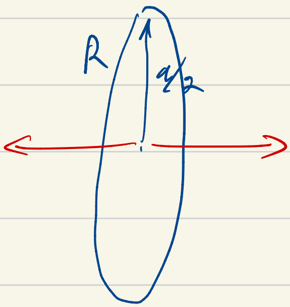
8.311. EM Theory

Pset 7, due April 27, 2022

[Name illegible]

(1)



$\vec{B}(t) = B_0 \cos(\omega t)\,\hat{z}$

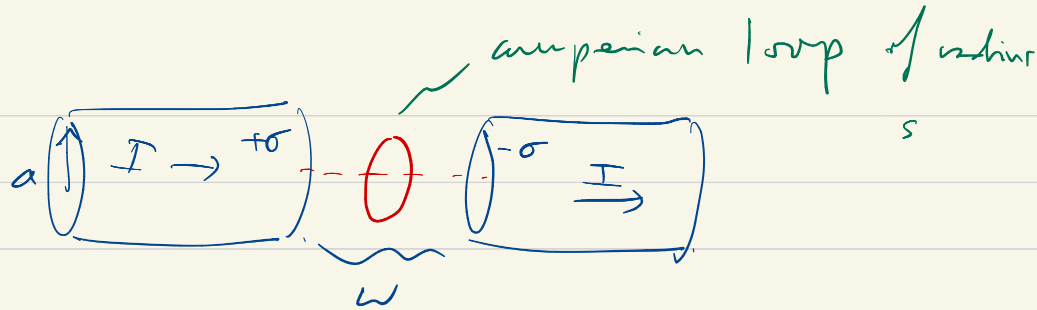
$$\Phi_B = \vec{B}\cdot\vec{A} = B_0 \cos(\omega t)\,\pi\left(\frac{a}{2}\right)^2$$

$\Rightarrow$ Lenz law:

$$\mathcal{E} = -\frac{\partial}{\partial t}\Phi_B = \omega B_0 \sin(\omega t)\,\pi\frac{a^2}{4}$$

Current $\boxed{i(t) = \frac{\mathcal{E}}{R} = \frac{\omega B_0 \pi a^2}{4R}\sin(\omega t)}$

2



From Ampère's law with Maxwell's addition ...

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt}\Phi_E$$

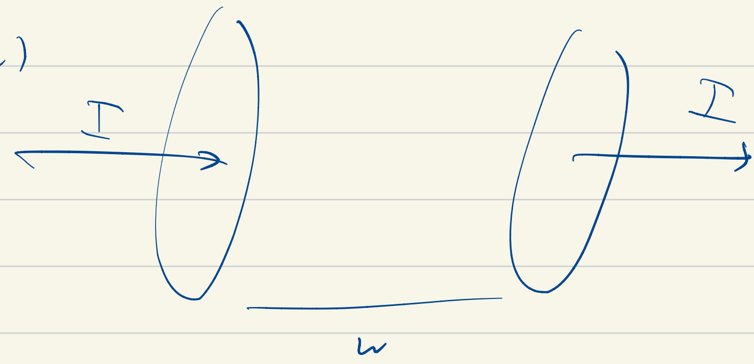
$Q(t)/\pi a^2$

$$= \mu_0 \varepsilon_0 \frac{d}{dt}\left(\frac{\sigma}{\varepsilon_0}\pi s^2\right)$$

$$B \cdot 2\pi s = \mu_0 \varepsilon_0 \frac{I}{\varepsilon_0}\frac{s^2}{a^2} = \frac{\mu_0 I s^2}{a^2}$$

