Lecture 3 - From Maxwell to QED

2.1 Maxwell's Equations

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{1}{\epsilon_0}\rho && (M1)\\ \vec{\nabla}\cdot\vec{B} &= 0 && (M2)\\ \vec{\nabla}\times\vec{E} &= -\frac{\partial}{\partial t}\vec{B} && (M3)\\ \vec{\nabla}\times\vec{B} &= \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}+\mu_0\vec{j} && (M4) \end{aligned}$$

- Lorentz equation ($v \ll c$):

$$m_\alpha \ddot{\vec{r}}_\alpha = q_\alpha\left(\vec{E}+\vec{v}_\alpha\times\vec{B}\right)$$

- Local conservation of charge (from M1 and M4)

$$\frac{\partial}{\partial t}\rho+\vec{\nabla}\cdot\vec{j}=0$$

$$\rho(\vec{r},t)=\sum_\alpha q_\alpha\,\delta(\vec{r}-\vec{r}_\alpha(t))$$

$$\vec{j}(\vec{r},t)=\sum_\alpha q_\alpha\vec{v}_\alpha(t)\delta(\vec{r}-\vec{r}_\alpha(t))$$

- Constants o Motion:

Total energy: $H=\sum_\alpha \frac{1}{2}m_\alpha v_\alpha^2+\frac{\epsilon_0}{2}\int \mathrm{d}^3r\left(\vec{E}^2+\vec{B}^2\right)$

Total Momentum: $\vec{P}=\sum_\alpha m_\alpha\vec{v}_\alpha+\epsilon_0\int \mathrm{d}^3r\vec{E}\times\vec{B}$

Total angular momentum: $\vec{J}=\sum_\alpha \vec{r}_\alpha\times m_\alpha\vec{v}_\alpha+\epsilon_0\int \mathrm{d}^3r\vec{r}\times\left(\vec{E}\times\vec{B}\right)$

2.2 Vector Potential

(M2): $\vec{\nabla}\cdot\vec{B}=0 \Leftrightarrow \boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$

(M3): $\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{\nabla} \times \vec{A} = -\vec{\nabla} \times \left(\frac{\partial}{\partial t}\vec{A}\right)$

$$\rightarrow \vec{\nabla} \times (\vec{E} + \tfrac{\partial}{\partial t}\vec{A}) = 0 \Leftrightarrow \boxed{\vec{E} + \frac{\partial}{\partial t}\vec{A} = -\vec{\nabla} U}$$

Given vector potential $\vec{A}$ and scalar potential U, (M2) and (M3) are automatically verified.

(M1) $\Rightarrow \Delta U = -\frac{1}{\epsilon_0}\rho - \vec{\nabla} \cdot \frac{\partial}{\partial t}\vec{A}$

(M4) $\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)\vec{A} = \mu_0 \vec{j} - \vec{\nabla}\left(\vec{\nabla} \cdot A + \frac{1}{c^2}\frac{\partial}{\partial t}U\right)$

State of the field given by $\vec{A}(\vec{r}, t_0)$ and $\frac{\partial}{\partial t}\vec{A}(\vec{r}, t_0)\ \forall \vec{r}$

- Gauge invariance:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla} f$$
$$U \rightarrow U' = U - \frac{\partial}{\partial t} f$$

Lorentz gauge: $\vec{\nabla} \cdot \vec{A} + \frac{1}{c^2}\frac{\partial}{\partial t}U = 0 \qquad \text{or} \qquad \partial_\mu A^\mu = 0$

(M1)+(M4):

$$\Box U = \frac{1}{\epsilon_0}\rho$$
$$\Box \vec{A} = \mu_0 \vec{j}$$
$$\text{or} \quad \partial_\nu \partial^\nu A^\mu = \mu_0 j^\mu$$

Here $\Box \equiv \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$, $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$, $A^\mu = \left(\frac{1}{c}U, \vec{A}\right)$, $j^\mu = \left(c\rho, \vec{j}\right)$

Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0$

(M1): $\Delta U = -\frac{1}{\epsilon_0}\rho$

Laplace equation for U. $U(\vec{r}, t)$ is completely specified by $\rho(\vec{r}, t)$!

