8.422 Problem Set 1

Due: Friday 5pm, February 17th, either via Canvas upload or on paper in envelope outside 26-255
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Office hour: TBA through Canvas announcement

1 Origin of Radiation Reaction

[6/20 points] We will investigate the origin of the damping term that we introduced in the Newton-Lorentz equation for the motion of an electron. We know that accelerated particles radiate, according to the Larmor formula for the radiated power of an accelerating charge

$$P = \frac{1}{6\pi\epsilon_0}\frac{q^2}{c^3}\left(\dot{\vec{v}}\right)^2$$

So we should account for this loss of energy by adding a *radiative reaction force* $\vec{F}_{\text{rad}}$. It should be zero if $\dot{\vec{v}} = 0$ since then there is no radiation, and it should be proportional to q^2 just as the radiated power, and since the force should be the same for positive and negative charges. Let us demand that the work done by this force on the particle is equal to the negative of the energy radiated in that time, i.e.

$$\int_{t_1}^{t_2} \vec{F}_{\text{rad}} \cdot \vec{v}\,\mathrm{d}t = -\int_{t_1}^{t_2} \frac{1}{6\pi\epsilon_0}\frac{q^2}{c^3}\dot{\vec{v}}\cdot\dot{\vec{v}}\,\mathrm{d}t$$

Assuming that the motion is periodic such that $(\dot{\vec{v}}\cdot\vec{v}) = 0$ at $t = t_1$ and $t = t_2$, find a candidate expression for the radiation force, in terms of q, c and $\vec{v}$ or its derivatives.
For a direct derivation of this force from Maxwell's equations, see Dalibard, Dupont-Roc, Cohen-Tannoudji, J. Physique **43** (1982) 1617-1638, and Exercise 5 in Photons & Atoms, posted on Canvas.
Beware of the $\cdot$ above the $\vec{v}$ in the previous formula, and integrate by parts:

$$\begin{aligned}\int_{t_1}^{t_2} \vec{F}_{rad}\cdot\vec{v}\,dt &= -\int_{t_1}^{t_2} \frac{q^2}{6\pi\epsilon_0 c^3}\dot{\vec{v}}\cdot\dot{\vec{v}}\,dt \\ &= \left[-\frac{q^2}{6\pi\epsilon_0 c^3}\dot{\vec{v}}\cdot\vec{v}\right]_{t_1}^{t_2} + \int_{t_1}^{t_2}\frac{q^2}{6\pi\epsilon_0 c^3}\ddot{\vec{v}}\cdot\vec{v}\,dt \\ &= \int_{t_1}^{t_2}\frac{q^2}{6\pi\epsilon_0 c^3}\ddot{\vec{v}}\cdot\vec{v}\,dt\end{aligned}$$

Since the motion is periodic, the boundary term is zero and we extract $\vec{F}_{rad} = \frac{q^2}{6\pi\epsilon_0 c^3}\ddot{\vec{v}}$.

2 Cross section for scattering of radiation in the Lorentz model

A classical electron with charge q and mass m is elastically bound to the origin by a restoring force $-m\omega_0^2\vec{r}$, and is set into forced motion by an incident monochromatic wave with frequency ω and emits into all space radiation of the same frequency. We will calculate the total scattering cross section $\sigma(\omega)$ of the electron and investigate its variation with ω. The electron undergoes forced motion along the z-axis of amplitude a and frequency ω, i.e. $z(t) = a\cos(\omega t)$. This leads to a time-averaged radiated power of

$$P = \frac{1}{3}\frac{q^2}{4\pi\epsilon_0}\frac{a^2\omega^4}{c^3} \tag{1}$$

(the factor of 1/2 compared to the previous problem is from the time-averaging of the oscillatory motion). We include the work done in radiating light using the radiation reaction force, which we can write

$$F_{\text{rad}} = \frac{2}{3}\frac{r_0}{\lambda\!\!\!^{-}_0}\frac{m\dddot{z}}{\omega_0}$$

where $\lambda\!\!\!^{-}_0 = c/\omega_0$ and r_0 is the classical electron radius given by $r_0 = \frac{q^2}{4\pi\epsilon_0 mc^2}$.

