MA355 Midterm 2

Colby College — Spring 2021

due Wednesday, April 28, by 8:30 EST

Please upload your solutions (preferably as a single PDF file) to Moodle.

Explain/justify all answers!

Problems

1. (25 points)
   (a) (15 points) A mathematics department has n identical combinatorics books, r identical algebra books, and t identical number theory books. In how many ways can these books be arranged on 3 distinct bookshelves?
   (b) In how many ways can we place k distinct books on n distinct shelves so that each shelf gets at least 3 books?

2. (25 points) Give a recurrence relation for the number of ways to divide $4n$ people into sets of four for games of bridge. (Don't worry about how they sit around the bridge table or who is the first dealer).

3. (30 points) (15 points for parts a-c, 15 points for part d.)
   (a) A *forest* is a set of trees, i.e. a graph in which each connected component is a tree. Let F be a forest on n vertices with k connected components. How many edges does F have?
   (b) A graph is *directed* if each of its edges is assigned a direction. A tree is *rooted* if one of its vertices is designated as the root. A *directed rooted tree* is a rooted tree whose edges are directed away from the root.
      1. Given a rooted tree, prove that there is a unique way to assign a direction to each edge so as to turn the tree into a directed rooted tree.
      2. Given a tree, how many directed rooted trees are there with the same set of vertices and edges?
   (c) A directed rooted forest is a forest each of whose components is a directed rooted tree. A directed rooted forest F is said to contain another directed rooted forest F' if F contains F' as a directed graph. Show that if F and F' are directed rooted forests on n vertices and F contains F' then the number of components of F is smaller than the number of components of F'.

(d) We say that $F_1, F_2, \ldots, F_k$ is a *refining sequence* if, for all $i \in \{1, 2, \ldots, k\}$, F_i is a directed rooted forest with vertex set $\{1, 2, \ldots, n\}$ having i components, and F_i contains F_{i+1}. Now fix F_k.

1. Find the number $N^*(F_k)$ of refining sequences ending in F_k.
2. Find the number $N(F_k)$ of directed rooted trees containing F_k.
3. How many trees with vertex set $\{1, 2, \ldots, n\}$ are there?

4. (20 points)

A hungry combinatorics professor is preparing for a long drive and decides to take n sandwiches with her. Although she has only roast beef available to put inside her sandwich, she has k different spreads she can put on the two sides of each sandwich. In how many ways can she make the sandwiches if she wants to have at least one of each variety of sandwich?