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Coherent + Squeezed States

a^+, a in H.O.

Recall the H.O. Hamiltonian can be written as

$$H = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = \hbar\omega\left(\hat{a}^+\hat{a} + \frac{1}{2}\right)$$

with
$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^+\right)$$
$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a} - \hat{a}^+\right)$$

The annihilation/creation ops. satisfy

$$[\hat{a},\hat{a}] = 0,\ [\hat{a}^+,\hat{a}^+] = 0,\ [\hat{a},\hat{a}^+] = 1$$

Coherent States

Consider the state obtained by acting with a translation on the vacuum:

$$\boxed{|C_{x_0}\rangle = T_{x_0}|0\rangle}$$

where $T_{x_0} \equiv e^{-\frac{i}{\hbar}x_0\hat{p}}$.

$|C_{x_0}\rangle$ is an example of a "coherent state".

Using BCH, one may show that

$$T_{x_0}^+\hat{p}\,T_{x_0} = \hat{p}$$
$$T_{x_0}^+\hat{x}\,T_{x_0} = \hat{x} + x_0$$

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thus
$$T_{x_0}^+ H T_{x_0} = H + m\omega^2 x_0\hat{x} + \frac{1}{2}m\omega^2 x_0^2$$

$$\Rightarrow \langle C_{x_0}|H|C_{x_0}\rangle = \frac{1}{2}\hbar\omega + \frac{1}{2}m\omega^2 x_0^2$$

This is exactly what we expect for a classical particle located at x_0

Time evolution in coherent state

Using $[H, a] = -\hbar\omega\hat{a}$

t-independent

we can compute the t-evolution of observables in a coherent state:
$$|C_{x_0}(t)\rangle = e^{-\frac{i}{\hbar}Ht}|C_{x_0}\rangle$$

1st
$$\langle C_{x_0}|\hat{x}|C_{x_0}\rangle = \langle 0|\hat{x}+x_0|0\rangle = x_0$$

2nd
$$\langle C_{x_0}|\hat{p}|C_{x_0}\rangle = \langle 0|\hat{p}|0\rangle = 0$$

So
$$\langle C_{x_0}(t)|\hat{x}|C_{x_0}(t)\rangle$$
$$= \langle C_{x_0}|e^{i\frac{H}{\hbar}t}\hat{x}e^{-\frac{iH}{\hbar}t}|C_{x_0}\rangle$$
$$\underset{(BCH)}{=} \langle C_{x_0}|\sqrt{\frac{\hbar}{2m\omega}}\,e^{i\frac{H}{\hbar}t}(\hat{a}+\hat{a}^+)e^{-iH/\hbar t}|C_{x_0}\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\langle C_{x_0}|\Big(\hat{a} + [i\tfrac{H}{\hbar}t, \hat{a}] + \dots\Big) + \Big(\hat{a}^+ + [\tfrac{iH}{\hbar}t, \hat{a}^+] + \dots\Big)|C_{x_0}\rangle$$

$\cos(\omega t) + i\sin(\omega t)$

$$= \sqrt{\frac{\hbar}{2m\omega}}\langle C_{x_0}|\left(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^+\right)|C_{x_0}\rangle$$

$\cos(\omega t) - i\sin(\omega t)$

$$= \langle C_{x_0}|\cos(\omega t)\cdot\hat{x} + \frac{1}{m\omega}\sin(\omega t)\cdot\hat{p}|C_{x_0}\rangle$$

$$= \underbrace{x_0\cdot\cos(\omega t)}$$

Exactly like a classical particle in x^2 potential

Similarly,

$$\langle C_{x_0}(t)|\hat{p}|C_{x_0}(t)\rangle$$
$$= -i\sqrt{\frac{\hbar m\omega}{2}}\langle C_{x_0}|e^{i\frac{H}{\hbar}t}(\hat{a}-\hat{a}^+)e^{-\frac{iH}{\hbar}t}|C_{x_0}\rangle$$
$$= -i\sqrt{\frac{\hbar\omega m}{2}}\langle C_{x_0}|e^{-i\omega t}\hat{a} - e^{i\omega t}\hat{a}^+|C_{x_0}\rangle$$
$$= \langle C_{x_0}|\cos(\omega t)\hat{p} - m\omega\sin(\omega t)\hat{x}|C_{x_0}\rangle$$

$$= -m\omega x_0\sin(\omega t)$$

just like the classical result!

Define $d \equiv \sqrt{\frac{\hbar}{m\omega}}$

then $|C_{x_0}\rangle = \exp\left(\frac{x_0}{\sqrt{2}d}(\hat{a}^+ - \hat{a})\right)|0\rangle$

Generalize this structure

$$|\alpha\rangle = \exp\left(\alpha\hat{a}^+ - \alpha^*\hat{a}\right)|0\rangle$$
$$= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\hat{a}^+}|0\rangle$$

"Zassenhaus formula"

Note that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Eigenstate of $\hat{a}$ (show on pset)

$\Rightarrow$ we find:

$$\langle\alpha|\hat{x}|\alpha\rangle = \frac{d}{\sqrt{2}}\langle\alpha|\hat{a}^+ + \hat{a}|\alpha\rangle = d\sqrt{2}\,\mathrm{Re}(\alpha)$$
$$\langle\alpha|\hat{p}|\alpha\rangle = \frac{i\hbar}{\sqrt{2}d}\langle\alpha|\hat{a}^+ - \hat{a}|\alpha\rangle = \frac{\hbar\sqrt{2}}{d}\mathrm{Im}(\alpha)$$

So in complete generality we may write:

$$\alpha = \frac{\langle\hat{x}\rangle}{\sqrt{2}d} + i\frac{d}{\sqrt{2}\hbar}\langle\hat{p}\rangle$$

Squeezed states

$$\underbrace{\exp\left(\frac{1}{2}\left(z^*\hat{a}^2 - z\hat{a}^{+2}\right)\right)}_{S(z)}|0\rangle = |S_z\rangle$$

For $z \in \mathbb{R}$
$$S(z)^+\hat{a}\,S(z) = \cosh z\,\hat{a} - \sinh z\,\hat{a}^+$$
$$S(z)^+\hat{a}^+S(z) = \cosh z\,\hat{a}^+ - \sinh z\,\hat{a}$$

$$\Rightarrow S(z)^+\hat{x}\,S(z) = e^{-z}\hat{x}$$
$$S(z)^+\hat{p}\,S(z) = e^{z}\hat{p}$$

$\rightarrow$ So we "squeeze" the uncertainty between $\hat{p}$ and $\hat{x}$!

if $z > 0$, $\hat{x}$ uncertainty $\downarrow$
$\hat{p}$ uncertainty $\uparrow$

if $z < 0$, $\hat{x}$ uncertainty $\uparrow$
$\hat{p}$ uncertainty $\downarrow$

Squeezed states still saturate uncertainty principle!