Today's plan

Lectures

- What is a symmetry?
- A crash course in group theory

Activities

- GeoGebra
- Brainstorming
- Breakout rooms

Note: I will be writing on top of these slides. I'll send you a link to the blank slides for now and upload the pdf with my written notes later today.

How to interact during the lectures

- This class is a safe (virtual) place. Questions are always welcome, no matter how trivial you may think they are.
- You can ask questions at any time during the lecture. You have a few options:
  - "Raise your hand" through Zoom and ask in person
  - Use the Zoom chat
  - Ask on sli.do (#W613)
- I will often ask you questions during the lectures. I **do not know** how well this is going to work online, but I'll still do it.

What is a symmetry?

a transformation of "A" that leaves "B" invariant

<u>ex:</u> $B =$ spiral $\quad A = \mathbb{R}^2$ (see geogebra)

<u>ex:</u> $S = \int \mathcal{L}\, dt \quad A =$ time $\quad B = S$

↳ Lagrangian

- "doing nothing" should be a symmetry (identity)
- undo symmetries (invertible)
- compose symmetries

Definition (group)

A group is a set G together with an operation $* : G \times G \to G$ satisfying the following properties:

- there is a special element $e \in G$, called the *identity*, such that

$$g * e = e * g = g, \quad \forall g \in G$$

- each element of G has an *inverse*, that is for each $g \in G$ there is an element $g^{-1} \in G$ such that

$$g^{-1} * g = g * g^{-1} = e$$

- the operation $*$ is *associative*, that is

$$a * (b * c) = (a * b) * c, \quad \forall a, b, c \in G.$$

Additionally, we say that the group G is *abelian* or *commutative* if

$$a * b = b * a, \quad \forall a, b \in G.$$

composition of functions

$f \circ (g \circ h) = (f \circ g) \circ h$

Notation:

- we often use ab for $a * b$
- technically the group is $(G, *)$

→ when there may be confusion

- $(\mathbb{Z},+)$, $(\mathbb{R},+)$, $(\mathbb{C},+)$ are abelian groups

  $e=0$ $\quad a*b=a+b$ $\quad a^{-1}=-a$

- $(\mathbb{R}\setminus\{0\},\cdot)$ $\quad e=1$ $\quad a*b=ab$ $\quad a^{-1}=\frac{1}{a}$

- $n\times n$ matrices (invertible) $\quad GL(n,\mathbb{R})=\{A\in M_n(\mathbb{R})\mid \det A\neq 0\}$

  with matrix multiplication $\quad e=\mathbb{1}_n$ (identity matrix)

- Circle group $\quad S^1=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2=1\}$

  $|z|=1$

  identify $(x,y)\in S^1$ with a complex number $z=x+iy$

  then we use complex number multiplication

  $\Rightarrow$ works because $|zw|=|z||w|=1$

Definition (subgroup)

Let G be a group with operation $*$. A subgroup of G is a subset $H \subseteq G$ that contains the identity element and is closed under the operation $*$ and under inversion, that is

- $e \in H$
- $a * b \in H$ for all $a, b \in H$
- $a \in H \implies a^{-1} \in H$

The notation $H \leq G$ is commonly used to indicate that H is a subgroup of G.

equiv. of subspace of vector space

vector spaces ARE groups!

Exercise

Prove the following consequences of the definition of a group:

1. the identity element is unique (if two elements satisfy the identity property, they are necessarily equal)
2. for each $g \in G$ the inverse g^{-1} is unique
3. $(g^{-1})^{-1} = g$
4. $(gh)^{-1} = h^{-1}g^{-1}$

Homomorphisms and isomorphisms

Definition (group homomorphism)

A group homomorphism is a map $\varphi : G \to H$ between two groups G and H such that

$$\varphi(a *_G b) = \varphi(a) *_H \varphi(b), \quad \forall a, b \in G.$$

Exercise

Prove that if $\varphi : G \to H$ is a group homomorphism, then

1. $\varphi(e_G) = e_H$ (Hint: look at $\varphi(e_G e_G)$)
2. $\varphi(g^{-1}) = \varphi(g)^{-1}$ for all $g \in G$.

↙ equiv to linear maps

→ multiply before or <u>after</u>

φ preserves group structure

Definition (kernel)

The kernel of a group homomorphism $\varphi : G \to H$ is the set

$$\ker\varphi = \{g \in G \mid \varphi(g) = e_H\}$$

of all the elements of G that are sent to the identity in H.

Note: $\varphi(e_G) = e_H$

$e_G \in \ker\varphi$

Proposition

A group homomorphism $\varphi : G \to H$ is injective if and only if its kernel is trivial, that is

$$\ker\varphi = \{e_G\}.$$

same as vect. spaces

Proof: $\Rightarrow$ if φ injective, at most one thing can be sent to e_H

$\ker\varphi = \{e_G\}$

$\Leftarrow$ let $a,b \in G$ with $\varphi(a) = \varphi(b) \Rightarrow e_H = \varphi(a)^{-1}\varphi(b)$

$\Rightarrow e_H = \varphi(a^{-1})\varphi(b) = \varphi(a^{-1}b) \Rightarrow a^{-1}b \in \ker\varphi \Rightarrow a^{-1}b = e_H \Rightarrow a = b$

Definition (isomorphism)

Two groups G and H are *isomorphic* (denoted by $G \cong H$) if there exists an invertible group homomorphism $\varphi : G \to H$. Such a map is called an *isomorphism* between G and H.

Exercise

Prove that $\mathbb{Z}_2$ is isomorphic to the subgroup $\{\mathbb{I}_n, -\mathbb{I}_n\} \leq \mathrm{GL}(n, \mathbb{C})$. While you are at it, prove that the latter is indeed a subgroup!

Exercise

I'll do you one better: prove that *any* group with only two elements is isomorphic to $\mathbb{Z}_2$.

← same as v. space version

Consider $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, \cdot)$

subgroup of $(\mathbb{R}\setminus\{0\}, \cdot)$

$\exp : x \in \mathbb{R} \to e^x \in \mathbb{R}_{>0}$

- invertible
- $\exp(x+y) = \exp(x)\exp(y)$

→ group homomorphism

$(\mathbb{R}, +) \cong (\mathbb{R}_{>0}, \cdot)$