MA355: Combinatorics Final (Prof. Friedmann)

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1. 9 and [9]

(a) There are $\boxed{2}$ partitions of 9 with all their parts of size 2 or 3:

$$\begin{aligned} 9 &= 3+2+2+2 \\ &= 3+3+3. \end{aligned}$$

(b) From Part (a), there are two kinds of admissible partitions of [9]: the ones with one size-3 block and three identical size-2 blocks, and the ones with three (identical) size-3 blocks.

To distribute [9] into three identical blocks of 3, there are

$$\frac{1}{3!}\binom{9}{3}\binom{6}{3}$$

ways. To distribute [9] into a block of 3 and three blocks of 2, we do the following: choose 3 out of 9 elements for the size-3 block, then distribute the 6 remaining elements across three identical size-2 blocks. There are

$$\binom{9}{3} \times \frac{1}{3!}\binom{6}{2}\binom{4}{2}$$

ways to do this. So, in total there are

$$\boxed{\frac{1}{3!}\binom{9}{3}\binom{6}{3} + \frac{1}{3!}\binom{9}{3}\binom{6}{2}\binom{4}{2}}$$

partitions of [9] such that all blocks have size 2 or 3.

2. F$_\text{i}$bonacci

(a) <u>Claim</u>: The number S_n of subsets $\mathcal{S}$ of $[n]$ such that $\mathcal{S}$ contains no two consecutive integers is $\boxed{F_{n+2}}$ for $n \geq 1$.

Proof. We first show that S_n follows a similar recurrence relation as the Fibonacci numbers. We have that $S_1 = |\{\{\varnothing\}, \{1\}\}| = 2$ and $S_2 = |\{\{\varnothing\}, \{1\}, \{2\}\}| = 3$. To find S_n for $n \geq 3$, we can split the subsets of $[n]$ into those that contain n and those that don't.

(i) Within the subsets that don't contain n, it is clear that there are S_{n-1} subsets with no consecutive integers.

(ii) Within the subsets that contain n, the admissible subsets are exactly the admissible subsets that don't contain $n-1$. We can get all of these subsets by appending n to each of the S_{n-2} admissible subsets of $[n-2]$.

Therefore, we have

$$S_n = S_{n-1} + S_{n-2}, \quad n \geq 3.$$

Now, the Fibonacci sequence starts with $F_1 = 1, F_2 = 1, F_3 = 2, \ldots$, while this sequence goes as $S_1 = 2, S_2 = 3, S_3 = 5, \ldots$. So, we shift the index by 2 to get

$$S_n = F_{n+2}, \quad n \geq 1$$

as claimed. □

(b) <u>Claim</u>: The number T_n of compositions of n into parts of size greater than 1 is $\boxed{F_{n-1}}$ if $n \geq 2$, with $T_1 = 0$.

Proof. We first show that T_n follows a similar recurrence relation as the Fibonacci numbers. We won't worry about the trivial case $T_1 = 0$ and start with $T_2 = 1$, $T_3 = 1$. To find T_n for $n \geq 3$, we first consider all T_{n+2} compositions of $n+2$ with parts size greater than 1. Some of these admissible compositions have parts of size 2, and some don't.

(i) For each of the admissible compositions of $n+2$ with at least a size-2 part, we can remove the first occurrence of the size-2 part, and obtain all admissible compositions for $n+2-2 = n$. There are T_n of these compositions.

(ii) For each of the admissible compositions of $n+2$ with parts of size greater than 2, we can simply subtract 1 from the first part and obtain all the admissible compositions of $n+2-1 = n+1$. There are T_{n+1} of these compositions.

