Pset 8

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Problem 7.20.* For the metallic coaxial cable with the circular cross-section (see Fig. 7.20 of the lecture notes), find the lowest non-TEM mode and calculate its cutoff frequency.

Solution: The analysis may repeat that of the hollow circular waveguide (Fig. 7.23a) in Sec. 7.7 of the lecture notes, up to the derivation of the Bessel equation (7.140) for the radial factor $\mathcal{R}(\rho)$ of the longitudinal field f – depending on the mode, either E_z or H_z. However, in the coaxial cable, the axial points with $\rho = 0$ are inaccessible for the field, and hence, instead of the simple solution given by Eq. (7.141), we have to look for its radial part in the form of a linear superposition of the Bessel functions of the fist and the second kind:

$$f_{nm} = \left[c_1 J_n(k_{nm}\rho) + c_2 Y_n(k_{nm}\rho)\right]\cos n(\varphi - \varphi_0), \qquad \text{with } n = 0, 1, 2,\ldots,$$

where the eigenvalues k_{nm} of the transverse wave number k_t have to be chosen to satisfy the boundary conditions at $\rho = a$ and $\rho = b$, and (for a non-TEM field) give a nonuniform longitudinal field, $f_{nm} \neq$ const.

For the E-modes, the boundary condition Eq. (7.124) yields the following system of two equations for the constants $c_{1,2}$:

$$c_1 J_n(k_{nm}a) + c_2 Y_n(k_{nm}a) = 0,$$
$$c_1 J_n(k_{nm}b) + c_2 Y_n(k_{nm}b) = 0.$$

These two linear, homogeneous equations are compatible if the denominator of the system equals zero. This requirement yields

$$J_n(k_{nm}a)Y_n(k_{nm}b) = J_n(k_{nm}b)Y_n(k_{nm}a),$$

where the integer index m (taking values 1, 2, 3,…) numbers the roots of this characteristic equation – for each fixed angular index n (taking values 0, 1, 2,…). Introducing the dimensionless variable $\xi_{nm} \equiv k_{nm}a$, we may rewrite this characteristic equation as

$$J_n(\xi_{nm})Y_n\left(\xi_{nm}\frac{b}{a}\right) = J_n\left(\xi_{nm}\frac{b}{a}\right)Y_n(\xi_{nm}). \qquad (*)$$

The table on the right shows the results for the product $(b/a - 1)\xi_{nm} \equiv k_{nm}(b - a)$, obtained by numerical solution of this equation for a few values of the ratio b/a, and a few lowest numbers n and m.[88] It shows that the product $k_{nm}(b - a)$ changes very slowly with this ratio, staying close to the asymptotic value $n\pi$ (reached at $b/a \to 1$), for all reasonable values of b/a.

b/a	$n = 0$, $m = 1$	$n = 1$, $m = 1$	$n = 0$, $m = 2$	$n = 1$, $m = 2$
4	3.073	3.336	6.243	6.403
2	3.123	3.197	6.273	6.312
1	π	π	2π	2π

This result may be readily understood physically: at the cutoff frequency $\omega_c = k_{nm}v$ (when $k_t = k = 2\pi/\lambda$), approximately m TEM half-wavelengths fit the distance between the external and internal conductors – almost independently of n, provided that the latter number is not too high. (Still, as could be expected, the axially-symmetric distribution with $n = 0$ gives the lowest k_t, and hence the lowest cutoff frequency.)

For the H-modes, we need to use another boundary condition, Eq. (7.126), that gives, instead of Eq. (*), a different characteristic equation:

$$J_n'(\xi_{nm})Y_n'\left(\xi_{nm}\frac{b}{a}\right) = J_n'\left(\xi_{nm}\frac{b}{a}\right)Y_n'(\xi_{nm}), \qquad (**)$$

where the prime sign means the derivative of the Bessel function over its whole argument. The table on the right gives results for a different dimensionless combination, $(1 + b/a)\xi_{nm} \equiv k_{nm}(a + b)$, which is more relevant (virtually parameter-independent) in this case, for two lowest modes with $n = 1$.[89] It shows that for any realistic b/a ratio, this combination is close to $2m$. Physically, this means that at the cutoff frequency, m TEM wavelengths $\lambda = 2\pi/k_t$ fit the average circumference $p = \pi(a + b)$ of the cable's cross-section.

