

ULTRACOLD FERMION GAS

- A Quick Guide -

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Preface

Greetings,

This is my personal “cheatsheet” for BEC1 experiment with strongly interacting Fermi gas (ironic, I know, but there’s a lot of history behind this irony). All information relevant to what I do in the lab plus what I find interesting is accumulated here.

Enjoy!

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Chapter 1

Ideal Fermi Gas

1.1 Thermodynamics of a non-relativistic gas

For a non-relativistic gas of particles with spin s in three dimensions with $\mathcal{E}(\vec{k}) = \hbar^2 k^2 / 2m$, we have

$$\begin{aligned}\beta P_\eta &= \frac{g}{\lambda^3} f_{5/2}^\eta(z) \\ n_\eta &= \frac{g}{\lambda^3} f_{3/2}^\eta(z) \\ \epsilon_\eta &= \frac{3}{2} P_\eta\end{aligned}$$

with $g = 2s + 1$ is the spin degeneracy factor, z is the fugacity parameter $e^{\beta\mu}$, and $\eta = -1$ for fermions and $\eta = 1$ for bosons, and

$$f_m^\eta(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{dx x^{m-1}}{z^{-1}e^x - \eta}.$$

The three equations above completely describe the thermodynamics of ideal quantum gases as a function of z . To rewrite these equations in terms of the density n_η , we have to express $f_m^\eta(z)$ in terms of n_η . To do this, we have to expand $f_m^\eta(z)$ in terms of z in the high temperature limit. The details of this procedure can be found in Chapter 7 of [2].

One of the main corollaries of the expansion is that the natural dimensionless expansion parameter

$$\frac{n_\eta \lambda^3}{g}$$

appears. Here, $\lambda = \sqrt{\hbar^2 / 2\pi m k_B T}$ is the de Broglie wavelength. For example,

$$P_\eta = n_\eta k_B T \left[1 - \frac{\eta}{2^{5/2}} \left(\frac{n_\eta \lambda^3}{g} \right) + \dots \right]$$

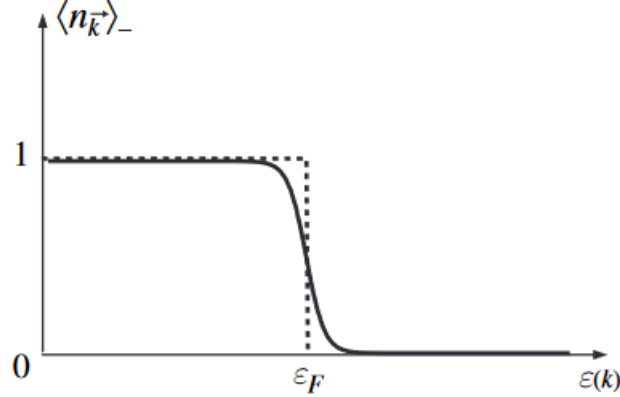
We see that quantum mechanical effects become important in the regime where $n_\eta \lambda^3 \geq g \sim 1$. This is called the **quantum degenerate limit**. Fermi and Boses gases behave very differently in this limit of high density and low temperature. In this document, we will focus mostly on fermions.

1.2 Thermodynamics of an ideal fermi gas

In the quantum degeneracy limit, fermions obey Fermi-Dirac statistics. The average occupation number as a function of momentum (or energy) is

$$\langle n(\vec{k}) \rangle = \frac{1}{e^{(\mathcal{E}(\vec{k}) - \mu)/k_B T} + 1} \xrightarrow{T \rightarrow 0} \begin{cases} 1 & \text{if } \mathcal{E} < \mu \\ 0 & \text{if } \mathcal{E} \geq \mu \end{cases}$$

Here μ is the chemical potential and $\mathcal{E}(\vec{k})$ is the energy. We see that μ is a limiting value for $\mathcal{E}(\vec{k})$ (at zero temperature), and call it the **Fermi energy** \mathcal{E}_F . At $T \rightarrow 0$, all one-particle states of energy less than \mathcal{E}_F are occupied, forming a **Fermi sea**.



For an ideal Fermi gas,

$$\mathcal{E}(\vec{k}) = \frac{\hbar^2 k^2}{2m} \implies \mathcal{E}_F := \frac{\hbar^2 k_F^2}{2m}$$

where k_F is the **Fermi wavenumber**. The Fermi wavenumber k_F can be computed from the number density as:

$$n = \int_{k < k_F} g \times \frac{d^3 \vec{k}}{(2\pi)^3} = g \int_{k < k_F} 4\pi k^2 \frac{dk}{(2\pi)^3} = \frac{g}{6\pi^2} k_F^3.$$

So,

$$\boxed{k_F = \left(\frac{6\pi^2 n}{g} \right)^{1/3} \implies \mathcal{E}_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2 n}{g} \right)^{2/3}}$$

