MA355 Final

Colby College — Spring 2021

due Tuesday, May 18, by 8:30 EST

Please upload your solutions (preferably as a single PDF file) to Moodle.

Explain/justify all answers!

Problems

1. (25 points)
   (a) (5 points) How many partitions of the integer 9 have all their parts of size 2 or 3?
   (b) (20 points)
   How many partitions of the set $[9]=\{1, 2, \ldots, 9\}$ have all their blocks of size 2 or 3? You may leave your answer expressed in terms of functions discussed in class (such as binomial coefficients, factorials, Stirling numbers, etc.). You don't need to give a numerical answer.

2. (25 points) The Fibonacci numbers are defined by $F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ if $n \geq 3$. Show how to express the following numbers in terms of the Fibonacci numbers.
   (a) (10 points) The number of subsets S of the set $[n] = \{1, 2, \ldots, n\}$ such that S contains no two consecutive integers.
   (b) (15 points) The number of compositions of n into parts of size greater than 1.

3. (25 points) Recall that $S(k, n)$ is the Stirling number of the second kind. In class we showed, among other things, that $S(k, 1) = 1$ and $S(k, k-1) = \binom{k}{2}$.
   (a) (5 points) Find a formula for $S(k, k-2)$.
   (b) (10 points) Find formulas for $S(k, 2)$ and $S(k, 3)$.
   (c) (10 points) Find a recurrence relation for the Stirling numbers that gives a formula for $S(k, n)$ in terms of $S(k', n-1)$ for $0 \leq k' < k$.

4. (25 points) Find the number of lattice paths from $(0, 0)$ to $(20, 30)$ that pass through $(8, 15)$ but that do not pass through $(14, 23)$. Express your answer in terms of familiar combinatorial numbers covered in the course; you don't have to give a final numerical answer.