QUANTUM FIELD THEORY & CONDENSED MATTER

(book by Shankar)

Feb 11, 2021

(I) THERMO - STAT MECH REVIEW

(a) Energy & Entropy in Thermo

Internal energy U: → state variable, i.e. sth which has a unique value associated w/ every state.

returns to same value if state is looped

→ 2 ways to change U: ⟶ $dU = -PdV$ (cons. of energy)
$\quad = \delta Q$. (heat)

→ get 1st law of thermo:

$$\boxed{dU = \delta Q - PdV}$$

Q is not a state variable, so we write δQ.

2nd law introduces entropy $\boxed{dS = \frac{\delta Q}{T}}$ → heat added reversibly

S is a state variable since $\oint dS = 0$ for a quasi-static cyclic process.

If Q is added reversibly,

→ $dU = TdS - PdV$ ⟹ $U = U(S,V)$

$T = \partial U/\partial S|_V$

$-P = \partial U/\partial V|_S$

$\to$ get
$$dS = \frac{1}{T}\,dU + \frac{P}{T}\,dV$$
$$S = S(U,V)$$
$$\frac{1}{T} = \left.\frac{\partial S}{\partial U}\right|_V , \quad \frac{P}{T} = \left.\frac{\partial S}{\partial V}\right|_U$$

$U(S,V) \to$ "fundamental relation".

For ideal gas...
$$P = \left.\frac{-\partial U}{\partial V}\right|_S = \frac{2}{3}\frac{U}{V}$$
$$T = \left.\frac{\partial U}{\partial S}\right|_V = \frac{2}{3nR}\,U$$

since $U(S,V) = C\left[\frac{e^{S/nR}}{V}\right]^{2/3}$

From these we find $\boxed{PV = nRT}$.

(b) Equilibrium as Max S

$\hookrightarrow$ S is max @ equilibrium

(c) Free Energy in Thermo

Suppose $V = $ const, so $U = U(S) \Rightarrow T = \frac{\partial U}{\partial S} = \frac{dU}{dS}$

$\Rightarrow T = T(S) \Rightarrow S = S(T)$

$$\boxed{\text{Free energy: } F(T) = U(S(T)) - T \cdot S(T)}$$

$\to U - ST$

we find that

$$\boxed{\frac{dF}{dT} = \dots = -S(T)}$$

Now, bring back V to get

$\to$ $F = F(T,V)$ $\quad -S = \left.\frac{\partial F}{\partial T}\right|_V$

$-P = \left.\frac{\partial F}{\partial V}\right|_T$ $\quad dF = -S\,dT - P\,dV$

(d) Equilibrium as Min of F

$F(V,T)$ has the same info as $U(V,S)$

(e) Microcanonical Distribution

Microstate: states described by maximal detail.

$\to$ A system can statistically be in ~~one~~ many microstates, each with probability p_i, with which observable O has value $O(i)$

$$\to \boxed{\langle O \rangle = \sum_i p_i O(i)}$$

Fluctuation (variance) is $\boxed{(\Delta O)^2 = \langle O^2 \rangle - \langle O \rangle^2}$

Fundamental postulate of statistical mech...

A macroscopic isolated system in thermal equilibrium is equally likely to be found in any of its accessible microstates

→ "equal weight probability"

↳ (Microcanonical distribution)

→ Boltzmann: Entropy of isolated system:

$$\boxed{S = k \ln \Omega}$$

k: $1.38 \cdot 10^{-23}\ JK^{-1}$; Ω: # microstates

ideal gas constant

$$\boxed{R = N_A k}$$

Avogadro's number

2 independent systems → $\left\{ \begin{array}{l} \Omega_{tot} = \Omega_1 \times \Omega_2 \\ S_{tot} = S_1 + S_2 . \end{array} \right\}$

(f) <u>Gibbs's Approach: Canonical Distribution</u>

Boltzmann gave description of system for a definite U.

Gibbs did the same, but for a definite T

→ Relative probability of system in state i of energy E_i is $e^{-\beta E(i)}$ where

$$\beta = \frac{1}{kT}.$$

$$\rightarrow \boxed{p(i) = \frac{e^{-\beta E(i)}}{\sum_i e^{-\beta E(i)}} = \frac{e^{-\beta E(i)}}{Z}}$$

where Z is the "partition function"

$$Z = \sum_i e^{-\beta E(i)}$$

Ex $E(x,p) = \frac{p^2}{2m} + \frac{1}{2} m\omega_0^2 x^2$

$$\rightarrow Z(\beta) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dp\, e^{-\beta\left(p^2/2m + \frac{1}{2} m\omega_0^2 x^2\right)}$$

In QM, we have

$$Z = \sum_i e^{-\beta E(i)} = \mathrm{Tr}\left(e^{-\beta H}\right)$$

$$= \int_{-\infty}^{\infty} dx\, \langle x| e^{-\beta H(x,p)} |x\rangle$$

this leads to the path integral

Ex $$\langle E\rangle = \frac{\sum_i E_i e^{-\beta E_i}}{Z} = -\frac{\partial \ln Z}{\partial \beta} = U$$

(g) Grand Canonical Distribution

↳ one that can exchange heat & particles.

In this case, $E(i)$ depends on chemical potential μ.

$$\rightarrow \quad p(E(i), N) = \frac{e^{-\beta(E(i)-\mu N)}}{Z}$$

where $$Z = \sum_{N} \sum_{i(N)} e^{-\beta(E(i)-\mu N)}$$

where
$$\langle N \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial \mu}\bigg|_{\beta}$$

$$-\frac{d \ln Z}{d\beta} = \langle U \rangle - \mu \langle N \rangle$$

Ex QM:

$$\langle N \rangle = n_{F/B} = \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

where $(+) \rightarrow$ fermions

$(-) \rightarrow$ bosons.

For more info on ~~stat~~ mech & thermo
$\rightarrow$ Review ~~P~~H 332 notes (on website)

(2) THE ISING MODEL in $d=0$ & $d=1$

(a) Ising model in $d=0$

- System has 2 d.o.f.: $\boxed{s_1 \,\&\, s_2 , \quad s_i = \pm 1.}$

  Config of the system is given by (s_1, s_2).

- Energy: $\boxed{E(s) = -J s_1 s_2 - B(s_1+s_2)}$

  models a system made of magnetic moments.

$J > 0 \Rightarrow$ ferromagnetic (E favors aligned spins)
$J < 0 \Rightarrow$ antiferromagnetic (- - antiparallel - -)

$B > 0$ always. Spins like to align w/ $\vec{B} = B\hat{z}$.

With these..

$$Z = \sum_{s_1, s_2} e^{K s_1 s_2 + h(s_1+s_2)}$$

where $K = \beta J$, $h = \beta B$.

$$Z = Z(K,h) = 2\cosh(2h)\cdot e^{K} + 2e^{-K}$$

high B / Low T $\Rightarrow$ State of min E dominate
$\rightarrow$ spins align w/ B.

$\beta \sim 0$ / high T $\Rightarrow$ equal weights $\rightarrow$ spins fluctuate!

To check this, look at "average magnetization"

$$M = \frac{s_1 + s_2}{2}$$

$$\langle M \rangle = \frac{1}{Z} \left\{ \sum_{s_1 s_2} \frac{1}{2}(s_1 + s_2) e^{K s_1 s_2 + h(s_1 + s_2)} \right\}$$

$$= \frac{1}{2Z} \frac{\partial Z(K,h)}{\partial h}$$

$$\rightarrow \langle M \rangle = \frac{1}{2} \frac{\partial \ln Z(K,h)}{\partial h}$$

Recall that free energy F is related to Z via

$$Z = e^{-\beta F}$$

$$\Rightarrow \langle M \rangle = \frac{1}{2} \frac{\partial [-\beta F(K,h)]}{\partial h}$$

With this, $-\beta F = \ln \left[2\cosh(2h) \cdot e^{K} + 2e^{-K} \right]$

so

$$\langle M \rangle = \frac{\sinh(2h)}{\cosh(2h) + e^{-2K}}$$

note that $\langle M \rangle \rightarrow 1$ as $h, K \rightarrow \infty$

$\langle M \rangle \rightarrow h$ as $K \rightarrow 0$

(as expected)

#

Suppose we want thermal average of a particular spin s_1. $\to$ need to add "source term" $\underline{h_1 s_1 + s_2 h_2}$ rather than $h(s_1+s_2)$.

$$\hookrightarrow \quad Z = \sum_{s_1, s_2} e^{K s_1 s_2 + h_1 s_1 + s_2 h_2} = e^{-\beta F(K, h_1, h_2)}$$

Can check that $\langle s_i \rangle = \frac{1}{Z}\frac{\partial Z}{\partial h_i} = \frac{\partial \ln Z}{\partial h_i} = \frac{\partial}{\partial h_i}\frac{(-\beta F(K,h_1,h_2))}{1}$

So,
$$\frac{\partial^2}{\partial h_1 \partial h_2}\left\{-\beta F(K, h_1, h_2)\right\} = \frac{\partial}{\partial h_1}\left(\frac{1}{Z}\frac{\partial Z}{\partial h_2}\right)$$

$$= \frac{1}{Z}\frac{\partial^2 Z}{\partial h_1 \partial h_2} - \frac{1}{Z^2}\frac{\partial Z}{\partial h_1}\frac{\partial Z}{\partial h_2}$$

$$= \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle$$

Let
$$\boxed{\langle s_1 s_2 \rangle_c = \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle}$$

correlation function

connected correlation function

What if h is uniform or zero? We would evaluate the derivative, then set $h_i = h$ or 0 $\forall i$.

