8.422 PSet 3 - Solutions

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The following solutions were prepared by me (Eric Wolf) for Pset 3 in the 2023 Spring adminstration of 8.422, Atomic and Optical Physics II. Any errors should be assumed to be my own, especially in sections marked 'Aside'.

Text in emphasis is excerpted from the problem set for the purpose of clarity.

1 Problem 1

Consider a classical light field. The classical expressions for first-order and second-order coherence are

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle}$$

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle^2}$$

where $\langle\rangle$ denotes the statistical average over many measurements. We may implement it as the time average $\lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)dt$, but you will not need this definition.

a) Prove that $\left|g^{(1)}(\tau)\right| \leq 1$

Using the given fact that the expectation $\langle f, g\rangle = \langle f^*(t)g(t)\rangle$ defines a scalar product, we may apply Cauchy's identity to the functions $f = E(t),\ g = E(t+\tau)$:

$$|\langle E(t), E(t+\tau)\rangle|^2 \leq \langle E(t), E(t)\rangle\langle E(t+\tau), E(t+\tau)\rangle$$

We divide over the terms on the right-hand side and explicitly write the expectations:

$$\frac{|\langle E^*(t)E(t+\tau)\rangle|^2}{\langle E^*(t)E(t)\rangle\langle E^*(t+\tau)E(t+\tau)\rangle} \leq 1 \tag{1}$$

At this point, we assume that $\langle E^*(t)E(t)\rangle = \langle E^*(t+\tau)E(t+\tau)\rangle$.

There are several ways to justify this. First, the spirit of the $g^{(1)}$ and $g^{(2)}$ functions as given here, which divide out the intensity expectation of the light at t, is to get an idea of how the stochastic fluctuations of the light behave without worrying about overall intensity - they aren't really "meant for" light whose average intensity has a time-dependence. If you wanted coherence functions for such light, you're better off using expressions akin to those in Problem 2, which divides by the geometric mean of the intensities at the two points of interest.

Second, in many experiments the correlations of light which give rise to interesting $g^{(1)}$ and $g^{(2)}$ functions will decay much faster than the overall intensity will change.

Third, it is actually somewhat difficult, using the rigorous definition given above, to *find* an electric field $E(t)$ which will satisfy $\langle E(t)^*E(t)\rangle \neq \langle E(t+\tau)^*E(t+\tau)\rangle$.

Given that we are willing to make this assumption, Equation 1 becomes

$$\frac{|\langle E^*(t)E(t+\tau)\rangle|^2}{|\langle E^*(t)E(t)\rangle|^2} \leq 1$$

whence, taking the square root, we obtain

$$\frac{|\langle E^*(t)E(t+\tau)\rangle|}{|\langle E^*(t)E(t)\rangle|} = |g^{(1)}(\tau)| \leq 1 \tag{2}$$

as required.

Aside:

In point of fact, $\langle f, g\rangle$ does *not* define an inner product on the set of, say, integrable "nice" functions of t; the condition that fails is that this "dot product" actually induces a *seminorm* on this vector space. That is,

$$\exists f(t) \neq 0 \text{ s.t.} \langle f, f\rangle = 0$$

Consider, for example, any function $f(t)$ for which $\int_{-\infty}^{\infty} |f(t)|^2 \in \mathbb{R}$. A Gaussian would have this problem, for instance. At least one approach for patching up this issue - roughly, demanding that any function in our inner product space has a positive minimum frequency below which all of its Fourier components are identically zero - imposes a sort of weak periodicity property on the function that automatically ensures that $\langle f(t)^* f(t)\rangle = \langle f^*(t+\tau)f(t+\tau)\rangle$.

b) Prove that for zero time delay, the second-order coherence obeys the identity $g^{(2)}(0) \geq 1$.

We use the provided hint; trivially,

$$\langle (I(t) - \langle I(t)\rangle)^2\rangle \geq 0$$

Expanding the LHS:

$$\langle I(t)^2 - 2I(t)\langle I(t)\rangle + \langle I(t)\rangle^2\rangle \geq 0$$

Using the fact that $\langle\langle g(t)\rangle f(t)\rangle = \langle g(t)\rangle\langle f(t)\rangle$ for arbitrary f, g, we have

$$\langle I(t)^2\rangle - 2\langle I(t)\rangle^2 + \langle I(t)\rangle^2 \geq 0$$

Adding over and dividing by $\langle I(t)\rangle^2$, we find

$$\frac{\langle I(t)^2\rangle}{\langle I(t)\rangle^2} \geq 1$$

which, given our definition of $g^{(2)}(0)$, implies in particular that

$$g^{(2)}(0) \geq 1 \tag{3}$$

as desired.

c) Using a similar argument, show that $g^{(2)}(\tau) \leq g^{(2)}(0)$.

We again write a trivial fact:

$$\langle (I(t+\tau) - I(t))^2\rangle \geq 0$$

Expanding, we obtain

$$\langle (I(t+\tau)^2\rangle - 2\langle (I(t)I(t+\tau))\rangle + \langle I(t)\rangle^2 \geq 0$$

In the spirit of the above, we assume that $\langle I(t+\tau)^2\rangle = \langle I(t)^2\rangle$, so that we obtain

$$2\langle I(t)I(t+\tau)\rangle \leq 2\langle I(t)^2\rangle$$

Dividing both sides by $2\langle I(t)\rangle^2$, we obtain

$$\frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} \leq \frac{\langle I(t)^2\rangle}{\langle I(t)\rangle^2}$$

and, recognizing the form of $g^{(2)}$,

$$g^{(2)}(\tau) \leq g^{(2)}(0)$$

as desired.

d) Consider chaotic classical light generated by an ensemble of ν atoms. The total electric field can be expressed as $E(t) = \sum_{i=1}^{\nu} E_i(t)$, where the phases of the E_i are random. Show that when ν is large,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$$

It helps to calculate a few quantities. First, we write

$$\langle E^*(t)E(t)\rangle = \sum_{i,j} \langle E_i^*(t)E_j(t)\rangle$$

using the linearity properties of the expectation value. Now, the assumed randomness of the phases gives that

$$\sum_{i,j} \langle E_i^*(t)E_j(t)\rangle = \sum_{i,j} \langle E_i^*(t)E_j(t)\rangle \delta_{i,j} = \sum_i \langle |E_i(t)|^2\rangle$$

That is, because the phases of the E_j are uncorrelated with E_i, we have that

$$\sum_{j\neq i} E_i^* E_j = 0$$

even without invoking the expectation.

