Radiation considered as a Reservoir   API
Master equation   Ch. IV

Einstein (1917):

$$\frac{dN_b}{dt} = -A_{b\to a} N_b + u(\omega)\left(B_{a\to b} N_a - B_{b\to a} N_b\right)$$

$$\frac{dN_a}{dt} = A_{b\to a} N_b + u(\omega)\left(B_{b\to a} N_b - B_{a\to b} N_a\right)$$

Let's justify using QED!

Use density matrix for atoms

$$\sigma = \mathrm{Tr}_R\, \rho$$

= partial trace over variables of radiation field of the global density matrix ρ.

$$\langle a|\sigma|b\rangle = \sigma_{ab} = \sum_\mu \langle a,\mu|\rho|b,\mu\rangle = \sum_\mu \rho_{a\mu,b\mu}$$

(roman letters a, b – atoms
greek " μ – field)

$\Rightarrow$ state of the atom described by density matrix

$$\begin{pmatrix} \sigma_{bb} & \sigma_{ba} \\ \sigma_{ab} & \sigma_{aa} \end{pmatrix}$$

$\sigma_{aa}, \sigma_{bb} \propto N_a, N_b$ – populations

σ_{ab}, σ_{ba} – coherences (evolve at $\frac{E_b - E_a}{\hbar}$ like dipole moment)

Two relevant timescales:

short: τ_c – fluctuations of the perturbation exerted by reservoir ($\mathcal{R}$) on particles ($\mathcal{A}$)

longer: T_R – characterizes rate of variation of $\mathcal{A}$.

We coarse-grain over $\Delta t \gg \tau_c$, but $\Delta t \ll T_R$.

For radiation field, $\tau_c \leq \frac{2\pi}{\omega_0}$ vacuum fluctuations

while $T_R \approx \frac{1}{\Gamma} \gg \tau_c$

(e.g. for spontaneous emission)

Also true if atom interacts with sufficiently broad light source with spectral width $\Delta\omega$. Then $\tau_c = \Delta\omega^{-1}$, while $T_R \propto (\text{intensity})^{-1}$, so $\tau_c \ll T_R$ can be fulfilled.

This is the case for black-body radiation!

Derivation of the Master equation

$$H = H_A + H_R + V$$

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}[H, \rho(t)]$$

$$\rightarrow \frac{d}{dt}\tilde{\rho}(t) = \frac{1}{i\hbar}[\tilde{V}(t), \tilde{\rho}(t)]$$ in interaction representation

$$\tilde{\rho}(t) = e^{i(H_A+H_R)t/\hbar}\rho(t)\, e^{-i(H_A+H_R)t/\hbar}$$ same for $\tilde{V}(t)$

Integrate: $$\tilde{\rho}(t+\Delta t) = \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^{t+\Delta t} dt'\,[\tilde{V}(t'), \tilde{\rho}(t')]$$

$$\Rightarrow \Delta\tilde{\rho}(t) = \tilde{\rho}(t+\Delta t) - \tilde{\rho}(t)$$

$$= \frac{1}{i\hbar}\int_t^{t+\Delta t} dt'\,[\tilde{V}(t'), \tilde{\rho}(t)] +$$

$$+ \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} dt''\,[\tilde{V}(t'), [\tilde{V}(t''), \tilde{\rho}(t'')]]$$

reduced density matrix in int. rep.: $\tilde{\sigma}(t) = Tr_R\, \tilde{\rho}(t)$

$$\Delta\tilde{\sigma}(t) = \frac{1}{i\hbar}\int_t^{t+\Delta t} dt'\, Tr_R[\tilde{V}(t'), \tilde{\rho}(t)] +$$

$$+ \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} dt''\, Tr_R[\tilde{V}(t'), [\tilde{V}(t''), \tilde{\rho}(t'')]]$$

This is exact.

