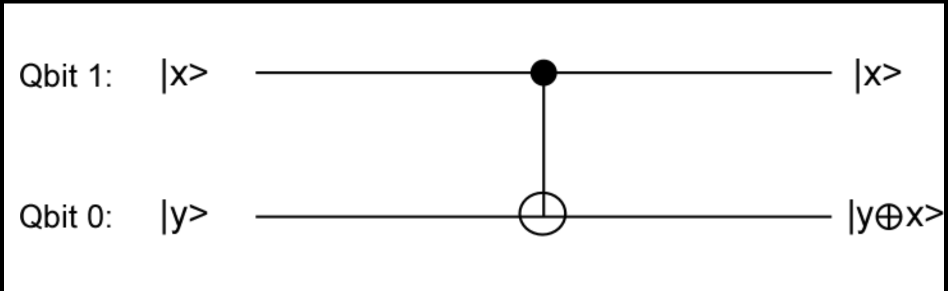
B. Phase oracles and “phase kickback”

Background: The Deutsch problem, the Deutsch-Jozsa problem, and the Bernstein-Vazirani problem are all problems that try to uncover the behavior of an “oracle.” The quantum algorithms that solve these problems with certainty after one query all use the technique of phase kickback.

Goal: To show how phase kickback works so that we can look at the other problems from a higher level.

1. Controlled-NOT (cNOT)

$$f : \{0,1\} \to \{0,1\}$$

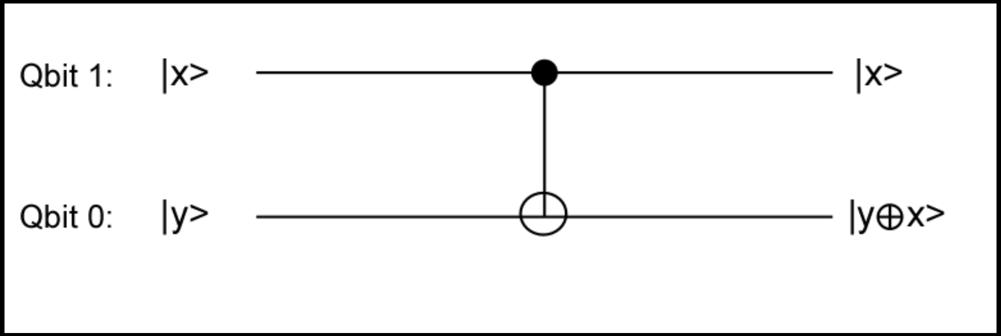


$$\mathbf{C}_{10}|x\rangle|y\rangle = |x\rangle|y \oplus x\rangle$$

Examples:

$$f : \{0,1\} \to \{0,1\}$$

$$\mathbf{C}_{10}|x\rangle|y\rangle = |x\rangle|y \oplus x\rangle$$



$$\begin{aligned}\mathbf{C}_{10}|x\rangle|0\rangle &= |x\rangle|0 \oplus x\rangle \\ &= |x\rangle|x\rangle\end{aligned}$$

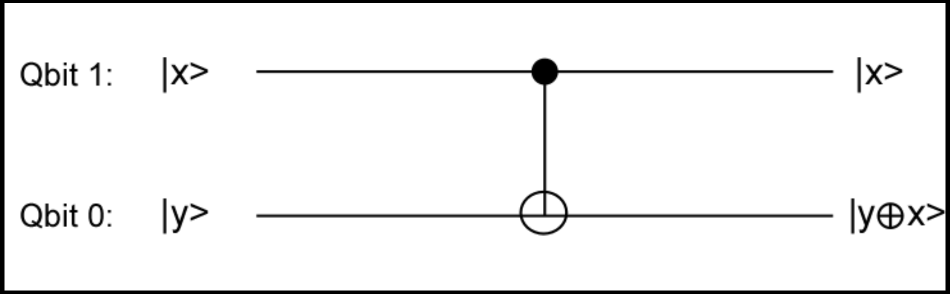
$$\begin{aligned}\mathbf{C}_{10}|x\rangle|1\rangle &= |x\rangle|1 \oplus x\rangle \\ &= |x\rangle|\bar{x}\rangle\end{aligned}$$

