QUANTUM FIELD THEORY

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Preface. These notes come from Prof. Patton's QFT notes, Peskin-Schröder QFT, and "Quantum field theory & condensed matter" - Shankar.

Enjoy!

Conventions

- $\hbar = c = 1$
- $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$
- $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1\ -1\ -1\ -1)$
- $p^2 = p_\mu p^\mu = E^2 - |p|^2 = mc^2 = m$
- $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$
- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$
- $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$
- $\varepsilon^{0123} = +1$; $\varepsilon_{0123} = -1$

  $\varepsilon^{1230} = -1$
- $E = i\frac{\partial}{\partial x^0}$, $\vec{p} = -i\vec{\nabla}$
- $p^\mu = i\partial^\mu$

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $$\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \cancel{<} 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx}\theta(x)$

- n-dimensional Dirac δ-fn:

$$\int d^n x \, \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(x)$$

$$\tilde{f}(x) = \int d^4k \, e^{ik\cdot x} f(x)$$

- $$\int d^4x \, e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k)$$

- <u>EM</u> $\Phi = \frac{Q}{4\pi r}$ $\leftarrow$ Coulomb potential

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \quad ; \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\Phi, \vec{A}) \quad ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\boxed{\text{Elements of classical Field Theory}}$$

- Lagrangian Field Theory:

$$S = \int \underline{L}\, dt = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

Lag-density

$$\left(\underline{L} = \mathcal{L}\, d^3x\right)$$

(Lagrangian)

Principle of least action:

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right]\right\}$$

(E-L eqn)

$$\Rightarrow \boxed{0 = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial \mathcal{L}}{\partial\phi}}$$

FTC $\rightarrow$ term vanishes @ boundary

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

Ex

$$\mathcal{L} = \phi^2 : \qquad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi \; ; \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow 2\phi = 0$$

$$\left. \begin{array}{l} \mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) \\ \Rightarrow \dfrac{\partial \mathcal{L}}{\partial \phi} = 0 \; ; \; \dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2\partial^\mu \phi \end{array} \right\} \Rightarrow \partial^\mu \phi = 0.$$

Ex

Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

- $\dfrac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$

- $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi.$

relativistic particle of mass m.

$$\text{E-L} \Rightarrow -m^2 \phi = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$\rightarrow \boxed{(\Box + m^2)\phi = 0}$$

Klein-Gordon Eqn.

Ex $\phi = e^{-ip \cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

Noether's Thm

Thm

$\boxed{\text{For every continuous symmetry, there exists a conserved current } j^\mu \text{ which implies a local conservation law}}$

$$\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j} \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

Ex Conservation of charge ...

(gauss' law)

$$Q = \int j^0 \, d^3x$$

$$\Rightarrow \frac{\partial Q}{\partial t} = \int \frac{dj^0}{dt} d^3x = -\int \vec{\nabla}\cdot\vec{j} \, d^3x$$

$$= -\oint \vec{j} \, d^2s$$

//

Idea Consider continuous transf. → infinitesimally...

(local)

$$(\star) \qquad \phi(x) \to \phi'(x) = \phi(x) + \underset{\text{small}}{\alpha} \Delta\phi(x)$$

$(\star)$ is a symmetry if EOM invariant under $(\star)$.

$\Leftrightarrow$ S is invariant.

$\Rightarrow$ $\mathcal{L}$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$,

for some J^μ.

Let us compare this expectation for $\Delta\mathcal{L}$ to the result obtained by varying the fields ...

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi)$$

$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right\}\right)}_{0}\Delta\phi$$

$$\Rightarrow \quad \Delta\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right\}$$

So $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi$ is the desired J^μ.

