Physics 8.321, Fall 2020
Homework #6

Due **Friday, November 12** by 8:00 PM.

1. [Sakurai and Napolitano Problem 16, Chapter 2 (page 151)]

Consider a function, known as the **correlation function**, defined by

$$C(t) = \langle x(t)x(0)\rangle\,,$$

where $x(t)$ is the position operator in the Heisenberg picture. Evaluate the correlation function explicitly for the ground state of a one-dimensional simple harmonic oscillator.

Answer: Start with noticing in Heisenberg picture we have $x(t) = x(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t)$, which yields

$$\begin{aligned} c(t) &= \langle x(t)x(0)\rangle \\ &= \langle x^2(0)\cos(\omega t) + \frac{p(0)x(0)}{m\omega}\sin(\omega t)\rangle. \end{aligned}$$

For the ground state of SHO the correlation function takes the form

$$\begin{aligned} c(t) &= \langle 0|x^2(0)\cos(\omega t)|0\rangle + \langle 0|\frac{p(0)x(0)}{m\omega}|0\rangle\sin(\omega t) \\ &= \frac{\hbar}{2m\omega}\cos(\omega t) - i\frac{\hbar}{2m\omega}\sin(\omega t) \\ &= \frac{\hbar}{2m\omega}e^{-i\omega t}, \end{aligned}$$

where we used

$$\begin{aligned} x(0) &= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \\ \Rightarrow x^2(0) &= \frac{\hbar}{2m\omega}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger). \end{aligned}$$

When sandwiched by ground state $|0\rangle$ on both side, only $aa^\dagger$ gives non-zero contribution, and is equal to

$$\langle 0|aa^\dagger|0\rangle = 1.$$

Also,

$$\begin{aligned} p(0) &= i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a) \\ \Rightarrow p(0)x(0) &= i\frac{\hbar}{2}(a^\dagger a^\dagger + a^\dagger a - aa^\dagger - aa), \end{aligned}$$

and similarly when this is sandwiched by the ground state only $aa^\dagger$ gives non vanishing contribution, showing the results we got above.

2. [Modified from Sakurai and Napolitano Problem 17, Chapter 2 (page 152)]

Consider again a one-dimensional simple harmonic oscillator. Do the following algebraically – that is, without using wavefunctions.

(a) Construct a linear combination of $|0\rangle$ and $|1\rangle$ such that $\langle x\rangle$ is as large as possible.

(b) Suppose the oscillator is in the state constructed in (a) at $t = 0$. What is the state vector for $t > 0$ in the Schrödinger picture? Evaluate the expectation value $\langle x\rangle$ as a function of time for $t > 0$, using (i) the Schrödinger picture and (ii) the Heisenberg picture. Evaluate $\langle p\rangle$ as a function of time as well and confirm Ehrenfest's theorem giving the classical equations of motion.

(c) Evaluate $\langle(\Delta x)^2\rangle$ as a function of time using either picture.

Answer:

(a) Assume $|\alpha\rangle = \cos\theta|1\rangle + \sin\theta e^{i\phi}|0\rangle$ without loss of generality and notice

$$\begin{aligned} x|\alpha\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger)(\cos\theta|1\rangle + \sin\theta e^{i\phi}|0\rangle) \\ &= \sqrt{\frac{\hbar}{2m\omega}}(\sqrt{2}\cos\theta|2\rangle + \sin\theta e^{i\phi}|1\rangle + \cos\theta|0\rangle), \end{aligned}$$

From this we obtain

$$\begin{aligned} \langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}(\cos\theta\sin\theta e^{i\phi} + \cos\theta\sin\theta e^{-i\phi}) \\ &= \sqrt{\frac{\hbar}{2m\omega}}\sin(2\theta)\cos\phi. \end{aligned}$$

Maximum occurs when, $\phi = 0$, and $\theta = \frac{\pi}{4}$, and we find

$$\boxed{|\alpha\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle.}$$

makes $\langle x\rangle$ as large as possible.

