Claim $\boxed{\hat{\gamma}_k \text{ is a fermion}}$

Pf $\{\hat{\gamma}_k, \hat{\gamma}_k^\dagger\} = \{ \hat{u}_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^\dagger, u_k \hat{c}_k^\dagger + v_k \hat{c}_{-k} \}$

$$= |u_k|^2 \underbrace{\{\hat{c}_k, \hat{c}_k^\dagger\}}_{1} + |v_k|^2 \underbrace{\{\hat{c}_{-k}^\dagger, \hat{c}_{-k}\}}_{1}$$

$$= |u_k|^2 + |v_k|^2 = 1.$$

$\square$.

With this new definition, we can write $\hat{H}_k$ (4x4) as

$$\hat{H}_k = \hat{\Psi}_k^\dagger H_k \hat{\Psi}_k$$

$$= \hat{\Psi}_k^\dagger U_k U_k^\dagger H_k U_k U_k^\dagger \hat{\Psi}_k$$

$$= \hat{\Phi}_k^\dagger \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix} \hat{\Phi}_k = \varepsilon_k \hat{\Phi}_k^\dagger \hat{\sigma}^z \hat{\Phi}_k$$

$(\hat{\gamma}_k^\dagger, \hat{\gamma}_{-k})$ $\begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}_{-k}^\dagger \end{pmatrix}$

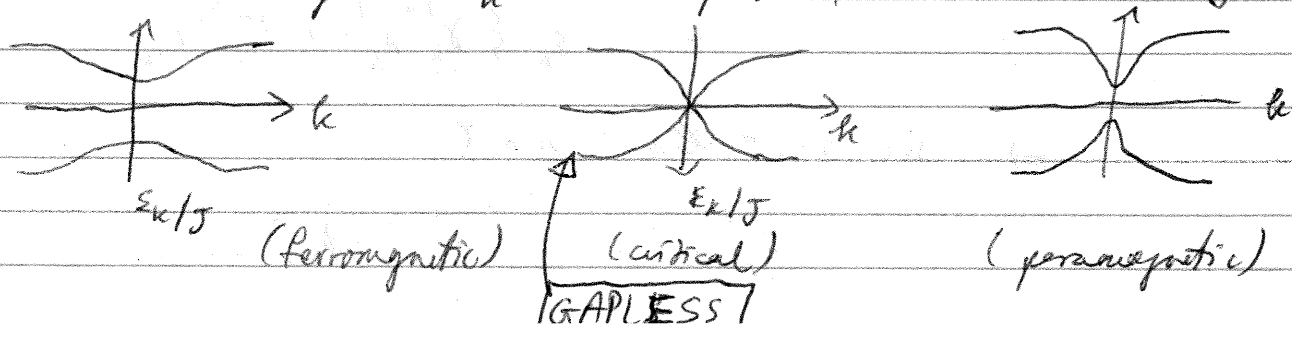
$$= \varepsilon_k \left(\hat{\gamma}_k^\dagger \hat{\gamma}_k - \hat{\gamma}_{-k} \hat{\gamma}_{-k}^\dagger \right)$$

anti-comm relation,

$$= \varepsilon_k \left(\hat{\gamma}_k^\dagger \hat{\gamma}_k + \hat{\gamma}_{-k}^\dagger \hat{\gamma}_{-k} - 1 \right)$$

$$\Rightarrow \boxed{\hat{H}_k = \varepsilon_k \left(\hat{\gamma}_k^\dagger \hat{\gamma}_k + \hat{\gamma}_{-k}^\dagger \hat{\gamma}_{-k} - 1 \right)}$$

The form of $\pm \varepsilon_k$ is important ...



- Critical point Gapless linear spectrum.
- Ferro vs Paramagnetic : indistinguishable, But topology is distinctly different

→ we'll see this later.

