Classical Poisson bracket: $\{x^i, p_j\} = \delta^i{}_j$
(p canonical momentum)

Quantization:

$$x^i \longrightarrow \hat{x}^i$$
$$p_j \rightarrow \hat{p}_j = -i\hbar \frac{\partial}{\partial x^i}$$ (note: p, not $m\dot{x}$ here)
$$H \longrightarrow \frac{p^2}{2m} - \frac{e}{2mc}(A\cdot p + p\cdot A) + \frac{e^2}{2mc^2} A^2 + e\phi.$$

Ehrenfest:

$$m\frac{d^2}{dt^2}\langle \vec{x}\rangle = \frac{d}{dt}\langle \vec{p}\rangle = \frac{1}{i\hbar}\langle [\vec{p}, H]\rangle$$

$$\Rightarrow \quad m\frac{d^2}{dt^2}\langle \vec{x}\rangle = \left\langle \vec{E}e + \frac{e}{2c}\left(\frac{d\vec{x}}{dt}\times\vec{B} - \vec{B}\times\frac{d\vec{x}}{dt}\right)\right\rangle$$

(Lorentz force law)

Gauge invariance of QM

Under a gauge transform, $A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$

Classically, canonical momentum $p_i = m\dot{x}^i - \frac{e}{c}A^i$ changes.
$x^i, \dot{x}^i$ remain fixed.

QM:

Schrödinger $$i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) = \left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + e\phi\right]\psi(\vec{x},t)$$

Take $$\vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$
$$\phi' = \phi - \frac{1}{c}\frac{\partial}{\partial t}\Lambda$$

Can rewrite Schrödinger

$$\left\{\left[i\hbar\frac{\partial}{\partial t} - e\phi\right] - \left[\frac{1}{2m}\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2\right]\right\}\psi(\vec{x},t) = 0.$$

Clear that $\psi'(\vec{x},t) = e^{\frac{ie}{\hbar c}\Lambda(t)}\psi(\vec{x},t)$

satisfies Schrödinger with $\phi \to \phi'$, $A \to A'$.

~~So~~ Since

$$\left[i\hbar\frac{\partial}{\partial t} - e\phi'\right]e^{\frac{ie}{\hbar c}\Lambda}\psi = e^{\frac{ie}{\hbar c}\Lambda}\left[i\hbar\frac{\partial}{\partial t} - e\phi\right]\psi$$

& $$\left[-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A'_i\right]e^{\frac{ie}{\hbar c}\Lambda}\psi = e^{\frac{ie}{\hbar c}\Lambda}\left[-i\hbar\frac{\partial}{\partial x^i} - \frac{e}{c}A_i\right]\psi.$$

So, under gauge transformations

$$\phi \to \phi' = \phi - \frac{1}{c}\frac{\partial}{\partial t}\Lambda$$
$$\vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda$$
$$\psi \to \psi' = e^{\frac{ie}{\hbar c}\Lambda}\psi.$$

No physical observables change, although, e.g.

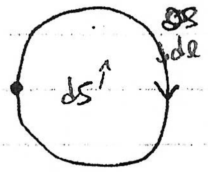
$\langle\vec{p}\rangle = \langle m\dot{\vec{x}} + \frac{e}{c}\vec{A}\rangle$ is gauge-dependent.

<u>Kinematical</u> momentum $\Pi^i = m\dot{x}^i$ is gauge-independent.

Aharanov – Bohm effect

[Another quantum effect arising from fields in regions not containing QM particle].

