COMPLEX ANALYSIS

- A Quick Guide -

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Preface

Greetings,

Complex Analysis: A Quick Guide to is compiled based on my MA352: Complex Analysis notes with professor Evan Randles. This guide is almost entirely based on *Complex Variables and Applications, Eighth edition* by Churchill and Brown.

Enjoy!

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Chapter 1

COMPLEX NUMBERS

1.1 Sums and Products

Let $z \in \mathbb{C}$, it is customary to write

$$z = x + iy = (x, y) \tag{1.1}$$

where

$$x = \mathrm{Re}(z) \in \mathbb{R} \qquad y = \mathrm{Im}(z) \in \mathbb{R}. \tag{1.2}$$

For $z_1, z_2 \in \mathbb{C}$,

$$z_1 = z_2 \iff \mathrm{Re}(z_1) = \mathrm{Re}(z_2) \wedge \mathrm{Im}(z_1) = \mathrm{Im}(z_2). \tag{1.3}$$

Addition works as we expect

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \tag{1.4}$$

So does multiplication

$$z_1 z_2 = (x_1, y_1)(x_2, y_2) = (x_1 x_2 - y - 1y_2, y_1 x_2 + x_1 y_2). \tag{1.5}$$

Of course,

$$i^2 = -1 = (-1, 0). \tag{1.6}$$

1.2 Basic Algebraic Properties

It is easy to see that complex number multiplication and addition are both commutative and associative:

$$z_1 + z_2 = z_2 + z_1, \qquad z_1 z_2 = z_2 z_1 \tag{1.7}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1 z_2) z_3 = z_1 (z_2 z_3). \tag{1.8}$$

The additive identity is $0 = (0, 0)$. The multiplicative identity is $1 = (1, 0)$. For $z = (x, y) \in \mathbb{C}$, the additive inverse is

$$-z = (-x, -y). \tag{1.9}$$

For any nonzero complex number $z = (x, y)$, there exists an multiplicative inverse z^{-1} such that $zz^{-1} = z^{-1}z = 1$. We can find that

$$z^{-1} = \left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2} \right). \tag{1.10}$$

The existence of the multiplicative inverse allows us to show that if a product of two complex numbers is zero, then at least one of them is zero. And of course, if two complex numbers are nonzero, then so is their product.

Subtraction and division are defined in terms of addition and multiplication. For $z_1 = (x_1, y_2)$ and $z_2 = (x_2, y_2) \neq 0$,

$$z_1 - z_2 = (x_1 - x_2, y_1 - y_2) \tag{1.11}$$

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = (x_1, y_2) \left(\frac{x_2}{x_2^2 + y_2^2}, \frac{-y_2}{x_2^2 + y_2^2} \right) = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} \right). \tag{1.12}$$

This formula can be difficult to remember, so here's way to obtain it:

$$\frac{z_1}{z_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)}. \tag{1.13}$$

1.3 Further Properties

By the distributive law, we can show that

$$\frac{z_1 + z_2}{z_3} = (z_1 + z_2) z_3^{-1} = z_1 z_3^{-1} + z_2 z_3^{-1} = \frac{z_1}{z_3} + \frac{z_2}{z_3}. \tag{1.14}$$

Beside some other expected properties involving quotients that follow, we also have the binomial formula. If z_1, z_2 are any two nonzero complex numbers, then

$$(z_1 + z_2)^n = \sum_{k=0}^{n} \binom{n}{k} z_1^k z_2 n - k \qquad (n = 1, 2, \dots) \tag{1.15}$$

1.4 Vectors and Moduli

It is natural to associate $z = (x, y)$ to a point of a plane with coordinates (x, y). The modulus of z is defined as

$$|z| = \sqrt{x^2 + y^2}. \tag{1.16}$$

The distance between two points z_1, z_2 is the same as the modulus of $z_1 - z_2$:

$$|z_2 - z_1| = \sqrt{(x_1 - x^2)^2 + (y_1 + y_2)^2}. \tag{1.17}$$

It is easy to see that

$$|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2 \tag{1.18}$$

so that

$$\mathrm{Re}(z) \leq |\mathrm{Re}(z)| \leq |z| \tag{1.19}$$
$$\mathrm{Im}(z) \leq |\mathrm{Im}(z)| \leq |z|. \tag{1.20}$$

Next, we have the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|. \tag{1.21}$$

An immediate consequence of this inequality is another inequality:

$$|z_1 + z_2| \geq ||z_1| - |z_2||. \tag{1.22}$$

To prove this, we simply write $|z_1| = |(z_1 + z_2) - z_2|$. The triangle inequality takes care of the rest of the proof.

In summary, we have

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|. \tag{1.23}$$

The triangle inequality can be generalized by induction to sums involving any *finite* number of terms:

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|. \tag{1.24}$$

1.5 Complex Conjugates

For $z = (x, y) \in \mathbb{C}$, the complex conjugate of z, denoted $\bar{z}$, is

$$\bar{z} = (x, -y). \tag{1.25}$$

We note

$$\bar{\bar{z}} = z, \qquad |\bar{z}| = |z|. \tag{1.26}$$

We can show that

$$\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2} \tag{1.27}$$

$$\overline{z_1 z_2} = \bar{z_1}\bar{z_2} \tag{1.28}$$

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z_1}}{\bar{z_2}}, \qquad (z_2 \neq 0) \tag{1.29}$$

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2} \tag{1.30}$$

$$\mathrm{Im}(z) = \frac{z - \bar{z}}{2i} \tag{1.31}$$

$$z\bar{z} = |z|^2 \tag{1.32}$$

$$|z_1 z_2| = |z_1||z_2| \tag{1.33}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \qquad (z_2 \neq 0). \tag{1.34}$$

1.6 Exponential Form

For any nonzero complex number $z = (x, y)$, the polar form is

$$z = x + iy = r\cos\theta + ir\sin\theta, \tag{1.35}$$

where $r = |z| \geq 0$. Note that for $z = 0$, the angle θ is not defined. Each value of θ is called an argument of z, denoted $\arg(z)$. However, because $\arg(z)$ is "multiple-valued," we define the *principal value* of $\arg(z)$, $\mathrm{Arg}(z)$ as

$$\arg(z) = \mathrm{Arg}(z) + 2n\pi, \qquad (n = 0, \pm 1, \pm 2, \dots) \tag{1.36}$$

Note that when z is a negative real number, $\mathrm{Arg}(z) = \pi$, not $-\pi$.

The polar form can also be re-written in a different way using Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{1.37}$$

With this,

$$z = re^{i\theta} = |z|e^{i\theta}. \tag{1.38}$$

1.7 Products and Powers in Exponential Forms

With a simple trigonometry identity, we can show that

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1 + \theta_2)}. \tag{1.39}$$

So,

$$z_1 z_2 = (r_1 r_2)e^{i(\theta_1 + \theta_2)}. \tag{1.40}$$

Similarly,

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}. \tag{1.41}$$

It is then easy to see that for $z \neq 0$,

$$z^{-1} = \frac{1}{z} = \frac{1}{r} e^{-i\theta}. \tag{1.42}$$

And of course, we can see that

$$z^n = r^n e^{in\theta}, \qquad (n = 0, \pm 1, \pm 2, \dots). \tag{1.43}$$

This can be verified by induction.

1.8 Arguments of Products and Quotients

For $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$,

$$z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}. \tag{1.44}$$

So,

$$\arg(z_1 + z_2) = \arg(z_1) + \arg(z_2). \tag{1.45}$$

1.9 Roots of Complex Numbers

Chapter 2

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