2nd edition
revised & enlarged
by D ter Haar

problems in electrodynamics

V.V. Batygin I.N. Toptygin

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This second edition contains nearly 900 problems on classical electrodynamics provided with answers or detailed solutions, including more than 200 problems on the special theory of relativity.

Chapters are devoted to vector and tensor calculus, electrostatics, steady currents, magnetostatics, electrical and magnetic properties of matter, quasi-stationary electromagnetic fields, propagation of electromagnetic waves, electromagnetic oscillations in finite bodies, special theory of relativity, relativistic mechanics, emission of electromagnetic waves, and interaction of charged particles with matter. Each section is prefaced by a short introduction.

This edition has been revised, supplemented, and edited by D ter Haar, and contains several new problems, including some taken from Oxford University examination papers, and new sections on superconductivity, coherence and interference, and x-ray diffraction, and a new chapter on plasma physics.

A large number of the problems follow naturally from a lecture course on electrodynamics, but some are concerned with topics which are not well covered by existing texts, such as the interaction of charged particles with matter, applications of conservation laws to the analysis of collision processes and particle disintegration, and ferromagnetic resonance.

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Other collections of problems with solutions

Problems in quantum mechanics, editor D ter Haar
third edition, revised and enlarged
1975, ISBN 0 85086 050 4, iv + 468 pages
This set of problems can be used either in conjunction with any modern textbook, or independently as advanced reading by students familiar with the basic ideas of quantum mechanics. For this third edition the problems have been rearranged, further ones included, and two new chapters added. There are now 456 problems arranged under the headings: one-dimensional motion; tunnel effect; commutation relations, Heisenberg relations, spreading of wave packets, operators; angular momentum, spin; central field of force; motion of particles in a magnetic field; atoms; molecules; scattering; creation and annihilation operators, density matrix, and relativistic wave equations. The emphasis is throughout on the building up of an understanding of basic ideas and on the familiarisation with basic techniques.

Problems in physical electronics, editors R L Ferrari, A K Jonscher
1973, ISBN 0 85086 038 5, ii + 471 pages
This collection of problems is intended for senior undergraduate and postgraduate students and lecturers in physics, electronics and electronic engineering, and also for self-study by practicing physicists and engineers. As in other volumes of this series, individual chapters have been written by specialists who have selected examples from real-life experience and presented them in a way which will deepen the insight of readers into various aspects of physical electronics. There are some 260 problems which are divided into twelve chapters dealing with general concepts; electron emission; motion of electrons in vacuum; vacuum devices; ionized gases; conduction in solids; dielectrics; solid state devices; optoelectronic devices; masers and lasers; electronic noise; and gyromagnetic media.

Problems in thermodynamics and statistical physics, editor P T Landsberg
1971, ISBN 0 85086 023 7, viii + 573 pages
This book is intended for teachers, undergraduates, and postgraduates in mathematics, physics, chemistry, and engineering. It covers a full range of topics in thermodynamics, statistical physics, and statistical mechanics and is divided into 28 chapters. Each chapter has been designed by an expert in the field with the result that the book presents a penetrating view of statistical physics and its uses. There are new and original presentations in various areas, and in several sections recent research work has been incorporated in book form for the first time. The chapters start with relatively easy problems which lead to progressively more difficult ones. Similarly, the early chapters are easier than the later ones, but in all cases full solutions are supplied. The book is therefore very suitable for self-study, if it is read broadly in sequence. The range covered is so extensive that the book can be a student's companion throughout his university career. At the same time teachers can turn to it for ideas and for inspiration.

Problems in solid state physics, editor H J Goldsmid
1968, ISBN 0 85086 000 8, vi + 466 pages
This volume is intended to complement standard texts used by final year undergraduate and postgraduate students in colleges and universities where the study of solid state physics is taken to some considerable depth. The sixteen chapters have been written by specialists and contain over 300 problems dealing with crystallography and crystal structures; growth of crystals; chemical bonds and lattice energy; elasticity of crystals; thermal properties of the crystal lattice; defects in crystals; dielectricity; diamagnetism and paramagnetism; ferromagnetism; antiferromagnetism and ferrimagnetism; magnetic resonance; electron theory of metals; energy band structures; properties of homogeneous semiconductors; semiconductor junctions; optical properties of solids; and superconductivity.

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Preface

This is the second English edition of the problem book by Batygin and Toptygin. It contains nearly nine hundred problems, including more than two hundred on the special theory of relativity and about seventy on vector and tensor analysis.

As compared with the first edition, several new problems have been included, as well as new sections on superconductivity, coherence and interference, and x-ray diffraction, and a new chapter on plasma physics. This edition also contains a few problems taken from Oxford University examination papers; I would like to thank Oxford University Press for permission to use them.

Let me finally mention that the second Russian edition was dedicated to the memory of Professor I M Shmushkevich.

Oxford,
October 1977 D ter Haar

Preface to the first Russian edition

This book contains about 750 problems on classical electrodynamics and its more important applications, including over 150 problems on the special theory of relativity, and about 70 problems on vector and tensor analysis.

In addition to problems illustrating fundamental concepts and laws of electrodynamics, which can be solved by purely mathematical methods, the collection includes a large number of more complicated problems (these are indicated by asterisks). Some of the solutions involve a considerable amount of effort, while others are purely theoretical in nature and follow from a lecture course on electrodynamics (propagation of waves in anisotropic and gyrotropic media, motion of charged particles in the electromagnetic field, representation of the electromagnetic field by a set of oscillators, and so on). Finally, there are problems which are concerned with topics which are not well covered by existing texts, for example, interaction of charged particles with matter (Chapter 13), applications of conservation laws to the analysis of collision processes and particle disintegration (Chapter 11), ferromagnetic resonance (Chapter 6), and so on.

The second part of the book gives answers and solutions to a large number of these problems.

Each section is prefaced by a short theoretical introduction in which the necessary formulae are given. These short introductions do not pretend to be complete; more extensive treatments will be found in the books listed in the bibliography.

The mathematical appendices review the basic properties of the δ-function and the cylindrical and spherical functions, which are necessary for the solution of the problems.

The present collection is based on lectures given in the Departments of Electronics and Physics and Mechanics of the Leningrad Polytechnic Institute. A large number of the problems were set to third and fourth year students.

In compiling this collection we have made frequent use of the well-known texts of L.D.Landau and E.M.Lifshits, I.E.Tamm, Ya.I.Frenkel', M.Abraham, and R.Becker, W.R.Smythe, J.A.Stratton, and others. A large number of monographs, review papers, and original papers were also consulted.

V.Batygin, I.Toptygin

Contents

problems

Vector and tensor calculus [1]

a Vector and tensor algebra. Transformations of vectors and tensors

A quantity in three-dimensional space, the value of which is unchanged under a rotation of the coordinate system, is called a scalar.

A vector in three-dimensional space is defined as a set of three quantities which transform in accordance with the rule

$$A_i' = \sum_{k=1}^{3} \alpha_{ik} A_k \tag{1.a.1}$$

when the system of coordinates is rotated. Here A_k are the components of the vector along the axes of the original system of coordinates, A_i' are the components along the axes of the rotated system, and α_{ik} are the transformation coefficients which are equal to the cosines of the angles between the kth axis of the original system and the ith axis of the rotated system.

We shall use the following summation rule: the summation sign will be omitted and summation will be understood to have been taken over all repeated subscripts. In accordance with this convention equation (1.a.1) may be rewritten in the form

$$A_i' = \alpha_{ik} A_k \ .$$

A tensor of rank 2 in three-dimensional space is defined as the nine-component quantity T_{ik} $(i, k = 1, 2, 3)$ which transforms in accordance with the rule

$$T_{ik}' = \alpha_{il} \alpha_{km} T_{lm} \ . \tag{1.a.2}$$

(Note that one must sum over l and m.) Similarly, a tensor of rank s in three-dimensional space is defined by the following transformation rule:

$$T_{ikl\dots r}' = \alpha_{ii'} \alpha_{kk'} \dots \alpha_{rr'} T_{i'k'l'\dots r'} \ . \tag{1.a.3}$$

In this expression the quantities T have s indices each.

Quantities which transform as vectors when the coordinate system is rotated may behave in two distinct ways under an inversion (i.e. the transformation $x' = -x$, $y' = -y$, $z' = -z$). Vectors whose components change sign on inversion of the coordinate system are known as polar vectors, or simply vectors. Vectors whose components do not change sign on inversion of the coordinate system are called pseudovectors or axial vectors. We shall not distinguish between covariant and contravariant components of vectors and tensors (see, for instance, Fock 1964) because this distinction is unimportant for the problems considered in this book.

[1] We refer to textbooks such as those by Smirnov (1964), Stratton (1941), Morse and Feshbach (1953), and Margenau and Murphy (1956) for more details of the general theory.

The vector product of two polar vectors is an example of an axial vector. Similarly, a tensor of rank s is referred to simply as a tensor if its components transform on inversion as the products of s coordinates, i.e. when the result of the transformation is to multiply them by $(-1)^s$. When the result of the inversion is to multiply the components by $(-1)^{s+1}$ then the tensor is referred to as a pseudotensor.

The array

$$\hat{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \tag{1.a.4}$$

is called the transformation matrix. The determinant whose elements are equal to the elements of a given matrix is called the determinant of that matrix. Thus, the determinant of the matrix in equation (1.a.4) is

$$|\hat{\alpha}| = \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{vmatrix} . \tag{1.a.5}$$

The sum $\hat{\alpha}+\hat{\beta}$ of two matrices is defined as the matrix $\hat{\gamma}$ whose elements are equal to the sums of the corresponding elements in the component matrices. Thus

$$\gamma_{ik} = \alpha_{ik} + \beta_{ik} . \tag{1.a.6}$$

The product $\hat{\alpha}\hat{\beta}$ of two matrices is defined as the matrix $\hat{\gamma}$ whose elements are obtained by multiplying the components α_{ik} and β_{ik} in accordance with the rule

$$\gamma_{ik} = \alpha_{il}\beta_{lk} . \tag{1.a.7}$$

(Note that there is a summation over l.) The matrix $\hat{\gamma}$ represents the transformation which results after the application of first the transformation corresponding to $\hat{\beta}$ and then the one corresponding to $\hat{\alpha}$.

The unit matrix is defined by

$$\hat{1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.a.8}$$

and describes the identity transformation $A_i' = A_i$. The elements of the unit matrix may be represented by the Kronecker symbol δ_{ik} which is such that

$$\delta_{ik} = \begin{cases} 1 & i = k , \\ 0 & i \neq k . \end{cases} \tag{1.a.9}$$

A matrix of the form

$$\hat{\alpha} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \tag{1.a.10}$$

is called a diagonal matrix. A matrix whose elements satisfy the condition

$$\alpha_{ik}\alpha_{il} = \delta_{kl} \tag{1.a.11}$$

is called an orthogonal matrix.

The reciprocal (or inverse) matrix $\hat{\alpha}^{-1}$ is defined by

$$\hat{\alpha}\hat{\alpha}^{-1} = \hat{\alpha}^{-1}\hat{\alpha} = \hat{1}\,, \tag{1.a.12}$$

and describes the reciprocal transformation, i.e. if $A_i' = \alpha_{ik}A_k$ then $A_k = \alpha_{ki}^{-1}A_i'$. Finally, the transposed matrix $\hat{\tilde{\alpha}}$ is obtained from $\hat{\alpha}$ by interchanging the columns and the rows, so that

$$\hat{\tilde{\alpha}} = \begin{pmatrix} \alpha_{11} & \alpha_{21} & \alpha_{31} \\ \alpha_{12} & \alpha_{22} & \alpha_{32} \\ \alpha_{13} & \alpha_{23} & \alpha_{33} \end{pmatrix}, \qquad \tilde{\alpha}_{ik} = \alpha_{ki}\,. \tag{1.a.13}$$

1.1 Find the cosine of the angle θ between two directions $\boldsymbol{n}$ and $\boldsymbol{n}'$ which are defined in a spherical system of coordinates by the angles ϑ, ϕ and ϑ', ϕ' respectively.

1.2 Prove the identities

$$\begin{aligned} ([A \wedge B] \cdot [C \wedge D]) &= (A \cdot C)(B \cdot D) - (A \cdot D)(B \cdot C)\,; \\ [[A \wedge B] \wedge [C \wedge D]] &= (A \cdot [B \wedge D])C - (A \cdot [B \wedge C])D \\ &= (A \cdot [C \wedge D])B - (B \cdot [C \wedge D])A\,. \end{aligned}$$

1.3 A set of three quantities a_i $(i = 1, 2, 3)$ is given in all Cartesian systems of coordinates and it is known that $a_i b_i$ is invariant with respect to rotations and reflections. Show that if b_i is a vector (pseudovector) then a_i is also a vector (pseudovector).

1.4 Show that if $a_i = T_{ik} b_k$ in every system of coordinates, where T_{ik} is a tensor of rank 2 while b_k is a vector, then a_i is also a vector.

1.5 Show that $\partial a_i/\partial x_k$ is a tensor of rank 2.

1.6 Show that if T_{ik} is a tensor of rank 2 and P_{ik} is a pseudotensor of rank 2, then $T_{ik}P_{ik}$ is a pseudoscalar.

1.7 Show that the symmetry of a tensor is a property which is invariant with respect to rotations, i.e. a tensor which is symmetric (skew-symmetric) in a given coordinate system remains symmetric (skew-symmetric) in all other systems which are rotated with respect to the original system.

1.8 Show that if the tensor S_{ik} is symmetric while the tensor A_{ik} is skew-symmetric, then

$$A_{ik}S_{ik} = 0\,.$$

1.9 Show that the sum of the diagonal components of a tensor of rank 2 is an invariant quantity.

1.10* In some cases it is convenient to use the cyclic components defined by

$$a_{\pm 1} = \mp \frac{(a_x \pm i a_y)}{\sqrt{2}}\,, \qquad a_0 = a_z$$

instead of the Cartesian components a_x, a_y, a_z of a vector. Express the scalar and vector products of two vectors in terms of their cyclic components. Find also the cyclic components of the position vector in terms of the spherical Legendre functions [2].

1.11* Find the components of the tensor ϵ_{ik}^{-1} which is reciprocal to ϵ_{ik}. Consider in particular the case where ϵ_{ik} is a symmetric tensor defined along the principal axes.

1.12 Suppose that in all coordinate systems the components of the vector $\boldsymbol{a}$ are linear functions of the components of another vector $\boldsymbol{b}$ so that $a_i = \epsilon_{ik} b_k$. Show that the quantity ϵ_{ik} is a tensor of rank 2. (More precisely, ϵ_{ik} is a tensor if $\boldsymbol{a}$ and $\boldsymbol{b}$ are both polar vectors or pseudovectors, whereas ϵ_{ik} is a pseudotensor if one of the vectors is polar and the other is axial.)

1.13 Show that the set of quantities $A_{ikl}B_{ik}$, where A_{ikl} is a tensor of rank 3 and B_{ik} is a tensor of rank 2, is a vector.

1.14 Find the transformation law for the set of volume integrals $T_{ik} = \int x_i x_k \, d^3 r$, which describes space rotations and reflections (x_i and x_k are the Cartesian coordinates).

1.15 Set up the transformation matrices for the basic unit vectors which represent the transformation from Cartesian coordinates to spherical coordinates and vice versa, and also the transformation from Cartesian coordinates to cylindrical coordinates and vice versa.

1.16 Write down the transformation matrices for the components of a vector which describe (a) the reflection of the three coordinate axes or (b) the rotation of the Cartesian system of coordinates about the z-axis through an angle ϕ.

1.17 Find the transformation matrix for the components of a vector, which describes the rotation of the coordinate axes defined by the Euler angles θ, ϕ, ψ (figure 1.17.1), by multiplying together the matrices corresponding to rotations about the z-axis through angle ϕ, about the line ON through an angle θ, and about the z'-axis through an angle ψ.

1.18 Find the matrix $\hat{D}\,(\theta, \phi, \psi)$ for the transformation of the cyclic coordinates of a vector (cf problem 1.10) when the coordinate system is rotated through the Euler angles θ, ϕ, and ψ (figure 1.17.1).

[2] Spherical functions are defined in appendix 2.

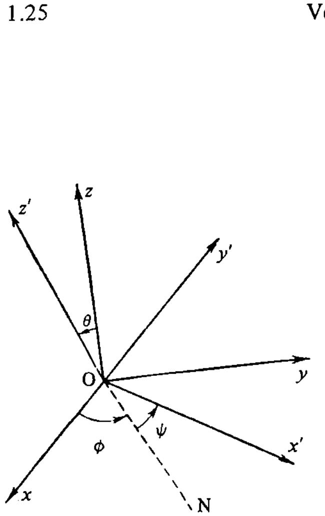


Figure 1.17.1.

1.19* Show that the matrix representing an infinitesimal rotation of a system of coordinates may be written in the form

$$\hat{\alpha} = \hat{1} + \hat{\epsilon}\,,$$

where $\hat{\epsilon}$ is a skew-symmetric matrix ($\epsilon_{ik} = -\epsilon_{ki}$). Elucidate the geometrical meaning of ϵ_{ik}.

1.20 Show that if $\hat{\alpha}$ is an orthogonal transformation matrix then the corresponding transposed matrix represents the inverse transformation.

1.21 Show that the matrix representing the reflection or rotation of the three basic unit vectors of a coordinate system is identical to the matrix describing the transformation of the components of a vector.

1.22* Show that when an even number of coordinate axes are reflected or rotated, the transformation determinant is equal to $+1$, whereas the corresponding result for an odd number of coordinate axes is -1. Transformations which have a determinant equal to $+1$ are called *proper* transformations and those with a determinant equal to -1 are called *improper* transformations.

1.23 Show that if the corresponding components of two vectors in one system of coordinates are respectively proportional to each other, then they are also proportional in any other system of coordinates (such vectors are said to be parallel).

1.24* A set of quantities e_{ikl} is given in all Cartesian systems of coordinates and has the following property: transposition of any two subscripts of e_{ikl} gives rise to a change of sign, and $e_{123} = 1$. Show that e_{ikl} is a pseudotensor of rank 3 (the completely antisymmetric unit tensor of rank 3).

1.25 Show that the components of a skew-symmetric tensor of rank 2 transform on rotation as the components of a vector.

1.26 Write down expressions for the components of the vector product of two vectors and for the curl of a vector in terms of the tensor e_{ikl}. Determine how these quantities transform on rotation or reflection.

1.27 Prove the following equations

(a) $e_{ikl}e_{lmn} = \delta_{im}\delta_{kn} - \delta_{in}\delta_{km}$; (b) $e_{ikl}e_{klm} = 2\delta_{im}$.

1.28 Rewrite the following expressions in an invariant vector form:
(a) $e_{inl}e_{irs}e_{lmp}e_{stp}a_n a_r b_m c_t$; (b) $e_{inl}e_{krs}e_{lmp}e_{stp}a_r a'_n b_k b'_i c_t c'_m$.

1.29 Show that

$$T_{ik}a_i b_k - T_{ik}a_k b_i = 2(\boldsymbol{\omega} \cdot [\boldsymbol{a} \wedge \boldsymbol{b}]) ,$$

where T_{ik} is an arbitrary tensor of rank 2, $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, and $\boldsymbol{\omega}$ is a vector equivalent to the skew-symmetric part of T_{ik}.

1.30 Express the product

$$(\boldsymbol{a} \cdot [\boldsymbol{b} \wedge \boldsymbol{c}])(\boldsymbol{a}' \cdot [\boldsymbol{b}' \wedge \boldsymbol{c}'])$$

in the form of a sum of terms containing only scalar products of vectors.
Hint. Use the theorem on the multiplication of determinants, or the pseudotensor e_{ikl} of rank 3 (cf problem 1.24).

1.31* Show that:
(a) the only vector whose components are identical in all systems of coordinates is the zero vector,
(b) any tensor of rank 2 whose components are identical in all systems of coordinates is proportional to δ_{ik},
(c) any rank 3 tensor whose components are identical in all systems of coordinates is proportional to e_{ikl}, and
(d) any rank 4 tensor whose components are identical in all systems of coordinates is proportional to $\delta_{ik}\delta_{lm} + \delta_{im}\delta_{kl} + \delta_{il}\delta_{km}$.

1.32* Suppose that $\boldsymbol{n}$ is a unit vector which is such that it is equally likely to lie along any direction in space. Find the average values of its components and of the products $\overline{n_i}$, $\overline{n_i n_k}$, $\overline{n_i n_k n_l}$, $\overline{n_i n_k n_l n_m}$ by using their transformation properties rather than by a direct evaluation of the corresponding integrals.

1.33 Find the averages over all directions of the following expressions:

$$\overline{(\boldsymbol{a} \cdot \boldsymbol{n})^2},\ \overline{(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{b} \cdot \boldsymbol{n})},\ \overline{(\boldsymbol{a} \cdot \boldsymbol{n})\boldsymbol{n}},\ \overline{[\boldsymbol{a} \wedge \boldsymbol{n}]^2},\ \overline{([\boldsymbol{a} \wedge \boldsymbol{n}] \cdot [\boldsymbol{b} \wedge \boldsymbol{n}])},$$
$$\overline{(\boldsymbol{a} \cdot \boldsymbol{n})(\boldsymbol{b} \cdot \boldsymbol{n})(\boldsymbol{c} \cdot \boldsymbol{n})(\boldsymbol{d} \cdot \boldsymbol{n})}$$

where $\boldsymbol{n}$ is a unit vector which is such that it is equally likely to lie along any direction and $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are constant vectors.
Hint. Use the results of the preceding problem.

1.34 Obtain all the possible independent invariant forms involving the polar vectors $\boldsymbol{n}$, $\boldsymbol{n}'$, and the pseudovector $\boldsymbol{l}$.

1.35 Find the independent pseudoscalars which can be constructed from (a) two polar vectors $\boldsymbol{n}$, $\boldsymbol{n}'$ and one pseudovector $\boldsymbol{l}$, and (b) three polar vectors $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, $\boldsymbol{n}_3$.

b Vector analysis

In an arbitrary orthogonal system of coordinates q_1, q_2, q_3 the square of an element of length is given by

$$\mathrm{d}l^2 = h_1^2\,\mathrm{d}q_1^2 + h_2^2\,\mathrm{d}q_2^2 + h_3^2\,\mathrm{d}q_3^2\,, \tag{1.b.1}$$

and an element of volume is given by

$$\mathrm{d}V = h_1 h_2 h_3\,\mathrm{d}q_1\,\mathrm{d}q_2\,\mathrm{d}q_3\,, \tag{1.b.2}$$

where

$$h_i = \left[\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2\right]^{1/2} \tag{1.b.3}$$

are functions of the coordinates. The various differential operations may then be written down as follows:

$$\left.\begin{aligned}
(\operatorname{grad}\Phi)_i &= \frac{1}{h_i}\frac{\partial\Phi}{\partial q_i}\,;\\
\operatorname{div}A &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}(h_2h_3A_1)+\frac{\partial}{\partial q_2}(h_1h_3A_2)+\frac{\partial}{\partial q_3}(h_1h_2A_3)\right]\,;\\
\operatorname{curl}A &= \begin{vmatrix} \dfrac{e_1}{h_2h_3} & \dfrac{e_2}{h_1h_3} & \dfrac{e_3}{h_1h_2} \\[2ex] \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\[2ex] h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}\,;\\
\nabla^2\Phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial q_2}\right)+\frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial q_3}\right)\right].
\end{aligned}\right. \tag{1.b.4}$$

In the formula for $\operatorname{curl}A$ the differential operators $\partial/\partial q_i$ operate on the elements of the last row of the determinant.

In spherical polar coordinates we have

$$\left.\begin{aligned}
&x = r\sin\vartheta\cos\phi\,, \qquad y = r\sin\vartheta\sin\phi\,, \qquad z = r\cos\vartheta\,;\\
&h_r = 1\,, \qquad h_\vartheta = r\,, \qquad h_\phi = r\sin\vartheta\,;\\
&\operatorname{grad}\Phi = \boldsymbol{e}_r\frac{\partial\Phi}{\partial r}+\frac{\boldsymbol{e}_\vartheta}{r}\frac{\partial\Phi}{\partial\vartheta}+\frac{\boldsymbol{e}_\phi}{r\sin\phi}\frac{\partial\Phi}{\partial\phi}\,;
\end{aligned}\right\} \tag{1.b.5}$$

Equations (1.b.5) continued over

$$\left.\begin{aligned}
\operatorname{div}A &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r)+\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}(A_\vartheta \sin\vartheta)+\frac{1}{r\sin\vartheta}\frac{\partial A_\phi}{\partial\phi}\ ;\\
(\operatorname{curl}A)_r &= \frac{1}{r\sin\vartheta}\left[\frac{\partial}{\partial\vartheta}(A_\phi\sin\vartheta)-\frac{\partial A_\vartheta}{\partial\phi}\right]\ ;\\
(\operatorname{curl}A)_\vartheta &= \frac{1}{r\sin\vartheta}\frac{\partial A_r}{\partial\phi}-\frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\ ;\\
(\operatorname{curl}A)_\phi &= \frac{1}{r}\frac{\partial(rA_\vartheta)}{\partial r}-\frac{1}{r}\frac{\partial A_r}{\partial\vartheta}\ ;\\
\nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\Phi}{\partial\vartheta}\right)+\frac{1}{r^2\sin^2\vartheta}\frac{\partial^2\Phi}{\partial\phi^2}\ .
\end{aligned}\right\} \tag{1.b.5}$$

In the cylindrical system of coordinates we have

$$\left.\begin{aligned}
&x = r\cos\phi\ , \qquad y = r\sin\phi\ , \qquad z = z\ ;\\
&h_r = 1\ , \qquad h_\phi = r\ , \qquad h_z = 1\ ;\\
&\operatorname{grad}\Phi = e_r\frac{\partial\Phi}{\partial r}+\frac{e_\phi}{r}\frac{\partial\Phi}{\partial\phi}+e_z\frac{\partial\Phi}{\partial z}\ ;\\
&\operatorname{div}A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r)+\frac{1}{r}\frac{\partial A_\phi}{\partial\phi}+\frac{\partial A_z}{\partial z}\ ;\\
&(\operatorname{curl}A)_r = \frac{1}{r}\frac{\partial A_z}{\partial\phi}-\frac{\partial A_\phi}{\partial z}\ ; \qquad (\operatorname{curl}A)_\phi = \frac{\partial A_r}{\partial z}-\frac{\partial A_z}{\partial r}\ ;\\
&(\operatorname{curl}A)_z = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)-\frac{1}{r}\frac{\partial A_r}{\partial\phi}\ ;\\
&\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}\ .
\end{aligned}\right\} \tag{1.b.6}$$

The following identities hold for any A and Φ

$$\operatorname{curl}\operatorname{grad}\Phi \equiv 0\ , \qquad \operatorname{div}\operatorname{curl}A \equiv 0\ , \qquad \operatorname{div}\operatorname{grad}\Phi = \nabla^2\Phi\ . \tag{1.b.7}$$

The following integral theorems may be used to transform volume, surface, and contour integrals.

Gauss' theorem:

$$\int_V \operatorname{div}A\ \mathrm{d}^3r = \oint_S (A\cdot\mathrm{d}^2S)\ , \tag{1.b.8}$$

where V is a given volume and S is a closed surface bounding V.

Stokes' theorem:

$$\oint_l (A\cdot\mathrm{d}l) = \int_S (\operatorname{curl}A\cdot\mathrm{d}^2S)\ , \tag{1.b.9}$$

where l is a closed contour and S is an arbitrary surface drawn through this contour.

In equations (1.b.8) and (1.b.9) the vector A should be a differentiable function of the coordinates.

1.36 Write down the cyclic components of the gradient in spherical coordinates (cf problem 1.10).

1.37 Use Cartesian, spherical, and cylindrical coordinates to obtain expressions for $\operatorname{div}\boldsymbol{r}$, $\operatorname{curl}\boldsymbol{r}$, $\operatorname{grad}(\boldsymbol{l}\cdot\boldsymbol{r})$, and $(\boldsymbol{l}\cdot\nabla)\boldsymbol{r}$ where $\boldsymbol{r}$ is the position vector and $\boldsymbol{l}$ a constant vector.

1.38 Use only spherical (or cylindrical) coordinates to find $\operatorname{curl}[\boldsymbol{\omega}\wedge\boldsymbol{r}]$ where $\boldsymbol{\omega}$ is a constant vector parallel to the z-axis.

1.39 Prove the following identities:

$$\begin{aligned}
&\operatorname{grad}(\varphi\psi) = \varphi\operatorname{grad}\psi + \psi\operatorname{grad}\varphi\;;\\
&\operatorname{div}(\varphi\boldsymbol{A}) = \varphi\operatorname{div}\boldsymbol{A} + (\boldsymbol{A}\cdot\operatorname{grad})\varphi\;;\\
&\operatorname{curl}(\varphi\boldsymbol{A}) = \varphi\operatorname{curl}\boldsymbol{A} - [\boldsymbol{A}\wedge\operatorname{grad}]\varphi\;;\\
&\operatorname{div}[\boldsymbol{A}\wedge\boldsymbol{B}] = (\boldsymbol{B}\cdot\operatorname{curl}\boldsymbol{A}) - (\boldsymbol{A}\cdot\operatorname{curl}\boldsymbol{B})\;;\\
&\operatorname{curl}[\boldsymbol{A}\wedge\boldsymbol{B}] = \boldsymbol{A}\operatorname{div}\boldsymbol{B} - \boldsymbol{B}\operatorname{div}\boldsymbol{A} + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}\;;\\
&\operatorname{grad}(\boldsymbol{A}\cdot\boldsymbol{B}) = [\boldsymbol{A}\wedge\operatorname{curl}\boldsymbol{B}] + [\boldsymbol{B}\wedge\operatorname{curl}\boldsymbol{A}] + (\boldsymbol{B}\cdot\nabla)\boldsymbol{A} + (\boldsymbol{A}\cdot\nabla)\boldsymbol{B}\,.
\end{aligned}$$

Hint. Use the operator ∇ and the differentiation and multiplication rules for vectors and not the components along the coordinate axes.

1.40 Prove the identities

$$\begin{aligned}
&(\boldsymbol{C}\cdot\operatorname{grad})(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot(\boldsymbol{C}\cdot\nabla)\boldsymbol{B}) + (\boldsymbol{B}\cdot(\boldsymbol{C}\cdot\nabla)\boldsymbol{A})\;;\\
&(\boldsymbol{C}\cdot\nabla)[\boldsymbol{A}\wedge\boldsymbol{B}] = [\boldsymbol{A}\wedge(\boldsymbol{C}\cdot\nabla)\boldsymbol{B}] - [\boldsymbol{B}\wedge[\boldsymbol{C}\cdot\nabla)\boldsymbol{A}]\;;\\
&(\nabla\cdot\boldsymbol{A})\boldsymbol{B} = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + \boldsymbol{B}\operatorname{div}\boldsymbol{A}\;;\\
&([\boldsymbol{A}\wedge\boldsymbol{B}]\cdot\operatorname{curl}\boldsymbol{C}) = (\boldsymbol{B}\cdot(\boldsymbol{A}\cdot\nabla)\boldsymbol{C}) - (\boldsymbol{A}\cdot(\boldsymbol{B}\cdot\nabla)\boldsymbol{C})\;;\\
&[[\boldsymbol{A}\wedge\nabla]\wedge\boldsymbol{B}] = (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} + [\boldsymbol{A}\wedge\operatorname{curl}\boldsymbol{B}] - \boldsymbol{A}\operatorname{div}\boldsymbol{B}\;;\\
&[[\nabla\wedge\boldsymbol{A}]\wedge\boldsymbol{B}] = \boldsymbol{A}\operatorname{div}\boldsymbol{B} - (\boldsymbol{A}\cdot\nabla)\boldsymbol{B} - [\boldsymbol{A}\wedge\operatorname{curl}\boldsymbol{B}] - [\boldsymbol{B}\wedge\operatorname{curl}\boldsymbol{A}]\,.
\end{aligned}$$

1.41 Write down expressions for $\operatorname{grad}\varphi(r)$, $\operatorname{div}\varphi(r)\boldsymbol{r}$, $\operatorname{curl}\varphi(r)\boldsymbol{r}$, and $(\boldsymbol{l}\cdot\nabla)\varphi(r)\boldsymbol{r}$.

1.42 Find the function $\varphi(r)$ which satisfies the condition $\operatorname{div}\varphi(r)\boldsymbol{r} = 0$.

1.43 Find the divergence and curl of the following vectors:

$$(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{b}\,,\qquad (\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{r}\,,\qquad [\boldsymbol{a}\wedge\boldsymbol{r}]\,,\qquad \varphi(r)[\boldsymbol{a}\wedge\boldsymbol{r}]\,,\qquad [\boldsymbol{r}\wedge[\boldsymbol{a}\wedge\boldsymbol{r}]]\,,$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors.

1.44 Evaluate:

$$\operatorname{grad}(\boldsymbol{A}(r)\cdot\boldsymbol{r})\,,\quad \operatorname{grad}(\boldsymbol{A}(r)\cdot\boldsymbol{B}(r))\,,\quad \operatorname{div}\varphi(r)\boldsymbol{A}(r)\,,\quad \operatorname{curl}\varphi(r)\boldsymbol{A}(r)\,,$$
$$(\boldsymbol{l}\cdot\nabla)\varphi(r)\boldsymbol{A}(r)\,.$$

1.45 Evaluate:

$$\operatorname{grad}\left[\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3}\right], \quad \operatorname{curl}\left\{\frac{[\boldsymbol{p}\wedge\boldsymbol{r}]}{r^3}\right\},$$

where $\boldsymbol{p}$ is a constant vector, by using the expressions for the gradient and the curl in terms of spherical coordinates. Give a geometrical interpretation of these vectors.

1.46 Show that

$$(\boldsymbol{A}\cdot\boldsymbol{\nabla})\boldsymbol{A} = -[\boldsymbol{A}\wedge\operatorname{curl}\boldsymbol{A}],$$

when $\boldsymbol{A}^2$ = constant.

1.47 Write down the components of the vector $\nabla^2\boldsymbol{A}$ along the axes of the spherical system of coordinates.

Hint. Use the identity

$$\nabla^2\boldsymbol{A} = -\operatorname{curl}\operatorname{curl}\boldsymbol{A} + \operatorname{grad}\operatorname{div}\boldsymbol{A}.$$

1.48 Write down the components of the vector $\nabla^2\boldsymbol{A}$ along the axes of the cylindrical system of coordinates.

1.49 Transform the volume integral $\int(\operatorname{grad}\varphi\cdot\operatorname{curl}\boldsymbol{A})\,\mathrm{d}^3r$ into a surface integral.

1.50 Evaluate the integrals

$$\oint\boldsymbol{r}(\boldsymbol{a}\cdot\boldsymbol{n})\,\mathrm{d}^2S, \qquad \oint(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{n}\,\mathrm{d}^2S$$

where $\boldsymbol{a}$ is a constant vector and $\boldsymbol{n}$ is the unit normal.

1.51 Transform the surface integrals

$$\oint\boldsymbol{n}\varphi\,\mathrm{d}^2S, \qquad \oint[\boldsymbol{n}\wedge\boldsymbol{a}]\,\mathrm{d}^2S, \qquad \oint(\boldsymbol{n}\cdot\boldsymbol{b})\boldsymbol{a}\,\mathrm{d}^2S,$$

where $\boldsymbol{b}$ is a constant vector and $\boldsymbol{n}$ the unit normal, into volume integrals.

1.52 Use one of the identities established in the preceding problem to derive the law of Archimedes by summing all the forces on the surface elements of a body immersed in a liquid.

1.53* Let $f(\boldsymbol{a},\boldsymbol{r})$ satisfy the condition

$$f(c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2, \boldsymbol{r}) = c_1 f(\boldsymbol{a}_1,\boldsymbol{r}) + c_2 f(\boldsymbol{a}_2,\boldsymbol{r}),$$

where c_1 and c_2 are arbitrary constants, and suppose further that $f(\boldsymbol{a},\boldsymbol{r})$ is a differentiable function of $\boldsymbol{r}$. Show that if V is an arbitrary volume, S is its bounding surface, and $\boldsymbol{n}$ is the unit normal to this surface then

$$\oint f(\boldsymbol{n},\boldsymbol{r})\,\mathrm{d}^2S = \int f(\boldsymbol{\nabla},\boldsymbol{r})\,\mathrm{d}^3r,$$

which is the generalised Gauss theorem. The operator ∇ in the integrand $f(\nabla, \boldsymbol{r})$ operates on $\boldsymbol{r}$ and lies on the left of all the variables

Hint. Express $\boldsymbol{n}$ in terms of the unit vectors of a Cartesian system of coordinates and use the relation (cf problem 1.51)

$$\int \frac{\partial \varphi}{\partial x} \mathrm{d}^3 r = \oint \varphi n_x \, \mathrm{d}^2 S .$$

1.54 Solve problems 1.50 and 1.51 with the aid of the generalised Gauss theorem proved in the preceding problem.

1.55 Transform the closed line integral $\oint \varphi \, \mathrm{d}\boldsymbol{l}$ into a surface integral.

1.56 Transform the closed line integral $\oint u \, \mathrm{d}f$ into a surface integral, where u and f are scalar functions of coordinates.

1.57 Prove the identity

$$\int (\boldsymbol{A} \cdot \operatorname{curl} \operatorname{curl} \boldsymbol{B} - \boldsymbol{B} \cdot \operatorname{curl} \operatorname{curl} \boldsymbol{A}) \, \mathrm{d}^3 \boldsymbol{r} = \oint (\{[\boldsymbol{B} \wedge \operatorname{curl} \boldsymbol{A}] - [\boldsymbol{A} \wedge \operatorname{curl} \boldsymbol{B}]\} \cdot \mathrm{d}^2 \boldsymbol{S}) .$$

1.58 The vector $\boldsymbol{A}$ satisfies the condition $\operatorname{div} \boldsymbol{A} = 0$ within the volume V, whereas on the surface S which bounds this volume $A_n = 0$. Show that

$$\int_V \boldsymbol{A} \, \mathrm{d}^3 \boldsymbol{r} = 0 .$$

1.59* Show that

$$\operatorname{div}_{\boldsymbol{R}} \int \frac{\boldsymbol{A}(\boldsymbol{r}) \, \mathrm{d}^3 \boldsymbol{r}}{|\boldsymbol{R} - \boldsymbol{r}|} = 0 ,$$

where $\boldsymbol{A}(\boldsymbol{r})$ is a vector satisfying the conditions of problem 1.58.

1.60 Prove the Gauss theorem

$$\int \frac{\partial T_{ik}}{\partial x_i} \mathrm{d}V = \oint T_{ik} \, \mathrm{d}S_i$$

for a three-dimensional tensor of rank 2.

Hint. Start with the Gauss theorem for a vector $A_i = T_{ik} a_k$ where $\boldsymbol{a}$ is an arbitrary constant vector.

1.61 Find the general form of the solution of Laplace's equation for the scalar functions (a) $f(r)$, (b) $f(\vartheta)$, and (c) $f(\phi)$ where r, ϑ, and ϕ are the spherical polar coordinates.

1.62 Repeat problem 1.61 for the cylindrical polar coordinates r, ϕ, and z.

1.63 Show that if the scalar function ψ is a solution of the equation $\nabla^2\psi + k^2\psi = 0$ and $\boldsymbol{a}$ is a constant vector, then the vector functions

$$\boldsymbol{L} = \operatorname{grad}\psi\,, \quad \boldsymbol{M} = \operatorname{curl}(\boldsymbol{a}\psi)\,, \quad \boldsymbol{N} = \operatorname{curl}\boldsymbol{M}$$

satisfy the equation

$$\nabla^2\boldsymbol{A} + k^2\boldsymbol{A} = 0\,.$$

1.64* The equation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1\,, \qquad a > b > c$$

represents an ellipsoid with semiaxes a, b, and c. The equations

$$\frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{z^2}{c^2+\xi} = 1\,, \qquad \xi \geqslant -c^2\,,$$

$$\frac{x^2}{a^2+\eta} + \frac{y^2}{b^2+\eta} + \frac{z^2}{c^2+\eta} = 1\,, \qquad -c^2 \geqslant \eta \geqslant -b^2\,,$$

$$\frac{x^2}{a^2+\zeta} + \frac{y^2}{b^2+\zeta} + \frac{z^2}{c^2+\zeta} = 1\,, \qquad -b^2 \geqslant \zeta \geqslant -a^2\,,$$

represent, respectively, an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets, which are confocal with the above ellipsoid. One of these surfaces, characterised by the quantities ξ, η, ζ, passes through each point in space. The quantities ξ, η, ζ are called the ellipsoidal coordinates of a point x, y, z. Find the transformation formulae relating ξ, η, ζ and x, y, z. Prove the orthogonality of the ellipsoidal system of coordinates and find the coefficients h_i of equation (1.b.3) and the Laplace operator in ellipsoidal coordinates.

1.65* When $a = b > c$, the ellipsoidal system of coordinates (cf preceding problem) degenerates into the so-called oblate spheroidal system. The coordinate ζ thus becomes a constant equal to $-a^2$ and should be replaced by another coordinate.

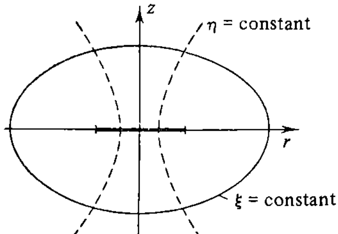


Figure 1.65.1.

The azimuthal angle ϕ in the xy-plane is chosen as the latter coordinate. The coordinates ξ and η are then defined by

$$\frac{r^2}{a^2+\xi}+\frac{z^2}{c^2+\xi}=1\,, \qquad \frac{r^2}{a^2+\eta}+\frac{z^2}{c^2+\eta}=1\,, \qquad r^2=x^2+y^2\,,$$

where $\xi \geqslant -c^2$, $-c^2 \geqslant \eta \geqslant -a^2$. The surfaces $\xi=$ constant represent oblate ellipsoids of revolution with the z-axis as the axis of rotation, and the surfaces $\eta=$ constant are hyperboloids of one sheet of revolution which are confocal with them (figure 1.65.1).

Find expressions for r and z in terms of the oblate spheroidal coordinates. Find also the coefficients h_i of equation (1.b.3) and the Laplace operator in terms of these coordinates.

1.66* The prolate spheroidal system of coordinates is obtained from the ellipsoidal system (cf problem 1.64) when $a > b = c$. The coordinate η then degenerates into a constant, and should be replaced by the azimuthal angle ϕ measured from the y-axis in the yz-plane. The coordinates ξ, ζ are defined by

$$\frac{x^2}{a^2+\xi}+\frac{r^2}{b^2+\xi}=1\,, \qquad \frac{x^2}{a^2+\zeta}+\frac{r^2}{b^2+\zeta}=1\,, \qquad r^2=y^2+z^2\,,$$

where $\xi \geqslant -b^2$, $-b^2 \geqslant \zeta \geqslant -a^2$.

The surfaces representing constant values of ξ and ζ are prolate ellipsoids and hyperboloids of two sheets of revolution (figure 1.66.1). Express x and r in terms of ξ and ζ. Find the coefficients h_i of equation (1.b.3) and the Laplace operator in terms of the variables ξ, ζ, ϕ.

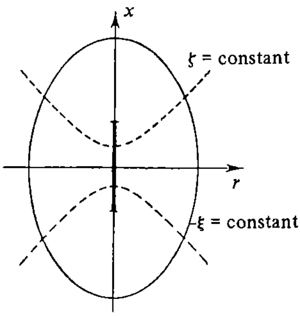


Figure 1.66.1.

1.67 Bispherical coordinates ξ, η, ϕ are related to the Cartesian coordinates by the expression

$$x=\frac{a\sin\eta\cos\phi}{\cosh\xi-\cos\eta}\,, \qquad y=\frac{a\sin\eta\sin\phi}{\cosh\xi-\cos\eta}\,, \qquad z=\frac{a\sinh\xi}{\cosh\xi-\cos\eta}\,,$$

where a is a constant parameter and $-\infty<\xi<\infty$, $0<\eta<\pi$, $0<\phi<2\pi$. Show that the coordinate surfaces $\xi=$ constant represent the spheres

$$x^2+y^2+(z-a\coth\xi)^2 = \left(\frac{a}{\sinh\xi}\right)^2,$$

the surfaces η = constant represent spindle-shaped surfaces of revolution about the z-axis, whose equation is

$$[(x^2+y^2)^{1/2}-a\cot\eta]^2+z^2 = \left(\frac{a}{\sin\eta}\right)^2,$$

and the surfaces ϕ = constant are semi-infinite planes diverging from the z-axis (figure 1.67.1). Show that these coordinate surfaces are orthogonal to each other and find the coefficients h_i of equation (1.b.3) and the Laplace operator in terms of these coordinates.

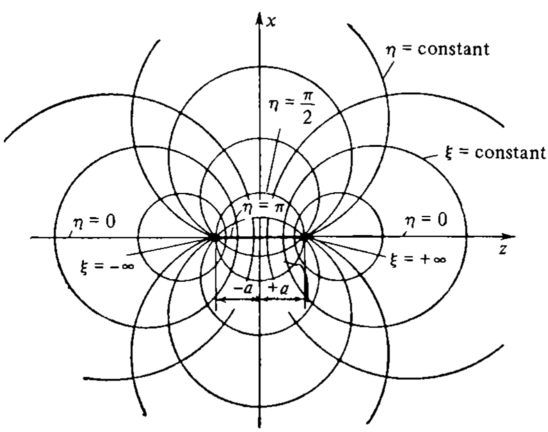


Figure 1.67.1.

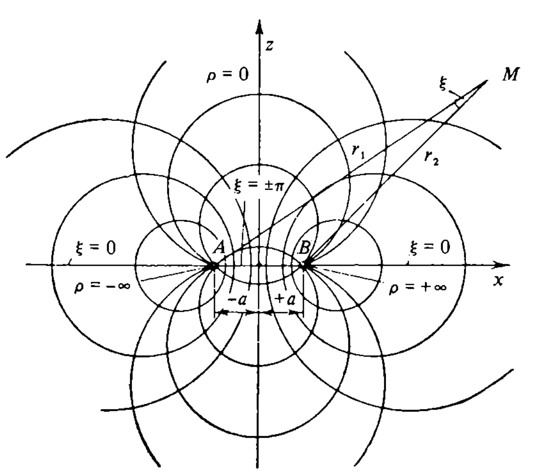


Figure 1.68.1.

1.68 The toroidal coordinates ρ, ξ, ϕ form an orthogonal system and are related to the Cartesian coordinates by the formulae

$$x = \frac{a \sinh\rho \cos\phi}{\cosh\rho - \cos\xi}, \qquad y = \frac{a \sinh\rho \sin\phi}{\cosh\rho - \cos\xi}, \qquad z = \frac{a \sin\xi}{\cosh\rho - \cos\xi},$$

where a is a constant parameter, $-\infty < \rho < \infty$, $-\pi < \xi < \pi$, and ϕ is an azimuthal angle lying between 0 and π.

Show that $\rho = \ln(r_1/r_2)$ (see figure 1.68.1 showing the planes $\phi =$ constant, $\phi + \pi =$ constant) and that ξ represents the angle between r_1 and r_2 ($\xi > 0$ when $z > 0$; $\xi < 0$ when $z < 0$). What is the form of the ρ and ξ coordinate surfaces? Find the coefficients h_i of equation (1.b.3).

Electrostatics in vacuo [1]

This chapter contains problems which are concerned with the determination of the potential $\varphi(\boldsymbol{r})$ and field strength $\boldsymbol{E}(\boldsymbol{r})$ from given volume, surface, and linear charge distributions. The distribution of point charges may be described by a volume density

$$\rho(\boldsymbol{r}) = \sum_i q_i \delta(\boldsymbol{r}-\boldsymbol{r}_i)$$

where q_i is the magnitude of the ith charge, $\boldsymbol{r}_i$ is the position vector of the ith charge, and $\delta(\boldsymbol{r}-\boldsymbol{r}_i)$ is the Dirac delta-function (cf appendix 1). For a static distribution the electric field satisfies the Maxwell equations

$$\operatorname{div}\boldsymbol{E} = 4\pi\rho\,, \qquad \operatorname{curl}\boldsymbol{E} = 0\,. \tag{2.0.1}$$

The integral form of the first of these equations is

$$\oint_S E_n\, \mathrm{d}^2S = 4\pi q\,, \tag{2.0.2}$$

where S is a closed surface; q is the total charge enclosed by this surface, and E_n the component of the electric field in the direction of the outward normal. The potential and the electric field strength are related by the expressions

$$\boldsymbol{E} = -\operatorname{grad}\varphi\,, \qquad \varphi(\boldsymbol{r}) = \int_{\boldsymbol{r}}^{\boldsymbol{r}_0} (\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{r})\,, \qquad \varphi(\boldsymbol{r}_0) = 0\,. \tag{2.0.3}$$

The potential φ satisfies the Poisson equation

$$\nabla^2\varphi = -4\pi\rho\,. \tag{2.0.4}$$

The potential is continuous and finite at all points at which there are no point charges. On a charged surface separating regions 1 and 2 (figure 2.0.1), $\varphi_1 = \varphi_2$. The normal derivatives of φ are discontinuous across a charged surface so that

$$E_{2n} - E_{1n} = 4\pi\sigma \quad \text{or} \quad \frac{\partial\varphi_1}{\partial n} - \frac{\partial\varphi_2}{\partial n} = 4\pi\sigma\,, \tag{2.0.5}$$

where the normal $\boldsymbol{n}$ is directed from region 1 into region 2.

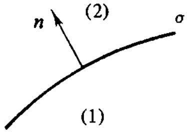


Figure 2.0.1.

[1] For details of the theory we refer to the text books by Smythe (1950), Stratton (1941), Jackson (1962), and Panofsky and Phillips (1962).

The change in the potential on passing through an electric double layer of strength (dipole moment per unit area) τ is given by (see, e.g. Panofsky and Phillips, 1962)

$$\frac{\partial \varphi_1}{\partial n} = \frac{\partial \varphi_2}{\partial n}, \qquad \varphi_2 - \varphi_1 = 4\pi\tau \tag{2.0.6}$$

where the normal $\boldsymbol{n}$ is directed from the negative to the positive side of the layer.

If the potentials corresponding to charge distributions ρ_1 and ρ_2 are φ_1 and φ_2 then the potential corresponding to the charge distribution $\rho = \rho_1 + \rho_2$ is $\varphi = \varphi_1 + \varphi_2$. This is the so-called superposition principle. An analogous result holds for the electric field $\boldsymbol{E}$. In particular, the superposition principle may be used to determine the potential due to a complicated charge distribution by integrating over elementary charges:

$$\varphi(\boldsymbol{r}) = \int \frac{\rho(\boldsymbol{r}')\,\mathrm{d}^3 r'}{|\boldsymbol{r}-\boldsymbol{r}'|}\,. \tag{2.0.7}$$

In the case of surface or line charge distributions, the volume integral in equation (2.0.7) must be replaced by the corresponding surface or line integrals, whereas in the case of a set of point charges it must be replaced by the appropriate summation. The latter remark applies also to all the formulae given below which contain volume integrals over charge distributions.

In the majority of cases, direct evaluation of the integral given by equation (2.0.7) is difficult. It is, therefore, frequently convenient to expand the integrand into a series in powers of x/r or x'/r, and then integrate term by term. This expansion may be obtained either in terms of Cartesian or spherical polar coordinates.

Cartesian coordinates (figure 2.0.2). When $r > a$ (a is the maximum distance of the charges from the origin O) we have (note the summation over repeated Greek indices)

$$\varphi(x, y, z) = \frac{q}{r} - p_\alpha \frac{\partial}{\partial x_\alpha}\frac{1}{r} + \frac{Q_{\alpha\beta}}{2!}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\frac{1}{r} - \frac{Q_{\alpha\beta\gamma}}{3!}\frac{\partial^3}{\partial x_\alpha \partial x_\beta \partial x_\gamma}\frac{1}{r} - \ldots \tag{2.0.8}$$

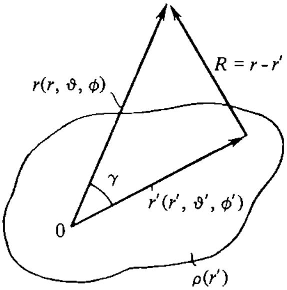


Figure 2.0.2.

The multipole moments q, p_α, $Q_{\alpha\beta}$, ... are given by the following volume integrals

$$q = \int \rho(\boldsymbol{r}')\,\mathrm{d}^3\boldsymbol{r}' \qquad \text{total charge,}$$

$$p_\alpha = \int \rho(\boldsymbol{r}')x'_\alpha\,\mathrm{d}^3\boldsymbol{r}' \qquad \text{dipole moment components,}$$

$$Q_{\alpha\beta} = \int \rho(\boldsymbol{r}')x'_\alpha x'_\beta\,\mathrm{d}^3\boldsymbol{r}' \qquad \text{quadrupole moment components.}$$

When the coordinate system is rotated, the quantities q, p_α, $Q_{\alpha\beta}$, ... transform, respectively, as scalars, vectors, tensors of rank 2, and so on. The second and third terms in the potential given by equation (2.0.8) may be written in the form

$$\varphi^{(p)} = \frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3}\,, \tag{2.0.9}$$

where $\boldsymbol{p} = (p_x, p_y, p_z)$ is the dipole moment of the system;

$$\varphi^{(Q)} = \frac{1}{2r^5}[(3x^2-r^2)Q_{xx} + (3y^2-r^2)Q_{yy} + (3z^2-r^2)Q_{zz}$$
$$+ 6xyQ_{xy} + 6xzQ_{xz} + 6yzQ_{yz}]\,.$$

Spherical polars. We shall use the expansion for $|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$ which is given in appendix 2 [equation (A2.15)]. Substituting this expansion into equation (2.0.7) we have for $r > r'$

$$\varphi(\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)^{1/2}\frac{Q_{lm}Y_{lm}(\vartheta,\phi)}{r^{l+1}} \qquad (r > r')\,, \tag{2.0.10}$$

where Q_{lm} is the multiple moment of order l, m and is given by

$$Q_{lm} = \left(\frac{4\pi}{2l+1}\right)^{1/2}\int \rho(\boldsymbol{r}')r'^l Y^*_{lm}(\vartheta',\phi')\,\mathrm{d}^3\boldsymbol{r}'\,. \tag{2.0.11}$$

If $r' > r$ then r and r' change places in equation (A2.15) of appendix 2 and

$$\varphi(\boldsymbol{r}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\frac{4\pi}{2l+1}\right)^{1/2} r^l Q'_{lm}Y_{lm}(\vartheta,\phi) \qquad (r < r')\,, \tag{2.0.12}$$

where

$$Q'_{lm} = \left(\frac{4\pi}{2l+1}\right)^{1/2}\int \frac{\rho(\boldsymbol{r}')}{r'^{l+1}}Y^*_{lm}(\vartheta',\phi')\,\mathrm{d}^3\boldsymbol{r}'\,. \tag{2.0.13}$$

If the point defined by the position vector $\boldsymbol{r}$ lies inside the charge distribution (see figure 2.0.2), then the region of integration in equation (2.0.7) must be divided into two parts by a sphere of radius r, centred on the origin O. The expansion given by equation (A2.15) of appendix 2 is then used inside the sphere and the expansion with $r(r')$ replaced by $r'(r)$ is used outside the sphere.

Real systems of charges are always bounded and their potential decreases at large distances at least as fast as $1/r$. However, when we consider the field near a middle part of a long cylinder or a finite flat body it is convenient to idealize the problem and assume the body to be infinite. The potential does in that case not decrease at infinity, but it describes the field correctly at distances which are small compared to the dimensions of the body.

The structure of the electrostatic field can be conveniently represented by lines of force and equipotential surfaces. The lines of force can be defined by the following set of differential equations involving the arbitrary orthogonal coordinates q_1, q_2, q_3:

$$\frac{h_1\,\mathrm{d}q_1}{E_1} = \frac{h_2\,\mathrm{d}q_2}{E_2} = \frac{h_3\,\mathrm{d}q_3}{E_3}\,, \tag{2.0.14}$$

where h_i are the coefficients given by equation (1.b.3). The equipotential surfaces are defined by $\varphi(r) = \text{constant}$.

Neutral points are defined as those points at which $E = 0$ at finite distances from systems of charges.

The energy of an electrostatic field may be computed from one of the following two formulae:

$$W = \frac{1}{8\pi}\int E^2\,\mathrm{d}^3r\,, \qquad W = \tfrac{1}{2}\int \rho\varphi\,\mathrm{d}^3r\,. \tag{2.0.15}$$

These formulae are equivalent if the charges are localised in a finite region of space and the integration extends over all space.

The energy of interaction between two systems of charges 1 and 2 is given by

$$U = \int \rho_1(r)\varphi_2(r)\,\mathrm{d}^3r = \int \frac{\rho_1(r_1)\rho_2(r_2)\,\mathrm{d}^3r_1\,\mathrm{d}^3r_2}{|r_1 - r_2|}\,. \tag{2.0.16}$$

The generalised ponderomotive forces may be obtained by differentiating U or W with respect to the corresponding generalised coordinates a_i:

$$F_i = -\frac{\partial U}{\partial a_i} \quad \text{or} \quad F_i = -\frac{\partial W}{\partial a_i}\,. \tag{2.0.17}$$

The generalised force is positive if it tends to increase the corresponding coordinate.

2.1 An infinite plane plate of thickness a is uniformly charged to a volume density ρ. Find the potential φ and the electric field strength $\boldsymbol{E}$.

2.2 An electric charge is distributed throughout space in such a way that its volume density is given by

$$\rho = \rho_0 \cos\alpha x \cos\beta y \cos\gamma z\,,$$

forming an infinite periodic lattice. Find the potential φ due to this distribution.

2.3 The plane $z = 0$ is charged to a density which varies in accordance with the periodic law

$$\sigma = \sigma_0 \sin\alpha x \sin\beta y ,$$

where σ_0, α, and β are constants. Find the potential φ due to this charge distribution.

2.4 An infinitely long cylinder of radius R carries a uniform charge density κ per unit length. Find the potential φ and the electric field strength $\boldsymbol{E}$ for this system.

2.5 Find the potential φ and the electric field strength $\boldsymbol{E}$ for a uniformly charged infinitely long straight filament.

2.6 Find the potential φ and the electric field strength $\boldsymbol{E}$ for a uniformly charged line segment of length $2a$, lying between $-a$ and $+a$ along the z-axis, when its total charge is q.

2.7 Find the form of the equipotential surfaces for the uniformly charged line segment considered in the preceding problem.

2.8 Find the potential φ and the electric field strength $\boldsymbol{E}$ for a sphere carrying a uniform volume charge distribution.

2.9 Find the potential φ and the electric field strength $\boldsymbol{E}$ for a sphere of radius R, carrying a uniform surface charge distribution, with total charge q.

2.10 A sphere of radius R is uniformly charged to a density ρ. The sphere contains an uncharged spherical cavity of radius R_1. The centres of the two spheres are separated by a distance a, where $a + R_1 < R$. Find the electric field strength $\boldsymbol{E}$ inside the cavity.

2.11 The space between two concentric spheres of radii R_1 and R_2 $(R_1 < R_2)$ is charged to a volume density given by $\rho = \alpha/r^2$. Find the total charge, q, the potential, φ, and the electric field strength, $\boldsymbol{E}$. Consider the limiting case $R_2 \to R_1$, assuming q to be constant.

2.12 Determine the energy W of the electrostatic field for the charge distributions considered in problems 2.8, 2.9, and 2.10. (Use two methods of computing the energy [see equation (2.0.15)].)

2.13 Consider a spherically symmetric charge distribution $\rho = \rho(r)$. By dividing the charge distribution into spherical shells, find the potential φ and the electric field strength $\boldsymbol{E}$ in terms of $\rho(r)$ (write down the potential φ and the field $\boldsymbol{E}$ in the form of an integral over r).

2.14 Use the results of problem 2.13 to solve problems 2.8 and 2.11.

2.15 In the ground state of a hydrogen atom, the electron charge is distributed with a density given by

$$\rho(r) = -\frac{e_0}{\pi a^3}\exp\left(-\frac{2r}{a}\right),$$

where $a = 5\cdot 29 \times 10^{-11}$ m (Bohr radius) and $e_0 = 4\cdot 80 \times 10^{-10}$ e.s.u. (the elementary charge). Find the potential φ_e and the field strength $\boldsymbol{E}_{er}$ due to the electron charge distribution, and also the total potential φ and the field strength $\boldsymbol{E}$ in the atom, assuming that the nucleus (proton) is localised at the origin. Give a rough sketch of the spatial variation of φ and $\boldsymbol{E}$.

Hint. It is convenient to use the method employed in the solution of problem 2.13.

2.16 Assume that the atomic nucleus may be looked upon as a uniformly charged sphere and find the maximum field strength due to this sphere. The radius of the nucleus is given by $R = 1\cdot 5 \times 10^{11} A^{1/3}$ m and its total charge is Ze_0, where A is the atomic weight, Z is the atomic number, and e_0 is the elementary charge.

2.17 Use the result of problem 2.13 to solve problem 2.9.

2.18 Two thin coaxial uniformly charged rings of equal radius R lie in parallel planes which are at a distance a from each other. The work which must be done in order to bring a point charge q from infinity to the centres of the two rings is respectively equal to A_1 and A_2. Find the total charges q_1 and q_2 carried by the rings.

2.19 Find the potential φ and the electric field strength $\boldsymbol{E}$ along the axis of a thin uniformly charged circular disc of radius R and total charge q. Show that the normal component of the field changes by $4\pi\sigma$ on passing through the surface of the disc. Consider the field at large distances from the disc.

2.20 A thin circular ring of radius R consists of two uniformly charged semicircular sections carrying charges q and $-q$. Find the potential φ and field strength $\boldsymbol{E}$ along the axis of the ring and in its immediate neighbourhood. Discuss the form of the field at large distances from the ring.

2.21 Express the potential φ due to a thin uniformly charged circular ring of radius R, carrying a total charge q, in terms of the complete elliptical integral of the first kind

$$K(k) = \int_0^{\frac{1}{2}\pi} \frac{d\beta}{(1-k^2\sin^2\beta)^{1/2}}\,.$$

Hint. Use the substitution $\phi' = \pi - 2\beta$ when integrating over the azimuth.

2.22 Use the general formula for the potential φ due to a thin circular ring (see problem 2.21) to determine the field (a) along the axis of the ring, (b) at large distances from the ring, and (c) in the immediate neighbourhood of the ring.

Hint. Use formula (8.113) from Gradshteyn and Ryzhik (1965) for case (c).

2.23 A sphere of radius R carries a surface charge $\sigma = \sigma_0 \cos\vartheta$. Find the potential φ due to this charge distribution by using the multipole expansion in terms of spherical polars.

2.24 The sources of an electric field are arranged in an axially symmetric distribution in such a way that there are no charges near the axis of symmetry. Express the potential φ and the electric field strength $\boldsymbol{E}$ near the axis of symmetry in terms of the potential φ and its derivatives along this axis.

2.25 Find the potential φ due to a thin uniformly charged circular ring with the aid of the multipole expansion in terms of spherical polars. The total charge q and the radius R are given.

2.26 Find the potential φ at large distances for the following systems of charges:

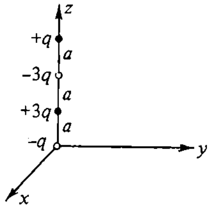
(a) point charges q, $-2q$, q lying along the z-axis at a distance a from each other (linear quadrupole);

(b) point charges $\pm q$ at the corners of a square of side a, arranged so that neighbouring charges have opposite signs, while a charge $+q$ is placed at the origin; the sides of the square are parallel to the x- and y-axes (planar quadrupole).

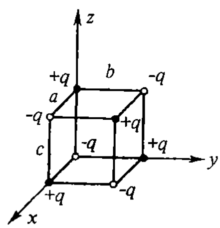
2.27 Find the potential φ at large distances from the following systems of charges:

(a) linear octupole (figure 2.27.1a),

(b) three-dimensional octupole (figure 2.27.1b).



(a)



(b)

Figure 2.27.1.

2.28 A point charge q is placed at a point whose spherical coordinates are r_0, ϑ_0, ϕ_0. Find the multipole expansion for the potential φ due to this charge.

2.29 An ellipsoid with semiaxes a, b, c is uniformly charged throughout its volume so that the total charge is q. Find the potential at large distances from the ellipsoid (retain terms up to the quadrupole term inclusively). Discuss the special cases of an ellipsoid of revolution with semiaxes $a = b$ and c and of a sphere ($a = b = c$). (Atomic nuclei having finite quadrupole moments may be approximately looked upon as ellipsoids of revolution.)

Hint. In integrating inside the ellipsoid use the generalised spherical coordinates $x = ar\sin\vartheta\cos\phi$, $y = br\sin\vartheta\sin\phi$, $z = cr\cos\vartheta$.

2.30 Two thin coaxial uniformly charged rings with radii a and b ($a > b$) and charges q and $-q$ respectively, lie in a given plane. Find the potential φ at a large distance from this system. Compare the result with the potential due to a linear quadrupole (see problem 2.26).

2.31* Show that the charge distribution

$$\rho = -(\boldsymbol{p}' \cdot \nabla)\delta(\boldsymbol{r})$$

represents an elementary dipole of moment $\boldsymbol{p}'$ at the origin. Discuss the result using the δ-function representation (appendix 1).

Hint. Start with the multipole expansion in terms of Cartesian coordinates.

2.32 Show that the charge distribution

$$\rho = q\prod_{i=1}^{n}(\boldsymbol{a}_i \cdot \nabla)\delta(\boldsymbol{r})$$

gives rise to the potential

$$\varphi(\boldsymbol{r}) = q\prod_{i=1}^{n}(\boldsymbol{a}_i \cdot \nabla)\frac{1}{r}\ .$$

2.33 Use the results of problem 2.26, and the fact that the quadrupole moment is a tensor of rank 2, to determine the potential φ at a large distance from a linear quadrupole whose orientation is defined by the polar angles γ, β. Indicate a second method of solving this problem.

2.34 The three-dimensional octupole shown in figure 2.27.1b is turned through an angle β about the z-axis. Find the potential φ at a large distance from it by transforming the components of the octupole moment. Compare with other methods of solution.

2.35 Find the potential φ at a large distance from a planar quadrupole which lies in a plane containing the z-axis (figure 2.35.1). Find the components of the quadrupole moment directly, and also by rotating the planar quadrupole discussed in problem 2.26(b).

2.36 A sphere of radius R is uniformly polarised to a dipole moment $\boldsymbol{P}$ per unit volume. Find the potential φ due to this sphere.

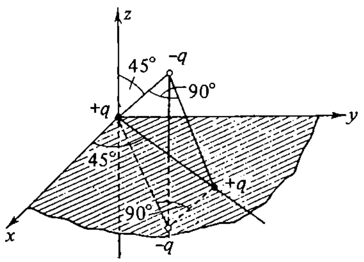


Figure 2.35.1.

2.37 A two-dimensional charge distribution is characterised by a charge density $\rho(r)$ which is independent of the z-coordinate. If $\rho(r) \neq 0$ in a finite region S in the xy-plane, then the potential φ outside the charge distribution may be expanded into a multipole expansion (two-dimensional multipoles). Find this expansion.

Hint. Use the result of problem 2.5, the superposition principle, and the expansion [see Gradshteyn and Ryzhik, 1965, formula (1.514)]

$$\ln(1+u^2-2u\cos\varphi) = -2\sum_{k=1}^{\infty}\frac{\cos k\varphi}{k}u^k ,$$

where $|u| < 1$.

2.38 Expand the potential φ due to a linear charge κ in terms of two-dimensional multipoles. The charged line is parallel to the z-axis and passes through the point (r_0, ϕ_0) in the xy-plane.

2.39 Find the potential φ at a large distance from two parallel linear charges κ and $-\kappa$ at a small distance a from each other (two-dimensional dipole).

2.40 A disc of radius R carries an electric double layer of strength τ = constant. Find the potential φ and the field strength $\boldsymbol{E}$ along the axis of symmetry perpendicular to the plane of the disc.

2.41 Find the electric field $\boldsymbol{E}$ due to an electric double layer of strength τ = constant, lying in the semi-infinite plane $y = 0$, $x > 0$. Compare with the magnetic field due to an infinite line current flowing along the z-axis. Solve the problem by
(a) direct integration of contributions due to double layer elements, and
(b) by determining the electrostatic potential first.

2.42 Find the equations of the lines of force due to the following system of point charges: $+q$ at $z = a$, $\pm q$ at $z = -a$. Sketch the lines of force. Are there any neutral points?

Hint. Owing to the symmetry of the problem the lines of force lie in the planes ϕ = constant, and E_z and E_r are independent of ϕ (cylindrical coordinates). The variables in the differential equation for the lines of force [see equation (2.0.14)] may be separated with the aid of the substitution

$$u = \frac{z+a}{r}, \qquad v = \frac{z-a}{r}.$$

2.43 Use the results of the preceding problem to find the equation for the lines of force due to a point dipole at the origin.

2.44 Find the equation for the lines of force due to a linear quadrupole [see problem 2.26(a)] and sketch the lines of force.

2.45 Show that the flux through a surface S due to a point charge q is equal to $q\Omega$ where Ω is the solid angle subtended by the surface at the point charge ($\Omega > 0$, if from this point one sees the negative side of the surface).

2.46 A charge q_1 lies on the axis of symmetry of a circular disc of radius A at a distance a from the plane of the disc. Find the charge q_2 which must be placed at a point located symmetrically relative to the disc in order that the flux through the disc in the direction towards the charge q_1 should be Φ.

2.47* Find the equation for the lines of force of a system of n colinear charges $q_1, q_2, ..., q_n$ placed at the points $z_1, z_2, ..., z_n$ on the z-axis without integrating the differential equations for the lines of force. Use the theorem of problem 2.45 for a force tube generated by rotating a line of force about the axis of symmetry.

2.48 Use the result of the preceding problem to find the equation for the lines of force due to a system of two point charges (cf problem 2.42) and a linear quadrupole (cf problem 2.44).

2.49 Uniformly charged filaments carrying charges κ_1 and $-\kappa_2$ per unit length are parallel and lie at a distance h from each other. Find the relation between κ_1 and κ_2 for which the family of equipotential surfaces associated with this system will include cylinders of finite radius and determine the radii and the position of the axes of the cylinders.

2.50 Point charges q_1 and $-q_2$ lie at a distance h from each other. Show that the equipotential surfaces associated with this sytem include a sphere of finite radius. Find the coordinates of the centre of the sphere and its radius. Find an expression for the potential φ on the surface of the sphere if the potential at infinity is zero.

2.51 Find the charge distribution giving rise to a potential of the form

$$\varphi(r) = \frac{q}{r}\exp(-\alpha r),$$

where α and q are constants.

2.52 Find the charge distribution giving rise to the potential

$$\varphi(r) = \frac{e_0}{a}\left(\frac{a}{r} + 1\right) \exp\left(-\frac{2r}{a}\right),$$

where e_0 and a are constants.

2.53 Find the energy of interaction U between the electron cloud and the nucleus in the hydrogen atom. Assume that the electron charge density is given by

$$\rho(r) = \frac{e_0}{\pi a^3} \exp\left(-\frac{2r}{a}\right),$$

where e_0 is the charge of the electron and a is a constant (the Bohr radius) (cf problem 2.15).

2.54 Under certain conditions it may be assumed that the electron clouds due to the two electrons in the helium atom are identical and may be described by the volume charge density

$$\rho(r) = -\left(\frac{8e_0}{\pi a^3}\right) \exp\left(-\frac{4r}{a}\right),$$

where a is the Bohr radius and e_0 is the charge of the electron. Find the energy of interaction U between the electrons in the helium atom using this approximation (which is in fact the zero-order approximation of perturbation theory).

