(1)

(82) Pf $R(m,n) \leq \underbrace{R(m-1,n)}_{A} + \underbrace{R(m,n-1)}_{B}$

→ Look at complete graph of $A+B$ vertices. Pick vertex X

Let R be the set of vertices connected to X by Red
G ... ... by Green

Then $A+B = |R| + |G| + 2$.

$\Rightarrow |R| \geq A$ or $|G| \geq B$.

If $|R| \geq A$ then → there is a Green K_n
or
→ together with X there is a red $K_{m-1+1} = K_m$

Similarly, for $|G| \geq B$ we have either a Red K_m
or Green $K_{n-1+1} = K_n$
(with X)

So $R(m,n) \leq R(m-1,n) + R(m,n-1)$ as desired.

by symmetry

83 a $R(4,4) \le R(3,4) + R(4,3) = 9 + 9 = \boxed{18}$

b 17 people.

number the people ccw from 0 to 16.

~~Suppose~~ ~~Contradiction~~ Suppose we can find a group of 4 mutual acq. Let 0 be in this group.

By symmetry, ~~the~~ ~~of~~ two other members of this ~~must be~~ are in $\{1, 2, 4, 8\}$.

Since 1 doesn't know 4 and so, we only actually have the following 3 possibilities for the first 3 people:

0 1 2

0 2 4

0 4 8

Can we find the fourth person? $\boxed{\text{NO}}$

0	knows	$\{1, 2, 4, 8, 9, 13, 15, 16\}$
1	knows	$\{2, 3, 5, 9, 10, 14, 16, 0\}$
2	knows	$\{3, 4, 6, 10, 11, 15, 0, 1\}$
4	knows	$\{5, 6, 8, 12, 13, 0, 1, 3\}$
8	knows	$\{9, 10, 12, 16, 0, 3, 5, 6\}$

For non-acquaintances: 0 doesn't know $\{3, 5, 6, 7\}$

→ 1 possibility for first three non acqs: 0 3 6

by previous method

But, ~~they all know~~ we find ~~that~~ this is $\boxed{\text{NOT possible}}$ either

c $\boxed{R(4,4) = 17 + 1 = 18}$

by part a & $R(4,4) > 17$ by b.

84 base case $m = n = 2$ (already proved last time)

Suppose true for $4 \leq m+n < k$.

To prove: true for $m+n = k$

By induction hypothesis,

$$R(m-1, n) \leq \binom{m+n-3}{m-2}$$

$$R(m, n-1) \leq \binom{m+n-3}{m-1}$$

$$\underset{\text{by 82}}{R(m,n) \leq} R(m-1, n) + R(m, n-1) \leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}$$

86 (a) if no coloring $N \leq M$ s.t. there is no monochromatic K_N ~~then~~ $\forall K_M$, then we have that there is no monochromatic K_N anywhere in K_M, so

$R(\underset{\text{red}}{n}, \underset{\text{or blue}}{n})$ ~~then~~ $> \#\,|K_M| = m$ $\Rightarrow$ $\boxed{R(n,n) > m}$

(b) If average < 1, then over all colorings of ~~K_m~~, there is at least 1 coloring which results in K_N not monochromatic for any N (This is obvious?)

$\rightarrow$ This implies $R(n,n) > m$ because then we can think of it as

If this average < 1 ~~as~~ the probability that ~~no~~ a monochromatic K_N exists, ~~and so~~ which means ~~that if the probability is < 1,~~

~~that~~ that there is a ~~...~~ coloring that gives no monochromatic K_N, and so $R(n,n) > m$.

