**Problem Set 1- Warming up Exercises**

Due: Friday 5pm, Feb 11, via Canvas upload or in envelope outside 26-255
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Office hours Wednesday Feb 9, 1-3pm, in 26-214 (CUA seminar room)

# 1 Driven harmonic oscillator [6 pts]

The driven damped harmonic oscillator is one which is driven by a sinusoidally varying applied force with amplitude $F_0$. It has a steady-state response at the driving frequency $\omega$. It also has the transient response of an undriven damped harmonic oscillator whose motion adds onto the steady-state solution to meet the initial conditions. The amplitude and phase vary with the detuning from resonance $\omega - \omega_0$. The resulting equation of motion

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t) \tag{1}$$

is typically solved by the complex exponential method where the complex equation of motion is found by changing $x \to z$ and $\cos\omega t \to e^{i\omega t}$. $x$ is the real part of the complex solution

$$\begin{aligned} x &= \mathrm{Re}\{z\} = \mathrm{Re}\left\{z_0 e^{i\omega t}\right\} \\ &= \mathrm{Re}\left\{\frac{F_0/m}{(\omega_0^2-\omega^2)+i\gamma\omega}e^{i\omega t}\right\} \\ &= \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}\mathrm{Re}\left\{e^{i\omega t+\phi}\right\} \end{aligned} \tag{2}$$

which can be written

$$x(\omega,t) = x_0(\omega)\cos(\omega t + \phi(\omega)) \tag{3}$$

a) i) The amplitude of the response is

$$x_0(\omega) = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \tag{4}$$

This is maximized for $\omega = \omega^*$ which can be found by setting $dx_0/d\omega = 0$, which eventually yields (for the underdamped case)

$$\begin{aligned} &\frac{d}{d\omega}\left[(\omega_0^2-\omega^2)^2+\gamma^2\omega^2\right]_{\omega=\omega^*} = 0 \\ &\omega^* = \begin{cases} \sqrt{\omega_0^2-\frac{\gamma^2}{2}}, & \gamma \le \sqrt{2}\omega_0 \\ 0, & \sqrt{2}\omega_0 < \gamma < 2\omega_0 \end{cases} \end{aligned} \tag{5}$$

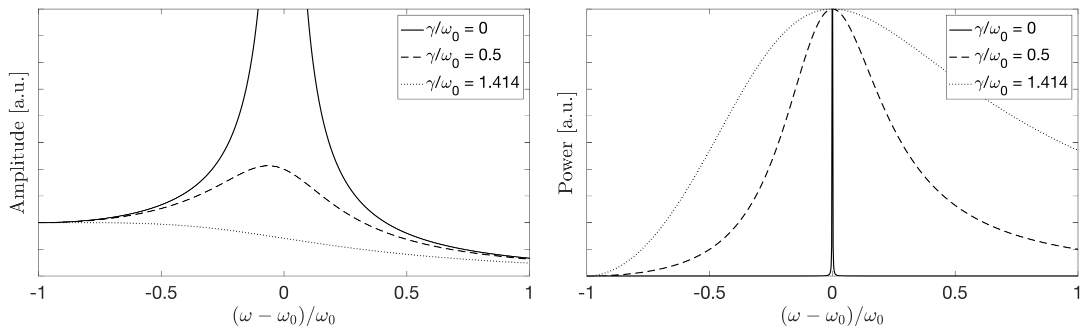


Figure 1: Amplitude (left) and normalized power (right) of underdamped driven harmonic oscillator for different $Q = \omega_0/\gamma$.

You can double-check these are maxima in $x_0$ by plotting them for various values of $\gamma/\omega_0$ as done in Fig. (1) or by taking the second derivative and find the ranges where the curvature is real and negative.

ii) The phase lag between the response and the drive is

$$\phi(\omega) = \tan^{-1}\left[\frac{\gamma\omega}{\omega^2 - \omega_0^2}\right]. \tag{6}$$

The lag of the response becomes $\pi/2$ when $\omega = \omega_0$.

iii) The power delivered from the drive to the oscillator (and dissipated by the damping), averaged over one cycle is

$$\begin{aligned} P(\omega) &= \langle F\dot{x}(\omega)\rangle \\ &= -F_0 x_0 \omega \langle \cos(\omega t)\sin(\omega t + \phi)\rangle \\ &= -F_0 x_0 \omega \langle \cos(\omega t)[\sin(\omega t)\cos(\phi) + \cos(\omega t)\sin(\phi)]\rangle \\ &= -F_0 x_0 \omega \sin(\phi)\langle \cos^2(\omega t)\rangle \\ &= \frac{\gamma F_0^2}{2m}\frac{\omega^2}{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2} \\ &= \left(\frac{1}{2}m\omega^2 x_0^2\right)\gamma. \end{aligned} \tag{7}$$

Power is dissipated via friction due to velocity. On resonance, the velocity is in phase with the driving force, resulting in the maximum power dissipated.

