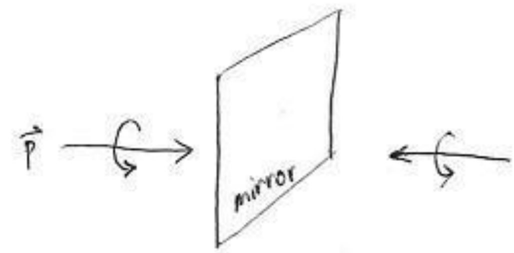
Discrete symmetries

Parity (P) flips the direction of spatial vectors

$$P : (t, \vec{x}) \to (t, -\vec{x})$$



momentum flips sign but spin remains the same.

As an operator on the creation and annihilation operators, we want

$$P^\dagger a^s_{\vec{p}} P = a^s_{-\vec{p}}$$
$$P^\dagger b^s_{\vec{p}} P = b^s_{-\vec{p}}$$

where P is some unitary operator, $P^\dagger P = P P^\dagger = 1$.

Taking the Hermitian conjugate, we also have

$$P^\dagger a^{s\dagger}_{\vec{p}} P = a^{s\dagger}_{-\vec{p}}$$
$$P^\dagger b^{s\dagger}_{\vec{p}} P = b^{s\dagger}_{-\vec{p}}$$

Actually this is a little too restrictive.
We could have

$$P^\dagger a^s_{\vec{p}} P = \eta_a a^s_{-\vec{p}}$$
$$P^\dagger b^s_{\vec{p}} P = \eta_b b^s_{-\vec{p}}$$

So long as $|\eta_a|^2 = |\eta_b|^2 = 1$. This is because all observables will have fermion operators in pairs and the phases η_a and η_b are not present in those:

$$P^\dagger a^{s'\dagger}_{\vec{p}'} a^s_{\vec{p}} P = a^{s'\dagger}_{-\vec{p}'} a^s_{-\vec{p}}$$
$$P^\dagger b^{s'\dagger}_{\vec{p}'} b^s_{\vec{p}} P = b^{s'\dagger}_{-\vec{p}'} b^s_{-\vec{p}}$$

Let us implement parity on $\psi(x)$:

$$P^\dagger \psi(x) P = \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(\eta_a a^s_{-\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + \eta_b^* b^{s\dagger}_{-\vec{p}} v^s(\vec{p}) e^{ip\cdot x} \right)$$

Let us define

$$\tilde{p} = (E_{\vec{p}}, -\vec{p})$$
$$\tilde{x} = (t, -\vec{x})$$

Note that

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix} \qquad \begin{matrix} \sigma = (1, \vec{\sigma}) \\ \bar{\sigma} = (1, -\vec{\sigma}) \end{matrix}$$

$$= \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\ \xi^s \\ \sqrt{\tilde{p}\cdot\sigma}\ \xi^s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u^s(-\vec{p}) = \gamma^0 u^s(-\vec{p})$$

$$v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\ \xi^s \\ -\sqrt{\tilde{p}\cdot\sigma}\ \xi^s \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \cdot v^s(-\vec{p})$$

$$= -\gamma^0 v^s(-\vec{p})$$

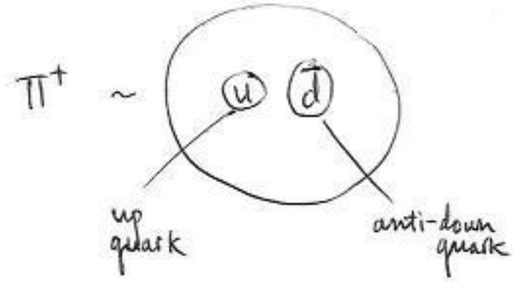
So therefore $\qquad (\tilde{p}\cdot\tilde{x} = p\cdot x)$

$$P^\dagger \psi(x) P = \gamma^0 \int \frac{d^3\vec{p}}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \sum_{s=1,2} \left(\eta_a a^s_{-\vec{p}} u^s(-\vec{p}) e^{-i\tilde{p}\cdot\tilde{x}} - \eta_b^* b^{s\dagger}_{-\vec{p}} v^s(-\vec{p}) e^{i\tilde{p}\cdot\tilde{x}} \right)$$

We see that if $\eta_a = -\eta_b^*$ then

$$P^\dagger \psi(x) P = \eta_a \gamma^0 \psi(\tilde{x})$$

Typically one chooses $\eta_a = +1$, $\eta_b = -1$. The choice is up to you. The relative minus sign between fermion and antifermion is the reason that pi-mesons have odd parity.



