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Course: **8.321 - Quantum Theory I**
Problem set: **#8**

1.

(a) By virtue of separation of variables, the energy must be given by

$$E = E_z + E_{xy}$$

where E_z is the energy from the infinite square well of length L and E_{xy} is the energy due to confinement in the annulus. We thus have

$$E_z = \frac{\hbar^2\pi^2 l^2}{2mL^2} = \frac{\hbar^2}{2m}\left(\frac{\pi l}{L}\right)^2, \qquad l = 1, 2, 3, \dots$$

The Schrödinger equation for the radial confinement is

$$-\frac{\hbar^2}{2m}\nabla^2\psi = E_{xy}\psi$$

By separation of variables we may say $\psi = \psi(\rho, \phi) = R(\rho)\Phi(\phi)$, so that

$$\frac{1}{\rho}\partial_\rho(\rho\partial_\rho(R\Phi)) + \frac{1}{\rho^2}\partial^2_\phi(R\Phi) = -\frac{2mE_{xy}}{\hbar^2}R\Phi \implies \frac{\rho}{R}\partial_\rho(\rho\partial_\rho R) + \frac{1}{\Phi}\partial^2_\phi(\Phi) = -\frac{2mE}{\hbar^2}\rho^2.$$

After putting

$$\frac{1}{\Phi}\partial^2_\phi\Phi = -m^2$$

where m is a natural number due to the single-valuedness of Φ, we have an equation for R:

$$\rho\partial_\rho(\rho\partial_\rho R) = \left(-\frac{2mE_{xy}}{\hbar^2}\rho^2 + m^2\right)R \implies \rho^2 R'' + \rho R' + \left(\frac{2mE_{xy}}{\hbar^2}\rho^2 - m^2\right)R = 0$$

whose solution are provided as a linear combination of Bessel functions of the first and second kind:

$$R = c_1 J_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho\right) + c_2 N_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho\right).$$

Using the boundary conditions $R(\rho_a) = R(\rho_b) = 0$ we may find c_1, c_2 from solving the system

$$c_1 J_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_a\right) + c_2 N_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_a\right) = 0$$
$$c_1 J_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_b\right) + c_2 N_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_b\right) = 0$$

The result are two equal ratios which relate the J, N's:

$$J_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_b\right)N_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_a\right) - N_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_a\right)J_m\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_b\right) = 0.$$

Let $E_{xy} = E_{mn}$ and $k_{mn} = \sqrt{\frac{2mE_{mn}}{\hbar^2}}$ be the nth root of the equation above. Then we have the full energy spectrum:

$$E = E_{xy} + E_z = \frac{\hbar^2}{2m}\left(k_{mn}^2 + \left(\frac{\pi l}{L}\right)^2\right)$$

where $l = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots$, as desired.

(b) In the presence of $\vec{B} = B\hat{z}$, we have that

$$-i\hbar\nabla \to -i\hbar\nabla - \frac{e}{c}\vec{A} \implies \nabla \to \nabla - \left(\frac{ie}{\hbar c}\right)\vec{A}$$

where

$$\vec{A} = \left(\frac{B\rho_a^2}{\rho}\right)\hat{\phi}$$

by virtue of Stokes's Theorem, as presented in the textbook. Here, the vector potential is such that $\nabla \times \vec{A} = \vec{B} = 0$ in the annulus region. Since $\vec{A}$ only has a nontrivial component in ϕ, the partial derivative with respect to ϕ now changes as

$$\partial_\phi \to \partial_\phi - \frac{ie}{\hbar c}\frac{B\rho_a^2}{2}$$

which modifies the $\Phi(\phi)$ equation to

$$\partial_\phi^2\Phi = -m^2\Phi \to \partial_\phi^2\Phi - \left(\frac{ie}{\hbar c}\right)B\rho_a^2\partial_\phi\Phi + \left[m^2 - \left(\frac{eB\rho_a^2}{2\hbar c}\right)^2\right]\Phi = 0$$

