QUANTUM FIELD THEORY

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Preface. These notes come from Prof. Patton's QFT notes, Peskin-Schröder QFT, and "Quantum field theory & Condensed matter" - Shankar.

Enjoy!

Conventions

- $\hbar = c = 1$
- $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$
- $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1\ -1\ -1\ -1)$
- $p^2 = p_\mu p^\mu = E^2 - |p|^2 = mc^2 = m$
- $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$
- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$
- $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$
- $\varepsilon^{0123} = +1 \; ; \; \varepsilon_{0123} = -1$

  $\varepsilon^{1230} = -1$
- $E = i\frac{\partial}{\partial x^0}\ , \quad \vec{p} = -i\vec{\nabla}$
- $p^\mu = i\partial^\mu$

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $$\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \cancel{<} 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx}\theta(x)$

- n-dimensional Dirac δ-fn:

$$\int d^n x \, \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(x)$$

$$\tilde{f}(x) = \int d^4k \, e^{ik\cdot x} f(x)$$

- $$\int d^4x \, e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k)$$

- EM $\Phi = \frac{Q}{4\pi r}$ ← Coulomb potential

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ ; \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\Phi, \vec{A}) \ ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of Classical Field Theory

Lagrangian Field Theory:

$$S = \int \underline{L}\, dt = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

Lag-density

$$(\underline{L} = \mathcal{L}\, d^3x)$$

(Lagrangian)

Principle of least action:

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \right\}$$

(E-L eqn)

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right] \right\}$$

$$\Rightarrow \boxed{0 = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi}}$$

FTC → term vanishes @ boundary

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

Ex

$$\mathcal{L} = \phi^2 : \quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi \, ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow 2\phi = 0$$

$$\left.\begin{aligned} &\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) \\ &\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = 0 \, ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2\partial^\mu \phi \end{aligned}\right\} \Rightarrow \partial^\mu \phi = 0.$$

Ex

Klein - Gordon field : (real)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\cdot \ \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\cdot \ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi .$$

$$E\text{-}L \Rightarrow \ -m^2 \phi = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$\rightarrow \boxed{(\Box + m^2)\phi = 0}$$

relativistic particle of mass m.

Klein - Gordon Eqn.

Ex $\phi = e^{-ip\cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

Noether's Thm

Thm

$$\boxed{\text{For every continuous symmetry, there exists a conserved current } j^\mu \text{ which implies a local conservation law}}$$

$$\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j} \iff \partial_\mu j^\mu = 0$$

Ex Conservation of charge ...

(gauss' law)

$$Q = \int j^0 \, d^3x$$

$$\Rightarrow \frac{\partial Q}{\partial t} = \int \frac{dj^0}{dt} d^3x = -\int \vec{\nabla}\cdot\vec{j} \, d^3x$$

$$= -\oint \vec{j} \, d^2 s$$

//

Idea Consider continuous transf. → infinitesimally... (local)

$$(\star) \qquad \phi(x) \to \phi'(x) = \phi(x) + \underbrace{\alpha}_{\text{small}} \Delta\phi(x)$$

$(\star)$ is a symmetry if EOM invariant under $(\star)$.

$\iff S$ is invariant.

$\Rightarrow \mathcal{L}$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$, for some J^μ.

Let us compare this expectation for $\Delta\mathcal{L}$ to the result obtained by varying the fields ...

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi)$$

$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right\}\right)}_{0}\Delta\phi$$

$$\Rightarrow \quad \Delta\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right\}$$

So $\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi$ is the desired $\mathcal{J}^\mu$.

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - \mathcal{J}^\mu.$$

Ex massless KG field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$$

Consider transformation:

$$\phi(x) \longrightarrow \phi(x) + \alpha \quad , \Delta\phi = 1$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = \partial^\mu\phi.$$

Check $\partial_\mu j^\mu = \partial_\mu\partial^\mu\phi = \Box\phi = 0$ since

$(m^2+\Box)\phi = 0$ & $m=0$

Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi .$$

again, EOM is

$$(m^2 + \Box)\phi = 0.$$

Symmetry: $\phi \to e^{i\alpha}\phi$.

For infinitesimal transf we have:

$$\alpha \Delta\phi = i\alpha\phi \qquad \text{(Taylor expand)}$$
$$\alpha \Delta\phi^\dagger = -i\alpha\phi^\dagger$$

One can check that

$$j^\mu = i\left[(\partial^\mu \phi^\dagger)\phi - \phi^\dagger(\partial^\mu \phi)\right]$$

is the conserved current.

$\boxed{\text{Ex}}$

Space-time translation:

Consider infi. translation:

$$x^\mu \to x^\mu - a^\mu$$

in field transforms this becomes:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Lagrangian is a scaler ⇒ must transform the same way:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Compare this to the eqn:

$$\mathcal{L} \longrightarrow \mathcal{L} + a \partial_\mu J^\mu,$$

we have $J^\mu = \delta^\mu_\nu \mathcal{L}$

→ apply them, we find:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\mu \phi) - \delta^\nu_\mu \mathcal{L}$$

Make μ explicit...

$$\boxed{T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L}}$$

STRESS-ENERGY TENSOR. (or Energy-Momentum tensor)

<u>Conserved charge</u> is the Hamiltonian...

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad (\text{time-translation})$$

momentum $\longrightarrow$ $$p^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi \, d^3x \quad (\text{spatial translation})$$

Ex Klein-Gordon field again...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 .$$

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - g^{00}\mathcal{L} = \ldots$$

$$= \frac{1}{2}(\partial_0\phi)(\partial^0\phi) - \frac{1}{2}(\partial_i\phi)(\partial^i\phi) + \frac{1}{2}m^2\phi^2$$

so

$$\mathcal{H} = \int T^{00} d^3x = \int \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right\} d^3x$$

note: all terms are positive ... (sum of squares)

$\rightarrow$ can't fall into arbitrary negative energy.

$$\boxed{\text{THE KLEIN-GORDON FIELD} \;\infty\; \text{HARMONIC OSCILLATOR}}$$

promote: ϕ, π to operators & impose suitable commutation relations...

Recall ...

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0 .$$

Harmonic oscillator: $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$

- creation: $a^{+} = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$

- $a^{+}a = \frac{1}{\omega}H - \frac{1}{2} \Leftrightarrow H = \omega\left(a^{+}a + \frac{1}{2}\right)$

  $\uparrow$

  # operator...

- $|0\rangle$, $a|0\rangle = 0$.

  $|n\rangle = \frac{1}{\sqrt{n!}}\underbrace{a^{+}\ldots a^{+}}_{n}|0\rangle$

- $[a^{+}a, a] = -a$

- $[a^{+}a, a^{+}] = a^{+}$

- $[H, a] = -\omega a$; $[H, a^{+}] = \omega a^{+}$

  a lowers by ω

  a^{+} raises by ω

- $Ha^{+}|n\rangle = (E_n + \omega)a^{+}|n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

For continuum system ... commutation relations become ..

$$\boxed{\begin{aligned} [\phi(x), \pi(y)] &= i\delta^{(3)}(x-y) \\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0. \end{aligned}}$$

Next, the Hamiltonian is now also an operator ...
To find spec(H), Fourier transf $\phi(x)$

$$\rightarrow \quad \phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \, e^{ip\cdot x} \phi(\vec{p},t)$$

Recall KG eqn: $(m^2 + \Box)\phi = 0$

$$\rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) \right\} \phi(\vec{p},t) = 0$$

$\Rightarrow$ This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}.$$

Now, $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2 \qquad (m=1)$

$\rightarrow$ know spectrum! $\quad (n+\frac{1}{2})\omega.$

$$\phi = \frac{1}{\sqrt{2\omega}}(a^\dagger + a) \quad ; \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$[a, a^\dagger] = 1.$$

Since it's more convenient to work in position space

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^{+} e^{-ip\cdot x} \right) \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^{+} e^{ip x} \right) \end{aligned}}$$

note

$$\begin{cases} a_p \text{ goes with } e^{+ip\cdot x} \\ a_p^{+} \text{ goes with } e^{-ip\cdot x}. \end{cases}$$

- Easy to show that

$$\begin{cases} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \end{cases}$$

- Can re-arrange...

