2) Observable quantities correspond to Hermitian operators whose eigenstates form a complete set

Observable quantity = something you can measure in an experiment.

[Note: book constructs $\mathcal{H}$ from estates of A; logic less clear as $\mathcal{H}_A \neq \mathcal{H}_B$ for some A, B]

3) An observable $H = H^\dagger$ defines the time evolution of the state in $\mathcal{H}$ through

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = i\hbar \lim_{\Delta t \to 0} \frac{|\psi(t+\Delta t)\rangle - |\psi(t)\rangle}{\Delta t} = H |\psi(t)\rangle.$$

(Schrödinger equation)

4) (Measurement & collapse postulate)

If an observable A is measured when the system is in a normalized state $|\alpha\rangle$, where A has an ON basis of eigenvectors $|a_i\rangle$ with eigenvalues a_i,

a) The probability of observing $A = a$ is

$$\sum_{j:\, a_j = a} |\langle a_j | \alpha\rangle|^2 = \langle \alpha | P_a | \alpha \rangle$$

where $P_a = \sum_{j:\, a_j = a} |a_j\rangle\langle a_j|$ is the projector onto the $A = a$ eigenspace.

b) If $A = a$ is observed, after the measurement the s[tate] becomes $|\alpha_a\rangle = P_a|\alpha\rangle = \sum_{j:\, a_j = a} |a_j\rangle\langle a_j|\alpha\rangle$

(normalized state is $|\tilde{\alpha}_a\rangle = |\alpha_a\rangle / \sqrt{\langle \alpha_a | \alpha_a \rangle}$.

Discussion of rule (4):

Simplest case: nondegenerate eigenvalues

$$|\alpha\rangle = \sum c_i |a_i\rangle , \qquad a_i \neq a_j .$$

Then probability of getting $A = a_i$ is $|c_i|^2$.

Norm of $\langle\alpha|\alpha\rangle = 1 \iff \sum |c_i|^2 = 1$.

After measuring $A = a_i$, state becomes $|\hat{\alpha}_{a_i}\rangle = |a_i\rangle$.

This postulate involves an
<u>irreversible</u>, <u>nondeterministic</u>, and <u>discontinuous</u>
change in the state of the system.

~~Leads to many confused discussions.~~

- source of considerable confusion
- less troublesome picture: path integrals.
- alternatives: non-local hidden variables ('t Hooft)?
  string theory → new principles (?)

For purposes of this course, take (4) as fundamental, though counterintuitive, postulate.

To discuss probabilities, need <u>ensembles</u>

Consequence of (4).
<u>Expectation value</u> of an observable A in state $|\alpha\rangle$ is

$$\langle A\rangle = \sum_i |c_i|^2 a_i = \langle\alpha|A|\alpha\rangle$$

since $A = \sum_i |a_i\rangle a_i \langle a_i|$.

Some further properties of S_i:

$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k$$

$$\{S_i, S_j\} = S_iS_j + S_jS_i = \frac{1}{2}\hbar^2\delta_{ij}$$

$$S^2 = \underline{S}\cdot\underline{S} = S_x^2 + S_y^2 + S_z^2 = \tfrac{3}{4}\hbar^2\mathbb{1} = \frac{3\hbar^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

And

$$[S^2, S_i] = 0.$$

Compatible vs. incompatible observables

Observables A, B are:

Compatible if $[A,B] = AB - BA = 0$

incompatible if $[A,B] \neq 0$.

Examples: S^2, S_z are compatible
S_x, S_y are not compatible.

Theorem: Compatible observables A, B can be simultaneously diagonalized, and have eigenvectors $|a_i, b_i\rangle$ with

$$A|a_i, b_i\rangle = a_i|a_i, b_i\rangle$$
$$B|a_i, b_i\rangle = b_i|a_i, b_i\rangle.$$

(Proof in last lecture: $AB|\alpha\rangle = a\,B|\alpha\rangle$ if $A|\alpha\rangle = a|\alpha\rangle$ so $B: \mathcal{H}_a \to \mathcal{H}$. diagonalize in each block.)

