1 Introduction

The following introduction to Electromagnetically Induced Transparency (EIT) is based on Stephen E. Harris' article in *PHYSICS TODAY*, 1997. Link, and various other sources including this well-written thesis by Furman of Reed College, this paper, and of course *Principles of Laser Spectroscopy and Quantum Optics* by Paul Berman.

In shortest possible terms, EIT is a technique which renders an otherwise opaque atomic medium transparent with electromagnetic radiation. The medium is typically a three-level atomic system, with specific requirements: two of the possible three transitions must be dipole-allowed (so transition rules satisfied) and one not dipole-allowed. The spectrum of the medium without (blue) and with (red) EIT is shown below:

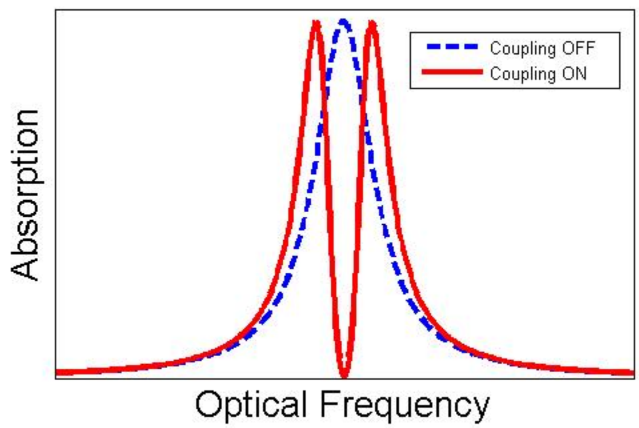

Figure 1

Notice how there is no absorption at resonance frequency with EIT. Typically, in the presence of a near-resonance field, a two-level atom with ground state $|1\rangle$ and excited state $|2\rangle$ will interact with the field, resulting in a non-zero probability amplitude of the excited state, $|P_2|^2 = \langle 2|2\rangle > 0$. If $|P_2(\delta\omega)|^2$ is the population of the excited state as a function of the detuning, then it follows the blue, Lorentzian line in the above figure. In EIT where there are two radiation fields, though, the energy levels of a three-level atom are altered. This in turn creates a window of frequencies at which the medium is transparent.

2 Derivation Overview

The following derivation is (heavily) inspired by Furman's derivation and the Heidelberg paper, but roughly condensed and sprinkled with a bit of my own narratives and insights here and there.

Let's consider a Λ configuration:

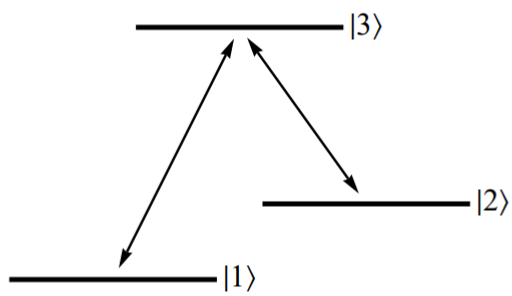

Figure 2

Let the energy of each state $|n\rangle$, $n = \{1, 2, 3\}$ be

$$E_n = \langle i| \hat{H} |n\rangle = \hbar\omega_n. \tag{1}$$

where $\hat{H}$ is the neutral atom Hamiltonian. Assume that the transition $|1\rangle \to |2\rangle$ is forbidden (just as shown in Figure 2). Since E_n and $|n\rangle$ are the eigenvalues and eigenstates of $\hat{H}$, respectively, we let the bare-atom Hamiltonian $\hat{H}_0$ be:

$$\hat{H}_0 = \hbar \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix}. \tag{2}$$

We can do this because the eigenstates $|n\rangle$ form an orthonormal basis. Next, let the applied fields be

$$\begin{aligned} \vec{E} &= \vec{E}_p \cos\Big(\omega_p t - \vec{k_p} \cdot \vec{r}\Big) + \vec{E}_c \cos\Big(\omega_c t - \vec{k_c} \cdot \vec{r}\Big) \\ &\approx \vec{E}_p \cos(\omega_p t) + \vec{E}_c \cos(\omega_c t) \\ &= \frac{\vec{E}_p}{2} \left(e^{i\omega_p t} - e^{-i\omega_p t}\right) + \frac{\vec{E}_c}{2} \left(e^{i\omega_c t} + e^{i\omega_c t}\right). \end{aligned} \tag{3}$$

where $\omega_p \approx \omega_{13} = \omega_3 - \omega_1$, and $\omega_c \approx \omega_{23} = \omega_3 - \omega_2$. The subscripts p and c means **probe** and **coupling**, respectively. We should also denote the relevant detuning $\delta_p = \omega_{13} - \omega_p$ and $\delta_c = \omega_{12} - \omega_c$. It makes sense to label our subscripts this way, because in the end we are interested in the probability amplitude of $|3\rangle$ as a function of the detuning δ_p of the probe beam from (bare atom) resonance.

