

Lecture 3 - From Maxwell to QED

2.1 Maxwell's Equations

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho \quad (M1)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (M2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{B} \quad (M3)$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c^2} \frac{\partial}{\partial t} \vec{E} + \mu_0 \vec{j} \quad (M4)$$

- Lorentz equation ($v \ll c$):

$$m_\alpha \ddot{\vec{r}}_\alpha = q_\alpha \left(\vec{E} + \vec{v}_\alpha \times \vec{B} \right)$$

- Local conservation of charge (from M1 and M4)

$$\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \vec{j} = 0$$

$$\rho(\vec{r}, t) = \sum_\alpha q_\alpha \delta(\vec{r} - \vec{r}_\alpha(t))$$

$$\vec{j}(\vec{r}, t) = \sum_\alpha q_\alpha \vec{v}_\alpha(t) \delta(\vec{r} - \vec{r}_\alpha(t))$$

- Constants of Motion:

$$\text{Total energy: } H = \sum_\alpha \frac{1}{2} m_\alpha v_\alpha^2 + \frac{\epsilon_0}{2} \int d^3r \left(\vec{E}^2 + \vec{B}^2 \right)$$

$$\text{Total Momentum: } \vec{P} = \sum_\alpha m_\alpha \vec{v}_\alpha + \epsilon_0 \int d^3r \vec{E} \times \vec{B}$$

$$\text{Total angular momentum: } \vec{J} = \sum_\alpha \vec{r}_\alpha \times m_\alpha \vec{v}_\alpha + \epsilon_0 \int d^3r \vec{r} \times (\vec{E} \times \vec{B})$$

2.2 Vector Potential

$$(M2): \vec{\nabla} \cdot \vec{B} = 0 \Leftrightarrow \boxed{\vec{B} = \vec{\nabla} \times \vec{A}}$$

$$(M3): \vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t} \vec{\nabla} \times \vec{A} = -\vec{\nabla} \times \left(\frac{\partial}{\partial t} \vec{A} \right)$$

$$\rightarrow \vec{\nabla} \times (\vec{E} + \frac{\partial}{\partial t} \vec{A}) = 0 \Leftrightarrow \boxed{\vec{E} + \frac{\partial}{\partial t} \vec{A} = -\vec{\nabla} U}$$

Given vector potential \vec{A} and scalar potential U , (M2) and (M3) are automatically verified.

$$(M1) \Rightarrow \Delta U = -\frac{1}{\epsilon_0} \rho - \vec{\nabla} \cdot \frac{\partial}{\partial t} \vec{A}$$

$$(M4) \Rightarrow \left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta \right) \vec{A} = \mu_0 \vec{j} - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial}{\partial t} U \right)$$

State of the field given by $\vec{A}(\vec{r}, t_0)$ and $\frac{\partial}{\partial t} \vec{A}(\vec{r}, t_0) \forall \vec{r}$

- Gauge invariance:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} f$$

$$U \rightarrow U' = U - \frac{\partial}{\partial t} f$$

$$\text{Lorentz gauge: } \vec{\nabla} \cdot \vec{A} + \frac{1}{c^2} \frac{\partial}{\partial t} U = 0 \quad \text{or} \quad \partial_\mu A^\mu = 0$$

(M1)+(M4):

$$\square U = \frac{1}{\epsilon_0} \rho$$

$$\square \vec{A} = \mu_0 \vec{j}$$

$$\text{or } \partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu$$

$$\text{Here } \square \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \Delta, \partial_\mu = \left(\frac{1}{c} \frac{\partial}{\partial t}, \vec{\nabla} \right), A^\mu = \left(\frac{1}{c} U, \vec{A} \right), j^\mu = (c\rho, \vec{j})$$

$$\text{Coulomb gauge: } \vec{\nabla} \cdot \vec{A} = 0$$

$$(M1): \Delta U = -\frac{1}{\epsilon_0} \rho$$

Laplace equation for U . $U(\vec{r}, t)$ is completely specified by $\rho(\vec{r}, t)$!

