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Massive Gravity Theories

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Abstract

We examine Fierz and Pauli's work in 1939 of adding a mass term to the Lagrangian for linearized gravity with the form $-\frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} + \lambda h^2)$ where $h_{\mu\nu}$ is a small perturbation away from flat spacetime and showing that the only value for λ which gives a ghost-free theory with the correct five degrees of freedom expected for a massive spin-2 particle such as the graviton is $\lambda = -1$ [1]. We start by rederiving the Lagrangian formulation for classical electrodynamics and giving examples of a ghost. Then we solve the linearized massless gravity equations to show their two degrees of freedom. We perform a Hamiltonian analysis for massless linearized gravity. Then we add the mass term to the Lagrangian, resolve the equations, find five degrees of freedom, and then perform a Hamiltonian analysis for massive gravity. We use the equations of motion and the Hamiltonian to show that $\lambda = -1$ is the only value for λ which gives a physical, ghost-free theory.

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Notes on notation

We use a Minkowski metric of the form

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{0.1}$$

We also use non-standard Einstein summation notation in parts of our analysis. We write terms such as

$$(h_{jj,0})^2. \tag{0.2}$$

Single indices in a squared term are meant to be summed over after expanding the entire expression, and repeated indices in a squared term are meant to be summed over once in each factor. So when we write $(h_{jj,0})^2$, we really mean

$$(h_{jj,0})^2 = -h^j{}_{j,0} h_k{}^{k,0} \tag{0.3}$$

with the standard convention of letters from the middle of the alphabet running from 1 to 3.

Except when it is clearer to do otherwise, we use the comma notation for derivatives. For 4-dimensional spacetime coordinates x^μ, the following notations are equivalent:

$$\frac{\partial}{\partial x^\mu} A_\nu = \partial_\mu A_\nu = A_{\nu,\mu}. \tag{0.4}$$

We use the standard convention of Greek letters running from 0 to 3.

Contents

1 Introduction

General relativity was developed at the beginning of the 20th century by Albert Einstein. It gives an accurate model for describing macroscopic gravitational interactions. However, general relativity is inconsistent with quantum mechanics and it is expected that the two will be combined into a new quantum theory of gravity. Additionally, cosmological models developed with general relativity, while accurate, rely on roughly 70% of the universe being made up of dark energy. Dark energy is required to make the models consistent with the accelerating expansion of the universe, but adding a mass to the graviton - a particle traditionally considered massless - could also explain this acceleration and potentially lead to an alternative to dark energy. Giving the graviton a small mass would weaken the effect of gravity at large distances and give similiar effects as acceleration in the context of conventional general relativity.

Massive gravity theories have a checkered past [13]. In 1939, Wolfgang Pauli and Markus Fierz experimented with adding mass to a linearized theory of gravity. They were able to find only a single way to add a mass term that created a physically correct theory [1]. Uniqueness in a theory is curious, and their result suggested that massive gravity theories warranted more investigation.

In 1970, van Dam, Veltman, and Zakharov claimed that the linear theory that Fierz and Pauli used could never agree with Einstein's original formulation of gravity. One important behavior of massive gravity is that as the mass of the graviton goes to zero, we should recover Einstein's original massless theory. The van Dam-Veltman-Zakharov, or the 'vDVZ', discontinuity was a supposed proof that linearized massive gravity lacks this behavior [2][3].

Two years later, in 1972, Arkady Vainshtein claimed to overturn the vDVZ discontinuity. He examined the full non-linear formulation of massive gravity and found that small masses for the graviton move the theory back towards Einstein's original massless gravity. The effect of the non-linearities - which Fierz and Pauli had neglected - maintains the required behavior of massive gravity acting like massless gravity for small masses [4]. So if the mass of the graviton is small, we simply may not have observed it yet.

In the same year, Boulware and Deser proved that any non-linear theory of massive gravity - any theory which uses Vainshtein's fix for the vDVZ discontinuity - must contain a 'ghost' [5]. A 'ghost' is any sort of internal inconsistency in the theory, such as a negative energy solution or a tachyonic solution, which renders the theory unphysical. The 'Boulware-Deser ghost' nullified Vainshtein's findings and killed all interest in massive gravity until the early 2000s.

There has been a flurry of activity in the last ten years. The Dvali-Gabadadze-Porrati model, or DGP model, is a proposed 5-dimensional theory of the universe with four regular dimensions and one infinite dimension. Researchers realized that the part of the theory involving the four standard dimensions mirrors the proposed massive gravity theory but lacks a ghost. This was encouraging, since it suggested that physical massive gravity is somehow feasible. In 2009, Claudia de Rham and Gregory Gabadadze demonstrated a way of constructing a massive gravity theory without the Boulware-Deser ghost, although it required a fifth dimension [6]. In 2010, de Rham, Gabadadze, and Tolley presented a four-dimensional massive gravity theory with no ghosts, and in 2011, Rachel Rosen and Fawad Hassan proved that the de Rham-Gabadadze-Tolley four dimensional massive model never has ghosts [7][8].

In this thesis, we attempt to prove more rigorously the results stated by Fierz and Pauli in 1939 [1]. We start in Section 2 with background information on Lagrangian and Hamiltonian formulations and the extension of these formulations to fields. Then we use classical electromagnetism as an example of varying the Lagrangian, obtaining and solving the equations of motion, and calculating the Hamiltonian. We also look at the Proca Lagrangian since it is an example of adding a mass term to an otherwise massless particle. In Section 3, we explain the idea of ghosts and give an example of one. In Section 4, we work through the equations of motion and the Hamiltonian for massless and massive linearized gravity, and in Section 5 we discuss the implications of the results obtained for massive gravity.

2 Field theory background

2.1 Lagrangian formulation

We start with an explanation of classical Lagrangian mechanics [10]. For a particular system, we define a quantity called the Lagrangian, denoted by L. For a classical system, the Lagrangian is defined as

$$L = T - V \tag{2.1}$$

where T is the kinetic energy of the system and V is the potential energy of the system. In general, we consider the Lagrangian to be a function of q_i, $\dot{q}_i$, and t, where i runs from $i = 1$ to $i = N$, $q_i = q_i(t)$ is one of N generalized coordinates describing the system, such as a position variable for a particle, $\dot{q}_i(t)$ is the first time derivative of the generalized coordinate q_i, and t is time.

Systems follow paths which make the difference between the average kinetic energy and the average potential energy 'stationary'. We can model this mathematically by introducing the action of a system, 'S', as

$$S = \int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2} (T - V) dt. \tag{2.2}$$

The action being 'stationary' means that a first order variation in the action does not affect the action. To go about finding the path, we assume that S is stationary for some L. If we then make a small, first order variation in L, by taking q_i slightly away from its value for the stationary path, this variation must make zero contribution to the action. We can allow second order and higher variations to affect the action because this is still consistent with the action being stationary for the path of the particle.

Here we show how to use the principle of stationary action to find the path of a system [11]. For a given action, the requirement of being stationary amounts to requiring that

$$\delta S = \delta \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \delta L = 0. \tag{2.3}$$

where the δ operator signifies a small, first order variation.

If we vary the generalized coordinates q_i of the Lagrangian by a small amount δq_i, we find that

$$\int_{t_1}^{t_2} L(q_i + \delta q_i, \frac{d}{dt}(q_i + \delta q_i))dt \tag{2.4}$$

$$= \int_{t_1}^{t_2} L(q_i, \dot{q}_i) + \delta L(q_i, \dot{q}_i)dt \tag{2.5}$$

$$= S + \delta S \tag{2.6}$$

Using the above condition shows that

$$0 = \delta S \tag{2.7}$$

$$= \int_{t_1}^{t_2} \delta L(q_i, \dot{q}_i)\, dt \tag{2.8}$$

$$= \int_{t_1}^{t_2} (\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i)\, dt \tag{2.9}$$

At this point we can integrate the second term by parts, yielding

$$= \int_{t_1}^{t_2} (\frac{\partial L}{\partial q_i}\delta q_i - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i)\, dt + [\frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i]_{t_1}^{t_2}. \tag{2.10}$$

Since we only consider variations away from paths starting and ending at the same points - this method is meaningless without this assumption - we know that $\delta q_i(t_1) = \delta q_i(t_2) = 0$, so the second term in the above equation vanishes.

Factoring out the δq_i gives

$$= \int_{t_1}^{t_2} (\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i})\delta q_i\, dt \tag{2.11}$$

which must equal zero. Since $\delta q_i \neq 0$ for all t, the terms inside the parentheses must be zero. So we find that the condition for the action to be stationary and thus the equation which gives a differential equation involving the generalized coordinates, the equation of motion, is

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \tag{2.12}$$

We get one equation for each generalized coordinate q_i, for a total of N 2^{nd} order equations of motion. These are called the Euler-Lagrange equations.

2.2 Legendre transform

An alternative to the Lagrangian formulation is called the Hamiltonian formulation. The Hamiltonian formulation is mathematically equivalent to the Lagrangian formulation. With the Lagrangian, we get N 2^{nd} order equations of motion; with the Hamiltonian, we get 2N 1^{st} order equations. The Hamiltonian uses the N generalized coordinates for a system and its N conjugate momenta. We get the Hamiltonian from the Lagrangian by using a technique called the Legendre transform. We first explain the Legendre transform for arbitrary functions.

The Legendre transform for a function $f(x, y)$ with respect to x converts $f(x, y)$ to a function $g(u, y)$ where $u = \frac{\partial f}{\partial x}$. We define $g(u, x, y)$ to be

$$g(u, x, y) = ux - f(x, y) \tag{2.13}$$

We can then write down the total derivative of $g(u, x, y)$,

$$dg = x\,du + u\,dx - \frac{\partial f}{\partial x}\,dx - \frac{\partial f}{\partial y}\,dy \tag{2.14}$$

Since u is arbitrary, we can define it to be whatever we like. Suppose we define it as

$$u = \frac{\partial f}{\partial x} = u(x, y) \tag{2.15}$$

Since f is a function of x and y, $\frac{\partial f}{\partial x}$ must be a function of x and y as well. With u defined as above, the new total derivative of $g = g(u(x, y), y)$ is

$$dg = x\,du - \frac{\partial f}{\partial y}\,dy \tag{2.16}$$

We see from 2.15 that we can write $x = x(u, y)$ by inverting the relationship. Then we see from 2.16 that g can be thought of as a function of u and y. So we can then pick out that

$$-x(u, y) = \frac{\partial g}{\partial u} \tag{2.17}$$

and

$$-v(u,y) = \frac{\partial g}{\partial y} = -\frac{\partial f}{\partial y} \tag{2.18}$$

Note the second equation involving $v = v(u,y)$ has u and y as dependent variables since $\frac{\partial f}{\partial y}$ is a function of $x = x(u,y)$. Including all the reconsidered functional dependencies for $g = g(u(x,y),y)$ gives the final equation for g as

$$g(u,y) = u\,x(u,y) - f(u(x,y),y) \tag{2.19}$$

Recall that we had originally defined $u(x,y)$ to be an arbitrary function - the function g gives the distance between the functions ux and $f(x,y)$. In fact, by setting $u = \frac{\partial f}{\partial x}$, we get the maximal distance between the curves ux and $f(x,y)$, since we can see that

$$\frac{d}{dx}(g(u(x,y),y)) = \frac{d}{dx}(ux - f(x,y)) = 0 \tag{2.20}$$

which implies that

$$u = \frac{\partial f}{\partial x}. \tag{2.21}$$

Note that here we take $g = g(x(u,y),y)$ so we need not worry about the chain rule acting on u.