(c) $$\text{mono}(c, N) = \begin{cases} 1 & N \text{ monochromatic for } c \\ 0 & \text{else.} \end{cases}$$

__Formula__: # colorings of $K_m = 2^{\binom{m}{2}}$

$$\boxed{\text{Average} = \sum_{c \text{ of } K_m} \sum_{\substack{\text{subsets} \\ N \text{ of} \\ \text{mono}(c,N)}} \frac{\text{mono}(c,N)}{\left\{\sum_{c \text{ of } K_m} 1\right\}}}$$

(d) $$\text{Average} = \frac{\sum_c \sum_{N \in \text{mono}(c,N)} \text{mono}(c,N)}{2^{\binom{m}{2}}}$$

$$= \frac{2 \cdot \binom{m}{n}}{2^{\binom{m}{2}}}$$

choose n out of m

2 ways of making it monochromatic.

Total # ways to color K_m.

$$= \boxed{2\binom{m}{n} 2^{-\binom{m}{2}}}$$

(e) $R(n,n) > m$ if this average is ≤ 1, i.e. $\binom{m}{n} \leq 2^{\binom{n}{2}-1}$

~~(f) ...~~ (f) next page...

(g) From (f) we have $f(n,n) > \sqrt[n]{n!\, 2^{\binom{n}{2}-1}}$

$\binom{n}{2} = \frac{n(n-1)}{2}$ goes like $\frac{n^2}{2}$.

get $\left[\sqrt{2\pi n}\; n^n e^{-n} \left(\frac{1}{2}\right)\left(\sqrt{2}\right)^{n^2}\right]^{1/n}$

as $n \to \infty$, $\left(\sqrt{2\pi n}\right)^{1/n} \to 1$

$\left(1/2\right)^{1/n} \to 1$

We get

$\sim \left(\frac{n}{e}\right) \cdot \left(\sqrt{2}\right)^n > \left(\sqrt{2}\right)^n$.

So $R(n,n) > \left(\sqrt{2}\right)^n$ as desired.

89 $S_n = 2S_{n-1}$. Solution to this is $S_n = 2^n \cdot K$ where K ~~[illegible]~~ is any number.

But by setting $S_0 = 2^0 \cdot K = 1 \Rightarrow K = 1$.

$\Rightarrow$ only $S_n = 2^n$ solves $\begin{cases} S_n = 2S_{n-1} \\ S_0 = 1 \end{cases}$

90 (a) 1^{st} order recurrence: $(2\cdot1), (2\cdot2), (2\cdot3), (2\cdot4),$

(i) Suppose 1^{st} order recurrence is $a_n = f(n)a_{n-1} + g(n)$ with $a_0 = a$ ~~[illegible]~~

~~Suppose $a_m = f'(m) a_{m-1} + g'(m) = f(m) a_{m-1} + g(m)$~~

~~Suppose we have a sequence b_n for which~~

$$\cancel{b_n = f(n)\, b_{n-1} + g(n)}$$

Suppose we have another sequence b_n for which

$$b_n = f(n)\, b_{n-1} + g(n) \quad , \quad b_0 = a$$

~~[illegible]~~ then

$$b_n = [f(n)]^n\, b_0 + g(n) \sum_{i=0}^{n-1} f^i(n)$$

$$= [f(n)]^n\, a + g(n) \sum_{i=0}^{n-1} f^i(n)$$

$$= a_n \quad \forall\, n \in \mathbb{N}^*$$

So $\{b_n\} \equiv \{a_n\}$.

"semi-colon"

95

$$\cancel{a_n = (50000)(n+1) + 3000\,n}$$

- $\boxed{a_n = a_{n-1} + 3000 \;;\; a_0 = 50000}$
- $\text{Total} = \sum_{i=0}^{n} a_i = 50000(n+1) + 3000\left(\sum_{i=0}^{n} i\right)$

$$= \boxed{50000(n+1) + 3000\,\frac{n(n+1)}{2}}$$

(96) $S_n = a_0 + \cdots + a_n$

where $\begin{cases} a_n = a_{n-1} + c \\ a_0 = a_0 \end{cases} \Rightarrow a_n = a_0 + c \cdot n$

$$S_n = \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_0 + ci = \boxed{a_0(n+1) + \frac{c(n)(n+1)}{2}}$$

(102) (a) $\boxed{\sum_{v} \deg(v) = 2\,\#\text{edges}}$

(b) if 1 edge →  or  then sum of degrees is indeed 2 ✓

BC

IH suppose true for k edges. Consider finite graph with $k+1$ edges ⇒ ~~...~~ an extra edge adds 2 to the total of degrees ()

so new sum $= \sum_{\checkmark} (\deg v) + 2 = 2[\#\text{edges} + 1]$

$= 2(k+1)$ ✓.