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p + a_{-p}^{+}) e^{ip\cdot x}. \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} (a_p - a_{-p}^{+}) e^{+ip\cdot x}. \end{cases}$$

→ get commutation relation between a_p:

$$\boxed{[a_p ; a_{p'}^{+}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}')}$$

Now, can check that

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{w_{p'}}{w_p}} \times e^{i(p\cdot x + p'\cdot x')} \times$$

$$\left([a^\dagger_{-p}, a_{p'}] - [a_p, a^\dagger_{-p}] \right)$$

$$= i\delta^{(3)}(x - x') \quad \checkmark$$

- Now, can express Hamiltonian in terms of ladder ops

Recall that $\longrightarrow$ KG field, lost time

$$H = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \right)$$

So ...

$$H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

with $\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a^\dagger_{-p}) e^{ip\cdot x}$ we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p + a^\dagger_{-p}) e^{ip\cdot x}$$

results in $\delta(p-p')$
$\Rightarrow p = p'$

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(p+p')\cdot x} \Big\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4}(a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'})$$

$$+ \frac{-p\cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}}(a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) \Big\}$$

Some $\delta^{(3)}$ will appear...

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a^\dagger_p] \right)$$

So

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3} \omega_p \left(a^\dagger_p a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)}$$

$\rightarrow$ can evaluate commutators...

$$\boxed{[H, a_p^\dagger] = \omega_p a_p^\dagger \ ; \quad [H, a_p] = -\omega a_p}$$

With H, can find momentum operator...

KG field $\longrightarrow$ from $p^i = \int d^3x\, T^{0i} = -\int \pi \partial_i \phi\, d^3x$, we get

$$\boxed{\begin{aligned} \vec{P} &= -\int d^3x\, \pi(x) \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a_p^\dagger a_p . \end{aligned}}$$

$a_p^\dagger$ creates momentum $\vec{p}$ & energy $\underset{E_p}{\omega_p} = \sqrt{|p|^2 + m^2}$.

Excitation: $a_p^\dagger a_q^\dagger \ldots |0\rangle \equiv$ "particles".

$\hookrightarrow$ <u>each excitation $a^\dagger_p$ is a particle.</u>

$\Rightarrow$ get particle statistics...

Consider 2-particle state... $a_p^\dagger a_q^\dagger |0\rangle$.

Since $[a_p^\dagger, a_q^\dagger] = 0$, we have

$$\boxed{a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle}$$

Klein Gordon particles follow Bose-Einstein stats.

- Normalization $\langle 0|0\rangle = 1$.

$$|p\rangle \propto a_p^\dagger |0\rangle.$$

This normalization $\longrightarrow$ $\langle q|p\rangle = (2\pi)^3 \delta^{(3)}(q-p)$ is NOT Lorentz inv

Pf Under a Lorentz boost $\rightarrow$ $p_3' = \gamma(p_3 + \beta E)$

$E' = \gamma(E + \beta p_3)$

using the identity $\rightarrow$ $\delta(f(x) - f(x_0)) = \dfrac{1}{|f'(x_0)|}\delta(x - x_0)$

we compute: $\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \cdot \left(\dfrac{dp_3'}{dp_3}\right)$

$\underbrace{\delta(p_1-q_1)\delta(p_2-q_2)}_{\text{same}}\underbrace{\delta(p_3-q_3)}_{\text{boosted}}$

$= \delta^{(3)}(p'-q') \cdot \gamma\left(1 + \beta\, dE/dp_3\right)$

$= \delta^{(3)}(p'-q') \dfrac{\gamma}{E}(E + \beta p_3)$

$= \delta^{(3)}(p'-q')(E'/E)$

So $$\boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\left(\frac{E'}{E}\right)}$$ $\rightarrow 1 \Leftrightarrow E = E_p = E'$

For normalization to work $\rightarrow$ use E_p ($= w_p$), not E.

$\rightarrow$ define: $\boxed{|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$

So that $$\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$$

Completeness relation...

1 particle $\rightarrow$ $$\boxed{1 = \int \frac{d^3p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$$

$\boxtimes$ Interpret $\phi(x)|0\rangle$ .... We know that $a_p^\dagger$ creates momentum p & energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|p\rangle \quad , \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

Pf. By defn...

annihilates.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|0\rangle \quad \square$$

$\rightarrow$ $\phi(x)|0\rangle$ is a lin. superposition of single-particle states

~~Note HW1, HW2 easy~~

that have well-defn momentum.

When nonrelativistic $\rightarrow E_p \simeq$ Constant!

$\Rightarrow$ [$\phi(x)$ acting on the vacuum, "creates a particle at position x".]

$\hookrightarrow$ Confirm this by computing.

$$\langle 0|\,\phi(x)\,|p\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\left(a_{p'}e^{ip'\cdot x} + a^\dagger_{p'}e^{-ip'\cdot x}\right)\sqrt{2E_p}\,a^\dagger_p|0\rangle$$

$$\rightarrow \boxed{\langle 0|\,\phi(x)\,|p\rangle = e^{ip\cdot x}}.$$

$\hookrightarrow$ Interpretation: position-space representation of the single-particle wfn of the state $|p\rangle$, just like...

$$\boxed{\langle x|p\rangle \propto e^{ip\cdot x}} \text{ in QM!}$$

$\langle 0|\phi(x) \sim \langle x|$ ... (don't take this literally, ofc).

Note HW1, HW2 are easy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time → we quantized KG field in the Schrödinger picture.

→ Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$\mathcal{U}(t) = e^{-iHt}$ is the time evolution.

$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle$. → State evolve in time

→ In the Heisenberg picture, ... Operators evolve in time

$$\mathcal{O}(t) = \mathcal{U}^{\dagger}(t)\, \mathcal{O}(0)\, \mathcal{U}(t).$$

So that

$$\langle \psi_1 | \mathcal{O}(t) | \psi_1 \rangle = \langle \psi_1(t) | \mathcal{O} | \psi_1(t) \rangle$$

↓ Heisenberg ↓ Schrödinger.

→ make the operators ϕ & π time-dependent like this

$$\boxed{\begin{aligned} \phi(x) \to \phi(x,t) &= e^{iHt}\phi(x)e^{-iHt} \\ \pi(x) \to \pi(x,t) &= e^{iHt}\pi(x)e^{-iHt} \end{aligned}}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\right]$$

$(\phi \leftrightarrow \phi)$
$\rightarrow$ only nontrivial term is 1^{st}.

$$= \int d^3x' \left(i\delta^{(3)}(x-x')\,\pi'(x,t)\right)$$

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t}\phi(x,t) = \pi(x,t)}$$

and

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right\}\right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')\left(-\nabla^2 + m^2\right)\phi(x',t)\right)$$

(integrate by parts here)

$$= -i\left(-\nabla^2 + m^2\right)\phi(x,t)$$

$$\rightarrow \boxed{\frac{\partial}{\partial t}\pi(x,t) = (m^2 - \nabla^2)\phi(x,t)}$$

Combining these 2 results we get...

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) = (\nabla^2 - m^2)\phi(x,t)}$$

rearranging this gives

$$\boxed{(\Box + m^2)\phi(x,t) = 0} \longrightarrow \text{just the KG eqn...}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

<u>Recall</u>: $H a_p = a_p (H - E_p)$ → from comm. rule.

$$\Rightarrow \begin{pmatrix}\text{proof by} \\ \text{induction}\end{pmatrix} \quad \boxed{H^n a_p = a_p (H - E_p)^n}$$

Similarly,
$$\boxed{H^n a_p^\dagger = a_p^\dagger (H + E_p)^n}$$

→ So, we have
$$\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}}$$
→ from above...

and
$$\boxed{e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{+iE_p t}}$$

→ Now... we want to write $\phi(x,t)$ in terms of these operators. (since $\phi(x)$, $\pi(x)$ is a combo of a & $a^\dagger$)

- We know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before...

$$\phi(x) = \phi(x,0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

Substitute this into $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$ we find ...

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-iE_pt} e^{ip\cdot x} + a_p^\dagger e^{iE_pt} e^{-ip\cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow \quad p\cdot x = E_p \cdot x^0 - \vec{p}\cdot\vec{x} \Big|_{p^0=E_p} = E_p t - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p}$$

So...

$$\boxed{\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right\} \Big|_{p^0=E_p}}$$

Similarly, we can find

$$\boxed{\pi(x,t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left(a_p e^{-i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right) \Big|_{p^0=E_p}}$$

from defn or from $\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$.

Note: we can also do everything, but starting from P and not H. But we won't worry about that.

Causality: Note that causality is broken ~~then~~ without the presence of fields.

$\rightarrow$ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture...

...the amplitude for a particle to propagate from y to x is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

"Two-point correlation function"

Let $\boxed{\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \text{ in terms of } a_p \text{ \& } a_p^\dagger \ldots |0\rangle$$

$$= \langle 0|\, a_p\, a_q^\dagger\, |0\rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

More explicitly...

