Final Exam Tickets Prep; MA353, S19

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1 Instructions

- Your answers should contain the same level of detail, and follow the same general guidelines as the solutions to the problems on the assignments. These guidelines are presented on our course web site.
- Part of your task here is to know the flow of the material as presented in the course, and to avoid potentially "circular" arguments.
  In particular you can refer to the results mentioned in other tickets, but only if these results precede (in <u>our</u> development of the subject) the one you are presenting.
  If you are unsure about this in a context of a particular problem, please talk to me a few days ahead of the test.
- If you find errors or potential errors or typos in the problems, please let me know right away!
- If you find, as a group, that the distribution of the problems across of the tickets would be more even if two particular problems were swapped, please let me know EARLY, and I will take it under consideration.

1 Ticket #1

Problem 1

Suppose that $\boldsymbol{V}$ a finite-dimensional vector space, and $\mathcal{E}_i : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ are idempotents such that

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \ldots + \mathcal{E}_n = \mathcal{I}_{\boldsymbol{V}}.$$

Prove that these idempotents are mutually annihilating.

Problem 2

Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are inner product spaces, and $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ is a linear function.

1. Argue that the following claims are equivalent.

   (a) $\mathcal{L}$ is an **isometry**; i.e. $\left\|\mathcal{L}(v)\right\|_{\boldsymbol{W}} = \|v\|_{\boldsymbol{V}}$, for all $v \in \boldsymbol{V}$.

   (b) $\left\langle \mathcal{L}(v), \mathcal{L}(z) \right\rangle_{\boldsymbol{W}} = \langle v, z \rangle_{\boldsymbol{V}}$, for all $v, z \in \boldsymbol{V}$.

   (c) $\mathcal{L}^* \mathcal{L} = \mathcal{I}_{\boldsymbol{V}}$.

   (d) $\left\|\mathcal{L}(v)\right\|_{\boldsymbol{W}} = 1$, for every unit vector $v \in \boldsymbol{V}$.

2. Argue that the following claims are equivalent.

   (a) $\mathcal{L}$ is a scalar multiple of an isometry.

   (b) $\left\|\mathcal{L}(v)\right\|_{\boldsymbol{W}} = \left\|\mathcal{L}(z)\right\|_{\boldsymbol{W}}$, for any unit vectors $v, z \in \boldsymbol{V}$.

   (c) $\mathcal{L}$ **preserves orthogonality**; i.e.

   $$\text{if } \langle v, z \rangle_{\boldsymbol{V}} = 0 \text{ then } \left\langle \mathcal{L}(v), \mathcal{L}(z) \right\rangle_{\boldsymbol{W}} = 0.$$

3. Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are fdips with orthonormal bases Γ and Ω respectively. Argue that $\mathcal{L}$ is an isometry if and only if the columns of $\left[\mathcal{L}\right]_{\Omega \leftarrow \Gamma}$ are orthonormal.

2 Ticket #2

Problem 3

Suppose that $\boldsymbol{V}$ a finite-dimensional vector space, and $\mathcal{E}_i : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ are idempotents such that

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 = \mathcal{I}_{\boldsymbol{V}}.$$

Prove that

$$Range\,(\mathcal{E}_1) \uplus Range\,(\mathcal{E}_2) \uplus Range\,(\mathcal{E}_3) \uplus Range\,(\mathcal{E}_4) = \boldsymbol{V},$$

and then argue that EVERY decomposition of $\boldsymbol{V}$ as a direct sum of four subspaces can be obtained this way.

Problem 4

Let $\boldsymbol{V}$ be an n-dimensional inner product space.

1. Suppose that Γ and Ω are two orthonormal bases of $\boldsymbol{V}$. Argue that $[\mathcal{I}]_{\Omega\leftarrow\Gamma}$ is a unitary matrix.
2. Argue that for each unitary matrix $\mathcal{U}$ in $\mathbb{M}_n$ there exist orthonormal bases Γ and Ω of $\boldsymbol{V}$ such that $\mathcal{U} = [\mathcal{I}]_{\Omega\leftarrow\Gamma}$.
3. Suppose that $\mathcal{A}, \mathcal{B} \in \mathbb{M}_n$. Argue that the following claims are equivalent.