Ex for uniform h...

$$\langle s_1 s_2 \rangle_c = \langle s_1 s_2 \rangle - \langle s_1 \rangle \langle s_2 \rangle = \left.\frac{\partial^2 \left[-\beta F(K, h_1, h_2)\right]}{\partial h_1 \partial h_2}\right|_{h_i = h \,\forall i}$$

High derivatives of $[-\beta F]$ gives fluctuations. Ex,

Magnetic susceptibility (χ) is the rate of change of the average magnetization with the applied field.

$$\rightarrow \quad \chi = \frac{1}{2}\frac{\partial \langle M \rangle}{dh} = \frac{1}{4}\frac{\partial^2[-\beta F]}{dh^2} = \langle M^2 \rangle - \langle M \rangle^2$$

"2" for 2 spins, which is the situation we're considering.

For N spins, we have (which we'll use later).

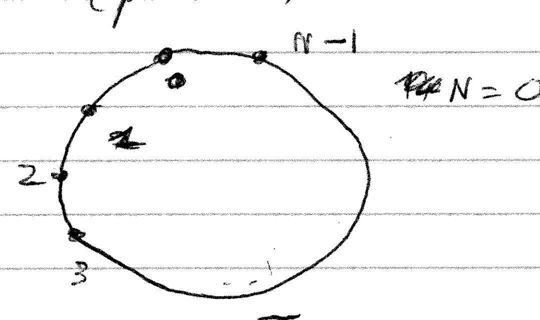
$$\chi = \frac{1}{N}\frac{\partial \langle M \rangle}{dh} = \frac{1}{N^2}\frac{\partial^2[-\beta F]}{dh^2} = \langle M^2 \rangle - \langle M \rangle^2$$

(b) Ising model with $d = 1$

→ 2 types of boundary conditions < open / periodic.

(open)

(periodic)



We eventually want to take the $\boxed{\text{thermodynamic limit } N \to \infty}$

The model is of "nearest neighbor type"

$$E = -J \sum_{i=0}^{N-1} s_i s_{i+1}$$ where we have set $B = 0$ (h)

From this, we get Z:

$$Z = \sum_{s_i = \pm 1} \exp\left\{ \sum_{i=0}^{N-1} K(s_i s_{i+1} - 1) \right\}$$

where $K = \beta J > 0$. We note that this term, which is spin-independent $(-K)$ is added to every site for convenience. This just shifts βF by NK.

Now, fix s_0 & define a relative spin variable.

$$t_i = s_i s_{i+1} .$$

With s_0 & $\{t_i\}$, we can construct the entire system.

$$Z = \sum_{t_i = \pm 1} \exp\left\{ \sum_{i=0}^{N-1} K(t_i - 1) \right\} = \sum_{t_i = \pm} \prod_{i=0}^{N-1} e^{K(t_i - 1)}$$

with which we find $\boxed{Z = 2(1 + e^{-2K})^N}$

! (the math takes some time to work out...)

2 choices of s_0.

Now, we want to look at the free energy per site in the thermodynamic limit $N \to \infty$.

$$f(K) = -\lim_{N \to \infty} \frac{1}{N} \ln Z$$ (recall $Z = e^{-\beta F}$)

we see that $f(K) \sim -\ln\left(1+e^{-2K}\right)$ upon drop pi
the $\ln 2/N$ in $N\to\infty$

#

What about correlation function? (sites i & $j > i$)

$$\langle s_j s_i \rangle = \frac{1}{Z} \sum_{s_k} \left\{ \exp\left[\sum_k K \left(s_k s_{k+1} - 1 \right) \right] \cdot s_j s_i \right\}$$

measure how likely s_i, s_j are on average point in the same direction.

well ...

$$s_j s_i = s_i s_j = s_i \overbrace{s_{i+1} s_{i+1}}^{1} \cdots s_{j-1} s_j$$

$$= t_i t_{i+1} \cdots t_{j-1} \quad , \text{ by the fact that } s_i^2 = 1$$

So,

$$\boxed{\langle s_j s_i \rangle = \langle t_i \rangle \langle t_{i+1} \rangle \cdots \langle t_{j-1} \rangle}$$

why?

$$\cancel{\langle s_j s_i \rangle = \frac{1}{Z} \left\{ \sum_{k} t_i t_{i+1} \cdots t_{j-1} \exp\left[\sum_k K (t_k - 1) \right] \right\}}$$

$$\cancel{\langle t_i \rangle = \frac{1}{Z} \left[\sum_{t_k = \pm 1} t_i \exp\left\{ \sum_k K (t_k - 1) \right\} \right]}$$

~~So note that there are $(j-i)$ terms in $\langle t_i \rangle \cdots \langle t_{j-1} \rangle$~~

Why?

Note that with this we have ...

$$\langle s_j s_i \rangle = \frac{1}{Z} \sum_{s_k} s_j s_i \exp\left\{ \Sigma_k K(s_k s_{k+1} - 1) \right\}$$

$$= \frac{1}{Z} \sum_{t_k} t_i \cdots t_{j-1} \exp\left\{ \Sigma_k K(t_k - 1) \right\}$$

This sum is completely factored. The sums ~~over~~ that don't have t_n multiplying the exponential cancel the cosh in $Z = 2(1+e^{-2K})^N$. So basically after carefully working this out we get

$$\langle s_j s_i \rangle = \underbrace{\langle t_i \rangle \langle t_{i+1} \rangle \cdots \langle t_{j-1} \rangle}_{j-i \text{ terms}}$$

Now, the average for each t is easy:

$$\langle t \rangle = \frac{1}{Z} \sum_{t=-1}^{+1} t \exp\left(\Sigma_k K(t_k - 1) \right) \qquad t = -1, +1$$

$$= \frac{1 e^{-0 \cdot K} - 1 e^{-2K}}{e^{0 \cdot K} + e^{-2K}} = \tanh K.$$

So $\langle s_j s_i \rangle = (\tanh K)^{j-i} = \exp\left[(j-i) \ln \tanh K \right]$

Now, choose $i > j$, we see that in general,

$$\langle s_j s_i \rangle = (\tanh K)^{|j-i|} = \exp\left\{ |j-i| \ln \tanh K \right\}$$

Note the $s_k s_{k+1} = t_k$ transform is called the "duality transformation"

At finite K, since $\tanh K < 1$

$$\boxed{\langle s_j s_i \rangle \to 0 \text{ as } |j-i| \to \infty \text{ exponentially}}$$

Note that $\langle s_j s_i \rangle$ only depends on the difference $|j-i|$

$\Rightarrow$ this is called $\boxed{\text{translational invariance}}$

(c) <u>The Monte Carlo method</u>

When $d > 1$, we can't do these calculations exactly

To do this, we need Monte Carlo...

$$\langle s_i s_j \rangle = \frac{\sum_c s_i s_j e^{-E(c)/kT}}{\sum_c e^{-E(c)/kT}} = \sum_c s_i s_j(c)\, P(c)$$

configuration   energy   probability

We won't worry about details here, but the point is that we make the computer generate a bunch of configurations c, and carry out the calculation.

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3 FROM STATISTICAL MECH TO QUANTUM MECH

Feb 12, 2021 We'll go through the same material, but through a different approach...

(a) Real-Time QM (as opposed to Euclidean QM) (imaginary time)

SE : $\boxed{i\hbar \frac{d}{dt}|\psi\rangle = H|\psi(t)\rangle}$

If H time independent then $|\psi(t)\rangle = \underbrace{U(t)}_{\text{propagator}}|\psi(0)\rangle$

$$\rightarrow \boxed{i\hbar \frac{d}{dt} U(t) = H U(t)}$$

Formal solution for $U(t)$ is $U(t) = e^{-iHt/\hbar}$

H is self-adjoint $\Rightarrow$ U is unitary : $\underline{U^\dagger U = I}$.

Spectral decomp: $U(t) = \sum_n |n\rangle\langle n| e^{-iE_n t/\hbar}$

where

$$H|n\rangle = E_n|n\rangle$$

matrix elements: $U(x', x, t)$ in $|x\rangle$ basis...

$$U(x', x, t) = \langle x'|U(t)|x\rangle$$

$$= \sum_n \psi_n(x')\psi_n^*(x)\, e^{-iE_n t/\hbar}$$

where $\psi_n(x) = \langle x|n\rangle$

Free particle fcns stuff... I won't worry about this since already did in PH431.

Now, what we just looked at is the Schrödinger picture

↳ Can also look at Heisenberg picture
→ operators where time dependence.

A Heisenberg operator $\Omega(t)$ is related to Schrö op. Ω by

$$\boxed{\Omega(t) = \mathcal{U}^\dagger(t)\,\Omega\,\mathcal{U}(t)}$$

from which we define the <u>time-ordered Green's fn</u>

$$\boxed{iG(t) = \langle 0|\, \mathcal{T}(\Omega(t)\Omega(0))\,|0\rangle}$$

ground state of H — time-order symbol

$\mathcal{T}$ is defined like this...

$$\mathcal{T}(\Omega(t_1)\Omega(t_2)) = \theta(t_2-t_1)\,\Omega(t_2)\,\Omega(t_1) + \theta(t_1-t_2)\,\Omega(t_1)\,\Omega(t_2)$$

Heaviside step fn.

Now, in condensed matter, we want a generalization of the above ground state expectation value ~~at~~ to one at finite temp β when the system can be in any state of energy E_n with Boltzmann weight:

$$\boxed{iG(t,\beta) = \frac{\sum_n e^{-\beta E_n}\langle n|\mathcal{T}(\Omega(t)\Omega(0))|n\rangle}{\sum_n e^{-\beta E_n}} = \frac{\mathrm{Tr}\left(e^{-\beta H}\mathcal{T}(\Omega(t)\Omega(0))\right)}{\mathrm{Tr}(e^{-\beta H})}}$$

Sometimes we need to compute the "retarded Green's fn" to calculate responses to external probes...

$$\boxed{iG_R(t,\beta) = \frac{\mathrm{Tr}\left[e^{-\beta H}\theta(t)[\Omega(t),\Omega(0)]\right]}{\mathrm{Tr}(e^{-\beta H})}}$$

We won't worry too much about this.