Recall

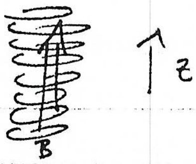


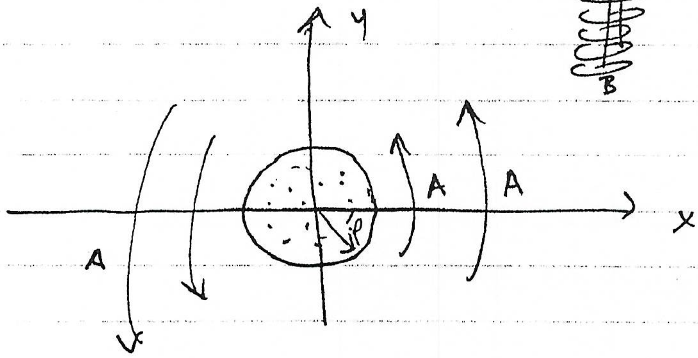
$$\int A \cdot dl = \int (\vec{\nabla} \times \vec{A}) \cdot ds$$

$$\left[\text{generally, } \int_\Sigma d\omega = \int_{\partial\Sigma} \omega \right]$$

for differential forms Σ p-manifold, ω (p-1) - form.

Consider an infinite solenoid

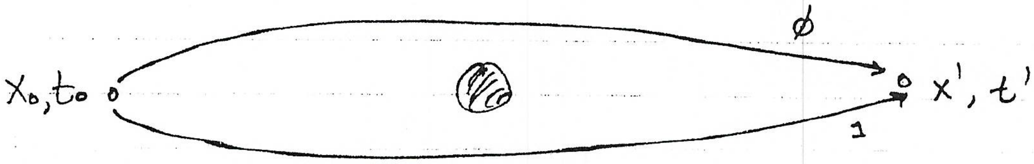




Calculate A: $2\pi R\, A_\theta = \int B \cdot ds = \Phi_B$ for $R > \rho$

so $A_\theta = \frac{1}{2\pi R} \Phi_B$ outside solenoid!

Consider a particle moving around the solenoid (impenetrable approximation)



Does B field affect interference pattern? (Yes!)

Use path integrals:

$$K(x_0,t_0;\, x',t') = \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]}$$

Consider paths of type ϕ:

$$e^{\frac{i}{\hbar} S(x(t))} = e^{\frac{i}{\hbar}\int dt\, \mathcal{L}(\dot{x},x)}$$

$$\mathcal{L}(\dot{x},x) = \frac{m}{2}\dot{x}^2 + \frac{e}{c}A_i\,\dot{x}^i - \cancel{e\phi}$$

Phase coming from A:

$$e^{\frac{ie}{\hbar c}\int A_i\, dx^i/dt\, dt} = e^{\frac{ie}{\hbar c}\int_{\text{path}} A_i\, dx^i}$$

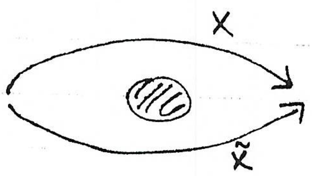
Note that $\int_x A_i dx^i = \int_{\tilde{x}} A_i d\tilde{x}^i$

if $x(t)$, $\tilde{x}(t)$ are topologically equivalent
(i.e., one can be deformed into the other without hitting solenoid)

Thus, <u>all</u> paths of type ϕ give a phase

$$e^{i\theta_0} = e^{\frac{ie}{\hbar c}\int_{\text{path }\phi} A_i dx^i}$$ from A.

Type 1 paths:



