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QUANTUM FIELD THEORY

Dec 13, 2020

Before. These notes come from Prof. Bostan's QFT notes, Peskin-Schroeder QFT, and "Quantum Field Theory & Condensed Matter" - Shankar.

Enjoy!

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Conventions

$$\hbar = c = 1$$

- $[Length] = [time] = [energy] = [mass]$

- $\gamma_{\mu\nu} = \gamma^{\mu\nu} = \text{diag}(1 - 1 - 1 - 1)$

- $p^2 = p_\mu p^\mu = E^2 - |\vec{p}|^2 = mc^2 = m$

- $x_\mu = g_{\mu\nu} x^\nu = (x^0, \vec{x})$

- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$

- $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$

- $\epsilon^{0123} = \pm 1 ; \epsilon_{0123} = -1$

- $\epsilon^{1230} = -1$

- $E = i \frac{\partial}{\partial x^0} \Rightarrow \vec{p} = i \vec{\nabla}$

- $p^\mu = i \partial^\mu$

• $\delta^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \delta^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \delta^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\delta^{ij} = \delta^{ij} + i\varepsilon^{ijk}\delta^k$

• $\delta^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \delta^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

• Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

• Dirac delta fn: $\delta(x) = \frac{d\theta(x)}{dx}$

• n -dimensional Dirac δ -fn.

$$\int d^n x \delta^n(x) = 1$$

• FT:

$$f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \tilde{f}(k)$$

$$\tilde{f}(x) = \int d^4 k e^{ik \cdot x} f(x)$$

• $\int d^4 x e^{ik \cdot x} = (2\pi)^4 \delta^{(4)}(k)$

• EM $\Phi = \frac{Q}{4\pi r}$ \leftarrow Coulomb potential

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- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\epsilon'^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\mathbf{E}, \mathbf{A}) ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

→

Elements of classical Field Theory

- ② Lagrangian Field Theory:

$$S = \int \underline{L} dt = \int d^4x \underline{L}(\phi, \partial_\mu \phi) \quad \begin{matrix} \text{lag-density} \\ \uparrow \end{matrix}$$

$(\underline{L} = L dx)$

Principle of least action:

$$\delta S = \delta \underline{S} \quad \begin{matrix} \text{(Lagrangian)} \\ \uparrow \end{matrix}$$

$$(\underline{\delta S}) = \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi + \frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta \phi \right] \right\}$$

$$\Rightarrow \boxed{0 = \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right] - \frac{\partial \underline{L}}{\partial \phi}}$$

FTC → term vanishes
at boundaries

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Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_m \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \right\}$$

Ex $\mathcal{L} = \dot{\phi}^2 : \quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\dot{\phi}, \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 0 \Rightarrow 2\dot{\phi} = 0$

$$\mathcal{L} = (\partial_m \phi) (\partial^m \phi) \quad \Rightarrow \quad \partial^m \phi = 0.$$

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = 0; \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 2\partial^m \phi$$

Ex Klein-Gordon field: (real)

$$\mathcal{L} = \frac{1}{2} (\partial_m \phi) (\partial^m \phi) - \frac{1}{2} m^2 \phi^2$$

$$\therefore \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\therefore \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = \partial^m \phi.$$

relativistic particle
of mass m

$$E-L \Rightarrow -m^2 \phi = \partial_m \partial^m \phi = \square \phi$$

$$\rightarrow \boxed{(\square + m^2) \phi = 0}$$

(Klein-Gordon Eqn.)

Ex $\phi = e^{-ipx} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

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Noether's Thm

Thm

For every continuous symmetry, there exists a conserved current j^μ which implies a local conservation law

$$\partial_\nu j^\nu = -\vec{\nabla} \cdot \vec{j} \quad \Leftrightarrow \quad \partial_\nu j^\nu = 0$$

Ex Conservation of charge -- (gauss' law)

$$\begin{aligned} Q &= \int j^0 d^3x \\ \Rightarrow \frac{dQ}{dt} &= \int \frac{dj^0}{dt} d^3x = - \int \vec{\nabla} \cdot \vec{j} d^3x \\ &= -\phi \vec{j} \cdot \vec{d}^2s \end{aligned}$$

Idea Consider continuous transf. \rightarrow infinitesimally (local)

$$(\star) \quad \phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

\uparrow
small

(\star) is a symmetry if EOM invariant under (\star).
 $\Rightarrow S$ is invariant.

$\Rightarrow L$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$
 for some J^μ .

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Let us compare this expectation for $D\phi$ to the result obtained by varying the fields ...

$$\alpha D\phi = \frac{\partial f}{\partial \phi} (\Delta \phi) + \left(\frac{\partial f}{\partial (\partial_\mu \phi)} \right) \partial_\mu (\Delta \phi)$$

$$= \alpha \partial_\mu \left(\frac{\partial f}{\partial (\partial_\mu \phi)} \Delta \phi \right) + \alpha \underbrace{\left(\frac{\partial f}{\partial \phi} - \partial_\mu \left\{ \frac{\partial f}{\partial (\partial_\mu \phi)} \right\} \right) \Delta \phi}_{0}$$

$$\Rightarrow D\phi = \partial_\mu \left\{ \frac{\partial f}{\partial (\partial_\mu \phi)} \Delta \phi \right\}$$

So $\frac{\partial f}{\partial (\partial_\mu \phi)} \Delta \phi$ is the desired J^μ .

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial f}{\partial (\partial_\mu \phi)} \Delta \phi - J^\mu$$

Ex massless KG field

$$L = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)$$

Consider transformation:

$$\phi(x) \rightarrow \phi(x) + \alpha, \quad \Delta \phi = 1$$

$$j^\mu = \frac{\partial f}{\partial (\partial_\mu \phi)} \Delta \phi = \partial^\mu \phi.$$

Check $\partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = \square \phi = 0$ since
 $(m^2 + \nabla^2) \phi = 0 \quad \uparrow m=0$

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Ex Complex KG field

$$L = (\partial_m \phi^+) (\partial^m \phi) - m^2 \phi^+ \phi.$$

again, EOM \Rightarrow

$$(m^2 + \Box) \phi = 0.$$

Symmetry: $\phi \rightarrow e^{i\alpha} \phi$.

For infinitesimal transf we have:

$$\begin{aligned}\alpha D\phi &= i\alpha \phi && \text{(Taylor expand)} \\ \alpha D\phi^+ &= -i\alpha \phi^+\end{aligned}$$

One can check that

$$j^\mu = i [(\partial^\mu \phi^+) \phi - \phi^+ (\partial^\mu \phi)]$$

is the conserved current.

[Ex]

Space-time translation:

Consider inf. translation:

$$x^\mu \rightarrow x^\mu - a^\mu$$

in field transform this becomes:

$$\phi(x) \rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

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Lagrangian is a scalar \Rightarrow must transform the same way:

$$L \rightarrow L + a^{\mu} \partial_{\mu} L = L + a^{\nu} \partial_{\mu} (\delta_{\nu}^{\mu} L)$$

Compare this to the eqn:

$$L \rightarrow L + a \partial_{\mu} J^{\mu},$$

we have

$$J^{\mu} = \delta_{\nu}^{\mu} L$$

\Rightarrow apply this, we find:

$$J^{\mu} = \frac{\partial L}{\partial(\partial_{\mu}\phi)} (\partial_{\mu}\phi) - \delta_{\nu}^{\mu} L$$

value μ explicit...

$$T_{\mu}^{\nu} = \frac{\partial L}{\partial(\partial_{\mu}\phi)} \partial_{\mu}\phi - \delta_{\mu}^{\nu} L$$

\hookrightarrow STRESS-ENERGY TENSOR, (or Energy-momentum tensor)

Conserved charge \Rightarrow the Hamiltonian

$$H = \int T^{00} d^3x = \int \pi d^3x \text{ (time-translation)}$$

$$\text{momentum} \rightarrow p^i = \int T^{0i} d^3x = - \int \pi \partial_i \phi d^3x \text{ (spatial translation)}$$

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Ex Klein-Gordon field again -

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.$$

$$T^{00} = \frac{\partial F}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} = \dots$$

$$= \frac{1}{2} (\partial_0 \phi) (\partial^0 \phi) - \frac{1}{2} (\partial_i \phi) (\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

So

$$H = \int T^{00} d^3x = \int \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} d^3x$$

(note: all terms are positive -- (sum of squares))

→ can't fall into arbitrary negative energy

THE KELIN-GORDON FIELD as HARMONIC OSCILLATOR

promote: ϕ, π to operators → impose suitable commutation relations

Reall ...

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0.$$

Harmonic oscillator: $H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2$

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Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}} \left(q\sqrt{mw} + ip \frac{1}{\sqrt{mw}} \right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}} \left(q\sqrt{mw} - ip \frac{1}{\sqrt{mw}} \right)$

- $a^\dagger a = \frac{1}{\omega} H - \frac{1}{2} \Rightarrow H = \omega(a^\dagger a + \frac{1}{2})$



operator...

- $|0\rangle$, $a|0\rangle = 0$.

- $|n\rangle = \frac{1}{\sqrt{n!}} \underbrace{a^\dagger \dots a^\dagger}_{n} |0\rangle$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -wa$; $[H, a^\dagger] = w a^\dagger$

a lowers by w

a^\dagger raises by w

- $H|n\rangle = (E_n + w)a^\dagger|n\rangle$

- $H|0\rangle = \frac{1}{2}w|0\rangle \rightarrow E_0 = \frac{1}{2}w$

- $E_n = \left(n + \frac{1}{2}\right)w$

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For continuous systems ... commutation relations become :-

$$\boxed{[\phi(x), \pi(y)] = i\delta^{(1)}(x-y)}$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0.$$

Next, the Hamiltonian is now also an operator.
To find spec(H), Fourier transform $\phi(x)$

$$\rightarrow \phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \phi(\vec{p}, t)$$

Recall KG eqn: $(m^2 + D)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{dt^2} + (|\vec{p}|^2 + m^2) \right\} \phi(\vec{p}, t) = 0$$

\rightarrow This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}$$

$$\text{Now, } H_{SHO} = \frac{1}{2}\vec{p}^2 + \frac{1}{2}m^2\phi^2 \quad (\text{m.e.)})$$

\rightarrow know spectrum' $(n + \frac{1}{2})\omega$.

$$\phi = \frac{1}{\sqrt{2\omega}} (at + a) ; \vec{p} = -i\sqrt{\frac{\omega}{2}} (a - at)$$

$$[a, a^\dagger] = 1$$

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Since it's more convenient to work in position space

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p e^{ip \cdot x} + a_p^\dagger e^{-ip \cdot x})$$

$$\pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p e^{ip \cdot x} - a_p^\dagger e^{-ip \cdot x})$$

Note

$\left. \begin{array}{l} a_p \text{ goes with } e^{+ip \cdot x} \\ a_p^\dagger \text{ goes with } e^{-ip \cdot x}. \end{array} \right\}$

9. Easy to show that

$$\left\{ \begin{array}{l} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}). \end{array} \right.$$

* Can re-arrange:

$$\left\{ \begin{array}{l} \phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p + a_{-p}^\dagger) e^{ip \cdot x} \end{array} \right.$$

$$\left. \begin{array}{l} \pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{-ip \cdot x}. \end{array} \right.$$

→ get commutation relation between a_p^\dagger :

$$[a_p^\dagger, a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$$

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Now, can check that

$$\begin{aligned} [\phi(x), \pi(x')] &= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{w_p}{w_{p'}}} x e^{ip \cdot x} (p \cdot x + p' \cdot x') \\ &\quad ([a_{-p}^\dagger, a_p] - [a_p, a_{-p}^\dagger]) \\ &= i \delta^{(3)}(x - x') \quad \checkmark \end{aligned}$$

• Now, can express Hamiltonian in terms of ladder ops

recall that

\rightarrow KG field, but
done

$$H = \int d^3 x \left\{ \frac{\partial L}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} \right\}$$

$$= \int d^3 x \left\{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define Π ... Turns out that

$$\Pi(x) = \frac{\partial f}{\partial (\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial f}{\partial \dot{\phi}} \right)$$

B --

$$H = \int d^3 x \left\{ \frac{1}{2} \Pi^2 + \frac{1}{2} (\nabla \phi) \cdot (\nabla \phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

with $\Pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{ip \cdot x}$ we get

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{w_p}} (a_p + a_{-p}^\dagger) e^{ip \cdot x}$$

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$$H = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(p+p') \cdot x} \left\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4} (a_p - a_p^\dagger) Y_{p'} (-a_{p'}^\dagger) + \frac{-p \cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}} (a_p + a_p^\dagger)(a_{p'} + a_{p'}^\dagger) \right\}$$

results in $C(p-p')$
 $\Rightarrow p = p'$

Some $\delta^{(3)}$
 will appear...

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

So

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

→ can evaluate commutators...

$$[H, a_p^\dagger] = \omega_p a_p^\dagger; [H, a_p] = -\omega_p a_p$$

With H , can find momentum operator...

kG field \rightarrow from $p^i = \int d^3x T^{0i} = - \int \nabla \phi d^3x$, we get

$$\begin{aligned} \vec{P} &= - \int d^3x \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3} \vec{p} a_p^\dagger a_p \end{aligned}$$

$$\frac{E_p}{\hbar} \rightarrow 0$$

a_p^\dagger creates momentum \vec{p} & energy $\omega_p = \sqrt{|\vec{p}|^2 + m^2}$.

Excitation: $a_p^\dagger a_q^\dagger \dots |0\rangle$ = "particles".

↪ each excitation at p is a particle.

\Rightarrow set particle statistics --

Consider 2-particle state $a_p^+ a_q^+ |0\rangle$.

Since $[a_p^+, a_q^+] = 0$, we have

$$[a_p^+ a_q^+ |0\rangle = a_q^+ a_p^+ |0\rangle]$$

\Rightarrow Klein Gordon particles follow Bose-Einstein stats.

• Normalization $\langle 0|0 \rangle = 1$.

$$|p\rangle \propto a_p^+ |0\rangle$$

This $\rightarrow \langle q|p \rangle = (2\pi)^3 \delta^{(3)}(q-p) \Rightarrow$ NOT Lorentz inv.

PF Under a Lorentz boost $\rightarrow p'_3 = \gamma(p_3 + \beta E)$

using the identity $\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$

$$\rightarrow \delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$$

$$\text{we can write: } \delta^{(3)}(p-q) = \delta^0(p'-q') \cdot \left(\frac{dp_3}{dp'_3}\right)$$

$$\begin{aligned} & \underbrace{\delta(p_1-q_1)\delta(p_2-q_2)\delta(p_3-q_3)}_{\text{Same boosted}} \\ &= \delta^{(3)}(p'-q') \cdot \gamma \left(1 + \beta \frac{dE}{dp_3}\right) \\ &= \delta^{(3)}(p'-q') \frac{\gamma}{E} (E + \beta p_3) \\ &= \delta^{(3)}(p'-q') (\gamma E) \end{aligned}$$

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$$\text{So } \boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \left(\frac{E'}{E}\right)}$$

$$1 \Leftrightarrow E = E_p = E'$$

For normalization to work \rightarrow use E_p , not E .

$$\rightarrow \text{define: } |p\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle$$

to find

$$\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$$

Completeness relation --

$$1 \xrightarrow{\text{particle}} \boxed{1 = \int \frac{dp^3}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$$

RS Interpret $\phi(x)|0\rangle$... we know that a_p^\dagger creates momentum p & energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{dp^3}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle , \quad a_p^\dagger |0\rangle = \frac{1}{\sqrt{2E_p}} |p\rangle$$

Pf. By defn --

$$\phi(x) = \int \frac{dp^3}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ipx} + a_p^\dagger e^{-ipx})$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{dp^3}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle \quad \square$$

$\rightarrow \phi(x)|0\rangle$ is a lin. superposition of single-particle states

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First here well-defn momentum.

When nonrelativistic $\rightarrow E_p \approx \text{constant}!$

\Rightarrow $\phi(x)$ acting on the vacuum, "creates a particle at position x ".

\hookrightarrow Confirm this by computing -

$$\langle 0 | \phi(x) | p \rangle = \langle 0 | \int d^3 p' \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p^\dagger e^{ip' \cdot x} + a_p^\dagger e^{-ip' \cdot x}) \sqrt{2E_p} a_p^\dagger$$

$$\boxed{\langle 0 | \phi(x) | p \rangle = e^{ip \cdot x}} .$$

\hookrightarrow Interpretation: position-space representation of the wave-particle wfns of the state $|p\rangle$, just like

$$\boxed{\langle n | p \rangle \propto e^{ip \cdot x}} \text{ in QM!}$$

$\langle 0 | \phi(x) \rangle \sim \langle 1 | \dots$ (don't take this literally, etc).

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Note Hw1, Hw2 are easy, so we'll skip for now.

ep (4), 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time \rightarrow we quantized KG field in the Schrödinger picture.

\rightarrow Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$U(t) = e^{-iHt}$ is the time evolution

$$|\Psi(t)\rangle = e^{-iHt} |\Psi(0)\rangle \xrightarrow{\text{state evolves in time}}$$

\rightarrow In the Heisenberg picture, ... Operators evolve in time

$$\hat{\theta}(t) = U^\dagger(t) \hat{\theta}(0) U(t).$$

So that

$$\langle \Psi, \hat{\theta}(t) \Psi \rangle = \langle \Psi, U(t) \hat{\theta}(0) U(t) \Psi \rangle$$

\downarrow

Heisenberg

\downarrow

Schrödinger.

\rightarrow make the operators $\phi \approx \pi$ time-dependent like this

$$\phi(x) \rightarrow \phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$$

$$\pi(x) \rightarrow \pi(x, t) = e^{iHt} \pi(x) e^{-iHt}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t} \theta = [\theta, H]$

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which gives, upon substituting in $\phi(x,t)$, $\Pi(x,t)$

$$\frac{i}{\partial t} \phi(x,t) = [\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}]$$

$$(\phi \leftrightarrow \phi) \Rightarrow \int d^3x' (-i\delta^{(3)}(x-x')\Pi'(x,t))$$

\rightarrow only non-trivial term is 2^{st} .

$$= i\Pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t} \phi(x,t) = \Pi(x,t)}$$

and

$$\frac{i}{\partial t} \Pi(x,t) = [\Pi(x,t), \int d^3x' \left\{ \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial\Pi)^2 + \frac{1}{2}m^2\phi^2 \right\}]$$

$$= \int d^3x' (-i\delta^{(3)}(x-x')(-\nabla^2 + m^2)\phi(x',t))$$

$$(\text{integrate by parts here}) = -i(-\nabla^2 + m^2)\phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \Pi(x,t) = (m^2 - \nabla^2)\phi(x,t)}$$

Combining these 2 results we get ...

$$\boxed{\frac{\partial^2}{\partial t^2} \phi(x,t) = (\nabla^2 - m^2)\phi(x,t)}$$

\hookrightarrow rearranging this gives

$$\boxed{(\nabla^2 + m^2)\phi(x,t) = 0} \rightarrow \text{just the KG eqn.}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

Recall: $H_{ap} = a_p (H - E_p) \rightarrow$ from comm. rule -

\rightarrow (proof by induction)

$$H^n a_p = a_p (H - E_p)^n$$

Similarly,

$$H^n a_p^+ = a_p^+ (H + E_p)^n$$

\rightarrow So, we have

$$e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t} \rightarrow \text{from above ...}$$

and

$$e^{iHt} a_p^+ e^{-iHt} = a_p^+ e^{+iE_p t}$$

\rightarrow Now -- we want to write $\phi(x,t)$ in terms of these operators. (since $\phi(x)$ is a comb. of a & a^+)

$\pi(x)$, we know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before - -

$$\phi(x) = \phi(x,0) = \int \frac{dp}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ip \cdot x} + a_p^+ e^{-ip \cdot x})$$

substitute this into $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$ we find

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$$\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow p \cdot x = E_p \cdot x - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p} = E_p t - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p}$$

so,

$$\boxed{\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p} \cdot \vec{x}} + a_p^\dagger e^{+i\vec{p} \cdot \vec{x}} \right\} \Big|_{p^0 = E_p}}$$

Similarly, we can find

$$\boxed{\Pi(x, t) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} (a_p e^{-i\vec{p} \cdot \vec{x}} - a_p^\dagger e^{+i\vec{p} \cdot \vec{x}}) \Big|_{p^0 = E_p}}$$

from defn or from $\Pi(x, t) = \frac{\partial}{\partial t} \phi(x, t)$.

Note we can also do everything, but starting from P and not Π . But we won't worry about that.



Causality Note that causality is broken ~~when~~ without the presence of fields.

→ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture --

-- the amplitude for a particle to propagate from y to x is given by

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

"Two-point correlation function"

Let $\boxed{D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle}$

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

= $\langle 0 |$ in terms of $a_p, a_p^\dagger \dots | 0 \rangle$

$$= \langle 0 | a_p^\dagger a_q^\dagger | 0 \rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)}$$

More explicitly --

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | e^{-ipx} e^{ip'y} a_p^\dagger a_{p'}^\dagger | 0 \rangle$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{1}{\sqrt{2E_p}} \right) \left(\frac{1}{\sqrt{2E_{p'}}} \right) e^{-ip \cdot x - ip' \cdot y} (2\pi)^3 \delta^{(3)}(p-p')$$

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)} \quad \square$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$.

① Suppose that $x-y = (t, \vec{v}, 0, 0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t}$$

$$(\text{timelike}) = \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2} e^{-iEt}$$

$$= e^{-imt} \xrightarrow[t \rightarrow \infty]{} \text{dominated by region above}$$

$$p \approx 0 =$$

② Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

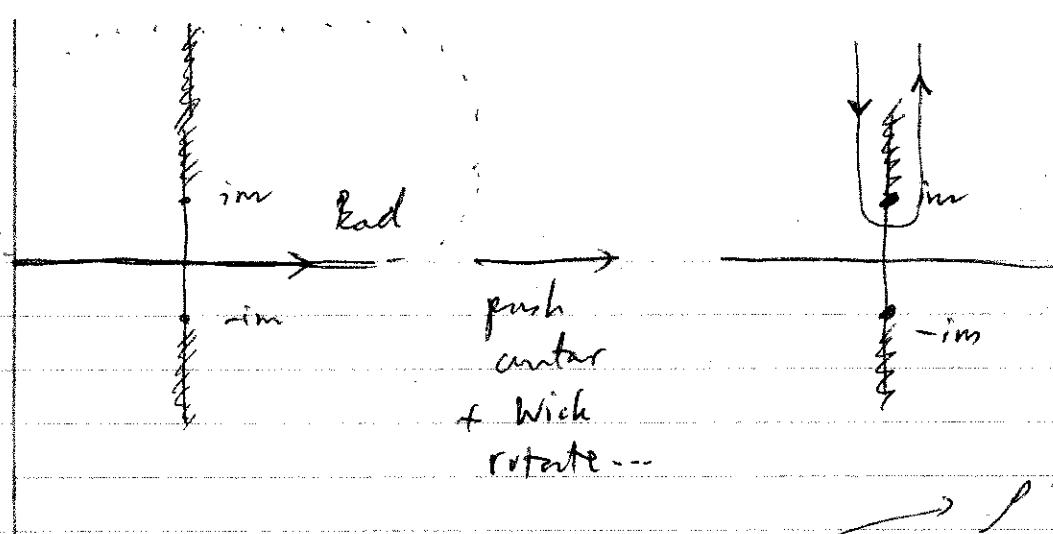
$$\mathcal{D}(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p} \cdot \vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_p} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ tim (singularity)

→ most charge current \rightarrow which rotates



To get

$$D(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty dp \frac{pe^{-ipr}}{\sqrt{p^2 - m^2}} \quad (\text{Wick Rotate})$$

$$\Rightarrow D(x-y) = \boxed{\frac{e^{-mr}}{r}} \rightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell...)

What does it mean for $D(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^m (x-y)_m = -(x-y)^2 < 0$ is spacelike, cannot have causality between $x-y$.

$D(x-y) \neq 0 \Rightarrow ???$ paradox?

\rightarrow No! To discuss causality, we should ask not whether particles can propagate over spacelike intervals -

-- but whether a "measurement" at one point can affect a measurement at another point where separation from the first is spacelike -

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement $\phi(x)$, call this $\phi(x)$ = a local measurement $\phi(y)$, called $\phi(y)$

So long as $[\phi(x), \phi(y)] = 0$, the 2 measurements don't affect one another.

~ measure the field $\phi @ x = \phi @ y$,

If $[\phi(x), \phi(y)] = 0$ when $(x-y)^2 < 0$ then we've proved

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\}$$

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \times \left[\left((a_p e^{-ip \cdot x} + a_{p'}^\dagger e^{ip \cdot x}), (a_p^\dagger e^{-ip \cdot y} + a_{p'} e^{ip \cdot y}) \right) \right]$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\}$$

$$(2\pi)^3 \delta^3(p-p') - (2\pi)^3 \delta^3(p-p')$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2E_p} \right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

$$\text{So } [\phi(x), \delta(y)] = D(x-y) - D(y-x).$$

Since $D(y-x)$ is Lorentz invariant, can take

$$y-x \rightarrow x-y. \text{ (possible b/c } (x-y)^2 < 0)$$

Note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

\rightarrow so this is why possible because $(x-y)^2 < 0$
(parabolic).

$$\text{So } D(y-x) = D(x-y)$$

$$\text{So, } [\phi(x), \phi(y)] = 0 \quad \checkmark$$

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~~The Klein-Gordon Propagator~~

Let's look at $[\phi(x), \phi(y)]$ in more details...

$[\phi(x), \delta(y)]$ is just a number

~~can write~~ $[\phi(x), \delta(y)] = \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$

$$\rightarrow \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right\}$$

(assuming $x^0 \neq y^0$)

$$= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip \cdot (x-y)} \right\} \Big|_{\substack{p^0 = E_p \\ p^0 = -E_0}} - \frac{i}{2E_p} e^{-ip \cdot (x-y)} \Big|_{\substack{p^0 = E_p \\ p^0 = -E_0}}$$

$$\Delta P^{\mu\nu} = \eta^{\mu\nu}$$

$$E_p^2 = p^2$$

$$i = \sqrt{-E_p}$$

B The Klein-Gordon Propagator

→ Before looking at this -- need to look at Green's Functions & Contour Integrals --

→ Consider function

$$\frac{1}{(p_0^0 - E_p)(p_0^0 + E_p)} \quad E_p = \sqrt{|p|^2 + m^2}.$$

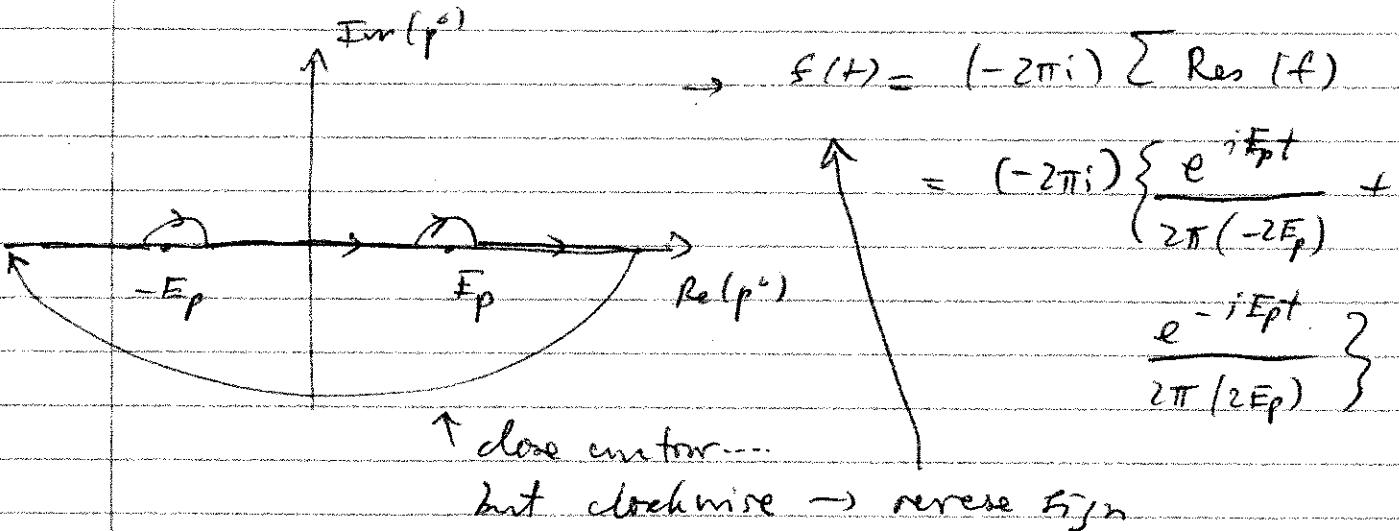
→ Poles at $p_0^0 = \pm E_p$.

Lab at FT:

$$f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}$$

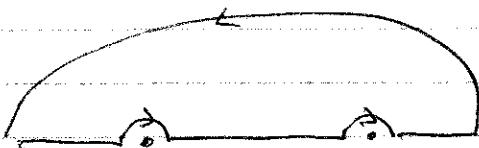
→ How to interpret this?

If $t > 0 \rightarrow$ ~~crosses poles~~



$$f(t) = \frac{-i}{2E_p} (e^{+iE_pt} - e^{-iE_pt}) \quad (t > 0)$$

If $t < 0$ close contours above poles



$$\rightarrow f(t) = 0$$

\rightarrow So, altogether, we have ...

$$f(t) = \int \frac{1}{2\pi} \frac{dp^0}{(p_0^0 E_p)(p^0 + E_p)} e^{-ip^0 t}$$

$$= \Theta(t) \left(\frac{-i}{2E_p} \right) (e^{-iE_p t} - e^{+iE_p t})$$

where $\Theta(t)$ is the Heaviside Step fn.

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

\rightarrow Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



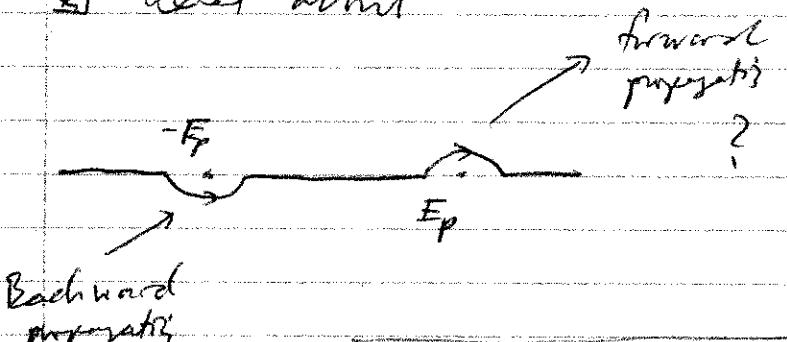
then we'll get

$$t > 0 \rightarrow f(t) = 0$$

$$t < 0 \rightarrow f(t) = \Theta(-t) \frac{i}{2E_p} (e^{-iE_p t} - e^{+iE_p t})$$

\rightarrow Advanced / Backward Propagating Green's fn.

What about



backward
propagation

$$\rightarrow \boxed{f(t) = \Theta(+)(-\frac{i}{2E_p})e^{-iE_pt} + \Theta(-)(\frac{-i}{2E_p})e^{+iE_pt}}$$

Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$

Consider this quantity...

$$\langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip(x-y)} + \frac{1}{-2E_p} e^{-ip(x-y)} \right\}$$

↑ pole @ $p_0 = E_p$ ↓ pole @ $p_0 = -E_p$

$\xrightarrow{\text{4D integral}} \int \frac{d^3 p}{(2\pi)^3} \int \frac{d^3 p'}{(2\pi)^3} \frac{-i}{p^2 - m^2} e^{-ip(x-y)}$

$f(+)$ before, where

$$(p^2 - E_p)(p^2 + E_p) = p'^2 - |p'|^2 - m^2 = p^2 - m^2$$

Now, call ...

$$D_R(x-y) = \theta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle$$

Then

$$\begin{aligned}
 \rightarrow (\square + m^2) D_R(x-y) &= \square D_R(x-y) + m^2 D_R(x-y) \\
 &= (\square \theta(x^0 - y^0)) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + 2(\partial_\mu \theta(x^0 - y^0)) \partial^\mu \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + \theta(x^0 - y^0) (\square + m^2) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &= -\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(y), \psi(y)] | 0 \rangle \quad \text{cancel} \\
 &\quad + 2\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle + 0 \\
 &= \delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \quad \text{milds} \\
 &= -i \delta^{(4)}(x-y) \quad \text{renormalization} \\
 &\qquad \qquad \qquad \downarrow \quad (\text{easy}) \\
 &\qquad \qquad \qquad -i \delta^{(3)}(x-y)
 \end{aligned}$$

So

$$(\square + m^2) D_R(x-y) = -i \delta^{(4)}(x-y)$$

$\rightarrow D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0 @ x^0 < y^0$

$\Rightarrow D_R(x-y) = \text{"Retarded" Green's fn}$

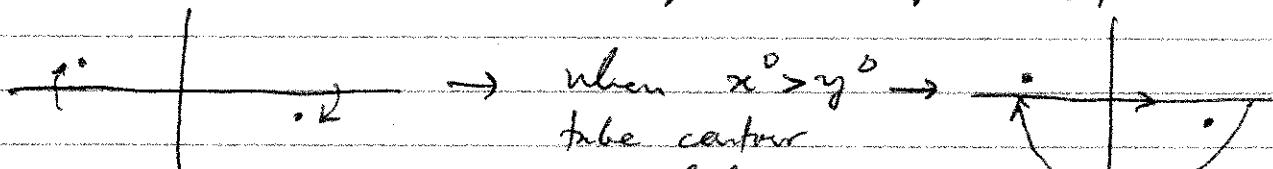
Now... As we have seen, there are many ways to take the contour...



→ Use the Feynman prescription instead

$$D_F(x-y) = \int d^4 p \frac{i}{(2\pi)^4} \frac{e^{-ip \cdot (x-y)}}{p^2 - m^2 + i\epsilon}$$

→ Convenient! B/c now poles are $p^0 = \pm(E_p - i\epsilon)$



when $x^0 < y^0$, → get same expression
but with $x \leftrightarrow y$.

So,

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \theta(x^0 - y^0) \langle 0 | \phi(1)(\phi(2)) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | \phi(1)(\phi(2)) | 0 \rangle$$

$$= \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

"time-ordering" symbol -

Time-ordering symbol \Rightarrow instructs us to place the operators & heat follows in order with the latest to the left -
 \rightarrow apply $(D + m^2)$ to last line, set D_F is Green's fn of Klein-Gordon Operator.

$$(\quad \quad \quad)$$

b) $D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator -

\hookrightarrow propagation amplitude

\rightarrow But we can't much calculation at this point just yet -

\rightarrow B/c we've only looked at the free kls theory

\rightarrow Field eqn in this case is linear \therefore there are no interactions -

\rightarrow this theory is too simple to make any predictions -

\rightarrow need perturbation -

One kind of interaction it is can also be solved

$$(\quad \quad \quad)$$

Particle Creation by a classical source

Consider the source $j(x)$

Result - Free field: $(D^2 + m^2)\phi = 0$

→ now... $(D^2 + m^2)\phi = j(x)$ Field ϕ is
 ↗ space time.

