8.421 Spring 2010 (Prof. Wolfgang Ketterle) **4/7/2010**

Midterm Quiz

Note: Some parts of problems can be solved without solving the earlier parts.

1. Magnetic moments and g-factors (18 points)

a. (4 p.) What is the magnetic moment of an atom in the following states: $^2S_{1/2}$, $^3S_1$, $^1D_2$, $^3P_0$?
(give the values without derivation)

b. You have two negatively charged particles with spins $\vec{s_k}$ and magnetic moment

$$\vec{\mu_k} = -g_k\,\mu_B\,\vec{s_k}/\hbar$$

The two spins couple to a total spin $\vec{S} = \vec{s_1} + \vec{s_2}$. For a weak external magnetic field B along the z-axis, find the possible values of the energy levels. Express the g-factor of the combined system by the individual g-factors g_i and the spin quantum numbers for the following situations:
(2 p.) $g_1 = g_2$
(6 p.) $g_1 = -g_2$
(6 p.) g_1 finite, $g_2=0$
Although you could solve the problem in general and then specialize to the three cases, it is recommended to do the simple calculations separately --- otherwise you may lose partial credit for the easier parts.

2. Spontaneous emission (13 points)

a. (2 p.)For weak monochromatic resonant excitation with Rabi frequency ω_R of an atom with natural linewidth Γ, what is the excitation rate (=rate of photon absorption) using lowest order perturbation theory?
(Note: The matrix element is $\hbar\,\omega_R/2$; you don't need to include numerical prefactors).

b. (4 p.)Using the result in a, explain how the excitation rate for a two-level system depends on the matrix element. Is this still true for broadband excitation?

c. (2 p.) In the limit of a very high (i.e. infinite) intensity, what is the rate of excitation? (give result without derivation). What is the cross section for photon absorption in this situation (assume monochromatic excitation)?

d. (5 p.) Specify the dominant multipole (such as E1 (electric dipole), E2, E3, .., M1, M2, M3 …) for spontaneous photon emission by an excited atomic electron in each of the following transitions. Assume simple hydrogenic wavefunctions without any relativistic or other corrections.
$2p_{1/2} \rightarrow 1s_{1/2}$
$2s_{1/2} \rightarrow 1s_{1/2}$
$3d_{3/2} \rightarrow 2s_{1/2}$
$2p_{3/2} \rightarrow 2p_{1/2}$
$3d_{3/2} \rightarrow 2p_{1/2}$

3. Hydrogen like atoms (17 points)

Consider an atom which consists of a lithium nucleus (Z=3) and (instead of an electron) a negative muon μ with a mass $m_\mu/m_e=207$.

a. (4 p.)Derive (using any model you want) the expression for the Bohr radius for the hydrogen atom. (you can ignore numerical factors)

b. (3 p.)For the muonic atom above, what is Bohr radius and the binding energy of the 1s state, relative to the hydrogen atom (neglect all nuclear mass corrections).

c. (3 p.)Compare atoms with the same nuclear charge, but with an electron or a muon in the 1s state. What is the ratio of the relativistic corrections to the binding energy?

d. (4 p.)Consider a lithium nucleus (Z=3) with three electrons? In units of the Rydberg constant, what is the ground state energy? What is the ground state energy, if one of the electrons is replaced by a negative muon? Neglect all interactions between electrons and muons.

e. (3 p.)Mention at least three phenomena which affect the energy of only s-electrons?

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1. a Magnetic moments

$^2S_{1/2}$	μ_B
$^3S_1$	$2\mu_B$
$^1D_2$	$2\mu_B$
$^3P_0$	0

b.
$$E = -\vec{\mu}\,\vec{B} = -(\vec{\mu}_1 + \vec{\mu}_2)\,\vec{B}$$

For weak Field
$$= \frac{\mu_B}{\hbar}\left(g_1\vec{S}_1 + g_2\vec{S}_2\right)\frac{\vec{S}}{|\vec{S}|}\;B\;\frac{M\hbar}{|\vec{S}|}$$

