$\boxed{\text{Ex 1}}$

① suppose $\begin{cases} g e = e g = g \\ g \tilde{e} = \tilde{e} g = g \end{cases} \quad \forall g \in G.$

Then $e \tilde{e} = \tilde{e}$ but also $e \tilde{e} = e \Rightarrow e = \tilde{e}$

② Let $g \in G$ and suppose $\begin{cases} g \cdot a = a \cdot g = e \\ g \cdot b = b \cdot g = e \end{cases}$

Then $b \cdot g \cdot a = b \cdot e = b$ but also $b \cdot g \cdot a = e \cdot a = a \Rightarrow a = b$

③ Since the inverse is unique and $g \cdot g^{-1} = g^{-1} \cdot g = 1$, then g is the inverse of g^{-1}

④ $\left.\begin{aligned} (h^{-1}g^{-1})(gh) &= h^{-1}g^{-1}gh = h^{-1}eh = h^{-1}h = e \\ (gh)(h^{-1}g^{-1}) &= ghh^{-1}g^{-1} = geg^{-1} = gg^{-1} = e \end{aligned}\right\} \Rightarrow (gh)^{-1} = h^{-1}g^{-1}$

$\boxed{\text{Ex 2}}$

① $\varphi(e_G) = \varphi(\overset{e_G}{e_G e_G}) = \varphi(e_G)\varphi(e_G)$

$\Rightarrow \varphi(e_G)^{-1}\varphi(e_G) = \varphi(e_G)^{-1}\varphi(e_G)\varphi(e_G) \Rightarrow e_H = e_H\,\varphi(e_G) = \varphi(e_G)$

② $\left.\begin{aligned} e_H &= \varphi(e_G) = \varphi(g^{-1}g) = \varphi(g^{-1})\varphi(g) \\ e_H &= \varphi(e_G) = \varphi(g g^{-1}) = \varphi(g)\varphi(g^{-1}) \end{aligned}\right. \Rightarrow \varphi(g^{-1}) = \varphi(g)^{-1}$

$\boxed{\text{Ex 3}}$ $\mathbb{Z}_2 = \{[0],[1]\}$ with $\begin{aligned} &[0]+[0] = [0] \\ &[0]+[1] = [1]+[0] = [1] \\ &[1]+[1] = [0] \end{aligned}$

let $\varphi : \mathbb{Z}_2 \to \{\mathbb{1}_n, -\mathbb{1}_n\}$

it must be $\varphi([0]) = \mathbb{1}_n$ since homomorphisms send identity to identity.

$\downarrow$

We want φ to be invertible $\Rightarrow$ $\varphi([1]) \neq \varphi([0])$ $\Rightarrow$ $\varphi([1]) = -\mathbb{1}_n$

is it a homomorphism?

$\varphi([0]+[0]) = \varphi([0]) = \mathbb{1}_n = \mathbb{1}_n \mathbb{1}_n = \varphi([0])\,\varphi([0])$ ✓

$\varphi([1]+[0]) = \varphi([1]) = -\mathbb{1}_n = (-\mathbb{1}_n)\mathbb{1}_n = \varphi([1])\,\varphi([0])$ ✓

$\varphi([0]+[1]) = \varphi([1]) = -\mathbb{1}_n = \mathbb{1}_n(-\mathbb{1}_n) = \varphi([0])\,\varphi([1])$ ✓

$\varphi([1]+[1]) = \varphi([0]) = \mathbb{1}_n = (-\mathbb{1}_n)(-\mathbb{1}_n) = \varphi([1])\,\varphi([1])$ ✓

$\rightarrow$ Yes $\Rightarrow$ $\mathbb{Z}_2 \cong \{\mathbb{1}_n, -\mathbb{1}_n\}$

$\boxed{\text{Ex 4}}$ What does a group G with two elements look like?

- one of the elements is the identity, call it 1
- the other element is <u>not</u> the identity, call it a

it must be $\begin{cases} 1*a = a*1 = a \\ 1*1 = 1 \end{cases}$

what about $a*a$? it must be in G, so it's either 1 or a

- if $a*a = a$ then there is no g in G such that $a*g = g*a = 1$

but a must have an inverse, so the only option is $\boxed{a*a=1}$

This looks just the same as $\mathbb{Z}_2$! In fact, choosing

$\varphi: G \rightarrow \mathbb{Z}_2$ such that $\varphi(1) = [0]$ and $\varphi(a) = [1]$ we get

$$\left.\begin{array}{l} \varphi \text{ invertible} \\ \varphi(1\cdot a) = \varphi(a) = [1] = [0]+[1] = \varphi(1)+\varphi(a) \\ \varphi(a\cdot 1) = \varphi(a) = [1] = [1]+[0] = \varphi(a)+\varphi(1) \\ \varphi(a\cdot a) = \varphi(1) = [0] = [1]+[1] = \varphi(a)+\varphi(a) \\ \varphi(1\cdot 1) = \varphi(1) = [0] = [0]+[0] = \varphi(1)+\varphi(1) \end{array}\right\} \Rightarrow G \cong \mathbb{Z}_2$$