Interaction between Light And Atoms

$$H = H_A + H_R + H_I$$

particles radiation interaction

$$H_I = -\frac{q}{m}\hat{\vec{p}}\cdot\hat{\vec{A}}(\vec{r}_0) + \frac{q^2}{2m}\left(\hat{\vec{A}}(\vec{r}_0)\right)^2$$

$\vec{r}_0$ = position of the nucleus

long wavelength approximation

Alternative Interaction (Göppert-Meyer Trafo.)

$$H_I' = -\hat{\vec{d}}\cdot\hat{\vec{E}}(\vec{r}_0)$$

Decomposition: $H_{I_1} = \frac{q}{m}\sum_l \sqrt{\frac{\hbar}{2\varepsilon_0 \omega V}}\,\hat{\vec{p}}\cdot\hat{\vec{\varepsilon}}_l\,(a_l + a_l^+)$

$$H_{I_2} = \frac{q^2}{2m}\frac{\hbar}{2\varepsilon_0 V}\sum_j\sum_l \frac{\vec{\varepsilon}_j\cdot\vec{\varepsilon}_l}{\sqrt{\omega_j\cdot\omega_l}}\left(a_j a_l^+ + a_j^+ a_l + a_j a_l + a_j^+ a_l^+\right)$$

choose $\vec{r}_0 = \vec{0}$ for convenience.

If problem involves spin: $H_S = g\mu_B\hat{\vec{S}}\cdot\hat{\vec{B}}(\vec{0})$

H_{Int} is linear in a_ℓ and $a_\ell^\dagger$

Basis: $|a, n_1 \ldots, n_j, \ldots\rangle$ and $|b, n_1 \ldots, n_j, \ldots\rangle$ eigenstates of $H_A + H_R$

Absorption: $|\varphi_i\rangle = |a, n_j\rangle$

$$|\varphi_f\rangle = |b, n_j' = n_j - 1\rangle$$

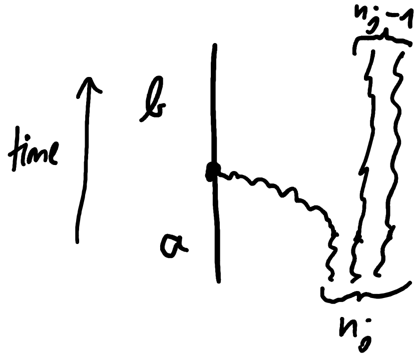
$$a_j |n_j\rangle = \sqrt{n_j}\,|n_j - 1\rangle$$

$$\langle b, n_j - 1 | \hat{H}_{Int} | a, n_j\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2\varepsilon_0 \omega_j V}}\langle b|\hat{\vec{p}}\cdot\vec{\varepsilon}_j|a\rangle\sqrt{n_j}$$

So prob. $P_{i\to f}(t) \propto |\langle\varphi_i|H_{Int}|\varphi_f\rangle|^2 \propto n_j \propto$ intensity.

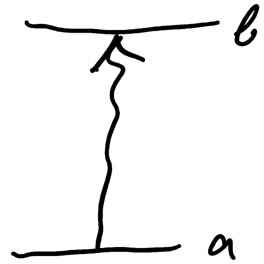
Significant probability:

$$E_a + n_j\hbar\omega_j = E_b + (n_j - 1)\hbar\omega_j$$



$E_b = E_a + \hbar\omega_j$

resonance



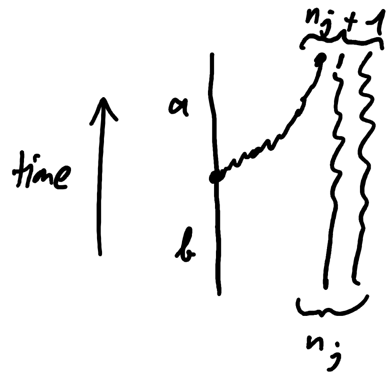
Emission:

$$a_j^+ |n_j\rangle = \sqrt{n_j+1}\, |n_j+1\rangle$$

$$\langle a, n_j+1 | H_{I_1} | b, n_j\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2\varepsilon_0 V\omega_j}}\,\langle a|\hat{\vec{p}}\cdot\vec{\varepsilon}|b\rangle \sqrt{n_j+1}$$

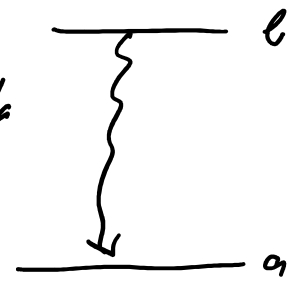
conservation of energy

$$E_a = E_b - \hbar\omega_j$$



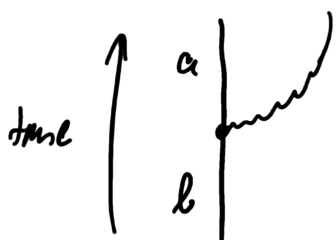
for large n_j, scattering proba $\propto n_j$

$\Rightarrow$ stimulated emission



$n_j = 0$ emission still possible:

$$\langle a, n_j = 1 | H_{I_1} | b, 0\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2\varepsilon_0\omega_j V}}\,\langle a|\hat{\vec{p}}\cdot\vec{\varepsilon}_j|b\rangle\cdot 1$$



