

# **MATRIX ANALYSIS**

## A Quick Guide

Huan Q. Bui

Colby College

PHYSICS & MATHEMATICS

Statistics

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## Preface

Greetings,

*Matrix Analysis: A Quick Guide to* is compiled based on my MA353: Matrix Analysis notes with professor Leo Livshits. The sections are based on a number of resources: *Linear Algebra Done Right* by Axler, *A Second Course in Linear Algebra* by Horn and Garcia, *Matrices and Linear Transformations* by Cullen, *Matrices: Methods and Applications* by Barnett, *Problems and Theorems in Linear Algebra* by Prasolov, *Matrix Operations* by Richard Bronson, and professor Leo Livshits' own textbook (in the making). Prerequisites: some prior exposure to a first course in linear algebra.

The development of this text will come in layers. The first layer, one that I am working on during the course of S'19 MA353, will be an overview of the key topics listed in the table of contents. As the semester progresses, I will be constantly updating the existing notes, as well as adding prof. Livshits' problems and my solutions to the problems. The second layer will come after the course is over, when concepts will have hopefully "come together."

I will decide how much narrative I should put into the text as text is developed over the semester. I'm thinking that I will only add detailed explanations wherever I find fit or necessary for my own studies. I will most likely keep the text as condensed as I can.

Enjoy!

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# 1 List of Special Matrices & Their Properties

1. **Hermitian/Self-adjoint:**  $H = H^\dagger$ . A Hermitian matrix is matrix that is equal to its own conjugate transpose:

$$H \text{ is Hermitian} \iff H_{ij} = \bar{H}_{ji}$$

## Properties 1.1.

- (a)  $H$  is Hermitian  $\iff \langle w, Hv \rangle = \langle Hw, v \rangle$ , where  $\langle, \rangle$  denotes the inner product.
- (b)  $H$  is Hermitian  $\iff \langle v, Hv \rangle \in \mathbb{R}$ .
- (c)  $H$  is Hermitian  $\iff$  it is *unitarily diagonalizable* with *real eigenvalues*.

**Unitary:**  $U^*U = UU^* = I = U^\dagger U = UU^\dagger$ . The real analogue of a unitary matrix is an orthogonal matrix. The following list contain the properties of  $U$ :

- (a)  $U$  preserves the inner product:

$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$

- (b)  $U$  is normal: it commutes with  $U^* = U^\dagger$ .
- (c)  $U$  is diagonalizable:

$$U = VDV^*,$$

where  $D$  is diagonal and unitary, and  $V$  is unitary.

- (d)  $|\det(U)| = 1$  (hence the real analogue to  $U$  is an orthogonal matrix)
- (e) Its eigenspaces are orthogonal.
- (f)  $U$  can be written as

$$U = e^{iH},$$

where  $H$  is a Hermitian matrix.

- (g) Any square matrix with unit Euclidean norm is the average of two unitary matrices.

2. **Idempotent:**  $M$  idempotent  $\iff M^2 = M$ .

- (a) Singularity: its number of independent rows (and columns) is less than its number of rows (and columns).
- (b) When an idempotent matrix is subtracted from the identity matrix, the result is also idempotent.

*“Proof”.*

$$[I - M][I - M] = I - M - M + M^2 = I - M - M + M = I - M.$$

□

- (c)  $M$  is idempotent  $\iff \forall n \in \mathbb{N}, A^n = A$ .
- (d) Eigenvalues: an idempotent matrix is always diagonalizable and its eigenvalues are either 0 or 1. (think “projection”)
- (e) Trace: the trace of an idempotent matrix equals the rank of the matrix and thus is always an integer. So

$$\text{tr}(A) = \dim(\text{Im } A).$$

3. **Nilpotent:** a nilpotent matrix is a square matrix  $N$  such that

$$N^k = 0$$

for some positive integer  $k$ . The smallest such  $k$  is sometimes called the **index** of  $N$ .

The following statements are equivalent:

- (a)  $N$  is nilpotent.
- (b) The minimal polynomial for  $N$  is  $x^k$  for some positive integer  $k \leq n$ .
- (c) The characteristic polynomial for  $N$  is  $x^n$ .
- (d) The only complex eigenvalue for  $N$  is 0.
- (e)  $\text{tr } N^k = 0$  for all  $k > 0$ .

**Properties 1.2.**

- (a) The degree of an  $n \times n$  nilpotent matrix is always less than or equal to  $n$ .
- (b)  $\det N = \text{tr}(N) = 0$ .
- (c) Nilpotent matrices are not invertible.
- (d) The only nilpotent diagonalizable matrix is the zero matrix.

## 2 List of Operations

1. **Conjugate transpose** is what its name suggests.
2. **Classical adjoint/Adjugate/adjunct** of a square matrix is the transpose of its cofactor matrix.



### 3 List of Algorithms

## 4 Complex Numbers

### 4.1 A different point of view

We often think of complex numbers as

$$a + ib$$

where  $a, b \in \mathbb{R}$  and  $i = \sqrt{-1}$ . While there is nothing “bad” about this way of thinking - in fact thinking of complex numbers as  $a + ib$  allows us to very quickly and intuitively do arithmetics operations on them - a “matrix representation” of complex numbers can give us some insights on “what we actually do” when we perform complex arithmetics.

Let us think of

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

as a different representation of the same object - the same complex number “ $a + ib$ .” Note that it does not make sense to say the matrix representation **equals** the complex number itself. But we shall see that a lot of the properties of complex numbers are carried into this matrix representation under interesting matricial properties.

$$\boxed{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \sim a + ib}$$

First, let us break the matrix down:

$$a + ib = a \times 1 + i \times b \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aI + b\mathcal{I}.$$

Right away, we can make some “mental connections” between the representations:

$$\begin{aligned} I &\sim 1 \\ \mathcal{I} &\sim i. \end{aligned}$$

Now, we know that complex number multiplications commute:

$$(a + ib)(c + id) = (c + id)(a + ib).$$

Matrix multiplications are not commutative. So, we might wonder whether commutativity holds under the this new representation of complex numbers. Well, the answer is yes. We can readily verify that

$$(aI + b\mathcal{I})(cI + b\mathcal{I}) = (cI + b\mathcal{I})(aI + b\mathcal{I}).$$

How about additions? Let's check:

$$(a + ib) + (c + id) = (a + c) + i(b + d) \sim \begin{pmatrix} a + c & -(b + d) \\ (b + d) & a + c \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Ah! Additions work. So, the new representation of complex numbers seems to be working flawlessly. However, we have yet to gain any interesting insights into the connections between the representations. To do that, we have to look into changing the form of the matrix. First, let's see what conjugation does:

$$(a + ib)^* = a - ib \sim \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^\top$$

Ah, so conjugation to a complex number in the traditional representation is the same as transposition in the matrix representations. What about the amplitude square? Let us call

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We have

$$(a + ib)(a - ib) \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = MM^\top = (a^2 + b^2)I = \det(M)I$$

Interesting. But observe that if  $\det(M) \neq 0$

$$\frac{1}{\det(M)} MM^\top = I.$$

This tells us that

$$M^\top = M^{-1},$$

where  $M^{-1}$  is the inverse of  $M$ , and, not surprisingly, it corresponds to the reciprocal to the complex number  $a + ib$ . We can readily show that

$$M^{-1} \sim (a + ib)^{-1} = \frac{1}{a^2 + b^2}(a - ib).$$

Remember that we can also think of a complex number as a column vector:

$$c + id \sim \begin{pmatrix} c \\ d \end{pmatrix}.$$

Let us look back at complex number multiplication under matrix representation:

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad) \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix}.$$

Multiplication actually works in this “mixed” way of representing complex numbers as well. Now, observe that what we just did was performing a linear transformation on a vector in  $\mathbb{R}^2$ . It is always interesting to look at the geometrical interpretation of this transformation. To do this, let us call  $N$  the “normalized” version of  $M$ :

$$N = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We immediately recognize that  $N$  is an orthogonal matrix. This means  $N$  is an orthogonal transformation (length preserving). Now, it is reasonable to define

$$\begin{aligned} \cos \theta &= \frac{a}{\sqrt{a^2 + b^2}} \\ \sin \theta &= \frac{b}{\sqrt{a^2 + b^2}}. \end{aligned}$$

We can write  $N$  as

$$N = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

which is a physicists’ favorite matrix: the rotation by  $\theta$ . So, let us write  $M$  in terms of  $N$ :

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \sqrt{a^2 + b^2} N = \sqrt{a^2 + b^2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

We can interpret  $M$  as a rotation by  $\theta$ , followed by a scaling by  $\sqrt{a^2 + b^2}$ . But what  $\sqrt{a^2 + b^2}$  exactly is just the “length” or the “amplitude” of the complex number  $a + ib$ , if we think of it as an arrow in a plane.

## 4.2 Relevant properties and definitions

1. The *modulus* of  $z = a + ib$  is the “amplitude” of  $z$ , denoted by  $|z| = \sqrt{a^2 + b^2} = z\bar{z}$ .
2. The modulus is *multiplicative*, i.e.

$$|wz| = |w||z|.$$

3. Triangle inequality:

$$|z + w| \leq |z| + |w|.$$

We can readily show this geometrically, or algebraically.

4. The *argument* of  $z = a + ib$  is  $\theta$ , where

$$\theta = \begin{cases} \tan^{-1} \left( \frac{b}{a} \right), & \text{if } a > 0 \\ \frac{\pi}{2} + k2\pi, k \in \mathbb{R} & \text{if } a = 0, b > 0 \\ -\frac{\pi}{2} + k2\pi, k \in \mathbb{R} & \text{if } a = 0, b < 0 \\ \text{Undefined} & \text{if } a = b = 0. \end{cases}$$

5. The *conjugate* of  $a+ib$  is  $a-ib$ . Conjugation is *additive* and *multiplicative*, i.e.

$$\begin{aligned} z + w &= \bar{z} + \bar{w} \\ \bar{w}z &= \bar{w}\bar{z}. \end{aligned}$$

Note that we can also show the multiplicative property with the matrix representation as well:

$$\bar{w}z \sim (WZ)^\top = Z^\top W^\top \sim \bar{z}\bar{w} = \bar{w}\bar{z}.$$

6. Euler's identity, generalized to de Moivre's formula:

$$z^n = r^n e^{in\theta}.$$

## 5 Vector Spaces & Linear Functions

### 5.1 Review of Linear Spaces and Subspaces

**Properties 5.1.** of linear spaces:

1. Commutativity and associativity of addition
2. Existence of an additively neutral element (null element). Zero multiples of elements give the null element:  $0 \cdot V = \mathbf{0}$
3. Every element has an (unique) additively antipodal element
4. Scalar multiplication distributes over addition
5. Multiplicative identity:
6.  $ab \cdot V = a \cdot (bV)$
7.  $(a + b)V = aV + bV$

$W$  is a subspace of  $V$  if

1.  $S \subseteq V$
2.  $S$  is non-empty
3.  $S$  is closed under addition and scalar multiplication

**Properties 5.2.** that are interesting/important/maybe-not-so-obvious:

1. If  $S$  is a subspace of  $V$  and  $S \neq V$  then  $S$  is a proper subspace of  $V$ .
2. If  $X$  is a subspace of  $Y$  and  $Y$  is a subspace of  $Z$ , then  $X$  is a subspace of  $Z$ .
3. Non-trivial linear (non-singleton) spaces are infinite.

### 5.2 Review of Linear Maps

Consider linear spaces  $V$  and  $W$  and elements  $v \in V$  and  $w \in W$  and scalars  $\alpha, \beta \in \mathbb{R}$ , a function  $F : V \rightarrow W$  is a linear map if

$$F[\alpha v + \beta w] = \alpha F[v] + \beta F[w].$$

**Properties 5.3.**

1.  $F[\mathbf{0}_V] = \mathbf{0}_W$
2.  $G[w] = (\alpha \cdot F)[w] = \alpha \cdot F[w]$
3. Given  $F : V \rightarrow W$  and  $G : V \rightarrow W$ ,  $H[v] = F[v] + G[v] = (F + G)[v]$  is called the sum of the functions  $F$  and  $G$ .

4. Linear combinations of linear maps are linear.
5. Compositions of linear maps are linear.
6. Compositions distributes over linear combinations of linear maps.
7. Inverses of linear functions (if they exist) are linear.
8. Inverse of a bijective linear function is a bijective linear function

### 5.3 Review of Kernels and Images

**Definition 5.1.** Let  $F : V \rightarrow W$  be given. The kernel of  $F$  is defined as

$$\ker(F) = \{v \in V \mid F[v] = \mathbf{0}_W\}.$$

**Properties 5.4.** Let  $F : V \rightarrow W$  a linear map be given. Also, consider a linear map  $G$  such that  $F \circ G$  is defined

1.  $F$  is null  $\iff \ker(F) = V \iff \text{Im}(F) = \mathbf{0}_W$
2.  $\ker(F)$  is a subspace of  $V$
3.  $\text{Im}(F)$  is a subspace of  $W$
4.  $F$  is injective  $\iff \ker(F) = \mathbf{0}_V$
5.  $\ker(F) \subseteq \ker(F \circ G)$ .
6.  $F$  injective  $\implies \ker(F) = \ker(F \circ G)$

**Definition 5.2.** Let  $F : V \rightarrow W$  be given. The image of  $F$  is defined as

$$\text{Im}(F) = \{w \in W \mid \exists v \in V, F[v] = w\}.$$

### 5.4 Atrices

**Definition 5.3. Atrix functions:** Let  $V_1, \dots, V_m \in \mathbf{V}$  be given. Consider  $f : \mathbb{R}^m \rightarrow \mathbf{V}$  be defined by

$$f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \sum_{i=1}^m a_i V_i.$$

We denote  $f$  by

$$(V_1 \quad V_2 \quad \dots \quad V_m).$$

We refer to the  $V_i$ 's as the **columns** of  $f$ , even though there doesn't have to be any columns. Basically,  $f$  is simply a function that takes in an ordered list of coefficients and returns a linear combination of  $V_i$  with the respective coefficients. A matrix is a special atrix. Not every atrix is a matrix.

**Properties 5.5.** of atrices

1. The  $V_i$ 's - the columns of an atrix - are the images of the standard basis tuples.
2.  $\mathbf{e}_j \in \ker(V_1 \dots V_m) \iff V_j = \mathbf{0}_V$ , where  $\mathbf{e}_j$  denotes a standard basis tuple with a 1 at the  $j^{\text{th}}$  position. To put in words, a kernel of an atrix contains a standard basis if and only if one of its columns is a null element.
3.  $\text{Im}(f) \equiv \text{Im}(V_1 \dots V_m) = \text{span}(V_1 \dots V_m)$
4.  $B$  is a null atrix  $\iff \ker(B) = \mathbb{R}^m \iff \text{Im}(B) = \mathbf{0}_V \iff V_j = \mathbf{0}_V \forall j = 1, 2, \dots, m$ .
5.  $f$  is a linear function  $\mathbb{R}^m \rightarrow V \iff f$  is an atrix function  $\mathbb{R}^m \rightarrow V$ .
6. An atrix  $A$  is bijective/invertible, then its inverse  $A^{-1}$  is a linear function, but is an atrix only if  $A$  is a matrix.
7. Linear combinations of atrices are atrices:

$$\begin{aligned} & \alpha \cdot (V_1 \quad \dots \quad V_m) + \beta \cdot (W_1 \quad \dots \quad W_m) \\ &= (\alpha V_1 + \beta W_1 \quad \dots \quad \alpha V_m + \beta W_m) \end{aligned}$$

8. Compositions of two atrices are NOT defined unless the atrix going first is a matrix. Consider  $F : \mathbb{R}^m \rightarrow W$  and  $G : \mathbb{R}^n \rightarrow T$ .  $F \circ G$  is only defined if  $T = \mathbb{R}^m$ . This makes  $G$  an  $n \times m$  matrix. It follows that the atrix  $F \circ G$  has the form

$$(f(g_1) \quad f(g_2) \quad \dots \quad f(g_m)).$$

9. Consider  $F : \mathbb{R}^m \rightarrow V$  and  $G : \mathbb{R}^k \rightarrow V$ .  $\text{Im}(F) \subseteq \text{Im}(G) \iff F = G \circ C$ , with  $C \in \mathbb{M}_{k \times m}$ , i.e.  $C$  is an  $k \times m$  matrix.
10. Consider an atrix  $A : \mathbb{R}^m \rightarrow V$ .  $A = (V_1 \dots V_m)$ .  $A$  is NOT injective.  
 $\iff \exists$  a non-trivial linear combination of the columns of  $A$  that gives  $\mathbf{0}_V$   
 $\iff A = [\mathbf{0}_v]$  or  $\exists j | V_j$  is linear combination of other columns of  $A$   
 $\iff$  The first column of  $A$  is  $\mathbf{0}_V$  or  $\exists j | V_j$  is a linear combination some of  $V_i, i < j$ .
11. If atrix  $A : \mathbb{R} \rightarrow V$  has a single column then it is injective if the column is not  $\mathbf{0}_V$ .

**Properties 5.6.** of elementary column operations for atrices. Elementary operations on the columns of  $F$  can be expressed as a composition of  $F$  and an appropriate elementary matrix  $E$ ,  $F \circ E$ .

1. Swapping  $i^{\text{th}}$  and  $j^{\text{th}}$  columns:  $F \circ E^{[i] \leftrightarrow [j]}$ .
2. Scaling the  $j^{\text{th}}$  column by  $\alpha$ :  $F \circ E^{\alpha \cdot [j]}$ .



3. Adjust the  $j^{th}$  column by adding to it  $\alpha \times i^{th}$  column:  $F \circ E^{[i] \xleftarrow{\pm} \alpha \cdot [j]}$ .
4. Elementary column operations do not change the injectivity nor image of  $F$ .
5. If a column of  $F$  is a linear combination of some of the other columns then elementary column operations can turn it into  $\mathbf{0}_V$ .
6. Removing/Inserting null columns or columns that are linear combinations of other columns does not change the image of  $F$ .
7. Given matrix  $A$ , it is possible to eliminate (or not) columns of  $A$  to end up with a matrix  $B$  with  $\text{Im}(B) = \text{Im}(A)$ .
8. If  $B$  is obtained from insertion of columns into matrix  $A$ , then  $\text{Im}(B) = \text{Im}(A) \iff$  the insert columns  $\in \text{Im}(A)$ .
9. Inserting columns to a surjective  $A$  does not destroy surjectivity of  $A$ . ( $A$  is already having extra or just enough columns)
10.  $A$  surjective  $\iff A$  is obtained by inserting columns (or not) into an invertible matrix  $\iff$  deleting some columns of  $A$  (or not) gives an invertible matrix.
11. If  $A$  is injective, then column deletion does not destroy injectivity. ( $A$  is already “lacking” or having just enough columns)
12. The new matrix obtained from inserting columns from  $\text{Im}(A)$  into  $A$  is injective  $\iff A$  is injective.

## 5.5 Linear Independence, Span, and Bases

### 5.5.1 Linear Independence

- $X_1 \dots X_m$  are linearly independent  
 $\iff F = [X_1 \dots X_m]$  injective  
 $\iff \sum a_i X_i = 0 \iff a_i = 0 \forall i$   
 $\iff \mathbf{0}_V \notin \{X_i\}$  and none are linear combinations of some of the others.  
 $\iff X_1 \neq \mathbf{0}_V$  and  $X_j$  is not a linear combination of any of  $X_i$ 's for  $i < j$ .

#### Properties 5.7.

1. The singleton list is linearly independent if its entry is not the null element
2. Sublists of a linearly independent list are linearly independent
3. List operations cannot create/destroy linearly independence.
4. If a linear map  $L : X \rightarrow W$  injective, then  $X_1 \dots X_m$  linearly independent  $\iff L(X_1) \dots L(X_m)$  linearly independent.

### 5.5.2 Span

#### Properties 5.8.

1. Spans are subspaces.  $\text{span}(X_1 \dots X_m), X_j \in V$  is a subspace of  $V$ .
2.  $\text{span}(X_1 \dots X_j) \subseteq \text{span}(X_1 \dots X_k)$  if  $j \leq k$ .
3. Adding elements to a list that spans  $V$  produces a list that spans  $V$ .
4. The following list operations do not change the span of  $V$ : removing the null/linearly dependent element, inserting a linearly dependent element, scaling element(s), adding (multiples) of an element to another element.
5. It is possible to reduce a list that spans to a list that spans AND have linearly independent elements.
6. Consider  $A : V \rightarrow W$ . If  $X_i$  span  $V$  then  $A(X_i)$  span  $\text{Im}(A)$ .  $i = 1, 2, \dots, m$ .
7. If  $A : V \rightarrow W$  invertible, then  $X_i$  span  $V \iff A(X_i)$  span  $W$ ,  $i = 1, 2, \dots, m$ .

### 5.5.3 Bases

#### Properties 5.9.

1. A list is a basis of  $V$  if the elements are linearly independent and they span  $V$ .
2. A singleton linear space has no basis.
3. Re-ordering the elements of a basis gives another basis.
4.  $\{X_1 \dots X_m\}$  is a basis of  $V$  if  $(X_1 \dots X_m)$  is injective AND  $\text{Im}(X_1 \dots X_m) = V$ , i.e.  $(X_1 \dots X_m)$  invertible.
5. If  $\{X_i\}$  is a basis of  $V$  and  $\{Y_i\}$  is a list of elements in  $W$ , then there exists a unique linear function  $L : V \rightarrow W$  satisfying

$$L[X_i] = Y_i$$

6. If  $A : V \rightarrow W$  bijective, then  $\{V_i\}$  forms a basis of  $V$  and  $\{A(X_i)\}$  forms a basis of  $W$ .
7. Elementary operations on bases give bases.

## 5.6 Linear Bijections and Isomorphisms

**Definition 5.4.** Let linear spaces  $V, W$  be given.  $V$  is **isomorphic** to  $W$  if  $\exists F : V \rightarrow W$  bijective. We say  $V \sim W$ .

**Properties 5.10.**

1. A non-zero scalar multiple of an isomorphism is an isomorphism.
2. A composition of isomorphism is an isomorphism.
3. “Isomorphism” behaves like an equivalence relation:
  - (a) Reflexivity:  $V \sim V$ .
  - (b) Symmetry: if  $V \sim W$  then  $W \sim V$ .
  - (c) Transitivity: if  $V \sim W$  and  $W \sim Z$  then  $V \sim Z$ .
4. Consider  $F : V \rightarrow W$  an isomorphism.
  - (a) Isomorphisms preserve linear independence.  $V_i$ ’s are linearly independent in  $V \iff F(V_i)$ ’s are linearly independent in  $W$ .
  - (b) isomorphisms preserve spanning.  $V_i$ ’s span  $V \iff F(V_i)$ ’s span  $W$ .
  - (c) Isomorphisms preserve bases.  $\{V_i\}$  is a basis of  $V \iff F\{V_i\}$  is a basis of  $W$ .
5. If  $V \sim W$  then  $\dim(V) = \dim(W)$  (finite or infinite).
6. If a linear map  $A : V \rightarrow W$  is given and  $A(X_i)$ ’s are linearly independent, then  $X_i$ ’s are linearly independent.
7. If a linear map  $A : V \rightarrow W$  is injective and  $\{X_i\}$  is a basis of  $V$  then  $\{A(X_i)\}$  is a basis of  $\text{Im}(A)$

## 5.7 Finite-Dimensional Linear Spaces

### 5.7.1 Dimension

**Properties 5.11.**

1.  $\mathbb{R}^m \sim \mathbb{R}^n \iff m = n$ .
2. Isomorphisms  $F : \mathbb{R}^n \rightarrow V$  are bijective atrices.
3. Isomorphisms  $G : W \rightarrow \mathbb{R}^m$  are inverses of bijective atrices.
4. Consider a non-singleton linear space  $V$ 
  - (a)  $V$  has a basis with  $n$  elements.
  - (b)  $V \sim \mathbb{R}^n$ .
  - (c)  $V \sim W$ , where  $W$  is any linear space with a basis of  $n$  elements.

5. If  $V$  is a linear space with a basis with  $n$  elements, then any basis of  $V$  has  $n$  elements.
6. Linear space  $V \sim W$  where  $W$  is  $n$ -dimensional if  $V$  is  $n$ -dimensional.
7. For a non-singleton linear space  $V$ ,  $V \sim \mathbb{R}^n \iff \dim(V) = n$ .
8.  $V \sim W \iff \dim(V) = \dim(W)$ .
9. If  $W$  is a subspace of  $V$ , then  $\dim(W) \leq \dim(V)$ . Equality holds when  $W = V$ .

### 5.7.2 Rank-Nullity Theorem

Let finite-dimensional linear space  $V$  and linear map  $F : V \rightarrow W$  be given. Then  $\text{Im}(F)$  is finite-dimensional and

$$\dim(\text{Im}(F)) + \dim(\ker(F)) = \dim(V)$$

A stronger statement: If a linear map  $F : V \rightarrow W$  has finite rank and finite nullity  $\iff V$  is finite-dimensional, then

$$\text{Im}(F) + \ker(F) = \dim(V).$$

## 5.8 Infinite-Dimensional Spaces

Consider a non-singleton linear space  $V$ . The following statements are equivalent:

1.  $V$  is infinite-dimensional.
2. Every linearly independent list in  $V$  can be enlarged to a strictly longer linearly independent set in  $V$ .
3. Every linearly independent list in  $V$  can be enlarged to an arbitrarily long (finite) linearly independent set in  $V$ .
4. There are arbitrarily long (finite) linearly independent lists in  $V$ .
5. There are linearly independent lists in  $V$  of any (finite) length.
6. No list of finitely many elements of  $V$  spans  $V$ .

## 6 Sums of Subspaces & Products of vector spaces

### 6.1 Direct Sums

**Definition 6.1.** Let  $U_j$ ,  $j = 1, 2, \dots, m$  are subspaces of  $V$ .  $\sum_1^m U_j$  is a *direct sum* if each  $u \in \sum U_j$  can be written in only one way as  $u = \sum_1^m u_j$ . The direct sum  $\sum_i^m U_j$  is denoted as  $U_1 \oplus \dots \oplus U_m$ .

**Properties 6.1.**

1. Condition for direct sum: If all  $U_j$  are subspaces of  $V$ , then  $\sum_1^m U_j$  is a direct sum  $\iff$  the only way to write 0 as  $\sum_1^m u_j$ , where  $u_j \in U_j$  is to take  $u_j = 0$  for all  $j$ .
2. If  $U, W$  are subspaces of  $V$  and  $U \cap W = \{0\}$  then  $U + W$  is a direct sum.
3. Let  $U_j$  be finite-dimensional and are subspaces of  $V$ .

$$U_1 \oplus \dots \oplus U_m \iff \dim(U_1 + \dots + U_m) = \sum_1^m \dim(U_j)$$

4. Let  $U_1, \dots, U_m$  be subspaces of  $V$ . Define a linear map  $\Gamma : U_1 \times \dots \times U_m \rightarrow U_1 + \dots + U_m$  by:

$$\Gamma(u_1, \dots, u_m) = \sum_1^m u_j.$$

$U_1 + \dots + U_m$  is a direct sum  $\iff \Gamma$  is injective.

Subspace addition is associative.

Subspace addition is commutative.

### 6.2 Products and Direct Sums

Here is a connection between direct sums and products of vector spaces. Suppose  $\mathbf{Z}_1$  and  $\mathbf{Z}_2$  are subspaces of  $\mathbf{W}$  and  $z_1 \in \mathbf{Z}_1$ ,  $z_2 \in \mathbf{Z}_2$ . Consider the function  $\boxtimes : \mathbf{Z}_1 \times \mathbf{Z}_2 \xrightarrow{\text{linear}} \mathbf{Z}_1 \oplus \mathbf{Z}_2 \hookrightarrow \mathbf{W}$  defined by

$$\boxtimes \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \triangleq z_1 + z_2. \tag{1}$$

We can show that

1.  $\boxtimes$  is a linear function.
2.  $\boxtimes$  is an isomorphism, i.e.,  $\mathbf{Z}_1 \times \mathbf{Z}_2 \sim \mathbf{Z}_1 \oplus \mathbf{Z}_2$ .