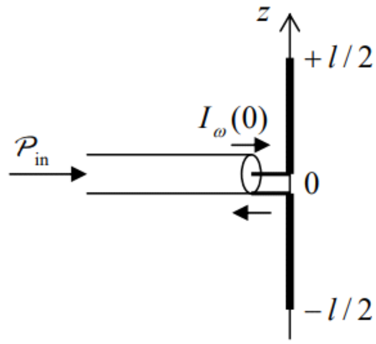
b/a	$n = 1$, $m = 1$	$n = 1$, $m = 2$
4	2.055	3.760
2	2.031	4.023
$\to 1$	$\to 2$	$\to 4$

Now we can compare the approximate values of the lowest k_{nm} for the E- and H-modes:

$$\left.\frac{\pi}{b-a}\right|_{\text{for } E_{01}} \leftrightarrow \left.\frac{2}{b+a}\right|_{\text{for } H_{11}},$$

so that (just as in the circular waveguide, see Sec. 7.7 of the lecture notes), the lowest non-TEM mode is H_{11}, with the cutoff frequency $\omega_c \approx 2v/(a + b)$, i.e. the TEM wavelength $\lambda_{max} \approx \pi(a + b)$. This result (which, at $a/b \to 0$, is close to Eq. (7.145) for the circular single-hole waveguide) is very important because it imposes a practical limit, $\lambda > \pi(a + b)$ for using coaxial cables as TEM transmission lines, in order to avoid unintentional excitation of the non-TEM modes on unavoidable small inhomogeneities.

Note, however, that in practical systems with long cables, the wavelength may be restricted even more severely by the wave attenuation effects – see Sec. 7.9 of the lecture notes.



Problem 8.3. Solve the dipole antenna radiation problem discussed in Sec. 8.2 of the lecture notes (see Fig. 8.3, partly reproduced on the right) for its optimal length $l = \lambda/2$, assuming[112] that the current distribution in each of its arms is sinusoidal:

$$I(z,t) = I_0 \cos\frac{\pi z}{l} \cos\omega t\,.$$

Solution: Since the main condition of the dipole approximation condition, $kl \ll 1$, is not satisfied for the antenna that long ($kl = (2\pi/\lambda)l = \pi$), we cannot use the formulas derived in Sec. 8.2, and need to return to the general expressions (8.17) for the retarded potentials. However, we may use the experience of Sec. 8.2, indicating that the fields in the far-field zone ($kr \gg 1$) may be more readily calculated from the vector-potential than from the scalar potential. Integrating Eq. (8.17b) over the antenna wire's cross-section, we get the following expression for the vector-potential in the free space surrounding the antenna: [113]

$$\mathbf{A}(\mathbf{r},t) = \mathbf{n}_z \frac{\mu_0 I_0}{4\pi} \int_{-l/2}^{+l/2} \cos\frac{\pi z'}{l} \cos\left[\omega\left(t - \frac{R}{c}\right)\right] \frac{dz'}{R} \equiv \mathbf{n}_z \frac{\mu_0 I_0}{4\pi} \mathrm{Re}\left[\int_{-l/2}^{+l/2} \cos\frac{\pi z'}{l} \exp\left\{-i\omega\left(t - \frac{R}{c}\right)\right\} \frac{dz'}{R}\right], \quad (*)$$

where $R = [\rho^2 + (z - z')^2]^{1/2}$, with ρ and z being the cylindrical coordinates of the observation point $\mathbf{r}$ in the reference frame shown in the figure above. At large distances, $r \gg l$, the argument of the complex exponent may be approximated as

$$\omega\left(t - \frac{R}{c}\right) \equiv \omega t - k\left[\rho^2 + (z - z')^2\right]^{1/2} \approx \omega t - k\left(\rho^2 + z^2 - 2zz'\right)^{1/2} \approx \omega t - kr\left(1 - \frac{zz'}{r^2}\right) \equiv (\omega t - kr) + kz'\cos\Theta\,,$$

where $k = \omega/c$ is the wave number, $r \equiv (\rho^2 + z^2)^{1/2}$, and $\Theta \equiv \cos^{-1}(z/r)$ is the angle between the direction toward the observation point and the antenna's axis z. Also, in the same limit, R in the denominator of

