Classical Mechanics III (8.09 and 8.309)

Assignment 5: Solutions

October 11, 2021

1. Canonical Transformations [12 points]

(a) [2 points] We can try $F = F_2(q, P) - Q_i P_i$:

$$\frac{\partial F_2}{\partial q_i} = p_i(q, P) = P_i \;\Rightarrow F_2 = \int P_i dq_i + g(p) = q_i P_i + g(p)$$

$$\frac{\partial F_2}{\partial P_i} = Q_i(q, P) = q_i = q_i + \frac{\partial g}{\partial P_i} \;\Rightarrow \frac{\partial g}{\partial P_i} = 0$$

and so taking $g = 0$, $F_2(q, P) = q_i P_i$ and $F = q_i P_i - Q_i P_i$ works. Alternatively, let's try $F = F_3(p, Q) + q_i p_i$:

$$\frac{\partial F_3}{\partial p_i} = -q_i(p, Q) = -Q_i \;\Rightarrow F_3 = -\int Q_i dp_i + h(Q) = -p_i Q_i + h(Q)$$

$$\frac{\partial F_2}{\partial Q_i} = -P_i(p, Q) = -p_i = -p_i + \frac{\partial h}{\partial Q_i} \;\Rightarrow \frac{\partial h}{\partial Q_i} = 0$$

so taking $h = 0$, $F_3(p, Q) = -p_i Q_i$ and $F = -p_i Q_i + q_i p_i$ works.

(b) [2 points] We can proceed as before:

$$\frac{\partial F_1}{\partial q} = p(q, Q) = Qt \;\Rightarrow F_1 = \int Qt dq + g(Q) = qQt + g(Q)$$

$$\frac{\partial F_1}{\partial Q} = -P(q, P) = qt = qt + \frac{\partial g}{\partial Q} \;\Rightarrow \frac{\partial g}{\partial Q} = 0$$

so taking $g = 0$, and $F_1(q, Q, t) = qQt$ works.

(c) [4 points] Let's treat the case $\ell = 0$ first: in this case $Q = q^k$, and it is impossible to express p and P solely in terms of q and Q. Therefore the generating function cannot take the form $F_1(q, Q, t)$. Now assuming $\ell \neq 0$, we see that $p = q^{-k/\ell} Q^{1/\ell}$ and $P = q^m p^n = q^{m - kn/\ell} Q^{n/\ell}$. We'll just take the equality of mixed derivatives here:

$$\left(\frac{\partial p}{\partial Q}\right)_{q,Q} = \frac{\partial}{\partial Q}\frac{\partial F_1}{\partial q} = \frac{\partial}{\partial q}\frac{\partial F_1}{\partial Q} = -\left(\frac{\partial P}{\partial q}\right)_{q,Q}$$

which gives

$$\frac{1}{\ell}q^{-k/\ell}Q^{1/\ell-1} = (-m - \frac{kn}{\ell})q^{m-kn/\ell-1}Q^{n/\ell}.$$

This gives us three equations in the variables k, ℓ, m, n:

$$\begin{aligned}
\frac{1}{\ell} &= -m - \frac{kn}{\ell} \\
-\frac{k}{\ell} &= m - \frac{kn}{\ell} - 1 \\
\frac{1}{\ell} - 1 &= \frac{n}{\ell}
\end{aligned}$$

which has the solutions $k = \ell + 1$, $m = -\ell$, and $n = 1 - \ell$, with $\ell \neq 0$.

(d) [4 points] Under this transformation, the new Hamiltonian must take the form

$$\begin{aligned}
K &= \frac{(\vec{P} - q\vec{A}')^2}{2m} + q\phi - q\frac{\partial f}{\partial t} \\
&= H - q\frac{\partial f}{\partial t}
\end{aligned}$$

since $\vec{p} - q\vec{A}$ remains unchanged. Therefore F_2 must take the form $F_2(\vec{x}, \vec{P}, t) = F_2'(\vec{x}, \vec{P}) - qf(\vec{x}, t)$. Taking the derivative with respect to $\vec{x}$,

