Classical Mechanics III (8.09)
Assignment 6: Solutions

October 17, 2021

1. Charged Particle in a Plane [12 points]

(a) [6 points] We will use cylindrical coordinates, assuming $B = B\hat{z}$. Then $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r} = \frac{1}{2}Br\hat{\theta}$. The Lagrangian for this system is

$$\begin{aligned} L &= \frac{1}{2}m\vec{v}^2 + q\vec{A}\cdot\vec{v} - \frac{1}{2}kr^2 \\ &= \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{qB}{2}r^2\dot{\theta} - \frac{1}{2}kr^2 \\ &= \frac{1}{2}\begin{pmatrix} \dot{r} & \dot{\theta} \end{pmatrix}\begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix}\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} + \frac{qB}{2}r^2\dot{\theta} - \frac{1}{2}kr^2 \end{aligned}$$

This is of the general form $L = L_0(\vec{q}) + \dot{\vec{q}}\cdot\vec{a} + \frac{1}{2}\dot{\vec{q}}^T\hat{T}\dot{\vec{q}}$ for which we know the Hamiltonian. Here $\vec{q} = \begin{pmatrix} r \\ \theta \end{pmatrix}$, $L_0 = -\frac{1}{2}kr^2$, $\vec{a} = \begin{pmatrix} 0 \\ qBr^2/2 \end{pmatrix}$, and $\hat{T} = \begin{pmatrix} m & 0 \\ 0 & mr^2 \end{pmatrix}$. Therefore

$$\begin{aligned} H &= \frac{1}{2}(\vec{p}-\vec{a})^T\hat{T}^{-1}(\vec{p}-\vec{a}) - L_0 \\ &= \frac{1}{2}\begin{pmatrix} p_r & p_\theta - \frac{qBr^2}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \frac{1}{mr^2} \end{pmatrix}\begin{pmatrix} p_r \\ p_\theta - \frac{qBr^2}{2} \end{pmatrix} + \frac{1}{2}kr^2 \\ &= \frac{p_r^2}{2m} + \frac{1}{2mr^2}\left(p_\theta - \frac{qBr^2}{2}\right)^2 + \frac{1}{2}kr^2. \end{aligned}$$

Since H is time-independent, it is conserved, and hence (in terms of Hamilton's characteristic function W)

$$H = \frac{1}{2m}\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W}{\partial \theta} - \frac{qBr^2}{2}\right)^2 + \frac{1}{2}kr^2 = \alpha_1.$$

Let's make use of the fact that θ is a cyclic coordinate. Then we immediately get

$$W = W_r(r,\alpha) + \alpha_2\theta$$

where $\alpha_2 = p_\theta$ is the constant value of the momentum conjugate to θ (note that $p_\theta = mr^2\dot{\theta} + \frac{qBr^2}{2}$

is not the same as the mechanical angular momentum). Hence

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2m}\left(\frac{\alpha_2}{r} - \frac{qBr}{2}\right)^2 + \frac{1}{2}kr^2 = \alpha_1.$$

Thus we have $W = W_r + \alpha_2\theta$ and

$$W_r = \pm\int\left[2m\alpha_1 - mkr^2 - \left(\frac{\alpha_2}{r} - \frac{qBr}{2}\right)^2\right]^{1/2} dr$$

(b) [6 points] From the results for the new coordinates we have two equations:

$$t + \beta_1 = \frac{\partial W}{\partial \alpha_1} = \pm m\int \frac{1}{\sqrt{2m\alpha_1 - mkr^2 - \left(\frac{\alpha_2}{r} - \frac{qBr}{2}\right)^2}} dr$$

$$\beta_2 = \frac{\partial W}{\partial \alpha_2} = \theta \pm \int \frac{\frac{qB}{2} - \frac{\alpha_2}{r^2}}{\sqrt{2m\alpha_1 - mkr^2 - \left(\frac{\alpha_2}{r} - \frac{qBr}{2}\right)^2}} dr$$

