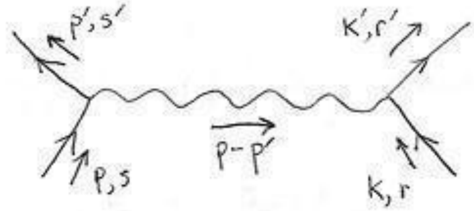
The sum over polarizations is a tricky point since physical photon have only 2 polarizations. More this later.

Coulomb potential

As with the Yukawa potential, we consider non-relativistic scattering of two different fermions with the same mass.



$$i\mathcal{M} = (-ie)^2 \left(\bar{u}^{s'}(p')\gamma^\mu u^s(p)\right) \frac{-i g_{\mu\nu}}{(p-p')^2 + i\varepsilon} \left(\bar{u}^{r'}(k')\gamma^\nu u^r(k)\right)$$

In the non-relativistic limit

$$\bar{u}^{s'}(p')\gamma^i u^s(p) = \begin{bmatrix} \xi^{s'\dagger} & \xi^{s'\dagger} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \begin{bmatrix} \xi^s \\ \xi^s \end{bmatrix} \times m$$

$$= 0 \qquad \text{for } i = 1, 2, 3$$

$$\bar{u}^{s'}(p')\gamma^0 u^s(p) = \begin{bmatrix}\xi^{s'\dagger} & \xi^{s'\dagger}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\xi^s\\ \xi^s\end{bmatrix} \times m$$
$$= 2m\,\delta^{s's}$$

Almost like the Yukawa model

$$i\mathcal{M} = \frac{ie^2}{-(\vec{p}-\vec{p}\,')^2}(2m)^2 \underbrace{g_{00}}_{=1} \delta^{s's}\delta^{r'r}$$

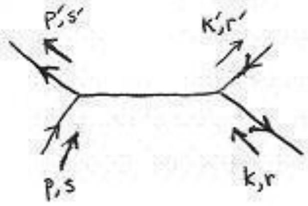
This is the same as the Yukawa model except an extra minus sign and $m_\phi = 0$.

So $V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r}$ where $\underbrace{\alpha = \frac{e^2}{4\pi} \simeq \frac{1}{137}}_{\text{fine structure constant}}$

Note: fermion–fermion scattering is repulsive, whereas Yukawa was attractive

Let us now consider fermion–antifermion scattering for Yukawa + QED.

Yukawa:



$$|\vec{p},s;\vec{k},r\rangle = \sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{k}}}\, a_{\vec{p}}^{s\dagger} b_{\vec{k}}^{r\dagger}|0\rangle$$

$$\langle \vec{k},r;\vec{p},s| = \langle 0|\, b_{\vec{k}}^{r} a_{\vec{p}}^{s} \sqrt{2E_{\vec{k}}}\sqrt{2E_{\vec{p}}}$$

$$\langle \vec{k}',r';\vec{p}',s'|\,\bar{\psi}\psi\phi\;\;\bar{\psi}\psi\phi\,|\vec{p},s;\vec{k},r\rangle$$

... overall minus sign

For the fermion we get

$$\bar{u}^{s'}(p')\,u^{s}(p) = 2m\,\delta^{s's} \quad \text{as before}$$

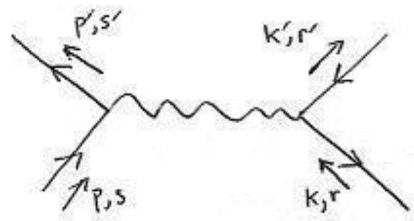
For the antifermion we get

$$\bar{v}^{r}(k)\,v^{r'}(k') = \begin{bmatrix}\xi^{-r\dagger} & -\xi^{-r\dagger}\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\xi^{-r'}\\ -\xi^{-r'}\end{bmatrix}\cdot m$$

$$= -2m\,\delta^{r'r}$$

So $\underset{\substack{\uparrow\uparrow\\ \text{fermion antifermion}}}{V_{f\bar{f}}(r)} = (-1)(-1)\underset{\substack{\uparrow\uparrow\\ \text{fermion-fermion}}}{V_{ff}(r)} = V_{ff}(r)$ (attractive)

For QED



Overall minus due to anticommutation just as in the Yukawa case.

