$\rightarrow \quad i\bar{\sigma}\cdot\partial\psi_L(x) - im\sigma^2\psi_L^*(x) = 0$

$\rightarrow$ leads to infinitesimal rotation...

$$\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Big\{\underbrace{i\bar{\sigma}\cdot\partial\psi_L(\Lambda^{-1}x) - im\sigma^2\psi_L^*(\Lambda^{-1}x)}_{=0}\Big\} = 0$$

$\rightarrow$ done! So Majorana eqn is invariant under infinitesimal rotations...

(b) Boosts (proceed in a similar way...)

Key

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= \bar{\sigma}^\mu - \frac{1}{2}\vec{\beta}\left\{\bar{\sigma}^\mu, \vec{\sigma}\right\}$$

$\rightarrow \quad \bar{\sigma}^\mu \rightarrow \bar{\sigma}^\mu - i\vec{\beta}\,[J_4^{0i}]^\mu{}_\nu\,\bar{\sigma}^\nu$

$\rightarrow \quad \left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = (\Lambda)^\mu{}_\nu\,\bar{\sigma}^\nu$ as before

so plug this back into the original eqn.

p 28, 2020

Dirac Matrices & Dirac Field Bilinears

Oct 2, 2020 Recall that $\bar{\psi}\psi$ is Lorentz scalar...

Recall that $\bar{\psi}\gamma^\mu\psi$ is also a 4-vector.

$\Rightarrow$ $\boxed{?}$ Consider $\bar{\psi}\Gamma\psi$, where Γ is any 4×4

→ can we decompose into terms that have definite transformation properties under the Lorentz group?

↳ Γ can be written as combo of 16-element basis defined by

$$
\begin{array}{llll}
1: & \mathbb{1} & & \rightarrow 1 \\
4: & \gamma^\mu & & \rightarrow 4C2 \\
6: & \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} & & \rightarrow 4C3 \\
4: & \gamma^{\mu\nu\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^{\sigma]} & & \rightarrow 4C2 \\
1: & \gamma^{\mu\nu\sigma\rho} = \gamma^{[\mu}\gamma^\nu\gamma^\sigma\gamma^{\rho]} & & \rightarrow 4C2
\end{array}
$$

16 total.

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisym tensor of successively higher rank.

→ Introduce

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$= -\frac{i}{4!}\varepsilon^{\mu\nu\sigma\rho}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho$$

0123 → 1

1023 → −1

↳ totally anti-symmetric

Note that $\rightarrow$ $\boxed{(\gamma^5)^2 = \mathbb{1}}$

$$\rightarrow \boxed{\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^3)^\dagger \cdots (\gamma^0)^\dagger \\ &= +i\,\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5\end{aligned}}$$

also

$$\boxed{\begin{aligned}\{\gamma^5, \gamma^\mu\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \quad (-1) \\ &= 0\end{aligned}}$$

and thus

$$\boxed{[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}\{\gamma^\mu, \gamma^\nu\}] = 0}$$

$\Rightarrow$ Eigenstates of γ^5 with different eigv don't mix under Lorentz transform.

$\rightarrow$ In basis, can write...

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{array}{l} \rightarrow \text{for } \psi_L \text{ (left-hd)} \\ \rightarrow \text{for } \psi_R \text{ (right-hd)} \end{array}$$

$\rightarrow$ a Dirac spinor with only L/R component is an eigenstate of γ^5 with eigv $(-1)/(1)$.

With γ^5, can rewrite the table of 4×4 matrices as

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}\{\gamma^\mu, \gamma^\nu\}$	tensor	6
$\gamma^\mu\gamma^5$	pseudo vector	4
γ^5	pseudo scalar	1
		16

pseudo-vector/scalar is due to the fact that they transform like vector/scalar under Lorentz transf, BUT with an additional sign change under parity-transf.

$\underline{\underline{Ex}}$ Parity transf: $\vec{x} \to -\vec{x}$

$$\hookrightarrow (x^0, x^i) \to (x^0, -x^i)$$

If instead $(x^0; \vec{x}) \to -(x^0; -\vec{x}) = (-x^0; \vec{x})$ under parity, we call this a pseudo-vector

$\to$ pseudo vect/scalar flips sign under parity transf.

$\to$ From vector & pseudo-vector we can form 2 currents out of Dirac field bilinears...

$$\boxed{\begin{aligned} j^\mu(x) &= \bar{\Psi}(x)\gamma^\mu\Psi(x) \to \text{vector current} \\ j^{\mu 5}(x) &= \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) \to \text{pseudo vector current} \end{aligned}}$$

Assume that Ψ satisfies Dirac eqn... $\bar{\Psi} = \Psi^\dagger\gamma^0$

$\to i\not{\partial}\Psi = m\Psi$ & $-i\not{\partial}\bar{\Psi} = m\bar{\Psi}$ (since $\bar{\gamma}^\mu = \gamma^\mu$)

$\to$ compute div of these currents...

$$\partial_\mu j^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu(\partial_\mu\Psi)$$

$$= im\bar{\Psi}\Psi + \bar{\Psi}(-im\Psi) = 0$$

$$\to \boxed{\partial_\mu j^\mu = 0}$$

→ j^μ is always <u>conserved</u> if $\psi(x)$ satisfies Dirac eqn

→ It is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha}\psi(x)$$

<u>Similarly</u>

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar\psi)\gamma^\mu\gamma^5\psi + \bar\psi\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= (\partial_\mu\bar\psi)\gamma^\mu\gamma^5\psi + (-1)\bar\psi\gamma^5\gamma^\mu\partial_\mu\psi$$

$$= im\bar\psi\gamma^5\psi + (-1)(-i)m\bar\psi\gamma^5\psi$$

→ $\boxed{\partial_\mu j^{\mu 5} = 2im\bar\psi\gamma^5\gamma}$ ⇝ "axial vector current

→ if $m=0$ then $\partial_\mu j^{\mu 5}$ is conserved.