$j(x)$ is nonzero only for a finite time interval

The associated lagrangian is

$$L = \frac{1}{2} (D^\mu \phi) (D_\mu \phi) - \frac{1}{2} m^2 \phi^2 + j(x) \phi(x)$$

If $j(x)$ is turned on for only a finite time, it is
 enough to solve

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{dp}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip \cdot x} + a_p^* e^{ip \cdot x})$$

With a source

$$\phi(x) = \phi_0(x) + i \int dy D_R(x-y) j(y)$$

we won't worry about this for now...

Some problems & Insights

(1) Classical EM (no sources) follow from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \epsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \quad (\vec{\nabla} \cdot \vec{E} = 0) \quad (r=0)$$

$$-\partial_t \vec{E} + \vec{\nabla} \times \vec{E} = 0 \quad (r=i)$$

(2) Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \right)$$

Derive E-L eqn:

$$i\partial_t \phi^\dagger - \frac{1}{2m} \nabla^2 \phi^\dagger = 0$$

$$-i\partial_t \phi - \frac{1}{2m} \nabla^2 \phi = 0$$

Now... write $\phi \rightarrow e^{-i\omega} \phi$, $\phi^+ \rightarrow e^{i\omega} \phi^+$

$$\sim \phi - i\omega \phi \\ \rightarrow D\phi \sim -i\omega$$

$$\sim \phi^+ + i\omega \phi^+ \\ \Delta \phi^+ \sim i\omega \phi^+$$

So that

$$j^1 = \frac{\partial f}{\partial (\partial_\mu \phi)} D\phi + \frac{\partial f}{\partial (\partial_\mu \phi^+)} D\phi^+$$



concurrent current -

↳ can find conjugate momenta:

$$\Pi(x) = \frac{\partial L}{\partial (\partial_\mu \phi)} \rightarrow \text{conjugate...}$$

→ can set Hamiltonian \rightarrow there's a Ham formula in book, but we
 $H = \int d^3x (\pi^+ \pi^- + (\nabla \phi)^2 / 2m^2 + m^2 \phi^+ \phi^-)$ won't worry abt this.

(3) If we take $(x-y)^2 = -r^2 \rightarrow$ can explicitly write

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2 \rightarrow D(x-y)$ can be written \rightarrow in terms of Bessel Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation ...

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

→ what happens to $\phi(x)$ under Λ ?

we require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform -- $\partial_\mu \phi(x) \rightarrow \partial_\mu (\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu (\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\rho_\mu (\partial_\rho \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho_\mu (\Lambda^{-1})^\sigma_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu \phi(x))^2 \rightarrow g^{\mu\nu} (\partial_\mu \phi'(x)) (\partial_\nu \phi'(x))$$

$$= g^{\mu\nu} [\partial_\mu (\phi(\Lambda^{-1}x))] [\partial_\nu (\phi(\Lambda^{-1}x))]$$

$$= \underbrace{g^{\mu\nu} \{(\Lambda^{-1})^\rho_\mu \partial_\rho \phi\}}_{= (\partial_\mu \phi)^2} \{(\Lambda^{-1})^\sigma_\nu \partial_\sigma \phi\} (\Lambda^{-1}x)$$

$$= (\partial_\mu \phi)^2 (\Lambda^{-1}x) \leftarrow = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi (\Lambda^{-1}x) \rightarrow \text{evaluated}$$

(37)

because

$$\phi(x) \rightarrow \phi(\tilde{x})$$

$$\partial_\mu \phi(x) \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)(\tilde{x})$$

$$(\partial_\mu \phi(x))^2 \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)^2 (\tilde{x})^\nu$$

so it is clear that

$$L \rightarrow L(\tilde{x})$$

↑

Lagrangian is Lorentz-invariant.

→ The action $S = \int d^4x L$ is also Lorentz inv.

→ also clear that EOM is also Lorentz inv.

$$(\square + m^2) \phi(x) = (\tilde{x})^\mu \partial_\mu (\tilde{x}^\nu \partial_\nu + m^2) \phi(\tilde{x})$$

$$= (2^\nu \partial_\nu + m^2) \phi(\tilde{x})$$

$$= 0$$

✓

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Sep 15, 2020

→ How do we find Lorentz-invariant theories, in general?

→ For simplicity, restrict attention to lin. transf

→ $\Phi_a = \phi \in \mathbb{C}^n$, → matrix giving Lorentz transform A.

$$\rightarrow \boxed{\Phi_a(x) \rightarrow \text{Mat}(A) \Phi_s(\tilde{x})}$$

n × n

The

→ most general nonlinear basis can be built
out of lines $\alpha \rightarrow$ suffice to consider M
only.

↳ for short, write $\Phi \mapsto M(\alpha) \Phi$.

→ What are the possible allowed $M(\alpha)$?

Ex $\{M(\alpha)\}$ form a group $M(\alpha') M(\alpha) = M(\alpha')$
 $\qquad\qquad\qquad \rightarrow \alpha'' \alpha = \alpha'$

→ the correspondence between $\alpha \in M$ must be
preserved under multiplication.

$\{\alpha\}$, Lorentz group $\longleftrightarrow \{M(\alpha)\} \rightarrow$ n-dim
representation of the
Lorentz group

↳ Q? What are the finite-dim matrix repr.
of the Lorentz group?

Ex in ∂M ... spin $\frac{1}{2} \rightarrow \{M\}$ are the 2×2 unitary
matrices with determinant 1

$$\rightarrow \boxed{U = e^{-i\vec{\theta}^i \vec{\sigma}^i/2}} \rightarrow \{\vec{\theta}^i\}_{i=1}^3$$

$$\begin{pmatrix} \vec{\theta} \\ \vec{\sigma} \end{pmatrix} \rightarrow \vec{\theta}$$

3 arbitrary parameters
& Pauli matrices

$$u(\vec{\theta}) := e^{-i\vec{\theta}^i \vec{\sigma}^i/2}$$

→ In the case for arbitrary spin representations--

$$\langle \psi(\vec{r}) \rangle = e^{-i\vec{\sigma} \cdot \vec{J}} \text{ where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i \sum_k \epsilon^{ijk} J^k$$

→ Check that this works for your $\frac{1}{2}$:

$$\left[\frac{\sigma^i}{2}, \frac{\sigma^k}{2} \right] = i \sum_l \epsilon^{ijk} \frac{\sigma^l}{2} \quad \checkmark$$

→ for spinless particles-- $\Psi(\vec{x})$ can be decomposed into
orbital angular momentum states $J=0, 1, 2, \dots$
(no intrinsic spin $\Rightarrow J=L$)

$$\vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$$

$$J^i = i \sum_l \epsilon^{ijk} x^k \nabla^l$$

$$\nabla^i = -\partial_i = -\frac{\partial}{\partial x^i}$$

But the cross product is special to 3D case.

→ write operator in antisymmetric tensor.

$$J^{ij} = -i(x^i \partial^j - x^j \partial^i)$$

→ reproduces
the cross
product.

so that $J^3 = J^{12}$, etc.

→ generate to 4D: → 6 operators that generate $\begin{cases} 3 boosts \\ 3 rotations \end{cases}$

$$J^{\mu\nu} = +i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

of the Lorentz group.

$\{ \rightarrow$ Spatial Rotations : $J^{\hat{k}} = i(x^0\partial^k - x^k\partial^0)$

\rightarrow Lorentz boosts along x^i axis : $J^{\hat{i}} = i(x^0\partial^i - x^i\partial^0)$

\rightarrow Now, want to get commutation rules.

\rightarrow compute the commutators of differential ops
to get

$$[J^{\mu\nu}, J^{\rho\rho}] = i(g^{\nu\rho}J^{\mu\rho} - g^{\mu\rho}J^{\nu\rho} - g^{\nu\mu}J^{\mu\rho} + g^{\mu\mu}J^{\nu\rho})$$

Ex 1 rotations : $J^{12} = -J^{21}$
 $J^{23} = -J^{32}$
 $J^{13} = -J^{31}$ } \Rightarrow 6 total
 3 boosters $J^{01} = -J^{10}$
 $J^{02} = -J^{20}$
 $J^{03} = -J^{30}$

Ex Consider the 4×4 matrix $(J^{\mu\nu})_{\alpha\beta}$ where
 μ, ν label which of the 6 matrices, while α, β
 label the component/matrix element :-

$$(J^{\mu\nu})_{\alpha\beta} = i(\delta_{\alpha}^{\mu}\delta_{\beta}^{\nu} - \delta_{\alpha}^{\nu}\delta_{\beta}^{\mu})$$

6. Verify that $(J^{\mu\nu})_{\alpha\beta}$ satisfies the comm.
 relation.

\rightarrow Here are matrix tent out an ordinary
 Lorentz 4-vectors --
 to see this

→ look at elements of the Lorentz group

$$U(\omega_{\mu\nu}) = \exp \left[-i \frac{\omega_{\mu\nu}}{2} J^{\mu\nu} \right]$$

infinitesimally \rightarrow

$$\begin{aligned} & \approx I + \frac{-i}{2} \omega_{\mu\nu} J^{\mu\nu} \\ & \approx \delta^\alpha_\beta + \frac{-i}{2} \omega_{\mu\nu} (J^{\mu\nu})^\alpha_\beta \end{aligned}$$

So, infinitesimally

$$V^\alpha \rightarrow \delta^\alpha_\beta + \frac{-i}{2} \omega_{\mu\nu} (J^{\mu\nu})^\alpha_\beta V^\beta$$

→ $\omega_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles.

$V_\alpha, V_\beta \rightarrow$ 4-meters

Ex 1 When $\omega_{12} = -\omega_{21} = \theta \Rightarrow \omega_{\mu\nu} = 0$ else, we get

$$[V^\mu] \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [V^\mu]$$

→ infinitesimal ROTATION on xy plane

Ex 2 when $\omega_0 = -\omega_{10} = \beta \Rightarrow$ get
 $\omega_{\mu\nu} = 0$ else

$$[V^\mu] \rightarrow \begin{pmatrix} 1 & \beta & 0 & 0 \\ \beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [V^\mu] \rightarrow$$

BOOST along x

THE DIRAC EQUATION

→ Now that we have seen one fid. representation of the Lorentz group

→ need to develop formalism for finding all other ~~formalisms~~ representations...
(problem 3.1)

→ focus on spin $\frac{1}{2}$ systems...

→ In this case, use Dirac's trick idea to -

Suppose we had a set of $4 \times n$ matrices γ^{μ} satisfying:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\mathbf{I}$$

Then we could write down an n -dim representation of the Lorentz algebra -

$$S^{\mu\nu} = \frac{i}{4} [\gamma^{\mu}, \gamma^{\nu}]$$

These matrices satisfy the commutation relation -

$$\rightarrow [S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

→ Verify that this trick works in 3D Euclidean space

so which use, $\gamma^0 = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & 0 \end{pmatrix} \rightarrow \{\gamma^0, \gamma^i\} = -2\delta^{ij}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4} [\delta^i, \delta^j] = \frac{1}{2} \sum_k i \epsilon^{ijk} \delta^k = J^i$$

→ Which is what we saw before \Rightarrow angular momentum.

$$\left. \begin{aligned} J^1 &= S^{12} = \frac{1}{2} \sigma^3 \\ J^2 &= S^{31} = \frac{1}{2} \sigma^2 \\ J^3 &= S^{23} = \frac{1}{2} \sigma^1 \end{aligned} \right\}$$

~~4~~

→ now, want $S^{\mu\nu}$ to be 4D Minkowski space.

→ Matrices $\delta^{\mu\nu}$ must be at least 4×4 .

→ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv.

Ex

$$\delta^\mu = \begin{pmatrix} 0 & \mathbb{I}_{2 \times 2} \\ \mathbb{I}_{2 \times 2} & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Weyl / "chiral" representation

→ In this case, the boost + rotation generators are ..

Boosts $\rightarrow S^{0i} = \frac{i}{4} [\delta^0, \delta^i] = \frac{-i}{2} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$

Rotations $\rightarrow S^{ij} = \frac{i}{4} [\delta^i, \delta^j] = \sum_k \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} 0^k & 0 \\ 0 & 0^k \end{pmatrix} = \sum_k \frac{1}{2} \epsilon^{ijk} \sum_p$

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at all transformations under which the ~~lagrangian~~^{Action's} ~~is~~ is "invariant"

→ In particular, we want \mathcal{S} to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\left\{ \begin{array}{l} \phi(x) \rightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x') \rightarrow \phi(\Lambda^{-1}x'). \end{array} \right. \begin{array}{l} \rightarrow \text{check that} \\ \mathcal{S} \text{ is invariant,} \end{array}$$

→ but this is very simple ... → There are many more transforms that leave \mathcal{S} Lorentz invariant.

→ How do we find all of them?

→ For simplicity, we'll just restrict ourselves to linear combinations of transformations

→ look at transformations of the form

$$\phi_a(x) \rightarrow \sum_b M_{ab}(A) \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \xrightarrow{A} M(A)\phi}$$

These matrices M must be "nice" in the sense that M must obey -

$$\text{if } \phi \rightarrow M(\Lambda') M(\Lambda) \phi = M(\Lambda' \Lambda) \phi$$

This says that $\{M\}$ (the collection of M 's) must be a representation of the Lorentz group.

What? So, recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

$$\rightarrow \boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

A representation π of a group G is a function π satisfying the property

$$\pi(g_1) \pi(g_2) = \pi(g_1 g_2)$$

$$\begin{matrix} \uparrow & \uparrow \\ e_g & e_{g'} \end{matrix} \quad \begin{matrix} \uparrow \\ e_{g'g} \end{matrix}$$

With this, it is clear that

$$\boxed{\{\Lambda\} \text{ Lorentz group} \Rightarrow \{M\} \text{ is a representation of } \{\Lambda\}}$$

So, what are these M ?

\rightarrow Ex $\boxed{\text{Rotation group for spin } \frac{1}{2} \text{ particles}}$

For spin $-\frac{1}{2}$, the most important nontrivial representation is the 2D representation.

→ There are unitary matrices with $\det = 1$

(2×2)

$$\Rightarrow \text{In general: } U = e^{-i \vec{\sigma} \cdot \vec{\theta}/2}$$

$\vec{\theta} \rightarrow \text{Pauli matrices}$

$\vec{\theta} \rightarrow \text{angle.}$

For infinitesimal rotations, we can write

$$U = I - \frac{i}{\hbar} \vec{\sigma} \cdot \vec{\theta} = I - \vec{\sigma} \cdot \vec{\theta}$$

{U} form a Lie-algebra of the L-group.

$\vec{\sigma}$ here are the "generators" of the Lie algebra

when {U} is a representation of the rotational group, we identify

$$\vec{\sigma} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{\sigma}$ as the quantum angular momentum operator

→ satisfies the commutation relation

$$[\sigma^i, \sigma^j] = i \epsilon^{ijk} \sigma^k$$

like the generators of $SO(3)$, namely the Pauli matrices.

→ finite rotations are formed by matrix exp.

$$R = \exp\{-i\theta^i \sigma^i\}$$

← →

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Back to present problem...

to get generators of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

$$(3D) \quad \vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\sigma})$$

$$(4D) \quad \boxed{J^{\mu\nu} = i(x^\mu \sigma^\nu - x^\nu \sigma^\mu)}$$

with commutation relations:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})}$$

→ any matrices that we represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i(\delta_\alpha^\mu \sigma^\nu - \delta_\beta^\mu \sigma^\nu)}$$

→ by symmetry, μ, ν take label which of the six matrices we want;

→ α, β label components -

The Dirac Eqn

What are the representations of the Lorentz group?
especially for spin- $\frac{1}{2}$?

Dirac's trick: if we have a set of $4 \times n$ matrices γ^μ which satisfies:

Dirac
algebra

$$\rightarrow [\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{I}_{n \times n}]$$

Then the n -dim representation of the Lorentz algebra:

$$S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$$

→ In other words, $S^{\mu\nu}$ satisfies:-

$$[\gamma^\mu, \gamma^\nu] = i(g^{\mu\nu}\gamma^0 + g^{\nu\mu}\gamma^0 - g^{\mu\nu}\gamma^0 - g^{\nu\mu}\gamma^0)$$

* Note that this trick works also in any dim.

e.g. take $\gamma^0 = i\sigma^3$ so that $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

$$\Rightarrow S^{ij} = \frac{1}{2} \epsilon^{ijk} S^k \quad \rightarrow \text{just as before.}$$

↑
2D representation of the rotation group.

$$\text{Spin } \frac{1}{2}: J^1 = J^{12} = \frac{1}{2}\sigma^3, J^2 = \frac{1}{2}\sigma^2 = S^{21}, J^3 = S^{23} = \frac{1}{2}\sigma^1$$

One such representation for the Dirac algebra is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4 \times 4} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4 \times 4}$$

Weyl / chiral representation.

get

Boosts $S^{0j} = \frac{i}{4} [\gamma^0, \gamma^j] = -\frac{i}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} 0 & \sigma^k \\ 0 & 0^k \end{pmatrix} = \frac{1}{2} \epsilon^{ijk} \epsilon^{lk}$$

Hermitian Defn

not restricted
but ψ is also
a
hermitian
field, not a
wfn

All 4-component field ψ that transforms under
boosts + rotations according to ψ is called
a Dirac spinor

S^{ij} are hermitian

S^{0i} are anti-hermitian

Since ψ is a chiral field, not a wfn

Now, what is the field eqn for ψ ?

→ try $(\square + m^2)\psi = 0 \leftarrow \text{KG field eqn.}$

But this obviously works because the representations are block-diagonal.

→ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~at~~ at transformation of θ matrices

In an expression we can think of...

$$E = 3 \Delta_{\frac{1}{2}} [4 \times 4] \Delta_{\frac{1}{2}} [i] \xrightarrow{\text{for spin } \frac{1}{2}}$$

where $\Delta_{\frac{1}{2}} = \exp \left\{ \frac{i}{2} w_{\mu\nu} S^{\mu\nu} \right\}$

$$\simeq 1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}$$

$$\Rightarrow [\delta^r] \rightarrow [\Delta_{\frac{1}{2}}] [\delta^u] [\Delta_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu} \right) \delta^u \left(1 - \frac{i}{2} w_{\alpha\beta} S^{\alpha\beta} \right)$$

$$= \dots \quad (\text{some terms of higher order cancelled})$$

$$= \delta^u - \frac{i}{2} w_{\alpha\beta} [\delta^u, S^{\alpha\beta}]$$

?

above a quick computation shows that

$$[\gamma^\mu, \gamma^\nu] = (\gamma^{\mu\nu})_\nu \gamma^\nu$$

where

$$\gamma^{\mu\nu} = +i(g^{\mu\nu}\delta_\nu^\rho - g^{\rho\nu}\delta_\nu^\mu)$$

So ...

$$\left[\gamma^\mu \rightarrow \gamma^\mu - \frac{i}{2} \omega_{\mu\rho} (\gamma^{\mu\nu})_\nu \gamma^\nu \right] = 1, \gamma^\mu 1,$$

$\rightarrow \gamma^\mu$ transforms like 4-vectors ...

$\Rightarrow \gamma^\mu$ are invariant under simultaneous rotations of
Polar vectors & spinor indices ...

\rightarrow can treat γ^μ or γ^ν as a vector index!

\rightarrow can add γ^μ into ∂_μ to form a Lorentz-
inv. differential operator ...

Dine eqn

$$(\gamma^\mu \partial_\mu - m) \psi = 0$$

check that this is Lorentz-inv:

Let $\psi(x) \rightarrow 1, \gamma^1(1^x)$ then

$$\gamma^\mu \partial_\mu \psi \rightarrow (\gamma^\mu 1, \gamma^1 \partial_\mu (1^x))$$

$$= i 1, \gamma^1 (1^x \gamma^\mu 1, \gamma^1) \cdot (1^1)_\mu (1^x) (1^x)$$

wavelet & transform

$$= i \Delta_{\frac{1}{2}} (\Delta)^{\alpha} \delta^m \cdot (\Delta)_{\mu}^{-\alpha} (\partial_2 \psi)(\tilde{x})$$

$$= i \Delta_{\frac{1}{2}} \delta^m \underbrace{(\Delta)^{\alpha} (\Delta)_{\mu}^{-\alpha}}_{\delta^{\alpha}_r} (\partial_2 \psi)(\tilde{x})$$

$$= i \Delta_{\frac{1}{2}} \delta^m \partial_m \psi(\tilde{x})$$

$$\Rightarrow i \delta^m \partial_m \psi(x) \rightarrow \Delta_{\frac{1}{2}} i \delta^m \psi(\tilde{x})$$

\rightarrow transforms the same way as $\psi(\tilde{x})$

clearer way:

$$\begin{aligned} \text{Let } [i \delta^m \partial_m - m] \psi(x) &\rightarrow [i \delta^m (\tilde{x})_{\mu} \partial_{\mu} - m] \Delta_{\frac{1}{2}} \psi(\tilde{x}) \\ &= \Delta_{\frac{1}{2}} \Delta_{\frac{1}{2}}^{-1} [i \delta^m \tilde{x}_{\mu} \partial_{\mu} - m] \Delta_{\frac{1}{2}} \psi(\tilde{x}) \\ &= \Delta_{\frac{1}{2}} \left\{ i \Delta_{\frac{1}{2}}^{-\alpha} (\tilde{x})_{\mu}^{\alpha} \partial_{\mu} - m \right\} \psi(\tilde{x}) \\ &= \Delta_{\frac{1}{2}} \left\{ i \delta^m \partial_m - m \right\} \psi(\tilde{x}) \\ &= 0 \quad \checkmark \end{aligned}$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i \delta^m \partial_m - m) \psi$$

$$\rightarrow 0 = (-i \delta^m \partial_m - m) (+i \delta^m \partial_m - m) \psi$$

$$= (\delta^m \delta^{\alpha} \partial_{\alpha} \partial_m + m^2) \psi = \dots$$

$$\begin{aligned}
 &= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2) \psi \\
 &= \left[\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2 \right] \psi \\
 &= \left[\frac{1}{2} \{ \gamma^\mu \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right] \psi \quad \xrightarrow{\text{R.G. eqn.}} \\
 &= (\gamma^\mu \partial_\mu \partial_\nu + m^2) \psi = (D + m^2) \psi = 0
 \end{aligned}$$

4

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger \psi$ doesn't work b/c under a boost,

$$\psi^\dagger \frac{1}{2} \Delta_{\frac{1}{2}} \psi \neq \psi^\dagger \psi \text{ since } \Delta_{\frac{1}{2}} = \exp \left\{ i \omega_{\mu\nu} S^{\mu\nu} \right\}$$

not unitary ... since not all $S^{\mu\nu}$ are terms.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, set

$$\bar{\psi} \rightarrow \bar{\psi} \frac{1}{2} \gamma^0 \approx \bar{\psi} \left(1 + i \omega_{\mu\nu} (S^{\mu\nu})^0 \right) \gamma^0$$

when $\omega_{\mu\nu} \ll 0 \Rightarrow \omega \neq 0$, $(S^{\mu\nu})^0 = (S^{\mu\nu})$

$$i (S^{\mu\nu} \leftrightarrow \gamma^0)$$

When $\mu=0$ or $\nu=0$, $(S^{\mu\nu})^+ = -S_{\mu\nu}$

S^μ anti-commutes w/ γ^0 .

$$\rightarrow \bar{\psi} \rightarrow \gamma^+ \left(1 + \frac{i}{2} w_{\mu\nu} (S^{\mu\nu})^+ \right) \gamma^0$$

$$= \underbrace{\gamma^+ \gamma^0}_{= \gamma^5} \left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu} \right)$$

$$= \bar{\psi} \left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu} \right) = \bar{\psi} \gamma_5^{-1} \text{ as desired.}$$

$$\rightarrow \boxed{\bar{\psi} \rightarrow \bar{\psi} \gamma_5^{-1}}$$

$$\text{and so } \boxed{\bar{\psi} \psi = \gamma^+ \gamma^0 \psi} \text{ is a Lorentz scalar.}$$

Similarly, can show that

$$\boxed{\bar{\psi} \gamma^\mu \psi} \text{ is a Lorentz vector.}$$

\rightarrow the correct Lorentz-invariant Dirac Lagrangian \rightarrow

$$\boxed{L_{\text{Dirac}} = \bar{\psi} (\not{D} - m) \psi}$$

$$\{ \text{E-L eqn for } \bar{\psi} \text{ gives } (\not{D} - m) \psi = 0$$

$$\text{E-L eqn for } \psi \text{ gives } -i \not{D}_\mu \bar{\psi} \not{\gamma}^\mu - m \bar{\psi} = 0$$

WEYL SPINOR

Recall that

$$S^{ij} = \frac{i}{2} \begin{pmatrix} \sigma^i & \sigma \\ \sigma & -\sigma^i \end{pmatrix}$$

$$S^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & \sigma \\ \sigma & \sigma^k \end{pmatrix}$$

Since block-diagonal \Rightarrow Dirac representation of the Lorentz group is reducible.

\rightarrow Can form 2-D representations by considering each block separately.

$$\rightarrow \text{write } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{\text{left-handed Weyl spinors}}$$

Under infinitesimal boost \vec{p} = rotation $\vec{\theta}$, these transform as

$$\psi_L \rightarrow (1 - i \vec{\theta} \cdot \vec{\sigma}/2 - i \vec{B} \cdot \vec{\sigma}/2) \psi_L$$

$$\psi_R \rightarrow (1 - i \vec{\theta} \cdot \vec{\sigma}/2 + i \vec{P} \cdot \vec{\sigma}/2) \psi_R$$

$$\rightarrow \text{Recall that } (\tanh(\vec{P}) = \frac{1}{c} \vec{v})$$

\rightarrow Transf of ψ_R is equiv to transf of ψ_L^\dagger

by unitary transf

$$\psi^* \rightarrow \left(1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}^*}{2} - \vec{p} \cdot \frac{\vec{\sigma}^*}{2}\right) \psi^*$$

noting that $\vec{\sigma}^2 \vec{\sigma}^* = -\vec{\sigma} \vec{\sigma}^*$ ($\vec{\sigma}^2 = \vec{\sigma}^2$)

$$\begin{pmatrix} 1 \\ (\vec{\sigma} \cdot i) \\ i \end{pmatrix} \quad \begin{pmatrix} 1 \\ (\vec{\sigma} \cdot i) \\ i \end{pmatrix}$$

we find:

$$\underbrace{\vec{\sigma}^2 \psi_L^*}_{\psi_L^*} \rightarrow \vec{\sigma}^2 \left[1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}^*}{2} - \vec{p} \cdot \frac{\vec{\sigma}^*}{2} \right] \psi_L^*$$

$$= \underbrace{\left[1 - i\vec{\sigma} \cdot \frac{\vec{\sigma}}{2} + \vec{p} \cdot \frac{\vec{\sigma}}{2} \right]}_{\text{like } \psi_R \text{ transform.}} \psi_L^*$$

$\underline{\text{So }} \vec{\sigma}^2 \psi_L^* \text{ transform like } \psi_R$

With $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, the Dirac eqn has form

$$(i\vec{\sigma}^m \partial_m - m) \Psi = 0 \Leftrightarrow \begin{pmatrix} -m & i(\vec{p}_0 + \vec{\sigma} \cdot \vec{\vec{v}}) \\ i(\vec{p}_0 - \vec{\sigma} \cdot \vec{\vec{v}}) & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

When $m=0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left\{ \begin{array}{l} i(\vec{p}_0 - \vec{\sigma} \cdot \vec{\vec{v}}) \psi_L = 0 \\ i(\vec{p}_0 + \vec{\sigma} \cdot \vec{\vec{v}}) \psi_R = 0 \end{array} \right\} \rightarrow \underline{\text{Welfl eqns.}}$$

\rightarrow important for neutrinos & weak force studies..

For convenience let us define -

$$\sigma^u = (1, \vec{\sigma}), \quad \bar{\sigma}^u = (1, -\vec{\sigma})$$

In flat

$$\sigma^u = \begin{pmatrix} 0 & \sigma^u \\ \bar{\sigma}^u & 0 \end{pmatrix}, \quad \bar{\sigma}^u = (1, \sigma^1, \sigma^2, \sigma^3)$$

With this, can simplify rotation. Dirac eqn becomes -

$$\begin{pmatrix} -m & i\sigma \cdot \vec{\sigma} \\ i\bar{\sigma} \cdot \vec{\sigma} & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$i(\vec{\sigma} + \bar{\sigma} \cdot \vec{\nabla}) \quad i(\vec{\sigma} - \bar{\sigma} \cdot \vec{\nabla})$$

∴ the Weyl eqns become -

$$(i\bar{\sigma} \cdot \vec{\sigma})\psi_L = 0$$

$$(i\sigma \cdot \vec{\sigma})\psi_R = 0$$

A hint
 $\vec{p}^* = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}$

Free-particle solution of Dirac Eqn

Since Dirac field Ψ satisfies KG eqn, Ψ can be written as a lin. comb of plane waves -

$$\Psi(x) = u(p) e^{-ip \cdot x}, \quad p^2 = m^2$$

Look only solutions with positive frequency --- that is
 $E_p = p^0 > 0$

Ψ solves Dirac eqn $\rightarrow (i\gamma^\mu \partial_\mu - m)\Psi = 0$

$$\rightarrow (i\gamma^\mu \partial_\mu - m) u(p) e^{-ip \cdot x} = 0$$

$$\rightarrow \boxed{(i\gamma^\mu p_\mu - m) u(p) = 0}$$

Get rest frame $\Rightarrow p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $\Lambda_{\frac{1}{2}}$.

In rest frame, we have

$$(i\gamma^\mu p_\mu - m) u(p) \rightarrow (m\gamma^0 - m) u(p_0) = m(\gamma^0 - 1) u(p_0) = 0$$

$$\rightarrow m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$$

$$\rightarrow \boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \xrightarrow{\text{two-component spinor}}}$$

just a factor ξ with norm. constraint.

$$\xi^\dagger \xi = 1$$

ξ

What are those ξ ?

Look at rotation generators.

$$\boxed{S^a = \frac{1}{2} \epsilon^{abc} (\sigma^b \sigma^c)}$$

In particular, $S^2 = S'^2 = \frac{1}{2} \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$

$$\text{So if } \left\{ \begin{array}{l} \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$$

Now, we're in rest frame, so $\rho^A = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Now, boost to frame where particle has velocity ...

$\vec{v} = v \cdot \vec{z}$. Let $\tanh(\gamma) = \frac{v}{c}$.

$$\text{Then } \begin{pmatrix} F \\ p^3 \end{pmatrix} = p^{-n} = \begin{pmatrix} \cosh n & 0 & 0 & \sinh(n) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh n & 0 & 0 & \cosh(n) \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh n \\ 0 \\ 0 \\ m \sinh n \end{pmatrix}$$

(infinitesimal $\frac{1}{2}$)

1. \rightarrow just the Lorentz transform -

→ In this form,

$$E = m \cosh \eta$$

Now, apply the same boost to $x(p)$...

$$u(p) = \frac{1}{2} \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \exp\left(\frac{-i}{2} \omega_{\text{av}} t^3\right)$$

$$= \overbrace{\exp\left(\frac{-i}{2} \tau \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}\right)}^{\sim 0.3} \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So, infinitesimally -

$$\exp \left\{ \frac{-i}{2} \eta \left(\sigma^3 \alpha \right) \right\} \approx \begin{pmatrix} \cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3 & 0 \\ 0 & \cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3 \end{pmatrix}$$

So Rest

$$u(p) \approx \sqrt{m} \begin{pmatrix} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) \xi \\ (\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) \xi \end{pmatrix}$$

Simplify - note that

$$\begin{aligned} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3)^2 &= \dots \\ &= \cosh \eta - \sinh \eta \sigma^3 \\ &= E/m - P^3/m \sigma^3 = \frac{p \cdot \sigma}{m} \end{aligned}$$

$$= \frac{p \cdot \sigma}{m} \text{ where } \sigma^4 = (1, \vec{0})$$

$$\text{So } \{ (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \sigma}{m}}$$

$$\text{So } \{ (\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \bar{\sigma}}{m}}$$

So -

$$u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix} \rightarrow \text{constant} = \text{valid for any arbitrary direction of } p$$

Fact $(p \cdot \sigma)(p \cdot \bar{\sigma}) = (p^0)^2 - \vec{p}^2 = p^2 = m^2.$

(61)

Now, back to example -

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow p \cdot \sigma = \dots = \begin{pmatrix} E-p^3 & 0 \\ 0 & E+p^3 \end{pmatrix}$$

$$\xrightarrow{\text{so}} \sqrt{p \cdot \sigma} = \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}$$

and $\widetilde{\sqrt{p \cdot \sigma}} = \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}$

Pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then $(\text{spin } \frac{1}{2})$

$$u(p) = \begin{pmatrix} \sqrt{E-p^3} (1) \\ \sqrt{E+p^3} (0) \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then $(\text{spin } \frac{-1}{2})$

$$u(p) = \begin{pmatrix} \sqrt{E+p^3} (0) \\ \sqrt{E-p^3} (1) \end{pmatrix}$$

In the massless limit $\rightarrow E \rightarrow p^3$ ($E = \sqrt{p^2 + (p^3)^2}$)

$$\rightarrow \boxed{u(p) = \begin{pmatrix} (0) \\ \sqrt{2E} (1) \end{pmatrix} \text{ spin } \frac{1}{2}}$$

$$u(p) = \begin{pmatrix} \sqrt{2E} (0) \\ (0) \end{pmatrix} \text{ spin } \frac{-1}{2}$$

These states: $u(p) = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, $u(p) = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ are eigenstates of the helicity operator

$$\boxed{h = \vec{p} \cdot \vec{\sigma} = \sum_i \frac{1}{2} p_i^i \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}} \quad \frac{1}{2} \vec{p} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

When $h = \frac{1}{2} \rightarrow$ call Right-handed

$h = -\frac{1}{2} \rightarrow$ call Left-handed

Note: Dirac helicity is frame-dependent... (for massive particle) — since can boost so that momentum is in the opposite direction,

(This won't happen for massless particles).

Rush to Weyl's eqn:

$$\left\{ \begin{array}{l} i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) \Psi_L = i(\vec{\sigma} \cdot \vec{\nabla}) \Psi_L = 0 \\ i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \Psi_R = i(\vec{\sigma} \cdot \vec{\nabla}) \Psi_R = 0 \end{array} \right.$$

Plug $\Psi = u(p) e^{-ip \cdot x} \sim$, $\partial_0 \rightarrow -iE$

$$\vec{\nabla} \rightarrow i\vec{p}$$

So, with $m=0$, $\vec{p} = E\vec{p}$.

$$\Rightarrow h = \frac{-1}{2}$$

$$\Rightarrow \text{get } \left\{ (E + E\vec{p} \cdot \vec{\sigma}) \Psi_L = 0 \Rightarrow (E)(1+2h) \Psi_L = 0 \right.$$

$$\left. (E - E\vec{p} \cdot \vec{\sigma}) \Psi_R = 0 \Rightarrow (E)(1-2h) \Psi_R = 0 \right. \Rightarrow h = \frac{1}{2}$$

$\Rightarrow \left\{ \begin{array}{l} \Psi_L \text{ is left-handed} \\ \Psi_R \text{ is right-handed} \end{array} \right. , \text{ as expected}$

~~if~~

Recap... $\Psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 > 0) \rightarrow \text{positive frequency}$
 $\rightarrow u(p) = \begin{pmatrix} \sqrt{p_0} \xi \\ \sqrt{p_0} \bar{\xi} \end{pmatrix} \rightarrow \text{spinor.}$

when $\Psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 < 0) \rightarrow \text{negative frequency}$

$$\Rightarrow (u(p)) = \dots = \begin{pmatrix} \sqrt{p_0} \xi \\ -\sqrt{p_0} \bar{\xi} \end{pmatrix}$$

~~-g~~

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = (\xi^+ \sqrt{p_0} \xi + \bar{\xi}^+ \sqrt{p_0} \bar{\xi}) \cdot \begin{pmatrix} \sqrt{p_0} \xi \\ \sqrt{p_0} \bar{\xi} \end{pmatrix}$$

$$= \xi^+ \underbrace{[(p_0 \xi) + (p_0 \bar{\xi})]}_{\text{depends on } p!} \xi$$

$$\Rightarrow u^\dagger u = 2E_p \xi^+ \xi \text{ which}$$

\rightarrow ~~also~~ $u^\dagger u$ is not a Lorentz-inv scalar
just like $\Psi^\dagger \Psi$.

