

QUANTUM FIELD THEORY

Sep 13, 2020

Before. These notes come from Prof. Paltin's QFT notes, Peskin-Schroeder QFT, and "Quantum Field Theory & Condensed Matter" - Shankar.

Enjoy!

4

Conventions

$$t = c = 1$$

$$[\text{length}] = [\text{time}] = [\text{energy}] = [\text{mass}]$$

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1 - 1 - 1 - 1)$$

$$p^2 = p_\mu p^\mu = E^2 - |\vec{p}|^2 = mc^2 = m$$

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, \vec{x})$$

$$x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

$$\epsilon^{0123} = +1 ; \quad \epsilon_{0123} = -1$$

$$\epsilon^{1230} = -1$$

$$E = i \frac{\partial}{\partial x^0}, \quad \vec{p} \rightarrow \vec{\nabla}$$

$$p^\mu = i \partial^\mu$$

(2)

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx} \theta(x)$

- n -dimensional Dirac δ -fn:

$$\int d^n x \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \tilde{f}(k)$$

$$\tilde{f}(k) = \int d^4 x e^{ik \cdot x} f(x)$$

- $\int d^4 x e^{ik \cdot x} = (2\pi)^4 \delta^{(4)}(k)$

- EM $\Phi = \frac{Q}{4\pi r} \leftarrow$ Coulomb potential

(3)

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi hc} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\vec{E}, \vec{A}), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of classical Field Theory

- Lagrangian Field Theory:

$$S = \int \underline{L} dt = \int \underline{L}(\phi, \partial_\mu \phi) d^4x \quad \left(\underline{L} = L d^4x \right)$$

Principle of least action:

$$0 = \delta S \quad (\text{Lagrangian})$$

$$(\text{E-L eqn}) = \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi + \frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta \phi \right] \right\}$$

$$\Rightarrow 0 = \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right] - \frac{\partial \underline{L}}{\partial \phi}$$

FTC \rightarrow term vanishes
@ Boundary

(4)

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_m \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \right\}$$

Ex $\mathcal{L} = \phi^2 : \quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi ; \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 0 \Rightarrow 2\phi = 0$

$$\begin{aligned} \mathcal{L} &= (\partial_m \phi) (\partial^m \phi) \\ \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} &= 0 ; \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 2\partial^m \phi \end{aligned} \quad \left. \right\} \Rightarrow \partial^m \phi = 0,$$

Ex Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2} (\partial_m \phi) (\partial^m \phi) - \frac{1}{2} m^2 \phi^2$$

$$\therefore \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = \partial^m \phi .$$

relativistic particle
of mass m .

$$E-L \Rightarrow -m^2 \phi = \partial_m \partial^m \phi = \square \phi$$

$$\rightarrow \boxed{(\square + m^2) \phi = 0}$$

(Klein - Gordon Eqn.)

$$\text{Ex } \phi = e^{-ip \cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$$

(5)

Noether's Thm

Thm

For every continuous symmetry, there exists a conserved current j^μ which implies a local conservation law

$$\partial_\nu j^\nu = -\vec{\nabla} \cdot \vec{j} \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

Ex Conservation of charge -- (gauss' law)

$$\begin{aligned} Q &= \int j^0 d^3x \\ \Rightarrow \frac{\partial Q}{\partial t} &= \int \frac{d j^0}{dt} d^3x = - \int \vec{\nabla} \cdot \vec{j} d^3x \\ &= - \oint \vec{j} d^2s \end{aligned}$$

Idea Consider continuous transf. \rightarrow infinitesimally (local)

$$(\star) \quad \phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

↑
small

(\star) is a symmetry if EOM invariant under (\star).

$\Rightarrow S$ is invariant.

$\Rightarrow L$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$,
for some J^μ .

(6)

Let us compare this expectation for ΔL to the result obtained by varying the fields ...

$$\begin{aligned}\alpha \Delta L &= \frac{\partial L}{\partial \phi} (\alpha \Delta \phi) + \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \partial_\mu (\alpha \Delta \phi) \\ &= \alpha \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right) + \alpha \underbrace{\left(\frac{\partial L}{\partial \phi} - \partial_\mu \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \right\} \right) \Delta \phi}_{0}\end{aligned}$$

$$\Rightarrow \Delta L = \partial_\mu \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right\}$$

So $\frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi$ is the desired J^μ .

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi - J^\mu$$

Ex massless KG field

$$L = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)$$

Consider transformation:

$$\phi(x) \rightarrow \phi(x) + \alpha, \quad \Delta \phi = 1$$

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi = \partial^\mu \phi.$$

Check $\partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = \square \phi = 0$ since
 $(m^2 + \nabla^2) \phi = 0 \Rightarrow m = 0$ \uparrow

(7)

Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^+) (\partial^\mu \phi) - m^2 \phi^+ \phi.$$

again, EOM \Rightarrow

$$(m^2 + \Box) \phi = 0.$$

Symmetry: $\phi \rightarrow e^{i\alpha} \phi$.

For infinitesimal transf we have:

$$\begin{aligned}\alpha D\phi &= i\alpha \phi && \text{(Taylor expand)} \\ \alpha D\phi^+ &= -i\alpha \phi^+\end{aligned}$$

One can check that

$$j^\mu = i [(\partial^\mu \phi^+) \phi - \phi^+ (\partial^\mu \phi)]$$

\Rightarrow the conserved current.

[Ex]

Space-time translation:

Consider inf. translation:

$$x^\mu \rightarrow x^\mu - a^\mu$$

\hookrightarrow in field transforms this becomes:

$$\phi(x) \rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

(8)

Lagrangian is a scalar \Rightarrow must transform the same way:

$$L \rightarrow L + a^{\mu} \partial_{\mu} L = L + a^{\nu} \partial_{\mu} (s_{\nu}^{\mu} L)$$

Compare this to the eqn:

$$L \rightarrow L + a \partial_{\mu} J^{\mu}$$

we have

$$J^{\mu} = \cancel{s}_{\nu}^{\mu} L$$

\Rightarrow apply this, we find:

$$J^{\mu} = \frac{\partial L}{\partial (\partial_{\mu} \phi)} (\partial_{\mu} \phi) - s_{\nu}^{\mu} L$$

value μ explicit...

$$\boxed{T_{\mu}^{\nu} = \frac{\partial L}{\partial (\partial_{\nu} \phi)} \partial_{\mu} \phi - \delta_{\mu}^{\nu} L}$$

\hookrightarrow STRESS-ENERGY TENSOR, (or Energy-momentum tensor)

Conserved charge \Rightarrow the Hamiltonian

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad (\text{time-translation})$$

$$\text{momentum} \rightarrow p^i = \int T^{0i} d^3x = - \int \mathcal{H} \partial_i \phi d^3x \quad (\text{spatial translation})$$

(9)

Ex Klein-Gordon field again

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.$$

$$T^{00} = \frac{\partial F}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} = \dots$$

$$= \frac{1}{2} (\partial_0 \phi) (\partial^0 \phi) - \frac{1}{2} (\partial_i \phi) (\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

so

$$H = \int T^{00} d^3x = \int \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} d^3x$$

(note: all terms are positive ... (sum of squares))

→ can't fall into arbitrary negative energy

THE KLEIN-GORDON FIELD as HARMONIC OSCILLATOR

promote: ϕ, π to operators \Rightarrow impose suitable commutation relations

Reall ...

$$[q_i, p_j] = i \hbar \delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0.$$

$$\text{Harmonic oscillator: } H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2$$

(10)

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}} \left(g\sqrt{m\omega} + ip \frac{1}{\sqrt{m\omega}} \right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}} \left(g\sqrt{m\omega} - ip \frac{1}{\sqrt{m\omega}} \right)$

- $a^\dagger a = \frac{1}{\omega} H - \frac{1}{2} \quad (\Rightarrow \quad H = \omega(a^\dagger a + \frac{1}{2})$



operator...

- $|0\rangle, a|0\rangle = 0.$

$$|n\rangle = \frac{1}{\sqrt{n!}} \underbrace{a^\dagger \dots a^\dagger}_{n} |0\rangle$$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -\omega a ; [H, a^\dagger] = \omega a^\dagger$

a lowers by ω

a^\dagger raises by ω

- $H a^\dagger |n\rangle = (E_n + \omega) a^\dagger |n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

(11)

For continuous system ... commutation relations become ..

$$[\phi(x), \pi(y)] = i\delta^{(1)}(x-y)$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0.$$

Next, the Hamiltonian is now also an operator...
To find $\text{spec}(H)$, Fourier transf $\phi(x)$

$$\rightarrow \phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \phi(\vec{p}, t)$$

Recall KG eqn: $(m^2 + D)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|\vec{p}|^2 + m^2) \right\} \phi(\vec{p}, t) = 0$$

\rightarrow This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}$$

$$\text{Now, } H_{SHO} = \frac{1}{2} \vec{p}^2 + \frac{1}{2} \omega^2 \phi^2 \quad (m=1)$$

\rightarrow know spectrum! $(n + \frac{1}{2})\omega$.

$$\phi = \frac{1}{\sqrt{2\omega}} (at + a) ; \vec{p} = -i\sqrt{\frac{\omega}{2}} (a - at)$$

$$[a, a^\dagger] = 1.$$

(12)

Since it's more convenient to work in position space

$$\boxed{\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p e^{ip \cdot x} + a_p^\dagger e^{-ip \cdot x})}$$

$$\boxed{\pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p e^{ip \cdot x} - a_p^\dagger e^{-ip \cdot x})}$$

Note

$$\left\{ \begin{array}{l} a_p \text{ goes with } e^{+ip \cdot x} \\ a_p^\dagger \text{ goes with } e^{-ip \cdot x}. \end{array} \right.$$

9 Easy to show that

$$\left\{ \begin{array}{l} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}). \end{array} \right.$$

* Can re-arrange...

$$\left\{ \begin{array}{l} \phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p + a_{-p}^\dagger) e^{ip \cdot x} \\ \pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{-ip \cdot x}. \end{array} \right.$$

→ set commutation relation between a_p :

$$\boxed{[a_p; a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')}$$

(13)

Now, can check that

$$\begin{aligned} [\phi(x), \pi(x')] &= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{w_p}{w_{p'}}} x \cdot e^{ip \cdot x} (p \cdot x + p' \cdot x') \\ &\quad \left([a_{-p}^\dagger, a_p] - [a_p, a_{-p}^\dagger] \right) \\ &= i \delta^{(3)}(x - x') \quad \checkmark \end{aligned}$$

Now, can express Hamiltonian in terms of ladder ops

recall that

KG field, but
done

$$H = \int d^3 x \left\{ \frac{\partial f}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \partial^0 f \right\}$$

$$= \int d^3 x \left\{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial f}{\partial (\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial f}{\partial \dot{\phi}} \right)$$

so ...

$$H = \int d^3 x \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi) \cdot (\nabla \phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

$$\text{with } \pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{ip \cdot x}$$

we get

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2 w_p}} (a_p + a_{-p}^\dagger) e^{-ip \cdot x}$$

(14)

$$H = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(p+p') \cdot x} \left\{ -\frac{\sqrt{w_p w_{p'}}}{4} (a_p - a_{-p}^\dagger) (a_{p'} + a_{-p'}^\dagger) \right. \\ \left. + \frac{-p \cdot p' + m^2}{4\sqrt{w_p w_{p'}}} (a_p + a_{-p}^\dagger) (a_{p'} + a_{-p'}^\dagger) \right\}$$

results in $C(p-p')$
 $\Rightarrow p = p'$

Some $S^{(3)}$
will appear...

$$= \int \frac{d^3p}{(2\pi)^3} w_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

Σ

$$H = \int \frac{d^3p}{(2\pi)^3} w_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

→ can evaluate commutators...

$$[H, a_p^\dagger] = w_p a_p^\dagger; [H, a_p] = -w_p a_p$$

With H , can find momentum operator...

kG field \rightarrow form $p^i = \int d^3x T^{0i} = - \int \nabla_i \phi d^3x$, we set

$$\tilde{P} = - \int d^3x \nabla_i \phi(x) \\ = \int \frac{d^3p}{(2\pi)^3} \tilde{p} a_p^\dagger a_p$$

$E_p \rightarrow 0$
 \parallel

a_p^\dagger creates momentum \tilde{p} & energy $w_p = \sqrt{|\tilde{p}|^2 + m^2}$.

Excitation: $a_p^\dagger a_q^\dagger \dots |0\rangle$ = "particles".

↳ such excitation at p is a particle.

\Rightarrow set particle statistics --

Consider 2-particle state $a_p^+ a_q^+ |0\rangle$.

Since $[a_p^+, a_q^+] = 0$, we have

$$a_p^+ a_q^+ |0\rangle = a_q^+ a_p^+ |0\rangle$$

\Rightarrow Klein Gordon particles follow Bose-Einstein state.

* Normalization $\langle 0|0 \rangle = 1$.

$$\langle p | \propto a_p^+ |0\rangle$$

This $\rightarrow \langle q | p \rangle = (2\pi)^3 \delta^{(3)}(q-p) \rightarrow$ NOT Lorentz inv

PF Under a Lorentz boost $\rightarrow p'_3 = \gamma(p_3 + \beta E)$

using the identity $E' = \gamma(E + \beta p_3)$

$$\rightarrow \delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(n - n_0)$$

$$\text{we can write: } \delta^{(3)}(p-q) = \delta^6(p'-q') \cdot \left(\frac{dp'_3}{dp_3} \right)$$

$$\begin{aligned} & \underbrace{\delta(p_1-q_1)}_{\text{same}} \underbrace{\delta(p_2-q_2)}_{\text{boosted}} \underbrace{\delta(p_3-q_3)}_{\text{boosted}} \\ &= \delta^{(3)}(p'-q') \cdot \gamma \left(1 + \beta \frac{dE}{dp_3} \right) \\ &= \delta^{(3)}(p'-q') \frac{\gamma}{E} (E + \beta p_3) \\ &= \delta^{(3)}(p'-q') \left(\frac{E'}{E} \right) \end{aligned}$$

(16)

$$\text{So } \boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \left(\frac{E'}{E}\right)} \rightarrow 1 \Leftrightarrow E = E_p = E'$$

For normalization to work \rightarrow use E_p , not E .

\rightarrow define: $|p\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle$

to find $\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$

Completeness relation ...

1 particle $\rightarrow \boxed{1 = \int \frac{d^3 p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$

RS Interpret $\phi(x)|0\rangle \dots$ we know that a_p^\dagger creates momentum p energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle , \quad a_p^\dagger |0\rangle = \frac{1}{\sqrt{2E_p}} |p\rangle$$

Pf. By defn ...

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ipx} + a_p^\dagger e^{-ipx})$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle \quad \square$$

$\rightarrow \phi(x)|0\rangle$ is a lin. superposition of single-particle states

Note Ch 8 & 9

(17)

Hint here: well-defn momentum.

When nonrelativistic $\rightarrow E_p \approx \text{constant}!$

\Rightarrow $\phi(x)$ acting on the vacuum, "creates a particle at position x ".

\hookrightarrow Confirm this by computing -

$$\langle 0 | \phi(x) | p \rangle = \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p^\dagger e^{ip \cdot x} + a_p^- e^{-ip \cdot x}) \sqrt{2E_p} a_p | p \rangle$$

$$\boxed{\langle 0 | \phi(x) | p \rangle = e^{ip \cdot x}}$$

\hookrightarrow Interpretation: position-space representation of the single-particle wf_n of the state $|p\rangle$, just like

$$\boxed{\langle n | p \rangle \propto e^{ip \cdot x}} \text{ in QM!}$$

$\langle 0 | \phi(x) | \sim x | \dots$ (don't take this literally, ofc).

Note Hw1, Hw2 are copy, so we'll skip for now.

ep 14, 2020

THE KLEIN-GORDON FIELD IN SPACETIME

Last time \rightarrow we quantized KG field in the Schrödinger picture.

\rightarrow Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$U(t) = e^{-iHt}$ is the time evolution.

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \xrightarrow{\text{state evolves in time}}$$

\rightarrow In the Heisenberg picture, ... Operators evolve in time

$$\theta(t) = U^\dagger(t) \theta(0) U(t).$$

to treat

$$\langle \psi_1 | \theta(t) | \psi_2 \rangle = \langle \psi_1(t) | \theta(t) | \psi_2(t) \rangle$$

\downarrow

Heisenberg

\downarrow

Schrödinger.

\rightarrow make the operators ϕ, π time-dependent like this

$$\phi(x) \rightarrow \phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$$

$$\pi(x) \rightarrow \pi(x, t) = e^{iHt} \pi(x) e^{-iHt}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t} \theta = [\theta, H]$

(19)

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$\frac{i}{\partial t} \phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\} \right]$$

$$(\phi \leftrightarrow \phi) \Rightarrow \int d^3x' \left(i\delta^{(3)}(x-x') \pi'(x,t) \right)$$

\rightarrow only continual term is 1^{st} .

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t} \phi(x,t) = \pi(x,t)}$$

and

$$\frac{i}{\partial t} \pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\pi)^2 + \frac{1}{2}m^2\phi^2 \right\} \right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x') (-\nabla^2 + m^2) \phi(x',t) \right)$$

(integrate by parts here)

$$= -i(-\nabla^2 + m^2) \phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \pi(x,t) = (m^2 - \nabla^2) \phi(x,t)}$$

Combining these 2 results we get ...

$$\boxed{\frac{\partial^2}{\partial t^2} \phi(x,t) = (\nabla^2 - m^2) \phi(x,t)}$$

\hookrightarrow rearranging this gives

$$\boxed{(\nabla^2 + m^2) \phi(x,t) = 0} \rightarrow \text{just the KG eqn...}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

Recall: $H_{\text{ap}} = a_p^{\dagger} (H - E_p) \rightarrow$ from comm. rule -

\Rightarrow (proof by induction)

$$H^n a_p = a_p (H - E_p)^n$$

Similarly,

$$H^n a_p^{\dagger} = a_p^{\dagger} (H + E_p)^n$$

\rightarrow So, we have

$$e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t} \rightarrow \text{from above...}$$

and

$$e^{iHt} a_p^{\dagger} e^{-iHt} = a_p^{\dagger} e^{+iE_p t}$$

\rightarrow Now -- we want to write $\phi(x, t)$ in terms of these operators. (since $\phi(x)$ is a comb of a & a^{\dagger})

$\pi(x)$
we know that $\phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before ...

$$\phi(x) = \phi(x, 0) = \int \frac{dp}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ip \cdot x} + a_p^{\dagger} e^{-ip \cdot x})$$

substitute this into $\phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$ we find

(21)

$$\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-ip \cdot x} + a_p^+ e^{ip \cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow p \cdot x = E_p \cdot x^0 - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p} = E_p t - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p}$$

So,

$$\boxed{\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p} \cdot \vec{x}} + a_p^+ e^{+i\vec{p} \cdot \vec{x}} \right\} \Big|_{p^0 = E_p}}$$

Similarly, we can find

$$\boxed{\Pi(x, t) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} (a_p e^{-i\vec{p} \cdot \vec{x}} - a_p^+ e^{+i\vec{p} \cdot \vec{x}}) \Big|_{p^0 = E_p}}$$

from defn or from $\Pi(x, t) = \frac{\partial}{\partial t} \phi(x, t)$.

Note we can also do everything, but starting from P and not H . But we won't worry about that.