Due to the single-valuedness of Φ, m in this case is not necessarily an integer. Letting

$$m^2 - \left(\frac{eB\rho_a^2}{2\hbar c}\right)^2 = m'^2,$$

we may repeat what we did before to find

$$J_{m'}\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_b\right)N_{m'}\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_a\right) - N_{m'}\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_a\right)J_{m'}\left(\sqrt{\frac{2mE_{xy}}{\hbar^2}}\rho_b\right) = 0.$$

Let $E_{xy} = E_{m'n}$ and $k_{m'n} = \sqrt{\frac{2mE_{m'n}}{\hbar^2}}$ be the nth root of the equation above. Then we have the full energy spectrum:

$$E = E_{xy} + E_z = \frac{\hbar^2}{2m}\left(k_{m'n}^2 + \left(\frac{\pi l}{L}\right)^2\right)$$

like before.

(c) Consider the ground state of both problems. In particular we look at the Φ solution. Ground state implies $m = 0$ and $m' = 0$. The normalized Φ solution when $B = 0$ is

$$\Phi(\phi) = 1$$

while the normalized Φ solution when $B \neq 0$ is

$$\Phi(\phi) = \exp\left(i\frac{eB\rho_a^2}{2\hbar c}\phi\right)$$

Due to the single-valuedness of Φ, we must have

$$\frac{eB\rho_a^2}{2\hbar c} = N$$

where N is an integer. So, we have "flux quantization":

$$\pi\rho_a^2 B = \frac{2\pi N\hbar c}{e}, \qquad N \in \mathbb{Z}$$

2.

(a) We this part we just compute:

$$\begin{aligned}
[\Pi_x, \Pi_y] &= \left[p_x - eA_x/c, p_y - eA_y/c\right] \\
&= \left[p_x - eA_x/c, p_y\right] + \left[p_x - eA_x/c, -eA_y/c\right] \\
&= \cancel{\left[p_x, p_y\right]} + (e/c)\left[-A_x, p_y\right] + (e/c)\left[p_x, -A_y\right] + \cancel{(e/c)^2[-A_x, -A_y]} \\
&= -i\hbar\frac{e}{c}\frac{\partial}{\partial y}A_x + i\hbar\frac{e}{c}\frac{\partial}{\partial x}A_y \\
&= i\hbar\frac{eB}{c}
\end{aligned}$$

(b) The new Hamiltonian is

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{\Pi_x^2}{2m} + \frac{\Pi_y^2}{2m}.$$

Let us put $\widetilde{\Pi}_x = (c/eB)\Pi_x$ so that $[\widetilde{\Pi}_x, \Pi_y] = i\hbar$. In these new variables the Hamiltonian becomes

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{e^2B^2}{c^2}\frac{\widetilde{\Pi}_x^2}{2m} + \frac{\Pi_y^2}{2m}$$

We may change our notation to make it more suggestive. Since $[\widetilde{\Pi}_x, \Pi_y] = i\hbar$, we may put $\widetilde{\Pi}_x = Y$, so that $[Y, \Pi_y] = i\hbar$. With this, the Hamiltonian is

$$\mathcal{H} = \frac{p_z^2}{2m} + \left[\frac{\Pi_y^2}{2m} + \frac{m}{2}\frac{e^2B^2}{m^2c^2}Y^2 + \frac{\Pi_y^2}{2m}\right]$$

The last two terms form a 1D QHO Hamiltonian. As a result, we immediately get the energy spectrum:

$$E = \frac{\hbar^2k^2}{2m} + \frac{\hbar|eB|}{mc}\left(n + \frac{1}{2}\right), \qquad n \in \mathbb{N}$$

as desired.

3.