1. *[2/20 points]* In the absence of incident radiation, the dynamical equation for the electron is written

$$m\ddot{z} = -m\omega_0^2 z + \frac{2}{3}\frac{r_0}{\lambda\!\!\!^{-}_0}\frac{m\dddot{z}}{\omega_0} \tag{2}$$

The radiation reaction force, proportional to $r_0/\lambda\!\!\!^{-}_0 \ll 1$, can be treated as a perturbation. Find the solutions of Eq. 2 of the form $e^{i\Omega t}$ and show that, to first order in $r_0/\lambda\!\!\!^{-}_0$, one has

$$\Omega = \pm\omega_0 + i\frac{\gamma_0}{2}$$

Give the expression for γ_0 as a function of r_0, ω_0 and c. What does the time $\tau_0 = \gamma_0^{-1}$ represent?
Hint: What is Ω to zero order in $r_0/\lambda\!\!\!^{-}_0$? To obtain the first order, replace Ω^3 by the cube of that zero order result, and approximate $\Omega^2 - \omega_0^2 \approx \pm 2\omega_0(\Omega \mp \omega_0)$.

It first doesn't hurt to evaluate the accuracy of our approximation. For the hydrogen atom where q and m are the charge and mass of the electron, we find $r_0 = 3\cdot 10^{-15}$m and $\lambda\!\!\!^{-}_0 = 1\cdot 10^{-8}$m if we take the ionization energy of the electron in the $1s^1$ ground state as a reference for $\hbar\omega_0$. The approximation is therefore very good for an atom.

Substituting $z = z(0)e^{i\Omega t}$, the equation becomes:

$$-m\Omega^2 = -m\omega_0^2 - i\frac{2}{3}\frac{r_0}{\lambda\!\!\!^{-}}\frac{m\Omega^3}{\omega_0}$$

To zero order in $r_0/\lambda\!\!\!^{-}$, $\Omega = \pm\omega_0$, substituting:

$$\Omega^2 = \omega_0^2 \pm i\frac{2}{3}\frac{r_0}{\lambda\!\!\!^{-}}\omega_0^2 = \omega_0^2\left(1 \pm i\frac{2}{3}\frac{r_0}{\lambda\!\!\!^{-}}\right)$$

$$\Omega \approx \pm\omega_0 + i\frac{\gamma_0}{2}$$

where $\gamma_0 = \frac{2r_0\omega_0}{3\lambda\!\!\!^{-}_0} = \frac{2r_0\omega_0^2}{3c}$. The time constant $\tau_0 = \gamma_0^{-1}$ is the characteristic time in which the envelope of the solution $z(t)$ has decayed by a factor $e^{-1/2}$ since the initial time.

2. *[1/20 points]* In the presence of an incident field polarized along the z-axis whose amplitude at the origin is $E\cos(\omega t)$, the dynamical equation for the electron is

$$m\ddot{z} = -m\omega_0^2 z + \frac{2}{3}\frac{r_0}{\lambda\!\!\!^{-}_0}\frac{m\dddot{z}}{\omega_0} + qE\cos(\omega t) \tag{3}$$

By making the Ansatz $z = \mathrm{Re}(z_0 e^{i\omega t})$, find the forced oscillatory motion of the electron. By replacing a^2 with $|z_0|^2$ in Eq. 1, find the expression for the power radiated by the electron P_{out}.

Just as a reminder, looking for complex solutions is valid since the differential equation is linear with constant coefficients. If $x(t)$ is solution of the equation written as is, and $y(t)$ solution of the equation translated by $T/4 = \pi/(2\omega)$, then $\bar{z}(t) = x(t) + iy(t)$ is solution of the full equation written now with $e^{i\omega t}$ instead of $\cos\omega t$: Searching $\bar{z}(t) = z_0 e^{i\omega t}$, the equation becomes:

$$-m\omega^2 z_0 e^{i\omega t} = -m\omega_0^2 z_0 e^{i\omega t} - i\frac{m\gamma_0\omega^3}{\omega_0^2} z_0 e^{i\omega t} + qEe^{i\omega t}$$

$$z_0 = \frac{qE}{m}\frac{1}{\omega_0^2 - \omega^2 + i\gamma_0\frac{\omega^3}{\omega_0^2}}$$

The power radiated by the electron is:

$$P_{out} = \frac{1}{3}\frac{q^2}{4\pi\epsilon_0}\frac{a^2\omega^4}{c^3} = \frac{1}{3}\frac{q^2}{4\pi\epsilon_0}\frac{\omega^4}{c^3}\frac{q^2E^2}{m^2}\frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma_0^2\frac{\omega^6}{\omega_0^4}}$$

3. *[1/20 points]* The incoming flux of energy (averaged over one period $2\pi/\omega$) from the incident wave, assumed to be plane, is $\phi_{\mathrm{in}} = \epsilon_0 cE^2/2$. From this and P_{out}, find the total cross section $\sigma(\omega)$. Express $\sigma(\omega)$ as a function of r_0^2, ω, ω_0 and γ_0.

