Another way to write Wick's theorem is

$$T\{\phi_1\phi_2\cdots\phi_n\}$$
$$= N\left\{\exp\left[\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \wick{\c\phi_i \c\phi_j}\,\frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}\right]\phi_1\phi_2\cdots\phi_n\right\}$$

Feynman diagrams

$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \text{all } \underset{\text{contractions}}{\text{possible}}\}$$

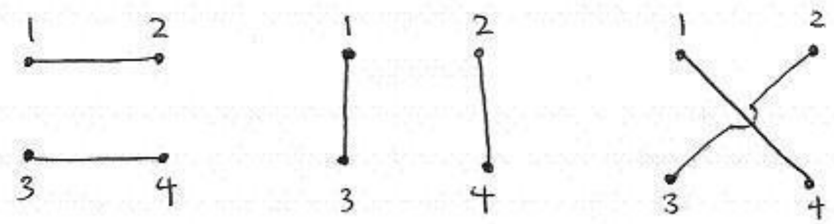
... but the only contribution to

$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle \text{ is when}$$

all the ϕ's are contracted

$$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle$$
$$= \wick{\c\phi_1\c\phi_2\,\c\phi_3\c\phi_4} + \wick{\c1\phi_1\c2\phi_2\c1\phi_3\c2\phi_4} + \wick{\c1\phi_1\c2\phi_2\c2\phi_3\c1\phi_4}$$

We can write this as "Feynman" diagrams



$\phi_1 \phi_2 \phi_3 \phi_4$ $\qquad$ $\phi_1 \phi_2 \phi_3 \phi_4$ $\qquad$ $\phi_1 \phi_2 \phi_3 \phi_4$

Let us consider something like

$$\langle 0 | T \{ \phi(x) \phi(y) \exp\left[-i \int_{-\infty}^{\infty} dt \, H_I(t) \right] \} | 0 \rangle$$

As a power series in the coupling λ,
the lowest order term is just

$$\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle =$$

$$= D_F(x-y)$$

At first order in λ, we have

$$\langle 0 | T \{ \phi(x) \phi(y) (-i) \int d^4z \, \frac{\lambda}{4!} \phi^4(z) \} | 0 \rangle$$

$$= -\frac{i\lambda}{4!}\int d^4z \, \langle 0 | T\{\phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z)\} | 0 \rangle$$

$$= -\frac{i\lambda}{4!}\int d^4z \left(\begin{array}{l} \phi(x)\phi(y)\cdot\left\{\phi(z)\phi(z)\phi(z)\phi(z) + \phi(z)\phi(z)\phi(z)\phi(z) \right. \\ \qquad\qquad\qquad \left. + \, \phi(z)\phi(z)\phi(z)\phi(z)\right\} \\ + \, \phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z) \\ + \quad \vdots \end{array} \right\} \text{12 such terms} \Bigg)$$

We can write this as

$$\propto \int d^4z \, D_F(x-y)\, D_F(z-z)\, D_F(z-z) \qquad\qquad \propto \int d^4z \, D_F(x-z)\, D_F(z-z)\, D_F(z-y)$$

three such terms $\qquad\qquad$ 12 such terms

How do we count these combinatorial factors?

Each H_I has 4 ϕ's: $\quad \phi(z)\phi(z)\phi(z)\phi(z)$

Clearly interchanging the contraction "ends" on these ϕ's will give the same amplitude ...

$$\cdots \ \phi(z)\,\phi(z)\,\phi(z)\,\phi(z) \ \cdots$$

$$\parallel$$

$$\cdots \ \phi(z)\,\phi(z)\,\phi(z)\,\phi(z) \ \cdots$$

$$\parallel$$

$$\cdots \ \phi(z)\,\phi(z)\,\phi(z)\,\phi(z) \ \cdots$$

$$\parallel$$

$$\vdots$$

(subtlety discussed later)

So for each H_I we "expect" a factor of $4!$ from the 4 identical ϕ's. This cancels the $\frac{1}{4!}$ in $\frac{\lambda}{4!}\phi^4$ and is the reason we put the $\frac{1}{4!}$ in the interaction.

In a diagram with more than one power of H_I, we can exchange all the contraction ends of one H_I with contraction ends of the other H_I:

$$\phi(z_1)\,\phi(z_1)\,\phi(z_1)\,\phi(z_1) \quad \phi(z_2)\,\phi(z_2)\,\phi(z_2)\,\phi(z_2)$$

$$\parallel$$

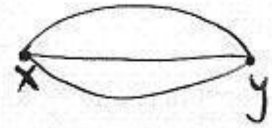
$$\phi(z_1)\,\phi(z_1)\,\phi(z_1)\,\phi(z_1) \quad \phi(z_2)\,\phi(z_2)\,\phi(z_2)\,\phi(z_2)$$

Since we integrate over z_1 + z_2, these give the same value for the amplitude. So for a diagram with n "internal vertices" (i.e., # of H_I's) we get a factor of $n!$. This cancels the $\frac{1}{n!}$ we get from the power series expansion of $\exp\{-i\int H_I(t)\,dt\}$. But there is a small subtlety....

