Classical Mechanics III (8.09 & 8.309) Fall 2021
Assignment 6

Massachusetts Institute of Technology
Physics Department
Mon. October 18, 2021

Due Mon. October 25, 2021
6:00pm

Announcements

This week we will continue our study of the Hamilton-Jacobi equations, and will discuss action-angle variables.

- On this problem set, **8.09 students** should do problems 1,2,3,5,6 and **8.309 students** should do problems 1,2,4,5,6.
- **Your two hour midterm is Tuesday evening, Nov.2, 7:30-9:30pm in our regular lecture classroom.** The midterm will cover the course material up to and including action angle variables. (It will not include perturbation theory.)
- Next week when you turn this problem set in there will not be another assignment posted. Instead I will post practice problems for the midterm. We will also schedule extra office hours for you the week of the midterm.

Reading Assignment

- The reading on Hamilton-Jacobi equations is **Goldstein** sections 10.1-10.5. The reading on Action-Angle Variables is **Goldstein** 10.6 and 10.8. You should also read section 10.7 pages 457-460 (only up to Eq.10.109).
- After we finish discussing action angle variables our next subject will be Perturbation Theory, for which the reading is **Goldstein** chapter 12, sections 12.1-12.3.

Problem Set 6

On this problem set you will explore the use of the Hamilton-Jacobi equations and action-angle variables. All of these problems are from Goldstein, or are related to a problem in Goldstein.

1. **Charged Particle in a Plane** [12 points, everyone] (Goldstein Ch.10 #6)

   A charged particle is constrained to move in a plane under the influence of a nonelectromagnetic central force potential $V = \frac{1}{2}kr^2$ with $k > 0$, and a constant magnetic field $\vec{B}$ perpendicular to the plane obtained from the vector potential

   $$\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}\,. \tag{1}$$

   (a) [6 points] Set up the Hamilton-Jacobi equation for Hamilton's characteristic function in plane polar coordinates. Separate the equation and reduce it to an integral.
   (b) [6 points] Solve for the motion when the canonical momentum $p_\theta = 0$ at time $t = 0$.

2. **A Time Dependent H** [10 points, everyone] (Goldstein Ch.10 #8)

   Suppose the potential in a problem of one degree of freedom is linearly dependent on time, such that the Hamiltonian has the form

   $$H = \frac{p^2}{2m} - mA\,t\,x\,, \tag{2}$$

   where m is the mass and A is a constant. Solve this problem using Hamilton's principal function S to find $x(t)$ and $p(t)$. Take the initial conditions at $t = 0$ to be $x = 0$ and $p = mv_0$. (If you get stuck, solve the problem a different way, and in doing so obtain a hint about the appropriate form of S. Then solve in the manner requested.)

3. **The $|x|$ Potential** [10 points, 8.09 ONLY] (Goldstein Ch.10 #13)

   A particle of mass m exhibits periodic motion in one dimension under the influence of a potential $V(x) = F|x|$ where $F > 0$ is a constant. Using action-angle variables, find the period of the motion as a function of the particle's energy. Check that your result has the correct dimensions.

4. **Two Potentials** [10 points, 8.309 ONLY] (Goldstein Ch.10 #13,#14)

   A particle of mass m exhibits periodic motion in one dimension under the influence of a potential $V(x)$. Using action-angle variables, find the period of the motion as a function of the particle's energy E. Check that your results have the correct dimensions.

   a) $V(x) = F|x|$ where $F > 0$ is a constant
   b) $V(x) = -k/|x|$ where $k > 0$ is a constant. Here assume $E < 0$.

5. **The** $\csc^2(x)$ **Potential** [18 points, everyone] (Goldstein Ch.10 #15)

   A particle of mass m and energy E moves in one dimension subject to the potential

   $$V(x) = a\ \csc^2\left(\frac{x}{x_0}\right), \tag{3}$$

   where a and x_0 are constants.

   (a) [2 points] Obtain an integral expression for Hamilton's characteristic function.

   (b) [4 points] Under what conditions can action-angle variables be used?

   (c) [8 points] Assume these conditions are met, find the frequency of oscillation as a function of energy by the action-angle method. (Hint: the integrals in section 10.8 of Goldstein may be useful. Show your steps.)

   (d) [4 points] Cross check your result in (c) by using the limit of small amplitude oscillations.

6. **A Three Dimensional Oscillator** [10 points, everyone] (Goldstein Ch.10 #20)

   Consider a three dimensional harmonic oscillator of mass m with unequal spring constants k_1, k_2, k_3 in the $(x, y, z) = (1, 2, 3)$ directions.

   (a) [3 points] By using separation of variables and introducing action-angle variables $J_{1,2,3}$ and $\mathrm{w}_{1,2,3}$, find the frequencies of the oscillator. You may use your knowledge of the action-angle variable solution for a one dimensional oscillator.

   (b) [3 points] The connection of $(\mathrm{w}_i, J_i)$to the original (x_i, p_i) variables is obtained from a straightforward generalization of the one-dimensional result:

   $$x = \left(\frac{J}{\pi\sqrt{km}}\right)^{1/2} \sin(2\pi\mathrm{w}), \qquad p = \left(\frac{J\sqrt{km}}{\pi}\right)^{1/2} \cos(2\pi\mathrm{w}).$$

   Using your knowledge that (x_i, p_i) are canonical variables, verify using Poisson brackets that your action-angle variables (w_i, J_i) from part (a) are also canonical variables. [Aside: This also follows directly from the fact that Hamilton's characteristic function, which we use to define the angle variables, is a F_2 type generating function.]

   (c) [4 points] When the oscillator has degeneracy it is more convenient to use a different set of canonical variables w_α and J_α with $\alpha = a, b, c$. Let

   $$J_a = J_1 + J_2 + J_3\,, \qquad J_b = J_1 + J_2\,, \qquad J_c = J_1\,,$$

   and derive expressions for $\mathrm{w}_{a,b,c}$ as a linear combination of $\mathrm{w}_{1,2,3}$ by demanding that $\{\mathrm{w}_\alpha, J_\alpha\}$ are canonical variables. Check that if $k_1 = k_2$ one of your angle variables $\mathrm{w}_{a,b,c}$ becomes conserved, and that if $k_1 = k_2 = k_3$ two of your angle variables become conserved.