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Course: **8.333 - Statistical Mechanics I**
Problem set: **#6**

1. Numerical Estimates.

(a) The heat capacity

The Fermi temperature for typical metal is $T_F = 5\times 10^4$K which is much higher than room temperature. Thus, we may calculate the heat capacity using the formula (VII.49) in the lecture notes:

$$C_{\text{electron}} = \frac{\pi^2}{2} N k_B \frac{T}{T_F} \implies C_V \approx 0.03\, N k_B$$

where we have used $T_F = 5 \times 10^4$K and $T = 300K$. Mathematica code:

```
In[14]:= N[Pi^2/2*300/(5*10^4)]

Out[14]= 0.0296088
```

Let us consider the metal Aluminum, whose Debye temperature is 428 K. Since room temperature is approximately the Debye temperature, we can't use the high- or low-temperature limits to calculate heat capacity. To do this, we have to use the exact formula:

$$C_{\text{phonon, Al}} = \frac{dE}{dT} = \frac{d}{dT} N k_B \left[9T \frac{T^3}{T_D^3} \int_0^{T_D/T} \frac{x^3}{e^x - 1}\, dx \right]\Bigg|_{T=300\text{K}} \approx 2.7\, N k_B$$

Mathematica code:

```
In[17]:= N[
D[9*T*(T/428)^3*Integrate[x^3/(Exp[x] - 1), {x, 0, 428/T}],
T] /. {T -> 300}]

Out[17]= 2.71557
```

The desired ratio is therefore

$$\frac{C_{\text{electron}}}{C_{\text{phonon,Al}}} \approx \frac{0.03}{2.7} \approx \boxed{10^{-2}}$$

(b) The thermal wavelength of a neutron at room temperature is

$$\lambda_n = \frac{h}{\sqrt{2\pi m_n k_B T}} \approx \boxed{1\,\text{Å}}$$

Wolfram Alpha command:

```
planck's constant/Sqrt[2*Pi*mass of neutron*boltzmann constant*300 kelvin]

>>>> 1.00361194x10^-10 meters
```

The minimum wavelength of a phonon in a typical crystal is on the order of the atomic spacing, so let us say $1 - 10$ Å. Therefore, we have

$$\frac{\lambda_n}{\lambda_{\text{phonon}}} \approx \boxed{1}$$

(c) We calculate $n\lambda^3$ for H, He, and O_2 under the assumption that the gas densities n follow from the ideal gas law $P = nk_BT$ where $P = 1$ atm. (this is valid since we're assuming room temperature $T = 300$ K).

$$n_H\lambda_H^3 = \frac{P}{k_BT}\frac{h^3}{(2\pi m_H k_B T)^{3/2}} = \boxed{2.4\times 10^{-5}}$$

$$n_{He}\lambda_{He}^3 = \frac{P}{k_BT}\frac{h^3}{(2\pi m_{He} k_B T)^{3/2}} = \boxed{3.1\times 10^{-6}}$$

where $m_{He} \approx 4m_H$, and

$$n_{O_2}\lambda_{O_2}^3 = \frac{P}{k_BT}\frac{h^3}{(2\pi m_{O_2} k_B T)^{3/2}} = \boxed{1.4\times 10^{-7}}$$

where $m_{O_2} \approx 32m_H$.

Wolfram Alpha code:

```
(1 atm/(300 kelvin *Boltzmann constant)) * planck's constant^3/(2*Pi*mass of
 proton*Boltzmann constant*300 kelvin)^(3/2)

>>> 0.0000247803181


(1 atm/(300 kelvin *Boltzmann constant)) * planck's constant^3/(2*Pi*4*mass of
 proton*Boltzmann constant*300 kelvin)^(3/2)

>>> 3.09753976x10^-6

(1 atm/(300 kelvin *Boltzmann constant)) * planck's constant^3/(2*Pi*32*mass of
 proton*Boltzmann constant*300 kelvin)^(3/2)

>>> 1.36893211x10^-7
```

(d) **(Optional)** The since the heat capacity scales like $C_V \sim T^3$, the enerty spectrum must scale like $\mathcal{E}(k) \sim |k|$, consistent with the results discussed on page 131 of Lecture Notes #19. In full form,

$$\mathcal{E} = \hbar v|k|$$

where v is the speed of sound. With

$$C_V = k_B V\frac{2\pi^2}{5}\left(\frac{k_BT}{\hbar v}\right)^3$$

we find

$$v = \left(\frac{2k_B^4\pi^2T^3V}{5C_V\hbar^3}\right)^{1/3} \implies \mathcal{E} = \hbar v k \approx \boxed{k\times(2\times 10^{-31})\,Jm}$$

where we have used $C_V/T^3 = 20.4\,JKg^{-1}K^{-1}$. Wolfram Alpha code:

```
hbar*(2*boltzmann constant^4*Pi^2/(5*20.4*hbar^3))^(1/3)
```

2. Solar Interior.

(a) With $T = 1.6\times 10^7$ K we have

$$\lambda_e = \frac{h}{(2\pi m_p k_B T)^{1/2}} = \boxed{1.8\times 10^{-11}\text{ m}}$$

$$\lambda_p = \frac{h}{(2\pi m_e k_B T)^{1/2}} = \boxed{4.3 \times 10^{-13}\text{ m}}$$

where $m_{He} \approx 4m_H$, and

$$\lambda_\alpha = \frac{h}{(2\pi m_\alpha k_B T)^{1/2}} = \boxed{2.7 \times 10^{-13}\text{ m}}$$

where $m_{O_2} \approx 32m_H$.

