# How to make big Hilbert spaces small

Huan Bui

ZGS, Oct 6, 2022

# Outline

- Motivation
- Compressing $|\Psi\rangle$ with SVD
- Matrix Product States (MPS)
- Ground state calculation (roughly)

# Motivation

$N$ sites, each with spin-1/2. Find ground state of:

$$\mathcal{H} = -J\sum_{i=1}^{N}\sigma_i^z\sigma_{i+1}^z - h\sum_{i=1}^{N}\sigma_i^x$$

Hilbert space dimension: $2^N$

Exact diagonalization O.K. for $N \lessapprox 20$ on laptop

$N \to \infty$: needle in the haystack

# Motivation

$N$ sites, each with spin-1/2. Find ground state of:

$$\mathcal{H} = -J \sum_{i=1}^{N} \sigma_i^z \sigma_{i+1}^z - h \sum_{i=1}^{N} \sigma_i^x$$

Hilbert space dimension: $2^N$

Exact diagonalization O.K. for $N \lessapprox 20$ on laptop

$N \to \infty$: needle in the haystack

$\boxed{!!}$ For many relevant Hamiltonians, haystack $\ll$ full Hilbert space

e.g. haystack $\sim$ subspace of states with low entanglement entropy

$\implies$ Clever parameterization + efficient algorithms = ☺?

# Compressing $|\Psi\rangle$?

$$|\Psi\rangle = \sum_{\{\sigma\}} \psi_{\sigma_1\sigma_2\ldots\sigma_N} |\sigma_1\sigma_2\ldots\sigma_N\rangle$$

# Compressing $|\Psi\rangle$?

$$|\Psi\rangle = \sum_{\{\sigma\}} \psi_{\sigma_1\sigma_2\ldots\sigma_N} |\sigma_1\sigma_2\ldots\sigma_N\rangle$$

- How to approximate $|\Psi\rangle$ well without storing $d^N$ coefficients?

# Compressing $|\Psi\rangle$?

$$|\Psi\rangle = \sum_{\{\sigma\}} \psi_{\sigma_1\sigma_2\ldots\sigma_N} |\sigma_1\sigma_2\ldots\sigma_N\rangle$$

- How to approximate $|\Psi\rangle$ well without storing $d^N$ coefficients?
- Possible to reduce entanglement entropy after approximation?

# Compressing $|\Psi\rangle$ with SVD

## Theorem (Singular value decomposition)

*For any $M$, there are unitaries $U, V$ for which $M = USV^\dagger$, with $S = diag(s_1, s_2, \dots)$.*

*$s_i$: singular values of $M \equiv$ eigenvalues of $\sqrt{M^\dagger M}$. $s_i \geq 0$*

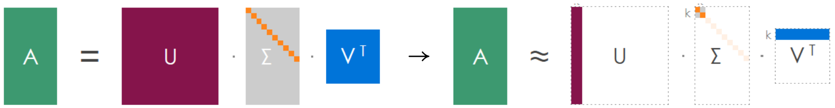
## Theorem (Low-rank approximation)

*The HS-distance from a rank-m matrix $A_{n\times n}$ to the nearest $n \times n$ matrix of rank $k \leq m$ is the square root of the sum of the squares of the smallest $n - k$ singular values of $A$.*

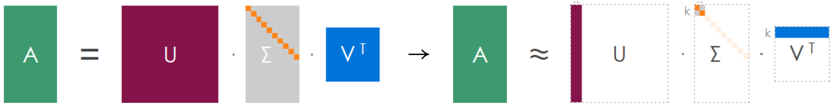
Hilbert-Schmidt norm:

$$\|A\|_{HS} = \sqrt{\sum |A_{ij}|^2} = \sqrt{\mathrm{Tr}(A^\dagger A)} = \sqrt{\sum s_i^2}$$

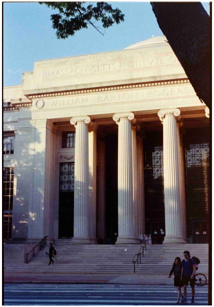
# Compressing $|\Psi\rangle$ with SVD



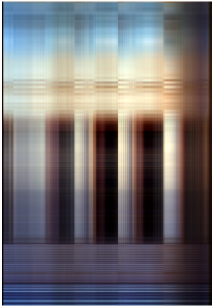
# Compressing $|\Psi\rangle$ with SVD




<u>Application</u>: image compression




$\rightarrow$



# Compressing $|\Psi\rangle$ with SVD

Idea: represent $|\Psi\rangle$ as a matrix, then SVD

# Compressing $|\Psi\rangle$ with SVD

Idea: represent $|\Psi\rangle$ as a matrix, then SVD

Split $N$ spins on a 1d chain into $L + R$:

$$|\Psi\rangle = \sum_{l,r} \psi_{lr} |l\rangle |r\rangle$$

$\psi_{lr}$ has two indices $\implies$ treat as a matrix (NOT an operator!)