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu .$$

Ex massless KG field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$$

Consider transformation:

$$\phi(x) \longrightarrow \phi(x) + \alpha \quad , \Delta\phi = 1$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = \partial^\mu\phi .$$

Check $\partial_\mu j^\mu = \partial_\mu\partial^\mu\phi = \Box\phi = 0$ since

$$(m^2 + \Box)\phi = 0 \quad \& \quad m = 0$$

Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi .$$

again, EOM is

$$(m^2 + \Box)\phi = 0.$$

Symmetry: $\phi \to e^{i\alpha}\phi$.

For infinitesimal transf we have:

$$\alpha D\phi = i\alpha\phi \qquad \text{(Taylor expand)}$$
$$\alpha D\phi^\dagger = -i\alpha\phi^\dagger$$

One can check that

$$j^\mu = i\left[(\partial^\mu\phi^\dagger)\phi - \phi^\dagger(\partial^\mu\phi)\right]$$

is the conserved current.

$\boxed{\text{Ex}}$

Space-time translation:

Consider infi. translation:

$$x^\mu \to x^\mu - a^\mu$$

in field transforms this becomes:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Lagrangian is a scaler ⇒ must transform the same way:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + a^{\mu}\partial_{\mu}\mathcal{L} = \mathcal{L} + a^{\nu}\partial_{\mu}(\delta^{\mu}_{\nu}\mathcal{L})$$

Compare this to the eqn:

$$\mathcal{L} \longrightarrow \mathcal{L} + a\,\partial_{\mu}J^{\mu},$$

we have
$$J^{\mu} = \delta^{\mu}_{\nu}\mathcal{L}$$

→ apply thm, we find:

$$J^{\mu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\phi)}(\partial_{\mu}\phi) - \delta^{\nu}_{\mu}\mathcal{L}$$

Make μ explicit...

$$\boxed{T_{\mu}^{\ \nu} = \frac{\partial\mathcal{L}}{\partial(\partial_{\nu}\phi)}\,\partial_{\mu}\phi - \delta_{\mu}^{\ \nu}\mathcal{L}}$$

→ STRESS - ENERGY TENSOR. (or Energy - Momentum tensor)

Conserved charge is the Hamiltonian...

$$H = \int T^{00}\,d^3x = \int \mathcal{H}\,d^3x \quad \text{(time-translation)}$$

momentum $\longrightarrow$ $$p^{i} = \int T^{0i}\,d^3x = -\int \pi\,\partial_i\phi\,d^3x \quad \text{(spatial translation)}$$

<u>Ex</u> Klein-Gordon field again...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 .$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} = \dots$$

$$= \frac{1}{2}(\partial_0 \phi)(\partial^0 \phi) - \frac{1}{2}(\partial_i \phi)(\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

<u>So</u>

$$H = \int T^{00} d^3x = \int \left\{ \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} d^3x$$

note: all terms are positive ... (sum of squares)

$\rightarrow$ can't fall into arbitrary negative energy.

$$\boxed{\text{THE KLEIN-GORDON FIELD} \sim \text{HARMONIC OSCILLATOR}}$$

<u>promote</u>: ϕ, π to operators .. impose suitable commutation relations...

Recall ....

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0 .$$

Harmonic oscillator: $H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2$

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$

- $a^\dagger a = \frac{1}{\omega}H - \frac{1}{2} \Leftrightarrow H = \omega\left(a^\dagger a + \frac{1}{2}\right)$

  $\uparrow$

  # operator...

- $|0\rangle$, $a|0\rangle = 0$.

  $|n\rangle = \frac{1}{\sqrt{n!}}\underbrace{a^\dagger \ldots a^\dagger}_{n}|0\rangle$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -\omega a$; $[H, a^\dagger] = \omega a^\dagger$

  a lowers by ω

  $a^\dagger$ raises by ω

- $Ha^\dagger|n\rangle = (E_n + \omega)a^\dagger|n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

For continuum system ... commutation relations become ..

$$\boxed{\begin{aligned} [\phi(x), \pi(y)] &= i\delta^{(3)}(x-y) \\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0. \end{aligned}}$$

