STATMECH 1

REVIEW

FINAL 2021

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Common Tactics for SM 1

Distinguish QM & CM partition fn ...

- $$\boxed{Z_c = \frac{1}{h^3}\int d^3p\, d^3q\, \exp(-\beta\mathcal{H})}$$

(N!) if id particles

one for each pair p, q

- $$\boxed{Z_{QM} = \mathrm{Tr}\left(\exp(-\beta\mathcal{H})\right)}$$

Sometimes this is a nice series, can be evaluated ...

From Z_q get Q_q → grand partition fn

↳ From Q_q get

$Q_q \to P(\{n_k\}) \to$ joint probability

$$= \boxed{\frac{1}{Q_q} \prod_{\vec{k}} \exp\left(-\beta(\varepsilon(\vec{k}) - \mu) n_{\vec{k}}\right)}$$

$\to \langle n_{\vec{k}} \rangle_\eta \to$ avg occupation number at energy $\vec{k}$ ...

$\eta \to (-1)$ F

$\eta \to (+1)$ B

$$= \frac{-\partial \ln Q_q}{\partial(\beta \varepsilon(\vec{k}))}$$

$$= \frac{1}{z^{-1} e^{\beta \varepsilon(k)} - \eta}$$

fugacity $\boxed{z = e^{\beta \mu}}$

From here, get average particle number & internal energy

$$\boxed{N_\eta = \sum_k \langle n_k \rangle_\eta} \; ; \; \boxed{E_\eta = \sum_k \varepsilon(\vec{k}) \langle n_{\vec{k}} \rangle_\eta}$$

- Spin degeneracy factor $g = 2s+1$
- $\varepsilon(k) = \hbar^2 k^2 / 2m$; $\sum_{\vec{k}} \to \int \frac{d^3k}{(2\pi)^3}$

To get $\boxed{\text{PRESSURE, DENSITY, ENERGY DENSITY}}$

$$\boxed{\beta P_\eta = \frac{\ln Q_\eta}{V}}$$

$$\boxed{\mathcal{E}_\eta = \frac{E_\eta}{V}}$$

$$\boxed{n_\eta \equiv \frac{N_\eta}{V}}$$

{pg. 188 of textbook}

. . .

Non relativistic gas

$$\boxed{\beta P_\eta = \frac{g}{\lambda^3} f^\eta_{5/2}(z)}$$

$$\boxed{n_\eta = \frac{g}{\lambda^3} f^\eta_{3/2}(z)}$$

$$\boxed{\mathcal{E}_\eta = \frac{3}{2} P_\eta}$$

where

$$\boxed{f_m^{\eta}(z) = \frac{1}{(m-1)!}\int_0^{\infty}\frac{dx\; x^{m-1}}{z^{-1}e^{x}-\eta}}$$

Two limits ... high & low temp.

High Temp

$$\boxed{f_m^{\eta}(z) \sim \sum_{\alpha=1}^{\infty} \eta^{\alpha+1}\frac{z^{\alpha}}{\alpha^{m}}}$$

→ get n, $\underline{P}$ in z ...

But then must find z ...

→ write z in terms of $\boxed{n\lambda^3}$

Then plug back in to find $P(n\lambda^3)$

Degenerate limit

QM important when

$$\boxed{n\lambda^3 \geqslant g}$$

spin degeneracy factor

→ high density, large thermal wavelength.

FERMI GAS

$$\boxed{\mu = k_B T \ln z}$$

Fermi energy

$$\boxed{\mathcal{E}_F(n) = \lim_{T\to 0} \mu}$$

$$\mathcal{E}_F(n) = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}$$

[?] How does fermi surface gets modified at low temp?

$\Rightarrow$ Sommerfeld expansion

$$\lim_{\beta\to\infty} f^-_m(z) \simeq \underbrace{\frac{(\ln z)^m}{m!}\left(1+\frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2}+\cdots\right)}$$

In the degenerate limit ...

$$\frac{n\lambda^3}{g} = f^-_{3/2}(z) = \qquad m = 3/2$$

$\Rightarrow$ get

$$\boxed{\ln z \simeq \beta\varepsilon_F\left(1-\frac{\pi^2}{12}\left(\frac{k_BT}{\varepsilon_F}\right)^2+\cdots\right)}$$

when $T=0 \Rightarrow \ln z = \beta\varepsilon_F$ ✓

$T \simeq 0 \rightarrow$ get 1st order correction

$$-\pi^2/12\left(k_BT/\varepsilon_F\right)^2\cdot\beta\varepsilon_F$$

Similarly can get pressure at low temp

$$\boxed{P_F = \left(\frac{2}{5}\right) n \, \varepsilon_F}$$

Fermi pressure.