 **Causality** Note that causality is broken when without the presence of fields.

→ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture --

-- the amplitude for a particle to propagate from $y \rightarrow x$ is given by

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

"Two-point correlation function"

Let $\boxed{D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle}$

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \langle 0 | \text{in terms of } a_p^+ a_q^- | 0 \rangle$$

$$= \langle 0 | a_p^+ a_q^- | 0 \rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)}$$

More explicitly --

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$\begin{cases} \textcircled{2} p' = \vec{p} \\ p'_0 = E_{p'} \end{cases}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | e^{-ipx} e^{ip' y} a_p^+ a_{p'}^- | 0 \rangle$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{1}{\sqrt{2E_p}} \right) \left(\frac{1}{\sqrt{2E_{p'}}} \right) e^{-ip \cdot x - ip' \cdot y} (2\pi)^3 \delta^{(3)}(p-p')$$

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip \cdot (x-y)} \quad \square$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$.

(1) Suppose that $x-y = (t, \vec{v}, 0, 0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t}$$

$$(\text{timelike}) = \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2} e^{-iEt}$$

$$= e^{-imt} \xrightarrow{\text{dominated by region above}} \text{dominated by region above}$$

$t - i\omega$

$p \approx 0 -$

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

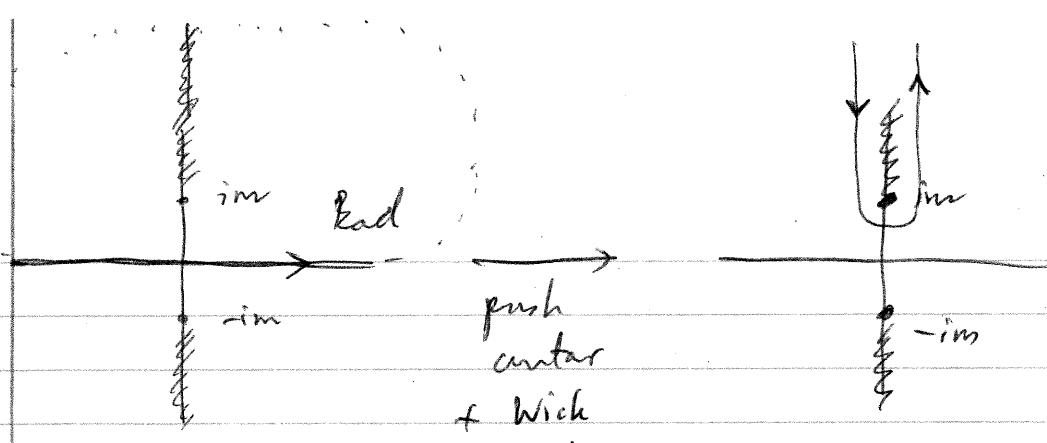
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p} \cdot \vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2E_p} \frac{e^{ipr} - e^{-ipr}}{2ipr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ tim (singularity) ...

→ must change contour... \rightarrow which rotate



To get

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty d\rho \frac{\rho e^{-i\rho r}}{\sqrt{\rho^2 - m^2}} \quad (\text{Wick rotate})$$

$$\Rightarrow \boxed{\mathcal{D}(x-y) = \frac{e^{-mr}}{r}} \rightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^m (x-y)_m = -(\vec{x}-\vec{y})^2 < 0$
is spacelike, cannot have causality between
 $x-y$.

$\mathcal{D}(x-y) \neq 0 \Rightarrow ??? \text{ paradox?}$

$\rightarrow \underline{\text{No!}}$ To discuss causality, we should ask not whether particles can propagate over spacelike intervals --

-- but whether a "measurement" at one point can affect a measurement at another point where separation from the first is spacelike --

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement $\phi(x)$, call this $\phi(x)$. or a local measurement $\phi(y)$, called $\phi(y)$

So long as $[\phi(x), \phi(y)] = 0$, the 2 measurements don't affect one another.

→ measure the field $\phi @ x + @ y$,

If $[\phi(x), \phi(y)] = 0$ when $(x-y)^2 < 0$ then we've good

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} \right. \\ \left. + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\}$$

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \times \left\{ (a_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}), \right. \\ \left. (a_p^\dagger e^{-ip \cdot y} + a_p e^{ip \cdot y}) \right]$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} \right. \\ \left. + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\} \\ (2\pi)^3 \delta^3(p - p') \quad -(2\pi)^3 \delta^3(p - p')$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2E_p} \right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

$$\text{So } [\phi(x), \phi(y)] = D(x-y) - D(y-x).$$

Since $D(y-x)$ is Lorentz invariant, can take

$$y-x \rightarrow x-y. \text{ (possible b/c } (x-y)^2 < 0)$$

Note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

\rightarrow so this is why possible because $(x-y)^2 < 0$
(negative).

$$\text{So } D(y-x) = D(x-y)$$

$$\text{So, } [\phi(x), \phi(y)] = 0 \quad \checkmark$$

~~24~~

~~The Klein-Gordon Propagator~~

Let's look at $[\phi(x), \phi(y)]$ in more details..

$[\phi(x), \phi(y)]$ is just a number

~~can write~~ $[\phi(x), \phi(y)] \equiv \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$

$$\rightarrow \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right\}$$

$$\Delta \text{ Poles}$$

$$E_p^2 = m^2$$

$$p^0 = \pm E_p$$

(assuming $x^0 > y^0$)

$$= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip \cdot (x-y)} \right|_{p^0=E_p} - \frac{1}{2E_p} e^{-ip \cdot (x-y)} \right\}_{p^0=-E_p}$$

= E_0

The Klein-Gordon Propagator

→ Before looking at this -- need to look at Green's Function & Contour Integrals --

→ Consider function

$$\frac{1}{(p_0^0 - E_p)(p_0^0 + E_p)} \quad E_p = \sqrt{|p|^2 + m^2}.$$

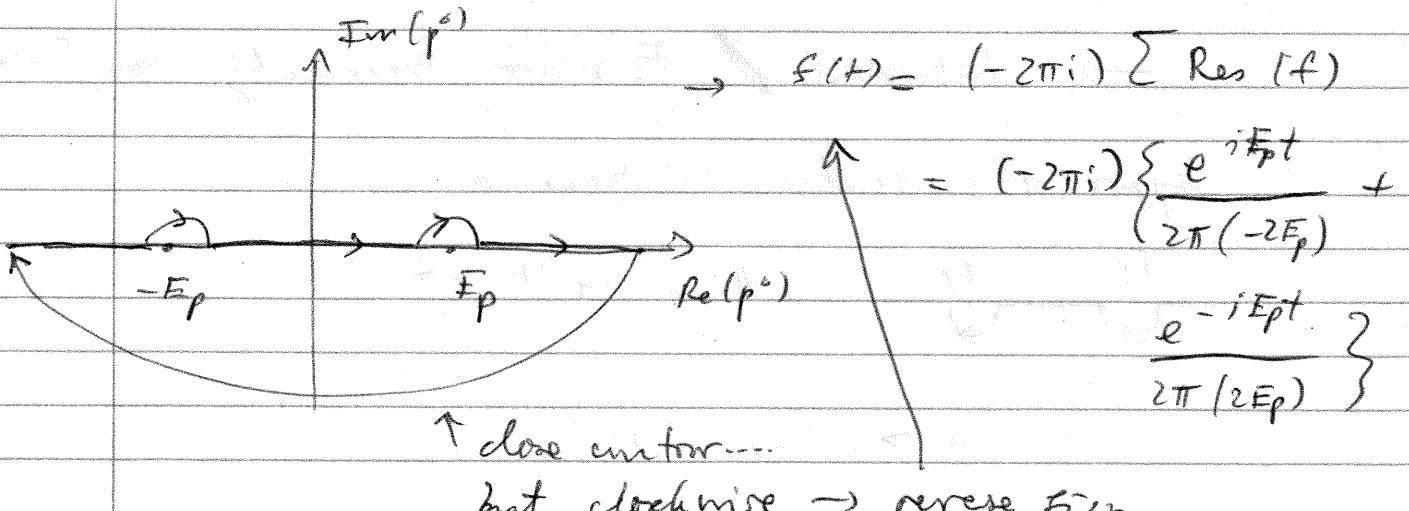
→ Poles at $p_0^0 = \pm E_p$.

Lab at FT:

$$f(t) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{e^{-ip^0 t}}{(p^0 - E_p)(p^0 + E_p)}$$

→ How to integrate this?

If $t > 0 \rightarrow$ ~~crosses poles~~



$$\rightarrow f(t) = \frac{-i}{2E_p} (e^{+iE_p t} - e^{-iE_p t}) \quad (t > 0)$$

If $t < 0$ close contours above poles



$$\rightarrow f(t) = 0.$$

→ So, altogether, we have ...

$$\begin{aligned} f(t) &= \int \frac{1}{2\pi} \frac{dp^0}{(p_0^0 E_p)(p^0 + E_p)} e^{-ip^0 t} \\ &= \Theta(t) \left(\frac{-i}{2E_p} \right) (e^{-iE_p t} - e^{+iE_p t}) \end{aligned}$$

where $\Theta(t)$ is the Heaviside Step fn ...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

→ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



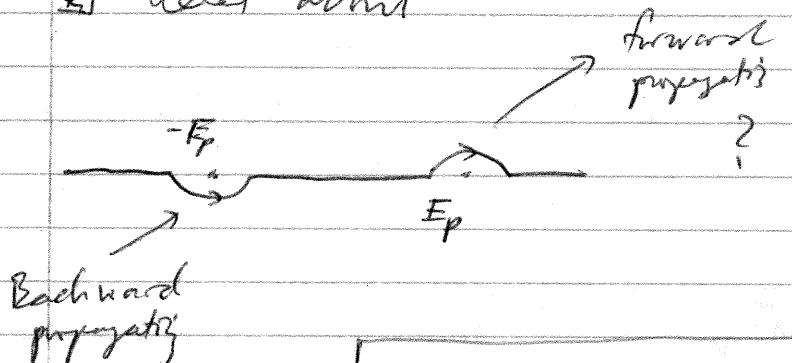
then we'll get

$$t > 0 \rightarrow f(t) = 0$$

$$t < 0 \rightarrow f(t) = \Theta(-t) \frac{i}{2E_p} (e^{-iE_p t} - e^{+iE_p t})$$

→ Advanced / Backward Propagating Green's fn.

What about



$$\rightarrow \boxed{f(t) = \Theta(+)(-\frac{i}{2E_p}) e^{-iE_pt} + \Theta(-)(-\frac{i}{2E_p}) e^{+iE_pt}}$$

Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$

Consider this quantity...

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip(x-y)} + \frac{1}{-2E_p} e^{-ip(x-y)} \right\} \\ &\quad \uparrow \quad \downarrow \\ &\quad \text{pole} \quad \text{pole @} \\ &\quad @ p_0 = E_p \quad p_0 = -E_p \end{aligned}$$

$$\begin{aligned} \text{(41) integral} \rightarrow &= \int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \underbrace{\frac{-i}{p^0 - m^2}}_{f(+)} e^{-ip(x-y)} \\ & \quad \underbrace{\qquad \qquad \qquad}_{\text{f(+)} \text{ before, where}} \end{aligned}$$

$$(p^0 - E_p)(p^0 + E_p) = p^{0^2} - |p|^2 - m^2 = p^2 - m^2$$

Now, call ...

$$D_R(x-y) = \theta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle$$

Then

$$\begin{aligned}
 \rightarrow (\square + m^2) D_R(x-y) &= \square D_R(x-y) + m^2 D_R(x-y) \\
 &= (\square \theta(x^0 - y^0)) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + 2(\partial_\mu \theta(x^0 - y^0)) \partial^\mu \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + \theta(x^0 - y^0) (\square + m^2) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &= -\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \quad \text{cancel} \\
 &\quad + 2\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle + 0 \\
 &= \delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \quad \text{milds} \\
 &= -i \delta^{(4)}(x-y) \quad \text{renormalization} \\
 &\quad \downarrow \quad \text{(easy)} \\
 &\quad -i \delta^{(3)}(x-y)
 \end{aligned}$$

So

$$(\square + m^2) D_R(x-y) = -i \delta^{(4)}(x-y)$$

$\rightarrow D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0 @ x^0 < y^0$

$\Rightarrow D_R(x-y) \equiv \text{"Retarded" Green's fn}$

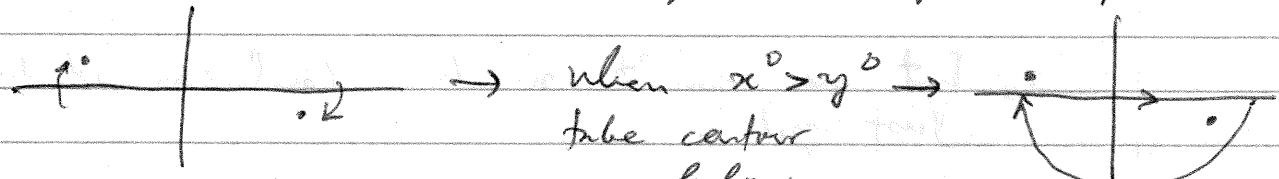
Now ... As we have seen, there are many ways to take the contour ...



→ Use the Feynman prescription instead

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

Convenient! B/c now poles are $p^0 = \pm(E_p - i\epsilon)$



when $x^0 < y^0 \rightarrow$
take contour above.

→ get same expression
but with $x \leftrightarrow y$.

So,

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \Theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \Theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle$$

$$= \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

"time-ordering" symbol -

Time-ordering symbol \Rightarrow instructs us to place the operators & heat follows in order with the latest to the left.

\rightarrow apply $(D + m^2)$ to last line, set D_F is Green's fn of Klein-Gordon Operator,

$$() \quad \overbrace{\hspace{10em}}^{\text{---}}$$

$D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator--

\hookrightarrow propagation amplitude

\rightarrow But we can't much calculation at this point just yet.

\rightarrow B/c we've only looked at the free K-G theory

\rightarrow Field eqn in this case is linear : there are no interactions--

\rightarrow this theory is too simple to make any predictions--

\rightarrow need perturbation --

One kind of interaction it's can also be solved



Particle Creation by a classical Source

Consider the source $j(x)$

Result... free field: $(D + m^2)\phi = 0$

→ now... $(D + m^2)\phi = j(x)$ Field ϕ is
 ↗ space time.

$j(x)$ is nonzero only for a finite time interval

The associated lagrangian is

$$L = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is
 enough to solve

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip_i x} + a_p^+ e^{ip_i x})$$

With a source...

$$\phi(x) = \phi_0(x) + i \int d^3y D_R(x-y)j(y)$$

We won't worry about this for now...

Some problems & Insights

(1) Classical EM (no sources) follow from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \epsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \quad \boxed{(\vec{\nabla} \cdot \vec{E} = 0) \quad (\nu = 0)}$$

\downarrow

$$\boxed{-\partial_t \vec{E} + \vec{\nabla} \times \vec{E} = 0} \quad (\nu = i)$$

(2) Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi^- - m^2 \phi^+ \phi^- \right)$$

Derive E-L eqn:

$$\boxed{i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ = 0}$$

$$\boxed{-i\partial_t \phi^- - \frac{1}{2m} \nabla^2 \phi^- = 0}$$

Now... write $\phi \rightarrow e^{-i\theta} \phi$, $\phi^+ \rightarrow e^{i\theta} \phi^+$

$$\begin{aligned} &\sim \phi - i\theta \phi \\ &\rightarrow \Delta \phi \sim -i\theta \end{aligned}$$

$$\begin{aligned} &\sim \phi^+ + i\theta \phi^+ \\ &\Delta \phi^+ \sim i\theta \end{aligned}$$

So that

$$j^{\mu} = \frac{\partial f}{\partial (\partial_{\mu} \phi)} \Delta \phi + \frac{\partial f}{\partial (\partial_{\mu} \phi^+)} \Delta \phi^+$$

↑
conserved current -

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial f}{\partial (\partial_x \phi)} \rightarrow \dots \text{conjugate\dots}$$

→ can get Hamiltonian → there's a formula in book,
but we worry abt this.

3) If we take $(x-y)^2 = -r^2 \rightarrow$ can implicitly write

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2 \rightarrow D(x-y)$ can be written in terms of Bessel Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation...

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

→ what happens to $\phi(x)$ under Λ ?

we require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform -- $\partial_\mu \phi(x) \rightarrow \partial_\mu (\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu (\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\rho_\mu (\partial_\rho \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho_\mu (\Lambda^{-1})^\sigma_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu \phi(x))^2 \rightarrow g^{\mu\nu} (\partial_\mu \phi'(x)) (\partial_\nu \phi'(x))$$

$$= g^{\mu\nu} [\partial_\mu (\phi(\Lambda^{-1}x))] [\partial_\nu (\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu} \{ (\Lambda^{-1})^\rho_\mu \partial_\rho \phi \} \{ (\Lambda^{-1})^\sigma_\nu \partial_\sigma \phi \} (\Lambda^{-1}x)$$

$$= (\partial_\mu \phi)^2 (\Lambda^{-1}x) \leftarrow = g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi (\Lambda^{-1}x) \rightarrow \text{evaluated}$$

(37)

Exercise

$$\phi(x) \rightarrow \phi(\tilde{x})$$

$$\partial_\mu \phi(x) \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)(\tilde{x})$$

$$(\partial_\mu \phi(x))^2 \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)^2 (\tilde{x})^\nu$$

So it is clear that

$$L \rightarrow L(\tilde{x})$$

↑

Lagrangian is Lorentz-invariant.

→ The action $S = \int d^4x L$ is also Lorentz inv.

→ also clear that EOM is also Lorentz inv.

$$\begin{aligned} (\square + m^2) \phi(x) &= (\tilde{x})^\mu \partial_\mu (\tilde{x})^\nu \partial_\nu + m^2 \phi(\tilde{x}) \\ &= (\partial^\mu \partial_\mu + m^2) \phi(\tilde{x}) \\ &= 0 \quad \checkmark \end{aligned}$$

20

Lep 10, 2020

→ How do we find Lorentz-invariant theories, in general?

→ For simplicity, restrict attention to lin. transf

→ $\phi_a = \phi \in \mathbb{C}^n$, → matrix giving Lorentz transf in A .

$$\rightarrow \boxed{\Phi_a(x) \rightarrow M_{ab}(A) \Phi_b(\tilde{x})}$$

$n \times n$

The

→ most general nonlinear draft can be built
out of linear ones \Rightarrow suffices to consider M
only.

↳ for short, write $\phi \mapsto M(\alpha) \phi$.

→ What are the possible allowed $M(\alpha)$?

◻ $\{M(\alpha)\}$ form a group $M(\alpha') M(\alpha) = M(\alpha')$
 $\curvearrowright \alpha'' \alpha = \alpha'$

→ the correspondence between $\alpha \in M$ must be
preserved under multiplication.

$\{1\}$ Lorentz group $\longleftrightarrow \{M(\alpha)\} \rightarrow$ n-dim
representation of the
Lorentz group

↳ [?] What are the finite-dim matrix reps
of the Lorentz group?

Ex in $\mathfrak{so}(4)$... spin $\frac{1}{2} \rightarrow \{M\}$ are the 2×2 unitary
matrices with determinant 1

$$\rightarrow \boxed{U = e^{-i\vec{\theta}^i \vec{\sigma}^i/2}} \rightarrow \{\vec{\theta}^i \vec{\sigma}^i/2\}$$

$\vec{\theta}^i$
 $\vec{\sigma}^i$

3 arbitrary parameters
& Pauli matrices.

$$\{U(\vec{\theta}) : e^{-i\vec{\theta}^i \vec{\sigma}^i/2}\}$$

→ In the case for arbitrary spin representations...

$$U(\vec{\theta}) = e^{-i \vec{\theta} \cdot \vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i \sum_l \epsilon^{ijk} J^l$$

→ Check that this works for spin $\frac{1}{2}$:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2} \right] = i \sum_l \epsilon^{jkl} \frac{\sigma^l}{2} \quad \checkmark$$

→ for spinless particles... $\psi(\vec{x})$ can be decomposed into orbital angular momentum states. $J=0, 1, 2, \dots$
(no intrinsic spin $\Rightarrow J=L$)

$$\bullet \quad \vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i \vec{\nabla})$$

$$\bullet \quad J^j = i \sum_l \epsilon^{jkl} x^k \nabla^l$$

$$\bullet \quad \nabla^l = -\partial_x^l = -\frac{\partial}{\partial x^l}$$

But the cross product is special to 3D case.