Examples:

$$f : \{0,1\} \rightarrow \{0,1\}$$

$$\mathbf{C}_{10}|x\rangle|0\rangle = |x\rangle|x\rangle$$
$$\mathbf{C}_{10}|x\rangle|1\rangle = |x\rangle|\bar{x}\rangle$$

$$|y\rangle \rightarrow |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$$



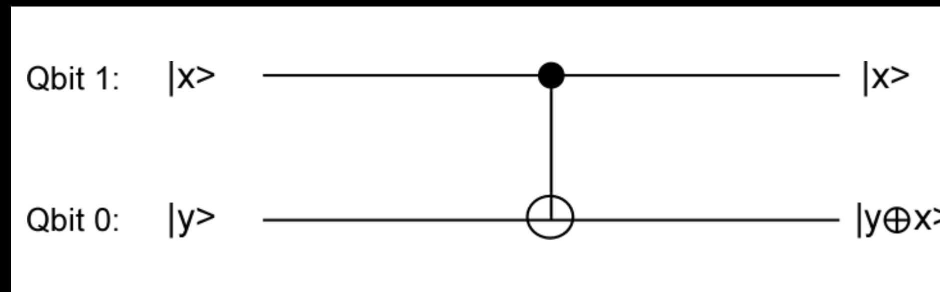
$$\begin{aligned}
\mathbf{C}_{10}|x\rangle|+\rangle &= \mathbf{C}_{10}|x\rangle\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\mathbf{C}_{10}|x\rangle|0\rangle + \mathbf{C}_{10}|x\rangle|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|x\rangle|x\rangle + |x\rangle|\bar{x}\rangle\right) \\
&= |x\rangle|+\rangle
\end{aligned}$$

Examples:

$$f : \{0,1\} \rightarrow \{0,1\}$$

$$\mathbf{C}_{10}|x\rangle|0\rangle = |x\rangle|x\rangle$$
$$\mathbf{C}_{10}|x\rangle|1\rangle = |x\rangle|\bar{x}\rangle$$

$$|y\rangle \rightarrow |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$\begin{aligned}
\mathbf{C}_{10}|x\rangle|-\rangle &= \mathbf{C}_{10}|x\rangle\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(\mathbf{C}_{10}|x\rangle|0\rangle - \mathbf{C}_{10}|x\rangle|1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(|x\rangle|x\rangle - |x\rangle|\bar{x}\rangle\right) \\
&= (-1)^{x}|x\rangle|-\rangle
\end{aligned}$$

Examples:

