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Course: **8.370 - QC**
Problem set: **#5**
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Collaborators/References: Prof. Shor (ideas for 3d), Christina Yu (density matrix approach for 4a)

## 1. Teleportation

Suppose the qubit Alice wants to teleport has state $|\psi\rangle_A = \alpha |0\rangle + \beta |1\rangle$. If Alice and Bob start with $(1/\sqrt{2})(|01\rangle_{AB} - |10\rangle_{AB})$ then Bob's states for each of Alice's measurement outcome and Bob's required unitary transformations are

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \rightarrow |\psi\rangle_B = +\alpha |1\rangle - \beta |0\rangle \implies \sigma_y$$

$$\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \rightarrow |\psi\rangle_B = -\alpha |0\rangle + \beta |1\rangle \implies \sigma_z$$

$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \rightarrow |\psi\rangle_B = +\alpha |1\rangle + \beta |0\rangle \implies \sigma_x$$

$$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \rightarrow |\psi\rangle_B = +\alpha |0\rangle + \beta |1\rangle \implies \mathrm{id}$$

Of course, another way to do this problem is to notice that the new EPR pair which Alice and Bob share is the original EPR pair transformed by $\sigma_y$. Therefore, Bob needs to apply a $\sigma_y$ to his original unitary transformations to obtain the new unitary transformations.

## 2. Deutsch-Jozsa

In the Deutsch-Jozsa algorithm, the probability for a state $|k\rangle$ to be measured is

$$\left| \frac{1}{2^n} \sum_{j=0}^{2^n-1} (-1)^{f(j)}(-1)^{j \cdot k} \right|^2$$

In the algorithm, we are interested in the probability of measuring $k = 0$, which is

$$\mathrm{Pr} = \left| \frac{1}{2^n} \sum_{j=0}^{2^n-1} (-1)^{f(j)} \right|^2.$$

For $f(j)$ constant, this quantity is 1. For $f(j)$ balanced, this quantity is 0.

In the case that $f(j) = 0$ for $r$ times and $f(j) = 1$ for $s$ times, the probability that the algorithm tells us that $f$ is constant is the probability of measuring $|0\rangle$, which is:

$$\mathrm{Pr} = \left| \frac{1}{2^n}(r - s) \right|^2 = \boxed{\frac{(r-s)^2}{2^{2n}}}$$

### 3. Two qubits

(a) It suffices to check that the states are mutually orthogonal:

$$\langle 00|01\rangle = \langle 0|0\rangle \langle 0|1\rangle = 0$$
$$\langle 00|1+\rangle = \langle 0|1\rangle \langle 0|+\rangle = 0$$
$$\langle 00|1-\rangle = \langle 0|1\rangle \langle 0|-\rangle = 0$$
$$\langle 01|1+\rangle = \langle 0|1\rangle \langle 1|+\rangle = 0$$
$$\langle 01|1-\rangle = \langle 1|1\rangle \langle 1|-\rangle = 0$$
$$\langle 1+|1-\rangle = \langle 1|1\rangle \langle +|-\rangle = 0.$$

Since the states also span the 4-dimensional Hilbert space for states of a 2-qubit system, the states provided form an orthonormal basis.

(b) Since each of the given states are product states, Alice and Bob can make sequential measurements to find what state their qubits are in originally. Consider the following procedure: Alice makes a measurement in the $\{|0\rangle, |1\rangle\}$ basis. If she sees $|0\rangle$, she calls Bob and tells him to measure his qubit in the $\{|0\rangle, |1\rangle\}$ basis. Else if Alice sees $|1\rangle$, then she calls and tells Bob to measure his qubit in the $\{|+\rangle, |-\rangle\}$ basis. They will be able to unambiguously identify the state they had in the beginning.

(c) Alice can always tell with certainty the state of her qubit by measuring in the $\{|0\rangle, |1\rangle\}$ basis. However, Bob cannot, simply because $|\pm\rangle$ is a superposition of $|0\rangle$ and $|1\rangle$ and vice versa. There is no measurement basis with which Bob (alone) can use to unambiguously identify the state of his qubit. One can also answer this question using density matrices, which I think is less convenient, as one has to assign probabilities to the possibilities given in the problem, etc.

(d) Yes. The strategy is as follows: Bob applies a CNOT on the two qubits that he has. The control qubit is that of his half of the EPR pair, and the target qubit is his qubit. Alice measures her qubit in the $\{|0\rangle, |1\rangle\}$ basis. If she sees that she has a $|0\rangle$ then she measures her half of the EPR pair in the $\{|0\rangle, |1\rangle\}$ basis. Else, she measures her half of the EPR pair in the $\{|+\rangle, |-\rangle\}$ basis. Meanwhile, Bob measures his qubits in the $\{|0\rangle, |1\rangle\}$ basis and his half of the EPR pair in the $\{|+\rangle, |-\rangle\}$ basis. To see how this works, we first consider the starting state. The four possible initial states of the problem are (here I am writing everything without normalization... I'm also dropping the kets to clarify everything):

$$|a\rangle = |0_A 0_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (00)_A(00)_B + (01)_A(01)_B$$
$$|b\rangle = |0_A 1_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (00)_A(10)_B + (01)_A(11)_B$$
$$|c\rangle = |1_A +_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (10)_A(+0)_B + (11)_A(+1)_B$$
$$|d\rangle = |1_A -_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (10)_A(-0)_B + (11)_A(-1)_B$$

