Review session

Time Evolution

Energy-time uncertainty

$\langle H\rangle = \sum_n |\langle \psi|\psi_n\rangle|^2 E_n$

Expect $\langle H\rangle$ to be linear over t

If $\langle H\rangle$ is extremum at $\langle H\rangle(t)=\langle H\rangle_0$ $\forall t$

then what is $\langle H\rangle$ is superposition of energy eigenstates

No time operator

because value will vary proportionally as there plus a $\sin\omega t$ and $\cos\omega t$

$\Delta t = \frac{\langle H\rangle_0}{\Delta\langle H\rangle}$

Popular interpretation

$\langle \Delta H^2\rangle \langle \Delta t^2\rangle \geq \hbar^2 \left(\frac{\langle H\rangle_0}{\Delta\langle H\rangle}\right)^2 = \frac{\hbar^2}{4}\left(\frac{\langle H\rangle_0}{\Delta\langle H\rangle}\right)^2$

$\Rightarrow \frac{\langle \Delta H\rangle}{\langle H\rangle_0}\langle \Delta t\rangle \geq \frac{\hbar}{2}$

"Wrong"

$\Rightarrow \Delta t \Delta E \geq \hbar/2$

physically makes sense. In small window of changes phases are close to state taken a less time to change!

Probability Flux

$P(x,t) = |\psi(x,t)|^2 = \sum_{n,m} c_n^* c_m e^{\frac{i}{\hbar}(E_n - E_m)t} \psi_n^*(x)\psi_m(x)$

can then by SE $\left[\frac{\partial P}{\partial t} = -\nabla\cdot\vec{J}\right](x,t)$ w/

$\vec{J}(x,t) = \frac{\hbar}{2m}\operatorname{Im}(\psi^*\nabla\psi)$

probability flux

With $\psi(x,t) = \sqrt{P(x,t)}\, e^{\frac{i}{\hbar}S(x,t)}$

$\vec{J}(x,t) = \frac{1}{m} P(x,t)\nabla S(x,t)$

Change of phase controls prob. flux

Ehrenfest thm

Ex: consider $H = \frac{p^2}{2m} + V(\vec{x})$

$\Rightarrow \frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\langle [x,H]\rangle = \frac{\langle p\rangle}{m}$ (Heisenberg)

$\Rightarrow \langle c\frac{dp}{dt}\rangle = \frac{i}{\hbar}\langle c\, H\rangle$

$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle -\nabla V(x)\rangle$

$\Rightarrow \left(m\frac{d^2\langle x\rangle}{dt^2} = \frac{d\langle p\rangle}{dt} = -\langle\nabla V(x)\rangle\right)$

classical EOM!

Then can derive classical system from quantum starting point!

Ehrenfest!

WKB Approx

Plug $\psi = \sqrt{P}\, e^{\frac{i}{\hbar}S}$ in SE (1-d case)

$O(\hbar)$: $-\frac{\partial S}{\partial t} = V + \frac{1}{2m}|\nabla S|^2$

Stationary state: $\Rightarrow \frac{1}{2m}|\nabla S|^2 = E - V$

$2d \Rightarrow S(x,t) = \int_{x_0}^{x} \sqrt{2m(E-V)}\, dx' + \frac{i}{\hbar}\int_0^t dt$

with $p = \sqrt{2m(E-V)}$

$O(\hbar)$: $\frac{\partial P}{\partial t} = \frac{1}{m}\frac{\partial}{\partial x}\left(P\frac{\partial S}{\partial x}\right) = 0$

$\Rightarrow P \propto \frac{1}{\sqrt{p}}$

Approx valid when

$\hbar\frac{\partial P}{\partial x} \ll \left(\frac{\partial S}{\partial x}\right)^2 \Leftrightarrow \left|\hbar\left(\frac{\partial p}{\partial x}\right)\right| \ll 1$

i.e. $\lambda = \frac{h}{p} \ll \frac{2\pi|V'|}{p^2}$

Classical limit when V constant

Clearly breaks down when $x \approx x_0$ $E = V(x)$

classical turning pt!

WKB Quantization

Unitarity of ψ in allowed region

$\Rightarrow \int_{x_1}^{x_2} \frac{dx}{\sqrt{2m(E-V(x))}}\, dx' = \left(n+\frac{1}{2}\right)\pi\hbar$

$[x_1, x_2]$ classical turning pts

Reduces to $n\pi\hbar$ case when $V=0$

Correlation Fcns

$\langle \psi(x_1)\psi(x_2)\rangle = \underbrace{\langle \psi(x_1)\psi(x_2)\rangle}_{\text{Heisenberg picture}}$

path integral: $\langle \psi(x_1)\psi(x_2)\rangle = \int \mathcal{D}[\psi(x)] \underbrace{\psi(x_1)\psi(x_2)}_{\text{integrate over all paths!}} e^{\frac{i}{\hbar}S[\psi]}$

Propagator

$K(x_1,t_1;x_2,t_2) \equiv \langle x_1|U(t_1,t_2)|x_2\rangle$

ψ_{final} is a Green's function for SE

Unitary SE: $\lim_{t_1\to t_2} K(x_1,t_1;x_2,t_2) = \delta(x_1-x_2)$

Path integral expression

$K(x^f,t_f;x^i,t_i) = \int_{x^i(t_i)=x^i}^{x^f(t_f)=x^f} \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}S[x]}$

Because classical path is $S=0$!

