So the problem of calculating correlation functions of Heisenberg fields can be reduced to calculating correlation functions of interacting picture fields with the free field ground state $|0\rangle$ :

$$\langle 0| T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\}|0\rangle$$

So we now study

$$\langle 0| T\{\phi_I(x)\phi_I(y)\}|0\rangle$$

$$\phi_I(x) = \underbrace{\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\, a_{\vec{p}}\, e^{-ip\cdot x}}_{\equiv\ \phi_I^+(x)} + \underbrace{\int\frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\, a^\dagger_{\vec{p}}\, e^{+ip\cdot x}}_{\equiv\ \phi_I^-(x)}$$

"positive frequency part" i.e., $e^{-iE_p t}$ "negative frequency part" i.e., $e^{+iE_p t}$

$\phi_I^+$ has annihilation operators only

$\phi_I^-$ has creation operators only

Note that

$$\underbrace{\phi_I^+(x)}_{\text{annihilations only}}|0\rangle = 0 \qquad \langle 0|\underbrace{\phi_I^-(x)}_{\text{creations only}} = 0$$

For $x^0 \geq y^0$,

$$T\{\phi_I(x)\,\phi_I(y)\} = \phi_I^+(x)\,\phi_I^+(y) + \phi_I^-(x)\,\phi_I^+(y) + \boxed{\phi_I^+(x)\,\phi_I^-(y)} + \phi_I^-(x)\,\phi_I^-(y)$$

$$= \phi_I^+(x)\,\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \boxed{\phi_I^-(y)\,\phi_I^+(x)} + \boxed{[\phi_I^+(x), \phi_I^-(y)]} + \phi_I^-(x)\,\phi_I^-(y)$$

What can we say about

$$[\phi_I^+(x), \phi_I^-(y)]\ ?$$

Note that it is a "number"... i.e., no creation or annihilation operators in it.

Also $\quad [\phi_I^+(x), \phi_I^-(y)] = \langle 0|[\phi_I^+(x), \phi_I^-(y)]|0\rangle$

$$= \langle 0|\,\phi_I^+(x)\,\phi_I^-(y)|0\rangle = \langle 0|\,\phi_I(x)\phi_I(y)|0\rangle$$

We can write

$$T\{\phi_I(x)\phi_I(y)\} = \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^-(y) + \langle 0|\phi_I(x)\phi_I(y)|0\rangle$$

notice that for each operator product, the $\phi^-$'s (creations) are on the left while the $\phi^+$'s (annihilations) are on the right

Let us define the operation N, called normal ordering.

$$N(f(a, a^\dagger))$$

... takes the string of $a + a^\dagger$'s and rearranges them so that the $a^\dagger$'s are on the left and $a$'s are on the right.

For example,

$$N(a^\dagger_{\vec{p}}\, a_{\vec{q}}) = a^\dagger_{\vec{p}}\, a_{\vec{q}}$$
$$N(a_{\vec{q}}\, a^\dagger_{\vec{p}}) = a^\dagger_{\vec{p}}\, a_{\vec{q}}$$

$$N(a_{\vec{p}}\, a_{\vec{q}}\, a^\dagger_{\vec{r}}) = a^\dagger_{\vec{r}}\, a_{\vec{p}}\, a_{\vec{q}}$$

ordering of $a_{\vec{p}}$ + $a_{\vec{q}}$ doesn't matter since they commute

It is worth mentioning that normal ordering is a <u>lexicographic</u> <u>convention</u> and not a true mathematical operation. Just because $A = B$, that doesn't mean $N(A) = N(B)$.

<u>Example</u> $N([a_{\vec{p}}, a^\dagger_{\vec{p}'}]) = N(a_{\vec{p}}\, a^\dagger_{\vec{p}'} - a_{\vec{p}}\, a^\dagger_{\vec{p}'})$

same $\qquad = 0$

But $N((2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')) = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')$

So when using normal ordering you have to specify the string of $a$'s and $a^\dagger$'s you will reorder.

