Problem Set 5

Due: Saturday 11:59pm, March 18th via Canvas upload
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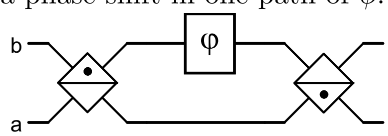
Office hour: TBA on Canvas.

1 Better Phase Measurements with Squeezed Vacuum

One application of squeezed light is to measure phase shifts with better precision than can be achieved with the same number of photons in a coherent state. In this problem, we employ a Mach-Zehnder interferometer with coherent and squeezed input states. This interferometer uses two 50/50 beamsplitters and has a phase shift in one path of ϕ:



and has output ports b_{out} and a_{out}.

(a) Calculate the output signal $\langle M \rangle = \langle b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out} \rangle$ as a function of ϕ and its the variance $\langle \Delta M^2 \rangle = \langle \Delta(b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out})^2 \rangle$ for the Mach-Zehnder interferometer with a coherent state and vacuum at its inputs ($|a\rangle = |\alpha\rangle$ and $|b\rangle = |0\rangle$). Calculate the Signal-to-Noise Ratio (SNR) for this measurement, $\frac{\langle M \rangle}{\sqrt{\langle \Delta M^2 \rangle}}$.

Note: The SNR should be calculated for the quantiy $M = b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}$

(b) Find the minimal detectible phase $\phi_{\min}$ for the Mach-Zehnder interferometer with a coherent state and vacuum at its inputs. (Use a small-angle approximation for ϕ.) The minimal detectible phase is the phase for which the SNR is 1.

(c) Repeat (a-b) using squeezed vacuum in port b ($|a\rangle = |\alpha\rangle$ and $|b\rangle = S(r)|0\rangle$). What degree of squeezing do you need to get factor of 2 increase in the phase resolution? Does squeezing change the average photon number in the device?

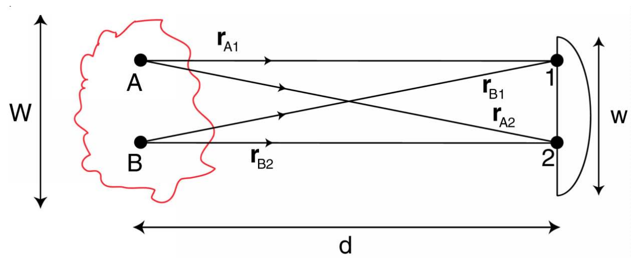
(d) The LIGO experiment is a long-arm Michelson interferometer that measures differences in the length of one arm compared to the other as a signature of gravitational waves. Model LIGO as the Mach-Zehnder interferometer studied in (a)-(c). If the interferometer uses 5W of 1064nm light, what is the minimal differential length change that can be detected with coherent state inputs? If each path is 4 km long, what is the minimal detectable strain? What if a 6dB squeezed vacuum state is used instead of a coherent vacuum state?

The LIGO team measured their sensitivity with squeezed vacuum. To learn about what they saw, see *Nature Photonics* **7**, 613–619 (2013). The first use of squeezed vacuum states in the direct measurement of gravitational waves was reported in Phys. Rev. Lett. **123**, 231107 (2019).

2 Hanbury Brown and Twiss Experiment with Atoms

This problem illustrates the coherence and collimation requirements for performing a Hanbury Brown and Twiss (HBT) experiment with atoms. In fact the HBT experiment was done for both

bosons (^{4}He) and fermions (^{3}He) by Jeltes and company in 2007 (T. Jeltes et al., Nature **445**, 402 (2007)). (Note: Ignore gravity in this problem.)
If a free particle starts at point A at time $t = 0$ with an amplitude (wavefunction) ψ_A, then the amplitude at another point 1 and time $t = \tau$ is proportional to $\psi_A e^{i(\mathbf{k}\cdot\mathbf{r}_{A1} - \omega\tau)}$, where $\mathbf{r}_{A1}$ is the vector from A to 1, $\mathbf{k}$ is the particle's wavevector, and $\hbar\omega$ is its total energy. This can be regarded as Huygen's principle for matter waves, and is a special case of the Feynman path integral formulation of quantum mechanics.