   (a) There exists a linear function $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ and orthonormal bases Γ and Ω of $\boldsymbol{V}$ such that

   $$\mathcal{A} = [\mathcal{L}]_{\Gamma\leftarrow\Gamma} \quad \text{and} \quad \mathcal{B} = [\mathcal{L}]_{\Omega\leftarrow\Omega}.$$

   (b) There exists a unitary matrix $\mathcal{U}$ in $\mathbb{M}_n$ such that

   $$\mathcal{B} = \mathcal{U}^{-1}\mathcal{A}\mathcal{U}.$$

3 Ticket #3

Problem 5

Suppose that $\boldsymbol{V}$ a vector space, and $\mathcal{E} \in \mathfrak{L}(\boldsymbol{V})$. Prove that the following claims are equivalent.

1. $\mathcal{E}^2 = \mathcal{E}$ (i.e. $\mathcal{E}$ is idempotent).
2. $\mathit{Range}\,(\mathcal{E}) = \{\ V \in \boldsymbol{V} \mid \mathcal{E}(V) = V\ \}$.
3. $\mathit{Range}\,(\mathcal{E}) = \mathit{Nullspace}\,(\mathcal{I} - \mathcal{E})$.
4. $\mathcal{I} - \mathcal{E}$ is idempotent.
5. $\mathit{Nullspace}\,(\mathcal{E}) = \mathit{Range}\,(\mathcal{I} - \mathcal{E})$.

Problem 6

Suppose that $\boldsymbol{V}$ is an fdips and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove that the following claims are equivalent.

1. $\mathcal{L}$ is normal.
2. $\boldsymbol{V}$ has an orthonormal basis comprised of the eigenvectors of $\mathcal{L}$.
3. $\mathcal{L}$ is a linear combination of projections that resolve the identity on $\boldsymbol{V}$.
4. $\mathcal{L}$ is a linear combination of mutually annihilating projections.

4 Ticket #4

Problem 7

Suppose that $\boldsymbol{V}$ is an fdips.

1. Suppose that $\mathcal{A} \in \mathfrak{L}(\boldsymbol{V})$ is normal. Prove that
$$\mathit{Range}\,(\mathcal{A}) = (\mathit{Nullspace}\,(\mathcal{A}))^{\perp} \ .$$
2. Prove that the following claims are equivalent for an idempotent $\mathcal{E} \in \mathfrak{L}(\boldsymbol{V})$.
   (a) $\mathcal{E}$ is positive semi-definite.
   (b) $\mathcal{E}$ is self-adjoint (i.e. is a projection).
   (c) $\mathcal{E}$ is normal.
   (d) $\mathit{Range}\,(\mathcal{E}) = (\mathit{Nullspace}\,(\mathcal{E}))^{\perp}$.

Problem 8

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove each of the following claims.

1. $\mathcal{L}$ is unitary if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ have modulus 1.
2. $\mathcal{L}$ is self-adjoint if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ are real.
3. $\mathcal{L}$ is positive semi-definite if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ are non-negative.
4. $\mathcal{L}$ is positive definite if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ are positive.

5 Ticket #5

Problem 9

Here $\boldsymbol{V}$ is an fdips.

1. Suppose that $\mathcal{E}_i : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ are projections such that

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 = \mathcal{I}_{\boldsymbol{V}} .$$

   Prove that

$$Range\,(\mathcal{E}_1) \ominus Range\,(\mathcal{E}_2) \ominus Range\,(\mathcal{E}_3) \ominus Range\,(\mathcal{E}_4) = \boldsymbol{V} ,$$

   and then argue that EVERY decomposition of $\boldsymbol{V}$ as an orthogonal direct sum of four subspaces can be obtained this way.

2. Suppose that $\mathcal{P} \in \mathfrak{L}(\boldsymbol{V})$ is a projection with range $\boldsymbol{W}$. Prove that for each $V \in \boldsymbol{V}$, $\mathcal{P}(V)$ is the element of $\boldsymbol{W}$ that is closest to V.

