Representations

Let G be a MLG. A finite dimensional complex vector space V is a **G-module** if there is a continuous action of G on V such that

- $g \triangleright (\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha(g\triangleright|\psi\rangle) + \beta(g\triangleright|\varphi\rangle)$
- $g \triangleright h \triangleright |\psi\rangle = (gh)\triangleright|\psi\rangle$

the map $G\times V \to V$ is continuous
$(g,|\psi\rangle) \mapsto g\triangleright|\psi\rangle$

($g\triangleright|\psi\rangle$ denotes the action of g on $|\psi\rangle$)

<u>Note</u>: if V is infinite-dimensional we have to be careful about its topology!

<u>Note 2</u>: The action of G on V is also called a representation of G on V (same thing, different point of view)

<u>example</u>: Let $V_n = \text{span}\{|n\rangle\}$ be a one-dimensional vector space with the action of $e^{i\theta}\in U(1)$ on $|n\rangle$ defined by

$$e^{i\theta}\triangleright|n\rangle = (e^{i\theta})^n|n\rangle = e^{in\theta}|n\rangle \qquad \text{for some fixed } n\in\mathbb{Z}$$

The action is extended to the other vectors by linearity. → $e^{i\theta}\triangleright\alpha|n\rangle$ is defined to be $\alpha(e^{i\theta}\triangleright|n\rangle)$

When $z,w\in U(1)$ we get

$$z\triangleright w\triangleright|n\rangle = z\triangleright w^n|n\rangle = z^n w^n|n\rangle = (zw)^n|n\rangle$$

→ only works because $n\in\mathbb{Z}$!

<u>example</u>: Let $V_n = \text{span}\{|n\rangle\}$ as before, with $n\in\mathbb{Z}$
We can extend the action defined for $U(1)$ to $\mathbb{C}\setminus\{0\} = GL(1,\mathbb{C})$ by defining

$$z\triangleright|n\rangle = z^n|n\rangle$$

- A G-module V is <u>unitary</u> if it has an inner product and G acts unitarily, that is $\langle\varphi|g\triangleright|\psi\rangle = \langle\psi|g^{-1}\triangleright|\varphi\rangle \quad \forall |\psi\rangle,|\varphi\rangle\in V$

→ with abuse of notation, we can say "$g^\dagger = g^{-1}$"

example: Define an inner product on V_n as $\langle n|n\rangle = 1$.

- For the action of $U(1)$ we get $\overline{\langle n|\, e^{i\theta} \triangleright |n\rangle} = \overline{\langle n|\, e^{in\theta}|n\rangle} = \overline{e^{in\theta}}\langle n|n\rangle = e^{-in\theta}$

  and $\langle n|(e^{i\theta})^{-1}\triangleright|n\rangle = \langle n|\, e^{-i\theta}\triangleright|n\rangle = \langle n|\, e^{-in\theta}|n\rangle = e^{-in\theta}$

  $\Rightarrow$ V_n is unitary

- For the action of $GL(1,\mathbb{C})$ we get $\langle n|\, z^{-1}\triangleright|n\rangle = z^{-n}$ (same steps as above)

  but $\overline{\langle n|\, z \triangleright|n\rangle} = \overline{\langle n|\, z^n|n\rangle} = \bar{z}^n \neq z^{-n}$ in general (unless $n=0$)

  $\Rightarrow$ V_n is not unitary

Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. A finite-dimensional complex vector space V is a $\mathfrak{g}$-module if there is an action of $\mathfrak{g}$ on V such that

- $X \triangleright (\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha(X\triangleright|\psi\rangle) + \beta(X\triangleright|\varphi\rangle)$
- $(\alpha X + \beta Y)\triangleright|\psi\rangle = \alpha(X\triangleright|\psi\rangle) + \beta(Y\triangleright|\psi\rangle)$
- $[X,Y]\triangleright|\psi\rangle = X\triangleright Y\triangleright|\psi\rangle - Y\triangleright X\triangleright|\psi\rangle$

- V is unitary if it has an inner product and

  $\overline{\langle\varphi|X\triangleright|\psi\rangle} = -\langle\psi|X\triangleright|\varphi\rangle$ "$X^\dagger = -X$" (action is anti-hermitian)

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example: $su(2) = \{X \in M_n(\mathbb{C}) \mid X^* = -X, tr(X) = 0\} = \text{span}\{X_1, X_2, X_3\}$

where $X_1 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ $X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $X_3 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

are a basis. Since $[\cdot,\cdot]$ is bi-linear, we only need to know what it does to the basis:

$$[X_1, X_2] = -X_3 \qquad [X_2, X_3] = -X_1 \qquad [X_3, X_1] = -X_2$$

If this looks <u>almost</u> familiar is because physicists like to multiply the Lie algebra elements by $-i$ (sometimes $+i$), so they would use $J_k = -i X_k$ instead. The reason why is that for a unitary module the action of J_k is hermitian (self-adjoint).

Note: $J_k \notin \underline{su}(2)$! When we use this trick we have to "complexify" $\underline{su}(2)$ to allow scalar multiplication by $\mathbb{C}$.

Now let $V = \mathbb{C}^2$ with $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

if we define, for $X \in \underline{su}(2)$, $\quad X \triangleright |\psi\rangle = \underbrace{X|\psi\rangle}_{\text{matrix-vector multiplication}}$

then
$$\begin{cases} X_3 \triangleright |\pm\rangle = \pm \frac{i}{2} |\pm\rangle \\ X_1 \triangleright |\pm\rangle = \frac{i}{2} |\mp\rangle \\ X_2 \triangleright |\pm\rangle = \mp \frac{1}{2} |\mp\rangle \end{cases}$$

and you can check that $[X_1, X_2] \triangleright |\psi\rangle = -X_3 \triangleright |\psi\rangle$, etc.

With the standard inner product $\quad \langle \pm | \mp \rangle = 0, \ \langle \pm | \pm \rangle = 1$

V is unitary (check it!)