$$\Rightarrow \boxed{\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

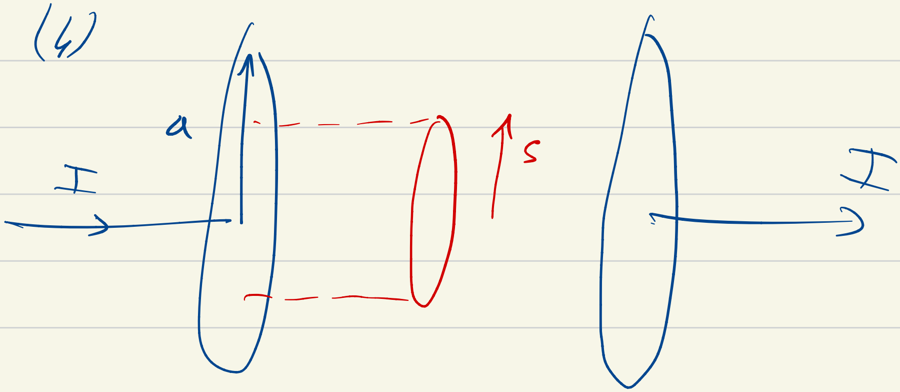
(3) (a)



$$\vec{E}_{in} = \frac{\sigma(t)}{\varepsilon_0}\hat{z} = \frac{Q(t)}{\pi a^2 \varepsilon_0}\hat{z}$$

$$= \boxed{\frac{It}{\pi a^2 \varepsilon_0}\hat{z}}$$

(b)



Intuitively, the displacement current is simply I multiplied by the ratio s^2/a^2 due to the Amperian loop

But ok... $I_d = \varepsilon_0 \cdot \frac{\partial}{\partial t}\Phi_E$

$$= \varepsilon_0 \cdot \frac{\partial}{\partial t}\left\{ \frac{It}{\pi a^2 \varepsilon_0} \cdot \underbrace{\pi s^2}_{A} \right\}$$

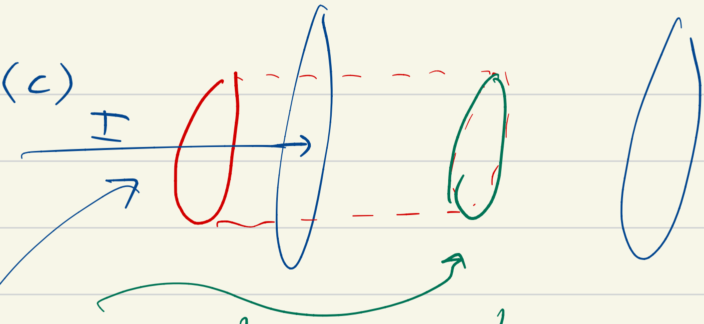
$$\boxed{I_d = Is^2/a^2}$$

Then $\vec{B}$ is found by Amp's law:

$$B \cdot 2\pi s = I_d \mu_0 = \frac{I\mu_0 s^2}{a^2}$$

$$\to \boxed{\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

(c)



amperian loop is here now ---

But calculate flux through here.

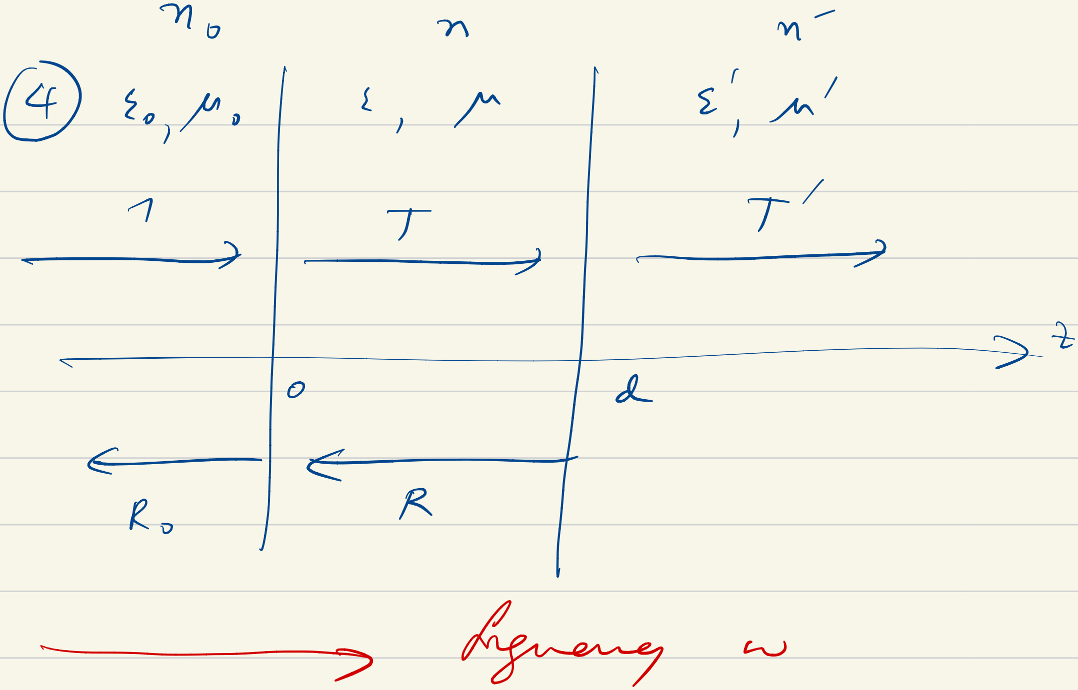
In this case --- the total current enclosed by the Amperian loop is the current thru the surface, which is simply

