Due: **Monday May 9 by 5pm**

Reading on linear response and disordered electrons: "Measurements and correlation functions" by P C Martin, David Tong's notes, Yoseph Imry "Introduction to Mesoscopic Physics", Jorgen Rammer "Quantum Transport Theory" Ch. 2, Ch. 3 and 8.512 class notes Linear response and Disordered electrons (active links)

The fluctuation-dissipation relation. Anderson localization.

1. [*Scattering near a phase transition (critical opalescence)*]

(a) [10pts] As discussed in class, when a probe, be it neutron, photon, etc., couples to some variable of the system at $\mathbf{r}$ and t, characterized by an operator $\phi(\mathbf{r}, t)$, the differential scattering crosssection is proportional to

$$\frac{d\sigma}{d\omega do} \sim S(q,\omega) = \int d^3r \int_{-\infty}^{\infty} dt e^{i\omega t - i\mathbf{qr}} \langle \phi(\mathbf{r}, t)\phi(0,0)\rangle \tag{1}$$

where $\langle ...\rangle$ denotes ensemble averaging. For this case, derive the fluctuation-dissipation theorem which tells us that the quantity $S(q,\omega)$ is proportional to the susceptibility associated with the quantity ϕ,

$$S(q,\omega) = -2[1 - e^{-\beta\hbar\omega}]^{-1}\chi''_\phi(\mathbf{q},\omega).$$

The coefficient of proportionality $[1 - e^{-\beta\hbar\omega}]^{-1}$ in the classical limit $\hbar\omega \ll k_BT$, varies as ω^{-1}. For the crosssection integrated over frequencies this gives $\frac{d\sigma}{do} \sim \int \frac{d\omega\chi''_\phi(\mathbf{q},\omega)}{\omega}$. Use the Kramers-Kronig relations to relate $\frac{d\sigma}{do}$ with the static susceptibility and show that, in this case, $\frac{d\sigma}{do} \sim \chi'_\phi(\mathbf{q}, 0)$.

(b) [10pts] Landau theory of phase transitions predicts that near the transition the static response function $\chi(\mathbf{q}, 0)$ is proportional to $\frac{1}{\mathbf{q}^2+\alpha(T-T_c)}$ with the correlation length $\xi = (\alpha(T-T_c))^{-1/2}$ diverging near the transition as $(T-T_c)^{-1/2}$. In the case of light scattering the maximum possible momentum transfer $\mathbf{Q} \approx 4\pi/\lambda_{\text{light}}$ is small compared to typical ξ values. Estimate $\chi(\mathbf{Q}, 0)$ for such Q and argue that the total scattering intensity diverges near the transition. This explanation of critical opalescence was first proposed by Ornstein and Zernike in 1914.

(c) [10pts] It is also interesting to consider dynamical response function associated with a phonon mode that softens near a phase transition. Such a response is generally represented by a harmonic oscillator model which gives a Lorentzian expression

$$\chi(\omega) \sim \frac{1}{\omega_0^2 - \omega^2 + 2i\omega\Gamma}$$

where Γ is damping. Generalize the argument of part b) to show that a soft mode ($\omega_0 \to 0$ at $T \to T_c$) also leads to critical opalescence. If the transition is driven by soft acoustic phonon then the total scattering associated with acoustic phonons, i.e. Brillouin scattering, should diverge. This behavior is illustrated in Fig.1.

2. [*Einstein relation for Brownian motion*]

Consider a Brownian particle in a fluid which executes a random walk due to thermal kicks and is also dragged by an external force $\mathbf{f} = -\nabla U(\mathbf{x})$. The equations of motion for such a particle read

$$\mathbf{v}(t) = \mu\mathbf{f} + \delta\mathbf{v}(t),$$

where μ is the drag coefficient, or mobility. The velocity fluctuations $\delta\mathbf{v}(t)$, describing thermal kicks, can be treated as delta-correlated: $\langle\delta\mathbf{v}(t)\delta\mathbf{v}(t')\rangle_T \to 0$ for the time separations $t-t'$ exceeding a microscopic time scale. In class we argued that the particle probability distribution obeys the diffusion equation with a drag term

$$\partial_t p(\mathbf{x}, t) = D\nabla^2 p(\mathbf{x}, t) - \nabla \cdot (\mu\mathbf{f}p(\mathbf{x}, t)), \quad D = \frac{1}{3}\int_0^\infty \langle\delta\mathbf{v}(t)\cdot\delta\mathbf{v}(0)\rangle_T dt \tag{2}$$

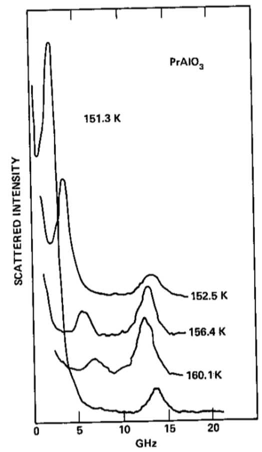

FIGURE 1. Transverse mode Brillouin spectra of PrAlO$_3$ approaching the cooperative Jahn-Teller transition at $T = 151K$. The soft mode propagates in the [101] direction with a polarization along [101]. The mode at 13 GHz is a TA mode which does not show softening. After P. Fleury, P Lazan, and L Van Uitert, Phys. Rev. Lett. 33, 492 (1974).

(a) [10pts] Find the steady-state solution of Eq.2 and determine the relation between D and μ for which this solution coincides with the Boltzmann probability distribution $p(\mathbf{x}) \sim \exp(-\beta U(\mathbf{x}))$ where $\beta = 1/k_B T$.

(b) [10pts] Derive the relation between μ and D you found in part a) from the Fluctuation-Dissipation Theorem.

3. [**Harper equation, duality and Anderson transition in 1D**]

In a tight-binding problem with a quasiperiodic potential,

$$\epsilon\psi_n = 2t' \cos(2\pi\omega n + \theta)\psi_n + t\psi_{n-1} + t\psi_{n+1} \tag{3}$$

the eigenstates can be either localized or delocalized depending on the ratio of t and t'. There is an Anderson transition when $t = t'$. To understand the origin of this behavior, we define a duality transformation connecting the real and reciprocal space as follows.

Consider Fourier-transformed wavefunction, $\psi_n = \int_{-\pi}^{\pi} \psi(p) e^{ipn} \frac{dp}{2\pi}$ and rewrite Schroedinger equation for ψ_p. Taking into account that a shift $n \to n \pm 1$ translates into multiplication by a phase factor $\psi(p) \to e^{\pm p}\psi(p)$, and conversely, the Fourier transform of $2\cos(2\pi\omega n + \theta)\psi_n$ is $e^{i\theta}\psi(p + 2\pi\omega) + e^{-i\theta}\psi(p - 2\pi\omega)$, we can write

$$\epsilon\psi(p) = 2t\cos(p)\psi(p) + t'\psi(p + 2\pi\omega) + t'\psi(p - 2\pi\omega) \tag{4}$$

where without loss of generality we set $\theta = 0$. After rescaling, $p = 2\pi\omega\tilde{p}$ this gives

$$\epsilon\psi(\tilde{p}) = 2t\cos(2\pi\omega\tilde{p})\psi(\tilde{p}) + t'\psi(\tilde{p} + 1) + t'\psi(\tilde{p} - 1) \tag{5}$$

(a) [10 pts] From the above, argue that there is localization when $t' > t$ and delocalization when $t' < t$.

(b) [10 pts] Confirm your conclusions by solving the problem numerically by using direct diagonalization of the Hamiltonian for a finite system as in Question 1. Please include your code in your submission.