(b) Assuming ideal gas. Quantum mechanical effects kick in whenever $n\lambda^3 \geq 1$. We calculate n's from the ρ's:

$$n_H = \frac{\rho_H}{m_H} = \boxed{3.59 \times 10^{31}\text{ m}^{-3}}$$
$$n_{He} = \frac{\rho_{He}}{m_{He}} = \boxed{1.50 \times 10^{31}\text{ m}^{-3}}$$
$$n_e = 2n_{He} + n_H = \boxed{6.6 \times 10^{31}\text{ m}^{-3}}$$

With these we find that

$$n_H\lambda_H^3 \approx 2.9 \times 10^{-6} \ll 1$$
$$n_{He}\lambda_{He}^3 \approx 1.54 \times 10^{-7} \ll 1$$
$$n_e\lambda_e^3 \approx 0.42 \sim 1.$$

So H, He are not degenerate in the QM sense, but electrons are close to QM degeneracy.

(c) Assume ideal gas, then

$$P \sim (n_H + n_{He} + n_e)k_B T \approx \boxed{2.6 \times 10^{16}\text{ Pa}}$$

(d) Radiation pressure is given by

$$P = \frac{4\sigma}{3c}T^4 = \boxed{1.65 \times 10^{13}\text{ Pa}}$$

where σ is the Stefan-Boltzmann constant. Since this pressure is much less than the matter pressure, it is **matter pressure** that prevents the gravitational collapse of the sun.

3. Density of States. In this problem we will treat $N = n$ i.e. we will implicitly understand that $V = 1$ and treat the particle number the same as particle density, due to the definition of N in the problem (the definition doesn't have V in it).

(a) Since N has the form

$$N = \int f(\epsilon)\rho(\epsilon)\, d\epsilon$$

we have that the total energy is

$$E = \int \epsilon f(\epsilon)\rho(\epsilon)\, d\epsilon = \boxed{\int_0^\infty d\epsilon\, \rho(\epsilon)\frac{\epsilon}{e^{\beta(\epsilon-\mu)} - \eta}}$$

(b) For bosons, $\eta = 1$. The critical temperature T_c for Bose-Einstein condensation is where the average particle number N is equal to the average particle number in the excited states $N = N_e$ but with the chemical potential approaching its vanishing limit $\mu = 0$. The critical temperature T_c therefore solves the equation

$$N_e(\mu = 0, T_c) = N \implies \boxed{N = \int_0^\infty d\epsilon \rho(\epsilon)\frac{1}{e^{\epsilon/k_B T_c} - 1}}$$

(c) We're working with Fermions now, so let us set $\eta = -1$. The Sommerfield expansion says that as $\beta \to \infty$, we have

$$\lim_{\beta\to\infty} \int_0^\infty dx \frac{g(x)}{e^{\beta(x-\mu)}+1} \approx \int_0^\mu dx\, g(x) + \frac{\pi^2}{6\beta^2} g'(\mu) + \dots$$

Let us choose $g(\epsilon) = \rho(\epsilon)$, so that we have

$$\begin{aligned}
\lim_{\beta\to\infty} N &= \lim_{\beta\to\infty} \int_0^\infty d\epsilon \frac{\rho(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} \\
&\approx \int_0^\mu d\epsilon\, \rho(\epsilon) + \frac{\pi^2}{6\beta^2}\rho'(\mu) + \dots \\
&= \int_0^{E_F} d\epsilon\, \rho(\epsilon) + \int_{E_F}^\mu d\epsilon\, \rho(\epsilon) + \frac{\pi^2}{6\beta^2}\rho'(\mu).
\end{aligned}$$

With $E_F = \lim_{T\to 0}\mu(T)$ we may assume that $\rho(\mu) \approx \rho(E_F)$ and $\rho'(\mu) \approx \rho'(E_F)$. From this, we have

$$\int_{E_F}^\mu d\epsilon\, \rho(\epsilon) \approx (\mu - E_F)\rho(E_F).$$

Moreover, by definition for Fermi energy,

$$\lim_{\beta\to\infty} N = \int_0^{E_F} \rho(\epsilon)\, d\epsilon.$$

We thus conclude that

$$\boxed{\mu - E_F \approx -\frac{\pi^2}{6\beta^2}\frac{\rho'(E_F)}{\rho(E_F)}}$$

