Matrix Theory in a Simple Quantum Adder

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Matrix Analysis

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Presentation layout

Quantum what?

Some ideas about quantum mechanics.
Bits and qubits. Quantum states. Measurement. Collapsing. Reversible.
Quantum computation? Information?
The big picture.

Terminology

Physics terms and math terms.

Recipe

What do we need to make this simple circuit?

- Tensor Products
- Unitary Operations

Tensor Products

The *tensor product* of $\mathbf{V} = \mathbb{C}^{\Sigma_1}$ and $\mathbf{W} = \mathbb{C}^{\Sigma_2}$ is

$$\mathbf{V} \otimes \mathbf{W} = \mathbb{C}^{\Sigma_1 \times \Sigma_2}.$$

Elementary tensors span $\mathbf{V} \otimes \mathbf{W}$. For $|v\rangle \in \mathbf{V}$ and $|w\rangle \in \mathbf{W}$,

$$|v\rangle \otimes |w\rangle \equiv |v\rangle\,|w\rangle \equiv |vw\rangle \in \mathbf{V} \otimes \mathbf{W}.$$

Example: Representing the classical number "1" with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Tensor Products (cont.)

span$(|00\rangle, |01\rangle, |10\rangle, |11\rangle) = \mathbf{V} \otimes \mathbf{W}$, where

$$|00\rangle = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^\top, |10\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top, |11\rangle = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

A generic state: For $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$,

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.$$

Not every element of $\mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|\psi\rangle, |\phi\rangle$ such that

$$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \to \textbf{Entangled}.$$

Tensor Products (cont.)

Bilinearity:

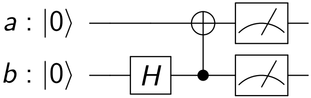
$$|a\rangle \otimes (\alpha\,|v\rangle + \beta\,|w\rangle) = \alpha\,|av\rangle + \beta\,|aw\rangle$$
$$(\alpha\,|v\rangle + \beta\,|w\rangle) \otimes |b\rangle = \alpha\,|vb\rangle + \beta\,|wb\rangle$$

For $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$, $\mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}\,|v\rangle) \otimes (\mathcal{B}\,|w\rangle)$$

$\rightarrow$ Useful when we apply a transformation to only one of two qubits.

Example: Entanglement



$$Hadamard = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}, CNOT_b = C_b = \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$$

$$\begin{aligned} C_b(I \otimes H)\left(\begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix}\right) &= C_b\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\right)\\ &= C_b\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0 & 0\end{bmatrix}^\top\\ &= \begin{bmatrix}1/\sqrt{2} & 0 & 0 & 1/\sqrt{2}\end{bmatrix}^\top \to \textbf{Entangled}\end{aligned}$$

Tensor Products (cont.)

Other properties:

- Associative
- Distributive
- Not commutative
- $(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$.
- $\mathrm{Tr}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{Tr}(\mathcal{A}) \cdot \mathrm{Tr}(\mathcal{B})$.
- $\det(\mathcal{A} \otimes \mathcal{B}) = (\det(\mathcal{A}))^m \cdot \det(\mathcal{B})^n$, where m is the dimension of $\mathcal{A}$ and n of $\mathcal{B}$.

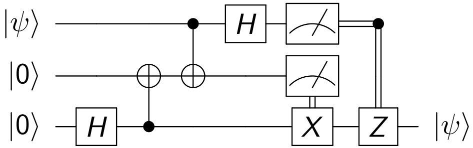
Unitary Operations

- Quantum Fourier Transform
- Control-phase
-

Control-phase gate

Again?

A sample quantum circuit.



Recap

What did we learn on the show tonight, Craig?
Q-circuit user guide [EF04]
quantum addition of classical numbers [CC16]
Mike and Ike [NC02]
Handbook of Linear Algebra [Hog07]
addition on quantum computer [Dra00]
QFT quick math [Bac]
Matrix analysis (where I read about unitary matrices) [HJ90]

References

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- Bryan Eastin and Steven T Flammia, *Q-circuit tutorial*, arXiv preprint quant-ph/0406003 (2004).
- Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge university press, 1990.
- Leslie Hogben, *Handbook of linear algebra*, Chapman and Hall/CRC, 2007.
- Michael A Nielsen and Isaac Chuang, *Quantum computation and quantum information*, 2002.