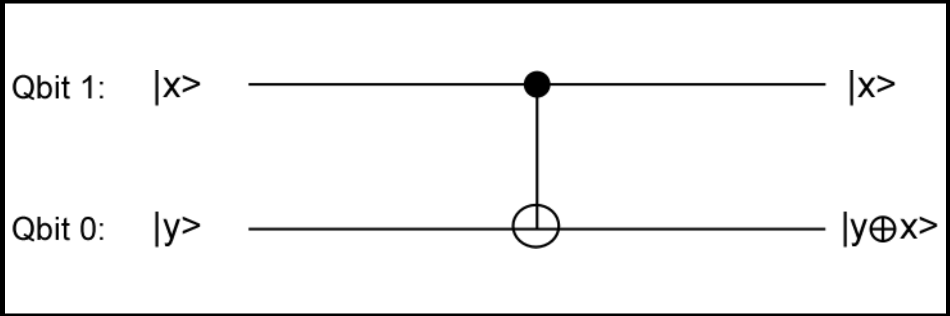
$$f : \{0,1\} \to \{0,1\}$$

$$\mathbf{C}_{10}|0\rangle|-\rangle = |0\rangle|-\rangle$$

$$\mathbf{C}_{10}|1\rangle|-\rangle = -|1\rangle|-\rangle$$

$$|x\rangle \to |\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

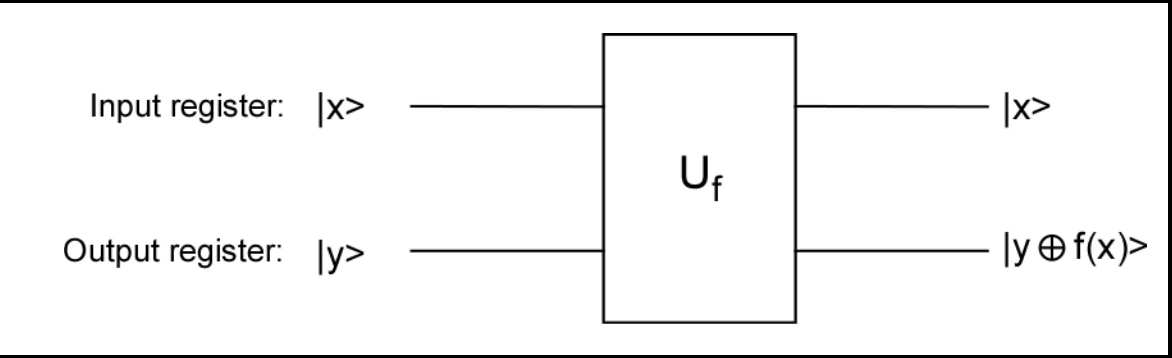
$$|y\rangle \to |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$\begin{aligned}
\mathbf{C}_{10}|\alpha\rangle|-\rangle &= \mathbf{C}_{10}\left\{\alpha_0|0\rangle + \alpha_1|1\rangle\right\}|-\rangle \\
&= \left\{\alpha_0\mathbf{C}_{10}|0\rangle|-\rangle + \alpha_1\mathbf{C}_{10}|1\rangle|-\rangle\right\} \\
&= \left\{\alpha_0|0\rangle|-\rangle - \alpha_1|1\rangle|-\rangle\right\} \\
&= \left\{\alpha_0|0\rangle - \alpha_1|1\rangle\right\}|-\rangle
\end{aligned}$$

2. Unitary for a 1-bit function f(x)

$$f : \{0, 1\} \rightarrow \{0, 1\}$$

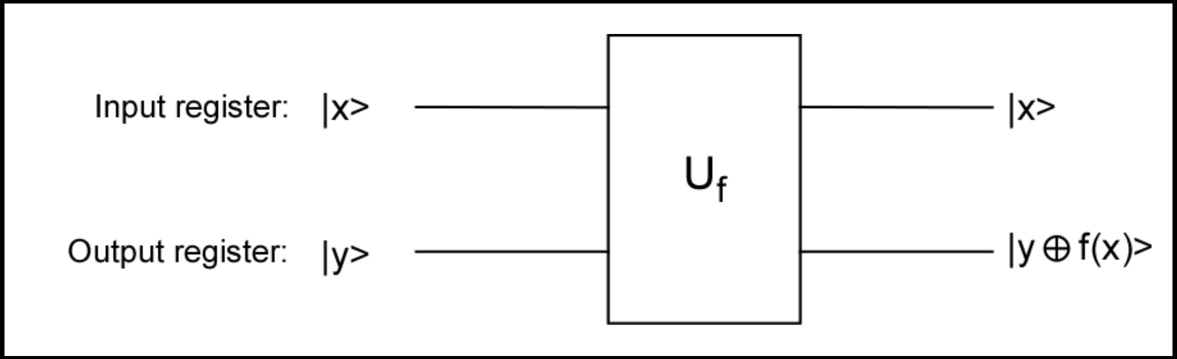


$$\mathbf{U}_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

Examples:

$$f : \{0,1\} \rightarrow \{0,1\}$$

$$\mathbf{U}_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$



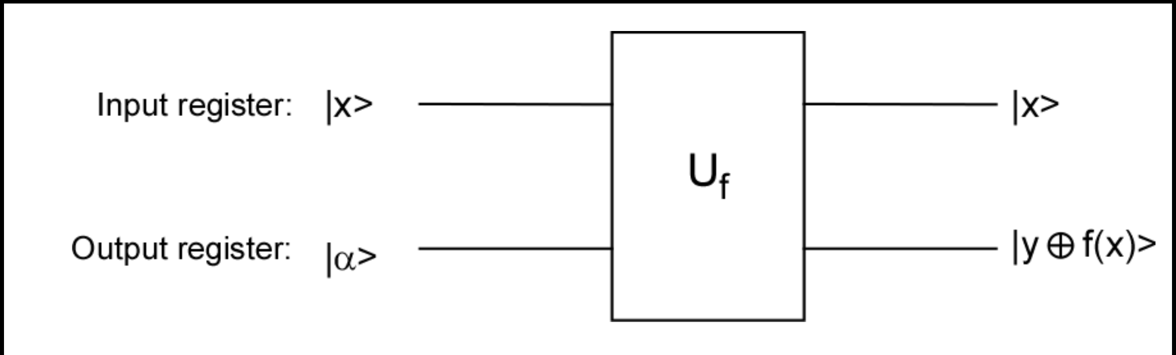
$$\begin{aligned}\mathbf{U}_f|x\rangle|0\rangle &= |x\rangle|0 \oplus f(x)\rangle \\ &= |x\rangle|f(x)\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{U}_f|x\rangle|1\rangle &= |x\rangle|1 \oplus f(x)\rangle \\ &= |x\rangle|\overline{f(x)}\rangle\end{aligned}$$

Examples:

$$f : \{0,1\} \to \{0,1\} \qquad \mathbf{U}_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

$$|y\rangle \to |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



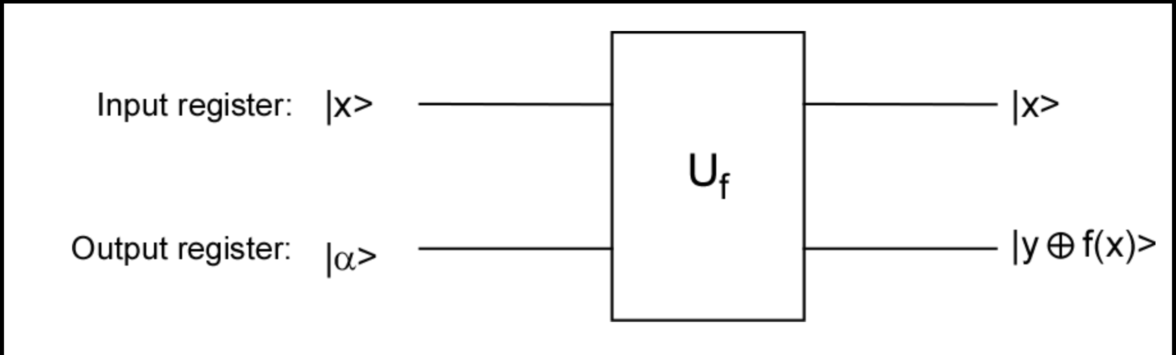
$$\mathbf{U}_f|x\rangle|-\rangle = \frac{1}{\sqrt{2}}\{\mathbf{U}_f|x\rangle|0\rangle - \mathbf{U}_f|x\rangle|1\rangle\}$$
$$= \frac{1}{\sqrt{2}}\{|x\rangle|0 \oplus f(x)\rangle - |x\rangle|1 \oplus f(x)\rangle\}$$

$$\mathbf{U}_f|x\rangle|-\rangle = \begin{cases} \frac{1}{\sqrt{2}}\{|x\rangle|0\rangle - |x\rangle|1\rangle\} & f(x) = 0 \\ \frac{1}{\sqrt{2}}\{|x\rangle|1\rangle - |x\rangle|0\rangle\} & f(x) = 1 \end{cases}$$

