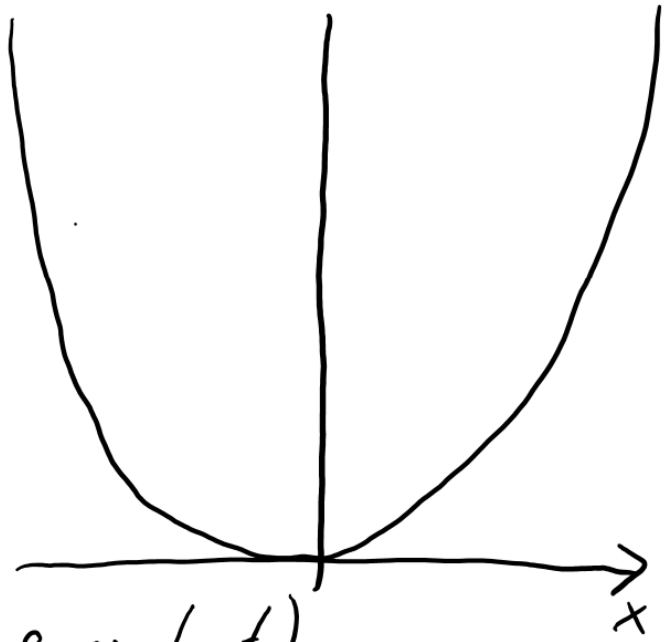
1. Classical Squeezing

Harmonic Oscillator:

$$V(x) = \frac{1}{2} m \omega^2 x^2$$

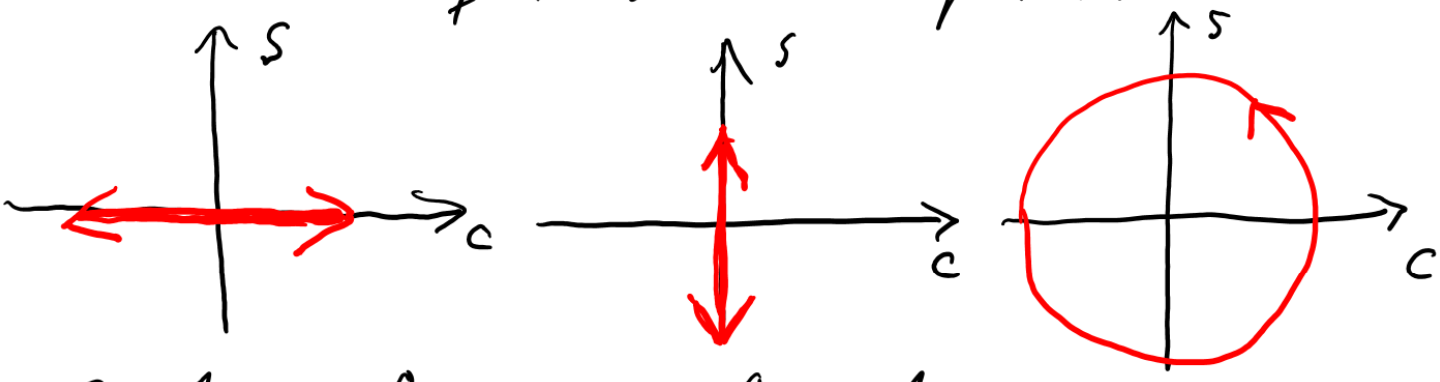
$$m \ddot{x} = -m\omega^2 x$$



Solution:

$$x(t) = \underbrace{c_0 \cos(\omega t)}_{c} + \underbrace{s_0 \sin(\omega t)}_{s}$$

two quadrature components



$c_0 = 1 \quad s_0 = 0$ $\qquad$ $c_0 = 0 \quad s_0 = 1$ $\qquad$ $c_0 = s_0$

Now: Add small parametric drive

$$V(x) = \frac{1}{2} m\omega^2 x^2 \left(1 + \varepsilon \sin(2\omega t)\right)$$

Now: Add small parametric drive:

$$V(x) = \frac{1}{2} m \omega^2 x^2 \left(1 + \varepsilon \sin(2\omega t)\right)$$

