Supplementary problems

1 (a) In how many ways may we pass out k identical pieces of candy to n children?

Solution: $\binom{n+k-1}{k}$ ∎

(b) In how many ways may we pass out k distinct pieces of candy to n children?

Solution: n^k ∎

(c) In how many ways may we pass out k identical pieces of candy to n children so that each gets at most one? (Assume $k \leq n$.)

Solution: $\binom{n}{k}$. ∎

(d) In how many ways may we pass out k distinct pieces of candy to n children so that each gets at most one? (Assume $k \leq n$.)

Solution: $n^{\underline{k}}$ ∎

(e) In how many ways may we pass out k distinct pieces of candy to n children so that each gets at least one? (Assume $k \geq n$.)

Solution: None of the above. ∎

(f) In how many ways may we pass out k identical pieces of candy to n children so that each gets at least one? (Assume $k \geq n$.)

Solution: $\binom{k-1}{n-1}$ ∎

2. The neighborhood betterment committee has been given r trees to distribute to s families living along one side of a street. Unless otherwise specified, it doesn't matter where a family plants the trees it gets.

(a) In how many ways can they distribute all of them if the trees are distinct, there are more families than trees, and each family can get at most one?

Solution: $s^{\underline{r}}$ ∎

(b) In how many ways can they distribute all of them if the trees are distinct and any family can get any number?

Solution: s^r ∎

(c) In how many ways can they distribute all the trees if the trees are identical, there are no more trees than families, and any family receives at most one?

Solution: $\binom{s}{r}$ ∎

(d) In how many ways can they distribute them if the trees are distinct, there are more trees than families, and each family receives at most one (so there could be some leftover trees)?

Solution: $\sum_{k=0}^{s} \binom{s}{k} r^{\underline{k}}$ or $\sum_{k=0}^{s} s^{\underline{k}} \binom{r}{k}$ ∎

(e) In how many ways can they distribute all the trees if they are identical and anyone may receive any number of trees?

Solution: $\binom{r+s-1}{r}$ ∎

(f) In how many ways can all the trees be distributed and planted if the trees are distinct, any family can get any number, and a family must plant its trees in an evenly spaced row along the road?

Solution: $s^{\overline{r}} = (r+s-1)^{\underline{r}}$ ∎

(g) Answer the question in Part 2f assuming that every family must get a tree.

Solution: $r!\binom{r-1}{s-1}$ ∎

(h) Answer the question in Part 2e assuming that each family must get at least one tree.

Solution: $\binom{r-1}{s-1}$ ∎

3. In how many ways can n identical chemistry books, r identical mathematics books, s identical physics books, and t identical astronomy books be arranged on three bookshelves? (Assume there is no limit on the number of books per shelf.)

Solution: $\frac{(n+r+s+t+2)!}{n!r!s!t!2!}$ ∎

➔ 4. One formula for the Lah numbers is

$$L(k,n) = \binom{k}{n}(k-1)^{\underline{k-n}}$$

Find a proof that explains this product.

Solution: First choose the n elements which will be the first member of the part they lie in. (This, in effect, labels the n parts.) Then assign the remaining $k-n$ elements to their parts by making an ordered function of $n-k$ objects to n recipients in $(n+(k-n)-1)^{k-n} = (k-1)^{k-n}$ ways. ∎

5. What is the number of partitions of n into two parts?

Solution: $n/2$ if n is even and $(n-1)/2$ if n is odd, equivalently, $\lfloor n/2 \rfloor$. ∎

6. What is the number of partitions of k into $k-2$ parts?

set is $S(k,n)n!$, because we have a one-to-one function from the blocks to the n-element set. ▮

➔144. How many labeled trees on n vertices have exactly 3 vertices of degree one? Note that this problem has appeared before in Chapter 2.

Solution: There are $\binom{n}{3}$ ways to choose the three vertices of degree 1. The remaining $n-3$ vertices must appear in the Prüfer code for the tree. We can think of the Prüfer code as a function from the $n-2$ places of the code onto the $n-2$ remaining vertices, so that there are $S(n-2,n-3)(n-3)!$ possible Prüfer codes. Thus we have $\binom{n}{3}\binom{n-2}{2}(n-3)! = n!(n-2)(n-3)/12$ labeled trees on n vertices. ▮

•145. Each function from a k-element set K to an n-element set N is a function from K onto *some* subset of N. If J is a subset of N of size j, you know how to compute the number of functions that map onto J in terms of Stirling numbers. Suppose you add the number of functions mapping onto J over all possible subsets J of N. What simple value should this sum equal? Write the equation this gives you.

