Assignment 5; MA353; Term: S19

Leo Livshits

Last modified at 15:25 on April 10, 2019

Problem 1

Suppose that $\mathfrak{F}$ is a commutative collection of linear operators on a (not necessarily finite-dimensional) vector space $\boldsymbol{V}$. Suppose that λ is an eigenvalue for some $\mathcal{A} \in \mathfrak{F}$, and let $\boldsymbol{E}_{\mathcal{A}}(\lambda)$ be the corresponding eigenspace of $\mathcal{A}$. Argue that this $\boldsymbol{E}_{\mathcal{A}}(\lambda)$ is an invariant subspace for every operator in $\mathfrak{F}$.

Fact 0.1 ☞ **All subspaces are complemented**

For every subspace $\boldsymbol{Z}$ of a (not necessarily finite-dimensional) vector space $\boldsymbol{V}$, there exists a subspace $\boldsymbol{W}$ of $\boldsymbol{V}$ such that

$$\boldsymbol{V} = \boldsymbol{Z} \dotplus \boldsymbol{W} .$$

Problem 2

Let us use the same set-up as in Problem 1, and let $\boldsymbol{W}$ be a subspace of $\boldsymbol{V}$ such that

$$\boldsymbol{V} = \boldsymbol{E}_{\mathcal{A}}(\lambda) \dotplus \boldsymbol{W} .$$

Argue that with respect to this decomposition, operators in $\mathfrak{F}$ have block matrix representation of the form

$$\begin{bmatrix} \mathcal{L} & \mathcal{M} \\ \mathcal{O} & \mathcal{K} \end{bmatrix},$$

where the $\mathcal{L}$'s form a commutative family in $\mathfrak{L}\left(\boldsymbol{E}_{\mathcal{A}}(\lambda), \boldsymbol{E}_{\mathcal{A}}(\lambda)\right)$, and the $\mathcal{K}$'s form a commutative family in $\mathfrak{L}(\boldsymbol{W}, \boldsymbol{W})$.

Problem 3 ☞ **Commuting collections of complex matrices are simultaneously △-izable**

Argue that every commutative family of operators on a finite-dimensional vector space over the complex numbers is simultaneously upper-triangularizable, and simultaneously lower-triangularizable.

Definition 0.2 ☞ **Conjugate transpose**

For every matrix $\mathcal{A} \in \mathbb{M}_k(\mathbb{C})$, let us write $\overline{\mathcal{A}}$ for the $k \times k$ matrix obtained by conjugating all entries of $\mathcal{A}$; in other words

$$\overline{\mathcal{A}}[\![i,j]\!] = \overline{\mathcal{A}[\![i,j]\!]}.$$

We shall write $\mathcal{A}^*$ for the matrix $\left(\overline{\mathcal{A}}\right)^T$. The reader should check that

$$\mathcal{A}^* = \overline{\mathcal{A}^T}\ .$$

Matrix $\mathcal{A}^*$ is said to be the **conjugate transpose** of $\mathcal{A}$, or the **adjoint** of $\mathcal{A}$.

Problem 4 ☞ **Transpose of a square matrix is similar to that matrix**

Use Jordan Cannonical Form Theorem to argue that the transpose $\mathcal{J}_{\lambda,n}^T$ of a Jordan block $\mathcal{J}_{\lambda,n}$ is similar to $\mathcal{J}_{\lambda,n}$, and then use this fact to argue that every $k \times k$ complex matrix is similar to its transpose.

Definition 0.3 ☞ **Hilbert-Schmidt / Frobenius norm**

For each matrix $\mathcal{A} \in \mathbb{M}_{n\times m}$ let $\left\|\mathcal{A}\right\|_2$ stand for the Euclidean norm of $\mathcal{A}$ interpreted as a folded element of $\mathbb{C}^{nm}$. In other words,

$$\left\|\mathcal{A}\right\|_2 := \sqrt{\sum_{i,j} (\mathcal{A}[\![i,j]\!])^2}.$$

$\left\|\mathcal{A}\right\|_2$ is said to be a **Hilbert-Schmidt norm** or a **Frobenius norm** of $\mathcal{A}$.