To extract thermodynamics properties of the degenerate Fermi gas in three dimensions (untrapped), we consider the high- z limit of $f_m^-(z)$. Again, Chapter 7 of [2] shows the details of the various expansions. The result is the **Sommerfeld expansion**:

$$\lim_{z \rightarrow \infty} f_m^-(z) = \frac{(\ln z)^m}{m!} \left[1 + \frac{\pi^2}{6} \frac{m(m-1)}{(\ln z)^2} + \frac{7\pi^4}{360} \frac{m(m-1)(m-2)(m-3)}{(\ln z)^4} + \dots \right]$$

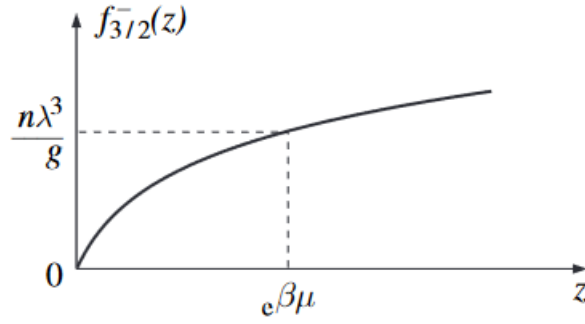
From the n_η equation in the previous section, we find that in the degenerate limit, the density and chemical potential are related by

$$\frac{n\lambda^3}{g} = f_{3/2}^-(z) = \frac{(\ln z)^{3/2}}{(3/2)!} \left[1 + \frac{\pi^2}{6} \frac{3}{2} \frac{1}{2} (\ln z)^{-2} + \dots \right] \gg 1$$

To lowest order, we have

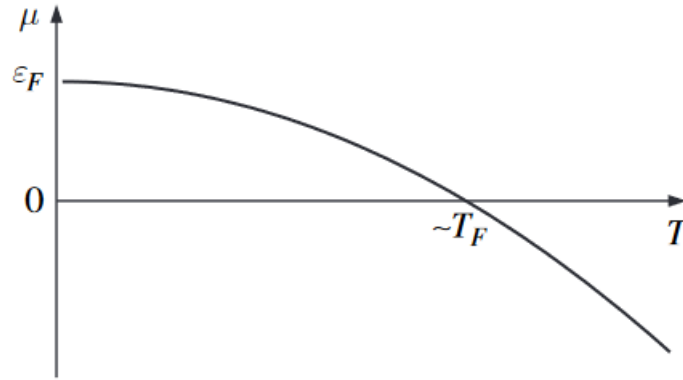
$$\lim_{T \rightarrow 0} \ln z = \left(\frac{3}{4\sqrt{\pi}} \frac{n\lambda^3}{g} \right)^{2/3} = \frac{\beta \hbar^2}{2m} \left(\frac{6\pi^2 n}{g} \right)^{2/3} = \beta \mathcal{E}_F \implies \lim_{T \rightarrow 0} \mu = \mathcal{E}_F$$

which is what we found earlier by looking at the mean occupation number for fermions.



At finite but small temperatures, we have corrections:

$$\ln z \approx \beta \mathcal{E}_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right]$$



We see that the chemical potential $\mu = k_B T \ln z$ is positive at low temperature and negative at high temperature. It changes sign at $T \sim T_F = \mathcal{E}_F/k_B$.

Using a similar approach, one finds the low-temperature expansion for pressure:

$$\beta P = \beta P_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right], \quad \boxed{P_F = \frac{2}{5} n \mathcal{E}_F}$$

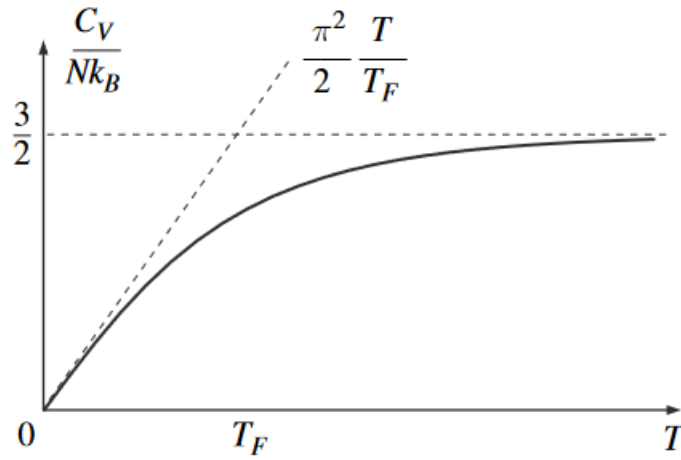
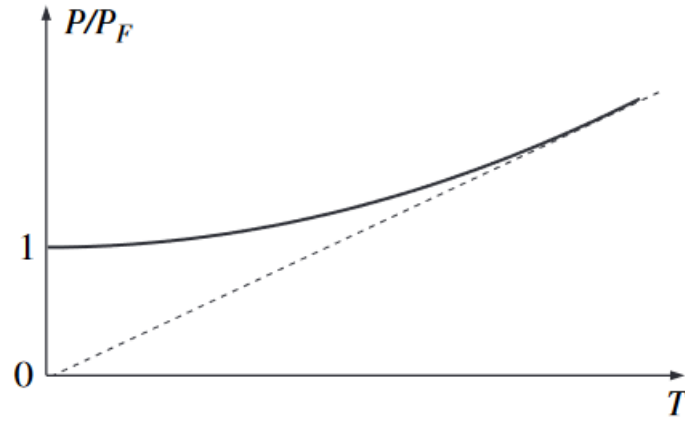
Here P_F is the **fermi pressure**. Note that the fermi gas at zero temperature has nonzero pressure and internal energy, unlike its classical counterpart.