\Rightarrow to make one such Lorentz-inv scalar, define

$$\bar{u}(p) = u^\dagger(p) \gamma^0$$



$$\bar{u}u = 2m \xi^+ \xi$$

Lorentz-inv
(indep of \vec{p})

L, wish $\bar{u}n = u^T \gamma^0 n = 2m \xi^+ \xi^- = 2m$

→ convenient to choose ONB spinors. ξ^1, ξ^2 .

This gives 2 linearly indep solutions for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ \sqrt{p \cdot \sigma} & \xi^s \end{pmatrix} \quad s=1,2}$$

Normalize:

$$\langle \bar{u}^r(p) u^s(p) \rangle = 2m \delta^{rs} \Leftrightarrow \bar{u}^r(p) u^s(p) = 2E_p \delta^{rs}$$

For the negative-freq solns, we get

$$\langle \bar{v}^r(p) v^s(p) \rangle = -2m \delta^{rs} \Leftrightarrow \bar{v}^r(p) v^s(p) = -2E_p \delta^{rs}$$

and

v, u are orthogonal to each other.

$$\langle \bar{u}^r(p) v^s(p) \rangle = \langle \bar{v}^r(p) u^s(p) \rangle = 0$$

†

Finally, talk about spin sumrs

→ useful when evaluating Feynman diagrams.

→ when we need to sum all spin- $\frac{1}{2}$ polarizations

Since $\{\xi^s\}$ form an ONB,

$$\sum_{s=1,2} \xi^s \xi^{s*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\sum_{s=1,2} u^s(p) \bar{u}^s(p) = \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \bar{\sigma}} \xi^s} \right) \cdot \left(\xi^{s*} \sqrt{p \cdot \bar{\sigma}}, \xi^{s*} \sqrt{p \cdot \sigma} \right)$$

$$\begin{aligned} &= \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \bar{\sigma}} \xi^s} \right) \cdot \left(s \sqrt{p \cdot \bar{\sigma}}, s \sqrt{p \cdot \sigma} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &\xrightarrow{\text{"completeness"}} = \begin{pmatrix} p \cdot \sigma & m \\ m & p \cdot \bar{\sigma} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \sqrt{p \cdot \sigma} \sqrt{p \cdot \bar{\sigma}} &= \sqrt{((p^0, \vec{p}) \cdot (1, -\vec{\sigma}))((p^0, \vec{p}) \cdot (1, \vec{\sigma}))} \\ &= \sqrt{(p \cdot \bar{\sigma})(p \cdot \sigma)} = \sqrt{((p^0, \vec{p}) \cdot (1, -\vec{\sigma}))((p^0, \vec{p}) \cdot (1, \vec{\sigma}))} \\ &= \sqrt{(\vec{p})^2 - p^2} = m. \end{aligned}$$

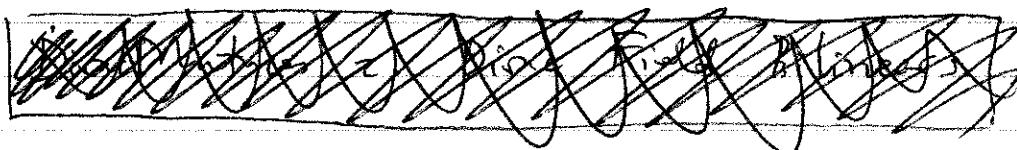
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$\sum_{s=1,2} u^s(p) \bar{u}^s(p) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \bar{\sigma} & m \end{pmatrix} = p \cdot \gamma + m I$	Feynman-slash notation
$\sum_{s=1,2} v^s(p) \bar{v}^s(p) = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \bar{\sigma} & -m \end{pmatrix} = p \cdot \gamma - m I$	

→ The comb. J-p occur so often that Feynman introduced the notation:

$$\not{p} = \gamma^\mu p_\mu = p_\mu \gamma^\mu$$

~~the~~



Excuse

Recall that $\psi = (\psi_L \psi_R)$

Let ψ_L^* be the complex conjugate of ψ_L .
The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

m = Majorana mass.

- ① Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.
- ② Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

a) In general, infinitesimal Lorentz transform in Ψ_L has the form

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} + \vec{\beta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

→ Rotation has the form:

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

$$\Rightarrow \sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L(x) \quad (\text{notes})$$

Lorentz transformation:

$$\Psi_L(x) \rightarrow A_{\frac{1}{2}} \Psi_L(A^{-1}x)$$

$$\partial_\mu \Psi_L(x) \rightarrow (A^{-1})^\mu_\nu \partial_\nu \Psi_L(A^{-1}x)$$

→ put these together:-

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(A^{-1}x)$$

$$\sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(A^{-1}x)$$

$$\rightarrow -im \sigma^2 \Psi_L^*(x) \rightarrow -im \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(A^{-1}x)$$

$$\text{Next, } i\vec{\sigma} \cdot \partial \Psi_L(x) = i\vec{\sigma}^M \partial_M \Psi_L(x)$$

$$\rightarrow i\vec{\sigma}^M (A^{-1})^\mu_\nu \partial_\mu \Psi_L(A^{-1}x) \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right)$$

$$= i\vec{\sigma}^M \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) (A^{-1})^\mu_\nu \partial_\mu \Psi_L(A^{-1}x)$$

we find: multiply:

$$1 = \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \left(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \quad (\text{not } x \text{ inv. not})$$

$$\rightarrow \rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^{\mu} (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})$$

$$\times (\Lambda')^{\alpha}_{\mu} \partial_{\alpha} \Psi_L(\Lambda' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \boxed{?} (\Lambda')^{\alpha}_{\mu} \partial_{\alpha} \Psi_L(\Lambda' x)$$

Want is $\boxed{?}$

$$\rightarrow \boxed{?} = (1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^{\mu} (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})$$

$$\approx \bar{\sigma}^{\mu} + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} \bar{\sigma}^{\mu} - i\vec{\theta} \cdot \vec{\sigma} \frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^{\mu} - \frac{i}{2} \vec{\theta} [\bar{\sigma}^{\mu}, \vec{\sigma}]$$

$\left. \begin{array}{c} \\ \\ \end{array} \right\}$ can show want

$$= \bar{\sigma}^{\mu} - i\vec{\theta} [J_{\mu}^{ij}] \vec{\sigma}^j$$



$$i(g^{ji}\delta_{\mu}^j - g^{ji}\delta_{\nu}^j)$$

$$\Rightarrow \boxed{?} = (\Delta_4)^{\mu}_{\nu} \bar{\sigma}^{\nu} \rightarrow \bar{\sigma}^{\mu} transforms like 4-vector$$

$$\rightarrow i\vec{\theta} \cdot \partial \Psi_L(x) \rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \Delta^{\mu}_{\nu} \bar{\sigma}^{\nu} (\Lambda')^{\alpha}_{\mu} \partial_{\alpha} \Psi_L(\Lambda' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \delta^{\alpha}_{\nu} \bar{\sigma}^{\nu} \partial_{\alpha} \Psi_L(\Lambda' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma} \cdot \partial \Psi_L(\Lambda' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma} \cdot \partial \Psi_L(\Lambda' x) \checkmark$$

$$\rightarrow i\vec{\sigma} \cdot \vec{r} \psi_c(x) - i m \vec{\sigma}^2 \psi_c^*(x) = 0$$

\rightarrow leads to infinitesimal rotation ...

$$(1 - i\vec{\beta} \cdot \frac{\vec{\sigma}}{2}) \left\{ i\vec{\sigma} \cdot \vec{r} \psi_c(x) - i m \vec{\sigma}^2 \psi_c^*(x) \right\} = 0$$

\Rightarrow done! So Majorana eqn is invariant under infinitesimal rotations.

~~q~~

① Boosts (proceed in a similar way ...)

Key

$$(1 - \vec{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \vec{\beta} \cdot \frac{\vec{\sigma}}{2})$$

$$= \bar{\sigma}^M - \frac{1}{2} \vec{\beta} \{ \bar{\sigma}^M, \vec{\sigma} \}$$

$$\Rightarrow \bar{\sigma}^M \rightarrow \bar{\sigma}^M - i \vec{\beta} [J^a]_x \bar{\sigma}^a$$

$$\rightarrow (1 - \vec{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \vec{\beta} \cdot \frac{\vec{\sigma}}{2}) = (1)_x \bar{\sigma}^M \text{ as before}$$

so plug this back into the original eqn

~~k~~

Sep 28, 2020

Dirac Matrices & Dirac Field Bilinears

Oct 2, 2020 Recall that $\bar{\psi}\psi$ is Lorentz scalar.

Recall that $\bar{\psi}\gamma^\mu\psi$ is also a 4-vector.

→ [?] Consider $\bar{\psi}\Gamma\psi$, where Γ is any 4×4

→ can we decompose Γ into terms that have definite transformation properties under the Lorentz group?

↳ Γ can be written as combo of 16-element basis defined by

$$\left\{ \begin{array}{ll} 1: & \mathbb{1} \rightarrow 1 \\ 4: & \gamma^\mu \rightarrow 4C2 \\ 6: & \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{\mu\nu} \gamma^3 \rightarrow 4C3 \\ 4: & \gamma^{\mu\nu\rho} = \gamma^{\mu\nu} \gamma^\rho \rightarrow 4C2 \\ 1: & \gamma^{\mu\nu\rho\sigma} = \gamma^{\mu\nu} \gamma^\rho \gamma^\sigma \rightarrow 4C2 \end{array} \right.$$

16 total:

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisymmetric tensor of successively higher ranks.

→ Introduction

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$= -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$$

$$\begin{aligned} 0123 &\rightarrow 1 \\ 7023 &\rightarrow -1 \end{aligned}$$

↳ totally anti-symmetric

$$\text{Note that } \rightarrow (8^5)^{\frac{2}{5}} = 11$$

$$\Rightarrow \overline{(y^s)^+} = -i(y^?)^+ - (y^e)^+$$

$$= + \gamma^0 \gamma^1 \gamma^2 \gamma^3 = - \gamma^3 \gamma^2 \gamma^1 \gamma^0 = \gamma^5$$

also

$$\{g^5, g^6\} = \cancel{g^0 g^1 g^2 g^3 g^4} + \cancel{g^1 g^2 g^3 g^4 g^5} (-1) = 0$$

Carol Haw

$$[g^\varepsilon, s^{\mu\nu}] = [g^\varepsilon, \frac{i}{2} \{ g^\mu, g^\nu \}] = 0$$

⇒ Eigenstates of \hat{S}^z with different eigenvalues don't mix under Lorentz-transf.m.

→ In basis, can write --

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \text{for } \gamma_L \text{ (left-hd)} \\ \rightarrow \text{for } \gamma_R \text{ (right-hd)}$$

→ a Dirac spinor with only LR component is an eigenstate of γ^5 with eigenv $(-1)/(1)$.

With δ^5 , can rewrite the table of 4×4 matrices as

$\frac{1}{2} g^{\mu\nu}$	scalar	1
δ^{MN}	vector	4
$\delta^M \gamma^S$	tensor	6
γ^S	pseudo vector	4
	pseudo scalar	1
		16

pseudo vector/scalar is due to the fact that they transform like vectors/scalars, BUT with an additional \rightarrow under Lorentz transf \rightarrow in charge under parity-transf.

Ex Parity transf: $\vec{x} \rightarrow -\vec{x}$

$$\downarrow (x^0, x^i) \rightarrow (x^0, -x^i)$$

If instead $(x^0, \vec{x}) \rightarrow -(x^0, \vec{x}) = (-x^0, \vec{x})$ under parity, we call this a pseudo-vector \rightarrow pseudo vector/scalar flips sign under parity transf.

\rightarrow From vector + pseudo-vector we can form 2 currents out of Dirac field bilinears

$j^{\mu}(x) = \bar{\psi}(x) \gamma^\mu \psi(x) \rightarrow$ vector current
$j^{\mu 5}(x) = \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x) \rightarrow$ pseudo-vector current

Assume that ψ satisfies Dirac eqn. $\bar{\psi} = \gamma^5 \psi$

$$\rightarrow i \not{D} \psi = m \psi \quad \rightarrow i \not{D} \bar{\psi} = m \bar{\psi} \quad (\text{since } \bar{\psi} = \gamma^5 \psi)$$

\rightarrow compute div of these currents.

$$\partial_\mu j^{\mu} = (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \bar{\psi} \gamma^\mu (\partial_\mu \psi)$$

$$= i m \bar{\psi} \psi + \bar{\psi} (-im \psi) = 0$$

$$\Rightarrow \boxed{\partial_\mu j^\mu = 0}$$

(77)

$\rightarrow j^m$ is always conserved if $\psi(x)$ satisfies
Dirac eqn

\rightarrow It is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{im\theta} \psi(x)$$

Similarity

$$\begin{aligned} \partial_m j^{m*} &= (\partial_m \bar{\psi}) \gamma^0 \gamma^5 \psi + \bar{\psi} \gamma^5 \gamma^0 \partial_m \psi \\ &= \underbrace{(\partial_m \bar{\psi}) \gamma^0 \gamma^5 \psi}_{= \text{im } \bar{\psi} \gamma^5 \psi} + (-1) \bar{\psi} \gamma^5 \underbrace{\partial_m \psi}_{= -im \bar{\psi} \gamma^5 \psi} \\ &= \text{im } \bar{\psi} \gamma^5 \psi + (-1)(-i)m \bar{\psi} \gamma^5 \psi \end{aligned}$$

$\rightarrow \boxed{\partial_m j^{m*} = 2 \text{im } \bar{\psi} \gamma^5 \psi} \rightarrow$ axial vector current

\rightarrow if $m=0$ then $\partial_m j^{m*}$ is conserved.

\rightarrow When $m=0$, j^m is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{im\theta} \psi(x)$$

(we worry about place the rest of this section in Weyl part II ...)

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QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi = \bar{\psi} (i \gamma^\mu \vec{\nabla}_\mu - m) \psi.$$

→ Canonical momentum conjugate to ψ is

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} = i \bar{\psi} \gamma^\mu = i \psi^+$$

→ Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \partial_\mu \psi - \mathcal{L} = i \bar{\psi} \gamma^\mu \partial_\mu \psi - i \bar{\psi} \gamma^\mu \partial_\mu \psi \\ &\quad - i \bar{\psi} \vec{\gamma} \cdot \vec{\nabla} \psi + m \bar{\psi} \psi \\ &= -i \bar{\psi} \vec{\gamma} \cdot \vec{\nabla} \psi + m \bar{\psi} \psi \end{aligned}$$

Thus, $\boxed{\mathcal{H} = \int \mathcal{H} d^3x = \int d^3x \bar{\psi} (-i \vec{\gamma} \cdot \vec{\nabla} + m) \psi}$

→ now let's figure out the constraints to make this a quantum field theory..

→ HOW NOT TO QUANTIZE THE DIRAC FIELD

This won't work!

Guess $[\psi_a(\vec{x}), i\psi_b^+(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}$

\uparrow spin components $(a, b = 1, 2, 3, 4)$

i.e.

$$[\psi_a(\vec{x}), \psi_b^+(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}.$$

In matrix notation...

$$[\psi_a(\vec{x}), \psi_b^+(\vec{y})] = \frac{1}{4 \times 4} \delta^{(3)}(\vec{x}-\vec{y})$$

$\downarrow \quad \downarrow$

$$\begin{bmatrix} \cdot & \cdots & \end{bmatrix}$$

Also guess $[\psi_a(\vec{x}), \psi_c(\vec{y})] = 0$

$$[\psi_a^+(\vec{x}), \psi_b^+(\vec{y})] = 0$$

No \neq real

$$[\psi(\vec{x}), \psi(\vec{y})] = [\psi(\vec{x}), \psi(\vec{y})] \gamma^0$$

$$= [\psi(\vec{x}), \psi^+(\vec{y})] \gamma^0 = \delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote -

(real) field $\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \hat{a}_p + \hat{a}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}$. (FT)

For complex field \rightarrow we get

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \hat{a}_p + \hat{b}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}$$

In the case of Dirac field, need spin degrees of freedom

Try --

$$\Psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{+i\vec{p}\cdot\vec{x}}$$

↑
Spin degrees
of freedom

Former components: $\Psi(x) = u(p) e^{i\vec{p}\cdot\vec{x}}$

$$2 \quad \Psi^+(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

Recall about u, v also solves Dirac eqn in the reverse
heat (in momentum space--)

$$p^\mu \delta_\mu u^r(p) = mu^r(p) = p^\mu \delta_\mu v^r(p) = -mv^r(p)$$

we can do the commutators --

$$[\hat{a}_p^r, \hat{a}_{p'}^{r'}] = (2\pi)^3 \delta^{rr'} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{b}_p^r, \hat{b}_{p'}^{r'}] = -(2\pi)^3 \delta^{rr'} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{a}_p^r, \hat{b}_{p'}^{r'}] = 0$$

the rest are all zero --

we find heat

\rightarrow as desired --

$$[\Psi_a(\vec{x}), \Psi_b(\vec{y})] = 0 = [\Psi_a^+(\vec{x}), \Psi_b^+(\vec{y})]$$

77

We also find that

$$[\Psi_a(\vec{x}), \Psi_b^*(\vec{y})] = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian.

$$H = \int d^3x \left[-i\vec{\nabla} \cdot \vec{\Phi} + m\vec{\Phi}^2 \right]$$

$$= \int d^3x \left\{ \Psi^* \underbrace{\vec{\nabla} \cdot [\vec{\Phi} + m]}_{\text{just const}} \Psi \right\}$$

Now, with $\vec{p}^r \partial_{\vec{p}} u^r(p) = mu^r(p)$

$$\rightarrow (\vec{p} \cdot \vec{\nabla} + m) u^r(p) = \vec{p}^r \vec{\nabla} u^r(p) = E_p u^r(p)$$

~~Similarly, SIC $\vec{p}^r \partial_{\vec{p}} v^r(p) = -mv^r(p)$~~

~~$\rightarrow (\vec{p} \cdot \vec{\nabla} + m) v^r(p) = -E_p v^r(p)$~~

So ...

$$\begin{aligned} \rightarrow [-i\vec{\nabla} \cdot \vec{\Phi} + m] \Psi &= [-i\vec{\nabla} \cdot \vec{\Phi} + m] \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_p^r u^r(p) + b_p^r v^r(p)] e^{i\vec{p} \cdot \vec{x}} \\ &= \vec{\nabla} \cdot \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ E_p a_p^r u^r(p) - E_p b_p^r v^r(p) \right\} e^{i\vec{p} \cdot \vec{x}} \end{aligned}$$

So ...

$$H = \int d^3x \left\{ \Psi^* \vec{\nabla} \cdot \vec{\Phi} + \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ - \right\} e^{i\vec{p} \cdot \vec{x}} \right\}$$

play in ...

$$\rightarrow H = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} E_p \left\{ a_p^r a_p^r - b_{+p}^r b_{+p}^{r+} \right\}$$

$$b_{+p}^{r+} b_{+p}^r + \text{const}$$

!

\rightarrow By creating more and more particles with b^+ , we can lower the energy indefinitely

\rightarrow This is bad...

\rightarrow So we should use Fermi-Dirac statistics instead \rightarrow anti-commutators instead of commutators

\rightarrow Requirements.

$$\{a_p^r, a_q^s\} = \{b_{+p}^r, b_{+q}^{s+}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q})$$

↑
no longer bosonic! all other
anti-commutators
are zero...

When this is true, we did ok!

$$\{\psi_a(x), \psi_b^+(y)\} = \delta^{(3)}(\vec{x} - \vec{y}) \delta_{ab}$$

$$\{\psi_a(x), \psi_b(y)\} = \{\psi_a^+(x), \psi_b^+(y)\} = 0$$

above we're using

$$\psi(x) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p^r u_r(\vec{p}) + b_{-p}^{r+} v_r(-\vec{p}) \right] e^{i\vec{p} \cdot \vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p \left(a_p^{rt} a_p^r - b_{-p}^{rt} b_{-p}^r \right) - b_{-p}^{rt} b_p^r + \text{const}$$

$$\Rightarrow \mathcal{H} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ a_p^{rt} a_p^r + b_{-p}^{rt} b_{-p}^r \right\}$$

now good, b/c E is bold below...

→ also can compute

$$\tilde{P} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} \tilde{p} \left(a_p^{rt} a_p^r + b_{-p}^{rt} b_{-p}^r \right)$$

To avoid sign confusion, we will usually write

$$\Psi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u_r^r(\vec{p}) e^{-ip \cdot \vec{x}} + b_{-p}^{rt} v_r^r(\vec{p}) e^{-ip \cdot \vec{x}} \right)$$

As a Heisenberg field,

$$\Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u_r^r(\vec{p}) e^{-ip \cdot \vec{x}} + b_{-p}^{rt} v_r^r(\vec{p}) e^{ip \cdot \vec{x}} \right)$$

where:

- a_p^r : annihilates particles
- a_p^{rt} : creates particles
- b_p^{rt} : annihilates anti-particles
- b_{-p}^r : creates anti-particles.

Vacuum state is $|0\rangle$ where

$$\begin{cases} \hat{a}_p^\dagger |0\rangle = 0 \\ \hat{b}_p^\dagger |0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ constant norm:

$$|\vec{p}, s\rangle = \sqrt{2E_p} |\hat{a}_p^\dagger |0\rangle$$

so that

$$|\vec{p}, s\rangle |\vec{q}, r\rangle = \sqrt{2E_p} \sqrt{2E_q} \delta^{(3)}(\vec{p} - \vec{q}) \delta^{rs}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lorentz}} \psi'(x) = \gamma \frac{1}{\sqrt{2}} \psi(\gamma^{-1} x)$$

$$\begin{aligned} \text{recall Rest with } \gamma &\rightarrow \exp\left\{-i\omega_{uv}\gamma^0/2\right\} \approx 1 - i\vec{\theta} \cdot \vec{\varepsilon} \\ \left\{ \begin{array}{l} \omega_2 = -\omega_{21} = \theta \\ S^{uv} = \frac{1}{2} [\vec{x}^u, \vec{x}^v] \end{array} \right. &= 1 - i\vec{\theta} \cdot \vec{\varepsilon} \end{aligned}$$

$$\rightarrow \text{and } \psi(\gamma^{-1} x) \approx (1 - \vec{\theta} \cdot \vec{\varepsilon}) \psi(x)$$

$$\vec{\varepsilon} = \vec{x} \times (-i\vec{\beta})$$

So we do $\psi \rightarrow \psi' + S\psi$ where

$$S\psi = \psi' - \psi = \left(\frac{i}{2}\vec{\theta} \cdot \vec{\varepsilon}\right)\psi(x) - \vec{\theta} \cdot (\vec{x} \times \vec{\beta}) \psi(x)$$

By Noether's Thm,

(81)

$$\vec{J}_{\text{total}} \text{ (total spin)} = \frac{1}{2} \vec{\epsilon} \times [\psi^\dagger (-i \vec{\sigma} \cdot \vec{\nabla}) \psi + \frac{1}{2} \bar{\psi} \vec{\sigma} \psi]$$

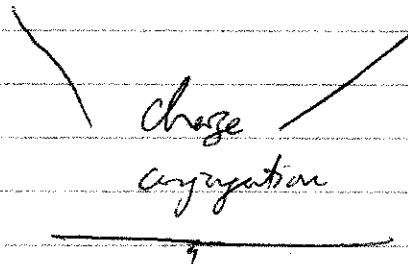
We won't worry about the rest of this section about propagators

→ we'll come back to them later when looking at Feynman diagrams.

DISCRETE SYMMETRIES OF THE DIRAC THEORY

Basically, we have

Parity — Time reversal



Recall that we before, we looked at implementation of continuous Lorentz transform -

→ found that $\gamma^1 \in \text{Lorentz group}$

$\exists U(1)$ unitary for which

$$U(1) \psi(x) U(1)^\dagger = \gamma_2 \psi(1x)$$

→ Now, we'll look about discrete symmetries on the Dirac field.

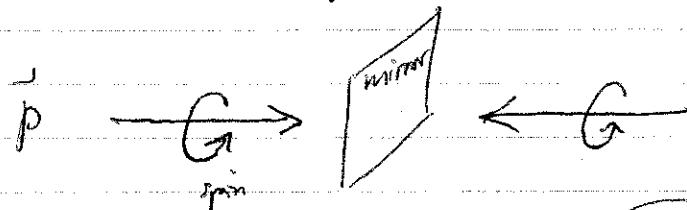
Apart from continuous Lorentz transformations, there are other spacetime-transformations for which the Lagrangian might remain invariant =

→ e.g. { time-reversal },
{ parity }.

[Parity] (P) : flips direction of spatial vectors

$$P: (t, \vec{x}) \rightarrow (t, -\vec{x})$$

→ mirror sym → change the handedness.



→ Note momentum flip sign, but spin is unchanged.

Time-reversal

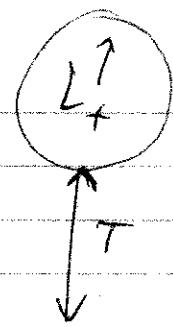
$$T: (t, \vec{x}) \rightarrow (-t, \vec{x})$$

P,T don't belong to the "proper" Lorentz group ↑

→ the full Lorentz group breaks into
4 disconnet subsets ...

()

(P3)



$$\xleftarrow{P} \quad \rightarrow$$

$$L^{\dagger} = PL_+^T$$

"orthochronous"

$$L_+^L = T P L_+^T$$

$$\xleftarrow{P} \quad \rightarrow$$

$$L^L = PTL_+^T$$

"non-orthochronous"

"proper"

"improper"

charge conjugation

\rightarrow intercharge particles & anti-particles.

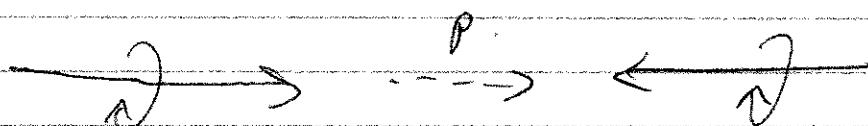
\hookrightarrow non-space-time.

Let's look at Parity.

Note that because $P: (t, \vec{x}) \rightarrow (t, -\vec{x})$

\rightarrow momentum flips sign

but not spin! \rightarrow what is P ? As an operator?



As an operator on creation/annihilation ops, we want

$$P^\dagger a_p^\dagger P = a_{-p}^\dagger \quad \& \quad P^\dagger b_p^\dagger P = b_{-p}^\dagger$$

where, as discussed, P must be unitary.

$$PP^t = P^tP = \mathbb{1}.$$

Taking adjoint, set

$$\boxed{P^t a_{\vec{p}}^s P = a_{-\vec{p}}^{s^t} \quad P^t b_{\vec{p}}^s P = b_{-\vec{p}}^{s^t}}$$

But there might be too restrictive --- we can get better constraints by requiring that

$$\boxed{P^t a_{\vec{p}}^s P = \eta_a a_{-\vec{p}}^{s^t} \quad P^t b_{\vec{p}}^s P = \eta_b b_{-\vec{p}}^{s^t}}$$

as long as $|\eta_a|^2 = |\eta_b|^2 = 1$ are phases!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel!

$$\boxed{P^t a_{\vec{p}}^{s^t} a_{\vec{p}}^s P = a_{-\vec{p}}^{s^t} a_{-\vec{p}}^s}$$

$$\boxed{P^t b_{\vec{p}}^{s^t} b_{\vec{p}}^s P = b_{-\vec{p}}^{s^t} b_{-\vec{p}}^s}$$

With this, let's ~~see~~ implement parity condition on $\psi(x)$

$\rightarrow P^t \psi P = ?$ (to find out what these η_a, η_b must be ...)

$$\hat{P}^t \chi(x) P = \int \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon_p}} \sum_{s=1,2} (\gamma_a a_{-\vec{p}}^s u^s(p) e^{-i\vec{p} \cdot \vec{x}} + \gamma_b^* b_{-\vec{p}}^s v^s(\vec{p}) e^{i\vec{p} \cdot \vec{x}})$$

Define $\begin{cases} \tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}) \end{cases}$

Note that

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad \text{where } \sigma = (1, \vec{\sigma}) \quad \bar{\sigma} = (1, -\vec{\sigma})$$

$$\begin{aligned} &= \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} \underbrace{u^s(-\vec{p})}_{\sim} \underbrace{\gamma^0}_{\sim} \\ &= \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} u^s(-\vec{p}) \end{aligned}$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\vec{p})}$$

and

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \dots = \begin{pmatrix} 0 & -\alpha \\ -\beta & 0 \end{pmatrix} v^s(-\vec{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(-\vec{p})}$$

With these, we find that

$$\tilde{p} \cdot \tilde{x} = p \cdot x$$

(86)

$$P^+ \bar{\psi}(x) P = \gamma^0 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(\eta_a a_s^\dagger u^s(p) e^{-ip \cdot \tilde{x}} + \eta_b^* b_s^\dagger v^s(-p) e^{ip \cdot \tilde{x}} \right)$$

Now, notice that if $\eta_a = \eta_b^*$ then it's "nice":

$$(\eta_a = \eta_b^*) \Rightarrow P \bar{\psi}(x) P = \eta_a \gamma^0 \bar{\psi}(\tilde{x}) \rightarrow \text{Proof}$$

In final form

$$\rightarrow \text{suffices to choose } \{\eta_a = 1 = -\eta_b\}$$

relative sign between fermions - antifermions --

-a-

Now, we fail to know how various Dirac field bilinears transform under parity -

Real... 5 of them:

$$\bar{\psi} \psi, \bar{\psi} \gamma^\mu \psi, ; \bar{\psi} [\gamma^\mu, \gamma^\nu] \psi$$

$$\bar{\psi} \gamma^\mu \gamma^\nu \psi, ; \bar{\psi} \gamma^\nu \psi.$$

to find these, first compute: $P \bar{\psi}(x) P$

$$\overline{P \bar{\psi}(x) P} = P^+ \bar{\psi}^+(x) \gamma^0 P = (P^+ \bar{\psi} P)^+ \gamma^0 \quad (\gamma^0 = \gamma^0)$$

$$\rightarrow = \eta_a^* (\gamma^0 \bar{\psi}(\tilde{x}))^+ \gamma^0 = \eta_a^* \bar{\psi}^+(\tilde{x}) \gamma^0 \gamma^0$$

$$\rightarrow \boxed{P^+ \bar{\psi} P = \eta_a^* \bar{\psi}(\tilde{x}) \gamma^0}$$

Wish this --

$$\begin{aligned} p^\dagger \bar{\psi} \psi p &= \underbrace{p^\dagger \bar{\psi}(x) p}_{(x)(x)} \underbrace{p^\dagger \psi(x) p}_{\text{II}} \\ &= \gamma_a \bar{\psi}(\tilde{x}) \gamma^0 \gamma_\mu \gamma^0 \psi(\tilde{x}) \\ &= |\gamma_a|^2 \bar{\psi}(\tilde{x}) \psi(\tilde{x}) \end{aligned}$$

Rules

$$p^\dagger \bar{\psi} \psi p(x) = \bar{\psi} \psi(\tilde{x}). \quad (\text{scalar})$$

linear.

can also show

$$\begin{aligned} p^\dagger \bar{\psi}(x) \gamma^\mu \psi p &= \bar{\psi} \gamma^0 \gamma^\mu \gamma^0 \psi(\tilde{x}) \\ (\text{vector field}) &= \begin{cases} + \bar{\psi} \gamma^\mu \psi(\tilde{x}) & \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi(\tilde{x}) & \mu = 1, 2, 3 \end{cases} \end{aligned}$$

$$p^\dagger (i \bar{\psi} \gamma^5 \psi) p = i \bar{\psi}(\tilde{x}) \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) = -i \bar{\psi} \gamma^5 \psi(\tilde{x})$$

~~$\bar{\psi} \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(\tilde{x})$~~ $\mu = 0$
 ~~$\bar{\psi} \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(\tilde{x})$~~ $\mu = 1, 2, 3$

$$\begin{aligned} p^\dagger \bar{\psi} \gamma^\mu \gamma^5 \psi p &= \bar{\psi}(\tilde{x}) \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(\tilde{x}) \\ (\text{pseudo}) &= \begin{cases} - \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) & \mu = 0 \\ + \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) & \mu = 1, 2, 3 \end{cases} \end{aligned}$$

(-)

Note The relative sign $-\eta_a = \eta_b^*$ is important.

for the relationship between fermion - anti-fermion

Consider antifermion - antifermion state -

$$a_p^{st} b_q^{st} |0\rangle \xrightarrow{P} P(a_p^{st} b_q^{st} |0\rangle)$$

$$= P^t (a_p^{st} b_q^{st}) P |0\rangle$$

$$= \underbrace{P^t a_p^{st} P P^t b_q^{st} P}_{\text{cancel}} |0\rangle$$

$$= (\eta_a) a_{-p}^{st} \eta_b b_{-q}^{st} |0\rangle$$

$$= -(\eta_b \eta_a^*) a_{-p}^{st} b_{-q}^{st} |0\rangle$$

$$= -a_{-p}^{st} b_{-q}^{st} |0\rangle$$

→ a state containing a fermion-antifermion pair gets an (-1) under parity does form extra

4

TIME REVERSAL

if T is unitary $\Rightarrow [T, H] = 0$

$$\rightarrow T^+ e^{iH} T = e^{iH} T^+ T = e^{iH}$$

→ no good ..

What if $T^2 T = -H$? or $[T, H] = 0$?

But this is no good either since implies that H is unbounded.

\rightarrow Assume this -

"Time-reversal is conjugate-linear/anti-linear"

Assume:

T is unitary

$$T^* T = c^* \quad (c \in \mathbb{C}).$$

$$[T, H] = 0$$

Wish those

$$T^* e^{-iHt} T = e^{-iHt} \quad \checkmark$$

\rightarrow time reversal: momentum

\downarrow

spin

are reversed

\rightarrow like watching a movie played backwards

$$G \xrightarrow{T} \xleftarrow{f}$$

Flipping momentum is easy.