→ write operators in antisymmetric tensor...

$$J^{ij} = -i(x^i \partial^j - x^j \partial^i) \quad \rightarrow \text{represents free cross product.}$$

$$\text{so that } J^3 = J^{12}, \text{ etc.}$$

→ generate to 4D: → 6 operators that generate 3 boosts, 3 rotations,

$$J^{\mu\nu} = +i(x^\mu \partial^\nu - x^\nu \partial^\mu) \quad \text{of the Lorentz group.}$$

$\{ \rightarrow$ Spatial Rotations: $J^{ik} = i(x^i \partial^k - x^k \partial^i)$

\rightarrow Lorentz boosts along x^j axis: $J^{ij} = i(x^i \partial^j - x^j \partial^i)$

\rightarrow Now, want to get commutation rules.

\rightarrow compute the commutators of differential ops

to get

$$[J^{mn}, J^{pq}] = i(g^{rp}J^{m\sigma} - g^{n\sigma}J^{m\rho} - g^{n\sigma}J^{mp} + g^{m\sigma}J^{np})$$

$$\left. \begin{array}{l} \text{Ex 1 rotations: } J^{12} = -J^{21} \\ J^{23} = -J^{32} \\ J^{13} = -J^{31} \end{array} \right\} \Rightarrow 6 \text{ total metrics...}$$

$$\left. \begin{array}{l} \text{3 boostr} \\ J^{01} = -J^{10} \\ J^{02} = -J^{20} \\ J^{03} = -J^{30} \end{array} \right\}$$

Ex Consider the 4×4 matrix $(J^{mn})_{\alpha\beta}$ where μ, ν label which of the 6 metrics, while α, β label the component/matrix element...

$$(J^{mn})_{\alpha\beta} = i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu)$$

\hookrightarrow can verify that $(J^{mn})_{\alpha\beta}$ satisfies the comm. relation...

\rightarrow These are matrices that act on ordinary Lorentz 4-vectors...

to see this...

→ Look at elements of the Lorentz group

$$U(w_{\mu\nu}) = \exp \left[-i \frac{\omega_{\mu\nu}}{2} J^{\mu\nu} \right]$$

infinitesimally \rightarrow

$$\begin{aligned} & \sim \mathbb{I} + \frac{-i}{2} w_{\mu\nu} J^{\mu\nu} \\ & \sim \delta_p^\alpha + \frac{-i}{2} w_{\mu\nu} (J^{\mu\nu})_p^\alpha \end{aligned}$$

So, infinitesimally...

$$V^\alpha \rightarrow \delta_p^\alpha + \frac{-i}{2} w_{\mu\nu} (J^{\mu\nu})_p^\alpha V^\beta$$

$w_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles.

$V_\alpha, V_\beta \rightarrow 4$ -vectors..

Ex 1 When $w_{12} = -w_{21} = \theta$, $w_{\mu\nu} = 0$ else, we get

$$[V^\mu] \rightarrow \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [V^\mu]$$

→ Infinitesimal ROTATION on xy plane.

Ex 2 when $w_{01} = -w_{10} = \beta \Rightarrow$ get
 $w_{\mu\nu} = 0$ else

$$[V^\mu] \rightarrow \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} [V^\mu] \rightarrow \boxed{\text{BOOST along } x}$$

THE DIRAC EQUATION

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to develop formalism for finding all other ~~formalisms~~ representations...
(problem 3.1)

→ focus on spin $\frac{1}{2}$ systems...

→ In this case, use Dirac's trick due to -

Suppose we had a set of 4 $n \times n$ matrices γ^{μ} satisfying:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\mathbf{I}$$

Then we could write down an n -dim representation of the Lorentz algebra...

$$S^{\mu\nu} = \frac{i}{4} [\gamma^{\mu}, \gamma^{\nu}]$$

These matrices satisfy the commutation relation...

$$\rightarrow [S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

→ Verify that this trick works in 3D Euclidean space

in which case, $\gamma^0 = \gamma^5$ → $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \sum_k i \epsilon^{ijk} \sigma^k = J^i$$

Which is what we saw before as angular momentum.

$$\left\{ J^1 = S^{12} = \frac{1}{2} \sigma^3 \right\}$$

$$\left\{ J^2 = S^{31} = \frac{1}{2} \sigma^2 \right\}$$

$$\left\{ J^3 = S^{23} = \frac{1}{2} \sigma^1 \right\}$$

→ now, want S^{mn} for 4D Minkowski space...

→ Matrices γ^m must be at least 4×4 .

→ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2 \times 2} \\ \mathbb{I}_{2 \times 2} & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

"Weyl" / "Chiral" representations.

→ In this case, the boost + rotation generators are ..

Boots
in

$$S^{0i} = \frac{i}{4} [\gamma^0, \gamma^i] = -\frac{i}{2} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Rotations
in

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \sum_k \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \sum_k \frac{1}{2} \epsilon^{ijk} \sum_l \sigma^l$$

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at all transformations under which the ~~law~~ ^{Action's} ~~gives~~ ~~it~~ is invariant

→ In particular, we want \mathcal{S} to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\left\{ \begin{array}{l} \phi(x) \rightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x^\mu) \rightarrow \phi(\Lambda^{\mu\nu} x^\nu). \end{array} \right. \rightarrow \begin{array}{l} \text{check that} \\ \mathcal{S} \text{ is invariant} \end{array}$$

→ But this is very simple ... ⇒ There are many more transformations that leave \mathcal{S} Lorentz invariant.

→ How do we find all of them?

→ For simplicity, we'll just restrict ourselves to linear combinations of transformations

→ Look at transformations of the form

$$\phi_a(x) \rightarrow \sum_b M_{ab}(\Lambda) \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \rightarrow M(\Lambda) \phi}$$

These matrices M must be "nice" in the sense that M must obey...

Ex

$$\boxed{\phi \rightarrow M(\Lambda') M(\Lambda) \phi = M(\Lambda' \Lambda) \phi}$$

This says that $\{M\}$ (the collection of M 's) must be a representation of the Lorentz group.

What?? So, recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

 \rightarrow

$$\boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

of a group to

A representation Π is a function π satisfying the property

$$\pi(g_1) \pi(g_2) = \pi(g_1 g_2)$$

 \uparrow \uparrow \uparrow g_1 $\in G$ $\in G$

With this, it is clear that

$$\boxed{\{\Lambda\} \text{ Lorentz group} \Rightarrow \{M\} \text{ is a representation of } \{\Lambda\}}$$

So... what are these M ?

 \rightarrow Ex

Rotation group for spin $1/2$ particles

For spin - $\frac{1}{2}$, the most important nontrivial representation is the 2D representation:

→ These are unitary matrices with $\det = 1$
 (2×2)

$$\Rightarrow \text{In general: } U = e^{-i \vec{\sigma} \cdot \vec{\theta}/2}$$

$\vec{\sigma}$ → Pauli matrices
 $\vec{\theta}$ → angle.

For infinitesimal rotations, we can write

$$U = I - i \frac{\vec{\sigma}}{\hbar} \cdot \vec{\theta} = I - \vec{\tau} \cdot \vec{\theta}$$

{U} form a Lie-algebra of the L-group.

$\vec{\tau}$ here are the "generators" of the Lie algebra

when {U} is a representation of the rotational group, we identify

$$\vec{\tau} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{\tau}$ is the quantum angular momentum operator

→ satisfies the commutation relation

$$[\vec{\tau}^i, \vec{\tau}^j] = i \epsilon^{ijk} \vec{\tau}^k$$

like the generators of $SO(3)$, namely the Pauli matrices -

→ finite rotations are formed by matrix exp.

$$R = \exp\left[-i\theta^i \hat{J}^i\right]$$

Angular momentum

Sep 27, 2020

Back to present problem...

to get generator of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

$$(3D) \quad \vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\sigma})$$

$$(4D) \quad \boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})}$$

→ any matrices that are to represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu)}$$

→ by symmetry, μ, ν take label which of the six matrices we want;

→ α, β label components.

The Dirac Eqn.

What are the representations of the Lorentz group?
especially for spin- $\frac{1}{2}$?

Dirac's trick: if we have a set of $4 \times n \times n$ matrices γ^μ which satisfies:

Dirac algebra

$$\rightarrow \boxed{\{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\gamma^{\mu\nu} \star I_{n \times n}}$$

Then the n -dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4} \{ \gamma^\mu, \gamma^\nu \}}$$

\rightarrow In other words, $S^{\mu\nu}$ satisfies:-

$$[\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = i(g^{\mu\rho} S^{\nu\sigma} - g^{\nu\rho} S^{\mu\sigma} - g^{\mu\sigma} S^{\nu\rho} + g^{\nu\sigma} S^{\mu\rho})$$

* Note that this trick works also in any dim.

e.g. take $\gamma^0 = i\sigma^3$ so that $\{ \gamma^i, \gamma^j \} = -2\delta^{ij}$

$$\Rightarrow \boxed{S^{ij} = \frac{1}{2} \epsilon^{ijk} S^k} \rightarrow \text{just as before.}$$

2D representation of the rotation group.

$$\text{Spin } \frac{1}{2}: J^1 = J^{12} = \frac{1}{2}\sigma^3; J^2 = \frac{1}{2}\sigma^2 = S^{21}; J^3 = S^{23} = \frac{1}{2}\sigma^1$$

One such representation for the Dirac algebra is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4 \times 4}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4 \times 4}$$

Weyl / chiral representation.

get

Boosts $S^{0j} = \frac{i}{4} [\gamma^0, \gamma^j] = \frac{-i}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} 0 & \sigma^k \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \epsilon^{ijk} \epsilon^{lk}$$

Hermitian Def'n

not ~~rotation~~
but Ψ is also

classical

field, not a
wfn

All 4-component field Ψ that transforms under boosts + rotations according to \rightarrow is called a Dirac spinor

S^{0j} are Hermitian

S^{ij} are anti-Hermitian

∴ fine b/c Ψ is a classical field, not a wfn.

Now, what is the field eqn for ψ ?

→ try $(\Box + m^2)\psi = 0 \leftarrow \text{KG field eqn.}$

But this obviously works because the representations are block-diagonal...

→ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~at~~ at transformation of δ matrices

In an expression we can think of...

$$[\dots] \Delta_{\frac{1}{2}} [4 \times 4] \Delta_{\frac{1}{2}} [\cdot, \cdot] \xrightarrow{\frac{1}{2} \text{ for spin } \frac{1}{2}}$$

where $\Delta_{\frac{1}{2}} = \exp \left\{ \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right\}$

$$\simeq 1 - \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}$$

$$\Rightarrow [\gamma^1] \rightarrow [\Delta_{\frac{1}{2}}] [\gamma^1] [\Delta_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right) \gamma^1 \left(1 - \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right)$$

$$= \dots \quad (\text{some terms of higher order cancelled...})$$

$$= \gamma^1 - \frac{i}{2} \omega_{\alpha\beta} \underbrace{[\gamma^1, S^{\alpha\beta}]}_{?}$$

above a quick computation shows that

$$[\gamma^\mu, \gamma^\nu] = (\gamma^{\mu\nu})_\nu \gamma^\nu$$

where

$$\gamma^{\mu\nu} = i(g^{\mu\nu}\gamma_5 - g^{\nu\mu}\gamma_5)$$

So ...

$$\boxed{\gamma^\mu \rightarrow \gamma^\mu - \frac{i}{2} \omega_{\alpha\beta} (\gamma^{\alpha\beta})_\nu \gamma^\nu = \Gamma_{\frac{1}{2}} \gamma^\mu \Gamma_{\frac{1}{2}}}$$

$\rightarrow \gamma^\mu$ transforms like 4-vectors ... !

$\Rightarrow \gamma^\mu$ are invariant under simultaneous rotations of
their vectors & spinor indices.

I can treat " μ " or γ^μ as a vector index!

\rightarrow can dot γ^μ into ∂_μ to form a Lorentz-

inv. differential operator ...

Dine eqn

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

check that this is Lorentz-inv:

Lit $\psi(x) \rightarrow \Gamma_{\frac{1}{2}} \psi(\Gamma' x)$ then

$$i\gamma^\mu \partial_\mu \psi \rightarrow (i\gamma^\mu \Gamma_{\frac{1}{2}}) \partial_\mu (\psi(\Gamma' x))$$

$$= i\Gamma_{\frac{1}{2}} (\Gamma_{\frac{1}{2}} \gamma^\mu \Gamma_{\frac{1}{2}}) \cdot (\Gamma')^\mu_\mu (\partial_\mu \psi)(\Gamma' x)$$

some Lorentz transform

$$\begin{aligned}
 &= i \Delta_{\frac{1}{2}} (\Delta)^{\mu}_{\nu} \gamma^{\nu} \cdot (\Delta)_{\mu}^{\alpha} (\partial_2 \psi)(\Delta' x) \\
 &= i \Delta_{\frac{1}{2}} \gamma^{\mu} \underbrace{(\Delta)^{\mu}_{\nu} (\Delta)_{\nu}^{\alpha}}_{\delta^{\alpha}_{\nu}} (\partial_2 \psi)(\Delta' x) \\
 &= i \Delta_{\frac{1}{2}} \gamma^{\mu} \partial_{\mu} \psi(\Delta' x)
 \end{aligned}$$

$$\Rightarrow i \gamma^{\mu} \partial_{\mu} \psi(x) \rightarrow \Delta_{\frac{1}{2}} i \gamma^{\mu} \psi(\Delta' x)$$

→ transforms the same way as $\psi(\Delta' x)$

Cleaner way:

$$\begin{aligned}
 \text{Let } [i \gamma^{\mu} \partial_{\mu} - m] \psi(x) &\rightarrow [\overbrace{i \gamma^{\mu} (\Delta')_{\mu}^{\nu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \psi(\Delta' x)] \Delta_{\frac{1}{2}} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \Delta_{\frac{1}{2}}^{-1} [\overbrace{i \gamma^{\mu} (\Delta')_{\mu}^{\nu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \psi(\Delta' x)] \Delta_{\frac{1}{2}} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \left\{ i \gamma^{\mu} \overbrace{(\Delta')_{\mu}^{\nu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \right\} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \left\{ i \gamma^{\nu} \partial_{\nu} - m \right\} \psi(\Delta' x) \\
 &= 0 \quad \checkmark
 \end{aligned}$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i \gamma^{\mu} \partial_{\mu} - m) \psi$$

$$\begin{aligned}
 \rightarrow 0 &= (-i \gamma^{\mu} \partial_{\mu} - m) (+i \gamma^{\nu} \partial_{\nu} - m) \psi \\
 &= (\gamma^{\mu} \gamma^{\nu} \partial_{\nu} \partial_{\mu} + m^2) \psi = ...
 \end{aligned}$$

$$\begin{aligned}
 &= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2) \psi \\
 &= \left[\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2 \right] \psi \\
 &= \left[\frac{1}{2} \{ \gamma^\mu \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right] \psi \quad \xrightarrow{\text{LG eqn.}} \\
 &= (g^{\mu\nu} \partial_\mu \partial_\nu + m^2) \psi = (D + m^2) \psi = 0
 \end{aligned}$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger \psi$ doesn't work b/c under a boost,

$$\psi^\dagger \frac{1}{\sqrt{2}} \Delta_{\frac{1}{2}} \psi \neq \psi^\dagger \psi \text{ since } \frac{1}{\sqrt{2}} = \exp \left\{ -i \omega \gamma^\mu S^\mu \right\}$$

not unitary ... since not all $S^{\mu\nu}$ are Herms.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, set

$$\bar{\psi} \rightarrow \bar{\psi} \frac{1}{\sqrt{2}} \gamma^0 \simeq \bar{\psi} \left(1 + i \frac{1}{2} \omega_{\mu\nu} (S^{\mu\nu})^+ \right) \gamma^0$$

when ~~assume~~ $\omega_0 \neq 0 \Rightarrow \omega \neq 0$, $(S^{\mu\nu})^+ = (S^{\mu\nu})^-$

$$i (S^{\mu\nu} \leftrightarrow \gamma^0)$$

When $\mu=0$ or $\nu=0$, $(S^{\mu\nu})^+ = -S_{\mu\nu}^\mu$

$S^{\mu\nu}$ anti-commutes w/ γ^0 .

$$\rightarrow \bar{\psi} \rightarrow \psi^+ \left(1 + \frac{i}{2} \gamma_\mu \nu (S^{\mu\nu})^+ \right) \gamma^0$$

$$= \underbrace{\psi^+}_{\gamma^0} \gamma^0 \left(1 + \frac{i}{2} \gamma_\mu \nu S^{\mu\nu} \right)$$

$$= \bar{\psi} \left(1 + \frac{i}{2} \gamma_\mu \nu S^{\mu\nu} \right) = \bar{\psi} \gamma_1^{-1} \text{ as desired.}$$

$$\rightarrow \boxed{\bar{\psi} \rightarrow \bar{\psi} \gamma_1^{-1}}$$

and so $\boxed{\bar{\psi} \psi = \psi^+ \gamma^0 \psi}$ is a Lorentz scalar.

Similarly, can show that

$$\boxed{\bar{\psi} \gamma^\mu \psi}$$
 is a Lorentz vector.

\rightarrow the correct Lorentz-invariant Dirac Lagrangian is

$$\boxed{L_{\text{Dirac}} = \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi}$$

{-L eqn for $\bar{\psi}$ gives $(\gamma^\mu \partial_\mu - m) \psi = 0$

{-L eqn for ψ gives $-i \partial_\mu \bar{\psi} \gamma^\mu - m \bar{\psi} = 0$

WEYL SPINOR

Recall that

$$\begin{aligned} S^{0j} &= \frac{-i}{2} \begin{pmatrix} \sigma^i & \alpha \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & \alpha \\ 0 & \sigma^k \end{pmatrix} \end{aligned}$$

Since block-diagonal \Rightarrow Dirac representation of the Lorentz group is reducible.

\rightarrow Can form 2-D representations by considering each block separately.

$$\rightarrow \text{write } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{\text{left-handed Weyl spinors}}$$

Under infinitesimal boost $\vec{\beta}$ + rotation $\vec{\theta}$, these transform as

$$\psi_L \rightarrow \left(1 - i \vec{\theta} \cdot \vec{\sigma}/2 - i \vec{\beta} \cdot \vec{\sigma}/2 \right) \psi_L$$

$$\psi_R \rightarrow \left(1 - i \vec{\theta} \cdot \vec{\sigma}/2 + i \vec{\beta} \cdot \vec{\sigma}/2 \right) \psi_R$$

$$\text{Recall that } (\tanh(\vec{\beta}) = \frac{1+i}{i})$$

\rightarrow Transf of ψ_R is equiv to transf of ψ_L^\pm

By writing down

$$\psi_L^* \rightarrow \left(1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}^*}{2} - \vec{\beta} \cdot \frac{\vec{\sigma}^*}{2} \right) \psi_L^*$$

noting that $\vec{\sigma}^* \vec{\sigma}^* = -\vec{\sigma} \vec{\sigma}$ ($\vec{\sigma}^2 = \vec{\sigma}^2$)

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we find.

$$\vec{\sigma}^2 \psi_L^* \rightarrow \vec{\sigma}^2 \left[1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}}{2} - \vec{\beta} \cdot \frac{\vec{\sigma}^*}{2} \right] \psi_L^*$$

$$= \left[1 - i\vec{\sigma} \cdot \frac{\vec{\sigma}}{2} + \vec{\beta} \cdot \frac{\vec{\sigma}}{2} \right] \psi_L^*$$

like ψ_R transform.