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

@ $p^0 = E_p$
$p^{0\prime} = E_{p'}$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0| e^{-ip\cdot x} e^{+ip'\cdot y} a_p a_{p'}^\dagger |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^{\cancel{6}}} \left(\frac{1}{\sqrt{2E_p}}\right)\frac{1}{\sqrt{2E_{p'}}} e^{-ip\cdot(x-p'y)} \cancel{(2\pi)^3} \delta^{(3)}(p-p')$$

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip\cdot(x-y)} \qquad \square.$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$...

(1) Suppose that $x-y = (t,0,0,0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2\sqrt{p^2+m^2}}\, e^{-i\sqrt{p^2+m^2}\,t}$$

(timelike)

$$= \frac{1}{4\pi^2}\int_m^\infty dE\, \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\underset{t\to\infty}{=} e^{-imt}.$$ → dominated by region where $p \approx 0$.

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

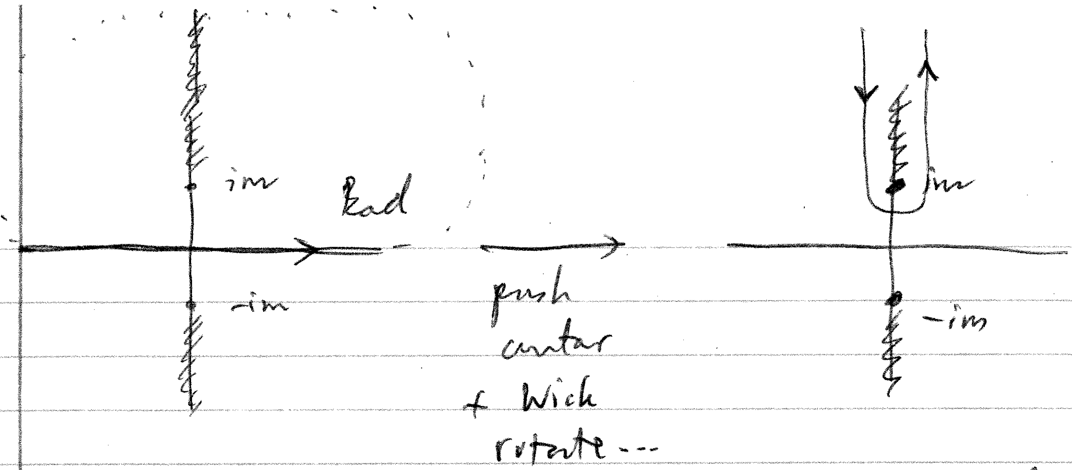
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3}\int_0^\infty dp\, \frac{p^2}{2E_p}\, \frac{e^{ipr}-e^{-ipr}}{ipr}$$

$$= \frac{-i}{2(2\pi)^2 r}\int_{-\infty}^{\infty} dp\, \frac{p\, e^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ $\pm im$ (singularity)...

→ must change contour... & Wick rotate



To get

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty d\rho \, \frac{\rho \, e^{-i\rho r}}{\sqrt{\rho^2 - m^2}}$$

$\rho = -ip$ (Wick rotate)

$$\Rightarrow \quad \boxed{\mathcal{D}(x-y) \underset{r\to\infty}{=} \frac{e^{-mr}}{r}} \longrightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^\mu (x-y)_\mu = -(\vec{x}-\vec{y})^2 < 0$ is spacelike, cannot have causality between x & y.

$\mathcal{D}(x-y) \neq 0 \Rightarrow$??? paradox?

→ <u>NO!</u> To discuss causality, we should ask not whether particles can propagate over spacelike intervals...

... but whether a "<u>measurement</u>" at one point can affect a measurement at another point whose separation from the first is spacelike -

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement @ x, call this $\phi(x)$. & a local measurement @ y, called $\phi'(y)$

So long as $[\phi(x), \phi'(y)] = 0$, the 2 measurements don't affect one another.

~ measure the field ϕ @ x & @ y,

If $[\phi(x), \phi(y)] = 0$ ~~when~~ $(x-y)^2 < 0$ then we're good

→ $$\cancel{[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \Big\{ e^{-ip\cdot x}[a_p, a^\dagger_{p'}] e^{ip'\cdot y} + e^{ip\cdot x}[a^\dagger_p, a_{p'}] e^{-ip'\cdot y} \Big\}}$$

→ $$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \times \Big\{ \Big[(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_{p'} e^{-ip'\cdot y} + a^\dagger_{p'} e^{ip'\cdot y}) \Big]$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6\, 2\sqrt{E_p E_{p'}}} \Big\{ e^{-ip\cdot x} \underbrace{[a_p, a^\dagger_{p'}]}_{(2\pi)^3 \delta^3(p-p')} e^{ip'\cdot y} + e^{ip\cdot x} \underbrace{[a^\dagger_p, a_{p'}]}_{-(2\pi)^3 \delta^3(p-p')} e^{-ip'\cdot y} \Big\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p}\right) \Big\{ e^{-ip(x-y)} - e^{-ip(y-x)} \Big\} = D(x-y) - D(y-x)$$

So $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$.

Since $D(y-x)$ is Lorentz invariant, can take

$y-x \rightarrow x-y$. (possible b/c $(x-y)^2 < 0$)

Note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

$\rightarrow$ so this is only possible because $(x-y)^2 < 0$ (spacelike).

So $D(y-x) = D(x-y)$

So, $[\phi(x), \phi(y)] = 0$ ✓

The Klein-Gordon Propagator

~~Let's look at $[\phi(x), \phi(y)]$ in more details...~~

~~$[\phi(x), \phi(y)]$ is just a number~~

~~$\Rightarrow$ can write $[\phi(x), \phi(y)] \equiv \langle 0| [\phi(x), \phi(y)] |0\rangle$~~

~~$\rightarrow \langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right\}$~~

~~(assuming $x^0 > y^0$)~~

~~$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_p} - \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0 = -E_p}$~~

Poles @ $p^2 = m^2$ $\Downarrow$ $p^0 = \pm E_p$

The Klein-Gordon Propagator

→ Before looking at this ... need to look at Green's Function & Contour Integrals ...

→ Consider function

$$\frac{1}{(p^0_0 - E_p)(p^0_0 + E_p)} \qquad E_p = \sqrt{|p|^2 + m^2}.$$

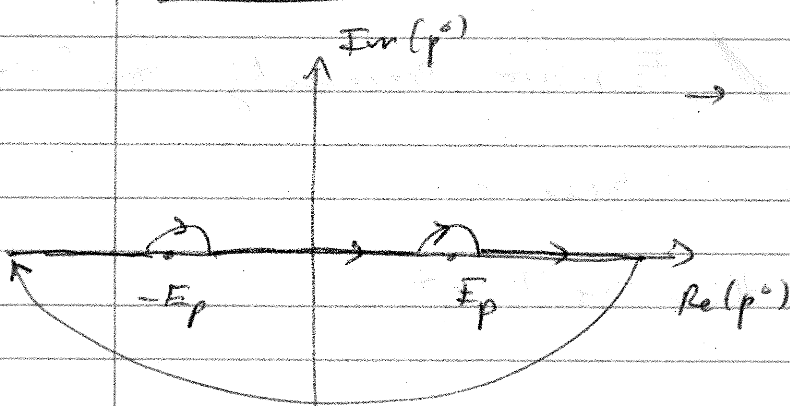
→ Poles at $p^0_0 = \pm E_p$.

Look at FT:

$$\boxed{f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}}$$

→ How to integrate this?

<u>If $t > 0$</u> →



$$\rightarrow f(t) = (-2\pi i) \sum \text{Res}(f)$$

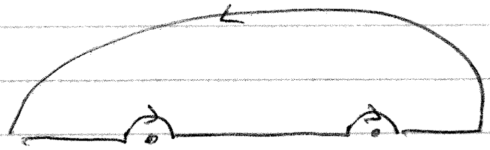
$$= (-2\pi i) \left\{ \frac{e^{iE_p t}}{2\pi(-2E_p)} + \frac{e^{-iE_p t}}{2\pi(2E_p)} \right\}$$

↑ close contour ...
but clockwise → reverse sign

→

$$\boxed{f(t) = \frac{-i}{2E_p} \left(e^{+iE_p t} - e^{-iE_p t} \right)} \qquad (t > 0)$$

<u>If $t<0$</u> close contour above poles



$\rightarrow$ $f(t) = 0$.

$\rightarrow$ So, altogether, we have...

