8.512 Recitation 10

- Pset 5 due Monday 04/18

Today: 1) Kubo formula for Hall conductivity
2) Friedel oscillations in 1D

Ref: David Tong's lecture notes on Quantum Hall Effect

1) Consider a 2D system subject to the perturbation

$$H_1(t) = -\vec{J}\cdot\vec{A}(t) = -\sum J_i A_i(t)$$

- Assume an AC background electric field $\vec{E}(t) = \vec{E}\, e^{-i\omega t}$ and choose the Weyl gauge $\varphi = 0$

$$\Rightarrow \quad \vec{E}(t) = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t}$$

$\Rightarrow$ in frequency space $\quad \vec{A} = \dfrac{\vec{E}}{i\omega}\, e^{-i\omega t}$

- Recall the spectral representation of Kubo's formula at $T=0$ $\quad (i=x,\ j=y)$

$$\chi_{xy}(\omega) = \frac{1}{\hbar}\sum_{n\neq 0}\left[\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{\omega-\omega_0+\omega_n} - \frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{\omega+\omega_0-\omega_n}\right]$$

- By def. of linear response

$$\langle J_x(\omega)\rangle = \chi_{xy}(\omega)\, A_y(\omega) = \frac{\chi_{xy}(\omega)}{i\omega}\, E_y(\omega) = \sigma_{xy}(\omega)\, E_y(\omega)$$

where we used Ohm's law to identify $\sigma_{xy}(\omega) = \dfrac{\chi_{xy}(\omega)}{i\omega}$

$$\Rightarrow \sigma_{xy}(\omega) = -\frac{i}{\hbar\omega}\sum_{n\neq 0}\left[\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \langle 0|J_x|n\rangle\langle n|J_y|0\rangle\right]$$

$$\Rightarrow \sigma_{xy}(\omega) = -\frac{i}{\hbar\omega}\sum_{n\neq 0}\left[\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle}{\omega-\omega_0+\omega_n} - \frac{\langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{\omega+\omega_0-\omega_n}\right]$$

- Next take the limit $\omega\to 0$. Strictly speaking, $\omega\to 0$ should be taken before $T\to 0$, but in this case the two limits commute.

$$\frac{1}{\omega-\omega_0+\omega_n} \approx \frac{1}{\omega_n-\omega_0} - \frac{\omega}{(\omega_n-\omega_0)^2} + O(\omega^2)$$

$$\frac{1}{\omega+\omega_0-\omega_n} \approx \frac{1}{\omega_0-\omega_n} - \frac{\omega}{(\omega_n-\omega_0)^2} + O(\omega^2)$$

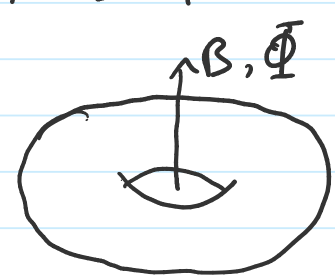
$$\sigma_{xy}(\omega\to 0) = \frac{i}{\hbar}\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{(\omega_n-\omega_0)^2} -$$

$$- \underbrace{\frac{i}{\hbar\omega}\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle + \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{\omega_n-\omega_0}}_{=0}$$

Because of rotational invariance $x\to y$, $y\to -x$

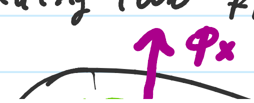
$$\Rightarrow \boxed{\sigma_{xy} = i\hbar\sum_{n\neq 0}\frac{\langle 0|J_y|n\rangle\langle n|J_x|0\rangle - \langle 0|J_x|n\rangle\langle n|J_y|0\rangle}{(E_n-E_0)^2}}$$

- Now consider a Hall system on a $L_x\times L_y$ torus T^2. Thread a uniform field $\vec{B}$ through the torus. Similarly to a SC, the flux is quantized according to Dirac quantization

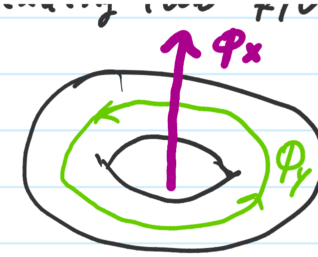


$$\Phi = B L_x L_y = n\frac{2\pi\hbar}{e} \equiv n\Phi_0 \quad , n\in\mathbb{Z}$$

- Additionally perturb the system by threading two fluxes in x and y directions



in x and y directions



$$\Rightarrow \quad A_x = \frac{\Phi_x}{L_x} \quad \text{and} \quad A_y = \frac{\Phi_y}{L_y} + Bx$$

$$H_1 = -J_x A_x - J_y A_y$$

- 1^{st} order perturbation theory in H_1

$$|0\rangle' = |0\rangle + \sum_{n\neq 0} \frac{\langle n|H_1|0\rangle}{E_n - E_0} |n\rangle$$

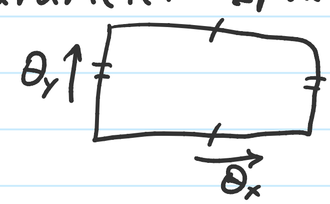
Infinitesimal change to $\Phi_{x,y}$ yields $\quad (|\psi_0\rangle \equiv |0\rangle)$:

$$\left|\frac{\partial \psi_0}{\partial \Phi_{x,y}}\right\rangle = -\frac{1}{L_{x,y}} \sum_{n\neq 0} \frac{\langle n|J_{x,y}|0\rangle}{E_n - E_0} |n\rangle$$

$$\Rightarrow \quad \sigma_{xy} = i\hbar \left[\left\langle \frac{\partial \psi_0}{\partial \Phi_y} \middle| \frac{\partial \psi_0}{\partial \Phi_x} \right\rangle - \left\langle \frac{\partial \psi_0}{\partial \Phi_x} \middle| \frac{\partial \psi_0}{\partial \Phi_y} \right\rangle \right]$$

- <u>Note</u>: The extra area factor $L_x L_y$ was used to convert between current and current density.

