Symmetries in Quantum Mechanics

Wigner's theorem. If $T: H \to H$ is an invertible transformation of an Hilbert space into itself that preserves transition amplitudes

$$\frac{|\langle T(\psi), T(\varphi)\rangle|^2}{\|T(\psi)\|^2\,\|T(\varphi)\|^2} = \frac{|\langle \psi, \varphi\rangle|^2}{\|\psi\|^2\,\|\varphi\|^2} \quad \text{for all } \psi, \varphi \in H$$

(bra-ket notation doesn't work here!)

then one of the following happens:

- T is linear and unitary (up to a multiplicative constant)
- T is anti-linear and anti-unitary (up to a multiplicative constant)

Since symmetries should (at the very least!) preserve transition amplitudes, then they should act a unitary or anti-unitary operators.

Since the identity is unitary, if G is a connected group of symmetries, by continuity G must act unitarily

(anti-unitary ones are used for time reversal)

This sure sounds like unitary representations!

Back to representations

if V is a G-module, we can make it into a $\mathrm{Lie}(G)$-module by defining

$$X \triangleright |\psi\rangle = \frac{d}{d\varepsilon}\left(e^{\varepsilon X} \triangleright |\psi\rangle\right)\Big|_{\varepsilon=0}$$

example: if $U(1)$ acts on $V_n = \text{span}\{|n\rangle\}$ as $e^{i\theta} \triangleright |n\rangle = e^{in\theta} |n\rangle$

and $i = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} e^{i\varepsilon} \in \underline{u}(1)$, then

$$i \triangleright |n\rangle = \frac{d}{d\varepsilon} e^{i\varepsilon} \triangleright |n\rangle \Big|_{\varepsilon=0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(e^{i\varepsilon} \triangleright |n\rangle - e^{i0} \triangleright |n\rangle\right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(e^{in\varepsilon} - 1\right)|n\rangle$$

$$= in |n\rangle$$

which is indeed an action of $\underline{u}(1)$ (check it!)

Note that the action of $\underline{u}(1)$ is anti-hermitian $\Rightarrow$ unitary $\underline{u}(1)$-module

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Does the converse work? In other words, do all the representations of Lie(G) come from representations of G?

Let's find out!

- $V_\alpha = \text{span}\{|\alpha\rangle\}$ $\alpha \in \mathbb{R}$ with $\underline{u}(1)$ acting as $i \triangleright |\alpha\rangle = i\alpha |\alpha\rangle$

  This is a unitary $\underline{u}(1)$ module, since

  $$\overline{\langle\alpha| i \triangleright |\alpha\rangle} = \overline{i\alpha \langle\alpha|\alpha\rangle} = -i\alpha \langle\alpha|\alpha\rangle = -\langle\alpha| i \triangleright |\alpha\rangle$$

  Can we say that $e^{i\theta} \triangleright |\alpha\rangle =$ "$\exp(i\theta \triangleright |\alpha\rangle)$"?

  Suppose we say that $e^{i\theta} \triangleright |\alpha\rangle = \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \triangleright |\alpha\rangle$

  $$i\theta \triangleright |\alpha\rangle = i\alpha\theta |\alpha\rangle$$

  $$(i\theta)^2 \triangleright |\alpha\rangle = i\theta \triangleright i\alpha\theta |\alpha\rangle = (i\alpha\theta)^2 |\alpha\rangle \qquad \text{etc.}$$

  $$\rightarrow \sum_{n=1}^{\infty} \frac{(i\theta)^n}{n!} \triangleright |\alpha\rangle = \left(\sum_{n=1}^{\infty} \frac{(i\alpha\theta)^n}{n!}\right) |\alpha\rangle = e^{i\theta\alpha} |\alpha\rangle$$

everything seems fine, but there's a problem: the choice of θ is ambiguous! $\quad e^{i\theta} = e^{i\theta + i2\pi k} \quad \forall k \in \mathbb{Z}$

standard choice in complex analysis

We need to make a choice, say $\theta \in (-\pi, \pi]$

$\rightarrow$ if $z \in U(1)$, $\quad z \triangleright |\alpha\rangle = e^{i\alpha \,\mathrm{Arg}(z)} |\alpha\rangle$ $\longrightarrow$ principal argument, returns something in $(-\pi, \pi]$ with $z = |z| e^{i\,\mathrm{Arg}(z)}$

All seems well. Is this a $U(1)$-module?

- linear ✓
- $z \triangleright w \triangleright |\alpha\rangle = z \triangleright e^{i\alpha\,\mathrm{Arg}(z)} |\alpha\rangle = e^{i\alpha\,\mathrm{Arg}(z)} e^{i\alpha\,\mathrm{Arg}(w)} |\alpha\rangle = e^{i\alpha(\mathrm{Arg}(z) + \mathrm{Arg}(w))} |\alpha\rangle$

but in general $\mathrm{Arg}(z) + \mathrm{Arg}(w) = \mathrm{Arg}(z+w) + 2\pi k$, $\quad k \in \{-1, 0, 1\}$ depending on what z and w are.

for example, $\quad e^{i\pi} \triangleright e^{i\pi} \triangleright |\alpha\rangle = e^{i\alpha 2\pi} |\alpha\rangle$

$$e^{i\alpha(\mathrm{Arg}(z) + \mathrm{Arg}(z))} = e^{i\alpha\,\mathrm{Arg}(z+w)} \underbrace{e^{i2\pi\alpha k}}_{\neq 1 \text{ if } k \neq 0,\ \alpha \notin \mathbb{Z}}$$

for example if $\alpha = 1/2$, $\quad e^{i\pi} \triangleright e^{i\pi} \triangleright |1/2\rangle = e^{i\pi} |1/2\rangle = -|1/2\rangle \neq (e^{i\pi} e^{i\pi}) \triangleright |1/2\rangle = |1/2\rangle$

Does this mean that things are broken?

No!

Remember that physical states are only defined up to a non-zero scalar $\quad (|\psi\rangle \sim \lambda |\psi\rangle \quad \text{if } \lambda \neq 0)$

$$\rightarrow \quad [e^{i\pi} \triangleright e^{i\pi} \triangleright |1/2\rangle] = [-|1/2\rangle] = [|1/2\rangle] = [e^{i2\pi} \triangleright |1/2\rangle]$$

and in general $\quad [z \triangleright w \triangleright |\alpha\rangle] = [(zw) \triangleright |\alpha\rangle]$