2.3 Hamiltonian formulation

For the Hamiltonian formulation, we apply a Legendre transform to the Lagrangian function with respect to $\dot{q}_i$. That is, we try to turn $L(q_i,\dot{q}_i,t)$ into a function $H(q_i,p_i,t)$ with $p_i = \frac{dL}{dq_i}$. We refer to p_i as the conjugate momenta. Compared to the previous section, $L(q_i,\dot{q}_i,t)$ is $f(x,y)$, $H(q_i,p_i,t)$ is $g(u,y)$, and $u(x,y)$ is $p_i(q_i,\dot{q}_i,t)$. The Legendre transform for 2N variables, excluding time, is completely analagous to the transform for the simple two variable function performed above. We transform $L(q_i,\dot{q}_i,t)$ as

$$H(q_i,\dot{q}_i,p_i,t) = \sum_{i=1}^{N} p_i\dot{q}_i - L(q_i,\dot{q}_i,t) \tag{2.22}$$

As above, we start with p_i being an arbitrary function, but we can eliminate the dependency of H on $\dot{q}_i$ if we pick a certain value for p_i. That particular value is

$$p_i = \frac{\partial L}{\partial \dot{q}_i}. \tag{2.23}$$

Then the total derivative of H gives

$$dH = \dot{q}_i\,dp_i + p_i\,d\dot{q}_i - \frac{\partial L}{\partial q_i}dq_i - \frac{\partial L}{\partial \dot{q}_i}\,d\dot{q}_i - \frac{\partial L}{\partial t}dt \tag{2.24}$$

$$= \dot{q}_i dp_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial t} dt \tag{2.25}$$

So we see that H should be thought of as $H(q_i, p_i, t)$. But we can also write the total derivative of H as

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt. \tag{2.26}$$

We will compare terms in the two forms of the total derivative, but we first make one more substitution. We re-use the Euler-Lagrange equation, which tells us that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0 \tag{2.27}$$

and which we can rewrite by substituting in our definition for p_i as

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = \frac{\delta H}{\delta q_i}. \tag{2.28}$$

Comparing terms in the two total derivatives above, we can also read off that

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{2.29}$$

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \tag{2.30}$$

So we see that the Hamiltonian replaces N 2^{nd} order equations in time involving q_i with 2N 1^{st} order equations in time involving q_i and p_i. We call p_i the conjugate momenta.

2.4 Field theories

We can describe any field theory with a slight modification to the above Lagrangian and Hamiltonian formulations. Using the Lagrangian and Hamiltonian for a classical field is analagous to using them for a classical system with N degrees of freedom with the condition that N goes to infinity. In this case, we define a Lagrangian density and define the traditional Lagrangian as the functional

$$L = \int_\Omega \mathcal{L} \, d^3\vec{x}. \tag{2.31}$$

Here Ω is the space considered. We calculate the action exactly as before, in equation 2.2. The only exception is replacing the generalized coordinates with the field being used. The procedures outlined above - calculating the Hamiltonian, the conjugate momenta, the equations of motion, etc - are all completely identical for field theories except for replacing the Lagrangian with the Lagrangian density and the Hamiltonian with the similarly defined Hamiltonian density.

The Lagrangian density can depend on a scalar field, a spinor, a vector field, or a tensor field, depending on the system being modeled by the Lagrangian. We model spin 0 particles, such as a pion, with a scalar field; spin $\frac{1}{2}$ particles such as quarks and leptons with spinors; spin 1 particles such as photons with vector fields; and spin 2 particles, such as gravitons, with 2-component tensor fields. These choices come due to the number of degrees of freedom for each particle. A massive spin particle with a total angular momentum quantum number of 'j' will have 2j+1 degrees of freedom. The photon is massless, so it only has two degrees of freedom, but relativity requires that we use a vector field. For example, varying a Lagrangian density for a 4-vector field A_μ using the coordinates $x^\mu = (ct, x, y, z)$ would result in

$$\frac{\partial L}{\partial A_\mu} - \frac{d}{dt}\frac{\partial L}{\partial(\partial_0 A_\mu)} = 0 \tag{2.32}$$

where ∂_0 is a time derivative. As another example, the conjugate momenta for the Hamiltonian density would become

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)}. \tag{2.33}$$

We detail the Lagrangian density treatment for electromagnetism and the photon in the next section.

2.5 Relativistic electromagnetism

In this section we introduce a way to repackage classical electromagnetism into tensor and vector quantities, allowing us to manipulate them in relativity. The process we will use to solve for the fields governing electromagnetism and for ensuring that it is a physical theory is similar to the process we will use in our analysis of massive gravity, so we present electromagnetism as an example.

Two of Maxwell's four equations tell us that the magnetic field $\vec{B}$ and the electric field $\vec{E}$ can be defined in terms of the magnetic vector potential $\vec{A}$ and the electric potential ϕ as

$$\vec{B} = \vec{\nabla} \times \vec{A} \tag{2.34}$$

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}. \tag{2.35}$$

We can combine ϕ and $\vec{A}$ into a four vector,

$$A^\mu = \left(\frac{\phi}{c}, \vec{A}\right), \tag{2.36}$$

where μ runs from 0 to 3.

Next we define the electromagnetic field tensor, $F_{\mu\nu}$, as

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{E^1}{c} & \frac{E^2}{c} & \frac{E^3}{c} \\ -\frac{E^1}{c} & 0 & -B^3 & B^2 \\ -\frac{E^2}{c} & B^3 & 0 & -B^1 \\ -\frac{E^3}{c} & -B^2 & B^1 & 0 \end{pmatrix}, \tag{2.37}$$

where $\vec{B} = (B^1, B^2, B^3)$ and $\vec{E} = (E^1, E^2, E^3)$

We can also notice that the components of the electromagnetic field tensor satisfy

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}. \tag{2.38}$$

in accordance with equations 2.34 and 2.35. We leave it to the reader to verify 2.38. So we see that we have packaged two of Maxwell's equations into the electromagnetic field tensor by using 2.38. We can package the other two Maxwell equations by defining a 4-current density,

$$j^\mu = (\rho c, \vec{J}), \tag{2.39}$$

where ρ is the total charge density and $\vec{J}$ is the three-dimensional current density and then seeing that the other two Maxwell equations, namely

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_o} \tag{2.40}$$

and

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_o \vec{J} \tag{2.41}$$

show up by setting

$$\partial_\mu F^{\mu\nu} = -\mu_o j^\nu. \tag{2.42}$$

We again leave it to the reader to verify that 2.40 and 2.41 are contained within 2.42.

By substituting 2.38 into 2.42, we obtain an equation of motion involving A^μ,

$$\Box A^\nu - A^\mu{}_{,\mu}{}^{\nu} = -\mu_o j^\nu \tag{2.43}$$

where we have defined $\Box = \partial_\mu \partial^\mu$ for simplicity. In the next section we show how gauge symmetries appear in this theory and affect the number of degrees of freedom.

2.6 Lagrangian formulation of electrodynamics and gauge symmetries

It turns out that Maxwell's equations can be modeled with a particular Lagrangian density which returns the same equation of motion as 2.43 for A^μ. For simplicity we will refer to the Lagrangian density as the Lagrangian. This Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu. \tag{2.44}$$

This Lagrangian models a massless (since there is no mass term) spin-1 particle, the photon. For the rest of this section we will consider a 'free theory' with no current source and take $j^\mu = 0$. This Lagrangian is also a gauge theory, meaning that it is invariant under a gauge transformation. A gauge transformation is the transformation of a vector by a 'gauge', an initially arbitrary scalar field. Mathematically, we perform a gauge transformation by defining a new quantity A'^μ as

$$A'^\mu = A^\mu + \chi^{,\mu}. \tag{2.45}$$

Here χ is the 'gauge function'. Since electromagnetism has gauge invariance, it is straightforward to show that, by using 2.38,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}F'^{\mu\nu}F'_{\mu\nu}. \tag{2.46}$$

where $F'^{\mu\nu}$ is 2.38 with A^μ replaced by A'^μ. We can vary the Lagrangian according to the procedures outlined in 2.1, resulting in the equation of motion for the 4-potential, A^μ, in the absence of a source current.

$$\Box A^\nu - A^\mu{}_{,\mu}{}^{\nu} = 0 \tag{2.47}$$

Next we use the gauge invariance of the Lagrangian and a clever choice of the gauge to make the equation of motion easier to solve. If we take ∂_μ of the gauge transformation 2.45 and rearrange the terms, we obtain

$$A^\mu{}_{,\mu} = A'^\mu{}_{,\mu} - \chi_{,\mu}{}^{\mu}. \tag{2.48}$$

Now we can make a clever choice for the gauge. We choose χ such that

$$-\chi_{,\mu}{}^{\mu} = A^\mu{}_{,\mu} \tag{2.49}$$

resulting in what we call the gauge condition,

$$A'^\mu{}_{,\mu} = 0 \tag{2.50}$$

In the new gauge, our equation of motion becomes, due to the gauge invariance,

$$\Box A'^\nu - A'^\mu{}_{,\mu}{}^{\nu} = 0, \tag{2.51}$$

which then simplifies to

$$\Box A^{\nu} = 0 \tag{2.52}$$

on using the gauge condition and dropping the primes. At this point we have spent the gauge symmetry by picking a clever gauge which resulted in a simplified equation of motion for A^{μ} and an additional gauge condition, 2.50. Now we are ready to attempt to solve the equation of motion. By inspection, we guess a solution with the form of a wave propagating in the direction of $\vec{k}$, the wave vector:

$$A_{\mu} = \epsilon_{\mu} e^{-ik_{\nu}x^{\nu}} \tag{2.53}$$

where $\epsilon_{\mu} = (\epsilon_o, \epsilon_1, \epsilon_2, \epsilon_3)$ is a 'polarization vector' and $k_{\nu} = (k_o, \vec{k})$ is the wavevector for the propagating wave. With this solution, satisfying the gauge condition 2.50 amounts to

$$k^{\mu}\epsilon_{\mu} = 0, \tag{2.54}$$

which we call the 'transverse condition', and satisfying the simplified equation of motion 2.52 amounts to

