Def: A matrix Lie group is a closed subgroup $G \leq GL(n,\mathbb{C})$ for some $n \in \mathbb{N}$ (closed w.r.t. the topology induced from $M_n(\mathbb{C})$)

→ using operator norm

$$\|A\| = \sup\left\{ \frac{\|Ax\|}{\|x\|} \,\middle|\, x \in \mathbb{C}^n \setminus \{0\} \right\}$$

examples

- $GL(n,\mathbb{C})$ general linear group over $\mathbb{C}$
- $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid \det A = 1\}$ special linear group over $\mathbb{C}$
- $GL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{C}) \mid \bar{A} - A = 0\}$ general linear group over $\mathbb{R}$
- $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}$ special linear group over $\mathbb{R}$
- $O(n) = \{A \in GL(n,\mathbb{R}) \mid A^t A = \mathbb{1}\}$ orthogonal group
- $SO(n) = \{A \in O(n) \mid \det A = 1\}$ special orthogonal group
- $U(n) = \{A \in GL(n,\mathbb{C}) \mid A^* A = \mathbb{1}\}$ unitary group
- $SU(n) = \{A \in U(n) \mid \det A = 1\}$ special unitary group

real Lie groups despite having complex matrices!

Some familiar examples:

- $U(1) = \{z \in \mathbb{C} \setminus \{0\} \mid |z|^2 = 1\}$ circle group
- $SO(2) = \left\{ \begin{pmatrix} \cos\vartheta & \sin\vartheta \\ -\sin\vartheta & \cos\vartheta \end{pmatrix} \,\middle|\, \vartheta \in [0, 2\pi) \right\}$ 2D rotation group

Def: A real or complex Lie algebra is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ with an operation $[\cdot,\cdot]: V \times V \to V$ (Lie bracket) satisfying

- $[\alpha x + \beta y, z] = \alpha[x,z] + \beta[y,z]$
- $[z, \alpha x + \beta y] = \alpha[z,x] + \beta[z,y]$
- $[y,x] = -[x,y]$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Def: Let $G \leq GL(n,\mathbb{C})$ be a MLG. The associated Lie algebra is the set $Lie(G) = \{X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in G \ \forall \varepsilon \in \mathbb{R}\}$ with Lie bracket $[X,Y] = XY - YX$

↳ matrix exponential $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$

Note: $Lie(G)$ is usually denoted by $\mathfrak{g}$

proof can be found in "extra" folder on google drive

Some more notions about MLGs:

- $G \leq GL(n, \mathbb{C})$ is called compact if it is closed and bounded (w.r.t. to $M_n(\mathbb{C})$)

  $\rightarrow$ $O(n)$, $SO(n)$, $U(n)$, $SU(n)$ are compact

- G connected if for every $g \in G$ there is a continuous path connecting it to the identity ($\gamma : [0,1] \rightarrow G$ with $\gamma(0) = e$, $\gamma(1) = g$)

- G simply connected if connected and every loop ($\gamma : [0,1] \rightarrow G$ with $\gamma(0) = \gamma(1)$) can be shrunk to a point ("no holes")

  $\rightarrow$ $U(1)$ is not simply connected

- if $X \in Lie(G)$ then $\{e^{\varepsilon X} \mid \varepsilon \in \mathbb{R}\}$ is a one-parameter subgroup of G.
  In fact, <u>every</u> (smooth) one-parameter subgroup of G is of this form.
  $\rightarrow$ X <u>generates</u> the subgroup $\{e^{\varepsilon X} \mid \varepsilon \in \mathbb{R}\}$

- if G is connected, every group element $g \in G$ can be written as

  $g = e^{X_1} e^{X_2} \cdots e^{X_k}$ with $X_i \in G$

  $\rightarrow$ G is generated by $Lie(G)$

- MLG homomorphism: continuous group homomorphism $\varphi : G \rightarrow H$

- MLG isomorphism: group isomorphism $\varphi : G \rightarrow H$ with φ and φ^{-1} continuous

<u>ex:</u> $\varphi : z \in U(1) \mapsto \begin{pmatrix} Re(z) & Im(z) \\ -Im(z) & Re(z) \end{pmatrix} \in SO(2)$ isomorphism

Representations

or representation of G

Let G be a MLG. A complex vector space V is a G-module if there is a continuous action of G on V → $g|\psi\rangle$ denotes g acting on $|\psi\rangle$

- $g(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha g|\psi\rangle + \beta g|\varphi\rangle$
- $g(h|\psi\rangle) = (gh)|\psi\rangle$

the map $G \times V \to V$ is continuous
$(g, |\psi\rangle) \mapsto g|\psi\rangle$

example: $V = \text{span}\{|n\rangle\}$ with action of $U(1)$:

$$e^{i\theta} \triangleright |n\rangle = e^{in\theta}|n\rangle \qquad n \in \mathbb{N}$$

symbol for "acting on", for when there is confusion.

- A G-module V is unitary if it has an inner product and $\overline{\langle\varphi|g|\psi\rangle} = \langle\psi|g^{-1}|\varphi\rangle$ with abuse of notation, $g^{\dagger} = g^{-1}$

- V is irreducible if the only invariant subspaces ($W \subseteq V$ closed such that $g|\psi\rangle \in W \ \forall g \in G, \forall|\psi\rangle \in W$) are V and $\{0\}$

Lie algebras:

Let $\mathfrak{g}$ be a Lie algebra. A complex vector space V is a $\mathfrak{g}$-module if there is an action ~~action~~ of $\mathfrak{g}$ on V

→ $X|\psi\rangle$ denotes X acting on $|\psi\rangle$

- $X(\alpha|\psi\rangle+\beta|\varphi\rangle)=\alpha X|\psi\rangle+\beta X|\varphi\rangle$
- $(\alpha X+\beta Y)|\psi\rangle=\alpha X|\psi\rangle+\beta Y|\psi\rangle$
- $[X,Y]|\psi\rangle=X(Y|\psi\rangle)-Y(X|\psi\rangle)$

V is unitary if $\overline{\langle\psi|X|\varphi\rangle}=-\langle\varphi|X|\psi\rangle$ "$X^\dagger=-X$"

---

if V is a G-module, we can make it into a $\mathrm{Lie}(G)$-module by defining

$$X|\psi\rangle=\frac{d}{d\varepsilon}e^{\varepsilon X}|\psi\rangle\Big|_{\varepsilon=0}$$

<u>example:</u> $\underline{u}(1)=\{i\theta\mid\theta\in\mathbb{R}\}=\mathrm{span}\{X\}$, $X=i$

using $V=\mathrm{span}\{|n\rangle\}$ with $e^{i\theta}\triangleright|n\rangle=e^{in\theta}|n\rangle$

we get $X|n\rangle=\frac{d}{d\varepsilon}e^{in\varepsilon}|n\rangle\Big|_{\varepsilon=0}=in|n\rangle$

Can we go from a $\mathrm{Lie}(G)$-module to a G-module?

$V=\mathrm{span}\{|\alpha\rangle\}$ with $X|\alpha\rangle=i\alpha|\alpha\rangle$, $\alpha\in\mathbb{R}$ is a unitary $u(1)$-module

if we define $e^{i\theta}\triangleright|\alpha\rangle=e^{\theta X}|\alpha\rangle=e^{i\alpha\theta}|\alpha\rangle$ for $\alpha=1/2$

we have $e^{i2\pi}\triangleright|\alpha\rangle=e^{i\pi}|\alpha\rangle=-|\alpha\rangle$ instead of $|\alpha\rangle$!

We'll revisit this next lecture.