Assignment 4; MA353; S19

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1 Problems

Problem 1

Suppose that $T \in \mathfrak{L}(\boldsymbol{V})$, and

$$Range\,(T) = \boldsymbol{W} + \boldsymbol{Z}\ ,$$

where $\boldsymbol{W}$ and $\boldsymbol{Z}$ are subspaces of $\boldsymbol{V}$. Argue that

$$\boldsymbol{V} = T^{-1}[\boldsymbol{W}] + T^{-1}[\boldsymbol{Z}]\ .$$

Problem 2

1. Suppose that $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ are non-trivial vector spaces, and for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \boldsymbol{V}_j \xrightarrow{\text{linear}} \boldsymbol{V}_i\ .$$

Let $\mathcal{L}$ be the block-matrix function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3 \longrightarrow \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3\ .$$

Suppose that it turns out that $\boldsymbol{V}_2 = \boldsymbol{V}_{2.1} \times \boldsymbol{V}_{2.2}$. Then $\mathcal{L}$ may be considered as a linear function on

$$\boldsymbol{V}_1 \times \boldsymbol{V}_{2.1} \times \boldsymbol{V}_{2.2} \times \boldsymbol{V}_3\ .$$

What is the corresponding block-matrix form of $\mathcal{L}$ and how does it relate to $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$? Justify your claims.

2. Suppose that $\boldsymbol{W}_1 \uplus \boldsymbol{W}_2 \uplus \boldsymbol{W}_3 = \boldsymbol{V}$ and the $\boldsymbol{W}_i$'s are non-trivial subspaces of $\boldsymbol{V}$. Suppose that for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \boldsymbol{W}_j \xrightarrow{\text{linear}} \boldsymbol{W}_i \ .$$

Let $\mathcal{L}$ be the block-matrix function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\uplus} : \boldsymbol{V} \longrightarrow \boldsymbol{V} \ .$$

Suppose that it turns out that $\boldsymbol{W}_1 = \boldsymbol{W}_{1.1} \uplus \boldsymbol{W}_{1.2}$. Then

$$\boldsymbol{V} = \boldsymbol{W}_{1.1} \uplus \boldsymbol{W}_{1.2} \uplus \boldsymbol{W}_2 \uplus \boldsymbol{W}_3 \ .$$

What is the block-matrix form of $\mathcal{L}$ with respect to this direct sum decomposition and how does it relate to $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\uplus}$? Justify your claims.

2 Polynomial Theory Preliminaries

The Fundamental Theorem of Algebra states that any polynomial of positive degree with complex coefficients can be expressed in exactly one way as a product

$$a\,(x-\lambda_1)^{\mu_1} \cdots (x-\lambda_k)^{\mu_k}$$

of positive powers of distinct monic linear polynomials and a non-zero complex number a. Of course here $\lambda_1, \ldots, \lambda_k$ are all of the distinct roots of the polynomial in question, and $\mu_1, \ldots, \mu_k$ are the respective **multiplicities** of the roots.

Two non-zero polynomials are said to be **relatively prime** if they have no common roots.

Note that a polynomial $a\,(x-\lambda_1)^{\mu_1} \cdots (x-\lambda_k)^{\mu_k}$ can also be expressed in a slightly redundant way as

$$a\,(x-\lambda_1)^{\mu_1} \cdots (x-\lambda_k)^{\mu_k}\,(x-\gamma_1)^0 \cdots (x-\gamma_m)^0 \ .$$

The advantage of this maneuver is this: when we have polynomials

$$f(x) = a\,(x-\lambda_1)^{\mu_1} \cdots (x-\lambda_k)^{\mu_k}$$

and

$$g(x) = b\,(x-\gamma_1)^{\nu_1}\cdots(x-\gamma_m)^{\nu_m}$$

we can express them both in a common form

$$c\,(x-\lambda_1)^{\rho_1}\cdots(x-\lambda_k)^{\rho_k}\,(x-\gamma_1)^{\rho_{k+1}}\cdots(x-\gamma_m)^{\rho_{k+m}}\quad,$$

with non-negative ρ's. (Of course the ρ's that yield f are usually not the same as those that yield g!)

Test Your Comprehension 2.1

Argue that a non-zero polynomial $a\,(x-\lambda_1)^{\mu_1}\cdots(x-\lambda_k)^{\mu_k}$ is a polynomial divisor of a non-zero polynomial $b\,(x-\lambda_1)^{\nu_1}\cdots(x-\lambda_k)^{\nu_k}$ exactly when

$$\mu_i \le \nu_i \quad \text{for all} \quad i.$$

Given non-zero polynomials

$$f(x) = a\,(x-\lambda_1)^{\mu_1}\cdots(x-\lambda_k)^{\mu_k} \quad \text{and} \quad g(x) = b\,(x-\lambda_1)^{\nu_1}\cdots(x-\lambda_k)^{\nu_k}\,,$$

the GCD(f, g) is the monic polynomial

$$(x-\lambda_1)^{\min(\mu_1,\nu_1)}\cdots(x-\lambda_k)^{\min(\mu_k,\nu_k)}\quad.$$

Test Your Comprehension 2.2 ☞ **GCD and relative primeness**

Argue that two non-zero polynomials are relatively prime exactly when their GCD is $\mathbb{1}$.

