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Course: **8.309 - Classical Mechanics III**
Problem set: **#5**

1. Canonical Transformations.

(a) Consider the following two generating functions:

$$F_2(q, P) = q_i P_i \qquad \text{and} \qquad F_3(p, Q) = -p_i Q_i$$

For F_2:

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i, \qquad Q_i = \frac{\partial F_2}{\partial P_i} = q_i$$

as desired.

And for F_3:

$$q_i = -\frac{\partial F_3}{\partial p_i} = Q_i, \qquad P_i = -\frac{\partial F_3}{\partial Q_i} = p_i$$

as desired.

(b) $Q = p/t, P = -qt$. We can choose the following generating function $\boxed{F_1(q, Q, t) = Qqt}$. Let's check that it works:

$$p = \frac{\partial F_1}{\partial q} = Qt \implies Q = \frac{p}{t}, \qquad P = -\frac{\partial F_1}{\partial Q} = qt.$$

(c) We want $F_1 = F_1(q, Q, t)$, so

$$P = -\frac{\partial F_1}{\partial Q} = q^m p^n \implies F_1 = -Qq^m p^n + g(q).$$

We also want

$$p = \frac{\partial F_1}{\partial q} = -Qmq^{m-1}p^n + g'(q) = Q^{1/l}q^{-k/l}.$$

This equality forces $g(q) = 0$, and so $\boxed{m = -1, l = 1, n = 0, k = 2}$. The associated transformations are

$$Q = q^2 p \qquad P = q^{-1}$$

(d) Under the gauge transformation $\vec{A} \to \vec{A}' = \vec{A} + \nabla f(\vec{x}, t)$, $\phi \to \phi' = \phi - \partial_t f(\vec{x}, t)$, we have

$$\mathcal{H}' = \frac{1}{2m}(\vec{P} - q\vec{A}')^2 + q\phi' = \frac{1}{2m}(\vec{P} - q\vec{A} - q\nabla f(\vec{X}, t))^2 + q\phi(\vec{X}, t) - q\frac{\partial}{\partial t}f(\vec{X}, t).$$

Since the $(\vec{p} - q\vec{A})$ term is unchanged, we must have

$$(\vec{P} - q\vec{A}')^2 = (\vec{P} - q\vec{A} - q\nabla f(\vec{X}, t))^2 = (\vec{p} - q\vec{A})^2.$$

which gives us one of the transformation:

$$\vec{P} = \vec{p} + q\nabla f(\vec{X}, t) \implies \vec{p} = \vec{P} - q\nabla f(\vec{X}, t)$$

The other transformation is simply $\vec{x} = \vec{X}$, and this gives

$$\mathcal{H}'(\vec{X}, \vec{P}, t) = \mathcal{H}(\vec{X}, \vec{P}, t) - q\frac{\partial}{\partial t}f(\vec{X}, t).$$

There is more than one way to show that the transformation

$$\vec{x} = \vec{X} \qquad \vec{p} = \vec{P} - q\nabla f(\vec{X}, t)$$

is canonical, but we will do this by checking that the fundamental Poissons brackets are correct:

$$\{X_i, X_j\}_{\vec{x}, \vec{p}} = \sum_k \frac{\partial X_i}{\partial x_k}\frac{\partial X_j}{\partial p_k} - \frac{\partial X_i}{\partial p_k}\frac{\partial X_j}{\partial x_k} = \sum_k \delta_{ik}\delta_{jk} - \delta_{ik}\delta_{jk} = 0$$

$$\{P_i, P_j\}_{\vec{x}, \vec{p}} = \sum_k \frac{\partial P_i}{\partial x_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial P_i}{\partial p_k}\frac{\partial P_j}{\partial x_k} = 0$$

$$\{X_i, P_j\}_{\vec{x}, \vec{p}} = \sum_k \frac{\partial X_i}{\partial x_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial X_i}{\partial p_k}\frac{\partial P_j}{\partial x_k} = \sum_k \delta_{ik}\delta_{jk} = \delta_{ij}.$$

