Classical Mechanics III 8.(3)09
Assignment 2: Solutions

September 21, 2021

1. Spherical Pendulum with Friction [8 points]

(a) [4 points] By Stokes' Law, the friction force acting on the spherical mass is $\vec{f} = -b\vec{v}$ for a constant b ($b = 6\pi a\eta$, where a is the radius of the ball and η is the viscosity of the liquid). The dissipation function $\mathcal{F}$ must satisfy $\vec{f} = -\vec{\nabla}_v \mathcal{F}$, giving (as we've seen in lecture)

$$\begin{aligned}\mathcal{F}(\vec{v}) &= \int_0^{\vec{v}} b\vec{v}' \cdot d\vec{v}' \\ &= \frac{b}{2}|\vec{v}|^2\end{aligned}$$

or in terms of spherical coordinates,

$$\mathcal{F} = \frac{b}{2}\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$$

where we've used the constraint that $\dot{r} = 0$.

(b) [4 points] With spherical coordinates (θ, ϕ) centered at the support of the pendulum (we don't take r as a coordinate since $r = \ell$ is fixed), the potential is $V = mgz = -mg\ell\cos\theta$ and the Lagrangian is

$$L = \frac{m}{2}\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + mg\ell\cos\theta.$$

The equations of motion are

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} + \frac{\partial\mathcal{F}}{\partial\dot{\theta}} &= m\ell^2\ddot{\theta} - m\ell^2\sin\theta\cos\theta\dot{\phi}^2 + mg\ell\sin\theta + b\ell^2\dot{\theta} = 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} + \frac{\partial\mathcal{F}}{\partial\dot{\phi}} &= \frac{d}{dt}\left(m\ell^2\sin^2\theta\dot{\phi}\right) + b\ell^2\sin^2\theta\dot{\phi} = 0.\end{aligned}$$

2. Bead Spiralling on a Helix [12 points]

(a) [8 points] Let us take as coordinates (r, θ, z). The Lagrangian is $L = \frac{m}{2}(\dot{z}^2 + \dot{r}^2 + r^2\dot{\theta}^2)$; but since we do not need the constraint forces in the r and z directions, we might as well eliminate r

using $r = bz$, giving us

$$L = \frac{m}{2}[(1+b^2)\dot{z}^2 + b^2 z^2 \dot{\theta}^2]$$

with two independent coordinates $\{\theta, z\}$ and one remaining constraint

$$g = \theta - az = 0.$$

We'll call the Lagrange multiplier Z_θ in anticipation of its interpretation as the constraint force (since the contraint equation has a linear θ term, $\partial g/\partial\theta = 1$, the force will be Z_θ). The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} - Z_\theta \frac{\partial g}{\partial z} = m(1+b^2)\ddot{z} - mb^2 z\dot{\theta}^2 + aZ_\theta = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - Z_\theta \frac{\partial g}{\partial \theta} = \frac{d}{dt}(mb^2 z^2 \dot{\theta}) - Z_\theta = 0$$

where

$$Z_\theta = Z_\theta \frac{\partial g}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \vec{F}_{cons} \cdot \frac{\partial \vec{x}_{particle}}{\partial \theta} = [\vec{x}_{particle} \times \vec{F}_{cons}]_z$$

is the constraint torque for the θ-rotation (i.e. the z-component of the torque). In any case, putting these two equations together we get

$$m(1+b^2)\ddot{z} - mb^2 z\dot{\theta}^2 + a\frac{d}{dt}(mb^2 z^2 \dot{\theta}) = 0$$

and inserting the constraint $\theta = az$ then gives, after simplification,

$$(1 + b^2 + a^2 b^2 z^2)\ddot{z} + a^2 b^2 z \dot{z}^2 = 0.$$

Dividing by $(1 + b^2 + a^2 b^2 z^2)\dot{z}$ gives

$$\frac{\ddot{z}}{\dot{z}} = \frac{a^2 b^2 z \dot{z}}{1 + b^2 + a^2 b^2 z^2} = \frac{z\dot{z}}{z_0^2 + z^2}$$

where we've defined

$$z_0 \equiv \frac{\sqrt{1+b^2}}{ab}.$$

We can now integrate the equation to get

$$\ln \dot{z} = -\frac{1}{2}\ln(z_0^2 + z^2) + C'$$

or

$$\dot{z} = \frac{C}{\sqrt{z_0^2 + z^2}}$$

for some constant of integration $C = e^{C'}$. The initial condition is $\dot{z} = v_0$ as $z = h$, which gives

