Quotient group and isomorphism theorem

Definition (normal subgroup)

Let G be a group. A subgroup $N \leq G$ is called *normal* if

$$gng^{-1} \in N, \quad \forall g \in G, \quad \forall n \in N.$$

The notation $N \trianglelefteq G$ is commonly used to indicate that N is a normal subgroup of G.

Definition (quotient group)

Let N be a normal subgroup of a group G. We can define an equivalence relation on G as

$$g \sim h \iff h^{-1}g \in N,$$

with equivalence classes

$$[g] = \{h \in G \mid h^{-1}g \in N\}.$$

The quotient group G/N (pronounced "G mod N") is the set of equivalence classes

$$G/N = \{[g] \mid g \in G\}$$

which is made into a group by defining

$$[g][h] = [gh], \quad [g]^{-1} = [g^{-1}], \quad e_{G/N} = [e_G].$$

Exercise

Show that the $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ is a normal subgroup of $(\mathbb{Z}, +)$ and that $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$.

Theorem (first isomorphism theorem)

Let $\varphi : G \to H$ be a group homomorphism. Then:

- $\operatorname{Im} \varphi$ *is a subgroup of* H
- $\ker \varphi$ *is a normal subgroup of* G
- $\operatorname{Im} \varphi$ *is isomorphic to the quotient group* $G / \ker \varphi$

Exercise

Prove the first two points of the isomorphism theorem.