(c) if 1 vertex : •

BC adding one edge = adds 2 degrees ✓

IH suppose true for n vertices. Now add 1 vertex to graph. suppose there are k edges to this vertex then

new sum → $\left(\sum_{v'} \deg(v')\right)' = \underbrace{\sum_{v} \deg(v)}_{\text{old sum}} + 2k = 2\,\#\text{old edges} + 2k$

$= 2(\#\text{old edges} + \underbrace{k}_{\text{new edges}})$ ✓

d) No induction...

Each edge connects exactly 2 vertices

$\Rightarrow$ each edge contributes exactly 2 to the sum of degrees...

103) $\boxed{\text{\# vertices of odd degree is even}}$

Pf

$$\underbrace{\sum_{v \in G} \deg(v)}_{\substack{\uparrow \\ \text{total} \\ \alpha\, 2 \cdot E \\ \text{(even)}}} = \underbrace{\sum_{\substack{v \in G \\ \deg(v) \\ \text{even}}} \deg(v)}_{\substack{\text{also even} \\ \text{because} \\ \deg(v) \text{ even} \\ \forall v \text{ here.}}} + \sum_{\substack{v \in G \\ \deg(v) \\ \text{odd}}} \deg(v)$$

$\Rightarrow \sum_{\substack{v \in G \\ \deg(v) \\ \text{odd}}} \deg(v)$ is also even.

This is true only if there is an even number of terms here

$\Rightarrow$ # vertices of odd deg is even.

(a) $\boxed{2n+2}$

Induction: CH_4 base case ✓

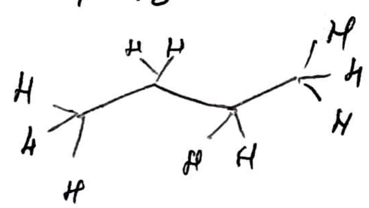
IH: Formula is C_nH_{2n+2} for $n \le k$

Look at $n = k+1$. An additional C breaks 2 C–H bonds and creates 3 $\Rightarrow$ +2 C–H bonds

→ new formula

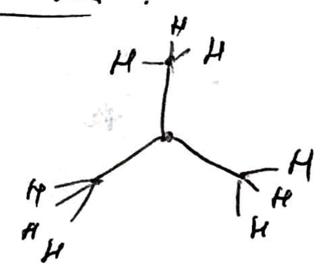
$$= C_{n+1}H_{2n+2+2} = C_{(n+1)}H_{2(n+1)+2}$$ ✓

(b) C_4H_{10}



(butane)

here's another:



(Iso-butane)

4 (a) $2n \to$ sets of 2 for tennis ...

Singly 4 p 53

$$\boxed{a_n = \frac{1}{n} \binom{2n}{2} \cdot a_{n-1}}$$

$$a_n = \frac{1}{n!}\binom{2n}{2}\binom{2(n-1)}{2}\cdots\binom{2}{2}$$

"permute" the groups further

pick 2 out of n

pick teams for $2(n-1)$

(b) $$a_n' = \frac{2^n}{n!}\binom{2n}{2}\binom{2(n-1)}{2}\cdots\binom{2}{2}$$ (server)

$$\Rightarrow \boxed{a_n' = \left(\frac{2}{n}\right)\binom{2n}{2} \cdot a_{n-1}'}$$