$$\begin{aligned}
\vec{p} &= \frac{\partial F_2}{\partial \vec{x}} \\
&= \frac{\partial F_2'}{\partial \vec{x}} - q\vec{\nabla} f
\end{aligned}$$

and matching this with $\vec{P} - q(\vec{A} + \vec{\nabla} f) = \vec{p} - q\vec{A}$, we see that we must take $\frac{\partial F_2'}{\partial \vec{x}} = \vec{P}$. The equation $\vec{Q} = \frac{\partial F_2}{\partial \vec{P}}$ doesn't provide an additional constraint; thus we can choose the generating function

$$F_2(\vec{x}, \vec{P}, t) = \vec{x} \cdot \vec{P} - qf(\vec{x}, t).$$

The transformation equations are

$$\vec{P} = \vec{p} + q\vec{\nabla} f \qquad \text{(as before)}$$

$$\vec{Q} = \frac{\partial F_2}{\partial \vec{P}} = \vec{x}.$$

2. Harmonic Oscillator [7 points]

(a) [2 points] We will simply evaluate the Possion bracket of $Q = p + iaq$ and $P = (p - iaq)/2ia$:

$$
\begin{aligned}
[Q,P] &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= (ia)\frac{1}{2ia} - (1)\frac{(-1)}{2} \\
&= 1
\end{aligned}
$$

Together with $[Q,Q]=0$ and $[P,P]=0$ this implies that the transformation to (Q,P) is canonical.

(b) [5 points] We have $QP = (p^2 + a^2q^2)/2ia$. Note that $H = \frac{p^2}{2m} + \frac{kx^2}{2} = (p^2 + (m\omega q)^2)/2m$, where $\omega = \sqrt{k/m}$. A suitable choice for a is therefore $a = m\omega$. Then

$$
Q = p + im\omega q, \qquad P = \frac{p - im\omega q}{2im\omega}.
$$

Since the canonical transformation is independent of time, the new Hamiltonian $K(Q,P)$ satisfies $K = H$, or

$$
K = i\omega QP.
$$

The equations of motions give

$$
\begin{aligned}
\dot{Q} = \frac{\partial K}{\partial P} = i\omega Q \quad &\Rightarrow \quad Q = Ae^{i\omega t} \\
\dot{P} = -\frac{\partial K}{\partial Q} = -i\omega P \quad &\Rightarrow \quad P = Be^{-i\omega t}
\end{aligned}
$$

where A, B are constants. Converting back to our original variables q and p,

$$
q = \frac{Q}{2im\omega} - P = \frac{A}{2im\omega}e^{i\omega t} - Be^{-i\omega t}
$$

$$
p = \frac{Q}{2} + im\omega P = \frac{A}{2}e^{i\omega t} + im\omega Be^{-i\omega t}
$$

As a check, notice that $p = m\dot{q}$, as we'd expect. Finally, if q, p are to be real (as is physically required), we can set $A = im\omega Ne^{i\delta}$ for real constants N, δ; it can then be easily verified that for q to be real we must have $B = -\frac{N}{2}e^{-i\delta}$, which then gives

$$
q = N\frac{e^{i(\omega t+\delta)} + e^{-i(\omega t+\delta)}}{2} = N\cos(\omega t + \delta)
$$

$$
p = im\omega N\frac{e^{i(\omega t+\delta)} - e^{-i(\omega t+\delta)}}{2} = -m\omega N\sin(\omega t + \delta)
$$

which is the general solution to the simple harmonic oscillator.

3. Poisson Brackets and Conserved Quantities [4 points]

Since $H = q_1 p_1 - q_2 p_2 + a q_1^2 + b q_2^2$, we have

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} = q_1 \quad , \quad \dot{q}_2 = \frac{\partial H}{\partial p_2} = -q_2$$
$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -p_1 - 2aq_1 \quad , \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2} = p_2 - 2bq_2 .$$

We can then compute the total time derivatives of $u_1 = (p_1 - aq_1)/q_2$ and $u_2 = q_1 q_2$ directly:

$$\begin{aligned}
\frac{du_1}{dt} &= \frac{\dot{p}_1 + a\dot{q}_1}{q_2} - \frac{(p_1 + aq_1)\dot{q}_2}{q_2^2} \\
&= \frac{-p_1 - 2aq_1 + aq_1}{q_2} + \frac{(p_1 + aq_1)q_2}{q_2^2} = 0 \\
\frac{du_2}{dt} &= \dot{q}_1 q_2 + q_1 \dot{q}_2 \\
&= q_1 q_2 - q_1 q_2 = 0.
\end{aligned}$$

(Notice that we're essentially computing the Poisson brackets of u_1 and u_2 with H here.)