### 6.3 Products of Vector Spaces

**Definition 6.2.** Product of vectors spaces

$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) : v_j \in V_j, j = 1, 2, \dots, m\}.$$

**Definition 6.3.** Addition on  $V_1 \times \cdots \times V_m$ :

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m).$$

**Definition 6.4.** Scalar multiplication on  $V_1 \times \cdots \times V_m$ :

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

**Properties 6.2.**

1. Product of vectors spaces is a vector space.

$$V_j \text{ are vectors spaces over } \mathcal{F} \implies V_1 \times \cdots \times V_m \text{ is a vector space over } \mathcal{F}.$$

2. Dimension of a product is the sum of dimensions:

$$\dim(V_1 \times \cdots \times V_m) = \sum_1^m \dim(V_j)$$

3. Vector space products are NOT commutative:

$$W \times V \neq V \times W.$$

However,

$$V \times W \sim W \times V.$$

4. Vector space products are NOT associative:

$$V \times (W \times Z) \neq (V \times W) \times Z$$

### 6.4 Rank-Nullity Theorem

Suppose  $Z_1$  and  $Z_2$  are subspaces of a finite-dimensional vector space  $W$ . Consider  $z_1 \in Z_1$ ,  $z_2 \in Z_2$ , and a function  $\phi : Z_1 \times Z_2 \rightarrow Z_1 + Z_2 \prec W$  defined by

$$\phi \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 + z_2.$$

First,  $\phi$  is a linear function, as it satisfies the linearity condition:

$$\phi \left( \alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \beta \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} \right) = \alpha \phi \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \beta \phi \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix}.$$

By rank-nullity theorem,

$$\dim(Z_1 \times Z_2) = \dim(Z_1 + Z_2) + \dim(\ker(\phi)).$$

But this is equivalent to

$$\dim(Z_1) + \dim(Z_2) = \dim(Z_1 + Z_2) + \dim(\ker(\phi))$$

The kernel of  $\phi$  is:

$$\ker(\phi) = \left\{ \begin{pmatrix} v \\ -v \end{pmatrix} \middle| v \in z \in Z_1, z \in Z_2 \right\} = \left\{ \begin{pmatrix} v \\ -v \end{pmatrix} \middle| v \in z \in Z_1 \cap Z_2 \right\}$$

We can readily verify that  $Z_1 \cap Z_2$  is a subspace of  $W$ . With this,  $\dim(\ker(\phi)) = \dim(Z_1 \cap Z_2)$ . So we end up with

$$\dim(Z_1 + Z_2) = \dim(Z_1) + \dim(Z_2) - \dim(Z_1 \cap Z_2).$$

### Properties 6.3.

1. When  $Z_1 \cap Z_2$  is trivial, then  $Z_1 + Z_2$  is direct.
2. When  $\dim(\ker(\phi)) = 0$ ,  $\phi$  is injective. But  $\phi$  is also surjective by definition, this implies  $\phi$  is a bijection, in which case

$$Z_1 \oplus Z_2 \sim Z_1 + Z_2.$$

## 6.5 Nullspaces & Ranges of Operator Powers

1. Sequence of increasing null spaces: Suppose  $T \in \mathcal{L}(V)$ , i.e.,  $T$  is some linear function mapping  $V \rightarrow V$ , then

$$\{0\} = \ker(T^0) \subset \ker(T^1) \subset \ker(T^2) \subset \cdots \subset \ker(T^k) \subset \ker(T^{k+1}) \subset \cdots$$

*Proof Outline.* Let  $k$  be a nonnegative integer and  $v \in \ker(T^k)$ . Then  $T^k v = 0$ , so  $T^{k+1} v = T(T^k v) = T(0) = 0$ , so  $v \in \ker T^{k+1}$ . So  $\ker(T^k) \subset \ker(T^{k+1})$ .  $\square$

2. Equality in the sequence of null spaces: Suppose  $m$  is a nonnegative integer such that  $\ker(T^m) = \ker(T^{m+1})$ , then

$$\ker(T^m) = \ker(T^{m+1}) = \ker(T^{m+2}) = \cdots$$

*Proof Outline.* We want to show

$$\ker(T^{m+k}) = \ker(T^{m+k+1}).$$

We know that  $\ker T^{m+k} \subset \ker T^{m+k+1}$ . Suppose  $v \in \ker T^{m+k+1}$ , then

$$T^{m+1}(T^k v) = T^{m+k+1} v = 0.$$

So

$$T^k v \in \ker T^{m+1} = \ker T^m.$$

So

$$0 = T^m(T^k v) = T^{m+k} v,$$

i.e.,  $v \in \ker T^{m+k}$ . So  $\ker T^{m+k+1} \subset \ker T^{m+k}$ . This completes the proof.  $\square$

3. Null spaces stop growing: If  $n = \dim(V)$ , then

$$\ker(T^n) = \ker(T^{n+1}) = \ker(T^{n+2}) = \dots$$

*Proof Outline.* To show:

$$\ker T^n = \ker T^{n+1}.$$

Suppose this is not true. Then the dimension of the kernel has to increase by at least 1 every step until  $n + 1$ . Thus  $\dim \ker T^{n+1} \geq n + 1 > n = \dim(V)$ . This is a contradiction.  $\square$

4.  $V$  is the direct sum of  $\ker(T^{\dim(V)})$  and  $\text{Im}(T^{\dim(V)})$ : If  $n = \dim(V)$ , then

$$V = \ker(T^n) \oplus \text{Im}(T^n).$$

*Proof Outline.* To show:

$$\ker T^n \cap \text{Im } T^n = \{0\}.$$

Suppose  $v \in \ker T^n \cap \text{Im } T^n$ . Then  $T^n v = 0$  and  $\exists u \in V$  such that  $v = T^n u$ . So

$$T^n v = T^{2n} u = 0.$$

So

$$T^n u = 0.$$

But this means  $v = 0$ .  $\square$

5. IT IS NOT TRUE THAT  $V = \ker(T) \oplus \text{Im}(T)$  in general.



## 6.6 Generalized Eigenvectors and Eigenspaces

**Definition 6.5.** Suppose  $T \in \mathcal{L}(V)$  and  $\lambda$  is an eigenvalue of  $T$ . A vector  $v \in V$  is called a **generalized eigenvector** of  $T$  corresponding to  $\lambda$  if  $v \neq 0$  and

$$(T - \lambda I)^j v = 0$$

for some positive integer  $j$ .

**Definition 6.6. Generalized Eigenspace:** Suppose  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbf{F}$ . The **generalized eigenspace** of  $T$  corresponding to  $\lambda$ , denoted  $G(\lambda, T)$ , is defined to be the set of all generalized eigenvectors of  $T$  corresponding to  $\lambda$ , along with the 0 vector.

**Properties 6.4.** 1. Suppose  $T \in \mathcal{L}(V)$  and  $\lambda \in \mathbf{F}$ . Then

$$G(\lambda, T) = \ker(T - \lambda I)^{\dim(V)}.$$

*Proof Outline.* Suppose  $v \in \ker(T - \lambda I)^{\dim(V)}$ . Then  $v \in G(\lambda, T)$ . So,  $\ker(T - \lambda I)^{\dim(V)} \subset G(\lambda, T)$ . Next, suppose  $v \in G(\lambda, T)$ . Then there is a positive integer  $j$  such that

$$v \text{ in } \ker(T - \lambda I)^j.$$

But if this is true, then

$$v \text{ in } \ker(T - \lambda I)^{\dim(V)},$$

since  $\ker(T - \lambda I)^{\dim(V)}$  is the largest possible kernel, in a sense.  $\square$

2. Linearly independent generalized eigenvectors: Let  $T \in \mathcal{L}(V)$ . Suppose  $\lambda_1, \dots, \lambda_m$  are distinct eigenvalues of  $T$  and  $v_1, \dots, v_m$  are corresponding generalized eigenvectors. Then  $v_1, \dots, v_m$  is linearly independent.

## 6.7 Nilpotent Operators

**Definition 6.7.** An operator is called **nilpotent** if some power of it equals 0.

**Properties 6.5.**

Nilpotent operator raised to dimension of domain is 0: Suppose  $N \in \mathcal{L}(V)$  is nilpotent. Then

$$N^{\dim(V)} = 0.$$

Matrix of a nilpotent operator: Suppose  $N$  is a nilpotent operator on  $V$ . Then there is a basis of  $V$  with respect to which the matrix of  $N$  has the form

$$\begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix};$$

here all entries on and below the diagonal are 0's.

## 6.8 Weyr Characteristic

## 7 Idempotents & Resolutions of Identity

**Definition 7.1.** An **Operator** is a linear function from a vector space to itself:

$$\mathcal{E} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}.$$

**Definition 7.2. Idempotents:** are operators with the property  $\mathcal{E}^2 = \mathcal{E}$ , i.e.,

$$\mathcal{E} \circ \mathcal{E} = \mathcal{E}.$$

Recall that if  $V = \mathbf{W} \oplus \mathbf{Z}$ , then there exists an idempotent  $\mathcal{E} \in \mathcal{L}(\mathbf{V})$  such that

$$\begin{aligned}\mathbf{W} &= \text{Im}(\mathcal{E}) \\ \mathbf{Z} &= \ker(\mathcal{E}).\end{aligned}$$

In fact, if  $\mathbf{V} = \mathbf{W} \oplus \mathbf{Z}$  then there exists at least **two** idempotents  $\mathcal{E}$ ,  $\mathbf{W} = \text{Im}(\mathcal{E})$ ,  $\mathbf{Z} = \ker(\mathcal{E})$  and  $\mathcal{F}$ ,  $\mathbf{W} = \ker(\mathcal{F})$ ,  $\mathbf{Z} = \text{Im}(\mathcal{F})$ .

A natural question to ask now is whether every idempotent  $\mathcal{E} \in \mathcal{L}(\mathbf{V})$  can be generated this way. The answer is **Yes**, but to answer to this, we need to show **existence** and **uniqueness**.

**Theorem 7.1.** If  $\mathcal{E} \circ \mathcal{E} = \mathcal{E}^2 = \mathcal{E}$ , then  $\mathbf{V} = \text{Im}(\mathcal{E}) \oplus \ker(\mathcal{E})$ .

*Proof.*

1. Show that  $\text{Im}(\mathcal{E}) + \ker(\mathcal{E}) = \mathbf{V}$ .

Let  $v \in \mathbf{V}$  be given, then  $v = \mathcal{E}(v) + v - \mathcal{E}(v)$ . Now, observe that

$$\mathcal{E}(v - \mathcal{E}(v)) = \mathcal{E}(v) - \mathcal{E}^2(v) = \mathcal{E}(v) - \mathcal{E}(v) = 0.$$

Therefore,  $\mathcal{E}(v) \in \text{Im}(\mathcal{E})$  and  $(v - \mathcal{E}(v)) \in \ker(\mathcal{E})$ . So,  $\text{Im}(\mathcal{E}) + \ker(\mathcal{E}) = \mathbf{V}$ .

2. Show that  $\text{Im}(\mathcal{E}) \oplus \ker(\mathcal{E}) = \mathbf{V}$ .

We want to show that  $\text{Im}(\mathcal{E}) \cap \ker(\mathcal{E}) = \{\mathbf{0}\}$ . So, let  $v \in \text{Im}(\mathcal{E}) \cap \ker(\mathcal{E})$  be given. Then  $v \in \text{Im}(\mathcal{E})$ , which implies  $\mathcal{E}(\mathcal{E}(x)) = \mathcal{E}(x) = v = \mathbf{0}$  for some  $x$ . Hence, the intersection is the trivial subspace.

□

□

**Properties 7.1.** A by product of this previous item is this fact. For any  $x \in \mathbf{V}$

$$\mathcal{E}(\mathcal{E}(x)) = \mathcal{E}(x) \iff \mathcal{E} = \mathcal{E}^2, \quad (2)$$

i.e.,  $\mathcal{E}$  is an idempotent exactly when it acts as an identity function on its own image.

**Theorem 7.2.** If  $\mathcal{E}^2 = \mathcal{E}$  and  $\mathcal{G}^2 = \mathcal{G}$  in  $\mathcal{L}(\mathbf{V})$  and

$$\begin{aligned}\text{Im}(\mathcal{G}) &= \text{Im}(\mathcal{E}) \\ \ker(\mathcal{G}) &= \ker(\mathcal{E})\end{aligned}$$

then

$$\mathcal{E} = \mathcal{G}.$$

*Proof.* Let  $\mathbf{W} = \text{Im}(\mathcal{E}) = \text{Im}(\mathcal{G})$  and  $\mathbf{Z} = \ker(\mathcal{E}) = \ker(\mathcal{G})$ . Note that  $\mathbf{W} \oplus \mathbf{Z} = \mathbf{V}$ . Consider  $v \in \mathbf{V}$ . Then

$$\mathcal{E}(v) = \mathcal{E}(w + z) = \mathcal{E}(w) + \mathcal{E}(z) = \mathcal{E}(w) + 0 = \mathcal{E}(w) = w,$$

where  $v = z + w$  is unique by directness. We can do the same thing with  $G$ , and get  $G = E$ . □

□

**Theorem 7.3.** For each decomposition  $V = \mathbf{W} \oplus \mathbf{Z}$ , there exists a unique idempotent whose image is  $\mathbf{W}$  and kernel is  $\mathbf{Z}$ . □

Now, notice that for  $v = w + z$ , we have  $v = w + z = \mathcal{E}(w) + \mathcal{F}(z) = \mathcal{E}(v) + \mathcal{F}(v) = Id(v)$ . So we have a little theorem:

**Theorem 7.4.** If  $\mathcal{E}$  and  $\mathcal{F}$  are idempotents such that  $\text{Im}(\mathcal{E}) = \ker(\mathcal{F})$  and  $\ker(\mathcal{E}) = \text{Im}(\mathcal{F})$  then

$$\mathcal{E} + \mathcal{F} = Id.$$

□

As a result we get another two results:

1. If  $\mathcal{E}^2 = \mathcal{E} \in \mathcal{L}(\mathbf{V})$  then  $(Id - \mathcal{E})^2 = (Id - \mathcal{E})$  also an idempotent.
- 2.

$$\begin{aligned}\text{Im}(\mathcal{E}) &= \ker(Id - \mathcal{E}) \\ \ker(\mathcal{E}) &= \text{Im}(Id - \mathcal{E})\end{aligned}$$

We say that in this case idempotents come in pairs. So, to informally summarize, if  $V = \mathbf{W} \oplus \mathbf{Z}$ , then there is a pair of idempotents. More concretely, if  $V = \mathbf{W} \oplus \mathbf{Z}$ , then there exist unique idempotents  $\mathcal{E}$  and  $\mathcal{F}$  such that  $\mathcal{E} + \mathcal{F} = Id$  and  $\mathbf{W} = \text{Im}(\mathcal{E}) = \ker(\mathcal{F})$  and  $\mathbf{Z} = \ker(\mathcal{E}) = \text{Im}(\mathcal{F})$ . The converse is also true. If there exist idempotents  $\mathcal{E}$  and  $\mathcal{F}$  such that if  $\mathcal{E} + \mathcal{F} = Id$  and  $\mathbf{W} = \text{Im}(\mathcal{E}) = \ker(\mathcal{F})$  and  $\mathbf{Z} = \ker(\mathcal{E}) = \text{Im}(\mathcal{F})$  then  $\mathbf{V} = \mathbf{W} \oplus \mathbf{Z}$ .

All is good, but what if the vector space  $\mathbf{V}$  is decomposed into three pieces as  $\mathbf{V} = \mathbf{W} \oplus \mathbf{Z} \oplus \mathbf{U}$ ? Well, we simply repeat what we have done before to

$\mathbf{V} = \mathbf{W} \oplus (\mathbf{Z} \oplus \mathbf{U})$ . We know that there exist a unique  $\mathcal{E}_{\mathbf{W}}$  such that  $\mathcal{E}_{\mathbf{W}}^2 = \mathcal{E}_{\mathbf{W}}$  and that  $\text{Im}(\mathcal{E}_{\mathbf{W}}) = \mathbf{W}$  and  $\ker(\mathcal{E}_{\mathbf{W}}) = \mathbf{U} \oplus \mathbf{Z}$ . Similarly, there exist a unique  $\mathcal{E}_{\mathbf{U}}$  such that  $\mathcal{E}_{\mathbf{U}}^2 = \mathcal{E}_{\mathbf{U}}$  and that  $\text{Im}(\mathcal{E}_{\mathbf{U}}) = \mathbf{U}$  and  $\ker(\mathcal{E}_{\mathbf{U}}) = \mathbf{W} \oplus \mathbf{Z}$ ; and there exist a unique  $\mathcal{E}_{\mathbf{Z}}$  such that  $\mathcal{E}_{\mathbf{Z}}^2 = \mathcal{E}_{\mathbf{Z}}$  and that  $\text{Im}(\mathcal{E}_{\mathbf{Z}}) = \mathbf{Z}$  and  $\ker(\mathcal{E}_{\mathbf{Z}}) = \mathbf{W} \oplus \mathbf{U}$ .

There are a few observations we can make:

1. Let  $v \in \mathbf{V}$  be given, then  $(\mathcal{E}_{\mathbf{W}} + \mathcal{E}_{\mathbf{U}} + \mathcal{E}_{\mathbf{Z}})(v) = (\mathcal{E}_{\mathbf{W}} + \mathcal{E}_{\mathbf{U}} + \mathcal{E}_{\mathbf{Z}})(w + u + z) = w + u + z = v$ . So  $\mathcal{E}_{\mathbf{W}} + \mathcal{E}_{\mathbf{U}} + \mathcal{E}_{\mathbf{Z}} = \text{Id}$ .
2.  $\mathcal{E}_{\mathbf{W}} \circ \mathcal{E}_{\mathbf{Z}}(v) = \mathcal{E}_{\mathbf{W}}(z) = \mathbf{0}_{\mathbf{V}}$ . So, in general,  $\mathcal{E}_i \circ \mathcal{E}_j = \mathcal{O}$  for  $i \neq j$  and  $\mathcal{E}_i$  if  $i = j$ .

What about the converse? Given the linear functions such that  $\mathcal{E}_j^2 = \mathcal{E}_j$  such that  $\text{Im}(\mathcal{E}_i) = \ker(\mathcal{E}_j + \mathcal{E}_k)$ , then does  $V = \mathbf{Z} \oplus \mathbf{W} + \mathbf{U}$  hold? The answer is yes! We can restate this for rigorously:

**Theorem 7.5.** Suppose  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$  are idempotents such that  $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \text{Id}$  and  $\mathcal{E}_i \circ \mathcal{E}_j = \delta_i^j \mathcal{E}_i$  then

$$\text{Im}(\mathcal{E}_1) \oplus \text{Im}(\mathcal{E}_2) \oplus \text{Im}(\mathcal{E}_3) = \mathbf{V}$$

and

$$\ker(\mathcal{E}_1) = \text{Im}(\mathcal{E}_2) \oplus \text{Im}(\mathcal{E}_3),$$

etc. □

The following two statements are equivalent:

1.  $\mathbf{W} \oplus \mathbf{Z} \oplus \mathbf{U} = \mathbf{V}$ .
2. There are idempotents  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \in \mathcal{L}(\mathbf{V})$  such that

$$\text{Im}(\mathcal{E}_1) = \mathbf{W}, \text{Im}(\mathcal{E}_2) = \mathbf{Z}, \text{Im}(\mathcal{E}_3) = \mathbf{U}$$

and

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \text{Id}$$

$$\mathcal{E}_j \circ \mathcal{E}_i = \delta_i^j \mathcal{E}_i$$

*Proof.* • The forward direction has been shown.

- We will show how (2) implies (1). We first show that  $\text{Im}(\mathcal{E}_1) + \text{Im}(\mathcal{E}_2) + \text{Im}(\mathcal{E}_3) = \mathbf{V}$ , then

$$\mathbf{V} \ni v = \text{Id}(v) = (\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3)(v) = \mathcal{E}_1(v) + \mathcal{E}_2(v) + \mathcal{E}_3(v) \in \text{Im}(\mathcal{E}_1) + \text{Im}(\mathcal{E}_2) + \text{Im}(\mathcal{E}_3).$$

It is straightforward to show that  $V = \text{Im}(\mathcal{E}_1) + \text{Im}(\mathcal{E}_2) + \text{Im}(\mathcal{E}_3)$ . To show directness, suppose that  $x_1 + x_2 + x_3 = \mathbf{0}$ ,  $x_i \in \text{Im}(\mathcal{E}_i)$ . We

want to show that  $x_i = \mathbf{0}$  for any  $i \in \{1, 2, 3\}$ . By construction, we have that

$$\mathcal{E}_1(x_1) + \mathcal{E}_2(x_2) + \mathcal{E}_3(x_3) = \mathbf{0},$$

so

$$\mathcal{E}_1(\mathcal{E}_1(x_1) + \mathcal{E}_2(x_2) + \mathcal{E}_3(x_3)) = \mathcal{E}_1(\mathbf{0}) = \mathbf{0}.$$

But this implies that

$$\mathcal{E}_1(\mathcal{E}_1(x_1)) = \mathcal{E}_1(x_1) = x_1 = \mathbf{0}.$$

Similarly, we get  $x_j = \mathbf{0}$  for all  $j$ . By one of the equivalent statements about directness, we get that  $\text{Im}(\mathcal{E}_1) \oplus \text{Im}(\mathcal{E}_2) \oplus \text{Im}(\mathcal{E}_3) = \mathbf{V}$ .

□

We can also (of course) look at the kernel. Observe that  $\mathcal{E}_1(\mathcal{E}_2(v)) = \mathbf{0}$ . So  $\text{Im}(\mathcal{E}_2) \subseteq \ker(\mathcal{E}_1)$ . Similarly,  $\text{Im}(\mathcal{E}_3) \subseteq \ker(\mathcal{E}_1)$ . So,  $\text{Im}(\mathcal{E}_3) + \text{Im}(\mathcal{E}_2) \subseteq \ker(\mathcal{E}_1)$ . But we also know that the images of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  intersect trivially, so this sum is direct. So we have  $\text{Im}(\mathcal{E}_3) \oplus \text{Im}(\mathcal{E}_2) \subseteq \ker(\mathcal{E}_1)$ . But we also know that  $V = \text{Im}(\mathcal{E}_1) \oplus \ker(\mathcal{E}_1) = \text{Im}(\mathcal{E}_1) \oplus \text{Im}(\mathcal{E}_2) \oplus \text{Im}(\mathcal{E}_3)$ , so

$$\text{Im}(\mathcal{E}_2) \oplus \text{Im}(\mathcal{E}_3) = \ker(\mathcal{E}_1)$$

as desired.

So, just a recap of what we have done so far, the following statements are equivalent for  $\mathcal{A} \in \mathcal{L}(\mathbf{V})$ :

1.  $\mathcal{A}^2 = \mathcal{A}$ .
2.  $\mathcal{A}(x) = x$  for  $x \in \text{Im}(\mathcal{A})$ .
3.  $(Id - \mathcal{A})^2 = (Id - \mathcal{A})$ .
4.  $\text{Im}(\mathcal{A}) = \ker(Id - \mathcal{A})$ .
5.  $\text{Im}(Id - \mathcal{A}) = \ker(\mathcal{A})$ .

## 8 Block-representations of operators

### 8.1 Coordinatization

Consider a finite-dimensional linear space  $V$  with basis  $\{V_i\}$ ,  $i = 1, 2, \dots, m$ . An element  $\tilde{V}$  in  $V$  can be expressed in exactly one way:

$$\tilde{V} = \sum_{i=1}^m a_i V_i,$$

where  $\{a_i\}$  is unique. We call  $\{a_i\}$  the coordinate tuple of  $\tilde{V}$  and  $a_i$ 's the coordinates.

#### Properties 8.1.

1. Inverse of a bijective matrix outputs the coordinates. Suppose  $A = [V_i]$ . Then

$$\tilde{V} = \sum_{i=1}^m a_i V_i \iff A^{-1}(Z) = (a_1 \quad \dots \quad a_m)^\top$$

### 8.2 Matricial representation of linear functions

#### 8.2.1 In Cartesian products

Let us start with a Cartesian product of two vector spaces and a linear function  $\mathcal{L}$  mapping this Cartesian product to itself  $\mathcal{L} : \mathbf{V} \times \mathbf{W} \xrightarrow{\text{linear}} \mathbf{V} \times \mathbf{W}$ , where a typical element in  $\mathbf{V} \times \mathbf{W}$  is given by

$$\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ w \end{pmatrix}.$$

Formally, we can represent the linear function  $\mathcal{L}$  as a block-matrix function

$$\left( \begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right) \begin{pmatrix} v \\ w \end{pmatrix} \triangleq \begin{pmatrix} F_{11}(v) + F_{12}(w) \\ F_{21}(v) + F_{22}(w) \end{pmatrix}.$$

So we have

$$\begin{aligned} F_{11} &: \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V} \\ F_{12} &: \mathbf{W} \xrightarrow{\text{linear}} \mathbf{V} \\ F_{21} &: \mathbf{V} \xrightarrow{\text{linear}} \mathbf{W} \\ F_{22} &: \mathbf{W} \xrightarrow{\text{linear}} \mathbf{W}. \end{aligned}$$

It can be illuminating if we show what the linear functions  $F_{ij}$  does in a block-matrix form:

$$\begin{array}{c|c|c} & \mathbf{V} & \mathbf{W} \\ \hline \mathbf{V} & \downarrow & \downarrow \\ \mathbf{W} & \downarrow & \downarrow \end{array}.$$

Now, since  $F_{ij}$  are all linear, the block-matrix function

$$\left( \begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right) : \mathbf{V} \times \mathbf{W} \rightarrow \mathbf{V} \times \mathbf{W}$$

is linear as well.

A natural question to ask would be: “Can every linear function be represented this way?” The answer is Yes, and we shall see how this is done.

Notice that

$$\mathcal{L} \begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} = \left( \begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right) \begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} F_{11}(v) \\ F_{21}(v) \end{pmatrix}. \quad (3)$$

In particular,

$$\Pi_{\mathbf{V}} \circ \mathcal{L} \begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} = F_{11}(v), \quad (4)$$

where  $\Pi_{\mathbf{V}} : \mathbf{V} \times \mathbf{W} \rightarrow \mathbf{V}$  is a coordinate projection from  $\mathbf{V} \times \mathbf{W}$  onto  $\mathbf{V}$ . In fact, we can also defined the linear function  $\gamma_{\mathbf{V}}$  that is a coordinate injection from  $\mathbf{V}$  into  $\mathbf{V} \times \mathbf{W}$ . So, given  $\mathcal{L} : \mathbf{V} \times \mathbf{W} \xrightarrow{\text{linear}} \mathbf{V} \times \mathbf{W}$ , we let:

$$F_{11} := \Pi_{\mathbf{V}} \circ \mathcal{L} \circ \gamma_{\mathbf{V}} \quad (5)$$

$$F_{12} := \Pi_{\mathbf{V}} \circ \mathcal{L} \circ \gamma_{\mathbf{W}} \quad (6)$$

$$F_{21} := \Pi_{\mathbf{W}} \circ \mathcal{L} \circ \gamma_{\mathbf{V}} \quad (7)$$

$$F_{22} := \Pi_{\mathbf{W}} \circ \mathcal{L} \circ \gamma_{\mathbf{W}}. \quad (8)$$

Now, we can easily check that  $\mathcal{L}$  is indeed

$$\left( \begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right). \quad (9)$$

We do this by considering  $\begin{pmatrix} v & w \end{pmatrix}^{\top} \in \mathbf{V} \times \mathbf{W}$ . We know that

$$F_{11}(v) + F_{12}(w) = \Pi_1 \circ \mathcal{L} \begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} + \Pi_1 \circ \mathcal{L} \begin{pmatrix} \mathbf{0} \\ w \end{pmatrix} = \Pi_1 \circ \mathcal{L} \begin{pmatrix} v \\ w \end{pmatrix} \quad (10)$$

$$F_{21}(v) + F_{22}(w) = \Pi_2 \circ \mathcal{L} \begin{pmatrix} v \\ \mathbf{0} \end{pmatrix} + \Pi_2 \circ \mathcal{L} \begin{pmatrix} \mathbf{0} \\ w \end{pmatrix} = \Pi_2 \circ \mathcal{L} \begin{pmatrix} v \\ w \end{pmatrix}. \quad (11)$$

So,

$$\begin{pmatrix} F_{11}(v) + F_{12}(w) \\ F_{21}(v) + F_{22}(w) \end{pmatrix} = \begin{pmatrix} \Pi_1 \circ \mathcal{L} \begin{pmatrix} v \\ w \end{pmatrix} \\ \Pi_2 \circ \mathcal{L} \begin{pmatrix} v \\ w \end{pmatrix} \end{pmatrix} = \mathcal{L} \begin{pmatrix} v \\ w \end{pmatrix}.$$

So, if we start with any  $\mathcal{L}$ , we can represent  $\mathcal{L}$  as a matrix of the linear functions  $F_{ij}$ .