→ When $m=0$, $j^{\mu 5}$ is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha\gamma^5}\psi(x).$$

(we worry about ~~that~~ the rest of this section in ~~Weyl~~ Peskin's ...)

QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi .$$

→ Canonical momentum conjugate to Ψ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger .$$

→ Hamiltonian is given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)}\partial_0\Psi - \mathcal{L} = \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi} - \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi}$$
$$- i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$
$$= -i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$

Thus,

$$\boxed{H = \int \mathcal{H} d^3x = \int d^3x\, \bar{\Psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\Psi}$$

→ now let's figure out the commutators to make this a quantum field theory...

→ HOW **NOT** TO QUANTIZE THE DIRAC FIELD

This won't work!

<u>Guess</u> $[\psi_a(\vec{x}), i\psi_b^\dagger(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$

spinor components $\quad (a, b = 1, 2, 3, 4)$

i.e. $$[\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

In matrix notation...

$$[\psi_a(\vec{x}), \psi_a^\dagger(\vec{y})] = \mathbb{1}_{4\times4}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix} \qquad [\,-\,-\,-\,-\,]$$

Also guess $[\psi_a(\vec{x}), \psi_b(\vec{y})] = 0$

$= [\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})] = 0$

Note that

$$[\psi(\vec{x}), \bar{\psi}(\vec{y})] = [\psi(x), \psi^\dagger(y)\gamma^0]$$

$$= [\psi(x), \psi^\dagger(y)]\gamma^0 = \gamma^0\delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote...

(real field) $$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{a_{\vec{p}} + a^\dagger_{-\vec{p}}\right\}e^{i\vec{p}\cdot\vec{x}} \quad \text{(FT)}$$

For <u>complex</u> field, we get

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{a_{\vec{p}} + b^\dagger_{-\vec{p}}\right\}e^{+i\vec{p}\cdot\vec{x}}$$

In the case of Dirac field, need spin degree of freedom...

Try ...

$$\Psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{r} u^r(\vec{p}) + b_{-\vec{p}}^{r} v^r(-\vec{p}) \right\} e^{+i\vec{p}\cdot\vec{x}}$$

Spin degrees of freedom

Fourier components: $\psi(x) = u(p)e^{ip}$

$$\psi^\dagger(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{r\dagger} u^{r\dagger}(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^{r\dagger}(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

Recall that u, v also solves Dirac eqn in the sense that (in momentum space...)

$$\boxed{p^\mu \gamma_\mu u^r(\vec{p}) = m u^r(p) \quad \& \quad p^\mu \gamma_\mu v^r(\vec{p}) = -m v^r(p)}$$

We can try the commutators ...

$$[a_{\vec{p}}^r, a_{\vec{p}'}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[b_{\vec{p}}^r, b_{\vec{p}'}^s] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[a_{\vec{p}}^r, a_{\vec{p}'}^s] = 0$$

$\vdots$

the rest are all zero...

We find that

as desired...

$$[\Psi_a(\vec{x}), \Psi_b(\vec{y})] = 0 = [\Psi_a^\dagger(\vec{x}), \Psi_b^\dagger(\vec{y})]$$

We also find that

$$[\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta_{ab}\, \delta^{(3)}(\vec{x}-\vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian...

$$H = \int d^3x \, [-i\bar{\psi}\,\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi]$$

$$= \int d^3x \, \{\psi^\dagger\gamma^0 [-i\vec{\gamma}\cdot\vec{\nabla} + m]\psi\}$$

just coefs

Now, with $p^\mu \gamma_\mu u^r(p) = m u^r(p)$

$$\longrightarrow (\vec{p}\cdot\vec{\gamma} + m)\, u^r(p) = p^0\gamma^0 u^r(p) = E_p \gamma^0 u^r(p)$$

Similarly, b/c $p^\mu\gamma_\mu v^{(r)}(p) = -m v^{(r)}(p)$

$$\longrightarrow (\vec{p}\cdot\vec{\gamma} + m)\, v^r(p) = -E_p\gamma^0 v^{(r)}(p).$$

So....

$$\rightarrow [-i\vec{\gamma}\cdot\vec{\nabla} + m]\psi = [-i\vec{\gamma}\cdot\vec{\nabla} + m]\sum_{r=1,2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_p^r u_p^r + b_{-p}^{r\dagger} v^r\right] e^{i\vec{p}\cdot\vec{x}}$$

$$= \gamma^0 \sum_{r=1,2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{E_p\, a_p^r u^r(\vec{p}) - E_p\, b_{-p}^{r\dagger} v^r(\vec{p})\right\} e^{i\vec{p}\cdot\vec{x}}$$

