Notes 8.370/18.435 Fall 2022

Lecture 14 Prof. Peter Shor

Today we talked more about density matrices.

There are two ways of obtaining density matrices. One is by having a probabilistic mixture of states, and the other is by starting with an entangled state and disregarding (or throwing away) part of it. We explained the first case on Friday. Today, we will explain the second case.

Let's do an example before we explain things in general. Suppose we have the state

$$\frac{2}{\sqrt{5}} \,|\, 00\rangle_{AB} + \frac{1}{\sqrt{5}} \,|\, 11\rangle_{AB}$$

shared between Alice and Bob. Now, suppose Alice measures her qubit but doesn't tell Bob the result. Bob will have a probabilistic mixture of quantum states, i.e., a density matrix.

FIrst, let's assume Alice measures her qubit in the $\{|\,0\rangle, |\,1\rangle\}$ basis. She gets the outcome $|\,0\rangle$ with probability $\frac{4}{5}$, in which case Bob also has the state $|\,0\rangle$, and she gets the state $|\,1\rangle$ with probability $\frac{1}{5}$, in which case Bob also has the state $|\,1\rangle$. Thus, Bob's density matrix (assuming he doesn't learn Alice's measurement result) is

$$\frac{4}{5} \,|\, 0\rangle\langle 0 \,| + \frac{1}{5} \,|\, 1\rangle\langle 1 \,| = \frac{4}{5}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}.$$

Now, let's assume that Alice measures in the $\{|\,+\rangle, |\,-\rangle\}$ basis. If Alice gets the result $|\,+\rangle$, we have

$$\frac{1}{\sqrt{2}}\Big(\langle 0\,| + \langle 1\,|\Big)\Big(\tfrac{2}{\sqrt{5}} \,|\, 00\rangle_{AB} + \tfrac{1}{\sqrt{5}} \,|\, 11\rangle_{AB}\Big) = \frac{2}{\sqrt{10}} \,|\, 0\rangle + \frac{1}{\sqrt{10}} \,|\, 1\rangle,$$

so Bob's state is $\frac{1}{5}\big(2\,|\,0\rangle + |\,1\rangle\big)$ with probability $\frac{4}{10} + \frac{1}{10} = \frac{1}{2}$.

A very similar calculation shows that if Alice gets $|\,-\rangle$, then Bob's state is $\frac{1}{5}\big(2\,|\,0\rangle - |\,1\rangle\big)$ with probability $\frac{1}{2}$.

Bob's density matrix in this case is

$$\frac{1}{2}\cdot\frac{1}{5}\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} + \frac{1}{2}\cdot\frac{1}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}.$$

Thus, in this case Bob's density matrix doesn't depend on the basis that Alice used for her measurement. We will later show that this is true in general.

The operation on matrices that gets a density matrix on a state space B (or A) from a density matrix on the joint state space of A and B. is called a *partial trace*. Suppose we have a density matrix ρ_{AB} on a joint system AB. If A is a qubit, we can express it as

$$\rho_{AB} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

where P, Q, R, S, are matrices on the quantum system B. Now, the partial trace over A is

$$\mathrm{Tr}_A \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = P + S \tag{1}$$

and the partial trace over B is

$$\mathrm{Tr}_B \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} \mathrm{Tr}\, P & \mathrm{Tr}\, Q \\ \mathrm{Tr}\, R & \mathrm{Tr}\, S \end{pmatrix}.$$

This generalizes in a straightforward way to the case where A is not a qubit. If A has dimension j and B has dimension k, then ρ_{AB} is a $j \times j$ array of $k \times k$ matrices. $\mathrm{Tr}_A \rho_{AB}$ is just the sum of the matrices along the diagonal, and to get $\mathrm{Tr}_B \rho_{AB}$, you take the trace of each of the j^2 matrices.