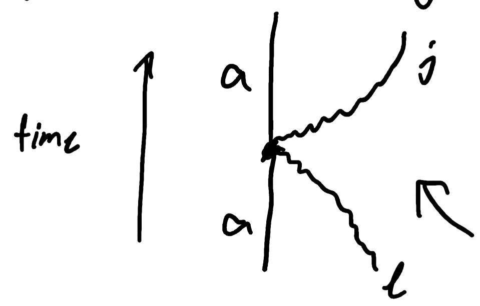
spontaneous emission

matrix element: $\langle a|\hat{\vec{p}}\cdot\vec{\varepsilon}_j|b\rangle = -i\frac{m}{\hbar}\langle a|[\hat{\vec{r}}\cdot\vec{\varepsilon}_j, H_A]|b\rangle$

$$= -i\frac{m}{\hbar}(E_b - E_a)\,\langle a|\hat{\vec{r}}\cdot\vec{\varepsilon}_j|b\rangle$$

$$= -i\frac{m\omega_0}{q}\,\langle a|\hat{\vec{d}}\cdot\vec{\varepsilon}_j|b\rangle$$

H_{I_2} : Elastic scattering, no change in atomic state



$a_j^+ a_l^+$ appearance of 2 photons

$a_j\, a_l$ disapperance of 2 photons

$\Rightarrow$ energetically forbidden, i.e. "very unlikely"

$a_j^+ a_l$ - elastic scattering, without change in frequency

amplitude:

$$\langle a, n_j = 1, n_l - 1 \mid H_{I_2} \mid a, n_j = 0, n_l \rangle = \frac{q^2}{m} \frac{\hbar}{2\varepsilon_0 V} \frac{\vec{\varepsilon}_j \cdot \vec{\varepsilon}_l}{\omega_j} \sqrt{n_l}$$

8.422 AMO II - Transition Amplitudes, Scattering S-Matrix

March 15, 2023

Transition amplitudes and S-matrix

Transition from state $|\psi_i\rangle$ to $|\psi_f\rangle$ under time evolution $U(t_f, t_i)$:

$$\langle\psi_f|U(t_f, t_i)|\psi_i\rangle$$

Transition amplitudes and S-matrix

Transition from state $|\psi_i\rangle$ to $|\psi_f\rangle$ under time evolution $U(t_f, t_i)$:

$$\langle\psi_f|U(t_f, t_i)|\psi_i\rangle$$

For perturbative calculations, split Hamiltonian $H = H_0 + V$ into an "unperturbed" part H_0 and an interaction V. Expand any state into eigenstates $|\varphi_n\rangle$ of H_0 and consider amplitudes

$$\langle\varphi_m|U(t_f, t_i)|\varphi_n\rangle$$

between eigenstates of H_0.

Transition amplitudes and S-matrix

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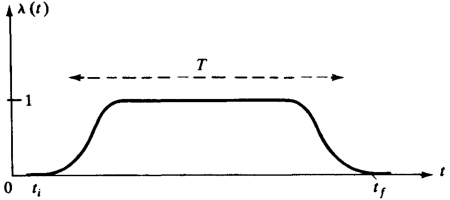
$$\langle\varphi_m|U(t_f, t_i)|\varphi_n\rangle$$

between eigenstates of H_0. Consider V being "switched on" at t_i, long before the collision and "off" at t_f, long after the collision, multiplying by $\lambda(t)$. The above transition amplitudes are, in the limit $t_i \to -\infty$ and $t_f \to +\infty$, elements of the scattering S-matrix.



Interaction representation

Split Hamiltonian as $H = H_0 + V$.
Unitary Transformation:

$$\boxed{T(t) = e^{iH_0 t/\hbar}}$$

Wavefunctions transform as: $|\tilde{\psi}(t)\rangle = e^{iH_0 t/\hbar}|\psi(t)\rangle$

Operators transform as: $\tilde{A}(t) = e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar}$

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Operators transform as: $\tilde{A}(t) = e^{iH_0 t/\hbar} A e^{-iH_0 t/\hbar}$

$$\begin{aligned} \Rightarrow i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\tilde{\psi}(t)\rangle &= -H_0|\tilde{\psi}(t)\rangle + e^{iH_0 t/\hbar}\left(H_0 + V\right)|\psi(t)\rangle \\ &= \tilde{V}(t)|\tilde{\psi}(t)\rangle \end{aligned}$$

$\Rightarrow |\tilde{\psi}(t)\rangle$ evolves only due to $\tilde{V}$.

