Fermionic Algebra and Fock Space

Earlier in class we saw how harmonic-oscillator-like bosonic commutation relations

$$\left[\hat{a}_\alpha, \hat{a}_\beta\right] \;=\; 0, \quad \left[\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\right] \;=\; 0, \quad \left[\hat{a}_\alpha, \hat{a}^\dagger_\beta\right] \;=\; \delta_{\alpha,\beta} \tag{1}$$

give rise to the bosonic Fock space in which the oscillator modes α correspond to single-particle quantum states $|\alpha\rangle$. In this note, we shall see how the fermionic anti-commutation relations

$$\left\{\hat{a}_\alpha, \hat{a}_\beta\right\} \;=\; 0, \quad \left\{\hat{a}^\dagger_\alpha, \hat{a}^\dagger_\beta\right\} \;=\; 0, \quad \left\{\hat{a}_\alpha, \hat{a}^\dagger_\beta\right\} \;=\; \delta_{\alpha,\beta} \tag{2}$$

give rise to the fermionic Fock space. Again, the modes α will correspond to the single-particle quantum states. For simplicity, I will assume discrete modes — for example, momenta (and spins) of a free particle in a big but finite box.

Hilbert Space of a Single Fermionic Mode

A single bosonic mode is equivalent to a harmonic oscillator; the relation $[\hat{a}, \hat{a}^\dagger] = 1$ gives rise to an infinite-dimensional Hilbert space spanning states $|n\rangle$ for $n = 0, 1, 2, 3, \ldots, \infty$. A single fermionic mode is different — its Hilbert space spans just two states, $|0\rangle$ and $|1\rangle$. In accordance with the Fermi statistics, multiple quanta in the same mode are not allowed.

To see how this works, note that the fermionic creation / annihilation operators $\hat{a}^\dagger$ and $\hat{a}$ satisfy not just the anti-commutation relation

$$\hat{a}\hat{a}^\dagger \;+\; \hat{a}^\dagger\hat{a} \;=\; 1 \tag{3}$$

between them but also

$$\{\hat{a}, \hat{a}\} \;=\; \{\hat{a}^\dagger, \hat{a}^\dagger\} \;=\; 0 \quad \Longleftrightarrow \quad \hat{a}\hat{a} = \hat{a}^\dagger\hat{a}^\dagger = 0. \tag{4}$$

As usual, the number of quanta is measured by the hermitian operator $\hat{n} = \hat{a}^\dagger\hat{a}$. For the bosons we also had $\hat{a}\hat{a}^\dagger = \hat{n} + 1$ but for the fermions we now have $\hat{a}^\dagger\hat{a} = 1 - \hat{n}$.

Consequently, for the fermions

$$\hat{n}(1-\hat{n}) \;=\; \hat{a}^\dagger\hat{a}\hat{a}\hat{a}^\dagger \;=\; 0 \quad \text{because } \hat{a}\hat{a} \;=\; 0, \tag{5}$$

which means that all the eigenvalues of $\hat{n}$ must satisfy $n(1-n)=0$. Thus, the only allowed occupation numbers for the fermions are $n=0$ and $n=1$.

The algebra of the fermionic creation / annihilation operators closes in the two-dimensional Hilbert space spanning $|n=0\rangle$ and $|n=1\rangle$. Specifically,

$$\hat{a}\,|0\rangle \;=\; 0, \tag{6.a}$$
$$\hat{a}^\dagger\,|0\rangle \;=\; |1\rangle\,, \tag{6.b}$$
$$\hat{a}\,|1\rangle \;=\; |0\rangle\,, \tag{6.c}$$
$$\hat{a}^\dagger\,|1\rangle \;=\; 0. \tag{6.d}$$

To see how this works, we first notice that $\hat{a}(1-\hat{n})=\hat{a}\hat{a}\hat{a}^\dagger=0$ (because $\hat{a}\hat{a}=0$) and $\hat{a}^\dagger\hat{n}=\hat{a}^\dagger\hat{a}^\dagger\hat{a}=0$ (because $\hat{a}^\dagger\hat{a}^\dagger=0$). Also, by definition of the eigenstates $|0\rangle$ and $|1\rangle$ of $\hat{n}$, $\hat{n}\,|1\rangle=|1\rangle$ and $(1-\hat{n})\,|0\rangle=|0\rangle$. Consequently,

$$\hat{a}\,|0\rangle \;=\; \hat{a}(1-\hat{n})\,|0\rangle \;=\; 0, \tag{6.a}$$
$$\hat{a}^\dagger\,|1\rangle \;=\; \hat{a}^\dagger\hat{n}\,|1\rangle \;=\; 0. \tag{6.d}$$

Next, we check that $\hat{a}^\dagger\,|0\rangle$ and $\hat{a}\,|1\rangle$ are eigenstates of $\hat{n}$ with respective eigenvalues 1 and 0 as in eqs. (6,b–c):

$$\begin{aligned}(\hat{n}-1)\big(\hat{a}^\dagger\,|0\rangle\big) \;&=\; -\hat{a}\hat{a}^\dagger\hat{a}^\dagger\,|0\rangle \;=\; 0 \quad \text{because } \hat{a}^\dagger\hat{a}^\dagger=0,\\ (\hat{n}-0)\big(\hat{a}\,|1\rangle\big) \;&=\; \hat{a}^\dagger\hat{a}\hat{a}\,|1\rangle \;=\; 0 \quad \text{because } \hat{a}\hat{a}=0.\end{aligned} \tag{7}$$