Assumptions concerning the reservoir

$$\tilde{\sigma}_R(t) = Tr_A\, \tilde{\rho}(t) \qquad \text{density operator for } R.$$

Approximately: $\tilde{\sigma}_R(t) \simeq \tilde{\sigma}_R(0) = \sigma_R$ (weak coupling to A)

Also: R is stationary $\Rightarrow [\sigma_R, H_R] = 0$.

so with $H_R|\mu\rangle = E_\mu|\mu\rangle$

$$\sigma_R = \sum_\mu p_\mu |\mu\rangle\langle\mu|$$

ex.: R in thermodynamic equilibrium at T

$$p_\mu = \frac{1}{Z}\, e^{-E_\mu/k_BT}$$

$$Z = \sum_\mu e^{-E_\mu/k_BT}$$

Assume that $V = -AR$

$$\rightarrow \tilde{V}(t) = -\tilde{A}(t)\,\tilde{R}(t)$$

$$\tilde{A}(t) = e^{iH_At/\hbar} A\, e^{-iH_At/\hbar}$$

$$\tilde{R}(t) = e^{iH_Rt/\hbar} R\, e^{-iH_Rt/\hbar}$$

Assume: $\langle R\rangle_R = tr(\sigma_R R) = tr(\sigma_R \tilde{R}(t)) = 0$

$$\Rightarrow tr_R[\sigma_R \tilde{V}(t)] = \tilde{A}(t)\, tr(\sigma_R \tilde{R}(t)) = 0$$

Two-time average: $g(t',t'') = Tr(\sigma_R \tilde{R}(t')\tilde{R}(t''))$

$Re\, g(t',t'')$ – correlation function for dynamics of fluctuations of R

$Im\, g(t',t'')$ – linear susceptibility

$$g(t',t'') = g(\tau) \qquad \tau = t' - t''$$

$$\Delta\tilde\sigma(t) = \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \, \cancel{Tr_R\left[\tilde V(t'), \tilde\rho(t)\right]} + \qquad \text{as } tr_R(\sigma_R \tilde R(t)) = 0$$

$$+ \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \, Tr_R\left[\tilde V(t'), \left[\tilde V(t''), \tilde\rho(t'')\right]\right]$$

if V is small and $\Delta t \ll T_R$, then here: $\tilde\rho(t'') \simeq \tilde\rho(t)$

Also, write $\tilde\rho(t) \simeq \tilde\sigma(t) \otimes \sigma_R$ (neglect correlations)

valid if $\tau_c \ll \Delta t \ll T_R$.

$$\frac{\Delta\tilde\sigma}{\Delta t} = -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \, tr_R\left[\tilde V(t'), \left[\tilde V(t''), \tilde\sigma(t)\otimes\sigma_R\right]\right]$$

Note $\displaystyle\frac{\Delta\tilde\sigma}{\Delta t} = \frac{\tilde\sigma(t+\Delta t) - \tilde\sigma(t)}{\Delta t} = \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \frac{d\tilde\sigma}{dt'}$

$=$ coarse-grained average of $\frac{d\tilde\sigma}{dt}$ over interval Δt.

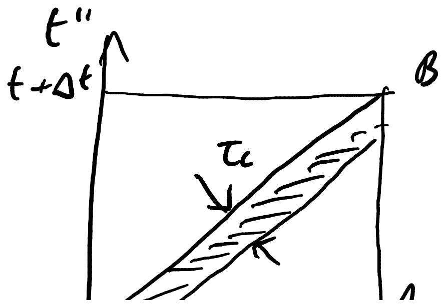
Since $\frac{\Delta\tilde\sigma}{\Delta t}$ depends only on $\tilde\sigma(t)$, coarse-grained evolution of $\mathcal{A}$ depends only on the present, not the past. (Markov process).

