1. (25 pts)

(a) An instructor has 13 students in her combinatorics class and wants to divide them into four groups with at least three students in each group. How many ways are there to do that? What if the instructor also picks a spokesperson for each group?

Answer: The setup requires that there must be 3 groups of three and 1 group of four students. The assignment procedure can be described as follows. Choose 4 out of 13 students for the first group, 3 out of the remaining 9 students for the second, 3 out of the remaining 6 students for the third group, and the rest go to the fourth group. Finally, permute the four groups. The total is thus

$$\Omega_{13} = \frac{1}{4!}\binom{13}{4}\binom{9}{3}\binom{6}{3}$$

Of course, we don't have to first select 4 students. We can check[1] that the same total is obtained by selecting the group of 4 second or last:

$$\Omega_{13} = \frac{1}{4!}\binom{13}{3}\binom{10}{4}\binom{6}{3} = \frac{1}{4!}\binom{13}{3}\binom{10}{3}\binom{7}{4}.$$

There are i ways to pick a spokesperson for a group with i members, so

$$\Omega_{13 + \text{pick spokesperson for each group}} = \Omega_{13} \times 4 \times 3 \times 3 \times 3 = \Omega_{13} \times 4 \times 3^3$$

(b) Five weeks after the beginning of the semester and right after the first midterm, the instructor divides the students into a new set of four groups, again with at least three students in each group. The instructor would like the new groups to be such that no two students who previously were in the same group are again in the same group. How many ways are there to do that?

Answer: Assume some group configuration was set for the first five weeks of the semester, say, Groups 1 to 4 where Group 4 has four students. In order to rearrange the groups as required, Group 4 must be completely separated into the four new groups. Thus, let each of these four students start a new group. This leaves us with 9 students. The procedure to send these students to new groups is as follows:

- Separate Group 3 into three of the four new groups ($4 \times 3!$ ways).

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[1]using a calculator or explicitly with pen and paper

- Send one student of Group 2 and one student from Group 1 into the new group that doesn't have anyone from Group 3 (3×3 ways).
- Send the remaining students of Group 2 into the three new groups that don't have any from Groups 1 and 2 ($3 \times 2!$ ways).
- Now, there is one new group with two students (the rest have three each). Send one student from the remaining two of Group 1 into this new group (2 ways).
- The pigeon hole principle guarantees that there are still two of the new groups that don't have anyone from Group 1. This means there are 2 ways to send the last student of Group 1.

We note that the order in which we did things (subsequently filling the new groups with students from Group 3, then 2 and 1) should not matter since the order of membership does not matter. Therefore, there is no extra symmetry factor[2], and the grand total is:

$$4 \times 3! \times 3 \times 3 \times 3 \times 2! \times 2 \times 2 = \boxed{5184}$$

(c) Answer all the above questions again, but for 12 students instead of 13.

Answer: Following a similar argument (picking 3 students for each subsequent group then permute), we find that

$$\boxed{\Omega_{12} = \frac{1}{4!}\binom{12}{3}\binom{9}{3}\binom{6}{3}}$$

and

$$\boxed{\Omega_{12 \text{ and pick spokesperson for each group}} = \Omega_{12} \times 3 \times 3 \times 3 \times 3 = \Omega_{12} \times 3^4}$$

Now, for the second part. Assume some group configuration was set for the first five weeks of the semester. The procedure to send these students to new groups is as follows:

- Choose a group (of three) and select one student from the remaining 9 to start four new groups (4×9 ways).
- Call the group selected in the previous step Group 1 and the group from which the student was selected Group 2. Send the remaining two students in Group 2 into two of the three available new groups – which are those with students from Group 1 ($3 \times 2!$ ways).
- Call the two remaining original groups Groups 3 and 4 (2 ways to do this). Pick two students from Group 3 and send them to the two new groups where each currently has only one student ($2 \times 3 \times 2!$ ways).
- Pick two students from Group 3 and send them to same two new groups from the previous item ($3 \times 2!$ ways).
- Send the remaining two students into the remaining two new groups ($2!$ ways).