So....

$$H = \int d^3x \left\{ \psi^\dagger \gamma^0\gamma^0 \sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\{\ldots\} e^{i\vec{p}\cdot\vec{x}} \right\}$$

plug in ...

$$\rightarrow \mathcal{H} = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} E_p \left\{ a_p^{r\dagger} a_p^r - \underbrace{b_{+p}^r b_{+p}^{r\dagger}}_{\substack{\| \\ b_{+p}^{r\dagger} b_{+\vec{p}}^r + \cancel{\text{const}}}} \right\}$$

$\boxed{!}$

$\rightarrow$ By creating more and more particles with $b^\dagger$, we can lower the energy indefinitely

$\rightarrow$ This is <u>bad</u>...

$\rightarrow$ So we should use Fermi-Dirac statistics instead $\rightarrow$ anti-commutators instead of commutators...

$\rightarrow$ Require that

$$\boxed{\{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

no longer bosonic! & all other anti-commutators are zero...

When this is true, we find that

$$\boxed{\begin{gathered} \{\Psi_a(\vec{x}), \Psi_b^\dagger(\vec{y})\} = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab} \\ \{\Psi_a(\vec{x}), \Psi_b(\vec{y})\} = \{\Psi_a^\dagger(x), \Psi_b^\dagger(y)\} = 0 \end{gathered}}$$

where we're using

$$\Psi(x) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_{\vec{p}} \left(a_p^{r\dagger} a_p^r - \underbrace{b_{-\vec{p}}^r b_{-\vec{p}}^{r\dagger}}_{-b_{-p}^{r\dagger} b_{-p}^r + \cancel{\text{const}}} \right)$$

$$\Rightarrow \boxed{\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{-\vec{p}}^{r\dagger} b_{-\vec{p}}^r \right\}}$$

now good, b/c E is bdd below...

→ also can compute...

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{+\vec{p}}^{r\dagger} b_{+\vec{p}}^r \right)}$$

To avoid sign confusion, we will usually write...

$$\boxed{\psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_{\vec{p}}^r u^r(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^{r\dagger} v^r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)}$$

As a Heisenberg field,

$$\boxed{\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u^r(\vec{p}) e^{-ip\cdot x} + b_p^{r\dagger} v^r(\vec{p}) e^{ip\cdot x} \right)}$$

where:

$$\begin{cases} a_p^r & : \text{annihilates particles} \\ a_p^{r\dagger} & : \text{creates particles} \\ b_p^r & : \text{annihilates anti-particles} \\ b_p^{r\dagger} & : \text{creates anti-particles.} \end{cases}$$

Vacuum state as $|0\rangle$ where

$$\begin{cases} a_p^s |0\rangle = 0 \\ b_p^s |0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ covariant norm:

$$\boxed{|\vec{p}, s\rangle = \sqrt{2E_p}\; a_p^{s\dagger} |0\rangle}$$

so that

$$\boxed{\langle \vec{p}, s | \vec{q}, r\rangle = \sqrt{2E_p}\sqrt{2E_{\vec{q}}}\; \delta^{(3)}(\vec{p}-\vec{q})\, \delta^{rs}}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lortz}} \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda_4^{-1} x)$$

recall that with $\rightsquigarrow \exp\{-i w_{\mu\nu} S^{\mu\nu}/2\} \approx \mathbb{1} - \frac{i}{2}\theta\cdot\Sigma$

$$\begin{cases} w_{12} = -w_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} \end{cases}$$

$= \mathbb{1} - \frac{i}{2}\vec{\theta}\cdot(\vec{\sigma} \ldots$

$\rightarrow$ and $\psi(\Lambda_4^{-1}x) \approx [1 - \theta\cdot\vec{J}]\,\psi(x)$

$\vec{J} = \vec{x} \times (-i\vec{\nabla})$

So ~~[illegible]~~ $\psi \to \psi + \delta\psi$ where

$$\delta\psi = \psi' - \psi = \left(-\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}\right)\psi(x) - \vec{\theta}\cdot(\vec{x}\times\vec{\nabla})\,\psi(x)$$

By Noether's thm,

$$\vec{J}_{total} \text{ (total spin)} = \int \frac{d^3x}{2}\left[\Psi^\dagger(-i\vec{x}\times\vec{\nabla})\Psi + \frac{1}{2}\Psi^\dagger\vec{\Sigma}\Psi\right].$$

We won't worry abt the rest of this section about propagators

$\rightarrow$ we'll come back to them later when looking at Feynman diagrams.

$\boxed{\text{DISCRETE SYMMETRIES OF THE DIRAC THEORY}}$

Basically, we have

Parity —— Time Reversal

Charge conjugation

Recall that before, we looked at implementation of continuous Lorentz transform -

$\rightarrow$ found that $\forall \Lambda \in$ Lorentz group

$\exists\, \mathcal{U}(\Lambda)$ unitary for which

$$\mathcal{U}(\Lambda)\Psi(x)\mathcal{U}^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\Psi(\Lambda x).$$

$\rightarrow$ Now, we'll talk about Discrete symmetries on the Dirac field.