Solution: The sum should equal the number of functions, n^k. Thus we get $\sum_{j=0}^{n}\binom{n}{j}S(k,j)j! = n^k$. By using the fact that $\binom{n}{j} = n^{\underline{j}}/j!$, this may be rewritten as $\sum_{j=0}^{n} n^{\underline{j}}S(k,j) = n^k$. ▮

∘146. In how many ways can the sandwiches of Problem 136 be placed into three distinct bags so that each bag gets at least one?

Solution: $S(9,3)\cdot 3! = 55,980$. ▮

∘147. In how many ways can the sandwiches of Problem 137 be placed into distinct bags so that each bag gets exactly three?

Solution: Choose three sandwiches for bag one in $\binom{9}{3}$ ways, three for bag two in $\binom{6}{3}$ ways and put the remainder in bag 3. This gives us $\binom{9}{3}\binom{6}{3} = \frac{9!}{3!3!3!} = 1680$ ways.

The $\frac{9!}{3!3!3!}$ suggests another solution. We can line up the sandwiches in 9! ways. We take the first three for bag one, the second three for bag two and the last three for bag 3. The order of the sandwiches in the bag does not matter though, so each there are 3!3!3! listings corresponding to each way of putting sandwiches in bags, giving us $\frac{9!}{3!3!3!}$ ways to put the sandwiches in bags. ▮

•148. In how many ways may we label the elements of a k element set with n distinct labels (numbered 1 through n) so that label i is used j_i times?

(If we think of the labels as $y_1, y_2, \ldots, y_n$, then we can rephrase this question as follows. How many functions are there from a k-element set K to a set $N = \{y_1, y_2, \ldots y_n\}$ so that each y_i is the image of j_i elements of K?) This number is called a *multinomial coefficient* and denoted by

$$\binom{k}{j_1, j_2, \ldots, j_n}.$$

Solution: If the j_is don't add to k, it is zero. Otherwise, $\binom{k}{j_1, j_2, \ldots, j_n} = \frac{k!}{j_1!j_2!\cdots j_n!}$. We get this either as the product of binomial coefficients

$$\binom{k}{j_1}\binom{k-j_1}{j_2}\binom{k-j_1-j_2}{j_3}\cdots\binom{j_n}{j_n},$$

or more elegantly, by lining up the elements of the domain in $k!$ ways, taking the first j_1 elements to y_1, the next j_2 elements to y_2 and so on. However the order of the j_i elements that go to y_i is irrelevant, so $j_1!j_2!\cdots j_n!$ lists all correspond to the same function, giving us $\frac{k!}{j_1!j_2!,\cdots j_n!}$ functions. ∎

149. Explain how to compute the number of functions from a k-element set K to an n-element set N by using multinomial coefficients.

Solution: Add the multinomial coefficients $\binom{k}{j_1, j_2, \ldots, j_n}$ over all possible nonnegative values of the j_is that add to k. To see why, let $N = \{y_1, y_2, \ldots, y_n\}$ and apply the definition of multinomial coefficients. ∎

150. Explain how to compute the number of functions from a k-element set K onto an n-element set N by using multinomial coefficients.

Solution: Add the multinomial coefficients $\binom{k}{j_1, j_2, \ldots, j_n}$ in which each j_i is positive. To see why, let $N = \{y_1, y_2, \ldots, y_n\}$ and note that we are counting functions that send at least one element of K to each element y_i. ∎

•151. What do multinomial coefficients have to do with expanding the kth power of a multinomial $x_1 + x_2 + \cdots + x_n$? This result is called the *multinomial theorem.*

Solution: When we use the distributive law to multiply out $(x_1 + x_2 + \cdots + x_n)^k$, we will get a sum of a bunch of terms of the form bag we have a number, and the multiset of numbers of apples in the various bags is what determines our distribution of apples into identical bags. A multiset of positive integers that add to n is called a **partition** of n. Thus the partitions of 3 are 1+1+1, 1+2 (which is the same as 2+1) and 3. The number of partitions of k is denoted by $P(k)$; in computing the partitions of 3 we showed that $P(3) = 3$. It is traditional to use Greek letters like λ (the Greek letter λ is pronounced LAMB duh) to stand for partitions; we might write $\lambda = 1, 1, 1$, $\gamma = 2, 1$ and $\tau = 3$ to stand for the three partitions of three. We also write $\lambda = 1^3$ as a shorthand for $\lambda = 1, 1, 1$, and we write $\lambda \dashv 3$ as a shorthand for "λ is a partition of three."