b) As the damping is decreased, the amplitude of the motion increases, thereby increasing the dissipated power on resonance. In the limit of no damping ($\gamma = 0$), the power dissipated on resonance becomes infinite because the amplitude blows up. The lineshape of the response becomes that of a Dirac delta function as seen in the right panel of Fig. (1).

c) The steady-state average energy $E$ stored in the oscillator is given by the average energy in the system per cycle. The virial theorem states that in a harmonic oscillator, the average kinetic energy is equal to that of the average potential energy, or $\langle K \rangle = \langle V \rangle$. We can write the total stored energy as

$$\begin{aligned} E &= 2\langle V \rangle = m\omega_0^2 \langle x^2 \rangle \\ &= m\omega_0^2 x_0^2 \langle \cos^2(\omega_0 t + \phi) \rangle \\ &= \frac{1}{2} m\omega_0^2 x_0^2. \end{aligned} \tag{8}$$

Given that the period of motion is $T$, the energy lost per radian is

$$\begin{aligned} E_{\text{lost}} &= PT/(2\pi) = P/\omega_0 \\ &= \frac{\gamma F_0^2}{2m} \frac{\omega_0}{(\omega_0^2 - \omega^2)^2 + \gamma^2 \omega_0^2} \\ &= \left( \frac{1}{2} m\omega_0^2 x_0^2 \right) \frac{\gamma}{\omega_0}. \end{aligned} \tag{9}$$

The quality factor $Q$ is the ratio of the energy stored to the energy lost:

$$Q = \frac{E}{E_{\text{lost}}} = \frac{\omega_0}{\gamma}. \tag{10}$$

## 2 Harmonically bound electron - Lorentz model [8 pts]

In the following, $\omega_0$ refers to the resonance frequency of the atom and $\omega$ the frequency of the drive.

a) The steady-state dipole moment $d(t) = ex(t)$ can be found by using the amplitude response of a driven harmonic oscillator without damping, given by Eq. (4) for $\gamma = 0$. We get

$$d(t) = ex(t) = \frac{e^2}{m} \frac{1}{\omega_0^2 - \omega^2} \mathcal{E} \cos(\omega t) = ex_0 \cos(\omega t) \tag{11}$$

where we define the amplitude of motion $x_0$

$$x_0 = \frac{e\mathcal{E}}{m} \frac{1}{\omega_0^2 - \omega^2}. \tag{12}$$

b) Say $x_0$ is the amplitude of motion and consider the resonant case $\omega = \omega_0$. Although our expression for $x_0$ that we found naively in part (a) diverges, we will see that there is inherent damping in the system, resulting in a finite amplitude, which we will call $x_0$ (but is NOT necessarily the expression we found in the previous part).

Using Eq. (11) and averaging over one cycle, we find the dissipated power is

$$\begin{aligned} P &= \frac{1}{6\pi\epsilon_0 c^3}\left\langle \left|\omega_0^2 e x_0 \cos(\omega_0 t)\right|^2 \right\rangle \\ &= \frac{1}{6\pi\epsilon_0 c^3}\frac{\omega_0^4 e^2 x_0^2}{2}. \end{aligned} \tag{13}$$

The energy lost per radian is

$$E_{\text{lost}} = P/\omega_0 = \frac{1}{6\pi\epsilon_0 c^3}\frac{\omega_0^3 e^2 x_0^2}{2}. \tag{14}$$

c) Using Eq. (8), the stored energy is $E_{\text{stored}} = m\omega_0^2 x_0^2/2$. The damping term is

$$\Gamma_{\text{rad}} = \frac{e^2\omega_0^2}{6\pi\epsilon_0 m c^3}. \tag{15}$$

d) The quality factor is $Q = \omega_0/\Gamma_{\text{rad}}$ as found in Eq. (10), yielding

$$Q = \frac{6\pi\epsilon_0 m c^3}{e^2\omega_0} = \frac{3}{2}\frac{c}{r_0\omega_0} = \frac{3}{2}\frac{\lambda\!\!\!^{-}_0}{r_0}. \tag{16}$$

e) The Lorentz model yields $Q = 4.99 \times 10^7$ and $\Gamma_{\text{rad}} = 2\pi \times 10\,\text{MHz}$, which is very close to the linewidth of sodium D2.

# 3 Quantum harmonic oscillator [6 pts]

a) The operators for position and momentum are

$$\begin{aligned} X &= \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)/2 \\ P &= i\sqrt{\frac{m\hbar\omega}{2}}(a^\dagger - a)/2. \end{aligned} \tag{17}$$

This shows that $\langle n|X|n\rangle = \langle n|P|n\rangle = 0$, i.e. that both the average position and momentum are zero.

