Electromagnetic Theory [Due Wed. March 2, 2022]

Problem Set #3 (8.311)

Problem 2.37. Calculate the 2D Green's function for the free spaces:

(i) outside a round conducting cylinder, and
(ii) inside a round cylindrical hole in a conductor. (30 pts)

Problem 3.2. A plane thin ring of radius R is charged with a constant linear density λ. Calculate the exact distribution of the electrostatic potential along the symmetry axis of the ring, and prove that at large distances, $r >> R$, the three leading terms of its multipole expansion are indeed correctly described by Eqs. (3.3)-(3.4) of the lecture notes.

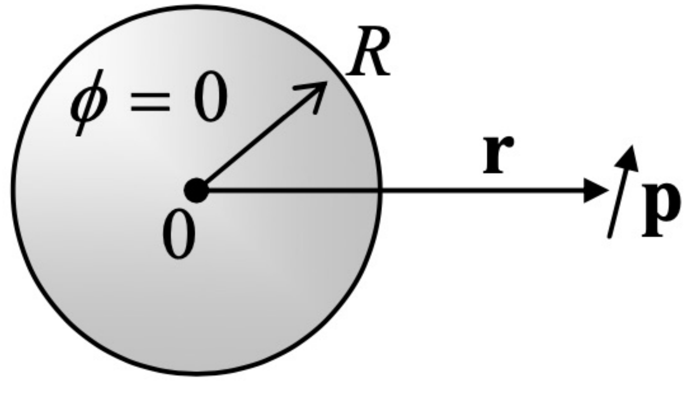
(50 pts)

Problem 3.6. An electric dipole is located above an infinite, grounded conducting plane. Calculate:

(i) the distribution of the induced charge in the conductor,
(ii) the dipole-to-plane interaction energy, and
(iii) the force and the torque acting on the dipole. (40 pts)

Problem 3.7. Calculate the net charge Q induced in a grounded conducting sphere of radius R by a dipole $\mathbf{p}$ located at point $\mathbf{r}$ outside the sphere – see the figure on the right.

(30 pts)



Problem 2.37. Calculate the 2D Green's function for the free spaces:

(i) outside a round conducting cylinder, and
(ii) inside a round cylindrical hole in a conductor.

Solutions:

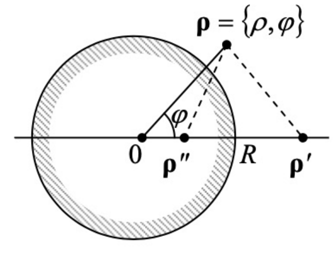
(i) Taking into account the solution of Task (i) of the previous problem,

$$G(\boldsymbol{\rho},\boldsymbol{\rho}')\Big|_{\text{unlimited space}} = -2\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'| + \text{const}$$

(where $\boldsymbol{\rho} = \{\rho, \varphi\}$ is the 2D radius vector of a point), and inspired by the solution of the similar 3D problem (see Eqs. (2.197)-(2.198) of the lecture notes), let us try to look for the solution of our current problem in a similar form:

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = -2\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'| + 2a\ln|\boldsymbol{\rho}-\boldsymbol{\rho}''| + b \equiv -\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + a\ln|\boldsymbol{\rho}-\boldsymbol{\rho}''|^2 + b, \qquad (*)$$

where $\boldsymbol{\rho}''$ is the charge image point, whose distance from the center is related to that of the actual source point $\boldsymbol{\rho}'$ (see the figure on the right) by the inversion relation similar to the first of Eqs. (2.198):



$$\rho'' = \frac{R^2}{\rho'}.$$

The coefficients a (physically, the relative magnitude of the image charge) and b are still to be determined.