∘157. Find all partitions of 4 and find all partitions of 5, thereby computing $P(4)$ and $P(5)$.

Solution: $4 = 1+1+1+1$, $4 = 2+1+1$, $4 = 2+1$, $4 = 3+1$, $4 = 4$, so that $P(4) = 5$. $5 = 1+1+1+1+1$, $5 = 2+1+1+1$, $5 = 2+2+1$, $5 = 3+1+1$, $5 = 3+2$, $5 = 4+1$, $5 = 5$, so that $P(5) = 7$. ∎

3.3.1 The number of partitions of k into n parts

A *partition of the integer k into n parts* is a multiset of n positive integers that add to k. We use $P(k, n)$ to denote the number of partitions of k into n parts. Thus $P(k, n)$ is the number of ways to distribute k identical objects to n identical recipients so that each gets at least one.

∘158. Find $P(6, 3)$ by finding all partitions of 6 into 3 parts. What does this say about the number of ways to put six identical apples into three identical bags so that each bag has at least one apple?

Solution: $6 = 4+1+1$, $6 = 3+2+1$, $6 = 2+2+2$, so $P(6, 3) = 3$. This says there are three ways to put six identical apples into three identical bags so that each bag gets at least one apple. ∎

3.3.2 Representations of partitions

∘159. How many solutions are there in the positive integers to the equation $x_1 + x_2 + x_3 = 7$ with $x_1 \geq x_2 \geq x_3$?

Solution: This problem is asking for $P(7, 3)$ and suggests an organized way to go about finding it: list the partitions starting with the largest part and work down. $7 = 5+1+1$, $7 = 4+2+1$, $7 = 3+3+1$, $7 = 3+2+2$, and if we have three numbers that add to seven, one must be larger than two, so there are four such solutions. ∎

160. Explain the relationship between partitions of k into n parts and lists $x_1, x_2, \ldots, x_n$ of positive integers that add to k with $x_1 \geq x_2 \geq \ldots \geq x_n$. Such a representation of a partition is called a *decreasing list* representation of the partition.

Solution: There is a bijection between partitions of k into n parts and lists, in non-increasing order, of n positive integers that add to k, because each multiset of numbers that adds to k can be listed in non-increasing order in exactly one way. ∎

∘161. Describe the relationship between partitions of k and lists or vectors $(x_1, x_2, \ldots, x_n)$ such that $x_1 + 2x_2 + \ldots kx_k = k$. Such a representation of a partition is called a *type vector* representation of a partition, and it is typical to leave the trailing zeros out of such a representation; for example $(2, 1)$ stands for the same partition as $(2, 1, 0, 0)$. What is the decreasing list representation for this partition, and what number does it partition?

Solution: The type vector of a partition of k is a way of representing the multiplicity function of the multiset of integers that adds to k. Thus there is a bijection between type vectors and partitions. The decreasing list representation of the partition with type vector $(2, 1)$ is 2,1,1. This is a partition of 4 ∎

162. How does the number of partitions of k relate to the number of partitions of $k + 1$ whose smallest part is one?

Solution: They are equal, because if we take two different partitions of k and increase the multiplicity of 1 in each (by one), they are still different; also if we take two different partitions of $k + 1$ that have parts of size one, and decrease the multiplicity of 1 in each (by one), they are still different. ∎

When we write a partition as $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_n$, it is customary to write the list of λ_is as a decreasing list. When we have a type vector $(t_1, t_2, \ldots, t_m)$ for a partition, we write either $\lambda = 1^{t_1}2^{t_2}\cdots m^{t_m}$ or $\lambda = m^{t_m}(m-1)^{t_{m-1}}\ldots 2^{t_2}1^{t_1}$. Henceforth we will use the second of these. When we write $\lambda = \lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_n^{i_n}$, we will assume that $\lambda_i > \lambda_{i+1}$.

3.3.3 Ferrers and Young Diagrams and the conjugate of a partition

The decreasing list representation of partitions leads us to a handy way to visualize partitions. Given a decreasing list $(\lambda_1, \lambda_2, \ldots \lambda_n)$, we draw a figure each row has a unique middle element, and each is shorter than the one above (by at least two squares) to reverse the process. Thus we have a bijection. ∎

166. Explain the relationship between the number of partitions of k into even parts and the number of partitions of k into parts of even multiplicity, i.e. parts which are each used an even number of times as in (3,3,3,3,2,2,1,1).