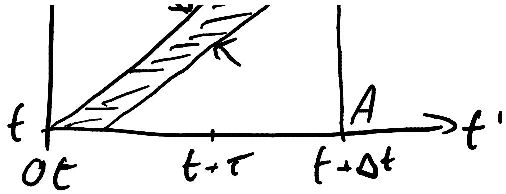
Since $\tilde V = -\tilde S \tilde R$, the trace involves terms like

$$tr_R\left(\sigma_R \tilde R(t') \tilde R(t'')\right) = g(t', t'') \to 0 \text{ for } |t'-t''| \gg \tau_c$$

Use $\displaystyle\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' = \int_0^{\Delta t} d\tau \int_{t+\tau}^{t+\Delta t} dt' \approx \int_0^{\infty} d\tau \int_t^{t+\Delta t} dt'$

(for integration over $g(\tau)$)




t
A
t'
0t
t+τ
t+Δt

$$g(\tau) = Tr\left(\sum_\mu \{ p_\mu |\mu\rangle\langle\mu| \tilde{R}(\tau)\tilde{R}(0) \} \right)$$

$$= \sum_\mu p_\mu \langle\mu| \tilde{R}(\tau)\tilde{R}(0)|\mu\rangle$$

$$= \sum_\mu \sum_\nu p_\mu |R_{\mu\nu}|^2 e^{i\omega_{\mu\nu}\tau}$$

$R_{\mu\nu} = \langle\mu|R|\nu\rangle$

$\omega_{\mu\nu} = \omega_\mu - \omega_\nu$

$\omega_\mu = E_\mu/\hbar$

$$g(-\tau) = g(\tau)^*$$

$g(\tau)$ = sum over oscillating terms, interfere destructively for large τ.

Assume $g(\tau) \to 0$ over characteristic time τ_c.

i.e.: R is stationary and exerts on $\mathcal{A}$ a "force" that fluctuates about zero with short correlation τ_c.

$$\frac{\Delta\tilde{\sigma}}{\Delta t} = -\frac{1}{\hbar^2}\int_0^\infty d\tau\, \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \Big\{ g(\tau)\big[\tilde{A}(t')\tilde{A}(t'-\tau)\tilde{\sigma}(t) - \tilde{A}(t'-\tau)\tilde{\sigma}(t)\tilde{A}(t')\big] + g(-\tau)\big[\tilde{\sigma}(t)\tilde{A}(t'-\tau)\tilde{A}(t') - \tilde{A}(t')\tilde{\sigma}(t)\tilde{A}(t'-\tau)\big]\Big\}$$

Now introduce basis of states

$$H_A |a\rangle = E_a |a\rangle$$

$$\frac{\Delta\tilde{\sigma}_{ab}}{\Delta t} = \sum_{cd} \gamma_{abcd}(t)\,\tilde{\sigma}_{cd}(t)$$

$$\gamma_{abcd}(t) = -\frac{1}{\hbar^2}\int_0^\infty d\tau\,\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \times$$

$$\times \Big\{ g(\tau)\big[\delta_{bd}\sum_n \tilde{A}_{an}(t')\tilde{A}_{nc}(t'-\tau) - \tilde{A}_{ac}(t'-\tau)\tilde{A}_{db}(t')\big] + g(-\tau)\big[\delta_{ac}\sum_n \tilde{A}_{dn}(t'-\tau)\tilde{A}_{nb}(t') - \tilde{A}_{ac}(t')\tilde{A}_{db}(t'-\tau)\big]\Big\}$$

dependence on t' comes from $\tilde{A}_{an}(t') \sim \exp(i\omega_{an}t')$

$\rightarrow$ all terms inside brackets vary as $\exp(i(\omega_{ab}-\omega_{cd})t')$

$$\Rightarrow \frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\,\exp(i(\omega_{ab}-\omega_{cd})t') = e^{i(\omega_{ab}-\omega_{cd})t} f[(\omega_{ab}-\omega_{cd})\Delta t]$$

$$f(x) = e^{i\frac{x}{2}}\frac{\sin(\frac{x}{2})}{x/2}$$

$\rightarrow$ secular approximation: retain terms with $|\omega_{ab}-\omega_{cd}| \ll \frac{1}{\Delta t}$.