What abt flipping spin? We know that

In some basis --

$$\xi(\uparrow) = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} (1)$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} (0)$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s=1, 2$ & define

reversed
spin

$$\xi^{-s} = -i\sigma^2 (\xi^s)^*$$

↳ This is the flipped spinor

It is clear that

$$\begin{aligned} \xi^{-s} &= -i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (\xi(\uparrow), \xi(\downarrow))^* \\ &= (\xi(\downarrow), -\xi(\uparrow))^* \end{aligned}$$

where $\xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^1$

$$\xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^2$$

→ This is convenient since our time reversal op. involves complex conjugation --

$$\rightarrow \text{Can show: } [\tilde{u}^s(-\vec{p})] = \begin{pmatrix} \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{st} \\ \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{st} \end{pmatrix}$$

So if we use the identity ...

$$\{\sqrt{p \cdot \sigma} \xi^2 = \sigma^2 \sqrt{p \cdot \sigma^2}\} \quad (\text{from using } \sigma^2 = -\bar{\sigma}^2 \sigma^2)$$

then we get

$$\begin{aligned} u^{-s}(\tilde{p}) &= \begin{pmatrix} \sqrt{p \cdot \sigma} (-i\sigma^2) \xi^{s\#} \\ \sqrt{p \cdot \bar{\sigma}} (-i\sigma^2) \xi^{s\#} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2) \sqrt{p \cdot \sigma^2} \xi^{s\#} \\ (-i\sigma^2) \sqrt{p \cdot \bar{\sigma}^2} \xi^{s\#} \end{pmatrix} \\ &= (-i) \begin{pmatrix} \sigma^2 & 0 \\ 0 & \bar{\sigma}^2 \end{pmatrix} [u^s(p)]^* = -\sigma^2 \bar{\sigma}^2 [u^s(p)]^* \\ &\Rightarrow u^{-s}(\tilde{p}) = -\sigma^2 \bar{\sigma}^2 [u^s(p)]^* \quad \begin{array}{l} \text{element-wise} \\ \text{cnpt conjugation} \end{array} \end{aligned}$$

similarly,

$$v^{-s}(\tilde{p}) = -\sigma^2 \bar{\sigma}^2 [v^s(p)]^*$$

in this relation, v^{-s} contains

$$\xi^{(-s)} = -\xi^s$$

a 360° flip

introduces

a $(-)$ sign.

~~ket state flip~~

~~but~~

Now we can define time reversal operation as
the creation - annihilation operator

here's
can't
act here → $T a_p^s T = \bar{a}_{-\bar{p}}^s$ & $T b_p^s T = \bar{b}_{-\bar{p}}^s$ ↑ flip \vec{p} ↓ flip momentum

above $a_{-\bar{p}}^s = (a_{\bar{p}}^\downarrow, -a_{\bar{p}}^\uparrow)$
we now define $b_{-\bar{p}}^s = (b_{\bar{p}}^\uparrow, -b_{\bar{p}}^\downarrow)$ just like what we did with $\zeta^s = (\zeta(\uparrow), -\zeta(\downarrow))$

if $a_p^s = (a_p^\uparrow, a_p^\downarrow)$ analogous to what we did before
 $b_p^s = (b_p^\uparrow, b_p^\downarrow)$

With this, let's evaluate $T \Psi(x) T$:

$$\rightarrow T^\dagger \Psi(x) T = \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1,2} T^\dagger (a_p^s u_p^s(p) e^{-ip \cdot x} + b_p^{s\dagger} v_p^s(p) e^{+ip \cdot x}) T$$

$$= \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1}^2 \left\{ a_{-\bar{p}}^s [u_{\bar{p}}^s(p)]^\dagger e^{-i\bar{p} \cdot x} + b_{-\bar{p}}^{s\dagger} [v_{\bar{p}}^s(p)]^\dagger e^{-i\bar{p} \cdot x} \right\}$$

where under T , $= \gamma^1 \gamma^2 \gamma^3 \Psi(x_T)$, $x_T = (-\bar{x}, \bar{x})$

$$T^\dagger a_p^s \rightarrow a_{-\bar{p}}^s$$

$$T^\dagger u_p^s(p) = -i\gamma_5 \gamma_1 \gamma_2 \gamma_3 u_{\bar{p}}^s(p)$$

$$\rightarrow T^\dagger b_p^s(p) = i\gamma_5 \gamma_1 \gamma_2 \gamma_3 b_{\bar{p}}^{s\dagger}(p)$$

$$T^\dagger e^{-ip \cdot x} T = \gamma^1 e^{+i\bar{p} \cdot x}; T^\dagger u_p^s T = [u_{\bar{p}}^s]^*$$

note sign
choose here (93)

Because $\{\tilde{u}^i(p)\}^* = \delta_{ij} u^j(\tilde{p})$, we have

$$\begin{aligned} T^+ \gamma(x) T &= \gamma' \gamma^3 \int \frac{d^3 \tilde{p}}{(2\pi)^3 \sqrt{2E_{\tilde{p}}}} \sum_{s=1}^2 \left\{ a_{\tilde{p}}^{-s} \tilde{u}^s(\tilde{p}) e^{i\tilde{p}(t_1, \tilde{x})} \right. \\ &\quad \left. + b_{\tilde{p}}^{-s} \tilde{v}^{-s}(\tilde{p}) e^{-i\tilde{p}(t_1, \tilde{x})} \right\} \\ &= \gamma' \gamma^3 \gamma(-t, x) \\ &= -\tilde{p}(-t, x) \end{aligned}$$

$$\Rightarrow \boxed{T^+ \gamma(x, t) T = \gamma' \gamma^3 \gamma(x, -t)}$$

Next, can check the action of T on bilinears...

$$\begin{aligned} T^+ \bar{\gamma} T &= T^+ \bar{\gamma} \gamma^0 T = T^+ \bar{\gamma}^+ \gamma^0 \xrightarrow{\text{real}} \\ &= (\gamma' \gamma^3 \gamma(x_T))^+ \gamma^0 = \gamma^+ (x_T) \gamma^3 \gamma^1 \gamma^0 \\ &\equiv \gamma^+ (x_T) \gamma^3 \gamma^1 \gamma^0 \quad \begin{matrix} \uparrow & \uparrow \\ -\gamma^3 & -\gamma^1 \end{matrix} \\ &= + \gamma^+ (x_T) \gamma^0 \gamma^3 \gamma^1 \end{aligned}$$

$$\Rightarrow \boxed{T^+ \bar{\gamma} T = -\bar{\gamma}(x_T) \gamma^1 \gamma^3}$$

with this, can compute the rest---

$$\underline{\text{Scalar}} \quad \boxed{T \bar{\gamma} \gamma T = \bar{\gamma} (-\gamma' \gamma^3) \underbrace{|\gamma' \gamma^3|}_{11} \gamma(x_T) = \bar{\gamma}(x_T) \gamma(x_T)}$$

Pseudoscalar \rightarrow set (-)

$$\boxed{T i \bar{\gamma} \gamma^5 \gamma T = -i \bar{\gamma} (-\gamma' \gamma^3) \cancel{(\gamma' \gamma^3)} \gamma(x_T) = -i \bar{\gamma}(x_T) \gamma^5 \gamma(x_T)}$$

Vector

$$T^+ \bar{\psi}^m \gamma^5 T = \bar{\psi}(-\gamma^1 \gamma^3)(\gamma^m)^+ (\gamma^1 \gamma^3) \psi$$

(*)

$$= \begin{cases} + \bar{\psi} \gamma^m \gamma^5 \psi & m=0 \\ - \bar{\psi} \gamma^m \gamma^5 \psi & m=1, 2, 3 \end{cases}$$

This makes sense... Recall that $\bar{\psi} \gamma^0 \psi$ is
the charge density

↳ $\bar{\psi} \gamma^0 \psi$ should be the same under T -

$$\text{as we saw: } T^+ \bar{\psi} \gamma^0 \gamma^5 T = \bar{\psi} \gamma^0 \psi.$$

but current density (time-dy) must reverse sign

$$\rightarrow T^+ \bar{\psi} \gamma^5 \gamma^5 T = - \bar{\psi} \gamma^5 \psi \quad \checkmark.$$

γ^5

Charge Conjugation - Matter-anti-matter flip

↳ anti-particle \rightarrow particle are swapped.

↳ spin + momentum are the same.

$$\text{Let } \left\{ \begin{array}{l} C^\dagger \alpha_p^\pm C = b_p^\pm \\ C^\dagger b_p^\pm C = \alpha_p^\pm \end{array} \right\} \rightarrow \text{ignore phases.}$$

How should C act on $\psi(x)$?

First, look at relation ...

$$\begin{aligned} (\nu^s(p))^{\pm} &= \left(\frac{\sqrt{p \cdot \sigma} (-i\sigma^2) \xi^{s\pm}}{\sqrt{p \cdot \bar{\sigma}} (-i\sigma^2) \xi^{s\pm}} \right)^{\pm} = \left(\frac{-i\sigma^2 \sqrt{p \cdot \sigma} \xi^{s\pm}}{i\sigma^2 \sqrt{p \cdot \bar{\sigma}} \xi^{s\pm}} \right)^{\pm} \\ &= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \bar{\sigma}} \xi^s} \right)^{\pm} = \text{[redacted]} \end{aligned}$$

→ set

$$\begin{cases} u^s(p) = -i\sigma^2 (\nu^s(p))^{\pm} \\ v^s(p) = -i\sigma^2 (u^s(p))^{\pm} \end{cases}$$

$$\begin{aligned} \rightarrow C^+ \gamma(x) C &= \int \frac{dp}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^2 \left\{ -i\sigma^2 b_p^s (v^s(p))^* e^{-ip \cdot x} \right. \\ &\quad \left. - i\sigma^2 a_p^{s\pm} (u^s(p))^* e^{ip \cdot x} \right\} \\ &= -i\sigma^2 \gamma^*(x) = -i\sigma^2 (\gamma^+)^T = -i(\bar{\gamma} \circ \gamma^2)^T \end{aligned}$$

$$\Rightarrow [C^+ \gamma(x) C = -i(\bar{\gamma} \circ \gamma^2)^T] \rightarrow C \text{ is a unitary op.}$$

On bilinear ... first, find

$$\bar{\gamma}^+ = (\gamma^+) \circ \gamma^0 = \gamma^+$$

$$\begin{cases} C^+ \bar{\gamma}(x) C = C^+ \gamma^+ \gamma^0 C = \underbrace{C^+ \gamma^+}_{\gamma^+} \gamma^0 = -i \gamma^+ \gamma^0 \\ = (-i \gamma^2 \gamma)^T \gamma^0 = (-i \gamma^0 \gamma^2 \gamma)^T \end{cases}$$

Next ...

$$C^+ \bar{\gamma} \gamma C = (-i(\gamma^0 \gamma^2 \gamma)^T) (-i(\bar{\gamma} \gamma^0 \gamma^2)^T) = \dots =$$

$$\begin{aligned} &= -[(-i \bar{\gamma} \gamma^0 \gamma^2)(-i \gamma^0 \gamma^2 \gamma)]^T = +\bar{\gamma} \gamma^0 \gamma^2 \gamma^2 \gamma \\ &= +\bar{\gamma} \gamma^0 \gamma^2 \gamma^2 \gamma^0 \gamma = +\bar{\gamma} \gamma \end{aligned}$$

PC

So - $C^T \bar{\gamma}^4 c = \bar{\gamma}^T \gamma \rightarrow$ scalar

vector

$$C^T \bar{\gamma}^5 \gamma^5 c = i (-i \gamma^0 \gamma^2 \gamma)^\dagger \gamma^5 (-i \gamma^0 \gamma^2)^\dagger = i \bar{\gamma}^5 \gamma^5$$

pseudo-scalar

$$C^T \bar{\gamma}^m \gamma^m c = - \bar{\gamma}^m \gamma^m \quad (I'll \text{ skip the derivations... to save time})$$

pseudo-vector

$$C^T \bar{\gamma}^m \gamma^5 c = + \bar{\gamma}^m \gamma^5 \gamma^5 \gamma$$

Summary

	$\bar{\gamma} \gamma$	$i \bar{\gamma} \gamma^5 \gamma$	$\bar{\gamma} \gamma^m \gamma$	$\bar{\gamma} \gamma^m \gamma^5 \gamma$			
P	+1	-1	$(-1)^m$	$-(-1)^m$	$(-1)^m (-1)^v$	$(-1)^m$	$(-1)^m$
T	+1	-1	$(-1)^m$	$(-1)^m$	$-(-1)^m (-1)^v$	$-(-1)^m$	$-(-1)^m$
C	+1	+1	-1	+1	-1	-1	-1
CPT	+1	+1	-1	-1	+1	+1	-1

Notice that

$L = \bar{\gamma} / (i \gamma^0 \gamma^2 - m) \gamma$ is invariant under C, P, T separately

→ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

Problem 5

↳ (to be continued--)

Invariance under CPT is required for any Lorentz invariant local Hamilton op.

Correlation functions for Dirac fields

$\langle 0 | \bar{\psi}_A(x) \psi_B(y) | 0 \rangle \rightarrow$ Dirac propagation amplitudes -

only "a" term contributes only "a*" term contributes

Result -

$$\rightarrow \bar{\psi}_A(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ a_A^{S+} u_A^S(p) e^{-ip \cdot x} + b_A^{S+} \bar{u}_A^S(p) e^{-ip \cdot x} \right\}$$

$$\rightarrow \bar{\psi}_B(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ b_B^{S+} \bar{u}_B^S(p) e^{-ip \cdot x} + a_B^{S+} u_B^S(p) e^{ip \cdot x} \right\} \quad \langle 0 |$$

$$\text{where } \{a_A^S, a_B^{S+}\} = \{b_A^S, b_B^{S+}\} = (2\pi)^3 \delta^{(3)}(p-q)/8$$

$$\rightarrow \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_S u_A^S(p) \bar{u}_B^S(p)}_{\delta(p-y)} e^{-ip(x-y)}$$

$$= (i\gamma_x + m) \underbrace{\int \frac{d^3 p}{(2\pi)^3 / 2E_p}}_{AB} e^{-ip(x-y)}$$

$$(p+m)_{AB}$$

$$\boxed{\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = (i\gamma_x + m)_{AB} D(x-y)}$$

$$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \bar{\psi}_A^s(p) \psi_B^s(p) e^{-ip(x-y)}$$

↑ ↑
 6 terms 6 terms
 contribute contribute

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2E_p} \right) (\phi - m)_{AB} e^{-ip(x-y)}$$

$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = - (i\partial_x + m)_{AB} \delta(y-x)$

Feynman Propagator

$$S_F^{AB}(x-y) = \begin{cases} \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle & \text{for } x^0 > y^0 \\ -\langle 0 | \bar{\psi}_B(y) \bar{\psi}_A(x) | 0 \rangle & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0 | T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \} | 0 \rangle$$

↑
--- time-ordering ---

where $T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \}$

$$= \theta(x^0 - y^0) \bar{\psi}_A(x) \bar{\psi}_B(y)$$

$$= \theta(y^0 - x^0) \bar{\psi}_B(y) \bar{\psi}_A(x)$$

↓ minus sign for Fermions ↓

(99)

let's check the calculations.

$$\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | \left\{ \sum_s a_{Ap}^s \bar{u}_A^s(p) e^{-ipx} + b_{Ap}^{s\dagger} \bar{u}_A^s(p) e^{-ipx} \right\}$$

$$\times \left\{ \sum_s b_{Bp'}^{s\dagger} \bar{u}_B^s(p') e^{-ip'y} + a_{Bp'}^s \bar{u}_B^s(p') e^{-ip'y} \right\} | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \langle 0 | \sum_s a_{Ap}^s \bar{u}_A^s(p) \sum_s a_{Bp'}^{s\dagger} \bar{u}_B^s(p') e^{-i(p-x-p'y)}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \sum_s \langle 0 | \bar{u}_A^s(p) \bar{u}_B^s(p') | 0 \rangle (2\pi)^3 \delta^{(3)}(p-p')$$

$$\times e^{i(p-x-p'y)}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_s \bar{u}_A^s(p) \bar{u}_B^s(p)}_{(\phi + m)_{AB}} e^{-ip(x-y)}$$

$$(\phi + m)_{AB} = (\gamma^m p_m + m)_{AB} \quad \begin{matrix} \text{(spin sym)} \\ \text{relations} \end{matrix}$$

$$= (\gamma^m p_m + m)_{AB} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \quad \checkmark$$

Similarly, we can get the other relations too.

-9-

Oct 5, 2020

(1) Real Dirac bispinor field -

$$\psi(\vec{x}) = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(p) + b_{-\vec{p}}^{r\dagger} v^r(p) \right] e^{i \vec{p} \cdot \vec{x}}$$

$$\text{Use } \{a_{\vec{p}}^r, a_{\vec{p}'}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{p}'}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(\vec{p}-\vec{p}')}$$

and all other anti-comm = 0, derive the following:

$$\{\psi_a(\vec{x}), \psi_b^+(\vec{y})\} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y})$$

(2) The momentum operator or the Noether charge associated with spatial translation.

$$\hat{P} = -i \int d^3x \psi^+(\vec{x}) \vec{\nabla} \psi(\vec{x})$$

Show that

$$\hat{P} = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \vec{p} (a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r)$$

Oct 10, 2020

(1) Well..

$$\{\psi_a(x), \psi_b^+(y)\}$$

$$= \psi_a(x) \psi_b^+(y) + \psi_b^+(y) \psi_a(x)$$

~~$$\frac{1}{(2\pi)^3} \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^3} \frac{1}{2\sqrt{2E_{p_1} E_{p_2}}} (p_1 \vec{x} + p_2 \vec{y})$$~~

To keep things clean -

$\Psi_a(x) \Psi_b^t(y) + \Psi_b(x) \Psi_a^t(y)$ which involves the factor.

$$\begin{aligned}
 & a \sum_{r=1}^2 \sum_{s=1}^2 \left[\hat{a}_{p_a}^{tr} u^r(p_a) + \hat{b}_{-p_a}^{ts} v^s(-p_a) \right] \left[\hat{a}_{p_b}^{rs} u^r(p_b) + \left(\hat{b}_{-p_b}^{rt} v^t(-p_b) \right)^+ \right] \\
 & + \sum_{r=1}^2 \sum_{s=1}^2 \left[\left(\hat{a}_{p_b}^{rs} u^r(p_b) \right)^+ + \left(\hat{b}_{-p_b}^{rt} v^t(-p_b) \right)^+ \right] \left[\hat{a}_{p_a}^{tr} u^r(p_a) + \hat{b}_{-p_a}^{ts} v^s(-p_a) \right] \\
 & = \sum_{r,s=1}^2 \left\{ \hat{a}_{p_a}^{tr}, \hat{a}_{p_b}^{st} \right\} u^r(p_a) u^s(p_b) + \left\{ \hat{b}_{p_a}^{tr}, \hat{b}_{p_b}^{ts} \right\} v^r(p_a) v^s(p_b) \\
 & = \sum_{r,s,t}^2 (i\pi)^3 \delta^{rs} \delta^{(2)}(p_a - p_b) \left\{ u^r(p_a) u^s(p_b) + v^r(p_a) v^s(p_b) \right\}
 \end{aligned}$$

$\Rightarrow \{ \Psi_a(x), \Psi_b^t(y) \}$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^2 \left\{ u^r(p) u^r(p) + v^r(p) v^r(-p) \right\}$$

Now, we want to convert $u^r \rightarrow \bar{u}^r$

\rightarrow need γ^0 . In particular, recall that $\gamma^0 = 1$ and $\overline{u^r(p)\gamma^0} = \bar{u}^r(p)$ \Rightarrow we have

$$\{ \Psi_a(x), \Psi_b^t(y) \} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^2 \left\{ u_p^r \bar{u}_p^r \gamma^0 + v_p^r \bar{v}_p^r \gamma^0 \right\}$$

$$= \left(\frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \right) \sum_{r=1}^2 \left\{ u_p^r \bar{u}_p^r + \frac{\sqrt{v_p^r}}{p^0 - p} \right\} \gamma^0 \quad (\text{spin sum})$$

$$= \left(\frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \right) (p_+ \gamma^0 + m \gamma^0 + p_- \gamma^0 - m \gamma^0) \gamma^0 |_{\bar{u}_a^r}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} [2\cancel{\vec{p}} \cdot \vec{r}] \delta^3$$

recall that
only $\vec{p} \rightarrow -\vec{p}$ (102)

$$\text{Rather } = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left\{ E_p \vec{r} - \vec{p} \cdot \vec{r} + E_p \vec{x} + \vec{p} \cdot \vec{x} \right\} \delta^3$$

$$= \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta_{ab} \delta^{(3)}(\vec{x}-\vec{y}).$$

$$\boxed{\delta_{ab} \left\{ \chi_a(x), \chi_b^\dagger(y) \right\} = \delta_{ab} \delta^{(3)}(x-y)}$$

-4

~~$$(2) \text{ Let } \vec{p} = -i \int d^3 x \vec{x} \chi^\dagger(\vec{x}) \vec{\nabla} \chi(\vec{x}).$$~~

Step 1 must

$$p = (E, \vec{p})$$

~~$$\vec{p} = \sum_{i=1}^3 \int \frac{d^3 p}{(2\pi)^3} \vec{p} \left(\frac{a_i^+}{E_p} u_p^i + \frac{b_i^-}{E_p} b_p^i \right).$$~~

~~$$i\vec{p} \cdot \vec{x} = \vec{a} \cdot \vec{u} + i\vec{b} \cdot \vec{v}$$~~

Well,

~~$$\chi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{i\vec{p} \cdot \vec{x}} \left\{ a_p^r u_p^r + b_p^r v_p^r \right\}.$$~~

~~$$\delta \chi(x) = \sum_{r=1}^3 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (i\vec{p})$$~~

$$(e) \Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u_p^s e^{-ip \cdot x} + b_p^{s^\dagger} v_p^s e^{ip \cdot x} \right)$$

$$\rightarrow \nabla \Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (i\vec{p}) \left\{ a_p^s u_p^s e^{-ip \cdot x} - b_p^{s^\dagger} v_p^s e^{ip \cdot x} \right\}$$

$$\Psi(x) = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{r=1}^{\infty} \left\{ b_q^r u_q^r(q) e^{-iq \cdot x} + a_q^{r^\dagger} v_q^r e^{iq \cdot x} \right\}$$

$$\rightarrow \int d^3 x (-i) \nabla \Psi(x) \quad \delta^{(3)}(\vec{p} - \vec{q})$$

$$= \int d^3 x \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_q}} e^{i\vec{x} \cdot (\vec{p} - \vec{q})} \vec{p} \times \sum_s \sum_r$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \frac{1}{E_p} \vec{p} \times \left\{ a \sum_{s,r=1}^2 \left(a_p^{s^\dagger} a_p^s \vec{v}_p^r u_p^r - b_p^{r^\dagger} b_p^r \vec{v}_p^s u_p^s \right) \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{2E_p}{2E_p} \vec{p} \left(a_p^{s^\dagger} a_p^s - b_p^{r^\dagger} b_p^r \right) \quad \text{cancel zero terms} = 0$$

$$= \int \frac{d^3 p}{(2\pi)^3} \vec{p} \left(a_p^{s^\dagger} a_p^s - b_p^{r^\dagger} b_p^r \right) \quad \text{ignore...}$$

$$\text{Finally} \rightarrow \{ b_p^r, b_p^{r^\dagger} \} = (2\pi)^3 \delta^{rr} (\vec{p} \cdot \vec{p})$$

$$\Rightarrow -b_p^r b_p^{r^\dagger} = b_p^{r^\dagger} b_p^r - (2\pi)^3 \delta^{rr} \delta(\vec{p} \cdot \vec{p})$$

$$\Rightarrow \vec{p} = \int d^3 x \Psi^+(x) (-i\vec{p}) \Psi(x) \quad \text{momentum op}$$

$$\boxed{\vec{p} = \int \frac{d^3 p}{(2\pi)^3} \vec{p} \left(a_p^{s^\dagger} a_p^s + b_p^{r^\dagger} b_p^r \right)}$$

More problems

① Let U be the following unitary op:

$$U = \exp \left\{ -i\frac{\pi}{2} \sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r*} - b_p^{r*}) \right\}$$

Investigate the effect of U on a_p^r , b_p^r . I.e.
compute $U^+ a_p^r U = ?$

What type of transform does U produce?

$$\xrightarrow{-i\epsilon} X^+$$

Well...

$$U^+ a_p^r U = \exp \left\{ +i\frac{\pi}{2} \left(\sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r*} - b_p^{r*}) \right) \right\}$$

$$X \leftarrow \exp \left\{ i\frac{\pi}{2} \sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r*} - b_p^{r*}) \right\}$$

\Rightarrow no good way to do this except for $\epsilon \rightarrow 0$.

Recall that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n$

$$\therefore U^+ a_p^r U = \left(\sum_{n=0}^{\infty} \frac{(X^t)^n}{n!} \right) a_p^r \left(\sum_{m=0}^{\infty} \frac{(X)^m}{m!} \right) = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} (X^t)^n a_p^r X^m$$

but note that U unitary iff X hermitian.

$$\rightarrow X^t = X \rightarrow U^+ a_p^r U = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} X^n a_p^r X^m$$

Weierstrass theorem:

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]]$$

2 Identity relating comm & anti-comm: ---

$$[ABC, C] = ABC - CAB$$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B, C\} - \{A, C\}B$$

We still need to compute $\{X, g^s\}$

$$U^\dagger g^s U = g^s + [X, g^s] + \frac{1}{2!} [X, [X, g^s]] + \dots$$

→ need to compute

$$[X, g^s] = X \tilde{g}^s - \tilde{g}^s X = \dots = ?$$

$$= \left(\frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left\{ (a_p^s - b_p^{s+}) (a_p^s - b_p^s) \right. \right. \tilde{g}^s \\ \left. \left. - \left\{ \tilde{g}^s (a_p^{s+} - b_p^{s+}) (a_p^s - b_p^s) \right\} \right\} \right)$$

$$= \frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left[(a_p^{s+} - b_p^{s+}) \left\{ a_p^s - b_p^s \right\} \tilde{g}^s \right. \\ \left. - \left\{ a_p^{s+} - b_p^{s+}, \tilde{g}^s \right\} (a_p^s - b_p^s) \right] \xrightarrow{D}$$

$$= \frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left[- (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q}) (a_p^s - b_p^s) \right]$$

$$= - \frac{i\pi}{2} (a_q^s - b_q^s)$$

Next turn,

$$\begin{aligned} [X, [X, a_q^r]] &= [X, -\frac{i\pi}{2}(a_q^r - b_q^r)] \\ &= \frac{-i\pi}{2}[X, a_q^r] + \frac{i\pi}{2}[X, b_q^r] = \dots \\ &= 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r) \end{aligned}$$

$$\text{So } e^X a_p^r e^{-X} = a_p^r - \frac{i\pi}{2}(a_q^r - b_q^r) + 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r)$$

what's next?

+ ?

each step $\rightarrow +2\left(\frac{i\pi}{2}\right)$ + alt (-) sign.

$$\begin{aligned} \rightarrow u a_p^r u &= a_p^r - \frac{i\pi}{2}(a_p^r - b_p^r) + \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2(a_p^r - b_p^r) \\ &\quad - \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3(a_p^r - b_p^r) + \frac{1}{4!} \left(\frac{i\pi}{2}\right)^4(-) \\ &= a_p^r \left\{ 1 - \frac{i\pi}{2} + \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2 - \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3 + \frac{6}{4!} \left(\frac{i\pi}{2}\right)^4 + \dots \right\} \\ &\quad + b_p^r \left\{ \frac{i\pi}{2} - \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2 + \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3 - \frac{6}{4!} \left(\frac{i\pi}{2}\right)^4 + \dots \right\} \\ &= a_p^r \left\{ 1 - \frac{1}{2} \cdot 2 \right\} + b_p^r \cdot \left\{ \frac{1}{2} - 2 \right\} \\ &= b_p^r \Rightarrow \boxed{n^+ a_p^r n^- = b_p^r} \end{aligned}$$

$\rightarrow u$ corresponds to charge conjugation!

(2) Using Dirac annihilation + creation ops
construct P for which

$$P^\dagger a_p^r P = a_{-p}^r \quad P^\dagger b_p^r P = b_{-p}^r$$

Last time, we find that $\xrightarrow{\text{target}}$
 $[X, a_p^r] = -\left(\frac{i\pi}{2}\right) (a_p^r - b_p^r)$

\Rightarrow we want X for which

$$[X, a_p^r] = -\left(\frac{i\pi}{2}\right) (a_p^r - a_{-p}^r).$$

$$[X, b_p^r] = -\left(\frac{i\pi}{2}\right) (b_p^r + b_{-p}^r).$$

\Rightarrow Let

$$P = \exp \left\{ -\frac{i\pi}{2} \sum_{n=1}^2 \int \frac{d^3 p}{(2\pi)^3} \left\{ a_p^{\dagger} (a_p^r - a_{-p}^r) + b_p^{\dagger} (b_p^r + b_{-p}^r) \right\} \right\}$$

↑

check this, like last time

\Rightarrow should work! c

Olv a_p^r only int. w/ 1st term $a^{\dagger}()a = 0$

$$a^{\dagger}() = \delta^{ij}(-)$$

Same with b_p^r ✓

Interacting Fields = Feynman Diagrams

To get better description of the real world, need to include interactions in the theory.

To preserve causality, new terms may involve products of fields at the same spacetime point!

↳ $\phi^4(x)$ ✓, but not $\phi(x)\phi(y)$

$$\rightarrow H_{\text{int}} = \int d^3x \, H_{\text{int}}[\phi(x)] = - \int d^3x \, \partial_\mu^\perp [\phi(x)]$$

→ insist that H_{int} is a ln. only of the fields, not of their derivatives.

→ Common ex in perturb. physics = QFT:

$$L = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4$$

Note

— $\Pi(x)$ is still $\partial_\mu \phi(x)$ since there are not any terms involving $\partial_\mu \phi$ interaction.

λ : dimensionless "coupling constant".

→ In general, strong interactions possesses long range.

However, no matter what the true physics looks like at high momenta or short distances, the low momentum / long distance physics is well-approximated by an "effective" FT.

with "renormalizable" interactions.

→ these interactions where coupling constant are has dimensions $\boxed{d > 0}$

$[Mass]^d$ where $d > 0$.

$$\text{Ex } -\frac{1}{2} m^2 \phi^2 \sim \frac{2}{4!} \phi^4 \text{ same dim}$$

→ $\lambda \sim [E_{mass}]^0 \rightarrow \text{renormalizable.}$

But $-\frac{\lambda_6}{6!} \phi^6 \rightarrow \underline{\text{not renormalizable.}}$

Since $\lambda \sim [E_{mass}]^{-2}$.

Perturbation Expansion

$$\text{Let } H = H_0 + H_{\text{int}} \rightsquigarrow = \int d^3 p \frac{2}{4!} \phi^4(x)$$

\uparrow
KG, free

→ we will generate power series in λ .

At any t_0 , we can write

$$\phi(t_0, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{i\vec{p} \cdot \vec{x}} + a_p^\dagger e^{-i\vec{p} \cdot \vec{x}} \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \tilde{a}_p^\dagger e^{i\vec{p} \cdot \vec{x}} + \tilde{a}_p e^{-i\vec{p} \cdot \vec{x}} \right\}$$

a_p^\dagger
where we've let
 a_p absorb e^{iEt_0}

The Heisenberg field is then given by:

$$\rightarrow \boxed{\phi(t, \vec{x}) = e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)}}$$

If there's no interaction then we have

$$\rightarrow \boxed{\phi_{\text{free}}(t, \vec{x}) = e^{iH_0(t-t_0)} \phi_{\text{free}}(t_0, \vec{x}) e^{-iH_0(t-t_0)}}$$

$$= \int \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p^+ e^{-ip \cdot x} + a_p^- e^{+ip \cdot x} \right\} \Big|_{\begin{array}{l} x_0^0 = t-t_0 \\ p^0 = E_p \end{array}}$$

Define this to be $\phi_I(t, \vec{x})$, the interaction picture field

the interaction picture field = Heisenberg field
when $\lambda = 0$.

Now, look at Heisenberg field ...

$$\begin{aligned} \phi(t, \vec{x}) &= e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)} \underbrace{e^{-iH_0(t-t_0)} \phi_I(t, \vec{x}) e^{+iH_0(t-t_0)}}_{U(t, t_0)} e^{-iH(t-t_0)} \\ &= U^+(t, t_0) \phi_I(t, \vec{x}) U(t, t_0) \end{aligned}$$

\rightarrow Evolve the operator on $\phi_I(t, \vec{x})$

Time evolution operator

OR

Evolve the state by $U(t, t_0) \rightarrow U|\phi\rangle$...

→ now we want to express $U(t, t_0)$ entirely in ϕ_I

To do this, note that $U(t, t_0)$ solves SE:

$$\begin{aligned} \frac{\partial U(t, t_0)}{\partial t} &= e^{iH_0(t-t_0)} (H - H_0) e^{-iH_0(t-t_0)} \\ &= e^{iH_0(t-t_0)} \underbrace{H_{\text{int}} e^{-iH_0(t-t_0)}}_{\text{Hint}} \\ &= e^{iH_0(t-t_0)} \underbrace{e^{-iH_0(t-t_0)} iH_0(t-t_0) e^{-iH_0(t-t_0)}}_{\text{Hint } e^{-iH_0(t-t_0)}} \\ &= H_I(t) U(t, t_0) \end{aligned}$$

where

$$H_I(t) = e^{iH_0(t-t_0)} \underbrace{H_{\text{int}} e^{-iH_0(t-t_0)}}_{\text{Hint}}$$

$$= \int d^3x \frac{i}{4!} \phi_I^4 [3]$$

$$= \int d^3x e^{iH_0(t-t_0)} \underbrace{\frac{i}{4!} \phi^4}_{\text{Hint}} \bar{e}^{-iH_0(t-t_0)}$$

$$= \int d^3x \frac{i}{4!} \phi^4 \checkmark$$

→ this is the Hamiltonian in the interaction picture.

so since U solves the SE:

$$i\partial_t U(t, t_0) = H_I(t) U(t, t_0),$$

U must look like

$$U(t, t_0) \sim \exp \{-iH_I t\}$$

More carefully, we actually have that

$$U(t, t_0) = T \exp \left\{ -i \int_{t_0}^t dt' H_I(t') \right\}$$

Lyman's formula

time-ordering symbol

Why T ? Why ordering? \Rightarrow B/c $H(t_1) \neq H(t_2)$ when $t_1 \neq t_2$.

" T " puts the latest operators on the left.

hence $i \partial_t U(t, t_0) = \underline{\underline{H_I(t)}} U(t, t_0)$.

As a power series in \mathcal{T} :

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt' H_I(t') + \frac{(-i)^2}{2!} \int_{t_0}^t \int_{t_0}^{t'} dt_1 dt_2 T \{ H_1(t_1) H_2(t_2) \} + \dots$$

$$+ \frac{(-i)^3}{3!} \int_{t_0}^t \int_{t_0}^{t'} \int_{t_0}^{t''} dt_1 dt_2 dt_3 T \{ H_1(t_1) H_2(t_2) H_3(t_3) \} + \dots$$

Note "the time-ordering of the exponential is just ~~the time-ordering~~ the Taylor series of the terms time-ordered".

\rightarrow Now, we want to generalize $U(t, t_0)$ to $U(t, t')$

referendum

This generalization is natural -

$$U(t, t') = T \left\{ \exp \left[-i \int_{t'}^t dt'' H_2(t'') \right] \right\} \quad (t > t')$$

Then we see that b/c both t, t' are variables -- we find :

$$\hat{\mathcal{D}}_t U(t, t') = H_t(t) U(t, t')$$

$$\hat{\mathcal{D}}_{t'} U(t, t') = -U(t, t') H_{t'}(t').$$

and thus -

$$U(t, t') = e^{iH_0(t-t')} e^{-iH(t-t')} e^{-iH_0(t-t')}$$

so U is unitary.