Solution: The number of partitions of k into even parts equals the number of partitions of parts of even multiplicity, because if we take the Young diagram of a partition of k into even parts and conjugate it, the resulting diagram has columns of even length. Thus the difference in heights of two successive columns is an even number, but this difference is the multiplicity of one of the parts of the conjugate. Further the height of the last column of a partition is the multiplicity of the first part. Since the multiplicity of any part of a partition is either the difference in height of two successive columns of the Young diagram or the height of the last column, then each part of the conjugate has even multiplicity. This bijection can be reversed, because if all the differences in height of the columns are even and the height of the last column is even, then when we conjugate this partition, the last row will be an even length, and all differences in length of the rows will be even, so all the parts of the resulting partition will be even. ∎

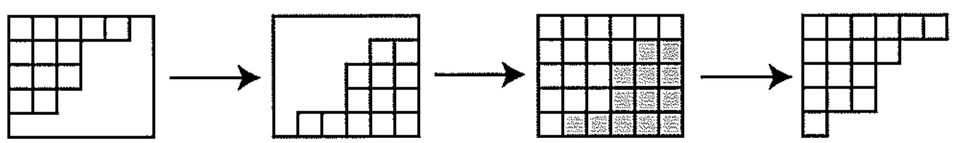
→167. Show that the number of partitions of k into four parts equals the number of partitions of $3k$ into four parts of size at most $k-1$ (or $3k-4$ into four parts of size at most $k-2$ or $3k+4$ into four parts of size at most k).

Solution: Think about putting the Young diagram of the partition into the upper left corner of a rectangle that is k units wide and four units high. Subdivide the rectangle into $4k$ squares of unit area. The Young diagram covers k of these squares. The uncovered squares are in rows of length $r_1 \leq r_2 \leq r_3 \leq r_4$. Thus if we list these lengths in the opposite order, we have a decreasing list representation of a partition of $3k$. Even r_1 will have to be positive, because the first part of the original partition will be at most $k-3$. The part size will be at most $k-1$ because r_4 must be less than k since the smallest part of the original partition is at least 1. To get partitions of $3k+4$, use a rectangle of width $k+1$, and to get partitions of $3k-4$, use a rectangle of width $k-1$. Since the first row of the Young diagram has at most

$k-3$ squares, we will still have four nonzero parts in the partition that results. ▮

168. The idea of conjugation of a partition could be defined without the geometric interpretation of a Young diagram, but it would seem far less natural without the geometric interpretation. Another idea that seems much more natural in a geometric context is this. Suppose we have a partition of k into n parts with largest part m. Then the Young diagram of the partition can fit into a rectangle that is m or more units wide (horizontally) and n or more units deep. Suppose we place the Young diagram of our partition in the top left-hand corner of an m' unit wide and n' unit deep rectangle with $m' \geq m$ and $n' \geq n$, as in Figure 3.3.

Figure 3.3: To complement the partition (5,3,3,2) in a 6 by 5 rectangle: enclose it in the rectangle, rotate, and cut out the original Young diagram.



(a) Why can we interpret the part of the rectangle not occupied by our Young diagram, rotated in the plane, as the Young diagram of another partition? This is called the *complement* of our partition in the rectangle.

Solution: If we fill the rectangle with unit squares, those not in the Young diagram of the original partition λ will fall into rows. The lengths of the rows are nonnegative, and are nondecreasing as we move down. Therefore, after we rotate through 180 degrees, these same rows will be listed in the opposite order, lined up along the left sides, and will have non-increasing length. Thus they will be the Young diagram of a partition. ▮

(b) What integer is being partitioned by the complement?

Solution: The integer being partitioned will be $m'n' - k$. ▮

(c) What conditions on m' and n' guarantee that the complement has the same number of parts as the original one?

Solution: If $m' > m$ and $n' = n$, then the two partitions will have the same number of parts, because we will have a nonzero number of empty squares at the end of each row of the Young diagram of λ. If $m' = m$ and $n'-n$ is the multiplicity of the largest part of λ, they will have the same number of parts. Otherwise, their numbers of parts will differ. ∎

(d) What conditions on m' and n' guarantee that the complement has the same largest part as the original one?

Solution: If $n' > n$ and $m = m'$, then the two partitions will have the same largest part. If $n' = n$ and $m' - m$ is the smallest part of λ, then they will have the same largest part. Otherwise, their largest parts will differ. ∎

(e) Is it possible for the complement to have both the same number of parts and the same largest part as the original one?