$\Box \vec{A} = \mu_0 \vec{j} - \frac{1}{c^2}\vec{\nabla}\frac{\partial}{\partial t}U$

2.3 Electrodynamics in reciprocal space

$\vec{\mathcal{E}}(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}^3 r\, \vec{E}(\vec{r}, t) e^{-i\vec{k}\cdot\vec{r}}$

$\vec{E}(\vec{r}, t) = \frac{1}{(2\pi)^{3/2}}\int \mathrm{d}k^3\, \vec{\mathcal{E}}(\vec{k}, t) e^{i\vec{k}\cdot\vec{r}}$

same for $\vec{E} \leftrightarrow \vec{\mathcal{E}}$, $\vec{B} \leftrightarrow \vec{\mathcal{B}}$, $\vec{A} \leftrightarrow \vec{\mathcal{A}}$, $U \leftrightarrow \mathcal{U}$, $\rho \leftrightarrow \rho$, $\vec{j} \leftrightarrow \vec{j}$

- Field equations become <u>local</u> in reciprocal space:

$$
\begin{aligned}
i\vec{k} \cdot \vec{\mathcal{E}} &= \frac{1}{\epsilon_0}\rho && (R1)\\
i\vec{k} \cdot \vec{\mathcal{B}} &= 0 && (R2)\\
i\vec{k} \times \vec{\mathcal{E}} &= -\dot{\vec{\mathcal{B}}} && (R3)\\
i\vec{k} \times \vec{\mathcal{B}} &= \frac{1}{c^2}\dot{\vec{\mathcal{E}}} + \mu_0\vec{j} && (R4)
\end{aligned}
$$

- Continuity equation:

$$i\vec{k} \cdot \vec{j} + \dot{\rho} = 0$$

- Fields and potentials:

$$\vec{\mathcal{B}} = i\vec{k} \times \vec{A}$$

$$\vec{\mathcal{E}} = -\dot{\vec{\mathcal{A}}} - i\vec{k}\mathcal{U}$$

- Gauge transformation:

$$
\begin{aligned}
\vec{\mathcal{A}} \to \vec{\mathcal{A}}' &= \vec{\mathcal{A}} + i\vec{k}f\\
\mathcal{U} \to \mathcal{U}' &= \mathcal{U} - \dot{f}
\end{aligned}
$$

- Equations for potentials:

$$k^2\mathcal{U} = \tfrac{1}{\epsilon_0}\rho + i\vec{k} \cdot \dot{\vec{\mathcal{A}}}$$

$$\tfrac{1}{c^2}\ddot{\vec{\mathcal{A}}} + k^2\vec{\mathcal{A}} = \mu_0\vec{j} - i\vec{k}(i\vec{k} \cdot \vec{\mathcal{A}} + \tfrac{1}{c^2}\dot{\mathcal{U}})$$

- Longitudinal and transverse fields:
  - Longitudinal field:

$$
\begin{aligned}
\vec{\nabla} \times \vec{V}_{||}(\vec{r}) &= 0\\
i\vec{k} \times \vec{\mathcal{V}}_{||}(\vec{k}) &= 0
\end{aligned}
$$

  - Transverse field:

$$\vec{\nabla} \cdot \vec{V}_\perp(\vec{r}) = 0$$
$$i\vec{k} \cdot \vec{\mathcal{V}}_\perp(\vec{k}) = 0$$

in reciprocal space, decomposition is simple: (not so in real space!)

$$\begin{aligned} \vec{\mathcal{V}}(\vec{k}) &= \vec{\mathcal{V}}_{||}(\vec{k}) + \vec{\mathcal{V}}_\perp(\vec{k}) \\ \vec{\mathcal{V}}_{||}(\vec{k}) &= \vec{\kappa}(\vec{\kappa} \cdot \vec{\mathcal{V}}(\vec{k})) \qquad \text{parallel to}\, \vec{k}\, \forall\, \vec{k} \\ \vec{\mathcal{V}}_\perp(\vec{k}) &= \vec{\mathcal{V}}(\vec{k}) - \vec{\mathcal{V}}_{||}(\vec{k}) = (\vec{\kappa} \times \vec{\mathcal{V}}) \times \vec{\kappa} \qquad \perp \text{ to}\, \vec{k} \forall \vec{k} \end{aligned}$$

Return to Maxwell:

(M2) $\Rightarrow \vec{\mathcal{B}}_{||} = \vec{0} = \vec{B}_{||}$

$\Rightarrow$ The magnetic field is purely transverse!