Test Your Comprehension 0.4

Argue that

$$\left\|\mathcal{A}\right\|_2 = \sqrt{Trace\,(\mathcal{A}^*\mathcal{A})} = \sqrt{Trace\,(\mathcal{A}\mathcal{A}^*)}\ .$$

Fact 0.5 ☞ **Some fundamental limits**

For any $\alpha > 1$,

$$\lim_{n\to\infty} \frac{n^{\text{fixed power}}}{\alpha^n} = 0\ .$$

For example,

$$\lim_{n\to\infty} \frac{n^{10357}}{\alpha^n} = 0\ ,$$

and

$$\lim_{n\to\infty} \frac{n^{10357\pi}}{\alpha^n} = 0\ .$$

Problem 5

1. Argue that

$$\lim_{n\to\infty} \left(5\alpha^n + 4n\alpha^{n-1} + 3\tfrac{n(n-1)}{2!}\alpha^{n-2} + 2\tfrac{n(n-1)(n-2)}{3!}\alpha^{n-3} + \tfrac{n(n-1)(n-2)(n-3)}{4!}\alpha^{n-4}\right) = \begin{cases} 0, & \text{if } 0 \le \alpha < 1 \\ \infty, & \text{if } \alpha \ge 1 \end{cases}\ .$$

2. Argue that for $m \geq 2$

$$\lim_{n\to\infty} \left\| \left(\mathcal{J}_{\lambda,m}\right)^n \right\|_2 = \begin{cases} 0, & \text{if } |\lambda| < 1 \\ \infty, & \text{if } |\lambda| \geq 1 \end{cases} .$$

Note that $\mathcal{J}_{\lambda,m} = \lambda\mathcal{I} + \mathcal{N}$, where $\mathcal{N}$ is a nice cyclic nilpotent of order m, so that

$$\left(\mathcal{J}_{\lambda,m}\right)^n = \left(\lambda\mathcal{I} + \mathcal{N}\right)^n = \lambda^n\mathcal{I} + \binom{n}{1}\lambda^{n-1}\mathcal{N} + \binom{n}{2}\lambda^{n-2}\mathcal{N}^2 + \cdots .$$

You may want to start with small m first, and calculate some $\left(\mathcal{J}_{\lambda,m}\right)^n$'s using *Mathematica* . . .

Problem 6

1. Use logarithms and L'Hopital's Rule to argue that for any $\alpha > 0$,

$$\lim_{x\to\infty} \left(5+4x\alpha^{-1}+3\frac{x(x-1)}{2!}\alpha^{-2}+2\frac{x(x-1)(x-2)}{3!}\alpha^{-3}+\frac{x(x-1)(x-2)(x-3)}{4!}\alpha^{-4}\right)^{\frac{1}{x}}=1 .$$

2. Argue that

$$\lim_{n\longrightarrow\infty} \left(\left\| \left(\mathcal{J}_{\lambda,m}\right)^n \right\|_2\right)^{\frac{1}{n}} = |\lambda| .$$

Extra Credit Problem 1

1. Suppose that $[x_n], [y_n], [z_n], [u_n]$ are sequences of positive numbers such that

$$\begin{aligned} \left[(x_n)^{\frac{1}{n}} \right] &\longrightarrow \alpha \\ \left[(y_n)^{\frac{1}{n}} \right] &\longrightarrow \beta \\ \left[(z_n)^{\frac{1}{n}} \right] &\longrightarrow \gamma \\ \left[(u_n)^{\frac{1}{n}} \right] &\longrightarrow \delta . \end{aligned}$$

   (a) Evaluate the limit of

$$\left[(\alpha^n + \beta^n + \gamma^n + \delta^n)^{\frac{1}{n}} \right] .$$

(b) Evaluate the limit of

$$\left[(x_n + y_n + z_n + u_n)^{\frac{1}{n}} \right].$$

2. Suppose that $\mathcal{A} = \mathcal{J}_{\lambda_1, m_1} \oplus \mathcal{J}_{\lambda_2, m_2} \oplus \cdots \oplus \mathcal{J}_{\lambda_{23}, m_{23}}$. Evaluate the limit

$$\lim_{n \longrightarrow \infty} \left(\left\| \mathcal{A}^n \right\|_2 \right)^{\frac{1}{n}}.$$