Next, the Hamiltonian is now also an operator...
To find spec(H), Fourier transf $\phi(x)$

$$\rightarrow \quad \phi(x,t) = \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \phi(\vec{p},t)$$

Recall KG eqn: $(m^2 + \Box)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) \right\} \phi(\vec{p},t) = 0$$

$\Rightarrow$ This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}.$$

Now, $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2 \qquad (m=1)$

$\rightarrow$ know spectrum! $\quad (n+\frac{1}{2})\omega.$

$$\phi = \frac{1}{\sqrt{2\omega}}(a^\dagger + a) \quad ; \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$[a, a^\dagger] = 1.$$

Since it's more convenient to work in position space

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right) \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^\dagger e^{-ip x} \right) \end{aligned}}$$

note

$$\begin{cases} a_p \text{ goes with } e^{+ip\cdot x} \\ a_p^\dagger \text{ goes with } e^{-ip\cdot x}. \end{cases}$$

- Easy to show that

$$\begin{cases} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \end{cases}$$

- Can re-arrange...

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p + a_{-p}^\dagger \right) e^{ip\cdot x} \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p - a_{-p}^\dagger \right) e^{+ip\cdot x}. \end{cases}$$

→ get commutation relation between $a_{\vec{p}}$:

$$\boxed{[a_p ; a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

Now, can check that

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{\omega_{p'}}{\omega_p}} \times e^{i(p\cdot x + p'\cdot x')} \times$$

$$\left([a^\dagger_{-p}, a_{p'}] - [a_p, a^\dagger_{-p'}] \right)$$

$$= i\delta^{(3)}(x - x') \quad \checkmark$$

- Now, can express Hamiltonian in terms of ladder ops

Recall that → KG field, last time

$$H = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \right)$$

So ...

$$H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

with $\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} (a_p - a^\dagger_{-p}) e^{ip\cdot x}$ we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p + a^\dagger_{-p}) e^{ip\cdot x}$$

results in $\delta(p-p')$

$\Rightarrow p = p'$

$$\mathcal{H} = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(p+p')\cdot x} \left\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4} (a_p - a_{-p}^\dagger)(a_{p'} - a_{-p'}^\dagger) \right.$$

$$\left. + \frac{-p\cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}} (a_p + a_{-p}^\dagger)(a_{p'} + a_{-p'}^\dagger) \right\}$$

Some $\delta^{(3)}$ will appear...

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)$$

So

$$\boxed{\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)}$$

$\rightarrow$ can evaluate commutators...

$$\boxed{[H, a_p^\dagger] = \omega_p a_p^\dagger \; ; \; [H, a_p] = -\omega a_p}$$

With $\mathcal{H}$, can find momentum operator...

KG field $\longrightarrow$ from $P^i = \int d^3x\, T^{0i} = -\int \pi \partial_i \phi\, d^3x$, we get

$$\boxed{\begin{aligned} \vec{P} &= -\int d^3x\, \pi(x) \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a_p^\dagger a_p \end{aligned}}$$

$a_p^\dagger$ creates momentum $\vec{p}$ & energy $\underset{E_p}{\omega_p} = \sqrt{|p|^2 + m^2}$. $\nearrow 0$

Excitation: $a_p^\dagger a_q^\dagger \ldots |0\rangle \equiv$ "particles".

$\hookrightarrow$ each excitation $a_p^\dagger$ is a particle.

$\Rightarrow$ get particle statistics...

Consider 2-particle state... $a_p^\dagger a_q^\dagger |0\rangle$.

Since $[a_p^\dagger, a_q^\dagger] = 0$, we have

$$\boxed{a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle}$$

Klein Gordon particles follow Bose-Einstein stats.