### 8.2.2 In direct sums

We don't often work with Cartesian products. So we want to use the idea developed above to break a vector space into direct sums. Suppose that we are given  $V = \mathbf{Z} \oplus \mathbf{W}$ . Let us use the bad notation

$$\begin{pmatrix} w \\ z \end{pmatrix}_+ = w + z,$$

so that we can mimic our previous idea. Recall the idea

$$\begin{array}{c|c|c} & \mathbf{V} & \mathbf{W} \\ \hline \mathbf{V} & \downarrow & \downarrow \\ \mathbf{W} & \downarrow & \downarrow \end{array},$$

given

$$\begin{aligned} F_{11} &: \mathbf{W} \rightarrow \mathbf{W} \\ F_{12} &: \mathbf{Z} \rightarrow \mathbf{W} \\ F_{21} &: \mathbf{W} \rightarrow \mathbf{Z} \\ F_{22} &: \mathbf{Z} \rightarrow \mathbf{Z}, \end{aligned}$$

we can define a linear function

$$\left( \begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right) : \mathbf{V} \rightarrow \mathbf{V}$$

by

$$\left( \begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right) \begin{pmatrix} w \\ z \end{pmatrix}_+ = \begin{pmatrix} F_{11}(w) + F_{12}(z) \\ F_{21}(w) + F_{22}(z) \end{pmatrix}_+.$$

Now, we want to find something that is similar to  $\Pi_i$  defined before. Since we are dealing with direct sums rather than Cartesian products, **idempotents** are perfect candidates. In fact, if

$$\mathcal{L} = \left( \begin{array}{c|c} F_{11} & F_{12} \\ \hline F_{21} & F_{22} \end{array} \right),$$

then

$$\mathcal{L} \begin{pmatrix} w \\ \mathbf{0} \end{pmatrix}_+ = \begin{pmatrix} F_{11}(w) \\ F_{21}(w) \end{pmatrix}_+$$

So it is easy to see that

$$\begin{aligned} \mathcal{E} \circ \mathcal{L} \begin{pmatrix} w & \mathbf{0} \end{pmatrix}^\top &= F_{11}(w) \in \mathbf{W} \\ (Id - \mathcal{E}) \circ \mathcal{L} \begin{pmatrix} w & \mathbf{0} \end{pmatrix}^\top &= F_{21}(w) \in \mathbf{Z}. \end{aligned}$$

The same argument applies to find  $F_{12}$  and  $F_{22}$ . So we can once again check that

$$\mathcal{L} = \left( \begin{array}{c|c} \mathcal{E} \circ \mathcal{L} \Big|_{\mathbf{W}} & \mathcal{E} \circ \mathcal{L} \Big|_{\mathbf{Z}} \\ \hline (Id - \mathcal{E}) \circ \mathcal{L} \Big|_{\mathbf{W}} & (Id - \mathcal{E}) \circ \mathcal{L} \Big|_{\mathbf{Z}} \end{array} \right).$$

So, if we have a linear function  $\mathcal{L} : \mathbf{V} \rightarrow \mathbf{V}$  and with respect to the decomposition  $V = \mathbf{W} \oplus \mathbf{Z}$  and that  $\mathcal{L}$  has the block form  $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$  then it means

$$A := \mathcal{E} \circ \mathcal{L} \Big|_{\mathbf{W}} : \mathbf{W} \xrightarrow{\text{linear}} \mathbf{W},$$

and so on, where  $\mathcal{E}$  is an idempotent with  $\text{Im}(\mathcal{E}) = \mathbf{W}$  and  $\ker(\mathcal{E}) = \mathbf{Z}$ .

There is a little caveat: direct sum is commutative, BUT here the order of  $\mathbf{W}$  and  $\mathbf{Z}$  matters in the definition of the block-matrix representation of  $\mathcal{L}$ .

**Example 8.1.** Consider the linear function  $G : \mathbf{V} \rightarrow \mathbf{V}$  an idempotent, such that  $\mathbf{V} = \text{Im}(G) \oplus \ker(G)$ . So,

$$G = \left( \begin{array}{c|c} Id \Big|_{\text{Im}(G)} & \mathcal{O} \\ \hline \mathcal{O} & \mathcal{O} \end{array} \right).$$

### 8.3 Properties of Block-matrix representation

We will show in the homework the following properties:

1. **Addition:** If

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and

$$S = \begin{pmatrix} P & Q \\ R & T \end{pmatrix}$$

then

$$L + S = \begin{pmatrix} A + P & B + Q \\ C + R & D + T \end{pmatrix}.$$

It is not too hard to show why this is true:

$$[L + S]_{11}(w) = \mathcal{E}[L + S](w) = \mathcal{E} \circ L(w) + \mathcal{E} \circ S(w) = A(w) + P(w) = [A + P](w).$$

2. **Scaling:**

$$\alpha \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \left( \begin{array}{c|c} \alpha A & \alpha B \\ \hline \alpha C & \alpha D \end{array} \right).$$

3. **Composition** also works nicely:

$$\left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \circ \left( \begin{array}{c|c} P & Q \\ \hline R & T \end{array} \right) = \left( \begin{array}{c|c} A \circ P + B \circ R & A \circ Q + B \circ T \\ \hline C \circ P + D \circ R & C \circ Q + D \circ T \end{array} \right). \quad (12)$$

It is also quite straightforward to show why this is true. Consider  $[L \circ S]_{11}$ :

$$\begin{aligned} [L \circ S]_{11}(w) &= \mathcal{E}[L \circ S](w) \\ &= \mathcal{E} \circ L[\mathcal{E} + (Id. - \mathcal{E})] \circ S(w) \\ &= \mathcal{E} \circ L \circ \mathcal{E} \circ S(w) + \mathcal{E} \circ L(Id. - \mathcal{E}) \circ S(w) \\ &= \mathcal{E} \circ L \circ P(w) + \mathcal{E} \circ L \circ R(w) \\ &= A \circ P(w) + B \circ R(w) \\ &= [A \circ P + B \circ R](w). \end{aligned}$$

4. Suppose

$$L = \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right).$$

If we pick a basis  $\beta$  for  $\mathbf{W}$  and  $\gamma$  for  $\mathbf{Z}$  then we can write the atrix

$$[L]_{\beta||\gamma \leftarrow \beta||\gamma} = [c_i \dots c_k].$$

where  $\beta||\gamma$  is a concatenation of the bases and the columns:

$$\begin{aligned} c_i &= [L(b_i)]_{\beta||\gamma} \\ &= \left[ \left( \begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \begin{pmatrix} b_i \\ \mathbf{0} \end{pmatrix} \right]_{\beta||\gamma} \\ &= \left[ \begin{pmatrix} A(b_i) \\ C(b_i) \end{pmatrix} \right]_{\beta||\gamma} \\ &= \begin{bmatrix} [A(b_i)]_{\beta} \\ [C(b_i)]_{\gamma} \end{bmatrix} \end{aligned}$$

where  $b_i$  is the  $i^{th}$  element of the basis set  $\beta$ . If we repeat this procedure for all  $b_i$ 's and  $g_i$ 's, we will get:

$$[L]_{\beta||\gamma \leftarrow \beta||\gamma} = \left( \begin{array}{c|c} [A]_{\beta \leftarrow \beta} & [B]_{\gamma \leftarrow \beta} \\ \hline [C]_{\beta \leftarrow \gamma} & [D]_{\gamma \leftarrow \gamma} \end{array} \right)$$

where a block, say  $[C]_{\beta \leftarrow \gamma}$  is generated by

$$[C]_{\gamma \leftarrow \beta} = \begin{bmatrix} [C(b_1)]_{\gamma} & [C(b_1)]_{\gamma} & \dots & [C(b_n)]_{\gamma} \end{bmatrix}.$$

## 8.4 Equality of rank and trace for idempotents; Resolution of Identity Revisited

Consider an illuminating example with an idempotent matrix:  $\mathcal{E} = \mathcal{E}^2 \in \mathbb{M}_{n \times n}$ . We know that  $\text{Im}(\mathcal{E}) \oplus \ker(\mathcal{E}) = \mathbb{C}^n$ . So,  $\mathcal{E} : \text{Im}(\mathcal{E}) \oplus \ker(\mathcal{E}) \xrightarrow{\text{linear}} \text{Im}(\mathcal{E}) \oplus \ker(\mathcal{E})$  can be represented as

$$\mathcal{E} = \left( \begin{array}{c|c} \mathcal{E}_{11} & \mathcal{E}_{12} \\ \hline \mathcal{E}_{21} & \mathcal{E}_{22} \end{array} \right)$$

where

$$\begin{aligned} \mathcal{E}_{11} &= \mathcal{E} \circ \mathcal{E} \Big|_{\text{Im}(\mathcal{E})} = Id. \\ \mathcal{E}_{12} &= \mathcal{E} \circ \mathcal{E} \Big|_{\ker(\mathcal{E})} = \mathcal{O} \\ \mathcal{E}_{21} &= (Id. - \mathcal{E}) \circ \mathcal{E} \Big|_{\text{Im}(\mathcal{E})} = \mathcal{O} \\ \mathcal{E}_{22} &= (Id. - \mathcal{E}) \circ \mathcal{E} \Big|_{\ker(\mathcal{E})} = \mathcal{O}, \end{aligned}$$

i.e.,

$$\mathcal{E} \sim \left( \begin{array}{c|c} Id. & \mathcal{O} \\ \hline \mathcal{O} & \mathcal{O} \end{array} \right).$$

The idea we have just explored in the beginning is that we can pick a basis for  $\text{Im}(\mathcal{E})$  and  $\ker(\mathcal{E})$  and concatenate to get a basis for the whole space so that we can write

$$[\mathcal{E}]_{\gamma_1 || \gamma_2 \leftarrow \gamma_1 || \gamma_2} = \left[ \begin{array}{c|c} [Id.]_{\gamma_1 \leftarrow \gamma_1} & [\mathcal{O}]_{\gamma_2 \leftarrow \gamma_1} \\ \hline [\mathcal{O}]_{\gamma_1 \leftarrow \gamma_2} & [\mathcal{O}]_{\gamma_2 \leftarrow \gamma_2} \end{array} \right].$$

Now since

$$\begin{aligned} [Id.]_{\gamma_1 \leftarrow \gamma_1} &= I \\ [\mathcal{O}]_{\gamma_i \leftarrow \gamma_j} &= 0, \end{aligned}$$

we can write

$$\mathcal{E} = \left( \begin{array}{c|c} I & 0 \\ \hline 0 & 0 \end{array} \right).$$

We have a theorem:

**Theorem 8.1.** Every idempotent  $\mathcal{E} \in \mathbb{M}_n$  is similar to matrix of the form

$$\begin{bmatrix} 1 & & \cdots & 0 \\ & 1 & & \\ \vdots & & \ddots & \vdots \\ 0 & & & 0 \end{bmatrix}$$

where the  $k$  1's along the diagonal is  $\text{rank}(\mathcal{E})$ , which also happens to be the trace of  $\mathcal{E}$ , i.e.,

$$\dim(\text{Im}(\mathcal{E})) = \text{rank}(\mathcal{E}) = \text{Tr}(\mathcal{E}).$$

□

Consequently, we have:

**Properties 8.2.** 1. Since  $\text{rank}(\mathcal{E}) = \text{Tr}(\mathcal{E})$ , if  $\text{Tr}(M)$  is not an integer or negative, then  $M$  is not an idempotent.

2. Suppose we have a resolution of identity of  $m$  idempotents:

$$\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_m = I_{\mathbb{C}^n}$$

then

$$\text{Tr}(\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_m) = \text{Tr}(I_{\mathbb{C}^n}) = n,$$

i.e.,

$$\sum_{i=1}^m \text{Tr}(\mathcal{E}_i) = n,$$

i.e.,

$$\sum_{i=1}^m \text{rank}(\mathcal{E}_i) = n.$$

3. Consider the same resolution of identity. Let us consider

$$\text{Im}(\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_m)$$

and

$$\text{Im}(\mathcal{E}_1) + \text{Im}(\mathcal{E}_2) + \cdots + \text{Im}(\mathcal{E}_m).$$

Next, consider

$$(\mathcal{E}_1 + \mathcal{E}_2 + \cdots + \mathcal{E}_m)(x) = \mathcal{E}_1(x) + \mathcal{E}_2(x) + \cdots + \mathcal{E}_m(x) \in \sum_{i=1}^m \text{Im}(\mathcal{E}_i)$$

Therefore,

$$\mathbb{C}^n = \text{Im}(I_{\mathbb{C}^n}) = \text{Im}\left(\sum_{i=1}^m \mathcal{E}_i\right) \subseteq \sum_{i=1}^m \text{Im}(\mathcal{E}_i) \subseteq \mathbb{C}^n.$$

So,

$$\sum_{i=1}^m \text{Im}(\mathcal{E}_i) = \mathbb{C}^n.$$

But we also know that

$$\sum_{i=1}^m \dim(\text{Im}(\mathcal{E}_i)) = \sum_{i=1}^m \text{rank}(\mathcal{E}_i) = n.$$

So, we have a direct sum:

$$\boxed{\bigoplus_{i=1}^m \text{Im}(\mathcal{E}_i) = \mathbb{C}^n}$$

In particular,

$$\dim\left(\bigoplus_{i=2}^m \text{Im}(\mathcal{E}_i)\right) = n - \dim(\text{Im}(\mathcal{E}_1)) = \dim(\ker(\mathcal{E}_1)).$$

But we also know that

$$\bigoplus_{i=2}^m \text{Im}(\mathcal{E}_i) \supseteq \text{Im}\left(\sum_{i=2}^m \mathcal{E}_i\right) = \text{Im}(I - \mathcal{E}_1) = \ker(\mathcal{E}_1).$$

But since

$$\dim(\ker(\mathcal{E}_1)) = \dim\left(\bigoplus_{i=2}^m \text{Im}(\mathcal{E}_i)\right),$$

it must be true that

$$\boxed{\ker(\mathcal{E}_1) = \bigoplus_{i=2}^m \text{Im}(\mathcal{E}_i)}$$

This implies

$$\text{Im}(\mathcal{E}_2) \subseteq \ker(\mathcal{E}_1),$$

which means

$$\mathcal{E}_1 \circ \mathcal{E}_2 = \mathcal{O}.$$

In general, for  $i \neq j$

$$\boxed{\mathcal{E}_i \circ \mathcal{E}_j = \mathcal{O}_{n \times n}}$$

This leads us to the next item:

4. If  $\mathcal{E}_1, \mathcal{E}_2, \dots, \mathcal{E}_m$  are idempotents and  $\sum_{i=1}^m \mathcal{E}_i = I$  then

$$\begin{cases} \mathcal{E}_i \circ \mathcal{E}_j = \mathcal{O} \\ \bigoplus_{i=1}^m \text{Im}(\mathcal{E}_i) = \mathbb{C}^n. \end{cases}$$

## 8.5 Direct sums of operators

In the last subsection, we have worked with idempotents and resolution of identity. Now, suppose we have any linear function  $\mathcal{L} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$ , where  $\mathbf{V}$  is finite dimensional. Suppose we have a non-trivial  $\mathcal{L}$ -invariant proper subspace  $\mathbf{W}$  of  $\mathbf{V}$ . Then we can write

$$\mathbf{V} = \mathbf{W} \oplus \square.$$

Let  $v_1, v_2, \dots, v_m \in \mathbf{W}$  a basis of  $\mathbf{W}$ , and  $v_{m+1}, v_{m+2}, \dots, v_n$  be a basis of  $\square$ . We immediately know that

$$\text{span}(v_1, v_2, \dots, v_m) \oplus \text{span}(v_{m+1}, v_{m+2}, \dots, v_n) = \mathbf{V}.$$

Now, with respect to the decomposition  $\mathbf{V} = \mathbf{W} \oplus \square$ , we can represent  $\mathcal{L}$  as

$$\begin{array}{c|c|c} & \mathbf{W} & \square \\ \hline \mathbf{W} & \mathcal{L}_{11} & \mathcal{L}_{12} \\ \square & \mathcal{L}_{21} & \mathcal{L}_{22} \end{array}$$

where, like what we have done before

$$\begin{aligned} \mathcal{L}_{11} &= \mathcal{E} \circ \mathcal{L} \Big|_{\mathbf{W}} \\ \mathcal{L}_{12} &= \mathcal{E} \circ \mathcal{L} \Big|_{\square} \\ \mathcal{L}_{21} &= (Id. - \mathcal{E}) \circ \mathcal{L} \Big|_{\mathbf{W}} \\ \mathcal{L}_{22} &= (Id. - \mathcal{E}) \circ \mathcal{L} \Big|_{\square}. \end{aligned}$$

But notice that since  $\mathbf{W} \in \mathcal{L}\text{at}(\mathcal{L})$

$$\mathcal{L}(w) \in \mathbf{W} = \text{Im}(\mathcal{E}) = \ker(Id. - \mathcal{E}),$$

i.e.,

$$(Id. - \mathcal{E}) \circ \mathcal{L} \Big|_{\mathbf{W}} = \mathcal{O}.$$

As a little aside, the converse is also true:

$$\begin{aligned}
& (Id. - \mathcal{E}) \circ \mathcal{L} \Big|_{\mathbf{W}} = \mathcal{O} \\
\implies \text{Im} \left( \mathcal{L} \Big|_{\mathbf{W}} \right) & \subseteq \ker(Id. - \mathcal{E}) = \text{Im}(\mathcal{E}) = \mathbf{W} \\
\implies \mathbf{W} & \in \mathfrak{Lat}(\mathcal{L}).
\end{aligned}$$

So, we have established that

$$\begin{aligned}
\mathcal{L}_{12} = \mathcal{O} & \iff \begin{array}{c|c|c} & \mathbf{W} & \square \\ \hline \mathbf{W} & & \\ \square & \mathcal{O} & \end{array} \iff \mathbf{W} \in \mathfrak{Lat}(\mathcal{L}) \\
\mathcal{L}_{21} = \mathcal{O} & \iff \begin{array}{c|c|c} & \mathbf{W} & \square \\ \hline \mathbf{W} & & \mathcal{O} \\ \square & & \end{array} \iff \square \in \mathfrak{Lat}(\mathcal{L}).
\end{aligned}$$

So, if  $\mathbf{W}, \square \in \mathfrak{Lat}(\mathcal{L})$  and  $\mathbf{W} \oplus \square = \mathbf{V}$  then  $\mathcal{L}$  can be represented by

$$\begin{array}{c|c|c} & \mathbf{W} & \square \\ \hline \mathbf{W} & \mathcal{A} & \mathcal{O} \\ \square & \mathcal{O} & \mathcal{B} \end{array}.$$

But of course the question is whether we know there is an invariant subspace of  $\mathcal{L}$  to start. While we don't really have a clue as to how to *find* one for any linear function  $\mathcal{L}$ , we can surely *construct* one. For  $v_0 \neq \mathbf{0} \in \mathbf{V}$ . We want to construct  $\mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$  such with  $v_0 \in \mathbf{W}$ , i.e.,

$$v_0 \in \mathbf{W}, \mathcal{L}(v_0) \in \mathbf{W}, \dots, \mathcal{L}^n(v_0) \in \mathbf{W}.$$

Say  $\mathcal{L}^k(v_0)$  is that first one that is a linear combination of previous ones, i.e.,

$$\mathcal{L}^k(v_0) = \sum_{i=0}^{k-1} a_i \mathcal{L}^i(v_0)$$

then, say for  $k = 3$ , we have

$$(\mathcal{L}^3 - a_0 \mathcal{L}^0 - a_1 \mathcal{L}^1 - a_2 \mathcal{L}^2)(v_0) = \mathbf{0}.$$

By the fundamental theorem of algebra, we can factor out the “polynomial” above to get

$$(\mathcal{L} - r_1 I)(\mathcal{L} - r_2 I)(\mathcal{L} - r_3 I)(v_0) = \mathbf{0}.$$

But since  $v_0 \neq \mathbf{0}$ , at least one of  $\mathcal{L} - r_i I$  is NOT injective, i.e., for some  $\hat{v} \neq \mathbf{0}$ ,

$$\mathcal{L}(\hat{v}) = r_i \hat{v}.$$

We have just shown that  $r_i$  is an eigenvalue for  $\mathcal{L}$  and  $\hat{v}$  is an eigenvector for  $\mathcal{L}$ .



**Theorem 8.2.** Every linear function on  $\mathbb{C}$  on finite dimensional vector space has an eigenvector.  $\square$

It follows immediately that

**Theorem 8.3.** Every linear function on  $\mathbb{C}$  on finite dimensional vector space has a one-dimensional invariant subspace.  $\square$

## 9 Invariant subspaces

### 9.1 Triangularization

### 9.2 Reducing subspaces

## 10 Polynomials applied to operators

10.1 Minimal polynomials at a vector

10.2 Existence of eigenvalues

10.3 Global minimal polynomials

10.4 (Minimal) polynomials of block- $\Delta^r$  operators

10.5 Minimal polynomials of block-diagonal operators

## 11 Eigentheory

### 11.1 Minimal polynomials and the spectrum

### 11.2 Spectral Mapping Theorem

## 12 Diagonalization

### 12.1 Spectral resolutions

### 12.2 Compressions to invariant subspaces

## 13 Simultaneous triangularization and diagonalization for commuting families

## 14 Primary decomposition over $\mathbb{C}$ and generalized eigenspaces

## 15 Cyclic decomposition and Jordan form

15.1 Square roots of operators

15.2 Similarity of a matrix and its transpose

15.3 Similarity of a matrix and its conjugate

15.4 Jordan forms of  $AB$  and  $BA$

15.5 Power-convergent operators

15.6 Power-bounded operators

15.7 Row-stochastic matrices



## 16 Determinant & Trace

### 16.1 Classical adjoints

### 16.2 Cayley-Hamilton theorem

### 16.3 General Laplace expansions

### 16.4 Cauchy-Binet formula

## 17 Inner products and norms

- 17.1 Riesz representation theorem
- 17.2 Adjoints
- 17.3 Grammians
- 17.4 Orthogonal complements and orthogonal decompositions
- 17.5 Ortho-projections
- 17.6 Closest point solutions
- 17.7 Gram-Schmidt and orthonormal bases

## 18 Isometries and unitary operators

## 19 Ortho-triangularization

## 20 Spectral resolutions

**21 Ortho-diagonalization; self-adjoint and normal operators; spectral theorems**

## **22 Positive (semi-)definite operators**

### **22.1 Classification of inner products**

### **22.2 Positive square roots**



## 23 Polar decomposition

## **24 Single value decomposition**

### **24.1 Spectral/operator norm**

### **24.2 Singular values and approximation**

### **24.3 Singular values and eigenvalues**

## 25 Problems and Solutions

### 25.1 Problem set 1 (under corrections)

**Problem. 1. Equivalent formulations of directness of a subspace sum:**

Suppose that  $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{315}$  are subspaces of a vector space  $\mathbf{W}$ . Argue that the following claims are equivalent.

1. The subspace sum  $\mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_{315}$  is direct.
2. If  $x_i \in \mathbf{V}_i$  and  $x_1 + x_2 + \dots + x_{315} = \mathbf{0}_{\mathbf{W}}$ , then  $x_i = \mathbf{0}_{\mathbf{W}}$ , for every  $i$ .
3. If  $x_i, y_i \in \mathbf{V}_i$  and  $x_1 + x_2 + x_3 + \dots + x_{315} = y_1 + y_2 + y_3 + \dots + y_{315}$ , then  $x_i = y_i$ , for every  $i$ .
4. For any  $i$ , no non-null element of  $\mathbf{V}_i$  can be express as a sum of the elements of the other  $\mathbf{V}_j$ 's.
5. For any  $i$ , no non-null element of  $\mathbf{V}_i$  can be expressed as a sum of the elements of the preceding  $\mathbf{V}_j$ 's.

**Problem. 2. Sub-sums of direct sums are direct:** Suppose that  $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{315}$  are subspaces of a vector space  $\mathbf{W}$ , and the subspace sum  $\mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_{315}$  is direct.

1. Suppose that for each  $i$ ,  $\mathbf{Z}_i$  is a subspace of  $\mathbf{V}_i$ . Argue that the subspace sum  $\mathbf{Z}_1 + \mathbf{Z}_2 + \dots + \mathbf{Z}_{315}$  is direct.
2. Argue that the sum  $\mathbf{V}_2 + \mathbf{V}_5 + \mathbf{V}_7 + \mathbf{V}_{12}$  is also direct.

**Problem. 3. Associativity of directness of subspace sums:** Suppose that

$$\mathbf{Y}, \mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_5, \mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{12}, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_4, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{53}$$

are subspaces of a vector space  $\mathbf{W}$ . Argue that the following claims are equivalent.

1. The subspace sum

$$\mathbf{Y} + \mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_5 + \mathbf{U}_1 + \mathbf{U}_2 + \dots + \mathbf{U}_{12} + \mathbf{Z}_1 + \mathbf{Z}_2 + \dots + \mathbf{Z}_4 + \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_{53}$$

is direct.

2. The subspace sums

$$[\mathbf{V} :=] \mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_5$$

$$[\mathbf{U} :=] \mathbf{U}_1 + \mathbf{U}_2 + \dots + \mathbf{U}_{12}$$

$$[\mathbf{Z} :=] \mathbf{Z}_1 + \mathbf{Z}_2 + \dots + \mathbf{Z}_4$$

$$[\mathbf{X} :=] \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_{53}$$

$$\mathbf{Y} + \mathbf{V} + \mathbf{U} + \mathbf{X} + \mathbf{Z}$$

are all direct.

**Problem. 4. Direct sums preserve linear independence:** Suppose that  $\mathbf{U}, \mathbf{V}, \mathbf{W}$  are subspaces of a vector space  $\mathbf{Z}$ , and the sum  $\mathbf{U}, \mathbf{V}, \mathbf{W}$  is direct.

1. Suppose that  $U_1, U_2, \dots, U_{13}$  is a linearly independent list in  $\mathbf{U}$ ,  $V_1, V_2, \dots, V_6$  is a linearly independent list in  $\mathbf{V}$ , and  $W_1, W_2, \dots, W_{134}$  is a linearly independent list in  $\mathbf{W}$ . Argue that the concatenated list

$$U_1, U_2, \dots, U_{13}, V_1, V_2, \dots, V_6, W_1, W_2, \dots, W_{134}$$

is linearly independent.

2. Suppose that  $U_1, U_2, \dots, U_{13}$  is a basis of  $\mathbf{U}$ ,  $V_1, V_2, \dots, V_6$  is a basis of  $\mathbf{V}$ , and  $W_1, W_2, \dots, W_{134}$  is a basis of  $\mathbf{W}$ . Argue that the concatenated list

$$U_1, U_2, \dots, U_{13}, V_1, V_2, \dots, V_6, W_1, W_2, \dots, W_{134}$$

is a basis of  $\mathbf{U} \oplus \mathbf{V} \oplus \mathbf{W}$ .

**Problem. 5.** Suppose that  $x_1, x_2, \dots, x_{14}$  are non-null elements of a linear space  $\mathbf{W}$ .

1. Argue that the following claims are equivalent.

- (a)  $x_1, x_2, \dots, x_{14}$  are linearly independent.
- (b) The subspace sum

$$\text{span}(x_1) + \text{span}(x_2) + \cdots + \text{span}(x_{14})$$

is direct.