$\underline{\text{So }} \vec{\sigma}^2 \psi_L^*$ transform like ψ_R ..

With $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, the Dirac eqn has form.

$$(i\vec{\sigma}^m \partial_m - m) \Psi = 0 \Leftrightarrow \begin{pmatrix} -m & i(\vec{\sigma}_0 + \vec{\sigma} \cdot \vec{\beta}) \\ i(\vec{\sigma}_0 - \vec{\sigma} \cdot \vec{\beta}) & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

When $m=0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left\{ \begin{array}{l} i(\vec{\sigma}_0 - \vec{\sigma} \cdot \vec{\beta}) \psi_L = 0 \\ i(\vec{\sigma}_0 + \vec{\sigma} \cdot \vec{\beta}) \psi_R = 0 \end{array} \right\} \rightarrow \underline{\text{Welfl eqns.}}$$

\rightarrow important for neutrinos & weak force studies..

For convenience let us define -

$$\sigma^{\mu} = (1, \vec{\sigma}) ; \quad \bar{\sigma}^{\mu} = (1, -\vec{\sigma})$$

So that $\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$ $\sigma^{\mu} = (1, \vec{\sigma}, \vec{\sigma}^2, \vec{\sigma}^3)$

With this, can simply rotation. Dirac eqn becomes -

$$\begin{pmatrix} -m & i\vec{\sigma} \cdot \vec{\alpha} \\ i\vec{\sigma} \cdot \vec{\alpha} & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$i(\vec{\alpha} + \vec{\sigma} \cdot \vec{\nabla}) \psi_L = 0$$

$$i(\vec{\alpha} - \vec{\sigma} \cdot \vec{\nabla}) \psi_R = 0$$

∴ the Weyl eqns become :

$$(i\vec{\sigma} \cdot \vec{\alpha}) \psi_L = 0$$

$$(i\vec{\sigma} \cdot \vec{\alpha}) \psi_R = 0$$

A hint

$$p^* = \sqrt{p^2 + m^2} = E_p$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p) e^{-ip \cdot x} , \quad p^2 = m^2$$

Look only solutions with positive frequency ... that is
 $E_p = p^0 > 0 \dots$

Ψ solves Dirac eqn $\rightarrow (i\gamma^\mu \partial_\mu - m)\Psi = 0$

$$\rightarrow (i\gamma^\mu \partial_\mu - m) u(p) e^{-ip \cdot x} = 0$$

$$\rightarrow \boxed{(i\gamma^\mu p_\mu - m) u(p) = 0}$$

Get rest frame $\Rightarrow p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $A_{1/2}$.

In rest frame, we have

$$(i\gamma^\mu p_\mu - m) u(p) \rightarrow (m\gamma^0 - m) u(p_0) = m(\gamma^0 - 1) u(p_0) = 0$$

$$\Rightarrow m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$$

$$\rightarrow \boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \xrightarrow{\text{two-component spinor}}}$$

just a factor ξ with norm. constraint.

$$\xi^\dagger \xi = 1,$$

$\cancel{\alpha}$

What are those ξ ?

Look at rotation generators ...

$$\boxed{s^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

$$\text{In particular, } S^2 = S'^2 = \frac{1}{2} \begin{pmatrix} 6^2 & 0 \\ 0 & 0^2 \end{pmatrix}$$

$$\text{So if } \left\{ \begin{array}{l} S = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ S = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$$

$$\text{Now, we're in rest frame, so } p' = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, boost to frame where particle has velocity ...

$$\vec{v} = v \cdot \hat{z} \cdot \circ \quad \text{Let } \tanh(\eta) = \frac{v}{c}.$$

↗ "rapidity"

$$\text{Then } \begin{pmatrix} E \\ p^3 \end{pmatrix} = p' = \begin{pmatrix} \cosh \eta & 0 & 0 & \sinh \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{minh})$$

(infinitesimal $\frac{1}{2}$)

$\frac{1}{2} \rightarrow$ just the Lorentz transform.

$$\rightarrow \text{In this frame, } \left\{ \begin{array}{l} E = m \cosh \eta \\ p^3 = m \sinh \eta \end{array} \right.$$

Now, apply the same boost to $\alpha(p)$...

$$\begin{aligned} \alpha(p) &= \frac{1}{2} \sqrt{m} \begin{pmatrix} s \\ s \end{pmatrix} \rightarrow \left(\frac{1}{2} \right) = \exp \left(\frac{-i}{2} \omega_{\mu\nu} S^{\mu\nu} \right) \\ &= \exp \left(\frac{-i}{2} \gamma \begin{pmatrix} 0^3 & 0 \\ 0 & -0^3 \end{pmatrix} \right) \sqrt{m} \begin{pmatrix} s \\ s \end{pmatrix} \\ &\quad \text{~} \uparrow i \cdot (0^3 - s) \end{aligned}$$

So, infinitesimally -

$$\exp \left\{ \frac{-i}{2} \eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right\} \approx \begin{pmatrix} \cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3 & 0 \\ 0 & \cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3 \end{pmatrix}$$

So Rest

$$u(p) \approx \sqrt{m} \begin{pmatrix} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) \xi \\ (\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) \xi \end{pmatrix} \quad \text{---}$$

Simplify ... note that

$$\begin{aligned} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3)^2 &= \dots \\ &= \cosh \eta - \sinh \eta \sigma^3 \\ &= \frac{E}{m} - \frac{P^3}{m} \sigma^3 = \frac{p \cdot \sigma}{m} \\ &= \frac{p^\mu \sigma^\mu}{m} \quad \text{where } \sigma^\mu = (1, \vec{\sigma}) \end{aligned}$$

So ... $\{ (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \sigma}{m}}$

and $(\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \bar{\sigma}}{m}}$

So -
$$\boxed{u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}} \rightarrow \text{current = valid for any arbitrary direction of } p.$$

Fact
$$(p \cdot \sigma)(p \cdot \bar{\sigma}) = (p^\mu)^2 - \vec{p}^2 = p^2 = m^2.$$

(61)

Now, back to example

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow p \cdot \sigma = \dots = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$$

$$\sqrt{p \cdot \sigma} = \begin{pmatrix} \sqrt{E - p^3} & 0 \\ 0 & \sqrt{E + p^3} \end{pmatrix}$$

and

$$\sqrt{p \cdot \bar{\sigma}} = \begin{pmatrix} \sqrt{E + p^3} & 0 \\ 0 & \sqrt{E - p^3} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E - p^3} (1) \\ \sqrt{E + p^3} (0) \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then (spin $-\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E + p^3} (0) \\ \sqrt{E - p^3} (1) \end{pmatrix}$$

In the massless limit, $E \rightarrow p^3$ ($E^2 = \sqrt{mc^2 + (p^3)^2}$)

$$\Rightarrow \boxed{u(p) = \begin{pmatrix} (0) \\ \sqrt{2E} (1) \end{pmatrix} \text{ spin } \frac{1}{2}}$$

$$\boxed{u(p) = \begin{pmatrix} \sqrt{2E} (0) \\ (0) \end{pmatrix} \text{ spin } -\frac{1}{2}}$$

These states: $u(p) = \sqrt{2E} \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$, $u(p) = \sqrt{2E} \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$ are eigenstates of the helicity operator

$$\boxed{h = \vec{p} \cdot \vec{S} = \sum_i \frac{1}{2} p_i^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}} = \frac{1}{2} \vec{p} \cdot \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

When $\{ h = \frac{1}{2} \Rightarrow \text{call Right-handed}$

$\{ h = -\frac{1}{2} \Rightarrow \text{call Left-handed}$

Note: Dirac helicity is frame-dependent... (for massive particle). — since can boost so that momentum is in the opposite direction,

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\left\{ \begin{array}{l} i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) \psi_L = i(\vec{\sigma} \cdot \vec{\partial}) \psi_L = 0 \\ i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \psi_R = i(\vec{\sigma} \cdot \vec{\partial}) \psi_R = 0 \end{array} \right.$$

Plug $\psi = u(p) e^{-ip \cdot x} \sim$, $\partial_0 \rightarrow -iE$

$$\vec{\nabla} \rightarrow i\vec{p}$$

↓, with $m=0$, $\tilde{p} = E\vec{p}$.

$$\Rightarrow h = \frac{-1}{2}$$

$$\Rightarrow \text{get } \left\{ (E + E\vec{p} \cdot \vec{\sigma}) \psi_L = 0 \Rightarrow (E)(1+2h) \psi_L = 0 \right.$$

$$\left. (E - E\vec{p} \cdot \vec{\sigma}) \psi_R = 0 \Rightarrow (E)(1-2h) \psi_R = 0 \right. \Rightarrow h = \frac{1}{2}$$

$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}$, as expected

#

Recap... $\psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 > 0) \rightarrow \text{positive frequency}$

 $\Rightarrow u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix} \rightarrow \text{spinor.}$

when $\psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 < 0) \rightarrow \text{negative frequency}$

$$\Rightarrow (u(p)) = \dots \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ -\sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix}$$

#

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = (\xi^+ \sqrt{p \cdot \sigma} \xi^+ \sqrt{p \cdot \bar{\sigma}}) \cdot \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix}$$

$$= \xi^+ \left[(p \cdot \sigma) + (p \cdot \bar{\sigma}) \right] \xi$$

$$\Rightarrow u^\dagger u = 2E_p \xi^+ \xi \quad \xrightarrow{\text{depends on } p!}$$

\sim ~~also~~ $u^\dagger u$ is not a Lorentz-inv scalar.
just like $\psi^\dagger \psi$.

\Rightarrow to make one such Lorentz-inv scalar, define

$$\bar{u}(p) = u^\dagger(p) \gamma^0$$



$$\bar{u}u = 2m \xi^+ \xi \quad \begin{matrix} \text{Lorentz-inv} \\ (\text{indep of } \vec{p}) \end{matrix}$$

$$\text{L}, \text{ wish after } \bar{u}n = u^r \gamma^0 n = 2m \xi^+ \xi^- = 2m$$

→ convenient to choose ONB spinors, ξ^1, ξ^2 .

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ \sqrt{p \cdot \bar{\sigma}} & \xi^s \end{pmatrix} \quad s=1,2}$$

Normalize:

$$\boxed{\bar{u}^r(p) u^s(p) = 2m \delta^{rs} \Leftrightarrow u^{r\dagger}(p) u^s(p) = 2E_p \delta^{rs}}$$

For the negative-freq solns, we get

$$\boxed{\bar{v}^r(p) v^s(p) = -2m \delta^{rs} \Leftrightarrow v^{r\dagger}(p) v^s(p) = +2E_p \delta^{rs}}$$

and

v, u are orthogonal to each other...

$$\boxed{\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0}$$

†

Finally, talk about spin sumrs

→ useful when evaluating Feynman diagrams.

→ when we need to sum all spin- $\frac{1}{2}$ polarizations

Since $\{\xi^s\}$ form an ONB,

$$\sum_{s=1,2} \xi^s \xi^{s*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\begin{aligned} \sum_{s=1,2} n^s(p) \bar{n}^s(p) &= \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \sigma} \xi^s} \right) \cdot \left(\xi^{s*} \sqrt{p \cdot \sigma}, \xi^{s*} \sqrt{p \cdot \sigma} \right) \\ &= \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \sigma} \xi^s} \right) \cdot \left(\xi^{s*} \sqrt{p \cdot \sigma}, \xi^{s*} \sqrt{p \cdot \sigma} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \xrightarrow{\text{"completeness"}} &= \begin{pmatrix} p \cdot \sigma & m \\ m & p \cdot \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} &= \sqrt{(p \cdot \sigma + p \cdot \sigma - p \cdot \sigma + p \cdot \sigma) m m} \\ &= \sqrt{(p \cdot \sigma)(p \cdot \sigma)} = \sqrt{((p^0, \vec{p}) \cdot (1, -\vec{\sigma})) ((p^0, \vec{p}) \cdot (1, \vec{\sigma}^2))} \\ &= \sqrt{(p^0)^2 - p^2} = m. \end{aligned}$$

$\frac{p}{\cancel{p}}$

$$\boxed{\sum_{s=1,2} n^s(p) \bar{n}^s(p) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix} = p \cdot \gamma + m I} \quad \begin{array}{l} \text{Feyn-} \\ \text{man's} \\ \text{slash} \\ \text{notation} \end{array}$$

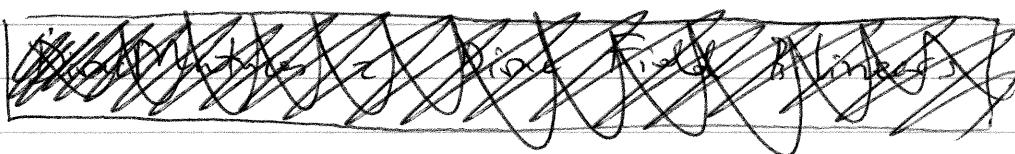
$\frac{p}{\cancel{p}}$

$$\boxed{\sum_{s=1,2} n^s(p) \bar{n}^s(p) = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \sigma & -m \end{pmatrix} = p \cdot \gamma - m I}$$

→ The combos $\partial \cdot p$ occur so often that Feynman introduced the notation:

$$\not{p} = \partial^\mu p_\mu = p_\mu \partial^\mu$$

#

Exercise

Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L .
The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

m = Majorana mass.

- (a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.
- (b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

a) In general, infinitesimal Lorentz transform on $\Psi_L(x)$ has the form

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} + \tilde{\rho} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

→ Rotation has the form:

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

$$\Rightarrow \sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L(x) \quad (\text{notes})$$

Lorentz transformed:

$$\Psi_L(x) \rightarrow \Lambda_{\frac{1}{2}} \Psi_L(\Lambda^{-1}x)$$

$$\partial_\mu \Psi_L(x) \rightarrow (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x)$$

→ put these together ...

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(\Lambda^{-1}x)$$

$$\sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(\Lambda^{-1}x)$$

$$\Rightarrow -im \sigma^2 \Psi_L^*(x) \rightarrow -im \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(\Lambda^{-1}x)$$

$$\text{Next, } i\vec{\sigma} \cdot \partial \Psi_L(x) = i\vec{\sigma}^\mu \partial_\mu \Psi_L(x)$$

$$\Rightarrow i\vec{\sigma}^\mu (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x) \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right)$$

$$= i\vec{\sigma}^\mu \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x)$$

we find: multiply:

$$1 = \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \left(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \quad (\text{rot} \times \text{inv.-rot})$$

$$\Rightarrow \rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\mu (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \\ \times (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \boxed{?} (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

Want is $\boxed{?}$

$$\rightarrow \boxed{?} = (1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\mu (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})$$

$$\approx \bar{\sigma}^\mu + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} \bar{\sigma}^\mu - i\vec{\theta} \cdot \vec{\sigma} \frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^\mu - \frac{i}{2} \vec{\theta} [\bar{\sigma}^\mu, \frac{\vec{\sigma}}{2}]$$

\downarrow
 \downarrow can show want

$$= \bar{\sigma}^\mu - i\vec{\theta} [J_\mu^{\alpha\beta}] \bar{\sigma}^\nu$$

\downarrow

$$i(g^{\mu\nu} \delta_\nu^\alpha - g^{\mu\nu} \delta_\nu^\alpha)$$

$$\Rightarrow \boxed{?} = (\Delta_4)^\mu_\nu \bar{\sigma}^\nu \rightarrow \bar{\sigma}^\mu transforms like 4-vector$$

$$\Rightarrow i\vec{\theta} \cdot \partial \Psi_L(x) \Rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \Delta_\nu^\mu \bar{\sigma}^\nu (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \delta_\nu^\alpha \bar{\sigma}^\nu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\nu \partial_\nu \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma} \cdot \partial \Psi_L(\Delta' x)$$

✓

$$\Rightarrow i\bar{\sigma} \cdot \partial \psi_c(x) - im \bar{\sigma}^2 \psi_c^*(x) = 0$$

\rightarrow leads to infinitesimal rotations ...

$$(1 - i\tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \underbrace{\{ i\bar{\sigma} \cdot \partial \psi_c(\tilde{x}) - im \bar{\sigma}^2 \psi_c^*(\tilde{x}) \}}_{=0} = 0$$

\Rightarrow done! So Majorana eqn is invariant under infinitesimal rotations.

\rightarrow

① Bosons (proceed in a similar way ...)

Key

$$(1 - \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \tilde{\beta} \cdot \frac{\vec{\sigma}}{2})$$

$$= \bar{\sigma}^M - \frac{1}{2} \tilde{\beta} \{ \bar{\sigma}^M, \bar{\sigma} \}$$

$$\Rightarrow \bar{\sigma}^M \rightarrow \bar{\sigma}^M - i\tilde{\beta} [\bar{\sigma}^M]_r \bar{\sigma}^r$$

$$\Rightarrow (1 - \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) = (1)_r \bar{\sigma}^r \text{ as before}$$

so plug this back into the original eqn.

$\cancel{*}$

Sep 28, 2020

Dirac Matrices & Dirac Field Bilinears

Oct 2, 2020 Recall that $\bar{\psi}\psi$ is Lorentz scalar...

Recall that $\bar{\psi}\gamma^\mu\psi$ is also a 4-vector.

⇒ $\boxed{?}$ Consider $\bar{\psi}\tilde{\Gamma}\psi$, where $\tilde{\Gamma}$ is any 4×4
 → can we decompose $\tilde{\Gamma}$ into terms that have
 definite transformation properties under the Lorentz
 group?

↳ $\tilde{\Gamma}$ can be written as combo of 16-element basis
 defined by

$$\left. \begin{array}{lll}
 1: & \mathbb{1} & \rightarrow 1 \\
 4: & \gamma^\mu & \rightarrow 4C2 \\
 6: & \gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \equiv \gamma^{\mu\nu\rho\sigma} & \rightarrow 4C3 \\
 4: & \gamma^{\mu\nu\rho} = \gamma^{\mu\rho}\gamma^\nu & \rightarrow 4C2 \\
 1: & \gamma^{\mu\nu\rho\sigma} = \gamma^{\mu\rho}\gamma^\nu\gamma^\sigma & \rightarrow 4C2
 \end{array} \right\}$$

16 total.

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisymmetric tensor of successively higher ranks

→ Introduction

$$\bar{\gamma}^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma$$

$$\begin{aligned}
 0123 &\rightarrow 1 \\
 7023 &\rightarrow -1
 \end{aligned}$$

↳ totally
anti-symmetric

Note that $\rightarrow \boxed{(8^5)^2 = 11}$

$$\rightarrow \overline{(Y^s)^+} = -i(Y^?)^+ \dots = (Y^e)^+$$

$$= + \gamma^2 \gamma^2 \gamma^1 \gamma^0 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5$$

also

$$\{g^s, g^m\} = i g^0 g^1 g^2 g^3 g^m + \underbrace{i g^m g^0 g^1 g^2 g^3}_{(-1)} = 0$$

and Hens

$$[\gamma^5, \gamma^{\mu\nu}] = [\gamma^5, \frac{i}{4} \{ \gamma^\mu, \gamma^\nu \}] = 0$$

\Rightarrow Eigenstates of \hat{r}^i with different eigenvalues don't mix under Lorentz transform.

