Review Problems

The second in-class test will take place on Friday **10/29/21**from **2:30 to 4:00** pm. There will be a recitation with test review on Wednesday **10/27/21**.

The problems presented here are to help you review the topics that will be covered in the test. The questions appearing in the test will be inspired by (but not identical to) these problems, as well as those in problem sets #2 and #3 (including any optional ones).

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The test is 'open book,' and the following formula sheet will accompany the test:

Physical Constants

Electron mass	$m_e \approx 9.1 \times 10^{-31} kg$	Proton mass	$m_p \approx 1.7 \times 10^{-27} kg$
Electron Charge	$e \approx 1.6 \times 10^{-19} C$	Planck's const./2π	$\hbar \approx 1.1 \times 10^{-34} Js^{-1}$
Speed of light	$c \approx 3.0 \times 10^8 ms^{-1}$	Stefan's const.	$\sigma \approx 5.7 \times 10^{-8} Wm^{-2}K^{-4}$
Boltzmann's const.	$k_B \approx 1.4 \times 10^{-23} JK^{-1}$	Avogadro's number	$N_0 \approx 6.0 \times 10^{23} mol^{-1}$

Conversion Factors

$1atm \equiv 1.0 \times 10^5 Nm^{-2}$ $\quad$ $1\mathring{A} \equiv 10^{-10} m$ $\quad$ $1eV \equiv 1.1 \times 10^4 K$

Thermodynamics

$dE = TdS + đW$ $\quad$ For a gas: $đW = -PdV$ $\quad$ For a wire: $đW = Jdx$

Mathematical Formulas

$\int_0^\infty dx\ x^n\ e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$ $\quad$ $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$

$\int_{-\infty}^{\infty} dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2} \exp\left[-\frac{\sigma^2 k^2}{2}\right]$ $\quad$ $\lim_{N\to\infty} \ln N! = N \ln N - N$

$\left\langle e^{-ikx}\right\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle$ $\quad$ $\ln \left\langle e^{-ikx}\right\rangle = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \langle x^n \rangle_c$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ $\quad$ $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$

Surface area of a unit sphere in d dimensions $\quad$ $S_d = \frac{2\pi^{d/2}}{(d/2-1)!}$

1. *One dimensional gas:* A thermalized gas particle is suddenly confined to a one–dimensional trap. The corresponding mixed state is described by an initial density function $\rho(q,p,t=0)=\delta(q)f(p)$, where $f(p)=\exp(-p^2/2mk_BT)/\sqrt{2\pi mk_BT}$.

(a) Starting from Liouville's equation, derive $\rho(q,p,t)$ and sketch it in the (q,p) plane.

(b) Derive the expressions for the averages $\left\langle q^2\right\rangle$ and $\left\langle p^2\right\rangle$ at $t>0$.

(c) Suppose that hard walls are placed at $q=\pm Q$. Describe $\rho(q,p,t\gg\tau)$, where τ is an appropriately large relaxation time.

(d) A "coarse–grained" density $\tilde{\rho}$, is obtained by ignoring variations of ρ below some small resolution in the (q,p) plane; e.g., by averaging ρ over cells of the resolution area. Find $\tilde{\rho}(q,p)$ for the situation in part (c), and show that it is stationary.

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2. *Evolution of entropy:* The normalized ensemble density is a probability in the phase space Γ. This probability has an associated entropy $S(t)=-\int d\Gamma\rho(\Gamma,t)\ln\rho(\Gamma,t)$.

(a) Show that if $\rho(\Gamma,t)$ satisfies Liouville's equation for a Hamiltonian $\mathcal{H}$, $dS/dt=0$.

(b) Using the method of Lagrange multipliers, find the function $\rho_{\max}(\Gamma)$ which maximizes the functional $S[\rho]$, subject to the constraint of fixed average energy, $\langle\mathcal{H}\rangle=\int d\Gamma\rho\mathcal{H}=E$.

(c) Show that the solution to part (b) is stationary, i.e. $\partial\rho_{\max}/\partial t=0$.

(d) How can one reconcile the result in (a), with the observed increase in entropy as the system approaches the equilibrium density in (b)? (Hint: Think of the situation encountered in the previous problem.)