This means that $\hat{a}^\dagger\,|0\rangle \propto$ some $|1\rangle$ and $\hat{a}\,|1\rangle \propto$ some $|0\rangle$, but we need to make sure that applying $\hat{a}$ to $\hat{a}^\dagger\,|0\rangle$ we get back to the same state $|0\rangle$ we stated from, and likewise applying $\hat{a}^\dagger$ to $\hat{a}\,|1\rangle$ brings us back to the original $|1\rangle$:

$$\begin{aligned} \hat{a}\big(|1\rangle = \hat{a}^\dagger\,|0\rangle\big) \;&=\; \hat{a}\hat{a}^\dagger\,|0\rangle \;=\; (1-\hat{n})\,|0\rangle \;=\; \text{same}\;\;|0\rangle\,, \\ \hat{a}^\dagger\big(|0\rangle = \hat{a}\,|1\rangle\big) \;&=\; \hat{a}^\dagger\hat{a}\,|1\rangle \;=\; \hat{n}\,|1\rangle \;=\; \text{same}\;\;|1\rangle\,. \end{aligned} \tag{8}$$

Finally, to make sure there are no numerical factors in eqs. (6,b–c) let's check the normalization: if $|1\rangle = \hat{a}^\dagger\,|0\rangle$ then $\langle 1|1\rangle = \langle 0|\,\hat{a}\hat{a}^\dagger\,|0\rangle = \langle 0|\,(1-\hat{n})\,|0\rangle = 1\times\langle 0,0\rangle$ and likewise, if $|0\rangle = \hat{a}\,|1\rangle$ then $\langle 0,0\rangle = \big\langle \hat{a}^\dagger\big|\,\hat{a}\,|1\rangle = \langle 1|\,\hat{n}\,|1\rangle = 1\times\langle 1|1\rangle$. In other words, both eqs. (6,b–c) as written are consistent with normalized states $\langle 0|0\rangle = \langle 1|1\rangle = 1$.

Multiple Fermionic Modes

Now consider multiple fermionic creation and annihilation operators $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$ that are hermitian conjugates of each other and satisfy the anti-commutation relations (2). For each mode α we define the occupation number operator

$$\hat{n}_\alpha \;\stackrel{\text{def}}{=}\; \hat{a}^\dagger_\alpha\hat{a}_\alpha\,. \tag{9}$$

All these operators commute with each other; moreover, each $\hat{n}_\alpha$ commutes with creation and annihilation operators for all the other modes $\beta\neq\alpha$. Indeed, using the Leibniz rules for commutators and anti-commutators

$$\begin{aligned} [A,BC] \;&=\; [A,B]C \;+\; B[A,C] \;=\; \{A,B\}C \;-\; B\{A,C\}, \\ [AB,C] \;&=\; A[B,C] \;+\; [A,C]B \;=\; A\{B,C\} \;-\; \{A,C\}B, \\ \{A,BC\} \;&=\; [A,B]C \;+\; B\{A,C\} \;=\; \{A,B\}C \;-\; B[A,C], \\ \{AB,C\} \;&=\; A[B,C] \;+\; \{A,C\}B \;=\; A\{B,C\} \;-\; [A,C]B, \end{aligned} \tag{10}$$

we obtain

$$\begin{aligned} \big[\hat{n}_\alpha,\hat{a}_\beta\big] \;&=\; \big[\hat{a}^\dagger_\alpha\hat{a}_\alpha,\hat{a}_\beta\big] \;=\; \hat{a}^\dagger_\alpha\big\{\hat{a}_\alpha,\hat{a}_\beta\big\} \;-\; \big\{\hat{a}^\dagger_\alpha,\hat{a}_\beta\big\}\hat{a}_\alpha \;=\; \hat{a}^\dagger_\alpha\times 0 \;-\; \delta_{\alpha\beta}\times\hat{a}_\alpha \\ &=\; -\delta_{\alpha\beta}\times\hat{a}_\beta \;\to\; 0 \quad \text{for } \beta\neq\alpha, \end{aligned} \tag{11}$$

$$\begin{aligned} \big[\hat{n}_\alpha,\hat{a}^\dagger_\beta\big] \;&=\; \big[\hat{a}^\dagger_\alpha\hat{a}_\alpha,\hat{a}^\dagger_\beta\big] \;=\; \hat{a}^\dagger_\alpha\big\{\hat{a}_\alpha,\hat{a}^\dagger_\beta\big\} \;-\; \big\{\hat{a}^\dagger_\alpha,\hat{a}^\dagger_\beta\big\}\hat{a}_\alpha \;=\; \hat{a}^\dagger_\alpha\times\delta_{\alpha\beta} \;-\; 0\times\hat{a}_\alpha \\ &=\; +\delta_{\alpha\beta}\times\hat{a}_\beta \;\to\; 0 \quad \text{for } \beta\neq\alpha, \end{aligned} \tag{12}$$