From here can get internal energy

$$\boxed{E = \frac{3}{2} PV = \ldots}$$

Then get HEAT CAPACITY

$$C_V = \frac{\partial E}{\partial T} = \frac{\pi^2}{2} N k_B \frac{T}{T_F}$$

linear scaling

ε_F / k_B ⊃ Fermi temp

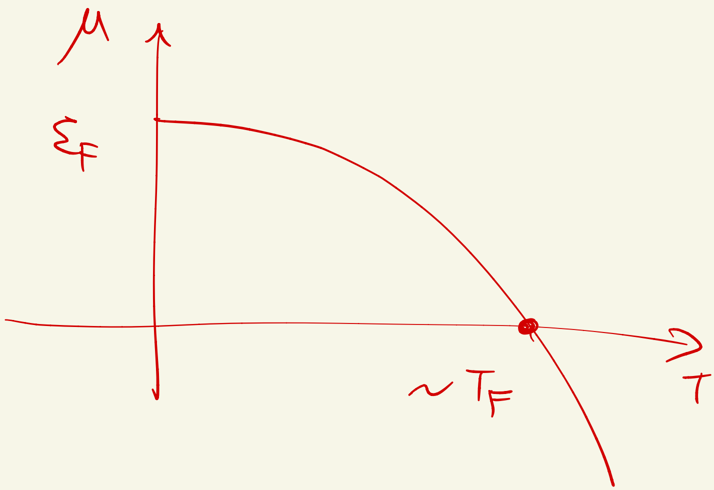
↳ linear scaling valid in all dimensions, valid also for interacting Fermi gas –

One thing to note about Fermi gas

$\boxed{\text{Chemical potential } \mu}$

μ can be positive / negative

In fact ...



Bose gas (degenerate)

Look at small T ...

$\boxed{\text{$\mu$ is always negative, with limiting value $=0$ @ $T=0$}}$

Use important identity

$$\boxed{\frac{d}{dz} f_m^{+}(z) = \frac{1}{z} f_{m-1}^{+}}$$

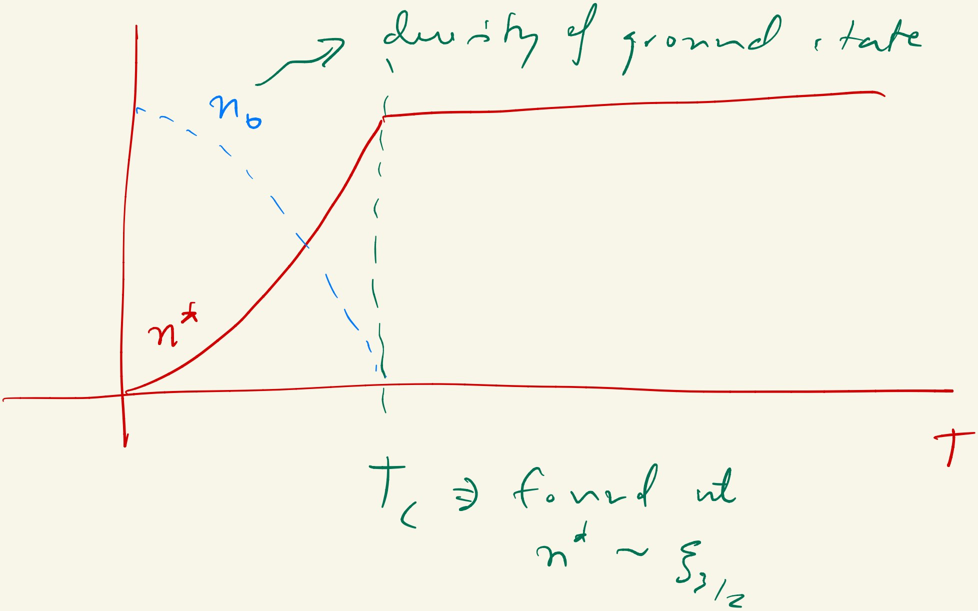
How to find condensation point T_c?

