Spring, 2021

Physics 312: Physics of Fluids

Assignment #7 (Solutions)

Background Reading

Friday, Mar. 26: Tritton 9.1, 9.2,
Kundu & Cohen 9.1 - 9.5

Monday, Mar. 29: Tritton 9.3,
Kundu & Cohen 9.6

Wednesday, Mar. 31: Tritton 9.4, 9.5,
Kundu & Cohen 9.12

Informal Written Reflection

Due: Thursday, April 1 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, April 2 (in class)

1. In this problem, we'll take a look at viscous smoothing of a discontinuity in the velocity field. This is our first opportunity to consider *unsteady* laminar flow…

(a) We learned in class that, for steady shear flows with parallel streamlines, the Navier-Stokes equation reduces to a very simple form,

$$\nu \frac{\partial^2 u}{\partial y^2} = 0,$$

when there are no imposed pressure gradients. The unsteady version of this problem is instructive because, in this case, the Navier-Stokes equation becomes the diffusion equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

Consider the following flow field, which has a discontinuity in velocity at $y = 0$:

$$u(y, 0) = \begin{cases} +U & \text{if } y > 0 \\ -U & \text{if } y < 0. \end{cases}$$

We intuitively expect that viscous effects will smooth over this discontinuity... In sections 9.7 and 9.8, Kundu and Cohen derive the following solution to this problem:

$$u(\eta) = U \operatorname{erf}(\eta) = U \frac{2}{\sqrt{\pi}} \int_0^{\eta} \exp(-\alpha^2)\, \mathrm{d}\alpha,$$

where $\operatorname{erf}(\eta)$ is known as the *error function* and $\eta = y/\sqrt{4\nu t}$ is a dimensionless combination of the vertical coordinate and time. Does this solution behave physically the way we expect it to?

(Hint: $\operatorname{erf}(\eta)$ is known to Wolfram Alpha and other similar programs. Plotting this function for different choices of t and comparing the results will help you answer this question...)

(b) In a previous assignment, we learned that a general solution to the one-dimensional diffusion equation can be written as a convolution:

$$u(y, t) = \int_{-\infty}^{\infty} G(y - y', t)\, u(y', 0)\, \mathrm{d}y',$$

$$\text{where} \quad G(y-y') = \frac{1}{\sqrt{4\pi\nu t}} \exp\left(-\frac{(y-y')^2}{4\nu t}\right).$$

Show, for the discontinuous initial condition $u(y,0)$ given above, that this general solution reduces to $u(\eta) = U\mathrm{erf}(\eta)$, as expected. Thus, viscous smoothing really is a diffusion problem!

(Hint: This is a tricky change of variables calculation! Start with $\alpha = (y-y')/\sqrt{4\nu t}$ and be careful with the limits of integration. Note also that

$$\int_0^\infty \exp(-\alpha^2)\, \mathrm{d}\alpha = \frac{\sqrt{\pi}}{2}$$

and that this identity may come in handy more than once!)

(c) This phenomenon can also be described as *vorticity* diffusion. Show that the diffusion equation can be rewritten

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2},$$

where ω is the only nonzero component of the vorticity. Using your expression for ω and the above result for $u(\eta)$, show that the vorticity field is described by

$$\omega(y,t) = -\frac{U}{\sqrt{\pi\nu t}} \exp\left(-\frac{y^2}{4\nu t}\right).$$

How does this function change shape over time? How does the *total* vorticity of the flow change over time?

(d) As with $u(y,t)$ we can rederive our result for the vorticity field directly from a convolution integral. First, using the relationship between circulation and vorticity, show that our flow discontinuity at $y = 0$ (our initial condition on velocity) can be represented as a "sheet" vortex: $\omega(y) = -2U\delta(y)$. Combining this result with the appropriate convolution integral, you should be able to reproduce your answers in (b).

(Hint: Start by looking through Kundu and Cohen, section 3.8.)

Solution:

(a) Plotting $\mathrm{erf}(y/\sqrt{4\nu t})$ for different values of t should convince you that, as t approaches 0, this solution looks more and more like the discontinuous initial condition $u(y, 0)$ given in the problem. Likewise, as t gets large, the solution gets flatter and flatter. So this error function solution really does describe the smoothing of a discontinuity! (Viscous fluids do not like to have discontinuities in their velocity fields, in other words!)

