PY 711 Fall 2010
Homework 1: Due Tuesday, August 31

1. (5 points) Classical electromagnetism with no external sources can be derived from the action
$$S = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \tag{1}$$
where
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$
Derive the Euler-Lagrange equations of this action, treating the components A_μ as the dynamical variables. Write the equations in terms of electric and magnetic fields by identifying $E^i = -F^{0i} = F^{i0}$ and $\sum_k \epsilon^{ijk}B^k = -F^{ij} = F^{ji}$, where ϵ^{ijk} is the antisymmetric Levi-Civita symbol with $\epsilon^{123} = 1$.

2. Consider the action for a free non-relativistic complex field,
$$S = \int d^3x dt \left[\frac{i}{2}\phi^*\frac{\partial\phi}{\partial t} - \frac{i}{2}\phi\frac{\partial\phi^*}{\partial t} - \frac{1}{2m}\left(\vec{\nabla}\phi^*\right)\cdot\left(\vec{\nabla}\phi\right)\right], \tag{3}$$
where m is the particle mass.

   (a) (5 points) Derive the Euler-Lagrange equations for this action. It is easiest to use the shortcut which treats ϕ and ϕ^* as separate variables.

   (b) (5 points) Derive the conserved Noether current j^μ associated with the symmetry transformation
$$\phi \to e^{-i\theta}\phi, \tag{4}$$
$$\phi^* \to e^{i\theta}\phi^*, \tag{5}$$
where θ is a real constant. Since the system is non-relativistic the time component, j^0, and spatial components, $\vec{j}$, will look somewhat different.

1. CLASSICAL ELECTROMAGNETISM WITH NO EXTERNAL SOURCES CAN BE DERIVED FROM THE ACTION

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

15/15

WHERE

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

DERIVE THE EULER-LAGRANGE EQUATIONS OF THIS ACTION, TREATING THE COMPONENTS A_μ AS THE DYNAMICAL VARIABLES. WRITE THE EQUATIONS IN TERMS OF ELECTRIC AND MAGNETIC FIELDS BY IDENTIFYING $E^i = -F^{0i} = F^{i0}$ AND $\sum_k \epsilon^{ijk} B^k = -F^{ij} = F^{ji}$, WHERE ϵ^{ijk} IS THE ANTISYMMETRIC LEVI-CIVITA SYMBOL WITH $\epsilon^{123} = 1$.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}$

Since there are 2 $g^{\alpha\beta}$, made up of only ± 1, the only difference is that the indices become superscripts. $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$= -\frac{1}{4} \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\nu A_\mu \partial^\nu A^\mu \right)$$

✓

$$= -\frac{1}{2} \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu \right)$$

The simplification comes from the fact that both μ and ν are summed in each term, so they can be switched. So the 4 terms are really only 2.

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2} (2\partial^\mu A^\nu - 2\partial^\nu A^\mu)$$

$$= \partial^\nu A^\mu - \partial^\mu A^\nu$$

$$= -F^{\mu\nu} = F^{\nu\mu}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\mu} = 0$$

✓

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = 0}$$

$$\partial_\mu F^{\mu\nu} = 0$$

$$\partial_\mu F^{\mu 0} = \left(\partial_0 \cancel{F^{00}}^{\to 0} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30}\right)$$

$$= \left(\partial_x E^x + \partial_y E^y + \partial_z E^z\right)$$

$$= \boxed{\vec{\nabla} \cdot \vec{E} = 0} \qquad \text{(Gauss' law } (\rho = 0))$$

✓

$$\partial_\mu F^{\mu 1} = \partial_0 F^{01} + \partial_1 \cancel{F^{11}}^{\to 0} + \partial_2 F^{21} + \partial_3 F^{31}$$

$$= -\partial_t E^x - \partial_y\left(\Sigma \epsilon^{21k} B^k\right) - \partial_3\left(\Sigma \epsilon^{31k} B^k\right)$$

$$= -\partial_t E^x - (-\partial_y B_z) - (\partial_z B_y)$$

$$= -\partial_t E^x + (\vec{\nabla} \times \vec{B})^x$$

$$\partial_\mu F^{\mu 2} = \partial_0 F^{02} + \partial_1 F^{12} + \partial_2 \cancel{F^{22}}^{\to 0} + \partial_3 F^{32}$$

$$= -\partial_t E^y - \partial_x B^z + \partial_z B^x$$

$$= -\partial_t E^y + (\vec{\nabla} \times \vec{B})^y$$

$$\partial_\mu F^{\mu 3} = \partial_0 F^{03} + \partial_1 F^{13} + \partial_2 F^{23} + \partial_3 \cancel{F^{33}}^{\to 0}$$