(M1)$\Rightarrow \vec{\mathcal{E}}_{||}(\vec{k}) = -\frac{i}{\epsilon_0}\rho(\vec{k})\frac{\vec{k}}{k^2}$

That is a product of two functions of $\vec{k}$ whose Fourier transforms are

$\rho(\vec{k}) \leftrightarrow \rho(\vec{r})$

$$-\frac{i}{\epsilon_0}\frac{\vec{k}}{k^2} \leftrightarrow \frac{(2\pi)^{3/2}}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

The Fourier transform of a product is a convolution:

$$\begin{aligned} \Rightarrow \vec{E}_{||}(\vec{r}, t) &= \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3r'\, \rho(\vec{r}', t)\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0}\sum_\alpha q_\alpha \frac{\vec{r} - \vec{r}_\alpha(t)}{|\vec{r} - \vec{r}_\alpha(t)|^3} \quad (1) \end{aligned}$$

Longitudinal E-field = instantaneous Coulomb field of ρ.

Result independent of gauge!

Q: Does this mean perturbations can travel faster than light?

> Answer: No! Only the total E-field matters, and $E_\perp$ makes E_{total} purely retarded!

Note: i)

$$\vec{\mathcal{E}} = -\dot{\vec{\mathcal{A}}} - i\vec{k}\mathcal{U}$$

$$\Rightarrow \vec{\mathcal{E}}_\perp = -\dot{\vec{\mathcal{A}}}_\perp \qquad \Rightarrow \vec{E}_\perp = -\dot{\vec{A}}_\perp$$

$$\vec{\mathcal{E}}_{||} = -\dot{\vec{\mathcal{A}}}_{||} - i\vec{k}\mathcal{U} \quad \Rightarrow \vec{E}_{||} = -\dot{\vec{A}}_{||} - \vec{\nabla}U$$

In the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0 \Rightarrow i\vec{k} \cdot \vec{\mathcal{A}} = 0 \Rightarrow \vec{\mathcal{A}}_{||} = 0$.

$$\vec{E}_\perp = -\dot{\vec{A}}_\perp \qquad \text{transverse field given by} \vec{A}$$

$$\vec{E}_{||} = -\vec{\nabla}U \qquad \text{longitudinal field given by} U$$

From $\vec{E}_{||}$ we find $U = \frac{1}{4\pi\epsilon_0}\int \mathrm{d}^3r' \frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$ Coulomb potential.

ii) gauge transformation does not change $\vec{A}_\perp$!

transverse vector potential $\vec{A}_\perp$ is gauge invariant:

$$\vec{A}'_\perp = \vec{A}_\perp$$

iii) generally, in any gauge:

$$\vec{\mathcal{E}}_\perp = -\dot{\vec{\mathcal{A}}}_\perp$$

$$\vec{\mathcal{B}} = \vec{\mathcal{B}}_\perp = i\vec{k} \times \vec{\mathcal{A}}_\perp$$

Transverse fields only depend on $\vec{\mathcal{A}}_\perp$.

iv) longitudinal part of (M4) gives:

$$\dot{\vec{\mathcal{E}}}_{||} + \frac{1}{\epsilon_0}\vec{j}_{||} = 0$$

$$\Rightarrow i\vec{k} \cdot \dot{\vec{\mathcal{E}}}_{||} + \frac{1}{\epsilon_0} i\vec{k} \cdot \vec{j}_{||} = i\vec{k} \cdot \dot{\vec{\mathcal{E}}} + \frac{1}{\epsilon_0} i\vec{k} \cdot \vec{j} = \frac{\dot{\rho}}{\epsilon_0} + \frac{i\vec{k}\cdot\vec{j}}{\epsilon_0} = 0$$

$$\Rightarrow \dot{\rho} + \vec{\nabla} \cdot \vec{j} = 0$$

This is just the continuity equation, so this does not add anything new.

