The Gross-Pitaevsky (GP) equation (skip)

So far we only considered spatially uniform BEC with no phase gradients. Our approach can be readily extended to allow for the BEC wavefunction variation in space and time. Using the Heisenberg equations of motion for the field $\psi(x,t)$ we write

$$i\hbar\partial_t\psi = [\psi, H] = \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu + U(x)\right)\psi + g\psi^\dagger\psi^2$$

[we added an external potential $U(x)$]. Taking expectation value with respect to the system state, and ignoring the out-of-condensate φ contributions, we replace $\langle\varphi_0|\psi|\varphi_0\rangle = \psi_0(x,t)$. This gives the GP equation for the BEC wavefunction

$$i\hbar\partial_t\psi_0 = \left(-\frac{\hbar^2}{2m}\nabla^2 - \mu + U(x)\right)\psi_0 + g\psi_0^*\psi_0^2$$

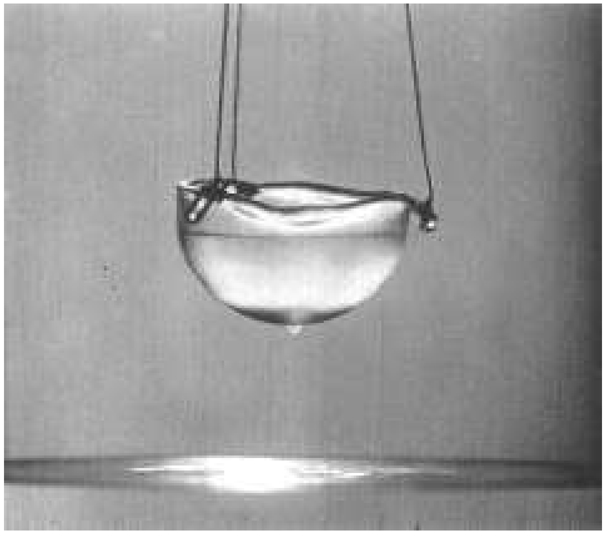
Trapped BEC (skip)

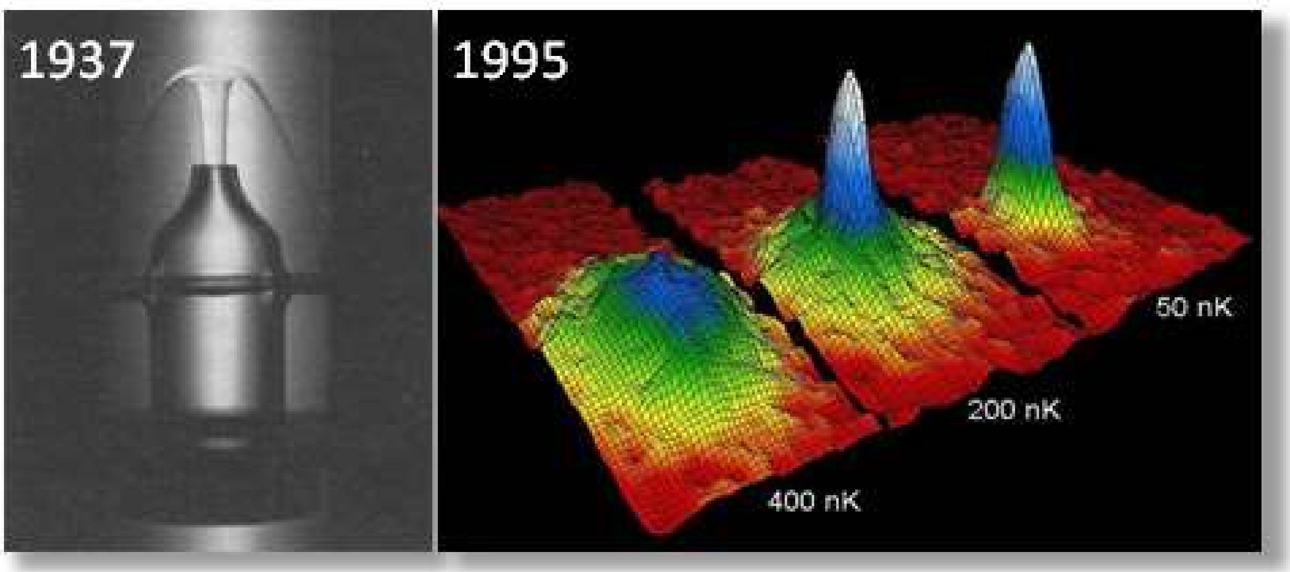
A seminal application of the GP equation is to describe BEC in trapped cold gases. In this case $U(x)$ is a smooth parabolic trap potential created optically or by a B field. The BEC profile can be inferred from minimizing the GP energy

$$\int d^3x \left[\frac{\hbar^2}{2m}\nabla\psi_0^*\nabla\psi_0 + (U(x)-\mu)|\psi_0|^2 + \frac{g}{2}|\psi_0|^4\right]$$