2.55 The centres of two spheres carrying charges q_1 and q_2 are at a distance a from each other ($a > R_1 + R_2$, where R_1 and R_2 are the radii of the spheres). The charges form a spherically symmetric distribution. Find the energy of interaction U and the force F between the spheres.

2.56 A soap bubble hanging from the end of an open tube contracts under the action of its surface tension (the coefficient of surface tension is γ). Assuming that the dielectric strength of air, i.e. the field strength at which breakdown occurs, is E_0, investigate whether the contraction may be prevented by placing a large charge on the bubble. What is the minimum equilibrium radius R of the bubble?

2.57* Two thin parallel coaxial rings of radii a and b carry uniformly distributed charges q_1 and q_2. The distance between the planes of the rings is c. Determine the energy of interaction U and the force F between the rings.

2.58 Find the force $\boldsymbol{F}$ and the couple N on an electric dipole of moment $\boldsymbol{P}$ due to a point charge q.

2.59 A dipole with a moment $\boldsymbol{p}_1$ is placed at the origin while another dipole with a moment $\boldsymbol{p}_2$ is placed at a point whose position vector is $\boldsymbol{r}$.

Find the energy of interaction U and the force $\boldsymbol{F}$ between the two dipoles. For which orientation of the dipoles does the force become a maximum?

2.60 A system of charges can be described by a volume density $\rho(\boldsymbol{r})$ and occupies a finite region in the neighbourhood of a given point O. The system is placed in an external electric field, which in the neighbourhood of this point may be represented by the following series for its potential:

$$\varphi_1(\boldsymbol{r}) = \sum_{l,\,m} \left(\frac{4\pi}{2l+1}\right)^{1/2} a_{lm} r^l Y_{lm}(\vartheta, \phi) \, .$$

Find the energy of interaction U of the system with the external field by expressing it in terms of the a_{lm} and the multipole moments Q_{lm} of the system (cf problem 3.38).

Electrostatics of conductors and dielectrics [1]

a Basic concepts and methods of electrostatics

The electrostatic field in a dielectric is characterised by the electric field $\boldsymbol{E}$ and the induction vector $\boldsymbol{D}$ which satisfy the following equations:

$$\left.\begin{aligned} &\mathrm{curl}\boldsymbol{E} = 0 \qquad \text{or} \qquad \oint_l E_l \,\mathrm{d}l = 0 \\ &\mathrm{div}\boldsymbol{D} = 4\pi\rho \qquad \text{or} \qquad \oint_S D_n \,\mathrm{d}^2S = 4\pi q \end{aligned}\right\} \tag{3.a.1}$$

where ρ is the density of free charges in the dielectric and q is the total free charge enclosed by the surface S. The density of bound charges in a dielectric may be expressed in terms of the polarisation $\boldsymbol{P}$ (electric dipole moment per unit volume, which is produced by the bound charges):

$$\rho_b = -\mathrm{div}\boldsymbol{P} . \tag{3.a.2}$$

The polarisation $\boldsymbol{P}$ can be expressed in terms of $\boldsymbol{E}$ and $\boldsymbol{D}$:

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P} . \tag{3.a.3}$$

For isotropic dielectrics and sufficiently small fields

$$\boldsymbol{D} = \epsilon\boldsymbol{E} , \tag{3.a.4}$$

where ϵ is the permittivity of the medium. In anisotropic dielectrics, ϵ is a tensor of rank 2, i.e.

$$D_i = \epsilon_{ik} E_k , \tag{3.a.5}$$

where the summation over k is again implied.

The electric field vector may be derived from a scalar potential φ:

$$\boldsymbol{E} = -\mathrm{grad}\varphi , \qquad \varphi(\boldsymbol{r}) = \int_{\boldsymbol{r}}^{\boldsymbol{r}_0} (\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}) , \tag{3.a.6}$$

where $\boldsymbol{r}$ is the position vector of the point of observation and $\varphi(\boldsymbol{r}_0) = 0$.

The potential satisfies the equation

$$\mathrm{div}(\epsilon \,\mathrm{grad}\varphi) = -4\pi\rho . \tag{3.a.7}$$

In those regions where the dielectric is uniform, the latter equation reduces to the Poisson equation

$$\nabla^2\varphi = -\frac{4\pi\rho}{\epsilon} . \tag{3.a.8}$$

[1] For details of the theory we refer to the textbooks by Stratton (1941), Smythe (1950), Jackson (1962), Panofsky and Phillips (1962), Landau and Lifshitz (1960), and Robinson (1973).

The boundary conditions at a boundary separating two different dielectrics are (the indices t and n indicate the components which are, respectively, tangential and normal to the boundary surface)

$$\boldsymbol{E}_{1t} = \boldsymbol{E}_{2t}\,, \qquad D_{2n} - D_{1n} = 4\pi\sigma\,, \tag{3.a.9}$$

or

$$\varphi_1 = \varphi_2\,, \qquad \epsilon_1\frac{\partial\varphi_1}{\partial n} - \epsilon_2\frac{\partial\varphi_2}{\partial n} = 4\pi\sigma\,. \tag{3.a.10}$$

The boundary conditions given by equation (3.a.9) hold both for isotropic and anisotropic media. The unit normal $\boldsymbol{n}$ is directed from the first medium into the second, $\boldsymbol{t}$ is a unit vector tangential to the surface, and σ is the surface density of free charges. The surface density of bound charges σ_b on the separation boundary is given by

$$\sigma_b = P_{1n} - P_{2n}\,. \tag{3.a.11}$$

The basic problem of electrostatics is to find the electric potential φ. It may be solved in various ways. The main method is to solve the differential equations (3.a.7) and (3.a.8) subject to the boundary conditions (3.a.9) or (3.a.10). It is sometimes possible to choose a set of fictitious point charges for which the potential in a particular region satisfies the same differential equation and the same boundary conditions as the given set of charges (method of images). In many cases it is a simple matter to find the image system (see problems 3.14, 3.18, 3.25, and 3.27).

The field vanishes inside a conductor placed in a constant electric field and hence the boundary conditions on the surface of a conductor are of the form

$$\boldsymbol{E}_t = 0\,, \qquad \varphi = \text{constant}\,. \tag{3.a.12}$$

If a certain region of space is occupied by a dielectric with permittivity ϵ, and the electrostatic field is known in all space, then as $\epsilon \to \infty$ the field becomes identical with the field produced when the dielectric is replaced by a conductor of the same geometric form.

The problem of the determination of the electric field due to a given bounded set of charged conductors placed in a dielectric has a unique solution if either the total charge or the potential of each conductor is known. In the first of these two cases, the boundary condition given by (3.a.12) must be supplemented by

$$q = \int_S \sigma\, \mathrm{d}^2S = -\frac{1}{4\pi}\oint_S \epsilon\frac{\partial\varphi}{\partial n}\,\mathrm{d}^2S\,, \tag{3.a.13}$$

where q is the charge on the conductor and the integral is evaluated over the surface of the conductor.

The capacitance C of a capacitor is defined as the ratio of the charge on one of its electrodes to the potential difference between the electrodes:

$$C = \frac{q}{\varphi_1 - \varphi_2} \,. \tag{3.a.14}$$

The capacitance of an isolated conductor is defined as the ratio of the charge on the conductor to its potential (it may be assumed that the potential vanishes at infinity).

The energy of an electrostatic field which is localised in a volume V is given by the integral

$$W = \int_V w \, d^3\boldsymbol{r} \,, \tag{3.a.15}$$

where $w = (\boldsymbol{D} \cdot \boldsymbol{E})/8\pi$ is the energy density and the integral is taken over the volume V.

When a dielectric of volume V and permittivity ϵ_2 is introduced into an isotropic dielectric with permittivity ϵ_1, then the energy of the electrostatic field is changed by an amount given by

$$U = \frac{1}{8\pi} \int_V (\epsilon_1 - \epsilon_2)(\boldsymbol{E}_2 \cdot \boldsymbol{E}_1) \, d^3\boldsymbol{r} \,, \tag{3.a.16}$$

where $\boldsymbol{E}_1$ is the initial field and $\boldsymbol{E}_2$ the final field (it is assumed that the sources of $\boldsymbol{E}_1$ are maintained unaltered). The quantity U may be looked upon as the energy of interaction of the dielectric body with the external field $\boldsymbol{E}_1$.

The force per unit volume, due to the electric field, on an isotropic dielectric whose permittivity depends only on its mass density τ is given by the following equation:

$$\boldsymbol{f} = \rho \boldsymbol{E} - \frac{1}{8\pi} E^2 \operatorname{grad} \epsilon + \frac{1}{8\pi} \operatorname{grad} \left(E^2 \frac{d\epsilon}{d\tau} \tau \right) \,. \tag{3.a.17}$$

The body forces acting on free and bound charges in a given volume V may be replaced by the equivalent system of surface stresses applied to the surface S of the volume V and given by

$$\boldsymbol{F} = \int_V \boldsymbol{f} \, d^3\boldsymbol{r} = \oint_S \boldsymbol{T}_n \, d^2S \,, \tag{3.a.18}$$

where $\boldsymbol{T}_n$ is the surface force per unit area applied to an element having a unit outward normal $\boldsymbol{n}$.

The surface stresses are described by a stress tensor T_{ik}, and the quantity $\boldsymbol{T}_n$ in equation (3.a.18) is the component of T_{ik} along the outward normal $\boldsymbol{n}$ to the surface element d^2S, so that

$$(\boldsymbol{T}_n)_i = T_{ik} n_k \,, \qquad T_{ik} = \frac{\epsilon}{4\pi} E_i E_k - \frac{1}{8\pi} E^2 \left(\epsilon - \frac{\partial \epsilon}{\partial \tau} \tau \right) \delta_{ik} \,. \tag{3.a.19}$$

The terms containing $(\partial\epsilon/\partial\tau)_T$ in equations (3.a.17) and (3.a.19) are not, in general, small and represent electrostriction. However, in calculating the resultant of the forces acting on a dielectric this term does not contribute to the final result and may be ignored (see problems 3.12 and 3.13). The stress tensor (3.a.19) may then be replaced by the simpler (Maxwell) tensor

$$T_n = \frac{\epsilon}{4\pi}(E_n \boldsymbol{E} - \tfrac{1}{2}\boldsymbol{n}E^2) . \tag{3.a.20}$$

The force per unit area on the surface of a conductor is given by

$$f_s = T_n' = \boldsymbol{n}\frac{\epsilon E^2}{8\pi} = \tfrac{1}{2}\sigma \boldsymbol{E} . \tag{3.a.21}$$

In a liquid dielectric which is in equilibrium in an electric field the electric stresses are balanced by the hydrostatic pressure. The equilibrium condition is then given by

$$p\boldsymbol{n} + T_n = \text{constant} , \tag{3.a.22}$$

where $p(\tau)$ is the pressure in the liquid and τ is the density. In particular, near the boundary between a liquid and the atmosphere ($\epsilon = 1$) the pressure $p(\tau)$ is greater than the atmospheric pressure by an amount given by

$$p(\tau) - p_{\text{atm}} = \frac{\tau E^2}{8\pi}\frac{\partial\epsilon}{\partial\tau} - \frac{\epsilon - 1}{8\pi}(\epsilon E_n^2 + E_t^2) , \tag{3.a.23}$$

where $\boldsymbol{E}$ is the electric field strength in the liquid, and $\boldsymbol{E}_n$ and $\boldsymbol{E}_t$ are the normal and tangential components of $\boldsymbol{E}$. Equation (3.a.23) describes the dependence of the pressure in a liquid near its surface on the electric field strength. The pressure inside the liquid (gas) is given by

$$\int_{p_0}^{p} \frac{dp}{\tau(p)} = \frac{E^2}{8\pi}\frac{\partial\epsilon}{\partial\tau} \tag{3.a.24}$$

where p_0 is the pressure at the point where $E = 0$. For an incompressible liquid

$$p - p_0 = \frac{\tau E^2}{8\pi}\frac{\partial\epsilon}{\partial\tau} . \tag{3.a.25}$$

3.1 A point charge q is placed on the planar separation boundary between two homogeneous infinite dielectrics with permittivities ϵ_1 and ϵ_2. Find the potential φ, the field strength $\boldsymbol{E}$, and the induction $\boldsymbol{D}$.

3.2 A point charge q lies on a straight line which is the line of intersection of three planes, the angles between the planes being α_1, α_2, and α_3 ($\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$). The space between each pair of planes is filled with homogeneous dielectrics with permittivities ϵ_1, ϵ_2, and ϵ_3. Find the potential φ, the electric field strength $\boldsymbol{E}$, and the induction $\boldsymbol{D}$.

3.3 The centre of a conducting sphere carrying a charge q lies on the plane boundary between two infinite homogeneous dielectrics with permittivities ϵ_1 and ϵ_2. Find the potential φ of the electric field and the charge distribution, σ, on the sphere.

3.4 The space between the electrodes of a spherical capacitor is partly filled with a dielectric which occupies a region subtending a solid angle Ω at the centre of the spheres. Find the capacitance C assuming that the radii of the electrodes are a and b, and the permittivity is ϵ.

3.5 The permittivity of a medium filling the space between the electrodes of a spherical capacitor whose radii are a and b is given by

$$\epsilon(r) = \begin{cases} \epsilon_1 = \text{constant}\,, & a \leqslant r < c\,, \\ \epsilon_2 = \text{constant}\,, & c \leqslant r \leqslant b\,, \end{cases}$$

where $a < c < b$. Find the capacitance, C, of the capacitor, the distribution of bound charges, σ_b, and the total bound charge in the dielectric.

3.6 A spherical capacitor whose electrodes have radii a and b is filled with a dielectric whose permittivity is given by $\epsilon(r) = \epsilon_0 a^2/r^2$ where r is the distance from the centre. Show that the capacitance of this capacitor is equal to the capacitance of a plane parallel capacitor filled with a homogeneous dielectric with permittivity ϵ_0, electrode area $4\pi a^2$, and distance between the electrodes equal to $b-a$ (neglect edge effects).

3.7 A plane parallel capacitor is filled with a dielectric the permittivity of which is given by $\epsilon = \epsilon_0(x+a)/a$, where a is the distance between the electrodes, S is the area of the plates, and the x-axis is perpendicular to the plates. Neglecting edge effects, find the capacitance C and the distribution of bound charges when a potential difference V is applied between the planes.

3.8 (a) Find the force f_0 per unit area on the two electrodes of a plane parallel capacitor in vacuo when the distance between the plates is a and the potential difference between them is V.

(b) Find the new value f of this force when the charged capacitor is separated from the battery and filled with a liquid dielectric of permittivity ϵ, or when a plate of solid dielectric of the same permittivity and thickness just smaller than a is introduced between the plates so as not to touch them.

(c) Find the force of attraction f between the plates when the capacitor is first filled with the liquid dielectric, or when the solid dielectric plate is inserted into it, before the capacitor is charged from the battery.

3.9 The electrodes of a plane parallel capacitor are at a distance h_1 from each other and have the form of rectangles with sides a and b. A plate of permittivity ϵ is placed between the electrodes in such a way that it is parallel to them. The plate is in the form of a parallelepiped of thickness h_2 and base area $a \times b$. The plate is not completely inserted into the capacitor so that a length $a-x$ of it remains outside. Find the force F with which the plate is attracted by the capacitor in the following two cases:

(a) a potential difference V is maintained across the capacitor, and

(b) each of the electrodes carries a constant charge q.

Ignore edge effects.

3.10* A plane parallel condenser is placed in an incompressible liquid of permittivity ϵ and density τ so that its plates are vertical. Find the distance h to which the liquid will rise in the capacitor when the distance between the plates is d and the potential difference between them is V.

Hint. Use equations (3.a.23) and (3.a.25).

3.11 Find the direction of the Maxwell stress T_n' which acts on an area d^2S whose normal $\boldsymbol{n}$ makes an angle ϑ with the direction of the electric field $\boldsymbol{E}$. Find also the magnitude of T_n' and the electrostrictional stress T_n''.

3.12 Two identical point charges q are placed in a homogeneous liquid dielectric of permittivity ϵ at a distance a from each other. Use the Maxwell or the total stress tensor to find the force $\boldsymbol{F}$ acting on each of the charges. What are the components making up the force of interaction $q^2/(a^2\epsilon)$ between the two charges? For comparison, calculate the forces on:

(a) the plane of symmetry perpendicular to the line joining the charges, and

(b) the surface of a small sphere centred on one of the charges.

3.13 An uncharged conducting sphere of radius R and mass m floats in a liquid of permittivity ϵ and density τ, so that three quarters of its volume are immersed. Find the potential φ_0 at which the sphere has to be maintained in order to ensure that half of its volume will be immersed. Solve the problem by using (a) the Maxwell stress tensor and (b) the total stress tensor including the electrostriction term.

3.14 A point charge q is placed at a point A which is at a distance a from the plane separating two semi-infinite homogeneous dielectrics whose

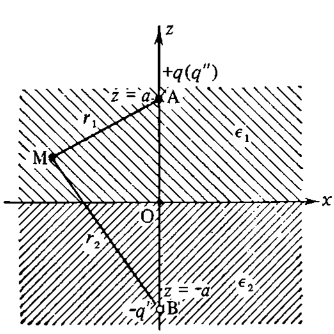


Figure 3.14.1.

permittivities are ϵ_1 and ϵ_2 (figure 3.14.1). Find the potential φ by the method of images.

Hint. The solution should be sought in the form

$$\varphi_1 = \frac{q}{\epsilon_1 r_1} - \frac{q'}{\epsilon_1 r_2}, \qquad z \geqslant 0,$$

$$\varphi_2 = \frac{q''}{\epsilon_2 r_1}, \qquad z < 0,$$

where q' and q'' are the required effective charges at the points B and A respectively.

3.15 Find the density σ_b of bound surface charges on the plane separating two homogeneous dielectrics with permittivities ϵ_1 and ϵ_2, which is due to a point charge q (see problem 3.14). Investigate what happens when ϵ_2 tends to infinity and give a physical interpretation of the result.

3.16 Find the force F on the point charge of problem 3.14 (image force). Solve the problem by a number of methods, including the Maxwell stress tensor method. Give a qualitative description of the motion of the charge if it is free to move through the dielectric.

3.17* Two semi-infinite homogeneous dielectrics with permittivities ϵ_1 and ϵ_2 are in contact along an infinite plane. Two charges q_1 and q_2 are placed at equal distances a from this plane along a straight line perpendicular to it. Find the forces F_1 and F_2 acting on each of the charges. Why are the two forces not equal?

3.18 A point charge q is placed in a homogeneous dielectric at a distance a from the plane boundary of a semi-infinite conductor. Find the potential φ in the dielectric, the distribution σ of charges induced in the metal, and the force F on the charge q.

3.19 A point charge q is placed between two earthed intersecting metal planes which are at an angle α_0 to each other. Use the method of images to find the electric field and discuss the special cases $\alpha_0 = 90°, 60°$, and $45°$.

3.20 An electric dipole of moment $\boldsymbol{p}$ is placed in a homogeneous dielectric near the plane boundary of a semi-infinite conductor. Find the potential energy of interaction U of the dipole with the induced charges, and the force $\boldsymbol{F}$ and couple $\boldsymbol{N}$ on the dipole.

3.21* A homogeneous sphere of radius a and permittivity ϵ_1 is placed in a homogeneous dielectric medium of permittivity ϵ_2. At a large distance from the sphere the electric field $\boldsymbol{E}_0$ in the dielectric is uniform. Find the potential φ everywhere and sketch the lines of force for the two cases $\epsilon_1 > \epsilon_2$ and $\epsilon_1 < \epsilon_2$. Find also the distribution of bound charges.

3.22 An infinite dielectric was at first homogeneous and uniformly polarised (polarisation vector $\boldsymbol{P}$ = constant). A spherical cavity was then introduced into it. Determine the change in the electric field $\Delta\boldsymbol{E}$ in the cavity in the following two cases:

(a) the introduction of the cavity does not affect the polarisation in the surrounding dielectric[2],

(b) the change in the field gives rise to the change in the polarisation $[\boldsymbol{P} = (\epsilon - 1)\boldsymbol{E}/4\pi]$.

3.23 An uncharged metal sphere of radius R is introduced into a uniform electric field $\boldsymbol{E}_0$. The permittivity of the surrounding medium is ϵ_0 = constant. Determine the resulting potential φ and density σ of surface charges on the sphere.

3.24* Two equal point charges $q_1 = q_2 = q$ are placed at a distance a from each other in a solid dielectric of permittivity ϵ_1. The charges lie at the centres of small spherical cavities of radius R. Find the forces acting on the charges and compare them with the electric stresses in the plane of symmetry perpendicular to the line joining the charges.

3.25* A conducting sphere of radius R is placed in the field of a point charge q which is at a distance a from the centre of the sphere $(a > R)$. The system is immersed in a homogeneous dielectric of permittivity ϵ. Find the potential φ and the charge distribution σ induced on the sphere in the following two cases:

(a) the potential of the sphere is maintained at V and the potential at infinity is zero, and

(b) the charge on the sphere is Q.

Write down the potential in the form of the sum of the potentials due to a number of point-charge images.

Hint. Use the expansion for the solution of Laplace's equation in terms of spherical harmonics (appendix 2) and the expansion for the field due to a point charge which was obtained in problem 2.28.

3.26 A conductor maintained at a potential V contains a spherical cavity of radius R which is filled with a dielectric having a permittivity ϵ. A point charge q is placed at a distance a from the centre of the cavity $(a < R)$. Find the field in the cavity and the equivalent system of image charges.

3.27 An earthed conducting plane has a hemispherical boss of radius a. The centre of the sphere lies on the plane. A point charge q is placed on the axis of symmetry of the system at a distance $b > a$ from the plane. Use the method of images to find the potential φ and also the charge Q induced on the boss.

[2] This occurs in electrets which consist of polar molecules of fixed orientation.

3.28 A conducting sphere of radius R_1 is placed in a homogeneous dielectric of permittivity ϵ_1. The sphere contains a spherical cavity of radius R_2 which is filled with a homogeneous dielectric of permittivity ϵ_2. A point charge q is placed at a distance a from the centre of the cavity ($a < R_2$). Find the potential φ in the whole of space.

3.29* A dielectric sphere of radius R and permittivity ϵ_1 is placed in a homogeneous dielectric ϵ_2. A point charge q is located at a distance a ($>R$) from the centre of the sphere. Find the potential φ in the whole of space. By passing to the limit, find the field for a conducting sphere. Find also the force acting on the charge q due to the polarisation of the sphere produced by it. How does this force change if we place another identical point charge at a spot symmetric with respect to the centre of the dielectric sphere?

3.30 Solve the preceding problem for the case where the point charge q is placed inside the dielectric sphere ϵ_1 ($a < R$). Discuss the special case $a = 0$ (charge at the centre of the sphere).

3.31* An insulated metal sphere of radius a is placed inside a hollow metal sphere of radius b. The distance between the centres of the two spheres is c where $c \ll a$, $c \ll b$. The total charge on the inner sphere is q. Find the charge distribution σ on the inner sphere and the force F acting on it. Terms involving second and higher powers of c may be neglected.

3.32 A spherical capacitor is formed by two nonconcentric spheres (see preceding problem). Calculate, in the first nonvanishing approximation, the correction ΔC to the capacitance which is due to the departure from exact concentricity.

3.33 Find the energy of interaction U between a point charge q and an earthed conducting sphere of radius R. The charge is at a distance a from the centre of the sphere and the system is placed in a homogeneous dielectric medium of permittivity ϵ. Find also the force F on the point charge.

3.34 A point charge q is placed in a dielectric at a distance a from the centre of a conducting insulated sphere of radius R which carries a total charge Q. Find the energy of interaction U between the charge and the sphere and the force F on the point charge.

3.35 What are the conditions which have to be satisfied by a test charge q (in the sense of its magnitude and position in space) in order that it can be used to investigate the field due to a system of charges situated on conductors and dielectrics, and in particular, the field due to a charged sphere in a homogeneous dielectric?

3.36* An electric dipole $\boldsymbol{p}$ is placed in a homogeneous dielectric at a distance r from the centre of an earthed conducting sphere of radius R.

Determine the system of images which is equivalent to the induced charges. Find also the energy of interaction U between the dipole and the sphere, and the force F and couple N on the dipole. Discuss the limiting case $r \to R$ $(r > R)$.

3.37 An electric dipole $\boldsymbol{p}$ is placed at the centre of a spherical cavity of radius R in a conductor. Find the distribution of charges σ induced on the surface of the cavity. Determine the field $\boldsymbol{E}'$ due to these charges inside the cavity.

3.38* The potential at a point O in a homogeneous dielectric with permittivity ϵ may be written down in the form

$$\varphi_1 = \sum_{l,m} \left(\frac{4\pi}{2l+1}\right)^{1/2} a_{lm}\, r^l\, Y_{lm}(\vartheta, \phi)\,.$$

The dielectric is then disturbed so that it is no longer homogeneous and neutral in the neighbourhood of O (e.g., an inclusion in the form of a conductor or a dielectric with permittivity $\epsilon_1 \neq \epsilon$ is placed in the region). As a result, the potential outside the disturbed region becomes $\varphi = \varphi_1 + \varphi_2$ where

$$\varphi_2 = \sum_{l,m} \left(\frac{4\pi}{2l+1}\right)^{1/2} \epsilon^{-1} r^{-(l+1)} Q_{lm}\, Y_{lm}(\vartheta, \phi)\,.$$

The potential φ_2 is due to free and bound charges in the region of the disturbance (the factor involving ϵ is introduced for convenience). Find the potential energy of interaction U between the inclusion and the external field φ_1.

Hint. Consider the electric stresses acting on a surface enclosing the inclusion. Use the result of problem 2.60.

3.39 Find the energy of interaction U_0 of the inclusion considered in the preceding problem with the external field in the case where, owing to the rapid convergence of the series, terms beyond $l = 1$ may be neglected (slowly varying external field). Give the result in a vector form. Find the force $\boldsymbol{F}$ and the couple $\boldsymbol{N}$ on the inclusion using this approximation.

3.40 Show that an uncharged dielectric body of permittivity ϵ_0 which is placed in a dielectric of permittivity ϵ is attracted towards regions of higher electric field strength, if $\epsilon_0 > \epsilon$, and repelled from such regions, if $\epsilon_0 < \epsilon$.

Hint. Use equation (3.a.16).

3.41 In general, the components of the dipole moment $\boldsymbol{p}$ induced in a dielectric body placed in an external uniform field $\boldsymbol{E}$ may be written down in the form $p_i = \beta_{ik} E_k$ where β_{ik} is the symmetric polarisability tensor. What orientation will the body tend to assume in the external uniform field? The body is uncharged and the tensor β_{ik} is such that $\beta_{ik}x_ix_k$ is positive definite, i.e. $\beta_{ik}x_ix_k > 0$ where x_i ($i = 1, 2, 3$) is an arbitrary nonvanishing vector.

3.42 A cylinder made from a dielectric with permittivity ϵ_1 is placed in a homogeneous liquid dielectric with permittivity ϵ_2. What will be the orientation of the cylinder when the system is placed in a uniform external field? What would be the orientation of a thin disc in the liquid dielectric?

3.43 Find the force F on a dielectric sphere due to a point charge q (cf the conditions of problem 3.29). Consider the limiting case of a conducting sphere. Solve the problem by two methods: the method of problem 3.38 and by means of equation (3.a.16).

3.44 An electrostatic field is produced by two conducting cylinders with parallel axes, radii R_1 and R_2, and charges $\pm\kappa$ per unit length. The distance between the axes is $a > R_1 + R_2$. Find the mutual capacitance C_{mut} of the system per unit length [$C_{\text{mut}} = \kappa/(\varphi_1 - \varphi_2)$ where φ_1 and φ_2 are the potentials of the cylinders].

Hint. Use the result of problem 2.49.

3.45 The axes of two identical conducting cylinders of radius R are placed at a distance a from each other. The cylinders carry a charge of $\pm\kappa$ per unit length. Find the distribution of charges σ on the surfaces of the two cylinders.

3.46 A capacitor consists of two cylindrical conducting surfaces with radii R_1 and $R_2 > R_1$. The distance between the axes is $a < R_2 - R_1$. Find the capacitance C of the system.

3.47 Determine the potential φ due to a point charge in a homogeneous anisotropic medium characterised by a permittivity tensor ϵ_{ik}.

3.48 A plane parallel plate made from an anisotropic homogeneous dielectric having a permittivity tensor ϵ_{ik} is placed in a vacuum. A uniform electric field $\boldsymbol{E}_0$ is present outside the plate. Using the boundary conditions for the field vector determine the field $\boldsymbol{E}$ inside the plate.

3.49 Find the capacitance C of a plane parallel capacitor with electrode area equal to S when the distance between the plates is a and the space between them is filled with an anisotropic dielectric with a permittivity ϵ_{ik}. Neglect edge effects.

3.50 Find the change in the direction of the lines of force associated with an electric field $\boldsymbol{E}$ on passing from vacuum into an anisotropic dielectric. Use the results of problem 3.48.

b Coefficients of potential and capacitance

The potentials V_i of a system of n conductors are homogeneous linear functions of the charges q_k on the conductors:

$$V_i = \sum_{k=1}^{n} s_{ik} q_k \qquad (i = 1, 2, 3, ..., n). \tag{3.b.1}$$

The quantities s_{ik} are called the coefficients of potential. They depend on the relative positions, form, and geometrical dimensions of the conductors, and also on the permittivity of the surrounding medium. The matrix $\hat{s}$ is symmetric:

$$s_{ik} = s_{ki}. \tag{3.b.2}$$

The quantity s_{ik} is the potential of the ith conductor when the kth conductor carries a charge $q_k = 1$ while the remaining conductors remain uncharged. All the s_{ik} are positive quantities.

It is clear that the charges on the conductors are homogeneous linear functions of their potentials so that

$$q_i = \sum_{k=1}^{n} c_{ik} V_k \qquad (i = 1, 2, 3, ..., n). \tag{3.b.3}$$

The quantities c_{ik} are called the coefficients of capacitance. Moreover, $c_{ii} > 0$ (self-capacitances) and $c_{ik} = c_{ki} < 0$ when $i \neq k$ (mutual capacitances).

The quantity c_{ik} is the charge on the ith conductor when all conductors other than the kth conductor are earthed while the kth conductor is maintained at a potential $V_k = 1$. The matrices s_{ik} and c_{ik} are each other's inverses.

In the case of an isolated conductor the only finite coefficient of capacitance is c_{11} and this is referred to simply as the capacitance of the conductor. The capacitance of a capacitor [equation (3.a.14)] may be expressed in terms of the coefficients of capacitance of its electrodes (cf problem 3.52). The energy of a system of conductors is given by

$$W = \tfrac{1}{2} \sum_{i,k} c_{ik} V_i V_k = \tfrac{1}{2} \sum_{i,k} s_{ik} q_i q_k. \tag{3.b.4}$$

The generalised force F_a corresponding to a generalised coordinate a is given by

$$F_a = -\tfrac{1}{2} \sum_{i,k} \frac{\partial s_{ik}}{\partial a} q_i q_k = +\tfrac{1}{2} \sum_{i,k} \frac{\partial c_{ik}}{\partial a} V_i V_k. \tag{3.b.5}$$

Green's reciprocity theorem is useful in the solution of electrostatic problems. This theorem reads as follows: If total charges $q_1, q_2, q_3, ..., q_n$ on n conductors produce potentials $V_1, V_2, V_3, ..., V_n$ and if charges

$q_1', q_2', q_3', \ldots, q_n'$ produce potentials $V_1', V_2', V_3', \ldots, V_n'$, then

$$\sum_{i=1}^{n} q_i V_i' = \sum_{i=1}^{n} q_i' V_i \,. \tag{3.b.6}$$

3.51 Prove Green's reciprocity theorem [equation (3.b.6)]. Use it to prove that $s_{ik} = s_{ki}$.

3.52 A given system consists of two conductors at a large distance from all other conductors. The conductor 1 is placed inside the hollow conductor 2. Express the capacitances C and C' of the capacitor and the isolated conductor forming this system in terms of its coefficients of capacitance. Show that the mutual capacitance of conductor 1 and any other conductor which lies outside conductor 2 vanishes.

3.53 Express the coefficients of potential s_{ik} in terms of the coefficients of capacitance c_{ik} in the case of a system of two conductors.

3.54 The capacitances of two isolated conductors are C_1 and C_2. The conductors are in vacuo at a distance r from each other, where r is large compared with the linear dimensions of the conductors. Show that the coefficients of capacitance of the system are given by

$$c_{11} = C_1\left(1+\frac{C_1C_2}{r^2}\right), \qquad c_{12} = -\frac{C_1C_2}{r}, \qquad c_{22} = C_2\left(1+\frac{C_1C_2}{r^2}\right).$$

Hint. Determine the coefficients of potential to within terms of the order of $1/r$.

3.55 The coefficients of capacitance of a system of two conductors are c_{11}, c_{22}, and $c_{12} = c_{21}$. Find the capacitance C of a capacitor formed by the two conductors.

3.56 Four identical conducting spheres are placed at the corners of a square. Sphere 1 carries a charge q while the other spheres are uncharged. Sphere 1 is then connected by means of a thin wire in turn to spheres 2, 3, and 4 (cyclic numeration) until equilibrium is established in each case. Find the charge on each of the conductors at the end of the operation. The coefficients of potential may be looked upon as given.

3.57 Three identical spheres of radius a are placed at the corners of an equilateral triangle with sides $b \gg a$. To start with all the spheres carry equal charges q. They are then earthed in turn until equilibrium is established. What is the charge of each sphere at the end of the process?

3.58 The self-capacitance of two conductors, which are lying in a dielectric with permittivity ϵ, at potentials V_1 and V_2 is C_1 and C_2 respectively. The distance between the conductors is much larger than their linear dimensions. Find the force F between them.

3.59 A closed conducting surface at a potential V_1 encloses a conductor at a potential V_0. At the same time, the potential at a point P in the space between the two surfaces is V_P. The conductors are then earthed and a charge q is placed at the point P. Find the charges induced on the conductors.

3.60 Show that the geometrical locus of points from which a unit point charge induces in an earthed conductor a charge of equal magnitude is an equipotential surface in the field of the conductor in the absence of the point charge.

3.61 Two conductors with self-capacitances c_{11} and c_{22} and mutual capacitance c_{12} form a part of a system of insulated conductors and are connected by a thin wire. Find the self-capacitance of the subsystem of joined conductors and the coefficients of mutual capacitance between it and the remaining conductors.

3.62 Two identical spherical capacitors with inner and outer radii a and b are insulated and placed at a large distance from each other. The inner spheres are given charges q and q_1, after which the outer spheres are joined by a wire. Find an approximate expression for the change ΔW in the energy of the system.

3.63 The thin outer electrode of a spherical capacitor is earthed and a small aperture is cut in it. An insulated wire is then passed through this aperture so that the inner sphere becomes connected to a third conductor which lies at a large distance from the capacitor. The self-capacitance of the third conductor is C and the total charge carried by the inner sphere and the distant conductor is q. Find the force F on the third conductor assuming that the radius of the outer sphere is b and that of the inner sphere is a.

3.64* A conductor is charged by placing it in contact with the plate of an electrophorus. The conductor is then removed and the plate is recharged to a charge Q. On first contact the charge given to the conductor is q. What is the charge received by the conductor after a very large number of contacts?

c Special methods of electrostatics

This section contains electrostatic problems which are mathematically more difficult. The various methods which may be used to solve electrostatic problems are discussed in several books, such as the ones listed at the beginning of this chapter. In this section only some of these methods are illustrated, namely, the method of curvilinear coordinates (in the case of elliptical surfaces and the surfaces of two spheres), the methods of images, integral transformations, and inversion. These methods are explained in the solutions of the various problems (for example, problems

3.65, 3.67, 3.77, 3.81, 3.83, and 3.87). We shall confine our attention here to a brief account of the method of inversion.

Inversion is defined as the transformation of space in which each point is transformed into a conjugate point relative to some suitably chosen sphere of inversion of radius R. If the spherical coordinates (with origin at the centre of the sphere of inversion) of the original point are r, ϑ, and ϕ, then the spherical coordinates of the inverse point are $r' = R^2/r$, ϑ, and ϕ. In vector form,

$$\boldsymbol{r}' = \frac{R^2\boldsymbol{r}}{r^2}, \quad \text{or} \qquad \boldsymbol{r} = \frac{R^2\boldsymbol{r}'}{r'^2}. \tag{3.c.1}$$

Inversion is a conformal transformation. Inversion transforms a sphere into a sphere. If, in particular, the centre of inversion lies on the sphere to be transformed, then the latter becomes a plane, and vice versa.

Laplace's equation is invariant with respect to inversion, i.e. if a function $\varphi(\boldsymbol{r})$ is a solution of Laplace's equation in the original space then

$$\varphi'(\boldsymbol{r}') = \frac{r}{R}\varphi(\boldsymbol{r}) = \frac{R}{r'}\varphi\left(\frac{R^2}{r'^2}\boldsymbol{r}'\right) \tag{3.c.2}$$

is a solution of the equation in the inverted space.

The basic problem which may be solved by the method of inversion may be formulated as follows: it is required to find the field due to a system of earthed conductors and point charges q_i placed at points with position vectors $\boldsymbol{r}_i$. The potential at infinity is V = constant. In order to solve the problem let us use the method of inversion to transform the surfaces of the conductors into a simpler form.

The point charges q_i are then replaced by

$$q_i' = \frac{R}{r_i}q_i, \tag{3.c.3}$$

which lie at the points

$$\boldsymbol{r}_i' = R^2\frac{\boldsymbol{r}_i}{r_i^2}.$$

Moreover, a charge

$$q_0 = -RV \tag{3.c.4}$$

appears at the point $r' = 0$. The electrostatic problem, that is, the determination of the potential $\varphi'(\boldsymbol{r}')$, is then solved in the inverted system. The potential $\varphi(\boldsymbol{r})$ may then be obtained by means of the reciprocal transformation.

3.65* A conducting ellipsoid carrying a charge q and having semiaxes a, b, and c is placed in a homogeneous dielectric of permittivity ϵ. Find the potential φ and the capacitance C of the ellipsoid and also the surface charge density σ.

Hint. Use the ellipsoidal coordinates (see problem 1.64). The potential should be sought in the form $\varphi(\xi)$.

3.66 Using the results of the preceding problem find the potentials and the capacitances of a prolate and an oblate ellipsoid of revolution. Consider the special cases of a thin long rod and a thin disc. The capacitance C and the potential φ of the prolate ellipsoid should also be found using the result of problem 2.7.

3.67* A conducting ellipsoid carrying a charge q is placed in a vacuum in a uniform external field $\boldsymbol{E}_0$ which is parallel to one of the axes of the ellipsoid. Find the potential φ of the resultant electric field.

Hint. Use the ellipsoidal coordinates of problem 1.64. The boundary conditions on the surface of the ellipsoid ($\xi = 0$) can only be satisfied if the dependence of the potential φ' due to the induced charges on the coordinates η, ζ is the same as for the external field: $\varphi' = \varphi_0(\xi, \eta, \zeta)F(\xi)$.

3.68 The field strength in a plane parallel capacitor is E_0. The earthed electrode carries a boss in the form of one half of a prolate ellipsoid of revolution whose axis of symmetry is perpendicular to the plates. The distance between the plates is large compared with the linear dimensions of the boss. Find the potential φ at a point between the plates, and the ratio of the maximum value E_{max} of the field to the field E_0. (The result of this problem explains how a lightning conductor works.)

3.69 An uncharged conducting ellipsoid is placed in a uniform external field $\boldsymbol{E}_0$ which is arbitrarily oriented relative to the axes of the ellipsoid. Find the total potential φ. Discuss the potential at a large distance from the ellipsoid by expressing it in terms of the depolarisation coefficients

$$n^{(x)} = \frac{abc}{2}\int_0^\infty \frac{ds}{(s+a^2)R_s}, \qquad n^{(y)} = \frac{abc}{2}\int_0^\infty \frac{ds}{(s+b^2)R_s},$$

$$n^{(z)} = \frac{abc}{2}\int_0^\infty \frac{ds}{(s+c^2)R_s} \qquad \{R_s = [(s+a^2)(s+b^2)(s+c^2)]^{1/2}\}.$$

3.70 Find expressions for the depolarisation coefficients introduced in the preceding problem for a prolate ellipsoid of revolution ($a > b = c$). Discuss the special cases of a very elongated ellipsoid (rod) and an almost spherical ellipsoid.

3.71 Find the depolarisation coefficients for an oblate conducting ellipsoid ($a = b > c$). Discuss the special case of a disc.

3.72* A dielectric ellipsoid with semiaxes a, b, and c is placed in a uniform external field $\boldsymbol{E}_0$. The permittivity of the ellipsoid and of the homogeneous dielectric which surrounds it are ϵ_1 and ϵ_2, respectively. Find the potential φ of the resultant electric field (use the hint given in problem 3.67). Find the field strength $\boldsymbol{E}$ inside the ellipsoid and also the potential φ_2 at large distances from the ellipsoid by expressing it in terms of the components of the polarisability of the ellipsoid along its principal axes.

3.73 An ellipsoid of revolution with permittivity ϵ_1 is placed in a uniform external field $\boldsymbol{E}_0$ in a homogeneous dielectric with permittivity ϵ_2. Find the energy U of the ellipsoid in this field and the couple N acting on it. Discuss also the case of a conducting ellipsoid of revolution.

3.74* Show that when a conducting liquid sphere is given a sufficiently large charge it becomes unstable. Find the critical charge q_c assuming that the radius of the drop is R and the surface tension is γ.

Hint. Compare the energy of a spherical drop with the energy of a deformed drop in the shape of a prolate ellipsoid of revolution. The surface area of this ellipsoid is given by

$$S = 2\pi b^2 + \frac{2\pi b a^2}{(a^2-b^2)^{1/2}} \arccos\frac{b}{a} \qquad (a > b = c)\,.$$

3.75* A uniform electric field $\boldsymbol{E}_0 \parallel z$ in the half-space $z < 0$ is bounded by an earthed conducting plane $z = 0$ which carries a circular aperture of radius a. Find the potential φ in the whole of space. Consider the special case of large distances from the aperture with $z > 0$.

Hint. Use the oblate spheroidal coordinates (see problem 1.65) with $c = 0$. The solution should be sought in the form $\varphi = -E_0 z F(\xi)$.

3.76 Find the distribution of charges σ on the conducting plane of the preceding problem.

3.77* A point charge q is placed between two earthed conducting half-planes OA and OB which are at an angle β to each other. The point charge is at N($\boldsymbol{r}_0$) in figure 3.77.1. The cylindrical coordinates of the point charge are $(r_0, \gamma, 0)$, the z-axis lies along the line of intersection of the two planes, and the azimuthal angle ϕ is measured from the plane OA. Show that the potential $\varphi(r, \phi, z)$ may be written in the form

$$\varphi(r, \phi, z) = \int_0^\infty \varphi_k(r, \phi)\cos kz \,\mathrm{d}k\,,$$

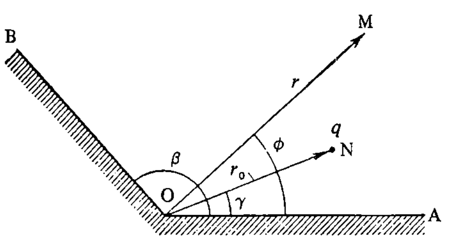


Figure 3.77.1.

where

$$\varphi_k(r, \phi) = \frac{8q}{\beta}\begin{cases} \sum_{n=1}^{\infty} K_{n\pi/\beta}(kr_0) I_{n\pi/\beta}(kr) \sin\frac{n\pi\gamma}{\beta}\sin\frac{n\pi\phi}{\beta}, & r < r_0, \\ \sum_{n=1}^{\infty} I_{n\pi/\beta}(kr_0) K_{n\pi/\beta}(kr) \sin\frac{n\pi\gamma}{\beta}\sin\frac{n\pi\phi}{\beta}, & r > r_0, \end{cases}$$

and I_ν and K_ν are the cylinder functions.

Hint. Use equation (A3.11) of appendix 3.

3.78 Show that the potential due to a point charge placed between the two planes of the preceding problem may be expressed in the form

$$\varphi(r, \phi, z) = \frac{q}{\beta(2rr_0)^{1/2}}\int_\eta^\infty \left\{\frac{\sinh(\pi\zeta/\beta)}{\cosh(\pi\zeta/\beta) - \cos[\pi(\phi-\gamma)/\beta]} - \frac{\sinh(\pi\zeta/\beta)}{\cosh(\pi\zeta/\beta) - \cos[\pi(\phi+\gamma)/\beta]}\right\}\frac{d\zeta}{(\cosh\zeta - \cosh\eta)^{1/2}},$$

where

$$\cosh\eta = \frac{r_0^2 + r^2 + z^2}{2rr_0}, \qquad \eta > 0.$$

Hint. Use the following formulae

$$\int_0^\infty K_\nu(kr) I_\nu(kr_0)\cos kz \, dk = \frac{1}{2(2rr_0)^{1/2}}\int_\eta^\infty \frac{\exp(-\xi\nu)\, d\xi}{(\cosh\xi - \cosh\eta)^{1/2}}$$

and

$$\sum_{n=1}^{\infty} p^n \cos nx = \tfrac{1}{2}\left(\frac{1-p^2}{1-2p\cos x + p^2} - 1\right).$$

3.79 Find the potential φ due to a charge q, having cylindrical coordinates $r_0, \gamma, z = 0$, which is lying close to the conducting half-plane $\phi = 0$.

Hint. Use the result of problem 3.78. In order to evaluate the integral use the substitution $\cosh\frac{1}{2}\zeta = \cosh\frac{1}{2}\eta\cosh u$ where $0 < u < \infty$.

3.80 Find the distribution of surface charge σ near the line of intersection of the two conducting planes of figure 3.77.1. Assume that the system lies in the field of an arbitrary charge distribution.

Hint. Consider the case of a single charge near the system; use the result of problem 3.77, the expansion given by equation (A3.6) of appendix 3, and the formula

$$\int_0^\infty K_\nu(k\rho) k^\nu \cos kz \, dk = 2^{\nu-1}\sqrt{\pi}\,\Gamma(\nu+\tfrac{1}{2})\frac{\rho^\nu}{(\rho^2+z^2)^{\nu+1/2}}.$$

3.81* A point charge q is placed at a distance a from a homogeneous plane parallel dielectric plate of thickness c. Find the electric field using the fact that both the product

$J_0(kr_1)\exp(\pm kz)$

and the integral

$$\int_0^\infty A(k)J_0(kr_1)\exp(\pm kz)\,\mathrm{d}k$$

satisfy the Laplace equation [r_1, z are cylindrical coordinates, J_0 is the zero-order Bessel function and $A(k)$ is an arbitrary function of k].

Hint. Use the result

$$\frac{1}{(r_1^2+z^2)^{1/2}} = \int_0^\infty \exp(-k|z|)J_0(kr_1)\,\mathrm{d}k\,,$$

which is derived in appendix 3.

3.82 A plane parallel dielectric plate of thickness $\frac{1}{2}a$ and permittivity ϵ is placed between the plates of a plane parallel capacitor which are at a distance a from each other. The dielectric is in contact with one of the plates, both of which are earthed. A charge q which may be looked upon as a point charge is then placed on the surface of the dielectric. Find the potential φ in the capacitor and discuss its form in the neighbourhood of the charge. Use the method of images.

3.83* The radii of the electrodes of a nonconcentric spherical capacitor are a_1 and a_2 and the distance between their centres is b $(a_1+b<a_2)$; the outer sphere is earthed and the inner sphere is maintained at a potential V. Find the potential φ inside the capacitor. Find also the capacitance C of the spherical capacitor.

Hint. Use bispherical coordinates (see problem 1.67). Use the substitution

$$\varphi = (2\cosh\xi - 2\cos\eta)^{1/2}\psi\,,$$

carry out the separation of variables in the equation for ψ, and use the results of appendix 2, especially equation (A2.16).

3.84 Find the capacitance of a nonconcentric spherical capacitor assuming that the distance between the centres is small ($b \ll a_1, b_2$) to within terms of the order of b^2 by using the results of the preceding problem (cf problem 3.32).

3.85 The distance between the centres of two conducting spheres of radii a_1 and a_2 is b ($b > a_1+a_2$). Find the coefficients of capacitance c_{ik} of the system using bispherical coordinates.

3.86 The two conducting spheres of the preceding problem are at a large distance from each other ($b \gg a_1, a_2$). Find the coefficients of capacitance c_{ik} to within terms of the order of b^{-4}.

3.87* Two conducting spheres of equal radii a touch each other. Find the capacitance C of the system by the inversion method. Find also the potential φ due to the system when the spheres carry a total charge q.

Hint. Use the results of problem 3.82.

3.88 Use the inversion method to determine the field of an earthed sphere of radius R with a point charge q placed at a distance $a > R$ from its centre (see problem 3.25).

Hint. The potential of a uniformly charged sphere in the absence of the point charge may be regarded as known.

3.89* The surface of a conductor is formed by two spheres of radii R_1 and R_2 which intersect on a circle of radius a. Find the capacitance C of the conductor using the solution of problem 3.78 (two intersecting planes) and the inversion method.

Hint. The surface of the conductor can be described in terms of toroidal coordinates (see problem 1.68) by the equations

$$\xi = \xi_1 = \text{constant}\,, \qquad \xi = \xi_2 = \text{constant}\,,$$

$$\left(\sin\xi_1 = \pm\frac{a}{R_1}\,, \qquad \sin\xi_2 = \pm\frac{a}{R_2}\right).$$

It is sufficient to consider the transformation of the coordinates in the plane perpendicular to the line of intersection of the two planes and passing through the centre of inversion, which should be taken to lie on the circle of intersection of the two spheres. In order to determine the charge q on the conductor for a given potential use the fact that the field at large distances from the conductor is of the form $\varphi = (q/r) - V$ where $-V$ is the potential at infinity.

3.90 Find the capacitance C of the following conductors:

(a) a hollow spherical segment of radius R which subtends an angle 2θ at the centre;

(b) a hemisphere of radius R.

3.91 A conductor is formed by two spheres each of radius a whose surfaces intersect at an angle $\frac{1}{2}\pi$. Find the capacitance C of the conductor.

Steady currents [1]

The distribution of steady currents in a conducting medium with specific conductivity $\gamma(\boldsymbol{r})$ may be described with the aid of a current density $\boldsymbol{j}(\boldsymbol{r})$ which satisfies the equation

$$\operatorname{div}\boldsymbol{j} = 0 . \tag{4.0.1}$$

This equation is a consequence of the law of conservation of charge. The current density in a medium is proportional to the sum of the electric field $\boldsymbol{E}$ and the external field (e.m.f.) $\boldsymbol{E}_\mathrm{e}$ which includes nonelectrical effects:

$$\boldsymbol{j} = \gamma(\boldsymbol{E}+\boldsymbol{E}_\mathrm{e}) . \tag{4.0.2}$$

This expression is equivalent to Ohm's law.

The electric field $\boldsymbol{E}$ and the current distribution $\boldsymbol{j}$ in a conductor can be conveniently described by a scalar potential φ (just as in electrostatics). The potential is defined by $\boldsymbol{E} = -\operatorname{grad}\varphi$. It follows from this definition and from equations (4.0.1) and (4.0.2) that the basic differential equation for φ is

$$\operatorname{div}(\gamma \operatorname{grad}\varphi) = \operatorname{div}\gamma\boldsymbol{E}_\mathrm{e} . \tag{4.0.3}$$

On surfaces across which γ or $\boldsymbol{j}_\mathrm{e} = \gamma\boldsymbol{E}_\mathrm{e}$ are discontinuous, equation (4.0.3) is replaced by the boundary conditions

$$\gamma_2 E_{2n} - \gamma_1 E_{1n} = j_{\mathrm{e}\,1n} - j_{\mathrm{e}\,2n} , \tag{4.0.4}$$

$$\varphi_1 = \varphi_2 . \tag{4.0.5}$$

On the surfaces of insulators ($\gamma = 0$) the condition given by equation (4.0.4) becomes

$$j_n = 0 , \quad \text{or} \quad \gamma E_n + j_{\mathrm{e}\,n} = 0 . \tag{4.0.6}$$

If the medium consists of a number of homogeneous regions and contains no external sources of e.m.f. then within each such region

$$\nabla^2 \varphi_k = 0 , \tag{4.0.7}$$

while on the separation boundary of the ith and kth regions

$$\varphi_i = \varphi_k , \qquad \gamma_i \frac{\partial \varphi_i}{\partial n} = \gamma_k \frac{\partial \varphi_k}{\partial n} \qquad \text{(no summation!)} . \tag{4.0.8}$$

It is clear from equations (4.0.3) to (4.0.8) that there is a close correspondence between the basic, current problem and the analogous problem in electrostatics. The solution of the current problem may be obtained by solving the corresponding electrostatic problem (and vice

[1] For details of the theory we refer to the textbooks by Smythe (1950) or Landau and Lifshitz (1960).

versa) providing the following replacements are made:

$$\begin{aligned} &\epsilon \to \gamma\,, \qquad && D \to -\gamma\,\mathrm{grad}\varphi\,, \\ &4\pi\rho \to -\mathrm{div}j_{\mathrm{e}}\,, \qquad && 4\pi\sigma \to j_{\mathrm{e}\,1n} - j_{\mathrm{e}\,2n}\,. \end{aligned} \tag{4.0.9}$$

The methods of electrostatics (see chapter 3) can therefore be used to solve the basic current problem. Thus, if a medium with finite conductivity $\gamma(\boldsymbol{r})$ contains perfect conductors ($\gamma \to \infty$) then the potential on each of the surfaces must satisfy the condition

$$\varphi_{\mathrm{el}} = \text{constant}\,. \tag{4.0.10}$$

The current problem is then analogous to the electrostatic problem of a system of conductors placed in a dielectric. As in the electrostatic case, there are two versions of the problem, namely, (a) the potentials of the electrodes (conductors) are given, $\varphi_k = V_k$; and (b) the currents leaving the electrodes are given so that

$$J_k = \oint_{S_k} j_n\, \mathrm{d}^2 S_k = -\oint \gamma \frac{\partial \varphi}{\partial n}\, \mathrm{d}^2 S_k\,. \tag{4.0.11}$$

It is clear from the latter expression that the analogous quantities are the charge on the kth conductor q_k in the electrostatic problem and the current $J_k/(4\pi)$ from the kth electrode in the current problem.

The potentials V_k of the electrodes are linear functions of the currents J_k leaving the electrodes:

$$\left.\begin{aligned} V_1 &= R_{11}J_1 + R_{12}J_2 + \ldots + R_{1n}J_n\,, \\ V_2 &= R_{21}J_1 + R_{22}J_2 + \ldots + R_{2n}J_n\,, \\ &\ldots\ldots\ldots\ldots\ldots \\ V_n &= R_{n1}J_1 + R_{n2}J_2 + \ldots + R_{nn}J_n\,. \end{aligned}\right\} \tag{4.0.12}$$

The coefficients R_{ik} are called the coefficients of resistance. They are independent of the potentials V_k and the currents J_k, and are determined exclusively by the geometry of the electrodes and the form of conductivity function γ. The coefficients of resistance are analogous to the coefficients of potential in the electrostatic problem (see problem 4.15).

The distribution of currents in quasi-linear conductors which is often encountered in practice may be determined with the aid of Kirchhoff's laws (see e.g. Smythe, 1950). A convenient method of considering complicated circuits involving quasi-linear conductors is the method of circulating currents [see e.g. Smythe (1950); in chapter 7 of that book one can find a large number of problems about current distributions].

4.1 An accumulator with a low internal resistance and e.m.f. $\mathcal{E}$ is unable to supply a given instrument with a current J for a long period of time. In order to extend the life of the accumulator the instrument and the accumulator are connected in parallel and then joined through a resistance

R to DC mains. The voltage V of the mains is subject to fluctuations which are such that $V_1 \leqslant V \leqslant V_2$ where $V_1 > V_2 > \mathcal{E}$. The resistance R is chosen so that when $V = V_1$ no current is drawn from the accumulator, $J_1 = 0$. Find the current J_2 drawn from the accumulator when $V = V_2$.

4.2 Find the parameters of the coil of a moving coil galvanometer which gives a maximum deflection for a given external e.m.f. and external resistance R (series connection). The deflection of the galvanometer is proportional to the number of turns n in the coil and the current J in the circuit. Since the volume occupied by the coil is limited, it may be assumed that nS is approximately constant, where S is the cross section of the wire from which the coil is made.

4.3 A square net made of a wire of uniform cross section consists of n^2 identical square cells. The resistance of one side of each of these cells is r. The current enters at one of the corners of the net and leaves at the opposite corner. Find the resistance R of the entire net for $n = 2$, 3, and 4.

Hint. In order to reduce the number of circulating currents make use of the symmetry of the circuit.

4.4* A telegraph line (figure 4.4.1) is suspended from n insulators at $A_1, A_2, \ldots, A_n$ (the earth forms the return lead). The sections AA_1, A_1A_2, ..., A_nA_{n+1} have the same resistance R and the insulators have an infinite resistance when dry. When the insulators are damp there is leakage to earth through them and the resistance of each of the insulators is then r. At one end of the line there is a battery with an internal resistance R_i and e.m.f. $\mathcal{E}$ while the final element is connected to earth through the resistance R_a. Find the current in each of the sections of the line and also the current flowing through the load R_a. By how much should the e.m.f. of the battery be increased in order that the current through the load should be the same both in the case of dry and damp insulation? Consider the special case of $R_a = 0$.

Hint. Consider circulating currents in the circuits formed by the section $A_{k-1}A_k$ of the line and the leakage paths for the insulators A_{k-1} and A_k. The solution of the resulting differential equation is the hyperbolic cosine.

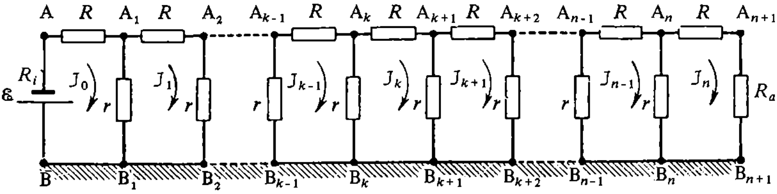


Figure 4.4.1.

4.5* An underground cable has a constant resistance ρ per unit length. The insulation of the cable is imperfect and such that the leakage conductivity per unit length is constant and equal to $1/\rho'$ with the earth acting as the return lead. Find the differential equation that describes the distribution of steady currents in the cable. Find the relation between the current $J(x)$ in the cable and the potential difference $\varphi(x)$ between the central conductor and the earth.

Hint. Use equation (4.4.1) of the solution of problem 4.4.

4.6* A battery of e.m.f. $\mathcal{E}$ and internal resistance R_i is connected between one end of an underground cable and the earth. The length of the cable is a and its resistance per unit length and conductivity per unit length are ρ and $1/\rho'$, respectively. The other end of the cable is connected to earth through a resistance R_a. Find the current distribution $J(x)$ along the cable. In particular, consider the special case $R_i = R_a = 0$. Check the result by finding the solution for the case where there is no leakage.

Hint. Use the differential equations obtained in problem 4.5, or equation (4.4.7) of the solution of problem 4.4.

4.7 Two plane parallel conducting plates are placed in contact with the two electrodes of a plane parallel capacitor. The thickness of the two plates is h_1 and h_2 and their conductivities and permittivities are, γ_1, γ_2, and ϵ_1, ϵ_2, respectively. The plates of the capacitor are made of a material whose conductivity is much greater than γ_1 and γ_2 and a constant potential difference V is maintained across them. Find the electric field E, the induction D, and the current density j in the plates, and also the densities of free and bound charges on all three separation boundaries.

4.8 Find the law of refraction for current lines across a smooth separation boundary between two media with conductivities γ_1 and γ_2, respectively.

4.9* A constant current J flows through an infinitely long straight conductor of radius a and conductivity γ. The conductor is surrounded by a thick cylindrical envelope which is coaxial with it and serves as the return lead. The inner radius of the envelope is b and the outer radius $c \to \infty$. Find the electric potential φ and the magnetic field H in the whole of space and determine the distribution of surface charges, σ. Assume that the permittivity of the medium between the conductors is ϵ.

4.10 Three conductors of circular cross section and the same radius r are connected in series forming a closed ring. The lengths of the conductors are $l_0, l_1, l_2 \gg r$ and their conductivities are $\gamma_0, \gamma_1, \gamma_2$. An external e.m.f. $\mathcal{E}_0$, which is independent of time, is uniformly distributed throughout the first of the three conductors. Find the electric field E and the distribution of charges inside the ring.

4.11 Find the energy flux S through the surfaces of the three conductors considered in problem 4.10. Hence deduce Joule's law.

4.12 The current distribution in a three-dimensional conductor with conductivity γ is such that the electric field strength and, therefore, the current density are constant on any equipotential surface. Show that the resistance of the conductor is given by the same formula as the resistance of a quasi-linear conductor with a variable cross section. Note that these conditions are analogous to the conditions under which the electrostatic Gauss theorem may be used in the corresponding electrostatic problem.

4.13 Using the result of the preceding problem find the resistance R for the following conductors:

(a) a spherical capacitor with plate radii a and b $(a < b)$, filled with a homogeneous medium of conductivity γ,

(b) a similar capacitor filled with two homogeneous layers with conductivities γ_1 and γ_2, which are separated by a spherical boundary of radius c (the layer with conductivity γ_1 is in contact with the inner electrode),

(c) a cylindrical capacitor with plate radii a and b $(a < b)$, and length l which is filled with a medium of conductivity γ (neglect edge effects).

4.14 A particular system is earthed with the aid of a perfectly conducting sphere of radius a. One half of the sphere is in contact with the ground (ground conductivity γ_1 = constant). The layer of earth which is in immediate contact with the sphere has an artificially increased conductivity γ_2. Find the resistance R to earth of the earthing device assuming that the outer radius of the special layer next to the sphere is b.

4.15* A system of perfect conductors (electrodes) is placed in a medium with conductivity $\gamma(r)$ and permittivity $\epsilon(r)$. The medium is such that $\gamma(r)/\epsilon(r)$ = constant at all points. This condition can also be formulated differently: The space between the perfect conductors is filled, instead of with a medium with a conductivity γ, with a dielectric medium with a permittivity ϵ which is everywhere proportional to γ, so that ϵ/γ = constant. Find the relation between the coefficients of potential s_{ik} and the coefficients of resistance R_{ik} of the system of conductors. What is the relation between the charges q_k on the electrodes and the currents J_k leaving them?

4.16 A capacitor of arbitrary form is filled with a homogeneous dielectric of permittivity ϵ. Find its capacitance C if it is known that when it is filled with a homogeneous conductor having a conductivity γ its DC resistance is R.

4.17 A system of electrodes is characterised by the coefficients of resistance R_{ik}. Find the amount of heat Q liberated per unit time in the space between the electrodes given that the currents J_k leaving the electrodes are known.

4.18 Two perfectly conducting spheres of radii a and b are placed in a homogeneous medium with conductivity γ and dielectric permittivity ϵ. The distance between the centres of the spheres is l and a current J enters one of the spheres and is taken out through the other. Find the resistance

$$R = \frac{V_a - V_b}{J}$$

of the medium between the spheres, where V_a and V_b are the potentials of the two spheres and J is the current leaving the sphere of radius a.

Hint. Express R_{ik} in terms of the coefficients of capacitance c_{ik} of the system (see problem 3.85).

4.19 The ends of a given circuit are earthed by means of two perfectly conducting spheres (radii a_1 and a_2). One half of each of the spheres is in contact with the earth which acts as the second lead. The distance between the spheres is $l \gg a_1, a_2$ and the conductivity of the earth is γ. Find the resistance R between the spheres.

4.20 Solve the preceding problem in the case where the two spheres are replaced by identical ellipsoids of revolution of volume V and eccentricity e_0. The axes of revolution of the ellipsoids are perpendicular to the earth's surface and their centres lie on this surface. Which of the two systems is a more efficient earthing device (i.e. ensures a lower resistance)?

4.21* A plane electrode $x = 0$ is capable of emitting an unlimited number of particles of charge e and mass m when an electric field is applied to it. The particles leave the plate with zero initial velocity and are accelerated towards another plane electrode which is parallel to the first electrode and is at a distance a from it. The potential difference between the electrodes is φ_0. The emission from the first electrode continues until the field due to the space charge produced between the plates compensates the external field so that $(\partial\varphi/\partial x)_{x=0} = 0$ on the surface of the first electrode. Find the steady-state current j between the electrodes as a function of the potential difference φ_0 between them.

Hint. The potential in the space between the electrodes obeys the Poisson equation $\nabla^2\varphi = -4\pi\rho$, $\rho = j/v$ where v is the velocity of the particles at a particular point.

Magnetostatics [1]

In the case of a constant magnetic field Maxwell's equations are of the form

$$\operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j}\,, \qquad \operatorname{div} \boldsymbol{B} = 0\,, \tag{5.0.1}$$

where $\boldsymbol{B}$ is the magnetic induction, $\boldsymbol{H}$ is the magnetic field strength, and $\boldsymbol{j}$ is the current density. The quantity c is equal to the ratio of the electromagnetic to electrostatic units of charge ($c = 3 \times 10^8$ m s^{-1}). In isotropic diamagnetics and paramagnetics $\boldsymbol{B}$ and $\boldsymbol{H}$ are related by

$$\boldsymbol{B} = \mu \boldsymbol{H}\,, \tag{5.0.2}$$

where μ is the magnetic permeability of the material (a scalar). In the case of anisotropic materials the permeability is a tensor of rank 2. The density of molecular currents in a material placed in a magnetic field may be expressed in terms of the magnetisation vector $\boldsymbol{M}$ (magnetic moment per unit volume):

$$\boldsymbol{j}_{\mathrm{mol}} = c \operatorname{curl} \boldsymbol{M}\,. \tag{5.0.3}$$

The three vectors $\boldsymbol{M}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ are related by

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi \boldsymbol{M}\,. \tag{5.0.4}$$

The main methods which may be used to determine the magnetic field in a nonferromagnetic medium are as follows:

(a) *The use of the Biot–Savart law.* The magnetic field due to a current element $J\,\mathrm{d}\boldsymbol{l}$ in a vacuum or a homogeneous medium is given by

$$\mathrm{d}\boldsymbol{H} = \frac{J}{cr^3}[\mathrm{d}\boldsymbol{l} \wedge \boldsymbol{r}]\,. \tag{5.0.5}$$

According to the superposition principle, the total field at a given point may be obtained by integrating equation (5.0.5) with respect to all the current elements.

(b) *Direct integration.* Equations (5.0.1) and (5.0.2) may be integrated subject to the boundary conditions

$$(\boldsymbol{n} \cdot [\boldsymbol{B}_2 - \boldsymbol{B}_1]) = 0 \qquad [\boldsymbol{n} \wedge (\boldsymbol{H}_2 - \boldsymbol{H}_1)] = \frac{4\pi}{c}\boldsymbol{i}\,, \tag{5.0.6}$$

where $\boldsymbol{i}$ is the surface current density and the normal $\boldsymbol{n}$ is drawn from the first region into the second. If the current distribution is axially

[1] For details of the theory refer to the textbooks by Stratton (1941), Smythe (1950), Landau and Lifshitz (1960, 1975), Panofsky and Phillips (1962), Jackson (1962), or Robinson (1973).

symmetric then the following integral form of the first of the two equations in equation (5.0.1) is useful:

$$\oint H_l \, \mathrm{d}l = \frac{4\pi}{c} J \, . \tag{5.0.7}$$

The closed integral in this expression is evaluated over an arbitrary closed path and J is the total current flowing through an arbitrary surface drawn through this path.

(c) *The vector potential method.* The vector potential A is defined by

$$\boldsymbol{B} = \mathrm{curl} \boldsymbol{A} \, , \tag{5.0.8}$$

with the subsidiary condition

$$\mathrm{div} \boldsymbol{A} = 0 \, . \tag{5.0.9}$$

In all regions in which the magnetic material is homogeneous, A satisfies the equation

$$\nabla^2 \boldsymbol{A} = -\frac{4\pi\mu}{c} \boldsymbol{j} \, . \tag{5.0.10}$$

The boundary conditions for the vector potential may be deduced from those for $\boldsymbol{B}$ and $\boldsymbol{H}$ [equation (5.0.6)].

The vector potential due to a given current distribution in a homogeneous medium with a given permeability μ may be written down in the form of the volume integral

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu}{c} \int \frac{\boldsymbol{j}(\boldsymbol{r}') \, \mathrm{d}^3 r'}{|\boldsymbol{r} - \boldsymbol{r}'|} \, . \tag{5.0.11}$$

The corresponding expression for a linear current may be obtained by making the following substitution $\boldsymbol{j} \, \mathrm{d}^3 r' \to J \, \mathrm{d}\boldsymbol{l}'$. At large distances from a given current distribution equation (5.0.11) assumes the simpler form

$$\boldsymbol{A} = \frac{[\boldsymbol{m} \wedge \boldsymbol{r}]}{r^3} \, , \tag{5.0.12}$$

where the magnetic dipole moment $\boldsymbol{m}$ is given by

$$\boldsymbol{m} = \frac{1}{2c} \int [\boldsymbol{r}' \wedge \boldsymbol{j}] \, \mathrm{d}^3 r' \, , \tag{5.0.13}$$

and it is assumed that $\mu = 1$.

(d) *The scalar potential method.* In all regions of space where $\boldsymbol{j} = 0$, we have $\mathrm{curl} \boldsymbol{H} = 0$, and hence

$$\boldsymbol{H} = -\mathrm{grad} \, \psi \, , \tag{5.0.14}$$

where ψ is a scalar potential which, for $\mu =$ constant, satisfies the Laplace equation. However, the scalar potential defined in this way will not in general be a single-valued point function. The scalar potential is used in problems 5.14, 5.15, etc.

Actual current distributions are bounded in space, and the current densities, potentials, and field strengths of such systems vanish at infinity. However, in a number of cases it turns out to be convenient to consider infinite conductors which have currents that produce nonvanishing fields at infinity. The results obtained in that way describe correctly the field close to the main parts of a finite conductor at distances small compared to the conductor's length.

The energy of the magnetic field localised in a volume V is given by the following integral

$$W = \frac{1}{8\pi}\int(\boldsymbol{H}\cdot\boldsymbol{B})\,\mathrm{d}^3r\,, \tag{5.0.15}$$

which is evaluated over the volume V. If a given system of currents has finite dimensions then its total energy may be calculated with the aid of the formula

$$W = \frac{1}{2c}\int(\boldsymbol{A}\cdot\boldsymbol{j})\,\mathrm{d}^3r\,, \tag{5.0.16}$$

in which the integration is carried out over the volume occupied by the currents.

