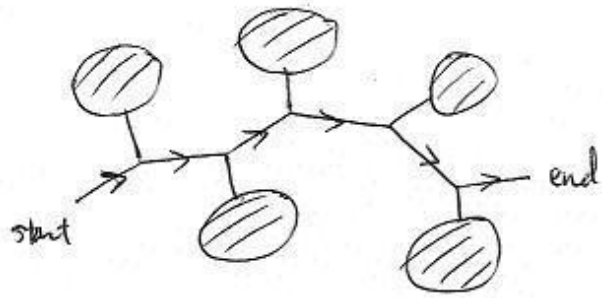


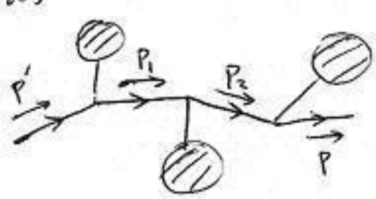
If the end is an outgoing fermion write down a

$\bar{u}(p)$

If the end is an incoming antifermion write down a

$\bar{v}(p)$

Write down the fermion propagators you encounter as you follow the particle number arrow backwards



$$\bar{u}(p)\,\frac{i(\not{p}_2+m)}{p_2^2-m^2+i\varepsilon}\,\frac{i(\not{p}_1+m)}{p_1^2-m^2+i\varepsilon}\,\cdots$$

Note: if $\underset{\xrightarrow[p]{}}{\longrightarrow}$ write $\dfrac{i(\not{p}+m)}{p^2-m^2+i\varepsilon}$ ← for internal lines you might as well use this momentum convention

if $\underset{\xleftarrow[p]{}}{\longrightarrow}$ write $\dfrac{i(-\not{p}+m)}{p^2-m^2+i\varepsilon}$

Last step:

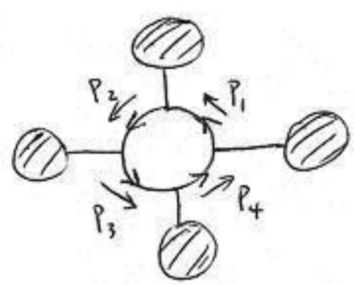
If the start is an incoming fermion write a

$u(p')$

If the start is an outgoing anti-fermion write a

$v(p')$

If the fermion line forms a closed loop...



...Take the trace of product of propagators going backwards along particle number arrow. [Why backwards? Because in English we tend to write from left to right but operators acting in succession go from right to left.] The multiply by $\times(-1)$ for each closed loop.

So the diagram above gets a

$$(-1) \times \mathrm{Tr}\left[\frac{i(\not{p}_4+m)}{p_4^2-m^2+i\varepsilon}\,\frac{i(\not{p}_3+m)}{p_3^2-m^2+i\varepsilon}\,\frac{i(\not{p}_2+m)}{p_2^2-m^2+i\varepsilon}\,\frac{i(\not{p}_1+m)}{p_1^2-m^2+i\varepsilon}\right]$$

Why the minus sign?

$$\bar\psi\psi\;\bar\psi\psi\;\bar\psi\psi\;\bar\psi\psi$$

... no criss-crosses but $\bar\psi_b\psi_a = -\psi_a\bar\psi_b$

Why the trace? Because we sum over the spinor indices.

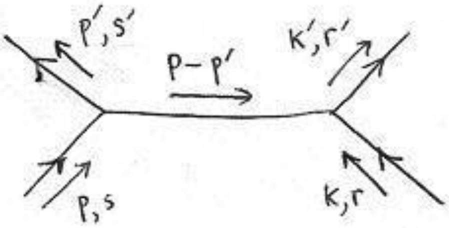
Yukawa potential

We consider non-relativistic scattering of two different fermions (just to keep things simple). They interact via exchange of scalar particle.

Ignoring $\mathcal{O}\left(\frac{\vec{p}^{\,2}}{m^2}\right)$ corrections, the momenta are

$$p = (m, \vec{p}\,), \quad k = (m, \vec{k}\,)$$
$$p' = (m, \vec{p}\,'), \quad k' = (m, \vec{k}\,')$$

[Two different kinds of fermions but the same mass, m. Example: neutron + proton]



$$i\mathcal{M} = (-ig)^2 \frac{i}{(p-p')^2 - m_\phi^2 + i\varepsilon} \left(\bar{u}^{s'}(p')\, u^s(p)\right)\left(\bar{u}^{r'}(k')\, u^r(k)\right)$$

In the non-relativistic limit,

$$(p-p')^2 = (m-m)^2 - (\vec{p}-\vec{p}\,')^2 + \mathcal{O}\left(\frac{(\vec{p}^{\,2})^2}{m^4}\right)$$
$$= -(\vec{p}-\vec{p}\,')^2$$

and $$u^s(p) = \sqrt{m}\begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} + \mathcal{O}\left(\frac{|\vec{p}|}{m}\right)$$

So $\bar{u}^{s'}(p') u^s(p) = (\sqrt{m}) (\xi^{s'\dagger} \;\; \xi^{s'\dagger}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix} (\sqrt{m})$

$$= 2m\,\delta^{s's}$$

Therefore

$$i\mathcal{M} = \frac{i g^2}{(\vec{p}-\vec{p}\,')^2 + m_\phi^2} (2m)\,\delta^{s's} (2m)\,\delta^{r'r}$$

