We now take a lengthy detour on traces of gamma matrices and contractions of gamma matrices.

Recall that $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ (Weyl representation)

So $\mathrm{Tr}\,\gamma^\mu = 0$.

We can prove this without resorting to specific representations

$$\mathrm{Tr}\,\gamma^\mu = \mathrm{Tr}\,\gamma^5\gamma^5\gamma^\mu = \mathrm{Tr}\,\gamma^5\gamma^\mu\gamma^5 \quad (\text{since } \mathrm{Tr}\,AB = \mathrm{Tr}\,BA)$$

$$= -\mathrm{Tr}\,\gamma^5\gamma^5\gamma^\mu$$

$$= -\mathrm{Tr}\,\gamma^\mu \Rightarrow \mathrm{Tr}\,\gamma^\mu = 0.$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = \tfrac{1}{2}\mathrm{Tr}\left[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right]$$

$$= \tfrac{1}{2}\mathrm{Tr}\left[2g^{\mu\nu}\cdot \underbrace{\mathbb{1}_{4\times4}}_{\text{identity matrix}}\right] = 4g^{\mu\nu}$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = \mathrm{Tr}\left[2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\rho\gamma^\sigma\right]$$

$$= \mathrm{Tr}\left[2g^{\mu\nu}\gamma^\rho\gamma^\sigma - \gamma^\nu(2g^{\mu\rho})\gamma^\sigma + \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\right]$$

$$= \mathrm{Tr}\left[2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + \gamma^\nu\gamma^\rho(2g^{\mu\sigma}) - \underline{\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu}\right]$$

Since $\mathrm{Tr}\left[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu\right] = \mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right]$,

we conclude that

$$2\,\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = \mathrm{Tr}\left[2g^{\mu\nu}\gamma^\rho\gamma^\sigma - 2g^{\mu\rho}\gamma^\nu\gamma^\sigma + 2g^{\mu\sigma}\gamma^\nu\gamma^\rho\right]$$

So $\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$

There are symbolic math programs that do more complicated strings of γ matrices.

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

Note that $\mathrm{Tr}\,\gamma^5 = \mathrm{Tr}\,\gamma^0\gamma^0\gamma^5 = \mathrm{Tr}\,\gamma^0\gamma^5\gamma^0$

$= -\mathrm{Tr}\,\gamma^0\gamma^0\gamma^5 = -\mathrm{Tr}\,\gamma^5$

So $\mathrm{Tr}\,\gamma^5 = 0$.

Also $\mathrm{Tr}\left[\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{n\text{ odd}}\gamma^5\right] = 0$, since

$\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5\right] = (-1)^n\,\mathrm{Tr}\left[\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right]$

anticommute

trace is cyclical $\mathrm{tr}(AB) = \mathrm{tr}(BA)$

$= -\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5\right]$.

Similarly $\mathrm{Tr}\left[\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{n\text{ odd}}\right] = \mathrm{Tr}\left[\gamma^5\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right]$

$= (-1)^n\,\mathrm{Tr}\left[\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}\gamma^5\right] = -\mathrm{Tr}\left[\gamma^5\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right]$

cyclical

$= -\mathrm{Tr}\left[\gamma^{\mu_1}\cdots\gamma^{\mu_n}\right]$. And so $\mathrm{Tr}\left[\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{n\text{ odd}}\right] = 0$

Some useful results:

$$\mathrm{Tr}\left[1_{4\times4}\right] = 4$$

$$\mathrm{Tr}\,[\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{\text{odd}}] = 0$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\right] = 4g^{\mu\nu}$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4g^{\mu\nu}g^{\rho\sigma} - 4g^{\mu\rho}g^{\nu\sigma} + 4g^{\mu\sigma}g^{\nu\rho}$$

$$\mathrm{Tr}\left[\gamma^5\right] = 0$$

$$\mathrm{Tr}\,[\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{\text{odd}}\gamma^5] = 0$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^5\right] = 0$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right] = -4i\varepsilon^{\mu\nu\rho\sigma} \leftarrow \varepsilon^{0123} = 1$$

completely antisymmetric

Another useful result is

$$\mathrm{Tr}\left[\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}\right] = \mathrm{Tr}\left[\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\right]$$

(reverse order)

Proof: In connection with charge conjugation we encounter a matrix...