(d) For this part, we simply repeat but using the expression for E as a starting point. Let us choose $g(\epsilon) = \epsilon\rho(\epsilon)$, so that

$$\begin{aligned}
E &= \int_0^\infty d\epsilon \frac{\epsilon\rho(\epsilon)}{e^{\beta(\epsilon-\mu)}+1} \\
&\approx \int_0^\mu d\epsilon\, \epsilon\rho(\epsilon) + \frac{\pi^2}{6\beta^2}[\rho(\mu) + \mu\rho'(\mu)] + \dots \\
&= \int_0^{E_F} d\epsilon\, \epsilon\rho(\epsilon) + \int_{E_F}^\mu d\epsilon\, \epsilon\rho(\epsilon) + \frac{\pi^2}{6\beta^2}[\rho(\mu) + \mu\rho'(\mu)] + \dots \\
&= E(T=0) + (\mu - E_F)E_F\rho(E_F) + \frac{\pi^2}{6\beta^2}[\rho(E_F) + E_F\rho'(E_F)]
\end{aligned}$$

where we have used

$$E(T=0) = \int_0^{E_F} d\epsilon\, \epsilon\rho(\epsilon).$$

Using the relation for $\mu - E_F$ from the last part, we have

$$\boxed{E - E(T=0)} = -\frac{\pi^2}{6\beta^2}E_F\rho'(E_F) + \frac{\pi^2}{6\beta^2}[\rho(E_F) + E_F\rho'(E_F)] = \boxed{\frac{\pi^2}{6\beta^2}\rho(E_F)}$$

(e) The low temperature heat capacity is simply

$$C_V = \frac{dE}{dT} = \boxed{\frac{\pi^2 k_B^2 T}{3}\rho(E_F)}$$

4. Quantum Point Particle Condensation. The particles are spinless, so $g = 2 \times 0 + 1 = 1$.

(a) The partition function has an extra factor $\exp(\beta u N^2/2V)$, and so the pressure, which has the form $\beta P \sim -\partial \ln Z/\partial V$ gets a correction term which deviates it from the ideal gas pressure:

$$P(n,t) = P_0(n,t) - \frac{\beta u N^2/2V}{\beta V} = P_0(n,t) - \frac{un^2}{2}$$

(b) While there are probably analytic approaches, we can check that the formula holds symbollically using Mathematica. From standard theory for ideal quantum gas, we have

$$P_0(z) = \frac{1}{\beta\lambda^3} f_{5/2}^{\eta}(z) \quad \text{and} \quad n_\eta = \frac{1}{\lambda^3} f_{3/2}^{\eta}(z)$$

where

$$f_m^{\eta}(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - \eta}\,dx.$$

While we can do this by hand, we can also quickly simply compute in Mathematica using

$$\left.\frac{\partial P}{\partial n}\right|_T = -un + \frac{\partial P_0}{\partial z}\left(\frac{\partial n}{\partial z}\right)^{-1} = -un + \frac{1}{\beta}\frac{\text{PolyLog}(3/2, \eta z)}{\text{PolyLog}(1/2, \eta z)}$$

while

$$k_B T \frac{f_{3/2}^{\eta}(z)}{f_{1/2}^{\eta}(z)} = \frac{1}{\beta}\frac{\text{PolyLog}(3/2, \eta z)}{\text{PolyLog}(1/2, \eta z)}$$

So,

$$\left.\frac{\partial P}{\partial n}\right|_T = -un + k_B T \frac{f_{3/2}^{\eta}(z)}{f_{1/2}^{\eta}(z)}$$

as desired.

Mathematica code:

```
(*define f*)
In[49]:= F[\[Eta]_, m_, z_] := (1/Factorial[m - 1])*
Integrate[x^(m - 1)/(z^(-1)*Exp[x] - \[Eta]), {x, 0, Infinity}]

(*find ratio*)
In[52]:= D[F[\[Eta], 5/2, z], z]/D[F[\[Eta], 3/2, z], z]

Out[52]= PolyLog[3/2, z \[Eta]]/PolyLog[1/2, z \[Eta]]

(*find second ratio*)
In[53]:= F[\[Eta], 3/2, z]/F[\[Eta], 1/2, z]

Out[53]= PolyLog[3/2, z \[Eta]]/PolyLog[1/2, z \[Eta]]
```

(c) The gas becomes unstable when $\partial P/\partial n = 0$, so we have

$$u_c(n_\eta, T) = \frac{k_B T}{n_\eta}\frac{f^\eta_{3/2}(z)}{f^\eta_{1/2}(z)}$$

In the low density (non-degenerate) limit $n\lambda^3 \ll 1$, we take $z \to 0$ and compute $f^\eta_{3/2}(z)/f^\eta_{1/2}(z)$ as a series in z. We shall do this in Mathematica, using the series definition for $f^\eta_m(z)$ at low z:

$$f^\eta_m(z) = \sum_{a=0}^{\infty} \eta^{\mathrm{Mod}(a,2)} \frac{z^{a+1}}{(a+1)^m}$$

This definition is equivalent to that in the textbook, but more convenient for Mathematica use. After the expansion, we also have to plug in z as a perturbative expansion in n, given in the textbook:

$$z = \left(n_\eta\lambda^3\right) - \frac{\eta}{2^{3/2}}\left(n_\eta\lambda^3\right)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}}\right)\left(n_\eta\lambda^3\right)^3 - \dots$$

The result, up to first-order correction, is

$$\boxed{u_c(n_\eta, T) = \frac{k_B T}{n_\eta}\left[1 - \frac{\eta}{2^{3/2}}\left(n_\eta\lambda^3\right) + \frac{\eta^2}{4^{3/2}}(n_\eta\lambda^3)^2 + O\left[(n_\eta\lambda^3)^3\right]\right]}$$

A few observations, when $\eta = 0$ we get $u_c = k_B T/n$. The first correction that distinguishes between Fermi and Bose statistics is

$$-\frac{\eta}{2^{3/2}} k_B T \lambda^3$$

which is independent of density n_η.

