Matrices in Quantum Computing

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Matrix Analysis

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Presentation layout

1. Background
2. Motivation
3. Some Matrix Theory
4. Example: A 2-Qubit Entangler
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Qubits & Quantum Gates

Qubit: A quantum system with measurable eigenstates $|0\rangle$ and $|1\rangle$,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \to \text{like a Classical Bit.}$$

But before measurement,

$$\text{Wavefunction} : |\psi\rangle = a\,|0\rangle + b\,|1\rangle \in \mathbb{C}^2, \quad |a|^2 + |b|^2 = 1.$$

Probabilistic:

$$P(|\psi\rangle \to |0\rangle) = |a|^2 \quad P(|\psi\rangle \to |1\rangle) = |b|^2.$$

Quantum gate: unitary transformation on $|\psi\rangle$ of one or many qubits.

Multiple Qubits

How to express two qubits, $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$ as one *composite* state?

$$|\psi_1\psi_2\rangle \stackrel{?}{\sim} |\psi_1\rangle\,, |\psi_2\rangle$$

What if there are more than two $|\psi_i\rangle$'s $\in \mathbf{V}_i$'s

$$|\psi_1\psi_2\ldots\psi_n\rangle \stackrel{?}{\sim} |\psi_1\rangle\,, |\psi_2\rangle\,, \ldots, |\psi_n\rangle?$$

Questions:

- Is there a vector space that contains $|\psi_1\psi_2\ldots\psi_n\rangle$?
- What is the vector space containing $|\psi_1\psi_2\ldots\psi_n\rangle$?
- How does $|\psi_1\psi_2\ldots\psi_n\rangle$ change w.r.t $\mathcal{A}_1\,|\psi_1\rangle$ where $\mathcal{A}_1 \in \mathfrak{L}(\mathbf{V})$?
- What about for $\mathcal{A}_1\,|\psi_1\rangle\,, \ldots \mathcal{A}_n\,|\psi_n\rangle$, where $\mathcal{A}_i \in \mathfrak{L}(\mathbf{V})$?

Tensor Product

Postulate (QM): [NC02]

The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

For $|\psi_1\rangle \in \mathbf{V}_1, \ldots, |\psi_n\rangle \in \mathbf{V}_n$,

$$|\psi_1 \ldots \psi_n\rangle \in \mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n,$$

where the joint state $|\psi_1 \ldots \psi_n\rangle$ is given by

$$|\psi_1 \ldots \psi_n\rangle = |\psi_1\rangle\,|\psi_2\rangle \ldots |\psi_n\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle\,.$$

$|\psi_1 \ldots \psi_n\rangle$ is an *elementary tensor* in $\mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n$.

Not all $|\phi\rangle \in \mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n$ are elementary.

Tensor Product: Definition

What is this "$\otimes$" object?

Definition [Kam]

The *tensor product* of $\mathbf{V}$ and $\mathbf{W}$ is a vector space $\mathbf{V} \otimes \mathbf{W}$ with the *bilinear map* $\phi : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{V} \otimes \mathbf{W}$, such that for every vector space $\mathbf{X}$ and every bilinear map $f : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{X}$, there exists a *unique linear map* $\hat{f} : \mathbf{V} \otimes \mathbf{W} \longrightarrow \mathbf{X}$ such that $f = \hat{f} \circ \phi$.

In other words...

Giving the $\hat{f} : \mathbf{V} \otimes \mathbf{W} \xrightarrow{\text{linear}} \mathbf{X}$ is the same as giving $f : \mathbf{V} \times \mathbf{W} \xrightarrow{\text{bilinear}} \mathbf{X}$.