$$= -\partial_t E^z + \partial_x B^y - \partial_y B^x$$

$$= -\partial_t E^z + (\vec{\nabla} \times \vec{B})^z$$

So, $\partial_\mu F^{\mu i} = -\partial_t E^i + (\vec{\nabla} \times \vec{B})^i = 0$ ✓

$$\boxed{-\partial_t E + \vec{\nabla} \times \vec{B} = 0}$$

5/5

2. CONSIDER THE ACTION FOR A FREE NON-RELATIVISTIC COMPLEX FIELD

$$S = \int d^3x \, dt \left(\frac{i}{2} \phi^* \frac{\partial \phi}{\partial t} - \frac{i}{2} \phi \frac{\partial \phi^*}{\partial t} - \frac{1}{2m} (\vec{\nabla} \phi^*) \cdot (\vec{\nabla} \phi) \right)$$

WHERE m IS THE PARTICLE MASS.

a. DERIVE THE EULER-LAGRANGE EQUATIONS FOR THIS ACTION. IT IS EASIEST TO USE THE SHORTCUT WHICH TREATS ϕ AND ϕ^* AS SEPARATE VARIABLES.

$$\mathcal{L} = \frac{i}{2} \phi^* \frac{\partial \phi}{\partial t} - \frac{i}{2} \phi \frac{\partial \phi^*}{\partial t} - \frac{1}{2m} (\vec{\nabla} \phi^*) \cdot (\vec{\nabla} \phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{i}{2} \partial_t \phi^* \qquad \frac{\partial \mathcal{L}}{\partial \phi^*} = \frac{i}{2} \partial_t \phi$$

✓ ✓

$$\frac{\partial \mathcal{L}}{\partial (\partial_t \phi)} = \frac{i}{2} \phi^* \qquad \frac{\partial \mathcal{L}}{\partial (\partial_t \phi^*)} = -\frac{i}{2} \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\nabla \phi)} = \frac{-1}{2m} \nabla \phi^* \qquad \frac{\partial \mathcal{L}}{\partial (\nabla \phi)} = \frac{-1}{2m} \nabla \phi$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\partial_t \left(\frac{i}{2} \phi^* \right) + \nabla \left(\frac{-1}{2m} \nabla \phi^* \right) + \frac{i}{2} \partial_t \phi^* = 0$$

✓

$$\partial_t \left(-\frac{i}{2} \phi \right) + \nabla \left(\frac{-1}{2m} \nabla \phi \right) - \frac{i}{2} \partial_t \phi = 0$$

$$\boxed{\begin{aligned} i \partial_t \phi^* - \frac{1}{2m} \nabla^2 \phi^* &= 0 \\ -i \partial_t \phi - \frac{1}{2m} \nabla^2 \phi &= 0 \end{aligned}}$$

These two equations are complex conjugates of one another, which is what we should have expected.

✓ 5/5

2 CONTINUED

b. DERIVE THE CONSERVED NOETHER CURRENT j^μ ASSOCIATED WITH THE SYMMETRY TRANSFORMATION

$$\phi \to e^{-i\theta}\phi$$
$$\phi^* \to e^{i\theta}\phi^*$$

WHERE θ IS A REAL CONSTANT. SINCE THE SYSTEM IS NON-RELATIVISTIC THE TIME COMPONENT, j^0, AND SPATIAL COMPONENTS, $\vec{j}$, WILL LOOK SOMEWHAT DIFFERENT.

$$\phi \to e^{-i\theta}\phi \cong \phi - i\theta\phi \qquad (\Delta\phi = -i\phi)$$
$$\phi^* \to e^{i\theta}\phi^* \approx \phi^* + i\theta\phi^* \qquad (\Delta\phi^* = i\phi^*)$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^*$$

$$j^0 = \left(\frac{i}{2}\phi^*\right)(-i\phi) + \left(\frac{-i}{2}\phi\right)(i\phi^*)$$

$$\boxed{j^0 = \phi^*\phi}$$

✓

$$\vec{j} = \left(\frac{-1}{2m}\nabla\phi^*\right)(-i\phi) + \left(\frac{-1}{2m}\nabla\phi\right)(i\phi^*)$$

$$\boxed{\vec{j} = \frac{i}{2m}\left(\phi\nabla\phi^* - \phi^*\nabla\phi\right)}$$