In the Born approximation the non-relativistic scattering amplitude is related to the potential by

$$\langle \vec{p}\,' | iT | \vec{p} \rangle = -i\,\tilde{V}(\vec{q})\,(2\pi)\,\delta(E_{\vec{p}\,'} - E_{\vec{p}})$$

where $\vec{q} = \vec{p} - \vec{p}\,'$

non-relativistic normalization

We deduce that

$$\tilde{V}(\vec{q}) = -\frac{g^2}{(\vec{p}-\vec{p}\,')^2 + m_\phi^2}$$

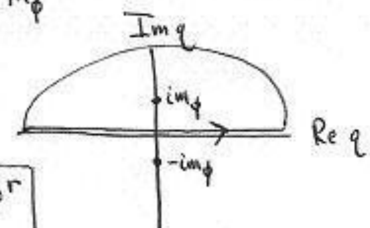
We have removed the $(2m)^2$ since this came from the relativistic normalization.

So
$$V(\vec{x}) = \int \frac{d^3\vec{q}}{(2\pi)^3} \frac{-g^2}{\vec{q}^{\,2}+m_\phi^2} e^{i\vec{q}\cdot\vec{x}} \qquad (r=|\vec{x}|)$$

$$= -\frac{g^2}{8\pi^3}\int_0^\infty dq\, q^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \,\frac{e^{iqr\cos\theta}}{q^2+m_\phi^2}$$

$$= -\frac{g^2}{4\pi^2}\int_0^\infty dq\, \frac{e^{iqr}-e^{-iqr}}{iqr}\,\frac{q^2}{q^2+m_\phi^2}$$

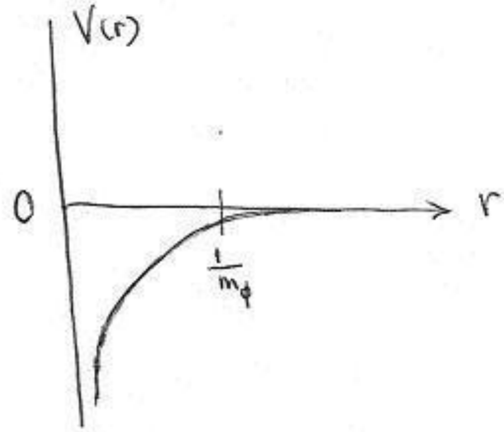
$$= -\frac{g^2}{i4\pi^2 r}\int_{-\infty}^{\infty} dq\, \frac{e^{iqr}\,q}{q^2+m_\phi^2}$$



$$= (2\pi i)\left(-\frac{g^2}{i4\pi^2 r}\right)\frac{e^{-m_\phi r}}{2} = \boxed{-\frac{g^2 e^{-m_\phi r}}{4\pi r}}$$

Yukawa potential

Short-range interaction



Preview of Quantum Electrodynamics (QED)

Gauge theories present a special challenge. We will simply write down the Feynman rules for now. We will derive things later.

We consider a massless vector particle called the photon and its quantum field $A_\mu(x)$.

Let

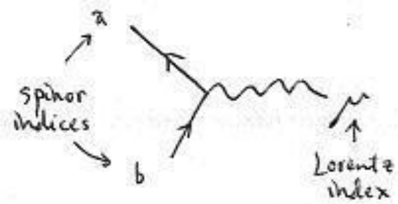
$$H_{int} = e \int d^3\vec{x}\ \bar{\psi} \gamma^\mu \psi A_\mu$$

The vector particle has spin 1. Just as the fermion came with spinor indices $u^s(p)$, the photon has a polarization vector $\varepsilon^\mu(p)$.

The Feynman rules in momentum space...

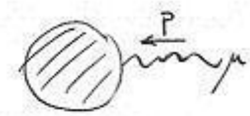
Photon propagator

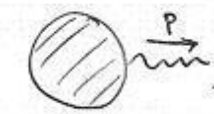
$$\frac{-i g_{\mu\nu}}{q^2 + i\varepsilon}$$



$$= -ie\gamma^{\mu}_{ab}$$

External photons:

$$A_\mu \, |\vec{p}, \varepsilon\rangle =$$  $$= \varepsilon_\mu(p)$$

$$\langle \vec{p}, \varepsilon | \, A_\mu =$$  $$= \varepsilon^*_\mu(p)$$

In Lorentz gauge we require $\partial_\mu A^\mu = 0$.
Then the field equations, $\partial_\mu F^{\mu\nu} = 0$, becomes

$$\partial_\mu \partial^\mu A^\nu = 0$$

So each A^ν $(\nu = 0,1,2,3)$ satisfies the Klein-Gordon equation with zero mass.

We can write

$$A_\mu(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{r=0}^{3} \left(a^r_{\vec{p}} \, \varepsilon^r_\mu(p) e^{-ip\cdot x} + a^{r\dagger}_{\vec{p}} \, \varepsilon^{r*}_\mu(p) e^{ip\cdot x} \right)$$

where $p^0 = E_{\vec{p}} = \sqrt{\vec{p}^{\,2}}$