Two regimes: 1) small particle number $N = \int d^3x|\psi_0|^2$ (with weak coupling $g > 0$), the BEC profile matches the 1-particle ground state; 2) for large N, the BEC swells due to particle repulsion, rendering the ∇ term small and giving $\psi_0(x) = (\mu - U(x))/g$ for $U(x) < \mu$, zero else (the so-called Thomas-Fermi approximation).

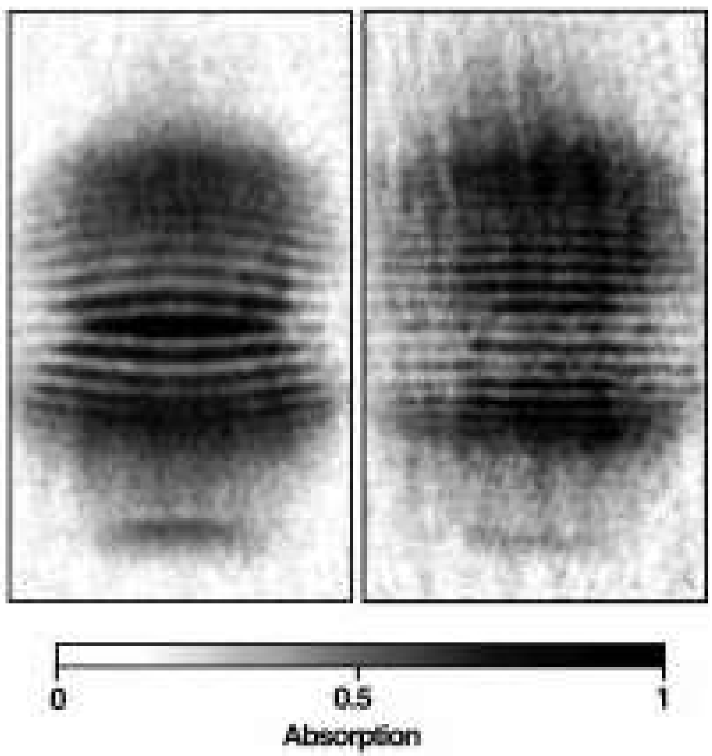
BEC in p space (1937) and in x space (1995)





Quantum coherence of matter waves: interference of two BEC's (despite the overall phase not measurable)

Start w splitting an atom trap in half with a laser beam. Then cool the sodium atoms in the two halves of the trap to form two independent BECs. At this point, quickly turn off the trap, allowing the atoms to fall and expand freely. As the two condensates began to overlap with one another, interference fringes formed (MIT, 1996).



Hydrodynamics of superfluids

Recall one-particle QM, the probability current is given by the w.f. phase gradient: $\rho(x,t) = \psi^*\psi$, $\partial_t\rho = -\nabla\cdot\boldsymbol{j}$,

$$\boldsymbol{j} = \frac{\hbar}{2mi}\psi^*\nabla\psi + \mathrm{c.c.} = \frac{\hbar}{m}|\psi|^2\nabla\theta$$

Today: macroscopic current, hydrodynamic quantity describing SF. Takes the same algebraic form as in one-particle QM
The continuity relation from the GP eqn $\nabla\boldsymbol{j} + \partial_t\rho = 0$,

$$\boldsymbol{j} = \frac{\hbar}{2im}\psi_0^*\nabla\psi_0 + \mathrm{c.c.} = \rho\frac{\hbar}{m}\nabla\theta, \quad \rho = |\psi_0|^2$$

- Velocity potential: $\boldsymbol{j} = n_s\boldsymbol{v}_s$, $\boldsymbol{v}_s = \nabla\varphi$, $\varphi = \frac{\hbar}{m}\theta$.
- Transverse modes (shear flows with $\nabla\times\boldsymbol{v}_s \neq 0$) not allowed at zero viscosity.
- The current-phase relation is completely general, follows from our many-body approach by decomposing quantum field into the condensate (order parameter) and out-of-condensate parts, $\psi(x,t) = \psi_0(x,t) + \delta\psi(x,t)$, $\psi_0(x,t) = \sqrt{n_s}e^{i\theta}$

Hydrodynamics of superfluids

In a true ground state, BEC is at rest, $\boldsymbol{v}_s = 0$. Describe the low-energy excitations?