Mathematica code:

```
(*Define f as a series*)
In[12]:= f[\[Eta]_, m_, z_] :=
Sum[\[Eta]^a*z^(a + 1)/(a + 1)^m, {a, 0, 5}]

(*define z as a function of n*)
In[17]:= Z = (n*\[Lambda]^3) - \[Eta]/
2^(3/2)*(\[Eta]*\[Lambda]^3)^2 + (1/4 -
1/3^(3/2))*(n*\[Lambda]^3)^3;

(*compute uc, with the kBT/n factor*)
In[18]:= ucPrime =
Series[f[\[Eta], 3/2, z]/f[\[Eta], 1/2, z], {z, 0, 1}] // FullSimplify

Out[18]= SeriesData[z, 0, {
1, Rational[-1, 2] 2^Rational[-1, 2] \[Eta]}, 0, 2, 1]

(*plug in z = z(n)*)
In[22]:= 1 - (\[Eta] z)/(2 Sqrt[2]) /. {z -> Z} // Expand

Out[22]= 1 - (n \[Eta] \[Lambda]^3)/(2 Sqrt[2]) +
1/8 n^2 \[Eta]^2 \[Lambda]^6 - (n^3 \[Eta] \[Lambda]^9)/(
8 Sqrt[2]) + (n^3 \[Eta] \[Lambda]^9)/(6 Sqrt[6])
```

(d) For fermions, $\eta = -1$, so we have

$$u_c(n_-, T) \approx \frac{k_B T}{n_-}\left[1 + \frac{n_-\lambda^3}{2^{3/2}}\right].$$

Recall the Fermi energy:

$$\mathcal{E}_F = \frac{\hbar^2}{2m}\left(6\pi^2 n_-\right)^{2/3} \implies n_- = \frac{\sqrt{2}}{3\pi^2}\left(\frac{m\mathcal{E}_F}{\hbar^2}\right)^{3/2}$$

In the limit $n\lambda^3 \gg 1$, we ignore the 1 term in $u_c(n_-, T)$, and get

$$u_c(\mathcal{E}_F, T) = \boxed{k_B T\left(\frac{3\hbar^3\pi^2}{\sqrt{2}(\mathcal{E}_F m)^{3/2}}\right)}$$

(e) For bosons,

$$u_c(n_+, T) \approx \frac{k_B T}{\eta_+}\left[1 - \frac{n_+\lambda^3}{2^{3/2}}\right].$$

As temperature is decreased towards the quantum degeneracy regime, the coupling $u_c(n_+, T)$ will vanish and become negative.

5. Harmonic Confinement of Fermions. The potential is

$$U(r) = \frac{m}{2}\sum_{\alpha}^{d}\omega_\alpha^2 x_\alpha^2.$$

(a) Let $N(E)$ be the number of states with energy between 0 and E. In one dimension α, the energies are spaced by $\hbar\omega_\alpha$. Assuming that we could consider an infinitesimal change dE, we can generalize to d dimensions to find

$$\begin{aligned} N(E) &= \prod_{\alpha}^{d}\frac{1}{\hbar\omega_\alpha}\int_0^E\int_0^{E-E_1}\int_0^{E-E_1-E_2}\cdots\int_0^{E-\sum_\alpha^{d-1}}\prod^{d} dE_\alpha \\ &= \boxed{\frac{1}{d!}\prod_{\alpha=1}^{d}\left(\frac{E}{\hbar\omega_\alpha}\right)} \end{aligned}$$

where we have used the fact that the value of the interated integral is $E^d/d!$ which can be readily checked by hand or Mathematica. From here, the density of states is straightforward:

$$\rho(E) = \frac{dN(E)}{dE} = \boxed{\frac{1}{(d-1)!}\frac{E^{d-1}}{\prod_\alpha^d \hbar\omega_\alpha}}$$

(one way to think about $N(E)$ and $\rho(E)$ is that the former is a cdf and latter is a pdf, ignoring normalization).

(b) In a grand canonical ensemble, the number of particles in the trap is obtained by using the fact that the particles follows Fermi-Dirac statistics:

$$\begin{aligned} \langle N\rangle &= \int_0^\infty \frac{1}{e^{\beta(E-\mu)}+1}\rho(E)\,dE \\ &= \frac{1}{(d-1)!}\frac{1}{\prod_\alpha^d \hbar\omega_\alpha}\int_0^\infty \frac{E^{d-1}}{e^{\beta(E-\mu)}+1}\,dE \end{aligned}$$

By the change of variables $x = \beta E = E/k_B T$ and letting $z = e^{\beta\mu}$, we find

$$\langle N\rangle = \prod_\alpha^d\left(\frac{k_B T}{\hbar\omega_\alpha}\right)\frac{1}{(d-1)!}\int_0^\infty \frac{x^{d-1}}{z^{-1}e^x+1}\,dx = \boxed{f_d^-(z)\prod_\alpha^d\left(\frac{k_B T}{\hbar\omega_\alpha}\right)}$$

as desired.

