Midterm #2

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7 <u>Poisson brackets</u>

(a) $I = \int d\Gamma\, A \{B, C\}$

$$= \int d\Gamma\, A \left\{ \frac{\partial B}{\partial \vec{q}} \frac{\partial C}{\partial \vec{p}} - \frac{\partial B}{\partial \vec{p}} \frac{\partial C}{\partial \vec{q}} \right\}$$

Integration by parts $\rightarrow$
$$= \int d\Gamma \left\{ -\frac{\partial C}{\partial \vec{p}} \cdot \frac{\partial A}{\partial \vec{q}} \cdot B + B \frac{\partial A}{\partial \vec{p}} \frac{\partial C}{\partial \vec{q}} \right\}$$

$$= \int d\Gamma\, B \left\{ \frac{\partial C}{\partial \vec{q}} \frac{\partial A}{\partial \vec{p}} - \frac{\partial C}{\partial \vec{p}} \cdot \frac{\partial A}{\partial \vec{q}} \right\}$$

$$= \int d\Gamma\, B \{C, A\} \quad \checkmark$$

$\square$

(b) If $C(\Gamma) = F(A(\Gamma))$ then

$$\int d\Gamma\, A \{B, C\} = \int d\Gamma\, B \{C, A\} = \int d\Gamma\, B \{F(A), A\}$$
$$= 0$$

part (a)

Since
$$\{F(A), A\} = \frac{\partial F(A)}{\partial \vec{q}} \cdot \frac{\partial A}{\partial \vec{p}} - \frac{\partial F(A)}{\partial \vec{p}} \cdot \frac{\partial \vec{A}}{\partial \vec{q}}$$

$$= F' \frac{\partial A}{\partial \vec{q}} \cdot \frac{\partial A}{\partial \vec{p}} - F' \frac{\partial A}{\partial \vec{p}} \cdot \frac{\partial A}{\partial \vec{q}} = 0 .$$

$\square$

Note that we could also do part (a) by using a Poisson bracket identity ... (Leibniz rule)

$$I = \int d\Gamma \, A \{B, C\}$$

$$= \int d\Gamma \left(\{AB, C\} - \{A, C\} B \right)$$

Now $\int d\Gamma \{AB, C\}$

$$= \int d\Gamma \, \frac{\partial (AB)}{\partial \vec{q}} \frac{\partial C}{\partial \vec{p}} - \frac{\partial (AB)}{\partial \vec{p}} \cdot \frac{\partial C}{\partial \vec{q}}$$

$$(\text{.. by parts}) = -\int d\Gamma \, AB \frac{\partial^2 C}{\partial \vec{p} \partial \vec{q}} + AB \frac{\partial C}{\partial \vec{p} \partial \vec{q}} = 0$$

So

$$I = \int d\Gamma \, A \{B, C\} = -\int d\Gamma \, B \{A, C\}$$

$$= \int d\Gamma \, B \{C, A\}$$

c $$\partial_t \rho = \{\mathcal{H}, \rho\}$$

$$S(t) = -\int d\Gamma\, \rho(\Gamma,t) \ln \rho(\Gamma,t)$$

then $$\frac{dS}{dt} = \frac{\partial S}{\partial t} = -\int d\Gamma\, (\partial_t \rho) \ln \rho(\Gamma,t) + \frac{\rho(\Gamma,t)}{\rho(\Gamma,t)} (\partial_t \rho(\Gamma,t))$$

$$= -\int d\Gamma\, (\partial_t \rho)(\ln \rho(\Gamma,t) + 1)$$

$\int d\Gamma \{\rho, \mathcal{H}\} = 0$ by integration by parts

$$= -\int d\Gamma\, \{\mathcal{H}, \rho\}(\ln \rho(\Gamma,t) + 1)$$

$$= -\int d\Gamma\, \{\mathcal{H}, \rho\} \ln \rho(\Gamma,t)$$

Assuming

$\mathcal{H} = \vec{P}^2/2m + \mathcal{U}$

$$= -\int d\Gamma\, (\ln \rho) \left\{ \frac{\partial \mathcal{U}}{\partial \vec{q}} \cdot \frac{\partial \rho}{\partial \vec{p}} - \frac{\vec{p}}{m} \frac{\partial \rho}{\partial \vec{q}} \right\}$$

