Problem Set 4

Due: Saturday 11:59am, March 11th via Canvas upload
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Office hour: TBA on Canvas

1 Squeezing Hamiltonian

In lecture we saw that the inverted harmonic oscillator Hamiltonian $H = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2$ generates squeezing of phase space distributions. Let us investigate the quantum version of the problem. $\hat{p}$ and $\hat{x}$ are now operators. Let's first go to dimensionless variables $\tilde{p} = \hat{p}\, l_0/\hbar$ and $\tilde{x} = \hat{x}/l_0$ with $l_0 = \sqrt{\hbar/m\omega}$ the (inverted) harmonic oscillator length scale ($\omega > 0$), and a (dimensionless) commutator $[\tilde{x}, \tilde{p}] = i$. So the Hamiltonian is

$$H = \frac{\hbar\omega}{2}\left(\tilde{p}^2 - \tilde{x}^2\right).$$

Let us simplify further by introducing new operators $p = \frac{\tilde{p}-\tilde{x}}{\sqrt{2}}$ and $x = \frac{\tilde{p}+\tilde{x}}{\sqrt{2}}$. A quick check shows that $[x, p] = i$, so this is a canonical transformation. I call them x and p for notational simplicity, but they are operators, and they are rotated 45 degrees with respect to the original $\tilde{x}$,$\tilde{p}$ pair. In terms of these new, dimensionless operators, the Hamiltonian takes on the form (apart from a constant offset that we decide to ignore)

$$H = \hbar\omega\, x\, p$$

a) Given any wavefunction $\psi_0(x)$ at $t = 0$, show using Schrödinger's equation $i\hbar\frac{\partial\psi}{\partial t} = H\,\psi$ that at later times $\psi(x,t) = \psi_0(x(t))$ and give $x(t)$.
Hint: Use $p = -i\frac{\partial}{\partial x}$. Further hint: $x(t)$ will be given by the time-reversed evolution of the coordinate x in the classical problem.

b) Given the Fourier transform of $\psi_0(x)$, denoted $\tilde{\psi}_0(p)$, find similarly the evolution of the momentum space wavefunction $\tilde{\psi}(p,t)$, showing that it is given by $\tilde{\psi}_0(p(t))$ where you should give $p(t)$.
Hint: What is the representation of the operator x in momentum space?

c) Simple application: Given the ground state wavefunction of a (non-inverted) harmonic oscillator $\psi_0(x) = \frac{1}{\pi^{1/4}}e^{-x^2/2}$, i.e. the wavefunction of the vacuum in x representation, find the width Δx of the time-evolved wavefunction after time t (say in units of the original width at $t = 0$).

d) Analogous question for the momentum space wavefunction of the vacuum, $\tilde{\psi}_0(p) = \frac{1}{\pi^{1/4}}e^{-p^2/2}$: What is its width Δp after time t, in units of the original width at $t = 0$?

e) Quickly make sure Heisenberg is happy: Check $\Delta x\,\Delta p = \frac{1}{2}$ (i.e. is unchanged from the value at $t = 0$).

f) Rewrite the Hamiltonian, which in proper symmetrized form is $H = \hbar\omega\,(xp + px)/2$, using the usual creation and annihilation operators $a = \frac{x+ip}{\sqrt{2}}$ and $a^\dagger = \frac{x-ip}{\sqrt{2}}$ which obey $\left[a, a^\dagger\right] = 1$. You will find a familiar form that appeared in lecture.

2 Disentangling the Squeezing operator

In class we encountered the slightly horrific-looking expression

$$e^{\frac{r}{2}\left(a^{\dagger 2}-a^2\right)}\left|0\right\rangle = \frac{1}{\sqrt{\cosh(r)}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}\tanh(r)^n\left|2n\right\rangle \tag{1}$$

showing that the squeezed vacuum is a) not empty and b) contains only even number states. Not ever daunted by anything, we will prove this result, and it will be using an almost magical technique of wide applicability. The difficulty of applying the squeezing operator to a state is that it contains, in the exponent, operators a^2 and $a^{\dagger 2}$ that do not commute with each other. Terms like $(a^{\dagger 2} - a^2)^n$ would first have to be brought to normal order via commutation relations before we could easily apply it on the vacuum. This seems daunting. But let's not despair and see what types of operators arise under such commutations.

a) Show that

$$\begin{aligned}\left[a^2, a^{\dagger 2}\right] &= 4a^\dagger a + 2\\ \left[a^2, a^\dagger a\right] &= 2a^2\\ \left[a^{\dagger 2}, a^\dagger a\right] &= -2a^{\dagger 2}\end{aligned}$$