2. Argue that the following claims are equivalent.

- (a)  $x_1, x_2, \dots, x_{14}$  is a basis of  $\mathbf{W}$ .
- (b)  $\mathbf{W} = \text{span}(x_1) \oplus \text{span}(x_2) \oplus \cdots \oplus \text{span}(x_{14})$

## 25.2 Problem set 2

### Problem. 1. Finite unions of subspaces are rarely a subspace

Suppose that  $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_n$  are subspaces of a vector space  $\mathbf{V}$ . Prove that the following are equivalent:

1.  $\mathbf{W}_1 \cup \mathbf{W}_2 \cup \dots \cup \mathbf{W}_n$  is a subspace of  $\mathbf{V}$ .
2. One of the  $\mathbf{W}_i$ 's contains all the others.

### Solution. 1.

- [1.  $\implies$  2.] : Suppose 2. is false and define

$$\mathbf{S} = \bigcup_{i=1}^{n_0} \mathbf{W}_i$$

a subspace of  $\mathbf{V}$ , where none of the  $\mathbf{W}_i$ 's  $\prec \mathbf{V}$  contains all the others, and  $n_0$  is minimal. If  $n_0 = 1$ , then the implication 1.  $\implies$  2. is true because  $\mathbf{W}_1$  contains itself. Therefore, in order for this implication to fail,  $n_0 \geq 2$ .

Because  $n_0$  is minimal,  $\mathbf{S}$  cannot be obtained from a union of less than  $n_0$  of the  $\mathbf{W}_i$ 's. Therefore, each  $\mathbf{W}_i$  contains an element  $w_i$  that does not belong to any other  $\mathbf{W}_j$ 's.

Take  $w_2 \in \mathbf{W}_2 \setminus \mathbf{W}_1$  and  $w_1 \in \mathbf{W}_1$  such that  $w_1 \notin \bigcup_{j \neq 1}^{n_0} \mathbf{W}_j$ . It follows that  $w_1, w_2 \in \mathbf{S}$  because  $\mathbf{W}_1, \mathbf{W}_2 \subseteq \mathbf{S}$ . And since  $\mathbf{S}$  is a subspace,  $nw_1 + w_2 \in \mathbf{S}$  for any  $n \in \mathbb{C}$  by closure under addition.

Consider a collection of  $n_0$  linear combinations of  $w_1$  and  $w_2$ , defined as

$$T = \{w_1 + w_2, 2w_1 + w_2, \dots, n_0 w_1 + w_2\} \subseteq \mathbf{S}.$$

Consider a typical element of  $T$ :  $nw_1 + w_2 \in \mathbf{S}$ ,  $n \in \{1, 2, \dots, n_0\}$ . Suppose  $nw_1 + w_2 \in \mathbf{W}_1$ . By closure under addition,

$$(nw_1 + w_2) - nw_1 = w_2 \in \mathbf{W}_1.$$

But this contradicts the choice of  $w_2$ . Therefore,  $nw_1 + w_2 \notin \mathbf{W}_1$ , for any  $n \in \{1, 2, \dots, n_0\}$ . It follows that  $T \subseteq \bigcup_{i=2}^{n_0} \mathbf{W}_i$ .

By construction,  $T$  has  $n_0$  elements, while  $\bigcup_{i=2}^{n_0} \mathbf{W}_i$  has  $(n_0 - 1)$   $\mathbf{W}_i$ 's. By the pigeonhole principle, a  $\mathbf{W}_k$  of  $\mathbf{W}_2, \dots, \mathbf{W}_{n_0}$  must contain  $nw_1 + w_2$  for two distinct values of  $n$ . Let these values be  $n_a$  and  $n_b$ . By closure under addition, we have

$$(n_a w_1 + w_2) - (n_b w_1 + w_2) = (n_a - n_b)w_1 \in \mathbf{W}_k.$$

But since  $n_a \neq n_b$ ,  $w_1 \in \mathbf{W}_k$ . This contradicts our initial choice of  $w_1$  and thus rules out the possibility that the implication 1.  $\implies$  2. fails to hold. Therefore, the implication 1.  $\implies$  2. must be true.



- [2.  $\implies$  1.] : Without loss of generality, assume that  $\mathbf{W}_1$  contains all the other  $\mathbf{W}_j$ 's. It follows that

$$\bigcup_{i=1}^{n_0} \mathbf{W}_i = \mathbf{W}_1.$$

Since  $\mathbf{W}_1$  is a subspace of  $\mathbf{V}$ ,  $\bigcup_{i=1}^{n_0} \mathbf{W}_i$  is also a subspace of  $\mathbf{V}$ . Therefore, 2.  $\implies$  1. must hold.

**Problem. 2. Images of pre-images and pre-images of images:** Is there a  $3 \times 3$  matrix  $\mathcal{A}$  such that

$$\mathbf{W} := \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$$

is an invariant subspace for  $\mathcal{A}$ , and

$$\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] \subsetneq \mathbf{W} \subsetneq \mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]?$$

Justify your answer.

**Solution. 2.**

First, we observe that  $\dim(\mathbf{W}) = 2$ , because a basis set of  $\mathbf{W}$ ,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

has two elements. Next, consider the vector space  $\mathbb{C}^3$  over the complex numbers. Since  $\mathbf{W}$  is a subspace and any element of  $\mathbf{W}$  is contained in  $\mathbb{C}^3$  by construction,  $\mathbf{W} \prec \mathbb{C}^3$ . In fact,  $\mathbf{W} \subsetneq \mathbb{C}^3$  because  $\mathbb{C}^3 \ni (0 \ 0 \ 1)^\top \notin \mathbf{W}$ .

By the condition  $\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] \subsetneq \mathbf{W} \subsetneq \mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]$ , it must be true that

$$\dim(\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]]) < \dim(\mathbf{W}) < \dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) .$$

Now, since  $\dim(\mathbf{W}) = 2$  and  $\mathcal{A}$  is a  $3 \times 3$  matrix (i.e., neither  $\dim(\ker(\mathcal{A}))$  nor  $\dim(\text{Im}(\mathcal{A}))$  can exceed  $3^{(\dagger)}$ ), it is required that  $\dim(\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]]) \leq 1$  and  $\dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) = 3$ .

Because  $\mathbf{W}$  is invariant under  $\mathcal{A}$ ,  $\mathcal{A}[\mathbf{W}] \subseteq \mathbf{W}$ . It follows that  $\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]] \subseteq \mathcal{A}^{-1}[\mathbf{W}]$ . Therefore,

$$3 = \dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) \leq \dim(\mathcal{A}^{-1}[\mathbf{W}]) .$$

It follows, by fact  $(\dagger)$ , that

$$3 \leq \dim(\mathcal{A}^{-1}[\mathbf{W}]) \leq 3,$$

which implies  $\dim(\mathcal{A}^{-1}[\mathbf{W}]) = 3 = \dim(\mathbb{C}^3)$ . Consider  $v \in \mathcal{A}^{-1}[\mathbf{W}]$ .  $v$  can have the form  $(x \ y \ z)^\top$ ,  $x, y, z \in \mathbb{C}$ , so  $v \in \mathbb{C}^3$ . This implies  $\mathcal{A}^{-1}[\mathbf{W}] \subseteq \mathbb{C}^3$ . But since their dimensions are both 3,  $\mathcal{A}^{-1}[\mathbf{W}] = \mathbb{C}^3$ . It follows that

$$\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] = \mathcal{A}[\mathbb{C}^3] \supseteq \mathcal{A}[\mathbf{W}],$$

where the second relation comes from the fact that  $\mathbf{W}$  is a subspace of  $\mathbb{C}^3$ . Hence,

$$1 \geq \dim(\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]]) \geq \dim(\mathcal{A}[\mathbf{W}]),$$

which implies  $\dim(\mathcal{A}[\mathbf{W}]) = 0$  or  $1$ .

So, such a  $3 \times 3$  matrix  $\mathcal{A}$  that satisfies the given condition is

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

First,  $\mathbf{W}$  is invariant under  $\mathcal{A}$  because take any  $(x \ y \ 0)^\top \in \mathbf{W}$ ,  $\mathcal{A}((x \ y \ 0)^\top) = (x \ 0 \ 0)^\top \in \mathbf{W}$ . This implies  $\mathcal{A}[\mathbf{W}] \subseteq [\mathbf{W}]$  (a).

Second, it is easy to see that the pre-image of  $\mathbf{W}$  under  $\mathcal{A}$  is  $\mathbb{C}^3$ , since take any  $(x \ y \ z)^\top \in \mathbb{C}^3$ , we have  $\mathcal{A}((x \ y \ z)^\top) = (x \ 0 \ 0)^\top \in \mathbf{W}$ . But observe that  $\text{Im}(\mathcal{A})$  is a proper subset of  $\mathbf{W}$ , because any  $(x \ y \ 0)^\top \in \mathbf{W}$  with  $y \neq 0$  is not contained in  $\text{Im}(\mathcal{A})$ . This implies  $\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] \subsetneq \mathbf{W}$  (b).

Lastly, as we have pointed out, because  $\dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) = 3$  and  $\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]] \prec \mathbb{C}^3$ ,  $\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]] = \mathbb{C}^3 \supsetneq \mathbf{W}$  (c).

From (a), (b), (c), we see that this  $3 \times 3$  matrix  $\mathcal{A}$  works as desired.

**Problem. 3. Invariant subspaces form a “lattice”:**

1. Argue that  $\mathfrak{Lat}(\mathcal{L})$  is closed under intersection; i.e., that the intersection of any two invariant subspaces for  $\mathcal{L}$  is also an invariant subspace for  $\mathcal{L}$ .
2. Argue that  $\mathfrak{Lat}(\mathcal{L})$  is closed under subspace sums; i.e., that a subspace sum of two invariant subspaces for  $\mathcal{L}$  is again an invariant subspace for  $\mathcal{L}$ .
3. Show that an image under  $\mathcal{L}$  of an invariant subspace for  $\mathcal{L}$  is again an invariant subspace for  $\mathcal{L}$ .
4. Show that a pre-image under  $\mathcal{L}$  of an invariant subspace for  $\mathcal{L}$  is again an invariant subspace for  $\mathcal{L}$ .

**Solution. 3.**

1. Let subspaces  $\mathbf{V}, \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$  be given.  $\mathfrak{Lat}(\mathcal{L})$  is closed under intersection if  $\mathbf{V} \cap \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ . Consider  $u \in \mathbf{V} \cap \mathbf{W}$ , then  $u \in \mathbf{V}$  and  $u \in \mathbf{W}$ . It follows that  $\mathcal{L}(u) \in \mathcal{L}[\mathbf{V}]$  and  $\mathcal{L}(u) \in \mathcal{L}[\mathbf{W}]$ .

Now, since  $\mathcal{L}[\mathbf{V}] \subseteq \mathbf{V}$  and  $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$  because  $\mathbf{V}, \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ , we have  $\mathcal{L}(u) \in \mathbf{V}$  and  $\mathcal{L}(u) \in \mathbf{W}$ , which implies  $\mathcal{L}(u) \in \mathbf{V} \cap \mathbf{W}$ . Since this implication holds for any  $u \in \mathbf{V} \cap \mathbf{W}$ ,  $\mathcal{L}[\mathbf{V} \cap \mathbf{W}] \subseteq \mathbf{V} \cap \mathbf{W}$ . Therefore,  $\mathbf{V} \cap \mathbf{W}$  is invariant under  $\mathcal{L}$ ; i.e.,  $\mathbf{V} \cap \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ . This completes the argument.

2. Let subspaces  $\mathbf{V}, \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$  be given.  $\mathfrak{Lat}(\mathcal{L})$  is closed under subspace sums if  $\mathbf{V} + \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ . Consider  $v \in \mathbf{V}, w \in \mathbf{W}$ . Then  $v + w \in \mathbf{V} + \mathbf{W}$ . Next, consider  $l \in \mathcal{L}[\mathbf{V} + \mathbf{W}]$  given by

$$l = \mathcal{L}(v + w) = \mathcal{L}(v) + \mathcal{L}(w) = v' + w',$$

where  $v' \in \mathbf{V}$  and  $w' \in \mathbf{W}$  because  $\mathbf{V}, \mathbf{W}$  are invariant under  $\mathcal{L}$ . So  $l \in \mathbf{V} + \mathbf{W}$ . This implies  $\mathcal{L}[\mathbf{V} + \mathbf{W}] \subseteq \mathbf{V} + \mathbf{W}$ ; i.e.,  $\mathfrak{Lat}(\mathcal{L})$  is closed under subspace sums.

3. Let  $\mathbf{V} \in \mathfrak{Lat}(\mathcal{L})$  be given. By definition,  $\mathcal{L}[\mathbf{V}] \subseteq \mathbf{V}$ . It follows that  $\mathcal{L}[\mathcal{L}[\mathbf{V}]] \subseteq \mathcal{L}[\mathbf{V}]$ . So,  $\mathcal{L}[\mathbf{V}]$  is invariant under  $\mathcal{L}$ .

4. Let  $\mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$  be given. Claim:  $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \subseteq \mathcal{L}^{-1}[\mathbf{W}]$ .

It follows from the definition of the pre-image of  $\mathbf{W}$  under  $\mathcal{L} : \mathbf{V} \rightarrow \mathbf{W}$  that  $\mathcal{L}^{-1}[\mathbf{W}] = \{v \in \mathbf{V} | \mathcal{L}(v) \in \mathbf{W}\}$ . This implies  $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \subseteq \mathbf{W}$  (i).

Next, consider  $w \in \mathbf{W} \subseteq \mathbf{V}$ . By definition,  $\mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]] = \{v \in \mathbf{V} | \mathcal{L}(v) \in \mathcal{L}[\mathbf{W}]\}$ , which implies  $w \in \mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]]$ , because  $\mathcal{L}(w) \in \mathcal{L}[\mathbf{W}]$ . Therefore,

$$\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]] \text{ (ii).}$$

Moreover, because  $\mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ ,  $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$  (iii).

From (i), (ii), and (iii) we have

$$\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \stackrel{(i)}{\subseteq} \mathbf{W} \stackrel{(ii)}{\subseteq} \mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]] \stackrel{(iii)}{\subseteq} \mathcal{L}^{-1}[\mathbf{W}].$$

Therefore,  $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \subseteq \mathcal{L}^{-1}[\mathbf{W}]$ , verifying the claim.

**Problem. 4. Cyclic invariant subspaces** For a given  $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$  and a fixed  $v_0 \in \mathbf{V}$ , define

$$\mathbf{P}(\mathcal{L}, v_0) := \left\{ (a_0\mathcal{L}^0 + a_1\mathcal{L}^1 + a_2\mathcal{L}^2 + \cdots + a_k\mathcal{L}^k)(v_0) \mid k \geq 0, a_i \in \mathbb{C} \right\}.$$

1. Argue that  $\mathbf{P}(\mathcal{L}, v_0)$  is a subspace of  $\mathbf{V}$ .
2. Argue that  $\mathbf{P}(\mathcal{L}, v_0)$  is invariant under  $\mathcal{L}$ .
3. Argue that  $\mathbf{P}(\mathcal{L}, v_0)$  is the smallest invariant subspace for  $\mathcal{L}$  that contains  $v_0$ . We refer to  $\mathbf{P}(\mathcal{L}, v_0)$  as **the cyclic invariant subspace for  $\mathcal{L}$  generated by  $v_0$** .
4. Argue that  $\mathbf{P}(\mathcal{L}, v_0)$  is 1-dimensional exactly when  $v_0$  is an eigenvector of  $\mathcal{L}$ .
5. Argue that a subspace  $\mathbf{W}$  of  $\mathbf{V}$  is invariant under  $\mathcal{L}$  exactly when it is a union of cyclic invariant subspaces for  $\mathcal{L}$ .

**Solution. 4.**

1. Consider a typical element  $P(\mathcal{L}, v_0) \in \mathbf{P}(\mathcal{L}, v_0)$ . Since  $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$ ,  $\mathcal{L}^i(v_0) \in \mathbf{V}$  for all  $i$  and  $v_0 \in \mathbf{V}$ . By closure under scalar multiplication and addition, for any  $a_i \in \mathbb{C}$ ,  $v_0 \in \mathbf{V}$ , and  $k \geq 0$ ,

$$P(\mathcal{L}, v_0) = \sum_{i=0}^k a_i \mathcal{L}^i(v_0) \in \mathbf{V}.$$

Therefore  $\mathbf{P}(\mathcal{L}, v_0) \subseteq \mathbf{V}$  (i).

With  $a_i = 0$  for all  $i$ ,  $\mathbf{P}(\mathcal{L}, v_0) \ni P(\mathcal{L}, v_0) = \sum_{i=1}^k 0 \cdot \mathcal{L}^i(v_0) = \mathbf{0}_{\mathbf{P}}$ . So  $\mathbf{P}(\mathcal{L}, v_0)$  contains the null element (ii).

Consider  $P_1(\mathcal{L}, v_0) = \sum_{i=0}^k b_i \mathcal{L}^i(v_0)$  and  $P_2(\mathcal{L}, v_0) = \sum_{j=0}^l c_j \mathcal{L}^j(v_0)$ , with  $b_i, c_j \in \mathbb{C}$  for all  $i, j$ . Without loss of generality, assume that  $k \geq l \geq 0$ . It is clear that  $\mathbf{P}(\mathcal{L}, v_0)$  is closed under addition because,

$$\begin{aligned} P_1(\mathcal{L}, v_0) + P_2(\mathcal{L}, v_0) &= \sum_{i=0}^k b_i \mathcal{L}^i(v_0) + \sum_{j=0}^l c_j \mathcal{L}^j(v_0) \\ &= \sum_{i=0}^k b_i \mathcal{L}^i(v_0) + \sum_{i=0}^k c_i \mathcal{L}^i(v_0) \quad \text{with } c_i = 0 \text{ for all } i > l \\ &= \sum_{i=0}^k (b_i + c_i) \mathcal{L}^i(v_0) \\ &= \sum_{i=0}^k d_i \mathcal{L}^i(v_0) \in \mathbf{P}(\mathcal{L}, v_0), \end{aligned}$$

where  $d_i = b_i + c_i \in \mathbb{C}$ . (iii).

It is also clear that  $\mathbf{P}(\mathcal{L}, v_0)$  is closed under scalar multiplication because given  $\mu \in \mathbb{C}$ ,

$$\mu P_1(\mathcal{L}, v_0) = \mu \sum_{i=0}^k b_i \mathcal{L}^i(v_0) = \sum_{i=0}^k \mu b_i \mathcal{L}^i(v_0) = \sum_{i=0}^k e_i \mathcal{L}^i(v_0) \in \mathbf{P}(\mathcal{L}, v_0),$$

where  $e_i = \mu b_i \in \mathbb{C}$  (iv).

By (i), (ii), (iii), and (iv),  $\mathbf{P}(\mathcal{L}, v_0)$  is a subspace of  $\mathbf{V}$ .

2. Let  $P(\mathcal{L}, v_0)$  be given. It suffices to show that  $\mathcal{L}(P(\mathcal{L}, v_0)) \in \mathbf{P}(\mathcal{L}, v_0)$ , for all  $P(\mathcal{L}, v_0) \in \mathbf{P}(\mathcal{L}, v_0)$ .

$$\mathcal{L}(P(\mathcal{L}, v_0)) = \mathcal{L} \left( \sum_{i=0}^k a_i \mathcal{L}^i(v_0) \right) = \sum_{i=0}^k a_i \mathcal{L}^{i+1}(v_0),$$

where  $a_i \in \mathbb{C}$  and  $k \geq 0$ . Define new coefficients  $b_j = a_i$  where  $j = i + 1$  and  $b_0 = 0$ , then

$$\mathcal{L}(P(\mathcal{L}, v_0)) = \sum_{j=1}^{k+1} b_j \mathcal{L}^j(v_0) = \sum_{j=0}^{k+1} b_j \mathcal{L}^j(v_0) \in \mathbf{P}(\mathcal{L}, v_0).$$

Therefore,  $\mathcal{L}[\mathbf{P}(\mathcal{L}, v_0)] \subseteq \mathbf{P}(\mathcal{L}, v_0)$ ; i.e.,  $\mathbf{P}(\mathcal{L}, v_0)$  is invariant under  $\mathcal{L}$ .

3. Let  $\mathbf{P}(\mathcal{L}, v_0)$  be given.  $\mathbf{P}(\mathcal{L}, v_0)$  is an invariant subspace under  $\mathcal{L}$  that contains  $v_0$  by definition. Suppose  $\mathbf{Q}(\mathcal{L}, v_0)$  is some invariant subspace under  $\mathcal{L}$  that also contains  $v_0$ . It suffices to show  $\mathbf{P}(\mathcal{L}, v_0) \subseteq \mathbf{Q}(\mathcal{L}, v_0)$  for any such  $\mathbf{Q}(\mathcal{L}, v_0)$ .

Consider  $\mathcal{L}^j(v_0)$  for some  $j \geq 0$ .  $\mathcal{L}^j(v_0) \in \mathbf{P}(\mathcal{L}, v_0)$  by definition. We claim that  $\mathcal{L}^j(v_0) = q \in \mathbf{Q}(\mathcal{L}, v_0)$  for any non-negative  $j$ . Assume that this is true. The base case where  $j = 0$  is true since  $v_0 \in \mathbf{Q}(\mathcal{L}, v_0)$ . The inductive case is also true because

$$\mathcal{L}^{(j+1)}(v_0) = \mathcal{L}(\mathcal{L}^j(v_0)) = \mathcal{L}(q) \in \mathbf{Q}(\mathcal{L}, v_0)$$

where the last relation comes from the fact that  $\mathbf{Q}(\mathcal{L}, v_0)$  is invariant under  $\mathcal{L}$ . Therefore, by the principle of induction,  $\mathcal{L}^j(v_0) \in \mathbf{Q}(\mathcal{L}, v_0)$  for any non-negative  $j$ . This implies  $\mathbf{P}(\mathcal{L}, v_0) \subseteq \mathbf{Q}(\mathcal{L}, v_0)$  for any such  $\mathbf{Q}(\mathcal{L}, v_0)$ . Thus  $\mathbf{P}(\mathcal{L}, v_0)$  must be the smallest invariant subspace under  $\mathcal{L}$  that contains  $v_0$ .

4. (a) ( $\implies$ ) : Let a  $\mathbf{P}(\mathcal{L}, v_0)$  be given such that  $\dim(\mathbf{P}(\mathcal{L}, v_0)) = 1$ . Consider the subspace  $\text{span}(v_0)$ . We know that  $\dim(\text{span}(v_0)) = 1$ . Consider  $v \in \text{span}(v_0)$ . Then  $v$  can be expressed as  $v = av_0$  where  $a \in \mathbb{C}$ . It follows immediately that  $v \in \mathbf{P}(\mathcal{L}, v_0)$ . Therefore,  $\text{span}(v_0) \subseteq \mathbf{P}(\mathcal{L}, v_0)$ . But because  $\dim(\mathbf{P}(\mathcal{L}, v_0)) = 1 = \dim(\text{span}(v_0))$ ,  $\mathbf{P}(\mathcal{L}, v_0) = \text{span}(v_0)$ ; i.e., all elements of  $\mathbf{P}(\mathcal{L}, v_0)$  are some scalar multiples of  $v_0$ .

Now, it suffices to show  $\mathcal{L}(v_0) = \lambda v_0$ , where  $\lambda \in \mathbb{C}$ . Consider  $P(\mathcal{L}, v_0) = \mathcal{L}^0(v_0) + \mathcal{L}(v_0) \in \mathbf{P}(\mathcal{L}, v_0) = \text{span}(v_0)$ ,  $v_0 \neq \mathbf{0}_V$ . By closure under addition,

$$(\mathcal{L}^0(v_0) + \mathcal{L}(v_0)) - v_0 = (\mathcal{L}^0(v_0) + \mathcal{L}(v_0)) - \mathcal{L}^0(v_0) = \mathcal{L}(v_0) \in \text{span}(v_0).$$

This implies  $\mathcal{L}(v_0) = \lambda v_0$  for some  $\lambda \in \mathbb{C}$ ; i.e.,  $v_0$  is an eigenvector of  $\mathcal{L}$ .

- (b) ( $\impliedby$ ) : If  $v_0$  is an eigenvector of  $\mathcal{L}$ , then  $\mathcal{L}^i(v_0) = \lambda^i v_0$ , where  $\lambda^i \in \mathbb{C}$  is the  $\mathcal{L}$ 's  $v_0$ -eigenvalue raised to the  $i^{\text{th}}$  power. It follows that a typical element of  $\mathbf{P}(\mathcal{L}, v_0)$  is

$$P(\lambda, v_0) = \sum_{i=0}^k a_i \mathcal{L}^i(v_0) = \left( \sum_{i=0}^k a_i \lambda^i \right) v_0 = \mu v_0,$$

where  $\mu = \sum_{i=0}^k a_i \lambda^i \in \mathbb{C}$ . Therefore,  $\mathbf{P}(\mathcal{L}, v_0) = \text{span}(v_0)$ ; i.e.,  $\dim(\mathbf{P}(\mathcal{L}, v_0)) = 1$ . This completes the argument.

5. (a) ( $\implies$ ) : Let  $\mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$  be given. Then  $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$ . Let  $w_1 \in \mathbf{W}$  be given. We first show that  $\mathcal{L}^k(w_1) \in \mathbf{W}$  for all non-negative  $k$  by induction. Assume that  $\mathcal{L}^k(w_1) = v_1 \in \mathbf{W}$  for all non-negative  $k$  holds. The base case where  $k = 0$  is true since  $\mathcal{L}^0(w_1) = w_1 \in \mathbf{W}$ . The inductive case is also true because

$$\mathcal{L}^{(k+1)}(w_1) = \mathcal{L}(\mathcal{L}^k(w_1)) = \mathcal{L}(v_1) \in \mathbf{W}.$$

By the principle of induction,  $\mathcal{L}^k(w_1) \in \mathbf{W}$  for all  $w_1 \in \mathbf{W}$  and  $k \geq 0$  ( $\dagger$ ). Now, consider  $P \in \mathbf{P}(\mathcal{L}, w_1)$ . By the definition of  $\mathbf{P}(\mathcal{L}, w_1)$ ,  $P$  is a linear combination of  $\mathcal{L}^0(w_1), \dots, \mathcal{L}^k(w_1)$  for some  $k \geq 0$ . So, by ( $\dagger$ ),  $P \in \mathbf{W}$ . This implies  $\mathbf{P}(\mathcal{L}, w_1) \subseteq \mathbf{W}$ .

We can repeat the argument above, starting with  $w_2, w_3, \dots \in \mathbf{W}$ , and conclude that  $\mathbf{P}(\mathcal{L}, w_2) \subseteq \mathbf{W}$ ,  $\mathbf{P}(\mathcal{L}, w_3) \subseteq \mathbf{W}$ , and so on. This means  $\mathbf{W}$  contains a union of  $\mathbf{P}(\mathcal{L}, w_1), \mathbf{P}(\mathcal{L}, w_2), \mathbf{P}(\mathcal{L}, w_3), \dots$ . Moreover, observe the fact that any element  $w_n \in \mathbf{W}$  can be used to generate a  $\mathbf{P}(\mathcal{L}, w_n)$  that is contained in the union of itself and the other  $\mathbf{P}(\mathcal{L}, w_m)$ 's, which is ultimately contained in  $\mathbf{W}$ . Therefore,  $\mathbf{W}$  is itself a union of cyclic invariant subspaces for  $\mathcal{L}$ .



(b) (  $\Leftarrow$  ) : Let  $\mathbf{W} = \mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n$  be a subspace of  $\mathbf{V}$ , where  $\mathbf{P}_i$  denotes  $\mathbf{P}(\mathcal{L}, v_i)$  and  $v_i$  is an arbitrary element of  $\mathbf{V}$ . By problem 1, because  $\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n$  is a subspace of  $\mathbf{V}$ , a  $\mathbf{P}_i$ ,  $1 \leq i \leq n$ , must contain all the other  $\mathbf{P}_j$ 's. Without loss of generality, assume that such a  $\mathbf{P}_i$  is  $\mathbf{P}_1$ . It follows that  $\mathbf{W} = \mathbf{P}_1$ . By part 2. of this problem, we know that  $\mathbf{P}_1$  is invariant under  $\mathcal{L}$ . Therefore,  $\mathbf{W}$  is invariant under  $\mathcal{L}$ .