(a) $\vec{A} = (-yB, 0, 0)$ leads to $\Pi_x = \hat{p}_x + \frac{eB}{c}\hat{y}$ and $\Pi_y = \hat{p}_y$. The Hamiltonian is

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2m}\left(\hat{p}_x + \frac{eB}{c}\hat{y}\right)^2 = \frac{p_z^2}{2m} + \frac{p_y^2}{2m} + \frac{me^2B^2}{2m^2c^2}\left(\frac{c}{eB}\hat{p}_x + \hat{y}\right)^2$$

We notice that $\hat{p}_x$ commutes with this Hamiltonian, and so we may replace p_x with $\hbar k_x$ to get

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{p_y^2}{2m} + \frac{me^2B^2}{2m^2c^2}\left(\frac{\hbar c k_x}{eB} + \hat{y}\right)^2$$

The term $p_z^2/2m$ is ancillary, and so is $p_x^2/2m$ which does not explicitly appear in the Hamiltonian above since k_x, same as $k_z = k$, is a constant of motion. The full wavefunction is therefore

$$\Psi_{k,n}(x,y,z) = e^{i(k_x x + kz)}\phi_n\left(y + \frac{\hbar c k_x}{eB}\right)$$

where ϕ_n are the eigenstates of the QHO with frequency $\omega = eB/mc$, $n \in \mathbb{N}$.

(b) $\vec{A} = (-yB/2, xB/2, 0)$ leads to $\Pi_x = \hat{p}_x + \frac{eB}{2c}\hat{y}$ and $\Pi_y = \hat{p}_y - \frac{eB}{2c}\hat{x}$. The Hamiltonian becomes

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{1}{2m}\left[\left(\hat{p}_x + \frac{eB}{2c}\hat{y}\right)^2 + \left(\hat{p}_y - \frac{eB}{2c}\hat{x}\right)^2\right]$$

Inspired by the approach on Wikipedia, let us ignore the z part for now and go dimensionless (since they're a lot of factors flying around) so that

$$\mathcal{H} = \frac{1}{2}\left[\left(-i\partial_x - \frac{y}{2}\right)^2 + \left(-i\partial_y + \frac{x}{2}\right)^2\right]$$

Let us define two new operators:

$$a_\pm = \frac{1}{\sqrt{2}}(a_x \pm i a_y)$$

where

$$a_x = \frac{x}{2} + \partial_x \quad \text{and} \quad a_y = \frac{y}{2} + \partial_y$$

(which can be obtained by going dimensionless with the usual definition of ladder operators). From here, we can readily check that

$$[a_+, a_+^\dagger] = [a_-, a_-^\dagger] = 1$$

and that the Hamiltonian is in fact

$$\mathcal{H} = a_-^\dagger a_- + \frac{1}{2}.$$

Proof. **While it is possible, I won't check this because I'm already extremely low on time.** □

In any case, from these two sets of ladder operators, we see that the eigenstates are specified by two quantum numbers n_-, n_+.

$$\begin{aligned}
a_-^\dagger |n_-, n_+\rangle &= \sqrt{n_- + 1}\,|n_- + 1, n_+\rangle \\
a_- |n_-, n_+\rangle &= \sqrt{n_-}\,|n_- - 1, n_+\rangle \\
a_+^\dagger |n_-, n_+\rangle &= \sqrt{n_+ + 1}\,|n_-, n_+ + 1\rangle \\
a_+ |n_-, n_+\rangle &= \sqrt{n_+}\,|n_-, n_+ - 1\rangle
\end{aligned}$$

The eigenstates are

$$|n_-, n_+\rangle = \frac{(a_-^\dagger)^{n_-}}{\sqrt{n_-!}}\frac{(a_+^\dagger)^{n_+}}{\sqrt{n_+!}}\,|0,0\rangle$$

Not sure what to do from here...

(c) $\vec{A} = (0, xB, 0)$ leads to $\Pi_y = \hat{p}_y - \frac{eB}{c}\hat{x}$ and $\Pi_x = \hat{p}_x$. The Hamiltonian is

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{1}{2m}\left(\hat{p}_y - \frac{eB}{c}\hat{x}\right)^2 = \frac{p_z^2}{2m} + \frac{p_y^2}{2m} + \frac{me^2B^2}{2m^2c^2}\left(\frac{c}{eB}\hat{p}_y - \hat{x}\right)^2$$

We notice that $\hat{p}_y$ commutes with this Hamiltonian, and so we may replace p_y with $\hbar k_y$ to get

