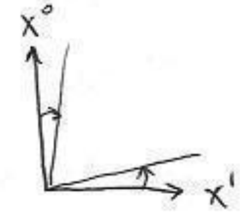
Case II: $\omega_{01} = -\omega_{10} = \eta$. This gives a boost η in the x^1-direction



$$\begin{bmatrix} V^\mu \end{bmatrix} \rightarrow \begin{bmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^\mu \end{bmatrix}$$

For infinitesmal η,

$$\begin{bmatrix} V^\mu \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \eta & 0 & 0 \\ \eta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V^\mu \end{bmatrix}$$

So in this case $-i[J^{01}V]^\mu = g^{\mu 0}V^1 - g^{\mu 1}V^0$

The general case, we deduce, is

$$\left[J^{\mu\nu}_{\text{four-vector}} \right]^\alpha{}_\beta = i\left(g^{\mu\alpha}\delta^\nu_\beta - g^{\nu\alpha}\delta^\mu_\beta \right)$$

Dirac equation

$\{A, B\} = AB + BA$
anticommutator

Suppose we can find $n \times n$ matrices

$$\gamma^\mu \quad (\mu = 0, 1, 2, 3)$$

such that

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \cdot \mathbb{1}$$

$\uparrow$ $n \times n$ identity matrix

then

$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ satisfies

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho} S^{\mu\sigma} + g^{\mu\sigma} S^{\nu\rho} - g^{\mu\rho} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\rho})$$

The relation $\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu}$ is called the Dirac algebra.

Let us first look at spatial components.

We try $\gamma^j = i\sigma^j \quad j=1,2,3$ (just for spatial indices)

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$(\sigma^1)^2 = (\sigma^2)^2 = (\sigma^3)^2 = 1$$

$$\begin{aligned} \sigma^1\sigma^2 &= i\sigma^3 = -\sigma^2\sigma^1 \\ \sigma^2\sigma^3 &= i\sigma^1 = -\sigma^3\sigma^2 \\ \sigma^3\sigma^1 &= i\sigma^2 = -\sigma^1\sigma^3 \end{aligned}$$

$$\{\sigma^j, \sigma^k\} = 2\delta^{jk}$$
$$[\sigma^j, \sigma^k] = 2i\varepsilon^{jkl}\sigma^l$$

We check that

$$\begin{aligned} \{\gamma^j, \gamma^k\} = (-1)\{\sigma^j, \sigma^k\} &= -2\delta^{jk} \\ &= 2g^{jk} \quad \checkmark \end{aligned}$$

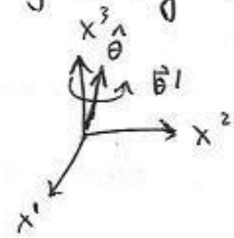
The elements of the Lorentz algebra...

$$S^{jk} = \frac{i}{4}[\gamma^j, \gamma^k] = -\frac{i}{4}[\sigma^j, \sigma^k] = \frac{1}{2}\varepsilon^{jkl}\sigma^l$$

This agrees with what expect for spin-$\frac{1}{2}$

$$\begin{aligned} J^3 &= S^{12} = \frac{\sigma^3}{2} \\ J^2 &= S^{31} = \frac{\sigma^2}{2} \\ J^1 &= S^{23} = \frac{\sigma^1}{2} \end{aligned}$$

Rotation by $\vec{\theta}$ given by $\exp[-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}]$



But we have not yet addressed the full Dirac algebra.

In the "Weyl" or "chiral" representation

$$\gamma^0 = \left[\begin{array}{c|c} 0 & 1 \\ \hline \underbrace{1}_{\begin{pmatrix}1&0\\0&1\end{pmatrix}} & \underbrace{0}_{\begin{pmatrix}0&0\\0&0\end{pmatrix}} \end{array}\right] \qquad \gamma^i = \left[\begin{array}{c|c} 0 & \sigma^i \\ \hline -\sigma^i & 0 \end{array}\right]$$

So for example $\gamma^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

In the Weyl representation

$$S^{0j} = \frac{i}{4}[\gamma^0, \gamma^j] = \frac{i}{4}[\gamma^0\gamma^j - \gamma^j\gamma^0]$$

$$= \frac{i}{4}\left[\begin{pmatrix} -\sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix} - \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}\right] = -\frac{i}{2}\begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$$

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{i}{4}[\gamma^i\gamma^j - \gamma^j\gamma^i]$$

$$= \frac{i}{4}\left[\begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} - \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}\right]$$

$$= \frac{1}{2}\begin{bmatrix} \varepsilon^{ijk}\sigma^k & 0 \\ 0 & \varepsilon^{ijk}\sigma^k \end{bmatrix} = \frac{1}{2}\varepsilon^{ijk}\underbrace{\begin{bmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{bmatrix}}_{\equiv \Sigma^k}$$

The four-components objects that transform according to the Lorentz algebra $S^{\mu\nu}$ are called Dirac (bi)spinors

Note that S^{ij} (rotations) are Hermitian while S^{0j} (boosts) are anti-Hermitian

Let's look at the transformation of γ matrices...