**Problem. 5. Invariant subspaces of commuting operators:** Suppose that linear functions  $\mathcal{L}, \mathcal{M} \in \mathfrak{L}(\mathbf{V})$  commute; i.e.,

$$\mathcal{L} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{L}.$$

1. Argue that for any non-negative integer  $k$ ,  $\text{Im}(\mathcal{M}^k)$  and  $\ker(\mathcal{M}^k)$  are invariant subspaces for  $\mathcal{L}$ .
2. Argue that every eigenspace of  $\mathcal{M}$  is an invariant subspace for  $\mathcal{L}$ .
3. By giving a general example (with justification, of course!) show that for each  $n > 1$  there are commuting matrices  $\mathcal{A}$  and  $\mathcal{B}$  in  $\mathbb{M}_n$  such that

$$\mathfrak{Lat}(\mathcal{A}) \neq \mathfrak{Lat}(\mathcal{B}).$$

**Solution. 5.**

1. Because  $\mathcal{L} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{L}$ ,  $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$  for any non-negative  $k$ . We can verify this by a short proof by induction. Assume that  $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$  holds for any non-negative  $k$ . The base case where  $k = 0$  is true, for  $\mathcal{L} \circ \mathcal{M}^0 = \mathcal{L} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{L} = \mathcal{M}^0 \circ \mathcal{L}$ , where  $\mathcal{I}$  is the identity operator. The inductive case is also true because

$$\mathcal{L} \circ \mathcal{M}^{k+1} = \mathcal{L} \circ \mathcal{M}^k \circ \mathcal{M} = \mathcal{M}^k \circ \mathcal{L} \circ \mathcal{M} = \mathcal{M}^k \circ \mathcal{M} \circ \mathcal{L} = \mathcal{M}^{k+1} \circ \mathcal{L}.$$

Therefore, by the principle of induction,  $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$  is indeed true for all  $k \geq 0$ .

(a) To show:  $\mathcal{L}[\text{Im}(\mathcal{M}^k)] \subseteq \text{Im}(\mathcal{M}^k)$ .

Since  $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$ ,  $\mathcal{L} \circ \mathcal{M}^k[\mathbf{V}] = \mathcal{M}^k \circ \mathcal{L}[\mathbf{V}]$ ; i.e.,  $\mathcal{L}[\text{Im}(\mathcal{M}^k)] = \mathcal{M}^k[\text{Im}(\mathcal{L})]$ . But since  $\text{Im}(\mathcal{L}) \subseteq \mathbf{V}$  (because  $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$ ),

$$\mathcal{M}^k[\text{Im}(\mathcal{L})] \subseteq \mathcal{M}^k[\mathbf{V}] = \text{Im}(\mathcal{M}^k).$$

It follows that  $\mathcal{L}[\text{Im}(\mathcal{M}^k)] \subseteq \text{Im}(\mathcal{M}^k)$ . So,  $\text{Im}(\mathcal{M}^k)$  is an invariant subspace for  $\mathcal{L}$ .

(b) To show:  $\mathcal{L}[\ker(\mathcal{M}^k)] \subseteq \ker(\mathcal{M}^k)$ .

Since  $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$ ,

$$\mathcal{L} \circ \mathcal{M}^k[\ker(\mathcal{M}^k)] = \{\mathbf{0}_V\} = \mathcal{M}^k \circ \mathcal{L}[\ker(\mathcal{M}^k)].$$

It follows that  $\mathcal{L}[\ker(\mathcal{M}^k)] \subseteq \ker(\mathcal{M}^k)$ . This completes the argument.

2. Let  $\mathbf{E}_\lambda$ , the eigenspace of  $\mathcal{M}$  associated with eigenvalue  $\lambda$ , be given. Consider  $e \in \mathbf{E}_\lambda$ , we have

$$\mathcal{M} \circ \mathcal{L}(e) = \mathcal{L} \circ \mathcal{M}(e) = \mathcal{L}(\lambda e) = \lambda \mathcal{L}(e).$$

This implies  $\mathcal{L}(e) \in \mathbf{E}_\lambda$ . Since this relation holds for any  $e \in \mathbf{E}_\lambda$ ,  $\mathcal{L}[\mathbf{E}_\lambda] \subseteq \mathbf{E}_\lambda$ . Therefore,  $\mathbf{E}_\lambda$  is an invariant subspace for  $\mathcal{L}$  for any eigenvalue  $\lambda$ . This proves the claim that *every* eigenspace of  $\mathcal{M}$  is an invariant subspace for  $\mathcal{L}$ .

3. It suffices to find commuting matrices  $\mathcal{A}$  and  $\mathcal{B}$  such that there exists an invariant subspace under  $\mathcal{A}$ ,  $\mathbf{J} \in \mathfrak{Lat}(\mathcal{A})$ , but  $\mathbf{J} \notin \mathfrak{Lat}(\mathcal{B})$ .

For  $n = 2$ , consider

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

$\mathcal{A}$  and  $\mathcal{B}$  commute:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Consider  $\mathbf{J} = \text{span}\{(1 \ 1)^\top\}$ .  $\mathbf{J} \in \mathfrak{Lat}(\mathcal{A})$  since  $\mathcal{A}[\mathbf{J}] = \{\mathbf{0}\} \subseteq \mathbf{J}$ . However, consider  $j = (1 \ 1)^\top \in \mathbf{J}$ , we have

$$\mathcal{B}(j) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \text{span}\{(1 \ 1)^\top\},$$

which implies  $\mathbf{J} \notin \mathfrak{Lat}(\mathcal{B})$ . Therefore,  $\mathfrak{Lat}(\mathcal{A}) \neq \mathfrak{Lat}(\mathcal{B})$ .

Observe that we can give a more general example for any  $n > 1$  based on the  $2 \times 2$  example above. Let  $\mathcal{A}$  be an  $n \times n$  zero matrix, and  $\mathcal{B}$  be an  $n \times n$  matrix of the form

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}.$$

It is clear that  $\mathcal{A}$  and  $\mathcal{B}$  commute, since  $\mathcal{A}$  is the zero matrix:

$$\mathcal{A}\mathcal{B} = [0]_{n \times n} = \mathcal{B}\mathcal{A}.$$

Consider  $j = (1 \ 1 \ \dots \ 1)^\top \in \mathbf{J} = \text{span}\{(1 \ 1 \ \dots \ 1)^\top\} \subseteq \mathbb{C}^n$ . It is clear that  $\mathcal{B}(j) = (1 \ 0 \ \dots \ 0)^\top \notin \mathbf{J}$  but  $\mathcal{A}(j') = \mathbf{0} \in \mathbf{J}$  for any  $j' \in \mathbf{J}$ . So,  $\mathbf{J} \in \mathfrak{Lat}(\mathcal{A})$  but  $\mathbf{J} \notin \mathfrak{Lat}(\mathcal{B})$ ; i.e.,

$$\mathfrak{Lat}(\mathcal{A}) \neq \mathfrak{Lat}(\mathcal{B}).$$

**Problem. 6. Invariant subspace chains:** Suppose that  $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$  and  $\mathbf{W}$  is an invariant subspace of  $\mathcal{L}$ .

1. Argue that the following inclusions hold:

$$\cdots \subseteq \mathcal{L}^3[\mathbf{W}] \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \mathcal{L}^{-3}[\mathbf{W}] \subseteq \cdots$$

Note that by Problem 3, all of the subspaces listed in the chain are invariant under  $\mathcal{L}$ .

2. Argue that the following implications hold:

$$\mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}] \implies \mathcal{L}^m[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}] \text{ for any } m \geq k,$$

and

$$\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+1)}[\mathbf{W}] \implies \mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-m}[\mathbf{W}] \text{ for any } m \geq k,$$

and then explain why the chain in part 1 stabilizes in both directions, and has at most  $\dim(\mathbf{V})$  proper inclusions.

3. Argue that any subspace  $\mathbf{M}$  that falls between two consecutive subspaces in the chain shown in part 1. is also invariant under  $\mathcal{L}$ .

**Solution. 6.**

1. (a) To show:  $\cdots \subseteq \mathcal{L}^3[\mathbf{W}] \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$ .

Assume that  $\mathcal{L}^{k+1}[\mathbf{W}] \subseteq \mathcal{L}^k[\mathbf{W}]$  for all  $k \geq 0$ . The base case  $k = 0$  is true because  $\mathbf{W}$  is an invariant subspace under  $\mathcal{L}$ . The inductive case is also true because

$$\mathcal{L}^{k+2}[\mathbf{W}] = \mathcal{L}[\mathcal{L}^{k+1}[\mathbf{W}]] \subseteq \mathcal{L}[\mathcal{L}^k[\mathbf{W}]] = \mathcal{L}^{k+1}[\mathbf{W}],$$

where the second relation follows from assumption. By principle of induction,  $\cdots \subseteq \mathcal{L}^3[\mathbf{W}] \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$ .

- (b) To show:  $\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \mathcal{L}^{-3}[\mathbf{W}] \subseteq \cdots$

Assume that  $\mathcal{L}^{-k}[\mathbf{W}] \subseteq \mathcal{L}^{-k-1}[\mathbf{W}]$  for all  $k \geq 0$ . The base case where  $k = 0$  is true: Given  $w \in \mathbf{W} \subseteq \mathbf{V}$ ,  $\mathcal{L}(w) \in \mathbf{W}$  because  $\mathbf{W}$  is invariant under  $\mathcal{L}$ . But since  $\mathcal{L}^{-1}[\mathbf{W}] = \{v \in \mathbf{V} \mid \mathcal{L}(v) \in \mathbf{W}\}$  and  $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$ ,  $w \in \mathcal{L}^{-1}[\mathbf{W}]$ . Therefore,  $\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}]$  (i).

The inductive case is also true:

$$\mathcal{L}^{-k-1}[\mathbf{W}] = \mathcal{L}^{-1}[\mathcal{L}^{-k}[\mathbf{W}]] \subseteq \mathcal{L}^{-1}[\mathcal{L}^{-k-1}[\mathbf{W}]] \subseteq \mathcal{L}^{-k-2}[\mathbf{W}].$$

So by the principle of induction,  $\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \cdots$  (ii).

From (i) and (ii),

$$\cdots \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \cdots$$

2. (a) Assume the hypothesis that  $\mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$  for some  $k \geq 0$ . We can prove by induction that  $\mathcal{L}^{k+j}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$  for all non-negative  $j$ . The base case where  $j = 0$  is true. The inductive case is also true since:

$$\mathcal{L}^{k+j+1}[\mathbf{W}] = \mathcal{L}[\mathcal{L}^{k+j}[\mathbf{W}]] = \mathcal{L}[\mathcal{L}^k[\mathbf{W}]] = \mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}].$$

By the principle of induction,  $\mathcal{L}^m[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$  must hold for any  $m \geq k \geq 0$  if  $\mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$ .

- (b) Assume that hypothesis that  $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+1)}[\mathbf{W}]$  for all  $k \geq 0$ . Also assume that  $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+j)}[\mathbf{W}]$  for all non-negative  $j$ . The base case where  $j = 0$  is true. In inductive case is also true since:

$$\begin{aligned} \mathcal{L}^{-(k+j+1)}[\mathbf{W}] &= \mathcal{L}^{-1}[\mathcal{L}^{-(k+j)}[\mathbf{W}]] \\ &= \mathcal{L}^{-1}[\mathcal{L}^{-k}[\mathbf{W}]] \\ &= \mathcal{L}^{-(k+1)}[\mathbf{W}] \\ &= \mathcal{L}^{-k}[\mathbf{W}]. \end{aligned}$$

By the principle of induction,  $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-m}[\mathbf{W}]$  must hold for  $m \geq k$  if  $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+1)}[\mathbf{W}]$ .

- (c) It follows from part 1. that

$$\dots \leq \dim(\mathcal{L}[\mathbf{W}]) \leq \dim(\mathbf{W}) \leq \dim(\mathcal{L}^{-1}[\mathbf{W}]) \leq \dots$$

Because  $\dim(\mathcal{L}^{-k}[\mathbf{W}]) \leq \dim(\mathbf{V})$  for any  $k \geq 0$ , there exists an  $\mathcal{L}^{-h}[\mathbf{W}]$ ,  $h \geq 0$ , that is an invariant subspace in the chain with the highest dimension where  $h$  is minimal. It follows that  $\dim(\mathcal{L}^{-(h+1)}[\mathbf{W}]) = \dim(\mathcal{L}^{-h}[\mathbf{W}])$  is maximal. But because  $\mathcal{L}^{-h}[\mathbf{W}] \subseteq \mathcal{L}^{-(h+1)}[\mathbf{W}]$ ,  $\mathcal{L}^{-h}[\mathbf{W}] = \mathcal{L}^{-(h+1)}[\mathbf{W}]$ . As a result,  $\mathcal{L}^{-h}[\mathbf{W}] = \mathcal{L}^{-j}[\mathbf{W}]$  for any  $j \geq h$ , which follows from (b). Therefore, the chain in part 1. stabilizes in the “inclusion” direction.

A similar argument can be given to show that the chain in part 1. also stabilizes in the “inclusion by” direction. Since  $0 \leq \dim(\mathcal{L}^l[\mathbf{W}])$  for any  $l \geq 0$ , there exists an  $\mathcal{L}^l[\mathbf{W}]$  in the chain with the lowest dimension but  $l$  is minimal. It follows that  $\dim(\mathcal{L}^{(l+1)}[\mathbf{W}]) = \dim(\mathcal{L}^l[\mathbf{W}])$  is minimal. But because  $\mathcal{L}^{(l+1)}[\mathbf{W}] \subseteq \mathcal{L}^l[\mathbf{W}]$ ,  $\mathcal{L}^{(l+1)}[\mathbf{W}] = \mathcal{L}^l[\mathbf{W}]$ . As a result,  $\mathcal{L}^k[\mathbf{W}] = \mathcal{L}^l[\mathbf{W}]$  for any  $k \geq l$ , which follows from (a). Therefore, the chain in part 1. stabilizes in the “inclusion by” direction as well.

We claim that for every *proper* inclusion, there is a dimension loss. Revisiting  $\mathcal{L}^{-h}[\mathbf{W}]$ , we know that  $\dim(\mathcal{L}^{-h}[\mathbf{W}]) \leq \dim(\mathbf{V})$ . Consider the first proper inclusion:

$$\dots \mathcal{L}^{-(h-j)}[\mathbf{W}] \subsetneq \mathcal{L}^{-(h-j+1)}[\mathbf{W}] \subseteq \dots \subseteq \mathcal{L}^{-h}[\mathbf{W}] \subseteq \dots$$

This implies  $\dim(\mathcal{L}^{-(h-j)}[\mathbf{W}]) < \dim(\mathcal{L}^{-h}[\mathbf{W}]) \leq \dim(\mathbf{V})$ . Suppose  $\dim(\mathcal{L}^{-h}[\mathbf{W}]) = \dim(\mathbf{V})$ , i.e., maximal, then

$$\dim(\mathcal{L}^{-(h-j)}[\mathbf{W}]) \leq \dim(\mathbf{V}) - 1.$$

By assuming equality is satisfied and consider the next proper inclusion in the chain, a similar argument shows that the invariant subspace of interest has at most  $\dim(\mathbf{V}) - 2$  dimensions. It follows that, at the  $\dim(\mathbf{V})^{\text{th}}$  proper inclusion, the considered invariant subspace has at most  $\dim(\mathbf{V}) - \dim(\mathbf{V}) = 0$  dimensions; i.e. the subspace is  $\{\mathbf{0}_{\mathbf{V}}\}$ . At this point, no further proper inclusion is possible. Therefore, the chain in part 1. has at most  $\dim(\mathbf{V})$  proper inclusion.

3. Let  $\mathcal{L}^{(k+1)}[\mathbf{W}] \subseteq \mathbf{M} \subseteq \mathcal{L}^k[\mathbf{W}]$  be given for some nonnegative integer  $k$ .  $\mathbf{M}$  is a subspace. We have that

$$\mathcal{L}[\mathbf{M}] \subseteq \mathcal{L}[\mathcal{L}^k[\mathbf{W}]] = \mathcal{L}^{(k+1)}[\mathbf{W}] \subseteq \mathbf{M}^{(i)}.$$

Let  $\mathcal{L}^{-h}[\mathbf{W}] \subseteq \mathbf{M} \subseteq \mathcal{L}^{-(h+1)}[\mathbf{W}]$  be given for some nonnegative integer  $h$ . We have that

$$\mathcal{L}[\mathbf{M}] \subseteq \mathcal{L}[\mathcal{L}^{-(h+1)}[\mathbf{W}]] = \mathcal{L}[\mathcal{L}^{-1}[\mathcal{L}^{-h}[\mathbf{W}]]] \subseteq \mathcal{L}[\mathcal{L}^{-1}[\mathbf{M}]].$$

But because  $\mathcal{L}^{-1}[\mathbf{M}] = \{v \in \mathbf{V} | \mathcal{L}(v) \in \mathbf{M}\}$ ,  $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{M}]] \subseteq \mathbf{M}$ . Therefore, we have  $\mathcal{L}[\mathbf{M}] \subseteq \mathbf{M}^{(ii)}$ .

From (i) and (ii),  $\mathbf{M}$  is invariant under  $\mathcal{L}$  if  $\mathbf{M}$  is a subspace that falls between two consecutive subspaces in the chain shown in part 1..

### 25.3 Problem set 3

**Problem. 1.** Suppose that  $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$  are non-trivial vector spaces, and for each  $i, j \in \{1, 2, 3\}$ ,

$$\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i.$$

1. Argue that

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$$

is a linear function.

2. Argue that every linear function  $\mathcal{T} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \rightarrow \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$  can be expressed as

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times},$$

with  $\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i$ , in exactly one way.

3. If the function

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \rightarrow \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$$

is defined similarly, find (with proof) the block-matrix form of the functions

$$2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times}$$

and

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times}.$$

**Solution. 1.**

1. (a) It suffices to show that  $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$  satisfies the linearity conditions.

Consider

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$$

where  $a_j, b_j \in \mathbf{V}_j$  for  $j \in \{1, 2, 3\}$ . By definition,

$$\begin{aligned} & \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_{11}(a_1) + \mathcal{L}_{12}(a_2) + \mathcal{L}_{13}(a_3) \\ \mathcal{L}_{21}(a_1) + \mathcal{L}_{22}(a_2) + \mathcal{L}_{23}(a_3) \\ \mathcal{L}_{31}(a_1) + \mathcal{L}_{32}(a_2) + \mathcal{L}_{33}(a_3) \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{11}(b_1) + \mathcal{L}_{12}(b_2) + \mathcal{L}_{13}(b_3) \\ \mathcal{L}_{21}(b_1) + \mathcal{L}_{22}(b_2) + \mathcal{L}_{23}(b_3) \\ \mathcal{L}_{31}(b_1) + \mathcal{L}_{32}(b_2) + \mathcal{L}_{33}(b_3) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_{11}(a_1) + \mathcal{L}_{12}(a_2) + \mathcal{L}_{13}(a_3) + \mathcal{L}_{11}(b_1) + \mathcal{L}_{12}(b_2) + \mathcal{L}_{13}(b_3) \\ \mathcal{L}_{21}(a_1) + \mathcal{L}_{22}(a_2) + \mathcal{L}_{23}(a_3) + \mathcal{L}_{21}(b_1) + \mathcal{L}_{22}(b_2) + \mathcal{L}_{23}(b_3) \\ \mathcal{L}_{31}(a_1) + \mathcal{L}_{32}(a_2) + \mathcal{L}_{33}(a_3) + \mathcal{L}_{31}(b_1) + \mathcal{L}_{32}(b_2) + \mathcal{L}_{33}(b_3) \end{pmatrix} \\ &= \begin{pmatrix} [\mathcal{L}_{11}(a_1) + \mathcal{L}_{11}(b_1)] + [\mathcal{L}_{12}(a_2) + \mathcal{L}_{12}(b_2)] + [\mathcal{L}_{13}(a_3) + \mathcal{L}_{13}(b_3)] \\ [\mathcal{L}_{21}(a_1) + \mathcal{L}_{21}(b_1)] + [\mathcal{L}_{22}(a_2) + \mathcal{L}_{22}(b_2)] + [\mathcal{L}_{23}(a_3) + \mathcal{L}_{23}(b_3)] \\ [\mathcal{L}_{31}(a_1) + \mathcal{L}_{31}(b_1)] + [\mathcal{L}_{32}(a_2) + \mathcal{L}_{32}(b_2)] + [\mathcal{L}_{33}(a_3) + \mathcal{L}_{33}(b_3)] \end{pmatrix}, \text{ by the linearity of } \mathcal{L}_{ij} \\ &= \begin{pmatrix} \mathcal{L}_{11}(a_1 + b_1) + \mathcal{L}_{12}(a_2 + b_2) + \mathcal{L}_{13}(a_2 + b_2) \\ \mathcal{L}_{21}(a_1 + b_1) + \mathcal{L}_{22}(a_2 + b_2) + \mathcal{L}_{23}(a_2 + b_2) \\ \mathcal{L}_{31}(a_1 + b_1) + \mathcal{L}_{32}(a_2 + b_2) + \mathcal{L}_{33}(a_2 + b_2) \end{pmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \left( \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right). \end{aligned}$$

Therefore, the additivity condition is satisfied.

- (b) Next, we consider the scaling condition. Let  $c \in \mathbb{C}$  be given,



$$\begin{aligned}
& \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \left( c \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) \\
&= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}_{11}(ca_1) + \mathcal{L}_{12}(ca_2) + \mathcal{L}_{13}(ca_3) \\ \mathcal{L}_{21}(ca_1) + \mathcal{L}_{22}(ca_2) + \mathcal{L}_{23}(ca_3) \\ \mathcal{L}_{31}(ca_1) + \mathcal{L}_{32}(ca_2) + \mathcal{L}_{33}(ca_3) \end{pmatrix} \\
&= \begin{pmatrix} c\mathcal{L}_{11}(a_1) + c\mathcal{L}_{12}(a_2) + c\mathcal{L}_{13}(a_3) \\ c\mathcal{L}_{21}(a_1) + c\mathcal{L}_{22}(a_2) + c\mathcal{L}_{23}(a_3) \\ c\mathcal{L}_{31}(a_1) + c\mathcal{L}_{32}(a_2) + c\mathcal{L}_{33}(a_3) \end{pmatrix}, \text{ by the linearity of } \mathcal{L}_{ij} \\
&= c \begin{pmatrix} \mathcal{L}_{11}(a_1) + \mathcal{L}_{12}(a_2) + \mathcal{L}_{13}(a_3) \\ \mathcal{L}_{21}(a_1) + \mathcal{L}_{22}(a_2) + \mathcal{L}_{23}(a_3) \\ \mathcal{L}_{31}(a_1) + \mathcal{L}_{32}(a_2) + \mathcal{L}_{33}(a_3) \end{pmatrix}.
\end{aligned}$$

Therefore, the scalar multiplication condition is also satisfied.

From (a) and (b),  $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$  is a linear function.

2. (a) Existence: Let  $\Pi_i$  be a coordinate projection on  $\mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$  onto  $\mathbf{V}_i$  defined in Definition 0.2. Let  $\gamma_i$  be a coordinate injection of  $\mathbf{V}_i$  into  $\mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$  also defined in Definition 0.2.

Consider the block-matrix  $\mathcal{L}_\times$ ,

$$\mathcal{L}_\times = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times.$$

Let the linear function  $\mathcal{T} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$  be given. Let each  $\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i$  be defined as

$$\mathcal{L}_{ij} = \Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{V}_j}.$$

Then we have

$$\begin{aligned} \mathcal{L}_\times &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \\ &= \begin{bmatrix} \Pi_1 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_1 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_1 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \\ \Pi_2 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_2 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_2 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \\ \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_3 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \end{bmatrix}_\times. \end{aligned}$$

Consider  $(w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \in \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$  with  $w_1 \in \mathbf{V}_1$ .

$$\begin{aligned} &\begin{bmatrix} \Pi_1 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_1 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_3} \\ \Pi_2 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_2 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_3} \\ \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_3 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \end{bmatrix}_\times \begin{pmatrix} w_1 \\ \mathbf{0}_\mathbf{V} \\ \mathbf{0}_\mathbf{V} \end{pmatrix} \\ &= \begin{bmatrix} \Pi_1 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} (w_1) \\ \Pi_2 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} (w_1) \\ \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} (w_1) \end{bmatrix} \\ &= \begin{bmatrix} \Pi_1 \circ \mathcal{T} (w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \\ \Pi_2 \circ \mathcal{T} (w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \\ \Pi_3 \circ \mathcal{T} (w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \end{bmatrix}. \end{aligned}$$

Let  $\mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)$  be  $\begin{pmatrix} a & b & c \end{pmatrix}^\top \in \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$ . By definition,

$$\begin{aligned} & \begin{bmatrix} \Pi_1 \circ \mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)^\top \\ \Pi_2 \circ \mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)^\top \\ \Pi_3 \circ \mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)^\top \end{bmatrix} \\ &= \begin{bmatrix} \Pi_1\left(\begin{pmatrix} a & b & c \end{pmatrix}^\top\right) \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ \Pi_2\left(\begin{pmatrix} a & b & c \end{pmatrix}^\top\right) \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \\ \Pi_3\left(\begin{pmatrix} a & b & c \end{pmatrix}^\top\right) \end{bmatrix} \\ &= \begin{bmatrix} a \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ b \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathcal{T}\left(\begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}\right) \end{aligned}$$

Therefore,

$$\mathcal{L}_\times \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix} = \mathcal{T} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}.$$

In general, the equality above also holds for any  $v = w_1 + w_2 + w_3 \in \mathbf{V}$  with  $w_i \in \mathbf{V}_i$ , since  $v$  can be expressed as

$$v = \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{\mathbf{V}} \\ w_2 \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \\ w_3 \end{pmatrix},$$

and the linearity of  $\mathcal{L}_\times$  and  $\mathcal{T}$  ensures that  $\mathcal{L}_\times(v) = \mathcal{T}(v)$ . Hence,  $\mathcal{T}$  can be expressed as the matrix  $\mathcal{L}_\times$  whose elements  $\mathcal{L}_{ij}$ 's are defined as

$$\mathcal{L}_{ij} = \Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{V}_j}.$$

- (b) Uniqueness: We observe that all of the steps in the “Existence part” are reversible. This means if we were given the linear map  $\mathcal{T}$ , we could consider applying  $\mathcal{T}$  to  $\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top, \begin{pmatrix} \mathbf{0}_{\mathbf{V}} & w_2 & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top$ , and  $\begin{pmatrix} \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} & w_3 \end{pmatrix}^\top \in \mathbf{V}$ ; and by reversing the steps in part (a), we know that  $\mathcal{T}$  can be expressed as a  $3 \times 3$  block-matrix whose elements must have the form  $\Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{V}_j} = \mathcal{L}_{ij}$ .

From parts (a) and (b), the linear function  $\mathcal{T}$  can be expressed as the matrix  $\mathcal{L}_\times$  in exactly one way.

3. Let

$$\mathcal{L} := \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$$

and

$$\mathcal{M} := \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$$

be given.

