Questions/Ideas #2 (*to be continued*)

November 14, 2019
Huan Bui

Hi Evan, here's a layout of this document:

- Part 1 focuses on the change of coordinates $\xi \to \phi(x^\mu)$ and its Jacobian. I will consider only the integral over $\mathbb{R}^2$ and will try to discuss a few things about the $\mathbb{R}^d$ integral. Here's a summary of this part:
  1. For a given E such that $P(t^E s) = tP(s)$, we can't in general have a coordinate transformation $\xi \to \phi(t^\mu)$ such that $|\det J(\phi)| = \mathrm{C} \times t^{\cdots}$ where C is a constant that is only dependent on the dimension n of $\mathbb{R}^n$. I will show this for the $n = 2$ case, but we can see it is even more difficult to make $|J(\phi)|$ independent of angles when $n > 2$.
  2. I then consider the transformation $\xi \to t^E s$ such that $P(t^E s) = tP(s) = t$. This raises some concerns about the bijectivity of ϕ but makes the integral much easier to handle. I will consider the $n = 4$, $E = \mathrm{diag}(1/2, 1/4)$ case and see what I can generalize from there.
  3. I consider the integral
$$\int_{\mathbb{R}^d} e^{iP(\xi) - ix\cdot\xi}\, d\xi. \tag{1}$$
- In Part 2, I consider some oscillatory integrals and in what sense they converge.

---

1. The integral over $\mathbb{R}^2$ we went over on Friday, Nov 8 has the form
$$I \equiv \int_{\mathbb{R}^2} f(\xi)\, d\xi_1 d\xi_2. \tag{2}$$
Let $E = \mathrm{diag}(1/d_1, 1/d_2)$, which corresponds to $P(\xi) = \xi_1^{d_1} + \xi_2^{d_2}$. Suppose all d_i's are even, and that $1/d_1 \geq 1/d_2$. On Friday, we considered the transformation
$$\xi \to \phi(t,\theta) \equiv t^E s = t^E \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \begin{bmatrix} t^{1/d_1}\cos\theta \\ t^{1/d_2}\sin\theta \end{bmatrix}. \tag{3}$$
The Jacobian of this transformation is
$$J = \det \begin{bmatrix} \frac{t^{-1+1/d_1}\cos\theta}{d_1} & -t^{1/d_1}\sin\theta \\ \frac{t^{-1+1/d_2} - \sin\theta}{d_2} & t^{1/d_2}\cos\theta \end{bmatrix} = t^{\mathrm{tr}\,E-1}\left(\frac{d_1\sin^2\theta + d_2\cos^2\theta}{d_1 d_2}\right). \tag{4}$$
We can consider two cases $d_1 = d_2$ and $d_1 \neq d_2$
   (a) ***Case 1:*** If $d_1 = d_2 = d$ then
$$J = \frac{t^{-1+2/d}}{d^2} = \frac{t^{\mathrm{tr}\,E-1}}{d^2}. \tag{5}$$
   We have discussed how $\phi : (0,\infty)\times[0,2\pi] \to \mathbb{R}^2$ is bijective. So the original integrals becomes
$$\begin{aligned} I &= \int_{\mathbb{R}^2} f(\xi)\, d\xi_1 d\xi_2 \\ &= \int_0^\infty \int_0^{2\pi} f(t^E s)|J|\, dt d\theta \\ &= \int_0^\infty \int_0^{2\pi} f\begin{bmatrix} t^{1/d}\cos\theta \\ t^{1/d}\sin\theta \end{bmatrix} \cdot \left(\frac{t^{\mathrm{tr}\,E-1}}{d^2}\right) dt d\theta. \end{aligned} \tag{6}$$

When $f(\xi) = e^{iP(\xi)} \to e^{iP(t^E s)} = e^{itP(s)}$ we have

$$I = \frac{1}{d^2} \int_0^\infty \int_0^{2\pi} t^{\operatorname{tr} E - 1} e^{it(\cos^d \theta + \sin^d \theta)} \, dt d\theta \tag{7}$$

i. *Case 1.1:* When $d = 2$, $\operatorname{tr} E = 1$, and so

$$I = 2\pi \int_0^\infty e^{it} \, dt. \tag{8}$$

This integral diverges (which I could see upon differentiating under the integral sign). So, I interpret this as a Fourier transform (up to factors of 2π) of the step function

$$g(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} \tag{9}$$

evaluated at $\omega = 1$, because

$$\mathcal{F}[g](\omega)\bigg|_1 \propto \int_{-\infty}^{\infty} g(t) e^{i \cdot 1 \cdot 1} \, dt = \underbrace{\int_{-\infty}^{0} g(t) e^{it} \, dt}_{0} + \int_0^\infty g(t) \cdot e^{it} \, dt = \int_0^\infty e^{it} \, dt. \tag{10}$$