$C = v_0\sqrt{z_0^2+h^2}$. To evaluate Z_θ, we calculate

$$\begin{aligned}
Z_\theta &= \frac{d}{dt}(mb^2z^2\dot\theta) = \frac{d}{dt}(mab^2z^2\dot z) \\
&= \frac{d}{dt}\left(\frac{mab^2Cz^2}{\sqrt{z_0^2+z^2}}\right) = \frac{2mab^2Cz\dot z}{\sqrt{z_0^2+z^2}} - \frac{mab^2Cz^3\dot z}{(z_0^2+z^2)^{3/2}} \\
&= \frac{mab^2C^2z(2z_0^2+z^2)}{(z_0^2+z^2)^2} \\
&= \frac{mab^2v_0^2z(z_0^2+h^2)(2z_0^2+z^2)}{(z_0^2+z^2)^2}, \qquad \text{where } z_0^2 = \frac{1+b^2}{a^2b^2}.
\end{aligned}$$

(b) [4 points] The energy is

$$\begin{aligned}
E &= \frac{m}{2}[(1+b^2)\dot z^2 + b^2z^2\dot\theta^2] = \frac{m}{2}\dot z^2(1+b^2+a^2b^2z^2) \\
&= \frac{mC^2}{2(z_0^2+z^2)}(1+b^2+a^2b^2z^2) \\
&= \frac{ma^2b^2C^2}{2} \qquad \text{(recalling that } z_0^2 = \frac{1+b^2}{a^2b^2}\text{)} \\
&= \frac{1}{2}ma^2b^2v_0^2(z_0^2+h^2),
\end{aligned}$$

which is a constant. However, the angular momentum

$$p_\theta = \frac{\partial L}{\partial\dot\theta} = mb^2z^2\dot\theta = mab^2z^2\dot z = \frac{mab^2v_0z^2\sqrt{z_0^2+h^2}}{\sqrt{z_0^2+z^2}}$$

is an increasing function of z , as we can check:

$$\begin{aligned}
\frac{d}{dz}\frac{z^2}{\sqrt{z_0^2+z^2}} &= \frac{2z}{\sqrt{z_0^2+z^2}} - \frac{z^3}{(z_0^2+z^2)^{3/2}} \\
&= \frac{z(2z_0^2+z^2)}{(z_0^2+z^2)^{3/2}} > 0.
\end{aligned}$$

Aside: we can also note that

$$p_z = \frac{\partial L}{\partial\dot z} = \frac{mv_0\sqrt{z_0^2+h^2}}{\sqrt{z_0^2+z^2}}$$

decreases as z increases.

3. Hoop Rolling on a Cylinder [12 points]

We will pick coordinates $\{\rho,\theta,\phi\}$, where ρ is the distance between the center of the hoop and the center of the sphere, θ is the angle to the contact point measured from the vertical, and ϕ is the angle the hoop has rotated by around its center. (The angular coordinate for rotations about the z-axis can be neglected.) The constraints are

$$\rho = R + r, \quad \text{or } g_1 = \dot\rho = 0$$

$$g_2 = \rho\dot\theta - r\dot\phi = 0 \quad \text{(the contact point is instantaneously at rest)}$$

(The constraints are in semiholonomic form; we could instead have written them in holonomic form by integrating g_2.) We now write the Lagrangian (recalling the moment of inertia of the hoop is mr^2):

$$L = \frac{m}{2}(\dot\rho^2 + \rho^2\dot\theta^2) + \frac{m}{2}r^2\dot\phi^2 - mg\rho\cos\theta.$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\rho}\right) - \frac{\partial L}{\partial\rho} - \Sigma_i\lambda_i\frac{\partial g_i}{\partial\dot\rho} = m\ddot\rho - m\rho\dot\theta^2 + mg\cos\theta - \lambda_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\theta}\right) - \frac{\partial L}{\partial\theta} - \Sigma_i\lambda_i\frac{\partial g_i}{\partial\dot\theta} = \frac{d}{dt}(m\rho^2\dot\theta) - mg\rho\sin\theta - \lambda_2\rho = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\phi}\right) - \frac{\partial L}{\partial\phi} - \Sigma_i\lambda_i\frac{\partial g_i}{\partial\dot\phi} = mr^2\ddot\phi + \lambda_2 r = 0$$

