Quiz 2

Tuesday, March 1, 2022 8:06 AM

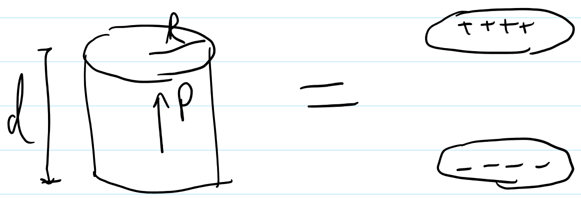
1. Consider a dielectric cylinder of radius R and length d with uniform polarization $\boldsymbol{P}$ along the axis of the cylinder.
(a) calculate the electric field $\boldsymbol{E}(\boldsymbol{r})$ at distances much larger than R and d. (10 pts)
(b) calculate the electric field inside the cylinder in the limit $d \ll R$. (10 pts)
Bonus: find $\boldsymbol{E}(\boldsymbol{r})$ at all $\boldsymbol{r}$ for arbitrary R, d (20 pts)

2. This problem is similar to one of the problems in Quiz 1, but please pay attention that the question asks to find the exact solution for arbitrary ratio d/R.

Calculate the mutual capacitance per unit length between two parallel cylindrical conductors that are of radius R, infinitely long, and separated at a distance d.
(30 pts)
Hint: image charge method

Solution.

1.



$\sigma_1 = P$ two charged disks

$\sigma_2 = -P$

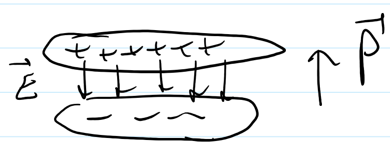
(a) At large distance, the system is equivalent to a dipole with dipole moment $\vec{p} = \sigma \pi R^2 d \, \hat{z}$

ϕ field of such dipole

$$\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\sigma \pi R^2 d \, \hat{z}\cdot\hat{r}}{r^3}$$

$$\vec{E}(\vec{r}) = -\nabla \phi(\vec{r}) = \frac{\sigma \pi R^2 d}{4\pi\varepsilon_0} \frac{3\vec{r} z - \hat{z} r^2}{r^5}$$

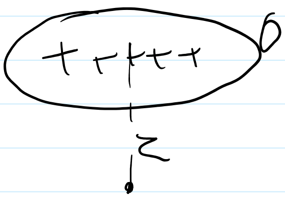
(b) for $d \ll R$



the system is equivalent to a plane capacitor

$$\vec{E} = -\frac{\sigma}{\varepsilon_0}\hat{z} = -\vec{P}/\varepsilon_0$$

(c) electric field of a charged disk



$$E_z(z) = \frac{\sigma z}{2\epsilon_0}\int_0^R \frac{r}{(r^2+z^2)^{3/2}}\,dr$$

$$= \frac{\sigma}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{(z^2+R^2)^{1/2}}\right)$$

Sum over two disks of opposite charge

$$E_z(z) = \frac{+P}{2\epsilon_0}\left\{\left(\frac{1}{z-\frac{d}{2}} - \frac{1}{\left[\left(z-\frac{d}{2}\right)^2+R^2\right]^{1/2}}\right) - \left(\frac{1}{z+\frac{d}{2}} - \frac{1}{\left[\left(z+\frac{d}{2}\right)^2+R^2\right]^{1/2}}\right)\right\}$$

2.

1.8.1 The capacitance of oppositely charged parallel cylinders

A two-dimensional problem, similar to the equal and opposite point charges above, is that of two parallel line charges of opposite sign. From Gauss's law, or from Sec. 1.5, the field due to a line charge (charge q per unit length) is equal to $2q/r$ at distance r from the line. The potential due to line charges q and $-q$ at a point distant by r_a and r_b from the two line charges is $2q\ln r_b/r_a$. The potential is constant when r_b/r_a is constant. As we shall see, this is on circles in planes perpendicular to the parallel line charges. The potential is thus constant on cylinders with axes parallel to the line charges. We take the line charges to intersect the $[x, y]$ plane at $[\pm\ell, 0]$; then

$$r_a^2 = (x-\ell)^2 + y^2, \quad r_b^2 = (x+\ell)^2 + y^2. \tag{1.56}$$

For constant $r_b/r_a = e^u$, we find

$$(x - \ell\coth u)^2 + y^2 = \ell^2/\sinh^2 u. \tag{1.57}$$

Suppose the cylinders have radii a, b, and their axes are separated by distance c. The cylinder surfaces are at u_a and $-u_b$, where $\sinh u_a = \ell/a$, $\sinh u_b = \ell/b$; their centers are at $x_a = \ell\coth u_a$, $x_b = -\ell\coth u_b$. Hence the distance between the cylinder axes is

$$c = \ell(\coth u_a + \coth u_b) = \sqrt{\ell^2 + a^2} + \sqrt{\ell^2 + b^2}. \tag{1.58}$$