I can also interpret I as a Laplace transform of some other function, but I don't think we gain anything from doing that.

ii. *Case 1.2:* When $d = 2m > 2$ then the original integral becomes

$$I = \frac{1}{(2m)^2} \int_0^\infty \int_0^{2\pi} t^{\operatorname{tr} E - 1} e^{it(\cos^{2m} \theta + \sin^{2m} \theta)} \, dt d\theta. \tag{11}$$

Base on a number of tests in Mathematica I think these integrals converge, but I can't seem to deduce any pattern. For example:

$$\begin{aligned}
d = 4 : I &= \frac{2 + 2i}{16} \sqrt{2\pi} \, K \left[\frac{1}{2} \right] \\
d = 6 : I &= \frac{(2\pi^2) \, {}_2F_1 \left[\frac{1}{6}, \frac{2}{3}, 1, \frac{9}{25} \right] (-3.50882 - 2.02582i)}{36 \Gamma \left[\frac{-1}{3} \right]} \\
d = 8 : I &= \frac{1}{2} \sqrt[8]{-1} \, \Gamma \left[\frac{9}{8} \right]^2 \\
d = 10 &: \ldots
\end{aligned}$$

(b) ***Case 2:*** If $1/d_1 > 1/d_2$ and both d_1, d_2 are even, then the Jacobian

$$J = t^{\operatorname{tr} E - 1} \left(\frac{d_1 \sin^2 \theta + d_2 \cos^2 \theta}{d_1 d_2} \right) \tag{12}$$

is always dependent on θ. Thus, the parameterization

$$\phi(t, \theta) = t^E \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} \tag{13}$$

now becomes disadvantageous because the integral I now contains t and θ in both the exponent and scaling factor $|J(\phi)|$. I think this requires us to consider a new parameterization

$$\Phi(t, \theta) = t^E s(\theta) \tag{14}$$

where $P[s(\theta)] = 1$. This does not guarantee the angle-independence of the volume element, but we can always integrate out the angles separately.

i. *Case 1:* Consider for example $d_1 = 2, d_2 = 4$. With this, $P(\xi) = \xi_1^2 + \xi_2^4$, and we know very well $E = \mathrm{diag}(1/2, 1/4)$. We can consider the parameterization $\Phi : (0, \infty) \times [0, \pi] \to \mathbb{R}^2$

$$\Phi(t,\theta) = \begin{cases} t^E \begin{bmatrix} \sin(2\theta) & \sqrt{\cos(2\theta)} \end{bmatrix}^\top & t \in (0,\infty), \theta \in [0, \pi/4) \\ t^E \begin{bmatrix} \sin(2\theta) & -\sqrt{-\cos(2\theta)} \end{bmatrix}^\top & t \in (0,\infty), \theta \in [\pi/4, 3\pi/4] \\ t^E \begin{bmatrix} \sin(2\theta) & \sqrt{\cos(2\theta)} \end{bmatrix}^\top & t \in (0,\infty), \theta \in (3\pi/4, \pi] \end{cases} . \tag{15}$$

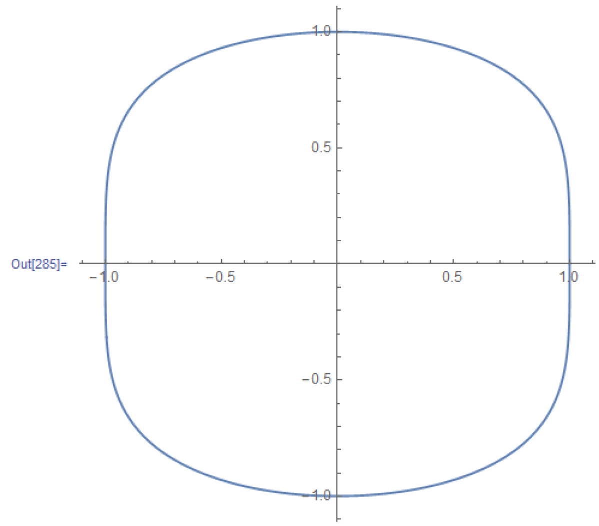

Figure 1: $\Phi(1,\theta)$, for $\theta \in [0, \pi]$.

