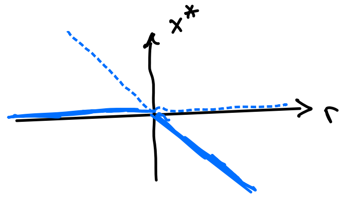
8.(3)09 Final Exam Solutions 2021

13

1. (a) 2

transcritical bifurcation diagram



4 (b)
- Dimples on a golf ball reduce the drag force
  2. (or increase the range, also OK)
- They do this by allowing turbulent boundary layer to
  2. form for smaller R (smaller velocity)

2 (c) $\dot{x} = \mu x + x^3 - x^5$, varying μ

5 (d) $H = H_0 + \Delta H$.

- Assume H_0 exhibits periodic behavior with period τ
- ΔH causes constant variables under H_0 to change in time
- variable that returns to initial value after $t=\tau$ is periodic
- variable that shifts after $t=\tau$ is secular

or

eg. H_0 = Harmonic osc with $\Delta H \propto x^4$. Here amplitude J is periodic & phase shift β is secular.

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2. 16

(a) 8

Variables:	r	v	f	ρ	η	g
dimension:	m	m/s	1/s	kg/m³	$\frac{kg}{m^3}\frac{m^2}{s} = \frac{kg}{m \cdot s}$	m/s²

$\rho\, \nu = \eta$

$6-3 = 3$ dimless ratios

2 are arguements of F : $f = \dfrac{v}{r} F\left[\dfrac{v r \rho}{\eta}, \dfrac{g r}{v^2}\right]$

R

$r \to \frac{r}{10}$, $v \to \frac{v}{\sqrt{10}}$, $\rho \to 10\sqrt{10}\,\rho$ $\therefore$ $\boxed{f \to \sqrt{10}\, f}$

(b) (8)

$H = \frac{p^2}{2m} + k|x| = E$ turning points $x = \pm E/k \equiv \pm x_0$

oscillation

$$J = \oint p\,dx = 2\int_{-x_0}^{+x_0} dx\,\sqrt{2m}\,\sqrt{E-k|x|}$$

$$= 4\int_0^{x_0} dx\,\sqrt{2m}\,(E-kx)^{1/2} = -\frac{8}{3k}\sqrt{2m}\,(E-kx)^{3/2}\Big|_0^{x_0} = \frac{8\sqrt{2m}\,E^{3/2}}{3k}$$

$$\therefore \quad E = \left(\frac{3k}{8\sqrt{2m}}\right)^{2/3} J^{2/3} \qquad \nu = \frac{\partial E}{\partial J} = \left(\frac{3k}{8\sqrt{2m}}\right)^{2/3}\frac{2}{3}J^{-1/3}$$

$$= \frac{\cancel{3}k}{8\sqrt{2m}}\,\frac{2}{\cancel{3}}\,E^{-1/2} = \frac{k}{4\sqrt{2m}}\,E^{-1/2}$$

so $\boxed{\tau = \frac{1}{\nu} = \frac{4\sqrt{2mE}}{k}}$

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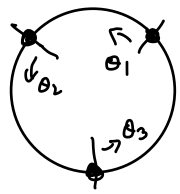
3 30

(a) (8)



$$L = T - V$$

$$T = \frac{m}{2}R^2\left(\dot\theta_1^2+\dot\theta_2^2\right) + \frac{(2m)}{2}R^2\dot\theta_3^2$$ 3.

displacement small angle, springs stretched by $s_{ij} = R(\theta_j-\theta_i)$

$$V = \frac{k}{2}\left(s_{12}^2+s_{23}^2+s_{31}^2\right] = \frac{kR^2}{2}\left[(\theta_1-\theta_2)^2+(\theta_2-\theta_3)^2+(\theta_3-\theta_1)^2\right]$$

5.

$$= \frac{kR^2}{2}\left[2\theta_1^2+2\theta_2^2+2\theta_3^2-2\theta_1\theta_2-2\theta_1\theta_3-2\theta_2\theta_3\right]$$

(b) (8)

$$L = \frac{1}{2}\dot{\vec\eta}^T\cdot\hat T\cdot\dot{\vec\eta} - \frac{1}{2}\vec\eta^T\cdot\hat V\cdot\vec\eta \qquad \text{with } \vec\eta = \begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix}$$

$$\hat T = mR\begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}, \qquad \hat V = kR^2\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}$$

$$\vec\eta = \vec a\,e^{-i\omega t} \qquad (\hat V - \omega^2\hat T)\cdot\vec a = 0 = \left(\frac{\hat V}{kR^2} - \frac{\omega^2}{kR^2}\hat T\right)$$