In literature you will sometimes see

$$: f(a, a^\dagger) : \quad \text{instead of}$$

$$N(f(a, a^\dagger))$$

Let us sider general $x^0, y^0$ (not only $x^0 \geq y^0$)

Then we can write

$$T\{\phi_I(x)\phi_I(y)\} = N\{\phi_I(x)\phi_I(y)\} + \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 \geq y^0 \\ [\phi_I^+(y), \phi_I^-(x)] & \text{for } x^0 \leq y^0 \end{cases}$$

Let us define $\contraction{}{\phi}{{}_I(x)}{\phi}\phi_I(x)\phi_I(y)$ (the "contraction" of $\phi_I(x)$ + $\phi_I(y)$)

as $\begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 \geq y^0 \\ [\phi_I^+(y), \phi_I^-(x)] & \text{for } x^0 \leq y^0 \end{cases}$

Note that $$\contraction{}{\phi}{{}_I(x)}{\phi}\phi_I(x)\phi_I(y) = \begin{cases} \langle 0|\phi_I(x)\phi_I(y)|0\rangle & \text{for } x^0 \geq y^0 \\ \langle 0|\phi_I(y)\phi_I(x)|0\rangle & \text{for } x^0 \leq y^0 \end{cases}$$

So $$\contraction{}{\phi}{{}_I(x)}{\phi}\phi_I(x)\phi_I(y) = \langle 0|T(\phi_I(x)\phi_I(y))|0\rangle$$

$$= D_F(x-y)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip(x-y)}$$

Okay, we have

$$T\{\phi(x)\,\phi(y)\} = N\{\phi(x)\,\phi(y)\} + \contraction{}{\phi}{(x)\,}{\phi}\phi(x)\,\phi(y)$$

(leaving the "I" subscript implicit)

$$\left(\text{note}\quad \contraction{}{\phi}{(x)\,}{\phi}\phi(x)\,\phi(y) = \contraction{}{\phi}{(y)\,}{\phi}\phi(y)\,\phi(x)\right)$$

## Wick's Theorem

Claim: $T\{\phi(x_1)\cdots\,\phi(x_n)\} =$

$$N\{\phi(x_1)\cdots\phi(x_n) + \text{"all possible contractions"}\}$$

For example

$$T\{\phi(x_1)\,\phi(x_2)\,\phi(x_3)\,\phi(x_4)\}$$

use shorthand $\phi_n \leftrightarrow \phi(x_n)$

$$\begin{aligned}
= N\{&\phi_1\phi_2\phi_3\phi_4 \\
&+ \contraction{}{\phi}{_1}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1\phi_2}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1\phi_2\phi_3}{\phi}\phi_1\phi_2\phi_3\phi_4 \\
&+ \contraction{\phi_1}{\phi}{_2}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{\phi_1}{\phi}{_2\phi_3}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{\phi_1\phi_2}{\phi}{_3}{\phi}\phi_1\phi_2\phi_3\phi_4 \\
&+ \contraction{}{\phi}{_1}{\phi}\contraction{\phi_1\phi_2}{\phi}{_3}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1\phi_2}{\phi}\bcontraction{\phi_1}{\phi}{_2\phi_3}{\phi}\phi_1\phi_2\phi_3\phi_4 + \contraction{}{\phi}{_1\phi_2\phi_3}{\phi}\contraction[0.5ex]{\phi_1}{\phi}{_2}{\phi}\phi_1\phi_2\phi_3\phi_4\ \}
\end{aligned}$$

The meaning of $N\{\overline{\phi_1 \phi_2}\,\phi_3 \phi_4\}$ for example

is $\overline{\phi_1 \phi_2}\, N\{\phi_3 \phi_4\} = D_F(x_1 - x_2)\, N\{\phi_3 \phi_4\}$

The proof:

Proof by induction. We have shown it for $n=2$. Assume it true for $n-1$.

Let us define

$$W(\phi_1, \cdots, \phi_n) \equiv N\{\phi_1 \phi_2 \cdots \phi_n + \text{"all possible contractions"}\}$$

We want to show that $W(\phi_1, \cdots, \phi_n) = T\{\phi_1 \cdots \phi_n\}$

Without loss of generality consider the case

$$x_1^0 \geq x_2^0 \cdots > x_n^0$$

Then
$$T\{\phi_1 \cdots \phi_n\} = \phi_1 T\{\phi_2 \cdots \phi_n\}$$
$$= \phi_1 W(\phi_2, \cdots, \phi_n) \longleftarrow \text{by induction hypothesis for } n-1$$
$$= \underbrace{\phi_1^- W(\phi_2, \cdots, \phi_n) + W(\phi_2, \cdots \phi_n)\phi_1^+}_{X} + \underbrace{[\phi_1^+, W]}_{Y}$$

Note that both $X$ + $Y$ are normal ordered.
$X$ contains all contractions in $W(\phi_1, \cdots, \phi_n)$ which does not contract $\phi_1$ with anything.
$Y$ contains all contractions in $W(\phi_1, \cdots, \phi_n)$ which does contract $\phi_1$ with something.

Therefore $T(\phi_1 \cdots \phi_n) = W(\phi_1, \cdots, \phi_n)$.
By induction it holds for all $n$.