$$k_{\mu}k^{\mu} = 0, \tag{2.55}$$

which we call the 'massless condition'. The name of the condition comes from requiring that the mass of the particle modeled by A^{μ} is massless, which we show next. In special relativity, the energy-momentum relation reduces to

$$E^2 - \vec{p}^2 = m^2 \tag{2.56}$$

using units in which c = 1. The requirement of 2.55 reduces to

$$(k^o)^2 = \vec{k}^2. \tag{2.57}$$

Since we write the 4-momentum p^{μ} as

$$p^{\mu} = (E, \vec{p}) = (\hbar k^o, \hbar \vec{k}), \tag{2.58}$$

2.55 combined with 2.56 implies that

$$0 = m^2, \tag{2.59}$$

forcing the photon, the quanta for the field A^{μ}, to be massless. Next we choose, for simplicity, a wavevector which propagates in the z-direction. So we choose k^{μ} to have the form

$$k^{\mu} = (k^o, 0, 0, k). \tag{2.60}$$

Due to the massless condition, we see that $k^o = k$. The transverse condition, 2.54, requires that

$$\epsilon_\mu k^\mu = 0 = k\epsilon_o + k\epsilon_3, \tag{2.61}$$

so

$$\epsilon_o = -\epsilon_3. \tag{2.62}$$

Although we have chosen a particular wavevector k^μ, this condition always allows us to solve for one of the components of ϵ_μ in terms of the other three. So the transverse condition will always reduce the number of independent entries for ϵ_μ, and equivalently A_μ, by one. Our new polarization vector is then

$$\epsilon_\mu = (\epsilon, \epsilon_1, \epsilon_2, -\epsilon) \tag{2.63}$$

where we have relabeled $\epsilon_o = -\epsilon_3 = \epsilon$. By inspection, the most general solution A_μ is a linear combination of the three independent solutions,

$$A_\mu^{(1)} = \epsilon_\mu^1 e^{-ik_\nu x^\nu} \tag{2.64}$$

$$A_\mu^{(2)} = \epsilon_\mu^2 e^{-ik_\nu x^\nu} \tag{2.65}$$

$$A_\mu^{(3)} = \epsilon_\mu^3 e^{-ik_\nu x^\nu} \tag{2.66}$$

where

$$\epsilon_\mu^{(1)} = (0, 1, 0, 0) \tag{2.67}$$

$$\epsilon_\mu^{(2)} = (0, 0, 1, 0) \tag{2.68}$$

$$\epsilon_\mu^{(3)} = (1, 0, 0, -1). \tag{2.69}$$

At this point we have shown that the gauge condition 2.50 gives us the transverse condition 2.54 which then allows us to write one of the components of the polarization vector ϵ_μ in terms of its other components. This reduces the number of independent entries in ϵ_μ and equivalently A_μ by one. Thus we have used the gauge symmetry to reduce the number of degrees of freedom for A_μ by one, from its initial four to three. Next we show that one of these degrees of freedom is unphysical and produces an auxiliary field. This will reduce the number of degrees of freedom for A_μ by one again, giving a final number of degrees of freedom of two.

Recalling the initial definition of the electromagnetic field tensor in terms of A_μ and using our third solution A_μ^3, we see that

$$F_{01}^{(3)} = \frac{E^1}{c} = A_{1,0}^{(3)} - A_{0,1}^{(3)} \tag{2.70}$$

$$= -i(k_0\epsilon_1^{(3)} - k_1\epsilon_0^{(3)})e^{-ik_\nu x^\nu} = 0. \tag{2.71}$$

This can be repeated for the other entries of $F^{(3)}_{\mu\nu}$ and will show that for the solution $A^{(3)}_\mu$, $\vec{B} = \vec{E} = 0$. It is straightforward to show that $A^{(1)}_\mu$ and $A^{(2)}_\mu$ give nonzero $\vec{E}$ and $\vec{B}$ fields.

We can also investigate how A^3_μ behaves in the equation of motion. If we take the original equation of motion, set $\mu = 0$, and split the summed indices into 0 and j where $j = 1, 2, 3$, we see that we get

$$\Box A_0 - A_{\nu,0}{}^{\nu} = 0 \tag{2.72}$$

$$= A_{0,0}{}^{0} + A_{0,j}{}^{j} - A_{0,0}{}^{0} - A_{j,0}{}^{j} = 0 \tag{2.73}$$

$$= A_{0,j}{}^{j} - A_{j,0}{}^{j} = 0. \tag{2.74}$$

We could, in principle, solve this equation for A_0. However, A_0 has no second time derivative acting on it, so we wouldn't find any harmonic wave motion for A_0 by itself. Recalling that we originally defined A_0 as the electric scalar potential, it is possible to show that the above equation gives Gauss's law, $\vec{\nabla} \cdot \vec{E} = 0$, which describes a nonpropagating $\vec{E}$ field. For some region with zero charge density but still under the influence of some exterior source charge or a region of non-zero charge density, this does not imply that $\vec{E} = 0$. In a free theory, however, we assume that there is no source charge anywhere which from Gauss's law implies that $\vec{E} = 0$ everywhere. Since $\vec{E} = -\vec{\nabla}\phi = -\vec{\nabla} A_0 = 0$, a propagating solution must have $A_0 = 0$, so we discard the solution $A^{(3)}_\mu$ as unphysical. We end up with two physical propagating wave solutions $A^{(1)}_\mu$ and $A^{(2)}_\mu$, giving us two degrees of freedom, which is the number we expect for a massless spin-1 particle such as the photon.

2.7 Proca Lagrangian

We can consider an alternative to the classical electrodynamics Lagrangian called the Proca Lagrangian, which adds a mass term to the original Lagrangian. Since we will later add a mass term to a linearized massless gravity theory, we present the Proca Lagrangian as an example for exploring the effects of adding a mass term to a previously massless theory. The Proca Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A^\mu_\mu \tag{2.75}$$

This Lagrangian now models a massive spin-1 particle. Adding the mass term removes the gauge symmetry which is present in classical electromagnetism, which prevents us from removing a degree of freedom by fixing a gauge. However, the Proca Lagrangian has a 'constraint' equation built into it which reduces the number of degrees of freedom from its initial four to three, which is correct for a massive spin-1 particle (j=1 so $m_j = -1, 0, 1$). We can vary the Lagrangian to get

$$\Box A^\mu - A^\nu{}_{,\nu}{}^{\mu} + m^2 A^\mu = 0. \tag{2.76}$$

Taking ∂_μ of this equation of motion yields the constraint

$$A^\mu{}_{,\mu} = 0, \tag{2.77}$$

which is the same as the gauge condition that we found in classical electrodynamics. This constraint reduces the equation of motion down to

$$\Box A_\mu + m^2 A_\mu = 0. \tag{2.78}$$

Thus the two equations that we need to satisfy with a solution are 2.78 and 2.77. As before, we guess a propagating wave solution with the form

$$A_\mu = \epsilon_\mu e^{-ik_\alpha x^\alpha}. \tag{2.79}$$

Using this on 2.78 gives

$$k_\alpha k^\alpha = m^2, \tag{2.80}$$

which shows that the solutions to A_μ are all massive. The constraint equation 2.77 requires

$$k_\mu \epsilon^\mu = 0, \tag{2.81}$$

which is the same transverse condition that we found in classical electromagnetism. Taking the same wavevector as before, $k_\mu = (k_0, 0, 0, k_3)$, and a generic polarization vector $\epsilon_\mu = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3)$, we see that the transverse condition requires that

$$\epsilon_\mu = (\frac{k_3}{k_0}\epsilon_3, \epsilon_1, \epsilon_2, \epsilon_3), \tag{2.82}$$

which gives three linearly independent solutions,

$$A_\mu^{(1)} = \epsilon_\mu^1 e^{-ik_\nu x^\nu} \tag{2.83}$$

$$A_\mu^{(2)} = \epsilon_\mu^2 e^{-ik_\nu x^\nu} \tag{2.84}$$

$$A_\mu^{(3)} = \epsilon_\mu^3 e^{-ik_\nu x^\nu} \tag{2.85}$$

where

$$\epsilon_\mu^{(1)} = (0, 1, 0, 0) \tag{2.86}$$

$$\epsilon_\mu^{(2)} = (0, 0, 1, 0) \tag{2.87}$$

$$\epsilon_\mu^{(3)} = (\frac{k_3}{k_0}, 0, 0, 1). \tag{2.88}$$

We find three massive degrees of freedom for the Proca Lagrangian, which is the number we expect for a massive spin-1 particle. The Proca Lagrangian starts with a potential four degrees of freedom. It lacks a gauge symmetry but has a constraint equation which allows us to remove one degree of freedom, leaving us with three total degrees of freedom.

2.8 Hamiltonian for classical electrodynamics

We can also examine the Hamiltonian for classical electrodynamics. We start with the Lagrangian,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}, \tag{2.89}$$

and expand it by substituting in 2.38 for $F_{\mu\nu}$ and $F^{\mu\nu}$. This results in

$$\mathcal{L} = \frac{1}{2}(A_{j,0})(A_{j,0}) + \frac{1}{2}(A_{0,j})(A_{0,j}) - \frac{1}{2}(A_{k,j})(A_{k,j}) - (A_{j,0})(A_{0,j}), \tag{2.90}$$

where we have split μ into its 0 and j components. Then the conjugate momenta are

$$\pi^\mu = \frac{\delta\mathcal{L}}{\delta(A_{\mu,0})}. \tag{2.91}$$

2.90 shows that

$$\pi^0 = \frac{\delta\mathcal{L}}{\delta(A_{0,0})} = 0 \tag{2.92}$$

and

$$\pi^j = \frac{\delta\mathcal{L}}{\delta(A_{j,0})} = A_{j,0} - A_{0,j}. \tag{2.93}$$

The Hamiltonian is

$$\mathcal{H} = \pi^\mu(A_{\mu,0}) - \mathcal{L} \tag{2.94}$$
$$= \pi^0(A_{0,0}) + \pi^j(A_{j,0}) - \mathcal{L} \tag{2.95}$$
$$= \pi^j(A_{j,0}) - \mathcal{L}. \tag{2.96}$$

Using the conjugate momenta 2.93 and canceling terms leaves

$$\mathcal{H} = \frac{1}{2}(\pi^j)^2 + \frac{1}{2}(A_{k,j})^2 - \frac{1}{2}(A_{k,j})(A_{j,k}) + \pi^j(A_{0,j}). \tag{2.97}$$

Next we use a clever simplification by seeing that

$$F_{jk}^2 = (A_{k,j} - A_{j,k})(A_{k,j} - A_{j,k}) \tag{2.98}$$
$$= 2((A_{k,j})^2 - (A_{k,j})(A_{j,k}) \tag{2.99}$$

which lets us simplify the Hamiltonian to

$$\mathcal{H} = \frac{1}{2}(\pi^j)^2 + \frac{1}{4}F_{jk}^2 + \pi^j(A_{0,j}). \tag{2.100}$$