In an expression we can think of

$$[\cdot\ \cdot\ \cdot\ \cdot] \underset{\Lambda^{-1}_{\frac{1}{2}}}{\nearrow} \left[\qquad\right] \underset{\Lambda_{\frac{1}{2}}}{\uparrow} \begin{bmatrix} \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

($\frac{1}{2}$-notation for spinors)

spinors as transforming or the matrix as transforming. For example

$$[\gamma^\mu] \to [\Lambda_{\frac{1}{2}}^{-1}][\gamma^\mu][\Lambda_{\frac{1}{2}}]$$

$$\approx \left(1+\tfrac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)\gamma^\mu\left(1-\tfrac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$$= \gamma^\mu - \tfrac{i}{2}\left[\gamma^\mu, \omega_{\alpha\beta}S^{\alpha\beta}\right]$$

$$= \gamma^\mu - \tfrac{i}{2}\omega_{\alpha\beta}\left[\gamma^\mu, S^{\alpha\beta}\right]$$

We find

$$[\gamma^\mu, S^{\alpha\beta}] = \left[J_4^{\alpha\beta}\right]^\mu{}_\nu \gamma^\nu$$

(for four-vectors)

earlier we found $\left[J_4^{\alpha\beta}\right]^\mu{}_\nu = i(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu)$

So $\quad \gamma^\mu \to \gamma^\mu - \tfrac{i}{2}\omega_{\alpha\beta}\left[J_4^{\alpha\beta}\right]^\mu{}_\nu \gamma^\nu$

Hence γ^μ transforms like a four-vector.

In shorthand

$$\Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = (\Lambda_4)^\mu{}_\nu \gamma^\nu$$

for four-vectors

So we expect an equation such as

$$[i\gamma^\mu \partial_\mu - m]\psi(x) = 0 \qquad \text{(Dirac equation)}$$

to be Lorentz invariant. Let us check...

$$\psi(x) \to \Lambda_{\frac{1}{2}} \psi(\Lambda_4^{-1} x)$$

recall that the argument transforms according to Λ^{-1}

$$\text{So } i\gamma^\mu \partial_\mu \psi(x) \to \underbrace{i\gamma^\mu \Lambda_{\frac{1}{2}}}_{=\,\Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1} i\gamma^\mu \Lambda_{\frac{1}{2}}} \underbrace{\partial_\mu [\psi(\Lambda_4^{-1}x)]}_{=\,(\Lambda_4^{-1})^\alpha{}_\mu (\partial_\alpha \psi)(\Lambda_4^{-1}x)}$$

$$\underbrace{\Lambda_{\frac{1}{2}} \Lambda_{\frac{1}{2}}^{-1}}_{=1} i\gamma^\mu \Lambda_{\frac{1}{2}} = i\Lambda_{\frac{1}{2}} (\Lambda_4)^\mu{}_\nu \gamma^\nu$$

Since $(\Lambda_4^{-1})^\alpha{}_\mu (\Lambda_4)^\mu{}_\nu = \delta^\alpha{}_\nu$, we find

$$i\gamma^\mu \partial_\mu \psi(x) \to \Lambda_{\frac{1}{2}} i\gamma^\mu \partial_\mu \psi(\Lambda_4^{-1}x)$$

So it transforms the same way as $\psi(\Lambda_4^{-1}x)$.

So if $[i\gamma^\mu \partial_\mu - m]\psi(x) = 0$

then in a new frame

$$\Lambda_{\frac{1}{2}}\, [i\gamma^\mu \partial_\mu - m]\, \psi(\Lambda_4^{-1} x) = 0$$

$$\Rightarrow [i\gamma^\mu \partial_\mu - m]\, \psi(\Lambda_4^{-1} x) = 0$$

Same in all frames

Suppose $(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$

Multiply by $(-i\gamma^\mu \partial_\mu - m)$ on the left

$$\underbrace{(-i\gamma^\mu \partial_\mu - m)(i\gamma^\mu \partial_\mu - m)}_{\gamma^\mu \gamma^\nu \partial_\nu \partial_\mu + m^2}\psi(x) = 0$$

$$\Rightarrow \left[\tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\partial_\nu\partial_\mu + m^2\right]\psi(x) = 0$$

(since $\partial_\nu \partial_\mu = \partial_\mu \partial_\nu$)

$$\Rightarrow \left[\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\nu\partial_\mu + m^2\right]\psi(x) = 0$$

$$\Rightarrow \left[\tfrac{1}{2} 2 g^{\mu\nu}\partial_\nu\partial_\mu + m^2\right]\psi(x) = 0$$

$$\Rightarrow (\partial_\mu \partial^\mu + m^2)\psi(x) = 0$$

Klein-Gordon equation