Anywho, we proceed. We write $g^{(1)}$ first:

$$g^{(1)}(\tau)\frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle}$$

We write

$$\langle E^*(t)E(t+\tau)\rangle = \sum_{i,j} \langle E_i^*(t)E_j(t+\tau)\rangle = \sum_i E_i^*(t)E_i(t+\tau)$$

so that we have

$$g^{(1)}(\tau) = \frac{\sum_i \langle E_i^*(t) E_i(t+\tau)\rangle}{\sum_i \langle |E_i(t)|^2\rangle}$$

and ultimately

$$\left|g^{(1)}(\tau)\right|^2 = \frac{\left(\sum_i \langle E_i^*(t) E_i(t+\tau)\rangle\right)\left(\sum_j \langle E_j(t) E_j^*(t+\tau)\rangle\right)}{\left(\sum_i \langle |E_i(t)|^2\rangle\right)\left(\sum_j \langle |E_j(t)\rangle|^2\right)} \tag{4}$$

Now let us go about evaluating $g^{(2)}$. It behooves us to simplify the sum which will appear in the numerator of $g^{(2)}$:

$$\begin{aligned} &\langle E^*(t) E^*(t+\tau) E(t+\tau) E(t)\rangle \\ &= \left\langle \sum_{i,j,k,l} E_i^*(t) E_j^*(t+\tau) E_k(t+\tau) E_l(t) \right\rangle \end{aligned}$$

Now, the random phase properties of the ensemble imply that any term for which one does not have

$$\{i,j\} = \{k,l\}$$

will sum up to zero. One cannot, in particular, have $k = l,\ i = j \neq k$; here, the frequencies may well be such that the term does not oscillate, but the random phases will still ensure that the sum is zero.

Then we write our sum in three parts:

$$\begin{aligned} &\left\langle \sum_{i,j,k,l} E_i^*(t) E_j^*(t+\tau) E_k(t+\tau) E_l(t) \right\rangle = \\ &\left\langle \sum_{i=j=k=l} E_i^*(t) E_j^*(t+\tau) E_k(t+\tau) E_l(t) \right\rangle + \\ \left\langle \sum_{i=k\neq j=l} E_i^*(t) E_j^*(t+\tau) E_k(t+\tau) E_l(t) \right\rangle + &\left\langle \sum_{i=l\neq j=k} E_i^*(t) E_j^*(t+\tau) E_k(t+\tau) E_l(t) \right\rangle \end{aligned} \tag{5}$$

Now, let us busy ourselves with the two terms on the second line first:

$$\begin{aligned} &\left\langle \sum_{i=k\neq j=l} E_i^*(t) E_j^*(t+\tau) E_k(t+\tau) E_l(t) \right\rangle = \\ &\left\langle \sum_{i\neq j} E_i^*(t) E_i(t+\tau) E_j^*(t+\tau) E_j(t) \right\rangle \\ &= \sum_{i\neq j} \langle E_i^*(t) E_i(t+\tau)\rangle \langle E_j^*(t+\tau) E_j(t)\rangle \end{aligned} \tag{6}$$

where we have exploited the fact that different emitters are assumed to be uncorrelated[1] to "split" the four-field expectation into a product of two two-field expectations. Crucially, we couldn't have done this for the term where all four E's come from the same emitter.

In an exactly similar fashion, we may find

[1] Ok, a given pair of emitters may happen to be correlated; the identity is valid when summed over an uncorrelated ensemble, though.

$$\left\langle \sum_{i=l\neq j=k} E_i^*(t)E_j^*(t+\tau)E_k(t+\tau)E_l(t) \right\rangle$$
$$= \sum_{i\neq j}\langle |E_i(t)|^2\rangle\langle |E_j(t+\tau)|^2\rangle = \sum_{i\neq j}\langle |E_i(t)|^2\rangle\langle |E_j(t)|^2\rangle \tag{7}$$

where, as before, we have assumed that the expectation of $|E(t)|^2$ is equal to that of $|E(t+\tau)|^2$.

Now, when divided by $\langle E^*(t)E(t)\rangle^2$, Equation 6 *almost* becomes equal to Equation 4, while Equation 7 *almost* becomes equal to 1. The problem in both cases is that Equations 6 and 7 are missing the terms where $i = j$. Quite separately, there is the single-emitter four-field term of Equation 5 to worry about.

The solution here is to realize that the terms which are "missing" from Equations 7 and 6 are of order ν, whereas the remaining terms as well as the divisor $\langle E^*(t)E(t)\rangle^2$ which will appear in $g^{(2)}$ are of order ν^2. Likewise, the extra term of Equation 5 is of order ν.

Thus, for a sufficiently large ensemble of emitters whose distribution of self-coherence properties are not too pathological[2], we will have that

$$g^{(2)}(\tau) = \frac{\left\langle \sum_{i,j,k,l} E_i^*(t)E_j^*(t+\tau)E_k(t+\tau)E_l(t)\right\rangle}{\langle E^*(t)E(t)\rangle^2}$$
$$\sim \frac{\left(\sum_i \langle E_i^*(t)E_i(t+\tau)\rangle\right)\left(\sum_j \langle E_j(t)E_j^*(t+\tau)\rangle\right)}{\left(\sum_i \langle |E_i(t)|^2\rangle\right)\left(\sum_j \langle |E_j(t)|^2\rangle\right)} + \frac{\sum_{i,j}\langle |E_i(t)|^2\rangle\langle |E_j(t)|^2\rangle}{\langle E^*(t)E(t)\rangle^2} \tag{8}$$
$$= |g^{(1)}(\tau)|^2 + 1$$

where the $\sim$ in Equation 8 indicates asymptotic equality for large ν.