- Total Energy:

$$\frac{\epsilon_0}{2}\int \mathrm{d}^3 r\, \vec{E}\cdot\vec{E} = \frac{\epsilon_0}{2}\int \mathrm{d}^3 k\, \vec{\mathcal{E}}^*\cdot\vec{\mathcal{E}}$$

(Parseval-Plancherel Theorem)

$$\vec{\mathcal{E}} \to \vec{\mathcal{E}}_\perp + \vec{\mathcal{E}}_{||}, \qquad \vec{\mathcal{E}}_\perp\cdot\vec{\mathcal{E}}_{||} = 0$$

$$\Rightarrow \frac{\epsilon_0}{2}\int \mathrm{d}^3 r\, \vec{E}^2 = \frac{\epsilon_0}{2}\int \mathrm{d}^3 k\, \left|\vec{\mathcal{E}}_{||}(\vec{k})\right|^2 + \frac{\epsilon_0}{2}\int \mathrm{d}^3 k\, \left|\vec{\mathcal{E}}_{\perp}(\vec{k})\right|^2$$

$$H_{\text{long}} = \frac{\epsilon_0}{2}\int \mathrm{d}^3 k \left|\vec{\mathcal{E}}_{||}(\vec{k})\right|^2 = \frac{\epsilon_0}{2}\int \mathrm{d}^3 r\, \vec{E}_{||}^2(\vec{r})$$

$$\begin{aligned} H_{\text{trans}} &= \frac{\epsilon_0}{2}\int \mathrm{d}^3 k \left(\left|\vec{\mathcal{E}}_{\perp}(\vec{k})\right|^2 + c^2\left|\vec{\mathcal{B}}(\vec{k})\right|^2\right) \\ &= \frac{\epsilon_0}{2}\int \mathrm{d}^3 r \left(\left|\vec{E}_{\perp}(\vec{r})\right|^2 + c^2\vec{B}^2(\vec{r})\right) \end{aligned}$$

H_{long} is the Coulomb energy:

$$H_{\text{long}} = \frac{1}{2\epsilon_0}\int \mathrm{d}^3 k\, \rho^*(\vec{k})\,\frac{\rho(\vec{k})}{k^2} = \frac{1}{8\pi\epsilon_0}\iint \mathrm{d}^3 r\,\mathrm{d}^3 r'\,\frac{\rho(\vec{r})\rho(\vec{r}\,')}{(|\vec{r}-\vec{r}\,'|)}$$

For a system of charges:

$$\begin{aligned} \rho(\vec{r}) &= \sum_\alpha q_\alpha \delta(\vec{r}-\vec{r}_\alpha(t)) \\ \rho(\vec{k}) &= \sum_\alpha \frac{q_\alpha}{(2\pi)^{3/2}} e^{-i\vec{k}\cdot\vec{r}_\alpha} \end{aligned}$$

$$\begin{aligned} H_{\text{long}} = V_{\text{Coulomb}} &= \sum_\alpha \frac{q_\alpha^2}{2\epsilon_0(2\pi)^3}\int \mathrm{d}^3 k \frac{1}{k^2} + \sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{2\epsilon_0(2\pi)^3}\int \mathrm{d}^3 k \frac{e^{-i\vec{k}\cdot(\vec{r}_\alpha-\vec{r}_\beta)}}{k^2} \\ &= \sum_\alpha \epsilon^\alpha_{\text{Coulomb}} + \frac{1}{8\pi\epsilon_0}\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\vec{r}_\alpha-\vec{r}_\beta|} \end{aligned}$$

The first term

$$\epsilon^{\alpha}_{\text{Coulomb}} = \frac{q_\alpha^2}{2\epsilon_0(2\pi)^3}\int \mathrm{d}^3k\frac{1}{k^2} = \frac{q_\alpha^2}{4\pi^2\epsilon_0}k_c$$

is the Coulomb self energy of the particle α, infinite unless we introduce a cut-off k_c. Physically, k_c should be the momentum scale at which our non-relativistic treatment breaks down, so $k_c \approx \frac{1}{\lambda_{\text{Compton}}}$, with $\lambda_{\text{Compton}} = \frac{\hbar}{m_e c}$.

The second term is the Coulomb interaction between particles.

Finally we obtain the Hamiltonian:

$$\boxed{H = \sum_\alpha \frac{1}{2}m_\alpha^2\dot{\vec{r}}_\alpha^{\,2} + V_{\text{Coulomb}} + H_{\text{trans}}}$$

- Total Momentum:

Let's use again $\vec{E} \to \vec{E}_\perp + \vec{E}_{||}$ in the definition of momentum, and find the parts:

$$\vec{P}_{\text{long}} = \epsilon_0\int \mathrm{d}^3r\vec{E}_{||}(\vec{r})\times\vec{B}(\vec{r}) = \epsilon_0\int \mathrm{d}^3k\,\vec{\mathcal{E}}^*_{||}(\vec{k})\times\vec{\mathcal{B}}(\vec{k})$$

$$\vec{P}_{\text{trans}} = \epsilon_0\int \mathrm{d}^3r\vec{E}_{\perp}(\vec{r})\times\vec{B}(\vec{r}) = \epsilon_0\int \mathrm{d}^3k\,\vec{\mathcal{E}}^*_{\perp}(\vec{k})\times\vec{\mathcal{B}}(\vec{k})$$