✓

5/5

CHECK $\partial_\mu j^\mu = 0$

$$\Rightarrow \quad \partial_t(\phi^*\phi) + \frac{i}{2m}\left(\phi\nabla^2\phi^* - \phi^*\nabla^2\phi\right) = 0$$

$$-i\phi\,\partial_t\phi^* - i\phi^*\partial_t\phi + \frac{1}{2m}\left(\phi\nabla^2\phi^* - \phi^*\nabla^2\phi\right) = 0$$

$$\phi^*\underbrace{\left(-i\partial_t\phi - \frac{1}{2m}\nabla^2\phi\right)}_{=0} - \phi\underbrace{\left(i\partial_t\phi^* - \frac{1}{2m}\nabla^2\phi^*\right)}_{=0} = 0$$

$$0 = 0 \ ✓$$

$\partial_\mu j^\mu = 0 \to$ conserved current.

PY 711 Solutions 1

1. $$\mathcal{L} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$\frac{\partial \mathcal{L}}{\partial A_\mu} = 0 \,, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = -F^{\nu\mu}$$

you get four terms
$+\frac{1}{4}F^{\mu\nu} - \frac{1}{4}F^{\nu\mu} + \frac{1}{4}F^{\mu\nu} - \frac{1}{4}F^{\nu\mu}$

Euler – Lagrange equations give

$$\partial_\nu F^{\nu\mu} = 0$$

For $\mu = 0$: $\quad \underbrace{\partial_0 F^{00}}_{=0} + \partial_i \underbrace{F^{i0}}_{=E^i} = 0 \rightarrow \vec{\nabla}\cdot\vec{E} = 0$

For $\mu = i$: $\quad \partial_0 \underbrace{F^{0i}}_{=-E^i} + \partial_j \underbrace{F^{ji}}_{=\sum_k \varepsilon^{ijk} B^k} = 0 \Rightarrow \frac{\partial \vec{E}}{\partial t} = \vec{\nabla}\times\vec{B}$

2. $$\mathcal{L} = \frac{i}{2}\phi^* \frac{\partial\phi}{\partial t} - \frac{i}{2}\phi\frac{\partial\phi^*}{\partial t} - \frac{1}{2m}(\vec{\nabla}\phi^*)\cdot(\vec{\nabla}\phi)$$

(a) $$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{i}{2}\frac{\partial\phi^*}{\partial t} \,, \quad \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial t}\right)} = \frac{i}{2}\phi^* \,, \quad \frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\phi)} = -\frac{1}{2m}\vec{\nabla}\phi^*$$

$$\frac{\partial\mathcal{L}}{\partial\phi^*} = \frac{i}{2}\frac{\partial\phi}{\partial t} \,, \quad \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi^*}{\partial t}\right)} = -\frac{i}{2}\phi \,, \quad \frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\phi^*)} = -\frac{1}{2m}\vec{\nabla}\phi$$

Euler – Lagrange equations :

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = 0$$

$$\frac{\partial\mathcal{L}}{\partial\phi^*} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\right) = 0$$

$$-\frac{i}{2}\frac{\partial\phi^*}{\partial t} - \frac{i}{2}\frac{\partial\phi^*}{\partial t} + \frac{\vec{\nabla}^2}{2m}\phi^* = 0$$

$$\frac{i}{2}\frac{\partial\phi}{\partial t} + \frac{i}{2}\frac{\partial\phi}{\partial t} + \frac{\vec{\nabla}^2}{2m}\phi = 0$$

complex conjugate equations

Suffices to write

$$i\frac{\partial\phi}{\partial t} = -\frac{\vec{\nabla}^2}{2m}\phi$$

(b)

$$\phi \to e^{-i\theta}\phi \qquad \phi^* \to e^{i\theta}\phi^*$$

$$\approx \phi - i\theta\phi \qquad \approx \phi^* + i\theta\phi^*$$

$$\Rightarrow \Delta\phi = -i\phi \qquad \Rightarrow \Delta\phi^* = i\phi^*$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\Delta\phi^*$$

$$j^0 = \left(\frac{i}{2}\phi^*\right)(-i\phi) + \left(-\frac{i}{2}\phi\right)(+i\phi^*)$$

$$= \phi^*\phi$$

$$\vec{j} = \left(-\frac{1}{2m}\vec{\nabla}\phi^*\right)(-i\phi) + \left(-\frac{1}{2m}\vec{\nabla}\phi\right)(i\phi^*)$$

$$= -\frac{i}{2m}\left(\phi^*\vec{\nabla}\phi - \phi\vec{\nabla}\phi^*\right)$$