Symmetry factors

It is best to consider the simplest diagram with the most general problem, and so we consider a simpler $\frac{\lambda}{3!}\phi^3$ theory for the moment.

At second order in λ, $\langle 0|\, T\left\{\exp\left[-i\int_{-\infty}^{\infty} H_I(t)\,dt\right]\right\}|0\rangle$ gives something $\propto \int \langle 0|\phi(x)\phi(x)\phi(x)\;\phi(y)\phi(y)\phi(y)|0\rangle \cdot d^4x\, d^4y$



$D_F(x-y)\,D_F(x-y)\,D_F(x-y)$

Naively we expect $2!$ from interchanging x + y and $3! \times 3! = 36$ from interchanging the ϕ's at x and ϕ's at y. So we expect 72 such terms. But in fact there are only <u>six</u>:

first ϕ_x — first ϕ_y :

$$\begin{cases} \phi_x \phi_x \phi_x \quad \phi_y \phi_y \phi_y \\ \phi_x \phi_x \phi_x \quad \phi_y \phi_y \phi_y \end{cases}$$

first ϕ_x — second ϕ_y :

$$\begin{cases} \phi_x \phi_x \phi_x \quad \phi_y \phi_y \phi_y \\ \phi_x \phi_x \phi_x \quad \phi_y \phi_y \phi_y \end{cases}$$

first ϕ_x — third ϕ_y :

$$\begin{cases} \phi_x \phi_x \phi_x \quad \phi_y \phi_y \phi_y \\ \phi_x \phi_x \phi_x \quad \phi_y \phi_y \phi_y \end{cases}$$

We have overshot by factor of $\frac{72}{6} = 12$. Why?

Let us start with

$$\phi_x \phi_x \phi_x \quad \phi_y \phi_y \phi_z$$

Consider what happens if we exchange the first + second ϕ_x's and exchange the first + second ϕ_y's (simultaneously):

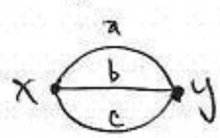
$$\phi_x \phi_x \phi_x \ \phi_y \phi_y \phi_y \longrightarrow \phi_x \phi_x \phi_x \ \phi_y \phi_y \phi_y \quad \underline{\underline{\text{same}}}$$

Notice that it gives the same thing!

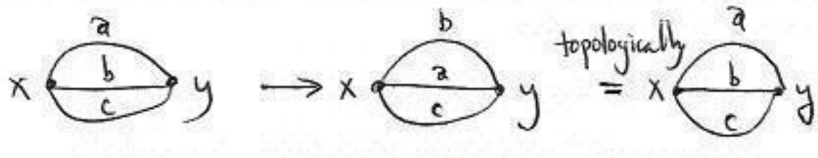
Also interchanging all the ϕ_x's with ϕ_y's won't do anything either...

$$\phi_x \phi_x \phi_x \ \phi_y \phi_y \phi_y \longrightarrow \phi_y \phi_y \phi_y \ \phi_x \phi_x \phi_x \quad \underline{\underline{\text{same}}}$$

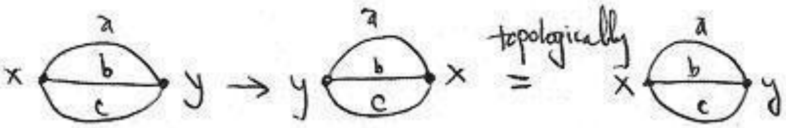
We can see what is going on by labelling the vertices + propagators in our Feynman diagram:



We see that interchanging the propagators a, b, c does not change the topology of the diagram:



Also exchanging the internal vertices $x \leftrightarrow y$ does not alter the diagram:



So our diagram has a permutation symmetry group. The symmetry group has $\underset{\substack{\uparrow \\ x \leftrightarrow y}}{2!} \times \underset{\substack{\uparrow \\ \{a,b,c\} \\ \text{permutations}}}{3!} = 12$ elements.

This is called the symmetry factor, S. In our case $S = 12$. The number of diagrams is

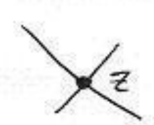
$$\underset{\substack{\uparrow \\ 12}}{\frac{1}{S}} \cdot \underbrace{(3!)(3!)(2!)}_{\substack{\text{what we would} \\ \text{have thought naively} \\ \text{to be the number of diagrams}}} = 6$$

Feynman Rules (in position space)

For each propagator

$$= D_F(x-y)$$

For each internal vertex



$$= (-i\lambda) \int d^4 z$$

note: no $\frac{1}{4!}$ here

Divide by symmetry factor S.