2 Problem 2

Consider light in a single mode of the radiation field. The quantum-mechanical expressions for first-order and second-order coherence are

$$g^{(1)}(\vec{r}_1, t_1, \vec{r}_2, t_2) = \frac{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^+(\vec{r}_2, t_2)\rangle}{\sqrt{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^+(\vec{r}_1, t_1)\rangle\langle \hat{E}^-(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_2, t_2)\rangle}}$$

$$g^{(2)}(\vec{r}_1, t_1, \vec{r}_2, t_2; \vec{r}_2, t_2, \vec{r}_1, t_1) = \frac{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^-(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_1, t_1)\rangle}{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^+(\vec{r}_1, t_1)\rangle\langle \hat{E}^-(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_2, t_2)\rangle}$$

where

$$\hat{E}^+(\vec{r}, t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\, a e^{-i(\omega t - \vec{k}\cdot\vec{r})}$$
$$\hat{E}^-(\vec{r}, t) = -i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\, a^\dagger e^{i(\omega t - \vec{k}\cdot\vec{r})}$$

a) Using these expressions, show that the second-order coherence may be written as

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2}$$

[2] It would be bad if there were potentially emitters in the ensemble whose intensities $I(t)$ become arbitrarily close to a train of δ functions, for instance.

Let ϵ denote the prefactor of the electric fields. Then we may manipulate the given expression for the second-order coherence, grouping together the phase factors etc:

$$g^{(2)}(\vec{r}_1, t_1, \vec{r}_2, t_2; \vec{r}_2, t_2, \vec{r}_1, t_1) = \frac{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^-(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_1, t_1)\rangle}{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^+(\vec{r}_1, t_1)\rangle\langle \hat{E}^-(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_2, t_2)\rangle}$$

$$g^{(2)}(\vec{r}_1, t_1, \vec{r}_2, t_2; \vec{r}_2, t_2, \vec{r}_1, t_1) = \frac{(i)^4\epsilon^4 e^{i(\omega t_1 - \vec{k}\cdot\vec{r}_1)(1-1)}e^{i(\omega t_1 - \vec{k}\cdot\vec{r}_2)(1-1)}}{i^4\epsilon^4 e^{i(\omega t_1 - \vec{k}\cdot\vec{r}_1)(1-1)}e^{i(\omega t_2 - \vec{k}\cdot\vec{r}_2)(1-1)}}\frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle\langle a^\dagger a\rangle}$$

Since this big prefactor is just one, we write

$$g^{(2)}(\vec{r}_1, t_1, \vec{r}_2, t_2; \vec{r}_2, t_2, \vec{r}_1, t_1) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} \tag{9}$$

which is the first prescribed form. We note that this result is independent of the $\vec{r}$ and t, so we will write it as $g^{(2)}$ from now on. We also note that

$$a^\dagger a^\dagger a a = a^\dagger(a a^\dagger - 1)a = (a^\dagger a a^\dagger a) - a^\dagger a = n^2 - n$$

whence we obtain the form

$$g^{(2)} = \frac{\langle n^2 - n\rangle}{\langle n\rangle^2} = \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2} \tag{10}$$

also as required.

b) Using the expression you just derived, show that, for light in a number state $|n\rangle$ where $n \geq 2$, $|g^{(1)}| = 1$ and $g^{(2)} = 1 - \frac{1}{n}$, independent of space-time separation. What is $g^{(1)}$ and $g^{(2)}$ for $n = 0$ and $n = 1$?

It behooves us to write a general expression for $g^{(1)}$;

$$\begin{aligned} g^{(1)}(\vec{r}_1, t_1, \vec{r}_2, t_2) &= \frac{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^+(\vec{r}_2, t_2)\rangle}{\sqrt{\langle \hat{E}^-(\vec{r}_1, t_1)\hat{E}^+(\vec{r}_1, t_1)\rangle\langle \hat{E}^-(\vec{r}_2, t_2)\hat{E}^+(\vec{r}_2, t_2)\rangle}} \\ &= \frac{-i^2\epsilon^2}{\epsilon^2}e^{-i(\omega(t_2 - t_1) - \vec{k}\cdot(\vec{r}_2 - \vec{r}_1))}\frac{\langle a^\dagger a\rangle}{\sqrt{\langle a^\dagger a\rangle\langle a^\dagger a\rangle}} \end{aligned}$$

and ultimately

$$g^{(1)}(\vec{r}_1, t_1, \vec{r}_2, t_2) = e^{-i(\omega(t_2 - t_1) - \vec{k}\cdot(\vec{r}_2 - \vec{r}_1))} \tag{11}$$

so that whereas the phase of $g^{(1)}$ varies in space and time, the absolute value always satisfies

$$|g^{(1)}| = 1 \tag{12}$$

Aside:

It may seem curious that such simple forms exist for $g^{(1)}$ and $g^{(2)}$, given that we have made no assumptions on the underlying statistics of the light yet. However, note that we are dealing here only with a single *mode* of the light field. Given that one is dealing with a specific mode, one has very rigid rules for how the fields evolve from one point in space to the next, and really the only degree of freedom left is the distribution of the photon number within the mode. This corresponds to the varied values of $g^{(2)}$ we will obtain; $g^{(1)}$ only deals with phase coherence, though, and this will carry only a trivial dependence on space and time for a single mode. If we wanted to have nontrivially spatially- or temporally-varying coherences, we would need to consider multiple modes. $\square$

Now, $|g^{(1)}| = 1$ has been shown generically. For a number state $|n\rangle$, one has

$$g^{(2)}_n = \frac{\langle n|n^2|n\rangle - \langle n|n|n\rangle}{\langle n|n|n\rangle^2} = \frac{n^2 - n}{n^2} = 1 - \frac{1}{n} \tag{13}$$

This formula is valid for $n \geq 1$; in particular, it tells us that $g^{(2)} = 0$ for $n = 1$. This makes sense; $g^{(2)}$ roughly measures the probability of finding another photon in our field, given that we find a first, and for a number state $n = 1$, you *never* find two photons.