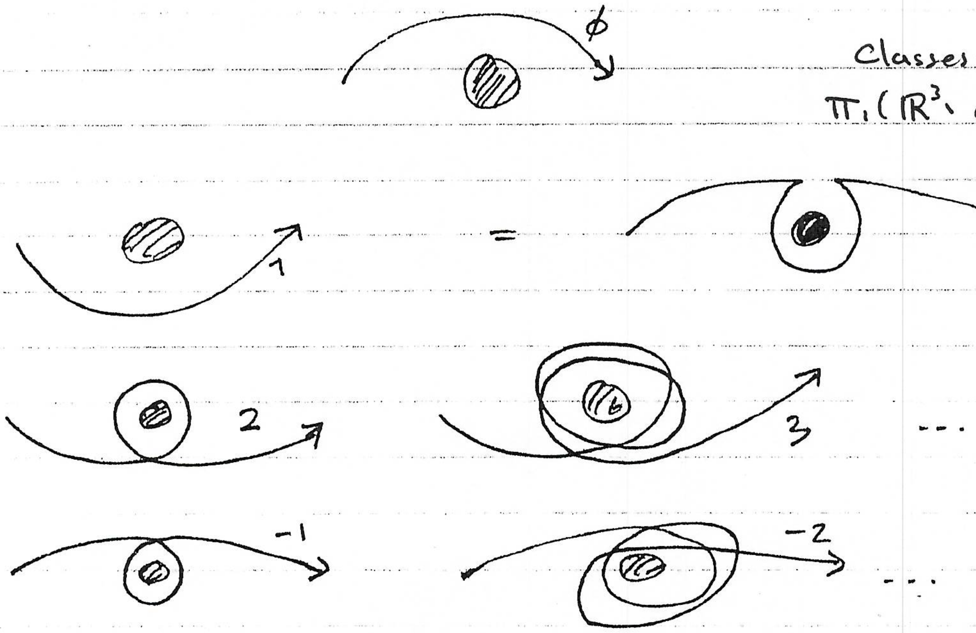
$$\int_{\tilde{x}} A_i d\tilde{x}^i - \int_x A_i dx^i = \Phi_B$$

so $$e^{i\theta_1} = e^{i\theta_0 + \frac{ie}{\hbar c}\Phi_B}.$$

Classify topologically inequivalent paths:

Classes: $k \in \mathbb{Z}$

$\pi_1(\mathbb{R}^3 \setminus \text{cylinder}) = \mathbb{Z}$



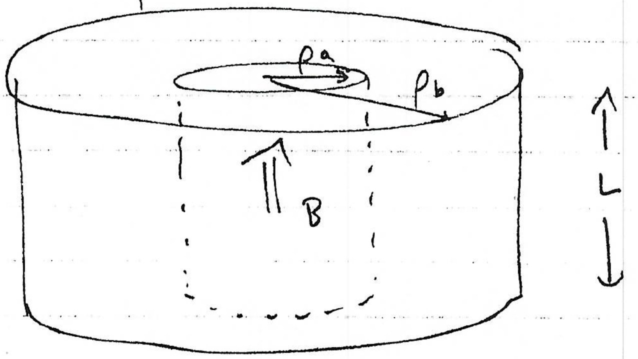
Total propagator:

$$K = \sum_{N=-\infty}^{\infty} \mathcal{D}[x_N(t)]\, e^{\frac{ie}{\hbar c}\int_0 A_i\, dx^i + \frac{ieN}{\hbar c}\Phi_B + \frac{i}{\hbar}\int_{x(t)} M\dot{x}^2}$$

$N =$ "winding #"

Interference clearly affected by Φ_B.
(paths of type ϕ, 1 dominate)

Static version of problem



Flux in core, particle in region $\rho_a < r < \rho_b$.

Energy levels depend on B (HW).

Magnetic Monopoles

In a source-free region, Maxwell's equations read:

{field notation} [form notation]

$$\left\{\begin{array}{l}\vec{\nabla}\cdot\vec{E}=0\\ \vec{\nabla}\times\vec{B}=\frac{1}{c}\frac{\partial}{\partial t}\vec{E}\end{array}\right\} \qquad \partial_\mu F^{\mu\nu}=0 \qquad [d{*}F=0]$$

Since

$$F_{\mu\nu}=\partial_{[\mu}A_{\nu]} \qquad [F=dA]$$

$$\Downarrow$$

$$\left\{\begin{array}{l}\vec{\nabla}\cdot\vec{B}=0\\ \vec{\nabla}\times\vec{E}=-\frac{1}{c}\frac{\partial}{\partial t}\vec{B}\end{array}\right\} \qquad \partial_{[\rho}F_{\mu\nu]}=\partial_{[\rho}\partial_\mu A_{\nu]}=0 \qquad [dF=ddA=0]$$

Equations are invariant under

$$\left\{\begin{array}{l}\vec{E}\longleftrightarrow-\vec{B}\\ \vec{B}\longrightarrow\vec{E}\end{array}\right\} \qquad F_{\mu\nu}\leftrightarrow\tfrac{1}{2}\varepsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma} \qquad [F\longleftrightarrow{*F}]$$