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[2]A more explicitly way to do this part is to consider 3 group choices for the first step, then 2 group choices for the second step, and divide the final answer by 3!, which gives us the same answer. The reason is that the group labels are arbitrary and membership doesn't care about the order of admission.

In this problem, we took a more careful approach where we have methodically considered the number of ways each subsequent original group can be picked. As a result, we have to divide everything by 1/4! to account for the fact that membership doesn't care about which group gets separated first:

$$\frac{1}{4!}(4 \times 9 \times 3 \times 2! \times 2 \times 3 \times 2! \times 3 \times 2! \times 2!) = \boxed{1296}$$

2. (25 pts)

(a) Let (a_i, b_i, c_i) for $1 \leq i \leq 9$ be nine vectors in $\mathbb{R}^3$ with integer coordinates. Show that two of these vectors have a sum whose coordinates are all even integers.

Answer: Let us assign to each vector $v_i = (a_i, b_i, c_i)$ a corresponding *configuration vector* defined by

$$C(v_i) = C((a_i, b_i, c_i)) = (a_i \bmod 2, b_i \bmod 2, c_i \bmod 2).$$

Because $(m + n) \bmod 2 \equiv m \bmod 2 + n \bmod 2$ for all $m, n \in \mathbb{Z}$, it is easy to see that $v_i + v_j$ has only even integer coordinates (i.e., $C(v_i + v_j) \equiv (0, 0, 0)$) exactly when $C(v_i) = C(v_j)$.

Since each of a_i, b_i, c_i can only be even or odd, $C(v_i)$ can only be one of $2^3 = 8$ possible values for every v_i. Thus, by virtue of the pigeon hole principle, given 9 v_i's, there must be at least two vectors v_m, v_n ($m \neq n$) for which $C(v_m) = C(v_n)$. From the preceding paragraph, $v_n + v_m$ has only even integer values, and Part (a) is proved.

(b) Show that this result is best possible, i.e. the conclusion may fail if we have only eight vectors.

Answer: Consider the following eight vectors $(a_i, b_i, c_i)^\top$ concatenated into a matrix:

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_8 \\ b_1 & b_2 & \ldots & b_8 \\ c_1 & c_2 & \ldots & c_8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

The conclusion from Part (a) fails here: The sum of any two vectors contains at least one instance of the entry $0 + 1 = 1$, which is an odd integer.

3. (25 pts) Chess is a game played on an 8×8 board with alternating light and dark squares. The rows of the board are called "ranks" and are numbered 1 through 8. There are two players, white and black, and each player has a set of 16 pieces: 8 pawns, 2 rooks, 2 bishops, 2 knights, 1 queen, and 1 king. Count the number of possible starting positions for this game given the following constraints:

(a) All the white pieces are placed in the first and second ranks, and all the black pieces are placed in the seventh and eighth ranks.

(b) The black starting position is a mirror image of the white starting position.

(c) All the white pawns are placed in the second rank.

(d) The white king is placed on any square between the two white rooks (not necessarily adjacent to either of them).

(e) The white bishops are placed on opposite-color squares.

Answer: By (a), we don't have to worry about possible symmetry factors due to the placement of the white and dark pieces relative to the orientation of the chessboard[3]. By (b), it suffices to just arrange the white pieces. By (c), we don't have to worry about arranging the pawns. By (d) and (e), we don't to worry about arranging the 2 white knights and 1 white queen either (we will arrange them at the end). This leaves us with 5 pieces: 1 white king, 2 white rooks, and 2 white bishops (all in the first rank). Let the first rank be represented by the following eight slots, numbered 1 to 8. Without missing/gaining a symmetry factor, assume a black-white pattern:

$$\underline{1b}\ \underline{2w}\ \underline{3b}\ \underline{4w}\ \underline{5b}\ \underline{6w}\ \underline{7b}\ \underline{8w}$$

By (d), the white king can only be in slots (2) through (7). By a two-fold symmetry, we will only consider the following three possible positions for the white king: (2), (3), and (4).