Further, for $t_1 \geq t_2 \geq t_3$,

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3)$$

$$U(t_1, t_3) [U(t_2, t_3)]^\dagger = U(t_1, t_2)$$

Now, let $|0\rangle$ be gnd state of H_0

$|S\rangle$ be gnd state of H

$|n\rangle$ be ~~gnd~~ label all $|E_n\rangle$ of H .

Then, $(E_0 = \langle \psi_0 | H | \psi_0 \rangle)$

$$\langle e^{-iHT} | \psi_0 \rangle = e^{-iE_0 T} |\psi_0\rangle \langle \psi_0 | \psi_0 \rangle + \sum_{n \neq 0} e^{-iE_n T} |n\rangle \langle n | \psi_0 \rangle$$

Assume $H | \psi_0 \rangle = 0$ consider $T \rightarrow \infty$ limit.

↓

$$T \rightarrow \infty (1 - i\varepsilon)$$

Then $e^{-iE_n T}$ dies slowest for $n=0$, and so ...

$$\rightarrow e^{-iHT} |\psi_0\rangle \rightarrow e^{-iE_0 T} |\psi_0\rangle \langle \psi_0 | \psi_0 \rangle \quad \text{assume } \langle \psi_0 | \psi_0 \rangle \neq 0$$

S

$$|\psi_0\rangle = \lim_{T \rightarrow \infty (1 - i\varepsilon)} (e^{-iE_0 T} \langle \psi_0 |) e^{-iHT} |\psi_0\rangle$$

Now, since T large, we can shift it by a small constant ...

$$|\psi_0\rangle = \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0 (T+t_0)} \langle \psi_0 | \right\}^{-1} e^{-iH(T+t_0)} |\psi_0\rangle$$

$$= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0 (t_0 - (-T))} \langle \psi_0 | \right\}^{-1} \underbrace{e^{-iH(t_0 - (-T))}}_{U(t_0, -T)} |\psi_0\rangle$$

$$= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0 (t_0 - (-T))} \langle \psi_0 | \right\}^{-1} \underbrace{e^{-iH(t_0 - (-T))} e^{-iH_0 (-T - t_0)}}_{e^{-iH_0 (-T - t_0)}} |\psi_0\rangle$$

$$= |\psi_0\rangle \text{ since } H_0 |\psi_0\rangle = 0$$

$$\Rightarrow |\psi_0\rangle = \lim_{T \rightarrow \infty (1 - i\varepsilon)} \frac{U(t_0, -T) |\psi_0\rangle}{e^{-iE_0 (t_0 - (-T))} \langle \psi_0 | \psi_0 \rangle}$$

Similarly,

$$\langle \alpha | = \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, t_0) \phi(x) }{e^{-iE_0(T-t_0)} \langle 0 | \alpha \rangle}$$

$$\langle 0 | u(T, t_0)$$

$$e^{-iE_0(T-t_0)} \langle 0 | \alpha \rangle$$

So, putting them together gives a correlation function.

For $x^0 > y^0 > t_0$, we have

$$\begin{aligned} \langle \alpha | \phi(x) \phi(y) | \alpha \rangle &= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, t_0) \phi(x) }{e^{-iE_0 T} \langle 0 | \alpha \rangle} \\ &\quad \times (u(x^0, t_0))^+ \phi_I(x) u(x^0, t_0) \\ &\quad \times (u(y^0, t_0))^+ \phi_I(y) u(y^0, t_0) \times \end{aligned}$$

$$\frac{u(t_0 - T)}{e^{-iE_0 T} \langle 0 | \alpha \rangle}$$

$$= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, x^0) \phi_I(x) u(x^0, y^0) \phi_I(y) u(y^0, -T) | 0 \rangle}{| \langle 0 | \alpha \rangle |^2 e^{-iE_0(2T)}}$$

awkward...

so divide the whole thing by $1 = \langle \alpha | \alpha \rangle$

$$1 = \langle \alpha | \alpha \rangle = \frac{\langle 0 | u(T, t_0) u(t_0, -T) | 0 \rangle}{| \langle 0 | \alpha \rangle |^2 e^{-iE_0(2T)}} \rightarrow n(T, -T)$$

To get (for $x^0 > y^0$)

$$\langle \alpha | \phi(x) \phi(y) | \alpha \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, x^0) \phi_I(x) u(x^0, y^0) \phi_I(y) u(y^0, -T) | 0 \rangle}{\langle 0 | u(T, -T) | 0 \rangle}$$

So, we have shown, by replacing \hat{U}_S with Feynman's formula (w/ fine-ordering)

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

$$= \lim_{T \rightarrow \infty(1-\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) \} \exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} | 0 \rangle}{\langle 0 | T \exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} | 0 \rangle}$$

L

So, looks like the term

$$\exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} \text{ as expected & can be found, so } \checkmark$$

Wick's theorem

→ So, we have reduced the problem of calculating correlation functions to evaluating

$$\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \dots \phi_I(x_n) \} | 0 \rangle$$

→ This is the vacuum exp-value of time-ordered products of finite number of field operators.

$n=2 \rightarrow$ get Feynman operator.

$n>2 \rightarrow$ can use ~~for~~ brute force, but there are also ways to simplify calculations.

Now, we study

$$\langle 0 | T \{ \phi_I(x) \phi_I(y) \} | 0 \rangle$$

Recall that

$$\phi_I(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon_p}} \left\{ a_p^\dagger e^{-ip \cdot x} + a_p^\dagger e^{+ip \cdot x} \right\}$$

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$$\text{Call } \phi_I^+(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon_p}} a_p^\dagger e^{-ip \cdot x}$$

$$\text{and } \phi_I^-(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon_p}} a_p^\dagger e^{+ip \cdot x}$$

which is enough to

$$\phi_I^+(x)|0\rangle = 0, \quad \langle 0|\phi_I^-(x) = 0.$$

↑ ↑
only has annihilation ops only has creation ops

→ For $x^0 > y^0$, $\rightarrow x^0 > y^0$

$$\begin{aligned} \Gamma \{ \phi_I^+(x) \phi_I^-(y) \} &= \phi_I^+(x) \phi_I^-(y) + \phi_I^-(x) \phi_I^+(y) \\ &\quad + \phi_I^+(x) \phi_I^-(y) + \phi_I^-(x) \phi_I^+(y) \end{aligned}$$

$$= \phi_I^+(x) \phi_I^-(y) + \phi_I^-(x) \phi_I^+(y) + \phi_I^-(x) \phi_I^+(y) + \phi_I^-(x) \phi_I^-(y)$$

$$\left. + [\phi_I^+(x), \phi_I^-(y)] \right]$$

every of these terms has the form $a^\dagger a^\dagger a^\dagger a^\dagger$

i.e. creation ops always on the left.

→ "Normal order" → less vanishing vacuum expectation

What can we say about the commutator?

It's a number, there is no creation/annihilation op's in it!

$$[\phi_I^+(x), \phi_I^-(y)] = \langle 0 | [\phi_I^+(x), \phi_I^-(y)] | 0 \rangle$$

$$= \langle 0 | \phi_I^+(x) \phi_I^-(y) | 0 \rangle - \langle 0 | \phi_I^-(y) \phi_I^+(x) | 0 \rangle.$$

With this, we can write

$$T\{ \phi_I^+(x) \phi_I^-(y) \} = \phi_I^+(x) \phi_I^-(y) + \phi_I^-(x) \phi_I^+(y) + \phi_I^-(y) \phi_I^+(x)$$

$$+ \phi_I^+(x) \phi_I^-(y) + \langle 0 | \phi_I^+(x) \phi_I^-(y) | 0 \rangle$$

Now, define the normal ordering symbol "N"

s.t. N takes the string a-at's and rearranges them so that at's are on the left

$$\text{ex. } N(a^\dagger_p a^\dagger_p) = a^\dagger_p a^\dagger_p$$

$$N(a^\dagger_p a^\dagger_p) = a^\dagger_p a^\dagger_p \quad \text{ordering for } a^\dagger_p, a^\dagger_p$$

$$N(a^\dagger_p a^\dagger_q a^\dagger_p) = a^\dagger_p a^\dagger_p a^\dagger_q \quad \text{doesn't matter}$$

since they commute

\Rightarrow Note N is not a well-defined mathematical operation

$$\text{e.g. } N(\Sigma_{\vec{p}, \vec{q}} f) \neq N((\omega)^{1/2} (\vec{p} - \vec{q}))$$

\rightarrow it is only a lexicographic convention.

Now, let us consider general x^0, y^0 , then

$$T\{\phi_I^+(x)\phi_I^-(y)\} = N\{\phi_I^+(x), \phi_I^-(y)\}$$

$$+ \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 > y^0 \\ [\phi_I^+(y), \phi_I^-(x)] & \text{for } x^0 < y^0 \end{cases}$$

Let us define the commutator of $\phi_I(x), \phi_I(y)$ as

$$\boxed{\phi_I^+(x)\phi_I^-(y) = \begin{cases} \sum [\phi_I^+(x), \phi_I^-(y)] & x^0 > y^0 \\ \sum [\phi_I^+(y), \phi_I^-(x)] & x^0 < y^0 \end{cases}}$$

Then notice that, from our previous derivation,

$$\boxed{\phi_I^+(x)\phi_I^-(y) = \begin{cases} \langle 0 | T\{\phi_I^+(x)\phi_I^-(y)\} | 0 \rangle & x^0 > y^0 \\ \langle 0 | \phi_I^-(y)\phi_I^+(x) | 0 \rangle & y^0 > x^0. \end{cases}}$$

So,

$$\left\{ \begin{aligned} \phi_I^+(x)\phi_I^-(y) &= \langle 0 | T\{\phi_I^+(x)\phi_I^-(y)\} | 0 \rangle \\ &= D_F(x-y) \quad \leadsto \text{Feynman propagator} \end{aligned} \right.$$

$$= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}$$

With this, we have that

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→ carry out the I^{\rightarrow} subscript

$$T\{\phi(x)\phi(y)\} = N\left\{ \phi(x)\phi(y) + \underbrace{\phi(x)\phi(y)}_{\phi(y)\phi(x)} \right\}$$

$$\rightarrow T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y)\} + \text{"contraction"}$$

In fact, the generalization of this is called
Wick's Theorem

$$T\{\phi(x_1)\phi(x_2) \dots \phi(x_m)\}$$

$$= N\{\phi(x_1)\phi(x_2) \dots \phi(x_m) + \text{all possible contractions}\}$$

$$\underline{\text{Ex}} \quad T\{\phi_1 \phi_2 \phi_3 \phi_4\}$$

($\phi_n = \phi(x_n)$)

$$= N\{\phi_1 \phi_2 \phi_3 \phi_4 +$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4}$$

$$\overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4}$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} \}$$

What does $N\{\phi_1 \phi_2 \phi_3 \phi_4\}$ mean?

$$N\{\phi_1 \phi_2 \phi_3 \phi_4\} = \langle \phi_1 \phi_2 \rangle N\{\phi_3 \phi_4\}$$

$$= D_f(x_1 - x_2) N\{\phi_3 \phi_4\}$$

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Proof \rightarrow prove by induction $n=2$ is good
(Feynman)

\rightarrow assume this holds for $n-1$

Let $W(\phi_1 \dots \phi_n) = N\{\phi_1 \phi_2 \dots \phi_n + \text{all possible contr.}\}$

To prove $W(\phi_1 \dots \phi_n) = T\{\phi_1, \phi_2, \dots, \phi_n\}$,

W/ ℓ /0/g: let $x_1^{\circ} \geq x_2^{\circ} \geq \dots \geq x_n^{\circ}$

$$L_T \{ \phi_1, \dots, \phi_n \} = \underbrace{\phi_1^+ W(\phi_2, \dots, \phi_n)}_{X} + \underbrace{[\phi_1^+, W]}_{Y}$$

$$\text{Let } X = \phi_1^* W + W \phi_1^+ ; Y = [\phi_1^+, W] ,$$

$X+Y$ are normal ordered: X contains all contractions in $W(\phi_1, \dots, \phi_n)$ which does not contract ϕ_i w/ anything.

γ contains all contractions in $W(\phi_1, \phi_n)$ which contracts ϕ_j with something -

$$so \quad T(\phi_1, \phi_2, \dots, \phi_n) = w(\phi_1, \dots, \phi_n)$$

(we won't worry too much about this proof)

→ the main idea is the theorem itself

In any case, we have another way to explicitly write out the result of Wick's theorem:

$$T\{\phi_1, \phi_2, \dots, \phi_n\} = N \left\{ \exp \left[\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \phi_i \phi_j \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j} \right] \phi_1 \dots \phi_n \right\}$$

↑
we'll see why later on.

→

Feynman Diagrams

Wick's theorem allows us to write

$$\langle 0 | T\{\phi_1, \dots, \phi_n\} | 0 \rangle$$

in terms of sums and products of Feynman propagators.

→ Now, we will develop the diagrammatic expressions.

Recall that

$$T\{\phi_1, \phi_2, \phi_3, \phi_4\} = N \{ \phi_1 \phi_2 \phi_3 \phi_4 + \text{all possible contractions} \}$$

But the only contribution to

$\langle 0 | T\{\phi_1, \phi_2, \phi_3, \phi_4\} | 0 \rangle$ is where all the ϕ_i 's are contracted.

∴ This b/c whenever things are in normal order, the exp value vanishes. $\rightarrow N(\phi_1 \phi_2 \phi_3 \phi_4) = \phi_1 \phi_2 N(1, 1)$

→ to "escape" from normal order, ϕ_i 's have to be contracted

This means that

$$\langle T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} \rangle = \phi_1 \phi_2 \phi_3 \phi_4 + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\perp} + \phi_1 \phi_2 \phi_3 \phi_4$$

→ can write this as Feynman diagrams.

$$\langle T \{ \phi_1 \phi_2 \phi_3 \phi_4 \} \rangle = \begin{array}{c} 1 \quad 2 \\ \nearrow \quad \searrow \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ | \quad | \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \searrow \quad \nearrow \\ 3 \quad 4 \end{array}$$

Interpretation

Particles are created at 2 spacetime points, each propagates to one of the other points, then get annihilated.

→ total amplitude of the process is the sum of the diagrams.

Well... what about something like...

$$\langle 0 | T \{ \phi(x) \phi(y) \} \exp \left\{ -i \int_{-\infty}^{\infty} dt H_I(t) \right\} \} | 0 \rangle ?$$

Well... as a power series in λ , the lowest order term is

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle = \delta_{xy}(x-y).$$

$$1^{\text{st}} \text{ order } \langle 0 | T \{ \phi(x) \phi(y) \} (-i) \left(\int_{-\infty}^{\infty} dt H_I(t) \right) | 0 \rangle$$

$$= \langle 0 | T \{ \phi(x) \phi(y) (-i) \int d^4 z \gamma_1 \gamma_4(z) \} | 0 \rangle$$

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$$\begin{aligned}
 &= -\frac{i\gamma}{4!} \int d^4z \langle 0 | T \{ \phi(x) \phi(y) \phi(z) \phi(\tau) \phi(+) \phi(-) \} | 0 \rangle \\
 &= -\frac{i\gamma}{4!} \int d^4z \left\{ \phi(x) \phi(y) \cdot \left\{ \phi(z) \phi(+) \phi(+) \phi(\tau) + \phi_z \phi_+ \phi_+ \phi_\tau \right. \right. \\
 &\quad \left. \left. + \phi_z \phi_+ \phi_+ \phi_\tau \right\} \right. \\
 &\quad \left. = \phi(x) \phi(y) \phi(z) \phi(\tau) \phi(+) \phi(-) \right\} \\
 &\qquad \qquad \qquad \rightarrow 12 \text{ terms that are identical}
 \end{aligned}$$

$$\begin{array}{c} x \quad y \\ \hline \end{array} + \begin{array}{c} x \quad y \\ \diagdown \quad \diagup \\ z \end{array} \rightarrow \begin{array}{c} \text{propagator} \\ D_F(x-z) D_F(y-z) \\ D_F(z-z) \end{array}$$

$$\int d^4z D_F(x-y) D_F(z-z) D_F(y-z) \quad \begin{matrix} 1 \\ 12 \text{ of these.} \end{matrix}$$

\hookrightarrow each contraction of D_F is a line.

each generation point is a dot.

\rightarrow but need to distinguish "external" and "internal" points:



Each internal point is associated w/ a factor of $-i\gamma \int d^4z$, with combinatorial factor...

How do we count these combinatorial factors?

well.. Each H_F has 4 ϕ_I 's. $\phi_{(z)} \phi_{(\bar{z})} \phi_{(t)} \phi_{(\bar{t})}$

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→ interchanging free subtraction "end" will give the same amplitude.

→ so for each H_F we expect a factor of 4!

→ cancels out the $\frac{1}{4!}$ in $\frac{1}{4!} \phi^4$.

* In a diagram with more terms are powers of H_F

We can exchange all the subtraction ends of one H_F with recombination ends of the other H_F .

↳ Since we integrate over all z_1, z_2 → this gives the same amplitude.

⇒ For a diagram with n "internal vertices",

(i.e. # of H_F 's), we get a factor of $n!$ This cancels the $\frac{1}{n!}$ factor from Taylor series expansion

of $\exp \{ -i \int H_F d\tau \} \cdot \}$

~~→ Can be a small subtlety but in~~

For example... consider the \mathcal{I}^3 term:

(
)

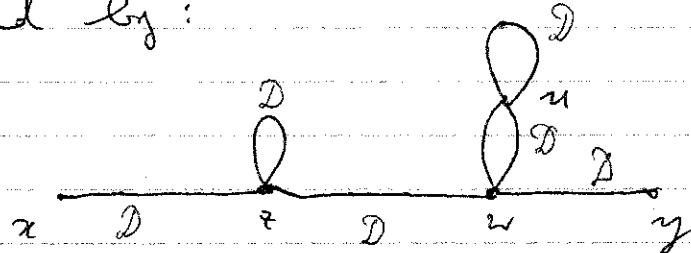
$$\begin{aligned}
 & \langle 0 | \phi(x) \phi(y) \frac{1}{3!} \left(\frac{-i\gamma}{4!} \right)^3 \int d^4 z d^4 w d^4 u D_F(x-z) D_F(z-w) D_F(w-y) D_F^2(w-u) D_F(u-v) \rangle \\
 & = \frac{1}{3!} \left(\frac{-i\gamma}{4!} \right)^3 \int d^4 z d^4 w D_F(x-z) D_F(z-w) D_F(w-y) D_F^2(w-u) D_F(u-v)
 \end{aligned}$$

The number of contractions that give this same expression is 75.

$$3! \times (4,3) \times (4,3,2,1) \times (9,3) \times (1/2)$$

↑ ↑ ↑ ↑ ↑
 introduce introduce introduce ... introduce
 vertices placement placement ... of w-m
 of contraction of contraction for w of w-m
 into z into w vertex vertex
 vertex vertex

Represented by:



Now... there is a subtlety here with all of this, and that is symmetry factors.



Gauge factors

→ Best to consider the simplest diagram with the most general problem.

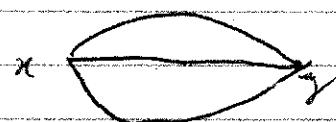
→ consider $\frac{2}{3!} \phi^3$ theory --

At 2nd order in λ , $\langle 0 | T \{ \exp \int_{-\infty}^{\infty} H_1 dt \} | 0 \rangle$

gives 5th like

$$\langle 0 | \phi(x) \phi(x) \phi(x) \phi(y) \phi(y) | 0 \rangle d^9x d^9y$$

↳ Feynman diagram is



$$D_F(x-y) D_F(x-y) D_F(x-y)$$

Nairally we expect $2!$ from interchanging x and y .

and $3! \times 3! = 36$ from interchanging the ϕ 's at x and ϕ 's at y .

→ Expect 72. But we actually only 6.

$$\overbrace{\phi_x^1 \phi_y^1} \rightarrow \left\{ \overbrace{\phi_x^1 \phi_x^2 \phi_x^3 \phi_y^4 \phi_y^5} \atop \overbrace{\phi_x^2 \phi_x^3 \phi_y^4 \phi_y^5 \phi_y^6} \right\} \text{ total } = 6.$$

$$\text{Gauge by } \phi_x^1 - \phi_y^2 \quad (\times 2) \\ \phi_x^1 - \phi_y^3 \quad (\times 2)$$

→ we have overshot by 12, because

$$\cancel{\phi_x \phi_y \phi_z \phi_y \phi_y} = \cancel{\phi_x \phi_x \phi_y \phi_y \phi_y}$$

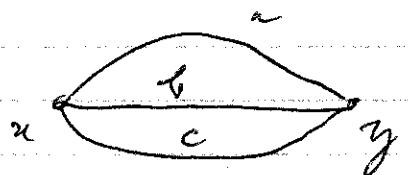
about 100 ways to do

(1) simultaneously swap $\overset{1^{\text{st}}}{x} = \overset{2^{\text{nd}}}{x}$
 $\overset{1^{\text{st}}}{x} = \overset{2^{\text{nd}}}{y}$

$$\cancel{\phi_x \phi_x \phi_y \phi_y \phi_y} = \phi_x \phi_x \phi_x \phi_y \phi_y \phi_y$$

(2) erasing all x 's = y 's doesn't do anything either ...

We can see what's going on by labeling the vertices = propagators ...



$a \leftrightarrow b$ don't
 $x \leftrightarrow y$ change
 the diagram

⇒ diagram has a permutation symmetry
 $x \leftrightarrow y$ also don't change the diagram

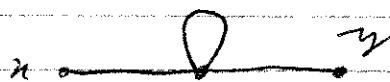
This has $2! \times 3! = 12$ elements; which is
 ten times we overshot with ...

This is called the symmetry factor 5.

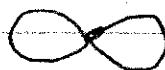
The number of diagrams or flows

$$\frac{1}{5} (3!) (3!) (2!) = 6.$$

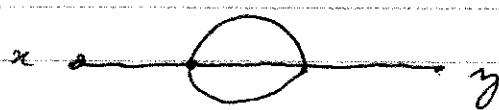
Some examples



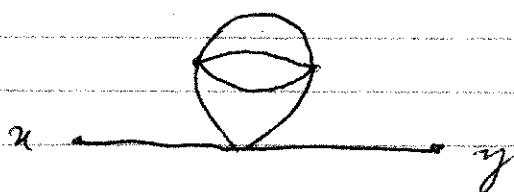
$$s = 2 \quad (\text{new}).$$



$$s = 2 \cdot 2 \cdot 2 = 8$$



$$s = 3! = 6$$



$$s = 3! \cdot 2 = 12$$

Now, we are ready to state the Feynman rules for position space.

Well start this rule by us find

$$\langle 0 | T \{ \phi(x) \phi(y) \} \exp \left\{ -i \int_{-\infty}^{\infty} p_4(t) dt \right\} | 10 \rangle$$

= (sum all possible diagrams with)
the external points

where each diagram is built out of propagators
vertices
external pts

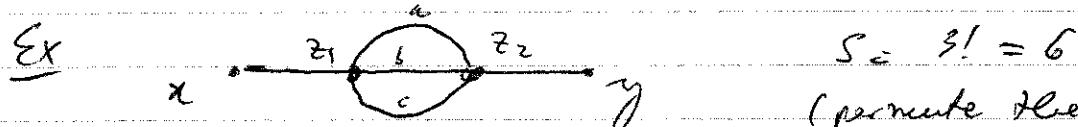
Feynman Rules for ϕ^4 theory

① For each propagator $\frac{x}{z} = D_F(x-z)$

② For each vertex $\times \frac{1}{2} = (-i\gamma) \int d^4 z$

③ For each external point $\frac{1}{x} = 1$

④ Divide by symmetry factor. $\frac{1}{4!}$
no here



$$S = 3! = 6$$

(permute the 3 propagators)
Ann $z_1 \rightarrow z_2$

Amplitude:

$$\left(\frac{-i\gamma}{1}\right)^2 \cdot \frac{1}{6} \cdot \int d^4 z_1 d^4 z_2 D_F(x-z_1) D_F(z_1-z_2) D_F(z_2-y)$$

Interpretation

- Each of the vertex factor $(-i\gamma)$ is the amplitude for the emission and/or absorption of particles at a vertex.

- The integral $\int d^4 z$ instructs us to sum over all points where fermi process can occur.

→ This is the principle of superposition!

- $\int d^4 z$ is addition of amplitudes

↳ Feynman rules tell us to multiply the amplitudes for each

independent part of the process.

Now, in most calculations, it is simpler to express the Feynman rules in terms of momenta.

→ We want momentum-space Feynman diagrams.

To do this, we write $D_F(x-y)$ in Fourier space

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

Now, we present this in a diagram by assigning a 4-momentum p to each propagator.

When 4 lines meet at a vertex, we get

$$\begin{array}{c} p_4 \\ \times \\ p_1 \\ \diagdown \\ p_3 \end{array} \rightarrow \int d^4 z e^{-ip_1 z} e^{-ip_2 z} e^{-ip_3 z} e^{+ip_4 z} = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)$$

i.e. momentum is conserved at each vertex.

The delta functions can be used to perform integrals for the propagators.

→ From here we get momentum-space Feynman rules.

6

Momentum-space Feynman rules

① Each propagator $\rightarrow = \frac{i}{p^2 - m^2 + i\epsilon}$

② Each vertex  $= -i\gamma$

③ Each external point  $= e^{-ip \cdot x}$

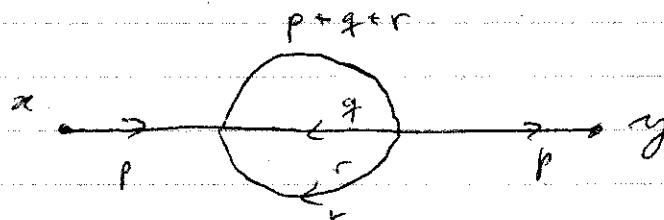
④ Impose momentum conservation at each vertex

⑤ Integrate over each ~~so~~ undetermined momenta

$$\int \frac{d^4 p}{(2\pi)^4}$$

⑥ Divide by symmetry factor.

Ex



$$\begin{aligned}
 &= \left(\frac{-i\gamma}{3}\right)^2 \frac{1}{6} \cdot \int \left\{ \left(\frac{i}{p^2 - m^2 + i\epsilon} \right)^2 \frac{i}{(q+p+r)^2 - m^2 + i\epsilon} \frac{i}{q^2 - m^2 + i\epsilon} \right. \\
 &\quad \left. \cdot \frac{i}{r^2 - m^2 + i\epsilon} \right\} \frac{d^4 q}{(2\pi)^4} \frac{d^4 r}{(2\pi)^4}.
 \end{aligned}$$

There is one subtlety, however...

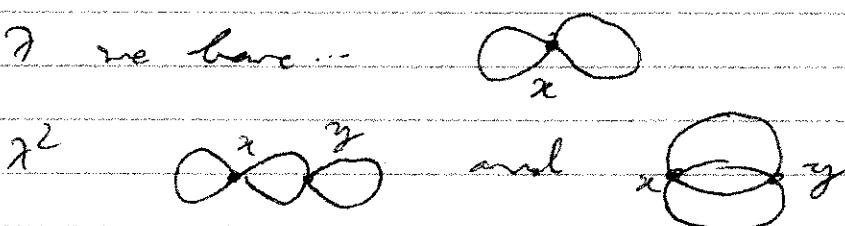
Consider diagrams without external vertices.

↳ Here are diagrams from the form:

$$\langle 0 | T \{ \text{exp} (-i \int_{-\infty}^{\infty} H_0 dt) \} | 0 \rangle$$

→ These are called "vacuum diagrams".

At order β^2 we have ...



and

$(\infty_x \infty_y) \rightarrow$ disconnected diagram

⇒ There is 5-2 for $(\infty_x \infty_y)$

In general, a vacuum diagram has connected subdiagrams V_i which appear n_i times

v_1

v_1

v_2

v_3



copies

$$n_1 = 2$$



$$n_2 = 1$$

$$n_3 = 3$$

The amplitude for the total diagram is the product:

$$\boxed{T \left(\prod_{i=1}^3 V_i^{n_i} \right)}$$

Now, the sum over all connected diagrams can be written as

$$\sum_{\text{all possible connected pieces}} \sum_{\substack{\text{all} \\ \in \{n\}}} (\text{value of connected piece}) \times \left\{ \prod_i \frac{1}{n_i!} (v_i)^{n_i} \right\}$$

The sum of the unconnected pieces factors out, giving

$$= (\sum_{\text{connected}}) \times \sum_{\substack{\text{all} \\ \in \{n\}}} \left(\prod_i \frac{1}{n_i!} (v_i)^{n_i} \right)$$

sum all the value of the connected pieces

$$\text{Now... } \sum_{\substack{\text{all} \\ \in \{n\}}} \left(\prod_i \frac{1}{n_i!} (v_i)^{n_i} \right)$$

$$= \prod_i \sum_{\substack{\text{all} \\ \in \{n\}}} \frac{1}{n_i!} (v_i)^{n_i}$$

$$= \prod_i \exp(v_i)$$

$$= \exp\left(\sum_i v_i\right)$$

$$\rightarrow \boxed{\sum_{\text{all diagrams}} = \sum_{\text{connected}} \times \exp\left(\sum_{\text{disconnected}}\right)}$$

Now, recall that the sum 'all vacuum diagrams' is just going to be

$$\exp(\sum V_i).$$

$$\Rightarrow \langle 0 | T \{ \exp(-i \int_{-\infty}^{\infty} H_I(t) dt) \} | 0 \rangle = \exp(\sum V_i)$$

and as we have agreed --

$$\begin{aligned} \langle 0 | T \{ \phi_x(x) \phi(y) \exp \left\{ -i \int_{-\infty}^{\infty} H_I(t) dt \right\} \} | 0 \rangle \\ = (\text{connected}) \times \exp(\sum V_i). \end{aligned}$$

\Rightarrow And so we have that

$$\langle S | T \{ \phi(x) \phi(y) \} | S \rangle = \lim \frac{\langle 0 | T(\phi(x) \phi(y)) \exp(iH_0 t) | 0 \rangle}{\langle 0 | T(\exp(-iH_0 t)) | 0 \rangle}$$

= \sum all connected diagrams with 2 external pts.

More generally --

$$\frac{\langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \exp \left(-i \int_{-\infty}^{\infty} H_I(t) dt \right) \} | 0 \rangle}{\langle 0 | T \{ \exp \left(-i \int_{-\infty}^{\infty} H_I(t) dt \right) \} | 0 \rangle}$$

= \sum connected diagrams with
end points x_1, \dots, x_n

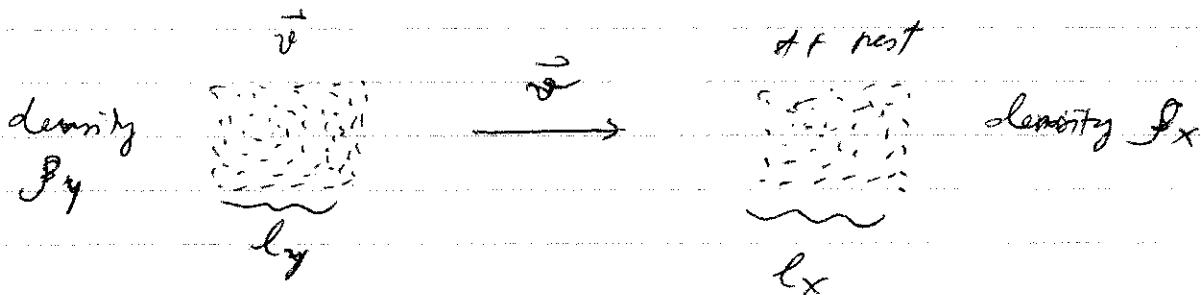
Cross section & the S-matrix

Now that we have a formula for computing the n-point correlation function...

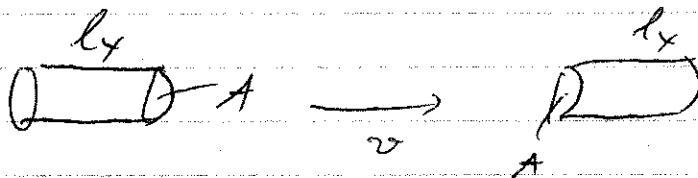
→ Next task is to compute quantities that can be measured
 ↳ cross section & decay rates.

The cross section :

Consider collision of 2 beams of particles with relatively well-defined momenta.



- { f_y, f_x are observed from rest.
- { f_y, f_x are densities at rest. Let A be the cross sectional area of overlap.



$$\text{Total \# of particles} \dots N_x = f_x l_x A$$

$$N_y = f_y l_y A$$

→ Total # of scatterings is proportional to $N_x N_y$.

Let total number of scatterings be

$$N_x \cdot N_y = \left(\frac{\sigma}{A}\right)$$

↑ probability one particular X particle & Y collide.

Call σ the effective area or "cross section" of the scattering process.

Let $N_x = 1$, then

$$\boxed{\sigma = \frac{\text{total # scatterings}}{p_x \cdot p_y}}$$

For small time interval...

$$\sigma = \frac{\text{total # scatterings} / \Delta t}{p_y \cdot (p_x / \Delta t)} \rightarrow \begin{array}{l} \text{scattering rate} \\ \text{particle flux.} \end{array}$$

The differential cross section is the portion of σ in which the final particle momentum lie inside some window of momenta.

↪ write this as

$$\frac{d\sigma}{d\vec{p}_f - d^3\vec{p}_f}, \text{ so that}$$

$\uparrow \quad \uparrow$

final particle momenta

$$\int \frac{d\sigma}{d^3 p_1 \cdots d^3 p_n} \cdot d^3 \vec{p}_1 \cdots d^3 \vec{p}_n = \int d\sigma = 0.$$

Now, if there are only 2 final particles then
there are only two free parameters --

Why? two spatial momenta \rightarrow 6 degrees

4-momentum conservation \rightarrow 4 ~~free~~ constraints

\Rightarrow can take these two degrees to be
orient. angles $\theta = \phi$

\rightarrow Then we can measure $\boxed{\frac{d\sigma}{d\Omega}(\theta, \phi)}$

where $d\Omega$ is the solid angle differential

$$d\Omega = d\cos\theta d\phi$$

"Differential cross section" refers to $\frac{d\sigma}{d\Omega}$

Let's look at one example -- Consider a periodic
box with length L in all orders -

Spatial momentum mode we now discrete:

$$\vec{k} = \frac{2\pi}{L} (n_x, n_y, n_z) . \quad n_i \in \mathbb{Z} .$$

Have comm. relation:

$$[a_{\vec{k}}, a_{\vec{k}'}^\dagger] = \delta_{\vec{k}, \vec{k}'} V \rightarrow V = V^3 \text{ (volume)}$$

$\Rightarrow V \rightarrow \infty$

$$\hat{a}_{\vec{k}, \vec{k}'}^\dagger \cdot V = \iiint_{-\infty}^{\infty} dx_1 dx_2 dx_3 e^{i(\vec{k} - \vec{k}') \cdot \vec{x}}$$

$$\rightarrow \iiint_{-\infty}^{\infty} dx_1 dx_2 dx_3 e^{i(\vec{k} - \vec{k}') \cdot \vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}').$$

In this box...

