of Fig. 1(a) is particularly suitable for the study of the effects of friction. The laws of viscous friction can be easily verified by attaching a small sphere (up to 0.5 in. in diameter) to the end of the needle P and/or by changing the mass of the pendulum and/or by adding glycerine to water. However, care must be taken to keep the speed of the needle always slower than a few inches per second. On the contrary, forces that are almost proportional to the square of the speed become important, the decay of the amplitude is no longer exponential, and higher mathematics is required to interpret the results.[2] The disposition of Fig. 1(b) has been used to introduce the concept of normal modes.[3] By coupling the rigid pendulums $M_1, \ldots, M_n$ with light springs, one can show that appropriate combinations of the displacements exist that change harmonically with time, no matter what the starting conditions are. The recordings of the motion of two pendulums coupled by a spring and of their normal modes are shown in Fig. 5. The apparatus of Fig. 2 allows one to introduce concepts like complex response, energy stored and dissipated, These concepts are usually demonstrated with electrical experiments which are much easier to execute. However, from a didactic point of view, the direct observation of the behavior of a mechanical system is certainly worth the increase of experimental difficulties. All the experiments, designed for and used in a first year physics laboratory since 1973, were intended to be quantitative and the students were required to test the relationships between the parameters of the experiments (lengths, masses, elastic constants of springs, fluid viscosity, etc.) and the times (periods and decay times) measured on the recordings. Gratifying agreement has always been obtained.

[1] N. R. Isenor, Am. J. Phys. **37**, 1159 (1969).
[2] B. J. Miller, Am. J. Phys. **42**, 298 (1974); F. S. Crawford, Am. J. Phys. **43**, 276 (1975).
[3] F. S. Crawford, *Waves—Berkeley Physics Course* (McGraw-Hill, New York, 1965), Vol. III, Chap. 1.

More on the prehistory of the Laplace or Runge–Lenz vector

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In a recent note[1] I remarked that Hamilton was reputed to have discovered independently the vector constant of motion properly described as the Laplace vector, and that I would appreciate a specific reference in Hamilton's works. As I had hoped, a number of readers responded, citing a July 1845 paper of Hamilton[2] along with a few recent published references to this paper.[3,4] Hamilton had invented quaternions in 1843, and in the next few years he was busy elaborating his new theory and applying it to various problems in mathematics and physics. In July 1845 he delivered a paper to the Royal Irish Academy entitled "Applications of Quaternions to Some Dynamical Questions" in the course of which he derived the existence of a new constant of the motion for the Kepler problem, a constant which he subsequently called the "eccentricity vector." In 1846 he communicated to the same body[5] his investigations on the locus of the velocity vector, when drawn from the origin, as the particle moves through its orbit. To this locus he gave the now-customary name of "hodograph." He showed that for the elliptical orbits of the inverse square law of attraction the hodograph is a circle, a result that can be quickly demonstrated with the help of the Laplace or "eccentricity" vector. The cross product of the angular momentum $\mathbf{L}$ with the Laplace vector $\mathbf{A}$,

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu k\, \mathbf{r}/r,$$

leads to the relation

$$\mathbf{p} = (\mathbf{B}/L^2) - (\mu k/rL^2)\, \mathbf{r} \times \mathbf{L},$$

where $\mathbf{B}$ is the constant vector $\mathbf{A} \times \mathbf{L}$ normal to the semimajor axis. If the x axis is taken to be along the semimajor axis, the equation of the hodograph is then given by

$$p_x{}^2 + (p_y - A/L)^2 = (\mu k/L)^2.$$

The momentum hodograph is therefore a circle of radius $\mu k/L$ with center located a distance A/L from the center of force along a direction normal to $\mathbf{A}$. It is seen that the constant nature of the Laplace vector and the circular character of the hodograph are closely linked. Both Hamilton[6] and Tait[7] in their monographs on quaternions prove that $\mathbf{A}$ is constant as a preliminary to demonstrating that the hodograph is circular.

The connection between the Laplace vector and the hodograph opens up a broader area for studying the history of the Laplace vector. In the latter half of the 19th century the hodograph was a commonly discussed topic in British treatises on dynamics. Maxwell,[8] Thomson and Tait,[9] and Routh[10] all prove the circularity of the hodograph for inverse-square-law forces and all, in effect, trip over the constant Laplace vector in the course of the proof. Gibbs, the arch-foe of quaternions, has considerable discussion on hodographs in his *Vector Analysis,* but interestingly enough he does not give the hodograph for the Kepler problem. However a few pages on[11] he derives the constant Laplace vector on his way to obtaining the equation for the orbit. The derivation is probably the earliest in modern vector notation and antedates Runge's similar proof by at least 20 years.