$$\square \vec{A} = \mu_0 \vec{j} - \frac{1}{c^2} \vec{\nabla} \frac{\partial}{\partial t} U$$

2.3 Electrodynamics in reciprocal space

$$\vec{\mathcal{E}}(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3r \vec{E}(\vec{r}, t) e^{-i\vec{k} \cdot \vec{r}}$$

$$\vec{E}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \vec{\mathcal{E}}(\vec{k}, t) e^{i\vec{k} \cdot \vec{r}}$$

same for $\vec{E} \leftrightarrow \vec{\mathcal{E}}, \vec{B} \leftrightarrow \vec{\mathcal{B}}, \vec{A} \leftrightarrow \vec{\mathcal{A}}, U \leftrightarrow \mathcal{U}, \rho \leftrightarrow \rho, \vec{j} \leftrightarrow \vec{j}$

- Field equations become local in reciprocal space:

$$i\vec{k} \cdot \vec{\mathcal{E}} = \frac{1}{\epsilon_0} \rho \quad (R1)$$

$$i\vec{k} \cdot \vec{\mathcal{B}} = 0 \quad (R2)$$

$$i\vec{k} \times \vec{\mathcal{E}} = -\dot{\vec{\mathcal{B}}} \quad (R3)$$

$$i\vec{k} \times \vec{\mathcal{B}} = \frac{1}{c^2} \dot{\vec{\mathcal{E}}} + \mu_0 \vec{j} \quad (R4)$$

- Continuity equation:

$$i\vec{k} \cdot \vec{j} + \dot{\rho} = 0$$

- Fields and potentials:

$$\vec{\mathcal{B}} = i\vec{k} \times \vec{\mathcal{A}}$$

$$\vec{\mathcal{E}} = -\dot{\vec{\mathcal{A}}} - i\vec{k} \mathcal{U}$$

- Gauge transformation:

$$\vec{\mathcal{A}} \rightarrow \vec{\mathcal{A}}' = \vec{\mathcal{A}} + i\vec{k} f$$

$$\mathcal{U} \rightarrow \mathcal{U}' = \mathcal{U} - \dot{f}$$

- Equations for potentials:

$$k^2 \mathcal{U} = \frac{1}{\epsilon_0} \rho + i\vec{k} \cdot \dot{\vec{\mathcal{A}}}$$

$$\frac{1}{c^2} \ddot{\vec{\mathcal{A}}} + k^2 \vec{\mathcal{A}} = \mu_0 \vec{j} - i\vec{k} (i\vec{k} \cdot \vec{\mathcal{A}} + \frac{1}{c^2} \dot{\mathcal{U}})$$

- Longitudinal and transverse fields:

- Longitudinal field:

$$\vec{\nabla} \times \vec{V}_{||}(\vec{r}) = 0$$

$$i\vec{k} \times \vec{\mathcal{V}}_{\perp}(\vec{k}) = 0$$

- Transverse field:

$$\vec{\nabla} \cdot \vec{V}_\perp(\vec{r}) = 0$$

$$i\vec{k} \cdot \vec{V}_\perp(\vec{k}) = 0$$

in reciprocal space, decomposition is simple: (not so in real space!)

$$\vec{V}(\vec{k}) = \vec{V}_\parallel(\vec{k}) + \vec{V}_\perp(\vec{k})$$

$$\vec{V}_\parallel(\vec{k}) = \vec{\kappa}(\vec{\kappa} \cdot \vec{V}(\vec{k})) \quad \text{parallel to } \vec{k} \forall \vec{k}$$

$$\vec{V}_\perp(\vec{k}) = \vec{V}(\vec{k}) - \vec{V}_\parallel(\vec{k}) = (\vec{\kappa} \times \vec{V}) \times \vec{\kappa} \quad \perp \text{ to } \vec{k} \forall \vec{k}$$