The value of $g^{(2)}$ for $n = 0$ is a different beast. We may be tempted to call it $-\infty$, but note that the middle expression in Equation 13 is of the form $\frac{0}{0}$ for a state where $n = 0$.

Sometimes, we can massage such forms away by taking a limit. This is not the case here, however, for we can choose to regard the vacuum in three ways:

1. It is a number state $|n = 0\rangle$
2. It is a coherent state $|\alpha = 0\rangle$
3. It is a thermal state $\rho_{T=0}$

We can't really take a limit for the discrete number states, but we can very easily take a limit of $g^{(2)}$ as $|\alpha| \to 0$ for the coherent states or $T \to 0$ for the thermal states. In subsequent problems, we will find that $g^{(2)} = 1$ for all coherent states with nonzero α, while $g^{(2)} = 2$ for all thermal states with nonzero T. Any notion of assigning a $g^{(2)}$ value vacuum to the vacuum by taking it as a limit of states for which we can calculate $g^{(2)}$ must then fail; $g^{(2)}$ is simply not defined for $n = 0$.

By the by, this difficulty does not crop up for the *first*-order coherence $g^{(1)}$; there's no obstacle to saying that $g^{(1)}$ for the vacuum is of the form 11.

c) Show that, for light in a coherent state $|\alpha\rangle$, $|g^{(1)}| = 1$ and $|g^{(2)}| = 1$.

The identity for $g^{(1)}$ has already been shown in Equation 12. For $g^{(2)}$, we write using Equation 9 that

$$g^{(2)}_{|\alpha\rangle} = \frac{\langle\alpha|a^\dagger a^\dagger a a|\alpha\rangle}{\langle\alpha|a^\dagger a|\alpha\rangle^2} = \frac{|\alpha|^4}{(|\alpha|^2)^2} = 1 \tag{14}$$

as desired.

d) Show that, for chaotic light with density matrix

$$\hat{\rho} = (1 - e^{-\frac{\hbar\omega}{k_B T}}) \sum_n e^{-\frac{n\hbar\omega}{k_B T}} |n\rangle \langle n|$$

$|g^{(1)}| = 1$ *and* $g^{(2)} = 2$.

The identity for $g^{(1)}$ has already been shown. For $g^{(2)}$, we need to evaluate both

$$\langle n\rangle = \mathrm{Tr}(\hat{n}\rho)$$

as well as

$$\langle n^2\rangle = \mathrm{Tr}(\hat{n}^2\rho)$$

We are free to do this in the number basis, whereupon our expectation values become

$$\langle n\rangle = (1 - e^{-\frac{\hbar\omega}{k_B T}}) \sum_n n e^{-\frac{n\hbar\omega}{k_B T}}$$

Let us write $\nu \equiv \frac{\hbar\omega}{k_B T}$. Then

$$\sum_n n e^{-n\frac{\hbar\omega}{k_B T}}$$
$$= \sum_n n e^{-n\nu}$$
$$= -\frac{\mathrm{d}}{\mathrm{d}\nu} \sum_n e^{-n\nu}$$
$$= -\frac{\mathrm{d}}{\mathrm{d}\nu} \frac{1}{1-e^{-\nu}}$$
$$= \frac{e^{-\nu}}{(1-e^{-\nu})^2}$$

whence

$$\langle n \rangle = (1-e^{-\nu}) \frac{e^{-\nu}}{(1-e^{-\nu})^2} = \frac{e^{-\nu}}{1-e^{-\nu}} = \frac{1}{e^{\nu}-1}$$

as we expect for bosons. Likewise, we may write

$$\langle n^2 \rangle = (1-e^{-\nu}) \sum_n n^2 e^{-n\nu}$$

As above,

$$\sum_n n^2 e^{-n\nu} = \frac{\mathrm{d}^2}{\mathrm{d}\nu^2} \sum_n e^{-n\nu}$$
$$= \frac{\mathrm{d}^2}{\mathrm{d}\nu^2} \frac{1}{1-e^{-\nu}} = \frac{\mathrm{d}}{\mathrm{d}\nu} \left(-\frac{e^{-\nu}}{(1-e^{-\nu})^2} \right)$$
$$= \frac{e^{-\nu}}{(1-e^{-\nu})^2} + 2\frac{e^{-2\nu}}{(1-e^{-\nu})^3}$$

whence

$$\langle n^2 \rangle = (1-e^{-\nu}) \sum_n n^2 e^{-n\nu} = \frac{e^{-\nu}}{1-e^{-\nu}} + 2\left(\frac{e^{-\nu}}{(1-e^{-\nu})} \right)^2$$

or ultimately, as derived in class,

$$\langle n^2 \rangle = \langle n \rangle + 2\langle n \rangle^2 \tag{15}$$

for a thermal state.

Using Equation 15 and Equation 10, we may immediately write for the thermal state that

$$g^{(2)} = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} = \frac{\big(\langle n \rangle + 2\langle n \rangle^2\big) - \langle n \rangle}{\langle n \rangle^2} = 2$$

as required.

e) Compute $g^{(2)}(\tau)$ for the following states as a function of α:

$$|\psi_+\rangle = \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2}\sqrt{1+e^{-2|\alpha|^2}}}$$

$$|\psi_-\rangle = \frac{|\alpha\rangle - |-\alpha\rangle}{\sqrt{2}\sqrt{1-e^{-2|\alpha|^2}}}$$

We first verify the normalizations. Observe that

$$\langle\psi_\pm|\psi_\pm\rangle = \frac{1}{2(1 \pm e^{-2|\alpha|^2})}\left(\langle\alpha|\alpha\rangle + \langle-\alpha|-\alpha\rangle \pm (\langle\alpha|-\alpha\rangle + \langle-\alpha|\alpha\rangle)\right)$$
$$\langle\psi_\pm|\psi_\pm\rangle = \frac{1}{2(1 \pm e^{-2|\alpha|^2})}\left(2 \pm 2\mathrm{Re}\,\langle\alpha|-\alpha\rangle\right)$$

Recall from the development in a previous pset that

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}}e^{-\frac{|\beta|^2}{2}}e^{\alpha^*\beta}$$