To find the rms values of position and momentum we note that

$$\begin{aligned}\langle(\Delta X)^2\rangle &= \langle n|X^2|n\rangle - (\langle n|X|n\rangle)^2 = \langle n|X^2|n\rangle \\ \langle(\Delta P)^2\rangle &= \langle n|P^2|n\rangle - (\langle n|P|n\rangle)^2 = \langle n|P^2|n\rangle.\end{aligned} \tag{18}$$

Using

$$\begin{aligned}X^2 &= \frac{\hbar}{2m\omega}(a^{\dagger 2} + aa^\dagger + a^\dagger a + a^2) \\ P^2 &= -\frac{m\hbar\omega}{2}(a^{\dagger 2} - aa^\dagger - a^\dagger a + a^2)\end{aligned} \tag{19}$$

and

$$\langle n|aa^\dagger + a^\dagger a|n\rangle = \langle n|2a^\dagger a + 1|n\rangle = 2n + 1 \tag{20}$$

we obtain the rms values

$$\begin{aligned}x_{\text{rms}} &= \langle\Delta X\rangle = \sqrt{\frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)} \\ p_{\text{rms}} &= \langle\Delta P\rangle = \sqrt{m\hbar\omega\left(n + \frac{1}{2}\right)}.\end{aligned} \tag{21}$$

b) According to the virial theorem for a potential $V \sim x^2$, we have $\langle V\rangle = \langle K\rangle = E/2$, where

$$\begin{aligned}\langle V\rangle &= \frac{1}{2}m\omega^2\langle X^2\rangle \\ \langle K\rangle &= \frac{1}{2m}\langle P^2\rangle.\end{aligned} \tag{22}$$

From part (a) we see that

$$\langle V\rangle_n = \left(n + \frac{1}{2}\right)\frac{\hbar\omega}{2} = \frac{E_n}{2} = \langle K\rangle_n \tag{23}$$

demonstrating that the previous results are consistent with the virial theorem.

c) Defining $\sigma = \sqrt{\hbar/(m\omega)}$, we have

$$\begin{aligned}\psi_0(x) &= \left(\frac{1}{\sigma\sqrt{\pi}}\right)^{1/2}\exp\left(-\frac{x^2}{2\sigma^2}\right) \\ \psi_1(x) &= \left(\frac{2}{\sigma\sqrt{\pi}}\right)^{1/2}\frac{x}{\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)\end{aligned} \tag{24}$$

which are sketched in Fig. (2).

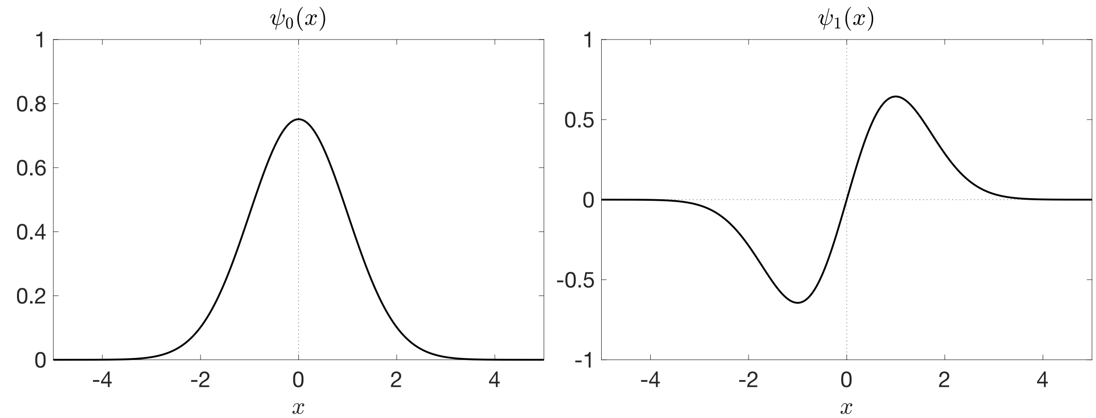


Figure 2: Harmonic oscillator wavefunctions setting $\sigma = 1$.

d) For $n = 0$, part (a) shows that

$$\begin{aligned} \text{rms size} &\equiv q_{\text{rms}} = \sqrt{\frac{\hbar}{2m\omega}} \\ \text{rms velocity} &\equiv \frac{p_{\text{rms}}}{m} = \sqrt{\frac{\hbar\omega}{2m}}. \end{aligned} \tag{25}$$

For a particle in state $n_x = 0,\ n_y = 0,\ n_z = 0$, these quantities are

$$\begin{aligned} \text{rms size} &= \sqrt{x_{\text{rms}}^2 + y_{\text{rms}}^2 + z_{\text{rms}}^2} = \sqrt{3}\, q_{\text{rms}} = \sqrt{\frac{3\hbar}{2m\omega}} \\ \text{rms velocity} &= \sqrt{v_{x,\text{rms}}^2 + v_{y,\text{rms}}^2 + v_{z,\text{rms}}^2} = \sqrt{3}\, \frac{p_{\text{rms}}}{m} = \sqrt{\frac{3\hbar\omega}{2m}}. \end{aligned} \tag{26}$$

Sodium has a mass of $23\,\text{amu} = 3.8 \times 10^{-26}\,\text{kg}$, so with $\omega = 2\pi \times 10^2\,\text{rad/s}$ the values are

$$\begin{aligned} \text{rms size} &= 2.6 \times 10^{-6}\,\text{m} \\ \text{rms velocity} &= 1.6 \times 10^{-3}\,\text{m/s}. \end{aligned} \tag{27}$$