$$
\begin{aligned}
\big[\hat{n}_\alpha, \hat{a}^\dagger_\beta \hat{a}_\gamma\big] \;&=\; \hat{a}^\dagger_\beta \big[\hat{n}_\alpha, \hat{a}_\gamma\big] \;+\; \big[\hat{n}_\alpha, \hat{a}^\dagger_\beta\big]\hat{a}_\gamma \;=\; -\hat{a}^\dagger_\beta \times \delta_{\alpha\gamma}\hat{a}_\gamma \;+\; \delta_{\alpha\beta}\hat{a}^\dagger_\beta \times \hat{a}_\gamma \\
&=\; \big(\delta_{\alpha\beta} - \delta_{\alpha\gamma}\big)\hat{a}^\dagger_\beta \hat{a}_\gamma \;\to\; 0 \quad \text{for } \beta = \gamma, \qquad (13) \\
\big[\hat{n}_\alpha, \hat{n}_\beta\big] \;&=\; \big[\hat{n}_\alpha, \hat{a}^\dagger_\beta \hat{a}_\beta \text{ (for same } \beta)\big] \;=\; 0. \qquad (14)
\end{aligned}
$$

The fact that all the $\hat{n}_\alpha$ commute with each other allows us to diagonalize all of them at once. This gives us the occupation-number basis of states $|\text{set of all } n_\alpha\rangle$ for the whole Hilbert space of the theory. Similar to the bosonic case, we may use $\hat{a}^\dagger_\alpha$ and $\hat{a}_\alpha$ operators to raise or lower any particular n_α without changing the other occupation numbers n_β; this means that all the occupation numbers may take any allowed values independently from each other. However, the only allowed values of the fermionic occupation numbers are 0 and 1 — multiple quanta in the same mode are not allowed.

Note that for a finite set of M modes the fermionic Hilbert space has a finite dimension 2^M. This fact is important for understanding the ground state degeneracies of fermionic fields in some non-trivial backgrounds that have zero-energy fermionic modes: For M zero modes independent from their hermitian conjugates, the ground level of the whole QFT has 2^M degenerate states.

Fermionic Fock Space

Now suppose there is an infinite but discrete set of fermionic modes α corresponding to some 1–particle quantum states $|\alpha\rangle$ with wave functions $\phi_\alpha(\mathbf{x})$. (By abuse of notations, I am including the spin and the other non-spatial quantum numbers into $\mathbf{x} = (x, y, z, \text{spin}, \textit{etc.})$.) In this case, the fermionic Hilbert space

$$
\mathcal{F} \;=\; \bigotimes_\alpha \mathcal{H}_{\text{mode}\,\alpha}\, (\text{spanning } \; |n_\alpha = 0\rangle \; \text{ and } \; |n_\alpha = 1\rangle) \qquad (15)
$$

has infinite dimension and we may interpret is as a *Fock space* or arbitrary number of identical fermions. Indeed, let

$$
\hat{N} \;=\; \sum_\alpha \hat{n}_\alpha \qquad (16)
$$

count the net number of fermionic quanta in all the modes, $N = 0, 1, 2, 3 \ldots, \infty$. Let's reorganize $\mathcal{F}$ into the eigenblocks of $\hat{N}$:

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_N = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \cdots. \tag{17}$$

The $\mathcal{H}_0$ block here spans a unique state with $N = 0$, namely the vacuum state $|\text{vac}\rangle = |\text{all } n_\alpha = 0\rangle$. The $\mathcal{H}_1$ block spans states with a single $n_\alpha = 1$ while all the other $n_\beta = 0$. Similar to the bosonic case, we may identify such states $|n_\alpha = 1; \text{other } n = 0\rangle = \hat{a}^\dagger_\alpha |\text{vac}\rangle$ with the single-particle states $|\alpha\rangle$ and hence the $\mathcal{H}_1$ block of $\mathcal{F}$ with the Hilbert space of a single particle.

The $\mathcal{H}_2$ block of $\mathcal{F}$ spans states

$$\Big|n_\alpha = n_\beta = 1; \text{other } n = 0\Big\rangle = \hat{a}^\dagger_\alpha \hat{a}^\dagger_\beta |\text{vac}\rangle \tag{18}$$

with $\alpha \neq \beta$ and only such states — the fermionic Fock space does not allow states $|n_\alpha = 2; \text{other } n = 0\rangle$ with doubly occupied modes. Note that in eq. (18) exchanging $\alpha \leftrightarrow \beta$ results in the same physical state but with an opposite sign (because $\hat{a}^\dagger_\alpha$ and $\hat{a}^\dagger_\beta$ anti-commute). To be precise, we define

$$|\alpha, \beta\rangle \stackrel{\text{def}}{=} \hat{a}^\dagger_\beta \hat{a}^\dagger_\alpha |\text{vac}\rangle = -|\beta, \alpha\rangle. \tag{19}$$

Likewise, the $\mathcal{H}_3$ block spans states

$$|\alpha, \beta, \gamma\rangle = \hat{a}^\dagger_\gamma \hat{a}^\dagger_\beta \hat{a}^\dagger_\alpha |\text{vac}\rangle \tag{20}$$

for 3 *different* modes α, β, γ, the $\mathcal{H}_4$ block spans states

$$|\alpha, \beta, \gamma, \delta\rangle = \hat{a}^\dagger_\delta \hat{a}^\dagger_\gamma \hat{a}^\dagger_\beta \hat{a}^\dagger_\alpha |\text{vac}\rangle \tag{21}$$

for 4 *different* modes $\alpha, \beta, \gamma, \delta$, *etc.*, *etc.* In all cases, the order of the modes $\alpha, \beta, \gamma, \ldots$ does not matter physically but affects the overall sign of the state,

$$|\text{any permutation of } \alpha, \beta, \ldots, \omega\rangle = |\alpha, \beta, \ldots, \omega\rangle \times (-1)^{\text{parity of the permutation}}. \tag{22}$$

Thus, each $\mathcal{H}_N$ (for $N \geq 2$) is a Hilbert space of N identical Fermions.