In particular, then,

$$\langle\alpha|-\alpha\rangle = e^{-|\alpha|^2}e^{\alpha^*(-\alpha)} = e^{-2|\alpha|^2} = \mathrm{Re}(\langle\alpha|-\alpha\rangle)$$

whence our overlap becomes

$$\langle\psi_\pm|\psi_\pm\rangle = \frac{1}{2 \pm 2e^{-2|\alpha|^2}}(2 \pm 2e^{-2|\alpha|^2}) = 1$$

and our normalization is correct - at least, unless $\alpha = 0$ and we are dealing with ψ_-.
Let $N_\pm \equiv \frac{1}{\sqrt{2(1 \pm e^{-2|\alpha|^2})}}$. Observe that

$$a\,|\psi_\pm\rangle = \alpha\frac{N_\pm}{N_\mp}\psi_\mp$$

and hence also

$$aa\,|\psi_\pm\rangle = \alpha^2\,|\psi_\pm\rangle$$

Then in particular,

$$\langle\psi_\pm|a^\dagger a^\dagger aa|\psi_\pm\rangle = |\alpha|^4\,\langle\psi_\pm|\psi_\pm\rangle = |\alpha|^4$$

However,

$$\langle\psi_\pm|a^\dagger a|\psi_\pm\rangle = \frac{N_\pm^2}{N_\mp^2}|\alpha|^2\,\langle\psi_\mp|\psi_\mp\rangle = \frac{N_\pm^2}{N_\mp^2}|\alpha|^2$$

We then obtain, leveraging Equation 9,

$$g_\pm^{(2)} = \frac{\langle\psi_\pm|a^\dagger a^\dagger aa|\psi_\pm\rangle}{\langle\psi_\pm|a^\dagger a|\psi_\pm\rangle^2} = \frac{|\alpha|^4}{\left(\frac{N_\pm^2}{N_\mp^2}|\alpha|^2\right)^2} = \frac{N_\mp^4}{N_\pm^4} = \frac{(1 \pm e^{-2|\alpha|^2})^2}{(1 \mp e^{-2|\alpha|^2})^2} \tag{16}$$

By inspection, we see that $g_+^{(2)} > 1$ for all (nonzero) α, whereas $g_-^{(2)} < 1$ for all nonzero α. Then the state $|\psi_-\rangle$ shows non-classical coherence.

3 Problem 3

Let the beamsplitter operator B, acting on angle θ on modes a and b, be defined by

$$B = e^{\theta(a^\dagger b - b^\dagger a)}$$

a) We can show that B conserves the total photon number and leaves coherent states as coherent states. Prove that B leaves $n_a + n_B = a^\dagger a + b^\dagger b$ unchanged. Prove also that $B^\dagger B = I$.

This is straightforward. First, note the well-known fact that, if A is Hermetian, then e^{iA} is unitary; this is ultimately a consequence of the rules of complex conjugation and the BCH lemma.

Observe further that we can write

$$B = e^{\theta(a^\dagger b - b^\dagger a)} = e^{i\left(-i\theta(a^\dagger b - b^\dagger a)\right)} = e^{iA}$$

with

$$A \equiv -i\theta(a^\dagger b - b^\dagger a) \tag{17}$$

Observe also that

$$A^\dagger = i\theta\left(a^\dagger b - b^\dagger a\right)^\dagger = -i\theta\left(a^\dagger b - b^\dagger a\right) = A$$

so that A is Hermetian, and hence B is unitary.

We may now show that A commutes with $n_A + n_B$. In particular, it suffices to show that

$$[a^\dagger b - b^\dagger a, n_A + n_B] = 0$$

This is, however, straightforward. We write

$$[a^\dagger b, n_A + n_B] = [a^\dagger b, a^\dagger a + b^\dagger b] = [a^\dagger, a^\dagger a]b + a^\dagger[b, b^\dagger b]$$

We note that

$$[a^\dagger, a^\dagger a] = a^\dagger a^\dagger a - a^\dagger a a^\dagger = a^\dagger[a^\dagger, a] = -a^\dagger$$

while

$$[b, b^\dagger b] = bb^\dagger b - b^\dagger bb = [b, b^\dagger]b = b$$

whence we obtain

$$[a^\dagger b, n_A + n_B] = -a^\dagger b + a^\dagger b = 0$$

By exactly analogous arguments, we would find that $[-b^\dagger a, n_A + n_B] = 0$, so that overall we have $[A, n_A + n_B] = 0$, as required.

If this is true, however, we know (e.g. by leveraging the series definition of e^A) that $n_A + n_B$ is unchanged by B acting on a state ψ:

$$\langle \psi B^\dagger | n_A + n_B | B\psi\rangle$$
$$= \langle \psi | B^\dagger (n_A + n_B) B | \psi\rangle$$

for arbitrary ψ. However,

$$B^\dagger(n_A + n_B)B = e^{-iA}(n_A + n_B)e^{iA} \underbrace{=}_{\text{because } [A, n_A + n_B] = 0} (n_A + n_B)e^{-iA}e^{iA} = n_A + n_B$$

so that $n_A + n_B$ is left unchanged by the operator B, as required.

b) Let $|\alpha\rangle$ be a coherent state. Compute $B\,|0\rangle_b\,|\alpha\rangle_a$, and show that the output is a tensor product of coherent states for all θ.

To prove this, we use the following trivial lemma:

Lemma: For an arbitrary unitary U and operator O, the state $|U\rangle\,\psi$ is an eigenstate of O if and only if the state $|\psi\rangle$ is an eigenstate of $U^\dagger OU$. Also, the eigenvalues are the same.

The proof is in one line:

$$OU\,|\psi\rangle = vU\,|\psi\rangle \leftrightarrow U^\dagger OU\,|\psi\rangle = U^\dagger vU\,|\psi\rangle = v\,|\psi\rangle$$

With this lemma in hand, it suffices to show that $|0\rangle_b\,|\alpha\rangle_a$ is an eigenstate of $B^\dagger aB$ and $B^\dagger bB$; the eigenvalues will tell us the "composition" of the coherent state.

