8.512 Recitation 4

- Pset 2 due 02/28 at 5pm, on Canvas

Today: 1) London theory for electrodynamics in SC
2) Landau mean-field theory
3) Ginzburg-Landau theory of SC
4) Type-I and II SC
5) Fluxoid quantization and vertices

- Ref: "Theory of Superconductivity" by C. Timm, Ch. 5-6

1) Idea: Superconductor ≈ Charged superfluid (2 fluid model)

normal fluid : $\vec{j}_n = \sigma_n \vec{E}$, $\sigma_n = \frac{e^2 n_n \tau}{m}$ (Drude)

superfluid : $\vec{j}_s = -e n_s \vec{v}_s$, $\frac{d\vec{v}_s}{dt} = -\frac{e\vec{E}}{m}$ (no drag)

$$\Rightarrow \boxed{\frac{\partial \vec{j}_s}{\partial t} = \frac{e^2 n_s}{m} \vec{E}}$$ 1st London Eq.

Assumptions: $n_{n,s}$ are uniform in space and constant in time

$$\frac{\partial}{\partial t} \vec{\nabla} \times \vec{j}_s = \frac{e^2 n_s}{m} \vec{\nabla} \times \vec{E} = -\frac{e^2 n_s}{mc} \frac{\partial \vec{B}}{\partial t} \quad \text{(Maxwell eq.)}$$

$$\Rightarrow \vec{\nabla} \times \vec{j}_s = -\frac{e^2 n_s}{mc} \vec{B} + \vec{C} \leftarrow \text{init. conditions}$$

- To recover Meissner's effect, London Brothers postulated that $\vec{C} = 0$ always:

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$$\boxed{\vec{\nabla}\times\vec{j}_s = -\frac{e^2 n_s}{mc}\vec{B}}$$

2nd London Eq.

- Ampere's law $\qquad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}(\vec{j}_s+\vec{j}_n)$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \frac{4\pi}{c}\left(-\frac{e^2 n_s}{mc}\vec{B}+\sigma_n\vec{\nabla}\times\vec{E}\right)$$

$$= \frac{4\pi}{c}\left(-\frac{e^2 n_s}{mc}\vec{B}-\frac{\sigma_n}{c}\cancel{\frac{\partial\vec{B}}{\partial t}}\right)$$

stationary state

$$-\cancel{\vec{\nabla}(\vec{\nabla}\cdot\vec{B})}^{\,0}+\nabla^2\vec{B} = \frac{4\pi e^2 n_s}{mc^2}\vec{B}$$

$$\Rightarrow \nabla^2\vec{B} = \frac{1}{\lambda_L^2}\vec{B} \quad \text{with} \quad \lambda_L=\sqrt{\frac{mc^2}{4\pi e^2 n_s}}$$

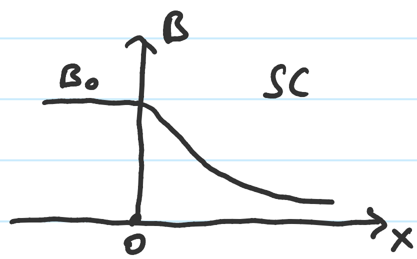
London penetration depth

Ex: SC half plane $(x\geq 0)$

Let $\vec{B}=B_0\hat{y}$ at the surface.

$\Rightarrow \quad \vec{B}=B_0\, e^{-x/\lambda_L}\hat{y} \quad$ for $x\geq 0$

$\vec{j}_s = -\frac{cB_0}{4\pi\lambda_L}e^{-x/\lambda_L}\hat{z}$ for $x\geq 0$



$\vec{j}_s$ is the screening current necessary to expel the magnetic field from the bulk SC.

