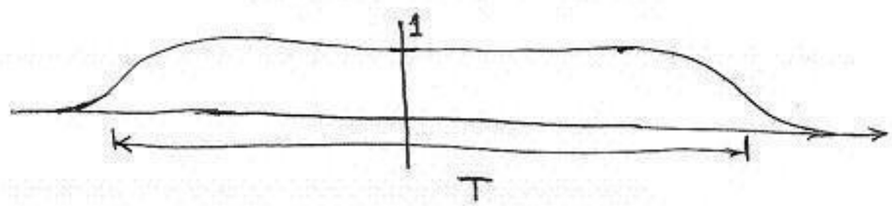
at some early time then slowly switching on the interactions, and then slowly switching off the interactions at some late time.

In other words $H_I(t) \rightarrow H_I(t) f(t)$

where $f(t)$ looks like



So $\int_{-\infty}^{\infty} f(t)\, dt = T$, $\int_{-\infty}^{\infty} (f(t))^2\, dt = T$

Let $S = T \exp\left\{-i \int_{-\infty}^{\infty} dt\, H_I(t) f(t)\right\}$

We define the S-matrix as

$$\langle \text{final} | S | \text{initial} \rangle$$

where $|\text{initial}\rangle$ is a free particle state with momenta $\vec{k}_i^I$ + energies E_i^I

and |final> is a free particle state with

momenta $\vec{k}_i^F$ + energies E_i^F.

Let us concern ourselves with the nontrivial part of the S-matrix,

$$\langle \text{final} | S - 1 | \text{initial} \rangle .$$

As $T \to \infty$, $V \to \infty$ we can write this amplitude as

$$\langle \text{final} | S - 1 | \text{initial} \rangle$$

$$= i \cdot \mathcal{M} \cdot (2\pi)^4 \, \delta(E^F_{tot} - E^I_{tot}) \, \delta^{(3)}(\vec{k}^F_{tot} - \vec{k}^I_{tot})$$

(i is part of standard definition for $\mathcal{M}$)

where $\mathcal{M}$ is a function of the momenta

For finite T and finite V we instead have

$$\langle \text{final} | S-1 | \text{initial} \rangle$$

$$= i\mathcal{M} \int_{-\infty}^{\infty} f(t)\, e^{i(E^F_{tot} - E^I_{tot})t}\, \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V$$

So $\left| \langle \text{final} | S-1 | \text{initial} \rangle \right|^2$

$$= |\mathcal{M}|^2 \cdot \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} \cdot V^2 \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) f(t')\, e^{i(E^F_{tot} - E^I_{tot})(t'-t)}\, dt'\, dt}$$

as $T \to \infty$ this clearly gives some constant times $\delta(E^F_{tot} - E^I_{tot})$.

What is this constant? To get let us integrate with respect to E^F_{tot}.

$$\longrightarrow \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big[\underbrace{\int_{-\infty}^{\infty} dE^F_{tot}\, e^{i(E^F_{tot} - E^I_{tot})(t'-t)}}_{2\pi\, \delta(t'-t)} \Big] f(t) f(t')\, dt'\, dt$$

$$= 2\pi \int_{-\infty}^{\infty} f^2(t)\, dt = 2\pi \cdot T$$

So the constant is $2\pi \cdot T$ and

$$\left| \langle \text{final} | S-1 | \text{initial} \rangle \right|^2 = |\mathcal{M}|^2 (2\pi)\, \delta(E^F_{tot} - E^I_{tot})\, \delta_{\vec{k}^F_{tot}, k^I_{tot}} V^2 T$$

$$|\langle \text{final} | S-1 | \text{initial} \rangle|^2$$

$$= |\mathcal{M}|^2 (2\pi)\, \delta(E^F_{tot} - E^I_{tot})\, \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}}\, V^2 T$$

We have been using relativistic normalizations for our states

$$\langle \text{initial} | \text{initial} \rangle = \prod_i (2E^I_i \cdot \overset{\text{becomes } (2\pi)^3 \delta^{(3)}(0) \text{ as } V \to \infty}{V})$$

$$\langle \text{final} | \text{final} \rangle = \prod_i (2E^F_i \cdot V)$$

To get the transition probability per unit time

$$\frac{\text{probability}}{\text{time}} = \frac{1}{T} \frac{|\langle \text{final} | S-1 | \text{initial} \rangle|^2}{\langle \text{final} | \text{final} \rangle \langle \text{initial} | \text{initial} \rangle}$$

$$= \frac{|\mathcal{M}|^2 (2\pi)\, \delta(E^F_{tot} - E^I_{tot})\, \delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}}\, V^2}{\prod_i (2E^F_i \cdot V) \prod_i (2E^I_i \cdot V)}$$