Recall the constraint force keeping the hoop on the sphere is

$$\begin{aligned} Q_\rho &= \vec{F}_{cons}\cdot\frac{\partial\vec{x}_{cons}}{\partial\rho} = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot\rho}\right) - \frac{\partial L}{\partial\rho} \\ &= \Sigma_i\lambda_i\frac{\partial g_i}{\partial\dot\rho} = \lambda_1, \end{aligned}$$

where $\vec{x}_{cons}$ is the spatial location on which the constraint force acts, i.e. the contact point. So only $\lambda_1 \geq 0$ is physically possible; the hoop leaves the sphere when $\lambda_1 = 0$.

Let us now solve the system of equations. The equation for ϕ gives $\lambda_2 = -mr\ddot\phi = -m\rho\ddot\theta$, with the constraint $g_2 = 0$. Inserting this into the equation for θ, and using the constraint equation $\dot\rho = 0$, gives

$$\ddot\theta = \frac{g}{2\rho}\sin\theta.$$

Multiplying both sides by $\dot\theta$ and then integrating then gives $\frac{d}{dt}\frac{\dot\theta^2}{2} = -\frac{d}{dt}\frac{g}{2\rho}\cos\theta$, or with the initial condition $\dot\theta = 0$ when $\theta = 0$,

$$\dot\theta^2 = \frac{g}{\rho}(1-\cos\theta).$$

Finally, inserting this formula into the equation for ρ (and setting $\ddot\rho = 0$) gives

$$\lambda_1 = mg(2\cos\theta - 1).$$

The hoop leaves the sphere when $\lambda_1 = 0$, i.e. when $\theta = \pi/3$.

4. An Infinite Rope [8 points]

Let the mass density of the rope be ρ. Let us put the zero potential plane at ground level; then the parts of the rope that hang vertically from the pulleys at $x = x_1$ and $x = x_2$ do not change with the rope's shape; hence these parts contribute a constant amount to the total potential energy. We therefore need only minimize the gravitational potential energy of the part of the rope hanging between the pulleys ($x_1 < x < x_2$). Noting that the curve is single valued in the x variable, we can write

$$\begin{aligned} V &= \int_{x_1}^{x_2} \rho g y(x)\, ds, \qquad \text{where } ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \dot{y}(x)^2}dx \\ &= \rho g \int_{x_1}^{x_2} y(x)\sqrt{1 + \dot{y}(x)^2}dx \\ &\equiv \rho g \int_{x_1}^{x_2} f(y, \dot{y})dx \end{aligned}$$

where $f(y, \dot{y}) \equiv y\sqrt{1 + \dot{y}^2}$ satisfies the Euler-Lagrange equation $\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) - \frac{\partial f}{\partial y} = 0$, or

$$\frac{d}{dx}\frac{y\dot{y}}{\sqrt{1 + \dot{y}^2}} = \sqrt{1 + \dot{y}^2}.$$

To solve this equation we can use the following trick: f behaves like a Lagrangian L and is x-independent (the equivalent of t-independent for a Lagrangian); hence the corresponding Hamiltonian must be conserved, i.e.

$$\begin{aligned} \dot{y}\frac{\partial f}{\partial \dot{y}} - f &= \frac{y\dot{y}^2}{\sqrt{1 + \dot{y}^2}} - y\sqrt{1 + \dot{y}^2} \\ &= -\frac{y}{\sqrt{1 + \dot{y}^2}} = -C \end{aligned}$$

for some nonnegative constant C, since $y \geq 0$ (we'll see in a short while that our solution satisfies this constraint). Thus we have

$$\dot{y} = \pm\sqrt{(y/C)^2 - 1}$$

which can be integrated to give, for some constant x_0,

$$\frac{x - x_0}{C} = \pm \cosh^{-1}\left(\frac{y}{C}\right)$$

or (since $\cosh u = \cosh(-u)$, we can drop the $\pm$)

