Answer all problems, but note that the first parts of each problem may be easier than subsequent parts. Therefore, make sure to proceed to the next problem if you get stuck.

You may find the following information helpful:

Physical Constants

Electron mass	$m_e \approx 9.1 \times 10^{-31}$kg	Proton mass	$m_p \approx 1.7 \times 10^{-27}$kg
Electron Charge	$e \approx 1.6 \times 10^{-19}$C	Planck's constant/2π	$\hbar \approx 1.1 \times 10^{-34}$Js
Speed of light	$c \approx 3.0 \times 10^8$ms^{-1}	Stefan's constant	$\sigma \approx 5.7 \times 10^{-8}Wm^{-2}\,{}^\circ$K^{-4}
Boltzmann's constant	$k_B \approx 1.4 \times 10^{-23} JK^{-1}$	Avogadro's number	$N_0 \approx 6.0 \times 10^{23}$mol^{-1}
Gravitational constant	$G \approx 6.7 \times 10^{-11}$Nm^{2}kg^{-2}		

Conversion Factors

$1atm \equiv 1.0 \times 10^5 Nm^{-2}$ $\qquad 1\mathring{A} \equiv 10^{-10} m$ $\qquad 1eV \equiv 1.1 \times 10^4\,{}^\circ K$

Thermodynamics

$dE = TdS + dW$ $\qquad$ For a gas: $dW = -PdV$ $\qquad$ For a film: $dW = \sigma dA$

Mathematical Formulas

$\lim_{x\to 0} \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \mathcal{O}\left(x^4\right)$ $\qquad \lim_{x\to\infty} \ln(1+x) = \ln x + \frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)$

$\lim_{x\to\infty} \coth x = 1 + 2e^{-2x} + \mathcal{O}\left(e^{-4x}\right)$ $\qquad \lim_{x\to 0} \coth x = \frac{1}{x} + \frac{x}{3} + \mathcal{O}\left(x^2\right)$

$\int_0^\infty dx\ x^n\ e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$ $\qquad (\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$

$\int_{-\infty}^\infty dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2} \exp\left[-\frac{\sigma^2 k^2}{2}\right]$ $\qquad \lim_{N\to\infty} \ln N! = N\ln N - N$

$\left\langle e^{-ikx}\right\rangle = \sum_{n=1}^\infty \frac{(-ik)^n}{n!} \langle x^n\rangle$ $\qquad \ln\left\langle e^{-ikx}\right\rangle = \sum_{n=1}^\infty \frac{(-ik)^n}{n!} \langle x^n\rangle_c$

$f_m^\eta(z) = \frac{1}{(m-1)!}\int_0^\infty dx \frac{x^{m-1}}{z^{-1}e^x - \eta} = \sum_{\alpha=1}^\infty \eta^{\alpha+1}\frac{z^\alpha}{\alpha^m}$ $\qquad \frac{df_m^\eta}{dz} = \frac{1}{z} f_{m-1}^\eta$ $\qquad f_1^\eta(z) = -\eta\ln(1-\eta z)$

$\zeta_m \equiv f_m^+(1)$ $\qquad \zeta_{3/2} \approx 2.612$ $\qquad \zeta_2 = \frac{\pi^2}{6}$ $\qquad \zeta_{5/2} \approx 1.341$ $\qquad \zeta_3 \approx 1.202$ $\qquad \zeta_4 = \frac{\pi^4}{90}$

$\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{m!}\left[1 + \frac{\pi^2}{6} m(m-1)(\ln z)^{-2} + \cdots\right]$ $\qquad f_m^-(1) = \left(1 - 2^{1-m}\right)\zeta_m$

1. *Free fermions:* Consider a grand canonical ensemble of non-interacting *fermions* with chemical potential μ. The one–particle states are labelled by a ℓ, and have energies $\mathcal{E}(\ell)$.

(a) What is the joint probability $P(\{n_\ell\})$, of finding a set of occupation numbers $\{n_\ell\}$, of the one–particle states?

(b) What are the average occupation numbers $\left\{\langle n_\ell\rangle_-\right\}$? Recast your answer to part (a) as a function of only the quantities $\{n_\ell\}$ and $\left\{\langle n_\ell\rangle_-\right\}$.

(c) A random variable has a set of discrete outcomes with probabilities p_n, where $n = 1, 2, \cdots, M$. What is the entropy of this probability distribution? What is the maximum possible entropy?

(d) What is the entropy of the probability distribution for fermion occupation numbers in part (b), and comment on its zero temperature limit.

(e) Calculate the variance of the total number of particles $\left\langle N^2\right\rangle_c$, and comment on its zero temperature behavior.

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2. *Non-degenerate quantum gas:* The aim of this problem is to compute *the first correction to classical behavior* due to quantum (Fermi or Bose) statistics. Consider a gas of density $n = N/V$, composed of identical particles with spin degeneracy g, and kinetic energy $\mathcal{H} = \sum_i \frac{\vec{p}_i^{\,2}}{2m}$ in three dimensions, with $d \equiv n\lambda^3/g \ll 1$, where $\lambda = h/\sqrt{2\pi m k_B T}$.

(a) Write the expressions for density and pressure (<u>no derivation necessary</u>) in the grand canonical ensemble, in terms of $z = e^{\beta\mu}$, de Broglie wavelength λ, and functions $f_m^\eta(z)$.

(b) Using the above results, find the expression for the pressure $P(n, T)$ at temperature T at order of n^2 (i.e. including the second virial coefficient).

(c) Find the expression for the energy density E/V to order of n^2.

(d) Find the expression for the heat capacity C_V/N including the first quantum correction, and comment on its behavior for bosons and fermions.

(e) Calculate the expansivity $\alpha_P \equiv \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_{P,N}$ to the same order.

(f) Compute the heat capacity C_P/N including the first quantum correction.

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3. *Trap sites in a Bose gas:* Using an 'optical lattice', a density ρ of trapping sites is created within a volume containing a density n of identical and non-interacting bosons. Each trap site can either be empty, or contain one (and only one) Bose particle. Assume that a trapped particle is completely localized to the site and can be treated as a classical particle of energy $\epsilon = -w < 0$ (i.e., energy is lowered by w upon trapping). We further assume that the trapping sites have no effect on the spectrum of the non-trapped particles, which can be treated as an ideal gas with single particle states of kinetic energy $(\hbar\vec{k})^2/2m$.

(a) In a grand canonical ensemble of chemical potential μ and temperature T, what is the density n_t of trapped particles.

(b) What is the density n_g of gas particles as a function of μ, T, the mass m and spin degeneracy g of the bosons?

(c) Obtain the critical (total) density $n^*(T)$ for the onset of Bose–Einsten condensation, and sketch it for a small value of w.

(d) Write the expression for the total energy density in the Bose condensed phase.

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