We plug in: $\vec{\mathcal{E}}_{||} = -\frac{i}{\epsilon_0}\rho(\vec{k})\frac{\vec{k}}{k^2}$; $\qquad \vec{\mathcal{B}} = i\vec{k}\times\vec{\mathcal{A}}$

$$\begin{aligned}\vec{P}_{\text{long}} &= \int \mathrm{d}^3k\,\rho^*\frac{\vec{k}}{k^2}\times(i\vec{k}\times\vec{\mathcal{A}})\\ &= \int \mathrm{d}^3k\rho^*[\vec{\mathcal{A}} - \vec{\kappa}(\vec{\kappa}\cdot\vec{\mathcal{A}})]\\ &= \int \mathrm{d}^3k\,\rho^*\vec{\mathcal{A}}_\perp = \int \mathrm{d}^3r\,\rho\vec{A}_\perp = \sum_\alpha q_\alpha\vec{A}_\perp(\vec{r}_\alpha)\end{aligned}$$

Independent of gauge, as $\vec{A}_\perp$ is independent of gauge.

total momentum:

$$\vec{P} = \sum_\alpha \left(m_\alpha \dot{\vec{r}}_\alpha + q_\alpha \vec{A}_\perp(\vec{r}_\alpha)\right) + \vec{P}_{\text{trans}}$$

The three terms are 1) the mechanical momentum, 2) the longitudinal field momentum and 3) the transverse field momentum.

$$\vec{P} = \sum_\alpha \vec{p}_\alpha + \vec{P}_{\text{trans}}$$

$$\vec{p}_\alpha = m_\alpha \dot{\vec{r}}_\alpha + q_\alpha \vec{A}_\perp(\vec{r}_\alpha)$$

In Coulomb gauge, $\vec{p}_\alpha$ is the canonical (or generalized) momentum to the coordinate $\vec{r}_\alpha$ of particle α.

Total energy:

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left[\vec{p}_\alpha - q_\alpha \vec{A}_\perp(\vec{r}_\alpha)\right]^2 + V_{\text{Coulomb}} + H_{\text{trans}}$$

This is precisely the Hamiltonian of the system in the coulomb gauge!

Total angular momentum:

$$\vec{J} = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha + \epsilon_0 \int \mathrm{d}^3 r\, \vec{r} \times (\vec{E}_\perp \times \vec{B})$$

2.4 Normal Variables

Maxwell's equations for transverse fields:

(M3): $\dot{\vec{\mathcal{B}}} = -i\vec{k} \times \vec{\mathcal{E}}_\perp$

$\Rightarrow \boxed{\vec{k} \times \dot{\vec{\mathcal{B}}} = ik^2 \vec{\mathcal{E}}_\perp}$

(M4): $\boxed{\dot{\vec{\mathcal{E}}}_\perp = ic^2 \vec{k} \times \vec{\mathcal{B}} - \frac{1}{\epsilon_0}\vec{j}_\perp(\vec{k}, t)}$

These are two coupled equations for $\vec{\mathcal{E}}_\perp$, $\vec{k} \times \vec{\mathcal{B}}$, that are local in reciprocal space.

Using $\vec{\mathcal{B}} = i\vec{k} \times \vec{\mathcal{A}} = i\vec{k} \times \vec{\mathcal{A}}_\perp$ we have

$\vec{k} \times \vec{\mathcal{B}} = -ik^2\vec{\mathcal{A}}_\perp$

Therefore we have, with $\omega = ck$

$$\boxed{\begin{aligned}\dot{\vec{\mathcal{A}}}_\perp &= -\vec{\mathcal{E}}_\perp \\ \dot{\vec{\mathcal{E}}}_\perp &= \omega^2\vec{\mathcal{A}}_\perp - \frac{1}{\epsilon_0}\vec{j}_\perp(\vec{k},t)\end{aligned}}$$

This is two coupled differential equations, just like

$$\boxed{\begin{aligned}\dot{\vec{x}} &= \frac{\vec{p}}{m} \\ \frac{\dot{\vec{p}}}{m} &= -\omega_0^2\vec{x} + \frac{q}{m}\vec{E}(t)\end{aligned}}$$

So we read off the correspondence $\vec{x} \equiv \vec{\mathcal{A}}_\perp$ and $\frac{\vec{p}}{m} \equiv -\vec{\mathcal{E}}_\perp$.