After having formed these equations we can now plug in our initial condition $\alpha_2 = p_\theta = 0$. In this case the two integrals are the same up to constant prefactors. For the first equation we have

$$t + \beta_1 = \pm m\int\left[2m\alpha_1 - m^2\left(\frac{k}{m} + \frac{q^2B^2}{4m^2}\right)r^2\right]^{-1/2} dr = \pm\frac{1}{\omega}\sin^{-1}(r\sqrt{\frac{m\omega^2}{2\alpha_1}})$$

where we note the similarity to the harmonic oscillator example, where we've defined $\omega = \sqrt{\frac{k}{m} + \frac{q^2B^2}{4m^2}}$. Inverting this (and taking the plus sign; the minus sign only adds a constant to β_1) we have

$$r = \sqrt{\frac{2\alpha_1}{m\omega^2}}\sin\big(\omega(t + \beta_1)\big).$$

Since we have done the integral we can use it in the second equation to give

$$\beta_2 = \theta \pm \frac{qB}{2}\int\left[2m\alpha_1 - \left(mk + \frac{q^2B^2}{4}\right)r^2\right]^{-1/2} dr = \theta + \frac{qB}{2m}(t + \beta_1)\,.$$

Thus

$$\theta = -\frac{qB}{2m}t + \beta'$$

for some constant β'. We have that the radial distance undergoes simple harmonic motion with angular frequency $\omega = \sqrt{\frac{k}{m} + \frac{q^2B^2}{4m^2}}$, while θ decreases linearly, i.e. the particle travels in a clockwise fashion with constant angular velocity. [Aside: To see this in mathematica try using (where we set

$\omega = 1$ and $2\alpha_1/m\omega^2 = 1$ and $\beta' = 0$ for convenience):

$Animate[ParametricPlot[Evaluate[\{Sin[t+a]Cos[alpha], Sin[t+a]Sin[alpha]\}/.\ alpha{-}{>}$
$-0.3t\ /.\ a{-}{>}1.2], \{t, 0, tm\}, PlotStyle- > \{Thick, Blue\}, PlotRange- >$
$\{\{-1.3, 3.5\}, \{-3, 3\}\}], \{tm, 0, 70\}]$

The orbital motion $r = r(\theta)$ traces out "flower" patterns. Here the constant $\beta_1 = 1.2$, and the constant $qB/(2m) = 0.3 < \omega = 1$, and you may change these values to see how it effects the closure of the orbit. The use of "Animate" allows us to see what happens as time increases.]

2. A Time Dependent H [10 points]

The Hamilton-Jacobi equation is $H(x, \frac{\partial S}{\partial x}, t) + \frac{\partial S}{\partial t} = 0$, or in our case

$$\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 - mAtx + \frac{\partial S}{\partial t} = 0$$

The form of the Hamilton-Jacobi equation suggests that we try a solution of the form

$$S(x, \alpha, t) = f(\alpha, t)x + g(\alpha, t)$$

since after substitution of this form the partial derivative terms and hence the resulting equation is only linear in x. Plugging this in,

$$\frac{f^2}{2m} - mAtx + \frac{\partial f}{\partial t}x + \frac{\partial g}{\partial t} = 0.$$

This equation is satisfied if

$$\frac{\partial f}{\partial t} = mAt$$

$$\frac{\partial g}{\partial t} = -\frac{f^2}{2m}$$

which can be straightfowardly integrated to give $f = \frac{1}{2}mAt^2 + \alpha$ (for some constant α) and

$$g = -\frac{1}{2m}\left(\frac{m^2A^2t^5}{20} + \frac{mA\alpha t^3}{3} + \alpha^2 t\right)$$

(we've dropped an additive constant from g, since it is just an additive constant in S.) Therefore

$$S(x, \alpha, t) = \left(\frac{1}{2}mAt^2 + \alpha\right)x - \frac{mA^2t^5}{40} - \frac{A\alpha t^3}{6} - \frac{\alpha^2 t}{2m}.$$