Problem 10

Suppose that $\boldsymbol{V}$ is a finite-dimensional vector space, and $\mathcal{L}$ is a diagonalizable linear operator on $\boldsymbol{V}$ with

$$\sigma_{\mathbb{C}}(\mathcal{L}) = \{\lambda_1, \lambda_2, \dots, \lambda_k\} .$$

1. Suppose that

$$\mathcal{L} = \gamma_1 \mathcal{F}_1 + \gamma_2 \mathcal{F}_2 + \cdots + \gamma_m \mathcal{F}_m ,$$

   for some idempotents $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ which resolve the identity on $\boldsymbol{V}$, and some *distinct* complex numbers $\gamma_1, \gamma_2, \dots, \gamma_m$. Argue that $m = k$, that

$$\sigma_{\mathbb{C}}(\mathcal{L}) = \{\lambda_1, \lambda_2, \dots, \lambda_k\} = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$$

   and that $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ are the atomic spectral idempotents of $\mathcal{L}$.

2. Explain how one discerns the atomic spectral idempotents and the eigenspaces of $\mathcal{L}$ from any $\mathcal{L}$-eigenbasis of $\boldsymbol{V}$. Justify your claims.

6 Ticket #6

Problem 11

Suppose that $\boldsymbol{V}$ is a vector space, and $T \in \mathfrak{L}(\boldsymbol{V})$. Prove that

$$\boldsymbol{V} = T^{-1}[\boldsymbol{W}] + T^{-1}[\boldsymbol{Z}] \,,$$

whenever $\boldsymbol{W}$ and $\boldsymbol{Z}$ are subspaces of $\boldsymbol{V}$ such that

$$\mathit{Range}\,(T) = \boldsymbol{W} + \boldsymbol{Z} \,.$$

Problem 12

Here $\boldsymbol{V}$ is a 15-dimensional vector space, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove that

$$\boldsymbol{V} = \mathit{Nullspace}\left(\mathcal{L}^{15}\right) \oplus \mathit{Range}\left(\mathcal{L}^{15}\right) \,.$$

Problem 13

Suppose that $\boldsymbol{V}$ a finite-dimensional vector space, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove that for any polynomial p,

$$\sigma_{\mathbb{C}}\big(p(\mathcal{L})\big) = \{\ p(\lambda) \mid \lambda \in \sigma_{\mathbb{C}}(\mathcal{L})\ \} \,.$$

7 Ticket #7

Problem 14

Suppose that p_1 and p_2 are relatively prime non-constant polynomials of degrees m and n respectively. Consider the function

$$\Psi : \mathbb{P}_{n-1} \times \mathbb{P}_{m-1} \longrightarrow \mathbb{P}_{m+n-1}$$

defined by

$$\Psi\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) := f \cdot p_1 - g \cdot p_2 \ .$$

Verify each of the following claims.

1. Ψ is a linear function.
2. Ψ is injective.
3. Ψ is surjective.
4. There exist polynomials q_1 and q_2 such that

   $$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1} \ ,$$

   where $\mathbb{1}$ is the constantly 1 polynomial.

Problem 15

Here $\boldsymbol{V}$ is an fdips and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove each of the following claims.

1. $\mathcal{L}$ is self-adjoint exactly when

   $$\langle \mathcal{L}(X), X \rangle \in \mathbb{R}, \quad \text{for all} \quad X \in \boldsymbol{V} .$$

2. If $\mathcal{L}$ is self-adjoint, so is $\mathcal{M}^*\mathcal{L}\mathcal{M}$.
3. If $\mathcal{L}$ is positive semi-definite, so is $\mathcal{M}^*\mathcal{L}\mathcal{M}$.
4. If $\boldsymbol{W}$ is an fdips and $\mathcal{M} \in \mathfrak{L}(\boldsymbol{V}, \boldsymbol{W})$, then $\mathcal{M}^*\mathcal{M}$ and $\mathcal{M}\mathcal{M}^*$ are positive semi-definite.

8 Ticket #8

Problem 16

Suppose that $\boldsymbol{V}$ is a finite-dimensional vector space, and $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_n$ are subspaces of $\boldsymbol{V}$. Prove that the following claims are equivalent.