Transition amplitudes in the interaction picture

Given

$$|\psi(t_f)\rangle = U(t_f, t_i)|\psi(t_i)\rangle$$

one has

$$|\tilde{\psi}(t_f)\rangle = \tilde{U}(t_f, t_i)|\tilde{\psi}(t_i)\rangle$$

with the time evolution operator in the interaction representation:

$$\tilde{U}(t_f, t_i) = e^{iH_0 t_f/\hbar} U(t_f, t_i) e^{-iH_0 t_i/\hbar}$$

Since $i\hbar \frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = (H_0 + V)|\psi(t)\rangle$, we have:

$$\boxed{U(t_f, t_i) = U_0(t_f, t_i) + \frac{1}{i\hbar}\int_{t_i}^{t_f} \mathrm{d}t\, U_0(t_f, t) V U(t, t_i)}$$

with $U_0(t_f, t_i) = e^{-iH_0(t_f - t_i)/\hbar}$. This satisfies $U(t_i, t_i) = \mathbb{1}$ and

$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t_f} U(t_f, t_i) = (H_0 + V) U(t_f, t_i)$$

Perturbative expansion

$$\boxed{U(t_f, t_i) = U_0(t_f, t_i) + \frac{1}{i\hbar} \int_{t_i}^{t_f} \mathrm{d}t \, U_0(t_f, t) V U(t, t_i)}$$

Let's expand in powers of V. Writing formally

$$\begin{aligned} U &= U_0 + U_0 V U \\ &= U_0 + U_0 V U_0 + U_0 V U_0 V U \\ &= \ldots \end{aligned}$$

$$\Rightarrow U(t_f, t_i) = U_0(t_f, t_i) + \sum_{n=1}^{\infty} U^{(n)}(t_f, t_i)$$

with

$$\begin{aligned} U^{(n)}(t_f, t_i) &= \left(\frac{1}{i\hbar}\right)^n \int_{t_f \geq \tau_n \ldots \tau_2 \geq \tau_1 \geq t_i} \mathrm{d}\tau_n \ldots \mathrm{d}\tau_2 \mathrm{d}\tau_1 \times \\ &\quad \times e^{-iH_0(t_f - \tau_n)/\hbar} V \ldots V e^{-iH_0(\tau_2 - \tau_1)/\hbar} V e^{-iH_0(\tau_1 - t_i)/\hbar} \end{aligned}$$

Interaction representation

$$U(t_f, t_i) = U_0(t_f, t_i) + \sum_{n=1}^{\infty} U^{(n)}(t_f, t_i)$$

with

$$\begin{aligned} U^{(n)}(t_f, t_i) &= \left(\frac{1}{i\hbar}\right)^n \int_{t_f \geq \tau_n \ldots \tau_2 \geq \tau_1 \geq t_i} \mathrm{d}\tau_n \ldots \mathrm{d}\tau_2 \mathrm{d}\tau_1 \times \\ & \quad \times e^{-iH_0(t_f - \tau_n)/\hbar} V \ldots V e^{-iH_0(\tau_2 - \tau_1)/\hbar} V e^{-iH_0(\tau_1 - t_i)/\hbar} \end{aligned}$$

Interaction representation:

$$\tilde{U}(t_f, t_i) = \mathbb{1} + \sum_{n=1}^{\infty} \tilde{U}^{(n)}(t_f, t_i)$$

$$\tilde{U}(t_f, t_i) = \left(\frac{1}{i\hbar}\right)^n \int_{t_f \geq \tau_n \ldots \tau_2 \geq \tau_1 \geq t_i} \mathrm{d}\tau_n \ldots \mathrm{d}\tau_2 \mathrm{d}\tau_1 \tilde{V}(\tau_n) \ldots \tilde{V}(\tau_2) \tilde{V}(\tau_1)$$

Scattering Matrix

Let's find the scattering matrix S_{fi}, the matrix element of $\tilde{U}(t_f, t_i)$ between the eigenstates $\langle\varphi_f|$ and $|\varphi_i\rangle$ of H_0:

$$\begin{aligned} S_{fi} &= \langle\varphi_f|\tilde{U}(t_f, t_i)|\varphi_i\rangle \\ &= \delta_{fi} + \sum_{n=1}^{\infty} S_{fi}^{(n)} \\ S_{fi}^{(n)} &= \langle\varphi_f|\tilde{U}^{(n)}(t_f, t_i)|\varphi_i\rangle \end{aligned}$$

First order:

$$\begin{aligned} S_{fi}^{(1)} &= \frac{1}{i\hbar}\int_{t_i}^{t_f} \mathrm{d}\tau_1\, V_{fi} e^{i(E_f - E_i)\tau_1/\hbar} \qquad && V_{fi} = \langle\varphi_f|V|\varphi_i\rangle \text{ time-independent} \\ &= \frac{V_{fi}}{i\hbar}\int_{-T/2}^{T/2} \mathrm{d}t\, e^{i(E_f - E_i)t/\hbar} && t_i = -T/2, t_f = +T/2 \\ &= -2\pi i\, V_{fi}\, \delta^{(T)}(E_f - E_i) \end{aligned}$$

Scattering Matrix

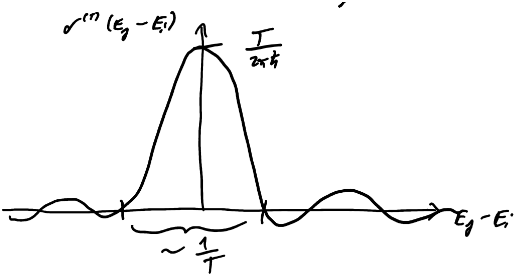
First order:

$$S_{fi}^{(1)} = -2\pi i\, V_{fi}\, \delta^{(T)}(E_f - E_i)$$

with

$$\delta^{(T)}(E_f - E_i) = \frac{1}{2\pi} \int_{-T/2}^{T/2} \frac{\mathrm{d}\tau_1}{\hbar}\, e^{i(E_f - E_i)\tau_1/\hbar}$$

$$= \frac{1}{\pi} \frac{\sin\left(\frac{E_f - E_i}{\hbar} \frac{T}{2}\right)}{E_f - E_i}$$



