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Problem set: #9

Due: Friday, April 15, 2022.

## 1. Transition Lifetimes and Blackbody Radiation.

(a) (i) The rate of *absorption* is given by the product of Einstein's *B* coefficient and the average number of photons per mode  $\langle n \rangle_{\omega_0}$  where  $\omega_0$  is angular frequency associated with the (dominant) transition with  $\lambda_0 = 590$  nm.

$$W = B\langle n \rangle \hbar \omega_0 \rho(\omega_0) = B\langle n \rangle \hbar \omega_0 \frac{\omega_0^2}{\pi^2 c^3} = 1/60 \text{ s}^{-1}.$$

where  $\rho(\omega_0)$  is the density of states. From lecture, we know that the Einstein's *B* coefficient can be written in terms of the (known) Einstein's  $A = \Gamma_0 = 1/\tau$  coefficient:

$$B = \frac{\pi^2 c^3}{\hbar \omega_0^3} A = \frac{\pi^2 c^3}{\hbar \omega_0^3} \frac{1}{\tau}$$

With this, we can plug in the numbers to find

$$\langle n \rangle = W \tau \approx 2.67 \times 10^{-10}$$
.

(ii) For blackbody radiation, we have

$$\langle n \rangle = \frac{1}{e^{\hbar \omega_0/k_BT} - 1} \implies T \approx 1100 \text{ K}.$$

We see that in order for the absorption rate to reach 1 photon per minute, the blackbody temperature has to be  $\sim 1100$  K, which is way above room temperature (of course the higher the temperature, the higher the absorption rate, and vice versa). As a result, there is no need to shield the vacuum system from room temperature radiation or cool the vacuum system to cryogenic temperatures (as expected).

- (b) Here we estimate the lifetime of hydrogen in the F = 1 hyperfine level of the 1S state.
  - (i)  $F = 1 \rightarrow F = 0$  is a magnetic dipole transition.
  - (ii) To estimate the lifetime of the F = 1 state, we may assume that the (magnetic dipole) transition matrix element is  $\mu_B$ . From here, we work entirely in the CGS unit system to find

$$\Gamma_0 = \frac{4\omega_0^3 \mu_B^2}{3\hbar c^3} \approx 2.91 \times 10^{-15} \text{ s}^{-1}$$

where the numerical values for the fundamental constants can be found on Wikipedia. The lifetime is

$$\tau = \frac{1}{\Gamma_0} \approx 1.1 \times 10^7 \text{ years.}$$

- (c) Now we look at a hydrogen BEC in the F = 1 state.
  - (i) To find the average number of photons per mode from blackbody radiation at the 21 cm line at 300 K and 4 K, we simply calculate

$$\langle n \rangle = \frac{1}{e^{\hbar \omega / k_B T} - 1}$$

at the corresponding temperatures and angular frequency. The answers are

$$T = 300 \text{ K}, \qquad \langle n \rangle \approx 4375$$

$$T = 4 \text{ K}, \qquad \langle n \rangle \approx 57.84.$$

(ii) Similar to what we did before (but reversed), we can find the transition rates at T=300 K and T=4 K. We will also need the lifetime  $\tau \approx 1.1 \times 10^7$  years for this calculation.

$$T = 300 \text{ K},$$
  $W = \langle n \rangle / \tau \approx 1.27 \times 10^{-11} \text{ s}^{-1}$   
 $T = 4 \text{ K},$   $W = \langle n \rangle / \tau \approx 1.68 \times 10^{-13} \text{ s}^{-1}$ 

- (iii) It is clear that we should not be concerned about blackbody radiation from the environment limiting our experiment with hydrogen in the F = 1 state if we need a trapping times on the order of seconds/minutes.
- (d) The lifetime is much longer for the case of a magnetic dipole transition in hydrogen where there are more photons per mode mainly because the lifetime  $\tau$  scales as  $1/\omega_0^3$ . The ratio between the sodium wavelength of 590 nm versus the 21 cm of hydrogen is  $\sim 10^{-6}$ . This gives a reduction factor of  $10^{-18}$ . On top of this, there is also another factor of  $\alpha^2$  reduction when replacing the electric with magnetic dipole matrix element.