The next two simplifications involve noting that

$$\pi^j = F_{0j} = -F^{0j} = E^j \tag{2.101}$$

and

$$\frac{1}{4}F_{jk}^2 = \frac{1}{4}F_{jk}F^{jk} = \frac{1}{2}\vec{B}^2. \tag{2.102}$$

Using both of these simplifications on 2.100 yields

$$\mathcal{H} = \frac{1}{2}(\vec{E}^2 + \vec{B}^2) + \pi^j(A_{0,j}) \tag{2.103}$$

The first term is the energy density for an electromagnetic wave - remember that $\mathcal{H}$ is the Hamiltonian density, not the Hamiltonian itself. The second term's significance isn't immediately apparent. If we integrate this second term by parts, we find that

$$\int_\infty \pi^j(A_{0,j})\, d^4x = \pi^j A_0\Big|_{-\infty}^{\infty} - \int_\infty ({\pi^j}_{,j})A_0 d^4x \tag{2.104}$$

Since classical fields must vanish at $+/-\infty$, the first term on the right hand side vanishes. Note that integration by parts on a multiplication of two classical fields lets us move a derivative from one field to the other at the expense of a minus sign - we will use this technique extensively in the following sections. However, Gauss's law tells us that

$${\pi^j}_{,j} = \vec{\nabla} \cdot \vec{E} = 0 \tag{2.105}$$

in a free theory. So both terms on the right in 2.104 vanish, showing that $\pi^j(A_{0,j}) = 0$. Thus the final Hamiltonian is

$$H = \int \mathcal{H}\, d^4x = \int \frac{1}{2}(\vec{E}^2 + \vec{B}^2) \geq 0 \tag{2.106}$$

which must be positive due to the squared terms. Next we will examine the Hamiltonian for the Proca Lagrangian and find that it is also non-negative. Non-negativity of the Hamiltonian indicates a ghost-free theory; we discuss ghosts in Section 3.

2.9 Hamiltonian for Proca Lagrangian

Here we examine the Hamiltonian for the the Proca Lagrangian, following the same procedure as for classical electrodynamics. The Proca Lagrangian is the same as the electrodynamics Lagrangian with the exception of the mass term. The mass term has no derivatives in it, so the conjugate momenta are unchanged by the mass term. Thus we can simply take the unsimplified Hamiltonian for electrodynamics, 2.97, and add the mass term to the end of it. This yields

$$\mathcal{H} = \frac{1}{2}(\pi^j)^2 + \frac{1}{2}(A_{k,j})^2 - \frac{1}{2}(A_{k,j})(A_{j,k}) + \pi^j(A_{0,j}) - \frac{1}{2}m^2A_0^2 + \frac{1}{2}m^2A_j^2 \tag{2.107}$$

upon splitting the mass term into its 0 and j components. Next we use the equations of motion to simplify this more. We rewrite 2.78 as

$$-F^{\mu\nu}{}_{,\nu} + m^2 A^\mu = 0, \tag{2.108}$$

recalling that $F^{\nu\mu} = A^{\mu,\nu} - A^{\nu,\mu}$ and $F^{\mu\nu}$ is antisymmetric. For $\mu = 0$, we get

$$-F^{0j}{}_{,j} + m^2 A^0 = 0, \tag{2.109}$$

which simplifies to

$$\pi^j{}_{,j} = -m^2 A^0 \tag{2.110}$$

on using $F^{0j} = -\pi^j$. Then we consider the two terms in the Hamiltonian involving A_0, namely

$$\pi^j (A_{0,j}) - \frac{1}{2} m^2 A_0^2, \tag{2.111}$$

integrate the first term by parts, and use $\pi^j{}_{,j} = -m^2 A^0$ to find

$$\pi^j (A_{0,j}) - \frac{1}{2} m^2 A_0^2 = \frac{1}{2} m^2 A_0^2. \tag{2.112}$$

This is the first substitution we make in the Proca Hamiltonian. The second is the same as in classical electrodynamics. Recall that we found

$$F_{jk}^2 = 2((A_{k,j})^2 - (A_{k,j})(A_{j,k}) \tag{2.113}$$

in the previous section. We see that $\frac{1}{2}((A_{k,j})^2 - (A_{k,j})(A_{j,k})$ in 2.107, so 2.113 is the second substitution we make in the Proca Hamiltonian. Applying both 2.112 and 2.113 to 2.107 gives

$$\mathcal{H} = \frac{1}{2}(\pi^j)^2 + \frac{1}{4} F_{jk}^2 + \frac{1}{2} m^2 A_0^2 + \frac{1}{2} m^2 A_j^2, \tag{2.114}$$

which is non-negative just like the Hamiltonian for electrodynamics.

3 Ghosts

'Ghosts' in a field theory are propagating components of the field which have a negative energy. Since the Hamiltonian represents the total energy, we can check for ghosts by asking whether the Hamiltonian is positive or negative, or examining the Hamiltonian to see which terms and specifically which components of the field make negative energy contributions. The most straightforward way to do this is to try to write the Hamiltonian in terms of squared terms, so that we know with certainty whether each term contributes positively or negatively to the Hamiltonian. We have already seen for the case of classical electromagnetism that its Hamiltonian is positive - this means that classical electrodynamics is a physical, ghost-free theory. In this section, we will first present a physical, ghost-free theory for a massive spin-0 particle, represented by a scalar field. Since this theory is ghost-free, a natural guess for a massive spin-1 particle might be to simply replace the scalar field with a vector field. However, we will see that this guess turns out to have ghosts and is thus not a physical theory.

For a massive spin-0 particle, we use the Lagrangian

$$\mathcal{L} = -\frac{1}{2}(\phi_{,\mu})(\phi^{,\mu}) - \frac{1}{2}m^2\phi^2 \tag{3.1}$$

where ϕ is the scalar field for the particle and m is its mass. This is called the Klein-Gordon Lagrangian. The conjugate momentum is then

$$\pi = \frac{\delta\mathcal{L}}{\delta(\phi_{,0})} = \phi_{,0} \tag{3.2}$$

so the Hamiltonian is, after splitting μ into its 0 and j components and using the conjugate momenta,

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\phi_{,j})^2 + \frac{1}{2}m^2\phi^2 \geq 0. \tag{3.3}$$

This theory has a positive Hamiltonian so we see no ghosts in this theory. If we want to represent a massive spin-1 particle, the logical guess based on the success of the massive spin-0 particle might be a Lagrangian of the form

$$\mathcal{L} = -\frac{1}{2}(A_{\nu,\mu})(A^{\nu,\mu}) - \frac{1}{2}m^2 A_\mu A^\mu. \tag{3.4}$$

By splitting μ into its 0 and j components, we can calculate the conjugate momenta π^0 and π^j to be

$$\pi^0 = \frac{\delta\mathcal{L}}{\delta(A_{0,0})} = -A_{0,0} \tag{3.5}$$

$$\pi^j = \frac{\delta\mathcal{L}}{\delta(A_{j,0})} = A_{j,0}. \tag{3.6}$$

Then, with some simplification, we can write the Hamiltonian density as

$$\mathcal{H} = -\frac{1}{2}(\pi^0)^2 + \frac{1}{2}(\pi^j)^2 - \frac{1}{2}(A_{0,j})^2 + \frac{1}{2}(A_{k,j})^2 - \frac{1}{2}m^2A_0^2 + \frac{1}{2}m^2A_j^2 \tag{3.7}$$

The sign for this Hamiltonian is not definite. However, note that if $A_0 = 0$, the Hamiltonian becomes positive. All the terms with A_0 make negative contributions to the energy. So we conclude that the A_0 component of A_μ is a ghost. Note this is still true even in the massless case, that is for $m = 0$. This theory also does not have a gauge symmetry and it has no constraint equation like the Proca Lagrangian, so the equations of motion for this Lagrangian yield four degrees of freedom - massive spin-1 particles should have three degrees of freedom.

4 Linearized gravity

Gravitational energy is propagated via gravitational radiation, or gravity waves. Since Einstein's field equations are extremely nonlinear, calculating the motion of these gravity waves exactly is almost impossible. Instead we resort to a linearized version of Einstein's field equations for a weak gravitational field. We start by presenting the Lagrangian for the full nonlinear Einstein equations and then show how it can be simplified by using a metric for a weak gravitational field and leaving only quadratic terms in the perturbation, yielding linearized Einstein equations [12, pp. 644 - 649]. The Lagrangian for the full nonlinear Einstein equations is

$$\mathcal{L} = \frac{1}{\kappa^2} R\sqrt{-g} \tag{4.1}$$

which can be rewritten as

$$\mathcal{L} = \frac{1}{\kappa^2} g^{\mu\nu}\sqrt{-g}(\Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta} - \Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\beta}{}_{\alpha\beta}), \tag{4.2}$$

where $\Gamma^{\beta}{}_{\mu\alpha}$ is the connection coefficient and $\kappa^2 = \frac{16\pi G}{c^4}$. We want to find the Einstein equations linearized about a small perturbation, so we need a Lagrangian which is quadratic in the perturbation. We first introduce a weak field metric with the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{4.3}$$

with $|h| \ll 1$. Noting that the two terms in the parentheses in 4.2 are quadratic, we need to rewrite the connection coefficients to be linear in $h_{\mu\nu}$. Using 4.3, we calculate the connection coefficient, dropping any terms that are quadratic in $h_{\mu\nu}$. We find

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{\kappa}{2}\eta^{\alpha\beta}(h_{\nu\beta,\mu} + h_{\beta\mu,\nu} - h_{\mu\nu,\beta}). \tag{4.4}$$

Inserting this linearized connection coefficient and 4.3 back into 4.2, we find

$$\mathcal{L} = -\frac{1}{4}\eta^{\mu\nu}[(h_{\alpha}{}^{\beta}{}_{,\mu} + h^{\beta}{}_{\mu,\alpha} - h_{\mu\alpha}{}^{,\beta})(h_{\beta}{}^{\alpha}{}_{,\nu} + h^{\alpha}{}_{\nu,\beta} - h_{\nu\beta}{}^{,\alpha}) - (h_{\nu}{}^{\alpha}{}_{,\mu} + h^{\alpha}{}_{\mu,\nu} - h_{\mu\nu}{}^{,\alpha})h^{\beta}{}_{\beta,\alpha}] + ... \tag{4.5}$$

where we have ignored the terms that are higher order than quadratic in $h_{\mu\nu}$. This can be simplified and multiplied through by a factor of two to obtain the Lagrangian we will use for linearized Einstein equations,

$$\mathcal{L}_0 = -\frac{1}{2}h_{\mu\nu,\lambda}h^{\mu\nu,\lambda} + h_{\nu\lambda,\mu}h^{\mu\nu,\lambda} - h^{\mu\nu}{}_{,\mu}h_{,\nu} + \frac{1}{2}h_{,\lambda}h^{,\lambda}. \tag{4.6}$$

Since we will later add a mass term to this Lagrangian, we will call this Lagrangian the 'massless Lagrangian' $\mathcal{L}_0$.