Scattering matrix:

$$S_{fi} = \langle \varphi_f | \tilde{U}(t_f, t_i) | \varphi_i \rangle$$
$$= \delta_{fi} + \sum_{n=1}^{\infty} S_{fi}^{(n)}$$
$$S_{fi}^{(n)} = \langle \varphi_f | \tilde{U}^{(n)}(t_n, t_i) | \varphi_i \rangle$$

First order

$$S_{fi}^{(1)} = \frac{1}{i\hbar} \int_{t_i}^{t_f} d\tau_1 \, V_{fi} \, e^{i(E_f - E_i)\tau_1/\hbar}$$

$V_{fi} = \langle \varphi_f | V | \varphi_i \rangle$
time-independent

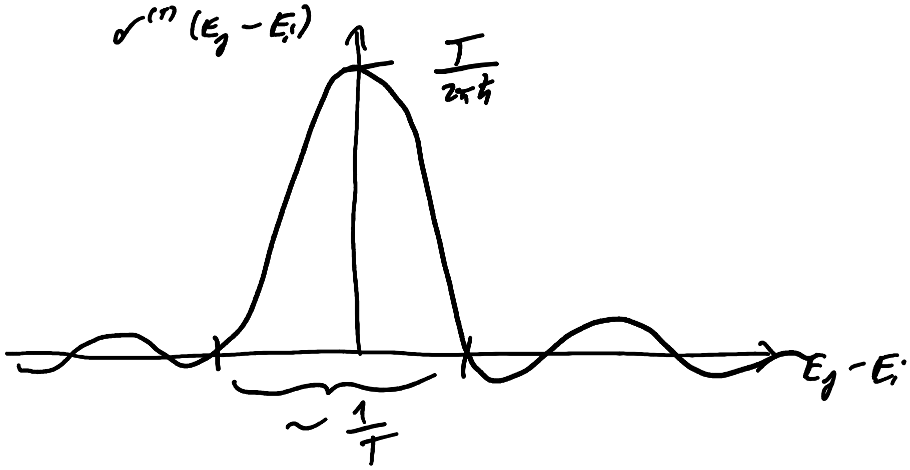
$$= \frac{V_{fi}}{i\hbar} \int_{-T/2}^{T/2} dt \, e^{i(E_f - E_i)t/\hbar}$$

$$S_{fi}^{(1)} = -2\pi i \, V_{fi} \, \delta^{(T)}(E_f - E_i)$$

with $\delta^{(T)}(E_f - E_i) = \frac{1}{2\pi} \int_{-T/2}^{T/2} \frac{d\tau_1}{\hbar} e^{i(E_f - E_i)\tau_1/\hbar}$

$$= \frac{1}{\pi} \frac{\sin (E_f - E_i)\frac{T}{2\hbar}}{E_f - E_i}$$

diffraction function



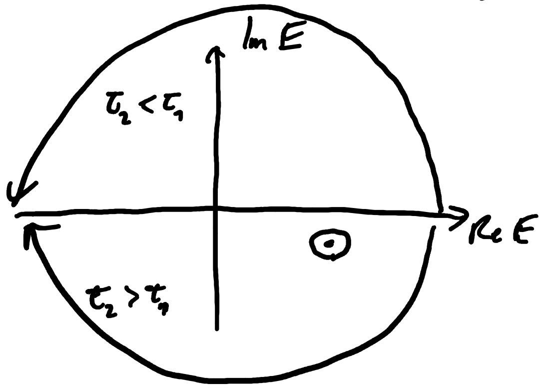
2^{nd} order:

$$S_{fi}^{(2)} = \left(\frac{1}{i\hbar}\right)^2 \int\limits_{\frac{T}{2} \geq \tau_2 \geq \tau_1 \geq -\frac{T}{2}} d\tau_1\, d\tau_2 \quad \sum_k V_{fk} V_{ki}\, e^{i(E_f - E_k)\tau_2/\hbar}\, e^{i(E_k - E_i)\tau_1/\hbar}$$

$\sum_k$ – sum over eigenstates of H_0

to get rid of restriction on time, use $\Theta(\tau_2 - \tau_1)$

$$e^{-iE_k(\tau_2 - \tau_1)/\hbar}\, \Theta(\tau_2 - \tau_1) = \lim_{\eta \to 0^+} -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-iE(\tau_2 - \tau_1)/\hbar}}{E + i\eta - E_k}\, dE$$



