PY 711 Fall 2010
Homework 7: Due Tuesday, October 12

1. (8 points) Let U be the following unitary operator defined in terms of Dirac annihilation and creation operators:

$$U = \exp\left[-\frac{i\pi}{2}\sum_{r=1,2}\int\frac{d^3\vec{p}}{(2\pi)^3}\left(a^{r\dagger}_{\vec{p}} - b^{r\dagger}_{\vec{p}}\right)\left(a^r_{\vec{p}} - b^r_{\vec{p}}\right)\right]. \tag{1}$$

Investigate the effect of this unitary transformation upon the annihilation operators by explicitly calculating $U^\dagger a^r_{\vec{p}} U$ and $U^\dagger b^r_{\vec{p}} U$. Which type of transformation does the unitary operator U produce?

2. (7 points) Using Dirac annihilation and creation operators, explicitly construct a unitary operator P which implements the parity transformation,

$$P^\dagger a^r_{\vec{p}} P = a^r_{-\vec{p}}, \tag{2}$$

$$P^\dagger b^r_{\vec{p}} P = -b^r_{-\vec{p}}. \tag{3}$$

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1. LET U BE THE FOLLOWING UNITARY OPERATOR DEFINED IN TERMS OF DIRAC CREATION AND ANNIHILATON OPERATORS:

$$U = \exp\left[-\frac{i\pi}{2}\sum_r \int \frac{d^3p}{(2\pi)^3} (a_{\vec p}^{r+} - b_{\vec p}^{r+})(a_{\vec p}^{r} - b_{\vec p}^{r})\right]$$

INVESTIGATE THE EFFECT OF THIS UNITARY TRANSFORMATION UPON THE ANNIHILATION OPERATORS BY EXPLICITLY CALCULATING $U^+ a_{\vec p}^r U$ AND $U^+ b_{\vec p}^r U$. WHICH TYPE OF TRANSFORMATION DOES THE UNITARY OPERATOR U PRODUCE?

$$U^+ = \exp\left[+\frac{i\pi}{2}\sum_r \int \frac{d^3p}{(2\pi)^3} \left((a_{\vec p}^{r+} - b_{\vec p}^{r+})(a_{\vec p}^{r} - b_{\vec p}^{r})\right)^+\right]$$

$$= \exp\left[+\frac{i\pi}{2}\sum_r \int \frac{d^3p}{(2\pi)^3} (a_{\vec p}^{r+} - b_{\vec p}^{r+})(a_{\vec p}^{r} - b_{\vec p}^{r})\right]$$

$$U^+ a_{\vec p}^r U = e^X a_{\vec p}^r e^{-X}$$

where $X = \frac{i\pi}{2}\sum_s \int \frac{d^3q}{(2\pi)^3} (a_{\vec q}^{s+} - b_{\vec q}^{s+})(a_{\vec q}^{s} - b_{\vec q}^{s})$

Baker-Hausdorff Lemma (Sakurai pg 96)

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots$$ ✓

Also need the identity

$$[AB,C] = A\{B,C\} - \{A,C\}B$$
$$= ABC + ACB - ACB - CAB$$
$$= ABC - CAB = [AB,C]$$ ✓

In addition, recall

$$\{a_{\vec p}^r, a_{\vec q}^{s+}\} = (2\pi)^3 \delta^{rs}\delta^3(\vec p - \vec q) = \{b_{\vec p}^r, b_{\vec q}^{s+}\}$$

All other combinations have anticommutators equal to zero.

1 CONTINUED

$$\left[\frac{i\pi}{2} \sum_s \int \frac{d^3q}{(2\pi)^3} \left(a_{\vec{q}}^{s+} - b_{\vec{q}}^{s+}\right)\left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right),\ a_{\vec{p}}^{r} \right]$$

$$= \frac{i\pi}{2} \sum_s \int \frac{d^3q}{(2\pi)^3} \Big(\left(a_{\vec{q}}^{s+} - b_{\vec{q}}^{s+}\right) \cancelto{0}{\left\{ \left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right), a_{\vec{p}}^{r} \right\}}$$

$$- \left\{ \left(a_{\vec{q}}^{s+} - b_{\vec{q}}^{s+}\right), a_{\vec{p}}^{r} \right\} \left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right) \Big)$$