The internal energy (density) is obtained from the third equation in the previous section:

$$\boxed{\epsilon = \frac{E}{V} = \frac{3}{2} P = \frac{3}{5} n k_B T_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right]}$$

From here, one finds the low-temperature heat capacity:

$$C_V = \frac{dE}{dT} = \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F} \right) + \mathcal{O} \left(\frac{T}{T_F} \right)^3$$



Note: the linear scaling of C_V as $T \rightarrow 0$ is a general feature of a fermi gas and is valid in all dimensions. The physical interpretation is as follows: At small temperatures, all single-particle states are occupied. As a result, only particles within a distance of about $k_B T$ of the fermi energy can be thermally excited. This represents only a small fraction (T/T_F) of the fermions. Each excited particle gains $k_B T$, leading to a change in E of $k_B T N (T/T_F)$. So $C_V \propto dE/dT \sim N k_B T/T_F$ which is linear in T . This conclusion is also **valid for interacting fermi gases**.

1.3 Ideal Fermi gas in a harmonic trap

See next chapter. This section is more naturally treated in conjunction with a number of related topics.

Chapter 2

Interacting Fermi Gas

2.1 Hyperfine Structure

2.2 Collisional Properties

2.3 Cooling and Trapping techniques

2.4 RF Spectroscopy

2.5 Feshbach resonance

This section is adapted from [1] and [3].

2.5.1 Scattering theory

It is necessary to have some basic understanding of scattering theory before having an intuitive picture of Feshbach resonance. Our main focus here is **low-energy scattering**, so we will not discuss Born perturbative series expansion method for high-energy scattering and instead look at method of partial waves.

Let's begin. There are three types of scattering processes: elastic, inelastic, and absorption. We will be focusing on **elastic scattering** where both the energy and particle number are conserved.

Scattering cross section

The main concept in scattering theory is the **differential cross section**, $d\sigma$, which characterizes scattering processes. $d\sigma$ has units of area. Consider a stream of particles with flux j (in units of particles per unit time per unit area) hitting some detector which measures the number of particles $N d\Omega$ per unit time scattered into an element of solid angle $d\Omega$. We must then have

$$j d\sigma = N d\Omega.$$

Rearranging gives

$$d\sigma = \frac{N}{j} d\Omega$$

From here we can integrate to find the **total differential cross section**:

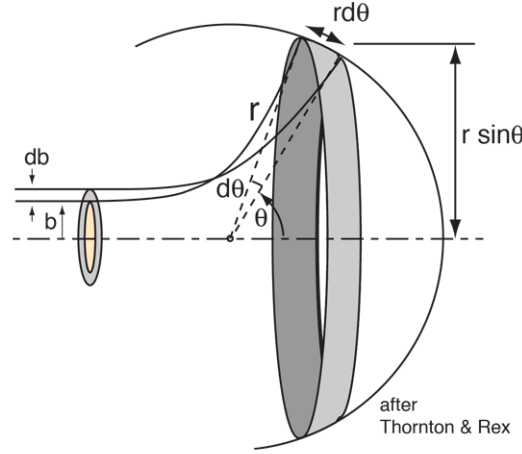
$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \frac{d\sigma}{d\Omega}.$$

As we will see, this quantity depends on the energy of incoming particles. The cross section is an effective area for collision.

Classical scattering

In most cases, we have concerned with scattering against a **central potential** $V(r)$. Consider the quantity $N d\Omega$ again. This quantity is the number of particles per unit time scattered between θ and $\theta + d\theta$ (large grey area). By conservation of particle number, it must be equal to the flux j multiplied by the area of the small grey area given by $2\pi b db$:

$$N d\Omega = 2\pi \sin \theta d\theta N = 2\pi b db j$$



By azimuthal symmetry, we see that $b = b(\theta)$. This quantity is called the **impact parameter**. It is the perpendicular distance to the closest approach if the projectile were undeflected. For every deflection angle θ , there is an associated impact parameter. Hence $b = b(\theta)$.

