

Exercise

$$L(q, \dot{q}, t) = \frac{1}{2} (m \dot{q}^2 - \kappa q^2) e^{\alpha t} \quad (\alpha \in \mathbb{R})$$

$$T_\varepsilon(t) = t + \varepsilon$$

$$Q_\varepsilon(q) = q e^{-\frac{\varepsilon \alpha}{2}}$$

$$\tilde{t} = T_\varepsilon(t)$$

$$\tilde{q}(\tilde{t}) = Q_\varepsilon(q(t))$$

- ① show that T_ε and Q_ε are one-parameter subgroups
- ② show that $\delta S = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \tilde{S}[\tilde{q}] = 0$ (without using E-L eqs)
- ③ Find the associated conserved quantity
- ④ Find the Euler-Lagrange equations
- ⑤ Show that the conserved quantity is indeed conserved if the E-L eqs hold

Note: for point ② use the fact that

$$\delta S = \int_a^b \left[L(q, \dot{q}, t) \frac{d}{dt} \delta t + \frac{\partial L}{\partial t} \delta t + \frac{\partial L}{\partial q} \delta q + \frac{\partial L}{\partial \dot{q}} \left(\frac{d}{dt} \delta q - \dot{q} \frac{d}{dt} \delta t \right) \right] dt$$

and substitute $L, \delta t, \delta q$