$\rightarrow$ T_c is the lowest temp at which

$$n = \text{upper bound for } n_x = \frac{g}{\lambda^3} \zeta_{3/2}$$

$$\rightarrow \quad \boxed{T_c = \frac{h^2}{2\pi m k_B} \left(\frac{n}{g\, \zeta_{3/2}} \right)^{2/3}}$$

Here's a picture to illustrate the point



$T \leq T_c \Rightarrow z = \underline{1}$ (since $\mu = 0$ now)

→ limiting density $n^* < n$

$n_0 = n - n^*$ → ground state density

→ <u>condensation</u>

BEC $\rightsquigarrow$ discontinuous (1^{st} order) & continuous (2^{nd} order) transition

$\rightarrow \left\{ \begin{array}{l} \text{finite } \underline{\text{latent heat}} \\ \underline{\text{divergent compressibility}} \end{array} \right.$

(see. Kardar 197)

Latent heat

$$\boxed{L = T_c v^* \left.\frac{dP}{dT}\right|_{\text{coexistence}}}$$

$1/n^* \quad \int \quad \propto \xi_{3/2}$

Compressibility

$$\boxed{\kappa_T = \frac{1}{n} \left.\frac{\partial n}{\partial P}\right|_T}$$

use

$$\cdot \ \frac{\partial P}{\partial z} = \frac{g k_B T}{\lambda^3} \frac{f^+_{3/2}(z)}{z}$$

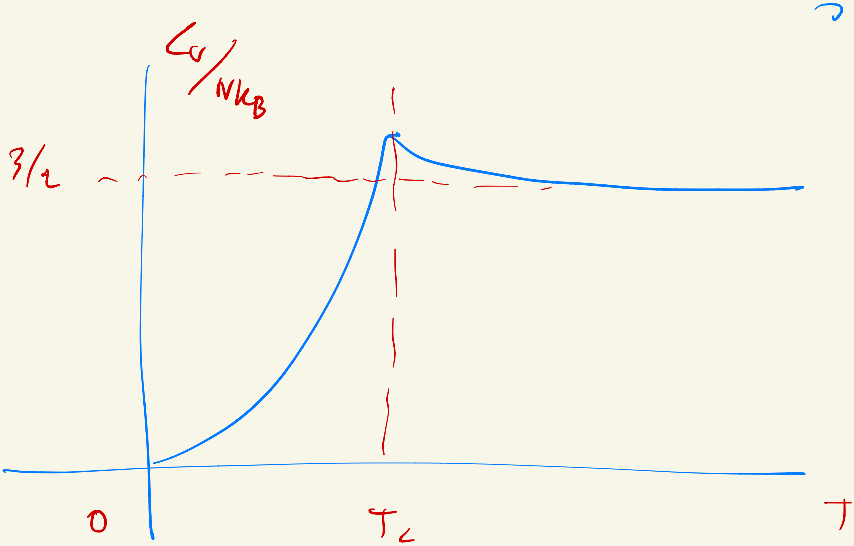
$$\cdot \ \frac{\partial n}{\partial z} = \frac{g}{\lambda^3} \frac{f^+_{1/2}(z)}{z}$$

$$= \frac{f^+_{1/2}(z)}{n k_B T f^+_{3/2}(z)}$$

$\longrightarrow$ diverges @ transition since $f^+_{1/2}(1) = \infty$

Also get heat capacity ...

↳ details in the book, but idea is



in general

$$\boxed{C_V \sim T^{d/2}}$$

high temp

$$\frac{C_V}{Nk_B} \sim \frac{3}{2}\left(1 + \frac{n\lambda^3}{2^{7/2}} + \dots\right)$$

low temp

$$\frac{C_V}{Nk_B} \sim \frac{15}{4}\,\frac{\zeta_{5/2}}{\zeta_{3/2}}\left(\frac{T}{T_c}\right)^{3/2}$$

GENERAL TACTICS

DENSITY OF STATES

This is always tricky somehow...

↳ appear when replacing sums with integrals...