# Compressing $|\Psi\rangle$ with SVD

Idea: represent $|\Psi\rangle$ as a matrix, then SVD

Split $N$ spins on a 1d chain into $L + R$:

$$|\Psi\rangle = \sum_{l,r} \psi_{lr} |l\rangle |r\rangle$$

$\psi_{lr}$ has two indices $\implies$ treat as a matrix (NOT an operator!)

Apply SVD: $\psi_{lr} = [\mathbf{U}\ \mathbf{D}\ \mathbf{V}]_{lr}$

$\mathbf{U}, \mathbf{V}$ are unitary. $\mathbf{D} = \mathrm{diag}(s_1, s_2, \dots)$:

$$\begin{aligned} s_i\text{'s} &= \text{singular values of } \psi_{lr} \\ &= \text{eigenvalues of } \sqrt{\psi^\dagger \psi} = \sqrt{\rho} \implies s_i^2 = \text{eigenvalues of } \rho \end{aligned}$$

# Compressing $|\Psi\rangle$ with SVD

After SVD:

$$
\begin{aligned}
|\Psi\rangle &= \sum_{l,r} \sum_{i} \mathbf{U}_{li} \mathbf{D}_{ii} \mathbf{V}_{ir} \, |l\rangle \, |r\rangle \\
&= \sum_{i} \sum_{l,r} \mathbf{U}_{li} \mathbf{D}_{ii} \mathbf{V}_{ir} |l\rangle |r\rangle \\
&= \sum_{i} s_i \, |i\rangle_L \, |i\rangle_R \quad \leftarrow \text{ Schmidt decomposition}
\end{aligned}
$$

# Compressing $|\Psi\rangle$ with SVD

After SVD:

$$
\begin{aligned}
|\Psi\rangle &= \sum_{l,r} \sum_{i} \mathbf{U}_{li} \mathbf{D}_{ii} \mathbf{V}_{ir} |l\rangle |r\rangle \\
&= \sum_{i} \sum_{l,r} \mathbf{U}_{li} \mathbf{D}_{ii} \mathbf{V}_{ir} |l\rangle |r\rangle \\
&= \sum_{i} s_i |i\rangle_L |i\rangle_R \quad \leftarrow \text{ Schmidt decomposition}
\end{aligned}
$$

Can simply read off reduced density matrices:

$$
\rho_L = \psi\psi^\dagger = \sum_i s_i^2 |i\rangle_L \langle i|_L \qquad \rho_R = \psi^\dagger\psi = \sum_i s_i^2 |i\rangle_R \langle i|_R
$$

# Compressing $|\Psi\rangle$ with SVD

After SVD:

$$\begin{aligned} |\Psi\rangle &= \sum_{l,r}\sum_{i}\mathbf{U}_{li}\mathbf{D}_{ii}\mathbf{V}_{ir}\,|l\rangle\,|r\rangle \\ &= \sum_{i}\sum_{l,r}\mathbf{U}_{li}\mathbf{D}_{ii}\mathbf{V}_{ir}|l\rangle|r\rangle \\ &= \sum_{i} s_i\,|i\rangle_L\,|i\rangle_R \;\leftarrow\; \text{Schmidt decomposition} \end{aligned}$$

Can simply read off reduced density matrices:

$$\rho_L = \psi\psi^\dagger = \sum_i s_i^2\,|i\rangle_L\,\langle i|_L \qquad \rho_R = \psi^\dagger\psi = \sum_i s_i^2\,|i\rangle_R\,\langle i|_R$$

Normalization:

$$\mathrm{Tr}\left(\psi^\dagger\psi\right) = \sum_i s_i^2 = 1 \implies s_i^2\text{: probability for i}^{\text{th}}\text{ Schmidt state pair}$$

# Compressing $|\Psi\rangle$ with SVD

Example:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle \right)$$

# Compressing $|\Psi\rangle$ with SVD

Example:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)$$

Matrixify and SVD:

$$|\Psi\rangle = \sum_{ij} \psi_{ij} |i\rangle |j\rangle \quad \text{with} \quad [\psi_{ij}] = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

# Compressing $|\Psi\rangle$ with SVD

Example:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)$$

Matrixify and SVD:

$$|\Psi\rangle = \sum_{ij} \psi_{ij} |i\rangle |j\rangle \quad \text{with} \quad [\psi_{ij}] = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$[\psi_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{D} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

# Compressing $|\Psi\rangle$ with SVD

Why SVD and Schmidt decomposition?