Here, the Alice's second qubit comes from her half of the EPR pair. Bob's second qubit comes from his half of the EPR pair. Now, Bob performs a CNOT where the control qubit is from his half of the EPR pair. After this application, the possible states become:

$$|a'\rangle = |0_A 0_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (00)_A(00)_B + (01)_A(11)_B$$
$$|b'\rangle = |0_A 1_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (00)_A(10)_B + (01)_A(01)_B$$
$$|c'\rangle = |1_A +_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (10)_A(+0)_B + (11)_A(+1)_B$$
$$|d'\rangle = |1_A -_B\rangle (|0_A 0_B\rangle + |1_A 1_B\rangle) = (10)_A(-0)_B - (11)_A(-1)_B$$

The table below shows the Bob's resulting states after Alice makes her measurement:

|  | $|a'\rangle$ | $|b'\rangle$ | $|c'\rangle$ | $|d'\rangle$ |
|---|---|---|---|---|
| $\langle 00|_A$ | $|00\rangle_B$ | $|10\rangle_B$ | $\varnothing$ | $\varnothing$ |
| $\langle 01|_A$ | $|11\rangle_B$ | $|01\rangle_B$ | $\varnothing$ | $\varnothing$ |
| $\langle 1+|_A$ | $\varnothing$ | $\varnothing$ | $|++\rangle_B$ | $|--\rangle_B$ |
| $\langle 1-|_A$ | $\varnothing$ | $\varnothing$ | $|+-\rangle_B$ | $|-+\rangle_B$ |

Notice that if Bob measures his first qubit (which is the qubit he was given originally) in the $\{|0\rangle, |1\rangle\}$ basis and his second qubit (which is his half of the EPR pair) in the $\{|+\rangle, |-\rangle\}$ basis, then after communicating with Alice after the measurement, they can work out which of the four possibilities $(|a\rangle, |b\rangle, |c\rangle, |d\rangle)$ they were given originally, based on the this table.

For example, if Bob measures $|0\rangle$ on his first qubit and Alice measures $|01\rangle$, then they know they were given $|0\rangle |1\rangle$, which becomes $|b\rangle$ after we concatenate it with the EPR pair.

## 4. GHZ

Alice, Bob, and Charlie hold the GHZ state:

$$\frac{1}{\sqrt{2}} (|000\rangle_{ABC} + |111\rangle_{ABC})$$

(a) In order to arrange so that Alice and Charlie share the Bell pair $(1/\sqrt{2})(|00\rangle + |11\rangle)$, Alice, Bob, and Charlie must somehow "factor" Bob's qubit out of the GHZ state. This requires a controlled-unitary gate or a classical controlled-operation whose control bit is Alice's or Charlie's qubit and target bit is Bob's. This is not possible if the trio are not allowed to communicate. One can also answer this part using density matrices as follows:

The density matrix associated with the GHZ state is

$$\rho_{ACB} = |GHZ\rangle \langle GHZ| = \frac{1}{2} (|000\rangle \langle 000| + |000\rangle \langle 111| + |111\rangle \langle 000| + |111\rangle \langle 111|).$$

Now, the reduced density matrix representing Alice's and Charlie's qubits is given by tracing out Bob's part from $\rho_{ACB}$:

$$\rho_{AC} = \langle 0|_B \rho_{ACB} |0\rangle_B + \langle 1|_B \rho_{ACB} |1\rangle_B = \frac{1}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|)$$

Notice that this density matrix is associated with a *mixed state*, which is a classical probability distribution of pure states. It is for this reason that Alice and Charlie cannot share an EPR pair if they cannot communicate with Bob.

(b) For Alice and Charlie to have the Bell pair state in the problem, Bob first applies a Hadamard on his qubit, so that the system becomes

$$\frac{1}{\sqrt{2}} (|0 + 0\rangle + |1 - 1\rangle) = \frac{1}{2} (|000\rangle + |010\rangle + |101\rangle - |111\rangle).$$

Now Bob measures his qubit in the $\{|0\rangle, |1\rangle\}$ basis. The system after this measurement is in one of the two states:

$$\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle)_{AC} |0\rangle_B \qquad \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle)_{AC} |1\rangle_B$$

with equal probability. If Bob measures $|0\rangle$, he does nothing, Alice and Charlie already share the correct EPR state. If Bob measures $|1\rangle$, then he lets either Charlie or Alice knows so that one of them applies a $\sigma_z$ on their qubit, so that $(|00\rangle - |11\rangle)_{AC} \rightarrow (|00\rangle + |11\rangle)_{AC}$.