$$= \frac{i\pi}{2} \sum_s \int \frac{d^3q}{(2\pi)^3} \left(-(2\pi)^3 \delta^{rs} \delta^3(\vec{p}-\vec{q}) \left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right)\right)$$

$$= -\left(\frac{i\pi}{2}\right)\left(a_{\vec{p}}^{r} - b_{\vec{p}}^{r}\right)$$

$$[X,[X,Y]] = \left[\frac{i\pi}{2} \sum_s \int \frac{d^3q}{(2\pi)^3} \left(a_{\vec{q}}^{s+} - b_{\vec{q}}^{s+}\right)\left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right),\ \left(-\frac{i\pi}{2}\right)\left(a_{\vec{p}}^{r} - b_{\vec{p}}^{r}\right) \right]$$

$$= -\left(\frac{i\pi}{2}\right)^2 \sum_s \int \frac{d^3q}{(2\pi)^3} \Big(\left(a_{\vec{q}}^{s+} - b_{\vec{q}}^{s+}\right) \cancelto{0}{\left\{ \left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right), \left(a_{\vec{p}}^{r} - b_{\vec{p}}^{r}\right) \right\}}$$

$$- \left\{ \left(a_{\vec{q}}^{s+} - b_{\vec{q}}^{s+}\right), \left(a_{\vec{p}}^{r} - b_{\vec{p}}^{r}\right) \right\} \left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right) \Big)$$

$$= +\left(\frac{i\pi}{2}\right)^2 \sum_s \int \frac{d^3q}{(2\pi)^3} \left(\left(\left\{ a_{\vec{q}}^{s+}, a_{\vec{p}}^{r} \right\} + \left\{ b_{\vec{q}}^{s+}, b_{\vec{p}}^{r} \right\} \right) \left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right) \right)$$

$$= \left(\frac{i\pi}{2}\right)^2 \sum_s \int \frac{d^3q}{(2\pi)^3} \left(2(2\pi)^3 \delta^{rs} \delta^3(\vec{p}-\vec{q}) \left(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}\right)\right)$$

$$= 2\left(\frac{i\pi}{2}\right)^2 \left(a_{\vec{p}}^{r} - b_{\vec{p}}^{r}\right)$$

Each subsequent commutator will have another factor of $2\left(\frac{i\pi}{2}\right)$ and alternating negative signs, since each is the commutator will always be between the exponent of U^+, which isn't changing, and $\left(a_{\vec{p}}^{r} - b_{\vec{p}}^{r}\right)$.

1 CONTINUED

So, we see that

$$U^\dagger a_{\vec p}^r U = a_{\vec p}^r - \left(\frac{i\pi}{2}\right)(a_{\vec p}^r - b_{\vec p}^r) + \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 2\,(a_{\vec p}^r - b_{\vec p}^r)$$

$$- \frac{1}{3!}\left(\frac{i\pi}{2}\right)^3 4\,(a_{\vec p}^r - b_{\vec p}^r) + \cdots$$

$$= a_{\vec p}^r \left(1 - \frac{i\pi}{2} + \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 2 - \frac{1}{3!}\left(\frac{i\pi}{2}\right)^3 4 + \cdots\right)$$

$$+ b_{\vec p}^r \left(\frac{i\pi}{2} - \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 2 + \frac{1}{3!}\left(\frac{i\pi}{2}\right)^3 4 + \cdots\right)$$

$$= a_{\vec p}^r \left(1 - \frac{1}{2}\left(i\pi - \frac{1}{2!}(i\pi)^2 + \frac{1}{3!}(i\pi)^3 - \cdots\right)\right)$$

$$+ b_{\vec p}^r \left(\frac{1}{2}\left(i\pi - \frac{1}{2!}(i\pi)^2 + \frac{1}{3!}(i\pi)^3 - \cdots\right)\right)$$

$$= a_{\vec p}^r \left(1 - \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n!}(i\pi)^n(-1)^{n-1}\right) + b_{\vec p}^r \left(\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}(i\pi)^n(-1)^{n-1}\right)$$

$$\sum_{n=1}^{\infty} \frac{1}{n!}(i\pi)^n(-1)^{n-1} = 2$$

$$= a_{\vec p}^r \left(1 - \frac{1}{2}(2)\right) + b_{\vec p}^r \left(\frac{1}{2}(2)\right)$$

