Tues., Sept. 7

Last time we showed that if an upper index x^μ

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu$$

then the lower index x_μ

$$x_\mu \to (\Lambda^{-1})^\nu{}_\mu x_\nu$$

Then we clearly see

$$x_\mu x^\mu \to \left[(\Lambda^{-1})^\alpha{}_\mu x_\alpha\right]\left[\Lambda^\mu{}_\beta x^\beta\right]$$

$$= x_\alpha \Big[\underbrace{(\Lambda^{-1})^\alpha{}_\mu (\Lambda)^\mu{}_\beta}_{\substack{\text{product of the matrices} \\ \Lambda^{-1}\times\Lambda = \text{identity}}}\Big] x^\beta$$

$$= x_\alpha \left[\delta^\alpha_\beta\right] x^\beta = \underbrace{x_\alpha x^\alpha}_{\text{invariant}}$$

The Lorentz group is the group of transformations that leaves the metric tensor $g_{\mu\nu}$ the same. In other words, in all inertia frames

$$g_{\mu\nu} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

The metric tensor has two lower indices

$$g_{\mu\nu} \to (\Lambda^{-1})^{\alpha}{}_{\mu} (\Lambda^{-1})^{\beta}{}_{\nu}\, g_{\alpha\beta}$$

If we think of Λ^{-1} and g as 4×4 matrices then

$$\underset{4\times4}{[g]} \to \underset{4\times4}{(\Lambda^{-1})^T}\ \underset{4\times4}{[g]}\ \underset{4\times4}{(\Lambda^{-1})}$$

So invariance of the metric means that

$$\begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} = (\Lambda^{-1})^T \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix} (\Lambda^{-1})$$

The set of all matrices satisfying this criterion forms a group.

Group: Set of elements $g_1, g_2, \cdots \in G$

1) Closed under multiplication
$g_i \cdot g_j \in G$ for all $g_i, g_j \in G$
2) Existence of an identity $e \in G$
such that $e \cdot g = g \cdot e = g$
for elements g of the group

3) Existence of an inverse g^{-1} for each element g such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

4) Associative . $(g_i \cdot g_j) \cdot g_k = g_i \cdot (g_j \cdot g_k)$

Let's back up for a moment. The group of $N \times N$ real matrices M which satisfy

$$M^T M = \text{identity matrix}$$

is called the special orthogonal group of $N \times N$ matrices or $SO(N)$.

The group of rotations in three dimensional space is $SO(3)$.

Our group of Lorentz transformation is almost like $SO(4)$. Except we have

$$M^T \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix} M = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}$$

instead of

$$M^T \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix} M = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}.$$

The Lorentz group is called $SO(3,1)$. Note that it includes the $SO(3)$ subgroup of rotations in three dimensional space.

Let us consider the $SO(3)$ subgroup of rotations a bit deeper. Recall from quantum mechanics that there exists a spin s (where s is an integer or half integer) representation of the rotation group. The dimension of the representation is $n = 2s+1$.

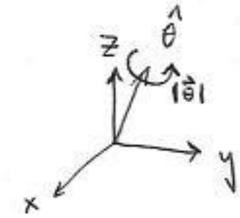
Let us start with the spin-$\frac{1}{2}$ representation. This is a two-dimensional representation which can parameterized as

$$U(\vec{\theta}) = e^{-\frac{i\vec{\theta}\cdot\vec{\sigma}}{2}}$$

$\vec{\sigma}$ are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This corresponds with a rotation of $|\vec{\theta}|$ radians about the $\hat{\theta}$ axis.



For any Lie group [elements form a differentiable manifold and group operation is smooth]

the transformations in the neighborhood of the identity determine the Lie algebra of the group.

These transformations are the exponential of i times an element of the Lie algebra

$$U(\vec{\theta}) = e^{-i\left(\frac{\vec{\theta}\cdot\vec{\sigma}}{2}\right)}$$

spin-$\frac{1}{2}$ representation of Lie algebra

In the general case, arbitrary spin, we have

$$U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}}$$

where the $\vec{J}$ components satisfy

$$[J^j, J^k] = i \sum_{l=1,2,3} \varepsilon^{jkl} J^l$$

$$\left[\begin{array}{ll} \varepsilon^{123} = 1 & \text{antisymmetric} \\ \varepsilon^{213} = -1 & \text{tensor} \\ \vdots & \end{array} \right)$$

For the spin-$\frac{1}{2}$ representation we can check that indeed

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2} \right] = i \sum_{l=1,2,3} \varepsilon^{jkl} \frac{\sigma^l}{2}$$

For the spin-s representation, $\vec{J}$ are $2s+1$ by $2s+1$ matrices.

