QUANTUM FIELD THEORY

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Preface. These notes come from Prof. Patton's QFT notes, Peskin-Schröder QFT, and "Quantum field theory & Condensed matter" - Shankar.

Enjoy!

Conventions

- $\hbar = c = 1$
- $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$
- $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1\ -1\ -1\ -1)$
- $p^2 = p_\mu p^\mu = E^2 - |p|^2 = mc^2 = m$
- $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$
- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$
- $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{\partial x^0}, \nabla\right)$
- $\varepsilon^{0123} = +1 \;;\; \varepsilon_{0123} = -1$

  $\varepsilon^{1230} = -1$
- $E = i\dfrac{\partial}{\partial x^0} \;,\; \vec{p} = -i\vec{\nabla}$
- $p^\mu = i\partial^\mu$

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $$\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx} \theta(x)$

- n-dimensional Dirac δ-fn:

$$\int d^n x \, \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \tilde{f}(x)$$

$$\tilde{f}(x) = \int d^4 k \, e^{ik \cdot x} f(x)$$

- $$\int d^4 x \, e^{ik \cdot x} = (2\pi)^4 \delta^{(4)}(k)$$

- EM $\Phi = \frac{Q}{4\pi r} \leftarrow$ Coulomb potential

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \; ; \; \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\Phi, \vec{A}) \; ; \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\boxed{\text{Elements of classical Field Theory}}$$

Lagrangian Field Theory:

Lag-density

$$S = \int \underline{L}\, dt = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

$$\left(\underline{L} = \mathcal{L}\, d^3x\right)$$

Principle of least action:

(Lagrangian)

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right\}$$

(E-L eqn)

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right]\right\}$$

$$\Rightarrow \boxed{0 = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi}}$$

FTC → term vanishes @ boundary

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

Ex $\mathcal{L} = \phi^2$: $\quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi \; ; \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow 2\phi = 0$

$$\left. \begin{aligned} &\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) \\ &\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = 0 \; ; \; \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2\partial^\mu \phi \end{aligned} \right\} \Rightarrow \partial^\mu \phi = 0.$$

Ex Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

- $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$
- $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi .$

$$\text{E-L} \Rightarrow -m^2 \phi = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$\rightarrow \boxed{(\Box + m^2)\phi = 0}$$

Klein - Gordon Eqn.

relativistic particle of mass m.

Ex $\phi = e^{-ip\cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

Noether's Thm

Thm

$$\boxed{\text{For every continuous symmetry, there exists a conserved current } j^\mu \text{ which implies a local conservation law}}$$

$$\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j} \iff \partial_\mu j^\mu = 0$$

Ex Conservation of charge...

(gauss' law)

$$Q = \int j^0 \, d^3x$$

$$\Rightarrow \frac{\partial Q}{\partial t} = \int \frac{dj^0}{dt} d^3x = -\int \vec{\nabla}\cdot\vec{j} \, d^3x$$

$$= -\oint \vec{j} \, d^2 s$$

Idea Consider continuous transf. → infinitesimally... (local)

$$(\star) \qquad \phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

small

$(\star)$ is a symmetry if EOM invariant under $(\star)$.

$\Leftrightarrow$ S is invariant.

$\Rightarrow$ $\mathcal{L}$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$.

for some J^μ.

Let us compare this expectation for $\Delta\mathcal{L}$ to the result obtained by varying the fields ...

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi)$$

$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right\}\right)}_{0}\Delta\phi$$

$$\Rightarrow \quad \Delta\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right\}$$

So $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi$ is the desired $\mathcal{J}^\mu$.

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu .$$

Ex massless KG field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$$

Consider transformation:

$$\phi(x) \longrightarrow \phi(x) + \alpha \quad , \Delta\phi = 1$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = \partial^\mu\phi .$$

Check $\partial_\mu j^\mu = \partial_\mu\partial^\mu\phi = \Box\phi = 0$ since

$$(m^2 + \Box)\phi = 0 \ ; \ m = 0$$

Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^+)(\partial^\mu \phi) - m^2 \phi^+ \phi .$$

again, EOM is

$$(m^2 + \Box)\phi = 0.$$

Symmetry: $\phi \to e^{i\alpha}\phi$.

For infinitesimal transf we have:

$$\alpha \Delta \phi = i\alpha\phi \qquad \text{(Taylor expand)}$$
$$\alpha \Delta \phi^+ = -i\alpha\phi^+$$

One can check that

$$j^\mu = i\left[(\partial^\mu \phi^+)\phi - \phi^+(\partial^\mu \phi)\right]$$

is the conserved current.

$\boxed{\text{Ex}}$

Space-time translation:

Consider infi. translation:

$$x^\mu \to x^\mu - a^\mu$$

in field transform this becomes:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Lagrangian is a scaler ⇒ must transform the same way:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Compare this to the eqn:

$$\mathcal{L} \longrightarrow \mathcal{L} + a \partial_\mu J^\mu,$$

we have $J^\mu = \delta^\mu_\nu \mathcal{L}$

→ apply them, we find:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\mu \phi) - \delta^\nu_\mu \mathcal{L}$$

make μ explicit...

$$\boxed{T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta_\mu^\nu \mathcal{L}}$$

STRESS-ENERGY TENSOR. (or Energy-Momentum tensor)

<u>Conserved charge</u> is the Hamiltonian...

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad \text{(time-translation)}$$

momentum $\longrightarrow$ $p^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi \, d^3x$ (spatial translation)

Ex Klein-Gordon field again...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2.$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - g^{00}\mathcal{L} = \cdots$$

$$= \frac{1}{2}(\partial_0\phi)(\partial^0\phi) - \frac{1}{2}(\partial_i\phi)(\partial^i\phi) + \frac{1}{2}m^2\phi^2$$

So

$$H = \int T^{00}\,d^3x = \int \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\vec\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\} d^3x$$

note: all terms are positive ... (sum of squares)

→ can't fall into arbitrary negative energy.

$\boxed{\text{THE KLEIN-GORDON FIELD as HARMONIC OSCILLATOR}}$

promote: ϕ, π to operators & impose suitable commutation relations...

Recall ...

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0.$$

Harmonic oscillator: $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$

- $a^\dagger a = \frac{1}{\omega}H - \frac{1}{2} \Leftrightarrow H = \omega\left(a^\dagger a + \frac{1}{2}\right)$

  $\uparrow$

  # operator...

- $|0\rangle$, $a|0\rangle = 0$.

  $|n\rangle = \frac{1}{\sqrt{n!}}\underbrace{a^\dagger \ldots a^\dagger}_{n}|0\rangle$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -\omega a$; $[H, a^\dagger] = \omega a^\dagger$

  a lowers by ω

  $a^\dagger$ raises by ω

- $Ha^\dagger|n\rangle = (E_n + \omega)a^\dagger|n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

For continuum system ... commutation relations become ..

$$\boxed{\begin{aligned}[\phi(x), \pi(y)] &= i\delta^{(3)}(x-y) \\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0.\end{aligned}}$$

Next, the Hamiltonian is now also an operator ...
To find spec(H), Fourier transf $\phi(x)$

$$\rightarrow \quad \phi(x,t) = \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot x} \phi(\vec{p},t)$$

Recall KG eqn: $(m^2 + \Box)\phi = 0$

$$\rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) \right\} \phi(\vec{p},t) = 0$$

$\Rightarrow$ This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}.$$

Now, $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2 \qquad (m=1)$

$\rightarrow$ know spectrum! $\quad (n+\frac{1}{2})\omega.$

$$\phi = \frac{1}{\sqrt{2\omega}}(a^\dagger + a) \quad ; \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$[a, a^\dagger] = 1.$$

Since it's more convenient to work in position space

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right) \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^\dagger e^{-ip x} \right) \end{aligned}}$$

note

$$\begin{cases} a_p \text{ goes with } e^{+ip\cdot x} \\ a_p^\dagger \text{ goes with } e^{-ip\cdot x}. \end{cases}$$

- Easy to show that

$$\begin{cases} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \end{cases}$$

- Can re-arrange...

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p + a_{-p}^\dagger \right) e^{ip\cdot x} . \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p - a_{-p}^\dagger \right) e^{+ip\cdot x} . \end{cases}$$

→ get commutation relation between a_p:

$$\boxed{[a_p ; a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

Now, can check that

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{-i}{2}\right)\sqrt{\frac{\omega_{p'}}{\omega_p}} \times e^{i(p\cdot x + p'\cdot x')} \times$$

$$\left([a^\dagger_{-p}, a_{p'}] - [a_p, a^\dagger_{-p'}]\right)$$

$$= i\delta^{(3)}(x-x') \quad \checkmark$$

- Now, can express Hamiltonian in terms of ladder ops

Recall that $\longrightarrow$ KG field, last time

$$H = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0\phi - g^{00}\mathcal{L} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0\phi)} = \partial^0\phi(x) \longrightarrow \left(\text{like } p = \frac{\partial \mathcal{L}}{\partial \dot{q}}\right)$$

So ...
$$H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2}m^2\phi^2 \right\}$$

with $\pi(x) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_p}{2}}\,(a_p - a^\dagger_{-p})\, e^{ip\cdot x}$ we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p + a^\dagger_{-p})\, e^{ip\cdot x}$$

$\rightarrow$ results in $\delta(p-p')$

$\Rightarrow p = p'$

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(p+p')\cdot x} \left\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4} (a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'}) \right.$$

$$\left. + \frac{-p\cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}} (a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) \right\}$$

Some $\delta^{(3)}$ will appear...

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p] \right)$$

So

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p] \right)}$$

$\rightarrow$ can evaluate commutators...

$$\boxed{[H, a^\dagger_p] = \omega_p a^\dagger_p \ ; \ [H, a_p] = -\omega a_p}$$

With H, can find momentum operator...

KG field $\longrightarrow$ from $p^i = \int d^3x\, T^{0i} = -\int \pi \partial_i \phi\, d^3x$, we get

$$\boxed{\begin{aligned} \vec{P} &= -\int d^3x\, \pi(x) \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_p a_p . \end{aligned}}$$

$a^\dagger_p$ creates momentum $\vec{p}$ & energy $\underset{E_p}{\omega_p} = \sqrt{|p|^2 + m^2}$.

Excitation: $a^\dagger_p a^\dagger_q \ldots |0\rangle \equiv$ "particles".

$\hookrightarrow$ <u>each excitation $a^\dagger_p$ is a particle.</u>

$\Rightarrow$ get particle statistics...

Consider 2-particle state... $a_p^\dagger a_q^\dagger |0\rangle$.

Since $[a_p^\dagger, a_q^\dagger] = 0$, we have

$$\boxed{a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle}$$

$\rightarrow$ Klein Gordon particles follow Bose-Einstein stats.

- <u>Normalization</u> $\langle 0|0\rangle = 1$.

$$|p\rangle \propto a_p^\dagger |0\rangle .$$

This normalization $\longrightarrow$ $\langle q|p\rangle = (2\pi)^3 \delta^{(3)}(q-p)$ is NOT Lorentz inv

<u>Pf</u> Under a Lorentz boost $\rightarrow$ $p_3' = \gamma(p_3 + \beta E)$

$$E' = \gamma(E + \beta p_3)$$

using the identity $\rightarrow$

$$\delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|}\delta(x - x_0)$$

<u>we compute</u>: $\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\cdot\left(\dfrac{dp_3'}{dp_3}\right)$

$\underbrace{\delta(p_1-q_1)\delta(p_2-q_2)}_{\text{same}}\underbrace{\delta(p_3-q_3)}_{\text{boosted}}$

$$= \delta^{(3)}(p'-q')\cdot\gamma\left(1+\beta\, dE/dp_3\right)$$

$$= \delta^{(3)}(p'-q')\frac{\gamma}{E}(E+\beta p_3)$$

$$= \delta^{(3)}(p'-q')(E'/E)$$

$1 \Leftrightarrow E = E_p = E'$

So $$\boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\left(\frac{E'}{E}\right)}$$

For normalization to work → use E_p ($\equiv w_p$), not E.

→ define: $\boxed{|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$

So that $$\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$$

Completeness relation...

1 particle → $$\boxed{1 = \int \frac{d^3p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$$

Ex Interpret $\phi(x)|0\rangle$ .... we know that $a_p^\dagger$ creates momentum p & energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|p\rangle \quad , \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

Pf. By defn...

annihilates.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|0\rangle \quad \square$$

→ $\phi(x)|0\rangle$ is a lin. superposition of single-particle states

~~Note HW1, HW2 easy~~

that have well-defn momentum.

When nonrelativistic $\rightarrow E_p \simeq$ Constant!

$\Rightarrow$ [$\phi(x)$ acting on the vacuum, "creates a particle at position x".]

↳ Confirm this by computing.

$$\langle 0|\,\phi(x)\,|p\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \left(a_{p'} e^{ip'\cdot x} + a_{p'}^\dagger e^{-ip'\cdot x}\right) \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

$$\rightarrow \boxed{\langle 0|\,\phi(x)\,|p\rangle = e^{ip\cdot x}}.$$

↳ Interpretation: position-space representation of the single-particle wfn of the state $|p\rangle$, just like ...

$\boxed{\langle x|p\rangle \propto e^{ip\cdot x}}$ in QM!

$\langle 0|\,\phi(x) \sim \langle x| \ldots$ (don't take this literally, ofc).

Note HW1, HW2 are easy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time → we quantized KG field in the Schrödinger picture.

→ Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$$\mathcal{U}(t) = e^{-iHt} \text{ is the time evolution.}$$

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle. \longrightarrow \underline{\text{State}} \text{ evolve in time}$$

→ In the Heisenberg picture, ... $\underline{\text{Operators}}$ evolve in time

$$\mathcal{O}(t) = \mathcal{U}^{\dagger}(t)\, \mathcal{O}(0)\, \mathcal{U}(t).$$

So that

$$\underset{\text{Heisenberg}}{\langle \psi_1 | \mathcal{O}(t) | \psi_1 \rangle} = \underset{\text{Schrödinger.}}{\langle \psi_1(t) | \mathcal{O} | \psi_1(t) \rangle}$$

→ make the operators ϕ & π time-dependent like this

$$\boxed{\begin{aligned} \phi(x) \to \phi(x,t) &= e^{iHt}\, \phi(x)\, e^{-iHt} \\ \pi(x) \to \pi(x,t) &= e^{iHt}\, \pi(x)\, e^{-iHt} \end{aligned}}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

$(\phi \leftrightarrow \phi)$
$\rightarrow$ only nontrivial term is 1st.

$$= \int d^3x' \left(i\delta^{(3)}(x-x')\, \pi'(x,t) \right)$$

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t}\phi(x,t) = \pi(x,t)}$$

and

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')(-\nabla^2 + m^2)\phi(x',t) \right)$$

(integrate by parts here)

$$= -i(-\nabla^2 + m^2)\phi(x,t)$$

$$\rightarrow \boxed{\frac{\partial}{\partial t}\pi(x,t) = (m^2 - \nabla^2)\phi(x,t)}$$

Combining these 2 results we get...

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) = (\nabla^2 - m^2)\phi(x,t)}$$

rearranging this gives

$$\boxed{(\Box + m^2)\phi(x,t) = 0} \longrightarrow \text{just the KG eqn...}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

<u>Recall</u>: $H a_p = a_p (H - E_p)$ → from comm. rule.

$$\Longrightarrow \begin{pmatrix}\text{proof by}\\ \text{induction}\end{pmatrix} \quad \boxed{H^n a_p = a_p (H-E_p)^n}$$

Similarly, $\boxed{H^n a_p^+ = a_p^+ (H+E_p)^n}$

→ So, we have

$$\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}} \rightarrow \text{from above...}$$

and

$$\boxed{e^{iHt} a_p^+ e^{-iHt} = a_p^+ e^{+iE_p t}}$$

→ Now... we want to write $\phi(x,t)$ in terms of these operators. (since $\phi(x)$, $\pi(x)$ is a combo of a & a^+)

- We know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before...

$$\phi(x) = \phi(x,0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^+ e^{-ip\cdot x} \right)$$

Substitute this into $\phi(x,t) = e^{iHt}\phi(x) e^{-iHt}$ we find ...

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-iE_pt} e^{ip\cdot x} + a_p^\dagger e^{iE_pt} e^{-ip\cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow \quad p\cdot x = E_p \cdot x^0 - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p} = E_pt - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p}$$

So...

$$\boxed{\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right\}\Bigg|_{p^0=E_p}}$$

Similarly, we can find

$$\boxed{\pi(x,t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left(a_p e^{-i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right)\Bigg|_{p^0=E_p}}$$

from defn or from $\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$.

<u>Note</u> we can also do everything, but starting from P and not H. But we won't worry about that.

<u>Causality</u> Note that causality is broken ~~then~~ without the presence of fields.

$\rightarrow$ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture...

...the amplitude for a particle to propagate from y to x is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

"Two-point correlation function"

Let $\boxed{\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \text{ in terms of } a_p \text{ \& } a_p^\dagger \ldots |0\rangle$$

$$= \langle 0|\, a_p\, a_q^\dagger\, |0\rangle =$$

$$= (2\pi)^3\, \delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{-ip\cdot(x-y)}$$

More explicitly...

@ $p^0 = E_p$
$p^{0\prime} = E_{p'}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_{p'}}}\langle 0|\, e^{-ip\cdot x}\, e^{+ip'\cdot y}\, a_p a_{p'}^\dagger\, |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_{p'}}}\right) e^{-ip\cdot x + ip'\cdot y}\, (2\pi)^3\, \delta^{(3)}(p-p')$$

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\cdot e^{-ip\cdot(x-y)} \qquad \square.$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$...

(1) Suppose that $x-y = (t, 0, 0, 0)$, then

$$D(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t}$$

(timelike)

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\underset{t\to\infty}{=} e^{-imt} .$$

→ dominated by region where $p \approx 0$.

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

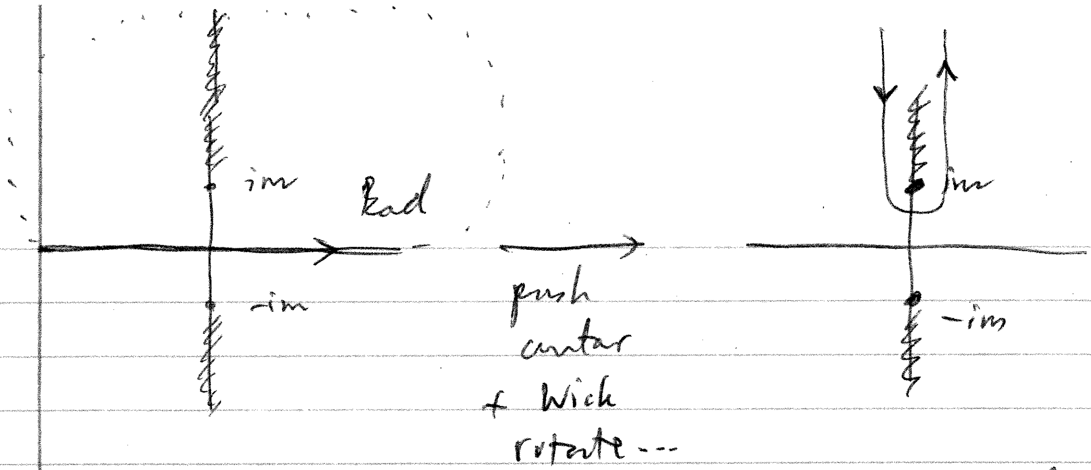
$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_p} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{p e^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ $\pm im$ (singularity)...

→ must change contours... & Wick rotate



To get

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \, \frac{\rho\, e^{-i\rho r}}{\sqrt{\rho^2 - m^2}}$$

$\rho = -ip$ (Wick rotate)

$$\Rightarrow \quad \boxed{\mathcal{D}(x-y) \underset{r\to\infty}{=} \frac{e^{-mr}}{r}} \longrightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^\mu (x-y)_\mu = -(\vec{x}-\vec{y})^2 < 0$ is spacelike, cannot have causality between x & y.

$\mathcal{D}(x-y) \neq 0 \Rightarrow$??? paradox?

$\rightarrow$ <u>NO!</u> To discuss causality, we should ask not whether particles can propagate over spacelike intervals ...

... but whether a "<u>measurement</u>" ~~at~~ at one point can affect a measurement at another point whose separation from the first is spacelike -

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement @ x, call this $\Phi(x)$. & a local measurement @ y, called $\Phi'(y)$

So long as $[\Phi(x), \Phi'(y)] = 0$, the 2 measurements don't affect one another.

~ measure the field ϕ @ x & @ y,

If $[\phi(x), \phi(y)] = 0$ ~~when~~ $(x-y)^2 < 0$ then we're good

→ $$\cancel{[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \Big\{ e^{-ip\cdot x}[a_p, a^\dagger_{p'}] e^{ip'\cdot y} + e^{ip\cdot x}[a^\dagger_p, a_{p'}] e^{-ip'\cdot y} \Big\}}$$

→ $$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \times \Big\{ \Big[(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_{p'} e^{-ip'\cdot y} + a^\dagger_{p'} e^{ip'\cdot y}) \Big]$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \Big\{ e^{-ip\cdot x} \underbrace{[a_p, a^\dagger_{p'}]}_{(2\pi)^3 \delta^3(p-p')} e^{ip'\cdot y} + e^{ip\cdot x} \underbrace{[a^\dagger_p, a_{p'}]}_{-(2\pi)^3 \delta^3(p-p')} e^{-ip'\cdot y} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p}\right) \Big\{ e^{-ip(x-y)} - e^{-ip(y-x)} \Big\} = D(x-y) - D(y-x)$$

So $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$.

Since $D(y-x)$ is Lorentz invariant, can take

$y-x \rightarrow x-y$. (possible b/c $(x-y)^2 < 0$)

note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

$\rightarrow$ so this is only possible because $(x-y)^2 < 0$ (spacelike).

So $D(y-x) = D(x-y)$

So, $[\phi(x), \phi(y)] = 0$ ✓

The Klein-Gordon Propagator

Let's look at $[\phi(x), \phi(y)]$ in more details...

$[\phi(x), \phi(y)]$ is just a number

$\Rightarrow$ can write $[\phi(x), \phi(y)] \equiv \langle 0| [\phi(x), \phi(y)] |0\rangle$

$$\rightarrow \langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right\}$$

(assuming $x^0 > y^0$)

△ Poles @ $p^2 = m^2$ ⇕ $p^0 = \pm E_p$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_p} - \frac{1}{2E_p} e^{-ip(x-y)} \Big|_{p^0 = -E_p} \right.$$

The Klein-Gordon Propagator

→ Before looking at this ... need to look at Green's Function & Contour Integrals...

→ Consider function

$$\frac{1}{(p_0^0 - E_p)(p_0^0 + E_p)} \qquad E_p = \sqrt{|p|^2 + m^2}.$$

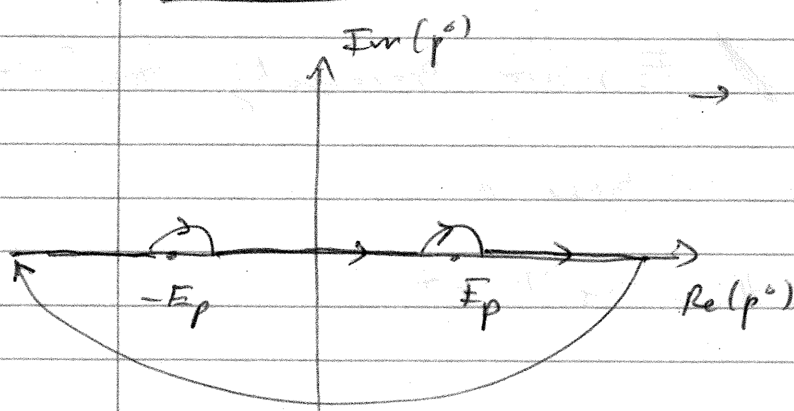
→ Poles at $p_0^0 = \pm E_p$.

Look at FT:

$$\boxed{f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}}$$

→ How to integrate this?