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{me^2B^2}{2m^2c^2}\left(\frac{\hbar c k_y}{eB} - \hat{x}\right)^2$$

The term $p_z^2/2m$ is ancillary, and so is $p_y^2/2m$ which does not explicitly appear in the Hamiltonian above since k_y, same as $k_z = k$, is a constant of motion. The full wavefunction is therefore

$$\Psi_{k,n}(x, y, z) = e^{i(k_y y + kz)}\phi_n\left(x - \frac{\hbar c k_y}{eB}\right)$$

where ϕ_n are the eigenstates of the QHO with frequency $\omega = eB/mc$, $n \in \mathbb{N}$.

4. Not sure if I can complete this problem because I'm running super low on time...

Let us pick the vector potential $\vec{A} = (0, xB, 0)$ from Part (c) to do this problem. We have $\vec{A}$ reproduces $\vec{B} = (0, 0, B)$, as wanted. The electric field is given by $\vec{E} = (E, 0, 0)$, and so we may pick the associated scalar potential to be $\phi(x, y, z) = Ex$. The Hamiltonian is therefore,

$$\begin{aligned}
\mathcal{H} &= \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{me^2B^2}{2m^2c^2}\left(\frac{\hbar c k_y}{eB} - \hat{x}\right)^2 + eE\hat{x} \\
&= \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{me^2B^2}{2m^2c^2}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right)^2 + eE\left(\hat{x} - \frac{\hbar c k_y}{eB}\right) + eE\frac{\hbar c k_y}{eB} \\
&= \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{m}{2}\left[\frac{e^2B^2}{m^2c^2}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right)^2 + \frac{2eE}{m}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right) + \frac{2}{m}\frac{E\hbar c k_y}{B}\right] \\
&= \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{m}{2}\left[\frac{e^2B^2}{m^2c^2}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right)^2 + 2\frac{eB}{mc}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right)\frac{cE}{B} + \frac{c^2E^2}{B^2} - \frac{c^2E^2}{B^2} + \frac{2}{m}\frac{E\hbar c k_y}{B}\right] \\
&= \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{m}{2}\left[\frac{e^2B^2}{m^2c^2}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right)^2 + 2\frac{eB}{mc}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right)\frac{cE}{B} + \frac{c^2E^2}{B^2}\right] + \frac{m}{2}\left[-\frac{c^2E^2}{B^2} + \frac{2}{m}\frac{E\hbar c k_y}{B}\right].
\end{aligned}$$

At this point we may drop the last term because we can always redefine the scalar potential ϕ so that they (which are constants and do not contribute to the dynamics of the problem) vanish. We therefore get

$$\mathcal{H} = \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{m}{2}\left[\frac{eB}{mc}\left(\hat{x} - \frac{\hbar c k_y}{eB}\right) + \frac{cE}{B}\right]^2 = \frac{p_z^2}{2m} + \frac{p_x^2}{2m} + \frac{m}{2}\left[\frac{eB}{mc}\left(\hat{x} - \frac{\hbar c k_y}{eB} + \frac{c^2mE}{eB^2}\right)\right]^2$$

The resulting eigenstates are thus

$$\Psi_{k,n}(x, y, z) = e^{i(k_y y + k_z z)}\phi_n\left(x - \frac{\hbar c k_y}{eB} + \frac{c^2mE}{eB^2}\right)$$

where, as before, ϕ_n denotes the harmonic oscillator eigenstates which frequency $\omega_c = eB/mc$.