(b) Imaginary-time QM

let $t = -i\tau$

then we have SE in img time:

$$\boxed{-\hbar\frac{d}{d\tau}|\Psi(\tau)\rangle = H|\Psi(\tau)\rangle}$$

The propagator $\boxed{\mathcal{U}(t) = e^{-iHt/\hbar} \Rightarrow e^{-H/\hbar\,\tau} = \mathcal{U}(\tau)}$

now $\mathcal{U}(\tau)$ is Hermitian, not unitary.

and in energy eigenbasis...

$$\boxed{\mathcal{U}(\tau) = \sum_n |n\rangle\langle n|\, e^{-E_n \tau/\hbar}} \quad \text{w/ } H|n\rangle = E_n|n\rangle$$

Now, since $\mathcal{U}(\tau)$ is Hermitian & not unitary (oscillatory), $\mathcal{U}(\tau)$ kills all but its ground state projector...

$$\rightarrow \lim_{\tau\to\infty} \mathcal{U}(\tau)\,|\psi(0)\rangle \longrightarrow |0\rangle\langle 0|\psi(0)\rangle\, e^{-E_0\tau/\hbar}$$

☒ what about Heisenberg operators?

Recall that $\Omega(t) = \mathcal{U}^\dagger(t)\,\Omega\,\mathcal{U}(t)$.

In τ, we have...

$$\boxed{\Omega(\tau) = e^{H/\hbar\,\tau}\,\Omega\, e^{-H\tau/\hbar}}$$

now note that in t: $\Omega^\dagger(t)$ & $\Omega(t)$ are adjoints

i.e. $(\Omega^\dagger)(t) = [\Omega(t)]^\dagger$

<u>But</u>

$$\Omega^\dagger(\tau) = e^{H/\hbar\,\tau}\,\Omega^\dagger\, e^{-H\tau/\hbar} \neq [\Omega(t)]^\dagger = e^{-H\tau/\hbar}\,\Omega^\dagger\, e^{H\tau/\hbar}$$

Now, time-ordered product...

$$\mathcal{T}[\Omega(\tau_2)\Omega(\tau_1)] = \theta(\tau_2-\tau_1)\Omega(\tau_2)\Omega(\tau_1) + \theta(\tau_1-\tau_2)\Omega(\tau_1)\Omega(\tau_2)$$

Same def as before, with exp value

$$\boxed{G(\tau_2-\tau_1) = -\langle 0|\, \mathcal{T}(\Omega(\tau_2)\Omega(\tau_1))\,|0\rangle \quad \text{in gnd state.}}$$

In all state @ Boltzmann weight:

$$\boxed{G(\tau_2-\tau_1, \beta) = -\frac{\mathrm{Tr}\{e^{-\beta H}\mathcal{T}(\Omega(\tau_2)\Omega(\tau_1))\}}{\mathrm{Tr}(e^{-\beta H})}.}$$

(c) The TRANSFER MATRIX

Now, let us come back to the $d=1$ Ising model and look at what QM comes out of it.

Recall the partition fn (without the duality trs fn)

$$\boxed{Z = \sum_{s_i}\prod_i e^{K(s_i s_{i+1}-1)}}$$

now look at each

$$e^{K(s_i s_{i+1}-1)} \longrightarrow \text{2 indices, 2 values of } s_i \rightarrow \text{4 total.}$$

Define $T_{s's} = \exp[K(s's-1)]$ $\longrightarrow$ elements of the 2×2 transfer matrix.

where
$$\boxed{\begin{aligned} T_{++} &= T_{--} = 1 \\ T_{+-} &= T_{-+} = e^{-2k} \end{aligned} \Rightarrow T = \begin{pmatrix} 1 & e^{-2k} \\ e^{-2k} & 1 \end{pmatrix} = I + e^{-2k}\sigma_1}$$

σ_x

note that T is Real - Hermitian.

With this can rewrite Z:

$$Z = \sum_{s_i,\ i=1,\dots,N-1}' T_{s_N s_{N-1}} T_{s_{N-1} s_{N-2}} \dots T_{s_2 s_1} T_{s_1 s_0}$$

But note that following thing... when multiplying matrices A & B,

$$(AB)_{ij} = \sum_{\cancel{k}} A_{ij} \cancel{B}_{jk}$$

$$\Rightarrow Z = \sum_{s_i,\ i=1,\dots,N-1} T_{s_N s_{N-1}} \dots T_{s_2 s_1} T_{s_1 s_0}$$

$$= \sum_{s_1=-1}^{1} \sum_{s_2=-1}^{1} \dots \sum_{s_{N-1}=-1}^{1} T_{s_N s_{N-1}} \dots T_{s_2 s_1} T_{s_1 s_0}$$

Isolate $\displaystyle\sum_{s_2=-1}^{1} T_{s_3 s_2} T_{s_2 s_1} = (T^2)_{s_1 s_3} \dots$

If we keep "collapsing" like this, we get

$$\boxed{Z = \langle s_N | T^N | s_0 \rangle}$$ (for fixed boundary condition)

If we sum over the first & last spins to get free boundary conditions, we get

$$Z = \sum_{s_0} \sum_{s_N} \langle s_N | T^N | s_0 \rangle$$

Now, if we're looking at periodic BC, $s_N = s_0$, and so

$$\boxed{Z = \sum_{s_0} \langle s_0 | T^N | s_0 \rangle = \mathrm{Tr}(T^N)}$$

$K = \beta J$

where recall that $T = I + e^{-2K} \sigma_x$

With this, we can show that the free energy per site is insensitive to boundary conditions as $N \to \infty$.

Recall the formula: $f(K) = -\lim_{N\to\infty} \frac{1}{N} \ln Z$.

Let us write $T = \lambda_0 |0\rangle\langle 0| + \lambda_1 |1\rangle\langle 1|$, by spectral decomp.

$$\Longrightarrow \quad T^N = \lambda_0^N |0\rangle\langle 0| + \lambda_1^N |1\rangle\langle 1|$$

Now, use thm "Perron - Frobenius" associated

$$\left[\text{A sqr matrix with positive entries will have a non-degenerate largest eigv \& eigvector with strictly positive components}\right]$$

$\Rightarrow$ Assume that T is such a matrix, for finite K.

$\therefore$ that $\lambda_0 > \lambda_1$. Then we have

$$\lim_{N\to\infty} T^N \simeq \lambda_0^N \left[|0\rangle\langle 0| + \underbrace{\mathcal{O}\left(\frac{\lambda_1}{\lambda_0}\right)^N}_{\text{vanishes since } \lambda_1 < \lambda_0.} \right]$$

and so

$$Z = \langle s_N | T^N | s_0 \rangle \simeq \langle s_N | 0 \rangle \langle 0 | s_0 \rangle \lambda_0^N \left(1 + \cancel{\mathcal{O}\left(\lambda_1/\lambda_0\right)^N}^{\,0}\right)$$

and so

$$-f = \lim_{N\to\infty} \frac{1}{N} \ln Z$$

$$= \lim_{N\to\infty} \left[\frac{1}{N} \ln \left\{ \langle s_N | 0 \rangle \langle 0 | s_0 \rangle \lambda_0^N \left(1 + \mathcal{O}(\ldots)^N\right) \right\} \right]$$

$$\simeq \lim_{N\to\infty} \left[\ln \lambda_0 + \frac{1}{N} \ln \left\{ \langle s_N | 0 \rangle \langle 0 | s_0 \rangle \right\} + \ldots \right]$$

$$\to \ln \lambda_0 \quad \text{so} \quad \boxed{f \longrightarrow -\ln \lambda_0} \quad \text{(OBC)}$$

which is independent of the bd spins so long as

$\langle 0 | s_0 \rangle, \langle s_N | 0 \rangle \neq 0$, which is assured by the Perron-Frob thm which states that all components of the dominate eigvector are positive.

\#

This can be done for periodic BC as well.

If $Z = \mathrm{Tr}(T^N) \Rightarrow f \to -\ln \lambda_0$ as before

(PBC)

Next, we want to compute the correlation fns using this formulation. How do we do this?

Feb 15, 2021

To do this, we need to work out some identities

First, look at $T = I + e^{-2K}\sigma_1$ (σ_x)

and

$$e^{K^*\sigma_1} = (\cosh K^*) I + (\sinh K^*)\sigma_1$$
$$= \cosh K^* (I + \tanh K^*)\sigma_1$$

$\tanh K = e^{-2K}$

now choose K^* such that $\boxed{\tanh K^* = e^{-2K}}$ then

$K^*(K)$

$$T = I + e^{-2K}\sigma_1 =$$
$$= I + (\tanh K^*)\sigma_1 = \boxed{\frac{e^{K^*\sigma_1}}{\cosh K^*} = T}$$

now, $\cosh K^*$ will often get dropped out of averages... and note that $K^* = K^*(K)$.

↳ $\boxed{K^* \text{ is the "dual" of } K}$

- Next, look at eig(T)...

$$\lambda_0 = e^{K^*}\ ;\ \lambda_1 = e^{-K^*}\ ,\ \frac{\lambda_1}{\lambda_0} = e^{-2K^*}$$

and

$$|0\rangle, |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$$

Now, consider $\langle s_j s_i \rangle$ where $j > i$. Assume that s_0, s_N are fixed. What is $\langle s_j s_i \rangle$?

→ Claim $\boxed{\langle s_j s_i \rangle = \dfrac{\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}}$

Pf: Don't care about the denominator, b/c it's just Z.