Oct 20, 2020

3.1. Ground state & Excited states of the Ising model

Recall that with $\vec{\Phi}_k^+ = \vec{\Psi}_k^+ \mathcal{U}_k = (\hat{\gamma}_k^+, \hat{\gamma}_{-k})$ we have

$$\hat{H}_k = \vec{\Psi}_k^+ \mathcal{U}_k \mathcal{U}_k^+ \mathcal{H}_k \mathcal{U}_k \mathcal{U}_k^+ \vec{\Psi}_k$$

$$= \hat{\vec{\Phi}}_k^+ \begin{pmatrix} \varepsilon_k & \\ & -\varepsilon_k \end{pmatrix} \hat{\vec{\Phi}}_k = (\varepsilon_k) \{ \hat{\gamma}_k^+ \hat{\gamma}_k - \hat{\gamma}_{-k} \hat{\gamma}_{-k}^+ \}$$

$$= (\varepsilon_k) \{ \hat{\gamma}_k^+ \hat{\gamma}_k + \hat{\gamma}_{-k}^+ \hat{\gamma}_{-k} - \mathbb{1} \}.$$

From here, we notice that if $|\phi_g\rangle$ denotes the gnd state of $\hat{\mathcal{H}}_k$ then

$$\hat{\mathcal{H}}_k |\phi_g\rangle = -\varepsilon_k |\phi_g\rangle$$

$$= -\varepsilon_k \{ \hat{\gamma}_k^+ \hat{\gamma}_k + \hat{\gamma}_{-k}^+ \hat{\gamma}_{-k} - \mathbb{1} \} |\phi_g\rangle$$

$\Rightarrow$ we must have that

$$(\hat{\gamma}_k^+ \hat{\gamma}_k + \hat{\gamma}_{-k}^+ \hat{\gamma}_{-k}) |\phi_g\rangle = 0$$

This occurs if $|\phi_\gamma\rangle$ annihilates the $\gamma_k^\dagger$'s for all k. (positive or negative)

→ These are called the "Bogoliubov vacuum".

$$\boxed{\gamma_k^\dagger |\phi_\gamma\rangle = 0 \qquad \forall k}$$

Even | # of particles
Odd |

In general, one can define 2 such states, one in the $p=0$ (ABC) and one in the $p=1$ (PBC) sector (odd)

However, the "winner" between the two is the actual global ground state is the one in the $p=0$ (even) sector

→ Energy of this gnd state is simply

$$\boxed{E_0^{ABC} = -\sum_{k>0}^{ABC} \epsilon_k}$$

Explicitly, the ground state is given by

$$\boxed{|\phi_\gamma\rangle^{ABC} \propto \prod_{k>0}^{ABC} \gamma_{-k}\gamma_k |0\rangle}$$

s.t. $\gamma_k^\dagger |0\rangle = 0$

where $|0\rangle$ is the vacuum for the original fermions

$$c_k^\dagger |0\rangle = 0.$$

Now, let's expand this ...

$$\prod_{k>0} \hat{\gamma}_{-k} \hat{\gamma}_k |0\rangle \overset{\text{by defn of } \hat{\gamma}_k}{=} \prod_{k>0} \left\{ u^*_{-k} \hat{c}_{-k} + v^*_{-k} \hat{c}^\dagger_k \right\} \left\{ \cancel{u^*_k \hat{c}_k} + v^*_k \hat{c}^\dagger_{-k} \right\} |0\rangle$$

c-no.

$$= \prod_{k>0} \left\{ u^*_{-k} \hat{c}_{-k} + v^*_{-k} \hat{c}^\dagger_k \right\} \left\{ v^*_k \hat{c}^\dagger_{-k} |0\rangle \right\}$$

$$= \prod_{k>0} v^*_k \Big(\underbrace{\hat{c}_{-k} \hat{c}^\dagger_{-k}}_{\mathbb{1}} u^*_{-k} + v^*_{-k} \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \Big) |0\rangle$$

$$= \prod_{k>0} v^*_k \left(u^*_{-k} + v^*_{-k} \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right) |0\rangle$$

$$= \prod_{k>0} v^*_k \left\{ u^*_k - v^*_k \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right\} |0\rangle$$

where we have used $\begin{cases} u_k = u_{-k} \\ v_k = -v_{-k} . \end{cases}$

With this, we find $|\phi_\gamma\rangle$ in terms of $\hat{c}_k$ & coefs ...
after normalising ...

$$\boxed{\begin{aligned} |\phi_\gamma\rangle^{ABC} &= \prod_{k>0}^{ABC} \left(u^*_k - v^*_k \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right) |0\rangle \\ &= \prod_{k>0}^{ABC} \left(u_k + v_k \hat{c}^\dagger_k \hat{c}^\dagger_{-k} \right) |0\rangle \end{aligned}}$$

note that we can $u^*_k \to u_k$ by choosing a phase in which

u_k is real & v_k is purely imaginary.