Since Eq. (*) satisfies the 2D Laplace equation (at $\boldsymbol{\rho} \neq \boldsymbol{\rho}', \boldsymbol{\rho}''$) for any a and b, these coefficients may be found (and thus the solution (*) proved) by requiring the Dirichlet boundary condition

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = 0, \quad \text{for } \boldsymbol{\rho} \equiv \{R, \varphi\},$$

to be satisfied at any point at the boundary, i.e. for any polar angle φ. Referring φ to the direction from the center toward points $\boldsymbol{\rho}'$ and $\boldsymbol{\rho}''$ (so that $\boldsymbol{\rho}' = \{\rho', 0\}$ and $\boldsymbol{\rho}'' = \{\rho'', 0\}$), and applying the basic trigonometry to the figure above, we may spell out this condition as

$$-\ln\left(\rho'^2 + R^2 - 2R\rho'\cos\varphi\right) + a\ln\left(\rho''^2 + R^2 - 2R\rho''\cos\varphi\right) + b \equiv$$

$$-\ln\left(\rho'^2 + R^2 - 2R\rho'\cos\varphi\right) + a\ln\left[\left(\frac{R^2}{\rho'}\right)^2 + R^2 - 2R\left(\frac{R^2}{\rho'}\right)\cos\varphi\right] + b \equiv$$

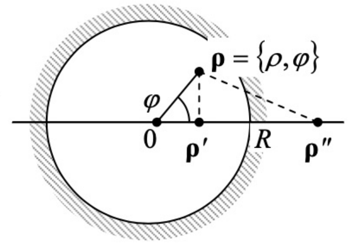
$$-\ln\left(\rho'^2 + R^2 - 2R\rho'\cos\varphi\right) + a\ln\left[\frac{R^2}{\rho'^2}\left(R^2 + \rho'^2 - 2R\rho'\cos\varphi\right)\right] + b = 0.$$

It is evident that this condition is satisfied if $a = 1$ and $b = -\ln(R^2/\rho'^2) \equiv 2\ln(\rho'/R)$, so that, finally, the Green's function is

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = -\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2 + \ln|\boldsymbol{\rho}-\boldsymbol{\rho}''|^2 - \ln\frac{R^2}{\rho'^2} \equiv 2\ln\frac{\rho'}{R}\frac{|\boldsymbol{\rho}-\boldsymbol{\rho}''|}{|\boldsymbol{\rho}-\boldsymbol{\rho}'|}. \qquad (**)$$

In terms of electrostatics, the result $a = 1$ may be interpreted as the image charge (or rather its linear density) having, in contrast to the 3D cases, the same magnitude as the real charge density, and the opposite charge. This equality kills the logarithmic divergence of Green's function (pertinent to the unlimited free space) at large distances.

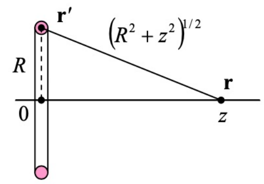
(ii) Repeating the solution of Task (i) for this geometry, we see that Eq. (**) is valid for the "inner" problem as well, but now with $\rho' < R < \rho''$ (see the figure on the right). In particular, on the central line of the cylindrical hole (i.e. at $\boldsymbol{\rho} = 0$),



$$G(0,\boldsymbol{\rho}') = 2\ln\frac{\rho''}{R} = 2\ln\frac{R}{\rho'} > 0.$$

Problem 3.2. A plane thin ring of radius R is charged with a constant linear density λ. Calculate the exact distribution of the electrostatic potential along the symmetry axis of the ring, and prove that at large distances, $r >> R$, the three leading terms of its multipole expansion are indeed correctly described by Eqs. (3.3)-(3.4) of the lecture notes.

Solution: Due to the axial symmetry of the system, the exact calculation of its electrostatic potential from Eq. (1.38), integrated over the cross-section of the ring, is elementary (see the figure on the right):



$$\phi = \frac{\lambda}{4\pi\varepsilon_0}\int_{\text{ring}}\frac{dr'}{|r-r'|} = \frac{\lambda}{4\pi\varepsilon_0}\int_0^{2\pi}\frac{Rd\varphi'}{\left(R^2+z^2\right)^{1/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{2\pi R}{\left(R^2+z^2\right)^{1/2}} \equiv \frac{\lambda R}{2\varepsilon_0 z}f(\xi),$$

where

$$\xi \equiv \left(\frac{R}{z}\right)^2, \quad \text{and} \quad f(\xi) \equiv (1+\xi)^{-1/2}.$$