These historical sidelights make it seem likely that the Laplace vector was not foreshadowed in Newton's *Principia* (except tenuously through the very existence of a closed orbit). Hamilton and his followers were all close students of Newton. Indeed Hamilton in an 1845 communication[12] remarked that he had "recently resumed study of a part of Sir Isaac Newton's *Principia*." None of these authors however suggest any connection with Newton's work, even though the circular hodograph theorem can be stated geometrically, in the manner so favored by Newton. In 1854

Percival Frost published a student's guide to Book I, Secs. I–III, of Newton's *Principia*. It evidently proved popular for it went through at least five editions by 1900. Frost discusses the hodograph and its circular form for elliptical orbits, but ascribes it entirely to Hamilton, with no mention of Newton.[13] It would appear thus that Newton did not discovery the vector constant of motion.[14]

Note added in proof. It has been called to my attention[15] that Professor Otto Volk[16] has traced the history of the Laplace–Runge–Lenz vector almost a full century further back than Laplace's publication. The interesting story he has uncovered deserves a brief summary here. In Newton's day, the usual goal was to discover the gravitational force law given the fact that planets and comets move in conic sections. The "inverse problem" (which we today would call rather the direct problem), of finding the orbits given the inverse-square law of force, was apparently first tackled in the early decades of the eighteenth century. Jakob Hermann, a disciple of the Bernoullis, in 1710 published in an obscure Italian journal[17] a direct integration of the orbit equation using the (then) new techniques of Leibniz's calculus. The magnitude of the Laplace vector appears as a constant of integration in the process. Hermann further clearly recognized the relation of the constant to the eccentricity of the conic sections. In the same year Hermann gave the result wider circulation by summarizing his paper in a letter to Johann I. Bernoulli which was published in the *Histoires et Memoires de l'Academie Royale des Sciences.*[18] The same volume of the *Memoires* contains a rather acerbic answering letter by Bernoulli,[19] who generalizes Hermann's derivation to allow for arbitrary orientation of the orbit in its plane. In effect, Bernoulli's proof gave the directions of the Laplace vector as well as its magnitude. Priority clearly belongs to Hermann and Johann Bernoulli, but in view of the long traditional association with Laplace it would seem most fitting to refer to the constant of motion as the Hermann–Bernoulli–Laplace vector.[20]

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2W. R. Hamilton, Proc. R. Irish Acad. **3** (1847), Appendix No. III, p. xxxvi ff, especially p. xxxix.

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5W. R. Hamilton, Ref. 2, p. 344 ff.

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7P. G. Tait, *An Elementary Treatise on Quaternions* (Clarendon, Oxford, 1867). In the enlarged 3rd ed. (Cambridge U. P., Cambridge, 1890), the page reference is Chap. XI, p. 283.

8J. C. Maxwell, *Matter and Motion* (Cambridge U. P., Cambridge, 1877; Dover, New York, 1952), p. 108.

9W. Thomson and P. G. Tait, *Treatise on Natural Philosophy* (Cambridge U.P., Cambridge, 1879); reprinted under the title *Principles of Mechanics and Dynamics* (Dover, New York, 1962), p. 26 ff.

10E. J. Routh, *A Treatise on Dynamics of a Particle* (Cambridge U. P., Cambridge, 1898; Dover, New York, 1960), p. 252 ff.

11J. W. Gibbs and E. B. Wilson, *Vector Analysis* (Scribners, New York, 1901), p. 135.

12W. R. Hamilton, Ref. 3, p. 309.

13P. Frost, *Newton's Principia, Sections I, II, III, with Notes and Illustrations* (Cambridge U. P., Cambridge, 1854). In the 2nd ed. of 1863 the page reference is p. 221.

14I am indebted to C. E. Wulfman, C. D. Collinson, and M. J. Laird who kindly sent me the references to Hamilton and to some of the other sources.

15Professor J. Stickforth, Technical University of Braunschweig, very kindly told me of Professor Volk's researches.

16O. Volk, Preprint No. 4 (Mathematisches Institut der Universität, Würzburg, 1975).

17J. Hermann, *Giornale de Letterati D'Italia* (1710), Vol. 2, pp. 447–467.

18J. Hermann, *Histoires de L'Academie Royale des Sciences avec les Mémoires de Mathematique et Physique* [Paris, (1710) 1712], pp. 102–103, and pp. 519–523. In the more accessible 1713 Amsterdam reprint the page numbers are 135–136 and 682–685.

19J. I. Bernoulli, *Histoires de L'Academie Royale des Sciences avec les Mémoires de Mathematique et Physique* [Paris, (1710) 1712], pp. 523–544. Also pp. 685–703, Amsterdam reprint of 1713.

20I am grateful to my colleague Dr. F. L. DiMaggio for his painstaking translation of Hermann's Italian paper.

On a misuse of superposition in circuit analysis

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To illustrate the power of the principles of symmetry and superposition, the intriguing problem of the resistance between adjacent nodes of an infinite, two-dimensional, square mesh of identical resistances is increasingly widely cited.[1-3] It is perhaps appropriate therefore to call attention to an error in one published solution.[4]

It correctly equates the specified circuit of Fig. 1(a) to that of Fig. 1(b), but then it assumes that the current in any branch of the circuit of Fig. 1(b) is the sum of those in the corresponding branches of Figs. 2(a) and 2(b). The assumption is valid only if the sources have infinite internal

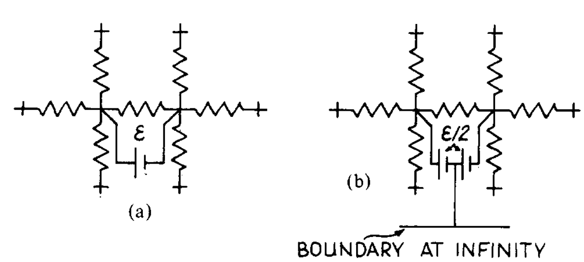

Fig. 1. (a) Original circuit. (b) Equivalent circuit.