$$y = C \cosh\left(\frac{x - x_0}{C}\right).$$

The boundary conditions give us two equations that allow us to solve for x_0 and C:

$$y_i = C \cosh\left(\frac{x_i - x_0}{C}\right), \qquad i = 1, 2.$$

Aside: If we instead do the integral for V over y by treating x as a function of y, i.e. setting

$$\begin{aligned} V &= \int_{y_1}^{y_2} \rho g y \, ds \\ &= \rho g \int_{y_1}^{y_2} y\sqrt{1+\dot{x}(y)^2} dy \end{aligned}$$

then this becomes equivalent to the problem of minimum surface of revolution problem (Goldstein p. 40). All the considerations made by Goldstein thus carry over to our problem as well.

5,6. A Falling Ladder for 8.09 or 8.309 [20 points]

(a) [4 points(8.09), 3 points(8.309)] We will take as our coordinates simply $\{x, y, \theta\}$, where (x, y) are the Cartesian coordinates of the center of mass of the ladder and θ is the angle the ladder makes with the floor. The moment of inertia of the ladder about the CM is (note the mass density of the ladder is just $\rho = M/L$)

$$I = 2\int_0^{L/2} \rho r^2 dr = \frac{ML^2}{12}$$

and so decomposing the motion of the ladder into the motion of the CM and the rotation of the ladder around the CM, we have

$$T = \frac{M}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{ML^2}{12}\dot{\theta}^2.$$

Setting the potential to be zero at $y = 0$, we get $V = Mgy$. Hence

$$L = \frac{M}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{ML^2}{12}\dot{\theta}^2 - Mgy.$$

The constraints for the ladder to be touching the wall and floor, in that order, are

$$f_w = x - \frac{L}{2}\cos\theta = 0$$

and

$$f_f = y - \frac{L}{2}\sin\theta = 0.$$

(b) [3 points(8.09), 2 points(8.309)] The equations of motion with Lagrange multipliers are thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} - \lambda_w \frac{\partial f_w}{\partial x} = M\ddot{x} - \lambda_w = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} - \lambda_f \frac{\partial f_f}{\partial y} = M\ddot{y} + Mg - \lambda_f = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \lambda_w \frac{\partial f_w}{\partial \theta} - \lambda_f \frac{\partial f_f}{\partial \theta} = \frac{ML^2}{12}\ddot{\theta} - \frac{L}{2}\sin\theta\lambda_w + \frac{L}{2}\cos\theta\lambda_f = 0$$

where $\lambda_w \frac{\partial f_w}{\partial x} = \lambda_w$ is the constraint force on the wall, and $\lambda_f \frac{\partial f_f}{\partial y} = \lambda_f$ is the constraint force on the floor.

(c) [5 points(8.09), 3 points(8.309)] Inserting the Lagrange equations for x and y into the one for θ gives

$$\frac{ML^2}{12}\ddot{\theta} - \frac{ML}{2}\sin\theta\ddot{x} + \frac{ML}{2}\cos\theta(\ddot{y}+g) = 0.$$

Differentiating the constraint equations $f_w = 0$ and $f_f = 0$ twice, we obtain

$$\begin{aligned} \dot{x} &= -\frac{L}{2}\sin\theta\dot{\theta} \quad , \quad \ddot{x} = -\frac{L}{2}\cos\theta\dot{\theta}^2 - \frac{L}{2}\sin\theta\ddot{\theta} \\ \dot{y} &= \frac{L}{2}\cos\theta\dot{\theta} \quad , \quad \ddot{y} = -\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{L}{2}\cos\theta\ddot{\theta}. \end{aligned}$$

Therefore

$$\frac{ML^2}{12}\ddot{\theta} - \frac{ML}{2}\sin\theta\left(-\frac{L}{2}\cos\theta\dot{\theta}^2 - \frac{L}{2}\sin\theta\ddot{\theta}\right) + \frac{ML}{2}\cos\theta\left(-\frac{L}{2}\sin\theta\dot{\theta}^2 + \frac{L}{2}\cos\theta\ddot{\theta} + g\right) = 0$$