SVD compression $\equiv$ make states with low entanglement entropy

# Compressing $|\Psi\rangle$ with SVD

Why SVD and Schmidt decomposition?

SVD compression ≡ make states with low entanglement entropy

How? von Neumann entanglement entropy between $L$ and $R$:

$$S(\rho_L) = -\mathrm{Tr}[\rho_L \ln \rho_L] = -\mathrm{Tr}[\rho_R \ln \rho_R] = S(\rho_R)$$

# Compressing $|\Psi\rangle$ with SVD

Why SVD and Schmidt decomposition?

SVD compression ≡ make states with low entanglement entropy

How? von Neumann entanglement entropy between $L$ and $R$:

$$S(\rho_L) = -\operatorname{Tr}[\rho_L \ln \rho_L] = -\operatorname{Tr}[\rho_R \ln \rho_R] = S(\rho_R)$$

Eigenvalues of $\rho_L, \rho_R$ are exactly $\{s_i^2\}$.

$$\rho_L = \psi\psi^\dagger = \sum_i s_i^2 \, |i\rangle_L \langle i|_L \qquad \rho_R = \psi^\dagger\psi = \sum_i s_i^2 \, |i\rangle_R \langle i|_R$$

# Compressing $|\Psi\rangle$ with SVD

Why SVD and Schmidt decomposition?

SVD compression ≡ make states with low entanglement entropy

How? von Neumann entanglement entropy between $L$ and $R$:

$$S(\rho_L) = -\operatorname{Tr}[\rho_L \ln \rho_L] = -\operatorname{Tr}[\rho_R \ln \rho_R] = S(\rho_R)$$

Eigenvalues of $\rho_L, \rho_R$ are exactly $\{s_i^2\}$.

$$\rho_L = \psi\psi^\dagger = \sum_i s_i^2 \left|i\right\rangle_L \left\langle i\right|_L \qquad \rho_R = \psi^\dagger\psi = \sum_i s_i^2 \left|i\right\rangle_R \left\langle i\right|_R$$

So,
$$S = S(\rho_L) = S(\rho_R) = -\sum_i^{\sim 2^{N/2}} s_i^2 \ln s_i^2 \to -\sum_i^{m} s_i^2 \ln s_i^2$$

Drop small $s_i$'s $\implies$ reduce $S$ and exponential compression, $m \sim \mathcal{O}(100)$

# Compressing $|\Psi\rangle$ with SVD

Example:

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle \right)$$

Matrixify and SVD

$$|\Psi\rangle = \sum_{ij} \psi_{ij} |i\rangle |j\rangle \quad \text{with} \quad [\psi_{ij}] = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$[\psi_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{D} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

# Compressing $|\Psi\rangle$ with SVD

<u>Example</u>:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\rangle - |\uparrow\downarrow\rangle\right)$$

Matrixify and SVD

$$|\Psi\rangle = \sum_{ij} \psi_{ij} |i\rangle |j\rangle \quad \text{with} \quad [\psi_{ij}] = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{pmatrix}$$

$$[\psi_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{D} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Entanglement entropy is 0 $\implies$ not entangled (makes sense)

# But wait...

We need $|\Psi\rangle$ to compress. But we want to find such a $|\Psi\rangle$ for some $H$.

# But wait...

We need $|\Psi\rangle$ to compress. But we want to find such a $|\Psi\rangle$ for some $H$.

# MPS and DMRG

MPS: Matrix product state $\leftarrow$

DMRG: Density matrix renormalization group

# MPS and DMRG

MPS: Matrix product state $\leftarrow$

DMRG: Density matrix renormalization group

MPS:

- expresses wavefunctions as products of matrices
- natural way to generate states with low entanglement entropy

# MPS and DMRG

MPS: Matrix product state $\leftarrow$

DMRG: Density matrix renormalization group

MPS:

- expresses wavefunctions as products of matrices
- natural way to generate states with low entanglement entropy

DMRG:

- numerical variational technique for finding ground states
- works with MPS, not $|\Psi\rangle$
- most efficient method for 1d systems

# MPS

**Every $|\Psi\rangle$ can be written as an MPS.** From Schmidt decomposition:

$$|\Psi\rangle = \sum_i s_i \, |i\rangle_L \, |i\rangle_R = \sum_{i_m} s_i \, |i_m\rangle_L \, |i_m\rangle_R$$

$m$ : position on the 1d chain where the $L - R$ split occurs.