5. We can start by using known expressions for Y_l^m. At the end of this problem I will solve for Y_l^m explicitly (using the eigenvalue equations) and from there check that they match with what we have here.

$$Y_2^m(\theta, \phi) = \sqrt{\frac{5}{4\pi}\frac{(2-m)!}{(2+m)!}}P_2^m(\cos\theta)e^{im\phi}, \qquad m = -2, -1, 0, 1, 2$$

With

$$P_2^m(\cos\theta) = \frac{(-1)^m}{8}(1-\cos^2\theta)^{m/2}\frac{d^{2+m}}{d\cos^{2+m}\theta}(\cos^2\theta - 1)^2$$

we find

$$\begin{aligned}
P_2^{-2}(\cos\theta) &= \frac{1}{8}\sin^2\theta \\
P_2^{-1}(\cos\theta) &= \frac{1}{2}\cos\theta\sin\theta \\
P_2^{0}(\cos\theta) &= \frac{1}{4}(1+3\cos 2\theta) \\
P_2^{1}(\cos\theta) &= -3\cos\theta\sin\theta \\
P_2^{2}(\cos\theta) &= 3\sin^2\theta
\end{aligned}$$

From here, we find that

$$\begin{aligned}
Y_2^{-2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{-2i\phi}\sin^2\theta \implies Y_2^{-2}(x,y,z) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{-2i\arctan(y/x)}(1-z^2) \\
Y_2^{-1}(\theta,\phi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{-i\phi}\sin\theta\cos\theta \implies Y_2^{-1}(x,y,z) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{-i\arctan(y/x)}z\sqrt{1-z^2} \\
Y_2^{0}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right) \implies Y_2^{0}(x,y,z) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3z^2-1\right) \\
Y_2^{1}(\theta,\phi) &= -\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\phi}\sin\theta\cos\theta \implies Y_2^{-1}(x,y,z) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\arctan(y/x)}z\sqrt{1-z^2} \\
Y_2^{2}(\theta,\phi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\phi}\sin^2\theta \implies Y_2^{-2}(x,y,z) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\arctan(y/x)}(1-z^2)
\end{aligned}$$

where we have used

$$\theta = \arccos z \qquad \text{and} \qquad \phi = \arctan(y/x)$$

Finally,

$$\begin{aligned}
\sum_m \left|Y_2^m\right|^2 &= \frac{1}{16}\frac{15}{2\pi}(1-z^2)^2 + \frac{1}{4}\frac{15}{2\pi}z^2(1-z^2) + \frac{1}{16}\frac{5}{\pi}(3z^2-1)^2 + \frac{1}{4}\frac{15}{2\pi}z^2(1-z^2) + \frac{1}{16}\frac{15}{2\pi}(1-z^2)^2 \\
&= \boxed{\frac{5}{4\pi}}
\end{aligned}$$

as expected from Unsöld's Theorem. Mathematica code:

```
In[60]:= 2*(1/16)*(15/(2*Pi))*(1 - z^2)^2 + (1/2)*15/2/Pi*
z^2*(1 - z^2) + (1/16)*5/Pi*(3*z^2 - 1)^2 // FullSimplify

Out[60]= 5/(4 \[Pi])
```

Now, we justify our answers above for solving for Y_2^m explicitly. To this end, we put $Y_2^m(\theta,\phi) = \Theta(\theta)\Phi(\phi)$. Using separation of variables, we have a system of equations:

$$\Phi'' = -m^2\Phi \qquad \text{and} \qquad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[2(2+1) - \frac{m^2}{\sin^2\theta}\right]\Theta = 0.$$

We choose $\Phi = e^{im\phi}$ where $m \in \mathbb{Z}$ to guarantee single-valuedness. For the Θ equation, we may change variables to $z = \cos\theta$, so that the differential equation for Θ reduces to

$$(1-z^2)\frac{d^2\Theta}{dz^2} - 2z\frac{d\Theta}{dz} + \left[2(2+1) - \frac{m^2}{1-z^2}\right]\Theta = 0$$

Solving in Mathematica gives

$$\Theta_{m=0}(z) = C_1(3z^2 - 1)$$
$$\Theta_{m=\pm 1}(z) = \pm C_2 z\sqrt{1 - z^2}$$
$$\Theta_{m=\pm 2}(z) = C_3(1 - z^2).$$

