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FINAL EXAM

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① Tight Binding Model

ⓐ Since $\mathcal{H}$ symmetric...

$$\mathcal{H} = \begin{pmatrix} 0 & t & & & t \\ t & 0 & t & & \\ & t & 0 & \ddots & \\ & & \ddots & \ddots & \\ & & & t & 0 \\ t & & & & 0 \end{pmatrix}$$

due to boundary condition

and

$$T = \begin{pmatrix} 0 & & & & 1 \\ 1 & 0 & & & \\ & 1 & \ddots & & \\ & & 1 & \ddots & \\ & & & \ddots & \\ & & & 1 & 0 \end{pmatrix}$$

So we see that

$$H = tT + tT^T$$

$$\Rightarrow \quad TH = tT^2 + tTT^T$$

$$HT = tT^2 + tT^TT$$

since $TT^T = T^TT$

$$\Rightarrow \quad [H, T] = 0$$

Alternatively, we could see that

Because $T|i\rangle = |i+1\rangle$

$\rightarrow \langle i| T^{+} = \langle i+1|$

matrix element $\rightarrow$ $\Rightarrow \langle i| T^{+} \mathcal{H} T |j\rangle =$

$$= \langle i+1| \mathcal{H} |j+1\rangle = \begin{cases} t & \text{if } i \neq j+1 \\ 0 & \text{else} \end{cases}$$

with resp to boundary conditions ...

$\dots$

$$= \langle i| \mathcal{H} |j\rangle$$

So $T^{+} \mathcal{H} T = \mathcal{H}$

$\rightarrow$ $HT = TH$

$\rightarrow$ $\boxed{[H, T] = 0}$ ✓

Now T is unitary $\Rightarrow$ eigenvalues are complex numbers with modulus 1.

$[H, T] = 0 \Rightarrow$ the energy basis ~~diag~~ simultaneously diagonalize $\mathcal{H}, T$ ...

- let us try $|\theta\rangle = \sum_n e^{in\theta} |n\rangle$ ... claim that this is a simultaneous eigenket to $\mathcal{H}$ & T

~~Thus~~ It is obvious that $|\theta\rangle$ is an eigenket of $\mathcal{H}$, so we just check ...

- $T|\theta\rangle = \sum_n e^{in\theta} |n+1\rangle$

$$= \sum_n e^{i(n-1)\theta} |n\rangle$$

$$= e^{i\theta} |\theta\rangle \longrightarrow \text{sign trouble?}$$

So we have found the eigenvalues for T. ✓

(b) From $T|\theta\rangle = e^{i\theta}|\theta\rangle$

We have

$$T^N|\theta\rangle = e^{i\theta N}|\theta\rangle$$

we want

$$T = T^{N+1}$$

$$\Rightarrow \quad e^{i\theta} = e^{i\theta(N+1)} \Rightarrow e^{i\theta N} = 1$$

So $\boxed{\theta = \frac{2\pi}{N}}$

(c)

As claimed before

$$|\theta\rangle = \sum_n e^{in\theta}|n\rangle$$

The energy spectrum is therefore found by

apply $\mathcal{H}$ to this state ...

$$\mathcal{H} \sum_n e^{in\theta} |n\rangle$$

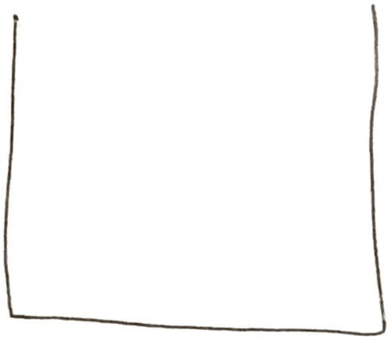
$$= t \sum_n e^{in\theta} |n+1\rangle + t \sum_n e^{in\theta} |n-1\rangle$$

$$= t \sum_n \left(e^{in\theta - i\theta} + e^{in\theta + i\theta} \right) |n\rangle$$

$$= \underbrace{2t\cos\theta}_{\longrightarrow \text{energy spectrum.}} \sum e^{in\theta} |n\rangle$$

$$\theta = \frac{2\pi}{N}$$

(2) Particle in a box



L

$m = \hbar = 1$

(a) Particle in a box of length L

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$n = 1, 2, 3$$

Energy : $E_n = \dfrac{\pi^2 \hbar^2 n^2}{2mL^2}$

for $|1\rangle$; $|2\rangle$

$$\psi_1(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right) \qquad E_1 = \cancel{\pi^2\hbar^2} = \frac{\pi^2}{2L^2}$$

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right) \qquad E_2 = \frac{4\pi^2}{2L^2}$$

(b) $|s\rangle = \alpha|1\rangle + \beta|2\rangle$, $\alpha, \beta \in \mathbb{R}$.