We can take the constant α to be the new momentum; then the corresponding (conserved) coordinate is

$$\beta = \frac{\partial S}{\partial \alpha} = x - \frac{At^3}{6} - \frac{\alpha t}{m}$$

or

$$x = \frac{A}{6}t^3 + \frac{\alpha}{m}t + \beta.$$

The old momentum is given by

$$p = \frac{\partial S}{\partial x} = \frac{1}{2}mAt^2 + \alpha$$

and $p = m\dot{x}$, as expected. Finally, plugging in the initial conditions $x(t=0) = 0$ and $p(t=0) = mv_0$ we get $\alpha = mv_0$ and $\beta = 0$, so

$$x = \frac{A}{6}t^3 + v_0 t$$

$$p = \frac{mA}{2}t^2 + mv_0.$$

3. The $|x|$ potential [10 points, 8.09 ONLY]

The energy is given by

$$H = \frac{p^2}{2m} + F|x| = E,$$

where E is a constant. The turning points of the motion are when $p = 0$, i.e. when $x = \pm E/F$. Clearly both x and p are oscillating, so the motion is a libration (oscillation). We can now integrate over the whole period to get

$$\begin{aligned} J &= \oint p\,dx \\ &= 2\int_{-E/F}^{E/F} \sqrt{2m(E - F|x|)}dx \\ &= 4\int_{0}^{E/F} \sqrt{2m(E - Fx)}dx \\ &= -\frac{8}{3F}\sqrt{2m}[(E - Fx)^{3/2}]_{x=0}^{x=E/F} \\ &= \frac{8\sqrt{2m}E^{3/2}}{3F} \end{aligned}$$

Here the factor of 2 in the second line is from integrating from $-E/F$ to E/F and then from E/F to $-E/F$ (which gives the same contribution since the momentum flips sign). Hence

$$H = E = \left(\frac{3F}{8\sqrt{2m}}\right)^{2/3} J^{2/3}.$$

The time derivative of angle variable $w = \frac{\partial W}{\partial J}$ is the frequency

$$\begin{aligned} \nu = \dot{w} = \frac{\partial H}{\partial J} &= \frac{2}{3}\left(\frac{3F}{8\sqrt{2m}}\right)^{2/3} J^{-1/3} \\ &= \frac{F}{4\sqrt{2m}}E^{-1/2} \end{aligned}$$

and therefore the period is $\tau = \nu^{-1} = \frac{4\sqrt{2mE}}{F}$.

Let's check units: $[F] = [E]/[x]$, so $[\tau] = \frac{[m]^{1/2}[E]^{1/2}}{[E]/[x]} = \frac{[m]^{1/2}[x]}{[E]^{1/2}} = \frac{[m]^{1/2}[x]}{[mx^2/t^2]^{1/2}} = [t]$, which checks.

4. Two Potentials [10 points, 8.309 ONLY]

(a) See solution to Problem 3.

(b) The energy is given by

$$H = \frac{p^2}{2m} - \frac{k}{|x|} = E,$$

where E is a constant. The momentum is given by

$$p = \pm\sqrt{2m}\left(E + \frac{k}{|x|}\right)^{1/2}.$$

The turning points of the motion are when $p = 0$, i.e. when $x = \pm\frac{k}{E} = \pm\frac{k}{\tilde{E}}$ where $\tilde{E} \equiv -E > 0$. Verifying this, we have $|x| = \frac{k}{\tilde{E}}$, thus $(E + \frac{k}{|x|}) = (-\tilde{E} + \tilde{E}) = 0$. Clearly both x and p are oscillating, so the motion is a libration (oscillation). We can now integrate over the whole period to get

$$\begin{aligned} J &= \oint p\, dx \\ &= 2\int_{-k/E}^{k/E} \sqrt{2m}\left(-\tilde{E} + \frac{k}{|x|}\right)^{1/2} dx \\ &= 4\int_{0}^{+k/\tilde{E}} \sqrt{2m}\left(-\tilde{E} + \frac{k}{|x|}\right)^{1/2} dx \end{aligned}$$

Here the factor of 2 in the second line is from integrating from $-k/E$ to k/E and then from k/E to $-k/E$ (which gives the same contribution since the momentum flips sign). The factor of 4 and lower limit of 0 in the third line comes from the fact that the integral is an even function.