$$\boxed{U^\dagger a_{\vec p}^r U = b_{\vec p}^r}$$

1 CONTINUED

$U^{\dagger} b_{\vec{p}}^{r} U$ will be similar

$$\left[\frac{i\pi}{2} \sum_s \int \frac{d^3q}{(2\pi)^3} (a_{\vec{q}}^{s\dagger} - b_{\vec{q}}^{s\dagger})(a_{\vec{q}}^{s} - b_{\vec{q}}^{s}),\ b_{\vec{p}}^{r} \right]$$

$$= \frac{i\pi}{2} \sum_s \int \frac{d^3q}{(2\pi)^3} \Big((a_{\vec{q}}^{s\dagger} - b_{\vec{q}}^{s\dagger}) \{ (a_{\vec{q}}^{s} - b_{\vec{q}}^{s}), b_{\vec{p}}^{r} \}$$

$$- \{ (a_{\vec{q}}^{s\dagger} - b_{\vec{q}}^{s\dagger}), b_{\vec{p}}^{r} \} (a_{\vec{q}}^{s} - b_{\vec{q}}^{s}) \Big)$$

$$= \frac{i\pi}{2} \sum_s \int \frac{d^3q}{(2\pi)^3} \left((2\pi)^3 \delta^{rs} \delta^3(\vec{p} - \vec{q}) (a_{\vec{q}}^{s} - b_{\vec{q}}^{s}) \right)$$

✓

$$= + \frac{i\pi}{2} (a_{\vec{p}}^{r} - b_{\vec{p}}^{r})$$

So, apparently it will be off by a negative sign from the calculations with $a_{\vec{p}}^{r}$, since I've already calculated the next commutator.

$$U^{\dagger} b_{\vec{p}}^{r} U = b_{\vec{p}}^{r} + \left(\frac{i\pi}{2}\right)(a_{\vec{p}}^{r} - b_{\vec{p}}^{r}) - \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 (a_{\vec{p}}^{r} - b_{\vec{p}}^{r})(2)$$

$$+ \frac{1}{3!}\left(\frac{i\pi}{2}\right)^3 (a_{\vec{p}}^{r} - b_{\vec{p}}^{r})(4) - \cdots$$

$$= b_{\vec{p}}^{r} \left(1 - \frac{i\pi}{2} + \frac{1}{2!}\left(\frac{i\pi}{2}\right)^2 (2) - \frac{1}{3!} 4 \left(\frac{i\pi}{2}\right)^3 + \cdots \right)$$

$$+ a_{\vec{p}}^{r} \left(\frac{i\pi}{2} - \frac{1}{2!} 2 \left(\frac{i\pi}{2}\right)^2 + \frac{1}{3!} 4 \left(\frac{i\pi}{2}\right)^3 - \cdots \right)$$

$$= b_{\vec{p}}^{r} \left(1 - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} (i\pi)^n (-1)^{n-1} \right) + a_{\vec{p}}^{r} \left(\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n!} (i\pi)^n (-1)^{n-1} \right)$$

$$\boxed{U^{\dagger} b_{\vec{p}}^{r} U = a_{\vec{p}}^{r}}$$

✓

Since $U^{\dagger} a_{\vec{p}}^{r} U = b_{\vec{p}}^{r}$ and $U^{\dagger} b_{\vec{p}}^{r} U = a_{\vec{p}}^{r}$, U produces a charge conjugation transformation, turning particles into antiparticles and vice versa.

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2. USING DIRAC ANNIHILATION AND CREATION OPERATORS, EXPLICITLY CONSTRUCT A UNITARY OPERATOR P WHICH IMPLEMENTS THE PARITY TRANSFORMATION

$$P^\dagger a^r_{\vec p} P = a^r_{-\vec p}$$

$$P^\dagger b^r_{\vec p} P = -b^r_{-\vec p}.$$

Let's assume that P is also an exponential, similar to U. We can use the Baker-Hausdorff lemma again to find the form of P.