So, the transformation is indeed canonical. To find $F_2(\vec{x}, \vec{P}, t)$, we require that

$$p_i = \frac{\partial F_2}{\partial x_i} = P_i - q\frac{\partial}{\partial x_j}f(\vec{x}, t) \qquad \text{and} \qquad X_i = \frac{\partial F_2}{\partial P_i} = x_i$$

where we have used $\vec{x} = \vec{X}$ to bring the independent variables to $\vec{x}$ and $\vec{P}$. The first equation implies

$$F_2(\vec{x}, \vec{P}, t) = x_i P_i - f(\vec{x}, t) + g(\vec{P}).$$

With this and the second equation we have

$$X_i = x_i - \frac{\partial}{\partial P_i}g(\vec{P}).$$

We can just set $g(\vec{P}) = 0$. With these, we may write

$$\boxed{F_2(\vec{x}, \vec{P}, t) = x_i P_i - qf(\vec{x}, t) = \vec{x}\cdot\vec{P} - qf(\vec{x}, t)}$$

2. Harmonic Oscillator.

(a)

$$\{Q, Q\}_{\vec{q}, \vec{p}} = \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial p} = 0$$

$$\{P, P\}_{\vec{q}, \vec{p}} = \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = 0$$

$$\{Q, P\}_{\vec{q}, \vec{p}} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{ia}{2ia} - \frac{-ia}{2ia} = 1.$$

Thus, the given transformation is indeed canonical.

(b) From the given transformations, we could solve for q, p in terms of P, Q:

$$q = \frac{1}{2ia}(Q - 2iaP) \qquad \text{and} \qquad p = \frac{1}{2}(Q + 2iaP).$$

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The linear harmonic oscillator Hamiltonian can thus be written as

$$\mathcal{H} = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}$$

$$= \frac{1}{2m}\left[\frac{1}{2}(Q + 2iaP)\right]^2 + \frac{m\omega^2}{2}\left[\frac{1}{2ia}(Q - 2iaP)\right]^2$$

$$= \frac{1}{8m}\left[(Q + 2iaP)^2 - \frac{m^2\omega^2}{a^2}(Q - 2iaP)^2\right].$$

An obvious choice for a would be $\boxed{a = m\omega}$, which gives

$$\mathcal{H} = \frac{1}{8m}\left[(Q + 2im\omega P)^2 - (Q - 2im\omega P)^2\right] = \frac{1}{8m}8im\omega QP = i\omega QP.$$

It is clear that the canonical transformation is not explicitly time-dependent, therefore the generating function is not time-dependent. As a result, $\boxed{K = H = i\omega QP}$. Now we solve for the equations of motion:

$$\dot{Q} = \frac{\partial K}{\partial P} = i\omega Q \implies Q(t) = Q_0 e^{i\omega t}$$

$$\dot{P} = -\frac{\partial K}{\partial Q} = -i\omega P \implies P(t) = P_0 e^{-i\omega t},$$

where

$$Q_0 = p(0) + im\omega q(0)$$

$$P_0 = \frac{p(0) - im\omega q(0)}{2im\omega}.$$

From $Q(t), P(t)$, we can invert to solve for $q(t), p(t)$:

$$\boxed{p(t)} = \frac{1}{2}(Q(t) + 2im\omega P(t))$$

$$= \frac{1}{2}\left((p(0) + im\omega q(0))e^{i\omega t} - 2im\omega \frac{p(0) - im\omega q(0)}{2im\omega}e^{-i\omega t}\right)$$

$$= \frac{1}{2}\left((p(0) + im\omega q(0))(\cos\omega t + i\sin\omega t) - (p(0) - im\omega q(0))(\cos\omega t - i\sin\omega t)\right)$$

$$= \boxed{p(0)\cos\omega t - m\omega q(0)\sin\omega t}$$

and

$$\boxed{q(t)} = \frac{1}{2im\omega}(Q(t) - 2im\omega P(t))$$

$$= \frac{1}{2ia}((p(0) + im\omega q(0))e^{i\omega t} - 2im\omega \frac{p(0) - im\omega q(0)}{2im\omega}e^{-i\omega t})$$

$$= \frac{1}{2ia}\left((p(0) + im\omega q(0))(\cos\omega t + i\sin\omega t) - (p(0) - im\omega q(0))(\cos\omega t - i\sin\omega t)\right)$$

$$= \boxed{q(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t}$$

3. Poisson Brackets and Conserved Quantities. Given

$$\mathcal{H} = q_1 p_1 - q_2 p_2 + a q_1^2 + b q_2^2$$

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where a, b are constants. To show that $u_1 = (p_1 + aq_1)/q_2$ and $u_2 = q_1 q_2$ are constants of motion we check that $du_1/dt = 0 = du_2/dt$. Since $\partial u_1/\partial t = \partial u_2/\partial t = 0$, it suffices to check that $\{u_1, \mathcal{H}\} = \{u_2, \mathcal{H}\} = 0$:

$$\{u_1, \mathcal{H}\} = \left(\frac{\partial u_1}{\partial q_1} \frac{\partial \mathcal{H}}{\partial p_1} - \frac{\partial u_1}{\partial p_1} \frac{\partial \mathcal{H}}{\partial q_1} \right) + \left(\frac{\partial u_1}{\partial q_2} \frac{\partial \mathcal{H}}{\partial p_2} - \frac{\partial u_1}{\partial p_2} \frac{\partial \mathcal{H}}{\partial q_2} \right)$$

$$= \left[\frac{a}{q_2} q_1 - \frac{1}{q_2}(p_1 + 2aq_1) \right] + \left[-\frac{(p_1 + aq_1)}{q_2^2}(-q_2) - 0 \right]$$

$$= \frac{1}{q_2} \left[aq_1 - p_1 - 2aq_1 + p_1 + aq_1 \right]$$

$$= 0 \quad \checkmark$$

$$\{u_2, \mathcal{H}\} = \left(\frac{\partial u_2}{\partial q_1} \frac{\partial \mathcal{H}}{\partial p_1} - \frac{\partial u_2}{\partial p_1} \frac{\partial \mathcal{H}}{\partial q_1} \right) + \left(\frac{\partial u_2}{\partial q_2} \frac{\partial \mathcal{H}}{\partial p_2} - \frac{\partial u_2}{\partial p_2} \frac{\partial \mathcal{H}}{\partial q_2} \right)$$

$$= \left[q_2 q_1 - 0 \right] + \left[q_1(-q_2) \right]$$

$$= 0. \quad \checkmark$$

4. Angular Momentum and the Laplace-Runge-Lenz vector.

(a) To avoid unnecessary repeated indices we will replace i, j, k by a, b, c when needed:

$$\boxed{\{x_i, L_j\}} = \{x_i, \epsilon_{jka} x_k p_a\} = \sum_l \frac{\partial x_i}{\partial x_l} \frac{\partial \epsilon_{jka} x_k p_a}{\partial p_l} - \frac{\partial x_i}{\partial p_l} \frac{\partial \epsilon_{jka} x_k p_a}{\partial x_l} = \sum_l \delta_{il} \epsilon_{jki} x_k \delta_{al} = \epsilon_{jki} x_k = \boxed{\epsilon_{ijk} x_k}$$

$$\boxed{\{p_i, L_j\}} = \{p_i, \epsilon_{jka} x_k p_a\} = \sum_l \frac{\partial p_i}{\partial x_l} \frac{\partial \epsilon_{jka} x_k p_a}{\partial p_l} - \frac{\partial p_i}{\partial p_l} \frac{\partial \epsilon_{jka} x_k p_a}{\partial x_l} = \sum_l -\delta_{il} \epsilon_{jka} p_a \delta_{kl} = -\epsilon_{jia} p_a = \boxed{\epsilon_{ijk} p_k}$$

Next,

$$\{L_i, L_b\} = \{\epsilon_{ijk} x_j p_k, \epsilon_{bca} x_c p_a\}$$

$$= \sum_l \frac{\partial \epsilon_{ijk} x_j p_k}{\partial x_l} \frac{\partial \epsilon_{bca} x_c p_a}{\partial p_l} - \frac{\partial \epsilon_{ijk} x_j p_k}{\partial p_l} \frac{\partial \epsilon_{bca} x_c p_a}{\partial x_l}$$

$$= \sum_l \delta_{jl} \epsilon_{ijk} p_k \delta_{al} \epsilon_{bca} x_c - \delta_{kl} \epsilon_{ijk} x_j \delta_{cl} \epsilon_{bca} p_a$$

$$= \epsilon_{ijk} \epsilon_{bcj} p_k x_c - \epsilon_{ijk} \epsilon_{bka} x_j p_a$$

$$= -\epsilon_{ikj} \epsilon_{bcj} p_k x_c + \epsilon_{ijk} \epsilon_{bak} x_j p_a$$

$$= -(\delta_{ib} \delta_{kc} - \delta_{ic} \delta_{kb}) p_k x_c + (\delta_{ib} \delta_{ja} - \delta_{ia} \delta_{jb}) x_j p_a$$

$$= -\delta_{ib} p_k x_k + x_i p_b + \delta_{ib} x_j p_j - x_b p_i$$

$$= x_i p_b - x_b p_i.$$

Putting back $b = j$ we get the final result

$$\boxed{\{L_i, L_j\}} = x_i p_j - x_j p_i = (\delta_{iu} \delta_{jv} - \delta_{iv} \delta_{ju}) x_u p_v = \epsilon_{ijk} \epsilon_{uvk} x_u p_v = \epsilon_{ijk} \epsilon_{kuv} x_u p_v = \boxed{\epsilon_{ijk} L_k}$$