(b) Time evolution in Schrödinger picture yields the state

$$\begin{aligned} |\alpha, t\rangle &= e^{-\frac{i}{\hbar}Ht}\left(\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|0\rangle\right) \\ &= \frac{1}{\sqrt{2}}e^{-i\frac{3}{2}\omega t}|1\rangle + \frac{1}{\sqrt{2}}e^{-i\frac{1}{2}\omega t}|0\rangle. \end{aligned}$$

Applying x similar to above

$$x|\alpha, t\rangle = \sqrt{\frac{\hbar}{2m\omega}}\frac{1}{\sqrt{2}}\left(\sqrt{2}e^{-i\frac{3}{2}\omega t}|2\rangle + e^{-i\frac{1}{2}\omega t}|1\rangle + e^{-i\frac{3}{2}\omega t}|0\rangle\right),$$

we find $\langle x\rangle$ as a function of time is

$$\boxed{\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t.}$$

In the Heisenberg picture the operator x evolves as,

$$\langle x(t)\rangle = \langle x(0)\rangle \cos(\omega t) + \frac{\langle p(0)\rangle}{m\omega}\sin(\omega t)$$

For the state $|\alpha\rangle$ we found above, we have

$$\langle x(0)\rangle = \sqrt{\frac{\hbar}{2m\omega}}, \quad \langle p(0)\rangle = 0.$$

which yields

$$\boxed{\langle x\rangle = \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t.}$$

Similarly to how we evaluate $\langle x\rangle$ above, we have

$$p|\alpha,t\rangle = i\sqrt{\frac{m\hbar\omega}{2}}\frac{1}{\sqrt{2}}\left(\sqrt{2}e^{-i\frac{3}{2}\omega t}|2\rangle + e^{-i\frac{1}{2}\omega t}|1\rangle - e^{-i\frac{3}{2}\omega t}|0\rangle\right),$$

This yields

$$\langle p(t)\rangle = -\sqrt{\frac{\hbar m\omega}{2}}\sin\omega t$$

which we already used above for $t = 0$. From this it follows that

$$m\frac{d^2}{dt^2}\langle x\rangle = \frac{d}{dt}\langle p\rangle = -\langle\frac{d}{dx}V(x)\rangle = -m\omega^2\langle x\rangle,$$

confirming Ehrenfest's theorem.

(c) Again, in Schrödinger picture:

$$x^2 = \frac{\hbar}{2m\omega}(aa + aa^\dagger + a^\dagger a + a^\dagger a^\dagger)$$

Since $|\alpha,t\rangle$ is linear superposition of $|1\rangle$ and $|0\rangle$, the aa, and $a^\dagger a^\dagger$ give 0 when sandwiched by $|\alpha,t\rangle$. Noticing $aa^\dagger = 1 + a^\dagger a$, we have

$$\begin{aligned}
\langle\alpha,t|x^2|\alpha,t\rangle &= \frac{\hbar}{2m\omega}(1 + 2\langle\alpha,t|a^\dagger a|\alpha,t\rangle) \\
&= \frac{\hbar}{2m\omega}\left(1 + 2\times(\frac{1}{\sqrt{2}})^2\right) \\
&= \frac{\hbar}{2m\omega}\times 2.
\end{aligned}$$

This produces

$$\begin{aligned}
\langle(\triangle x)^2\rangle &= \langle x^2\rangle - \langle x\rangle^2 \\
&= \frac{\hbar}{2m\omega}\times 2 - \frac{\hbar}{2m\omega}\cos^2\omega t \\
&= \boxed{\frac{\hbar}{2m\omega}(2 - \cos^2\omega t).}
\end{aligned}$$

3. Consider a simple harmonic oscillator of frequency ω which begins in the state

$$|\psi(0)\rangle = c_0 e^{\phi_0 a^\dagger} |0\rangle$$

where $\phi_0 = \alpha + i\beta$ is an arbitrary complex number and $c_0 = \exp(-|\phi_0|^2/2)$.

(a) Solve the equation of motion for $|\psi(t)\rangle$.

(b) Evaluate $\langle x\rangle$, $\langle p\rangle$ as functions of time.

(c) Describe the wavefunction associated with $|\psi(t)\rangle$ in terms of modulus $\rho(x)$ and phase $S(x)$. Give the physical interpretation of the modulus and phase. Describe qualitatively what happens to the wavefunction over time. Compare with the time-development of a free particle given an initial Gaussian state.

Answer:

(a) First notice we can write this state as

$$\begin{aligned} |\psi(0)\rangle &= c_0 e^{\phi_0 a^\dagger}|0\rangle \\ &= \sum_n^\infty c_0 \frac{\phi_0^n}{n!}(a^\dagger)^n|0\rangle \\ &= \sum_n^\infty c_0 \frac{\phi_0^n}{\sqrt{n!}}|n\rangle. \end{aligned}$$