Probability that particle is in the left half is

$$P(\text{left half}) = \int_0^{L/2} (\alpha\psi_1 + \beta\psi_2)^*(\alpha\psi_1 + \beta\psi_2)\, dx$$

$=$ (mathematica)

$$= \frac{1}{6}\left(3\alpha^2 + 3\beta^2 + \frac{16\alpha\beta}{\pi}\right)$$

Now $\alpha^2 + \beta^2 = 1$, so

$$P(\text{left half}) = \frac{1}{6}\left(3 + \frac{16\alpha\beta}{\pi}\right)$$

$$= \frac{1}{2} + \frac{8}{3}\frac{\alpha\beta}{\pi}$$

maximized iff $\boxed{\alpha = \beta = 1/\sqrt{2}}$ (so that $\alpha^2 + \beta^2 = 1$)

So $\boxed{|s\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)}$

(c) <u>Time evolution ...</u>

let $|s\rangle = \alpha|1\rangle + \beta|2\rangle$ be at $t=0$

$$|s(t)\rangle = e^{-iHt/\hbar}|s\rangle = \alpha e^{-iE_1 t}|1\rangle + \beta e^{-iE_2 t}|2\rangle$$

<u>but</u> $\Psi(t) = \alpha e^{-i\pi^2 t/2L^2}|1\rangle + \beta e^{-i\,4\pi^2/2L^2\, t}|2\rangle$

Probability that the particle is on the right half

$$P\begin{pmatrix}\text{right}\\ \text{half}\end{pmatrix} = \int_{L/2}^{L} \Psi^*_{(t)}\Psi(t,x)\,dx$$

$$= \int_{L/2}^{L} \left(\alpha e^{-i\pi^2 t/2L^2}\psi_1 + \beta e^{-i\,4\pi^2/2L^2\, t}\psi_2\right)^* \left(\alpha e^{-i\pi^2 t/2L^2}\psi_1 + \beta e^{-i\,4\pi^2/2L^2\, t}\psi_2\right)dx$$

$$= \frac{1}{2}(\alpha^2+\beta^2) - \frac{8\alpha\beta}{3\pi}\cos\left(\frac{3\pi^2 t}{2L^2}\right)$$

Assume α, β fixed, then P(right) maxed

when $\frac{3\pi^2 t}{2L^2} = \frac{\pi}{2} \Rightarrow \boxed{t} = \frac{\pi}{2}\cdot\frac{2L^2}{3\pi^2} = \boxed{\frac{L^2}{3\pi}}$

(3) Parity & Charge Conj

(a) P is unitary, $P^2 = 1$

So eigenvalues of P are $\boxed{\pm 1}$

(b) $|\tau^+\rangle$; $|\theta^+\rangle$; $|\pi^+\pi^+\pi^-\rangle$; $|\pi^+\pi^0\rangle$

$$|\tau^+\rangle \xrightarrow{P} \lambda_\tau |\tau^+\rangle \qquad |\theta^+\rangle \xrightarrow{P} \lambda_\theta |\theta^+\rangle \qquad |\pi^+\pi^+\pi^-\rangle \xrightarrow{P} -|\pi^+\pi^+\pi^-\rangle \qquad |\pi^+\pi^0\rangle \xrightarrow{P} |\pi^+\pi^0\rangle$$

initially, $|\tau^+\rangle$

Then $e^{-i\mathcal{H}t/\hbar} |\tau^+\rangle = \alpha |\pi^+\pi^+\pi^-\rangle + \dots$