Force: $-V'(x) = -m\omega^2 x(t)\left(1 + \varepsilon \sin(2\omega t)\right)$

$$\overset{!}{=} m \ddot{x}(t)$$

Assuming ε small will not cause our $\varepsilon = 0$ solution to change much during one period, we make the Ansatz:

$$x(t) = c_0(t)\cos(\omega t) + s_0(t)\sin(\omega t)$$

where $c_0(t)$ and $s_0(t)$ are slowly varying envelopes such that $\frac{d}{dt} c_0 \ll \omega c_0$.

$$\Rightarrow \dot{x}(t) = -c_0 \omega \sin(\omega t) + s_0 \omega \cos(\omega t) + \dot{c}_0 \cos(\omega t) + \dot{s}_0 \sin(\omega t)$$

$$\ddot{x}(t) = -\omega^2 x(t) - 2\omega \dot{c}_0 \sin(\omega t) + 2\omega \dot{s}_0 \cos(\omega t) + \text{terms involving } \ddot{c}_0, \ddot{s}_0 \ll \omega \dot{c}_0, \omega \dot{s}_0.$$

$$\Rightarrow m\ddot{x} = -\cancel{m\omega^2 x(t)} - 2\omega \dot{c}_0 \sin(\omega t) + 2\omega \dot{s}_0 \cos(\omega t)$$

$$= -V'(x) = -m\omega^2 x(t)\left(\cancel{1} + \varepsilon \sin(2\omega t)\right)$$

$$\Rightarrow -2\omega \dot{c}_0 \sin(\omega t) + 2\omega \dot{s}_0 = -m\omega^2 \varepsilon x(t) \sin(2\omega t)$$

$$x(t)\sin(2\omega t) = \left(c_0\cos(\omega t) + s_0\sin(\omega t)\right)\sin(2\omega t)$$

$$= c_0\cos(\omega t)\sin(2\omega t) + s_0\sin(\omega t)\sin(2\omega t)$$

$$= \frac{c_0}{2}\left(\sin(\omega t) + \sin(3\omega t)\right) + \frac{s_0}{2}\left(\cos(\omega t) - \cos(3\omega t)\right)$$

$$= \frac{c_0}{2}\sin(\omega t) + \frac{s_0}{2}\cos(\omega t) + \text{terms oscillating at } 3\omega$$

drop fast oscillations

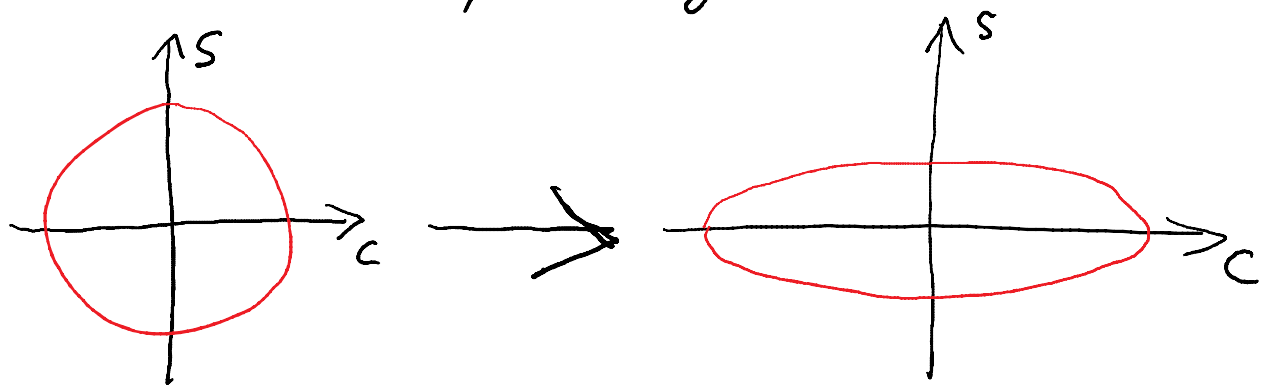
$$\Rightarrow -2\omega\dot{c}_0\sin(\omega t) + 2\omega\dot{s}_0\cos(\omega t) = -\omega^2\frac{\varepsilon}{2}\left(c_0\sin(\omega t) + s_0\cos(\omega t)\right)$$

$$\Rightarrow \quad \dot{c}_0 = \frac{\omega\varepsilon}{4}c_0 \qquad \dot{s}_0 = -\frac{\omega\varepsilon}{4}s_0$$

$$\Rightarrow \quad c_0(t) = e^{\frac{\varepsilon\omega t}{4}} \qquad s_0(t) = e^{-\frac{\varepsilon\omega t}{4}}$$

cos-quadrature amplified! sin-quadrature damped!