Similarly, the LCM(f, g) is the monic polynomial

$$(x-\lambda_1)^{\max(\mu_1,\nu_1)}\cdots(x-\lambda_k)^{\max(\mu_k,\nu_k)}\quad.$$

Test Your Comprehension 2.3

1. Argue that GCD(f, g) is a monic polynomial divisor of f and g of the highest degree, and every other polynomial divisor of f and g divides GCD(f, g).
2. Argue that LCM(f, g) is a monic polynomial of the smallest degree that is divisible by both f and g, and every other polynomial divisible by f and g is divisible by LCM(f, g).

Test Your Comprehension 2.4

1. By TYC 2.3, for any non-zero polynomials f and g,

$$f = p \cdot \mathrm{GCD}(f, g) \quad \text{and} \quad g = q \cdot \mathrm{GCD}(f, g)\ ,$$

for some non-zero polynomials p and q. Argue that p and q are relatively prime.

2. By TYC 2.3, for any non-zero polynomials f and g,

$$\mathrm{LCM}(f, g) = \hat{p} \cdot f \quad \text{and} \quad \mathrm{LCM}(f, g) = \hat{q} \cdot g\ ,$$

for some non-zero polynomials $\hat{p}$ and $\hat{q}$. Argue that $\hat{p}$ and $\hat{q}$ are relatively prime.

3 More Problems

Problem 3

Suppose that relatively prime polynomials p_1 and p_2 have degrees 6 and 11 respectively. Consider the function

$$\Psi : \mathbb{P}_{10} \times \mathbb{P}_5 \longrightarrow \mathbb{P}_{16}$$

defined by

$$\Psi\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) := f \cdot p_1 - g \cdot p_2\ .$$

Verify each of the following claims.

1. Ψ is a linear function.
2. Ψ is injective.
3. Ψ is surjective.
4. There exist polynomials q_1 and q_2 such that

$$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}\ ,$$

where $\mathbb{1}$ is the constantly 1 polynomial.

Hint: 2. Show that Ψ has a trivial kernel. This is the tricky part. Argue that if $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ were a non-zero element in the kernel then neither f nor g would be zero, and all the roots of p_2 would have to be roots of f with equal or greater multiplicities. Argue that the degree of f does not allow for that.
3. Rank-Nullity.

Problem 4

1. What would you do to prove the last claim of problem 3 in a general case of non-zero relatively prime polynomials p_1 and p_2? You do not need to carry out the proof, but you DO need to set it all up along the lines of Problem 3. Make sure you cover all of the cases!
2. Prove that the following claims are equivalent:
   (a) Non-zero polynomials p_1 and p_2 are relatively prime.
   (b) There exist polynomials q_1 and q_2 such that
   $$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1} \ .$$
3. Argue that for any non-zero polynomials f and g there exist polynomials q_1 and q_2 such that
   $$q_1 \cdot f + q_2 \cdot g = \mathrm{GCD}(f, g) \ .$$
4. Argue that the following claims are equivalent for any non-zero polynomials f, g and h.
   (a) There exist polynomials q_1 and q_2 such that
   $$q_1 \cdot f + q_2 \cdot g = h \ .$$
   (b) h is a polynomial multiple of $\mathrm{GCD}(f, g)$.

Definition 3.1

A subspace $\boldsymbol{W}$ of the vector space $\mathbb{P}$ of all complex polynomials is said to be **an ideal** in $\mathbb{P}$, if $\boldsymbol{W}$ has an "absorption" property with respect to multiplication:

$$\left.\begin{array}{r} f \in \mathbb{P} \\ g \in \boldsymbol{W} \end{array}\right\} \Longrightarrow f \cdot g \in \boldsymbol{W} \ .$$

For example,

$$\{ \ p \in \mathbb{P} \mid p(0) = 0 \ \}$$

is an ideal in $\mathbb{P}$, as are $\mathbb{P}$, $\{\mathbb{0}\}$, and

$$\left\{ \ p \in \mathbb{P} \mid p \text{ is a polynomial multiple of } 3 + 7x - 8x^2 + x^7 \ \right\} \ .$$

Problem 5

Suppose that $\boldsymbol{W}$ is a non-$\{\mathbb{0}\}$ ideal in $\mathbb{P}$. Then $\boldsymbol{W}$ contains some monic polynomials (why?) and among these there must be some of the smallest degree, say $n_{_o}$. Let $p_{_o}$ be one such. (It is entirely possible that $p_{_o} = \mathbb{1}$.)

1. Suppose that p is a non-zero polynomial in $\boldsymbol{W}$, and using the Division Algorithm for Polynomials (see Axler p.121) we write

$$p = q \cdot p_{_o} + r \ ,$$

   where $q, r \in \mathbb{P}$ and $\deg r < \deg p_{_o}$.* Argue that $r \in \boldsymbol{W}$.

2. Use the result of part 1 to argue that every polynomial p in $\boldsymbol{W}$ is a polynomial multiple of $p_{_o}$.
3. Argue that $p_{_o}$ is the only monic polynomial of the smallest degree $n_{_o}$ in $\boldsymbol{W}$, and that

$$\boldsymbol{W} = \{ \ q \cdot p_{_o} \ | \ q \in \mathbb{P} \ \} \ .$$

This polynomial $p_{_o}$ is said to be **the generator** of the ideal $\boldsymbol{W}$.

*Here we use the convention that the degree of the constantly zero polynomial is "$-\infty$".