Prop: if $G \leq GL(n, \mathbb{C})$ is a matrix Lie group then

$\mathrm{Lie}(G) = \{X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in G, \forall \varepsilon \in \mathbb{R}\}$ with $[X,Y] = XY - YX$ is a Lie algebra

proof:

- $[\cdot,\cdot]$ is anti-commutative, bilinear, and satisfies Jacobi identity
- 0 matrix belongs to $\mathrm{Lie}(G)$ since $e^{\varepsilon 0} = e^0 = \mathbb{1} \in G \quad \forall \varepsilon \in \mathbb{R}$
- if $X \in \mathrm{Lie}(G)$ and $\alpha \in \mathbb{R}$ then $e^{\varepsilon \alpha X} \in G \quad \forall \varepsilon \in \mathbb{R}$ since $\varepsilon\alpha \in \mathbb{R}$
  $\Rightarrow \alpha X \in \mathbb{R}$
- if $X, Y \in \mathrm{Lie}(G)$ then $e^{\varepsilon(X+Y)} = \lim_{k \to \infty} \Big(\underbrace{e^{\varepsilon X/k}}_{\in G} \underbrace{e^{\varepsilon Y/k}}_{\in G} \Big)^k$
  which is a limit of products of elements of G. since G is closed, the limit converges to something in G $\Rightarrow$ $X+Y \in \mathrm{Lie}(G)$
- This means that $\mathrm{Lie}(G)$ is a subspace of $M_n(\mathbb{C})$ (as a real vector space)

$\rightarrow$ $\mathrm{Lie}(G)$ is a real Lie algebra.