$$P^\dagger a^r_{\vec p} P = e^X a^r_{\vec p} e^{-X} = a^r_{\vec p} + [X, a^r_{\vec p}] + \frac{1}{2!}[X,[X,a^r_{\vec p}]] + \dots = a^r_{-\vec p}$$

$$P^\dagger b^r_{\vec p} P = e^X b^r_{\vec p} e^{-X} = b^r_{\vec p} + [X, b^r_{\vec p}] + \frac{1}{2!}[X,[X,b^r_{\vec p}]] + \dots = -b^r_{-\vec p}$$

In problem 1, we found that

$$U^\dagger a^r_{\vec p} U = a^r_{\vec p}\left(1 - \frac{1}{2}\sum_{n=1}\frac{(i\pi)^n}{n!}(-1)^{n-1}\right) + b^r_{\vec p}\left(\frac{1}{2}\sum_{n=1}\frac{(i\pi)^n}{n!}(-1)^{n-1}\right)$$

and

$$\left[\frac{i\pi}{2}\sum\int\frac{d^3p}{(2\pi)^3}(a^{r\dagger}_{\vec p} - b^{r\dagger}_{\vec p})(a^r_{\vec p} - b^r_{\vec p}), a^s_{\vec q}\right] = -\left(\frac{i\pi}{2}\right)(a^s_{\vec q} - b^s_{\vec q})$$

Let's try to find an X that gives

$$[X, a^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)(a^r_{\vec p} - a^r_{-\vec p})$$

✓

$$[X, b^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)(b^r_{\vec p} + b^r_{-\vec p})$$

where the difference in sign is necessary to get the correct sign for the transformation.

Since the a and b stay separate, I will keep them separate in the operator (no terms like $a^r_{\vec p} b^r_{\vec p}$ etc.). This way, the a terms will be ignored when transforming $b^r_{\vec p}$ and the b terms will be ignored when transforming $a^r_{\vec p}$. ✓

If the commutators follow the same pattern as in problem 1 (extra factor of $-2\left(\frac{i\pi}{2}\right)$ on each subsequent commutator), we'll end up with the same sums as in problem 1 and should have our operator.

2 CONTINUED

The simplest operator to try first would be

$$P = \exp\Big[-\frac{i\pi}{2} \underbrace{\sum \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec p}^{r+}(a_{\vec p}^{r} - a_{-\vec p}^{r}) + b_{\vec p}^{r+}(b_{\vec p}^{r} + b_{-\vec p}^{r})\right)}_{\text{this must be hermitian. See last page.}}\Big]$$

Look at $a_{\vec p}^{r}$ first.

$$\left[+\frac{i\pi}{2}\sum\int\frac{d^3q}{(2\pi)^3}\left(a_{\vec q}^{s+}(a_{\vec q}^{s} - a_{-\vec q}^{s}) + b_{\vec q}^{s+}(b_{\vec q}^{s} + b_{\vec q}^{s}\right),\ a_{\vec p}^{r}\right]$$

$$= \left[\frac{i\pi}{2}\sum\int\frac{d^3q}{(2\pi)^3}\left(a_{\vec q}^{s+}(a_{\vec q}^{s} - a_{-\vec q}^{s})\right),\ a_{\vec p}^{r}\right]$$

$$+ \cancel{\left[\frac{i\pi}{2}\sum\int\frac{d^3q}{(2\pi)^3}\left(b_{\vec q}^{s+}(b_{\vec q}^{s} + b_{-\vec q}^{s})\right),\ a_{\vec p}^{r}\right]} \rightarrow 0$$

(after expansion into anticommutators, $\{a_{\vec p}^{r}, b_{\vec q}^{s}\} = 0$, $\{a_{\vec p}^{r}, b_{\vec q}^{s+}\} = 0$)

$$= \left(\frac{i\pi}{2}\right)\sum\int\frac{d^3q}{(2\pi)^3}\left(a_{\vec q}^{s+}\underbrace{\cancel{\{(a_{\vec q}^{s} - a_{-\vec q}^{s}),\ a_{\vec p}^{r}\}}}_{\to 0} - \{a_{\vec q}^{s+},\ a_{\vec p}^{r}\}(a_{\vec q}^{s} - a_{-\vec q}^{s})\right)$$

$$= \left(\frac{i\pi}{2}\right)\sum\int\frac{d^3q}{(2\pi)^3}\left(-(2\pi)^3\delta^{rs}\delta^3(\vec p - \vec q)\,(a_{\vec q}^{s} - a_{-\vec q}^{s})\right)$$

$$= -\left(\frac{i\pi}{2}\right)(a_{\vec p}^{r} - a_{-\vec p}^{r})$$

Now let's check that the next commutator has an extra factor of $-2\left(\frac{i\pi}{2}\right)$. Since I know the b part falls out, I will ignore it from the start.