# MPS

**Every $|\Psi\rangle$ can be written as an MPS.** From Schmidt decomposition:

$$|\Psi\rangle = \sum_i s_i \, |i\rangle_L \, |i\rangle_R = \sum_{i_m} s_i \, |i_m\rangle_L \, |i_m\rangle_R$$

$m$ : position on the 1d chain where the $L - R$ split occurs.

$|i_m\rangle_L$ can be built recursively:

$$|i_1\rangle_L = \sum_{\sigma_1} A^{\sigma_1}_{i_1} \, |\sigma_1\rangle$$

# MPS

**Every $|\Psi\rangle$ can be written as an MPS.** From Schmidt decomposition:

$$|\Psi\rangle = \sum_i s_i \, |i\rangle_L \, |i\rangle_R = \sum_{i_m} s_i \, |i_m\rangle_L \, |i_m\rangle_R$$

$m$ : position on the 1d chain where the $L - R$ split occurs.

$|i_m\rangle_L$ can be built recursively:

$$|i_1\rangle_L = \sum_{\sigma_1} A^{\sigma_1}_{i_1} \, |\sigma_1\rangle$$

$$|i_2\rangle_L = \sum_{i_1,\sigma_2} A^{\sigma_2}_{i_2,i_1} \, |i_1\rangle_L \, |\sigma_2\rangle = \sum_{i_1,\sigma_1,\sigma_2} A^{\sigma_1}_{i_1} A^{\sigma_2}_{i_2,i_1} \, |\sigma_1\sigma_2\rangle = \sum_{\sigma_1,\sigma_2} (A^{\sigma_1} A^{\sigma_2})_{1,i_2} \, |\sigma_1\sigma_2\rangle$$

# MPS

**Every $|\Psi\rangle$ can be written as an MPS.** From Schmidt decomposition:

$$|\Psi\rangle = \sum_i s_i |i\rangle_L |i\rangle_R = \sum_{i_m} s_i |i_m\rangle_L |i_m\rangle_R$$

$m$ : position on the 1d chain where the $L - R$ split occurs.

$|i_m\rangle_L$ can be built recursively:

$$|i_1\rangle_L = \sum_{\sigma_1} A^{\sigma_1}_{i_1} |\sigma_1\rangle$$

$$|i_2\rangle_L = \sum_{i_1,\sigma_2} A^{\sigma_2}_{i_2,i_1} |i_1\rangle_L |\sigma_2\rangle = \sum_{i_1,\sigma_1,\sigma_2} A^{\sigma_1}_{i_1} A^{\sigma_2}_{i_2,i_1} |\sigma_1\sigma_2\rangle = \sum_{\sigma_1,\sigma_2} (A^{\sigma_1} A^{\sigma_2})_{1,i_2} |\sigma_1\sigma_2\rangle$$

$$|i_3\rangle_L = \sum_{\sigma_1,\sigma_2,\sigma_3} (A^{\sigma_1} A^{\sigma_2} A^{\sigma_3})_{1,i_3} |\sigma_1\sigma_2\sigma_3\rangle$$

$$\vdots$$

# MPS

Schmidt states:

$$|i_m\rangle_L = \sum_{\{\sigma\}_1^m} (A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_m})_{1,i_m} |\sigma_1 \sigma_2 \dots \sigma_m\rangle$$

$$|i_m\rangle_R = \sum_{\{\sigma\}_{m+1}^N} (B^{\sigma_{m+1}} B^{\sigma_{m+2}} \dots B^{\sigma_N})_{i_m,1} |\sigma_{m+1} \sigma_{m+2} \dots \sigma_N\rangle$$

# MPS

Schmidt states:

$$|i_m\rangle_L = \sum_{\{\sigma\}_1^m} (A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_m})_{1,i_m} |\sigma_1 \sigma_2 \dots \sigma_m\rangle$$

$$|i_m\rangle_R = \sum_{\{\sigma\}_{m+1}^N} (B^{\sigma_{m+1}} B^{\sigma_{m+2}} \dots B^{\sigma_N})_{i_m,1} |\sigma_{m+1} \sigma_{m+2} \dots \sigma_N\rangle$$