$$C \equiv \gamma^0\gamma^2 \text{ with the properties}$$

$$C^2 = (\gamma^0\gamma^2)(\gamma^0\gamma^2) = 1\,,\quad C^\dagger = \gamma^{2\dagger}\gamma^{0\dagger} = C$$

$$C^\dagger\gamma^\mu C = -(\gamma^\mu)^T$$

So $\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}]$

$$= \mathrm{Tr}[C^\dagger\gamma^{\mu_1}C C^\dagger\gamma^{\mu_2}C\cdots C^\dagger\gamma^{\mu_n}C]$$

$$= (-1)^n\,\mathrm{Tr}[(\gamma^{\mu_1})^T(\gamma^{\mu_2})^T\cdots(\gamma^{\mu_n})^T]$$

$$= (-1)^n\,\mathrm{Tr}[\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}] \quad \left(\begin{array}{c}\text{since}\\ \mathrm{Tr}\,M = \mathrm{Tr}\,M^T\end{array}\right)$$

Since the trace vanishes unless n is even,

$$\mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_n}] = \mathrm{Tr}[\gamma^{\mu_n}\cdots\gamma^{\mu_2}\gamma^{\mu_1}].$$

Some useful contraction identities

$$\gamma_\mu\gamma^\mu = \gamma^\mu\gamma^\nu g_{\mu\nu} = \tfrac{1}{2}(\gamma^\mu\gamma^\nu + \underbrace{\gamma^\nu\gamma^\mu}_{\substack{\text{since}\\ g_{\mu\nu}\text{ is symmetric}}})\,g_{\mu\nu}$$

$$= \tfrac{1}{2}\,2g^{\mu\nu}g_{\mu\nu} = 4$$

$$\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(2g^{\nu\mu} - \gamma^\mu\gamma^\nu) = +2\gamma^\nu - 4\gamma^\nu$$

$$= -2\gamma^\nu$$

Other results:

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu$$

Back to our calculation of $|\mathcal{M}|^2$ (unpolarized).

We had one term that had

$\mathrm{Tr}\left[\gamma^\mu (\not{p}+m_e)\gamma^\nu(\not{p}'-m_e)\right]$. The trace is zero unless we have an even number of γ's. So we have

$$\mathrm{Tr}\left[\gamma^\mu \underset{\substack{\uparrow \\ p_\alpha \gamma^\alpha}}{\not{p}}\, \gamma^\nu \underset{\substack{\uparrow \\ p'_\beta \gamma^\beta}}{\not{p}'}\right] - \mathrm{Tr}\left[\gamma^\mu \gamma^\nu\right] m_e^2$$

$$= 4 p^\mu p'^\nu - 4 g^{\mu\nu} p\cdot p' + 4 p^\nu p'^\mu - 4 g^{\mu\nu} m_e^2$$

$$= 4\left[p^\mu p'^\nu + p^\nu p'^\mu - g^{\mu\nu}(p\cdot p' + m_e^2)\right]$$

Similarly,

$$\mathrm{Tr}\left[\gamma_\mu (\not{k}'-m_\mu)\gamma_\nu(\not{k}+m_\mu)\right]$$

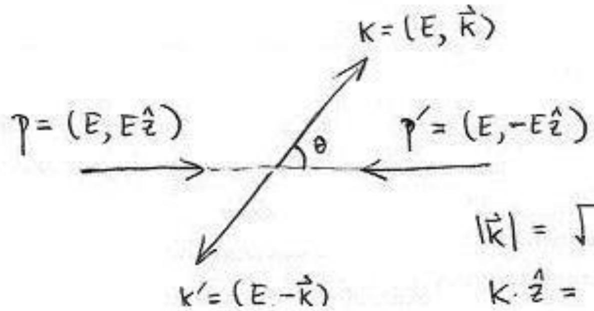
$$= 4\left[k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu}(k'\cdot k + m_\mu^2)\right]$$

So $$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4e^4}{(q^2)^2}\left[\begin{aligned} &2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) \\ &-2(k'\cdot k)(p\cdot p' + m_e^2) \\ &-2(p\cdot p')(k'\cdot k + m_\mu^2) \\ &+4\,(k'\cdot k + m_\mu^2)(p\cdot p' + m_e^2)\end{aligned}\right]$$

If the energy is sufficiently high we can neglect m_e. So we set $m_e = 0$.