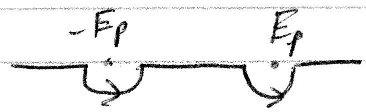
$$\boxed{\begin{aligned} f(t) &= \int \frac{1}{2\pi} \frac{dp^0}{(p^0-E_p)(p^0+E_p)} e^{-ip^0 t} \\ &= \Theta(t)\left(\frac{-i}{2E_p}\right)\left(e^{-iE_p t} - e^{+iE_p t}\right) \end{aligned}}$$

where $\Theta(t)$ is the Heaviside Step fn...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 . \end{cases}$$

$\rightarrow$ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



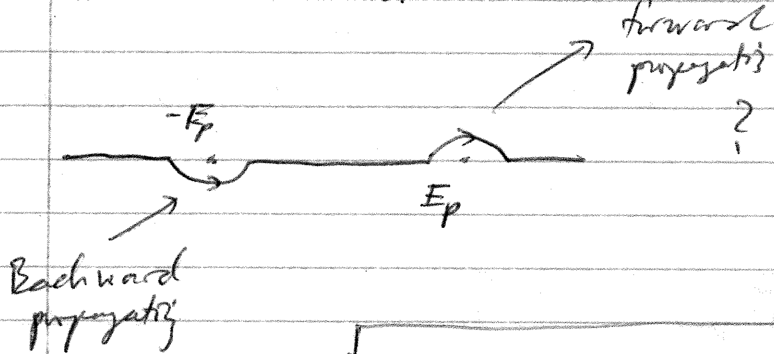
, then we'll get

$t > 0 \rightarrow f(t) = 0$

$t < 0 \rightarrow f(t) = \Theta(-t)\dfrac{i}{2E_p}\left(e^{-iE_p t} - e^{iE_p t}\right)$

$\rightarrow$ Advanced / Backward Propagating Green's fn.

☒ What about



$$\longrightarrow \quad \boxed{f(t) = \Theta(t)\left(\frac{-i}{2E_p}\right)e^{-iE_pt} + \Theta(-t)\left(\frac{-i}{2E_p}\right)e^{+iE_pt}}$$

↑ Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$ ...

Consider this quantity...

$$\langle 0|\,[\phi(x),\phi(y)]\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\underbrace{\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{ \frac{1}{2E_p}e^{-ip(x-y)} + \frac{1}{-2E_p}e^{-ip(x-y)} \right\}$$

pole @ $p_0 = E_p$ pole @ $p_0 = -E_p$

4D integral →

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3}\underbrace{\int \frac{dp^0}{2\pi i}\,\frac{-1}{p^2-m^2}\,e^{-ip(x-y)}}_{f(t)\text{ before, where}}$$

$$(p^0 - E_p)(p^0 + E_p) = p^{0\,2} - |p|^2 - m^2 = p^2 - m^2 .$$

Now, call ...

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle}$$

Then

$$(\Box+m^2)D_R(x-y) = \Box D_R(x-y) + m^2 D_R(x-y)$$

$$= (\Box\theta(x^0-y^0))\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ 2(\partial_\mu\theta(x^0-y^0))\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ \theta(x^0-y^0)(\Box+m^2)\underbrace{\langle 0|[\phi(x),\phi(y)]|0\rangle}_{\text{constant}}$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$+ 2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0$$

$$= \delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

needs verification (easy)

$$= -i\delta^{(4)}(x-y)$$

$$-i\delta^{(3)}(x-y)$$

So

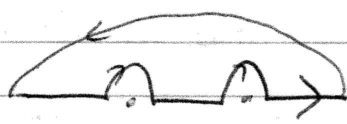
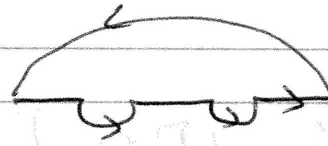
$$\boxed{(\Box+m^2)D_R(x-y) = -i\delta^{(4)}(x-y)}$$

$\longrightarrow$ $D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y)=0$ @ $x^0<y^0$

$\Rightarrow$ $D_R(x-y) \equiv$ "Retarded" Green's fn

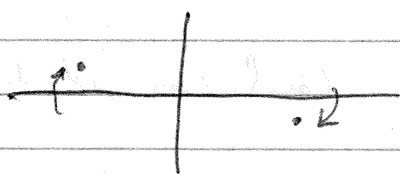
Now ... As we have seen, there are many ways to take the contour...

 or  or  etc

→ Use the Feynman prescription instead

$$\boxed{D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}}$$

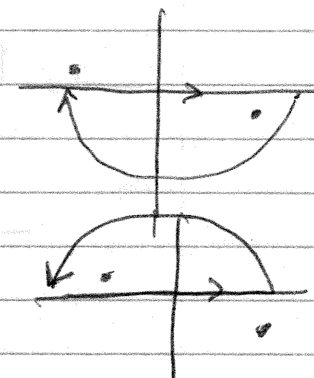
→ Convenient! B/c now poles are $p^0 = \pm(E_p - i\varepsilon)$



→ when $x^0 > y^0$ → take contour below

when $x^0 < y^0$ → take contour above.

→ get same expression but with $x \leftrightarrow y$.



So,

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$\equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

"time-ordering" symbol.

<u>Time-ordering symbol</u> $\Rightarrow$ instructs us to place the operators that follows in order with the latest to the left.

→ apply $(\Box + m^2)$ to last line, get D_F is Green's fn of Klein-Gordon Operator.

↳ $D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator...

↳ propagation amplitude

→ But we can't much calculation at this point just yet.

→ B/c we've only looked at the <u>free</u> KG theory

→ field eqn in this case is linear: there are <u>no interactions</u>...

→ this theory is too simple to make any predictions...

→ need perturbation ....

One kind of interaction that can also be solved ...

↳

Particle Creation by a classical Source

Consider the source $j(x)$

Recall... free field: $(\Box + m^2)\phi = 0$

→ now... $(\Box + m^2)\phi = j(x)$ fixed fn in space time.

$j(x)$ is nonzero only for a finite time interval

The associated Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is easiest to solve...

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

With a source...

$$\phi(x) = \phi_0(x) + i\int d^4y\, D_R(x-y)\, j(y)$$

⋮

We won't worry about this for now...

Some problems & Insights

(1) Classical EM (no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \varepsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} ; \quad \vec{\nabla} \cdot \vec{E} = 0 \qquad (\nu = 0)$$

$$-\partial_t \mathcal{E} + \vec{\nabla} \times \vec{E} = 0 \qquad (\nu = i)$$

(2) Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi\right)$$

Derive E-L eqn:

$$\boxed{\begin{aligned} i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ &= 0 \\ -i\partial_t \phi - \frac{1}{2m} \nabla^2 \phi &= 0 \end{aligned}}$$

now... write $\phi \to e^{-i\theta}\phi$, $\phi^\dagger \to e^{i\theta}\phi^\dagger$

$$\sim \phi - i\theta\phi \qquad \sim \phi^\dagger + i\theta\phi^\dagger$$

$$\to \Delta\phi \sim -i\theta \qquad \Delta\phi^\dagger \sim i\phi^\dagger$$

So that

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\Delta\phi^\dagger$$

conserved current.

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \to \text{... compute ...}$$

→ can get Hamiltonian... there's a Ham. formula in book, but we won't worry abt this.

$$\mathcal{H} = \int d^3x \left(\pi^\dagger \pi + (\nabla\phi)^\dagger(\nabla\phi) + m^2 \phi^\dagger \phi \right)$$

(3) If we take $(x-y)^2 = -r^2$ → can implicitly write

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2$ → $D(x-y)$ can be written ~~as~~ in terms of Bessels Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation... $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu}$

→ what happens to $\phi(x)$ under Λ?

we require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform ... $\partial_\mu \phi(x) \to \partial_\mu(\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu(\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\nu{}_\mu (\partial_\nu\phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu\phi(x))^2 \to g^{\mu\nu}(\partial_\mu\phi'(x))(\partial_\nu\phi'(x))$$

$$= g^{\mu\nu}[\partial_\mu(\phi(\Lambda^{-1}x))][\partial_\nu(\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu}\{(\Lambda^{-1})^\rho{}_\mu \partial_\rho\phi\}\{(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma\phi\}(\Lambda^{-1}x)$$

$$= (\partial_\mu\phi)^2(\Lambda^{-1}x) \leftarrow = g^{\rho\sigma}\partial_\rho\phi\,\partial_\sigma\phi\,(\Lambda^{-1}x) \to \text{evaluated}$$

Recall

$$\boxed{\begin{aligned} \phi(x) &\to \phi(\Lambda^{-1}x) \\ \partial_\mu \phi(x) &\to (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x) \\ (\partial_\mu \phi(x))^2 &\to \cancel{g^{\rho\sigma}} \; (\partial_\mu \phi)^2(\Lambda^{-1}x) \end{aligned}}$$

So it is clear that

$$\boxed{\mathcal{L} \longrightarrow \mathcal{L}(\Lambda^{-1}x)}$$

Lagrangian is Lorentz-invariant.