$E = E_k - i\eta$

If $\tau_2 < \tau_1$, close via $\mathrm{Im}\, E > 0$
$\rightarrow$ no residue

If $\tau_2 > \tau_1$, close via $\mathrm{Im}\, E < 0$
and pick up residue

$$S_{fi}^{(2)} = \left(\frac{1}{i\hbar}\right)^2 \left(-\frac{1}{2\pi i}\right) \int_{-T/2}^{T/2} d\tau_2 \int_{-T/2}^{T/2} d\tau_1 \int_{-\infty}^{\infty} dE\, e^{i(E_f - E)\tau_2/\hbar}\, e^{-i(E_i - E)\tau_1/\hbar}\, W_{fi}(E)$$

$$W_{fi}(E) = \lim_{\eta \to 0^+} \sum_k \frac{V_{fk} V_{ki}}{E + i\eta - E_k}$$

time integrals each yield $2\pi\hbar\, \delta^{(T)}(E_f - E)$ and $2\pi\hbar\, \delta^{(T)}(E_i - E)$

All E_i, E_f, E must be $\simeq$ equal to within $\frac{\hbar}{T}$.

We will assume $W_{fi}(E)$ varies only slightly with E over this range.

$$S_{fi}^{(2)} = -\frac{1}{2\pi i} \frac{4\pi^2 \hbar^2}{(i\hbar)^2} \left[\lim_{\eta \to 0} \sum_k \frac{V_{fk} V_{ki}}{E_i - E_k + i\eta}\right] \cdot \underbrace{\int dE\, \delta^{(T)}(E - E_i)\, \delta^{(T)}(E - E_f)}_{= \delta^{(T)}(E_i - E_f)}$$

Summarizing:

$$S_{fi} = \delta_{fi} - 2\pi i\, \delta^{(T)}(E_f - E_i)\left[V_{fi} + \lim_{\eta\to 0^+} \sum_k \frac{V_{fk} V_{ki}}{E_i - E_k + i\eta}\right] + \mathcal{O}(V^3)$$

$$S_{fi} = \delta_{fi} - 2\pi i\, \delta(E_f - E_i)\ \mathcal{T}_{fi}$$

$$\mathcal{T}_{fi} = \langle \varphi_f | V | \varphi_i \rangle + \langle \varphi_f | V \frac{1}{E_i - H + i\eta} V | \varphi_i \rangle$$

check: $\frac{1}{A} = \frac{1}{B} + \frac{1}{B}(B-A)\frac{1}{A}$

$$A = E_i - H + i\eta$$
$$B = E_i - H_0 + i\eta$$

$$\Rightarrow \frac{1}{E_i - H + i\eta} = \frac{1}{E_i - H_0 + i\eta} + \frac{1}{E_i - H_0 + i\eta} V \frac{1}{E_i - H + i\eta}$$

Transition Probabilities:

a) $|\varphi_f\rangle$ different $|\varphi_i\rangle$

$$P_{fi}(T) = |S_{fi}|^2 = 4\pi^2 \left[\delta^{(T)}(E_i - E_f)\right]^2 \times$$

$$\times \left| V_{fi} + \lim_{\eta\to 0^+} \sum_k \frac{V_{fk} V_{ki}}{E_i - E_k + i\eta} + \dots \right|^2$$

$$\left|\delta^{(T)}(E_f - E_i)\right|^2 = \frac{1}{\pi^2} \frac{\sin^2((E_f - E_i)T/2\hbar)}{(E_f - E_i)^2}$$

$$\int_{-\infty}^{\infty} dE_f \left(\delta^{(T)}(E_f - E_i)\right)^2 = \frac{T}{2\pi\hbar}$$

b) Two discrete states:

$$P_{fi}(T) = \frac{4 |V_{fi}|^2}{(E_f - E_i)^2} \sin^2\left(\frac{(E_f - E_i)T}{2\hbar}\right)$$

exact formula:

$$P_{fi}(T) = \frac{4 |V_{fi}|^2}{(E_f - E_i)^2 + 4 |V_{fi}|^2} \sin^2\left(\frac{T}{2\hbar} \sqrt{(E_f - E_i)^2 + 4|V_{fi}|^2}\right)$$

same! To lowest order in V_{fi}

$E_f = E_i$: $\quad P_{fi}(T) = |V_{fi}|^2 T^2 / \hbar^2$

grows like $T^2 \to$ beginning of Rabi oscillation

Atom - light interaction: Assume only discrete states:

$$|\varphi_i\rangle = |a, n_j\rangle$$
$$|\varphi_f\rangle = |b, n_j - 1\rangle$$

Neglect spontaneous emission

$$\langle\varphi_f|\hat{H}_{I_1}|\varphi_i\rangle = \frac{\hbar\Omega_1}{2}$$

$$\Omega_1 = -\frac{q}{m}\frac{2}{\sqrt{2\varepsilon_0\hbar\omega_j\cdot V}}\langle b|\hat{\vec{p}}\cdot\vec{\varepsilon}_j|a\rangle\sqrt{n_j}$$

$$P_{i\to f} = \frac{\Omega_1^2}{\Omega_1^2+\delta^2}\sin^2\left(\sqrt{\Omega_1^2+\delta^2}\,\frac{T}{2}\right)$$

detuning $\delta = \omega_j - \frac{E_b - E_a}{\hbar}$

Transition into a continuum:

$$dP(E_f, \beta, T) = \int_{\substack{E \in dE_f \\ \beta \in d\beta_f}} dE\, d\beta\, \rho(E,\beta)\, |\langle E,\beta|\tilde{U}(T)|\varphi_i\rangle|^2$$

$$= \int dE\, d\beta \quad 4\pi^2 |v(E,\beta;\varphi_i)|^2 \underbrace{\left[\delta^{(T)}(E-E_i)\right]^2}_{\approx \frac{T}{2\pi\hbar}\, \delta^{(T)}(E-E_i)}$$