$$\left[\frac{i\pi}{2}\sum\int\frac{d^3q}{(2\pi)^3}\left(a_{\vec q}^{s+}(a_{\vec q}^{s} - a_{-\vec q}^{s})\right),\ \left(-\frac{i\pi}{2}\right)(a_{\vec p}^{r} - a_{-\vec p}^{r})\right]$$

$$= -\left(\frac{i\pi}{2}\right)^2\sum\int\frac{d^3q}{(2\pi)^3}\Big(a_{\vec q}^{s+}\underbrace{\cancel{\{(a_{\vec q}^{s} - a_{-\vec q}^{s}),\ a_{\vec p}^{r} - a_{-\vec p}^{r}\}}}_{\to 0}$$

$$- \{a_{\vec q}^{s+},\ (a_{\vec p}^{r} - a_{-\vec p}^{r})\}(a_{\vec q}^{s} - a_{-\vec q}^{s})\Big)$$

$$= \left(\frac{i\pi}{2}\right)^2\sum\int\frac{d^3q}{(2\pi)^3}\left(\left((2\pi)^3\delta^{rs}\delta^3(\vec p - \vec q) - (2\pi)^3\delta^{rs}\delta^3(-\vec p - \vec q)\right)(a_{\vec q}^{s} - a_{-\vec q}^{s})\right)$$

$$= \left(\frac{i\pi}{2}\right)^2\left((a_{\vec p}^{r} - a_{-\vec p}^{r}) - (a_{-\vec p}^{r} - a_{\vec p}^{r})\right)$$

good

$$= 2\left(\frac{i\pi}{2}\right)^2(a_{\vec p}^{r} - a_{-\vec p}^{r})$$

2 CONTINUED

So, the pattern does hold.

$$P^{+} a_{\vec{p}}^{r} P = a_{\vec{p}}^{r} - \frac{1}{2}(i\pi)(a_{\vec{p}}^{r} - a_{-\vec{p}}^{r}) + \frac{1}{2!}\frac{1}{2}(i\pi)^2(a_{\vec{p}}^{r} - a_{-\vec{p}}^{r}) + \dots$$

$$= a_{\vec{p}}^{r}\Big(1 - \underbrace{\frac{1}{2}\sum_{n=1}\frac{(i\pi)^n}{n!}(-1)^{n-1}}_{=+2}\Big) - a_{-\vec{p}}^{r}\Big(\underbrace{\frac{1}{2}\sum_{n=1}\frac{(i\pi)^n}{n!}(-1)^{n}}_{=-2}\Big)$$

$$\boxed{P^{+} a_{\vec{p}}^{r} P = a_{-\vec{p}}^{r}}$$

Now, we need to check that $b_{\vec{p}}^{r}$ transforms correctly. Again, since I know the a terms will drop out, I will ignore them.

$$\left[\frac{i\pi}{2}\sum\int\frac{d^3q}{(2\pi)^3}\left(b_{\vec{q}}^{s+}(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s})\right), b_{\vec{p}}^{r}\right]$$

$$= \left(\frac{i\pi}{2}\right)\sum\int\frac{d^3q}{(2\pi)^3}\left(b_{\vec{q}}^{s+}\cancelto{0}{\{(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s}), b_{\vec{p}}^{r}\}} - \{b_{\vec{q}}^{s+}, b_{\vec{p}}^{r}\}(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s})\right)$$

$$= \left(\frac{i\pi}{2}\right)\sum\int\frac{d^3q}{(2\pi)^3}\left(-(2\pi)^3\delta^{rs}\delta^3(\vec{p}-\vec{q})(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s})\right)$$

$$= -\left(\frac{i\pi}{2}\right)(b_{\vec{p}}^{r} + b_{-\vec{p}}^{r})$$

$$\left[\frac{i\pi}{2}\sum\int\frac{d^3q}{(2\pi)^3}\left(b_{\vec{q}}^{s+}(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s})\right), -\left(\frac{i\pi}{2}\right)(b_{\vec{p}}^{r} + b_{-\vec{p}}^{r})\right]$$