The goal now is to uncouple these equations, at least for the free field, $\vec{j}_\perp = 0$.

Normal variables for the position / momentum case:

$$\alpha = \mathcal{N}(x + i\frac{p}{m\omega_0})$$
$$\beta = \mathcal{N}(x - i\frac{p}{m\omega_0}) = \alpha^*$$

equations of motion: $\dot{\alpha} = \mathcal{N}(\dot{x} + i\frac{\dot{p}}{m\omega_0}) = \mathcal{N}(\frac{p}{m} - i\omega_0 x + i\frac{q}{m\omega_0}E) = -i\omega_0\alpha + i\frac{q\mathcal{N}}{m\omega_0}E$

$$\boxed{\dot{\alpha} = -i\omega_0\alpha + i\frac{q\mathcal{N}}{m\omega_0}E}$$

quantization:

$\alpha \rightarrow a$

$\beta \rightarrow a^\dagger$

$\left[a, a^\dagger\right] = -\mathcal{N}^2\frac{i}{m\omega_0}2\left[x,p\right] = \mathcal{N}^2\frac{2\hbar}{m\omega_0} \stackrel{!}{=} 1$

Therefore $\mathcal{N} = \sqrt{\frac{m\omega_0}{2\hbar}} = \frac{1}{\sqrt{2}}\frac{1}{a_{\text{h.o.}}}$.

Using our correspondence $\vec{x} \equiv \vec{\mathcal{A}}_\perp$ and $\frac{\vec{p}}{m} \equiv -\vec{\mathcal{E}}_\perp$ we find

$$\boxed{\begin{aligned}\vec{\alpha}(\vec{k},t) &= \mathcal{N}(k)\left(\vec{\mathcal{A}}_\perp - i\frac{\vec{\mathcal{E}}_\perp}{\omega_0}\right)\\ \vec{\beta}(\vec{k},t) &= \mathcal{N}(k)\left(\vec{\mathcal{A}}_\perp + i\frac{\vec{\mathcal{E}}_\perp}{\omega_0}\right)\end{aligned}}$$

In terms of $\vec{\mathcal{B}} = i\vec{k}\times\vec{\mathcal{A}}_\perp \Rightarrow \vec{\mathcal{A}}_\perp = i\frac{\vec{k}}{k^2}\times\vec{\mathcal{B}} = \frac{i}{k}\vec{\kappa}\times\vec{\mathcal{B}}$.

$$\boxed{\begin{aligned}\vec{\alpha}(\vec{k},t) &= \frac{i}{ck}\mathcal{N}(k)\left(c\vec{\kappa}\times\vec{\mathcal{B}} - \vec{\mathcal{E}}_\perp\right)\\ \vec{\beta}(\vec{k},t) &= \frac{i}{ck}\mathcal{N}(k)\left(c\vec{\kappa}\times\vec{\mathcal{B}} + \vec{\mathcal{E}}_\perp\right)\end{aligned}}$$

Note: $\vec{E}_\perp$ is real, so $\vec{\mathcal{E}}^*(\vec{k},t) = \vec{\mathcal{E}}(-\vec{k},t)$ and thus

$$\begin{aligned}\alpha^*(\vec{k},t) &= -\frac{i}{ck}\mathcal{N}(k)\left(c\vec{\kappa}\times\vec{\mathcal{B}}^*(\vec{k}) - \vec{\mathcal{E}}^*_\perp(\vec{k})\right)\\ &= -\frac{i}{ck}\mathcal{N}(k)\left(c\vec{\kappa}\times\vec{\mathcal{B}}(-\vec{k}) - \vec{\mathcal{E}}_\perp(-\vec{k})\right)\\ &= -\frac{i}{ck}\mathcal{N}(k)\left(-c(-\vec{\kappa})\times\vec{\mathcal{B}}(-\vec{k}) - \vec{\mathcal{E}}_\perp(-\vec{k})\right)\\ &= +\vec{\beta}(-\vec{k},t) \qquad (2)\end{aligned}$$

So $\vec{\beta}(\vec{k},t)$ is not independent of $\vec{\alpha}(\vec{k},t)$.