- Normalization $\langle 0|0\rangle = 1$.

$$|p\rangle \propto a_p^\dagger |0\rangle .$$

This normalization $\longrightarrow$ $\langle q|p\rangle = (2\pi)^3 \delta^{(3)}(q-p)$ is NOT Lorentz inv

Pf Under a Lorentz boost $\rightarrow$ $p_3' = \gamma(p_3 + \beta E)$

$$E' = \gamma(E + \beta p_3)$$

using the identity $\rightarrow$ $\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$

we compute: $\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \cdot \left(\frac{dp_3'}{dp_3}\right)$

$\underbrace{\delta(p_1-q_1)\delta(p_2-q_2)}_{\text{same}}\underbrace{\delta(p_3-q_3)}_{\text{boosted}}$

$$= \delta^{(3)}(p'-q') \cdot \gamma\left(1 + \beta \, dE/dp_3\right)$$

$$= \delta^{(3)}(p'-q') \frac{\gamma}{E}(E + \beta p_3)$$

$$= \delta^{(3)}(p'-q')(E'/E)$$

So $\boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\left(\frac{E'}{E}\right)}$ → $1 \Leftrightarrow E = E_p = E'$

For normalization to work → use $\underset{\overset{|||}{w_p}}{E_p}$, not E.

→ define: $\boxed{|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$

So that $\boxed{\langle p|q\rangle = 2E_p\,(2\pi)^3\,\delta^{(3)}(p-q)}$

Completeness relation...

1 particle → $\boxed{1 = \int \frac{d^3p}{(2\pi)^3}\, |p\rangle \frac{1}{2E_p} \langle p|}$

Ex Interpret $\phi(x)|0\rangle$ .... we know that $a_p^\dagger$ creates momentum p & energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x} |p\rangle \quad , \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

Pf. By defn...

annihilates.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|0\rangle \quad \square$$

→ $\phi(x)|0\rangle$ is a lin. superposition of single-particle states

~~Note HW1, HW2 easy~~

that have well-defn momentum.

When nonrelativistic $\rightarrow$ $E_p \approx$ constant!

$\Rightarrow$ [$\phi(x)$ acting on the vacuum, "creates a particle at position x".]

↳ Confirm this by computing.

$$\langle 0|\,\phi(x)\,|p\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\left(a_{p'}e^{ip'\cdot x} + a^\dagger_{p'}e^{-ip'\cdot x}\right)\sqrt{2E_p}\,a_p^\dagger|0\rangle$$

$$\rightarrow \boxed{\langle 0|\,\phi(x)\,|p\rangle = e^{ip\cdot x}}.$$

↳ Interpretation: position-space representation of the single-particle wfn of the state $|p\rangle$, just like ...

$\boxed{\langle x|p\rangle \propto e^{ip\cdot x}}$ in QM!

$\langle 0|\,\phi(x) \sim \langle x|$ ... (don't take this literally, ofc).

Note HW1, HW2 are easy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time → we quantized KG field in the Schrödinger picture.

→ Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$$\mathcal{U}(t) = e^{-iHt} \text{ is the time evolution.}$$

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle. \rightarrow \underline{\underline{\text{State}}} \text{ evolve in time}$$

→ In the Heisenberg picture,... $\underline{\text{Operators}}$ evolve in time

$$\mathcal{O}(t) = \mathcal{U}^\dagger(t)\, \mathcal{O}(0)\, \mathcal{U}(t).$$

So that

$$\langle \psi_1 | \mathcal{O}(t) | \psi_1 \rangle = \langle \psi_1(t) | \mathcal{O} | \psi_1(t) \rangle$$

↓ Heisenberg ↓ Schrödinger.

→ make the operators ϕ & π time-dependent like this

$$\boxed{\begin{aligned} \phi(x) \rightarrow \phi(x,t) &= e^{iHt}\, \phi(x)\, e^{-iHt} \\ \pi(x) \rightarrow \pi(x,t) &= e^{iHt}\, \pi(x)\, e^{-iHt} \end{aligned}}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\right]$$

$$= \int d^3x' \left(i\delta^{(3)}(x-x')\,\pi'(x,t)\right)$$

$(\phi \leftrightarrow \phi)$
→ only nontrivial term is 1st.