→ look at numerator, which looks like...

$\langle s_N | T^N s_j s_i | s_0 \rangle$

now, insert ~~$\sum_{i\in}$~~ ~~$I \Leftarrow$~~ ~~$\frac{1}{\sqrt{2}}\{(\cdots)\}$~~

$I = \cancel{\ \ }\ (|+\rangle\langle+| + |-\rangle\langle-|)$

identity

eigenbasis of σ_3

(since we call $\{|0\rangle, |1\rangle\}$ basis for σ_1)

With this we get

$\langle s_N | T\, I\, T \ldots I\, T\, s_j s_i | s_0 \rangle$, which $= \langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle$

Why? ~~Now~~, Reading from right to left, we just get sum over Boltzmann weight until we get to site ~~s_i~~ i

$$\begin{cases} \sigma_3 |s_i\rangle = s_i |s_i\rangle \\ T|s_0\rangle = |s_1\rangle, \text{ and so on,} \end{cases}$$

~~σ_3~~ $\Rightarrow \sigma_3 T |s_{i-1}\rangle = \sigma_3 |s_i\rangle = s_i |s_i\rangle$

↝ pull out a factor of s_i ✓.

And keep going for site j as well

$\rightarrow$ get $\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle$

$= \langle s_N | T^N s_j s_i | s_0 \rangle$. ✓

So,

$$\langle s_j s_i \rangle = \frac{\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^i | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}$$

But we can write this differently...

Define the "Heisenberg Operators" by

$$\boxed{\sigma_3(n) = T^{-n} \sigma_3 T^n}$$

the site index n plays the role of discrete integer-value time, and T is the time-evolution operator / propagator for one unit of Euclidean (imag) time

With this,

$$\boxed{\langle s_j s_i \rangle = \frac{\langle s_N | T^N \sigma_3(j) \sigma_3(i) | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}}$$

note that this kinda makes sense.

$\rightarrow$ Now, consider $N \to \infty$ & look at sites $j > i$ where j, i are far from the end points (0 & N).

Then we can approximate $T^\alpha \simeq \lambda_0^\alpha |0\rangle\langle 0|$, $\left(\alpha = N, N-j, i\right)$

- In this limit, we have (from the defn)

$$\langle s_j s_i \rangle = \frac{\langle s_N | T^{N-j} \sigma_3 T^{j-i} \sigma_3 T^{i} | s_0 \rangle}{\langle s_N | T^N | s_0 \rangle}$$

$$= \frac{\langle s_N | \lambda_0^{N-j} |0\rangle\langle 0| \sigma_3 T^{j-i} \sigma_3 \lambda_0^{i} |0\rangle\langle 0| s_0 \rangle}{\langle s_N | \underset{\lambda_0^N |0\rangle\langle 0|}{\underset{\downarrow}{T^N}} | s_0 \rangle}$$

projector to $|0\rangle$

projector to $|0\rangle$

$$\text{(formally)} \simeq \frac{\langle s_N | \lambda_0^N |0\rangle\langle 0| T^{-j} \sigma_3 T^{j} T^{-i} \sigma_3 T^{i} \overset{|0\rangle\langle 0|}{\cancel{|0\rangle\langle 0|}} s_0 \rangle}{\langle s_N | \lambda_0^N |0\rangle\langle 0| s_0 \rangle}$$

$$= \langle 0 | \sigma_3(j)\, \sigma_3(i) | 0 \rangle$$

So,
$$\boxed{\langle s_j s_i \rangle = \langle 0 | \sigma_3(j)\, \sigma_3(i) | 0 \rangle}$$

and dependence on the boundary dropped out.

Now, if $i > j$, we'll get $\sigma_3(i)\sigma_3(j)$, so, more precisely ...

$$\boxed{\langle s_j s_i \rangle = \langle 0 | T\{ \sigma_3(j)\, \sigma_3(i) \} | 0 \rangle}$$

as before:

$$T[\sigma_3(j)\sigma_3(i)] = \theta(j-i)\, \sigma_3(j)\sigma_3(i) + \theta(i-j)\, \sigma_3(i)\sigma_3(j)$$

Note: we're associating i & j with time! Take notice of this...

→ Now, let us evaluate the expression

$\langle s_j s_i \rangle = \langle 0 | \sigma_3(j) \sigma_3(i) | 0 \rangle$ → eigv of σ_1. $= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

in terms of the eigenvectors of T.

- Before this, let's look at the <u>mean magnetization</u>

$\langle s_i \rangle = \langle \cancel{s_j} s_i \rangle = \langle 0 | I \cdot \sigma_3(i) | 0 \rangle$

1

$= \langle 0 | \sigma_3 | 0 \rangle$ (check this!)

by ↑ approx T

$\overset{\langle s \rangle}{\|} = \langle s \rangle$

⇒ $\boxed{\langle s_i \rangle = \langle 0 | \sigma_3 | 0 \rangle}$ ⇒ independent of i so long as

(by $i \ll N$, $i \gg 0$) i is <u>far from the ends</u>.

↳ This of course fails @ 0 temp ⇒ $K^* = 0$, at which $T = I$ → Perron-Frobenius thm no longer holds.

#

→ Now, back to evaluating $\langle s_j s_i \rangle = \langle 0 | \sigma_3(j) \overset{T(\ \nearrow\ )}{\sigma_3(i)} | 0 \rangle$.

We note that even though it appears that $\langle s_j s_i \rangle$ only depends on the gnd state, we in fact have to know the full state even in the $N = \infty$ limit.

In any case, look at $j > i$ and insert the complete set of eigv of T (which is also $\{|0\rangle, |1\rangle\}$ between $\sigma_3(j)$ & $\sigma_3(i)$.

...

$$\langle s_j s_i \rangle = \langle 0 | \sigma_3(j) \{ |0\rangle\langle 0| + |1\rangle\langle 1| \} \sigma_3(i) |0\rangle$$

$$= \underbrace{\langle 0 | \sigma_3(j) |0\rangle}\underbrace{\langle 0|\sigma_3(i)|0\rangle}$$

$\langle 0|\sigma_3|0\rangle = 0$

$i = j = 0$

$$+ \langle 0 | \sigma_3(j) |1\rangle\langle 1| \sigma_3(i) |0\rangle$$

This is actually $\sigma_3(0)$ but we'll keep it here

$$= \langle s \rangle^2 + \langle 0 | \sigma_3(j) |1\rangle\langle 1| \sigma_3(i) |0\rangle$$

$$= \langle s\rangle^2 + \langle 0 | T^{-j} \sigma_3(0) T^{j-i} |1\rangle\langle 1| \sigma_3(0) T^{i} |0\rangle$$

$$= \langle s \rangle^2 + \left(\frac{\lambda_1}{\lambda_0}\right)^{j-i} \left| \langle 0 | \sigma_3 |1\rangle \right|^2$$

$$\Rightarrow \langle s_j s_i \rangle_c = \langle s_j s_i \rangle - \langle s \rangle^2$$

connected corr. fn.

$$= \left(\frac{\lambda_1}{\lambda_0}\right)^{j-i} \left| \langle 0 | \sigma_3 |1\rangle \right|^2$$

$$= e^{-2K^*(j-i)} \left| \langle 0 | \sigma_3 |1\rangle \right|^2$$

$$\Rightarrow \langle s_j s_i \rangle_c = [\tanh K]^{|j-i|} \left| \langle 0 | \sigma_3 |1\rangle \right|^2$$

$i < j$ or $j < i$ → we'll find same thing

replace $j-i$ by $|j-i|$, without loss of generality

So, in general, $\langle s_j s_i \rangle_c = [\tanh K]^{|j-i|} \left| \langle 0 | \sigma_3 |1\rangle \right|^2$

$$\rightarrow \boxed{\text{as } |j-i| \to \infty, \quad \langle s_j s_i \rangle \to \langle s \rangle^2}$$

With this, we can define the $\boxed{\text{correlation length}}$ (ξ)

For general models, not necessarily of Ising spins or $d=1$, we'll find that

$$\lim_{|j-i|\to\infty} \langle s_j s_i \rangle_c = \frac{e^{-|j-i|/\xi}}{|j-i|^{d-2+\eta}}$$

correlation length

some number

distance between 2 spins

The exponential decay dominates the power law asymptotically

$\rightarrow$ can extract ξ from the expression above to find

$$\xi^{-1} = \lim_{|j-i|\to\infty}\left[\frac{-\ln\langle s_j s_i\rangle_c}{|j-i|}\right]$$

Can check this:

$$\ln\langle s_j s_i\rangle_c = -\frac{|j-i|}{\xi} - \ln\left[|j-i|^{d-2+\eta}\right]$$

As $|j-i|\to\infty \rightarrow |j-i| \gg \ln|j-i|$

So we have

$$-\xi = \lim_{|j-i|\to\infty}\left\{\frac{-\ln\langle s_j s_i\rangle_c}{|j-i|}\right\}$$

Now, $\langle s \rangle = \langle 0 | \sigma_3 | 0 \rangle = \langle 0 | 1 \rangle = 0$

So, $\langle s_j s_i \rangle_c = \langle s_j s_i \rangle = \exp\{ |j-i| \tanh K \}$

So,

$$\xi^{-1} = \lim_{|j-i| \to \infty} \left\{ \frac{-\ln \langle s_j s_i \rangle_c}{|j-i|} \right\}$$

$$= -\ln \tanh K = \boxed{2K^* = \xi^{-1}}$$

→ Why is the correlation length ξ defined in terms of $\langle s_j s_i \rangle_c$?

↳ To see this, we turn on the magnetic field
→ $h > 0$ ⇒ $\langle s \rangle \neq 0$ (mean magnetization no longer zero).