$\Rightarrow$ Squeezing!



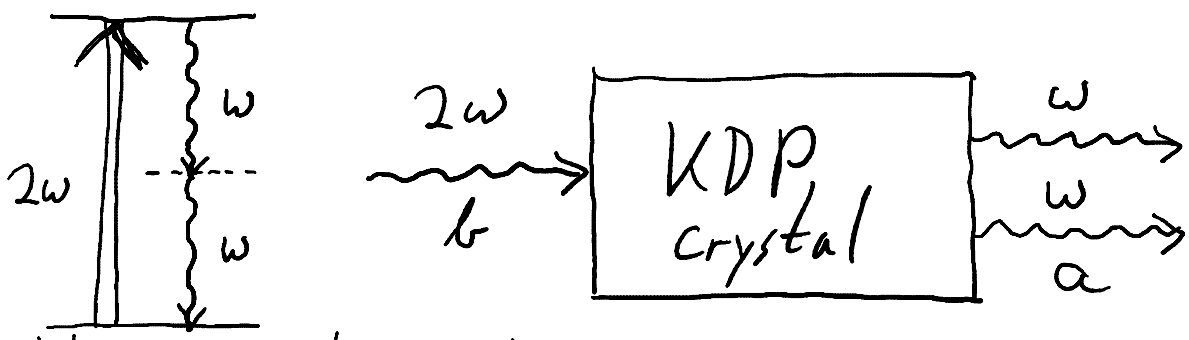
Quantum Squeezed States

Quantum harmonic oscillator
+ parametric drive at 2ω
$\Rightarrow$ Squeezed States

Requires non-linear optics to mix 2ω drive with ω oscillator.

$\Rightarrow$ Needs interaction between photons!

For example: OPO: optical parametric oscillator



$$H = a a b^\dagger + a^\dagger a^\dagger b$$

Let b be a strong coherent state

$$b|\beta\rangle = \beta|\beta\rangle$$

$$b^\dagger|\beta\rangle \approx \beta^*|\beta\rangle \longrightarrow \text{a classical field, a c-number}$$

choose $\beta = \frac{r}{2} e^{i\varphi}$

$$H = \frac{r}{2}\left[a^2 e^{i\varphi} + a^{\dagger 2} e^{-i\varphi}\right]$$

time evolution $e^{-iHt} \Rightarrow$ definition of

Squeezing Operator $S(r) \underset{\varphi=\frac{\pi}{2}}{=} \exp\left(\frac{r}{2}\left(a^2 - a^{\dagger 2}\right)\right)$

Squeezing Operator $\left(z = r\exp\left(i\varphi - \frac{\pi}{2}\right)\right)$ $\boxed{S(z) = \exp\left(\frac{1}{2}\left(z a^2 - z^* a^{+2}\right)\right)}$

$$S(z) = e^{iA} \quad \text{with} \quad A = -\frac{i}{2}\left(z a^2 - z^* a^{+2}\right)$$

$S(z)$ is unitary since A is hermitian.

What does $S(z)$ do?

Remember translation operator $D(\alpha) = \exp(\alpha a^+ - \alpha^* a)$

$$D(\alpha)|\beta\rangle = |\beta + \alpha\rangle.$$

Now $S(z) = \exp\left(\frac{1}{2}\left(z a^2 - z^* a^{+2}\right)\right) = D\left(-\frac{z^*}{2} a^+\right)$

Classically, that means each point α in phase space is translated by an amount $-\frac{z^*}{2}\alpha^*$.