This is a $\boxed{\text{BCS form}}$

What about the PBC sector state? I.e. what about the ground state of the odd-fermion sector?

↳ Since a BCS-paired state is always fermion-even, the unpaired $\hat{H}_{k=0,\pi}$ must contribute exactly one fermion in the gnd state.

Suppose $h>0$, the gnd state has $\begin{cases} \hat{n}_{k=0} \to 1 \\ \hat{n}_{k=\pi} \to 0 \end{cases}$ (since gnd state)

↳ ~~Recall that~~ and so...

$$\cancel{\hat{H}_k = 2(h - J\cos k)(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k} \hat{c}_{-k}^\dagger)}$$

$$\cancel{- 2J \sin k (ie^{-2i\phi} \hat{c}_k^\dagger \hat{c}_{-k}^\dagger + ie^{2i\phi} \hat{c}_{-k} \hat{c}_k)}$$

~~when $k = 0, \pi \Rightarrow \hat{H}_k = 2(h \mp J)(\hat{c} \ldots)$~~

~~→ also, we have that the gnd state energy has an extra term~~

$$\cancel{\boxed{\delta E_{0,\pi} = \min(\hat{H}_{0,\pi}) = -2J}}$$

Now, recall that

$$\hat{H}_{k=0,\pi} = -2J(\overset{1}{\hat{n}_0} - \overset{0}{\hat{n}_\pi}) + 2h(\overset{1}{\hat{n}_0} + \overset{0}{\hat{n}_\pi} - 1)$$

So, the ~~gnd~~ we get $\hat{H}$ a term of the form $\boxed{\delta E_{0,\pi} = \min(\hat{H}_{0,\pi}) = -2J}$

single fermion

The PBC ground state is therefore

$$\boxed{\begin{aligned} |\phi_\gamma\rangle^{PBC} &= \hat{c}^{\dagger}_{k=0} \prod_{\pi > k > 0}^{PBC} \left(u_k^* - v_k^* \hat{c}_k^{\dagger} \hat{c}_{-k}^{\dagger}\right)|0\rangle \\ &= \hat{\gamma}_0 \prod_{0<k<\pi}^{PBC} \left(u_k^* - v_k^* \hat{c}_k^{\dagger} \hat{c}_{-k}^{\dagger}\right)|0\rangle \end{aligned}}$$

where we defined $\hat{\gamma}_0 = \hat{c}^{\dagger}_{k=0}$; $\hat{\gamma}_\pi = \hat{c}_{k=\pi}$

for the unpaired states.

The corresponding energy is

$$\boxed{E_0^{PBC} = -2J - \sum_{0<k<\pi}^{PBC} \varepsilon_k}$$ (as expected)

What happens in the thermodynamic limit $L \to \infty$??

→ We would expect that the energy per site $e_0 = \frac{E_0}{L}$, the gnd state energy tends to an integral...

Expect: $$e_0 = -\lim_{L\to\infty} \frac{1}{L} \sum_{k>0}^{ABC} \varepsilon_k = -\int_0^{\pi} \frac{dk}{2\pi}\, \varepsilon_k$$

But ---

There is some subtlety when one treats the boundary points at 0 & π

notice that for E_0^{ABC} involves $\frac{L}{2}$ k-points in $(0,\pi)$

but E_0^{PBC} involves $\frac{L}{2}-1$ in $(0,\pi)$

+ an extra term $-2J$

It turns out that the energy splitting

$\boxed{\Delta E_0 = E_0^{PBC} - E_0^{ABC}}$ when $-J<h<J$ (ferro. ordered)

decays exponentially fast when $L\to\infty$

$\Rightarrow$ the two sectors ABC, PBC provide the required double degeneracy of the ferromag phase so long as $|h|<J$.

When $h=0$ for instance (easy to see...)

$\Rightarrow$ On the contrary, ΔE_0 is finite in the quantum disordered region $|h|>J$

$\rightarrow \boxed{\Delta E_0 = 2(|h|-J)}$

and goes to zero as a power law $\sim \frac{\pi}{2L}$

at the critical points $h_c = \pm J$

(figures)

What about excited states?