A system of two identical fermions has an antisymmetric wavefunction of two arguments, $\psi(\mathbf{x}_1, \mathbf{x}_2) = -\psi(\mathbf{x}_2, \mathbf{x}_1)$. A complete basis for such wavefunctions comprises antisymmetric tensor products or single-particle wave-functions

$$\phi_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2) \;=\; \frac{\phi_\alpha(\mathbf{x}_1)\phi_\beta(\mathbf{x}_2) \,-\, \phi_\beta(\mathbf{x}_1)\phi_\alpha(\mathbf{x}_2)}{\sqrt{2}} \;=\; -\phi_{\alpha\beta}(\mathbf{x}_2, \mathbf{x}_1). \tag{23}$$

Note that such wave functions are not only antisymmetric in $\mathbf{x}_1 \leftrightarrow \mathbf{x}_2$ but also separately antisymmetric in $\alpha \leftrightarrow \beta$, $\phi_{\beta\alpha}(\mathbf{x}_1, \mathbf{x}_2) = -\phi_{\alpha\beta}(\mathbf{x}_1, \mathbf{x}_2)$, so we may identify them as wave functions of two-fermions states $|\alpha, \beta\rangle = \hat{a}^\dagger_\beta \hat{a}^\dagger_\alpha |\mathrm{vac}\rangle = -|\beta, \alpha\rangle \in \mathcal{H}_2$.

Likewise, a wavefunction of N identical fermions is totally antisymmetric in its N arguments,

$$\psi(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \;=\; \psi(\text{any permutation of } \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N) \times (-1)^{\text{parity of permutation}}. \tag{24}$$

A complete basis for such wavefunctions obtains from totally antisymmetrized products of N different single-particle wave-functions

$$\begin{aligned}
\phi_{\alpha_1,\ldots,\alpha_N}(\mathbf{x}_1, \ldots, \mathbf{x}_N) \;&=\; \frac{1}{\sqrt{N!}} \sum_{(\tilde{\alpha}_1,\ldots,\tilde{\alpha}_N)}^{\substack{\text{all permutations} \\ \text{of } (\alpha_1,\ldots,\alpha_N)}} \phi_{\tilde{\alpha}_1}(\mathbf{x}_1) \times \cdots \times \phi_{\tilde{\alpha}_N}(\mathbf{x}_N) \times (-1)^{\text{parity}} \\
&=\; \frac{1}{\sqrt{N!}} \det \begin{vmatrix}
\phi_{\alpha_1}(\mathbf{x}_1) & \phi_{\alpha_2}(\mathbf{x}_1) & \cdots & \phi_{\alpha_N}(\mathbf{x}_1) \\
\phi_{\alpha_1}(\mathbf{x}_2) & \phi_{\alpha_2}(\mathbf{x}_2) & \cdots & \phi_{\alpha_N}(\mathbf{x}_2) \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{\alpha_1}(\mathbf{x}_N) & \phi_{\alpha_2}(\mathbf{x}_N) & \cdots & \phi_{\alpha_N}(\mathbf{x}_N)
\end{vmatrix}.
\end{aligned} \tag{25}$$

The *Slater's determinant* here is not only antisymmetric in $(\mathbf{x}_1, \ldots, \mathbf{x}_N)$ but also antisymmetric with respect to the single-particle states $(\alpha_1, \ldots, \alpha_N)$, so we may identify it as a wave-function of the N-fermion state

$$|\alpha_1, \alpha_2, \ldots, \alpha_N\rangle \;=\; \hat{a}^\dagger_{\alpha_N} \cdots \hat{a}^\dagger_{\alpha_2} \hat{a}^\dagger_{\alpha_1} |\mathrm{vac}\rangle \in \mathcal{N}_N\,. \tag{26}$$

To complete the wave-function picture of the Fermionic Fock space, let me spell out the action of the creation operators $\hat{a}^\dagger_\alpha$ and the annihilation operators $\hat{a}_\alpha$. For any

N-fermions state $|N;\psi\rangle$ with a totally-antisymmetric wave function $\psi(\mathbf{x}_1,\ldots,\mathbf{x}_N)$, the state $|N+1;\psi'\rangle = \hat{a}^\dagger_\alpha\,|N;\psi\rangle$ has a totally antisymmetric function of $N+1$ variables

$$\psi'(\mathbf{x}_1,\ldots,\mathbf{x}_{N+1}) \;=\; \frac{1}{\sqrt{N+1}}\sum_{i=1}^{N+1}(-1)^{N+1-i}\phi_\alpha(\mathbf{x}_i)\times\psi(\mathbf{x}_1,\ldots,\cancel{\mathbf{x}_i},\ldots,\mathbf{x}_{N+1}) \tag{27}$$

while the state $|N-1,\psi''\rangle = \hat{a}_\alpha\,|N,\psi\rangle$ has a totally antisymmetric function of $N-1$ variables

$$\psi''(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}) \;=\; \sqrt{N}\int d^3\mathbf{x}_N\,\phi^*_\alpha(\mathbf{x}_N)\times\psi(\mathbf{x}_1,\ldots,\mathbf{x}_{N-1},\mathbf{x}_N). \tag{28}$$

The proof of these formulae is left out as an optional exercise to the students.