Eq. (*) may be approximately replaced with r, and moved out of the integral over z'.[114] These simplifications allow us to reduce Eq. (*), in the far-field zone, to

$$\mathbf{A}(\mathbf{r},t) \approx \mathbf{n}_z \frac{\mu_0 I_0}{4\pi r} \mathrm{Re}\left[\exp\{i(kr-\omega t)\}\int_{-l/2}^{+l/2} \cos\frac{\pi z'}{l} \exp\{-ikz'\cos\Theta\} dz'\right], \qquad \text{at } kr, kl >> 1,$$

and to work out of the remaining integral analytically:

$$\int_{-l/2}^{+l/2} \cos\frac{\pi z'}{l}\exp\{-ikz'\cos\Theta\}dz' \equiv \int_{-\pi/2k}^{+\pi/2k} \cos kz' \exp\{-ikz'\cos\Theta\}dz' \equiv \frac{1}{2}\sum_{\pm}\int_{-\pi/2k}^{+\pi/2k}\exp\{ik(\pm 1-\cos\Theta)z'\}dz'$$

$$= \frac{1}{2}\sum_{\pm}\frac{\exp\{ik(\pm 1-\cos\Theta)z'\}}{ik(\pm 1-\cos\Theta)}\Bigg|_{kz'=-\pi/2}^{kz'=+\pi/2} \equiv \sum_{\pm}\frac{\sin\left(\pm\frac{\pi}{2}-\frac{\pi}{2}\cos\Theta\right)}{k(\pm 1-\cos\Theta)} \equiv \sum_{\pm}\frac{\pm\cos\left(\frac{\pi}{2}\cos\Theta\right)}{k(\pm 1-\cos\Theta)} \equiv 2\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{k\sin^2\Theta}.$$

Since this result is purely real, the vector-potential equals

$$\mathbf{A}(\mathbf{r},t) = \mathbf{n}_z \frac{\mu_0 I_0}{2\pi}\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{\sin^2\Theta}\frac{\cos(\omega t - kr)}{kr} \equiv \mathbf{n}_z \frac{\mu_0 I(0, t-r/c)}{2\pi kr}\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{\sin^2\Theta}.$$

The structure of this expression is exactly the same as that of Eq. (8.23), with the following replacement:

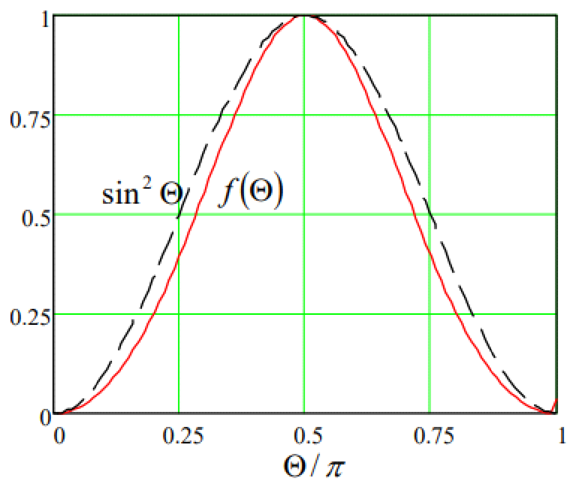
$$\dot{\mathbf{p}}(t) \to \mathbf{n}_z \frac{2I(0,t)}{k}\frac{\cos\left(\frac{\pi}{2}\cos\Theta\right)}{\sin^2\Theta},$$

and we may repeat all the transitions from that formula to Eq. (8.26) to get the following radiation power density, i.e. the radial component of the Poynting vector:

$$S_r = \frac{Z_0}{(4\pi cr)^2}\left(\frac{2\dot{I}(0,t)}{k}\right)^2 f(\Theta) = \frac{Z_0 I_0^2 \cos^2(\omega t - kr)}{4\pi^2 r^2} f(\Theta),$$

where

$$f(\Theta) \equiv \frac{\cos^2[(\pi/2)\cos\Theta]}{\sin^2\Theta}.$$



This expression shows that the angular distribution of the radiation, shown in the figure on the right, is somewhat (though not too much) different from that ($\sin^2\Theta$) of a radiating dipole – in particular, of a short dipole antenna analyzed in Sec. 8.2. Because of that, its solid-angle integral,