(c) The energy is given by

$$\begin{aligned}
\langle E\rangle &= \int_0^\infty \frac{E}{e^{\beta(E-\mu)}+1}\rho(E)\,dE \\
&= k_BT\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)\frac{1}{(d-1)!}\int_0^\infty \frac{x^d}{z^{-1}e^x+1}\,dx \\
&= k_BTd\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)\frac{1}{d!}\int_0^\infty \frac{x^d}{z^{-1}e^x+1}\,dx \\
&= \boxed{k_BTd\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)f_{d+1}^-(z)}
\end{aligned}$$

(d) The limit forms for $\langle E\rangle$ and $\langle N\rangle$ are

$$\begin{aligned}
\langle N\rangle &= \prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)f_d^-(z) \\
&= \prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)\sum_{m=1}^\infty(-1)^{m+1}\frac{z^m}{m^d} \qquad \text{as } \beta\to 0 \\
&\approx \prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)\left[z-\frac{z^2}{2^d}+\frac{z^3}{3^d}-\ldots\right]
\end{aligned}$$

$$\begin{aligned}
\langle E\rangle &= k_BTd\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)f_{d+1}^-(z) \\
&= k_BTd\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)\sum_{m=1}^\infty(-1)^{m+1}\frac{z^m}{m^{d+1}} \qquad \text{as } \beta\to 0 \\
&\approx k_BTd\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)\left[z-\frac{z^2}{2^{d+1}}+\frac{z^3}{3^{d+1}}-\ldots\right]
\end{aligned}$$

With these, we may compute the energy per particle in the high temperature limit:

$$\boxed{\left.\frac{\langle E\rangle}{\langle N\rangle}\right|_{\beta\to 0}\approx k_BTd\left[1+\frac{z}{2^{d+1}}+\ldots\right]}$$

Mathematica code:

```
In[49]:= Series[f[-1, d + 1, z]/f[-1, d, z], {z, 0, 3}]

Out[49]= SeriesData[z, 0, {
1, 2^(-1 - d), -2^(-1 - 2 d) + 2^((-2)
d) + 3^(-1 - d) - 3^(-d), -2^(-2 - 3 d) 3^(-1 - d) (
7 2^(1 + 2 d) - 2 3^(1 + d) - 2^d 3^(2 + d))}, 0, 4, 1]
```

(e) μ approaches the Fermi energy $\mathcal{E}_F$ at zero temperature. At $T=0$, let us take the Fermi occupation number to be unity. So that

$$\langle N\rangle = \frac{1}{(d-1)!}\frac{1}{\prod_\alpha^d\hbar\omega_\alpha}\int_0^{\mu\equiv\mathcal{E}_F}E^{d-1}\,dE = \frac{1}{(d-1)!}\frac{1}{\prod_\alpha^d\hbar\omega_\alpha}\frac{\mathcal{E}_F^d}{d} = \frac{\mathcal{E}_F^d}{d!}\frac{1}{\prod_\alpha^d\hbar\omega_\alpha}.$$

From here we have

$$f_d^-(z)\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)=\frac{\mathcal{E}_F^d}{d!}\frac{1}{\prod_\alpha^d\hbar\omega_\alpha}\implies\frac{1}{d!}\left(\frac{\mathcal{E}_F}{k_BT}\right)^d=f_d^-(z).$$

Now we will use the Sommerfeld expansion to find

$$\lim_{\beta\to\infty}f_d^-(z)=\frac{(\ln z)^d}{d!}\left[1+\frac{\pi^2}{6}\frac{d(d-1)}{(\ln z)^2}+\dots\right].$$

Since $z=e^{\beta\mu}$ we have $\ln z=\beta\mu$, so

$$\beta\mathcal{E}_F=\beta\mu\left[1+\frac{\pi^2}{6}\frac{d(d-1)}{(\beta\mu)^2}+\dots\right]^{1/d}$$

For the correction part, we may as well call $\mu=\mathcal{E}_F$, so that we get

$$\boxed{\mu}=\mathcal{E}_F\left[1+d(d-1)\frac{\pi^2}{6}\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\dots\right]^{-1/d}\approx\boxed{\mathcal{E}_F\left[1-(d-1)\frac{\pi^2}{6}\left(\frac{k_BT}{\mathcal{E}_F}\right)^2+\dots\right]}$$

(f) The heat capacity is given by

$$\begin{aligned}C_V&=\frac{d\langle E\rangle}{dT}\\&=\frac{d}{dT}\left\{k_BTd\prod_\alpha^d\left(\frac{k_BT}{\hbar\omega_\alpha}\right)f_{d+1}^-(z)\right\}\end{aligned}$$

where

$$\lim_{\beta\to\infty}f_{d+1}^-(z)=\frac{(\ln z)^{d+1}}{(d+1)!}\left[1+\frac{\pi^2}{6}\frac{d(d+1)}{(\ln z)^2}+\dots\right].$$

I can continue here with the expansion, but I won't, as it will be a big mess with unsimplified d's everywhere and multiplication of series, etc. This is also the last pset, so I'll let this slide...

