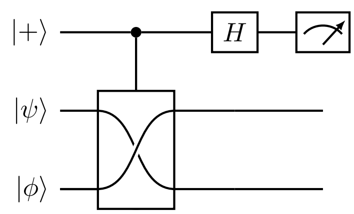
Homework 9   8.370/18.435/2.111   Due 12/07

1. The SWAP test.

   This is a test for figuring out whether two pure quantum states $|\phi\rangle$ and $|\psi\rangle$ are the same.

   Suppose we have the quantum circuit:



   where the gate is a controlled SWAP (CSWAP), a unitary gate that operates as follows:

$$\begin{aligned}&\text{CSWAP}\,|0\rangle\,|\phi\rangle\,|\psi\rangle = |0\rangle\,|\phi\rangle\,|\psi\rangle,\\ &\text{CSWAP}\,|1\rangle\,|\phi\rangle\,|\psi\rangle = |1\rangle\,|\psi\rangle\,|\phi\rangle,\end{aligned}$$

   (a) If $|\phi\rangle = |\psi\rangle$, what is the probability that we observe $|0\rangle$ on the top wire?

   (b) If $\langle\phi|\psi\rangle = 0$, what is the probability that we observe $|0\rangle$ on the top wire?

   (c) Now, even though it wasn't designed to be used this way, suppose that we apply the SWAP test with the inputs being two identical density matrices,

$$\rho_1 = \rho_2 = \sqrt{p}\,|0\rangle\langle 0| + \sqrt{1-p}\,|1\rangle\langle 1|\,.$$

   What is the probability that we observe $|0\rangle$ on the top wire?

Solution: After the circuit, the state is

$$\frac{1}{\sqrt{2}}\left(|0\rangle\,|\psi\rangle\,|\phi\rangle + |1\rangle\,|\phi\rangle\,|\psi\rangle\right)$$

Applying a Hadamard gate and observing a $|0\rangle$ is the same as observing a $|+\rangle$, so the probability that we observe $|+\rangle$ is

$$\begin{aligned}\Big|\frac{1}{\sqrt{2}}\langle +|\left(|0\rangle\,|\psi\rangle\,|\phi\rangle + |1\rangle\,|\phi\rangle\,|\psi\rangle\right)\Big|^2 &= \Big|\frac{1}{2}\left(|\psi\rangle\,|\phi\rangle + |\phi\rangle\,|\psi\rangle\right)\Big|^2\\ &= \frac{1}{4}\Big(\langle\psi|\,\langle\phi| + \langle\phi|\,\langle\psi|\Big)\Big(|\psi\rangle\,|\phi\rangle + |\phi\rangle\,|\psi\rangle\Big)\\ &= \frac{1}{4}\Big(2 + \langle\psi|\phi\rangle\,\langle\phi|\psi\rangle + \langle\phi|\psi\rangle\,\langle\psi|\phi\rangle\Big)\\ &= \frac{1}{2}\Big(1 + |\,\langle\psi|\phi\rangle\,|^2\Big)\end{aligned}$$

If $|\psi\rangle = |\phi\rangle$, then $\langle\psi|\phi\rangle = 1$, and the probability of seeing $|0\rangle$ is 1.
If $\langle\psi|\phi\rangle = 0$, the probability is $\frac{1}{2}$. When the input is density matrices, we can

think of a density matrix as a source of qubits, outputting $|0\rangle$ with probability p and $|1\rangle$ with probability $1-p$. When both inputs are $|0\rangle$ or $|1\rangle$, which happens with probability $p^2+(1-p)^2$, we see $|0\rangle$ with probability 1. If two inputs are different, which happens with probability $2p(1-p)$, we see $|0\rangle$ with probability $1/2$. Therefore the overall probability is

$$p^2+(1-p)^2+p(1-p)=1-p(1-p).$$

2. Suppose k and ℓ are two odd numbers. There is a generalization of the nine-qubit code to a $k\ell$-qubit code that has the following codewords:

$$|0\rangle_L = \frac{1}{2^{\ell/2}}\Big(|\underbrace{000\ldots0}_{k}\rangle + |\underbrace{111\ldots1}_{k}\rangle\Big)^{\otimes\ell},$$
$$|1\rangle_L = \frac{1}{2^{\ell/2}}\Big(|\underbrace{000\ldots0}_{k}\rangle - |\underbrace{111\ldots1}_{k}\rangle\Big)^{\otimes\ell}.$$