Let's look at excited states in the $p=0$ (even) sector...

→ Consider the state $\hat{\gamma}^{\dagger}_{k_1}|\phi_{\gamma}\rangle^{ABC}$.

Have that

$$\hat{\gamma}^{\dagger}_{k_1}|\phi_\gamma\rangle^{ABC} = \hat{\gamma}^{\dagger}_{k_1}\prod_{k>0}^{ABC}\left(u_k^* - v_k^*\,\hat{c}^{\dagger}_k\hat{c}^{\dagger}_{-k}\right)|0\rangle$$

$$= \left(-v_{-k_1}\hat{c}_{-k_1} + u_{-k_1}\hat{c}^{\dagger}_{k_1}\right)\prod_{k>0}^{ABC}\left(u_k^* - v_k^*\,\hat{c}^{\dagger}_k\hat{c}^{\dagger}_{-k}\right)|0\rangle$$

→ $$\boxed{\hat{\gamma}^{\dagger}_{k_1}|\phi_\gamma\rangle^{ABC} = \hat{c}^{\dagger}_{k_1}\prod_{\substack{k>0\\ k\neq|k_1|}}^{ABC}\left(u_k^* - v_k^*\,\hat{c}^{\dagger}_k\hat{c}^{\dagger}_{-k}\right)|0\rangle}$$

⇒ $\hat{\gamma}^{\dagger}_{k_1}$ transforms the Cooper pair at momentum $(|k_1|, -|k_1|)$ into an unpaired fermion in the state

$$\hat{c}^{\dagger}_{k_1}|0\rangle :$$

$\hat{c}^{\dagger}_{k_1}$ ⇒ this cost an extra energy $+\varepsilon_{k_1}$ over the gnd state.

→ There's a problem with parity here because a single unpaired fermion changes the overall fermion parity.

→ Lowest allowed state must involve 2 creation ops: $\hat{\gamma}^{\dagger}_{k_1}$, $\hat{\gamma}^{\dagger}_{k_2}$, with $k_1 \neq |k_2|$;

$$\rightarrow \boxed{\hat{\gamma}^\dagger_{k_1}\hat{\gamma}^\dagger_{k_2}|\phi_\gamma\rangle^{ABC} = \hat{c}^\dagger_{k_1}\hat{c}^\dagger_{k_2}\prod^{ABC}_{\substack{k>0 \\ k\neq|k_1|,|k_2|}}\left(u_k^* - v_k^*\hat{c}^\dagger_k\hat{c}^\dagger_{-k}\right)|0\rangle}$$

The energy of such an excitation is $\boxed{E_0^{ABC} + \varepsilon_1 + \varepsilon_2}$.

<u>Fun fact</u> : if we consider the special case $\hat{\gamma}^\dagger_{k_1}\hat{\gamma}^\dagger_{-k_1}$ we find that

$$\hat{\gamma}^\dagger_{k_1}\hat{\gamma}^\dagger_{-k_1}|\phi_\gamma\rangle^{ABC} = \left(v_{k_1} + u_{k_1}\hat{c}^\dagger_{k_1}\hat{c}^\dagger_{-k_1}\right)\prod^{ABC}_{\substack{k>0 \\ k\neq|k_1|}}\left(u_k^* - v_k^*\hat{c}^\dagger_k\hat{c}^\dagger_{-k}\right)|0\rangle$$

$\Rightarrow$ $\hat{\gamma}^\dagger_{k_1}\hat{\gamma}^\dagger_{-k_1}$ transforms the Cooper pair at momentum $(|k_1|, -|k_1|)$ into the corresponding anti-bonding pair :

↳ Explicitly ...

$$\underbrace{\left(u_{k_1}^* - v_{k_1}^*\hat{c}^\dagger_{k_1}\hat{c}^\dagger_{-k_1}\right)|0\rangle}_{\text{Cooper pair @ } (|k_1|, -|k_1|)} \xrightarrow{\hat{\gamma}^\dagger_{k_1}\hat{\gamma}^\dagger_{-k_1}} \underbrace{\left(v_{k_1} + u_{k_1}\hat{c}^\dagger_{k_1}\hat{c}^\dagger_{-k_1}\right)|0\rangle}_{\text{anti-bonding pair}}$$

this can be checked by substituting in the defn of $\hat{\gamma}^\dagger_{\pm k_1}$.