Examples:

$$f : \{0,1\} \to \{0,1\} \qquad \mathbf{U}_f|x\rangle|y\rangle = |x\rangle|y \oplus f(x)\rangle$$

$$|y\rangle \to |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$\mathbf{U}_f|x\rangle|-\rangle = \begin{cases} \frac{1}{\sqrt{2}}\{|x\rangle|0\rangle - |x\rangle|1\rangle\} & f(x) = 0 \\ -\frac{1}{\sqrt{2}}\{|x\rangle|0\rangle - |x\rangle|1\rangle\} & f(x) = 1 \end{cases}$$

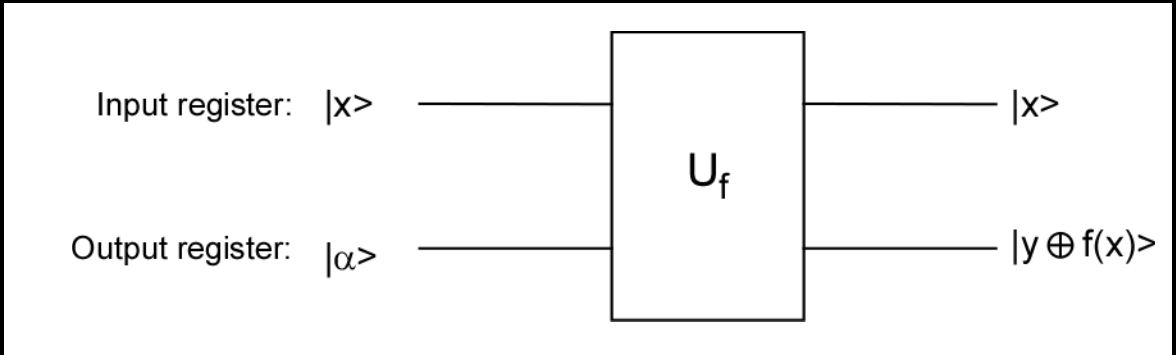
$$\boxed{\mathbf{U}_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle}$$

Examples:

$$f : \{0,1\} \to \{0,1\} \qquad \mathbf{U}_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$$

$$|x\rangle \to |\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

$$|y\rangle \to |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$\mathbf{U}_f|\alpha\rangle|-\rangle = \alpha_0(-1)^{f(0)}|0\rangle|-\rangle + \alpha_1(-1)^{f(1)}|1\rangle|-\rangle$$
$$= \left\{\alpha_0(-1)^{f(0)}|0\rangle + \alpha_1(-1)^{f(1)}|1\rangle\right\}|-\rangle$$

Examples:

$$f : \{0,1\} \rightarrow \{0,1\} \qquad \mathbf{U}_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$$

$$|x\rangle \rightarrow |\alpha\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

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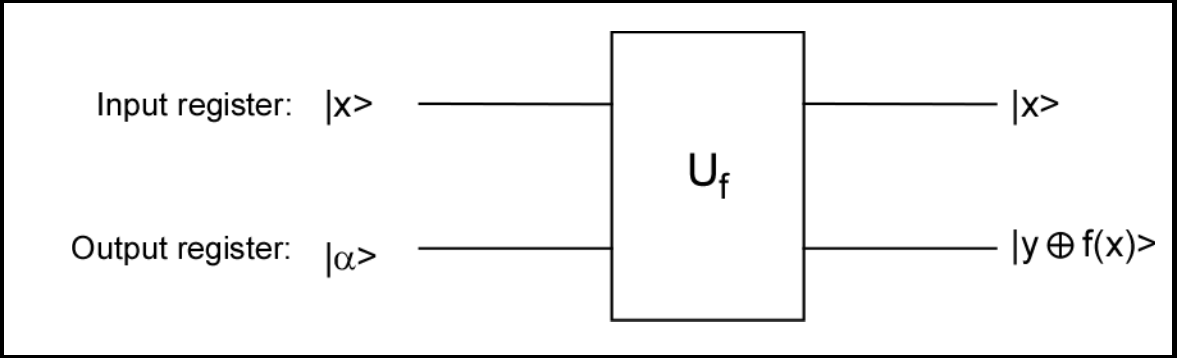


$$\mathbf{U}_f|\alpha\rangle|-\rangle = \alpha_0(-1)^{f(0)}|0\rangle|-\rangle + \alpha_1(-1)^{f(1)}|1\rangle|-\rangle$$
$$= \left\{\alpha_0(-1)^{f(0)}|0\rangle + \alpha_1(-1)^{f(1)}|1\rangle\right\}|-\rangle$$