$\Rightarrow$ transition probability per unit time:

$$dw(E_f, \beta_f) = \frac{1}{T}\, dP(E_f, \beta_f, T)$$

$$= \frac{2\pi}{\hbar} \int dE\, d\beta\, \rho(E,\beta)\, |v(E,\beta;\varphi_i)|^2\, \delta^{(T)}(E-E_i)$$

$$\boxed{\frac{dw}{d\beta} = \frac{2\pi}{\hbar} |v(E_f = E_i; \beta_f; \varphi_i)|^2\, \rho(E_f = E_i, \beta_f)}$$

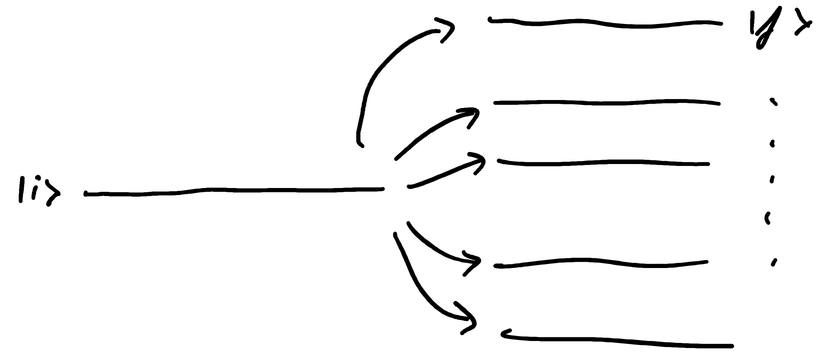
Fermi's Golden rule

Transition between continua:

divide $\frac{dw}{d\Omega}$ by incoming flux $\phi_i = \frac{c}{L^3} n_e$

$$\frac{d\Gamma}{d\Omega} = c\, \frac{n_e}{L^3}\, \frac{d\sigma}{d\Omega} = \phi_i\, \frac{d\sigma}{d\Omega}$$

$\frac{d\sigma}{d\Omega}$ – scattering cross section per unit angle



P_{fi}

A $\quad P_{fi} \sim T$?

B $\quad P_{fi} \sim T^2$?

C $\quad P_{fi} \sim$ more complicated ?

$$S_{fi} = \delta_{fi} - 2\pi i \, \delta^{(T)}(E_f - E_i) V_{fi} + \dots$$

$$\forall f \neq i: \; P_{fi} = |2\pi i \, \delta^{(T)}(E_f - E_i) \, V_{fi}|^2$$

$$= T \cdot \frac{2\pi}{\hbar} |V_{fi}|^2 \, \delta^{(T)}(E_f - E_i)$$

$$E_f \approx E_i: \quad P_{if} = T^2 \frac{|V_{fi}|^2}{\hbar^2}$$

Quadratic in T!
Beginning of Rabi oscillation

$P_{ii} = ?$

A $\quad P_{ii} = 1 - \# T$

B $\quad P_{ii} = 1 - \# T^2$

C $\quad$ more complicated

$$P_{ii} = 1 - \sum_{f \neq i} P_{fi}$$

$$= 1 - T \underbrace{\frac{2\pi}{\hbar} \sum_{f \neq i} |V_{fi}|^2 \, \delta^{(T)}(E_f - E_i)}_{\Gamma}$$

$$= 1 - \Gamma T$$ linear in T!

($= e^{-\Gamma T}$ for small T) beginning of exponential decay

$$S_{fi} = \delta_{fi} - 2\pi i\, \delta^{(T)}(E_i - E_f)\left[V_{fi} + \sum_k \frac{V_{ik} V_{kf}}{E_i - E_k + i\eta} \ldots \right]$$

$$P_{ii}(T) = \left| 1 - 2\pi i\, \delta^{(T)}(0) \left[\underset{\substack{\downarrow \\ 0 \text{ incorporate into } H_0}}{V_{ii}} + \sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} + \ldots \right] \right|^2$$

$$= \left| 1 - 2\pi i \frac{T}{2\pi\hbar} \sum_k \frac{|V_{ki}|^2}{E_i - E_k + i\eta} + \ldots \right|^2$$

$$= \underline{\left| 1 - i T \frac{1}{\hbar} \sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} + \ldots \right|^2}$$

Introduce states $|k\rangle$ as a continuum:

$$|V_{ik}|^2 \longrightarrow |V(E)|^2$$

$$\sum_k \longrightarrow \int dE \int d\beta\, \rho(E,\beta)$$

$$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} \longrightarrow \int dE\, d\beta\, \rho(E,\beta) \frac{|V(E)|^2}{E_i - E + i\eta}$$

