# Cover Page

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(1)

(a) $dE = TdS - PdV$

$$\Rightarrow dS = \frac{dE + PdV}{T} = \frac{1}{T}\{dE + PdV\}$$

Now, $dE = d(DpV) = D[Vdp + PdV]$

So, $dS = \frac{1}{T}\{D[Vdp + PdV] + PdV\}$

$$= \frac{1}{T}\{DVdp + P(D+1)dV\}$$

So, $\boxed{dS(p,V) = \frac{1}{cp^n}\{DVdp + P(D+1)dV\}}$

(b) • Since $S$ is a function of state, we have

$$\frac{\partial^2 S}{\partial p\,\partial V} = \frac{\partial^2 S}{\partial V\,\partial p},$$

• Now, $dS(p,V) = \left.\frac{\partial S}{\partial p}\right|_V dp + \left.\frac{\partial S}{\partial V}\right|_p dV$

So
$$\frac{\partial S}{\partial p} = \frac{DV}{T} = \frac{DV}{cp^n} = \frac{DV}{c}p^{-n} \Rightarrow \frac{\partial^2 S}{\partial p\,\partial V} = \frac{D}{c}p^{-n}$$

$$\frac{\partial S}{\partial V} = \frac{P(D+1)}{T} = \frac{P(D+1)}{cp^n} = \left(\frac{D+1}{c}\right)p^{-n+1} \Rightarrow \frac{\partial^2 S}{\partial V\,\partial p} = \frac{D+1}{c}(1-n)p^{-n}$$

$$\Rightarrow D = (D+1)(1-n) \Rightarrow \boxed{D = \frac{1-n}{n}}$$

(c) Form of adiabatic curves as $p_s(V)$.

Adiabatic $\Rightarrow$ $dQ = 0$ $\Rightarrow$ $dE = -PdV$

$$\Rightarrow D\{PdV + VdP\} = -PdV$$

$$\Rightarrow DV\,dP = -(D+1)PdV$$

$$\Rightarrow \int \frac{dP}{P} = \int -\underbrace{\left(\frac{D+1}{D}\right)}_{\mathcal{D}} \frac{dV}{V}$$

$$\Rightarrow \ln P_f/P_i = -\mathcal{D} \ln \frac{V_f}{V_i} = \cancel{\mathcal{D} \ln \frac{V_i}{V_f}}$$

$$\Rightarrow P_f/P_i = \frac{V_f^{-\mathcal{D}}}{V_i^{-\mathcal{D}}} \Rightarrow P_f V_f^{+\mathcal{D}} = \text{constant} = G$$

$\Rightarrow$ Form of adiabatic ~~area~~ curve is ...

$$p = GV^{-\mathcal{D}} \text{ where } \mathcal{D} = \frac{D+1}{D} = \frac{1}{1-n}$$

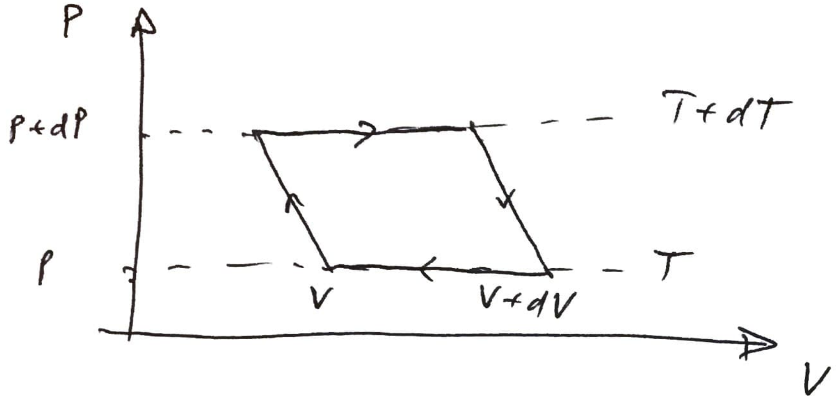
$$\boxed{p_s(V) = GV^{1/1-n}}$$

constant.