   How many bit errors can this code correct? How many phase errors can this code correct? How many syndrome bits do you need to measure to correct the bit errors? How many syndrome bits do you need to measure to correct the phase errors? (For the last two questions, I am asking how many bits are in the syndrome you compute, not how many bits you need to XOR to find each bit of the syndrome.)

   Solution: This is the concatenation of a bit-error-correcting repetition code of length k and a phase-error correcting code of length ℓ. This will thus correct $\lfloor\frac{1}{2}(k-1)\rfloor$ bit errors and $\lfloor\frac{1}{2}(\ell-1)\rfloor$ phase errors, where $\lfloor t\rfloor$ means the greatest integer less than or equal to t. To correct bit errors, you have to measure $k-1$ syndrome bits for each set of k qubits, so you need to measure $\ell(k-1)$ syndrome bits altogether. To correct the phase errors, you only need to measure $\ell-1$ syndrome bits. This is $\ell k-\ell+\ell-1=\ell k-1$ syndrome bits total, which is the right number since the quantum error correcting code encodes 1 qubit.

3. Suppose I encode a qubit $|\psi\rangle=\alpha|0\rangle+\beta|1\rangle$ with the 7-qubit Hamming code, with code C_1 having generator matrix G and C_2 having generator matrix H, as given in class:

$$G=\begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1\\ 1&1&1&1&1&1&1 \end{pmatrix}, \qquad H=\begin{pmatrix} 0&0&0&1&1&1&1\\ 0&1&1&0&0&1&1\\ 1&0&1&0&1&0&1 \end{pmatrix}.$$

   The encoded state is stored in a noisy memory, and two errors occur (this is one more error than the code is designed to correct). When you decode the state with errors, what one-qubit state will you obtain if:

   (a) there is a σ_x error on qubit 3 and a σ_z error on qubit 6,

Solution: Note that quantum Hamming code and all CSS codes treat X and Z errors separately. Since Hamming code can correct one bit of error, we can correct the X error on qubit 3 and Z error on qubit 6 independently, and the state we obtain is still $|\psi\rangle$.

(b) there is a σ_x error on qubit 3 and a σ_y error on qubit 6.

Solution: We treat a Y error as a X error and a Z error on the same qubit. Since there is only one Z error, it can be corrected. Now given X errors on qubit 3 and 6, we have

$$[0, 0, 1, 0, 0, 1, 0]H^T = [1, 0, 1].$$

This corresponds to qubit 5, so after decoding the state experienced the operator $IIXIXXI$. One can check that this operator maps $|0\rangle_L$ to $|1\rangle_L$ and vice versa, which means the final state is $\alpha\,|1\rangle + \beta\,|0\rangle$.

4. There is a CSS code with distance 2 (which thus can detect one error, but not correct any) with the following codes C_1 and C_2:

$$C_1 = \{0000, 0011, 0101, 1001, 0110, 1010, 1100, 1111\}, \qquad C_2 = \{0000, 1111\}.$$

(a) Write down the four codewords of this code.