"Maxwell duality"

Including (static) sources:

$$\vec{\nabla}\cdot\vec{E}=4\pi\rho.$$

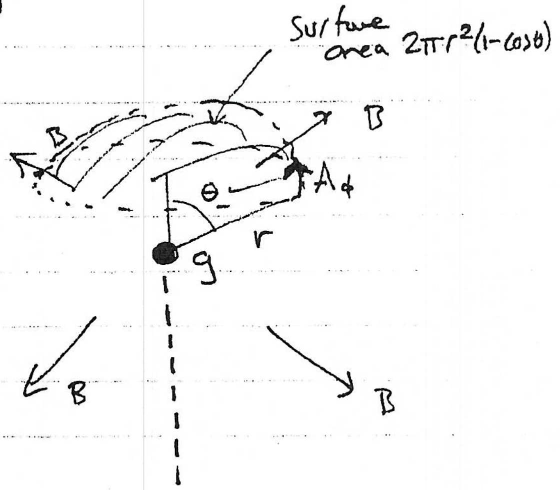
What about magnetic charge $\vec{\nabla}\cdot\vec{B}=4\pi\rho_M$?

Note: $\vec{\nabla}\cdot\vec{B}=\emptyset$ when $\vec{B}=\vec{\nabla}\times\vec{A}$ $[F=dA]$,

so need to generalize notion of vector potential, to get ρ_m.

say we have a magnetic charge g, so

$$\vec{B} = \frac{g}{r^2}\hat{r}.$$



What is $\vec{A}$?

Can try $\vec{A} = A\hat{\phi}$

$$\int \vec{A}\cdot d\vec{l} = 2\pi r \sin\theta\, A$$

$$= \frac{g}{r^2}\cdot 2\pi r^2(1-\cos\theta)$$

[Ex. show $\nabla\times\vec{A} = \vec{B}$ above]

so perhaps $\boxed{\vec{A} = \frac{g(1-\cos\theta)}{r\sin\theta}\hat{\phi}}$ (?). [valid for $\theta < \pi$]

Singular on z^- axis ("Dirac string")

$\int dA \overset{?}{=} 4\pi g$

Need to use another solution in region outside z^+:

$$\tilde{A} = -\frac{g(1+\cos\theta)}{r\sin\theta}\hat{\phi} \qquad [\theta > 0]$$

$\vec{A}, \tilde{A}$ related by a gauge transformation on $0 < \theta < \pi$.

combining local charts $\Rightarrow$ global picture

Mathematically: "Connection on a $U(1)$ fiber bundle over $\mathbb{R}^3 \setminus \{0\}$ with first Chern class 1".

Geometrically: circle over each point in space connected in topologically nontrivial fashion.

[POU useful for nonabelian gauge theories, Kaluza-Klein theories]

Classically, only $\vec{B}$ is physical, not $\vec{A}$, so "Dirac string" does not pose any obvious problems.

Quantum mechanically,

Recall $e^{\frac{ie}{\hbar c}\int_P A_\mu dx^\mu}$ enters propagator for charged particle moving along path P.

If $\oint A_\mu dx^\mu = 2\pi n \cdot \frac{\hbar c}{e}$, it is not observable. Since position is gauge choice, this must be the case.

Thus, we have

$$4\pi g = 2\pi n \cdot \frac{\hbar c}{e},$$

so

$$g = n \cdot \frac{\hbar c}{2|e|} \cong n\left(\frac{137}{2}\right)|e|.$$

Magnetic charge is <u>quantized</u> in units of $\frac{\hbar c}{2|e|}$.

Turning around, assume $\exists$ magnetic monopole of charge g.

Then any electric charge is

$$e = n\frac{\hbar c}{2|g|}.$$

Can explain why proton charge $= |e|$ (known to 4×10^{-19}).

Many models of fundamental physics (GUT's, etc...) predict monopoles

No monopoles seen yet in nature [except, possibly, 1].