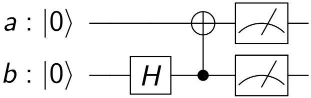
Matrices in Quantum Computing

Huan Q. Bui

Matrix Analysis

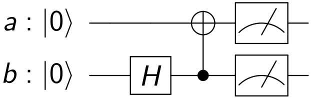
Professor Leo Livshits

CLAS, May 2, 2019


a : |0⟩
b : |0⟩
H

Presentation layout

Background



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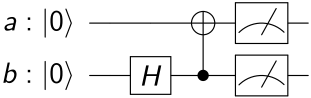

Components:

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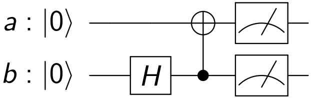

Components:

1. Quantum bits - Qubits

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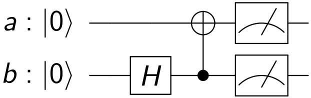

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1. Quantum bits - Qubits
2. Quantum gates: single and multiple-qubit gates

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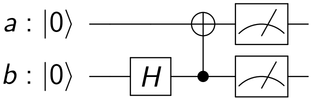
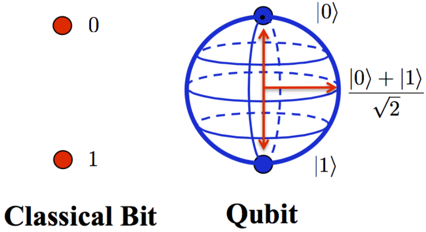


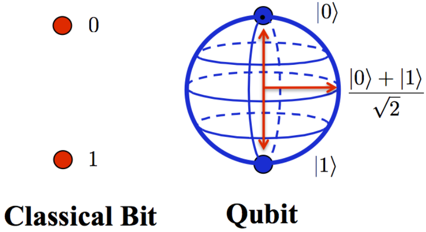
Components:

1. Quantum bits - Qubits
2. Quantum gates: single and multiple-qubit gates
3. Measurement

Quantum Bits - Qubits

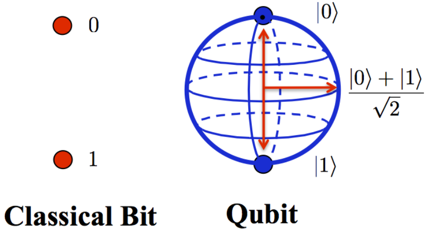


Quantum Bits - Qubits



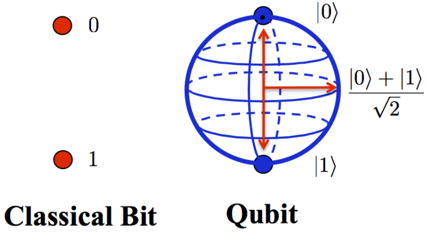
$$a\begin{bmatrix}1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$$

Quantum Bits - Qubits



$$a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad |a|^2 + |b|^2 = 1$$

Quantum Bits - Qubits



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$$\boxed{a\,|0\rangle + b\,|1\rangle}$$

Quantum Gates

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$\rightarrow$ linear transformations on one or many qubits.

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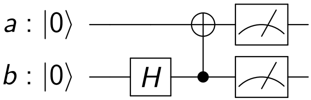

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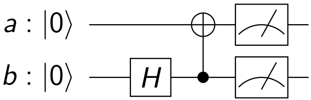

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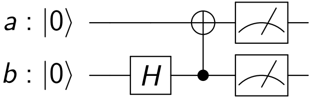

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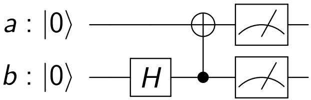

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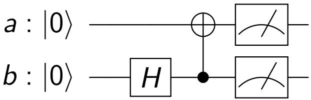

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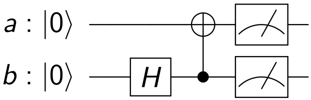