Thanks to the relations (28) and (27), the Fock-space formulae for the additive one-body operators work similarly to the bosonic case: If in N-fermion Hilbert spaces

$$\hat{A}_{\rm tot} \;=\; \sum_{i=1}^{N}\hat{A}_1(i^{\rm th}) \tag{29}$$

where each $\hat{A}_1(i^{\rm th})$ acts only on the $i^{\rm th}$ particle, then in the Fock space

$$\hat{A}_{\rm tot} \;=\; \sum_{\alpha,\beta}\langle\alpha|\,\hat{A}_1\,|\beta\rangle\times\hat{a}^\dagger_\alpha\hat{a}_\beta\,. \tag{30}$$

For example, for free non-relativistic electrons in a box with $\alpha=(\mathbf{p},s)$ we have

$$\begin{aligned}
\hat{H}_{\rm tot} \;&=\; \sum_{\mathbf{p},s}\frac{\mathbf{p}^2}{2m}\times\hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s}\,,\\
\hat{\mathbf{P}}_{\rm tot} \;&=\; \sum_{\mathbf{p},s}\mathbf{p}\times\hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s}\,,\\
\hat{\mathbf{S}}_{\rm tot} \;&=\; \sum_{\mathbf{p},s,s'}\left\langle\tfrac12,s'\right|\hat{\mathbf{S}}\left|\tfrac12,s\right\rangle\times\hat{a}^\dagger_{\mathbf{p},s'}\hat{a}_{\mathbf{p},s}\,.
\end{aligned} \tag{31}$$

Likewise, the two-body additive operators that act in N-fermion spaces as

$$\hat{B}_{\text{tot}} \;=\; \tfrac{1}{2}\sum_{i\neq j}\hat{B}_2(i^{\text{th}},j^{\text{th}}) \tag{32}$$

in the Fock space become

$$\begin{aligned} \hat{B}_{\text{tot}} \;&=\; \tfrac{1}{2}\sum_{\alpha,\beta,\gamma,\delta} B_{\alpha,\beta,\gamma,\delta}\times\hat{a}_\alpha^\dagger\hat{a}_\beta^\dagger\hat{a}_\delta\hat{a}_\gamma \quad \langle\!\langle\,\text{note the order!}\,\rangle\!\rangle \\ \text{where } B_{\alpha,\beta,\gamma,\delta} \;&=\; \left(\langle\alpha|\otimes\langle\beta|\right)\hat{B}_2\left(|\gamma\rangle\otimes|\delta\rangle\right). \end{aligned} \tag{33}$$

For example, a spin-blind potential $V_2(\mathbf{x}_1-\mathbf{x}_2)$ becomes

$$\begin{aligned} \hat{V}_{\text{tot}} \;&=\; \tfrac{1}{2}\sum_{i\neq j}V_s(\mathbf{x}_i-\mathbf{x}_j) \\ &=\; \frac{1}{2L^3}\sum_{\mathbf{q}}W(\mathbf{q})\sum_{\mathbf{p}_1,\mathbf{p}_2}\sum_{s_1,s_2}\hat{a}^\dagger_{\mathbf{p}_1+\mathbf{q},s_1}\hat{a}^\dagger_{\mathbf{p}_2-\mathbf{q},s_2}\hat{a}_{\mathbf{p}_2,s_2}\hat{a}_{\mathbf{p}_1,s_1} \\ \text{where } W(\mathbf{q}) \;&=\; \int\! d^3\mathbf{x}\, V_2(\mathbf{x})e^{-i\mathbf{q}\mathbf{x}}. \end{aligned} \tag{34}$$

Note that while the formulae for this operator in the bosonic and the fermionic Fock spaces have similar forms, the actual operators are quite different because the two Fock spaces have different algebras of the creation and annihilation operators and different quantum states (symmetric vs. antisymmetric). Thus, the physical effect of similar $V_2(\mathbf{x}_1-\mathbf{x}_2)$ potentials for the fermions and for the bosons may be quite different from each other.

Fermionic Particles and Holes

Consider a system of fermions with a one-body Hamiltonian of the form

$$\hat{H} \;=\; \sum_\alpha \mathcal{E}_\alpha\hat{a}_\alpha^\dagger\hat{a}_\alpha \;+\; E_0\,. \tag{35}$$

When **all** particle energies $\mathcal{E}_\alpha$ are positive, the ground state of the system is the vacuum state $|\text{vac}\rangle$ with all $n_\alpha=0$. In terms of the creation and annihilation operators, $|\text{vac}\rangle$ can be identified as the unique state killed by all the annihilation operators, $\hat{a}_\alpha\,|\text{vac}\rangle = 0\ \forall\alpha$. The excited states of the Hamiltonian (35) are N-particle states which obtain by applying creation operators to the vacuum, $|\alpha_1,\ldots,\alpha_N\rangle = \hat{a}^\dagger_{\alpha_N}\cdots\hat{a}^\dagger_{\alpha_1}|\mathrm{vac}\rangle$; the energy of such a state is $E = E_0 + \mathcal{E}_{\alpha_a} + \cdots \mathcal{E}_{\alpha_N} > E_0$.