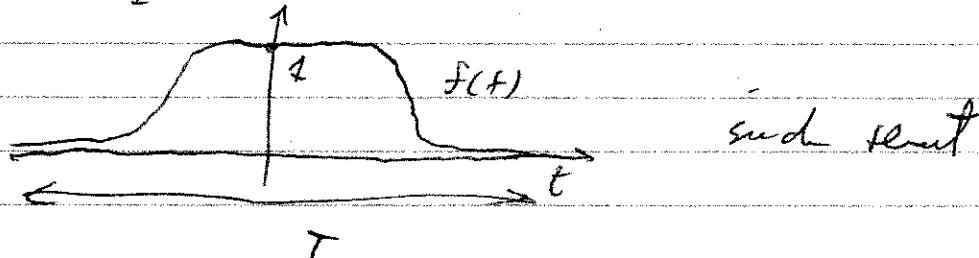
$$\begin{aligned} \phi(x) &= \sum_{\vec{k}} \frac{(2\pi/l)^3}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} (a_{\vec{k}} e^{-ik \cdot x} + a_{\vec{k}}^\dagger e^{i\vec{k} \cdot \vec{x}}) \\ &= \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}}} (a_{\vec{k}} e^{-ik \cdot x} + a_{\vec{k}}^\dagger e^{i\vec{k} \cdot \vec{x}}) \end{aligned}$$

Oct 25, 2020

Now imagine starting with free field theory.

at some early time then switching on the interaction
shortly, and then slowly switching off the interactions.

i.e. $H_i(t) \rightarrow H_i(t) f(t)$ where $f(t)$ looks like



such that $\int_{-\infty}^{\infty} f(t) dt = T$, $\int_{-\infty}^{\infty} (f(t))^2 dt = T$

$$\text{Let } S = T \exp \left\{ -i \int_{-\infty}^{\infty} dt H_I(t) f(t) \right\}$$

Define the S matrix as

$$\langle \text{final} | S | \text{initial} \rangle$$

where $|\text{initial}\rangle$ is a free particle state with momentum \vec{k}_I^I & energy E_I^I

and $|\text{final}\rangle$ is a free particle state with momentum \vec{k}_F^F & energy E_F^F

Now look at $S-I$:

$$\langle \text{final} | S-I | \text{initial} \rangle$$

as $T \rightarrow \infty$, $V \rightarrow \infty$ we can write this amplitude as

$$\langle \text{final} | S-I | \text{initial} \rangle = i \cdot N \cdot (2\pi)^4 \delta(E_F^F - E_I^I) \prod_j \delta^{(2)}(\vec{k}_{tot}^F - \vec{k}_{tot}^I)$$

fn of the momenta

For finite T and finite V we instead have

$\langle \text{Final } | S - I | \text{Initial} \rangle$

$$= i M \int_{-\infty}^{\infty} f(t) e^{i(E_{\text{tot}}^F - E_{\text{tot}}^I)t} \cdot \delta_{k_{\text{tot}}^F, k_{\text{tot}}^I} \cdot V$$

$S,$

$$|\langle \text{Final } | S - I | \text{Initial} \rangle|^2 = |M|^2 \cdot \delta_{k_{\text{tot}}^F, k_{\text{tot}}^I} \cdot V^2$$

$$\cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) f(t') e^{i(E_{\text{tot}}^F - E_{\text{tot}}^I)(t-t')} dt dt'$$

as $T \rightarrow \infty$, this is some constant
times $\delta(E_{\text{tot}}^F - E_{\text{tot}}^I)$)

What is this constant? ←

To get this \rightarrow integrate w.r.t E_{tot}^F .

$$\rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} dE_{\text{tot}}^F e^{-i(E_{\text{tot}}^F - E_{\text{tot}}^I)(t-t')} \right\} f(t) f(t') dt dt'$$

$2\pi \delta(t-t')$

$$= 2\pi \int_{-\infty}^{\infty} f^2(t) dt = 2\pi \cdot T.$$

So the constant is $2\pi \cdot T$ and ∞

$$|\langle \text{Final } | S - I | \text{Initial} \rangle|^2 = |M|^2 (2\pi) \delta(E_{\text{tot}}^F - E_{\text{tot}}^I) \delta_{k_{\text{tot}}^F, k_{\text{tot}}^I} \cdot V^2 T$$

Now, we have been using relative normalizations

$$\left\{ \begin{array}{l} \langle \text{final} | \text{initial} \rangle = \prod_i (2E_i^F \cdot v) \\ \langle \text{final} | \text{initial} \rangle = \prod_i (2E_i^F \cdot v) \end{array} \right. \quad \text{becomes} \quad (2\pi)^3 \delta^{(3)}(0) \text{ as } V \rightarrow \infty$$

To get transition probability per unit time -

$$\frac{\text{probability}}{\text{time}} = \frac{1}{T} \frac{|\langle \text{final} | S - I | \text{initial} \rangle|^2}{\langle \text{final} | \text{final} \rangle \langle \text{initial} | \text{initial} \rangle}$$

$$= \frac{|M|^2 (2\pi)^3 \delta(E_{tot}^F - E_{tot}^I) \delta(\vec{k}_{tot}^F, \vec{k}_{tot}^I) \cdot V^2}{\prod_i (2E_i^F \cdot v) \prod_i (2E_i^I \cdot v)}$$

$$\prod_i (2E_i^F \cdot v) \prod_i (2E_i^I \cdot v)$$

$$\text{As } V \rightarrow \infty, \delta(\vec{k}_{tot}^F, \vec{k}_{tot}^I) \cdot V \rightarrow (2\pi)^3 \delta^{(3)}(\vec{k}_{tot}^F - \vec{k}_{tot}^I)$$

If we sum over final states in some window then we have

$$\sum_{\vec{k}_1^F, \dots, \vec{k}_{n_F}^F} \frac{1}{(2E_1^F \cdot v)} \dots \frac{1}{(2E_{n_F}^F \cdot v)} \frac{|M|^2 (2\pi)^3 \delta(\vec{k}_{tot}^F - \vec{k}_{tot}^I) \cdot V}{(2E_1^I \cdot v) \dots (2E_{n_I}^I \cdot v)}$$

$$\text{As } V \rightarrow \infty, \frac{d^3 \vec{k}_1^F}{(2\pi)^3 2E_1^F} \dots \frac{d^3 \vec{k}_{n_F}^F}{(2\pi)^3 2E_{n_F}^F} \quad \left\{ \begin{array}{l} n_I = \# \text{ initial particles} \\ n_F = \# \text{ final particles} \end{array} \right.$$

Consider single particle decay ... ($n_F = 2$)

The total decay rate is $\Gamma = \int dP$ where

$$dP = \frac{1}{2E^F} \left(\prod_{i=1}^{n_F} \frac{d^3 \vec{k}_i^F}{(2\pi)^3 2E_i^F} \right) |M|^2 (2\pi)^4 \delta^{(4)}(\vec{k}_{tot}^F - \sum_i \vec{k}_i^F)$$

For a 2-particle initial state, the cross section is given by

$$\sigma = \frac{\text{probability}}{\text{time} \cdot \text{flux density}}$$

flux density = relative velocity between beam and target
 \times density of incoming beam in lab frame.

We have normalized probability for one incoming beam particle \rightarrow density = $1/V$, and flux.

$$\text{flux} = \frac{1/\vec{v}_A - 1/\vec{v}_B}{V} \rightarrow \vec{v}_A, \vec{v}_B = \text{velocity of particles in lab frame}$$

$$\text{So } d\sigma = \left(\prod_{i=1}^{n_F} \frac{d^3 \vec{k}_i^F}{(2\pi)^3 2E_i^F} \right) (2\pi)^4 \delta^{(4)}(\vec{k}_{tot}^F - \sum_i \vec{k}_i^F) \frac{|M|^2}{2E_A E_B |\vec{v}_A - \vec{v}_B|}$$

call this $d\Gamma_{n_F}$

Now consider special case: 2 final particles ($\eta_F = 2$)
in CM frame

$$\vec{p}_1 \text{ other } \eta_F = 2 \quad p_1 = |\vec{p}_1|$$

$$-\vec{p}_1 \rightarrow d\pi_2 = \int \frac{dp_1 \, d\Omega}{(2\pi)^3 2E_1 2E_2} (2\pi) \delta(E_{\text{cm}} - E_1 - E_2)$$

where final particle energies are $E_1 = \sqrt{(\vec{p}_1)^2 + m^2}$, $E_2 = \sqrt{|\vec{p}_2|^2 + m^2}$

~~$$d\pi_2 = \int \frac{dp_1}{16\pi^2 E_1 E_2} d\Omega$$~~

$$\rightarrow d\pi_2 = \int \frac{d\Omega}{16\pi^2} \int_0^\infty \frac{p_1^2 \delta(-E_{\text{cm}} + \sqrt{(\vec{p}_1)^2 + m^2} + \sqrt{(\vec{p}_2)^2 + m^2})}{\sqrt{(\vec{p}_1)^2 + m^2} \cdot \sqrt{(\vec{p}_2)^2 + m^2}} dp_1$$

Recall that $\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}$

$$\frac{dE_1}{dp_1} = \frac{d\sqrt{(\vec{p}_1)^2 + m^2}}{dp_1} = \frac{1}{2} \frac{2p_1}{\sqrt{(\vec{p}_1)^2 + m^2}} = \frac{p_1}{E_1}$$

Likewise $\frac{dE_2}{dp_2} = \frac{d\sqrt{(\vec{p}_2)^2 + m^2}}{dp_2} = \frac{p_2}{E_2}$

$$\therefore d\pi_2 = \int \frac{d\Omega}{16\pi^2} \frac{p_1^2}{E_1 E_2 \left(\frac{p_1}{E_1} + \frac{p_2}{E_2} \right)} \Bigg|_{p_1 \text{ s.t. } E_{\text{cm}} = E_1 + E_2}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_1 + E_2} = \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_{\text{cm}}}$$

S for 2 particles \rightarrow 2 particle ...

$$\left(\frac{dc}{dr} \right)_{\text{com}} = \frac{16 \pi^2 \mu^2 \cdot (M)^2}{2 E_A E_B (\vec{r}_A - \vec{r}_B) \cdot 16 \pi^2 E_{\text{com}}} \quad (E_{\text{com}} = E_A + E_B)$$

Nur, it is conventional to define the T -matrix as

$$S = 1 + iT$$

$$\text{Claim } \langle \vec{p}_1^F, \dots, \vec{p}_{n_F}^F | iT | \vec{p}_A, \vec{p}_B \rangle$$

$$= \lim_{t \rightarrow \infty} \langle \vec{p}_1^F, \dots, \vec{p}_{n_F}^F | T \exp \left[i \int_{-\infty}^{\infty} dt \mathcal{H}(t) \right] | \vec{p}_A, \vec{p}_B \rangle_{\text{free}}^*$$

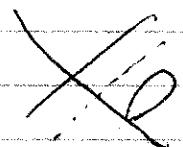
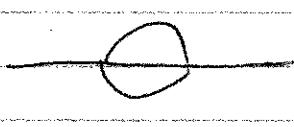
where $*$ = connected diagrams only
+ "asymptotic" diagrams only

"asymptotic" \equiv diagrams can't be broken into
two connected pieces by cutting one
internal line
(i.e. 1 particle irreducible)

ex

(not asymptotic)

(asymptotic)



The claim won't be proven now, but the stem
is similar as before...

$$|\alpha\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} (e^{-iE_0 T} |\alpha\rangle_0)^{-1} e^{-iH T} |\alpha\rangle_0$$

and we would like something similar...

$$\langle \tilde{p}_1, \dots, \tilde{p}_n \rangle \propto \lim_{T \rightarrow \infty(1-i\epsilon)} e^{-iHT} \langle \tilde{p}_1, \dots, \tilde{p}_n \rangle_{\text{free}}$$

For now, we'll just take this claim as true...

→ Note that

relativistic
normalization

$$\phi_I^+(x) |\vec{p}\rangle_{\text{free}} = \underbrace{\int \frac{d^3 k}{(2\pi)^3 \sqrt{2E_k}}}_{\text{relativistic normalization}} a_k^\dagger e^{-ik \cdot x} \sqrt{2E_p} a_p^\dagger |\alpha\rangle_0$$

$$= e^{-ip \cdot x} |\alpha\rangle_0$$

We can think of shifting the commutator of $\phi_I^+(x)$
with the a_p^\dagger from $|\vec{p}\rangle_{\text{free}}$

→ suggest the relation

$$\boxed{\phi_I^+(x) |\vec{p}\rangle_{\text{free}}} = \cancel{a_p^\dagger} e^{-ip \cdot x}$$

Now drop the "free" subscript - really...

$$\langle \tilde{p} | \phi_I^+(x) = e^{+ip \cdot x} \langle 0 | + \text{c.c.}$$

$$\langle \tilde{p} | \phi_I^+(x) = e^{+ip \cdot x}$$

→ Set Feynman Rules in position space with external lines

Propagator: $\frac{x-y}{D_F(x-y)}$

Internal vertex: $\times_z \quad (-i\lambda) \int d^4 z$

Each external line $\text{p} \leftarrow e^{-ip} x$

Divide by symmetry factor S .

Feynman rules in momentum space with external line

Propagator: $\frac{p}{p^2 - m^2 + i\epsilon}$

Internal vertex: $\times \quad -i\lambda = \text{momentum conservation}$

External line

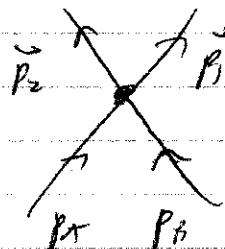
$\text{p} \leftarrow$ no extra factor (just 1)

Integrate over all consumed momenta & divide by S .

~~4~~

S₁ $\langle \tilde{p}_1, \tilde{p}_2 | i\tau | \tilde{p}_A, \tilde{p}_B \rangle$ at lowest order...

well -



Feynman amplitude:

$$iM = -i\lambda$$

So

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{(\vec{p}^{\text{final}} / M)^2}{2E_A E_B |\vec{v}_A - \vec{v}_B| 16\pi^2 E_{\text{cm}}}$$

Let $\rho = |\vec{p}^{\text{final}}| = |\vec{p}_{\text{tot}}| = |\vec{p}_A|$ all same since masses are all same

$$E_{\text{cm}} = 2E_A = 2\sqrt{\rho^2 + m^2}$$

$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2\rho}{E_A}$$

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{\pi^2 \rho}{\frac{1}{2} (2E_A)(2E_B) 16\pi^2 E_{\text{cm}}} = \frac{\pi^2}{64\pi^2 E_{\text{cm}}^2}$$

This is spherically symmetric, so

$$\sigma_{\text{tot}} = (4\pi) \frac{\pi^2}{64\pi^2 E_{\text{cm}}^2} \cdot \frac{1}{2} \rightarrow \text{particles in final state are identical so need a } 1/2 \text{ factor.}$$

→

$$\sigma_{\text{tot}} = \frac{\pi^2}{32\pi^2 E_{\text{cm}}^2}$$

→ our first QFT cross section!

Feynman Rules for Fermions

3rd 26
2020

so far we've discussed only the ϕ^4 theory --

→ need to generalize results to theories containing fermions

→ need to generalize defn. of time ordering
2 normal ordering symbols to include fermions --

Recall --

$$\langle T\{\psi_a(x)\psi_b(y)\}\rangle = \begin{cases} \psi_a(x)\psi_b(y) & x^0 > y^0 \\ -\psi_b(y)\psi_a(x) & x^0 < y^0 \end{cases}$$

The Feynman propagator, under this defn is

$$\begin{aligned} S_F(x-y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{\gamma(p+m)}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \\ &= \langle 0 | T\{\psi_a(x)\psi_b(y)\} | 0 \rangle \end{aligned}$$

(recall that $p = \not{p} = \not{\partial} + \not{m}$)

Generalize of T to more than two fermion fields --

$$T\{\psi_1\psi_2\psi_3\psi_4\} = \begin{cases} \psi_1\psi_2\psi_3\psi_4 & \text{if } x_1^0 > x_2^0 > x_3^0 > x_4^0 \\ -\psi_2\psi_1\psi_3\psi_4 & \text{if } x_2^0 > x_1^0 > x_3^0 > x_4^0 \\ -\psi_3\psi_2\psi_1\psi_4 & \text{if } x_3^0 > x_2^0 > x_1^0 > x_4^0 \\ \vdots & \end{cases}$$

Rule : $x(-1)$ if odd # permutations

$x(+1)$ if even # permutations.

Similarly, for normal ordering symbol -

$$N \{ a_{p_1}^{\dagger} a_{p_2}^{\dagger} a_{p_3}^{\dagger} a_{p_4}^{\dagger} \} = (-1)^3 a_{p_4}^{\dagger} a_{p_3}^{\dagger} a_{p_2}^{\dagger} a_{p_1}^{\dagger}$$

$x(-1)$ if odd permutation of fields

$x(+1)$ if even

With these, can generalize to get Wick's theorem -

1st case: 2 line fields $T \{ \overline{\psi_a(x)} \psi_b(y) \}$

$$T \{ \overline{\psi_a(x)} \psi_b(y) \} = N \{ \overline{\psi_a(x)} \psi_b(y) \} + \overline{\psi_a(x)} \psi_b(y)$$

$$\text{where } \overline{\psi_a(x)} \overline{\psi_b(y)} = \begin{cases} \{ \overline{\psi_a^+(x)}, \overline{\psi_b^-(y)} \} & x^0 > y^0 \\ -\{ \overline{\psi_b^-(y)}, \overline{\psi_a^+(x)} \} & x^0 < y^0 \end{cases}$$

$$= \langle 0 | T \{ \overline{\psi_a(x)} \overline{\psi_b(y)} \} | 0 \rangle$$

$$= \overline{S_F(x-y)}$$

$$= -\overline{\psi_b(y)} \psi_a(x)$$

When recall that

$\psi^+, \bar{\psi}^+$ are the positive frequency part of $\psi, \bar{\psi}$
 \rightarrow i.e. part with annihilation operators.

$\psi^-, \bar{\psi}^-$... "negative" creation operators.

Also note that:

$$\boxed{\psi_a(x) \bar{\psi}_b(y) = \bar{\psi}_b(x) \psi_a(y) = 0}$$

Just as we proved Wick's theorem for bosons, we can show the same for fermions.

$$T\{\psi_1 \bar{\psi}_2 \psi_3 \bar{\psi}_4 \dots\} = N[\psi_1 \bar{\psi}_2 \psi_3 \bar{\psi}_4 + \text{all possible combinations}]$$

where we note that an expression such as

$$N[\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4] = - \bar{\psi}_3 \psi_3 N[\psi_2 \bar{\psi}_4]$$

gets a minus sign since the $\bar{\psi}_3$ must loop over the ψ_2 .

Helpful hint for any fully contracted quantity, count the number of times the contraction lines cross-over tells you if the # of perm. is odd/even.

Ex $\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6 \rightarrow$ even

$\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6 \rightarrow$ odd.

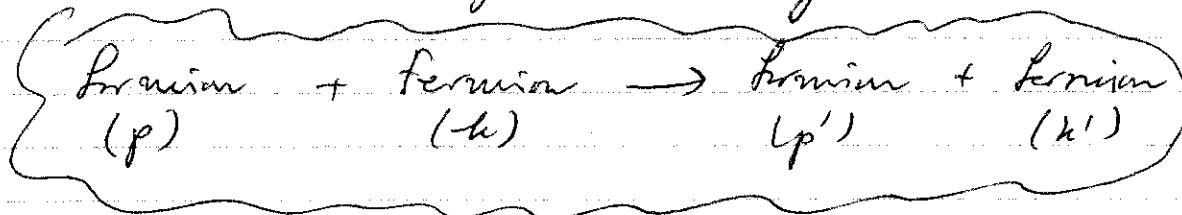
Yukawa Theory

Now we consider the simplest theory with fermions.

$$H_{\text{Yukawa}} = H_{\text{Dirac}} + H_{\text{plain-Lagrangian}} + \int d^3x g \bar{\psi} \gamma^\mu \phi$$

Simplest model of QED. We will carefully work out the rules of calculations for Yukawa theory before going to QED.

We will consider two-particle scattering reaction:



The leading contribution comes from the H_I^2 term of the S-matrix:

$$\langle p', k' | T \left\{ \frac{1}{2!} (-ig) \int d^3x \bar{\psi}_I \gamma^\mu \psi_I (-ig) \int d^3y \bar{\psi}_2 \gamma_\mu \psi_2 \right\} | p, k \rangle$$

Now we Wick's theorem to reduce this to N-product of contractions -> can act on uncontracted fields

Represent this as the contraction:

$$Y_I(x) |p, s\rangle = \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_s \underbrace{\left(\frac{s}{2p'} u^s(p') e^{-ip' \cdot x} \sqrt{2E_p} \right)}_{\text{contraction}} \underbrace{\left(\bar{u}^s(p) \right)}_{\text{operator}} |p, s\rangle$$

$$= e^{-ip \cdot x} u^s(p) |p, s\rangle$$

Fermion state with momentum \vec{p} , spin s

Define:

$$\text{So } \boxed{\langle \vec{p}, s | \Psi_F(x) | \vec{p}, s \rangle = e^{-ip \cdot x} u^s(p)}$$

$$\text{Similarly, } \langle \vec{k}, s | \Psi_F(x) | \vec{k}, s \rangle = e^{-ik \cdot x} \bar{u}^s(k)$$

$$\left\{ \begin{array}{l} \langle \vec{p}, s | \Psi_F(x) = e^{+ip \cdot x} \bar{u}^s(p) \\ \langle \vec{k}, s | \Psi_F(x) = e^{+ik \cdot x} v^s(k) \end{array} \right.$$

h. typically, a contribution to the matrix element is
see animation ...

$$\langle \vec{p}', \vec{k}' | \frac{1}{2!} (\text{fig}) \int d^4x \bar{u}^s(\vec{q}) \delta(-i\vec{q}) \int d^4y \bar{u}^s(\vec{q}) \delta(\vec{p}' - \vec{q})$$

Up to a (-) sign, the value of this quantity is

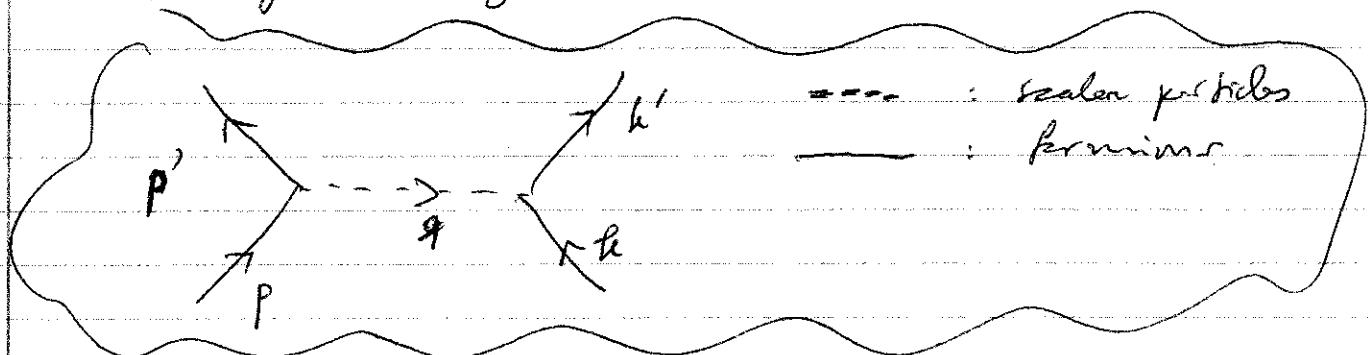
$$\begin{aligned} J &= (-i\vec{q})^2 \underbrace{\int \frac{d^4q}{(2\pi)^4}}_{\text{Int.}} \frac{i}{q^2 - m_\phi^2} (2\pi)^4 \delta^{(4)}(\vec{p}' - \vec{p} + \vec{q}) \\ &\quad \times (2\pi)^4 \delta^{(4)}(\vec{k}' - \vec{k} - \vec{q}) \bar{u}(\vec{p}') u(\vec{p}) \bar{u}(\vec{k}') u(\vec{k}) \\ &\quad \phi - \phi \quad \vec{p}' \leftrightarrow \vec{q} \quad 4_p \quad \vec{k}' - \vec{q} \quad 4 - k \end{aligned}$$

over all intermediate momenta.

→ Upon using the δ functions, $J = iM (2\pi)^4 \delta^{(4)}(\vec{p})$

$$\text{where } M = \frac{-i\vec{q}^2}{q^2 - m_\phi^2} \bar{u}(\vec{p}') u(\vec{p}) \bar{u}(\vec{k}') u(\vec{k})$$

The Feynman diagram for this is ...



Feynman rules for fermions in momentum space

① Propagators: $\overline{\phi(x)\phi(y)} = \frac{q}{\text{---}} = \frac{i}{q^2 - m_\phi^2 + i\varepsilon}$

$$\overline{\psi(x)\bar{\psi}(y)} = \frac{P}{\text{---}} = \frac{i(p+m)}{p^2 - m^2 + i\varepsilon}$$

② Vertices = $(-ig)$

③ External leg contractions: ④ $\overline{\phi(q)} = \frac{q}{\text{---}} = 1$

⑤ $\overline{\psi(p,s)} = \frac{q}{\text{---}} = 1$

⑥ $\overline{\psi(p,s)} = \frac{P}{\text{---}} = u^*(p)$
 (fermions)

⑦ $\overline{\psi(p,s)} = \frac{P}{\text{---}} = \bar{u}^*(p)$

⑧ $\overline{\psi(k,s)} = \frac{k}{\text{---}} = \bar{v}^*(k)$
 (anti-fermion)

⑨ $\overline{\psi(k,s)} = \frac{k}{\text{---}} = v^*(k)$

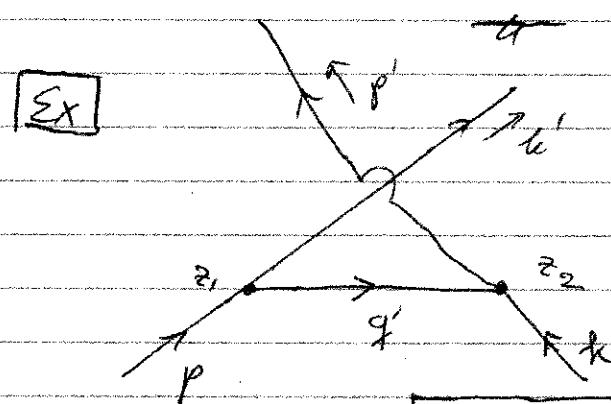
⑩ Momentum conservation at each vertex

⑪ Integrate over intermediate momenta
 ⑫ Figure out sign of diagram

Note { initial state have momentum pointing in
} final out

external particle / antiparticle (\Rightarrow)  same direction
opposite direction

Need to compute: There are 2 cross-ways, so (+1).



$$M = \int d^4 z_1 d^4 z_2 \langle k', p' | \bar{q}_1 q_1, \phi_1, \bar{q}_2 q_2, \phi_2 | \tilde{p}, \tilde{k} \rangle$$

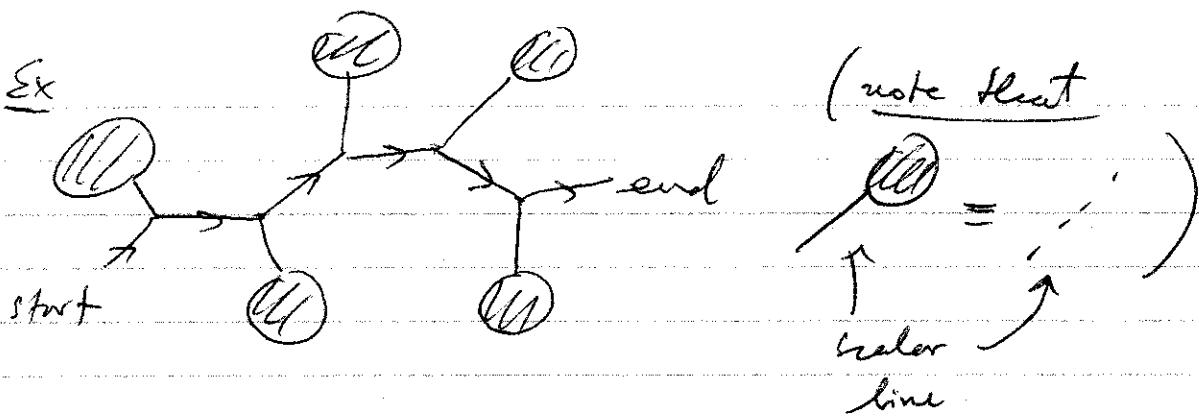
$$\text{cross-cross} = 3 \Rightarrow (-1)$$

$$\rightarrow M = (-ig)^2 (-1) \cdot \frac{i}{q'^2 - m_\phi^2 + i\epsilon} (\bar{u}(k') u(p)) (\bar{u}(p) u(k))$$

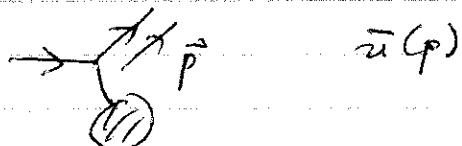
2 vertices 3cc $\langle \bar{u}'/\bar{q} \quad q/p \rangle \langle p'/\bar{q} \quad q/k \rangle$

$$\text{where } q' = p - k'.$$

{}{} Tips for each fermion line that doesn't close into loop, follow the particle number arrow to the end



If the end is an outgoing fermion write down as



If the end is an incoming antifermion --



Write down fermion propagators you encounter as you follow the particle number arrow backwards

$$u(p-bar) \xrightarrow{p_2 + m} \frac{i(p_3 + m)}{p_3^2 - m^2 + i\epsilon}$$

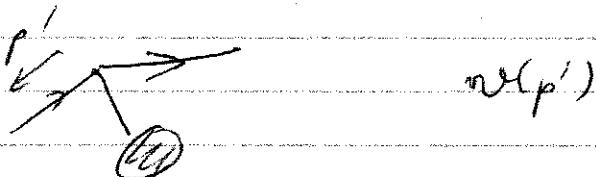
Note if $\xrightarrow{\quad} \xrightarrow{\quad} \frac{i(p+m)}{p^2 - m^2 + i\epsilon}$

If $\xrightarrow{\quad} \xleftarrow{\quad} , \frac{i(-p+m)}{p^2 - m^2 + i\epsilon}$

If the start is an incoming fermion



If --- incoming anti fermion



If the fermion lines form a closed loop

$$\begin{aligned}
 & \text{Diagram: A loop of four fermion lines labeled } p_1, p_2, p_3, p_4 \text{ with arrows indicating direction.} \\
 & = \overline{\psi} \psi \overline{\psi} \psi \\
 & = (-1) + \left\{ \overline{\psi} \gamma^5 \psi \overline{\psi} \gamma^5 \psi \right\} \\
 & = (-1) + \left\{ S_F S_F S_F S_F \right\}
 \end{aligned}$$

→ a closed loop always gives a factor of (-1) in the trace of a product of Dirac matrices.

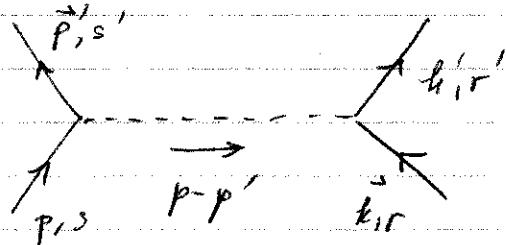
"trace" because we sum over the spinor indices.

The Yukawa Potential

Constitutes non-relativistic scattering of 2 different fermions
 \rightarrow interact via exchange of a scalar particle.

Ignore $O(\vec{p}/m^2)$ corrections, momenta are

$$\left\{ \begin{array}{l} p = (m, \vec{p}) , \quad k = (m, \vec{k}) \\ p' = (m, \vec{p}') , \quad k' = (m, \vec{k}') \end{array} \right.$$



$$= (m - m)^2 - (\vec{p}' - \vec{p})^2 + O(\dots)$$

$$(p' - p)^2 = -|\vec{p}' - \vec{p}|^2 + O(p'')$$

$$u^s(p) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^r \end{pmatrix}, \text{ etc.}$$

$$\text{unless } \xi^s \xi^r = \delta_{sr}$$

$$\text{Spinor products: } \left\{ \begin{array}{l} \bar{u}^s(p') u^s(p) = 2m \xi^{s+} \xi^s = 2m s^{ss} \\ \bar{u}^r(k) u^r(k) = 2m \xi^{r+} \xi^r = 2m s^{rr} \end{array} \right.$$

\rightarrow the spin of each particle is conserved.

$$\rightarrow \text{amplitude: } iM = \frac{i\theta^2}{(\vec{p}' - \vec{p})^2 + m^2} (\bar{u}^s(p') u^s(p)) (\bar{u}^r(k) u^r(k))$$

$$iM = \frac{\delta g}{|\vec{p}' - \vec{p}|^2 + m^2} \frac{2m s^{rr} 2m s^{ss}}{2m s^{rr} 2m s^{ss}}$$

$$\text{Oscillations} \quad iM = \frac{i\bar{s}^2}{(\vec{p}' - \vec{p})^2 + m_q^2} \quad 2m\delta^{ss'} 2m\delta^{rr'}$$

Compared with the Born approximation to the scattering amplitude in nonrelativistic scattering theory (Section 15.1), in form of the potential $V(x)$:

$$\langle p' | iT | p \rangle = -i \tilde{V}(\vec{q}) (2\pi) \delta(E_{p'} - E_p)$$

(where $\vec{q} = \vec{p}' - \vec{p}$)
 where does this come from?

(cf. Griffiths Chapter 21 on the Born approximation)

→ check out Born's approximation to scattering amplitude in nonrelativistic quantum mechanics

Nov 1, 2020

so we find that

$$\text{Amplitude} = iM = \frac{i\bar{s}^2}{(\vec{p}' - \vec{p})^2 + m_q^2} \quad 2m\delta^{ss'} 2m\delta^{rr'} \xrightarrow{\text{FT}} \mathcal{F}\{V(r)\}$$

$$\therefore \langle p' | iT | p \rangle = -i \tilde{V}(\vec{q}) (2\pi) \delta(E_{p'} - E_p)$$

$$(\vec{q} = \vec{p}' - \vec{p})$$

→ for Yukawa interaction.