For the fermion we get

$$\bar{u}^{s'}(p')\gamma^\mu u^s(p) \to \bar{u}^{s'}(p')\gamma^0 u^s(p)$$
$$= 2m\,\delta^{s's}$$

For the antifermion we get

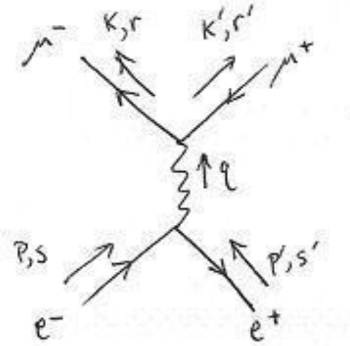
$$\bar{v}^{r}(k)\gamma^\mu v^{r'}(k') \to \bar{v}^{r}(k)\gamma^0 v^{r'}(k')$$
$$= \left[\xi^{r\dagger}\ -\xi^{r\dagger}\right]\begin{bmatrix}\xi^{r'} \\ -\xi^{r'}\end{bmatrix}\cdot m$$
$$= 2m\,\delta^{r'r}$$

So $\quad V_{f\bar{f}}(\mathbf{r}) = -V_{ff}(r) \qquad$ (attractive)

Exchange particle	ff or $\bar{f}\bar{f}$	$f\bar{f}$
scalar (Yukawa)	attractive	attractive
vector (QED)	repulsive	attractive
tensor (gravity)	attractive	attractive

Chapter 5 (QED)

$e^+e^- \to \mu^+\mu^-$ One of simplest possible processes in QED. We will calculate the unpolarized cross-section.



$$i\mathcal{M} = (-ie)^2 \left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma^\nu v^{r'}(k')\right)\frac{-ig_{\mu\nu}}{q^2+i\varepsilon}$$

$$= \frac{ie^2}{(p+p')^2}\left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)$$

can drop $+i\varepsilon$ since $(p+p')^2 > 0$ for physical p, p'

We need $|\mathcal{M}|^2$.

Notice that $(\bar{v}\gamma^\mu u)^* = (\overline{\bar{v}\gamma^\mu u}) = \bar{u}\,\bar{\gamma}^\mu v$
$= \bar{u}\,\gamma^\mu v$

Recall that $\bar{M} = \gamma^0 M^\dagger \gamma^0$
and so $\bar{\gamma}^0 = \gamma^0\gamma^{0\dagger}\gamma^0 = \gamma^0$
$\bar{\gamma}^i = \gamma^0\gamma^{i\dagger}\gamma^0 = \gamma^i$

Therefore
$$|\mathcal{M}|^2 = \frac{e^4}{((p+p')^2)^2}$$
$$\times \left(\bar{v}^{s'}(p')\gamma^\mu u^s(p)\right)\left(\bar{u}^r(k)\gamma_\mu v^{r'}(k')\right)$$
$$\times \left(\underbrace{\bar{u}^s(p)\gamma^\nu v^{s'}(p')}_{\text{conjugate of line above}}\right)\left(\underbrace{\bar{v}^{r'}(k')\gamma_\nu u^r(k)}_{\text{conjugate of line above}}\right)$$

Useful to rewrite as

$$\frac{e^4}{((p+p')^2)^2}\left(\underbrace{\bar{v}^{s'}(p')\gamma^\mu u^s(p)}\,\underbrace{\bar{u}^s(p)\gamma^\nu v^s(p')}_{\text{conjugates}}\right)\left(\underbrace{\bar{u}^{r}(k)\gamma_\mu v^{r'}(k')}\,\underbrace{\bar{v}^{r'}(k')\gamma_\nu u^r(k)}_{\text{conjugates}}\right)$$

$$= \frac{e^4}{((p+p')^2)^2}\,\mathrm{Tr}\left[\gamma^\mu u^s(p)\bar{u}^s(p)\gamma^\nu v^s(p')\bar{v}^{s'}(p')\right]\mathrm{Tr}\left[\gamma_\mu v^{r'}(k')\bar{v}^{r'}(k')\gamma_\nu u^r(k)\bar{u}^r(k)\right]$$

Looks complicated, but this will simplify as we sum over final spins + average over initial spins.

Four possible initial spins ($2\,e^-$ spins, $2\,e^+$ spins) and so we divide by four.

$$\frac{1}{4}\sum_{s,s',r,r'}|\mathcal{M}|^2 = \frac{e^4}{4((p+p')^2)^2}\,\mathrm{Tr}\left[\gamma^\mu(\not{p}+m_e)\gamma^\nu(\not{p}'-m_e)\right] \times \mathrm{Tr}\left[\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu(\not{k}+m_\mu)\right],$$

where we used

$$\sum_s u^s(p)\bar{u}^s(p) = \not{p}+m$$

$$\sum_s v^s(p)\bar{v}^s(p) = \not{p}-m$$

(we use same u, v notation for e^-/e^+ and μ^-/μ^+ ... need to remember which get an m_e + which get an m_μ)