After time evolving it we produce

$$\begin{aligned} |\psi(t)\rangle &= \sum_n^\infty c_0 \frac{\phi_0^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}E_n t}|n\rangle \\ &= \sum_n^\infty c_0 \frac{\phi_0^n}{\sqrt{n!}} e^{-\frac{i}{\hbar}(n+\frac{1}{2})\hbar\omega t}|n\rangle \\ &= e^{-i\frac{\omega}{2}t}\sum_n^\infty c_0 \frac{\phi_0^n}{\sqrt{n!}}(e^{-i\omega t})^n|n\rangle \\ &= \boxed{e^{-i\frac{\omega}{2}t} c_0 e^{\phi_0 e^{-i\omega t} a^\dagger}|0\rangle.} \end{aligned}$$

(b) Let

$$|\psi(t)\rangle = e^{-i\frac{\omega}{2}t} c_0 e^{\phi_0 e^{-i\omega t} a^\dagger}|0\rangle \equiv e^{-i\frac{\omega}{2}t} c_0 |\phi_0 e^{-i\omega t}\rangle$$

and $|\phi_0 e^{-i\omega t}\rangle$ is an eigenstate of a:

$$\begin{aligned} a|\phi_0 e^{-i\omega t}\rangle &= \phi_0 e^{-i\omega t}|\phi_0 e^{-i\omega t}\rangle \\ \langle\phi_0 e^{-i\omega t}|a^\dagger &= \langle\phi_0 e^{-i\omega t}|(\phi_0 e^{-i\omega t})^* \end{aligned}$$

we also have

$$\langle\phi_0 e^{-i\omega t}|\phi_0 e^{-i\omega t}\rangle = e^{|\phi_0 e^{-i\omega t}|^2} = e^{|\phi_0|^2} = \frac{1}{|c_0|^2}$$

Now recalling

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$
$$p = i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)$$

We have

$$\begin{aligned}
\Rightarrow \langle x(t)\rangle &= |c_0|^2\langle\phi_0 e^{-i\omega t}|\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)|\phi_0 e^{-i\omega t}\rangle \\
&= |c_0|^2\sqrt{\frac{\hbar}{2m\omega}}\langle\phi_0 e^{-i\omega t}|(\phi_0 e^{-i\omega t})^* + \phi_0 e^{-i\omega t}|\phi_0 e^{-i\omega t}\rangle \\
&= \boxed{\sqrt{\frac{\hbar}{2m\omega}} \times 2\,\mathrm{Re}(\phi_0 e^{-i\omega t})} \\
&= \boxed{\sqrt{\frac{\hbar}{2m\omega}} \times 2\,|\phi_0|\cos(\theta - \omega t)} \quad \text{where } \phi_0 = |\phi_0|e^{i\theta} \\
&= \boxed{2\sqrt{\frac{\hbar}{2m\omega}}(\alpha\cos\omega t + \beta\sin\omega t)} \\
\Rightarrow \langle p(t)\rangle &= |c_0|^2\langle\phi_0 e^{-i\omega t}|i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)|\phi_0 e^{-i\omega t}\rangle \\
&= |c_0|^2 i\sqrt{\frac{m\hbar\omega}{2}}\langle\phi_0 e^{-i\omega t}|(\phi_0 e^{-i\omega t})^* - \phi_0 e^{-i\omega t}|\phi_0 e^{-i\omega t}\rangle \\
&= \boxed{\sqrt{\frac{m\hbar\omega}{2}} \times 2\,\mathrm{Im}(\phi_0 e^{-i\omega t})} \\
&= \boxed{\sqrt{\frac{m\hbar\omega}{2}} \times 2\,|\phi_0|\sin(\theta - \omega t)} \\
&= \boxed{2\sqrt{\frac{m\hbar\omega}{2}}(\beta\cos\omega t - \alpha\sin\omega t)}
\end{aligned}$$

We also get

$$\boxed{\phi_0 e^{-i\omega t} = \sqrt{\frac{m\omega}{2\hbar}}\langle x(t)\rangle + i\frac{1}{\sqrt{2m\hbar\omega}}\langle p(t)\rangle}$$

Notice that $\langle x(t)\rangle$ and $\langle p(t)\rangle$ (1) oscillate (2) with phase difference $\frac{\pi}{2}$, exactly like a *classical* simple harmonic oscillator.