- Hamiltonian for weakly perturbed BEC: $H = \int d^3r \left(\frac{m}{2}\rho_s v_s^2 + \frac{\kappa}{2}\delta\rho^2\right)$. The stiffness κ defines the chemical potential (internal pressure) $\delta\mu = \kappa\delta\rho$.
- In a harmonic approximation expect phonons: longitudinal pressure waves in the fluid with an acoustic spectrum. The continuity relation and $\boldsymbol{v} = \frac{\hbar}{m}\nabla\theta$ predict $\partial_t\delta\rho = -\rho_0\nabla\boldsymbol{v}_s = -\rho_0\frac{\hbar}{m}\nabla^2\theta$.
- This gives the wave equation provided $\delta\rho \sim \partial_t\theta$. Or, $\frac{\kappa}{\rho_0}\delta\rho_s = \hbar\partial_t\theta$.
- This is known as Josephson relation (by extension from superconductivity). Follows from the QM energy-phase relation $\delta\mu = \hbar\partial_t\theta$.
- Wave equation for acoustic phonons: $\frac{\hbar\rho_0}{\kappa}\ddot{\theta} = \rho_0\frac{\hbar}{m}\nabla^2\theta$ with the sound velocity $s = \sqrt{\kappa/m}$

Application to the thermomechanical (fountain) effect

The fountain effect

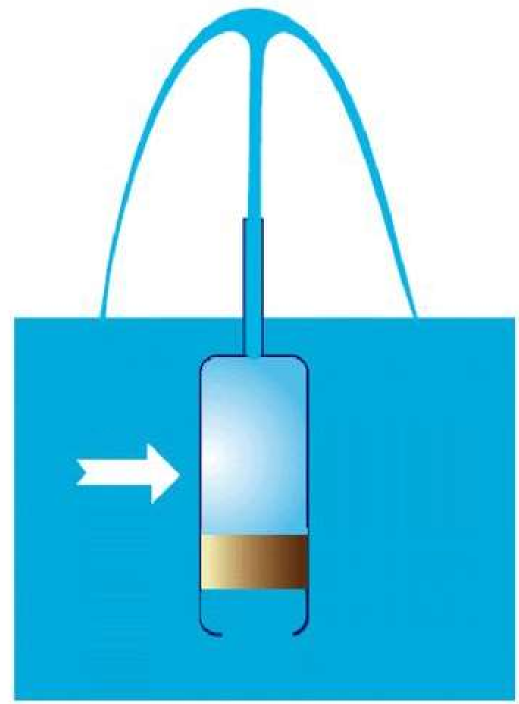
An extreme example of the thermomechanical effect is the Fountain Effect, discovered by Jack Allen at St Andrews University in 1938

The superleak in this case is a wide tube containing fine compressed powder.

One end is open to the He-II bath and the other is joined to a vertical capillary

When the powder is heated, superfluid flows into the superleak with such speed that He-II is forced out of the capillary as a jet.

A very small amount of heat will produce a jet 30-40cm high



Lecture 15 Superconductivity and Superfluidity

Condensate depletion by local heating (optical source) sets up a growing phase difference between the warmer and colder regions. Vertical phase gradient ramps up in time and drives a non-stop superflow

