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 Course: **8.421 - AMO I**
 Problem set: **#9**
 Due: Friday, April 15, 2022.

1. Transition Lifetimes and Blackbody Radiation.

- (a) (i) The rate of *absorption* is given by the product of Einstein's B coefficient and the average number of photons per mode $\langle n \rangle_{\omega_0}$ where ω_0 is angular frequency associated with the (dominant) transition with $\lambda_0 = 590$ nm.

$$W = B\langle n \rangle = 1/60 \text{ s}^{-1}.$$

From lecture, we know that the Einstein's B coefficient can be written in terms of the (known) Einstein's $A = \Gamma_0 = 1/\tau$ coefficient:

$$B = \frac{\pi^2 c^3}{\hbar \omega_0^3} A = \frac{\pi^2 c^3}{\hbar \omega_0^3} \frac{1}{\tau}$$

With this, we can plug in the numbers to find

$$\langle n \rangle = \frac{W}{B} = \frac{W \hbar \omega_0^3 \tau}{\pi^2 c^3} \approx 1.0324 \times 10^{-15}.$$

- (ii) For blackbody radiation, we have

$$\langle n \rangle = \frac{1}{e^{\hbar \omega_0 / k_B T} - 1} \implies T \approx 707.2 \text{ K}.$$

We see that in order for the absorption rate to reach 1 photon per minute, the blackbody temperature has to be ~ 700 K, which is way above room temperature (of course the higher the temperature, the higher the absorption rate, and vice versa). As a result, there is no need to shield the vacuum system from room temperature radiation or cool the vacuum system to cryogenic temperatures (as expected).

- (b) Here we estimate the lifetime of hydrogen in the $F = 1$ hyperfine level of the $1S$ state.

- (i) $F = 1 \rightarrow F = 0$ is a magnetic dipole transition.
 (ii) To estimate the lifetime of the $F = 1$ state, we may assume that the (magnetic dipole) transition matrix element is μ_B . From here, we work entirely in the CGS unit system to find

$$\Gamma_0 = \frac{4\omega_0^3 \mu_B^2}{3\hbar c^3} \approx 2.91 \times 10^{-15} \text{ s}^{-1}$$

where the numerical values for the fundamental constants can be found on Wikipedia. The lifetime is

$$\tau = \frac{1}{\Gamma_0} \approx 1.1 \times 10^7 \text{ years}.$$

- (c) Now we look at a hydrogen BEC in the $F = 1$ state.

- (i) To find the average number of photons per mode from blackbody radiation at the 21 cm line at 300 K and 4 K, we simply calculate

$$\langle n \rangle = \frac{1}{e^{\hbar \omega / k_B T} - 1}$$

at the corresponding temperatures and angular frequency. The answers are

$$\begin{aligned} T = 300 \text{ K}, \quad \langle n \rangle &\approx 4375 \\ T = 4 \text{ K}, \quad \langle n \rangle &\approx 57.84. \end{aligned}$$

- (ii) Similar to what we did before (but reversed), we can find the transition rates at $T = 300$ K and $T = 4$ K. We will also need the lifetime $\tau \approx 1.1 \times 10^7$ years for this calculation.

$$\begin{aligned} T = 300 \text{ K}, \quad W &= 1.484 \times 10^{11} \text{ s}^{-1} \\ T = 4 \text{ K}, \quad W &= 1.962 \times 10^9 \text{ s}^{-1}. \end{aligned}$$

- (iii) It is clear that we should be concerned about blackbody radiation from the environment limiting our experiment with hydrogen in the $F = 1$ state if we need a trapping times on the order of seconds/minutes.
- (d) The lifetime is much longer for the case of a magnetic dipole transition in hydrogen where there are more photons per mode mainly because the lifetime τ scales as $1/\omega_0^3$. The ratio between the sodium wavelength of 590 nm versus the 21 cm of hydrogen is $\sim 10^{-6}$. This gives a reduction factor of 10^{-18} . On top of this, there is also another factor of α^2 reduction when replacing the electric with magnetic dipole matrix element.