With the perturbation from the radiation, the new Hamiltonian of the atom is:

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \tag{4}$$

where, with $\hat{\rho} \equiv qd$ being the dipole moment operator:

$$\hat{H}_{1,ij} = -E\hat{\rho}_{ij}, \tag{5}$$

where E is the field strength. We will see how E relates to the Rabi rate Ω later. Now, a state does not dipole-interact with itself, hence $\rho_{ii} = 0$. In addition, since the transition $|1\rangle \to |2\rangle$ is forbidden, $\rho_{12} = \rho 21 = 0$. It follows that:

$$\begin{aligned} \hat{H} &= \hat{H}_0 + \hat{H}_1 \\ &= \hbar \begin{pmatrix} \omega_1 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & \omega_3 \end{pmatrix} - E \begin{pmatrix} 0 & 0 & \rho_{13} \\ 0 & 0 & \rho 23 \\ \rho_{31} & \rho_{32} & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

Now, what we have been working with so far are the time-independent eigenstates $|n\rangle$. In the following steps we shall bring in the time-dependent parts. To do this, we invoke the unitary matrix $U(t)$, which transforms $|n\rangle$ into full time-dependent wavefunctions:

$$U(t) = e^{iH_0 t/\hbar} \begin{pmatrix} e^{i\omega_1 t} & 0 & 0 \\ 0 & e^{i\omega_2 t} & 0 \\ 0 & 0 & e^{i\omega_3 t} \end{pmatrix}. \tag{7}$$

Obviously, if we apply $\hat{U}(t)$ to $\hat{H}_0$ there wouldn't be anything interesting, since $\hat{U}(t) = d/dt\hat{H}_0$. But we can apply $\hat{U}(t)$ to $\hat{H}_1$. The change of basis rule gives

$$\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t) = -E \begin{pmatrix} 0 & 0 & \rho_{13}e^{-i\omega_{13}t} \\ 0 & 0 & \rho_{23}e^{-i\omega_{23}t} \\ \rho_{31}e^{i\omega_{13}t} & \rho_{32}e^{i\omega_{32}t} & 0 \end{pmatrix}. \tag{8}$$

Multiplying E into $\hat{H}_1$ and applying rotating wave approximation to result gives the non-zero matrix elements:

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{13} = -\frac{1}{2}E_p\rho_{13}e^{i(\omega_p-\omega_{13})t} \tag{9}$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{23} = -\frac{1}{2}E_c\rho_{23}e^{i(\omega_c-\omega_{23})t} \tag{10}$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{31} = -\frac{1}{2}E_p\rho_{31}e^{i(\omega_p-\omega_{31})t} \tag{11}$$

$$\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger\right)_{32} = -\frac{1}{2}E_c\rho_{32}e^{i(\omega_c-\omega_{32})t} \tag{12}$$

Transforming $\hat{H}_1$ back to the vector space of $|n\rangle$ gives:

$$\begin{aligned} \hat{H}_1 &= \hat{U}^\dagger(t)\left(\hat{U}(t)\hat{H}_1\hat{U}^\dagger(t)\right)\hat{U}(t) \\ &= -\frac{1}{2}\begin{pmatrix} 0 & 0 & E_p\rho_{13}e^{i\omega_p t} \\ 0 & 0 & E_c\rho_{23}e^{i\omega_c t} \\ E_p\rho_{31}e^{-i\omega_p t} & E_c\rho_{32}e^{-i\omega_c t} & 0 \end{pmatrix}. \end{aligned} \tag{13}$$

Now, to put the dipole moment in terms of the Rabi frequency Ω:

$$\Omega_p = \frac{E_p|\rho_{13}|}{\hbar} = \frac{E_p\rho_{13}e^{-i\phi_p}}{\hbar} \tag{14}$$

$$\Omega_c = \frac{E_c|\rho_{23}|}{\hbar} = \frac{E_c\rho_{23}e^{-i\phi_c}}{\hbar} \tag{15}$$