Plugging in $z = \cos\theta$ and normalizing we find the same solution as before:

$$Y_2^{-2}(\theta, \phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{-2i\phi}\sin^2\theta$$
$$Y_2^{-1}(\theta, \phi) = \frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{-i\phi}\sin\theta\cos\theta$$
$$Y_2^{0}(\theta, \phi) = \frac{1}{4}\sqrt{\frac{5}{\pi}}\left(3\cos^2\theta - 1\right)$$
$$Y_2^{1}(\theta, \phi) = -\frac{1}{2}\sqrt{\frac{15}{2\pi}}e^{i\phi}\sin\theta\cos\theta$$
$$Y_2^{2}(\theta, \phi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}e^{2i\phi}\sin^2\theta$$

where the normalization condition is

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \left|\Theta(\theta)\Phi(\phi)\right|^2 \sin\theta = 1$$

Mathematica code for solving ODE's and finding normalization constants:

```
(*m= +/- 2*)
DSolve[(1 - x^2)*D[y[x], {x, 2}] -
2*x*D[y[x], x] + (2*(2 + 1) - 4/(1 - x^2))*y[x] == 0, y[x], x]


(*m = +/- 1*)
DSolve[(1 - x^2)*D[y[x], {x, 2}] -
2*x*D[y[x], x] + (2*(2 + 1) - 1/(1 - x^2))*y[x] == 0, y[x], x]

(*m = 0*)
DSolve[(1 - x^2)*D[y[x], {x, 2}] -
2*x*D[y[x], x] + (2*(2 + 1) - 0/(1 - x^2))*y[x] == 0, y[x], x]

(*Normalization*)
(*m=0*)
In[92]:= 2*Pi*Integrate[((-1 + 3 Cos[t]^2))^2*Sin[t], {t, 0, Pi}]

Out[92]= (16 \[Pi])/5

(*m= +/- 1*)
In[93]:= 2*Pi*Integrate[(Sin[t]*Cos[t])^2*Sin[t], {t, 0, Pi}]

Out[93]= (8 \[Pi])/15

(*m = =/- 2*)
In[94]:= 2*Pi*Integrate[(Sin[t]^2)^2*Sin[t], {t, 0, Pi}]

Out[94]= (32 \[Pi])/15
```

6. By separation of variables, we write

$$\Psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

The 3D Schrödinger's equation in spherical coordinates reads

$$-\frac{\hbar^2}{2m}\left(\frac{1}{r^2}\partial_r(r^2\partial_r(R\Theta\Phi)) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta(R\Theta\Phi)) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2(R\Theta\Phi)\right) + (V(r) - E)R\Theta\Phi = 0$$

Let $R = u(r)/r$ and diving both sides of the equation by $R\Theta\Phi/r^2$, we find

$$-\frac{\hbar^2}{2m}\left[r^2\frac{u''}{u} + \left(\frac{1}{\sin^2}\frac{\Phi''}{\theta\Phi} + \frac{\cos\theta\Theta' + \Theta''}{\sin\theta}\right)\right] = r^2(E - V(r))$$

Rearranging gives

$$\frac{r^2u''}{u} + \frac{2m}{\hbar^2}r^2(E - V(r)) = -\frac{1}{\sin^2}\frac{\Phi''}{\theta\Phi} - \frac{\cos\theta\Theta' + \Theta''}{\sin\theta} = \lambda$$

We may rewrite the angular equation as

$$\frac{1}{Y}\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta Y\right) + \frac{1}{Y}\frac{1}{\sin^2\theta}\partial_\phi^2 Y = -\lambda$$

where $Y(\theta,\phi) = \Theta(\theta)\Phi(\phi)$. The solutions are of course the spherical harmonics $Y = Y_l^m(\theta,\phi)$ and $\lambda = l(l+1)$. With this, we come back to the radial equation to find

$$\frac{r^2u''}{u} + \frac{2m(E - V(r))}{\hbar^2}r^2 = l(l+1)$$

Rearranging gives

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right]u(r) = Eu(r)$$

as desired.

