SC482: FINAL

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$\boxed{\text{Question 1}}$ $X_i \sim Exp(\theta)$

$$f(x|\theta) = \frac{1}{\theta} e^{-x/\theta} \quad ; \; x > 0.$$

(A) Find mle...

$$\mathcal{L}(\theta) = \frac{1}{\theta^n} e^{-\Sigma x_i/\theta}$$

$$\ell(\theta) = \ln \mathcal{L}(\theta) = -n \ln \theta - \frac{1}{\theta} \Sigma x_i$$

$$\partial_\theta \ell(\theta) = 0 \Rightarrow \frac{-n}{\cancel{\theta}} + \frac{1}{\theta^{\cancel{2}}} \Sigma x_i = 0$$

$$\Rightarrow \hat{\theta} = \Sigma x_i / n \Rightarrow \boxed{\hat{\theta} = \bar{X}}$$

(B) Is $\bar{X}$ efficient?

- $E[\hat{\theta}] = E[\bar{X}] = \frac{1}{n} E[\Sigma x_i] = \frac{1}{n} \Sigma E[x_i]$

$$= \frac{1}{n} \cdot n \cdot \theta = \theta \Rightarrow \hat{\theta} \text{ unbiased.}$$

- $Var(\hat{\theta}) = \frac{1}{n^2} Var(\Sigma x_i) = \frac{1}{n^2} \Sigma Var(x_i) = \frac{n\theta^2}{n^2} = \frac{\theta^2}{n}.$

- $I(\theta) = n I(\hat{\theta}) = Var(\partial_\theta \ell(\theta)) = Var\left[\frac{-n}{\theta} + \frac{1}{\theta^2} \Sigma x_i\right]$

$$= Var\left[\frac{1}{\theta^2} \Sigma x_i\right] = \frac{1}{\theta^4} \Sigma Var(x_i) = \frac{1}{\theta^4} \cdot n\theta^2 = \frac{n}{\theta^2}$$

$$\Rightarrow \underline{Var(\hat{\theta}) = \frac{1}{I(\theta)}}, \; \underline{\hat{\theta} \text{ unbiased}} \Rightarrow \boxed{\hat{\theta} \text{ is efficient}}$$

(CRLB)

(c) Find MVUE for θ

- Hence $\frac{\sum X_i}{n}$ as ~~suffi~~ unbiased estimator for θ.

- Now, by factorization theorem --

$$\mathcal{L}(x|\theta) = \frac{1}{\theta^n} \exp\left[-\frac{1}{\theta}\sum x_i\right]$$

$\sum x_i$ is a sufficient statistic for θ.

- Because $\bar{X}$ is a function of the sufficient statistic $\sum x_i$ and $\bar{X}$ is an unbiased estimator for θ.

$\Rightarrow$ Rao – Blackwell says $\bar{X}$ is the MVUE for θ.

Alternatively, since $\sum x_i$ is both sufficient & complete (pdf is a member of the scalar exp class) and $\frac{\sum X_i}{n} = \bar{X}$ is an unbiased estimator for θ.

$\Rightarrow$ $\bar{X}$ is the MVUE for θ.

Question 2 $X_i \sim \text{Ray}(\theta)$

$$f(x) = \frac{2x}{\theta} e^{-x^2/\theta} \quad ; \quad x > 0$$

(A) Find sufficient statistic for θ

$$\mathcal{L}(\theta) = \left(\frac{2}{\theta}\right)^n \left(\prod^n x_i\right) \exp\left\{\frac{-1}{\theta} \sum x_i^2\right\}$$

$$\begin{cases} K_1(Y_1, \theta) = \left(\frac{2}{\theta}\right)^n \exp\left\{\frac{-1}{\theta} \sum x_i^2\right\} \\ K_2(X_i \dots) = \prod^n x_i \end{cases}$$

by factorization theorem $\Rightarrow$ $\boxed{Y_1 = \sum^n X_i^2}$ is one sufficient statistic.