which after simplification gives

$$\ddot{\theta} = -\frac{3}{2}\frac{g}{L}\cos\theta.$$

Multiplying both sides by $\dot{\theta}$,

$$\ddot{\theta}\dot{\theta} = -\frac{3}{2}\frac{g}{L}\cos\theta\dot{\theta}$$

and integrating, we get

$$\frac{\dot{\theta}^2}{2} = -\frac{3}{2}\frac{g}{L}(\sin\theta - \sin\theta_0)$$

where the constant $\frac{3}{2}\frac{g}{L}\sin\theta_0$ was chosen to satisfy the boundary condition $\dot{\theta} = 0$ when $\theta = \theta_0$. Noting that θ must decrease with time (i.e. $\dot{\theta} < 0$) we take the negative root,

$$\frac{d\theta}{dt} = \dot{\theta} = -\sqrt{\frac{3g}{L}}\sqrt{\sin\theta_0 - \sin\theta}$$

which can be inverted then integrated to give the desired result

$$t(\theta) = -\sqrt{\frac{L}{3g}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\sqrt{\sin\theta_0 - \sin\theta'}}.$$

(An alternative way of obtaining the formula for $\dot{\theta}^2$ is to use conservation of the energy function.)
(d) [4 points(8.09), 2 points(8.309)] The constraint force on the wall is

$$\lambda_w = M\ddot{x} = -\frac{ML}{2}(\cos\theta\dot{\theta}^2 + \sin\theta\ddot{\theta}).$$

Using our expressions from (b) for $\dot{\theta}^2$ and $\ddot{\theta}$, we have

$$\begin{aligned} \lambda_w &= -\frac{ML}{2}\left(\frac{3g}{L}\cos\theta(\sin\theta_0 - \sin\theta) - \frac{3}{2}\frac{g}{L}\sin\theta\cos\theta\right) \\ &= \frac{3MG}{4}\cos\theta(3\sin\theta - 2\sin\theta_0). \end{aligned}$$

The ladder leaves the wall at $\lambda_w = 0$, which happens when $\theta = \theta_c \equiv \sin^{-1}(\frac{2}{3}\sin\theta_0)$. At this angle the top of the ladder is at a height $L\sin\theta = \frac{2}{3}L\sin\theta_0$.

(e) [4 points(8.09), 2 points(8.309)] Once the ladder leaves the wall, the constraint $f_w = 0$ no longer holds, and we can set $\lambda_w = 0$ in our equations of motion. The new equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = M\ddot{x} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} - \lambda_f\frac{\partial f_f}{\partial y} = M\ddot{y} + Mg - \lambda_f = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} - \lambda_f\frac{\partial f_f}{\partial \theta} = \frac{ML^2}{12}\ddot{\theta} + \frac{L}{2}\cos\theta\lambda_f = 0$$

along with the constraint

$$f_f = y - \frac{L}{2}\sin\theta = 0.$$

(f) [8.309 ONLY, 4 points] We have

$$\lambda_f = M(\ddot{y} + g) = \frac{ML}{2}(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}) + Mg.$$

Before the ladder leaves the wall, $\theta > \theta_c$, we can use our previous expressions for $\dot{\theta}^2$ and $\ddot{\theta}$ obtained in (b) to derive

$$\begin{aligned}
\lambda_f &= \frac{ML}{2}\left(-\frac{3g}{L}\sin\theta(\sin\theta_0 - \sin\theta) - \frac{3}{2}\frac{g}{L}\cos^2\theta\right) + Mg \\
&= \frac{ML}{2}\left(-\frac{3g}{L}\sin\theta(\sin\theta_0 - \sin\theta) - \frac{3}{2}\frac{g}{L}(1 - \sin^2\theta)\right) + Mg \\
&= \frac{3Mg}{4}\sin\theta(3\sin\theta - 2\sin\theta_0) + \frac{Mg}{4}
\end{aligned}$$

which is always positive, since the term in the brackets $3\sin\theta - 2\sin\theta_0 \geq 0$ when the ladder is still in contact with the wall.