Ex

$$N = \sum_n \frac{1}{z^{-1} e^{\beta \varepsilon_n} - \eta} = \int_0^\infty d\varepsilon \, \rho(\varepsilon) \frac{1}{z^{-1} e^{\beta \varepsilon} - \eta}$$

$$\rho(\varepsilon) = \frac{dN}{d\varepsilon}.$$

$$\sum_n \rightsquigarrow \int dn = \int \underbrace{\left(\frac{dn}{d\varepsilon}\right)}_{\rightsquigarrow\, \rho(\varepsilon)} d\varepsilon$$

That's it!

HIGH VS. LOW TEMP EXPANSIONS

Assume that we have N

$$N = \int d\varepsilon\, \rho(\varepsilon) \frac{1}{z^{-1} e^{\beta\varepsilon} - \eta}$$

For Fermions, remember that

So, how to find $\mu - \varepsilon_F$ at low T?

$\Rightarrow$ $\boxed{\text{Sommerfeld exp}}$

LOW TEMP $\Rightarrow$ SOMMERFELD EXPANSION

$$= \int_0^\infty dx \frac{g(x)}{e^{\beta(x-\mu)}+1} \simeq \int_0^\mu dx\, g(x) + \frac{\pi^2}{6\beta^2} g'(\mu)$$

Expand $\mu \simeq \varepsilon_F + \varepsilon$

get

$$\mu - \varepsilon_F \simeq -\frac{\pi^2}{6\beta^2} \frac{\rho'(\varepsilon_F)}{\rho(\varepsilon_F)}$$

$$N = \int_0^\infty d\varepsilon \frac{\rho(\varepsilon)}{e^{\beta\varepsilon - \beta\mu}+1}$$

$$\simeq \int_0^\mu d\varepsilon\, \rho(\varepsilon) + \frac{\pi^2}{6\beta^2} \rho'(\varepsilon_F)$$

$$= \underbrace{\int_0^{E_F} d\varepsilon\, \rho(\varepsilon)}_{} + \int_{E_F}^\mu d\varepsilon\, \rho(\varepsilon) + \frac{\pi^2}{6\beta^2} \rho'(\varepsilon_F)$$

$$N = N + (\mu - \varepsilon_F)\rho(\varepsilon_F) + \frac{\pi^2}{6\beta^2}\rho'(\varepsilon_F)$$

itself

Compressibility & Condensation

Compressibility

$$\kappa = \frac{\partial v}{\partial P} = \frac{-N}{n^2}\frac{\partial n}{\partial V} = \frac{1}{n}\frac{\partial n}{\partial P}$$

$$\boxed{\text{STABILITY} \longleftrightarrow \kappa > 0}$$

Expansions of $f_m^\eta(z)$

Tricks

$$\boxed{\frac{d}{dz} f_m^\eta(z) = \frac{1}{z} f_{m-1}^\eta(z)}$$

Say we have sth like

$$\boxed{u_c = \frac{k_B T}{n} \frac{f_{3/2}^\eta(z)}{f_{\cancel{z}/2}^\eta(z)}}$$

Want to get this in terms of $n\lambda^3$ ...

→ need to solve for z ......

now this is only valid in the

<u>non degenerate limit</u> $(n\lambda^3 \ll 1)$

$$z \sim \frac{n\lambda^3}{g} - \frac{\eta}{2^{3/2}}\left(\frac{n\lambda^3}{g}\right)^2 + \cdots$$

Then, since $z \sim n$ in the nondeg limit

→ use results from non-relativistic gas

section to find z rate

$$f_m^{\eta}(z) \simeq \sum_{\alpha=1}^{\infty} \eta^{\alpha+1} \frac{z^{\alpha}}{\alpha^m}$$

sub in $z = \frac{n \lambda^3}{g} - \dots$

→ get 1st order correction

Then get term by term, like this

✓

In the degenerate limit, however, we'll have

Sommerfeld expansion

$$n_c = \frac{k_B T}{h} \frac{f_{3/2}^{n}(z)}{f_{1/2}^{n}(z)} = \frac{k_B T}{h} \frac{\int}{} = \dots$$

$$\lim_{z \to \infty} f_m^{-}(z) = \frac{(\ln z)^m}{m!}\left[1 + \frac{\pi^2}{6} \frac{m(m-1)}{(\ln z)^2} + \dots\right]$$