From the equation above, we obtain

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$$

Example: For hard sphere scattering, the impact parameter is

$$b(\theta) = R \sin \alpha = R \sin \left(\frac{\pi - \theta}{2} \right) = -R \cos(\theta/2).$$

This gives

$$\frac{d\sigma(\theta)}{d\Omega} = \frac{R^2}{4} \implies \sigma = \pi R^2,$$

as expected. The cross section for hard sphere scattering is simply the cross sectional area of the sphere. \square

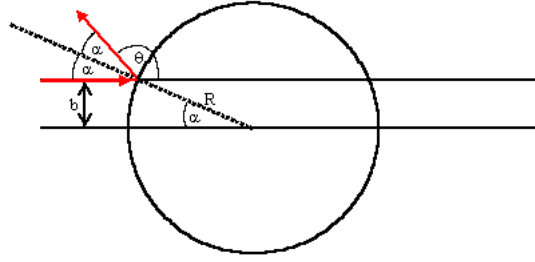
Example: Here we look at classical Coulomb scattering with the central potential:

$$V(r) = \frac{\kappa}{r}.$$

In this case,

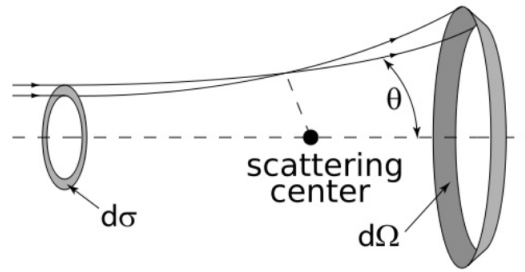
$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right| = \frac{\kappa^2}{16E^2} \frac{1}{\sin^4 \theta/2}.$$

This is the famous **Rutherford formula**. \square



Quantum scattering

The setup here is same as before. We are interested in the steady state of the scattering problem (think of this as having a constant stream of incoming particles/waves and ignore the "encounter period"). The total



wavefunction of incoming and scattered particles must solve the Schrödinger equation:

$$E\psi(\vec{r}) = - \left[\frac{\hbar^2}{2m} \nabla^2 + V(r) \right] \psi(\vec{r}).$$

Let the incoming wave be a plane wave

$$\psi_{\text{inc}} = e^{i\vec{k}\cdot\vec{r}}$$

Then the outgoing wave at large r may be written as

$$\psi_{\text{out}} = f(\theta) \frac{e^{ikr}}{r}.$$

And so the total wavefunction may be written as

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + f(\theta) \frac{e^{ikr}}{r}, \quad r \gg L$$

where L is some effective range of the potential. By writing down the full wavefunction in the partial wave expansion, one may find that

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos \theta)$$

where

$$f_l(k) = \frac{e^{2i\delta_l(k)} - 1}{2ik} = \frac{e^{i\delta_l(k)}}{k} \sin \delta_l$$

are the **partial wave scattering amplitudes**, which are defined by the **phase shifts** $\delta_l(k)$.

Now we want to relate the scattering amplitude to cross section. To do this, we have to find the particle flux. By definition:

$$\vec{j} = -i \frac{\hbar}{m} \text{Re} [\psi^* \nabla \psi] \approx \frac{\hbar \vec{k}}{m} + \frac{\hbar k}{m} \hat{e}_r \frac{|f(\theta)|^2}{r^2} + \dots$$

after neglecting rapidly fluctuation contributions. With this, we can compute:

$$N d\Omega = (\vec{j} \cdot \hat{e}_r) r^2 d\Omega = \frac{\hbar k}{m} |f(\theta)|^2 d\Omega + \dots$$

But we also know that this flux $N d\Omega$ must be equal to the incoming flux $j_I d\sigma = (\hbar k/m) d\sigma$ (plane wave incoming flux). So,

$$N d\Omega = j_I d\sigma \implies d\sigma = |f(\theta)|^2 d\Omega \implies \boxed{\frac{d\sigma}{d\Omega} = |f(\theta)|^2}$$

From here and the definition for $f(\theta)$, we can calculate the total cross section:

$$\sigma_{\text{tot}} = \int d\sigma = \int |f(\theta)|^2 d\Omega = \dots = 4\pi \sum_l (2l+1) |f_l(k)|^2 = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l(k),$$

where we have used the definition of $f_l(k)$ in the last equality. A consequence of this is that

$$\text{Im } f(0) = \frac{k}{4\pi} \sigma_{\text{tot}}.$$

This is the **optical theorem**, which encapsulate particle conservation.

In summary, the quantum scattering cross section is fully characterized by the scattering amplitude $f(\theta)$, which implicitly depends on the energy $E = E(k)$ in its partial wave expansion. The partial wave amplitudes $f_l(k)$ in these expansions are ultimately defined by the phase shifts $\delta_l(k)$.

The next step is therefore to find the phase shifts. To do this we need the **method of partial waves**, where we start by decomposing the full wavefunction into, well, partial waves:

$$\psi(\vec{r}) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos \theta).$$

From the Schrödinger equation for the scattering wavefunction with $E = \hbar^2 k^2 / 2m$ (energy of the incoming wave), we find the radial equation

$$\left[-\frac{\hbar^2}{2m} \left(\partial_r^2 + \frac{2}{r} \partial_r - \frac{l(l+1)}{r^2} \right) + V(r) \right] R_l(r) = \frac{\hbar^2 k^2}{2m} R_l(r).$$

Rearranging this gives

$$\left[\partial_r^2 + \frac{2}{r} \partial_r - \frac{l(l+1)}{r^2} - U(r) + k^2 \right] R_l(r) = 0$$

with $U(r) = 2mV(r)/\hbar^2$.