## 2. Saturation Intensity.

(a) We first manipulate the Einstein A coefficient so that it is written in terms of the oscillator strength f, the fine structure constant  $\alpha$  and the transition frequency  $\omega$ . The oscillator strength is given by

$$f_{21} = \frac{2m\omega_{21}}{3\hbar} \frac{1}{2J_1 + 1} \sum_{m_1, m_2} |\langle J_1 m_1 | \vec{r} | J_2 m_2 \rangle|^2 = \frac{2m\omega_{21}}{3\hbar} \frac{S_{12}}{2J_1 + 1}.$$

where  $S_{12}$  is the line strength, while

$$A_{12} = \frac{4\omega^3 e^2}{3\hbar c^3} \sum_{m_2} |\langle 1m_1 | \vec{r} | 2m_2 \rangle|^2 = \frac{4\omega^3 e^2}{3\hbar c^3} \frac{S_{12}}{2J_1 + 1}.$$

From here we have that

$$A_{12} = \frac{4e^2\omega_{21}^3}{3\hbar c^3} \frac{3\hbar f_{21}}{2m\omega_{21}} = \frac{2e^2\omega_{21}^2 f_{21}}{mc^3} = \frac{2\hbar\omega_{21}^2 f_{21}}{mc^2} \frac{e^2}{\hbar c} = \frac{2\alpha\hbar\omega_{21}^2 f_{21}}{mc^2}.$$

Assuming  $f_{21} = 1$  we may use this formula to estimate the lifetime of sodium. Plugging in the numerical values for the constants above (in CGS units), we find

$$A \approx 1.917 \times 10^8 \,\text{s}^{-1} \implies \tau = \frac{1}{A} \approx 5.2 \,\text{ns}.$$

More precisely, if we call *f* the *absorption* oscillator strength, then we actually have

$$A = \frac{2\alpha\hbar\omega^2}{mc^2} \frac{2J_1 + 1}{2J_2 + 1} f.$$

Assuming that we're working the D lines of sodium, we will take  $J_1 = 1/2$  and  $J_2 = 1/2$  and 3/2. From Steck's, we find that f = 0.64 for the  $3^2S_{1/2} \rightarrow 3^2P_{3/2}$  transition and f = 0.32 for the  $3^2S_{1/2} \rightarrow 3^2P_{1/2}$  transition. Plugging in the numbers we find that

$$\tau_{1/2 \to 3/2} \approx \tau_{1/2 \to 3/2} \approx 16 \text{ ns},$$

as expected.

(b) The saturation intensity for the principal transition in sodium, with  $\sigma_0 = 3\lambda^2/2\pi$ , is:

$$I_{\rm sat} = \frac{\hbar \omega A}{2\sigma_0} \approx 63.3 \, {\rm W/m^2}.$$

where we are using A from the previous part and ignoring fine and hyperfine structure.

## 3. Saturation of Atomic Transitions.

(a) Here we consider a two-state atom with  $R_{ge} = R_{eg} = R$  the stimulated absorption/emission rate and  $A = \Gamma$  the spontaneous emission rate. Define the saturation parameter s as  $s = 2R/\Gamma$ . In equilibrium, we have

$$\Gamma N_b + R N_b - R N_a = 0 \implies \frac{N_b}{N_a} = \frac{R}{R+\Gamma} = \frac{s\Gamma/2}{s\Gamma/2+\Gamma} = \frac{s}{s+2}.$$

(b) The equilibrium spontaneous emission rate per atom  $AN_b$  can be expressed in terms of  $\Gamma$  and s and the total density  $N = N_a + N_b = 1$ . From part (a), we have

$$\frac{N_b}{N} = \frac{N_b}{N_b + N_a} = \frac{N_b/N_a}{N_b/N_a + 1} = \frac{s}{s + (s + 2)} = \frac{1}{2} \frac{s}{s + 1} = N_b.$$