$$J \equiv \oint_{4\pi} f(\Theta)d\Omega = 2\pi\int_0^{\pi} f(\Theta)\sin\Theta d\Theta,$$

is also slightly different – it equals approximately 7.658 instead of $8\pi/3 \approx 8.378$ for $\sin^2\Theta$. As a result, the total average radiation power is

$$\mathscr{P} \equiv r^2 \oint_{4\pi} \overline{S}_r d\Omega \approx Z_0 \frac{J}{4\pi^2}\frac{I_0^2}{2}.$$

Just as was done in sec. 8.2 for the short antenna, we may recalculate this result into the antenna's impedance as "seen" by the generator (or transmission line) feeding it:

$$\operatorname{Re} Z_A = \frac{J}{4\pi^2} Z_0 \approx \frac{7.658}{4\pi^2} Z_0 \approx 73.1\,\Omega.$$

This impedance is very close to one of the coaxial cable standard values (75 Ω), making its matching with antenna straightforward. (If this statement is not clear, please revisit the discussion at the end of Sec. 7.5 of the lecture notes.) Again, all these results are only valid with an accuracy of a few percent, because they are based on the sinusoidal (rather than self-consistently calculated) distribution of the current along the antenna's length.

Problem 9.20

(a) Use the binomial expansion for the square root in Eq. 9.126:

$$\kappa \cong \omega\sqrt{\frac{\epsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\epsilon\omega}\right)^2 - 1\right]^{1/2} = \omega\sqrt{\frac{\epsilon\mu}{2}}\frac{1}{\sqrt{2}}\frac{\sigma}{\epsilon\omega} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}.$$

So (Eq. 9.128) $d = \frac{1}{\kappa} \cong \frac{2}{\sigma}\sqrt{\frac{\epsilon}{\mu}}$. qed

For pure water, $\begin{cases} \epsilon = \epsilon_r\epsilon_0 = 80.1\,\epsilon_0 \quad \text{(Table 4.2)}, \\ \mu = \mu_0(1+\chi_m) = \mu_0(1-9.0\times10^{-6}) \cong \mu_0 \quad \text{(Table 6.1)}, \\ \sigma = 1/(2.5\times10^5) \quad \text{(Table 7.1)}. \end{cases}$

So $d = (2)(2.5\times10^5)\sqrt{\frac{(80.1)(8.85\times10^{-12})}{4\pi\times10^{-7}}} = \boxed{1.19\times10^4\text{ m.}}$

(b) In this case $(\sigma/\epsilon\omega)^2$ dominates, so (Eq. 9.126) $k \cong \kappa$, and hence (Eqs. 9.128 and 9.129) $\lambda = \frac{2\pi}{k} \cong \frac{2\pi}{\kappa} = 2\pi d$, or $d = \frac{\lambda}{2\pi}$. qed

Meanwhile $\kappa \cong \omega\sqrt{\frac{\epsilon\mu}{2}}\sqrt{\frac{\sigma}{\epsilon\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{(10^{15})(4\pi\times10^{-7})(10^7)}{2}} = 8\times10^7$; $d = \frac{1}{\kappa} = \frac{1}{8\times10^7} = 1.3\times10^{-8} = \boxed{13\text{ nm.}}$ So the fields do not penetrate far into a metal—which is what accounts for their opacity.

(c) Since $k \cong \kappa$, as we found in (b), Eq. 9.134 says $\phi = \tan^{-1}(1) = 45°$. qed

Meanwhile, Eq. 9.137 says $\frac{B_0}{E_0} \cong \sqrt{\epsilon\mu\frac{\sigma}{\epsilon\omega}} = \boxed{\sqrt{\frac{\sigma\mu}{\omega}}}$. For a typical metal, then, $\frac{B_0}{E_0} = \sqrt{\frac{(10^7)(4\pi\times10^{-7})}{10^{15}}} = \boxed{10^{-7}\text{ s/m.}}$ (In vacuum, the ratio is $1/c = 1/(3\times10^8) = 3\times10^{-9}$ s/m, so the magnetic field is comparatively about 100 times larger in a metal.)