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$$H|0\rangle = H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

Multiple Qubits

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Qubit 1: $a\,|0\rangle + b\,|1\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$ Qubit 2: $c\,|0\rangle + d\,|1\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$

Multiple Qubits

$$\text{Qubit 1: } a\left|0\right\rangle + b\left|1\right\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \text{Qubit 2: } c\left|0\right\rangle + d\left|1\right\rangle = \begin{bmatrix} c \\ d \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxtimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a \begin{bmatrix} c \\ d \end{bmatrix} \\ b \begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}.$$

Multiple Qubits

Do this for the basis states

$$|0\rangle \boxtimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad |0\rangle \boxtimes |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad |1\rangle \boxtimes |0\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad |1\rangle \boxtimes |1\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

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Notation:

$$|00\rangle = |0\rangle \boxtimes |0\rangle \qquad |01\rangle = |0\rangle \boxtimes |1\rangle$$
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Can see that we have a basis for describing the combined state.

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxtimes \begin{bmatrix} c \\ d \end{bmatrix} = ac\,|00\rangle + ad\,|01\rangle + bc\,|10\rangle + bd\,|11\rangle \,.$$

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Ex: $p(x) \cdot q(y)$ is a "combined state." But there are NO $p(x), q(y)$ s.t.

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even though $xy + 1$ is a legitimate "combined state."

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If

$$\mathcal{A} = \begin{bmatrix} m & n \\ o & p \end{bmatrix} \quad \text{and } \mathcal{B} = \begin{bmatrix} q & r & s \\ t & u & v \\ w & x & y \end{bmatrix}$$

Kronecker Product

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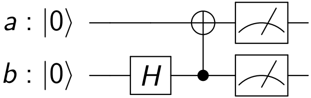
$$= \begin{bmatrix} mq & mr & ms & nq & nr & ns \\ mt & mu & mv & nt & nu & nv \\ mw & mx & ms & nw & nx & ny \\ oq & or & os & pq & pr & ps \\ ot & ou & ov & pt & pu & pv \\ ow & ox & oy & pw & px & py \end{bmatrix}$$

Kronecker Product

Check that $I\left|0\right\rangle \boxtimes H\left|0\right\rangle = (I \otimes H)\left|00\right\rangle$:



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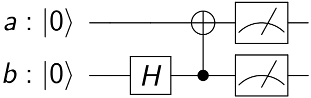

LHS:

$$I\,|0\rangle \boxtimes H\,|0\rangle$$

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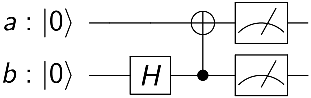

LHS:

$$I\left|0\right\rangle \boxtimes H\left|0\right\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \boxtimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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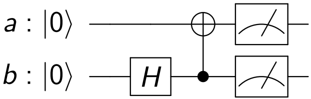

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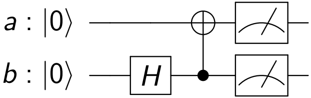

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Some properties & Elementariness revisited

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3. Associative
4. NOT commutative. Ex: $\left|01\right\rangle \neq \left|10\right\rangle$.

Some properties & Elementariness revisited

$\otimes$ and $\boxtimes$ are very much alike.

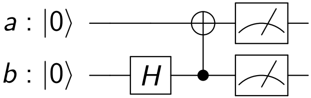
1. Bilinear
2. Distributive.

$$\begin{bmatrix} a \\ b \end{bmatrix} \boxtimes \begin{bmatrix} c \\ d \end{bmatrix} = (a\left|0\right\rangle + b\left|1\right\rangle) \boxtimes (c\left|0\right\rangle + d\left|1\right\rangle)$$
$$= ac\left|00\right\rangle + ad\left|01\right\rangle + bc\left|10\right\rangle + bd\left|11\right\rangle.$$

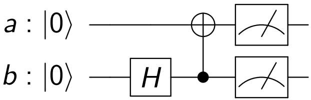
3. Associative
4. NOT commutative. Ex: $\left|01\right\rangle \neq \left|10\right\rangle$.
5. Elementariness.