(b) Using $\alpha = (y - y')/\sqrt{4\nu t}$ to change variables, the convolution integral becomes

$$u(y,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\alpha^2)\, u(\alpha)\, \mathrm{d}\alpha,$$

$$\text{where} \quad u(\alpha) = \begin{cases} -U & \text{if } \alpha > \eta \\ +U & \text{if } \alpha < \eta \end{cases}.$$

Thus, assuming η is positive (which we may as well),

$$\begin{aligned} u(y,t) &= \frac{U}{\sqrt{\pi}} \int_{-\infty}^{\eta} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha - \frac{U}{\sqrt{\pi}} \int_{\eta}^{\infty} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha \\ &= \frac{U}{\sqrt{\pi}} \left(\frac{\pi}{2} + \int_{0}^{\eta} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha \right) - \frac{U}{\sqrt{\pi}} \left(\frac{\pi}{2} - \int_{0}^{\eta} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha \right) \\ &= U \frac{2}{\sqrt{\pi}} \int_{0}^{\eta} \mathrm{e}^{-\alpha^2}\, \mathrm{d}\alpha \;=\; U \mathrm{erf}(\eta). \end{aligned}$$

(c) For unidirectional flow in the x-direction, the only nonzero contribution to the vorticity is

$$\omega = -\frac{\partial u}{\partial y}.$$

In this case, taking the curl of the Navier-Stokes equation gives us a vorticity equation,

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2}.$$

Combining this with our similarity solution $u(\eta)$, we get:

$$\begin{aligned}\omega(y,t) &= -\frac{\partial}{\partial y}u(y,t) = -\frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}u(\eta)\\ &= -\frac{U}{\sqrt{\pi\nu t}}\frac{\partial}{\partial\eta}\int_0^{\eta}\exp(-\alpha^2)\,\mathrm{d}\alpha = -\frac{U}{\sqrt{\pi\nu t}}\exp(-\frac{y^2}{4\nu t}).\end{aligned}$$

Since this is a Gaussian, we know it gets fatter and flatter over time. We also know from its normalization that integrating over all y would give us a *constant* total vorticity of $-2U$.

(d) The vorticity equation found in part (b) has the form of a diffusion equation and, therefore, has a solution of the form

$$\begin{aligned}\omega(y,t) &= \int_{-\infty}^{\infty} G(y-y',t)\,\omega(y',0)\,\mathrm{d}y',\\ \text{where}\quad G(y-y') &= \frac{1}{\sqrt{4\pi\nu t}}\exp(-\frac{(y-y')^2}{4\nu t}).\end{aligned}$$

We obtain the initial vorticity distribution $\omega(y,0)$ associated with the initial discontinuity in velocity by computing the circulation Γ around a box of side length L (with sides parallel to the $y=0$ plane):

$$\Gamma = \oint_C \mathbf{u}\cdot\mathrm{d}\mathbf{s} = 2LU.$$

The alternative definition of Γ given in Kundu and Cohen (3.18) gives us (with the minus sign coming from the direction of $\boldsymbol{\omega}$):

$$\Gamma = \int_A \boldsymbol{\omega}\cdot\mathrm{d}\mathbf{A} = -\int_A \omega_0\,\delta(y)\,\mathrm{d}A = -\omega_0 L$$

Thus, the velocity discontinuity can be thought of as a sheet vortex: $\omega(y) = -2U\delta(y)$. Substituting this for the initial vorticity distribution in the convolution integral, we recover our answer from part (b).

2. In this problem, we will work through Kundu and Cohen's (rather challenging!) derivation of drag due to low Re flow over a sphere...

   (a) The streamfunction formulation of the Navier-Stokes equation for creeping flow in spherical coordinates (Kundu and Cohen, equation (9.64)) is written

   $$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0.$$

   (This looks messy but it's still a massive simplification.) Where does this equation come from? Don't worry about every step of the derivation. Just highlight the key moves: What happened to the pressure term? Where are the velocities? What's this function ψ? And so on.