$$\frac{\Delta\tilde{\sigma}_{ab}}{\Delta t} = \sum_{c,d}^{(sec)} e^{i(\omega_{ab}-\omega_{cd})t} R_{abcd}\,\tilde{\sigma}_{cd}(t)$$

with $R_{abcd} = -\frac{1}{\hbar^2} \int_0^\infty d\tau \, \{ g(\tau) [\delta_{bd} \sum_n A_{an} A_{nc} e^{i\omega_{cn}\tau} -$

$- A_{ac} A_{db} e^{i\omega_{ca}\tau}] +$

$+ g(-\tau) [\delta_{ac} \sum_n A_{dn} A_{nb} e^{i\omega_{nd}\tau} -$

$- A_{ac} A_{db} e^{i\omega_{bd}\tau}] \}$

switch to Schrödinger picture $\sigma_{ab}(t) = e^{-i\omega_{ab}t} \tilde{\sigma}_{ab}(t)$

$$\frac{d\sigma_{ab}(t)}{dt} = -i\omega_{ab}\sigma_{ab}(t) + e^{-i\omega_{ab}t} \frac{d\tilde{\sigma}_{ab}(t)}{dt}$$

$$\Rightarrow \boxed{\frac{d\sigma_{ab}(t)}{dt} = -i\omega_{ab}\sigma_{ab}(t) + \sum_{c,d}^{(sec)} R_{abcd}\,\sigma_{cd}(t)}$$

linear differential system with time-independent coeff.

$R_{abcd} \sim \frac{1}{T_R}$ where T_R is evolution time of A.

Physical content:

1. Evolution of populations

$$\frac{d\sigma_{aa}}{dt} = \sum_c R_{aacc}\,\sigma_{cc}$$

($\omega_{aa} = 0$, ignore coherences with very low free evolution $|\omega_{cd}| \ll \frac{1}{\Delta t}$)

$$\underset{a \neq c}{R_{aacc}} = \frac{1}{\hbar^2} \int_{-\infty}^{\infty} d\tau \; g(\tau) |A_{ac}|^2 e^{i\omega_{ca}\tau}$$

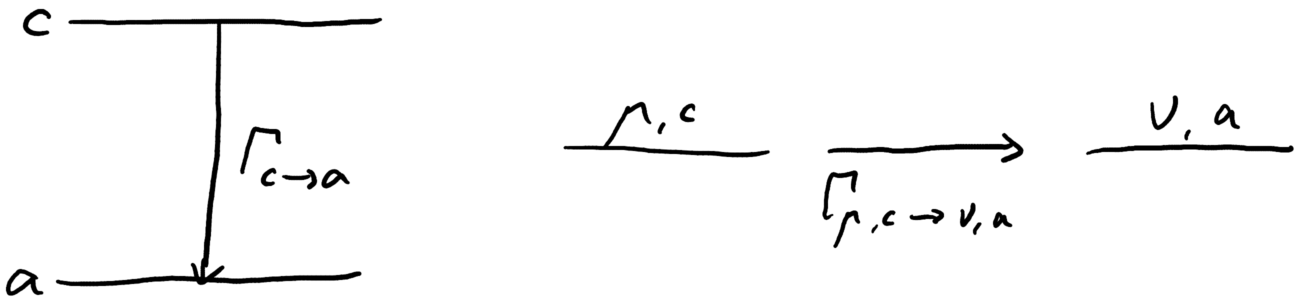
$$= \frac{1}{\hbar^2} \sum_\mu p_\mu \sum_\nu \int_{-\infty}^{\infty} d\tau \; e^{i(\omega_{\mu\nu}+\omega_{ca})\tau} |A_{ac}|^2 |R_{\mu\nu}|^2$$

$\int d\tau$ gives $2\pi\hbar\, \delta(E_\mu + E_c - E_\nu - E_a)$

$|A_{ac}|^2 |R_{\mu\nu}|^2 = |\langle \nu, a | V | \mu, c \rangle|^2$.