I would still have $\lambda_1 > \lambda_0$, and the same argument would still show that ξ is determined by $\langle s_j s_i \rangle_c$.

→ So, what does $\langle s_j s_i \rangle$ tell us?

Well... $\langle s_j s_i \rangle$ is the Exp Val of product of 2 spins. If they fluctuate independently, then $\langle s_j s_i \rangle$ is as likely to be -1 or 1 ⇒ $\langle s_j s_i \rangle = 0$.

→ But they aren't independent! Why?

① $h > 0$ ⇒ spins tend to align. (even @ $K = 0$)
Even when $K = 0$, at which the probability distr for s_i, s_j factorizes into 2 indep dist
→ $\langle s_j s_i \rangle = \langle s \rangle^2$, independent of $|j-i|$

(2) Now with $K>0$, the aligned spins will generate via the K term additional internal field parallel to h, which enhances $\langle s\rangle$.

⇒ We are looking for additional correlations in the "fluctuations" on top of the average $\langle s\rangle$.

→ Can define new correlation fn.

$$\langle s_j s_i\rangle_{new} = \Big\langle \big(s_j - \langle s\rangle\big)\big(s_i - \langle s\rangle\big)\Big\rangle$$

$$= \langle s_j s_i\rangle - \cancel{\langle s_j\rangle\langle s\rangle - \langle s_j\rangle\langle s\rangle + \langle s\rangle^2}$$

$$= \langle s_j s_i\rangle_c \quad ✓$$

which is what we defined earlier. Note too that we're using <u>translational invariance</u> in the cancellation.

⇒ In general, we will have $\boxed{\text{"clustering"}}$, of which the Ising model is an example.

↳ [Clustering]: when the connected correlation between 2 vars A & B die asymptotically.

i.e.

$$\lim_{|j-i|\to\infty} \langle A_j B_i\rangle = \langle A_j\rangle\langle B_i\rangle$$

→ joint pdf must asymptotically factorize.

→ This will come up again when we talk about phase transitions.

(i) The Hamiltonian

T : the transfer matrix, plays the role of the time-evolution operator (in img time), when we define the Heisenberg ops.

$$\sigma_3(j) = T^{-j}\sigma_3 T^{j}$$

Now... if we identify T ... with ... $\mathcal{U}(\tau) = e^{-H\tau}$
then what is τ? $\rightarrow$ One step in the discrete time lattice.
Well... now, we introduce a Hamiltonian eg

$$\boxed{T = e^{-H}}$$

real-Hermitian

This H is dim-less because unit of time is unity.

Note that T is real, Hermitian $\Rightarrow$ symmetric

$\rightarrow$ H also real, Herm $\Rightarrow$ symmetric.

$\Rightarrow$ T, H share eigenbasis. $\{|0\rangle, |1\rangle\}$

We have...

$$T = e^{K^*\sigma_1} \qquad H = -K^*\sigma_1$$

In summary,

$|0\rangle$: $T|0\rangle = \lambda_0|0\rangle = e^{K^*}|0\rangle$
$H|0\rangle = E_0|0\rangle = -K^*|0\rangle$

$f = -\ln\lambda_0$
$= -K^* = E_0$

$|1\rangle$: $T|1\rangle = \lambda_1|1\rangle = e^{-K^*}|1\rangle$
$H|1\rangle = E_1|1\rangle = K^*|1\rangle$

A few remarks on generality!

(1) $\langle s_j s_i \rangle_c$ depends only on the ratio ~~of~~ λ_1/λ_0, which are of eigv of T, and falls exp-ly with distance with coef $2K^{*}$.

Now, $2K^{*} = K^{*} - (-K^{*})$, which is the gap to the 1st excited state of H where $T = e^{-H}$.

$$\Rightarrow \boxed{\xi^{-1} = E_1 - E_0 = m} \text{ is very general}$$

"mass gap"

(2) $\langle s_j s_i \rangle_c$ also depends on $|\langle 0 | \sigma_3 | 1 \rangle|^2$. $(\lambda_0 > \lambda_1)$

gnd state

next exc state.

$\Rightarrow$ This is also a general feature.
If $\langle 0 | \sigma_3 | 1 \rangle = 0$, then we must go up till we find a state that is connected to the gnd state by σ_3.

Here, $\langle 0 | \sigma_3 | 1 \rangle = \langle 0 | 0 \rangle = 1 \neq 0$.

If T is bigger than 2×2, the sum over states will have more than 2 terms,

$\rightarrow$ $\langle s_j s_i \rangle_c$ will be Σ of decaying exps, and a unique ξ will emerge only asymptotically when the small mass gap dominates!

(ii) Turning on h

Now, add the magnetic field back into the Hamiltonian

~~$\mathcal{H} =$~~ Σ → add term $h \sum_i s_i$.

What is the transfer matrix now?

→ Suppose we define $T = e^{K^* \sigma_1} e^{h \sigma_3} \equiv T_K T_h$

then T reproduces Boltzmann weight, but is not symmetric (Hermitian).

→ To fix, write

$$h \sum_i s_i = \frac{1}{2} h \sum_i (s_i + s_{i+1})$$

to get

$$\boxed{T = T_h^{1/2} \, T_K \, T_h^{1/2}}$$

which is symmetric. (Recall that T's defined via its elements $T_{\pm\pm}$, $T_{\mp\pm}$)

Define $T = e^{-\mathcal{H}}$. Before, when $h = 0$, $\mathcal{H} = -K^* \sigma_1$.
But here, ~~now~~ we can't find an explicit formula for $\mathcal{H}$...

Ex The eigenvalues of T are: $T = \begin{pmatrix} e^{h} & e^{-2K} \\ e^{-2K} & e^{h} \end{pmatrix}$

$$\text{eigv}(T) = \frac{1}{2} e^{-h-2K} \left(e^{2K} + e^{2h+2K} \mp \sqrt{\begin{matrix} 4e^{2h} + e^{4K} \\ -2e^{2h+4K} \\ + e^{4h+4K} \end{matrix}} \right)$$

<u>Calculate magnetisation</u> recall that mean magnetisation

= magnetisation per site, which is related to the free energy...

Recall that $M = \frac{-\partial F}{\partial h} \Rightarrow \langle s \rangle = \frac{M}{N} = \frac{-1}{N}\frac{\partial F}{\partial h}$

$= \begin{pmatrix} e^{h} & e^{-2k} \\ e^{-2k} & e^{-h} \end{pmatrix}$

where $\frac{F}{N} = f = -\ln \lambda_0 \longrightarrow$ larger eigv.

$$= -\ln\left\{ \frac{1}{2} e^{-2h+2K}(-1+e^{2h}) \times \left[e^{2K}(-1+e^{2h}) + \sqrt{4e^{2h}+e^{4K}(-1+e^{2h})^2} \right] \right\}$$

$$\Rightarrow m = \langle s \rangle = \frac{-1}{N}\frac{\partial F}{\partial h} = -\frac{\partial f}{\partial h} = \frac{e^{2K}(-1+e^{2h})}{\sqrt{4e^{2h}+e^{4K}(-1+e^{2h})^2}}$$

= (simplification)

$$\boxed{\langle s \rangle = \frac{\sinh(h)}{\sqrt{e^{-4K}+\sinh^2(h)}}}$$ ✓

note that when $h = 0$, $\langle s_i \rangle = 0$ as expected

$\boxed{?}$ Can we evaluate $\langle s \rangle \overset{?}{=} \langle s_j \rangle$ ~~by~~ by definition?

<u>NO</u>, because we no longer have translational invariance! $\langle s \rangle \neq \langle s_j \rangle$

↳ <u>We have to use</u> $\frac{\partial f}{\partial h}$ instead. (IMPORTANT)

(d) Classical to Quantum Mapping: Dictionary

(1) Schrödinger operators are σ_3 | Ω

Heisenberg ops are $\sigma_3(j)$ | $\Omega(t)$

in stat mech | before, in QM

(2) Transfer matrix T | Propagator in imaginary time ($\Delta\tau$)

$$T \Longleftrightarrow \mathcal{U}(\Delta\tau)$$

Units: $[\Delta\tau] = 1$.

With this,

$$Z(s_i, s_f) = \langle s_N = s_f | T^N | s_0 = s_i \rangle \Longleftrightarrow \langle s_f | \mathcal{U}(\Delta\tau) | s_i \rangle$$

matrix elements corresponding to propagator $\mathcal{U}$.

(3) Heisenberg - Schrö:

$$\sigma_3(j) = T^{-j} \sigma_3 T^{j} \Longleftrightarrow \mathcal{U}^{-1}(j\Delta\tau)\, \sigma_3\, \mathcal{U}(j\Delta\tau)$$

$$= \sigma_3(\tau = j\Delta\tau)$$

(4) Hamiltonian: $T = e^{-H} = e^{-H\Delta\tau}$, from which we have that the dominant eigv of T is the gnd state of H. ($|0\rangle$)

So it's important to remember that

$$\boxed{\begin{aligned} |0\rangle &= \text{"excited state" of } T \text{ (dominant eigvect)} \\ &= \text{ground state of the Hamiltonian } H \end{aligned}}$$ (transfer matrix)

5 Correlation function in $N \to \infty$ (thermo limit)

= gnd state expectation value of the time-ordered product of the corresponding Heisenberg op:

$$\boxed{\langle s_j s_i \rangle \iff \langle 0 | \mathcal{T} \{ \sigma_3(j) \sigma_3(i) \} | 0 \rangle}$$

gnd state of H

excited state of T

—//—

IV. QUANTUM TO STATISTICAL MECHANICS

Here we'll go from Imaginary time QM to classical SM.
by retracing the path that led to the transfer matrix

$\Rightarrow$ and map the problem of <u>Quantum Stat Mech</u>.