- London eqs. can be summarized as

$$\vec{j}_s = -\frac{e^2 n_s}{mc}\vec{A}$$

Charge conservation $\vec{\nabla}\cdot\vec{j}_s=0 \Rightarrow \vec{\nabla}\cdot\vec{A}=0$ London gauge

- QM justification for London eqs:

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$$\vec{p} = m\vec{v} + \frac{e}{c}\vec{A}$$

if $\vec{A}=0$, GS has $\langle \vec{p} \rangle = 0$ (Bloch thm)
if $\vec{A} \neq 0$ and the wavefunction is "rigid", i.e. retains GS property $\langle \vec{p} \rangle = 0$, then

$$\langle \vec{v}_s \rangle = -\frac{e}{mc}\vec{A}$$

$$\Rightarrow \vec{j}_s = n_s e \langle \vec{v}_s \rangle = -\frac{e^2 n_s}{mc}\vec{A}$$

Rigidity is later justified in BCS due to non-zero gap...

• <u>Note</u>: All of the above is valid under re-definition of charge carriers:

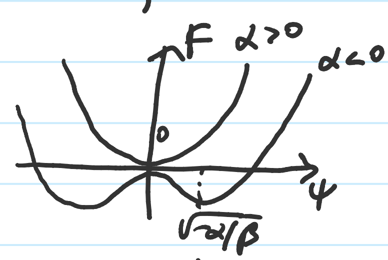
$e \to 2e$
$m \to 2m$
$n_s \to \frac{n_s}{2}$
$\frac{e^2 n_s}{mc} \to \frac{e^2 n_s}{mc}$

We will find this is indeed the case later on.

2) Landau theory = mean-field theory for order param ψ
$|\psi|^2 \sim n_s$ is small

• Expand free energy close to phase transition, taking symmetry into account:

$$F(\psi, T) = \alpha(T)|\psi|^2 + \frac{\beta(T)}{2}|\psi|^4 + \ldots$$



If $\alpha \geq 0$, min $F(\psi)$ at $\psi = 0$ $\Rightarrow n_s = 0$ (normal conductor)
If $\alpha < 0$, min $F(\psi)$ at $|\psi| = \sqrt{-\frac{\alpha}{\beta}}$ $\Rightarrow n_s \neq 0$ (SC)

$\Rightarrow$ phase transition at $\alpha = 0$ $\Rightarrow$ expand $\alpha = a(T - T_c)$

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$\beta = \text{const}$

$|\psi| \sim \sqrt{T_c - T}$ characteristic of mean-field

$n_s \sim (T_c - T)$

3) Ginzburg - Landau theory:

a) Include gradients to account for non-uniform ψ

b) Add vector potential $\vec{A}$ and magnetic field energy density $\frac{B^2}{8\pi}$

$$F = \int d^3r \left[\alpha |\psi|^2 + \frac{\beta}{2} |\psi|^4 + \frac{1}{2m_*} \left| \left(i\hbar \vec{\nabla} + \frac{q}{c} \vec{A} \right) \psi \right|^2 + \frac{B^2}{8\pi} \right]$$

where the mass m_* and charge q of the carriers are phenomenological constants.

- Treat ψ and ψ^* as independent variables
  Use integration by parts, e.g. $\int d^3r \, \nabla\psi \nabla\psi^* = -\int d^3r (\nabla^2\psi)\psi^*$

$$\Rightarrow \frac{\delta F}{\delta \psi^*} = 0 \quad \Rightarrow \quad \boxed{\frac{1}{2m_*} \left(i\hbar \vec{\nabla} + \frac{q}{c} \vec{A} \right)^2 \psi + \alpha\psi + \beta |\psi|^2 \psi = 0}$$

- Similarly, using $\vec{\nabla} \times \vec{A} = \vec{B}$ we find

$$\frac{\delta F}{\delta A} = \frac{F(A + \delta A) - F(A)}{\delta A} = 0 \Rightarrow -i \frac{q\hbar}{2m^* c} \left(\psi \nabla\psi^* - \psi^* \nabla\psi \right) + \frac{q^2}{m_* c^2} |\psi|^2 \vec{A} + \frac{\vec{\nabla} \times \vec{B}}{4\pi} = 0$$

$$\boxed{\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \vec{B} = i \frac{q\hbar}{2m_*} \left(\psi \nabla\psi^* - \psi^* \nabla\psi \right) - \frac{q^2}{m_* c} |\psi|^2 \vec{A}}$$

Ginzburg - Landau eqs.