As $V \to \infty$, $\delta_{\vec{k}^F_{tot}, \vec{k}^I_{tot}} V \to (2\pi)^3 \delta^{(3)}(\vec{k}^F_{tot} - \vec{k}^I_{tot})$

If we sum over final states in some window then we have

$$\underbrace{\sum_{\vec{k}^F_1, \ldots \vec{k}^F_{n_F}} \frac{1}{(2E^F_1 \cdot V)} \cdots \frac{1}{(2E^F_{n_F} \cdot V)}}_{\text{As } V \to \infty,\ \frac{d^3\vec{k}^F_1}{(2\pi)^3 2E^F_1} \cdots \frac{d^3\vec{k}^F_n}{(2\pi)^3 2E^F_{n_F}}} \frac{|\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(k^F_{tot} - k^I_{tot}) \cdot V}{(2E^I_1 \cdot V) \cdots (2E^I_{n_I} \cdot V)}$$

n_I = # of initial particles
n_F = # of final particles

Let's consider a single particle decay ($n_I = 1$).
The total decay rate is $\Gamma = \int d\Gamma$ where

$$d\Gamma = \frac{1}{2E^I}\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3 2E_i^F}\right) |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(K_{tot}^F - K_{tot}^I)$$

For a two-particle initial state, the cross-section is given by

$$\sigma = \frac{\text{probability}}{\text{time} \cdot \text{flux density}}$$

The flux density is the relative velocity between the beam and target times density of incoming beam in the laboratory frame. We have normalized our probability for one incoming beam particle, so density $= \frac{1}{V}$, and

$$\text{flux} = \frac{|\vec{v}_A - \vec{v}_B|}{V}$$ $\vec{v}_A, \vec{v}_B$ velocities of particles in the laboratory frame

So
$$d\sigma = \underbrace{\left(\prod_{i=1}^{n_F} \frac{d^3\vec{k}_i^F}{(2\pi)^3 2E_i^F}\right)(2\pi)^4 \delta^{(4)}(K_{tot}^F - K_{tot}^I)}_{\equiv\, d\Pi_{n_F}} \frac{|\mathcal{M}|^2}{2E_A 2E_B |\vec{v}_A - \vec{v}_B|}$$

Let's consider a special case... two final particles ($n_F = 2$) in the center of mass frame.

$$\int d\Pi_2 = \int \frac{dp_1\, p_1^2\, d\Omega}{(2\pi)^3\, 2E_1\, 2E_2} (2\pi)\, \delta(-E_{cm} + E_1 + E_2)$$

$p_1 \equiv |\vec{p}_1|$

initial center of mass energy

[where the final particle energies are $E_1 = \sqrt{\vec{p}_1^{\,2} + m_1^2}$, $E_2 = \sqrt{\vec{p}_2^{\,2} + m_2^2}$]

$$= \int \frac{d\Omega}{16\pi^2} \int_0^\infty \frac{p_1^2\, \delta(-E_{cm} + \sqrt{p_1^2+m_1^2} + \sqrt{p_1^2+m_2^2})}{\sqrt{p_1^2+m_1^2}\,\sqrt{p_1^2+m_2^2}}\, dp_1$$

[recall $\underset{(x_0 \text{ is simple zero})}{\delta(f(x))} = \dfrac{\delta(x-x_0)}{|f'(x_0)|}$]

$$\frac{dE_1}{dp_1} = \frac{d\sqrt{p_1^2+m_1^2}}{dp_1} = \frac{1}{2}\frac{2p_1}{\sqrt{p_1^2+m_1^2}} = \frac{p_1}{E_1}$$

$$\frac{dE_2}{dp_1} = \frac{p_1}{E_2}$$

So
$$\int d\Pi_2 = \int \frac{d\Omega}{16\pi^2} \frac{p_1^2}{E_1 E_2 \left(\frac{p_1}{E_1} + \frac{p_1}{E_2}\right)} \Bigg|_{p_1 \text{ chosen so } E_1+E_2 = E_{cm}}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_1+E_2} = \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_{cm}}$$