Math excursion: $\displaystyle \frac{1}{x + i\eta} = \frac{1}{x + i\eta} \frac{x - i\eta}{x - i\eta} = \frac{x}{x^2 + \eta^2} - i \frac{\eta}{x^2 + \eta^2}$

$$\lim_{\eta\to 0} \int dx\, f(x) \frac{x}{x^2+\eta^2} \approx \lim_{\eta \to 0} \left(\int_{-\infty}^{-\eta} dx + \int_{\eta}^{\infty} dx \right) \frac{f(x)}{x} + \int_{-\eta}^{\eta} dx\, f(x) \frac{x}{x^2+\eta^2}$$

$$= P \int_{-\infty}^{\infty} dx \frac{f(x)}{x} + f(0) \underbrace{\int_{-\eta}^{\eta} dx \frac{x}{x^2+\eta^2}}_{0}$$

So $\displaystyle \lim_{\eta\to 0} \int dx \frac{x f(x)}{x^2+\eta^2} = P \int dx \frac{f(x)}{x}$

$$\lim_{\eta \to 0} \int dx\, f(x) \underbrace{\frac{\eta}{x^2+\eta^2}}_{\substack{\text{peaked at } x=0 \\ \text{height: } \frac{1}{\eta};\ \text{width} \sim \eta}} \approx f(0) \int_{-\infty}^{\infty} dx \frac{\eta}{x^2+\eta^2} = \pi f(0)$$

$$\Rightarrow \lim_{\eta \to 0} - \frac{i\eta}{x^2+\eta^2} = - i\pi\, \delta(x)$$

$$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} = \mathcal{P} \int dE\, d\beta\, \rho(E,\beta) \frac{|V(E,\beta)|^2}{E_i - E}$$

$$- i\pi \int dE\, d\beta\, \rho(E,\beta)\, |V(E,\beta)|^2\, \delta(E_i - E)$$

$$= \Delta - i\pi \int d\beta\, \rho(E_i,\beta)\, |V(E_i,\beta)|^2$$

$$= \Delta - i \frac{\hbar \Gamma}{2}$$

$$P_{ii} = \left| 1 - i T \frac{1}{\hbar} \left(\Delta - i \hbar \frac{\Gamma}{2} \right) \right|^2$$

$$= \left| 1 - T \frac{\Gamma}{2} - i T \frac{\Delta}{\hbar} \right|^2$$

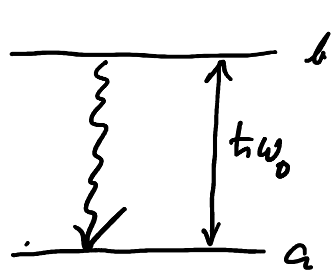
$$= 1 - \Gamma T + \text{higher order terms}$$

Non-perturbative

$$S_{ii} = \underbrace{e^{-\Gamma T/2}}_{\text{decay}} \; \underbrace{e^{-i \Delta T/\hbar}}_{\text{Level shift}}$$

$$P_{ii} = |S_{ii}|^2 = e^{-\Gamma T} \quad \text{exponential decay.}$$

Spontaneous Emission



Continuum of final states

$$\vec{k}_j = \frac{2\pi}{L}\left(n_x \vec{e}_x + n_y \vec{e}_y + n_z \vec{e}_z\right)$$

$$\omega_j = c k_j$$

Fermi's Golden Rule:

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar} \left| \langle a, \vec{k}\vec{\varepsilon} \,|\, H_I \,|\, b, 0\rangle \right|^2 \rho(\vartheta, \varphi, \hbar\omega = \hbar\omega_0)$$

$$\langle a, \vec{k}\vec{\varepsilon} \,|\, -\hat{\vec{d}}\cdot\hat{\vec{E}} \,|\, b, 0\rangle = \langle a|\hat{\vec{d}}|b\rangle \cdot \langle \vec{k}\vec{\varepsilon}\,|\, -i \sum_\ell \mathcal{E}_\ell \vec{\varepsilon}_\ell \left(a_\ell - a_\ell^\dagger\right)|0\rangle$$

$(\hbar c k = \hbar\omega_0)$

$$= i\, d\, \vec{e}_z \cdot \mathcal{E}_k \vec{\varepsilon}$$

$$= i \sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}}\; d \left(\vec{\varepsilon}\cdot\vec{e}_z\right)$$

density of states: $dN = \rho\, dE\, d\Omega$ $\qquad E = \hbar c k$

$$= \left(\frac{L}{2\pi}\right)^3 d^3k = \left(\frac{L}{2\pi}\right)^3 k^2\, dk\, d\Omega$$

$$= \frac{V}{(2\pi)^3}\frac{E^2}{(\hbar c)^3}\, dE\, d\Omega$$

$$\Rightarrow \quad \rho(\vartheta, \varphi, E) = \frac{V}{(2\pi)^3}\frac{E^2}{(\hbar c)^3}$$

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar}\,\frac{\hbar\omega_0}{2\varepsilon_0 \cancel{V}}\, d^2 \left(\vec{\varepsilon}\cdot\vec{e}_z\right)^2 \cdot \frac{\cancel{V}}{(2\pi)^3}\frac{(\hbar\omega_0)^2}{(\hbar c)^3}$$

$$= \frac{1}{8\pi^2\varepsilon_0}\,\frac{\omega_0^3}{\hbar c^3}\, d^2 \left(\vec{\varepsilon}\cdot\vec{e}_z\right)^2$$

Sum up over all directions ϑ, φ and all polarizations.