To evaluate further, we make substitutions letting

$$x = \frac{k}{\tilde{E}}x',\ dx = \frac{k}{\tilde{E}}dx'.$$

We then have

$$\begin{aligned} J &= \frac{4k\sqrt{2m}}{\tilde{E}}\sqrt{\tilde{E}}\int_0^1 (-1 + \frac{1}{x'})^{1/2} dx' \\ &= \frac{4k\sqrt{2m}}{\tilde{E}}\sqrt{\tilde{E}}(\pi/2) \\ &= \frac{2\pi k\sqrt{2m}}{\sqrt{\tilde{E}}} \end{aligned}$$

Therefore

$$\tilde{E} = (2\pi)^2(2mk^2)J^{-2} = -H, \text{ and } \mathrm{H} = -8\pi^2\mathrm{mk}^2\mathrm{J}^{-2}.$$

The time derivative of angle variable $w = \frac{\partial W}{\partial J}$ is the frequency

$$\begin{aligned} \nu = \dot{w} = \frac{\partial H}{\partial J} &= (8\pi^2 mk^2)(+2)J^{-3} \\ &= \frac{(-E)^{3/2}}{\sqrt{2m}k\pi} \end{aligned}$$

and therefore the period is $\tau = \nu^{-1} = \frac{\sqrt{2m}k\pi}{(-E)^{3/2}}$.

Aside: Note that as $x \to 0$ that the potential energy $V \to -\infty$, while the kinetic energy $T \to +\infty$ to keep $E = T + V$ fixed. The singular behavior at $x = 0$ is integrable (behaving as a square-root singularity in the velocity), so the particle transverses the singular region in a finite time.

5. The $\csc^2(x)$ Potential [18 points]

(a) [2 points] H is conserved, so we can take $H = \alpha_1$. Therefore Hamilton's characteristic function $W(q, \alpha)$ is determined by

$$\frac{1}{2m}\left(\frac{\partial W}{\partial x}\right)^2 + a\csc^2(\frac{x}{x_0}) = \alpha_1$$

which gives

$$W = \pm \int \sqrt{2m\Big(\alpha_1 - a\csc^2(\frac{x}{x_0})\Big)}dx$$

(b) [4 points] The variables (x, p) need to undergo libration or rotation. For $a < 0$ the potential $V = a\csc^2(x/x_0)$ has no minimum except for when $V = -\infty$ (as can be checked by simply plotting the function, or by differentiating twice), so the motion is unbounded and cannot be periodic. Hence action-angle variables can only be used when $a > 0$.

For $a > 0$, we can analyze what kind of motion should result. $V = \infty$ at $x = n\pi x_0$ for integer n, and hence the particle is confined to a well at $(n-1)\pi x_0 < x < n\pi x_0$ for some n. Since all wells are equivalent, we'll assume the particle is confined to $x \in (0, \pi x_0)$. The minimum of V is $V(\pi x_0/2) = a$, so we must have $E \geq a$.

The motion is clearly a libration, since the particle moves back and forth between two turning points. The turning points $x_1 < x_2$ are given when E is all potential energy ($p(x_1) = p(x_2) = 0$),

$$E = a\csc^2(x_1/x_0) = a\csc^2(x_2/x_0).$$

(c) [8 points] We have

$$
\begin{aligned}
J &= \oint p dx \\
&= 2\int_{x_1}^{x_2} \sqrt{2m\Big(E - a\csc^2(\frac{x}{x_0})\Big)}\ dx = 4\int_{\pi x_0/2}^{x_2} \sqrt{2m\Big(E - a\csc^2(\frac{x}{x_0})\Big)}\ dx \\
&= 4\int_{\pi x_0/2}^{x_2} \sqrt{2mE\Big(1 - \sin^2(\frac{x_2}{x_0})\csc^2(\frac{x}{x_0})\Big)}\ dx \qquad \text{(we used the result from (a) above, also } p(x_2) = 0\text{)} \\
&= 4\sqrt{2mE}\int_{\pi x_0/2}^{x_2} \csc(\frac{x}{x_0})\sqrt{\sin^2(\frac{x}{x_0}) - \sin^2(\frac{x_2}{x_0})}\ dx \\
&= 4\sqrt{2mE}\int_{\pi x_0/2}^{x_2} \csc(\frac{x}{x_0})\sqrt{\cos^2(\frac{x_2}{x_0}) - \cos^2(\frac{x}{x_0})}\ dx \\
&= 4\sqrt{2mE}\int_{\pi/2}^{\theta_2} \csc(\theta)\sqrt{\cos^2(\theta_2) - \cos^2(\theta)}\ d\theta
\end{aligned}
$$