1. The subspace sum $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}$ is direct.
2. If $x_i \in \boldsymbol{V}_i$ and
$$x_1 + x_2 + x_3 + \ldots + x_{315} = 0_{\boldsymbol{W}} \ ,$$
then $x_i = 0_{\boldsymbol{W}}$, for every i.
3. If $x_i, y_i \in \boldsymbol{V}_i$ and
$$x_1 + x_2 + x_3 + \ldots + x_{315} = y_1 + y_2 + y_3 + \ldots + y_{315} \ ,$$
then $x_i = y_i$, for every i.
4. For any i, no non-null element of $\boldsymbol{V}_i$ can be expressed as a sum of the elements of the other $\boldsymbol{V}_j$'s.
5. For any i, no non-null element of $\boldsymbol{V}_i$ can be expressed as a sum of the elements of the preceding* $\boldsymbol{V}_j$'s.
6. $\dim\left(\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}\right) = \dim\left(\boldsymbol{V}_1\right) + \dim\left(\boldsymbol{V}_2\right) + \ldots + \dim\left(\boldsymbol{V}_{315}\right)$.

*Here "preceding" refers to the order of the list $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_{315}$.

Problem 17

Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are fdips, and $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ is a linear function. Argue that

1. $\left\|\sqrt{\mathcal{L}^*\mathcal{L}}(X)\right\| = \left\|\mathcal{L}(X)\right\|$, for all $X \in \boldsymbol{V}$ and
$\left\|\sqrt{\mathcal{L}\mathcal{L}^*}(Y)\right\| = \left\|\mathcal{L}^*(Y)\right\|$, for all $Y \in \boldsymbol{W}$.
2. $\mathit{Nullspace}\left(\sqrt{\mathcal{L}^*\mathcal{L}}\right) = \mathit{Nullspace}\left(\mathcal{L}\right)$ and
$\mathit{Nullspace}\left(\sqrt{\mathcal{L}\mathcal{L}^*}\right) = \mathit{Nullspace}\left(\mathcal{L}^*\right)$.
3. $\mathit{Range}\left(\sqrt{\mathcal{L}^*\mathcal{L}}\right) = \mathit{Range}\left(\mathcal{L}^*\right)$ and
$\mathit{Range}\left(\sqrt{\mathcal{L}\mathcal{L}^*}\right) = \mathit{Range}\left(\mathcal{L}\right)$.

9 Ticket #9

Problem 18

Suppose that $\boldsymbol{W}$ is a non-$\{\mathbb{O}\}$ ideal in the vector space $\mathbb{P}$. Prove that there exists a unique monic polynomial p_o such that

$$\boldsymbol{W} = \{\ q \cdot p_o \mid q \in \mathbb{P}\ \}\ .$$

Problem 19

1. Here $\boldsymbol{V}$ is an fdips and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove that $\sqrt{\mathcal{L}^*\mathcal{L}}$ is unitarily equivalent to $\sqrt{\mathcal{L}\mathcal{L}^*}$, and that $\mathcal{L}^*\mathcal{L}$ is unitarily equivalent to $\mathcal{L}\mathcal{L}^*$.

2. Suppose that $\mathcal{A} \in \mathbb{M}_{m\times n}$. Since $\sqrt{\mathcal{A}^*\mathcal{A}}$ and $\sqrt{\mathcal{A}\mathcal{A}^*}$ are positive semi-definite, each is unitarily similar to a positive semi-definite diagonal matrix; say $\mathcal{D}$ and $\mathcal{F}$, respectively. Prove that $\mathcal{D}$ and $\mathcal{F}$ have exactly the same positive diagonal entries and each of these is repeated the same number of times in $\mathcal{D}$ as in $\mathcal{F}$.

10 Ticket #10

Problem 20

Suppose that $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_n$ are subspaces of a vector space $\boldsymbol{V}$. Prove that the following are equivalent:

1. $\boldsymbol{V}_1 \cup \boldsymbol{V}_2 \cup \cdots \cup \boldsymbol{V}_n$ is a subspace of $\boldsymbol{V}$;
2. One of the $\boldsymbol{V}_i$'s contains all the others.