- If ψ is uniform, we recover London eq.

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$$\vec{j} = -\frac{q^2}{m_* c}|\psi|^2\vec{A}$$

if we identify $\begin{cases} q = 2e \\ m_* = 2m \\ |\psi|^2 = \frac{1}{2}n_s \end{cases}$ (see justification based on flux quantization)

- Penetration depth

$$\boxed{\lambda = \sqrt{\frac{mc^2}{4\pi e^2 n_s}} = \sqrt{\frac{mc^2}{8\pi e^2(-\frac{\alpha}{\beta})}} \sim \frac{1}{\sqrt{T_c - T}}}$$

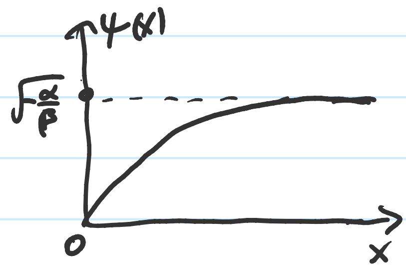
- Let $\vec{A}=0$ and consider SC half-plane from before

$$\begin{cases} -\frac{\hbar^2}{2m_*}\psi'' + \alpha\psi + \beta|\psi|^2\psi = 0 \\ \psi(x=0) = 0 \\ \psi(x=\infty) = \sqrt{-\frac{\alpha}{\beta}} \end{cases}$$

Look for real solutions $f(x) = \sqrt{-\frac{\alpha}{\beta}}\,\psi(x)$

$$-\frac{\hbar^2}{2m_*\alpha}f'' + f - f^3 = 0$$

$$f(x) = \tanh\left(\frac{x}{\sqrt{2}\,\xi}\right)$$



where ξ is the coherence length

$$\boxed{\xi = \sqrt{-\frac{\hbar^2}{2m_*\alpha}} \sim \frac{1}{\sqrt{T_c - T}}}$$

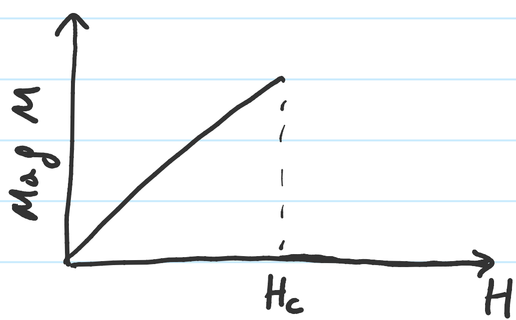
(lengthscale on which ψ varies)

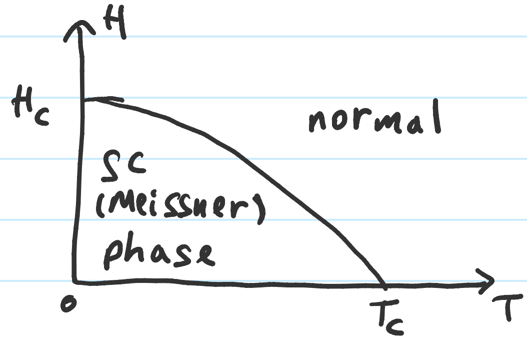
- We can now define the temperature-independent G-L parameter

$$\kappa = \frac{\lambda}{\xi}$$

4) Consider SC in applied external field $\vec{H}$.