Set $R_{aacc} = \Gamma_{c \to a}$
$\quad a \neq c$

$$\Gamma_{c \to a} = \frac{2\pi}{\hbar} \sum_{\mu} p_{\mu} \sum_{\nu} |\langle \nu, a | V | \mu, c \rangle|^2 \delta(E_{\mu} + E_c - E_{\nu} - E_a)$$



$\Gamma_{c \to a}$ – average over all possible initial states μ of the reservoir (weighted by p_μ) and summed over all the final states ν of the reservoir, with $\delta(\,)$ expressing energy conservation

= Fermi's Golden Rule

$$R_{aaaa} = -\sum_{n \neq a} \Gamma_{a \to n}$$

$$\Rightarrow \quad \frac{d\sigma_{aa}}{dt} = -\sigma_{aa} \sum_{n \neq a} \Gamma_{a \to n} + \sum_{c \neq a} \sigma_{cc} \Gamma_{c \to a}$$

$$\frac{d\sigma_{aa}}{dt} = \sum_{c \neq a} \left(\sigma_{cc} \Gamma_{c \to a} - \sigma_{aa} \Gamma_{a \to c} \right)$$

$\Rightarrow$ balance of transfer of populations

$$\sum_a \frac{d}{dt} \sigma_{aa} = 0$$

i) Steady state: $\sigma_{aa}^{st}\Gamma_{a\to c} = \sigma_{cc}^{st}\Gamma_{c\to a}$

detailed balance

ii) If reservoir R is in thermal equilibrium,

$$e^{-E_a/k_BT}\Gamma_{a\to c} = e^{-E_c/k_BT}\Gamma_{c\to a}.$$

$\Rightarrow$ populations $\sigma_{aa} \longrightarrow e^{-E_a/k_BT}$ in steady state.

$\Rightarrow$ A reaches thermodyn. equilibrium if R is in thermo. equilibrium. Same temperature T!

Coherences:

Non-deg. case:

$$\frac{d}{dt}\sigma_{ab} = -i\omega_{ab}\sigma_{ab} + R_{abab}\sigma_{ab}$$

$$R_{abab} = -\frac{1}{\hbar^2}\int_0^\infty d\tau \Big\{ g(\tau)\Big[\sum_n |A_{an}|^2 e^{i\omega_{an}\tau} - A_{aa}A_{bb}\Big] + g(-\tau)\Big[\sum_n |A_{bn}|^2 e^{-i\omega_{bn}\tau} - A_{aa}A_{bb}\Big]\Big\}$$

$$\longrightarrow \quad R_{abab} = -\Gamma_{ab} - i\Delta_{ab}$$

$$\Delta_{ab} = \Delta_a - \Delta_b$$

$$\Delta_a = \frac{1}{\hbar}\mathcal{P}\sum_\mu p_\mu \sum_\nu \sum_n \frac{|\langle \nu, n|V|\mu, a\rangle|^2}{E_\mu + E_a - E_\nu - E_n}$$

Shift of $|a\rangle$

$$\Gamma_{ab} = \Gamma_{ab}^{\text{non-ad.}} + \Gamma_{ab}^{\text{ad.}}$$

$$\Gamma_{ab}^{\text{non ad}} = \frac{1}{2}\Big(\sum_{n\neq a}\Gamma_{a\to n} + \sum_{n\neq b}\Gamma_{b\to n}\Big)$$

$$\Gamma_{ab}^{ad.} = \frac{2\pi}{\hbar} \sum_\mu p_\mu \sum_\nu \delta(E_\mu - E_\nu) \times$$

$$\times \left(\frac{1}{2} |\langle \nu, a|V|\mu, a\rangle|^2 + \frac{1}{2} |\langle \mu, b|V|\nu, b\rangle|^2 - \right.$$

$$\left. - \mathrm{Re} \langle \mu, a|V|\nu, a\rangle\langle \nu, b|V|\mu, b\rangle \right).$$