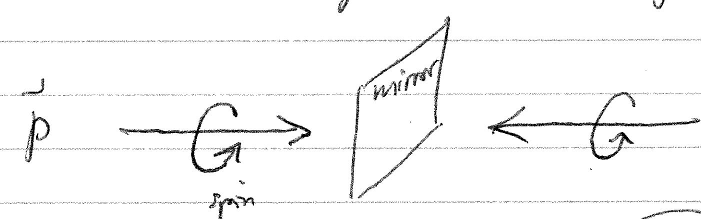
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant:

$\to$ e.g. $\left\{ \begin{matrix} \text{time-reversal} \\ \text{parity} \end{matrix} \right\}$.

Parity (P) : flips direction of spatial vectors

$$\boxed{P: (t, \vec{x}) \longrightarrow (t, -\vec{x})}.$$

$\rightsquigarrow$ mirror sym $\to$ change the handedness.



$\rightsquigarrow$ Note: momentum flips sign, but spin is unchanged.

Time-reversal

$$\boxed{T: (t, \vec{x}) \to (-t, \vec{x})}$$

P, T don't belong to the "proper" Lorentz group $L_+^\uparrow$

$\longrightarrow$ the full Lorentz group breaks into 4 disconnected subsets ...

$\hookrightarrow$

$$\begin{tikzcd}
L_+^\uparrow \arrow[r, leftrightarrow, "P"] \arrow[d, leftrightarrow, "T"] & L_-^\uparrow = PL_+^\uparrow \arrow[d, leftrightarrow, "T"] \\
L_+^\downarrow = TL_+^\uparrow \arrow[r, leftrightarrow, "P"] & L_-^\downarrow = PTL_+^\uparrow
\end{tikzcd}$$

"orthochronous"

"non-orthochronous"

"proper" "improper"

$\boxed{\text{charge conjugation}}$ → interchange particles & anti-particles.

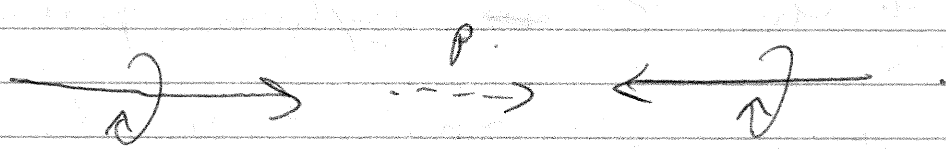
↳ non-space-time.

Let's look at $\boxed{\text{Parity}}$.

Note that because $P: (t, \vec{x}) \to (t, -\vec{x})$

→ momentum flips sign

but not spin! ⇒ What is P? As an operator?



As an operator on creation/annihilation ops, we want

$$\boxed{P^\dagger a_{\vec{p}}^s P = a_{-\vec{p}}^s \quad \& \quad P^\dagger b_{\vec{p}}^s P = b_{-\vec{p}}^s}$$

where, as discussed, P must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, get

$$\boxed{P^\dagger a^{s\dagger}_{\vec{p}} P = a^{s\dagger}_{-\vec{p}} \quad \& \quad P^\dagger b^{s\dagger}_{+\vec{p}} P = b^{s\dagger}_{-\vec{p}}}$$

But these might be too restrictive ... we can get better constraints by requiring that.

$$P^\dagger a^s_{\vec{p}} P = \eta_a a^s_{-\vec{p}} \quad \& \quad P^\dagger b^s_{\vec{p}} P = \eta_b b^s_{-\vec{p}}$$

as long as $|\eta_a|^2 = |\eta_b|^2 = \mathbb{1}$ are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel:

$$\left\{\begin{aligned} P^\dagger a^{s'\dagger}_{\vec{p}} a^s_{\vec{p}} P &= a^{s'\dagger}_{-\vec{p}} a^s_{-\vec{p}} \\ P^\dagger b^{s'\dagger}_{\vec{p}} b^{s'}_{\vec{p}} P &= b^{s\dagger}_{-\vec{p}} b^s_{-\vec{p}}. \end{aligned}\right\}$$

With this, let's implement parity condition on $\Psi(x)$

$$\rightarrow P^\dagger \Psi P = ?$$

(to find out what these η_a, η_b must be ...)

$$P^\dagger \psi(x) P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(\eta_a a^s_{-\vec{p}} u^s(p) e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^* b^{s\dagger}_{-\vec{p}} v^s(p) e^{i\tilde{p}\cdot\tilde{x}} \right)$$

Define
$$\begin{cases} \tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}). \end{cases}$$

Note that

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} \quad \text{where } \sigma = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma} = (\mathbb{1}, -\vec{\sigma})$$

$$= \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\, \xi^s \\ \sqrt{\tilde{p}\cdot\sigma}\, \xi^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\, \xi^s \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\, \xi^s \end{pmatrix}}_{u^s(-\vec{p})} = \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}}_{\gamma^0} u^s(-\vec{p})$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\vec{p})}$$

and
$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} = \cdots = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} v^s(-\vec{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(p)}$$

With these, we find that ...