$\Rightarrow$ this costs an energy of $2\varepsilon_{k_1}$.

- From here, we can construct all excited states for the even ($p=0$) sector... $\rightarrow$ by applying an even number of $\hat{\gamma}_k^\dagger$ to $|\phi_\gamma\rangle^{ABC}$

    where each $\hat{\gamma}_k^\dagger$ costing an energy ε_k (makes sense...)

$\rightarrow$ In the occupation number representation (Fock) we have...

$$\boxed{\begin{aligned} |\Psi_{\{n_k\}}\rangle &= \prod_k^{ABC} \left(\hat{\gamma}_k^\dagger\right)^{n_k} |\phi_\gamma\rangle^{ABC} \quad \text{with } n_k = 0,1 \ \& \ \sum_k^{ABC} n_k \text{ even.} \\ E_{\{n_k\}} &= E_0 + \sum_k^{ABC} n_k \varepsilon_k \end{aligned}}$$

$\rightarrow$ Note that there are a total of 2^{L-1} such states, as required.

In the $p=1$ sector, we must be careful... We should still apply an even number of $\hat{\gamma}_k^\dagger$ to the gnd state $|\phi_\gamma\rangle^{PBC}$ (to preserve fermion parity), including in the choice of unpaired ops $\hat{\gamma}_0^\dagger$, amounting to removing the fermion from the $k=0$ state, and $\hat{\gamma}_\pi^\dagger$ amounting to creating a fermion in the $k=\pi$ state.

Next, we look at how we can relate all this back to the spin representation...

→ $\boxed{\text{Relationship with the spin representation}}$

→ here we relate the spectrum in the fermionic representation & the corresponding physics in the original spin representation.

→ here let us fix $\chi = 1$.

→ look at classical Ising model...

$$\hat{H}_{cl} = -J \sum_{j=1}^{L} \sigma_j^x \sigma_{j+1}^x$$

→ there are 2 degenerate gnd states...

$$|++\ldots+\rangle \quad \& \quad |--\ldots-\rangle$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(1, \pm 1)^T$ denote the 2 eigenstates of σ^x with eigv ± 1.

Recall parity op: $\hat{P} = \prod_{j=1}^{L} \sigma_j^z$; and that $\sigma^z|\pm\rangle = |\mp\rangle$

we have... $\begin{cases} \hat{P}|++\ldots+\rangle = |--\ldots-\rangle \\ \hat{P}|--\ldots-\rangle = |++\ldots+\rangle \end{cases}$

⇒ 2 eig states of $\hat{P}$ must be

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|++\ldots+\rangle \pm |--\ldots-\rangle\right) \rightarrow \hat{P}|\Psi_\pm\rangle = \mp|\Psi_\pm\rangle$$

These two opposite parity states $|\Psi_\pm\rangle$ must be rep. by 2 fermionic gnd states belonging to the ABC/PBC sectors, which are degenerate when $h=0$.

→ Now, let $h \neq 0$ but $|h| \ll J$. Let us consider...

$$\hat{\mathcal{H}}_{OBC} = -J \sum_{j=1}^{L-1} \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x - h \sum_{j=1}^{L} \hat{\sigma}_j^z.$$

At lowest-perturbative order in $|h|/J$, the 2 lowest-energy states have the same form: $\prod|+\rangle$, $\prod|-\rangle$

or $|\Psi_+\rangle$, $|\Psi_-\rangle$.

→ to get higher excitations, consider the domain-wall configs of the form...

$$|\ell\rangle = |\underbrace{-- \ldots -}_{1 \to \ell \text{ sites}} ++ \ldots +\rangle, \quad \ell = 1 \ldots L-1$$

For $h=0$, all $|\ell\rangle$'s are degenerate, and separated from the 2 gnd states by a gap ($2J$).

~~→ can study the effect of small transverse field by perturbation theory...~~

~~The Hamiltonian restricted to the $L-1$ (dim) subspace of the domain-wall excitations has the form~~

$$\cancel{\hat{\mathcal{H}}_{eff} = 2J \sum_{\ell=1}^{L-1} |\ell\rangle\langle\ell| - h \sum_{\ell=1}^{L-2} (|\ell\rangle\langle \ell+1| + h.c.)}$$

let's estimate the separation between the two gnd states originating from the $h=0$ doublet, when $h>0$.