   By definition $\sigma(\omega) = P_{out}/\phi_{in}$, so:

$$\sigma(\omega) = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0^2}\frac{\omega^4}{c^4}\frac{q^2}{m^2}\frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma_0^2\frac{\omega^6}{\omega_0^4}} = \frac{8\pi}{3}r_0^2\frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + \gamma_0^2\frac{\omega^6}{\omega_0^4}}$$

4. *[1/20 points]* Assume $\omega \ll \omega_0$ (Rayleigh scattering). Show that $\sigma(\omega)$ is then proportional to a power of ω, which you should give.

   For $\omega \ll \omega_0$, we have:

$$\sigma(\omega) = \frac{8\pi}{3}r_0^2\left(\frac{\omega}{\omega_0}\right)^4$$

   Therefore the cross section is proportional to ω to the fourth power.

5. *[1/20 points]* Assume $\omega_0 \ll \omega \ll c/r_0$ (Thomson scattering). Show that $\sigma(\omega)$ is equal to a constant.

   The condition $\omega \ll c/r_0$ ensures that the damping term is negligible in front of ω^4 as $\omega^4/(\gamma_0^2\omega^6/\omega_0^4) \gg \omega_0^4 r_0^2/(\gamma_0^2 c^2) = (3/2)^2$. So for $\omega_0 \ll \omega \ll c/r_0$ we have:

$$\sigma(\omega) = \frac{8\pi}{3} r_0^2$$

Therefore the cross section is equal to a constant given by the classical radius of the electron.

6. *[1/20 points]* Assume finally ω near ω_0 (resonant scattering). Show that the variation of $\sigma(\omega)$ with $\omega - \omega_0$ exhibits a resonance. What is the width of the resonance? What is the value of the cross section $\sigma(\omega_0)$ at resonance?

   For $\omega \approx \omega_0$, then $\omega^2 - \omega_0^2 \approx 2\omega_0(\omega - \omega_0)$ and we have:

$$\sigma(\omega) = \frac{2\pi}{3} r_0^2 \frac{\omega_0^2}{(\omega - \omega_0)^2 + \gamma_0^2/4}$$

   The half-width of the resonance is γ_0. On resonance, the cross section is $\sigma(\omega_0) = \sigma_0 = \frac{8\pi}{3} r_0^2 \frac{\omega_0^2}{\gamma_0^2} = 6\pi \lambdabar_0^2$.

   We see that the cross section is much bigger on resonance than it is for the Thomson scattering case. Their ratio is $\sigma_{res}/\sigma_{Th} = (\omega_0/\gamma_0)^2 = (3\lambdabar_0/2r_0)^2$ which is to a numerical prefactor the square of the one justifying our approximation. There is over 13 orders of magnitude difference between the two if we reuse our numbers.

3 Classical Model of the Light Force

Later in class, we will discuss light forces from the full quantum perspective. Here you will perform a (semi-)classical derivation based on the Lorentz model: Assume that a hydrogenic atom can be modeled classically as an electron harmonically bound to a nucleus, with a resonant frequency ω_0 and damping coefficient γ. The nucleus is fixed at position $\vec{r}_0$ while the electron's position is denoted by $\vec{r}$. Now suppose the atom is illuminated with an electromagnetic wave of the form

$$\vec{E}(\vec{r}, t) = \hat{\epsilon} E(\vec{r}, t) = \hat{\epsilon} E_0(\vec{r}) \cos(\omega t + \theta(\vec{r})) \tag{4}$$

where $\theta(\vec{r})$ is the phase of the wave as a function of position $\vec{r}$ at time $t = 0$. The dipole moment of the electron+nucleus (charge $\times$ distance) may be written as

$$\vec{d}(\vec{r}, t) = \vec{d}_0(u \cos(\omega t + \theta(\vec{r})) + v \sin(\omega t + \theta(\vec{r})) \tag{5}$$

with u and v the components in and out of phase with the driving field, respectively.
The force of the light on the atom is

$$\vec{F} = (\vec{d} \cdot \hat{\epsilon}) \nabla E(\vec{r}, t) \tag{6}$$

1. *[2/20 points]* **Time averaged force**

   Make the dipole approximation that $\vec{E}(\vec{r}) \approx \vec{E}(\vec{r}_0)$. Show that the time averaged force is

$$\left\langle \vec{F} \right\rangle = \frac{1}{2}(\hat{d} \cdot \hat{\epsilon})(u \nabla E_0(\vec{r}_0) - v E_0(\vec{r}_0) \nabla \theta(\vec{r}_0)) \tag{7}$$

This expression turns out to be exactly analogous to the quantum-mechanically derived force. The first term is the dipole (stimulated) force, and the second term is the scattering (spontaneous) force.

