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Course: **8.321 - Quantum Theory I**
Problem set: **#3**

1.

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

(a) To show that AB commute, we simply compute their commutator:

$$[A,B] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, A and B commute.

(b) Notice that $\text{rank}(A) = 1$. So A must have eigenvalue of zero with multiplicity of two. The other eigenvalue is 2 by inspection, where the corresponding eigenvector is $(1,0,1)^\top$. The other two 0-eigenvectors must span the subspace orthogonal to $(1,0,1)^\top$. We may choose $(0,1,0)^\top$ and $(-1,0,1)^\top$.

To find the eigenvalues of B we may use the traditional method of characteristic polynomials.

$$0 = \det(B - \lambda\mathbb{I}) = -6 - \lambda + 4\lambda^2 - \lambda^3 \implies 0 = (\lambda-3)(\lambda-2)(\lambda+1).$$

The corresponding eigenvectors are

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\vec{x}_1 = 3\vec{x}_1 \implies \vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\vec{x}_2 = 2\vec{x}_2 \implies \vec{x}_2 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}\vec{x}_3 = -1\vec{x}_3 \implies \vec{x}_3 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

(c) It is clear that $(1,0,1)^\top$ is a simultaneous eigenvector of A and B. Also notice that the eigenvectors $\vec{x}_2$ and $\vec{x}_3$ of B are orthogonal to each other and to $(1,0,1)^\top$. This means $\vec{x}_2$ and $\vec{x}_3$ span the subspace associated with the eigenvalue zero for A. Thus, $\vec{x}_2, \vec{x}_3$ are eigenvectors of A and it suffices to normalize $\vec{x}_1, \vec{x}_2, \vec{x}_3$ to form a unitary matrix:

$$\boxed{U = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix}}$$

Simultaneous diagonalization of A and B:

$$U^\dagger A U = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$U^\dagger B U = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

as desired.

2. N spin-1/2 particles in

$$\mathcal{H} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)} \otimes \cdots \otimes \mathcal{H}_2^{(n)}.$$

where each $\mathcal{H}_2^{(i)}$ is two-dimensional.

(a) The dimension of $\mathcal{H}$ is 2^n.

(b) $S_z = S_z^{(1)} + S_z^{(2)} + \cdots + S_z^{(n)}$. There are $\binom{n}{i}$ product (eigen)states with i particles in $|\uparrow\rangle$ and $(n-i)$ particles in $|\downarrow\rangle$. For the product state with i particles in $|\uparrow\rangle$, the corresponding eigenvalue is

$$\lambda = \frac{\hbar}{2} i - \frac{\hbar}{2}(n-i) = \frac{\hbar}{2}(2i - n), \qquad i = 0, 1, 2, \ldots, n$$

So, the spectrum of S_z is

$$\sigma(S_z) = \left\{ \frac{n\hbar}{2}, \frac{(n-2)\hbar}{2}, \ldots, \frac{-(n-2)\hbar}{2}, \frac{-n\hbar}{2} \right\}$$

There are $n+1$ distinct eigenvalues. The multiplicity of each λ_i is $\binom{n}{i}$ where λ_i is the eigenvalue associated with the product state with i spins in $|\uparrow\rangle$.

As a sanity check, the sum of the multiplicities must be 2^n. This is the case here due to a well-known combinatorial relation:

$$\sum_{i=0}^{n} \binom{n}{i} = (1+1)^n = 2^n.$$

(c) $I = \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} + \mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)} + \cdots + \mathbf{S}^{(N-1)} \cdot \mathbf{S}^{(N)} + \mathbf{S}^{(N)} \cdot \mathbf{S}^{(1)}$. We claim that $[I, S_z] = 0$ and shall prove this by induction.

(d)

3.