Some properties and Elementariness revisited

Ex:

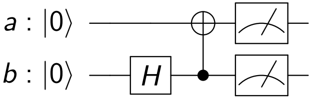


Ex:



The Control-NOT gate:

Ex:

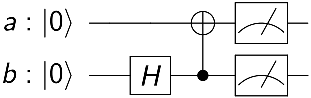


The Control-NOT gate:

$$CNOT_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Some properties and Elementariness revisited

Ex:

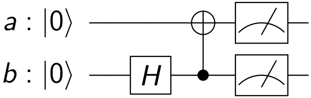


The Control-NOT gate:

$$CNOT_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{cases} |00\rangle \to |00\rangle \\ |10\rangle \to |10\rangle \\ |01\rangle \to |11\rangle \\ |11\rangle \to |01\rangle \end{cases}$$

Some properties and Elementariness revisited

Ex:



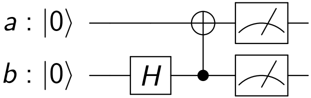
The Control-NOT gate:

$$CNOT_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \longrightarrow \begin{cases} |00\rangle \to |00\rangle \\ |10\rangle \to |10\rangle \\ |01\rangle \to |11\rangle \\ |11\rangle \to |01\rangle \end{cases}$$

Also called "entangled."

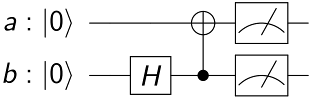
Entanglement Circuit

Time to decode:



Entanglement Circuit

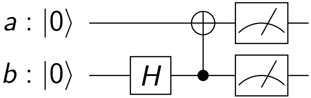
Time to decode:



1 Step 1:

Entanglement Circuit

Time to decode:



1 Step 1:

$$a : |0\rangle \to |0\rangle$$

$$b : |0\rangle \to \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

$$\left|a' b'\right\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^{\top}$$

Entanglement Circuit

2 Step 2:

Entanglement Circuit

2 Step 2:

$$CNOT_b \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}$$

Entanglement Circuit

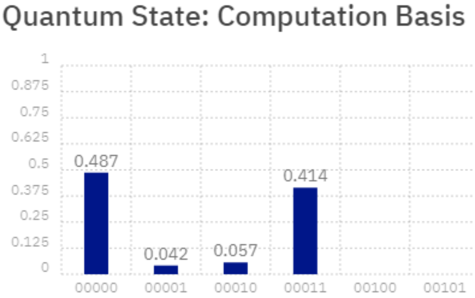
2 Step 2:

$$CNOT_b \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which is:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \leftarrow \textbf{Entangled}$$

Simulation on IBM-Q



Tensor Product

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$\otimes$ and $\boxtimes$ are really "the same!"

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Why tensor product?

Tensor Product

$\otimes$ and $\boxtimes$ are really "the same!" $\rightarrow$ Tensor products.

Why tensor product?

Postulate (QM):

The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

Tensor Product

Tensor Product

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\hat{f}", "\textit{linear}"'] & & \mathbf{X} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[urr, "f", "\textit{bilinear}"'] & &
\end{tikzcd}
$$

Tensor Product

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\hat{f}", "linear"'] & \mathbf{X} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[ur, "f", "bilinear"'] &
\end{tikzcd}
$$

Roughly speaking...