$$P^\dagger \bar{\psi}(x) P = P^\dagger \psi^\dagger(x) P \gamma^0 = (P^\dagger \psi(x) P)^\dagger \gamma^0$$

$$= \eta_a^* (\gamma^0 \psi(\tilde{x}))^\dagger \gamma^0 = \eta_a^* \psi^\dagger(\tilde{x}) \gamma^0 \gamma^0 = \eta_a^* \bar{\psi}(\tilde{x}) \gamma^0$$

$$(\gamma^{0\dagger} = \gamma^0)$$

Therefore $P^\dagger \bar{\psi}(x) \psi(x) P = P^\dagger \bar{\psi}(x) P P^\dagger \psi(x) P$

$$= |\eta_a|^2 \bar{\psi}(\tilde{x}) \psi(\tilde{x})$$

Can also show

$$P^\dagger \bar{\psi} \gamma^0 \psi P = \bar{\psi} \gamma^0 \psi(\tilde{x})$$

$$P^\dagger \bar{\psi} \gamma^i \psi P = \bar{\psi}(\tilde{x}) \gamma^0 \gamma^i \gamma^0 \psi(\tilde{x}) = -\bar{\psi} \gamma^i \psi(\tilde{x})$$

$$P^\dagger (i \bar{\psi} \gamma^5 \psi) P = i \bar{\psi}(\tilde{x}) \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) = -i \bar{\psi} \gamma^5 \psi$$

$$P^\dagger \bar{\psi} \gamma^\mu \gamma^5 \psi P = \bar{\psi}(\tilde{x}) \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(\tilde{x}) = \begin{cases} -\bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) & \mu = 0 \\ \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) & \mu = 1,2,3 \end{cases}$$

$\bar{\psi}\psi$ scalar
$i\bar{\psi}\gamma^5\psi$ pseudoscalar
$\bar{\psi}\gamma^\mu\psi$ vector (+ for $\mu=0$, − for $\mu=1,2,3$)
$\bar{\psi}\gamma^\mu\gamma^5$ axial vector (− for $\mu=0$, + for $\mu=1,2,3$)

Time Reversal

Suppose there exists a linear unitary operator T for time reversal. Then if $[T, H] = 0$

$$\Rightarrow T^\dagger e^{-iHt} T = e^{-iHt}$$

No good! We want direction of time reversed. Another possibility...

$$T^\dagger H T = -H$$
$$\text{or } \{T, H\} = 0$$

No good either since this implies that the eigenvalues of H and $-H$ are the same. This would mean H is unbounded below.

We instead assume something a little weird...
time reversal is conjugate-linear or "anti-linear."

$$T^\dagger \underset{\substack{\uparrow \\ \text{complex number}}}{c} T = c^*$$

We assume that $[T, H] = 0$ and $T^\dagger T = T T^\dagger = 1$.

Then we get $T^\dagger e^{-iHt} T = e^{+iHt}$.

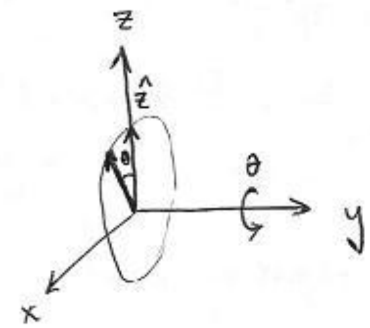
Time reversal is like watching a movie backwards.
Momentum reverses and spin reverses.

"Reversing" the spin is a bit complicated. Let us study it in detail.

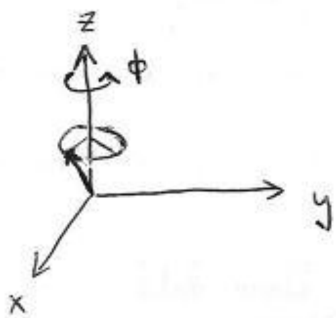
$$\underset{\substack{\text{spin-up} \\ z\text{-axis}}}{\xi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \qquad \underset{\substack{\text{spin-down} \\ z\text{-axis}}}{\xi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

We can find spin-up and spin down along the

(θ, ϕ) direction by rotating by angle θ



about the y axis and then rotating by angle ϕ



The transformation is

$$M(\theta, \phi) = \exp\left[-i\phi \frac{\sigma^3}{2}\right] \exp\left[-i\theta \frac{\sigma^2}{2}\right]$$

We can compute exp using a Taylor series

$$\exp\left[-i\theta \frac{\sigma^2}{2}\right] = 1 - \frac{i\theta}{2}\sigma^2 - \frac{\left(\frac{\theta}{2}\right)^2}{2!} + \frac{i\left(\frac{\theta}{2}\right)^3}{3!}\sigma^2 + \dots$$

odd terms give $-i \sin\frac{\theta}{2}\, \sigma^2$

even terms give $\cos\frac{\theta}{2}$

$$\exp\left[-i\theta\frac{\sigma^2}{2}\right] = \cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\,\sigma^2$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$\exp\left[-i\phi\frac{\sigma^3}{2}\right] = \cos\frac{\phi}{2} - i\sin\frac{\phi}{2}\,\sigma^3$$

$$= \begin{bmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{bmatrix}$$

So $$M(\theta,\phi) = \begin{bmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} & -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} & e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{bmatrix}$$

Therefore spin-up in the (θ,ϕ) direction is

$$\xi^1 = M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix} e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\sin\frac{\theta}{2} \end{pmatrix}$$

spin down in the (θ,ϕ) direction is

$$\xi^2 = M\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix} -e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2} \\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2} \end{pmatrix}$$

Interesting bit of trivia ...

