MA439: Functional Analysis
Tychonoff Spaces: Exercises 5, 6, 12, 13, 14 on p.31, Ben Mathes

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Exercise 1 (Ex 5, p.31). *In an arbitrary topological space $\mathcal{X}$, we say that a sequence (x_i) converges to x (and write $x_i \to x$) if, for every open set G containing x, the sequence (x_i) is eventually in G. Prove that $x_i \to x$ if and only if, for every subbasic open set S containing x, (x_i) is eventually in S.*

Proof. ($\Longrightarrow$) Suppose that $x_i \to x$ in $\mathcal{X}$. Let some subbasic open set S be given. Because S is subbasic open, it is open. Thus, because $x_i \to x$, (x_i) is eventually in S.

($\Longleftarrow$) Let a sequence (x_i) be given. Suppose that for every subbasic open set $S \subseteq \mathcal{X}$ containing x, (x_i) is eventually in S. And, let O_x be an open set containing x. It follows that there is some $k \in \mathbb{N}_+$ for which $x \in \cap_{i=1}^k S_i \subseteq O_x$, where each S_i is a subbasic open set containing x. From here, it is clear that there is some positive integer N for which $x_n \in S_i$ for all $i = 1, \dots, k$ whenever $n \geq N$. Thus, (x_i) is eventually in $\cap_{i=1}^k S_i \subseteq O_x$. So, $x_i \to x$. $\square$

Exercise 2 (Ex 6, p.31). *In arbitrary topological spaces, the neighborhood filter $\mathcal{F}_x$ of a point x is defined to be the collection of all subsets that contain an open set containing x, and we again define $\mathcal{F} \to x$ to mean $\mathcal{F}_x \subseteq \mathcal{F}$. Prove that $F \to x$ if and only if every subbasic open set containing x is in $\mathcal{F}$.*

Proof. ($\Longrightarrow$) Suppose that $\mathcal{F} \to x$. Let S be a subbasic open set containing x. S necessarily contains an open subset containing x, so $S \in \mathcal{F}_x \subseteq \mathcal{F}$.

($\Longleftarrow$) Consider the neighborhood filter $\mathcal{F}_x$ and some $F \in \mathcal{F}_x$. F contains an open set containing x. Thus, there is some $k \in \mathbb{N}_+$ for which $F \supseteq \cap_{i=1}^k S_i$ where S_i's are subbasic open sets. Now, because each $S_i \in \mathcal{F}$, $\cap_{i=1}^k S_i \in \mathcal{F}$ (since $\mathcal{F}$ is a filter). So, $F \in \mathcal{F}$. Thus, $\mathcal{F}_x \subseteq \mathcal{F}$, i.e., $\mathcal{F} \to x$. $\square$

Exercise 3 (Ex 12, p.31). *Assume that $S = p_k^{-1}(G)$ is a subbasic open set in a product space $\prod_i \mathcal{X}_i$. Prove that $S = p_k^{-1}(p_k(S))$, and if $p_k(E) \subseteq p_k(S)$, then $E \subseteq S$.*

Proof. When $S = p_k^{-1}(G) = \dots \times G \times \dots$, we have that $p_k(S) = p_k(p_k^{-1}(G)) = G$. So $S = p_k^{-1}(G) = p_k^{-1}(p_k(S))$. Next, assume that $p_k(E) \subseteq p_k(S)$, then because p_k is a projection, $p_k^{-1}(p_k(E)) \subseteq p_k^{-1}(p_k(S)) = S$. Also, because p_k is a projection, $E \subseteq p_k^{-1}(p_k(E))$. So $E \subseteq S$. $\square$

Exercise 4 (Ex 13, p.31). *Prove that a topological space is compact if and only if every open covering by basic open sets has a finite subcover.*

Proof. ($\Longrightarrow$) Let $(\mathcal{X}, \tau)$ be a topological space. Assume that $\mathcal{X}$ is compact, then every open covering has a finite subcover. In particular, every open covering with basic open sets has a finite subcover.

($\Longleftarrow$) Let $(\mathcal{X}, \tau)$ be a topological space. Let a base $\mathcal{B}$ and an open covering $\mathcal{C}$ be given. For each $x \in \mathcal{X}$, pick $O_x \subseteq \mathcal{C}$ such that $x \in O_x$ and pick $B_x \in \mathcal{B}$ for which $x \in B_x \subseteq O_x$. Assume that every open covering with basic open sets has a finite subcover, then the collection $\mathcal{C}_{\mathcal{B}} = \{B_x : x \in \mathcal{X}\}$ has a finite subcover $\{B_{x_1}, \dots, B_{x_N}\}$. Consequently the collection $\{O_{x_1}, \dots, O_{x_N}\}$ is a finite subcover in $\mathcal{C}$. Thus, $\mathcal{C}$ has a finite subcover. $\square$

Exercise 5 (Ex 14, p.31, Alexander's Subbase Theorem)**.** *Prove that a topological space is compact if and only if every open covering by subbasic open sets has a finite subcover. (This requires the axiom of choice.)*

Proof. ($\implies$) Let $(\mathcal{X}, \tau)$ be a topological space. Assume that $\mathcal{X}$ is compact, then every open covering has a finite subcover. In particular, every open covering with subbasic open sets has a finite subcover.

($\impliedby$) Let $(\mathcal{X}, \tau)$ be a topological space. Let a subbase $\mathcal{B}$ and an open covering $\mathcal{C}$ be given. Assume that $\mathcal{X}$ is not compact yet every subbasic cover from $\mathcal{B}$ has a finite subcover. By the axiom of choice, choose an open cover $\mathcal{C}$ without a finite subcover is that **maximal**. Observe that $\mathcal{B} \cap \mathcal{C}$ cannot cover $\mathcal{X}$, since otherwise there would be a finite subcover coming from $\mathcal{B}$. With this, pick an $x \in \mathcal{X} \setminus \bigcup(\mathcal{B} \cap \mathcal{C})$. Since $\mathcal{C}$ is a cover, choose an $O_x \in \mathcal{C}$ such that $x \in O_x$. Since $\mathcal{B}$ is a subbase, there are $\{B_1, \ldots, B_k\} \subseteq \mathcal{B}$ for which $x \in \cap_{i=1}^{k} B_i \subseteq O_x$. Now, by the choice of x, $B_i \notin \mathcal{B} \cap \mathcal{C}$ for all $i = 1, 2, \ldots, k$.

Since $\mathcal{C}$ is the maximal open cover for which there is no finite subcover, the collection $\mathcal{C} \cup \{B_i\}$ must have a finite subcover for each $i = 1, 2, \ldots, k$. Thus,l let this be $\{C_1^i, C_2^i, \ldots, C_{n^i}^i\} \cup \{B_i\}$. It follows that $\{C_1^i, C_2^i, \ldots, C_{n^i}^i\}_{i=1}^{n} \cup \{\cup_{i=1}^{n} B_i\}$ is a finite open covering of $\mathcal{X}$. As a result, $\{C_1^i, C_2^i, \ldots, C_{n^i}^i\}_{i=1}^{n} \cup \{U\}$ is also a finite cover for $\mathcal{X}$. But notice that $U \in \mathcal{C}$, so this finite cover is made up entirely of elements of $\mathcal{C}$. This is a contradiction. So, $\mathcal{X}$ must be compact. [1] $\square$

[1] Source: James Keesling, Dept. of Mathematics, Univ. of Florida