(A does not change state while interacting with R)

If coherences are present $\left(|\omega_{cd} - \omega_{ab}| \ll \frac{1}{\Delta t}\right)$:

$$R_{abcd} = \frac{2\pi}{\hbar} \sum_\mu p_\mu \sum_\nu \langle \nu, a|V|\mu, c\rangle\langle \mu, d|V|\nu, b\rangle \times$$

$$\times \delta(E_\mu + E_c - E_\nu - E_a).$$

Application to a two-level atom coupled to the radiation field
(atom infinitely heavy here)

$$V = -\vec{d}\cdot\vec{E}_\perp(\vec{0})$$
$$= -\vec{d}\sum_i \sqrt{\frac{\hbar\omega_i}{2\varepsilon_0 V}}\,\vec{\varepsilon}_i\,(a_i - a_i^+)$$

$$\sigma_R = |0\rangle\langle 0|$$
$$\Gamma_{a\to b} = 0$$
$$\Gamma_{b\to a} = \frac{2\pi}{\hbar}\sum_{\vec{k}\vec{\varepsilon}} |\langle a;\vec{k}\vec{\varepsilon}|V|b;0\rangle|^2\,\delta(\hbar\omega - \hbar\omega_{ba})$$
$$= \Gamma = \frac{1}{\tau} \qquad \tau - \text{lifetime of } b.$$

$$\frac{d}{dt}\sigma_{bb} = -\Gamma\sigma_{bb}$$
$$\frac{d}{dt}\sigma_{aa} = +\Gamma\sigma_{bb}$$

same as Einstein with $n(0) = 0$!

$$\frac{d}{dt}\sigma_{ba} = -i(\omega_{ba} + \Delta_{ba})\sigma_{ba} - \frac{\Gamma}{2}\sigma_{ba}$$

$$\Delta_{ba} = \Delta_b - \Delta_a$$

$$\Delta_b = \frac{1}{\hbar}\,\mathcal{P}\sum_{\vec{k}\vec{\varepsilon}} \frac{|\langle a,\vec{k}\vec{\varepsilon}|V|b;0\rangle|^2}{\hbar\omega_{ba} - \hbar\omega}$$

$$\Delta_a = \frac{1}{\hbar}\,\mathcal{P}\sum_{\vec{k}\vec{\varepsilon}} \frac{|\langle b,\vec{k}\vec{\varepsilon}|V|a;0\rangle|^2}{-\hbar\omega_{ba} - \hbar\omega}$$

$\Gamma^{\text{ad.}} = 0$ as V has no diagonal elements in $|a\rangle$ or $|b\rangle$

Weak broadband radiation:

$$\sigma_R = \sum_{\{n_i\}} p(n_1, \dots, n_i, \dots) |n_1 \dots n_i \dots\rangle\langle n_1 \dots n_i \dots|$$

$$\Gamma' = \Gamma_{a\to b} = \frac{2\pi}{\hbar} \sum_{\{n_i\}} p(n_1, \dots, n_i, \dots) \times$$

$$\times \sum_{\{n_i'\}} |\langle b; n_1', \dots, n_i', \dots | V | a; n_1, \dots, n_i, \dots\rangle|^2 \delta(E_{final} - E_{init})$$

conservation of energy: $n_i' = n_i - 1$ only possible.