The magnetic energy of a quasi-linear conductor carrying a current J may be expressed in terms of the self-inductance L of the conductor:

$$W = \frac{1}{2c^2}LJ^2\,. \tag{5.0.17}$$

The self-inductance may also be written down in the form of a double integral over the volume occupied by the conductor:

$$L = \frac{1}{J^2}\int\int\frac{(\boldsymbol{j}(\boldsymbol{r})\cdot\boldsymbol{j}(\boldsymbol{r}'))}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3r\,\mathrm{d}^3r'\,. \tag{5.0.18}$$

The energy of interaction between two current-carrying conductors is given by

$$W_{12} = \frac{1}{4\pi}\int(\boldsymbol{H}_1\cdot\boldsymbol{B}_2)\,\mathrm{d}^3r = \frac{1}{c}\int(\boldsymbol{j}_1\cdot\boldsymbol{A}_2)\,\mathrm{d}^3r_1 \qquad (W_{12} = W_{21})\,, \tag{5.0.19}$$

where the first integral is evaluated over the whole of space and the second over the volume of one of the conductors. In the case of quasi-linear currents the energy may be expressed in terms of the coefficient of mutual inductance L_{12}:

$$W_{12} = \frac{1}{c^2}L_{12}J_1J_2\,, \tag{5.0.20}$$

which may also be rewritten in the form

$$W_{12} = \frac{1}{c} J_1 \Phi_{12} , \tag{5.0.21}$$

where Φ_{12} is the magnetic flux due to the first conductor which is intercepted by the second and is given by

$$\Phi_{12} = \int (\boldsymbol{B}_2 \cdot \mathrm{d}^2 \boldsymbol{S}_1) = \oint (\boldsymbol{A}_2 \cdot \mathrm{d}\boldsymbol{l}_1) = \frac{1}{c} L_{12} J_2 . \tag{5.0.22}$$

The mutual inductance may be obtained from the expression for the energy (5.0.20), from the expression for the flux (5.0.22), or, in the case of linear currents, from the formula

$$L_{12} = \mu \oint \oint \frac{(\mathrm{d}\boldsymbol{l}_1 \cdot \mathrm{d}\boldsymbol{l}_2)}{r_{12}} . \tag{5.0.23}$$

The generalised forces F_i acting between two fixed currents may be obtained by differentiating the energy of interaction W_{12} (or the quantity $U_{12} = -W_{12}$, which is known as the potential function) with respect to the corresponding generalised coordinates:

$$F_i = \frac{\partial W_{12}}{\partial q_i} = -\frac{\partial U_{12}}{\partial q_i} . \tag{5.0.24}$$

The forces may also be obtained from Ampère's formula

$$\mathrm{d}\boldsymbol{F} = \frac{J}{c} [\mathrm{d}\boldsymbol{l} \wedge \boldsymbol{B}] , \tag{5.0.25}$$

where $\mathrm{d}\boldsymbol{l}$ is a circuit element carrying a current J and $\mathrm{d}\boldsymbol{F}$ is the force on this element due to the external magnetic induction $\boldsymbol{B}$.

The forces which act on currents and magnetic materials may be calculated with the aid of the Maxwell stress tensor

$$T_{lm} = \frac{\mu}{4\pi} (H_l H_m - \tfrac{1}{2} H^2 \delta_{lm}) , \tag{5.0.26}$$

which is analogous to the stress tensor used in chapter 3 for the electric field.

In those regions of space which are occupied by ferromagnetics the relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ is a nonlinear one. It is not even single-valued (hysteresis) and the solution of magnetostatic problems becomes exceedingly difficult. However, in discussing permanent magnets it is frequently assumed that the relation between $\boldsymbol{B}$ and $\boldsymbol{H}$ is linear as before:

$$\boldsymbol{B} = \mu \boldsymbol{H} + 4\pi \boldsymbol{M}_0 , \tag{5.0.27}$$

where $\boldsymbol{M}_0$ is a 'constant', i.e. independent of $\boldsymbol{H}$, magnetisation, which is looked upon as a given function of the coordinates. Ferromagnetics obeying equation (5.0.27) are called idealised ferromagnetics.

5.1 A conductor of radius a is surrounded by a thin conducting shell of radius b. The two conductors are coaxial and carry equal but opposite currents J. Determine the magnetic field $\boldsymbol{H}$ due to this system at all points in space. Solve the problem by two methods, namely, by integrating the differential form of Maxwell's equations and by using the integral form of these equations which is given by equation (5.0.7).

5.2 Determine the magnetic field $\boldsymbol{H}$ and the magnetic induction $\boldsymbol{B}$ due to a constant current J flowing through an infinite cylindrical conductor of radius a. Assume that the permeabilities of the conductor and the surrounding material are μ_0 and μ, respectively. Solve the problem by the simplest method, i.e. with the aid of the integral form of Maxwell's equations, and also by introducing the vector potential $\boldsymbol{A}$.

5.3 Solve the preceding problem for a hollow cylindrical conductor of inner and outer radii a and b respectively.

5.4 A straight and infinitely long strip of width a carries a current J which is uniformly distributed across the width of the strip. Find the magnetic field $\boldsymbol{H}$ and verify the results by considering the limiting case of large distances from the strip.

5.5 Two thin infinitely long parallel plates carry equal and opposite currents J. The width of each plate is a and the distance between them b. Find the force f per unit length on each plate.

5.6 Find the vector potential $\boldsymbol{A}$ of the magnetic field $\boldsymbol{H}$ due to two parallel line currents J flowing in opposite directions. The distance between the two currents is $2a$.

5.7 Determine the magnetic field $\boldsymbol{H}$ due to two parallel planes carrying equal surface current densities i = constant. Consider the two cases when the currents flow (a) in the same direction and (b) in opposite directions.

5.8 Determine the magnetic field $\boldsymbol{H}$ in a cylindrical cavity inside an infinitely long cylindrical conductor. The radius of the cavity is a and of the conductor is b, and the distance between their parallel axes is d $(b > a+d)$. Assume that the current J is uniformly distributed across the cross section of the conductor.

Hint. Use the principle of the superposition of fields.

5.9* Find the vector potential $\boldsymbol{A}$ and the magnetic field $\boldsymbol{H}$ due to a thin ring of radius a which carries a current J. The surrounding medium is uniform and has a magnetic permeability μ. Express the results in terms of elliptical integrals.

Hint. Use the method employed in the solution of problem 2.21.

5.10* Show that if a given magnetic field is axially symmetric and may be represented by a vector potential with components $A_\phi(r, z)$, $A_r = A_z = 0$ (cylindrical coordinates), then the equation for the lines of magnetic induction is

$$rA_\phi(r, z) = \text{constant} .$$

Hint. Consider the flux of magnetic induction inside the tube produced by rotating one of the lines of induction about the axis of symmetry (cf solution of problem 2.47).

5.11 Express the magnetic field $\boldsymbol{H}$ and the vector potential A of an axially symmetric field outside its sources in terms of the magnetic field strength $H(z)$ along the axis of symmetry.

5.12 Determine the magnetic field $\boldsymbol{H}$ along the axis of a solenoid in the form of a closely wound cylindrical coil. The height of the cylinder is h, its radius is a, the number of turns per unit length is n, and the current in the coil is J.

5.13* A sphere of radius a is uniformly charged over its surface to a total charge e and is rotating about one of its diameters with a constant angular velocity ω. Find the magnetic field inside and outside the sphere. Express the magnetic field $\boldsymbol{H}$ outside the sphere in terms of the magnetic moment $\boldsymbol{m}$ of the sphere.

5.14 Find the scalar potential ψ of the magnetic field due to an infinitely long straight wire carrying a current J. Calculate the components of the magnetic field.

5.15* Find the scalar potential ψ of the magnetic field due to a closed linear circuit carrying a current. Solve the problem (a) by integrating the Laplace equation and (b) by using the following well-known expression for the vector potential:

$$\boldsymbol{A} = \frac{J}{c}\oint\frac{\mathrm{d}\boldsymbol{l}}{r} .$$

Hint. In part (a) write the solution of the Laplace equation as an integral over a closed surface.

5.16 Find the force $\boldsymbol{F}$ and the couple $\boldsymbol{N}$ on a closed thin conductor carrying a current and placed in a uniform magnetic field $\boldsymbol{H}$. The conductor is of an arbitrary form. Solve the problem in two ways, namely, by direct integration of the forces and moments of forces on current elements, and also by using the potential function. Express the result in terms of the magnetic moment $\boldsymbol{m}$.

5.17 Find the potential function U for two small current loops with magnetic moments $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. Find the force $\boldsymbol{F}$ and the couple N on the two currents. Consider the special case $\boldsymbol{m}_1$ parallel to $\boldsymbol{m}_2$.

5.18 Show that the forces between two small current loops are such that the corresponding magnetic moments tend to set themselves parallel to each other and to the line joining their centres.

5.19 Find the potential function u_{21} (per unit length) for two parallel infinitely long straight line currents J_1 and J_2, and the force f per unit length between them.

5.20 A square frame carrying a current J_2 is placed so that two of its parallel sides are parallel to a long straight wire carrying a current J_1 (figure 5.20.1). The length of each side of the frame is a. Find the force $\boldsymbol{F}$ and the couple N, with respect to the axis OO′, on the frame.

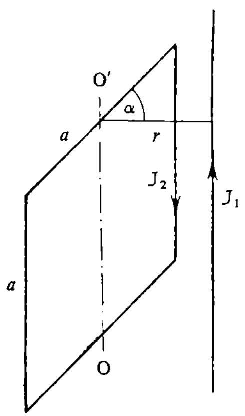


Figure 5.20.1.

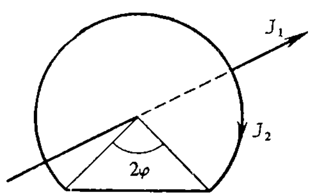


Figure 5.21.1.

5.21 A frame carrying a current J_2 consists of the arc of a circle which subtends an angle $2(\pi - \varphi)$. The ends of the arc are joined by a cord as shown in figure 5.21.1. A straight wire carrying a current J_1 passes through the centre of the circle and is perpendicular to the plane of the frame. Find the couple N on the frame.

5.22 A thin conducting cylindrical shell of radius b contains a coaxial wire of radius a and magnetic permeability μ_0. The space between the shell and the wire is filled with material of permeability μ. Find the self-inductance $\mathcal{L}$ per unit length of the line.

5.23 A line consists of two thin coaxial cylindrical shells of radii a and b $(a < b)$. The space between the two shells is filled with a material of permeability μ. Find the self-inductance $\mathcal{L}$ per unit length of the line.

5.24 A long straight conductor and a ring of radius a lie in the same plane. The distance between the wire and the centre of the ring is b. Find the mutual inductance L_{12} and the force F between the two conductors if the current through the wire is J_1 and the current through the ring is J_2.

5.25* Two thin rings of radii a and b are placed so that their planes are perpendicular to the line joining their centres. Assuming that the distance between the centres is l, find the mutual inductance L_{12} and express the result in terms of elliptical integrals. Consider the limiting cases $l \gg a, b$ and $a \approx b \gg l$.

5.26 Find the force F between the two circular currents in the preceding problem.

5.27 Find the self-inductance $\mathcal{L}$ per unit length of an infinite cylindrical solenoid with a closely wound coil and arbitrary cross section (not necessarily circular). Assume that the cross sectional area is S and the number of turns per unit length is n.

5.28 Find the self-inductance L of a coil of thin wire with n turns per unit length. The coil has a circular cross section of radius a and a finite length h ($h \gg a$). Calculate L to within terms of the order of a/h.

5.29 Find the self-inductance L of a toroidal solenoid. The radius of the toroid is b, the cross section of the toroid is circular (radius a), and the total number of turns is N. Determine the self-inductance per unit length of the toroid in the limiting case $b \to \infty$ (N/b = constant). Solve the same problem for a toroidal solenoid of rectangular cross section (sides a and h). How does the self-inductance change if the uniformly distributed current, while retaining its direction, flows not along a wire wound around the torus, but directly along the hollow shell of the torus?

5.30 Determine the self-inductance $\mathcal{L}$ per unit length of a 'line' consisting of two straight wires of radii a and b which are at a distance h from each other. The currents in the two wires are equal and opposite.

5.31* Show that the self-inductance of a thin closed conductor of circular cross section is approximately given by[2]

$$L = \frac{\mu_0 l}{2} + L',$$

where μ_0 is the magnetic permeability of the conductor, l is its length, and L' is the mutual inductance of two linear circuits, one of which lies along the axis AMB of the quasi-linear conductor and the other along the

[2] The two terms in this formula may be called the internal and external self-inductance, respectively, since they represent the magnetic energy stored inside and outside the conductor.

line CND, along which an arbitrary open surface S bounded by the axis cuts the surface of the conductor (figure 5.31.1).

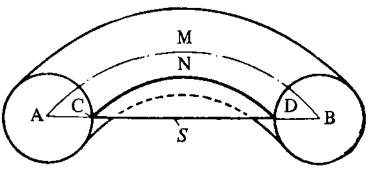


Figure 5.31.1.

5.32 Determine the self-inductance L of a thin ring of wire having a radius b. The radius of the wire is $a \ll b$.

Hint. Use the formula given in the preceding problem and the results of problem 5.25.

5.33 Determine the mutual inductance L_{12} of two thin parallel wires of length a placed at a distance l from each other and lying along the parallel sides of a rectangle. (The mutual inductance has no direct physical meaning since the currents in these wires cannot be closed. However, this inductance may be used to find the inductance of a closed circuit with parallel straight sections; see problems 5.34 and 5.35.)

5.34 Determine the coefficient of mutual inductance L_{12} of two identical squares of side a placed at a distance l from each other and coinciding with the opposite faces of a rectangular parallelepiped. Find the force F between them assuming that $\mu = 1$ everywhere.

5.35 Determine the self-inductance L of a square of side b which is made of a thin wire of radius $a \ll b$. The magnetic permeabilities in the surrounding space and of the material of the wire are μ and $\mu_0 = 1$ respectively.

Hint. Use the formulae obtained in the solution of problems 5.31 and 5.33.

5.36 Determine the magnetic moment $\boldsymbol{m}$ of a charged sphere rotating about one of its diameters. Consider the cases of uniform volume and surface charge distributions.

5.37* The current density associated with the spin magnetic moment of the electron in the hydrogen atom is given by $\boldsymbol{j} = c\,\mathrm{curl}[\rho(r)\boldsymbol{a}]$, where $\boldsymbol{a}$ is a constant vector, c is the ratio of the electromagnetic to the electrostatic units of charge, and ρ is the charge density in the atom, which depends only on the radial distance r and is zero at infinity. Show that the magnetic field at the origin is $\frac{8}{3}\pi\rho(0)\boldsymbol{a}$.

Hint. Use the results of problem 1.32.

5.38 Reduce the determination of the magnetic field due to given currents in a nonuniform, nonferromagnetic medium to the corresponding problem in electrostatics. To do this, the magnetic field should be written in the form $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}'$, where $\boldsymbol{H}_0$ is the 'primary' field which would be produced by the same current distribution in empty space and $\boldsymbol{H}'$ is the field due to the presence of the magnetic material. Introduce the scalar potential ψ for $\boldsymbol{H}'$ and obtain an equation for ψ and the corresponding boundary conditions.

5.39 A current-carrying circuit lies on the plane separating two media with magnetic permeabilities μ_1 and μ_2. Determine the magnetic field strength $\boldsymbol{H}$ in the whole of space assuming that the field produced by the circuit in the absence of the media is known.

5.40 An infinite straight wire carrying a current $\mathcal{J}$ is parallel to the plane separating two media with magnetic permeabilities μ_1 and μ_2. The distance from the wire to the separation boundary is a. Determine the magnetic field.

Hint. Use the method of images as in electrostatic problems (chapter 3).

5.41 A sphere of radius a and magnetic permeability μ is placed in a uniform magnetic field $\boldsymbol{H}_0$. Determine the resulting magnetic field $\boldsymbol{H}$, the induced magnetic moment $\boldsymbol{m}$, and the current density $\boldsymbol{j}_{\mathrm{mol}}$ which is equivalent to the magnetisation of the sphere.

5.42* An anisotropic nonferromagnetic sphere is placed in a uniform magnetic field. Find the resulting field $\boldsymbol{H}$ and the moment N of the forces acting on the sphere.

5.43 An infinitely long hollow cylindrical shell with internal and external diameters a and b, respectively, is placed in an external uniform magnetic field $\boldsymbol{H}_0$ which is perpendicular to its axis. The magnetic permeabilities of the cylinder and of the surrounding medium are μ_1 and μ_2 respectively. Find the field strength H in the cavity. Consider in particular the case $\mu_1 \gg \mu_2$.

5.44 A hollow sphere of internal and external radii a and b, respectively, is placed in an external uniform magnetic field $\boldsymbol{H}_0$. The magnetic permeabilities of the sphere and of the surrounding medium are μ_1 and μ_2 respectively. Find the field H in the cavity. Consider in particular the case $\mu_1 \gg \mu_2$.

5.45 An infinitely long straight wire of radius a and magnetic permeability μ_1 is placed in an external uniform magnetic field $\boldsymbol{H}_0$ in a medium of magnetic permeability μ_2. The field is transverse to the wire and the wire carries a constant current $\mathcal{J}$. Find the resulting magnetic field inside and outside the wire.

5.46 Determine the scalar and vector potentials ψ and A in a bounded region in which the magnetisation is known to be $M(r)$. Show by direct calculation that the vectors $\boldsymbol{B} = \mathrm{curl}\boldsymbol{A}$ and $\boldsymbol{H} = -\mathrm{grad}\,\psi$ are related by

$$\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}.$$

5.47 A body of arbitrary form is uniformly magnetised. Show that the scalar potential due to this body may be written in the form

$$\psi = -(\boldsymbol{M} \cdot \mathrm{grad}\,\varphi)$$

where $\boldsymbol{M}$ is the magnetisation and φ is the electrostatic potential due to a uniformly charged body (with density $\rho = 1$) of the same form and dimensions.

5.48 A straight wire carrying a current J is parallel to the axis of an infinitely long circular cylinder and at a distance b from it. The radius of the cylinder is a where $a < b$ and the magnetic permeability of the cylinder is μ. Find the force f per unit length of the wire[3].

Hint. Use the method of images.

5.49 A straight wire carrying a current J lies inside an infinitely long cylindrical cavity cut in a uniform magnetic medium. The wire is parallel to the axis of the cylinder and lies at a distance b from it. The radius of the cylinder is a and the magnetic permeability of the medium is μ. Determine the force f per unit length of the wire.

5.50 A current J flows through a straight wire lying along the z-axis. Three half planes at angles α_1, α_2, and α_3 to each other $(\alpha_1+\alpha_2+\alpha_3 = 2\pi)$ intersect along the z-axis and the spaces between them are filled with magnetic media with permeabilities μ_1, μ_2, μ_3, respectively. Determine the magnetic field H_i $(i = 1, 2, 3)$ in the three regions.

5.51* Find the magnetic field due to a uniformly magnetised permanent magnet of spherical form. The magnetic permeability of the sphere is μ, and of the external medium is μ_2.

5.52* Find the field due to an infinitely long uniformly magnetised cylinder of radius a. The magnetisation vector $\boldsymbol{M}_0$ is perpendicular to the axis of the cylinder and the magnetic permeabilities of the cylinder and of the surrounding medium are μ_1 and μ_2 respectively.

5.53 A uniformly magnetised sphere (idealised ferromagnetic) is placed in an external uniform magnetic field $\boldsymbol{H}_0$. Find the resulting magnetic field and the couple $\boldsymbol{N}$ acting on the sphere. The magnetic permeability of the sphere is μ_1 and of the surrounding medium is $\mu_2 = 1$.

5.54 A small permanent magnet having a magnetic moment $\boldsymbol{m}$ is placed in vacuo near the plane boundary of a medium of magnetic permeability μ.

[3] From the results of problems 5.40 and 5.48 one can easily obtain the solution to the electrostatic problem of determining the field produced by a charged filament.

Find the force F and the couple N acting on the permanent magnet.
Hint. Use the method of images.

5.55 An ellipsoid with permeability μ is placed in a uniform magnetic field $\boldsymbol{H}_0$. Find the internal field and the magnetic moment of the ellipsoid.

5.56* An ellipsoid made from an anisotropic material of permeability μ_{ik} is placed in a uniform external magnetic field $\boldsymbol{H}_0$. Find the internal magnetic field $\boldsymbol{H}_1$ in the ellipsoid.

Electrical and magnetic properties of matter[1]

a Polarisation of matter in a constant field

In general, the polarisation vector $\boldsymbol{P}$ (electric dipole moment per unit volume) is a nonlinear function of the electric field $\boldsymbol{E}$. However, in isotropic dielectrics and for sufficiently low fields, the polarisation is proportional to the field strength:

$$\boldsymbol{P} = \alpha \boldsymbol{E} . \tag{6.a.1}$$

The dielectric susceptibility α is determined by the properties of the dielectric and is in general a function of temperature. The relation between α and the permittivity ϵ is

$$\epsilon = 1 + 4\pi\alpha . \tag{6.a.2}$$

It is found for all substances that in a constant field $\alpha > 0$, $\epsilon > 1$. In anisotropic dielectrics ϵ and α are tensors of rank 2.

In a sufficiently rarified medium (gas) the polarisability α is proportional to the number of particles N per unit volume:

$$\alpha = N\beta , \qquad \epsilon = 1 + 4\pi N\beta , \tag{6.a.3}$$

where β is the average polarisability per molecule. This result is obtained if it is assumed that the field $\mathcal{E}$ acting on a molecule is equal to the average field $\boldsymbol{E}$. In the case of a dense material the difference between the two fields must be taken into account. Thus, it is found that for nonpolar substances, i.e. substances whose molecules do not have a permanent dipole moment in the absence of the field, and are either distributed at random or form a crystalline lattice with cubic symmetry, the field acting on a molecule can be written as

$$\mathcal{E} = \boldsymbol{E} + \tfrac{4}{3}\pi \boldsymbol{P} . \tag{6.a.4}$$

When the latter equation holds, equations (6.a.3) are replaced by

$$\alpha = \frac{N\beta}{1 - \frac{4}{3}\pi N\beta} , \qquad \frac{\epsilon - 1}{\epsilon + 2} = \tfrac{4}{3}\pi N\beta . \tag{6.a.4$'$}$$

Equations (6.a.4) and (6.a.4′) are known as the Lorenz–Lorentz relations.

The magnetisation vector $\boldsymbol{M}$ (magnetic dipole moment per unit volume) is frequently proportional to the magnetic field strength:

$$\boldsymbol{M} = \chi \boldsymbol{H} = \frac{\mu - 1}{4\pi} \boldsymbol{H} . \tag{6.a.5}$$

The magnetic permeability μ and the magnetic susceptibility χ are determined by the properties of the medium and are functions of

[1] For details of the theory see the textbooks by Fröhlich (1958), Landau and Lifshitz (1960), Born and Wolf (1970), London (1950), Shoenberg (1952), and Pines (1963).

temperature. In contrast to the dielectric susceptibility, the magnetic susceptibility may be either positive or negative. Materials with $\chi > 0$ are called paramagnetics and those with $\chi < 0$ are called diamagnetics. The relation between $\boldsymbol{M}$ and $\boldsymbol{H}$ for ferromagnetics is nonlinear and not single-valued.

In solving the problems given in this section, it will be necessary to use Boltzmann's formula[2] in addition to the equations of mechanics and electrodynamics. This formula gives the distribution of noninteracting particles in an external field:

$$\mathrm{d}N(q_i) = C\exp\left[-\frac{U(q_i)}{kT}\right]\mathrm{d}V\,, \tag{6.a.6}$$

where $U(q_i)$ is the potential energy of the ith particle in the external field, q_i are the generalised coordinates characterising the position and the orientation of the particles, $\mathrm{d}V$ is a volume element in q-space, $\mathrm{d}N(q_i)$ is the number of particles in $\mathrm{d}V$, $k = 1{\cdot}38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ is Boltzmann's constant, T is the absolute temperature, and C is a normalising constant determined from the normalisation condition

$$\int \mathrm{d}N(q_i) = N_0\,, \tag{6.a.7}$$

where N_0 is the total number of particles in the system and the integral is evaluated over the volume occupied by the system.

When the polar angles ϑ, ϕ defining the orientation of a molecule are chosen as the generalised coordinates, the volume element may be written $\mathrm{d}V = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\phi$.

6.1 The charge density in the electron cloud of a hydrogen atom is described by the function

$$\rho(r) = -\frac{e_0}{\pi a_0^3}\exp\left(-\frac{2r}{a_0}\right),$$

where e_0 is the electronic charge and a_0 is a constant (Bohr radius). Calculate the polarisability β of the hydrogen atom in a weak external field, assuming that the distortion of the electron cloud may be neglected. Repeat your calculation on the assumption that the electron cloud is distributed uniformly in a sphere of radius a_0.

6.2 An atom with a spherically symmetric charge distribution is placed in a uniform magnetic field $\boldsymbol{H}$. Show that the additional field around the nucleus which is due to the diamagnetic current is given by

$$\Delta H = -\frac{eH}{3mc^2}\varphi(0)\,,$$

[2] For a derivation of this formula see textbooks on statistical mechanics (e.g. Landau and Lifshitz 1969, ter Haar 1966, Pathria 1972).

where $\varphi(0)$ is the electrostatic potential due to the atomic electrons and e/m is the charge to mass ratio for an electron.

6.3* A molecule consists of two atoms at a distance a from each other. The atoms are spherically symmetric and their polarisabilities are β' and β''. Find the polarisability tensor for the molecule assuming that the atomic radii are small compared to a. Discuss in particular the case $\beta' = \beta''$.

6.4 Starting from the law of conservation of energy, show that the polarisability tensor for a molecule in a constant field is symmetric.

6.5 A dielectric consists of identical molecules having zero dipole moments in the absence of an external field. The polarisability tensor β_{ik} for each molecule is known. Find the polarisability α of the dielectric. Consider the two cases where (a) all the molecules are oriented in the same direction, and (b) the orientation of the molecules is random. Use the Lorenz-Lorentz formula to allow for the difference between the field acting on the molecule and the average field. [Note that case (a) may occur for crystalline and amorphous solids and case (b) for gases, liquids, and solids. However, in distinction to a gas, a solid is a system of strongly interacting particles and therefore the concept of isolated molecules may have no physical significance for solids.]

6.6* If the polarisability of a molecule is a function of its orientation, then the energy of interaction between the molecule and an external field will also depend on the orientation. Hence, in addition to the distorting polarisation mechanism, there is also a directional mechanism which will come into play even when the molecule is nonpolar. This will give rise to a temperature dependence of the permittivity of a substance consisting of randomly oriented nonpolar molecules. Investigate this effect in the case of a diatomic gas in a weak constant electric field, and calculate the polarisability α of the dielectric assuming that the longitudinal and transverse polarisabilities of the gas are β_1 and β_2 respectively.

6.7 Two gas molecules have dipole moments p_1 and p_2 and lie at a distance R from each other. Owing to collisions with other molecules, their orientation will change and the probability that they will assume a given mutual orientation will be given by Boltzmann's formula (6.a.6) with U equal to the energy of interaction between the two dipoles. Assuming that $U \ll kT$, show that the magnitude of U averaged over the Boltzmann distribution is

$$U(R) = -\frac{2p_1^2p_2^2}{3kTR^6} .$$

Hint. In averaging over the directions of the dipoles, use the formulae obtained in problem 1.32. These formulae should also be used in the next problem.

6.8 A molecule having an electric dipole moment p interacts with a nonpolar molecule having a polarisability β. Show that the energy of interaction averaged over all possible orientations is

$$U(R) = -\frac{\beta p^2}{R^6}$$

where R is the distance between the molecules (see hint to the preceding problem).

6.9* In general, a dielectric placed in a constant electric field will possess higher order moments in addition to the dipole moment represented by the polarisation vector $\boldsymbol{P}$. Find the volume and surface charge densities equivalent to a quadrupole polarisation Q_{ik} (the Q_{ik} are the components of the quadrupole moment per unit volume of the dielectric).

6.10 The permittivity of polar substances, for which the Lorenz–Lorentz formula does not hold, may be calculated by the following approximate method due to Onsager. Consider a small spherical cavity containing a single molecule and suppose that the cavity is surrounded by a homogeneous dielectric of permittivity ϵ. The cavity is evacuated and the field within it is equal to the effective field acting on the molecule. This field is determined by solving the macroscopic equations of electrostatics. Use this method to find the relation between the permittivity ϵ of the medium and the polarisability β of its molecules.

6.11* Consider a system consisting of particles with charge e and mass m, each of which moves at a certain fixed distance a from a given (its own) centre. The system is in a magnetic field in a state of statistical equilibrium. Show that the total magnetic susceptibility of the system is zero.

6.12* An ionised gas consists of ions (charge Ze, mean concentration N_0) and electrons (charge e, mean concentration n_0). The gas as a whole is electrically neutral ($ZN_0 = n_0$) and is in a state of statistical equilibrium at a temperature T. Assuming that the gas may be described in terms of classical statistics and that the energy of interaction between the particles is small compared with the thermal energy kT, find the charge density in the neighbourhood of an ion.

6.13 An infinite conducting plate bounded by the planes $x = h$, $x = -h$ is placed in a constant uniform electric field E_0 which is at right angles to the plate. The plate as a whole is electrically neutral, the mean concentration of free charges is N_0 and the permittivity is ϵ. Assuming that the change in the concentration under the action of the applied field is small ($|N-N_0| \ll N_0$), find the field distribution inside the plate and determine the thickness of the layer in which the 'surface' charge is concentrated.

6.14 A layer of an electrolyte lies between two infinite plane electrodes $x = h$, $x = -h$ at which potentials φ_0 and $-\varphi_0$ are maintained. The electrolyte consists of two types of ions having charges $+e$ and $-e$, and the mean concentration in the absence of the external field is N_0. Assuming that the permittivity of the electrolyte is ϵ, determine the potential at any point between the electrodes.

Hint. Use the method employed in the solution of problem 6.12.

b Polarisation of matter in a variable field

In the case of a variable electromagnetic field, the electric induction at a time t depends on the values of the field at all the preceding instants of time:

$$D(t) = E(t)+\int_{-\infty}^{t} f(t-u)E(u)\,\mathrm{d}u\,, \tag{6.b.1}$$

where $f(t-u)$ is a function determined by the properties of the medium. A similar formula will of course hold for the magnetic vectors. The direct proportionality between the induction and the field strength will only hold for the Fourier components of these vectors, i.e. for fields which are sinusoidal functions of time:

$$D_\omega = \epsilon(\omega)E_\omega\,, \qquad B_\omega = \mu(\omega)H_\omega\,. \tag{6.b.2}$$

The permittivity ϵ and the magnetic permeability μ are then functions of the frequency of the field. In order to calculate the frequency dependence, some assumptions must be made about the microscopic structure of matter. A rigorous theory of the polarisation of matter can only be developed on the basis of quantum mechanics, since classical mechanics and electrodynamics are unable to explain the structure of matter. However, the classical oscillator model of an atom does yield a number of important qualitative predictions about the behaviour of matter in a variable field. According to this model the force on an atomic electron is

$$F = -kr\,, \tag{6.b.3}$$

where r is the distance of the electron from the nucleus and k is a constant. In order to account for the dissipation of electromagnetic energy, it is also necessary to introduce a 'frictional' force which acts on the electron and is proportional to its velocity:

$$F_{\mathrm{fr}} = -\eta\dot{r}\,. \tag{6.b.4}$$

The permittivity deduced for this oscillator model is given by

$$\epsilon(\omega) = 1+\frac{\omega_{\mathrm{p}}^2}{\omega_0^2-\omega^2-\mathrm{i}\gamma\omega}\,, \tag{6.b.5}$$

where $\omega_{\mathrm{p}}^2 = 4\pi e^2N/m$, $\omega_0 = k/m$, $\gamma = \eta/m$, N is the number of atoms per unit volume and e and m are the charge and mass of the electron respectively.

Figure 6.b.1 shows the real (ϵ') and imaginary (ϵ'') parts of ϵ as functions of the frequency ω. The imaginary part is responsible for the absorption of electromagnetic energy and becomes appreciable only near the eigenfrequency ω_0 of the oscillators. In this region ϵ' decreases with increasing ω (anomalous dispersion). In the remaining regions ϵ' increases with increasing ω (normal dispersion).

According to quantum theory, equation (6.b.5) must be replaced by the somewhat similar formula

$$\epsilon(\omega) = 1+\omega_{\mathrm{p}}^2 \sum_i \frac{f_i}{\omega_i^2-\omega^2-\mathrm{i}\gamma_i\omega}, \tag{6.b.6}$$

where ω_{p}, f_i, ω_i, γ_i are constants.

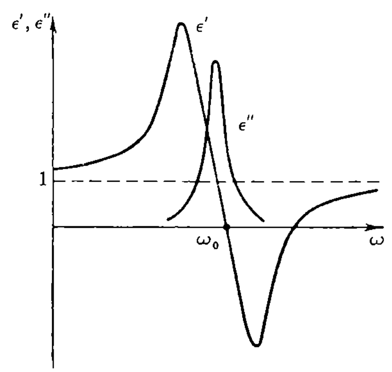


Figure 6.b.1.

6.15 An artificial dielectric consists of identical perfectly conducting metal spheres of radius a which are randomly distributed in vacuo. The average number of spheres per unit volume is N. An electromagnetic wave is propagated through the dielectric. Neglecting the difference between the field acting on each sphere and the average field, determine the permittivity ϵ and the magnetic permeability of the dielectric. Under what conditions can the dielectric be looked upon as a continuous medium?

Hint. The electric and magnetic polarisabilities of perfectly conducting spheres are calculated in problems 3.23 and 7.40.

6.16* Determine the permittivity of a conducting sphere assuming that the ions are fixed and the effect of bound electrons may be neglected. Energy dissipation should be allowed for by introducing the 'frictional' force $-\eta\dot{r}$ acting on the electrons whose concentration is N. Find the relation between the coefficient η and the conductivity.

6.17* A gaseous dielectric in a state of statistical equilibrium at a temperature T consists of molecules for which the principal values of the polarisability tensor are $\beta^{(1)} = \beta$, $\beta^{(2)} = \beta^{(3)} = \beta'$ where β and β' are functions of frequency ω. The dielectric is acted upon by a constant uniform electric field $\boldsymbol{E}_0$ and the concentration of the molecules is N. Find the permittivity tensor for the dielectric assuming that

$$E(t) = \mathcal{E}\exp(-\mathrm{i}\omega t)\,,$$

where $\mathcal{E} \ll E_0$.

6.18 A gaseous dielectric consists of polar molecules whose electric dipole moment in the absence of an external field is p_0. The principal values of the polarisability tensor of a molecule in a variable field are $\beta^{(1)} = \beta$, $\beta^{(2)} = \beta^{(3)} = \beta'$, where the x_1-axis is parallel to p_0. The dielectric is placed in an electric field having a constant component $\boldsymbol{E}_0$ and an alternating component $E(t) = \mathcal{E}\exp(-\mathrm{i}\omega t)$. Neglecting the orienting effect of the alternating field, and also the orienting effect associated with the anisotropy of the molecular polarisation in the constant field, find the permittivity tensor for the alternating field if the temperature is T, and the particle density N.

6.19* A system of charges (molecule) is placed in a sinusoidal electromagnetic field. Show that if there is no dissipation of electromagnetic energy in the system, then the polarisability tensor is Hermitian, i.e. $\beta_{ik} = \beta_{ki}^*$.

Hint. Use equation (7.a.7) to calculate the work done by the field.

6.20 Show that if a tensor β_{ik} is Hermitian, then in a suitably chosen coordinate system it may be written in the form

$$\beta_{ik} = \beta^{(i)}\delta_{ik} + \mathrm{i}e_{ikl}g_l\,,$$

where e_{ikl} is the skew-symmetric unit tensor of rank 3 (see problem 1.24), and $\boldsymbol{g}$ is a real vector (gyration vector). Note that media in which the gyration vector is finite are called gyrotropic. The propagation of electromagnetic waves in gyrotropic media is considered in section b of chapter 8.

6.21 Find the polarisability β_{ik} of an atom in the field of a plane monochromatic wave and a weak constant external magnetic field $\boldsymbol{H}_0$. Use the oscillator model and the method of successive approximations. Neglect the effect of the magnetic component of the wave and losses of electromagnetic energy. Determine also the gyration vector $\boldsymbol{g}$.

6.22 Use the oscillator model of an atom to find the permittivity tensor $\epsilon_{ik}(\omega)$ for a dielectric containing N atoms per unit volume, when the dielectric is placed in an arbitrary constant magnetic field $\boldsymbol{H}_0$. The dissipation of electromagnetic energy and the effects of the magnetic component of the plane wave may be neglected. When will the exact solution become identical with the approximate solution of the preceding problem?

Hint. In integrating the equations of motion of the electron use the cyclic coordinates:

$$x_{\pm 1} = \mp\frac{1}{\sqrt{2}}(x \pm \mathrm{i}y)\,, \qquad x_0 = z\,.$$

6.23* Find the permittivity tensor for a plasma in an external constant magnetic field H, given that the average electron density is N. Positive ions may be regarded as fixed and energy losses should be taken into account by introducing a 'frictional force' $-\eta\dot{\boldsymbol{r}}$.

6.24 Suppose that in the plasma considered in the preceding problem there is also a constant electric field $\boldsymbol{E}$. Use an approximation which is linear in H_0 to find the relation between the current density $\boldsymbol{j}$ and the electric field $\boldsymbol{E}$. Find also the electrical conductivity tensor.

Hint. Solve the equations of motion by the method of successive approximations.

6.25* Find the permittivity of an ionised gas placed in a constant magnetic field, taking into account the motion of positive ions but assuming that the ion mass is much greater than the electron mass. Discuss the dependence of the permittivity on the frequency ω and compare the result with the case where the ions may be considered as fixed. Assume that the concentration of ions and electrons is N.

Hint. Consider the equations of motion for an electron and an ion. The 'frictional' force on an electron is $-\eta(\dot{\boldsymbol{r}}-\dot{\boldsymbol{R}})$, the frictional force on an ion is $-\eta(\dot{\boldsymbol{R}}-\dot{\boldsymbol{r}})$, and $\boldsymbol{r}$ and $\boldsymbol{R}$ are the position vectors of the electron and ion respectively.

6.26 Suppose that in an infinite homogeneous medium there is only one special direction (e.g. the direction of an external field). Suppose further that T_{ik} is a tensor parameter of this medium, e.g. the permittivity or the magnetic permeability. It is clear that the components of the tensor T_{ik} should be invariant with respect to any rotation of the coordinate system about the special direction. Derive the limitations which are imposed by this invariance condition on the form of the tensor T_{ik}.

6.27 In some cases the function $f(t)$ which determines the relation between $\boldsymbol{D}$ and $\boldsymbol{E}$ [see equation (6.b.1)] is of the form $f(t) = f_0\exp(-t/\tau)$ where f_0 and τ are constants[(3)]. Show that

$$\epsilon(\omega) = 1+\frac{\epsilon_0 - 1}{1-\mathrm{i}\omega\tau}\,,$$

where ϵ_0 is the static value of the permittivity.

[(3)] This relation corresponds, for instance, to the model of a substance consisting of rigid dipoles. It neglects the polarisability of the electron shells.

6.28* Assuming that the polarisation of a particular medium can only arise after an electric field has been applied to it, show that the real and imaginary parts of the permittivity $\epsilon(\omega) = \epsilon'(\omega)+i\epsilon''(\omega)$ are given by

$$\epsilon'(\omega) = 1+\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{\epsilon''(\omega')}{\omega'-\omega}\,d\omega' , \qquad \epsilon''(\omega) = -\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{\epsilon'(\omega')-1}{\omega'-\omega}\,d\omega' .$$

These are the Kramers–Kronig dispersion relations. The symbol $\fint$ denotes the principal value of the integral. Note that in the case of metals, for which $\epsilon(\omega)$ may have a singularity at $\omega = 0$, the second of these formulae is of the form

$$\epsilon''(\omega) = -\frac{1}{\pi}\fint_{-\infty}^{\infty}\frac{\epsilon'(\omega')-1}{\omega'-\omega}\,d\omega'+\frac{4\pi\sigma}{\omega} ,$$

where σ is the static conductivity of the metal.

Hint. Consider the polarisation $\boldsymbol{P}(t)$ due to a field $\boldsymbol{E}(t) = \boldsymbol{E}_0\delta(t)$. Use equation (A1.17) of appendix 1.

6.29 Use the Kramers–Kronig dispersion relations of the preceding problem to determine the real part $\epsilon'(\omega)$ of the permittivity, assuming that the imaginary part is given by

$$\epsilon''(\omega) = \frac{(\epsilon_0-1)\omega\tau}{1+\omega^2\tau^2} ,$$

where ϵ_0 and τ are constants.

6.30 Prove the following sum rule for the imaginary part of the dielectric permittivity:

$$\int_0^{\infty}\left[\operatorname{Im}\frac{1}{\epsilon(\omega)}\right]\omega\,d\omega = -\tfrac{1}{2}\pi\omega_p^2 , \qquad \int_0^{\infty}[\operatorname{Im}\epsilon(\omega)]\omega\,d\omega = \tfrac{1}{2}\pi\omega_p^2 ,$$

where $\omega_p = (4\pi Ne^2/m)^{1/2}$ is the plasma frequency.

Hint. Bear in mind that $\operatorname{Re}\epsilon(\omega)$ is an even and $\operatorname{Im}\epsilon(\omega)$ an odd function of ω on the real axis. Use the asymptotic expression $\epsilon(\omega) = 1-\omega_p^2/\omega^2$, which is valid everywhere in the upper ω-halfplane as $|\omega| \to \infty$.

6.31 Show that if one wants to describe the electromagnetic field in matter one needs to introduce, apart from the average electric and magnetic fields $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$, only a single induction vector $\boldsymbol{D}'(\boldsymbol{r}, t)$ (and not two, $\boldsymbol{D}$ and $\boldsymbol{H}$, as usual):

$$\boldsymbol{D}'(\boldsymbol{r}, t) = \boldsymbol{E}(\boldsymbol{r}, t)+4\pi\int_{-\infty}^{t}\boldsymbol{j}'(\boldsymbol{r}, t')\,dt' ,$$

where $\boldsymbol{j}'(\boldsymbol{r}, t')$ is the averaged current density induced in the matter; it satisfies the equation of continuity $\operatorname{div}\boldsymbol{j}'+\partial\rho'/\partial t = 0$, where ρ' is the average charge density in the matter. By averaging the vacuum equations write down the Maxwell equations for $\boldsymbol{E}$, $\boldsymbol{B}$, and $\boldsymbol{D}'$ in the medium. The external charge and current densities $\boldsymbol{j}$ and ρ are given quantities.

c Ferromagnetic resonance

Classical theories are unable to supply a self-consistent explanation of ferromagnetism. Ferromagnetism is largely due to the intrinsic (spin) magnetic moments of electrons and the specific interaction forces between them, which are all of quantum-mechanical origin. However, some ferromagnetic phenomena may be discussed on the basis of a semiclassical theory. Among these phenomena is ferromagnetic resonance.

Ferromagnetic resonance is established under the following conditions: a constant magnetic field acting on the magnetic moment of an atom or a single electron gives rise to the Larmor precession of the moment about the direction of the field. This motion is eventually damped out because of the conversion of the Larmor precession energy into thermal energy. If the external field is sufficiently large, all the elementary magnetic moments become parallel to the external field. The ferromagnetic is then referred to as saturated and, correspondingly, the magnetic moment per unit volume is called the saturation magnetisation. In this section we shall always assume that the ferromagnetic is in fact magnetised to saturation. If in addition to the constant field there is also an alternating magnetic field, which is perpendicular to the constant field, then the alternating field will tend to maintain the precessional motion and when its frequency becomes equal to the precession frequency, ferromagnetic resonance will set in.

The motion of the magnetisation vector in a ferromagnetic is described by the Landau–Lifshitz equation (Landau and Lifshitz, 1935) which may be obtained as follows. The couple acting on a magnetic moment $\boldsymbol{m}$ of a particle (atom or single electron) placed in a magnetic field $\boldsymbol{H}_{\mathrm{eff}}$ is $N = [\boldsymbol{m} \wedge \boldsymbol{H}_{\mathrm{eff}}]$. Since the rate of change of the angular momentum $\boldsymbol{K}$ is equal to the sum of the moments of the external forces we have

$$\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}t} = N = [\boldsymbol{m} \wedge \boldsymbol{H}_{\mathrm{eff}}] \,. \tag{6.c.1}$$

According to quantum mechanics, the relation between the magnetic moment and the angular momentum of an electron is

$$\boldsymbol{m} = -\gamma \boldsymbol{K}\,, \qquad \gamma = \frac{e_0}{mc}\,,$$

where e_0 and m are the charge and mass of the electron and c is the velocity of light in vacuo. By using this relation and taking the average of both sides of equation (6.c.1) over an infinitely small volume, we obtain the Landau–Lifshitz equation

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = -\gamma[\boldsymbol{M} \wedge \boldsymbol{H}_{\mathrm{eff}}]\,, \tag{6.c.2}$$

where M is the magnetisation vector and H_{eff} is the average magnetic field acting on each elementary magnetic moment. In an infinite isotropic medium magnetised to saturation

$$H_{\text{eff}} = H - \lambda M + q\nabla^2 M \,, \tag{6.c.3}$$

where H is the average magnetic field in the medium and λ, q are constants. The second term in this expression is the so-called Weiss molecular field which does not contribute to equation (6.c.2) because $[M \wedge M] = 0$. The third term is only important when M is a very rapid function of the coordinates. We shall not consider such changes in M in this section and will therefore assume that $H_{\text{eff}} = H$.

In order that equation (6.c.2) should describe electromagnetic energy losses in the medium as well as the other effects described above, an additional dissipative term must be added to it. It is usual to assume that H_{eff} includes a 'frictional field' $-p\, \mathrm{d}M/\mathrm{d}t$. Equation (6.c.2) then becomes

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\gamma\left[M \wedge \left(H - p\frac{\mathrm{d}M}{\mathrm{d}t}\right)\right] , \tag{6.c.4}$$

where p is the loss parameter. If the losses are small and the total magnetic field is the sum of a constant field H_0 and a variable field $h(t)$, where $|h| \ll H_0$, then equation (6.c.4) may be simplified to read

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\gamma[M \wedge H] + \omega_r(\chi_0 H - M) \,, \tag{6.c.5}$$

where $\chi_0 = M_0/H_0$, $\omega_r = p\gamma^2 M_0^2/\chi_0$, and $M_0 = |M|$ is the saturation magnetisation. The Landau–Lifshitz equation is the starting point in the solution of problems on ferromagnetic resonance. In recent years ferromagnetics with very low conductivity (ferrodielectrics, ferrites) have found extensive application in ultrahigh frequency electronics. The propagation of electromagnetic waves in ferrites is discussed in chapters 8 and 9.

6.32 Find the law of motion of the magnetisation vector M in an infinite ferrite medium magnetised to saturation. Assume that there are no losses and that the magnetic field H in the medium is constant and uniform.

6.33 Solve the preceding problem in the case of finite losses using the Landau–Lifshitz equation in the form given by equation (6.c.5). Assume that the angle between M and H is small and that $\omega_r \ll \omega_0 = \gamma H_0$.

6.34* Suppose that in an infinite ferromagnetic medium there is a constant uniform magnetic field H_0 and a high frequency field $h\exp(-\mathrm{i}\omega t)$ where h = constant. Assuming that $h \ll H_0$ and neglecting energy losses, find the forced oscillations of the magnetisation vector M using an approximation which is linear in h. (The eigenoscillations, i.e. the Larmor precession under the action of the constant field $\boldsymbol{H}_0$, will be damped out as the result of losses which are present in all real systems.)

6.35 Use the result of the preceding problem to find the magnetic susceptibility and permeability tensors χ_{ik} and μ_{ik} for a high-frequency field. Find the components of the tensor μ_{ik} as functions of the constant magnetic field H_0 for $M_0 = 1 \cdot 6 \times 10^{-2}$ T and $\nu = \omega/(2\pi) = 9375$ MHz ($\lambda = 3 \cdot 2 \times 10^{-2}$ m). Investigate the resonance character of the variation in these quantities and determine H_0 at resonance.

6.36* The magnetic field in an infinite ferrite medium magnetised to saturation consists of a constant component $H_0 = H_z$ and a circularly polarised alternating component $H_x = h \cos \omega t$, $H_y = h \sin \omega t$, $h =$ constant. Find the exact solution of the Landau–Lifshitz equation corresponding to the forced precession of the vector $\boldsymbol{M}$ at the frequency ω of the external field. The energy dissipation may be ignored.

6.37 Solve problem 6.34 taking energy losses into account. Use the Landau–Lifshitz equation in the form given by equation (6.c.5).

6.38 Use the result of the preceding problem to find the magnetic permeability tensor μ_{ik} for a high-frequency field. Obtain expressions for the real and imaginary parts of the components of this tensor. Find the dependence of both parts of the components of the permeability tensor on a constant magnetic field for $M_0 = 1 \cdot 6 \times 10^{-2}$ T, $\omega_r = 3 \times 10^9$ s^{-1}, and $\nu = \omega/(2\pi) = 9375$ MHz. Determine the resonance value of the constant field, i.e. the value at which the imaginary parts of the components of the permeability tensor are a maximum.

6.39 Determine the half-width ΔH_0 of the resonance curve for the imaginary parts of the components of the magnetic permeability tensor, assuming that $\omega_r \ll \omega$. The half-width is defined as the distance between the ordinates $\mu'' = \mu_{\text{res}}$ and $\mu'' = \frac{1}{2}\mu_{\text{res}}$.

6.40* Find the Larmor precession frequency ω_k in an ellipsoidal ferromagnetic specimen in which energy losses may be neglected. The specimen is in an external uniform magnetic field $\boldsymbol{H}_0$ which is parallel to one of the axes of the ellipsoid. Assume that the departure of the magnetisation vector $\boldsymbol{M}$ from the equilibrium position is small.

Hint. The Landau–Lifshitz equation will now include the internal field $\boldsymbol{H}_1$. This field will differ from the external field $\boldsymbol{H}_0$ owing to the demagnetising effect which is a consequence of the form of the body:

$$\boldsymbol{H}_1 = \boldsymbol{H}_0 - \boldsymbol{H}_{\text{dm}}, \qquad H_{k\,\text{dm}} = 4\pi N_{kl} M_l ,$$

where N_{kl} is the demagnetisation tensor (see problem 5.55).

6.41 Solve the preceding problem taking energy losses into account. Retain only those terms which are linear in ω_r.

6.42* Discuss forced oscillations in a small ellipsoidal specimen in the presence of energy losses. Determine the components of the magnetic susceptibility tensor for a high-frequency field, assuming that the amplitude h of the field is small compared with the constant field H_0.

6.43 In some ferromagnetic media (antiferromagnetics) the resultant magnetisation $\boldsymbol{M}$ consists of two parts, so that $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2$ where $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are due to ions lying at different sites of the crystal lattice and forming two magnetic sublattices. In the equilibrium state $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are antiparallel so that $M = |M_1 - M_2|$. When an external magnetic field is introduced, each of the vectors experiences the molecular Weiss field [see equation (6.c.3)] and the two vectors cease to be antiparallel. Find the natural precession frequencies assuming that $\lambda |M_1 - M_2| \gg H_0$ where H_0 is the external field and λ is the molecular Weiss constant. Assume that the departures of the vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ from their equilibrium positions are small.

d Superconductivity

A consistent theory of superconductivity must be a quantal one. However, one can use classical arguments to construct a phenomenological electrodynamics of superconductors.

The Maxwell equations retain their form also when there are superconductors present. The only difference with the case of normal conductors lies in the material equations connecting the fields with the currents.

At temperatures much lower than the temperature of the transition to the superconducting state, but not too close to the absolute zero, the local theory suggested by F and H London (1950) is applicable. The total charge and current densities in a superconductor consist, according to that theory, of a normal part (ρ_n, $\boldsymbol{j}_n$) and a superconducting part (ρ_s, $\boldsymbol{j}_s$):

$$\rho = \rho_s + \rho_n \,, \qquad \boldsymbol{j} = \boldsymbol{j}_s + \boldsymbol{j}_n \,,$$

where $\boldsymbol{j}_s = -en_s\boldsymbol{v}_s$ with n_s the density of the superconducting electrons and $\boldsymbol{v}_s$ their velocity, $\boldsymbol{j}_n = \sigma \boldsymbol{E}$ with σ the electrical conductivity. The continuity equation is satisfied separately for the normal and the superconducting components.

The basic property of the electrons which correspond to superconductivity is that as they move through a superconductor they are—in contrast to the normal electrons—not scattered by the thermal oscillations of the lattice or by impurities. The motion of the superconducting electrons is thus determined solely by their interaction with the electromagnetic field and is described by the equation

$$\frac{\mathrm{d}\boldsymbol{v}_s}{\mathrm{d}t} \equiv \frac{\partial \boldsymbol{v}_s}{\partial t} + (\boldsymbol{v}_s \cdot \nabla)\boldsymbol{v}_s = -\frac{e}{m}\left\{\boldsymbol{E} + \frac{1}{c}[\boldsymbol{v}_s \wedge \boldsymbol{H}]\right\}.$$

Any change in the magnetic field leads, because of electromagnetic induction, to the occurrence of a surface current, the value of which is uniquely determined by the field. This current compensates the magnetic field inside the superconductor. The superconductor therefore behaves magnetically as a perfect diamagnetic. From the equation of motion of the superconducting electrons and the Maxwell equations one can derive Londons' material equation:

$$c\,\mathrm{curl}(\Lambda \boldsymbol{j}_s)+\boldsymbol{H} = 0\,, \tag{6.d.1}$$

where $\Lambda = mn_s e^2$ is the London parameter.

This equation describes the diamagnetism of the superconducting electrons. Using equation (6.d.1) the equation of motion of the superconducting electrons becomes

$$\frac{\partial \boldsymbol{j}_s}{\partial t} - \frac{1}{\Lambda}\boldsymbol{E} = 0\,, \tag{6.d.2}$$

where the small term ∇v_s^2 has been dropped. The material equations (6.d.1) and (6.d.2) considered together with the Maxwell equations completely determine the electrodynamics of superconductors in the low frequency region.

We shall everywhere in this section assume that $\epsilon = \mu = 1$. Moreover, we shall often not stipulate in the statement of the problems (see problems 6.44 to 6.50) that the superconductor is characterised by a London parameter Λ or by the depth the magnetic field penetrates into the superconductor,

$$\delta = \left(\frac{\Lambda c^2}{4\pi}\right)^{1/2}, \tag{6.d.3}$$

which can be used instead of it. For most superconductors $\delta \approx 10^{-7}$ to 10^{-8} m.

6.44 Write down the Maxwell equations and the material equations which describe a static electromagnetic field in a superconductor. Derive the equations which in that case describe the current and magnetic field distributions.

6.45 A superconductor fills the half-space $x \geqslant 0$, while $x \leqslant 0$ is a vacuum. In the vacuum there is a uniform magnetic field $\boldsymbol{H}_0$ parallel to y. Find the magnetic field and current distributions in the superconductor in the static case.

6.46 Find the force per unit area of the surface of the superconductor considered in the preceding problem. What is the direction of that force?

6.47 A superconducting plate of thickness $2a$ is positioned symmetric with respect to the $x = 0$ plane and is placed in a uniform magnetic field $\boldsymbol{H}_0$ parallel to y. Find the magnetic field distribution in the bulk of the plate and also the average magnetic moment per unit volume.

6.48 An infinitely long circular superconducting cylinder is placed in a uniform magnetic field $\boldsymbol{H}_0$ parallel to z; the axis of the cylinder is parallel to the field. Find the magnetic field distribution inside the cylinder and the average magnetic moment per unit volume.

6.49 A superconducting sphere of radius a is placed in a uniform magnetic field H_0. Determine the current distribution in the sphere and the magnetic field everywhere in space. Consider the limiting cases $a \gg \delta$ and $a \ll \delta$.

6.50 A current J flows through an infinitely long straight conductor of circular cross section (radius a). Find the distribution of the current density j over the cross section of the conductor and the magnetic field everywhere in space.

6.51 A superconductor is in the shape of a coil of arbitrary cross section. A current concentrated in a thin surface layer flows through it. Prove that the magnetic flux through a surface based on a contour drawn inside the conductor vanishes if the current density on the contour vanishes. Use the material equation (6.d.2) and the Maxwell equations.

6.52 A superconducting plane coil of self-inductance L through which a current J flows, enters completely a uniform magnetic field $\boldsymbol{H}_0$. Determine the current J' which will flow along the coil after this has happened. The area of an axial cross section of the coil is S. The normal to the plane of the coil makes an angle ϑ with the direction of $\boldsymbol{H}_0$.

6.53 A conducting coil of self-inductance L is in the normal state in an external magnetic field (the magnetic flux through the contour of the coil equals Φ_0). Afterwards the temperature is lowered and the coil changes to the superconducting state. What current will flow through the coil if the external magnetic field is now excluded?

Quasi-stationary electromagnetic fields [1]

a Quasi-stationary phenomena in linear conductors

If the period of an alternating electromagnetic field is much greater than the time taken by the electromagnetic wave to travel through a given system, i.e.

$$T \gg \frac{l}{c}, \qquad \omega \ll \frac{c}{l}, \tag{7.a.1}$$

where c is the velocity of light and l is a characteristic linear dimension of the system, then the velocity of propagation of electromagnetic disturbances through the system may be assumed to be infinite. The corresponding approximation is known as the quasi-stationary approximation. The quasi-stationary approximation occasionally gives good results even when this condition is not satisfied, e.g. in the theory of long transmission lines.

In the quasi-stationary approximation, the current in a closed circuit consisting of a source of e.m.f. $\mathcal{E}(t)$, a capacitance C, an inductance L, and a resistance R, satisfies the following differential equations

$$J = \frac{dq}{dt}, \qquad \frac{1}{c^2}L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q = \mathcal{E}(t),$$

where q is the charge on one of the capacitor plates. When the e.m.f. is of the form $\mathcal{E}(t) = \mathcal{E}_0\exp(-i\omega t)$, then in steady state the current is proportional to the e.m.f. so that

$$J = \frac{\mathcal{E}}{Z}, \qquad Z = R+i\left(\frac{1}{\omega C}-\frac{\omega L}{c^2}\right). \tag{7.a.2}$$

The quantity Z is called the complex impedance of the circuit.

The eigenfrequency ω_0 of the oscillations in a circuit consisting of a capacitance C and a self-inductance L is given by Thomson's formula

$$\omega_0 = \frac{c}{(LC)^{1/2}}. \tag{7.a.3}$$

In the case of a circuit consisting of a number of branches, the differential equations giving the currents in the various branches may be set up with the aid of Kirchhoff's laws.

The e.m.f. induced in a linear circuit due to a change in the magnetic flux intercepted by it is given by Faraday's law

$$\mathcal{E}_{\text{ind}} = -\frac{1}{c}\frac{d\Phi}{dt}, \tag{7.a.4}$$

[1] For more details see the books by Landau and Lifshitz (1960), Smythe (1950), Stratton (1941), Ramo and Whinnery (1944), and Brillouin (1946).

where Φ is the flux of magnetic induction through the circuit. The change in flux may be due to a change in the magnetic field or to the motion or a deformation of the circuit. If there are a number of inductively coupled circuits then the total flux of magnetic induction through the ith circuit Φ_i is given by

$$\Phi_i = \frac{1}{c}\sum_k L_{ik} J_k , \tag{7.a.5}$$

where J_k is the current in the kth circuit, L_{ik} $(i \neq k)$ is the mutual inductance between the ith and the kth circuits, and $L_{ii} \equiv L_i$ is the self-inductance of the ith circuit (see the introduction to chapter 5).

The generalised force acting on a conductor carrying a current in a quasi-stationary field is given by

$$F_i = \left(\frac{\partial W}{\partial q_i}\right)_J , \tag{7.a.6}$$

where W is the magnetic energy of the system, q_i is the generalised coordinate, and the differentiation is carried out at constant current in the conductors. The magnetic energy can be expressed in terms of the currents and the coefficients of inductance by means of the same formulae as in the static case [see equations (5.0.17) and (5.0.20)].

The time average of the product of harmonic functions of time, $a(t) = a_0 \exp(-\mathrm{i}\omega t)$, may be calculated with the aid of the formulae

$$\overline{a^2(t)} = \tfrac{1}{2}|a|^2 , \qquad \overline{a(t)b(t)} = \tfrac{1}{2}\,\mathrm{Re}(ab^*) . \tag{7.a.7}$$

For example, the average dissipation of heat in a circuit may be calculated from the following formulae

$$Q = \tfrac{1}{2}\,\mathrm{Re}(\mathcal{E} J^*) = \tfrac{1}{2}|J|^2\,\mathrm{Re}Z . \tag{7.a.8}$$

7.1 A circular loop of wire of radius a is placed in a magnetic field H_0 and rotates with an angular velocity ω about a diameter which is perpendicular to H_0. Find the current in the loop $J(t)$, the retarding couple $N(t)$, and the average power $\overline{P}$ which is required to maintain the rotation.

7.2 A plane LCR circuit of area S rotates with an angular velocity ω in a constant magnetic field H_0 about an axis lying in the plane of the circuit and perpendicular to H_0. Determine the average retarding couple $\overline{N}$ on the circuit.

7.3 The current through one of two inductively coupled circuits is $J(t) = J_0 \exp(-\mathrm{i}\omega t)$. The inductance and resistance of the two circuits are given. Express the average generalised force on the circuits in terms of the derivative of the mutual inductance with respect to the generalised coordinate q_i.

7.4 Two identical circuits each have a resistance R and inductance L. and one of them includes a source of e.m.f. $\mathcal{E}(t) = \mathcal{E}_0 \exp(-\mathrm{i}\omega t)$. The mutual inductance between the circuits is L_{12}. Determine the average force $\overline{F}$ on each of the circuits. Express the result in terms of the derivative of the mutual inductance with respect to the corresponding coordinate.

7.5 Determine the natural frequencies of electrical oscillations in the two circuits shown in figure 7.5.1 when the coupling between the two circuits is achieved through a capacitance C $[Z = \mathrm{i}/(\omega C)]$.

Hint. Set up a system of algebraic equations for the currents and put the determinant of the system equal to zero.

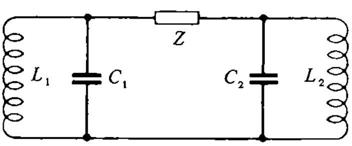


Figure 7.5.1.

7.6 Solve the preceding problem in the case when the coupling between the two circuits is through an inductance (see figure 7.5.1; $Z = -\mathrm{i}\omega L/c^2$).

7.7 Find the natural frequencies of oscillation $\omega_{1,2}$ in two inductively coupled circuits with capacitances C_1, C_2, inductances L_1, L_2, and mutual inductance L_{12}.

7.8 Two circuits are coupled to each other through a resistance R (see figure 7.5.1, with $Z = R$). Find the natural frequencies of oscillation of the two circuits when the coupling is small, i.e. R is large.

7.9 A circuit consisting of an inductance L_1, a capacitance C_1, and a resistance R_1, includes an external source of e.m.f. $\mathcal{E}(t) = \mathcal{E}_0 \exp(-\mathrm{i}\omega t)$. The circuit is inductively coupled to another circuit whose parameters are L_2, C_2, R_2. The mutual inductance between the two circuits is L_{12}. Find the currents J_1 and J_2 in the two circuits. Consider in particular the case where the second circuit is purely inductive ($R_2 = 0$, $C_2 = \infty$) and determine the frequency ω at which the current J_1 is a maximum.

7.10 Find the complex impedance Z of the two-terminal network shown in figure 7.10.1

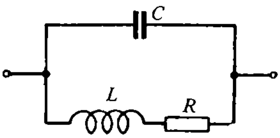


Figure 7.10.1.

7.11 A capacitor is filled with a dielectric whose permittivity is given by

$$\epsilon = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}$$

(ionised gas; see problem 6.16). In the absence of the medium the capacitance is C_0. Show that the complex impedance of such a capacitor is equal to the impedance of the two-terminal network shown in figure 7.10.1 when the parameters L, C, and R of this circuit are suitably chosen. Find L, C, and R.

7.12 Determine the average energy $\overline{W}$ stored per unit time in the capacitor described in the preceding problem. Find also the heat loss Q per unit time and express both quantities in terms of the potential difference between the capacitor plates $U = U_0 \exp(-i\omega t)$.

7.13 A capacitor is filled with a medium having a permittivity

$$\epsilon = 1 + \frac{\omega_p^2}{\omega_0^2 - i\gamma\omega - \omega^2}$$

[dielectric with losses; see equation (6.b.5)]. In the absence of the dielectric the capacitance is C_0. Find the parameters C, C_1, L, and R of the two-terminal network shown in figure 7.13.1, when its AC impedance is equal to the impedance of the above capacitor.

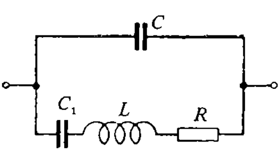


Figure 7.13.1.

7.14 Find the average energy $\overline{W}$ per unit time stored in the capacitor considered in the preceding problem and the average heat loss Q per unit time. Assume that the potential difference between the capacitor plates is $U = U_0 \exp(-i\omega t)$.

7.15 An oscillatory circuit consists of a capacitance C and an inductance L. At a certain instant of time a battery of constant e.m.f. $\mathcal{E}$ and an internal resistance R is connected across the plates of the capacitor. Find the current flowing through the inductance as a function of time. Investigate the dependence of this current on L, C, and R.

7.16 A rectangular voltage pulse $U_1(t) = U_0$ for $0 \leqslant t \leqslant T$ and $U_1(t) = 0$ for $t < 0$, $t > T$ is applied to a circuit consisting of a resistance R and a capacitance C connected in series. Find the voltage across the resistance.

7.17 A rectangular pulse $U_1(t) = U_0$ for $0 \leqslant t \leqslant T$ and $U_1(t) = 0$ for $t < 0$ and $t > T$ is applied across a circuit consisting of a resistance R and an inductance L connected in series. Find the voltage $U_2(t)$ across the inductance.

7.18 A circuit consists of a plane parallel capacitor having a capacitance C and a resistance R (figure 7.18.1). One wants to produce a field between the capacitor plates which will increase linearly from zero to E_0 during a time T and then decrease linearly to zero over an equal interval of time. Determine the form of the pulse which must be applied across the input of this circuit in order to produce the required variation.

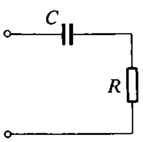


Figure 7.18.1.

7.19 An e.m.f. $\mathcal{E}(t) = \mathcal{E}_0 \cos(\omega t + \varphi_0)$ is connected at $t = 0$ across a circuit consisting of a resistance R and an inductance L. Find the current $J(t)$ in the circuit and determine the phase angle φ_0 for which there will be no transients.

7.20* An artificial transmission line consists of N identical sections ($N \gg 1$) and is open at each end (figure 7.20.1). Find the natural frequencies of the system.

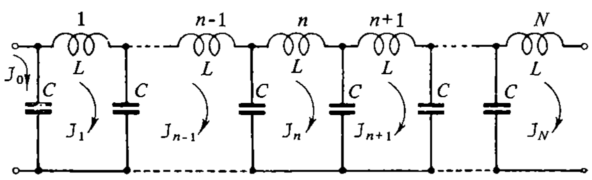


Figure 7.20.1.

7.21 Assuming that the total number of natural frequencies of a long artificial transmission line (see the preceding problem) is large, find the number of oscillations in a frequency interval $\Delta\omega$.

7.22* A long artificial transmission line consisting of $2N$ alternately identical sections (figure 7.22.1) is open at each end. Investigate the spectrum of natural frequencies of the system.

7.23* A long artificial transmission line consists of N identical sections (figure 7.23.1) whose impedances are given by

$$Z_1 = -\mathrm{i}\left(\frac{\omega}{c^2}L_1 - \frac{1}{\omega C_1}\right), \qquad Z_2 = -\mathrm{i}\left(\frac{\omega}{c^2}L_2 - \frac{1}{\omega C_2}\right).$$

A potential difference U_1 is applied across one end of the line while the other end remains on open circuit. Find the potential difference U_2 between the points a and b.

Hint. The solution should be sought in the form of a difference equation involving the current J_n in the nth section with $J_n = q^n \times$ constant.

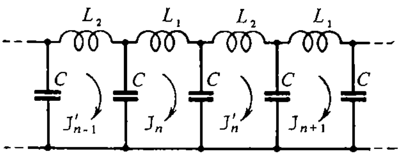


Figure 7.22.1.

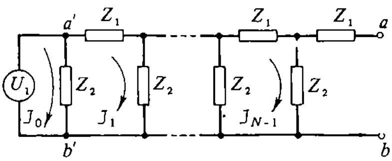


Figure 7.23.1.

7.24 Use the results of the preceding problem to investigate the transmission coefficient $K = U_2/U_1$ as a function of frequency for $N \gg 1$. Find the frequency interval in which K is appreciably different from zero.

7.25 From a consideration of a long artificial transmission line with lumped parameters (problem 7.20) obtain a differential equation for the current in a long line with uniformly distributed parameters by taking the appropriate limit.

7.26 A long lossless transmission line with continuously distributed parameters and length l is on open circuit at each end. Find the spectrum of natural oscillations of the system and compare it with the spectrum of the lumped-parameter line considered in problem 7.20.

7.27* An e.m.f. connected to a closed circuit gives rise to a current $J(t) = J_0 \exp(-\mathrm{i}\omega t)$ in the circuit. Find the general expression for the complex impedance of the circuit without neglecting the delay in the system.

7.28 Find the correction to the inductance and resistance $R_r(\omega)$ of a circular circuit of radius a in the first nonvanishing approximation (see the preceding problem). Show that $R_r(\omega)$ is the coefficient of proportionality between the average energy emitted per unit time and the r.m.s. current in the circuit.

b Eddy currents and skin effect

If a conductor placed in an external magnetic field satisfies the conditions given by equation (7.a.1), then at each instant of time the field near the conductor must satisfy the following equations of magnetostatics:

$$\operatorname{div} \boldsymbol{B} = 0\,, \qquad \operatorname{curl} \boldsymbol{H} = 0\,, \tag{7.b.1}$$

and the equation

$$\operatorname{curl} \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} . \tag{7.b.2}$$

When the conductivity σ of a conductor is sufficiently high ($\sigma/\omega \gg \epsilon'$ where ϵ' is the real part of the permittivity), the field inside the conductor is described by equation (7.b.2) and

$$\operatorname{div} \boldsymbol{B} = 0 , \qquad \operatorname{curl} \boldsymbol{H} = \frac{4\pi}{c}\sigma \boldsymbol{E} . \tag{7.b.3}$$

From equations (7.b.2) and (7.b.3) one can obtain second-order equations for the vectors $\boldsymbol{E}$, $\boldsymbol{H}$. In the case of a homogeneous medium these equations are of the form

$$\nabla^2 \boldsymbol{H} = \frac{4\pi\mu\sigma}{c^2}\frac{\partial \boldsymbol{H}}{\partial t} , \qquad \nabla^2 \boldsymbol{E} = \frac{4\pi\mu\sigma}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} . \tag{7.b.4}$$

On the boundaries separating two conductors, or a conductor and a dielectric, the field components must satisfy the boundary conditions

$$B_{1n} = B_{2n} , \qquad H_{1\tau} = H_{2\tau} , \qquad E_{1\tau} = E_{2\tau} . \tag{7.b.5}$$

The quantity $\delta = c/(2\pi\mu\sigma\omega)^{1/2}$ (the thickness of the skin layer) characterises the depth of penetration of the field into the conductor (ω is the frequency of the field). In the case of a strong skin effect it may be assumed that the field does not penetrate into the conductor, in which case $\boldsymbol{H} = 0$ inside the conductor, whereas outside the conductor and at its surface the relation between the surface current $\boldsymbol{i}$ and the field is

$$\boldsymbol{H} = \frac{4\pi}{c}[\boldsymbol{i} \wedge \boldsymbol{n}] . \tag{7.b.6}$$

Owing to the appearance of eddy currents, a conductor placed in a magnetic field assumes a magnetic moment even if its permeability is $\mu = 1$. This magnetic moment may be conveniently characterised by the magnetic polarisability tensor β_{ik} which is defined by

$$m_i = \beta_{ik} H_{0k} , \tag{7.b.7}$$

where $\boldsymbol{m}$ is the magnetic moment of the body and $\boldsymbol{H}_0$ is a periodic external magnetic field. The tensor β_{ik} is symmetric ($\beta_{ik} = \beta_{ki}$) and its components are in general complex and depend on the frequency.