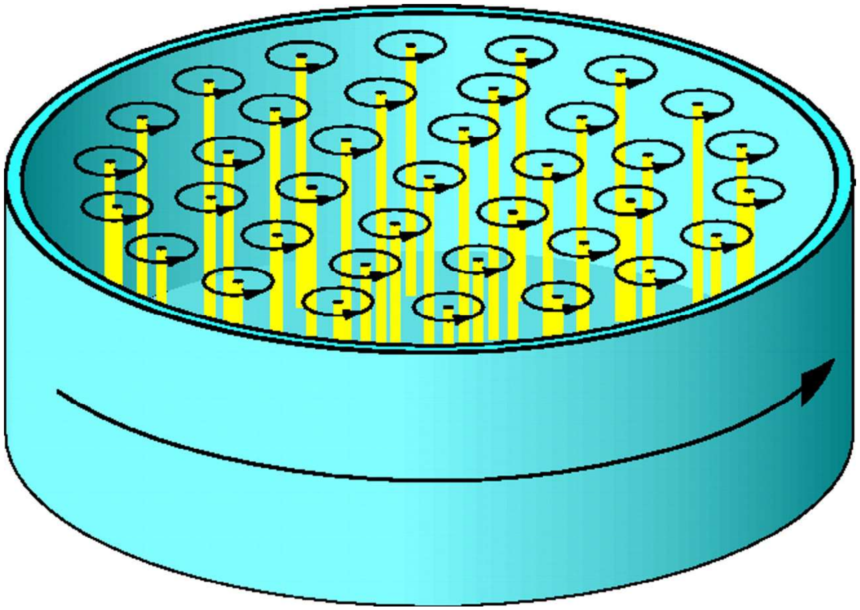
- Naively, the relation $\boldsymbol{v}_s \sim \nabla\theta$ does not allow rotation, since $\mathrm{curl}\,\mathrm{grad} = 0$
- Actually, a superfluid may rotate "as a whole" when it contains a finite density of vortices – filaments that carry quantized vorticity. Vortices are stabilized by the order parameter phase winding by a multiple of 2π. Phase winding provides topological protection to the vortices and defines the vorticity quantum.
- Persistent frictionless flows in an annulus and other closed systems: protected by the topology of phase winding $\oint \nabla\theta dl = 2\pi m$.

Vorticity quantization (Onsager '49, Feynman '55)

Investigate vortices $\psi(x) = e^{i\theta(x-x_0)}\sqrt{\rho(x-x_0)}$

- Current $\boldsymbol{j} = \rho \boldsymbol{v}_s$, the SF velocity $\boldsymbol{v}_s = \frac{\hbar}{m}\nabla\theta$
- Quantized phase winding for topological defects $\oint d\theta = 2\pi n$, i.e. $\psi(x)$ is single-valued, $\Delta\theta = 2\pi n$ with integer $n = 0, \pm 1, \pm 2 \ldots$
- Quantized vorticity $\oint d\boldsymbol{s}\boldsymbol{v}_s = 2\pi n \frac{\hbar}{m}$ (the Onsager-Feynman quantization condition)
- If $n \neq 0$, contour encircles point where $\rho = 0$
- line of zeros at vortex core
- quantized vorticity: line of zeros can only
  (i) terminate at boundary, or
  (ii) form loops

Vortices in a rotating superfluid



- Rotation of a superfluid is not uniform but takes place via a lattice of quantized vortices, whose cores (yellow) are parallel to the axis of rotation.
- To get a vortex inside a rotating vessel, an external force (say, gravity for 4He) has to work against the attraction to the boundary (can model by an image vortex of an opposite vorticity). This will be the origin of the critical value for the rotation speed that determines the threshold for vortices to start entering a vessel.

The number of vortices in a rotating superfluid

Velocity of a rotating classical fluid:

$$\boldsymbol{v}(\boldsymbol{r}) = \Omega \times \boldsymbol{r}, \quad \Omega \parallel z, \quad \boldsymbol{r} \perp z$$