Let $\omega_0^2 = \frac{k}{m}$

$$\underbrace{\begin{pmatrix}2-\omega^2/\omega_0^2 & -1 & -1\\ -1 & 2-\frac{\omega^2}{\omega_0^2} & -1\\ -1 & -1 & 2-2\frac{\omega^2}{\omega_0^2}\end{pmatrix}}_{\equiv\,\hat B}\cdot\vec a = 0$$

(c) (14) 10 pts solution & 4 pts pictures

Either Physical Intuition + bit math or Math

Intuition

- $\theta_1 = \theta_2 = \theta_3$ everyone rotates together $\boxed{\vec{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}}$ ← no need to normalize, but could

3. $\hat{B}\begin{pmatrix}1\\1\\1\end{pmatrix} = \left(\frac{-\omega^2}{\omega_0^2}\right)\begin{pmatrix}1\\1\\2\end{pmatrix}$ so take $\boxed{\omega = 0}$

- should have solution with $\theta_3 = 0$, $\theta_1 = -\theta_2$

3. $\boxed{\vec{a} = \begin{pmatrix}1\\-1\\0\end{pmatrix}}$, $\hat{B}\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}3-\omega^2/\omega_0^2\\-3+\omega^2/\omega_0^2\\0\end{pmatrix} = 0 \therefore \boxed{\omega = \sqrt{3}\,\omega_0 = \sqrt{\frac{3k}{m}}}$

- final solution, use orthogonality $\vec{a}^{(k)}\cdot\hat{T}\cdot\vec{a}^{(l)} = 0 \quad k \neq l$

4. $\hat{T} \propto \begin{pmatrix}1&&0\\&1&\\0&&2\end{pmatrix}$ so $\vec{a} = \begin{pmatrix}1\\1\\\alpha\end{pmatrix}$ orthogonal $\begin{pmatrix}1\\-1\\0\end{pmatrix}$

$$(1\,1\,1)\begin{pmatrix}1&&\\&1&\\&&2\end{pmatrix}\begin{pmatrix}1\\1\\\alpha\end{pmatrix} = 0 = (1\,1\,2)\begin{pmatrix}1\\1\\\alpha\end{pmatrix} = 2+2\alpha \therefore \alpha = -1$$

$\boxed{\vec{a} = \begin{pmatrix}1\\1\\-1\end{pmatrix}}$, $\hat{B}\cdot\vec{a} = \begin{pmatrix}2-\omega^2/\omega_0^2\\2-\omega^2/\omega_0^2\\-4+2\omega^2/\omega_0^2\end{pmatrix} = 0 \Rightarrow \boxed{\omega = \sqrt{2}\,\omega_0 = \sqrt{\frac{2k}{m}}}$

$\boxed{\text{or}}$ could guess $\begin{pmatrix}1\\1\\\alpha\end{pmatrix}$ & try it $\hat{B}\begin{pmatrix}1\\1\\\alpha\end{pmatrix} = \begin{pmatrix}1-\omega^2/\omega_0^2-\alpha\\1-\omega^2/\omega_0^2-\alpha\\-2+2\alpha-2\frac{\omega^2}{\omega_0^2}\alpha\end{pmatrix} = 0$

$$\Rightarrow \alpha = 1-\omega^2/\omega_0^2, \quad 0 = -\cancel{2}+2(\cancel{1}-\omega^2/\omega_0^2) - 2\frac{\omega^2}{\omega_0^2}(1-\omega^2/\omega_0^2)$$
$$= (-\omega^2/\omega_0^2)(4-2\omega^2/\omega_0^2) \therefore \omega = \sqrt{2}\omega_0 = \sqrt{\frac{2k}{m}}$$
$$\& \quad \alpha = -1$$

Math

$$\det\hat{B} = 0 = (2-\omega^2/\omega_0^2)\left[\left(2-\frac{\omega^2}{\omega_0^2}\right)\left(2-2\frac{\omega^2}{\omega_0^2}\right)-1\right] + \underbrace{\left[-2+\frac{2\omega^2}{\omega_0^2}-1\right]-\left[1+2-\frac{\omega^2}{\omega_0^2}\right]}_{-3(2-\omega^2/\omega_0^2)}$$