If $t > 0$ → ~~[illegible]~~



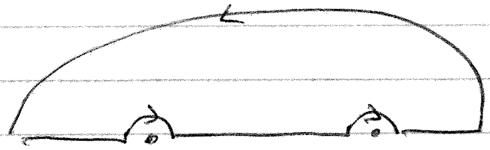
→ $f(t) = (-2\pi i) \sum \text{Res}(f)$

$$= (-2\pi i) \left\{ \frac{e^{iE_p t}}{2\pi(-2E_p)} + \frac{e^{-iE_p t}}{2\pi(2E_p)} \right\}$$

↑ close contour ...
but clockwise → reverse sign

→ $$\boxed{f(t) = \frac{-i}{2E_p}\left(e^{+iE_p t} - e^{-iE_p t} \right)} \qquad (t > 0)$$

If $t<0$ close contours above poles



$\rightarrow \quad f(t) = 0.$

$\rightarrow$ So, altogether, we have...

$$f(t) = \int \frac{1}{2\pi} \frac{dp^0}{(p^0-E_p)(p^0+E_p)} e^{-ip^0 t}$$

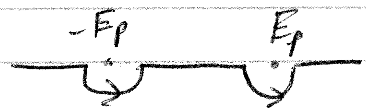
$$= \Theta(t)\left(\frac{-i}{2E_p}\right)\left(e^{-iE_p t} - e^{+iE_p t}\right)$$

where $\Theta(t)$ is the Heaviside Step fn...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

$\longrightarrow$ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



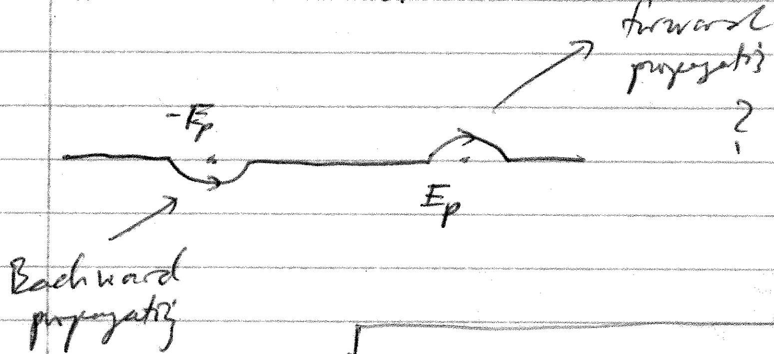
, then we'll get

$t > 0 \rightarrow f(t) = 0$

$t < 0 \rightarrow f(t) = \Theta(-t)\,\dfrac{i}{2E_p}\left(e^{-iE_p t} - e^{iE_p t}\right)$

$\rightarrow$ Advanced / Backward Propagating Green's fn.

☐ What about



$$\longrightarrow \quad \boxed{f(t) = \Theta(t)\left(\frac{-i}{2E_p}\right)e^{-iE_pt} + \Theta(-t)\left(\frac{-i}{2E_p}\right)e^{+iE_pt}}$$

↑ Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$ ...

Consider this quantity...

$$\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\underbrace{\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{ \frac{1}{2E_p}e^{-ip(x-y)} + \frac{1}{-2E_p}e^{-ip(x-y)} \right\}$$

pole @ $p_0 = E_p$ pole @ $p_0 = -E_p$

4D integral →

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3}\underbrace{\int \frac{dp^0}{2\pi i}\frac{-1}{p^2 - m^2}e^{-ip(x-y)}}_{f(t)\text{ before, where}}$$

$$(p^0 - E_p)(p^0 + E_p) = p^{0\,2} - |p|^2 - m^2 = p^2 - m^2.$$

Now, call ...

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle}$$

Then

$$(\Box + m^2) D_R(x-y) = \Box D_R(x-y) + m^2 D_R(x-y)$$

$$= (\Box\theta(x^0-y^0))\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ 2(\partial_\mu \theta(x^0-y^0))\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ \theta(x^0-y^0)(\Box+m^2)\underbrace{\langle 0|[\phi(x),\phi(y)]|0\rangle}_{\cancel{\text{constant}}}$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$+ 2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0$$

needs verification (easy)

$$= \delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$-i\delta^{(3)}(x-y)$$

$$= -i\delta^{(4)}(x-y)$$

So

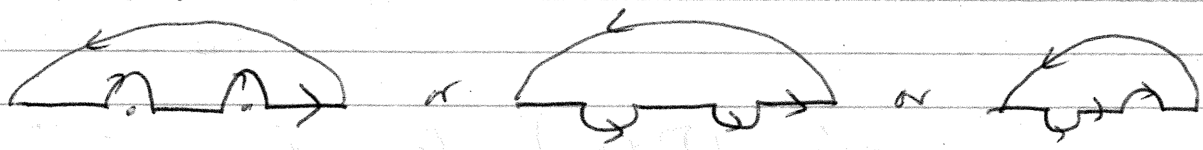
$$\boxed{(\Box+m^2)D_R(x-y) = -i\delta^{(4)}(x-y)}$$

$\longrightarrow$ $D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0$ @ $x^0 < y^0$

$\Rightarrow$ $D_R(x-y) \equiv$ "Retarded" Green's fn

Now ... As we have seen, there are many ways to take the contour ...

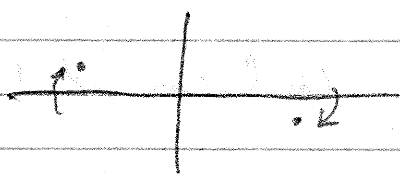


or ... or ... etc

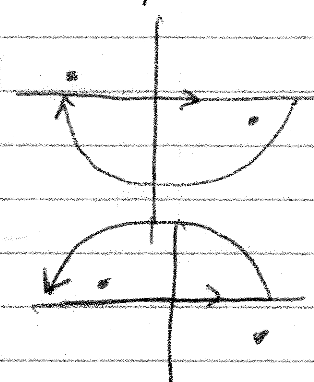
$\longrightarrow$ Use the Feynman prescription instead

$$\boxed{D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}}$$

$\rightarrow$ Convenient! B/c now poles are $p^0 = \pm(E_p - i\varepsilon)$



$\longrightarrow$ when $x^0 > y^0 \longrightarrow$ take contour below



when $x^0 < y^0 \longrightarrow$ take contour above.

$\rightarrow$ get same expression but with $x \leftrightarrow y$.

So,

$$\boxed{\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases} \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &\equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \end{aligned}}$$

"time-ordering" symbol.

Time-ordering symbol $\Longrightarrow$ instructs us to place the operators that follows in order with the latest to the left -

→ apply $(\Box + m^2)$ to last line, get D_F is Green's fn of Klein-Gordon Operator.

→ $D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator...

→ propagation amplitude

→ But we can't much calculation at this point just yet.

→ B/c we're only looked at the free KG theory

→ field eqn in this case is linear : there are no interactions...

→ this theory is too simple to make any predictions...

→ need perturbation ....

One kind of interaction that can also be solved ...

→

Particle Creation by a classical Source

Consider the source $j(x)$

Recall... free field: $(\Box + m^2)\phi = 0$

now... $(\Box + m^2)\phi = j(x)$ fixed fn in space time.

$j(x)$ is nonzero only for a finite time interval

The associated Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is easiest to solve...

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

With a source...

$$\phi(x) = \phi_0(x) + i\int d^4y\, D_R(x-y)\, j(y)$$

$\vdots$

We won't worry about this for now...

Some problems & Insights

1. **Classical EM** (no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \varepsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \; ; \quad \vec{\nabla} \cdot \vec{E} = 0 \qquad (\nu = 0)$$

$$-\partial_t \mathcal{E} + \vec{\nabla} \times \vec{E} = 0 \qquad (\nu = i)$$

2. **Complex scalar field**

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi \right)$$

Derive E-L eqn:

$$\boxed{\begin{aligned} & i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ = 0 \\ & -i\partial_t \phi - \frac{1}{2m} \nabla^2 \phi = 0 \end{aligned}}$$

now... write $\phi \Rightarrow e^{-i\theta}\phi$, $\phi^\dagger \to e^{i\theta}\phi^\dagger$

$$\sim \phi - i\theta\phi \qquad\qquad \sim \phi^\dagger + i\theta\phi^\dagger$$

$$\to \Delta\phi \sim -i\theta \qquad\qquad \Delta\phi^\dagger \sim i\phi^\dagger$$

So that

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\Delta\phi^\dagger$$

conserved current.

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \to \text{... compute ...}$$

→ can get Hamiltonian... — there's a Ham. formula in book, but we won't worry abt this.

$$\mathcal{H} = \int d^3x\,(\pi^\dagger\pi + (\nabla\phi)^\dagger(\nabla\phi) + m^2\phi^\dagger\phi)$$

(3) If we take $(x-y)^2 = -r^2$ → can implicitly write

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2$ → $D(x-y)$ can be written ~~as~~ in terms of Bessels Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation... $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu}$

→ what happens to $\phi(x)$ under Λ?

we require that $\phi'(x') = \phi(x)$

$$\to \boxed{\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform ... $\partial_\mu \phi(x) \to \partial_\mu(\phi(\Lambda^{-1}x))$

$$\to \boxed{\partial_\mu(\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\to (\partial_\mu \phi(x))^2 \to g^{\mu\nu}(\partial_\mu \phi'(x))(\partial_\nu \phi'(x))$$

$$= g^{\mu\nu}[\partial_\mu(\phi(\Lambda^{-1}x))][\partial_\nu(\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu}\{(\Lambda^{-1})^\rho{}_\mu \partial_\rho \phi\}\{(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \phi\}(\Lambda^{-1}x)$$

$$= (\partial\phi)^2(\Lambda^{-1}x) \leftarrow = g^{\rho\sigma}\partial_\rho \phi \, \partial_\sigma \phi \, (\Lambda^{-1}x) \to \text{evaluated}$$

Recall
$$\boxed{\begin{aligned} \phi(x) &\rightarrow \phi(\Lambda^{-1}x) \\ \partial_\mu \phi(x) &\rightarrow (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x) \\ (\partial_\mu \phi(x))^2 &\rightarrow (\partial_\mu \phi)^2(\Lambda^{-1}x) \end{aligned}}$$

So it is clear that

$$\boxed{\mathcal{L} \longrightarrow \mathcal{L}(\Lambda^{-1}x)}$$

Lagrangian is Lorentz-invariant.

→ The action $S = \int d^4x\, \mathcal{L}$ is also Lorentz inv.

→ also clear that EOM is also Lorentz inv.

$$(\Box + m^2)\,\phi'(x) = \left((\Lambda^{-1})^\mu{}_\nu \partial_\mu (\Lambda^{-1})^{\sigma\mu} \partial_\sigma + m^2\right)\phi(\Lambda^{-1}x)$$
$$= (\partial^\sigma \partial_\sigma + m^2)\,\phi(\Lambda^{-1}x)$$
$$= 0 \quad ✓$$

Sep 20, 2020

→ How do we find Lorentz-invariant theories, in general?

→ for simplicity, restrict attention to lin. transf

→ $\phi_a = v \in \mathbb{C}^n$, → matrix giving Lorentz transform Λ.

→ $$\boxed{\Phi_a(x) \Rightarrow \underbrace{M_{ab}(\Lambda)}_{n\times n}\, \Phi_b(\Lambda^{-1}x)}$$

→ the most general nonlinear transf can be built out of linear ones ⇒ suffices to consider M only.

↳ for short, write $\Phi \longmapsto M(\Lambda)\Phi$.

→ what are the possible allowed $M(\Lambda)$?

Obs $\{\Lambda\}$ form a group $M(\Lambda'')M(\Lambda) = M(\Lambda')$
$\Lambda''\Lambda = \Lambda'$

→ the correspondence between Λ & M must be preserved under multiplication.

$\{\Lambda\}$ Lorentz group $\longmapsto \{M(\Lambda)\} \to$ n-dim representation of the Lorentz group

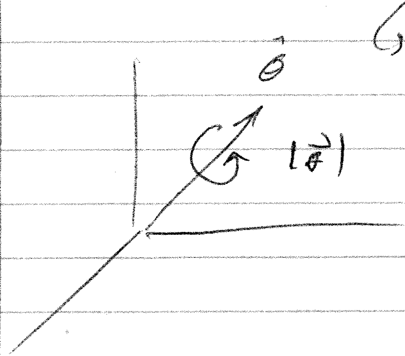
↳ $\boxed{?}$ what are the finite-dim matrix reps of the Lorentz group?

Ex in QM... spin $\frac{1}{2}$ → $\{M\}$ are the 2×2 unitary matrices with determinant 1.

→ $\boxed{\mathcal{U} = e^{-i\theta^i\sigma^i/2}}$ → $\sum_i \theta^i\sigma^i/2$

3 arbitrary parameters

& pauli matrices.



$\mathcal{U}(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2}$

→ In the case for arbitrary spin representations...

$$\mathcal{U}(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i\sum_\ell \epsilon^{jk\ell} J^\ell$$

→ check that this works for spin 1/2:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i\sum_\ell \epsilon^{jk\ell}\frac{\sigma^\ell}{2} \quad \checkmark$$

→ for spinless particles... $\Psi(\vec{x})$ can be decomposed into orbital angular momentum states $J = 0, 1, 2, \ldots$ (no intrinsic spin ⇒ $J = L$)

- $\vec{J} = \vec{L} = \vec{x}\times\vec{p} = \vec{x}\times(-i\vec{\nabla})$
- $J^j = i\sum_\ell \epsilon^{jk\ell} x^k \nabla^\ell$
- $\nabla^\ell = -\nabla_\ell = -\dfrac{\partial}{\partial x^\ell}$

But the cross product is special to 3D case.

→ write operator as antisymmetric tensor...

$$\boxed{J^{ij} = -i(x^i\nabla^j - x^j\nabla^i)}$$ → reproduces the cross product.

so that $J^3 = J^{12}$, etc.

→ generate to 4D: → 6 operators that generate { 3 boosts, 3 rotations, of the Lorentz group.

$$\boxed{J^{\mu\nu} = +i(x^\mu\partial^\nu - x^\nu\partial^\mu)}$$

→ Spatial Rotations: $J^{jk} = i(x^j \partial^k - x^k \partial^j)$

→ Lorentz boosts along x^j axis: $J^{0j} = i(x^0 \partial^j - x^j \partial^0)$

→ Now, want to get commutation rules...
→ compute the commutators of differential ops to get

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\sigma} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)}$$

Ex 3 rotations: $J^{12} = -J^{21}$
$J^{23} = -J^{32}$
$J^{13} = -J^{31}$

3 boosts $J^{01} = -J^{10}$
$J^{02} = -J^{20}$
$J^{03} = -J^{30}$

→ 6 total generating matrices...

Ex Consider the 4x4 matrices $(J^{\mu\nu})_{\alpha\beta}$ where μ, ν label which of the 6 matrices, while α, β label the component / matrix element...

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)}$$

→ Can verify that $(J^{\mu\nu})_{\alpha\beta}$ satisfies the comm. relation...

→ These are matrices that act on ordinary Lorentz 4-vectors...
to see this...

→ Look at elements of the Lorentz group...

$$\mathcal{U}(\omega_{\mu\nu}) = \operatorname{Exp}\left[\frac{-i}{2}\,\omega_{\mu\nu} J^{\mu\nu}\right]$$

infinitesimally →
$$\simeq \mathbb{I} + \frac{-i}{2}\omega_{\mu\nu} J^{\mu\nu}$$
$$\simeq \delta^\alpha_\beta + \frac{-i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta$$

So, infinitesimally...

$$V^\alpha \longrightarrow \delta^\alpha_\beta + \frac{-i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta\, V^\beta$$

→ $\omega_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles...

$V_\alpha, V_\beta \longrightarrow$ 4-vectors...

<u>Ex 1</u> When $\omega_{12} = -\omega_{21} = \theta$, $\omega_{\mu\nu} = 0$ else, we get

$$[V^\mu] \longrightarrow \begin{array}{c|c} & \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{array}{c} t \\ x \\ y \\ z \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} [V^\mu]$$

→ Infinitesimal $\boxed{\text{ROTATION on } xy \text{ plane.}}$

<u>Ex 2</u> When $\omega_{01} = -\omega_{10} = \beta \Rightarrow$ get
$\omega_{\mu\nu} = 0$ else

$$[V^\mu] \overset{\sim}{\longrightarrow} \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} [V^\mu] \rightarrow \boxed{\text{BOOST along } x}$$

THE DIRAC EQUATION

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to develop formalism for finding all other ~~formalisms~~ representations...
(Problem 3.1)

→ focus on spin $\frac{1}{2}$ systems...

→ In this case, use Dirac's trick ~~due to~~ -

Suppose we had a set of 4 $n \times n$ matrices γ^μ satisfying:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \times I}$$

Then we could write down an n-dim representation of the Lorentz algebra...

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

These matrices satisfy the commutation relation...

→ $[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$

→ Verify that this trick works in 3D Euclidean space:

in which case, $\boxed{\gamma^j = i\sigma^j}$ → $\boxed{\{\gamma^i, \gamma^j\} = -2\delta^{ij}}$

→ The matrices representing the Lorentz algebra are then

(23)

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\sum_k \varepsilon^{ijk}\sigma^k = J^l$$

Which is what we saw before $\Rightarrow$ angular momentum.

$$\left\{ \begin{array}{l} J^3 = S^{12} = \frac{1}{2}\sigma^3 \\ J^2 = S^{31} = \frac{1}{2}\sigma^2 \\ J^1 = S^{23} = \frac{1}{2}\sigma^1 \end{array} \right\}$$

→ now, want $S^{\mu\nu}$ for 4D Minkowski space...

→ matrices γ^μ must be at least 4×4.

→ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2\times2} \\ \mathbb{I}_{2\times2} & 0 \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

"Weyl" / "Chiral" representation.

→ In this case, the boost & rotation generators are...

Boosts → $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

(sum) sum

Rotations → $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \sum_k \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \sum_k \frac{1}{2}\varepsilon^{ijk}\Sigma^k$

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at <u>all</u> transformations under which the ~~Lagrangian~~ Action S is "invariant"

→ In particular, we want S to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\begin{cases} \phi(x) \longrightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x^\mu) \rightarrow \phi(\Lambda^{-1\,\mu}{}_{\nu}\, x^\nu). \end{cases}$$

→ (check that S is invariant)

→ But this is very simple ... ⇒ There are <u>many</u> more transforms that leave S Lorentz invariant.

→ How do we find all of them?

⇒ For simplicity, we'll just restrict ourselves to <u>linear ~~combinations~~ transformations</u>

→ look at transformations of the form.

$$\phi_a(x) \longrightarrow \sum_b M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \longrightarrow M(\Lambda)\,\phi}$$

These matrices M must be "nice" in the sense that M must obey...

$$\boxed{\phi \rightarrow M(\Lambda'')M(\Lambda)\phi = M(\Lambda''\Lambda)\phi}$$

This says that $\{M\}$ (the collection of M's) must be a <u>representation</u> of the Lorentz group.

<u>What??</u> So... recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

$$\rightarrow \boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

<u>A representation</u> π of a group G is a function satisfying the property

$$\underbrace{\pi(g_1)}_{\in G}\ \underbrace{\pi(g_2)}_{\in G} = \underbrace{\pi(g_1 g_2)}_{\in G}$$

With this, it is clear that

$$\boxed{\{\Lambda\}\ \text{Lorentz group} \Longrightarrow \{M\} \text{ is a representation of } \{\Lambda\}}$$

So... what are these M's?

→ <u>Ex</u> Rotation group for spin $\frac{1}{2}$ particles

For spin-$\frac{1}{2}$, the most important nontrivial representation is the 2D representation.

↳ these are unitary matrices with det = 1
(2x2)

$\Rightarrow$ In general: $U = e^{-i\vec{\sigma}\cdot\vec{\theta}/2}$

$\vec{\sigma} \rightarrow$ Pauli matrices
$\vec{\theta} \rightarrow$ angle.

For infinitesimal rotations, we can write

$$U = 1 - \frac{i}{\hbar}\vec{J}\cdot\vec{\theta} \equiv 1 - \vec{J}\cdot\vec{\theta}$$

$\{U\}$ form a Lie-algebra of the L-group.

$\vec{J}$ here are the "generator" of the Lie algebra

when $\{U\}$ is a representation of the rotational group, we identify

$$\vec{J} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{J}$ as the quantum angular momentum operator

→ satisfies the commutation relation

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

like the generators of SO(3), namely the Pauli matrices...

→ finite rotations are formed by matrix exp.

$$R = \exp\left[-i\theta^i \sigma^i\right]$$

Sep 27, 2020

Back to present problem...

to get generator of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

(3D) $\quad \vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$

(4D) $\quad \boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)}$$

→ any matrices that are ~~re~~ represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$$

→ by symmetry, μ, ν ~~table~~ label which of the six matrices we want;

α, β label components.

The Dirac Eqn

What are the representations of the Lorentz group?

, especially for spin-$\frac{1}{2}$?

<u>Dirac's trick</u> if we have a set of 4 $n \times n$ matrices γ^μ which satisfies:

Dirac algebra →
$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} * I_{n\times n}}$$

Then the n-dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

→ In other words, $S^{\mu\nu}$ satisfies ...

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho})$$

(*) Note that this trick works ~~~~ in any dim.

e.g. take $\gamma^j = i\sigma^j$ so that $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

$$\Rightarrow S^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma^k$$ → just as before.

2D representation of the rotation group.

Spin $\frac{1}{2}$: $J^3 = J^{12} = \frac{1}{2}\sigma^3$; $J^2 = \frac{1}{2}\sigma^2 = S^{31}$, $J^1 = S^{23} = \frac{1}{2}\sigma^1$

One such representation for the Dirac algebra is

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4\times 4} \; ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4\times 4}}$$

Weyl / chiral representation.

get

Boosts $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

Hermitian Defn not ~~[illegible]~~, but Ψ is also classical field, not a wfn

All 4-component field Ψ that transforms under boosts & rotations according to is called a Dirac spinor

S^{ij} are Hermitian
S^{0i} are anti-Hermitian

fine b/c Ψ is a classical field, not a ~~k~~ wfn.

Now, what is the field eqn for ψ?

$\rightarrow$ try $(\Box + m^2)\psi = 0$ $\leftarrow$ KG field eqn.

But this obviously works because the representation are block-diagonal...

$\rightarrow$ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~the~~ at transformation of γ matrices

In an expression we can think of...

$$[\cdots]\,\Lambda_{\frac{1}{2}}^{-1}\,[4\times4]\,\Lambda_{\frac{1}{2}}\,[\vdots]$$

$\frac{1}{2}$: for spin $\frac{1}{2}$

where $\Lambda_{\frac{1}{2}} = \exp\left\{\frac{-i}{2}\,\omega_{\mu\nu}S^{\mu\nu}\right\}$

$$\simeq 1 - \frac{i}{2}\,\omega_{\mu\nu}S^{\mu\nu}$$

$$\Rightarrow \quad [\gamma^\mu] \longrightarrow [\Lambda_{\frac{1}{2}}^{-1}][\gamma^\mu][\Lambda_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2}\,\omega_{\mu\nu}S^{\mu\nu}\right)\gamma^\mu\left(1 - \frac{i}{2}\,\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$= \ldots$ (some terms of higher order cancelled ...)

$$= \gamma^\mu - \frac{i}{2}\,\omega_{\alpha\beta}\underbrace{[\gamma^\mu, S^{\alpha\beta}]}_{?}$$

where a quick computation shows that

$$\boxed{[\gamma^\mu, S^{\alpha\beta}] = (J^{\alpha\beta})^\mu{}_\nu \gamma^\nu}$$

where

$$J^{\alpha\beta} = +i\left(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu\right)$$

So...

$$\boxed{\gamma^\mu \longrightarrow \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}(J^{\alpha\beta})^\mu{}_\nu \gamma^\nu} = \Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}$$

→ γ^μ transforms like 4-vectors...!

⇒ γ^μ are invariant under simultaneous rotations of their <u>vectors</u> & <u>spinor</u> indices...

→ can treat "μ" on γ^μ as a vector index!

→ can dot γ^μ into ∂_μ to form a Lorentz-inv. differential operator...

<u>Dirac eqn</u>

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}$$

Check that this is Lorentz-inv:

Let $\Psi(x) \longrightarrow \Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$ then

$$i\gamma^\mu\partial_\mu\Psi \to \left(i\gamma^\mu\Lambda^{+1}_{\frac{1}{2}}\right)\left(\partial_\mu\left(\Psi(\Lambda^{-1}x)\right)\right)$$

$$= i\Lambda_{\frac{1}{2}}\left(\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\right)\cdot(\Lambda^{-1})^\alpha{}_\mu(\partial_\alpha\Psi)(\Lambda^{-1}x)$$

~~same~~ Lorentz transform

$$= i\Lambda_{\frac{1}{2}} (\Lambda)^{\mu}{}_{\nu}\, \gamma^{\nu} \cdot (\Lambda^{-1})^{\alpha}{}_{\mu}\, (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}}\, \gamma^{\mu}\, \underbrace{(\Lambda)^{\mu}{}_{\nu}(\Lambda)^{\alpha}{}_{\mu}}_{\delta^{\alpha}_{\nu}}\, (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}}\, \gamma^{\mu} \partial_\mu \Psi(\Lambda^{-1}x)$$

$$\Rightarrow \quad i\gamma^\mu \partial_\mu \Psi(x) \rightarrow \Lambda_{\frac{1}{2}}\, i\gamma^\mu \Psi(\Lambda^{-1}x)$$

→ transforms the same way as $\Psi(\Lambda^{-1}x)$.