Example

$$S = 3! = 6$$

(permute the three propagators from z_1 to z_2)

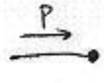
$$\text{Amplitude} = \frac{(-i\lambda)^2}{6} \int d^4 z_1\, d^4 z_2\, D_F(x-z_1) \left(D_F(z_1-z_2)\right)^3 D(z_2-y)$$

Feynman Rules (in momentum space)

For each propagator

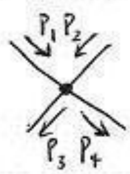
$$= \frac{i}{p^2 - m^2 + i\varepsilon}$$

For each external vertex



$$= e^{-ip\cdot x}$$

For each interal vertex



$= -i\lambda$ and momentum conservation

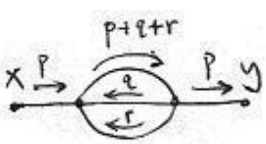
(i.e., $(2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)$)

Integrate over all momenta that are unconstrained...

$$\int \frac{d^4 p}{(2\pi)^4}$$

Divide by symmetry factor S.

Example

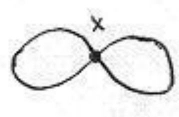


$$= \frac{(-i\lambda)^2}{6} \int \left\{ \left(\frac{i}{p^2 - m^2 + i\varepsilon} \right)^2 \frac{i}{(p+q+r)^2 - m^2 + i\varepsilon} \, \frac{i}{q^2 - m^2 + i\varepsilon} \, \frac{i}{r^2 - m^2 + i\varepsilon} \, \frac{d^4 q \, d^4 r}{(2\pi)^4 (2\pi)^4} \right\}$$

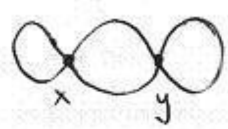
Let us consider diagrams without external vertices... diagrams that contribute to

$$\langle 0 | T \exp\{-i \int_{-\infty}^{\infty} dt \, H_I(t)\} | 0 \rangle$$

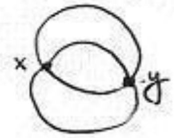
These are called "vacuum" diagrams. At order λ we have vacuum diagram



At order λ^2 we have



and

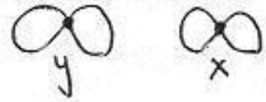


But at order λ^2 we also have the disconnected diagram

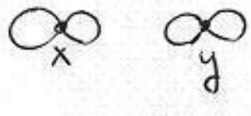
The amplitude for this disconnected diagram is the product of the amplitudes for the connected pieces.

However, note that there is a symmetry factor

$S = 2$, since

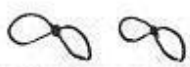


is topologically equivalent to

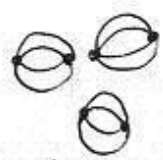


In general a vacuum diagram has say connected subdiagrams V_i which appear n_i times:

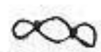
Example: V_1 V_2 V_3



$n_1 = 2$



$n_2 = 3$



$n_3 = 1$

We will use V_i to represent both the diagram and the corresponding amplitude.

The amplitude for the total diagram is then

$$\prod_i \left(\frac{1}{n_i!} V^{n_i} \right)$$

So the sum over all vacuum diagrams can be written as

$$\left(1+\frac{(V_1)^1}{1!}+\frac{(V_1)^2}{2!}+\frac{(V_1)^3}{3!}+\cdots\right)\left(1+\frac{(V_2)^1}{1!}+\frac{(V_2)^2}{2!}+\frac{(V_2)^3}{3!}+\cdots\right)$$

$\times \cdots$ ← for all V_i

Notice how each monomial in this product has the form

$$\frac{(V_1)^{n_1}}{n_1!}\frac{(V_2)^{n_2}}{n_2!}\cdots$$

for some unique $n_1, n_2, \ldots$

We can write this product as

$$\prod_i \left(\sum_{n_i=0}^{\infty}\left(\frac{1}{n_i!}V_i^{n_i}\right)\right)$$

$$= \prod_i e^{V_i} = \exp\left(\sum_i V_i\right)$$

Last time we showed that

$$\langle 0 | T \{ \exp[-i \int_{-\infty}^{\infty} H_I(t)\, dt] \} | 0 \rangle$$

= sum of all vacuum diagrams

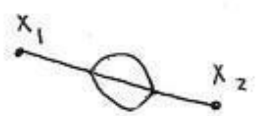
= exponential of all connected vacuum diagrams

Note that the n-point function

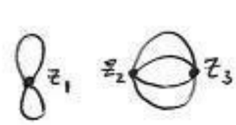
$$\langle 0 | T \{ \phi(x_1) \cdots \phi(x_n) \exp[-i \int_{-\infty}^{\infty} H_I(t)\, dt] \} | 0 \rangle$$

$$= \left[\begin{array}{l} \text{sum of connected} \\ \text{diagrams with endpoints} \\ x_1, x_2, \cdots x_n \end{array} \right] \times \left[\begin{array}{l} \text{sum of all} \\ \text{vacuum diagrams} \end{array} \right]$$

Example:



connected diagram with endpoints x_1, x_2



vacuum diagram

Therefore

$$\frac{\langle 0 | T \{ \phi(x_1) \cdots \phi(x_n) \exp[-i \int_{-\infty}^{\infty} H_I(t)\, dt] \} | 0 \rangle}{\langle 0 | T \{ \exp[-i \int_{-\infty}^{\infty} H_I(t)\, dt] \} | 0 \rangle}$$

= sum of connected diagrams with endpoints $x_1, x_2, \cdots x_n$