Full wavefunction:

$$\boxed{|\Psi\rangle = \sum_{\{\sigma\}} A^{\sigma_1} A^{\sigma_2} \dots A^{\sigma_m} \; S \; B^{\sigma_{m+1}} B^{\sigma_{m+2}} \dots B^{\sigma_N} |\sigma_1 \sigma_2 \dots \sigma_N\rangle}$$

Matrix dimensions:

$$\underbrace{(1 \times d), (d \times d^2), \dots,}_{A} \underbrace{(d^{N/2} \times d^{N/2})}_{S}, \underbrace{\dots (d^2 \times d), (d \times 1)}_{B}$$

# MPS

Example:

$$|\mathrm{GHZ}_4\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{\{\sigma\}} A_1^{\sigma_1} A_2^{\sigma_2} A_3^{\sigma_3} A_4^{\sigma_4} |\sigma_1 \sigma_2 \sigma_3 \sigma_4\rangle$$

# MPS

Example:

$$|\mathrm{GHZ}_4\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{\{\sigma\}} A_1^{\sigma_1} A_2^{\sigma_2} A_3^{\sigma_3} A_4^{\sigma_4} |\sigma_1\sigma_2\sigma_3\sigma_4\rangle$$

where

$$A_1^0 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad A_2^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_3^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_4^0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A_1^1 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad A_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_3^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_4^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

# MPS & Hilbert space decimation

Idea: Motivated by SVD, not allow matrix dimensions to exceed a fixed $D$

# MPS & Hilbert space decimation

Idea: Motivated by SVD, not allow matrix dimensions to exceed a fixed $D$

From:

$$(1 \times d), (d \times d^2), \ldots, (d^{N/2} \times d^{N/2}), \ldots (d^2 \times d), (d \times 1)$$

To:

$$(1 \times d), (d \times d^2), \ldots (D \times D), (D \times D), (D \times D) \ldots (d^2 \times d), (d \times 1)$$

# MPS & Hilbert space decimation

Idea: Motivated by SVD, not allow matrix dimensions to exceed a fixed $D$

From:

$$(1 \times d), (d \times d^2), \ldots, (d^{N/2} \times d^{N/2}), \ldots (d^2 \times d), (d \times 1)$$

To:

$$(1 \times d), (d \times d^2), \ldots (D \times D), (D \times D), (D \times D) \ldots (d^2 \times d), (d \times 1)$$

To generalize, can make all matrices to $D \times D$, so that

$$\boxed{|\Psi\rangle = \sum_{\{\sigma\}} \mathrm{Tr}\left[A_1^{\sigma_1} A_2^{\sigma_2} \ldots A_m^{\sigma_m} A_{m+1}^{\sigma_{m+1}} A_{m+2}^{\sigma_{m+2}} \ldots A_N^{\sigma_N}\right] |\sigma_1 \sigma_2 \ldots \sigma_N\rangle}$$

# MPS

Example:

$$|\mathrm{GHZ}_4\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{\{\sigma\}} \mathrm{Tr}\left[A^{\sigma_1} A^{\sigma_2} A^{\sigma_3} A^{\sigma_4}\right] |\sigma_1 \sigma_2 \sigma_3 \sigma_4\rangle$$

# MPS

Example:

$$|\mathrm{GHZ}_4\rangle = \frac{|0000\rangle + |1111\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}} \sum_{\{\sigma\}} \mathrm{Tr}\left[A^{\sigma_1} A^{\sigma_2} A^{\sigma_3} A^{\sigma_4}\right] |\sigma_1 \sigma_2 \sigma_3 \sigma_4\rangle$$

where

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad A^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Can see that

$$\mathrm{Tr}\left[A^{\sigma_1} A^{\sigma_2} A^{\sigma_3} A^{\sigma_4}\right] = 1 \iff \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$$

# So far

- SVD & Schmidt decomposition allows for compressing $|\Psi\rangle$

# So far

- SVD & Schmidt decomposition allows for compressing $|\Psi\rangle$
- From Schmidt decomposition to MPS for general $|\Psi\rangle$

## So far

- SVD & Schmidt decomposition allows for compressing $|\Psi\rangle$
- From Schmidt decomposition to MPS for general $|\Psi\rangle$
- Decimate Hilbert space by keeping matrices in MPS at $(D \times D)$

