Assignment 2; MA353; S19

Leo Livshits

Last modified at 10:10 on February 22, 2019

1 Preliminaries

Definition 1.1

When $\boldsymbol{V}$ is a vector space, we write $\mathfrak{L}(\boldsymbol{V})$ for the linear space of all linear functions $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$.

For any subspace $\boldsymbol{W}$ of $\boldsymbol{V}$, and any such $\mathcal{L}$, we write

$$\mathcal{L}[\boldsymbol{W}] := \{ \ \mathcal{L}(w) \ | \ \ w \in \boldsymbol{W} \ \} = Range\left(\mathcal{L}_{|_{\boldsymbol{W}}}\right),$$

where $\mathcal{L}_{|_{\boldsymbol{W}}} : \boldsymbol{W} \longrightarrow \boldsymbol{V}$ is the restriction of the function $\mathcal{L}$ to a (new) domain $\boldsymbol{W}$.

Since it is easy to see that $\mathcal{L}_{|_{\boldsymbol{W}}}$ is a linear function, $\mathcal{L}[\boldsymbol{W}]$ is a subspace of $\boldsymbol{V}$. We refer to $\mathcal{L}[\boldsymbol{W}]$ as **the image of $\boldsymbol{W}$ under $\mathcal{L}$**.

We will write $\mathcal{L}^2[\boldsymbol{W}]$ for

$$\mathcal{L}[\mathcal{L}[\boldsymbol{W}]] \text{ , etc.}$$

As usual we set $\mathcal{L}^0$ to be the identity function on $\boldsymbol{V}$, so that

$$\mathcal{L}^0[\boldsymbol{W}] = \boldsymbol{W} \ .$$

In a similar fashion we define

$$\mathcal{L}^{-1}[\boldsymbol{W}] := \{ \ v \in \boldsymbol{V} \ | \ \mathcal{L}(v) \in \boldsymbol{W} \ \} \ .$$

The student is encouraged to check that $\mathcal{L}^{-1}[\boldsymbol{W}]$ is a subspace of $\boldsymbol{V}$.

We refer to $\mathcal{L}^{-1}[\boldsymbol{W}]$ as **the pre-image of $\boldsymbol{W}$ under $\mathcal{L}$**.

We write $\mathcal{L}^{\scriptscriptstyle -2}[\boldsymbol{W}]$ for

$$\mathcal{L}^{\scriptscriptstyle -1}[\mathcal{L}^{\scriptscriptstyle -1}[\boldsymbol{W}]]\text{ , etc.}$$

$\mathcal{L}^{-2}[\boldsymbol{W}]$ is another way of referring to $\mathcal{L}^{\scriptscriptstyle -2}[\boldsymbol{W}]$, and the same goes for other powers.

Note that $\mathcal{L}$ may not be invertible and so* $\mathcal{L}^{-2}[\boldsymbol{W}]$ should NOT be interpreted as $\left(\mathcal{L}^{-1}\right)^2[\boldsymbol{W}]$, etc. This explains my use of a (non-standard) notation $\mathcal{L}^{\scriptscriptstyle -2}[\boldsymbol{W}]$ here.

---

* In contrast to a valid formula

$$\mathcal{L}^{\scriptscriptstyle 2}[\boldsymbol{W}] = \mathcal{L}[\mathcal{L}[\boldsymbol{W}]] = (\mathcal{L})^2\,[\boldsymbol{W}]\,.$$

Definition 1.2

A subspace $\boldsymbol{W}$ of $\boldsymbol{V}$ is said to be **invariant** under $\mathcal{L}$ whenever $\mathcal{L}_{|_{\boldsymbol{W}}} : \boldsymbol{W} \longrightarrow \boldsymbol{W}$; in other words, when

$$\mathcal{L}[\boldsymbol{W}] \subseteq \boldsymbol{W}\ .$$

Note that $\boldsymbol{V}$ and $\{O_{\boldsymbol{V}}\}$ are invariant under every $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$.

The collection of all invariant subspaces for $\mathcal{L}$ is denoted by $\mathfrak{Lat}(\mathcal{L})$.

Test Your Comprehension 1.3

Argue that for any $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$, the subspaces $\boldsymbol{V}$, $\{0_{\boldsymbol{V}}\}$, *Range* $(\mathcal{L})$ and *Nullspace* $(\mathcal{L})$ are invariant under $\mathcal{L}$. *

---

*We will improve on this in the problems below.