4. Angular Momentum and the Laplace-Runge-Lenz vector [13 points]

(a) [4 points] We have

$$[x_i, L_j] = \epsilon_{jk\ell}[x_i, x_k p_\ell] = \epsilon_{jk\ell} x_k \delta_{i\ell} = \epsilon_{ijk} x_k$$

$$[p_i, L_j] = \epsilon_{jk\ell}[p_i, x_k p_\ell] = -\epsilon_{jk\ell} p_\ell \delta_{ik} = \epsilon_{ij\ell} p_\ell = \epsilon_{ijk} p_k .$$

Using these we can compute

$$\begin{aligned}
[L_i, L_j] &= \epsilon_{ik\ell}[x_k p_\ell, L_j] \\
&= \epsilon_{ik\ell}(x_k [p_\ell, L_j] + p_\ell [x_k, L_j]) \\
&= \epsilon_{ik\ell}(\epsilon_{\ell jm} x_k p_m + \epsilon_{kjn} p_\ell x_n) \\
&= (\delta_{ij}\delta_{km} - \delta_{im}\delta_{jk}) x_k p_m + (\delta_{j\ell}\delta_{in} - \delta_{ij}\delta_{\ell n}) p_\ell x_n \\
&= \delta_{ij} \vec{x} \cdot \vec{p} - x_j p_i + x_i p_j - \delta_{ij} \vec{p} \cdot \vec{x} \\
&= x_i p_j - x_j p_i = \epsilon_{ijk} L_k .
\end{aligned}$$

Finally,

$$\begin{aligned}
[L_i, \vec{L}^2] &= [L_i, L_j L_j] = 2L_j [L_i, L_j] \\
&= 2\epsilon_{ijk} L_j L_k = 0
\end{aligned}$$

where the last equality holds because $\epsilon_{ijk} L_j L_k = \epsilon_{ikj} L_k L_j = -\epsilon_{ijk} L_k L_j$, where we first renamed the indices $j \leftrightarrow k$ and then used the antisymmetry of the Levi-Civita symbol.

Note that there is a relation

$$[\vec{F}, \vec{L} \cdot \hat{n}] = \hat{n} \times \vec{F}$$

for a system vector $\vec{F}$. This gives another, easier way to do the problem.

(b) [7 points] Let us first calculate the Poisson brackets $[p_i, H]$ and $[r_i, H]$:

$$[p_i, H] = [p_i, -\frac{k}{r}] = k\frac{\partial}{\partial r_i}\left(\frac{1}{r}\right) = -\frac{k}{r^2}\frac{\partial r}{\partial r_i} = -\frac{kr_i}{r^3}$$

$$[r_i, H] = [r_i, \frac{\vec{p}^2}{2\mu}] = \frac{1}{2\mu}\frac{\partial \vec{p}^2}{\partial p_i} = \frac{p_i}{\mu}$$

We now have

$$\begin{aligned}
[A_i, H] &= \epsilon_{ijk}[p_j L_k, H] - \mu k[\frac{r_i}{r}, H] \\
&= \epsilon_{ijk}p_j[L_k, H] + \epsilon_{ijk}L_k[p_j, H] - \frac{\mu k}{r}[r_i, H] - \mu k r_i[\frac{1}{r}, H] \\
&= 0 - \epsilon_{ijk}L_k\frac{kr_j}{r^3} - \frac{kp_i}{r} - \mu k r_i\frac{\partial}{\partial r_\ell}\left(\frac{1}{r}\right)\cdot\frac{\partial H}{\partial p_\ell}
\end{aligned}$$