$h=0 \Rightarrow \prod|+\rangle$ & $\prod|-\rangle$ are degenerate.

& this doublet is separated from other states by $2J$.

Now... $\prod|+\rangle$ & $\prod|-\rangle$ are coupled only at order L in perturbation theory...

↳ b/c we need to flip L spins using $\hat{\sigma}_i^z$ to couple one to another.

→ Expect their splitting to be

$$\Delta E_L \sim (h/J)^L \rightarrow \text{exponentially small in } L \text{ for small } |h|.$$

⇒ the resulting eigenstates $|\psi_\pm(h)\rangle$, even and odd approach the 2 eigenstates $|\psi_\pm\rangle$

where again... $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}\left(\prod|+\rangle \pm \prod|-\rangle\right)$.

for $h \to 0$.

⇒ $\boxed{\text{In the thermodynamic limit, break } \mathbb{Z}_2 \text{ symmetry.}}$

At any finite size we have the symmetry-preserving gnd states $|\psi_\pm(h)\rangle \to |\psi_\pm\rangle$ as $h \to 0$.

↳ there are superpositions of macroscopically ordered states $|\pm\rangle_h = \frac{1}{\sqrt{2}}\left(|\psi_+(h)\rangle \pm |\psi_-(h)\rangle\right)$.

→ There can be explicit symmetry breaking in the subspace generated by $|\Psi_\pm(h)\rangle$ <u>only</u> in the thermodynamic limit in which

↳ the 2 macrostates are degenerate & the slightest perturbation selects one of the two macroscopically ordered superpositions $|\pm\rangle_h$.

~~Symmetry~~

Nov 11, 2020

$\boxed{\text{Nambu formalism for the general case}}$

Summary of what we've seen so far..

In the ordered case, $\mathcal{H}$ can be diagonalized by

1) Fourier transform : reducing problem to a collection of (2×2) pseudo spin 1/2 problems
2) Bogoliubov transform ✓

→ In the disordered case, we do kind of the same thing, but we won't be able to reduce to 2×2 problems in a simple way.

Instead, we'll need the Nambu formalism.

Define a column vector $\hat{\Psi}$ and its Hermitian conjugate row vector $\hat{\Psi}^{+}$, each of length $2L$ by

$$\hat{\Psi} = \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_L \\ \hat{c}_1^{+} \\ \vdots \\ \hat{c}_L^{+} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{c}} \\ \hat{\mathbf{c}}^{+} \end{pmatrix}$$

$$\hat{\Psi}^{+} = \begin{pmatrix} \hat{\mathbf{c}}^{+} & \hat{\mathbf{c}} \end{pmatrix}$$

OR

$$\hat{\Psi}_j = \hat{c}_j \;;\; \hat{\Psi}_{j+L} = \hat{c}_j^{+}$$
$$\hat{\Psi}_j^{+} = \hat{c}_j^{+} \;;\; \hat{\Psi}_{j+L}^{+} = \hat{c}_j \qquad j \le L$$

where the $\mathbf{c}_j$'s are from the Jordan-Wigner transf.

Warning

$\hat{\Psi}_j$ satisfies the standard fermionic anti-commutation rule

$$\{ \hat{\Psi}_j, \hat{\Psi}_{j'}^{+} \} = \delta_{j,j'} \quad \forall j \le 2L$$

But note that

$$\{ \hat{\Psi}_j, \hat{\Psi}_{j+L} \} = 1 \quad \forall j \le L$$

↳ we'll worry abt this later

Next, we introduce the SWAP matrix ($2L \times 2L$):

$$\$ = \begin{pmatrix} 0_{L\times L} & \mathbb{1}_{L\times L} \\ \mathbb{1}_{L\times L} & 0_{L\times L} \end{pmatrix}$$

Next, consider a general fermionic quadratic form

$$\hat{H} = \hat{\Psi}^\dagger \mathbb{H} \hat{\Psi} = (\hat{c}^\dagger\ \hat{c}) \begin{pmatrix} A & B^* \\ -B^* & -A \end{pmatrix} \begin{pmatrix} \hat{c} \\ \hat{c}^\dagger \end{pmatrix}$$

- $\mathbb{H}$ is Hermitean
- A ~~& B are~~ also Hermitean ($A = A^\dagger$)
- B is anti-symmetric. ($B = -B^T$)

Note

(particle-hole symmetry: $\mathbb{H}\$ = -\$\mathbb{H}^*$)

we won't worry about this now...