Since P is a symm of $\mathcal{H}$ ... $PH = HP$

So $P e^{-i\mathcal{H}t/\hbar} |\tau^+\rangle = -\alpha |\pi^+\pi^+\pi^-\rangle + \dots$

$= e^{-i\mathcal{H}t/\hbar} P |\tau^+\rangle = e^{-i\mathcal{H}t/\hbar} \cdot \lambda_\tau |\tau^+\rangle$

$= \lambda_\tau \alpha |\pi^+\pi^+\pi^-\rangle + \dots$

So $-\alpha = \lambda_\tau \alpha \Rightarrow \boxed{\lambda_\tau = -1}$

Note that by the same argument, we'll have

$+\beta = (-1)\beta$ where β is the coef ~~of~~ for $|\pi^+\pi^0\rangle$

So the prob. that the system is in $|\pi^+\pi^0\rangle$ is

$\boxed{\text{zero}}$

(c) $|\theta^+\rangle$ initially... then by similar argument, we find that

$\gamma = \eta_\theta \gamma \Rightarrow \boxed{\eta_\theta = 1}$

So $\langle\tau^+|\theta^+\rangle = \langle\tau^+|P^+P|\theta^+\rangle$

$= -\langle\tau^+|\theta^+\rangle$

$\Rightarrow \boxed{\langle\tau^+|\theta^+\rangle = 0}$

(1) $C^2 = 1$, $[C, P] = 0$

$$|K^0\rangle \perp |\bar{K}^0\rangle$$

$$\begin{cases} CP\,|K^0\rangle = -|\bar{K}^0\rangle \\ CP\,|\bar{K}^0\rangle = -|K^0\rangle \end{cases}$$

$$K_1 = \alpha|K^0\rangle + \beta|\bar{K}^0\rangle$$

$$CP\,K_1 = -\alpha|\bar{K}^0\rangle - \beta|K^0\rangle = \alpha|K^0\rangle + \beta|\bar{K}^0\rangle$$

$$\Rightarrow \quad \alpha = -\beta \; ; \; -\beta = \alpha \quad \checkmark$$

So pick

$$\boxed{K_1 = \frac{1}{\sqrt{2}}\left(|K^0\rangle - |\bar{K}^0\rangle\right)}$$

$\cancel{CPK_2} =$ $\quad K_2 = a|K^0\rangle + b|\bar{K}^0\rangle$

$$CP\,K_2 = -a|\bar{K}^0\rangle - b|K^0\rangle = -a|K^0\rangle - b|\bar{K}^0\rangle$$

$$\Rightarrow \quad a = b \; ; \; -a = -b$$

$$\Rightarrow \boxed{K_2 = \frac{1}{\sqrt{2}}\left(|K^0\rangle + |\bar{K}^0\rangle\right)}$$

(e) $\mathcal{H}_0 = -g \cdot CP$. $\quad CP \leftrightarrow \mathcal{H}_0$

initially in $|\bar{K}^0\rangle$

$e^{-i\mathcal{H}t}|\bar{K}^0\rangle = \cancel{e^{-i\mathcal{H}t}|\bar{K}^0\rangle = -e^{-i\mathcal{H}t}|\bar{K}^0\rangle}$

$= e^{+igCP\,t/\hbar}|\bar{K}^0\rangle = |\bar{K}^0\rangle + \cdots$

$|K^0\rangle$ observed with probability

$\boxed{\cancel{\cos^2(gt/\hbar)}}$ since @ $t=0$ we just have $|\bar{K}^0\rangle$

$\boxed{\sin^2(gt/\hbar)}$

(f)

if $|K_2\rangle$ observed after initially in $|K_1\rangle$

Then CP is <u>not</u> a symmetry of V

b/c if it is, then $\mathcal{H} = \mathcal{H}_0 + V$ has a CP symmetry and $e^{-i\mathcal{H}t/\hbar}$ preserves this symmetry, but $|K_1\rangle$ and $|K_2\rangle$ have <u>different</u> CP symmetries.