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t}\phi(x,t) = \pi(x,t)}$$

and

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')(-\nabla^2+m^2)\phi(x',t)\right)$$

(integrate by parts here)

$$= -i(-\nabla^2+m^2)\phi(x,t)$$

$$\rightarrow \boxed{\frac{\partial}{\partial t}\pi(x,t) = (m^2-\nabla^2)\phi(x,t)}$$

Combining these 2 results we get...

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) = (\nabla^2 - m^2)\phi(x,t)}$$

↳ rearranging this gives

$$\boxed{(\Box + m^2)\phi(x,t) = 0} \longrightarrow$$ just the KG eqn...

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

<u>Recall</u>: $H a_p = a_p (H - E_p)$ → from comm. rule.

$\Rightarrow$ (proof by induction) $\boxed{H^n a_p = a_p (H-E_p)^n}$

Similarly, $\boxed{H^n a_p^\dagger = a_p^\dagger (H+E_p)^n}$

→ So, we have

$$\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}}$$ → from above...

and

$$\boxed{e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{+iE_p t}}$$

→ Now... we want to write $\phi(x,t)$ in terms of these operators. (since $\phi(x)$, $\pi(x)$ is a combo of a & $a^\dagger$)

- We know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before...

$$\phi(x) = \phi(x,0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

Substitute this into $\phi(x,t) = e^{iHt}\phi(x) e^{-iHt}$ we find

...

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-iE_pt} e^{ip\cdot x} + a_p^+ e^{iE_pt} e^{-ip\cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow p\cdot x = E_p \cdot x^0 - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p} = E_pt - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p}$$

So...

$$\boxed{\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p}\cdot\vec{x}} + a_p^+ e^{+i\vec{p}\cdot\vec{x}} \right\}\Big|_{p^0=E_p}}$$

Similarly, we can find

$$\boxed{\pi(x,t) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_p}{2}} \left(a_p e^{-i\vec{p}\cdot\vec{x}} - a_p^+ e^{+i\vec{p}\cdot\vec{x}} \right)\Big|_{p^0=E_p}}$$

from defn or from $\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$.

<u>Note</u> we can also do everything, but starting from P and not $\mathcal{H}$. But we won't worry about that.

<u>Causality</u> Note that causality is broken ~~when~~ without the presence of fields.

$\rightarrow$ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture...

...the amplitude for a particle to propagate from y to x is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

"Two-point correlation function"

Let $\boxed{D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle}$

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \text{ in terms of } a_p \text{ \& } a_p^\dagger \ldots |0\rangle$$

$$= \langle 0|\, a_p\, a_q^\dagger\, |0\rangle =$$

$$= (2\pi)^3\, \delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

More explicitly...

@ $p^0 = E_p$, $p^{0\prime} = E_{p'}$

$$D(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0|\, e^{-ip\cdot x}\, e^{+ip'\cdot y}\, a_p a_{p'}^\dagger\, |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^{\cancel{6}}} \left(\frac{1}{\sqrt{2E_p}}\right)\frac{1}{\sqrt{2E_{p'}}}\, e^{-ip\cdot x + ip'\cdot y}\, \cancel{(2\pi)^3}\, \delta^{(3)}(p-p')$$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip\cdot(x-y)} \qquad \square .$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$...

(1) Suppose that $x-y = (t,0,0,0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}}\, e^{-i\sqrt{p^2+m^2}\,t}$$

(timelike)
$$= \frac{1}{4\pi^2}\int_m^\infty dE\, \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\underset{t\to\infty}{=} e^{-imt}.$$ → dominated by region where $p \approx 0$.