Cleaner way:

Let $[i\gamma^\mu \partial_\mu - m]\Psi(x) \rightarrow [i\gamma^\mu (\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m]\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$

$$= \underbrace{\Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}}[i\gamma^\mu \Lambda^{-1\,\nu}{}_{\mu}\partial_\nu - m]\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}}\left\{ i\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m \right\}\Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}}\left\{ i\gamma^\nu \partial_\nu - m \right\}\Psi(\Lambda^{-1}x)$$

$$= 0 \quad ✓$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i\gamma^\mu \partial_\mu - m)\Psi$$

$$\rightarrow \quad 0 = (-i\gamma^\mu \partial_\mu - m)(+i\gamma^\nu \partial_\nu - m)\Psi$$

$$= (\gamma^\mu \gamma^\nu \partial_\nu \partial_\mu + m^2)\Psi \quad = \ldots$$

$$= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi$$

$$= \left[\frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu + m^2\right]\psi$$

$$= \left[\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2\right]\psi$$

KG eqn.

$$= (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi = (\Box + m^2)\psi = 0$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger\psi$ doesn't work b/c under a boost,

$$\psi^\dagger \Delta_{\frac{1}{2}}^\dagger \Delta_{\frac{1}{2}} \psi \neq \psi^\dagger\psi \quad \text{since } \Delta_{\frac{1}{2}} = \exp\{-i\omega_{\mu\nu} S^{\mu\nu}\}$$

not unitary ... since not all $S^{\mu\nu}$ are Herm.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, get

$$\bar{\psi} \to \psi^\dagger \Delta_{\frac{1}{2}}^\dagger \gamma^0 \cong \psi^\dagger\left(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right)\gamma^0$$

when ~~.....~~ $\mu \neq 0$ & $\nu \neq 0$, $(S^{\mu\nu})^\dagger = (S^{\mu\nu})$

& $(S^{\mu\nu} \leftrightarrow \gamma^0)$

when $\mu = 0$ or $\nu = 0$, $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$

$\Rightarrow$ $S^{\mu\nu}$ anti-commutes w/ γ^0.

$$\rightarrow \quad \bar{\Psi} \rightarrow \Psi^\dagger \left(1 + \frac{i}{2} w_{\mu\nu} (S^{\mu\nu})^\dagger\right) \gamma^0$$

$$= \underbrace{\Psi^\dagger \gamma^0}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right)$$

$$= \bar{\Psi}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right) = \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \quad \text{as desired}$$

$$\rightarrow \quad \boxed{\bar{\Psi} \rightarrow \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1}}$$

and so $\boxed{\bar{\Psi}\Psi = \Psi^\dagger \gamma^0 \Psi}$ is a Lorentz scalar.

Similarly, can show that

$\boxed{\bar{\Psi} \gamma^\mu \Psi}$ is a Lorentz vector.

$\Rightarrow$ the correct Lorentz-inv Dirac Lagrangian is

$$\boxed{\mathcal{L}_{Dirac} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi}$$

$$\begin{cases} \text{E-L eqn for } \bar{\Psi} \text{ gives } (i\gamma^\mu \partial_\mu - m)\Psi = 0 \\ \text{E-L eqn for } \Psi \text{ gives } -i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \end{cases}$$

WEYL SPINOR

Recall that

$$\boxed{\begin{aligned} S^{0i} &= \frac{-i}{2}\begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix} \\ S^{ij} &= \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix}\sigma^k & 0\\ 0 & \sigma^k\end{pmatrix}\end{aligned}}$$

Since block-diagonal $\Rightarrow$ Dirac representation of the Lorentz group is reducible.

$\rightarrow$ Can form 2-D representations by considering each block separately...

$\rightarrow$ write $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$ $\psi_L \rightarrow$ Left-handed, $\psi_R \rightarrow$ RH — Weyl spinors

Under infinitesimal boost $\vec{\beta}$ & rotation $\vec{\theta}$, these transform as

$$\boxed{\begin{aligned} \psi_L &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_L \\ \psi_R &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_R .\end{aligned}}$$

Recall that $\left(\tanh(|\vec{\beta}|) = \frac{|\vec{v}|}{c}\right)$.

$\rightarrow$ Transf of ψ_R is equiv to transf of ψ_L^*

By noting that

$$\psi_L^* \longrightarrow \left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

noting that $\underset{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{\sigma^2}\,\vec{\sigma}^* = -\vec{\sigma}\,\underset{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{\sigma^2}$ $\quad(\sigma^2 = \sigma^y)$

we find:

$$\sigma^2\psi_L^* \longrightarrow \sigma^2\left[1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right]\psi_L^*$$

$$= \underbrace{\left[1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right]}_{\text{like } \psi_R \text{ transform.}}\sigma^2\psi_L^*$$

So $\sigma^2\psi_L^*$ transform like ψ_R ...

With $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$, the Dirac eqn has form.

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0 \iff \begin{pmatrix}-m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = 0}$$

When $m = 0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left.\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0\\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0\end{cases}\right\} \longrightarrow \text{Weyl eqns.}$$

$\rightarrow$ important for neutrinos & weak force studies...

For convenience let us define...

$$\boxed{\sigma^\mu \equiv (1, \vec{\sigma}) \quad ; \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})}$$

so that

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$

With this, can simplify notation... Dirac eqn becomes...

$$\boxed{\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0}$$

$i\sigma\cdot\partial = i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})$

$i\bar{\sigma}\cdot\partial = i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})$

& the Weyl eqns become:

$$\boxed{\begin{aligned} (i\bar{\sigma}\cdot\partial)\psi_L &= 0 \\ (i\sigma\cdot\partial)\psi_R &= 0 \end{aligned}}$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p)\, e^{-ip\cdot x} \quad , \quad p^2 = m^2$$

so that $p^0 = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}$.

Look only solutions with positive frequency ... that is
$E_p = p^0 > 0$ ....

ψ solves Dirac eq^n $\Rightarrow$ $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\Leftrightarrow \quad (i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$$

$$\rightarrow \quad \boxed{(i\gamma^\mu p_\mu - m)\, u(p) = 0}$$

go to rest frame $\Rightarrow$ $p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $\Lambda_{\frac{1}{2}}$.

In rest frame, we have

$$(i\gamma^\mu p_\mu - m)\, u(p) \rightarrow (m\gamma^0 - m)\, u(p_0) = m(\gamma^0 - 1)\, u(p_0) = 0$$

$$\Rightarrow \quad m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$$

$$\rightarrow \quad \boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}} \quad \text{two-component spinor}$$

just a factor with norm. constraint.

$$\xi^\dagger \xi = 1.$$

<u>What are these ξ?</u>

Look at <u>rotation</u> generators...

$$\boxed{S^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

In particular, $S^z = S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}$

So if $\left\{ \begin{array}{l} \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \text{gives spin-}z = \frac{+1}{2} \\ \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \to \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$

Now, we're in rest frame, so $p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Now, boost to frame where particle has velocity...

$\vec{v} = v \cdot \vec{z}$. Let $\underbrace{\tanh(\eta)}_{\text{"rapidity"}} = \frac{v}{c}$.

Then

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = p^\mu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh(\eta) \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ 0 \\ 0 \\ m\sinh\eta \end{pmatrix}$$

(infinitesimal $\Lambda_{\frac{1}{2}}$)

$\Lambda_{\frac{1}{2}} \to$ just the Lorentz transform.

$\to$ In this frame, $\left\{ \begin{array}{l} E = m\cosh\eta \\ p^3 = m\sinh\eta \end{array} \right.$

Now, apply the same boost to $u(p)$...

$$u(p) = \Lambda_{\frac{1}{2}} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad \left(\Lambda_{\frac{1}{2}}\right) = \exp\left(\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

$$= \exp\left(\frac{-1}{2}\eta\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right) \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$\sigma^3 \quad i/ \left(\sigma^3 \quad \sigma^3\right)$

So, infinitesimally ...

$$\exp\left\{\frac{-1}{2}\eta\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right\} \simeq \begin{pmatrix}\cosh \eta/2 - \sinh \eta/2\,\sigma^3 & 0\\ 0 & \cosh\eta/2 + \sinh\frac{\eta}{2}\sigma^3\end{pmatrix}$$

So that

$$u(p) \simeq \sqrt{m}\begin{pmatrix}(\cosh \eta/2 - \sinh \eta/2\,\sigma^3)\,\xi\\ (\cosh \eta/2 + \sinh \eta/2\,\sigma^3)\,\xi\end{pmatrix} \ldots$$

Simplify ... note that

$$\left(\cosh\frac{\eta}{2} - \sinh\frac{\eta}{2}\sigma^3\right)^2 = \ldots$$

$$= \cosh\eta - \sinh\eta\,\sigma^3$$

$$= E/m - p^3/m\,\sigma^3 = \frac{p\cdot\sigma}{m}$$

$$= \frac{p_\mu\sigma^\mu}{m} \quad \text{where } \sigma^\mu = (1,\vec{\sigma})$$

$p_\mu\sigma^\mu/m$

So... $\left\{\begin{aligned}&\left(\cosh\eta/2 - \sinh\eta/2\,\sigma^3\right) = \sqrt{\frac{p\cdot\sigma}{m}}\\ &\left(\cosh\eta/2 + \sinh\eta/2\,\sigma^3\right) = \sqrt{\frac{p\cdot\bar{\sigma}}{m}}\end{aligned}\right.$

So ... $\boxed{u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi\\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix}}$ $\longrightarrow$ compact & valid for any arbitrary direction of $\vec{p}$.

Fact $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0)^2 - \vec{p}^{\,2} = p^2 = m^2.$

Now, back to example...

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow \quad p\cdot\sigma = \cdots = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$$

so

$$\sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}$$

and

$$\sqrt{p\cdot\bar{\sigma}} = \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}$$

pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then (spin $\frac{-1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

In the massless limit, $E \to p^3$ $\left(E^2 = \sqrt{m^2c^4 + (p^3)^2}\right)$

$$\rightarrow \quad \boxed{\begin{array}{ll} u(p) = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} & \text{spin } \frac{1}{2} \\ u(p) = \begin{pmatrix} \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} & \text{spin } \frac{-1}{2} \end{array}}$$

These states: $u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $u(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are eigenstates of the helicity operator

$$\boxed{h \equiv \hat{p}\cdot\vec{S} = \sum_i \frac{1}{2}\hat{p}^i\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}} = \frac{1}{2}\hat{p}\cdot\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

When $\begin{cases} h = \frac{1}{2} \Rightarrow \text{call Right-handed} \\ h = \frac{-1}{2} \Rightarrow \text{call Left-handed}\end{cases}$

Note: Dirac helicity is frame-dependent... (for massive particle). — since can boost so that momentum is in the opposite direction.

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = i(\bar{\sigma}\cdot\partial)\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = i(\sigma\cdot\partial)\psi_R = 0\end{cases}$$

Plug $\psi = u(p)e^{-ip\cdot x}$ in, $\partial_0 \longrightarrow -iE$

$\vec{\nabla} \longrightarrow i\vec{p}$.

So, with $m=0$, $\vec{p} = E\hat{p}$.

$\Rightarrow$ get $\begin{cases} (E + E\hat{p}\cdot\sigma)\psi_L = 0 \Rightarrow (E)(1+2h)\psi_L = 0 \quad \Rightarrow h = \frac{-1}{2} \\ (E - E\hat{p}\cdot\sigma)\psi_R = 0 \Rightarrow (E)(1-2h)\psi_R = 0 \quad \longrightarrow h = \frac{1}{2}\end{cases}$

$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}$, as expected

//

Recap... $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 > 0)$ → positive frequency

$\rightarrow \quad u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$ → spinor.

when $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 < 0)$ → negative frequency

$\Rightarrow \quad (u(p)) = \ldots = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$

//

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = \left(\xi^\dagger \sqrt{p\cdot\sigma},\ \xi^\dagger\sqrt{p\cdot\bar{\sigma}}\right)\cdot\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$$

$$= \xi^\dagger\left[(p\cdot\sigma) + (p\cdot\bar{\sigma})\right]\xi$$

→ depends on p!

$$\Rightarrow \quad u^\dagger u = 2E_p\, \xi^\dagger\xi$$

→ $u^\dagger u$ is not a Lorentz-inv scalar. just like $\psi^\dagger\psi$.

⇒ to make one such Lorentz-inv scalar, define

$$\boxed{\bar{u}(p) = u^\dagger(p)\,\gamma^0} \Rightarrow \boxed{\bar{u}u = 2m\,\xi^\dagger\xi}$$ Lorentz-inv (indep of $\vec{p}$)

So, with ~~also~~ $\bar{u}u = u^\dagger \gamma^0 u = 2m \underbrace{\xi^\dagger \xi}_{1} = 2m$

$\rightarrow$ convenient to choose ONB spinors, ξ^1 & ξ^2.

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix} \qquad s = 1,2}$$

Normalize:

$$\bar{u}^r(p)\, u^s(p) = 2m\,\delta^{rs} \Leftrightarrow u^{r\dagger}(p)\, u^s(p) = 2E_p\,\delta^{rs}$$

For the negative-freq solns, we get

$$\bar{v}^r(p)\, v^s(p) = -2m\,\delta^{rs} \Leftrightarrow v^{r\dagger}(p)\, v^s(p) = +2E_p\,\delta^{rs}$$

and

v, u are orthogonal to each other...

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$$

Finally, talk about $\boxed{\text{spin sums}}$

$\rightarrow$ useful when evaluating Feynman diagrams.

$\rightarrow$ when we need to sum all spin-$\frac{1}{2}$ polarization

Since $\{\xi^s\}$ form an ONB, ...

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\sigma} \right)$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar\sigma} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

"completeness" $\longrightarrow$

$$= \begin{pmatrix} p\cdot\sigma & m \\ m & p\cdot\bar\sigma \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix}$$

$$\left(\begin{aligned} \sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\sigma} &= m \\ &= \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)} = \sqrt{\left((p^0,\vec p)\cdot(1,-\vec\sigma)\right)\left((p^0,\vec p)\cdot(1,\vec\sigma)\right)} \\ &= \sqrt{(p^0)^2 - p^2} = m. \end{aligned} \right)$$

So,

$$\boxed{\begin{aligned} &\sum_{s=1,2} u^s(p)\,\bar u^s(p) = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix} = \overbrace{p\cdot\gamma}^{\not p} + m\,I \\ &\sum_{s=1,2} v^s(p)\,\bar v^s(p) = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar\sigma & -m \end{pmatrix} = \underbrace{p\cdot\gamma}_{\not p} - m\,I \end{aligned}}$$

Feynman's slash notation

→ The combo $\gamma \cdot p$ occur so often that Feynman introduced the notation:

$$\boxed{\not{p} \equiv \gamma^\mu p_\mu = p_\mu \gamma^\mu}$$

#

~~[illegible] Dirac Field Bilinears~~

Exercise

Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L. The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

$$m = \text{Majorana mass.}$$

(a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.

(b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

(a) In general, infinitesimal Lorentz transforms on ψ_L has the form

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

→ Rotation has the form:

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

$$\Rightarrow \sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L(x) \quad \text{(notes)}$$

Lorentz transform:

$$\psi_L(x) \to \Lambda_{\frac{1}{2}}\psi_L(\Lambda^{-1}x)$$

$$\partial_\mu\psi_L(x) \longrightarrow (\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

→ put these together...

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(\Lambda^{-1}x)$$

$$\sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

$$\rightarrow \quad -im\sigma^2\psi_L^*(x) \rightarrow -im\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

Next, $i\bar{\sigma}\cdot\partial\psi_L(x) = i\bar{\sigma}^\mu\partial_\mu\psi_L(x)$

$$\Rightarrow i\bar{\sigma}^\mu(\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= i\bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)(\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

use trick: multiply:

$$1 \equiv \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right) \quad \text{(rot × inv\_rot)}$$

$$\Rightarrow \to i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\overbrace{\left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)}$$

$$\times\ (\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\boxed{?}\,(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

what is $\boxed{?}$

$$\to \boxed{?} = \left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\vec{\sigma}/2\right)$$

$$\cong \bar{\sigma}^{\mu} + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\bar{\sigma}^{\mu} - i\vec{\theta}\cdot\bar{\sigma}^{\mu}\frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^{\mu} - \frac{i}{2}\vec{\theta}\left[\bar{\sigma}^{\mu},\vec{\sigma}\right]$$

can show that

$$= \bar{\sigma}^{\mu} - i\theta\left[J_4^{ij}\right]^{\mu}_{\ \nu}\bar{\sigma}^{\nu}$$

$$\downarrow$$

$$i\left(g^{i\mu}\delta^{j}_{\nu} - g^{j\mu}\delta^{i}_{\nu}\right)$$

$\Rightarrow \boxed{?} = (\Lambda_4)^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu} \to \bar{\sigma}^{\mu}$ transforms like 4-vector

$$\Rightarrow i\bar{\sigma}\cdot\partial\Psi_L(x) \Rightarrow i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Lambda^{\mu}_{\ \nu}\bar{\sigma}^{\nu}(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\delta^{\alpha}_{\nu}\,\bar{\sigma}^{\nu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\nu}\,\partial_{\nu}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}\cdot\partial\Psi_L(\Lambda^{-1}x)$$

✓

$\rightarrow \quad i\bar{\sigma}\cdot\partial\psi_L(x) - im\sigma^2\psi_L^*(x) = 0$

$\rightarrow$ leads to infinitesimal solution...

$$\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Big\{ \underbrace{i\bar{\sigma}\cdot\partial\psi_L(\Lambda^{-1}x) - im\sigma^2\psi_L^*(\Lambda^{-1}x)}_{=0} \Big\} = 0$$

$\Rightarrow$ done! So Majorana eqn is invariant under infinitesimal rotations...

(b) <u>Boosts</u> (proceed in a similar way...)

<u>Key</u>

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= \bar{\sigma}^\mu - \frac{1}{2}\vec{\beta}\left\{\bar{\sigma}^\mu, \vec{\sigma}\right\}$$

$$\rightarrow \quad \bar{\sigma}^\mu \rightarrow \bar{\sigma}^\mu - i\beta\left[\mathcal{J}_4^{0i}\right]^\mu{}_\nu\,\bar{\sigma}^\nu$$

$\rightarrow$

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = (\Lambda)^\mu{}_\nu\,\bar{\sigma}^\nu \quad \text{as before}$$

so plug this back into the original eqn.

Sep 28, 2020

Dirac Matrices & Dirac Field Bilinears

Oct 2, 2020

Recall that $\bar{\Psi}\Psi$ is Lorentz scalar...

Recall that $\bar{\Psi}\gamma^\mu\Psi$ is also a 4-vector.

$\Rightarrow$ $\boxed{?}$ Consider $\bar{\Psi}\Gamma\Psi$, where Γ is any 4x4

→ can we decompose into terms that have definite transformation properties under the Lorentz group?

↳ Γ can be written as combo of 16-element basis defined by

$$
\begin{aligned}
1:&\quad \mathbb{1} & &\rightarrow 1\\
4:&\quad \gamma^\mu & &\rightarrow 4C2\\
6:&\quad \gamma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu,\gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} & &\rightarrow 4C3\\
4:&\quad \gamma^{\mu\nu\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^{\sigma]} & &\rightarrow 4C2\\
1:&\quad \gamma^{\mu\nu\sigma\rho} = \gamma^{[\mu}\gamma^\nu\gamma^\sigma\gamma^{\rho]} & &\rightarrow 4C2
\end{aligned}
$$

16 total.

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisym tensor of successively higher rank.

→ Introduce

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$= \frac{-i}{4!}\varepsilon^{\mu\nu\sigma\rho}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho$$

0123 → 1

1023 → -1

↳ totally anti-symmetric

Note that $\rightarrow$ $\boxed{(\gamma^5)^2 = \mathbb{1}}$

$$\rightarrow \boxed{\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^3)^\dagger \cdots (\gamma^0)^\dagger \\ &= +i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5\end{aligned}}$$

also

$$\boxed{\begin{aligned}\{\gamma^5,\gamma^\mu\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\underbrace{\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3}_{(-1)} \\ &= 0\end{aligned}}$$

and thus

$$\boxed{[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}\{\gamma^\mu,\gamma^\nu\}] = 0}$$

$\Rightarrow$ Eigenstates of γ^5 with different eigv don't mix under Lorentz transfm.

$\rightarrow$ In basis, can write...

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{array}{l} \rightarrow \text{for } \psi_L \text{ (left-hd)} \\ \rightarrow \text{for } \psi_R \text{ (right-hd)} \end{array}$$

$\rightarrow$ a Dirac spinor with only L/R component is an eigenstate of γ^5 with eigv $(-1)/(1)$.

With γ^5, can rewrite the table of 4×4 matrices as

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu,\gamma^\nu]$	tensor	6
$\gamma^\mu\gamma^5$	pseudo vector	4
γ^5	pseudo scalar	1
		16

pseudo-vector/scalar is due to the fact that they transform like vector/scalar, BUT with an additional sign change under parity-transf.
under Lorentz transf

Ex Parity transf: $\vec{x} \to -\vec{x}$

$\hookrightarrow (x^0, x^i) \to (x^0, -x^i)$

If instead $(x^0; \vec{x}) \to -(x^0; -\vec{x}) = (-x^0; \vec{x})$ under parity, we call this a <u>pseudo-vector</u>

$\to$ pseudo vect/scalar flips sign under parity transf.

$\to$ From vector & pseudo-vector we can form 2 currents out of Dirac field bilinears...

$$\boxed{\begin{aligned} j^\mu(x) &= \bar{\Psi}(x)\gamma^\mu\Psi(x) \to \text{vector current} \\ j^{\mu 5}(x) &= \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) \to \text{pseudo vector current} \end{aligned}}$$

Assume that Ψ satisfies Dirac eqn... $\bar{\Psi} = \Psi^\dagger\gamma^0$

$\longrightarrow i\not{\partial}\Psi = m\Psi$ & $-i\not{\partial}\bar{\Psi} = m\bar{\Psi}$ (since $\bar{\gamma}^\mu = \gamma^\mu$)

$\to$ compute div of these currents...

$$\partial_\mu j^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu(\partial_\mu\Psi)$$

$$= im\bar{\Psi}\Psi + \bar{\Psi}(-im\Psi) = 0$$

$$\to \boxed{\partial_\mu j^\mu = 0}$$

→ j^{μ} is always <u>conserved</u> if $\psi(x)$ satisfies Dirac eqn

→ It is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha}\psi(x)$$

<u>Similarly</u>

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + (-1)\bar\psi\gamma^5\gamma^\mu\partial_\mu\psi$$

$$= im\bar\psi\gamma^5\psi + (-1)(-i)m\bar\psi\gamma^5\psi$$

→ $\boxed{\partial_\mu j^{\mu 5} = 2im\bar\psi\gamma^5\gamma}$ ⇝ "axial vector current"

→ if $m=0$ then $\partial_\mu j^{\mu 5}$ is conserved.

→ When $m=0$, $j^{\mu 5}$ is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha\gamma^5}\psi(x).$$

(we worry about ~~the~~ the rest of this section in ~~Weyl~~ Peskin's ...)

—//—

QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi .$$

→ Canonical momentum conjugate to Ψ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger .$$

→ Hamiltonian is given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)}\partial_0\Psi - \mathcal{L} = \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi} - \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi}$$

$$- i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$

$$= -i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$

Thus,

$$\boxed{H = \int \mathcal{H}\, d^3x = \int d^3x\, \bar{\Psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\Psi}$$

→ now let's figure out the commutators to make this a quantum field theory...

→ HOW NOT TO QUANTIZE THE DIRAC FIELD

This won't work!

<u>Guess</u> $$[\psi_a(\vec{x}),\ i\psi_b^\dagger(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

spinor components $(a, b = 1, 2, 3, 4)$

i.e. $$[\psi_a(\vec{x}),\ \psi_b^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

In matrix notation ...

$$[\psi_a(\vec{x}),\ \psi_a^\dagger(\vec{y})] = \mathbb{1}_{4\times4}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix} \qquad [\ -\ -\ -\ -\ ]$$

Also guess $$[\psi_a(\vec{x}),\ \psi_b(\vec{y})] = 0$$

$$= [\psi_a^\dagger(\vec{x}),\ \psi_b^\dagger(\vec{y})] = 0$$

Note that

$$[\psi(\vec{x}),\ \bar{\psi}(\vec{y})] = [\psi(x),\ \psi^\dagger(y)\gamma^0]$$

$$= [\psi(x),\ \psi^\dagger(y)]\gamma^0 = \gamma^0\delta^{(3)}(\vec{x}-\vec{y})$$

With these ... we recall that for bosons we wrote ...

(real field) $$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\vec{a}_p + \vec{a}^\dagger_{-p}\right\} e^{i\vec{p}\cdot\vec{x}}. \quad \text{(FT)}$$

↳ For <u>complex</u> field, we get

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\vec{a}_p + \vec{b}^\dagger_{-p}\right\} e^{+i\vec{p}\cdot\vec{x}}.$$

In the case of Dirac field, need spin degree of freedom...

Try ...

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{\,r} u^r(\vec{p}) + b_{-\vec{p}}^{\,r} v^r(-\vec{p}) \right\} e^{+i\vec{p}\cdot\vec{x}}$$

Spin degrees of freedom

Fourier components: $\psi(x) = u(p) e^{ip}$

$$\psi^\dagger(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{\,r\dagger} u^{r\dagger}(\vec{p}) + b_{-\vec{p}}^{\,r\dagger} v^{r\dagger}(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

Recall that u, v also solves Dirac eqn in the sense that (in momentum space...)

$$\boxed{p^\mu \gamma_\mu u^r(\vec{p}) = m u^r(p) \quad \& \quad p^\mu \gamma_\mu v^r(\vec{p}) = -m v^r(p)}$$

We can try the commutators ...

$$[a_{\vec{p}}^{\,r}, a_{\vec{p}'}^{\,s\dagger}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[b_{\vec{p}}^{\,r}, b_{\vec{p}'}^{\,s}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[a_{\vec{p}}^{\,r}, a_{\vec{p}'}^{\,s}] = 0$$

⋮

the rest are all zero...