$$= -\left(\frac{i\pi}{2}\right)^2\sum\int\frac{d^3q}{(2\pi)^3}\Big(b_{\vec{q}}^{s+}\cancelto{0}{\{(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s}), (b_{\vec{p}}^{r} + b_{-\vec{p}}^{r})\}}$$

$$- \{b_{\vec{q}}^{s+}, (b_{\vec{p}}^{r} + b_{-\vec{p}}^{r})\}(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s})\Big)$$

$$= \left(\frac{i\pi}{2}\right)^2\sum\int\frac{d^3q}{(2\pi)^3}\left(\left((2\pi)^3\delta^{rs}\delta^3(\vec{p}-\vec{q}) + (2\pi)^3\delta^{rs}\delta^3(-\vec{p}-\vec{q})\right)(b_{\vec{q}}^{s} + b_{-\vec{q}}^{s})\right)$$

$$= \left(\frac{i\pi}{2}\right)^2\left((b_{\vec{p}}^{r} + b_{-\vec{p}}^{r}) + (b_{-\vec{p}}^{r} + b_{\vec{p}}^{r})\right)$$

$$= 2\left(\frac{i\pi}{2}\right)^2(b_{\vec{p}}^{r} + b_{-\vec{p}}^{r})$$

2 CONTINUED.

Again, our pattern holds

$$P^\dagger b_{\vec{p}}^r P = b_{\vec{p}}^r - \frac{1}{2}(i\pi)\left(b_{\vec{p}}^r + b_{-\vec{p}}^r\right) + \frac{1}{2!}\,\frac{1}{2}(i\pi)^2\left(b_{\vec{p}}^r + b_{-\vec{p}}^r\right) + \ldots$$

$$= b_{\vec{p}}^r\left(1 - \frac{1}{2}\sum_{n=1}\frac{(i\pi)^n}{n!}(-1)^{n-1}\right) + b_{-\vec{p}}^r\left(\frac{1}{2}\sum_{n=1}\frac{(i\pi)^n}{n!}(-1)^{n-1}\right)$$

$$\boxed{P^\dagger b_{\vec{p}}^r P = -b_{\vec{p}}^r}$$

So, this operator does perform the parity transformation correctly.

$$\boxed{P = \exp\left[\frac{-i\pi}{2}\sum\int\frac{d^3p}{(2\pi)^3}\left(a_{\vec{p}}^{r\dagger}\left(a_{\vec{p}}^r - a_{-\vec{p}}^r\right) + b_{\vec{p}}^{r\dagger}\left(b_{\vec{p}}^r + b_{-\vec{p}}^r\right)\right)\right]}$$

✓

Is $\sum\int\frac{d^3p}{(2\pi)^3}\left(a_{\vec{p}}^{r\dagger}\left(a_{\vec{p}}^r - a_{-\vec{p}}^r\right) + b_{\vec{p}}^{r\dagger}\left(b_{\vec{p}}^r + b_{-\vec{p}}^r\right)\right)$ Hermitian?

$$\left(\sum\int\frac{d^3p}{(2\pi)^3}\left(a_{\vec{p}}^{r\dagger}\left(a_{\vec{p}}^r - a_{-\vec{p}}^r\right) + b_{\vec{p}}^r\left(b_{\vec{p}}^r + b_{-\vec{p}}^r\right)\right)\right)^\dagger$$

$$= \sum\int\frac{d^3p}{(2\pi)^3}\left(\left(a_{\vec{p}}^{r\dagger} - a_{-\vec{p}}^{r\dagger}\right)a_{\vec{p}}^r + \left(b_{\vec{p}}^{r\dagger} + b_{-\vec{p}}^{r\dagger}\right)b_{\vec{p}}^r\right)$$

$$= \sum\int\frac{d^3p}{(2\pi)^3}\left(a_{\vec{p}}^{r\dagger}a_{\vec{p}}^r - \underbrace{a_{-\vec{p}}^{r\dagger}a_{\vec{p}}^r} + b_{\vec{p}}^{r\dagger}b_{\vec{p}}^r + \underbrace{b_{-\vec{p}}^{r\dagger}b_{\vec{p}}^r}\right)$$

Since we are integrating over all $\vec{p}$, the signs on these momenta can both switch and everything remains the same.