2 Problems

Problem 1 ☞ Finite unions of subspaces are rarely a subspace

Suppose that $\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_n$ are subspaces of a vector space $\boldsymbol{V}$. Prove that the following are equivalent:

1. $\boldsymbol{W}_1 \cup \boldsymbol{W}_2 \cup \cdots \cup \boldsymbol{W}_n$ is a subspace of $\boldsymbol{V}$;
2. One of the $\boldsymbol{W}_i$'s contains all the others.

Hint: [1. $\Rightarrow$ 2.]: Suppose this implication is not always true. Then there must be a smallest n for which it fails; let us call it n_0. Since the implication is trivially true for $n = 1$, such an n_0 must be at least 2, and the implication does hold true for all smaller n.
Let $\boldsymbol{W}_1, \boldsymbol{W}_2, \ldots, \boldsymbol{W}_{n_0}$ be a collection of subspaces of a vector space $\boldsymbol{V}$ such that $\boldsymbol{W}_1 \cup \boldsymbol{W}_2 \cup \cdots \cup \boldsymbol{W}_{n_0}$ is a subspace of $\boldsymbol{V}$, but none of the $\boldsymbol{W}_i$'s contains all the others.
Argue that no $\boldsymbol{W}_i$ is contained in the union of the other $\boldsymbol{W}_j$'s.
Then argue that each $\boldsymbol{W}_i$ contains an element w_i that does not belong to any other $\boldsymbol{W}_j$'s.
Argue that $w_1 + w_2$ belongs to $\boldsymbol{W}_1 \cup \boldsymbol{W}_2 \cup \cdots \cup \boldsymbol{W}_n$.
Argue that $w_1 + w_2$ belongs to $\boldsymbol{W}_2 \cup \cdots \cup \boldsymbol{W}_n$.
Repeat with $2w_1 + w_2$, $3w_1 + w_2$, $4w_1 + w_2, \ldots, n_0 w_1 + w_2$.
Argue that one of $\boldsymbol{W}_2, \ldots, \boldsymbol{W}_{n_0}$ contains $kw_1 + w_2$ for two distinct values of k.
Infer a contradiction.

Problem 2 ☞ Images of pre-images and pre-images of images

Is there a 3×3 matrix $\mathcal{A}$ such that

$$\boldsymbol{W} := \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \;\middle|\; x, y \in \mathbb{C} \right\}$$

is an invariant subspace for $\mathcal{A}$, and

$$\mathcal{A}[\mathcal{A}^{-1}[\boldsymbol{W}]] \subsetneq \boldsymbol{W} \subsetneq \mathcal{A}^{-1}[\mathcal{A}[\boldsymbol{W}]]\ ?$$

Justify your answer.

Problem 3 ☞ Invariant subspaces form a "lattice"

1. Argue that $\mathfrak{Lat}(\mathcal{L})$ is closed under intersection; i.e. that the intersection of any two invariant subspaces for $\mathcal{L}$ is also an invariant subspace for $\mathcal{L}$.
2. Argue that $\mathfrak{Lat}(\mathcal{L})$ is closed under subspace sums; i.e. that a subspace sum of two invariant subspaces for $\mathcal{L}$ is again an invariant subspace for $\mathcal{L}$.
3. Show that an image under $\mathcal{L}$ of an invariant subspace for $\mathcal{L}$ is again an invariant subspace for $\mathcal{L}$.
4. Show that a pre-image under $\mathcal{L}$ of an invariant subspace for $\mathcal{L}$ is again an invariant subspace for $\mathcal{L}$.

Mathematical lattice is a partially ordered set such that every pair of elements A and B has a "join" and a "meet" within the lattice. The join is the smallest of the elements that dominate both A and B (with respect to the order), and the meet is the largest of the elements dominated by both A and B.
A collection of all subspaces of a vector space $\boldsymbol{V}$ is partially ordered by the operation of inclusion, and is a lattice with respect to this operation, with the meet of two subspaces being their intersection, and the join being their subspace sum.
In this problem you are verifying that $\mathfrak{Lat}(\mathcal{L})$ is a sub-lattice of the lattice of all subspaces of $\boldsymbol{V}$. It is common to refer to $\mathfrak{Lat}(\mathcal{L})$ as "the lattice of the invariant subspaces of $\mathcal{L}$".