We find that

as desired...

$$[\psi_a(\vec{x}), \psi_b(\vec{y})] = 0 = [\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})]$$

We also find that

$$\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian...

$$\mathcal{H} = \int d^3x \left[-i\bar{\psi}\,\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi\right]$$

$$= \int d^3x \left\{\psi^\dagger\gamma^0\left[-i\vec{\gamma}\cdot\vec{\nabla}+m\right]\psi\right\}$$

just coefs

Now, with $p^\mu\gamma_\mu u^r(p) = m u^r(p)$

$$\longrightarrow \quad (\vec{p}\cdot\vec{\gamma}+m)\,u^r(p) = p^0\gamma^0 u^r(p) = E_p\gamma^0 u^r(p)$$

Similarly, b/c $\quad p^\mu\gamma_\mu v^{(r)}(p) = -m v^{(r)}(p)$

$$\longrightarrow \quad (\vec{p}\cdot\vec{\gamma}+m)\,v^r(p) = -E_p\gamma^0 v^{(r)}(p).$$

So ....

$$\longrightarrow \quad \left[-i\vec{\gamma}\cdot\vec{\nabla}+m\right]\psi = \left[-i\vec{\gamma}\cdot\vec{\nabla}+m\right]\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_p^r u_p^r + b_{-p}^{r\dagger} v^r\right]e^{i\vec{p}\cdot\vec{x}}$$

$$= \gamma^0\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{E_p\, a_p^r u^r(\vec{p}) - E_{+p}\, b_{-p}^{r\dagger} v^r(\vec{p})\right\}e^{i\vec{p}\cdot\vec{x}}$$

So....

$$\mathcal{H} = \int d^3x\left\{\psi^\dagger\gamma^0\gamma^0\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\cdots\right\}e^{i\vec{p}\cdot\vec{x}}\right\}$$

plug in ...

$$\to \quad \mathcal{H} = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} E_p \left\{ a_p^{r\dagger} a_p^r - \underbrace{b_{+p}^r b_{+p}^{r\dagger}}_{=\, b_{+p}^{r\dagger} b_{+\vec{p}}^r + \cancel{\text{const}}} \right\}$$

$\boxed{!}$

$\to$ By creating more and more particles with $b^\dagger$, we can lower the energy indefinitely

$\to$ This is bad...

$\to$ So we should use Fermi-Dirac statistics instead $\to$ anti-commutators instead of commutators...

$\to$ Require that

$$\boxed{\{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = \{b_{+\vec{p}}^r, b_{+\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

no longer bosonic! , all other anti-commutators are zero...

When this is true, we find that

$$\boxed{\begin{aligned} &\{\Psi_a(\vec{x}), \Psi_b^\dagger(\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab} \\ &\{\Psi_a(\vec{x}), \Psi_b(\vec{y})\} = \{\Psi_a^\dagger(x), \Psi_b^\dagger(y)\} = 0 \end{aligned}}$$

where we're using

$$\Psi(x) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_{\vec p} \left(a_p^{r\dagger} a_{\vec p}^r - \underbrace{b_{-\vec p}^r b_{-\vec p}^{r\dagger}}_{-b_{-p}^{r\dagger} b_{-p}^r + \cancel{const}} \right)$$

$$\Rightarrow \boxed{\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ a_{\vec p}^{r\dagger} a_{\vec p}^r + b_{-\vec p}^{r\dagger} b_{-\vec p}^r \right\}}$$

now good, b/c E is bdd below...

→ also can compute...

$$\boxed{\vec P = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} \vec p \left(a_{\vec p}^{r\dagger} a_{\vec p}^r + b_{+\vec p}^{r\dagger} b_{+\vec p}^r \right)}$$

To avoid sign confusion, we will usually write...

$$\boxed{\psi(\vec x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_{\vec p}^r u^r(\vec p) e^{+i\vec p\cdot\vec x} + b_{\vec p}^{r\dagger} v^r(\vec p) e^{-i\vec p\cdot\vec x} \right)}$$

As a Heisenberg field,

$$\boxed{\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u^r(\vec p) e^{-i\vec p\cdot\vec x} + b_p^{r\dagger} v^r(\vec p) e^{i\vec p\cdot\vec x} \right)}$$

where:

$$\begin{cases} a_p^r & : \text{annihilates particles} \\ a_p^{r\dagger} & : \text{creates particles} \\ b_p^r & : \text{annihilates anti-particles} \\ b_p^{r\dagger} & : \text{creates anti-particles.} \end{cases}$$

Vacuum state as $|0\rangle$ where

$$\begin{cases} a_p^s |0\rangle = 0 \\ b_p^s |0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ covariant norm:

$$\boxed{|\vec{p}, s\rangle = \sqrt{2E_p}\; a_p^{s\dagger} |0\rangle}$$

so that

$$\boxed{\langle \vec{p}, s | \vec{q}, r\rangle = \sqrt{2E_p}\sqrt{2E_{\vec{q}}}\; \delta^{(3)}(\vec{p}-\vec{q})\, \delta^{rs}}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lortz}} \psi'(x) = \Lambda_{\frac{1}{2}}\, \psi(\Lambda_4^{-1} x)$$

recall that with $\quad \rightsquigarrow \exp\left\{ -i\, w_{\mu\nu} S^{\mu\nu}/2 \right\} \simeq 1 - \frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}$

$$\begin{cases} w_{12} = -w_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \end{cases} \qquad = 1 - \frac{i}{2}\vec{\theta}\cdot\begin{pmatrix}\vec{\sigma} & \\ & \end{pmatrix}$$

$\rightarrow$ and $\psi(\Lambda_4^{-1} x) \simeq \left[1 - \theta\cdot\vec{J}\right]\psi(x)$

$$\vec{J} = \vec{x} \times (-i\vec{\nabla})$$

So $\psi \to \psi + \delta\psi$ where

$$\delta\psi = \psi' - \psi = \left(-\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}\right)\psi(x) - \vec{\theta}\cdot(\vec{x}\times\vec{\nabla})\,\psi(x)$$

By Noether's thm,

$$\vec{J}_{total} \text{ (total spin)} = \int \frac{d^3x}{2} \left[\psi^\dagger (-i\vec{x} \times \vec{\nabla}) \psi + \frac{1}{2} \psi^\dagger \vec{\Sigma} \psi \right].$$

We won't worry abt the rest of this section about propagators

$\hookrightarrow$ we'll come back to them later when looking at Feynman diagrams.

DISCRETE SYMMETRIES OF THE DIRAC THEORY

Basically, we have

Parity —— Time Reversal

Charge conjugation

Recall that before, we looked at implementation of continuous Lorentz transform –

$\to$ found that $\forall \Lambda \in$ Lorentz group

$\exists\, \mathcal{U}(\Lambda)$ unitary for which

$$\mathcal{U}(\Lambda)\psi(x)\mathcal{U}^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1} \psi(\Lambda x).$$

$\to$ Now, we'll talk about Discrete symmetries on the Dirac field.

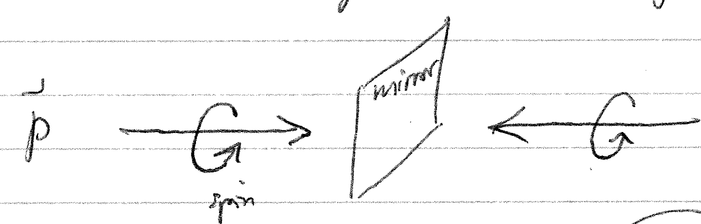
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant:

→ e.g. $\left\{ \begin{array}{c} \text{time-reversal} \\ \text{parity} \end{array} \right\}$.

Parity (P) : flips direction of spatial vectors

$$\boxed{P : (t, \vec{x}) \longrightarrow (t, -\vec{x})}.$$

⇝ mirror sym → change the handedness.



⇝ Note: momentum flips sign, but spin is unchanged.

Time-reversal

$$\boxed{T : (t, \vec{x}) \to (-t, \vec{x})}$$

P, T don't belong to the "proper" Lorentz group $L_+^\uparrow$

→ the full Lorentz group breaks into 4 disconnected subsets ...

↳

$$\begin{tikzcd} L_+^{\uparrow} \arrow[r, leftrightarrow, "P"] \arrow[d, leftrightarrow, "T"] & L_-^{\uparrow} = P L_+^{\uparrow} \arrow[d, leftrightarrow, "T"] \\ L_+^{\downarrow} = T P_+^{\uparrow} \arrow[r, leftrightarrow, "P"] & L_-^{\downarrow} = PT L_+^{\uparrow} \end{tikzcd}$$

"orthochronous"

"non-orthochronous"

"proper" "improper"

$\boxed{\text{charge conjugation}}$ → interchange particles & anti-particles.

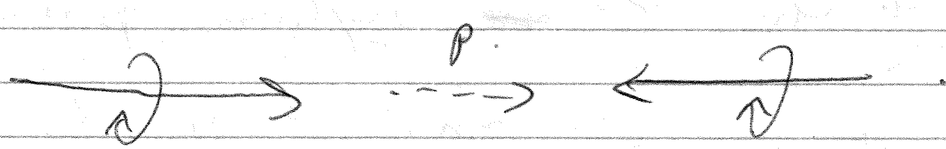
↳ non-space-time.

Let's look at $\boxed{\text{Parity}}$.

Note that because $P: (t, \vec{x}) \to (t, -\vec{x})$

→ momentum flips sign

<u>but</u> not spin! ⇒ what is P? As an operator?



As an operator on creation/annihilation ops, we want

$$\boxed{P^{\dagger} a_{\vec{p}}^{s} P = a_{-\vec{p}}^{s} \quad \& \quad P^{\dagger} b_{\vec{p}}^{s} P = b_{-\vec{p}}^{s}}$$

where, as discussed, P must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, get

$$\boxed{P^\dagger a^{s\dagger}_{\vec{p}} P = a^{s\dagger}_{-\vec{p}} \quad \& \quad P^\dagger b^{s\dagger}_{+\vec{p}} P = b^{s\dagger}_{-\vec{p}}}$$

But these might be too restrictive ... we can get better constraints by requiring that.

$$P^\dagger a^s_{\vec{p}} P = \eta_a a^s_{-\vec{p}} \quad \& \quad P^\dagger b^s_{\vec{p}} P = \eta_b b^s_{-\vec{p}}$$

as long as $|\eta_a|^2 = |\eta_b|^2 = \mathbb{1}$ are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel:

$$\left\{\begin{aligned} P^\dagger a^{s'\dagger}_{\vec{p}} a^s_{\vec{p}} P &= a^{s'\dagger}_{-\vec{p}} a^s_{-\vec{p}} \\ P^\dagger b^{s'\dagger}_{\vec{p}} b^{s'}_{\vec{p}} P &= b^{s\dagger}_{-\vec{p}} b^s_{-\vec{p}}. \end{aligned}\right\}$$

With this, let's ~~use~~ implement parity condition on $\Psi(x)$

$$\rightarrow P^\dagger \Psi P = ?$$

(to find out what these η_a, η_b must be ....

$$P^{\dagger}\psi(x)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\Big(\eta_a\, a^s_{-\vec{p}}\, u^s(p)\, e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^{*}\, b^{s\dagger}_{-\vec{p}}\, v^s(\vec{p})\, e^{i\tilde{p}\cdot\tilde{x}}\Big)$$

<u>Define</u>
$$\begin{cases}\tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}).\end{cases}$$

<u>Note that</u>
$$u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix} \quad \text{where } \sigma = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma} = (\mathbb{1}, -\vec{\sigma})$$

$$= \begin{pmatrix}\sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi^s \\ \sqrt{\tilde{p}\cdot\sigma}\,\xi^s\end{pmatrix}$$

$$= \begin{pmatrix}0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}\overbrace{\begin{pmatrix}\sqrt{\tilde{p}\cdot\sigma}\,\xi^s \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}}^{u^s(-\vec{p})} = \overbrace{\begin{pmatrix}0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}}^{\gamma^0} u^s(-\vec{p})$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\vec{p})}$$

and
$$v^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix} = \cdots = \begin{pmatrix}0 & -\mathbb{1} \\ -\mathbb{1} & 0\end{pmatrix} v^s(-\vec{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(p)}$$

With these, we find that ...

$\tilde{p}\cdot\vec{x} = p\cdot x$

$$P^\dagger \psi(x) P = \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(\eta_a a^s_{-p} u^s(-p) e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^* b^{s\dagger}_{-p} v^s(-p) e^{i\tilde{p}\cdot\tilde{x}} \Big)$$

now, notice that if $\eta_a = \eta_b^\dagger$ then it's "nice":

$$\boxed{(\eta_a = \eta_b^\dagger) \Rightarrow P\psi^\dagger(x)P = \eta_a \gamma^0 \psi(\tilde{x})}$$ → P-transf in final form

→ suffices to choose $\eta_a = 1 = -\eta_b$.

relative sign between fermions & antifermions ...

Now, useful to know how various Dirac field bilinears transform under parity ...

Recall ... 5 of them:

$$\bar\psi\psi, \quad \bar\psi\gamma^\mu\psi, \quad i\bar\psi[\gamma^\mu, \gamma^\nu]\psi$$

$$\bar\psi\gamma^\mu\gamma^5\psi, \quad i\bar\psi\gamma^5\psi.$$

to find these, first compute: $P\bar\psi(x)P$ ...

$$P^\dagger\bar\psi(x)P = P^\dagger\psi^\dagger(x)\gamma^0 P = (P^\dagger\psi P)^\dagger\gamma^0 \qquad (\gamma^{0\dagger} = \gamma^0)$$

$$= \eta_a^* (\gamma^0\psi(\tilde{x}))^\dagger\gamma^0 = \eta_a^* \psi^\dagger(\tilde{x})\gamma^0\gamma^0$$

$$\rightarrow \boxed{P^\dagger\bar\psi P = \eta_a^* \bar\psi(\tilde{x})\gamma^0}$$

With this---

$$P^\dagger \underset{(x)}{\bar\Psi}\underset{(x)}{\Psi} P = \underbrace{P^\dagger \bar\Psi(x) P}\overset{\mathbb{1}}{\widetilde{P^\dagger}} \Psi(x) P .$$

$$= \eta_a^* \bar\Psi(\tilde{x})\gamma^0 \eta_a \gamma^0 \Psi(\tilde{x})$$

$$= |\eta_a|^2 \bar\Psi(\tilde{x}) \Psi(\tilde{x})$$

Scalar $\boxed{P^\dagger \bar\Psi \Psi P(x) = \bar\Psi\Psi(\tilde{x})}$ (scalar) bilinear.

can also show...

$$\boxed{\begin{aligned} P^\dagger \underbrace{\bar\Psi(x)\gamma^\mu\Psi} P &= \bar\Psi \gamma^0\gamma^\mu\gamma^0 \Psi(\tilde{x}) \\ \text{(vector)} \quad &= \begin{cases} + \bar\Psi\gamma^\mu\Psi(\tilde{x}) & \mu = 0 \\ - \bar\Psi\gamma^\mu\Psi(\tilde{x}) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

$$P^\dagger (i\bar\Psi\gamma^5\Psi) P = i\bar\Psi(\tilde{x})\gamma^0\gamma^5\gamma^0\Psi(\tilde{x}) = -i\bar\Psi\gamma^5\Psi(\tilde{x})$$

pseudo scalar $\to$ (−)

$$\cancel{= \begin{cases} -\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde{x}) & \mu=0 \\ \bar\Psi\gamma^\mu\gamma^5\Psi(\tilde{x}) & \mu=1,2,3 \end{cases}}$$

$$P^\dagger \bar\Psi\gamma^\mu\gamma^5\Psi(x) P = \bar\Psi(\tilde{x})\gamma^0\gamma^\mu\gamma^5\gamma^0\Psi(\tilde{x})$$

pseudo vector.

$$= \begin{cases} -\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde{x}) & \mu = 0 \\ +\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde{x}) & \mu = 1,2,3 \end{cases}$$

(−)

Note the relative sign: $\boxed{-\eta_a = \eta_b^*}$ is important for the relationship between fermions & anti-fermions

Consider ~~and~~ fermion - antifermion state...

$$a_p^{s\dagger} b_q^{s\dagger}|0\rangle \xrightarrow{P} P\left(a_p^{s\dagger} b_q^{s\dagger}|0\rangle\right)$$

$$= P^\dagger\left(a_p^{s\dagger} b_q^{s\dagger}\right)P\,|0\rangle$$

$$= P^\dagger a_p^{s\dagger} P P^\dagger b_q^{s\dagger} P\,|0\rangle$$

$$= \eta_a\, a_{-p}^{s\dagger}\, \eta_b\, b_{-q}^{s\dagger}\,|0\rangle$$

$$= -(\eta_b \eta_b^*)\, a_{-p}^{s\dagger}\, b_{-q}^{s\dagger}\,|0\rangle$$

$$= -\,a_{-p}^{s\dagger}\, b_{-q}^{s\dagger}\,|0\rangle$$

→ a state containing a fermion-antifermion pair gets an extra (-1) under parity transform.

$\boxed{\text{TIME REVERSAL}}$

if T is unitary & $[T, H] = 0$

→ $T^\dagger e^{iHt} T = e^{iHt} T^\dagger T = e^{iHt}$

→ no good ...

What if $T^\dagger H T = -H$? or $\{T, H\} = 0$?

But this is no good either since implies that
H is unbounded ...

→ Assume this ...

"Time-reversal is conjugate-linear / anti-linear"

Assume:

T is unitary

$T^\dagger c T = c^*$ $\quad (c \in \mathbb{C})$.

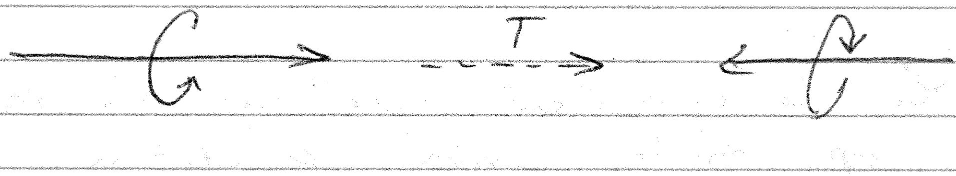
$[T, H] = 0$

With these ...

$$T^\dagger e^{-iHt} T = e^{iHt} \checkmark$$

→ time reversal: momentum & spin are reversed

↳ like watching a movie played backwards



Flipping momentum is easy.

What abt flipping spinor's? We know that

In some basis...

$$\xi(\uparrow) = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s = 1, 2$ & define

reversed spin

$$\boxed{\xi^{-s} = -i\sigma^2 (\xi^s)^*.}$$

σ^y

This is the flipped spinor...

It is clear that

$$\xi^{-s} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow)^*, \xi(\downarrow)^*)^*$$

$$= (\xi(\downarrow), -\xi(\uparrow))^*$$

where
$$\begin{cases} \xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{1*} \\ \xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{2*} \end{cases}$$

→ This is convenient since our time reversal op. involves complex conjugation...

→ Can show:
$$\boxed{u^{-s}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\,\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\,\xi^{s*} \end{pmatrix}}$$

So if we use the identity ...

$$\boxed{\sqrt{\tilde{p}\cdot\sigma}\,\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}}$$

(prove using $\sigma^2\vec{\sigma} = -\vec{\sigma}^*\sigma^2$

then we set

$$u^{-s}(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\,\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\,\xi^{s*} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2)\sqrt{\tilde{p}\cdot\sigma^*}\,\xi^{s*} \\ (-i\sigma^2)\sqrt{\tilde{p}\cdot\bar{\sigma}^*}\,\xi^{s*} \end{pmatrix}$$

$$= (-i)\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(p)]^* = -\gamma^1\gamma^3[u^s(p)]^*$$

$\sigma^2 = \sigma^y$

$$\Rightarrow \quad \boxed{u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^*}$$

element-wise complex conjugation

Similarly, ...

$$\boxed{v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^*}$$

in this relation, v^{-s} contains $\boxed{\xi^{-(-s)} = -\xi^s}$

a 360° flip introduces a $(-)$ sign.

~~[illegible]~~

Now we can define time reversal operation on the creation - annihilation operators...

spins don't matter here ...

$$\boxed{T^\dagger a^s_{\vec p} T = a^{-s}_{-\vec p} \quad \& \quad T^\dagger b^s_{\vec p} T = b^{-s}_{-\vec p}}$$

→ flip spin

→ flip momentum

where we now define

$$a^{-s}_{-\vec p} = (a^{\downarrow}_{-\vec p}, -a^{\uparrow}_{-\vec p})$$
$$b^{-s}_{-\vec p} = (b^{\downarrow}_{-\vec p}, -b^{\uparrow}_{-\vec p})$$

just like what we did with $\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow))$

$$\text{if} \quad a^s_{\vec p} = (a^{\uparrow}_{\vec p}, a^{\downarrow}_{\vec p})$$
$$b^s_{\vec p} = (b^{\uparrow}_{\vec p}, b^{\downarrow}_{\vec p})$$

analogous to what we did before...

With this, let's evaluate $T^\dagger \Psi(x) T$:

$$\rightarrow \quad T^\dagger \Psi(x) T = \int \frac{d^3p}{(2\pi)^3} \underbrace{\tfrac{1}{\sqrt{2E_p}}} \sum_{s=1,2} T^\dagger \left(a^s_{\vec p} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_p v^s(p) e^{+ip\cdot x} \right) T$$

$$= \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^{2} \left\{ a^{-s}_{-\vec p} [u^s(p)]^* e^{ip\cdot x} + b^{-s\dagger}_{-p} [v^s(p)]^* e^{-ip\cdot x} \right]$$

• where under T, $\quad = \gamma^1 \gamma^3 \Psi(x_T), \; x_T = (-t, \vec x)$

• $a^s_p \xrightarrow{T} a^{-p}_{-p}$

~~$u^s(p)$~~ $\cancel{u^{-s}(\tilde p) = -\gamma_1\gamma_3 [u^s(p)]^*}$

• $\rightarrow \cancel{\gamma_1\gamma_3 u^{-s}(\tilde p) = [u^s(p)]^*}$

• $T^\dagger e^{-ip\cdot x} T = e^{+ip\cdot x} \; ; \; T^\dagger u^s_p T = [u^s_p]^*$

note sign change here

Because $\{u^s(p)\}^* = \gamma_1\gamma_3 u^{-s}(\tilde{p})$, we have

$$T^\dagger \psi(x) T = \gamma^1\gamma^3 \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^{2} \left\{ a_{\tilde{p}}^{-s} u^{-s}(\tilde{p}) e^{i\tilde{p}(t,-\vec{x})} + b_{\tilde{p}}^{-s\dagger} v^{-s}(\tilde{p}) e^{-i\tilde{p}(t,-\vec{x})} \right\}$$

$$= \gamma^1\gamma^3 \psi(-t, x) \qquad = -\tilde{p}(-t, \vec{x})$$

$$\Rightarrow \boxed{T^\dagger \psi(x,t) T = \gamma^1\gamma^3 \psi(x,-t)}$$

Next, can check the action of T on bilinears...

$$T^\dagger \bar{\psi} T = T^\dagger \psi^\dagger \gamma^0 T = T^\dagger \psi^\dagger T \gamma^0 \quad \text{real}$$

$$= (\gamma^1\gamma^3 \psi(x_T))^\dagger \gamma^0 = \psi^\dagger(x_T) \gamma^{3\dagger} \gamma^{1\dagger} \gamma^0$$

$$-\gamma^3 \quad -\gamma^1$$

$$= \psi^\dagger(x_T) \gamma^3\gamma^1\gamma^0$$

$$= +\psi^\dagger(x_T)\gamma^0 \gamma^3\gamma^1$$

$$\Rightarrow \boxed{T^\dagger \bar{\psi} T = -\bar{\psi}(x_T)\gamma^1\gamma^3}$$

with this, can compute the rest....

Scalar $\boxed{T \bar{\psi}\psi(x) T = \bar{\psi}\underbrace{(-\gamma^1\gamma^3)(\gamma^1\gamma^3)}_{\mathbb{1}}\psi(x_T) = \bar{\psi}(x_T)\psi(x_T)}$

Pseudoscalar $\to$ get $(-)$ "pseudo"

$$\boxed{T^\dagger i\bar{\psi}\gamma^5\psi T = -i\bar{\psi}(-\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3)\psi(x_T) = -i\bar{\psi}(x_T)\gamma^5\psi(x_T)}$$

Vector

$$\boxed{\begin{aligned} T^\dagger \bar{\Psi}\gamma^\mu \Psi T\,(x) &= \bar{\Psi}(-\gamma^1\gamma^3)(\gamma^\mu)^*(\gamma^1\gamma^3)\Psi \\ &= \begin{cases} +\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu=0 \\ -\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu=1,2,3. \end{cases} \end{aligned}}$$

This makes sense... Recall that $\bar{\Psi}\gamma^0\Psi$ is the charge density

↳ $\bar{\Psi}\gamma^0\Psi$ should be the same under T -

as we saw: $T^\dagger\bar{\Psi}\gamma^0\Psi T = \bar{\Psi}\gamma^0\Psi$.

but current density (time-dep) must reverse si gn

→ $T^\dagger\bar{\Psi}\gamma^i\Psi T = -\bar{\Psi}\gamma^i\Psi$ ✓.