Then we prove the following facts:

$$B^\dagger aB = a\cos(\theta) + b\sin(\theta) \tag{18}$$

$$B^\dagger bB = b\cos(\theta) - a\sin(\theta) \tag{19}$$

To prove this, we use the following form of the BCH formula:

$$e^A a e^{-A} = \sum_{j=0}^{\infty} \frac{1}{j!}\{A, a\}_j$$

where the "iterated commutator" $\{A, a\}_j$ is defined by $\{A, a\}_0 \equiv a$ and $\{A, a\}_{n+1} = [A, \{A, a\}_n]$.

Let us define $V \equiv -iA$, with A defined as in Equation 17, so that

$$V = -\theta(a^\dagger b - b^\dagger a)$$

Then it follows that

$$B^\dagger aB = \left(e^{iA}\right)^\dagger a e^{iA} = e^{-iA} a e^{iA} = e^V a e^{-V} = \sum_{j=0}^{\infty} \frac{1}{j!}\{V, a\}_j \tag{20}$$

We observe that

$$\{V, a\}_j = (-\theta)^j\,\{a^\dagger b - b^\dagger a, a\}_j$$

Now, since

$$[a^\dagger b - b^\dagger a, a] = [a^\dagger b, a] - [b^\dagger a, a] = b[a^\dagger, a] - 0 = -b$$

and

$$[a^\dagger b - b^\dagger a, b] = [a^\dagger b, b] - [b^\dagger a, b] = 0 - [b^\dagger, b]a = a$$

we can then fairly quickly find, e.g. by induction, that

$$\{V, a\}_j = \begin{cases} a(-\theta)^j & j \equiv 0 \bmod 4 \\ -b(-\theta)^j & j \equiv 1 \bmod 4 \\ -a(-\theta)^j & j \equiv 2 \bmod 4 \\ b(-\theta)^j & j \equiv 3 \bmod 4 \end{cases} \tag{21}$$

From Equations 20 and 21, we may write that

$$B^\dagger a B = \sum_{j=0}^{\infty} \frac{1}{j!}\{V, a\}_j$$

$$= \underbrace{a\sum_{k=0}^{\infty} \frac{(-\theta)^{2k}}{2k!}(-1)^k}_{\text{even terms}} \underbrace{-b\sum_{k=0}^{\infty} \frac{(-\theta)^{2k+1}}{(2k+1)!}(-1)^k}_{\text{odd terms}}$$

$$= a\cos(-\theta) - b\sin(-\theta) = a\cos(\theta) + b\sin(\theta)$$

as claimed by Equation 18.

By analogous arguments to the above, it straightforward to see that

$$\{V, b\}_j = \begin{cases} b(-\theta)^j & j \equiv 0 \bmod 4 \\ a(-\theta)^j & j \equiv 1 \bmod 4 \\ -b(-\theta)^j & j \equiv 2 \bmod 4 \\ -a(-\theta)^j & j \equiv 3 \bmod 4 \end{cases} \tag{22}$$

We then obtain, by analogy to Equation 20,

$$B^\dagger b B = \sum_j \frac{1}{j!}\{V, b\}_j$$

$$= \underbrace{b\sum_{k=0}^{\infty} \frac{(-\theta)^{2k}}{(2k)!}(-1)^k}_{\text{even terms}} \underbrace{+a\sum_{k=0}^{\infty} (-1)^k \frac{(-\theta)^{2k+1}}{(2k+1)!}}_{\text{odd terms}}$$

$$= b\cos(-\theta) + a\sin(-\theta) = b\cos(\theta) - a\sin(\theta)$$

as claimed in Equation 19.

With these in hand, along with our lemma, observe that since

$$B^\dagger a B \left|0\right\rangle_b \left|\alpha\right\rangle_a = (a\cos(\theta) + b\sin(\theta)) \left|0\right\rangle_b \left|\alpha\right\rangle_a = \alpha\cos(\theta) \left|0\right\rangle_b \left|\alpha\right\rangle_a$$

and

$$B^\dagger b B \left|0\right\rangle_b \left|\alpha\right\rangle_a = (b\cos(\theta) - a\sin(\theta)) \left|0\right\rangle_b \left|\alpha\right\rangle_a = -\alpha\sin(\theta) \left|0\right\rangle_b \left|\alpha\right\rangle_a$$

we have that the state $B\left|0\right\rangle_b \left|\alpha\right\rangle_a$ is a simultaneous eigenstate of a with eigenvalue $\alpha\cos(\theta)$ and of b with eigenvalue $-\alpha\sin(\theta)$. This is enough to fully specify this state as the tensor product of two coherent states; we write

$$B\left|0\right\rangle_b \left|\alpha\right\rangle_a = \left|-\alpha\sin(\theta)\right\rangle_b \left|\alpha\cos(\theta)\right\rangle_a \tag{23}$$

as required.

c) There is a close connection between the Lie group $SU(2)$ and the algebra of two coupled harmonic oscillators, which is useful for understanding B. Let's define

$$s_z \equiv a^\dagger a - b^\dagger b$$

$$s_+ \equiv a^\dagger b$$

$$s_- \equiv ab^\dagger$$

and let $s_\pm = \frac{1}{2}(s_x \pm i s_y)$. *What is* $B(\theta)$ *in spin space? What is* $a^\dagger a + b^\dagger b$ *in spin space? Show that* s_x, s_y, s_z *have the same commutation relations as the Pauli matrices. This relationship also explains why* $a^\dagger a + b^\dagger b$ *is invariant: it is the Casimir operator of this algebra.*

We begin by writing

$$s_x = s_+ + s_- = a^\dagger b + b^\dagger a \tag{24}$$

$$s_y = -i(s_+ - s_-) = -i(a^\dagger b - b^\dagger a) \tag{25}$$

With Equation 25 in hand, we can write

$$B(\theta) = e^{\theta(a^\dagger b - b^\dagger a)} = e^{i\theta(-i(a^\dagger b - b^\dagger a))} = e^{i\theta s_y} \tag{26}$$

The form of Equation 26, which expresses B as the exponential of $i\theta$ times what will be shown to be an angular momentum operator, suggests that it can be interpreted as a rotation by θ about the y axis in spin space.[3]

Let us now turn our attention to $n_A + n_B$. It is useful to get some insight into what it might be *before* starting our manipulations, so let's review some known facts:

1. It is left unchanged by the beamsplitter operator B, which we know represents a rotation about the y axis in spin space.
2. It isn't just equal to s_y
3. If $n_A + n_B = N$, then the number of possible values of $s_z = n_A - n_B$ is equal to $N + 1$

Taken together, the first two points above suggest that $n_A + n_B$ is somehow related to S^2, the other obvious quantity besides s_y that is conserved by rotation about s_y. The third point suggests that, since the number of s_z values for fixed total angular momentum quantum number s is $2s + 1$, we may have the relationship $N = \frac{s}{2}$.