→ In basis, can write -

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \text{for } \Psi_L \text{ (left-hd)} \\ \rightarrow \text{for } \Psi_R \text{ (right-hd),}$$

\rightarrow a Dirac spinor with only L/R component is an eigenstate of γ^5 with eig $\sqrt{(-1)/(1)}$.

With δ^5 , can rewrite the table of 4×4 matrices as

$\frac{1}{2}$	scalar	1
γ^m	vector	4
$\delta^{MV} = \frac{1}{2}\{\gamma^m, \gamma^n\}$	tensor	6
$\gamma^M \gamma^S$	pseudo vector	4
γ^S	pseudo scalar	<u>1</u>
		16

pseudo-vector/scalar is due to the fact that they transform like vector/scalar, BUT with an additional under Lorentz transf \rightarrow in charge under parity-transf.

Ex Parity transf: $\vec{x} \rightarrow -\vec{x}$

$$\hookrightarrow (x^0, x^i) \rightarrow (x^0, -x^i)$$

If instead $(x^0, \vec{x}) \rightarrow -(x^0, \vec{x}) = (-x^0, \vec{x})$ under parity, we call this a pseudo-vector

\rightarrow pseudo vector/scalar flips sign under parity transf.

\rightarrow From vector + pseudo-vector we can form 2 currents out of Dirac field bilinears -

$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x) \rightarrow$ vector current
$j^{\mu 5}(x) = \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x) \rightarrow$ pseudo vector current

Assume that ψ satisfies Dirac eqn.. $\bar{\psi} = \psi^\dagger \gamma^0$

$$\rightarrow i \not{D} \psi = m \psi \quad \rightarrow i \not{D} \bar{\psi} = m \bar{\psi} \quad (\text{Given } \not{D} = \not{\partial}^0,$$

\rightarrow compute div of these currents -

$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi}) \not{\partial}^\mu \psi + \bar{\psi} \not{\partial}^\mu (\partial_\mu \psi)$$

$$= i m \bar{\psi} \psi + \bar{\psi} (-im \psi) = 0$$

$$\rightarrow \boxed{\partial_\mu j^\mu = 0}$$

$\rightarrow j^m$ is always conserved if $\psi(x)$ satisfies
Dirac eqn

\rightarrow It is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{i\alpha} \psi(x)$$

Similarity

$$\begin{aligned}\partial_m j^{ms} &= (\partial_m \bar{\psi}) \gamma^m \gamma^5 \psi + \cancel{\bar{\psi} \gamma^m \gamma^5 \partial_m \psi} \\ &= (\partial_m \bar{\psi}) \gamma^m \gamma^5 \psi + (-1) \bar{\psi} \gamma^5 \cancel{\gamma^m \partial_m} \psi \\ &= \text{im } \bar{\psi} \gamma^5 \psi + (-1)(-i) m \bar{\psi} \gamma^5 \psi\end{aligned}$$

$\rightarrow \boxed{\partial_m j^{ms} = 2 \text{im } \bar{\psi} \gamma^5 \psi} \rightarrow$ axial vector current

\rightarrow if $m=0$ then $\partial_m j^{ms}$ is conserved.

\rightarrow When $m=0$, j^m is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{i\alpha \gamma^5} \psi(x)$$

(we worry about the rest of this section in ~~Wojciech~~ Pashkin's ...)

-4

QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\psi} (\gamma \not{d} - m) \psi = \bar{\psi} (\gamma^0 \partial_0 + \vec{\gamma} \cdot \vec{\partial}) \psi - m \bar{\psi} \psi .$$

→ Canonical momentum conjugate to ψ is

$$\frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = \gamma^0 \bar{\psi} \gamma^0 = \gamma^0 \bar{\psi} \gamma^0 = i \psi^+ .$$

→ Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} \partial_0 \psi - \mathcal{L} = i \bar{\psi} \gamma^0 \partial_0 \psi - i \bar{\psi} \gamma^0 \partial_0 \psi \\ &\quad - i \bar{\psi} \vec{\gamma} \cdot \vec{\partial} \psi + m \bar{\psi} \psi \\ &= -i \bar{\psi} \vec{\gamma} \cdot \vec{\partial} \psi + m \bar{\psi} \psi \end{aligned}$$

Thus,

$$\mathcal{H} = \int \mathcal{H} d^3x = \int d^3x \bar{\psi} (-i \vec{\gamma} \cdot \vec{\partial} + m) \psi$$

→ now let's figure out the commutators to make this a quantum field theory...

→ DO NOT QUANTIZE THE DIRAC FIELD

This won't work!

Guess $\left[\psi_a(\vec{x}), i\psi_b^+(\vec{y}) \right] = i\delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}$

↑ spin ↑
components

$(a, b = 1, 2, 3, 4)$

i.e.

$$\left[\psi_a(\vec{x}), \psi_b^+(\vec{y}) \right] = \delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}.$$

In matrix notation ...

$$\left[\psi_a(\vec{x}), \psi_b^+(\vec{y}) \right] = \mathbf{1}_{4 \times 4} \delta^{(3)}(\vec{x}-\vec{y})$$

↓ ↓
[:] [---]

Also guess $\left[\psi_a(\vec{x}), \psi_b(\vec{y}) \right] = 0$

$$\left[\psi_a^+(\vec{x}), \psi_b^+(\vec{y}) \right] = 0$$

No. & Next

$$\left[\psi(\vec{x}), \psi(\vec{y}) \right] = \left[\psi(\vec{x}), \psi^+(\vec{y}) \right] \gamma^0$$

$$= \left[\psi(\vec{x}), \psi^+(\vec{y}) \right] \gamma^0 = \gamma^0 \delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote -

(real) field $\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \vec{a}_p + \vec{a}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}$. (FT)

For complex field \rightarrow we get

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \vec{a}_p + \vec{b}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}.$$

In the case of Dirac field, need spin degrees of freedom.

Try --

$$\Psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{+i\vec{p} \cdot \vec{x}}$$

↑
Spin degrees of freedom

Former components: $\Psi(\vec{x}) = u(p) e^{i\vec{p} \cdot \vec{x}}$

$$2 \quad \Psi^+(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{-i\vec{p} \cdot \vec{x}}$$

Recall about u, v also solves Dirac eqn in the reverse
heat (in momentum space --)

$$p^m \delta_m u^r(p) = mu^r(p) \quad p^m \delta_m v^r(p) = -mv^r(p)$$

We can by the commutators --

$$[\hat{a}_p^r, \hat{a}_{p'}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{b}_p^r, \hat{b}_{p'}^{s\dagger}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{a}_p^r, \hat{b}_{p'}^{s\dagger}] = 0$$

The rest are all zero --

We find heat \rightarrow as desired --

$$[\Psi_a(\vec{x}), \Psi_b(\vec{y})] = 0 = [\Psi_a^+(\vec{x}), \Psi_b^+(\vec{y})]$$

77

We also find that

$$\{\Psi_a(\vec{x}), \Psi_b^+(\vec{y})\} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian ...

$$H = \int d^3x \left[-i\vec{\nabla} \cdot \vec{\psi} + m\vec{\psi}\vec{\psi} \right]$$

$$= \int d^3x \left\{ \psi^0 \underbrace{\left[-i\vec{\nabla} \cdot \vec{\psi} + m \right]}_{\text{just const}} \psi \right\}$$

just const

$$\text{Now, with } p^m \partial_\mu u^r(p) = mu^r(p)$$

$$\rightarrow (\vec{p} \cdot \vec{\nabla} + m) u^r(p) = p^0 \delta^0 u^r(p) = E_p \delta^0 u^r(p)$$

$$\text{Similarly, SIC } p^m \partial_\mu v^r(p) = -mv^r(p)$$

$$(\vec{p} \cdot \vec{\nabla} + m) v^r(p) = -E_p \delta^0 v^r(p).$$

So ...

$$\rightarrow [-i\vec{\nabla} \cdot \vec{\psi} + m] \psi = [-i\vec{\nabla} \cdot \vec{\psi} + m] \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_p^r u_p^r + b_p^r v_p^r] e^{ip \cdot \vec{x}}$$

$$= \psi^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ E_p a_p^r u_p^r(p) - E_p b_p^r v_p^r(p) \right\} e^{ip \cdot \vec{x}}$$

So ...

$$H = \int d^3x \left\{ \psi^+ \psi^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \dots \right\} e^{ip \cdot \vec{x}} \right\}$$

play in ...

$$\rightarrow H = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} E_p \left\{ a_p^r a_p^r - b_{+p}^r b_{+p}^{r+} \right\}$$

\downarrow
 $b_{+p}^{r+} b_{+p}^r + \text{const}$

!

→ By creating more and more particles with b_{+p}^r , we can lower the energy indefinitely

→ This is bad...

→ So we should use Fermi-Dirac statistics instead → anti-commutators instead of commutators...

Requirement.

$$\left\{ a_p^r, a_q^{s+} \right\} = \left\{ b_{+p}^r, b_{+q}^{s+} \right\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q})$$

↑
 no longer harmonic! ↗ all other
 anti-commutators
 are zero...

When this is true, we find that

$$\left\{ \psi_a(\vec{x}), \psi_b^+(\vec{y}) \right\} = S^{(3)}(\vec{x} - \vec{y}) \delta_{ab}$$

$$\left\{ \psi_a(\vec{x}), \psi_b(\vec{y}) \right\} = \left\{ \psi_a^+(\vec{x}), \psi_b^+(\vec{y}) \right\} = 0$$

where we're using

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p^r u_r(\vec{p}) + b_{-p}^{r+} v_r(-\vec{p}) \right] e^{i\vec{p} \cdot \vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p \left(a_p^r a_p^r - b_{-p}^r b_{-p}^r \right) - b_{-p}^r b_p^r + \text{const}$$

$$\Rightarrow \mathcal{H} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ a_p^r a_p^r + b_{-p}^r b_{-p}^r \right\}$$

now good, b/c E is bold below...

→ also can compute

$$\tilde{P} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} \tilde{p} \left(a_p^r a_p^r + b_{-p}^r b_{-p}^r \right)$$

To avoid sign confusion, we will usually write

$$\Psi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u^r(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + b_{-p}^r v^r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)$$

As a Heisenberg field,

$$\Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u^r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} + b_{-p}^r v^r(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} \right)$$

where:

- | | |
|------------------------------------------------|-------------------------------------------------------------------------------------------------------------|
| a_p^r
a_p^{r+}
b_p^r
b_p^{r+} | : annihilates particles
: creates particles
: annihilates anti-particles
: creates anti-particles. |
|------------------------------------------------|-------------------------------------------------------------------------------------------------------------|

Vacuum state as $|0\rangle$ where

$$\begin{cases} \hat{a}_p^\dagger |0\rangle = 0 \\ \hat{b}_p^\dagger |0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ conserved norm:

$$|\vec{p}, s\rangle = \sqrt{2E_p} \hat{a}_p^\dagger |0\rangle$$

so that

$$|\vec{p}, s\rangle |\vec{q}, r\rangle = \sqrt{2E_p} \sqrt{2E_q} \delta^{(3)}(\vec{p} - \vec{q}) \delta^{rs}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lorentz}} \psi'(x) = \gamma \frac{1}{\sqrt{2}} \psi(\gamma^{-1} x)$$

recall that with $\omega_{12} = -\omega_{21} = \theta$

$$\begin{cases} \omega_{12} = -\omega_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \end{cases} \Rightarrow \exp\left\{-i\omega_{\mu\nu} \gamma^\nu \frac{\gamma^\mu}{2}\right\} = 1 - i\vec{\theta} \cdot \vec{\gamma}$$

$$= 1 - i\vec{\theta} \cdot \vec{\gamma}$$

$$\rightarrow \text{and } \psi(\gamma^{-1} x) \approx [1 - \vec{\theta} \cdot \vec{\gamma}] \psi(x)$$

$$\vec{\gamma} = \vec{x} \times (-i\vec{\nabla})$$

so we'd $\psi \rightarrow \psi + S\psi$ where

$$S\psi = \psi' - \psi = \left(\frac{i}{2}\vec{\theta} \cdot \vec{\gamma}\right)\psi(x) - \vec{\theta} \cdot (\vec{x} \times \vec{\nabla})\psi(x)$$

By Noether's Thm,

$$\vec{J}_{\text{total}} \text{ (total spin)} = \int \frac{d^3x}{2} \left[\bar{\psi}^\dagger (-i\vec{\gamma} \cdot \vec{\nabla}) \psi + \frac{1}{2} \bar{\psi}^\dagger \vec{\Sigma} \psi \right].$$

~~to~~

We won't worry about the rest of this section about propagators

\rightarrow we'll come back to them later when looking at Feynman diagrams.

~~to~~

DISCRETE SYMMETRIES OF THE DIRAC THEORY

Basically, we have

Parity — Time reversal

Charge conjugation

~~to~~

Recall that we before, we looked at implementation of continuous Lorentz transform -

\rightarrow found that $\gamma_1 \in$ Lorentz group

$\exists U(1)$ unitary for which

$$U(1) \psi(x) U(1)^\dagger = \gamma_2' \psi(\gamma_1 x).$$

\rightarrow Now, we'll look about discrete symmetries on the Dirac field.

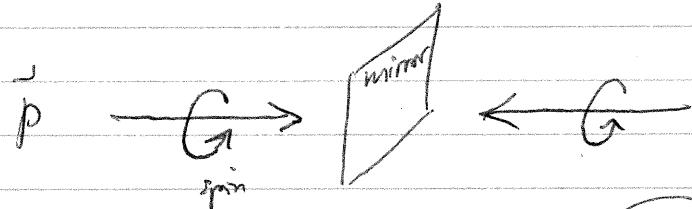
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant:

→ e.g. { time-reversal },
{ parity }.

[Parity] (P) : flips direction of spatial vectors

$$P: (t, \vec{x}) \rightarrow (t, -\vec{x})$$

↔ mirror sym → change the handedness.



→ Note momentum flip sign, but spin is unchanged.

[Time-reversal]

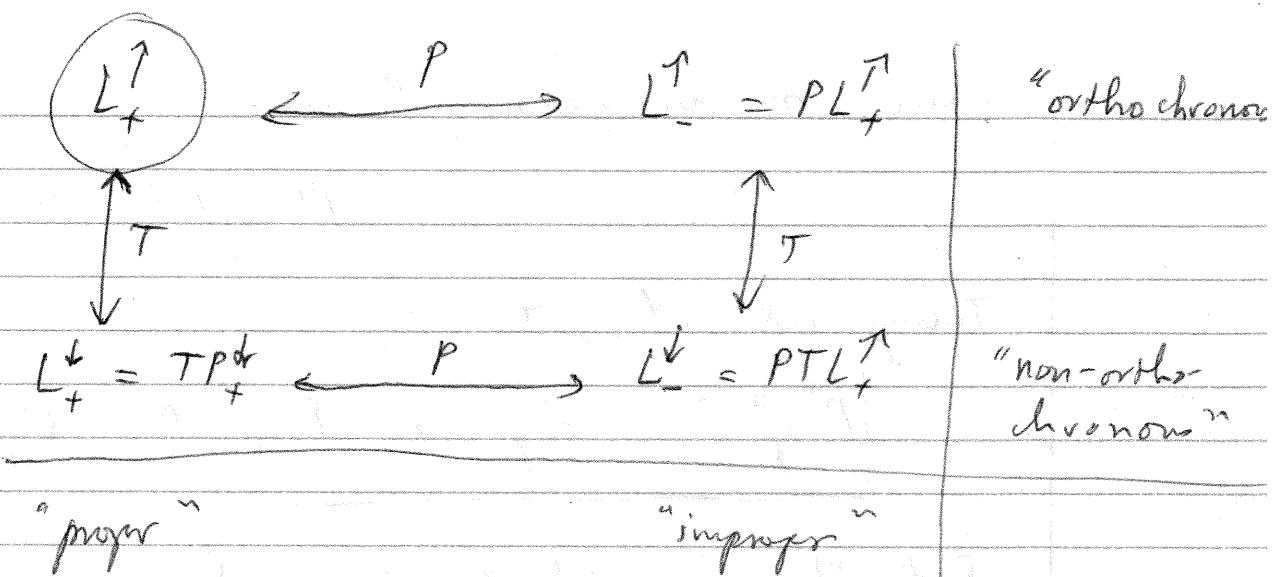
$$T: (t, \vec{x}) \rightarrow (-t, \vec{x})$$

P,T don't belong to the "proper" Lorentz group L_+

→ the full Lorentz group breaks into 4 disjoint subsets ...

(L)

(03)



charge conjugation \rightarrow intercharge particles & anti-particles.

\hookrightarrow non-space-time.

Let's look at Parity.

Note that because $P: (t, \vec{x}) \rightarrow (t, -\vec{x})$

\rightarrow momentum flips sign

but not spin! \rightarrow what is P ? As an operator?

$$\xrightarrow{\text{---}} \xrightarrow{\text{---}} \xleftarrow{\text{---}} \xleftarrow{\text{---}}$$

As an operator on creation/annihilation ops, we want

$$P^\dagger a_{\vec{p}}^s P = a_{\vec{p}}^s \quad \& \quad P^\dagger b_{\vec{p}}^s P = b_{-\vec{p}}^s$$

where, as discussed, P must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, set

$$\boxed{P^\dagger \tilde{a}_p^s P = a_{-\vec{p}}^{s\dagger} \quad P^\dagger \tilde{b}_p^s P = b_{-\vec{p}}^{s\dagger}}$$

But there might be too restrictive --- we can get better constraints by requiring that:

$$\boxed{P^\dagger \tilde{a}_p^s P = \eta_a a_{-\vec{p}}^{s\dagger} \quad P^\dagger \tilde{b}_p^s P = \eta_b b_{-\vec{p}}^{s\dagger}}$$

as long as $\eta_a^2 = (\eta_b)^2 = 1$ are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel:

$$\left\{ \begin{array}{l} P^\dagger \tilde{a}_p^s \tilde{a}_{\vec{p}}^{s\dagger} P = a_{-\vec{p}}^{s\dagger} a_{-\vec{p}}^s \\ P^\dagger \tilde{b}_p^s \tilde{b}_{\vec{p}}^{s\dagger} P = b_{-\vec{p}}^{s\dagger} b_{-\vec{p}}^s \end{array} \right.$$

With this, let's ~~see~~ implement parity condition on $\psi(x)$

$$\rightarrow P^\dagger \psi P = ? \quad \left(\begin{array}{l} \text{to find out what these} \\ \eta_a + \eta_b \text{ must be...} \end{array} \right)$$

$$P^t \chi(x) P = \int \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2\tilde{p}}} \sum_{s=1,2} (\gamma_a^s a_{-\vec{p}}^s u^s(p) e^{-i\tilde{p} \cdot \vec{x}} + \gamma_b^s b_{-\vec{p}}^s v^s(\vec{p}) e^{i\tilde{p} \cdot \vec{x}})$$

Define $\begin{cases} \tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}) \end{cases}$

Note that

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad \text{where } \sigma = (1, \vec{\sigma}) \quad \bar{\sigma} = (1, -\vec{\sigma})$$

$$\begin{aligned} &= \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} u^s(-\tilde{p}) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u^s(-\tilde{p}) \end{aligned}$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\tilde{p})}$$

and

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \dots = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} v^s(-\tilde{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(-\tilde{p})}$$

With these, we find that

$$\tilde{p} \cdot \tilde{x} = p \cdot x$$

(86)

$$P^+ \bar{\psi}(x) P = 8^0 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(\gamma_a \frac{a^s}{-p} u^s(-p) e^{-ip \cdot \tilde{x}} + \gamma_b^* \frac{b^s}{-p} v^s(-p) e^{ip \cdot \tilde{x}} \right)$$

Now, notice that if $\gamma_a = \gamma_b^*$ then it's "nice":

$$(\gamma_a = \gamma_b^*) \Rightarrow P \bar{\psi}(x) P = \gamma_a 8^0 \bar{\psi}(\tilde{x}) \quad \rightarrow P_{\text{transf}} \text{ in final form}$$

\rightarrow sufficient to choose $\gamma_a = 1 = -\gamma_b^*$

relative sign between fermions - antifermions --

-4

Now, useful to know how various Dirac field bilinears transform under parity ...