$$I\, s^2/a^2$$

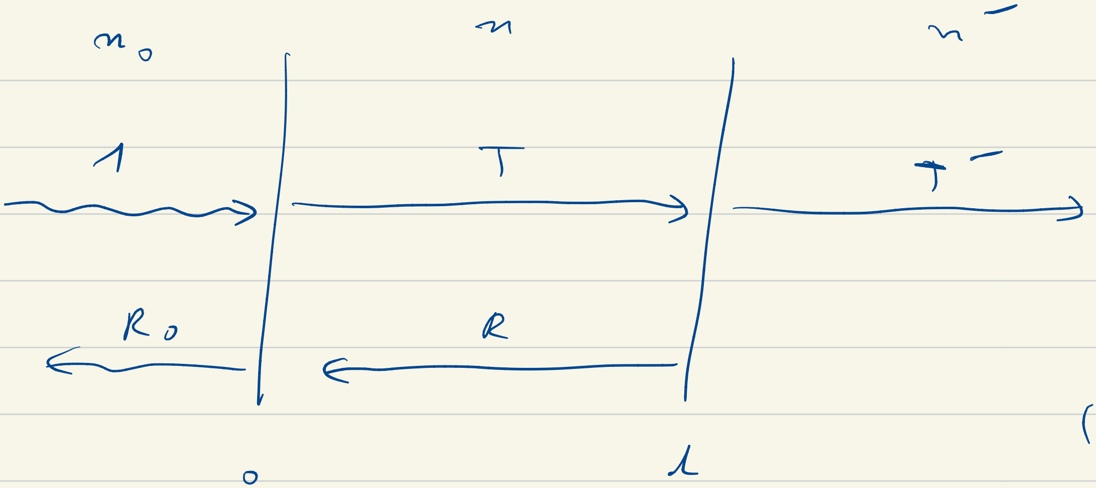
So we end up with

$$\boxed{\vec{B} = \frac{\mu_0 I s}{2\pi a^2}\,\hat{\phi}}$$

like before



- To do this... have to match boundary conditions



(2)

@ $z = 0$ ... $1 + R_0 = T + R$

@ $z = d$ ... $Te^{ikd} + Re^{-ikd} = T'e^{ik'd}$

@ $z = 0$ $\qquad \dfrac{1 - R_0}{Z_0} = \dfrac{T - R}{Z} \qquad (7.67)$

@ $z = d$ $\qquad \dfrac{Te^{ikd} - Re^{-ikd}}{Z} = \dfrac{T'e^{ik'd}}{Z'}$

- Only care about R_0 in relation to constants of the problem, r_0

can solve for R_0 in terms of Z's...