# So far

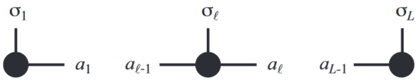
- SVD & Schmidt decomposition allows for compressing $|\Psi\rangle$
- From Schmidt decomposition to MPS for general $|\Psi\rangle$
- Decimate Hilbert space by keeping matrices in MPS at $(D \times D)$
- Smaller Hilbert space ($ND^2$ vs $d^N$) and MPS corresponds to $|\Psi\rangle$ with lower entanglement entropy

# So far

- SVD & Schmidt decomposition allows for compressing $|\Psi\rangle$
- From Schmidt decomposition to MPS for general $|\Psi\rangle$
- Decimate Hilbert space by keeping matrices in MPS at $(D \times D)$
- Smaller Hilbert space ($ND^2$ vs $d^N$) and MPS corresponds to $|\Psi\rangle$ with lower entanglement entropy
- MPS: natural way to describe ground states of relevant Hamiltonians

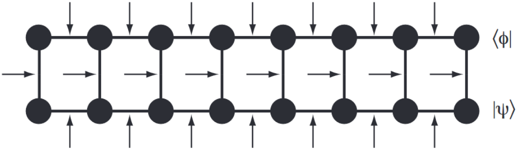
# Graphical notation

Matrices:




MPS:


Overlap for two MPS's (arrows indicate sum over indices)



# Graphical notation

MPO: Matrix product operators

If $\langle \sigma_1 \sigma_2 \ldots \sigma_N | \Psi \rangle = A^{\sigma_1} A^{\sigma_2} \ldots A^{\sigma_N}$, then

$$\langle \sigma_1 \ldots \sigma_N | \, \hat{O} | \sigma_1' \ldots \sigma_N' \rangle = W^{\sigma_1 \sigma_1'} W^{\sigma_2 \sigma_2'} \ldots W^{\sigma_{N-1} \sigma_{N-1}'} W^{\sigma_N \sigma_N'}$$

or

# Graphical notation

<u>MPO</u>: Matrix product operators

If $\langle \sigma_1 \sigma_2 \ldots \sigma_N | \Psi \rangle = A^{\sigma_1} A^{\sigma_2} \ldots A^{\sigma_N}$, then

$$\langle \sigma_1 \ldots \sigma_N | \hat{O} | \sigma_1' \ldots \sigma_N' \rangle = W^{\sigma_1 \sigma_1'} W^{\sigma_2 \sigma_2'} \ldots W^{\sigma_{N-1} \sigma_{N-1}'} W^{\sigma_N \sigma_N'}$$

or

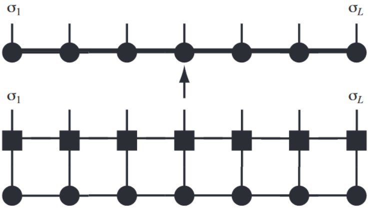
$$\hat{O} = \sum_{\{\sigma, \sigma'\}} W^{\sigma_1 \sigma_1'} W^{\sigma_2 \sigma_2'} \ldots W^{\sigma_{N-1} \sigma_{N-1}'} W^{\sigma_N \sigma_N'} |\sigma\rangle \langle \sigma'|$$

$\implies$ can calculate $\langle \Psi | H | \Psi \rangle$ in MPS language

# Graphical notation

MPO on MPS:

$$\hat{O}\,|\Psi\rangle = \sum_{\{\sigma,\sigma'\}} (W^{\sigma_1\sigma_1'} W^{\sigma_2\sigma_2'} \dots)(A^{\sigma_1} A^{\sigma_2} \dots)\,|\sigma\rangle = \sum_{\{\sigma\}} N^{\sigma_1} N^{\sigma_2} \dots |\sigma\rangle$$



# Ground state search

Variationally extremize

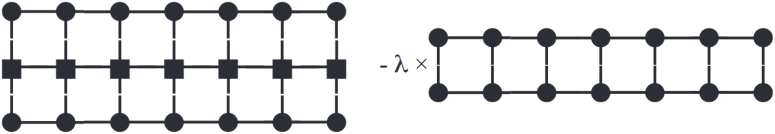
$$\langle\Psi| H |\Psi\rangle - \lambda \langle\Psi|\Psi\rangle, \text{ so that } |\Psi\rangle \to |\Psi_g\rangle, \ \lambda \to E_0$$