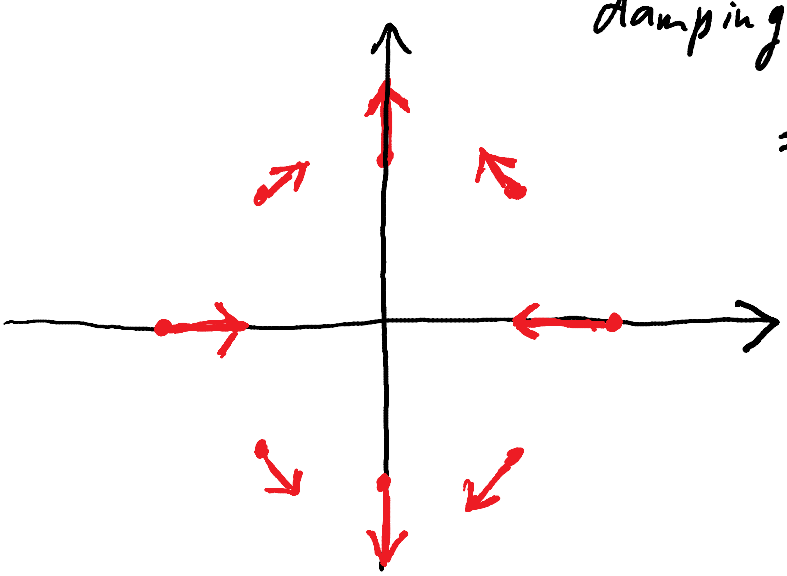
For example, choose $z \equiv 2\varepsilon$ real.

Translation of $\alpha = x + iy$ by $-\varepsilon\alpha^* = -\varepsilon(x - iy)$

means $x + iy \longrightarrow x(1-\varepsilon) + iy(1+\varepsilon)$

damping x amplifying y.

$\Rightarrow$ Squeezing!



Let's see this in the quantum description with operators $\hat{x} = \frac{a + a^\dagger}{\sqrt{2}}$ $\hat{p} = i\,\frac{a^\dagger - a}{\sqrt{2}}$

Quadrature components

$$[\hat{x}, \hat{p}] = i \quad \text{so} \quad \boxed{\Delta x \Delta p \geq \frac{1}{2}}$$

Recall that in a coherent state basis,

$$\Delta x^2 = \frac{1}{2} \quad \text{and} \quad \Delta p^2 = \frac{1}{2}.$$

We will show that for squeezed states, while necessarily $\Delta x \Delta p \geq \frac{1}{2}$ must still hold, we can have either $\Delta x^2 < \frac{1}{2}$ or $\Delta p^2 < \frac{1}{2}$.

For this, define $t \equiv S(z)\, a\, S^\dagger(z) = e^{iA}\, a\, e^{-iA}$

$$= a + [iA, a] + \frac{1}{2!}[iA, [iA, a]] + \ldots$$

$$t = a + z^* a^\dagger + \frac{1}{2!}|z|^2 a + \frac{1}{3!}|z|^2 z^* a^\dagger + \frac{1}{4!}|z|^4 a + \ldots$$

$$= a\left(1 + \frac{1}{2!}r^2 + \frac{1}{4!}r^4 + \ldots\right) + a^\dagger e^{i\varphi}\left(r + \frac{1}{3!}r^3 + \frac{1}{5!}r^5 + \ldots\right)$$

$$\Rightarrow \boxed{\begin{aligned} t &= a\cosh(r) + a^\dagger e^{i\varphi}\sinh(r) \\ t^\dagger &= a^\dagger \cosh(r) + a\, e^{-i\varphi}\sinh(r) \end{aligned}} \qquad (z = re^{i\varphi})$$

Bogoliubov transformation

Properties of transformed operators:

$t = \mu^* a - \nu a^+$ with $|\mu|^2 - |\nu|^2 = 1$

Commutation rule $[t, t^+] = 1$

just like $[a, a^+] = 1$ $\Rightarrow$ same algebra!

$\Rightarrow$ Introduce squeezed states as eigenstates of t:

$$t|\tau\rangle_t = \tau|\tau\rangle_t$$

$$|\tau\rangle_t = D(\tau)|0\rangle_t$$

$$D(\tau) = \exp(\tau t^+ - \tau^* t)$$

$$t|0\rangle_t = 0 \qquad \text{squeezed vacuum}$$

Squeezed vacuum $\boxed{|0\rangle_t = S(z)|0\rangle}$

Check: $t|0\rangle_t = t\,S(z)|0\rangle = S(z)\,a\,S^+(z)S(z)|0\rangle$

$= S(z)\,a|0\rangle = 0$ ✓

As for coherent states $|\tau\rangle = D(\tau)|0\rangle_t$

Squeezed Quadratures:

$$\hat{X}_t = \frac{t + t^+}{\sqrt{2}} \qquad \hat{p}_t = i\,\frac{t^+ - t}{\sqrt{2}}$$