The time average of the heat liberated inside a conductor may be calculated from

$$Q = \int \overline{(\boldsymbol{j} \cdot \boldsymbol{E})}\, \mathrm{d}^3 r = \int \sigma \overline{E^2}\, \mathrm{d}^3 r \tag{7.b.8}$$

or

$$Q = -\frac{c}{4\pi}\oint(\overline{[E \wedge H]} \cdot \mathrm{d}^2 S)\,. \tag{7.b.9}$$

In the first of these formulae the integral is evaluated over the volume of the conductor, whereas in the second formula it is evaluated over its surface. The heat Q can also be expressed in terms of the imaginary part of the magnetic polarisability tensor ($\beta_{ik} = \beta'_{ik} + \mathrm{i}\beta''_{ik}$); in the case of a field which is a harmonic function of time, we have

$$Q = \frac{\omega}{2}\beta''_{ik}\,\mathrm{Re}(H_{0i}H^*_{0k})\,. \tag{7.b.10}$$

7.29 A coil, in which a current $J_0 \exp(-\mathrm{i}\omega t)$ is flowing, is wound on a wide plate which has a conductivity σ, magnetic permeability μ, and is bounded by the planes $x = \pm h$. The number of turns, which are parallel to one another, per unit length is n and the thickness of the coil is very small. Neglecting edge effects, determine the real part of the amplitude of the magnetic field inside the plate. Investigate the limiting cases of a strong ($\delta \ll h$) and weak ($\delta \gg h$) skin effect.

7.30* An infinitely long metal cylinder with conductivity σ and magnetic permeability μ is placed so that its axis coincides with the axis of an infinite solenoid of circular cross section, which carries a current $J = J_0 \exp(-\mathrm{i}\omega t)$. Find the magnetic and electric field in the whole of space, and also the current distribution j in the cylinder, assuming that the radius of the cylinder a, the radius of the solenoid b, and the number of turns per unit length, n, are known.

7.31 A conducting cylinder is placed in the uniform magnetic field $H = H_0 \exp(-\mathrm{i}\omega t)$ which is parallel to its axis. By using the results of the preceding problem, investigate the current distribution j inside the cylinder, in the limit of low and high frequencies.

7.32 Calculate the amount of heat Q liberated per unit time per unit length of the cylinder considered in problem 7.30. Investigate the limit of low and high frequencies.

7.33 Find the magnetic polarisability β (per unit length) of a cylinder of radius a and conductivity σ placed in an alternating magnetic field of frequency ω which is parallel to its axis. Assume that the magnetic permeability is $\mu = 1$ and consider the limit of high and low frequencies.

7.34* A metal cylinder of radius a and conductivity σ is placed in an external uniform magnetic field $\boldsymbol{H} = \boldsymbol{H}_0 \exp(-\mathrm{i}\omega t)$ which is perpendicular to its axis. Assuming that the magnetic permeability is $\mu = 1$, find the resultant field and the current density $\boldsymbol{j}$ in the cylinder.

Hint. Express $\boldsymbol{E}$, $\boldsymbol{H}$ in terms of the vector potential $\boldsymbol{A}$ and integrate the differential equation for $\boldsymbol{A}$.

7.35 Find the energy dissipation per unit length of a perfectly conducting circular cylinder placed in an alternating magnetic field of frequency ω which is perpendicular to the axis of the cylinder.

7.36* An infinitely long cylinder of radius a and conductivity σ is placed in a circularly polarised magnetic field, which is perpendicular to its axis,

$$\boldsymbol{H}_0(t) = (\boldsymbol{H}_{01} + \mathrm{i}\boldsymbol{H}_{02}) \exp(-\mathrm{i}\omega t),$$

$\boldsymbol{H}_{01}$ and $\boldsymbol{H}_{02}$ are mutually perpendicular vectors and are of equal length $H_{01} = H_{02} = H_0$ [the vector $\boldsymbol{H}_0(t)$ describes a circle of radius H_0 in the plane perpendicular to the axis of the cylinder]. Find the average couple $\overline{N}$ per unit length of the cylinder ($\mu = 1$).

7.37 An infinitely long cylinder rotates about its axis with an angular velocity ω in a constant uniform magnetic field $\boldsymbol{H}_0$ which is perpendicular to the axis. Find the retarding couple $\overline{N}$ per unit length of the cylinder.

7.38* An infinitely long metal cylinder of radius a, conductivity σ, and permeability μ is placed in a constant and uniform magnetic field H_0 which is parallel to its axis. The external field is then switched off and is maintained at zero. Find the magnetic field inside the cylinder as a function of time.

7.39 A metal sphere of radius a, conductivity σ, and magnetic permeability μ is placed in a uniform alternating magnetic field $H_0(t)$, with $H_0(t) = H_0 \exp(-\mathrm{i}\omega t)$. Assuming that the frequency is low, find the distribution of eddy currents in the sphere and the average power Q absorbed by it. Use the first-order approximation.

7.40 A metal sphere of radius a, conductivity σ, and magnetic permeability μ is placed in a uniform magnetic field which varies with a frequency ω. Find the resultant field $\boldsymbol{H}$ and the average power Q absorbed by the sphere at high frequencies.

Hint. In calculating the field outside the sphere, assume that the field inside it is zero, i.e. neglect the penetration depth δ in comparison with the radius a. In determining the field inside the sphere assume that its surfaee is plane.

7.41* A conducting ellipsoid is placed in a uniform alternating magnetic field. Determine the magnetic polarisability of the ellipsoid in the case of a strong skin effect, i.e. assume that the depth of penetration of the field into the conductor can be neglected. Consider the limiting cases of a thin circular disc and a long thin rod.

7.42* A sphere of radius a and conductivity σ is placed in a uniform magnetic field $H(t) = H_0 \exp(-\mathrm{i}\omega t)$. Find the resultant magnetic field and the distribution of eddy currents in the sphere in the general case of an arbitrary frequency. Verify that in the limiting cases of a strong and weak skin effect the general solutions become identical with those obtained in problems 7.39 and 7.40, respectively (assume for simplicity that $\mu = 1$).

7.43 Find the average power, Q, absorbed by a conducting sphere in a uniform alternating magnetic field of arbitrary frequency.

7.44 Find the active resistance R of a thin cylindrical conductor of length l, radius a, and conductivity σ in the presence of the skin effect, assuming that the permeability is $\mu = 1$. Investigate the limit of low and high frequencies.

7.45 A layer of another metal is deposited on the surface of a cylindrical conductor of radius a and conductivity σ_1. The thickness of the layer is h and its conductivity is σ_2 ($h \ll a$). Find the active resistance R of the composite conductor to alternating currents, assuming that the thickness of the skin layer is small compared with a ($\mu = 1$).

7.46 An infinitely long hollow cylinder of internal radius a and wall thickness h ($h \ll a$) is placed in a uniform magnetic field $H_0(t)$, with $H_0(t) = H_0 \exp(-\mathrm{i}\omega t)$, which is parallel to its axis. Find the amplitude H' of the magnetic field in the cavity and investigate its dependence on ω.

Hint. Since $h \ll a$, the field inside the shell may be determined on the assumption that the shell is plane.

7.47 An alternating current $J = J_0 \exp(-\mathrm{i}\omega t)$ flows along a hollow cylindrical conductor of average radius a, conductivity σ, magnetic permeability μ, and thickness $h \ll a$. Find the current distribution j across the conductor and its active resistance R per unit length. Find also the condition which must be satisfied in order to ensure that the resistance of a hollow conductor is very nearly equal to the resistance of a solid conductor of equal external radius.

Hint. Neglect the curvature of the surface of the conductor.

7.48* A linear current J flows inside a metal tube at a distance l from its axis. The radius of the tube is a, the wall thickness is $h \ll a$, and the conductivity of the walls is σ ($\mu = 1$). The current J and the distance l are both arbitrary functions of time, but are such that at all times $l \ll a$. Assuming that equation (7.a.1) is satisfied, determine the force f per unit length on the current J due to eddy currents induced in the cylindrical shell in the case of a weak skin effect ($h \ll \delta$).

7.49* Solve the preceding problem in the case of a strong skin effect ($h \gg \delta$).

Propagation of electromagnetic waves [1]

a Plane waves in a homogeneous medium. Reflection and refraction. Wavepackets

In the absence of charges and currents, the electromagnetic field vectors in a dielectric satisfy the equations

$$\mathrm{curl}\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}\,, \tag{8.a.1}$$

$$\mathrm{curl}\boldsymbol{H} = \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}\,, \tag{8.a.2}$$

$$\mathrm{div}\boldsymbol{D} = 0\,, \tag{8.a.3}$$

$$\mathrm{div}\boldsymbol{B} = 0\,. \tag{8.a.4}$$

In a nondispersive medium the field vectors are related by

$$\boldsymbol{D} = \epsilon \boldsymbol{E}\,, \qquad \boldsymbol{B} = \mu \boldsymbol{H}\,, \tag{8.a.5}$$

where ϵ is the permittivity and μ the magnetic permeability. If the losses in electromagnetic energy are negligible then ϵ and μ are real. In the case of a homogeneous medium equations (8.a.1) to (8.a.5) may be combined to yield the following second-order equation for $\boldsymbol{E}$ and $\boldsymbol{H}$:

$$\nabla^2\boldsymbol{E} - \frac{1}{v_{\mathrm{ph}}^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0\,, \qquad \nabla^2\boldsymbol{H} - \frac{1}{v_{\mathrm{ph}}^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = 0\,, \tag{8.a.6}$$

where $v_{\mathrm{ph}} = c/(\epsilon\mu)^{1/2}$ can be identified with the phase velocity of the waves described by equations (8.a.6).

In general, equations (8.a.5) will only hold for monochromatic field components and the permittivity and permeability will be complex functions of frequency (dispersion). The imaginary parts of the permittivity and permeability determine the dissipation of electromagnetic energy in the medium.

In a conducting medium in which the field is varying slowly, so that the relation between the current and the field is of the form $j = \sigma E$, where σ is the static conductivity, equation (8.a.2) must be replaced by

$$\mathrm{curl}\boldsymbol{H} = \frac{4\pi}{c}\sigma E + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t}\,. \tag{8.a.7}$$

The latter relation will again assume the form of equation (8.a.2) if one introduces the complex permittivity, which at low frequencies is

[1] For details of the theory we refer to the textbooks by Landau and Lifshitz (1960, 1975), Born and Wolf (1970), Stratton (1941), Smythe (1950), Jackson (1962), Panofsky and Phillips (1962), Alfvén and Fälthammer (1963), and Stroke (1969), and the review paper by Akhiezer, Bar'yakhtar, and Kaganov (1961).

given by

$$\epsilon(\omega) = \epsilon' + \mathrm{i}\frac{4\pi\sigma}{\omega} , \tag{8.a.8}$$

where ϵ' and σ are the static values of the permittivity and conductivity. At high frequencies the permittivity of a conducting medium is also a complex function of frequency.

In the case of good conductors (metals) the second term in equation (8.a.8) is very large and hence at low frequencies

$$\epsilon(\omega) = \mathrm{i}\frac{4\pi\sigma}{\omega} . \tag{8.a.9}$$

If the frequency is such that the depth of penetration of the field into the metal is much smaller than the radius of curvature of its surface, and also much less than the wavelength in the surrounding medium, then the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$ near the surface of the conductor are related by

$$\boldsymbol{E}_\tau = \zeta[\boldsymbol{H}_\tau \wedge \boldsymbol{n}] , \tag{8.a.10}$$

where $\boldsymbol{n}$ is an inward unit normal and ζ is the surface impedance which depends on frequency and the properties of the metal, and is defined by

$$\zeta = \left(\frac{\mu}{\epsilon}\right)^{1/2} \tag{8.a.11}$$

Equation (8.a.10) will hold provided $|\zeta| \ll 1$. It may be used as an approximate boundary condition for the determination of the field outside a conductor.

If the field is a simple periodic function of time, the medium is homogeneous, and $\mu = 1$, then the electric field will satisfy the equation

$$\nabla^2\boldsymbol{E} + \frac{\epsilon\omega^2}{c^2}\boldsymbol{E} - \operatorname{grad}\operatorname{div}\boldsymbol{E} = 0 , \tag{8.a.12}$$

and the magnetic field $\boldsymbol{H}$ will be given in terms of $\boldsymbol{E}$ by equation (8.a.1).

A plane monochromatic wave propagating in the direction of the wavevector $\boldsymbol{k}$, where $k = 2\pi/\lambda$ and λ is the wavelength, is described by

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp\{\mathrm{i}[(\boldsymbol{k}\cdot\boldsymbol{r}) - \omega t]\} . \tag{8.a.13}$$

The wave amplitude $\boldsymbol{E}_0 = \boldsymbol{E}' + \mathrm{i}\boldsymbol{E}''$ is in general a complex vector. Moreover, $\boldsymbol{E}_0 \perp \boldsymbol{k}$, i.e. the waves are transverse. Depending on the magnitude and direction of the real vectors $\boldsymbol{E}'$ and $\boldsymbol{E}''$, the wave will be plane, circularly, or elliptically polarised.

Plane monochromatic waves with a well-defined frequency and a well-defined polarisation are a mathematical idealisation. The waves which we call monochromatic are, in fact, always quasi-monochromatic. We can consider them as superpositions of monochromatic waves with frequencies in some range $\Delta\omega$. In any given point of space such a wave can be described by the function $\boldsymbol{E}_0(t)\exp(-\mathrm{i}\omega t)$, where ω is some average frequency in the range $\Delta\omega$, and $\boldsymbol{E}_0(t)$ is a function which varies much more slowly than $\exp(-\mathrm{i}\omega t)$. Moreover, one is often (and in the optical band as a rule) dealing with the simultaneous observation of radiation from many independent sources, with phase differences which vary randomly. These waves will be nonmonochromatic and only partially polarised.

One can consider in a unique way the polarisation both of monochromatic (and completely polarised) and of nonmonochromatic (partially polarised) waves. The polarisation and intensity of these waves can be characterised by the tensor

$$I_{ik} = \overline{E_{0i}E_{0k}^*}\ , \tag{8.a.14}$$

where the averaging is performed over the moment of the observation and over an ensemble of independent sources, and $i, k = 1, 2$ characterise the two basic directions in the xy-plane (here $\boldsymbol{k}$ is parallel to z). The polarisation tensor is Hermitean: $I_{ik} = I_{ki}^*$. It can be written as follows:

$$I_{ik} = I_1 e_i^{(1)} e_k^{(1)*} + I_2 e_i^{(2)} e_k^{(2)*}\ , \tag{8.a.15}$$

where I_1 and I_2 are positive quantities and $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$ are mutually orthogonal complex vectors, normalised by the condition $(\boldsymbol{e}^{(i)} \cdot \boldsymbol{e}^{(k)*}) = \delta_{ik}$ and characterising two basic polarisation states of a partially polarised wave. It is clear from equation (8.a.15) that such a wave can be considered to be an incoherent[(2)] superposition of two basic elliptically polarised waves. The shape and orientation of the polarisation ellipses of these waves are described by the vectors $\boldsymbol{e}^{(1)}$ and $\boldsymbol{e}^{(2)}$. The polarisation ellipses are similar, and their corresponding axes are mutually perpendicular. The quantities I_1 and I_2 are the intensities of the basic waves. The total intensity of the wave $I = \overline{\boldsymbol{E}_0\boldsymbol{E}_0^*} = I_1 + I_2 = \mathrm{Tr}(I_{ik})$. The quantities I_i and $\boldsymbol{e}^{(i)}$ can be determined from the set of equations

$$I_{ik} e_k = I e_i\ . \tag{8.a.16}$$

The ratio

$$P = \frac{I_1 - I_2}{I_1 + I_2} = 1 - \rho \qquad (I_2 \leqslant I_1) \tag{8.a.17}$$

is called the degree of polarisation of a partially polarised wave, and $\rho = I_2/I_1$ is its degree of depolarisation. For a completely polarised wave $P = 1$ $(\rho = 0)$, and for an unpolarised wave $P = 0$ $(\rho = 1)$, $I_1 = I_2 = \frac{1}{2}I$, and the polarisation tensor becomes

$$I_{ik} = \tfrac{1}{2}\delta_{ik}\ .$$

(2) Oscillations with phase differences which change randomly are said to be incoherent.

When a plane wave falls on a plane separation boundary between two media, the angles θ_0, θ_1, and θ_2, which are the angles of incidence, reflection, and refraction (figure 8.a.1), are related by

$$\theta_1 = \theta_0 , \qquad \frac{\sin\theta_2}{\sin\theta_0} = \frac{n_1}{n_2} , \qquad n_i = (\epsilon_i)^{1/2} , \tag{8.a.18}$$

where $n_{1,2}$ are the refractive indices of the first and second media (we assume that $\mu_1 = \mu_2 = 1$).

The amplitudes of the reflected wave (E_1, H_1) and the refracted wave (E_2, H_2) can be expressed in terms of the amplitude (E_0, H_0) of the incident wave. The corresponding relationships are known as Fresnel's formulae:

(a) If E_0 is normal to the plane of incidence, then

$$E_1 = \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)} E_0 , \qquad E_2 = \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_2 + \theta_0)} E_0 . \tag{8.a.19}$$

(b) If H_0 is normal to the plane of incidence, then

$$H_1 = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} H_0 , \qquad H_2 = \frac{\sin 2\theta_0}{\sin(\theta_0 + \theta_2)\cos(\theta_0 - \theta_2)} H_0 . \tag{8.a.20}$$

The angle θ_2 is given in terms of the permittivities of the two media by equations (8.a.18). Equations (8.a.18) to (8.a.20) retain their form even when ϵ_2 is complex, but then the angle θ_2 will also be complex and will not have a simple geometrical interpretation. The treatment of a complex θ_2 is discussed in problem 8.22.

The reflection coefficient R is defined as the ratio of the time average of the reflected energy flux to the time average of the incident energy flux.

The superposition of plane monochromatic waves with different wavevectors and frequencies is often referred to as a wave group, or wavepacket. A wavepacket is described by

$$\Psi(\boldsymbol{r}, t) = \int \psi(\boldsymbol{k}) \exp\{i[(\boldsymbol{k} \cdot \boldsymbol{r}) - \omega t]\} \, dk_x \, dk_y \, dk_z , \tag{8.a.21}$$

where $\Psi(\boldsymbol{r}, t)$ is a Cartesian component of $\boldsymbol{E}$ or $\boldsymbol{H}$. The function $\psi(\boldsymbol{k})$ is called the amplitude function. The maximum of the amplitude of a wavepacket moves through space with the group velocity $v_g = d\omega/dk$.

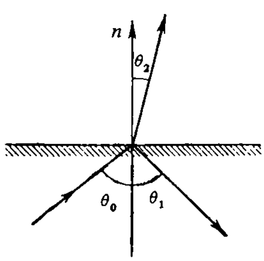


Figure 8.a.1.

8.1 Two plane monochromatic linearly polarised waves of the same frequency propagate along the z-axis. The first wave is polarised along the x-axis and has an amplitude a, and the second is polarised along the y-axis and has an amplitude b. The phase of the second wave leads the phase of the first wave by χ. Find the polarisation of the resultant wave.

8.2 In the preceding problem, consider the dependence of the polarisation on the phase difference χ when $a = b$.

8.3 Two monochromatic waves of equal frequency are circularly polarised in opposite directions. They are in phase and propagate in the same direction. Their amplitudes are a and b. Determine the type of polarisation for different values of the ratio a/b (a and b may be chosen to be real).

8.4 Express the degree of polarisation P of a plane wave in terms of the components I_{ik} of the polarisation tensor. What is the condition which the components I_{ik} must satisfy in order that the wave is completely polarised?

Hint. Use equation (8.a.15) and the orthogonality of the basic polarisation vectors.

8.5 Verify that a partially polarised electromagnetic wave can always be considered as the sum of an unpolarised and a completely polarised wave. To do that show that the polarisation tensor (8.a.15) can in the general case be written in the form

$$I_{ik} = \tfrac{1}{2}I(1-P)\delta_{ik} + \tfrac{1}{2}PI_{ik}^{\text{pol}} ,$$

where the determinant of the tensor I_{ik}^{pol} vanishes so that this tensor describes a completely polarised wave. The first term in this expansion corresponds to the unpolarised wave, where $I = I_1 + I_2$ is the total intensity and P the degree of polarisation.

8.6 A plane monochromatic wave with intensity I propagates along the z-axis and is polarised with a polarisation ellipse with semiaxes a and b. The semimajor axis a makes an angle ϑ with the x-axis. Find the polarisation tensor and consider possible special cases.

8.7* An electromagnetic wave is formed by the superposition of two incoherent 'almost monochromatic' waves of the same intensity I and with approximately equal frequencies and wavevectors. Both waves are linearly polarised and the directions of polarisation are defined in the plane perpendicular to the wavevector by the unit vectors e^1 $(1, 0)$ and e^2 $(\cos\vartheta, \sin\vartheta)$. Find the polarisation tensor I_{ik} of the resultant partially polarised wave and its degree of polarisation. Elucidate the nature of the polarisation of this wave (see problem 8.6).

8.8 Solve the preceding problem in the case where the two intensities are unequal ($I_1 \neq I_2$) and the two directions of polarisation are at an angle of $\frac{1}{4}\pi$.

8.9 The Hermitian polarisation tensor of an electromagnetic wave is given by

$$I_{ik} = \tfrac{1}{2}I\left(\delta_{ik} + \sum_{l=1}^{3} \xi_l \hat{\tau}_{ik}^{(l)}\right) = \tfrac{1}{2}I\begin{pmatrix} 1+\xi_3 & \xi_1 - \mathrm{i}\xi_2 \\ \xi_1 + \mathrm{i}\xi_2 & 1-\xi_3 \end{pmatrix},$$

where I is the total intensity of the wave, ξ_i are real parameters which are such that $\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \leqslant 1$ (Stokes parameters), and $\hat{\tau}^{(l)}$ are the matrices

$$\hat{\tau}^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \hat{\tau}^{(2)} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \hat{\tau}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Explain the physical significance of the parameters ξ_i. To do this, express the degree of depolarisation ρ in terms of ξ_i and determine the polarisation of the two component waves into which a partially polarised wave may be resolved under the following three conditions:

(a) $\xi_1 \neq 0$, $\quad \xi_2 = \xi_3 = 0$;
(b) $\xi_2 \neq 0$, $\quad \xi_1 = \xi_3 = 0$;
(c) $\xi_3 \neq 0$, $\quad \xi_1 = \xi_2 = 0$.

8.10 The electric field $\boldsymbol{E}$ of an electromagnetic wave in which the real and imaginary components of the complex wavevector $\boldsymbol{k}$ are in different directions, is linearly polarised. Determine the mutual disposition of the vectors $\boldsymbol{E}_0$, $\boldsymbol{H}'$, $\boldsymbol{H}''$, $\boldsymbol{k}'$, and $\boldsymbol{k}''$, where $\boldsymbol{H}'$, $\boldsymbol{H}''$ correspond to the real and imaginary parts of the complex amplitude $\boldsymbol{H}_0$ and $\boldsymbol{k}'$ and $\boldsymbol{k}''$ are the real and imaginary parts of the wavevector $\boldsymbol{k}$. Find the locus of the end point of the vector $\boldsymbol{H}$ at a given point in space.

Solve the same problem in the case where the vector $\boldsymbol{H}$ is linearly polarised.

8.11 A circularly polarised monochromatic wave falls obliquely on the plane boundary of a dielectric. Determine the polarisation of the reflected and refracted waves.

8.12* A beam of almost monochromatic unpolarised light is incident on the plane boundary of a dielectric. Find the polarisation tensors $I_{ik}^{(1)}$, $I_{ik}^{(2)}$ and the depolarisation coefficients ρ_1, ρ_2 of the reflected and refracted light.

8.13 An unpolarised, almost monochromatic beam of light is incident on the plane boundary of separation between two dielectrics. Determine the coefficient of reflection R and the depolarisation coefficients $\rho_{1,2}$ of the reflected and refracted light, if the angle of incidence is equal to Brewster's angle.

8.14 Derive the Fresnel formulae for the case where an electromagnetic wave is incident on the plane boundary of a conducting medium with a small surface impedance ζ.

8.15 Find the reflection coefficient R of a metal surface with a small surface impedance $\zeta = \zeta' + i\zeta''$. Find the angle of incidence θ_0 at which the reflection coefficient is a minimum.

8.16 A linearly polarised wave is incident on the plane boundary of a conducting medium with a low surface impedance ζ. Determine the polarisation of the reflected wave if the glancing angle of incidence is equal to the angle Φ_0 determined in the solution of the preceding problem.

8.17 A linearly polarised plane wave is incident at an angle θ_0 on the surface of a metal. The direction of the electric vector is at an angle $\frac{1}{4}\pi$ to the plane of incidence. The experimentally determined ratio of the transverse and longitudinal (with respect to the plane of incidence) components of the reflected wave is found to be $E_{\parallel 1}/E_{\perp 1} = \tan\rho$ and the phase difference between the components δ is such that

$$\frac{E_{\parallel 1}}{E_{\perp 1}} = \tan\rho \exp(i\delta) .$$

Express the real part of the refractive index n' and the absorption coefficient n'' in terms of ρ, δ, and θ_0 where $n' + in'' = \zeta^{-1}$, if ζ is the surface impedance and $|n'^2 - n''^2| \gg \sin^2\theta_0$.

8.18 Find the reflection coefficient R of the plane surface of a conductor at normal incidence, in the limiting case of low conductivity [cf equation (8.a.8)].

8.19* Show that a linearly polarised wave will in general become elliptically polarised after total reflection from the surface of a dielectric. Under what conditions will the polarisation become circular?

8.20 Investigate the transport of energy in the case of total internal reflection. Find the energy flux along the separation boundary, and also in the perpendicular direction, in the medium from which the reflection takes place. Determine the lines of the Poynting vector S.

8.21 A plane monochromatic wave is incident upon the plane separating two dielectrics with permittivities ϵ_1 and ϵ_2. Describe the nature of the field on both sides of the boundary in the case of glancing incidence (angle of incidence $\theta_0 \to \frac{1}{2}\pi$).

8.22* An electromagnetic wave is incident obliquely from a dielectric on the plane boundary of a conducting medium. Find the direction of propagation, the attenuation, and the phase velocity v_{ph} in the conducting medium.

8.23* A dielectric layer of permittivity ϵ_2 is bounded by the planes $z = 0$ and $z = a$ and lies between dielectric media with permittivities ϵ_1 and ϵ_3 ($\mu_1 = \mu_2 = \mu_3 = 1$). An electromagnetic wave falls normally on the surface of the layer from the region $z < 0$. Find the thickness of the layer corresponding to minimum reflection, and the ratio of ϵ_1, ϵ_2, and ϵ_3 for which there will be no reflection.

8.24* A plane wave falls normally from a vacuum on the boundary of a dielectric. Investigate the effect of the 'sharpness' of the boundary on the reflection coefficient, assuming that the permittivity is given by

$$\epsilon(z) = \epsilon - \frac{\Delta\epsilon}{\exp(z/a)+1}, \qquad \epsilon = 1+\Delta\epsilon,$$

where ϵ and $\Delta\epsilon$ are constants. Investigate the special cases of large and small a.

Hint. Use the substitutions $\xi = -\exp(-z/a)$ and $E(\xi) = \xi^{-ika}\psi(\xi)$ in the differential equation for $E(z)$ [cf equation (8.a.12)] where $\psi(\xi)$ satisfies the hypergeometric equation [see Gradshteyn and Ryzhik (1965), equation (9.151)].

8.25* In the absence of absorption, the permittivity of a plasma is given by (cf problem 6.16)

$$\epsilon = 1 - \frac{4\pi e^2 N}{m\omega^2}.$$

Discuss the propagation of electromagnetic waves in a plasma whose concentration is described by the linear function $N(z) = N_0 z$. Consider the case where a plane monochromatic wave is incident normally on a nonhomogeneous layer of plasma (this may be of importance in the propagation of radio waves in the ionosphere).

Hint. Solve the equation for $E(z)$ by expanding the required function into a Fourier integral.

8.26 Construct a one-dimensional wavepacket Ψ at time $t = 0$ in the case where the amplitude is given by the Gaussian function

$$a(k) = a_0 \exp\left[-\frac{(k-k_0)^2}{(\Delta k)^2}\right]$$

and a_0, k_0 and Δk are constants. Find the relation between the width of the packet Δx and the range of wavenumbers Δk which contribute significantly to the wavepacket.

8.27 A wavepacket Ψ is formed by the superposition of plane waves of different frequency. The amplitude function is given by the Gaussian curve

$$a(\omega) = a_0 \exp\left[-\frac{(\omega-\omega_0)^2}{(\Delta\omega)^2}\right]$$

where a_0, ω_0, and $\Delta\omega$ are constants. Find the amplitude of the packet as a function of time at the point $x = 0$. Determine the relation between the length of the wavepacket Δt and the frequency range $\Delta\omega$.

8.28 A given object is illuminated by light of wavelength λ and is viewed through a microscope. Find the minimum possible linear dimension Δx_{min} of the object which is allowed by the condition $\Delta x \Delta k \geqslant 1$.

8.29 The position of an object is determined by means of radar. What is the limiting accuracy with which this determination may be carried out if the wavelength is λ and the distance to the object is l?

8.30 Investigate the form and the motion of a wavepacket obtained as a result of the superposition of plane waves with equal amplitudes a_0 and with wavevectors lying in the range $|k_0 - k| \leqslant q$, where k_0 and q are constants. Replace the actual law of dispersion $\omega(k)$ by the approximate relation

$$\omega(k) = \omega(k_0) + \left(\frac{d\omega}{dk}\bigg|_0 \cdot [k - k_0]\right) .$$

8.31* Investigate the spreading of a one-dimensional wavepacket in a dispersive medium. Assume that the amplitude function is given by the Gaussian curve $a(k) = a_0 \exp[-\alpha(k-k_0)^2]$, and retain all terms up to and including the second-order term in the expansion of ω in terms of k.

8.32 Find the phase velocity v_{ph} and the group velocity v_{g} for a medium whose permittivity is given by

$$\epsilon(\omega) = 1 + \frac{\omega_{\text{p}}^2}{\omega_0^2 - \omega^2} .$$

Consider only the cases of large and small frequencies (compared with ω_0) and assume that $\mu = 1$.

8.33 Determine the rate of transport of energy by a one-dimensional wavepacket in a dispersive medium. Show that this velocity is equal to a group velocity v_{g}.

Hint. The rate of transport of energy v is given by the relation $\bar{\gamma} = v\bar{w}$, where

$$\bar{w} = \frac{1}{16\pi}\left[\frac{d(\omega\epsilon)}{d\omega}EE^* + \frac{d(\omega\mu)}{d\omega}HH^*\right]$$

is the average energy density in the dispersive medium (see Landau and Lifshitz, 1960) and $\bar{\gamma}$ is the average energy flux.

b Plane waves in anisotropic and gyrotropic media

When the electrical and magnetic properties of a medium are different in different directions so that the permittivity and the magnetic permeability of such a medium are tensors, the medium is referred to as optically anisotropic. Optical anisotropy may be a consequence of the crystalline structure of the body, but it may also be produced by an external electric field (cf problems 6.17 and 6.18) or by an external mechanical effect. In the absence of an external magnetic field, the tensors $\epsilon_{ik}(\omega)$ and $\mu_{ik}(\omega)$ are symmetric:

$$\epsilon_{ik} = \epsilon_{ki}\,, \qquad \mu_{ik} = \mu_{ki}\,. \tag{8.b.1}$$

We shall not consider the effects associated with the spatial nonuniformity of the field, which leads to a dependence of the permittivity and the magnetic permeability on the wavevector $\boldsymbol{k}$ (see Landau and Lifshitz, 1960, and also problem 8.48).

In an anisotropic medium, two plane monochromatic waves of the same frequency and polarised linearly in perpendicular planes may be propagated in a given direction with different phase velocities. The directions along which both waves have the same velocity of propagation are called optic axes. The direction of propagation of a wave, which is defined by the normal to the wave surface, is not in general the same as the ray direction, i.e. the direction of the Poynting vector.

Crystals in which two of the principal values of the permittivity tensor are equal $[\epsilon^{(1)} = \epsilon^{(2)} = \epsilon_\perp,\ \epsilon^{(3)} = \epsilon_\parallel]$ are known as uniaxial. Their optic axis lies along the $x_3 = z$-axis. The wavevectors of the two waves propagating at an angle θ to the optic axis are then given by

$$k_1 = \frac{\omega}{c}(\epsilon_\perp\mu)^{1/2}\,, \qquad k_2 = \frac{\omega}{c}\left(\frac{\epsilon_\perp\epsilon_\parallel\mu}{\epsilon_\perp \sin^2\theta + \epsilon_\parallel \cos^2\theta}\right)^{1/2} \tag{8.b.2}$$

The first of these waves is known as the ordinary wave. The induction vector $\boldsymbol{D}$ and the electric field vector $\boldsymbol{E}$ of this wave are parallel to each other and perpendicular to the wavevector $\boldsymbol{k}_1$ and the plane containing the wavevector and the optic axis (principal plane). The second wave is known as the extraordinary wave. In this wave the induction vector $\boldsymbol{D}$ lies in the principal plane and is perpendicular to the corresponding wavevector $\boldsymbol{k}_2$. The vector $\boldsymbol{E}$ lies in the principal plane but is not parallel to $\boldsymbol{D}$.

In the presence of an external constant magnetic field, the tensors ϵ_{ik} and μ_{ik} are no longer symmetric. In nonabsorbing media, to which we shall confine our attention in this section, these tensors are Hermitian, i.e.

$$\epsilon_{ik} = \epsilon_{ki}^*\,, \qquad \mu_{ik} = \mu_{ki}^*\,. \tag{8.b.3}$$

The relation between the field strengths and the inductions is then of the form (cf problem 6.20)

$$\boldsymbol{D} = \hat{\epsilon}'\boldsymbol{E} + \mathrm{i}[\boldsymbol{E} \wedge \boldsymbol{g}_\mathrm{e}]\,, \qquad \boldsymbol{B} = \hat{\mu}'\boldsymbol{H} + \mathrm{i}[\boldsymbol{H} \wedge \boldsymbol{g}_\mathrm{m}]\,, \tag{8.b.4}$$

where $\boldsymbol{g}_\mathrm{e}$ and $\boldsymbol{g}_\mathrm{m}$ are the gyration vectors (electric and magnetic respectively) and $\hat{\epsilon}'\boldsymbol{E}$ is a vector with components equal to $\epsilon'_{ik}E_k$. Media for which the field vectors are given by equation (8.b.4) are known as gyrotropic media.

Two plane waves of the same frequency can propagate in a gyrotropic medium in a given direction with different phase velocities. These waves are elliptically polarised in opposite directions, and the polarisation ellipses have the same ratio of axes but are rotated relative to each other by $\frac{1}{2}\pi$.

The boundary conditions on the surface of an anisotropic or gyrotropic body are the same as on the separation boundary between two isotropic media [cf equations (3.a.9) and (5.0.6)].

8.34 The direction of propagation of the extraordinary wave in a uniaxial crystal is at an angle θ to the optic axis. Determine the angle α between the wavevector $\boldsymbol{k}$ and the electric field $\boldsymbol{E}$, and also the angle ϑ between the ray direction (Poynting vector) and the optic axis of the crystal.

8.35 A plane wave is incident on the plane surface of a uniaxial crystal placed in a vacuum. The optic axis of the crystal is at right angles to its surface. Find the directions of propagation of the ordinary and extraordinary rays in the crystal, if the angle of incidence is θ_0.

8.36 Solve the preceding problem in the case where the optic axis of the crystal is parallel to its surface and makes an angle α with the plane of incidence.

8.37 A plane monochromatic wave propagates in an infinite ferrite medium which is magnetised to saturation at angle θ to a constant magnetic field. The magnetic permeability of the ferrite is given by the tensor[3]

$$\mu_{ik} = \begin{pmatrix} \mu_\perp & -\mathrm{i}\mu_\mathrm{a} & 0 \\ \mathrm{i}\mu_\mathrm{a} & \mu_\perp & 0 \\ 0 & 0 & \mu_\parallel \end{pmatrix},$$

where the z-axis is parallel to the constant magnetic field. Assuming that the permittivity of the ferrite ϵ may be looked upon as a scalar[4], find the phase velocities of propagation $v_{1,2}$.

8.38 A plane monochromatic wave propagates in a dielectric placed in a constant uniform magnetic field. Assuming that the permeability of the dielectric is $\mu = 1$, and that the permittivity tensor (see problem 6.22) is of the form

$$\epsilon_{ik} = \begin{pmatrix} \epsilon_\perp & -\mathrm{i}\epsilon_\mathrm{a} & 0 \\ \mathrm{i}\epsilon_\mathrm{a} & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix},$$

find the phase velocities of propagation.

8.39 Investigate the polarisation of waves which can propagate in an infinite ferrite medium magnetised to saturation. Consider the special cases (a) where the propagation occurs along the constant magnetic field and (b) where the propagation takes place at right angles to this field.

(3) The dielectric permittivity tensor of a gaseous dielectric in a uniform external field has the same form (see problem 6.22).

(4) This is so because the constant magnetic field has a much greater effect on the magnetic properties of the ferrite than on the electrical properties.

8.40 A dielectric is placed in an external magnetic field. A plane monochromatic wave propagates in the direction of the magnetic field (the z-axis) and is linearly polarised at the point $z = 0$. Determine the polarisation of the wave at points $z \neq 0$.

Hint. Use the permittivity tensor obtained in the solution of problem 6.22

8.41 A plane, circularly polarised wave is incident normally on the plane boundary of a ferrite placed in a vacuum. The ferrite is magnetised in the direction of incidence of the wave. Determine the polarisation and the amplitude of the reflected and transmitted waves.

Hint. Use the boundary conditions for the field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$.

8.42 Solve the preceding problem in the case where the incident wave is linearly polarised.

8.43* An artificial dielectric consists of thin perfectly conducting circular discs which are similarly oriented and placed in a vacuum. A constant magnetic field $\boldsymbol{H}_0$ is applied at right angles to the planes of the discs and a plane electromagnetic wave propagates in the direction parallel to the magnetic field. Determine the phase velocities of propagation, assuming that the dielectric may be looked upon as a continuous medium.

Hint. Take into account the Hall effect which appears as a result of the presence of the external magnetic field.

8.44 A plane wave is incident normally upon a plane lattice formed by thin, parallel, infinitely long conductors. The distances between the conductors and their thickness are much smaller than the wavelength. What effect will the lattice have upon the propagation of waves with different polarisations?

8.45 Consider the possibility of the propagation of longitudinal oscillations in a medium with a dielectric permittivity $\epsilon(\omega)$. The electric field vector $\boldsymbol{E}$ is parallel to the wavevector for such oscillations. Give the conditions for which the damping of such oscillations will be small. At what frequency are longitudinal oscillations possible in a plasma (its dielectric permittivity was calculated in problem 6.16)?

8.46 The region $x < 0$ is occupied by a plasma with dielectric permittivity $\epsilon(\omega) = 1 - \omega_p^2/\omega^2$ (see problem 6.16) while the region $x > 0$ is a vacuum. Show that along the plasma–vacuum boundary a surface wave can propagate in which the field strength is damped exponentially with distance from the boundary. Find the frequency for which such a wave is possible and its polarisation. Restrict the discussion to that of a slow wave ($v_{ph} = \omega/k \ll c$).

8.47 An ionised gas is placed in a constant magnetic field. A transverse plane wave propagates in the direction parallel to the field. Find the phase velocities of propagation. Consider the special case of low frequencies ($\omega \to 0$) and the nature of the electromagnetic waves, taking into account the motion of the positive ions.

Hint. Use the expression, which was obtained in problem 6.25, for the permittivity tensor of an ionised gas in a constant magnetic field.

8.48 Determine the magnetic permeability tensor $\mu_{ik}(\omega, \boldsymbol{k})$ of a ferrodielectric without neglecting the term $q\nabla^2\boldsymbol{M}$ for the effective magnetic field in equation (6.c.3). To do this, consider the motion of the magnetisation vector under the action of a plane monochromatic wave. The ferrodielectric is magnetised to saturation by a constant magnetic field $\boldsymbol{H}_0$.

Hint. Confine the analysis to the case of small amplitudes and linearise the equation of motion of the magnetisation vector.

8.49 Find the dispersion relation for electromagnetic waves propagating in an isotropic ferrodielectric magnetised to saturation without neglecting the term $q\nabla^2\boldsymbol{M}$ in the expression for the effective field $\boldsymbol{H}_{\text{eff}}$ [equation (6.c.3)]. Show that three types of waves with different dispersion relations can propagate in such a medium. Determine the explicit form of the dispersion relation $\omega(\boldsymbol{k})$ for these waves, subject to the condition $\omega^2\epsilon/(ck)^2 \ll 1$. Estimate the relative magnitude of the electric and magnetic fields for this branch of the oscillations.

8.50 Determine the surface impedance ζ of a ferromagnetic conductor placed in a constant magnetic field which is parallel to its surface. Use the magnetic permeability tensor given in problem 8.37 and assume the components of the electrical conductivity tensor are $\sigma_{11} = \sigma_{22} = \sigma_1$, $\sigma_{33} = \sigma_3$, $\sigma_{12} = -\sigma_{21} = -\mathrm{i}\sigma_2$, $\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0$.

Hint. The surface impedance is a tensor of rank 2 and should be determined from [cf equation (8.a.10)]

$$E_{\tau i} = \zeta_{ik}[\boldsymbol{H}_\tau \wedge \boldsymbol{n}]_k ,$$

where $i, k = 1, 2, \boldsymbol{E}_{\tau i}$ and $\boldsymbol{H}_\tau$ are the tangential components of the field vectors near the surface of the conductor, and $\boldsymbol{n}$ is a unit normal.

8.51 Solve the preceding problem in the case where the constant magnetic field is normal to the surface of the ferromagnetic conductor.

c Scattering of electromagnetic waves by macroscopic bodies. Diffraction

The diffraction of electromagnetic waves by conducting or dielectric bodies is described by the solutions of Maxwell's equations subject to appropriate boundary conditions. An exact solution is only possible in a very limited number of cases (cf problems 8.54 and 8.59). However, an approximate solution may frequently be obtained.

If the linear dimensions of the body are small in comparison with the wavelength, then the electromagnetic field near the body may be looked upon as uniform. A body placed in a uniform periodic field will assume electric and magnetic moments which will be the same functions of time as the external field. The scattered wave appears as the result of the emission of radiation by these alternating moments. The scattering of electromagnetic waves by bodies of small linear dimensions may therefore be reduced to the determination of the dipole moments induced in them. The radiation fields are given in terms of the dipole moments by equations (12.a.17) and (12.a.20).

The differential cross section for scattering into a solid angle $\mathrm{d}\Omega$ is defined by

$$\mathrm{d}\sigma_{\mathrm{s}} = \frac{\mathrm{d}I(\vartheta, \alpha)}{\overline{\gamma}_0}, \tag{8.c.1}$$

where $\mathrm{d}I = \overline{\gamma}\,\mathrm{d}^2S = \overline{\gamma}r^2\,\mathrm{d}\Omega$ is the time average of the intensity emitted into the solid angle $\mathrm{d}\Omega$ and $\overline{\gamma}$ and $\overline{\gamma}_0$ are the average energy flux densities of the scattered and incident radiation. The energy flux density is described by the Poynting vector

$$\boldsymbol{S} = \frac{c}{4\pi}[\boldsymbol{E} \wedge \boldsymbol{H}]. \tag{8.c.2}$$

The absorption cross section is equal to the ratio of the average energy absorbed by a body per unit time to the average energy flux density in the incident wave:

$$\sigma_{\mathrm{a}} = \frac{Q}{\overline{\gamma}_0}. \tag{8.c.3}$$

In the opposite limiting case, i.e. when the wavelength is much smaller than the linear dimensions of the body, one can use the methods of geometrical optics. In the case of diffraction of an electromagnetic wave by an aperture in an infinite opaque screen, the amplitude of the diffracted field, in the geometrical optics approximation, is given by

$$u_{\mathrm{P}} = \frac{k}{2\pi\mathrm{i}}\int\frac{u}{R}\exp(\mathrm{i}kR)\,\mathrm{d}^2S_n. \tag{8.c.4}$$

This formula may be derived from Huygens' principle. The quantity u_{P} is the field at a point P in front of the screen (figure 8.c.1), u is the field at

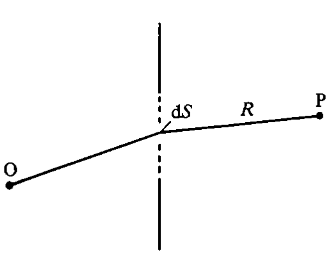


Figure 8.c.1.

an element of area d^2S of the aperture (this field is assumed to be the same as in the absence of the screen, i.e. undistorted by the screen), d^2S_n are the components of the area vector $\mathbf{d^2S}$ in the direction of the ray reaching d^2S from the source of light O, R is the distance between d^2S and the point of observation P, and k is the modulus of the wavevector.

The source of light O and the point of observation P may be either at finite or infinite distances from the screen. When both O and P, or only one of them are at a finite distance from the screen, the phenomenon is known as Fresnel diffraction. When both O and P are at very large distances from the screen, then the rays of light reaching the aperture and the rays leaving it may be regarded as parallel, and this case is known as Fraunhofer diffraction. Equation (8.c.4) may then be written in the approximate form

$$u_P = \frac{u_0 \exp(ikR_0)}{2\pi i R_0} \int \exp\{i([\boldsymbol{k}-\boldsymbol{k}'] \cdot \boldsymbol{r})\} d^2S_n \,, \tag{8.c.5}$$

where $\boldsymbol{k}$ and $\boldsymbol{k}'$ are the wavevectors of the incident and diffracted radiation, R_0 is the distance from the aperture to the point of observation, and u_0 is the field amplitude within the aperture. The intensity of the diffracted wave is proportional to the square of the modulus of the amplitude $|u_P|^2$.

Two screens are defined as complementary if the apertures in one correspond to identical opaque regions in the other, and vice versa. The Babinet principle holds for such screens. Suppose that the wave fields at a given point with two complementary screens are given by u_1 and u_2, and u is the wave field in the absence of the two screens. We then have

$$u_1 + u_2 = u \,. \tag{8.c.6}$$

Equations (8.c.4) and (8.c.5) take no account of the polarisation of the waves, i.e. the amplitude u is looked upon as a scalar. The diffraction formula which takes into account the vector nature of the electromagnetic field is

$$\boldsymbol{E}_P = -\frac{ik}{4\pi R} \int \{[\boldsymbol{n}_0 \wedge \boldsymbol{H}] - \boldsymbol{n}(\boldsymbol{n} \cdot [\boldsymbol{n}_0 \wedge \boldsymbol{H}]) + [\boldsymbol{n} \wedge [\boldsymbol{n}_0 \wedge \boldsymbol{E}]]\} \exp(ikr)\, d^2S \,, \tag{8.c.7}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the fields within the aperture, $\boldsymbol{E}_P$ is the electric field at a large distance from the screen (in the so-called wave zone), $\boldsymbol{n}$ is a unit vector in the direction of propagation of the diffracted wave, $\boldsymbol{n}_0$ is a unit normal to the plane of the aperture and is drawn towards the point of observation, r is the distance of the element of area d^2S from the point of observation, and R is the distance of the point of observation from the origin of coordinates, which is taken to lie in the plane of the aperture. The magnetic field in the wave zone is given in terms of the electric field by the usual formula $\boldsymbol{H}_P = [\boldsymbol{n} \wedge \boldsymbol{E}_P]$.

8.52* A plane monochromatic wave is incident on an infinitely long, circular, perfectly conducting cylinder of radius a, in a direction at right angles to its axis. The cylinder lies in vacuo, and the vector $\boldsymbol{E}_0$ of the incident wave is parallel to the axis of the cylinder. Determine the resultant field, the distribution of currents on the surface of the cylinder, and the total current $\mathcal{J}$ flowing along the cylinder.

8.53 Find the differential scattering cross section $d\sigma_s$ for the cylinder considered in problem 8.52. Find also the total scattering cross section σ_s.

8.54* A plane monochromatic wave is incident on a perfectly conducting, circular cylinder so that its magnetic vector $\boldsymbol{H} = \boldsymbol{H}_0 \exp\{i[(\boldsymbol{k}\cdot\boldsymbol{r}) - \omega t]\}$ is parallel, and the wavevector $\boldsymbol{k}$ is perpendicular, to the axis of the cylinder. The cylinder lies in vacuo. Find the resultant electromagnetic field In particular, discuss the special case of a thin cylinder ($ka \ll 1$) and determine the differential and total scattering cross sections $d\sigma_s$ and σ_s for the cylinder.

8.55 Let $d\sigma_\parallel$ and $d\sigma_\perp$ be the differential scattering cross sections of an infinitely long cylinder for plane waves with the electric vector $\boldsymbol{E}$ parallel and perpendicular to the axis of the cylinder, respectively. Find the differential scattering cross section $d\sigma_s'$ in the case of a wave whose electric vector $\boldsymbol{E}$ is at an angle φ to the axis of the cylinder, and also the differential scattering cross section $d\sigma_s''$ for an unpolarised wave.

Hint. Use the principle of superposition of waves.

8.56 An unpolarised plane wave is scattered by a perfectly conducting thin cylinder ($ka \ll 1$). Determine the degree of depolarisation ρ of the scattered waves as a function of the angle of scattering.

8.57* Solve problem 8.54 in the case of diffraction of a plane wave by an infinitely long cylinder without assuming that the cylinder is perfectly conducting, but regarding its surface impedance ζ as small. Use the approximate boundary condition given by equation (8.a.10).

8.58 Determine the average loss of energy Q and the absorption cross section σ_a per unit length of the cylinder considered in the preceding problem. Investigate the limiting case $ka \ll 1$ and explain your results.

8.59* Investigate the diffraction of a plane monochromatic wave by a dielectric cylinder. The cylinder lies in vacuo and has a radius a, permittivity ϵ, and magnetic permeability μ. The wave is incident at right angles to a generator of the cylinder and the vector $\boldsymbol{E}$ is perpendicular to its axis. Determine the resultant field.

8.60* A linearly polarised, plane monochromatic wave is scattered by a sphere of radius a, which is much smaller than the wavelength λ. Express the components of the scattered electromagnetic wave in the wave zone in terms of the electric and magnetic polarisabilities of the sphere. Determine the differential scattering cross section.

Hint. In view of the fact that $a \ll \lambda$, the external field near the sphere may be looked upon as uniform, and it is sufficient to consider the emission of radiation by the induced electric and magnetic dipole moments $\boldsymbol{p}$ and $\boldsymbol{m}$.

8.61 Calculate the differential and total scattering cross sections $d\sigma_s$ and σ_s and also the degree of depolarisation ρ of the secondary radiation when an unpolarised wave is scattered by a sphere of radius a which is much smaller than the wavelength λ. Express the result in terms of the electric and magnetic polarisabiliities β_e and β_m of the sphere.

8.62 By using the results of the preceding problem, determine the differential and total scattering cross sections $d\sigma_s$ and σ_s for unpolarised light and a small dielectric sphere of permittivity ϵ ($\mu = 1$), and also determine the degree of depolarisation of the scattered radiation. Give a sketch showing these quantities as functions of the scattering angle θ. Indicate the limits of applicability of the solutions. Solve the same problem for a perfectly conducting sphere with $\mu = 1$.

8.63 A plane monochromatic wave is incident at an angle $\frac{1}{2}\pi - \alpha$ on a perfectly conducting thin disc whose radius a is much smaller than the wavelength λ. Determine the differential and total scattering cross sections $d\sigma_s$ and σ_s for different polarisations of the incident wave and also the scattering cross section for an unpolarised wave.

8.64 A uniform dielectric of permittivity ϵ ($\mu = 1$) contains a cavity in the form of a thin disc of radius a and thickness $2h$. Unpolarised light of wavelength $\lambda \gg a$ is incident normally on one of the planes defining the cavity. Find the differential and total scattering cross sections $d\sigma_s$ and σ_s.

8.65* Find the differential and total cross sections for the scattering of a plane wave of wavelength λ by a perfectly conducting cylinder of length $2h$ and radius $a \ll h$, $h \ll \lambda$. Investigate the various types of polarisation of the incident wave. The cylinder may be approximated by a prolate ellipsoid of revolution with semiaxes a and h.

Hint. Use the solutions of problems 3.70, 3.72, and 7.41.

8.66 Solve the preceding problem for a dielectric cylinder whose length $2h$ is much less than the wavelength λ inside the cylinder.

8.67 A plane monochromatic wave $\boldsymbol{E}_0 \exp[i(\boldsymbol{k} \cdot \boldsymbol{r}) - i\omega t]$ is scattered by a dielectric sphere of radius a with a polarisability $(\epsilon - 1)/4\pi \ll 1$ ($\mu = 1$). Because of the small polarisability the polarisation of the sphere is in first approximation proportional to the field of the incident wave. Determine the differential scattering cross section and the degree of depolarisation ρ of the scattered radiation. Describe the nature of the scattering in the case of a very large sphere ($ka \gg 1$).

8.68 Determine the total scattering cross section σ_s for the dielectric sphere discussed in the preceding problem in the limit $ka \gg 1$ and compare with the case when $ka \ll 1$.

8.69* A plane monochromatic wave is scattered by a system of charges, for example, a macroscopic body. The electric field at large distances from the scatterer is of the form

$$\boldsymbol{E} = E_0 \left[\boldsymbol{e} \exp(\mathrm{i}kz) + \boldsymbol{F}(\boldsymbol{n}) \frac{\exp(\mathrm{i}kr)}{r}\right],$$

where $\boldsymbol{n} = \boldsymbol{r}/r$, $\boldsymbol{e} = \boldsymbol{E}_0/E_0$, $k = \omega/c$, $\boldsymbol{E}_0$ is the amplitude of the incident wave, and $\boldsymbol{F}(\boldsymbol{n})$ is the scattering amplitude which characterises the properties of the scatterer and is a function of frequency. Show that

$$\sigma_t = \frac{4\pi}{k} \operatorname{Im}(\boldsymbol{e} \cdot \boldsymbol{F}(\boldsymbol{n}_0)),$$

where $\sigma_t = \sigma_s + \sigma_a$ is the total cross section for the interaction of the wave with the system of charges, σ_s is the scattering cross section, σ_a is the absorption cross section, and $\boldsymbol{F}(\boldsymbol{n}_0)$ is the forward-scattering amplitude.

8.70* A plane monochromatic wave is incident on a macroscopic particle whose linear dimensions are much smaller than the wavelength λ. The electric and magnetic polarisabilities are $\beta_e = \beta_e' + \mathrm{i}\beta_e''$ and $\beta_m = \beta_m' + \mathrm{i}\beta_m''$. Since the polarisabilities are complex, the scattering is accompanied by absorption. Calculate the absorption cross section.

Hint. The energy absorbed per unit time is equal to the flux of the Poynting vector through the surface of a large sphere surrounding the particle.

8.71 Calculate the cross section for the absorption of electromagnetic waves by a conducting sphere with a small surface impedance $\zeta = \zeta' + \mathrm{i}\zeta''$. The radius of the sphere, b, is small compared with the wavelength λ.

8.72 A plane monochromatic wave is incident on a macroscopic body of given absorption and differential scattering cross sections σ_a and $d\sigma_s/d\Omega$. Express the time average of the force $\overline{F}$ acting on the body in terms of these cross sections.

8.73* Determine the average force $\overline{F}$ acting on a small sphere of radius a placed in the field of a plane monochromatic wave. Consider the examples of a perfectly conducting sphere and a dielectric sphere of permittivity ϵ and magnetic permeability $\mu = 1$. The amplitude of the incident wave is E_0.

8.74 A point source of light is placed on a line which passes through the centre of a circular opaque screen of radius a and is perpendicular to it. Assuming that the geometrical optics approximation may be used ($\lambda \ll a$), find the intensity of light I at a point P on the axis.

8.75 In the preceding problem, consider diffraction at the complementary screen, i.e. a circular aperture in an infinite opaque screen.

8.76 A parallel beam of light is incident normally on a circular aperture in an opaque screen. Find the intensity distribution along the axis.

8.77 Find the angular distribution of the intensity of light in the case of Fraunhofer diffraction at an annular aperture (radii $a > b$) in an infinite opaque screen. Assume that the beam is incident normally on the plane of the aperture and consider the special case of diffraction by a circular aperture.

8.78 Find the angular distribution of the intensity of light for oblique incidence of a parallel beam on a circular aperture (Fraunhofer diffraction).

8.79 A plane linearly polarised wave is incident normally on the rectangular aperture $-a \leqslant x \leqslant a, -b \leqslant y \leqslant b$ in an infinite, thin screen. The amplitudes of the electric and magnetic fields are $E_y = E_0$, $H_x = -E_0, H_y = E_x = 0$. Determine the field radiated from the aperture and also the angular distribution of the radiation.

8.80 A plane linearly polarised wave $E_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}) - \mathrm{i}\omega t]$ is incident normally on a circular aperture of radius a in an infinite, thin screen. Determine the field of the radiation emitted from the aperture and the angular intensity distribution.

d Coherence and interference

Detectors of electromagnetic radiation in the optical band react to the intensity I of the radiation, which is a quadratic function of the field components, averaged over time:

$$I = \overline{|u|^2} = \frac{1}{T}\int_0^T |u|^2 \, \mathrm{d}t \, .$$

This averaging expresses the fact that the time T over which detectors operate is not less than 10^{-10} s (exceptionally 10^{-13} s) whereas a characteristic period for optical oscillations is 10^{-15} to 10^{-16} s.

Therefore one can observe only those interference pictures that are sufficiently stable over a time interval which is longer than T. This complicates the observation of the interference of waves in the optical range.

Thermal, luminescent, or bremsstrahlung light sources consist, as a rule, of a large number of independent (incoherent) radiators that emit light which varies in phase and polarisation. Almost complete agreement in phase and polarisation is reached in quantum optical generators (lasers) in which the stimulated emission of light plays the main role. However, there are also, in this case, fluctuations in phase and polarisation, owing to the spontaneous emission and the scattering by different fluctuating inhomogeneities.

For an observation of a stable interference picture one usually has recourse to splitting the wave field of each of the independent radiators (and of the source as a whole) into several beams. If the wavepackets formed after the splitting intersect again, having traversed different optical paths, an interference picture may appear in the region where they intersect, provided well-defined coherence conditions are satisfied.

These conditions reduce to the requirement that interference pictures from different independent sources do not blot each other out. One can distinguish the simplest two cases of coherence:

(1) *Temporal coherence.* Interference of wavepackets can occur only if the time τ of the retardation of one of the packets is shorter than the lifetime Δt of a separate radiator. As to order of magnitude $\Delta t \sim 1/\Delta\nu$, where $\Delta\nu = \Delta\omega/2\pi$ is the spectral range of the frequencies emitted by the atoms (see problems 8.84 to 8.86). Instead of the coherence time, Δt, one can consider the longitudinal size, $l_\parallel$, of the coherence region (coherence length):

$$l_\parallel = c\Delta t \sim \frac{c}{\Delta\nu} \sim \frac{\lambda^2}{\Delta\lambda}\,, \tag{8.d.1}$$

where λ is the wavelength of the emitted quasi-monochromatic wave and $\Delta\lambda$ the spread in the wavelengths, which is connected with the spectral width through the relation $\Delta\lambda = (\lambda^2/c)\Delta\nu$.

(2) *Spatial coherence.* If the source is extended, the interference pictures from independent radiators, which are sufficiently far from one another in the source, can blot each other out by being superimposed upon one another. The field retains coherence in the vicinity of points for a region with transverse dimensions of the order

$$l_\perp \sim \frac{\lambda}{\Delta\vartheta} \sim \lambda\frac{R}{L}\,, \tag{8.d.2}$$

where $\Delta\vartheta$ is the angular size of the source, L the transverse size of the source, and R the distance of the source from the point of observation. The longitudinal size $l_\parallel$ of the coherence region is determined by equation (8.d.1).

The quantity

$$\Delta V = l_\perp^2 l_\parallel \sim \left(\frac{R}{L}\right)^2\left(\frac{\lambda}{\Delta\lambda}\right)\lambda^3 \tag{8.d.3}$$

is called the coherence volume.

We call the average number of photons (light quanta) which cross a coherence area $l_\perp^2$ during the coherence time $\Delta t = 1/\Delta\nu$ the degeneracy parameter δ of the radiation:

$$\delta = l_\perp^2\frac{\gamma}{\hbar\omega}\Delta t = \frac{\Delta V\gamma}{c\hbar\omega}\,, \tag{8.d.4}$$

where γ is the energy flux density of the radiation in the frequency range $\Delta\nu$, $\hbar\omega = 2\pi\hbar\nu$ is the energy of one photon, and $\hbar = 1\cdot05 \times 10^{-34}$ J s is Dirac's constant (Planck's constant divided by 2π). The degeneracy parameter characterises an important property of quantum emitters: the capacity for stimulated or induced emission. This property consists in the fact that the intensity of radiation from emitters in an electromagnetic field is proportional to $1+\delta$ and increases with increasing δ.

Let the field $u(r, t)$ at the point of observation r at time t be expressed, according to Huygens' principle, in terms of the fields at the points r_1, r_2 at times $t-t_1$, $t-t_2$:

$$u(r, t) = A_1 u(r_1, t-t_1) + A_2 u(r_2, t-t_2)\,. \tag{8.d.5}$$

Here $t_1 = s_1/c$, $t_2 = s_2/c$, $s_1 = |r-r_1|$, $s_2 = |r-r_2|$, and A_1, A_2 are factors which depend on the geometry involved and the dimensions of apertures that are positioned close to the points with radius vectors r_1 and r_2.

We can write the observed average intensity at r at the time t for a stationary regime in the form

$$I(r) = \overline{u^*(r, t)u(r, t)} = I_1(r) + I_2(r) + 2[I_1(r)I_2(r)]^{1/2}\,\mathrm{Re}\,\gamma(r_1, r_2, \tau)\,. \tag{8.d.6}$$

In this formula $\tau = (s_1 - s_2)/c$ and the quantities

$$I_i(r) = |A_i|^2\,\overline{|u(r_i, t-t_i)|^2} = |A_i|^2 I(r_i)$$

are the intensities at the point r, if only the ith aperture were open. The function $\gamma(r_1, r_2, \tau)$ is the so-called complex degree of coherence (or the coefficient of partial coherence) and is defined as follows:

$$\gamma(r_1, r_2, \tau) = \frac{\Gamma(r_1, r_2, \tau)}{[I_1(r)I_2(r)]^{1/2}}\,, \tag{8.d.7}$$

where

$$\Gamma(r_1, r_2, \tau) = \overline{u(r_1, t)u^*(r_2, t+\tau)} \tag{8.d.8}$$

is the correlation function of the fields at r_1 and r_2 at times t and $t+\tau$. The case of spatial coherence corresponds to $\tau = 0$.

The concept of a correlation function and the definitions (8.d.7) and (8.d.8) retain their meaning independently of the way of studying the coherent properties of the field by means of two apertures described here. We can split a light beam from a point source into two beams with intensities I_1 and I_2 in any way and achieve a delay of one of them by a time τ relative to the other. If we combine these beams again later on and observe the intensity of the resulting field, averaged over t, in a small region around the point r, this intensity will be described by an equation such as equation (8.d.6), the correlation function will be given by equation (8.d.8), and the coefficient of partial coherence by equation (8.d.7) with $r_1 = r_2 = r$. The function $\Gamma(r, r, \tau)$ is called the autocorrelation function of the field in the point with radius vector r at the times t and $t+\tau$.

The partial coherence coefficient satisfies the inequalities

$$0 \leqslant |\gamma(r_1, r_2, \tau)| \leqslant 1 .$$

The lower limit of these inequalities corresponds to incoherent light for which $I(r) = I_1(r)+I_2(r)$, whereas, on the other hand, the upper limit corresponds to coherent light. We take the Michelson visibility $B(r)$ as a measure of the sharpness of the interference bands:

$$\begin{aligned} B(r) &= \frac{I_{\max}-I_{\min}}{I_{\max}+I_{\min}} \\ &= |\gamma(r_1, r_2, \tau)|^2 \frac{2(I_1 I_2)^{1/2}}{I_1+I_2} . \end{aligned} \tag{8.d.9}$$

The position of the maxima of the averaged intensity is determined by the condition

$$\arg\gamma(r_1, r_2, \tau) = 2n\pi , \qquad n = 0, \pm 1, \pm 2, \ldots .$$

If in the field of a coherent light wave there are some objects that scatter this wave, in the field where the scattered wave overlaps with the basic—'reference'—wave an interference picture is formed with an intensity which in each point of the region depends both on the intensities and on the phase differences of the scattered and the reference waves. This picture can be mapped on a photographic plate, and afterwards one can use the photographic plate as a diffraction lattice that transmits coherent light through it. The intensity I' of the light passing through the developed photographic plate in a given point (x, y) of it, when the plate is illuminated with light with an intensity distribution $I(x, y)$, is proportional to $I(x, y)$:

$$I'(x, y) = T(x, y)I(x, y)$$

and depends on the blackening of the photographic plate, characterised by the 'transmissivity' $T(x, y)$. The transmissivity depends on the intensity $I_0(x, y)$ of the primary field that causes the blackening and on the contrast of the photographic emulsion, which is characterised by the law

$$T(x, y) \propto [I_0(x, y)]^{-\frac{1}{2}\gamma} ,$$

where γ is the contrast coefficient (gamma) of the emulsion.

A photographic plane on which the picture of the interference of a reference wave with the wave scattered by an object is represented is called a hologram. It turns out that when coherent light is transmitted through a hologram, behind it a three-dimensional picture is formed of the original object. The process of reconstructing the original wave field in this way is called holography (see, e.g. Stroke, 1969) and is illustrated by problems 8.97 to 8.101.

Finally we give some astronomical constants that are used in the solutions of the problems:

Average distance from the Earth to the Sun:	$1 \cdot 50 \times 10^8$ km;
Solar diameter:	$1 \cdot 39 \times 10^6$ km;
1 light year:	$9 \cdot 46 \times 10^{12}$ km;
1 pc (parsec):	$30 \cdot 8 \times 10^{12}$ km.

8.81 Derive the estimate, equation (8.d.2), for the transverse coherence dimension $l_\perp$. Start from the fact that interference pictures produced by emitters in different points of an extended quasi-monochromatic source of diameter L should not blot each other out within the limits of the coherence region. The distance from the source is R, and the wavelength λ.

8.82 Derive the estimate (8.d.4) for the degeneracy parameter δ.

8.83 A quasi-monochromatic source has a transverse dimension L and emits light with a wavelength λ. Estimate the order of magnitude of the solid angle $d\Omega$ within which this radiation is coherent.

8.84 What are the transverse and longitudinal dimensions and the solid angle and volume of coherence of the radiation emitted by sodium atoms in the solar atmosphere? On the Earth one observes a spectral line of wavelength $\lambda_0 = 5 \times 10^{-7}$ m, while the atomic mass is $m = 3 \cdot 7 \times 10^{-26}$ kg. The main contribution to the width of the spectral line comes from the thermal motion of the atoms (the temperature $T \approx 6000$ K).

Hint. The Doppler width of a spectral line is

$$\gamma_D = \left[\frac{8\pi^2 kT}{m\lambda_0^2}\right]^{1/2},$$

where k is the Boltzmann constant (see problem 12.74).

8.85 How are the results of the preceding problem changed, if one observes from Earth a star of the same type as the Sun at a distance of 10 light years?

8.86 Determine the longitudinal and transverse dimensions, and also the volume of coherence in the immediate neighbourhood, of a laser operating at a wavelength $\lambda_0 = 5 \times 10^{-7}$ m with a frequency spread $\Delta\nu = 10^2$ Hz. The diameter of the mirrors is $D = 5 \times 10^{-2}$ m.

8.87 Find the degeneracy parameter δ of the emission from a perfect black body at temperature T. Make numerical estimates for $\lambda = 10^{-2}$ m and $\lambda = 5 \times 10^{-7}$ m with $T = 273$ K and for $\lambda = 5 \times 10^{-7}$ m with $T = 10000$ K.

Hint. The spectral energy density of the emission by a black body is

$$I_\nu = \frac{8\pi^2 \hbar \nu^3}{c^3[\exp(2\pi\hbar\nu/kT) - 1]},$$

where $k = 1 \cdot 38 \times 10^{-23}$ J K^{-1} is Boltzmann's constant.

8.88 Find the degeneracy parameter for the laser considered in problem 8.86. The power of the radiation is 200 W. What is the effective temperature to which this value of δ corresponds?

8.89 Give the relation between the autocorrelation function

$$\Gamma(\boldsymbol{r}, \boldsymbol{r}, \tau) = \overline{u(\boldsymbol{r}, t)u^*(\boldsymbol{r}, t+\tau)}$$

and the power spectrum $I(\omega)$ of the radiation. The intensity of the radiation is

$$I = \overline{u^*(t)u(t)} = \int_0^\infty I(\omega)\,\mathrm{d}\omega\,.$$

8.90 Find the autocorrelation function of the radiation, if the line transmission is narrow and has a rectangular form in a range of width $\Delta\omega$ around ω_0. The intensity of the radiation is I.

8.91 In a Young interference experiment one observes an interference picture in the region where the beams which are diffracted by two openings overlap (figure 8.91.1). The openings are at a distance D from one another at the points (0, 0) and (x, y). The light source is extended and its dimensions are much larger than D. It is at a distance R from the openings ($R \gg D$). The light is sufficiently monochromatic so that for each of the independent emitters the condition of temporal coherence is satisfied. Express the coefficient of partial coherence in terms of the intensity distribution $I(x, y)$ of the radiation over the cross section of the light source.

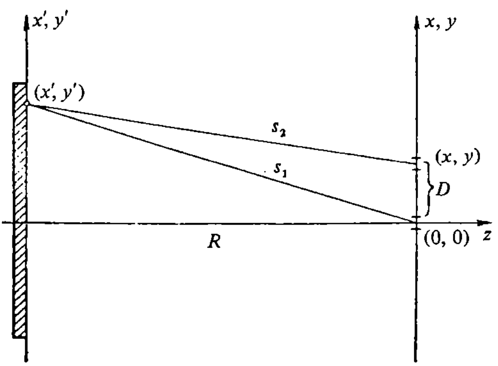


Figure 8.91.1.

8.92 The Michelson stellar interferometer is a version of the Young interference scheme in which the distance between the openings can be varied. Find the dependence of the visibility B of the interference bands in the Michelson interferometer on the distance D between the openings and on the wavelength λ for the following two cases:

(a) A binary is observed, i.e. a system of two stars which are close to one another, at an angular distance α from each other. Each of the stars can be considered to be a point light source. Assume the luminosity of the two stars to be the same.

(b) A single star is observed which has large dimensions and subtends an angular diameter α (consider this star to be a uniformly emitting disc).

8.93 In the Michelson stellar interferometer, that is considered in the preceding problem, light enters from a binary or from a single star of large dimensions. When the distance D between the openings is increased the visibility of the interference bands becomes weaker and for some value $D = D_0$ becomes zero. Determine: (a) the distance ρ between the components of the binary in Capella, which is at a distance $R = 44{\cdot}6$ light years from us, if $D_0 = 7{\cdot}08 \times 10^{-1}$ m, and the observations are at a wavelength $\lambda = 5 \times 10^{-7}$ m; (b) the diameter d of Betelgeuze, which is at a distance of 652 light years, if $D_0 = 7{\cdot}2$ m and $\lambda = 6 \times 10^{-7}$ m.

Hint. The first nonvanishing root of the Bessel function $J_1(x)$ is equal to $x_1 = 3{\cdot}8317$.

8.94 In the Hanbury Brown-Twiss interferometer (figure 8.94.1) one detects independently, and afterwards multiplies and registers, the light intensity that comes from two incoherent point sources which are far from one another or from different points of a single extended source. One can consider the waves coming from the sources to be plane waves (with wavevectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$) with randomly fluctuating amplitudes and phases. Show that, by observing the correlation between intensities, one can use the Hanbury Brown-Twiss interferometer to measure the angular distance between sources.