Return to Maxwell:

$$(M2) \Rightarrow \vec{B}_\parallel = \vec{0} = \vec{B}_\parallel$$

\Rightarrow The magnetic field is purely transverse!

$$(M1) \Rightarrow \vec{E}_\parallel(\vec{k}) = -\frac{i}{\epsilon_0} \rho(\vec{k}) \frac{\vec{k}}{k^2}$$

That is a product of two functions of \vec{k} whose Fourier transforms are

$$\rho(\vec{k}) \leftrightarrow \rho(\vec{r})$$

$$-\frac{i}{\epsilon_0} \frac{\vec{k}}{k^2} \leftrightarrow \frac{(2\pi)^{3/2}}{4\pi\epsilon_0} \frac{\vec{r}}{r^3}$$

The Fourier transform of a product is a convolution:

$$\begin{aligned} \Rightarrow \vec{E}_\parallel(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0} \int d^3r' \rho(\vec{r}', t) \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \sum_\alpha q_\alpha \frac{\vec{r} - \vec{r}_\alpha(t)}{|\vec{r} - \vec{r}_\alpha(t)|^3} \quad (1) \end{aligned}$$

Longitudinal E-field = instantaneous Coulomb field of ρ .

Result independent of gauge!

- Q: Does this mean perturbations can travel faster than light?

Note: i)

$$\begin{aligned}\vec{\mathcal{E}} &= -\dot{\vec{\mathcal{A}}} - i\vec{k}\mathcal{U} \\ \Rightarrow \vec{\mathcal{E}}_{\perp} &= -\dot{\vec{\mathcal{A}}}_{\perp} & \Rightarrow \vec{E}_{\perp} &= -\dot{\vec{A}}_{\perp} \\ \vec{\mathcal{E}}_{\parallel} &= -\dot{\vec{\mathcal{A}}}_{\parallel} - i\vec{k}\mathcal{U} & \Rightarrow \vec{E}_{\parallel} &= -\dot{\vec{A}}_{\parallel} - \vec{\nabla}U\end{aligned}$$

In the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow i\vec{k} \cdot \vec{\mathcal{A}} = 0 \Rightarrow \vec{\mathcal{A}}_{\parallel} = 0$.

$$\begin{aligned}\vec{E}_{\perp} &= -\dot{\vec{A}}_{\perp} & \text{transverse field given by } \vec{A} \\ \vec{E}_{\parallel} &= -\vec{\nabla}U & \text{longitudinal field given by } U\end{aligned}$$

From \vec{E}_{\parallel} we find $U = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\vec{r}', t)}{|\vec{r} - \vec{r}'|}$ Coulomb potential.

ii) gauge transformation does not change \vec{A}_{\perp} !

transverse vector potential \vec{A}_{\perp} is gauge invariant:

$$\vec{A}'_{\perp} = \vec{A}_{\perp}$$

iii) generally, in any gauge:

$$\begin{aligned}\vec{\mathcal{E}}_{\perp} &= -\dot{\vec{\mathcal{A}}}_{\perp} \\ \vec{\mathcal{B}} &= \vec{\mathcal{B}}_{\perp} = i\vec{k} \times \vec{\mathcal{A}}_{\perp}\end{aligned}$$

Transverse fields only depend on $\vec{\mathcal{A}}_{\perp}$.

iv) longitudinal part of (M4) gives:

$$\begin{aligned}\dot{\vec{\mathcal{E}}}_{\parallel} + \frac{1}{\epsilon_0} \vec{j}_{\parallel} &= 0 \\ \Rightarrow i\vec{k} \cdot \dot{\vec{\mathcal{E}}}_{\parallel} + \frac{1}{\epsilon_0} i\vec{k} \cdot \vec{j}_{\parallel} &= i\vec{k} \cdot \dot{\vec{\mathcal{E}}} + \frac{1}{\epsilon_0} i\vec{k} \cdot \vec{j} = \frac{\dot{\rho}}{\epsilon_0} + \frac{i\vec{k} \cdot \vec{j}}{\epsilon_0} = 0 \\ \Rightarrow \dot{\rho} + \vec{\nabla} \cdot \vec{j} &= 0\end{aligned}$$

This is just the continuity equation, so this does not add anything new.