- Type-I SC $\quad \kappa < \frac{1}{\sqrt{2}} \quad \Rightarrow$ uniform state





- Gibbs free energy $\quad G = F - \int d^3r \, \frac{\vec{H}\cdot\vec{B}}{4\pi}$

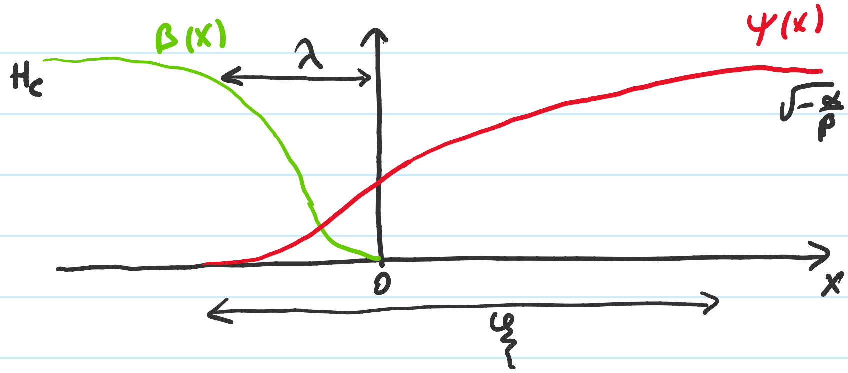
SC : $\vec{B}=0$, $|\psi| = \sqrt{-\frac{\alpha}{\beta}}$, $\quad \frac{G_s}{V} = \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 = -\frac{\alpha^2}{2\beta}$

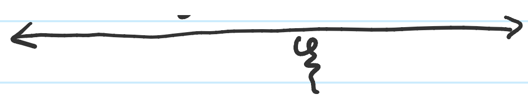
normal : $\vec{B} = \vec{H}$, $|\psi| = 0$, $\quad \frac{G_n}{V} = \frac{H^2}{8\pi} - \frac{H^2}{4\pi} = -\frac{H^2}{8\pi}$

System is SC if $\quad G_s < G_n \quad \Leftrightarrow \quad H < H_c$

$$\boxed{H_c = \sqrt{4\pi \frac{\alpha^2}{\beta}}}$$

- Domain wall between SC and normal phase

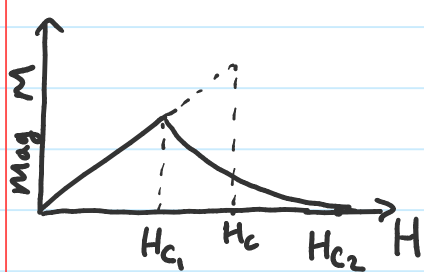


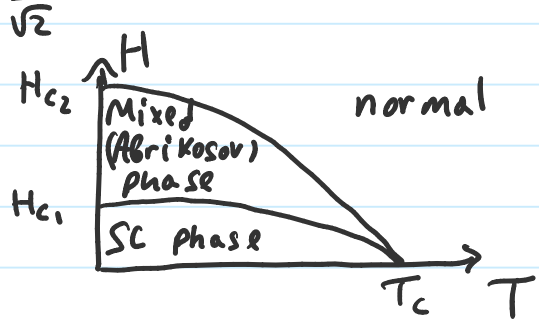


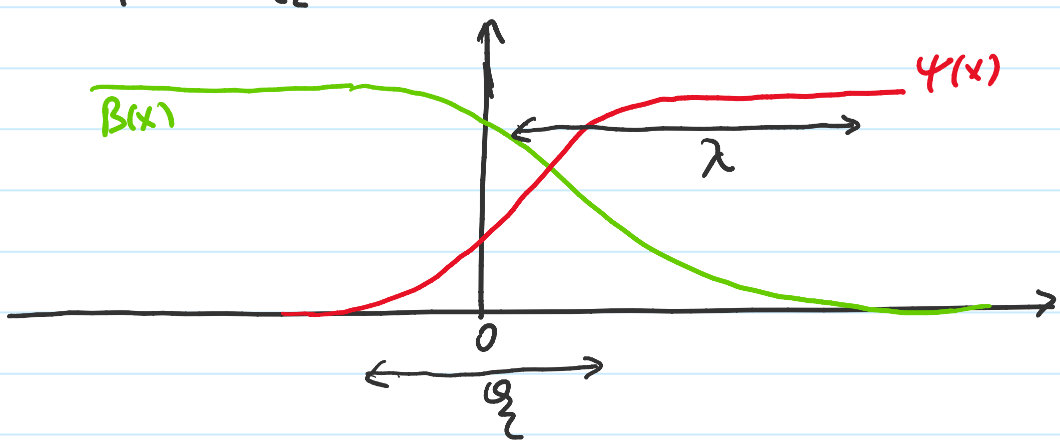
Domain wall energy is given by diff. in cost of expelling the magnetic field and gain due to SC condensation. For type-I, $\xi > \lambda$ and this energy is positive.

