1.2 Mathematical Preliminaries

1.2.1 Hilbert spaces

First postulate of QM:

* The state of a QM system at time t is given by a vector (ray) $|\alpha\rangle$ in a complex Hilbert space $\mathcal{H}$.
[will state more precisely soon.]

Vector spaces

A vector space V is a collection of objects ("vectors") $|\alpha\rangle$ having the following properties:

A1: $|\alpha\rangle + |\beta\rangle$ gives a unique vector $|\gamma\rangle$ in V.

A2: (commutativity) $|\alpha\rangle + |\beta\rangle = |\beta\rangle + |\alpha\rangle$

A3: (associativity) $(|\alpha\rangle + |\beta\rangle) + |\gamma\rangle = |\alpha\rangle + (|\beta\rangle + |\gamma\rangle)$

A4: $\exists$ vector $|\phi\rangle$ such that $|\phi\rangle + |\alpha\rangle = |\alpha\rangle$ $\forall |\alpha\rangle$

A5: For all $|\alpha\rangle$ in V, $-|\alpha\rangle$ is also in V so that $|\alpha\rangle + (-|\alpha\rangle) = |\phi\rangle$.

[(A1-A5): V is a commutative group under $+$]

For some field F (i.e., $\mathbb{R}$, $\mathbb{C}$, with $+$, $*$ defined) scalar multiplication of any $c \in F$ with any $|\alpha\rangle \in V$ gives a vector $c|\alpha\rangle \in V$.
Scalar multiplication has the following properties

M1: $c(d|\alpha\rangle) = (cd)|\alpha\rangle$

M2: $1|\alpha\rangle = |\alpha\rangle$

M3: $c(|\alpha\rangle + |\beta\rangle) = c|\alpha\rangle + c|\beta\rangle$

M4: $(c+d)|\alpha\rangle = c|\alpha\rangle + d|\alpha\rangle$.

V is called a "Vector space over F".

$F = \mathbb{R}$: "real v.s."
$F = \mathbb{C}$: "complex v.s.".

Examples of vector spaces

a) Euclidean D-dimensional space is a real V.S. $\begin{pmatrix} a_1 \\ \vdots \\ a_D \end{pmatrix}$

b) State space of spin ½ particle is 2D cpx V.S.
states: $c_+|+\rangle + c_-|-\rangle$, $c_\pm \in \mathbb{C}$
[note: certain states phys. equiv.]

c) Space of functions $f: [0, L] \to \mathbb{C}$

Henceforth we always take $F = \mathbb{C}$

Subspaces

$V \subset W$ is a subspace of a V.S. W if V satisfies all props of V.S. (with same $+$, $c*$ as W) and is a subset of W.

[suffices for V to be closed under $+$, scalar mult.]

Ray

A ray in V is a 1D subspace $\{c|\alpha\rangle\}$.

Linear independence & bases

$|\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_n\rangle$ are <u>linearly independent</u> iff

$$c_1|\alpha_1\rangle + c_2|\alpha_2\rangle + \ldots + c_n|\alpha_n\rangle = 0$$

has only $c_1 = c_2 = \cdots = c_n = 0$ as a solution.

If $|\alpha_1\rangle, \ldots, |\alpha_n\rangle$ in V are lin. independent, but all sets of $n+1$ vectors are lin. dependent, then

- V is <u>n-dimensional</u> (n may be finite, countably ∞, or uncountably ∞.)
- $|\alpha_1\rangle, \ldots, |\alpha_n\rangle$ form a <u>basis</u> for the space V.

If $|\alpha_1\rangle, \ldots, |\alpha_n\rangle$ are a basis for V, then any vector $|\beta\rangle$ can be expanded in the basis as

$$|\beta\rangle = \sum_i c_i |\alpha_i\rangle. \qquad \text{(Thm.)}$$

Unitary spaces

A complex vector space V is a <u>unitary space</u> (a.k.a. inner product space)

if given $|\alpha\rangle, |\beta\rangle \in V$ there is an inner product $\langle\alpha|\beta\rangle \in \mathbb{C}$ with the following properties:

I1 : $\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle^*$

I2 : $\langle \alpha | (|\beta\rangle + |\beta'\rangle) = \langle \alpha | \beta \rangle + \langle \alpha | \beta' \rangle$

I3 : $\langle \alpha | (c |\beta\rangle) = c \langle \alpha | \beta \rangle$

I4 : $\langle \alpha | \alpha \rangle \geq 0$

I5 : $\langle \alpha | \alpha \rangle = 0$ iff $|\alpha\rangle = |\emptyset\rangle$.