(integration by parts) $$= \int d\Gamma\, \rho \frac{\partial \mathcal{U}}{\partial \vec{q}} \cdot \frac{1}{\rho} \cdot \frac{\partial \rho}{\partial \vec{p}} - \int d\Gamma\, \rho \frac{\vec{p}}{m} \frac{1}{\rho} \frac{\partial \rho}{\partial \vec{q}}$$

(by parts again) $$= -\int d\Gamma\, \rho \frac{\partial^2 \mathcal{U}}{\partial \vec{p}\, \partial \vec{q}} + \int d\Gamma\, \rho \frac{\partial}{\partial \vec{q}} \vec{P}/m$$

$$= 0 + 0$$

$$= 0 \Rightarrow \boxed{\frac{dS(t)}{dt} = 0}$$

✓

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$$\langle A \rangle(t) = \int d\Gamma\, \rho(\Gamma, t)\, A(\Gamma)$$

$$\frac{d\langle A\rangle}{dt} = \quad \text{since} \quad \underline{\partial_t A(\Gamma) = 0}.$$

$$= \int d\Gamma\, A \cdot \frac{\partial \rho}{\partial t}(\Gamma) \qquad \partial_t \rho = \{\mathcal{H}, \rho\}$$

$$= \sum_{\alpha=1}^{3N} \int d\Gamma\, A \times \left(\frac{\partial \rho}{\partial p_\alpha} \frac{\partial \mathcal{H}}{\partial q_\alpha} - \frac{\partial \rho}{\partial q_\alpha} \frac{\partial \mathcal{H}}{\partial p_\alpha} \right)$$

Integration by parts...

$$= -\sum_{\alpha=1}^{3N} \int d\Gamma\, \rho \left\{ \overbrace{\left[\frac{\partial A}{\partial p_\alpha} \frac{\partial \mathcal{H}}{\partial q_\alpha} - \frac{\partial A}{\partial q_\alpha} \frac{\partial \mathcal{H}}{\partial p_\alpha} \right]}^{\{\mathcal{H}, A\}} + \cancelto{0}{A \left(\frac{\partial^2 \mathcal{H}}{\partial p_\alpha \partial q_\alpha} - \frac{\partial^2 \mathcal{H}}{\partial q_\alpha \partial p_\alpha} \right)} \right\}$$

$$= -\int d\Gamma\, \rho\, \{\mathcal{H}, A\} = \langle \{A, \mathcal{H}\} \rangle$$

✓

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(a) Boltzmann Eqns

$$\begin{cases} \mathcal{L}_a f_a = C_{aa} + C_{ab} + C_{ac} \\ \mathcal{L}_b f_b = C_{bb} + C_{ba} + C_{bc} \\ \mathcal{L}_c f_c = C_{cc} + C_{ca} + C_{cb} \end{cases}$$

where $\mathcal{L}_\alpha = \partial_t + \frac{\vec{p}_\alpha}{m_\alpha} \cdot \vec{\nabla}$

(b) No collision $\Rightarrow$ $C_{\alpha\beta} = \delta_{\alpha\beta} C_{\alpha\alpha}$

→ zeroth order solutions are simply the local equilibrium solutions:

$$\boxed{f^0_\alpha(\vec{p}, \vec{q}, t) = \frac{n_\alpha(\vec{q}, t)}{[2\pi k_B T(\vec{q}, t) m]^{3/2}} \exp\left\{ - \frac{(\vec{p} - m\vec{u}(\vec{q}, t))^2}{2 m k_B T(\vec{q}, t)} \right\}}$$

for $\alpha = a, b, c$

More generally, $\boxed{f^{(0)}_\alpha \propto A_\alpha \exp(-\beta \mathcal{H}_\alpha)}$

c Suppose only (a), (b) interact and (c) does nothing, then f_c stays the same, i.e. $f_c' = 0$ since there's no collision with (c).