Now suppose for a moment that all the particle energies $\mathcal{E}_\alpha$ are negative instead of positive. In this case, adding particles decreases the energy, so the ground state of the system is not the vacuum but rather the full-to-capacity state

$$|\mathrm{full}\rangle \;=\; |\mathrm{all}\ n_\alpha = 1\rangle \;=\; \prod_{\mathrm{all}\,\alpha} \hat{a}^\dagger_\alpha\,|\mathrm{vac}\rangle \tag{36}$$

with energy

$$E_{\mathrm{full}} \;=\; E_0 \;+\sum_{\mathrm{all}\,\alpha}\mathcal{E}_\alpha\,. \tag{37}$$

Never mind whether the sum here is convergent; if it is not, we may add an infinite constant to the E_0 to cancel the divergence. What's important to us here are the energy difference between this ground state and the excited states.

The excited states of the system are not completely full but have a few *holes*. That is, $n_{\alpha_1} = \cdots = n_{\alpha_N} = 0$ for some N modes $(\alpha_1,\ldots,\alpha_N)$ while all the other $n_\beta = 1$. The energy of such a state is

$$E \;=\; E_0 \;+\sum_{\beta\neq\alpha_a,\ldots,\alpha_N}\mathcal{E}_\beta \;=\; E_{\mathrm{full}} \;-\sum_{i=1}^{N}\mathcal{E}_{\alpha_i} \;>\; E_{\mathrm{full}}\,. \tag{38}$$

In other words, an un-filled hole in mode α carries a positive energy $-\mathcal{E}_\alpha$.

In terms of the operator algebra, the $|\mathrm{full}\rangle$ state is the unique state killed by all the creation operators, $\hat{a}^\dagger_\alpha|\mathrm{full}\rangle = 0\ \forall\alpha$. The holes can be obtained by acting on the $|\mathrm{full}\rangle$ state with the annihilation operators that remove one particle at a time. Thus,

$$|1\,\mathrm{hole\,at}\,\alpha\rangle \;=\; \Big|\hat{n}_\alpha = 0;\,\mathrm{other}\,n = 1\Big\rangle \;=\; \hat{a}_\alpha\,|\mathrm{full}\rangle \tag{39}$$

and likewise

$$|N\,\mathrm{holes\,at}\,\alpha_1,\ldots,\alpha_{\mathrm{N}}\rangle = \;\hat{a}_{\alpha_N}\cdots\hat{a}_{\alpha_1}\,|\mathrm{full}\rangle\,. \tag{40}$$

Altogether, when the ground state is $|\mathrm{full}\rangle$, the creation and the annihilation operators exchange their roles. Indeed, the $\hat{a}_\alpha$ make extra holes in the full or almost-full states while the $\hat{a}_\alpha^\dagger$ operators annihilates those holes (by filling them up). Also, the algebraic definition of the $|\text{full}\rangle$ and $|\text{vac}\rangle$ states are related by this exchange: $\hat{a}_\alpha\,|\text{vac}\rangle = 0\ \forall\alpha$ vs. $\hat{a}_\alpha^\dagger\,|\text{full}\rangle = 0\ \forall\alpha$.

To make this exchange manifest, let us define a new family of fermionic creation and annihilation operators,

$$\hat{b}_\alpha \;=\; \hat{a}_\alpha^\dagger\,, \quad \hat{b}_\alpha^\dagger \;=\; \hat{a}_\alpha\,. \tag{41}$$

Unlike the bosonic commutation relations (1), the fermionic anti-commutation relations (2) are symmetric between $\hat{a}$ and $\hat{a}^\dagger$, so the $\hat{b}_\alpha$ and $\hat{b}_\alpha^\dagger$ satisfy exactly the same anti-commutation relations as the $\hat{a}_\alpha$ and $\hat{a}_\alpha^\dagger$,

$$\begin{pmatrix} \left\{\hat{a}_\alpha, \hat{a}_\beta\right\} \;=\; 0 \\ \left\{\hat{a}_\alpha^\dagger, \hat{a}_\beta^\dagger\right\} \;=\; 0 \\ \left\{\hat{a}_\alpha, \hat{a}_\beta^\dagger\right\} \;=\; \delta_{\alpha\beta} \end{pmatrix} \iff \begin{pmatrix} \left\{\hat{b}_\alpha, \hat{b}_\beta\right\} \;=\; 0 \\ \left\{\hat{b}_\alpha^\dagger, \hat{b}_\beta^\dagger\right\} \;=\; 0 \\ \left\{\hat{b}_\alpha, \hat{b}_\beta^\dagger\right\} \;=\; \delta_{\alpha\beta} \end{pmatrix}. \tag{42}$$

Physically, the $\hat{b}_\alpha^\dagger$ operators *create holes* while the $\hat{b}_\alpha$ operators *annihilate holes*, and the holes obey exactly the same Fermi statistics as the original particles. In condensed-matter terminology, the holes are *quasi-particles*, but the only distinction between the quasi-particles and the true particles is that the later may exist outside the condensed matter. When viewed from the inside of condensed matter, this distinction becomes irrelevant.