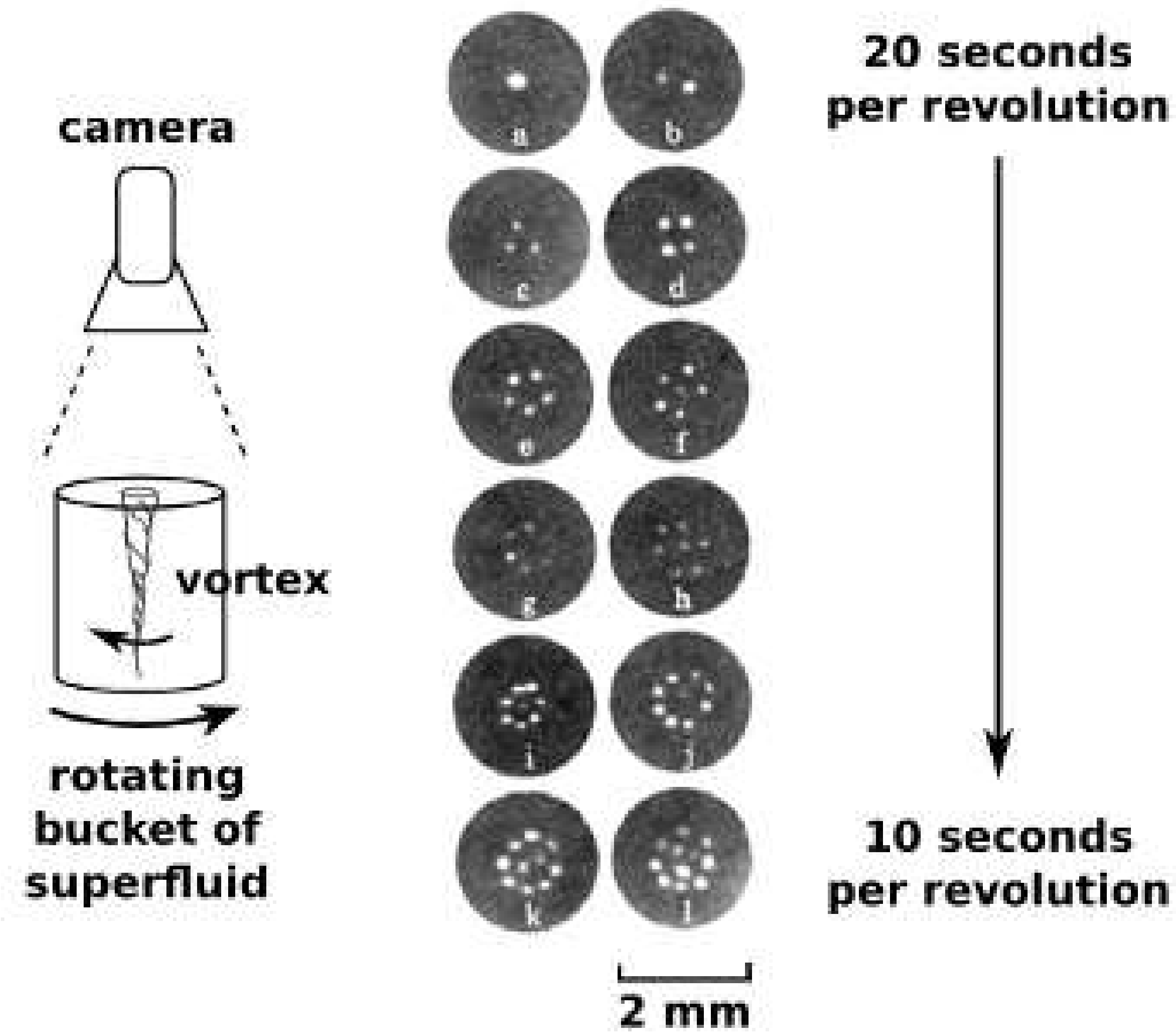
which gives constant vorticity $\nabla \times \boldsymbol{v} = 2\Omega$.
For vortices positioned at $\boldsymbol{r} = \boldsymbol{r}_i$, the vorticity is

$$\nabla \times \boldsymbol{v} = \sum_i \frac{2\pi\hbar}{m} \delta(\boldsymbol{r} - \boldsymbol{r}_i)$$

This mimics uniform rotation when the average number of vortices equals $n = 2\Omega \frac{m}{h} A$, where A is the area.

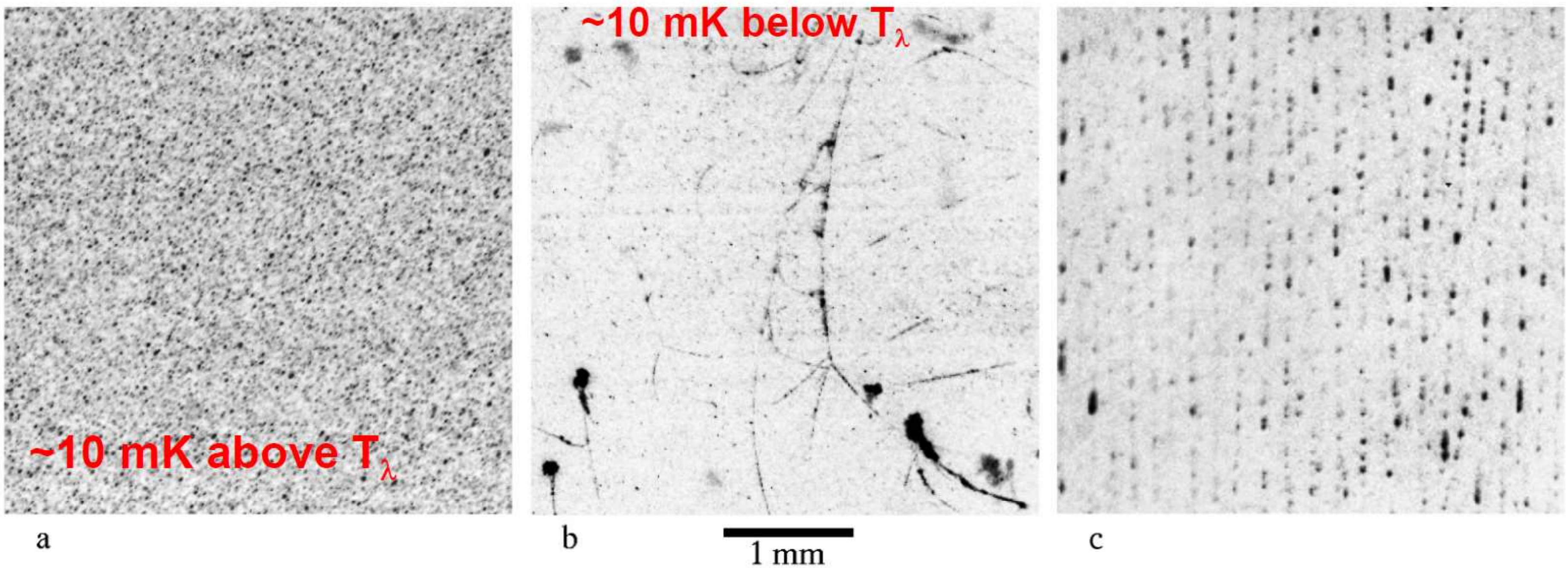
For a rotating vessel of radius R vortices enter one by one when $\Omega = \Omega_n$, with