Now using $\epsilon_{ijk}L_k = r_i p_j - r_j p_i$, $\frac{\partial}{\partial r_\ell}\left(\frac{1}{r}\right) = -\frac{r_\ell}{r^3}$, and $\frac{\partial H}{\partial p_\ell} = \frac{p_\ell}{\mu}$, we can continue:

$$\begin{aligned}
[A_i, H] &= (r_j p_i - r_i p_j)\frac{kr_j}{r^3} - \frac{kp_i}{r} + \frac{kr_i r_\ell p_\ell}{r^3}, \qquad \text{but } r_j r_j = r^2 \\
&= \frac{kp_i r^2}{r^3} - \frac{kr_i r_j p_j}{r^3} - \frac{kp_i}{r} + \frac{kr_i r_\ell p_\ell}{r^3} \\
&= 0.
\end{aligned}$$

(c) [2 points] Taking the square of $\vec{A} = \vec{p}\times\vec{L} - \mu k\vec{r}/r$,

$$\begin{aligned}
\vec{A}^2 &= (\vec{p}\times\vec{L})^2 - 2\mu k(\vec{p}\times\vec{L})\cdot\frac{\vec{r}}{r} + \mu^2 k^2\frac{\vec{r}^2}{r^2} \\
&= \vec{p}^2\vec{L}^2 - 2\frac{\mu k}{r}(\vec{r}\times\vec{p})\cdot\vec{L} + \mu^2 k^2 \\
&= (\vec{p}^2 - \frac{2\mu k}{r})\vec{L}^2 + \mu^2 k^2 \\
&= \mu^2 k^2 + 2\mu H\vec{L}^2.
\end{aligned}$$

The second equality uses that $(\vec{p}\times\vec{L})^2 = \vec{p}^2\vec{L}^2$ since $\vec{p}$ and $\vec{L}$ are perpendicular, and also $(\vec{p}\times L)\cdot\vec{r} = (\vec{r}\times\vec{p})\cdot\vec{L}$.

5. An Exponential Potential [13 points]

This set of solutions will deal with both signs of p, even though the problem statement only asks for the case $p > 0$.

(a) [6 points] We want a time-independent generating function $F_2(x, P)$ that gives $K = P^2$ as the new Hamiltonian. Since the generating function is time-independent we must have $K = H$, or

$$P^2 = p^2 + e^x \Rightarrow p = \pm\sqrt{P^2 - e^x}$$

Moreover, the derivates of F_2 are fixed:

$$\begin{aligned}\frac{\partial F_2}{\partial x} &= p = \pm\sqrt{P^2 - e^x} \\ \frac{\partial F_2}{\partial P} &= Q\end{aligned}$$

We can integrate the first equation:

$$F_2(x, P) = \pm \int \sqrt{P^2 - e^x} dx + g(P),$$

where the plus sign gives p positive and vice versa. Since the second equation doesn't give us a constraint (it is just the transformation equation for Q), $g(P)$ is arbitrary; we might as well set it to zero. To integrate the above equation, let's make the substitution $e^x = P^2 \sin^2 x'$, $0 \leq x' \leq \pi/2$; then $e^x dx = 2P^2 \sin x' \cos x' dx'$, or $dx = 2 \cot x' dx'$. Thus

$$\begin{aligned}F_2 &= \pm \int 2P \cos x' \cot x' dx' \\ &= \pm \int 2P \left(\frac{1}{\sin x'} - \sin x' \right) dx' \\ &= \pm 2P \big[\ln \tan \frac{x'}{2} + \cos x' \big]\end{aligned}$$

where the last equality holds if we let P and p always have the same sign. We now need to express this expression in terms of x and P. Note that