Problem 21

Here $\boldsymbol{V}$ is an fdips and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove that there is a unitary $\mathcal{U} \in \mathfrak{L}(\boldsymbol{V})$ such that

$$\mathcal{L} = \mathcal{U}\sqrt{\mathcal{L}^*\mathcal{L}}\ ,$$

and that for any such $\mathcal{U}$, we also have

$$\mathcal{L} = \sqrt{\mathcal{L}\mathcal{L}^*}\ \mathcal{U}\ .$$

11 Ticket #11

Problem 22

Here $\boldsymbol{V}$ is a finite-dimensional vector space. For a given $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$ and a fixed $V_\circ \in \boldsymbol{V}$, define

$$\boldsymbol{P}(\mathcal{L}, V_\circ) := \left\{ \left(a_0\mathcal{I}_{\boldsymbol{V}} + a_1\mathcal{L} + a_2\mathcal{L}^2 + \cdots + a_k\mathcal{L}^k\right)(V_\circ) \;\middle|\; k \geq 0, \quad a_i \in \mathbb{C} \right\} .$$

1. Argue that $\boldsymbol{P}(\mathcal{L}, V_\circ)$ is a subspace of $\boldsymbol{V}$.
2. Argue that $\boldsymbol{P}(\mathcal{L}, V_\circ)$ is invariant under $\mathcal{L}$.
3. Argue that $\boldsymbol{P}(\mathcal{L}, V_\circ)$ is contained in every invariant subspace for $\mathcal{L}$ that contains $V_\circ$. We refer to $\boldsymbol{P}(\mathcal{L}, V_\circ)$ as **the cyclic invariant subspace for $\mathcal{L}$ generated by $V_\circ$** .
4. Argue that $\boldsymbol{P}(\mathcal{L}, V_\circ)$ is 1-dimensional exactly when $V_\circ$ is an eigenvector of $\mathcal{L}$.
5. Argue a subspace $\boldsymbol{W}$ of $\boldsymbol{V}$ is invariant under $\mathcal{L}$ exactly when it is a union of cyclic invariant subspaces for $\mathcal{L}$.

Problem 23

Develop (with proof) an extension of the Polar Decomposition theorem to non-square matrices.

12 Ticket #12

Problem 24

Here $\boldsymbol{V}$ is an fdips.

1. Prove that every commuting collection in $\mathfrak{L}(\boldsymbol{V})$ is simultaneously ortho-triangularizable.

2. Prove that every commuting collection of diagonalizable linear functions in $\mathfrak{L}(\boldsymbol{V})$ is simultaneously diagonalizable.

Problem 25

State and prove Riesz Representation theorem for linear functionals on an fdips.

13 Ticket #13

Problem 26

Here $\boldsymbol{V}$ is a finite-dimensional vector space. Prove that the following claims about $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$ are equivalent.

1. There is a basis of $\boldsymbol{V}$ comprised entirely of the eigenvectors of $\mathcal{L}$.
2. $\boldsymbol{V}$ is the direct sum of all distinct eigenspaces of $\mathcal{L}$.
3. The sum of the dimensions of the distinct eigenspaces of $\mathcal{L}$ equals the dimension of $\boldsymbol{V}$.
4. $\mathcal{L}$ is a linear combination of idempotents that resolve the identity on $\boldsymbol{V}$.
5. $\mathcal{L}$ is a linear combination of mutually annihilating idempotents in $\mathfrak{L}(\boldsymbol{V})$.
6. If $\mathcal{A} = [\mathcal{L}]_{\Gamma \leftarrow \Gamma}$ for some basis Γ of $\boldsymbol{V}$, then $\mathcal{A}$ is similar to a diagonal matrix.
7. The minimal polynomial of $\mathcal{L}$ factors into distinct linear factors.

Problem 27

Suppose that $\boldsymbol{V}$ is a finite-dimensional vector space. Prove that there is a function $\Psi : \boldsymbol{V} \times \boldsymbol{V} \longrightarrow \mathbb{C}$ which defines an inner product on $\boldsymbol{V}$.

Loosely speaking, you are proving that every finite-dimensional vector space is an fdips.

14 Ticket #14

Problem 28

Here $\boldsymbol{V}$ is a finite-dimensional vector space, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$.

1. Argue that for each $V \in \boldsymbol{V}$ there is a unique monic polynomial p_V of smallest degree such that $p(\mathcal{L})$ annihilates V.