④ <u>Thermal Field Double state</u>

$$|\psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_{n=0} e^{-\frac{\beta E_n}{2}} |n\rangle_L \otimes |n\rangle_R$$

(a) $$\mathcal{U}(t) = \exp\left(\frac{-i}{\hbar}(\mathcal{H}^R + \mathcal{H}^L)t\right)$$

$$= \exp\left[\frac{-i}{\hbar}\left[\mathbb{1} \otimes \mathcal{H}^R + \mathcal{H}^L \otimes \mathbb{1}\right]t\right]$$

Note that

$$\left[\mathbb{1} \otimes \mathcal{H}^R, \mathcal{H}^L \otimes \mathbb{1}\right] = 0$$

So,

$$\mathcal{U}(t) = \exp\left\{\frac{-i}{\hbar}\mathcal{H}^R t\right\} \exp\left\{\frac{-i}{\hbar}\mathcal{H}^L t\right\}$$

Technically,

$$\mathcal{U}(t) = \exp\left\{\frac{-i}{\hbar}\mathcal{H}^L t\right\} \otimes \exp\left\{\frac{-i}{\hbar}\mathcal{H}^R t\right\}$$

So this is the answer... I can't say if the answer in the problem is correct or not because of ambiguous notation

(3) $a^R_H(t) = \mathcal{U}(t)^+ a^R \mathcal{U}(t)$

$a^L_H(t) = \mathcal{U}^+(t)\, a^L_S\, \mathcal{U}(t)$

EOM for $a^L_H(t)$?

$$\frac{d}{dt} a^L_H(t) = \frac{\partial \mathcal{U}^+}{\partial t} \cdot a^L_S\, \mathcal{U} + \mathcal{U}^+ a^L_S \frac{\partial \mathcal{U}}{\partial t}$$

$$= \frac{-1}{i\hbar} \mathcal{U}^+ (\mathcal{H}^L + \mathcal{H}^R)\, \mathcal{U}\mathcal{U}^+ a^L_S\, \mathcal{U}$$

$$+ \mathcal{U}^+ a^L_S \cdot \frac{1}{i\hbar} \mathcal{U}\mathcal{U}^+ (\mathcal{H}^L + \mathcal{H}^R)\, \mathcal{U}$$

$$= \frac{1}{i\hbar} [a^L_H(t), \underbrace{\mathcal{U}^+ (\mathcal{H}^L + \mathcal{H}^R)\, \mathcal{U}}_{\mathcal{H}^L + \mathcal{H}^R}]$$

$$= \frac{1}{i\hbar} [a^L_H(t), \mathcal{H}^L + \mathcal{H}^R]$$

$$= \frac{1}{i\hbar} [a^L_H(t), \mathcal{H}^L]$$

(c) <u>Solve to find $a^L_H(t)$</u>

Evaluate EOM RHS ...

$$\frac{1}{i\hbar}\left[a^L_H(t), H^L\right]$$

$$= \frac{1}{i\hbar}\left[U^\dagger a^L_S U, \hbar\omega\left(a^{L\dagger}_S a^L_S + \tfrac{1}{2}\right)\right]$$

$$= \frac{1}{i\hbar} U^\dagger \left[a^L_S, \hbar\omega\left(a^{L\dagger}_S a^L_S + \tfrac{1}{2}\right)\right] U$$

$$\cancel{= \frac{1}{i\hbar} U^\dagger \left[a^L_S, \hbar\omega\left(a^{L\dagger}_S a^L_S\right)\right] U}$$

$$= \frac{1}{i\hbar}\left(\hbar\omega\, a^L_S\right) = -i\omega \underbrace{U^\dagger a^L_S U}_{a^L_H}$$

So $$\boxed{\frac{d}{dt} a^L_H(t) + i\omega\, a^L_H = 0}$$

Solution to this ODE is

$$\boxed{a^L_H(t)} = a^L_H(0)\, e^{-i\omega t} = \boxed{a^L_S\, e^{-i\omega t}}$$

since $U(0) = \mathbb{1}$.