So, we find

$$\Gamma N_b = \frac{1}{2} \frac{\Gamma s}{s+1}.$$

Recalling the optical Bloch equations, we know that

$$N_b = \frac{\Omega^2/\Gamma^2}{1 + 2\Omega^2/\Gamma^2}.$$

Thus we may identify s with  $2\Omega^2/\Gamma^2$ . The scattering cross section has the form

$$\sigma = \frac{\sigma_0}{1 + I/I_{\rm sat}} = \frac{\sigma_0}{1 + 2\Omega^2/\Gamma^2}.$$

where we have ignored the (small) detuning term and used the definition of  $I_{\text{sat}}$ . We can immediately see that  $\sigma(s)$  bleaches out as  $\sigma(s=0)/(1+s)$ , as desired.

(c) For s = 1, the energy density  $\langle w \rangle_{SAT}$  per unit frequency is given by

$$\langle w \rangle = \frac{R}{B},$$

where *B* is the Einstein *B* coefficient. With s = 1, we find that

$$s = 1 = \frac{2R}{\Gamma} \implies R = \frac{\Gamma}{2} = B\langle w \rangle \implies \langle w \rangle = \frac{\Gamma}{2R}.$$

 $\langle w \rangle$  is independent of the atomic dipole matrix element since both the  $A = \Gamma$  and B Einstein coefficients are proportional to the matrix element (modulus) squared.

(d) With

$$\frac{\Gamma}{B} = \frac{8\pi\hbar}{\lambda^3} = \frac{\hbar\omega^3}{\pi^2c^3},$$

we find

$$\langle w \rangle_{\text{SAT}} = \frac{\hbar \omega^3}{2\pi^2 c^3}.$$

The mean occupation number n per photon mode for s = 1 is given by

$$\langle n \rangle = \frac{B \langle w \rangle_{\text{SAT}}}{\Gamma} = \frac{B}{\Gamma} \frac{\Gamma}{2B} = \frac{1}{2}.$$

(e) We have laser light of intensity  $I_0$  and Lorentzian lineshape centered at the atomic transition frequency  $\omega_0$  with FWHM  $\Gamma' \gg \Gamma$ . The energy density of this beam per frequency interval at  $\omega_0$  is

$$S(\omega_0) = \frac{I(\omega_0)}{c} = I_0 \frac{1}{\pi c} \frac{\Gamma'/2}{(\omega_0 - \omega_0)^2 + (\Gamma'/2)^2} = \frac{2I_0}{\pi c \Gamma'}.$$

We now want  $I_s$  such that

$$s=1=\frac{2R}{\Gamma}=\frac{2B\langle w\rangle}{\Gamma}=2S(\omega_0)\frac{B}{\Gamma}=\frac{4I_s}{\pi c\Gamma'}\frac{\pi^2c^3}{\hbar\omega^3}=\frac{4\pi c^2}{\Gamma'\hbar\omega^3}I_s\implies I_s=\frac{\Gamma'\hbar\omega^3}{4\pi c^2}.$$

(f) The stimulated broadband absorption rate is given by

$$R = B\langle w \rangle = B \frac{2I_0}{\pi c \Gamma'} = \frac{2I_0}{\pi c \Gamma'} \frac{\Gamma \pi^2 c^3}{\hbar \omega^3} = \frac{2\pi \Gamma c^2 I_0}{\hbar \omega^3 \Gamma'} = \frac{\omega_R^2}{\Gamma'},$$

as desired. Here, we have used the result seen on the last problem set:

$$\omega_R^2 = \frac{2\pi\Gamma c^2 I_0}{\hbar\omega^3}.$$

For s = 1, we have  $I_0 = I_s$ , so

$$\omega_R^2 = \frac{2\pi\Gamma c^2}{\hbar\omega^3} \frac{\Gamma'\hbar\omega^3}{4\pi c^2} = \frac{\Gamma\Gamma'}{2}.$$

(g) If we set  $\Gamma' = \Gamma$ , then we get exactly the saturation intensity of a monochromatic laser beam and the Rabi frequency at saturation. This is not surprising since when setting  $\Gamma' = \Gamma$  we are no longer in the broadband regime.