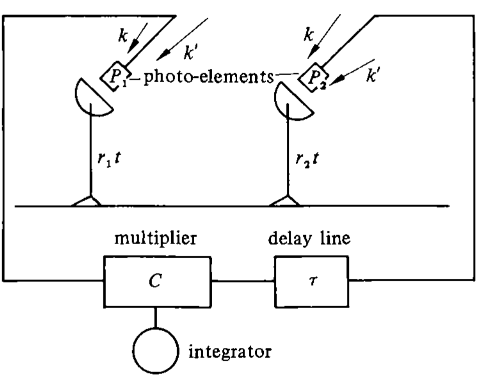


Figure 8.94.1.

8.95 A plane wave of wavelength λ is almost normally incident upon the lateral surface of a thin prism with vertex angle $\alpha \ll 1$ and refractive index n. Find the x-dependence (figure 8.95.1) of the phase shift that the wave acquires in the flat layer ABCD, which is partly occupied by the prism.

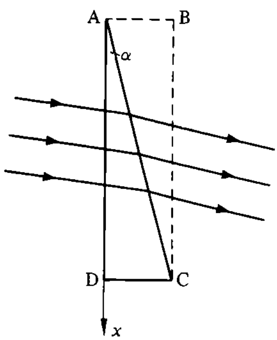


Figure 8.95.1.

8.96 A plane wave is incident upon a thin converging or diverging lens with radii of curvature R_1 and R_2 and refractive index n (figure 8.96.1). The wavelength is λ and the angle between the wavevector and the optical axis is small. Find the x-dependence of the phase shift that the wave acquires in the plane layer ABCD, which is partly occupied by the lens.

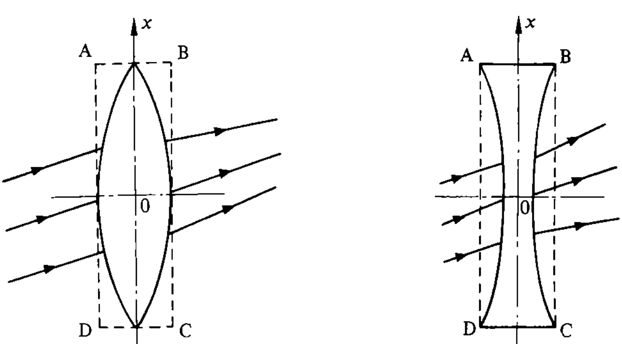


Figure 8.96.1.

8.97 A monochromatic plane wave of wavelength λ from a laser is incident upon a Fresnel double mirror (figure 8.97.1) with an angle $\vartheta \ll 1$ between the planes of the mirrors. An interference wave field is formed in the region where the two plane waves which come from the double mirror overlap. A photographic plate is placed in that region at an angle $\vartheta_1 \ll 1$ to the front of one of the waves. A system of transparent and dark interference bands is formed on the plate. Describe the wave field which is produced behind the plate, if, after it has been developed, one transmits through it a plane wave, normal to its surface, from the same laser.

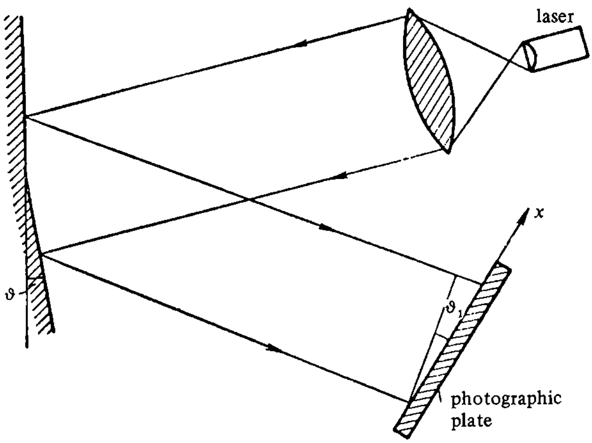


Figure 8.97.1.

8.98 A monochromatic plane wave passes simultaneously through a prism and an aperture in an opaque screen and is incident upon a photographic plate at a distance f (figure 8.98.1). The prism is a thin one with refracting angle $\alpha \ll 1$ and refractive index n. Find the intensity distribution of the field on the photographic plate due to the interference between the 'reference' plane wave (that part of the wave which has passed through the prism and was deflected by it) and the wave diffracted by the aperture (the angle of diffraction is assumed to be small).

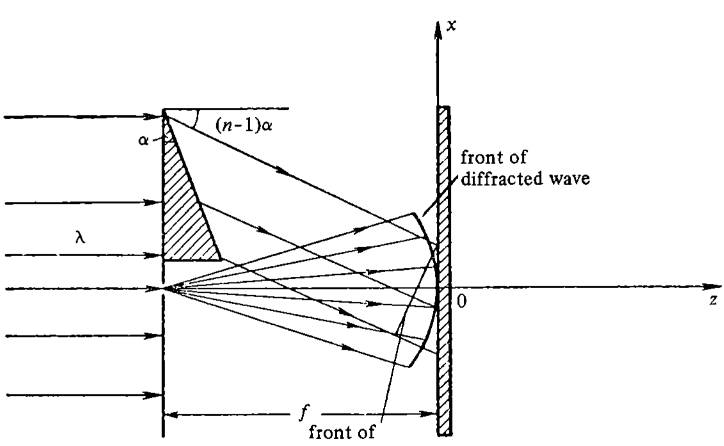


Figure 8.98.1.

8.99 Find the transmission distribution $T(x)$ through a hologram that is obtained under the conditions described in the preceding problem. Assume that when the hologram was produced the intensity of the reference wave was large compared with the intensity of the wave passing through the aperture. Trace the establishment of the initial wavefronts when a plane monochromatic wave $u_0 = A_0' \exp[i(kz - \omega t)]$, which is incident normally, passes through this hologram (the wavelength is the same as that of the original wave). In particular, trace the appearance of the point image of the original aperture.

Hint. The wave field behind the hologram can be obtained simply by multiplying the wave $u_0(x)$, which is incident upon the hologram, by the transmissivity $T(x)$. For an interpretation of the expressions obtained turn to the solutions of problems 8.95 and 8.96.

8.100 One produces by the set-up considered in the preceding two problems a hologram of two apertures at a distance $2D$ from one another in the plane of the prism. Using this hologram one can produce the image of the two apertures. Find this image and elucidate when it will be amplified.

Hint. For the establishment of the image one can illuminate the hologram with light of a wavelength λ' which is different from the wavelength λ used to obtain the hologram.

8.101 Determine the resolving power of the hologram which is obtained in a set-up such as the one considered in problem 8.98. The hologram is produced on a photographic plate with emulsion grain size d.

e X-ray diffraction

When considering the scattering of X-rays by macroscopic bodies it is important that the wavelength λ is comparable to the size a of the atoms. In condensed media the interatomic distances are of the same order of magnitude, whereas these distances are much larger than a in gases. Because of this it is impossible to average over physically small volume elements which contain many atoms. However, when the frequency of the X-rays is large compared to the characteristic atomic frequencies $\omega_{at} \sim v_{at}/c$ one may consider the electrons in the medium to be free. Since for free, nonrelativistic electrons the equations of motion in an external electromagnetic field can easily be integrated, one can evaluate the current induced by the field and determine the dielectric permittivity as a function of position, $\boldsymbol{r}$:

$$\epsilon(\boldsymbol{r}) = 1 - \frac{4\pi e^2 n(\boldsymbol{r})}{m\omega^2} . \tag{8.e.1}$$

Here $n(\boldsymbol{r})$ is the electron density in the body, which is determined by the laws of quantum mechanics and is averaged over a statistical distribution of the states of thermal motion of the atoms.

The Maxwell equations have their usual form (8.a.1) to (8.a.4) with the dielectric permittivity (8.e.1) and magnetic permeability $\mu = 1$, provided that $4\pi e^2 n/m\omega^2 \ll 1$.

Consider a plane wave $E_0 \exp[i(k_0 \cdot r) - i\omega t]$ with an X-ray frequency $\omega \gg \omega_{at}$, which is incident upon some body of finite size. In order that the incident radiation can be considered to be a plane polarised wave it is necessary that the size of the body is small compared to the coherence length, which was defined in the preceding section. In that case we have for the differential scattering cross section of a linearly polarised wave (see section c of chapter 8) for the definition of cross section

$$d\sigma = r_0^2 \sin^2\theta \left| \int n(r) \exp[i(q \cdot r)] \, d^3r \right|^2 d\Omega , \tag{8.e.2}$$

where $r_0 = e^2/mc^2$ is the classical electron radius, k the wavevector of the scattered wave, $k = k_0 = \omega/c$, θ the angle between E_0 and k, $d\Omega$ an element of solid angle in the direction k, and $q = k_0 - k$ is the wavevector transfer. The quantity q is connected with the scattering angle ϑ of the wave (i.e. the angle between k_0 and k) through the formula

$$q = 2\frac{\omega}{c} \sin\tfrac{1}{2}\vartheta = \frac{4\pi}{\lambda} \sin\tfrac{1}{2}\vartheta . \tag{8.e.3}$$

The scattering cross section of an unpolarised X-ray wave is

$$d\sigma = \tfrac{1}{2} r_0^2 (1 + \cos^2\vartheta) \left| \int n(r) \exp[i(q \cdot r)] \, d^3r \right|^2 d\Omega . \tag{8.e.4}$$

The condition for the applicability of equations (8.e.2) and (8.e.4) is the requirement that the total cross section $\sigma = \int_{(4\pi)} d\sigma$ is small compared to the cross-sectional area of the sample as a whole.

In the case of X-ray diffraction by a perfect single crystal the cross sections (8.e.2) ro (8.e.4) display a number of steep maxima, the position of which is given by the von Laue equation

$$k_0 - k = 2\pi g , \tag{8.e.5}$$

where g is a reciprocal lattice vector. If the elementary crystal cell has the shape of a rectangular parallelepiped of edge lengths a_1, a_2, and a_3, we have

$$g = \left(\frac{n_1}{a_1}, \frac{n_2}{a_2}, \frac{n_3}{a_3} \right),$$

where n_1, n_2, and n_3 are arbitrary integers.

The integral of the form occurring in equations (8.e.2) or (8.e.4) integrated over the volume V_{at} of a single atom is called the atomic formfactor:

$$F_{at}(q) = \int_{V_{at}} n_{at}(r) \exp[i(q \cdot r)] \, d^3r . \tag{8.e.6}$$

The atomic formfactor is simply the Fourier transform of the distribution $n_{at}(r)$ of the electrons in the atom and knowing it we can find $n_{at}(r)$ by taking the inverse Fourier transform.

X-ray diffraction is considered in detail in the books by Landau and Lifshitz (1960) and Bacon (1966).

8.102 Elucidate under what conditions the scattering cross section for X-rays by bodies of finite size takes the form of the scattering cross section by free charges (Thomson formula). Write down the appropriate expressions for the cross sections. The number of atoms in the body is N and the number of electrons in each atom is Z.

8.103 The electron density distribution in a Z-electron atom can be approximated by the expression $n_{at}(r) = n_{0\,at} \exp(-r/a)$, in which $n_{0\,at} = Z/\pi a^3$, $a = a_0/Z^{1/3}$, $a_0 = 0{\cdot}529 \times 10^{-10}$ m is the Bohr radius. Find the differential cross section for the scattering of an X-ray wave by a monatomic gas containing N atoms, with the assumption that the distribution of the atoms is completely random.

8.104 Find the cross section for the scattering of X-rays by a volume of gas containing n diatomic atoms. The atoms in a molecule are identical and are at a fixed distance R from one another. Assume that the formfactor $F_{at}(q)$ of an atom which is part of the molecular structure is the same as for an isolated atom.

8.105 What is the change in the cross section for the scattering of X-rays by a volume of a diatomic gas, considered in the preceding problem, if the thermal oscillations of the atoms in the molecule are taken into account?

Hint. Assume that the distance R between the atoms is distributed around an average value $R_0 \gg b$ according to the law

$$\mathrm{d}W_x = \frac{1}{b\sqrt{\pi}} \exp\left(-\frac{x^2}{b^2}\right) \mathrm{d}x\,,$$

where $x = R - R_0$, $b = (2kT/\mu\omega^2)^{1/2}$, T is the temperature, μ the reduced mass, and ω the frequency of the eigenoscillations of the atoms in the molecule.

8.106 Derive the von Laue equation (8.e.5) and the Bragg condition $k \sin\frac{1}{2}\vartheta = \pi|\boldsymbol{g}|$, where $|\boldsymbol{g}|$ is the length of a reciprocal lattice vector, by considering the interference of waves scattered by separate centres of a perfect crystalline lattice.

8.107 Find the cross section for the scattering of X-rays by a perfect single crystal consisting of N identical atoms with formfactors $F_{at}(q)$ (assume that these formfactors are the same as for the case of isolated atoms). The elementary cell has the form of a cube with edge length a, and the crystal has the form of a rectangular parallelepiped of edge lengths L_1, L_2, L_3, which are parallel to the edges of the elementary cell.

Determine the position of the main maxima and verify that the von Laue equation (8.e.5) is satisfied. Find the magnitude of the cross section in those maxima.

8.108 A crystal consists of cubic elementary cells of edge length a and has the form of a right prism with a right isosceles triangle as base (the legs of the triangle $L_1 = L_2$, the lateral side is L_3). Determine the positions of the main maxima and find the magnitude of the cross section in these maxima.

8.109 Find the intensity distribution in the diffraction spot near one of the main maxima when X-rays are scattered by a single crystal (see problem 8.107). The wavevector of the incident X-rays is parallel to the edge L_3 and $k \gg 1/a$. Determine the width of the diffraction maximum and the total cross section corresponding to the scattering within the limits of a single diffraction spot.

8.110 Evaluate the intensity distribution in a diffraction spot around a main maximum for an arbitrary angle of incidence and arbitrary ratio between k and $1/a$. The X-rays are scattered by a single crystal in the form of a rectangular parallelepiped with edge lengths L_1, L_2, and L_3 (see problem 8.107).

8.111 Solve the preceding problem for the case of scattering by a spherical single crystal sample (of radius R).

9

Electromagnetic oscillations in finite bodies[1]

A part of space bounded on all sides by metal walls is called a cavity resonator. A system of standing waves with definite frequencies ω (eigenfrequencies of the resonator) may exist within the resonator. When the resonator is not filled with a dielectric and its walls are perfectly conducting, the standing wave system may be obtained by solving the equations

$$\nabla^2 \boldsymbol{E} + \frac{\omega^2}{c^2}\boldsymbol{E} = 0\,, \qquad \operatorname{div}\boldsymbol{E} = 0\,, \tag{9.0.1}$$

subject to the boundary condition

$$\boldsymbol{E}_\tau = 0\,. \tag{9.0.2}$$

The resonator eigenfunctions[2] $\boldsymbol{E}_\nu$ corresponding to different eigenfrequencies ω_ν are mutually orthogonal. The eigenfunctions corresponding to a single frequency (of which there can be several; see problems 9.20 and 9.22) can also be chosen to be mutually orthogonal. We normalise them to 4π:

$$\int (\boldsymbol{E}_{\nu'} \cdot \boldsymbol{E}_\nu)\, \mathrm{d}^3\boldsymbol{r} = 4\pi\delta_{\nu\nu'}\,, \tag{9.0.3}$$

where the integral is taken over the volume of the resonator. The eigenfunctions $\boldsymbol{H}_\nu$, which can be expressed in terms of the $\boldsymbol{E}_\nu$ by means of the Maxwell equations, have the same properties.

Because of energy losses in the walls or in the matter filling the resonator, and also because of the emission of energy into the space outside, the eigenoscillations of real resonators are damped. The energy losses of a given type of oscillation are characterised by the quality (Q), Q_ν, which is determined by the equation

$$Q_\nu = \frac{\tilde{\omega}_\nu W_\nu}{P_\nu} \quad \text{or} \quad Q_\nu = \frac{\tilde{\omega}_\nu}{2\gamma_\nu}\,. \tag{9.0.4}$$

Here W_ν is the energy stored in the resonator; P_ν is the time-averaged energy loss; $\tilde{\omega}_\nu$ is the resonance frequency which may be different from the resonance frequency of an ideal resonator; and γ_ν is the damping rate.

In contrast to a resonator, a waveguide takes the form of a cavity (duct) of indefinite length. Travelling waves propagate along the axis of the waveguide (z-axis), while standing waves are set up in the transverse direction. In general, the waves in a waveguide will not be transverse.

(1) For details of the theory we refer to the books by Landau and Lifshitz (1960), Jackson (1962), or Panofsky and Phillips (1962).

(2) The index ν stands for the set of four quantities which uniquely determine the resonator eigenoscillation ('mode').

Waves in which $E_z \neq 0, H_z = 0$ are called electric-type or E-waves, whereas waves with $H_z \neq 0, E_z = 0$ are called magnetic-type or H-waves. Purely transverse waves are only possible in waveguides whose cross sections are multiply connected.

The type of wave which may propagate in a given waveguide is determined by solving Maxwell's equations subject to the appropriate boundary conditions. A wave travelling along the axis of a waveguide may be described by

$$E(r, t) = \mathrm{E}(x, y) \exp[\mathrm{i}(kz - \omega t)], \qquad H(r, t) = \mathrm{H}(x, y) \exp[\mathrm{i}(kz - \omega t)]$$

where ω is the frequency and k is the component of the wavevector in the direction of the axis of the waveguide. The quantity k is also called the propagation constant.

In the case of electric-type waves (E-waves) $\mathrm{H}_z = 0$ and E_z is a solution of the equation

$$\nabla^2 \mathrm{E}_z + \kappa^2 \mathrm{E}_z = 0, \tag{9.0.5}$$

where $\kappa^2 = \omega^2 \epsilon \mu / c^2 - k^2$, κ is the transverse component of the wavevector, ϵ and μ are the permittivity and magnetic permeability of the dielectric filling the waveguide, and the boundary condition on the waveguide wall is

$$\mathrm{E}_z = 0. \tag{9.0.6}$$

In the case of magnetic-type waves (H-waves) $\mathrm{E}_z = 0$ and H_z is the solution of

$$\nabla^2 \mathrm{H}_z + \kappa^2 \mathrm{H}_z = 0, \tag{9.0.7}$$

which satisfies the boundary condition on the waveguide wall

$$\mathrm{E}_\tau = 0 \quad \text{or} \quad \frac{\partial \mathrm{H}_z}{\partial n} = 0. \tag{9.0.8}$$

The symbol ∇^2 in equations (9.0.5) and (9.0.6) is the two-dimensional Laplace operator. Strictly speaking, the boundary conditions (9.0.6) and (9.0.8) hold only for waveguides whose walls are perfectly conducting.

The transverse components of the vectors E and H may be expressed in terms of the longitudinal components of these vectors with the aid of Maxwell's equations.

An E-wave or an H-wave of a given type (i.e. with a given κ) can only propagate in a waveguide with a simply connected cross section when its frequency is greater than a certain limiting frequency ω_0. The corresponding 'vacuum wavelength' $\lambda_0 = 2\pi c/\omega_0$ is of the order of the transverse linear dimensions of the waveguide. When $\omega < \omega_0$, the propagation constant k is purely imaginary and wave propagation is impossible. However, even when $\omega > \omega_0$ the propagation constant k is in general complex. This is due to the fact that the walls of the waveguide have a finite conductivity and energy dissipation takes place within them, with the result that the electromagnetic wave is attenuated in accordance with an $\exp(-\alpha z)$ law. The attenuation coefficient α, which is the imaginary part of k, is equal to the energy dissipated per unit time within the walls of the waveguide per unit length divided by twice the energy flux along the waveguide. When the surface impedance $\zeta = \zeta' + i\zeta''$ is small, the attenuation coefficient for E-waves is approximately

$$\alpha = \frac{\omega\zeta'}{2\kappa k c}\oint |\nabla E_z|^2 \,dl \Big/ \int |E_z|^2 \,d^2S\,, \tag{9.0.9}$$

and the corresponding result for H-waves is

$$\alpha = \frac{c\kappa^2\zeta'}{2k\omega}\oint \left[|H_z|^2 + \left(\frac{k^2}{\kappa^4}\right)|\nabla H_z|^2\right] dl \Big/ \int |H_z|^2 \,d^2S\,, \tag{9.0.10}$$

where E_z and H_z are the field components for $\zeta = 0$, i.e. for perfectly conducting walls, dl is an element of the perimeter of the transverse cross section of the waveguide, and d^2S is an area element of the cross section.

9.1 Determine the types of waves which can propagate in a rectangular waveguide with perfecting conducting walls and sides a and b. Find the corresponding dispersion relation and the field configurations, i.e. the dependence of the field components on the coordinates.

9.2 Determine the attenuation coefficients α of the various types of wave in a rectangular waveguide, given that the surface impedance of the waveguide walls is ζ.

9.3 An infinite layer of a dielectric material of permittivity ϵ and magnetic permeability μ fills the region $-a \leqslant x \leqslant a$. Show that the layer will act as a waveguide, i.e. the field of a travelling electromagnetic wave will be largely concentrated within the layer. Determine the types of wave which may propagate in such a waveguide. Confine the analysis to the special case where the field vectors are independent of the y coordinate.

9.4 A dielectric layer with permittivity ϵ and magnetic permeability μ fills the region $0 \leqslant x \leqslant a$ and is in contact with a perfectly conducting solid on one side. The region $x > a$ is evacuated. Determine the types of electromagnetic wave, with amplitudes decreasing with distance from the layer, which may propagate along the layer. Compare the possible types of wave with the wave system obtained in the preceding problem.

9.5 Find the possible types of wave in a circular waveguide of radius a assuming that the walls are perfectly conducting. Determine the limiting frequency ω_0 for this waveguide.

9.6 By making use of the result of the preceding problem, find the attenuation coefficients α of the various types of wave in a circular waveguide, with the assumption that the surface impedance is ζ.

9.7 Determine the phase and group velocities v_{ph} and v_g of waves in rectangular and circular waveguides with perfectly conducting walls and also the dependence of the wave velocities on $\lambda = 2\pi c/\omega$.

9.8 Determine the phase and group velocities v_{ph} and v_g of waves in a waveguide by a geometrical method. To do this, consider the simplest $H_{1\,0}$ wave in a rectangular waveguide, resolve it into plane waves, and investigate the reflection of these waves from the waveguide walls.

9.9 Investigate the structure of the transverse electromagnetic wave in a perfectly conducting coaxial line of inner and outer radii a and b respectively. Determine the average energy flux $\overline{\gamma}$ along the line. Investigate the limiting case of a single, perfectly conducting cylindrical conductor.

9.10 Determine the possible types of transverse electromagnetic waves in a coaxial line with perfectly conducting walls of radii a and $b > a$.

9.11 Determine the value of the attenuation coefficient α of the transverse electromagnetic wave in a coaxial line of radii a and $b > a$ and surface impedance $\zeta = \zeta' + i\zeta''$.

Hint. Use the relation between the attenuation coefficient and the energy losses given at the beginning of this chapter.

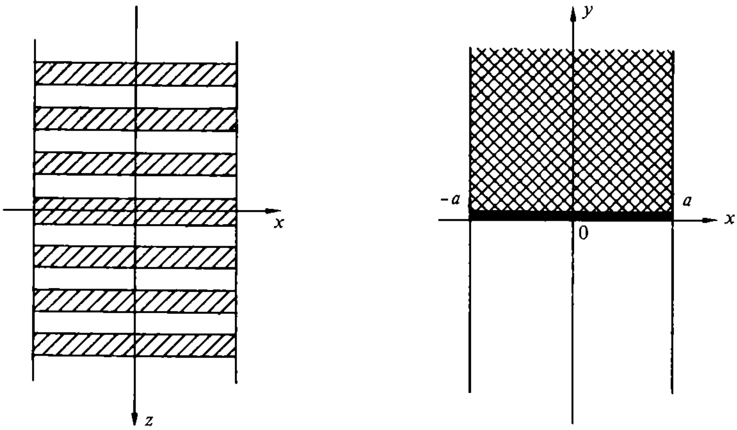
9.12* Discuss the propagation of an axially symmetric E-wave along an infinitely long cylindrical conductor having a finite conductivity and placed in a vacuum. Determine the phase velocity and show that in the case of a perfectly conducting cylinder the wave will become identical with the transverse electromagnetic wave (see problem 9.9). Use the approximate boundary condition given by equation (8.a.10).

9.13 An axially symmetric E-wave propagates in a circular waveguide of radius b which is partly filled with a dielectric. The dielectric has a permittivity ϵ and occupies the region $a \leqslant r \leqslant b$. If $a \ll b$, determine the phase velocity as a function of frequency and find the limiting frequency. Under what conditions will the phase velocity be less than c? Consider the limiting case of a waveguide filled completely by the dielectric.

9.14 A ladder diaphragm (figure 9.14.1a) consisting of thin metallic strips oriented along the x-axis is placed in the plane $y = 0$ between two perfectly conducting planes $x = \pm a$ (figure 9.14.1b). The distances between the strips and their width are small compared to the wavelength. The region $y > 0$ above the ladder diaphragm is filled with a dielectric of permittivity ϵ, the region $y < 0$ contains air. Find the possible types of travelling waves which can propagate along the z-axis in such a system. What is the connection between the wavevector and the frequency of these waves?

Hint. For sufficiently long wavelengths the ladder diaphragm can be considered as an anisotropic conducting plane with a conductivity in the

x-direction which is infinite and a conductivity in the z-direction which is equal to zero.



(a) (b)

Figure 9.14.1.

9.15 A rectangular waveguide with sides a and b and perfectly conducting walls is filled with a ferrodielectric. A constant magnetic field is applied at right angles to the wider wall of the waveguide (along the y-axis). The permittivity and magnetic permeability tensors of the ferrodielectric are given by

$$\epsilon_{ik} = \begin{pmatrix} \epsilon_\perp & 0 & -i\epsilon_a \\ 0 & \epsilon_\parallel & 0 \\ i\epsilon_a & 0 & \epsilon_\perp \end{pmatrix}, \qquad \mu_{ik} = \begin{pmatrix} \mu_\perp & 0 & -i\mu_a \\ 0 & \mu_\parallel & 0 \\ i\mu_a & 0 & \mu_\perp \end{pmatrix}$$

(see the solution of problem 6.35). Determine the components of the electromagnetic field, the propagation constant, and the limiting frequency when the field is independent of y.

9.16 The electric and magnetic fields in a waveguide with perfectly conducting walls are described by

$$\boldsymbol{E}_0 = \mathsf{E}_0(x, y) \exp[i(k_0 z - \omega t)], \qquad \boldsymbol{H}_0 = \mathsf{H}_0(x, y) \exp[i(k_0 z - \omega t)].$$

When a dielectric core in the form of a cylinder of arbitrary cross section is inserted into the waveguide with its longitudinal axis parallel to the axis of the waveguide, the fields are given by

$$\boldsymbol{E} = \mathsf{E}(x, y) \exp[i(kz - \omega t)], \qquad \boldsymbol{H} = \mathsf{H}(x, y) \exp[i(kz - \omega t)].$$

The dielectric may in general be characterised by the tensorial parameters ϵ_{ik}, μ_{ik}. Show, with the aid of Maxwell's equations, that the change in propagation constant is given by

$$\Delta k = k - k_0 = \frac{\omega \int_{\Delta S} (\Delta\epsilon_{ik} E_k E_{0i}^* + \Delta\mu_{ik} H_k H_{0i}^*)\, \mathrm{d}^2 S}{c \int_S (\{[E_0^* \wedge H] + [E \wedge H_0^*]\} \cdot e_z)\, \mathrm{d}^2 S},$$

where $\Delta\epsilon_{ik} = \epsilon_{ik} - 1$, $\Delta\mu_{ik} = \mu_{ik} - 1$. The integral in the numerator is evaluated over the cross section of the dielectric rod (ΔS) and the integral in the denominator is taken over the cross section of the waveguide (S).

9.17 A ferrodielectric plate of thickness $d \ll a$ is inserted into a rectangular waveguide with perfectly conducting walls (figure 9.17.1). The plate is magnetised along the axis of the waveguide. Use the formula obtained in the preceding problem to determine the change Δk in the propagation constant for $H_{1\,0}$ waves to within terms of the order of d. Assume that the permittivity of the wall is a scalar, and the magnetic permeability tensor is as given in problem 8.37.

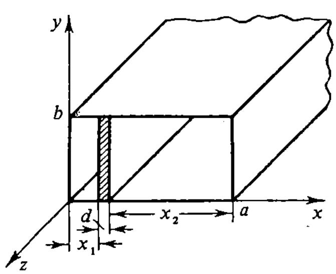


Figure 9.17.1.

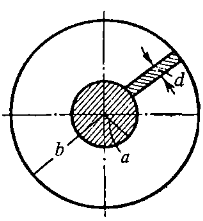


Figure 9.18.1.

9.18 A thin ferrite plate ($d \ll a, b$) is inserted into the coaxial waveguide shown in figure 9.18.1 and is magnetised in the direction of the axis of the waveguide. Find the change Δk in the propagation constant of the transverse electromagnetic wave.

Hint. Use the method employed in the preceding problem to find the amplitude of the disturbed fields.

9.19 Solve the preceding problem for a constant magnetising field H_0 applied at right angles to the axis of the waveguide. Consider two directions of this field, namely, (a) H_0 perpendicular to the longer side of the plate and (b) H_0 perpendicular to the shorter side of the plate.

9.20 Determine the eigenfrequencies and normalised eigenfunctions of a cavity resonator with perfectly conducting walls. The resonator is in the form of a rectangular $a \times b \times h$ parallelepiped. The eigenfunctions should be chosen such that they are mutually orthogonal.

9.21 Determine the number of eigenoscillations $\Delta N(\omega)$ per frequency interval $\Delta\omega$ in the hollow resonator considered in the preceding problem.

Assume that $\Delta\omega \ll \omega$ and $\Delta N \gg 1$, and that the total volume of the resonator is V.

9.22 A given resonator is in the form of a right-circular cylinder of height h and radius a. Assuming that the walls are perfectly conducting, find the frequency of the eigenoscillations. Consider the E and H oscillations.

9.23 Two circular metallic plates of radius R are at a small distance d from one another and form a capacitor. The plates of the capacitor are closed by a conductor of thickness $2a$ in the form of a ring of radius b (figure 9.23.1). Find the eigenfrequency of the oscillations of such an 'open resonator', by assuming that one may apply a quasi-stationary approximation. Assume all conductors to be perfectly conducting.

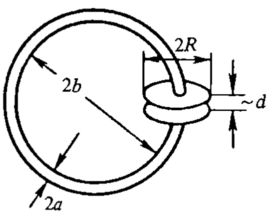


Figure 9.23.1.

9.24 Find the eigenfrequency ω_0 of the oscillations of the system shown in figure 9.24.1, assuming that the corresponding wavelength λ_0 is large compared to the dimensions of the system. Neglect energy losses and edge effects.

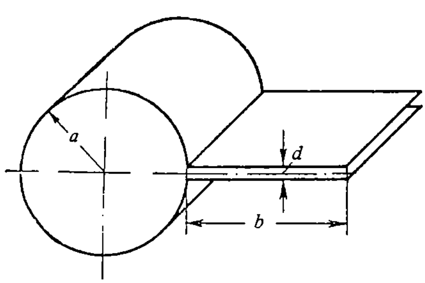


Figure 9.24.1.

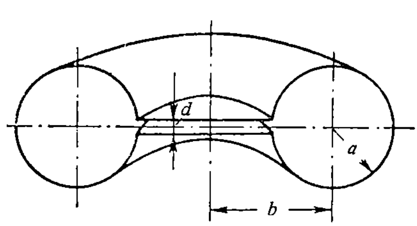


Figure 9.25.1.

9.25 To decrease energy losses due to radiation one uses, instead of an open oscillating contour (see figure 9.23.1), a closed resonator consisting of a toroidal cavity joined to a plane capacitor with circular plates (its cross section and dimensions are shown in figure 9.25.1). Find the eigenfrequency ω_0 of the basic kind of oscillations of such a resonator in the quasi-stationary approximation. Under what conditions is this approximation applicable? Assume the walls of the resonator to be perfectly conducting.

9.26 Solve the preceding problem for a toroidal resonator with a cavity of rectangular cross section (figure 9.26.1).

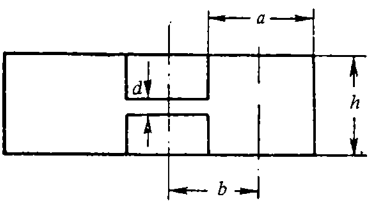


Figure 9.26.1.

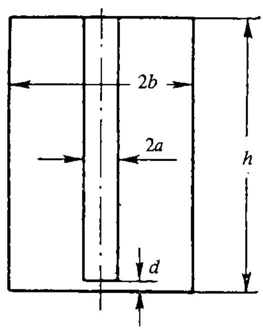


Figure 9.27.1.

9.27 A resonator is a circular cylinder (height h, inside radius b) along the axis of which there is fitted a perfectly conducting rod of radius a (figure 9.27.1). The walls of the cylinder are also perfectly conducting. Between the rod and one of the end planes of the cylinder there is a gap d. Find the eigenfrequencies of the electromagnetic oscillations which are transverse with respect to the axis of the system, assuming that the wavelength of these oscillations is much longer than the gap width d (but not longer than the height h of the cylinder). What is the change in the spectrum of the oscillations as $d \to 0$?

9.28 Let the eigenfrequencies ω_ν and the eigenfunctions $\boldsymbol{E}_\nu$, $\boldsymbol{H}_\nu$ of the oscillations of a resonator with perfectly conducting walls be known. Calculate the change in the eigenfrequencies caused by the finite conductivity of the resonator walls. The surface impedance ζ of the walls is small.

Hint. Look for the solution of the Maxwell equations in the form

$$\boldsymbol{E}(\boldsymbol{r}, t) = \sum_\nu q_\nu(t)\boldsymbol{E}_\nu(\boldsymbol{r}), \qquad \boldsymbol{H}(\boldsymbol{r}, t) = \sum_\nu p_\nu(t)\boldsymbol{H}_\nu(\boldsymbol{r}),$$

where q_ν and p_ν are unknown functions of the time. Derive equations for q_ν and p_ν up to terms linear in ζ and study their solution.

9.29 A cavity resonator has the shape of a cube of edgelength a. The conductivity of the wall is σ and the magnetic permeability $\mu = 1$. Evaluate the Q of the resonator for an arbitrary kind of oscillation. How does it depend on the frequency? At what frequencies do the resonance properties of the system disappear?

9.30 A cavity resonator with walls of surface impedance ζ is excited by an external current $j(r)\exp(-i\omega t)$ flowing inside the resonator. The frequency ω of the current is close to one of the eigenfrequencies of the resonator. Find the electromagnetic field excited in the resonator and the way it depends on the frequency ω near resonance.

Hint. Use the method developed in problem 9.28.

9.31 An open resonator in the infra-red band consists of two parallel circular mirrors of diameter D, at a distance L opposite to one another (figure 9.31.1). Let the eigenoscillation of such a system be realised in the form of two waves with $\lambda \ll L, D$ propagating at right angles to the planes of the mirrors toward one another and forming a standing electromagnetic wave.

Give an order of magnitude estimate of the Q of such a resonator in the geometrical optics approximation. Take into account the energy losses at the reflection from the mirrors (reflection coefficient R) and the radiation through the side surface, bearing in mind that there is diffraction. Take as the parameters of the resonator: $D = L = 10^{-2}$ m; $R = 0{\cdot}95$; $\lambda = 3 \times 10^{-6}$ m.

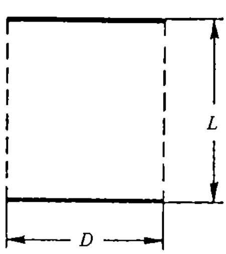


Figure 9.31.1.

9.32 The mirrors of the open resonator considered in the preceding problem are slightly nonparallel. The angle between their planes $\beta \ll 1$. Estimate the additional losses through radiation and the corresponding contribution to the Q-value of the resonator caused by the fact that the mirrors are not parallel. What values of the angle β are allowable without essentially changing the total Q of the resonator?

9.33 In a resonator formed by two parallel mirrors (see figure 9.31.1) the eigenoscillations with $\lambda \ll L, D$ are realised in the form of standing waves in the space between the mirrors. Consider that kind of oscillation in which the wavevector of the standing wave makes a small angle ϑ with the normal to the planes of the mirrors.

(a) Find the condition determining possible values of ϑ for a given λ.

(b) Give an order of magnitude estimate of the Q of the resonator as a function of the angle ϑ. Consider different ratios between the losses in the mirrors and the losses through radiation.

Special theory of relativity [1]

a Lorentz transformations

The space and time coordinates in two inertial reference frames S and S' are related by the Lorentz transformation formulae

$$x = \gamma(x' + Vt'), \qquad y = y', \qquad z = z', \qquad t = \gamma\left(t' + \frac{Vx'}{c^2}\right), \tag{10.a.1}$$

where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = V/c$. It is assumed that S' moves with a constant velocity V along the Ox-axis relative to S in such a way that the coordinate axes remain parallel to each other, the x- and x'-axes coincide, and the origins of S and S' coincide at times $t = t' = 0$. The inverse Lorentz transformation is obtained by changing the sign of V:

$$x' = \gamma(x - Vt), \qquad y' = y, \qquad z' = z, \qquad t' = \gamma\left(t - \frac{Vx}{c^2}\right). \tag{10.a.2}$$

The quantities $x_0 = ct, x_1 = x, x_2 = y, x_3 = z$ are the coordinates of the world point

$$x_i = (ct, r). \tag{10.a.3}$$

Any four quantities A_0, A_1, A_2, A_3, which transform when we change from one inertial system of reference to another in the same way as the coordinates and the time, i.e. according to the formulae

$$A_0 = \gamma(A'_0 + \beta A'_1), \qquad A_1 = \gamma(A'_1 + \beta A'_0),$$
$$A_2 = A'_2, \qquad A_3 = A'_3, \tag{10.a.4}$$

form a four-dimensional vector (four-vector, 4-vector) A_i, $i = 0, 1, 2, 3$. The three-dimensional vector $A = (A_1, A_2, A_3)$ is called the spatial component of the 4-vector A_i and the quantity A_0 is the time component.

The scalar product of two four-dimensional vectors is defined as follows:

$$A_iB_i = A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3. \tag{10.a.5}$$

As before (see chapter 1) we shall assume that one sums over repeated indices, which now take on the values 0, 1, 2, 3. The term with the index 0 is now taken with a plus sign, and the terms with indices 1, 2, 3 are given a minus sign. This sign convention will be used in what follows.

The squares of 4-vectors A_i^2 ($\equiv A_iA_i$), defined in accordance with equation (10.a.5), and their scalar products A_iB_i have the same values in all inertial systems of reference (they are invariant under Lorentz transformations). The 4-vector A_i is called space-like if $A_i^2 < 0$ and time-like if $A_i^2 > 0$.

[1] For details of the theory we refer to the books by Fock (1964), Landau and Lifshitz (1960, 1975), Bergmann (1958), Jackson (1962), Pauli (1958), Heitler (1944), Panofsky and Phillips (1962), and Pathria (1974).

The invariant quantity

$$s_{12} = [c^2(t_1 - t_2) - (r_1 - r_2)^2]^{1/2} , \tag{10.a.6}$$

is called the interval between two events with coordinates (r_1, t_1) and (r_2, t_2).

The time indicated by a clock moving together with a given object is called the proper time of the object. If the object moves relative to the frame S with a velocity V, then the proper time interval $d\tau$ may be expressed in terms of the time interval dt in the frame S, and is given by

$$d\tau = dt \left(1 - \frac{V^2}{c^2}\right)^{1/2} \tag{10.a.7}$$

The quantity $dt(1-\beta^2)^{1/2}$ is invariant under a Lorentz transformation.

If a given rod has a length l_0 at rest, then when it moves with a velocity v in the direction of its longitudinal axis, its length as measured by an observer at rest is given by

$$l = l_0 \left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{10.a.8}$$

The four-dimensional velocity (four-velocity) is defined as the 4-vector whose components are given by

$$u_i = \frac{dx_i}{d\tau} = \left[\frac{c}{(1-v^2/c^2)^{1/2}}, \frac{v}{(1-v^2/c^2)^{1/2}}\right] , \tag{10.a.9}$$

where $\boldsymbol{v} = d\boldsymbol{r}/dt$ is the ordinary particle velocity. It is clear from equation (10.a.9) that

$$u_i^2 = c^2 . \tag{10.a.10}$$

The 4-velocity transforms in accordance with equation (10.a.4) just like any other 4-vector.

The components of ordinary velocity are not the space components of any particular 4-vector, and transform in accordance with the following expressions ($V \parallel x$):

$$v_x = \frac{v'_x + V}{1 + v'_x V/c^2} , \qquad v_y = \frac{v'_y(1 - V^2/c^2)^{1/2}}{1 + v'_x V/c^2} , \qquad v_z = \frac{v'_z(1 - V^2/c^2)^{1/2}}{1 + v'_x V/c^2} . \tag{10.a.11}$$

If the particle velocity is at angles ϑ and ϑ' to the x-axis in S and S' respectively, then

$$\tan\vartheta = \frac{v'(1 - V^2/c^2)^{1/2} \sin\vartheta'}{v' \cos\vartheta' + V} , \qquad v' = (v_x'^2 + v_y'^2 + v_z'^2)^{1/2} . \tag{10.a.12}$$

The four-dimensional acceleration of a particle is the 4-vector with components

$$w_i = \frac{du_i}{d\tau} = \frac{d^2 x_i}{d\tau^2} . \tag{10.a.13}$$

The wavevector $\boldsymbol{k}$ and the frequency ω of a plane electromagnetic wave are the components of the wave 4-vector

$$k_i = \left(\frac{\omega}{c}, \boldsymbol{k}\right) . \tag{10.a.14}$$

It follows that the phase of a plane wave $\varphi = -k_i x_i$ is an invariant.

It follows from equation (10.a.4) that the transformation formulae for the angle ϑ between a light ray and the x-axis are

$$\tan\vartheta = \frac{\sin\vartheta'}{\gamma(\cos\vartheta' + \beta)} \quad \text{or} \quad \cos\vartheta = \frac{\cos\vartheta' + \beta}{1 + \beta\cos\vartheta'} . \tag{10.a.15}$$

Section a of chapter 11 deals with problems concerning the Lorentz transformation for energy, momentum, and force.

10.1 Suppose that a system S' moves relative to a system S with a velocity V along the x-axis. When clocks at S' (x_0', y_0', z_0') and S (x_0, y_0, z_0) pass each other they indicate times t_0' and t_0 respectively. Write down the Lorentz transformation formulae for this case.

10.2 A system S' moves relative to a system S with a velocity V. Show that when clocks in S and S' are compared, the clock in one of the systems, whose readings are successively compared with the readings of two clocks in the second frame (synchronised within that frame), will always lag behind. Express one of the two time intervals in terms of the other. (The readings of the moving clocks are compared when they pass each other.)

10.3 The length of a rod moving in the direction of its longitudinal axis in a given reference frame may be found by determining the time interval taken by the rod to travel past a point fixed in the frame and by multiplying it by the velocity of the rod. Show that this method of measuring the length of a rod will yield the usual Lorentz contraction.

10.4 A system S' moves relative to a system S with a velocity V. Clocks at rest at the origins of the two frames are found to be synchronised ($t = t' = 0$) when they pass each other. Find the coordinates in each system of the world point, which has the property that clocks at rest in S and S' indicate the same time $t = t'$ at this point. Determine the law of motion of the world point.

10.5 To measure time let us use a periodic process of reflecting a light 'spot' alternately from two mirrors mounted at the ends of a rod of length l. One period is the time it takes the 'spot' to move from one mirror to the other and back. The light clocks are fixed in the system S' and oriented parallel to the direction of the motion. Use the postulate of the constancy of the velocity of light to show that the interval of eigentime $d\tau$ can be expressed in terms of the time interval dt in the system S through equation (10.a.7).

10.6 Solve the preceding problem for the case when the light clocks are oriented at right angles to the direction of the relative velocity.

10.7 A 'train' A′B′ has a length $l_0 = 8 \cdot 64 \times 10^8$ km in the system in which it is at rest. The train travels past a 'platform' AB having an equal length in its own rest frame with the velocity $V = 240000$ km s^{-1}. Identical synchronised clocks are placed at A, B, and A′, B′. When the head of the 'train' (B′) becomes level with the beginning of the 'platform' (A), the corresponding clocks both indicate 12 h 00 min. Answer the following questions: (a) Is there a reference frame in which all four clocks will show 12 h 00 min? (b) What will be the time shown by the four clocks when A′ becomes level with A? (c) What will be the time shown by the clocks when the head of the 'train' B′ becomes level with B?

10.8 Determine the time (as measured by a clock at rest on the earth's surface) taken by a rocket to reach Proximus Centauri and return to earth with a constant velocity $v = (0 \cdot 9999)^{1/2} c$, if the distance to the star is four light years[(2)]. Use the result to determine the time interval which may be used as a basis for estimating the amount of food and other supplies for the journey. Calculate the kinetic energy of the rocket if its mass is 10 tonnes.

10.9 Two rulers, each having a length l_0 at rest, move in opposite directions with uniform velocities along the x-axis. An observer at rest relative to one of the rulers notes that the time interval between instants at which the left and right ends of the rulers pass each other is Δt. What is the relative velocity of the two rulers? In what order will the ends of the rulers pass each other for observers at rest relative to either of the rulers, and also for an observer with respect to whom both rulers move with equal velocities in opposite directions?

10.10 Derive the Lorentz transformation formulae without assuming that the velocity of S' relative to S is parallel to the x-axis. Give the result in a vector form.

Hint. Resolve the position vector $\boldsymbol{r}$ into the longitudinal and transverse components relative to the direction of the relative velocity $\boldsymbol{V}$ and use the Lorentz transformation (10.a.1).

10.11 Write down the Lorentz transformation formulae for an arbitrary 4-vector $A_i = (A_0, \boldsymbol{A})$ without assuming that the velocity $\boldsymbol{V}$ of the system S' relative to S is parallel to the x-axis.

10.12 Derive the velocity addition formulae when the velocity $\boldsymbol{V}$ of a system S' relative to S has an arbitrary direction. Give the result in a vector form.

10.13 Three reference frames, S, S', and S'' are such that S'' moves relative to S' with a velocity V' in the direction parallel to the x'-axis, and

(2) A light year is defined as the distance travelled by a light beam in vacuo in one year (see introduction to section 8, d).

S' moves relative to S with a velocity V in the direction parallel to the x-axis. The corresponding axes of all the three systems remain parallel. Find the Lorentz transformation connecting S'' and S and use it to obtain a formula for the addition of parallel velocities.

10.14 Prove the formula

$$\left(1-\frac{v^2}{c^2}\right)^{1/2} = \frac{(1-v'^2/c^2)^{1/2}(1-V^2/c^2)^{1/2}}{1+(\boldsymbol{v}'\cdot\boldsymbol{V})/c^2},$$

where $\boldsymbol{v}$ and $\boldsymbol{v}'$ are the velocities of a particle relative to frames S and S', and $\boldsymbol{V}$ is the velocity of S' relative to S.

10.15 Prove the relation

$$v = \frac{\{(\boldsymbol{v}'+\boldsymbol{V})^2-[\boldsymbol{v}'\wedge\boldsymbol{V}]^2/c^2\}^{1/2}}{1+(\boldsymbol{v}'\cdot\boldsymbol{V})/c^2},$$

where $\boldsymbol{v}$ and $\boldsymbol{v}'$ are the velocities of a particle relative to frames S and S', and $\boldsymbol{V}$ is the velocity of S' relative to S.

10.16 Three consecutive transformations of the system of reference are carried out: (a) A transition from the system S to a system S' moving with respect to S with a velocity $\boldsymbol{V}$, parallel to the x-axis; (b) A transition from the system S' to a system S'', moving with respect to S' with a velocity $\boldsymbol{v}$, parallel to the y'-axis; (c) A transition from the system S'' to a system S''' moving with respect to S'' with a velocity equal to the relativistic sum of the velocities $-\boldsymbol{v}$ and $-\boldsymbol{V}$. (We draw attention to the fact that the resulting velocity depends on the order in which the velocities are added.) Show that, as one should expect, the system S''' is at rest with respect to S and $t''' = t$, but that S''' is rotated with respect to S over a certain angle in the xy-plane (Thomas precession). Evaluate the angle φ of the Thomas precession.

Hint. Use the formulae in the general form of a Lorentz transformation (see problem 10.10) and of velocity addition (see problem 10.12), and write these formulae in terms of their components along Cartesian axes.

10.17 Two rulers, each of which has a length l_0 in its own rest frame, move towards each other with equal velocities v relative to a given reference system. Find the length l of each of the rulers in the reference frame in which the other ruler is at rest.

10.18 Two electron beams travel along the same straight line, but in opposite directions, with velocities $v = 0{\cdot}9c$ relative to the laboratory system. Find the relative velocity V of the electrons (a) as measured by an observer at rest in the laboratory and (b) by an observer moving together with one of the electron beams.

10.19 The effects which occur during a collision between two elementary particles are independent of the uniform motion of the two particles as a

whole. Such effects depend only on their relative velocity. A given relative velocity may be communicated to the colliding particles in two ways (we shall assume for simplicity that they have the same mass):
(a) an accelerator is used to accelerate one of the particles to an energy $\mathcal{E}$ while the other is at rest, or (b) both particles are accelerated towards each other by identical machines until they reach an energy $\mathcal{E}_0 < \mathcal{E}$. Compare the two energies $\mathcal{E}$ and $\mathcal{E}_0$ and consider in particular the ultrarelativistic case.

10.20 Find the transformation formulae for the acceleration $\dot{\boldsymbol{v}}$ in the case when the system S' moves relative to the system S with an arbitrary velocity $\boldsymbol{V}$. Express your result in vector form.

10.21 Express the components of the four-dimensional acceleration w_i in terms of the ordinary acceleration $\dot{\boldsymbol{v}}$ and the velocity $\boldsymbol{v}$. Find w_i^2. Is the four-dimensional acceleration space-like or timelike?

10.22 Express the acceleration $\dot{\boldsymbol{v}}'$ of a particle in the inertial frame in which it is instantaneously at rest in terms of its acceleration $\dot{\boldsymbol{v}}$ in the laboratory system. Consider the cases when $\boldsymbol{v}$ varies only in magnitude or only in direction.

10.23 A relativistic particle performs a 'constant-acceleration' one-dimensional motion (its acceleration $\dot{v} \equiv w$ is constant in the eigenframe of reference). Find the time dependence of the velocity $v(t)$ and of the coordinate $x(t)$ in the laboratory frame of reference, given that the initial velocity is v_0 and the initial position x_0. Consider, in particular, the nonrelativistic and the ultrarelativistic limits.

Hint. Use the result of the preceding problem.

10.24 The rocket considered in problem 10.8 is accelerated from rest to a velocity $v = (0{\cdot}9999)^{1/2}c$. The acceleration of the rocket in the system in which it is instantaneously at rest is $|\dot{\boldsymbol{v}}| = 20 \text{ m s}^{-2}$. Determine the time taken by the rocket to reach this velocity as measured by clocks in the frame in which it was originally at rest, and clocks at rest in the rocket.

Hint. Neglect the effect of inertial forces on the clocks in the rocket, i.e. determine the sum of the proper time intervals $d\tau = dt(1-v^2/c^2)^{1/2}$ in the inertial frames in which the rocket is successively instantaneously at rest.

10.25 A particle moves with velocity $\boldsymbol{v}$ and acceleration $\dot{\boldsymbol{v}}$ so that after a short time interval δt its velocity in the laboratory frame S is changed by the amount $\delta \boldsymbol{v} = \dot{\boldsymbol{v}}\delta t$. Let S' be the inertial system which is instantaneously comoving with the particle at time t and S'' the same system at time $t+\delta t$. Use the Lorentz transformation to show up to terms linear in $\delta \boldsymbol{v}$ that the coordinates and time in these systems are connected by the relations

$$\boldsymbol{r}'' = \boldsymbol{r}' + [\Delta\boldsymbol{\varphi} \wedge \boldsymbol{r}'] - t'\Delta\boldsymbol{v}\,, \qquad t'' = t' - \frac{(\boldsymbol{r}' \cdot \Delta\boldsymbol{v})}{c^2}\,, \tag{10.25.1}$$

where

$$\Delta \boldsymbol{v} = \gamma\left[\delta \boldsymbol{v} + (\gamma - 1)\frac{(\boldsymbol{v} \cdot \delta \boldsymbol{v})}{v^2}\boldsymbol{v}\right], \qquad \Delta \varphi = (\gamma - 1)\frac{[\delta \boldsymbol{v} \wedge \boldsymbol{v}]}{v^2}. \qquad (10.25.2)$$

What is the geometrical meaning of the transformation (10.25.1)? What is the form of equations (10.25.2) when $v \ll c$, in the first nonvanishing approximation?

Hint. It is convenient to consider a chain of transformations $S'' \to S \to S'$, and to make use of the formulae given in the solution of problem 10.10.

10.26 A system S' moves with a velocity $\boldsymbol{V}$ relative to a system S. The velocities of two bodies relative to S are respectively $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Find the angle α between the velocities of the two bodies as measured in S and S'.

Hint. Use the results of problems 10.12 and 10.14.

10.27 What happens to the angle between the two velocities in the preceding question when the velocity of S' relative to S tends to the velocity of light c?

10.28 At a particular instant of time the direction of the rays of light from a star are at an angle ϑ to the orbital velocity $\boldsymbol{v}$ of the Earth in the frame in which the Sun is at rest. Find the change in the direction of the Earth–star line after a period of six months without assuming that v/c is small. This phenomenon is known as the aberration of light.

10.29 Find the form of the apparent curve, described by a star on the celestial sphere, which is due to the annual aberration. Assume that the polar coordinates of the star in the frame in which the sun is at rest are ϑ, ϕ, where the polar axis is drawn at right angles to the plane of the earth's orbit. The orbital velocity of the earth may be assumed to be very much smaller than the velocity of light.

10.30 A beam of light subtends a solid angle $d\Omega$ in a given reference system. Find the change in this angle when it is measured in another inertial reference system.

10.31 Assuming that the stars in that part of the galaxy which is nearest to us are distributed uniformly, determine their distribution $dN/d\Omega'$ as measured by an observer in a rocket moving with a velocity approaching the velocity of light.

10.32 Find the transformation formulae for the frequency ω and the wavevector $\boldsymbol{k}$ of a plane monochromatic light wave between two inertial systems. Assume that the direction of the relative velocity $\boldsymbol{V}$ is arbitrary.

10.33 Find the frequency ω of a light wave as observed in the case of the transverse Doppler effect (the direction of propagation is at right angles to the direction of motion of the source in the system in which the detector of the radiation is at rest). Find the direction of propagation of the wave in the system in which the source is at rest.

10.34 The wavelength of light emitted by a source in the system in which it is at rest is λ_0. Find the wavelength λ measured by an observer approaching the source with a velocity V, and an observer moving with the same velocity away from the source.

10.35 A source emitting light of frequency ω_0 isotropically in all directions in its own frame of reference moves uniformly in a straight line relative to an observer with velocity V and at the moment of closest approach it is proceeding away from the observer at an impact parameter d. The number of photons emitted per unit time per unit solid angle (photon intensity flux) is equal to J_0 in the rest frame of the source. Find the way the frequency ω and the photon flux intensity J registered by the observer depend on the angle between the beam direction and the direction of the velocity V. At what angles $\theta = \theta_0$ are the registered frequency and photon flux intensity the same as ω_0 and J_0? What fraction of photons is registered by the observer in the ranges $0 \leqslant \theta \leqslant \theta_0$ and $\theta_0 \leqslant \theta \leqslant \pi$? Draw the functions $\omega(\theta)$ and $J(\theta)$ for $V/c = \frac{1}{3}$ and $V/c = \frac{4}{5}$. What happens to these functions as $V/c \to 1$?

10.36 Find the angular distribution of the light intensity I (energy of the light emitted per unit time into a unit solid angle) and also the total light flux from the light source considered in the preceding problem.

Hint. Each photon has an energy $\hbar\omega$, where $\hbar$ is Planck's constant divided by 2π (Dirac's constant).

10.37 A mirror moves with a velocity V in a direction at right angles to its plane. Find the law of reflection of a plane monochromatic wave from the mirror, and also the change in the frequency on reflection. Consider in particular the case $V \to c$.

10.38 Solve the preceding problem in the case when the mirror travels without rotation in a direction parallel to its own plane.

10.39 An opaque cube of edgelength l_0 in its own rest frame moves with velocity V relative to an observer (see figure 10.39.1). The observer

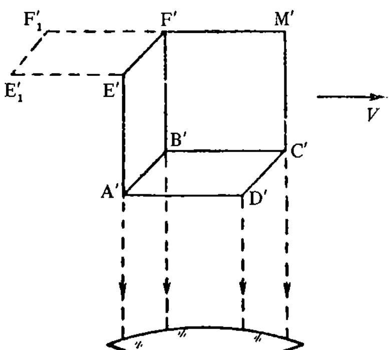


Figure 10.39.1.

photographs it at the moment when a beam of light emitted from the surface of the cube enters the objective of a camera at a right angle to the direction of motion (in the frame of the camera). The cube is seen under a small solid angle so that one may assume that the rays from the different points of the cube are parallel.

What is the shape of the image on the photographic plate? Make a sketch of the image, indicating on it the corners and edges of the cube as they are photographed. Evaluate their relative lengths. Which is the fixed object the image of which is equivalent to the photograph? What would be the shape of the image of a moving cube, if the Galileo transformations were valid?

10.40 A thin rod M′N′ is fixed in the system of reference S' and in that reference system has a length l_0 and is oriented as shown in figure 10.40.1. The system S' moves with a velocity $V \parallel Ox$ relative to a photographic plate AB which is at rest in the system S. At the time when the rod passes the photographic plate there is a short light flash which is such that the light rays are incident normally upon the xz-plane, the plane of the plate.

(a) What is the length l of the image on the plate? Can it be equal to or exceed l_0?

(b) At what angle of inclination α' are only the end points of the rod photographed?

(c) What is the angle of inclination, α, of the rod with respect to the x-axis?

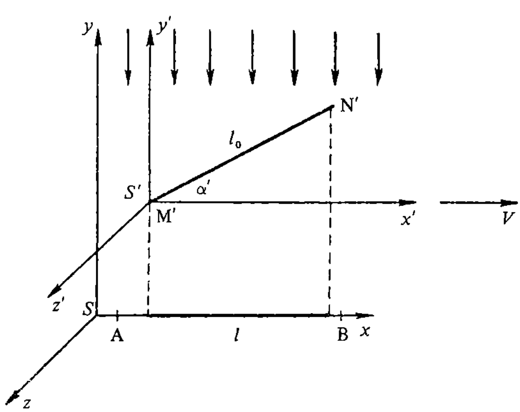


Figure 10.40.1.

10.41 A sphere moving with a velocity V is photographed by a fixed observer under a small solid angle. The rays of light from the sphere are incident in a parallel beam on the objective of the camera and the beam is at a right angle to the direction of the velocity V. What is the shape of the image on the photographic plate?

Hint. Consider the sphere as a collection of thin discs moving parallel to their planes and construct the image of each disc.

10.42 Let a moving opaque cube be photographed by a fixed observer at the moment when the light rays coming from the cube make an arbitrary angle α with the direction of the velocity V of the cube (in the frame of the observer). The solid angle under which the cube is seen is small so that the rays form a parallel beam which is normally incident on the surface of the photographic plate (figure 10.42.1). Show that the photograph must be the same as the photograph of a fixed cube which is turned at an angle. Find the angle of rotation for different values of V and fixed α. For what value of V will the single face A′B′ be photographed? When will the single face B′C′ be photographed?

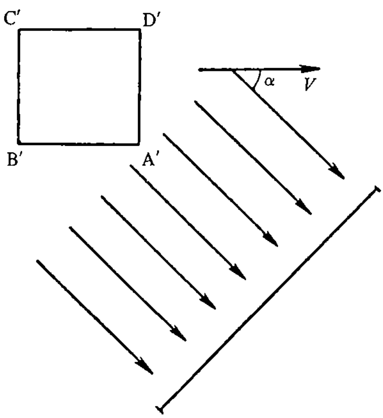


Figure 10.42.1.

10.43 Set up the wave 4-vector describing the propagation of a plane monochromatic wave in a medium which has a refractive index n and moves with a velocity V (the phase velocity of the wave in a stationary medium may be taken to be $v' = c/n$). Find the transformation formulae for the frequency, the angle between the wavevector and the direction in which the medium moves, and the phase velocity.

10.44 A plane wave propagates in a medium moving with a velocity V in the direction of displacement of the medium. The wavelength in vacuo is λ. Find the velocity v of the wave in the laboratory system (Fizeau experiment). The refractive index n is determined in the system S' in which the medium is at rest and depends on the wavelength λ' in this system. Neglect terms of second order in V/c.

b Four-dimensional vectors and tensors

On changing from one inertial frame (S') to another (S) the components of a 4-vector transform according to the formulae

$$A_i = \alpha_{ik} A'_k \,, \tag{10.b.1}$$

where the transformation matrix $\hat{\alpha}$ has the form (remember that the summation rule is such that summation over repeated indices is understood to be taken with a plus sign for the term corresponding to the index 0 and with a minus sign for the terms corresponding to the indices 1, 2, 3)

$$\hat{\alpha} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ \beta\gamma & -\gamma & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{10.b.2}$$

It corresponds to the transformation (10.a.1), in which the corresponding coordinate axes in the frames S and S' are parallel, the relative velocity is along the x-axis, and the origins are the same at $t = t' = 0$.

The transformation matrix satisfies the relations

$$\alpha_{il}\alpha_{kl} = g_{ik} \,, \qquad \alpha_{li}\alpha_{lk} = g_{ik} \,, \tag{10.b.3}$$

where g_{ik} is the metric tensor, which has the form

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{10.b.4}$$

The signs in the main diagonal of the metric tensor correspond to the signs in equation (10.a.5) for the scalar product of two 4-vectors.

The transformation which is the inverse of equation (10.b.1) can be written in the form:

$$A'_i = \alpha_{ki} A_k \,. \tag{10.b.5}$$

The coordinates of a world point $x_0 = ct$, $x_1 = x$, $x_2 = y$, $x_3 = z$ form a 4-vector and transform according to equations (10.b.1) and (10.b.5).

When we perform two successive Lorentz transformations the corresponding matrices multiply according to the normal matrix multiplication rule (see chapter 1, section a).

A four-dimensional tensor (4-tensor) of rank N is a set of 4^N quantities $T_{ik\ldots l}$ which, when we change to another inertial system of reference, transform as the products of the corresponding components of a 4-vector $A_i, A_k, \ldots, A_l$:

$$T_{ik\ldots l} = \alpha_{ip}\alpha_{kr}\ldots\alpha_{ls} T'_{pr\ldots s} \,. \tag{10.b.6}$$

The determinant $|\alpha_{ik}|$of the matrix $\hat{\alpha}$ of the Lorentz transformation may be equal to -1 [proper Lorentz transformation, such as equation

(10.a.1)] or to +1 (improper transformation). Any proper Lorentz transformation can be reduced to a transformation of the form (10.a.1) and a rotation in space; such transformations can be considered to be rotations in four-dimensional space. Improper Lorentz transformations include reflections of one coordinate or of three coordinates.

A pseudotensor of rank N is defined as the set of 4^N quantities $P_{ik...l}$ which transform in accordance with the formulae

$$P_{ik...l} = \alpha_{ip}\alpha_{kr}...\alpha_{ls}|\alpha_{mn}|P_{pr...s} \, . \tag{10.b.7}$$

The skew-symmetric unit pseudotensor of rank 4 (see problem 10.50) is an example of a pseudotensor. Its components e_{iklm} are defined by the following conditions: (a) e_{iklm} changes sign when any pair of subscripts is reversed, and (b) $e_{0123} = 1$. It follows that the components of e_{iklm} will vanish when at least two of the subscripts are the same, or will be equal to ± 1 when the subscripts are all different.

10.45 Prove the relations

$$A_i = g_{ik}A_k \, , \qquad A_iB_i = A_ig_{ik}B_k \, , \qquad g_{ik}g_{kl} = g_{il} \, , \qquad g_{ii} = 4 \, ,$$

where g_{ik} is the metric tensor (10.b.4), while A_i and B_i are four-vectors. When summing over repeated indices one should use the sign rules given after equation (10.a.5).

10.46 Show that the tensor g_{ik} given by equation (10.b.4) has the same form in all inertial frames.

10.47 Show that the components A_1, A_2, A_3 of the four-dimensional tensor $A_i = (A_0, A_1, A_2, A_3)$ transform on space rotation as the components of the three-dimensional vector $\boldsymbol{A} = (A_1, A_2, A_3)$ and also that the component A_0 is a three-dimensional scalar.

10.48 Find the three-dimensional tensors into which a 4-tensor of rank 2 is found to divide as a result of space rotations.

10.49 Show that the components of a skew-symmetric 4-tensor of rank 2 transform on space rotation as the components of two independent three-dimensional vectors.

10.50 Show that the quantity e_{iklm}, which was defined in the introduction to this section, does in fact transform as a pseudotensor.

10.51 Show that

(a) $e_{iklm}e_{lmrs} = 2(g_{is}g_{kr} - g_{ir}g_{ks})$,

(b) $e_{iklm}e_{klmn} = 6g_{in}$,

where e_{iklm} and g_{ik} were defined in the introduction to this section.

10.52 Show that

$$e_{iklm}e_{lmrs}A_iB_kC_rD_s = 2(A_iD_i)(B_kC_k) - 2(A_iC_i)(B_kD_k) \, .$$

10.53 Construct a 4-vector from the partial derivatives $\partial\varphi/\partial x_i$ ($i = 0, 1, 2, 3$), where φ is a scalar. Find an expression for the components of ∇_i, the four-dimensional gradient operator.

10.54 Construct a 4-tensor T_{ik} from the partial derivatives $\partial A_i/\partial x_k$ (i, k = 0, 1, 2, 3), where A_i is a 4-vector. Show that the 4-divergence $\partial A_i/\partial x_i$ is an invariant.

10.55 Find the transformation laws for the quantities:

(a) A_i^2; (b) $T_{ik}A_k$, if A_i is a 4-vector and T_{ik} a 4-tensor.

10.56 Two 4-vectors A_i and B_i are called parallel if

$$\frac{A_0}{B_0} = \frac{A_1}{B_1} = \frac{A_2}{B_2} = \frac{A_3}{B_3}\ .$$

Show that the ratio of the corresponding components of parallel 4-vectors is invariant under a Lorentz transformation.

10.57 Find the number of different components of a 4-tensor of rank 3 which is skew-symmetric with respect to the transposition of any pair of subscripts. Show that the components transform on rotation as the components of a four-dimensional pseudovector.

10.58 Three reference systems S, S', and S'' are such that S'' moves relative to S' with a velocity V' in the direction parallel to the x'-axis and S' moves relative to S with a velocity V in the direction of the x-axis. Corresponding axes of the three systems remain parallel. By multiplying together the corresponding matrices, find the matrix for the transformation from S'' to S. Hence deduce the formula for the addition of parallel velocities [see equation (10.a.11)].

10.59 Write down the Lorentz transformation (10.a.1) in terms of the variables $x_1, x_2, x_3, x_0 = ct$ by expressing the magnitude of the relative velocity V in terms of the angle α given by $\tanh\alpha = V/c$.

10.60 A system S' moves relative to a system S with a velocity V ($\tanh\alpha = V/c$) in the direction defined by the polar angles ϑ, ϕ. The corresponding axes of S and S' remain parallel. Find the matrix $\hat{g}$ for the transformation from S' to S by multiplying together the simple transformation matrices.

c Relativistic electrodynamics

In this section we shall summarise the fundamental formulae of relativistic electrodynamics in vacuo. The three-dimensional current density $\boldsymbol{j} = \rho\boldsymbol{v}$ and the charge density ρ form the current density 4-vector

$$j_i = (c\rho, \boldsymbol{j})\ . \qquad (10.c.1)$$

The electric and magnetic fields are the components of the skew-symmetric electromagnetic-field 4-tensor

$$F_{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix} . \tag{10.c.2}$$

The field components in two systems S and S', whose x- and x'-axes are parallel to the relative velocity of the two systems, transform in accordance with the following formulae

$$\begin{aligned} E_x &= E'_x , \quad & E_y &= \gamma(E'_y + \beta H'_z) , \quad & E_z &= \gamma(E'_z - \beta H'_y) ; \\ H_x &= H'_x , \quad & H_y &= \gamma(H'_y - \beta E'_z) , \quad & H_z &= \gamma(H'_z + \beta E'_y) . \end{aligned} \tag{10.c.3}$$

The quantities

$$H^2 - E^2 = \text{invariant} , \qquad (E \cdot H) = \text{invariant} \tag{10.c.4}$$

are invariant under the Lorentz transformation. The vector and scalar potentials A and φ form the potential 4-vector

$$A_i = (\varphi, A) . \tag{10.c.5}$$

The components of the energy-momentum tensor in a vacuum are defined by

$$T_{ik} = \frac{1}{4\pi}(-F_{il}F_{kl} + \tfrac{1}{4}g_{ik}F_{lm}F_{lm}) . \tag{10.c.6}$$

The nine space components of the tensor T_{ik} form the three-dimensional Maxwell stress tensor

$$T_{\alpha\beta} = \frac{1}{4\pi}(-E_\alpha E_\beta - H_\alpha H_\beta) + \frac{1}{8\pi}(E^2 + H^2)\delta_{\alpha\beta} . \tag{10.c.7}$$

The space-time components of T_{ik} are proportional to the components of the energy flux density S and the field momentum density g:

$$\left.\begin{aligned} T_{0\alpha} &= \frac{1}{c}S_\alpha , \qquad & S &= \frac{c}{4\pi}[E \wedge H] , \\ T_{04} &= cg_\alpha , \qquad & g &= \frac{1}{4\pi c}[E \wedge H] = \frac{1}{c^2}S . \end{aligned}\right\} \tag{10.c.8}$$

The time component of T_{ik} is related to the field energy density w by the formula

$$T_{00} = w = \frac{1}{8\pi}(E^2 + H^2) . \tag{10.c.9}$$

The divergence of the tensor T_{ik} determines the volume density of forces $f_i = [(\boldsymbol{v} \cdot \boldsymbol{f})/c, \boldsymbol{f}]$ acting on the charges:

$$\frac{\partial T_{ik}}{\partial x_k} = f_i = \frac{1}{c} F_{ik} j_k \ . \tag{10.c.10}$$

Let us consider now the formulae of electrodynamics in the presence of media. The field vectors $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$, and $\boldsymbol{H}$ form two skew-symmetric four-dimensional tensors of rank 2, i.e. the field tensor

$$F_{ik} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{10.c.11}$$

and the induction tensor

$$H_{ik} = \begin{pmatrix} 0 & -D_x & -D_y & -D_z \\ D_x & 0 & -H_z & H_y \\ D_y & H_z & 0 & -H_x \\ D_z & -H_y & H_x & 0 \end{pmatrix} . \tag{10.c.12}$$

The polarisation and magnetisation vectors $\boldsymbol{P}$ and $\boldsymbol{M}$ form the following 4-tensor,

$$M_{ik} = \begin{pmatrix} 0 & P_x & P_y & P_z \\ -P_x & 0 & -M_z & M_y \\ -P_y & M_z & 0 & -M_x \\ -P_z & -M_y & M_x & 0 \end{pmatrix} . \tag{10.c.13}$$

The formulae $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$ and $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{M}$ may be joined into the single relation

$$H_{ik} = F_{ik} - 4\pi M_{ik} \ . \tag{10.c.14}$$

The 4-force f_i per unit volume of the medium is given by

$$f_i = \left\{\frac{1}{c}[Q + (\boldsymbol{f} \cdot \boldsymbol{v})], \boldsymbol{f}\right\}, \tag{10.c.15}$$

where $\boldsymbol{f}$ is the pondermotive force per unit volume and Q is the Joule heat liberated per unit volume per unit time.