4.1 Linearized massless gravity

In this section, we work through the linearized massless gravity theory to find its solutions and number of degrees of freedom [9, pp. 169-177]. For a massless spin-2 particle such as the graviton, we expect two degrees of freedom. We start by varying the massless Lagrangian for linearized gravity to find the equations of motion,

$$\Box h_{\mu\nu} - h^{\sigma}{}_{\mu,\sigma\nu} - h^{\sigma}h{}_{\nu,\sigma\mu} + h_{,\mu\nu} + \eta_{\mu\nu}h^{\rho\lambda}{}_{,\rho\lambda} - \eta_{\mu\nu}\Box h = 0. \tag{4.7}$$

We can simplify this equation of motion by defining a new quantity,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}, \tag{4.8}$$

which simplifies the equation of motion to

$$\bar{h}_{\mu\nu,\alpha}^{\;\;\;\;\;\alpha} + (\eta_{\mu\nu}\bar{h}^{\alpha\beta}{}_{,\alpha\beta} - \bar{h}^{\alpha}_{\nu,\mu\alpha} - \bar{h}^{\alpha}_{\mu,\nu\alpha}) = 0. \tag{4.9}$$

Next we can apply a diffeomorphism transformation with the form

$$x^{\mu'} = x^{\mu} + \epsilon^{\mu}(x^{\alpha}). \tag{4.10}$$

Diffeomorphisms are mappings of a manifold to itself, and they are a symmetry of general relativity. Mathematically, they are similar to infinitesimal general coordinate transformations. The diffeomorphism symmetry can be treated like a gauge symmetry, and it will ultimately let us simplify the equations of motion by introducing a gauge condition. ϵ^{μ} is small and on the same order as $h_{\mu\nu}$, so we will discard any terms in the following equations which are quadratic in ϵ^{μ} or involve an ϵ^{μ} multiplied with an $h^{\mu\nu}$. The coordinate transformation matrix for the gauge transformation is then

$$X^{\mu'}_{\nu} = \delta^{\mu}_{\nu} + \epsilon^{\mu}{}_{,\nu}. \tag{4.11}$$

We can get the inverse transformation matrix by

$$X^{\mu}_{\nu'} = \delta^{\mu}_{\nu} - \epsilon^{\mu}{}_{,\nu}, \tag{4.12}$$

since then $X^{\mu}_{\nu'}X^{\nu'}_{\alpha} = \delta^{\mu}_{\alpha}$.

Using this coordinate transformation on $h^{\mu\nu}$ and $\bar{h}^{\mu\nu}$ then yields

$$h^{\mu'\nu'} = h^{\mu\nu} - \epsilon^{\mu,\nu} - \epsilon^{\nu,\mu} \tag{4.13}$$

$$\bar{h}^{\mu'\nu'} = \bar{h}^{\mu\nu} - \epsilon^{\mu,\nu} - \epsilon^{\nu,\mu} + \eta^{\mu\nu}\epsilon^{\alpha}{}_{,\alpha} \tag{4.14}$$

The Lagrangian 4.6 has gauge invariance under the transformation 4.13. Our next step is to investigate choices for the gauge $\epsilon^{\mu}(x^{\alpha})$. We see that

$$\bar{h}^{\mu'\alpha'}{}_{,\alpha'} = \bar{h}^{\mu'\alpha'}{}_{,\beta}X^{\beta}_{\alpha'} = \bar{h}^{\mu'\alpha'}{}_{,\beta}(\delta^{\beta}_{\alpha} - \epsilon^{\beta}{}_{,\alpha'}) \tag{4.15}$$

$$= \bar{h}^{\mu'\alpha'}{}_{,\beta}\delta^{\beta}_{\alpha} = \bar{h}^{\mu'\alpha'}{}_{,\alpha} \tag{4.16}$$

Note that we discard the $\bar{h}^{\mu'\alpha'}_{,\beta}\epsilon^{\beta}_{,\alpha'}$ term since it is negligibly small. At this point we can use 4.14 to show that

$$\bar{h}^{\mu'\alpha'}{}_{,\alpha} = \bar{h}^{\mu\alpha}{}_{,\alpha} - \epsilon^{\mu,}{}_{,\alpha}{}^{\alpha}. \tag{4.17}$$

Now we can pick ϵ^{μ} such that

$$\epsilon^{\mu}{}_{,\alpha}{}^{\alpha} = \bar{h}^{\mu\alpha}{}_{,\alpha}. \tag{4.18}$$

With this clever choice for ϵ^{μ}, we see that from 4.17,

$$\bar{h}^{\mu'\alpha'}{}_{,\alpha} = 0. \tag{4.19}$$

which from 4.15 is equivalent to

$$\bar{h}^{\mu'\alpha'}{}_{,\alpha'} = 0. \tag{4.20}$$

This simplifies the equation of motion 4.9 down to

$$\bar{h}_{\mu'\nu',\alpha'}{}^{\alpha'} = 0. \tag{4.21}$$

Due to the gauge invariance of the Lagrangian and thus the equations of motion, we can drop the primes here. This tells us that

$$\bar{h}_{\mu\nu,\alpha}{}^{\alpha} = 0. \tag{4.22}$$

Again, because of the gauge invariance, we can also drop the primes on 4.20, which gives

$$\bar{h}^{\mu\alpha}{}_{,\alpha} = 0. \tag{4.23}$$

Just as in the primed case, 4.22 is only true when 4.23 is true. We call 4.23 the gauge condition, since it came from our choice of gauge. The final condition we note is that after picking a gauge, seeing its effect on the equations of motion, and finding the gauge condition, 4.23 and 4.18 together imply that

$$\epsilon^{\mu}{}_{,\alpha}{}^{\alpha} = 0. \tag{4.24}$$

This is from the 'residual gauge freedom' - that is, the gauge is not completely fixed by 4.18. In summary, we find three conditions which need to be satisfied for a solution of $\bar{h}^{\mu\alpha}$ - the simplified equation of motion, 4.22, the gauge condition 4.23, and the condition for the gauge ϵ^{μ}, 4.24.

Now we can attempt to solve 4.22. Since it has the form of the wave equation, we guess a solution of the form

$$\bar{h}^{\mu\nu} = A^{\mu\nu}\cos(k_\alpha x^\alpha). \tag{4.25}$$

Plugging this solution into 4.22 implies that

$$k^\mu k_\mu = 0 \tag{4.26}$$

Since 4.23 must always be true, we also see that

$$A^{\mu\nu}k_\nu = 0. \tag{4.27}$$

Since $\bar{h}^{\mu\nu}$ is symmetric, it has only ten independent entries to begin with. 4.27 lets us write four entries of $A^{\mu\nu}$ in terms of the rest of its entries, which reduces the number of independent entries in $\bar{h}^{\mu\nu}$ by four, leaving six indepedent entries. 4.24 will also give us four additional conditions, which gives $\bar{h}^{\mu\nu}$ a total of two independent entries after accounting for all the necessary conditions.

Next we fix the wavevector and define

$$k^\mu = (k, 0, 0, k) \tag{4.28}$$

$$k_\mu = (-k, 0, 0, k). \tag{4.29}$$

4.27 gives four equations,

$$A^{00}k - A^{03}k = 0, \tag{4.30}$$

$$A^{10}k - A^{13}k = 0, \tag{4.31}$$

$$A^{20}k - A^{23}k = 0, \tag{4.32}$$

$$A^{30}k - A^{33}k = 0, \tag{4.33}$$

which collectively tell us that $A^{\mu 0} = A^{\mu 3}$. From symmetry, we know that $A^{\mu\nu} = A^{\nu\mu}$. Using both of these conditions, we can write $A^{\mu\nu}$ in terms of A^{00},A^{01},A^{02},A^{11},A^{12}, and A^{22}:

$$A^{\mu\nu} = \begin{pmatrix} A^{00} & A^{01} & A^{02} & A^{00} \\ A^{01} & A^{11} & A^{12} & A^{01} \\ A^{02} & A^{12} & A^{22} & A^{02} \\ A^{00} & A^{01} & A^{02} & A^{00} \end{pmatrix} \tag{4.34}$$

Now we examine 4.24. We guess a solution of similar form:

$$\epsilon^\mu = \xi^\mu \sin(k_\alpha x^\alpha) \tag{4.35}$$

Since the wavevector k^μ is null, this solution automatically satisfies 4.24. Then,

$$\epsilon^\mu{}_{,\nu} = \xi^\mu k_\nu \cos(k_\alpha x^\alpha) \tag{4.36}$$

Next we return to 4.14. Since

$$\bar{h}^{\mu'\nu'} = A^{\mu'\nu'} \cos(k_{\alpha'} x^{\alpha'}), \tag{4.37}$$

we can use 4.14 to see that

$$A^{\mu'\nu'} = A^{\mu\nu} - \xi^{\mu} k^{\nu} - k^{\mu} \xi^{\nu} + \eta^{\mu\nu}(\xi^{\alpha} k_{\alpha}). \tag{4.38}$$

Note that we take $\cos(k_{\alpha'} x^{\alpha'}) \approx \cos(k_{\alpha} x^{\alpha})$ since they differ only in the first order in a Taylor expansion. Next we can use 4.43 and our wavevector k^{μ} to get values for $A^{\mu'\nu'}$. We find that

$$\begin{array}{cc} A^{0'0'} = A^{00} - k(\xi^0 + \xi^3) & A^{1'1'} = A^{11} - k(\xi^0 - \xi^3) \\ A^{0'1'} = A^{01} - k\xi^1 & A^{1'2'} = A^{12} \\ A^{0'2'} = A^{02} - k\xi^2 & A^{2'2'} = A^{22} - k(\xi^0 - \xi^3) \end{array}. \tag{4.39}$$

We can eliminate some of these entries by cleverly picking values for ξ^{μ}. Picking

$$\begin{array}{cc} \xi^0 = \frac{(2A^{00} + A^{11} + A^{22})}{4k} & \xi^1 = \frac{A^{01}}{k} \\ \xi^2 = \frac{A^{02}}{k} & \xi^3 = \frac{(2A^{00} - A^{11} - A^{22})}{4k} \end{array} \tag{4.40}$$

leaves $A^{0'0'} = A^{0'1'} = A^{0'2'} = 0$ and $A^{1'1'} = -A^{2'2'}$. Due to the gauge invariance, we can drop the primes and rewrite our matrix $A^{\mu\nu}$ as

$$A^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & 0 \\ 0 & A^{12} & -A^{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.41}$$

By inspection, this matrix is a linear combination of the two matrices

$$e^{\mu\nu}_{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.42}$$

and

$$e^{\mu\nu}_{(2)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.43}$$

We call these matrices 'linear polarization matrices'.