$$\hat{X}_t = \frac{1}{\sqrt{2}}\, S(z)\,(a^+ + a)\,S^+(z)$$

choose $\varphi = 0$: $= \frac{1}{\sqrt{2}}\,(a^+ + a)\,(\cosh(r) + \sinh(r))$

$$= \hat{x}\, e^{r}$$

$$\hat{p}_t = \frac{i}{\sqrt{2}}\, S(z)\,(a^+ - a)\,S^+(z)$$

$$= \frac{i}{\sqrt{2}}\,(a^+ - a)\,(\cosh(r) - \sinh(r))$$

$$= \hat{p}\, e^{-r}$$

Let's calculate Δx and Δp for a squeeted state $|\tau\rangle$:

Note that $\quad {}_t\langle\tau|\hat{x}_t|\tau\rangle_t = \frac{1}{\sqrt{2}}\langle\tau|t + t^+|\tau\rangle = \frac{1}{\sqrt{2}}(\tau^* + \tau)$

and $\quad {}_t\langle\tau|\hat{x}_t^2|\tau\rangle_t = \frac{1}{2}\langle\tau|t^{+2} + tt^+ + t^+t + t^2|\tau\rangle$

$$= \frac{1}{2}\left(\tau^{*2} + 2|\tau|^2 + \tau^2 + 1\right)$$

So $\quad \Delta\hat{x}_t^2 = {}_t\langle\tau|\hat{x}_t^2|\tau\rangle_t - {}_t\langle\tau|\hat{x}_t|\tau\rangle_t^2 = \frac{1}{2}$

$$\Delta\hat{p}_t^2 = \frac{1}{2}$$

(algebraically, all the same for t, $|\tau\rangle_t$ as a, $|\alpha\rangle$)

But since $\hat{x} = \hat{x}_t e^{-r}$ and $\hat{p} = \hat{p}_t e^{r}$ we have

$$\langle \tau | \hat{x} | \tau \rangle_t = e^{-r} \langle \tau | \hat{x}_t | \tau \rangle_t$$

$$\langle \tau | \hat{x}^2 | \tau \rangle_t = e^{-2r} \langle \tau | \hat{x}_t^2 | \tau \rangle_t$$

So $\Delta x^2 = \langle \tau | \hat{x}^2 | \tau \rangle_t - \left(\langle \tau | \hat{x} | \tau \rangle_t \right)^2$

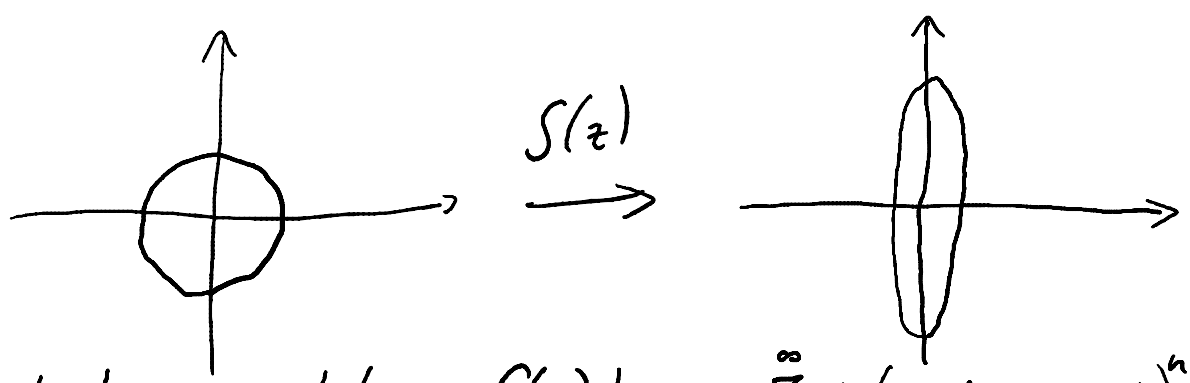
$$= \frac{1}{2} e^{-2r}$$

$$\Delta p^2 = \frac{1}{2} e^{2r}$$

So $\Delta x \Delta p = \frac{1}{2}$ still $\Rightarrow$ squeezed states are minimum uncertainty states

But $\Delta x \ll \frac{1}{2}$ and $\Delta p \gg \frac{1}{2}$.

