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1

MA355: COMBINATORICS HW 5

76

(a) To prove $a^{m+n} = a^m a^n$

Base: $m=n=0 \Rightarrow a^{0+0} = 1 = \underbrace{a^0}_{1} \cdot \underbrace{a^0}_{1}$ ✓

- Let $m=0$. ~~To prove $a^n =$~~
  Then $a^{m+n} = a^n = \underbrace{a^0}_{1} \cdot a^n$ ✓

Similarly, statement holds for $n=0$.

- Suppose true for $m, n-1$ and $m-1, n$ then

$$a^{m+n-1} = a^m a^{n-1}$$

(doesn't matter)

$$a^{m-1+n} = a^{m-1} a^n$$

Multiply by a given $a a^{m+n-1} = a^{m+n} = a a^{m-1+n}$

$$a a^{m+n-1} = a^m a^{n-1} \cdot a = a^m a^n \qquad a a^{m-1+n} = a a^{m-1} a^n = a^m a^n \checkmark$$

(b) $a^{mn} = a^{\overbrace{m+m+\cdots+m}^{n \text{ times}}}$

$= (a a^{m-1}) a^{\overbrace{m+\cdots+m}^{n-1}}$ (by defn)

$\vdots$

$= \underbrace{(a^m)\cdots(a^m)}_{n \text{ times}} = (a^m)^n.$

We can of course make things rigorous by induction: Fix m and assume true for $n-1$, then complete the proof by ~~definition~~ the given.

78 **Defn** $\sum_{i=1}^{n} a_i = a_n + \sum_{i=1}^{n-1} a_i$

Distr law : $b(a+c) = ba + bc$.

To prove $b\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ba_i$

To do this we write $b\sum_{i=1}^{n} a_i = b\left[a_n + \sum_{i=1}^{n-1} a_i\right]$

$= ba_n + b\sum_{i=1}^{n-1} a_i$.

Base case : $n = 1 \rightarrow$ holds trivially.

Ind. Hypothesis : holds upto $n-1 \Rightarrow b\sum_{i=1}^{n-1} a_i = \sum_{i=1}^{n-1} ba_i$

So, $b\sum_{i=1}^{n} a_i = ba_n + \sum_{i=1}^{n-1} ba_i$) defn

$= \sum_{i=1}^{n-1+1} ba_i = \sum_{i=1}^{n} ba_i$ ✓.

80 Partition form of product principle: (☆)

"Partition of finite set S into m blocks, each of size n, then S has mn size."

General form

"Let S be a set of fns $f: [n] \to X$. Supp that

- k_1 choices for $f(1)$
- for each choice of $f(1), f(2), \dots f(i-1)$ there are k_i choices for $f(i)$

Then # fns in the set S is $k_1 k_2 \cdots k_n$."

Pf Proof by induction. Base case: $n = 1$ then trivially ~~...~~ we have k_1 choices for $f(1)$ by hypothesis

For $n = 2$, the partition form of the product principle guarantees that the statement also holds.

→ Assume that true for $n \geq 2$, WTS true for $n+1$

↳ by inductive hypothesis, # functions $[n] \to X$ is $k_1 k_2 \cdots k_n$. Now, ~~..~~ look at the set of functions $[n+1] \to X$. By hypothesis we have

$\# f(1) = k_1 \quad \dots \quad \# f(n) = k_n, \quad \# f(n+1) = k_{n+1}$

size of each block. → { For each choice of $f(n+1)$, we have $k_1 k_2 \cdots k_n$ choices for $f: [n] \to X$.

We have (k_{n+1}) "blocks", so by (☆), we have total $= k_1 k_2 \cdots k_n k_{n+1}$

Lugg 11 p 31

n identical ping pong balls.

Paint R, G, W, B.

Line n balls in a line: $\circ\ \circ\ \circ \ldots \circ\ \circ\ \circ$.

Need to put n balls into k bins, each with at least 1 ball...

↳ need $(k-1)$ "separators" for $(n-1)$ "gaps" between the balls.

But I don't think we require at least 1 ball per bin.

→ In this case we have a total of $\underline{n-1+k}$ gaps

→ choose $(k-1)$ out of $(n-1+k)$

→ Answer is $\boxed{\binom{n+k-1}{k-1} = \frac{(n+k-1)!}{n!\,(k-1)!}}$

with $k=4$, we have

$$\# = \frac{(n+4-1)!}{n!\,(4-1)!} = \frac{(n+3)!}{n!\,3!}$$