Before we get too carried away, we should verify that the $s_{x,y,z}$ commute in the same fashion as the Pauli matrices:

$$[s_x, s_y] = [a^\dagger b + b^\dagger a, -i(a^\dagger b - b^\dagger a)] = -i\left([a^\dagger b, -b^\dagger a] + [b^\dagger a, a^\dagger b]\right) = 2i[a^\dagger b, b^\dagger a] = 2i(a^\dagger b b^\dagger a - b^\dagger a a^\dagger b)$$

$$\begin{aligned}
&= 2i(a^\dagger a b b^\dagger - a a^\dagger b^\dagger b) \\
&= 2i(a^\dagger a(b^\dagger b + 1) - (a^\dagger a + 1) b^\dagger b) \\
&= 2i((a^\dagger a b^\dagger b + a^\dagger a - a^\dagger a b^\dagger b - b^\dagger b) \\
&= 2i(a^\dagger a - b^\dagger b) = 2i s_z
\end{aligned}$$

Also,

$$\begin{aligned}
[s_z, s_x] &= [a^\dagger a - b^\dagger b, a^\dagger b + b^\dagger a] \\
&= [a^\dagger a, a^\dagger] b + [a^\dagger a, a] b^\dagger - a^\dagger [b^\dagger b, b] - a [b^\dagger b, b^\dagger]
\end{aligned}$$

Now, note that

$$[a^\dagger a, a] = a^\dagger a a - a a^\dagger a = [a^\dagger, a] a = -a$$

[3] As it turns out, it's actually a rotation by 2θ, for reasons that'll become clear below.

$$[a^\dagger a, a^\dagger] = a^\dagger a a^\dagger - a^\dagger a^\dagger a = a^\dagger [a, a^\dagger] = a^\dagger$$

so that

$$[s_z, s_x] = a^\dagger b - a b^\dagger + a^\dagger b - a b^\dagger$$
$$= 2(a^\dagger b - a b^\dagger) = 2i(-i(a^\dagger b - b^\dagger a))$$

and hence

$$[s_z, s_x] = 2i s_y$$

Finally,

$$[s_y, s_z] = -i[a^\dagger b - b^\dagger a, a^\dagger a - b^\dagger b]$$
$$= -i\left([a^\dagger, a^\dagger a]b - a^\dagger [b, b^\dagger b] - [a, a^\dagger a]b^\dagger + a[b^\dagger, b^\dagger b]\right)$$
$$= -i\left(-a^\dagger b - a^\dagger b - a b^\dagger - a b^\dagger\right)$$
$$= 2i\left(a^\dagger b + b^\dagger a\right)$$

And hence

$$[s_y, s_z] = 2i s_x$$

and the required commutation relations are shown.

Aside:

The commutation relations we have just shown are true of the Pauli matrices, but of course there are analogous relationships for angular momentum operators in general:

$$[L_j, L_k] = i\hbar\epsilon_{jkl} L_l$$

Thus, if we divide out an $\hbar$, we find that the s_x etc. satisfy the angular momentum algebra's requirements up to a factor of 2; we should actually expect e.g. $\frac{s_x}{2}$ to behave like an angular momentum. $\square$

With this in hand, we're equipped to examine $n_A + n_B$. Let's evaluate

$$\frac{1}{4}s^2 = \frac{1}{4}\left(s_x^2 + s_y^2 + s_z^2\right)$$
$$= \frac{1}{4}\left((a^\dagger b + b^\dagger a)^2 - (a^\dagger b - b^\dagger a)^2 + (n_A - n_B)^2\right)$$
$$= \frac{1}{4}\left(2a^\dagger b b^\dagger a + 2b^\dagger a a^\dagger b + (n_A - n_B)^2\right)$$
$$= \frac{1}{4}\left(2a^\dagger a b^\dagger b + 2a^\dagger a + 2a^\dagger a b^\dagger b + 2b^\dagger b + (n_A - n_B)^2\right)$$
$$= \frac{1}{4}\left(4n_A n_B + 2n_A + 2n_B + (n_A - n_B)^2\right)$$
$$= \frac{1}{4}\left((n_A + n_B)^2 + 2(n_A + n_B)\right)$$
$$= \left(\frac{n_A + n_B}{2}\right)^2 + \frac{n_A + n_B}{2}$$

or, letting $N = n_A + n_B$,

$$= \left(\frac{N}{2} + 1\right)\frac{N}{2}$$

We then see that, when we interpret the s_x properly, $\frac{N}{2}$ represents the total angular momentum quantum number of a subspace, such that

$$\frac{1}{4}s^2 = j(j+1), \; j = \frac{N}{2}$$

d) How does a beamsplitter transform an input photon number state? Let $B = B(\frac{\pi}{4})$ be a 50/50 beamsplitter, such that

$$BaB^\dagger = \frac{1}{\sqrt{2}}(a+b)$$
$$BbB^\dagger = \frac{1}{\sqrt{2}}(b-a)$$

Compute $B|0\rangle|n\rangle$, where the first label is mode b and the second label is mode a. Note that the result is not $|\frac{n}{2}\rangle|\frac{n}{2}\rangle$, since n is a photon number eigenstate and not a coherent state. What photon number states have the largest amplitude? How sharp is the distribution for $n = 10$ and $n = 100$, or as a function of n, if a general solution exists?