Anyhow, from the hole point of view, the $|\text{full}\rangle$ state is the *hole vacuum* — the unique state with no holes at all, algebraically defined by $\hat{b}_\alpha\,|\text{full}\rangle = 0\ \forall\alpha$. The excitations are N-hole states obtained by acting with hole-creation operators $\hat{b}_\alpha^\dagger$ on the hole-vacuum, $|\text{holes at}\,\alpha_1,\ldots,\alpha_{\text{N}}\rangle \;=\; \hat{b}_{\alpha_N}^\dagger\cdots\hat{b}_{\alpha_1}^\dagger\,|\text{full}\rangle$. And the Hamiltonian operator (35) of the system becomes

$$\begin{aligned} \hat{H} \;&=\; E_0 \;+\; \sum_\alpha \mathcal{E}_\alpha \left(\hat{a}_\alpha^\dagger\hat{a}_\alpha \;=\; \hat{b}_\alpha\hat{b}_\alpha^\dagger \;=\; 1 - \hat{b}_\alpha^\dagger\hat{b}_\alpha\right) \\ &=\; E_{\text{full}} \;+\; \sum_\alpha (-\mathcal{E}_\alpha)\hat{b}_\alpha^\dagger\hat{b}_\alpha\,, \end{aligned} \tag{43}$$

in accordance with individual holes having positive energies $-\mathcal{E}_\alpha > 0$.

Thus far, I made no assumptions about the one-particle states corresponding to modes α. Quite often, they are eigenstates of some conserved quantum numbers such as momentum or spin (or rather $\hat{S}_z$). When one makes a hole by removing a particle from mode $(\mathbf{p}, s)$, the net momentum of the system changes by $-\mathbf{p}$ while the net S_z changes by $-s$, so one can say that the hole in that mode has momentum $-\mathbf{p}$ and $S_z = -s$. Consequently, the hole operators are usually defined as

$$\hat{b}_{\mathbf{p},s} \;=\; \hat{a}^{\dagger}_{-\mathbf{p},-s}\,, \quad \hat{b}^{\dagger}_{\mathbf{p},s} \;=\; \hat{a}_{-\mathbf{p},-s}\,, \tag{44}$$

which leads to

$$\hat{\mathbf{P}}_{\text{tot}} \;=\; \mathbf{P}_{\text{full}} \;+\; \sum_{\mathbf{p},s} \mathbf{p} \times \hat{b}^{\dagger}_{\mathbf{p},s}\hat{b}_{\mathbf{p},s} \tag{45}$$

and likewise

$$\hat{S}^z_{\text{tot}} \;=\; S^z_{\text{full}} \;+\; \sum_{\mathbf{p},s} s \times \hat{b}^{\dagger}_{\mathbf{p},s}\hat{b}_{\mathbf{p},s}\,. \tag{46}$$

Finally, consider a system where the energies $\mathcal{E}_\alpha$ take both signs: positive for some modes α but negative for other modes. For example, a free fermion gas with a positive chemical potential μ and free-energy operator

$$\hat{H} \;=\; \sum_{\mathbf{p},s} \left(\frac{\mathbf{p}^2}{2m} - \mu\right) \hat{a}^{\dagger}_{\mathbf{p},s}\hat{a}_{\mathbf{p},s}\,. \tag{47}$$

has positive $\mathcal{E}$ for $|\mathbf{p}| > p_f$ but negative $\mathcal{E}$ for $|\mathbf{p}| < p_f$ where p_f is the Fermi momentum defined by the threshold $(p_f^2/2m) - \mu = 0$. For this system the ground state is the *Fermi sea* where

$$n_{\mathbf{p},s} \;=\; \Theta(|\mathbf{p}| < p_F) \;=\; \begin{cases} 1 & \text{for } |\mathbf{p}| < p_f, \\ 0 & \text{for } |\mathbf{p}| > p_f. \end{cases} \tag{48}$$

In terms of the creation and annihilation operators, the Fermi sea is the state

$$|\text{FS}\rangle \;=\; \prod_{\mathbf{p},s}^{|\mathbf{p}|<p_f \text{ only}} \hat{a}^{\dagger}_{\mathbf{p},s}\,|\text{vac}\rangle \tag{49}$$

which satisfies

$$\hat{a}_{\mathbf{p},s}\,|\mathrm{FS}\rangle \;=\; 0 \text{ for } |\mathbf{p}|>p_f \quad\text{and}\quad \hat{a}^\dagger_{\mathbf{p},s}\,|\mathrm{FS}\rangle \;=\; 0 \text{ for } |\mathbf{p}|<p_f\,. \tag{50}$$

We may treat this state as a quasi-particle vacuum if we redefine all the operators killing the $|\mathrm{FS}\rangle$ as annihilation operators. Thus, we define

$$\hat{b}_{\mathbf{p},s} \;=\; \hat{a}^\dagger_{-\mathbf{p},-s}\,, \quad \hat{b}^\dagger_{\mathbf{p},s} \;=\; \hat{a}_{-\mathbf{p},-s} \quad \textbf{for } |\mathbf{p}|<p_F \textbf{ only} \tag{51}$$

but *keep the original* $\hat{a}_{\mathbf{p},s}$ *and* $\hat{a}^\dagger_{\mathbf{p},s}$ *operators for momenta outside the Fermi surface.* Despite the partial exchange, the complete set or creation and annihilation operators satisfies the fermionic anticommutation relations:

$$\begin{aligned} \textbf{all } \{\hat{a},\hat{a}\} \;&=\; \{\hat{b},\hat{b}\} \;=\; \{\hat{a},\hat{b}\} \;=\; 0,\\ \textbf{all } \{\hat{a}^\dagger,\hat{a}^\dagger\} \;&=\; \{\hat{b}^\dagger,\hat{b}^\dagger\} \;=\; \{\hat{a}^\dagger,\hat{b}^\dagger\} \;=\; 0,\\ \textbf{all } \{\hat{a},\hat{b}^\dagger\} \;&=\; \{\hat{b}^\dagger,\hat{a}\} \;=\; 0, \end{aligned} \tag{52}$$