Solution: For the two partitions to have the same number of parts, either $m' = m$ or $n' = n$. If $m' = m$ and they have the same largest part, then $n' > n$. But this is consistent with $n' - n$ being the multiplicity of the largest part of λ. Thus they can have the same number of parts and the same largest part if $m' = m$ and $n' - n$ is the multiplicity of the largest part of λ, or similarly if $n = n'$ and $m' - m$ is the smallest part of λ. ∎

(f) If we complement a partition in an m' by n' box and then complement that partition in an m' by n' box again, do we get the same partition that we started with?

Solution: If we complement a partition in an m' by n' box and then complement that partition in *the same rectangle*, then we get the original partition back. ∎

➔169. Suppose we take a partition of k into n parts with largest part m, complement it in the smallest rectangle it will fit into, complement the result in the smallest rectangle it will fit into, and continue the process until we get the partition 1 of one into one part. What can you say about the partition with which we started?

Solution: Let us call the process of enclosing λ in the smallest rectangle possible and then forming the complement in that rectangle *encomplementation* (this is short for *en*closure and *complementation* and is not a standard term—there is no standard term for this operation) and call the result of it the *encomplement* of λ. The result of two encomplementations on the Young diagram of a partition is to remove all rows of maximum length and all columns of maximum length from the Young diagram. Thus the description of the result of an even number

$2j$ of encomplementations is straightforward; we remove all the rows of the j largest distinct lengths and all columns of the j largest distinct lengths. So if an even number of encomplementations brings us to a partition with one block of size one, we should be able to describe the original partition fairly easily. To deal with the result of an odd number of encomplementations, we ask what happens if we encomplement just once. If the complement of λ in the smallest rectangle in which if fits has one square, then $\lambda = \lambda_1^{n_1}\lambda_1 - 1$. Thus we are asking for the partitions which, after an even number of encomplementations, give us either the partition with one block or a partition of the form $\lambda_1^{n_1}(\lambda_1 - 1)$. First we ask what kind of partition results in the second one after two encomplementations. If we get $\lambda_1^{n_1}(\lambda_1 - 1)$ from two encomplementations, the partition we started with had the form

$$\lambda_0^{n_0}(\lambda_1 + \lambda_2)^{n_1}(\lambda_1 + \lambda_2 - 1)\lambda_2^{n_2}.$$

If we get $\lambda_1^{n_1}(\lambda_1 - 1)$ from four encomplementations, then we started with a partition of the form

$$\lambda_{-1}^{n_{-1}}(\lambda_0 + \lambda_3)^{n_0}(\lambda_1 + \lambda_2 + \lambda_3)^{n_1}(\lambda_1 + \lambda_2 + \lambda_3 - 1)(\lambda_2 + \lambda_3)^{n_3}\lambda_3^{n_3}.$$

From this pattern we see that a partition that results in $\lambda_1^{n_1}(\lambda_1 - 1)$ after $2j$ encomplementations has the form

$$\lambda_{1-j}^{n_{1-j}}\lambda_{2-j}^{n_{2-j}}\cdots\lambda_0^{n_0}\lambda_1'^{\,n_1}(\lambda_1' - 1)\lambda_2^{n_2}\cdots\lambda_{j+1}^{n_{j+1}}, \tag{3.3}$$

where $\lambda_i > \lambda_{i+1}$ and $\lambda_0 > \lambda_1' > \lambda_2 + 1$.

On the other hand, a partition λ that results in 1 after two encomplementations has the form $\lambda_0^{n_0}(\lambda_1 + 1)\lambda_1^{n_1}$, and so a partition that results in 1 after j encomplementations is of the form

$$\lambda_{1-j}^{n_{1-j}}\lambda_{2-j}^{n_{2-j}}\cdots\lambda_0^{n_0}(\lambda_1 + 1)\lambda_1^{n_1}\lambda_2^{n_2}\cdots\lambda_j^{n_j}, \tag{3.4}$$

where $\lambda_i > \lambda_{i+1}$ and $\lambda_0 > \lambda_1 + 1$. Thus a partition results in a single part of size 1 after some number of encomplementations if and only if it has the form of Equation 3.3 or Equation 3.4. ∎

170. Show that $P(k, n)$ is at least $\frac{1}{n!}\binom{k-1}{n-1}$.

Solution: The number of compositions of k into n parts is $\binom{k-1}{n-1}$. We can divide the compositions into blocks, where two compositions are in the same block if and only if one is a rearrangement of the other.