Now, let's look ~~also~~ back at the Ising case:

$$\left\{ \begin{aligned} \hat{\mathbb{H}}_{p=0,1} = -\sum_{j=1}^{L} \left(J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1} + h.c. \right) \\ + \sum_{j=1}^{L} h_j \left(\hat{c}_j \hat{c}_j^\dagger - \hat{c}_j \hat{c}_j^\dagger \right) \end{aligned} \right\}$$

with boundary condition $\hat{c}_{L+1} = (-1)^{p+1} \hat{c}_1$

Note that J_j, h_j are real.

Okay now that we have $H_{p=0}$ $(2L\times 2L)$ and $H_{p=1}$ $(2L\times 2L)$, for each ~~symmetry~~ parity sector ... $p = 0, 1$

since each $\hat{c}_j$ is 2×2

→ The corresponding $2L\times 2L$ matrices H_p are ~~un~~ real & symmetric.

→ A is real & symmetric, B is real & antisymmetric

So,

$$H = \begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \xrightarrow{\text{PBC}} H_p = \begin{pmatrix} A & B \\ \underbrace{-B}_{B^T} & \underbrace{-A}_{-A^T} \end{pmatrix}$$

The structure of A, B are given by

$$\boxed{\begin{aligned} &\begin{cases} A_{j,j} = h_j \\ A_{j,j+1} = A_{j+1,j} = -\dfrac{J_j^+}{2} = \dfrac{-J_j}{2} \end{cases} \\ &\text{and} \qquad\qquad \underline{\text{else} = 0} \\ &\begin{cases} B_{j,j} = 0 \\ B_{j,j+1} = -B_{j+1,j} = -\dfrac{J_j^-}{2} = -\dfrac{x J_j}{2} \end{cases} \end{aligned}}$$

where $J_j^x = J_j(1+x)/2$; $J_j^y = J_j(1-x)/2$

In the PBC-spin case, we get additional matrix elements

$$A_{L,1} = A_{1,L} = (-1)^p \frac{J_L^+}{2} = (-1)^p \frac{J_L}{2}$$

$$B_{L,1} = -B_{1,L} = (-1)^p \frac{J_L^-}{2} = (-1)^p \frac{x J_L}{2}$$

both depending on the fermion parity p.

The OBC case is recovered by setting $J_L = 0$, which makes $H_1 = H_0$. → note that there are no longer the $H_{p=0,1}$ as we've seen before...

$\boxed{\text{Now let us diagonalize } \hat{H}}$

The Bogoliubov-de Gennes Eqn

Consider the eigv problem

$$H \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix} = \begin{pmatrix} A & B \\ -B^* & -A^* \end{pmatrix} \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix} = \varepsilon_\mu \begin{pmatrix} u_\mu \\ v_\mu \end{pmatrix}$$

where u, v are L-dimensional

μ : index referring to the μ-th eigv.

This gives the Bogoliubov-de Gennes eqn

$$\begin{cases} A u_\mu + B v_\mu = \varepsilon_\mu u_\mu \\ -B^* u_\mu - A^* v_\mu = \varepsilon_\mu v_\mu \end{cases}$$

Easy to show that if $(u_\mu, v_\mu)^T$ is eigv with eigv ε_μ

then $(v_\mu^* \; u_\mu^*)^T$ is eigv with eigv $-\varepsilon_\mu$.

In Ising case, A, B are real, so solutions ~~are~~ can be taken to be real...