$\tilde{p}\cdot\tilde{x} = p\cdot x$

$$P^\dagger \psi(x) P = \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(\eta_a a^s_{-p} u^s(-p) e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^* b^s_{-p} v^s(-p) e^{i\tilde{p}\cdot\tilde{x}} \Big)$$

now, notice that if $\eta_a = \eta_b^*$ then it's "nice":

$$\boxed{(\eta_a = \eta_b^*) \Rightarrow P\psi^\dagger(x)P = \eta_a \gamma^0 \psi(\tilde{x})}$$ → P-transf in final form

→ suffices to choose $\eta_a = 1 = -\eta_b$.

relative sign between fermions & antifermions...

Now, useful to know how various Dirac field bilinears transform under parity...

Recall... 5 of them:

$$\bar\psi\psi, \quad \bar\psi\gamma^\mu\psi, \quad i\bar\psi[\gamma^\mu,\gamma^\nu]\psi$$

$$\bar\psi\gamma^\mu\gamma^5\psi, \quad i\bar\psi\gamma^5\psi.$$

to find these, first compute: $P\bar\psi(x)P$...

$$P^\dagger\bar\psi(x)P = P^\dagger\psi^\dagger(x)\gamma^0 P = (P^\dagger\psi P)^\dagger\gamma^0 \qquad (\gamma^{0\dagger} = \gamma^0)$$

$$= \eta_a^*(\gamma^0\psi(\tilde{x}))^\dagger\gamma^0 = \eta_a^*\psi^\dagger(\tilde{x})\gamma^0\gamma^0$$

$$\rightarrow \boxed{P^\dagger\bar\psi P = \eta_a^*\bar\psi(\tilde{x})\gamma^0}$$

With this---

$$P^\dagger \underset{(x)(x)}{\bar\Psi\Psi} P = \underbrace{P^\dagger \bar\Psi(x) P}\overset{\mathbb{1}}{\widetilde{P^\dagger}}\, \Psi(x) P.$$

$$= \eta_a^* \bar\Psi(\tilde x)\gamma^0 \eta_a \gamma^0 \Psi(\tilde x)$$

$$= |\eta_a|^2 \bar\Psi(\tilde x)\Psi(\tilde x)$$

Scalar $\boxed{P^\dagger \bar\Psi\Psi P(x) = \bar\Psi\Psi(\tilde x)}$ (scalar) bilinear.

can also show...

$$\boxed{\begin{aligned} &P^\dagger \underbrace{\bar\Psi(x)\gamma^\mu\Psi} P = \bar\Psi\gamma^0\gamma^\mu\gamma^0\Psi(\tilde x) \\ &\text{(vector) } \cancel{\text{bilinear}} \quad = \begin{cases} +\bar\Psi\gamma^\mu\Psi(\tilde x) & \mu = 0 \\ -\bar\Psi\gamma^\mu\Psi(\tilde x) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

$$\boxed{\begin{aligned} &P^\dagger (i\bar\Psi\gamma^5\Psi) P = i\bar\Psi(\tilde x)\gamma^0\gamma^5\gamma^0\Psi(\tilde x) = -i\bar\Psi\gamma^5\Psi(\tilde x) \\ &\text{pseudo scalar} \to (-) \qquad \cancel{= \begin{cases} -\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu=0 \\ \bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu=1,2,3 \end{cases}} \end{aligned}}$$

$$\boxed{\begin{aligned} &P^\dagger\bar\Psi\gamma^\mu\gamma^5\Psi(x)P = \bar\Psi(\tilde x)\gamma^0\gamma^\mu\gamma^5\gamma^0\Psi(\tilde x) \\ &\text{pseudo vector} \quad = \begin{cases} -\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 0 \\ +\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

$\downarrow$

$(-)$

<u>Note</u> the relative sign: $\boxed{-\eta_a = \eta_b^*}$ is important.
for the relationship between fermions - anti-fermi

<u>Consider</u> ~~an~~ fermion - antifermion state...

$$a_p^{s\dagger} b_q^{s\dagger} |0\rangle \xrightarrow{P} P\left(a_p^{s\dagger} b_q^{s\dagger}|0\rangle\right)$$

$$= P^\dagger\left(a_p^{s\dagger} b_q^{s\dagger}\right) P\,|0\rangle$$

$$= \underbrace{P^\dagger a_p^{s\dagger} P}P^\dagger b_q^{s\dagger} P\,|0\rangle$$

$$= \eta_a\, a_{-p}^{s\dagger}\, \eta_b\, b_{-q}^{s\dagger}\,|0\rangle$$

$$= -\left(\eta_b \eta_b^*\right) a_{-p}^{s\dagger} b_{-q}^{s\dagger}\,|0\rangle$$

$$= -\,a_{-p}^{s\dagger} b_{-q}^{s\dagger}\,|0\rangle$$

$\rightarrow$ a state containing a fermion - antifermion pair gets an (−1) under parity transform.
extra

$\boxed{\text{TIME REVERSAL}}$

if T is unitary & $[T, H] = 0$

$\rightarrow$ $T^\dagger e^{iHt}\, T = e^{iHt}\, T^\dagger T = e^{iHt}$

$\rightarrow$ no good...

What if $T^\dagger H T = -\cancel{H}$? or $\{T, H\} = 0$?

But this is no good either since implies that H is unbounded ...

→ Assume this ...