$$\Omega_n = \frac{h}{2\pi m R^2} n, \quad n = 0, 1, 2, 3 \ldots$$

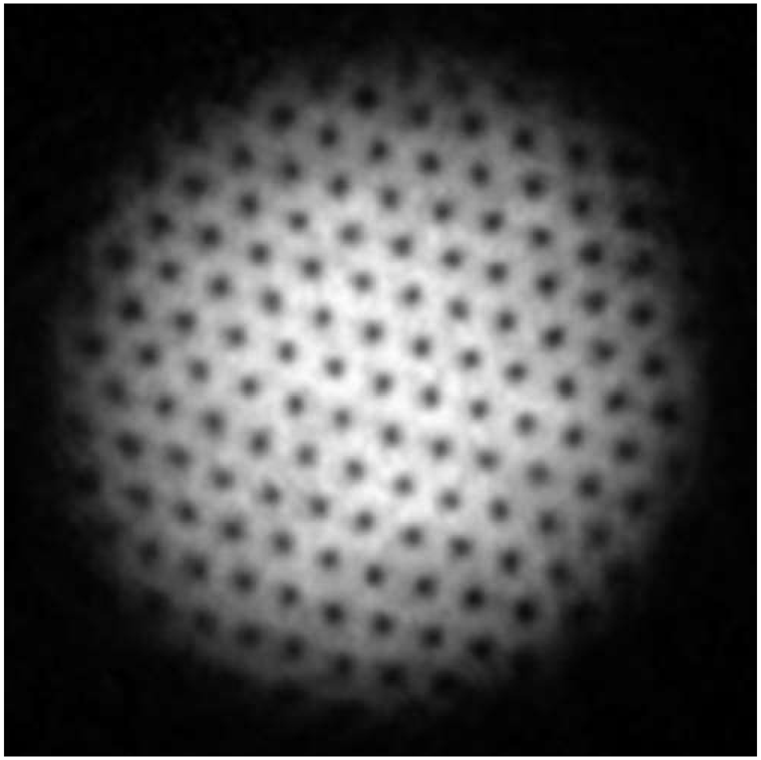

camera
vortex
rotating
bucket of
superfluid
a
b
c
d
e
f
g
h
i
j
k
l
2 mm
20 seconds
per revolution
10 seconds
per revolution

