Exam 1 8.370/18.435/2.111 Four problems.

Your Name: ____________________________________

Overall comments: Overall, most students did well on the first two problems and made mistakes in the third and fourth problem. The exact rubric is written after each problem solution. The two most common grade range is 18-24, and then 27-32.

1. Suppose we have two qubits on system AB in the state

$$\frac{1}{\sqrt{2}}\big(\,|00\rangle_{AB} + |11\rangle_{AB}\,\big)$$

(a) We apply a measurement on the first qubit, A, using the basis

$$\{\alpha\,|0\rangle + \beta\,|1\rangle\,, -\beta\,|0\rangle + \alpha\,|1\rangle\}\,,$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 1$. What are the probabilities of each outcome, and what is the resulting state of system B?

Solution: Let $|v_1\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, |v_2\rangle = \alpha\,|0\rangle - \beta\,|1\rangle$. The probability of measuring $|v_1\rangle$ is

$$\begin{aligned}\langle EPR|\,|v_1\rangle\,\langle v_1|\,|EPR\rangle &= (\frac{\alpha}{\sqrt{2}}\,\langle 0| + \frac{\beta}{\sqrt{2}}\,\langle 1|)(\frac{\alpha}{\sqrt{2}}\,|0\rangle + \frac{\beta}{\sqrt{2}}\,|1\rangle)\\ &= (\alpha^2+\beta^2)/2 = 1/2.\end{aligned}$$

The resulting state (on system B) would be $\alpha\,|0\rangle + \beta\,|1\rangle = |v_1\rangle$. Similarly, the probability of measuring $|v_2\rangle$ is $1/2$, and the resulting state would be $|v_2\rangle$.

(b) Now, starting with the same state, suppose we apply a measurement on the first qubit, A, but this time using the basis

$$\left\{\frac{1}{\sqrt{2}}\,|0\rangle + i\frac{1}{\sqrt{2}}\,|1\rangle\,, \frac{1}{\sqrt{2}}\,|0\rangle - i\frac{1}{\sqrt{2}}\,|1\rangle\right\},$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha^2 + \beta^2 = 1$. what are the probabilities of each outcome, and what is the resulting state of system B for each outcome?

Solution: Let $|v_1\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{i}{\sqrt{2}}\,|1\rangle\,, |v_2\rangle = \frac{1}{\sqrt{2}}\,|0\rangle - \frac{i}{\sqrt{2}}\,|1\rangle$. The probability of measuring $|v_1\rangle$ is

$$\begin{aligned}\langle EPR|\,|v_1\rangle\,\langle v_1|\,|EPR\rangle &= (\frac{1}{2}\,\langle 0| + \frac{i}{2}\,\langle 1|)(\frac{1}{2}\,|0\rangle + \frac{-i}{2}\,|1\rangle)\\ &= \frac{1}{4} + \frac{1}{4} = 1/2.\end{aligned}$$

The resulting state (on system B) would be $\frac{1}{\sqrt{2}}\,|0\rangle + \frac{-i}{\sqrt{2}}\,|1\rangle = |v_2\rangle$. Similarly, the probability of measuring $|v_2\rangle$ is $1/2$, and the resulting state would be $|v_1\rangle$.

Rubric: For this problem, each part is worth 5 points. Most students did well on this problem.

(a) The most common mistake is in part (b), students forget to take the conjugate and get $\frac{1}{2}\,|0\rangle + \frac{i}{2}\,|1\rangle$ as the resulting state when measuring $|v_1\rangle$.
Points off: 2

(b) Another common mistake is lack of, or wrong normalization of the resulting state or the probabilities.

Points off: 1

(c) Some submissions calculated the probability of measurement in part (b) as zero, due to mistakes with conjugates. These submissions typically get **6 points**, depending on part (a).

(d) If a submission gets part (b) completely wrong, it usually gets **5 points**, depending on part (a).

2. (a) What is the density matrix ρ representing an equal mixture (probability $\frac{1}{2}$ each) of the quantum states:

$$|+\rangle \otimes |0\rangle \quad \text{and} \quad |0\rangle \otimes |-\rangle$$

Solution: Here we have

$$\begin{aligned}
\rho &= \frac{1}{2}(|+\rangle \otimes |0\rangle)(\langle +| \otimes \langle 0|) + \frac{1}{2}(|0\rangle \otimes |-\rangle)(\langle 0| \otimes \langle -|) \\
&= \frac{1}{4}(|00\rangle\langle 00| + |00\rangle\langle 10| + |10\rangle\langle 00| + |10\rangle\langle 10|) + \frac{1}{4}(|00\rangle\langle 00| - |00\rangle\langle 01| - |01\rangle\langle 00| + |01\rangle\langle 01|) \\
&= \frac{1}{4}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \frac{1}{4}\begin{bmatrix} 2 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}$$

(b) What is $\mathrm{Tr}_A\rho$, where ρ is as in part (a)?

Solution: Note that $\mathrm{Tr}_A(|ab\rangle\langle cd|) = \langle a|c\rangle\, |b\rangle\langle d|$. Applying this to our expression in part (a), we have

$$\begin{aligned}
\mathrm{Tr}_A(\rho) &= \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 0|) + \frac{1}{4}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1|) \\
&= \frac{1}{4}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}.
\end{aligned}$$

(c) What is $\mathrm{Tr}_B\rho$?