Thus the solution $\bar{h}^{\mu\nu} = A^{\mu\nu}\cos(k_{\alpha} x^{\alpha})$ is a linear combination of two solutions and we can write it most generally as

$$\bar{h}^{\mu\nu} = (\alpha e^{\mu\nu}_{(1)} + \beta e^{\mu\nu}_{(2)})\cos(k_\alpha x^\alpha). \tag{4.44}$$

We have found two independent degrees of freedom for the solution to our original equation of motion. This is the correct number of independent degrees of freedom for a massless spin-2 particle.

4.2 Linearized massive gravity

As explained in the introduction, the addition of a mass term to the Lagrangian for the linearized field equations has potential for solving open questions in cosmology. Since we want an equation of motion which is linear in $h^{\mu\nu}$, we can have a term that is at most quadratic in the Lagrangian. The only two ways to write a quadratic scalar term from $h^{\mu\nu}$ are $h_{\mu\nu}h^{\mu\nu}$ and $h^2 = h^\alpha_\alpha h^\sigma_\sigma$. Thus the most general quadratic scalar term would be a linear combination of these possibilities. We could write it as

$$(\alpha h_{\mu\nu}h^{\mu\nu} + \beta h^2), \tag{4.45}$$

but this introduces two parameters α and β. We can simplify this to one parameter by writing instead

$$\alpha(h_{\mu\nu}h^{\mu\nu} + \frac{\beta}{\alpha}h^2) \tag{4.46}$$

and choosing $\alpha = m^{\frac{1}{2}}$, where m is the mass of the graviton and $\frac{\beta}{\alpha} = \lambda$. Our new Lagrangian is then the Lagrangian for the massless graviton plus the above mass term. This mass term is called the 'Fierz-Pauli' mass term after Fierz and Pauli, and the claim is that there is only one λ value, specifically $\lambda = -1$, for which we get the right number of degrees of freedom and no ghosts. We write the massive Lagrangian $\mathcal{L}_M$ as

$$\begin{aligned}\mathcal{L}_M &= \mathcal{L}_0 - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} + \lambda h^2) \\ &= -\frac{1}{2}h_{\mu\nu,\lambda}h^{\mu\nu,\lambda} + h_{\nu\lambda,\mu}h^{\mu\nu,\lambda} - h^{\mu\nu}{}_{,\mu}h_{,\nu} + \frac{1}{2}h_{,\lambda}h^{,\lambda} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} + \lambda h^2)\end{aligned} \tag{4.47}$$

Introducing the mass term to the Lagrangian breaks the gauge symmetry, which means that we can't use the gauge condition and residual gauge freedom to reduce the number of degrees of freedom down to 2 as we did for the massless case. However, there are two constraints, one of which has the same effect as the gauge condition, which will ultimately give us five degrees of freedom. We can vary the Lagrangian 4.86 to give us the new equation of motion,

$$\Box h_{\mu\nu} - h^\sigma{}_{\mu,\sigma\nu} - h^\sigma{}_{\nu,\sigma\mu} + h_{,\mu\nu} + \eta_{\mu\nu}h^{\rho\lambda}{}_{,\rho\lambda} - \eta_{\mu\nu}\Box h - m^2 h_{\mu\nu} - m^2\lambda\eta_{\mu\nu}h = 0. \tag{4.48}$$

If we take the trace of this, that is multiply by $\eta^{\mu\nu}$, we find

$$\Box h - h^{\alpha\beta}{}_{,\alpha\beta} = -\frac{1}{2}m^2 h(1+4\lambda). \tag{4.49}$$

We also take ∂^μ of 4.48 to find a constraint equation. This is similar to the Proca Lagrangian case and actually creates the exact same condition as the gauge condition for the massless case. Taking ∂^μ of 4.48 gives

$$\Box h_{\mu\nu}{}^{,\mu} - h^{\sigma}{}_{\mu,\nu}{}^{\mu}{}_{\sigma} - \Box h^{\sigma}{}_{\nu,\sigma} + \Box h_{,\nu} + h^{\alpha\beta}{}_{,\nu\alpha\beta} - \Box h_{,\nu} - m^2 h_{\mu\nu}{}^{,\nu} - m^2\lambda h_{,\nu} = 0 \tag{4.50}$$

which, after canceling terms, reduces to

$$h_{\mu\nu}{}^{,\nu} = -\lambda h_{,\mu}. \tag{4.51}$$

Taking ∂^μ yields

$$h_{\mu\nu}{}^{,\mu\nu} = -\lambda \Box h. \tag{4.52}$$

We can use 4.52 on 4.49 to find

$$\Box h + \lambda \Box h = -\frac{1}{2}m^2(1+4\lambda)h \tag{4.53}$$

$$(1+\lambda)\Box h = -\frac{1}{2}m^2(1+4\lambda)h. \tag{4.54}$$

This leaves two cases. If $\lambda = -1$, then we see that $h = 0$, but letting $\lambda \neq -1$ allows $h \neq 0$. Here we will examine the $\lambda = -1$ case, which is called the Fierz-Pauli mass term [1]. See Section 6 for the case when $\lambda \neq -1$. Setting $\lambda = -1$ affects our condition 4.51, which reduces to

$$h_{\mu\nu}{}^{,\nu} = 0. \tag{4.55}$$

This is exactly the same condition that the gauge fixing created in the massless case. So we can use 4.55, which reduces the number of degrees of freedom by four, together with $h = 0$, which will reduce the number of degrees of freedom by one, to cut the number of degrees of freedom down to five, the expected number for a massive spin-2 particle, from an initial ten. As with the massless case, we guess a solution of the form

$$h_{\mu\nu} = A_{\mu\nu}\cos(k_\alpha x^\alpha) \tag{4.56}$$

We work in the rest frame of the massive object for simplicity, so we pick wavevectors $k_\mu = (-m, 0, 0, 0)$ and $k^\mu = (m, 0, 0, 0)$. 4.55 tells us

$$A_{\mu\nu}k^\nu = 0 \tag{4.57}$$

which gives $A_{\mu 0} = 0$.

Using this condition on $A_{\mu 0}$, we can rewrite the matrix $A_{\mu\nu}$ as

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & A^{13} \\ 0 & A^{12} & A^{22} & A^{23} \\ 0 & A^{13} & A^{23} & A^{33} \end{pmatrix}. \tag{4.58}$$

Note here that we have retained the same constants from $A^{\mu\nu}$ but adjusted signs accordingly for $A_{\mu\nu}$. The other condition we have is that $h = 0$. This amounts to

$$h = A_{\mu}{}^{\mu} \cos(k_\alpha x^\alpha) = 0 \tag{4.59}$$

$$= (-A_{00} + A_{11} + A_{22} + A_{33}) \cos(k_\alpha x^\alpha). \tag{4.60}$$

Since $A_{00} = 0$, we see that

$$A^{33} = -(A^{11} + A^{22}). \tag{4.61}$$

This effectively reduces the number of degrees of freedom by one more. We rewrite $A_{\mu\nu}$ again in its final form as

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A^{11} & A^{12} & A^{13} \\ 0 & A^{12} & A^{22} & A^{02} \\ 0 & A^{13} & A^{23} & -(A^{11} + A^{22}) \end{pmatrix}. \tag{4.62}$$

So we end up with five independent entries. By inspection, we can write the five 'linear polarization matrices' as

$$e^{(1)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.63}$$

$$e^{(2)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.64}$$

$$e^{(3)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.65}$$

$$e^{(4)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{4.66}$$

$$e^{(5)}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{4.67}$$

As with the massless case, the most general solution for $h_{\mu\nu} = A_{\mu\nu}\cos(k_\alpha x^\alpha)$ is then

$$h_{\mu\nu} = (\alpha e^{(1)}_{\mu\nu} + \beta e^{(2)}_{\mu\nu} + \gamma e^{(3)}_{\mu\nu} + \delta e^{(4)}_{\mu\nu} + \epsilon e^{(5)}_{\mu\nu})\cos(k_\alpha x^\alpha) \tag{4.68}$$

where α, β, γ, δ, and ϵ are constants. So we have found a solution with five independent degrees of freedom, which is what we expect for a massive spin-2 particle such as the graviton.

4.3 Hamiltonian for linearized massless gravity

Since we want to ensure that the massive case has no ghosts, we need to look at the Hamiltonian and see if there are any propagating components of $h^{\mu\nu}$ with the wrong sign for the energy. First we will examine the massless case. We state the massless Lagrangian again here for reference

$$\mathcal{L}_0 = -\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\lambda h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \tag{4.69}$$

.

Next we split it into $\mu = 0$ and $\mu = j$ components, cancel like terms, and use integrations by parts to find that

$$\begin{aligned} \mathcal{L} = \frac{1}{2}(h_{jk,0})^2 + (h_{0k,j})^2 - \frac{1}{2}(h_{jk,l})^2 - (h_{0k,j})(h_{0j,k}) - 2(h_{0j,k})(h_{jk,0}) + (h_{kj,l})(h_{lj,k}) \\ + (h_{0k,k})(h_{jj,0}) - \frac{1}{2}(h_{jj,0})(h_{kk,0}) - (h_{00,l})(h_{jj,l}) + \frac{1}{2}(h_{jj,l})(h_{kk,l}) + (h_{0j,j})(h_{kk,0}) \\ + (h_{jk,j})(h_{00,k}) - (h_{jk,j})(h_{ll,k}) \end{aligned} . \tag{4.70}$$

Next we can extract three sets of equations of motion for $\mu\nu = 00$, $\mu\nu = 0j$, and $\mu\nu = jk$. Varying the above form of the Lagrangian with respect to h_{00} yields

$$h_{jj,kk} - h_{jk,jk} = 0, \tag{4.71}$$

varying with respect to h_{0j} yields

$$-h_{0j,kk} + h_{0k,jk} + h_{jk,0k} - h_{kk,0j} = 0, \tag{4.72}$$

and varying with respect to h_{jk} yields,

$$\begin{aligned} &h_{jk,00} - h_{jk,ll} - (h_{0j,0k} + h_{0k,0j}) + h_{lk,lj} + h_{lj,lk} \\ &+ \delta_{jk}(2h_{0l,0l} - h_{ll,00} - h_{00,ll} + h_{mm,ll} - h_{lm,lm}) + h_{00,jk} - h_{ll,jk} = 0^{.} \end{aligned} \tag{4.73}$$

It is straightforward to check these with the general equation of motion 4.7 by placing $\mu\nu = 00$, $\mu\nu = 0j$, or $\mu\nu = jk$ and comparing the result with the appropriate equation of motion above.