where $\theta = x/x_0$ and $\theta_2 = x_2/x_0$. This integral looks just like Eq. (10.132) of Goldstein now, with the substitution $\cos^2\theta_2 \leftrightarrow \sin^2 i$; for completeness we'll do the integral here. If we make the substitution $\cos\theta = \cos\theta_2 \sin\psi$, this reduces to

$$J = 4x_0\sqrt{2mE}\int_0^{\pi/2} \frac{\cos^2\theta_2\cos^2\psi}{1 - \cos^2\theta_2\sin^2\psi} d\psi$$

and then making the substitution $u = \tan\psi$, we get

$$
\begin{aligned}
J &= 4x_0\sqrt{2mE}\cos^2\theta_2 \int_0^\infty \frac{du}{(1+u^2)(1+u^2\sin^2\theta_2)} \\
&= 4x_0\sqrt{2mE}\int_0^\infty \left(\frac{1}{1+u^2} - \frac{\sin^2\theta_2}{1+u^2\sin^2\theta_2}\right) du \\
&= 4x_0\sqrt{2mE}\left(\int_0^\infty \frac{du}{1+u^2} - \sin\theta_2\int_0^\infty \frac{du'}{1+u'^2}\right) \\
&= 2\pi x_0\sqrt{2mE}(1 - \sin\theta_2)
\end{aligned}
$$

where we used $\int_0^\infty \frac{du}{1+u^2} = \frac{\pi}{2}$. Now recalling $E = a\csc^2\theta_2$, we have finally

$$J = 2\pi x_0\sqrt{2m}(\sqrt{E} - \sqrt{a})$$

or

$$E = \left(\frac{J}{2\pi x_0\sqrt{2m}} + \sqrt{a}\right)^2$$

The frequency of oscillation is

$$
\begin{aligned}
\nu &= \frac{\partial E}{\partial J} = \frac{1}{\pi x_0\sqrt{2m}}\left(\frac{J}{2\pi x_0\sqrt{2m}} + \sqrt{a}\right) \\
&= \frac{1}{2\pi}\sqrt{\frac{2E}{m x_0^2}}.
\end{aligned}
$$

(d) [4 points] We need to expand the potential about the minimum $x_{min} = \pi x_0/2$. We have $V(x_{min}) = a$,

$$V'(x) = -\frac{2a}{x_0}\cot(\frac{x}{x_0})\csc^2(\frac{x}{x_0}) \Rightarrow V'(x_{min}) = 0$$

$$V''(x) = \frac{2a}{x_0^2}\csc^4(\frac{x}{x_0}) + \frac{4a}{x_0^2}\cot^2(\frac{x}{x_0})\csc^2(\frac{x}{x_0}) \Rightarrow V''(x_{min}) = \frac{2a}{x_0^2}.$$

For $V(x) = a + \frac{V''(x_{min})}{2}(x - x_{min})^2 + O((x - x_{min})^3)$, the frequency of small oscillations is given by

$$\nu_{HO} = \frac{1}{2\pi}\sqrt{\frac{V''(x_{min})}{m}} = \frac{1}{2\pi}\sqrt{\frac{2a}{mx_0^2}}.$$

For small oscillations $E = a\csc^2(x_2/x_0) = a + O((x_2 - x_0)^2)$, and hence $\nu = \nu_{HO} + O((x_2 - x_0)^2)$, as expected.

6. A Three Dimensional Oscillator [10 points]

(a) [3 points] The Hamiltonian of the system is

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + \frac{1}{2}(k_1 x_1^2 + k_2 x_2^2 + k_3 x_3^2)$$

Since H is conserved, and is clearly separable, we can write $H = E = E_1 + E_2 + E_3$ where the E_i's are constants such that

$$E_i = \frac{p_i^2}{2m} + \frac{1}{2}k_i x_i^2$$

From the one-dimensional system we know that

$$J_i = \frac{2\pi E_i}{\omega_i}, \quad \omega_i = \sqrt{\frac{k_i}{m}}$$

(A quick way to see this is to check that

$$J = \oint p dq = \iint_{enclosed} dp dq$$

is the enclosed area of the orbit in phase space, which in this case is an ellipse with semiaxes $p_{lim} = \sqrt{2mE_i}$ and $q_{lim} = \sqrt{2E_i/k}$, with area $J_i = \pi p_{lim} q_{lim} = 2\pi E_i\sqrt{m/k_i}$.)