2. Saturation Intensity.

- (a) We first manipulate the Einstein A coefficient so that it is written in terms of the oscillator strength f , the fine structure constant α and the transition frequency ω . The oscillator strength is given by

$$f_{21} = \frac{2m\omega_{21}}{3\hbar} \frac{1}{2J_1 + 1} \sum_{m_1, m_2} |\langle J_1 m_1 | \vec{r} | J_2 m_2 \rangle|^2 = \frac{2m\omega_{21}}{3\hbar} \frac{S_{12}}{2J_1 + 1}.$$

where S_{12} is the line strength, while

$$A_{12} = \frac{4\omega^3 e^2}{3\hbar c^3} \sum_{m_2} |\langle 1m_1 | \vec{r} | 2m_2 \rangle|^2 = \frac{4\omega^3 e^2}{3\hbar c^3} \frac{S_{12}}{2J_1 + 1}.$$

From here we have that

$$A_{12} = \frac{4e^2 \omega_{21}^3}{3\hbar c^3} \frac{3\hbar f_{21}}{2m\omega_{21}} = \frac{2e^2 \omega_{21}^2 f_{21}}{mc^3} = \frac{2\hbar \omega_{21}^2 f_{21}}{mc^2} \frac{e^2}{\hbar c} = \frac{2\alpha \hbar \omega_{21}^2 f_{21}}{mc^2}.$$

Assuming $f_{21} = 1$ we may use this formula to estimate the lifetime of sodium. Plugging in the numerical values for the constants above (in CGS units), we find

$$A \approx 1.917 \times 10^8 \text{ s}^{-1} \implies \tau = \frac{1}{A} \approx 5.2 \text{ ns}.$$

More precisely, if we call f the *absorption* oscillator strength, then we actually have

$$A = \frac{2\alpha \hbar \omega^2}{mc^2} \frac{2J_1 + 1}{2J_2 + 1} f.$$

Assuming that we're working the D lines of sodium, we will take $J_1 = 1/2$ and $J_2 = 1/2$ and $3/2$. From Steck's, we find that $f = 0.64$ for the $3^2S_{1/2} \rightarrow 3^2P_{3/2}$ transition and $f = 0.32$ for the $3^2S_{1/2} \rightarrow 3^2P_{1/2}$ transition. Plugging in the numbers we find that

$$\tau_{1/2 \rightarrow 3/2} \approx \tau_{1/2 \rightarrow 3/2} \approx 16 \text{ ns},$$

as expected.

- (b) The saturation intensity for the principal transition in sodium, with $\sigma_0 = 3\lambda^2/2\pi$, is:

$$I_{\text{sat}} = \frac{\hbar \omega A}{2\sigma_0} \approx 38.8 \text{ mW/cm}^2.$$

where we are using A from the previous part and ignoring fine and hyperfine structure.

3. Saturation of Atomic Transitions.

- (a) Here we consider a two-state atom with $R_{ge} = R_{eg} = R$ the stimulated absorption/emission rate and $A = \Gamma$ the spontaneous emission rate. Define the saturation parameter s as $s = 2R/\Gamma$. In equilibrium, we have

$$\Gamma N_b + RN_b - RN_a = 0 \implies \frac{N_b}{N_a} = \frac{R}{R + \Gamma} = \frac{s\Gamma/2}{s\Gamma/2 + \Gamma} = \frac{s}{s + 2}.$$

- (b) The equilibrium spontaneous emission rate per atom AN_b can be expressed in terms of Γ and s and the total density $N = N_a + N_b = 1$. From part (a), we have

$$\frac{N_b}{N} = \frac{N_b}{N_b + N_a} = \frac{N_b/N_a}{N_b/N_a + 1} = \frac{s}{s + (s + 2)} = \frac{1}{2} \frac{s}{s + 1} = N_b.$$