Then
$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{4e^4}{(q^2)^2}\left[2(p\cdot k)(p'\cdot k') + 2(p\cdot k')(p'\cdot k) + 2m_\mu^2 (p\cdot p')\right]$$

Now let's go the center of mass frame...



$(m_e = 0)$

$|\vec{k}| = \sqrt{E^2 - m_\mu^2}$

$k\cdot\hat{z} = |\vec{k}|\cos\theta$

$$q^2 = (p+p')^2 = (2E)^2 - 0^2 = 4E^2 \qquad p\cdot p' = E^2 + E^2 = 2E^2$$
$$p\cdot k = E^2 - E|\vec{k}|\cos\theta \qquad p\cdot k' = E^2 + E|\vec{k}|\cos\theta$$
$$p'\cdot k' = E^2 - E|\vec{k}|\cos\theta \qquad p'\cdot k = E^2 + E|\vec{k}|\cos\theta$$

So
$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{8e^4}{16E^4}\left[E^2(E - |\vec{k}|\cos\theta)^2 + E^2(E + |\vec{k}|\cos\theta)^2 + 2m_\mu^2 E^2\right]$$
$$= \frac{e^4}{E^4}\left[E^2(E^2 + \underbrace{|\vec{k}|^2}_{E^2 - m_\mu^2}\cos^2\theta) + m_\mu^2 E^2\right]$$
$$= \frac{e^4}{E^4}\left[E^4 + E^4\cos^2\theta - E^2 m_\mu^2\cos^2\theta + m_\mu^2 E^2\right]$$
$$= e^4\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

For two body scattering into a two body final state,

$$\frac{d\sigma}{d\Omega} = \frac{1}{2E_A 2E_B |V_A - V_B|} \frac{|\vec{k}|}{16\pi^2 E_{cm}} \times \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

In our case the electron mass is being neglected, $m_e \simeq 0$, and so the positron + electron are moving at nearly the speed of light,

$$|V_A - V_B| = 2.$$

So $$\frac{d\sigma}{d\Omega} = \frac{1}{2E\cdot 2E\cdot 2} \cdot \frac{\sqrt{E^2 - m_\mu^2}}{16\pi^2 E_{cm}} \cdot e^4\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$(E_A = E_B = E = \frac{E_{cm}}{2})$

$$= \frac{e^4}{256\pi^2 E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

Integrating over $d\Omega$ gives

$$\sigma_{tot} = \frac{e^4 \cdot 2\pi}{256\pi^2 E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[2\left(1 + \frac{m_\mu^2}{E^2}\right) + \frac{2}{3}\left(1 - \frac{m_\mu^2}{E^2}\right)\right]$$

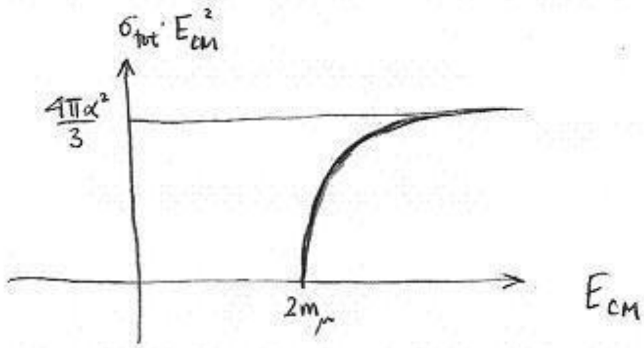
$$= \frac{e^4}{64\pi E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[\frac{4}{3}\left(1 + \frac{1}{2}\frac{m_\mu^2}{E^2}\right)\right]$$

Conventional to write this as

$$\sigma_{tot} = \frac{4\pi\alpha^2}{3E_{cm}^2}\sqrt{1-\frac{m_\mu^2}{E}}\left(1+\frac{1}{2}\frac{m_\mu^2}{E^2}\right)$$

where $\alpha = \frac{e^2}{4\pi}$

When $E_{cm} < 2m_\mu$, the cross section is zero since there isn't enough energy to produce a muon + antimuon pair.



As $E_{cm} \to \infty$, $\sigma_{tot} \to \frac{4\pi\alpha^2}{3E_{cm}^2}$

$$\frac{d\sigma}{d\Omega} \to \frac{\alpha^2}{4E_{cm}^2}\left(1+\cos^2\theta\right)$$

Quarks are Dirac fermions which participate in the strong interactions. Although quarks