We see that

$$P(\xi) \to P(t^E s) = tP(s) = t\left[\sin^2(2\theta) + \left(\pm\sqrt{\pm\cos(2\theta)}\right)^4\right] = t. \tag{16}$$

With this, the original integral becomes

$$I = \int_0^\infty \int_0^{\pi/4} e^{it} |J(\Phi)|\, dt d\theta + \int_0^\infty \int_{\pi/4}^{3\pi/4} e^{it} |J(\Phi)|\, dt d\theta + \int_0^\infty \int_{3\pi/4}^{\pi} e^{it} |J(\Phi)|\, dt d\theta. \tag{17}$$

Next we find what $|J(\Phi)|$ is for each integral. We consider two cases: $t^E(\sin(2\theta), \sqrt{\cos(2\theta)})^\top$ and $t^E(\sin(2\theta), -\sqrt{-\cos(2\theta)})^\top$.

When $\theta \in [0, \pi/4) \cup (3\pi/4, \pi]$, we have

$$|J(\Phi)| = \left| \det \begin{bmatrix} \frac{\sin(2\theta)}{2\sqrt{t}} & 2\sqrt{t}\cos(2\theta) \\ \frac{\sqrt{\cos(2\theta)}}{4t^{3/4}} & -\frac{\sqrt[4]{t}\sin(2\theta)}{\sqrt{\cos(2\theta)}} \end{bmatrix} \right| = \frac{1}{2\sqrt[4]{t}\sqrt{\cos(2\theta)}} \tag{18}$$

We should check if the θ integrals converge in these cases (they do):

$$\int_0^{\pi/4} \frac{1}{\sqrt{\cos(2\theta)}}\, d\theta = \int_{\pi/4}^{3\pi/4} \frac{1}{\sqrt{\cos(2\theta)}}\, d\theta = \frac{K\left(\frac{1}{2}\right)}{\sqrt{2}}. \tag{19}$$

When $\theta \in [\pi/4, 3\pi/4]$, we have

$$|J(\Phi)| = \left| \det \begin{bmatrix} \frac{\sin(2\theta)}{2\sqrt{t}} & 2\sqrt{t}\cos(2\theta) \\ -\frac{\sqrt{-\cos(2\theta)}}{4t^{3/4}} & -\frac{\sqrt[4]{t}\sin(2\theta)}{\sqrt{-\cos(2\theta)}} \end{bmatrix} \right| = \frac{1}{2\sqrt[4]{t}\sqrt{-\cos(2\theta)}}. \tag{20}$$

We check if the θ integral converges in this case (it does):

$$\int_{\pi/4}^{3\pi/4} \frac{1}{\sqrt{-\cos(2\theta)}}\, d\theta = -i\left(\sqrt{2}K\left(\frac{1}{2}\right) - 2K(2)\right). \tag{21}$$

Thus, it is possible to integrate out all the angle elements to find

$$I = \Omega \int_0^\infty t^{\operatorname{tr} E - 1} e^{it}\, dt = \Omega \int_0^\infty t^{-1/4} e^{it}\, dt. \tag{22}$$

where Ω is constant.

At this point, we can use van der Corput's lemma to show I is bounded. We can also evaluate this directly in Mathematica to find

$$I = \Omega(-1)^{3/8}\Gamma\left(\frac{3}{4}\right). \tag{23}$$

ii. In general... (*to be continued*)

iii. What happens when we look at the integral

$$I(x) = \int_{\mathbb{R}^d} e^{iP(\xi) - ix\cdot\xi}\, d\xi = \int_0^\infty \int_{\mathcal{S}} e^{iP(t^E s) - ix\cdot t^E s}\, d\xi = \int_0^\infty \int_{\mathcal{S}} e^{it - ix\cdot t^E s}\, d\xi \tag{24}$$

where $P(s) = 1$? One strategy is to expand the $x \cdot t^E s$ term and see what dominates

$$I(x) = \sum_{n=0}^\infty \int_0^\infty \int_{\mathcal{S}} e^{it} \frac{\left(-ix \cdot t^E s\right)^n}{n!}\, d\xi. \tag{25}$$

Is there a way to get some kind of bounds for each term in the sum? When $x = 0$, $x \ll 1$, etc. or when $s \in \mathcal{S}$, what can we say about high-n terms in the expansion of $I(x)$?

2. Some oscillatory integral stuff...