Next we work on obtaining the Hamiltonian for the massless case. We start by calculating the conjugate momenta from 4.70. We see

$$\pi^{00} = \frac{\delta \mathcal{L}}{\delta(h_{00,0})} = 0, \tag{4.74}$$

$$\pi^{0j} = \frac{\delta \mathcal{L}}{\delta(h_0 j, 0)} = 0, \tag{4.75}$$

and

$$\pi^{jk} = \frac{\delta \mathcal{L}}{\delta(h_{jk,0})} = h_{jk,0} - (h_{0k,j} + h_{0j,k}) + \delta_{jk}(2h_{l0,l} - h_{ll,0}). \tag{4.76}$$

Then we can rearrange to see that

$$h_{jk,0} = \pi_{jk} + h_{0k,j} + h_{0j,k} - \delta_{jk}(2h_{0l,l} - h_{ll,0}). \tag{4.77}$$

We also take the trace of π^{jk}, which is

$$\pi_{ll} = h_{ll,0} - h_{0l,l} - h_{0l,l} + 3(2h_{0l,l} - h_{ll,0}) \tag{4.78}$$

$$= -2h_{0l,l} + 4h_{0l,l}, \tag{4.79}$$

so

$$(2h_{0l,l} - h_{ll,0}) = \frac{1}{2}\pi_{ll}. \tag{4.80}$$

Then we use this result in our equation for π_{jk} and rewrite it as

$$\pi_{jk} = h_{jk,0} - (h_{0k,j} + h_{0j,k}) + \frac{1}{2}\delta_{jk}\pi_{ll}. \tag{4.81}$$

We can also rewrite $h_{jk,0}$ as

$$h_{jk,0} = \pi_{jk} + h_{0k,j} + h_{0j,k} - \frac{1}{2}\delta_{jk}\pi_{ll}. \tag{4.82}$$

We will use these two results in the Hamiltonian since we need to replace $h_{jk,0}$ and $h_{ll,0}$ with the conjugate momentum π_{jk}.

The Hamiltonian is

$$\mathcal{H}_0 = \pi^{\mu\nu} h_{\mu\nu,0} - \mathcal{L}, \tag{4.83}$$

which reduces to

$$\mathcal{H}_0 = \pi^{jk} h_{jk,0} - \mathcal{L} \tag{4.84}$$

since π^{00} and π^{0j} are 0.

The next step is to replace $h_{jk,0}$ and $h_{ll,0}$ with the conjugate momentum π_{jk} in the Lagrangian $\mathcal{L}$ using 4.82 and 4.80 and to substitute in for the $h_{jk,0}$ term in the first term on the right hand side of the Hamiltonian. After making these substitutions and making a number of cancellations, we find

$$\begin{aligned} \mathcal{H}_0 =& \frac{1}{2}(\pi_{jk})^2 - \frac{1}{4}(\pi_{ll})^2 + 2h_{0k,j}\pi_{jk} + \frac{1}{2}(h_{jk,l})^2 - h_{jk,l}h_{lj,k} + \\ & h_{00,l}h_{kk,l} - \frac{1}{2}h_{jj,l}h_{kk,l} - h_{jk,j}h_{00,k} + h_{jk,j}h_{ll,k} \end{aligned}. \tag{4.85}$$

This is the Hamiltonian for the massless case before using any equations of motion. This is important to note because up to this point the massive and massless Hamiltonians have the same form for the non-mass terms (that is, for terms without an 'm' in them).

In general relativity it is known that the full Hamiltonian for the nonlinear theory is zero. Since the Hamiltonian is the generator of time translation, and since time in general relativity does not have an absolute definition, it makes sense that the Hamiltonian is zero for general relativity. It is possible to simplify this form of the Hamiltonian for the linearized theory more, but we were unsuccessful in explicitly showing that it is non-negative, which would be proof of the absence of ghosts in linearized massless gravity. However, we will assume that $\mathcal{H}_{\prime} = 0$ here since it holds in the nonlinear theory and assume that the linearized massless theory does not contain ghosts. At this point we will move on to linearized massive gravity and show how the Hamiltonian changes with the addition of a mass term to the massless linearized Lagrangian.

4.4 Hamiltonian for linearized massive gravity

Here we do the Hamiltonian analysis for the addition of the mass term with an unspecified λ. After working out the general case with an unspecified λ, we will see that $\lambda = -1$, the Fierz-Pauli mass term, is the only physical mass term. The massive Lagrangian for linearized gravity is

$$\begin{aligned}\mathcal{L}_M = \mathcal{L}_0 - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} + \lambda h^2) = -\frac{1}{2}h_{\mu\nu,\lambda}h^{\mu\nu,\lambda} + h_{\nu\lambda,\mu}h^{\mu\nu,\lambda} - h^{\mu\nu}{}_{,\mu}h_{,\nu} + \\ \frac{1}{2}h_{,\lambda}h^{,\lambda} - \frac{1}{2}m^2(h_{\mu\nu}h^{\mu\nu} + \lambda h^2)\end{aligned} \quad . \tag{4.86}$$

We use the simplified result for the $\mathcal{L}_0$ component and expand the massive portion out into $\mu = 0$ and $\mu = j$ components as usual. This gives

$$\begin{aligned}\mathcal{L}_M = &\frac{1}{2}(h_{jk,0})^2 + (h_{0k,j})^2 - \frac{1}{2}(h_{jk,l})^2 - (h_{0k,j})(h_{0j,k}) - 2(h_{0j,k})(h_{jk,0}) + (h_{kj,l})(h_{lj,k}) \\ &+ (h_{0k,k})(h_{jj,0}) - \frac{1}{2}(h_{jj,0})(h_{kk,0}) - (h_{00,l})(h_{jj,l}) + \frac{1}{2}(h_{jj,l})(h_{kk,l}) + (h_{0j,j})(h_{kk,0}) \\ &+ (h_{jk,j})(h_{00,k}) - (h_{jk,j})(h_{ll,k}) - \frac{1}{2}m^2[(1+\lambda)h_{00}^2 - 2h_{0j}^2 + h_{jk}^2 - 2\lambda h_{00}h_{ll} + \lambda h_{ll}^2]\end{aligned} \quad . \tag{4.87}$$

We get the equations of motion as before by varying $\mathcal{L}_M$. Varying with respect to h_{00} gives

$$h_{jj,kk} - h_{jk,jk} - m^2(1+\lambda)h_{00} + m^2\lambda h_{ll} = 0, \tag{4.88}$$

varying with respect to h_{0j} yields

$$-h_{0j,kk} + h_{0k,jk} + h_{jk,0k} - h_{kk,0j} + m^2 h_{0j} = 0, \tag{4.89}$$

and varying with respect to h_{jk} yields

$$\begin{aligned}&h_{jk,00} - h_{jk,ll} - (h_{0j,0k} + h_{0k,0j}) + h_{lk,lj} + h_{lj,lk} + \delta_{jk}(2h_{0l,0l} - h_{ll,00} - h_{00,ll} + h_{mm,ll} - h_{lm,lm}) \\ &+ h_{00,jk} - h_{ll,jk} + m^2 h_{jk} - m^2\lambda\delta_{jk}h_{00} + m^2\lambda\delta_{jk}h_{ll} = 0.\end{aligned} \tag{4.90}$$

The conjugate momenta for the massive Lagrangian are the same as for the massless Lagrangian since the mass term doesn't contain any ∂_0 terms. This is convenient because it means that the only change in the massive Hamiltonian $\mathcal{H}_M$ compared to the massles Hamiltonian $\mathcal{H}_0$ is from the change in the Lagrangian from $\mathcal{L}_0$ to $\mathcal{L}_M$. So all we need to do is take $\mathcal{H}_0$ and include the mass term at the end. This gives

$$\begin{aligned}\mathcal{H}_M = \pi^{\mu\nu}h_{\mu\nu,0} - \mathcal{L}_M = &\frac{1}{2}(\pi_{jk})^2 - \frac{1}{4}(\pi_{ll})^2 + 2h_{0k,j}\pi_{jk} + \frac{1}{2}(h_{jk,l})^2 \\ &- h_{jk,l}h_{lj,k} + h_{00,l}h_{kk,l} - \frac{1}{2}h_{jj,l}h_{kk,l} - h_{jk,j}h_{00,k} + h_{jk,j}h_{ll,k} \\ &+ \frac{1}{2}m^2[(1+\lambda)h_{00}^2 - 2h_{0j}^2 + h_{jk}^2 - 2\lambda h_{00}h_{ll}h + \lambda h_{ll}^2]\end{aligned} \quad . \tag{4.91}$$

Next we use four substitutions to change the Hamiltonian to a form which makes it easier to spot the requirement on λ for avoiding ghosts. The first of these substitutions involves the conjugate momentum π_{jk}, from 4.81. If we take ∂_j of this, we get

$$\pi_{jk,j} = h_{jk,0j} - (h_{0k,jj} + h_{0j,jk}) + \frac{1}{2}\delta_{jk,j}\pi_{ll,j}. \tag{4.92}$$

Using 4.80,

$$\pi_{jk,j} = h_{jk,0j} - h_{0k,jj} - h_{0j,jk} + \delta_{jk}(2h_{0l,lj} - h_{ll,0j})$$

$$\pi_{jk,j} = h_{jk,0j} - h_{0k,jj} - h_{0j,jk} + 2h_{0l,lk} - h_{ll,0k}$$

$$\pi_{jk,j} = h_{jk,0j} - h_{0k,jj} + h_{0l,lk} - h_{ll,0k}$$

From one of our equations of motion, 4.89,

$$\pi_{jk,j} = -m^2 h_{0k}. \tag{4.93}$$

We can make this look like a term in the Hamiltonian 4.91 by multiplying on the left by $2h_{0k,j}$ and integrating by parts to move the ∂_j on to the π_{jk} at the expense of a minus sign. This gives

$$2h_{0k,j}\pi_{jk} = 2m^2(h_{0k})^2. \tag{4.94}$$

which will be the first substitution. The second substitution comes from rewriting two terms from the Hamiltonian by using integration by parts and the first equation of motion 4.88. We see that

$$h_{00,l}h_{kk,l} - h_{jk,j}h_{00,k} = h_{00}[-h_{kk,ll} + h_{jk,jk}] \tag{4.95}$$

by using integration by parts, and by using 4.88,

$$h_{00}[-h_{kk,ll} + h_{jk,jk}] = h_{00}[-m^2(1+\lambda)h_{00} + m^2\lambda h_{ll}] \tag{4.96}$$

$$= \frac{1}{2}m^2[-2(1+\lambda)h_{00}^2 + 2\lambda h_{00}h_{ll}]. \tag{4.97}$$

This will be the second substitution. The third substitution involves the first equation of motion again. So, from 4.88,

$$h_{jj,kk} - h_{jk,jk} = m^2(1+\lambda)h_{00} - m^2\lambda h_{ll}. \tag{4.98}$$

Next we multiply by h_{ll} and integrate by parts. So

$$-h_{ll,k}h_{jj,k} + h_{ll,j}h_{jk,l} = m^2(1+\lambda)h_{00}h_{ll} - m^2\lambda h_{ll}^2. \tag{4.99}$$