→ The action $S = \int d^4x \, \mathcal{L}$ is also Lorentz inv.

→ also clear that EOM is also Lorentz inv.

$$(\Box + m^2)\,\phi'(x) = \left((\Lambda^{-1})^\mu{}_\nu \partial_\mu (\Lambda^{-1})^{\sigma\mu} \partial_\sigma + m^2\right)\phi(\Lambda^{-1}x)$$

$$= (\partial^\sigma \partial_\sigma + m^2)\,\phi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Sep 20, 2020

→ How do we find Lorentz-invariant theories, in general?

→ for simplicity, restrict attention to lin. transf

→ $\phi_a = v \in \mathbb{C}^n$,

→ $\boxed{\Phi_a(x) \Rightarrow \underbrace{M_{ab}(\Lambda)}_{n\times n} \Phi_b(\Lambda^{-1}x)}$

matrix giving Lorentz transform Λ.

→ the most general nonlinear transf can be built out of linear ones ⇒ suffices to consider M only.

↳ for short, write $\Phi \mapsto M(\Lambda)\Phi$.

→ what are the possible allowed $M(\Lambda)$?

Obs $\{\Lambda\}$ form a group $M(\Lambda'')M(\Lambda) = M(\Lambda')$

$\Lambda''\Lambda = \Lambda'$

→ the correspondence between Λ & M must be preserved under multiplication.

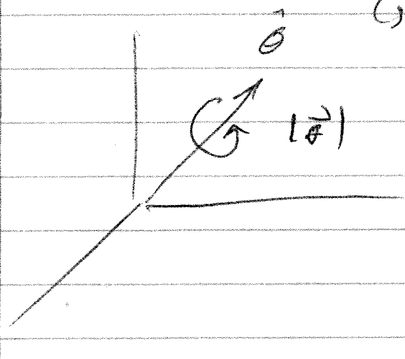
$\{\Lambda\}$ Lorentz group $\longmapsto \{M(\Lambda)\} \to$ n-dim representation of the Lorentz group

↳ $\boxed{?}$ what are the finite-dim matrix reps of the Lorentz group?

Ex in QM... spin $\frac{1}{2}$ → $\{M\}$ are the 2×2 unitary matrices with determinant 1.

→ $\boxed{U = e^{-i\theta^i\sigma^i/2}}$ → $\sum_i \theta^i \sigma^i/2$

3 arbitrary parameters & Pauli matrices.



$U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2}$

→ In the case for arbitrary spin representations...

$$\mathcal{U}(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i \sum_l \epsilon^{ijl} J^l$$

→ check that this works for spin 1/2:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i \sum_l \epsilon^{jkl} \frac{\sigma^l}{2} \quad \checkmark$$

→ for spinless particles... $\psi(\vec{x})$ can be decomposed into orbital angular momentum states $J = 0, 1, 2, \ldots$
(no intrinsic spin ⇒ $J = L$)

- $\vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$
- $J^j = i \sum_l \epsilon^{jkl} x^k \nabla^l$
- $\nabla^l = -\nabla_l = -\dfrac{\partial}{\partial x^l}$

But the cross product is special to 3D case.

→ write operator as antisymmetric tensor...

$$\boxed{J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)}$$ → reproduces the cross product.

So that $J^3 = J^{12}$, etc.

→ generalize to 4D: → 6 operators that generate { 3 boosts, 3 rotations, of the Lorentz group.

$$\boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

→ Spatial Rotations: $J^{jk} = i(x^j \partial^k - x^k \partial^j)$

→ Lorentz boosts along x^j axis: $J^{0j} = i(x^0 \partial^j - x^j \partial^0)$

→ Now, want to get commutation rules...
→ compute the commutators of differential ops

to get

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\sigma} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)}$$

Ex 3 rotations:

$$\left.\begin{aligned} J^{12} &= -J^{21} \\ J^{23} &= -J^{32} \\ J^{13} &= -J^{31} \\ \text{3 boosts} \quad J^{01} &= -J^{10} \\ J^{02} &= -J^{20} \\ J^{03} &= -J^{30} \end{aligned}\right\} \rightarrow \text{6 total generating matrices...}$$

Ex Consider the 4x4 matrices $(J^{\mu\nu})_{\alpha\beta}$ where μ, ν label which of the 6 matrices, while α, β label the component / matrix element...

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)}$$

→ can verify that $(J^{\mu\nu})_{\alpha\beta}$ satisfies the comm. relation...

→ These are matrices that act on ordinary Lorentz 4-vectors...

to see this...

→ Look at elements of the Lorentz group...

$$\mathcal{U}(\omega_{\mu\nu}) = \operatorname{Exp}\left[\frac{-i}{2}\omega_{\mu\nu}J^{\mu\nu}\right]$$

infinitesimally →

$$\simeq \mathbb{I} + \frac{-i}{2}\omega_{\mu\nu}J^{\mu\nu}$$

$$\simeq \delta^{\alpha}_{\beta} + \frac{-i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}_{\ \beta}$$

So, infinitesimally...

$$V^{\alpha} \longrightarrow \delta^{\alpha}_{\beta} + \frac{-i}{2}\omega_{\mu\nu}(J^{\mu\nu})^{\alpha}_{\ \beta}V^{\beta}$$

→ $\omega_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles...

$V_{\alpha}, V_{\beta} \longrightarrow$ 4-vectors...

<u>Ex 1</u> When $\omega_{12} = -\omega_{21} = \theta$, $\omega_{\mu\nu} = 0$ else, we get

$$[V^{\mu}] \longrightarrow \begin{array}{c|c} & \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{array}{c} t \\ x \\ y \\ z \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} [V^{\mu}]$$

→ Infinitesimal $\boxed{\text{ROTATION on } xy \text{ plane.}}$

<u>Ex 2</u> When $\omega_{01} = -\omega_{10} = \beta \Rightarrow$ get
$\omega_{\mu\nu} = 0$ else

$$[V^{\mu}] \overset{\sim}{\longrightarrow} \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} [V^{\mu}] \longrightarrow \boxed{\text{BOOST along } x}$$

THE DIRAC EQUATION

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to ~~develop~~ formalism for finding all other ~~formalisms~~ representations ...

(problem 3.1)

→ focus on spin $\frac{1}{2}$ systems ...

→ In this case, use Dirac's trick ~~due to~~ -

Suppose we had a set of 4 $n \times n$ matrices γ^μ satisfying :

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times I}$$

then we could write down an n-dim representation of the Lorentz algebra ...

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

these matrices satisfy the commutation relation ...

→ $[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$

→ Verify that this trick works in 3D Euclidean space:

in which case, $\boxed{\gamma^j = i\sigma^j}$ → $\boxed{\{\gamma^i, \gamma^j\} = -2\delta^{ij}}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\sum_k \varepsilon^{ijk}\sigma^k = J^l$$

Which is what we saw before $\Rightarrow$ angular momentum.

$$\begin{cases} J^3 = S^{12} = \frac{1}{2}\sigma^3 \\ J^2 = S^{31} = \frac{1}{2}\sigma^2 \\ J^1 = S^{23} = \frac{1}{2}\sigma^1 \end{cases}$$

$\rightarrow$ now, want $S^{\mu\nu}$ for 4D Minkowski space...

$\rightarrow$ matrices γ^μ must be at least 4×4.

$\rightarrow$ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2\times2} \\ \mathbb{I}_{2\times2} & 0 \end{pmatrix} \; ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

"Weyl" / "Chiral" representation.

$\rightarrow$ In this case, the boost & rotation generators are...

Boosts $\longrightarrow$ $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

(sum) sum

Rotations $\longrightarrow$ $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \sum_k \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \sum_k \frac{1}{2}\varepsilon^{ijk}\Sigma^k$

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at ~~all~~ all transformations under which the Action S ~~Lagrangian~~ is "invariant"

→ In particular, we want S to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\begin{cases} \phi(x) \longrightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x^\mu) \rightarrow \phi(\Lambda^{-1\,\mu}{}_\nu\, x^\nu). \end{cases} \rightarrow \left(\text{check that } S \text{ is invariant}\right.$$

→ But this is very simple ... ⇒ There are many more transforms that leave S Lorentz invariant.

→ How do we find all of them?

⇒ For simplicity, we'll just restrict ourselves to linear ~~combinations~~ transformations

→ look at transformations of the form.

$$\phi_a(x) \longrightarrow \sum_b M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \longrightarrow M(\Lambda)\, \phi}$$

These matrices M must be "nice" in the sense that M must obey...

$$\boxed{\phi \to M(\Lambda'')M(\Lambda)\phi = M(\Lambda''\Lambda)\phi}$$

This says that $\{M\}$ (the collection of M's) must be a representation of the Lorentz group.