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3. *The Vlasov equation* is obtained in the limit of high particle density $n=N/V$, or large inter-particle interaction range λ, such that $n\lambda^3\gg 1$. In this limit, the collision terms are dropped from the left hand side of the equations in the BBGKY hierarchy.

The BBGKY hierarchy

$$\left[\frac{\partial}{\partial t}+\sum_{n=1}^{s}\frac{\vec{p}_n}{m}\cdot\frac{\partial}{\partial\vec{q}_n}-\sum_{n=1}^{s}\left(\frac{\partial U}{\partial\vec{q}_n}+\sum_{l}\frac{\partial\mathcal{V}(\vec{q}_n-\vec{q}_l)}{\partial\vec{q}_n}\right)\cdot\frac{\partial}{\partial\vec{p}_n}\right]f_s$$
$$=\sum_{n=1}^{s}\int dV_{s+1}\frac{\partial\mathcal{V}(\vec{q}_n-\vec{q}_{s+1})}{\partial\vec{q}_n}\cdot\frac{\partial f_{s+1}}{\partial\vec{p}_n},$$

has the characteristic time scales

$$\begin{cases}\dfrac{1}{\tau_U}\sim\dfrac{\partial U}{\partial\vec{q}}\cdot\dfrac{\partial}{\partial\vec{p}}\sim\dfrac{v}{L},\\[2mm] \dfrac{1}{\tau_c}\sim\dfrac{\partial\mathcal{V}}{\partial\vec{q}}\cdot\dfrac{\partial}{\partial\vec{p}}\sim\dfrac{v}{\lambda},\\[2mm] \dfrac{1}{\tau_X}\sim\displaystyle\int dx\frac{\partial\mathcal{V}}{\partial\vec{q}}\cdot\frac{\partial}{\partial\vec{p}}\frac{f_{s+1}}{f_s}\sim\frac{1}{\tau_c}\cdot n\lambda^3,\end{cases}$$

where $n\lambda^3$ is the number of particles within the interaction range λ, and v is a typical velocity. The Boltzmann equation is obtained in the dilute limit, $n\lambda^3 \ll 1$, by disregarding terms of order $1/\tau_X \ll 1/\tau_c$. The Vlasov equation is obtained in the dense limit of $n\lambda^3 \gg 1$ by ignoring terms of order $1/\tau_c \ll 1/\tau_X$.

(a) Assume that the N body density is a product of one particle densities, i.e. $\rho = \prod_{i=1}^N \rho_1(\mathbf{x}_i, t)$, where $\mathbf{x}_i \equiv (\vec{p}_i, \vec{q}_i)$. Calculate the densities f_s, and their normalizations.

(b) Show that once the collision terms are eliminated, all the equations in the BBGKY hierarchy are equivalent to the single equation

$$\left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m}\cdot\frac{\partial}{\partial \vec{q}} - \frac{\partial U_{\text{eff}}}{\partial \vec{q}}\cdot\frac{\partial}{\partial \vec{p}}\right] f_1(\vec{p},\vec{q},t) = 0,$$

where

$$U_{\text{eff}}(\vec{q},t) = U(\vec{q}) + \int d\mathbf{x}'\mathcal{V}(\vec{q}-\vec{q}\,')f_1(\mathbf{x}',t).$$

(c) Now consider N particles confined to a box of volume V, with no additional potential. Show that $f_1(\vec{q},\vec{p}) = Ng(\vec{p})/V$ is a stationary solution to the Vlasov equation *for any* $g(\vec{p})$. Why is there no relaxation towards equilibrium for $g(\vec{p})$?

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4. *Two component plasma:* Consider a *neutral* mixture of N ions of charge $+e$ and mass m_+, and N electrons of charge $-e$ and mass m_-, in a volume $V = N/n_0$.