$$e^{x/2} = P \sin x' = 2P \sin \frac{x'}{2} \cos \frac{x'}{2}$$

$$\sqrt{P^2 - e^x} + P = P \cos x' + P = 2P \cos^2 \frac{x'}{2}$$

and taking the ratio between the two,

$$\frac{1}{\tan \frac{x'}{2}} = Pe^{-x/2} + \sqrt{P^2 e^{-x} - 1}.$$

Using this and $\sqrt{P^2 - e^x} = P \cos x'$, we get finally

$$\begin{aligned}F_2 &= \pm\big[2\sqrt{P^2 - e^x} - 2P \ln(Pe^{-x/2} + \sqrt{P^2 e^{-x} - 1})\big] \\ &= \pm\big[2\sqrt{P^2 - e^x} - 2P \cosh^{-1}(Pe^{-x/2})\big] \\ &= \pm\big[2\sqrt{P^2 - e^x} - 2P \tanh^{-1}(\sqrt{P^2 - e^x}/P)\big]\end{aligned}$$

where the last two lines are equally acceptable forms (they follow from identities of the inverse hyperbolic functions).

Note that we are unable to choose a single generating function that covers both the cases $p > 0$ and $p < 0$; we need to switch from the plus sign to the minus when the motion of the particle goes from $p > 0$ and $p < 0$. One could try to resolve this problem by choosing the sign for P such that

P and p always have the same sign, but this is problematic because P would then be discontinuous (it flips sign when p changes sign). It appears that some quantity always behaves in a discontinuous manner between the cases $p > 0$ and $p < 0$. (Note to grader: the problem only asks to treat the case $p > 0$.)

(b) [3 points] We have

$$\begin{aligned} Q &= \frac{\partial F_2}{\partial P} = \pm\left[\frac{2P}{\sqrt{P^2 - e^x}} - 2\cosh^{-1}(Pe^{-x/2}) - \frac{2Pe^{-x/2}}{\sqrt{P^2 e^{-x} - 1}}\right] \\ &= \mp 2\cosh^{-1}(Pe^{-x/2}) = \mp 2\cosh^{-1}(\sqrt{p^2 + e^x}\, e^{-x/2}) \\ &= \mp 2\cosh^{-1}(\sqrt{p^2 e^{-x} + 1}) \end{aligned}$$

where $Q < 0$ if $p > 0$ and vice versa. Alternatively,

$$\begin{aligned} Q &= \mp 2\cosh^{-1}(Pe^{-x/2}) = \mp 2\tanh^{-1}(\sqrt{P^2 - e^x}/P) \\ &= \mp 2\tanh^{-1}(|p|/\sqrt{p^2 + e^x}) \\ &= -2\tanh^{-1}(p/\sqrt{p^2 + e^x}). \end{aligned}$$

For P, since we already chose $P > 0$, we immediately have

$$P = \sqrt{p^2 + e^x}.$$

(c) [4 points] From the above we have

$$\cosh(\frac{Q}{2}) = Pe^{-x/2}$$

or

$$x = 2\ln\Big(\frac{P}{\cosh(\frac{Q}{2})}\Big).$$

Also as before,

$$\begin{aligned} p &= \pm\sqrt{P^2 - e^x} \\ &= \pm\sqrt{P^2 - P^2/\cosh^2(\frac{Q}{2})} \\ &= \pm P|\tanh(\frac{Q}{2})| \\ &= -P\tanh(\frac{Q}{2}) \end{aligned}$$

since we already know from (b) that $\tanh(Q/2)$ and p have opposite signs. Now from the Hamiltonian $K = P^2$, we see that Q is a cyclic coordinate and hence P is conserved. Moreover,

$$\dot{Q} = \frac{\partial K}{\partial P} = 2P.$$

Therefore for some constants a and b,

$$P(t) = a, \qquad Q(t) = 2(at + b).$$

Plugging this back into our previous formulas for x and p, we see that

$$x(t) = 2\ln\left(\frac{a}{\cosh(at+b)}\right), \qquad p(t) = -a\tanh(at+b).$$

6. Projectile with Hamilton-Jacobi [11 points]

Let $+y$ point vertically upwards. We have

$$H = \frac{p_x^2 + p_y^2}{2m} + mgy = \alpha_1$$

where we've already applied conservation of energy (since H is time-independent). In terms of Hamilton's characteristic function W, this is equivalent to

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2\right] + mgy = \alpha_1.$$

Assume now W is separable: $W = W_x(x, \alpha) + W_y(y, \alpha)$. Then

$$\frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2 + \left[\frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2 + mgy\right] = \alpha_1.$$