A <u>complete set of commuting observables</u> (CSCO) is a set of observables $\{A, B, C, \dots\}$ such that all observables in the set commute:

$$[A,B] = [A,C] = [B,C] = \dots = 0$$

and such that for any $a, b, \dots$ at most one solution exists to the eigenvalue equations

$$A|\alpha\rangle = a|\alpha\rangle$$
$$B|\alpha\rangle = b|\alpha\rangle$$
$$\vdots$$

<u>Tensor product spaces</u> (useful for many-particle systems) [+ quantum computing, ...]

Given two Hilbert spaces $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$, with ~~bases~~ complete ON bases $|\phi_i^{(1)}\rangle, |\phi_j^{(2)}\rangle$, the tensor product

$$\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$$

is the Hilbert space with ON basis

$$|\phi_{i,j}\rangle = |\phi_i^{(1)}\rangle \otimes |\phi_j^{(2)}\rangle$$

and inner product

$$\langle \phi_{i,j} | \phi_{k,l}\rangle = \langle \phi_i^{(1)} | \phi_k^{(1)}\rangle_1 \langle \phi_j^{(2)} | \phi_l^{(2)}\rangle_2 .$$

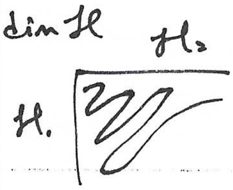
[e.g. 2 qubits: $|++\rangle, |+-\rangle, |-+\rangle, |--\rangle$]

If $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ have dimensions N, M, then $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ has dimension NM.

If $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ seperable, $\mathcal{H}$ is separable.

[in particular, if either or both of $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}$ have countable basis, & both countable or finite $\mathcal{H}$ has countable basis.]

Note: Finite product of countable-∞ dim $\mathcal{H}^{(i)} \rightarrow$ countable ∞ dim $\mathcal{H}$



Tensor products of kets and operators

Kets:

If
$$|\alpha\rangle = \sum c_i |\phi_i^{(1)}\rangle \in \mathcal{H}^{(1)}$$
$$|\beta\rangle = \sum d_j |\phi_j^{(2)}\rangle \in \mathcal{H}^{(2)}$$
are kets in $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$.

then
$$|\alpha\rangle \otimes |\beta\rangle = \sum_{i,j} c_i d_j |\phi_{i,j}\rangle \in \mathcal{H}$$
is in $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$.

[Note: not all vectors in $\mathcal{H}$ are of tensor product form.
Ex. $|\phi_{1,1}\rangle + |\phi_{1,2}\rangle + |\phi_{2,1}\rangle$]

Operators:

If A, B are operators on $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$,
then we can construct

$A \otimes B$ as an operator on $\mathcal{H} = \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$
through
$$(A \otimes B)|\phi_{i,j}\rangle = \left(A|\phi_i^{(1)}\rangle\right) \otimes \left(B|\phi_j^{(2)}\rangle\right).$$

[defines $A \otimes B$ on all of $\mathcal{H}$ by linearity]

- If A, B are observables, then $A \otimes B$ is an observable.

Simple class of operators on $\mathcal{H}$:

$$A \otimes \mathbb{1}, \qquad \mathbb{1} \otimes B.$$

If A, B act on $\mathcal{H}^{(1)}$, $\mathcal{H}^{(2)}$, will often refer to these as just A, B when context is clear.

~~Note~~ $\cancel{\sum A \otimes \mathbb{1}, \mathbb{1} \otimes B}$

Useful relation:

$$(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD).$$

Note: $[(A \otimes \mathbb{1}), (\mathbb{1} \otimes B)] = 0.$

Notation: in many books, tensor product symbol $\otimes$ is omitted

$|\alpha\rangle \otimes |\beta\rangle \Rightarrow |\alpha\rangle|\beta\rangle$

$A \otimes B \Rightarrow A\,B.$

CSCO's in tensor product spaces

If $\{A_1, A_2, \dots, A_k\}$ are a CSCO for $\mathcal{H}^{(1)}$,
& $\{B_1, \dots, B_\ell\}$ are a CSCO for $\mathcal{H}^{(2)}$,

then $\{A_1, \dots, A_k, B_1, \dots B_\ell\}$ are a CSCO for $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$

[Ex of notation $A_1 = A_1 \otimes \mathbb{1}$.]

