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MA355 HW2 Due Friday 26/2/2021

32 (c) There are 2 ways to do this...

	...	0	0	1	−2	3	−4	...
	...	0	0	1	−1	1	−1	...
n=0	...	0	0	1	0	0	0	...
n=1	...	0	0	1	1	0	0	...
=2	...	0	0	1	2	1	0	...
⋮	...	0	0	1	3	3	1	...
	...	⋮	⋮	⋮	⋮	6	⋮	...

(k=(0) (1) (2) (3))

→ extend the k=0 column so that

$$\boxed{\binom{n}{k} = 1 \ \forall\ k=0 \text{ and } n \in \mathbb{Z}.}$$

and build the rest of the table using the pascal eqn.

OR

...	1	0	0			
...	−2	1	0	0	0	0 ...
...	1	−1	1	0	0	0 ...
...		0	0	1	0	0 ...
...		0	0	1	1	0 ...
...		0	0	1	2	1 ...
		⋮	⋮	⋮	⋮	⋮

(−2) (−1) k=(0) (1) (2) ...

→ extend the $\underline{n=k \text{ diagonal}}$ so that

$$\boxed{\binom{n}{k} = 1 \ \forall\ n=k \in \mathbb{Z}.}$$

and build the rest of the table by Pascal eqn.

because it only "makes sense" to define $\binom{-1}{0} = 1$ or 0.

☆ Not sure how to fully justify this... How do we generalize $\frac{n!}{k!(n-k)!}$ so that it works $\forall\, n,k \in \mathbb{Z}$? instead of !

Do we use Γ functions ~~2~~ limits of Γ functions?

(33) (a) $\chi_{\{1,3\}}(1) = 1$ $\quad \chi_{\{1,3\}}(3) = 1$

$\chi_{\{1,3\}}(2) = 0$ $\quad \chi_{\{1,3\}}(4) = 0$

(b) $f(S) = \chi_S$ $\quad S \xrightarrow{f} \chi_S$.

- if $X \neq Y$ then $f(X) = \chi_X$
  $f(Y) = \chi_Y \neq \chi_X$

  $\rightarrow f$ is one-to-one.

- f is onto by definition.

So f is bijective.

(c) How many χ_S can we have? For any element in $[n]$, there are 2 choices: $i \in S$ or $i \notin S$.
$\rightarrow$ there are 2^n such χ_S.

So there must also be 2^n subsets S of $[n]$.

(37)

- If $k > n$ then there's 1 way (if it's ok to have left over balls) or 0 ways (if left overs not allowed)

- If $k \leq n$ then it is $\binom{n}{k}$

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(a) $k!$ if $k \le n$. 0 else.

(b) $\frac{n!}{(n-k)!}$ if $k \le n$, 0 else.

(c) Let B_K be a block. Since B consists of just the k-element permutations of K for each K of S, we can map:

total # k-perm of S

each block has $k!$ perm

$$B_K \longrightarrow K.$$

(d) # of k-element subsets of n-element set $= \frac{n!}{(n-k)!} \cdot \frac{1}{k!} = \frac{n!}{(n-k)!\,k!}$.

40 (c) 1/ # play both − # not chosen $= \binom{5}{2}\left\{\binom{5}{2}\binom{3}{1} - \binom{4}{1}\binom{1}{1}\right\}$

$= 260$.

2/ # chosen as center + # chosen as fwd + # not chosen

$$= \binom{5}{2}\left\{\binom{1}{0}\binom{4}{2} + \binom{2}{1}\binom{4}{1} + \binom{2}{1}\binom{4}{2}\right\}$$

$$= 10\,(6 + 8 + 12) = 260.$$

3/ # can be center + # can be fwd − # neither

$$= \binom{5}{2}\left\{\binom{3}{1}\binom{4}{2} + \binom{2}{1}\binom{5}{2} - \binom{2}{1}\binom{4}{2}\right\}$$

$$= 10\,\{18 + 20 - 12\} = 260.$$

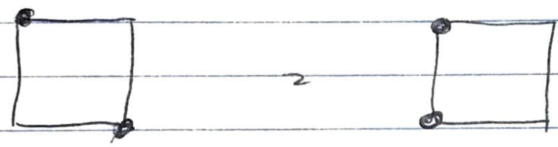
(43) This one is similar to the round/ (reflection) table problem, but there's symmetry factor of 2 for the cases where $n \geq 3$.

So # is $\begin{cases} \frac{(n-1)!}{2} & \text{for } n \geq 3 \\ 1 & \text{for } n = \{1, 2\} \end{cases}$

(45) $\boxed{\text{TWO}}$ We only color red beads... (red beads on the same edge, or not)

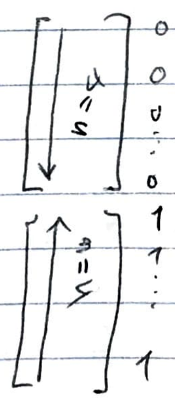
$\Downarrow$

2 ways.



$\boxed{\text{Sup 3}}$

To get grey code of length 5, we do the following: take the grey code for $n=4$, flip it and concatenate to the original grey code, then add 0 to the end of the top half, and 1 to the end of the bottom half



0000
0001
0011
0010
0110
0100
0101
0111
1111
1011
1001
~~[illegible]~~
1011
1010
1110
1100
1101
~~[illegible]~~

Ex

$n=2$	$\rightarrow$ $n=3$
00	000
01	010
11	110
10	100
	101
	111
	011
	001

$\rightarrow$ To prove that there exists a Grey code for sequences of length n, we just argue by induction. The base case is easy, and we have just shown the inductive case. --

Supp 4

~~Let a set S be given. Also S is of size n.~~

~~Generate a Grey code of length n for the sequences of length~~

We list out the subsets of S according to a Grey code starting at 0000...0 where a 0 at the i^{th} position means that the i^{th} element of S is not being chosen to be in the subset, and 1 to be "chosen"

(So 000...0 = ϕ
000...1 = $\{a\}$ where a is the 1^{st} element of S
and
so on

We can see that each subset of S can be represented by a sequence of 0, 1's in the Grey code!

Now, we see that we can count the # of 1's to determine if the set has even/odd # elements.

And observe that following each "even" sequence is necessarily an "odd" sequence, since one index has been flipped. & vice versa.

and there are 2^n total sequences...

→ ~~each even subset can be paired with~~

→ we can always pair up even & odd subsets

and so there has to be the same number of each kind.

(to do this bijectivity, we may flip one index, and so on ...)