Answer: $\boxed{P_s(V) = GV^{+1/n-1}}$

(d) Infinitesimal Carnot cycle in $(p, V)$ coords

"full" $\rightsquigarrow$


(e) The work done is $đW = dP\,dV$

~~$đQ = dE + PdV = TdS$ (expanding along constant pressure)~~

$\cancel{\Rightarrow DV\,dp + P(D+1)\,dV}$

$0$

$\cancel{\Rightarrow P(D+1)\,dV}$

Carnot Efficiency: $\eta = \dfrac{đW}{đQ} = \dfrac{dT}{T} = \dfrac{dP\,dV}{đQ}$

$$\Rightarrow \frac{1}{T}\, d(cp^n) = \frac{dP\,dV}{đQ}$$

$$\rightarrow \underbrace{\frac{1}{cp^n} \cdot cp^{n-1} n\, \cancel{dp}}_{} = \underbrace{\frac{\cancel{dP}\,dV}{Q}}_{} \Rightarrow \boxed{đQ = (dV)\left(P/n\right)}$$

$$\left(\frac{n}{p}\right) = \frac{dV}{Q}$$

(2)

(a) $$C_1 (T_1 - T_f) = C_2 (T_f - T_2)$$

$$C_1 T_1 - C_1 T_f = C_2 T_f - C_2 T_2$$

$$C_1 T_1 + C_2 T_2 = (C_2 + C_1) T_f$$

$$\Rightarrow \boxed{T_f = \frac{C_1 T_1 + C_2 T_2}{C_1 + C_2}}$$

$$\Delta S = S_f - S_i$$

$$= (S_{1,f} + S_{2,f}) - (S_{1,i} + S_{2,i})$$

where $T_f = \frac{C_1 T_1 + C_2 T_2}{C_1 + C_2}$

$$= \Delta S_1 + \Delta S_2 = \boxed{C_1 \ln \frac{T_f}{T_1} + C_2 \ln \frac{T_f}{T_2}}$$

Note can simplify, but expression still clunky

(b) If Carnot engine is used then $\Delta S = 0$

$$\Rightarrow C_1 \ln \frac{T_f}{T_1} = - C_2 \ln \frac{T_f}{T_2}$$

$$\frac{T_f^{C_1}}{T_1^{C_1}} = \frac{T_2^{C_2}}{T_f^{C_2}} \Rightarrow T_f^{C_1 + C_2} = T_1^{C_1} T_2^{C_2}$$

$$\Rightarrow \boxed{T_f = \sqrt[C_1 + C_2]{T_1^{C_1} T_2^{C_2}}}$$

Work done is gotten by conservation of Energy:

$$W = (C_1 T_1 + C_2 T_2) - (C_1 + C_2) T_f$$

$$\boxed{W = (C_1 T_1 + C_2 T_2) - (C_1 + C_2) \sqrt[C_1 + C_2]{T_1^{C_1} T_2^{C_2}}}$$

3. (a) $p(x) = \frac{1}{2b} \exp\left(-\frac{|x-a|}{b}\right) \quad x \in \mathbb{R}.$

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$$\boxed{\langle e^{-ikx} \rangle} = \int_{-\infty}^{\infty} e^{-ikx} p(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-ikx} \frac{1}{2b} \exp\left(-\frac{|x-a|}{b}\right) dx$$

$$= \int_{-\infty}^{a} e^{-ikx} \frac{1}{2b} \exp\left(\frac{+|x-a|}{b}\right) dx$$

$$+ \int_{a}^{\infty} \frac{1}{2b} \exp\left(\frac{-(x-a)}{b}\right) dx$$

$$= \frac{e^{-iak}}{2(1+ibk)} + \frac{i e^{-iak}}{2(i+bk)} = \boxed{\frac{e^{-iak}}{1+b^2k^2}}$$

$$\approx 1 - iak + \left(\frac{-a^2}{2} - b^2\right)k^2 + \dots$$

So $\boxed{\langle x \rangle = a}$

$\langle x^2 \rangle = a^2 + 2b^2 \Rightarrow \boxed{\sigma^2} = \langle x^2 \rangle - \langle x \rangle^2$

$= a^2 + 2b^2 - a^2$

$= \boxed{2b^2}$ Variance

(2) $$p(x) = \frac{|x|}{2a^2} \exp\left(-\frac{|x|}{a}\right) \qquad -\infty < x < \infty$$

$$\boxed{\langle e^{-ikx} \rangle} = \int_{-\infty}^{\infty} e^{-ikx} p(x)\, dx$$

$$= \int_{-\infty}^{0} \frac{-x}{2a^2} \exp\left(\frac{x}{a}\right) e^{-ikx} dx + \int_{0}^{\infty} \frac{+x}{2a^2} \exp\left(\frac{-x}{a}\right) dx$$

$$= \frac{-1}{2(i+ak)^2} + \frac{-1}{2(-i+ak)^2}$$

$$= \boxed{\frac{1-a^2k^2}{(1+a^2k^2)^2}} \approx 1 - 3a^2k^2 + 5a^4k^4 + \ldots$$

$$\boxed{\langle x \rangle = 0}$$

$$\langle x^2 \rangle = 6a^2 \Rightarrow \boxed{\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 = 6a^2}$$