$\boxed{\text{"Time-reversal is conjugate-linear/anti-linear"}}$

Assume:

T is unitary

$$T^\dagger c T = c^* \qquad (c \in \mathbb{C}).$$

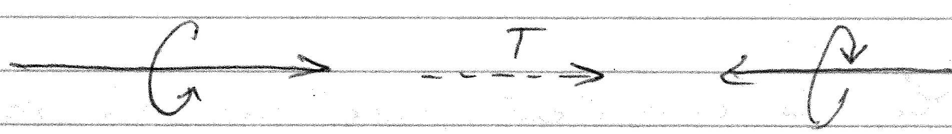
$$[T, H] = 0$$

With these ...

$$T^\dagger e^{-iHt} T = e^{iHt} \quad ✓$$

→ time reversal: momentum & spin are reversed

↳ like watching a movie played backwards



Flipping momentum is easy.

What abt flipping spinor s? We know that

In some basis...

$$\xi(\uparrow) = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s = 1, 2$ & define

reversed spin

$$\boxed{\xi^{-s} = -i\sigma^2 (\xi^s)^*.}$$

σ^y

This is the <u>flipped spinor</u>...

It is clear that

$$\xi^{-s} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow)^*, \xi(\downarrow)^*)^T$$

$$= (\xi(\downarrow), -\xi(\uparrow))^T$$

where
$$\left\{ \begin{aligned} \xi^2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{1*} \\ \xi^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{2*} \end{aligned} \right.$$

→ This is convenient since our time reversal op. involves complex conjugation...

→ Can show:
$$\boxed{u^{-s}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\xi^{s*} \end{pmatrix}}$$

So if we use the identity ...

$$\sqrt{\tilde{p}\cdot\sigma}\,\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}$$ (prove using $\sigma^2\vec{\sigma} = -\vec{\sigma}^*\sigma^2$)

then we set

$$u^{-s}(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\,\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\,\xi^{s*} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2)\sqrt{\tilde{p}\cdot\sigma^*}\,\xi^{s*} \\ (-i\sigma^2)\sqrt{\tilde{p}\cdot\bar{\sigma}^*}\,\xi^{s*} \end{pmatrix}$$

$$= (-i)\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(p)]^* = -\gamma^1\gamma^3[u^s(p)]^*$$

$\sigma^2 = \sigma^y$

$$\Rightarrow \boxed{u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^*}$$ element-wise complx conjugation

Similarly, ...

$$\boxed{v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^*}$$

in this relation, v^{-s} contains $\boxed{\xi^{-(-s)} = -\xi^s}$

a 360° flip introduces a $(-)$ sign.

~~[illegible]~~

Now we can define time reversal operation on the creation - annihilation operators...

spins don't matter here... $\longrightarrow$ $\boxed{T^\dagger a^s_{\vec p} T = a^{-s}_{-\vec p} \quad \& \quad T^\dagger b^s_{\vec p} T = b^{-s}_{-\vec p}}$ $\longrightarrow$ flip spin, $\longrightarrow$ flip momentum

where we now define
$$a^{-s}_{-\vec p} = (a^{\downarrow}_{-\vec p}, -a^{\uparrow}_{-\vec p})$$
$$b^{-s}_{-\vec p} = (b^{\downarrow}_{-\vec p}, -b^{\uparrow}_{-\vec p})$$
just like what we did with $\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow))$

if
$$a^s_{\vec p} = (a^{\uparrow}_{\vec p}, a^{\downarrow}_{\vec p})$$
$$b^s_{\vec p} = (b^{\uparrow}_{\vec p}, b^{\downarrow}_{\vec p})$$
analogous to what we did before...

With this, let's evaluate $T^\dagger\Psi(x)T$:

$$\rightarrow \quad T^\dagger\Psi(x)T = \int \frac{d^3p}{(2\pi)^3} \underbrace{\frac{1}{\sqrt{2E_p}}} \sum_{s=1,2} T^\dagger\left(a^s_{\vec p}\, u^s(p)\, e^{-ip\cdot x} + b^{s\dagger}_p v^s(p)\, e^{+ip\cdot x}\right)T$$
$$= \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} \Big\{ a^{-s}_{-\vec p} [u^s(p)]^* e^{ip\cdot x} + b^{-s\,\dagger}_{-p} [v^s(p)]^* e^{-ip\cdot x} \Big]$$

. where under T, $= \gamma^1\gamma^3\Psi(x_T)$, $x_T = (-t, \vec x$

- $a^s_p \xrightarrow{T} a^{-p}_{-p}$
- ~~$u^s(p)$~~ ~~$u^{-s}(\tilde p) = -\gamma_1\gamma_3[u^s(p)]^*$~~
- $\rightarrow$ ~~$\gamma_1\gamma_3 u^{-s}(\tilde p) = -[u^s(p)]^*$~~
- $T^\dagger e^{-ip\cdot x}T = e^{+ip\cdot x}$; $T^\dagger u^s_p T = [u^s_p]^*$

note sign change here

Because $\{u^s(p)\}^* = \gamma_1\gamma_3 u^{-s}(\tilde{p})$, we have

$$T^\dagger \psi(x) T = \gamma^1\gamma^3 \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^{2} \left\{ a^{-s}_{\tilde{p}} u^{-s}(\tilde{p}) e^{i\tilde{p}(t,-\vec{x})} + b^{-s}_{\tilde{p}} v^{-s}(\tilde{p}) e^{-i\tilde{p}(t,-\vec{x})} \right\}$$