Squeezed vacuum:



Explicit computation: $S(z)|0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{r}{2} (a^2 - a^{+2}) \right)^n |0\rangle$

$$= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \tanh^n(r) \, |2n\rangle$$

The more you squeeze, the more photons are needed! even!

E-field of a squeezed state:

$$\hat{E}(\vec{r}) = i\,\mathcal{E}_1 \left\{ a\, e^{i\vec{k}\cdot\vec{r}} - a^{+} e^{-i\vec{k}\cdot\vec{r}} \right\}$$

$$= i\,\mathcal{E}_1 \left\{ (a - a^{+}) \cos(\vec{k}\cdot\vec{r}) + i\,(a + a^{+}) \sin(\vec{k}\cdot\vec{r}) \right\}$$

$$= -\sqrt{2}\,\mathcal{E}_1 \left\{ \hat{p} \cos(\vec{k}\cdot\vec{r}) + \hat{x} \sin(\vec{k}\cdot\vec{r}) \right\}$$

$$= -\sqrt{2}\,\mathcal{E}_1 \left\{ \hat{p}_t\, e^{r} \cos(\vec{k}\cdot\vec{r}) + \hat{x}_t\, e^{-r} \sin(\vec{k}\cdot\vec{r}) \right\}$$

So $\langle \tau | \hat{E} | \tau \rangle_t = -\mathcal{E}_1 \mathcal{E}_1 \left\{ i(\tau^* - \tau)\, e^{r} \cos(\vec{k}\cdot\vec{r}) + (\tau^* + \tau)\, e^{-r} \sin(\vec{k}\cdot\vec{r}) \right\}$

same as for coherent state with

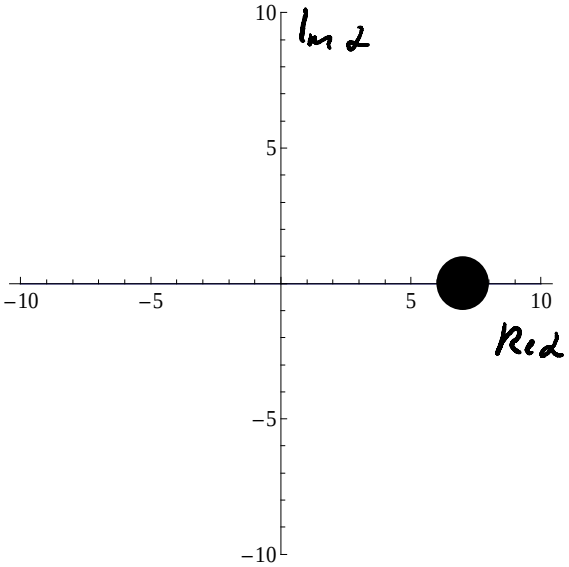
$$\alpha = \tau \cosh r - \tau^* \sinh r$$

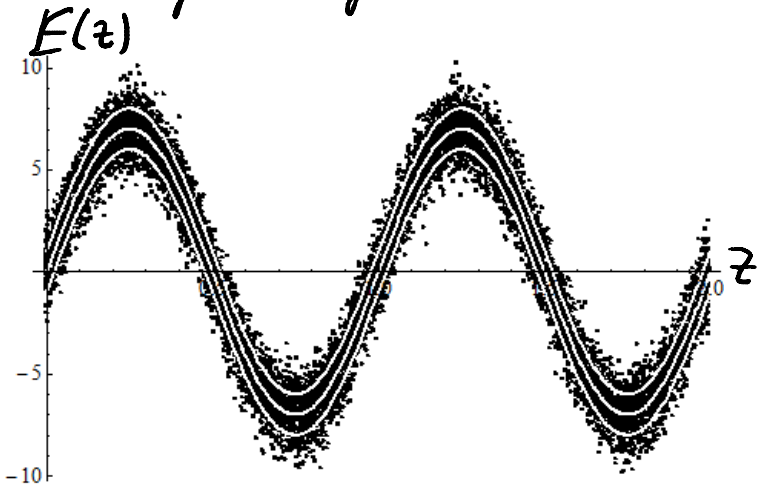
But: $\langle \tau | \hat{E}^2 | \tau \rangle_t = 2\,\mathcal{E}_e^2 \langle \tau | \left\{ \hat{p}_t^2\, e^{2r} \cos^2 + \hat{x}_t^2\, e^{-2r} \sin^2 + (\hat{p}_t \hat{x}_t + \hat{x}_t \hat{p}_t) \sin \cdot \cos \right\} | \tau \rangle_t$

$$= 2\,\mathcal{E}_1^2 \left\{ \frac{1}{2} e^{2r} \cos^2 + \frac{1}{2} e^{-2r} \sin^2 \right\} + \left(\langle \tau | \hat{E} | \tau \rangle_t \right)^2$$