10.61 Write down the transformation formulae for the field vectors $(\boldsymbol{E}, \boldsymbol{B})$, $(\boldsymbol{D}, \boldsymbol{H})$ and the polarisations $(\boldsymbol{P}, \boldsymbol{M})$ in the case where the S' system moves relative to the S system with an arbitrarily oriented velocity $\boldsymbol{V}$. Express the transformation formulae in a vector form.

Hint. Use the expression for the transformation coefficients in problem 10.60 and the asymmetry of the tensors F_{ik}, H_{ik}, and M_{ik}.

10.62 A uniform electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$ exists in the reference system S. Find the velocity of the system S' relative to S in which $\boldsymbol{E}'$ is parallel to $\boldsymbol{H}'$. Does this problem always have a solution, and if so, is it always unique? Find the absolute values of $\boldsymbol{E}'$ and $\boldsymbol{H}'$.

10.63 The electric and magnetic fields in a reference system S are mutually perpendicular ($\boldsymbol{E} \perp \boldsymbol{H}$). Find the velocity of the system S' relative to S, in which only the electric or only the magnetic field is present. Does this problem always have a solution, and if so, is it unique?

10.64 An infinitely long circular cylinder is uniformly charged with a density κ per unit length. A uniformly distributed current J flows in the direction of the axis of the cylinder. The permittivity and the magnetic permeability are equal to unity in all space. Find the reference system in which there is only an electric or only a magnetic field, and determine the magnitude of these fields.

10.65 The system of differential equations for the magnetic field lines of the form

$$[\mathrm{d}\boldsymbol{r} \wedge \boldsymbol{H}] = 0 \qquad (10.65.1)$$

is not relativistically invariant and does not retain its form when we change to another inertial frame.

(a) Show that for fields of a special form the set of equations

$$[\mathrm{d}\boldsymbol{r} \wedge \boldsymbol{H}] + c\boldsymbol{E}\,\mathrm{d}t = 0\,, \qquad (\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r}) = 0 \qquad (10.65.2)$$

can be considered as a relativistically invariant generalisation of the set (10.65.1).

(b) Elucidate the structure of the fields for which such a generalisation is possible, by considering the conditions of compatibility of equations (10.65.2). How many independent equations does the set (10.65.2) contain?

(c) What is the form of the integrability condition of the set (10.65.2)?

(d) Verify that the field lines determined by the set (10.65.2) move transversely with a velocity $\boldsymbol{u} = c[\boldsymbol{E} \wedge \boldsymbol{H}]/H^2$, i.e. that they move even in the case of static fields.

10.66 Show that the relativistically invariant set of equations for the electrical field lines which is the analogue of the set (10.65.2) of the preceding problem has the form

$$e_{iklm} F_{lm}\,\mathrm{d}x_k = 0\,. \qquad (10.66.1)$$

What are the conditions imposed upon $\boldsymbol{E}$ and $\boldsymbol{H}$ and also upon the charge and current distributions by the conditions that the set (10.66.1) is compatible and integrable? How do the field lines determined by the set (10.66.1) move?

10.67 Find the magnitude of the e.m.f. induced in a conductor moving in a magnetic field $\boldsymbol{B}$, with the aid either of the transformation formulae for the field strength or the transformation formulae for the potentials.

10.68 Find the potentials φ, $\boldsymbol{A}$ and fields $\boldsymbol{E}$, $\boldsymbol{H}$ due to a point charge e moving with a uniform velocity $\boldsymbol{V}$ by carrying out the Lorentz transformation from the reference system in which the charges are at rest.

10.69 Evaluate the total electromagnetic momentum, $\boldsymbol{G}$, of the field produced by a uniformly moving charge (see preceding problem).

10.70 Show that the electric field due to a uniformly moving point charge is 'compressed' in the direction of motion, and that the field E on the line of motion of the charge is reduced as compared with the Coulomb field. Is this reduction in the field consistent with the transformation formula $E_{\parallel} = E'_{\parallel}$?

10.71 An electric dipole having a moment $\boldsymbol{p}_0$ in the reference system in which it is at rest, moves with a uniform velocity $\boldsymbol{V}$. Find the potentials φ, $\boldsymbol{A}$ and the fields $\boldsymbol{E}$, $\boldsymbol{H}$ due to the dipole.

10.72 Derive the transformation formulae for the electric and magnetic dipole moments $\boldsymbol{p}$ and $\boldsymbol{m}$ of a polarised and magnetised body between the inertial system in which the body is at rest and another inertial system.

Hint. Use the transformation formulae for the polarisation and magnetisation vectors $\boldsymbol{P}$ and $\boldsymbol{M}$.

10.73 An uncharged wire loop in the form of a rectangle of sides a and b carries a current J' and moves with a uniform velocity V which is parallel to the side of length a. The wire has a finite cross section. Find the distribution of electric charges in the loop and its electric and magnetic moments in the laboratory system.

10.74 Deduce the relativistic transformation law for the Joule heat Q from the definition of the 4-vector of force.

10.75 Find the transformation formulae for the energy momentum tensor T_{ik} under the Lorentz transformation (10.a.1).

10.76 Find the sum of the diagonal elements (i.e. the trace) of the energy-momentum tensor (10.c.6).

10.77 An electromagnetic field exists only within a finite volume of space V in which there are no charges. Show that the total energy and momentum of the field transform as a 4-vector.

10.78 The total angular momentum of a system consisting of an electromagnetic field in vacuo and a set of point charges may be defined by[3]

$$K_{ik} = -\frac{1}{c}\int_t (x_i T_{kl} - x_k T_{il})\,\mathrm{d}^2 S_l + \sum (x_i p_k - x_k p_i)\,,$$

[3] It can be easily verified using the definition of the tensor T_{ik} that the space part $K_{\alpha\beta}$ of the tensor K_{ik} is a skew-symmetric tensor equivalent to the vector $\boldsymbol{K} = \int [\boldsymbol{r} \wedge \boldsymbol{g}]\,\mathrm{d}^3 r + \sum [\boldsymbol{r} \wedge \boldsymbol{p}]$, where $\boldsymbol{g} = [\boldsymbol{E} \wedge \boldsymbol{H}]/(4\pi c)$ is the field momentum density.

where the integral is taken over the entire hypersurface $x_0 = ct =$ constant. The summation is carried out over all the particles, and the values of x_i, p_k are taken at the points of intersection of the world lines of the corresponding charges and the hypersurface $x_0 =$ constant. Prove the conservation of the total angular momentum K_{ik} of the system, taking into account the fact that $\partial T_{ik}/\partial x_k = F_{ik} j_k / c$.

10.79 A particular system, which consists of particles and an electromagnetic field, occupies a finite evacuated region of space. By considering the balance of the total angular momentum $K_{\alpha\beta}$ of the system, find an expression for the flux density $\boldsymbol{R}$ of the angular momentum of the field. Use the expression for K_{ik} given in the preceding problem.

Relativistic mechanics [1]

a Energy and momentum

The momentum $\boldsymbol{p}$ of a relativistic particle is given by

$$\boldsymbol{p} = \frac{m\boldsymbol{v}}{(1-v^2/c^2)^{1/2}} , \tag{11.a.1}$$

where $\boldsymbol{v}$ is the velocity of the particle and m is its rest mass. The total energy $\mathcal{E}$ of a freely moving particle can be expressed in terms of the velocity:

$$\mathcal{E} = \frac{mc^2}{(1-v^2/c^2)^{1/2}} \tag{11.a.2}$$

or in terms of the momentum:

$$\mathcal{E} = c(p^2+m^2c^2)^{1/2} . \tag{11.a.3}$$

The kinetic energy T of a particle is given by

$$T = \mathcal{E} - mc^2 , \tag{11.a.4}$$

where $\mathcal{E}_0 = mc^2$ is the rest energy. The energy, the momentum, and the velocity of a particle are related by

$$\mathcal{E}\boldsymbol{v} = c^2\boldsymbol{p} . \tag{11.a.5}$$

The energy and the momentum of a particle are the time and space components of the energy-momentum 4-vector (4-momentum)

$$p_i = \left(\frac{\mathcal{E}}{c}, \boldsymbol{p}\right) . \tag{11.a.6}$$

In the transition from one inertial system into another, the momentum transforms in accordance with equation (10.a.4). The square of the 4-momentum is a relativistic invariant:

$$p_i^2 = \frac{\mathcal{E}^2}{c^2} - p^2 = m^2c^2 . \tag{11.a.7}$$

A particle is referred to as nonrelativistic if its kinetic energy is small, and ultrarelativistic if its kinetic energy is very large, as compared with the rest energy. The velocity of an ultrarelativistic particle approaches the velocity of light, and its energy is given by

$$\mathcal{E} = cp . \tag{11.a.8}$$

[1] For details of the theory we refer to the books by Landau and Lifshitz (1975), Fock (1964), Bergmann (1958), Akhiezer and Berestetskii (1965), Pathria (1974), Jackson (1962), Pauli (1958), and Okun (1965).

Particles with zero rest mass and zero rest energy (photons and neutrinos) are always ultrarelativistic and their velocity is exactly equal to the velocity of light c.

The energy and momentum of a photon in vacuo are given by

$$\mathcal{E} = \hbar\omega , \qquad p = \frac{\hbar\omega}{c} = \hbar k , \tag{11.a.9}$$

where ω is the frequency and $\hbar = 1 \cdot 05 \times 10^{-34}$ J s is Planck's constant divided by 2π.

The total energy and momentum of an isolated system of particles is conserved. It follows that if there is no interaction between the particles before and after a particular reaction (disintegration or collision) then the initial total 4-momentum $p_i^{(0)}$ is equal to the final total 4-momentum p_i:

$$\sum_a p_{ai}^{(0)} = \sum_b p_{bi} , \tag{11.a.10}$$

where the summation is carried out over all the particles existing before and after the reaction.

In collision theory it is convenient to use either the laboratory system S, or the centre of mass system S' (CM system) in which the total momentum $\boldsymbol{p}$ is zero. The useful device consisting of making use of the invariance of the squares of 4-momenta should be noted (see solutions to problems 11.34, 11.40, and 11.58).

There are two types of collisions, namely, elastic, in which there is no change in the internal state and therefore no change in the masses of particles, and inelastic, in which there is a change in the internal energy (mass) of the colliding particles, and creation or annihilation of particles may take place. In an inelastic collision between two particles, the total mass m_1+m_2 of the colliding particles differs from the total mass M_k of the particles emerging from the reaction by the amount given by

$$\Delta M = m_1+m_2-M_k , \tag{11.a.11}$$

which is called the mass defect. The quantity $Q = c^2\Delta M$ is known as the Q value of the reaction.

Reactions which proceed according to the scheme

$$a+b \to c+d , \tag{11.a.12}$$

i.e. those in which two particles are transformed into two other particles, are called two-particle reactions—a particular case of a two-particle reaction is the elastic scattering of one particle by another one. It is convenient to describe the kinematics of two-particle reactions by means of the invariant quantities s, t, and u:

$$s = (p_{ai}+p_{bi})^2 , \qquad t = (p_{ai}-p_{ci})^2 , \qquad u = (p_{ai}-p_{di})^2 , \tag{11.a.13}$$

where p_{ai}, ... are the 4-momenta of the particles which take part in the reaction. Any of the quantities s, t, and u can be expressed in terms of the two other ones using the relation[2]

$$s+t+u = (m_a^2+m_b^2+m_c^2+m_d^2)c^2 \,. \tag{11.a.14}$$

The kinematic plane in which the values of the variables s and t (or s, t, and u—see problem 11.56) are plotted gives a graphical representation of the kinematics of a two-particle reaction.

Many formulae of relativistic kinematics become simpler if we use a system of units in which the velocity of light $c = 1$. Masses, energies, and momenta are then measured in the same units, for example, MeV ($1\ \text{MeV} = 10^6\ \text{eV} = 10^{-3}\ \text{GeV} = 1{\cdot}602 \times 10^{-13}\ \text{J}$). In some problems of the present section we use such a system of units. In a number of cases the masses of elementary particles are measured in units of the electron mass m_e, i.e. one uses a system of units in which $m_e = 1$.

Table 11.a.1 gives the rest energy $\mathcal{E}_0$ of a number of elementary particles. Table 11.a.2 gives the binding energies of three light nuclei.

Table 11.a.1.

Particle	Mass	
	in units m_e	in MeV
Photon γ	0	0
Neutrino ν	0	0
Electron e^-, Positron e^+	1	0·511
Muons $\mu^\pm$	207	105·7
Pions $\pi^\pm$	273	139·6
π^0	264	135·0
Kaons $K^\pm$	966	493·8
$K^0, \overline{K}^0$	974	497·8
Proton p	1836	938·2
Neutron n	1839	939·5
Lambda hyperon Λ	2181	1115·4

Table 11.a.2.

Isotopes	B, MeV
^{2}He	2·23
^{4}He	28·11
^{7}Li	38·96

(2) We can, for instance, choose s and t as the two independent quantities. All other quantities, such as the energies and scattering angles in the laboratory frame and in the centre of mass frame, can be expressed in terms of them; see problems 11.51 to 11.53.

The binding energy is defined by

$$B = \Delta M c^2 = \sum \mathcal{E}_{0n} - \mathcal{E}_{0N} , \tag{11.a.15}$$

where $\mathcal{E}_{0n}$ is the rest energy of a nucleon and $\mathcal{E}_{0N}$ is the rest energy of the nucleus.

11.1 Express the momentum p of a relativistic particle in terms of its kinetic energy T.

11.2 Express the velocity $\boldsymbol{v}$ of a particle in terms of its momentum $\boldsymbol{p}$.

11.3 A particle of rest mass m has an energy $\mathcal{E}$. Find the velocity v of the particle. Discuss in particular the nonrelativistic and the ultrarelativistic limits.

11.4 Find approximate expressions for the kinetic energy T of a particle of rest mass m in terms of (a) its velocity v, and (b) its momentum p, to within terms of the order of v^4/c^4 and p^4/m^4c^4 respectively, where $v \ll c$.

11.5 Find the velocity v of a particle of rest mass m and charge e after it has traversed a potential difference V (assume that the initial velocity is zero). Consider the nonrelativistic and the ultrarelativistic situations.

11.6 Find the particle velocity v for: (a) electrons in an electronic valve ($\mathcal{E}$ = 300 eV), (b) electrons in a 300 MeV synchrotron, (c) protons in a 680 MeV synchrocyclotron, and (d) protons in a 10 GeV proton synchrotron.

11.7 The beam current in an accelerator is J and the kinetic energy at the output of the accelerator is T. Find the force F exerted by the beam on a target placed in its path, and the power W dissipated in the target.

11.8 A body moves with a relativistic velocity v through a gas containing N slowly moving particles of rest mass m per unit volume. Find the pressure p exerted by the gas on a surface element of the body which is perpendicular to its velocity.

11.9 In a linear accelerator, the particles are accelerated in the gaps between hollow cylindrical electrodes (the so-called drift tubes), and travel along the common axis of these electrodes. The acceleration is produced by a high-frequency electric field having a frequency ν = constant. Find the lengths of the drift tubes which will ensure that a particle of charge e and rest mass m will traverse all the accelerating gaps at instants of time at which the maximum voltage V_e acts across the gaps. Estimate also the total length of the accelerator with N drift tubes.

11.10 A beam of monochromatic μ-mesons produced in the upper atmosphere is incident vertically on the earth's surface(3). Find the ratio of the intensity of the beam at a height h above sea level to the intensity

(3) Of course, the problem is formulated in a simplified form.

at sea level, by assuming that the reduction in the beam intensity in the particular layer of air of thickness h occurs only as a result of the spontaneous disintegration of the μ-mesons (energy of the μ-mesons $\mathcal{E} = 4 \cdot 2 \times 10^8$ eV, $h = 3$ km, half-life of μ-meson at rest $\tau_0 = 2 \cdot 2 \times 10^{-6}$ s).

11.11 A reference system S' moves with a velocity V relative to a system S. A particle of rest mass m having an energy $\mathcal{E}'$ and velocity $\boldsymbol{v}'$ in S' moves at an angle ϑ' to the direction of V. Find the angle ϑ between the momentum $\boldsymbol{p}$ of the particle and the direction of V in the system S. Express the energy and the momentum of the particle in S in terms of ϑ', $\mathcal{E}'$ or ϑ', v'. Consider in particular the ultrarelativistic case $\mathcal{E}' \gg mc^2$, $V \approx c$. Show that in the ultrarelativistic case there will be a range of angles in which it is possible to use the approximate formula $\vartheta \approx \gamma^{-1} \tan\frac{1}{2}\vartheta$.

11.12 A system S' moves with a velocity V relative to a system S. The angular distribution of particles of equal energy $\mathcal{E}'$ in S' is described by the function $\mathrm{d}W/\mathrm{d}\Omega' = F'(\vartheta', \phi')$ where $\mathrm{d}W$ is the fraction of particles which move within the solid angle $\mathrm{d}\Omega'$ in the system S'. This function is usually normalised so that

$$\int \mathrm{d}W = \int F'(\vartheta', \phi')\, \mathrm{d}\Omega' = 1 ,$$

where the angle ϑ' is measured from the direction of V. Find the angular distribution of such particles in the system S. Consider in particular the ultrarelativistic case.

11.13 The number of particles $\mathrm{d}N$ in a volume element $\mathrm{d}^3\boldsymbol{r}$ with momentum components lying between p_x and $p_x + \mathrm{d}p_x$, p_y and $p_y + \mathrm{d}p_y$, and p_z and $p_z + \mathrm{d}p_z$ can be written in the form

$$\mathrm{d}N = f(\boldsymbol{r}, \boldsymbol{p}, t)\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{p} ,$$

where $\mathrm{d}^3\boldsymbol{p} = \mathrm{d}p_x\, \mathrm{d}p_y\, \mathrm{d}p_z$ is a volume element in momentum space and $f(\boldsymbol{r}, \boldsymbol{p}, t)$ is the distribution function, or particle number density in phase space. Find the way the distribution function $f(\boldsymbol{r}, \boldsymbol{p}, t)$ transforms under a Lorentz transformation.

11.14* Particles of type 1 having a velocity $\boldsymbol{v}_1$ in a system S are scattered by stationary particles of type 2. Find the transformation law for the scattering cross section $\mathrm{d}\sigma_{12}$ to a reference system S' in which particles of type 2 have a velocity $\boldsymbol{v}_2'$ and particles of type 1 a velocity $\boldsymbol{v}_1'$. Consider in particular the case where $\boldsymbol{v}_1'$ and $\boldsymbol{v}_2'$ are parallel.

Hint. The scattering cross section $\mathrm{d}\sigma_{12}$ is defined as the ratio of the number of particles scattered per unit time by a single scattering centre into a solid angle $\mathrm{d}\Omega$, to the intensity of incident particles $I_{12} = n_1 v_0$, where n_1 is the number of incident particles per unit volume and v_0 is the initial relative velocity of particles of type 1 and 2 (cf problem 10.18).

11.15 A π^0-meson moving with a velocity v decays in flight into two γ-rays. Find the angular distribution of the γ-rays, $\mathrm{d}W/\mathrm{d}\Omega$, in the laboratory system, assuming that the distribution in the rest system of the π^0-meson is spherically symmetric.

11.16 Express the energy of the π^0-meson considered in the preceding problem in terms of the ratio f of the number of γ-rays emitted into the forward and backward hemispheres.

11.17 A π^0-meson decays in flight into two γ-quanta. Show that the minimum angle of separation of the γ-quanta, $\vartheta_{\min}$, is given by the condition

$$\cos\tfrac{1}{2}\vartheta_{\min} = \frac{v}{c}$$

in the system of reference in which the pion velocity is v.

11.18* Find the dependence of the energy of the γ-rays appearing as a result of the disintegration of the π^0-meson (cf problem 11.15) on the angle ϑ between the direction of motion of the γ-ray and the direction of motion of the π^0-meson. Determine the energy spectrum of the γ-rays in the laboratory system.

Hint. It follows from the laws of conservation of energy and momentum that in the rest system of the π^0-meson, the energy of the γ-rays is $\mathcal{E}' = \frac{1}{2}mc^2$, where m is the rest mass of the π^0-meson.

11.19 Show that whatever the form of the energy spectrum of π^0-mesons, the energy spectrum of the γ-rays which are produced as a result of the disintegration of these mesons will, in the laboratory system, have a maximum at $\mathcal{E} = \mathcal{E}'$, $\mathcal{E}' = \frac{1}{2}mc^2$, where m is the rest mass of the π^0-meson. Express the rest mass m of the π^0-meson in terms of $\mathcal{E}_1$ and $\mathcal{E}_2$, which are two arbitrary values of the γ-ray energy on either side of the above maximum and correspond to equal values of the distribution function.

Hint. Use the γ-ray energy spectrum deduced in the preceding problem.

11.20 Determine the rest mass m of a particle given that it disintegrates into two particles of rest masses m_1 and m_2. It is known from experiment that the momenta of the two particles are p_1 and p_2 and the angle between their directions of motion is ϑ. Evaluate the mass of the charged pion, which decays into a muon and a neutrino, if experiment shows that the pion before the decay was at rest while the muon momentum after the decay was $29 \cdot 8$ MeV c^{-1}. The muon mass is given in table 11.a.1.

11.21 Determine the rest mass m_1 of a particle given that it is one of two particles produced as a result of the disintegration of a particle having a rest mass m and momentum p. Assume that the momentum p_2, the rest mass m_2, and the direction of motion ϑ_2 of the second particle produced in the disintegration are also known.

11.22 A particle of rest mass m_1 and velocity v collides with a stationary particle of rest mass m_2 and is absorbed by it. Find the rest mass m and the velocity V of the resultant particle.

11.23 A pion (rest mass m_π) at rest in the laboratory frame disintegrates into a muon (rest mass m_μ) and a neutrino. Find the momentum p_μ of the muon.

The muon has a mean lifetime τ, if at rest. Find the mean distance d between the points where the pion (at rest) and the muon disintegrate.

11.24 A body at rest of mass m_0 disintegrates into two parts of masses m_1 and m_2. Evaluate the kinetic energies T_1 and T_2 of the disintegration products. Find the way the disintegration energy is distributed in the rest frame of the disintegrating particle between (a) the α-particle and the daughter nucleus in the α-decay of ^{238}U; (b) the muon and the neutrino in pion decay ($\pi \to \mu+\nu$); (c) the γ-quantum and the recoil of the nucleus when a γ-quantum is emitted.

11.25 A particle of rest mass m splits into two particles of rest masses m_1 and m_2, moving with velocities v_1 and v_2 with respect to the frame in which the original particle was at rest. Derive a relation between m_1, m, v_1 and v_2.

A rocket of rest mass m_0 starts from rest relative to a frame S; at a later time it has a velocity v with respect to S and rest mass m. It is propagated by ejecting mass in a fixed direction with a constant velocity w relative to itself. Find its velocity as a function of m.

11.26 If in the case of the rocket considered in the preceding problem the mass is ejected at such a rate that the acceleration of the rocket, relative to a frame with respect to which it is instantaneously at rest, is a constant, f, find the mass of the rocket as a function of time. The mass is once again ejected at a constant velocity w relative to the rocket.

11.27 A particle at rest decays according to the reaction $a \to b+d$. Express the decay energy $Q_a = m_a - m_b - m_d$ $(c = 1)$ in terms of the kinetic energy T_b of one of the particles formed in the decay and the masses m_b and m_d. Evaluate the decay energy and the mass of the Σ^+ particle which decays as $\Sigma^+ \to n+\pi^+$, by making use of the experimental value $T_{\pi^+} = 91{\cdot}7$ MeV and the neutron and π^+ masses given in table 11.a.1. Do the same for the alternative decay of the Σ^+ as $\Sigma^+ \to p+\pi^0$, if in that case $T_p = 18{\cdot}8$ MeV.

11.28 A free excited nucleus at rest emits a γ-quantum; its excitation energy was $\Delta\mathcal{E}$. Find the frequency ω of the γ-quantum. The mass of the excited nucleus was m. What is the reason why $\omega \neq \Delta\mathcal{E}/\hbar$? How does the result change if the nucleus is rigidly attached to a crystalline lattice (Mössbauer effect)?

11.29* A particle a of mass m at rest decays as follows: $a \to a_1 + a_2 + a_3$ into three particles of masses m_1, m_2, m_3 with kinetic energies T_1, T_2, T_3. Study the kinematics of such a decay using a Dalitz plot. To do this, introduce the variables $x = (T_2 - T_3)/\sqrt{3}$, $y = T_1$ and consider the xy-plane. Each actual decay corresponds to a well-defined point in that plane.

(a) Show that the energy conservation law delimits a region in the x, y-plane which has the form of an equilateral triangle. Verify that the lengths of the perpendiculars drawn from the point that depicts the given decay onto the sides of the triangle are equal to the kinetic energies of the particles produced in the decay.

(b) Check that the two quantities x and y that we have introduced are sufficient to determine the magnitudes of the momenta of the particles produced in the decay as well as the angles between the momenta in the rest frame of the decaying particle.

(c) The conservation law for the three-dimensional momentum leads to the fact that not all points inside the triangle correspond to true decays. Find the region in the xy-plane inside which decays are kinematically possible for the particular case $m_2 = m_3 = 0$, $m_1 \neq 0$.

11.30 Construct the Dalitz plot (see preceding problem) for muon and kaon decays:

(a) $\mu^\pm \to e^\pm + 2\nu$; (b) $K^\pm \to \pi^0 + e^\pm + \nu$.

In the last process the electron is, as a rule, produced ultrarelativistically, so that one can neglect its rest mass. Determine the maximum energies of the particles.

11.31 Construct the Dalitz plot (see problem 11.29) for the following decay of a positive kaon:

$$K^+ \to \pi^- + \pi^+ + \pi^+ .$$

The decay energy $Q = m_K - 3m_\pi \approx 75$ MeV $< m_\pi$ $(c = 1)$ so that the pions which are produced can be approximately taken to be nonrelativistic. What is the maximum energy of each of the particles?

11.32 Construct the Dalitz plot (see problem 11.29) for the ω decay:

$$\omega \to \pi^+ + \pi^- + \pi^0 .$$

Assume the masses of the three pions to be the same; the decay energy is $Q = m_\omega - 3m_\pi \approx 360$ MeV, $m_\omega \approx 780$ MeV $(c = 1)$. What is the largest energy of each of the pions?

11.33* In problem 11.29 we gave the rules for the construction of a Dalitz plot for the decay into three particles. The probability $\mathrm{d}W$ for the decay has the form

$$\mathrm{d}W = \rho\, \mathrm{d}\Gamma .$$

Here ρ is a quantity which depends on the interaction forces leading to the decay and on the momenta of the particles, while $d\Gamma$ is a volume element in phase space Γ, which is determined by the integral

$$\Gamma = \int \frac{d^3\boldsymbol{p}_1}{\mathscr{E}_1}\frac{d^3\boldsymbol{p}_2}{\mathscr{E}_2}\frac{d^3\boldsymbol{p}_3}{\mathscr{E}_3}\delta(p_i - p_{1i} - p_{2i} - p_{3i}) ,$$

where p_i is the 4-momentum of the decaying particle [$p_i = (m, 0)$ if the particle is at rest when decaying], $p_{\alpha i} = (\mathscr{E}_\alpha, \boldsymbol{p}_\alpha)$ $(\alpha = 1, 2, 3)$ are the 4-momenta of the particles formed in the decay, and $d^3\boldsymbol{p}_\alpha$ is a volume element of the momentum space of the αth particle. The four-dimensional δ-function expresses the conservation law of 4-momentum in the decay and shows that the integration is only over those values of the momenta $\boldsymbol{p}_1$, $\boldsymbol{p}_2$, and $\boldsymbol{p}_3$ which are compatible with the energy and momentum conservation laws.

Express $d\Gamma$ in terms of dx and dy and show that the phase volume Γ can be expressed in terms of the area of the allowed region in the Dalitz plot, when we use appropriate units. Give the proof for the general case where all masses are different and nonvanishing.

11.34* A particle of rest mass m collides with a stationary particle of rest mass m_1. The collision results in the production of a number of particles with a total rest mass M. If $m + m_1 < M$, then at low kinetic energies of the incident particle the reaction will not proceed, because it would violate the law of conservation of energy. Determine the minimum kinetic energy of the incident particle at which the reaction becomes energetically possible, i.e. the threshold kinetic energy T_0 for the reaction.

11.35 Find the energy thresholds T_0 for the following reactions:
(a) π-meson production in nucleon–nucleon scattering ($N + N \rightarrow N + N + \pi$),
(b) photoproduction of π-mesons on nucleons ($N + \gamma \rightarrow N + \pi$),
(c) production of a K-meson and a Λ-hyperon ($\pi + N \rightarrow \Lambda + K$), and
(d) proton–antiproton pair production in the scattering of a proton of rest mass m_p by a nucleus of rest mass m. Consider in particular the case of proton–proton scattering. Estimate the threshold for the production of an antiproton in the collision between a proton and a nucleus of mass number A, assuming that $m \approx m_p A$.

11.36 Find an approximate expression for the energy threshold T_0 of reactions in which the change in the rest mass ΔM of the colliding particles is a small fraction of their total rest mass M (this situation occurs in the case of a collision between two nonrelativistic particles). Use this formula to determine the energy threshold T_0 of the following reactions: (a) photodisintegration of the deuteron ($\gamma + {}^2_1\mathrm{H} \rightarrow p + n$) and (b) the reaction ${}^4_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^7_3\mathrm{Li} + p$. Compare the approximate expressions with the exact expressions (see problem 11.34).

11.37 Show that the electron–positron pair production by γ-rays is only possible if a particle of finite rest mass participates in the reaction. (The intrinsic state of the particle remains unchanged in the reaction, its sole purpose being to take up a fraction of the energy and momentum, so that the corresponding conservation laws may be satisfied.) Find the threshold T_0 for the pair-production reaction.

11.38 Show that the annihilation of an electron–positron pair with the emission of a single γ-ray is forbidden by the conservation of energy-momentum, but the annihilation of a pair with the emission of two photons is not forbidden.

11.39 A particle of energy $\mathcal{E}$ and rest mass m_1 moves towards a stationary particle of rest mass m_2. Find the velocity v of the centre of mass relative to the laboratory system.

11.40* A particle of rest mass m_1 and energy $\mathcal{E}_0$ undergoes an elastic collision with a stationary particle (rest mass m_2). Express the angles of scattering ϑ_1, ϑ_2 of the particles in the laboratory system in terms of their energies $\mathcal{E}_1$, $\mathcal{E}_2$ after the collision.

11.41 Use the solution of the preceding problem to express the energy of the particles undergoing the elastic collision in terms of the scattering angles in the laboratory system.

11.42 An ultrarelativistic particle of mass m and energy $\mathcal{E}_0$ is elastically scattered by a fixed nucleus of mass $M \gg m$. Determine how the final energy $\mathcal{E}$ of the particle depends on the scattering angle ϑ.

11.43 Solve the preceding problem for the case of inelastic scattering of a particle by a nucleus. The excitation energy $\Delta\mathcal{E}$ of the nucleus in its rest frame satisfies the inequality $mc^2 \ll \Delta\mathcal{E} \ll Mc^2$.

11.44 A particle of rest mass m is elastically scattered by a stationary particle of equal rest mass. Express the kinetic energy T_1 of the scattered particle in terms of the kinetic energy T_0 of the incident particle and the scattering angle ϑ_1.

11.45 Use the result of problem 11.41 to find (in the nonrelativistic approximation) the dependence of the final kinetic energies T_1 and T_2 of the particles undergoing an elastic collision on the initial kinetic energy T_0 of the incident particle, and the scattering angles ϑ_1, ϑ_2 in the laboratory system. Assume that the target particle was stationary prior to the collision.

11.46 Particles of rest mass m_1 and m_2 undergo an elastic collision. Their velocities in the centre of mass system are v_1' and v_2', the scattering angle is ϑ', and the velocity of the centre of mass system relative to the laboratory system is V. Determine the angle χ between the directions of motion of the particles after the collision in the laboratory system. Consider in particular the case $m_1 = m_2$.

11.47 A photon of frequency ω_0 is scattered by a uniformly moving free electron. The momentum of the electron $\boldsymbol{p}_0$ is at an angle ϑ_0 to the direction of motion of the photon. Find the frequency ω of the scattered photon as a function of its direction of motion. Consider in particular the case where the electron was originally at rest (Compton effect).

11.48 A photon of energy $\hbar\omega_0$ is scattered by an ultrarelativistic electron of mass m and energy $\mathcal{E}_0 \gg \hbar\omega_0$. Find the maximum energy $\hbar\omega$ of the scattered photon.

11.49 Find the change in the energy of an electron when it collides with a photon, if the initial energy of the electron is $\mathcal{E}_0$ and that of the photon is $\hbar\omega_0$, and the angle between their momenta ϑ. Discuss the result. Under what conditions will electrons be accelerated owing to collisions with photons?

11.50 Express the invariant variables s, t, and u given by equation (11.a.13) in terms of the mass m, the absolute magnitude q of the momentum, and the scattering angle ϑ in the centre-of-mass frame for the case of elastic scattering of identical particles.

11.51 Let the particle b be at rest in the laboratory frame. Express the energy $\mathcal{E}_a$ of the particle a in the laboratory frame and also the energies $\mathcal{E}_a'$ and $\mathcal{E}_b'$ in the centre-of-mass frame in terms of the invariant variable s [see equation (11.a.13)]. Do the same for the absolute magnitudes of the three-dimensional momenta p_a and p' ($p_a' = p_b' = p'$). Use units in which the velocity of light $c = 1$.

11.52 Express the energies in the laboratory frame, $\mathcal{E}_c$ and $\mathcal{E}_d$, of the particles which are produced in a two-particle reaction in terms of the invariant variables given by equation (11.a.13).

11.53 Express the angle θ between the three-dimensional momenta $\boldsymbol{p}_a$ and $\boldsymbol{p}_c$ in the laboratory frame for the case of a two-particle reaction in terms of the invariant variables s, t, u given by equation (11.a.13). Express the angle θ' between the momenta $\boldsymbol{p}_a'$ and $\boldsymbol{p}_c'$ in terms of the same variables.

11.54 Construct the region of allowable values of the variables s and t [see equation (11.a.13)] for the reaction $\gamma + \mathrm{p} \rightarrow \pi^0 + \mathrm{p}$ (photo-production of a π^0-meson using a proton). What point of that region corresponds to the threshold of the reaction? What is the threshold value T_0 of the γ-quantum energy in the laboratory frame? What is the value of the kinetic energy T_π of the π^0-meson in the laboratory frame at threshold?

11.55 Two γ-quanta change into an electron–positron pair. The energy of one of them is given and is equal to $\mathcal{E}_0$. For what values $\mathcal{E}_2$ of the energy of the second quantum and for what value of the angle ϑ between their momenta is this reaction possible? Plot these values in an $\mathcal{E}_2$, $\cos\vartheta$-plane. Find also the region of allowable values of the variables s and t given by equation (11.a.13). Use as energy unit mc^2, where m is the electron mass.

11.56 Construct in the kinematic s, t-plane [see equation (11.a.13)] the physical regions corresponding to the following three processes:
(a) $\pi^+ + p \to \pi^+ + p$: elastic scattering;
(b) $\pi^- + \bar{p} \to \pi^- + \bar{p}$: elastic scattering of antiparticles;
(c) $\pi^+ + \pi^- \to p + \bar{p}$: proton–antiproton pair production.
The masses of all pions are m and of all nucleons M.

11.57 Use the law of conservation of energy-momentum to show that the emission and absorption of light by a free electron in a vacuum is impossible.

11.58* Show that the uniform motion of a free charged particle of rest mass m, charge e, and velocity $\boldsymbol{v}$ in a medium with a refractive index $n(\omega)$ may be accompanied by the emission of electromagnetic waves (Cherenkov effect[4]). Express the angle ϑ between the direction of propagation of the wave and the velocity $\boldsymbol{v}$ in terms of v, ω, and $n(\omega)$ (cf problem 13.2).

Hint. The energy and momentum of a photon in a medium with a refractive index $n(\omega)$ is $\mathscr{E} = \hbar\omega$ and $p = n(\omega)\hbar\omega/c$.

11.59 Show that a free electron moving with a velocity v in a medium, can absorb electromagnetic waves of frequency ω provided $v > c/n(\omega)$, where $n(\omega)$ is the refractive index.

11.60* A particle which, in general, may have a complicated structure and may contain a number of electric charges (for example an atom), moves with a uniform velocity $\boldsymbol{v}$ through a medium with a refractive index $n(\omega)$. The particle is at first in an excited state, and then emits a photon of frequency ω_0 (in the rest system) when it undergoes a transition to the ground state. The photon is observed in the laboratory system at an angle ϑ to the direction of motion of the particle. Find the frequency ω of the photon in the laboratory system (this is the Doppler effect in a refracting medium). Consider in particular the case $\omega_0 \to 0$.

Hint. Neglect terms which are of the second order in $\hbar$ and assume that $\hbar\omega_0 \ll mc^2$, where m is the rest mass of the particle.

11.61* The particle considered in the preceding problem moves with a uniform velocity through a medium. The particle is in the ground state and the remaining conditions are the same as those in the preceding problem. Show that the emission of radiation may still occur, accompanied by the excitation of the particle. Elucidate the conditions necessary for this behaviour, and find the frequency ω of the emitted radiation.

11.62 It follows from the energy and momentum conservation laws that Cherenkov emission of a single quantum of frequency ω is impossible if the refractive index $n(\omega) \leqslant 1$ (see problem 11.59). In particular, single-quantum Cherenkov emission of sufficiently hard photons is impossible,

[4] An analogous effect may occur for a neutral particle which has a nonvanishing electric or magnetic moment.

since for high frequencies $n(\omega) < 1$. Show that when a fast charged particle of energy $\mathcal{E}_0$ moves uniformly through a medium, two photons may be emitted simultaneously, one of which, with frequency ω_2, may be hard so that $n(\omega_2) \to 1$. Make clear the conditions which the frequency ω_1 of the other photon and the particle velocity v_0 ($\hbar\omega_1 \ll cp_0$) must satisfy in order that such a process is possible (hard Cherenkov emission). What is the highest frequency of the hard quantum?

11.63 Consider the kinematics of the hard Cherenkov emission (see preceding problem), assuming the electron to be ultrarelativistic, $\mathcal{E}_0 \gg mc^2$, and the angle ϑ_2 at which the hard quantum flies off to be small. Determine the maximum value of the energy $(\hbar\omega_2)_{\max}$ of the hard quantum which can be reached in this case; consider characteristic particular cases.

11.64 A crystalline lattice can take up momentum only in discrete amounts $q = 2\pi\hbar g$, where g is a reciprocal lattice vector. In the case of a crystalline lattice with an elementary cell in the form of a rectangular parallelepiped with edgelengths a_1, a_2, a_3, the vectors g are $g = (n_1/a_1, n_2/a_2, n_3/a_3)$, where n_1, n_2, and n_3 are arbitrary integers. Assuming that the crystal, which has a very large mass, cannot take up any energy from a particle, elucidate the nature of the angular distribution of particles scattered by a single crystal.

11.65 Using the relation $p_0 = 2\pi\hbar/\lambda_0$ between the momentum p_0 of the particle and the corresponding wavelength λ_0 derive the Bragg condition: $2a\sin\frac{1}{2}\vartheta = n\lambda_0$, where a is the distance between crystal planes and ϑ the scattering angle of the particles.

11.66 Elucidate the nature of the energy spectrum of bremsstrahlung quanta which occur when charged particles are scattered by a single crystal (cf problem 11.64). The angle between the direction of propagation of a bremsstrahlung quantum and the initial momentum of the particle is fixed and small, $\vartheta \ll 1$. The particle is ultrarelativistic, $\mathcal{E}_0 \gg mc^2$.

11.67 To an observer at the origin of a frame S, the rays from a distant star appear to make an angle ψ with the observer's positive x-axis. A second observer is moving with a velocity v along the x-axis. Find the frequency ν' measured by the second observer, if the frequency measured by the first one is ν.

Find the angle ψ' made by the rays from the star with the x-axis, as reckoned by the second observer.

b The motion of charged particles in an electromagnetic field

The force acting on a particle of charge e, which is moving with a velocity $\boldsymbol{v}$ in an electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$, is given by the Lorentz formula

$$\boldsymbol{F} = e\boldsymbol{E} + \frac{e}{c}[\boldsymbol{v} \wedge \boldsymbol{H}] . \qquad (11.b.1)$$

The change in the kinetic energy of the particle per unit time is given by

$$(\boldsymbol{F}\cdot\boldsymbol{v}) = e(\boldsymbol{E}\cdot\boldsymbol{v}) = \dot{\mathcal{E}} = \frac{d\mathcal{E}}{dt}\,, \tag{11.b.2}$$

where $\mathcal{E}$ is the energy of the particle (see section a).

The magnetic field does no work on the particle, since the force associated with it is always perpendicular to the particle velocity. The quantities $\boldsymbol{F}$ and $d\mathcal{E}/dt$ may be used to construct the following 4-vector (the Minkowski force)

$$F_i = \left[\frac{(\boldsymbol{F}\cdot\boldsymbol{v})}{c(1-v^2/c^2)^{1/2}}\,,\ \frac{\boldsymbol{F}}{(1-v^2/c^2)^{1/2}}\right]. \tag{11.b.3}$$

The 4-force may be expressed in terms of the electromagnetic field tensor: $F_i = (e/c)F_{ik}u_k$, where u_k is the 4-velocity of the particle and F_{ik} is the field tensor.

The four-dimensional form of the differential equation of motion of a particle is

$$\frac{dp_i}{d\tau} = eF_i \quad \text{or} \quad m\frac{du_i}{d\tau} = \frac{e}{c}F_{ik}u_k\,. \tag{11.b.4}$$

On resolving these equations along the space and the time axes, we obtain the equations of motion in the three-dimensional form, and the law of conservation of energy:

$$\dot{\boldsymbol{p}} = e\boldsymbol{E}+\frac{e}{c}[\boldsymbol{v}\wedge\boldsymbol{H}]\,, \qquad \dot{T} = e(\boldsymbol{v}\cdot\boldsymbol{E})\,, \tag{11.b.5}$$

where $T = \mathcal{E}-mc^2$ is the kinetic energy of the particle, $\boldsymbol{p}$ is its momentum, m is the rest mass, and the dot over the symbols indicates differentiation with respect to time t. The equations given by (11.b.5) hold for any velocity of the particle.

The relativistic Lagrange function for a charged particle in an electromagnetic field having a potential φ, $\boldsymbol{A}$ is of the form

$$L = -mc^2\left(1-\frac{v^2}{c^2}\right)^{1/2} - U\,. \tag{11.b.6}$$

In the nonrelativistic case we have

$$L = \tfrac{1}{2}mv^2 - U\,. \tag{11.b.7}$$

Here

$$U = -\frac{e}{c}(\boldsymbol{A}\cdot\boldsymbol{v})+e\varphi\,. \tag{11.b.8}$$

The quantity U plays the role of a potential energy of interaction between the particle and the external field. The equations of motion of the particle may then be written in the Lagrangian form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 , \tag{11.b.9}$$

where q_i, $\dot{q}_i$ are the generalised coordinates and velocities.

The current associated with the orbital motion of a point charged particle round a given centre may be characterised by the magnetic moment[(5)]

$$\boldsymbol{m} = \kappa \boldsymbol{l} , \tag{11.b.10}$$

where $\kappa = e/2mc$ is the gyromagnetic ratio, m is the mass of the particle, and $\boldsymbol{l} = [\boldsymbol{r} \wedge m\boldsymbol{v}]$ is the angular momentum. The particle will experience a couple $\boldsymbol{N} = [\boldsymbol{m} \wedge \boldsymbol{H}]$ in an external magnetic field $\boldsymbol{H}$, and this will produce a change in the angular momentum $\boldsymbol{l}$ which is given by $\mathrm{d}\boldsymbol{l}/\mathrm{d}t = \boldsymbol{N}$. It follows that the rate of change of the magnetic moment is given by

$$\frac{\mathrm{d}\boldsymbol{m}}{\mathrm{d}t} = \kappa[\boldsymbol{m} \wedge \boldsymbol{H}] . \tag{11.b.11}$$

In addition to the angular momentum and the magnetic moment associated with orbital motion, microparticles also exhibit an intrinsic (spin) angular momentum $\boldsymbol{s}$ and a magnetic moment $\boldsymbol{m}_0$ which are either parallel or antiparallel and are related by

$$\boldsymbol{m}_0 = \kappa_0 \boldsymbol{s} . \tag{11.b.12}$$

For an electron $\kappa_0 = e/mc < 0$, where e is the charge on the electron and m its mass. The rate of change of $\boldsymbol{m}_0$ is given by an equation similar in form to equation (11.b.11), except that κ is replaced by κ_0 and $\boldsymbol{m}$ by $\boldsymbol{m}_0$.

The neutron is not electrically charged but does, nevertheless, possess a spin magnetic moment $\boldsymbol{m}_0$. According to quantum theory, the direction of this moment in an external magnetic field $\boldsymbol{H}(\boldsymbol{r})$ can only be parallel or antiparallel to the field, and the initial orientation of the moment is conserved under certain conditions[(6)]. The motion of neutrons with magnetic moments parallel (or antiparallel) to the field can then be looked upon as the motion of classical particles in a force field with a potential energy

$$U = \mp m_0 H , \tag{11.b.13}$$

[(5)] The classical theory described below can only be applied to microparticles with certain reservations. A rigorous theory of motion of elementary magnetic moments can only be developed on the basis of the quantum theory. We use here κ for the gyromagnetic ratio, since γ is used for $(1-v^2/c^2)^{-1/2}$. There should be no confusion with the κ used in problems 11.72 and 11.73.

[(6)] This will be true if the angle of rotation of the field per unit time in the system in which the neutron is at rest is small compared with the precession frequency $\omega_{\mathrm{L}} = 2m_0H/\hbar$ of the magnetic moment $\boldsymbol{m}_0$ in the field $\boldsymbol{H}$ (adiabatic approximation).

where $H = |H(r)|$. The energy U is usually very small and hence a magnetic field will only have an appreciable effect on the motion of very slow (cold) neutrons.

11.68 Write down the relativistic equation of motion for a particle acted upon by a force $\boldsymbol{F}$ by expressing the momentum explicitly in terms of the velocity $\boldsymbol{v}$. Consider in particular the following cases: (a) the direction of the velocity remains constant, (b) the magnitude of the velocity remains constant, and (c) $v \ll c$.

11.69 Derive the expression relating the force $\boldsymbol{F}$ on a particle in the laboratory system and the force $\boldsymbol{F}'$ in the rest system. Assume that the velocity of the particle $\boldsymbol{v}$ is given.

11.70 A body of rest mass m is stationary relative to a rocket which moves with a relativistic velocity v in a circular orbit of radius R. Find the force F on the body as measured by an observer in the frame in which the body is instantaneously at rest.

11.71 Two charges e and e' move parallel to the x-axis with equal constant velocities $\boldsymbol{v}$. Using the results of problem 10.68, show that the electromagnetic force between the charges may be derived from the so-called convection potential[7] $\psi = (1-\beta^2)e/R$ where

$$R = \{(x_1-x_2)^2+(1-\beta^2)[(y_1-y_2)^2+(z_1-z_2)^2]\}^{1/2},$$

$\boldsymbol{r}_1, \boldsymbol{r}_2$ are the position vectors of the two charges, and the force $\boldsymbol{F}$ is given by $\boldsymbol{F} = -e' \operatorname{grad} \psi$. What will happen when $v \to c$?

11.72 Find the convection potential ψ for an infinitely long, uniformly charged, straight conductor. The linear density of charge in the frame in which the conductor is at rest is κ. The conductor moves without rotation at an angle α to its axis with a velocity v (in the laboratory frame). Consider in particular $\alpha = 0$ and $\alpha = \frac{1}{2}\pi$.

11.73 An infinitely long, uniformly charged, straight line carrying a charge κ per unit length in the frame in which it is at rest moves with a velocity v in the direction parallel to itself. A point charge at a distance r from the line moves with an equal velocity in the direction parallel to the line. Find the electromagnetic force F on the point charge for an arbitrary velocity v.

11.74 The distribution of electrons in a parallel beam is axially symmetric and is characterised by a volume charge density ρ in the reference frame in which the electrons are at rest. The electrons are accelerated by a

[7] The convection potential of a system of charges moving as a whole is defined as that function of the coordinates which, when differentiated, yields the components of the Lorentz force in the laboratory system per unit test charge moving together with the system of charges.

potential difference V and the total beam current is J. Find the electromagnetic force F on each of the electrons in the laboratory frame.

Hint. Use the result of the preceding problem.

11.75 Find the spread Δa over a path length L of the electron beam considered in the preceding problem, assuming that it is due to the mutual repulsion between the electrons. Assume that the broadening is small ($\Delta a \ll L$) and that the cross section of the beam is circular with a radius a.

11.76* A particle of charge e and rest mass m moves with an arbitrary velocity in a uniform constant electric field $\boldsymbol{E}$. At the initial instant of time, $t = 0$, the particle was at the origin and had a momentum $\boldsymbol{p}_0$. Determine the three-dimensional coordinates and the time t of the particle in the laboratory system as functions of its proper time τ. By eliminating τ express the three-dimensional coordinates as functions of t (the problem can also be solved directly by integrating the equations of motion in the three-dimensional form). Consider in particular the nonrelativistic and the ultrarelativistic limits.

11.77 Find the trajectory of a charged particle of charge e and rest mass m in a uniform constant electric field $\boldsymbol{E}$, by using the results of the preceding problem. Consider in particular the nonrelativistic limit.

11.78 Find the range l of a relativistic charged particle of charge e, rest mass m, and potential energy $\mathcal{E}$ in a retarding uniform electric field $\boldsymbol{E}$ which is parallel to the initial velocity of the particle.

11.79* A relativistic particle of charge e and rest mass m moves in a constant uniform magnetic field $\boldsymbol{H}$. At the initial instant of time, $t = 0$, the particle was at the point with position vector $\boldsymbol{r}_0$ and had a momentum $\boldsymbol{p}_0$. Determine the law of motion of the particle.

11.80* A nonrelativistic particle of charge e and rest mass m moves in constant crossed electric and magnetic fields $\boldsymbol{E} = (0, E_y, E_z)$ and $\boldsymbol{H} = (0, 0, H)$ respectively. At the initial instant of time, $t = 0$, the particle was at the origin and had a velocity $\boldsymbol{v} = (v_{0x}, 0, v_{0z})$. Determine the functions $x(t)$, $y(t)$, and $z(t)$ and sketch out the possible trajectories of the particle.

Hint. The substitution $u = x + iy$ will simplify the integration.

11.81 A relativistic particle moves in parallel uniform constant electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ ($\boldsymbol{E} \parallel \boldsymbol{H} \parallel z$). At $t = 0$ the particle was at the origin and had a momentum $\boldsymbol{p}_0 = (p_{0x}, 0, p_{0z})$. Determine the quantities x, y, z, t as functions of the proper time τ of the particle.

11.82 Determine the law of motion of a particle in mutually perpendicular uniform constant electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{H}$. Use two methods, namely, (a) the Lorentz formula, with the assumption that the motion of a particle in a purely electric or purely magnetic field is known (see problems 11.76 and 11.79), and (b) integrate equation (11.b.8).

11.83 Find the kinetic energy T of a particle as a function of the proper time τ for the types of motion considered in problems 11.76, 11.81, and 11.82.

11.84 A particle having a small initial velocity v_0 ($v_0 \ll c$) moves in crossed constant uniform electric and magnetic fields $\boldsymbol{E} = (0, E_y, E_z)$, $\boldsymbol{H} = (0, 0, H)$, $E \ll H$. Determine the law of motion of the particle by using the Lorentz transformation and assuming that its motion in parallel electric and magnetic fields is known (see problem 11.81). Use the results of problem 10.61 and compare with the result of problem 11.80.

11.85 Determine the law of motion of a particle of charge e and rest mass m in the field of a plane electromagnetic wave $\boldsymbol{E}(t')$, $\boldsymbol{H}(t')$, where $t' = t-(\boldsymbol{n}\cdot\boldsymbol{r})/c$ and $\boldsymbol{n}$ is a unit vector in the direction of propagation. Assume that at $t = 0$ the particle was at rest at the origin.

Hint. Note that the proper time τ of the particle is the same as the argument t' of the plane wave.

11.86 A particle of charge e and mass m is moving in the following field of an electromagnetic wave:

$$\boldsymbol{E} = \boldsymbol{e}_x a\cos\theta\ , \qquad \boldsymbol{B} = \boldsymbol{e}_y \frac{a}{c}\cos\theta\ ,$$

where $\theta = k(z-ct)$. Find the orbit of the particle, given that at $t = 0$ it was at rest at the origin.

11.87 A nonrelativistic charged particle of charge e and mass m passes through a two-dimensional electrostatic field the potential of which is $\varphi = k(x^2-y^2)$ where $k = \text{constant} > 0$ (i.e. a strong focusing lens). Assume that at $t = 0$ the particle was at the point (x_0, y_0, z_0) and the initial velocity v_0 was parallel to the z-axis. Determine the motion of the particle.

11.88 Find the differential equation of motion for a relativistic particle in an electromagnetic field, starting from the Lagrange function in terms of cylindrical coordinates.

Hint. In evaluating the time derivative in the Lagrange equations, note that the derivative is taken along the trajectories of the particles, so that r, ϕ, z should be looked upon as functions of time.

11.89* A potential difference V is maintained between the plates of a cylindrical capacitor whose radii are a and b ($a < b$). An axially symmetric magnetic field, which is parallel to the axis of the capacitor, is set up between the plates. The inner plate (cathode) emits electrons with zero initial velocity. Find the critical magnetic flux Φ_{cr} at which the electrons will no longer reach the anode.

11.90 A long straight cylindrical cathode of radius a carries a uniformly distributed current J and emits electrons with zero initial velocity. The cathode is surrounded by a long coaxial anode of radius b which is maintained at a potential V relative to the cathode. Find the minimum potential difference between the cathode and the anode which will ensure that the electrons will reach the anode in spite of the effect of the magnetic field due to the current J.

11.91 An infinitely long straight cylindrical conductor of radius a carries a current J. The surface of the conductor emits electrons whose initial velocity v_0 is parallel to the axis of the conductor. Find the maximum distance b from the axis of the conductor which the electrons can reach.

11.92 Solve problem 11.90 by using the Lorentz transformation from the frame in which there is only one field ($\boldsymbol{E}$ or $\boldsymbol{H}$).

Hint. Use the results of problems 10.64 and 11.91.

11.93* A relativistic particle of charge $-e$ and rest mass m moves in the field of a stationary point charge Ze. Find the equation of the trajectory of the particle. Investigate the possible trajectories when the angular momentum $K > Ze^2/c$.

Hint. Use the law of conservation of energy and the equations obtained in problem 11.88.

11.94 Investigate the possible trajectories of the particle considered in the preceding problem if $K \leqslant Ze^2/c$.

11.95* A relativistic particle of charge e and rest mass m moves in the field of a heavy point charge Ze. Find the trajectory of the particle and discuss the solution.

11.96 Show that when a particle moves in an attractive Coulomb field (see problem 11.93) its velocity will tend to c when $r \to 0$ $(Ze^2 \geqslant Kc)$.

11.97 Determine the trajectory of the relative motion of nonrelativistic particles of charges e, e', masses m_1, m_2, and energy $\mathcal{E}$. Investigate the solution.

11.98* Find the differential scattering cross section $\sigma(\theta)$ for nonrelativistic particles of charge e in the field of a stationary point charge e'. Assume that the velocity of the particles at a large distance from the scattering centre is v_0.

11.99 Determine the angle θ through which a relativistic particle of charge e, energy $\mathcal{E} > mc^2$, and angular momentum $K > |ee'|/c$ is deflected in the Coulomb field of a heavy stationary charge e' (see problems 11.93 and 11.95).

11.100 A relativistic particle of charge e, rest mass m, and velocity v_0 at large distances is scattered through a small angle by the Coulomb field of a fixed charge e'. Determine the differential scattering cross section $\sigma(\theta)$.

11.101 An electron of charge e and rest mass m travels through the evacuated space above a plane uncharged surface of a dielectric of permittivity ϵ. Initially the electron moves in a direction parallel to the surface of the dielectric with a velocity v and at a distance a from it. Find the projected distance x from the initial position to the point at which the electron will enter the dielectric.

11.102* While an electron is being accelerated in the betatron, the magnetic field is continuously increasing, giving rise to the induced e.m.f. which accelerates the electron in a stationary orbit. Show that in order to accelerate an electron in an orbit of constant radius, it is necessary that the total magnetic flux cutting the orbit should be twice the flux associated with a magnetic field which is uniform within the orbit and zero along the orbit (the 2:1 betatron rule).

11.103* Show that, to within terms of the order of v^2/c^2, the energy of the retarded interaction between two charged particles is of the form

$$U(t) = \frac{e_1 e_2}{R}\left\{1 - \frac{1}{2c^2}[(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2) + (\boldsymbol{v}_1 \cdot \boldsymbol{n})(\boldsymbol{v}_2 \cdot \boldsymbol{n})]\right\},$$

where $\boldsymbol{R}$ is the relative position vector of the two particles, $\boldsymbol{n} = \boldsymbol{R}/R$, and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are the velocities of the particles (this is the so-called Breit formula; an analogous expression is used in an approximate quantum description of the retarded interaction). All the quantities on the right-hand side of the equation are taken at time t.

Hint. Use the expansions for the Liénard–Wiechert potentials (see problem 12.36) and retain only those terms which are independent of the accelerations and their derivatives. Carry out a gradient transformation of the potentials in such a way that the scalar potential becomes identical with the Coulomb potential.

11.104 Find an approximate expression for the Lagrange function for two interacting particles, having charges e_1, e_2 and rest masses m_1, m_2, by taking into account the retarding effect to within terms of the order of v^2/c^2.

11.105 A particle having a magnetic moment $\boldsymbol{m}$ and gyromagnetic ratio κ is placed in a uniform magnetic field $\boldsymbol{H}$. Determine the motion of the magnetic moment of the particle.

11.106 A particle of charge e and mass m and with an intrinsic angular momentum (spin) $\boldsymbol{s}$ and a magnetic moment $\boldsymbol{m} = e\boldsymbol{s}/mc$ moves nonrelativistically in an external electrostatic central field $\varphi(r)$. Evaluate the interaction energy U of the spin and the external field in the first nonvanishing approximation in v/c, bearing in mind the Thomas precession of the instantaneously attached frame with an angular velocity

$$\boldsymbol{\omega}_{\mathrm{T}} = \frac{[\boldsymbol{v} \wedge \dot{\boldsymbol{v}}]}{2c^2}.$$

The appearance of the Thomas precession was explained in problem 10.25.

Hint. The rates of change of any vector $\boldsymbol{A}$ in a fixed and in a rotating frame of reference are connected by the relation

$$\left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_{\text{rotating}} + [\boldsymbol{\Omega} \wedge \boldsymbol{A}],$$

where $\boldsymbol{\Omega}$ is the rotational velocity (see, e.g. ter Haar, 1971).

11.107 Solve the preceding problem, assuming that the particle moves in a potential $V(r)$ which is not electrostatic. There is therefore no magnetic field in the comoving frame of reference.

11.108 A neutron (magnetic moment $\boldsymbol{m}_0$, kinetic energy T) enters a magnetic field defined by H = constant, $x \geqslant 0$, and $H = 0$, $x < 0$. Under what conditions will the neutron be reflected from the field?

11.109 Investigate the possible trajectories of a neutron (rest mass m, dipole moment $\boldsymbol{m}_0$) in the field of an infinitely long straight conductor carrying a current J.

11.110 A beam of cold neutrons (velocity v_0, magnetic moment $\boldsymbol{m}_0$, rest mass m) is scattered by the magnetic field of an infinitely long straight conductor carrying a current J. Determine the differential scattering length

$$l(\phi) = \left|\frac{\mathrm{d}s}{\mathrm{d}\phi}\right|,$$

where $s(\phi)$ is the impact parameter for which the neutron is scattered through an angle ϕ.

Hint. Use the method employed in the solution of problem 11.98.

Emission of electromagnetic waves[1]

a The Hertz vector and the multipole expansion

The problem of the determination of a variable electromagnetic field in a vacuum for a given distribution of charges $\rho(\boldsymbol{r}', t)$ and currents $\boldsymbol{j}(\boldsymbol{r}', t)$ may be solved by evaluating the retarded potentials

$$\varphi(\boldsymbol{r}, t) = \int \frac{\rho(\boldsymbol{r}', t-R/c)}{R} \mathrm{d}^3\boldsymbol{r}' , \tag{12.a.1}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{1}{c}\int \frac{\boldsymbol{j}(\boldsymbol{r}', t-R/c)}{R} \mathrm{d}^3\boldsymbol{r}' , \tag{12.a.2}$$

where $R = |\boldsymbol{r}-\boldsymbol{r}'|$, $\boldsymbol{r}$ is the position vector of the point at which the field is to be determined, $\boldsymbol{r}'$ is the position vector of the field source, and $\mathrm{d}^3\boldsymbol{r}'$ is a volume element of the field source. These potentials satisfy the d'Alembert equations:

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho , \tag{12.a.3}$$

$$\nabla^2\boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\frac{4\pi}{c}\boldsymbol{j} , \tag{12.a.4}$$

and are related by the Lorentz condition

$$\mathrm{div}\boldsymbol{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0 . \tag{12.a.5}$$

The number of unknown functions may be reduced if the potentials $\boldsymbol{A}(\boldsymbol{r}, t)$ and $\varphi(\boldsymbol{r}, t)$, which are related by equation (12.a.5), are replaced by the single vector function $\boldsymbol{Z}(\boldsymbol{r}, t)$ which is known as the Hertz vector, or the polarisation potential, and is related to $\boldsymbol{A}$ and φ by the expressions

$$\varphi = -\mathrm{div}\boldsymbol{Z} , \tag{12.a.6}$$

$$\boldsymbol{A} = \frac{1}{c}\frac{\partial\boldsymbol{Z}}{\partial t} . \tag{12.a.7}$$

The distribution of charges and currents can then be conveniently described with the aid of a single vector function $\boldsymbol{P}(\boldsymbol{r}', t)$ which is related to ρ and $\boldsymbol{j}$ by the expressions

$$\rho = -\mathrm{div}\boldsymbol{P} , \tag{12.a.8}$$

$$\boldsymbol{j} = \frac{\partial\boldsymbol{P}}{\partial t} . \tag{12.a.9}$$

[1] For details of the theory see, for instance, the books by Landau and Lifshitz (1975), Stratton (1941), Jackson (1962), Panofsky and Phillips (1962), Smythe (1950), Pauli (1958), Heitler (1944), and Ginzburg and Syrovatskii (1964).

This definition of $\boldsymbol{P}$ ensures that the continuity condition $\operatorname{div}\boldsymbol{j}+\partial\rho/\partial t = 0$ is satisfied. The quantity $\boldsymbol{P}$ is often referred to as the polarisation (but must not be confused with the polarisation of a dielectric).

The Hertz vector $\boldsymbol{Z}$ satisfies the d'Alembert equation

$$\nabla^2\boldsymbol{Z}-\frac{1}{c^2}\frac{\partial^2\boldsymbol{Z}}{\partial t^2} = -4\pi\boldsymbol{P}\,. \tag{12.a.10}$$

The vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are given by

$$\boldsymbol{E} = \operatorname{curl}\operatorname{curl}\boldsymbol{Z}-4\pi\boldsymbol{P}\,, \qquad \boldsymbol{H} = \frac{1}{c}\frac{\partial\operatorname{curl}\boldsymbol{Z}}{\partial t}\,. \tag{12.a.11}$$

In order to determine the electromagnetic field for given ρ and $\boldsymbol{j}$ with the aid of the Hertz vector, the polarisation vector $\boldsymbol{P}$ must first be determined from equations (12.a.8) and (12.a.9). Owing to the analogy between equations (12.a.3) to (12.a.4) and (12.a.10), the Hertz vector can then be expressed in terms of $\boldsymbol{P}$ in a way similar to that whereby φ and $\boldsymbol{A}$ are expressed in terms of ρ and $\boldsymbol{j}$:

$$\boldsymbol{Z}(\boldsymbol{r}, t) = \int\frac{\boldsymbol{P}(\boldsymbol{r}', t-R/c)}{R}\,\mathrm{d}^3\boldsymbol{r}'\,. \tag{12.a.12}$$

If a system of charges and currents is confined to a finite region, its linear dimensions are of the order of a, and if the wavelengths λ which contribute significantly to the spectral expansions of the potentials are such that

$$\frac{a}{\lambda} \ll 1 \quad \text{and} \quad \frac{a}{r} \ll 1\,, \tag{12.a.13}$$

then the integrands can be expanded in powers of a/λ and a/r. If one retains only the first term of this expansion then

$$\boldsymbol{Z}(\boldsymbol{r}, t) = \frac{\boldsymbol{p}(t')}{r}\,, \tag{12.a.14}$$

where $t' = t-(r/c)$ is the retarded time of the centre of the system. The quantity

$$\boldsymbol{p}(t') = \int\boldsymbol{r}'\rho(\boldsymbol{r}', t')\,\mathrm{d}^3\boldsymbol{r}' \tag{12.a.15}$$

is the electric dipole moment of the charge distribution (cf problems 12.20 and 12.21). The corresponding expression for $\boldsymbol{A}$ and φ can then be obtained with the aid of equations (12.a.6) and (12.a.7).

The field at large distances r from the system of charges is of particular interest. In this case we have the following inequality in addition to equations (12.a.13):

$$r' \ll \lambda \ll r\,. \tag{12.a.16}$$

This defines the so-called wave zone. The field can then be determined by expanding the vector potential in powers of a/λ. To within terms of the order of v^2/c^2, this expansion is of the form

$$A(r, t) = \frac{\dot{p}(t')}{cr} + \frac{\ddot{Q}(t')}{2c^2 r} + \frac{[\dot{m}(t') \wedge n]}{cr} , \tag{12.a.17}$$

where $n = r/r$ is a unit vector in the direction of propagation of the electromagnetic waves and where the dots represent differentiation with respect to t'. The magnetic dipole moment is given by

$$m = \frac{1}{2c}\int [r' \wedge j] \, d^3r' , \tag{12.a.18}$$

and the components of the quadrupole moment are

$$Q_\alpha = \sum_\beta Q_{\alpha\beta} n_\beta , \qquad Q_{\alpha\beta} = \int \rho(r') x'_\alpha x'_\beta \, d^3r' . \tag{12.a.19}$$

The form of the dependence of the vector potential in the wave zone on the distance r from the system is a characteristic feature. It ensures (see below) that the energy flux at infinity remains finite. This means that the vector potential describes the emission of electromagnetic energy.

The second and third terms of the vector potential, which represent the electric quadrupole and the magnetic dipole respectively, are smaller by a factor[(2)] of a/λ as compared with the first term (electric dipole) and may be neglected unless there are some special reasons responsible for a large reduction in the first term.

In the wave zone, the field in a sufficiently small region will be of the form of a plane wave travelling away from the source. The corresponding field strengths may be evaluated from the formulae

$$H = \frac{1}{c}[\dot{A} \wedge n] , \qquad E = [H \wedge n] . \tag{12.a.20}$$

The angular distribution of the emitted radiation may be characterised by the amount of energy flowing into a unit solid angle per unit time and this is given by

$$\frac{dI}{d\Omega} = \frac{c}{4\pi} H^2 r^2 . \tag{12.a.21}$$

The total intensity I is obtained by integrating equation (12.a.21) over all directions.

Using the expansion given by equation (12.a.17), the final expression for the total intensity is given by

$$I = \frac{2}{3c^3}(\ddot{p})^2 + \frac{1}{60c^5}\left[3\sum_{\alpha\beta} \dddot{Q}^2_{\alpha\beta} - \left(\sum_\beta \dddot{Q}_{\beta\beta}\right)^2\right] + \frac{2}{3c^3}(\ddot{m})^2 . \tag{12.a.22}$$

(2) If the radiating system is a particle moving with velocity v in a bounded region of size a, $a/\lambda \approx v/c$.

12.1 Use a direct substitution to verify that the retarded potentials satisfy the d'Alembert equation and the Lorentz condition.

12.2 Use the results of problem 1.32 to obtain the formula given by equation (12.a.22).

12.3 Write down the equations which are satisfied by the electromagnetic potentials φ and $\boldsymbol{A}$ if the Lorentz condition (12.a.5) is replaced by the condition $\operatorname{div}\boldsymbol{A} = 0$ (this is the so-called Coulomb gauge).

12.4 Prove that the electromagnetic fields are invariant under the gauge transformation

$$\boldsymbol{A} \to \boldsymbol{A} + \nabla\zeta\,, \qquad \varphi \to \varphi - \frac{1}{c}\frac{\partial\zeta}{\partial t}\,,$$

where ζ is an arbitrary function.

12.5 Prove that the gauge transformation (see preceding problem) and the Lorentz condition (12.a.5) are Lorentz invariant.

12.6 Let $\boldsymbol{A}$ and φ be the vector and scalar potentials in a frame S corresponding to the Coulomb gauge, so that $\operatorname{div}\boldsymbol{A} = 0$. Find the potentials $\boldsymbol{A}'$ and φ' corresponding to the Coulomb gauge in a frame S' moving with a velocity $\boldsymbol{v}$ with respect to S. Consider the case where the transformation from S to S' can be considered to be an infinitesimal Lorentz transformation [compare an article by France (1976)].

12.7 Consider the case of an electromagnetic wave described by the following potentials in the Coulomb gauge in a frame S:

$$\boldsymbol{A} = \boldsymbol{A}_0 \exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}) - \mathrm{i}\omega t]\,, \qquad \varphi = 0\,.$$

Find the potential $\boldsymbol{A}'$ in the frame S' moving with a velocity $\boldsymbol{v}$ with respect to S [see preceding problem and cf France (1976)]. Consider again the case where the transformation from S to S' can be considered to be an infinitesimal Lorentz transformation.

12.8 Show that when the Lorentz condition is satisfied then, in the wave zone, the scalar potential of a finite radiating system may be expressed in terms of the vector potential by the formula $\varphi = (\boldsymbol{n}\cdot\boldsymbol{A})$.

12.9 Using the solution of problem 10.79 [equations (10.79.2) and (10.79.3)], find an expression for the loss of angular momentum per unit time, $-\mathrm{d}\boldsymbol{K}/\mathrm{d}t$, by a system radiating as an electric dipole.

12.10 Find the equations for the lines of force of the electric and magnetic fields of a point electric dipole oscillator having a dipole moment $\boldsymbol{p} = \boldsymbol{p}_0 \cos\omega t$. Investigate the change in the field configuration between the wave zone and the region in the immediate neighbourhood of the oscillator.

Hint. If the polar axis lies along $\boldsymbol{p}_0$ then the electromagnetic field of the oscillator is of the form

$$E_r = \frac{2p_0\cos\vartheta}{r^2}\left[\frac{\cos(kr-\omega t)}{r} + k\sin(kr-\omega t)\right],$$

$$E_\vartheta = \frac{p_0\sin\vartheta}{r}\left[\left(\frac{1}{r^2}-k^2\right)\cos(kr-\omega t)+\frac{k}{r}\sin(kr-\omega t)\right],$$

$$E_\phi = H_r = H_\vartheta = 0\,,$$

$$H_\phi = -\frac{p_0k^2}{r}\cos(kr-\omega t)+\frac{p_0k}{r^2}\sin(kr-\omega t)\,.$$

12.11 Find the electromagnetic field $\boldsymbol{H}$, $\boldsymbol{E}$ due to a charge e moving with a uniform velocity on a circle of radius a. Assume that the motion is nonrelativistic, the angular velocity is ω, and the distance to the point of observation r is much greater than the radius a. Find the time average of the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and of the total intensity $\bar{I}$ of the radiation, and investigate its polarisation.