Therefore knowing $\vec{\alpha}(\vec{k},t)\forall\vec{k}$ is fully equivalent to knowing $\vec{\mathcal{E}}_\perp$ and $\vec{\mathcal{B}}$.

- Time evolution:

$$\boxed{\dot{\vec{\alpha}}(\vec{k},t) + i\omega_0\vec{\alpha}(\vec{k},t) = \frac{i\mathcal{N}(k)}{\epsilon_0\omega}\vec{j}_\perp(\vec{k},t)}$$

This equation is strictly equivalent to Maxwell's equations!

$$\begin{aligned}\vec{\mathcal{E}}_\perp(\vec{k},t) &= \frac{i}{2}\frac{\omega_0}{\mathcal{N}(k)}\left(\vec{\alpha}(\vec{k},t) - \vec{\alpha}^*(-\vec{k},t)\right)\\ \vec{\mathcal{B}}(\vec{k},t) &= \frac{i}{2}\frac{k}{\mathcal{N}(k)}\vec{\kappa}\times\left(\vec{\alpha}(\vec{k},t) + \vec{\alpha}^*(\vec{k},t)\right)\end{aligned}$$

So what we have is equivalent to a driven harmonic oscillator with eigenfrequency $\omega = ck$, driven by a source term proportional to $\vec{j}_\perp(\vec{k},t)$.

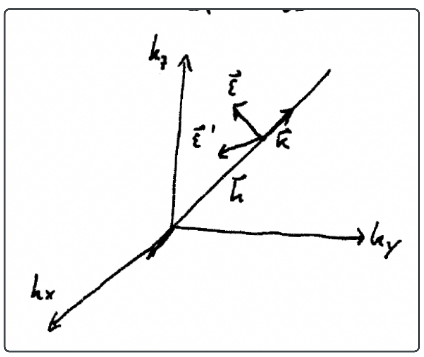
- Comments:
- $\vec{j}_\perp = 0 \Rightarrow \vec{\alpha}(\vec{k},t) = \alpha(\vec{k},0)e^{-i\omega_0 t} \qquad \text{free evolution}$

  $\Rightarrow$ the $\vec{\alpha}(\vec{k},t)$ are "normal variables"

- If $\vec{j}_\perp(\vec{k},t)$ is from an external source (i.e. independent of the $\vec{\alpha}(\vec{k},t)$), the $\vec{\alpha}(\vec{k},t)$ at the various $\vec{k}$ still evolve independently from each other.

  If $\vec{j}_\perp(\vec{k},t)$ are particles interacting with the field, then $\vec{j}_\perp$ depends on $\vec{\alpha}$, so generally the various `$\vec{\alpha}(\vkt,t)$` are coupled through $\vec{j}_\perp$. We thus need to solve the coupled problem: Maxwell and Lorentz equation.

- Notation: $\vec{\alpha}$ is (like $\vec{\mathcal{E}}_\perp$ and $\vec{\mathcal{B}}$) transverse. So for each $\vec{k}$, one may expand $\vec{\alpha}$ into two unit vectors $\vec{\epsilon}$ and $\vec{\epsilon}'$, normal to each other and both located in the plane normal to $\vec{\kappa}$.

  $\vec{\epsilon}\cdot\vec{\epsilon} = \vec{\epsilon}'\cdot\vec{\epsilon}' = \vec{\kappa}\cdot\vec{\kappa} = 1$

  $\vec{\epsilon}\cdot\vec{\epsilon}' = \vec{\epsilon}\cdot\vec{\kappa} = \vec{\epsilon}'\cdot\vec{\kappa} = 0$

  $$\Rightarrow \begin{aligned} \vec{\alpha}(\vec{k},t) &= \vec{\epsilon}\alpha_\epsilon(\vec{k},t) + \vec{\epsilon}'\alpha_{\epsilon'}(\vec{k},t) \\ &= \sum_\epsilon \vec{\epsilon}\alpha_\epsilon(\vec{k},t) \end{aligned}$$

  with $\alpha_\epsilon(\vec{k},t) = \vec{\epsilon}\cdot\vec{\alpha}(\vec{k},t)$ the component of $\vec{\alpha}$ along $\vec{\epsilon}$.