(c) ~~Sh~~ Find $f(\beta, w)$ for which

$$a_s^L |\Psi_B\rangle = f(\beta, w)\, a_s^{R\dagger} |\Psi_B\rangle$$

$$a_s^L |\Psi_B\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_{n=0} e^{-\beta E_n/2}\, a_s^L |n\rangle_L \otimes |n\rangle_R$$

$$= \frac{1}{\sqrt{Z_\beta}} \sum_{n=1} e^{-\beta E_n/2} \sqrt{n}\, |n-1\rangle_L \otimes |n\rangle_R$$

~~shift~~

$$= \frac{1}{\sqrt{Z_\beta}} \sum_{n=0} e^{-\beta E_{n+1}/2} |n\rangle_L \otimes |n+1\rangle_R \cdot \sqrt{n+1}$$

$$= \frac{1}{\sqrt{Z_\beta}} \sum_{n=0} e^{-\beta E_{n+1}/2} |n\rangle_L \otimes a_s^{R\dagger} |n\rangle_R$$

$$= a_s^{R\dagger} \frac{1}{\sqrt{Z_\beta}} \sum_{n=0} e^{-\beta \frac{E_{n+1}}{2}} |n\rangle_L \otimes |n\rangle_R$$

$$= a_s^{R\dagger} \cdot \underbrace{e^{\beta E_0/2}} |\Psi_B\rangle$$

$$e^{\beta E_0/2} = e^{\beta \hbar w/2 \cdot \frac{1}{2}} = \boxed{e^{\beta \hbar w/4} = f(\beta, w)}$$

(e) Squeezed states

no time left

(f)

(5) $J = S_1 + S_2 + S_3$

(a) possible values of j are $\boxed{\frac{1}{2}}$ and $\boxed{\frac{3}{2}}$

Can see this as follows...

$$J = \underbrace{S_1 + S_2}_{S} + S_3 \rightarrow s_3 = \frac{1}{2}$$

$S \downarrow$ has eigenvalues $\hbar^2 s(s+1)$

$s = \{0, 1\}$

So $\quad |s_3 - s| \leq j \leq |s_3 + s|$

$$\frac{1}{2} \leq j \leq 3/2$$

j has steps of 1

So $\boxed{j = \frac{1}{2}, \frac{3}{2}}$

$\uparrow\uparrow\uparrow \quad \uparrow\downarrow\uparrow$ that's it.

(b) State with $j = \frac{3}{2}$; $m = \frac{1}{2}$

<u>guess</u>

$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\left\{\left|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\right\rangle + \left|\tfrac{1}{2}\ \tfrac{-1}{2}\ \tfrac{1}{2}\right\rangle + \left|\tfrac{-1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle\right\}$$

<u>Solution</u> ... treating $S_1 + S_2$ as S then (Clebsch-Gordan)

$$\left|\tfrac{3}{2}\ \tfrac{1}{2}\right\rangle = \frac{1}{\sqrt{3}}\underbrace{\left|1,1\right\rangle}_{\text{first 2 spins}}\underbrace{\left|\tfrac{1}{2}\ \tfrac{-1}{2}\right\rangle}_{\text{3rd spin}} + \sqrt{\tfrac{2}{3}}\left|1,0\right\rangle\left|\tfrac{1}{2}\ \tfrac{+1}{2}\right\rangle$$

$$= \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\right\rangle + \sqrt{\tfrac{2}{3}}\,\frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\tfrac{-1}{2}\right\rangle + \left|\tfrac{-1}{2}\tfrac{1}{2}\right\rangle\right)\left|\tfrac{+1}{2}\right\rangle$$

$$= \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}\ \tfrac{1}{2}\ \tfrac{-1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\tfrac{1}{2}\ \tfrac{-1}{2}\ \tfrac{1}{2}\right\rangle + \frac{1}{\sqrt{3}}\left|\tfrac{-1}{2}\ \tfrac{1}{2}\ \tfrac{1}{2}\right\rangle$$

adding 2 particles... spin 1 and spin 1/2 ...