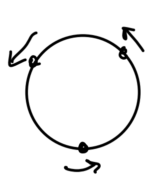
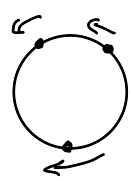
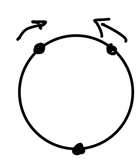
4. $$0 = (2-\omega^2/\omega_0^2)\left(\frac{2\omega^2}{\omega_0^2}\right)\left(\frac{\omega^2}{\omega_0^2}-3\right) \Rightarrow \omega = 0, \sqrt{2}\,\omega_0 \;\&\; \sqrt{3}\,\omega_0$$

$\boxed{\omega = 0}$ 2. $\begin{pmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{pmatrix}\vec{a} = 0 \therefore \vec{a} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

$\boxed{\omega = \sqrt{2}\,\omega_0}$ 2. $\begin{pmatrix}0&-1&-1\\-1&0&-1\\-1&-1&-2\end{pmatrix}\vec{a} = 0 \therefore \vec{a} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$

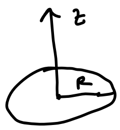
$\boxed{\omega = \sqrt{3}\,\omega_0}$ 2. $\begin{pmatrix}-1&-1&-1\\-1&-1&-1\\-1&-1&-4\end{pmatrix}\vec{a} = 0 \therefore \vec{a} = \begin{pmatrix}1\\-1\\0\end{pmatrix}$

Either method:

4. $\omega = 0$  $\omega = \sqrt{2}\,\omega_0$  $\omega = \sqrt{3}\,\omega_0$ 

4. Dumbell $\qquad \rho = \frac{M}{\pi R^2} \qquad \{r, \theta, z\}$ coords

(a) (12)



$$I_{zz} = \rho \int_0^R r\,dr \int_0^{2\pi} d\theta \,\{\underbrace{x^2+y^2}_{r^2}\} = \rho \frac{R^4}{4}(2\pi) = \frac{MR^2}{2}$$

$$I_{xx} = I_{yy} = \rho \int_0^{R^2} r\,dr \int_0^{2\pi} d\theta\,(x^2+0) = \rho \int_0^R r\,dr \int_0^{2\pi} d\theta\, r^2\cos^2\theta$$

$$= \rho \frac{R^4}{4}\pi = \frac{mR^2}{4}$$

$$I_{disk} = \frac{MR^2}{2}\begin{pmatrix} 1/2 & & 0 \\ & 1/2 & \\ 0 & & 1 \end{pmatrix}$$ ← full points for just writing it down

5.

For one disk use // axes to translate $\quad I^{(a)}_{ab} = I^{(cm)}_{ab} + M(\delta_{ab}\vec{R}^2 - R_a R_b)$

$$I^{(a)} = I_{disk} + ML^2 \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix}$$

For two disk double it

$$I^{dumbell} = 2 I_{disk} + 2ML^2 \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 0 \end{pmatrix} = \begin{pmatrix} \frac{MR^2}{2}+2ML^2 & 0 & 0 \\ 0 & \frac{MR^2}{2}+2ML^2 & 0 \\ 0 & 0 & MR^2 \end{pmatrix}$$

or $\qquad I_{xx} = I_{yy} = \frac{MR^2}{2} + 2ML^2 \quad \& \quad I_{zz} = MR^2$

(b) (7)

3. $\vec{\omega} = \omega(0, \sin\theta, \cos\theta)$

4. $\vec{L} = \hat{I}^{dumbell}\cdot\vec{\omega} = \omega\sin\theta\left(\frac{MR^2}{2}+2ML^2\right)\hat{y} + \omega\cos\theta\, MR^2\,\hat{z}$

(c) (7) Use Euler Equations with $(1,2,3) = (x\ y\ z)$

$\omega_1 = 0, \quad \omega_2 = \omega\sin\theta, \quad \omega_3 = \omega\cos\theta, \quad \omega = \alpha t, \quad \dot{\omega} = \alpha$

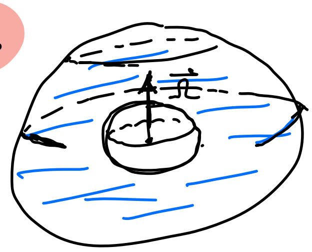
3. $\tau_1 = I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 = 0 - \left(\frac{MR^2}{2} + 2ML^2 - MR^2\right)\omega^2\sin\theta\cos\theta$