2. Argue that there exists $V \in \boldsymbol{V}$ such that p_V is the minimal polynomial of $\mathcal{L}$.

Problem 29

Argue that for each positive semi-definite linear operator $\mathcal{L}$ on an fdips $\boldsymbol{V}$ there is a unique positive semi-definite operator $\mathcal{R}$ on $\boldsymbol{V}$ such that

$$\mathcal{R}^2 = \mathcal{L}\ .$$

15 Ticket #15

Problem 30

Here $\boldsymbol{V}$ is a vector space, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Suppose that polynomials p_1 and p_2 are relatively prime monic polynomials and $p_1 p_2$ annihilates $\mathcal{L}$.

1. Prove that $\boldsymbol{V} = \mathit{Nullspace}\,(p_1(\mathcal{L})) \uplus \mathit{Nullspace}\,(p_2(\mathcal{L}))$, and that with respect to this decomposition $\mathcal{L}$ has a block form $\begin{bmatrix} \mathcal{A} & \mathcal{O} \\ \mathcal{O} & \mathcal{B} \end{bmatrix}$, where p_1 annihilates $\mathcal{A}$, and p_2 annihilates $\mathcal{B}$.

2. Argue that if we also know that $p_1 p_2$ is the minimal polynomial of $\mathcal{L}$, then p_1 is the minimal polynomial of $\mathcal{A}$, and p_2 is the minimal polynomial of $\mathcal{B}$.

Problem 31

Suppose that the partitioned square matrix

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}.$$

is normal, and has square diagonal blocks. Prove that two of the blocks have the same Frobenius norm.

Problem 32

Suppose that diagonal matrices $\mathcal{A}$ and $\mathcal{B}$ are similar, and that 4 appears as the diagonal entry of $\mathcal{A}$ exactly 6 times. Prove that 4 appears as the diagonal entry of $\mathcal{B}$ exactly 6 times.

16 Ticket #16

Problem 33

Here $\boldsymbol{V}$ is a vector space, and $\mathcal{L}, \mathcal{E} \in \mathfrak{L}(\boldsymbol{V})$, where $\mathcal{E}$ is idempotent.

1. Prove that the following claims are equivalent.
   (a) *Range* $(\mathcal{E})$ is invariant under $\mathcal{L}$.
   (b) $\mathcal{E}\mathcal{L}\mathcal{E} = \mathcal{L}\mathcal{E}$.
2. Prove that the following claims are equivalent.
   (a) *Range* $(\mathcal{E})$ and *Nullspace* $(\mathcal{E})$ are invariant under $\mathcal{L}$.
   (b) $\mathcal{E}$ and $\mathcal{L}$ commute.
3. Argue that if $\boldsymbol{V}$ is a direct sum of 12 invariant subspaces of $\mathcal{L}$, then with respect to this decomposition, $\mathcal{L}$ has a block-diagonal form.

Problem 34

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$. Prove that the following claims are equivalent.

1. $\mathcal{L}$ is positive semi-definite.
2. $\mathcal{L} = \mathcal{P}^2$ for some positive semi-definite operator $\mathcal{P}$ on $\boldsymbol{V}$.
3. $\mathcal{L} = \mathcal{M}^2$ for some self-adjoint operator $\mathcal{M}$ on $\boldsymbol{V}$.
4. $\mathcal{L} = \mathcal{T}^*\mathcal{T}$ for some operator $\mathcal{T}$ on $\boldsymbol{V}$.
5. $\mathcal{L} = \mathcal{K}\mathcal{K}^*$ for some operator $\mathcal{K}$ on $\boldsymbol{V}$.

Deduce that for a positive semi-definite $\mathcal{L}$,

$$\mathcal{L}(X) = O \iff \langle \mathcal{L}(X), X \rangle = 0.$$

17 Ticket #17

Problem 35

Here $\boldsymbol{V}$ is a finite-dimensional vector space, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$.