Solution: We have four cosets of C_2 in C_1, because $|C_1|/|C_2| = 8/2 = 4$. These are:

$$\begin{array}{ll} \{0000, 1111\} & \{1100, 0011\} \\ \{0101, 1010\} & \{0110, 1001\}. \end{array}$$

So the codewords are

$$\begin{array}{ll} \frac{1}{\sqrt{2}}\big(\,|0000\rangle + |1111\rangle\,\big) & \frac{1}{\sqrt{2}}\big(\,|1100\rangle + |0011\rangle\,\big) \\ \frac{1}{\sqrt{2}}\big(\,|0101\rangle + |1010\rangle\,\big) & \frac{1}{\sqrt{2}}\big(\,|0110\rangle + |1001\rangle\,\big) \end{array}$$

We have to identify qubits 1 and 2. Let's assign the codewords as:

$$\begin{array}{ll} |00\rangle_L = \frac{1}{\sqrt{2}}\big(\,|0000\rangle + |1111\rangle\,\big) & |01\rangle_L = \frac{1}{\sqrt{2}}\big(\,|1100\rangle + |0011\rangle\,\big) \\ |10\rangle_L = \frac{1}{\sqrt{2}}\big(\,|0101\rangle + |1010\rangle\,\big) & |11\rangle_L = \frac{1}{\sqrt{2}}\big(\,|0110\rangle + |1001\rangle\,\big) \end{array}$$

For the next three questions, you don't have to list what happens to all four codewords if you can give a systematic description of the effects of the operations.

(b) Suppose you apply a σ_x to qubits 1 and 2 of this code (assume the qubits are labeled 1, 2, 3, 4). Does this operation take codewords to states in the code? What states does it take the four codewords to?

Solution: This operation maps $|00\rangle_L$ to $|01\rangle_L$, $|01\rangle_L$ to $|00\rangle_L$, $|10\rangle_L$ to $|11\rangle_L$, $|11\rangle_L$ to $|10\rangle_L$. It is a logical X operator on the second qubit.

(c) Suppose you apply a σ_z to qubits 1 and 3 of this code (assume the qubits are labeled 1, 2, 3, 4). Does this operation take codewords to codewords? What states does it take the four codewords to?

Solution: This operation maps $|00\rangle_L$ to $|00\rangle_L$, $|01\rangle_L$ to $-|01\rangle_L$, $|10\rangle_L$ to $|10\rangle_L$, $|11\rangle_L$ to $-|11\rangle_L$. It is a logical Z operator on the second qubit.

(d) Suppose you apply H to all four qubits. Does this operation take the four codewords to states in the code? What states does it take the four codewords to?

Solution: Here we have

$$
\begin{aligned}
H^{\otimes 4}\frac{1}{\sqrt{2}}(|x\rangle + |1111-x\rangle) &= \frac{1}{4\sqrt{2}}\Big(\sum_{y\in\{0,1\}^4} (-1)^{x\cdot y}|y\rangle + (-1)^{1111\cdot y - x\cdot y}|y\rangle\Big)\\
&= \frac{1}{4\sqrt{2}}\Big(\sum_{y\in\{0,1\}^4} (-1)^{x\cdot y}(1+(-1)^{1111\cdot y})|y\rangle\Big)\\
&= \frac{1}{2\sqrt{2}}\Big(\sum_{y\in\{0,1\}^4, y\cdot 1111=0} (-1)^{x\cdot y}|y\rangle\Big)\\
&= \frac{1}{2}(|00\rangle_L + (-1)^{x\cdot 1100}|01\rangle_L + (-1)^{x\cdot 0101}|10\rangle_L + (-1)^{x\cdot 0110}|11\rangle_L).
\end{aligned}
$$

In other words,

$$
\begin{aligned}
H^{\otimes 4}|00\rangle_L &= \frac{1}{2}(|00\rangle_L + |01\rangle_L + |10\rangle_L + |11\rangle_L)\\
H^{\otimes 4}|01\rangle_L &= \frac{1}{2}(|00\rangle_L + |01\rangle_L - |10\rangle_L - |11\rangle_L)\\
H^{\otimes 4}|10\rangle_L &= \frac{1}{2}(|00\rangle_L - |01\rangle_L + |10\rangle_L - |11\rangle_L)\\
H^{\otimes 4}|11\rangle_L &= \frac{1}{2}(|00\rangle_L - |01\rangle_L - |10\rangle_L + |11\rangle_L).
\end{aligned}
$$