12.12 Investigate the effect due to interference on the emission of electromagnetic waves by a system of charges in the following situation. Two identical electric charges e move with a uniform velocity at an angular frequency ω on a circular orbit of radius a in such a way that they remain at the opposite ends of a diameter. Find the polarisation, the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$, and the intensity $\bar{I}$ of the radiation. What will be the effect of the removal of one of these charges? (cf the solution of the preceding problem).

12.13 How should the position of the particles in the preceding problem be adjusted on the circle in order that the intensities of the electric dipole and quadrupole radiations should be equal?

12.14 Two electric dipole oscillators vibrate with the same frequency ω but their phases differ by $\frac{1}{2}\pi$. The amplitudes of the dipole moments are both equal to p_0 and are at an angle φ to each other. Assuming that the distance between the oscillators is small compared with the wavelength, find the field $\boldsymbol{H}$ in the wave zone, the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$, and the average total intensity $\bar{I}$ of the emitted radiation.

12.15 Investigate the polarisation of the radiation field due to the oscillator system considered in the preceding problem, using the method employed in the solution of problem 8.1.

12.16* Find the time average of the energy flux $\bar{\gamma}$ at large distances from the charge considered in problem 12.11, retaining terms of the order of r^{-3}. Find the couple N on a totally absorbing spherical screen of large radius with the charge at its centre.

12.17 A uniformly magnetised sphere of radius a and magnetisation $\boldsymbol{M}$ rotates with a constant angular velocity ω about an axis passing through the centre of the sphere and making an angle φ with the direction of $\boldsymbol{M}$. Find the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ and investigate the polarisation of the field. Determine the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and total intensity $\bar{I}$ of the radiation.

12.18 A uniformly charged drop pulsates with its density remaining constant. The surface of the drop is described by

$$R(\vartheta) = R_0[1 + aP_2(\cos\vartheta)\cos\omega t]\,,$$

where $a \ll 1$. The total charge on the drop is q. Find the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and total intensity $\bar{I}$ of the radiation emitted.

12.19 An electric charge q is arranged in a continuous spherically symmetric distribution in a bounded region and executes radial pulsations. Find the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ outside the charge distribution.

12.20 Find expressions for the electric dipole $\boldsymbol{Z}_p$, electric quadrupole $\boldsymbol{Z}_Q$, and magnetic dipole $\boldsymbol{Z}_m$ terms in the expansion for the Hertz vector, which hold for an arbitrary dependence of the charges and currents on time at distances $r \gg a$ and $\lambda \gg a$ (the condition $r \gg \lambda$ need not be satisfied).

12.21 Find, in vector form, expressions for the electromagnetic field strengths due to electric and magnetic dipole oscillators at distances which are large compared with their dimensions.

Hint. When differentiating with respect to $\boldsymbol{r}$, note that the moments $\boldsymbol{p}$ and $\boldsymbol{m}$ should be taken at the retarded time $t' = t-(r/c)$ and are, therefore, functions of $\boldsymbol{r}$.

12.22 Find the angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and the total intensity $\bar{I}$ of the radiation from the open resonators considered (a) in problem 9.23; (b) in problem 9.24.

12.23 Two identical electric dipole moments lie along a given straight line and oscillate in antiphase with a frequency ω and amplitude $\boldsymbol{p}_0$. The distance between the centres a is such that $\lambda \gg a$. Find the electromagnetic field at distances $r \gg a$. Find also the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ of the radiation and its total intensity $\bar{I}$.

12.24* A standing current wave of amplitude $\mathcal{J}_0$ and frequency ω is excited in a linear antenna of length l with nodes at the ends. The number of current half waves in the antenna is m. Find the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ of the emitted radiation.

12.25 Find the total radiation intensity $\bar{I}$ and the radiation resistance $R = 2\bar{I}/\mathcal{J}_0^2$ of the antenna considered in the preceding problem.

Hint. The result can be expressed in terms of the integral cosine

$$\mathrm{Ci}(x) = C + \ln x + \int_0^x \frac{\cos t - 1}{t}\,\mathrm{d}t\,,$$

where $C = 0{\cdot}577$ is the Euler constant.

12.26 A travelling current wave $J = J_0 \exp[\mathrm{i}(k\xi - \omega t)]$ propagates in a linear antenna of length l, where $k = \omega/c$ and ξ is the coordinate of a point on the antenna. The loads at the ends of the antenna are chosen so that there are no reflected waves. Find the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ and the total intensity $\bar{I}$ of the radiation.

12.27* A standing current wave of the form $J = J_0 \exp(-\mathrm{i}\omega t)\sin n\phi'$ is excited in a circular wire loop of radius a. Find the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ in the wave zone.

12.28* The centres of two electric dipole oscillators of frequency ω and equal amplitudes $\boldsymbol{p}_0 \parallel x$ lie on the z-axis at equal distances from the origin and distances $a = \frac{1}{4}\lambda$ from each other. The phase difference between the oscillators is $\frac{1}{2}\pi$. Find the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ of the radiation.

12.29 A system of charges B having a charge density $\rho(\boldsymbol{r}, t)$ and current density $\boldsymbol{j}(\boldsymbol{r}, t)$ is reflected in the $z = 0$ plane in such a way that (a) each point $\boldsymbol{r} = (x, y, z)$ transforms into $\boldsymbol{r}' = (x, y, -z)$, and (b) the charge density changes sign so that $\rho(\boldsymbol{r}, t) = -\rho'(\boldsymbol{r}', t)$, with ρ' the charge density in the reflected system B'. Find the law of transformation on reflection of the current density $\boldsymbol{j}(\boldsymbol{r}, t)$ and the electric $(\boldsymbol{p}, \boldsymbol{Q})$ and magnetic $(\boldsymbol{m})$ moments of the system, and also the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$.

12.30 Show that the electromagnetic field due to an arbitrary system of charges B, placed near a perfectly conducting plane, may be obtained as a superposition of the fields due to B and a system B' reflected in this plane (see preceding problem). Consider in particular the radiation emitted by an electric dipole oscillator having a moment $\boldsymbol{p}(t) = \boldsymbol{p}_0 f(t)$, where $|\boldsymbol{p}_0| = 1$, $f(t)$ is an arbitrary function, and the dipole is at a distance $b \ll \lambda$ from the plane, and is at an angle $\varphi_0 =$ constant to it. Use the electric-dipole approximation.

12.31 An electric dipole oscillates with a frequency ω and an amplitude $\boldsymbol{p}_0$. It is placed at a distance of $\frac{1}{2}a$ from a perfectly conducting plane where $a \ll \lambda$ and $\boldsymbol{p}_0$ is parallel to the plane. Find the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ at distances $r \gg \lambda$ and the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$ of the emitted radiation.

12.32 Show that if a function $u(r, \vartheta, \phi)$ satisfies the Helmholtz equation $\nabla^2 u + k^2 u = 0$, then the Hertz potential for a monochromatic electric wave $(H_r = 0)$ of frequency $\omega = kc$ in a source-free space is of the form

$$\boldsymbol{Z} = u\boldsymbol{r} + \operatorname{grad}\chi$$

where $\chi = k^{-2}\partial(ru)/\partial r$. Find also the components of the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ along the axes of a spherical system of coordinates in terms of $u(r, \vartheta, \phi)$ (the function u is known as the Debye potential).

Hint. In showing that $\nabla^2 \boldsymbol{Z} + k^2 \boldsymbol{Z} = 0$, note that $\nabla^2\chi + k^2\chi + 2u = 0$.

12.33 Show that the field due to a bound electric dipole oscillator placed at the point $\boldsymbol{r}_0(\boldsymbol{r}_0 \parallel \boldsymbol{p}_0)$ and having a moment $\boldsymbol{p}_0 \exp(-\mathrm{i}\omega t)$ may be described by a Debye potential (see preceding problem) which is of the form

$$u = \frac{p_0}{r_0 R}\exp(\mathrm{i}kR),$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_0$.

Hint. The Hertz vector $\boldsymbol{Z} = u\boldsymbol{r} + \operatorname{grad}\chi$ corresponding to the potential u differs from the expression $(\boldsymbol{p}_0/R)\exp(\mathrm{i}kR)$ [see equation (12.a.14)] but leads to the same expressions for the fields $\boldsymbol{E}$, $\boldsymbol{H}$.

12.34 An electric dipole oscillator placed at a distance b from the centre of a perfectly conducting sphere of radius a has a dipole moment $p_0 \exp(-\mathrm{i}\omega t)$. The dipole lies along the line joining it to the centre of the sphere. By using the Debye potential u (see problem 12.32), find the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ and the average angular distribution $\overline{\mathrm{d}I/\mathrm{d}\Omega}$.

b The electromagnetic field of a moving point charge

A point charge e moving with a velocity $\boldsymbol{v}(t')$, whose position vector at time t' is $\boldsymbol{r}_0(t')$, will give rise to an electromagnetic field whose potentials at a time t at a point having a position vector $\boldsymbol{r}$ are given by the Liénard-Wiechert formulae

$$\varphi(\boldsymbol{r}, t) = \left.\frac{e}{R[1-(\boldsymbol{R}\cdot\boldsymbol{v})/c]}\right|_{t'}, \qquad \boldsymbol{A}(\boldsymbol{r}, t) = \left.\frac{e\boldsymbol{v}}{c[R-(\boldsymbol{R}\cdot\boldsymbol{v})/c]}\right|_{t'}, \tag{12.b.1}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}_0$. The retarded time t' is defined by

$$c(t-t') = |\boldsymbol{R}|. \tag{12.b.2}$$

The field strengths derived from the Liénard-Wiechert potentials are

$$\boldsymbol{E}(\boldsymbol{r}, t) = e\frac{(1-\beta^2)(\boldsymbol{n}-\boldsymbol{v}/c)}{[1-(\boldsymbol{n}\cdot\boldsymbol{v})/c]^3R^2} + \left.\frac{e[\boldsymbol{n}\wedge[(\boldsymbol{n}-\boldsymbol{v}/c)\wedge\dot{\boldsymbol{v}}]]}{c^2[1-(\boldsymbol{n}\cdot\boldsymbol{v})/c]^3R}\right|_{t'}, \tag{12.b.3}$$

$$\boldsymbol{H} = [\boldsymbol{n}\wedge\boldsymbol{E}]|_{t'},$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{R}}{R}, \qquad \beta = \frac{v}{c}.$$

The first term in the expression for $\boldsymbol{E}$ and the corresponding term of $\boldsymbol{H}$ describe a field which falls off as R^{-2} (quasi-stationary field) and moves together with the charge. The second term in $\boldsymbol{E}$ and the corresponding term in $\boldsymbol{H}$ describe a field which falls off as R^{-1} (radiation field). The energy flux associated with the latter field is independent of R. This means that the radiation field is not rigidly attached to the charge responsible for it. At large distances from the charge, i.e. in the wave zone, the quasi-stationary field is negligible compared with the radiation field. As can be seen from equation (12.b.3), the condition that the radiation field should appear is that the acceleration should be nonvanishing ($\dot{v} \neq 0$).

The intensity of radiation in the direction $n = R/R$ in the wave zone is given by

$$\frac{\mathrm{d}I_n(t)}{\mathrm{d}\Omega} = \frac{c}{4\pi}E^2(t)R^2 = \frac{e^2}{4\pi c^3}\left[\frac{2(n \cdot v)(v \cdot \dot{v})}{c[1-(n \cdot v)/c]^5} + \frac{\dot{v}^2}{[1-(n \cdot v)/c]^4} - \frac{(1-v^2/c^2)(n \cdot \dot{v})^2}{[1-(n \cdot v)/c]^6}\right]. \tag{12.b.4}$$

If the velocity, v, of the charge is small compared with the velocity of light, then the radiation field can be expanded into a multipole expansion and can be calculated with the aid of equations (12.a.17) to (12.a.22).

As a result of the emission of radiation, the accelerated particle loses energy and momentum which are transferred to the electromagnetic field. The loss of the ith component of the 4-vector of energy-momentum of the particle $p_i = (\mathcal{E}/c, p)$ per unit proper time τ may be expressed in terms of the 4-velocity u_i and the 4-acceleration w_i of the particles and is given by

$$-\frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{2e^2}{3c^3}w_k^2 u_i \,. \tag{12.b.5}$$

The energy lost by the particle per unit time in the laboratory system, i.e. the rate of loss of energy $\mathrm{d}\mathcal{E}/\mathrm{d}t'$, differs from equation (12.b.5) by the factor $\gamma = (1-v^2/c^2)^{-1/2}$, since $\mathrm{d}t' = \gamma\,\mathrm{d}\tau$. The total intensity of the radiation as determined with the aid of equation (12.b.4) is not in its turn identical to the rate of loss of energy (see problems 12.41 to 12.47).

The vector potential $A(R_0, t)$ due to a charge executing a periodic motion in a closed orbit $r = r_0(t')$ with a period $2\pi/\omega_0$ may be expanded into a Fourier series:

$$A(R_0, t) = \sum_{l=-\infty}^{\infty} A_l \exp(-\mathrm{i}\omega_0 lt) \,.$$

At large distances from the orbit, the Fourier component A_l is given by

$$A_l = e\frac{\exp(\mathrm{i}kR_0)}{cR_0T}\oint \exp\{\mathrm{i}[l\omega_0 t' - (k \cdot r_0(t'))]\}v(t')\,\mathrm{d}t' \,, \tag{12.b.6}$$

where

$$\omega_0 = \frac{2\pi}{T} \,, \qquad k = \frac{l\omega_0}{c}n \,.$$

The integral in the above expression may be evaluated over the whole trajectory of the charge.

Charged particles which experience a collision move with a finite acceleration and, therefore, emit electromagnetic energy. The law of motion of the colliding particles, and hence the radiation emitted on collision, depends on the form of the interaction and the impact parameter, s (if the potential energy of the interaction between the two particles is a function of the distance between them only). The energy emitted in all directions when a beam of particles is scattered can be conveniently characterised by the total emission

$$\kappa = 2\pi \int_0^\infty \Delta W(s) s \, \mathrm{d}s \, , \tag{12.b.7}$$

where $\Delta W(s)$ is the energy emitted in a single collision between two particles and s is the corresponding impact parameter.

The angular distribution of the radiation may be characterised by the differential emission $\mathrm{d}\kappa_n$ which is defined by

$$\frac{\mathrm{d}\kappa_n}{\mathrm{d}\Omega} = 2\pi \int_0^\infty \frac{\mathrm{d}[\Delta W_n(s)]}{\mathrm{d}\Omega} s \, \mathrm{d}s \, , \tag{12.b.8}$$

where $\mathrm{d}[\Delta W_n(s)]/\mathrm{d}\Omega$ is the energy emitted per unit solid angle in the direction $\boldsymbol{n}$ for a single collision of two particles with an impact parameter s, which is averaged with respect to the azimuth in the plane perpendicular to the particle beam. There is an analogous formula for the differential emission per unit frequency interval $\mathrm{d}\kappa_\omega/\mathrm{d}\omega$. When dipole radiation predominates, equation (12.b.8) becomes

$$\frac{\mathrm{d}\kappa_n}{\mathrm{d}\Omega} = \frac{1}{4\pi c^3}[A + BP_2(\cos\vartheta)] \, , \tag{12.b.9}$$

where $P_2(\cos\vartheta) = \frac{3}{2}\cos^2\vartheta - \frac{1}{2}$ is a Legendre polynomial (see appendix 2), ϑ is the polar angle between the direction $\boldsymbol{n}$ of the radiation and the direction of the particle beam, and the quantities A and B are given by

$$A = \frac{2}{3} \int_0^\infty 2\pi s \, \mathrm{d}s \int_{-\infty}^{+\infty} \ddot{\boldsymbol{p}}^2 \, \mathrm{d}t \, , \qquad B = \frac{1}{3} \int_0^\infty 2\pi s \, \mathrm{d}s \int_{-\infty}^{+\infty} (\ddot{\boldsymbol{p}}^2 - 3\ddot{p}_z^2) \, \mathrm{d}t \, . \tag{12.b.10}$$

The low frequency spectrum of the radiation emitted during a collision occupying a time interval τ may be found from ($\omega\tau \ll 1$)

$$\frac{\mathrm{d}\Delta W_\omega}{\mathrm{d}\omega} = \frac{2}{3\pi c^3}\left[\sum e(\boldsymbol{v}_2 - \boldsymbol{v}_1)^2\right] \, , \tag{12.b.11}$$

where the sum is evaluated over all the colliding particles and $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ are the velocities of the particles before and after the collision (v_1, $v_2 \ll c$).

12.35* Derive the Liénard–Wiechert potentials [see equation (12.b.1)] from the general formulae for the retarded potentials.

Hint. The charge distribution in a point particle may be characterised by a volume charge density $\rho(\boldsymbol{r}', t) = e\delta[\boldsymbol{r}' - \boldsymbol{r}_0(t)]$ where $\boldsymbol{r}_0(t)$ is the position vector of the particle at time t and e is its charge. In evaluating the volume integral with respect to $\mathrm{d}^3r' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$, it will be necessary to use the new variable $\boldsymbol{R}_1 = \boldsymbol{r}' - \boldsymbol{r}_0$.

12.36* Find the expansion for the Liénard–Wiechert potentials in powers of R/c by expanding the general expressions for the retarded potentials (12.a.1) and (12.a.2) in powers of $1/c$.

12.37 Derive the potentials due to a uniformly moving point charge from the Liénard–Wiechert potentials by expressing the retarded time t' in the latter in terms of the time t of observation of the field (cf problems 10.68 and 12.90).

12.38 Find the field strengths due to a uniformly moving point charge by making use of the general formulae (12.b.3). Express the field in terms of the time of observation, t, by eliminating the retarded time t' (see problem 10.68).

12.39 A charge e moves with a small velocity $\boldsymbol{v}$ and acceleration $\dot{\boldsymbol{v}}$ in a bounded region. Find approximate expressions for the electromagnetic field $\boldsymbol{E}$, $\boldsymbol{H}$ due to the particle at a point whose distance from the particle is large compared with the linear dimensions of the region within which the particle is moving. Determine the position of the boundary between the quasi-stationary zone and the wave zone.

12.40 Determine the angular distribution $\mathrm{d}I/\mathrm{d}\Omega$ and the total intensity I of the radiation emitted by the charge considered in the preceding problem.

12.41* Express the energy lost by radiation per unit solid angle by a particle, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'\,\mathrm{d}\Omega$, in terms of the intensity (flux) $\mathrm{d}I/\mathrm{d}\Omega$ in a given direction given by the Poynting vector. Solve the problem (a) analytically, by considering the relation between the retarded time, t', and the time of observation, t, and (b) geometrically, by considering the form of the region of space within which the electromagnetic energy emitted by the particle in a time $\mathrm{d}t'$ is localised.

12.42 Show that if a particle executes a periodic motion, then the average rate of energy loss per period is equal to the average intensity of the radiation.

12.43 Prove equation (12.b.4).

12.44 Find the total rate of emission of energy in all directions, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$, by a charged particle, and express it in terms of (a) the velocity $\boldsymbol{v}(t')$ and acceleration $\dot{\boldsymbol{v}}(t')$, and (b) the velocity $\boldsymbol{v}(t')$ and the external field $\boldsymbol{E}$, $\boldsymbol{H}$ producing the acceleration of the particle. Assume that the mass, m, and the charge, e, of the particle are given.

12.45 Express the rate of loss of momentum $-\mathrm{d}\boldsymbol{p}/\mathrm{d}t'$ by a radiating charged particle in terms of the total rate of loss of energy.

12.46 A radiating particle is observed from two reference systems which experience uniform relative motion. Compare the total rates of energy loss in the two systems.

12.47 The velocity $\boldsymbol{v}$ of a relativistic particle at a retarded time t' is parallel to its acceleration $\dot{\boldsymbol{v}}$. Find the instantaneous angular distribution, $\mathrm{d}I/\mathrm{d}\Omega$, of the emitted radiation, the total instantaneous intensity, I, and the total rate of emission of energy, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$. What is the nature of the angular distribution of the radiation in the ultrarelativistic case?

12.48 The velocity of a particle decreases from v_0 to zero in a time interval τ. Find the angular distribution of the radiation emitted during this time interval, assuming that the deceleration is constant. Determine the length of the pulse Δt as recorded by stationary instruments.

12.49 A relativistic particle of charge e, rest mass m, and momentum p moves in a circular orbit in a constant uniform magnetic field $\boldsymbol{H}$. The radius of the orbit is $a = cp/eH$. Find the total rate of loss of energy by the particle, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$.

12.50 An ultrarelativistic electron moves in a uniform magnetic field of field strength $\boldsymbol{H}$ along a helix. Its velocity $\boldsymbol{v}$ makes an angle θ with the vector $\boldsymbol{H}$. Find the energy loss per unit time of the electron, $-\mathrm{d}\mathcal{E}/\mathrm{d}t'$. Find also the radiative energy flux I through a fixed sphere of large radius which encloses the electron.

12.51 Find the instantaneous angular distribution $\mathrm{d}I/\mathrm{d}\Omega$ of the intensity of radiation emitted by a relativistic particle whose velocity at the retarded time t' is perpendicular to its acceleration. Sketch out the polar diagram for $v \ll c$ and $v \sim c$. Determine the directions in which the radiation intensity is zero.

12.52 A particle of charge e and rest mass m moves with a velocity v on a circular orbit in a constant uniform magnetic field $\boldsymbol{H}$. Find the average angular distribution of the intensity of the emitted radiation $\overline{\mathrm{d}I/\mathrm{d}\Omega}$. Investigate the ultrarelativistic case $v \sim c$.

Hint. Use the results of the preceding problem. Transform to polar coordinates with the origin at the centre of the circular trajectory and the polar axis parallel to $\boldsymbol{H}$. When evaluating the integral over the azimuthal angle use equations (3.428) from Gradshteyn and Ryzhik (1965).

12.53* Find the Fourier components $\boldsymbol{A}_n$, $\boldsymbol{H}_n$ of the radiation field due to a charge e moving on a circular orbit of radius a with a relativistic velocity v. Investigate the polarisation of the Fourier components.

Hint. Use equations (A3.11) and (A3.9) of appendix 3.

12.54 Explain the presence of higher harmonics in the spectrum of the field radiated by a charge moving with a constant velocity in a circular orbit (see preceding problem). What will be the change in the intensity of these harmonics when $\beta = v/c \to 0$? What will then be the form of the radiation field?

12.55* A charge e moves on a circular orbit of radius a with a velocity $v = \beta c$. Find the spectral expansion of the intensity of the radiation emitted in a given direction, $\mathrm{d}I_n/\mathrm{d}\Omega$.

12.56* N electrons move simultaneously on a given circular orbit (see problem 12.53). Consider the effect of the interference between the fields produced by these electrons on the intensity of the nth Fourier component of the emitted radiation. Consider the following special cases: (a) the distribution of the electrons over the circular orbit is random, (b) the electrons are uniformly distributed and lie at angular distances $2\pi/N$ from each other, and (c) the electrons are arranged in a bunch whose dimensions are small compared with the radius of the orbit (the result will then depend on the ratio of the wavelength to the dimensions of the bunch).

12.57* Two particles which have charges e_1, e_2 and masses m_1, m_2 $(e_1/m_1 \neq e_2/m_2)$ execute elliptical motion (see problem 11.97). Find the time average of the total intensity of the radiation emitted by them.

12.58 Find the average rate of loss of angular momentum $\mathrm{d}K/\mathrm{d}t$ of a system of two particles executing an elliptical motion (see preceding problem).

Hint. A general formula for the change in the angular momentum was obtained in the solution of problem 12.9.

12.59* Find the effective differential emission $\mathrm{d}\kappa_n/\mathrm{d}\Omega$ for the scattering of a beam of particles with charges e_1, masses m_1, and velocities v_0 by a charged particle of charge e_2 (of the same sign as e_1) and mass m_2.

Hint. In evaluating the integrals for A and B given by equation (12.b.9) replace the integration with respect to t by integration with respect to r where $\mathrm{d}t = \mathrm{d}r/\dot{r}$, $\dot{r} = v_0[1-(2a/r)-(s/r)^2]^{1/2}$. s is the impact parameter, and $2a$ is the minimum distance of approach of the two particles (this distance is reached when $s = 0$). Integrate first with respect to s and then with respect to r. In evaluating B, it will be necessary to use the equation for the trajectory of the relative motion, and this will be found in the solution of problem 11.97.

12.60* A particle of charge e_1 and mass m is scattered by another particle of charge e_2 and mass much greater than m. The impact parameter is s and the kinetic energy of the incident particle is large compared with the potential energy of interaction e_1e_2/r. The velocity v of the incident particle may, therefore, be looked upon as constant during the entire event and need not be small compared with the velocity of light. Find the angular distribution of the total emission $\mathrm{d}\Delta W_n/\mathrm{d}\Omega$ and consider in particular the case $\beta = v/c \ll 1$.

Hint. Use the general formula for the angular distribution of total radiation [equation (12.b.4)]. The acceleration of the particle $\dot{\boldsymbol{v}}$ should be expressed in terms of the Coulomb force acting upon it and also the velocity $\boldsymbol{v}$, using the formulae $\boldsymbol{v} = c^2\boldsymbol{p}/\mathcal{E}$ and $\dot{\boldsymbol{p}} = e_1e_2\boldsymbol{r}/r^3$.

12.61 Determine the total loss of energy ΔW and momentum $\Delta \boldsymbol{p}$ by the particle during the collision considered in the preceding problem. Obtain these quantities both by direct integration of the angular distribution found in the preceding problem, and by using the formulae obtained in the solution of problems 12.44 and 12.45.

12.62* A particle of charge e_1 and rest mass m collides with a heavy particle of charge e_2. The impact parameter s is large, so that the kinetic energy of the particle during the motion is large compared with its potential energy, and its velocity is much less than the velocity of light. Find the spectrum of the radiation emitted by the particle, $\mathrm{d}\Delta W_\omega/\mathrm{d}\omega$.

Hint. Use equation (A3.15) of appendix 3.

12.63 A beam of particles each of charge e and velocity $v \ll c$ is scattered by a perfectly hard sphere of radius a. Find the emission $\mathrm{d}\kappa_\omega$ in the frequency range $\mathrm{d}\omega$. Also determine the total emission κ.

12.64* A beam of particles having charges e_1 and masses m_1 is scattered by a particle of charge e_2 and mass m_2 ($e_1/m_1 = e_2/m_2$). Express the differential emission $\mathrm{d}\kappa_n/\mathrm{d}\Omega$ in terms of the components $Q_{\alpha\beta}$ of the quadrupole moment of the system. Give the result in a form analogous to equations (12.b.9) and (12.b.10).

12.65* Find the total emission κ for a beam of charged particles (charge e, rest mass m, velocity v_0) scattered by an identical particle.

c Interaction of charged particles with radiation

A radiating system of particles, transmitting energy and momentum to the radiation field, experiences a reaction due to this field. If the radiation is of the electric dipole type, then each particle of charge e experiences a radiative deceleration (radiative friction) and the corresponding force is given by

$$\boldsymbol{f} = \frac{2e}{3c^3}\dddot{\boldsymbol{p}}\,, \tag{12.c.1}$$

where $\boldsymbol{p}$ is the electric dipole moment of the system. For a single charge having a velocity $v \ll c$

$$\boldsymbol{f} = \frac{2e^2}{3c^3}\ddot{\boldsymbol{v}}\,. \tag{12.c.2}$$

In the ultrarelativistic case when $v \approx c$ the radiative friction is of the form

$$f_x = -\frac{2e^4}{3(mc^2)^4}[(E_y - H_z)^2 + (E_z + H_y)^2]\mathcal{E}^2 , \qquad (12.c.3)$$

where the x-axis is taken in the direction of the velocity of the particle, $\boldsymbol{E}$, $\boldsymbol{H}$ are the components of the external field in which the particle is moving, and $\mathcal{E} = mc^2/(1 - v^2/c^2)^{1/2}$ is the energy of the particle.

The radiative friction as given by equations (12.c.1) to (12.c.3) is only approximate. The concept of radiative friction can only be used if the corresponding force is small compared with other forces acting on the particle in its rest system. This condition is satisfied for a particle of charge e and mass m, moving in a given electromagnetic field $\boldsymbol{E}, \boldsymbol{H}$, provided

$$\lambda \gg r_0 , \qquad (12.c.4)$$

$$H \ll \frac{m^2c^4}{e^3} = \frac{e}{r_0^2} , \qquad (12.c.5)$$

where λ is the wavelength of the radiation emitted by the particle and $r_0 = e^2/mc^2 = 2{\cdot}8 \times 10^{-15}$ m is the classical radius of the electron. These conditions indicate that classical electrodynamics gives rise to internal contradictions at small distances (large frequencies) and in strong fields. It should, however, be noted that owing to quantum effects, classical electrodynamics ceases to hold even before these internal contradictions become important. This occurs at distances of the order of $\lambda_0 = \hbar/mc = 137r_0$ and in fields $H \sim e/\lambda_0 r_0 = m^2c^4/137e^3$.

An electromagnetic wave incident on a system of charges will give rise to an acceleration of these charges. As a result, the system becomes a source of secondary waves and scatters the incident radiation. The process of scattering may be characterised by the differential and total scattering cross sections which were defined in section b of chapter 8.

The electromagnetic field due to a moving charged particle possesses energy, momentum, and therefore mass (electromagnetic mass of the particle). The problem of the electromagnetic mass of elementary particles cannot be resolved on the basis of classical electrodynamics. However, the classical theory gives a satisfactory descriptive account of the *idea* of the electromagnetic mass, and problems 12.66 to 12.69 will illustrate the fundamentals of this theory and also the associated difficulties.

12.66* Find the momentum of the electromagnetic field due to a particle of charge e moving with a uniform velocity $\boldsymbol{v}$. The particle may be looked upon as a solid sphere of radius r_0 in its own rest system S' (in the system in which the velocity of the particle is v Lorentz contraction will occur). Introduce the electromagnetic rest mass m_0 of the particle, which is related to its field energy at rest by the Einstein relation, and discuss the difficulties which arise in this connection.

12.67 Find the energy W_m of the magnetic field and the total electromagnetic energy W of the particle considered in the preceding problem.

12.68* Find the force $\boldsymbol{F}$ which a charged, spherically symmetric particle exerts on itself when it undergoes an accelerated, translational motion at a low velocity $v \ll c$. Retarded effects and the Lorentz contraction may be neglected.

Hint. Calculate the resultant force on small elements of charge de of the particle, using the Liénard–Wiechert expression for the field strength (12.b.3).

12.69* Find the correct expression for the self-force $\boldsymbol{F}$ on a charged, spherically symmetric particle (see preceding problem). In deriving this expression, allow for the finite velocity of propagation of the interaction by including terms which are linear in the time of propagation of the interaction between the elements of the particle $(t' - t)$. Consider in particular the limiting case of a point particle. Estimate the contribution due to higher-order terms in this limiting situation.

12.70 Calculate the lifetime T which characterises the Rutherford model of the hydrogen atom, assuming that the electron in the atom moves and radiates as a classical particle. Assume that the electron moves towards the proton in a quasi-circular spiral, so that at each instant it radiates as if it were moving on a circular orbit, i.e. assume that the radius of the orbit is a slow function of time. Investigate the conditions under which this approximation is satisfactory. (Initial radius a of the atom may be taken to be 5×10^{-11} m.)

12.71 A relativistic particle of charge e and rest mass m moves on a circular orbit in a constant, uniform, magnetic field $\boldsymbol{H}$, and loses energy by the emission of radiation. Find the energy $\mathcal{E}(t)$ of the particle and the radius $r(t)$ of its orbit as a function of time, assuming that at time $t = 0$ the energy is $\mathcal{E}_0$ (compare the preceding problem).

12.72 An electron is accelerated in a betatron and travels on an orbit of constant radius a. The induced electric field responsible for the acceleration is due to an alternating magnetic field of frequency ω. Find the critical energy of the electron $\mathcal{E}_{cr}$ at which the energy lost by the emission of radiation is equal to the energy communicated to the electron by the induced electric field.

12.73* A particle of charge e and rest mass m is attracted towards a fixed point with a force $-m\omega_0^2 r$. Free oscillations commence in this harmonic oscillator at a time $t = 0$. Find the damping law of the oscillations by assuming that the radiative reaction is small. Determine the form of the spectrum of the oscillator and the width of a spectral line (natural width). Find the relation between the uncertainty in the energy $\hbar\omega$ of the emitted photons and the lifetime of the oscillator.

12.74 A gas consists of atoms of mass m. Owing to the thermal motion of the atoms and the Doppler effect, an observer at rest relative to the container will record a frequency which is different from the frequency ω_0 of the radiation emitted by these atoms at rest. Neglect the natural line width and find the form $dI_\omega/d\omega$ of the spectrum emitted by the gas at a temperature T.

Hint. The velocity distribution of the atoms may be assumed to be Maxwellian, i.e. given by

$$\frac{dN}{N} = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mv^2}{2kT}\right) dv_x\, dv_y\, dv_z\ ,$$

where dN/N is the fraction of molecules whose velocity v lies within the range $dv_x\, dv_y\, dv_z$ and k is the Boltzmann constant ($1{\cdot}38 \times 10^{-23}$ J K^{-1}). Since the velocity of the molecules is much smaller than the velocity of light, all terms of order higher than v/c may be neglected in the formula for the Doppler frequency change (see problem 10.32).

12.75 An emitting atom, which may be described by the harmonic oscillator model, is in translational motion in a gas. As a result of collisions with other atoms, there are discontinuous changes in its oscillations. The probability that the atom will not undergo a collision between time τ and $\tau + d\tau$ is given by $dW(\tau) = \frac{1}{2}\Gamma \exp(-\frac{1}{2}\Gamma_\tau)\, d\tau$, so that the average time between collisions is $\bar{\tau} = 2/\Gamma$. Neglecting the natural line width, find the form of the spectrum emitted by the oscillator, $dI_\omega/d\omega$.

12.76* A group of waves of total intensity $S = \int_0^\infty S_\omega\, d\omega$ (S is the total amount of energy passing through a unit area) is incident on a three-dimensional isotropic oscillator. The spectral width of the group is large compared with the natural width of a spectral line of the oscillator, γ, and the velocity of the electron is much smaller than the velocity of light. Find the energy absorbed by the oscillator from the wave group, taking into account the braking radiation (bremsstrahlung). What will be the effect on your result of the polarisation and the direction of propagation of the waves making up the group?

12.77 Find the total amount of energy ΔW absorbed by a one-dimensional oscillator of frequency ω_0 from a group of waves of spectral density S_ω in the following three cases: (a) a linearly polarised plane group of waves in which the $\boldsymbol{E}$ vector is at an angle ϑ to the axis of the oscillator, (b) an unpolarised plane group of waves travelling at an angle θ to the oscillator axis, and (c) an isotropic radiation field (plane waves with any polarisation and direction of propagation are incident on the oscillator with equal probability).

12.78* A linearly polarised wave is incident on an isotropic harmonic oscillator. Assuming that the velocity of the electron is much less than the velocity of light, find the differential and total scattering cross sections, $d\sigma/d\Omega$ and σ taking into account the radiative friction force. Consider in particular the cases of weakly and strongly bound electrons.

12.79 A plane, circularly polarised, electromagnetic wave is scattered by a free charge. Determine the scattered field $\boldsymbol{H}$ and investigate its polarisation. Find also the differential and total scattering cross sections, $d\sigma/d\Omega$ and σ.

12.80 An unpolarised plane wave is scattered by a free charge. Find the degree of depolarisation ρ of the scattered wave as a function of the scattering angle ϑ.

12.81* A linearly polarised wave is scattered by a free charge, which moves with a relativistic velocity $\boldsymbol{v}$ in the direction of propagation of the wave. Find the differential scattering cross section. Consider also the scattering of an unpolarised wave.

Hint. Use equation (12.b.4) and express $\dot{\boldsymbol{v}}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{H}$.

12.82* An isotropic, harmonic oscillator of frequency ω_0, charge e, and mass m is placed in a weak, uniform, constant magnetic field $\boldsymbol{H}$. Determine the motion of the oscillator, and investigate the polarisation of the radiation emitted by it. Note that this harmonic oscillator is a model of an atom in an external, magnetic field, and hence the solution of the problem will be equivalent to the classical theory of the Zeeman effect.

d Expansion of an electromagnetic field in terms of plane waves

An electromagnetic field is a function of the two independent variables $\boldsymbol{r}$ and t. In many instances it is convenient to use the Fourier expansion for the field. The following expansions are frequently employed.

(1) Expansion in terms of monochromatic waves:

$$f(\boldsymbol{r}, t) = \int_{-\infty}^{\infty} f_\omega(\boldsymbol{r}) \exp(-i\omega t)\, d\omega\,, \tag{12.d.1}$$

$$f_\omega(\boldsymbol{r}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\boldsymbol{r}, t) \exp(i\omega t)\, dt\,. \tag{12.d.1'}$$

(2) Expansion in terms of plane waves:

$$f(\boldsymbol{r}, t) = \int f_{\boldsymbol{k}}(t) \exp[i(\boldsymbol{k}\cdot\boldsymbol{r})]\, d^3k\,, \tag{12.d.2}$$

$$f_{\boldsymbol{k}}(t) = \frac{1}{(2\pi)^3}\int f(\boldsymbol{r}, t) \exp[-i(\boldsymbol{k}\cdot\boldsymbol{r})]\, d^3r\,, \tag{12.d.2'}$$

where

$$d^3\boldsymbol{k} \equiv dk_x\, dk_y\, dk_z\,.$$

(3) Expansion in terms of plane monochromatic waves:

$$f(r, t) = \int f_{k\omega} \exp[\mathrm{i}(k \cdot r) - \mathrm{i}\omega t] \, \mathrm{d}^3k \, \mathrm{d}\omega \,, \tag{12.d.3}$$

$$f_{k\omega} = \frac{1}{(2\pi)^4} \int f(r, t) \exp[-\mathrm{i}(k \cdot r) + \mathrm{i}\omega t] \, \mathrm{d}^3r \, \mathrm{d}t \,. \tag{12.d.3$'$}$$

It follows from Maxwell's equations that the frequency ω is a function of the wavevector k. The equation $\omega = \omega(k)$ is known as the dispersion relation. Since the field components $f(r, t)$ are real, we have

$$f_\omega = f^*_{-\omega} \,, \qquad f_k = f^*_{-k} \,, \qquad f_{k\omega} = f^*_{-k,-\omega} \,. \tag{12.d.4}$$

Equations (12.d.2) and (12.d.3) describe the field in all space. It follows that the integrals in these formulae are to be evaluated over all values of the wavevectors, and over all space.

Equations (A1.14) and (A1.15) of appendix 1 are very useful in connection with the Fourier expansions. In particular, equation (A1.15) and equation (12.d.4) may be used to show that

$$\left.\begin{aligned} \int_{-\infty}^{\infty} f^2(t) \, \mathrm{d}t &= 4\pi \int_0^{\infty} |f_\omega|^2 \, \mathrm{d}\omega \,, \\ \int_{-\infty}^{\infty} f^2(r, t) \, \mathrm{d}r &= (2\pi)^3 \iiint |f_k|^2 \, \mathrm{d}^3k \,. \end{aligned}\right\} \tag{12.d.5}$$

The expansion into plane monochromatic waves is of major importance in quantum electrodynamics where each of these waves is identified with a photon, i.e. with a particle moving with the velocity of light. The energy $\mathcal{E}$ and the momentum p of photons are related to the frequency ω and the wavevector k by the formulae

$$\mathcal{E} = \hbar\omega \,, \qquad p = \hbar k \,. \tag{12.d.6}$$

12.83 Prove equations (12.d.5).

12.84 Find the relation between the Fourier components of the fields $\boldsymbol{E}, \boldsymbol{H}$ and the potentials A, φ (consider all three forms of the Fourier expansion).

12.85 Write down Maxwell's equations for the Fourier components for the three forms of Fourier expansion considered above. Assume that the whole of space is filled by a homogeneous isotropic dispersive medium with permittivity $\epsilon(\omega)$ and magnetic permeability $\mu(\omega)$.

12.86 Write down the d'Alembert equations and the Lorentz condition for the Fourier components of the potentials $A(r, t)$ and $\varphi(r, t)$. Consider all three forms of Fourier expansion and assume that all space is filled with a homogeneous isotropic medium with parameters $\epsilon(\omega)$ and $\mu(\omega)$.

12.87* Find the plane-wave expansion for the potential φ of the Coulomb field due to a fixed point charge.

12.88 Find the plane-wave expansion for the electric field $\boldsymbol{E}$ due to a point charge e.

12.89 A point charge moves with a velocity $\boldsymbol{v}$ = constant in vacuo. Expand the potentials φ, $\boldsymbol{A}$ and the fields $\boldsymbol{E}$, $\boldsymbol{H}$ into plane monochromatic waves.

12.90* Find the potentials $\varphi(\boldsymbol{r}, t)$, $\boldsymbol{A}(\boldsymbol{r}, t)$ due to a point charge in uniform motion (see the solution of problem 10.68), by using the plane-wave expansions for these potentials obtained in the preceding problem.

Hint. In order to evaluate the integral with respect to $\boldsymbol{k}$ carry out the substitution $k_x \to k_x[1-(v^2/c^2)]^{-1/2}$, $k_y \to k_y$, $k_z \to k_z$. Assume that the x-axis is parallel to the direction of the velocity and use the plane-wave expansion for the field of a fixed point charge (see problem 12.87).

12.91* A neutral point system of charges moves in vacuo with a uniform velocity $\boldsymbol{v}$. Find the electromagnetic field potentials $\varphi(\boldsymbol{r}, t)$, $\boldsymbol{A}(\boldsymbol{r}, t)$ by using the Fourier expansion in terms of plane monochromatic waves and assuming that the electric and magnetic dipole moments of the system in the laboratory frame are given.

Hint. The electric charge density and the current density of the system are given by

$$\boldsymbol{j} = c\,\mathrm{curl}[\boldsymbol{m}\delta(\boldsymbol{r}-\boldsymbol{v}t)]+\frac{\partial}{\partial t}[\boldsymbol{p}\delta(\boldsymbol{r}-\boldsymbol{v}t)]\,,$$

$$\rho = -\mathrm{div}[\boldsymbol{p}\delta(\boldsymbol{r}-\boldsymbol{v}t)]\,.$$

12.92 Find the potentials of the field due to a uniformly moving magnetic dipole having a moment $\boldsymbol{m}_0$ in its own rest system (a) for $\boldsymbol{m}_0 \parallel \boldsymbol{v}$ and (b) for $\boldsymbol{m}_0 \perp \boldsymbol{v}$.

Hint. Use the transformation formulae of problem 10.73.

12.93 Find the field due to a uniformly moving electric dipole of moment $\boldsymbol{p}_0$ in its own rest system, using the results of problem 12.91 (see the solution of problem 10.71).

12.94 Show that the plane-wave components of the Fourier expansion of an irrotational vector are parallel to $\boldsymbol{k}$ (longitudinal), while the Fourier components of a solenoidal vector are perpendicular to $\boldsymbol{k}$ (transverse).

12.95* Find the equations which are satisfied in vacuo by the irrotational and solenoidal parts of the electromagnetic field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$. Show that the irrotational part of the electric field $\boldsymbol{E}_\parallel(\boldsymbol{r}, t)$ represents the instantaneous (unretarded) Coulomb field determined by the charge distribution at the same instant of time for which $\boldsymbol{E}_\parallel$ is defined.

12.96* Find the plane-wave expansion for the vector potential $\boldsymbol{A}(\boldsymbol{r}, t)$ in vacuo when $\rho = 0, \boldsymbol{j} = 0$ $(\varphi = 0)$. The field occupies all space. Write down the Fourier amplitudes of these waves in the form

$$A_{k\lambda}(t) = \frac{c}{\pi\sqrt{2}} q_{k\lambda}(t) \boldsymbol{e}_{k\lambda} \,,$$

where $\boldsymbol{e}_{k\lambda}$ is a unit vector which characterises the direction of polarisation of the given transverse wave, so that $(\boldsymbol{k} \cdot \boldsymbol{e}_{k\lambda}) = 0$ (see the beginning of section a, chapter 8). In this representation there will be two independent polarisation unit vectors ($\lambda = 1, 2$) for each $\boldsymbol{k}$. The unit vectors $\boldsymbol{e}_{k_1}$ and $\boldsymbol{e}_{k_2}$ are orthogonal, i.e. $(\boldsymbol{e}_{k_1} \cdot \boldsymbol{e}^*_{k_2}) = (\boldsymbol{e}^*_{k_1} \cdot \boldsymbol{e}_{k_2}) = 0$. Find the equations which are satisfied by the complex 'coordinates' $q_{k\lambda}(t)$. Express the fields $\boldsymbol{E}$, $\boldsymbol{H}$, the energy W, and the momentum $\boldsymbol{G}$ of the field in terms of $q_{k\lambda}$ and $\dot{q}_{k\lambda}$.

12.97* Using the results of the preceding problem, express the field vectors $\boldsymbol{A}$, $\boldsymbol{E}$, $\boldsymbol{H}$ in terms of the real oscillator coordinates

$$Q_{k\lambda} = a_{k\lambda} \exp(-\mathrm{i}\omega t) + a^*_{k\lambda} \exp(\mathrm{i}\omega t) \,.$$

Find also the energy W and the momentum $\boldsymbol{G}$ of the field in terms of these coordinates.

12.98* The electromagnetic radiation field may be described in terms of the oscillator coordinates $q_{k\lambda}$ (see problem 12.96). Write down the differential equations for the interaction between the field and a charged, nonrelativistic particle.

12.99* Find the rate of change in the energy of a radiation field $\mathrm{d}W/\mathrm{d}t$ due to the interaction of a particle with the field. Express the rate of change in terms of the oscillator coordinates $q_{k\lambda}$ and the forces $F_{k\lambda}(t)$ (see the solution of the preceding problem).

12.100* A particle of charge e executes a simple harmonic oscillation $\boldsymbol{r} = \boldsymbol{r}_0 \sin\omega_0 t$, where $\boldsymbol{r}_0$ = constant. Use the field oscillator method (see problem 12.98) to find the angular distribution and the total intensity I of the emitted radiation[3].

12.101 A particle of charge e moves with a constant angular velocity ω_0 on a circle of radius a_0. Using the field oscillator method (see problem 12.98), investigate the polarisation of the radiation emitted by the particle, and hence find the angular distribution and the total intensity of the radiation (cf problem 12.11).

12.102* A linearly polarised wave of frequency ω is incident on a harmonic oscillator whose frequency is ω_0. Using the field oscillator method, find the differential and total scattering cross sections, $\mathrm{d}\sigma/\mathrm{d}\Omega$

[3] The problem can, of course, be solved in a much simpler way (see section a of this chapter). The method suggested in the problem is closely related to that employed in the solution of the analogous problem in quantum electrodynamics.

and σ (neglect radiative friction). Investigate the polarisation of the scattered radiation.

12.103 Find by the field oscillator method the differential and total scattering cross sections, $d\sigma/d\Omega$ and σ, for linearly polarised, circularly polarised, and unpolarised monochromatic waves scattered by a free charge (cf problems 12.78 and 12.79).

12.104 A free charge scatters (a) an unpolarised wave of frequency ω, and (b) a circularly polarised wave. Investigate the polarisation of the radiated field, using the field oscillator method (see problems 12.78 and 12.79).

The radiation emitted during the interaction of charged particles with matter [1]

In this chapter we shall use the methods of classical macroscopic electrodynamics to discuss energy losses by fast particles in matter.

The macroscopic theory does not take into account the spatial dispersion of the permittivity and magnetic permeability, and may be used whenever the medium may be looked upon as continuous, i.e. when the incident particle interacts simultaneously with a large number of atoms. This means that the macroscopic equations may be used to determine the energy communicated to those electrons in the medium which are at sufficiently large distances r from the trajectory, so that $r \gg a$, where a is of the order of the interatomic distance. In condensed media, the quantity a is of the order of the linear dimensions of the atom ($\sim 10^{-10}$ m).

The velocity of the particle v should satisfy the condition $v \gg v_{at}$, where v_{at} is the average velocity of the atomic electrons. At lower velocities, the particle will transmit its energy mainly to electrons near its trajectory, where the macroscopic equations do not hold.

Energy losses due to ionisation and excitation of atoms in the medium are referred to as ionisation losses. For a particle moving through a plasma, a considerable proportion of the energy lost is used in the excitation of oscillations of the electron gas as a whole (longitudinal plasma waves; see problem 8.45).

The medium is also found to have an important effect on the emission of transverse electromagnetic waves by moving particles. Thus, if the velocity of a charged particle in a nonabsorbing dielectric exceeds the ambient phase velocity of light, then Cherenkov radiation will be produced. The theory of this effect was given by Tamm and Frank.

The electromagnetic field produced in a medium by a moving particle may be determined from Maxwell's equations. The charge and current densities for these equations may conveniently be written in the form $\rho = e\delta[\boldsymbol{r}-\boldsymbol{r}_0(t)]$ and $\boldsymbol{j} = e\dot{\boldsymbol{r}}_0\delta[\boldsymbol{r}-\boldsymbol{r}_0(t)]$, where e is the charge of the particle and $\boldsymbol{r}_0(t)$ is its position vector. The integration of Maxwell's equations for a dispersive medium is carried out by expanding the required quantities (field vectors) in terms of Fourier integrals with respect to the coordinates and time. A system of algebraic equations is then obtained for the Fourier components (see, for example, problem 13.1).

In order to find the energy of the Cherenkov radiation emitted per unit length of the trajectory of a particle, it is necessary to determine the electromagnetic field due to this particle in the medium, and to compute the energy flux through a cylindrical surface of unit length and infinite

(1) For details of the theory see, for instance, the books by Landau and Lifshitz (1960) or Ginzburg and Syrovatskii (1964) or the paper by Fermi (1940).

radius surrounding the trajectory of the particle. The time integral of this energy flux will give the total energy emitted by the particle per unit path length in the form of electromagnetic waves. If the radius a of the cylindrical surface is finite, then the time integral of the energy flux will include not only the Cherenkov radiation, but also the energy communicated to electrons at distances $r > a$ from the trajectory of the particle.

13.1* A particle of charge e moves with a constant velocity through a homogeneous isotropic medium of permittivity $\epsilon(\omega)$ and magnetic permeability $\mu = 1$. Determine the components of the electromagnetic field due to this particle.

13.2* A particle moves with a constant velocity $v = \beta c$ in a non-absorbing dielectric. With the aid of the results of the preceding problem, investigate the field due to this particle at a large distance from its trajectory. Show that a sufficiently fast particle will emit transverse electromagnetic waves (Cherenkov effect). Find the conditions under which this radiation will be emitted, and the total Cherenkov loss, w_{Ch}, per unit path length.

13.3 A particle of charge e moves with a constant velocity through a medium whose permittivity is given by the approximate formula

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega_0^2 - \omega^2} .$$

Determine the Cherenkov energy emitted per unit path length when the particle velocity satisfies the condition $v^2\epsilon_0 > c^2$, where ϵ_0 is the static permittivity. Find the range of angles within which the radiation is concentrated. Obtain a numerical estimate, assuming that

$$\omega_0 = 6 \times 10^{15}\ \mathrm{s}^{-1}, \qquad \epsilon_0 = 2, \qquad v = c .$$

13.4 Derive the condition $\cos\theta = (\beta n)^{-1}$, giving the direction of propagation of the Cherenkov radiation, by considering the interference between waves emitted by the particle at various points along its trajectory.

13.5 The Cherenkov radiation emitted by a particle may be looked upon as the result of a resonance between eigenoscillations in the medium and an impressed force associated with the moving particle. Find the condition for the appearance of the Cherenkov effect by comparing the frequencies of the eigenoscillations of the medium and those of the exciting force.

13.6 A relativistic particle of velocity v passes through a dielectric plate of thickness l in a direction at right angles to its plane. The refractive index of the plate is n and dispersion may be neglected. Find the length τ of the burst of Cherenkov radiation which an observer fixed with respect to the plate will register. Determine the energy flux I of Cherenkov radiation through the surface of the plate during the burst. Neglect edge effects.

13.7 Show that the minimum velocity $v_{\min}$ of a particle at which the Cherenkov effect becomes possible in a particular direction, satisfies the condition

$$v_{\min} \cos\theta = v_g(\omega_m) ,$$

where v_g is the group velocity of electromagnetic waves in the dielectric, ω_m is the frequency at which the refractive index has a maximum, and θ is the angle between the direction of emission and the velocity of the particle. The dielectric may be regarded as nonabsorbing.

13.8* A particle moves with a constant velocity $v = \beta c$ in a non-dispersive medium which has a permittivity ϵ and magnetic permeability μ. Determine the electromagnetic potentials φ and A. Consider the two situations $v < v_{\text{ph}}$ and $v > v_{\text{ph}}$, where v_{ph} is the phase velocity of electromagnetic waves in the medium under investigation.

13.9 A straight-line conductor lying along the x-axis is displaced along the y-axis with a constant velocity v in a nonabsorbing medium of permittivity $\epsilon(\omega)$ and magnetic permeability $\mu(\omega)$. The conductor is electrically neutral in the laboratory frame, and carries a current J in the direction of the x-axis[(2)]. Find the condition for the emission of Cherenkov radiation. Determine the total energy, w_{Ch}, of this radiation per unit path length, and find the retarding force f acting per unit length of the conductor due to the field produced by it.

Hint. The vector potential has the single component $A_x(y, z, t)$. When performing the inverse Fourier transformation use the method of going round the poles that was used in the preceding problem.

13.10 Two point charges e_1 and e_2 move with equal constant velocities v along the same straight line, but at a distance l (measured in the laboratory frame) from each other. Assuming that the motion takes place in a medium of permittivity $\epsilon(\omega)$ and magnetic permeability $\mu = 1$, find the energy of the Cherenkov radiation, w_{Ch}, emitted per unit path length. Consider the cases $e_1 = e_2 = e$, and $e_1 = -e_2 = e$. By passing to the limit, find the Cherenkov energy losses for a point electric dipole lying along the direction of motion.

13.11* Two point charges, $+e$ and $-e$, separated by a constant distance l, move with a constant velocity v in a medium of permittivity $\epsilon(\omega)$ and permeability $\mu = 1$. The line joining the charges is at an angle α to the direction of motion (l and α are measured in the laboratory system).

(2) Fast-moving, current-carrying particle beams may occur in accelerators or in certain forms of discharges.

Use the method employed in the preceding problem to find the Cherenkov energy, w_{Ch}, emitted per unit path length when l is very small.

13.12* A magnetic dipole moves with a constant velocity $v = \beta c$ in a nonabsorbing medium which has a permittivity $\epsilon(\omega)$ and permeability $\mu(\omega)$. The magnetic moment measured in the laboratory system is $\boldsymbol{m}$ and is parallel to the direction of the velocity. Determine the Cherenkov energy losses per unit path length, w_{Ch}.

Hint. Use the Fourier transformation to integrate the equations for the potentials. A moving magnetic dipole gives rise to a current $\boldsymbol{j}(\boldsymbol{r}, t) = c\,\mathrm{curl}\,\boldsymbol{m}\delta(\boldsymbol{r}-\boldsymbol{v}t)$.

13.13* A fast particle of charge e moves through a nonabsorbing dielectric of permittivity

$$\epsilon(\omega) = 1+\frac{\omega_{\mathrm{p}}^2}{\omega_0^2-\omega^2},$$

where $\omega_{\mathrm{p}}^2 = 4\pi e^2 N/m$. Determine the energy lost per unit path length at distances from the particle trajectory which are greater than the interatomic distance a. The parameter a should be chosen so that the macroscopic equations will hold in the region $r > a$. Explain the physical significance of the various terms in the expression for the energy loss.

13.14* A charged particle of velocity $v = \beta c$ moves through a plasma of permittivity (see problem 6.16)

$$\epsilon(\omega) = 1-\frac{\omega_{\mathrm{p}}^2}{\omega^2},$$

where $\omega_{\mathrm{p}}^2 = 4\pi e^2 N/m$. Find the energy losses per unit path length due to distant collisions, i.e. collisions for which $r > a$, where a is the distance at which the macroscopic equations begin to hold.

13.15* A point charge e moves in vacuo in a direction perpendicular to the boundary of a perfect conductor. Determine the spectral and the angular distribution of the radiation emitted when the charge enters the conductor. The acceleration of the charge due to its interaction with the image induced in the conductor may be neglected. The velocity of the charge is $v = \beta c$.

Hint. The field in the vacuum is due to the charge and its image moving towards each other with equal constant velocities. When the particle reaches the boundary of the conductor, its charge is instantaneously screened by the free electrons in the conductor, and this is equivalent to a sudden deceleration of the charge and its image at the same point, i.e. on the boundary of the conductor.

13.16* A point charge e has a velocity $v = \beta c$ and moves in a vacuum in a direction perpendicular to the boundary of a nonabsorbing dielectric of permittivity $\epsilon(\omega)$ ($\mu = 1$). Neglecting the acceleration of the charge due to image forces, determine the spectral and angular distribution of the radiation emitted into the vacuum ($z > 0$ in figure 13.16.1) when the particle enters the dielectric.

Hint. The charge and current densities produced by the moving particle should be replaced by the equivalent set of harmonic oscillators. In order to determine the field in the wave zone, use the reciprocity theorem $(\boldsymbol{p}_{\mathrm{B}} \cdot \boldsymbol{E}_{\mathrm{A}}(\mathrm{B})) = (\boldsymbol{p}_{\mathrm{A}} \cdot \boldsymbol{E}_{\mathrm{B}}(\mathrm{A}))$, where $\boldsymbol{E}_{\mathrm{B}}(\mathrm{A})$ is the field produced at the point A by the harmonic dipole oscillator $\boldsymbol{p}_{\mathrm{B}}$ at B, and $\boldsymbol{E}_{\mathrm{A}}(\mathrm{B})$ is the field produced at B by the oscillator $\boldsymbol{p}_{\mathrm{A}}$ at A. Fresnel's formulae may be used to calculate $\boldsymbol{E}_{\mathrm{A}}(\mathrm{B})$, since the point of observation A is at a large distance from the point at which the charge enters the dielectric (wave-zone approximation).

Plasma physics [1]

a The motion of separate particles in a plasma

Electrical and magnetic fields very strongly affect the motion of charged particles in a plasma. They are produced by the electrons and ions of the plasma and also by external sources. If there are few collisions between the particles in the plasma, during time intervals much shorter than the time between collisions each separate particle moves under the influence of the macroscopic fields $\boldsymbol{E}$ and $\boldsymbol{H}$ which exist in the plasma and their motion is described by the equations of motion from mechanics, (11.a.1) and (11.b.9). In the case of inhomogeneous and variable fields the integration of the exact equations is, as a rule, a complicated mathematical problem.

The picture of the particle motion can be appreciably simplified if the magnetic field is strong and varies slowly in space and time, while the electrical field is weak [see inequalities (14.a.6) to (14.a.6″)]. In that case one can take the action of the electrical field and also the spatial and temporal inhomogeneities of the magnetic field into account, with the use of perturbation theory. The motion of the particles proceeds as follows: at each moment in time the particle is rotating quickly about the direction of the magnetic field lines with the cyclotron frequency $ceH/\mathcal{E}$, where e is the particle charge and $\mathcal{E}$ is its energy. The centre around which the particle moves (guiding centre) travels along the magnetic field line and is also slowly displaced at right angles to it under the influence of the electrical field and the inhomogeneities in the magnetic field. Moreover, the transverse and longitudinal momenta of the particle change their absolute magnitude slowly.

The approximation corresponding to this picture of the particle motion is called the guiding-centre or drift approximation, and the motion of the guiding centre at right angles to the magnetic field lines is called drift. The equations in the drift approximation can be derived by averaging the exact equations of motion over the fast rotation of the particles around the magnetic field line, using inequalities (14.a.6) to (14.a.6″). The set of drift equations has the form

$$\dot{\boldsymbol{r}} = v_{\parallel}\boldsymbol{h} + \frac{c}{H^2}[\boldsymbol{E} \wedge \boldsymbol{H}] + \tfrac{1}{2}v_{\perp}R_{\perp}\left[\boldsymbol{h} \wedge \frac{\nabla H}{H}\right] + v_{\parallel}R_{\parallel}[\boldsymbol{h} \wedge (\boldsymbol{h} \cdot \nabla)\boldsymbol{h}]\,, \qquad (14.a.1)$$

$$\dot{p}_{\parallel} = +\tfrac{1}{2}p_{\perp}v_{\perp}\,\mathrm{div}\boldsymbol{h} + e(\boldsymbol{E} \cdot \boldsymbol{h})\,, \qquad (14.a.2)$$

$$\dot{p}_{\perp} = -\tfrac{1}{2}p_{\parallel}v_{\perp}\,\mathrm{div}\boldsymbol{h}\,. \qquad (14.a.3)$$

[1] For more details about the theory we refer to the books by Jackson (1962), Longmire (1963), Northrop (1963), Alfvén and Fälthammer (1963), Thompson (1962), van Kampen and Felderhof (1967), Akhiezer et al (1975), and the series edited by Leontovich (1965-1970).

Here $p_\parallel$ is the component of the particle momentum along the magnetic field $\boldsymbol{H}$, $p_\perp$ is the absolute magnitude of the momentum component at right angles to the magnetic field $\boldsymbol{H}$, $\boldsymbol{h} = \boldsymbol{H}/H$, the unit vector in the direction of the magnetic field,

$$R_\perp = \frac{cp_\perp}{eH}\,, \qquad R_\parallel = \frac{cp_\parallel}{eH}\,, \qquad v_\perp = \frac{p_\perp}{m}\,, \qquad v_\parallel = \frac{p_\parallel}{m}\,,$$

$$m = m_0\left(1 - \frac{v_\perp^2 + v_\parallel^2}{c^2}\right)^{-\frac{1}{2}}\,;$$

m_0 and e are the particle rest mass and charge. All field strengths on the right-hand sides of equations (14.a.1) to (14.a.3) are taken at the position of the guiding centre, and $\dot{\boldsymbol{r}}$ is the guiding-centre velocity.

The first term on the right-hand side of equation (14.a.1), $v_\parallel h$, describes the motion of the guiding centre along the magnetic field line, the second term describes the transverse motion under the action of the electrical field (electrical drift). The third and fourth terms give, respectively, the transverse drift due to the change in the magnetic field in magnitude and direction. If there are not only electrical and magnetic fields acting on the particle, but also a nonelectromagnetic force $\boldsymbol{F}$, one must add a term $c[\boldsymbol{F} \wedge \boldsymbol{H}]/eH^2$ to the right-hand side of equation (14.a.1) and a term $(\boldsymbol{F} \cdot \boldsymbol{h})$ to the right-hand side of equation (14.a.2).

Equations (14.a.2) and (14.a.3) allow us to find the change in the total particle energy with time:

$$\frac{\mathrm{d}}{\mathrm{d}t}(mc^2) = e(\boldsymbol{E} \cdot \boldsymbol{h})v_\parallel\,. \tag{14.a.4}$$

From these equations it also follows that

$$\frac{p_\perp^2}{H} = \text{constant}\,, \tag{14.a.5}$$

i.e. the quantity $I = p_\perp^2/H$ is an integral of motion. However, it is not an exact, but an approximate integral of motion, occurring because the electrical field is small and the magnetic field changes slowly. Such approximate integrals of motion are called adiabatic invariants.

Equations (14.a.1) to (14.a.3) are approximate equations of motion of the particle which are valid for slow spatial changes of E and H:

$$\left|R_\parallel\frac{\partial H}{\partial x}\right| \ll H\,, \qquad \left|R_\perp\frac{\partial H}{\partial x}\right| \ll H\,, \qquad \left|R_\parallel\frac{\partial E}{\partial x}\right| \ll E\,, \qquad \left|R_\perp\frac{\partial E}{\partial x}\right| \ll E\,, \tag{14.a.6}$$

where the coordinate x may be in any direction. Moreover, there is the condition that the electrical field is small:

$$\frac{cE}{H} \ll v \tag{14.a.6'}$$

and the condition that the electrical and magnetic fields change slowly with time:

$$\omega \ll \frac{ceH}{\mathcal{E}}, \qquad (14.a.6'')$$

where ω is a frequency, which is characteristic for the change in the field.

14.1 A uniform magnetic field $\boldsymbol{H}$ and an arbitrarily oriented force $\boldsymbol{F}$ act upon a nonrelativistic particle of charge e and mass m. Show that the component of the force $\boldsymbol{F}$ which is at right angles to $\boldsymbol{H}$ causes a uniform motion (drift) of the particle with a constant velocity

$$\boldsymbol{v}_{\mathrm{d}} = \frac{c}{eH^2}[\boldsymbol{F} \wedge \boldsymbol{H}]$$

at right angles to the magnetic field lines.

Explain the occurrence of the drift qualitatively by considering the particle trajectory and the forces acting on the particle at different points of the trajectory.

14.2 Show by direct calculation that the quantity $p_\perp^2/H$ is an adiabatic invariant[(2)] for the case of a magnetic field $H(t)$ which is uniform and constant in direction, but which has an absolute magnitude that varies slowly. To do this, evaluate the electrical field and integrate the equation describing the change in the transverse particle momentum $p_\perp$ with time, assuming that during a single cyclotron period the particle trajectory can be considered to be a circle coinciding with the electrical field lines.

14.3 A system of identical noninteracting particles is in a uniform magnetic field H and has an isotropic momentum distribution. All particles have the same energy $\mathcal{E}_0$. Afterwards the magnetic field increases adiabatically to a magnitude nH. Find the angular distribution $\mathrm{d}w(\vartheta)$ and the average square of the particle energy $\overline{\mathcal{E}^2}$ in the final state.

14.4* Let a magnetic field which is constant in direction vary weakly in space as far as its absolute magnitude is concerned. Show that this nonuniformity of the field in first approximation leads to a particle drift at right angles to the field with a velocity

$$\boldsymbol{v}_{\mathrm{d}} = \frac{v_\perp R_\perp}{2H^2}[\boldsymbol{H} \wedge \nabla H],$$

where $v_\perp$ is the absolute magnitude of the particle velocity component at right angles to the direction of the field, and $R_\perp = cp_\perp/eH$ is the particle Larmor radius [cf the general formula (14.a.1)].

14.5 Use the fact that the quantity $I = p_\perp^2/H$ is invariant to prove that in the drift approximation the magnetic flux through the orbit of the

(2) For a discussion of adiabatic invariants see Northrop (1963) or ter Haar (1971).

cyclotron rotation of the particle as well as the magnetic moment of a nonrelativistic particle that is produced by its cyclotron rotation are conserved. What are the additional conditions necessary for the magnetic moment of a relativistic particle to be conserved?

14.6 A particle moves in a weakly nonuniform constant magnetic field. By using the fact that the quantity $I = p_\perp^2/H$ is invariant and the energy conservation law, show that in the drift approximation a force $\boldsymbol{F}$ acts upon the particle in the direction of the magnetic field lines, and find the magnitude of that force. Express it in terms of the magnetic moment of the cyclotron rotation of the particle.

14.7 Between the regions I and II, in which the static magnetic field is uniform and equal to H, there is a region III in which the field is larger ('magnetic mirror'). The maximum value of the field is H_m, and the shape of the field lines is shown schematically in figure 14.7.1. A particle moves in the region I with a momentum $\boldsymbol{p}$ which at a certain instant makes an angle ϑ with the field lines. Assuming the change in the field to be slow, find the relation between ϑ, H, and H_m for which the particle will be reflected from the strong field region.

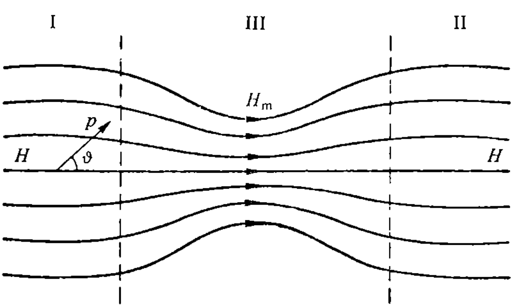


Figure 14.7.1.

14.8 The magnetic field structure in an adiabatic trap with an axially symmetric field has the shape shown schematically in figure 14.18.1.

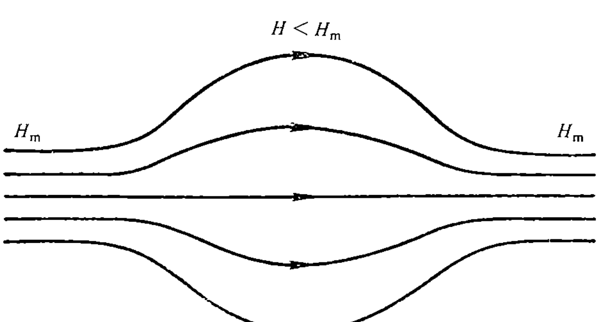


Figure 14.8.1.

A batch of particles is injected into the central part of the trap where the field strength is H; the velocity distribution of the particles is isotropic. What fraction of the particles will be retained in the trap after a long time?

14.9 In the trap with an axially symmetric field, as shown in figure 14.8.1, a set of particles is captured. The particles spend most of their time in the central part of the trap where the field is almost uniform. Let the field in the trap increase slowly with time in such a way that the shape of the magnetic lines of force do not change. Determine the change in the distance of the guiding centre of each of the particles from the axis of the trap.

14.10 In a uniform magnetic field of strength H there is a fixed point charge q. A particle of charge e and mass m, which at infinity has a longitudinal velocity component equal to $v_{\parallel}$, is scattered by the charge q. Assume that the drift approximation is applicable and neglect the change in the longitudinal velocity in the scattering process; find along which line of force the guiding centre of the particle will move after the scattering. Before the scattering it moves along the field line, the equation of which is $r = l$, $\phi = 0$, in a cylindrical system of coordinates with the z-axis going through the charge q and in the direction of the field.

14.11 The Earth's magnetic field is approximately the field of a point dipole with magnetic moment $\mu = 8 \cdot 1 \times 10^{15}$ T m^3. A proton with energy $\mathcal{E} = 50$ MeV is at a certain time in the plane of the magnetic equator at a distance of two Earth radii from the centre of the Earth and moves at right angles to the magnetic lines of force. Find the motion of the guiding centre of the proton in the drift approximation. After what time T has it turned completely around the Earth's sphere? What is the Larmor radius R of the proton? The Earth's radius is $r_* = 6380$ km, and its mass $M = 6 \times 10^{24}$ kg.

14.12* A proton is in the plane of the geomagnetic equator (see preceding problem) at a distance r from the centre of the Earth and its momentum makes an angle α with the direction of the magnetic field lines.

(a) Neglecting the gravitational field show that the guiding centre of the proton will not only move along the magnetic lines of force, but will also undergo an azimuthal drift. Find the angular drift velocity ω_{d}, and express it in terms of r and the geomagnetic latitude λ.

(b) Give the value λ_{m} corresponding to the points where the particles are reflected in the Earth's magnetic field.

(c) Find the conditions under which the proton can reach the surface of the Earth.

14.13 A bounded stationary beam of identical nonrelativistic particles of mass m, charge e, and with velocities v is incident upon a fixed particle of charge e' (figure 14.13.1). The particle density in the beam is n. Evaluate the force acting upon the fixed particle, neglecting the interactions between the incident particles. Elucidate the reason why this force becomes infinite as the beam radius $s_m \to \infty$. Is the force still infinite if e' is one of the charges in a neutral plasma?

Hint. Use the results of problem 11.98.

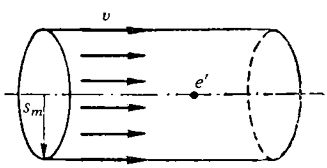


Figure 14.13.1.

14.14 A 'test' particle of charge e and mass m moves with a velocity v in a gas consisting of identical charged particles. Their mass is m', their charge is e', their density is n', and their velocity distribution function is $f(v)$ $\left[\int f(v)\,d^3v = n'\right]$. Write down an expression for the average force $\overline{F}(v)$ acting on the 'test' particle.

Hint. Use the result obtained in the solution to the preceding problem. Neglect the velocity dependence of the Coulomb logarithm.