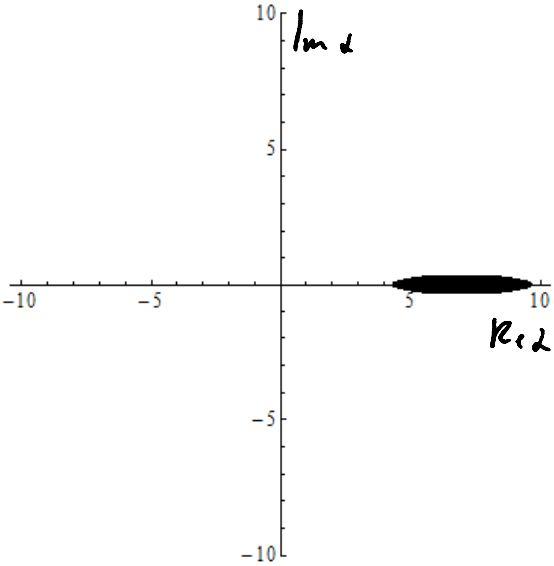
$$\Rightarrow \Delta E^2 = \mathcal{E}_e^2 \left(e^{2R} \cos^2(\vec{k}\cdot\vec{r}) + e^{-2r} \sin^2(\vec{k}\cdot\vec{r}) \right)$$