$\vec{\varepsilon}$, $\vec{\varepsilon}'$ and $\vec{k}/|\vec{k}|$ are ortho normal:

$$1 = \underbrace{(\vec{\varepsilon} \cdot \vec{e}_z)^2 + (\vec{\varepsilon}' \cdot \vec{e}_z)^2}_{1 - \cos^2\vartheta = \sin^2\vartheta} + \underbrace{\left(\frac{\vec{k}}{|\vec{k}|} \cdot \vec{e}_z\right)^2}_{\cos^2\vartheta}$$

$$\int d\Omega \sin^2\vartheta = 2\pi \int_{-1}^{1} d(\cos\vartheta) \sin^2\vartheta = 2\pi \int_{-1}^{1} du\,(1-u^2)$$

$$= 2\pi \left(2 - \frac{2}{3}\right) = \frac{8\pi}{3}$$

$$\boxed{\Gamma = \frac{d^2 \omega_0^3}{3\pi \varepsilon_0 \hbar c^3}}$$

$$d = q z_{ab} \quad \text{and} \quad \frac{q^2}{4\pi\varepsilon_0 \hbar c} = \alpha \approx \frac{1}{137}$$

$$\Gamma = \frac{4}{3} \alpha \frac{\omega_0^3 z_{ab}^2}{c^2}$$

What's the Q of the atomic oscillator?

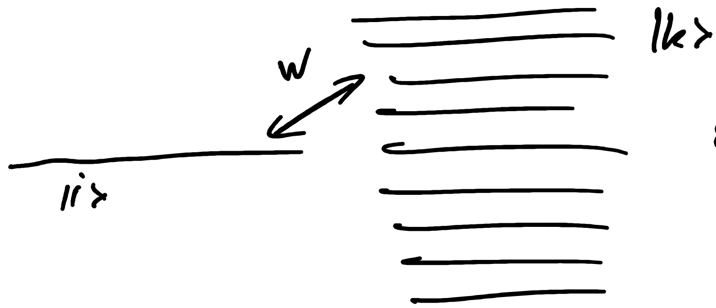
$$\frac{1}{Q} = \frac{\Gamma}{\omega_0} = \frac{4}{3} \alpha \underbrace{\frac{\omega_0^2 z_{ab}^2}{c^2}}_{\alpha^2} \qquad \hbar\omega_0 = \alpha^2 m c^2$$

$$z_{ab} = a_0 = \frac{\hbar^2}{me^2} = \frac{1}{\alpha} \frac{\hbar}{mc}$$

$$\frac{1}{Q} = \frac{\Gamma}{\omega_0} \approx \alpha^3$$

$$Q = \frac{1}{\alpha^3} \simeq \text{few } 10^6$$

Wigner – Weisskopf treatment of spont. emission



$$\mathcal{E}_k = k\,\varepsilon \quad \text{(say)}$$

Fermi's Golden Rule: $\Gamma = \frac{2\pi}{\hbar}|W|^2 \cdot \frac{1}{\varepsilon}$

Wave function: $|\psi(t)\rangle = \gamma_i(t)\,|i\rangle + \sum_{k=-\infty}^{\infty} \gamma_k(t)\, e^{-i\varepsilon k t/\hbar}\,|k\rangle$

$$\begin{cases} i\hbar \frac{d}{dt}\gamma_i(t) = W \sum\limits_{k=-\infty}^{k} \gamma_k(t)\, e^{-ik\varepsilon t/\hbar} \\ i\hbar \frac{d}{dt}\gamma_k(t) = W\, e^{ik\varepsilon t/\hbar}\,\gamma_i(t) \end{cases}$$

Initially $\gamma_k(0) = 0$ so

$$\gamma_k(t) = \frac{W}{i\hbar}\int_0^t dt'\, \gamma_i(t')\, e^{ik\varepsilon t'/\hbar}$$

$$\Rightarrow \quad \frac{d}{dt}\gamma_i(t) = -\frac{\Gamma}{2\pi\hbar}\int_0^t dt'\,\gamma_i(t') \underbrace{\left[\sum_{k=-\infty}^{\infty} \varepsilon\, e^{ik\varepsilon(t'-t)/\hbar}\right]}_{\simeq \int_{-\infty}^{\infty} dE\, e^{iE(t'-t)/\hbar} = 2\pi\hbar\,\delta(t'-t)}$$

$\tau = t' - t$

$$\frac{d}{dt}\gamma_i(t) = -\Gamma \int_{-t}^{0} d\tau\, \delta(\tau)\, \gamma_i(t+\tau)$$

$$= -\frac{\Gamma}{2}\,\gamma_i(t)$$

$$\Rightarrow \quad \gamma_i(t) = e^{-\frac{\Gamma}{2}t}$$

$$P_{ii}(t) = e^{-\Gamma t}$$