That means, crucially, that the (Lie) algebra of operators $(a^2, a^{\dagger 2}, a^\dagger a + \frac{1}{2})$ is *closed* under commutation. So by expanding the exponential and using commutation rules, will never create operators other than those three. The exponential of any combination of these "building blocks" is an element in the Lie group spanned by the exponentials of each block, so that we *must* be able to write

$$e^{\frac{r}{2}\left(a^{\dagger 2}-a^2\right)} = e^{\frac{u}{2}a^{\dagger 2}}e^{t\left(a^\dagger a+\frac{1}{2}\right)}e^{\frac{v}{2}a^2} \tag{2}$$

We just have to find the numbers u, t, v, which will all be functions of r. Now here comes the real magic: If we find any other Lie algebra whose three operators obey the same commutation relations, we can write down the very same relation, but might have a much easier time finding the missing numbers u, t and v! It turns out that in this case, we are dealing with the Lie algebra $su(2)$, and Pauli matrices, i.e. simple 2×2 matrices, will do a great job as stand-ins for the much more complicated photon creation and annihilation operators (which act on an infinite-dimensional Hilbert space).

b) Show that the replacement

$$\begin{aligned}a^\dagger a + \frac{1}{2} &\rightarrow \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\\ a^2 &\rightarrow -\sigma_- = -\sigma_x + i\,\sigma_y = \begin{pmatrix}0 & 0\\ -2 & 0\end{pmatrix}\\ a^{\dagger 2} &\rightarrow \sigma_+ = \sigma_x + i\,\sigma_y = \begin{pmatrix}0 & 2\\ 0 & 0\end{pmatrix}\end{aligned}$$

of the photon operators with 2×2 matrices yields the same commutation relations as we had above.

c) Write the matrix representation of the operator $\frac{r}{2}(a^{\dagger 2} - a^2)$ and carry out the exponentiation. This gives you the matrix representation for $e^{\frac{r}{2}(a^{\dagger 2}-a^2)}$.

d) Find the matrix representations for each of the factors $U \equiv e^{\frac{u}{2}a^{\dagger 2}}$, $T \equiv e^{t\left(a^\dagger a+\frac{1}{2}\right)}$ and $V \equiv e^{\frac{v}{2}a^2}$ and carry out the matrix product UTV.

e) Comparing term by term of the 2×2 matrices you found in c) and d), find the unknown parameters u, t and v. Write down your result for Eq. 2 in full glory. It's worth putting up on your wall.

f) Now simply prove Eq. 1 using your result from e).

Fun math fact: Using the results of problem 1 and 2, you have now proven the purely mathematical relation (using $r = \omega t$)

$$\psi_0(x(t)) = \frac{1}{\sqrt{\cosh(r)}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \tanh(r)^n \langle x|2n\rangle$$

or

$$\frac{1}{\sqrt{e^r}} e^{-\frac{x^2}{2\,e^{2r}}} = \frac{1}{\sqrt{\cosh(r)}} e^{-\frac{x^2}{2}} \sum_{n=0}^{\infty} \frac{1}{2^{2n} n!} \tanh(r)^n H_{2n}(x)$$

where we used that $\langle x|n\rangle = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{2^n n!}} e^{-x^2/2} H_n(x)$ with $H_n(x)$ the Hermite polynomials. As a physicist, you interpret this equation as saying that the squeezed vacuum consists of a superposition of all even-number photon states.
You can for yourself revisit problem 4, iv) of problem set 2, where you plotted the Q-distribution of a squeezed state. Truncate the sum at $n = 0, 1, 2, 3$, etc., and see how the squeezed state is built up from the (round and happy) vacuum from pairs of zeroes lining it. In the limit of $n \to \infty$ the zeroes will migrate to infinity, leaving a smooth squeezed state behind.

For more about Lie Algebras, I can highly recommend the book by Robert Gilmore, *Lie Groups, Physics, and Geometry - An Introduction for Physicists, Engineers and Chemists*, Cambridge University Press, 2008.

3 Generation of Squeezed States by Two-Photon Interactions

Consider a mode $(\vec{k}, \vec{\varepsilon})$ with wavevector $\vec{k}$ and polarization vector $\vec{\varepsilon}$ of the electromagnetic field with frequency ω whose Hamiltonian H is given by

$$H = \hbar\omega a^\dagger a + i\hbar\Lambda \left[(a^\dagger)^2 e^{-2i\omega t} - a^2 e^{2i\omega t}\right] \tag{3}$$

where $a^\dagger$ and a are the creation and annihilation operators of the mode.
The first term of Eq. 3 is the energy of the mode for the free field. The second term describes a two-photon interaction process such as parametric amplification (a classical wave of frequency 2ω generating two photons with frequency ω). Λ is a real quantity characterizing the strength of the interaction.
In this problem, you will show that this Hamiltonian produces squeezed vacuum and explore how it acts on coherent states.