- (a) Let  $\mathcal{L}$  be the matrix of the linear function  $\mathcal{T}$  above and  $\mathcal{M}$  be the matrix of a linear function  $\mathcal{K} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$ . Consider the expression  $2\mathcal{L}_{ij} + 3\mathcal{M}_{ij}$ . Let  $\mathcal{L}_{ij}$  be constructed as in Part 1 for the linear function  $\mathcal{T}$ . Let  $\mathcal{M}$  be constructed similarly for  $\mathcal{K}$ . Then, we have

$$\begin{aligned} 2\mathcal{L}_{ij} + 3\mathcal{M}_{ij} &= 2\Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{w}_j} + 3\Pi_i \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{w}_j} \\ &= \Pi_i \circ 2\mathcal{T} \circ \gamma_j \Big|_{\mathbf{w}_j} + \Pi_i \circ 3\mathcal{K} \circ \gamma_j \Big|_{\mathbf{w}_j} \\ &= \Pi_i \circ \left( 2\mathcal{T} \circ \gamma_j \Big|_{\mathbf{w}_j} + 3\mathcal{K} \circ \gamma_j \Big|_{\mathbf{w}_j} \right) \\ &= \Pi_i \circ (2\mathcal{T} + 3\mathcal{K}) \circ \gamma_j \Big|_{\mathbf{w}_j} \\ &= (2\mathcal{T} + 3\mathcal{K})_{ij}, \text{ by definition.} \end{aligned}$$

This shows that each element of the matrix of the linear function  $(2\mathcal{L} + 3\mathcal{K})$  is given by

$$\boxed{(2\mathcal{T} + 3\mathcal{K})_{ij} = 2\mathcal{L}_{ij} + 3\mathcal{M}_{ij}}$$

i.e.,

$$\begin{aligned} &2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times} \\ &= \begin{bmatrix} 2\mathcal{L}_{11} + 3\mathcal{M}_{11} & 2\mathcal{L}_{12} + 3\mathcal{M}_{12} & 2\mathcal{L}_{13} + 3\mathcal{M}_{13} \\ 2\mathcal{L}_{21} + 3\mathcal{M}_{21} & 2\mathcal{L}_{22} + 3\mathcal{M}_{22} & 2\mathcal{L}_{23} + 3\mathcal{M}_{23} \\ 2\mathcal{L}_{31} + 3\mathcal{M}_{31} & 2\mathcal{L}_{32} + 3\mathcal{M}_{32} & 2\mathcal{L}_{33} + 3\mathcal{M}_{33} \end{bmatrix}_{\times}. \end{aligned}$$

- (b) Let  $\mathcal{T}$ ,  $\mathcal{M}_{ij}$ ,  $\mathcal{K}$ , and  $\mathcal{L}_{ij}$  be defined similarly:  $\mathcal{L}$  is the matrix of the linear function  $\mathcal{T}$  and  $\mathcal{M}$  is the matrix of the linear function  $\mathcal{K}$ . Consider the expression  $\sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj}$ , we have

$$\begin{aligned} \sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj} &= \sum_{k=1}^3 \Pi_i \circ \mathcal{T} \circ \gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= \sum_{k=1}^3 \Pi_i \circ \mathcal{T} \circ \left( \gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \right) \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= \Pi_i \circ \mathcal{T} \circ \left( \sum_{k=1}^3 \gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \right) \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j}. \end{aligned}$$

Consider  $v = (w_1 \ w_2 \ w_3)^\top \in \mathbf{V}$  such that  $w_i \in \mathbf{W}_i$  for  $i \in \{1, 2, 3\}$ . We can show that the summation in the expression above is simply the identity function:

$$\begin{aligned} \left( \sum_{k=1}^3 \gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \right) (v) &= \gamma_1 \Big|_{\mathbf{W}_1} \circ \Pi_1(v) + \gamma_2 \Big|_{\mathbf{W}_2} \circ \Pi_2(v) + \gamma_3 \Big|_{\mathbf{W}_3} \circ \Pi_3(v) \\ &= \gamma_1(w_1) + \gamma_2(w_2) + \gamma_3(w_3) \\ &= \begin{pmatrix} w_1 \\ \mathbf{0}_V \\ \mathbf{0}_V \end{pmatrix} + \begin{pmatrix} \mathbf{0}_V \\ w_2 \\ \mathbf{0}_V \end{pmatrix} + \begin{pmatrix} \mathbf{0}_V \\ \mathbf{0}_V \\ w_3 \end{pmatrix} \\ &= \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ &= v. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj} &= \Pi_i \circ \mathcal{T} \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= \Pi_i \circ (\mathcal{T} \circ \mathcal{K}) \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= (\mathcal{T} \circ \mathcal{K})_{ij}. \end{aligned}$$

This shows that each element of the matrix of the linear function  $(\mathcal{T} \circ \mathcal{K})$  is given by

$$(\mathcal{T} \circ \mathcal{K})_{ij} = \sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj}$$

which gives the full block-matrix form:

$$\begin{aligned}
& \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \\
&= \begin{bmatrix} \sum_{k=1}^3 \mathcal{L}_{1k} \circ \mathcal{M}_{k1} & \sum_{k=1}^3 \mathcal{L}_{1k} \circ \mathcal{M}_{k2} & \sum_{k=1}^3 \mathcal{L}_{1k} \circ \mathcal{M}_{k3} \\ \sum_{k=1}^3 \mathcal{L}_{2k} \circ \mathcal{M}_{k1} & \sum_{k=1}^3 \mathcal{L}_{2k} \circ \mathcal{M}_{k2} & \sum_{k=1}^3 \mathcal{L}_{2k} \circ \mathcal{M}_{k3} \\ \sum_{k=1}^3 \mathcal{L}_{3k} \circ \mathcal{M}_{k1} & \sum_{k=1}^3 \mathcal{L}_{3k} \circ \mathcal{M}_{k2} & \sum_{k=1}^3 \mathcal{L}_{3k} \circ \mathcal{M}_{k3} \end{bmatrix}_{\times}
\end{aligned}$$

**Problem. 2.** Suppose that  $\mathbf{V} = \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3$  and the  $\mathbf{W}_i$ 's are non-trivial subspaces of  $\mathbf{V}$ . Suppose that for each  $i, j \in \{1, 2, 3\}$ ,

$$\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i.$$

1. Verify for formula

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\boxtimes} = \left( \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right)_{\boxtimes},$$

where we have used as few brackets as possible without losing clarity.

2. Argue that every  $T : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$  can be expressed as  $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}$ ,

with  $\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$ , in exactly one way.

3. If the function

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus} : \mathbf{V} \longrightarrow \mathbf{V}$$

is defined similarly, find (with proof) the corresponding black-matrix form of the functions

$$2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}$$

and

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}.$$

4. Suppose that  $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$  is a resolution of the identity, where  $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$ . If

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus},$$

express (with proof, of course)  $\mathcal{L}_{ij}$  in terms of  $\mathcal{L}, \mathcal{E}_k$ 's and  $\mathbf{W}_k$ 's.

5. Continuing with the set-up of part 4, find  $\mathcal{M}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$  such that

$$\mathcal{E}_1 = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}.$$

Then do the same for  $\mathcal{E}_2$  and  $\mathcal{E}_3$ .

**Solution. 2.**

1. By definition,

$$\begin{aligned}
\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\boxtimes} &= \begin{bmatrix} \boxtimes \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \boxtimes^{-1} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\boxtimes} \\
&= \begin{bmatrix} \boxtimes \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \boxtimes^{-1} \circ \boxtimes \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \boxtimes \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \boxtimes \left( \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \\
&= \left( \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right)_{\boxtimes}.
\end{aligned}$$

2. (a) Existence: Let the linear functions  $\mathcal{E}_i := \Pi_i \circ \boxtimes^{-1} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{W}_i$  and  $\mathcal{K}_i := \boxtimes \circ \gamma_i : \mathbf{W}_i \xrightarrow{\text{linear}} \mathbf{V}$  be given, where the  $\Pi_i$ 's and  $\gamma_i$ 's are defined the same way as in Problem 1.

Consider the block matrix  $\mathcal{L}_{\oplus}$

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

Let the linear function  $\mathcal{T} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$  be given. Let each  $\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$  be defined as:

$$\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{W}_j}.$$

By definition,

$$\begin{aligned}
\mathcal{L}_{ij} &= \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{W}_j} \\
&= \Pi_i \circ \boxtimes^{-1} \circ \mathcal{T} \circ \boxtimes \circ \gamma_j \Big|_{\mathbf{W}_j}.
\end{aligned}$$



Applying  $\mathcal{L}_\oplus$  to  $(w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \in \mathbf{V}$ , we get

$$\begin{aligned}
\mathcal{L} \begin{pmatrix} w_1 \\ \mathbf{0}_V \\ \mathbf{0}_V \end{pmatrix}_{\mathfrak{X}} &= \left[ \begin{array}{c} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} \circ \gamma_1 \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} \circ \gamma_1 \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} \circ \gamma_1 \end{array} \middle| \begin{array}{c} (w_1) \\ \mathbf{w}_1 \\ \mathbf{w}_1 \end{array} \right]_{\mathfrak{X}} \\
&= \left[ \begin{array}{c} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} (w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} (w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} (w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \end{array} \right]_{\mathfrak{X}} \\
&= \left[ \begin{array}{c} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \end{array} \right]_{\mathfrak{X}} \\
&= \mathfrak{X} \left( \begin{bmatrix} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \end{bmatrix} \right) \\
&= \mathfrak{X} \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1), \text{ by Part 2(a) of Problem 1} \\
&= \mathcal{T}(w_1)
\end{aligned}$$

By the definition of  $\mathfrak{X}$ , we can write this as

$$\mathcal{L}(w_1 + \mathbf{0}_V + \mathbf{0}_V) = \mathcal{L}(w_1) = \mathcal{T}(w_1).$$

In general, the equality above also holds if we start with any  $v = w_1 + w_2 + w_3 \in \mathbf{V}$  with  $w_i \in \mathbf{W}_i$  for  $i = \{1, 2, 3\}$ , because as we have shown in Part 2(a) of Problem 1,  $v$  can be written as

$$v = \begin{pmatrix} w_1 \\ \mathbf{0}_V \\ \mathbf{0}_V \end{pmatrix}_{\mathfrak{X}} + \begin{pmatrix} \mathbf{0}_V \\ w_2 \\ \mathbf{0}_V \end{pmatrix}_{\mathfrak{X}} + \begin{pmatrix} \mathbf{0}_V \\ \mathbf{0}_V \\ w_3 \end{pmatrix}_{\mathfrak{X}},$$

and the linearity of  $\mathcal{L}$  and  $\mathcal{T}$  ensures  $\mathcal{L}(v) = \mathcal{T}(v)$ . Hence,  $\mathcal{T}$  can be expressed as the matrix  $\mathcal{L}$  whose elements are defined as

$$\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j}.$$

- (b) Uniqueness: Once again, we observe that all of the steps in the “Existence” part are reversible. We can consider applying  $\mathcal{T}$  to  $v$ , expressed in Part 3(a), and by reversing the steps in Part (a) show that the  $3 \times 3$  block-matrix of  $\mathcal{T}$  must consist elements defined by

$$\mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} = \mathcal{L}_{\oplus ij} \text{ for } i, j \in \{1, 2, 3\}.$$

From parts (a) and (b), the linear function  $\mathcal{T}$  can be expressed as the matrix  $\mathcal{L}_\oplus$  defined in the problem statement in exactly one way.

3. (a) Let  $\mathcal{L}_\oplus = [\mathcal{L}_{\oplus ij}]$  denote the matrix  $\mathcal{L}_\oplus : \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3 \xrightarrow{\text{linear}} \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3$ . Let  $\mathcal{M}_\oplus = [\mathcal{M}_{\oplus ij}]$  be defined similarly for  $\mathcal{M}$ . By part 1, we can re-express the matrix

$$[(2\mathcal{L}_\oplus + 3\mathcal{M}_\oplus)_{ij}] = 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\oplus + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\oplus$$

as

$$\begin{aligned} & 2 \mathfrak{X} \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} + 3 \mathfrak{X} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} + \mathfrak{X} \circ 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ \left( 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} \right) \\ &= \mathfrak{X} \circ \left( 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \right) \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ \left( 2[\mathcal{L}_{\times ij}] + 3[\mathcal{M}_{\times ij}] \right) \circ \mathfrak{X}^{-1}, \end{aligned}$$

where the notation  $[A_{ij}]$  denotes the block-matrix of elements  $A_{ij}$  where  $A$  is a linear function, and the last relation:

$$[(2\mathcal{L}_\times + 3\mathcal{M}_\times)_{ij}] = 2[\mathcal{L}_{\times ij}] + 3[\mathcal{M}_{\times ij}].$$

follows from Problem 1. Then, we have

$$\begin{aligned} [(2\mathcal{L}_\oplus + 3\mathcal{M}_\oplus)_{ij}] &= \mathfrak{X} \circ \left( 2[\mathcal{L}_{\times ij}] + 3[\mathcal{M}_{\times ij}] \right) \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ [2\mathcal{L}_{\times ij}] \circ \mathfrak{X}^{-1} + \mathfrak{X} \circ [3\mathcal{M}_{\times ij}] \circ \mathfrak{X}^{-1} \\ &= 2[\mathcal{L}_{\oplus ij}] + 3[\mathcal{M}_{\oplus ij}]. \end{aligned}$$

So, in the block-matrix form:

$$\begin{aligned} & 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\oplus + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\oplus \\ &= \begin{bmatrix} 2\mathcal{L}_{11} + 3\mathcal{M}_{11} & 2\mathcal{L}_{12} + 3\mathcal{M}_{12} & 2\mathcal{L}_{13} + 3\mathcal{M}_{13} \\ 2\mathcal{L}_{21} + 3\mathcal{M}_{21} & 2\mathcal{L}_{22} + 3\mathcal{M}_{22} & 2\mathcal{L}_{23} + 3\mathcal{M}_{23} \\ 2\mathcal{L}_{31} + 3\mathcal{M}_{31} & 2\mathcal{L}_{32} + 3\mathcal{M}_{32} & 2\mathcal{L}_{33} + 3\mathcal{M}_{33} \end{bmatrix}_\oplus. \end{aligned}$$

- (b) Let  $\mathcal{L}_\oplus$  and  $\mathcal{M}_\oplus$  denote the same matrices in part (a), and let  $\mathcal{L}_\oplus$  represent the linear function  $\mathcal{T}$ ,  $\mathcal{M}_\oplus$  represent the linear function  $\mathcal{N}$ . Let the matrix  $[(\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij}]$  express the composition  $\mathcal{T} \circ \mathcal{N}$ . Its the elements  $[\mathcal{L}_\oplus \circ \mathcal{M}_\oplus]$  can be constructed as

$$(\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij} = \mathcal{E}_i \circ (\mathcal{L}_\oplus \circ \mathcal{M}_\oplus) \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} = \mathcal{E}_i \circ (\mathcal{T} \circ \mathcal{N}) \circ \mathcal{K}_j \Big|_{\mathbf{w}_j}, \quad (\dagger)$$

as we have done in Part 1. Consider the expression  $\sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj}$ . By part 1, we have

$$\mathcal{L}_{\oplus ik} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_k \Big|_{\mathbf{w}_k} \quad \text{and} \quad \mathcal{M}_{\oplus kj} = \mathcal{E}_k \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j}.$$

This gives

$$\begin{aligned} \sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj} &= \sum_{k=1}^3 \mathcal{E}_i \circ \mathcal{T} \circ \left( \mathcal{K}_k \Big|_{\mathbf{w}_k} \circ \mathcal{E}_k \right) \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} \\ &= \mathcal{E}_i \circ \mathcal{T} \circ \left( \sum_{k=1}^3 \mathcal{K}_k \Big|_{\mathbf{w}_k} \circ \mathcal{E}_k \right) \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} \end{aligned}$$

By definition, we have

$$\mathcal{K}_k \Big|_{\mathbf{w}_k} = \mathfrak{K} \circ \gamma_k \Big|_{\mathbf{w}_k} \quad \text{and} \quad \mathcal{E}_k = \Pi_k \circ \mathfrak{K}^{-1},$$

which imply

$$\begin{aligned} \sum_{k=1}^3 \mathcal{K}_k \Big|_{\mathbf{w}_k} \circ \mathcal{E}_k &= \sum_{k=1}^3 \mathfrak{K} \circ \gamma_k \Big|_{\mathbf{w}_k} \circ \Pi_k \circ \mathfrak{K}^{-1} \\ &= \mathfrak{K} \circ \left( \sum_{k=1}^3 \gamma_k \Big|_{\mathbf{w}_k} \circ \Pi_k \right) \circ \mathfrak{K}^{-1} \\ &= \mathfrak{K} \circ \mathfrak{K}^{-1}, \quad \text{by Part 3(b) of Problem 1} \\ &= \text{Identity.} \end{aligned}$$

It follows that,

$$\sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} = (\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij}, \quad \text{by } (\dagger).$$

Therefore, the matrix elements of  $[\mathcal{L}_\oplus \circ \mathcal{M}_\oplus]$  are given by

$$(\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij} = \sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj}$$

Since the full block-matrix expression for  $[\mathcal{L}_\oplus \circ \mathcal{M}_\oplus]$  is very similar to what we have found with  $\mathcal{L}_\times \circ \mathcal{M}_\times$ , we will not produce it here.

4. Let  $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$  be a resolution of identity, with  $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$ , be given. Furthermore, let the linear map  $\mathcal{L} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$  be given where

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

Claim:

$$\boxed{\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{L} \Big|_{\mathbf{W}_j} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i}$$

Consider applying  $\mathcal{L}$  to  $(w_1 \quad \mathbf{0}_{\mathbf{V}} \quad \mathbf{0}_{\mathbf{V}})_{\mathfrak{X}}^{\top} \in \mathbf{V}$ , we have

$$\mathcal{L} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \begin{pmatrix} \mathcal{L}_{11}(w_1) \\ \mathcal{L}_{21}(w_1) \\ \mathcal{L}_{31}(w_1) \end{pmatrix}_{\mathfrak{X}}.$$

To extract an element  $\mathcal{L}_{i1}$ , we can apply  $\mathcal{E}_i$  to this result:

$$\mathcal{E}_i \circ \mathcal{L} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \mathcal{E}_i \begin{pmatrix} \mathcal{L}_{11}(w_1) \\ \mathcal{L}_{21}(w_1) \\ \mathcal{L}_{31}(w_1) \end{pmatrix}_{\mathfrak{X}} = \mathcal{L}_{i1}(w_1).$$

This shows that by considering the restricted map  $\mathcal{L} \Big|_{\mathbf{W}_1}$ , we are able to

obtain the  $\mathcal{L}_{i1}$  element of  $\mathcal{L}$  by composing the idempotent  $\mathcal{E}_i$  with  $\mathcal{L} \Big|_{\mathbf{W}_1}$ .

So in general, if we consider the restricted map  $\mathcal{L} \Big|_{\mathbf{W}_j}$ , the same procedure will give us the element  $\mathcal{L}_{ij}$ . Hence, we have verified the claim that

$$\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{L} \Big|_{\mathbf{W}_j}.$$

5. Let

$$\mathcal{E}_1 = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}.$$

be given, where  $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$  and  $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$  is a resolution of identity. Continuing with the set-up of part 4, we have

$$\mathcal{M}_{ij} = \mathcal{E}_i \circ \mathcal{E}_1 \Big|_{\mathbf{W}_j}.$$

Since  $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$  is a resolution of identity, and  $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$  where  $\mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3 = \mathbf{V}$ ,  $\ker(\mathcal{E}_1) = \mathbf{W}_2 \oplus \mathbf{W}_3$ . This implies the restricted map  $\mathcal{E}_1 \Big|_{\mathbf{W}_j}$  gives  $\mathbf{0}_{\mathbf{V}}$  for  $j \neq 1$  and acts as an identity on  $\mathbf{W}_j$  if  $j = 1$ . Therefore, for  $j \neq 1$ ,  $M_{ij}$  is a zero function.

If  $j = 1$ , then

$$\mathcal{M}_{ij} = \mathcal{E}_i \circ \mathcal{E}_1 \Big|_{\mathbf{W}_1} = \mathcal{E}_i \Big|_{\mathbf{W}_1},$$

since  $\mathcal{E}_1$  acts as identity on  $\mathbf{W}_1$ . The same argument for the restricted map  $\mathcal{E}_i \Big|_{\mathbf{W}_1}$  shows that  $\mathcal{M}_{ij}$  is the zero function if  $i \neq 1$  and the identity on  $\mathbf{W}_1$  if  $i = 1$ . Therefore,

$$\text{For } \mathcal{E}_1 : \quad \mathcal{M}_{ij} = \begin{cases} Id_{\mathbf{W}_1}, & i, j = 1 \\ \mathcal{O}, & i \text{ or } j \neq 1 \end{cases}$$

The exact same procedure can be used to find the elements  $M_{ij}$  of  $\mathcal{E}_2$  and  $\mathcal{E}_3$ , simply replacing all 1's with 2's and 3's. So,

$$\begin{aligned} \text{For } \mathcal{E}_2 : \quad \mathcal{M}_{ij} &= \begin{cases} Id_{\mathbf{W}_2}, & i, j = 2 \\ \mathcal{O}, & i \text{ or } j \neq 2 \end{cases} \\ \text{For } \mathcal{E}_3 : \quad \mathcal{M}_{ij} &= \begin{cases} Id_{\mathbf{W}_3}, & i, j = 3 \\ \mathcal{O}, & i \text{ or } j \neq 3 \end{cases} \end{aligned}$$

**Problem. 3.**

1. Using the set-up of Definition 0.7, verify the identity

$$[\mathcal{T}(\mathcal{X})]_{\Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma} [\mathcal{X}]_{\Gamma}.$$

2. Use part 1 to prove that

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma} = \begin{bmatrix} [\mathcal{T}(v_1)]_{\Omega} & [\mathcal{T}(v_2)]_{\Omega} & \dots & [\mathcal{T}(v_n)]_{\Omega} \end{bmatrix}.$$

3. If  $\mathbf{U}$  is a finite-dimensional vector space with a coordinate system  $\Delta$ , and  $\mathcal{S} : \mathbf{Z} \xrightarrow{\text{linear}} \mathbf{U}$ , argue that

$$[\mathcal{S} \circ \mathcal{T}]_{\Delta \leftarrow \Gamma} = [\mathcal{S}]_{\Delta \leftarrow \Omega} [\mathcal{T}]_{\Omega \leftarrow \Gamma}.$$

4. Verify that in the case  $\mathcal{T}$  is invertible (in which case  $m = n$ ), so is  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ , and

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1}.$$

**Solution. 3.**

1. By the set-up in Definition 0.3, we have  $[\mathcal{T}]_{\Omega \leftarrow \Gamma} = [\ ]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma}$ , and  $[\mathcal{X}]_{\Gamma} = [\ ]_{\Gamma}(\mathcal{X})$ . Hence,

$$\begin{aligned} [\mathcal{T}]_{\Omega \leftarrow \Gamma} [\mathcal{X}]_{\Gamma} &= [\ ]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \circ [\mathcal{X}]_{\Gamma} \\ &= [\ ]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \circ [\ ]_{\Gamma}(\mathcal{X}) \\ &= [\ ]_{\Omega} \circ \mathcal{T} \circ (\mathcal{A}_{\Gamma} \circ [\ ]_{\Gamma})(\mathcal{X}) \\ &= [\ ]_{\Omega} \circ \mathcal{T}(\mathcal{X}) \\ &= [\ ]_{\Omega}(\mathcal{T}(\mathcal{X})) \\ &= [\mathcal{T}(\mathcal{X})]_{\Omega}. \end{aligned}$$

2. Claim:

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma} = \begin{bmatrix} [\mathcal{T}(v_1)]_{\Omega} & [\mathcal{T}(v_2)]_{\Omega} & \dots & [\mathcal{T}(v_n)]_{\Omega} \end{bmatrix}.$$

Let  $\mathcal{X}$  be  $v_i \in \{v_1, v_2, \dots, v_n\}$  a coordinate system of  $\Gamma$ . Then we can create a bijective matrix

$$A_{\Gamma} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

whose inverse is  $[\ ]_{\Gamma}$ . To find the columns of the matrix  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ , we want to apply  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$  to the standard basis vectors. To generate these standard basis vectors, we simply coordinatize  $\mathcal{X}$ :

$$[\mathcal{X}]_{\Gamma} = [v_i]_{\Gamma} = [\ ]_{\Gamma}(v_i) = A_{\Gamma}^{-1}(v_i) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^{-1}(v_i) = \vec{e}_i \in \mathbb{C}^n.$$

Thus, the  $i^{th}$  column of  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$  is given by

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma}(\vec{e}_i) = [\mathcal{T}]_{\Omega \leftarrow \Gamma}[v_i]_{\Gamma} = [\mathcal{T}(v_i)]_{\Gamma}, \text{ by Part 1.}$$

This verifies the claim.

3. Let the finite-dimensional vector space  $\mathbf{U}$  with coordinate system  $\Delta$  be given. Let  $\mathcal{S} : \mathbf{Z} \xrightarrow{\text{linear}} \mathbf{U}$  be given. Claim:

$$[\mathcal{S} \circ \mathcal{T}]_{\Delta \leftarrow \Gamma} = [\mathcal{S}]_{\Delta \leftarrow \Omega} [\mathcal{T}]_{\Omega \leftarrow \Gamma}.$$

Let  $\Delta = (w_1, w_2, \dots, w_l)$  be a coordinate system for  $\mathbf{U}$ . We also define the bijective matrix

$$A_{\Delta} := \begin{bmatrix} w_1 & w_2 & \dots & w_l \end{bmatrix} : \mathbb{C}^l \rightarrow \mathbf{U}.$$

Since  $A_{\Delta}$  is invertible, we can write  $[\ ]_{\Delta}$  for  $A_{\Delta}^{-1}$ .

By the set-up in definition 0.7, we have the following

$$\begin{aligned} [\mathcal{T}]_{\Omega \leftarrow \Gamma} &= [\ ]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ [\mathcal{S}]_{\Delta \leftarrow \Omega} &= [\ ]_{\Delta} \circ \mathcal{S} \circ \mathcal{A}_{\Omega}. \end{aligned}$$

It follows that

$$\begin{aligned} [\mathcal{S}]_{\Delta \leftarrow \Omega} [\mathcal{T}]_{\Omega \leftarrow \Gamma} &= [\ ]_{\Delta} \circ \mathcal{S} \circ \mathcal{A}_{\Omega} \circ [\ ]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ &= [\ ]_{\Delta} \circ \mathcal{S} \circ (\mathcal{A}_{\Omega} \circ [\ ]_{\Omega}) \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ &= [\ ]_{\Delta} \circ \mathcal{S} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ &= [\ ]_{\Delta} \circ (\mathcal{S} \circ \mathcal{T}) \circ \mathcal{A}_{\Gamma} \\ &= [\mathcal{S} \circ \mathcal{T}]_{\Delta \leftarrow \Gamma}, \end{aligned}$$

where we have used the fact that  $A_{\Omega}[\ ]_{\Omega} = A_{\Omega}A_{\Omega}^{-1}$  is the identity function. This completes the argument.

4. (a) To show:  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$  is invertible when  $\mathcal{T}$  is invertible.

The matrix  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$  can be expressed as

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma} = [\ ]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma}.$$

Since  $T$ ,  $[\ ]_{\Omega}$ , and  $\mathcal{A}_{\Gamma}$  are invertible, the above composition is also invertible, i.e.,  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$  is invertible.

(b) Let  $\mathcal{T}$  be invertible (in which case  $m = n$ ). Claim:

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1}.$$

Since  $\mathcal{T}$ ,  $\mathcal{A}_\Omega$ ,  $\mathcal{A}_\Gamma$  are all invertible, their inverses are defined. By the set-up in definition 0.7, the matrix  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$  can be written as the composition  $[\ ]_\Omega \circ \mathcal{T} \circ \mathcal{A}_\Gamma$  for the linear function  $T : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{Z}$ . Since  $\mathcal{T}^{-1} : \mathbf{Z} \xrightarrow{\text{linear}} \mathbf{V}$  is defined, we can express its matrix  $[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega}$  as

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\ ]_\Gamma \circ \mathcal{T}^{-1} \circ \mathcal{A}_\Omega. \quad (\dagger)$$

Consider the inverse of the matrix  $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ :

$$\begin{aligned} [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1} &= ([\ ]_\Omega \circ \mathcal{T} \circ \mathcal{A}_\Gamma)^{-1} \\ &= \mathcal{A}_\Gamma^{-1} \circ ([\ ]_\Gamma \circ \mathcal{T})^{-1} \\ &= \mathcal{A}_\Gamma^{-1} \circ \mathcal{T}^{-1} \circ [\ ]_\Omega^{-1} \\ &= [\ ]_\Gamma \circ \mathcal{T}^{-1} \circ \mathcal{A}_\Omega. \quad (\dagger\dagger) \end{aligned}$$

From  $(\dagger)$  and  $(\dagger\dagger)$ , we have

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1}$$

as claimed.



**Problem. 4.** Let us write  $\Gamma_0$  for the standard coordinate system  $(1, x, x^2, x^3)$  of the vector space  $\mathbb{P}_3$  of all polynomials of degree at most 3.