A 360° rotation for a spin-$\frac{1}{2}$ particles gives an overall minus sign ...

set $\phi = 0$, $\theta = 2\pi$

This gives a 2π rotation about the y-axis

$$M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos\pi\\ \sin\pi\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix},\quad M\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\sin\pi\\ \cos\pi\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

Similarly $\phi = 2\pi$, $\theta = 0$ gives a 2π rotation about the z-axis

$$M\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}e^{-i\pi}\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix},\quad M\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\e^{i\pi}\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

In order to do a spin flip we reverse the direction by taking $(\theta+\pi, \phi)$ (note that taking $(\theta-\pi,\phi)$ gives an overall minus sign difference)

$$\xi^1(\theta+\pi,\phi) = \begin{pmatrix}e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}+\frac{\pi}{2}\right)\\ e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right)\end{pmatrix} = \begin{pmatrix}-e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}\\ e^{i\frac{\phi}{2}}\cos\frac{\theta}{2}\end{pmatrix}$$

$$\xi^2(\theta+\pi,\phi) = \begin{pmatrix}-e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}+\frac{\pi}{2}\right)\\ e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}+\frac{\pi}{2}\right)\end{pmatrix} = \begin{pmatrix}-e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}\\ -e^{i\frac{\phi}{2}}\sin\frac{\theta}{2}\end{pmatrix}$$

So if $\xi^s = (\xi^\uparrow, \xi^\downarrow)$ then $\underset{\text{(spin-reversed)}}{\xi^{-s}} = (\xi^\downarrow, -\xi^\uparrow)$

You may also notice that

$$\xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{1*}$$

$$\xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{2*}$$

So we can write ξ^{-s} in a fancy way

$$\xi^{-s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{s*} \qquad s = 1,2$$

$$= -i\sigma^2 \xi^{s*}$$

This is convenient since our time reversal operator involves complex conjugation.

We can show

$$u^{-s}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2 \xi^{s*}) \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2 \xi^{s*}) \end{pmatrix} \qquad \tilde{p} = (E_{\vec{p}}, -\vec{p})$$

If we use the identity

$$\sqrt{\tilde{p}\cdot\sigma}\;\sigma^2 = \sigma^2 \sqrt{p\cdot\sigma^*} \qquad \text{(takes a few steps to prove)}$$

use $\sigma^2 \vec{\sigma} = -\vec{\sigma}^* \sigma^2$

$$u^{-s}(-\vec{p}) = -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(\vec{p})]^*$$

$$= -\gamma^1\gamma^3 [u^s(\vec{p})]^*$$

Similarly $\quad v^{-s}(-\vec{p}) = -\gamma^1\gamma^3 [v^s(\vec{p})]^*.$

Now can define the time reversal operation on the creation and annihilation operators

$$T^\dagger a^s_{\vec{p}} T = a^{-s}_{-\vec{p}}$$

$$T^\dagger b^s_{\vec{p}} T = b^{-s}_{-\vec{p}}$$

where $\quad a^{-s}_{-\vec{p}} = (a^{\downarrow}_{-\vec{p}}, -a^{\uparrow}_{-\vec{p}})$

$\qquad\quad b^{-s}_{-\vec{p}} = (b^{\downarrow}_{-\vec{p}}, -b^{\uparrow}_{-\vec{p}})$

So $\quad T^\dagger \psi(x) T =$

$$\int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s T^\dagger (a^s_{\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{ip\cdot x}) T$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \Big(a^{-s}_{-\vec{p}} \underbrace{[u^s(\vec{p})]^*}_{\gamma^1\gamma^3 u^{-s}(-\vec{p})} e^{ip\cdot x} + b^{-s\dagger}_{-\vec{p}} \underbrace{[v^s(\vec{p})]^*}_{\gamma^1\gamma^3 v^{-s}(-\vec{p})} e^{-ip\cdot x}\Big)$$

[$\gamma^1\gamma^3$ is the inverse of $-\gamma^1\gamma^3$]

$$= \gamma^1\gamma^3 \psi(x_T) \quad \text{where} \quad x_T = (-t, \vec{x})$$