(B) MVUE for θ

$$E[Y_1] = E\left[\sum^n X_i^2\right] = \sum E[X_i^2] = n\, E[X_i^2]$$

$$E[X_i^2] = \int_0^\infty \frac{2x^3}{\theta} e^{-x^2/\theta} dx \overset{x^2 = u}{=\!=} \int_0^\infty \frac{1}{\theta} u e^{-u/\theta} du = \theta$$

$$= \theta \qquad (= E_{Exp}[u] \; ; \; u \sim Exp(\theta))$$

$\Rightarrow E[Y_1] = n\theta \Rightarrow \boxed{\hat{\theta} = \frac{1}{n} \sum X_i^2} \Rightarrow E[\hat{\theta}] = \theta \Rightarrow \boxed{\frac{1}{n}\sum X_i^2}$ is the MVUE

~~unbias~~ unbiased est. for θ

$\hat{\theta}$ unbiased, and a function of sufficient Y_1

(c) <u>To show that the MVUE is unique...</u>

We know that

$Y_1 = \sum x_i^2$ is a sufficient statistic for θ.

and $\hat{\theta} = \frac{1}{n} Y_1 = \frac{1}{n}\sum X_i^2$ is a function of Y_1 and

and unbiased estimator for θ.

$\hat{\theta}$ is the unique MVUE for θ if $\{f_{Y_1}(y_1;\theta) : \theta \in \Omega\}$

is complete.

(Lehmann - Scheffé)

$\boxed{\text{i.e.}}$ we want to show that ~~ff~~ with $\{f_{Y_1}(y_1;\theta)\}$

if $E(u(Y_1)) = 0 \quad \forall \theta \in \Omega$

then $u(y_1) \equiv 0$ except on a set of points that has probability zero for each $f_{Y_1}(y_1;\theta)$ in the family.

$\square$

$\boxed{\text{Question 3}}$ Y is a single observation...

$$f(y|\theta) = \theta y^{\theta-1} \quad ; \quad 0 \leq y \leq 1.$$

4 Find most powerful test of $H_0: \theta = 1$
$H_a: \theta = 2$

Find form of rejection region $(\alpha = 0.05)$

Find specific values of test statistic for which H_0 is rejected

Well...

$$\frac{\overset{1}{\mathcal{L}(\theta_0)}}{\underset{2}{\mathcal{L}(\theta_A)}} = \frac{1}{2y} < K \Rightarrow y > K'$$

rejection region has the form: $\boxed{C = \{ y \in [0,1] ; y \geq c \}}$

for some constant c.

~~With~~ Integrating under the null... $\theta_0 = 1 \Rightarrow f(y|\theta) = 1$

$$\int_{K'}^{1} 1 \, dy = 0.05 \Rightarrow K' = 1 - 0.05 = \boxed{0.95}$$

$\Rightarrow$ We reject if $\boxed{y > 0.95 \Rightarrow C_{\alpha=0.05} = \{ y \in [0,1] ; y > 0.95 \}}$

This is the most powerful test for $H_0: \theta = 1$ vs. $H_a: \theta = 2$

(B) Is the test UMP for $H_a: \theta > 1$?

~~Obviously, the form of the rejection region DEPENDS on the specific value of the alternative parameter θ_A~~

~~Where~~

As long as $\theta_A > 1$, the form of the rejection region will not depend on the specific value of θ_A, because

$$\frac{L(\theta_0)}{L(\theta_A)} = \frac{1}{\theta_A y^{\theta_A - 1}} \sim \frac{1}{y^T} < K \text{ where } T > 0$$

$\Rightarrow$ still reject if $y > K'$.