- The white king is on (2). In this case, one white rook must be on (1) to keep (d) true.

$$\underline{R}\ \underline{K}\ \underline{3b}\ \underline{4w}\ \underline{5b}\ \underline{6w}\ \underline{7b}\ \underline{8w}$$

   The other white rook has 6 options. Once this rook is placed, the color ratio is two black to three white slots (or vice versa – it doesn't matter), which gives $2 \times 3 = 6$ arrangements for the white bishops. Without placing the white queen and white knights, we have $6 \times 6 = \boxed{36}$ configurations.

- The white king is on (3). There are two sub-cases:

   - If one white rook is on (1):

$$\underline{R}\ \underline{2w}\ \underline{K}\ \underline{4w}\ \underline{5b}\ \underline{6w}\ \underline{7b}\ \underline{8w}$$

     then there are two sub-sub-cases for the other white rook:

     ∗ If it is on (4), (6), or (8), then the color ratio is two black to three white slots, which gives $2 \times 3 = 6$ arrangements for the white bishops.

     ∗ If it is on (5) or (7), then one white bishop must go into the remaining black slot, and the other white bishop has 4 (white) options.

   In summary, there are $3 \times 6 + 2 \times 4 = 26$ arrangements in this first sub-case.

  – Or if one white rook is on (2):

$$\underline{1b}\ \underline{R}\ \underline{K}\ \underline{4w}\ \underline{5b}\ \underline{6w}\ \underline{7b}\ \underline{8w}$$

   then there are two sub-sub-cases for the other white rook. A nearly identical argument again gives a total of 26 possible arrangements.

  Therefore, without placing the queen and knights, there are $26 + 26 = \boxed{52}$ configurations if the white king is on (3).

- The white king is on (4). There are three sub-cases:

  – If one white rook is on (1):

$$\underline{R}\ \underline{2w}\ \underline{3b}\ \underline{K}\ \underline{5b}\ \underline{6w}\ \underline{7b}\ \underline{8w}$$

  – Or if one white rook is on (2):

$$\underline{1b}\ \underline{R}\ \underline{3b}\ \underline{K}\ \underline{5b}\ \underline{6w}\ \underline{7b}\ \underline{8w}$$

  – Or if one white rook is on (3):

$$\underline{1b}\ \underline{2w}\ \underline{R}\ \underline{K}\ \underline{5b}\ \underline{6w}\ \underline{7b}\ \underline{8w}$$

  The other white rook must be on either (5), (6), (7), or (8). In the first and third sub-cases, the color ratio after placing the second white rook is two black to three white slots or vice versa, and as a result there are $2 \times 3 = 6$ ways to place the two white bishops. In the second sub-case, if the second white rook is on (5) or (7), then as before, there are 6 ways to place the bishops; else, there are only four ways to place the bishops. In total, when the white king is on (4), there are $(4 \times 6) + (2 \times 6 + 2 \times 4) + (4 \times 6) = \boxed{68}$ possible configurations (without placing the queen and knights).

 In the remaining 3 slots, there are 3 ways to place the queen and two white knights. So, the grand total is:

$$2 \times 3 \times (36 + 52 + 68) = \boxed{936}$$

where the extra factor of 2 is due to the fact that we considered only half the possible slots for the white king (the other half is exactly the same, by symmetry).

4. (25 pts)

(a) A circle of 17 friends has the property that no matter how we choose two from these 17 friends, those two people correspond with each other on one of three given subjects. Prove that there are three friends among the circle of these 17 friends such that any two of the three of them correspond with each other on the same subject.