$V(\vec{q}) = -\frac{g^2}{ \vec{q} ^2 + m_q^2}$

With $\tilde{V}(q)$, we take inv FT to get V .

$$V(r) = \int \frac{d^3 q}{(2\pi)^3} \frac{-g^2}{|q|^2 + m_q^2} e^{i\vec{q} \cdot \vec{r}}$$

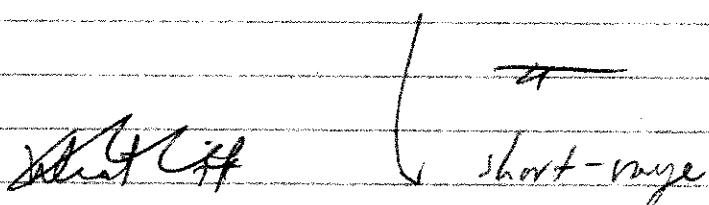
$$= \frac{-g^2}{(4\pi r)^3} \int_0^\infty \frac{1}{1 - \frac{q^2}{q^2 + m_q^2}} e^{iqr} - e^{-iqr} \frac{1}{q^2 + m_q^2}$$

$$= \frac{-g^2}{4\pi^2 i r} \int_{-\infty}^\infty dq \frac{q e^{iqr}}{q^2 + m_q^2}$$

evaluate w/ CIF

$$\Rightarrow V(r) = \frac{-g^2}{4\pi} \frac{e^{m_q r}}{r} \quad \rightarrow \text{attractive Yukawa potential}$$

range $1/m_q = \frac{\hbar}{m_q c}$, the Compton wavelength of the decayed boson. (Yukawa predicted the mass of the pion from this)



~~short~~ ~~range~~

short-range

$V(r)$

$1/m_q$

r

Review of QUANTUM ELECTRODYNAMICS

To get to QED, we go from Yukawa theory by
replace the scalar field particle ϕ with
a vector particle A_μ

→ replace Yukawa Lagrangian with

$$H_{\text{int}} = \int d^3x e \bar{\psi} \gamma^\mu \psi A_\mu.$$

→ Vector particle has spin 1

Recall that fermions come with spinor indices

$$\psi(p)$$

→ Photon has a polarization state $\epsilon^\mu(p)$

Feynman Rules (annihilation area) → (derive later)

$$\text{Photon propagator: } p^\mu p^\nu \rightarrow \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$$

$$\text{New vertex} \quad \left\{ \begin{array}{l} a \\ b \end{array} \right. \rightarrow \frac{\text{Dresser}}{\text{spinor indices}} \quad = -i \gamma^\mu_{ab} \quad (\text{Lorentz index})$$

$$\text{External photon lines} \quad A^\mu(\vec{p}, \epsilon) = \frac{p^\mu}{\vec{p}^\mu} = \epsilon^\mu(p)$$

$$\langle \vec{p}, \epsilon | A_\mu = \frac{p_\mu}{\vec{p}^\mu} = \epsilon_\mu(p)$$

To justify there also null flat in Lorentz gauge we have

$$\partial_\mu A^\mu = 0, \text{ in which case}$$

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow \boxed{\partial_\mu A^\nu = 0.}$$

h, such A^ν ($\nu = 0, 1, 2, 3$) satisfies the k6 eqn with zero mass.

→ In \vec{p} space - solutions are plane waves.

$$\left\{ \epsilon_{\mu}(p) e^{-ip^{\nu}x}, \text{ where } p^2 = 0 \text{ (massless)} \right. \\ \left. \epsilon_{\mu}(p) = 4\text{-vector} \right.$$

In Fourier amplitudes -

$$A_\mu(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{(2E_p)} \sum_{r=0}^3 (\hat{a}_p^r \epsilon_m^r(p) e^{-ip^{\nu}x} + \hat{a}_p^{r\dagger} \epsilon_m^{r*}(p) e^{ip^{\nu}x})$$

also $|p| = E_p = \sqrt{|\vec{p}|^2}$

where $r = 0, 1, 2, 3 \rightarrow$ basis for polarization vectors.

But notice that photons in the real world only have 2 polarizations -

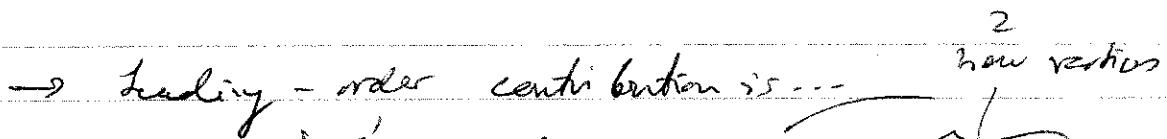
→ More on this later..

H

The Coulomb potential

Consider nonrelativistic scattering calculation of the previous section...

→ Look at scattering of 2 fermions of the same mass...

→ Leading-order contribution is ... 

$$iM = \frac{e^2}{(p-p')^2} \bar{u}(p') \gamma^\mu u(p) \frac{-iq\omega}{(p'-p)^2} \times \bar{u}(k') \gamma^\nu u(k)$$

↑
polaron line

In nonrelativistic limit...

$$\bar{u}^s(p') \gamma^\mu u^s(p) = [\xi^{st} \xi^{s+} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}] \begin{pmatrix} \xi^r \\ \xi^s \end{pmatrix}_{xm}$$

$$= 0 \quad \forall i = 1, 2, 3$$

and

$$\bar{u}^s(p') \gamma^\mu u^s(p) = (\xi^{st} \xi^{s+}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^r \\ \xi^s \end{pmatrix}_{xm}$$

$$= 2m\delta^{ss'}$$

So... almost like the Yukawa case...

$$iM \approx \frac{ie^2}{-(\vec{p}-\vec{p}')^2} (2m)^2 g_{00} \delta^{ss'} \delta^{rr'}$$

Same as Yukawa except the $(-)$ sign $\approx m_p = 0$.

→ this potential is repulsive! → the potential is
of the Coulomb potential.

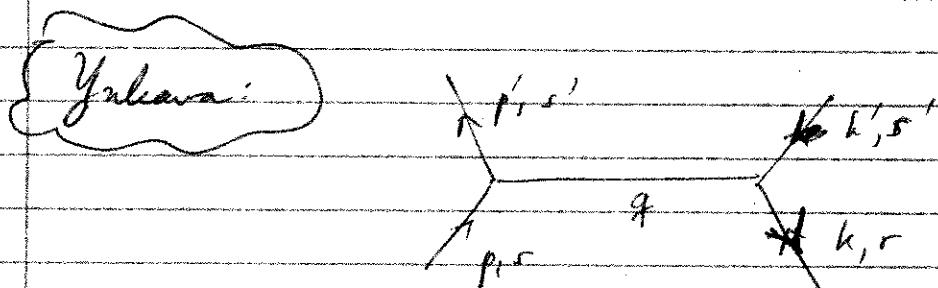
$$V(r) = \frac{e^2}{4\pi r} = \alpha$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

Fine structure constant

What about fermion-anti-fermion subtrig?

or Yukawa = QED?



$$|\tilde{p}, s; \tilde{k}, r\rangle = (\sqrt{2E_p} \sqrt{2E_k} a_p^\dagger b_k^\dagger) |0\rangle$$

$$|\tilde{p}', s'; \tilde{k}', r'\rangle = (\sqrt{2E_{p'}} \sqrt{2E_{k'}} b_{k'}^\dagger a_{p'}^\dagger)$$

and so ...

$$\langle \tilde{k}, r'; \tilde{p}', s' | \underbrace{\gamma^4 \phi \gamma^4 \phi}_{\text{}} | \tilde{p}, s; \tilde{k}, r \rangle \quad (3 \text{ cross-terms})$$

For fermions:

$$\bar{u}^s(p) u^s(p) = 2m \delta^{ss}$$

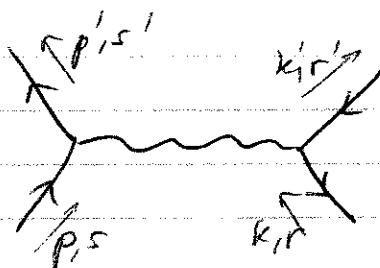
anti fermion

$$\bar{v}^r(k) v^r(k') = (\bar{s}^r - \bar{\bar{s}}^r) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{s}' \\ \bar{\bar{s}}' \end{pmatrix} x_m$$

$$= -2m \delta^{rr}$$

$$\text{so } V_{\bar{f}f}(r) = (-1)(-1) V_{ff}(r) = V_{ff}(r) \quad (\text{attractive})$$

QED



\rightarrow has an overall (-1) sign

For fermion - set $\bar{n}^s(p') \delta^{\mu} n^s(p) \rightarrow \bar{n}^s(p') \delta^{\mu} n^s(p)$
 $\approx 2m \delta^{ss}$

For antifermion ...

$$\bar{v}^r(k) \gamma^\mu v^{r'}(k') \rightarrow \bar{v}^r(k) \gamma^0 v^{r'}(k') = 2m \delta^{rr}$$

~~$$\frac{d}{r} V_{\bar{f}f}(r) = (-1) V_{ff}(r) = -V_{ff}(r)$$
 (attractive)~~

<u>Exchange particle</u>	<u>ff or $\bar{f}\bar{f}$</u>	<u>$f\bar{f}$</u>
scalar (Yukawa)	attractive	attractive
weak (QED)	repulsive	attractive
tensor (gravity)	attractive	attractive

Some problems...

(167)

[Problem] (normal ordering - applications in QM & coherent states.)

$$\rightarrow \text{look at QSHO: } H = \frac{1}{2} (p^2 + q^2)$$

where $\{p, q\} = -i$ as usual.

$$a = \frac{1}{\sqrt{2}} (q + ip)$$

$$a^\dagger = \frac{1}{\sqrt{2}} (q - ip)$$

Let $|Y_0\rangle$ be the ground state of H with normalization

$\langle Y_0 | Y_0 \rangle = 1$. For any $z \in \mathbb{C}$, we define the coherent state

$$|z\rangle = ce^{z a^\dagger} |Y_0\rangle$$

where c is such that $\langle z | z \rangle = 1$.

(a) What is c ?

$$\text{Well... } \langle z | z \rangle = 1 = \langle Y_0 | e^{z^* a} c^* c e^{z a^\dagger} | Y_0 \rangle$$

$$\text{Recall...} \quad = |c|^2 \langle Y_0 | e^{z^* a} e^{z a^\dagger} | Y_0 \rangle$$

$$\begin{aligned} (a^\dagger)^n |Y_0\rangle &= \sqrt{n!} |Y_n\rangle \\ &= |c|^2 \sum_{m,n} \langle Y_0 | \frac{(z^*)^n}{n!} \frac{(z a^\dagger)^m}{m!} | Y_0 \rangle \\ &= |c|^2 \sum_{m,n} (z^*)^n (z)^m \frac{1}{\sqrt{n! m!}} \langle Y_n | Y_m \rangle \\ &= \sum_n \frac{|z|^2}{n!} |c|^2 \\ &= e^{|z|^2} |c|^2 \end{aligned}$$

$$\Rightarrow \boxed{c = \exp \left\{ -|z|^2/2 \right\}}$$

③ What is $\langle z_1 z_2 \rangle$?

Answer (from text)

$$\langle z_1 z_2 \rangle = \exp \left\{ -\frac{|z_1|^2 - |z_2|^2}{2} \right\} \exp [z_1^* z_2]$$

④ Show that $|z\rangle$ is eig-state of a - field op.

↳ plug in a we commutes rule to

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \rightarrow \text{get}$$

$$\langle a|z\rangle = z|z\rangle$$

⑤ For $n, m \in \mathbb{Z}^+$, determine the normal-ordered expectation value

$$\langle z| : \phi^m z^n : |z\rangle$$

$$\rightarrow \text{use } p = \frac{1}{\sqrt{2}}(a - a^\dagger)$$

$$q = \frac{1}{\sqrt{2}}(a + a^\dagger)$$

then expand the normal ordering in binomial exp.

$$:p^m q^n: = \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^n \sum_{\ell=0}^m \binom{n}{k} \binom{m}{\ell} (1-a^\dagger)^k (a^\dagger)^\ell a^{n-k} a^{m-\ell}$$

$$= \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^n \sum_{\ell=0}^m \binom{n}{k} \binom{m}{\ell} (a^\dagger)^{k+\ell} a^{n+m-k-\ell} (-1)^\ell$$

$$\text{So ... with } a/\beta = z/\beta \Rightarrow \langle z | a \rangle = \langle z | z^{\frac{1}{\beta}} \rangle$$

$$\langle z | p^m z^n | \beta \rangle = \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} \langle z | (\alpha^+)^{k+l} \alpha^{n+m-k-l} | z \rangle (-i)^k$$

$$= \underbrace{\dots}_{=} (z^{\frac{1}{\beta}})^{n+l} z^{n+m-k-l} (-i)^k$$

$$= \frac{(-i)^n}{2^{(n+m)/2}} (z - z^2)^n (z + z^2)^m$$

$$= [\sqrt{z} \operatorname{Im}(z)]^n [\sqrt{z} \operatorname{Re}(z)]^m$$

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More problems

- (1) Consider a universe with 1 dim in five
 $d=1, 2, 3$ for space.

$$\begin{aligned} L &= \frac{1}{2} (\partial_x \phi_x)(\partial^x \phi_x) - \frac{1}{2} m_x^2 \phi_x^2 + \frac{1}{2} (\partial_y \phi_y)(\partial^y \phi_y) - \frac{1}{2} m_y^2 \phi_y^2 \\ &\quad + \frac{1}{2} (\partial_\Phi \Theta)(\partial^\Phi \Theta) - \frac{1}{2} M^2 \Theta^2 - 2 \Theta \phi_x \phi_y \end{aligned}$$

where $M > m_x + m_y$.

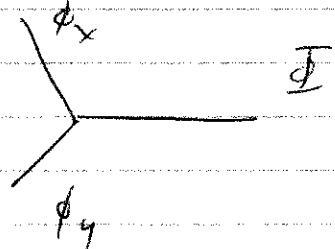
- (a) $d=3$: Find decay rate of the Θ particle into com
 frame to the lowest non-vanishing order in λ .

Do the same for $d=2, 1$.

Amplitude is given by

$$\langle \text{Final } | s=1 \text{ initial} \rangle = iM(2\pi) \frac{\delta(E_{\text{tot}}^F - E_{\text{tot}}^I)}{(2\pi)^4 \delta^4(k_x^F - k_x^I)}$$

To lowest non-trivial order



$$(iM = -i\gamma)$$

$$\text{In com., } \vec{k}_{\text{tot}}^F = \vec{0}; \quad E_{\text{tot}}^F = M$$

$$\text{Decay rate: } P = \frac{\text{probability}}{\text{time}} = \frac{1}{2M} \int d\Omega_2 |M|^2$$

where

$$\int d\Omega_2 = \int \frac{d^4 k_x}{(2\pi)^4 2E_x} \frac{d^4 k_y}{(2\pi)^4 2E_y} (2\pi) \delta(E_{\text{tot}}^F - M) (2\pi)^4 \delta^{(4)}(\vec{k}_{\text{tot}}^F)$$

$$= \int \frac{d^4 k_x}{(2\pi)^4 (2E_x)(2E_y)} k_x^{-d-1} (2\pi) \delta(E_{\text{tot}}^F - M)$$

$$\text{where } E_i = \sqrt{\vec{k}_i^2 + m_i^2}, \quad \frac{dE_i}{dk_i} = \frac{k_i}{E_i}$$

$$\rightarrow \int d\Omega_2 = \int d\Omega \frac{k_x^{d+1} (2\pi)}{(2\pi)^4 \cdot 4} \frac{1}{E_x E_y \left(\frac{k_x}{E_x} + \frac{k_y}{E_y} \right)} = \int d\Omega \frac{k_x^{d+2}}{(2\pi)^{d+1} 4M}$$

The value of k_x for which $E_{\text{tot}}^F = E_x + E_y = M$ is

$$E_{\text{tot}}^F = \frac{1}{2M} \sqrt{(M-m_x-m_y)(M-m_x+m_y)(M+m_x-m_y)} \\ \times (M+m_x+m_y)$$

$$\text{So } P = \left(\int d\Omega \right) \frac{\pi^2}{8M} \frac{u_x^{d-2}}{(2k)^{d-1}}$$

$$\text{when } d=3, \quad P = \frac{\pi^2 k_x}{8\pi M^2}$$

$$d=2, \quad P = \frac{\pi^2}{8M^2}$$

$$d=1, \quad P = \frac{\pi^2}{4k_x M^2}$$

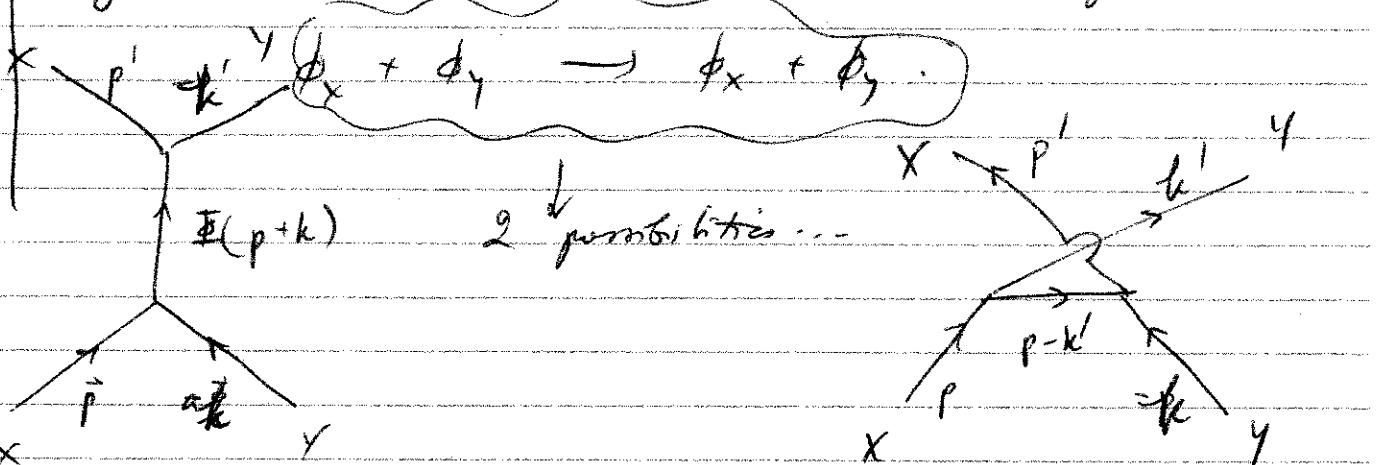
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More problems. Consider the same Lagrangian density ...

$$L = \frac{1}{2} (\partial_\mu \phi_x) \partial^\mu \phi_x - \frac{1}{2} m_x^2 \phi_x^2 + \frac{1}{2} (\partial_\mu \phi_y) \partial^\mu \phi_y - \frac{1}{2} m_y^2 \phi_y^2$$

$$\text{Find } \underbrace{S}_{\text{as a fn}} + \frac{1}{2} (\partial_\mu \mathcal{E}) (\partial^\mu \mathcal{E}) - \frac{1}{2} m^2 \mathcal{E}^2 - 2 \mathcal{E} \phi_x \phi_y.$$

Consider $d=3$. In com frame consider elastic scattering
of $p = (\vec{p}, 0)$ by $\phi_x + \phi_y$ to lowest non vanishing order in λ .



$$iM_1 = (-i\lambda)^2 \frac{i}{(\rho + h)^2 - M^2 + i\epsilon}$$

$$iM_2 = (-i\lambda)^2 \frac{i}{(\rho - h)^2 - M^2 + i\epsilon}$$

$$\rightarrow iM = iM_2 + iM_4 = -i\lambda^2 \left[\frac{1}{(\vec{p}-\vec{k})^2 - M^2 + i\epsilon} + \frac{1}{(\vec{p}+\vec{k})^2 - M^2 + i\epsilon} \right]$$

In com ... $\vec{p} = (E_x, \vec{p})$ $\vec{k} = (E_y, -\vec{p})$

$$\vec{p}' = (E_x, \vec{p}') \quad \vec{k}' = (E_y, -\vec{p}')$$

$$|\vec{p}| = |\vec{p}'|$$

where $E_{\text{cm}} = \sqrt{\vec{p}_x^2 + m^2}$, $E_{\text{com}} = E_x - E_y$.

We have ~ formula that says ...

$$\left(\frac{d\sigma}{ds} \right)_{\text{cm}} = \frac{1/|M|^2}{2E_x 2E_y |\vec{v}_x - \vec{v}_y| (16\pi^2) E_{\text{cm}}}$$

$$\text{where } v_x = \vec{p}/E_x \rightarrow v_y = -\vec{p}/E_y$$

$$\rightarrow |\vec{v}_x - \vec{v}_y| = |\vec{p}| \left(\frac{1}{E_x} + \frac{1}{E_y} \right)$$

$$\rightarrow 2E_x 2E_y |\vec{v}_x - \vec{v}_y| = 4|\vec{p}| (E_x + E_y)$$

$$\rightarrow \left(\frac{d\sigma}{ds} \right)_{\text{cm}} = \frac{|M|^2}{64\pi^2 (E_x + E_y)^2}$$

$$\text{now... } iM = -i\lambda^2 \left(\frac{1}{(\vec{p}-\vec{k})^2 - M^2 + i\epsilon} + \frac{1}{(\vec{p}+\vec{k})^2 - M^2 + i\epsilon} \right)$$

Using ...

$$\begin{aligned}
 (p-k')^2 &= (E_x - E_y)^2 - (\vec{p} + \vec{k}')^2 \\
 &= (E_x - E_y)^2 - (2\vec{p}^2 + 2\vec{k}'^2 \cos\theta) \\
 &= (E_x - E_y)^2 - 2\vec{p}^2 (1 + \cos\theta)
 \end{aligned}$$

2

$$(p+k)^2 = (E_x + E_y)^2,$$

we get

$$|M|^2 = \lambda^4 \left\{ \frac{1}{(E_x - E_y)^2 - 2\vec{p}^2(1+\cos\theta) - \mu^2} + \frac{1}{(E_x + E_y)^2 - \mu^2} \right\}$$

5

$$\frac{d\sigma}{d\Omega} = \frac{\lambda^4}{64\pi^2 (E_x + E_y)^2} \left[\frac{1}{(\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2})^2} \right]^2$$

6, explicitly ...

$$\left. \frac{d\sigma}{d\Omega} \right|_{cm} = \frac{\lambda^4}{64\pi^2 \left[(\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2})^2 \right]} \times$$

$$\left\{ \frac{1}{((\sqrt{p^2 + m_x^2} - \sqrt{p^2 + m_y^2}) - 2\vec{p}^2(1+\cos\theta) - \mu^2)} + \frac{1}{((\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2}) - \mu^2)} \right\}$$

4

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Trace Condition of γ^{μ}

Show that in Weyl's representation

$$\gamma^\mu = \begin{pmatrix} \alpha & \beta^\mu \\ \bar{\beta}^\mu & \bar{\alpha} \end{pmatrix}$$

$$\Rightarrow \boxed{\text{Tr } \gamma^\mu = 0}$$

$$\text{Tr } \gamma^\mu \gamma^\nu = \frac{1}{2} \text{Tr} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$

$$= \frac{1}{2} \text{Tr} (2g^{\mu\nu} \cdot \mathbb{1}_{4 \times 4}) = \boxed{4g^{\mu\nu} = \text{Tr } \gamma^\mu \gamma^\nu}$$

$$\begin{aligned} \text{Tr} (\underbrace{\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma}_{\text{sym}}) &= \text{Tr} [\underbrace{2g^{\mu\nu} \gamma^\rho \gamma^\sigma}_{\text{sym}} - \underbrace{\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma}_{\text{antisym}}] \\ &= \text{Tr} (2g^{\mu\nu} \gamma^\rho \gamma^\sigma - \gamma^\nu (2g^{\mu\rho}) \gamma^\sigma + \underbrace{\gamma^\mu \gamma^\rho \gamma^\sigma}_{\text{antisym}}) \\ &= \text{Tr} (2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + \gamma^\mu \gamma^\rho (2g^{\nu\sigma}) \\ &\quad - \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu) \end{aligned}$$

$$\Rightarrow 2\text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = \text{Tr} (2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + 2g^{\nu\sigma} \gamma^\mu \gamma^\rho)$$

$$\Rightarrow \boxed{\text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\nu\sigma} g^{\mu\rho}}$$

$$\text{Next... } \gamma^5 = \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma$$

$$\text{Tr } \gamma^5 = \text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = \text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma = -\text{Tr } \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \Rightarrow \boxed{\text{Tr } \gamma^5 = 0}$$

$$\text{Also, } \underbrace{\text{Tr}(\gamma^m \gamma^{m_2} \dots \gamma^{m_n} \gamma^s)}_{n \text{ odd}} = 0$$

Since by moving γ^s to the front, we get a factor of $(-1)^n$. if n odd then $(-1) \Rightarrow$ cancel to get 0

Similarly...

$$\text{Tr}(\gamma^m \dots \gamma^{m_n}) = 0 \quad \forall n \text{ odd}$$

More results

$$\cdot \text{Tr}(\mathbb{1}_{\alpha\beta\gamma\delta}) = 4$$

$$\cdot \text{Tr}(\gamma^m \dots \gamma^{m_n}) = 0, \quad n \text{ odd}$$

$$\cdot \text{Tr}(\gamma^u \gamma^v) = 4g^{uv}$$

$$\cdot \text{Tr}(\gamma^u \gamma^v \gamma^w \gamma^s) = 4g^{uv}g^{ws} - 4g^{us}g^{vw} + 4g^{ws}g^{vu}$$

$$\cdot \text{Tr}(\gamma^s) = 0$$

$$\cdot \text{Tr}(\underbrace{\gamma^m \dots \gamma^{m_n}}_{\text{odd}} \gamma^s) = 0$$

$$\cdot \text{Tr}(\gamma^u \gamma^v \gamma^s) = 0$$

$$\cdot \text{Tr}(\gamma^u \gamma^v \gamma^w \gamma^s) = -4i\varepsilon^{uvw} \rightarrow \text{completely antisymmetric}$$

More useful contraction identities

$$\cdot \delta_\mu \delta^\mu = \delta^\nu \delta^\mu g_{\mu\nu} = \frac{1}{2} (\delta^{\mu\nu} + \delta^{\nu\mu}) g_{\mu\nu} \\ = \frac{1}{2} 2g^{\mu\nu} g_{\mu\nu} = 4.$$

$$\cdot \delta_\mu \Gamma^\nu \gamma^\mu = \delta_\mu (2g^{\nu\mu} - \gamma^\mu \gamma^\nu) = +2\delta^\nu - 4\delta^\nu = -2\delta^\nu$$

$$\cdot \delta^\mu \delta^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho}$$

$$\cdot \delta^\mu \delta^\nu \delta^\rho \delta^\sigma \gamma_\mu = -2\delta^\nu \delta^\rho \gamma^\sigma$$

With this we're ready to look at chapter 5

Chapter 5: Elementary Processes of QED

$$e^+ e^- \rightarrow \mu^+ \mu^-$$

one of the most simple processes in QED

\rightarrow , we will calculate the unpolarized cross-section

$$iM = (1-ic)^2 (\bar{v}^r(p') \Gamma^{\mu r}(q)) (\bar{u}^r(k) \gamma^\nu v^r(k')).$$

$$\times \left(\frac{-ig_{\mu\nu}}{q^2 + i\varepsilon} \right)$$

$$= \frac{ie^2}{(p+p')^2} (\bar{v}^s(p') \delta^{\mu s}(p)) (\bar{u}^r(k) \gamma_\mu v^r(k'))$$

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\rightarrow need $|M|^2$. notice that

$$(\bar{v}^m v)^* = \overline{(\bar{v}^n u)} = \bar{u}^n v = \bar{u}^m v,$$

recall that $\bar{M} = \gamma^0 M^+ \gamma^0$

$$\text{and so } \bar{\gamma}^0 = \gamma^0 \gamma^0 \gamma^0 = \gamma^0$$

$$\bar{\gamma}' = \gamma^0 \gamma' \gamma^0 = \gamma'$$

$$\Rightarrow |M|^2 = \frac{e^4}{((p+p')^2)^2} \times (\bar{v}^s(p) \bar{v}^m u^s(p)) (\bar{u}^r(k) v_n v^r(k')) \\ \times (\bar{u}^s(q) \bar{v}^n v^s(p')) (\bar{v}^r(l) v_n v^r(k'))$$

$$= \frac{e^4}{((p+p')^2)^2} \left\{ \bar{v}^s(p) \bar{v}^m u^s(p) \bar{u}^s(q) \bar{v}^r s'(q') \right\} \\ \times \left\{ \bar{u}^r(k) v_n \bar{v}^s(k') v^r(k') \bar{v}_n v^r(k) \right\}$$

?

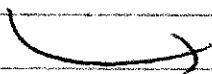
This simplifies when we throw away the spin info.

$$\rightarrow \text{Want to compute } \frac{1}{2} \sum_s \frac{1}{2} \sum_{s'} \sum_{r r'} |M(r, s' \rightarrow r, r')|^2$$

$$\text{Recall completeness: } \sum_s n^s(p) \bar{n}^s(p) = p + m$$

$$\sum_s v^s(p) \bar{v}^s(p) = p - m$$

With this, we find that



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$$\sum_{ss'rr'} \bar{v}^s(p') \gamma^m u^s(p) \bar{u}^s(p) \gamma^n v^r(p') \\ = \bar{v}(p') \gamma^m u^s(p) \bar{u}^s(p) \gamma^n v^r(p')$$

= $\ell(p) ?$

$$= \sum_{ss'} \bar{v}_a^s(p') \gamma^m_{ab} u_b^s(p) \bar{u}_c^s(p) \gamma^n_{cd} v_d^r(p')$$

$$= (p'-m)_{da} \gamma^m_{ab} (p+m)_{bc} \delta^b_{cd}$$

$$= \text{Tr} \{ (p'-m) \gamma^m (p+m) \gamma^m \}$$

So

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{e^4}{4(p+p')^4} \text{Tr} \{ (p'-m_e) \gamma^m (p+m_e) \gamma^m \} \times \text{Tr} \{ (k+m_e) \gamma_m (k'-m_e) \gamma_m \}$$

Now, look at each term---

$$\rightarrow \text{Tr} \{ \gamma^m (p+m_e) \gamma^m (p-m_e) \}$$

$$= \text{Tr} \{ \gamma^m p^\alpha \gamma^m p^\beta \} - \text{Tr} \{ \gamma^m \gamma^\nu \} m_e^2$$

\uparrow
 $p_\alpha \delta^\alpha \quad p_\beta \delta^\beta$

$$= 4 p^\mu p^\nu - 4 g^{\mu\nu} p^\mu p^\nu + 4 p^\nu p^\mu - 4 g^{\mu\nu} m_e^2$$

$$= 4 [p^\mu p^\nu + p^\nu p^\mu - g^{\mu\nu} (p^\mu p^\nu + m_e^2)]$$

Similarly, we

$$\begin{aligned} & \text{Tr} \{ \delta_{\mu}(k'-m) \delta_{\nu}(k+m_{\mu}) \} \\ &= 4 \left[k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu} (k' \cdot k + m_{\mu}^2) \right] \end{aligned}$$

So ...

$$\sum_{\text{spins}} \frac{1}{4} |M|^2 = \frac{4e^4}{q^4} \left\{ \begin{array}{l} 2(p \cdot k)(p' \cdot k') + 2(p \cdot k')(p' \cdot k) \\ - 2(k \cdot k')(p \cdot p' + m_e^2) \\ \rightarrow 2(p \cdot p')(k \cdot k + m_{\mu}^2) \\ + 4(k \cdot k + m_{\mu}^2)(p \cdot p + m_e^2) \end{array} \right\}$$

Neon energy is sufficiently high, can neglect
 ~~m_e~~ . $m_e \rightarrow$ set $m_e = 0$.

Get ...

$$\boxed{\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{4e^4}{q^4} \left\{ 2(p \cdot k')(p \cdot k) + 2(p \cdot k')(p' \cdot k) \right\} + 2m_{\mu}^2(p \cdot p')}$$

To get a more explicit formula \rightarrow go to cm frame

$p = (E, E\vec{\epsilon})$ $k = (E, \vec{k})$

$\xrightarrow{\text{unplanned}}$ $\xrightarrow{\text{?}}$ $p'(E, -E\vec{\epsilon})$ $|k| = \sqrt{E^2 - m_{\mu}^2}$

$k' = (E, -\vec{k})$ $k \cdot \vec{\epsilon} = |\vec{k}| \cos \theta$.

$$\begin{aligned} q^2 &= (p + p')^2 = 2E^2 - 0 = 4E^2 \\ p \cdot p' &= \frac{E^2 + E^2}{E^2 + E^2} = 2E^2 \end{aligned}$$

$$p \cdot k = E^2 - E(\vec{k}) \cos \theta \quad p' \cdot k' = E^2 + E(\vec{k}') \cos \theta$$

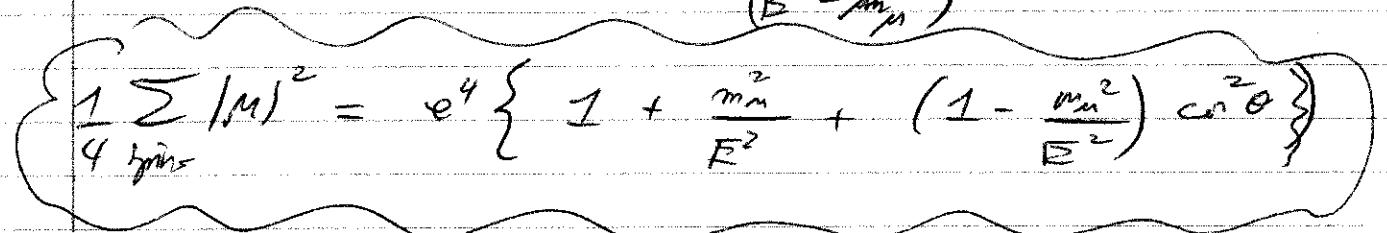
$$p' \cdot k' = E^2 - E(\vec{k}') \cos \theta \quad p' \cdot k = E^2 + E(\vec{k}) \cos \theta$$

So

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{e^4}{16E^4} \left\{ E^2(E - 1/\cos \theta)^2 + E^2(E + 1/\cos \theta)^2 + 2m_\mu^2 E^2 \right\}$$

$$= \frac{e^4}{E^4} \left\{ E^2/E^2 + \frac{1}{\cos^2 \theta} + m_\mu^2 E^2 \right\}$$

$(E^2 - m_\mu^2)$

$$\Rightarrow \frac{1}{4} \sum_{\text{spins}} |M|^2 = e^4 \left\{ 1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right\}$$


Now... for 2 body scattering into a 2-body system

$$\frac{d\sigma}{d\Omega} = \frac{1}{4F_A F_B (v_A - v_B)} \frac{1}{16\pi^2 E_{cm}} \times \frac{1}{4} \sum_{\text{spins}} |M|^2$$

where $m_e \approx 0$ and $\gamma \approx c$ are wrong
the speed of light

$$\rightarrow |v_A - v_B| \approx 2c = 2 \rightarrow = 2E_A = 2E_B = 2E$$

So

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E^2 \cdot 2} \frac{\sqrt{E^2 - m_\mu^2}}{16\pi^2 E_{cm}} \left\{ 1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right\}$$

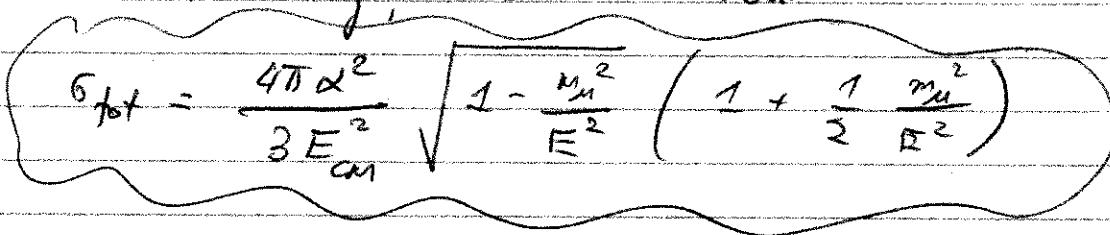
$$= \frac{e^4}{256\pi^2 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ 1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right) \cos^2 \theta \right\}$$

Integrate over $d\Omega$ gives ...

$$\sigma_{tot} = \frac{e^4 \cdot 2\pi}{288 T^2 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ 2 \left(1 + \frac{m_\mu^2}{E^2} \right) + \frac{2}{3} \left(1 - \frac{m_\mu^2}{E^2} \right) \right\}$$

$$= \frac{e^4}{64\pi E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ \frac{4}{3} \left(1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right) \right\}$$

Consequently we write this as ...