→ We can organize the eigv in a unitary (or orth. if solutions are real) $2L \times 2L$ matrix:

$$\mathbb{U} = \left(\begin{array}{c|c} u_1 \cdots u_L & v_1^* \cdots v_L^* \\ \hline v_1 \cdots v_L & u_1^* \cdots u_L^* \end{array}\right) = \begin{pmatrix} u, & V^* \\ V & u^* \end{pmatrix}$$

$$(u_n = \{u_1, u_2, \ldots u_L\})$$

where u, V are $L \times L$ matrices. With this, we find that $\mathbb{U}$ diagonalizes $\mathbb{H}$.

$$\boxed{\mathbb{U}^\dagger \mathbb{H} \mathbb{U} = \left(\begin{array}{c|c} \begin{matrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \ddots \\ & & & \varepsilon_L \end{matrix} & \mathcal{O} \\ \hline \mathcal{O} & \begin{matrix} -\varepsilon_1 & & \\ & \ddots & \\ & & -\varepsilon_L \end{matrix} \end{array}\right)}$$

Let's call this $\mathbb{E}_{diag}$.

If we define new Number Fermion operators $\hat{\Phi}, \hat{\Phi}^\dagger$ s.t.

$$\hat{\Psi} = \mathbb{U} \hat{\Phi}$$

then

$$\hat{H} = \hat{\Psi}^\dagger \mathbb{H} \hat{\Psi} = \hat{\Phi}^\dagger \mathbb{U}^\dagger \mathbb{H} \mathbb{U} \hat{\Phi} = \hat{\Phi}^\dagger \mathbb{E}_{diag} \hat{\Phi}$$

In this case,

$$\vec{\hat{\Phi}} = \begin{pmatrix} \hat{\gamma} \\ \hat{\gamma}^{+} \end{pmatrix} = \mathbb{U}^{+} \hat{\Psi} = \begin{pmatrix} U^{+} & V^{+} \\ V^{T} & U^{T} \end{pmatrix} \begin{pmatrix} \hat{c} \\ \hat{c}^{+} \end{pmatrix}$$

More explicitly ...

$$\boxed{\begin{cases} \hat{\gamma}_{\mu} = \sum_{j=1}^{L} \left(U^{*}_{j\mu} \hat{c}_j + V^{*}_{j\mu} \hat{c}_j^{+} \right) \\ \hat{\gamma}_{\mu}^{+} = \sum_{j=1}^{L} \left(V_{j\mu} \hat{c}_j + U_{j\mu} \hat{c}_j^{+} \right) \end{cases}}$$

(One can check that $\vec{\hat{\Phi}}, \hat{\Phi}$ are indeed fermion operators ...)

this can be inverted ... ($\hat{\Psi} = \mathbb{U} \hat{\Phi}$)

$$\boxed{\begin{cases} \hat{c}_j = \sum_{\mu} \left(U_{j\mu} \hat{\gamma}_{\mu} + V^{*}_{j\mu} \hat{\gamma}_{\mu}^{+} \right) \\ \hat{c}_j^{+} = \sum_{\mu} \left(V_{j\mu} \hat{\gamma}_{\mu} + U^{*}_{j\mu} \hat{\gamma}_{\mu}^{+} \right) \end{cases}}$$

So, in terms of $\hat{\gamma}_{\mu}$, $\hat{H}$ reads

$$\hat{H} = \sum_{\mu=1}^{L} \left(\varepsilon_{\mu} \hat{\gamma}_{\mu}^{+} \hat{\gamma}_{\mu} - \varepsilon_{\mu} \hat{\gamma}_{\mu} \hat{\gamma}_{\mu}^{+} \right)$$

$$= \sum_{\mu=1}^{L} 2 \varepsilon_{\mu} \left(\hat{\gamma}_{\mu}^{+} \hat{\gamma}_{\mu} - \frac{1}{2} \right)$$

The gnd state is that annihilated by all $\hat{\gamma}_\mu$...

$$\boxed{\hat{\gamma}_\mu |\phi_g\rangle = 0 \;\forall \mu \iff \hat{H}|\phi_g\rangle = E_0 |\phi_g\rangle \text{ with } E_0 = -\sum_{\mu=1}^{L} \varepsilon_\mu,}$$

The 2^L Fock states can be expressed as

$$\begin{cases} |\psi_{\{n_\mu\}}\rangle = \prod_{\mu=1}^{L} (\hat{\gamma}_\mu^\dagger)^{n_\mu} |\phi_g\rangle & n_\mu = \{0,1\} \\ E_{\{n_\mu\}} = E_0 + 2\sum_\mu n_\mu \varepsilon_\mu \end{cases}$$

<u>BCS - form of the ground state</u>