Then we'll get

$$n_c = n_c(\ln z)$$

But recall that

$$z = e^{\beta\mu} \longrightarrow e^{\beta\varepsilon_F}$$

in $n\lambda^3 >> 1$

$$\rightarrow \quad n_c = n_c(\beta\varepsilon_F) \quad ✓$$

Done (For Fermions)

For ~~Bosons~~, $\mu \to 0$ as $T \to 0$, so

look at

$$f^+_{3/2}(0) \simeq 0$$

$$f^+_{1/2}(0) \simeq \infty$$

$$\boxed{n_c \to 0}$$

Tricks to get Correct Density of States

Ex

$$E = \sum_{k} \hbar\omega(\vec{k})\left(\langle n_{\vec{k}}\rangle + \frac{1}{2}\right)$$

generic harmonic oscillator

$$= E_0 + A\int \frac{d^2k}{(2\pi)^2}\,\frac{\hbar\omega(\vec{k})}{e^{\beta\hbar\omega(\vec{k})}-1}$$

x/β

Now $\omega^2 = (\sigma/\rho)k^3$

Recall

$x = \beta\hbar\omega(k)$

$$\int \frac{d^3k}{(2\pi)^2}\,\frac{\varepsilon(k)}{e^{\beta\varepsilon(k)}-1}$$

$k = (\rho/\sigma)^{1/3}\,\omega^{2/3}$

$\rightarrow dk = (\rho/\sigma)^{1/3}\,\frac{2}{3}\,d\omega\cdot\omega^{-1/3}$

Then $$\frac{d^2k}{(2\pi)^2} = \frac{2\pi k\,dk}{(2\pi)^2}$$

$$= \frac{k\,dk}{2\pi}$$

$$= \frac{1}{2\pi}\left(\frac{\rho}{\sigma}\right)^{1/3}\omega^{2/3}\left(\frac{\rho}{\sigma}\right)^{1/3}\omega^{-1/3}\,\frac{2}{3}\,d\omega$$

$$= \frac{1}{2\pi}\left(\frac{\rho}{\sigma}\right)^{2/3}\frac{2}{3}\,\omega^{1/3}\,d\omega$$

Finally ... $\omega = \frac{x}{\beta\hbar}$

$$= \underbrace{\frac{1}{2\pi}\left(\frac{\rho}{\sigma}\right)^{2/3}\frac{2}{3}\left(\frac{x}{\beta\hbar}\right)^{1/3}}\,dx$$

$x^{4/3}$

Inter integral

$$= \int_0^\infty \frac{1}{2\pi}\left(\frac{\rho}{\sigma}\right)^{2/3}\frac{2}{3}\left(\frac{x}{\beta\hbar}\right)^{1/3}\left(\frac{x}{\beta}\right)\frac{1}{e^x-1}\,dx$$

$$= \text{stuff} \ldots$$

Have

$$\int_0^\infty \frac{x^{4/3}}{e^x - 1}\,dx$$

<u>Recall</u>

$$f_m^\eta(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - \eta}\,dx$$

Here $z = 1$; $m = 7/3$

$$\int_0^\infty \frac{x^{4/3}}{e^x - 1}\,dx = \left(\tfrac{7}{3} - 1\right)!\; f^{+}_{7/3}(1)$$

$$= \left(4/3\right)!\; \zeta_{7/3}$$

<u>Riemann zeta function</u>

Then done.

Finding chemical potential?

Suppose want to find $\mu = \mu(T, P)$.

what do you do?

$\rightarrow$ Use $\boxed{\beta P = \frac{\eta}{\lambda^3} f^{\eta}_{5/2}(z)}$

For non-degenerate, expand

$$f^{\eta}_{m}(z) = \sum_{\alpha=1}^{\infty} \eta^{\alpha+1} \frac{z^{\alpha}}{\alpha^{m}}$$

$$\sim z$$

$\Rightarrow$ Set $\beta P \simeq \frac{1}{\lambda^3} z = \frac{1}{\lambda^3} \ln(\beta\mu)$

$\Rightarrow$ get μ ✓

$$\mu = \mu(T, P)$$

How to calculate density n?

$$n = \frac{N}{A} = \frac{1}{A} \int d\varepsilon \, \rho(\varepsilon) \, p(\varepsilon) \quad \text{... Done}$$

Avg particle number

can be area / volume / length ...

get $f_m^{\eta}(z)$ ...