What?? So... recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

$$\to \quad \boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

A representation Π of a group G is a function satisfying the property

$$\Pi(\underset{\in G}{g_1})\,\Pi(\underset{\in G}{g_2}) = \Pi(\underset{\in G}{g_1 g_2})$$

With this, it is clear that

$$\boxed{\{\Lambda\} \text{ Lorentz group} \Longrightarrow \{M\} \text{ is a representation of } \{\Lambda\}}$$

So... what are these M's?

→ Ex Rotation group for spin $\frac{1}{2}$ particles

For spin-$\frac{1}{2}$, the most important nontrivial representation is the 2D representation.

↳ these are unitary matrices with det = 1
(2x2)

⇒ In general: $\mathcal{U} = e^{-i\vec{\sigma}\cdot\vec{\theta}/2}$

$\vec{\sigma} \rightarrow$ Pauli matrices
$\vec{\theta} \rightarrow$ angle.

For infinitesimal rotations, we can write

$$\mathcal{U} = 1 - \frac{i}{\hbar}\vec{J}\cdot\vec{\theta} \equiv 1 - \vec{J}\cdot\vec{\theta}$$

$\{\mathcal{U}\}$ form a Lie-algebra of the L-group.

$\vec{J}$ here are the "generator" of the Lie algebra

when $\{\mathcal{U}\}$ is a representation of the rotational group, we identify

$$\vec{J} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{J}$ as the quantum angular momentum operator

→ satisfies the commutation relation

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

like the generators of SO(3), namely the Pauli matrices...

→ finite rotations are formed by matrix exp.

$$R = \exp\left[-i\theta^i \sigma^i \right]$$

Sep 27, 2020

Back to present problem...

to get generator of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

(3D) $\vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$

(4D) $\boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)}$$

→ any matrices that are ~~pe~~ represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$$

→ by symmetry, μ, ν ~~table~~ label which of the six matrices we want;

, α, β label components.

The Dirac Eqn

What are the representations of the Lorentz group?
, especially for spin-$\frac{1}{2}$?

<u>Dirac's trick</u> if we have a set of 4 $n \times n$ matrices γ^μ which satisfies:

Dirac algebra $\rightarrow$ $$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} * I_{n\times n}}$$

Then the n-dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

$\rightarrow$ In other words, $S^{\mu\nu}$ satisfies ...

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho})$$

$\circledast$ Note that this trick works ~~~~ in any dim.

e.g. take $\gamma^j = i\sigma^j$ so that $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

$$\Rightarrow \quad S^{ij} = \frac{1}{2}\varepsilon^{ijk}S^k \quad \rightarrow \text{ just as before.}$$

2D representation of the rotation group.

Spin $\frac{1}{2}$: $J^3 = J^{12} = \frac{1}{2}\sigma^3$; $J^2 = \frac{1}{2}\sigma^2 = S^{31}$, $J^1 = S^{23} = \frac{1}{2}\sigma^1$

One such representation for the Dirac algebra is

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4\times 4} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4\times 4}}$$

Weyl / chiral representation.

get

Boosts $\quad S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

Hermitian Defn

not ~~unitary~~, but Ψ is also classical field, not a wfn

All 4-component field Ψ that transforms under boosts & rotations according to is called a Dirac spinor

S^{ij} are Hermitian
S^{0i} are anti-Hermitian

fine b/c Ψ is a classical field, not a wfn.

Now, what is the field eqn for ψ?

$\rightarrow$ try $(\Box + m^2)\psi = 0$ $\leftarrow$ KG field eqn.

But this obviously works because the representation are block - diagonal ...

$\rightarrow$ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~the~~ at transformation of γ matrices

In an expression we can think of...

$$[\cdots]\,\Lambda_{\frac{1}{2}}^{-1}\,[4\times 4]\,\Lambda_{\frac{1}{2}}\,[\vdots]$$

$\frac{1}{2}$: for spin $\frac{1}{2}$

where $\Lambda_{\frac{1}{2}} = \exp\left\{\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right\}$

$$\simeq 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\Rightarrow [\gamma^\mu] \rightarrow [\Lambda_{\frac{1}{2}}^{-1}][\gamma^\mu][\Lambda_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$$= \ldots \quad \text{(some terms of higher order cancelled ...)}$$

$$= \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}\underbrace{[\gamma^\mu, S^{\alpha\beta}]}_{?}$$

where a quick computation shows that

$$\boxed{[\gamma^\mu, S^{\alpha\beta}] = (\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma^\nu}$$

where

$$\mathcal{J}^{\alpha\beta} = +i\left(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu\right)$$

So...

$$\boxed{\gamma^\mu \longrightarrow \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^\mu{}_\nu \cancel{\gamma}^\nu} = \Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}$$

→ γ^μ transforms like 4-vectors...!

⇒ γ^μ are invariant under simultaneous rotations of their vectors & spinor indices...

→ can treat "μ" on γ^μ as a vector index!

→ can dot γ^μ into ∂_μ to form a Lorentz-inv. differential operator...

Dirac eqn

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}$$

Check that this is Lorentz-inv:

Let $\Psi(x) \longrightarrow \Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$ then

$$i\gamma^\mu\partial_\mu\Psi \rightarrow \left(i\gamma^\mu\Lambda^{+1}_{\frac{1}{2}}\right)\left(\partial_\mu\left(\Psi(\Lambda^{-1}x)\right)\right)$$

$$= i\Lambda_{\frac{1}{2}}\left(\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\right)\cdot(\Lambda^{-1})^\alpha{}_\mu(\partial_\alpha\Psi)(\Lambda^{-1}x)$$

same Lorentz transform

$$= i\Lambda_{\frac{1}{2}} (\Lambda)^{\mu}{}_{\nu} \gamma^{\nu} \cdot (\Lambda^{-1})^{\alpha}{}_{\mu} \; (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \underbrace{(\Lambda)^{\mu}{}_{\nu}(\Lambda)^{\alpha}{}_{\mu}}_{\delta^{\alpha}_{\nu}} (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \partial_\mu \Psi(\Lambda^{-1}x)$$

$$\Rightarrow \quad i\gamma^{\mu}\partial_\mu \Psi(x) \longrightarrow \Lambda_{\frac{1}{2}} i\gamma^{\mu}\Psi(\Lambda^{-1}x)$$

$\rightarrow$ transforms the same way as $\Psi(\Lambda^{-1}x)$.

Cleaner way:

Let $[i\gamma^{\mu}\partial_\mu - m]\Psi(x) \rightarrow [i\gamma^{\mu}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m]\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$

$$= \underbrace{\Lambda_{\frac{1}{2}}\Lambda^{-1}_{\frac{1}{2}}}[i\gamma^{\mu}\Lambda^{-1\,\nu}{}_{\mu}\partial_\nu - m]\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}}\left\{ i\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m \right\}\Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}}\left\{ i\gamma^{\nu}\partial_\nu - m \right\}\Psi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i\gamma^{\mu}\partial_\mu - m)\Psi$$

$$\rightarrow \quad 0 = (-i\gamma^{\mu}\partial_\mu - m)(+i\gamma^{\nu}\partial_\nu - m)\Psi$$

$$= (\gamma^{\mu}\gamma^{\nu}\partial_\nu\partial_\mu + m^2)\Psi \qquad = \ldots$$

$$= \left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2 \right) \psi$$

$$= \left[\frac{1}{2} \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \right) \partial_\mu \partial_\nu + m^2 \right] \psi$$

$$= \left[\frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right] \psi$$

KG eqn.

$$= \left(g^{\mu\nu} \partial_\mu \partial_\nu + m^2 \right) \psi = \left(\Box + m^2 \right) \psi = 0$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger \psi$ doesn't work b/c under a boost,

$\psi^\dagger \Delta_{\frac{1}{2}}^\dagger \Delta_{\frac{1}{2}} \psi \neq \psi^\dagger \psi$ since $\Delta_{\frac{1}{2}} = \exp\{-i\omega_{\mu\nu} S^{\mu\nu}\}$

not unitary ... since not all $S^{\mu\nu}$ are Herm.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, get

$$\bar{\psi} \rightarrow \psi^\dagger \Delta_{\frac{1}{2}}^\dagger \gamma^0 \cong \psi^\dagger \left(1 + \frac{i}{2} \omega_{\mu\nu} (S^{\mu\nu})^\dagger \right) \gamma^0$$

when ~~[illegible]~~ $\mu \neq 0$ & $\nu \neq 0$, $(S^{\mu\nu})^\dagger = (S^{\mu\nu})$

$\quad$ & $(S^{\mu\nu} \leftrightarrow \gamma^0)$

When $\mu = 0$ or $\nu = 0$, $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$

$\because$ $S^{\mu\nu}$ anti-commutes w/ γ^0.

$$\rightarrow \quad \bar{\Psi} \rightarrow \psi^\dagger \left(1 + \frac{i}{2} w_{\mu\nu} (S^{\mu\nu})^\dagger\right) \gamma^0$$

$$= \underbrace{\psi^\dagger \gamma^0}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right)$$

$$= \bar{\Psi}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right) = \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \quad \text{as desired}$$

$$\rightarrow \quad \boxed{\bar{\Psi} \rightarrow \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1}}$$

and so $\boxed{\bar{\Psi}\psi = \psi^\dagger \gamma^0 \psi}$ is a Lorentz scalar.