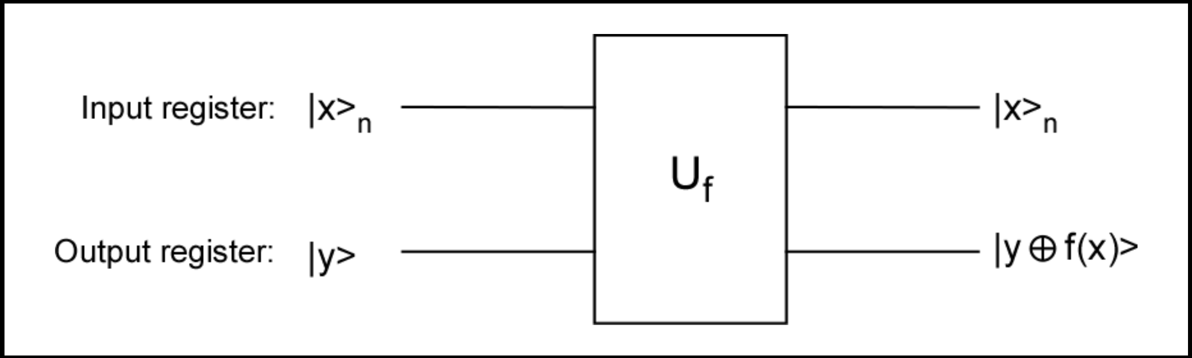
In Deutsch's problem both α's are $1/\sqrt{2}$

$$\mathbf{U}_f|\alpha\rangle|-\rangle = \begin{cases} \pm|+\rangle|-\rangle & f(x) = \text{constant} \\ \pm|-\rangle|-\rangle & f(x) = \text{balanced} \end{cases}$$

Applying a Hadamard to the input register gives $|0\rangle$ for a constant function and $|1\rangle$ for a balanced function.

3. Unitary for a n-bit to 1-bit function f(x)

$$f : \{0,1\}^n \rightarrow \{0,1\}$$

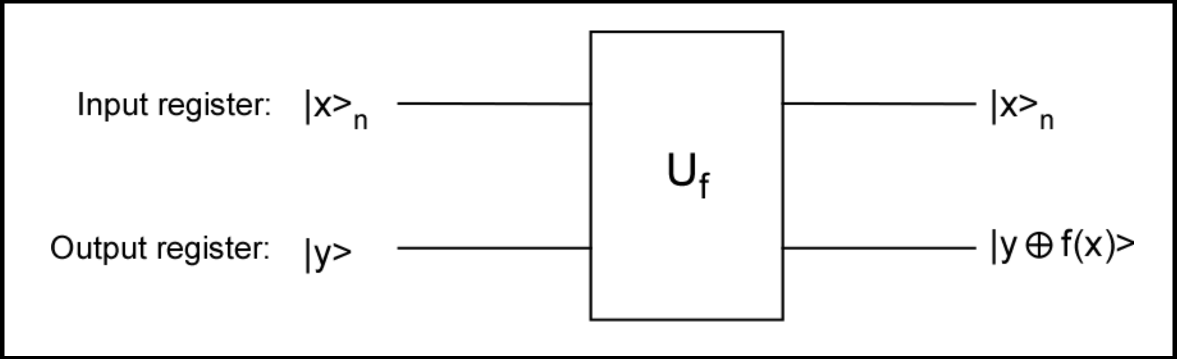


$$\mathbf{U}_f|x\rangle_n|y\rangle = |x\rangle_n|y \oplus f(x)\rangle$$

A 1-bit value

Examples:

$$f : \{0,1\}^n \to \{0,1\} \qquad \mathbf{U}_f|x\rangle_n|y\rangle = |x\rangle_n|y \oplus f(x)\rangle$$



$$\begin{aligned}\mathbf{U}_f|x\rangle_n|0\rangle &= |x\rangle_n|0 \oplus f(x)\rangle\\ &= |x\rangle_n|f(x)\rangle\end{aligned}$$