(Based on figure 19-5, in G. Baym, *Lectures on Quantum Mechanics, also on Canvas)*

(a) Correlation function

Assume we have a particle at A with amplitude ψ_A and one at B with amplitude ψ_B. The joint probability P of finding one particle at 1 and one at 2 is

$$P = \left| \psi_A e^{i\phi_{A1}} \psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}} \psi_B e^{i\phi_{B1}} \right|^2 \qquad (1)$$

and is proportional to the second-order coherence function $g^{(2)}(1,2)$. The $\pm$ is for bosons/fermions and makes the two-particle wavefunction symmetric/antisymmetric under the exchange of particles. Here, $\phi_{A1} = \mathbf{k}_A \cdot \mathbf{r}_{A1} - \omega\tau$ is the phase factor for the path from point A to detector 1, etc. Calculate P as a function of $\mathbf{r}_{21}$, the vector from point 2 to point 1 on the detector.

(b) Transverse Collimation

Assume you are given a source (e.g. a ball of trapped atoms) with transverse dimension W and detector with transverse dimension w where $|\mathbf{r}_{21}| \leq w$. The distance between source and detector d is much greater than all other distances. The transverse component of the phase factor in part (a) can be written: $\phi_t = (\mathbf{k}_A - \mathbf{k}_B)_t \cdot (\mathbf{r}_{21})_t$. Assume that the signal at the detector is mainly due to atoms with wavevectors distributed around $\mathbf{k}_0$. Argue that the transverse collimation required to see second order correlation effects can be expressed as $Ww \ll d\lambda_{dB}$, where λ_{dB} is the de Broglie wavelength corresponding to $\mathbf{k}_0$. (Hint: How does ϕ_t vary for atoms originating at different points in the source and being detected at different points on the detector?)

Consider a ^{6}Li MOT at 500 μK. Calculate the de Broglie wavelength. Assuming a MOT and detector of approximately equal size ($W \approx w$), estimate an upper bound on the MOT and detector size using $d = 10$ cm.

(c) Longitudinal Collimation

(i) The longitudinal component of the phase factor in part (a) can be written: $\phi_l = (\mathbf{k}_A - \mathbf{k}_B)_l \cdot (\mathbf{r}_{21})_l$. Assume a Gaussian distribution of wavevector differences $p(\mathbf{k}_A - \mathbf{k}_B) = e^{-|\mathbf{k}_A - \mathbf{k}_B|^2\gamma^2}$ where the width γ is related to the temperature of the atoms. Calculate $\langle P \rangle$ using this distribution and your result from part (a). Sketch $\langle P \rangle$ for both fermions and bosons, indicating the extent of $(\mathbf{r}_{21})_l$ over which the second order correlation effect can be seen. (Hint: Use the fact that $\phi_t \ll 2\pi$ from part (b) to simplify the integral.)

(ii) Now assume you have a pulsed source of atoms with longitudinal dimension L. Atoms are released at time $t = 0$ and detected at some later time $t = \tau$. Give geometric arguments to show that the wavevectors of detected atoms must obey $|\,(\mathbf{k}_A - \mathbf{k}_B)_l\,| \le \frac{mvL}{\hbar d}$, where the velocity $v = \frac{d}{\tau}$. This implies that the different velocity groups separate during the expansion, narrowing (by a factor $\frac{L}{d}$) the velocity distribution of atoms detected at any particular time.

Consider again the ^{6}Li MOT from part (b). Assuming $\tau = 0.1$s and $L \approx W$, estimate the necessary timing resolution of the detector in order to see second order correlation effects?

(d) Phase-Space Volume Enhancement

We now pull all the pieces together. The peak in $g^{(2)}(1,2)$ is visible for $(\mathbf{k}_A - \mathbf{k}_B) \cdot \mathbf{r}_{21} \le 2\pi$. This is equivalent to saying that we must detect atoms from within a single phase space cell, defined by $\delta p_x \delta x \le h$ (and likewise for y and z). In our trapped atom sample, the 3D volume of a phase space cell is $\delta x \delta y \delta z = (\lambda_{dB})^3$. Liouville's theorem says that as our ball of atoms expands, the number of phase space cells remains constant. Verify that, by using this pulsed source, the volume of a coherent phase space cell is increased by a factor d^3/W^2L by the time atoms reach the detector. What is the order of magnitude of this increase (assuming $L \approx W$)?

Estimate the average occupation of a cell of phase space for the ^{6}Li MOT from parts (b) and (c). Use the following numbers for the ^{6}Li MOT: 10^{10} atoms in 1 cm^3. How does this compare with the average occupation of a BEC or a degenerate Fermi cloud?