$\boxed{\text{Charge Conjugation - Matter-anti-matter flip}}$

↳ $\begin{cases} \text{anti particles + particles are swapped.} \\ \text{spin + momentum are the same.} \end{cases}$

Let $\left\{ \begin{aligned} C^\dagger a_p^s C &= b_p^s \\ C^\dagger b_p^s C &= a_p^s . \end{aligned} \right\}$ → ignore phases...

How should C act on $\Psi(x)$?

First, look at relation...

$$(v^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\,(-i\sigma^2)\xi^{s*} \\ \sqrt{p\cdot\bar\sigma}\,(-i\sigma^2)\xi^{s*} \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma^*}\,\xi^{s*} \\ i\sigma^2\sqrt{p\cdot\sigma^*}\,\xi^* \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} = \cancel{\ }$$

$\rightarrow$ ~~set~~

$$\boxed{\begin{aligned} u^s(p) &= -i\gamma^2(v^s(p))^* \\ v^s(p) &= -i\gamma^2(u^s(p))^* \end{aligned}}$$

$$\rightarrow \quad C^\dagger\psi(x)C = \int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sum_{s=1}^{2}\Big\{-i\gamma^2 b_p^s(v^s(p))^* e^{-ip\cdot x} - i\gamma^2 a_p^{s\dagger}(u^s(p))^* e^{ip\cdot x}\Big)$$

$$= -i\gamma^2\psi^*(x) = -i\gamma^2(\psi^\dagger)^T = -i(\bar\psi\gamma^0\gamma^2)^T$$

$$\Rightarrow \boxed{C^\dagger\psi(x)C = -i(\bar\psi\gamma^0\gamma^2)^T} \quad \rightarrow C \text{ is a unitary op.}$$

On bilinears... first, find $\quad \bar\psi^\dagger = (\psi^\dagger)^\dagger\gamma^0 = \psi\gamma^0$

$$\boxed{\begin{aligned} C^\dagger\bar\psi(x)C &= C^\dagger\psi^\dagger\gamma^0 C = \underbrace{C^\dagger\psi^\dagger C}\gamma^0 = -i\psi^T\gamma^2\gamma^0 \\ &= (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T \end{aligned}}$$

Next ....

$$C^\dagger\bar\psi\psi C = (-i(\gamma^0\gamma^2\psi)^T)(-i(\bar\psi\gamma^0\gamma^2)^T) = \cdots\cdots =$$

$$= -\left[(-i\bar\psi\gamma^0\gamma^2)(-i\gamma^0\gamma^2\psi)\right]^T = +\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi$$

$$= -+\bar\psi\gamma^0\gamma^2\gamma^2\gamma^0\psi = +\bar\psi\psi$$

So... $\boxed{C^\dagger \bar{\psi}\psi C = \bar{\psi}^\dagger \psi}$ $\longrightarrow$ scalar

vector $\boxed{C^\dagger i \bar{\psi}\gamma^5\psi C = i\,(-i\gamma^0\gamma^2\psi)^T\gamma^5\,(-i\bar{\psi}\gamma^0\gamma^2)^T = i\bar{\psi}\gamma^5\psi}$

pseudo-scalar $C^\dagger\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$

pseudo vector. $C^\dagger\bar{\psi}\gamma^\mu\gamma^5\psi C = +\bar{\psi}\gamma^\mu\gamma^5\psi$

(I'll skip the derivations... to save time)

<u>Summary</u>

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

notice that
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$ is invariant under C, P, T separately

$\rightarrow$ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

$\downarrow$

Invariance under CPT is required for any Lorentz invariant local Hermitian op.

$\boxed{\text{Problem 5}}$

$\hookrightarrow$ (to be continued...)

Correlation functions for Dirac fields

$$\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle \rightsquigarrow \text{Dirac propagation amplitudes...}$$

only "a" terms contributes — only "a^{+}" term contributes

Recall...

$$\rightarrow \psi_A(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left\{ a^s_{\vec{p}} u^s_A(p) e^{-ip\cdot x} + \cancel{b^{s\dagger}_{\vec{p}} v^s_A(p) e^{-ip\cdot x}} \right\}$$

$\langle 0 |$

$$\rightarrow \bar{\psi}_B(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left\{ \cancel{b^s_{\vec{p}} \bar{v}^s_B(p) e^{-ip\cdot x}} + a^{s\dagger}_{\vec{p}} \bar{u}^s_B(p) e^{ip\cdot x} \right\}$$

$| 0 \rangle$

where $\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}} \} = \{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}} \} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) \delta^{rs}$

$$\rightarrow \langle 0 | \psi_A(x) \bar{\psi}_B(x) | 0 \rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_s u^s_A(p) \bar{u}^s_B(p)}_{(\not{p}+m)_{AB}} e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{AB} \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

$$\boxed{\langle 0 | \psi_A(x) \bar{\psi}_B(x) | 0 \rangle = (i\not{\partial}_x + m)_{AB} \, D(x-y)}$$

- - -

$$\langle 0|\bar{\Psi}_B(y)\Psi_A(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s v_A^s(p)\bar{v}_B^s(p)}_{(\not{p}-m)_{AB}} e^{-ip(x-y)}$$

b terms contribute

$b^\dagger$ terms contribute

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_p}\right)(\not{p}-m)_{AB}\; e^{-ip(x-y)}$$

$$\boxed{\langle 0|\bar{\Psi}_B(y)\Psi_A(x)|0\rangle = -(i\not{\partial}_x + m)_{AB}\, D(y-x)}$$

Feynman Propagator

$$S_F^{AB}(x-y) = \begin{cases} \langle 0|\Psi_A(x)\,\bar{\Psi}_B(y)|0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\bar{\Psi}_B(y)\Psi_A(x)|0\rangle & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0|T\{\Psi_A(x), \bar{\Psi}_B(y)\}|0\rangle$$

--- time-ordering ---

where $T\{\Psi_A(x), \bar{\Psi}_B(y)\}$

$$= \theta(x^0-y^0)\,\Psi_A(x)\bar{\Psi}_B(y)$$

$$- \theta(y^0-x^0)\,\bar{\Psi}_B(y)\Psi_A(x)$$

$\boxed{\text{minus sign for Fermions}}$

Let's check the calculations...

$$\langle 0|\psi_A(x)\bar{\psi}_B(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_{p'}}}\langle 0|\left\{\sum_s a^s_{A\vec{p}}\, u^s_A(p)e^{-ip\cdot x} + \cancel{b^{s\dagger}_{A\vec{p}}\, v^s_A(p)e^{-ip\cdot x}}\right\}$$

$$\times\left\{\cancel{\sum_s b^s_{B\vec{p}'}\,\bar{v}^s_B(p')e^{-ip'\cdot y}} + a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{ip'\cdot y}\right\}|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\langle 0|\sum_s a^s_{A\vec{p}}\, u^s_A(p)\sum_s a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{-i(p\cdot x-p'\cdot y)}|0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\sum_s \underbrace{\langle 0|}_{1} u^s_A(p)\bar{u}^s_B(p')|0\rangle (2\pi)^3\delta^{(3)}(p-p')$$

$$\times e^{i(p\cdot x-p'\cdot y)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\bar{u}^s_B(p)}\, e^{-ip(x-y)}$$

$$(\not{p}+m)_{AB} = (\gamma^\mu p_\mu + m)_{AB} \quad \left(\frac{\text{spin sum}}{\text{relations}}\right)$$

$$= (i\not{\partial}_x+m)_{AB}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)} \;\checkmark$$

Similarly, we can get the other relation too...

Oct 5, 2020

(1) Recall Dirac bispinor field...

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a^r_{\vec{p}}\, u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}}\, v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Use $\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{p}'} \} = \{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{p}'} \} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$

and all other anti-comm $= 0$, derive the following:

$$\{ \psi_a(\vec{x}), \psi_b^\dagger(\vec{y}) \} = \delta_{ab}\, \delta^{(3)}(\vec{x}-\vec{y})$$

(2) The momentum operator is the Noether charge associated with spatial translation...

$$\vec{P} = -i \int d^3\vec{x}\, \psi^\dagger(x)\, \vec{\nabla} \psi(\vec{x})$$

Show that

$$\vec{P} = \sum_{r=1}^{2} \int \frac{d^3 p}{(2\pi)^3}\, \vec{p} \left(a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} + b^{r\dagger}_{\vec{p}} b^r_{\vec{p}} \right)$$

Oct 10, 2020

(1) Well...

$$\{ \psi_a(x), \psi_b^\dagger(y) \}$$

$$= \psi_a(x)\, \psi_b^\dagger(y) + \psi_b^\dagger(y)\, \psi_a(x)$$

$$\cancel{\sum_{r=1}^{2} \int \frac{d^3 p_a\, d^3 p_b}{(2\pi)^6\, 2\sqrt{E_{p_a} E_{p_b}^*}}\, e^{i(\vec{p}_a\cdot\vec{x} - \vec{p}_b\cdot\vec{y})} \left[a^r \right.}$$

To keep things clean...

$\psi_a(x)\,\psi_b^\dagger(y) + \psi_b^\dagger(x)\psi_a(x)$ which involves the factor...

$$\sum_{r=1}^{2}\sum_{s=1}^{2}\left[\hat{a}^r_{p_a} u^r(p_a) + \hat{b}^{r\dagger}_{-p_a} v^r(-p_a)\right]\left[(\hat{a}^s_{p_b} u^s(p_b))^\dagger + (\hat{b}^{s\dagger}_{-p_b} v^s(-p_b))\right]$$

$$+\sum_{r=1}^{2}\sum_{s=1}^{2}\left[(\hat{a}^s_{p_b} u^s(p_b))^\dagger + (\hat{b}^{s\dagger}_{-p_b} v^s(-p_b))^\dagger\right]\left[\hat{a}^r_{p_a} u^r(p_a) + \hat{b}^{r\dagger}_{-p_a} v^r(-p_a)\right]$$

$$= \sum_{r,s=1}^{2}\left\{\hat{a}^r_{p_a}, \hat{a}^{s\dagger}_{p_b}\right\} u^r(p_a)u^{s\dagger}(p_b) + \left\{\hat{b}^r_{p_a}, \hat{b}^{\dagger s}_{p_b}\right\} v^r(p_a)v^s(p_b)\Big\}$$

$$= \sum_{r,s=1}^{2}(2\pi)^3\delta^{r,s}\,\delta^{(3)}(p_a-p_b)\left\{u^r(p_a)u^{s\dagger}(p_b) + v^r(-p_a)v^{s\dagger}(-p_b)\right\}$$

$$\Rightarrow \left\{\psi_a(x), \psi_b^\dagger(y)\right\}$$
$$= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}(\vec{x}-\vec{y})}\sum_{r=1}^{2}\left\{u^r(p)u^{r\dagger}(p) + v^r(-p)v^{r\dagger}(-p)\right\}$$

Now, we want to convert $u^{s\dagger} \to \bar{u}$

$\to$ need γ^0. In particular, recall that $\gamma^0\gamma^0 = \mathbb{1}$

from page 63 $\Rightarrow$ and $\boxed{u^{s\dagger}(p)\gamma^0 = \bar{u}^s(p)}$, we have

$$\left\{\psi_a(x), \psi_b^\dagger(y)\right\} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}(\vec{x}-\vec{y})}\sum_{r=1}^{2}\left\{u^r_p \bar{u}^r_p\gamma^0 + v^r_{-\vec{p}}\bar{v}^r_{-\vec{p}}\gamma^0\right\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}(\vec{x}-\vec{y})}\sum_{r=1}^{2}\left\{u^r_p\bar{u}^r_p + v^r_{-\vec{p}}\bar{v}^r_{-\vec{p}}\right\}\gamma^0 \quad \left(\text{spin sum}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{i\vec{p}(\vec{x}-\vec{y})}\left(p_+\cdot\gamma + \cancel{m\mathbb{I}} + p_-\cdot\gamma - \cancel{m\mathbb{I}}\right)\gamma^0\Big|_{\delta_{a,b}}$$

$p_+\cdot\gamma \to E_p\gamma^0 - \cancel{\vec{p}\cdot\vec{\gamma}}$, $\quad p_-\cdot\gamma \to E_p\gamma^0 + \cancel{\vec{p}\cdot\vec{\gamma}}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \cancel{\left[2 p\cdot\gamma \right]\gamma^0} \rightarrow$$ recall that only $\vec{p} \to -\vec{p}$

Rather $$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left\{ E_p \cancel{\gamma^0} - \cancel{\vec{p}\cdot\vec{\gamma}} + E_p \cancel{\gamma^0} + \cancel{\vec{p}\cdot\vec{\gamma}} \right\} \cancel{\gamma^0}_{ab}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

So $$\boxed{\{ \psi_a(x),\ \psi_b^\dagger(y) \} = \delta_{ab}\,\delta^{(3)}(x-y)}$$

---

(2) Let

$$\vec{P} = -i \int d^3\vec{x}\ \psi^\dagger(\vec{x})\, \vec{\nabla}\psi(\vec{x}).$$

Show that

$p = (p^0, \vec{p})$

$$\vec{P} = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3}\, \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r \right).$$

$ip\cdot x = \ldots + i\vec{p}\cdot\vec{x}$

Well.

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{ip\cdot x} \sum_{r=1}^{2} \left\{ a_{\vec{p}}^r u_{\vec{p}}^r + b_{-\vec{p}}^{r\dagger} v_{-\vec{p}}^r \right\}.$$

So

$$\nabla\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (i\vec{p})$$

(e) $$\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_{\vec p}^s u_p^s e^{-ip\cdot x} + b_{\vec p}^{s\dagger} v_p^s e^{ip\cdot x} \right)$$

$$\rightarrow \nabla \Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (i\vec p) \left\{ a_{\vec p}^s u_p^s e^{-ip\cdot x} - b_{\vec p}^{s\dagger} v_p^s e^{ip\cdot x} \right\}$$

$$\Psi^\dagger(x) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{r=1}^{2} \left\{ b_q^r v^{\dagger r}(q) e^{-iqx} + a_q^{r\dagger} u_q^{\dagger r} e^{iqx} \right\}$$

$$\rightarrow \int d^3x (-i) \Psi^\dagger \nabla \Psi$$

$$= \int d^3x \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_q}} \underbrace{e^{ix(p-q)}}_{\delta^{(3)}(p-q)} \vec p \times \sum_s \times \sum_r$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \frac{1}{E_p} \vec p \left\{ \sum_{s,r=1}^{2} \left(a_{\vec p}^{r\dagger} a_{\vec p}^s \underbrace{u_p^{\dagger r} u_p^s}_{2E_p \delta^{rs}} - b_{\vec p}^r b_{\vec p}^{s\dagger} v_p^{\dagger r} v_p^s \right) \right\}$$

other uv terms = 0

$$= \int \frac{d^3p}{(2\pi)^3} \frac{2E_p}{2E_p} \vec p \left(a_p^{r\dagger} a_p^r - b_p^r b_p^{r\dagger} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec p \left(a_p^{r\dagger} a_p^r - b_p^r b_p^{r\dagger} \right)$$

Finally ... $\{ b_p^r, b_p^{r\dagger} \} = (2\pi)^3 \delta^{rr} (\vec p - \vec p)$ ignore ...

$$\Rightarrow -b_p^r b_p^{r\dagger} = b_p^{r\dagger} b_p^r - (2\pi)^3 \delta^{rr} \delta(\vec p - \vec p)$$

$$\Rightarrow \vec P = \int d^3x \, \Psi^\dagger(x) (-i\nabla) \Psi(x)$$

momentum op

$$\boxed{\vec P = \int \frac{d^3p}{(2\pi)^3} \vec p \left(a_p^{r\dagger} a_p^r + b_p^{r\dagger} b_p^r \right)}$$

More problems

(1) Let $\mathcal{U}$ be the following unitary op:

$$\mathcal{U} = \exp\left\{ \frac{-i\pi}{2} \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec p}^{r\,\dagger} - b_{\vec p}^{r\,\dagger} \right)\left(a_{\vec p}^{r} - b_{\vec p}^{r} \right) \right\}$$

investigate the effect of $\mathcal{U}$ on $a_{\vec p}^{r}$ & $b_{\vec p}^{r}$. I.e. compute

$$\mathcal{U}^\dagger a_{\vec p}^{r}\, \mathcal{U} \quad \& \quad \mathcal{U}^\dagger b_{\vec p}^{r}\, \mathcal{U} .$$

What type of transform does $\mathcal{U}$ produce?

Well...

$$\mathcal{U}^\dagger a_{\vec p}^{r}\, \mathcal{U} = \exp\Big\{ \underbrace{+\frac{i\pi}{2}\Big(\sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec p}^{r\,\dagger} - b_{\vec p}^{r\,\dagger} \right)\left(a_{\vec p}^{r} - b_{\vec p}^{r} \right)\Big)^\dagger}_{X^\dagger} \Big\}$$

$$\cdot\, a_{\vec p}^{r} \,\cdot\, \exp\Big\{ \underbrace{\frac{i\pi}{2} \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec p}^{r\,\dagger} - b_{\vec p}^{r\,\dagger} \right)\left(a_{\vec p}^{r} - b_{\vec p}^{r} \right)}_{X} \Big\}$$

$\Rightarrow$ no good way to do this except for power series,

Recall that $e^{x} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$, $e^{-x} = \sum_{n=0}^{\infty} \frac{1}{n!} (-x)^n$

So

$$\mathcal{U}^\dagger a_{\vec p}^{r}\, \mathcal{U} = \left(\sum_{n=0}^{\infty} \frac{(X^\dagger)^n}{n!} \right) a_{\vec p}^{r} \left(\sum_{m=0}^{\infty} \frac{(X)^m}{m!} \right) = \sum_{n,m=0}^{\infty} \frac{1}{n!\,m!} (X^\dagger)^n a_{\vec p}^{r} X^m$$

but notice that $\mathcal{U}$ unitary iff X hermitian.

$$\rightarrow X^\dagger = X \rightarrow \mathcal{U}^\dagger a_{\vec p}^{r}\, \mathcal{U} = \sum_{n,m=0}^{\infty} \frac{1}{n!\,m!} X^n a_{\vec p}^{r} X^m .$$

Useful theorem:

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots$$

& Identity relating comm & anti-comm:

$$[AB, C] = ABC - CAB$$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B,C\} - \{A,C\}B.$$

We ~~get~~ will need to compute $\cancel{X}$, $a_{\vec p}^{s}$

$$U^\dagger a_{\vec q}^{r} U = a_{\vec q}^{r} + [X, a_{\vec q}^{r}] + \frac{1}{2!}[X,[X,a_{\vec q}^{r}]] + \ldots$$

→ need to compute

$$[X, a_{\vec q}^{r}] = X a_{\vec q}^{r} - a_{\vec q}^{r} X = \ldots = \; ?$$

$$= \Big(+\frac{i\pi}{2}\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\{(a_{\vec p}^{s\dagger} - b_{\vec p}^{s\dagger})(a_{\vec p}^{s} - b_{\vec p}^{s})\, a_{\vec q}^{r}$$
$$- \{a_{\vec q}^{r}(a_{\vec p}^{s\dagger} - b_{\vec p}^{s\dagger})(a_{\vec p}^{s} - b_{\vec p}^{s})\}\Big)$$

$$= \frac{i\pi}{2}\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\Big[(a_{\vec p}^{s\dagger} - b_{\vec p}^{s\dagger})\underbrace{\{a_{\vec p}^{s} - b_{\vec p}^{s}, a_{\vec q}^{r}\}}_{\to 0}$$
$$- \{a_{\vec p}^{s\dagger} - b_{\vec p}^{s\dagger}, a_{\vec q}^{r}\}(a_{\vec p}^{s} - b_{\vec p}^{s})\Big]$$

$$= \frac{i\pi}{2}\sum_{s=1}^{2}\int \frac{d^3p}{(2\pi)^3}\Big[-(2\pi)^3\delta^{rs}\delta^{(3)}(\vec p - \vec q)(a_{\vec p}^{s} - b_{\vec p}^{s})\Big]$$

$$= -\frac{i\pi}{2}(a_{\vec q}^{r} - b_{\vec q}^{r})$$

Next term,

$$[X,[X,a^r_q]] = [X, \tfrac{-i\pi}{2}(a^r_q - b^r_q)]$$

$$= \tfrac{-i\pi}{2}[X, a^r_q] + \tfrac{i\pi}{2}[X, b^r_q] = \cdots$$

$$= 2\left(\tfrac{i\pi}{2}\right)^2 (a^r_q - b^r_q)$$

So... $e^X a^r_p e^{-X} = a^r_q - \frac{i\pi}{2}(a^r_q - b^r_q) + 2\left(\frac{i\pi}{2}\right)^2 (a^r_q - b^r_q)$

what's next? $+ \,.\,?$

each step $\rightarrow$ $+2\left(\frac{i\pi}{2}\right)$ & alt $(-)$ sign.

$$\rightarrow \quad \mathcal{U}^\dagger a^r_p \mathcal{U} = a^r_p - \frac{i\pi}{2}(a^r_p - b^r_p) + \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 (a^r_p - b^r_p),$$

$$- \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 (a^r_p - b^r_p) + \frac{1}{4!}\left(\frac{i\pi}{2}\right)^4 6\,(\ldots) + \cdots$$

$$= a^r_p \left\{1 - \frac{i\pi}{2} + \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 - \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 + \frac{6}{4}\left(\frac{i\pi}{2}\right)^4 + \cdots\right\}$$

check this!

$$+ \; b^r_p \left\{\frac{i\pi}{2} - \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 + \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 - \frac{6}{4}\left(\frac{i\pi}{2}\right)^4 + \cdots\right\}$$

$$= a^r_p \left\{1 - \frac{1}{2}\cdot 2\right\} + b^r_p \cdot \left\{\frac{1}{2}\cdot 2\right\}$$

$$= b^r_p \quad \rightarrow \quad \boxed{\mathcal{U}^\dagger a^r_p \mathcal{U} = b^r_p}$$

$\rightarrow$ <u>$\mathcal{U}$ corresponds to charge conjugation!</u>

2) using Dirac annihilation & creation ops, construct P for which

$$P^\dagger a^r_{\vec p} P = a^r_{-\vec p} \quad \& \quad P^\dagger b^r_{\vec p} P = b^r_{-\vec p}.$$

Last time, we find that

$$[X, a^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(a^r_{\vec p} - b^r_{\vec p}\right)$$ → target

→ we want X for which

$$\begin{cases} [X, a^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(a^r_{\vec p} - a^r_{-\vec p}\right). \\ [X, b^r_{\vec p}] = \cancel{-}\left(\frac{i\pi}{2}\right)\left(\cancel{a}b^r_{\vec p} + b^r_{-\vec p}\right). \end{cases}$$

→ Let

$$P = \exp\left\{\cancel{-}\frac{i\pi}{2}\sum_{r=1}^{2}\int\frac{d^3p}{(2\pi)^3}\left\{a^{r\dagger}_{\vec p}\left(a^r_{\vec p} - \cancel{a}a^r_{-\vec p}\right) + b^{r\dagger}_{\vec p}\left(b^r_{\vec p} + b^r_{-\vec p}\right)\right\}\right\}$$

check this, like last time

→ should work! :)

Obv a^r_p only int. w/ 1st term $a^\dagger(\,)a = 0$

$aa^\dagger(\,) = \delta^{\cdot\cdot}(-)$ ✓

Same with b^r_p ✓.

Interacting Fields & Feynman Diagrams

To get better description of the real world, need to include interactions in the theory.

To preserve causality, new terms may involve only products of fields at the same spacetime point!

↳ $\phi^4(x)$ ✓, but not $\phi(x)\phi(y)$.

→ $$H_{int} = \int d^3x \, \mathcal{H}_{int}[\phi(x)] = -\int d^3x \, \mathcal{L}_{int}[\phi(x)]$$

→ insist that H_{int} is a fn only of the fields, not of their derivatives.

→ Common ex in particle physics & CMT:

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4}$$

note: $\Pi(x)$ is still $\partial_0 \phi_0(x)$ since there are not any interaction terms involving $\partial_0 \phi$

λ: dimensionless "coupling constant".

→ In general, adding interactions produces divergences

However, no matter what the true physics looks like at high momenta or short distances, the low momentum / long distance physics is well-approximated by an "effective" FT.

with "renormalizable" interactions

→ those interactions where coupling constant are has dimensions $\boxed{d>0}$

$[\text{Mass}]^d$ where $d > 0$.

Ex $\quad -\frac{1}{2} m^2 \phi^2 \quad \& \quad \frac{\lambda}{4!}\phi^4$ same dim

→ $\lambda \sim [\text{mass}]^0$ → Renormalizable.

But $\quad -\frac{\lambda_6}{6!}\phi^6$ → not renormalizable.

since $\lambda \sim [\text{mass}]^{-2}$.

$\boxed{\text{Perturbation Expansion}}$

Let $H = H_0 + H_{int}$ $\rightsquigarrow$ $= \int d^3x \, \frac{\lambda}{4!}\phi^4(x)$

↑ KG, free

→ we will generate power series in λ.