(a) Show that the Vlasov equations for this two component system are

$$\begin{cases} \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m_+}\cdot\frac{\partial}{\partial \vec{q}} + e\frac{\partial \Phi_{\text{eff}}}{\partial \vec{q}}\cdot\frac{\partial}{\partial \vec{p}}\right] f_+(\vec{p},\vec{q},t) = 0 \\ \left[\frac{\partial}{\partial t} + \frac{\vec{p}}{m_-}\cdot\frac{\partial}{\partial \vec{q}} - e\frac{\partial \Phi_{\text{eff}}}{\partial \vec{q}}\cdot\frac{\partial}{\partial \vec{p}}\right] f_-(\vec{p},\vec{q},t) = 0 \end{cases},$$

where the effective Coulomb potential is given by

$$\Phi_{\text{eff}}(\vec{q},t) = \Phi_{\text{ext}}(\vec{q}) + e\int d\mathbf{x}'C(\vec{q}-\vec{q}\,')\left[f_+(\mathbf{x}',t) - f_-(\mathbf{x}',t)\right].$$

Here, Φ_{ext} is the potential set up by the external charges, and the Coulomb potential $C(\vec{q})$ satisfies the differential equation $\nabla^2 C = 4\pi\delta^3(\vec{q})$.

(b) Assume that the one particle densities have the stationary forms $f_\pm = g_\pm(\vec{p})n_\pm(\vec{q})$. Show that the effective potential satisfies the equation

$$\nabla^2\Phi_{\text{eff}} = 4\pi\rho_{\text{ext}} + 4\pi e\left(n_+(\vec{q}) - n_-(\vec{q})\right),$$

where ρ_{ext} is the external charge density.

(c) Further assuming that the densities relax to the equilibrium Boltzmann weights $n_\pm(\vec{q}) = n_0 \exp\left[\mp\beta e \Phi_{\text{eff}}(\vec{q})\right]$, leads to the self-consistency condition

$$\nabla^2 \Phi_{\text{eff}} = 4\pi \left[\rho_{\text{ext}} + n_0 e \left(e^{\beta e \Phi_{\text{eff}}} - e^{-\beta e \Phi_{\text{eff}}}\right)\right],$$

known as the *Poisson–Boltzmann equation.* Due to its nonlinear form, it is generally not possible to solve the Poisson–Boltzmann equation. By linearizing the exponentials, one obtains the simpler *Debye* equation

$$\nabla^2 \Phi_{\text{eff}} = 4\pi \rho_{\text{ext}} + \Phi_{\text{eff}}/\lambda^2.$$

Give the expression for the *Debye screening length* λ.

(d) Show that the Debye equation has the general solution

$$\Phi_{\text{eff}}(\vec{q}) = \int d^3\vec{q}' G(\vec{q} - \vec{q}')\rho_{\text{ext}}(\vec{q}'),$$

where $G(\vec{q}) = \exp(-|\vec{q}|/\lambda)/|\vec{q}|$ is the screened Coulomb potential.

(e) Give the condition for the self-consistency of the Vlasov approximation, and interpret it in terms of the inter-particle spacing?

(f) Show that the characteristic relaxation time ($\tau \approx \lambda/c$) is temperature independent. What property of the plasma is it related to?

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5. *Two dimensional electron gas in a magnetic field:* When donor atoms (such as P or As) are added to a semiconductor (e.g. Si or Ge), their conduction electrons can be thermally excited to move freely in the host lattice. By growing layers of different materials, it is possible to generate a spatially varying potential (work–function) which traps electrons at the boundaries between layers. In the following, we shall treat the trapped electrons as a gas of classical particles *in two dimensions.* If the layer of electrons is sufficiently separated from the donors, the main source of scattering is from electron–electron collisions.

(a) Write down heuristically (i.e. not through a step by step derivation), the Boltzmann equations for the densities $f_\uparrow(\vec{p}, \vec{q}, t)$ and $f_\downarrow(\vec{p}, \vec{q}, t)$ of electrons with up and down spins, in terms of the two cross-sections $\sigma \equiv \sigma_{\uparrow\uparrow} = \sigma_{\downarrow\downarrow}$, and $\sigma_\times \equiv \sigma_{\uparrow\downarrow}$, of *spin conserving* collisions.

(b) Show that $d\text{H}/dt \leq 0$, where $\text{H} = \text{H}_\uparrow + \text{H}_\downarrow$ is the sum of the corresponding H functions.

(c) Show that $d\text{H}/dt = 0$ for any $\ln f$ which is, *at each location*, a linear combination of quantities conserved in the collisions.