→ get solution with interaction between a & b...

$$\boxed{\begin{aligned} \mathcal{L}_a f_a &= C_{aa} + C_{ab} \\ \mathcal{L}_b f_b &= C_{bb} + C_{ba} \end{aligned} \qquad \mathcal{L}_c f_c = 0.}$$

In this case, f_a, f_b will have corrections from higher order terms ... For example, the first order correction is

$$\boxed{f_\alpha'(\vec{p},\vec{q},t) = f_\alpha^0\left(1 - \tau_\alpha \mathcal{L}_\alpha[\ln f_\alpha^0]\right)}$$ → first order correction

where $\alpha = a, b$.

we can work this out through tedious calculations...

(d) For (a) and (c), they will have the same form b/c they only interact with (b)

$\mathcal{L}_a f_a = C_{aa} + C_{ab}$ → same form as part (c)

$\mathcal{L}_c f_c = C_{cc} + C_{bc}$ → same form as part (c)

$\mathcal{L}_b f_b = C_{bb} + C_{ba} + C_{bc}$

(e) What are the slow hydrodynamic modes?

In the slow hydrodynamic modes, collision terms can be considered zero, and we get that the densities follow the Vlasov equations ...

In this case, we have that $f_\alpha^{(s)}$ satisfies

$$\left\{ \partial_t + \sum_{n=1}^{s} \left(\frac{\vec{p}_n}{m_\alpha} \cdot \frac{\partial}{\partial \vec{q}_n} - \frac{\partial U_{eff,\alpha}}{\partial \vec{q}_n} \cdot \frac{\partial}{\partial \vec{p}_n} \right) \right\} f_\alpha^{(s)} = 0$$

where $\mathcal{U}_{eff,\alpha} = \mathcal{U}_\alpha(\vec{q}) + N_\alpha \int dV' \mathcal{V}(\vec{q}-\vec{q}') \rho_1^\alpha (\vec{x}', t)$

here
$$\begin{aligned} \mathcal{V}_a &= \mathcal{V}_{ab} + \mathcal{V}_{aa} + \mathcal{V}_{ac} \\ \mathcal{V}_b &= \mathcal{V}_{bb} + \mathcal{V}_{bc} + \mathcal{V}_{ba} \quad \ldots \\ \mathcal{V}_c &= \mathcal{V}_{cb} + \mathcal{V}_{ca} + \mathcal{V}_{cc} \end{aligned}$$

(f) Consider N_α, N_b, N_c in a box of volume V.

Then we have

$$\mathcal{H}_{eff}^{(\alpha)} = \sum_{i=1}^{N} \left\{ \frac{\vec{p}_i^{\,2}}{2m} + \mathcal{U}_{eff}^{(\alpha)}(\vec{q}_i) \right\}$$

where

$$\mathcal{U}_{eff}^{(\alpha)} = 0 + N_\alpha \int dV' \mathcal{V}(\vec{q}-\vec{q}') \frac{1}{V} g_\alpha(\vec{p}) = \frac{N_\alpha}{V} \int d^3q \, \mathcal{V}(\vec{q})$$

where $\rho_1^{(\alpha)} = g_\alpha(\vec{p})/V$

Substituting this into the Vlasov eqn gives

$$\left(\partial_t + \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}} \right) \rho_1^{(\alpha)} = 0 \quad \to \text{stationary solution}$$

And so the equilibrium form for f_α is

$$\boxed{f_1^{(\alpha)}(\vec{p}, \vec{q}) = \frac{N_\alpha}{V} g^{(\alpha)}(\vec{p})}$$ where we have used $f_1^{(\alpha)} = N_\alpha \rho_1^{(\alpha)}$

(for any $g^{(\alpha)}(\vec{p})$) → some momentum dist.

In general,

We can recover $f^{(\alpha)}$ by independence

$$f_s^{(\alpha)} = \frac{N_\alpha !}{(N_\alpha - s)!} \prod_{n=1}^{s} \rho_1^{(\alpha)}(\vec{x}_n, t)$$

So

$$\boxed{f^{(\alpha)} = N_\alpha ! \prod_{n=1}^{s} g^{(\alpha)}(\vec{p}, \cancel{t})/V}$$

(setting $s = N$)

no more t dependence since we're at equilibrium.