Recall ... 5 of them:

$$\bar{\psi} \psi, \bar{\psi} \gamma^\mu \psi, ; \bar{\psi} [\gamma^\mu, \gamma^\nu] \psi$$

$$\bar{\psi} \gamma^\mu \gamma^\nu \psi, ; \bar{\psi} \gamma^\nu \psi.$$

\rightarrow find these, first compute: $P \bar{\psi}(x) P$ --

$$P^+ \bar{\psi}(x) P = P^+ \bar{\psi}^+(x) \gamma^0 P \stackrel{\curvearrowright}{=} (P^+ \bar{\psi} P)^+ \gamma^0 \quad (\gamma^0 = \gamma^0)$$

$$\rightarrow = \gamma_a^* (\gamma^0 \bar{\psi}(\tilde{x}))^+ \gamma^0 = \gamma_a^* \bar{\psi}^+(\tilde{x}) \gamma^0 \gamma^0$$

$$\rightarrow \boxed{P \bar{\psi} P = \gamma_a^* \bar{\psi}(\tilde{x}) \gamma^0}$$

With this --

$$\begin{aligned}
 p^+ \bar{\psi} \psi p &= \underbrace{p^+}_{(x)} \underbrace{\bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x)}_{(x)} \underbrace{p}_\mu \\
 &= \gamma_\mu \bar{\psi}(\tilde{x}) \gamma^0 \gamma_\mu \gamma^0 \psi(\tilde{x}) \\
 &= |\gamma_\mu|^2 \bar{\psi}(\tilde{x}) \psi(\tilde{x})
 \end{aligned}$$

scalar

$$p^+ \bar{\psi} \psi p(x) = \bar{\psi} \psi(\tilde{x}). \quad (\text{scalar})$$

scalar.

can also show --

$$\begin{aligned}
 p^+ \bar{\psi}(x) \gamma^\mu \psi p &= \bar{\psi} \gamma^0 \gamma^\mu \gamma^0 \psi(\tilde{x}) \\
 (\text{vector}) &= \left\{ \begin{array}{l} + \bar{\psi} \gamma^\mu \psi(\tilde{x}) \quad \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi(\tilde{x}) \quad \mu = 1, 2, 3 \end{array} \right.
 \end{aligned}$$

$$p^+ (i \bar{\psi} \gamma^5 \psi) p = i \bar{\psi}(\tilde{x}) \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) = -i \bar{\psi} \gamma^5 \psi(\tilde{x})$$

↑
pseudo
scalar
(-)

~~$$\begin{aligned}
 &\bar{\psi} \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) \quad \mu = 0 \\
 &\bar{\psi} \gamma^1 \gamma^5 \gamma^0 \psi(\tilde{x}) \quad \mu = 1 \\
 &\bar{\psi} \gamma^2 \gamma^5 \gamma^0 \psi(\tilde{x}) \quad \mu = 2 \\
 &\bar{\psi} \gamma^3 \gamma^5 \gamma^0 \psi(\tilde{x}) \quad \mu = 3
 \end{aligned}$$~~

$$p^+ \bar{\psi} \gamma^\mu \gamma^5 \psi p = \bar{\psi}(\tilde{x}) \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(\tilde{x})$$

↑
pseudo
vector.
(-)

$$\begin{aligned}
 &= \left\{ \begin{array}{l} - \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) \quad \mu = 0 \\ + \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) \quad \mu = 1, 2, 3 \end{array} \right.
 \end{aligned}$$

Note The relative sign: $-\gamma_a = \gamma_b^*$ is important.

for the relationship between fermion - anti - fermi

Consider ~~and~~ fermion - anti fermion state...

$$\begin{aligned}
 & a_p^{st} b_q^{st} |0\rangle \xrightarrow{P} P(a_p^{st} b_q^{st} |0\rangle) \\
 &= P^+ (a_p^{st} b_q^{st}) P |0\rangle \\
 &= \underbrace{P^+ a_p^{st} P P^+ b_q^{st} P}_{\gamma_a} |0\rangle \\
 &= (\gamma_a) a_{-p}^{st} \gamma_b b_{-q}^{st} |0\rangle \\
 &= -(\gamma_b \gamma_b^*) a_{-p}^{st} b_{-q}^{st} |0\rangle \\
 &= -a_{-p}^{st} b_{-q}^{st} |0\rangle
 \end{aligned}$$

→ a state containing a fermion-antifermion pair gets an (-1) under parity transformation.

extra

—

[TIME REVERSAL].

if T is unitary $\Rightarrow [T, H] = 0$

$$\rightarrow T^+ e^{iHt} T = e^{iHt + T^+ T} = e^{iHt}$$

→ no good...

What if $T^+ T = -H$? or $[T, H] = 0$?

But this \Rightarrow no good either since implies that H is unbounded ...

\rightarrow Assume this ...

"Time-reversal is conjugate-linear/anti-linear"

Assume:

T is unitary

$$T^* T = c^* \quad (c \in \mathbb{C})$$

$$[T, H] = 0$$

With those

$$T^* e^{-iHt} T = e^{-iHt} \quad \checkmark$$

\rightarrow Time-reversal:

momentum

\downarrow

spin

are reversed

\rightarrow like watching a movie played back-wards

$$G \xrightarrow{\quad} T \xrightarrow{\quad} \leftarrow \int$$

Flipping momentum is easy.

What abt flipping spinor? We know that

In some basis --

$$\xi(\uparrow) = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s=1, 2$ & define

reversed
spin

$$\xi^{-s} = -i\sigma^2 (\xi^s)^{\dagger}.$$

→ This is the flipped spinor → σ^z

It is clear that

$$\begin{aligned} \xi^{-s} &= -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow), \xi(\downarrow))^{\dagger} \\ &= (\xi(\downarrow), -\xi(\uparrow))^{\dagger} \end{aligned}$$

where $\xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{-1\dagger}$

$$\xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{-2\dagger}$$

→ This is convenient since our time reversal op. involves complex conjugation --

→ Can show: $\boxed{i\vec{\sigma}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{s\dagger} \\ \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{s\dagger} \end{pmatrix}}$

So if we use the identity ...

$$\{\sqrt{\tilde{p} \cdot \sigma} \gamma^2 = \sigma^2 \sqrt{\tilde{p} \cdot \sigma^2}\}$$

(prove using $\sigma^2 \bar{\sigma}^2 = -\bar{\sigma}^2 \sigma^2$)

then we get

$$\begin{aligned} u^{-s}(\tilde{p}) &= \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} (-i\sigma^2) \xi^{s\pm} \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} (-i\sigma^2) \xi^{s\mp} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2) \sqrt{\tilde{p} \cdot \sigma^2} \xi^{s\pm} \\ (-i\sigma^2) \sqrt{\tilde{p} \cdot \bar{\sigma}^2} \xi^{s\mp} \end{pmatrix} \\ &= (-i) \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} [u^s(p)]^\star = -\gamma^1 \gamma^3 [u^s(p)]^\star \end{aligned}$$

\uparrow
 $\sigma^2 = \sigma^2$

element-wise
cmplx conjugation

similarly,

$$v^{-s}(\tilde{p}) = -\gamma^1 \gamma^3 [\vartheta^s(p)]^\star$$

in this relation, v^{-s} contains

$$\xi^{(-s)} = -\xi^s$$

a 360° flip
introduces
a $(-)$ sign.

~~Introducing~~
~~Effect~~

Now we can define time reversal operation on the creation - annihilation operators ---

here \rightarrow $T^+ a_p^s T = \bar{a}_{-\vec{p}}^{-s}$ & $T^+ b_p^s T = \bar{b}_{-\vec{p}}^{-s}$

↑ flip \vec{p}
↓ flip momentum

where $a_{-\vec{p}}^{-s} = (a_{-\vec{p}}^\downarrow, -a_{-\vec{p}}^\uparrow)$
 we now define $b_{-\vec{p}}^{-s} = (b_{-\vec{p}}^\downarrow, -b_{-\vec{p}}^\uparrow)$

just like what we did with $\zeta^s = (\zeta(\uparrow), -\zeta(\downarrow))$

if $\left\{ \begin{array}{l} a_p^s = (a_p^\uparrow, a_p^\downarrow) \\ b_p^s = (b_p^\uparrow, b_p^\downarrow) \end{array} \right.$

analogous to what we did before...

With this, let's evaluate $T^\dagger \Psi(x) T$:

$$\begin{aligned} T^\dagger \Psi(x) T &= \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1,2} T^+ (a_p^s u_p^s(p) e^{-ip \cdot x} + b_p^{s+} v_p^s(p) e^{+ip \cdot x}) T \\ &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^2 \left\{ a_{-\vec{p}}^{-s} [u_{-\vec{p}}^s(p)]^* e^{-ip \cdot x} \right. \\ &\quad \left. + b_{-\vec{p}}^{-s} [v_{-\vec{p}}^s(p)]^* e^{+ip \cdot x} \right\} \end{aligned}$$

where under T , $= \gamma^1 \gamma^2 \Psi(x_T)$, $x_T = (-t, \vec{x})$

$\left\{ \begin{array}{l} a_p^s \xrightarrow{T} a_{-\vec{p}}^{-s} \\ b_p^s \xrightarrow{T} b_{-\vec{p}}^{-s} \end{array} \right.$

$\Rightarrow u_{-\vec{p}}^s(p) = \gamma_1 \gamma_2 u_p^s(p)$

$\Rightarrow v_{-\vec{p}}^s(p) = v_p^s(p)$

$\therefore T^\dagger e^{-ip \cdot x} T = \mathbb{E} e^{+ip \cdot x}; T^\dagger u_p^s T = [u_p^s]^*$

note sign here
choose ↑
93

Becare $\{u^s(p)\}^* = \gamma_1 \gamma_3 u^{-s}(\tilde{p})$, we have

$$\begin{aligned} T^+ \psi(x) T &= \gamma' \gamma^3 \int \frac{d^2 \tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^2 \left\{ a_{\tilde{p}}^{-s} u^{-s}(\tilde{p}) e^{i\tilde{p}(t_1, \tilde{x})} \right. \\ &\quad \left. + b_{\tilde{p}}^{-s} v^{-s}(\tilde{p}) e^{-i\tilde{p}(t_1, \tilde{x})} \right\} \\ &= \gamma' \gamma^3 \psi(-t, x) \\ &= -\tilde{\rho}(-t, \tilde{x}), \end{aligned}$$

$$\Rightarrow \boxed{T^+ \psi(x, t) T = \gamma' \gamma^3 \psi(x, -t)}$$

Next, can check the action of T on bilinears...

$$\begin{aligned} T^+ \bar{\psi} T &= T^+ \psi^+ \gamma^0 T = T^+ \psi^+ T \gamma^0 \xrightarrow{\text{real}} \\ &= (\gamma' \gamma^3 \psi(x_T))^+ \gamma^0 = \psi^+(x_T) \gamma^3 \gamma^1 \gamma^0 \\ &= \psi^+(x_T) \gamma^3 \gamma^1 \gamma^0 \quad \begin{matrix} \uparrow \\ -\gamma^3 \end{matrix} \quad \begin{matrix} \uparrow \\ -\gamma^1 \end{matrix} \\ &= +\psi^+(x_T) \gamma^0 \gamma^3 \gamma^1 \\ &\Rightarrow \boxed{T^+ \bar{\psi} T = -\bar{\psi}(x_T) \gamma^1 \gamma^3} \end{aligned}$$

with this, can compute the rest---

Scalar $\boxed{T \bar{\psi} \psi T = \bar{\psi} (-\gamma' \gamma^3) \underbrace{(\gamma' \gamma^3)}_{11} \psi(x_T) = \bar{\psi}(x_T) \psi(x_T)}$

Pseudoscalar \rightarrow set (-) \rightarrow "pseudo"

$$\boxed{T^+ \bar{\psi} \gamma^5 \psi T = -i \bar{\psi} (-\gamma' \gamma^3) (\gamma' \gamma^3) \psi(x_T) = -i \bar{\psi}(x_T) \gamma^5 \psi(x_T)}$$

Vector

$$\boxed{T^+ \bar{\psi} \gamma^\mu \psi T = \bar{\psi} (-\gamma^1 \gamma^3) (\gamma^\mu)^T (\gamma^1 \gamma^3) \psi}$$

(x)

$$= \begin{cases} + \bar{\psi} \gamma^\mu \psi (x_T) & \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi (x_T) & \mu = 1, 2, 3 \end{cases}$$

This makes sense... Recall that $\bar{\psi} \gamma^0 \psi$ is the charge density

↳ $\bar{\psi} \gamma^0 \psi$ should be the same under T -

as we saw: $T^+ \bar{\psi} \gamma^0 \psi T = \bar{\psi} \gamma^0 \psi$.

but current density (time-dy) must reverse sign

$$\rightarrow T^+ \bar{\psi} \gamma^5 \psi T = - \bar{\psi} \gamma^5 \psi \quad \checkmark.$$

→

Charge Conjugation - Matter-anti-matter flip

{ anti-particles \rightarrow particles are swapped.

{ spin + momentum are the same.

Let $\left\{ \begin{array}{l} C^\dagger \bar{a}_p^s C = b_p^- \\ C^\dagger \bar{b}_p^- C = \bar{a}_p^s \end{array} \right\} \rightarrow$ ignore phases...

How should C act on $\psi(x)$?

First, look at relation ...

$$(v^s(p))^{\pm} = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}} & (-i\gamma^2) \xi^{s\pm} \\ \sqrt{p\cdot\bar{\sigma}} & (-i\gamma^2) \xi^{s\pm} \end{pmatrix}^{\pm} = \begin{pmatrix} -i\gamma^2 \sqrt{p\cdot\bar{\sigma}} \xi^{s\pm} \\ i\gamma^2 \sqrt{p\cdot\bar{\sigma}} \xi^{s\pm} \end{pmatrix}^{\pm}$$

$$= \begin{pmatrix} 0 & -i\gamma^2 \\ -i\gamma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}} \xi^s \\ \sqrt{p\cdot\bar{\sigma}} \xi^s \end{pmatrix} = \cancel{\text{both}}$$

→ set

$$\boxed{u^s(p) = -i\gamma^2 (v^s(p))^{\pm}}$$

$$\boxed{v^s(p) = -i\gamma^2 (u^s(p))^{\pm}}$$

$$\rightarrow C^+ \psi(x) C = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^2 \left\{ -i\gamma^2 b_p^s (v^s(p))^* e^{-ip \cdot x} - i\gamma^2 a_p^{s\pm} (u^s(p))^{\pm} e^{ip \cdot x} \right\}$$

$$= -i\gamma^2 \psi^*(x) = -i\gamma^2 (\psi^+)^T = -i(\bar{\psi} \gamma^0 \gamma^2)^T$$

$$\Rightarrow \boxed{C^+ \psi(x) C = -i(\bar{\psi} \gamma^0 \gamma^2)^T} \rightarrow C \text{ is a unitary op.}$$

On bilinear ... first, find $\bar{\psi} = (\psi^+)^+ \gamma^0 = \psi^0$

$$\boxed{C^+ \bar{\psi} \psi^0 C = C^+ \psi^+ \gamma^0 C = \underbrace{C^+ \psi^+}_{\psi^0} \gamma^0 = -i \psi^T \gamma^0 \gamma^0}$$

$$= (-i \gamma^2 \psi)^T \gamma^0 = (-i \gamma^0 \gamma^2 \psi)^T$$

Next ...

$$C^+ \bar{\psi} \psi C = (-i(\bar{\psi} \gamma^0 \gamma^2 \psi)^T) (-i(\bar{\psi} \gamma^0 \gamma^2 \psi)^T) = \dots =$$

$$= -[(-i \bar{\psi} \gamma^0 \gamma^2)(-i \bar{\psi} \gamma^0 \gamma^2)]^T = +\bar{\psi} \gamma^0 \gamma^2 \gamma^0 \gamma^2 \psi$$

$$= +\bar{\psi} \gamma^0 \gamma^2 \gamma^2 \gamma^0 \psi = +\bar{\psi} \psi$$

(P)

$$\text{So } \boxed{C^\dagger \bar{\gamma}^4 C = \bar{\gamma}^\dagger \gamma} \rightarrow \text{reduces}$$

vector

$$\boxed{C_i^\dagger \bar{\gamma}^i \gamma^i C = i (-i \gamma^0 \gamma^2 \gamma)^T \gamma^i (-i \bar{\gamma}^0 \bar{\gamma}^2 \bar{\gamma})^T = i \bar{\gamma}^i \gamma^i}$$

pseudo-scalar

$$\boxed{C^\dagger \bar{\gamma}^m \gamma^m C = - \bar{\gamma}^m \gamma^m}$$

pseudo scalar

$$\boxed{C^\dagger \bar{\gamma}^m \gamma^m \gamma^i C = + \bar{\gamma}^m \gamma^m \gamma^i}$$

(I'll skip the derivations... to save time)

Summary

	$\bar{\gamma} \gamma$	$i \bar{\gamma} \gamma^5 \gamma$	$\bar{\gamma} \gamma^m \gamma$	$\bar{\gamma} \gamma^m \gamma^5 \gamma$	$\bar{\gamma} \gamma^m \gamma^5 \gamma$	∂_μ
P	+1	-1	$(-1)^m$	$-(-1)^m$	$(-1)^m (-1)^v$	$(-1)^m$
T	+1	-1	$(-1)^m$	$(-1)^m$	$-(-1)^m (-1)^v$	$-(-1)^m$
C	+1	+1	-1	+1	-1	+1
CPT	+1	+1	-1	-1	+1	-1

Notice that

$$L = \bar{\gamma} (i \gamma^\mu \partial_\mu - m) \gamma \text{ is invariant under } C, P, T \text{ separately}$$

→ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

Problem 5

↳ (to be continued...)

Invariance under CPT is required for any Lorentz invariant local Hermitian op.