Problem 4 ☞ **Cyclic invariant subspaces**

For a given $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$ and a fixed $v_{\circ} \in \boldsymbol{V}$, define

$$\boldsymbol{P}(\mathcal{L}, v_{\circ}) := \left\{ \left(a_0\mathcal{L}^0 + a_1\mathcal{L} + a_2\mathcal{L}^2 + \cdots + a_k\mathcal{L}^k\right)(v_{\circ}) \;\middle|\; k \geq 0, \quad a_i \in \mathbb{C} \right\} .$$

1. Argue that $\boldsymbol{P}(\mathcal{L}, v_{\circ})$ is a subspace of $\boldsymbol{V}$.
2. Argue that $\boldsymbol{P}(\mathcal{L}, v_{\circ})$ is invariant under $\mathcal{L}$.
3. Argue that $\boldsymbol{P}(\mathcal{L}, v_{\circ})$ is the smallest invariant subspace for $\mathcal{L}$ that contains $v_{\circ}$. We refer to $\boldsymbol{P}(\mathcal{L}, v_{\circ})$ as **the cyclic invariant subspace for $\mathcal{L}$ generated by $v_{\circ}$** .
4. Argue that $\boldsymbol{P}(\mathcal{L}, v_{\circ})$ is 1-dimensional exactly when $v_{\circ}$ is an eigenvector of $\mathcal{L}$.
5. Argue a subspace $\boldsymbol{W}$ of $\boldsymbol{V}$ is invariant under $\mathcal{L}$ exactly when it is a union of cyclic invariant subspaces for $\mathcal{L}$.

Problem 5 ☞ **Invariant subspaces of commuting operators**

Suppose that linear functions $\mathcal{L}, \mathcal{M} \in \mathfrak{L}(\boldsymbol{V})$ commute; i.e.

$$\mathcal{L} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{L} .$$

1. Argue that for any non-negative integer k, *Range* $\left(\mathcal{M}^k\right)$ and *Nullspace* $\left(\mathcal{M}^k\right)$ are invariant subspaces for $\mathcal{L}$.
2. Argue that every eigenspace of $\mathcal{M}$ is an invariant subspace for $\mathcal{L}$.
3. By giving a general example (with justification, of course!) show that for each $n > 1$ there are commuting matrices $\mathcal{A}$ and $\mathcal{B}$ in $\mathbb{M}_n$ such that
$$\mathfrak{Lat}(\mathcal{A}) \neq \mathfrak{Lat}(\mathcal{B}) \ .$$

Problem 6 ☞ **Invariant subspace chains**

Suppose that $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$ and $\boldsymbol{W}$ is an invariant subspace of $\mathcal{L}$.

1. Argue that the following inclusions hold:

$$\cdots \subseteq \mathcal{L}^3[\boldsymbol{W}] \subseteq \mathcal{L}^2[\boldsymbol{W}] \subseteq \mathcal{L}[\boldsymbol{W}] \subseteq \boldsymbol{W} \subseteq \mathcal{L}^{-1}[\boldsymbol{W}] \subseteq \mathcal{L}^{-2}[\boldsymbol{W}] \subseteq \mathcal{L}^{-3}[\boldsymbol{W}] \subseteq \cdots$$

   Note that by Problem 3, all of the subspaces listed in the chain are invariant under $\mathcal{L}$.

2. Argue that the following implications hold:

$$\mathcal{L}^{k+1}[\boldsymbol{W}] = \mathcal{L}^{k}[\boldsymbol{W}] \Longrightarrow \mathcal{L}^{m}[\boldsymbol{W}] = \mathcal{L}^{k}[\boldsymbol{W}] \text{ for any } m \geq k,$$

   and

$$\mathcal{L}^{-k}[\boldsymbol{W}] = \mathcal{L}^{-(k+1)}[\boldsymbol{W}] \Longrightarrow \mathcal{L}^{-k}[\boldsymbol{W}] = \mathcal{L}^{-m}[\boldsymbol{W}] \text{ for any } m \geq k\ ,$$

   and then explain why the chain in part 1 stabilizes in both directions, and has at most $\dim(\boldsymbol{V})$ proper inclusions.

3. Argue that any subspace $\boldsymbol{M}$ that falls between two consecutive subspaces in the chain shown in part 1,* is also invariant under $\mathcal{L}$.

*for example, if $\mathcal{L}^{-3}[\boldsymbol{W}] \subseteq \boldsymbol{M} \subseteq \mathcal{L}^{-2}[\boldsymbol{W}]$, etc.