Central object is the partition fn:

$$\boxed{Z_Q = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\beta\hbar H/\hbar} = \mathrm{Tr}\, \mathcal{U}(\tau = \beta\hbar)}$$

H: Quantum Hamiltonian

Z_Q is just the trace of the imaginary-time evolution operator for a time $\beta\hbar$.

We want to study a more general object:

$$\mathcal{U}(f, i; \tau) = \langle f | \mathcal{U}(\tau) | i \rangle$$

(a) From $\mathcal{U}$ to Z

Feb 18, 2021

Begin with the matrix element of the propagator

$$\mathcal{U}(f, i; \tau) = \langle f | \mathcal{U}(\tau) | i \rangle$$

We will set $\hbar \equiv 1$ for now.

(b) Example from Spin 1/2

Consider Hamiltonian

$$\boxed{\mathcal{H} = -B_1\sigma_1 - B_3\sigma_3}$$

which describes a spin-$\frac{1}{2}$ in a magnetic field

$$\vec{B} = \hat{i}B_1 + 0\hat{j} + B_3\hat{k} \quad (\text{in } x\text{-}z \text{ plane})$$

Now factor...

$$\longrightarrow \quad \mathcal{U}(\tau) = \left\{\mathcal{U}\left(\tau/N\right)\right\}^N$$

where

$$\mathcal{U}\left(\tau/N\right) \equiv T_\varepsilon = e^{-\varepsilon H}, \text{ with } \varepsilon = \tau/N$$

Put in $N-1$ intermediate ~~states~~ sums over σ_3 eigenstates & renaming $|i\rangle$, $|f\rangle$ to $|s_0\rangle$, $|s_N\rangle$, resp., we get

$$\boxed{\mathcal{U}(s_N, s_0) = \sum_{s_i}\prod_{i=0}^{N-1}\langle s_{i+1}|e^{-\varepsilon H}|s_i\rangle}$$

$$\downarrow$$

$$\langle s_N|\mathcal{U}(\tau)|s_0\rangle = \langle s_N|(e^{-\varepsilon H})^N|s_0\rangle \quad (\text{like before})$$

$$= \sum_{s_i}\prod_{i=0}^{N-1}\langle s_{i+1}|e^{-\varepsilon H}|s_i\rangle$$

So, $\mathcal{U}(s_N, s_0)$ is just the Z that we've seen before (for $N+1$ spins)

$\Longrightarrow$ the transfer matrix is defined by the elements:

$$\boxed{T_{s_{i+1}s_i} = \langle s_{i+1}|e^{-\varepsilon H}|s_i\rangle}$$

Now, what are the classical $\{s_i\}$ parameters?

$\Rightarrow$ to find them, we equate the general expression for the Boltzmann weight of the $\{s_i\}$ problem to the matrix elements of the QM problem --

$$e^{K(s's-1)\pm\frac{h}{2}(s'+s)+c} = \langle s'|e^{-\epsilon H}|s\rangle$$

$$= \langle s'|\, e^{+\epsilon(B_1\sigma_1 + B_3\sigma_3)}\,|s\rangle$$

Boltzmann weight

QM

$\boxed{?}$ why does the exponent on the LHS have such a form?

$\hookrightarrow$ Pf Let $T_{s's} = e^{R(s',s)}$.

Now, let $R(s',s) = \sum_{i,j} (s')^i (s)^j B_{ij}$

Since $\underline{s^2 = (s')^2 = 1}$ and that $T_{s's}$, $R(s',s)$

must both be symmetric in $s \leftrightarrow s'$,

$R(s',s)$ must have the form $Ass' + B(s'+s) + C$

$\#$

Now, we want to solve for K, h, c in terms of B_1, B_3 & c.

$\Rightarrow$ Choose $s = s' = \pm 1$ & $s = -s'$. Then we can show that

$$\begin{cases} \tanh(h) = \dfrac{B_3}{B}\tanh(\varepsilon B) \\ e^{-2k+c} = \dfrac{B_1}{B}\sinh(\varepsilon B) \\ e^{2c}(1-e^{-4K}) = 1 \end{cases}$$

$\det(e^{\varepsilon\vec{\sigma}\cdot\vec{B}}) = 1$

using the identity $\boxed{e^{\varepsilon\vec{\sigma}\cdot\vec{B}} = \cosh(\varepsilon B) + \frac{\vec{B}\cdot\vec{\sigma}}{B}\sinh(\varepsilon B)}$

Note that the classical parameters K, h, c depend on ε, the time slice $(\varepsilon = \tau/N)$

→ Changing ε will change the params K, h, c, but Z will remain constant.

How do we fix the problem of discrete time?
→ We have to take $N \to \infty$ & $\varepsilon \to 0$.

In Euclidean time QM, the lattice in time is an artifact that must be banished at the end by taking $N \to \infty$ or $\varepsilon = \tau/N \to 0$

More concretely... consider a typical situation where we want to evaluate...

$$G(\tau_2-\tau_1) = \frac{\mathrm{Tr}\{e^{-H\tau}[\mathcal{T}(\sigma_3(\tau_2)\sigma_3(\tau_1))]\}}{\mathrm{Tr}[e^{-H\tau}]}$$

If we map this to the PBC Ising model, we'll just set

$$\boxed{G(\tau_2-\tau_1) = \langle s_{i_2} s_{i_1} \rangle}$$

where ~~i_2~~ $\tau_1 = i_1\varepsilon$; $\tau_2 = i_2\varepsilon$

How do we work in the limit $\varepsilon\to 0$?

↳ In the infinitesimal limit of ε, $(T_\varepsilon = e^{-\varepsilon H})$

$$\boxed{\mathcal{U}(\tau/N) = \mathcal{U}(\varepsilon) = T_\varepsilon \longrightarrow I - \varepsilon H + O(\varepsilon^2)}$$

So the matrix elements of T_ε are basically those of εH.

With $H = -B_1\sigma_1 - B_3\sigma_3$, we have

$$e^{K(s's-1)+h/2(s'+s)+c} = \langle s'|e^{-\varepsilon H}|s\rangle$$

$$= \langle s'| I + \varepsilon B_1\sigma_1 + \varepsilon B_3\sigma_3 |s\rangle \quad (\varepsilon\to 0)$$

And thus, as $\varepsilon\to 0$

$$\boxed{\begin{aligned} T_{\pm\pm} &= e^{\pm h+c} = 1\pm\varepsilon B_3 \\ T_{+-} &= e^{-2K+c} = \varepsilon B_1 \end{aligned}}$$

Also, $T_{++}\cdot T_{--} = e^{2c} \simeq 1 \Rightarrow c=0$. So the parameters are

$$\boxed{h = \varepsilon B_3 \ ; \ e^{-2K} = \varepsilon B_1}$$

which agree with ~~[illegible]~~ the exact results earlier, in the $\varepsilon\to 0$ limit.

↳ (with sinh, tanh, etc)

(c) $\boxed{\text{The T-continuum limit of Fradkin - Susskind}}$

We won't worry too much about this, except noting that

- When $\epsilon \to 0$, the QM problem maps to classical, where the parameters h, e^{-2k} are $\propto \epsilon$, i.e. infinitesimal.

This corresponds with $T \simeq I$,

(d) $\boxed{\text{Two } N\to\infty \text{ limits - 2 Two Temperatures}}$

$N \to \infty$ has different meanings depending on what we hold constant.

- If T is fixed then $N \to \infty$ means taking the thermodynamic limit. In this limit, $T^N \simeq \lambda_0^N |0\rangle\langle 0|$ where λ_0 is the dominating eigv.

  If we let $T = e^{-H}$ then the system is dominated by the gnd state of H, which is $|0\rangle$.

- If we choose $T_\epsilon = \mathcal{U}(\epsilon)$ then the parameters in T_ϵ vary, but $Z = \mathrm{Tr}\, T^N$ or $\langle s_N | T^N | s_0\rangle$ is constant.

  ↳ Then system has finite extent in $\mathcal{I}$, but the pts in that time interval become dense as $N \to \infty$

  ↳ Physics is <u>NOT</u> dom. by max(eigv(T)). This holds only if $\mathcal{I} \to \infty$.

↳

8 TN

- If we want a quantum system @ $T=0$, i.e. $\beta \to \infty$
  - ↳ must let $\boxed{\bar{\beta} = \beta\hbar \to \infty \Rightarrow \text{gnd state dominates}}$
    - ↳ This can be a problem for Monte Carlo method b/c the # of pts increases really fast...

There are actually 2 temperatures when we map the quantum E&B partition fn to a classical Ising model

Temperatures

We actually have
- $\beta \Rightarrow$ controls length in τ (spatial extent of Ising problem)
- $K = J/kT$
  - ↳ temperature of the Ising model, which varies the parameters of the quantum problem.

Now, we go to the <u>Feynman Path Integral</u>!

V: THE FEYNMAN PATH INTEGRAL

(i) The Feynman Path Integral in Real Time

Consider time-indep Hamiltonian:

$$\boxed{\mathcal{H} = \frac{p^2}{2m} + V(x)}$$

From which we have:

$$\boxed{\begin{aligned} \exp\left\{\frac{-i\varepsilon}{\hbar}H\right\} &= \exp\left\{\frac{-i\varepsilon}{\hbar}\left[\frac{p^2}{2m} + V(x)\right]\right\} \\ &\simeq \exp\left\{\frac{-i\varepsilon}{\hbar}\cdot\frac{p^2}{2m}\right\}\exp\left\{\frac{-i\varepsilon}{\hbar}V(x)\right\} \end{aligned}}$$

because the commutators in ~~of~~

$$e^A e^B = \exp\left\{(A+B) + \frac{1}{2}[A,B] + \dots\right\}$$

Baker Campbell Hausdorff formula

are proportional to higher pwr of ε. i.e.

$$\frac{1}{2}[A,B], \dots = \mathcal{O}(\varepsilon^2)$$

$$\text{and } \varepsilon \to 0 \to [A,B]^n \to 0.$$

Next, we can split $V(x)$ into $\frac{V(x)}{2} + \frac{V(x)}{2}$ and write

$$H = \frac{V(x)}{2} + p^2/2m + V(x)/2$$

which mimicks writing the magnetic field term as $T_h^{7/2}\, T_K\, T_h^{7/2}$.