$\langle \alpha | \beta \rangle$ is a <u>sesquilinear</u> form (linear in $|\beta\rangle$, conj. lin. in $|\alpha\rangle$)

<u>Ex's.</u> a) $V = \mathbb{C}^N$, N-tuples $|z\rangle = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}$, $z_i \in \mathbb{C}$.

$$\langle z | w \rangle = z_1^* w_1 + z_2^* w_2 + \dots + z_N^* w_N$$

b) $V = \{ f : [0, L] \to \mathbb{C} \}$

$$\langle f | g \rangle = \int_0^L f^*(x)\, g(x)\, dx$$

Terminology:

$\sqrt{\langle \alpha | \alpha \rangle}$ = <u>norm</u> of $|\alpha\rangle$ (sometimes, $\| \, |\alpha\rangle \, \|$)

if $\langle \alpha | \beta \rangle = 0$, $|\alpha\rangle, |\beta\rangle$ are <u>orthogonal</u>

Dual spaces

For a (complex) vector space V, the dual space V^* is the set of linear functions $\langle\beta| : V \longrightarrow \mathbb{C}$.

Given the inner product $\langle\beta|\alpha\rangle$, can construct isomorphis[m] [defined by $f_{|\alpha\rangle} : |\beta\rangle \to \langle\alpha|\beta\rangle$ using inner product]

$$|\alpha\rangle \longleftrightarrow \langle\alpha|$$
$$V \longleftrightarrow \hat{V}^* \subset V^*$$
note: $$c|\alpha\rangle \longleftrightarrow c^*\langle\alpha|.$$

(Note: $\hat{V}^* = V^*$ cts dual for Hilbert space)

Physics notation (Dirac "bra-ket" notation)

$|\alpha\rangle \in V$ ket
$\langle\beta| \in V^*$ bra

Hilbert space

A space V is complete if every Cauchy sequence $\{|\alpha_n\rangle\}$ converges in V

$\forall \varepsilon\ \exists N : \| |\alpha_n\rangle - |\alpha_m\rangle \| < \varepsilon \quad \forall m,n > N$

(i.e., $\exists\ |\alpha\rangle : \lim_{n\to\infty} \| |\alpha\rangle - |\alpha_n\rangle \| = 0$.)

A complete (complex) unitary space is a Hilbert space.

Note: an example of an incomplete unitary space is the space of vectors with a finite number of nonzero entries. The sequence $\{(1, 0, 0, \ldots), (1, \tfrac{1}{2}, 0, 0, \ldots), (1, \tfrac{1}{2}, \tfrac{1}{3}, 0, \ldots), \vdots\}$ is Cauchy, but doesn't converge in V. This is not a Hilbert space.

A Hilbert space can be:

a) Finite dimensional (basis $|\alpha_1\rangle, \ldots, |\alpha_n\rangle$)
Ex. spin 1/2 particle in Stern-Gerlach expt.

b) Countably infinite dimensional (basis $|\alpha_1\rangle, |\alpha_2\rangle, \ldots$)
Ex. Quantum SHO

c) Uncountably infinite dimensional (basis $|\alpha_x\rangle$, $x \in \mathbb{R}$)

[Technical aside:

A space V is separable if $\exists$ countable set $D \subset V$ so that $\overline{D} = V$.
(D dense in V: $\forall \varepsilon, |x\rangle \in V \ \exists \ |y\rangle \in D$: $\| |x\rangle - |y\rangle \| < \varepsilon$)

(a), (b) are separable, (c) is not.

non-separable Hilbert spaces are very dicey mathematical

Generally, separability implicit in discussion
– e.g., label basis $|\alpha_i\rangle$, i takes discrete values

Orthonormal basis

An orthonormal basis is a basis $|\varphi_i\rangle$ with $\langle\varphi_i|\varphi_j\rangle = \delta_{ij}$

Any basis $|\alpha_1\rangle, |\alpha_2\rangle, \ldots$ can be made orthonormal by Schmidt orthonormalization

[Note: no pages 1-15, 1-16]

$$|\phi_1\rangle = \frac{|\alpha_1\rangle}{\sqrt{\langle\alpha_1|\alpha_1\rangle}}$$

$$|\alpha_2'\rangle = |\alpha_2\rangle - |\phi_1\rangle\langle\phi_1|\alpha_2\rangle$$

$$|\phi_2\rangle = \frac{|\alpha_2'\rangle}{\sqrt{\langle\alpha_2'|\alpha_2'\rangle}}$$

$$|\alpha_3'\rangle = |\alpha_3\rangle - |\phi_1\rangle\langle\phi_1|\alpha_3\rangle - |\phi_2\rangle\langle\phi_2|\alpha_3\rangle$$

$$\vdots$$

Gives $|\phi_i\rangle$ with $\underline{\langle\phi_i|\phi_j\rangle} = \delta_{ij}$.