- Total Energy:

$$\frac{\epsilon_0}{2} \int d^3r \vec{E} \cdot \vec{E} = \frac{\epsilon_0}{2} \int d^3k \vec{\mathcal{E}}^* \cdot \vec{\mathcal{E}}$$

(Parseval-Plancherel Theorem)

$$\vec{\mathcal{E}} \rightarrow \vec{\mathcal{E}}_{\perp} + \vec{\mathcal{E}}_{\parallel}, \quad \vec{\mathcal{E}}_{\perp} \cdot \vec{\mathcal{E}}_{\parallel} = 0$$

$$\Rightarrow \frac{\epsilon_0}{2} \int d^3r \vec{E}^2 = \frac{\epsilon_0}{2} \int d^3k \left| \vec{\mathcal{E}}_{\parallel}(\vec{k}) \right|^2 + \frac{\epsilon_0}{2} \int d^3k \left| \vec{\mathcal{E}}_{\perp}(\vec{k}) \right|^2$$

$$H_{\text{long}} = \frac{\epsilon_0}{2} \int d^3k \left| \vec{\mathcal{E}}_{\parallel}(\vec{k}) \right|^2 = \frac{\epsilon_0}{2} \int d^3r \vec{E}_{\parallel}^2(\vec{r})$$

$$\begin{aligned} H_{\text{trans}} &= \frac{\epsilon_0}{2} \int d^3k \left(\left| \vec{\mathcal{E}}_{\perp}(\vec{k}) \right|^2 + c^2 \left| \vec{\mathcal{B}}(\vec{k}) \right|^2 \right) \\ &= \frac{\epsilon_0}{2} \int d^3r \left(\left| \vec{E}_{\perp}(\vec{r}) \right|^2 + c^2 \vec{B}^2(\vec{r}) \right) \end{aligned}$$

H_{long} is the Coulomb energy:

$$H_{\text{long}} = \frac{1}{2\epsilon_0} \int d^3k \rho^*(\vec{k}) \frac{\rho(\vec{k})}{k^2} = \frac{1}{8\pi\epsilon_0} \iint d^3r d^3r' \frac{\rho(\vec{r})\rho(\vec{r}')}{(|\vec{r} - \vec{r}'|)}$$

For a system of charges:

$$\rho(\vec{r}) = \sum_{\alpha} q_{\alpha} \delta(\vec{r} - \vec{r}_{\alpha}(t))$$

$$\rho(\vec{k}) = \sum_{\alpha} \frac{q_{\alpha}}{(2\pi)^{3/2}} e^{-i\vec{k} \cdot \vec{r}_{\alpha}}$$

$$\begin{aligned} H_{\text{long}} = V_{\text{Coulomb}} &= \sum_{\alpha} \frac{q_{\alpha}^2}{2\epsilon_0(2\pi)^3} \int d^3k \frac{1}{k^2} + \sum_{\alpha \neq \beta} \frac{q_{\alpha}q_{\beta}}{2\epsilon_0(2\pi)^3} \int d^3k \frac{e^{-i\vec{k} \cdot (\vec{r}_{\alpha} - \vec{r}_{\beta})}}{k^2} \\ &= \sum_{\alpha} \epsilon_{\text{Coulomb}}^{\alpha} + \frac{1}{8\pi\epsilon_0} \sum_{\alpha \neq \beta} \frac{q_{\alpha}q_{\beta}}{|\vec{r}_{\alpha} - \vec{r}_{\beta}|} \end{aligned}$$

The first term

$$\epsilon_{\text{Coulomb}}^{\alpha} = \frac{q_{\alpha}^2}{2\epsilon_0(2\pi)^3} \int d^3k \frac{1}{k^2} = \frac{q_{\alpha}^2}{4\pi^2\epsilon_0} k_c$$

is the Coulomb self energy of the particle α , infinite unless we introduce a cut-off k_c . Physically, k_c should be the momentum scale at which our non-relativistic treatment breaks down, so $k_c \approx \frac{1}{\lambda_{\text{Compton}}}$, with $\lambda_{\text{Compton}} = \frac{\hbar}{m_e c}$.