Hence the system tends to minimize area of domain walls.

- Type-II SC $\qquad \kappa > \frac{1}{\sqrt{2}}$







- Now the domain wall energy is negative.

- To maximize total area of domain walls, the system tends to create vortices. However vortices can't be smaller than flux quantum Φ_0.

$$\Phi = \int d\vec{a}\cdot\vec{B} = \oint d\vec{s}\cdot\vec{A}\ , \quad \psi = \sqrt{n_s}\, e^{i\theta}$$

$$\vec{j}_s = \frac{i e\hbar}{2m_*}\left(\psi\nabla\psi^* - \psi^*\nabla\psi\right) - \frac{2e^2}{mc}|\psi|^2\vec{A}$$

$$\Rightarrow \Phi = \oint d\vec{s}\cdot\left(-\frac{mc}{e^2 n_s}\vec{j}_s + \frac{\hbar c}{2e}\nabla\theta\right)$$

$$\oint d\vec{s}\cdot\nabla\theta = 2\pi n \qquad n\in\mathbb{Z}$$

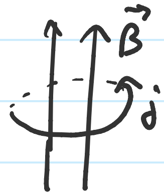
$\Rightarrow$ Fluxoid quantization

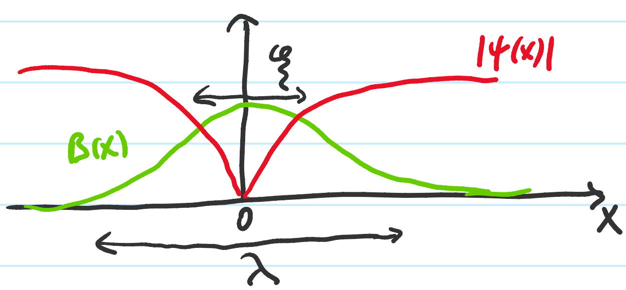
$$\Phi' = \Phi + \frac{mc}{e^2 n_s}\oint d\vec{s}\cdot\vec{j}_s = \frac{\hbar c}{2e}\cdot 2\pi n = n\frac{hc}{2e}$$

$$\boxed{\Phi_0 = \frac{hc}{2e}}$$ flux quantum

Deep inside the SC $\vec{j}_s = 0 \Rightarrow \Phi = \Phi' = n\Phi_0$

- The flux quantum penetrates the SC in a vortex line





- Vortex line energy per unit length ($\kappa \gg 1$)

$$E_v \approx \left(\frac{\Phi_0}{4\pi\lambda}\right)^2 \ln\kappa$$

First vortex appears at $H = H_{c_1}$

$$L E_v = \frac{1}{4\pi} \int d^3r \, \vec{H} \cdot \vec{B}$$

$$= \frac{H_{c1}}{4\pi} \int d^3r \, B = \frac{H_{c1}}{4\pi} L \Phi_0$$

$$\Rightarrow \boxed{H_{c1} = \frac{\Phi_0}{4\pi\lambda^2} \ln \kappa = \frac{H_c}{\sqrt{2}} \frac{\ln \kappa}{\kappa}} \quad (\kappa \gg 1)$$

• One can also show that $\boxed{H_{c2} = \sqrt{2} \, \kappa H_c}$

Hence $\kappa = \frac{1}{\sqrt{2}}$ sets the difference between Type-I and Type-II SC.