   (b) Guess a separable solution of the form $\psi(r,\theta) = f(r)\sin^2\theta$ and solve the boundary value problem for creeping flow over a rigid sphere. You should find (Kundu and Cohen, equation (9.68)):

   $$\psi(r,\theta) = Ur^2\sin^2\theta\left[\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right],$$

   Verify that this gives you:

   $$u_r = U\cos\theta\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3},\right]$$
   $$u_\theta = -U\sin\theta\left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3},\right].$$

   (c) Using the solution you found in (b), solve the creeping flow equation $\nabla p = \nabla^2\mathbf{u}$ for the pressure field. You should be able to reproduce (Kundu and Cohen, equation (9.70)):

   $$p - p_\infty = -\mu U\cos\theta\frac{3a}{2r^2}.$$

   (Hint: Be careful with differentiation in spherical coordinates! The appendices will help you identify all of the necessary terms.)

(d) Calculate the stress components σ_{rr} and $\sigma_{r\theta}$ at the surface of the sphere and integrate over the surface to rederive Stokes' famous drag law for creeping flow over a sphere:

$$D = 6\pi\mu a U.$$

(Hint: Once again, you will need the appendices. Look for the stress tensor components in spherical coordinates.)

Solution:

(a) The first step, as discussed in class, is to take the curl of the creeping flow equation. This produces a simpler equation for the vorticity (and kills off the pressure term):

$$\nabla^2 \boldsymbol{\omega} = 0,$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. Because the flow is axisymmetric, we can define the following streamfunction ψ as follows:

$$u_r = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}.$$

The lack of ϕ dependence in the flow means there is only one nonzero component of the vorticity, ω_ϕ. All we do from here is write out the ϕ component of the vorticity equation, in spherical coordinates and in terms of ψ. It takes work but, in the end at last, you arrive at Kundu and Cohen's equation (9.64).

(b) Plugging in our trial solution $\psi(r,\theta) = f(r)\sin^2\theta$ shows that the equation is indeed separable:

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 f\sin^2\theta \\
&\quad = \left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]\left[\sin^2\theta\frac{\partial^2 f}{\partial r^2} - 2\sin^2\theta\frac{f}{r^2}\right] \\
&\quad = \sin^2\theta\left[\frac{\partial^4 f}{\partial r^4} - \frac{4}{r^2}\frac{\partial^2 f}{\partial r^2} + \frac{8}{r^3}\frac{\partial f}{\partial r} - \frac{8}{r^4}f\right] = 0.
\end{aligned}$$

Dividing through by $\sin^2\theta$ gives you a purely radial differential equation for $f(r)$, whose solution is given in Kundu and Cohen:

$$f(r) = Ar^4 + Br^2 + Cr + \frac{D}{r}.$$

The far field boundary condition $f(\infty) = Ur^2/2$ requires $A = 0$ and $B = U/2$ (see pg. 323 in Kundu and Cohen). The boundary conditions at $r = a$ then lead to the known solutions for ψ. From there, simply use the definition of the streamfunction for axisymmetric flow to recover u_r and u_θ.

(c) Here, as in part (a), we need to be careful to include all the terms arising from the conversion to spherical coordinates. Thus, the radial piece of the creeping flow equation has the following form:

$$\begin{aligned}
\frac{\partial p}{\partial r} = \mu&\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\right]u_r \\
&- \frac{2\mu}{r^2}u_r - \frac{2\mu}{r^2\sin\theta}\frac{\partial}{\partial\theta}(u_\theta\sin\theta).
\end{aligned}$$

Plugging in our solutions for u_r and u_θ, we find

$$\frac{\partial p}{\partial r} = \mu U\cos\theta\frac{3a}{r^3}.$$

Likewise, working with the θ piece of the creeping flow equation (in spherical coordinates) and plugging in our solutions for u_r and u_θ, we find

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \mu U\sin\theta\frac{3a}{2r^3}.$$

Integrating either of these equations for the pressure gradient gives us the expected answer.

(d) As Kundu and Cohen explain (refer also to Figure 9.14), there are two non-vanishing contributions to the net force per unit area, one from pressure and one from the viscous stress acting tangential to the surface of the sphere:

$$-p\cos\theta - \sigma_{r\theta}\sin\theta = \frac{3\mu U}{2a}\cos^2\theta + \frac{3\mu U}{2a}\sin^2\theta = \frac{3\mu U}{2a}.$$

Multiplying this constant by the surface area of the sphere leaves us with Stokes' famous result for drag on a sphere.