Lecture 4 - Quantization of the electromagnetic field

2.5 Quantization of the e.m. field

Take cubic box of side length L , volume $V=L^3$, with periodic boundary conditions:

$$k_{ ext{x, y, z}} = rac{2\pi}{L} n_{ ext{x, y, z}}$$

- ••• Therefore $lpha_\epsilon(ec k,t) o lpha_{ec k,\epsilon}(t)$ or simply $lpha_i$ $(i=(ec k_i,ec\epsilon_i)).$
 - Correspondence

$$\int \mathrm{d}^3 k \sum_{\epsilon} f(ec{k},ec{\epsilon}) \leftrightarrow \left(rac{2\pi}{L}
ight)^3 \Sigma_i f(ec{k}_i,ec{\epsilon}_i)$$

• Analogy with harmonic oscillator:

One e.m. field mode
$$\dot{\mathcal{A}}_i = -\mathcal{E}_i$$

$$\dot{\mathcal{E}}_i = \omega_i^2 \mathcal{A}_i$$

$$\mathcal{E}_i = \omega_i^2 \mathcal{A}_i$$

$$\mathcal{E}_i = \frac{p}{m}$$

$$\mathcal{E}_i = \frac{p}{m}$$

$$\mathcal{E}_i = \frac{p}{m}$$

$$\mathcal{E}_i = -\frac{p}{m}$$

$$\mathcal{E}_i = -\frac$$

Quantization and Commutation relations:

One e.m. field mode
$$\mathcal{A}_i o \hat{\mathcal{A}}_i \ \mathcal{E}_i o \hat{\mathcal{E}}_i \ \left[\hat{\mathcal{A}}_i, \hat{\mathcal{E}}_i
ight] = -rac{V}{(2\pi)^3\epsilon_0}i\hbar \ \hat{a}_i ext{ annihilation operator} \ ext{associated to } lpha_i \ \left[\hat{a}_i, \hat{a}_i^{\dagger}
ight] = 1 ext{ for} \ \mathcal{N} = \sqrt{rac{\epsilon_0\omega_i}{2\hbar}rac{(2\pi)^3}{V}}$$

$$egin{aligned} x &
ightarrow \hat{x} \ p &
ightarrow \hat{p} \ [\hat{x},\hat{p}] = i\hbar \end{aligned}$$

 \hat{a} annihilation operator associated to α

$$\left[\hat{a},\hat{a}^{\dagger}
ight]=1$$
 for

$$\mathcal{N}=\sqrt{rac{m\omega}{2\hbar}}$$

- Physical Operators:
- Hamiltonian:

$$egin{aligned} H_i &= rac{\hbar \omega_i}{2} \left(lpha_i^* lpha_i + lpha_i lpha_i^*
ight) &\left(= \hbar \omega_i |lpha_i|^2
ight) & H &= rac{\hbar \omega}{2} \left(lpha^* lpha + lpha lpha^*
ight) \ \hat{H}_i &= rac{\hbar \omega_i}{2} \left(\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger
ight) & \hat{H} &= rac{\hbar \omega}{2} \left(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger
ight) \ \hat{H} &= \sum_i \hbar \omega_i \left(\hat{a}_i^\dagger \hat{a}_i + rac{1}{2}
ight) & \hat{H} &= \hbar \omega \left(\hat{a}^\dagger \hat{a} + rac{1}{2}
ight) \end{aligned}$$

• Momentum:

$$ec{P} = \sum_i rac{\hbar ec{k}_i}{2} \left(lpha_i^*lpha_i + lpha_ilpha_i^*
ight) \ ec{P} = \sum_i rac{\hbar ec{k}_i}{2} \left(\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i\hat{a}_i^\dagger
ight) \ ec{P} = \sum_i \hbar ec{k}_i \left(\hat{a}_i^\dagger \hat{a}_i + rac{1}{2}
ight) \ \mathrm{but} \ \sum_i ec{k}_i = 0 \ \mathrm{so} \ \hat{ec{P}} = \sum_i \hbar ec{k}_i \hat{a}_i^\dagger \hat{a}_i \ ec{a}_i^\dagger \hat{a}_i$$