At any t_0, we can write

$$\phi(t_0, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}\, e^{ip\cdot x} + a^\dagger_{\vec{p}}\, e^{-ip\cdot x} \right\}$$

$$\equiv \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \underset{\substack{|||\\ a_{\vec{p}}}}{\tilde{a}_{\vec{p}}}\, e^{i\vec{p}\cdot\vec{x}} + \tilde{a}^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right\}$$

where we've let $a_{\vec{p}}$ absorb $e^{\pm iEt_0}$

The Heisenberg field is then given by:

$$\rightarrow \boxed{\phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,x)\,e^{-iH(t-t_0)}}$$

If there's no interaction then we have

$$\rightarrow \boxed{\begin{aligned}\phi_{free}(t,\vec{x}) &= e^{iH_0(t-t_0)}\,\phi(t_0,x)\,e^{-iH_0(t-t_0)} \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right\}\end{aligned}} \Bigg|_{\substack{x^0 = t-t_0 \\ p^0 = E_{\vec{p}}}}$$

Define this to be $\phi_I(t,\vec{x})$, the interaction picture field

the interaction picture field $\equiv$ Heisenberg field when $\lambda = 0$.

Now, look at Heisenberg field...

$$\begin{aligned}\phi(t,\vec{x}) &= e^{iH(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH(t-t_0)} \\ &= \underbrace{e^{iH(t-t_0)}\,e^{-iH_0(t-t_0)}}\,\phi_I(t,\vec{x})\,e^{+iH_0(t-t_0)}\,e^{-iH(t-t_0)} \\ &= \mathcal{U}^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,\mathcal{U}(t,t_0)\end{aligned}$$

time evolution operator

$\rightarrow$ Evolve the operator on $\phi_I(t,\vec{x})$

OR

Evolve the states by $\mathcal{U}(t,t_0) \rightsquigarrow \mathcal{U}|\phi\rangle$ ...

→ now, we want to express $\mathcal{U}(t,t_0)$ entirely in ϕ_I

To do this, note that $\mathcal{U}(t,t_0)$ solves SE:

$$i\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int}\, e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$= H_I(t)\,\mathcal{U}(t,t_0)$$

where

$$H_I(t) = e^{iH_0(t-t_0)} \mathcal{H}_{int}\, e^{-iH_0(t-t_0)}$$

$$= \int d^3x\, \frac{\lambda}{4!}\phi_I^4 \quad \boxed{?}$$

$$= \int d^3x\, e^{iH_0(t-t_0)} \frac{\lambda}{4!}\phi^4\, e^{-iH_0(t-t_0)}$$

$$= \int d^3x\, \frac{\lambda}{4!}\phi_I^4 \quad \checkmark$$

→ this is the Hamiltonian in the interaction picture.

<u>So</u> since $\mathcal{U}$ solves the SE:

$$i\partial_t\,\mathcal{U}(t,t_0) = H_I(t)\,\mathcal{U}(t,t_0),$$

$\mathcal{U}$ must look like

$$\boxed{\mathcal{U}(t,t_0) \sim \exp\{-iH_I t\}}$$

More carefully, we actually have that

$$\boxed{\mathcal{U}(t,t_0) = T\exp\left\{-i\int_{t_0}^{t} dt'\, H_I(t')\right\}}$$

Dyson's formula

time-ordering symbol

Why T? Why ordering? $\Rightarrow$ B/c $H(t_1) \not\Leftrightarrow H(t_2)$ when $t_1 \neq t_2$.

"T" puts the <u>latest</u> operators on the left.

hence $i\partial_t\, \mathcal{U}(t,t_0) = \underline{H_I(t)}\, \mathcal{U}(t,t_0)$.

As a pwr series in λ:

$$\mathcal{U}(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt'\, H_I(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1\, dt_2\, T\{H_1(t_1) H_2(t_2)\}$$
$$+ \frac{(-i)^3}{3!}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} dt_1\, dt_2\, dt_3\, T\{H_1(t_1) H_2(t_2) H_3(t_3)\} + \cdots$$

Note "the time-ordering of the exponential is just ~~the time-ordering~~ the Taylor series of the terms time-ordered ..."

$\rightarrow$ Now, we want to generalize $\mathcal{U}(t,t_0)$ to $\mathcal{U}(t,t')$

reference time

This generalization is natural...

$$\boxed{\mathcal{U}(t,t') \equiv T\left\{\exp\left[-i\int_{t'}^{t} dt''\, \mathcal{H}_I(t'')\right]\right\} \quad (t \geq t')}$$

Then we see that b/c both t, t' are variables...
we find:

$$\boxed{\begin{aligned} i\partial_t\, \mathcal{U}(t,t') &= \mathcal{H}_I(t)\,\mathcal{U}(t,t') \\ i\partial_{t'}\, \mathcal{U}(t,t') &= -\,\mathcal{U}(t,t')\,\mathcal{H}_I(t'). \end{aligned}}$$

and thus...

$$\boxed{\mathcal{U}(t,t') = e^{iH_0(t-t_0^*)}\, e^{-iH(t-t')}\, e^{-iH_0(t-t_0^*)}}$$

So $\mathcal{U}$ is unitary.

Further, for $t_1 \geq t_2 \geq t_3$,

$$\mathcal{U}(t_1,t_2)\,\mathcal{U}(t_2,t_3) = \mathcal{U}(t_1,t_3)$$

$$\mathcal{U}(t_1,t_3)\left[\mathcal{U}(t_2,t_3)\right]^\dagger = \mathcal{U}(t_1,t_2)$$

Now, let $|0\rangle$ be gnd state of $\mathcal{H}_0$

$|\Omega\rangle$ be gnd state of $\mathcal{H}$

$|n\rangle$ be ~~gnd~~ label all $|E_n\rangle$ of $\mathcal{H}$.

Then, $(E_0 = \langle\Omega|H|\Omega\rangle)$

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Assume $H_0|0\rangle = 0$ & consider $T\to\infty$ limit.

$\updownarrow$

$T\to\infty(1-i\varepsilon)$

Then e^{-iE_nT} dies slowest for $n=0$, and so ...

$$\to e^{-iHT}|0\rangle \to e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$$

assume $\langle\Omega|0\rangle \neq 0$

So

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle}$$

Now, since T large, we can shift it by a small constant ...

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right\}^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right\}^{-1} \underbrace{e^{-iH(t_0-(-T))}}_{\mathcal{U}(t_0,-T)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right\}^{-1} e^{-iH(t_0-(-T))}\underbrace{e^{-iH_0(-T-t_0)}|0\rangle}_{=|0\rangle \text{ since } H_0|0\rangle=0}$$

$$\to |\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\mathcal{U}(t_0,-T)|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle}$$

Similarly,

$$\boxed{\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle}}$$

So, putting these together gives a correlation function...

For $x^0 > y^0 > t_0$, we have

$$\to \langle\Omega|\phi(x)\phi(y)|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle} \overbrace{(\mathcal{U}(x^0,t_0))^\dagger \phi_I(x)\mathcal{U}(x^0,t_0)}^{\phi(x)}$$

$$\times (\mathcal{U}(y^0,t_0))^\dagger \phi_I(y)\,\mathcal{U}(y^0,t_0)\times$$

$$\frac{\mathcal{U}(t_0,-T)}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,x^0)\phi_I(x)\,\mathcal{U}(x^0,y^0)\phi_I(y)\,\mathcal{U}(y^0,-T)|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}}$$

awkward...

so divide the whole thing by $1 = \langle\Omega|\Omega\rangle$

$$1 = \langle\Omega|\Omega\rangle = \frac{\langle 0|\overbrace{\mathcal{U}(T,t_0)\,\mathcal{U}(t_0,-T)}^{\mathcal{U}(T,-T)}|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}}$$

to get (for $x^0 > y^0$)

$$\boxed{\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,x^0)\phi_I(x)\,\mathcal{U}(x^0,y^0)\phi_I(y)\,\mathcal{U}(y^0,-T)|0\rangle}{\langle 0|\mathcal{U}(T,-T)|0\rangle}}$$

So, we have that, by replacing U's with Dyson's formula (w/ time-ordering) ...

$$\boxed{\langle \Omega | T\{\phi(x)\phi(y)\} |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\{\phi_I(x)\phi_I(y) \exp\{-i\int_{-T}^{T} dt\, H_I(t)\}\} |0\rangle}{\langle 0| T\exp\{-i\int_{-T}^{T} dt\, H_I(t)\}|0\rangle}}$$

So, looks like the term

$$\exp\left\{ -i\int_{-T}^{T} dt\, H_I(t) \right\}$$ ~~can~~ ~~[illegible]~~ can be found, so

Wick's theorem

→ So, we have reduced the problem of calc. correlation functions to evaluating

$$\boxed{\langle 0| T\{ \phi_I(x_1)\phi_I(x_2) \cdots \phi_I(x_n)\} |0\rangle}$$

→ This is the vacuum exp-value of time-ordered products of finite number of field operators.

$n = 2 \Rightarrow$ get Feynman operator.

$n > 2 \rightarrow$ can use brute force, but there are also ways to simplify calculations.

Now, we study $\boxed{\langle 0| T\{\phi_I(x)\phi_I(y)\}|0\rangle}$

Recall that

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$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}\, e^{-ip\cdot x} + a^\dagger_{\vec{p}}\, e^{+ip\cdot x} \right\}$$

Call $\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\vec{p}}\, e^{-ip\cdot x}$

and $\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a^\dagger_{\vec{p}}\, e^{+ip\cdot x}$

which is ~~useful~~ b/c

$$\phi_I^+(x)|0\rangle = 0, \qquad \langle 0|\phi_I^-(x) = 0.$$

only has annihilation ops — only has creation ops.

→ For $x^0 > y^0$, $x^0 > y^0$

$$T\{\phi_I(x)\phi_I(y)\} = \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y)$$

$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + [\phi_I^+(x), \phi_I^-(y)]$$

every of these terms has the form $a^\dagger_{\vec{p}} a^\dagger_{\vec{q}} a_{\vec{k}} a_{\vec{k}}$

i.e. creation ops always on the left.

→ "Normal order" → has vanishing vacuum exp value.

What can we say about the commutator?

↑ just a number, there is no creation/annihilation op. in it!

$$[\phi_I^+(x), \phi_I^-(y)] = \langle 0|[\phi_I^+(x), \phi_I^-(y)]|0\rangle$$

$$= \langle 0|\phi_I^+(x)\phi_I^-(y)|0\rangle = \langle 0|\phi_I(x)\phi_I(y)|0\rangle .$$

With these, we can write

$$T\{\phi_I(x)\phi_I(y)\} = \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^-(y) + \langle 0|\phi_I(x)\phi_I(y)|0\rangle$$

Now, <u>define</u> the "normal ordering symbol" N

s.t. N takes the string a, $a^\dagger$'s and rearranges them so that $a^\dagger$'s are on the left

Ex.

$$\begin{cases} N(a_{\vec p}^\dagger a_{\vec p}) = a_{\vec p}^\dagger a_{\vec p} \\ N(a_{\vec p} a_{\vec p}^\dagger) = a_{\vec p}^\dagger a_{\vec p} . \\ N(a_{\vec p} a_{\vec q} a_{\vec p}^\dagger) = a_{\vec p}^\dagger a_{\vec p} a_{\vec q} \end{cases}$$

ordering for $a_{\vec p}$, $a_{\vec q}$ doesn't matter since they commute.

→ <u>Note</u> N is not a well-defined mathematical operation

e.g. $N([a_{\vec p}, a_{\vec q}^\dagger]) \neq N((2\pi)^3\delta^{(3)}(\vec p - \vec q))$

→ it is only a lexicographic convention.

Now, let us consider general x^0, y^0, then

$$T\{\phi_I(x)\,\phi_I(y)\} = N\{\phi_I(x)\,\phi_I(y)\} + \begin{cases} [\phi_I^+(x),\ \phi_I^-(y)] & \text{for } x^0 > y^0 \\ [\phi_I^+(y),\ \phi_I^-(x)] & \text{for } x^0 < y^0 \end{cases}$$

→ Let ~~us~~ define the contraction of $\phi_I(x)$, $\phi_I(y)$ as

$$\overbrace{\phi_I(x)\,\phi_I(y)} = \begin{cases} [\phi_I^+(x),\ \phi_I^-(y)] & x^0 \geq y^0 \\ [\phi_I^+(y),\ \phi_I^-(x)] & x^0 < y^0 \end{cases}$$

then notice that, from our previous observation,

$$\overbrace{\phi_I(x)\,\phi_I(y)} = \begin{cases} \langle 0|\, \phi_I(x)\phi_I(y)\,|0\rangle & x^0 \geq y^0 \\ \langle 0|\, \phi_I(y)\phi_I(x)\,|0\rangle & y^0 > x^0. \end{cases}$$

So,

Feynman propagator

$$\overbrace{\phi_I(x)\phi_I(y)} = \langle 0|\, T\{\phi_I(x)\,\phi_I(y)\}\,|0\rangle$$

$$= D_F(x-y) \rightsquigarrow \text{Feynman propagator.}$$

$$= \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$

With these, we have that

→ leaving out the I's subscripts

$$T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y) + \overbracket{\phi(x)\phi(y)}\}$$

$$\overbracket{\phi(x)\phi(y)} = \overbracket{\phi(y)\phi(x)}$$

$$\to \quad T\{\phi(x)\phi(y)\} = N\{\phi(x)\phi(y)\} + \text{"contraction"}.$$

In fact, the generalization of this is called

<u>Wick's Theorem</u>

$$\boxed{\begin{aligned} &T\{\phi(x_1)\phi(x_2)\cdots\phi(x_m)\} \\ &= N\{\phi(x_1)\phi(x_2)\cdots\phi(x_m) + \text{all possible contractions}\} \end{aligned}}$$

<u>Ex</u> $T\{\phi_1\phi_2\phi_3\phi_4\}$ $\qquad (\phi_n = \phi(x_n))$

$$\begin{aligned} = N\{&\phi_1\phi_2\phi_3\phi_4 + \\ &+ \overbracket{\phi_1\phi_2}\phi_3\phi_4 + \phi_1\overbracket{\phi_2\phi_3}\phi_4 + \phi_1\phi_2\overbracket{\phi_3\phi_4} \\ &\overbracket{\phi_1\phi_2\phi_3\phi_4} + \overbracket{\phi_1\phi_2\phi_3}\phi_4 + \phi_1\overbracket{\phi_2\phi_3\phi_4} + \overbracket{\phi_1\phi_2}\,\overbracket{\phi_3\phi_4} \\ &+ \overbracket{\phi_1\phi_2\phi_3}\phi_4 \text{ with } \phi_1\underbracket{\phi_2\phi_3\phi_4} + \underbracket{\phi_1\overbracket{\phi_2\phi_3}\phi_4}\ \} \end{aligned}$$

what does $N\{\overbracket{\phi_1\phi_2}\phi_3\phi_4\}$ mean?

$$\begin{aligned} N\{\overbracket{\phi_1\phi_2}\phi_3\phi_4\} &= \overbracket{\phi_1\phi_2}\,N\{\phi_3\phi_4\} \\ &= D_F(x_1-x_2)\,N\{\phi_3\phi_4\}. \end{aligned}$$

<u>Proof</u> → prove by induction. $n=2$ is good (Feynman)

→ assume this holds for $n-1$.

Let $W(\phi_1 \dots \phi_n) = N\{\phi_1\phi_2 \dots \phi_n + \text{all possible contr.}\}$

To prove $W(\phi_1 \dots \phi_n) = T\{\phi_1\phi_2 \dots \phi_n\}$.

W/l/o/g: let $x_1^0 \geq x_2^0 \geq \dots \geq x_n^0$.

<u>Then</u> $T\{\phi_1 \dots \phi_n\} = \phi_1 T\{\phi_2 \dots \phi_n\}$

$$= \phi_1 W(\phi_2 \dots \phi_n) \quad \text{since } n-1 \text{ true.}$$

So $T\{\phi_1 \dots \phi_n\} = \underbrace{\phi_1^- W(\phi_2 \dots \phi_n) + W(\phi_2 \dots \phi_n)\phi_1^+}_{X}$

$$+ \underbrace{[\phi_1^+, W]}_{Y}$$

Let $X \equiv \phi_1^- W + W\phi_1^+$; $Y \equiv [\phi_1^+, W]$.

$X + Y$ are normal ordered: X contains all contractions in $W(\phi_1 \dots \phi_n)$ which does not contract ϕ_1 w/ anything.

Y contains all contractions in $W(\phi_1 \dots \phi_n)$ which contracts ϕ_1 with something.

So $T(\phi_1\phi_2 \dots \phi_n) = W(\phi_1 \dots \phi_n)$.

(we won't worry too much abt this proof.

→ the main idea is the theorem itself).

□

In any case, we have another way to explicitly write out the result of Wick's Thm:

$$T\{\phi_1 \phi_2 \cdots \phi_n\} = N\left\{ \exp\left[\frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{n} \overline{\phi_i \phi_j}\, \frac{\partial}{\partial \phi_i}\frac{\partial}{\partial \phi_j}\right] \phi_1 \cdots \phi_n \right\}$$

↑ we'll see why later on.

Feynman Diagrams

Wick's Thm allows ~~us~~ to write

$$\langle 0 | T\{\phi_1 \cdots \phi_n\} | 0 \rangle$$

in terms of sums and products of Feynman propagators.

→ Now, we will develop ~~a~~ diagrammatic expressions.

Recall that

$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \text{all possible contractions}\}$$

But the only contribution to

$\langle 0 | T\{\phi_1\phi_2\phi_3\phi_4\} | 0 \rangle$ is when all the ϕ's are contracted.

This b/c whenever things are in normal order, the exp value vanishes. → $N(\overline{\phi_1\phi_2}\phi_3\phi_4) = \overline{\phi_1\phi_2}\, N(\phi_3\phi_4$

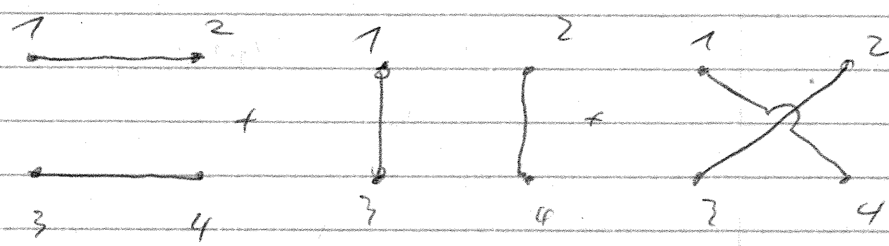
→ to "escape" from normal order, ϕ's have to be contracted

This means that

$$T\{\phi_1\phi_2\phi_3\phi_4\} = \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4$$

→ Can write this as Feynman diagrams, ..

$T\{\phi_1\phi_2\phi_3\phi_4\} =$



↳ Interpretation

Particles are created at 2 spacetime points, each propagates to one of the other points, then get annihilated.

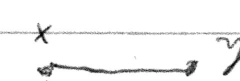
→ total amplitude of the process is the sum of the diagrams.

Well... what about something like...

$$\langle 0|\, T\left\{\phi(x)\,\phi(y)\,\exp\left\{-i\int_{-\infty}^{\infty} dt\, H_I(t)\right\}\right\} |0\rangle \;?$$

- Well... as a pwr series in λ, the lowest order term is

$$\langle 0|\, T\{\phi(x)\phi(y)\}|0\rangle = D_F(x-y).$$



- 1st order $\quad \langle 0|\, T\{\phi(x)\phi(y)\,(-i)\left(\int_{-\infty}^{\infty} dt\, H_I(t)\right)_{dt} |0\rangle$

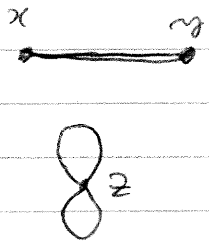
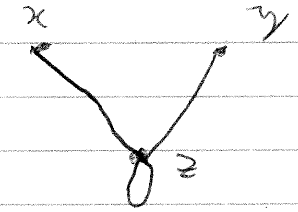
$$= \langle 0|\, T\{\phi(x)\phi(y)\,(-i)\int d^4z\, \tfrac{\lambda}{4!}\phi^4(z)\}\, |0\rangle$$

$$= \dots \frac{-i\lambda}{4!} \int d^4z \, \langle 0 | T \{ \phi(x) \phi(y) \overbrace{\phi(z) \phi(z) \phi(z) \phi(z)}^{\phi^4(z)} \} | 0 \rangle$$

$$= \frac{-i\lambda}{4!} \int d^4z \Big\{ \phi(x)\phi(y) \cdot \{ \phi(z)\phi(z)\phi(z)\phi(z) + \phi_z \phi_z \phi_z \phi_z + \phi_z \phi_z \phi_z \phi_z$$

$$+ \phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z) \Big\}$$

→ 12 terms that are identical

=  +  → propagators

γ $\qquad$ γ

$$\int d^4z \, D_F(x-z) D_F(z-y) \, D_F(z-z)$$

↑

12 of these.

$$\int d^4z \, D_F(x-y) D_F(z-z) D_F(z-z)$$

↳ each contraction D_F is a line.

each spacetime point is a dot.

→ But need to distinguish "external" and "internal" points.

↓ x, y $\qquad$ ↓ z

Each internal point is associated w/ a factor of $-i\lambda \int d^4z$, with combinatorial factor...

How do we count these combinatorial factors?

Oct, 19, 2020

→ well... Each H_I has 4 ϕ_I's : $\phi(z)\phi(z)\phi(z)\phi(z)$,

→ interchanging the contraction "ends" will give the same amplitude...

→ so for each H_I we expect a factor of 4!

→ cancels out the $\frac{1}{4!}$ in $\frac{\lambda}{4!}\phi^4$.

- In a diagram with more there are power of H_I

We can exchange all the contractions ends of one H_I with contraction ends of the other H_I.

→ Since we integrate over all z_1, z_2 ⇒ these give the same amplitude.

→ For a diagram with n "internal vertices",

(i.e. # of H_I's), we get a factor of $n!$ This cancels the $\frac{1}{n!}$ factor from Taylor series expansion.

of $\exp\{-i\int H_I(t)\,dt\}$.

→ ~~Chief there is a small subtlety... that~~ is

For example... consider the λ^2 term:

$w-y \quad x-z \quad z-z \quad z-w \quad w-u \quad w-u \quad u-u$

$$\langle 0|\phi(x)\phi(y)\,\frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,\phi\phi\phi\phi\int d^4w\,\phi\phi\phi\phi\int d^4u\,\phi\phi\phi\phi$$

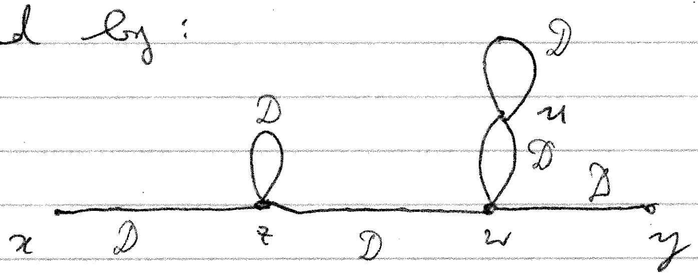
$$= \frac{1}{3!}\left(\frac{-i\lambda}{4!}\right)^3\int d^4z\,d^4w\,d^4u\,\mathcal{D}_F(x-z)\,\mathcal{D}_F(z-z)\,\mathcal{D}_F(z-w)\,\mathcal{D}_F(w-y)\,\mathcal{D}_F^2(w-u)\,\mathcal{D}_F(u-u)$$

The number of contractions that give this same expression is

$$3! \times (4.3) \times (4.3.2.1) \times (4.3) \times (1/2)$$

- $3!$: interchange vertices
- (4.3): placement of contractions into z vertex
- $(4.3.2.1)$: placement of contraction into w vertex
- (4.3): " ... for u vertex
- $(1/2)$: interchange of w-u interaction

Represented by:



Now... there is a subtlety that goes will all of this, and that's <u>symmetry factors</u>.

Symmetry Factors

→ Best to consider the simplest diagram with the most general problem.

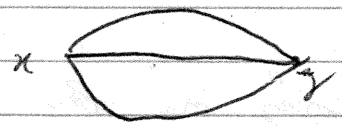
→ consider $\frac{\lambda}{3!}\phi^3$ theory ...

At 2^{nd} order in λ, $\langle 0 | T \left\{ \exp \int_{-\infty}^{\infty} \mathcal{H}_I(t)\, dt \right\} | 0 \rangle$

gives sth like

$$\int \langle 0 | \phi(x)\phi(x)\phi(x)\ \phi(y)\phi(y)\phi(y) | 0 \rangle \, d^4x\, d^4y$$

↳ Feynman diagram is



$$\mathcal{D}_F(x-y)\, \mathcal{D}_F(x-y)\, \mathcal{D}_F(x-y).$$

Naively we expect $2!$ from interchanging x-y.

and $3! \times 3! = 36$ from interchanging the ϕ's at x and ϕ's at y.