Solution: Note that $\mathrm{Tr}_B(|ab\rangle\langle cd|) = \langle b|d\rangle\, |a\rangle\langle c|$. Applying this to our expression in part (a), we have

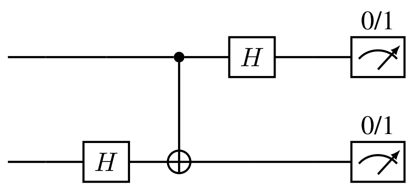
$$\begin{aligned}
\mathrm{Tr}_B(\rho) &= \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{4}(|0\rangle\langle 0| + |0\rangle\langle 0|) \\
&= \frac{1}{4}\begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}.
\end{aligned}$$

Rubric: For this problem, each part is worth 6, 2 and 2 points. Most students did well on this problem.

(a) A common mistake is wrong normalization of the density matrices.
Points off: 1

(b) In some cases, a student made a critical mistake in part (a), and as a result got wrong answers for part (b) and (c) as well. These submissions usually get **4 points**, the logic being that 2 points are given for (a), and 1 point is given for (b) and (c) for the right approaches. If any part has only the wrong answer but no justification, no points are given for that part.

3. Suppose we have the quantum circuit:



This circuit corresponds to a mesurement on the input: there is a measurement basis (not the standard one) such that each of the four possible outcomes, $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ of the circuit corresponds to one state of that basis, and measuring in this basis gives the same probabilities of outcomes as putting the state through the circuit. What are these states, and what is this correspondence?

Solution: We work backwards. If the final measurement result is 00 with probability 1, then reversing the circuit, we have

$$\begin{aligned}(I \otimes H)(\text{CNOT})(H \otimes I)\,|00\rangle &= (I \otimes H)(\text{CNOT})(\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|10\rangle) \\ &= (I \otimes H)(\frac{1}{\sqrt{2}}\,|00\rangle + \frac{1}{\sqrt{2}}\,|11\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(I \otimes H)(\text{CNOT})(H \otimes I)\,|01\rangle &= (I \otimes H)(\text{CNOT})(\frac{1}{\sqrt{2}}\,|01\rangle + \frac{1}{\sqrt{2}}\,|11\rangle) \\ &= (I \otimes H)(\frac{1}{\sqrt{2}}\,|01\rangle + \frac{1}{\sqrt{2}}\,|10\rangle) \\ &= \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle). \\ (I \otimes H)(\text{CNOT})(H \otimes I)\,|10\rangle &= (I \otimes H)(\text{CNOT})(\frac{1}{\sqrt{2}}\,|00\rangle - \frac{1}{\sqrt{2}}\,|10\rangle) \\ &= (I \otimes H)(\frac{1}{\sqrt{2}}\,|00\rangle - \frac{1}{\sqrt{2}}\,|11\rangle) \\ &= \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle). \\ (I \otimes H)(\text{CNOT})(H \otimes I)\,|11\rangle &= (I \otimes H)(\text{CNOT})(\frac{1}{\sqrt{2}}\,|01\rangle - \frac{1}{\sqrt{2}}\,|11\rangle) \\ &= (I \otimes H)(\frac{1}{\sqrt{2}}\,|01\rangle - \frac{1}{\sqrt{2}}\,|10\rangle) \\ &= \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle).\end{aligned}$$

It's easy to see that these states are orthonormal, and they are the measurement basis corresponding to this circuit.

Rubric: The key idea in this problem is to reverse the circuit. A relatively small portion of submissions got this idea, and most submissions did not.

(a) The most common mistake in this problem is that the submission did not reverse the circuit. Instead, the submission evaluated the circuit's value on $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, and claimed that the resulting four states is the measurement basis. These submission are typically given **x points**, where **x** equals to the number of states they calculated correctly, and $x \leq 4$.

(b) In other cases, the students realized that the circuit should be reversed. These submissions usually get at least **6 points**. Points are taken off depending on how many states are computed wrong using the reverse circuit.

4. Suppose we have a unitary transformation on one qubit that takes $\frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle$ to $|0\rangle$. What are the possible states it could take $\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle$ to?

Solution: Suppose we have unitary transform U. There are a few ways to solve this problem, but the most straightforward one is as follows. Unitary transforms preserve the inner product between vectors. In other words, if

$$|v\rangle = U(\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle) = \alpha |0\rangle + \beta |1\rangle \,,$$

then

$$\langle v|0\rangle = (\frac{2}{\sqrt{5}}\langle 0| + \frac{1}{\sqrt{5}}\langle 1|)(\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle) = 3/5.$$

Therefore, we must have $\alpha = 3/5$. This also tells us that $|\beta| = 4/5$. In fact, β can be any value of the form $\frac{4}{5}e^{i\phi}$, for $\phi \in [0, 2\pi]$. To have $U(\frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle) = \frac{3}{5}|0\rangle + \frac{4}{5}e^{i\phi}|1\rangle$, simply set

$$U = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 & 1 \\ e^{i\phi} & -2e^{i\phi}. \end{bmatrix}$$

One can find this U by solving linear equations, and it is easy to check that U is indeed unitary.

Rubric: There are many ways to do this problem, and as long as the answer is (roughly, to be defined later) correct with a valid approach, the submission will get **6 points**. That being said, only a small portion of submissions got this problem correct.

(a) In most cases, the submission did not make any concrete progress on the problem. The answer (if there is one) is incorrect. Some common incorrect answers are $|0\rangle$, or some $a|0\rangle + b|1\rangle$ where a, b are expressions with unsolved variables that are unclear how to solve. Most of these submissions quoted some facts about unitary matrices, in which case they get **1-2 points**.

(b) The most common mistake made in submissions that roughly got the right answer is that they concluded that the state is $\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$, without the $e^{i\phi}$ phase on $|1\rangle$.

Points off: 2.

(c) Another common mistake made in submissions that roughly got the right answer is that they did not compute what U to use to get the resulting state.

Points off: 1.