The second term is the Coulomb interaction between particles.

Finally we obtain the Hamiltonian:

$$H = \sum_{\alpha} \frac{1}{2} m_{\alpha}^2 \dot{\vec{r}}_{\alpha}^2 + V_{\text{Coulomb}} + H_{\text{trans}}$$

- Total Momentum:

Let's use again $\vec{E} \rightarrow \vec{E}_{\perp} + \vec{E}_{\parallel}$ in the definition of momentum, and find the parts:

$$\vec{P}_{\text{long}} = \epsilon_0 \int d^3r \vec{E}_{\parallel}(\vec{r}) \times \vec{B}(\vec{r}) = \epsilon_0 \int d^3k \vec{\mathcal{E}}_{\parallel}^*(\vec{k}) \times \vec{\mathcal{B}}(\vec{k})$$

$$\vec{P}_{\text{trans}} = \epsilon_0 \int d^3r \vec{E}_{\perp}(\vec{r}) \times \vec{B}(\vec{r}) = \epsilon_0 \int d^3k \vec{\mathcal{E}}_{\perp}^*(\vec{k}) \times \vec{\mathcal{B}}(\vec{k})$$

$$\text{We plug in: } \vec{\mathcal{E}}_{\parallel} = -\frac{i}{\epsilon_0} \rho(\vec{k}) \frac{\vec{k}}{k^2}; \quad \vec{\mathcal{B}} = i\vec{k} \times \vec{\mathcal{A}}$$

$$\begin{aligned} \vec{P}_{\text{long}} &= \int d^3k \rho^* \frac{\vec{k}}{k^2} \times (i\vec{k} \times \vec{\mathcal{A}}) \\ &= \int d^3k \rho^* [\vec{\mathcal{A}} - \vec{\kappa}(\vec{\kappa} \cdot \vec{\mathcal{A}})] \\ &= \int d^3k \rho^* \vec{\mathcal{A}}_{\perp} = \int d^3r \rho \vec{A}_{\perp} = \sum_{\alpha} q_{\alpha} \vec{A}_{\perp}(\vec{r}_{\alpha}) \end{aligned}$$

Independent of gauge, as \vec{A}_{\perp} is independent of gauge.

total momentum:

$$\vec{P} = \sum_{\alpha} \left(m_{\alpha} \dot{\vec{r}}_{\alpha} + q_{\alpha} \vec{A}_{\perp}(\vec{r}_{\alpha}) \right) + \vec{P}_{\text{trans}}$$

The three terms are 1) the mechanical momentum, 2) the longitudinal field momentum and 3) the transverse field momentum.

$$\vec{P} = \sum_{\alpha} \vec{p}_{\alpha} + \vec{P}_{\text{trans}}$$

$$\vec{p}_{\alpha} = m_{\alpha} \dot{\vec{r}}_{\alpha} + q_{\alpha} \vec{A}_{\perp}(\vec{r}_{\alpha})$$

In Coulomb gauge, \vec{p}_{α} is the canonical (or generalized) momentum to the coordinate \vec{r}_{α} of particle α .

Total energy:

$$H = \sum_{\alpha} \frac{1}{2m_{\alpha}} \left[\vec{p}_{\alpha} - q_{\alpha} \vec{A}_{\perp}(\vec{r}_{\alpha}) \right]^2 + V_{\text{Coulomb}} + H_{\text{trans}}$$

This is precisely the Hamiltonian of the system in the coulomb gauge!