Similarly, can show that

$\boxed{\bar{\Psi}\gamma^\mu \psi}$ is a Lorentz vector.

$\rightarrow$ the correct Lorentz-inv Dirac Lagrangian is

$$\boxed{\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi}$$

$$\begin{cases} \text{E-L eqn for } \bar{\Psi} \text{ gives } (i\gamma^\mu \partial_\mu - m)\Psi = 0 \\ \text{E-L eqn for } \Psi \text{ gives } -i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \end{cases}$$

WEYL SPINOR

Recall that

$$\boxed{\begin{aligned} S^{0i} &= \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \end{aligned}}$$

Since block-diagonal $\Rightarrow$ Dirac representation of the Lorentz group is reducible.

$\rightarrow$ Can form 2-D representations by considering each block separately...

$\rightarrow$ write $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\begin{matrix} \rightarrow \text{Left-handed} \\ \rightarrow \text{RH} \end{matrix}$ Weyl spinors

Under infinitesimal boost $\vec{\beta}$ & rotation $\vec{\theta}$, these transform as

$$\boxed{\begin{aligned} \psi_L &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_L \\ \psi_R &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_R . \end{aligned}}$$

Recall that $\left(\tanh(|\vec{\beta}|) = \frac{|v|}{c}\right)$.

$\rightarrow$ Transf of ψ_R is equiv to transf of ψ_L^*

By noting that

$$\psi_L^* \longrightarrow \left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

noting that $\underset{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}{\sigma^2}\vec{\sigma}^* = -\vec{\sigma}\,\underset{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}{\sigma^2}$ $\quad (\sigma^2 = \sigma^y)$

we find.

$$\underbrace{\sigma^2\psi_L^*}\longrightarrow \sigma^2\left[1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right]\psi_L^*$$

$$= \underbrace{\left[1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right]}_{\text{like } \psi_R \text{ transform.}}\underbrace{\sigma^2\psi_L^*}$$

So $\sigma^2\psi_L^*$ transform like ψ_R ...

With $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, the Dirac eqn has form.

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0 \iff \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0}$$

When $m = 0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left\{\begin{aligned} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L &= 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R &= 0 \end{aligned}\right\} \longrightarrow \text{Weyl eqns.}$$

$\rightarrow$ important for neutrinos & weak force studies...

For convenience let us define...

$$\boxed{\sigma^\mu \equiv (1, \vec{\sigma}) \quad ; \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})}$$

$\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$

so that

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

With this, can simplify notation.. Dirac eqn becomes...

$$\boxed{\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0}$$

$i\sigma\cdot\partial = i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})$

$i\bar{\sigma}\cdot\partial = i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})$

& the Weyl eqns become:

$$\boxed{\begin{aligned} (i\bar{\sigma}\cdot\partial)\psi_L &= 0 \\ (i\sigma\cdot\partial)\psi_R &= 0 \end{aligned}}$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p)\, e^{-ip\cdot x} \quad , \quad p^2 = m^2$$

so that

$$p^0 = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}.$$

Look only solutions with positive frequency ... that is
$E_p = p^0 > 0$ ....

ψ solves Dirac eqn $\rightarrow$ $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$\Leftrightarrow$ $(i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$

$\rightarrow$ $\boxed{(i\gamma^\mu p_\mu - m)\, u(p) = 0}$

go to rest frame $\Rightarrow$ $p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $\Lambda_{\frac{1}{2}}$.

In rest frame, we have

$(i\gamma^\mu p_\mu - m)\, u(p) \rightarrow (m\gamma^0 - m)\, u(p_0) = m(\gamma^0 - 1)\, u(p_0) = 0$

$\Rightarrow$ $m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$

$\rightarrow$ $\boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}}$ $\rightarrow$ two-component spinor

just a factor, with norm. constraint.

$\xi^\dagger \xi = 1$.

<u>What are these ξ?</u>

Look at <u>rotation</u> generators...

$$\boxed{S^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

In particular, $S^z = S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}$

So if $\left\{ \begin{array}{l} \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$

Now, we're in rest frame, so $p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Now, boost to frame where particle has velocity...

$\vec{v} = v \cdot \vec{z}$. Let $\tanh(\eta) = \frac{v}{c}$. "rapidity"

Then
$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = p^\mu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh(\eta) \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ 0 \\ 0 \\ m\sinh\eta \end{pmatrix}$$

(infinitesimal $\Lambda_{\frac{1}{2}}$)

$\Lambda_{\frac{1}{2}} \rightarrow$ just the Lorentz transform.

$\rightarrow$ In this frame, $\boxed{\begin{array}{l} E = m\cosh\eta \\ p^3 = m\sinh\eta \end{array}}$

Now, apply the same boost to $u(p)$...

$$u(p) = \Lambda_{\frac{1}{2}} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad (\Lambda_{\frac{1}{2}}) = \exp\left(\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

$$= \exp\left(\frac{-1}{2}\eta\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right) \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

S^{03} $\quad -i/\ (\sigma^3 \quad \sigma^3)$

So, infinitesimally ...

$$\exp\left\{\frac{-1}{2}\eta\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right\} \simeq \begin{pmatrix}\cosh\eta/2 - \sinh\eta/2\,\sigma^3 & 0\\ 0 & \cosh\eta/2 + \sinh\frac{\eta}{2}\sigma^3\end{pmatrix}$$

So that

$$u(p) \simeq \sqrt{m}\begin{pmatrix}(\cosh\eta/2 - \sinh\eta/2\,\sigma^3)\,\xi\\ (\cosh\eta/2 + \sinh\eta/2\,\sigma^3)\,\xi\end{pmatrix} \ldots$$

Simplify ... note that

$$\left(\cosh\eta/2 - \sinh\eta/2\,\sigma^3\right)^2 = \ldots$$

$$= \cosh\eta - \sinh\eta\,\sigma^3$$

$$= E/m - p^3/m\,\sigma^3 = \frac{p\cdot\sigma}{m}$$

$$= \frac{p_\mu\sigma^\mu}{m} \quad \text{where } \sigma^\mu = (1, \vec{\sigma})$$

$p_\mu\sigma^\mu/m$

So ...

$$\left(\cosh\eta/2 - \sinh\eta/2\,\sigma^3\right) = \sqrt{\frac{p\cdot\sigma}{m}}$$

&

$$\left(\cosh\eta/2 + \sinh\eta/2\,\sigma^3\right) = \sqrt{\frac{p\cdot\bar{\sigma}}{m}}$$

So ...

$$\boxed{u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi\\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix}} \longrightarrow \text{compact + valid for any arbitrary direction of } \vec{p}.$$

$\hookrightarrow$ **Fact** $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0)^2 - \vec{p}^{\,2} = p^2 = m^2.$

Now, back to example...

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow \quad p\cdot\sigma = \cdots = \begin{pmatrix} E-p^3 & 0 \\ 0 & E+p^3 \end{pmatrix}$$

so $$\sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}$$

and $$\sqrt{p\cdot\bar\sigma} = \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}$$

pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then (spin $-\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

In the massless limit, $E \to p^3$ $\left(E^2 = \sqrt{m^2c^4 + (p^3)^2}\right)$

$$\Rightarrow \boxed{\begin{array}{ll} u(p) = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} & \text{spin } \frac{1}{2} \\ u(p) = \begin{pmatrix} \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} & \text{spin } -\frac{1}{2} \end{array}}$$

These states: $u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $u(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are eigenstates of the helicity operator

$$\boxed{h \equiv \hat{p}\cdot\vec{S} = \sum_i \frac{1}{2}\hat{p}^i\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}} = \frac{1}{2}\hat{p}\cdot\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

When $\begin{cases} h = \frac{1}{2} \Rightarrow \text{call Right-handed} \\ h = \frac{-1}{2} \Rightarrow \text{call Left-handed}\end{cases}$

Note: Dirac helicity is frame-dependent... (for massive particle). — since can boost so that momentum is in the opposite direction.