# Ground state search

Variationally extremize

$$\langle\Psi| H |\Psi\rangle - \lambda \langle\Psi|\Psi\rangle, \text{ so that } |\Psi\rangle \to |\Psi_g\rangle, \ \lambda \to E_0$$

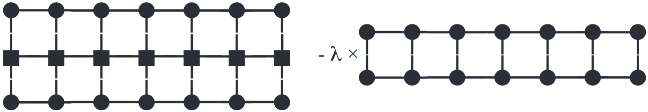
Graphically,



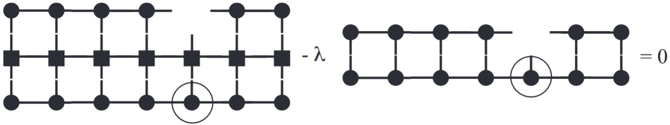
Idea: Keep matrices on all sites but one ($\ell$) constant, optimize elements of $M_\ell^{\sigma_\ell}$. Sweep through $\ell$.

# Ground state search

Minimizing




≡ solving (gradient descent)

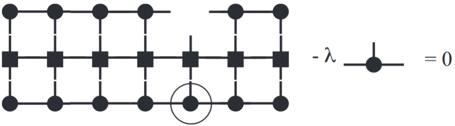



$\Longrightarrow$ a generalized eigenvalue problem, where we treat $M_\ell^{\sigma_\ell}$ as a vector $v$

Aside: left-/right-normalized

$$\sum_{\sigma_\ell} A^{\sigma_\ell \dagger} A^{\sigma_\ell} = I \qquad \sum_{\sigma_\ell} B^{\sigma_\ell \dagger} B^{\sigma_\ell} = I$$

If $|\Psi\rangle$ is both left- and right-normalized,




$\implies$ standard eigenvalue problem

# Ground state search

Algorithm:

- Start with initial guess for $|\Psi\rangle$, which is right-normalized $|\Psi\rangle \sim BBBBB\ldots$

# Ground state search

Algorithm:

- Start with initial guess for $|\Psi\rangle$, which is right-normalized $|\Psi\rangle \sim BBBBB\ldots$
- Calculate current state of network for sites $N-1$ through 1

# Ground state search

Algorithm:

- Start with initial guess for $|\Psi\rangle$, which is right-normalized $|\Psi\rangle \sim BBBBB\ldots$
- Calculate current state of network for sites $N-1$ through 1
- *Right-sweep*: From $\ell = 1$ through $N-1$: at $\ell$, solve for $M^{\sigma_\ell}$ then left-normalize into $A^{\sigma_\ell}$ by SVD; remaining matrices of the SVD are multiplied into $M^{\sigma_{\ell+1}}$

# Ground state search

Algorithm:

- Start with initial guess for $|\Psi\rangle$, which is right-normalized $|\Psi\rangle \sim BBBBB\ldots$
- Calculate current state of network for sites $N-1$ through 1
- *Right-sweep*: From $\ell = 1$ through $N-1$: at $\ell$, solve for $M^{\sigma_\ell}$ then left-normalize into $A^{\sigma_\ell}$ by SVD; remaining matrices of the SVD are multiplied into $M^{\sigma_{\ell+1}}$
- *Left-sweep*: From $\ell = L$ through $\ell = 2$: at $l$, solve for $M^{\sigma_\ell}$ then right-normalize into $B^{\sigma_\ell}$ by SVD; remaining matrices of the SVD are multiplied into $M^{\sigma_{\ell-1}}$

# Ground state search

Algorithm:

- Start with initial guess for $|\Psi\rangle$, which is right-normalized $|\Psi\rangle \sim BBBBB\ldots$
- Calculate current state of network for sites $N-1$ through 1
- *Right-sweep*: From $\ell = 1$ through $N-1$: at $\ell$, solve for $M^{\sigma_\ell}$ then left-normalize into $A^{\sigma_\ell}$ by SVD; remaining matrices of the SVD are multiplied into $M^{\sigma_{\ell+1}}$
- *Left-sweep*: From $\ell = L$ through $\ell = 2$: at $l$, solve for $M^{\sigma_\ell}$ then right-normalize into $B^{\sigma_\ell}$ by SVD; remaining matrices of the SVD are multiplied into $M^{\sigma_{\ell-1}}$
- Repeat right and left sweeps until convergence