After the ladder leaves the wall, Lagrange's equation for θ gives $\ddot{\theta} = -\frac{6}{ML}\cos\theta\lambda_f$, and hence inserting this into the equation for y we get

$$\begin{aligned}
\lambda_f = M(\ddot{y} + g) &= \frac{ML}{2}(-\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}) + Mg \\
&= \frac{ML}{2}(-\sin\theta\dot{\theta}^2 - \frac{6}{ML}\cos^2\theta\lambda_f) + Mg
\end{aligned}$$

or

$$\lambda_f(1 + 3\cos^2\theta) = Mg - \frac{ML}{2}\sin\theta\dot{\theta}^2 \qquad (*)$$

To proceed, we need to solve the equations of motion again for $\dot{\theta}^2$. This time we'll do it by using conservation of energy. The energy is given by

$$E = \frac{M}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{ML^2}{12}\dot{\theta}^2 + Mgy$$

We have $\ddot{x} = 0$ (from the Lagrange equation for x in part (d)) and so the $\dot{x}^2$ only contributes a constant to E. Thus we can set

$$\frac{M}{2}\dot{y}^2 + \frac{1}{2}\frac{ML^2}{12}\dot{\theta}^2 + Mgy = C$$

for some constant $C = E - \frac{M}{2}\dot{x}^2$. Inserting our previous expressions derived from the constraint for y and $\dot{y}$, we obtain

$$\frac{M}{2}\frac{L^2}{4}\cos^2\theta\dot{\theta}^2 + \frac{1}{2}\frac{ML^2}{12}\dot{\theta}^2 + \frac{1}{2}MgL\sin\theta = C$$

or

$$\dot{\theta}^2 = \frac{C' - 12\frac{g}{L}\sin\theta}{1 + 3\cos^2\theta}$$

for another constant $C' = 24C/(ML^2)$. Note that $E = \frac{MgL}{2}\sin\theta_0$ (initially all the energy is gravitational potential energy), and

$$\begin{aligned}
\dot{x} &= -\frac{L}{2}\sin\theta\dot{\theta}|_{\theta=\theta_c} \\
&= \frac{L}{2}\sin\theta_c\sqrt{\frac{3g}{L}}\sqrt{\sin\theta_0 - \sin\theta_c} \\
&= \frac{\sqrt{gL}}{3}\sin\theta_0\sqrt{\sin\theta_0} \qquad \text{(recalling that } \sin\theta_c = \frac{2}{3}\sin\theta_0)
\end{aligned}$$

and hence

$$\begin{aligned}
C' &= \frac{24}{ML^2}(E - \frac{M}{2}\dot{x}^2) \\
&= \frac{24}{ML^2}\left(\frac{MgL}{2}\sin\theta_0 - \frac{MgL}{18}\sin^3\theta_0\right) \\
&= \frac{12g}{L}\left(\sin\theta_0 - \frac{\sin^3\theta_0}{9}\right).
\end{aligned}$$

and therefore

$$\dot{\theta}^2 = \frac{12g}{L}\frac{\sin\theta_0 - \sin\theta - \sin^3\theta_0/9}{1 + 3\cos^2\theta}.$$

Inserting our formula for $\dot{\theta}^2$ into (*) gives

$$\begin{aligned}
\lambda_f(1 + 3\cos^2\theta) &= Mg - \frac{ML}{2}\sin\theta\dot{\theta}^2 \\
&= Mg - 6Mg\sin\theta\frac{\sin\theta_0 - \sin\theta - \sin^3\theta_0/9}{1 + 3\cos^2\theta}
\end{aligned}$$

which can be rearranged as

$$\begin{aligned}
\lambda_f(1 + 3\cos^2\theta)^2 &= Mg[1 + 3\cos^2\theta + 6\sin\theta(\sin\theta + \sin^3\theta_0/9 - \sin\theta_0)] \\
&= Mg(4 + 3\sin^2\theta - 6\sin\theta\sin\theta_0 + \frac{2}{3}\sin\theta\sin^3\theta_0) \\
&= Mg[1 + 3(\sin\theta - \sin\theta_0)^2 + 3(1 - \sin^2\theta_0) + \frac{2}{3}\sin\theta\sin^3\theta_0]
\end{aligned}$$

and since every term inside the brackets is manifestly nonnegative, we see that $\lambda_f > 0$ always: the ladder never loses contact with the floor.

NOTE: A numerical solution to part (f) is fine, such as if a result for λ_f is derived and then plotted to demonstrate that it is positive.

(g) [8.309 ONLY, 4 points] The solution is provided in the Mathematica notebook on the course webpage. It contains many more results than what was asked for. The requested plots are marked by having red descriptive text above the plot.