1. Use *Mathematica* and the fact that  $\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{\Gamma_0}$  is an isomorphism (and isomorphisms map bases to bases) to verify that

$$\Delta := (1 + x + x^2 + x^3, 1 + 2x + 4x^2 + 8x^3, \\ 1 + 3x + 9x^2 + 27x^3, 1 + 4x + 16x^2 + 64x^3)$$

and

$$\Omega := (x + x^2, x - 6x^3, 1 + 4x^2, 1 + 8x^3)$$

are also coordinate systems of  $\mathbb{P}_3$ .

2. Verify that general identity

$$\begin{aligned} [\mathcal{T}]_{\Delta \leftarrow \Omega} &= [\mathcal{I}_{\mathbb{P}_3}]_{\Delta \leftarrow \Gamma_0} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\ &= \left( [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta} \right)^{-1} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega}, \end{aligned}$$

for  $\mathcal{T} : \mathbb{P}_3 \xrightarrow{\text{linear}} \mathbb{P}_3$ .

3. Consider  $\mathcal{T} : \mathbb{P}_3 \rightarrow \mathbb{P}_3$  defined by

$$\mathcal{T}(p(x)) := xp'(2x) + p(x).$$

Prove that  $\mathcal{T}$  is a linear function and use part 2 to find  $[\mathcal{T}]_{\Delta \leftarrow \Omega}$ .

**Solution. 4.**

1. (a) Let

$$\Delta := (1 + x + x^2 + x^3, 1 + 2x + 4x^2 + 8x^3, \\ 1 + 3x + 9x^2 + 27x^3, 1 + 4x + 16x^2 + 64x^3)$$

be given. To show that  $\Delta$  is a coordinate system of  $\mathbf{V}$ , it suffices to show there exists an isomorphism  $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta} : \mathbb{C}_{\Delta_0}^4 \rightarrow \mathbb{C}_{\Gamma_0}^4$ .

We can construct  $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta}$  column-by-column by letting  $\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}_{\Gamma_0}$  act on the elements of  $\Delta$ , by Part 2 of Problem 3:

$$[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta} = \begin{bmatrix} [(1 + x + x^2 + x^3)]_{\Gamma_0} \\ [(1 + 2x + 4x^2 + 8x^3)]_{\Gamma_0} \\ [(1 + 3x + 9x^2 + 27x^3)]_{\Gamma_0} \\ [(1 + 4x + 16x^2 + 64x^3)]_{\Gamma_0} \end{bmatrix}^{\top} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}.$$

To show  $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta}$  is an isomorphism, it suffices to check if its determinant is nonzero, which can be done in *Mathematica*:

$$\det([\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta}) = 12 \neq 0.$$

Therefore,  $\Delta$  is a coordinate system of  $\mathbb{P}_3$ .

*Mathematica* code:

```
In[4]:= Det[{{1, 1, 1, 1}, {1, 2, 3, 4}, {1, 4, 9, 16},
             {1, 8, 27, 64}}]
Out[4]= 12
```

- (b) For  $\Omega := (x + x^2, x - 6x^3, 1 + 4x^2, 1 + 8x^3)$ , we repeat the procedure laid out above to determine whether  $\Omega$  is a coordinate system for  $\mathbb{P}_3$ .

$$[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Omega} = \begin{bmatrix} [(x + x^2)]_{\Gamma_0} \\ [(x - 6x^3)]_{\Gamma_0} \\ [(1 + 4x^2)]_{\Gamma_0} \\ [(1 + 8x^3)]_{\Gamma_0} \end{bmatrix}^\top = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & -6 & 0 & 8 \end{pmatrix}.$$

We can calculate the determinant of  $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Omega}$  in *Mathematica*:

$$\det([\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Omega}) = -32 \neq 0.$$

Therefore,  $\Omega$  is a coordinate system of  $\mathbb{P}_3$ .

*Mathematica* code:

```
In[6]:= Det[{{0, 0, 1, 1}, {1, 1, 0, 0},
             {1, 0, 4, 0}, {0, -6, 0, 8}}]
Out[6]= -32
```

2. Let  $\mathcal{T} : \mathbb{P}_3 \xrightarrow{\text{linear}} \mathbb{P}_3$  be given.

- (a) First identity:

$$\begin{aligned} & [\mathcal{I}_{\mathbb{P}_3}]_{\Delta \leftarrow \Gamma_0} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\ &= ([\ ]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Gamma_0}) \circ ([\ ]_{\Gamma_0} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma_0}) \circ ([\ ]_{\Gamma_0} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Omega}) \\ &= [\ ]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{T} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Omega} \\ &= [\ ]_{\Delta} \circ \mathcal{T} \circ \mathcal{A}_{\Omega} \\ &= [\mathcal{T}]_{\Delta \leftarrow \Omega}. \end{aligned}$$

- (b) The second identity follows from the fact that

$$\begin{aligned} ([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta})^{-1} &= ([\ ]_{\Gamma_0} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Delta})^{-1} \\ &= [\ ]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3}^{-1} \circ \mathcal{A}_{\Gamma_0} \\ &= [\ ]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Gamma_0}. \end{aligned}$$

So we have

$$\begin{aligned}
[\mathcal{T}]_{\Delta \leftarrow \Omega} &= [\mathcal{I}_{\mathbb{P}_3}]_{\Delta \leftarrow \Gamma_0} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\
&= ([\ ]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Gamma_0}) \circ [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\
&= ([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta})^{-1} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega}
\end{aligned}$$

3. Let  $\mathcal{T} : \mathbb{P}_3 \rightarrow \mathbb{P}_3$  be given.  $T$  is defined as

$$\mathcal{T}(p(x)) := xp'(2x) + p(x).$$

(a) Claim:  $\mathcal{T}$  is a linear function.

$\mathcal{T}$  is a linear function if it satisfies the linearity conditions. Consider  $p(x), q(x) \in \mathbb{P}_3$ . Then we have

$$\begin{aligned}
\mathcal{T}(p(x) + q(x)) &= x(p(2x) + q(2x))' + (p(x) + q(x)) \\
&= xp'(2x) + xq'(2x) + p(x) + q(x) \\
&= (xp'(2x) + p(x)) + (xq'(2x) + q(x)) \\
&= \mathcal{T}(p(x)) + \mathcal{T}(q(x)). \quad (\dagger)
\end{aligned}$$

Consider  $c \in \mathbb{C}$  and  $p(x) \in \mathbb{P}_3$ ,

$$\begin{aligned}
\mathcal{T}(cp(x)) &= x(cp(2x))' + cp(x) \\
&= c xp'(2x) + cp(x) \\
&= c(xp'(2x) + p(x)) \\
&= c\mathcal{T}(p(x)). \quad (\dagger\dagger)
\end{aligned}$$

From  $(\dagger)$  and  $(\dagger\dagger)$ ,  $\mathcal{T}$  is a linear function.

(b) Find  $[\mathcal{T}]_{\Delta \leftarrow \Omega}$ .

By part 2 of Problem 3

$$[\mathcal{T}]_{\Delta \leftarrow \Omega} = \left([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta}\right)^{-1} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega}.$$

In part 1, we have found the isomorphisms:

$$\begin{aligned}
\left([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta}\right)^{-1} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -13/3 & 3/2 & -1/6 \\ -6 & 19/2 & -4 & 1/2 \\ 4 & -7 & 7/2 & -1/2 \\ -1 & 11/6 & -1 & 1/6 \end{pmatrix} \\
[\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} &= \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & -6 & 0 & 8 \end{pmatrix}.
\end{aligned}$$

To find  $[\mathcal{T}]_{\Delta \leftarrow \Omega}$  we only need to find  $[\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0}$ . To do this, we use part 2 of Problem 3, which says,

$$[\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} = \begin{bmatrix} [\mathcal{T}(1)]_{\Gamma_0} & [\mathcal{T}(x)]_{\Gamma_0} & [\mathcal{T}(x^2)]_{\Gamma_0} & [\mathcal{T}(x^3)]_{\Gamma_0} \end{bmatrix},$$

where

$$\mathcal{T}(1) = x \frac{d}{dx} 2 + 1 = 1$$

$$\mathcal{T}(x) = x \frac{d}{dx} (2x) + x = 3x$$

$$\mathcal{T}(x^2) = x \frac{d}{dx} (2x)^2 + x^2 = 9x^2$$

$$\mathcal{T}(x^3) = x \frac{d}{dx} (2x)^3 + x^3 = 25x^3.$$

Therefore,

$$\begin{aligned} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} &= \begin{bmatrix} [T(1)]_{\Gamma_0} & [T(x)]_{\Gamma_0} & [T(x^2)]_{\Gamma_0} & [T(x^3)]_{\Gamma_0} \end{bmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}. \end{aligned}$$

Finally, we can compute  $[\mathcal{T}]_{\Delta \leftarrow \Omega}$ :

$$\begin{aligned} [\mathcal{T}]_{\Delta \leftarrow \Omega} &= \begin{pmatrix} 4 & -13/3 & 3/2 & -1/6 \\ -6 & 19/2 & -4 & 1/2 \\ 4 & -7 & 7/2 & -1/2 \\ -1 & 11/6 & -1 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & -6 & 0 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 12 & 58 & -88/3 \\ -15/2 & -93/2 & -150 & 94 \\ 21/2 & 54 & 130 & -96 \\ -7/2 & -39/2 & -37 & 97/3 \end{pmatrix}. \end{aligned}$$

*Mathematica code:*

```
A =
Inverse[{{1, 1, 1, 1}, {1, 2, 3, 4}, {1, 4, 9, 16}, {1, 8, 27, 64}}]

{{4, -(13/3), 3/2, -(1/6)}, {-6, 19/2, -4, 1/2}, {4, -7, 7/
2, -(1/2)}, {-1, 11/6, -1, 1/6}}

B = {{1, 0, 0, 0}, {0, 3, 0, 0}, {0, 0, 9, 0}, {0, 0, 0, 25}}

CC = {{0, 0, 1, 1}, {1, 1, 0, 0}, {1, 0, 4, 0}, {0, -6, 0, 8}}

In[7]:= A.B.CC

Out[7]= {{1/2, 12, 58, -(88/3)}, {-(15/2), -(93/2), -150, 94},
{21/2, 54, 130, -96}, {-(7/2), -(39/2), -37, 97/3}}
```

**Problem. 5.** Suppose that  $\mathbf{V} = \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3$  and  $\Gamma_i$  is a coordinate system of  $\mathbf{W}_i$ . Suppose that for each  $i, j \in \{1, 2, 3\}$ ,

$$\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i.$$

Let

$$\mathcal{L} := \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

Let  $\Delta$  be the concatenation  $\Gamma_1 || \Gamma_2 || \Gamma_3$  of the coordinate systems  $\Gamma_i$ . As we know,  $\Delta$  is a coordinate system of  $\mathbf{V}$ .

Prove that  $[\mathcal{L}]_{\Delta \leftarrow \Delta}$  equals the partitioned matrix

$$\begin{bmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} & [\mathcal{L}_{12}]_{\Gamma_1 \leftarrow \Gamma_2} & [\mathcal{L}_{13}]_{\Gamma_1 \leftarrow \Gamma_3} \\ [\mathcal{L}_{21}]_{\Gamma_2 \leftarrow \Gamma_1} & [\mathcal{L}_{22}]_{\Gamma_2 \leftarrow \Gamma_2} & [\mathcal{L}_{23}]_{\Gamma_2 \leftarrow \Gamma_3} \\ [\mathcal{L}_{31}]_{\Gamma_3 \leftarrow \Gamma_1} & [\mathcal{L}_{32}]_{\Gamma_3 \leftarrow \Gamma_2} & [\mathcal{L}_{33}]_{\Gamma_3 \leftarrow \Gamma_3} \end{bmatrix}.$$

**Solution. 5.** Consider a  $3 \times 3$  block-matrix, which consists of just the linear function  $\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$  at the  $i^{\text{th}}$  row and  $j^{\text{th}}$  column and the zero function everywhere else. Let  $[\mathcal{L}_{ij}]_{\oplus} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$  denote this matrix. By this construction, we have

$$\mathcal{L} = \sum_{i=1}^3 \sum_{j=1}^3 [\mathcal{L}_{ij}]_{\oplus} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

It follows that the matrix  $[\mathcal{L}]_{\Delta \leftarrow \Delta}$  of  $\mathcal{L}$  can be written as

$$\begin{aligned} [\mathcal{L}]_{\Delta \leftarrow \Delta} &= \left[ \sum_{i=1}^3 \sum_{j=1}^3 [\mathcal{L}_{ij}]_{\oplus} \right]_{\Delta \leftarrow \Delta} \\ &= \sum_{i=1}^3 \sum_{j=1}^3 [[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta} \end{aligned}$$

Consider the coordinate system  $\Gamma_k$  where  $k \in \{1, 2, 3\}$ . Let  $\Gamma_k$  be formed by a (finite) collection of basis elements, called  $g_{lk}$ 's. Consider the mapping  $[\mathcal{L}_{ij}]_{\oplus}(g_{lk})_{\oplus}$  and the matrix  $[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta}$ .

We observe that  $[\mathcal{L}_{ij}]_{\oplus}(g_{lk})_{\oplus}$  is  $\mathbf{0}_{\mathbf{V}}$  if  $k \neq j$  and  $\mathbf{\nabla}([\mathcal{L}_{ij}]_{\oplus}(g_{lj}) + \mathbf{0}_{\mathbf{V}} + \mathbf{0}_{\mathbf{V}})$  if  $j = k$ , where  $\mathbf{\nabla}$  is defined in Definition 0.4. What this says is that  $[\mathcal{L}_{ij}]_{\oplus}(g_{lk})_{\oplus}$  gives a column vector with  $\mathcal{L}_{ij}(g_{lj})$  on the  $i^{\text{th}}$  row and  $\mathbf{0}_{\mathbf{V}}$  everywhere else. This means that if we consider the coordinatization  $[[\mathcal{L}_{ij}]_{\oplus}(g_{lj})_{\oplus}]_{\Delta}$ , where  $\Delta = \Gamma_1 || \Gamma_2 || \Gamma_3$ , we will end up with a column vector with the zero function everywhere except  $[[\mathcal{L}_{ij}]_{\oplus}(g_{lj})]_{\Gamma_i}$  on the  $i^{\text{th}}$  row. ( $\dagger$ )

Now, we can construct  $[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta}$  column-by-column as

$$[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta} = \begin{bmatrix} [[[\mathcal{L}_{ij}]_{\oplus}(g_{11})]_{\oplus}]_{\Delta}^{\top} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{12})]_{\oplus}]_{\Delta}^{\top} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{13})]_{\oplus}]_{\Delta}^{\top} \\ \vdots & \vdots & \vdots \\ [[[\mathcal{L}_{ij}]_{\oplus}(g_{a1})]_{\oplus}]_{\Delta} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{b2})]_{\oplus}]_{\Delta} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{c3})]_{\oplus}]_{\Delta} \end{bmatrix}^{\top}$$

where  $a, b, c$  are the number of elements in  $\Gamma_1, \Gamma_2, \Gamma_3$ , respectively. From  $(\dagger)$ , we know that two of three matrix-blocks in the expression above will be the zero-block because  $j$  has a fixed value of either 1, 2, or 3. Consider the only non-zero block (located at the  $j^{th}$  column)

$$\begin{bmatrix} [[[\mathcal{L}_{ij}]_{\oplus}(g_{1j})]_{\oplus}]_{\Delta} \\ \vdots \\ [[[\mathcal{L}_{ij}]_{\oplus}(g_{aj})]_{\oplus}]_{\Delta} \end{bmatrix}^{\top} = [[[\mathcal{L}_{ij}]_{\oplus}(g_{1j})]_{\oplus}]_{\Delta} \quad \dots \quad [[[\mathcal{L}_{ij}]_{\oplus}(g_{aj})]_{\oplus}]_{\Delta}. \quad (\dagger\dagger)$$

Recalling that  $[[[\mathcal{L}_{ij}]_{\oplus}(g_{lj})]_{\oplus}]_{\Delta}$  is a column vector with  $\mathcal{L}_{ij}(g_{lj})$  on the  $i^{th}$  row and  $\mathbf{0}_{\mathbf{V}}$  everywhere else, it follows that we can represent this matrix block as one with two rows of zeros and an  $i^{th}$  row of

$$[\mathcal{L}_{ij}(g_{1j}) \quad \mathcal{L}_{ij}(g_{1j}) \quad \dots \quad \mathcal{L}_{ij}(g_{aj})],$$

which precisely expresses the matrix  $[\mathcal{L}_{ij}]_{\Gamma_i \leftarrow \Gamma_j}$ . Therefore, we can treat the expression  $(\dagger\dagger)$  as a 3-element column of the matrix  $[[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta}$  whose element on the  $i^{th}$  row and  $j^{th}$  column is  $[\mathcal{L}_{ij}]_{\Gamma_i \leftarrow \Gamma_j}$ , and the zero function everywhere else. Hence,

$$\begin{aligned} [\mathcal{L}]_{\Delta \leftarrow \Delta} &= \sum_{i=1}^3 \sum_{j=1}^3 [[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta} \\ &= \begin{bmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} & [\mathcal{L}_{21}]_{\Gamma_2 \leftarrow \Gamma_1} & [\mathcal{L}_{31}]_{\Gamma_3 \leftarrow \Gamma_1} \\ [\mathcal{L}_{12}]_{\Gamma_1 \leftarrow \Gamma_2} & [\mathcal{L}_{22}]_{\Gamma_2 \leftarrow \Gamma_2} & [\mathcal{L}_{32}]_{\Gamma_3 \leftarrow \Gamma_2} \\ [\mathcal{L}_{13}]_{\Gamma_1 \leftarrow \Gamma_3} & [\mathcal{L}_{23}]_{\Gamma_2 \leftarrow \Gamma_3} & [\mathcal{L}_{33}]_{\Gamma_3 \leftarrow \Gamma_3} \end{bmatrix}. \end{aligned}$$

This completes the argument.

An example can help justify this argument. Consider  $[\mathcal{L}_{11}]_{\oplus}$ ,

$$[\mathcal{L}_{11}] = \begin{bmatrix} \mathcal{L}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\oplus},$$

where we are using the 0's to denote the zero functions  $\mathbf{W}_i \rightarrow \mathbf{W}_j$ . Then consider  $g_{11}$  the first vector in the coordinate system  $\Gamma_1$ . We have

$$\begin{bmatrix} \mathcal{L}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\oplus} \begin{pmatrix} g_{11} \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \begin{pmatrix} \mathcal{L}_{11}(g_{11}) \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}}.$$

Now consider  $[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta}$ . This matrix can be constructed column-by-column with each column being  $[\mathcal{L}_{11}(g_{ij})]_{\Delta \leftarrow \Delta}$ , where  $j$  indicates from which  $\Gamma_j$   $g_{ij}$  is taken. We observe that if  $j$  is not 1, then  $\mathcal{L}_{11}(g_{ij}) = \mathbf{0}_{\mathbf{V}}$ . This means the all columns of  $[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta}$  are the zero columns unless  $j = 1$ . Consider the  $j = 1$  columns. We know that the coordinatization

$$\left[ \begin{pmatrix} \mathcal{L}_{11}(g_{11}) \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{K}} \right]_{\Delta \leftarrow \Delta} = \begin{pmatrix} [\mathcal{L}_{11}(g_{11})]_{\Gamma_1} \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix},$$

because the  $\mathcal{L}_{11}(g_{11}) \in \mathbf{W}_1$ . So it follows that the concatenation of the columns

$$\begin{pmatrix} [\mathcal{L}_{11}(g_{11})]_{\Gamma_1} & \cdots & \mathcal{L}_{11}(g_{a1})_{\Gamma_1} \\ \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}$$

in the matrix  $[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta}$  is nothing but

$$\begin{pmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} \\ \mathcal{O} \\ \mathcal{O} \end{pmatrix}.$$

So we have

$$[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta} = \begin{bmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix},$$

where  $\mathcal{O}$  denotes the zero function.

We can equivalently generate other examples for  $[\mathcal{L}_{ij}]_{\oplus}$ ,  $i, j \in \{1, 2, 3\}$ , and in the end add up the results (with different combinations of  $i, j$ 's of course) to get  $[\mathcal{L}]_{\Delta \leftarrow \Delta}$ , expressed in the form given in the problem statement as desired.

## 25.4 Problem set 4

### Problem. 1.

Suppose that  $T \in \mathcal{L}(\mathbf{V})$  and

$$\text{Im}(T) = \mathbf{W} + \mathbf{Z},$$

where  $\mathbf{W}$  and  $\mathbf{Z}$  are subspaces of  $\mathbf{V}$ . Argue that

$$\mathbf{V} = T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}].$$

### Solution.

Since  $\text{Im}(T) = \mathbf{W} + \mathbf{Z}$ ,  $T^{-1}[\mathbf{W} + \mathbf{Z}] = T^{-1}[\text{Im}(T)] = \mathbf{V}$ . Also,

$$\begin{aligned} T^{-1}[\mathbf{W}] &= \{w \in \mathbf{V} \mid T(w) \in \mathbf{W}\} \subseteq \mathbf{V} \\ T^{-1}[\mathbf{Z}] &= \{z \in \mathbf{V} \mid T(z) \in \mathbf{Z}\} \subseteq \mathbf{V}, \end{aligned}$$

which means

$$T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}] = \{w + z \in \mathbf{V} \mid T(w) \in \mathbf{W}, T(z) \in \mathbf{Z}, w, z \in \mathbf{V}\}.$$

It follows directly from this definition that

$$T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}] \subseteq \mathbf{V} = T^{-1}[\mathbf{W} + \mathbf{Z}] \quad (\dagger).$$

Consider  $v \in T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}]$ .  $v$  can be written as  $w + z$  where  $T(w) \in \mathbf{W}$  and  $T(z) \in \mathbf{Z}$ , which implies

$$T(v) = T(w + z) = T(w) + T(z) \in \mathbf{W} + \mathbf{Z},$$

where the second equality follows from the linearity of  $T$ . Hence,

$$v \in T^{-1}[\mathbf{W} + \mathbf{Z}]. \quad (\dagger\dagger)$$

From  $(\dagger)$  and  $(\dagger\dagger)$ ,  $\mathbf{V} = T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}]$  as desired.



**Problem. 2.**

1. Suppose that  $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$  are non-trivial vector spaces, and for each  $i, j \in \{1, 2, 3\}$ ,

$$\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i.$$

Let  $\mathcal{L}$  be the block-matrix function:

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \xrightarrow{\text{linear}} \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3.$$

Suppose that it turns out that  $\mathbf{V}_2 = \mathbf{V}_{2.1} \times \mathbf{V}_{2.2}$ . Then  $\mathcal{L}$  maybe considered as a linear function on

$$\mathbf{V}_1 \times \mathbf{V}_{2.1} \times \mathbf{V}_{2.2} \times \mathbf{V}_3.$$

What is the corresponding block-matrix form of  $\mathcal{L}$  and how does it relate

to  $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$  ? Justify your claims.

2. Suppose that  $\mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3 = \mathbf{V}$  and the  $\mathbf{W}_i$ 's are non-trivial subspaces of  $\mathbf{V}$ . Suppose that for each  $i, j \in \{1, 2, 3\}$ :

$$\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i.$$

Let  $\mathcal{L}$  be the block-matrix function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}.$$

Suppose it turns out that  $\mathbf{W}_1 = \mathbf{W}_{1.1} \oplus \mathbf{W}_{1.2}$ . Then

$$\mathbf{V} = \mathbf{W}_{1.1} \oplus \mathbf{W}_{1.2} \oplus \mathbf{W}_2 \oplus \mathbf{W}_3.$$

What is the block-matrix form of  $\mathcal{L}$  with respect to this direct sum de-

composition and how does it relate to  $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}$  ? Justify your claims.

**Solution.**

1. Recall from the previous problem set that the elements of the block matrix function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \xrightarrow{\text{linear}} \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3.$$

can be expressed as

$$\mathcal{L}_{ij} = \Pi_i \circ \mathcal{L} \circ \gamma_j,$$

where the  $\Pi_i$ 's and  $\gamma_j$ 's are coordinate projections from  $\mathbf{V}$  onto  $\mathbf{V}_i$  and injections from  $\mathbf{V}_j$  into  $\mathbf{V}$ . Let  $\mathbf{V}_2 = \mathbf{V}_{2.1} \times \mathbf{V}_{2.2}$ . We claim that the matrix-block form of  $\mathcal{L}$  in this case is

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{1|2.1} & \mathcal{L}_{1|2.2} & \mathcal{L}_{13} \\ \mathcal{L}_{2.1|1} & \mathcal{L}_{2.1|2.1} & \mathcal{L}_{2.1|2.2} & \mathcal{L}_{2.1|3} \\ \mathcal{L}_{2.2|1} & \mathcal{L}_{2.2|2.1} & \mathcal{L}_{2.2|2.2} & \mathcal{L}_{2.2|3} \\ \mathcal{L}_{31} & \mathcal{L}_{3|2.1} & \mathcal{L}_{3|2.2} & \mathcal{L}_{33} \end{bmatrix}_{\times}.$$

In particular, we claim the following relations between the matrix elements  $\mathcal{L}_{ij}$  in the former expression and the block-elements in the latter expression:

$$\mathcal{L}_{12} = [\mathcal{L}_{1|2.1} \quad \mathcal{L}_{1|2.2}], \mathcal{L}_{32} = [\mathcal{L}_{3|2.1} \quad \mathcal{L}_{3|2.2}],$$

$$\mathcal{L}_{21} = \begin{bmatrix} \mathcal{L}_{2.1|1} \\ \mathcal{L}_{2.2|1} \end{bmatrix}, \mathcal{L}_{23} = \begin{bmatrix} \mathcal{L}_{2.1|3} \\ \mathcal{L}_{2.2|3} \end{bmatrix}, \mathcal{L}_{22} = \begin{bmatrix} \mathcal{L}_{2.1|2.1} & \mathcal{L}_{2.1|2.2} \\ \mathcal{L}_{2.2|2.1} & \mathcal{L}_{2.2|2.2} \end{bmatrix}.$$

We can mimic what we have done to define  $\mathcal{L}_{ij}$  to define these new block-matrix elements. Let  $\delta_{n.m}$  be defined as the coordinate injection from  $\mathbf{V}_{n.m}$  into  $\mathbf{V}_2$ ,  $n, m \in \{1, 2\}$ . Also, define  $\rho_{n.m}$  as the coordinate projection from  $\mathbf{V}$  onto  $\mathbf{V}_{n.m}$ ,  $n, m \in \{1, 2\}$ . Consider the linear function  $\mathcal{L}_{22} : \mathbf{V}_2 \rightarrow \mathbf{V}_2$ , which we claim to be

$$\mathcal{L}_{22} = \begin{bmatrix} \mathcal{L}_{2.1|2.1} & \mathcal{L}_{2.1|2.2} \\ \mathcal{L}_{2.2|2.1} & \mathcal{L}_{2.2|2.2} \end{bmatrix}.$$

By exactly the same way we have defined the  $\mathcal{L}_{ij}$ 's of  $\mathcal{L}$  using the coordinate projections and injections, we have

$$\begin{aligned} \mathcal{L}_{2.1|2.1} &= \rho_{2.1} \circ \mathcal{L}_{22} \circ \delta_{2.1} \\ \mathcal{L}_{2.2|2.1} &= \rho_{2.2} \circ \mathcal{L}_{22} \circ \delta_{2.1} \\ \mathcal{L}_{2.1|2.2} &= \rho_{2.1} \circ \mathcal{L}_{22} \circ \delta_{2.2} \\ \mathcal{L}_{2.2|2.2} &= \rho_{2.2} \circ \mathcal{L}_{22} \circ \delta_{2.2} \end{aligned}$$

In terms of the linear function  $\mathcal{L}$ ,

$$\begin{aligned}\mathcal{L}_{2.1|2.1} &= \rho_{2.1} \circ \Pi_2 \circ \mathcal{L} \circ \gamma_2 \circ \delta_{2.1} \\ \mathcal{L}_{2.2|2.1} &= \rho_{2.2} \circ \Pi_2 \circ \mathcal{L} \circ \gamma_2 \circ \delta_{2.1} \\ \mathcal{L}_{2.1|2.2} &= \rho_{2.1} \circ \Pi_2 \circ \mathcal{L} \circ \gamma_2 \circ \delta_{2.2} \\ \mathcal{L}_{2.2|2.2} &= \rho_{2.2} \circ \Pi_2 \circ \mathcal{L} \circ \gamma_2 \circ \delta_{2.2}.\end{aligned}$$

Next, consider  $\mathcal{L}_{i2} = \Pi_i \circ \mathcal{L} \circ \gamma_2$ . We define

$$\begin{aligned}\mathcal{L}_{i|2.1} &= \Pi_i \circ \mathcal{L} \circ \gamma_2 \circ \delta_{2.1} \\ \mathcal{L}_{i|2.2} &= \Pi_i \circ \mathcal{L} \circ \gamma_2 \circ \delta_{2.2},\end{aligned}$$

for  $i \in \{1, 3\}$ . Finally, consider  $\mathcal{L}_{2j} = \Pi_2 \circ \mathcal{L} \circ \gamma_j$ . We define

$$\begin{aligned}\mathcal{L}_{2.1|j} &= \rho_{2.1} \circ \Pi_2 \circ \mathcal{L} \circ \gamma_j \\ \mathcal{L}_{2.2|j} &= \rho_{2.2} \circ \Pi_2 \circ \mathcal{L} \circ \gamma_j,\end{aligned}$$

for  $j \in \{1, 3\}$ .