Giving the $\hat{f} : \mathbf{V} \otimes \mathbf{W} \xrightarrow{\text{linear}} \mathbf{X}$ is the same as giving $f : \mathbf{V} \times \mathbf{W} \xrightarrow{\text{bilinear}} \mathbf{X}$.
$f = \hat{f} \circ \phi$

Tensor Product

If the target space $\mathbf{X}$ is $\mathbf{V} \otimes \mathbf{W}$. $\mathcal{L}$ is an operator on $\mathbf{V}$, $\mathcal{M}$ on $\mathbf{W}$,

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& & \\
\mathbf{V} \times \mathbf{W} \arrow[uu, hook, "\phi"] \arrow[uurr, "f", "\textit{bilinear}"'] & &
\end{tikzcd}
$$

Tensor Product

If the target space $\mathbf{X}$ is $\mathbf{V} \otimes \mathbf{W}$. $\mathcal{L}$ is an operator on $\mathbf{V}$, $\mathcal{M}$ on $\mathbf{W}$,

$$\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\hat{f}", "linear"'] & & \mathbf{V} \otimes \mathbf{W} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[urr, "f", "bilinear"'] & &
\end{tikzcd}$$

$$\mathcal{L}[v] \otimes \mathcal{M}[w]$$

Tensor Product

If the target space $\mathbf{X}$ is $\mathbf{V} \otimes \mathbf{W}$. $\mathcal{L}$ is an operator on $\mathbf{V}$, $\mathcal{M}$ on $\mathbf{W}$,

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\hat{f}", "linear"'] & \mathbf{V} \otimes \mathbf{W} \\
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\end{tikzcd}
$$

$$(\mathcal{L} \otimes \mathcal{M})(v \otimes w) \qquad \mathcal{L}[v] \otimes \mathcal{M}[w]$$

Tensor Product

If the target space $\mathbf{X}$ is $\mathbf{V} \otimes \mathbf{W}$. $\mathcal{L}$ is an operator on $\mathbf{V}$, $\mathcal{M}$ on $\mathbf{W}$,

$$\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\hat{f}", "\textit{linear}"'] & & \mathbf{V} \otimes \mathbf{W} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[urr, "f", "\textit{bilinear}"'] & &
\end{tikzcd}$$

$\rightarrow$ by uniqueness

$$(\mathcal{L} \otimes \mathcal{M})(v \otimes w) = \mathcal{L}[v] \otimes \mathcal{M}[w]$$

Tensor Product & Kronecker Product

Tensor Product & Kronecker Product

ν a basis for **V**, ω for **W** $\rightarrow$ can make a basis τ for $\mathbf{V} \otimes \mathbf{W}$

Tensor Product & Kronecker Product

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$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\mathcal{L}\otimes\mathcal{M}", "linear"'] \arrow[d, "\{\,\}_\tau"] & \mathbf{V} \otimes \mathbf{W} \\
\mathbb{C}^{nm} \arrow[r, "{[\mathcal{L}\otimes\mathcal{M}]_{\tau\leftarrow\tau}}", "linear"'] & \mathbb{C}^{nm} \arrow[u, "\mathcal{A}_\tau"]
\end{tikzcd}$$

Tensor Product & Kronecker Product

ν a basis for $\mathbf{V}$, ω for $\mathbf{W}$ $\rightarrow$ can make a basis τ for $\mathbf{V} \otimes \mathbf{W}$

$$\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\mathcal{L}\otimes\mathcal{M}", "linear"'] \arrow[d, "\{\}_\tau"] & \mathbf{V} \otimes \mathbf{W} \\
\mathbb{C}^{nm} \arrow[r, "{[\mathcal{L}\otimes\mathcal{M}]_{\tau\leftarrow\tau}}", "linear"'] & \mathbb{C}^{nm} \arrow[u, "\mathcal{A}_\tau"']
\end{tikzcd}$$

$$\boxed{[\mathcal{L} \otimes \mathcal{M}]_{\tau\leftarrow\tau} = [\mathcal{L}]_{\nu\leftarrow\nu} \otimes [\mathcal{M}]_{\omega\leftarrow\omega}}$$

Recap

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- How a 2-qubit entangling circuit works

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- Qubits, quantum gates as matrices

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References

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