To calculate $\{L_i \vec{L}^2\}$, we have to be more explicit with our index notation. To be explicit we will say $L_a L_a = L_a^2 \neq \vec{L}^2$ and use the "product rule" for Poisson brackets $\{uv, w\} = u\{v, w\} + \{u, w\}v$.

$$\{L_i, \vec{L}^2\} = \{L_i, L_i L_i + L_j L_j + L_k L_k\} = \{L_i, L_i L_i\} + \{L_i, L_j L_j\} + \{L_i, L_k L_k\}$$

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since $\{L_i, L_i\} = 0$, the first term on the RHS is zero, which leaves us with

$$\boxed{\{L_i, \vec{L}^2\}} = -\{L_j L_j, L_i\} - \{L_k L_k, L_i\}$$

$$= -L_j\{L_j, L_i\} - \{L_j, L_i\}L_j - L_k\{L_k, L_i\} + \{L_k, L_i\}L_k$$

$$= 2L_j \epsilon_{ijk} L_k + 2L_k \epsilon_{ika} L_a$$

$$= 2\epsilon_{ijk} L_j L_k - 2\epsilon_{iak} L_a L_k$$

$$= \boxed{0}$$

(b) **Note:** I tried to do this part by using the fact that $\{L_i, \mathcal{H}\} = 0$, but for some reason I got stuck when dealing with indices and how to sort out various terms. I then decided that since $\mathcal{H}$ has relatively simple derivatives in r_l and p_l it would probably be okay for me to proceed using the definition of $\{A_i, \mathcal{H}\}$. This method does not require assuming that $\{L_i, \mathcal{H}\}$ (even though it is true).

Since $\vec{A} = \vec{p} \times \vec{L} - \mu k \vec{r}/r$, to show that $\vec{A}$ is conserved we check that $\{\vec{A}, \mathcal{H}\} = 0$, i.e., $\{A_i, \mathcal{H}\} = 0$. To do this, we must first write everything in terms of components and computer their derivatives

$$\vec{A} = \vec{p} \times \vec{L} - \frac{\mu k \vec{r}}{r} \implies A_i = \epsilon_{ijk} p_j L_k - \frac{\mu k \vec{r}_i}{r}$$

$$= \epsilon_{ijk} \epsilon_{kab} p_j r_a p_b - \frac{\mu k r_i}{r}$$

$$= \epsilon_{ijk} \epsilon_{abk} p_j r_a p_b - \frac{\mu k r_i}{r}$$

$$= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja}) p_j r_a p_b - \frac{\mu k t_i}{r}$$

$$= r_i p_j p_j - p_j r_j p_i - \frac{\mu k r_i}{r}$$

$$= r_i \vec{p}^2 - p_j r_j p_i - \frac{\mu k r_i}{r}$$

With this,

$$\frac{\partial A_i}{\partial r_l} = \delta_{il}\vec{p}^2 - p_j p_i \delta_{jl} - \mu k \left(\delta_{il} \frac{1}{r} - \frac{r_i}{r^2} \frac{\partial r}{\partial r_l} \right)$$

$$= \delta_{il}\vec{p}^2 - p_l p_i - \mu k \left(\frac{\delta_{il}}{r} - \frac{r_i r_l}{r^3} \right)$$

$$= \delta_{il}\vec{p}^2 - p_l p_i - \frac{\mu k}{r} \left(\delta_{il} - \frac{r_i r_l}{r^2} \right)$$

where we have used the fact that

$$\frac{\partial r}{\partial r_l} = \frac{\partial \sqrt{r_l r_l}}{\partial r_l} = \frac{1}{\sqrt{r_l r_l}} 2\frac{1}{2} r_l = \frac{r_l}{r}.$$

We also need to find $\partial A_i / \partial p_l$:

$$\frac{\partial A_i}{\partial p_l} = 2r_i p_j \delta_{jl} - r_j \delta_{jl} p_i - p_j r_j \delta_{il} = 2r_i p_l - r_l p_i - p_j r_j \delta_{il}.$$