Provided that the potential is sufficiently short-ranged, the scattering wavefunction is a superposition of incoming and outgoing spherical waves:

$$R_l(r) \approx \frac{i}{2k} \sum_{l=0}^{\infty} i^l (2l+1) \left(\frac{e^{-i(kr-l\pi/2)}}{r} - e^{2i\delta_l(k)} \frac{e^{i(kr-l\pi/2)}}{r} \right).$$

In particular, for $l = 0$ (s -wave channel):

$$R_0(r) \approx \frac{1}{kr} e^{i\delta_0(k)} \sin(kr + \delta_0(k))$$

At low energy ($kL \ll 1$), the s -wave channel dominate.

Introducing the **reduced radial wavefunction**

$$u(r) = rR_0(r), \quad u(0) = 0$$

we get a simplified radial equation:

$$[\partial_r^2 - U(r) + k^2] u(r) = 0.$$

We now introduce the **scattering length** a_0 , defined by the condition:

$$u(a_0) = 0, \quad kL \ll 1$$

What, then, is a_0 in terms of the incoming wave momentum?

$$\begin{aligned} 0 &= u(a_0) \\ &= \sin(ka_0 + \delta_0(k)) \\ &= \sin(ka_0) \cos(\delta_0(k)) + \cos(ka_0) \sin(\delta_0(k)) \\ &= \sin \delta_0(k) [\cot \delta_0(k) \sin(ka_0) + \cos(ka_0)] \\ &\approx \sin \delta_0(k) [ka_0 \cot \delta_0(k) + 1] \end{aligned}$$

Solving this gives

$$a_0 = -\lim_{k \rightarrow 0} \frac{1}{k} \tan \delta_0(k)$$

To see why a_0 is called the scattering length, we can compute the scattering cross section at low energy:

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sin^2 \delta_0(k) \xrightarrow{k \rightarrow 0} \frac{4\pi}{k^2} \frac{(ka_0^2)}{1 + (ka_0)^2} \approx 4\pi a_0^2.$$

We see that a_0 **characterizes the effective size of the target**, and so its nomenclature is justified.

Example: Quantum hard-sphere scattering.

Consider the hard-sphere potential,

$$U(r) = \begin{cases} \infty & r < R \\ 0 & r > R \end{cases}$$

The boundary condition is $u(R) = 0$ (the sphere is impenetrable). From here, the low-energy scattering reduced radial wavefunction is

$$u(r) = A \sin(kr + \delta_0), \quad \delta_0 = -kR.$$

The scattering length is therefore

$$a_0 = -\lim_{k \rightarrow 0} \frac{1}{k} \tan(-kR) \approx R.$$

Since the phase shift is known, we can calculate the scattering amplitude:

$$f_0(k) = \frac{e^{ikR}}{k} \sin(kR).$$

The total scattering cross section is therefore

$$\sigma_{\text{tot}} \approx 4\pi \frac{\sin^2 kR}{k^2} \approx 4\pi R^2$$

Notice that the answer is $4\pi R^2$ as opposed to πR^2 like in the classical case. The factor of 4 enhancement is due to diffraction processes at the sharp potential. \square

Example: Scattering by attractive square well.

Here we consider a potential of the form

$$U(r) = -U_0 \theta(R - r), \quad U_0 > 0$$

where $\theta(\cdot)$ is the Heavyside step function. We solve the SE in two regions and get

$$u(r) = \begin{cases} C \sin(Kr) & r < R \\ \sin(kr + \delta_0) & r > R \end{cases}$$

where $K^2 = k^2 + U_0 > k^2$ and δ_0 is the scattering phase shift. By matching the solutions and their derivatives at R gives

$$C \sin(KR) = \sin(kR + \delta_0) \quad CK \cos(KR) = k \cos(kR + \delta_0) \implies K \cot(KR) = k \cot(kR + \delta_0)$$

From this, we get

$$\tan \delta_0(k) = \frac{k \tan(KR) - K \tan(kR)}{K + k \tan(kR) \tan(KR)}, \quad K^2 = k^2 + U_0.$$

When $kR \ll 1$ (low energy), $K \approx \sqrt{U_0}$, we find that the scattering length is

$$a_0 = -\lim_{k \rightarrow 0} \frac{1}{k} \tan \delta_0(k) \approx -R \left(\frac{\tan(KR)}{KR} - 1 \right).$$

Notice that for $KR < \pi/2$, the scattering length is negative. **What does the sign of the scattering length represent?** $a_0 > 0$ means the potential is repulsive, while $a_0 < 0$ means the potential is attractive. In this problem, the attractive square potential leads to the $l = 0$ phase shift

$$\delta_0 \approx -ka_0 \approx kR \left(\frac{\tan KR}{KR} - 1 \right)$$

and total cross section

$$\sigma_{\text{tot}} = \frac{4\pi}{k^2} \sin^2 \delta_0(k) \approx 4\pi R^2 \left(\frac{\tan KR}{KR} - 1 \right)^2, \quad K \approx \sqrt{U_0}$$

Let k be fixed and let U_0 be tunable.