Example of tensor products: Two spin-1/2 particles

Consider two spin-1/2 particles with Hilbert spaces $\mathcal{H}_2^{(1)}, \mathcal{H}_2^{(2)}$

The two-particle Hilbert space is

$$\mathcal{H} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)}.$$

A basis for $\mathcal{H}$ is:

$$|{++}\rangle = |+\rangle_1 \otimes |+\rangle_2$$
$$|{+-}\rangle = |+\rangle_1 \otimes |-\rangle_2$$
$$|{-+}\rangle = |-\rangle_1 \otimes |+\rangle_2$$
$$|{--}\rangle = |-\rangle_1 \otimes |-\rangle_2$$

Operators:

A complete set of commuting observables is

$$S_z^{(1)} = S_z^{(1)} \otimes \mathbb{1}$$
$$S_z^{(2)} = \mathbb{1} \otimes S_z^{(2)} .$$

Consider operators

$$S_z = S_z^{(1)} + S_z^{(2)} .$$

$$= \begin{pmatrix} \hbar & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\hbar \end{pmatrix}$$

$$S_z^{(1)} S_z^{(2)} = (S_z^{(1)} \otimes \mathbb{1})(\mathbb{1} \otimes S_z^{(2)})$$

$$= S_z^{(1)} \otimes S_z^{(2)}$$

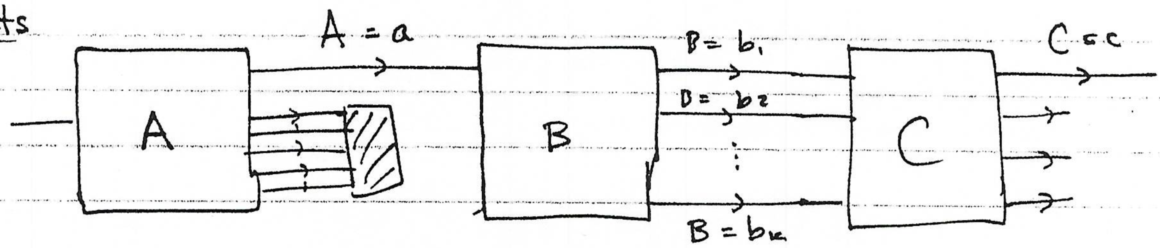
$$= \begin{pmatrix} \hbar/2 & & & \\ & \hbar/2 & & \\ & & -\hbar/2 & \\ & & & -\hbar/2 \end{pmatrix} \begin{pmatrix} \hbar/2 & & & \\ & -\hbar/2 & & \\ & & \hbar/2 & \\ & & & -\hbar/2 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix} .$$

Incompatible observables

If $[A,B] \neq 0$, then cannot simultaneously diagonalize A, B.

Experiments



(Assume A, B, C nondegenerate)

i) Allow all b_i to combine without measuring **B**

$$\underset{[B \text{ not measured}]}{\text{Probability}}(C=c) = |\langle c | a \rangle|^2$$

ii) Measure B, & allow all parts to combine

$$\text{Probability}(B=b_i) = |\langle b_i | a \rangle|^2$$

$$\text{Prob.}(C=c \text{ given } B=b_i) = |\langle c | b_i \rangle|^2$$

$$\underset{[\text{when } B \text{ measured}]}{\text{Prob}}(C=c) = \sum_i |\langle c | b_i \rangle|^2 |\langle b_i | a \rangle|^2$$

$$= \sum_i \langle c | b_i \rangle \langle b_i | a \rangle \langle a | b_i \rangle \langle b_i | c \rangle$$

$$= \sum_i z_i^* z_i \,, \qquad z_i = \langle a | b_i \rangle \langle b_i | c \rangle$$

know $(\sum z_i)(\sum z_i^*) = |\langle a|c\rangle|^2$.