The terms inside the square brackets is a function of the coordinate y only, while the term outside the bracket depends on the coordinate x only; it thus follows that the terms inside the square brackets form a constant independent of the coordinates,

$$\frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2 + mgy = \alpha_2$$

and also

$$\frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2 = \alpha_1 - \alpha_2.$$

The equation for W_x is straightforward to integrate:

$$W_x(x, \alpha) = \pm\sqrt{2m(\alpha_1 - \alpha_2)}\, x.$$

For W_y, we have instead that $\frac{\partial W_y}{\partial y} = \pm\sqrt{2m(\alpha_2 - mgy)}$, which gives

$$W_y(y, \alpha) = \mp\frac{2}{3}\frac{1}{mg}\sqrt{2m}(\alpha_2 - mgy)^{3/2}$$

and therefore putting the two contributions together gives

$$W = \pm\sqrt{2m(\alpha_1 - \alpha_2)}x \mp \frac{2}{3}\frac{1}{mg}\sqrt{2m}(\alpha_2 - mgy)^{3/2}$$

where the two sets of plus/minus signs can be taken independently.

We can now take the derivatives of W with respect to the new constant momenta α_1 and α_2:

$$t + \beta_1 = Q_1 = \frac{\partial W}{\partial \alpha_1} = \pm\sqrt{2m}\frac{1}{2\sqrt{\alpha_1 - \alpha_2}}x$$

$$\beta_2 = Q_2 = \frac{\partial W}{\partial \alpha_2} = \mp\sqrt{2m}\frac{1}{2\sqrt{\alpha_1 - \alpha_2}}x \mp \frac{1}{mg}\sqrt{2m(\alpha_2 - mgy)}$$

The first constant β_1 is easy to deal with: since $x = 0$ at $t = 0$, we immediately get $\beta_1 = 0$, giving the time dependence for x

$$x = \sqrt{\frac{2(\alpha_1 - \alpha_2)}{m}}t$$

(we take the positive root here). For the second equation, let us define $\beta' = mg\beta_2/\sqrt{2m}$, and further absorb one of the $\mp$ signs into β':

$$\beta' = \sqrt{\alpha_2 - mgy} \pm \frac{mg}{2\sqrt{\alpha_1 - \alpha_2}}\, x$$

or

$$\alpha_2 - mgy = \left(\beta' \mp \frac{mg}{2\sqrt{\alpha_1 - \alpha_2}}\, x\right)^2$$

$$y = \frac{\alpha_2}{mg} - \frac{\beta'^2}{mg} \pm \frac{\beta'}{\sqrt{\alpha_1 - \alpha_2}}x - \frac{mg}{4(\alpha_1 - \alpha_2)}x^2$$

This gives the trajectory of the projectile $y = y(x)$; to get the time-dependence for y, we plug in our previous time dependence for x, giving

$$y = \frac{\alpha_2 - \beta'^2}{mg} \pm \beta'\sqrt{\frac{2}{m}}t - \frac{gt^2}{2}.$$

Now to match the initial conditions. We've already matched $x(t = 0) = 0$; for the other three conditions,

$$y(t = 0) = 0 = \frac{\alpha_2 - \beta'^2}{mg} \qquad \Rightarrow \alpha_2 = \beta'^2$$

$$\frac{dx}{dt}(t = 0) = v_0 \cos\theta = \sqrt{\frac{2(\alpha_1 - \alpha_2)}{m}} \qquad \Rightarrow \sqrt{\alpha_1 - \alpha_2} = \sqrt{\frac{m}{2}}v_0 \cos\theta$$

$$\frac{dy}{dt}(t = 0) = v_0 \sin\theta = \pm\beta'\sqrt{\frac{2}{m}} \qquad \beta' = \sqrt{\frac{m}{2}}v_0 \sin\theta \text{ (take positive sign)}$$

which gives, after plugging everything in,

$$x(t) = (v_0 \cos\theta)t$$

$$y(t) = (v_0 \sin\theta)t - \frac{gt^2}{2}$$

$$y(x) = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2.$$

(Another way to deal with separability here is to notice that x is a cyclic coordinate.)