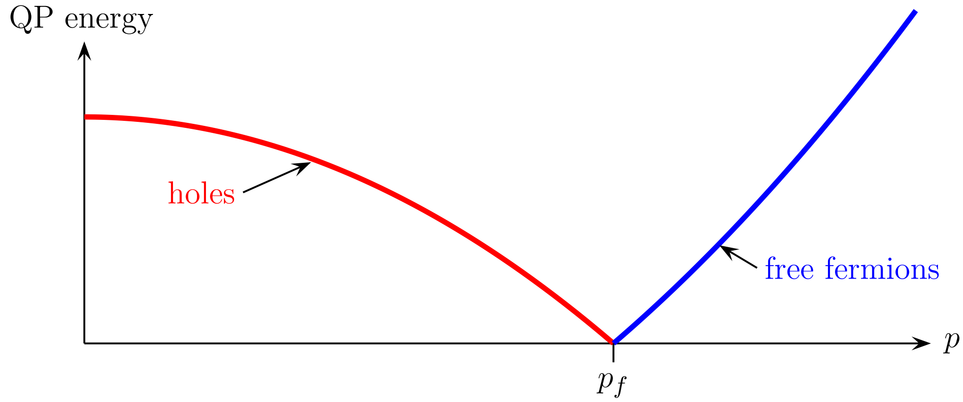
— provided we restrict the $\hat{b}_{\mathbf{p},s}$ and the $\hat{b}^\dagger_{\mathbf{p},s}$ to $|\mathbf{p}|<p_f$ only and the $\hat{a}_{\mathbf{p},s}$ and the $\hat{a}^\dagger_{\mathbf{p},s}$ to $|\mathbf{p}|>p_f$ only — while

$$\left\{\hat{a}_{\mathbf{p},s},\hat{a}^\dagger_{\mathbf{p}',s'}\right\} \;=\; \delta_{\mathbf{p},\mathbf{p}'}\delta_{s,s'} \quad\text{and}\quad \left\{\hat{b}_{\mathbf{p},s},\hat{b}^\dagger_{\mathbf{p}',s'}\right\} \;=\; \delta_{\mathbf{p},\mathbf{p}'}\delta_{s,s'}\,. \tag{53}$$

The Fermi sea $|\mathrm{FS}\rangle$ is the quasi-particle vacuum state of these fermionic operators — it is killed by all the annihilation operators $\hat{a}_{\mathbf{p},s}$ and $\hat{b}_{\mathbf{p},s}$ *in the set.* The two types of creation operators $\hat{a}^\dagger_{\mathbf{p},s}$ and $\hat{b}^\dagger_{\mathbf{p},s}$ create two distinct types of quasi-particles — respectively, the extra fermions above the Fermi surface and the holes below the surface. Both types of quasi-particles have positive energies. Indeed, in terms of our new fermionic operators, the Hamiltonian becomes

$$\hat{H} \;=\; E_{\mathrm{FS}} + \sum_{\mathbf{p},s}^{|\mathbf{p}|>p_f\text{ only}} \left(\frac{\mathbf{p}^2}{2m}-\mu>0\right)\times\hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} + \sum_{\mathbf{p},s}^{|\mathbf{p}|<p_f\text{ only}} \left(\mu-\frac{\mathbf{p}^2}{2m}>0\right)\times\hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s}\,. \tag{54}$$

Graphically,



Besides energies, all the quasi-particles have definite momenta, spins S^z, and charges,

$$\begin{aligned}
\hat{\mathbf{P}}_{\text{tot}} &= \sum_{\mathbf{p},s}^{|\mathbf{p}|>p_f \text{ only}} \mathbf{p} \times \hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} + \sum_{\mathbf{p},s}^{|\mathbf{p}|<p_f \text{ only}} \mathbf{p} \times \hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s}\,, \\
\hat{S}^z_{\text{tot}} &= \sum_{\mathbf{p},s}^{|\mathbf{p}|>p_f \text{ only}} s \times \hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} + \sum_{\mathbf{p},s}^{|\mathbf{p}|<p_f \text{ only}} s \times \hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s}\,, \\
\hat{Q}_{\text{tot}} &= \sum_{\mathbf{p},s}^{|\mathbf{p}|>p_f \text{ only}} (+q) \times \hat{a}^\dagger_{\mathbf{p},s}\hat{a}_{\mathbf{p},s} + \sum_{\mathbf{p},s}^{|\mathbf{p}|<p_f \text{ only}} (-q) \times \hat{b}^\dagger_{\mathbf{p},s}\hat{b}_{\mathbf{p},s} + Q_{\text{FS}}\,,
\end{aligned} \tag{55}$$

where q depends on the fermion species and the type of charge in question — for the electric charge in an electron gas $q = -e$ while for the baryon number in a degenerate neutron gas in a neuron star $q = +1$. In any case, the charge of a hole is always exactly opposite to the charge of a free particle.