- Mathematica...

$$R_0 = \frac{-i Z (Z_0 - Z') \cos(dk) + (Z^2 - Z_0 Z') \sin(dk)}{i Z (Z_0 + Z') \cos(dk) + (Z^2 + Z_0 Z') \sin(dk)}$$

Want destructive interference, so

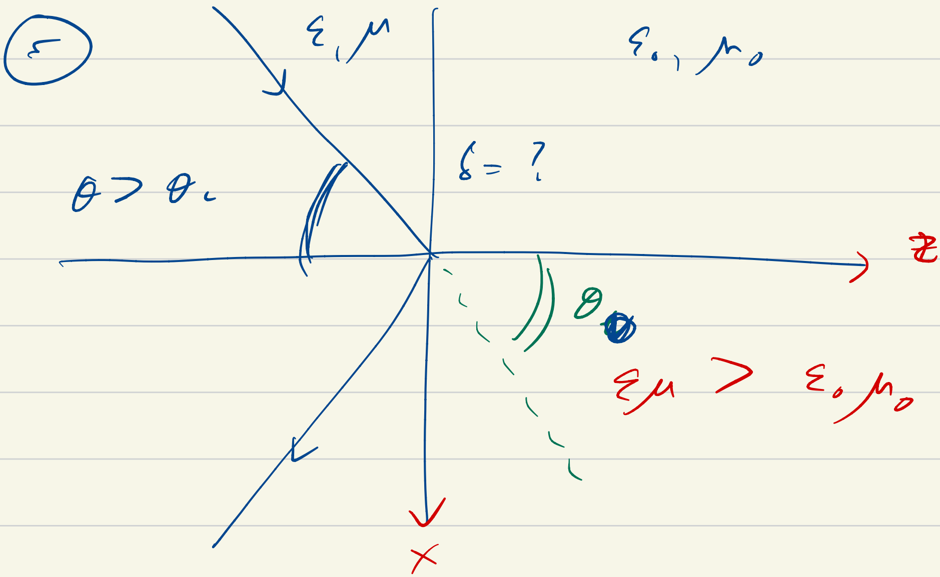
$$kd = \pi \left(m + \frac{1}{2}\right)$$

Can set $kd = \pi/2 \rightarrow \boxed{d = \frac{\pi}{2k} = \frac{\lambda}{4}}$

With this $R_0 = \dfrac{Z^2 - Z_0 Z'}{Z^2 + Z_0 Z'}$ min $= 0$

when $\boxed{Z = \sqrt{Z_0 Z'}}$

- Not sure how to calculate ε, μ from here as there is not enough information
- But ok ... if we assume μ's are constant then all the scaling goes to ε ...



- Monochromatic wave ...

$(\varepsilon, \mu) \quad E = E_0 \, e^{i(k_x x + k_z z - \omega t)}$

- On vacuum side ...

$$k_{0,x} = k_0 \sin\theta_0$$

$$k_{0,z} = k_0 \cos\theta_0 = k_0 \sqrt{1 - \sin^2\theta_0}$$

By Snell's law ...

$$k \sin\theta = k_0 \sin\theta_0$$

$$\Rightarrow \sin\theta_0 = \frac{k}{k_0} \sin\theta$$

$$\Rightarrow k_{0,z} = k_0 \sqrt{1 - \left(\frac{k}{k_0} \sin\theta\right)^2}$$

now $\theta > \theta_c = \sin^{-1}(k/k_0)$

$$\Rightarrow \frac{k}{k_0} \sin\theta > 1$$

$$\Rightarrow k_{0,z} \rightsquigarrow \text{imaginary}$$

Take $k_{0,z} = i/\delta =$

$$= k_0 \sqrt{1 - \left(\frac{k}{k_0} \sin\theta\right)^2}$$

So $$\delta = \frac{1}{k_0}\left\{\left(\frac{k}{k_0}\right)^2 \sin^2\theta - 1\right\}^{-1/2}$$

$$= \frac{\lambda_{vac}}{2\pi\, n_0}\left\{\left(\frac{n}{n_0}\right)^2 \sin^2\theta - 1\right\}^{-1/2}$$

$$\boxed{\delta = \frac{\lambda_{vac}}{2\pi}\left\{ n^2 \sin^2\theta - 1\right\}^{-1/2}}$$

Here I've taken $n_0 = 1$

This result doesn't depend on polarization, since we're working only with $\vec{k}$.