Therefore reexpressing the Hamiltonian in terms of J_i,

$$H = \frac{\omega_1}{2\pi}J_1 + \frac{\omega_2}{2\pi}J_2 + \frac{\omega_3}{2\pi}J_3.$$

The frequencies are

$$\nu_i = \dot{\mathrm{w}}_i = \frac{\partial H}{\partial J_i} = \frac{\omega_i}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k_i}{m}}$$

as expected.

(b) [3 points] The straightforward generalization is of course

$$x_i = \left(\frac{J_i}{\pi\sqrt{k_i m}}\right)^{1/2}\sin(2\pi \mathrm{w}_i), \quad p_i = \left(\frac{J_i\sqrt{k_i m}}{\pi}\right)^{1/2}\cos(2\pi \mathrm{w}_i).$$

We need to check either the Poisson bracket relations

$$\begin{aligned} [x_i, p_j]_{\vec{\mathrm{w}},\vec{J}} &= \delta_{ij} \\ [x_i, x_j]_{\vec{\mathrm{w}},\vec{J}} = [p_i, p_j]_{\vec{\mathrm{w}},\vec{J}} &= 0 \end{aligned}$$

or

$$\begin{aligned} [\mathrm{w}_i, J_j]_{\vec{x},\vec{p}} &= \delta_{ij} \\ [\mathrm{w}_i, \mathrm{w}_j]_{\vec{x},\vec{p}} = [J_i, J_j]_{\vec{x},\vec{p}} &= 0. \end{aligned}$$

Let's check the former set. It is obvious that $[x_i, p_j]_{\vec{\mathrm{w}},\vec{J}} = 0$ for $i \neq j$ since x_i depends only on (w_i, J_i) and p_j depends only on (w_j, J_j); similarly $[x_i, x_j]_{\vec{\mathrm{w}},\vec{J}} = [p_i, p_j]_{\vec{\mathrm{w}},\vec{J}} = 0$. Therefore we only need to check $[x_i, p_i]_{\vec{\mathrm{w}},\vec{J}} = 1$ (we are *not* using the summation convention here):

$$\begin{aligned} [x_i, p_i]_{\vec{\mathrm{w}},\vec{J}} &= \frac{\partial x_i}{\partial \mathrm{w}_i}\frac{\partial p_i}{\partial J_i} - \frac{\partial x_i}{\partial J_i}\frac{\partial p_i}{\partial \mathrm{w}_i} \\ &= 2\pi\sqrt{\frac{J_i}{\pi m\omega_i}}\cos(2\pi \mathrm{w}_i)\cdot\frac{1}{2}\sqrt{\frac{m\omega_i}{\pi J_i}}\cos(2\pi \mathrm{w}_i) + \frac{1}{2}\sqrt{\frac{1}{\pi m\omega_i J_i}}\cos(2\pi \mathrm{w}_i)\cdot 2\pi\sqrt{\frac{m\omega_i J_i}{\pi}}\cos(2\pi \mathrm{w}_i) \\ &= \cos^2(2\pi \mathrm{w}_i) + \sin^2(2\pi \mathrm{w}_i) = 1. \end{aligned}$$

Alternatively, we can check the Poisson brackets of $(\vec{\mathrm{w}}, \vec{J})$ with respect to $(\vec{x}, \vec{p})$. We first invert the relations to get

$$\mathrm{w}_i = \frac{1}{2\pi}\tan^{-1}\left(m\omega_i\frac{x_i}{p_i}\right), \quad J_i = \frac{\pi}{m\omega_i}(p_i^2 + (m\omega_i x_i)^2)$$

and then evaluate (the other Poisson brackes are zero, by the same argument as before)

$$\begin{aligned} [\mathrm{w}_i, J_i]_{q_i,p_i} &= \frac{\partial \mathrm{w}_i}{\partial x_i}\frac{\partial J_i}{\partial p_i} - \frac{\partial \mathrm{w}_i}{\partial p_i}\frac{\partial J_i}{\partial x_i} \\ &= \frac{1}{2\pi}\frac{m\omega_i/p_i}{1 + (m\omega_i x_i/p_i)^2}\cdot\frac{2\pi p_i}{m\omega_i} + \frac{1}{2\pi}\frac{m\omega_i x_i/p_i^2}{1 + (m\omega_i x_i/p_i)^2}\cdot\frac{2\pi m^2\omega_i^2 x_i}{m\omega_i} \\ &= \frac{1}{1 + (m\omega_i x_i/p_i)^2} + \frac{(m\omega_i x_i/p_i)^2}{1 + (m\omega_i x_i/p_i)^2} = 1 \end{aligned}$$

as expected.