# Ground state search

Algorithm, formalized: Subscript denotes the number of updates

$$
\begin{aligned}
M_0 B_0 B_0 B_0 &\xrightarrow{\text{diag}} M_1 B_0 B_0 B_0 \xrightarrow{\text{SVD}} A_1 M_0 B_0 B_0 \\
&\xrightarrow{\text{diag}} A_1 M_1 B_0 B_0 \xrightarrow{\text{SVD}} A_1 A_1 M_0 B_0 \\
&\xrightarrow{\text{diag}} A_1 A_1 M_1 B_0 \xrightarrow{\text{SVD}} A_1 A_1 A_1 M_0 \\
&\xrightarrow{\text{diag}} A_1 A_1 A_1 M_1 \xrightarrow{\text{SVD}} A_1 A_1 M_1 B_1 \\
&\xrightarrow{\text{diag}} A_1 A_1 M_2 B_1 \xrightarrow{\text{SVD}} A_1 M_1 B_2 B_1 \\
&\xrightarrow{\text{diag}} A_1 M_2 B_2 B_1 \xrightarrow{\text{SVD}} M_1 B_2 B_2 B_1 \\
&\vdots
\end{aligned}
$$

# DMRG

# DMRG

- Introduce left and right blocks $A$ and $B$, with one spin in each

# DMRG

- Introduce left and right blocks $A$ and $B$, with one spin in each
- Insert a pair between $A$ and $B$, so chain looks like $A \circ \circ B$

# DMRG

- Introduce left and right blocks $A$ and $B$, with one spin in each
- Insert a pair between $A$ and $B$, so chain looks like $A \circ \circ B$
- Diagonalize Hamiltonian to get $|\Psi_0\rangle$ and $E_0$

# DMRG

- Introduce left and right blocks $A$ and $B$, with one spin in each
- Insert a pair between $A$ and $B$, so chain looks like $A \circ \circ B$
- Diagonalize Hamiltonian to get $|\Psi_0\rangle$ and $E_0$
- Find reduced Hilbert space for $A\circ$ and $\circ B$. How?

# DMRG

- Introduce left and right blocks $A$ and $B$, with one spin in each
- Insert a pair between $A$ and $B$, so chain looks like $A \circ \circ B$
- Diagonalize Hamiltonian to get $|\Psi_0\rangle$ and $E_0$
- Find reduced Hilbert space for $A\circ$ and $\circ B$. How?
  - Schmidt-decompose $|\Psi_0\rangle$ and truncate
  - Get approximate $|\tilde{\psi}_0\rangle$ with rank $D$ (out of $Dd$)

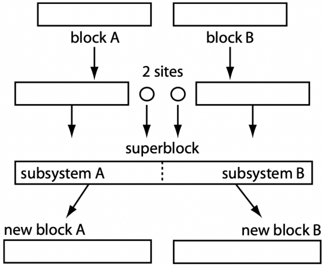
  $\implies$ $D$ most relevant states in $A\circ$, same for $\circ B$

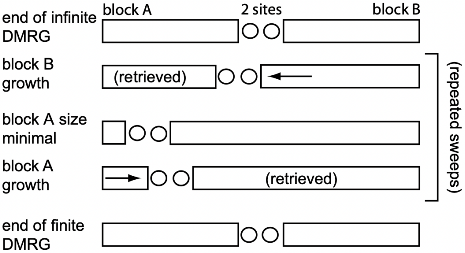
# DMRG

- Introduce left and right blocks $A$ and $B$, with one spin in each
- Insert a pair between $A$ and $B$, so chain looks like $A \circ \circ B$
- Diagonalize Hamiltonian to get $|\Psi_0\rangle$ and $E_0$
- Find reduced Hilbert space for $A\circ$ and $\circ B$. How?
  - Schmidt-decompose $|\Psi_0\rangle$ and truncate
  - Get approximate $|\tilde{\psi}_0\rangle$ with rank $D$ (out of $Dd$)

    $\implies$ $D$ most relevant states in $A\circ$, same for $\circ B$
- Repeat

# DMRG


block A
block B
2 sites
superblock
subsystem A
subsystem B
new block A
new block B


end of infinite DMRG
block A
2 sites
block B
block B growth
(retrieved)
block A size minimal
block A growth
(retrieved)
(repeated sweeps)
end of finite DMRG

# References

U. Schollwöck.
The density-matrix renormalization group.
*Rev. Mod. Phys.*, 77:259–315, Apr 2005.

Ulrich Schollwoeck.
The density-matrix renormalization group in the age of matrix product states.
*Annals of Physics*, 326:96–192, 2011.