We thus have $T_{n+2} = T_n + T_{n+1}$, and so re-indexing gives

$$T_n = T_{n-1} + T_{n-2}, \quad n \geq 3.$$

Now, the Fibonacci sequences starts with $F_1 = 1, F_2 = 1, F_3 = 2, \ldots$, while this sequence goes as $T_2 = 1, T_3 = 1, T_4 = 2, \ldots$. So, we shift the index by -1 to get

$$T_n = F_{n-1}, \quad n \geq 2,$$

with $T_1 = 0$, as claimed. □

3. S(t, i)rling.

(a) Claim:

$$\boxed{S(k,k-2) = \sum_{i=3}^{k}(i-2)\binom{i-1}{2}, \quad k \geq 2}$$

Proof. From Problem 134, we know that

$$S(k,n) = S(k-1,n-1) + nS(k-1,n).$$

With $n = k-2$, we have

$$\begin{aligned} S(k,k-2) &= S(k-1,k-3) + (k-2)S(k-1,k-2) \\ &= S(k-1,k-3) + (k-2)S(k-1,(k-1)-1) \\ &= S(k-1,k-3) + (k-2)\binom{k-1}{2}. \end{aligned}$$

Let S_k denote $S(k,k-2)$, then we have a recurrence relation

$$S_k = S_{k-1} + (k-2)\binom{k-1}{2}, \quad k \geq 2.$$

From here, we find a formula for S_k:

$$S(k,k-2) = S_k = \underbrace{S_2}_{=S(2,0)=0} + \sum_{i=3}^{k}(i-2)\binom{i-1}{2} = \sum_{i=3}^{k}(i-2)\binom{i-1}{2}.$$

as desired. I have checked this against the table we made for Problem 135. □

(b) Claim:

$$\boxed{S(k,2) = 2^{k-1} - 1, \quad k \geq 2}$$

and

$$\boxed{S(k,3) = \frac{1}{2}(1 + 3^{k-1} - 2^k), \quad k \geq 3}$$

Proof. From Problem 134, we know that

$$S(k,n) = S(k-1,n-1) + nS(k-1,n).$$

With $n = 2$, we have

$$\begin{aligned} S(k,2) &= S(k-1,1) + 2S(k-1,2) \\ &= 1 + 2S(k-1,2). \end{aligned}$$

Let S_k denote $S(k,2)$ then we have a first-order linear recurrence

$$S_k = 1 + 2S_{k-1}.$$

with $k \geq 2$ and $S_2 = S(2,2) = 1$. The formula for $S(k,2)$, due to Problem 98, is

$$S(k,2) = S_k = 2^{k-2}S_2 + 1 \times \left(\frac{2^{k-2}-1}{2-1}\right) = 2^{k-2} + 2^{k-2} - 1 = 2^{k-1} - 1, \quad k \geq 2,$$

as claimed[1]. I have checked this against the table we made for Problem 135. △

Next, we will use this result and Problem 134 to find $S(k,3)$:

$$\begin{aligned} S(k,3) &= S(k-1,2) + 3S(k-1,3) \\ &= (2^{k-2} - 1) + 3S(k-1,3). \end{aligned}$$

Let T_k denote $S(k,3)$, then we have a recurrence relation

$$T_k = (2^{k-2} - 1) + 3T_{k-1}$$

with $k \geq 3$ and $T_3 = S(3,3) = 1$. As far as I know, there's really no nice way to deal with this but brute force... By writing this out term by term, we find a rough formula for T_k:

$$\begin{aligned} T_3 &= S(3,3) = 1 \\ T_4 &= S(4,3) = (2^{4-2} - 1) + 3 \cdot 1 \\ T_5 &= S(5,3) = (2^{5-2} - 1) + 3(2^{4-2} - 1 + 3 \cdot 1) \\ T_6 &= S(6,3) = (2^{6-2} - 1) + 3[(2^{5-2} - 1) + 3(2^{4-2} - 1 + 3 \cdot 1)] \\ &\vdots \\ T_k &= S(k,3) = \sum_{i=4}^{k} 2^{i-2}3^{k-i} - \sum_{i=0}^{k-4} 3^i + 3^{k-3}. \end{aligned}$$