- Flux quantization implies that Φ is only defined mod Φ_0
$\Rightarrow \Phi_x, \Phi_y$ are periodic and their parameter space is also a torus T^2_Φ



Parametrize this torus by angles

$$\theta_{x,y} = \frac{2\pi \Phi_{x,y}}{\Phi_0} = \frac{e}{\hbar} \Phi_{x,y} \in [0, 2\pi)$$

- The Berry connection and curvature are defined as

$$a_{x,y} = -i \langle \psi_0 | \frac{\partial}{\partial \theta_{x,y}} | \psi_0 \rangle$$

$$\mathcal{F}_{xy} = \frac{\partial a_x}{\partial \theta_y} - \frac{\partial a_y}{\partial \theta_x} = -i\left[\frac{\partial}{\partial \theta_y}\langle \psi_0 | \frac{\partial \psi_0}{\partial \theta_x}\rangle - \frac{\partial}{\partial \theta_x}\langle \psi_0 | \frac{\partial \psi_0}{\partial \theta_y}\rangle\right]$$

- Putting everything together: $\sigma_{xy} = -\frac{e^2}{\hbar}\mathcal{F}_{xy}$
- Average over all flux configurations

$$\sigma_{xy} = -\frac{e^2}{\hbar}\int_{T^2_\Phi}\frac{d^2\theta}{(2\pi)^2}\mathcal{F}_{xy}$$

- One can show that this integral gives an integer

$$C = \left[\frac{1}{2\pi}\int_{T^2_\Phi} d^2\theta\, \mathcal{F}_{xy}\right] \in \mathbb{Z}$$ Chern number

$\Rightarrow$ Hall conductivity is quantized!

$$\boxed{\sigma_{xy} = -\frac{e^2}{h}C}$$ Integer Hall Effect

2) Consider the setup of problem 3 on pset 5:

$$\varphi(x) = e^{ikx} - e^{-ikx}$$

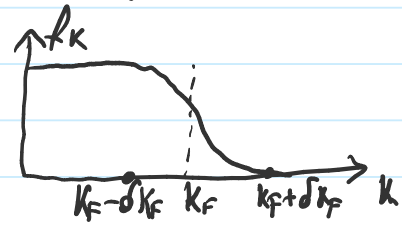
$$\langle n(x)\rangle = \int_0^\infty \frac{dk}{2\pi}|\varphi(x)|^2 f_k = \int_0^\infty \frac{dk}{2\pi}\cdot 4\sin^2(kx) f_k$$

- At $T=0$ we have $f_k = \Theta(k_F - k)$

$$\langle n(x)\rangle = \frac{k_F}{\pi}\left[1 - \frac{\sin(2k_F x)}{2k_F x}\right]$$

- At $0<T\ll\mu$ the Fermi surface is smeared in a window of width

$$\delta k_F = \frac{1}{\beta\hbar v_F} = \frac{m}{\beta\hbar^2 k_F} \ll k_F$$



- Split the integral

$$\langle n(x)\rangle = \int_0^\infty \frac{dk}{\pi}\left(1-\cos(2kx)\right)\Theta(k_F-\delta k_F-k) + \int_{-\delta k_F}^{+\delta k_F}\frac{dk}{\pi}\,\frac{1-\cos(2x(k_F+k))}{e^{\beta\frac{\hbar^2}{m}\cdot k_F k}+1}$$

where in the 2^{nd} integral we shifted k by k_F and used

$$(k+k_F)^2-k_F^2 \approx 2k_F k \qquad \text{for } |k|\ll k_F$$

$$\langle n(x)\rangle = \frac{k_F-\cancel{\delta k_F}}{\cancel{\pi}} - \frac{\sin(2x(k_F-\delta k_F))}{2\pi x} + \cancel{\frac{\delta k_F}{\pi}} - \frac{\delta k_F}{\pi}\int_{-1}^{1} d\tilde{k}\,\frac{\cos(2x(k_F+\delta k_F\tilde{k}))}{e^{\tilde{k}}+1}$$

where we changed integration variables to $\tilde{k} = \frac{k}{\delta k_F}$

- The last integral has a closed form solution in terms of β-functions (see Mathematica)

- After expanding these functions order-by-order for $\delta k_F \ll k_F$ we find

$$\langle n(x)\rangle = \frac{k_F}{\pi}\left[1 - \frac{\delta k_F}{k_F}\cdot\frac{\sin(2kx)}{\sinh(2\delta k_F x)}\right]$$

As $T\to 0$, $\delta k_F \to 0$ and we recover the prev. result!