We can now verify that  $\mathcal{L}$  can be represented as the 4-block-by-4-block as above. Consider

$$v = \begin{pmatrix} a \\ \begin{pmatrix} b \\ c \end{pmatrix} \\ d \end{pmatrix} \in \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$$

with  $\mathbf{V}_2 = \mathbf{V}_{2.1} \times \mathbf{V}_{2.2}$ .  $v$  can be written as

$$v = \begin{pmatrix} a \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \begin{pmatrix} b \\ \mathbf{0} \end{pmatrix} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \begin{pmatrix} \mathbf{0} \\ c \end{pmatrix} \\ \mathbf{0} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ \begin{pmatrix} \mathbf{0} \\ d \end{pmatrix} \end{pmatrix}.$$

We can check that  $\mathcal{L}(v)$  is the same as applying the 4-block-by-4-block matrix expression to  $v$  by going through every term in the above expression of  $v$ . For example, we can consider  $\begin{pmatrix} a & (\mathbf{0} \ \mathbf{0})^\top & \mathbf{0} \end{pmatrix}^\top$ . Then

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{1|2.1} & \mathcal{L}_{1|2.2} & \mathcal{L}_{13} \\ \mathcal{L}_{2.1|1} & \mathcal{L}_{2.1|2.1} & \mathcal{L}_{2.1|2.2} & \mathcal{L}_{2.1|3} \\ \mathcal{L}_{2.2|1} & \mathcal{L}_{2.2|2.1} & \mathcal{L}_{2.2|2.2} & \mathcal{L}_{2.2|3} \\ \mathcal{L}_{31} & \mathcal{L}_{3|2.1} & \mathcal{L}_{3|2.2} & \mathcal{L}_{33} \end{bmatrix}_\times \begin{pmatrix} a \\ \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11}(a) \\ \begin{pmatrix} \mathcal{L}_{2.1|1}(a) \\ \mathcal{L}_{2.2|1}(a) \end{pmatrix} \\ \mathcal{L}_{31}(a) \end{pmatrix}.$$

We can look at

$$\begin{aligned}\begin{pmatrix} \mathcal{L}_{2.1|1}(a) \\ \mathcal{L}_{2.2|1}(a) \end{pmatrix} &= \begin{pmatrix} \rho_{2.1} \circ \mathcal{L} \circ \gamma_1(a) \\ \rho_{2.2} \circ \mathcal{L} \circ \gamma_1(a) \end{pmatrix} \\ &= \begin{pmatrix} \rho_{2.1} \circ \Pi_2 \circ \mathcal{L}_{21}(a) \\ \rho_{2.2} \circ \Pi_2 \circ \mathcal{L}_{21}(a) \end{pmatrix} \\ &= \mathcal{L}_{21}(a), \quad \text{by the construction of } \rho_{2.1}, \rho_{2.2}.\end{aligned}$$

Hence

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{1|2.1} & \mathcal{L}_{1|2.2} & \mathcal{L}_{13} \\ \mathcal{L}_{2.1|1} & \mathcal{L}_{2.1|2.1} & \mathcal{L}_{2.1|2.2} & \mathcal{L}_{2.1|3} \\ \mathcal{L}_{2.2|1} & \mathcal{L}_{2.2|2.1} & \mathcal{L}_{2.2|2.2} & \mathcal{L}_{2.2|3} \\ \mathcal{L}_{31} & \mathcal{L}_{3|2.1} & \mathcal{L}_{3|2.2} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} a \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11}(a) \\ \mathcal{L}_{2.1|1}(a) \\ \mathcal{L}_{2.2|1}(a) \\ \mathcal{L}_{31}(a) \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11}(a) \\ \mathcal{L}_{21}(a) \\ \mathcal{L}_{31}(a) \end{pmatrix}.$$

We can repeat this checking process and find that the middle two “columns” of the 4-by-4 block expression correspond to the second column of the 3-by-3 block expression, and that the last column of the 4-by-4 expression corresponds to the the third column of the 3-by-3 expression.

This will complete our justification.

2. From the previous problem set, we know that

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} = \mathfrak{X} \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \mathfrak{X}^{-1}$$

where  $\mathfrak{X}$  is the isomorphism  $\mathbf{W}_1 \times \mathbf{W}_2 \times \mathbf{W}_3 \xrightarrow{\text{linear}} \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3$  defined in the last problem set. Since  $\mathbf{W}_1 = \mathbf{W}_{1.1} \oplus \mathbf{W}_{1.2}$ , we can define a similar isomorphism  $\clubsuit : \mathbf{W}_{1.1} \times \mathbf{W}_{1.2} \xrightarrow{\text{linear}} \mathbf{W}_{1.1} \oplus \mathbf{W}_{1.2}$  where for  $w_{1.1} \in \mathbf{W}_1$  and  $w_{1.2} \in \mathbf{W}_{1.2}$ ,

$$\clubsuit \begin{pmatrix} w_{1.1} \\ w_{1.2} \end{pmatrix} = w_{1.1} + w_{1.2}.$$

Also, define the linear functions  $\sigma_{1.m} := \rho_{1.m} \circ \clubsuit^{-1} : \mathbf{W}_1 \rightarrow \mathbf{W}_{1.m}$  and  $\omega_{1.m} := \clubsuit \circ \delta_{1.m} : \mathbf{W}_{1.m} \rightarrow \mathbf{W}_1$ , where  $\rho_{1.m}$  and  $\delta_{1.m}$  are linear functions defined similarly as in Part 1,  $m = \{1, 2\}$ .

Considering the previous part of this problem, we know that the block-matrix expression for  $\mathcal{L}_{\times}$  corresponding to  $\mathbf{W}_1 = \mathbf{W}_{1.1} \times \mathbf{W}_{1.2}$  is

$$\begin{bmatrix} \mathcal{L}_{1.1|1.1} & \mathcal{L}_{1.1|1.2} & \mathcal{L}_{1.1|2} & \mathcal{L}_{1.1|3} \\ \mathcal{L}_{1.2|1.1} & \mathcal{L}_{1.2|1.2} & \mathcal{L}_{1.2|2} & \mathcal{L}_{1.2|3} \\ \mathcal{L}_{2|1.1} & \mathcal{L}_{2|1.2} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{3|1.1} & \mathcal{L}_{3|1.2} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times},$$

where the matrix elements are defined in the same way as in Part 1. We claim that the 4-by-4 block-matrix expression for  $\mathcal{L}_{\oplus}$  is

$$\begin{bmatrix} \mathcal{L}_{1.1|1.1} & \mathcal{L}_{1.1|1.2} & \mathcal{L}_{1.1|2} & \mathcal{L}_{1.1|3} \\ \mathcal{L}_{1.2|1.1} & \mathcal{L}_{1.2|1.2} & \mathcal{L}_{1.2|2} & \mathcal{L}_{1.2|3} \\ \mathcal{L}_{2|1.1} & \mathcal{L}_{2|1.2} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{3|1.1} & \mathcal{L}_{3|1.2} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

Recall the linear functions  $\mathcal{E}_i := \Pi_i \circ \mathfrak{X}^{-1} : \mathbf{V} \rightarrow \mathbf{V}_i$  and  $\mathcal{K}_j := \mathfrak{X} \circ \gamma_j : \mathbf{W}_j \rightarrow \mathbf{V}$  from the last problem set. We have also defined

$$\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{L}_{\oplus} \circ \mathcal{K}_j.$$

We claim that

$$\mathcal{L}_{11} = \begin{bmatrix} \mathcal{L}_{1.1|1.1} & \mathcal{L}_{1.1|1.2} \\ \mathcal{L}_{1.2|1.1} & \mathcal{L}_{1.2|1.2} \end{bmatrix}, \mathcal{L}_{12} = \begin{bmatrix} \mathcal{L}_{1.1|2} \\ \mathcal{L}_{1.2|2} \end{bmatrix}, \mathcal{L}_{13} = \begin{bmatrix} \mathcal{L}_{1.1|3} \\ \mathcal{L}_{1.2|3} \end{bmatrix},$$

$$\mathcal{L}_{21} = \begin{bmatrix} \mathcal{L}_{2|1.1} & \mathcal{L}_{2|1.2} \end{bmatrix}, \mathcal{L}_{31} = \begin{bmatrix} \mathcal{L}_{3|1.1} & \mathcal{L}_{3|1.2} \end{bmatrix}.$$

The definitions of these matrix elements are similar to those in Part 1, except that we have to use the isomorphism  $\clubsuit$  to obtain the appropriate inputs for the  $\mathcal{L}_{ij}$ 's. Consider the elements of  $\mathcal{L}_{11}$ . We claim that

$$\mathcal{L}_{1.i|1.j} = \sigma_{1.i} \circ \mathcal{E}_1 \circ \mathcal{L}_{\oplus} \circ \mathcal{K}_1 \circ \omega_{1.j}.$$

For the elements of  $\mathcal{L}_{21}$  and  $\mathcal{L}_{31}$ , we claim that

$$\mathcal{L}_{i|1.j} = \mathcal{E}_i \circ \mathcal{L}_{\oplus} \circ \mathcal{K}_1 \circ \omega_{1.j}.$$

For the elements of  $\mathcal{L}_{12}$  and  $\mathcal{L}_{13}$ , we claim that

$$\mathcal{L}_{1.i|j} = \sigma_{1.i} \circ \mathcal{E}_1 \circ \mathcal{L}_{\oplus} \circ \mathcal{K}_j.$$

The justification for these definitions can be reproduced from the example in Part 1., except instead of consider a  $v$  in the Cartesian product, we now considering a  $v$  in the direct sum:

$$v = \begin{pmatrix} a & \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{\clubsuit} & 0 \end{pmatrix}_{\mathfrak{X}}^{\top},$$

which can be expressed as

$$\begin{pmatrix} a & \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{\clubsuit} & 0 \end{pmatrix}_{\mathfrak{X}}^{\top} + \begin{pmatrix} 0 & \begin{pmatrix} b \\ 0 \end{pmatrix}_{\clubsuit} & 0 \end{pmatrix}_{\mathfrak{X}}^{\top} + \begin{pmatrix} 0 & \begin{pmatrix} 0 \\ c \end{pmatrix}_{\clubsuit} & 0 \end{pmatrix}_{\mathfrak{X}}^{\top} + \begin{pmatrix} 0 & \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{\clubsuit} & d \end{pmatrix}_{\mathfrak{X}}^{\top}.$$

Verifying the above definitions is now only a matter of evaluating  $\mathcal{L}_{\oplus}$  applied to each term in this expression for  $v$  and compare the result to the 4-by-4 block representation applied to the same term. This will complete our justification.

**Problem. 3.**

Suppose that relatively prime polynomials  $p_1$  and  $p_2$  have degrees 6 and 11 respectively. Consider the function:

$$\psi : \mathbb{P}_{10} \times \mathbb{P}_5 \rightarrow \mathbb{P}_{16}$$

defined by

$$\psi \begin{pmatrix} f \\ g \end{pmatrix} := f \cdot p_1 - g \cdot p_2.$$

Verify each of the following claims.

1.  $\psi$  is a linear function.
2.  $\psi$  is injective.
3.  $\psi$  is surjective.
4. There exist polynomials  $q_1$  and  $q_2$  such that

$$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1},$$

where  $\mathbb{1}$  is the constantly 1 polynomial.

**Solution.**

1. (a) Consider

$$\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \in \mathbb{P}_{10} \times \mathbb{P}_5.$$

Then

$$\begin{aligned} \psi \begin{pmatrix} f \\ g \end{pmatrix} + \psi \begin{pmatrix} h \\ k \end{pmatrix} &= (f \cdot p_1 - g \cdot p_2) + (h \cdot p_1 - k \cdot p_1) \\ &= (f + h) \cdot p_1 - (g + k) \cdot p_2 \\ &= \psi \begin{pmatrix} f + h \\ g + k \end{pmatrix} \\ &= \psi \left( \begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} \right) \quad (\dagger) \end{aligned}$$

- (b) Consider  $c \in \mathbb{C}$ ,

$$\psi \begin{pmatrix} cf \\ cg \end{pmatrix} = cf \cdot p_1 - cg \cdot p_2 = c(f \cdot p_1 - g \cdot p_2) = c\psi \begin{pmatrix} f \\ g \end{pmatrix}. \quad (\dagger\dagger)$$

From  $(\dagger)$  and  $(\dagger\dagger)$ ,  $\psi$  is a linear function.

2. Assume (to get a contradiction) that  $\begin{pmatrix} f \\ g \end{pmatrix}$  is a non-zero element in  $\ker(\psi)$ .

Then

$$f \cdot p_1 - g \cdot p_2 = 0.$$

Since  $p_1, p_2$  are both non-zero (they are relatively prime), if either one of  $f$  or  $g$  were zero, it would force the other polynomial to be zero so that  $f \cdot p_1 - g \cdot p_2 = 0$ , contradicting the requirement that  $\begin{pmatrix} f \\ g \end{pmatrix}$  is non-zero element. Therefore, neither  $f$  nor  $g$  can be zero.

Since  $p_1$  and  $p_2$  are relatively prime (i.e., they have no common roots), the roots of  $p_2$  must also be roots of  $f$ . Furthermore, the roots of  $p_2$  must also be roots of  $f$  with equal or greater multiplicities, because otherwise there will at least one factor  $(x - \lambda_k)$  in the factorization of  $p_2$  that has a higher power (or multiplicity) than that in  $f$  (and not in  $p_1$ ), resulting in a non-zero  $f \cdot p_1 - g \cdot p_2$ .

This implies that the degree of  $f$  has to be at least  $\deg(p_2) = 11$ . But since  $\deg(f) \leq 10$ , we have a contradiction. By symmetry, we will also have a contradiction if we repeat this argument for  $g$  and  $p_1$ .

Hence, there is no such non-zero  $\begin{pmatrix} f \\ g \end{pmatrix}$  in the kernel of  $\psi$ , i.e.,  $\psi$  is injective.

3. Since both  $\mathbb{P}_{10} \times \mathbb{P}_5$  and  $\mathbb{P}_{16}$  are finite-dimensional vector spaces, we can use the Rank-Nullity Theorem:

$$\dim(\ker(\psi)) + \dim(\text{Im}(\psi)) = \dim(\mathbb{P}_{10} \times \mathbb{P}_5).$$

From part 1., we know  $\dim(\ker(\psi)) = 0$ , so

$$\dim(\text{Im}(\psi)) = \dim(\mathbb{P}_{10} \times \mathbb{P}_5) = \dim(\mathbb{P}_{10}) + \dim(\mathbb{P}_5) = 11 + 6 = 17 = \dim(\mathbb{P}_{16}).$$

Furthermore,  $\text{Im}(\psi) \prec \mathbb{P}_{16}$ . Hence,  $\text{Im}(\psi) = \mathbb{P}_{16}$ . So,  $\psi$  is surjective.

4. Since  $\psi$  is surjective, there exist polynomials  $f \in \mathbb{P}_{10}$  and  $g \in \mathbb{P}_5$  such that

$$\psi \begin{pmatrix} f \\ g \end{pmatrix} = 1 \in \mathbb{P}_{16} = \text{Im}(\psi),$$

i.e., the given  $f$  and  $g$  satisfy

$$f \cdot p_1 - g \cdot p_2 = 1.$$

Let  $q_1 = f \in \mathbb{P}_{10}$  and  $q_2 = -g \in \mathbb{P}_5$ , then

$$q_1 \cdot p_1 + q_2 \cdot p_2 = 1.$$

This verifies the existence of  $q_1$  and  $q_2$ .

**Problem. 4.**

1. What would you do to prove the last claim of problem 2 in a general case of non-zero relatively prime polynomials  $p_1$  and  $p_2$ ? You do not need to carry out the proof, but you DO need to set it all up along the lines of Problem 2. Make sure you cover all of the cases!
2. Prove that the following claims are equivalent.
  - (a) Non-zero polynomials  $p_1$  and  $p_2$  are relatively prime.
  - (b) There exist polynomials  $q_1$  and  $q_2$  such that

$$q_1 \cdot p_1 + q_2 \cdot p_2 = 1.$$

3. Argue that for any non-zero polynomials  $f$  and  $g$  there exist polynomials  $q_1$  and  $q_2$  such that

$$q_1 \cdot f + q_2 \cdot g = \gcd(f, g).$$

4. Argue that the following claims are equivalent for any non-zero polynomials  $f$ ,  $g$ , and  $h$ .
  - (a) There exist polynomials  $q_1$  and  $q_2$  such that

$$q_1 \cdot f + q_2 \cdot g = h.$$

- (b)  $h$  is a polynomial multiple of  $\gcd(f, g)$ .

**Solution.**

1. (a) Case 1: Both  $p_1$  and  $p_2$  are zero-degree polynomials.

In this case,  $p_1$  and  $p_2$  are two constant, non-zero polynomials. Let  $p_2(z) = C \in \mathbb{C}$  for any  $z \in \mathbb{C}$ ,  $C \neq 0$ . If we let  $q_1(z)$  be the constant 0 and  $q_2(z)$  be the constant  $1/C$  for any  $z \in \mathbb{C}$  then

$$q_1(z)p_1(z) + q_2(z)p_2(z) = 0 + \frac{1}{C}C = 0 + 1 = 1,$$

for any  $z \in \mathbb{C}$ , i.e.,  $q_1 \cdot p_1 + q_2 \cdot p_2 = 1$ . This shows there are polynomials  $q_1$  and  $q_2$  such that  $q_1 \cdot p_1 + q_2 \cdot p_2 = 1$  is satisfied.

- (b) Case 2: Exactly one of  $p_1$  and  $p_2$  is a zero-degree polynomial.

Assume that  $p_1$  is the non-zero zero-degree polynomial of the two, i.e., it is a non-zero constant polynomial. Let  $p_1(z) = C \in \mathbb{C}$ ,  $C \neq 0$ ,



for any  $z \in \mathbb{C}$ . If we let  $q_1(z)$  be the constant  $1/C$  and  $q_2(z)$  be the constant 0 for any  $z \in \mathbb{C}$  then

$$q_1(z)p_1(z) + q_2(z)p_2(z) = \frac{1}{C}C + 0 = 1 + 0 = 1,$$

for any  $z \in \mathbb{C}$ , i.e.,  $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$ . This shows there are polynomials  $q_1$  and  $q_2$  such that  $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$  is satisfied.

(c) Case 3: Both  $p_1$  and  $p_2$  have degree greater than 0.

To show the existence of  $q_1$  and  $q_2$  such that  $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$  we can construct a similar proof to that in Problem 3. Let the polynomials  $p_1$  and  $p_2$  be given with degrees  $m, n > 0$  respectively. Consider the function:

$$\psi : \mathbb{P}_{n-1} \times \mathbb{P}_{m-1} \rightarrow \mathbb{P}_{m+n-1}, \quad (13)$$

defined by

$$\psi \begin{pmatrix} f \\ g \end{pmatrix} := f \cdot p_1 + g \cdot p_2. \quad (14)$$

We can then show that  $\psi$  is a linear function. Next, we show that it is injective, which implies surjectivity by the Rank-Nullity Theorem. By the surjectivity of  $\psi$ , we know that given  $\mathbb{1} \in \text{Im}(\psi)$ , there must exist a pair  $f \in \mathbb{P}_{n-1}, -g \in \mathbb{P}_{m-1}$  such that  $\psi \begin{pmatrix} f \\ -g \end{pmatrix} = \mathbb{1}$ .

From parts (a), (b), and (c), we can conclude that there exist polynomials  $q_1, q_2$  such that for non-zero relatively prime polynomials  $p_1, p_2$ ,  $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$ .

2.
  - $[a. \implies b.]$  This implication is true by Part 1. of this problem.
  - $[b. \implies a.]$  Let polynomials  $q_1$  and  $q_2$  be given such that

$$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1},$$

where  $p_1$  and  $p_2$  are also polynomials. Claim: Non-zero  $p_1$  and  $p_2$  are relatively prime.

Assume that  $p_1$  and  $p_2$  are non-zero. From TYC 2.4, we can write  $p_1$  and  $p_2$  as

$$\begin{aligned} p_1 &= \hat{p}_1 \cdot \gcd(p_1, p_2) \\ p_2 &= \hat{p}_2 \cdot \gcd(p_1, p_2), \end{aligned}$$

where  $\hat{p}_1$  and  $\hat{p}_2$  are non-zero relatively prime polynomials. It follows that

$$q_1 \cdot \hat{p}_1 \cdot \gcd(p_1, p_2) + q_2 \cdot \hat{p}_2 \cdot \gcd(p_1, p_2) = 1,$$

i.e.,

$$(q_1 \cdot \hat{p}_1 + q_2 \cdot \hat{p}_2) \cdot \gcd(p_1, p_2) = 1,$$

which implies

$$\deg(q_1 \cdot \hat{p}_1 + q_2 \cdot \hat{p}_2) = \deg(\gcd(p_1, p_2)) = 0.$$

Therefore,  $\gcd(p_1, p_2)$  has to be a constant polynomial. Moreover, since  $\gcd(p_1, p_2)$  is monic by definition, it has to be the constantly 1 polynomial. Hence,  $p_1$  and  $p_2$  are relatively prime by definition.

3. Let non-zero polynomials  $f, g$  be given. Then we have

$$\begin{aligned} f &= \hat{f} \cdot \gcd(f, g) \\ g &= \hat{g} \cdot \gcd(f, g), \end{aligned}$$

where  $\hat{f}, \hat{g}$  are relatively prime. It follows that

$$\begin{aligned} q_1 \cdot f + q_2 \cdot g &= q_1 \cdot \hat{f} \cdot \gcd(f, g) + q_2 \cdot \hat{g} \cdot \gcd(f, g) \\ &= (q_1 \cdot \hat{f} + q_2 \cdot \hat{g}) \cdot \gcd(f, g). \end{aligned}$$

From Part 2., we know that for relatively prime polynomials  $\hat{f}$  and  $\hat{g}$ , there exist polynomials  $q_1, q_2$  such that

$$q_1 \cdot \hat{f} + q_2 \cdot \hat{g} = 1.$$

This means such  $q_1$  and  $q_2$  will give

$$(q_1 \cdot \hat{f} + q_2 \cdot \hat{g}) \cdot \gcd(f, g) = 1 \cdot \gcd(f, g) = \gcd(f, g).$$

This completes the argument.

4. • [ $a. \implies b.$ ] Let non-zero polynomials  $f, g, h$  be given such that

$$q_1 \cdot f + q_2 \cdot g = h,$$

for some polynomials  $q_1, q_2$ . From Part 3., we know that

$$\begin{aligned} h &= q_1 \cdot f + q_2 \cdot g = q_1 \cdot \hat{f} \cdot \gcd(f, g) + q_2 \cdot \hat{g} \cdot \gcd(f, g) \\ &= (q_1 \cdot \hat{f} + q_2 \cdot \hat{g}) \cdot \gcd(f, g), \end{aligned}$$

for relatively prime  $\hat{f}$  and  $\hat{g}$ . This implies  $h$  is a polynomial multiple of  $\gcd(p_1, p_2)$ .

- [b.  $\implies$  a.] Let  $h$  be written as  $k \cdot \gcd(f, g)$  where  $k$  is some polynomial. From Part 3., there exist polynomials  $s_1, s_2$  such that

$$s_1 \cdot f + s_2 \cdot g = \gcd(f, g).$$

Multiplying both sides by the polynomial  $k$ , we have

$$k \cdot s_1 \cdot f + k \cdot s_2 \cdot g = k \cdot \gcd(f, g) = h.$$

Hence, there exist polynomials  $q_1$  and  $q_2$  such that

$$q_1 \cdot f + q_2 \cdot g = h,$$

namely,

$$q_1 = k \cdot s_1$$

$$q_2 = k \cdot s_2.$$

**Problem. 5.**

Suppose that  $\mathbf{W}$  is a non- $\{0\}$  ideal in  $\mathbb{P}$ . Then  $\mathbf{W}$  contains some monic polynomials (why?) and among these there must be some of the smallest degree, say  $n_0$ . Let  $p_0$  be one such. (It is entirely possible that  $p_0 = 1$ .)

1. Suppose that  $p$  is a non-zero polynomial in  $\mathbf{W}$ , and using the Division Algorithm for Polynomials (see Axler p.121) we write

$$p = q \cdot p_0 + r,$$

where  $q, r \in \mathbb{P}$  and  $\deg(r) < \deg(p_0)$ . Argue that  $r \in \mathbf{W}$ .

2. Use the result of part 1 to argue that every polynomial  $p$  in  $\mathbf{W}$  is a polynomial multiple of  $p_0$ .
3. Argue that  $p_0$  is the only monic polynomial of the smallest degree  $n_0$  in  $\mathbf{W}$ , and that

$$\mathbf{W} = \{q \cdot p_0 \mid q \in \mathbb{P}\}.$$

This polynomial  $p_0$  is said to be **the generator** of the ideal  $\mathbf{W}$ .

**Solution.**

1. Let  $p \in \mathbf{W}$  a non-zero polynomial be given. Let  $p_0 \in \mathbf{W}$  be the monic polynomial with the smallest degree.  $\mathbf{W}$  is an ideal in  $\mathbb{P}$ , so for  $q \in \mathbb{P}$ ,  $q \cdot p_0 \in \mathbf{W}$ .  $\mathbf{W}$  is also subspace, so

$$r = p - q \cdot p_0 \in \mathbf{W}.$$

2. Let the above  $q, p_0, p, r$  be given again. We know that

$$p = q \cdot p_0 + r,$$

where  $\deg(r) < \deg(p_0)$ . Since  $\deg(p_0)$  minimal and  $p$  is non-zero,  $r$  has to be the constantly zero polynomial. Therefore,

$$p = q \cdot p_0 \in \mathbf{W},$$

i.e., every  $p \in \mathbf{W}$  is a polynomial multiple of  $p_0$ .

3. Let some  $p'_0$  be a monic polynomial of the smallest degree  $n_0$  in  $\mathbf{W}$  be given. Since  $p'_0 \in \mathbf{W}$ ,  $p'_0$  is a polynomial multiple of  $p_0$ , by the previous part. Let us write  $p'_0 = q \cdot p_0$  for some  $q \in \mathbb{P}$ . Since  $\deg(p'_0) = \deg(p_0) = n_0$ ,  $\deg(q) = 1$ , i.e.,  $q$  is a constant polynomial. Furthermore, since both  $p_0$  and  $p'_0$  are monic,  $q$  has to be 1. Hence,  $p'_0 = p_0$ , i.e.,  $p_0$  is unique.

It follows from the previous part that  $\mathbf{W}$  is a collection of elements that are polynomial multiples of a unique monic polynomial of the smallest degree, called  $p_0$ , i.e.,

$$\mathbf{W} = \{q \cdot p_0 \mid q \in \mathbb{P}\}.$$