The next step is to do the same for $\mathcal{H} = \vec{p}^2/2\mu - k/r$:

$$\frac{\partial \mathcal{H}}{\partial r_l} = -k \frac{\partial}{\partial r_l} \frac{1}{r} = \frac{k}{r^2} \frac{\partial r}{\partial r_l} = \frac{k r_l}{r^3}.$$

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$$\frac{\partial \mathcal{H}}{\partial p_l} = \frac{1}{2\mu} \frac{\partial p_i p_i}{\partial p_l} = \frac{1}{\mu} p_i \delta_{il} = \frac{p_l}{\mu}.$$

With these, we are ready:

$$\{A_i, \mathcal{H}\} = \sum_l \frac{\partial A_i}{\partial r_l} \frac{\partial \mathcal{H}}{\partial p_l} - \frac{\partial A_i}{\partial p_l} \frac{\partial \mathcal{H}}{\partial r_l}$$

$$= \sum_l \left(\delta_{il} \vec{p}^2 - p_l p_i - \frac{\mu k}{r} \left(\delta_{il} - \frac{r_i r_l}{r^2} \right) \right) \frac{p_l}{\mu} - (2 r_i p_l - r_l p_i - p_j r_j \delta_{il}) \frac{k r_l}{r^3}$$

$$= \frac{1}{\mu} \cancel{p_i} \vec{p}^2 - \frac{1}{\mu} \cancel{p_i} \vec{p}^2 - \frac{k}{r} \left(p_i - \frac{r_i r_l p_l}{r^2} \right) - \frac{k}{r^3} (2 r_i p_l r_l - r_l p_i r_l - p_j r_j r_i)$$

$$= -p_i \frac{k}{r} + r_i \frac{k \vec{r} \cdot \vec{p}}{r^3} - r_i \frac{2 k \vec{r} \cdot \vec{p}}{r^3} + p_i \frac{k}{r} + r_i \frac{k \vec{r} \cdot \vec{p}}{r^3}$$

$$= \left(-p_i \frac{k}{r} + p_i \frac{k}{r} \right) + \left(r_i \frac{k \vec{r} \cdot \vec{p}}{r^3} - r_i \frac{2 k \vec{r} \cdot \vec{p}}{r^3} + r_i \frac{k \vec{r} \cdot \vec{p}}{r^3} \right)$$

$$= 0.$$

So, the Laplace-Runge-Lenz vector is conserved, as desired.

(c) We first find what $\vec{L}^2$ is in terms of $\vec{x}, \vec{p}$:

$$\vec{L}^2 = L_i L_i$$

$$= \epsilon_{ijk} L r_j p_k \epsilon_{ibc} r_b p_c = \epsilon_{jki} \epsilon_{bci} r_j r_b p_k p_c$$

$$= (\delta_{jb} \delta_{kc} - \delta_{jc} \delta_{kb}) r_j r_b p_k p_c$$

$$= r_j r_j p_k p_k - r_j p_j r_k p_k$$

$$= \vec{r}^2 \vec{p}^2 - (\vec{r} \cdot \vec{p})^2$$

With this, we expand the LHS and simplify to get $\vec{A}^2$:

$$\mu^2 k^2 + 2\mu \mathcal{H} \vec{L}^2 = \mu^2 k^2 + 2\mu \left(\frac{\vec{p}^2}{2\mu} - \frac{k}{r} \right) [\vec{r}^2 \vec{p}^2 - (\vec{r} \cdot \vec{p})^2]$$

$$= \mu^2 k^2 + \left(\vec{p}^2 - \frac{2\mu k}{r} \right) [\vec{r}^2 \vec{p}^2 - (\vec{r} \cdot \vec{p})^2]$$

$$= \mu^2 k^2 + \vec{p}^2 \vec{r}^2 \vec{p}^2 - \vec{p}^2 (\vec{r} \cdot \vec{p})^2 - \frac{2\mu k}{r} \vec{r}^2 \vec{p}^2 + \frac{2\mu k}{r} (\vec{r} \cdot \vec{p})^2$$

$$= \vec{r}^2 \left(\vec{p}^2 \vec{p}^2 - \frac{2\mu k}{r} \vec{p}^2 + \frac{\mu^2 k^2}{r^2} \right) - 2(\vec{r} \cdot \vec{p})^2 \left(\vec{p}^2 - \frac{\mu k}{r} \right) + (\vec{r} \cdot \vec{p})^2 \vec{p}^2$$