Correlation functions for Dirac fields

$\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle \rightarrow$ Dirac propagation amplitudes
 ↓ ↗
 only "a" only "a"
 term contributes term contributes

Recall -

$$\rightarrow \bar{\psi}_A(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ a_A^S u_A^S(p) e^{-ip \cdot x} + b_A^{S+} v_A^S(p) e^{-ip \cdot x} \right\}$$

$$\rightarrow \bar{\psi}_B(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ b_B^S \bar{v}_B^S(p) e^{-ip \cdot x} + a_B^{S+} \bar{u}_B^S(p) e^{ip \cdot x} \right\}$$

where $\{a_A^S, a_B^{S+}\} = \{b_A^S, b_B^{S+}\} = (2\pi)^3 \delta^{(3)}(p-q)/8$

$$\rightarrow \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_S u_A^S(p) \bar{v}_B^S(p)}_{AB} e^{-ip(x-y)}$$

$$= (i\gamma_x - m) \underbrace{\int \frac{d^3 p}{(2\pi)^3 / 2E_p}}_{AB} e^{-ip(x-y)}$$

$$(p+m)_{AB}$$

$$\boxed{\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = (i\gamma_x + m)_{AB} D(x-y)}$$

$$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \underbrace{\bar{\psi}_A^s(p) \psi_B^s(p)}_{(\phi-m)_{AB}} e^{-ip(x-y)}$$

↑ ↑
 6 terms 6 terms
 contribute contribute

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2E_p} \right) (\phi-m)_{AB} e^{-ip(x-y)}$$

$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = - (i\partial_x + m)_{AB} \delta(y-x)$

Feynman Propagator

$$S_f^{AB}(x-y) = \begin{cases} \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle & \text{for } x^0 > y^0 \\ -\langle 0 | \bar{\psi}_B(y) \bar{\psi}_A(x) | 0 \rangle & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0 | T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \} | 0 \rangle$$

↑
--- time-ordering ---

where $T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \}$

$$= \theta(x^0 - y^0) \bar{\psi}_A(x) \bar{\psi}_B(y)$$

$$- \theta(y^0 - x^0) \bar{\psi}_B(y) \bar{\psi}_A(x)$$

minus sign for Fermions

(99)

let's check the calculations.

$$\langle 0 | \psi_A(x) \bar{\psi}_B(y) | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | \left\{ \sum_s a_{A\vec{p}}^s \bar{u}_A^s(p) e^{-ipx} + b_{A\vec{p}}^{s+} \bar{u}_A^{s+}(p) e^{-ipx} \right\}$$

$$\times \left\{ \sum_s b_{B\vec{p}}^{s+} \bar{u}_B^{s+}(p') e^{-ip'y} + a_{B\vec{p}}^s \bar{u}_B^s(p') e^{-ip'y} \right\} | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \langle 0 | \sum_s a_{A\vec{p}}^s \bar{u}_A^s(p) \sum_s a_{B\vec{p}}^{s+} \bar{u}_B^{s+}(p') e^{-i(p-x-p'y)}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \sum_s \langle 0 | u_A^s(p) \bar{u}_B^{s+}(p') | 0 \rangle (2\pi)^3 \delta^{(3)}(p-p') e^{i(p-x-p'y)}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_s u_A^s(p) \bar{u}_B^{s+}(p)}_{(p+m)_{AB}} e^{-ip(x-y)}$$

$$(p+m)_{AB} = (\gamma^m p_m + m)_{AB} \begin{pmatrix} \text{spin sum} \\ \text{relations} \end{pmatrix}$$

$$= (i)_{AB} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \quad \checkmark$$

Similarly, we can get the other relation too...

-g

Oct 5, 2020

(1) Recall Dirac bispinor field --

$$\psi(\vec{x}) = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(\vec{p}) \right] e^{i\vec{p} \cdot \vec{x}}$$

$$\text{Use } \{a_{\vec{p}}^r, a_{\vec{p}'}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{p}'}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(\vec{p}-\vec{p}')}$$

and all other anti-comm = 0, derive the following:

$$\{\psi_a(\vec{x}), \psi_b^+(\vec{y})\} = \delta_{ab} \delta^{(3)}(\vec{x}-\vec{y})$$

(2) The momentum operator is the Noether charge associated with spatial translation.

$$\vec{P} = -i \int d^3x \psi^+(\vec{x}) \vec{\nabla} \psi(\vec{x})$$

Show Keert

$$\vec{P} = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r \right)$$

Oct 10, 2020

(1) Well..

$$\{\psi_a(x), \psi_b^+(y)\}$$

$$= \psi_a(x) \psi_b^+(y) + \psi_b^+(y) \psi_a(x)$$

~~$$\frac{1}{2} \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^6} \frac{1}{2\sqrt{E_{p_1} E_{p_2}}} e^{i(\vec{p}_1 \vec{x} + \vec{p}_2 \vec{y})}$$~~

To keep things clean --

$\gamma_a(x) \gamma_c^+(y) + \gamma_b^+(x) \gamma_a(y)$ which involves the factor.

$$\begin{aligned}
 & a \sum_{r=1}^2 \sum_{s=1}^2 \left[\hat{a}_{p_a}^{1r} u^r(p_a) + \hat{b}_{-p_a}^{1st} v^r(-p_a) \right] \left[\left(\hat{a}_{p_a}^{1s} u^s(p_b) \right)^+ + \left(\hat{b}_{-p_b}^{1st} v^s(-p_b) \right)^+ \right] \\
 & + \sum_{r=1}^2 \sum_{s=1}^2 \left[\left(\hat{a}_{p_b}^{1s} u^s(p_b) \right)^+ + \left(\hat{b}_{-p_b}^{1st} v^s(-p_b) \right)^+ \right] \left[\hat{a}_{p_a}^{1r} u^r(p_a) + \hat{b}_{-p_a}^{1st} v^r(-p_a) \right] \\
 & = \sum_{r,s=1}^2 \left\{ \hat{a}_{p_a}^{1r} \hat{a}_{p_b}^{1st} \right\} u^r(p_a) u^s(p_b) + \left\{ \hat{b}_{p_a}^{1r} \hat{b}_{p_b}^{1st} \right\} v^r(p_a) v^s(p_b) \\
 & = \sum_{r,s=1}^2 (\text{?})^2 \delta^{r,s} \delta^{(1)}(p_a - p_b) \left\{ u^r(p_a) u^s(p_b) + v^r(p_a) v^s(-p_b) \right\}
 \end{aligned}$$

$$\Rightarrow \{ \Psi_a(x), \Psi_b^{\dagger}(y) \} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^2 \left\{ u^r(p) u^{r\dagger}(p) + v^r(f_p) v^{r\dagger}(-p) \right\}$$

Now, we want to convert $\bar{u} \rightarrow \bar{u}$

\rightarrow need γ^0 . In particular, recall that $\gamma^0 = 1$
 and $\boxed{u^5(p)\gamma^0 = \bar{u}^5(p)}$ \Rightarrow we have

$$\{ \psi_n(x), \psi_j^+(y) \} = \int \frac{d\vec{p}}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(x-y)} \sum_{r=1}^2 \{ u_p^r \bar{u}_p^{r+} \gamma^0 + v_p^r \bar{v}_p^{r+} \gamma^0 \}$$

$$= \left(\frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{ip(x-y)} \right) \sum_{r=1}^2 \left\{ \underbrace{\frac{u_r^r}{u_p^r}}_{1} + \underbrace{\frac{v_r^r}{v_p^r}}_{1} \right\} \gamma^r \quad (\text{spin } \downarrow m)$$

$$= \left(\frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left(P_+ \gamma + m \not{I} + P_- \gamma - m \not{I} \right) \gamma^0 \right|_{\text{dis}}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} [2\cancel{\langle \vec{p} \cdot \vec{r} \rangle} \delta^3]$$

recall that
only $\vec{p} \rightarrow -\vec{p}$ (102)

$$\text{Rather } = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left\{ E_p \cancel{\langle \vec{x} - \vec{p}, \vec{x} + \vec{p} \rangle} + E_p \cancel{\langle \vec{y} - \vec{p}, \vec{y} + \vec{p} \rangle} \right\} \delta^3$$

$$= \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}).$$

$$\boxed{\delta_{ab} \{ \psi_a(x), \psi_b^\dagger(y) \} = \delta_{ab} \delta^{(3)}(x-y)}$$

4

(2) ~~but~~ $\vec{p} = -i \int d^3 x \vec{x} \psi^\dagger(\vec{x}) \vec{\nabla} \psi(\vec{x})$.

then must

$$p = (p^+, \vec{p})$$

$$\vec{p} = \sum_{i=1}^2 \int \frac{d^3 p}{(2\pi)^3} \vec{p} \left(\frac{a^+}{p^+} a^r_p + b^+_p b^r_p \right).$$

$$i\vec{p} \cdot \vec{x} = \cancel{a^+ a^r} + i\vec{b}^+ \vec{b}^r$$

Well.

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{i\vec{p} \cdot \vec{x}} \left\{ a^r_p u^r_p + b^r_p v^r_p \right\}.$$

$$\delta \psi(x) = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (i\vec{p})$$

$$(2) \quad \Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u_p^s e^{-ip_x} + b_p^{s\dagger} v_p^s e^{ip_x} \right)$$

$$\rightarrow \nabla \Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (i\vec{p}) \left\{ a_p^s u_p^s e^{-ip_x} - b_p^{s\dagger} v_p^s e^{ip_x} \right\}$$

$$\hat{\Psi}(x) = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{r=1}^{\infty} \left\{ b_q^r v_q^r (q) e^{-iq_x} + a_q^{r\dagger} u_q^r (q) e^{iq_x} \right\}$$

$$\sim \int d^3 x (i) \nabla \hat{\Psi} D \Psi$$

$$= \int d^3 x \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_q}} e^{ix(p-q)} \hat{p}_x \times \sum_s \sum_r$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \frac{1}{E_p} \hat{p}_x \left\{ a \sum_{s,r=1}^2 \left(a_p^{s\dagger} a_p^s \hat{v}_p^r u_p^r - b_p^{r\dagger} b_p^r \hat{v}_p^s u_p^s \right) \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{2E_p}{2E_p} \hat{p}_x \left(a_p^{r\dagger} a_p^r - b_p^{r\dagger} b_p^r \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \hat{p}_x \left(a_p^{r\dagger} a_p^r - b_p^{r\dagger} b_p^r \right)$$

Finally ... $\{ b_p^r, b_p^{r\dagger} \} = (2\pi)^3 \delta^{rr} (\hat{p} - \hat{p})$

$$\Rightarrow -b_p^r b_p^{r\dagger} = b_p^{r\dagger} b_p^r - (2\pi)^3 \delta^{rr} \delta(\hat{p} - \hat{p})$$

$$\Rightarrow \hat{p}_x = \int d^3 x \Psi^+(x) (-i\partial_x) \Psi(x) \quad \rightarrow \text{momentum op}$$

$$\boxed{\hat{p}_x = \int \frac{d^3 p}{(2\pi)^3} \hat{p}_x \left(a_p^{r\dagger} a_p^r + b_p^{r\dagger} b_p^r \right)}$$

More problems

① Let \mathcal{U} be the following unitary op:

$$\mathcal{U} = \exp \left\{ -i\frac{\pi}{2} \sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r\dagger} - b_p^{r\dagger}) \right\}$$

Investigate the effect of \mathcal{U} on $a_p^r - b_p^{r\dagger}$. I.e.
compute $\mathcal{U}^\dagger a_p^r \mathcal{U} = n^r b_p^{r\dagger}$.

What type of transform does \mathcal{U} produce?
~~-x~~ $\xrightarrow{+X}$

Well...

$$\mathcal{U}^\dagger a_p^r \mathcal{U} = \exp \left\{ +i\frac{\pi}{2} \left(\sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r\dagger} - b_p^{r\dagger}) \right) \right\}$$

$\xleftarrow{+X} \exp \left\{ i\frac{\pi}{2} \sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r\dagger} - b_p^{r\dagger}) \right\}$

\Rightarrow no good way to do this except for powers,

Recall that $e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$, $e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} (-X)^n$

$$\begin{aligned} \mathcal{U}^\dagger a_p^r \mathcal{U} &\simeq \left(\sum_{n=0}^{\infty} \frac{(X^r)^n}{n!} \right) a_p^r \left(\sum_{m=0}^{\infty} \frac{(-X)^m}{m!} \right) = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} (X^r)^n (-X)^m \\ &= (X^r) a_p^r X^m \end{aligned}$$

but note that \mathcal{U} unitary iff X hermitian.

$$\rightarrow X^\dagger = X \rightarrow \mathcal{U}^\dagger a_p^r \mathcal{U} = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} X^n a_p^r X^m.$$

Weilil theorem:

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]]$$

2 Identity relating comm & anti-comm: ---

$$[AB, C] = ABC - CAB$$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B, C\} - \{A, C\}B.$$

We will need to compute \hat{X}, \hat{a}_q^\pm

$$U_q^\pm U = \hat{a}_q^\pm + [X, \hat{a}_q^\pm] + \frac{1}{2!} [X, [X, \hat{a}_q^\pm]] + \dots$$

→ need to compute

$$[X, \hat{a}_q^\pm] = X \hat{a}_q^\pm - \hat{a}_q^\pm X = \dots = ?$$

$$= \left(\frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left\{ (a_p^{s+} - b_p^{s+}) (a_p^s - b_p^s) \right\} \hat{a}_q^\pm \right. \\ \left. - \left\{ \hat{a}_q^\pm (a_p^{s+} - b_p^{s+}) (a_p^s - b_p^s) \right\} \right)$$

$$= \frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left[\left(a_p^{s+} - b_p^{s+} \right) \left\{ a_p^s - b_p^s \right\} \hat{a}_q^\pm \right. \\ \left. - \left\{ a_p^{s+} - b_p^{s+}, \hat{a}_q^\pm \right\} (a_p^s - b_p^s) \right]$$

$$= \frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left[- (2\pi)^3 s^{rs} 8^{cs} (\hat{p} - \hat{q}) (a_p^s - b_p^s) \right]$$

$$= - \frac{i\pi}{2} (a_q^s - b_q^s)$$

Next turn,

$$\begin{aligned} [X, [X, a_q^r]] &= [X, -\frac{i\pi}{2}(a_q^r - b_q^r)] \\ &= \frac{-i\pi}{2}[X, a_q^r] + \frac{i\pi}{2}[X, b_q^r] = \dots \\ &= 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r) \end{aligned}$$

$$So \dots e^X a_p^r e^{-X} = a_p^r - \frac{i\pi}{2}(a_q^r - b_q^r) + 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r)$$

what's next?

+ ?

each step $\rightarrow +2\left(\frac{i\pi}{2}\right)$ = alt (+) sign.

$$\begin{aligned} \rightarrow u^\dagger a_p^r u &= a_p^r - \frac{i\pi}{2}(a_q^r - b_q^r) + \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r) \\ &\quad - \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3(a_q^r - b_q^r) + \frac{1}{4!} \left(\frac{i\pi}{2}\right)^4 6(-) \end{aligned}$$

$$= a_p^r \left\{ 1 - \frac{i\pi}{2} + \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2 - \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3 + \frac{6}{4!} \left(\frac{i\pi}{2}\right)^4 + \dots \right\}$$

~~check this~~

$$\begin{aligned} &+ b_p^r \left\{ \frac{i\pi}{2} - \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2 + \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3 - \frac{6}{4!} \left(\frac{i\pi}{2}\right)^4 + \dots \right\} \\ &= a_p^r \left\{ 1 - \frac{1}{2} \cdot 2 \right\} + b_p^r \cdot \left\{ \frac{1}{2} - 2 \right\} \end{aligned}$$

$$= b_p^r \Rightarrow \boxed{u^\dagger a_p^r u^* = b_p^r}$$

$\rightarrow u$ corresponds to charge conjugation!

(2) Using Dirac annihilation creation ops, construct P for which

$$P^+ a_p^r P = a_{-p}^r \quad P^+ b_p^r P = b_{-p}^r.$$

Last time, we find that \rightarrow target

$$[X, a_p^r] = -\left(\frac{i\pi}{2}\right) (a_p^r - b_p^r)$$

\rightarrow we want X for which

$$\{ [X, a_p^r] = -\left(\frac{i\pi}{2}\right) (a_p^r - a_{-p}^r) .$$

$$[X, b_p^r] = -\left(\frac{i\pi}{2}\right) (b_p^r + b_{-p}^r) .$$

\rightarrow fit

$$P = \exp \left\{ -\frac{i\pi}{2} \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \left\{ a_p^{\dagger} (a_p^r - b_{-p}^r) + b_p^{\dagger} (b_p^r + b_{-p}^r) \right\} \right\}$$



check this, like last time

\rightarrow should work! \square

Olv a_p^r only int. w/ 1st term $a^{\dagger}()a = 0$

$$a^{\dagger}() = \delta^{--}(\omega) \quad \checkmark$$

Same with b_p^r \checkmark

Interacting Fields = Feynman Diagrams

To get better description of the real world, need to include interactions in the theory.

To preserve causality, new terms may involve products of fields at the same spacetime point!

↳ $\phi^4(x)$ ✓, but not $\phi(x)\phi(y)$.

$$\rightarrow H_{\text{int}} = \int d^3x \, H_{\text{int}}[\phi(x)] = - \int d^3x \, \partial_{\mu} \phi \partial^{\mu} \phi$$

→ insist that H_{int} is a func of the fields, not of their derivatives.

→ Common ex in perturb physics = CMT:

$$\boxed{L = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4}$$

Note

— $\Pi(x)$ is still $\partial_\mu \phi(x)$ since there are no new terms involving $\partial_\mu \phi$ interaction.

λ : dimensionless "coupling constant".

→ In general, adding interactions preserves invariance.

However, no matter what the true physics looks like at high momenta or short distances, the low momentum / long distance physics is well-approximated by an "effective" FT.

with "renormalizable" interactions.

→ these interactions where coupling constant are has dimensions $\boxed{d > 0}$

$[\text{Mass}]^d$ where $d > 0$.

$$\text{Ex} \quad -\frac{1}{2} m^2 \phi^2 = \frac{2}{4!} \phi^4 \text{ same dim}$$

→ $\lambda \sim [\text{Mass}]^0 \rightarrow \text{renormalizable.}$

But $-\frac{\lambda_6}{6!} \phi^6 \rightarrow \underline{\text{not renormalizable.}}$

(since $\lambda \sim [\text{Mass}]^{-2}$)

Perturbation Expansion

$$\text{Let } H = H_0 + f_{\text{int}} \rightsquigarrow = \int d^3 p \frac{1}{4!} \phi^4(x)$$

$$\uparrow$$

KG, free

→ we will generate power series in λ .

At any t_0 , we can write

$$\phi(t_0, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{i\vec{p} \cdot \vec{x}} + a_p^\dagger e^{-i\vec{p} \cdot \vec{x}} \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \tilde{a}_p e^{i\vec{p} \cdot \vec{x}} + \tilde{a}_p^\dagger e^{-i\vec{p} \cdot \vec{x}} \right\}$$

$$a_p^\dagger$$

where we've let
 a_p^\dagger absorb e^{iEt_0}

The Heisenberg field is then given by:

$$\rightarrow \boxed{\phi(t, \vec{x}) = e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)}}$$

If there's no interaction then we have

$$\rightarrow \boxed{\phi_{\text{free}}(t, \vec{x}) = e^{iH_0(t-t_0)} \phi_{\text{free}}(t_0, \vec{x}) e^{-iH_0(t-t_0)}}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p^+ e^{-ip \cdot x} + a_p^- e^{ip \cdot x} \right\} \Big|_{\substack{x_0 = t-t_0 \\ p^0 = E_p}}$$

Define this to be $\phi_I(t, \vec{x})$, the interaction picture field

The interaction picture field = Heisenberg field when $\lambda = 0$.

Now, look at Heisenberg field ..

$$\begin{aligned} \phi(t, \vec{x}) &= e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)} \\ &= \underbrace{e^{iH(t-t_0)} e^{-iH_0(t-t_0)}}_{\phi_I(t, \vec{x})} e^{+iH_0(t-t_0)} e^{-iH(t-t_0)} \\ &= U^+(t, t_0) \phi_I(t, \vec{x}) U(t, t_0) \end{aligned}$$

\rightarrow Evolve the operator on $\phi_I(t, \vec{x})$

Time evolution operator

OR

Evolve the state by $U(t, t_0) \rightarrow U|\phi\rangle ..$

→ now, we want to express $U(t, t_0)$ entirely in ϕ_I

To do this, note that $U(t, t_0)$ solves SE:

$$\begin{aligned} \frac{\partial}{\partial t} U(t, t_0) &= e^{iH_0(t-t_0)} (H - H_0) e^{-iH_0(t-t_0)} \\ &= e^{iH_0(t-t_0)} \text{ Hint } e^{-iH_0(t-t_0)} \\ &= \underbrace{e^{iH_0(t-t_0)} \text{ Hint } e^{-iH_0(t-t_0)}}_{e^{iH_0(t-t_0)} e^{-iH_0(t-t_0)}} e^{iH_0(t-t_0)} e^{-iH_0(t-t_0)} \\ &= H_I(t) U(t, t_0) \end{aligned}$$

where

$$H_I(t) = e^{iH_0(t-t_0)} \text{ Hint } e^{-iH_0(t-t_0)}$$

$$= \int d^3x \frac{\partial}{4!} \phi^4 \quad [3]$$

$$= \int d^3x e^{iH_0(t-t_0)} \overbrace{\frac{\partial}{4!} \phi^4}^{\rightarrow} \overbrace{e^{-iH_0(t-t_0)}}^{\leftarrow}$$

$$= \int d^3x \frac{\partial}{4!} \phi^4 \quad \checkmark$$

→ this is the Hamiltonian in the interaction picture.