Now, we simplify this as follows:

$$\begin{aligned} T_k &= \sum_{i=4}^{k} 2^{i-2}3^{k-i} - \frac{3^{k-3}-1}{3-1} + 3^{k-3} \\ &= 3^{k-2}\left[\sum_{i=4}^{k}\left(\frac{2}{3}\right)^{i-2} + \frac{1}{2\cdot 3}\right] + \frac{1}{2} \\ &= 3^{k-2}\left[\sum_{j=2}^{k-2}\left(\frac{2}{3}\right)^{j} + \frac{1}{2\cdot 3}\right] + \frac{1}{2} \\ &= 3^{k-2}\left[-1 - \frac{2}{3} + \frac{1-(2/3)^{k-1}}{1-2/3} + \frac{1}{2\cdot 3}\right] + \frac{1}{2} \\ &= \frac{1}{2}(1 + 3^{k-1} - 2^k), \qquad k \geq 3 \end{aligned}$$

as desired. I've also checked this against the table from Problem 135. △ □

[1] Here, recurrence begins at S_2, so the exponent in the formula only goes up to $k-2$.

(c) Claim[2]:

$$\boxed{S(k,n) = \sum_{i=1}^{k} S(k-i, n-1)\binom{k-1}{i-1}}$$

Proof. Intuitively, this formula makes sense. To put k distinct things into n identical boxes so that each box gets at least one, we can pick out a few things from k, put them into one box, and distribute the rest into the remaining $n-1$ boxes so that each gets at least one. As a result, $S(k,n)$ is the sum of the number of ways this can happen.

To be more precise, we can talk about $S(k,n)$ as the number of ways to partition the set $[k]$ into n non-empty parts $P_1, P_2, \ldots, P_n$. Fix the kth element in the part P_n (since each part must have at least one, and the parts are identical). We want to look at all possibilities for P_n. Suppose P_n must have i elements, then we need to add $(i-1)$ extra elements from the remaining $(k-1)$ elements to P_n. There are $\binom{k-1}{i-1}$ ways to do this. Next, we need to distribute the remaining $(k-i)$ elements into the remaining $(n-1)$ parts such that each part gets at least one. There are $S(k-i, n-1)$ ways to do this for each i. From here, we see that $S(k,n)$ is a sum of $S(k-i,n-1)\binom{k-1}{i-1}$ over all i's:

$$S(k,n) = \sum_{i=1}^{k} S(k-i, n-1)\binom{k-1}{i-1}.$$

□

[2]The claim is inspired by Supplementary Problem 11 on Page 76.

4. LattiC$_\mathbf{e}$ paths. We break the journey from $(0,0) \to (20,30)$ into $(0,0) \to (8,15)$ followed by $(8,15) \to (20,30)$. We can do this because the lattice walker can't move backwards (i.e., to the left or down). The number of paths P_1 from $(0,0)$ to $(8,15)$ is given by

$$P_1 = \binom{8+15}{8} = \binom{23}{8}.$$

Now we want to go from $(8,15)$ to $(20,30)$ but avoid $(14,23)$. Since a path from $(8,15)$ to $(20,30)$ either goes through $(14,23)$ or not, the number of paths from $(8,15)$ to $(20,30)$ is combination of paths through $(14,23)$ and not through $(14,23)$. There are:

$$\binom{(20-8)+(30-15)}{(30-15)} = \binom{27}{15}$$

paths from $(8,15)$ to $(20,30)$, while there are

$$\binom{(14-8)+(23-15)}{(23-15)}\binom{(20-14)+(30-23)}{(30-23)} = \binom{14}{8}\binom{13}{7}$$

paths from $(8,15)$ to $(20,30)$ that go through $(14,23)$. So, the number of paths from $(8,15)$ to $(20,30)$ that don't go through $(14,23)$ is

$$P_2 = \binom{27}{15} - \binom{14}{8}\binom{13}{7}.$$

With this, we combine the two parts of the journey to find that there are

$$P = P_1 \times P_2 = \boxed{\binom{23}{8} \times \left\{\binom{27}{15} - \binom{14}{8}\binom{13}{7}\right\}}$$

paths from $(0,0)$ to $(20,30)$ that go through $(8,15)$ but not $(14,23)$.