Let us expand this function $f(\xi)$ into the Taylor series in the argument ξ at point $\xi = 0$:

$$f(\xi) = (1+\xi)^{-1/2}\Big|_{\xi=0} + \frac{d}{d\xi}(1+\xi)^{-1/2}\Big|_{\xi=0}\xi + \frac{1}{2}\frac{d^2}{d\xi^2}(1+\xi)^{-1/2}\Big|_{\xi=0}\xi^2 + ... = 1 - \frac{1}{2}\xi + \frac{3}{8}\xi^2 + ...$$

At large distances from the ring, where ξ is small, the function $f(\xi)$ is well approximated by these three leading terms, so that

$$\phi \approx \frac{\lambda}{2\varepsilon_0}\left[\frac{R}{z} - \frac{1}{2}\left(\frac{R}{z}\right)^3 + \frac{3}{8}\left(\frac{R}{z}\right)^5\right], \qquad \text{at } z >> R. \qquad (**)$$

This expression should be compared with the first three terms of the quadrupole expansion (3.3):

$$\phi(\mathbf{r}) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r}Q + \frac{1}{r^3}\sum_{j=1}^{3} r_j p_j + \frac{1}{2r^5}\sum_{j,j'=1}^{3} r_j r_{j'} \mathscr{Q}_{jj'}\right), \qquad (***)$$

where

$$Q = \int \rho(\mathbf{r}')d^3r', \qquad p_j = \int \rho(\mathbf{r}')r'_j d^3r', \qquad \mathscr{Q}_{jj'} = \int \rho(\mathbf{r}')\left(3r'_j r'_{j'} - r'^2\delta_{jj'}\right)d^3r',$$

and $\rho(\mathbf{r}')$ is the electric charge density (per unit volume). In the standard polar coordinates, with the origin in the ring's center (see the figure above), the integration over the ring's length yields

$$Q = \lambda\int_0^{2\pi} Rd\varphi' = 2\pi R\lambda;$$

$$p_x = \lambda\int_0^{2\pi} R\cos\varphi'\, Rd\varphi' = 0, \qquad p_y = \lambda\int_0^{2\pi} R\sin\varphi'\, Rd\varphi' = 0, \qquad p_z = \lambda\int_0^{2\pi} 0\cdot Rd\varphi' = 0;$$

$$\mathscr{Q}_{xx} = \lambda\int_0^{2\pi}\left(3R^2\cos^2\varphi' - R^2\right)Rd\varphi' = \pi\lambda R^3, \qquad \mathscr{Q}_{yy} = \lambda\int_0^{2\pi}\left(3R^2\sin^2\varphi' - R^2\right)Rd\varphi' = \pi\lambda R^3,$$

$$\mathscr{Q}_{zz} = \lambda\int_0^{2\pi}\left(0 - R^2\right)Rd\varphi' = -2\pi\lambda R^3, \qquad \mathscr{Q}_{xy} = \mathscr{Q}_{yx} = \lambda\int_0^{2\pi} 3R^2\sin\varphi'\cos\varphi'\, Rd\varphi' = 0,$$

$$\mathscr{Q}_{xz} = \mathscr{Q}_{zx} = \lambda\int_0^{2\pi} 3\cdot 0\cdot R\cos\varphi'\, Rd\varphi' = 0, \qquad \mathscr{Q}_{yz} = \mathscr{Q}_{zy} = \lambda\int_0^{2\pi} 3\cdot 0\cdot R\sin\varphi'\, Rd\varphi' = 0.$$

As a sanity check, the quadrupole moment matrix calculated above has zero trace (the sum of their diagonal elements) – as it should, for any system, by the very definition of the tensor $\mathscr{Q}_{jj'}$:

$$\mathrm{Tr}(\mathscr{Q}) \equiv \sum_{j=1}^{3}\mathscr{Q}_{jj} \equiv \sum_{j=1}^{3}\int \rho(\mathbf{r})\left(3r_j^2 - r^3\right)d^3r = \int \rho(\mathbf{r})\sum_{j=1}^{3}\left(3r_j^2 - r^2\right)d^3r = \int \rho(\mathbf{r})\left(3r^2 - 3r^2\right)d^3r = 0.$$