Answer: We pick out one friend X from the group. The remaining 16 can be partitioned into Group 1, Group 2, and Group 3, where Group i contains those who correspond with X on Subject #i. Let us consider Group 1:

- If there is one pair in this group who also correspond with each other on #1, then together with X they form a group of three who all correspond on #1.
- Otherwise, all members of Group 1 correspond with each other only on #2 and #3. There are two sub-cases:
    - There is a group of three who correspond on #2 or #3 (and we're done)
    - Or, there is no such group of three. Now, according to the 2-color version of the Ramsey problems we did in class (specifically 65 and 66), this holds only if Group 1 has at most 5 friends.[4]

As a result of symmetry, we see that Group 2 and Group 3 must also have at most 5 friends to avoid a group of three who correspond on the same subject. Thus, the total size of the friend circle must be at most

$$|X| + \max |\text{Group 1}| + \max |\text{Group 2}| + \max |\text{Group 3}| = 1 + 5 + 5 + 5 = 16$$

in order to avoid such a group of three. Thus, with 17 friends, this is not possible.

Remark: We recognize that this problem is a 3-color version of the (2-color) Ramsey problems in our textbook. Here, the setup of the problem translates to "given a 3-color K_{17}." Accordingly, our goal translates to showing that there always exists a monochromatic K_3. This translation is useful for Part (b).

(b) Let

$$n_k = k! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots \frac{1}{k!} \right) + 1.$$

We color all edges of K_{n_k} with one of k colors. Prove that there will be a triangle with monochromatic edges (i.e., edges of a single color).

Answer: We prove this result by induction on k. For $k = 1$, $n_k = n_1 = 3$, and the statement holds (trivially) because we have a monochromatic ($k = 1$) triangle ($n = 3$). Just for assurance, we can consider $k = 2$, which gives $n_k = n_2 = 6$. The statement is true by virtue of Problems 65 and 66 in our textbook. With these, we now assume that the statement holds up to k where $k \geq 1$. We will show that it still holds for $k + 1$.

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[4]Reminder: If Group 1 has 6 friends who correspond only on Subjects #2 and #3, then we're guaranteed a group of at least three who correspond on either Subject #2 or #3, by Problems 65 and 66.

To this end, we follow our argument in Part (a). After picking out one vertex X in $K_{n_{k+1}}$, we partition the remaining $(n_{k+1} - 1)$ vertices into $(k + 1)$ groups[5], where again Group i contains all vertices connected to X by an edge of Color #i. Consider Group 1:

- If there are two vertices in Group 1 that are connected by an edge of Color #1, then together with X they form a K_3 of Color #1.
- Otherwise, all vertices in Group 1 are connected only by edges of k other colors. There are two sub-cases:
  - There is a monochromatic K_3 (and we're done)
  - Or, there is no monochromatic K_3. According to our k-color inductive hypothesis, this holds only if Group 1 has at most $(n_k - 1)$ vertices.[6]

By symmetry, we see that to avoid a monochromatic K_3 in $K_{n_{k+1}}$, the total number of vertices must be at most:

$$\underbrace{1}_{|X|} + \underbrace{(k + 1)}_{\text{\# of groups}} \times \underbrace{(n_k - 1)}_{\text{max size per group}} = 1 + (k + 1) \times k! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} \right)$$

$$= 1 + (k + 1)! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} \right).$$

As a result, when there is an extra vertex, i.e., when the number of vertices is

$$2 + (k+1)! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} \right) = 1 + (k+1)! \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!} + \frac{1}{(k + 1)!} \right) = n_{k+1},$$

where we have put $1 = (k + 1)!/(k + 1)!$, a monochromatic K_3 is inevitable. Thus, we are guaranteed a monochromatic K_3 in $K_{n_{k+1}}$, and we are done with the proof.

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[5]Note: n_k is always bigger than the number of partition blocks k just in case the reader is concerned by the possibility of having empty blocks due to having too few vertices.

[6]Otherwise, there is a monochromatic K_3.