(c) ~~$l = \ell d$~~ $j = 1/2$, $m = 1/2$

$\left| \frac{1}{2} \; \frac{1}{2} \right\rangle$ total spin $= \frac{1}{2} \Rightarrow$ can choose

$\uparrow$ total

$\Downarrow$ spin 0 +

$s_1 + s_2 = 0$

If $s_1 + s_2 = 0$ then

~~$s_1 \left| \frac{1}{2} \; \frac{1}{2} \right\rangle$~~ total ~~$|s_1 s_2\rangle \otimes |00\rangle$~~

For 2 particles, ~~there is~~ there is a pair in $|00\rangle$

$$|00\rangle = \frac{1}{\sqrt{2}}\left(\left| \tfrac{1}{2} \; \tfrac{-1}{2} \right\rangle - \left| \tfrac{-1}{2} \; \tfrac{1}{2} \right\rangle \right)$$

So there are ~~are~~ a few possibilities so that $m_{s\,total} = \frac{1}{2}$

$$\frac{1}{\sqrt{2}}\left(\left| \tfrac{1}{2} \; \tfrac{-1}{2} \right\rangle - \left| \tfrac{-1}{2} \; \tfrac{1}{2} \right\rangle \right) \left| \tfrac{1}{2} \right\rangle$$

or

$$\frac{1}{\sqrt{2}}\left| \tfrac{1}{2} \; \tfrac{1}{2} \; \tfrac{-1}{2} \right\rangle - \frac{1}{\sqrt{2}}\left| \tfrac{-1}{2} \; \tfrac{1}{2} \; \tfrac{1}{2} \right\rangle$$

or

$$\frac{1}{\sqrt{2}}\left| \tfrac{1}{2} \; \tfrac{1}{2} \; \tfrac{-1}{2} \right\rangle - \frac{1}{\sqrt{2}}\left| \tfrac{1}{2} \; \tfrac{-1}{2} \; \tfrac{1}{2} \right\rangle$$

or any linear combo of these...

(6) $\boxed{\text{Anharmonic Oscillator}}$

$$V = \frac{1}{2} w^2 x^2 + \lambda x^4 \qquad m = \hbar = 1$$

(a) $\tilde{V} = x^4$

Compute $\underline{\langle 0 | \tilde{V} | 0 \rangle}$

where

$$\psi_0(x) = \left(\frac{w}{\pi}\right)^{1/4} e^{-\omega x^2/2}$$

$$\langle 0 | \tilde{V} | 0 \rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, x^4\, \psi_0(x)\, dx$$

$$= \text{(mathematica)}$$

$$= \boxed{\frac{3}{4\omega^2}}$$

(b) From (a), first order correction to ground state energy is

$$\lambda \langle 0 | \tilde{V} | 0 \rangle = \boxed{\lambda \cdot \frac{3}{4\omega^2}}$$

(c) $\lambda = 0$ ground state is ψ_0.

We will use the theorem

$$\bar{\mathcal{H}} \equiv \frac{\langle \psi_0 | \mathcal{H} | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} \geq E_0$$

where $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} \omega^2 x^2 + \lambda x^4$

$$= -\frac{1}{2} \partial_x^2 + \frac{1}{2} \omega^2 x^2 + \lambda x^4$$

well...

$$\langle \psi_0 | \mathcal{H} | \psi_0 \rangle = \langle \psi_0 | \mathcal{H}_0 | \psi_0 \rangle + \lambda \langle \psi_0 | \tilde{V} | \psi_0 \rangle$$

$$= \frac{\omega}{2} + \lambda \cdot \frac{3}{4\omega^2}$$

So an upper bound for the true ground state energy for an arbitrary λ is

$$\boxed{\frac{\omega}{2} + \frac{3\lambda}{4\omega^2}}$$

Wait what? something wrong with the wording here ???

I'm not sure about the wording of this question?

(c) If we use $\psi_0 = e^{-ax^2}$ then

$$\frac{\langle \psi_0 | H | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle} = \frac{4a^2 + w^2}{8a} \quad \text{for } \lambda = 0$$

$$\text{or} \quad = \frac{4a^3 + 6\sqrt{2}\lambda + aw^2}{8a^2} \quad \text{for } \lambda \neq 0$$

$$= \frac{4a^2 + w^2}{8a} + \left(\frac{3\lambda}{2\sqrt{2}\, a^2} \right)$$

minimize this wrt a to get upper bound?

↳ solution for a is ugly... (cubic)

So agrees with answer in (3), sort of ?

if

→ minimal @ $a = w/2$

→ correction is $\dfrac{3\lambda}{2\sqrt{2}\left(\frac{w}{2}\right)^2} = \left(\dfrac{3\lambda}{\frac{\sqrt{2}}{2} w^2} \right)$

→ agrees with (5) if λ small...