8.(3)09 Solutions of Section 2

September 17, 2021

1 Particle on a cylinder

(a)

$$L = \frac{1}{2} m \left(R^2 \dot{\theta}^2 + \dot{z}^2 \right) - \frac{1}{2} k \left(R^2 + z^2 \right) \tag{1}$$

(b)

$$mR^2 \ddot{\theta} = 0 \tag{2}$$

$$m\ddot{z} + kz = 0 \tag{3}$$

p_θ and H are conserved.

(Notice that in this question, you cannot say the angular momentum is conserved. Actually, the angular momentum as a vector, is changing all the time, as long as the particle oscillates in the z direction. Only the z component of the angular momentum, p_θ, is conserved.)

(c)

Constant motion in θ, oscillatory motion in z. See the last example in 809MathematicaIntroduction.nb (material for section 1).

2 Point transformation

(a)

$$L = \frac{m}{2} \dot{\vec{r}}^T \cdot A \cdot \dot{\vec{r}} - \frac{k}{2} \vec{r}^T \cdot A \cdot \vec{r} \tag{4}$$

$$A = \begin{pmatrix} a, b \\ b, c \end{pmatrix} \tag{5}$$

(b)

Real symmetric matrices have real eigenvalues and orthogonal eigenvectors, and can be diagonalized with a real orthogonal matrix P.

$$A\vec{\mu}_1 = \lambda_1 \vec{\mu}_1, \;\; A\vec{\mu}_2 = \lambda_2 \vec{\mu}_2 \tag{6}$$

$$\Longrightarrow AP = P\Lambda, \text{ with } P = (\vec{\mu}_1, \, \vec{\mu}_2) \text{ and } \Lambda = \begin{pmatrix} \lambda_1, 0 \\ 0, \lambda_2 \end{pmatrix}. \tag{7}$$

You can solve $\lambda_{1,2}$ and $\vec{\mu}_{1,2}$ to be (You don't have to calculate $\vec{\mu}_{1,2}$ or P)

$$\lambda_{1,2} = \frac{a + c \pm \sqrt{a^2 - 2ac + c^2 + 4b^2}}{2}, \tag{8}$$

$$\vec{\mu}_{1,2} = \frac{1}{\sqrt{(\lambda_{1,2} - c)^2 + b^2}} \begin{pmatrix} \lambda_{1,2} - c \\ b \end{pmatrix}. \tag{9}$$

Since P is an orthogonal matrix ($P^T P = P P^T = I$), we have

$$A = P \Lambda P^T. \tag{10}$$

Defining $\vec{q}$ to be

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = P^T \vec{r}, \tag{11}$$

then we have

$$L = \frac{m}{2} \dot{\vec{q}}^T \Lambda \dot{\vec{q}} - \frac{k}{2} \vec{q}^T \Lambda \vec{q} \tag{12}$$

$$= \frac{m}{2} \left(\lambda_1 \dot{q}_1^2 + \lambda_2 \dot{q}_2^2 \right) - \frac{k}{2} \left(\lambda_1 q_1^2 + \lambda_2 q_2^2 \right) . \tag{13}$$

The equations of motion for q_1 and q_2 are

$$\ddot{q}_i + \omega^2 q_i = 0, \ (i = 1, 2, \ \omega = \sqrt{k/m}). \tag{14}$$

(c)

We can solve the equations of motion (or whatever the form you like)

$$\vec{q} = \begin{pmatrix} A \cos \omega t + B \sin \omega t \\ C \cos \omega t + D \sin \omega t \end{pmatrix} \tag{15}$$

In the first case,

$$A = \begin{pmatrix} 0, b \\ b, 0 \end{pmatrix}, \ P = P^T = \frac{1}{\sqrt{2}} \begin{pmatrix} -1, & 1 \\ 1, & 1 \end{pmatrix}, \ \vec{r} = P^T \vec{q}. \tag{16}$$

In the second case,

$$A = \begin{pmatrix} a, & 0 \\ 0, & -a \end{pmatrix}, \ P = I, \ \vec{r} = \vec{q}. \tag{17}$$

(d)

If $b^2 = ac$, one of the eigenvalues is zero, then the Lagrangian can be written as

$$L = \frac{m}{2} \lambda_1 \dot{q}_1^2 - \frac{k}{2} \lambda_1 q_1^2 . \tag{18}$$

There is only one independent coordinate.