(c) A quick way to obtain $\langle x|\psi(t)\rangle$ is to make use of the results of "squeezed states" we already computed in Problem (2), Problem set #5:

$$|\alpha, \beta, \gamma\rangle \equiv e^{\alpha} e^{\beta a^\dagger} e^{\gamma (a^\dagger)^2}|0\rangle$$

$$\Rightarrow \langle\alpha_1, \beta_1, \gamma_1|\alpha_2, \beta_2, \gamma_2\rangle = \frac{1}{\sqrt{1 - 4\gamma_1^*\gamma_2}} e^{\frac{\beta_1^*\beta_2 + \gamma_1^*\beta_2^2 + \gamma_2(\beta_1^*)^2}{1-4\gamma_1^*\gamma_2}} e^{\alpha_1^* + \alpha_2}$$

Now

$$
\begin{aligned}
|\psi(t)\rangle &= e^{-i\frac{\omega}{2}t}c_0 e^{\phi_0 e^{-i\omega t}a^\dagger}|0\rangle = e^{-i\frac{\omega}{2}t-|\phi_0|^2/2}e^{\phi_0 e^{-i\omega t}a^\dagger}|0\rangle \\
|x\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} e^{\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}xa^\dagger-\frac{1}{2}(a^\dagger)^2}|0\rangle
\end{aligned}
$$

In terms of squeezed states variables these states can be described by the following parameters

$$
\begin{aligned}
e^{\alpha_1} &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} \\
\beta_1 &= \sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x \\
\gamma_1 &= -\frac{1}{2} \\
e^{\alpha_2} &= e^{-i\frac{\omega}{2}t}c_0 \\
\beta_2 &= \phi_0 e^{-i\omega t} \\
\gamma_2 &= 0
\end{aligned}
$$

From this parametrization and what we have derived for squeezed states we see that

$$
\langle x|\psi(t)\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} c_0 e^{-i\frac{\omega}{2}t} \underbrace{e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} e^{\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x\phi_0 e^{-i\omega t}-\frac{1}{2}(\phi_0 e^{-i\omega t})^2}}_{\equiv I}
$$

With some algebra, I can be seen to equal to

$$
\begin{aligned}
I &= e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} e^{\sqrt{2}\sqrt{\frac{m\omega}{\hbar}}x\phi_0 e^{-i\omega t}-\frac{1}{2}(\phi_0 e^{-i\omega t})^2} \\
&= e^{-\frac{m\omega}{\hbar}\frac{x^2}{2}} e^{-\frac{1}{2}|\phi_0|^2\cos(2(\theta-\omega t))-i\frac{1}{2}|\phi_0|^2\sin(2(\theta-\omega t))+2\sqrt{\frac{m\omega}{2\hbar}}x\cos(\theta-\omega t)+i2\sqrt{\frac{m\omega}{2\hbar}}x|\phi_0|\sin(\theta-\omega t)} \\
&= c_0^{-1} e^{-\left(\sqrt{\frac{m\omega}{2\hbar}}x-|\phi_0|\cos(\theta-\omega t)\right)^2} e^{-i\frac{1}{2}|\phi_0|^2\sin(2(\theta-\omega t))+i2\sqrt{\frac{m\omega}{2\hbar}}x|\phi_0|\sin(\theta-\omega t)} \\
&= c_0^{-1} e^{-\frac{m\omega}{2\hbar}(x-\langle x(t)\rangle)^2} e^{-\frac{i}{\hbar}\frac{1}{2}\langle x(t)\rangle\langle p(t)\rangle+\frac{i}{\hbar}x\langle p(t)\rangle}
\end{aligned}
$$

where we used $\cos(2(\theta-\omega t)) = 2\cos^2(\theta-\omega t)-1$ at second step.
Carrying on:

$$
\begin{aligned}
\Rightarrow \langle x|\psi(t)\rangle &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}(x-\langle x(t)\rangle)^2} e^{-\frac{i}{\hbar}\frac{1}{2}\hbar\omega t-\frac{i}{\hbar}\frac{1}{2}\langle x(t)\rangle\langle p(t)\rangle+\frac{i}{\hbar}x\langle p(t)\rangle} \\
&\equiv \sqrt{\rho(x,t)}\, e^{\frac{i}{\hbar}S(x,t)} \\
\Rightarrow \sqrt{\rho(x,t)} &= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}(x-\langle x(t)\rangle)^2} \\
S(x,t) &= -\frac{1}{2}\hbar\omega t - \frac{1}{2}\langle x(t)\rangle\langle p(t)\rangle + x\langle p(t)\rangle
\end{aligned}
$$

We see that the *probability distribution*, $|\langle x|\psi(t)\rangle|^2$, keeps the original Gaussian shape and width for all times. The center, $\langle x(t)\rangle$ oscillates with time, and is independent from its momentum $\langle p(t)\rangle$. Furthermore, we have $\frac{\partial S}{\partial x} = \langle p(t)\rangle$.
On the other hand, for a free particle initially in a Gaussian state, its center moves at constant speed $\langle p\rangle/m$, and its width increases with time.