Plug equation (4) and (5) into equation (6), replacing $\vec{r}$ with $\vec{r}_0$ (since the atom is much smaller than the length scale over which the electric field varies in space):

$$\begin{aligned}\left\langle\vec{F}\right\rangle &= (\hat{d}\cdot\hat{\epsilon})\left(u\cos\left(\theta\left(\vec{r}_0\right)+\omega t\right)+v\sin\left(\theta\left(\vec{r}_0\right)+\omega t\right)\right)\times \\ &\quad \left(\nabla E_0(\vec{r}_0)\cos\left(\theta\left(\vec{r}_0\right)+\omega t\right)-E_0(\vec{r}_0)\nabla\theta\left(\vec{r}_0\right)\sin\left(\theta\left(\vec{r}_0\right)+\omega t\right)\right) \qquad (8)\end{aligned}$$

Cross terms $\sin(\theta(\mathbf{x}_0)+\omega t)\cos(\theta(\mathbf{x}_0)+\omega t)$ will average to zero over a cycle of the laser frequency. The other terms, $\sin^2(\theta(\mathbf{x}_0)+\omega t)$ and $\cos^2(\theta(\mathbf{x}_0)+\omega t)$ average to $\frac{1}{2}$ and produce the desired equation (7).

2. *[3/20 points]* **The potential picture**

Recalculate the time averaged force on the atom from the instantaneous potential energy of a dipole in an electric field. How does this answer differ from that of 1? Speculate as to why.

Using the dipole approximation, the instantaneous electrostatic potential energy from an <u>induced</u> electric dipole in the electric field is[1]

$$U = -\frac{1}{2}\vec{d}\cdot\vec{E}(\vec{r}_0,t) \qquad (9)$$

In contrast the potential energy of a permanent dipole does not have the factor of 1/2. Substituting in equations (4) and (5) gives

$$U = -\frac{1}{2}\vec{d}_0\cdot\hat{\epsilon}\left(\left(u\cos(\theta(\vec{r}_0)+\omega t)+v\sin(\theta(\vec{r}_0)+\omega t)\right)E_0(\vec{r}_0)\cos(\theta(\vec{r}_0)+\omega t)\right) \qquad (10)$$

Taking the gradient of the potential gives us a force.

$$\begin{aligned}&\vec{F} = -\nabla U = \\ &\frac{1}{2}\left(\vec{d}_0\cdot\hat{\epsilon}\right)\left[-u\sin\left(\theta\left(\vec{r}_0\right)+\omega t\right)\nabla\theta\left(\vec{r}_0\right)+v\cos\left(\theta\left(\vec{r}_0\right)+\omega t\right)\nabla\theta\left(\vec{r}_0\right)\right]E_0(\vec{r}_0)\cos(\theta(\vec{r}_0)+\omega t) \\ &\quad +\frac{1}{2}\left(\vec{d}_0\cdot\hat{\epsilon}\right)\left[u\cos\left(\theta\left(\vec{r}_0\right)+\omega t\right)+v\sin\left(\theta\left(\vec{r}_0\right)+\omega t\right)\right]\left[\nabla E_0(\vec{r}_0)\cos(\theta(\vec{r}_0)+\omega t)-\right. \\ &\qquad\qquad \left.E_0(\vec{r}_0)\sin(\theta(\vec{r}_0)+\omega t)\nabla\theta\left(\vec{r}_0\right)\right. \qquad (11)\end{aligned}$$

Averaging over an optical cycle

$$\vec{F} = \frac{1}{2}\left(\vec{d}_0\cdot\hat{\epsilon}\right)\left[\frac{v}{2}\nabla\theta\left(\vec{r}_0\right)E_0(\vec{r}_0)+\frac{u}{2}\nabla E_0(\vec{r}_0)-\frac{v}{2}\nabla\theta\left(\vec{r}_0\right)E_0(\vec{r}_0)\right] \qquad (12)$$

$$\vec{F} = \frac{1}{4}\left(\vec{d}_0\cdot\hat{\epsilon}\right)u\nabla E_0(\vec{r}_0) \qquad (13)$$

This approach fails to capture the physics of the spontaneous force, which acts out of phase with the electric field and cannot be expressed in terms of a conservative potential. The factor of 2 difference in the stimulated force comes from the fact that u is proportional to $E_0(\vec{r}_0)$ because the dipole is an induced one. For $u \propto E_0(\vec{r}_0)$, $\nabla(uE_0(\vec{r}_0) = 2u\nabla E_0(\vec{r}_0)$. The same holds for $v \propto E_0(\vec{r}_0)$, and this recovers the missing factor of 2.