Tensor Product: Construction

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\hat{f}", "\textit{linear}"'] & \mathbf{X} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[ur, "f", "\textit{multilinear}"'] &
\end{tikzcd}
$$

Tensor Product [CER]

Let $v_1, \ldots, v_n$ be a basis for $\mathbf{V}$ and $w_1, \ldots, w_m$ be a basis for $\mathbf{W}$,

- For $i \in [1, n]$, $j \in [1, m]$, $\{v_i \otimes w_j\}$ is a basis of $\mathbf{V} \otimes \mathbf{W}$:

$$v \otimes w = \sum_{i}^{n} \alpha_i v_i \otimes \sum_{j}^{m} \beta_j w_j = \sum_{i,j}^{n,m} \alpha_i \beta_j (v_i \otimes w_j)$$

- $\dim(\mathbf{V} \otimes \mathbf{W}) = \dim(\mathbf{V}) \dim(\mathbf{W}) = nm.$

Tensor Product

Let $\mathcal{A} \otimes \mathcal{B} \in \mathfrak{L}(\mathbf{V} \otimes \mathbf{W})$, where $\mathcal{A} \in \mathfrak{L}(\mathbf{V})$, and $\mathcal{B} \in \mathfrak{L}(\mathbf{W})$.

$$(\mathcal{A} \otimes \mathcal{B})(v \otimes w) \overset{?}{\sim} \mathcal{F}(v, w) \overset{\Delta}{=} \mathcal{A}(v) \otimes \mathcal{B}(w).$$

One way to see this...

$$\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\mathcal{A}\otimes\mathcal{B}"] & & \mathbf{V} \otimes \mathbf{W} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[urr, "\mathcal{F}"'] & &
\end{tikzcd}$$

By uniqueness,

$$(\mathcal{A} \otimes \mathcal{B}) \circ \phi = \mathcal{F} \iff \boxed{(\mathcal{A} \otimes \mathcal{B})(v \otimes w) = \mathcal{A}(v) \otimes \mathcal{B}(w)}$$

Tensor Product to Kronecker Product

Let Γ be a basis for $\mathbf{V} \otimes \mathbf{W}$, and $\{\,\}_\Gamma = \mathcal{A}_\Gamma^{-1}$ is the coordinatization from $\mathbf{V} \otimes \mathbf{W}$ to $\mathbb{C}^{nm}$, where $n = \dim(\mathbf{V})$, $m = \dim(\mathbf{W})$.

$$\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\mathcal{L}\otimes\mathcal{M}", "linear"'] \arrow[d, "\{\,\}_\Gamma"] & \mathbf{V} \otimes \mathbf{W} \\
\mathbb{C}^{nm} \arrow[r, "\{\mathcal{L}\otimes\mathcal{M}\}_{\Gamma\leftarrow\Gamma}", "linear"'] & \mathbb{C}^{nm} \arrow[u, "\mathcal{A}_\Gamma"']
\end{tikzcd}$$

Kronecker Product

$$[\mathcal{L} \otimes \mathcal{M}]_{\Gamma \leftarrow \Gamma} = [\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma}.$$

If

$$[\mathcal{L}]_{\Gamma \leftarrow \Gamma} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \quad \text{and} \quad [\mathcal{M}]_{\Gamma \leftarrow \Gamma} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

then the *Kronecker product* $[\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma}$ is defined as

$$\begin{aligned} [\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma} &= \begin{bmatrix} l_{11}\mathcal{M} & l_{12}\mathcal{M} \\ l_{21}\mathcal{M} & l_{22}\mathcal{M} \end{bmatrix} \\ &= \begin{bmatrix} l_{11} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{12} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \\ l_{21} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{22} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Kronecker Products

Doesn't care where scalar goes...

$$(\alpha\mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\alpha\mathcal{B}) = \alpha(\mathcal{A} \otimes \mathcal{B})$$

Associative:

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$$

Left-distributive:

$$\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = \mathcal{A} \otimes \mathcal{B} + \mathcal{A} \otimes \mathcal{C}$$

Right-distributive:

$$(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes \mathcal{B} + \mathcal{B} \otimes \mathcal{C}$$

Tensor Products: more basic properties

Other properties:

- Bilinear: linear in both arguments.
- Associative
- Distributive
- Not commutative
- $(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$.
- $\text{Tr}(\mathcal{A} \otimes \mathcal{B}) = \text{Tr}(\mathcal{A}) \cdot \text{Tr}(\mathcal{B})$.
- $\det(\mathcal{A} \otimes \mathcal{B}) = (\det(\mathcal{A}))^m \cdot \det(\mathcal{B})^n$, where $m = \text{size}(\mathcal{A}), n = \text{size}(\mathcal{B})$.