$\quad = \left(\frac{MR^2}{2} - 2ML^2\right)(\alpha t)^2 \sin\theta\cos\theta$

2. $\tau_2 = I_2\dot{\omega}_2 - (I_3 - I_1)\underbrace{\omega_3\omega_1}_{0} = \left(\frac{MR^2}{2} + 2ML^2\right)\alpha\sin\theta$

2. $\tau_3 = I_3\dot{\omega}_3 - (I_1 - I_2)\overbrace{\omega_1\omega_3}^{0} = MR^2\alpha\cos\theta$

5. 35



a) 4 $\vec{v} = \vec{\Omega} \times \vec{r}_1 = \Omega r_1 \, \hat{z} \times \hat{r}$

$\hat{z} = \cos\theta \, \hat{r} - \sin\theta \, \hat{\theta}$, $\hat{z} \times \hat{r} = -\sin\theta \, (-\hat{\phi})$

so $\boxed{\vec{v} = \Omega r_1 \sin\theta \, \hat{\phi}}$

b) 4 No dependence on ϕ : $\boxed{\begin{array}{l} \vec{v} = v_\phi(r,\theta) \, \hat{\phi} \\ p = p(r,\theta) \end{array}}$

c) 2 $R = \dfrac{uL}{\nu} = \dfrac{uL\rho}{\eta}$, $\begin{array}{l} L = r_1 \\ u = \Omega r_1 \end{array}$ $\Rightarrow$ $\boxed{R = \dfrac{\Omega r_1^2 \rho}{\eta}}$

[ok if $r_1 \to r_2$, or left with $\nu = \eta/\rho$]

d) 8 Look at $\nabla^2 \vec{v} = \Big[\underbrace{\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}}_{\text{only acts on } v_\phi} + \underbrace{\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}_{\text{only acts } \hat{\phi}} \Big] (v_\phi \hat{\phi})$

$$\therefore \nabla^2 (v_\phi \hat{\phi}) = \left(\nabla^2 v_\phi(r,\theta) \right) \hat{\phi} + \frac{v_\phi}{r^2 \sin^2\theta} \underbrace{\frac{\partial^2}{\partial\phi^2} \hat{\phi}}$$

$$= \frac{\partial}{\partial\phi} \left(-\cos\theta \, \hat{\theta} - \sin\theta \, \hat{r} \right)$$

$$= -\cos^2\theta \, \hat{\phi} - \sin^2\theta \, \hat{\phi}$$

$$= -\hat{\phi}$$

$$\nabla^2 (v_\phi \hat{\phi}) = \left[\nabla^2 v_\phi - \frac{v_\phi}{r^2 \sin^2\theta} \right] \hat{\phi}$$

so purely in $\hat{\phi}$ direction

$$\vec{\nabla} p = \eta \nabla^2 \vec{v}$$

$$\left(\hat{r} \frac{\partial}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial}{\partial \theta} \right) p(r,\theta) = \eta \nabla^2 \vec{v} \propto \hat{\phi} \qquad \therefore \quad \frac{\partial p}{\partial r} = 0, \quad \frac{\partial p}{\partial \theta} = 0$$

so pressure p = constant

e) 10 Direction $\vec{v} = \vec{\Omega} \times [\vec{\nabla} f(r)] = \Omega \hat{z} \times [\hat{r} f'(r)]$

$\hat{z} \times \hat{r} = \hat{\phi} \sin\theta$ $\Rightarrow$ $\vec{v} = \Omega f'(r) \sin\theta \, \hat{\phi}$ correct direction

Equation $\nabla^2 \vec{v} = 0 = \vec{\Omega} \times \underbrace{[\vec{\nabla} \nabla^2 f(r)]}_{\text{must be } 0}$ $\therefore$ $\nabla^2 f(r) = \text{constant} = K$

$$K = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) f(r) = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 f'(r) = \boxed{f''(r) + \frac{2}{r} f'(r) = K}$$