$$\Gamma' = \frac{2\pi}{\hbar} \sum_{n_1 \dots n_i \dots} p(n_1 \dots, n_i \dots) \times$$

$$\times \sum_i |\langle b; n_1 \dots, n_i - 1, \dots | V | a; n_1 \dots n_i \dots\rangle|^2 \delta(\hbar\omega_i - \hbar\omega_{ba})$$

$$= \frac{2\pi}{\hbar} \sum_i \left(\sum_{\{n_i\}} n_i\, p(n_1 \dots, n_i \dots) \right) \times$$

$$\times |\langle b; 0 | V | a; 1_i\rangle|^2 \delta(\hbar\omega_i - \hbar\omega_{ba})$$

$$\langle n_i \rangle = \sum_{\{n_i\}} n_i\, p(n_1 \dots, n_i \dots)$$

$$\Gamma_{b\to a} = \frac{2\pi}{\hbar} \sum_i (\langle n_i \rangle + 1) |\langle a; 1_i | V | b; 0\rangle|^2 \delta(\hbar\omega_i - \hbar\omega_{ba})$$

stimulated & spontaneous emission

$$= \Gamma + \Gamma'$$

$$\frac{d}{dt}\sigma_{bb} = -\Gamma\sigma_{bb} + \Gamma'(\sigma_{aa} - \sigma_{bb})$$

$$\frac{d}{dt}\sigma_{aa} = +\Gamma\sigma_{bb} + \Gamma'(\sigma_{bb} - \sigma_{aa})$$

= Einstein

For isotropic, unpolarized radiation, $\Gamma' = \Gamma \langle n(\omega_{ba}) \rangle$

$$\langle n(\omega_{ba}) \rangle = \frac{1}{e^{\hbar\omega_{ba}/kT} - 1}$$

In steady state: $$\frac{\sigma_{bb}}{\sigma_{aa}} = \frac{\Gamma'}{\Gamma + \Gamma'} = \frac{\langle n(\omega_{ba}) \rangle}{1 + \langle n(\omega_{ba}) \rangle}$$

$$\Rightarrow \quad \frac{\sigma_{bb}}{\sigma_{aa}} = e^{-\hbar\omega_{ba}/k_B T}$$

$\Rightarrow$ atom reaches thermal equilibrium

$$u(\omega)\, d\omega = \frac{\langle n(\omega) \rangle \hbar\omega}{L^3} \cdot \left(\frac{L}{2\pi}\right)^3 8\pi k^2 dk$$

$$\Rightarrow \quad u(\omega) = \frac{\hbar\omega^3 \langle n(\omega) \rangle}{\pi^2 c^3}$$

Coherences:

$$\frac{d}{dt}\sigma_{ba} = -i\left(\omega_{ba} + \Delta_{ba} + \Delta'_{ba}\right)\sigma_{ba} - \frac{1}{2}\left(\Gamma + 2\Gamma'\right)\sigma_{ba}$$

Γ' indicates additional damping due to absorption and stimulated emission

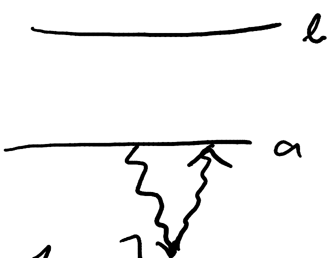
$\Delta'_{ba} = \Delta'_b - \Delta'_a$ additional shift produced by incident radiation

$$\hbar\Delta'_a = \mathcal{P}\sum_{n_1\ldots n_i\ldots} p(n_1,\ldots,n_i\ldots) \times$$

$$\times \sum_i \frac{|\langle b; n_1,\ldots,n_i-1,\ldots|V|a; n_1,\ldots,n_i,\ldots\rangle|^2}{\hbar\omega_i - \hbar\omega_{ba}}$$

$$\sim \mathcal{P}\int \frac{I(\omega)d\omega}{\omega-\omega_{ba}}$$

more correctly: include



$$\rightarrow \Delta'_a \sim \mathcal{P}\int d\omega\, I(\omega)\left[\frac{1}{\omega-\omega_{ba}} + \frac{1}{-\omega-\omega_{ba}}\right]$$

$$= 2\omega_{ba}\,\mathcal{P}\int d\omega \frac{I(\omega)}{\omega^2-\omega_{ba}^2}$$