- If $KR \ll 1$, then $a_0 < 0$, the potential is attractive.
- If $KR \rightarrow \pi/2$, both scattering length and cross section diverge. The influence of the target becomes effectively infinite in range. In this case, the square well just meets the criterion to host a single s -wave bound state.

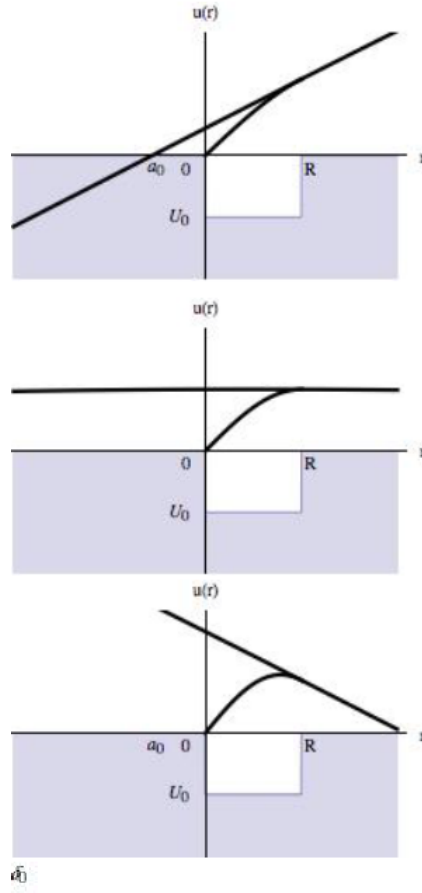


Figure 2.1: Attractive potential, infinite range potential, repulsive potential.

- As KR is increased, a_0 turns positive, and the potential becomes repulsive until $KR = \pi$ for which $\sigma_{\text{tot}} = 0$, and the process repeats.
- When $KR = n\pi$, the scattering cross section vanishes identically and the target becomes invisible. This is the **Ramsauer-Townsend effect**.

We are sensing that at $KR = n\pi/2$ where n is odd, there is a **resonance** phenomenon. And we would be right. In general, the l -th partial cross section is given by

$$\sigma_l = \frac{4\pi}{k^2} (2l+1) \frac{1}{1 + \cot^2 \delta_l(k)}, \quad \sigma_{\text{tot}} = \sum_l \sigma_l.$$

The functional form tells us that σ_l is peaked whenever $\cot \delta_l$ vanishes. In fact, as $\delta_l(k)$ is swept across odd multiples of $\pi/2$, the cross section exhibits a narrow peak as a function of energy. This is indeed a **resonance phenomenon**.

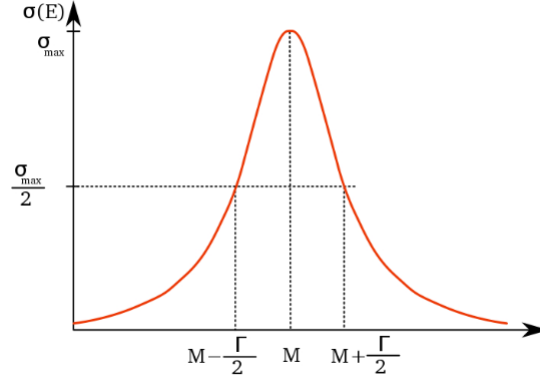
In practice, the metastable bound state formed by the particle in the potential has a lifetime defined by $\Gamma(E) = \hbar/\tau$, where $\Gamma(E)$ is the width of the resonance. Near the resonance, the phase shift is related to this width as

$$\cot \delta_l(k) = \frac{E_R - E}{\Gamma(E)/2}$$

where E_R is the resonance energy. It turns that if $\Gamma(E)$ varies slowly in energy, then the partial cross section near resonance is given by the **Breit-Wigner formula**

$$\sigma_l(E) = \frac{4\pi}{k^2} (2l+1) \frac{\Gamma^2(E_R)/4}{(E - E_R)^2 + \Gamma^2(E_R)/4}$$

which is a typical resonance profile.



Physically, at resonance, the amplitude of the wavefunction within the potential region is high and the probability of finding the scattered particle inside the well is correspondingly high.

SUMMARY: To understand low-energy quantum scattering, we consider a simple problem with incoming plane wave of the form e^{ikz} . The total wavefunction is then given by the superposition of this incoming wave and a spherical wave scaled by the **scattering amplitude** $f(\theta)$:

$$\psi = e^{ikz} + f(\theta) \frac{e^{ikr}}{r}.$$

From partial wave analysis, the total scattering amplitude can be written in terms of the **partial wave scattering amplitudes**, which is defined by the **phase shifts**:

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos \theta) = \sum_{l=0}^{\infty} (2l+1) \left[\frac{e^{i\delta_l(k)}}{k} \sin \delta_l(k) \right] P_l(\cos \theta)$$

The scattering amplitude completely characterizes the scattering phenomenon. The **differential cross section** is related to the scattering amplitude via

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2$$

from which the **total cross section** can be calculated.

The problem of characterizing a scattering phenomenon thus comes down to finding the **phase shifts**. To do this, we use the method of partial waves to decompose a wavefunction into the radial and angular part. The Schrödinger equation enforces the **reduced radial wavefunction** $u(r) = rR(r)$ to satisfy:

$$[\partial_r^2 - U(r) + k^2]u(r) = 0$$

for low-energy ($l = 0$) scattering. The **scattering length** a_0 , which characterizes the effective size of the target, is defined by the condition:

$$u(a_0) = 0.$$

Since $u(r) = \sin(kr + \delta_0(k))$, we have the definition:

$$a_0 = -\lim_{k \rightarrow 0} \frac{1}{k} \tan \delta_0(k)$$

from which we could verify that $\sigma_{\text{tot}} \approx 4\pi a_0^2$, which justifies the nomenclature for a_0 .

For the case of the square well, we consider changing the depth U_0 of the potential, so that $K^2 = k^2 + U_0$ is a variable. In the case that $KR \ll 1$, the scattering length is negative, signifying that the potential is repulsive. In the case that $KR = n\pi/2$ where n is odd, the scattering length and the cross section diverges, signifying that the potential has infinite effective range. As KR increases, a_0 becomes positive, and the potential becomes repulsive. Finally, in the case that $KR = n\pi$, the scattering length vanishes and the target becomes invisible.

As the scattering length goes from negative to infinity to positive, we go across a resonance. The resonance occurs when $KR = n\pi/2$, where n is odd. The partial cross section near resonance is given by the **Breit-Wigner formula**:

$$\sigma_l(E) = \frac{4\pi}{k^2} (2l+1) \frac{\Gamma^2(E_R)/4}{(E - E_R)^2 + \Gamma^2(E_R)/4}$$

which is a typical resonance profile.

2.5.2 Scattering theory for ultracold atoms

Ultracold atoms interact via the van der Waals potential, which can be well approximated by a square well. In the ultracold regime, $KR \ll 1$ and therefore results from the previous sections are valid. This and the forthcoming subsections follow from [3].

Consider a spherical well potential of radius R and depth V . This potential has scattering states for $E > 0$ and bound states for $E < 0$ when the depth is larger than a critical value V_c . Let $E = \hbar^2 k^2/m$ for $E > 0$ and $|E| = \hbar^2 k^2/m$ for $E < 0$. $V^2 = \hbar^2 K^2/m$ and $E_R = \hbar^2/mR^2$. The critical well depth is $V_c = (\pi^2/4)E_R$. New bound state appear when $K_n R = (2n+1)\pi/2$ at $V_n = (2n+1)^2 V_c$.

The ultracold condition $KR \ll 1$ is equivalent to $E \ll E_R$. As we have done before, the scattering length is given by

$$a = -\lim_{k \rightarrow 0} \frac{1}{k} \tan \delta_s(k) = R \left(1 - \frac{\tan KR}{KR} \right).$$

By expanding $k \cot \delta_s(k)$ in k^2 , we obtain

$$k \cot \delta_s(k) \approx -\frac{1}{a} + \frac{1}{2} r_{\text{eff}} k^2 + \dots$$

from which we obtain the effective range of the potential:

$$r_{\text{eff}} = R - \frac{1}{K^2 a} - \frac{1}{3} \frac{R^3}{a^2}$$

which is small and is on the order of R unless the well is very shallow ($K^2 R a \ll 1$) or $a < R$.

As we have seen before, the scattering length diverges whenever a new bound state enters the potential. When this happens, we know that the phase shift is approaching $\pi/2$.

Away from resonances, the scattering length is close to the **background scattering length** R . Close to a resonance, the scattering length diverges as

$$a \approx \frac{1}{KR(K - K_n)} \approx \frac{2\hbar^2}{mR(V - V_n)}.$$

The bound state has negative energy, which is given in terms of the scattering length:

$$E_B = -\frac{\hbar^2}{ma^2}.$$

For $a > 0$ (repulsive potential), the bound state energy depends on V like

$$E_B = -\frac{\hbar^2}{ma^2} = -\frac{1}{4} \frac{(V - V_n)^2}{E_R}.$$

Moral of the story: the binding energy depends quadratically on the “detuning” while the scattering length is inversely proportional to the “detuning.” In ultracold atom Feshbach resonances, the detuning is controlled by an external magnetic field.

2.5.3 Feshbach resonance

Ultracold atoms interact through the short-ranged van der Waals interaction, whose effective range can be tuned by allowing particles to form virtual bound state – a resonance. By adjusting the separation between entrance channel states and bound state through an external magnetic field, the system can be tuned through resonance. This allows the effective interaction to be tuned from repulsive to attractive simply by changing an external field.

Consider two fermionic atoms. Their spin wavefunction is given by the tensor product $|s_1, m_1\rangle \otimes |s_2, m_2\rangle$ where s is the total spin quantum number and s_i is the spin projection quantum number. Using Clebsch-Gordan coefficients, we can write these wavefunctions in the coupled basis as

$$\begin{aligned} |0, 0\rangle &= \frac{1}{\sqrt{2}} (\uparrow\downarrow - \downarrow\uparrow) \\ |1, 1\rangle &= \uparrow\uparrow \\ |1, 0\rangle &= \frac{1}{\sqrt{2}} (\uparrow\downarrow + \downarrow\uparrow) \\ |1, -1\rangle &= \downarrow\downarrow \end{aligned}$$

The first state is the **singlet state**, while the bottom three are the **triplet states**. Notice that the singlet state is the only state with an antisymmetric spin wavefunction. This means that it must have symmetric spatial wave function, and therefore the two fermions can exist close together in this state. On the other hand, the particles have symmetric spin wave function in the triplet states, so the spatial wavefunctions must be antisymmetric, i.e., the particles cannot exist close together.

The interaction between the atoms depends on whether the atoms are in the singlet or a triplet state. The singlet state and triplet states have different potentials. Due to the fact that the atoms can exist close together in the singlet state, the potential associated with this state, $V_S(r)$, is deeper. Here, r is the inter-atomic potential. The triplet state potential $V_T(r)$, on the other hand, is shallower.

If the atoms enter in a triplet configuration, and if there is no coupling between $V_S(r)$ and $V_T(r)$, then they simply scatter off each other and acquire some fixed phase shift. Same story with $V_S(r)$. However, there is a coupling between $V_S(r)$ and $V_T(r)$, namely the **hyperfine interaction** V_{hf} .

V_{hf} is not diagonal in the total electronic spin $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$ of the two atoms and thus provides a coupling between singlet and triplet potentials:

$$\begin{aligned} V_{\text{hf}} &= a_{\text{hf}} (\mathbf{s}_1 \cdot \mathbf{i}_1 + \mathbf{s}_2 \cdot \mathbf{i}_2) \\ &= \frac{a_{\text{hf}}}{2} [\mathbf{S}(\mathbf{i}_1 + \mathbf{i}_2) + (\mathbf{s}_1 - \mathbf{s}_2)(\mathbf{i}_1 + \mathbf{i}_2)] \\ &= V_{\text{hf}}^+ + V_{\text{hf}}^-. \end{aligned}$$

Notice that V_{hf}^- connects the singlet and triplet states. In particular, the operator $\mathbf{s}_1 - \mathbf{s}_2$ is antisymmetric in 1 and 2 and thus connects the symmetric (triplet) electronic spin states to antisymmetric (singlet) states. So, V_{hf} is off-diagonal in the triplet/singlet basis.

The singlet potential is a **closed channel** which means that the singlet continuum states are not available as final scattering states by energy conservation. We can simply think of singlet states as having $E < 0$, and so cannot have free particle states.

A Feshbach resonance occurs when the state that the atoms collide in is resonant with a bound state in the singlet potential. The energy difference between the incoming and the Feshbach bound state can be tuned by an external magnetic field due to their different magnetic moments.

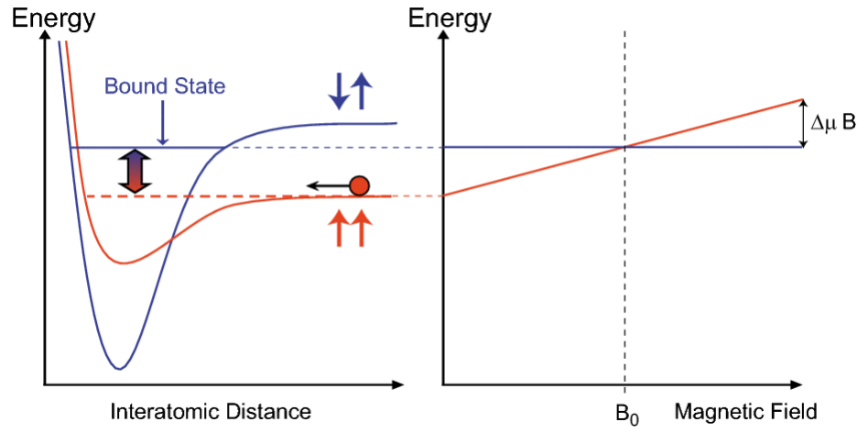


Figure 2.2: By tuning an external magnetic field, the energy of the incoming state can be made equal to the energy of a bound singlet state

A model for Feshbach resonances

Consider a simple model for Feshbach resonance where one bound state $|m\rangle$ in the closed channel and other states that are far detuned in energy. The continuum of plane waves of relative momentum \mathbf{k} between the two particles in the incoming channel are denoted $|k\rangle$.

2.5.4 FB resonance: Spherical Well Model

2.5.5 FB resonance: Coupled Square Well Model

2.5.6 An intuitive picture

2.6 Analysis of density distributions

Chapter 3

Strongly Interacting Fermi Gas

Bibliography

- [1] S. L. Campbell. Feshbach resonances in ultracold gases.
- [2] M. Kardar. *Statistical physics of particles*. Cambridge University Press, 2007.
- [3] W. Ketterle and M. W. Zwierlein. Making, probing and understanding ultracold fermi gases. *La Rivista del Nuovo Cimento*, 31(5):247–422, 2008.