So prob $(C=c)$ does not depend on measurement of B when

$$\sum_i z_i^* z_i = \left(\sum_i z_i^*\right)\left(\sum_j z_j\right).$$

Sufficent condition: only one $z_i \neq 0$,

so either $\langle a|b_i\rangle = 0$ or $\langle c|b_i\rangle = 0$
for all but one value of i

Sufficient condition: Either $[A,B]=0$ or $[B,C]=0$.

Dispersion

For A an observable, $|\alpha\rangle$ a state,

define $\Delta A = A - \langle A\rangle$

$\langle \Delta A^2\rangle$ is dispersion of A.

$$\langle \Delta A^2\rangle = \langle A^2 - 2A\langle A\rangle + \langle A\rangle^2\rangle$$

$$= \langle A^2\rangle - \langle A\rangle^2$$

is variance (a.k.a. mean square deviation) of A.

If $A|\alpha\rangle = a|\alpha\rangle$

$$\langle \Delta A^2 \rangle = a^2 - a^2 = 0.$$

Variance : measures "fuzziness" of state.

<u>Example:</u> In state $|+\rangle$

$$\langle \Delta S_z^2 \rangle = \left[(1 \;\; 0) \begin{pmatrix} \hbar^2/4 & \\ & \hbar^2/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$

$$- \left[(1 \;\; 0) \begin{pmatrix} \hbar/2 & \\ & -\hbar/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]^2$$

$$= \hbar^2/4 - \hbar^2/4 = 0.$$

$$\langle \Delta S_x^2 \rangle = (1 \;\; 0) \begin{pmatrix} \hbar^2/4 & \\ & \hbar^2/4 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$- \left[(1 \;\; 0) \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]^2$$

$$= \hbar^2/4.$$

Uncertainty relation

If A, B are observables,

$$\langle \Delta A^2 \rangle \langle \Delta B \rangle^2 \geq \frac{1}{4} \left| \langle [A,B] \rangle \right|^2$$

Proof:

Schwarz: (on $\Delta A|\alpha\rangle$, $\Delta B|\alpha\rangle$)

$$\left(\langle\alpha|\Delta A\right)\left(\Delta A|\alpha\rangle\right)\left(\langle\alpha|\Delta B\rangle\right)\left(\Delta B|\alpha\rangle\right)$$

$$\geq \left(\langle\alpha|\Delta A\right)\left(\Delta B|\alpha\rangle\right)\left(\langle\alpha|\Delta B\right)\left(\Delta A|\alpha\rangle\right)$$

$$= \left|\langle\alpha|\left(\tfrac{1}{2}[\Delta A, \Delta B] + \tfrac{1}{2}\{\Delta A, \Delta B\}\right)|\alpha\rangle\right|^2$$

$$= \frac{1}{4}\Big| \underbrace{\langle [A,B] \rangle}_{\substack{\text{imaginary} \\ ([A,B] \text{ skew-Hermitian}) \\ [\text{prob } 1-1.]}} + \underbrace{\langle \{\Delta A, \Delta B\} \rangle}_{\substack{\text{real} \\ (\{\Delta A, \Delta B\} \text{ Hermitian})}} \Big|^2$$

$$= \frac{1}{4}\left|\langle [A,B]\rangle\right|^2 + \frac{1}{4}\left|\langle \{\Delta A, \Delta B\}\rangle\right|^2$$

$$\geq \frac{1}{4}\left|\langle [A,B]\rangle\right|^2.$$

Example: In state $|\alpha\rangle = |+\rangle$,

$$\langle \Delta S_z^2 \rangle = 0$$
$$\langle \Delta S_x^2 \rangle = \hbar^2/4$$

$$\frac{1}{4}\left|\langle [S_z, S_x] \rangle\right|^2 = \frac{1}{4}\left|\langle S_y \rangle\right|^2 = 0.$$