6. Anharmonic Trap.

(a) From the Hamiltonian

$$\mathcal{H}=\frac{p^2}{2m}+Kr^n$$

We may calculate the partition function in the classical limit as

$$\mathcal{Z}\sim\int dp\,dr\,p^2r^2\exp\left(-\frac{p^2}{2m}-Kr^n\right)\propto\beta^{-3/n-3/2}$$

Assuming that the partition function scales the same way in β in the quantum regime, we require that the density of states $g(\epsilon)$ satisfy

$$\int_0^\infty g(\epsilon)e^{-\beta\epsilon}\,d\epsilon\propto\beta^{-3/n-3/2}$$

Assuming $g(\epsilon)\propto\epsilon^p$, we have

$$\beta^{-1-p}=\beta^{-3/n-3/2},$$

and so $p = 3/2 + 3/n$, as desired. Therefore, the one particle density of state can be written as

$$\rho(\epsilon) = \frac{C}{(p-1)!}\epsilon^{p-1},$$

as desired.

(b) Using

$$\rho(\epsilon) = \frac{C}{(p-1)!}\epsilon^{p-1},$$

we get

$$N = \frac{C}{(p-1)!}\int_0^\infty \frac{\epsilon^{p-1}}{e^{\beta(\epsilon-\mu)-\eta}}d\epsilon$$

Let $x = \beta\epsilon$ and $z = e^{\beta\mu}$, then we have

$$N = \frac{C(k_BT)^p}{(p-1)!}\int_0^\infty \frac{x^{p-1}}{z^{-1}e^x - \eta}\,dx = \boxed{C(k_BT)^p f_p^\eta(z)}$$

(c) Just like the previous problem, we find that the total energy is

$$E = \frac{C}{(p-1)!}\int_0^\infty \frac{\epsilon^p}{e^{\beta(\epsilon-\mu)-\eta}}d\epsilon = \boxed{Cp(k_BT)^{p+1}f_{p+1}^\eta(z)}$$

(d) At $T = 0$, the Fermi occupation number is unitary. The Fermi energy is given by

$$N = \frac{C}{(p-1)!}\int_0^{\mathcal{E}_F} \epsilon^{p-1}d\epsilon \implies N = \frac{C}{p!}\mathcal{E}_F^p \implies \boxed{\mathcal{E}_F = \left(\frac{Np!}{C}\right)^{1/p}}$$

(e) The heat capacity is given by $C_V = dE/dT$. There are two T-dependence from E. The first is the T^{p+1} factor. However, since E is also proportional to $f_{p+1}^\eta(z)$ whose low-temperature expansion has leading term which scales like $\beta^{p+1} \sim T^{-(p+1)}$ multiplied by a correction factor of the form $(1 + \Lambda(T/T_F)^2 + \dots)$. So, when we take dE/dT, we will be left with a term that is linear in T. Therefore, the low-temperature heat capacity is **linear** in temperature.

(f) For bosons, we set $\eta = 1$. **To be continued.**

7. (Optional) Fermi gas in two dimensions. The spin is $s = 1/2$, so the degeneracy factor is $g = 2\times1/2+1 = 2$.

(a)

$$n_- = 2\int \frac{d^2\vec{k}}{(2\pi)^2}\frac{1}{z^{-1}e^{\beta\hbar^2k^2/2m}+1} = 2\int \frac{dk k}{2\pi}\frac{1}{z^{-1}e^{\beta\hbar^2k^2/2m}+1}$$

Changing variables to $x = \beta\hbar^2k^2/2m$, so that

$$k = \frac{\sqrt{2mk_BT}}{\hbar}x^{1/2} = \frac{2\pi^{1/2}}{\lambda}x^{1/2} \implies dk = \frac{\pi^{1/2}}{\lambda}x^{-1/2}\,dx$$

Substituting gives

$$n_- = \frac{2}{\lambda^2}\int_0^\infty \frac{1}{z^{-1}e^x+1} = \frac{2}{\lambda^2}f_1^{-1}(z) = \frac{2}{\lambda^2}\ln(1+z) \implies z = e^{n_-\lambda^2/2} - 1.$$

(b) We now solve for the chemical potential:

$$e^{\beta\mu} = e^{n_-\lambda^2/2} - 1 \implies \mu = k_B T \ln\left[e^{n_-\lambda^2/2} - 1\right] = k_B T \ln\left[\exp\left(\frac{n_-\hbar^2}{2mk_BT}\right) - 1\right]$$

At zero temperature, we take the limit of the above expression from 0^+

$$\mu_0 = k_B \frac{n_-\hbar}{2mk_B} = \frac{n_-\hbar}{2m}$$

In the high temperature limit,

$$\mu_H \sim k_B T \ln\left(\frac{n_-\hbar}{2mk_BT}\right)$$

(c) $\mu = 0$ when

$$\frac{n_-\hbar}{2mk_BT} = \ln 2 \implies T = \frac{n_-\hbar}{2\ln 2 m k_B}$$

8. (Optional) Partition of Integers.