The reason that this is called a *partial trace* is that if we take the partial trace of a tensor product matrix, say $M_A \otimes M_B$, then

$$\begin{aligned} \mathrm{Tr}_A(M_A \otimes M_B) &= (\mathrm{Tr}_A M_A) M_B \\ \mathrm{Tr}_B(M_A \otimes M_B) &= (\mathrm{Tr}_B M_B) M_A. \end{aligned}$$

Let's take the partial trace for the example state we had earlier,

$$\frac{2}{\sqrt{5}} \,|\, 00 \rangle_{AB} + \frac{1}{\sqrt{5}} \,|\, 11 \rangle_{AB}$$

The density matrix is

$$\rho_{AB} = \frac{1}{5} \left(\begin{array}{cc|cc} 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{array} \right)$$

To get $\mathrm{Tr}_A \rho_{AB}$, we add up the 2×2 matrices along the diagonal, which gives

$$\mathrm{Tr}_A \rho_{AB} = \frac{1}{5} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

To get $\mathrm{Tr}_B \rho_{AB}$, we take the trace of each of the 2×2 matrices in each quadrant. This gives

$$\mathrm{Tr}_B\, \rho AB = \frac{1}{5} \left(\begin{array}{c|c} \mathrm{Tr} \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix} & \mathrm{Tr} \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \\ \hline \mathrm{Tr} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} & \mathrm{Tr} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \right) = \frac{1}{5} \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

For this case, this turns out to be the same density matrix as we obtained when we took the partial trace over A's system, because the original state $\frac{2}{\sqrt{5}} \,|\, 00 \rangle + \frac{1}{\sqrt{5}} \,|\, 11 \rangle$ is symmetric in A and B.

We now give another formula for the partial trace of a quantum state. This formula can be generalized to take the partial trace over system that is a tensor product of more than two subsystems. Suppose you have a basis $\{|\, e_i\rangle\}$ for system A. Then

$$\mathrm{Tr}\,_A \rho = \sum_{i=0}^{d-1} \langle e_i \,|\, \rho \,|\, e_i \rangle \tag{2}$$

Why does this represent the same matrix as the first formula? What we mean by $\langle e_0 \,|$ if the systems are qubits is

$$\langle e_0 \,| = (1, 0) \otimes I_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

And

$$\langle e_1 \,| = (0, 1) \otimes I_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, if

$$M = \begin{pmatrix} P & Q \\ R & S \end{pmatrix},$$

where P, Q, R, S, are 2×2 matrices, we have

$$\langle e_0 \,|\, M \,|\, e_0 \rangle = P$$

and

$$\langle e_1 \,|\, M \,|\, e_1 \rangle = S,$$

so summing them gives the formula of Eq. (1).

There is another way of seeing this. If $\{|\, e_i\rangle\}$ is the standard basis $\{|\, 0\rangle, |\, 1\rangle, \ldots, |\, d-1\rangle\}$, over system A, then $\langle e_i \,|\, \rho \,|\, e_i \rangle$ just picks out the entries with i in the first coordinate of the matrix ρ_{AB}, and summing these gives the sum of the matrices along the diagonal.

Now, we will show that it doesn't matter which basis you use in the formula (2) for partial trace, you get the same result for $\mathrm{Tr}_A\, \rho$. Suppose we have two orthonormal bases for system A, $\{|\, e_j\rangle\}$ and $\{|\, f_i\rangle\}$. We can express one basis in terms of the other:

$$|\, f_i\rangle = \sum \alpha_{ij} \,|\, e_j\rangle\,.$$

Since both bases are orthonormal, the length of a vector expressed in the basis $\{|\, e_j\rangle\}$ must be the same as the length in $\{|\, f_i\rangle\}$. This means that the matrix (α_{ij}) is unitary, since this change of basis preserves the lengths of vectors. Now, we do some algebra:

$$\begin{aligned} \sum_i \langle f_i \,|\, \rho \,|\, f_i \rangle &= \sum_i \sum_{j'} \sum_j \alpha^*_{ij'} \langle e_{j'} \,|\, \rho \,|\, e_j \rangle \, \alpha_{ij} \\ &= \sum_{j'} \sum_j \langle e_{j'} \,|\, \rho \,|\, e_j \rangle \left(\sum_i \alpha^*_{ij'} \alpha_{ij} \right) \end{aligned}$$

The term $\sum_i \alpha^*_{ij'}\alpha_{ij}$ is just the inner product of the columns j and j', which is $\delta(j, j')$, i.e., it is 1 if $j = j'$ and 0 otherwise. Thus, we have

$$\sum_i \langle f_i \,|\, \rho \,|\, f_i \rangle = \sum_j \langle e_j \,|\, \rho \,|\, e_j \rangle\,,$$

the formula we wanted to prove.

So this shows that no matter what measurement Alice makes, Bob has the same density matrix.