Homework 5 8.370/18.435/2.111 Due 10/26

1. Recall that to do teleportation, as described in class, Alice and Bob start with the state $\frac{1}{\sqrt{2}}(|00\rangle_{AB}+|11\rangle_{AB})$, and Alice measured the bit she wished to teleport and her half of the EPR pair in the Bell basis. She then tells the measurement result to Bob, and he applies a Pauli matrix according to the following table:

Alice's measurement result	Bob's unitary transform
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	id
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	σ_x
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	σ_z
$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	σ_y

Suppose Alice and Bob start with the Bell state $\frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$. What table should they use?

Solution: Following the argument in lecture notes 11, we know that

$$\frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB}) = -i(I \otimes \sigma_Y)\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}).$$

Since σ_Y is acting on Bob's qubit, suppose Alice gets $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ as her measurement, then we have

$$\begin{aligned}
&\frac{1}{2}(\langle 00|_{AA} + \langle 11|_{AA})(\alpha\,|0\rangle + \beta\,|1\rangle)_A(|01\rangle_{AB} - |10\rangle_{AB}) \\
&= \frac{-i}{2}\sigma_Y(\langle 00|_{AA} + \langle 11|_{AA})(\alpha\,|0\rangle + \beta\,|1\rangle)_A(|00\rangle_{AB} + |11\rangle_{AB})] \\
&= (-i)\sigma_Y(\frac{\alpha}{2}\,|0\rangle + \frac{\beta}{2}\,|1\rangle).
\end{aligned}$$

In other words, we see that if Alice measures $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, then she should apply σ_Y (the global phase of $-i$ can be ignored). The full table then looks like:

Alice's measurement result	Bob's unitary transform
$\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$	σ_y
$\frac{1}{\sqrt{2}}(\|01\rangle + \|10\rangle)$	σ_z
$\frac{1}{\sqrt{2}}(\|00\rangle - \|11\rangle)$	σ_x
$\frac{1}{\sqrt{2}}(\|01\rangle - \|10\rangle)$	id

2. The Deutsch-Jozsa algorithm distinguishes between functions which are balanced and functions which are constant. Suppose it is handed a function which is neither balanced nor constant; that is, suppose the function has r values for which $f(x) = 0$ and s values for which $f(x) = 1$, where $r + s = 2^n$. What is the probability that the Deutsch-Jozsa algorithm will output that the function is constant?

Solution: We follow the analysis in lecture notes. Let O_f be the phase oracle for the function f, then from lecture we have

$$H^{\otimes n} O_f H^{\otimes n}\,|0\rangle = \frac{1}{2^n}\sum_{j=0}^{2^n-1}\sum_{k=0}^{2^n-1}(-1)^{f(j)}(-1)^{j\cdot k}\,|k\rangle\,.$$

The amplitude of $|0^n\rangle$ is

$$\frac{1}{2^n}\sum_{j=0}^{2^n-1}(-1)^{f(j)} = (r-s)/2^n.$$

So the probability of measuring $|0\rangle$ and for the algorithm to report a constant function is $(r-s)^2/4^n$.

3. (20 points)

Consider the following four states in the two-qubit system where one qubit is held by Alice and the other by Bob:

$$|0\rangle_A \otimes |0\rangle_B\,,\ \ |0\rangle_A \otimes |1\rangle_B\,,\ \ |1\rangle_A \otimes |+\rangle_B\,,\ \ |1\rangle_A \otimes |-\rangle_B\,,$$

(a) Explain why these four states form an orthonormal basis.

Solution: The first two states are orthogonal to the last two, because the first component of the tensor products are orthogonal: $|0\rangle_A$ is orthogonal to $|1\rangle_A$. The first and second states are orthogonal because $|0\rangle_B$ is orthogonal to $|1\rangle_B$. The third and fourth states are orthogonal because $|+\rangle_B$ is orthogonal to $|-\rangle_B$.

(b) Suppose Alice and Bob are in different laboratories connected only by a telephone line. Show that they can identify these four states unambiguously.

Solution: First, Alice measures her state in the $\{|0\rangle\,,|1\rangle\}$ basis. She telephones Bob and tells him the result. If Alice's qubit was $|0\rangle$, Bob measures in the $\{|0\rangle\,,|1\rangle\}$ basis. If her qubit was $|1\rangle$, he measures in the $\{|+\rangle\,,|-\rangle\}$ basis. This procedure distinguishes all four states.

(c) Now, suppose Alice and Bob are forced to make simultaneous measurements. That is, they agree beforehand on some measurement strategy. They go to their labs and make measurements (during this time they cannot communicate with each other). They then pick up the telephone and try to decide which of the four original states they had.

They would like to find a way to identify these four states unambiguously under this restriction. Unfortunately, it is the case that they cannot. Give as good an explanation as you can for why they cannot.

Solution: Since the two states are unentangled, when Alice and Bob measures together, Bob has to decide precisely whether the state is in $|+\rangle\,,|-\rangle$ basis or in $|0\rangle\,,|1\rangle$ basis with one measurement. This is impossible, as we know that Bob cannot determine with certainty whether a state is in $|0\rangle$ or in $|+\rangle$.

(d) Now, suppose Alice and Bob share an EPR pair in the state $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$. They again are required to measure their states simultaneously (possibly involving the EPR pair in their measurements) and then communicate afterwards to determine their states. Can they identify their four states unambiguously now?

Hint: it might help for Alice or Bob to apply CNOT gates to their qubits.