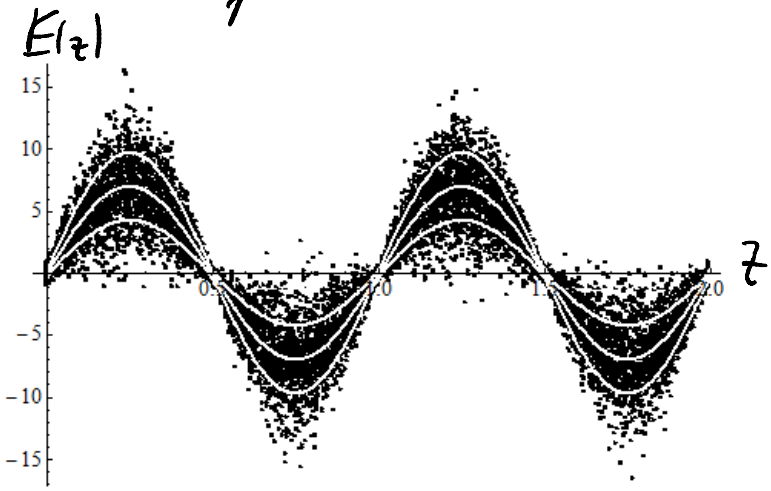
E-field of Coherent State





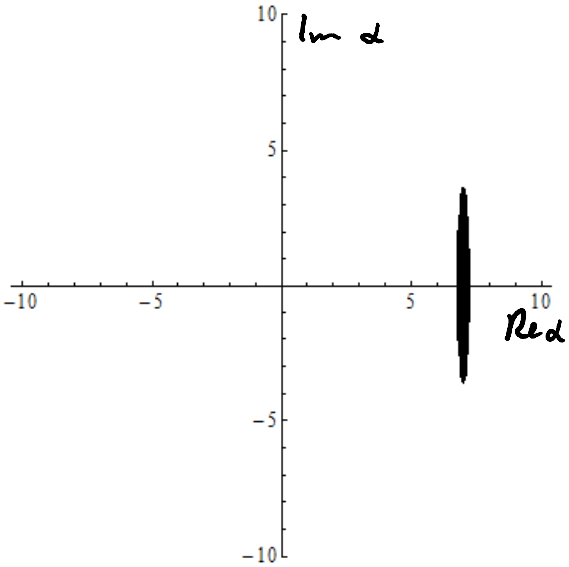
Squeezed State $r = -1$

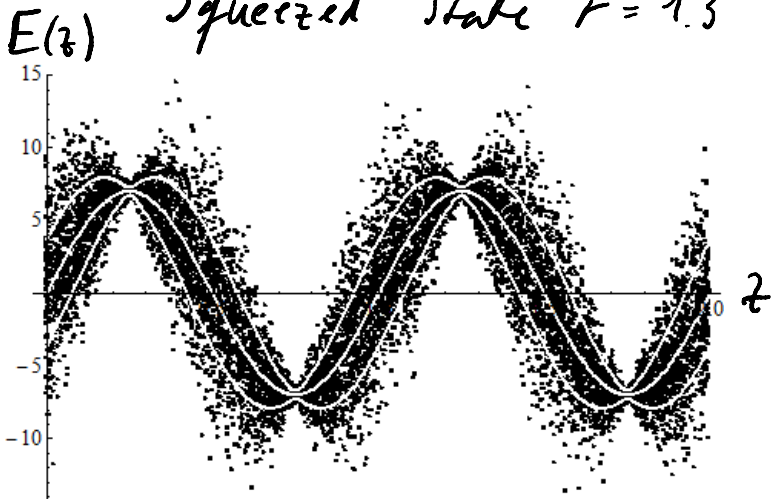




$\Rightarrow$ Phase Squeezed

Squeezed State $r = 1.3$





$\Rightarrow$ Amplitude Squeezed

Photon number: ${}_t\langle\tau|\hat{N}|\tau\rangle_t = {}_t\langle\tau|a^+a|\tau\rangle_t$

$$a = t\cosh r - t^+\sinh r$$
$$a^+ = t^+\cosh r - t\sinh r \qquad (\varphi = 0)$$

$$\hat{N} = a^+a$$
$$= t^+t\cosh^2 r + tt^+\sinh^2 r - (t^2 + t^{+2})\sinh\cdot\cosh$$
$$= t^+t(\cosh^2 r + \sinh^2 r) - (t^2 + t^{+2})\sinh\cdot\cosh + \sinh^2 r$$

$$\Rightarrow {}_t\langle\tau|\hat{N}|\tau\rangle_t = |\tau|^2(\cosh^2 r + \sinh^2 r) - (\tau^2 + \tau^{*2})\sinh\cdot\cosh + \sinh^2 r$$
$$= |\tau\cosh r - \tau^*\sinh r|^2 + \sinh^2 r$$
$$\Rightarrow \boxed{{}_t\langle\tau|\hat{N}|\tau\rangle_t = |\alpha|^2 + \underline{\underline{\sinh^2 r}}}$$

with $\alpha = \tau\cosh r - \tau^*\sinh r$

$\Rightarrow$ Average photon number in a squeezed state is <u>larger</u> than in a coherent state of same average E-field

$$\Rightarrow {}_t\langle 0|\hat{N}|0\rangle_t = \sinh^2 r$$

The average photon number in the squeezed vacuum is <u>not</u> zero!

Fluctuation in the photon number (pick τ, α' real)

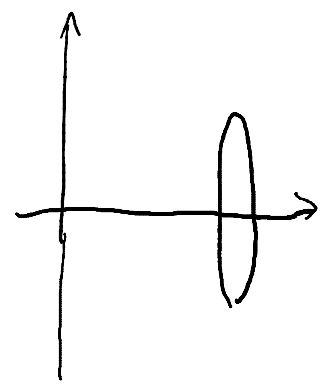
$$\Delta N^2 = \ldots \text{algebra} \ldots = \alpha'^2 e^{-2r} + 2\sinh^2 r \cdot \cosh^2 r$$

again with $\alpha' = \tau \cosh r - \tau^* \sinh r$

For $r > 0$ and $\alpha'^2 \gg e^{2r}$, the second term is negligible and one finds

$$\Delta N^2 \approx |\alpha'|^2 e^{-2r} \approx \langle \hat{N} \rangle e^{-2r}$$

So $\dfrac{\Delta N}{\langle \hat{N} \rangle} = \dfrac{1}{\sqrt{\langle \hat{N} \rangle}} e^{-r} \ll \dfrac{1}{\sqrt{\langle \hat{N} \rangle}}$



$\Rightarrow$ A number-squeezed state

Measurements of the phase, related to the conjugate quadrature, will instead have increased fluctuations.