1. Suppose that $\boldsymbol{V}$ is a direct sum of 12 subspaces, and with respect to this decomposition, $\mathcal{L}$ has a block-upper-triangular form. Argue that the minimal polynomials of the diagonal blocks of $\mathcal{L}$ divide the minimal polynomial of $\mathcal{L}$, and that the minimal polynomial of $\mathcal{L}$ divides the product of the minimal polynomials of the diagonal blocks of $\mathcal{L}$.
2. Suppose that $\boldsymbol{V}$ is a direct sum of 12 subspaces, and with respect to this decomposition, $\mathcal{L}$ has a block-diagonal form. Argue that the minimal polynomial of $\mathcal{L}$ is the least common multiple of the minimal polynomials of the diagonal blocks of $\mathcal{L}$.

Problem 36

Here $\boldsymbol{V}$ is an fdips, and $V_1, V_2, \ldots, V_{23} \in \boldsymbol{V}$. Prove that the Gramian matrix corresponding to $V_1, V_2, \ldots, V_{23}$ is invertible if and only if $V_1, V_2, \ldots, V_{23}$ are linearly independent.

Problem 37

Prove that for any (not necessarily square) matrix $\mathcal{A}$, the Frobenius norm of $\mathcal{A}$ equals the square root of the sum of the squares of the singular values of $\mathcal{A}$.

18 Ticket #18

Problem 38

An $n \times n$ matrix $\mathcal{A}$ is said to be power-bounded, if the sequence

$$\|A\|_{HS}, \|A^2\|_{HS}, \|A^3\|_{HS}, \ldots$$

is bounded. Argue that the following are equivalent for a 7×7 Jordan block $\mathcal{J}_\lambda$.

1. $\mathcal{J}_\lambda$ is power bounded.
2. $\lim_{k\to\infty} \left\|(\mathcal{J}_\lambda)^k\right\|_{HS} = 0$.
3. $|\lambda| < 1$.

Problem 39

1. Argue that the following claims are equivalent for a square matrix $\mathcal{A}$.
   (a) $\mathcal{A}$ is normal.
   (b) $\mathcal{A}$ is unitarily equivalent to a diagonal matrix.
2. Argue that the following claims are equivalent for a square matrix $\mathcal{B}$.
   (a) $\mathcal{B}$ is unitary.
   (b) $\mathcal{B}$ is unitarily equivalent to a diagonal matrix with diagonal entries of modulus 1.
3. Argue that the following claims are equivalent for a square matrix $\mathcal{C}$.
   (a) $\mathcal{C}$ is self-adjoint.
   (b) $\mathcal{C}$ is unitarily equivalent to a diagonal matrix with real entries.

19 Ticket #19

Problem 40

Here $\mathcal{J}_\lambda$ is a 7×7 Jordan block. Argue that

$$\lim_{k\to\infty} \left\| (\mathcal{J}_\lambda)^k \right\|_{HS}^{\frac{1}{k}} = |\lambda| \ .$$

Problem 41

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L} \in \mathfrak{L}(\boldsymbol{V})$.

1. Prove that

$$Nullspace\left(\mathcal{L}^*\right) = Range\left(\mathcal{L}\right)^\perp \ .$$

2. Prove that

$$Nullspace\left(\mathcal{L}^*\right) = Nullspace\left(\mathcal{L}\right) \ ,$$

whenever $\mathcal{L}$ is normal, and conclude that in this case

$$\boldsymbol{V} = Range\left(\mathcal{L}\right) \oplus Nullspace\left(\mathcal{L}\right) \ .$$

20 Ticket #20

Problem 42

Here $\boldsymbol{V}$ is an fdips. Prove that for every subspace $\boldsymbol{W}$ of $\boldsymbol{V}$, we have

$$\boldsymbol{V} = \boldsymbol{W} \uplus \boldsymbol{W}^{\perp}.$$

Problem 43

State and prove both forms of the Singular Value Decomposition theorem for square matrices, and then extend the theorem to non-square matrices (with proof).

21 Extra Credit Problem

Extra Credit Problem 1

Prove that the distance (measured via Frobenius/Hilbert-Schmidt norm) from a given matrix $\mathcal{A} \in \mathbb{M}_n$ of rank m to the nearest $n \times n$ matrix of rank k (not exceeding m), is the square root of the sum of the squares of the smallest $n - k$ singular values of $\mathcal{A}$.