$$= (r_i r_i) \left(\vec{p} - \frac{\mu k}{r} \right)^2 - 2(r_i p_i)(\vec{r} \cdot \vec{p}) \left(\vec{p}^2 - \frac{\mu k}{r} \right) + (\vec{r} \cdot \vec{p}) p_i p_i$$

$$= \underbrace{\left[r_i \left(\vec{p} - \frac{\mu k}{r} \right) - (\vec{r} \cdot \vec{p}) p_i \right]^2}_{A_i}$$

$$= \vec{A}^2 \quad \checkmark$$

where we have used the previously derived result:

$$A_i = r_i \vec{p}^2 - p_j r_j p_i - \frac{k \mu r_i}{r} = r_i \left(\vec{p}^2 - \frac{\mu k}{r} \right) - (\vec{r} \cdot \vec{p}) p_i.$$

5. An Exponential Potential.

(a) $\mathcal{H} = p^2 + e^x$. We want $K = P^2 = H = p^2 + e^x$. Since $p > 0$, we have $p = \sqrt{P^2 - e^x}$. With this,

$$p = \frac{\partial F_2}{\partial x} = \sqrt{P^2 - e^x} \implies F_2(x, P) = 2\sqrt{P^2 - e^x} - 2P \tanh^{-1}\left(\frac{\sqrt{P^2 - e^x}}{P}\right) + g(P).$$

Since we don't have any requirements from $X = X(x, P)$, we may as well set $g(P) = 0$, so that

$$\boxed{F_2(x, P) = 2\sqrt{P^2 - e^x} - 2P \tanh^{-1}\left(\frac{\sqrt{P^2 - e^x}}{P}\right)}$$

Mathematica code:

```
F2 = Integrate[Sqrt[P^2 - E^x], x]

>>>> 2 Sqrt[-E^x + P^2] - 2 P ArcTanh[Sqrt[-E^x + P^2]/P]
```

(b) The transformation equations can be obtained from F_2. First, assuming $P > 0$ we have that

$$\boxed{P(x, p) = \sqrt{p^2 + e^x}}$$

Next,

$$X(x, p) = \frac{\partial F_2}{\partial P} = \frac{\partial}{\partial P}\left[2\sqrt{P^2 - e^x} - 2P\,\mathrm{arctanh}\left(\frac{\sqrt{P^2 - e^x}}{P}\right)\right] = -2\,\mathrm{arccoth}\left(\frac{P}{\sqrt{P^2 - e^x}}\right)$$

Substituting in the result for P we find

$$\boxed{X(x, p)} = -2\,\mathrm{arccoth}\left(\frac{\sqrt{p^2 + e^x}}{\sqrt{p^2 + e^x - e^x}}\right) = \boxed{-2\,\mathrm{arccoth}\left(\frac{\sqrt{p^2 + e^x}}{p}\right)}$$

Mathematica code:

```
D[F2, P] // FullSimplify

>>>> -2 ArcCoth[P/Sqrt[-E^x + P^2]]
```

Assuming $P < 0$ gives us another set of transformation equations:

$$\boxed{P(x, p) = -\sqrt{p^2 + e^x}} \qquad \boxed{X(x, p) = -2\,\mathrm{arccoth}\left(\frac{-\sqrt{p^2 + e^x}}{p}\right)}$$

(c) To determine $x(t), p(t)$ we may solve for the Hamilton EOMs for K first:

$$\dot{P} = -\frac{\partial K}{\partial X} = 0 \implies P = \pm\sqrt{p^2 + e^x} = \text{constant} = \pm P_0 := \pm\sqrt{p(0)^2 + e^{x(0)}}$$

$$\dot{X} = \frac{\partial K}{\partial P} = 2P \implies X = 2Pt + X_0$$

$$\implies -2\,\mathrm{arccoth}\left(\frac{\pm\sqrt{p^2 + e^x}}{p}\right) = -2\,\mathrm{arccoth}\left(\frac{\pm P_0}{p(t)}\right) = \pm 2P_0 t + X_0$$

$$\implies p(t) = \mp P_0 \tanh\left(\frac{X_0}{2} \pm t P_0\right)$$

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From this, we can invert to find $x(t)$:

$$e^{x(t)} = P^2 - p^2 = P_0^2 - P_0^2 \tanh^2\left(\frac{X_0}{2} \pm tP_0\right) \implies x(t) = \ln\left[P_0^2 - P_0^2 \tanh^2\left(\frac{X_0}{2} \pm tP_0\right)\right]$$

where P_0 is already defined above and $X_0 = -2\text{arccoth}(\pm P_0/p(0))$. We also answer whether the signs matter i.e., whether there is a "preferred" set of transformation equations. Suppose that $P > 0$, then

$$p(t) = -P_0 \tanh\left(\frac{X_0}{2} + tP_0\right).$$

At $t = 0$, we have $p(0) = -P_0 \tanh(X_0/2)$ where $X_0 = -2\text{arccoth}(P_0/p(0)) < 0$, and so $p(t) > 0$. ✓