Then possibly look at expansions later.

Find critical values @ condensation...

To do this ... the $f_m^{\eta}(z)$ term must have unity argument ... (usually)

so $z = 1$ or $\underbrace{e^{\beta\mu} e^{\beta \text{ other stuff}} = 1}$

$\Downarrow$

solve for other stuff

$3/2$

More tricks

Abel-Plana Formula (Pset 5)

can be used to show that

$$\boxed{\sum_{\ell=0}^{\infty} (2\ell+1) e^{-u\ell(\ell+1)} = \frac{1}{u} + \frac{1}{3} + \frac{u}{15} + \mathcal{O}(u^2)}$$

related to partition fn of QM rotor...

Recall that $\mathcal{H} = L^2/2I$

$\hookrightarrow$ in eigvs: $= \dfrac{\hbar^2 \ell(\ell+1)}{2I}$

$$\rightarrow \boxed{\mathcal{Z} = \sum_{\ell=0}^{\infty} (2\ell+1) \exp\left\{ -\frac{\beta\hbar^2}{2I}\ell(\ell+1) \right\}}$$

with degeneracy $(2\ell+1)$

$\rightarrow$ from here, everything else follows...

Boson joint probability

$$\boxed{P(\{n_{\vec q}\}) = \prod_{\vec q} (1 - z_{\vec q})(z_{\vec q})^{n_{\vec q}}}$$

$$\Downarrow$$

$$\langle e^{ikn_{\vec q}} \rangle = \sum_{n_{\vec q}=0}^{\infty} e^{ikn} P(n)$$

Fixed $\vec q$

$$= \sum_{n=0}^{\infty} \cancel{\prod_{\vec q}} (1 - z_{\vec q}) \left[z_{\vec q} e^{ik} \right]^n$$

$$= \sum_{n=0}^{\infty} \left(1 - e^{\beta(\mu-\varepsilon)}\right)\left(e^{\beta(\mu-\varepsilon)+ik}\right)^n$$

(geom series)

$$\boxed{\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}}$$

$$= \frac{1 - e^{\beta(\mu-\varepsilon)}}{1 - e^{\beta(\mu-\varepsilon)+ik}}$$

From here get <u>mean</u> + <u>variance</u>

$$\langle e^{ikn} \rangle = \frac{1 - e^{\beta(\mu-\epsilon)}}{1 - e^{\beta(\mu-\epsilon+ik)}}$$

$$= \frac{1 - z_q^{\rightharpoonup}}{1 - z_q^{\rightharpoonup} e^{ik}}$$

$\downarrow$

look at cumulants ...

$\rightarrow$ series expand for

$\log \langle e^{ikn} \rangle$ around 0 in powers of k

$$\sim ik \underbrace{\boxed{\frac{z_q}{1-z_q}}}_{\substack{\text{mean} \\ \langle n_q^0 \rangle_c}} - \frac{k^2}{2} \underbrace{\boxed{\frac{z}{(1-z)^2}}}_{\substack{\text{variance} \\ \langle \Delta n_q^2 \rangle_c}} + \dots$$

General Techniques For Finding Mean, Variance...

Let X be a r.v. following $p(x)$

- Then want to find $\langle x \rangle$, $\langle x \rangle_c$, etc

Then can use the cumulant expansion trick → which comes from MGF ---

- First find Characteristic function---

$$\langle e^{ikx} \rangle = \int p(x) e^{ikx} dx$$

$$= FT[p(x)](k)$$

- From here, take log → then expand in powers of k ...

$$\boxed{\log \langle e^{ikx} \rangle}$$

$$\text{Log}\langle e^{ikx}\rangle = \sum_{n=1}^{\infty} (K_n) \frac{(it)^n}{n!}$$

This is the n^{th} cumulant that we want

<u>Cumulants</u>

$$\begin{cases} K_1 = \langle x \rangle & \text{(mean)} \\ K_2 = \langle x^2 \rangle_c = \langle x^2 \rangle - \langle x \rangle^2 & \text{(var)} \\ K_3 = \ldots & \text{(skew)} \end{cases}$$

<u>Idea</u>: Defn

$$\boxed{\text{The Cumulant GF is the log of the MFG}}$$

Density matrix

Can get pretty tricky...