$= -\tilde{p}(-t, \vec{x})$

$$= \gamma^1\gamma^3 \psi(-t, x)$$

$$\Rightarrow \boxed{T^\dagger \psi(x,t) T = \gamma^1\gamma^3 \psi(x,-t)}$$

Next, can check the action of T on bilinears...

$$T^\dagger \bar{\psi} T = T^\dagger \psi^\dagger \gamma^0 T = T^\dagger \psi^\dagger T \gamma^0 \quad \text{real}$$

$$= (\gamma^1\gamma^3 \psi(x_T))^\dagger \gamma^0 = \psi^\dagger(x_T) \gamma^{3\dagger} \gamma^{1\dagger} \gamma^0$$

$-\gamma^3 \quad -\gamma^1$

$$= \psi^\dagger(x_T) \gamma^3\gamma^1\gamma^0$$

$$= +\psi^\dagger(x_T)\gamma^0 \underbrace{\gamma^3\gamma^1}$$

$$\Rightarrow \boxed{T^\dagger \bar{\psi} T = -\bar{\psi}(x_T)\gamma^1\gamma^3}$$

with this, can compute the rest....

Scalar $\boxed{T \bar{\psi}\psi(x) T = \bar{\psi}\underbrace{(-\gamma^1\gamma^3)(\gamma^1\gamma^3)}_{\mathbb{1}}\psi(x_T) = \bar{\psi}(x_T)\psi(x_T)}$

Pseudoscalar $\to$ get $(-)$

$$\boxed{T^\dagger i\bar{\psi}\gamma^5\psi T = -i\bar{\psi}(-\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3)\psi(x_T) = -i\bar{\psi}(x_T)\gamma^5\psi(x_T)}$$

"pseudo"

Vector

$$\boxed{\begin{aligned} T^\dagger \bar{\Psi}\gamma^\mu \Psi T_{(x)} &= \bar{\Psi}(-\gamma^1\gamma^3)(\gamma^\mu)^*(\gamma^1\gamma^3)\Psi \\ &= \begin{cases} +\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu = 0 \\ -\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu = 1,2,3. \end{cases} \end{aligned}}$$

This makes sense... Recall that $\bar{\Psi}\gamma^0\Psi$ is the charge density

↳ $\bar{\Psi}\gamma^0\Psi$ should be the same under T.

as we saw: $T^\dagger\bar{\Psi}\gamma^0\Psi T = \bar{\Psi}\gamma^0\Psi$.

But current density (time-dep) must reverse si[g]n

$\rightarrow \quad T^\dagger\bar{\Psi}\gamma^i\Psi T = -\bar{\Psi}\gamma^i\Psi$ ✓.

$\boxed{\text{Charge Conjugation - Matter-anti-matter flip}}$

↳ anti particles & particles are swapped.
spin & momentum are the same.

Let $\begin{Bmatrix} C^\dagger a_p^s C = b_p^s \\ C^\dagger b_p^s C = a_p^s. \end{Bmatrix} \rightarrow$ ignore phases...

How should C act on $\Psi(x)$?

First, look at relation...

$$(v^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\ (-i\sigma^2)\xi^{s*} \\ \sqrt{p\cdot\bar\sigma}\ (-i\sigma^2)\xi^{s*} \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma^*}\,\xi^{s*} \\ i\sigma^2\sqrt{p\cdot\sigma^*}\,\xi^* \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} =$$

$\longrightarrow$ set

$$\boxed{\begin{aligned} u^s(p) &= -i\gamma^2(v^s(p))^* \\ v^s(p) &= -i\gamma^2(u^s(p))^* \end{aligned}}$$

$$\rightarrow\quad C^\dagger\psi(x)C = \int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sum_{s=1}^{2}\Big\{-i\gamma^2 b_p^s\,(v^s(p))^*\,e^{-ip\cdot x} - i\gamma^2 a_p^{s\dagger}\,(u^s(p))^*\,e^{ip\cdot x}\Big)$$

$$= -i\gamma^2\psi^*(x) = -i\gamma^2(\psi^\dagger)^T = -i(\bar\psi\gamma^0\gamma^2)^T$$

$$\Rightarrow\ \boxed{C^\dagger\psi(x)C = -i(\bar\psi\gamma^0\gamma^2)^T} \quad\rightarrow\ C \text{ is a unitary op.}$$

On bilinears... first, find $\quad \bar\psi^\dagger = (\psi^\dagger)^\dagger\gamma^0 = \psi\gamma^0$

$$\boxed{\begin{aligned} C^\dagger\bar\psi(x)C &= C^\dagger\psi^\dagger\gamma^0 C = \underbrace{C^\dagger\psi^\dagger C}\gamma^0 = -i\psi^T\gamma^2\gamma^0 \\ &= (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T \end{aligned}}$$

Next....