(d) Show that the streaming terms in the Boltzmann equation are zero for any function that depends only on the quantities conserved by the one body Hamiltonians.

(e) Show that momentum $\vec{L} = \vec{q} \times \vec{p}$, is conserved during, and away from collisions, for magnetic fields perpendicular to the layer.

(f) Write down the most general form for the equilibrium distribution functions for particles confined to a circularly symmetric potential.

(g) How is the result in part (g) modified by including scattering from magnetic and non-magnetic impurities?

(h) Do conservation of spin and angular momentum lead to new hydrodynamic equations?

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6. *The Lorentz gas* describes non-interacting particles colliding with a fixed set of scatterers. It is a good model for scattering of electrons from donor impurities. Consider a uniform two dimensional density n_0 of fixed impurities, which are hard circles of radius a.

(a) Show that the differential cross section of a hard circle scattering through an angle θ is

$$d\sigma = \frac{a}{2} \sin \frac{|\theta|}{2}\, d\theta,$$

and calculate the total cross section.

(b) Write down the Boltzmann equation for the one particle density $f(\vec{q}, \vec{p}, t)$ of the Lorentz gas (including only collisions with the fixed impurities). *(Ignore the electron spin.)*

(c) Using the definitions $\vec{F} \equiv -\partial U / \partial \vec{q}$, and

$$n(\vec{q}, t) = \int d^2\vec{p} f(\vec{q}, \vec{p}, t), \qquad \text{and} \qquad \langle g(\vec{q}, t) \rangle = \frac{1}{n(\vec{q}, t)} \int d^2\vec{p} f(\vec{q}, \vec{p}, t) g(\vec{q}, t),$$

show that for any function $\chi(|\vec{p}|)$, we have

$$\frac{\partial}{\partial t}(n\langle \chi \rangle) + \frac{\partial}{\partial \vec{q}} \cdot \left(n \left\langle \frac{\vec{p}}{m} \chi \right\rangle \right) = \vec{F} \cdot \left(n \left\langle \frac{\partial \chi}{\partial \vec{p}} \right\rangle \right).$$

(d) Derive the conservation equation for local density $\rho \equiv mn(\vec{q}, t)$, in terms of the local velocity $\vec{u} \equiv \langle \vec{p}/m \rangle$.

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7. *Thermal conductivity:* Consider a classical gas between two plates separated by a distance w. One plate at $y = 0$ is maintained at a temperature T_1, while the other plate at $y = w$ is at a different temperature T_2. The gas velocity is zero, so that the initial zeroth order approximation to the one particle density is,

$$f_1^0(\vec{p}, x, y, z) = \frac{n(y)}{[2\pi m k_B T(y)]^{3/2}} \exp\left[-\frac{\vec{p} \cdot \vec{p}}{2m k_B T(y)} \right].$$

(a) What is the necessary relation between $n(y)$ and $T(y)$ to ensure that the gas velocity $\vec{u}$ remains zero? (Use this relation between $n(y)$ and $T(y)$ in the remainder of this problem.)

(b) Using Wick's theorem, or otherwise, show that

$$\left\langle p^2\right\rangle^0 \equiv \left\langle p_\alpha p_\alpha\right\rangle^0 = 3\left(mk_BT\right), \quad \text{and} \quad \left\langle p^4\right\rangle^0 \equiv \left\langle p_\alpha p_\alpha p_\beta p_\beta\right\rangle^0 = 15\left(mk_BT\right)^2,$$

where $\langle\mathcal{O}\rangle^0$ indicates local averages with the Gaussian weight f_1^0. Use the result $\left\langle p^6\right\rangle^0 = 105(mk_BT)^3$ in conjunction with symmetry arguments to conclude

$$\left\langle p_y^2 p^4\right\rangle^0 = 35\left(mk_BT\right)^3.$$

(c) The zeroth order approximation does not lead to relaxation of temperature/density variations related as in part (a). Find a better (time independent) approximation $f_1^1(\vec{p}, y)$, by linearizing the Boltzmann equation in the single collision time approximation, to

$$\mathcal{L}\left[f_1^1\right] \approx \left[\frac{\partial}{\partial t} + \frac{p_y}{m}\frac{\partial}{\partial y}\right] f_1^0 \approx -\frac{f_1^1 - f_1^0}{\tau_K},$$

where τ_K is of the order of the mean time between collisions.