On the other hand if we pick the $P < 0$ transformation then

$$p(t) = P_0 \tanh\left(\frac{X_0}{2} - tP_0\right).$$

At $t = 0$, we have $p(0) = P_0 \tanh(X_0/2)$ where $X_0 = -2\text{arccoth}(-P_0/p(0)) > 0$, and so $p(t) > 0$. ✓

Therefore, either transformation works, so we may as well choose the positive transformation can obtain the following solution:

$$\boxed{p(t) = -P_0 \tanh\left(\frac{X_0}{2} + tP_0\right) \qquad \text{and} \qquad x(t) = \ln\left[P_0^2 - P_0^2 \tanh^2\left(\frac{X_0}{2} + tP_0\right)\right]}$$

where

$$X_0 = -2\text{arccoth}\left(\frac{\sqrt{p(0) + e^{x(0)}}}{p(0)}\right) \qquad \text{and} \qquad P_0 = \sqrt{p(0)^2 + e^{x(0)}}$$

Mathematica code:

```
(* Positive solution *)
Solve[-2 ArcCoth[Z/p[t]] == 2*Z*t + C, p[t]] // FullSimplify

p[t] -> -Z Tanh[C/2 + tZ]

(* Negative solution *)
Solve[-2 ArcCoth[-Z/p[t]] == -2*Z*t + C, p[t]] // FullSimplify

p[t] -> Z Tanh[C/2 - tZ]
```

6. Projectile with Hamilton-Jacobi.

Note: Upon finishing this problem I realized that x is a cyclic coordinate, and so I could have called $p_x = \alpha_2$ a constant and simplied the solution a tiny bit. However, I didn't, and instead calling α_2 the kinetic energy in the x-direction. This also works, but I suppose the solution isn't as "clean" as it could have been. In any case, I believe that both solutions are equally good, so I won't make further modifications to mine.

Our (Hamiltonian) coordinates will be p_x, p_y, x, y. We will seek four constants $\alpha_1, \alpha_2, \beta_1, \beta_2$. The Hamiltonian is simply

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy.$$

The Hamilton-Jacobi equation says

$$\mathcal{H}\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, t\right) + \frac{\partial S}{\partial t} = 0$$

which in our case becomes

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0 \implies \frac{1}{2m}\left[\left(\frac{\partial S_{xy}}{\partial x}\right)^2 + \left(\frac{\partial S_{xy}}{\partial y}\right)^2\right] + mgy = \alpha_1 = -\frac{\partial S_t}{\partial t}$$

where we have used the fact that $p_i = \partial S / \partial x_i$ and that S is separable in spatial and time coordinates: $S = S_{xy} + S_t$ where S_{xy} has no t-dependence and S_t has no spatial dependence. From here we find that

$$S_t = -\alpha_1 t.$$

We can now identify α_1 as the total (conserved) energy, but we'll only keeping that in mind for now. We now solve the spatial part once again by separation of variables

$$\frac{1}{2m}\left[\left(\frac{\partial S_{xy}}{\partial x}\right)^2 + \left(\frac{\partial S_{xy}}{\partial y}\right)^2\right] + mgy = \alpha_1 \implies \frac{1}{2m}\left(\frac{\partial S_{xy}}{\partial x}\right)^2 = \alpha_1 - \frac{1}{2m}\left(\frac{\partial S_{xy}}{\partial y}\right)^2 - mgy = \alpha_2$$

where α_2 is another constant. By separability we may as well write $S_{xy} = S_x + S_y$ where S_x has no y-dependence and S_y has no x-dependence. From here we find that

$$\frac{\partial S_x}{\partial x} = \pm\sqrt{2m\alpha_2} \implies S_x = \pm \int \sqrt{2m\alpha_2}\, dx$$

and

$$\frac{\partial S_y}{\partial y} = \pm\sqrt{2m(\alpha_1 - \alpha_2 - mgy)} \implies S_y = \pm \int \sqrt{2m(\alpha_1 - \alpha_2 - mgy)}\, dy.$$