→ Expect 72. But are actually only 6:

$$\phi_x^1 \ \ \phi_y^1 \ \rightarrow \left\{ \begin{array}{l} \phi_x \phi_x \phi_x \ \ \phi_y \phi_y \phi_y \\ \phi_x \phi_x \phi_x \ \ \phi_y \phi_y \phi_y \end{array} \right.$$

Similarly for $\phi_x^1 - \phi_y^2$ $(\times 2)$

$\phi_x^1 - \phi_y^3$ $(\times 2)$

total = 6.

→ we have <u>overshot by 12</u>, because

$$\cancel{\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y = \phi_x\phi_x\phi_x\phi_y\phi_y\phi_y}$$

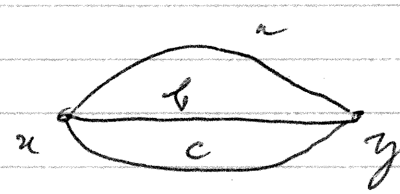
~~also interchanging $x \leftrightarrow y$~~

(1) Simultaneously swap 1st & 2nd x
& 1st & 2nd y

$$\phi_x\phi_x\phi_x\phi_y\phi_y\phi_y = \phi_x\phi_x\phi_x\phi_y\phi_y\phi_y$$

(2) exchanging all x's & y's doesn't do anything either...

We can see what's going on by labelling the vertices & propagators ...



$a \leftrightarrow b$, $\updownarrow$ c $\nearrow$ don't change the diagram

$x \leftrightarrow y$ also doesn't change the diagram

⇒ diagram has a permutation symmetry

This has $2! \times 3! = 12$ <u>elements</u>; which is the factor we overshot with...

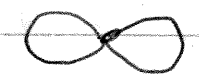
This is called the symmetry factor S.

The number of diagrams is then

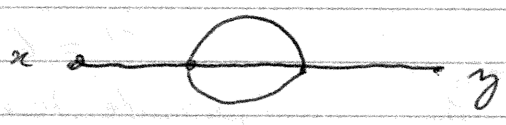
$$\frac{1}{s}(3!)(3!)(2!) = 6.$$

Some examples

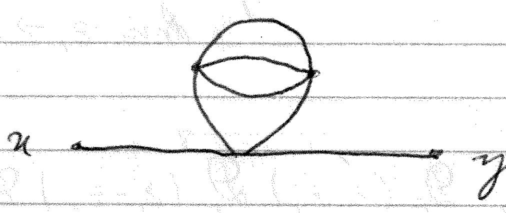
$s = 2$ $(x \leftrightarrow y)$.



$s = 2 \cdot 2 \cdot 2 = 8$



$s = 3! = 6$



$s = 3! \cdot 2 = 12$

Now, we are ready to state the Feynman rules for position space...

Recall that this rule lets us find

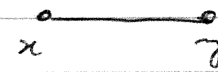
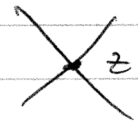
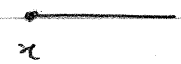
$$\langle 0| T\{\phi(x)\phi(y) \exp\{-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\,dt\}\} |0\rangle$$

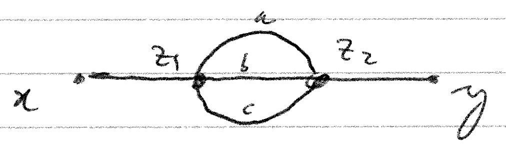
$$= \left(\text{sum of all possible diagrams with two external points}\right)$$

where each diagram is built out of
- propagators
- vertices
- external pts

...

Feynman Rules for ϕ^4 theory

1. For each propagator  $= \mathcal{D}_F(x-y)$
2. For each vertex  $= (-i\lambda)\int d^4z$
3. For each external point  $= 1$
4. Divide by symmetry factor. no $\frac{1}{4!}$ here

Ex 

$S = 3! = 6$
(permute the 3 propagators from $z_1 \to z_2$)

Amplitude:

$$\left(\frac{-i\lambda}{1}\right)^2 \cdot \frac{1}{6} \cdot \int d^4z_1\, d^4z_2\, \mathcal{D}_F(x-z_1)\, \mathcal{D}_F^3(z_1-z_2)\, \mathcal{D}_F(z_2-y)$$

Interpretation

- Each of the vertex factor $(-i\lambda)$ is the amplitude for the emission and/or absorption of particles at a vertex.

- The integral $\int d^4z$ instructs us to sum over all points where this process can occur.

$\rightarrow$ this is the principle of superposition!

- $\int d^4z$ is addition of amplitudes

$\hookrightarrow$ Feynman rules tell us to multiply the amplitudes for each independent part of the process.

Now, in most calculations, it is simpler to express the Feynman rules in terms of momentum

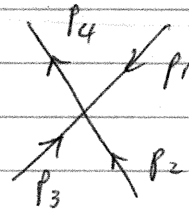
→ We want <u>momentum-space Feynman diagrams</u>

To do this, we write $D_F(x-y)$ in Fourier space

$$\boxed{D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} e^{-ip\cdot(x-y)}}$$

Now, we present this in a diagram by assigning a 4-momentum p to each propagator...

When 4 lines meet at a vertex, we get



$$\longrightarrow \int d^4z \, e^{-ip_1 z} e^{-ip_2 z} e^{-ip_3 z} e^{+ip_4 z}$$

$$= (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)$$

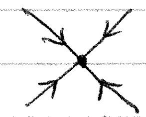
i.e. <u>momentum is conserved</u> at each vertex.

The delta functions can be used to perform integrals for the propagators...

→ From here we get momentum space Feynman rules.

Momentum-space Feynman rules

① Each propagator $\xrightarrow[p]{} \; = \dfrac{i}{p^2 - m^2 + i\varepsilon}$

② Each vertex  $= -i\lambda$

③ Each external point $x \; \cdot \!\xleftarrow[p]{} \; = e^{-ip\cdot x}$

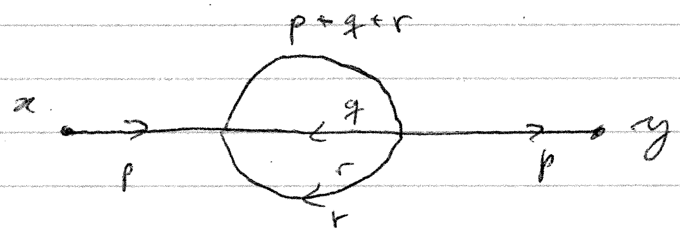
④ Impose momentum conservation at each vertex

⑤ Integrate over each undetermined momentum

$$\int \frac{d^4 p}{(2\pi)^4}$$

⑥ Divide by symmetry factor.

Ex



$$= \left(\frac{-i\lambda}{\cancel{4!}}\right)^2 \frac{1}{6} \cdot \int \left\{ \left(\frac{i}{p^2 - m^2 + i\varepsilon}\right)^2 \frac{i}{(q+p+r)^2 - m^2 + i\varepsilon} \; \frac{i}{q^2 - m^2 + i\varepsilon} \right.$$

$$\left. \cdot \frac{i}{r^2 - m^2 + i\varepsilon} \right\}, \; \frac{d^4 q}{(2\pi)^4} \; \frac{d^4 r}{(2\pi)^4} \, .$$

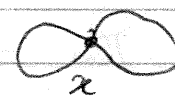
There is one subtlety, however...

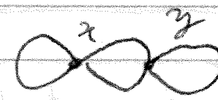
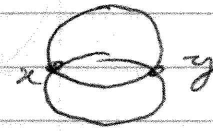
Consider diagrams without external vertices.

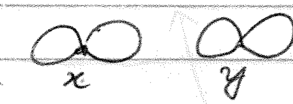
↳ these are diagrams from the form:

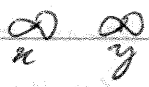
$$\langle 0 | \, T \left\{ \exp \left(-i \int_{-\infty}^{\infty} H_I(t) \, dt \right) \right\} | 0 \rangle$$

→ these are called "vacuum diagrams".

At order λ we have... 

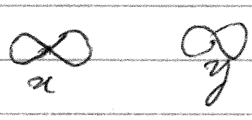
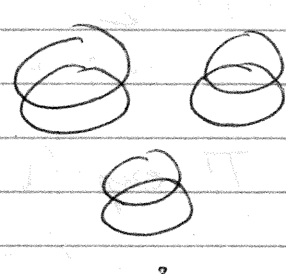
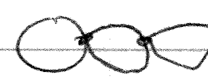
λ^2  and 

and () → disconnected diagram.

⟹ there is $S = 2$ for ()

In general, a vacuum diagram has connected subdiagrams V_i which appears n_i times

<u>Ex</u> V_1 V_2 V_3

\# copies ⟶ $n_1 = 2$ $n_2 = 3$ $n_3 = 1$

The amplitude for the total diagram is the product:

$$\boxed{\prod_i \left(\frac{1}{n_i!} V_i^{n_i} \right)}$$

Now, ~~so~~, the sum over all ~~vacuum~~ diagrams can be written as

$$\boxed{\sum_{\substack{\text{all possible}\\ \text{connected}\\ \text{pieces}}} \sum_{\substack{\text{all}\\ \{n_i\}}} \left(\begin{array}{c}\text{value of}\\ \text{connected}\\ \text{piece}\end{array}\right) \times \left\{ \prod_i \frac{1}{n_i!} (V_i)^{n_i} \right\}}$$

The sum of the connected pieces factors out, giving

$$\boxed{= \left(\sum \text{connected}\right) \times \sum_{\substack{\text{all}\\ \{n_i\}}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)}$$

sum all the values of the connected pieces

now...

$$\sum_{\{n_i\}} \left(\prod_i \frac{1}{n_i!} (V_i)^{n_i} \right)$$

$$= \prod_i \sum_{\{n_i\}} \frac{1}{n_i!} (V_i)^{n_i}$$

$$= \prod_i \exp(V_i)$$

$$= \exp\left(\sum_i V_i\right)$$

$$\boxed{\sum \text{all diagrams} = \sum \text{connected} \times \exp\left(\sum \text{disconnected}\right)}$$

Now, recall that the sum of all <u>vacuum</u> diagrams is just going to be

$$\exp\left(\sum_i V_i\right).$$

$$\Rightarrow \langle 0| T\left\{\exp\left(-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\,dt\right)\right\}|0\rangle = \exp\left(\sum_i V_i\right)$$

and as we have argued...

$$\langle 0| T\left\{\phi_I(x)\phi(y)\exp\left\{-i\int_{-\infty}^{\infty} \mathcal{H}_I(t)\,dt\right\}\right\}|0\rangle$$

$$= \left(\sum \text{connected}\right) \times \exp\left(\sum_i V_i\right).$$

$\Rightarrow$ And so we have that

$$\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim \frac{\langle 0| T\{(\phi(x)\phi(y))\{\exp\int iH(t)dt\}|0\rangle}{\langle 0| T(\exp(-i\int H(t)dt))}$$

$= \sum{}'$ all connected diagrams with 2-external pts.

More generally...

$$\boxed{\frac{\langle 0| T\left\{\phi(x_1)\dots\phi(x_n)\exp\left(-i\int_{-\infty}^{\infty}\mathcal{H}_I(t)\,dt\right)\right\}|0\rangle}{\langle 0| T\left\{\exp\left(-i\int_{-\infty}^{\infty}\mathcal{H}_I(t)\,dt\right)\right\}|0\rangle} = \sum \text{connected diagrams with endpoints } x_1 \dots x_n}$$

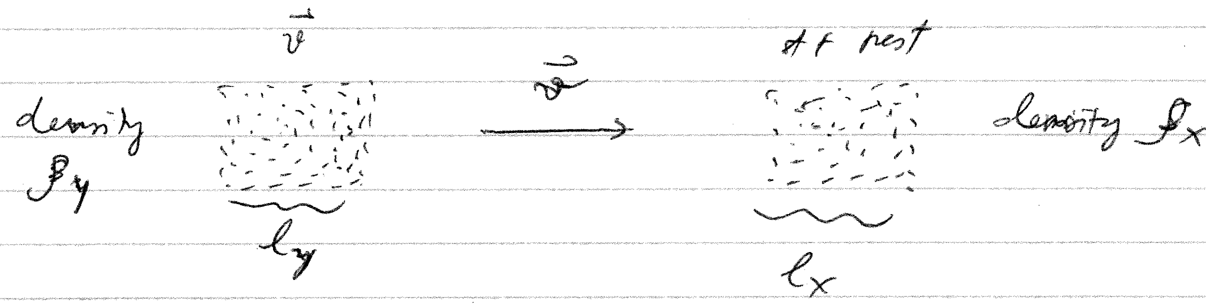
Cross section & the S-matrix

Now that we have a formula for computing the n-point correlation function...

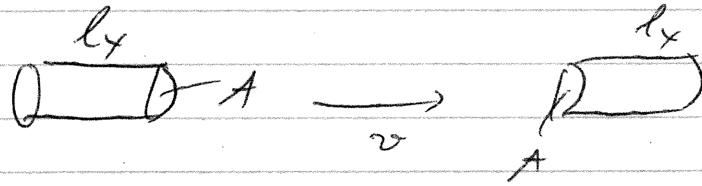
→ Next task is to compute quantities that can be measured
↳ cross section & decay rates.

The cross section:

Consider collision of 2 beams of particles with relatively well-defined momenta.



{ l_y, l_x are observed from rest.
{ ρ_y, ρ_x are densities at rest. Let A be the cross sectional area of overlap...



Total # of particles... $N_x = \rho_x l_x A$

$N_y = \rho_y l_y A$

→ Total # of scatterings is proportional to $N_x N_y$..

Let total number of scatterings be

$$N_X \cdot N_Y \cdot \left(\frac{\sigma}{A}\right)$$

probability one particular X particle & Y collide.

Call σ the effective area or "cross section" of the scattering process.

Let $N_X = 1$, then

$$\boxed{\sigma = \frac{\text{total \# scatterings}}{\rho_Y \cdot l_Y}}$$

For small time interval...

$$\sigma = \frac{\text{total \# scatterings} / \Delta T}{\rho_Y \cdot (l_Y / \Delta T)}$$

scattering rate

particle flux.

The differential cross section is the portion of σ in which the final particle momenta lie inside some window of momenta.

↳ write this as

$$\frac{d\sigma}{d^3\vec{p}_1 \cdots d^3\vec{p}_n} \text{, so that}$$

final particle momenta.

$$\int \frac{d\sigma}{d^3\vec{p}_1 \cdots d^3\vec{p}_n} \, d^3\vec{p}_1 \cdots d^3\vec{p}_n = \int d\sigma = \sigma .$$

Now, if there are only 2 final particles then there are only two free parameters ...

<u>Why?</u> two spatial momenta → 6 params

4-momentum conservation → 4 ~~params~~ constraints

⇒ can take these two params to be orient. angles θ & ϕ

→ Then we can measure $\boxed{\frac{d\sigma}{d\Omega}(\theta, \phi)}$

where $d\Omega$ is the solid angle differential

$$\boxed{d\Omega = d\cos\theta \, d\phi}$$

"Differential cross section" refers to $\frac{d\sigma}{d\Omega}$

Let's look at an example ... Consider a periodic box with length L on all sides.

Spatial momentum modes are now discrete:

$$\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z) . \qquad n_i \in \mathbb{Z} .$$

these canon. relation:

$$\boxed{[a_{\vec{k}}, a^{\dagger}_{\vec{k}'}] = \delta_{\vec{k},\vec{k}'} V} \longrightarrow V = L^3 \text{ volume}$$

as $V \to \infty$

$$\delta_{\vec{k},\vec{k}'} V = \int_0^L\int_0^L\int_0^L dx_1\, dx_2\, dx_3\; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$\to \int_0^\infty\int_0^\infty\int_0^\infty dx_1\, dx_2\, dx_3\; e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$= (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}').$$

In this box ...

$$\phi(x) = \sum_{\vec{k}} \frac{(2\pi/L)^3}{(2\pi)^3\sqrt{2E_{\vec{k}}}}\left(a_{\vec{k}} e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} e^{+ik\cdot x}\right)$$

$$= \sum_{k} \frac{1}{\sqrt{2E_k}}\left(a_{\vec{k}} e^{-ik\cdot x} + a^{\dagger}_{\vec{k}} e^{+ik\cdot x}\right)$$

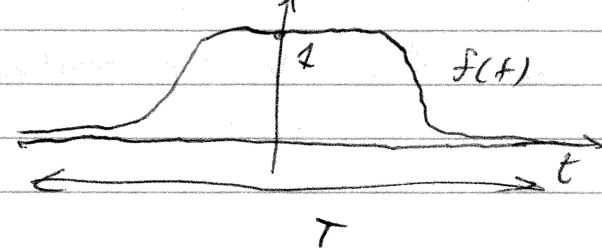
Oct 25, 2020

S-Matrix

Now imagine starting with free field theory...

at some early time then switching on the interaction slowly and then slowly switching off the interactions...

i.e. $H_I(t) \longrightarrow H_I(t)\, f(t)$ where $f(t)$ looks like



such that

Such that $\int_{-\infty}^{\infty} f(t)\,dt = T$, $\int_{-\infty}^{\infty} (f(t))^2\,dt = T$

Let $S = T \exp\left\{ -i \int_{-\infty}^{\infty} dt \; H_I(t) f(t) \right\}$

Define the S matrix as

$$\langle \text{final} | S | \text{initial} \rangle$$

where $|\text{initial}\rangle$ is a free particle state with momenta $\vec{k}_i^{\,I}$ & energies $E_i^{\,I}$

and $|\text{final}\rangle$ is a free particle state with momenta $\vec{k}_f^{\,F}$ + energies $\vec{E}_i^{\,F}$

Now look at $S - I$:

$$\langle \text{final} | S - I | \text{initial} \rangle$$

- as $T \to \infty$, $V \to \infty$ we can write this amplitude as

$$\langle \text{final} | S - I | \text{initial} \rangle = i \cdot \mathcal{M} \cdot (2\pi)^4 \delta(E_{tot}^F - E_{tot}^I)\, \delta^{(3)}(\vec{k}_{tot}^F - \vec{k}_{tot}^I)$$

fn of the momenta

- For finite T and finite V we instead have

$\langle \text{final} | S - I | \text{initial} \rangle$

$$= i\mathcal{M} \int_{-\infty}^{\infty} f(t)\, e^{i(E_{tot}^F - E_{tot}^I)t} \cdot \delta_{\vec{k}_{tot}^F, \vec{k}_{tot}^I} \cdot V$$

So,

$$|\langle \text{final} | S - I | \text{initial} \rangle|^2 = |\mathcal{M}|^2 \cdot \delta_{\vec{k}_{tot}^F, \vec{k}_{tot}^I} \cdot V^2$$

$$\cdot \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) f(t')\, e^{i(E_{tot}^F - E_{tot}^I)(t'-t)}\, dt'\, dt}$$

as $T \to \infty$, this is some constant times $\delta(E_{tot}^F - E_{tot}^I)$

What is this constant?

To get this $\to$ integrate w.r.t. E_{tot}^F..

$$\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underbrace{\left\{ \int_{-\infty}^{\infty} dE_{tot}^F\, e^{i(E_{tot}^F - E_{tot}^I)(t'-t)} \right\}}_{2\pi\delta(t-t')} f(t) f(t')\, dt\, dt'$$

$$= 2\pi \int_{-\infty}^{\infty} f^2(t)\, dt = 2\pi \cdot T.$$

So the constant is $2\pi \cdot T$ and so

$$|\langle \text{final} | S - I | \text{initial} \rangle|^2 = |\mathcal{M}|^2 (2\pi)\, \delta(E_{tot}^F - E_{tot}^I)\, \delta_{\vec{k}_{tot}^F, \vec{k}_{tot}^I}\, V^2 T$$

Now, we have been using relativistic normalizations

$$\left\{ \begin{array}{l} \langle \text{initial} \mid \text{initial} \rangle = \prod_i \left(2E_i^I \cdot V \right) \quad \text{becomes } (2\pi)^3 \delta^{(3)}(0) \text{ as } V \to \infty \\ \\ \langle \text{final} \mid \text{initial} \rangle = \prod_i \left(2E_i^F \cdot V \right) \end{array} \right.$$

To get transition probability per unit time ...

$$\frac{\text{probability}}{\text{time}} = \frac{1}{T} \frac{\left| \langle \text{final} \mid S - I \mid \text{initial} \rangle \right|^2}{\langle \text{final} \mid \text{final} \rangle \langle \text{initial} \mid \text{initial} \rangle}$$

$$= \frac{|\mathcal{M}|^2 (2\pi) \delta\left(E^F_{tot} - E^I_{tot} \right) \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V^2}{\prod_i \left(2E_i^F \cdot V \right) \prod_i \left(2E_i^I \cdot V \right)}$$

As $V \to \infty$, $\delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V \Rightarrow (2\pi)^3 \delta^{(3)}\left(\vec{k}^F_{tot} - \vec{k}^I_{tot} \right)$

If we sum over final states in some window then we have

$$\underbrace{\sum_{\vec{k}^F_1, \dots \vec{k}^F_{n_F}} \frac{1}{(2E_1^F \cdot V)} \cdots \frac{1}{(2E^F_{n_F} \cdot V)}} \frac{|\mathcal{M}|^2 (2\pi)^4 \delta\left(k^F_{tot} - k^F_{tot} \right) \cdot V}{(2E^I_1 \cdot V) \cdots (2E^I_{n_I} \cdot V)}$$

As $V \to \infty$, $\dfrac{d^3 \vec{k}_1^F}{(2\pi)^3 2E_1^F} \cdots \dfrac{d^3 k^F_{n_F}}{(2\pi)^3 2E^F_{n_F}}$

$$\left\{ \begin{array}{l} n_I = \#\ \text{initial particles} \\ n_F = \#\ \text{final particles} \end{array} \right.$$

Consider single particle decay... ($n_I = 1$)

The total decay rate is $\Gamma = \int d\Gamma$ where

$$d\Gamma = \frac{1}{2E^I}\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3 2E_i^F}\right)|M|^2 (2\pi)^4 \delta^{(4)}(k_{tot}^F - k_{tot}^I)$$

For a 2-particle initial state, the cross section is given by

$$\sigma = \frac{\text{probability}}{\text{time} \cdot \text{flux density}}$$

flux density = relative velocity between beam and target × density of incoming beam in lab frame.

We have normalized probability for one incoming beam particle → density = $1/V$, and flux...

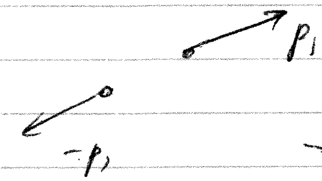
$$\text{flux} = \frac{|\vec{V}_A - \vec{V}_B|}{V}$$

$\vec{V}_A, \vec{V}_B$ = velocity of particles in lab frame.

So

$$d\sigma = \underbrace{\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3 2E_i^F}\right)(2\pi)^4 \delta^{(4)}(k_{tot}^F - k_{tot}^I)}_{\text{call this } d\Pi_{n_F}} \frac{|M|^2}{2E_A E_B |\vec{V}_A - \vec{V}_B|}$$

Now consider special case: 2 final particles ($n_f = 2$) in com frame



then $n_f = 2$ $\qquad p_1 = |\vec{p}_1|$

$$\rightarrow \int d\Pi_2 = \int \frac{dp_1\, p_1^2\, d\Omega}{(2\pi)^3\, 2E_1\, 2E_2}\,(2\pi)\,\delta(E_{com} - E_1 - E_2)$$

where final particle energies are $\left\{ \begin{array}{l} E_1 = \sqrt{|\vec{p}_1|^2 + m_1^2} \\ E_2 = \sqrt{|\vec{p}_2|^2 + m_2^2} \end{array} \right.$

So

$$= \cancel{\int d\Pi_2 = \int d\Omega \frac{p_1^2}{16\pi^2 E_1 E_2}}$$

$$\rightarrow = \int d\Pi_2 = \int \frac{d\Omega}{16\pi^2} \int_0^\infty \frac{p_1^2\, \delta\left(-E_{com} + \sqrt{|\vec{p}_1|^2 + m_1^2} + \sqrt{|\vec{p}_2|^2 + m_2^2}\right)}{\sqrt{|\vec{p}_1|^2 + m_1^2} \cdot \sqrt{|\vec{p}_2|^2 + m_2^2}}\, dp_1$$

Recall that $\delta(f(x)) = \dfrac{\delta(x - x_0)}{|f'(x_0)|}$

$$\frac{dE_1}{dp_1} = \frac{d\sqrt{|\vec{p}_1|^2 + m_1^2}}{dp_1} = \frac{1}{2}\, \frac{2p_1}{\sqrt{|\vec{p}_1|^2 + m_1^2}} = \frac{p_1}{E_1}$$

Likewise $\dfrac{dE_2}{dp_2} = \text{- - -} = \dfrac{p_2}{E_2}$.