The outer relation gives the value of ℓ in terms of a, b, c:

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The outer relation gives the value of ℓ in terms of a, b, c:

$$\ell^2 = \frac{(c+a+b)(c+a-b)(c-a+b)(c-a-b)}{4c^2}. \tag{1.59}$$

Figure 1.3 illustrates the two-cylinder geometry.

Since $r_b/r_a = e^u$, $V = 2qu$. The cylinder surfaces are at u_a and $-u_b$; the potential difference is therefore $V_a - V_b = 2q(u_a + u_b) = 2q(\operatorname{arcsinh} \ell/a + \operatorname{arcsinh} \ell/b)$. Thus the capacitance per unit length of the cylinder pair is

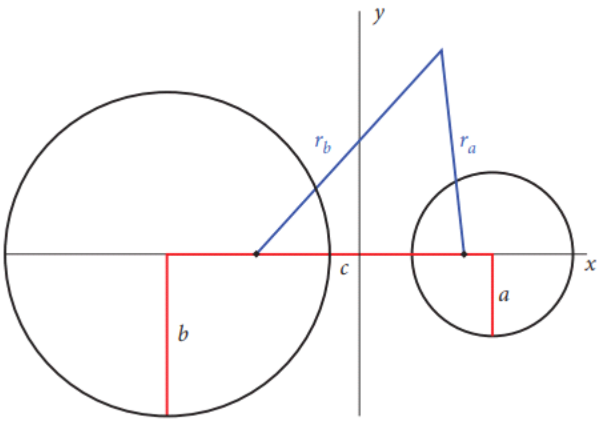

FIG. 1.3
Lengths in the two-cylinder problem: a, b are the radii of the two cylinders, and c is the distance between the cylinder axes. The figure is drawn for $b = 2a$, $c = 4a$. The bipolar coordinate lengths r_a, r_b are distances to a field point from the points $[\pm\ell, 0]$. The cylinder surfaces correspond to constant $u = \ln r_b/r_a$.

$$C(q, -q) = \frac{1}{2 \operatorname{arcsinh} \dfrac{\ell}{a} + 2 \operatorname{arcsinh} \dfrac{\ell}{b}}. \tag{1.60}$$

This problem is discussed in detail in Sec. 7.2; the solution given there is in bicylindrical coordinates u, v, one of which was used here to simplify the algebra. The first of the expressions given in (1.55) is equivalent to (1.60), as shown in Sec. 7.2.

A related problem is that of a *charged cylinder alongside an earthed conducting plane.* Suppose that cylinder of radius a is near a conducting surface at $x = 0$. Line charges $\pm q$ at $[\pm\ell, 0]$ will give zero potential on the $x = 0$ plane, and potential $V_a = 2qu_a = 2q \operatorname{arcsinh} \ell/a$ on the cylinder. Let d be the distance of the cylinder axis from the conducting plane ($d = c/2$ in our previous notation). The image of our cylinder is another cylinder of radius a with axis at $[-d, 0]$, so $\ell^2 = d^2 - a^2$ from (1.58). The capacitance per unit length of the cylinder-plane combination is therefore

$$C = \frac{q}{2qu_a} = \frac{1}{2\ln\left(\dfrac{d}{a} + \sqrt{\dfrac{d^2}{a^2} - 1}\right)}. \tag{1.61}$$

When the cylinder is far from the plane on the scale of its radius, the capacitance tends to

$$C = \frac{1}{2\ln\dfrac{2d}{a}} + O\left(\frac{a^2}{d^2}\right) \qquad (d \gg a). \tag{1.62}$$

The opposite limit is that of a cylinder close to the conducting plane. We set $d = a + s$ and expand in powers of s, the closest cylinder-plane separation, to find

$$C = \sqrt{\frac{a}{8s}} + O\left(\sqrt{\frac{s}{a}}\right) \qquad (a \gg s). \tag{1.63}$$