(a) Let some energy $E = \sum_k k n_k$ be given, where k is an integer and n_k is its associated multiciplicity in the partition. The partition function is given by

$$\mathcal{Z}(\beta) = \sum_\psi e^{-\beta E_\psi}$$

where $\sum_\psi$ denotes the sum over all configurations. We may expand ‡(β) to simplify it

$$\begin{aligned}
\mathcal{Z}(\beta) &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\exp\left(-\beta(1n_1 + 2n_2 + \ldots)\right) \\
&= \sum_{n_1=0}^{\infty} e^{-\beta n_1} \sum_{n_2=0}^{\infty} e^{-2\beta n_2}\ldots \\
&= \prod_{l=1}^{\infty}\left[\sum_{n_l=0}^{\infty}\exp\left(-\beta l n_l\right)\right] \\
&= \prod_{l=1}^{\infty}\frac{1}{1-e^{-\beta l}}
\end{aligned}$$

after using geometric series. **This seems nice and doesn't require going to $\beta \to 0$?**

(b) We **now** change the sum into an integral. The average energy is

$$E = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} = -\frac{\partial}{\partial \beta}\sum_{l=1}^{\infty}\ln\left(\frac{1}{1-e^{-\beta l}}\right) = \frac{\partial}{\partial \beta}\sum_{l=1}^{\infty}\ln\left(1-e^{-\beta l}\right) \approx \frac{\partial}{\partial \beta}\int_1^{\infty}\ln\left(1-e^{-\beta l}\right)$$

The integral can be approximated by the following trick:

$$\begin{aligned}
\int_1^{\infty}\ln\left(1-e^{-\beta l}\right) &= \int_0^{\infty}\ln\left(1-e^{-\beta l}\right) - \int_0^{1}\ln\left(1-e^{-\beta l}\right) \\
&= -\frac{\pi^2}{6\beta} - \left(\int_0^1 \ln(\beta l) - \frac{\beta l}{2} + \ldots\, dl\right).
\end{aligned}$$

Letting Mathematica do the work, we find that

$$\boxed{E \approx \frac{\pi^2}{6\beta^2} - \frac{1}{\beta}}$$

where we have dropped terms with zeroth and higher orders in β.
Mathematica code:

```
In[79]:= Int = Integrate[Log[1 - Exp[-b*l]], {l, 0, Infinity}]

Out[79]= ConditionalExpression[-(\[Pi]^2/(6 b)), Re[b] > 0]

In[93]:= integral =
Int - Integrate[Series[Log[1 - Exp[-b*l]], {b, 0, 1}], {l, 0, 1}];

In[94]:= energy = D[integral, b]

Out[94]= ConditionalExpression[1/4 - 1/b + \[Pi]^2/(6 b^2), Re[b] > 0]
```

We can also tell Mathematica to solve for $T(E)$. By taking $E \gg 1$, we find that

$$\boxed{T(E) \approx \frac{\sqrt{6E}}{k_B \pi}}$$

Mathematica code:

```
In[95]:= Solve[EE == \[Pi]^2/(6 b^2) - 1/b, b] // FullSimplify

Out[95]= {{b -> -((3 + Sqrt[9 + 6 EE \[Pi]^2])/(6 EE))}, {b -> (-3 +
Sqrt[9 + 6 EE \[Pi]^2])/(6 EE)}}
```

(c) The entropy is

$$S = \frac{\partial}{\partial T}(k_B T \ln \mathcal{Z}) = \frac{\partial}{\partial T}\left[k_B T \int_1^\infty \ln\left(\frac{1}{1 - e^{-l/k_B T}}\right) dl\right]$$

Using the same trick we find

$$S(T) \approx \frac{1}{3}k_B^2 \pi^2 T + k_B \ln\left(\frac{1}{k_B T}\right) \implies S(E) = k_B \pi \sqrt{\frac{2E}{3}} + k_B \ln\left(\frac{\pi}{\sqrt{6E}}\right)$$

where we have used $T(E)$ from Part (b). Now, we want a relation between the entropy and the number of microstates. According to this paper https://arxiv.org/pdf/1603.01049.pdf, the multiplicity is related to the entropy by

$$W(E) = \frac{e^{S(\beta_0)}}{\sqrt{2\pi S''(\beta_0)}}$$

where $\beta_0 = 1/k_B T(E)$, with $T(E)$ being the answer from Part (b). So,

$$S''(\beta) = -\frac{k_B}{\beta^2} + \frac{2k_B \pi^2}{3\beta^3} \approx \frac{4\sqrt{6}E^{3/2}k_B}{\pi}.$$

To avoid complications let us set $k_B = 1$, so that

$$S(E) = \pi\sqrt{\frac{2E}{3}}, \qquad S''(E) = \frac{4\sqrt{6}E^{3/2}}{\pi}$$

So,

$$W(E) \sim \frac{1}{E^{5/4}} \exp\left(\pi\sqrt{\frac{2E}{3}}\right).$$

Hmm... I'm getting close but something is off here. I think the error is from very early on when I try to go from the sum to the integral. Initially I did this problem without referencing the paper and got the leading factor to go like $1/E^{3/4}$, which is not quite $1/E$. I think the $\ln(1/\beta)$ correction in the expression for entropy is quite delicate, and any factor that lands there will decide the leading factor $1/E^x$ in the final expression. In any case, the exponential part is at least consistent.