So $$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^2}\, \frac{p_1^2}{E_1 E_2 \left(\frac{p_1}{E_1} + \frac{p_2}{E_2}\right)} \Bigg|_{p_1 \text{ s.t. } E_{com} = E_1 + E_2}$$

$$= \int \frac{d\Omega}{16\pi^2}\, \frac{p_1}{E_1 + E_2} = \int \frac{d\Omega}{16\pi^2}\, \frac{p_1}{E_{com}}$$

So for 2 particles $\longrightarrow$ 2 particles ...

$$\left(\frac{d\sigma}{d\Omega}\right)_{com} = \frac{|\vec{p}^{\,final}| \cdot |\mathcal{M}|^2}{2E_A E_B |\vec{v}_A - \vec{v}_B| \cdot 16\pi^2 E_{com}} \qquad (E_{com} = E_A + E_B)$$

Now, it is conventional to define the T-matrix as

$$\boxed{S = 1 + iT}$$

Claim $\langle \vec{p}_1^{\,F}, \ldots, \vec{p}_{n_F}^{\,F} | iT | \vec{p}_A, \vec{p}_B \rangle$

$$= \left({}_{final}\langle \vec{p}_1^{\,F}, \ldots \vec{p}_{n_F}^{\,F} | T\exp\left[i \int_{-\infty}^{\infty} dt\, H_I(t) \right] | \vec{p}_A, \vec{p}_B \rangle_{free} \right)_{\star}$$

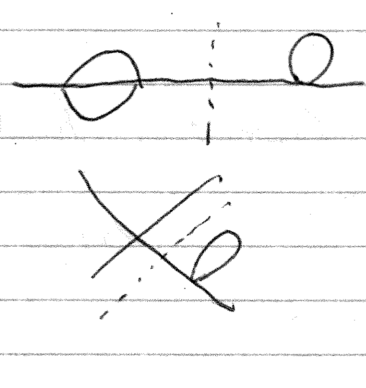
where $\star$ = connected diagrams only + "amputated" diagrams only

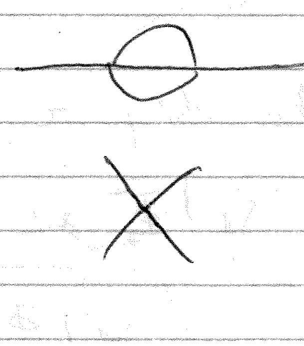
"amputated" $\equiv$ diagrams can't be broken into disconnected pieces by cutting one internal line

(i.e. 1 particle irreducible)

Ex

(not amputated) (amputated)





The claim won't be proven now, but the idea is similar as before...

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

and we would like something similar...

$$|\vec{p}_1 \ldots \vec{p}_{n_F}\rangle \propto \lim_{T\to\infty(1-i\varepsilon)} e^{-iHT}|\vec{p}_1,\ldots,\vec{p}_n\rangle_{\text{free}}$$

For now, we'll just take this claim as true...

→ Note that

$$\phi_I^+(x)|\vec{p}\rangle_{\text{free}} = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_k}}\, a_{\vec{k}}\, e^{-ik\cdot x} \underbrace{\sqrt{2E_p}}_{\substack{\text{relativistic}\\ \text{normalization}}}\, a_{\vec{p}}^\dagger|0\rangle$$

$$= e^{-ip\cdot x}|0\rangle$$

We can think of taking the commutator of $\phi_I^+(x)$ with the $a_p^\dagger$ from $|p\rangle_{\text{free}}$

→ suggests the notation

$$\boxed{\phi_I^+(x)\, |\vec{p}\rangle_{\text{free}} = e^{-ip\cdot x}}$$

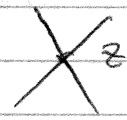
Now drop the "free" subscript & finally...

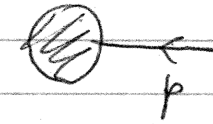
$\langle\vec{p}|\Phi_I^-(x) = e^{+ip\cdot x}\langle 0|$ & define

$$\boxed{\langle\vec{p}|\Phi(x) = e^{ip\cdot x}}$$

→ Set Feynman Rules in position space with external lines

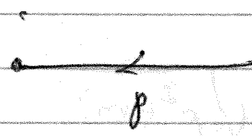
Propagator :  $D_F(x-y)$

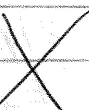
Internal vertex :  $(-i\lambda)\int d^4z$

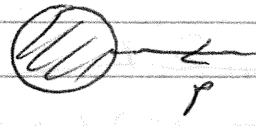
Each external line  $e^{-ip\cdot x}$

Divide by symmetry factor S.

Feynman Rules in momentum space with external line

Propagator :  $\dfrac{i}{p^2-m^2+i\varepsilon}$

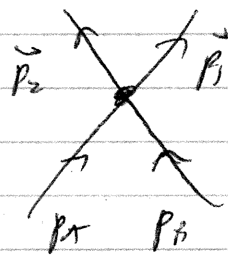
Internal vertex :  $-i\lambda$ & momentum conservation

External line  no extra factor (just 1)

Integrate over all unconstrained momenta & divide by S.

Ex $\langle \vec{p}_1, \vec{p}_2 | iT | \vec{p}_A, \vec{p}_B \rangle$ at lowest order...

well...



Feynman amplitude:

$$i\mathcal{M} = -i\lambda$$

So

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\vec{p}^{\,final}|\,|\mathcal{M}|^2}{2E_A E_B\,|\vec{v}_A - \vec{v}_B|\,16\pi^2 E_{cm}}$$

Let $p = |\vec{p}^{\,final}| = |\vec{p}_A| = |\vec{p}_B|$ all same since masses are all same

$$E_{cm} = 2E_A = 2\sqrt{p^2+m^2}$$

$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2p}{E_A}.$$

So

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^2 p}{\frac{2p}{E_A}(2E_A)(2E_B)16\pi^2 E_{cm}} = \frac{\lambda^2}{64\pi^2 E_{cm}^2}$$

This is spherically symmetric, so

$$\sigma_{tot} = (4\pi)\frac{\lambda^2}{64\pi^2 E_{cm}^2}\cdot\frac{1}{2}$$

$\longrightarrow$ particles in final state are identical so need a 1/2 factor...

$$\rightarrow \boxed{\sigma_{tot} = \frac{\lambda^2}{32\pi^2 E_{cm}^2}}$$

$\rightarrow$ <u>our first QFT cross section!</u>

Feynman Rules for Fermions

Oct 26
2020

So far we've discussed only the ϕ^4 theory...

→ need to generalize results to theories containing fermions.

→ need to generalize defn. of time-ordering & normal ordering symbols to include fermions...

Recall...

$$T\{\psi_a(x)\overline{\psi_b(y)}\} = \begin{cases} \psi_a(x)\overline{\psi_b(y)} & x^0 > y^0 \\ -\overline{\psi_b(y)}\psi_a(x) & x^0 < y^0 \end{cases}$$

The Feynman propagator, under this defn is

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(\not{p}+m)}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}$$

$$= \langle 0| T\{\psi_a(x)\overline{\psi_b(y)}\} |0\rangle$$

(Recall that $\not{p} = \gamma^\mu p_\mu$

Generalize of T to more than two fermion fields...

$$T\{\psi_1\psi_2\psi_3\psi_4\} = \begin{cases} \psi_1\psi_2\psi_3\psi_4 & \text{if } x_1^0 > x_2^0 > x_3^0 > x_4^0 \\ -\psi_2\psi_1\psi_3\psi_4 & \text{if } x_2^0 > x_1^0 > x_3^0 > x_4^0 \\ -\psi_3\psi_2\psi_1\psi_4 & \text{if } x_3^0 > x_2^0 > x_1^0 > x_4^0 \\ \vdots & \end{cases}$$

Rule: $\times(-1)$ if odd # permutations.

$\times(+1)$ if even # permutations.

Similarly, for normal ordering symbol...

$$N\{a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3} a^\dagger_{\vec{p}_4}\} = (-1)^3\, a^\dagger_{\vec{p}_4} a_{\vec{p}_1} a_{\vec{p}_2} a_{\vec{p}_3}$$

$\times(-1)$ if odd permutation of fields

$\times(+1)$ if even

With these, can generalize to get Wick's theorem...

1st case: 2 Dirac fields $T\{\psi_a(x)\bar{\psi}_b(y)\}$

$$T\{\psi_a(x)\bar{\psi}_b(y)\} = N\{\psi_a(x)\bar{\psi}_b(y)\} + \overbrace{\psi_a(x)\bar{\psi}_b(y)}$$

where $\overbrace{\psi_a(x)\bar{\psi}_b(y)} \equiv \begin{cases} \{\psi_a^+(x), \bar{\psi}_b^-(y)\} & x^0 > y^0 \\ -\{\bar{\psi}_b^-(y), \psi_a^+(x)\} & x^0 < y^0 \end{cases}$

$$= \langle 0| T\{\psi_a(x)\bar{\psi}_b(y)\}|0\rangle$$

$$= S_F(x-y)$$

$$= -\overbrace{\bar{\psi}_b(y)\psi_a(x)}$$

where recall that

$\psi^+, \bar{\psi}^+$ are the positive frequency part of $\psi, \bar{\psi}$

→ i.e. part with annihilation operators.

$\psi^-, \bar{\psi}^-$ ... " negative " creation operators.

Also ~~the~~ note that:

$$\boxed{\psi_a(x)\,\psi_b(y) = \bar{\psi}_a(x)\,\bar{\psi}_b(y) = 0}$$

→ Just as we proved Wick's thm for bosons, we can show the same for fermions

$$T\{\psi_1 \bar{\psi}_2 \psi_3 \cdots\} = N\left[\psi_1 \bar{\psi}_2 \psi_3 \cdots + \text{all possible contractions}\right]$$

where we note that an expression such as

! $$N[\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4] = -\psi_1 \bar{\psi}_3\, N[\psi_2 \bar{\psi}_4]$$

gets a minus sign since the $\bar{\psi}_3$ must leap over the ψ_2

<u>Helpful hint</u> for any <u>fully</u> contracted quantity, count the number of times the contraction lines criss-cross tells you if the # of perm. is odd/even...

<u>Ex</u> $\psi_1 \psi_2 \psi_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6$ → even

$\psi_1 \psi_2 \psi_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6$ → odd.

Yukawa theory

↳ Next we consider the simplest theory with fermions.

$$\boxed{H_{Yukawa} = H_{Dirac} + H_{Klein-Gordon} + \int d^3x \, g \bar{\Psi}\Psi\phi}$$

↑ simplified model of QED. We will carefully work out the rules of calculations for Yukawa theory before going to QED.

We will consider two-particle scattering reaction:

fermion (p) + fermion (k) ⟶ fermion (p') + fermion (k')

The leading contribution comes from the H_I^2 term of the S-matrix:

$$_0\langle p', k'| \, T\left\{ \frac{1}{2!}(-ig)\int d^4x \, \bar{\Psi}_I \Psi_I \phi_I \, (-ig)\int d^4y \, \bar{\Psi}_I \Psi_I \phi_I \right| p, k\rangle_0$$

Now use Wick's thm to reduce this to N-product of contractions → then act on uncontracted fields

Represent this as the contraction:

$$\Psi_I(x)|\vec{p}, s\rangle = \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}'}}} \sum_{s'} a^{s'}_{\vec{p}'} u^{s'}(p') e^{-ip'\cdot x} \underbrace{\sqrt{2E_{\vec{p}}}\, a^{s\dagger}_{\vec{p}} |0\rangle}_{|p,s\rangle}$$

$$= \underline{e^{-ip\cdot x} u^s(p)|0\rangle}$$

↑ fermion state with momentum $\vec{p}$, spin s

Define:

So $\boxed{\psi_I(x)\,|\vec{p}, s\rangle = e^{-ip\cdot x}\, u^s(p)}$

Similarly,

$$\bar{\psi}_I(x)\,|\vec{k}, s\rangle = e^{-ik\cdot x}\, \bar{v}^s(k)$$

$$\langle \vec{p}, s|\, \bar{\psi}_I(x) = e^{+ip\cdot x}\, \bar{u}^s(p)$$

$$\langle \vec{k}, s|\, \psi_I(x) = e^{+ik\cdot x}\, v^s(k)$$

So, typically, a contribution to the matrix element is the contraction ...

$$\langle \vec{p}', \vec{k}'|\, \frac{1}{2!}(-ig)\int d^4x\, \bar{\psi}\psi\phi\,(-ig)\int d^4y\, \bar{\psi}\psi\phi\, |\vec{p}, \vec{k}\rangle$$

Up to a (−) sign, the value of this quantity is

$$J = (-ig)^2 \int \frac{d^4q}{(2\pi)^4}\, \frac{i}{q^2 - m_\phi^2}\, (2\pi)^4 \delta^{(4)}(p' - p + q)$$

$$\times\, (2\pi)^4 \delta^{(4)}(k' - k - q)\, \bar{u}(p')\, u(p)\, \bar{u}(k')\, u(k)$$

$\phi - \phi$

$\vec{p}' \leftrightarrow \bar{\psi}$ $\quad \psi\text{-}p$ $\quad \vec{k}' - \bar{\psi}$ $\quad \psi - k$

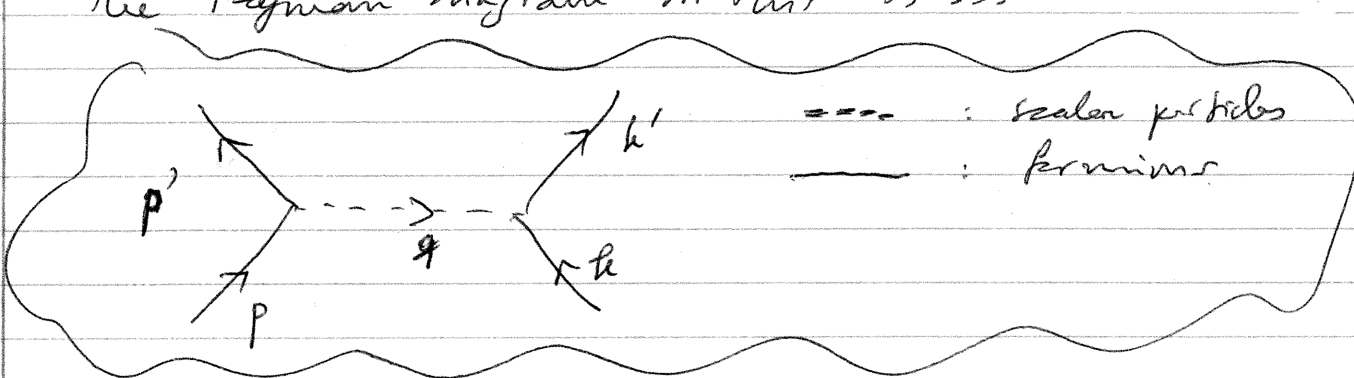
int. over all intermediate momenta.

where $q = p - p' = k' - k$.

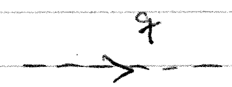
→ Upon using the δ functions, $J = iM(2\pi)^4 \delta^{(4)}(\Sigma p)$

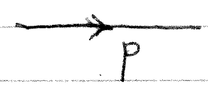
where $iM = \dfrac{-ig^2}{q^2 - m_\phi^2}\, \bar{u}(p')\, u(p)\, \bar{u}(k')\, u(k)$

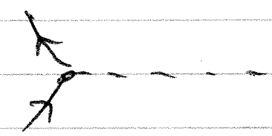
The Feynman diagram for this is ...

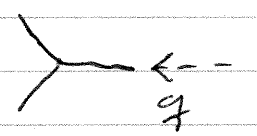


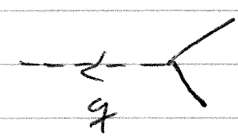
Feynman rules for fermions in momentum space

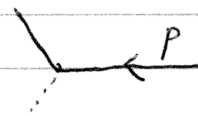
① Propagators: $\overbrace{\phi(x)\phi(y)}$ =  $= \dfrac{i}{q^2 - m_\phi^2 + i\varepsilon}$

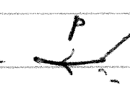
$\overbrace{\Psi(x)\bar{\Psi}(y)}$ =  $= \dfrac{i(\not{p} + m)}{p^2 - m^2 + i\varepsilon}$

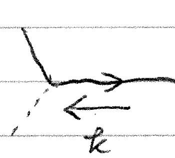
② Vertices  $= (-ig)$

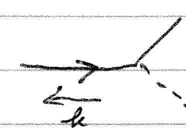
③ External leg contractions: ⓐ $\overbrace{\phi|q\rangle}$ =  $= 1$

ⓑ $\overbrace{\langle q|\phi}$ =  $= 1$

ⓒ $\overbrace{\Psi|\vec{p},s\rangle}$ =  $= u^s(p)$ (fermions)

ⓓ $\overbrace{\langle\vec{p},s|\bar{\Psi}}$ =  $= \bar{u}^s(p)$

ⓔ $\overbrace{\bar{\Psi}|\vec{k},s\rangle}$ =  $= \bar{v}^s(k)$ (antifermion)

ⓕ $\overbrace{\langle k,s|\Psi}$ = 

④ Mom. consv @ each vertex

⑤ Integrate over intermediate mom.!

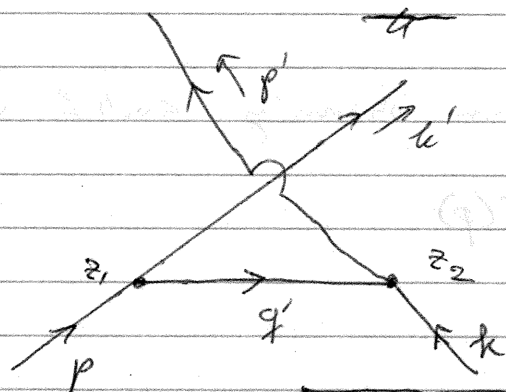
⑥ Figure out sign of diagram

Note: initial states have momentum pointing <u>in</u>
final ______ <u>out</u>

external particle / antiparticle $\Rightarrow$ same direction / opposite direction

Back to example: there are 2 criss-cross, so $(+1)$.

$\boxed{Ex}$



$$M = \int d^4 z_1\, d^4 z_2 \, \langle k', p' |\, \bar\psi_1 \psi_1 \phi_1 \, \bar\psi_2 \psi_2 \phi_2 \,| \vec p, \vec k \rangle$$

criss-cross $= 3 \Rightarrow (-1)$

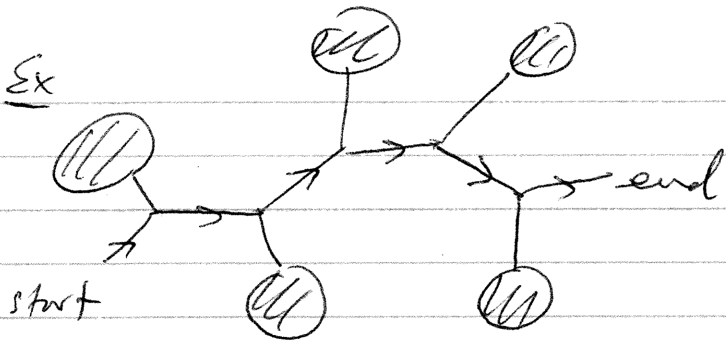
$$\to M = (-ig)^2 (-1) \cdot \frac{i}{q'^2 - m_\phi^2 + i\varepsilon} \left(\bar u(k')\, u(p)\right)\left(\bar u(p')\, u(k)\right)$$

2 vertices, 3 c-c, ϕ-ϕ, $\langle k' | \bar\psi$, $\psi | p\rangle$, $\langle p' | \bar\psi$, $\psi | k \rangle$

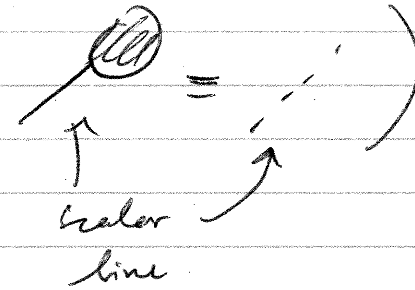
where $q' = p - k'$.

Tips: for each fermion line that doesn't close into loop, follow the particle number arrow to the end

Ex

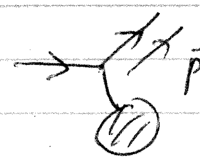


(note that



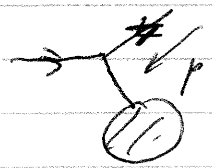
)

If the end is an outgoing fermion write down as



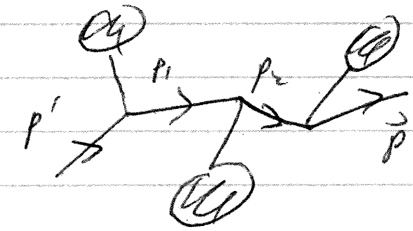
$\bar{u}(p)$

If the end is an incoming antifermion ---

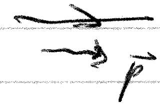


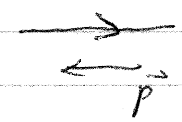
$\bar{v}(p)$

Write down fermion propagators you encounter as you follow the particle number arrow backwards

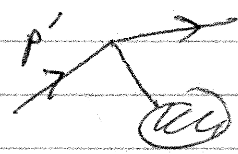


$$\bar{u}(p)\,\frac{i(\not{p}_2+m)}{p_2^2-m^2+i\epsilon}\cdot\frac{i(\not{p}_3+m)}{p_3^2-m^2+i\epsilon}\cdots$$

Note if  , $\dfrac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$

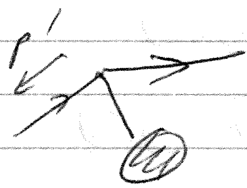
if  , $\dfrac{i(-\not{p}+m)}{p^2-m^2+i\varepsilon}$

If the start is an incoming fermion...



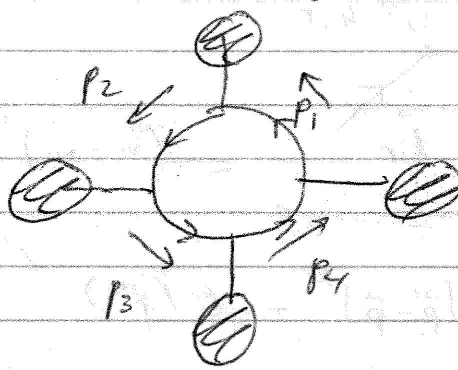
$u(p')$

If ... incoming antifermion



$v(p')$

If the fermion lines form a closed loop



$$= \bar\psi\psi\,\bar\psi\psi\,\bar\psi\psi\,\bar\psi\psi$$

$$= (-1)\,\mathrm{tr}\left\{\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\,\psi\bar\psi\right\}$$

$$= (-1)\,\mathrm{tr}\left\{S_F\,S_F\,S_F\,S_F\right\}$$

→ a closed loop always gives a factor of (-1) & the trace of a product of Dirac matrices.

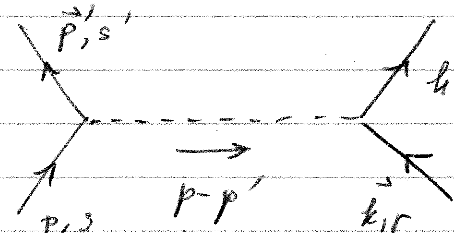
→ "trace" because we sum over the spinor indices.

The Yukawa Potential

Consider non-relativistic scattering of 2 different fermions → interact via exchange of a scalar particle.

Ignore $\mathcal{O}(\vec{p}^{\,2}/m^2)$ corrections, momenta are

$$\begin{cases} p = (m, \vec{p}) , & k = (m, \vec{k}) \\ p' = (m, \vec{p}\,') , & k' = (m, \vec{k}') \end{cases}$$



$$= (m-m)^2 - (\vec{p}\,' - \vec{p})^2 + \mathcal{O}(\cdots)$$

$$(p'-p)^2 = -|\vec{p}\,' - \vec{p}|^2 + \mathcal{O}(\vec{p}^{\,4})$$

$$u^s(p) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} , \text{ etc.}$$

where $\xi^{s'\dagger} \xi^s = \delta_{ss'}$

Spinor products: $\begin{cases} \bar{u}^{s'}(p') u^s(p) = 2m \xi^{s'\dagger} \xi^s = 2m \delta^{ss'} \\ \bar{u}^{r'}(k) u^r(k) = 2m \xi^{r'\dagger} \xi^r = 2m \delta^{rr'} \end{cases}$

→ the spin of each particle is conserved.

⇒ amplitude: $i\mathcal{M} = \dfrac{-ig^2}{(p'-p')^2 + m_\phi^2} (\bar{u}^{s'}(p') u^s(p))(\bar{u}^{r'}(k') u^r(k))$

$$\boxed{i\mathcal{M} = \frac{ig^2}{|\vec{p}\,' - \vec{p}|^2 + m_\phi^2} 2m \delta^{rr'} 2m \delta^{ss'}}$$

Okay so $\quad iM = \dfrac{ig^2}{(\vec{p}-\vec{p}\,')^2 + m_\phi^2} \; 2m\,\delta^{ss'}\, 2m\,\delta^{rr'}$

Compared with the Born approximation to the scattering amplitude in nonrelativistic ~~scattering~~ ~~interaction~~ ~~..~~ QM, in terms of the potential $V(x)$:

$$\langle p'|\,iT\,|p\rangle = -i\,\tilde{V}(\vec{q})\,(2\pi)\,\delta(E_{\vec{p}\,'} - E_{\vec{p}})$$

where $\vec{q} = \vec{p}\,' - \vec{p}$.

where does this come from?

(cf. Griffiths Chapter 11 on the Born approximation)