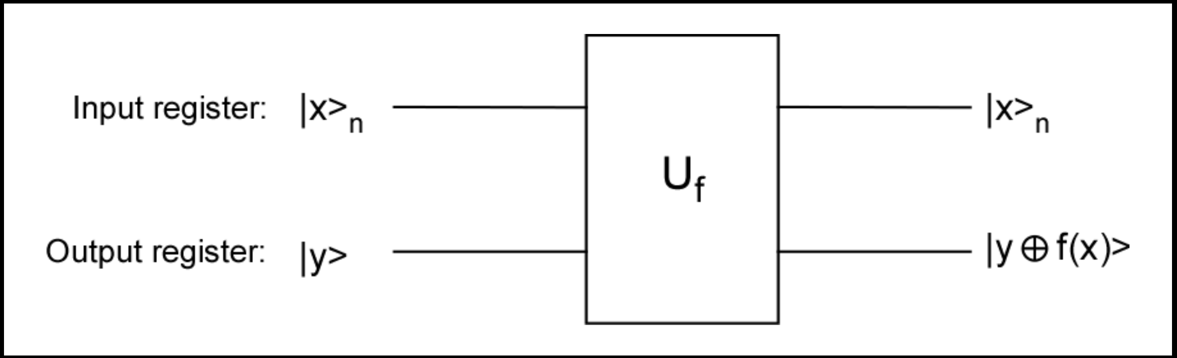
$$\begin{aligned}\mathbf{U}_f|x\rangle_n|1\rangle &= |x\rangle_n|1 \oplus f(x)\rangle\\ &= |x\rangle_n|\overline{f(x)}\rangle\end{aligned}$$

Exactly the same behavior as for a single Qbit input register

Examples:

$$f : \{0,1\}^n \to \{0,1\} \qquad \mathbf{U}_f|x\rangle_n|y\rangle = |x\rangle_n|y \oplus f(x)\rangle$$

$$|y\rangle \to |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



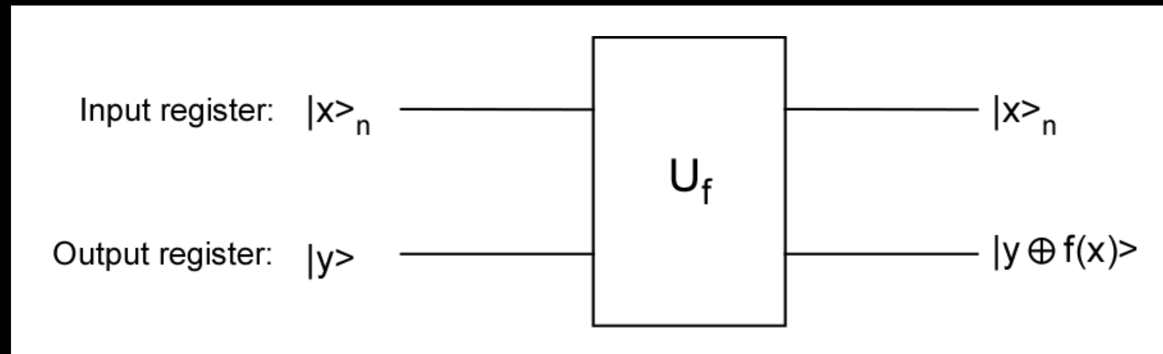
$$\mathbf{U}_f|x\rangle_n|-\rangle = \frac{1}{\sqrt{2}}\{\mathbf{U}_f|x\rangle_n|0\rangle - \mathbf{U}_f|x\rangle_n|1\rangle\}$$
$$= \frac{1}{\sqrt{2}}\{|x\rangle_n|0 \oplus f(x)\rangle - |x\rangle_n|1 \oplus f(x)\rangle\}$$