(c) [4 points] The new action variables are

$$J_a = J_1 + J_2 + J_3, \quad J_b = J_1 + J_2, \quad J_c = J_1.$$

The new conjugate angle variables must satisfy $[\mathrm{w}_\alpha, J_\beta] = \delta_{\alpha\beta}$ (let Greek indices run over a, b, c

and Latin indices run over $1, 2, 3$), where

$$\begin{aligned}[\mathrm{w}_\alpha, J_\beta] &= \sum_i \left(\frac{\partial \mathrm{w}_\alpha}{\partial \mathrm{w}_i} \frac{\partial J_\beta}{\partial J_i} - \frac{\partial \mathrm{w}_\alpha}{\partial J_i} \frac{\partial J_\beta}{\partial \mathrm{w}_i} \right). \\ &= \sum_i \frac{\partial \mathrm{w}_\alpha}{\partial \mathrm{w}_i} \frac{\partial J_\beta}{\partial J_i}\end{aligned}$$

since $\partial J_\beta / \partial \mathrm{w}_i = 0$ in our case. Thus we must demand that

$$\begin{cases} [\mathrm{w}_a, J_a] &= \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_1} + \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_2} + \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_3} = 1 \\ [\mathrm{w}_a, J_b] &= \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_1} + \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_2} = 0 \\ [\mathrm{w}_a, J_c] &= \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_1} = 0 \end{cases} \Longrightarrow \begin{cases} \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_1} &= 0 \\ \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_2} &= 0 \\ \frac{\partial \mathrm{w}_a}{\partial \mathrm{w}_3} &= 1 \end{cases} \Longrightarrow \mathrm{w}_a = \mathrm{w}_3$$

$$\begin{cases} [\mathrm{w}_b, J_a] &= \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_1} + \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_2} + \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_3} = 0 \\ [\mathrm{w}_b, J_b] &= \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_1} + \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_2} = 1 \\ [\mathrm{w}_b, J_c] &= \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_1} = 0 \end{cases} \Longrightarrow \begin{cases} \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_1} &= 0 \\ \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_2} &= 1 \\ \frac{\partial \mathrm{w}_b}{\partial \mathrm{w}_3} &= -1 \end{cases} \Longrightarrow \mathrm{w}_b = \mathrm{w}_2 - \mathrm{w}_3$$

$$\begin{cases} [\mathrm{w}_c, J_a] &= \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_1} + \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_2} + \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_3} = 0 \\ [\mathrm{w}_c, J_b] &= \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_1} + \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_2} = 0 \\ [\mathrm{w}_c, J_c] &= \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_1} = 1 \end{cases} \Longrightarrow \begin{cases} \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_1} &= 1 \\ \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_2} &= -1 \\ \frac{\partial \mathrm{w}_c}{\partial \mathrm{w}_3} &= 0 \end{cases} \Longrightarrow \mathrm{w}_c = \mathrm{w}_1 - \mathrm{w}_2.$$

In summary, the new angle variables are

$$\mathrm{w}_a = \mathrm{w}_3, \quad \mathrm{w}_b = \mathrm{w}_2 - \mathrm{w}_3, \quad \mathrm{w}_c = \mathrm{w}_1 - \mathrm{w}_2.$$

Since $\dot{\mathrm{w}}_i = \frac{1}{2\pi}\sqrt{\frac{k_i}{m}}$, if $k_1 = k_2$ we have $\dot{\mathrm{w}}_1 = \dot{\mathrm{w}}_2$ and hence $\dot{\mathrm{w}}_c = \dot{\mathrm{w}}_1 - \dot{\mathrm{w}}_2 = 0$ and w_c is conserved. Similarly if $k_1 = k_2 = k_3$ then $\dot{\mathrm{w}}_b = \dot{\mathrm{w}}_2 - \dot{\mathrm{w}}_3 = 0$ and w_b is conserved as well.