Problem Set 1- Warming up Exercises

Due: Friday 5pm, Feb 11, via Canvas upload or in envelope outside 26-255
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Office hours Wednesday Feb 9, 1-3pm, in 26-214 (CUA seminar room)

1 Driven harmonic oscillator [6 pts]

Recall the equation of motion of the driven harmonic oscillator:

$$\ddot{x} + \gamma \dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t) \tag{1}$$

Let us investigate some properties of its solution:

a) The frequency of the underdamped solution of the undriven oscillator is $\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$. For the driven, underdamped oscillator, calculate the driving frequency at which

   i) the amplitude of the response becomes maximal,
   ii) the phase lag between the response and the drive becomes $\pi/2$, and
   iii) the power delivered from the drive to the oscillator (and dissipated by the damping), averaged over one cycle, becomes maximal.

   Explain in words why the dissipated power becomes maximal at that frequency.

b) When driven far from resonance, the power dissipated in the damped oscillator increases linearly with the damping γ, but on resonance varies as γ^{-1}. Why does reducing the damping increase the power dissipated on resonance?

c) For resonant driving, give the steady-state average energy E stored in the oscillator and also the energy dissipated in one cycle E_{lost}. Relate the ratio to the quality factor Q.

2 Harmonically bound electron - Lorentz model [8 pts]

Consider a harmonically bound electron of charge e and mass m, with its natural frequency ω equal to the absorbed or emitted frequency when the atom makes a transition between two states (this is the only ingredient coming from quantum mechanics). To start with, let it be undamped and driven by an electric field $\mathcal{E}\cos(\omega t)$, so that the force

$$F = e\mathcal{E}\cos(\omega t)$$

The resulting motion can be regarded as giving an oscillating dipole moment.

a) Give the resulting steady-state dipole moment $d(t) = ex(t)$. The expression will be useful in later chapters of the course when dealing with **oscillator strengths**.

Even in the absence of other kinds of damping, the motion will be damped because of what is called **radiation damping**. From classical electrodynamics, we know that any accelerated charge will emit radiation. For example, an electron in a circular orbit around a proton is constantly being accelerated, and therefore loses energy. This is why such a "planetary" model for the hydrogen atom failed until Bohr postulated stationary orbits where the electron does not lose energy. The total power radiated by an accelerated electron in the full solid angle of 4π is (see e.g. Griffiths, Introduction to Electrodynamics, Eq. 11.70)

$$P = \frac{1}{6\pi\epsilon_0 c^3} \left|\ddot{d}\right|^2$$

b) Calculate the energy lost per orbital cycle E_{lost} assuming an amplitude x_0 of the motion.

c) Obtain the quality factor Q from the ratio of the energy stored in the oscillator to the energy lost (see 1c, beware of 2πs), and from it, obtain the damping rate Γ_{rad}.

d) Express the quality factor in terms of the classical radius of the electron

$$r_0 = \frac{e^2}{4\pi\epsilon_0 mc^2}$$

(the Coulomb energy of two electrons a distance r_0 apart equals the electron's rest energy) and $\lambda\!\!\!^{-} = \lambda/(2\pi) = c/\omega$, the (reduced) wavelength of emitted radiation.

e) What is Q and Γ_{rad} from this estimate for the sodium D2 line ($\lambda = 589\,\text{nm}$)? Write the result as $\Gamma_{\text{rad}} = 2\pi \times \ldots\,\text{Hz}$. Consult the web for the natural linewidth of the sodium D2 line. How does it compare with the result from the Lorentz model?

3 Quantum harmonic oscillator [6 pts]

Consider a one-dimensional harmonic oscillator of mass m and frequency ω which is in number state $|n\rangle$. Recall how operators for position and momentum are related to the raising and lowering operators and answer the following questions:

a) Find both the average and the rms position and momentum (four things in total).

b) Check your results using energy and the virial theorem.

c) Sketch the wavefunction $\psi_n(x)$ for $n = 0$ and $n = 1$.

d) If a sodium (Na) atom is confined in the $|0, 0, 0\rangle$ state of a 3D harmonic oscillator trap with oscillation frequencies in each dimension $\omega = 2\pi \times 100\,\text{Hz}$, what is its rms size and velocity?