Check Solution $f(r) = Ar^2 + B/r$

$$f'(r) = 2Ar - B/r^2, \quad \frac{2}{r} f'(r) = 4A - \frac{2B}{r^3}$$

$$f''(r) = 2A + 2B/r^3,$$

$$f'' + \frac{2}{r} f' = 6A = \text{constant} \checkmark$$

? f Boundary Conditions ① $\vec{v} = 0$ for $\vec{r} = \vec{r}_2$ outer sphere
② $\vec{v} = \Omega r_1 \sin\theta \, \hat{\phi}$ for $\vec{r} = \vec{r}_1$ inner sphere

$$\vec{v} = \Omega \sin\theta \left(2Ar - B/r^2\right) \hat{\phi}$$

for ① need $2Ar_2^3 = B$, ② needs $2Ar_1 - \frac{B}{r_1^2} = r_1$

So $2A\left(1 - \frac{r_2^3}{r_1^3}\right) = 1$, $\boxed{A = \frac{1}{2\left(1 - r_2^3/r_1^3\right)}, \quad B = \frac{r_2^3}{1 - r_2^3/r_1^3}}$

$$\Rightarrow \boxed{\vec{v} = \hat{\phi}\, \Omega r \sin\theta \frac{\left(1 - \frac{r_2^3}{r^3}\right)}{\left(1 - \frac{r_2^3}{r_1^3}\right)}}$$

limit $r_2 \to \infty$ $\quad \vec{v} = \hat{\phi}\, \Omega r \sin\theta \frac{r_1^3}{r^3} = \hat{\phi}\, \Omega \sin\theta \frac{r_1^3}{r^2}$

falls off ∞ $r \to \infty$ as expected

6. 30 "Non-linear Attractions"

(a) 16

$\dot{x} = xy - 1$

$\dot{y} = y(1-y^2)$

$= y(1-y)(1+y)$

Fixed pts

$xy = +1$ ← can't have $y=0$

$y(1-y^2) = 0$ so $y = \pm 1$

& $x = 1/y$

$= \pm 1$

2 $\boxed{(x^*, y^*) = (1,1) \text{ or } (-1,-1)}$

$\boxed{(1,1)}$ let $x = 1+u$, $y = 1+v$

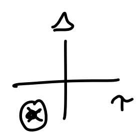
$\dot{u} = (1+u)(1-v) - 1 = u + v + \cdots$

$\dot{v} = (1+v)(-v)(2+v) = -2v + \cdots$

5.

$M = \begin{pmatrix} 1 & 1 \\ 0 & -2 \end{pmatrix}$ $\tau = \mathrm{tr}\, M = -1$, $\Delta = \det M = -2$, $\tau^2 - 4\Delta = 1 + 8 = 9$

$\lambda_\pm = \frac{-1 \pm 3}{2} = \boxed{+1, -2}$ Saddle node



$\lambda_+ = +1$ $\begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$ $\therefore$ $\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ eigenvector

$\lambda_- = -2$ $\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$ → $\boxed{\frac{1}{\sqrt{10}}\begin{pmatrix} 1 \\ -3 \end{pmatrix}}$ eigenvector

$\boxed{(-1,-1)}$ let $x = -1+u$, $y = -1+v$

$\dot{u} = (-1+u)(-1+v) - 1 = -u - v + \cdots$

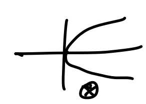
$\dot{v} = (-1+v)(2-v)(v) = -2v + \cdots$

5.

$M = \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}$, $\tau = -3$, $\Delta = +2$

$\tau^2 - 4\Delta = 9 - 8 = +1 > 0$

Stable node

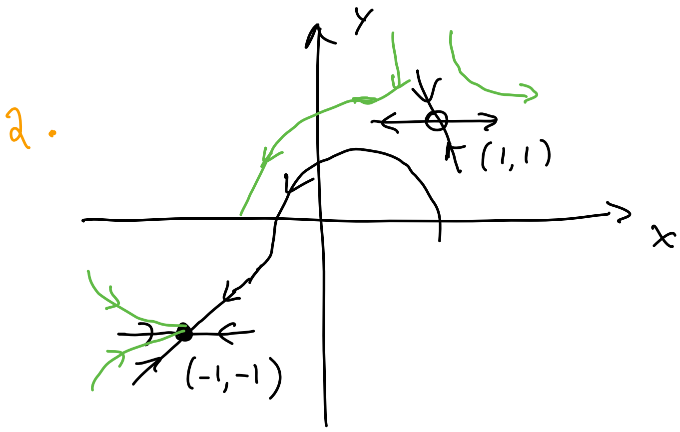


$\lambda_\pm = \frac{-3 \pm 1}{2} = \boxed{-1, -2}$

$\lambda_+ = -1$ $\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$ $\therefore$ $\boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ eigenvector