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0,\vec{r})$ then

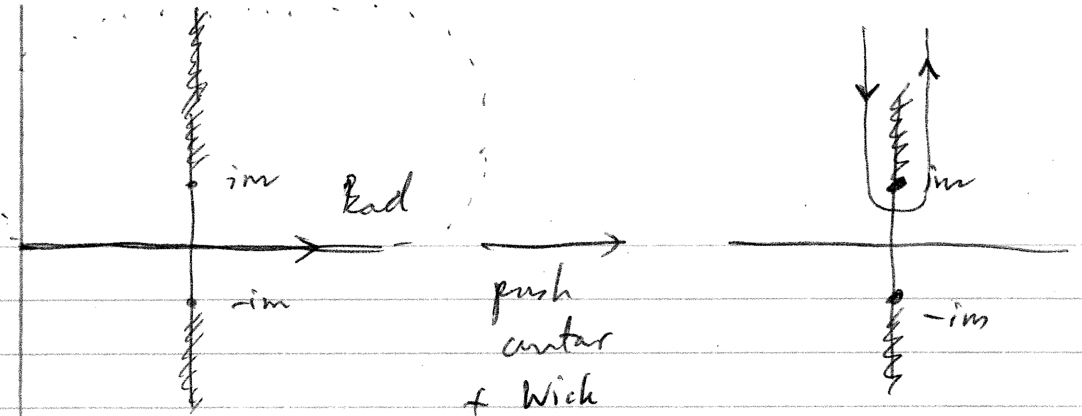
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3}\, \frac{1}{2E_p}\, e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2E_p}\, \frac{e^{ipr}-e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^\infty dp\, \frac{p\,e^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ $\pm im$ (singularity)...

→ must change contour... & Wick rotate



To get

$$D(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}}$$

$\rho = -ip$ (Wick rotate)

$$\Rightarrow \boxed{D(x-y) \underset{r\to\infty}{=} \frac{e^{-mr}}{r}} \longrightarrow \text{non zero!}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $D(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^\mu (x-y)_\mu = -(\vec{x}-\vec{y})^2 < 0$ is spacelike, cannot have causality between x & y.

$D(x-y) \neq 0 \Rightarrow$??? paradox?

→ NO! To discuss causality, we should ask not whether particles can propagate over spacelike intervals ...

... but whether a "measurement" at one point can affect a measurement at another point whose separation from the first is spacelike –

$\rightarrow$ No violation of causality b/c no signal can be transmitted $\rightarrow$ no info is exchanged.

$\rightarrow$ Suppose we have a local measurement @ x, call this $\phi(x)$. & a local measurement @ y, called $\phi'(y)$

So long as $[\phi(x), \phi'(y)] = 0$, the 2 measurements don't affect one another.

$\sim$ measure the field ϕ @ x & @ y,

If $[\phi(x), \phi(y)] = 0$ ~~when~~ $(x-y)^2 < 0$ then we're good

$$\rightarrow \cancel{[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x}[a_p, a^\dagger_{p'}] e^{ip'\cdot y} + e^{ip\cdot x}[a^\dagger_p, a_{p'}] e^{-ip'\cdot y} \right\}}$$

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \times \left\{ \left[(a_p e^{-ip\cdot x} + a^\dagger_p e^{ip\cdot x}), (a_{p'} e^{-ip'\cdot y} + a^\dagger_{p'} e^{ip'\cdot y}) \right] \right.$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6\, 2} \frac{1}{\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x} \underbrace{[a_p, a^\dagger_{p'}]}_{(2\pi)^3 \delta^3(p-p')} e^{ip'\cdot y} + e^{ip\cdot x} \underbrace{[a^\dagger_p, a_{p'}]}_{-(2\pi)^3 \delta^3(p-p')} e^{-ip'\cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p}\right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

So $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$.

Since $D(y-x)$ is Lorentz invariant, can take

$y-x \rightarrow x-y$. (possible b/c $(x-y)^2 < 0$)

Note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

$\rightarrow$ so this is only possible because $(x-y)^2 < 0$ (spacelike).

So $D(y-x) = D(x-y)$

So, $[\phi(x), \phi(y)] = 0$ ✓

The Klein-Gordon Propagator