Solution: Let us denote Alice's qubit as A, Bob's qubit as B, and their shared EPR pair as A_eB_e. Consider the following strategy. Suppose Bob applies a CNOT gate with control on B and target on B_e. The states they have in the four cases are:

Initial State	After CNOT
$\|0\rangle_A \otimes \|0\rangle_B$	$\|0\rangle_A \otimes \frac{1}{\sqrt{2}}(\|000\rangle + \|110\rangle)_{A_eB_eB}$
$\|0\rangle_A \otimes \|1\rangle_B$	$\|0\rangle_A \otimes \frac{1}{\sqrt{2}}(\|011\rangle + \|101\rangle)_{A_eB_eB}$
$\|1\rangle_A \otimes \|+\rangle_B$	$\|1\rangle_A \otimes \frac{1}{\sqrt{2}}(\|0++\rangle + \|0--\rangle + \|1++\rangle - \|1--\rangle)_{A_eB_eB}$
$\|1\rangle_A \otimes \|-\rangle_B$	$\|1\rangle_A \otimes \frac{1}{\sqrt{2}}(\|0-+\rangle + \|0+-\rangle + \|1+-\rangle - \|1-+\rangle)_{A_eB_eB}$

Suppose now that Alice measures A in $|0\rangle, |1\rangle$ basis. If she gets $|0\rangle$, she can measure A_e in $|0\rangle, |1\rangle$ basis. If Bob also measures B_e in $|0\rangle, |1\rangle$ basis, we see that if $B = |0\rangle$ initially, then Alice's measurement of A_e and Bob's measurement of B_e must agree. If $B = |1\rangle$ initially, then their measurements must disagree.

If Alice measures $A = |1\rangle$, she can measure A_e in the $|+\rangle, |-\rangle$ basis. If Bob measures B in $|+\rangle, |-\rangle$ basis, we see that if $B = |+\rangle$ initially, then Alice's measurement of A_e and Bob's measurement of B must agree. If $B = |-\rangle$ initially, then their measurements must disagree.

To sum up, after the CNOT gate, Bob should measure B in $|+\rangle, |-\rangle$ basis and B_e in $|0\rangle, |1\rangle$ basis. Alice should measure A in $|0\rangle, |1\rangle$ basis, and if she gets $|0\rangle$, measures A_e in $|0\rangle, |1\rangle$ basis; if she gets $|1\rangle$, measures A_e in $|+\rangle, |-\rangle$ basis. They can then share their measurement results over classical information and determine the initial state with certainty.

I should provide some motivation to this strategy. The idea here is that when Alice measures A in the $|0\rangle, |1\rangle$ basis, she would know which basis is Bob's qubit in. However, they cannot communicate during the measurements. Therefore, before Bob measures his qubit B in one of the two bases, he should try to save some information about B in the EPR pair. This is done by the CNOT gate, which entangled B with the EPR pair. Alice and Bob can then try to measure their EPR pairs in various bases to try to recover this information.

It is worth noting that this protocol would not work if, say, we don't have an EPR pair, but instead Bob just have another qubit at state $|0\rangle$, and Bob tries to CNOT B with this "ancilla" qubit. In the case where $B = |+\rangle$, this will create an EPR state. If Bob later measures B in $|+\rangle, |-\rangle$ basis, he would only get $|+\rangle$ with probability $1/2$. In other words, the information about whether his qubit is in $|+\rangle$ state or not is lost.

4. Suppose Alice, Bob, and Charlie hold the GHZ state

$$\frac{1}{\sqrt{2}}\big(|000\rangle_{ABC} + |111\rangle_{ABC} \big).$$

(a) Show that if they are not allowed to communicate, there is no way for them to arrange for Alice and Charlie to end up sharing the EPR state

$$\frac{1}{\sqrt{2}}\big(|00\rangle_{AC} + |11\rangle_{AC} \big).$$

Solution: If you trace out Bob, Alice and Charlie have the density matrix

$$\frac{1}{2}\big(|00\rangle\langle 00| + |11\rangle\langle 11| \big).$$

This is a separable state, because it is a mixture of the two tensor product states $|00\rangle$ and $|11\rangle$. Alice and Bob cannot generate entanglement from this state. (They can make this state without using any entanglement—Alice flips a coin, and if it's heads, she and Chalie both prepare $|0\rangle$, and if it's tails, they both prepare $|1\rangle$. So if they could generate entanglement from this state, they could generate entanglement without any quantum communication.)

(b) Show if they are allowed to communicate classical information, but not quantum states, Alice and Charlie can end up with the EPR state

$$\frac{1}{\sqrt{2}}\big(|00\rangle_{AC} + |11\rangle_{AC} \big).$$

Solution: Bob measures his qubit in the $\{|+\rangle, |-\rangle\}$ basis. I'll skip the computations, but what happens is that with probability $\frac{1}{2}$, Alice, Bob, and Charlie hold the states

$$\frac{1}{\sqrt{2}}\big(|00\rangle_{AC} + |11\rangle_{AC}\big)|+\rangle_B$$

and with probability $\frac{1}{2}$ they hold the states

$$\frac{1}{\sqrt{2}}\big(|00\rangle_{AC} - |11\rangle_{AC}\big)|-\rangle_B$$

If Bob's result is $|+\rangle$, he tells Alice and Charlie to do nothing. If his result is $|-\rangle$, then one of Alice and Charlie needs to apply σ_z to their qubit. This will give the desired state.