7. The Hamiltonian is

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}\omega^2x^2 + \frac{1}{2}\omega^2y^2.$$

It is clear that the Hamiltonian can be written as

$$\mathcal{H} = \hbar\omega\left(a_x^\dagger a_x + a_y^\dagger a_y + 1\right)$$

where, as usual,

$$\begin{aligned} a_i &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{r}_i + \frac{i}{m\omega}\hat{p}_i\right) \\ a_i^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{r}_i - \frac{i}{m\omega}\hat{p}_i\right) \end{aligned}$$

Since $n_i = a_i^\dagger a_i = 0, 1, 2, \ldots$ we can see that for each energy level $\hbar\omega(n_x + n_y + 1)$ has a degeneracy of $n_x + n_y + 1$. Now we wish to write J_z in terms of the creation and annihilation operators. This can be done by working from the definition:

$$\begin{aligned} J_z &= xp_y - yp_x \\ &= \frac{i\hbar}{2}(a_x^\dagger + a_x)(a_y^\dagger - a_y) - \frac{i\hbar}{2}(a_y^\dagger + a_y)(a_x^\dagger - a_x) \\ &= i\hbar(a_x a_y^\dagger - a_x^\dagger a_y) \end{aligned}$$

Let us introduce

$$a_\pm = \frac{1}{\sqrt{2}}(a_x \pm i a_y)$$

which satisfy nice properties:

$$[a_\pm, a_\pm^\dagger] = \frac{1}{2}[a_x \pm i a_y, a_x^\dagger \mp i a_y^\dagger] = 1.$$

Moreover, we also have

$$a_\pm^\dagger a_\pm = \frac{1}{2}(a_x^\dagger \mp i a_y^\dagger)(a_x \pm i a_y) = \frac{1}{2}(a_x^\dagger a_x + a_y^\dagger a_y \pm i a_x^\dagger a_y \mp i a_x a_y^\dagger)$$

from which we find

$$\mathcal{H} = \hbar\omega \left(a_x^\dagger a_x + a_y^\dagger a_y + 1 \right) = \hbar\omega(a_+^\dagger a_+ + a_-^\dagger a_- + 1)$$

and

$$J_z = i\hbar(a_x a_y^\dagger - a_x^\dagger a_y) = \hbar(a_-^\dagger a_- - a_+^\dagger a_+)$$

If we define $N_\pm = a_\pm^\dagger a_\pm$ then we have

$$\mathcal{H} = \hbar\omega(N_+ + N_- + 1) \qquad \text{and} \qquad J_z = \hbar(N_- - N_+).$$

Using the exact same analysis we did with a_x (or a_y for that matter) where we say

$$\mathcal{H} a_x^\dagger |E\rangle = \cdots = (E + \hbar\omega)(a_x^\dagger |E\rangle)$$

using the commutation relations for $a, a^\dagger$, we find that the spectra of $N_\pm$ are also nonnegative integers. Moreover, since $[N_+, N_-] = 0$, they are simultaneously diagonalizable. This means that specifying a pair of eigenvalues of N_-, N_+ uniquely determines the simultaneous eigenvector for N_-, N_+ and completely specifies energy eigenstate. This basically says that $\{N_-, N_+\}$ is a CSCO. $\{\mathcal{H}, J_z\}$ is a CSCO follows from the fact that $\mathcal{H}, J_z$ are essentially linearly independent linear combinations of $\{N_-, N_+\}$ (there's a subslety here since $\mathcal{H}$ has an offset $\hbar\omega\mathbb{I}$, but since the identity operator commutes with everything we're okay).

Finally, suppose that the system has energy $\hbar\omega(n+1)$. We would like to know what the possible eigenvalues of J_z are. Well, since $n = n_+ + n_-$, we have $(n+1)$ cases

$$\begin{aligned}
&n_+ = n, n_- = 0 \implies n_- - n_+ = -n \\
&n_+ = n-1, n_- = 1 \implies n_- - n_+ = -(n+2) \\
&\vdots \\
&n_+ = 0, n_- = n \implies n_- - n_+ = n
\end{aligned}$$

This implies that there are $(n+1)$ possible eigenvalues for J_z whose values are

$$m = n_- - n_+ \in \{-n, -n+2, \ldots, n-2, n\}$$