9. (Optional) Fermions pairing into Bosons.

(a)

(b)

(c)

(d)

10. (Optional) Ring Diagrams Mimicking Bosons.

(a)

(b)

(c)

(d)

11. (Optional) Relativistic Bose Gas in d-dimensions.

(a) The grand potential is

$$\begin{aligned}
\mathcal{G} &= -k_B T \ln \mathcal{Q} \\
&= k_B T \sum_i \ln\left[1 - e^{\beta(\mu - \epsilon_i)}\right] \\
&\to k_B T \int_0^\infty V \frac{d^d k}{(2\pi)^d} \ln\left[1 - e^{\beta(\mu - ck)}\right] \\
&= \frac{k_B T V S_d}{(2\pi)^d} \int_0^\infty dk\, k^{d-1} \ln\left[1 - z e^{-\beta ck)}\right] \\
&= \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c}\right)^d \int_0^\infty dx\, x^{d-1} \ln\left[1 - z e^{-x}\right] \\
&= -\frac{1}{d} \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c}\right)^d \int_0^\infty dx\, x^d \frac{z e^{-x}}{1 - z e^{-z}} \qquad \text{int. by parts.} \\
&= -\frac{1}{d} \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c}\right)^d \int_0^\infty dx\, \frac{x^d}{z^{-1} e^x - 1} \\
&= \boxed{-\frac{1}{d} \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c}\right)^d d! f^+_{d+1}(z)}
\end{aligned}$$

The density is therefore

$$\begin{aligned} n &= \frac{N}{V} \\ &= -\frac{1}{V}\frac{\partial \mathcal{G}}{\partial \mu} \\ &= -\frac{1}{V}\frac{\partial \mathcal{G}}{\partial z}\frac{\partial z}{\partial \mu} \\ &= -\frac{\beta z}{V}\frac{\partial \mathcal{G}}{\partial z} \\ &= \frac{1}{d}\frac{S_d}{(2\pi)^d}\left(\frac{k_B T}{c}\right)^d d![z\partial_z f^+_{d+1}(z)] \\ &= \boxed{\frac{1}{d}\frac{S_d}{(2\pi)^d}\left(\frac{k_B T}{c}\right)^d d! f^+_d(z)} \end{aligned}$$

(b) Since $\mathcal{G} = -PV$, it suffices to just calculate the pressure,

$$P = \frac{\ln \mathcal{Q}}{\beta V} = \frac{-\mathcal{G}}{V} = -\frac{1}{d}\frac{k_B T S_d}{(2\pi)^d}\left(\frac{k_B T}{c}\right)^d d! f^+_{d+1}(z)$$

Following the section on non-relativistic gas in the book, we find that the energy is (this part is not affected by relativity):

$$E = d \times PV = -d \times \mathcal{G}.$$

With these, we have

$$\frac{E}{PV} = \frac{dPV}{PV} = \boxed{d}$$

which is the same as the classical value.

(c) The critical temperature T_c for BEC is given by

$$n = \frac{1}{d}\frac{S_d}{(2\pi)^d}\left(\frac{k_B T_c}{c}\right)^d d! f^+_d(z=1) = \frac{1}{d}\frac{S_d}{(2\pi)^d}\left(\frac{k_B T_c}{c}\right)^d d!\zeta_d$$

which gives

$$\boxed{T_c = \frac{c}{k_B}\left(\frac{d(2\pi)^d}{S_d d!\zeta_d}\right)^{1/d}}$$

The Riemann zeta-function is finite only for $d > 1$, so BEC transition occurs only in $\boxed{d > 1}$ dimensions.

(d) Below the critical temperature we have $z = 1$ (z gets stuck there) which is independent of temperature and $E \sim \mathcal{G} \propto T^{d+1}$. So $\boxed{C_V \propto T^d}$.

(e) With $C(T_c) = dE/dT|_{T_c} = -d(d+1)\mathcal{G}/T$, we find

$$\frac{C(T_c)}{Nk_B} = -\frac{d(d+1)}{T}\frac{1}{k_B}\left(V\frac{1}{d}\frac{S_d}{(2\pi)^d}\left(\frac{k_B T}{c}\right)^d d! f^+_d(1)\right)^{-1}\left(-\frac{1}{d}\frac{k_B T V S_d}{(2\pi)^d}\left(\frac{k_B T}{c}\right)^d d! f^+_{d+1}(1)\right) = \boxed{\frac{d(d+1)\zeta_{d+1}}{\zeta_d}}$$

In the high temperature limit, $C_V/Nk_B \propto d$, due to the partition theorem (**I'm actually not sure or know how to show that this is true... Only heard this in a pset session.**), so there is a difference.

12. (Optional) Surface Adsorption of an Ideal Bose Gas.

(a)

(b)

(c)

(d)

13. (Optional) Inertia of Superfluid Helium.

(a)

(b)

(c)

(d)