So since U solves the SE:

$$\frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0),$$

U must look like

$$U(t, t_0) \sim \exp \{-iH_I t\}$$

More carefully, we actually have that

$$U(t, t_0) = T \exp \left\{ -i \int_{t_0}^t dt' H_I(t') \right\}$$

Lyman's formula

time-ordering symbol.

Why T ? Why ordering? \Rightarrow B/c $H(t_1) \not\rightarrow H(t_2)$ when $t_1 \neq t_2$.

" T " puts the latest operators on the left.

hence $i \partial_t U(t, t_0) = \underline{\underline{H_I(t)}} U(t, t_0)$.

As a power series in λ :

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt' H_I(t') + \frac{(-i)^2}{2!} \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 T \{ H_1(t_1) H_2(t_2) \} + \dots$$

$$+ \frac{(-i)^3}{3!} \int_{t_0}^t \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 dt_3 T \{ H_1(t_1) H_2(t_2) H_3(t_3) \} + \dots$$

Note "the time-ordering of the exponential is just ~~the time-ordering~~ the Taylor series of the terms time-ordered ...".

\rightarrow Now, we want to generalise $U(t, t_0)$ to $U(t, t')$

↑
referendum

This generalization is natural -

$$U(t, t') = T \left\{ \exp \left[-i \int_{t'}^t dt'' H_I(t'') \right] \right\} \quad (t \geq t')$$

Then we see that b/c both t, t' are variables -- we find :

$$i\partial_t U(t, t') = H_I(t) U(t, t')$$

$$i\partial_{t'} U(t, t') = -U(t, t') H_I(t').$$

and thus --

$$U(t, t') = e^{iH_0(t-t')} e^{-iH(t-t')} e^{-iH_0(t-t')}$$

so U is unitary.

Further, for $t_1 \geq t_2 \geq t_3$,

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3)$$

$$U(t_1, t_3) [U(t_2, t_3)]^\dagger = U(t_1, t_2)$$

Now, let $|0\rangle$ be gnd state of H_0

$|S\rangle$ be gnd state of H

$|n\rangle$ be gnd label all $|E_n\rangle$ of H .

Then, $(E_0 = \langle \psi_0 | H | \psi_0 \rangle)$

$$\langle e^{-iHT} | \psi_0 \rangle = e^{-iE_0 T} |\psi_0\rangle \langle \psi_0 | \psi_0 \rangle + \sum_{n \neq 0} e^{-iE_n T} |n\rangle \langle n | \psi_0 \rangle$$

Assume $H_0 |0\rangle = 0$ in consider $T \rightarrow \infty$ limit.

↑

Then $e^{-iE_n T}$ dies slowest for $n=0$, and so --

$$T \rightarrow \infty (1 - i\varepsilon)$$

$$\rightarrow e^{iHT} |\psi_0\rangle \rightarrow e^{-iE_0 T} |\psi_0\rangle \langle \psi_0 | \psi_0 \rangle \quad \text{assume } \langle \psi_0 | \psi_0 \rangle \neq 0$$

So

$$|\psi_0\rangle = \lim_{T \rightarrow \infty (1 - i\varepsilon)} (e^{-iE_0 T} \langle \psi_0 | \psi_0 \rangle)^{-1} e^{-iHT} |\psi_0\rangle$$

Now, since T large, we can shift it by a small constant --

$$\begin{aligned} |\psi_0\rangle &= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0(T+t_0)} \langle \psi_0 | \psi_0 \rangle \right\}^{-1} e^{-iH(T+t_0)} |\psi_0\rangle \\ &= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0(t_0 - (-T))} \langle \psi_0 | \psi_0 \rangle \right\}^{-1} e^{-iH(t_0 - (-T))} |\psi_0\rangle \\ &= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0(t_0 - (-T))} \langle \psi_0 | \psi_0 \rangle \right\}^{-1} e^{-iH(t_0 - (-T))} e^{-iH_0(-T-t_0)} |\psi_0\rangle \\ &= |\psi_0\rangle \text{ since } H_0 |\psi_0\rangle = 0. \end{aligned}$$

$$\Rightarrow |\psi_0\rangle = \lim_{T \rightarrow \infty (1 - i\varepsilon)} \frac{U(t_0, -T) |\psi_0\rangle}{e^{-iE_0(t_0 - (-T))} \langle \psi_0 | \psi_0 \rangle}$$

Similar

$$\langle \Omega \rangle = \lim_{T \rightarrow \infty} (1 - i\epsilon)$$

$\langle \text{lotu}(T, t_0) \rangle$

$$e^{-iE_0(T-t_0)}\langle 0|S^z \rangle$$

So, putting these together gives a correlation function -

For $x^\circ > y^\circ > t_\circ$, we have

$$\rightarrow \langle x | \phi(x) \phi(x) | x \rangle$$

$\overbrace{\quad\quad\quad}^{\phi(x)}$

$$= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, t_0) | (u(x^0, t_0))^+ \phi(x) u(x^0, t_0) | 0 \rangle}{e^{-iE_0 t / (T-t_0)} \langle 0 | x \rangle}$$

$$x \left(u(y^o, t_o) \right)^+ \phi_I(y) u(y^o, t_o) x$$

$$= \lim_{T \rightarrow \infty (1+i\varepsilon)} \frac{\langle 0 | u(T, x^0) \phi_F(y) u(x^0, y^0) \phi_I(y) u(y^0, -T) | 0 \rangle}{|\langle 0 | \phi \rangle|^2 e^{-i E_0 (2T)}}$$

acknowledgments

so divide the whole thing by $1 = \langle x/x \rangle$

$$I = \langle \sigma_1 \sigma_2 \rangle = \frac{\langle 0| \bar{u}(T, t_0) u(t_0, -T) | 0 \rangle}{\langle 0| \bar{\sigma}_2 | 0 \rangle^2} e^{-\int F_D(2T)} \rightarrow m(T, -T)$$

To get (for $x^o > y^o$)

$$\langle \alpha | \phi(x) \phi(y) | \alpha \rangle = \lim_{T \rightarrow \infty} \frac{\langle 0 | u(T, x^0) \phi_I(x) u(x^0, y^0) \phi_I(y) u(y^0, -T) | 0 \rangle}{\langle 0 | u(-T, -T) | 0 \rangle}$$

So, we have shown, by replacing U^* with Dyson's formula (w/ time-ordering)

$$\boxed{\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle} = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) \} \exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} | 0 \rangle}{\langle 0 | T \exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} | 0 \rangle}$$

So, looks like the term

$$\exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} \text{ can be treated & can be found, so } \swarrow$$

Wick's theorem

→ So, we have reduced the problem of calculating correlation functions to evaluating

$$\boxed{\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \dots \phi_I(x_n) \} | 0 \rangle}$$

→ This is the vacuum exp-value of time-ordered products of finite number of field operators.

$n=2 \rightarrow$ get Feynman operator.

$n>2 \rightarrow$ can use ~~for~~ brute force, but there are also ways to simplify calculations.

Now, we study

$$\boxed{\langle 0 | T \{ \phi_I(x) \phi_I(y) \} | 0 \rangle}$$

Recall that

$$\phi_I(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ a_p^- e^{-ip \cdot x} + a_p^+ e^{+ip \cdot x} \right\}$$

Oct 11,
2020

$$\text{Call } \phi_I^+(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} a_p^- e^{-ip \cdot x}$$

$$\text{and } \phi_I^-(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} a_p^+ e^{+ip \cdot x}$$

which is useful b/c

$$\phi_I^+(x)|0\rangle = 0, \quad \langle 0|\phi_I^-(x) = 0.$$

only has annihilation ops

only has creation ops

For $x^0 > y^0$,

$$\Gamma \{ \phi_I^+(x) \phi_I^-(y) \} = \phi_I^+(x) \phi_I^+(y) + \phi_I^-(x) \phi_I^-(y)$$

$$+ \phi_I^+(x) \phi_I^-(y) + \phi_I^-(x) \phi_I^+(y)$$

$$= \phi_I^+(x) \phi_I^+(y) + \phi_I^-(y) \phi_I^+(x) + \phi_I^-(x) \phi_I^+(y) + \phi_I^-(x) \phi_I^-(y)$$

$$+ [\phi_I^+(x), \phi_I^-(y)]$$

every of these terms has the form $a^\dagger a^\dagger a^\dagger a^\dagger$

i.e. creation ops always on the left.

\rightarrow "Normal order" \rightarrow less vanishing vacuum expectation value

What can we say about the commutator?

It's just a number, there's no creation/annihilation op's in it!

$$\begin{aligned} [\phi_I^+(x), \phi_I^-(y)] &= \langle 0 | [\sum \phi_I^+(x), \phi_I^-(y)] | 0 \rangle \\ &= \langle 0 | \phi_I^+(x) \phi_I^-(y) | 0 \rangle = \langle 0 | \phi_I^-(y) \phi_I^+(x) | 0 \rangle. \end{aligned}$$

With this, we can write

$$\begin{aligned} T\{\phi_I^+(x) \phi_I^-(y)\} &= \overbrace{\phi_I^+(x) \phi_I^+(y)} + \overbrace{\phi_I^-(x) \phi_I^+(y)} + \overbrace{\phi_I^-(y) \phi_I^+(x)} \\ &\quad + \overbrace{\phi_I^-(x) \phi_I^-(y)} + \langle 0 | \phi_I^-(x) \phi_I^-(y) | 0 \rangle \end{aligned}$$

Now, define the normal ordering symbol "N"

s.t. N takes the string of at's and rearranges them so that at's are on the left

$$\text{ex. } \left\{ \begin{array}{l} N(a_p^\dagger a_p^\dagger) = a_p^\dagger a_p^\dagger \\ N(a_p^\dagger a_p^\dagger) = a_p^\dagger a_p^\dagger \cdot \text{ ordering for } a_p^\dagger, a_p^\dagger \\ N(a_p^\dagger a_p^\dagger a_p^\dagger) = a_p^\dagger a_p^\dagger a_p^\dagger \text{ doesn't matter} \end{array} \right. \begin{array}{l} \text{since they commute.} \\ \text{since they commute.} \end{array}$$

\Rightarrow Note N is not a well-defined mathematical operation

$$\text{e.g. } N(\sum a_p^\dagger a_p^\dagger) \neq N((2a)^3 f^{(3)}(\vec{p} - \vec{q}))$$

\rightarrow it is only a lexicographic convention.

Now, let us consider general x^0, y^0 , then

$$T\{\phi_I(x)\phi_I(y)\} = N\{\phi_I(x), \phi_I(y)\}$$

$$+ \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 > y^0 \\ (\phi_I^+(y), \phi_I^-(x)) & \text{for } x^0 < y^0 \end{cases}$$

→ Let us define the continuation of $\phi_I(x), \phi_I(y)$ as

$$\phi_I^+(x)\phi_I^-(y) = \begin{cases} \sum \phi_I^+(x), \phi_I^-(y) & x^0 > y^0 \\ \sum \phi_I^+(y), \phi_I^-(x) & x^0 < y^0 \end{cases}$$

Then notice that, from our previous derivation,

$$\boxed{\phi_I^+(x)\phi_I^-(y)} = \begin{cases} \langle 0 | T\{\phi_I^+(x)\phi_I^-(y)\} | 0 \rangle & x^0 > y^0 \\ \langle 0 | \phi_I^-(y)\phi_I^+(x) | 0 \rangle & y^0 > x^0. \end{cases}$$

So,

$$\left. \begin{aligned} \phi_I^+(x)\phi_I^-(y) &= \langle 0 | T\{\phi_I^+(x)\phi_I^-(y)\} | 0 \rangle \\ &= D_F(x-y) \quad \sim \text{Feynman propagator} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \end{aligned} \right\}$$

With this, we have that

(20)

→ carry out the I^r subscript

$$T \{ \phi(x) \phi(y) \} = N \left\{ \phi(x) \phi(y) + \underbrace{\phi(x) \phi(y)}_{\phi(y) \phi(x)} \right\}$$

$$\rightarrow T \{ \phi(x) \phi(y) \} = N \{ \phi(x) \phi(y) \} + \text{"contraction"}$$

In fact, the generalization of this is called
Wick's Theorem

$$T \{ \phi(x_1) \phi(x_2) \dots \phi(x_m) \}$$

$$= N \{ \phi(x_1) \phi(x_2) \dots \phi(x_m) + \text{all possible contractions} \}$$

$$\text{Ex } T \{ \phi_1 \phi_2 \phi_3 \phi_4 \}$$

$$(\phi_n = \phi(x_n))$$

$$= N \{ \phi_1 \phi_2 \phi_3 \phi_4 +$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4}$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4}$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} \}$$

What does $N \{ \phi_1 \phi_2 \phi_3 \phi_4 \}$ mean?

$$N \{ \phi_1 \phi_2 \phi_3 \phi_4 \} = \overbrace{\phi_1 \phi_2}^{\phi_1 \phi_2} N \{ \phi_3 \phi_4 \}$$

$$= D_f(x_1 - x_2) N \{ \phi_3 \phi_4 \}.$$

Proof \rightarrow prove by induction. $n=2$ is good
(Feynman)

\rightarrow assume this holds for $n-1$.

Let $W(\phi_1 \dots \phi_n) = N\{\phi_1 \phi_2 \dots \phi_n + \text{all possible contractions}\}$

To prove $W(\phi_1 \dots \phi_n) = T\{\phi_1 \phi_2 \dots \phi_n\}$.

W/l/o/j: let $x^0 \geq x_1^0 \geq \dots \geq x_n^0$.

Then $T\{\phi_1 \dots \phi_n\} = \phi_1 T\{\phi_2 \dots \phi_n\}$ since
 $\phi_1 \in W(\phi_1 \dots \phi_n)$ done.

$$\text{So } T\{\phi_1 \dots \phi_n\} = \underbrace{\phi_1^+ W(\phi_2 \dots \phi_n)}_X + \underbrace{W(\phi_2 \dots \phi_n) \phi_1^+}_Y + [\phi_1^+, W]$$

Let $X = \phi_1^+ W + W \phi_1^+$; $Y = [\phi_1^+, W]$.

$X+Y$ are normal ordered: X contains all contractions in $W(\phi_1 \dots \phi_n)$ which don't contact ϕ_1 with anything.

Y contains all contractions in $W(\phi_1 \dots \phi_n)$ which contracts ϕ_1 with something.

$$\text{So } T(\phi_1 \phi_2 \dots \phi_n) = W(\phi_1 \dots \phi_n).$$

(we won't worry too much abt this proof.)

\rightarrow the main idea is the theorem itself). \square

In any case, we have another way to explicitly write out the result of Wick's Theorem:

$$T\{\phi_1, \phi_2, \dots, \phi_n\} = N \left\{ \exp \left[\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \phi_i \phi_j \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j} \right] \phi_1 \dots \phi_n \right\}$$

↑
we'll see why later on.

~~it~~

Feynman Diagrams

Wick's Theorem allows us to write

$$\langle 0 | T\{\phi_1, \dots, \phi_n\} | 0 \rangle$$

in terms of sums and products of Feynman propagators.

→ Now, we will develop ~~the~~ diagrammatic expressions.

Recall that

$$T\{\phi_1, \phi_2, \phi_3, \phi_4\} = N \left\{ \phi_1 \phi_2 \phi_3 \phi_4 + \text{all possible contractions} \right\}$$

But the only contribution to

$\langle 0 | T\{\phi_1, \phi_2, \phi_3, \phi_4\} | 0 \rangle$ is when all the ϕ 's are contracted -

↳ This b/c whenever things are in normal order, the exp value vanishes - $\rightarrow N(\bar{\phi}_1 \phi_2 \phi_3 \phi_4) = \bar{\phi}_1 \phi_2 N(1, \phi_3, \phi_4)$

→ to "escape" from normal order, ϕ 's have to be contracted

This means that

$$\{T\{\phi_1 \phi_2 \phi_3 \phi_4\}\} = \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{L}} + \phi_1 \phi_2 \phi_3 \phi_4$$

→ can write this as Feynman diagrams...

$$T\{\phi_1 \phi_2 \phi_3 \phi_4\} = \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ | \quad | \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ | \quad | \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ | \quad | \\ 3 \quad 4 \end{array}$$

↳ Interpretation

Particles are created at 2 spacetime points, each propagates to one of the other points, then get annihilated.

→ total amplitude of the process is the sum of the diagrams.

Well... what about something like...

$$\langle 0 | T\{\phi(x) \phi(y)\} \exp\left\{-i \int_{-\infty}^{\infty} dt H_I(t)\right\} \rangle | 0 \rangle ?$$

Well... as a power series in λ , the lowest order term is

$$\langle 0 | T\{\phi(x) \phi(y)\} | 0 \rangle = D_\phi(x-y) \cdot \xrightarrow{x} \xrightarrow{y}$$

$$\text{1st order } \langle 0 | T\{\phi(x) \phi(y)\} (-i) \left(\int_{-\infty}^{\infty} dt H_I(t) \right) | 0 \rangle$$

$$= \langle 0 | T\{\phi(x) \phi(y) (-i) \int d^4 z \bar{\phi}_q(z) \phi^q(z)\} | 0 \rangle$$

(124)

$$\begin{aligned}
 &= -\frac{i\gamma}{4!} \int d^4 z \langle 0 | T \{ \phi(x) \phi(y) \phi(z) \phi(\tau) \phi(1\tau) \phi(2) \} | 0 \rangle \\
 &= -\frac{i\gamma}{4!} \int d^4 z \left\{ \phi(x) \phi(y) \cdot \left\{ \phi(z) \phi(\tau) \phi(+) \phi(+) \phi(1\tau) + \phi_z \phi_z \phi_2 \phi_2 \right. \right. \\
 &\quad \left. \left. + \phi_z \phi_+ \phi_+ \phi_2 \right\} \right. \\
 &\quad \left. + \phi(x) \phi(y) \phi(1z) \phi(2) \phi(1z) \phi(2) \right\} \\
 &\qquad \qquad \qquad \xrightarrow{\text{12 terms, but are identical}}
 \end{aligned}$$

$$= \begin{array}{c} x \\ \text{---} \\ y \end{array} + \begin{array}{c} x \\ \swarrow \quad \searrow \\ y \\ \text{---} \\ z \end{array} \xrightarrow{\text{1 propagator}} \int d^4 z \mathcal{D}_F(x-z) \mathcal{D}_F(y-z)$$

$$\int d^4 z \mathcal{D}_F(x-y) \mathcal{D}_F(z-z) \mathcal{D}_F(1\tau z)$$

↑
12 of these.

→ each contraction \mathcal{D}_F is a line.

each quantum point is a dot.

→ but need to distinguish "external" and "internal" points.

\downarrow
 x, y

\downarrow
 z

Each internal point is associated w/ a factor of $-i\gamma \int d^4 z$, with combinatorial factor...

How do we count these combinatorial factors?