Tuesday, October 12

Tuesday, October 12, 2021 10:32 AM

Projection Ops

If observable A is measured, possible outcomes are eigenvalues λ_i with degeneracies n_i

We may find an orthormal basis for the eigenspace of A with eigenvalue λ_i. Denote these vectors by $|\lambda_i; l\rangle$ where l runs from 1 to n_i.

Orthonormality: $\langle \lambda_i; l | \lambda_i; l' \rangle = \delta_{ll'}$

[Also $\langle \lambda_i; l | \lambda_j; \tilde{l} \rangle = 0$ if $\lambda_i \neq \lambda_j$].

Projector onto λ_i eigenspace:

$$M_{\lambda_i} = \sum_{l=1}^{n_i} |\lambda_i; l\rangle\langle \lambda_i; l|$$

Exercise: show $M_{\lambda_i} \cdot M_{\lambda_i} = M_{\lambda_i}$ and $M_{\lambda_i}^{+} = M_{\lambda_i}$

Probability of outcome λ_i when measuring A in state $|\psi\rangle$ is

$$\boxed{\langle \psi | M_{\lambda_i} | \psi \rangle}$$

State of system after outcome λ_i is measured:

$$\boxed{\frac{M_{\lambda_i}|\psi\rangle}{\sqrt{\langle \psi | M_{\lambda_i} | \psi \rangle}}}$$

Mathematica Exercises

- Code up all single site Pauli Operators for a 3 qubit system
- Compute $[\sigma_x^{(1)}, \sigma_y^{(2)}]$ $(A = 0)$

  → What $[\sigma_a^{(i)}, \sigma_b^{(j)}]$? $(A = 2i\,\delta^{ij} \epsilon_{abc} \sigma_c^{(j)})$

- Code up a general state

  $$|v\rangle = \sum_{abc} V_{abc} |a\rangle|b\rangle|c\rangle$$

  in 8×1 vector form

  $(A = \{V_{111}, V_{112}, V_{121}, V_{122}, V_{211}, V_{212}, V_{221}, V_{222}\}^T)$

- Compute $(\mathbb{1}_2 \otimes \mathbb{1}_2 \otimes \sigma_x^{(3)})|v\rangle$

  $(A = \{V_{112}, V_{111}, V_{122}, V_{121}, V_{212}, V_{211}, V_{222}, V_{221}\}^T)$

- Compute Eigensystem of $\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(3)}$

[A: $\lambda = -3$, $\vec{v} = \frac{1}{\sqrt{2}}\{0,0,0,1,0,0,-1,0\}^T = \frac{1}{\sqrt{2}}(|+-\rangle_{13} - |-+\rangle_{13}) \otimes |-\rangle_2$

or $\vec{v} = \frac{1}{\sqrt{2}}\{0,1,0,0,-1,0,0,0\}^T = \frac{1}{\sqrt{2}}(|+-\rangle_{13} - |-+\rangle_{13}) \otimes |+\rangle_2$

$\lambda = 1$, $\vec{v} = \{0,0,0,0,0,0,0,1\}^T = |---\rangle$

or $\{0,0,0,0,0,1,0,0\}^T = |-+-\rangle$

or $\{1,0,0,0,0,0,0,0\}^T = |+++\rangle$

or $\{0,0,1,0,0,0,0,0\}^T = |+-+\rangle$

or $\{0,1,0,0,1,0,0,0\}^T = \frac{1}{\sqrt{2}}(|+-\rangle_{13} + |-+\rangle_{13}) \otimes |+\rangle_2$

or $\{0,0,0,1,0,0,1,0\}^T = \frac{1}{\sqrt{2}}(|+-\rangle_{13} + |-+\rangle_{13}) \otimes |-\rangle_2$]

→ Is it possible to write the $\lambda = -3$ eigenstates as $v_1 \otimes v_2 \otimes v_3$?

[A: no, these are entangled!]

- Starting w/ $|+++\rangle$, measure either $\sigma_z^{(3)}$ or $\sigma_x^{(3)}$, then $\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(3)}$

  → what is prob(−3) for last measurement in each case?

  → What important qualitative difference would we have if we compared $\sigma_z^{(2)}, \sigma_x^{(2)}$ in intermediate step?

[A: $\underline{\sigma_z^{(3)}}$: $p = 1$ get $+1$: state $\{1,0,0,0,0,0,0,0\}^T$

$\underline{\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(3)}}$: $p = 0$ to get $\lambda = -3$

$\underline{\sigma_x^{(3)}}$: $p = 1/2$ get $+$ → state $\frac{1}{\sqrt{2}}\{1,1,0,0,0,0,0,0\}^T$

$p = 1/2$ get $-$ → state $\frac{1}{\sqrt{2}}\{1,-1,0,0,0,0,0,0\}^T$

If $\sigma_x^{(3)} = +$: $p = 1/4$ for $\lambda = -3$

If $\sigma_x^{(3)} = -$: $p = 1/4$ for $\lambda = -3$

⇒ overall $\lambda = -3$ with $p = \frac{1}{2}\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = 1/4$.]

A: If used $\sigma_z^{(2)}$ v.s. $\sigma_x^{(2)}$, both commute with $\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(3)}$, so measurement outcome is undisturbed

⇒ $p = 0$ for $\lambda = -3$ in both cases!