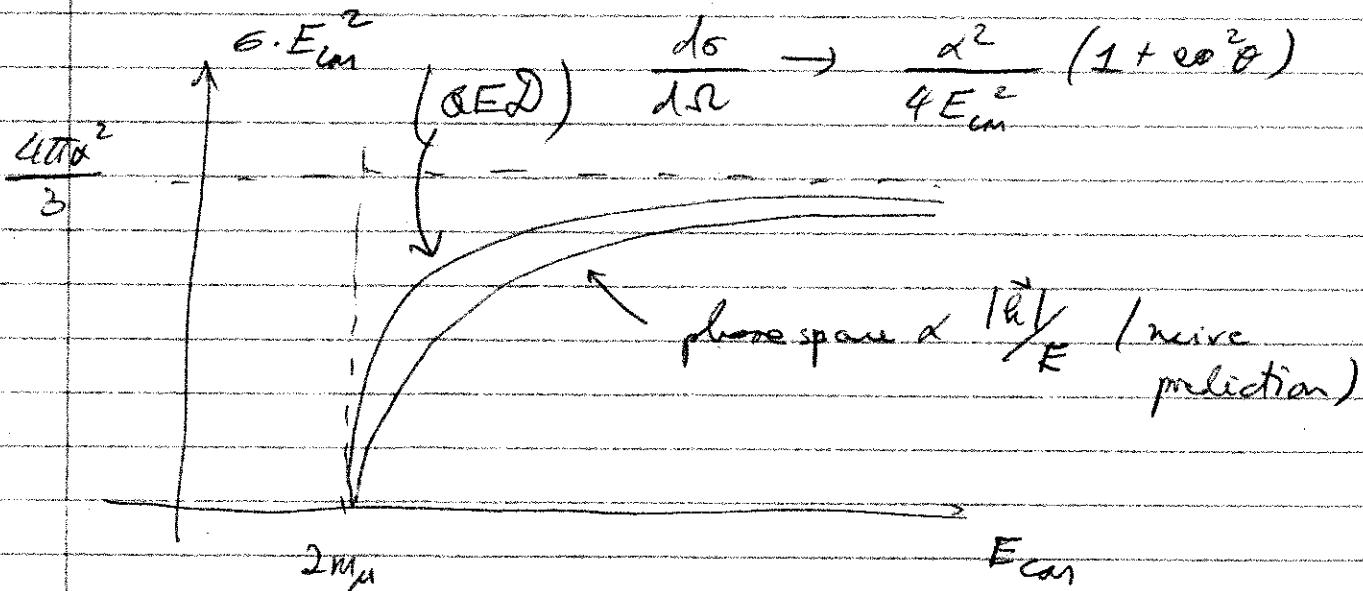


$$\sigma_{tot} = \frac{4\pi \alpha^2}{3 E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right)$$

where $\alpha = e^2/4\pi$

When $E_{cm} < 2m_\mu \rightarrow$ cross section is zero
there won't enough energy to produce
a muon + anti-muon pair.

$$\text{As } E_{cm} \rightarrow \infty \rightarrow \sigma_{tot} \rightarrow \frac{4\pi \alpha^2}{3 e^2 C_m^2}$$



Summary of method

- ① Draw Feynman diagram for the process
 - ② Use Feynman rules to calculate M
 - ③ Calculate $|M|^2$, averaged or sum over all spins using the completeness relation.
 - ④ Evaluate traces using the trace theorem, collect terms & simplify as much as possible.
 - ⑤ Pick a frame. Draw a picture of kinematics. Express all 4-momentum vectors in terms of variables such as $E = \theta$.
 - ⑥ Plug $|M|^2$ into cross section formula & integrate over all phase space variables to find σ_{tot} .
- sf

Production of Quarks - Anti-quark pairs

An important cross section is:

$$e^+ e^- \rightarrow \text{hadrons}$$

↑

many number of strongly interacting
particles.

$\text{QCD} \rightarrow$ all hadrons are composed of three fermions
called "quarks" which come in many "flavors".

↳ each quark has

mass	electric charge
additional quantum number	(color)

↳ simplest process in QCD is

$$e^+ e^- \rightarrow q\bar{q} \quad (\text{quark-anti-quark pair})$$

To modify the $e^+ e^- \rightarrow \mu^+ \mu^-$ to get here, we
need 3 things:

- Replace the massless e with $Q/\epsilon l \rightarrow$ quark
charge
- Color each quark 3 times, one for each color
- Include strong interaction within the produced
quark - anti-quark

First 2 charges are easy: masses + charge are known.

u, c, t quarks: $Q = 2/3$

d, s, b quarks: $Q = -1/3$

\rightarrow finally find $M \propto \text{size}^2 \rightarrow$ insert Q^2 .

\rightarrow counting: multiply everything by 3.

Finally, in light energy limit, strong interaction can be neglected (Nobel 2004)

$\rightarrow \sigma_{\text{tot}}$ for $e^+ e^- \rightarrow g\bar{g}$ looks like ...

$$\sigma_{\text{tot}} \rightarrow R = \frac{4\pi r^2}{8E_{\text{cm}}^2} \cdot (3Q^2) \underset{\uparrow}{=} (3Q^2)R$$

$$\frac{86.8 \text{ nbarn}}{(E_{\text{cm}}, \text{in GeV})^2}$$

\rightarrow Asymptotically,

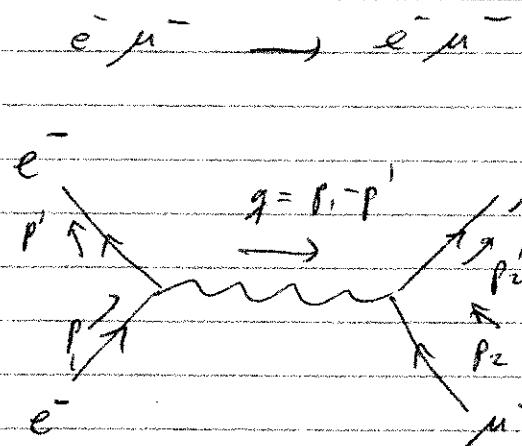
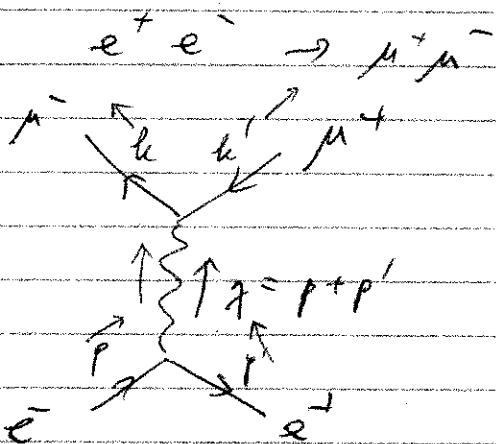
$$\sigma(e^+ e^- \rightarrow \text{hadrons}) \xrightarrow[E \rightarrow \infty]{} 3 \cdot (2, Q^2) R$$

(there's more to this ... but we won't be noisy about this too much)

(all a lot of experimental details.)

Crossing symmetry $e^+ e^- \rightarrow \mu^+ \mu^-$

Consider a related process: $e^- \bar{\mu}^- \rightarrow e^- \mu^-$



$$\sum_{\text{spins}} \frac{1}{4} |M|^2 = \frac{8e^4}{q^4} \left\{ (p \cdot h)(p' \cdot u') + (p \cdot u')(p' \cdot h) + m_\mu^2 (p \cdot p') \right\}$$

$$iM = \frac{(ie)}{q^2} (\bar{u}(q_1) \gamma^\mu u(p_1)) \times (\bar{u}(q_2) \gamma^\nu u(p_2))$$

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{q^2} [(p_1 \cdot p_2)(p_1 \cdot p_2) + (p_1 \cdot p_2)(p_1 \cdot p_2) - m_\mu^2 (p_1 \cdot p_1)]$$

$$+ (p_1 \cdot p_2)(p_1 \cdot p_2) - m_\mu^2 (p_1 \cdot p_1)]$$

cancel...

$$p \longleftrightarrow p_1$$

$$h \longleftrightarrow p'_1$$

$$p' \longleftrightarrow -p'$$

$$h' \longleftrightarrow -p_2$$

canceling \rightarrow antisym.

particle \rightarrow antiparticle

neutrinos \rightarrow -neutrinos

The kinematics here will be completely different.

$$\begin{array}{c}
 \text{e}^{-} p_1 = (h, \vec{k}) \\
 \text{e}^{-} p_1 = (h, k\hat{z}) \\
 \text{e}^{-} \mu^{-} \quad \mu^{-} \\
 \checkmark \qquad \qquad \qquad p_2 = (E, -k\hat{z}) \\
 \text{e}^{-} p_2' = (E, -\vec{k}) \\
 \left. \begin{array}{l} E^2 = h^2 + m_h^2 \\ \vec{k} \cdot \hat{z} = h \cos \theta \\ E + h = E_{\text{cm}} \end{array} \right\}
 \end{array}$$

So let's evaluate the $p_1 p_2$, $p_1' p_2'$ -- combos etc.

$$\frac{1}{4} \sum |p_i|^2 = \frac{2e^4}{a^2(1-\cos\theta)^2} \left\{ (E+h)^2 + (E+h \cos\theta)^2 - m_h^2 (1-\cos\theta) \right\}$$

$$\frac{(ds)}{(ds)_\text{cm}} = \frac{|p_i|^2}{64\pi^2 (E+h)^2} \quad (\text{which is obtained from the 1-massless particle limit of the 2-body 2-body } \frac{ds}{ds_\text{cm}})$$

$$\frac{(ds)}{(ds)_\text{cm}} = \frac{\alpha^2}{2h^2 (E+h)^2 (1-\cos\theta)^2} \left\{ (E+h)^2 + (E+h \cos\theta)^2 - m_h^2 (1-\cos\theta) \right\}$$

$$\text{where } h = \sqrt{E^2 - m_h^2}.$$

When $E \rightarrow \infty$, can set $m_h = 0$, and

$$\frac{(ds)}{(ds)_\text{cm}} \rightarrow \frac{\alpha^2}{2E_{\text{cm}}^2 (1-\cos\theta)^2} (4 + (1+\cos\theta)^2)$$

Note that $\left(\frac{ds}{ds} - \frac{1}{\alpha^4} \right)$ converges as $\theta \rightarrow 0$

This is because of the photon propagator being nearly a shell ($q^2 \approx 0$).

The same result can be seen in non-relativistic quantum scattering.

↳ Divergent cross section is due to the fact that Coulomb force has $\propto r^{-1}$.

General Cross Symmetry

→ The general section is an example of cross symmetry ...

→ For a sub-particle ...

$$\mu(\phi q) + X \rightarrow Y = \mu(X \rightarrow Y + \bar{\phi} q)$$

↳ flip sign of ϕ and you get the new amplitude.

→ For fermion spinors, there is an additional minus sign for the regularized spin sum since

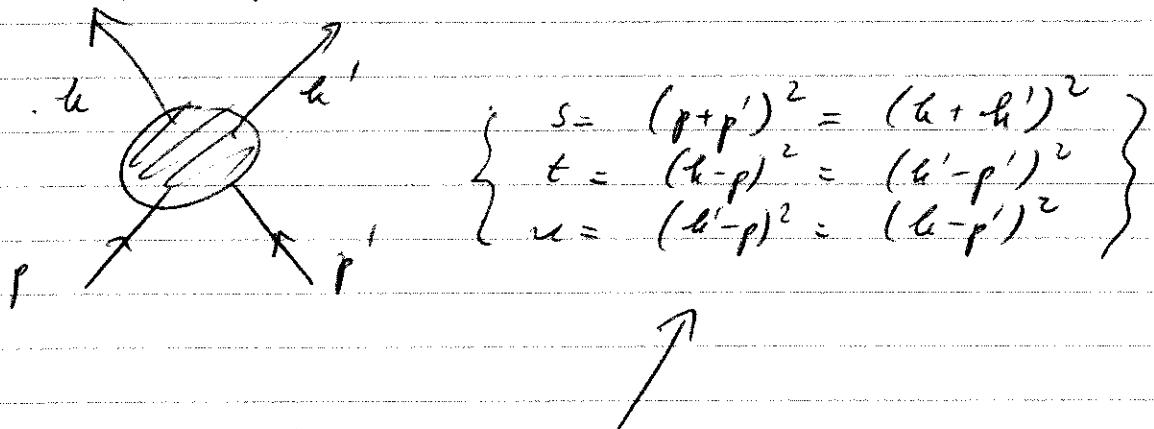
$$\sum_{\text{spins}} \bar{u}(p) u(q) = p + m \text{ while}$$

$$\sum_{\text{spins}} \bar{v}(-p) v(q) = -p - m = -(p + m)$$

→ $\times (-1)$ for each flipped fermion as well.

Mandelstam variables. (Convenient for crossing symmetry)

2 body \rightarrow 2 body scattering -



(Mandelstam variables)

With this, we can write -

$e^- \rightarrow e^+ \mu^- \mu^+$ and $\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{s^2} \left\{ \left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2 \right\}$

$e^- \rightarrow e^+ \quad e^+ e^- \rightarrow \mu^+ \mu^-$

and

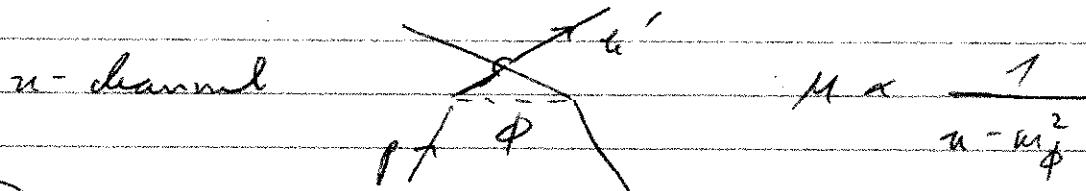
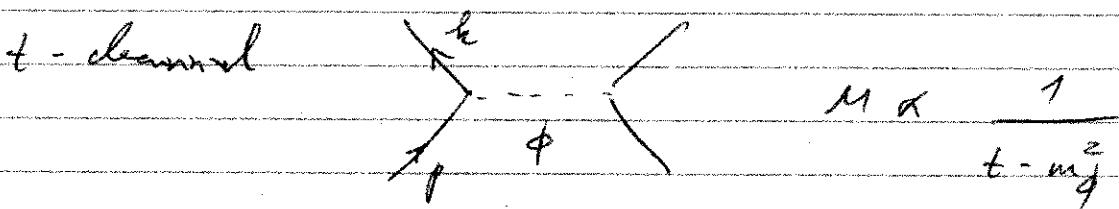
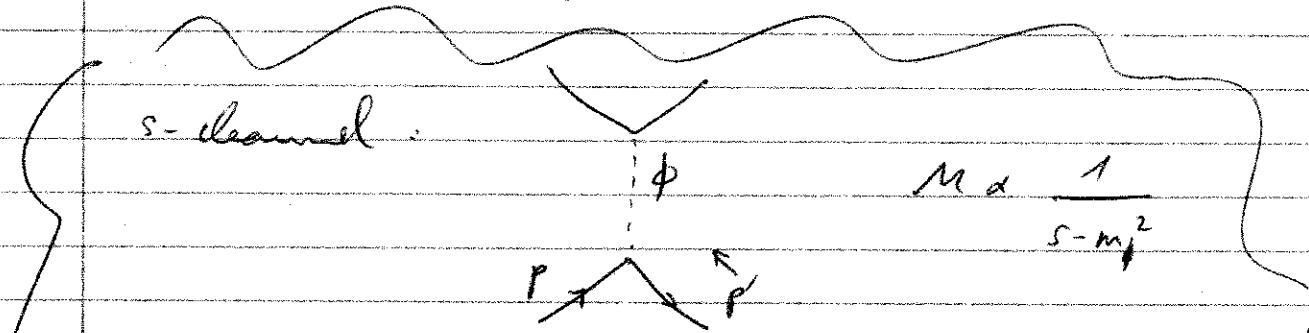
$e^- \rightarrow \mu^- \quad \mu^- \rightarrow \mu^- \quad \text{and} \quad \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{t^2} \left\{ \left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2 \right\}$

$e^- \mu^- \rightarrow e^- \mu^-$

(189)

When a 2 body - 2 body diagram contains only one virtual particle --

↳ eventually decided that job is being in certain "channels"



To get better feel -- let's look at what happens in com frame --

$$\rightarrow h = (E_{\mathfrak{F}P})$$

$$p_i(E, \vec{p}_i)$$

$$f = (E, -\rho \vec{e})$$

$$\vec{b}' = (E, -\vec{p})$$

$$\left\{ \begin{array}{l} s = (p+p')^2 = (h'+h)^2 = (2E)^2 = E_{cm}^2 \\ t = (h-p)^2 = -p^2 \sin^2 \theta - p^2 (\cos \theta - 1)^2 = -p^2 (1 - \cos \theta) \\ u = (h'-p)^2 = -p^2 \sin^2 \theta - p^2 (\cos \theta + 1)^2 = -2p^2 (1 + \cos \theta) \end{array} \right.$$

~~Other terms are zero~~

We see that $t \rightarrow 0$ as $\theta \rightarrow 0$
 $u \rightarrow 0$ as $\theta \rightarrow \pi$

Also note that ...

$$s+t+u = \sum_{i=1}^4 m_i^2$$

square of mass of incoming/outgoing particles

This is due in general - just add anything up w/ no momentum conservation.

$$(p+p'-h'-h)^2 = 0.$$

~~t~~

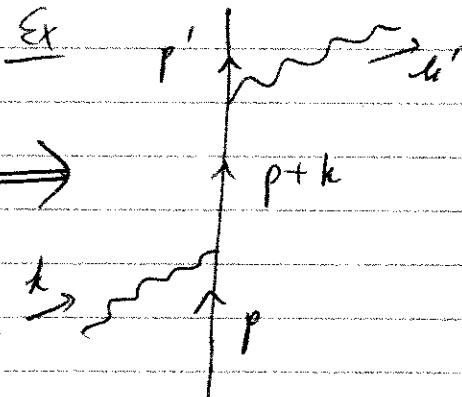
Coupton scattering $e^\gamma \rightarrow e^\gamma$

\rightarrow X-ray + R-ray or electrons...

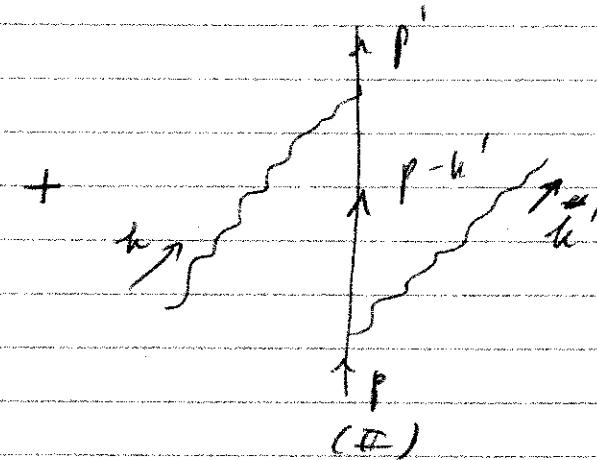
We will calculate T_{tot} to lowest order in α .

(we will be using all the techniques we learned so far)

+ how to deal with external photons.



2 diagrams
at lowest order



Let $\epsilon_x(h)$ + $\epsilon_h(h')$ be the photon polarization vectors

$$iM = (I) + (II)$$

$$= \bar{u}(p') (-ie\gamma^\mu) \epsilon_x^\pm(h') \frac{i(p+k+m)}{(p+k)^2 - m^2} (-ie\gamma^\nu) \epsilon_x(h) u(p)$$

$$+ \bar{u}(p') (-ie\gamma^\nu) \epsilon_h(h) \frac{i(p-k+m)}{(p-k')^2 - m^2} (-ie\gamma^\mu) \epsilon_x^\pm(h') u(p)$$

$$= -ie^2 \epsilon_x^\pm(h') \epsilon_h(h) \bar{u}(p') \left\{ \frac{\gamma^\mu (p+k+m) \gamma^\nu}{(p+k)^2 - m^2} + \frac{\gamma^\nu (p-k'+m) \gamma^\mu}{(p-k')^2 - m^2} \right\} u(p)$$

now note that $p^2 = m^2 = 0$, $k^2 = 0$

$$\rightarrow (p+k)^2 - m^2 = 2p \cdot k$$

$$(p-k')^2 - m^2 = -2p \cdot k'$$

and finally ...

$$\begin{aligned}
 (\rho + m) \gamma^\nu u(p) &= (2\rho^\nu - \delta^\nu_\mu + \delta_m^\nu) u(p) \\
 &= 2\rho^\nu u(p) - \underbrace{\gamma^\nu (\rho - m) u(p)}_{\text{order } \delta p} \\
 &= 2\rho^\nu u(p) \quad \text{order } \delta p \text{ of } u(p)
 \end{aligned}$$

h.

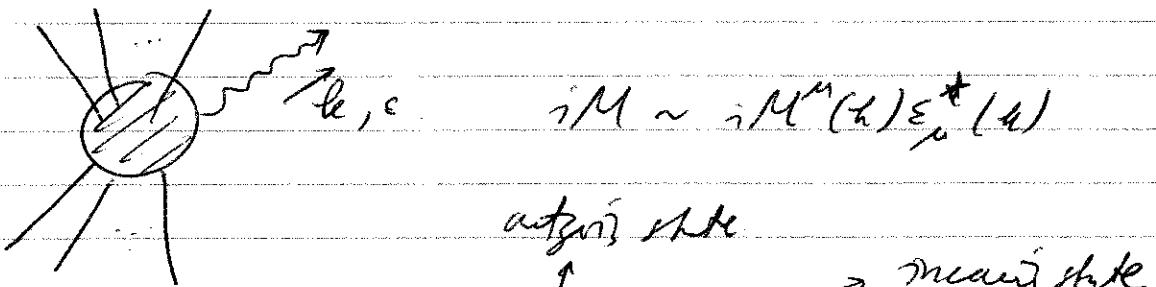
$$iM = -ie^2 \epsilon_\mu^{*\dagger}(k') \epsilon_\nu(k) \bar{u}(p') \left\{ \frac{\delta'' k' \delta'' + 2\delta'' p'}{2p \cdot k} + \frac{-\delta' k' \delta' + 2\delta' p'}{-2p \cdot k'} \right\} u(p)$$

Next, Photon Polarization + Ward Identity

Interaction for QED has the form

$$e \int d^4x j^\mu A_\mu \quad \text{where } j^\mu \text{ is the conserved electric charge current.}$$

Consider a process with outgoing photon with momentum k & polarization ϵ .



where $M''(k) \propto \int d^4x e^{ikx} \langle f | j^\mu(x) | i \rangle$.

(4)

Now, since $\partial_\mu j^\mu(x) = 0$...

$$\begin{aligned}
 k_\mu M^\mu(k) &\propto \int d^4x e^{ik\cdot x} \langle h(j^\mu(x))/i \rangle \\
 &= -i \int d^4x \partial_\mu (e^{ik\cdot x}) \langle f|j^\mu(x)/i \rangle \\
 &= i \int d^4x e^{ik\cdot x} \langle f| \partial_\mu j^\mu(x)/i \rangle \\
 &= 0.
 \end{aligned}$$

→ This is an example of the Ward identity in QED.

→ When you replace the polarization vector by any external photon (moving or at rest) by the momentum $k_\mu \rightarrow$ the amplitude vanishes.

$$[k_\mu M^\mu(k) = 0]$$



More on this later...

→ This is essentially a statement of current conservation.

↳ which is a consequence of gauge symmetry of QED.

mm

What can we say about the polarization sum

$$\sum_i \epsilon_i^{\mu}(\boldsymbol{k}) \epsilon_i^{\nu}(\boldsymbol{k}) ?$$

polarizations

↳ Consider an example - Let $\boldsymbol{k} = (k, 0, 0, k)$

→ 2 physical polarizations are usually taken as

$$\epsilon_1^{\mu} = (0, 1, 0, 0)$$

$$\epsilon_2^{\mu} = (0, 0, 1, 0) \quad \checkmark$$

$$\rightarrow \sum_i \epsilon_i^{\mu} \epsilon_i^{\nu} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} / \omega$$

But this is ugly - we can find a more familiar form -

Note that polarization enters calculation in the form

$$\sum_i |\epsilon_i^{\mu}(\boldsymbol{k}) M^{\mu}(k)|^2$$

which is

$$\sum_i \epsilon_i^{\mu}(\boldsymbol{k}) \checkmark \epsilon_i^{\nu}(\boldsymbol{k}) M_{\mu}(\boldsymbol{k}) M_{\nu}^*(\boldsymbol{k})$$

$$= |M_1(k)|^2 + |M_2(k)|^2$$

$$= |M_1(k)|^2 + |M_2(k)|^2 + |M_3(k)|^2 - |M_0(k)|^2$$

since $\boldsymbol{k} = (k, 0, 0, k)$

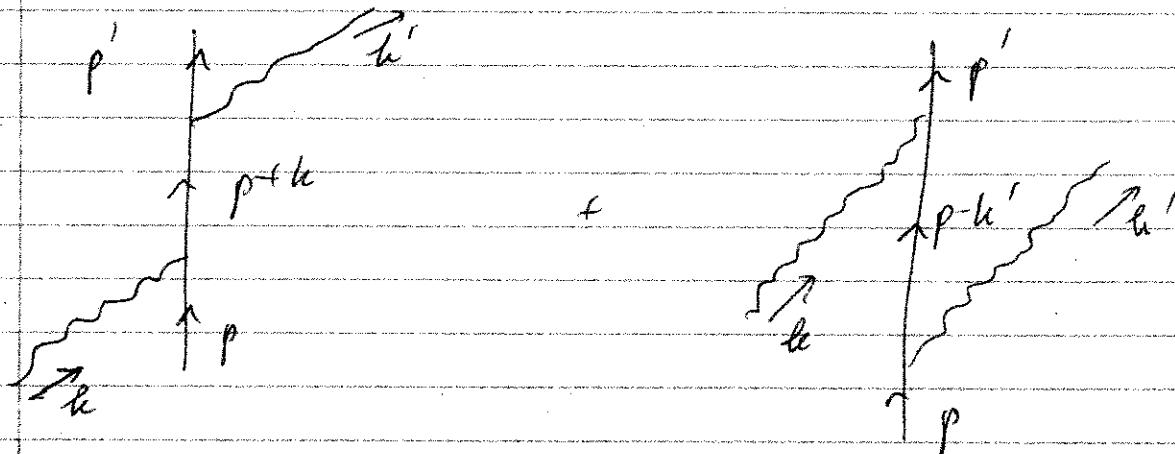
$$= -g_{\mu\nu} M^{\mu}(k) M^{\nu}(k) = k^{\mu} M_{\mu} = 0 \quad (\text{Ward identity})$$

So we can take $\left\{ \sum_i \epsilon_i^{\mu\nu}(k) \epsilon_i^\nu(k) = -g^{\mu\nu} \right.$

even though this is not really an equality...

Now back to Compton scattering...

\hookrightarrow Hop Klein-Nishina Formula



Recall that we found

$$iM = -ie^2 \bar{\epsilon}_\mu^{\mu\nu}(k) \epsilon_\nu(k) \bar{n}(p) \left\{ \frac{\gamma^\mu k^\nu + 2\gamma^\nu p^\mu}{2p \cdot k} + \frac{\gamma^\nu k^\mu - 2\gamma^\mu p^\nu}{2p \cdot k'} \right\}$$

\hookrightarrow now, we square, sum & divide by # of initial spins (2 electrons \times 2 photons = 4)

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{e^4}{4} \underbrace{\gamma_{\mu\nu} \gamma_{\rho\sigma} \times}_{\text{factors}} \{ (p+m) \left\{ \frac{\gamma^\mu k^\nu + 2\gamma^\nu p^\mu}{2p \cdot k} + \frac{\gamma^\nu k^\mu - 2\gamma^\mu p^\nu}{2p \cdot k'} \right\} \}$$

$$\times (p+m) \times \left\{ \frac{\gamma^\mu k^\nu + 2\gamma^\nu p^\mu}{2p \cdot k} + \frac{\gamma^\nu k^\mu - 2\gamma^\mu p^\nu}{2p \cdot k'} \right\}$$

(from
polarization sums)

There's a lot of work & we don't do all details here ...

$$\text{Use: } \partial (\gamma^\mu k^\nu \gamma_\mu \partial_\nu k_\lambda) = -2k(\gamma^\mu k^\nu \partial_\mu \gamma_\nu)$$

$$\text{since } \gamma^\mu \partial_\mu = -2\partial$$

$$\text{Next: } \gamma^\mu k^\nu \partial_\mu \gamma_\nu = -2k \partial k$$

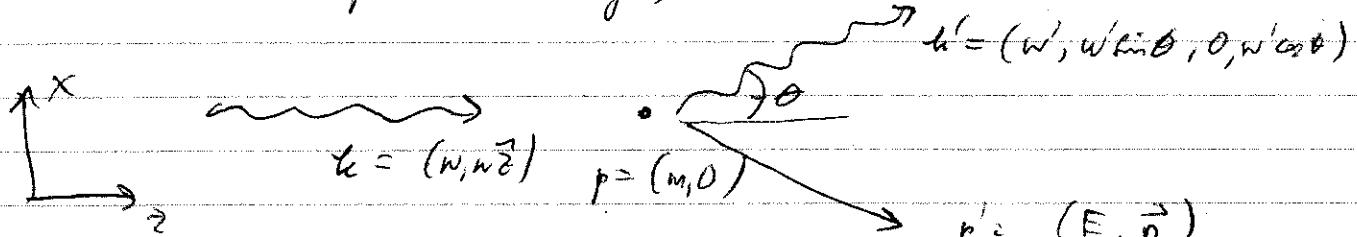
so ...

$$\begin{aligned} &= +4\partial [\gamma^\mu k^\nu \partial_\mu \gamma_\nu] = 4 \cdot 4 ((p \cdot k)(p \cdot k) - (q \cdot p)(k \cdot k) \\ &\quad + (p \cdot k)(p \cdot k)) \\ &= 32(p \cdot k)(p \cdot k) \end{aligned}$$

In the end, we get

$$\frac{1}{4} \sum_{\text{spins}} (M)^2 = 2e^4 \left\{ \frac{p \cdot k'}{p \cdot k} + \frac{p \cdot k}{p' \cdot k} + \frac{2m^2}{(p \cdot k)(p \cdot k')} \right. \\ \left. + m^4 \left(\frac{1}{p \cdot k} - \frac{1}{p' \cdot k'} \right)^2 \right\}$$

Now, from Compton scattering, the frame is --



$$\begin{aligned} p'^2 &= m^2 \\ \Rightarrow m^2 &= (p')^2 = (p+k-k')^2 = p^2 + k^2 + k'^2 + 2p \cdot (k-k') \\ &= m^2 + 2p \cdot (k-k') - 2k \cdot k' - 2k \cdot k' \end{aligned}$$

$$= m^2 + 2m(w-w') - 2mw'(1-\cos\theta)$$

$$\text{So } 2m(w-w') \approx 2mw'(1-\cos\theta) = 0$$

$$\Rightarrow \left\{ \begin{array}{l} w' = \frac{1}{1 + \frac{w(1-\cos\theta)}{m}} \\ \downarrow \\ \end{array} \right. \quad \begin{array}{c} w \\ \hline 1 + \frac{w(1-\cos\theta)}{m} \end{array}$$

Cayley's formula for the shift in the photon wavelength -

→ Next, have to work out the phase space integrals

$$\int dT_2 = \int \frac{d^3 h'}{(2\pi)^3 2w'} \frac{1}{2E'} (2\pi)^4 \delta^{(4)}(k'+p'-k-p)$$

$$= \int \frac{d^3 h'}{(2\pi)^3 2w' 2E'} (2\pi) \delta(w' + E' - w - m) \quad (\vec{p} = \vec{k} + \vec{p}' - \vec{k}')$$

$$= \int \frac{d^3 h'}{(2\pi)^3 2w' 2E'} (2\pi) \delta\left(w' + \sqrt{w^2 + m^2 - 2wm\cos\theta + m^2} - w - m\right)$$

$$\text{now note } P(f(x)) = \frac{\delta(x-x_0)}{|f'(x_0)|}$$

$$= \int \frac{w'^2 dw' d\Omega}{(2\pi)^3 4w'E'} (2\pi) \frac{\delta(w' - \sqrt{1 + \frac{w^2}{m^2}(1-\cos\theta)})}{\left| 1 + \frac{w' - w\cos\theta}{E'} \right|} \rightarrow \sqrt{-}$$

$$= \frac{1}{4\pi} \int_{-1}^1 \frac{dw' \sin\theta}{\left| 1 + \frac{w' - w\cos\theta}{E'} \right| E'} = \frac{1}{8\pi} \int_{-1}^1 \frac{dw' \sin\theta}{(E' + w' - w\cos\theta)}$$

But $E' + w' = \text{total energy} = w + m$ (initial energy)

So, we have:

$$\frac{1}{8\pi} \int_{-1}^1 \frac{dw' \theta w'}{|w+m-w\cos\theta|} = \frac{w'}{8\pi} \int_0^\pi \frac{d\cos\theta}{m+w(1-\cos\theta)}$$

Now, $v_{\text{relativ}} = 0$, so $|v_A - v_B| = 1$

$$\text{So } \frac{d\sigma}{d\cos\theta} = \frac{1}{2w} \frac{1}{2m} \frac{w'}{8\pi(m+w(1-\cos\theta))} \frac{1}{4 \sin^2 \theta}$$

Since $\frac{1}{m+w(1-\cos\theta)} = \frac{w'}{mw}$, we have

$$\frac{d\sigma}{d\cos\theta} \geq \frac{1}{32\pi} \frac{w'^2}{m^2 w^2} \left\{ 2e^4 \left(\frac{p \cdot h'}{p \cdot h} + \frac{p \cdot h}{p \cdot h'} + 2m \left(\frac{1}{p \cdot h} - \frac{1}{p \cdot h'} \right) + m^2 \left(\frac{1}{p \cdot h} - \frac{1}{p \cdot h'} \right)^2 \right) \right\}$$

Using $p \cdot h = mw$ & $p \cdot h' = m \cdot w'$

$$\begin{aligned} \frac{d\sigma}{d\cos\theta} &= \frac{1}{32\pi} \frac{w'^2}{m^2 w^2} \left\{ 2e^4 \left(\frac{w'}{w} + \frac{w}{w'} + 2m \left(\frac{1}{w} - \frac{1}{w'} \right) + m^2 \left(\frac{1}{w} - \frac{1}{w'} \right)^2 \right) \right. \\ &= \left(\frac{\pi \alpha^2}{m^2} \left(\frac{w'}{w} \right)^2 \left\{ \frac{w'}{w} + \frac{w}{w'} - 6m^2 \theta^2 \right\} \right) \end{aligned}$$

Klein-Nishina formula (1929)

Thomson cross

When $w \rightarrow 0$, $w'/w \rightarrow 1$ → section for

$$\frac{d\sigma}{d\cos\theta} \rightarrow \frac{\pi \alpha^2}{m^2} (1 + w^2 \theta^2), \quad \sigma_{\text{Th}} = \frac{8\pi \alpha^2}{3m^2}$$

scattering of
classical EM
by free e^-

Radiative Corrections

Soft Bremsstrahlung \rightarrow low freq radiation when
 e^- and/or goes sudden acceleration