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$$= \sum\int\frac{d^3p}{(2\pi)^3}\left(a_{\vec{p}}^{r\dagger}a_{\vec{p}}^r - a_{\vec{p}}^{r\dagger}a_{-\vec{p}}^r + b_{\vec{p}}^{r\dagger}b_{\vec{p}}^r + b_{\vec{p}}^{r\dagger}b_{-\vec{p}}^r\right)$$

$$= \sum\int\frac{d^3p}{(2\pi)^3}\left(a_{\vec{p}}^{r\dagger}\left(a_{\vec{p}}^r - a_{-\vec{p}}^r\right) + b_{\vec{p}}^{r\dagger}\left(b_{\vec{p}}^r + b_{-\vec{p}}^r\right)\right)$$

✓

Since taking the Hermitian conjugate left this unchanged, it is indeed Hermitian, and P is unitary.

PY711 Solutions #7

1. We start with some general observations. Consider two operators S, T such that $[S,T] = cT$ for some constant c. Then

$$ST = TS + cT = T(S+c)$$
$$S^2T = SST = ST(S+c) = T(S+c)^2$$

In general we can prove by induction that $S^nT = T(S+c)^n$. So

$$e^{+iS}T = T\,e^{+i(S+c)}$$
$$\Rightarrow e^{+iS}T\,e^{-iS} = e^{+ic}\,T$$

For our problem $S = \frac{\pi}{2}\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\,(a_{\vec{p}}^{r\dagger} - b_{\vec{p}}^{r\dagger})(a_{\vec{p}}^{r} - b_{\vec{p}}^{r})$.

Let $c_{\vec{p}}^r = \frac{1}{\sqrt{2}}(a_{\vec{p}}^r - b_{\vec{p}}^r)$ and $d_{\vec{p}}^r = \frac{1}{\sqrt{2}}(a_{\vec{p}}^r + b_{\vec{p}}^r)$. Note that

$$\{c_{\vec{p}}^r, c_{\vec{p}'}^{r'\dagger}\} = (2\pi)^3\,\delta^{rr'}\delta^{(3)}(\vec{p}-\vec{p}') = \{d_{\vec{p}}^r, d_{\vec{p}'}^{r'\dagger}\}$$

and all other commutators vanish. Notice that

$$S = \pi\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\,c_{\vec{p}}^{r\dagger}c_{\vec{p}}^r,$$

which is π times the number of "c"-type quanta.

We have

$$[S, c_{\vec{p}}^r] = \pi\sum_{r'=1,2}\int\frac{d^3p'}{(2\pi)^3}\left(c_{\vec{p}'}^{r'\dagger}c_{\vec{p}'}^{r'}c_{\vec{p}}^r - c_{\vec{p}}^r c_{\vec{p}'}^{r'\dagger}c_{\vec{p}'}^{r'}\right)$$
$$= \pi\sum_{r'=1,2}\int\frac{d^3p'}{(2\pi)^3}\Big[(2\pi)^3\delta^{rr'}\delta^{(3)}(\vec{p}-\vec{p}')(-c_{\vec{p}'}^{r'}) + c_{\vec{p}'}^{r'\dagger}c_{\vec{p}}^{r}c_{\vec{p}'}^{r'} + c_{\vec{p}'}^{r'\dagger}c_{\vec{p}}^{r}c_{\vec{p}'}^{r'}\Big]$$

cancel

$$= -\pi\, c_{\vec{p}}^r$$

Also $[S, d_{\vec{p}}^r] = 0$. Therefore $e^{iS}c_{\vec{p}}^r e^{-iS} = e^{-i\pi}c_{\vec{p}}^r = -c_{\vec{p}}^r$, and $e^{iS}d_{\vec{p}}^r e^{-iS} = d_{\vec{p}}^r$. Since $a_{\vec{p}}^r = \frac{1}{\sqrt{2}}(c_{\vec{p}}^r + d_{\vec{p}}^r)$, $b_{\vec{p}}^r = \frac{1}{\sqrt{2}}(-c_{\vec{p}}^r + d_{\vec{p}}^r)$,

$$e^{iS}a_{\vec{p}}^r e^{-iS} = \frac{1}{\sqrt{2}}(-c_{\vec{p}}^r + d_{\vec{p}}^r) = b_{\vec{p}}^r$$
$$e^{iS}b_{\vec{p}}^r e^{-iS} = \frac{1}{\sqrt{2}}(c_{\vec{p}}^r + d_{\vec{p}}^r) = a_{\vec{p}}^r.$$

The transformation corresponds with charge conjugation.

