g(0)$

$F(1) = g(1), \ldots$

But hard for large N.

But if we can show that for any $M \in \mathbb{N}$ assuming that $F(M) = g(M)$ implies that $F(M+1) = g(M+1)$, then we're done!

We know $F(0) = g(0)$ so the assumption is valid for $M = 0$ and therefore we must have $F(0+1) = g(0+1)$. But now assumption is valid for $M=1$ so $F(2) = g(2)$ and so on...