If $\{|\phi_i\rangle\}$ are an orthonormal basis then for all $|\alpha\rangle$,

$$|\alpha\rangle = \sum_i C_i|\phi_i\rangle, \qquad C_i = \langle\phi_i|\alpha\rangle.$$

Can write as $|\alpha\rangle = \sum_i |\phi_i\rangle\langle\phi_i|\alpha\rangle$.

(completeness relation)

<u>Schwarz inequality</u>

$$\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle \geq |\langle\alpha|\beta\rangle|^2 \qquad \forall\ |\alpha\rangle, |\beta\rangle.$$

<u>Pf.</u> write $|\delta\rangle = |\alpha\rangle + \lambda|\beta\rangle$

$$\langle\delta|\delta\rangle = \langle\alpha|\alpha\rangle + \lambda\langle\alpha|\beta\rangle + \lambda^*\langle\beta|\alpha\rangle + |\lambda|^2\langle\beta|\beta\rangle$$

set $\lambda = -\dfrac{\langle\beta|\alpha\rangle}{\langle\beta|\beta\rangle}$

$$\rightarrow \langle\delta|\delta\rangle = \langle\alpha|\alpha\rangle - \frac{|\langle\alpha|\beta\rangle|^2}{\langle\beta|\beta\rangle} \geq 0 \quad \square$$

1.2.2 Operators

Second postulate of QM:
* Observables are self-adjoint (Hermitian) operators on $\mathcal{H}$

Linear operators

A linear operator from a vs V to a v.s. W is a transformation $A: V \to W$ such that

$$A(|\alpha\rangle + |\beta\rangle) = A|\alpha\rangle + A|\beta\rangle \quad \forall \; |\alpha\rangle, |\beta\rangle \in V$$
$$\& \; A(c|\alpha\rangle) = cA|\alpha\rangle$$

write $A = B$ iff $A|\alpha\rangle = B|\alpha\rangle \quad \forall |\alpha\rangle$

Assume V a Hilbert space $\mathcal{H}$ $(\cong \mathcal{H}^*)$, $A: \mathcal{H} \to \mathcal{H}$ a linear operator

A acts on V^* through $(\langle\beta|A)|\alpha\rangle = \langle\beta|(A|\alpha\rangle)$
[acts on right on bras]

Outer product: $|\beta\rangle\langle\alpha|$ simple class of ops

$$(|\beta\rangle\langle\alpha|)|\gamma\rangle = |\beta\rangle\langle\alpha|\gamma\rangle$$

[Note: physics notation hides math defs e.g. $\langle\alpha|\beta\rangle$ i.p vs lin Fun]

A is bounded iff $\displaystyle\sup_{\substack{|\alpha\rangle \in \mathcal{H} \\ |\alpha\rangle \neq 0}} \frac{\langle\alpha|A|\alpha\rangle}{\langle\alpha|\alpha\rangle} < \infty$

An Unbounded operator A generally has a domain of definition $\mathrm{Dom}(A)$

ex. $\mathcal{H} = \mathcal{L}^2(\mathbb{R}) = \{f: \mathbb{R} \to \mathbb{C} : \int_{-\infty}^{\infty} f^* f < \infty\}$

$e^{-x^2} \in \mathcal{H}$. consider $\mathcal{O}$ = mult by e^{x^2}

$\mathcal{O} \cdot e^{-x^2} = 1$ is not in $\mathcal{H}$, so $e^{-x^2} \notin \mathrm{Dom}(e^{x^2})$

generally $A: \underset{\text{lin subspace}}{\mathrm{Dom}(A)} \longrightarrow \mathcal{H}$ [unfortunately relevant: e.g. X]

Adjoint (generalizes $M^\dagger = (M^T)^*$)

For a bounded operator A, define $A^\dagger$ (Hermitian adjoint of A)

$$\langle\beta|\left(A^\dagger|\alpha\rangle\right) = \langle\beta'|\alpha\rangle, \quad |\beta'\rangle = A|\beta\rangle.$$

i.e. $\langle\alpha|A \longleftrightarrow A^\dagger|\alpha\rangle$ using $\mathcal{H} \cong \mathcal{H}^*$

Ex. $\left(|\beta\rangle\langle\alpha|\right)^\dagger = |\alpha\rangle\langle\beta|$

Follows that $\langle\alpha|A^\dagger|\beta\rangle = \left(\langle\beta|A|\alpha\rangle\right)^*$

Hermitian operators

A is Hermitian (symmetric) if $A = A^\dagger$

= self-adjoint if bounded

[unbounded: A SA if symm + $\mathrm{Dom}(A) = \mathrm{Dom}(A^\dagger)$]

[note: unbounded A can have mult self-adj extensions
e.g. Dirac in mag. monopole bg. See Reed + Simon, Jackiw...]

In general, need self-adjoint for eg spectral theorem, but we will be sloppy henceforth...