  The $\left\{\alpha_\epsilon(\vec{k},t)\right\}$ form a complete set of independent variables

  $\dot{\alpha}_\epsilon + i\omega\alpha_\epsilon = \frac{i\mathcal{N}(k)}{\epsilon_0\omega}\vec{\epsilon}\cdot\vec{j}$

- The energy can be expressed in terms of the α_ϵ as

$$\begin{aligned}
H_{\text{trans}} &= \frac{\epsilon_0}{2}\int \mathrm{d}^3 r \left(\vec{E}_\perp^2(\vec{r}) + c^2\vec{B}^2(\vec{r})\right) \\
&= \frac{\epsilon_0}{2}\int \mathrm{d}^3 k \left(\vec{\mathcal{E}}_\perp^*(\vec{k})\vec{\mathcal{E}}_\perp(\vec{k}) + c^2\vec{\mathcal{B}}^*(\vec{k})\vec{\mathcal{B}}(\vec{k})\right) \\
&= \epsilon_0 \int \mathrm{d}^3 k \frac{\omega_0^2}{4\mathcal{N}^2(k)} \left(\vec{\alpha}^*(\vec{k})\cdot\vec{\alpha}(\vec{k}) + \vec{\alpha}(-\vec{k})\cdot\vec{\alpha}^*(-\vec{k})\right) \\
&\stackrel{!}{=} \int \mathrm{d}^3 k \sum_\epsilon \frac{\hbar\omega}{2}\left(\alpha_\epsilon^*(\vec{k},t)\alpha_\epsilon(\vec{k},t) + \alpha_\epsilon(\vec{k},t)\alpha_\epsilon^*(\vec{k},t)\right)
\end{aligned}$$

We choose therefore $\mathcal{N}^2 = \frac{\epsilon_0\omega^2}{2\hbar\omega} = \frac{\epsilon_0\omega}{2\hbar}$ or

$$\mathcal{N} = \sqrt{\frac{\epsilon_0\omega}{2\hbar}}$$

This will also ensure that $\left[a(\vec{k}), a^\dagger(\vec{k})\right] = \delta(\vec{k}-\vec{k}')$

Momentum:

$$\vec{P}_{\text{trans}} = \int \mathrm{d}^3 k \sum_\epsilon \frac{\hbar\vec{k}}{2}\left(\alpha_\epsilon^*\alpha_\epsilon + \alpha_\epsilon\alpha_\epsilon^*\right)$$

Fields:

$$\vec{E}_\perp(\vec{r},t) = i\int \mathrm{d}^3 k \sum_\epsilon \mathcal{E}_\omega \left(\alpha_\epsilon(\vec{k},t)\vec{\epsilon}e^{i\vec{k}\cdot\vec{r}} - \alpha_\epsilon^*(\vec{k},t)\vec{\epsilon}e^{-i\vec{k}\cdot\vec{r}}\right)$$

$$\vec{B}(\vec{r},t) = i\int \mathrm{d}^3 k \sum_\epsilon \mathcal{B}_\omega \left(\alpha_\epsilon(\vec{k},t)\vec{\kappa}\times\vec{\epsilon}\, e^{i\vec{k}\cdot\vec{r}} - \alpha_\epsilon^*(\vec{k},t)\vec{\kappa}\times\vec{\epsilon}\, e^{-i\vec{k}\cdot\vec{r}}\right)$$

with $\mathcal{E}_\omega = \sqrt{\frac{\hbar\omega}{2\epsilon_0(2\pi)^3}}$ and $\mathcal{B}_\omega = \frac{\mathcal{E}_\omega}{c}$.

For free fields $\vec{j}_\perp = \vec{0}$: $\alpha_\epsilon(\vec{k},t) = \alpha_\epsilon(\vec{k})e^{-i\omega t}$, so

$$\vec{E}_\perp(\vec{r},t) = i\int \mathrm{d}^3 k \sum_\epsilon \mathcal{E}_\omega\alpha_\epsilon(\vec{k})\vec{\epsilon}e^{i(\vec{k}\cdot\vec{r}-\omega t)} + \text{c.c.}$$

Notation:

$$\vec{E}_\perp^+(\vec{r},t) = i\int \mathrm{d}^3k \sum_\epsilon \mathcal{E}_\omega \alpha_\epsilon(\vec{k})\vec{\epsilon}\, e^{i\vec{k}\cdot\vec{r}}$$