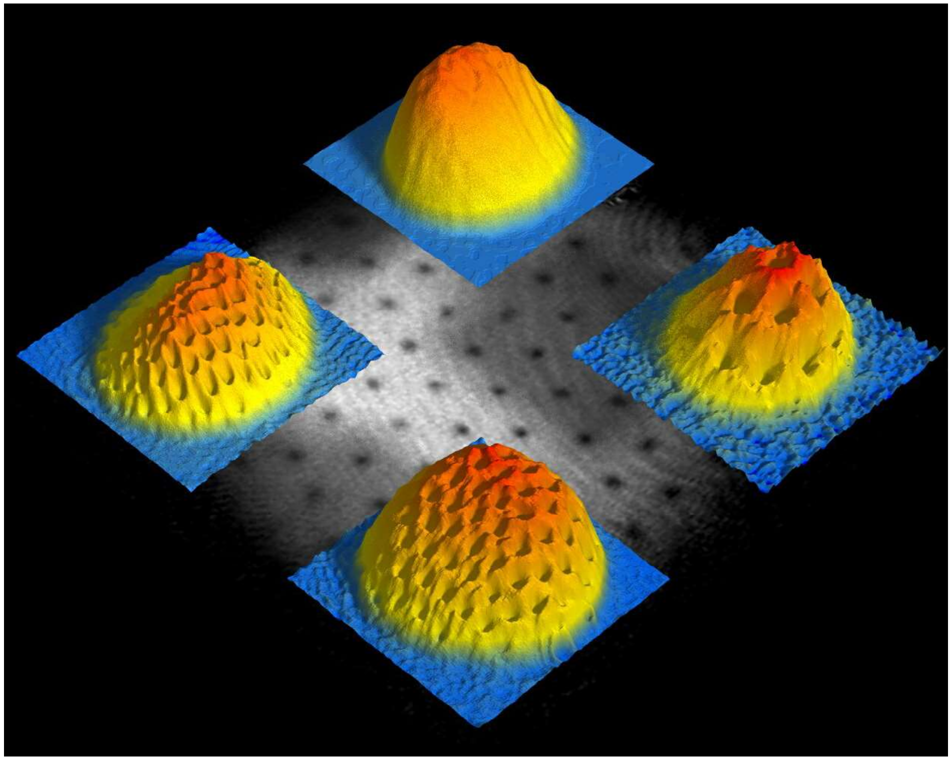
Imaging vortex filaments in 4He

Video: Crazy pool vortex (Physics Girl Youtube channel)
Live vortices in 4He imaged by nanoparticle decoration technique: Vortices, particles and superfluidity (clickable) by K. R. Sreenivasan



Vortices in a cold-atom BEC





Vortex soln of GP eqn with core along the z axis (skip)

Linear vortex with n vorticity quanta: in cylindrical coordinates r, φ, z, seek solution to GP eqn with the condensate phase $\theta = n\varphi$:

$$\psi(r,\varphi) = |\psi_0| e^{in\varphi} f(r), \quad |\psi_0| = \sqrt{\mu/g},$$

such that $f(r \to 0) = 0$, $f(r \to \infty) = 1$.

Nondimensionalize GP eqn: the healing length $\xi = \frac{\hbar}{\sqrt{2m\mu}}$,

$$s^2 \frac{\partial^2 f}{\partial s^2} + s\frac{\partial f}{\partial s} + (s^2 - n^2) f - s^2 f^3 = 0, \quad s = r/\xi$$

Solve numerically, asymptotic behavior $f(s \gg 1) = 1 - n^2/s^2$,

$f(s \ll 1) = Cs^{|n|}$.

Vortex energy and stability

- Phase varies at all distances, small and large; modulus varies only at $r \lesssim \xi$ in vortex core
- Vortex energy given as the kinetic energy of the fluid $E = \int d^3r \frac{1}{2}\rho \mathbf{v}^2 = \int d^3r \frac{1}{2}\rho \frac{\hbar^2(\nabla\varphi)^2}{m}$. For a vortex line of length L with n vorticity quanta:

$$E = L \int_0^\infty 2\pi r dr \frac{\hbar^2 \rho}{2m} \frac{n^2}{r^2} \approx n^2 L \frac{\pi \hbar^2 \rho}{m} \ln \frac{L}{\xi}$$

  Vortex core radius of order ξ (a few $\AA$ for 4He, a fraction of a micron for cold gases)
- Vortex energy $\sim n^2$, thus $n > 1$ vortices unstable, break up into $n = 1$ vortices.
- Single-quantum vortices, $n = 1$, have positive energy but are stable (topological protection)

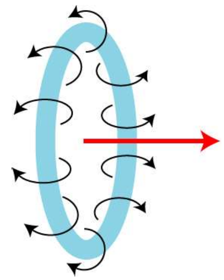
Vortex-vortex interactions

- Vortices of equal sign repel (see above)
- Two vortices of opposite signs a distance R apart. The energy $E = \int d^3r \frac{1}{2}\rho \mathbf{v}^2$ is estimated as 2 times one-vortex contribution for $r \lesssim R$:

$$E = 2L \int_0^{\sim R} 2\pi r dr \frac{\hbar^2 \rho}{2m} \frac{n^2}{r^2} \approx 2L \frac{\pi \hbar^2 \rho}{m} \ln \frac{R}{\xi}$$

- Vortex energy grows with R. Therefore the vortex and antivortex attract.
- Energy $\propto L$ the system size, but is finite for a vortex ring.