The full solution is therefore

$$S = S_x + S_y + S_t = \pm \int \sqrt{2m\alpha_2}\, dx \pm \int \sqrt{2m(\alpha_1 - \alpha_2 - mgy)}\, dy - \alpha_1 t.$$

From here, we solve for the constants β_1, β_2:

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = -t \pm \int \frac{m}{\sqrt{2m(\alpha_1 - \alpha_2 - mgy)}}\, dy$$

$$\beta_2 = \frac{\partial S}{\partial \alpha_2} = \pm \int \frac{m}{\sqrt{2m\alpha_2}}\, dx \mp \int \frac{m}{\sqrt{2m(\alpha_1 - \alpha_2 - gmy)}}\, dy = \pm \int \frac{m}{\sqrt{2m\alpha_2}}\, dx \mp (\beta_1 + t),$$

where the simplification in the second equation is by virtue of the first equation. Solving the first equation for y we find

$$t + \beta_1 = \pm\frac{1}{mg}\sqrt{2m(\alpha_1 - \alpha_2 - mgy)} \implies y(t) = \frac{\alpha_1 - \alpha_2}{mg} - \frac{1}{2}g(t + \beta_1)^2.$$

Inverting the second equation we find

$$x(t) = \pm\frac{\sqrt{2m\alpha_2}}{m}\left[\beta_2 \pm (\beta_1 + t)\right].$$

It remains to solve for $\alpha_1, \alpha_2, \beta_1, \beta_2$ based on initial conditions. We know that $\boxed{\alpha_1 = E}$, the total energy, which is just the initial kinetic energy (as the object was launched from the ground), so $E = \alpha_1 = mv_0^2/2$. At $t = 0$, we have

$$x(0) = \pm\frac{\sqrt{2m\alpha_2}}{m}[\beta_2 \pm \beta_1] = y(0) = \frac{mv_0^2/2 - \alpha_2}{mg} - \frac{1}{2}g\beta_1^2 = 0.$$

So $\beta_2 = \mp\beta_1$. Since $x(0) = 0, \dot{x}(0) > 0$, we may take the positive solution:

$$x(t) = \frac{\sqrt{2m\alpha_2}}{m}t \implies \dot{x}(0) = v_0\cos\theta = \frac{\sqrt{2m\alpha_2}}{m} \implies \boxed{\alpha_2 = \frac{mv_0^2\cos^2\theta}{2}}$$

We then solve for β_1 from $y(0) = 0$:

$$\frac{mv_0^2/2 - mv_0^2\cos^2\theta/2}{mg} = \frac{1}{2}g\beta_1^2 \implies \beta_1 = \pm\sqrt{\frac{v_0^2\sin^2\theta}{g^2}}.$$

In order for $\dot{y}(0) > 0, \beta_2 < 0$. Thus, we take the negative solution for β_2:

$$\dot{y}(0) = -g\beta_1 > 0 \implies \beta_1 < 0 \implies \boxed{\beta_1 = -\frac{v_0\sin\theta}{g}}$$

Putting everything back we get the equations for $x(t)$ and $y(t)$, using the fact that $\beta_2 \pm \beta_1 = 0$:

$$\boxed{x(t) = v_0t\cos\theta} \quad \checkmark \qquad \boxed{y(t) = \frac{v_0^2\sin^2\theta}{2g} - \frac{1}{2}g\left(t - \frac{v_0\sin\theta}{g}\right)^2 = v_0t\sin\theta - \frac{1}{2}gt^2} \quad \checkmark$$

From here, it is easy to find the momentum coordinates:

$$p_x = \frac{\partial S}{\partial x} = \sqrt{2m\alpha_2} = mv_0\cos\theta \quad \checkmark$$

$$p_y = \frac{\partial S}{\partial y} = \sqrt{2m(\alpha_1 - \alpha_2 - mgy)} = m(v_0\sin\theta - gt) \quad \checkmark$$

Finally, we want to solve for the equation of the trajectory, i.e. we want to solve for $y(x)$. To do this, we solve for t in terms of x:

$$t(x) = \frac{x}{v_0\cos\theta}$$

and plug it into the equation for $y(t)$:

$$\boxed{y(x) = \frac{v_0\sin\theta x}{v_0\cos\theta} - \frac{1}{2}g\left(\frac{x^2}{v_0^2\cos^2\theta}\right) = x\tan\theta - \frac{gx^2}{2v_0^2\cos^2\theta}} \quad \checkmark$$