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = i(\bar{\sigma}\cdot\partial)\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = i(\sigma\cdot\partial)\psi_R = 0\end{cases}$$

Plug $\psi = u(p)e^{-ip\cdot x}$ in, $\partial_0 \longrightarrow -iE$

$\vec{\nabla} \longrightarrow i\vec{p}$.

So, with $m = 0$, $\vec{p} = E\hat{p}$.

$$\Rightarrow \text{get} \begin{cases} (E + E\underbrace{\hat{p}\cdot\sigma}_{})\psi_L = 0 \Rightarrow (E)(1+2h)\psi_L = 0 \Rightarrow h = \frac{-1}{2} \\ (E - E\hat{p}\cdot\sigma)\psi_R = 0 \Rightarrow (E)(1-2h)\psi_R = 0 \longrightarrow h = \frac{1}{2}\end{cases}$$

$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}$, as expected

//

Recap... $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 > 0)$ → positive frequency

$\rightarrow u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$ → spinor.

when $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 < 0)$ → negative frequency

$\Rightarrow (u(p)) = \ldots = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$

//

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = \left(\xi^\dagger \sqrt{p\cdot\sigma}, \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)\cdot\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger \left[(p\cdot\sigma) + (p\cdot\bar\sigma)\right]\xi$$

→ depends on p!

$$\Rightarrow u^\dagger u = 2E_p\, \xi^\dagger\xi \; \cancel{\quad}$$

→ ~~~~ $u^\dagger u$ is not a Lorentz-inv scalar. just like $\psi^\dagger\psi$.

$\Rightarrow$ to make one such Lorentz-inv scalar, define

$$\boxed{\bar u(p) = u^\dagger(p)\gamma^0} \Rightarrow \boxed{\bar u u = 2m\,\xi^\dagger\xi}$$ Lorentz-inv (indep of $\vec p$)

So, with ~~...~~ $\bar{u}u = u^\dagger \gamma^0 u = 2m \underbrace{\xi^\dagger \xi}_{1} = 2m$

$\to$ convenient to choose ONB spinors. ξ^1 & ξ^2.

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^s \end{pmatrix} \qquad s = 1,2}$$

Normalize:

$$\bar{u}^r(p)\, u^s(p) = 2m\,\delta^{rs} \Leftrightarrow u^{r\dagger}(p)\, u^s(p) = 2E_p\, \delta^{rs}$$

For the negative-freq solns, we get

$$\bar{v}^r(p)\, v^s(p) = -2m\,\delta^{rs} \Leftrightarrow v^{r\dagger}(p)\, v^s(p) = +2E_p\, \delta^{rs}$$

and

v, u are orthogonal to each other...

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$$

$\nparallel$

Finally, talk about $\boxed{\text{spin sums}}$

$\to$ useful when evaluating Feynman diagrams.

$\to$ when we need to sum all spin-$\frac{1}{2}$ polarization

Since $\{\xi^s\}$ form an ONB, ...

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\cdot\sigma} \right)$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

"completeness" $\longrightarrow$

$$= \begin{pmatrix} p\cdot\sigma & m \\ m & p\cdot\bar{\sigma} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$$\left(\begin{aligned} \sqrt{p\cdot\bar{\sigma}}\,\sqrt{p\cdot\sigma} &= \cancel{\tfrac{1}{2}\left((p\cdot\bar{\sigma})+(p\cdot\sigma)\right)^2 - p\cdot\sigma - p\cdot\bar{\sigma}} \\ &= \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} = \sqrt{\left((p^0,\vec{p})\cdot(1,-\vec{\sigma})\right)\left((p^0,\vec{p})\cdot(1,\vec{\sigma})\right)} \\ &= \sqrt{(p^0)^2 - p^2} = m. \end{aligned} \right)$$

So,

$$\boxed{\begin{aligned} \sum_{s=1,2} u^s(p)\,\bar{u}^s(p) &= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \overbrace{p\cdot\gamma}^{\not{p}} + m I \\ \sum_{s=1,2} v^s(p)\,\bar{v}^s(p) &= \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = \underbrace{p\cdot\gamma}_{\not{p}} - m I \end{aligned}}$$

{ Feynman's slash notation

→ The combo $\gamma \cdot p$ occur so often that Feynman introduced the notation:

$$\boxed{\not p \equiv \gamma^\mu p_\mu = p_\mu \gamma^\mu}$$

~~[illegible] Matrices & Dirac Field Bilinears~~

Exercise

Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L. The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

$$m = \text{Majorana mass.}$$

(a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.

(b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

(a) In general, infinitesimal Lorentz transforms on ψ_L has the form

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

→ Rotation has the form:

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

$$\Rightarrow \sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L(x) \quad \text{(notes)}$$

Lorentz transform:

$$\psi_L(x) \rightarrow \Lambda_{\frac{1}{2}}\psi_L(\Lambda^{-1}x)$$

$$\partial_\mu\psi_L(x) \longrightarrow (\Lambda^{-1})^\alpha_{\ \mu}\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

→ put these together...

$$\psi_L(x) \rightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(\Lambda^{-1}x)$$

$$\sigma^2\psi_L^*(x) \rightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

$$\rightarrow \quad -im\sigma^2\psi_L^*(x) \rightarrow -im\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

Next, $i\bar{\sigma}\cdot\partial\psi_L(x) = i\bar{\sigma}^\mu\partial_\mu\psi_L(x)$

$$\Rightarrow i\bar{\sigma}^\mu(\Lambda^{-1})^\alpha_{\ \mu}\,\partial_\alpha\psi_L(\Lambda^{-1}x)\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= i\bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)(\Lambda^{-1})^\alpha_{\ \mu}\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

use trick: multiply:

$$1 \equiv \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right) \quad \text{(rot} \times \text{inv\_rot)}$$

$$\Rightarrow \; \to i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\overbrace{\left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)}$$

$$\times \; (\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\boxed{?}\,(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

what is $\boxed{?}$

$$\to \boxed{?} = \left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\vec{\sigma}/2\right)$$

$$\simeq \bar{\sigma}^{\mu} + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\bar{\sigma}^{\mu} - i\vec{\theta}\cdot\bar{\sigma}^{\mu}\frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^{\mu} - \frac{i}{2}\vec{\theta}\left[\bar{\sigma}^{\mu},\vec{\sigma}\right]$$

can show that

$$= \bar{\sigma}^{\mu} - i\theta\left[J_4^{ij}\right]^{\mu}_{\ \nu}\bar{\sigma}^{\nu}$$

$$\downarrow$$

$$i\left(g^{i\mu}\delta^{j}_{\nu} - g^{j\mu}\delta^{i}_{\nu}\right)$$

$$\Rightarrow \boxed{?} = (\Lambda_4)^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu} \;\to\; \bar{\sigma}^{\mu} \text{ transforms like 4-vector}$$

$$\Rightarrow \; i\bar{\sigma}\cdot\partial\Psi_L(x) \Rightarrow i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Lambda^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu}(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\delta^{\alpha}_{\nu}\,\bar{\sigma}^{\nu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\nu}\,\partial_{\nu}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}\cdot\partial\Psi_L(\Lambda^{-1}x)$$

✓

$$\rightarrow \quad i\bar{\sigma}\cdot\partial\psi_L(x) - im\,\sigma^2\psi_L^*(x) = 0$$

$\rightarrow$ leads to infinitesimal rotation...

$$\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Big\{ \underbrace{i\bar{\sigma}\cdot\partial\psi_L(\Lambda^{-1}x) - im\,\sigma^2\psi_L^*(\Lambda^{-1}x)}_{=0} \Big\} = 0$$

$\Rightarrow$ done! So Majorana eqn is invariant under infinitesimal rotations...

(b) <u>Boosts</u> (proceed in a similar way...)

<u>Key</u>

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= \bar{\sigma}^\mu - \frac{1}{2}\vec{\beta}\left\{\bar{\sigma}^\mu, \vec{\sigma}\right\}$$

$$\rightarrow \quad \bar{\sigma}^\mu \longrightarrow \bar{\sigma}^\mu - i\beta\left[J_4^{0i}\right]^\mu{}_\nu\,\bar{\sigma}^\nu$$

$$\rightarrow \quad \left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = (\Lambda)^\mu{}_\nu\,\bar{\sigma}^\nu \quad \text{as before}$$

so plug this back into the original eqn.

Sep 28, 2020

Dirac Matrices & Dirac Field Bilinears