In general though we can consider combinations of several irreducible spin representations. Consider for example the wavefunction for a spinless particle

$$\psi(\vec{x})$$

We know that ψ can be decomposed in orbital angular momentum states $J = 0, 1, 2, \cdots$ (since no intrinsic spin, $J = L$).

However we also know from quantum mechanics that $\vec{J}$ can be written as differential operator on the wavefunction

$$\vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$$

If we are a bit more sophisticated we can write

$$J^j = i \sum_{l=1,2,3} \varepsilon^{jkl} x^k \nabla^l \qquad [\nabla^l = -\nabla_l = -\tfrac{\partial}{\partial x^l}]$$

Here is where three dimensions is a bit special. There are three components for J^j $j = 1, 2, 3$. The reason there are three components is not because there are three axes, but because there are pairs of axes.

Rotation between x^1 + x^2 $\rightarrow$ J^3

Rotation between x^2 + x^3 $\rightarrow$ J^1

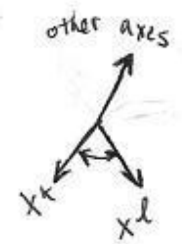
Rotation between x^3 + x^1 $\rightarrow$ J^2

So in two dimensions there is only one type of rotation (between x^1 + x^2). In four dimensions there are six types of rotations (x^1 + x^2, x^1 + x^3, x^1 + x^4, x^2 + x^3, x^2 + x^4, x^3 + x^4).

So in general dimensions we define a two-index object for angular momentum

$$J^{kl} = i\left(x^k \nabla^l - x^l \nabla^k\right)$$

(rotations between x^k + x^l)



A straightforward derivation of the Lie algebra for $SO(3,1)$ is a bit tedious. So we try to guess starting from $J^{k\ell}$ in four dimensions for $SO(4)$ and using properties of upper + lower Lorentz indices.

We now have to be careful with upper + lower indices. Note that

$$\partial^\mu \equiv g^{\mu\nu} \frac{\partial}{\partial x^\nu}$$

And so for spatial indices

$$\partial^i = -\partial_i = -\frac{\partial}{\partial x^i}$$

Whereas for the time index

$$\partial^0 = \partial_0 = \frac{\partial}{\partial x^0}$$

We make a guess...

For the Lorentz group we try

$$J^{\mu\nu} = i\,(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

with $J^{jk} = i\,(x^j \partial^k - x^k \partial^j)$

for spatial rotations between the x^j & x^k axes

and $J^{0j} = i\,(x^0 \partial^j - x^j \partial^0)$

for Lorentz boosts along the x^j axis

The reason for the missing minus sign is because of minus we get in $\partial^i = -\partial_i = -\frac{\partial}{\partial x^i}$.

With some work one finds (homework)

$$[J^{\mu\nu}, J^{\rho\sigma}] = i\,[\,\overset{\text{inner indices}}{g^{\nu\rho}}\,J^{\mu\sigma} + \overset{\text{outer indices}}{g^{\mu\sigma}}\,J^{\nu\rho} - \underset{\text{first indices}}{g^{\mu\rho}}\,J^{\nu\sigma} - \underset{\text{second indices}}{g^{\nu\sigma}}\,J^{\mu\rho}\,]$$

This is the Lorentz algebra

[Lie algebra of $SO(3,1)$]

There are 3 rotations $J^{12} = -J^{21}$

$J^{23} = -J^{32}$

$J^{31} = -J^{13}$

and 3 boosts $J^{01} = -J^{10}$

$J^{02} = -J^{20}$

$J^{03} = -J^{30}$

Any matrices $[J^{\mu\nu}]_{ij}$ satisfying the commutation relation above provides a representation of the Lorentz algebra.

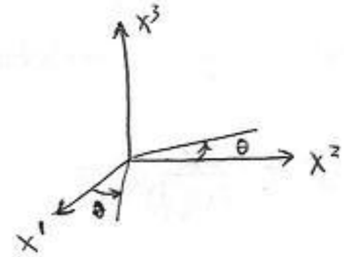
The elements of the Lorentz group near the identity can be written as

$$U(\omega_{\mu\nu}) = \exp\Big[-\tfrac{i}{2}\overbrace{\omega_{\mu\nu}}^{\text{antisymmetric tensor}} J^{\mu\nu}\Big]$$

the factor of $\frac{1}{2}$ is because sum over μ + ν, gives two copies of J^{ij} and J^{0i}

Let us try to figure out $[J^{\mu\nu}]_{ij}$ for Lorentz four vectors.

Case I: $\omega_{12} = -\omega_{21} = \theta$ This gives a rotation of the x^1 axis into the x^2 axis.



$$[V^\mu] \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [V^\mu]$$

For infinitesimal θ,

$$[V^\mu] \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} [V^\mu]$$

So in this case $-i[J^{12} V]^\mu = g^{\mu 1} V^2 - g^{\mu 2} V^1$