So, we find

$$\Gamma N_b = \frac{1}{2} \frac{\Gamma s}{s + 1}.$$

Recalling the optical Bloch equations, we know that

$$N_b = \frac{\Omega^2/\Gamma^2}{1 + 2\Omega^2/\Gamma^2}.$$

Thus we may identify s with $2\Omega^2/\Gamma^2$. The scattering cross section has the form

$$\sigma = \frac{\sigma_0}{1 + I/I_{\text{sat}}} = \frac{\sigma_0}{1 + 2\Omega^2/\Gamma^2}.$$

where we have ignored the (small) detuning term and used the definition of I_{sat} . We can immediately see that $\sigma(s)$ bleaches out as $\sigma(s = 0)/(1 + s)$, as desired.

- (c) For $s = 1$, the energy density $\langle w \rangle_{\text{SAT}}$ per unit frequency is given by

$$\langle w \rangle = \frac{R}{B},$$

where B is the Einstein B coefficient. With $s = 1$, we find that

$$s = 1 = \frac{2R}{\Gamma} \implies R = \frac{\Gamma}{2} = B\langle w \rangle \implies \langle w \rangle = \frac{\Gamma}{2B}.$$

$\langle w \rangle$ is independent of the atomic dipole matrix element since both the $A = \Gamma$ and B Einstein coefficients are proportional to the matrix element (modulus) squared.

- (d) With

$$\frac{\Gamma}{B} = \frac{8\pi\hbar}{\lambda^3} = \frac{\hbar\omega^3}{\pi^2 c^3},$$

we find

$$\langle w \rangle_{\text{SAT}} = \frac{\hbar\omega^3}{2\pi^2 c^3}.$$

The mean occupation number n per photon mode for $s = 1$ is given by

$$\langle n \rangle = \frac{B\langle w \rangle_{\text{SAT}}}{\Gamma} = \frac{B}{\Gamma} \frac{\Gamma}{2B} = \frac{1}{2}.$$

- (e) We have laser light of intensity I_0 and Lorentzian lineshape centered at the atomic transition frequency ω_0 with FWHM $\Gamma' \gg \Gamma$. The energy density of this beam per frequency interval at ω_0 is

$$S(\omega_0) = \frac{I(\omega_0)}{c} = I_0 \frac{1}{\pi c} \frac{\Gamma'/2}{(\omega_0 - \omega_0)^2 + (\Gamma'/2)^2} = \frac{2I_0}{\pi c \Gamma'}.$$

We now want I_s such that

$$s = 1 = \frac{2R}{\Gamma} = \frac{2B\langle w \rangle}{\Gamma} = 2S(\omega_0) \frac{B}{\Gamma} = \frac{4I_s}{\pi c \Gamma'} \frac{\pi^2 c^3}{\hbar \omega^3} = \frac{4\pi c^2}{\Gamma' \hbar \omega^3} I_s \implies I_s = \frac{\Gamma' \hbar \omega^3}{4\pi c^2}.$$

- (f) The stimulated broadband absorption rate is given by

$$R = B\langle w \rangle = B \frac{2I_0}{\pi c \Gamma'} = \frac{2I_0}{\pi c \Gamma'} \frac{\Gamma \pi^2 c^3}{\hbar \omega^3} = \frac{2\pi \Gamma c^2 I_0}{\hbar \omega^3 \Gamma'} = \frac{\omega_R^2}{\Gamma'},$$

as desired. Here, we have used the result seen on the last problem set:

$$\omega_R^2 = \frac{2\pi \Gamma c^2 I_0}{\hbar \omega^3}.$$

For $s = 1$, we have $I_0 = I_s$, so

$$\omega_R^2 = \frac{2\pi \Gamma c^2}{\hbar \omega^3} \frac{\Gamma' \hbar \omega^3}{4\pi c^2} = \frac{\Gamma \Gamma'}{2}.$$

- (g) If we set $\Gamma' = \Gamma$, then we get exactly the saturation intensity of a monochromatic laser beam and the Rabi frequency at saturation. This is not surprising since when setting $\Gamma' = \Gamma$ we are no longer in the broadband regime.