Total angular momentum:

$$\vec{J} = \sum_{\alpha} \vec{r}_{\alpha} \times \vec{p}_{\alpha} + \epsilon_0 \int d^3r \vec{r} \times (\vec{E}_{\perp} \times \vec{B})$$

2.4 Normal Variables

Maxwell's equations for transverse fields:

$$(M3): \dot{\vec{B}} = -i\vec{k} \times \vec{\mathcal{E}}_{\perp}$$

$$\Rightarrow \boxed{\vec{k} \times \dot{\vec{B}} = ik^2 \vec{\mathcal{E}}_{\perp}}$$

$$(M4): \boxed{\dot{\vec{\mathcal{E}}}_{\perp} = ic^2 \vec{k} \times \vec{B} - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{k}, t)}$$

These are two coupled equations for $\vec{\mathcal{E}}_{\perp}$, $\vec{k} \times \vec{B}$, that are local in reciprocal space.

Using $\vec{B} = i\vec{k} \times \vec{\mathcal{A}} = i\vec{k} \times \vec{\mathcal{A}}_{\perp}$ we have

$$\vec{k} \times \vec{\mathcal{B}} = -ik^2 \vec{\mathcal{A}}_{\perp}$$

Therefore we have, with $\omega = ck$

$$\begin{aligned} \dot{\vec{\mathcal{A}}}_{\perp} &= -\vec{\mathcal{E}}_{\perp} \\ \dot{\vec{\mathcal{E}}}_{\perp} &= \omega^2 \vec{\mathcal{A}}_{\perp} - \frac{1}{\epsilon_0} \vec{j}_{\perp}(\vec{k}, t) \end{aligned}$$

This is two coupled differential equations, just like

$$\begin{aligned} \dot{\vec{x}} &= \frac{\vec{p}}{m} \\ \frac{\dot{\vec{p}}}{m} &= -\omega_0^2 \vec{x} + \frac{q}{m} \vec{E}(t) \end{aligned}$$

So we read off the correspondence $\vec{x} \equiv \vec{\mathcal{A}}_{\perp}$ and $\frac{\vec{p}}{m} \equiv -\vec{\mathcal{E}}_{\perp}$.

The goal now is to uncouple these equations, at least for the free field, $\vec{j}_{\perp} = 0$.

Normal variables for the position / momentum case:

$$\begin{aligned} \alpha &= \mathcal{N}(x + i \frac{p}{m\omega_0}) \\ \beta &= \mathcal{N}(x - i \frac{p}{m\omega_0}) = \alpha^* \end{aligned}$$

$$\begin{aligned} \text{equations of motion: } \dot{\alpha} &= \mathcal{N}(\dot{x} + i \frac{\dot{p}}{m\omega_0}) = \mathcal{N}(\frac{p}{m} - i\omega_0 x + i \frac{q}{m\omega_0} E) = \\ &= -i\omega_0 \alpha + i \frac{q\mathcal{N}}{m\omega_0} E \end{aligned}$$

$$\dot{\alpha} = -i\omega_0 \alpha + i \frac{q\mathcal{N}}{m\omega_0} E$$

quantization:

$$\alpha \rightarrow a$$

$$\beta \rightarrow a^{\dagger}$$

$$[a, a^{\dagger}] = -\mathcal{N}^2 \frac{i}{m\omega_0} 2[x, p] = \mathcal{N}^2 \frac{2\hbar}{m\omega_0} \stackrel{!}{=} 1$$

$$\text{Therefore } \mathcal{N} = \sqrt{\frac{m\omega_0}{2\hbar}} = \frac{1}{\sqrt{2}} \frac{1}{a_{\text{h.o.}}}.$$