$\underline{\lambda_- = -2}$ $\quad \begin{pmatrix} 1 & 1 \\ 6 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad \boxed{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ eigenvector

2.



b

6

$$\dot{r} = \mu r - r^3 = r(\mu - r^2) = r(\sqrt{\mu} - r)(\sqrt{\mu} + r)$$

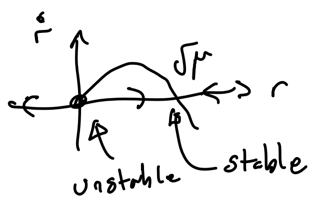
$$\dot{\theta} = \omega + b r^2 \qquad \omega > 0, \; b > 0$$

fixed pt $\quad r = 0 \qquad \underline{\mu > 0}$ unstable $\qquad \underline{\mu < 0}$ stable $\qquad$ 2

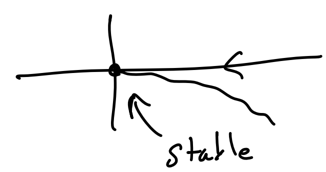
limit cycle $\quad r = \sqrt{\mu} \quad$ needs $\mu > 0$, & stable $\qquad$ 1

from

$\mu > 0$



$\mu < 0$



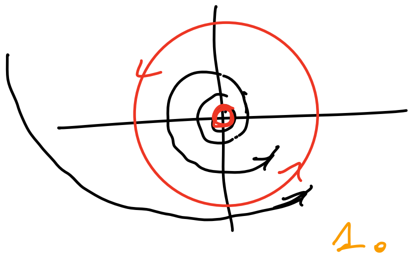
OR $\quad$ linear analysis in r, $\quad \dot{r} = \mu r \quad$ stable $\mu < 0$, unstable $\mu > 0$

$$r = \sqrt{\mu} + \Delta r, \quad \dot{\Delta r} = \sqrt{\mu}\,(2\sqrt{\mu} + \cancel{\Delta r})(-\Delta r) = -\sqrt{\mu}\,\Delta r$$

stable

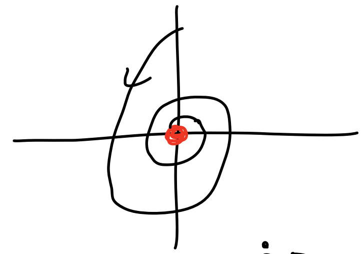
5 (b) cont

$\mu > 0$



1.

$\mu < 0$



1.

$\dot{\theta} > 0$
clockwise

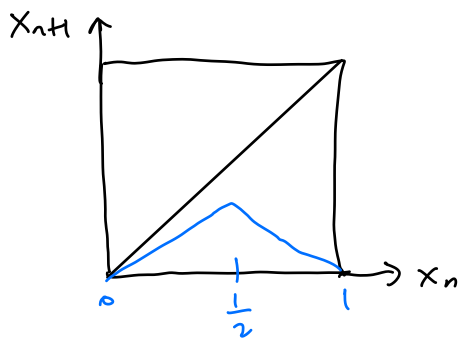
1. $\therefore$ $\boxed{\mu = 0 \quad \text{is supercritical Hopf bifurcation}}$

(c) (8)

$$X_{n+1} = \begin{cases} r x_n & 0 \le x_n \le \tfrac{1}{2} \\ r(1-x_n) & \tfrac{1}{2} \le x_n \le 1 \end{cases} = f(x_n)$$

$\boxed{0 < r < 1}$

4.

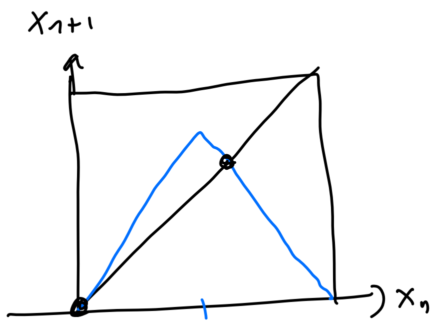


only $x = 0$ is fixed point

$$f'(x_n) = \begin{Bmatrix} r \\ -r \end{Bmatrix}$$

so $|f'(0)| = r < 1 \implies$ stable

$\boxed{1 < r < 2}$



4.

- $x = 0$ is fixed pt
  $x = r(1-x)$.
- $x = \frac{r}{1+r}$ is too

$$f'(x) = \begin{Bmatrix} r \\ -r \end{Bmatrix}$$

$$|f'(x)| = r > 1$$

<u>both unstable</u>