QUESTIONS #1

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Huan Bui

Hi Evan, I tried to evaluate some (seemingly doable) integrals for the d-dimensional problem after our discussion and came up with some questions.

1. When we write the integral
$$\int_{\mathbb{R}^d} e^{-iP(\xi)-ix\cdot\xi}\,d\xi,$$
how is the measure $d\xi$ defined? When, say, $d=2$, is $d\xi$ just equal to $dxdy$ in the euclidean basis?

2. Under some change of basis, say
$$\xi \to t^E\eta,$$
does this measure change as
$$d\xi \to t^{\operatorname{tr} E} d\eta \sim t^{\operatorname{tr} E}\, dt d\Omega(\eta)$$
as I would expect?

3. I looked at your "Some thoughts/questions in algebraic geometry" and found an example of a nondegenerate homogeneous $P(\xi):\mathbb{R}^2\to\mathbb{C}$ given by
$$P(\xi) = -i\xi_1 + \xi_2^2$$
with respect to the euclidean basis with $\operatorname{diag}(1,1/2)\in \operatorname{Exp}(P)$. Assuming $d\xi = d\xi_1 d\xi_2$ in the euclidean basis and letting $x=0$, I attempted to evaluate
$$\int_{\mathbb{R}^d} e^{-iP(\xi)}\,d\xi = \int_{\mathbb{R}^d} e^{-i(i\xi_1+\xi_2^2)}\,d\xi_1 d\xi_2.$$
I recognized that we don't need to worry about "integrating out the angular element" since $P(\xi)$ doesn't have any cross terms. So I rewrote this as
$$\int_{\mathbb{R}} e^{\xi_1}\,d\xi_1 \underbrace{\int_{\mathbb{R}} e^{-i\xi_2^2}\,d\xi_2}_{(1-i)\sqrt{\pi/2}}.$$
The first integral doesn't converge, even as an improper Riemann integral. Does this mean there must be other restrictions other than $P(\xi)$ being nondegenerate homogeneous for $H_P^1(\xi_1,\xi_2)$ to exist?

4. I guess this is not a question. I tried to evaluate the following integral after the change of variables from $\xi \to t^E \eta$, where η is such that $P(\eta) = 1$:

$$\int_0^\infty e^{-iP(\eta)t} t^{\operatorname{tr} E}\, dt.$$

where I'm leaving out the "angular integration Ω" for now. It turned out that

$$\int_0^\infty e^{-it} t^{\operatorname{tr} E}\, dt = -ie^{\left(-\frac{1}{2} i\pi \operatorname{tr} E\right)} \Gamma(\operatorname{tr} E + 1)$$

so long as $\operatorname{tr} E \neq 0$ and $-1 < \operatorname{tr} E$. This all assumes that $x = 0$ in

$$\int_{\mathbb{R}^d} e^{-iP(\xi) - ix\cdot\xi}\, d\xi = \int_{\mathbb{R}^d} e^{-it - ix\cdot t^E \eta} t^{\operatorname{tr} E}\, dt d\Omega(\eta).$$

I'm running some test cases with $x \neq 0$, assuming $x \cdot t^E \eta$ to be just some linear combination of the powers of t. Most of them seem to converge. But of course I'll have to worry about integrating over all η as well.