However, we won't do that here b/c there's no gain.

- Anyway... we have to compute the following...

$$\langle x' | \left(e^{-i\epsilon H/\hbar}\right)^N | x\rangle \cong \langle x' | \underbrace{\left(e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V(x)}{\hbar}}\right) \cdots}_{N \text{ times}} e^{\frac{-i\epsilon V(x)}{\hbar}} | x\rangle \quad (\epsilon \to 0)$$

time-~~evolution~~ operator

→ now insert $I = \int_{-\infty}^{\infty} |x\rangle\langle x| dx$ between each of the

factor $\mathcal{U}(t/N) = \exp\left\{\frac{-i\epsilon p^2}{2m\hbar}\right\} \exp\left\{\frac{-i\epsilon V(x)}{\hbar}\right\}$ ~~gives~~.

→ What does this give? Consider $N=3$...

$$\mathcal{U}(x_3, x_0; t) = \langle x_3 | e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V(x)}{\hbar}} \int |x_2\rangle\langle x_2| dx_2\, e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V(x)}{\hbar}}$$

$$\cdot \int |x_1\rangle\langle x_1| dx_1\, e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V(x)}{\hbar}} |x_0\rangle$$

$$= \int \prod_{n=1}^{2} dx_n \, \langle x_3 | e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V}{\hbar}} |x_2\rangle$$

$$\cdot \langle x_2 | e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V}{\hbar}} |x_1\rangle \langle x_1 | e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V}{\hbar}} |x_0\rangle$$

Now, consider the matrix element $\langle x_n | e^{\frac{-i\epsilon p^2}{2m\hbar}} e^{\frac{-i\epsilon V(x)}{\hbar}} |x_{n-1}\rangle$

note that $|x_{n-1}\rangle$ is an eigenvector of $V(X)$, so

$$e^{-i\varepsilon V(X)/\hbar}|x_{n-1}\rangle = |x_{n-1}\rangle \cdot e^{-i\varepsilon V(x_{n-1})/\hbar}$$

So,

$$\langle x_n| \exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\} \exp\left\{\frac{-i\varepsilon V(X)}{\hbar}\right\} |x_{n-1}\rangle$$

$$= \underbrace{\langle x_n| \exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\} |x_{n-1}\rangle}_{} \exp\left\{\frac{-i\varepsilon V(x_{n-1})}{\hbar}\right\},$$

this is now just the free particle propagator from x_{n-1} to x_n in time ε.

We know that

$$\boxed{\begin{aligned} \mathcal{U}_{free}(x_n, x_{n-1}; \varepsilon) &= \langle x_n| \exp\left\{\frac{-i\varepsilon P^2}{2m}\right\} |x_{n-1}\rangle \\ &= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \cdot \exp\left\{\frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon}\right\} \end{aligned}}$$

Why? Recall that $I = \int_{-\infty} \frac{dp}{2\pi\hbar} |p\rangle\langle p|$, and so

$$\langle x_n| \exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\} |x_{n-1}\rangle = \int_{\infty} \frac{dp}{2\pi\hbar} \langle x_n|p\rangle\langle p| \exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\} |x_{n-1}\rangle$$

$$= \frac{1}{2\pi\hbar} \int \frac{dp}{1} e^{ipx_n/\hbar} \cdot \langle p| \exp\left\{\frac{-i\varepsilon P^2}{2m\hbar}\right\} |x_{n-1}\rangle$$

$$= \frac{1}{2\pi\hbar} \int dp\, e^{ipx_n/\hbar} \exp\left\{\frac{+i\varepsilon p^2}{2m\hbar}\right\} \langle p|x_{n-1}\rangle$$

$$= \frac{1}{2\pi\hbar}\int dp \; e^{ip(x_n - x_{n-1})/\hbar} \exp\left\{\frac{-i\varepsilon p^2}{2m\hbar}\right\}$$

= (Mathematica) Feynman Gaussian Integral trick ...)

$$= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \exp\left\{\frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon}\right\} \quad ✓$$

□

<u>Remark</u>: note that we didn't expand $e^{-i\varepsilon p^2/2m\hbar}$ out to order ε because $\hat{p}^2$ has ~~a~~ singular matrix elements between $\langle x_n|$ & $|x_{n-1}\rangle$.

Note that $\langle x_n| \exp\left\{\frac{-i\varepsilon p^2}{2m\hbar}\right\} |x_{n-1}\rangle$ has not a series expansion in ε.

With this, we have the following ...

$$\boxed{\begin{aligned}&\langle x_n| \exp\left\{\frac{-i\varepsilon p^2}{2m\hbar}\right\} \exp\left\{\frac{-i\varepsilon V(x)}{\hbar}\right\} |x_{n-1}\rangle \\ &\qquad = \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \cdot \exp\left\{\frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon}\right\} \cdot \exp\left\{\frac{-i\varepsilon}{\hbar} V(x_{n-1})\right\}\end{aligned}}$$

So, by collecting all these factors, we have for general N

$$\boxed{\begin{aligned}\mathcal{U}(x_N, x_0; t) &= \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} \left\{\int \prod_{n=1}^{N-1} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2} dx_n\right\} \\ &\quad \times \exp\left\{\sum_{n=1}^{N} \frac{im(x_n - x_{n-1})^2}{2\hbar\varepsilon} - \frac{i\varepsilon}{\hbar} V(x_{n-1})\right\}\end{aligned}}$$

Now, rewrite

$$\exp\left(\sum_{n=1}^{N}\left\{\frac{im(x_n-x_{n-1})^2}{2\hbar\epsilon}-\frac{i\epsilon}{\hbar}V(x_{n-1})\right\}\right)$$

$$=\exp\left(\frac{i\epsilon}{\hbar}\sum_{n=1}^{N}\left\{\frac{m(x_n-x_{n-1})^2}{2\epsilon^2}-V(x_{n-1})\right\}\right)$$

kinetic − potential = $\mathcal{L}$

so that we can obtain the continuum version as $\underline{\epsilon\to 0}$:

$$\boxed{\mathcal{U}(x,x';t)=\int[\mathcal{D}x]\,\exp\left\{\frac{i}{\hbar}\int_0^t\mathcal{L}(x,\dot{x})\,dt\right\}}$$

where

$$\int[\mathcal{D}x]=\lim_{N\to\infty}\left(\frac{m}{2\pi\hbar i\epsilon}\right)^{1/2}\int\left[\prod_{n=1}^{N-1}\left(\frac{m}{2\hbar i\epsilon\pi}\right)^{1/2}dx_n\right]$$

and the Lagrangian $\mathcal{L}(x,\dot{x})$ is the fn from classical mech

$$\mathcal{L}(x,\dot{x})=\underbrace{\frac{1}{2}m\dot{x}^2}_{\text{kinetic}}-\underbrace{V(x)}_{\text{potential}}=K-V$$

The quantity $S[x(t)]=\int_{t_1}^{t_2}\mathcal{L}\,dt$ is called the <u>action</u>

functional

Schematically, we write

$$\boxed{\mathcal{U}(x',x;t)=\int[\mathcal{D}x]\,e^{\frac{i}{\hbar}S}}$$

action.

Stationary points of S contribute the most to the propagator amplitude $\Rightarrow$ obey the Euler - Lagrange Eqn:

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}}$$

principle of least action.

As $\hbar \to 0$, we may forget all but the classical path and its coherent neighbors & write

$$\boxed{\mathcal{U}(x', x; t) \simeq A e^{i/\hbar S_c}}$$

where S_c is the action evaluated @ classical path.
A is the prefactor that reflects the sum over other paths.

↳ we can have saddle-point approximations to help evaluate these (just like method of stationary phase in osc. integrals)

↳ We won't worry abt this for now.

We'll also skip some of the calculations & examples here.

(b) $\boxed{\text{The Feynman Phase Space Path Integral}}$

To do this, we look back at

$$\langle x_N | \underbrace{\left(e^{-i\varepsilon p^2/2m\hbar} e^{-i\varepsilon V(x)/\hbar}\right) \cdots \quad e^{-i\varepsilon V(x)/\hbar}}_{N\text{-times } e^{-i\varepsilon H/\hbar}} | x_0 \rangle$$

Introduce resolution of identities :

$$I = \int_{\infty} |x\rangle\langle x|\, dx = \int_{\infty} \frac{dp}{2\pi\hbar} |p\rangle\langle p| .$$

where ~~$|x\rangle\langle p|$~~ $\langle x|p\rangle = e^{ip\cdot x/\hbar}$

- Let $N = 3$. Then we insert 3 resolutions of identity in p and 2 in terms of x alternatingly ...