Vortex rings: self-propelled, carry energy and momentum



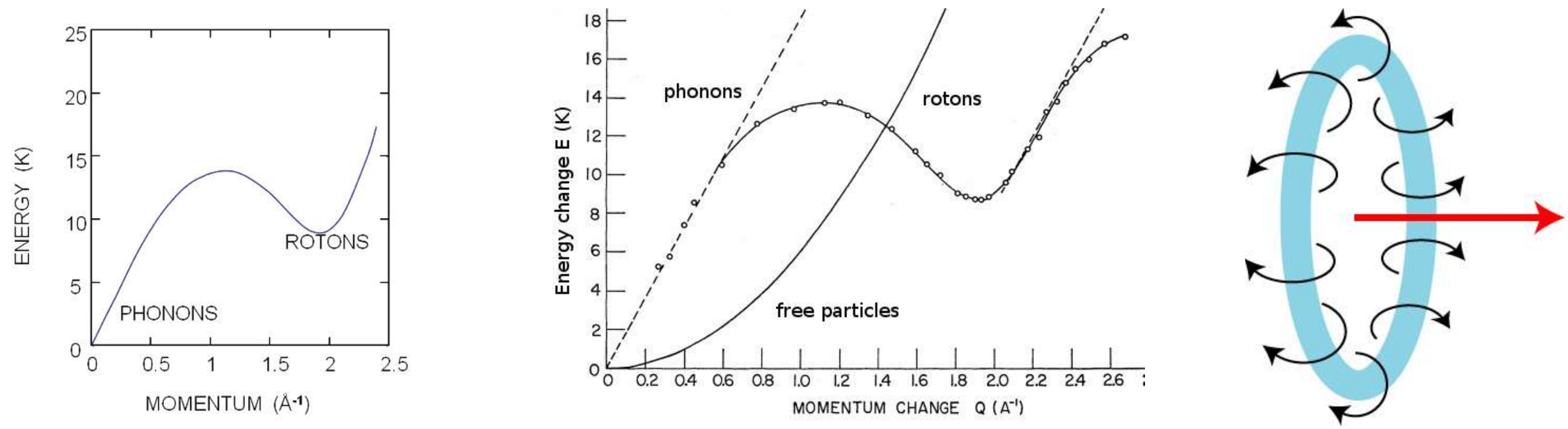
- Velocity distribution around the ring is described by a Biot-Savart law (mimics B field of a current loop)

$$\nabla \times v = \frac{2\pi\hbar}{m} \oint ds \boldsymbol{\ell} \delta(\boldsymbol{r} - \boldsymbol{r}(s)), \quad \nabla \cdot \boldsymbol{v} = 0$$

  Self-propelled with velocity $v = \frac{\hbar}{2m} \frac{\ln(R/\xi)}{R}$, where R is the ring radius.
- Predicts the total momentum $p \sim R^2 \hbar \rho_s$
- Dispersion relation $E(p) \sim \sqrt{p} \ln \frac{1}{p}$. "Group velocity" $v = dE/dp \sim \frac{\ln R}{R}$ agrees with the Biot-Savart law.

Vortices as low-energy excitations (Feynman)

- Contrast to phase-waves (Goldstone modes): gapless, but cannot be created for subcritical velocity (Landau criterion)
- Vortex excitations disrupt phase coherence, destroy SF, however: topologically protected, extended strings; a large energy cost to create, $\Delta E = \frac{\pi\hbar^2}{m} nL \log\frac{L}{a}$,where L a vortex filament length
- Feynman: microscopic vortex rings as a representation for Landau's rotons
- Finite threshold for vortices to start entering a rotating vessel, or a moving fluid: an external force (say, gravity for 4He) has to work against the attraction to the boundary (can model by an image vortex of an opposite vorticity)



Left: Schematic dispersion of elementary excitations in 4He indicating the phonon and roton parts of the dispersion. Center: Dispersion curve of excitations in 4He measured by neutron scattering [D. G. Henshaw and A. D. B. Woods, Phys. Rev. 121, 1266 (1961)]. Right: Feynman's vortex ring model of elementary excitations in strongly-interacting superfluids.

Vortex-driven topological phase transitions (Berezinskii-Kosterlitz-Thouless)

Brief summary (clickable) Superfluidity of 4He films: