Spring, 2021

Physics 312: Physics of Fluids

Assignment #5

Background Reading

Friday, Mar. 12: Tritton 5.4,
Kundu & Cohen 4.1 - 4.3

Monday, Mar. 15: Kundu & Cohen 4.7

Wednesday, Mar. 17: Tritton 5.6, 5.7,
Kundu & Cohen 4.10, 4.11

Informal Written Reflection

Due: Thursday, March 18 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, March 19 (in class)

1. In class, we used an arbitrary *fixed* volume to derive the continuum expression of mass conservation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0.$$

We could instead have derived this expression by considering an arbitrary *material* volume. Show that this alternative derivation leads to the same result. (This is Kundu and Cohen, Chapter 4, Problem 2.)

(Hint: You'll use an integral theorem strategy like the one we used in class except, since you're now dealing with a material volume instead of a fixed volume, the theorem takes a slightly different form. The version you'll need is called the *Reynolds transport theorem* in Kundu and Cohen, Chapter 4...)

2. In class, we used an arbitrary *material* volume to derive the continuum expression of momentum conservation,

$$\rho \frac{D}{Dt} u_i = \rho g_i + \frac{\partial}{\partial x_j} \tau_{ij}.$$

We could instead have derived this expression by considering an arbitrary *fixed* volume...

(a) The i-th component of total momentum of the fluid contained in a fixed volume V is given by

$$M_i = \int_V \rho u_i \, \mathrm{d}V.$$

The i-th component of total force of the fluid contained in this volume is given by

$$F_i = \int_V \left[\rho g_i + \frac{\partial}{\partial x_j} \tau_{ij} \right] \mathrm{d}V.$$

The *momentum principle* (for a fixed volume) relates these two expressions in the following way:

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t} M_i + \int_A \rho u_i u_j n_j \, \mathrm{d}A.$$

Can you interpret the expression $\rho u_j n_j \, \mathrm{d}A$ (appearing in the last term) as a flux? Use your answer to develop a physical interpretation of the momentum principle. Explain your ideas carefully.

(b) Use the momentum principle to rederive our main result (this is Kundu and Cohen, Chapter 4, Problem 4):

$$\rho \frac{D}{Dt} u_i = \rho g_i + \frac{\partial}{\partial x_j} \tau_{ij}.$$

(Hint: You'll need the continuity equation and a theorem telling you how to take a time derivative of an integral over a fixed volume.)

3. The solution to the one-dimensional diffusion equation,

$$\frac{\partial}{\partial t}\phi(x,t) = k\frac{\partial^2}{\partial x^2}\phi(x,t),$$

can be written as a *convolution*:

$$\phi(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\phi(y,0)\,\mathrm{d}y,$$

where $\phi(y,0)$ is an initial condition for $\phi(x,t)$ (y is just a dummy variable) and the convolution kernel G has the form of a Gaussian,

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-\frac{x^2}{4kt}).$$

Note that $\phi(x,t) = G(x,t)$ if the initial condition is a delta function ($\phi(y,0) = \delta(y)$). How does $G(x,t)$ change shape as t increases? How does the integral of $G(x,t)$ over all x vary with time?...Use your answers to these questions to interpret the behavior of solutions to the one-dimensional diffusion equation.