(a) Write the equation of motion for $a(t)$ using the Heisenberg picture. Take

$$a(t) = b(t)e^{-i\omega t}. \tag{4}$$

What are the equations of motion for $b(t)$ and $b^\dagger(t)$?

(b) The contribution of the mode $(\vec{k}, \vec{\varepsilon})$ to the electric field is

$$\vec{E}(\vec{r}, t) = i\mathcal{E}_\omega \vec{\varepsilon}\left[a(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger(t)e^{-i\vec{k}\cdot\vec{r}}\right] \tag{5}$$

where $a(t)$ is the solution of Eq. 4. Show that

$$b_P(t) = \frac{b(t) + b^\dagger(t)}{2} \qquad \text{and} \qquad b_Q(t) = \frac{b(t) - b^\dagger(t)}{2i} \tag{6}$$

represent physically two quadrature components of the field. Find equations of motion for $b_P(t)$ and $b_Q(t)$ and give their solutions, assuming that $b_P(0)$ and $b_Q(0)$ are known. Give the corresponding solutions for $b(t)$ and $b^\dagger(t)$ and express them in terms of $b(0)$ and $b^\dagger(0)$.

(c) Assume that at $t = 0$, the electromagnetic field is in the vacuum state. Calculate the mean number of photons $\langle N\rangle$ in the mode $(\vec{k}, \vec{\varepsilon})$ at time t and the dispersion $\Delta b_P(t)$ and $\Delta b_Q(t)$ of the two quadrature components of the field. Explain the results.

(d) In class, we defined squeezed vacuum with parameter $z = re^{i\phi}$ as

$$|0_z\rangle = S(z)|0\rangle = \exp\left[\frac{1}{2}z^* a^2 - \frac{1}{2}z a^{\dagger 2}\right]|0\rangle = \frac{1}{\sqrt{\cosh r}}\sum_{n=0}^{\infty}\frac{\sqrt{(2n)!}}{2^n n!}e^{i2n\phi}(\tanh r)^n|2n\rangle \tag{7}$$

Show that the two-photon interaction in Eq. 3 produces this state (and this operator) when you apply it for a time $t = t_0$. What is z?

Note: For this problem, you are expected to show that $S(z) = \exp\left[\frac{1}{2}z^* b^2 - \frac{1}{2}z b^{\dagger 2}\right]$ and find z, which should be real.

(e) Plot the Q function of the states that arise at time t under this two-photon interaction.

(i) Squeezed vacuum: $Q_1(\alpha) = |\langle\alpha|S(z)|0\rangle|^2$

(ii) A displaced squeezed state: $Q_2(\alpha) = |\langle\alpha|D(\beta)S(z)|0\rangle|^2$

(iii) A squeezed coherent state: $Q_3(\alpha) = |\langle\alpha|S(z)D(\beta)|0\rangle|^2$

Here, S is the squeezing operator defined in Eq. 7 and $D = \exp\left[\alpha a^\dagger - \alpha^* a\right]$ is the displacement operator. Compare plots made with $z = 0.2$, 1.2 and 4. Compare the displaced squeezed state $Q_2(\alpha)$ and the squeezed coherent state $Q_3(\alpha)$. Does the two-photon interaction in Eq. 3 create create amplitude or phase squeezing?

A note and some hints: There will be some ugly math for this question. We want you to understand the states that are produced by this squeezing, so you can just compute $Q(\alpha)$ numerically to plot it.

To compute it analytically, you can make use of the following relations (using $z = r$ real):

$$\begin{aligned}
S(r)|0\rangle &= \frac{1}{\pi}\frac{e^{r/2}}{\sqrt{e^{2z}-1}}\int_{-\infty}^{\infty} d\alpha\, e^{-[\alpha^2/(e^{2r}-1)]}|\alpha\rangle \\
D(\gamma)S(r) &= S(r)D(\gamma_+) \\
S(r)D(\gamma) &= D(\gamma_-)S(r)
\end{aligned}$$

where $\gamma_{\pm}(r) = (\cosh r)\gamma \pm (\sinh r)\gamma^{*}$.

Note: In the above integral over α, the phase dependence has already been integrated out and α is real. You can also find the analytic solution using the number state representation for $S(z)|0\rangle$.