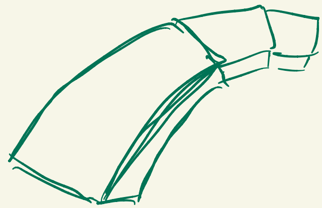
So just use book definition...

Remember that there are $\boxed{3}$ types of QM ensembles

- Microcanonical (system isolated)
- Canonical (contact w/ heat reservoir)
- Grand canonical (contact – heat + particle reservoir)

Microcanonical

$$\rho(E) = \frac{\delta(\mathcal{H} - E)}{\Omega(E)}$$



$$\langle n|\rho(E)|m\rangle = \frac{1}{\Omega(E)} \begin{cases} 1 & \text{if } E_n = E, \ m=n \\ 0 & \end{cases}$$

in this case (Final 2013)

$$\rho(\vec{p}) = \frac{1}{\Omega(E)} \begin{cases} 1 & \text{if } E < \frac{\vec{p}^2}{2m} < \delta E \\ 0 & \text{else} \end{cases}$$

$$\Omega(E) = \frac{V}{h^3}(4\pi)p^2\delta p$$

$\uparrow$

phasespace vol

$$p = \sqrt{2mE}$$

$$\delta p = \sqrt{\frac{m}{2E}}\,\delta E$$

$$\rightarrow \Omega(E) = \frac{4\pi V}{h^3}\sqrt{2mE}\,m\,\delta E$$

To get corresponding density matrix in $\vec{x}$ basis ...

use $$\boxed{\langle \vec{p} | \vec{x} \rangle = \frac{1}{\sqrt{V}} e^{-i\vec{p}\cdot\vec{x}/\hbar}}$$

$$\langle \vec{x} | \rho | \vec{x} \rangle = \sum \langle x | \vec{p} \rangle \rho \langle \vec{p} | \vec{x} \rangle$$

$$= \ldots$$

(see solution for technical details)

Mechanical Stability

- Sometimes, for BEC, use compressibility, see where it diverges ...

- more classically ... need to find (n_c, T_c) for ideal gas beyond which mechanical stability fails

...

$\Rightarrow$ usually look at

$$\boxed{\begin{aligned} \partial P / \partial n &= 0 \\ \partial^2 P / \partial n^2 &= 0 \end{aligned}}$$

$\rightarrow$ solve simultaneously to get (n_c, T_c)

$\rightarrow$ Pressure isotherms

Pairing of Fermions to Bosons

$$e^- + e^- \longrightarrow \text{boson} \quad (-\varepsilon)$$

Then

if $n_e = \ldots$

Then to get bosons...

$m_b = 2m_e$

get extra $-\varepsilon$ in $\varepsilon(k)$

$$\rightarrow \boxed{\xi_b(k) = -\zeta + \frac{\hbar^2 k^2}{2m_b}}$$

$2m_e$

Also $z \longrightarrow z' = e^{2\beta\mu} = z^2 \ldots$

$\rightarrow$ get $f^+_{3/2}(z^2 y)$ $\boxed{y = e^{-\beta\varepsilon}}$

from pairing

$\boxed{\text{BEC in } d \text{ dimensions}}$

(see solution)

→ full text there ...

Possible question in exam in

BEC in 2d or 1d

→ can transition occur etc

Virial Coeffs

↳ all done before,
use textbook to work
this out ...

Fermi Energy, Temperature

↳ for $\varepsilon(\vec{k}) = \frac{\hbar^2 k^2}{2m}$

→ formula given in textbook

and

$$N = g \int_0^{\varepsilon < \varepsilon_F} \underbrace{\langle n_\varepsilon \rangle}_{\text{more like } \rho(\varepsilon)} \, d\varepsilon$$

probability and stuff (see text)

Saddle point approx

See webpage (www.nbi.dk)

↓

all info there

Standard N-particle (id)

Z ..

$$\boxed{Z_N = \frac{1}{N!} Z^N}$$

$$Z = \left(\frac{V}{\lambda^3}\right)$$

for ideal gas in 3d