[1] arXiv:physics/9902072 [physics.atom-ph]

3. *[2/20 points]* **Dipole moment of the model atom**

Now we will solve explicitly for the dipole moment of the model atom. In complex notation (see class notes), the equation of motion is

$$m\frac{\partial^2 \vec{r}^+}{\partial t^2} + m\gamma\frac{\partial \vec{r}^+}{\partial t} + m\omega_0^2 \vec{r}^+ = -e\hat{\epsilon}E_0^+(\vec{r}_0)e^{i(\theta(\vec{r}_0)+\omega t)} \tag{14}$$

where $\vec{r} = \vec{r} - \vec{r}_0$. Solve this equation to find (the real) $\vec{d} = -e\vec{r}$. Substitute the quadrature components of $\vec{d}$ into the force equation from part (a) to find that

$$\vec{F} = -\frac{e^2}{2m\omega_0}\frac{\delta\nabla E_0^2 + \gamma E_0^2\nabla\theta}{4\delta^2 + \gamma^2} \tag{15}$$

where $\delta = \omega - \omega_0$. Make the approximation that $\omega \approx \omega_0$. [Note: To convince yourself that signs are correct, you can go to DC ($\omega \to 0$) and verify that dipoles are attracted by electric field maxima. Also, for a traveling wave $\theta(\vec{r}) = -\vec{k}\cdot\vec{r}$, and you can see that the spontaneous force is in the direction of wave propagation, $\vec{k}$, as it should be.]

Since the only vector input to equation (14) is the polarization direction $\hat{\epsilon}$ of the electric field, this will be the unit vector for $\vec{r}$ as well. The homogeneous solution decays away and is irrelevant here. The particular solution can be found by using the trial function:

$$\vec{r} = r_0\hat{\epsilon}E_0(\vec{r}_0)e^{i(\theta(\vec{r}_0)+\omega t)} \tag{16}$$

Substituting this into equation (14) gives

$$(-m\omega^2 + m\gamma + m\omega_0^2)r_0\hat{\epsilon}E_0(\vec{r})e^{i(\theta(\vec{r})+\omega t)} = -e\hat{\epsilon}E_0(\vec{r})e^{i(\theta(\vec{r})+\omega t)} \tag{17}$$

$$r_0 = \frac{-\frac{e}{m}}{\omega_0^2 - \omega^2 + i\Gamma\omega} \tag{18}$$

For $\omega = \delta + \omega_0$ and $\delta \ll \omega_0$ we can approximate $\omega_0^2 - \omega^2 + i\Gamma\omega \approx (-2\delta + i\Gamma)\omega_0$. The dipole moment can now be written as

$$\vec{d} = -e\vec{r} = -\frac{e^2}{m\omega_0}\frac{2\delta + i\Gamma}{4\delta^2 + \Gamma^2}\hat{\epsilon}E_0(\vec{r})e^{i(\theta(\vec{r})+\omega t)} \tag{19}$$

If we take the the real part of our solution $\vec{d}$ and compare with equation (5), we can identify the real quadrature components u and v (set $\vec{d} = \hat{\epsilon}$):

$$Re(\vec{d}) = -e\vec{r} = -\frac{e^2}{m\omega_0}\frac{1}{4\delta^2 + \Gamma^2}\hat{\epsilon}E_0(\vec{r})(2\delta\cos((\vec{r}) + \omega t) - \Gamma\sin((\vec{r}) + \omega t)) \tag{20}$$

$$u = -\frac{e^2}{m\omega_0}\frac{2\delta}{4\delta^2 + \Gamma^2}E_0(\vec{r}) \tag{21}$$

$$v = -\frac{e^2}{m\omega_0}\frac{\Gamma}{4\delta^2 + \Gamma^2}E_0(\vec{r}) \tag{22}$$

We can use these in equation (7) and the identity $E\nabla E = \frac{1}{2}\nabla(E^2)$ to get equation (15).