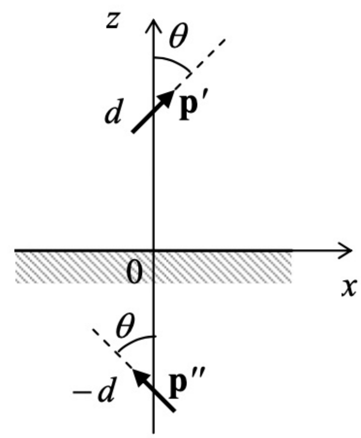
Now plugging these results into Eq. (***) written for z axis, i.e. with $r_1 = r_2 = 0$, and $r = r_3 = z$,

$$\phi(z) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{z}Q + \frac{1}{z^2}p_z + \frac{1}{2z^3}\mathscr{Q}_{zz}\right),$$

we get

$$\phi(z) \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{z}2\pi R\lambda + \frac{1}{z^2}\cdot 0 - \frac{1}{2z^3}2\pi\lambda R^3\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{2\pi R\lambda}{z} - \frac{\pi\lambda R^3}{z^3}\right).$$

Comparing this expression with Eq. (**), we see that the quadrupole approximation exactly describes the three leading (two non-zero) terms of that expansion. Evidently, for this simple geometry the exact calculation was much simpler, and readily gave one more (higher-order) term, but in more complex cases the multipole expansion may be the only way to carry out an (approximate) analytical calculation.



Problem 3.6. An electric dipole is located above an infinite, grounded conducting plane. Calculate:

(i) the distribution of the induced charge in the conductor,
(ii) the dipole-to-plane interaction energy, and
(iii) the force and the torque acting on the dipole.

Solutions:

(i) The problem may be solved by the introduction of a dipole image, at the same distance d below the plane, and with the same dipole moment magnitude p as the original dipole, but reflected in the vertical plane perpendicular to that containing the dipole moment vector (see the figure above). The simplest way to understand this fact is to represent the dipole in the approximate form of two point charges, (+q) and (-q), slightly displaced along the direction of the dipole moment vector, and then construct the dipole image from the mirror images of these point charges in the conducting plane. However, so far this is just a guess, not a proof; let us prove this fact.

The net field of these two dipoles evidently satisfies the Poisson equation in the upper half-space, so that the only thing we have to prove is that it also satisfies the boundary condition ($\phi = 0$) on the plane surface. Let us generalize Eq. (3.7) of the lecture notes to the system of two dipoles (calling them $\mathbf{p}'$ and $\mathbf{p}''$, see the figure above), with the following Cartesian components:

$$p_x' = -p_x'' = p\sin\theta, \quad p_y' = p_y' = 0, \quad p_z' = p_z'' = p\cos\theta, \qquad (*)$$

and located, respectively, at points $\mathbf{r}'$ and $\mathbf{r}''$ with the following Cartesian coordinates:

$$x' = x'' = 0, \quad y' = y'' = 0, \quad z' = -z'' = d.$$

(Here x is the coordinate within the vertical plane that contains the vectors $\mathbf{p}'$ and $\mathbf{p}''$, i.e. in the plane of our drawing, while the y-axis is perpendicular to that plane.) In these coordinates, the generalization yields

$$\phi = \frac{1}{4\pi\varepsilon_0}\left[\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}'}{|\mathbf{r}-\mathbf{r}'|^3} + \frac{(\mathbf{r}-\mathbf{r}'')\cdot\mathbf{p}''}{|\mathbf{r}-\mathbf{r}''|^3}\right] = \frac{p}{4\pi\varepsilon_0}\left\{\frac{(z-d)\cos\theta + x\sin\theta}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} + \frac{(z+d)\cos\theta - x\sin\theta}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}.$$

This equation shows that the potential indeed vanishes everywhere on the surface ($z = 0$), thus proving our guess.