$$C^\dagger\bar\psi\psi C = (-i(\gamma^0\gamma^2\psi)^T)(-i(\bar\psi\gamma^0\gamma^2)^T) = \cdots\cdots =$$

$$= -\left[(-i\bar\psi\gamma^0\gamma^2)(-i\gamma^0\gamma^2\psi)\right]^T = +\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi$$

$$= -+\bar\psi\gamma^0\gamma^2\gamma^2\gamma^0\psi = +\bar\psi\psi$$

So... $\boxed{C^\dagger \bar{\psi}\psi C = \bar{\psi}\psi}$ $\longrightarrow$ scalar

vector $\boxed{C^\dagger i\bar{\psi}\gamma^5\psi C = i(-i\gamma^0\gamma^2\psi)^T\gamma^5(-i\bar{\psi}\gamma^0\gamma^2)^T = i\bar{\psi}\gamma^5\psi}$

pseudo-scalar $C^\dagger\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi$

pseudo vector. $C^\dagger\bar{\psi}\gamma^\mu\gamma^5\psi C = +\bar{\psi}\gamma^\mu\gamma^5\psi$

(I'll skip the derivations... to save time)

Summary

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

notice that

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$ is invariant under C, P, T separately

$\rightarrow$ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

$\downarrow$

Invariance under CPT is required for any Lorentz invariant local Hermitian op.

Problem 5

$\hookrightarrow$ (to be continued...)

Correlation functions for Dirac fields

$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle \rightsquigarrow$ Dirac propagation amplitudes...

only "a" terms contribute — only "$a^{\dagger}$" term contributes

Recall...

$\langle 0|$

$$\rightarrow \psi_A(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{a^s_{Ap}u^s_A(p)e^{-ip\cdot x} + \cancel{b^{s\dagger}_{Ap}v^s_A(p)e^{-ip\cdot x}}\right\}$$

$$\rightarrow \bar{\psi}_B(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{\cancel{b^s_{Bp}\bar{v}^s_B(p)e^{-ip\cdot x}} + a^{s\dagger}_{Ap}\bar{u}^s_B(p)e^{ip\cdot x}\right\}$$

$\rightarrow |0\rangle$

where $\{a^r_p, a^{s\dagger}_q\} = \{b^r_p, b^{s\dagger}_q\} = (2\pi)^3\delta^{(3)}(p-q)\delta^{rs}$

$$\rightarrow \langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\bar{u}^s_B(p)}_{(\not{p}+m)_{AB}}e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{AB}\underbrace{\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}}$$

$$\boxed{\langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = (i\not{\partial}_x + m)_{AB}\,\mathcal{D}(x-y)}$$

- - -

$$\langle 0|\bar{\psi}_B(y)\psi_A(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s v^s_A(p)\bar{v}^s_B(p)}_{(\not{p}-m)_{AB}}e^{-ip(x-y)}$$

$\bar{\psi}_B(y)$: b terms contribute

$\psi_A(x)$: $b^\dagger$ terms contribute

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_p}\right)(\not{p}-m)_{AB}\, e^{-ip(x-y)}$$

$$\boxed{\langle 0|\bar{\psi}_B(y)\psi_A(x)|0\rangle = -(i\not{\partial}_x + m)_{AB}\, D(y-x)}$$

Feynman Propagator

$$S_F^{AB}(x-y) = \begin{cases} \langle 0|\psi_A(x)\bar{\psi}_B(y)|0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\bar{\psi}_B(y)\psi_A(x)|0\rangle & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0|T\{\psi_A(x), \bar{\psi}_B(y)\}|0\rangle$$

... time-ordering ...

where $T\{\psi_A(x), \bar{\psi}_B(y)\}$

$$= \theta(x^0-y^0)\,\psi_A(x)\bar{\psi}_B(y)$$
$$- \theta(y^0-x^0)\,\bar{\psi}_B(y)\psi_A(x)$$

$\boxed{\text{"minus sign for Fermions"}}$

Let's check the calculations...

$$\langle 0|\psi_A(x)\bar{\psi}_B(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_{p'}}}\langle 0|\left\{\sum_s a^s_{A\vec{p}}\, u^s_A(p)e^{-ip\cdot x} + \cancel{b^{s\dagger}_{A\vec{p}}\, v^s_A(p)e^{ip\cdot x}}\right\}$$

$$\times\left\{\sum_s \cancel{b^s_{B\vec{p}'}\,\bar{v}^s_B(p')e^{-ip'\cdot y}} + a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{ip'\cdot y}\right\}|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\langle 0|\sum_s a^s_{A\vec{p}}\,u^s_A(p)\sum_s a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{-i(p\cdot x-p'\cdot y)}|0\rangle$$

$$= \int \frac{d^3p\,d^3p'}{(2\pi)^6}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\sum_s\langle 0|u^s_A(p)\bar{u}^s_B(p')|0\rangle\,(2\pi)^3\delta^{(3)}(p-p')$$
$$\times\, e^{i(p\cdot x-p'\cdot y)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\,\bar{u}^s_B(p)}_{}\; e^{-ip(x-y)}$$

$$(\not{p}+m)_{AB} = (\gamma^\mu p_\mu + m)_{AB} \quad \left(\frac{\text{spin sum}}{\text{relations}}\right)$$

$$= (i\not{\partial}_x+m)_{AB}\int\frac{d^3p}{(2\pi)^3\,2E_p}\,e^{-ip(x-y)}. \checkmark$$

Similarly, we can get the other relation too...