$\Rightarrow$ $\boxed{\text{YES, the test is still UMP}}$

(c) Find power for $H_0: \theta_0 = 1$, $H_A: \theta_A = 2$

$$\text{Power} = P(y > 0.95 \mid \theta = 2) = \int_{0.95}^{1} 2y \, dy = y^2 \Big|_{0.95}^{1} =$$

$$= 1 - 0.95^2 =$$

$$= 0.0975$$

D <u>Likelihood ratio test</u> $H_0: \theta_0 = 1$; $H_A: \theta_A > 1$.

$$f(y|\theta) = \theta y^{\theta-1}, \quad 0 \le y \le 1$$

Null space : $\{\theta : \theta = 1\}$

Alternative space: $\{\theta : \theta > 1\}$

Find the mle for θ:

$$\mathcal{L}(\theta) = f(y|\theta) = \theta y^{\theta - 1}$$

$$\to \ln(\mathcal{L}(\theta)) = \ell(\theta) = \ln\theta + (\theta - 1)\ln y$$

$$\to \partial_\theta \ell(\theta) = \frac{1}{\theta} + \ln y = 0 \Leftrightarrow \boxed{\hat{\theta} = \frac{-1}{\ln(y)}}$$

$$\to \boxed{\Lambda = \frac{\mathcal{L}(\hat{\theta}_0)}{\mathcal{L}(\hat{\theta}_A)} = \frac{\theta_0 y^{\theta_0 - 1}}{\hat{\theta}_A y^{\hat{\theta}_A - 1}} = \frac{1}{\frac{-1}{\ln(y)} \cdot y^{-1/\ln y - 1}}}$$

Asymptotically ... $-2\ln\Lambda \sim \chi^2(1)$

$\to$ reject if $\boxed{-2\ln\Lambda > \chi^2_{1,\,0.05} = 3,84.}$

~~table~~ $\Leftrightarrow$ if $\Lambda < 0.14667$. Let $z = \ln y$

$$\to \Lambda = \frac{-z}{(e^z)^{-1/z - 1}} = \frac{-z}{e^{-1-z}} < 0.14667 \Leftrightarrow$$ this is a transcendent-al equation

... I can't find the values for y with which we reject H_0 ~~[illegible]~~ b/c this is transcendental

(7 bis)

... we reject if $-2\ln\Lambda > \chi^2_{1,\,0.05} = 3.84$

this means if $\Lambda < 0.14667$.

now...

$$\Lambda = \frac{1}{\frac{-1}{\ln y}\, y^{-1/\ln y - 1}} = \frac{-\ln y}{1} \cdot y^{\left(1/\ln y + 1\right)}$$

we want to find y such that $\Lambda < 0.14667$.

$(0 \le y \le 1)$

to do this, we can ask mathematica:

to find the intersection of the graphs $\Lambda(y)$ and the line $y = 0.14667$.

From there, we can find where $\Lambda(y) < 0.14667$.

...

$\Rightarrow$ This is a transcendental equation, so we can't solve this by hand.

$\boxed{\text{Question 4}}$ $X \sim Poi(\lambda)$

$$p(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!} \quad ; \quad x = 0, 1, 2, \ldots$$

(A) Exp class ...

$$p(x|\lambda) = \exp\left\{-\lambda + x\ln\lambda - \ln(X!)\right\} \; ; \; \begin{matrix} x \in \mathbb{N}_+ \\ \lambda \in \mathbb{R}^+ \cup \infty \end{matrix}$$

$\Downarrow$

$p \in$ Exponential class of dist.

$$\begin{cases} p(\theta) = p(\lambda) = + \ln(\lambda) \\ k(x) = X \\ H(x) = -\ln(X!) \\ q(\theta) = q(\lambda) = -\lambda \end{cases}$$

(B) Show that the complete sufficient statistic for this dist also belongs to the exp family ..

- Note that $Poi(\lambda) \in$ regular exponential class of pmf

$\Rightarrow$ the statistic $Y_1 = \sum K(x_i) = X$ is a complete sufficient statistic for λ. (theorem)

- If we have a sample of iid $X_i \sim Poi(\lambda)$ then $Y_1 = \sum^n X_i$ is a complete sufficient statistic for λ. (Theorem).