Particles in EM fields

Start w potential by constant

Lets see effect on measurable quantities

$|\psi(x)|^2 \to \sum_n |\psi_n(x)|^2$

$\Delta E = \text{phase}$

Recall EM described by gauge field A_μ. Fundamental: $\int_{\text{loop}} \oint \nabla\cdot\vec{A}$

$A_\mu = (-\phi, \vec{A})$

$F_{0i} = -E_i \Rightarrow E_i = -\frac{\partial\phi}{\partial x^i} - \frac{\partial A^i}{\partial t}$

$F_{ij} = \epsilon_{ijk} B^k \Rightarrow B^i = \epsilon^{ijk}\partial_j A_k$ $(\nabla\times\vec{A})$

$F_{\mu\nu}$ is invariant by $A_\mu \to A_\mu + \partial_\mu\Lambda$

Gauge invariance

$\frac{1}{\hbar}\int_{x_1}^{x_2} \frac{\partial S}{\partial x} dx = \int_{x_1}^{x_2} A_x(x')\, dx' = e\phi$

at canonical momentum: $P_x = \frac{\partial S}{\partial x} = m\dot{x} + \frac{e}{c}A^x$

$\Rightarrow \left(H = \frac{p^2}{2m} - \frac{e}{c}\frac{\dot{x}}{2m}(p - \frac{e}{c}A^x) + e\phi\right)$

Take SE w/ $A' = A + \nabla\Lambda$, $\phi' = \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t}$

Find $\psi'(x) = e^{\frac{ie}{\hbar c}\Lambda}\psi(x,t)$

solves the new SE

So gauge xform is

$\vec{\phi} \to \vec{\phi} - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

$\vec{A} \to \vec{A} + \nabla\Lambda$

$\psi \to e^{\frac{ie}{\hbar c}\Lambda}\psi$

No physical observables change, but e.g. the canonical momentum $\vec{P} = m\dot{\vec{x}} + \frac{e}{c}\vec{A}$ is gauge-dependent

Symmetries in QM

Rotational symm. grp? $\Rightarrow SO(3)$

3×3 matrix SE. $R^T R = 1$, $\det R = 1$

(preserve lengths + orientation)

But recall neutron interferometer

Shared $\vec{x}$ pt. not always trivial

$\xrightarrow{\text{Rot}} \begin{pmatrix}|\psi_1\rangle + \frac{|\psi_2\rangle}{\sqrt{2}} \\ |\psi_2\rangle - \frac{|\psi_1\rangle}{\sqrt{2}}\end{pmatrix} = -|\psi_1(x,0)\rangle$

$|\psi(x,t)\rangle = \begin{pmatrix}|\psi_1(x,t)\rangle \\ |\psi_2(x,t)\rangle\end{pmatrix} = -|\psi(x,0)\rangle$

Lie Algebra's Reps

Rep: $D: g \to B(V)$

$D(g)\cdot H\cdot D(g)^{-1} = H$, $D(g)^{-1} = D(g)^\dagger D(g)$

Lie algebra

Consider rot. $D(\vec{\theta}) = \text{rotation}$ about $\vec{\theta}$

$\vec{\theta} = \vec{\theta}\,\hat{\vec{\theta}}$

$A^T M = \mathbb{1} \Rightarrow A + A^T = 0$, $\det A$ is antisymm.

General Lie algebra properties:

i) $[\sum_i X_i, \sum_j Y_j] = \sum_{ij} [X_i, Y_j]$

ii) C_i, C_j commute in A, B

iii) $[A, \sum_i C_i] = \sum_i [A, C_i]$

iv) Jacobi ...

For $SO(3)$ and $SU(2)$:

$[\sum_i X_i, \sum_j Y_j] = \sum_{ij} [X_i, Y_j]$

Reps of SU(2)

Def: $J^2 = J_x^2 + J_y^2 + J_z^2$

$J_\pm = J_x \pm i J_y$

Find $[J^2, J_z] = 0$

$[J_z, J_\pm] = \pm\hbar J_\pm$

$[J_+, J_-] = 2\hbar J_z$

Simultaneously diagonalize J^2, J_z

$J^2|m,\lambda\rangle = \lambda|m,\lambda\rangle$, $J_z|m,\lambda\rangle = \hbar m|m,\lambda\rangle$

Find $J_z(J_+|m,\lambda\rangle) = c|m+1,\lambda\rangle$

Finite rep?

need bound $\Rightarrow \exists j$ where $J_+ = J_-$

$\Rightarrow$ Reps labeled by $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ (spin-j)

i.e. $J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle$, $J_z|j,m\rangle = \hbar m|j,m\rangle$

$J_\pm|j,m\rangle = \sqrt{j(j+1) - m(m\pm 1)}\,\hbar\,|j,m\pm 1\rangle$

Addition of Angular momentum

Consider 2 part. system $|j_1\rangle \otimes |j_2\rangle$

total ang. mom.