Now the induced surface charge density may be calculated from Eq. (2.3) of the lecture notes as

$$\sigma = -\varepsilon_0 \frac{\partial\phi}{\partial z}\Big|_{z=0},$$

giving

$$\sigma = \frac{p}{2\pi}\frac{(2d^2 - x^2 - y^2)\cos\theta - 3dx\sin\theta}{(x^2+y^2+d^2)^{5/2}}.$$

(ii) The potential energy of interaction between the actual dipole and its image (i.e. the conducting plane) may be calculated using Eqs. (3.16) of the lecture notes, with the additional factor ½, because the image dipole is induced by the actual one – see the discussion following that formula in the lecture notes. Taking $r = 2d$, and using Eqs. (*), we get

$$U_{\text{int}} = -\frac{1}{8\pi\varepsilon_0}\frac{p^2}{(2d)^3}\left(1+\cos^2\theta\right). \qquad (**)$$

Note that for any angle θ, the interaction energy is negative, with its magnitude increasing at $d \to 0$, i.e. the dipole is always attracted to a conductor.

(iii) Now we can use Eq. (**) to calculate the force and the torque acting of the dipole. As should be clear from the symmetry of that expression (namely, its independence on the horizontal position of the dipole), the force has only one nonvanishing component,

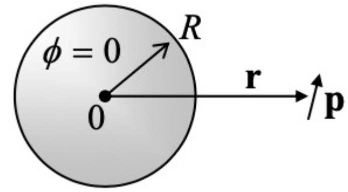
$$F_z = -\frac{\partial U_{\text{int}}}{\partial d} = -\frac{1}{4\pi\varepsilon_0}\frac{3p^2}{16d^4}\left(1+\cos^2\theta\right) < 0,$$

so that the force is directed toward the plane. The torque vector also has only one Cartesian component, perpendicular to the plane of drawing:

$$\tau_y = -\frac{\partial U_{\text{int}}}{\partial \theta} = -\frac{1}{4\pi\varepsilon_0}\frac{p^2}{16d^3}\sin 2\theta .$$

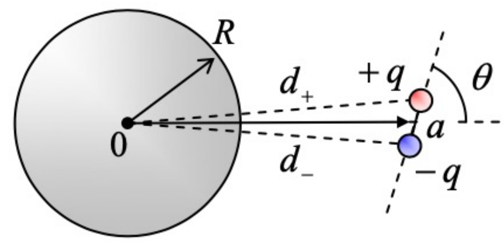
(Alternatively, this result may be obtained from Eq. (3.17) of the lecture notes.) It is interesting that the torque disappears at $\theta = 0$, π, and $\pm\pi/2$, i.e. in all positions in which the dipole moment is aligned with the field created by its image. Of these configurations, only the former two, $\theta = 0$ (both dipoles up), and $\theta = \pi$ (both dipoles down), are stable with respect to dipole rotation, because they correspond to the minima of the interaction energy (**).[68]

Problem 3.7. Calculate the net charge Q induced in a grounded conducting sphere of radius R by a dipole $\mathbf{p}$ located at point $\mathbf{r}$ outside the sphere – see the figure on the right.



Solution: According to Eq. (3.9) of the lecture notes, the dipole $\mathbf{p}$ may be represented as a limiting case of a couple of equal but opposite charges $\pm q$, displaced by $\mathbf{a} = \mathbf{p}/q$, at $a \to 0$, but $q \to \infty$, so that $p = qa = \text{const}$. For our problem's geometry, in the first approximation in the small parameter $a/r << 1$, the distances of the components of such a pair from the sphere's center are

$$d_{\pm} = r \pm \frac{a}{2}\cos\theta,$$



where θ is the angle between the vectors $\mathbf{r}$ and $\mathbf{p}$ – see the figure on the right. Now applying Eq. (2.199) to each charge of the pair, we get

$$Q = -\frac{R}{d_+}q + \frac{R}{d_-}q = Rq\left[-\frac{1}{r + (a/2)\cos\theta} + \frac{1}{r - (a/2)\cos\theta}\right].$$

The limit of this expression at $a/r \to 0$,

$$Q \to \frac{Rqa}{r^2}\cos\theta \equiv \frac{Rp}{r^2}\cos\theta \equiv \frac{R\mathbf{r}\cdot\mathbf{p}}{r^3},$$

gives the solution of our problem.