Physics 8.321, Fall 2021
Homework #4

Due **Friday, October 22** by 8:00 PM.

1. [Sakurai and Napolitano Problem 21, Chapter 1 (page 63)]

Evaluate the x-p uncertainty product $\langle(\Delta x)^2\rangle\langle(\Delta p)^2\rangle$ for a one-dimensional particle confined between rigid walls,

$$V = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{otherwise} \end{cases}$$

Do this for both the ground and excited states.

2. [Sakurai and Napolitano Problem 22, Chapter 1 (page 63)]

Estimate the rough order of magnitude of the length of time that an ice pick can be balanced on its point if the only limitation is that set by the Heisenberg uncertainty principle. Assume that the point is sharp and that the point and the surface on which it rests are hard. You may make approximations that do not alter the general order of magnitude of the result. Assume reasonable values for the dimensions and weight of the ice pick. Obtain an approximate numerical result and express it *in seconds*

3. Let $H = \frac{p^2}{2m} + V(x)$ be the Hamiltonian for a one-dimensional quantum system with discrete eigenstates $H|a\rangle = E_a|a\rangle$. Show the following results:

(a) $\sum_{a'} |\langle a|x|a'\rangle|^2(E_{a'} - E_a) = \frac{\hbar^2}{2m}$.

(b) $\langle a|p|a'\rangle = \frac{im}{\hbar}(E_a - E_{a'})\langle a|x|a'\rangle$
and hence $\sum_{a'} |\langle a|x|a'\rangle|^2(E_{a'} - E_a)^2 = \frac{\hbar^2}{m^2}\langle a|p^2|a\rangle$.

(c) Generalize to 3 dimensions and show the quantum virial theorem
$\langle a|\frac{p^2}{2m}|a\rangle = \frac{1}{2}\langle a|\mathbf{x} \cdot \nabla V(\mathbf{x})|a\rangle$.

4. A particle of mass m is in a 1D potential $V(x) = v\delta(x - a) + v\delta(x + a)$ where $v < 0$.

(a) Find the wave function for a bound state with even parity ($\psi(x) = \psi(-x)$).

(b) Find an expression for the energy for even parity states, and determine how many such states exist.

(c) Solve for the even parity bound state energy when $\frac{ma|v|}{\hbar^2} \ll 1$.

(d) Repeat parts (a) and (b) for odd parity ($\psi(x) = -\psi(-x)$). For what values of v are there bound states?

(e) Find the even and odd parity state binding energies for $\frac{ma|v|}{\hbar^2} \gg 1$, and explain physically why these energies move closer together as $a \to \infty$