2. Let us define the operators

$$c^r_{\vec p} = \frac{1}{\sqrt2}(a^r_{\vec p} - a^r_{-\vec p}), \quad d^r_{\vec p} = \frac{1}{\sqrt2}(a^r_{\vec p} + a^r_{-\vec p})$$
$$f^r_{\vec p} = \frac{1}{\sqrt2}(b^r_{\vec p} + b^r_{-\vec p}), \quad g^r_{\vec p} = \frac{1}{\sqrt2}(b^r_{\vec p} - b^r_{-\vec p})$$

Then
$$\{c^r_{\vec p}, c^{r'\dagger}_{\vec p'}\} = \{g^r_{\vec p}, g^{r'\dagger}_{\vec p'}\} = (2\pi)^3 \delta^{rr'}[\delta^{(3)}(\vec p - \vec p') - \delta^{(3)}(\vec p + \vec p')]$$
$$\{d^r_{\vec p}, d^{r'\dagger}_{\vec p'}\} = \{f^r_{\vec p}, f^{r'\dagger}_{\vec p'}\} = (2\pi)^3 \delta^{rr'}[\delta^{(3)}(\vec p - \vec p') + \delta^{(3)}(\vec p + \vec p')]$$

Note also that $c^r_{-\vec p} = -c^r_{\vec p}$, $d^r_{-\vec p} = d^r_{\vec p}$, $f^r_{-\vec p} = -f^r_{\vec p}$, $g^r_{-\vec p} = -g^r_{\vec p}$.
All other anticommutators vanish.

Let $S_c = \frac{\pi}{2}\sum_{r=1,2}\int\frac{d^3\vec p}{(2\pi)^3} c^{r\dagger}_{\vec p} c^r_{\vec p}$ and $S_f = \frac{\pi}{2}\sum_{r=1,2}\int\frac{d^3\vec p}{(2\pi)^3} f^{r\dagger}_{\vec p} f^r_{\vec p}$.

Then S_c and S_f commute with each other and also

$$[S_c, c^r_{\vec p}] = \frac{\pi}{2}(-c^r_{\vec p} + c^r_{-\vec p}) = -\pi c^r_{\vec p}$$
$$[S_c, d^r_{\vec p}] = [S_c, f^r_{\vec p}] = [S_c, g^r_{\vec p}] = 0$$
$$[S_f, f^r_{\vec p}] = \frac{\pi}{2}(-f^r_{\vec p} + f^r_{-\vec p}) = -\pi f^r_{\vec p}$$
$$[S_f, c^r_{\vec p}] = [S_f, d^r_{\vec p}] = [S_f, g^r_{\vec p}] = 0$$

So let $U = e^{-iS_c - iS_f}$. Then

$$U^\dagger c^r_{\vec p} U = e^{-i\pi} c^r_{\vec p} = -c^r_{\vec p}, \quad U^\dagger d^r_{\vec p} U = d^r_{\vec p},$$
$$U^\dagger f^r_{\vec p} U = e^{-i\pi} f^r_{\vec p} = -f^r_{\vec p}, \quad U^\dagger g^r_{\vec p} U = g^r_{\vec p},$$

This produces the parity transformation required,

$$U^\dagger a^r_{\vec p} U = a^r_{-\vec p}, \quad U^\dagger b^r_{\vec p} U = -b^r_{-\vec p}$$

Written more explicitly

$$U = \exp\Big[-\frac{i\pi}{4}\sum_{r=1,2}\int\frac{d^3\vec p}{(2\pi)^3}(a^{r\dagger}_{\vec p} - a^{r\dagger}_{-\vec p})(a^r_{\vec p} - a^r_{-\vec p})$$
$$- \frac{i\pi}{4}\sum_{r=1,2}\int\frac{d^3\vec p}{(2\pi)^3}(b^{r\dagger}_{\vec p} + b^{r\dagger}_{-\vec p})(b^r_{\vec p} + b^r_{-\vec p})\Big]$$