$\vec{J}_1 = \vec{J}_1 \otimes \mathbb{1}_2$

$\vec{J}_2 = \mathbb{1}_1 \otimes \vec{J}_2$

Basis states: $|j_1, m_1\rangle \otimes |j_2, m_2\rangle$

are not eigenstates of J^2

want to diagonalize J_1^2, J_2

Ex: $j_1 = j_2 = 1$, max momentum

$|1,1\rangle \to m=1$

$|1,1\rangle_1 |1,1\rangle_2 \to m=2$

$|-1\rangle \to m=-1$

Find $|1,1\rangle = |1,1\rangle_1 |1,1\rangle_2 \Rightarrow |2,2\rangle$

$J_-|1,1\rangle = \sqrt{(1+1)-(1-1)}\,\hbar\,|1,0\rangle$

$J_-|1,1\rangle = \sqrt{(1+1)-(1-1)}\,\hbar\,|1,0\rangle$

(then w/ $|1,0\rangle|1,1\rangle$ apply J_- to get $|2,1\rangle$)

Always $\exists$ state with maximal m

$m = j_1 + j_2 \Rightarrow |m=j_1+j_2, m=j_2\rangle$

$= |j_1, m=j_1\rangle \otimes |j_2, m=j_2\rangle$

apply J_- to get

Perturbation Theory

Simple one exactly solve. Hard. Unperturbed: $H = H_0 + \lambda V$

ψ solve for $H(\lambda) = E(\lambda)\psi$ as power series in λ

Non degenerate

Inexact	Exact
$H(\lambda)\psi = E(\lambda)\psi$	$H(\lambda) = E_n(\lambda)$
$\langle\psi|\psi\rangle = 1$ fixed	$\langle\psi_n(\lambda)|\psi_n(\lambda)\rangle = 1$

Assume SE non-degenerate

write $|\psi\rangle = |\psi_0\rangle + \lambda|\psi_1\rangle + \lambda^2|\psi_2\rangle + \ldots$

$E_n = E_n^0 + \lambda E_n^1 + \lambda^2 E_n^2 + \ldots$

expand $H(\lambda)\psi = E_n(\lambda)\psi$

λ^0: $H_0|\psi_0\rangle = E_n^0|\psi_0\rangle$

λ^1: $H_0|\psi_1\rangle + V|\psi_0\rangle = E_n^0|\psi_1\rangle + E_n^1|\psi_0\rangle$

λ^2: $H_0|\psi_2\rangle + V|\psi_1\rangle = E_n^0|\psi_2\rangle + E_n^1|\psi_1\rangle + E_n^2|\psi_0\rangle$

inner prod w/ $\langle\psi_0|$ $\langle\psi_0|\psi_0\rangle = 1$

$\Rightarrow \sum_{k=0}^{n} E_k^n = \langle\psi_0^{(n)}|V|\psi_0^{(n-1)}\rangle$

Low order

$E_n^1 = \langle\psi_0^{(0)}|V|\psi_0^{(0)}\rangle$

$\Rightarrow E_n^1 = E_n^{(0)} + \lambda\langle\psi_0^{(0)}|V|\psi_0^{(0)}\rangle + O(\lambda^2)$

$|\psi_0^{(1)}\rangle = \sum_{k\neq n} \frac{|\psi_k^{(0)}\rangle\langle\psi_k^{(0)}|V|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}$

Degenerate

Suppose one exactly solve H_0.

Unperturbed: $H = H_0 + \lambda V$

ψ solve for $H(\lambda) = E(\lambda)\psi$ as power series in λ

$\psi = \sum_{n=0}^{\infty} \lambda^n \psi_n$

$E = \sum_{n=0}^{\infty} \lambda^n E_n$

Assume E_n non-degenerate

λ^0: $H_0|\psi_0\rangle = E_n^0|\psi_0\rangle$

λ^1: $H_0|\psi_1\rangle + V|\psi_0\rangle = E_n^0|\psi_1\rangle + E_n^1|\psi_0\rangle$

λ^2: $H_0|\psi_2\rangle + V|\psi_1\rangle = E_n^0|\psi_2\rangle + E_n^1|\psi_1\rangle + E_n^2|\psi_0\rangle$

inner prod w/ $\langle\psi_0|$ $\langle\psi_0|\psi_0\rangle = 1$

$\Rightarrow \sum_{k=0}^{n} E_k^n = \langle\psi_0^{(n)}|V|\psi_0^{(n-1)}\rangle$

Low order

$E_n^1 = \langle\psi_0^{(0)}|V|\psi_0^{(0)}\rangle$

$\Rightarrow E_n = E_n^{(0)} + \lambda\langle\psi_0^{(0)}|V|\psi_0^{(0)}\rangle + O(\lambda^2)$

$|\psi_0^{(1)}\rangle = \sum_{k\neq n} \frac{|\psi_k^{(0)}\rangle\langle\psi_k^{(0)}|V|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}$