In keeping with the hint, let us first recognize that

$$|0\rangle|n\rangle = \frac{1}{\sqrt{n!}}a^{\dagger n}|0\rangle|0\rangle$$

Now, we may not have much intuition about the operation of the beamsplitter operator on states, but the knowledge of how it acts on coherent states at least makes it easy to see that

$$B^\dagger|0\rangle|0\rangle = |0\rangle|0\rangle$$

We may now immediately see that

$$B|0\rangle|n\rangle = \frac{1}{\sqrt{n!}}Ba^{\dagger n}|0\rangle|0\rangle = \frac{1}{\sqrt{n!}}Ba^{\dagger n}B^\dagger|0\rangle|0\rangle$$

Or, inserting a whole lot of identities $I = B^\dagger B$ in clever places inside of the stack of $a^{\dagger n}$:

$$B|0\rangle|n\rangle = \frac{1}{\sqrt{n!}}(Ba^\dagger B^\dagger)^n|0\rangle|0\rangle$$
$$B|0\rangle|n\rangle = \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}(a^\dagger + b^\dagger)\right)^n|0\rangle|0\rangle$$

and now, using the binomial expansion,

$$B|0\rangle|n\rangle = \frac{1}{2^{\frac{n}{2}}\sqrt{n!}}\sum_{j=0}^{n}\binom{n}{j}a^{\dagger j}b^{\dagger(n-j)}|0\rangle|0\rangle$$

Then applying the creation operators and noting that

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

we obtain a simple form:

$$\boxed{B|0\rangle|n\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{j=0}^{n}\sqrt{\binom{n}{j}}|n-j\rangle|j\rangle} \tag{27}$$

Perhaps the probability of finding j photons in mode a after the beamsplitter is applied is even more informative:

$$\boxed{P_j = |\langle j, n-j|B|0,n\rangle|^2 = 2^{-n}\binom{n}{j}} \tag{28}$$

That is, the photon number in modes a and b is exactly binomially distributed; in statistics language, we would say that the number of photons in mode a is a binomial distribution with n trials and probability $p = \frac{1}{2} = q$ of success (or failure). The photons in b are likewise binomially distributed, but please note that the two distributions are perfectly correlated; one always has that $n_A + n_B = n$.

It is well known from statistics that the mean of a binomial distribution is given by $np = \frac{n}{2}$ here; where n is even, this will be the peak of our discrete distribution. It is also known that the variance of a binomial distribution is given by $\sigma^2 = npq$, so here we have that the width of the peak in the distribution of n_A is given by $\sigma = \sqrt{\frac{n}{4}} = \frac{\sqrt{n}}{2}$. In particular, for $n = 10$, the width is $\frac{\sqrt{10}}{2} \approx 1.6$, while for $n = 100$ the width is $\frac{\sqrt{100}}{2} = 5$.

Aside:
A word of caution: this binomial distribution that we have calculated is compatible with a mental picture where each of the n photons goes into the beamsplitter and, independently of all the others, makes a decision to either go into mode a or b. This picture is useful here, but it should not be taken too far; in particular, we shouldn't imagine that sending n photons into *each* mode of the beamsplitter would give an identical output distribution to sending $2n$ photons into mode a. See, for example, the well-known Hong-Au-Mandel effect, where sending a single photon into each arm guarantees that the two total photons come out in the same mode. □

4 Problem 4

Let $a^\dagger$, a and $b^\dagger, b$ be the creation and annihilation operators for two modes of light input to a beamsplitter, and let the unitary transformation performed by the beamsplitter be

$$a_1 = UaU^\dagger = \frac{a+b}{\sqrt{2}}$$

$$b_1 = UbU^\dagger = \frac{a-b}{\sqrt{2}}$$

For light input in state $|\psi\rangle$, the output of a coincidence circuit for the HBT experiment is

$$V_\psi = V_0 \langle \psi, 0|a_1^\dagger a_1 b_1^\dagger b_1|\psi, 0\rangle$$

where V_0 is some proportionality constant and $|\psi, 0\rangle$ denotes a state with $|\psi\rangle$ in mode a and $|0\rangle$ in mode b. Show that V_ψ gives a measure of $g^{(2)}(\tau)$,

$$g^{(2)}(\tau) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2}$$

up to an additive offset and normalization.

This is straightforward enough. Let us express our V_ψ in terms of the original operators:

$$V_\psi = V_0 \langle \psi, 0|a_1^\dagger a_1 b_1^\dagger b_1|\psi, 0\rangle$$

$$= \frac{V_0}{4} \langle \psi, 0|(a^\dagger + b^\dagger)(a+b)(a^\dagger - b^\dagger)(a-b)|\psi, 0\rangle$$

$$= \frac{V_0}{4} \langle \psi, 0| a^\dagger (a+b)(a^\dagger - b^\dagger) a |\psi, 0\rangle$$

where we have exploited the fact that mode b is just the vacuum. Proceeding,

$$= \frac{V_0}{4} \langle \psi, 0| a^\dagger \left(aa^\dagger - ab^\dagger + ba^\dagger - bb^\dagger \right) a |\psi, 0\rangle$$

Again leveraging the fact that mode b is in a vacuum state, and also that a and b commute, we may write

$$= \frac{V_0}{4} \langle \psi, 0| a^\dagger \left(aa^\dagger - bb^\dagger \right) a |\psi, 0\rangle$$

$$= \frac{V_0}{4} \langle \psi, 0| a^\dagger \left(a^\dagger a - b^\dagger b \right) a \rangle$$

$$= \frac{V_0}{4} \langle \psi, 0| a^\dagger a^\dagger a a - n_B |\psi, 0\rangle$$

$$\boxed{V_\psi = \frac{V_0}{4} \langle \psi, 0| a^\dagger a^\dagger a a |\psi, 0\rangle} \tag{29}$$

Accordingly, we find that V_ψ is proportional to the numerator of $g^{(2)}$; if we were to divide it by the total intensity squared, $I^2 = \langle a^\dagger a \rangle^2$, we would obtain $g^{(2)}$, as desired.