(d) Use f_1^1, along with the averages obtained in part (b), to calculate h_y, the y component of the heat transfer vector, and hence find K, the coefficient of thermal conductivity.

(e) What is the temperature profile, $T(y)$, of the gas in steady state?

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8. *Electron emission:* When a metal is heated in vacuum, electrons are emitted from its surface. The metal is modeled as a classical gas of noninteracting electrons held in the solid by an abrupt potential well of depth ϕ (the work function) relative to the vacuum.

(a) What is the relationship between the initial and final velocities of an escaping electron?

(b) In thermal equilibrium at temperature T, what is the probability density function for the velocity of electrons?

(c) If the number density of electrons is n, calculate the current density of thermally emitted electrons.

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9. *Light and matter:* In this problem we use kinetic theory to explore the equilibrium between atoms and radiation.

(a) The atoms are assumed to be either in their ground state a_0, or in an excited state a_1, which has a higher energy ε. By considering the atoms as a collection of N fixed two-state systems of energy E (i.e. ignoring their coordinates and momenta), calculate the ratio n_1/n_0 of densities of atoms in the two states as a function of temperature T.

Consider photons γ of frequency $\omega = \varepsilon/\hbar$ and momentum $|\vec{p}| = \hbar\omega/c$, which can interact with the atoms through the following processes:

(i) *Spontaneous emission:* $a_1 \to a_0 + \gamma$.

(ii) *Adsorption:* $a_0 + \gamma \to a_1$.

(iii) *Stimulated emission:* $a_1 + \gamma \to a_0 + \gamma + \gamma$.

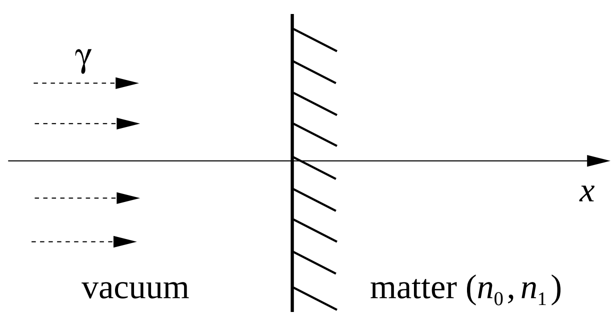
Assume that spontaneous emission occurs with a probability σ_{sp}, and that adsorption and stimulated emission have corresponding constant (angle-independent) probabilities (cross-sections) of σ_{ad} and σ_{st}, respectively.

(b) Write down the Boltzmann equation governing the density f of the photon gas, treating the atoms as fixed scatterers of densities n_0 and n_1.

(c) Find the equilibrium density $f_{\text{eq.}}$ for the photons of the above frequency.

(d) According to Planck's law, the density of photons at a temperature T depends on their frequency ω as $f_{\text{eq.}} = \left[\exp\left(\hbar\omega/k_BT\right) - 1\right]^{-1}/h^3$. What does this imply about the above cross sections?

(e) Consider a situation in which light shines along the x axis on a collection of atoms whose boundary coincides with the $x = 0$ plane, as illustrated in the figure.



Clearly, f will depend on x (and p_x), but will be independent of y and z. Adapt the Boltzmann equation you propose in part (b) to the case of a uniform incoming flux of photons with momentum $\vec{p} = \hbar\omega\hat{x}/c$. What is the *penetration length* across which the incoming flux decays?

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10. *Moments of momentum:* Consider a gas of N classical particles of mass m in thermal equilibrium at a temperature T, in a box of volume V.

(a) Write down the equilibrium one particle density $f_{\text{eq.}}(\vec{p}, \vec{q})$, for coordinate $\vec{q}$, and momentum $\vec{p}$.

(b) Calculate the joint characteristic function, $\left\langle \exp\left(-i\vec{k}\cdot\vec{p}\right)\right\rangle$, for momentum.

(c) Find all the joint cumulants $\left\langle p_x^\ell p_y^m p_z^n \right\rangle_c$.

(d) Calculate the joint moment $\langle p_\alpha p_\beta \left(\vec{p}\cdot\vec{p}\right)\rangle$.

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