So for two particles $\rightarrow$ two particles,

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\vec{p}^{\,\text{final}}|\; |\mathcal{M}|^2}{2E_A\, 2E_B\, |\vec{v}_A - \vec{v}_B|\, 16\pi^2\, E_{cm}} \qquad (E_{cm} = E_A + E_B)$$

It is conventional to define the T-matrix... $S = 1 + iT$

Claim:

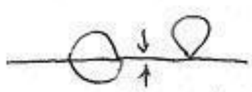
$$\langle \vec{p}_1^F, \cdots \vec{p}_{n_F}^F | iT | \vec{p}_A, \vec{p}_B \rangle$$

$$= \left({}_{\text{free}}\langle \vec{p}_1^F, \cdots \vec{p}_{n_F}^F | T \exp\left[-i \int_{-\infty}^{\infty} dt \, H_I(t) \right] | \vec{p}_A, \vec{p}_B \rangle_{\text{free}} \right)_{\star}$$

$\star$ = connected diagrams only + "amputated" diagrams only

"Amputated" means that the diagram can't be broken into disconnected pieces by cutting one internal line. This is also called one-particle irreducible (1PI).

Not Amputated



Amputated

The claim won't be proven until later in the course, but the idea is similar to before...

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle$$

and we would like something similar...

$$|\vec{p}_1, \vec{p}_2 \cdots \vec{p}_n\rangle \propto \lim_{T\to\infty(1-i\varepsilon)} e^{-iHT} |\vec{p}_1, \vec{p}_2 \cdots \vec{p}_n\rangle_{free}$$

But this is tricky. More on this in a couple of chapters ahead. For now we just take the claim as true pending later verification, though we will find some subtleties and fix them at that time.

Note that

$$\phi_I^+(x)\,|\vec{p}\rangle_{free} = \int \frac{d^3\vec{k}}{(2\pi)^3\sqrt{2E_{\vec{k}}}}\, a_{\vec{k}}\, e^{-ik\cdot x}\, \overbrace{\sqrt{2E_{\vec{p}}}\, a^\dagger_{\vec{p}}\, |0\rangle}^{\text{relativistic normalization}}$$

$$= e^{-ip\cdot x}\,|0\rangle$$

We can think of taking the commutator of $\phi_I^+(x)$ with the $a^\dagger_{\vec{p}}$ from $|\vec{p}\rangle_{free}$. This suggests the notation

$$\overline{\phi_I(x)\,|\vec{p}\rangle}_{\text{free}} = e^{-ip\cdot x}$$

We now drop the "free" subscript. Similarly

$$\langle\vec{p}|\,\phi_I^-(x) = e^{+ip\cdot x}\langle 0|$$

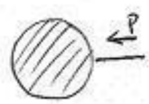
and we define

$$\overline{\langle\vec{p}|\,\phi_I}(x) = e^{+ip\cdot x}$$

Feynman Rules in position space with external lines

For each propagator $x \longrightarrow y$ $\quad D_F(x-y)$

For each internal vertex  $\quad (-i\lambda)\int d^4z$

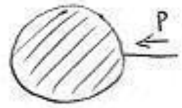
For each external line  $\quad e^{-ip\cdot x}$

Divide by symmetry factor S

Feynman Rules in momentum space with external lines

For each propagator $\xrightarrow[\;p\;]{}$ $\quad \dfrac{i}{p^2-m^2+i\varepsilon}$

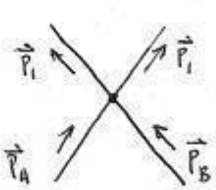
For each internal vertex  $-i\lambda$ and momentum conservation

For each external line 

no extra factor (i.e, 1)

Integrate over all unconstrained momenta and divide by symmetry factor S.

Example $\langle \vec{p}_1, \vec{p}_2 | \, iT \, | \vec{p}_A, \vec{p}_B \rangle$ at lowest order



Feynman amplitude
$i\mathcal{M} = -i\lambda$

So $$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\vec{p}^{\,final}| \, |\mathcal{M}|^2}{2E_A \, 2E_B \, |\vec{v}_A - \vec{v}_B| \, 16\pi^2 E_{cm}}$$

Let $p = |\vec{p}^{\,final}| = |\vec{p}_A| = |\vec{p}_B|$ all same since masses are all the same

$$E_{cm} = 2E_A = 2E_B = 2\sqrt{p^2+m^2}$$

$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2p}{E_A}$$