Rearranging a bit gives

$$h_{jk,j}h_{ll,k} = (h_{jj,l})^2 + m^2(1+\lambda)h_{00}h_{ll} - m^2\lambda h_{ll}^2, \tag{4.100}$$

which is the third substitution. The final substitution involves only the massless portion of the Hamiltonian. We will use the Levi-Civita symbol, ϵ_{ijk}, which has the property

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (i,j,k) \text{ is (1,2,3), (3,1,2), or (2,3,1)} \\ -1, & \text{if } (i,j,k) \text{ is (1,3,2), (3,2,1), or (2,1,3)} \\ 0, & \text{otherwise} \end{cases} . \tag{4.101}$$

There is an identity associted with the Levi-Civita symbol which we will use. It says that

$$\epsilon_{ijk}\epsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{4.102}$$

If we examine $(\epsilon_{ijk}h_{kl,j})^2$, we can see that

$$(\epsilon_{ijk}h_{kl,j})^2 = \epsilon_{ijk}h_{kl,j}\epsilon_{imn}h_{nl,m} \tag{4.103}$$

$$= \epsilon_{ijk}\epsilon_{imn}h_{kl,j}h_{nl,m}. \tag{4.104}$$

Using the identity, we get

$$= h_{kl,j}^2 - h_{kl,j}h_{jl,k} \tag{4.105}$$

Rearranging, dividing by 2, and integrating the second term in the above equation by parts twice, we find

$$\frac{1}{2}(h_{jk,l})^2 = \frac{1}{2}(\epsilon_{ijk}h_{kl,j})^2 + \frac{1}{2}(h_{kl,k})^2. \tag{4.106}$$

This is the fourth substitution we will make. So applying all four substitutions - 4.94, 4.96, 4.100, and 4.106 - to the massive Hamiltonian 4.91 gives

$$\begin{aligned}\mathcal{H}_M = \frac{1}{2}(\pi_{jk})^2 - \frac{1}{4}(\pi_{ll})^2 + \frac{1}{2}(\epsilon_{ijk}h_{kl,j})^2 - \frac{1}{2}h_{jk,k}^2 - \frac{1}{2}h_{jj,l}h_{kk,l} + \\ (h_{jj,l})^2 + \frac{1}{2}m^2[-(1+\lambda)h_{00}^2 + 2(1+\lambda)h_{00}h_{ll} - \lambda h_{ll}^2 + 2h_{0j}^2 + h_{jk}^2].\end{aligned} \tag{4.107}$$

where we have also integrated the $h_{jk,l}h_{lj,k}$ term by parts twice to replace it with $(h_{jk,l})^2$.

Since $h = -h_{00} + h_{jj}$, adding and subtracting a $\frac{1}{2}m^2h_{ll}^2$ lets us complete the square in the massive portion and find our final form for the massive Hamiltonian,

$$\begin{aligned}\mathcal{H}_M = \frac{1}{2}(\pi_{jk})^2 - \frac{1}{4}(\pi_{ll})^2 + \frac{1}{2}(\epsilon_{ijk}h_{kl,j})^2 - \frac{1}{2}h_{jk,k}^2 + \\ \frac{1}{2}h_{jj,l}^2 + \frac{1}{2}m^2[-(1+\lambda)h^2 + h_{ll}^2 + 2h_{0j}^2 + h_{jk}^2].\end{aligned} \tag{4.108}$$

Except for the terms in the square brackets at the end, $\mathcal{H}_M$ is equivalent to to $\mathcal{H}_0$. Since we are assuming that $\mathcal{H}_0 = 0$, we are interested only in the terms in the square brackets when looking for ghosts in linearized massive gravity, which we do in the next section.

5 Discussion

Does massive gravity have ghosts? For a physical theory, we should have five degrees of freedom with no ghosts since we are considering the graviton, a massive spin 2 particle. It depends on the choice of λ. We claim that $\lambda = -1$ is the only choice for λ which satisfies these requirements. First we examine the $\lambda = -1$ case.

For $\lambda = -1$, we find that the massive portion, the terms multiplied by a m^2, of $\mathcal{H}_M$ is non-negative. The massive portion would reduce to

$$\frac{1}{2}m^2[h_{ll}^2 + 2h_{0j}^2 + h_{jk}^2] \tag{5.1}$$

which is non-negative.

But what about the rest of Hamiltonian? We need the entire Hamiltonian to be non-negative to be sure of having no ghosts in the theory. It can be shown that the full, non-linearized massless Hamiltonian is equal to 0. Since the mass terms are only additive in the Lagrangian and thus only affect the equations of motion 4.71, 4.72, and 4.73 by adding additional terms involving an m^2, we can be sure that the collection of terms in the massive Hamiltonian $\mathcal{H}_M$ without an m^2 is equivalent to the Hamiltonian for the massless theory. That is, $\mathcal{H}_0$ will reduce to the terms without an m^2 in 4.108 with a bit of manipulation and using the massless equations of motion. This means that as long as we keep the massive portion of the massive Hamiltonian non-negative, we should have a ghost-free theory. $\lambda = -1$ satisfies this requirement. We can also check the sign of the energy that 4.7 gives us. If $\lambda = -1$, then 4.54 tells us that $h = 0$ and so 4.51 tells us that $h_{\mu\nu}{}^{,\nu} = 0$. This reduces 4.7 to

$$\Box h_{\mu\nu} = m^2 h_{\mu\nu}. \tag{5.2}$$

Recall that for units where $\hbar = 1$ and $c = 1$, $k^\mu = (E, \vec{p})$. Plugging in the solution $h_{\mu\nu} = A_{\mu\nu}\cos(k_\alpha x^\alpha)$ to the above equation and canceling the $\cos(k_\alpha x^\alpha)$ terms gives us

$$E^2 - \vec{p}^2 = m^2 \tag{5.3}$$

which is consistent with the energy-momentum relation.

So for $\lambda = -1$, we get a physical mass for the propagating modes and a non-negative massive portion of the Hamiltonian. Are there other λ values which would give us the same behavior?

For $\lambda \neq -1$, 4.54 would give us that $h \neq 0$ and in particular that

$$\Box h = \frac{-(1+4\lambda)}{2(1+\lambda)} m^2 h. \tag{5.4}$$

We can write this a bit more compactly by defining $M^2 = \frac{1+4\lambda}{2(1+\lambda)} m^2$, so

$$\Box h = -M^2 h. \tag{5.5}$$

Then the energy relation we get is

$$E^2 - \vec{p}^2 = -M^2 \tag{5.6}$$

which has the wrong sign for the energy. We could 'fix' this and try to change the sign in front of M^2. This would require

$$\lambda < -\frac{1}{4} \text{ and } \lambda > -1 \tag{5.7}$$

which is possible to satisfy, but this would make the sign in front of h^2 in the massive portion of the Hamiltonian negative. Alternatively, we could also 'fix' the sign in front of M^2 by requiring

$$\lambda > -\frac{1}{4} \text{ and } \lambda < -1 \tag{5.8}$$

but these are clearly incompatible. We can't both keep the massive portion of the Hamiltonian non-negative and also have the right sign for the energy in the energy-momentum relation. Furthermore, we end up with the wrong number of degrees of freedom. Losing the $h = 0$ condition which only comes from $\lambda = -1$ leaves only the $h_{\mu\nu}{}^{\nu} = -\lambda h_{,\mu}$ relation. While we can use this relation to reduce the number of degrees of freedom from ten down to six, we still have one degree of freedom too many. $\Box h = -M^2 h$ shows that this sixth degree of freedom, h, propagates, and we have just shown that it would have the wrong sign for the energy. Thus we see the appearance of a ghost mode if $\lambda \neq 1$. The only value for λ which keeps the Hamiltonian positive, has the correct number of degrees of freedom, and only propagates modes with the correct sign for the energy is $\lambda = -1$, which is the same result that Fierz and Pauli found.

6 Conclusion

We considered the question of whether it is possible to add a mass term to a linearized theory of gravity and how this mass term affects the theory. We first re-derived the Euler-Lagrange equations, showed how the Hamiltonian is a Legendre transform applied to the Lagrangian, and how the Hamiltonian converts N 2^{nd} order equations into 2N 1^{st} order equations. Then we extended the classical Lagrangian and Hamiltonian formulations to field theory and worked through a Lagrangian treatment of classical electrodynamics. The vector field solution for classical electrodynamics has two independent degrees of freedom which represent the photon's two possible polarizations. A third degree of freedom appears, but it doesn't propagate and gives zero $\vec{E}$ and $\vec{B}$ fields, so we call it an unphysical degree of freedom and discard it.

We explained that ghosts are propagating field solutions to an equation of motion with the wrong sign for the energy - hence they are unphysical and we need to keep them out of potential theories. We can check for ghosts by examining the Hamiltonian - if the Hamiltonian is guaranteed non-negative, the theory is ghost free. We can usually show this by writing the Hamiltonian as a sum of squared terms which are all positive. We showed that the scalar field - the 'Klein-Gordon' Lagrangian - for a massive spin-0 particle is ghostless, and hence a physical theory. Trying to extend the Klein-Gordon Lagrangian to a massive spin-1 particle by replacing the scalar field with a vector field doesn't work because it introduces a ghost - specifically the A_0 component. We saw this by looking at its Hamiltonian and seeing that all the components in the Hamiltonian with A_0 are negative. We also examined the Proca Lagrangian, which models a massive spin-1 particle and is a useful example of adding a mass term to a Lagrangian, and found that it is a ghost-free theory.

We then considered the case of linearized general relativity. The Lagrangian we used assumes a weak field with small perturbations $h_{\mu\nu}$ added to a flat spacetime $\eta_{\mu\nu}$ and linearizes Einstein's field equations around $h_{\mu\nu}$. We showed how the diffeomorphism for this Lagrangian acts as a gauge transformation and reduces the number of degrees of freedom from ten down to six. The residual gauge freedom reduces the number of degrees of freedom by another four, resulting in a final two degrees of freedom. This is the number of degrees of freedom we would expect for a massless spin-2 particle. We were not able to show explicitly that the massless linearized gravity theory has a non-negative Hamiltonian. However, since the Hamiltonian is zero in the full non-linear theory, we assume this holds as well in the linearized version.

Then we moved on to the massive linearized gravity Lagrangian. We added the most general mass term to the linearized gravity Lagrangian since we can add only quadratic terms to keep the equations of motion linear in $h_{\mu\nu}$. We keep the mass term general by including a parameter called λ. Fierz and Pauli claimed that $\lambda = -1$ is the only value for λ which keeps the theory ghost-free and physical. The resulting theory, that is with $\lambda = -1$, is called Fierz-Pauli gravity. We solved the massive Lagrangian for $\lambda = -1$ and found five degrees of freedom, which is the number of degrees of freedom expected for a massive spin-2 particle. We carried out a Hamiltonian analysis for the massive Lagrangian and found that the only way to keep massive portion of the massive Hamiltonian non-negative and have only five propagating degrees of freedom is to have $\lambda = -1$. If $\lambda \neq -1$, we see the appearance of an additional ghost mode.

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