This gives

$$\boxed{\begin{aligned} \mathcal{U}(x_3, x_0; t) = \int [\mathcal{D}p\, \mathcal{D}x]\, &\langle x_3| e^{-i\varepsilon p^2/2m\hbar} |p_3\rangle\langle p_3| e^{-i\varepsilon V(x)/\hbar} |x_2\rangle \cdot \\ &\times \langle x_2| e^{-i\varepsilon p^2/2m\hbar} |p_2\rangle\langle p_2| e^{-i\varepsilon V(x)/\hbar} |x_1\rangle \\ &\times \langle x_1| e^{-i\varepsilon p^2/2m\hbar} |p_1\rangle\langle p_1| e^{-i\varepsilon V(x)/\hbar} |x_0\rangle \end{aligned}}$$

where

$$\int [\mathcal{D}p\, \mathcal{D}x] = \underbrace{\int_{\infty} \cdots \int_{\infty}}_{2N-1 \text{ times}} \prod_{n=1}^{N} \frac{dp_n}{2\pi\hbar} \prod_{n=1}^{N-1} dx_n$$

- Observe that evaluating each matrix element is very easy b/c we're only working with eigenvalues & eigvector

~~→ Gather all terms and take the $\varepsilon \to 0$ limit we get~~

Gather all terms to get the following ...

$$\mathcal{U}(x,x',t) = \int [\mathcal{D}p\,\mathcal{D}x] \exp\left\{ \sum_{i=1}^{N} \left(\frac{-i\varepsilon}{2m\hbar} p_n^2 + \underbrace{\frac{i}{\hbar} p_n (x_n - x_{n-1})}_{\sim p\dot{x}} - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right) \right\}$$

- As $N \to \infty$ i.e. $\varepsilon \to 0$ we can write in ~~continuous~~ time (after manipulating ε like before ---) we get

$$\sum_{i=1}^{N} \left(\frac{-i\varepsilon p_n^2}{2m\hbar} + \frac{i}{\hbar} p_n (x_n - x_{n-1}) - \frac{i\varepsilon}{\hbar} V(x_{n-1}) \right)$$

$$= \frac{i\varepsilon}{\hbar} \sum_{i=1}^{N} \left[\left(\frac{-p_n^2}{2m} - V(x_{n-1}) \right) + \frac{p_n (x_n - x_{n-1})}{\varepsilon} \right]$$

- ε acts like time, so, as $N \to \infty$; i.e. $\varepsilon \to 0$,

$$\begin{cases} 1^{st} \text{ term} \Rightarrow \text{becomes } \mathcal{H}(x,p) \\ 2^{nd} \text{ term} \Rightarrow \text{becomes } p\dot{x} \end{cases}$$

So, in the continuum limit,

$$\boxed{\mathcal{U}(x,x',t) = \int [\mathcal{D}p\,\mathcal{D}x] \exp\left\{ \frac{i}{\hbar} \int_0^t [p\dot{x} - \mathcal{H}(x,p)]\,dt \right\}}$$

$x = x(t)$ $\quad$ $p = p(t)$

"phase space path integral"

Note that since p is quadratic in the exponent, we can actually integrate it out, and get the usual "configuration space" path ~~integral~~ as expected ...

(c) $\boxed{\text{The Feynman Path Integral for Imag. Time}}$

Repeat the same process, but with imaginary time, we get

$$\mathcal{U}(x, x'; \tau) = \langle x | \mathcal{U}(\tau) | x' \rangle$$

$$= \int \underbrace{[\mathcal{D}x]}_{\lim_{N\to\infty} \left(\frac{m}{2\hbar\pi\varepsilon}\right)^{1/2} \prod_{i=0}^{N-1} \left(\frac{m}{2\pi\hbar\varepsilon}\right)^{1/2} dx_i} \exp\left[-\frac{1}{\hbar}\int_0^{\tau} \mathcal{L}_E(x, \dot{x})\, d\tau\right]$$

$$\boxed{\mathcal{L}_E = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)}$$

$\mathcal{L}_E$ is the Euclidean Lagrangian.

= Sum of KE & real-time potential.

⊛ ⟹ particle actually sees the potential flipped upside-down

Similarly, we can define the Euclidean action...

$$S_E = \int \mathcal{L}_E \, d\tau$$

Principle of least action still applies...

$e^{-S_E/\hbar}$ is biggest when S_E smallest, ...

(d) Classical – Quantum Connection

- Path integral from $x_0 \to x_N$ is identical in form to a classical partition function of a system with $N+1$ coordinates x_n, with BC: x_0, x_N fixed.

  ↳ x_n: intermediate state labels of the quantum problem = classical variables summed over in the partition function.

- S_E (Euclidean action) $\equiv$ Energy in the partition fn.

- The role of $\hbar$ is played by T.
  As $\hbar$ (or T) $\to 0$, the sum / integral over all configs is dominated by the min of S_E, or energy & fluctuations are suppressed.

- $K_1 = m/2\hbar\varepsilon$; $K_2 = m\omega^2\varepsilon/2\hbar$

- $\varepsilon \to 0$ in the QM problem, so classical params must also approach limits $K_1 \to \infty$, $K_2 \to 0$.

- The single QM degree of freedom is traded for a 1-D array of classical d.o.f (spin ± 1).

  ↳ Dimensionality goes up by one as we go from QM → CM.

—//—

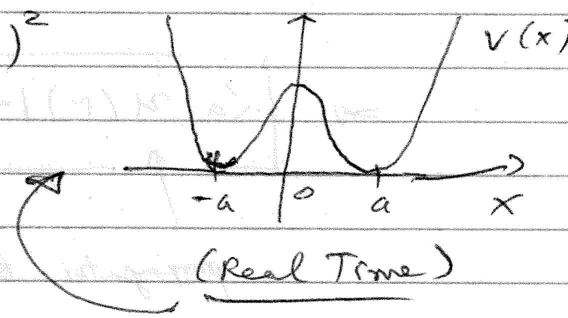
(e) Tunneling by Euclidean Path Integrals

Recall that we can approximate path integrals by looking @ contribution from the classical path as $\hbar \to 0$.

→ But if $\exists$ Barrier ⇒ we get no classical path.
→ can't find tunneling amplitudes.

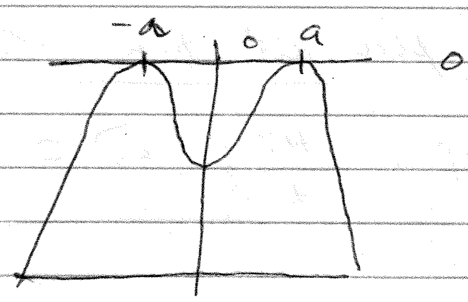
→ However in Euclidean dynamics, the potential is turned upside down ...
↳ Tunneling up can be found by this formalism.

$\boxed{\text{Ex}}$ let $V(x) = A^2(x^2 - a^2)^2$

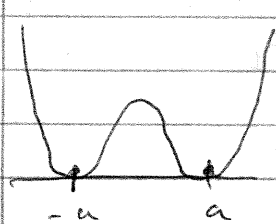


(Real Time)

In img. time:



In real time, we have a degeneracy in gnd state



→ call them $\boxed{|\pm a\rangle}$.

$$\mathcal{H} = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$$

→ Let's shift reference so that $\underline{E_0 = 0}$.

then the energy levels will be split into

$$E = \pm H_{+-} = \pm H_{-+} = \pm \langle a|H|-a\rangle .$$

and the eigenstates will be $|S\rangle$ & $|A\rangle$

symmetric comb of $|\pm a\rangle$

antisymmetric comb of $|\pm a\rangle$.

$\Rightarrow$ There's no classical path between $-a$ & $+a$ in real time, but there is one in img time (so we invert the potential).

$\Rightarrow$ Consider ...

$$\Rightarrow \boxed{\langle a|\mathcal{U}(\tau)|-a\rangle = \langle a|\exp\left\{\frac{-H\tau}{\hbar}\right\}|-a\rangle}$$

propagator from $-\tau/2 \rightarrow \tau/2$

Now ... pick out term linear in τ ...

$$\boxed{\langle a|\exp\left\{\frac{-H\tau}{\hbar}\right\}|-a\rangle \simeq 0 - \frac{1}{\hbar}\tau\langle a|H|-a\rangle + O(\tau^2)}$$

same H as in real time.

Now, from the semiclassical approximation we also know that - up to the term linear in τ ...

$$\boxed{\langle a|e^{-H\tau/\hbar}|-a\rangle \simeq \tau e^{-S_{cl}/\hbar}}$$

Why? See p. 56 of the book for ideas ...

So, we infer that

$$\boxed{\langle a| H |-a\rangle \simeq e^{-S_{cl}/\hbar}}$$

where

$$S_{cl} = \int (T+V)\,d\tau \quad \text{because} \quad E_0 = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V = T-V \quad \text{conserved}$$

$$= \int 2T\,d\tau \quad \longleftarrow \text{want 0-energy solution} \rightarrow T = V$$

$$= \int m\dot{x}\dot{x}\,d\tau$$

$$= \int_{-a}^{a} p(x)\,dx = \int_{-a}^{a} \sqrt{2mV(x)}\,dx$$

$\Rightarrow$ Tunneling amplitude is $e^{-\frac{1}{\hbar}\int_{-a}^{a}\sqrt{2mV(x)}\,dx}$ which agrees with Schrödinger's approach.

Note that S_{cl} does not depend on $\tau_0 \Rightarrow$ every path has the same action.

Sum over everything gives...

$$\langle a| U(\tau) |-a\rangle \simeq \tau e^{-S_{cl}/\hbar}$$

compare this to $\langle a| e^{-\frac{1}{\hbar}\tau H} |-a\rangle \simeq 0 - \frac{\tau}{\hbar}\langle a|H|-a\rangle + O(\tau^2)$

$$\Rightarrow \boxed{H_{+-} \simeq e^{-\frac{1}{\hbar}S_{cl}}} \quad \text{From here we have...}$$

$$|S\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + |-a\rangle\right) \quad ; \quad |A\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle - |-a\rangle\right)$$

$$E_S = -e^{-\frac{1}{\hbar}S_{cl}} \quad ; \quad E_A = +e^{-\frac{1}{\hbar}S_{cl}}$$

(f) Spontaneous Symmetry Breaking