Using our correspondence $\vec{x} \equiv \vec{\mathcal{A}}_{\perp}$ and $\frac{\vec{p}}{m} \equiv -\vec{\mathcal{E}}_{\perp}$ we find

$$\begin{aligned}\vec{\alpha}(\vec{k}, t) &= \mathcal{N}(k) \left(\vec{\mathcal{A}}_{\perp} - i \frac{\vec{\mathcal{E}}_{\perp}}{\omega_0} \right) \\ \vec{\beta}(\vec{k}, t) &= \mathcal{N}(k) \left(\vec{\mathcal{A}}_{\perp} + i \frac{\vec{\mathcal{E}}_{\perp}}{\omega_0} \right)\end{aligned}$$

In terms of $\vec{\mathcal{B}} = i\vec{k} \times \vec{\mathcal{A}}_{\perp} \Rightarrow \vec{\mathcal{A}}_{\perp} = i \frac{\vec{k}}{k^2} \times \vec{\mathcal{B}} = \frac{i}{k} \vec{\kappa} \times \vec{\mathcal{B}}$.

$$\begin{aligned}\vec{\alpha}(\vec{k}, t) &= \frac{i}{ck} \mathcal{N}(k) \left(c\vec{\kappa} \times \vec{\mathcal{B}} - \vec{\mathcal{E}}_{\perp} \right) \\ \vec{\beta}(\vec{k}, t) &= \frac{i}{ck} \mathcal{N}(k) \left(c\vec{\kappa} \times \vec{\mathcal{B}} + \vec{\mathcal{E}}_{\perp} \right)\end{aligned}$$

Note: \vec{E}_{\perp} is real, so $\vec{\mathcal{E}}^*(\vec{k}, t) = \vec{\mathcal{E}}(-\vec{k}, t)$ and thus

$$\begin{aligned}\alpha^*(\vec{k}, t) &= -\frac{i}{ck} \mathcal{N}(k) \left(c\vec{\kappa} \times \vec{\mathcal{B}}^*(\vec{k}) - \vec{\mathcal{E}}_{\perp}^*(\vec{k}) \right) \\ &= -\frac{i}{ck} \mathcal{N}(k) \left(c\vec{\kappa} \times \vec{\mathcal{B}}(-\vec{k}) - \vec{\mathcal{E}}_{\perp}(-\vec{k}) \right) \\ &= -\frac{i}{ck} \mathcal{N}(k) \left(-c(-\vec{\kappa}) \times \vec{\mathcal{B}}(-\vec{k}) - \vec{\mathcal{E}}_{\perp}(-\vec{k}) \right) \\ &= +\vec{\beta}(-\vec{k}, t)\end{aligned}\quad (2)$$

So $\vec{\beta}(\vec{k}, t)$ is not independent of $\vec{\alpha}(\vec{k}, t)$.

Therefore knowing $\vec{\alpha}(\vec{k}, t) \forall \vec{k}$ is fully equivalent to knowing $\vec{\mathcal{E}}_{\perp}$ and $\vec{\mathcal{B}}$.

- Time evolution:

...

$$\dot{\vec{\alpha}}(\vec{k}, t) + i\omega_0 \vec{\alpha}(\vec{k}, t) = \frac{i\mathcal{N}(k)}{\epsilon_0 \omega} \vec{j}_{\perp}(\vec{k}, t)$$

This equation is strictly equivalent to Maxwell's equations!

$$\begin{aligned}\vec{\mathcal{E}}_{\perp}(\vec{k}, t) &= \frac{i}{2} \frac{\omega_0}{\mathcal{N}(k)} \left(\vec{\alpha}(\vec{k}, t) - \vec{\alpha}^*(-\vec{k}, t) \right) \\ \vec{\mathcal{B}}(\vec{k}, t) &= \frac{i}{2} \frac{k}{\mathcal{N}(k)} \vec{\kappa} \times \left(\vec{\alpha}(\vec{k}, t) + \vec{\alpha}^*(\vec{k}, t) \right)\end{aligned}$$

So what we have is equivalent to a driven harmonic oscillator with eigenfrequency $\omega = ck$, driven by a source term proportional to $\vec{j}_{\perp}(\vec{k}, t)$.