$$\mathbf{U}_f|x\rangle_n|-\rangle = \begin{cases} \frac{1}{\sqrt{2}}\{|x\rangle_n|0\rangle - |x\rangle_n|1\rangle\} & f(x) = 0 \\ \frac{1}{\sqrt{2}}\{|x\rangle_n|1\rangle - |x\rangle_n|0\rangle\} & f(x) = 1 \end{cases}$$

Examples:

$$f : \{0,1\}^n \to \{0,1\} \qquad \mathbf{U}_f|x\rangle_n|y\rangle = |x\rangle_n|y \oplus f(x)\rangle$$

$$|y\rangle \to |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$\mathbf{U}_f|x\rangle_n|-\rangle = \begin{cases} \frac{1}{\sqrt{2}}\{|x\rangle_n|0\rangle - |x\rangle_n|1\rangle\} & f(x) = 0 \\ -\frac{1}{\sqrt{2}}\{|x\rangle_n|0\rangle - |x\rangle_n|1\rangle\} & f(x) = 1 \end{cases}$$

$$\boxed{\mathbf{U}_f|x\rangle_n|-\rangle = (-1)^{f(x)}|x\rangle_n|-\rangle}$$

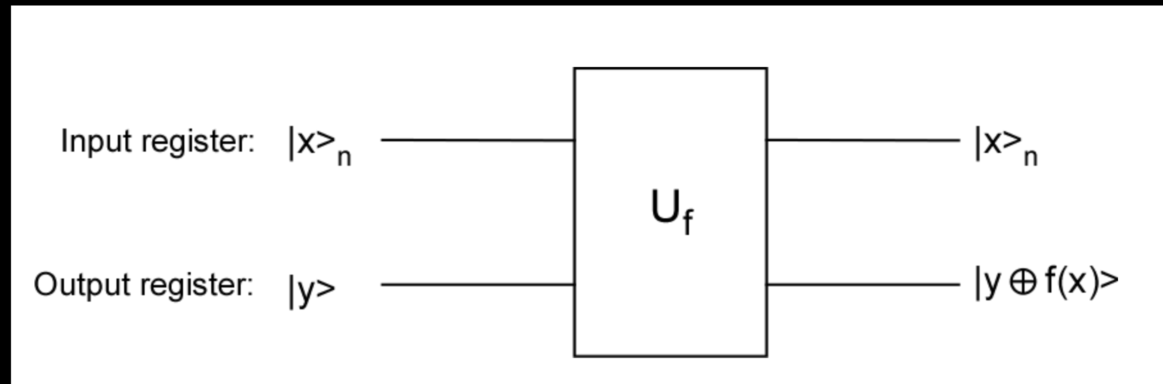
Exactly the same behavior as for a 1-Qbit input register

Examples:

$$f : \{0,1\}^n \to \{0,1\} \qquad \mathbf{U}_f|x\rangle_n|y\rangle = |x\rangle_n|y \oplus f(x)\rangle$$

$$|x\rangle \to |\psi\rangle = \sum_{0 \le x < 2^n} \alpha_x |x\rangle$$

$$|y\rangle \to |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



$$\mathbf{U}_f|\psi\rangle|-\rangle \sum_{0 \le x < 2^n} \alpha_x \mathbf{U}_f|x\rangle|-\rangle$$

$$\mathbf{U}_f|\psi\rangle|-\rangle = \sum_{0 \le x < 2^n} \alpha_x (-1)^{f(x)} |x\rangle|-\rangle$$

$$= \left\{ \sum_{0 \le x < 2^n} \alpha_x (-1)^{f(x)} |x\rangle \right\} |-\rangle$$