• Electric Field:

$$\hat{ec{E}}(ec{r}) = i \sum_{i} \mathcal{E}_{i} \left(ec{\epsilon}_{i} \hat{a}_{i} e^{i ec{k}_{i} ec{r}} - ec{\epsilon}_{i} \hat{a}_{i}^{\dagger} e^{-i ec{k}_{i} ec{r}}
ight)$$

with
$$\mathcal{E}_i = \sqrt{rac{\hbar \omega_i}{2\epsilon_0 V}}$$

Comment:

Within the Lagrangian formalism one sees that the momentum conjugate to $\mathcal{A}_{\perp\epsilon}$ is $\Pi_{\epsilon}=\epsilon_0\dot{\mathcal{A}}_{\perp\epsilon}=-\epsilon_0\mathcal{E}_{\perp\epsilon}$. The canonical commutation relations are then

$$\left[{\cal A}_{\epsilon}(ec{k}), \Pi^{\dagger}_{\epsilon'}(ec{k}')
ight] = i\hbar \delta_{\epsilon\epsilon'} \delta(ec{k} - ec{k}')$$

This agrees with $[\; {\cal A}_i, {\cal E}_j] = -rac{V}{(2\pi)^3} rac{i\hbar}{\epsilon_0} \delta_{ij}$ as

$$1=\int \mathrm{d}^3k\,\delta(ec{k}-ec{k}')\leftrightarrow rac{(2\pi)^3}{V}\sum_krac{V}{(2\pi)^3}\delta_{kk'}$$
 or $\delta(ec{k}-ec{k}')\leftrightarrow rac{V}{(2\pi)^3}\delta_{kk'}$

2.6 Total Hamiltonian and Momentum:

$$H = \sum_lpha rac{1}{2m_lpha} \left(ec p_lpha - q_lpha ec A_ot (ec r_lpha)
ight)^2 + \sum_lpha \left(-g_lpha rac{q_lpha}{2m_lpha}
ight) ec S_lpha \cdot ec B(ec r_lpha)$$

\$\$+V_{\rm Coulomb} + H_{\rm R}\$\$

 $\$ Coulomb}=\sum_\alpha\epsilon^\alpha_{\rm Coulomb}+\frac{1} {8\pi_0}\

 $\$ coulomb}^\alpha = $\frac{q_\alpha^2}{2\epsilon_0^2}^3$ \intk \frac{1} {k^2} = \frac{q_\alpha^2}{4\pi^2}^2

 $H_{\rm R} = \frac{0}{2} \int_0^2 \int_0^2 |\int_0^2 |\int_0^2 |\int_0^2 |\int_0^2 \int_0^2 \int_0^2$

Total Momentum:

 $s=\sqrt{P} = \sum_{\alpha + \beta} + \sqrt{P}_{\rm R}$

 $\$ \vec{P}_{\rm R} = \sum_i \hbar \vk_i\, \hat{a}_i^\dagger \hat{a}_i\$\$

 $H = H_{\rm P} + H_{\rm R} + H_{\rm I}$

Particle Hamiltonian: $$H_{\rm P} = \sum_{\alpha \in \mathbb{N}} + V_{\rm D} = \sum_{\alpha \in \mathbb{N}$

Interaction:

$$H_{\rm I} = H_{\rm II} + H_{\rm II} + H_{\rm II}^S$$

 $H_{\rm II} = - \sum_{\alpha \in \{q_\alpha\}}{m_\alpha} \$

 $\H_{\rm I1}^S = -\sum_{\alpha \in \{q_\alpha\}}{2 m_\alpha} \$ \cdot \vec{B}(\vr_\alpha)\$\$

 $H_{\rm 12} = \sum_{\alpha_2} \frac{q_\alpha}{2m_\alpha} \$

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