Entangling 2 qubits

- Entanglement, intuitively (or not)
- Entanglement, mathematically.
- Recipe for a 2-qubit entangler.
- Running on IBM-Q.

Composite State as a Kronecker Product

Example: Representing the classical numbers "1" and "0" with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$0_2 \equiv |00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top, |11\rangle = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

In fact, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis for $\otimes^2\mathbb{C}^2$, the 2-qubit system.

Entanglement

Not every $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|c\rangle, |d\rangle \in \mathbb{C}^2$ such that

$$\begin{aligned} |c\rangle \otimes |d\rangle = |\beta_{00}\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \\ &= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \to \textbf{Entangled} \end{aligned}$$

Examples: Bell states, also entangled [CMTH]

$$\begin{aligned} |\beta_{10}\rangle &= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|11\rangle \\ |\beta_{01}\rangle &= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \\ |\beta_{11}\rangle &= \frac{1}{\sqrt{2}}|01\rangle - \frac{1}{\sqrt{2}}|10\rangle \end{aligned}$$

"Entangled" operators

For operators: $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$, $\mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}\,|v\rangle) \otimes (\mathcal{B}\,|w\rangle).$$

Not all $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ can be written as $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$.

Example:

$$CNOT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

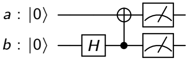
Recipe

What do we need to entangle two qubits?

- Hadamard gate
- CNOT gate
- Measure

2-Qubit Entanglement Circuit

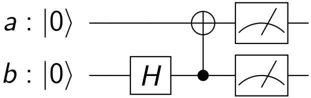
[EF04]



$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{cases} |00\rangle \rightarrow |00\rangle \\ |10\rangle \rightarrow |10\rangle \\ |01\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |01\rangle \end{cases}$$

Entanglement (cont.)



$$C_b(I \otimes H)\left(\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \begin{bmatrix}1\\0\end{bmatrix}_b\right) = C_b\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_b\right)$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$$

$\rightarrow$ **Entangled**

Entanglement (cont.)

Notice:

$$(I\,|0\rangle) \otimes (H_b\,|0\rangle) = (I \otimes H_b)(|0\rangle \otimes |0\rangle)$$

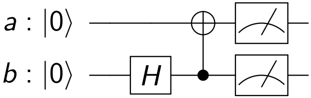
$$\begin{bmatrix}1\\0\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} & \mathcal{O}\\ \mathcal{O} & \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

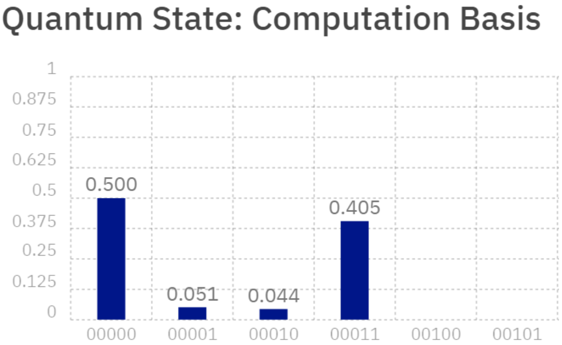
$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}^\top = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}^\top$$

$\rightarrow$ Possible to write H as $I \otimes H_b$. Not possible for $CNOT_b$.

Simulation on IBM-Q

Entanglement circuit, revisited





Recap

- Representing multi-qubit systems with tensor products.
- Representing multi-qubit operators with tensor products.
- Entanglement, mathematically.
- 2-qubit entangler, mathematically.
- Entanglement on IBM-Q.

References

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