$\to$ We know that $\boxed{\sum X_i \sim Poi(\sum\lambda) = Poi(n\lambda)}$. So Y_1 is also a member of the exponential family, by a similar argument (to (A)). $\square$

Explicitly ...

$$p(Y = \cancel{x_i} \mid \lambda) = \frac{e^{-n\lambda}(n\lambda)^{y}}{y!}$$

$$= \exp\left\{-n\lambda + y\ln(n\lambda) - \ln(y!)\right\}$$

$y \in \mathbb{N}$; $n\lambda \in \mathbb{R}^{+} \cup \infty$

$$\begin{cases} p(n\lambda) = \ln(n\lambda) \\ k(y) = y \\ H(y) = -\ln(y!) \\ q(n\lambda) = -n\lambda \end{cases}$$

→ so, see that $p_Y(y \mid n\lambda)$ is also a member of the exponential class ...

Question 5 $X_1, \dots X_n$; $X_i \sim Uni(0,\theta)$

(A) MLE for $E[X_1]$ & $Var(X_1)$

$$E[X_1] = \frac{\theta}{2} \quad ; \quad Var[X_1] = \frac{\theta^2}{12}.$$

→ to find mle for θ → need mle for $\hat{\theta}$.

we know that $\hat{\theta} = \max_i (X_i) = Y_n$

So, by invariance property of mle...

$$\widehat{E[X_1]} = \frac{\hat{\theta}}{2} = \frac{\max(X_i)}{2} = \boxed{\frac{Y_n}{2}}$$

$$\widehat{Var[X_i]} = \frac{\hat{\theta}^2}{12} = \frac{\max^2(X_i)}{12} = \boxed{\frac{Y_n^2}{12}}$$

(B) Find minimal sufficient statistic for θ

$$f(x_1 \dots x_n) = \frac{1}{\theta^n} I(X > 0) I(x < \theta)$$

By factorization ... $\begin{cases} K_1(Y_n, \theta) = \frac{1}{\theta^n} I(Y_n < \theta) \\ K_2[X_1 \dots X_n] = I\{X_{(1)} > 0\} \end{cases}$

We see that $Y_{(n)}$ (the max) is sufficient for θ.

→ Since we can no longer reduce ... → Y_n is minimal sufficient for θ

(c) Show... if $\theta > 0$ then Y_n is complete...

We have $g_{(n)}(y) = n y^{n-1} \theta^{-n}$; $0 \leq y \leq \theta$

$= \left(\frac{n}{\theta^n}\right) y^{n-1}$

Suppose $u(y)$ is a function s.t. $E(u(Y)) = 0$.

then

$$E[u(y)] = \int_0^\theta u(y)\, n y^{n-1} \theta^{-n}\, dt = 0$$

taking ∂_θ ... we get

$$0 = u(\theta)\, \underbrace{n \cdot \theta^{n-1} \cdot \theta^{-n}}_{\neq 0} = 0$$ (fundamental theorem of calc)

$\Rightarrow u(\theta) = 0$ identically $\forall \theta > 0 \Rightarrow \forall y$ too

So, the family of $f_{Y_{(n)}}(y)$ is complete.

$\Rightarrow$ $\boxed{Y_{(n)} \text{ is a complete (minimal) sufficient statistic for } \theta.}$

D) Show: if $\theta > 1$ then Y_n is not complete

if $\theta > 1$, then by a similar argument we will get

$$0 = u(\theta)\, n\theta^{n-1}\theta^{-n} = (u(\theta)) \cdot \frac{n}{\theta} = 0 \quad \forall \theta.$$

Now, $\theta > 1 \rightarrow u(\theta) = 0 \;\forall\; \theta > 1$ only.

$\Rightarrow$ The condition $E(u(y)) = 0$ hence only requires

$u(y) = 0 \;\forall\; y > 1$ only, not $\forall y \in \operatorname{Supp}\{f\}$.

$\Rightarrow$ $\boxed{Y_n \text{ is not (necessarily) complete.}}$

Question 6

(A) True

(B) True ... (mle's are asymptotically efficient)

(C) False

(D) True

(E) True

(F) False (e.g. $U(0,\theta)$ ... mle is good but regular conditions are not satisfied).

(G) True