M quantum number for S_z

$$= \mu_B B M g$$

with $g = \left(g_1\vec{S}_1 + g_2\vec{S}_2\right)\vec{S} \big/ S(S+1)\hbar^2$

$g_1 = g_2 \Rightarrow g = g_1$

$g_1 = -g_2 \qquad g = g_1\left(\vec{S}_1 - \vec{S}_2\right)\left(\vec{S}_1 + \vec{S}_2\right) \big/ S(S+1)\hbar^2$

$$= g_1\left(\vec{S}_1^{\,2} - \vec{S}_2^{\,2}\right) \big/ S(S+1)\hbar^2$$

$$= g_1\,\frac{S_1(S_1+1) - S_2(S_2+1)}{S(S+1)}$$

$g_2 = 0$

$$g = g_1 \vec{s}_1 \vec{S} / S(S+1)\hbar^2$$

$$= g_1 \underbrace{\vec{s}_1 (\vec{s}_1 + \vec{s}_2)} / S(S+1)\hbar^2$$

$$\vec{s}_1^{\,2} + \vec{s}_1 \vec{s}_2 = \vec{s}_1^{\,2} + \frac{1}{2}\left(\vec{S}^2 - \vec{s}_1^{\,2} - \vec{s}_2^{\,2}\right)$$
$$= \frac{1}{2}\left(\vec{S}^2 + \vec{s}_1^{\,2} - \vec{s}_2^{\,2}\right)$$

$$g = \frac{g_1\left(S(S+1) + s_1(s_1+1) - s_2(s_2+1)\right)}{2\,S(S+1)}$$

2. Spontaneous emission

Fermi's Gold Rule

a) Exc. rate $= \underbrace{(\text{Matrix element})^2}_{\propto\, w_R^2} \cdot \underbrace{\text{Density of states}}_{\frac{1}{\Gamma}}$

$$= \frac{w_R^2}{\Gamma}$$ (prefactor is actually 1)

b) $\Gamma \propto (\text{Matrix element})^2$

$\Rightarrow$ exc. rate independent of matrix element

Broadband: $\Gamma \rightarrow$ Spectral width, exc rate $\propto$ (Matrix element)^2

c) Exc. rate $\approx \Gamma/2$

$$\sigma \longrightarrow 0 \qquad \left(\sigma = \frac{\sigma_0}{1+S}\right)$$

d) Multipole L requires

$$|j_i - j_f| \leq L \leq j_i + j_f$$

Parity $\Delta P = (-1)^L$ electric multipole

$(-1)^{L+1}$ magnetic multipole

$2p_{1/2} \rightarrow 1s_{1/2}$ $L=1$ $\Delta P = -$ $\Rightarrow$ $E1$

$2s_{1/2} \rightarrow 1s_{1/2}$ $L=1$ $\Delta P = +$ $(M1)$

It looks like M1 would be possible, however, 1s and 2s wavefunctions are orthogonal.
Transition allowed only in case of relativistic corrections, or as 2 photon transition

$3d_{3/2} \rightarrow 1s_{1/2}$ $L=1,2$ $\Delta P = +$ $(M1)$ or $E2$

$2p_{3/2} \rightarrow 2p_{1/2}$ $L=1,2$ $\Delta P = +$ $M1$ or $E2$

$3d_{3/2} \rightarrow 2p_{1/2}$ $L=1,2$ $\Delta P = -$ $E1$ or $(M2)$

Full credit for correct Ei and Mi

3)

4

a) Pot. energy $\sim -\frac{e^2}{a_0}$

Kinetic energy $\sim \left(\frac{\hbar}{a_0}\right)^2 \frac{1}{m}$

Minimize total energy

$$-\frac{e^2}{a_0} + \frac{\hbar^2}{a_0^2\, m} \quad \Rightarrow \quad a_0 = \frac{\hbar^2}{m e^2}$$

b) muonic atom $\quad a_\mu = \frac{\hbar^2}{m_\mu Z e^2} = \underbrace{\frac{m_e}{m_\mu}\frac{1}{Z}}_{\frac{1}{207 \cdot 3}} a_H$

621 × smaller

Rydberg constant $\quad R_\mu = \frac{1}{2}\frac{Ze}{a_\mu} = R_H \cdot \underbrace{\frac{Z^2 m_\mu}{m_e}}_{3^2 \cdot 207}$

1s binding energy larger by $3^2 \cdot 207$

c) Fine structure constant $\frac{e^2}{\hbar c}$ independent of mass $\Rightarrow$ relative relativistic corrections are the same

OR: Energies scale as m_μ $\Rightarrow$ Kinetic energy $\frac{1}{2} m_\mu v^2$ implies that v independent of mass d)

Li $\quad Ryd|_{Li} = 9\, Ryd|_{H}$

$$E|_{Li} = E_{1s} + E_{1s} + E_{2s}$$

$$= -9\, Ryd\left(1 + 1 + \frac{1}{4}\right)$$

One muon

$$E = E_{1s,e^-} + E_{1s,e^-} + E_{1s,\mu^-}$$

no Pauli exclusion

$$= -9\, Ryd\,(1 + 1 + 207)$$

e) s- electrons

- Lamb shift
- Darwin term of Fine structure
- Finite nuclear size isotope shift
- Hyperfine interaction contact term

Note: Quantum defect is small, but non zero for non-s electrons