14.15 A test particle of charge e and mass m moves in a medium consisting of randomly distributed, fixed, infinitely heavy, identical particles of charge e' and density n. What is the change with time in energy and momentum of the test particle owing to the effect of the average force exerted by the medium?

14.16 The particles in a medium have velocities which have the same absolute magnitude v_0 and which are distributed with spherical symmetry; their charge and mass are e and m respectively. Evaluate the average force $\overline{F}$ acting on a test particle of charge e' and mass m' moving with a velocity v.

14.17 Solve the preceding problem for the case where the particles in the medium move with velocities v_0 which all have the same magnitude and direction.

14.18* The electrons in a plasma show random thermal motion and have, as well as this, an organised velocity component which occurs due to the action of a uniform electrical field $\boldsymbol{E}$ produced by an external source. Give an order-of-magnitude estimate of the dependence of the average friction force $\overline{F}$ on the ordered velocity $\boldsymbol{u}$, assuming that the friction is caused by collisions with fixed ions. Show that $\overline{F}$ as function of $\boldsymbol{u}$ has a maximum and estimate the magnitude of $\overline{F}_{max}$. How will the electron gas behave under the action of an electrical field E when $E < \overline{F}_{max}/e$ and when $E > \overline{F}_{max}/e$?

b Collective motions in a plasma

A plasma, i.e. an ionised gas or a conducting liquid, consists of free charges. When one applies electrical and magnetic fields to such a system, macroscopic motion of the substance may result. In turn, the macroscopic motions may lead to the appearance of an electromagnetic field. A plasma is therefore, as a rule, a system in which the matter and the electromagnetic field interact strongly. The analysis of the behaviour of such a system is very complicated and a complete finished theory of the behaviour of a real plasma does not exist at present.

If the mean free path of the plasma particles is much shorter than the characteristic dimensions of the region in which the plasma moves one can use hydrodynamical equations in which electromagnetic forces are taken into account to describe the motion of the plasma. The electromagnetic field is described by the Maxwell equations, neglecting the displacement current as compared to the conduction current, which is valid if the field varies sufficiently slowly in time. This is called the magneto-hydrodynamical approximation. It is valid for a sufficiently dense medium in which the mean free path is short owing to the frequent collisions of the particles with one another. However, the hydrodynamical approximation can also be applied for a description of the motion of a collisionless (rarefied) plasma across a strong magnetic field. The radius of the cyclotron rotation of the particles around the magnetic lines of force plays the role of the mean free path in that case.

The equations of magneto-hydrodynamics for an incompressible conducting liquid can be written in the following form:

$$\frac{\partial \boldsymbol{v}}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla\left(p+\frac{H^2}{8\pi}\right)+\frac{1}{4\pi\rho}(\boldsymbol{H}\cdot\nabla)\boldsymbol{H}+\frac{\eta}{\rho}\nabla^2\boldsymbol{v}\,, \tag{14.b.1}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = \operatorname{curl}[\boldsymbol{v}\wedge\boldsymbol{H}]+\frac{c^2}{4\pi\sigma}\nabla^2\boldsymbol{H}\,, \tag{14.b.2}$$

$$\operatorname{div}\boldsymbol{v} = 0\,, \tag{14.b.3}$$

$$\operatorname{div}\boldsymbol{H} = 0\,. \tag{14.b.4}$$

Here $\boldsymbol{v}(\boldsymbol{r}, t)$ is the hydrodynamical (averaged) velocity of the matter; $\rho = \text{constant}$ is its density, p its pressure, σ its conductivity, and η its viscosity coefficient.

The current density and the electrical field in a moving liquid can be found from the Maxwell equation $\operatorname{curl}\boldsymbol{H} = 4\pi\boldsymbol{j}/c$ and Ohm's law, which in a moving medium has the form

$$\boldsymbol{j} = \sigma\left(\boldsymbol{E}+\frac{1}{c}[\boldsymbol{v}\wedge\boldsymbol{H}]\right)\,. \tag{14.b.5}$$

If the conductivity is very high ($\sigma \to \infty$) the last term on the right-hand side of equation (14.b.2) plays an insignificant role and we can write that equation in the form

$$\frac{\partial H}{\partial t} = \operatorname{curl}[v \wedge H] . \tag{14.b.6}$$

The magnetic lines of force are in this case 'frozen in' in the matter: when the matter moves, the lines of force move together with the particles of the medium which are situated on them. Thus the magnetic flux through any contour which moves with the liquid remains constant.

If the conductivity is low or the velocity v small, we can neglect the $\operatorname{curl}[v \wedge H]$ term in equation (14.b.2) and the equation becomes the same as equation (7.b.4).

Charge separation processes in the plasma and displacement currents become important when the frequencies corresponding to the changes in the fields are high. If we neglect electromagnetic energy losses the dielectric permittivity of the plasma has the form

$$\epsilon(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2} , \tag{14.b.7}$$

where the quantity

$$\omega_{\mathrm{p}} = \left(\frac{4\pi n e^2}{m}\right)^{1/2} \tag{14.b.8}$$

(n is the electron density, m is the electron mass, and e is the electron charge) is called the Langmuir frequency, or the plasma frequency, or the frequency of the plasma oscillations. It characterises the frequency of the oscillations of the electrons relative to the ions. Such oscillations arise for any charge distribution in the plasma (see problem 14.26). The correct description of a plasma in the case of fast changing fields is given by using the Maxwell equations together with Boltzmann's kinetic equation, but a discussion of that case lies outside the framework of the present book.

14.19* A viscous, incompressible, conducting liquid moves between two fixed parallel planes in the direction of the z-axis under the action of a constant pressure gradient $\mathrm{d}p/\mathrm{d}z$ = constant. The conductivity is σ, the viscosity η, and the distance between the planes $2a$. A constant, uniform, external magnetic field H_0 is applied at right angles to the planes in the direction of the x-axis. Evaluate the x-dependence of the velocity of the liquid and the extra magnetic field occurring in the moving liquid. Analyse the result for large and small values of H_0.

14.20 A viscous incompressible liquid lies between the parallel planes $x = \pm a$. The plane $x = -a$ moves with a velocity $-v_0$ and the plane $x = a$ moves with a velocity v_0, both in the z-direction. There is no pressure gradient, and the electrical conductivity σ and the viscosity η are given. A uniform magnetic field H_0 is applied at right angles to the planes. Evaluate the velocity of the liquid and the additional magnetic field in it.

14.21 Along a cylindrical column of a hot plasma of radius a a current J is flowing which is distributed across the cross section with a density $j(r)$. What is the r-dependence of the plasma density, if it balances the magnetic pressure produced by the current flowing along the column?

Let the plasma be isothermal and let it satisfy the perfect gas equation of state. Express the current strength J in terms of the plasma temperature T and the total number of particles N of one sign per unit length of the plasma column. Neglect viscosity and consider a stationary plasma state with $v = 0$.

14.22 What should be the distribution of the current across the cross section of the plasma column (see preceding problem) in order that the pressure of the plasma be constant across the column?

14.23 Plasma is ejected isotropically in all directions from the surface of a sphere of radius a which is rotating about its diameter with constant angular velocity Ω. The plasma velocity is constant and directed along the radius. Close to the surface of the sphere there is a magnetic field which, in the frame of reference that rotates with the sphere, has the value $\boldsymbol{H}(a, \vartheta, \phi) = \boldsymbol{H}_0(\vartheta, \phi)$, where ϕ is measured in the plane perpendicular to the axis of rotation. The energy density in the plasma is large compared to the energy density of the magnetic field so that we can neglect the effect of the field on the motion of the plasma. By assuming that the magnetic field is frozen in the plasma, find the way it varies with position and time in the region $r > a$ in the fixed frame of reference[3].

14.24 Find the shape of the lines of force of the interplanetary magnetic field in Parker's model, considered in the preceding problem. Determine the magnitude of the magnetic field and the angle θ between the lines of force and the radial direction on the Earth's orbit for the following values of the parameters: solar radius $a = 0{\cdot}7 \times 10^6$ km, average magnetic field at the surface of the Sun $H_0 \approx \approx 80$ A m^{-1}, radius of the Earth's orbit $r_0 \approx 1{\cdot}5 \times 10^8$ km, velocity of the solar wind $v = 300$ km s^{-1}, and angular velocity of the solar rotation $\Omega = 2{\cdot}7 \times 10^{-6}$ rad s^{-1}.

14.25 A uniform magnetic field $\boldsymbol{H}$, parallel to the axis of a plasma cylinder, and a radial electrical field $\boldsymbol{E}$ act upon the plasma cylinder. Evaluate that part of the energy of the system which is connected with the electrical field, taking into account the electrical drift of the plasma. Use the expression obtained for the energy to determine the transverse dielectric permittivity $\epsilon_\perp$ of a plasma in a magnetic field.

[3] Parker has used the model considered in this problem to describe the interplanetary magnetic field produced by solar plasma currents (solar wind).

14.26 A quasi-neutral plasma lies between the planes $x = \pm d$. Let there be at a certain moment a charge separation such that all electrons are in the plane $x = d$ and all ions are in the plane $x = -d$. Owing to the electrostatic forces the charges will start to perform oscillations. Neglect particle collisions and find the frequency ω of these oscillations, given that the average density of particles of one sign is n.

14.27 Find the depth of penetration into a plasma of an electromagnetic field for different frequencies. To do this, consider normal incidence of an electromagnetic wave onto a plane boundary of a plasma and calculate the reflection coefficient R and the transverse electrical field in the plasma $E(r, t)$. Take formula (14.b.7) for the dielectric permittivity.

14.28* Find the dielectric permittivity of a collisionless plasma, taking the thermal motion of the electrons into account. In order to do this, integrate the equations of motion for an electron in an external field $\boldsymbol{E} = \boldsymbol{E}_0 \exp[\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r}) - \mathrm{i}\omega t]$, evaluate the current density produced by a single particle, and average this expression over an initial equilibrium distribution of positions and velocities, assumed to be Maxwellian. Restrict yourself to the linear approximation in the electrical field strength $\boldsymbol{E}$ and neglect the motion of the ions. The average density of the electrons is n and the plasma temperature is T (in energy units).

14.29 The longitudinal dielectric permittivity of a plasma is of the form

$$\epsilon_{\parallel} = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + \frac{3k^2}{\omega^2} v_{\parallel}^2\right),$$

if the thermal motion of the particles is taken into account; here $v_{\parallel}^2 = T/m$ and the second term within the brackets is small compared to unity. Evaluate the phase and group velocities of longitudinal plasma waves.

14.30 At time $t = 0$ the charge neutrality in a plasma is violated and as a result there appears a volume charge with density $\rho(\boldsymbol{r}, 0)$.

(a) Evaluate the charge density $\rho(\boldsymbol{r}, t)$ for $t > 0$ by making use of equation (14.b.7) for the dielectric permittivity of the plasma.

(b) Describe qualitatively the change in this result if the thermal motion of the plasma particles is taken into account. Use the $\epsilon_{\parallel}$ given in the preceding problem to perform an actual calculation, with the choice of $\rho(\boldsymbol{r}, 0)$ in the form

$$\rho(\boldsymbol{r}, 0) = \rho_0 \frac{x}{x_0} \exp\left(-\frac{x^2}{x_0^2}\right),$$

where ρ_0 and x_0 are constants.

14.31 Consider the plasma model in which a gas of negatively charged electrons, with charge $-e$, move in a neutralising background of uniform charge density $n_0 e$, where n_0 is the average electron number density. If an additional infinitesimal point charge q is put at the origin, find the change $\varphi(r)$ in the electrostatic potential, for the case where the system is at temperature T (cf problem 6.12).

14.32 Discuss the conditions under which the Debye screened potential derived in the preceding problem is valid.

14.33 Find the electron number density for the case when in the plasma described by the model of problem 14.31 an additional charge q is put at the origin.

14.34 Find the total number n_{exc} of excess electrons in the Debye sphere under the conditions of the preceding problem.

14.35 Find the energy E_{int} of the interaction between the additional charge q and the charge in the Debye sphere under the conditions of problem 14.31.

14.36 Find the energy E_{Deb} of the charge density in the Debye sphere under the conditions of problem 14.31.

14.37 Estimate the average number of electrons n_{D} in the Debye sphere under the conditions of problem 14.31.

14.38* Consider an electromagnetic wave of intensity $E_0 \exp(ikx - i\omega t)$ that propagates through a plasma which, for the sake of simplicity, is taken to be one-dimensional. Discuss qualitatively how this wave will lose energy to the electrons in the plasma; the velocity distribution of the electrons may be assumed to be Maxwellian. Hence find qualitatively how the damping of such a wave depends on the temperature T and the wavenumber k (Landau damping).

solutions

Vector and tensor calculus

a Vector and tensor algebra. Transformations of vectors and tensors

1.1 $\cos\theta = (\boldsymbol{n}\cdot\boldsymbol{n}') = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\phi-\phi')$.

1.3 Since b_i $(i = 1, 2, 3)$ are the components of a vector, it follows that when the system of coordinates is rotated, $b_i' = \alpha_{ik}b_k$. Substituting b_i' into $a_i'b_i'$ = invariant and comparing with $a_k b_k$ = invariant, we find that $a_k = \alpha_{ik}a_i'$, a_k transform as the components of a vector when the system of coordinates is rotated. Since the invariant (a scalar) does not change sign on reflection, then the components a_i and b_i should either both change sign (polar vectors) or should both remain unaltered (pseudovectors).

1.10 $[\boldsymbol{a}\wedge\boldsymbol{b}]_0 = \mathrm{i}(a_{-1}b_{+1} - a_{+1}b_{-1})$, $\qquad [\boldsymbol{a}\wedge\boldsymbol{b}]_{\pm1} = \pm\mathrm{i}(a_0 b_{\pm1} - a_{\pm1}b_0)$,

$$(\boldsymbol{a}\cdot\boldsymbol{b}) = \sum_{\mu=1}^{\mu=-1}(-1)^{\mu}a_{-\mu}b_{\mu}\,, \qquad r_{\mu} = \left(\frac{3}{4\pi}\right)^{1/2} rY_{1\mu}\,.$$

1.11 The tensor reciprocal to ϵ_{ik} satisfies the conditions

$$\epsilon_{ik}\epsilon_{kl}^{-1} = \delta_{il}\,. \tag{1.11.1}$$

The above is a set of algebraic equations involving the components of the reciprocal tensor ϵ_{ik}^{-1}. The solutions are of the form

$$\epsilon_{ik}^{-1} = \frac{\Delta_{kl}}{|\epsilon|}\,, \tag{1.11.2}$$

where Δ_{ik} is the cofactor of the element ϵ_{ik} in the determinant $|\epsilon|$. It follows from equation (1.11.2) that the necessary condition for the existence of the reciprocal tensor is $|\epsilon| \neq 0$. Bearing in mind the well-known property of determinants $\Delta_{ki}\epsilon_{kl} = \delta_{il}|\epsilon|$, we see that the reciprocal tensor must, in addition to equation (1.11.1), also satisfy the conditions

$$\epsilon_{ik}^{-1}\epsilon_{kl} = \delta_{il}\,.$$

If ϵ_{ik} is a symmetric tensor defined along the principal axes: $\epsilon_{ik} = \epsilon^{(i)}\delta_{ik}$ (no summation over i), then

$$\epsilon_{ik}^{-1} = \frac{1}{\epsilon^{(i)}}\delta_{ik}\,.$$

1.14 The T_{ik} form a tensor of rank 2.

1.15 In the transformation $\boldsymbol{e}_i \to \boldsymbol{e}_i'$, the coefficients $\alpha_{ik} = (\boldsymbol{e}_i'\cdot\boldsymbol{e}_k)$ in $\boldsymbol{e}_i' = \alpha_{ik}\boldsymbol{e}_k$ represent the components of the new unit vectors along the old unit vectors. On evaluating the components (figures 1.15.1 and

1.15.2) we obtain the following transformation matrices:

(a) transformation from Cartesian to spherical coordinates:

$$\hat{\alpha} = \begin{pmatrix} \sin\vartheta\cos\phi & \sin\vartheta\sin\phi & \cos\vartheta \\ \cos\vartheta\cos\phi & \cos\vartheta\sin\phi & -\sin\vartheta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix},$$

$$\hat{\alpha}^{-1} = \begin{pmatrix} \sin\vartheta\cos\phi & \cos\vartheta\cos\phi & -\sin\phi \\ \sin\vartheta\sin\phi & \cos\vartheta\sin\phi & \cos\phi \\ \cos\vartheta & -\sin\vartheta & 0 \end{pmatrix};$$

(b) transformation from Cartesian to cylindrical coordinates:

$$\hat{\alpha} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \hat{\alpha}^{-1} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

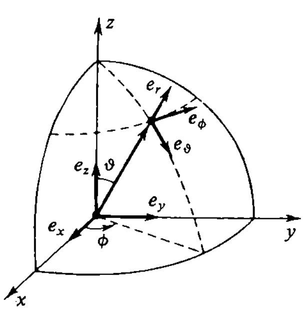


Figure 1.15.1

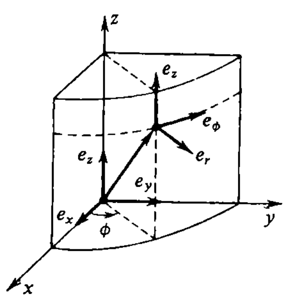


Figure 1.15.2.

1.16 Let $\hat{g}$ represent the matrix relating the components of the vector in coordinate systems S' and S ($A_i' = g_{ik}A_k$). We then have:

(a) for the case of reflection

$$\hat{g}_- = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix};$$

(b) for the case of rotation

$$\hat{g}(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where ϕ is positive in the clockwise direction (on looking in the direction of the positive z-axis).

1.17 Using the results of the preceding problem we have

$$\hat{g}(\theta,\phi,\psi) = \hat{g}(\psi)\hat{g}(\theta)\hat{g}(\phi) =$$

$$\begin{pmatrix} \cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\sin\phi\cos\psi & -\sin\phi\sin\psi + \cos\theta\cos\phi\cos\psi & \sin\theta\cos\psi \\ \sin\phi\sin\theta & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

1.18

$$\hat{D}(\theta,\phi,\psi) =$$

$$\begin{bmatrix} \tfrac{1}{2}(1+\cos\theta)\exp[\mathrm{i}(\phi+\psi)] & -\dfrac{\mathrm{i}}{\sqrt{2}}\sin\theta\exp(\mathrm{i}\psi) & -\tfrac{1}{2}(1-\cos\theta)\exp[\mathrm{i}(\psi-\phi)] \\ -\dfrac{\mathrm{i}}{\sqrt{2}}\sin\theta\exp(\mathrm{i}\phi) & \cos\theta & -\dfrac{\mathrm{i}}{\sqrt{2}}\sin\theta\exp(-\mathrm{i}\phi) \\ -\tfrac{1}{2}(1-\cos\theta)\exp[\mathrm{i}(\phi-\psi)] & -\dfrac{\mathrm{i}}{\sqrt{2}}\sin\theta\exp(-\mathrm{i}\psi) & \tfrac{1}{2}(1+\cos\theta)\exp[-\mathrm{i}(\phi+\psi)] \end{bmatrix}$$

1.19 Since the matrix corresponding to rotation through zero angle (identity transformation) is the unit matrix $\hat{1}$, it follows that for a rotation through a small angle $|\epsilon_{ik}| \ll 1$. In order to prove the relation $\epsilon_{ik} = -\epsilon_{ki}$, we can use the invariance of $r^2 = \delta_{ik}x_i x_k$ with respect to rotation. Since $x_i' = \alpha_{ik}x_k = x_i + \epsilon_{ik}x_k$, it follows that to within terms of the first order we have

$$r'^2 = r^2 + 2\epsilon_{ik}x_i x_k \ .$$

It follows from the invariance of r^2 that

$$\epsilon_{ik}x_i x_k = 0$$

for arbitrary x_i, and this is only possible when $\epsilon_{ik} = -\epsilon_{ki}$. Consider now the vector $\boldsymbol{\delta\varphi}$ with components $\delta\varphi_i = \frac{1}{2}e_{ikl}\epsilon_{kl}$. We then have

$$\boldsymbol{r}' = \boldsymbol{r} + [\boldsymbol{\delta\varphi} \wedge \boldsymbol{r}] \ ,$$

and it is clear that $\boldsymbol{\delta\varphi}$ is a vector representing an infinitesimal rotation whose direction is parallel to the axis of rotation and whose magnitude is equal to the angle of rotation.

1.22 The proof is the same for any number of transformations. Let the transformation matrix be $\hat{\alpha}$ and its determinant $|\hat{\alpha}|$. Since the matrix $\hat{\alpha}$ is orthogonal, there are n^2 equations of the form $\alpha_{ik}\alpha_{lk} = \delta_{il}$. Since the left-hand sides of these equations contain the elements of the determinant which is equal to the product of two determinants $|\hat{\alpha}|$, we have

$$|\hat{\alpha}||\hat{\alpha}| = |\hat{1}| = 1\ , \quad \text{or} \quad |\hat{\alpha}|^2 = 1\ .$$

Hence,

$$|\hat{\alpha}| = \pm 1\ .$$

It will now be shown that for rotations $|\hat{\alpha}| = +1$. For a rotation through a zero angle (identity) $|\hat{\alpha}| = |\hat{1}| = 1$. Since the elements of the matrix $\hat{\alpha}$ are continuous functions of the parameters defining the rotation (e.g. the Euler angles; see answer to problem 1.17), it follows that the result $|\hat{\alpha}| = 1$ will also hold for finite rotations.

The form of $|\hat{\alpha}|$ which corresponds to reflections is

$$|\hat{\alpha}| = \begin{vmatrix} \pm 1 & 0 & 0 & \dots \\ 0 & \pm 1 & 0 & \dots \\ 0 & 0 & \pm 1 & \dots \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix} .$$

The negative signs of the diagonal elements correspond to reflected axes. It is clear that $|\hat{\alpha}| = +1$ for an even number of such axes, and $|\hat{\alpha}| = -1$ for an odd number.

1.24 Of all the 27 quantities e_{ikl}, only six have nonzero values. The remainder have at least two identical subscripts, and in view of the skew-symmetry, they are all equal to zero ($e_{iik} = -e_{iik} = 0$). The finite components are

$$e_{123} = e_{312} = e_{231} = -e_{321} = -e_{213} = -e_{132} = 1 .$$

Consider now the expression $\alpha_{1i}\alpha_{2k}\alpha_{3l}e_{ikl}$. Recalling the definition of a third-order determinant and the definition of e_{ikl}, we may rewrite this expression in the form

$$\alpha_{1i}\alpha_{2k}\alpha_{3l}e_{ikl} = |\hat{\alpha}| = +1 = e'_{123} .$$

On interchanging two subscripts on the left, e.g. 1 and 2, we have

$$\alpha_{2i}\alpha_{1k}\alpha_{3l}e_{ikl} = -\alpha_{1k}\alpha_{2i}\alpha_{3l}e_{kil} = -e'_{123} = e'_{213} \dots .$$

It is clear from these equations that the e_{ikl} transform on rotation as a tensor of rank 3. The quantities e_{ikl} remain unaltered on reflection and hence they form an axial tensor of rank 3. This tensor has the interesting property that its components are the same in all coordinate systems.

1.25 The tensor A_{ik} may be written in the form of the table

$$A_{ik} = \begin{pmatrix} 0 & A_{12} & -A_{31} \\ -A_{21} & 0 & A_{23} \\ A_{31} & -A_{23} & 0 \end{pmatrix} .$$

Let $A_{23} = A_1$, $A_{31} = A_2$, and $A_{12} = A_3$. These three equations may also be written in the form $A_i = \frac{1}{2}e_{ikl}A_{kl}$, where e_{ikl} is the completely antisymmetric unit tensor of rank 3 (see preceding problem). Since A_{kl} is a tensor of rank 2, it follows that the quantities A_i ($i = 1, 2, 3$) form a vector. The vector A_i is called the dual of the tensor A_{ik}.

1.26

$$[A \wedge B]_i = e_{ikl} A_k B_l \,, \qquad \mathrm{curl}_i A = e_{ikl} \frac{\partial A_l}{\partial x_k} \,.$$

$[A \wedge B]$ and curl$A$ may be looked upon as skew-symmetric tensors of rank 2, or as their dual vectors whose components do not change sign on reflection (pseudovectors).

1.28 (a) $a^2(b \cdot c) + (a \cdot b)(a \cdot c)$; (b) $([[a \wedge b] \wedge c] \cdot [[a' \wedge b'] \wedge c'])$.

1.30 $(a \cdot a')(b \cdot b')(c \cdot c') + (a \cdot b')(b \cdot c')(c \cdot a') + (b \cdot a')(c \cdot b')(a \cdot c')$
$-(a \cdot c')(c \cdot a')(b \cdot b') - (a \cdot b')(b \cdot a')(c \cdot c') - (b \cdot c')(c \cdot b')(a \cdot a')$.

1.31 Consider the proof for a vector and a tensor of rank 2:

(a) Since the components of the vector should, by definition, be the same in all systems of coordinates, it follows that for any rotation $A_i' = A_i$, i.e.

$$A_x' = A_x \,, \qquad A_y' = A_y \,, \qquad A_z' = A_z \,. \tag{1.31.1}$$

Let us rotate the system of coordinates about the z-axis through an angle π. Since, in general, the transformation formulae for rotations are $A_i' = \alpha_{ik} A_k$, we find that

$$A_x' = -A_x \,, \qquad A_y' = -A_y \,, \qquad A_z' = A_z \,. \tag{1.31.2}$$

Equations (1.31.1) and (1.31.2) are consistent only if $A_x = A_y = 0$. Next, consider the rotation of the coordinate system about the x-axis through an angle π. As before, we can show that $A_z = 0$, i.e. the vector $A = 0$ if its components are independent of the choice of the coordinate system.

(b) An arbitrary tensor of rank 2 may be written down in the form of a sum of a symmetric and a skew-symmetric tensor:

$$T_{ik} = S_{ik} + A_{ik} \,.$$

The skew-symmetric tensor is equivalent to a pseudovector (see problem 1.25), and in view of the above property of a vector, its components are independent of the choice of the coordinate system only if they are equal to zero. We shall therefore consider the symmetric tensor S_{ik}.

Let us select a coordinate system in which S_{ik} has the diagonal form $\lambda^{(i)} \delta_{ik}$. If the $\lambda^{(i)}$ are not all equal, then the components of the tensor will depend on the choice of the axes, i.e. on which subscript (1, 2, or 3) is used to represent a given axis. The components of the tensor will be independent of the choice of the axes only if $\lambda^{(1)} = \lambda^{(2)} = \lambda^{(3)} = \lambda$. The tensor will then be of the form $\lambda \delta_{ik}$.

1.32 The required mean values are given by the following integrals

$$\overline{n_i} = \frac{1}{4\pi} \int n_i \, \mathrm{d}^2\omega \,, \qquad \overline{n_i n_k} = \frac{1}{4\pi} \int n_i n_k \, \mathrm{d}^2\omega \,, \ldots \,. \tag{1.32.1}$$

However, instead of the direct evaluation of these integrals it is more convenient to use a method based upon the transformation properties of the quantities under consideration. It is clear that the quantities $\overline{n_i}$, $\overline{n_i n_k}$, and so on, are tensors of rank 1, 2, 3, 4, ..., respectively. On the other hand, it follows from their definition [equation (1.32.1)] that these quantities should be independent of the choice of the coordinate system. They are therefore expressed in terms of tensors whose components are independent of the choice of the coordinate system.

Consider $\overline{n_i}$. Since the only vector whose components are independent of the choice of the coordinate system is the zero vector (see problem 1.31), it follows that $\overline{n_i} = 0$.

The tensor $\overline{n_i n_k}$ should be capable of being expressed in terms of a symmetric tensor of rank 2 whose components are independent of the choice of the coordinate system. The only tensor satisfying this condition is δ_{ik}.

We therefore have

$$\overline{n_i n_k} = \lambda \delta_{ik} ,$$

where λ can be determined by contracting[1] the tensor:

$$\overline{n_i n_i} = \overline{n^2} = 1 = 3\lambda , \qquad \lambda = \tfrac{1}{3} .$$

Similarly,

$$\overline{n_i n_k n_l} = 0 ,$$

$$\overline{n_i n_k n_l n_m} = \tfrac{1}{15}(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl}) .$$

1.33 $\tfrac{1}{3}a^2$, $\quad \tfrac{1}{3}(\boldsymbol{a} \cdot \boldsymbol{b})$, $\quad \tfrac{1}{3}\boldsymbol{a}$, $\quad \tfrac{2}{3}a^2$, $\quad \tfrac{2}{3}(\boldsymbol{a} \cdot \boldsymbol{b})$,
$\tfrac{1}{15}[(\boldsymbol{a} \cdot \boldsymbol{b})(\boldsymbol{c} \cdot \boldsymbol{d}) + (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) + (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c})]$.

1.34 n^2, n'^2, l^2, $(\boldsymbol{n} \cdot \boldsymbol{n}')$, $([\boldsymbol{n} \wedge \boldsymbol{n}'] \cdot \boldsymbol{l})$, $(\boldsymbol{n} \cdot \boldsymbol{l})^2$, $(\boldsymbol{n}' \cdot \boldsymbol{l})^2$, $(\boldsymbol{n} \cdot \boldsymbol{l})(\boldsymbol{n}' \cdot \boldsymbol{l})$.

1.35 $(\boldsymbol{n} \cdot \boldsymbol{l})$, $(\boldsymbol{n}' \cdot \boldsymbol{l})$, $(\boldsymbol{n}_1 \cdot [\boldsymbol{n}_2 \wedge \boldsymbol{n}_3])$.

b Vector analysis

1.36

$$\nabla_{\pm 1} = \mp \frac{1}{\sqrt{2}} \exp(\pm i\phi) \left(\sin\vartheta \frac{\partial}{\partial r} + \frac{\cos\vartheta}{r} \frac{\partial}{\partial \vartheta} \pm \frac{i}{r \sin\vartheta} \frac{\partial}{\partial \phi} \right) ,$$

$$\nabla_0 = \cos\vartheta \frac{\partial}{\partial r} - \frac{\sin\vartheta}{r} \frac{\partial}{\partial \vartheta} .$$

1.37 $\operatorname{div}\boldsymbol{r} = 3$, $\operatorname{curl}\boldsymbol{r} = 0$, $\operatorname{grad}(\boldsymbol{l} \cdot \boldsymbol{r}) = \boldsymbol{l}$, $(\boldsymbol{l} \cdot \nabla)\boldsymbol{r} = \boldsymbol{l}$.

1.38 $\operatorname{curl}[\boldsymbol{\omega} \wedge \boldsymbol{r}] = 2\boldsymbol{\omega}$.

[1] Contracting of a tensor consists in the summation over two indices which are the same.

1.41

$$\operatorname{grad}\varphi(r) = \frac{\boldsymbol{r}}{r}\varphi'\,; \qquad \operatorname{div}\varphi(r)\boldsymbol{r} = 3\varphi + r\varphi'\,;$$

$$\operatorname{curl}\varphi(r)\boldsymbol{r} = 0\,; \qquad (\boldsymbol{l}\cdot\nabla)\varphi(r)\boldsymbol{r} = \boldsymbol{l}\varphi + \frac{\boldsymbol{r}(\boldsymbol{l}\cdot\boldsymbol{r})}{r}\varphi'\,.$$

Here, and also below, differentiation with respect to r is indicated by a prime.

1.42

$$\varphi(r) = \frac{\text{const}}{r^3}\,.$$

1.43

$$\operatorname{div}(\boldsymbol{r}\cdot\boldsymbol{a})\boldsymbol{b} = (\boldsymbol{a}\cdot\boldsymbol{b}), \qquad \operatorname{curl}(\boldsymbol{r}\cdot\boldsymbol{a})\boldsymbol{b} = [\boldsymbol{a}\wedge\boldsymbol{b}],$$

$$\operatorname{div}(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{r} = 4(\boldsymbol{a}\cdot\boldsymbol{r}), \qquad \operatorname{curl}(\boldsymbol{a}\cdot\boldsymbol{r})\boldsymbol{r} = [\boldsymbol{a}\wedge\boldsymbol{r}],$$

$$\operatorname{div}[\boldsymbol{a}\wedge\boldsymbol{r}] = 0, \qquad \operatorname{curl}[\boldsymbol{a}\wedge\boldsymbol{r}] = 2\boldsymbol{a},$$

$$\operatorname{div}\varphi(r)[\boldsymbol{a}\wedge\boldsymbol{r}] = 0, \qquad \operatorname{curl}\varphi(r)[\boldsymbol{a}\wedge\boldsymbol{r}] = (2\varphi + r\varphi')\boldsymbol{a} - \frac{\boldsymbol{r}(\boldsymbol{a}\cdot\boldsymbol{r})}{r}\varphi',$$

$$\operatorname{div}[\boldsymbol{r}\wedge[\boldsymbol{a}\wedge\boldsymbol{r}]] = -2(\boldsymbol{a}\cdot\boldsymbol{r}), \qquad \operatorname{curl}[\boldsymbol{r}\wedge[\boldsymbol{a}\wedge\boldsymbol{r}]] = 3[\boldsymbol{r}\wedge\boldsymbol{a}].$$

1.44

$$\operatorname{grad}(\boldsymbol{A}(r)\cdot\boldsymbol{r}) = \boldsymbol{A} + \frac{\boldsymbol{r}}{r}(\boldsymbol{r}\cdot\boldsymbol{A}'),$$

$$\operatorname{grad}(\boldsymbol{A}(r)\cdot\boldsymbol{B}(r)) = \frac{\boldsymbol{r}}{r}\{(\boldsymbol{A}'\cdot\boldsymbol{B}) + (\boldsymbol{A}\cdot\boldsymbol{B}')\},$$

$$\operatorname{div}\varphi(r)\boldsymbol{A}(r) = \frac{\varphi'}{r}(\boldsymbol{r}\cdot\boldsymbol{A}) + \frac{\varphi}{r}(\boldsymbol{r}\cdot\boldsymbol{A}')\,,$$

$$\operatorname{curl}\varphi(r)\boldsymbol{A}(r) = \frac{\varphi'}{r}[\boldsymbol{r}\wedge\boldsymbol{A}] + \frac{\varphi}{r}[\boldsymbol{r}\wedge\boldsymbol{A}']\,,$$

$$(\boldsymbol{l}\cdot\nabla)\varphi(r)\boldsymbol{A}(r) = \frac{(\boldsymbol{l}\cdot\boldsymbol{r})}{r}(\varphi'\boldsymbol{A} + \varphi\boldsymbol{A}')\,.$$

1.45

$$-\operatorname{grad}\left[\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3}\right] = \operatorname{curl}\left\{\frac{[\boldsymbol{p}\wedge\boldsymbol{r}]}{r^3}\right\}\,;$$

the components of this vector along the basic unit vectors $\boldsymbol{e}_r$, $\boldsymbol{e}_\vartheta$, $\boldsymbol{e}_\phi$ are respectively equal to

$$\frac{2p\cos\vartheta}{r^3}\,, \qquad \frac{p\sin\vartheta}{r^3}\,, \qquad 0\,.$$

The vector lines are formed by the intersection of the two families of surfaces $\phi = C_1$, $r = C_2\sin\vartheta$.

1.47

$$(\nabla^2 A)_r = \nabla^2 A_r - \frac{2}{r^2}A_r - \frac{2}{r^2 \sin\vartheta}\frac{\partial}{\partial\vartheta}(\sin\vartheta A_\vartheta) - \frac{2}{r^2\sin\vartheta}\frac{\partial A_\phi}{\partial\phi} ,$$

$$(\nabla^2 A)_\vartheta = \nabla^2 A_\vartheta - \frac{A_\vartheta}{r^2\sin^2\vartheta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\vartheta} - \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\phi}{\partial\phi} ,$$

$$(\nabla^2 A)_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\vartheta} + \frac{2}{r^2\sin\vartheta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\vartheta}{r^2\sin^2\vartheta}\frac{\partial A_\vartheta}{\partial\phi} .$$

1.48

$$(\nabla^2 A)_r = \nabla^2 A_r - \frac{A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\phi}{\partial\phi} ,$$

$$(\nabla^2 A)_\phi = \nabla^2 A_\phi - \frac{A_\phi}{r^2} + \frac{2}{r^2}\frac{\partial A_r}{\partial\phi} ,$$

$$(\nabla^2 A)_z = \nabla^2 A_z .$$

1.49

$$\int(\mathrm{grad}\varphi \cdot \mathrm{curl}A)\,\mathrm{d}^3r = \oint([A \wedge \mathrm{grad}\varphi] \cdot \mathrm{d}^2S) = \quad (\varphi\,\mathrm{curl}A \cdot \mathrm{d}^2S) .$$

1.50 Here, as in a number of other cases, it is convenient to consider the scalar product of the integrals with a constant vector $\boldsymbol{c}$:

$$\left(c \cdot \oint r(a \cdot n)\,\mathrm{d}^2S\right) = \oint(c \cdot r)a_n\,\mathrm{d}^2S$$
$$= \int \mathrm{div}[(c \cdot r)a]\,\mathrm{d}^3r = (a \cdot c)\int \mathrm{d}^3r = (a \cdot c)V .$$

Since $\boldsymbol{c}$ is an arbitrary vector it follows that $\int(a \cdot n)r\,\mathrm{d}^2S = aV$. Similarly, $\oint(a \cdot r)n\,\mathrm{d}^2S = aV$.

1.51

$$\oint n\varphi\,\mathrm{d}^2S = \int \mathrm{grad}\varphi\,\mathrm{d}^3r , \qquad \oint[n \wedge a]\,\mathrm{d}^2S = \int \mathrm{curl}a\,\mathrm{d}^3r ,$$

$$\oint(n \cdot b)a\,\mathrm{d}^2S = \int(b \cdot \nabla)a\,\mathrm{d}^3r .$$

1.55 Using the method of problem 1.50, it is easy to show that

$$\oint \varphi\,\mathrm{d}\boldsymbol{l} = \int[n \wedge \mathrm{grad}\varphi]\,\mathrm{d}^2S ,$$

where $\boldsymbol{n}$ is the unit normal to the surface.

1.56

$$\int([\mathrm{grad}\,u \wedge \mathrm{grad}\,f]\cdot \boldsymbol{n})\,\mathrm{d}^2S\,.$$

1.61

(a) $A+\dfrac{B}{r}$; (b) $A+B\ln\tan\dfrac{\vartheta}{2}$; (c) $A+B\phi$.

1.62 (a) $A+B\ln r$; (b) $A+B\phi$; (c) $A+Bz$.

1.64

$$\left.\begin{aligned} x &= \pm\left[\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(b^2-a^2)(c^2-a^2)}\right]^{1/2}, \\ y &= \pm\left[\frac{(\xi+b^2)(\eta+b^2)(\zeta+b^2)}{(c^2-b^2)(a^2-b^2)}\right]^{1/2}, \\ z &= \pm\left[\frac{(\xi+c^2)(\eta+c^2)(\zeta+c^2)}{(a^2-c^2)(b^2-c^2)}\right]^{1/2}; \end{aligned}\right\} \qquad (1.64.1)$$

$$h_1 = \frac{[(\xi-\eta)(\xi-\zeta)]^{1/2}}{2R_\xi}, \qquad h_2 = \frac{[(\eta-\zeta)(\eta-\xi)]^{1/2}}{2R_\eta},$$

$$h_3 = \frac{[(\zeta-\xi)(\zeta-\eta)]^{1/2}}{2R_\zeta}.$$

$$\nabla^2 = \frac{4}{(\xi-\eta)(\xi-\zeta)(\eta-\zeta)}\left[(\eta-\zeta)R_\xi\frac{\partial}{\partial\xi}\left(R_\xi\frac{\partial}{\partial\xi}\right)+(\zeta-\xi)R_\eta\frac{\partial}{\partial\eta}\left(R_\eta\frac{\partial}{\partial\eta}\right)\right.$$
$$\left.+(\xi-\eta)R_\zeta\frac{\partial}{\partial\zeta}\left(R_\zeta\frac{\partial}{\partial\zeta}\right)\right],$$

where $R_u = [(u+a^2)(u+b^2)(u+c^2)]^{1/2}$. It is clear from equation (1.64.1) that to each set of values of ξ, η, ζ there correspond eight sets of values of x, y, z.

The orthogonality of the ellipsoidal coordinate system can be proved by finding $\mathrm{grad}\,\xi$, $\mathrm{grad}\,\eta$, $\mathrm{grad}\,\zeta$ and then evaluating the scalar products $(\mathrm{grad}\,\xi\cdot\mathrm{grad}\,\eta)$ and so on, which all turn out to be equal to zero. The quantities $\mathrm{grad}\,\xi$, $\mathrm{grad}\,\eta$, $\mathrm{grad}\,\zeta$ may be found directly from the equations defining ξ, η, ζ by taking the gradient of both sides of these equations and using equation (1.64.1).

1.65

$$z = \pm\left[\frac{(\xi+c^2)(\eta+c^2)}{c^2-a^2}\right]^{1/2}, \qquad r = \left[\frac{(\xi+a^2)(\eta+a^2)}{a^2-c^2}\right]^{1/2};$$

$$h_1 = \frac{(\xi-\eta)^{1/2}}{2R_\xi}, \qquad h_2 = \frac{(\xi-\eta)^{1/2}}{2R_\eta}, \qquad h_3 = r,$$

where

$$R_\xi = [(\xi+a^2)(\xi+c^2)]^{1/2}\,, \qquad R_\eta = [(\eta+a^2)(-\eta-c^2)]^{1/2}\,.$$

$$\nabla^2 = \frac{4}{\xi-\eta}\left[R_\xi \frac{\partial}{\partial\xi}\left(R_\xi \frac{\partial}{\partial\xi}\right) + R_\eta \frac{\partial}{\partial\eta}\left(R_\eta \frac{\partial}{\partial\eta}\right)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\,.$$

1.66

$$x = \pm\left[\frac{(\xi^2+a^2)(\zeta+a^2)}{a^2-b^2}\right]^{1/2}, \qquad r = \left[\frac{(\xi+b^2)(\zeta+b^2)}{b^2-a^2}\right]^{1/2};$$

$$h_1 = \frac{(\xi-\zeta)^{1/2}}{2R_\xi}\,, \qquad h_2 = r\,, \qquad h_3 = \frac{(\xi-\zeta)^{1/2}}{2R_\zeta}$$

where

$$R_\xi = [(\xi+a^2)(\xi+b^2)]^{1/2}\,, \qquad R_\zeta = [(\zeta+a^2)(-\zeta-b^2)]^{1/2}\,;$$

$$\nabla^2 = \frac{4}{\xi-\zeta}\left[R_\xi \frac{\partial}{\partial\xi}\left(R_\xi \frac{\partial}{\partial\xi}\right) + R_\zeta \frac{\partial}{\partial\zeta}\left(R_\zeta \frac{\partial}{\partial\zeta}\right)\right] + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}\,.$$

1.67

$$h_\xi = h_\eta = \frac{a}{\cosh\xi-\cos\eta}\,, \qquad h_\phi = \frac{a\sin\eta}{\cosh\xi-\cos\eta}\,;$$

$$\nabla^2 = \frac{(\cosh\xi-\cos\eta)^3}{a^2}\left[\frac{\partial}{\partial\xi}\left(\frac{1}{\cosh\xi-\cos\eta}\frac{\partial}{\partial\xi}\right)\right.$$
$$\left.+\frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\frac{\sin\eta}{\cosh\xi-\cos\eta}\frac{\partial}{\partial\eta}\right) + \frac{1}{\sin^2\eta(\cosh\xi-\cos\eta)}\frac{\partial^2}{\partial\phi^2}\right].$$

1.68 The surfaces ρ = constant are the toroids

$$[(x^2+y^2)^{1/2}-a\cot\rho]^2+z^2 = \left(\frac{a}{\sinh\rho}\right)^2,$$

and the surfaces ξ = constant are the spherical segments

$$(z-\arctan\xi)^2+x^2+y^2 = \left(\frac{a}{\sin\xi}\right)^2,$$

$$h_\rho = h_\xi = \frac{a}{\cosh\rho-\cos\xi} \qquad h_\phi = \frac{a\sinh\rho}{\cosh\rho-\cos\xi}\,.$$

Electrostatics in vacuo

2.1

$$\varphi_1 = -2\pi\rho z^2\,, \qquad \boldsymbol{E}_1 = -4\pi\rho z \boldsymbol{e}_z \qquad \left(|z| < \frac{a}{2}\right),$$

$$\varphi_2 = -\tfrac{1}{2}\pi\rho a(4|z|-a)\,, \qquad \boldsymbol{E}_2 = -2\pi\rho a\frac{z}{|z|}\boldsymbol{e}_z \qquad \left(|z| > \frac{a}{2}\right),$$

where the z-axis is perpendicular to the surface of the plate.

2.2

$$\varphi(x, y, z) = \frac{4\pi\rho_0}{\alpha^2+\beta^2+\gamma^2}\cos\alpha x\,\cos\beta y\,\cos\gamma z\,.$$

2.3

When $z > 0$: $\qquad \varphi = \dfrac{2\pi\sigma_0}{\lambda}\exp(-\lambda z)\sin\alpha x\,\sin\beta y\,;$

when $z < 0$: $\qquad \varphi = \dfrac{2\pi\sigma_0}{\lambda}\exp(\lambda z)\sin\alpha x\,\sin\beta y\,, \qquad \lambda = (\alpha^2+\beta^2)^{1/2}\,.$

The exponential decrease in the potential along the z-axis is due to the fact that the plane contains charges of different sign.

2.4 The simplest method of solution involves the electrostatic Gauss theorem. In the solution involving integration of the Poisson equation, the Laplace operator must be expressed in terms of cylindrical coordinates. Owing to the symmetry of the system, φ is a function of r only. For the volume charge distribution we find

$$\varphi_1 = \int_r^R \frac{2\kappa r}{R^2}\,\mathrm{d}r = \kappa\left(1-\frac{r^2}{R^2}\right), \qquad E_1 = \frac{2\kappa r}{R^2} \qquad (r \leqslant R)\,;$$

$$\varphi_2 = \int_r^R \frac{2\kappa}{r}\,\mathrm{d}r = -2\kappa\ln\frac{r}{R}\,, \qquad E_2 = \frac{2\kappa}{r} \qquad (r \geqslant R)\,.$$

In the case of the surface charge distribution

$$\varphi_1 = 0\,, \qquad \varphi_2 = -2\kappa\ln\frac{r}{R}\,.$$

2.5

$$\varphi = -2\kappa\ln r\,, \qquad E = \frac{2\kappa}{r}\,,$$

where κ is the charge per unit length. The arbitrary constant in the potential is chosen so that $\varphi = 0$ when $r = 1$.

2.6

$$\varphi(x, y, z) = \frac{q}{2a}\ln\left|\frac{z-a+[(z-a^2)+x^2+y^2]^{1/2}}{z+a+[(z+a)^2+x^2+y^2]^{1/2}}\right|.$$

2.7 We use the notation

$$z_1 = z+a\,, \qquad z_2 = z-a\,,$$

$$r_{1,2} = [x^2+y^2+z_{1,2}^2]^{1/2}\,, \qquad C = \frac{z_1+r_1}{z_2+r_2}\,.$$

It follows from the solution of the preceding problem that

$$r_1+r_2 = 2a\frac{C+1}{C-1} = \text{constant} \tag{2.7.1}$$

(remember that $z_1-z_2 = 2a$).

Equation (2.7.1) shows that the equipotential surfaces are ellipsoids of revolution whose foci lie at the ends of the segment.

2.8

$$\varphi_1(r) = \frac{q}{R}\left(\frac{3}{2}-\frac{r^2}{2R^2}\right)\,, \qquad \boldsymbol{E}_1 = \frac{q\boldsymbol{r}}{R^3} \qquad (r \leqslant R)\,;$$

$$\varphi_2(r) = \frac{q}{r}\,, \qquad \boldsymbol{E}_2 = \frac{q\boldsymbol{r}}{r^3} \qquad (r \geqslant R)\,.$$

2.9

$$\varphi_1(r) = \frac{r}{R}\,, \qquad \boldsymbol{E}_1 = 0 \qquad (r < R)\,;$$

$$\varphi_2(r) = \frac{q}{r}\,, \qquad \boldsymbol{E}_2 = \frac{q\boldsymbol{r}}{r^3} \qquad (r > R)\,.$$

2.10 The electric field in the cavity is uniform and is given by

$$\boldsymbol{E} = \tfrac{4}{3}\pi\rho\boldsymbol{r}-\tfrac{4}{3}\pi\rho(\boldsymbol{r}-\boldsymbol{a}) = \tfrac{4}{3}\pi\rho\boldsymbol{a}\,.$$

2.11 $q = 4\pi\alpha(R_2-R_1)\,;$

$$E_1 = 0\,, \qquad \varphi_1 = \frac{q}{R_2-R_1}\ln\frac{R_2}{R_1} \quad \text{when} \quad r \leqslant R_1\,;$$

$$E_2 = \frac{q(r-R_1)}{R_1-R_1)r^2}\,,$$

$$\varphi_2 = \frac{q}{R_2-R_1}\left(1-\ln\frac{r}{R_2}-\frac{R_1}{r}\right) \quad \text{when} \quad R_1 \leqslant r \leqslant R_2\,;$$

$$E_3 = \frac{q}{r^2}\,; \qquad \varphi_3 = \frac{q}{r} \quad \text{when} \quad r \geqslant R_2\,.$$

The field of a sphere with a uniform surface charge distribution is obtained by letting $R_2 \to R_1 \equiv R$ keeping q constant.

2.12 The energy W for the charge distributions of problems 2.8, 2.9, and 2.11 is given by

$$W = \frac{3q^2}{5R}\,, \qquad W = \frac{q^2}{2R}\,, \qquad W = \frac{q^2}{R_2-R_1}-\frac{q^2R_1}{(R_2-R_1)^2}\ln\frac{R_2}{R_1}\,,$$

respectively.

By comparing the contributions to the energy W given by the integrals $\int_0^R$ and $\int_R^\infty$ it may be shown that most of the energy is localized outside the charge distribution (83% in the case of a uniformly charged sphere).

2.13

$$\varphi(r) = \frac{4\pi}{r}\int_0^r \rho(r')r'^2\,\mathrm{d}r' + 4\pi\int_r^\infty \rho(r')r'\,\mathrm{d}r' \ ;$$

$$E(r) = \frac{4\pi r}{r^3}\int_0^r \rho(r')r'^2\,\mathrm{d}r' \ .$$

2.15 The field due to the electron cloud in the atom is given by

$$\varphi_e(r) = -\frac{e_0}{r}\left[1-\exp\left(-\frac{2r}{a}\right)\right] + \frac{e_0}{a}\exp\left(-\frac{2r}{a}\right) ;$$

$$E_{er} = -\frac{e_0}{r^2}\left[1-\left(\frac{2r}{a}+1\right)\exp\left(-\frac{2r}{a}\right)\right] + \frac{2e_0}{a^2}\exp\left(-\frac{2r}{a}\right) .$$

The potential and the magnitude of the field strength of the total electric field in the atom are given by

$$\varphi(r) = \varphi_e(r) + \frac{e_0}{r} \ ;$$

$$E_r = \frac{e_0}{r^2}\left(\frac{2r}{a}+1\right)\exp\left(-\frac{2r}{a}\right) + \frac{2e_0}{a^2}\exp\left(-\frac{2r}{a}\right) .$$

2.16 The field strength is a maximum on the surface of the nucleus:

$$E_{\max} = \frac{Ze_0}{R^2} = 6{\cdot}4 \times 10^{20}\frac{Z}{A^{2/3}}\,\mathrm{V\ m^{-1}} \ .$$

2.17 Use the fact that the surface density is given by

$$\rho(r, \vartheta, \phi) = \sigma(\vartheta, \phi)\delta(r-a) \ .$$

2.18

$$q_{1,2} = \frac{R(R^2+a^2)^{1/2}}{qa^2}[(R^2+a^2)^{1/2}A_{1,2} - RA_{2,1}] \ .$$

2.19

$$\varphi = \frac{2q}{R^2}[(R^2+z^2)^{1/2} - |z|] \ ,$$

$$E_x = E_y = 0 \ , \qquad E_z = \frac{2q}{R^2}\left[\frac{z}{|z|} - \frac{z}{(R^2+z^2)^{1/2}}\right] ,$$

where z is the coordinate of the point of observation measured from the plane of the disc.

2.20 If the positively charged half of the ring occupies the region $x > 0$ in the xy-plane, then for $x, y \ll (R^2+z^2)/R$ it is found, after expanding the integrand $\int(\kappa/r_{12})\,dl$ into a series, that

$$\varphi = \frac{4qRx}{\pi(R^2+z^2)^{3/2}} ,$$

and hence

$$E_x = -\frac{4qR}{\pi(R^2+z^2)^{3/2}} , \qquad E_y = 0 , \qquad E_z = \frac{12qRxz}{\pi(R^2+z^2)^{5/2}} .$$

When $z \gg R$ the field is the same as the field due to an electric dipole which lies along the x-axis and has a moment equal to $4qR/\pi$.

2.21 Owing to the symmetry of the system, the potential will be independent of the azimuthal angle ϕ and hence the xz-plane may be drawn through the point of observation without loss of generality (figure 2.21.1). Therefore,

$$r_{12} = (r^2+R^2-2rR\sin\vartheta\cos\phi')^{1/2}$$

and

$$\varphi(r,\vartheta) = 2\kappa R\int_0^{\pi}\frac{d\phi'}{(r^2+R^2-2rR\sin\vartheta\cos\phi')^{1/2}} ,$$

where $\kappa = q/(2\pi R)$.

Substituting $\phi' = \pi-2\beta$ and writing

$$k^2 = \frac{4rR\sin\vartheta}{r^2+R^2+2rR\sin\vartheta} ,$$

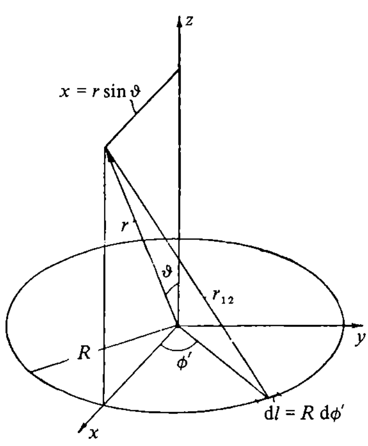


Figure 2.21.1.

we have

$$\varphi(r, \vartheta) = \frac{4\kappa R}{(r^2+R^2+2rR\sin\vartheta)^{1/2}}\int_0^{\frac{1}{2}\pi}\frac{\mathrm{d}\beta}{(1-k^2\sin^2\beta)^{1/2}} = \frac{2k\kappa}{(rR\sin\vartheta)^{1/2}}K(k)\,.$$

2.22

(a) $\varphi = \dfrac{q}{(R^2+z^2)^{1/2}}$,

where z is the distance of the point of observation from the plane of the ring.

(b) $\varphi = \dfrac{q}{r}$.

(c) Let r' be the distance from the point of observation to the ring. For $r' \ll R$ we have

$$1-k^2 \approx \frac{r'^2}{4R^2}\,, \qquad K(k) = \ln\frac{8R}{r'}\,, \qquad \varphi(r) = -2\kappa\ln r' + \text{constant}\,,$$

which is the same result as for a linear charge.

2.23

$$\varphi_1 = \tfrac{4}{3}\pi\sigma_0 r\cos\vartheta \qquad (r \leqslant R)\,, \qquad \varphi_2 = \frac{4\pi\sigma_0 R^3}{3r^2}\cos\vartheta \qquad (r \geqslant R).$$

Inside the sphere the electric field is uniform and has an intensity $E_{1z} = -\frac{4}{3}\pi\sigma_0$; outside the sphere the field is the same as for a dipole with a moment $\frac{4}{3}\pi\sigma_0 R^3$.

2.24 Owing to the axial symmetry of the field, the Laplace equation in terms of cylindrical coordinates (the polar axis along the axis of symmetry of the system) is of the form

$$\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r}\frac{\partial\varphi}{\partial r}+\frac{\partial^2\varphi}{\partial z^2} = 0\,. \qquad (2.24.1)$$

We shall seek the solution of equation (2.24.1) in the form of a power series

$$\varphi(r, z) = \sum_{n=0}^{\infty} a_n(z)r^n\,, \qquad a_0(z) = \varphi(0, z) \equiv \Phi(z)\,, \qquad (2.24.2)$$

where $\Phi(z)$ is the potential along the axis of symmetry of the system. Substituting equation (2.24.2) into equation (2.24.1), rearranging and equating to zero the coefficients of the resulting series, we obtain the recurrence relations for the coefficients $a_n(z)$, and hence

$$\varphi(r, z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{r}{2}\right)^{2n}\Phi^{(2n)}(z) = \Phi(z)-\frac{r^2}{4}\Phi''(z)+\dots\,,$$

$$E_r = -\frac{\partial\varphi}{\partial r} = \frac{r}{2}\Phi''(z)+\dots\,, \qquad E_\phi = 0\,, \qquad E_z = -\frac{\partial\varphi}{\partial z} = -\Phi'(z)+\dots\,.$$

2.25 The multipole moments to be computed are

$$Q_{lm} = \left(\frac{4\pi}{2l+1}\right)^{1/2} \int \kappa R^l Y_{lm}^*\left(\frac{\pi}{2}, \phi\right) R\, d\phi,$$

$$Q'_{lm} = \left(\frac{4\pi}{2l+1}\right)^{1/2} \int \frac{\kappa}{R^{l+1}} Y_{lm}^*\left(\frac{\pi}{2}, \phi\right) R\, d\phi.$$

Using equations (A2.1) and (A2.5) of appendix 2 we find that

$$\varphi(r, \vartheta) = \frac{q}{r}\sum_{n=0}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!}\left(\frac{R}{r}\right)^{2n} P_{2n}(\cos\vartheta), \quad \text{when}, \quad r > R;$$

$$\varphi(r, \vartheta) = \frac{q}{R}\sum_{n=0}^{\infty}(-1)^n \frac{(2n-1)!!}{(2n)!!}\left(\frac{r}{R}\right)^{2n} P_{2n}(\cos\vartheta), \quad \text{when} \quad r < R;$$

$$x!! \equiv x(x-2)\ldots \begin{cases} 1, & \text{if } x \text{ is odd,} \\ 2, & \text{if } x \text{ is even.} \end{cases}$$

Both formulae hold at $r = R$ ($\vartheta \neq \frac{1}{2}\pi$).

2.26

(a) $$\varphi \approx qa^2\frac{3z^2-r^2}{r^5} = 2qa^2\frac{P_2(\cos\vartheta)}{r^3};$$

(b) $$\varphi \approx \frac{3qa^2\sin^2\vartheta\cos\phi\sin\phi}{r^3}.$$

2.27

(a) $$\varphi \approx \frac{6qa^3P_3(\cos\vartheta)}{r^4} = qa^3\frac{15\cos^3\vartheta - 9\cos\vartheta}{r^4};$$

(b) $$\varphi \approx \frac{15qabcxyz}{r^7} = \frac{15qabc\sin^2\vartheta\cos\vartheta\sin\phi\cos\phi}{r^4},$$

2.28

$$\varphi(r, \vartheta, \phi) = \sum_{l,m}\frac{4\pi}{2l+1}\frac{r^l}{r_0^{l+1}} Y_{lm}^*(\vartheta_0, \phi_0)Y_{lm}(\vartheta, \phi) \quad \text{when} \quad r < r_0;$$

$$\varphi(r, \vartheta, \phi) = \sum_{l,m}\frac{4\pi}{2l+1}\frac{r_0^l}{r^{l+1}} Y_{lm}^*(\vartheta_0, \phi_0)Y_{lm}(\vartheta, \phi) \quad \text{when} \quad r > r_0.$$

2.29

$$\varphi(x, y, z) \approx \frac{q}{r} + q\frac{a^2(3x^2-r^2)+b^2(3y^2-r^2)+c^2(3z^2-r^2)}{10r^5}.$$

In the case of an ellipsoid of revolution

$$a = b \quad \text{and} \quad \varphi(r, \vartheta) = \frac{q}{r} + q\frac{c^2-a^2}{5}\frac{P_2(\cos\vartheta)}{r^3}.$$

In the case of a sphere

$$a = b = c \quad \text{and} \quad \varphi = \frac{q}{r}.$$

2.30 In spherical coordinates (polar axis along the axis of symmetry of the system and the origin at the centre of the rings) the potential is

$$\varphi(r, \vartheta) = -\frac{q(a^2-b^2)}{2}\frac{P_2(\cos\vartheta)}{r^3} .$$

This expression represents the potential of a linear quadrupole in which two charges of $-q$ each are at a distance of $\frac{1}{2}(a^2-b^2)^{1/2}$ from the central charge of $2q$.

2.31 Since $\delta(r) = 0$ everywhere except at $r = 0$, we have

$$q = -\int(p' \cdot \nabla)\delta(r)\,\mathrm{d}^3r = -\oint(p' \cdot n)\delta(r)\,\mathrm{d}^2S = 0 .$$

Also,

$$p_\alpha = -\int x_\alpha(p' \cdot \nabla)\delta(r)\,\mathrm{d}^3r = -\int x_\alpha p'_n\frac{\partial\delta(r)}{\partial x_n}\,\mathrm{d}^3r = \int p'_n\frac{\partial x_\alpha}{\partial x_n}\delta(r)\,\mathrm{d}^3r.$$

The latter transformation involves integration by parts. We sum over the repeated subscript n. The resulting surface integral becomes equal to zero since $\delta(r) = 0$ when $r \neq 0$. In view of the definition of the δ-function we have

$$p_\alpha = p'_n\frac{\partial x_\alpha}{\partial x_n} = p'_n\,\delta_{\alpha n} = p'_\alpha .$$

All the multipole moments of higher order are proportional to the components of r at $r = 0$ and are therefore equal to zero. For example, in the case of the quadrupole moment

$$\begin{aligned}Q_{\alpha\beta} &= -\int x_\alpha x_\beta p'_n\frac{\partial\delta(r)}{\partial x_n}\,\mathrm{d}^3r = \int\delta(r)p'_n\frac{\partial(x_\alpha x_\beta)}{\partial x_n}\,\mathrm{d}^3r\\ &= [p'_\alpha x_\beta + p'_\beta x_\alpha]_{r=0} = 0 .\end{aligned}$$

2.32 On integrating by parts n times, we have

$$\varphi(r) = q(-1)^n\int\delta(r')\prod_i(a_i \cdot \nabla')\frac{1}{|r-r'|}\,\mathrm{d}^3r' = q\prod_i(a_i \cdot \nabla)\frac{1}{r} .$$

2.33 The simplest method is to use the formula

$$\varphi = \frac{qa^2}{r^5}(3z'^2-r^2)$$

(cf solution to problem 2.26) and express z' in terms of x, y, and z (figure 2.33.1). The final result is

$$\varphi = \frac{qa^2}{r^5}[3(x\sin\gamma\cos\beta+y\sin\gamma\sin\beta+z\cos\gamma)^2-r^2] .$$

This result may also be obtained by using the fact that the components of the quadrupole moment form a tensor of rank 2. In the coordinate system (x', y', z') the components of the quadrupole moment are

$$Q'_{xx} = Q'_{yy} = Q'_{xy} = Q'_{xz} = Q'_{yz} = 0\,, \qquad Q'_{zz} = 2qa^2\,.$$

The transformation matrix is of the form

$$\hat{\alpha} = \begin{pmatrix} \cos\gamma\cos\beta & -\sin\beta & \sin\gamma\cos\beta \\ \cos\gamma\sin\beta & \cos\beta & \sin\gamma\sin\beta \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix}.$$

This matrix may then be used to evaluate the components of $Q_{\alpha\beta}$ in the (x, y, z) system with the use of the expression

$$Q_{\alpha\beta} = \sum_{\gamma,\delta} \alpha_{\alpha\gamma}\alpha_{\beta\delta}\, Q'_{\gamma\delta}\,,$$

and equation (2.0.8).

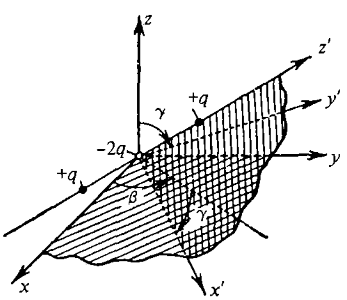


Figure 2.33.1.

2.34

$$\varphi = \frac{15qabcz}{2r^7}[(y^2-x^2)\sin 2\beta + 2xy\cos 2\beta].$$

2.35

$$\varphi = \frac{qa^2}{4r^3}(3\sin^2\vartheta\sin 2\phi - 3\cos 2\vartheta - 1).$$

2.36 According to the superposition principle we have

$$\varphi(\boldsymbol{r}) = \int_V \frac{(\boldsymbol{P}\cdot\{\boldsymbol{r}-\boldsymbol{r}'\})}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\,\mathrm{d}^3r' = \int(\boldsymbol{P}\cdot\mathrm{grad}')\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^3r'\,.$$

Application of Gauss' theorem to this expression yields

$$\varphi(\boldsymbol{r}) = \int_S \frac{P_n}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}^2S\,,$$

where S is the inner surface of the polarised sphere and $P_n = P\cos\vartheta$.

If the solution of problem 2.23 is used, it is found that

$$\varphi_1 = \frac{4\pi Pr}{3}\cos\vartheta \qquad (r \leqslant R)\,, \qquad \varphi_2 = \frac{4\pi PR^3}{3r^2}\cos\vartheta \qquad (r \geqslant R)\,.$$

2.37

$$\varphi(r) = -2\kappa\ln r + 2\sum_{n=1}^{\infty}\frac{A_n\cos n\phi + B_n\sin n\phi}{nr^n}\,,$$

where $\kappa = \int\rho(r')\,d^2S'$ is the total charge per unit length,

$$A_n = \int\rho(r')r'^n\cos n\phi'\,d^2S' \quad \text{and} \quad B_n = \int\rho(r')r'^n\sin n\phi'\,d^2S'$$

are the two-dimensional multipole moments of order n. It follows from these formulae that the potential of a dipole in the two-dimensional case is given by

$$\varphi = 2\frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^2}\,,$$

where $\boldsymbol{p} = \int\rho(r')\boldsymbol{r}'\,d^2S'$ is the dipole moment per unit length of the distribution and $\boldsymbol{r}$ is the position vector in the xy-plane.

2.38

$$\varphi(r,\phi) = -2\kappa\ln r + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r_0}{r}\right)^n\cos n(\phi-\phi_0)\,, \quad \text{when} \quad r > r_0\,,$$

$$\varphi(r,\phi) = -2\kappa\ln r + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{r}{r_0}\right)^n\cos n(\phi-\phi_0)\,, \quad \text{when} \quad r < r_0\,.$$

2.39

$$\phi(r) \approx \frac{2\kappa a}{r}\cos\phi = \frac{2(\boldsymbol{p}\cdot\boldsymbol{r})}{r^2}\,,$$

where $\boldsymbol{p}$ is the dipole moment per unit length, $\boldsymbol{r}$ is the position vector in the xy-plane ($r \gg a$), and the z-axis lies along one of the linear charges.

2.40 Along the axis of symmetry of the disc (the z-axis is drawn from the negative to the positive side of the disc),

$$\varphi(z) = \tau\Omega = 2\pi\tau\left[1 - \frac{|z|}{(R^2+z^2)^{1/2}}\right]\frac{z}{|z|}\,;$$

$$E_x = E_y = 0\,, \qquad E_z = \frac{2\pi a^2\tau z}{|z|(a^2+z^2)^{3/2}}\,.$$

2.41 (a) In cylindrical coordinates

$$E_\phi = \frac{2\tau}{r}\,, \qquad E_r = E_z = 0\,;$$

(b) $\varphi = 2\tau(\pi-\phi)\,, \qquad E_\phi = -\dfrac{1}{r}\dfrac{\partial\varphi}{\partial\phi} = \dfrac{2\tau}{r}\,; \qquad E_r = E_z = 0\,.$

The field $\boldsymbol{E}$ is identical in form with the magnetic field due to a straight line current $I = \tau c$.

2.42 The equation of the lines of force is

$$(z+a)[(z+a)^2+r^2]^{-\frac{1}{2}} \pm (z-a)[(z-a)^2+r^2]^{-\frac{1}{2}} = C\,,$$

where C is a constant. The case of charges of different sign is illustrated in figure 2.42.1a. Figure 2.42.1b illustrates the case of two charges of the same kind. The neutral point occurs at $r = 0$, $z = 0$, for the case of two charges of the same kind (figure 2.42.1b).

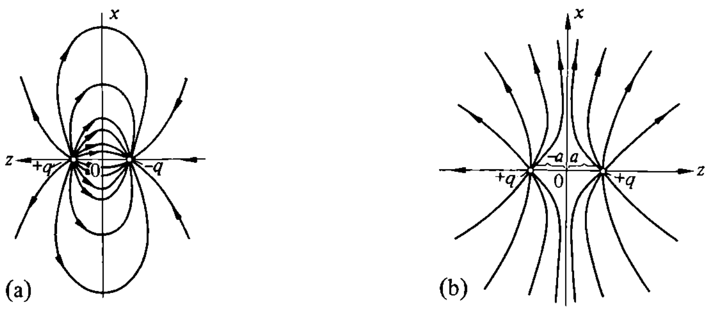


Figure 2.42.1.

2.43 It is convenient to use spherical coordinates. By letting a tend to zero, expanding into a series, and neglecting terms of the order of a^2 and higher, it is found that

$$r = C\sin^2\vartheta\,.$$

2.45 $r = C(\sin^2\vartheta|\cos\vartheta|)^{\frac{1}{2}}$, $\qquad C = $ constant.

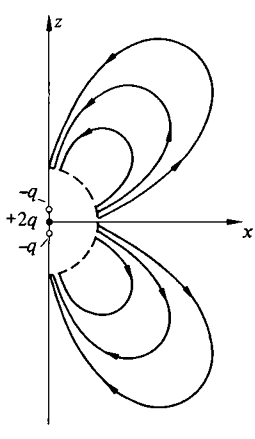


Figure 2.45.1.

It should be remembered that in the case of a quadrupole of finite dimensions this formula will only hold at large distances (figure 2.45.1).

2.46

$$q_2 = \frac{\Phi + \sqrt{2}(\sqrt{2}-1)\pi q_1}{\sqrt{2}(\sqrt{2}-1)\pi} .$$

2.47 Consider a force tube which is obtained by rotating a line of force about the z-axis. Application of the electrostatic Gauss theorem to the volume bounded by the surface of this tube and two $z =$ constant planes which do not contain charges, shows that the flux through any cross section normal to the axis of the tube,

$$\Phi(z) = \sum_i q_i \Omega_i(z)$$

(cf problem 2.45), is independent of z (when z lies between z_k and z_{k+1}). Here $\Omega_i(z) = 2\pi(\pm 1 - \cos\alpha_i)$ is the solid angle subtended by the negative side of the cross section at the point z_i at which the charge q_i is located, and α_i is the angle between the z-axis and the position vector of a point on the periphery of the normal cross section whose coordinates are r, z. The positive sign should be taken for $z > z_i$ and the negative sign for $z < z_i$. If on varying z, the normal cross section of the tube passes through a charge q_k, then $\Phi(z)$ will change discontinuously by $\pm 4\pi q_k$, but $\sum_i q_i \cos\alpha_i$ will remain constant. Writing $\cos\alpha_i$ in terms of z, z_i, and r, we obtain the required equation for the family of lines of force:

$$\sum_i \frac{q_i(z-z_i)}{[r^2+(z-z_i)^2]^{1/2}} = C , \qquad C = \text{constant} .$$

2.49 Consider the cylindrical coordinate system whose z-axis lies along the axis of the cylinder (figure 2.49.1). Instead of the condition $\varphi =$ constant on the surface S of the cylinder, it is more convenient to use the condition that $\partial\varphi/\partial\phi = 0$ on the surface S, which is a consequence of it. After differentiation we have

$$\frac{\kappa_1 x_1}{R^2 + x_1^2 - 2Rx_1\cos\phi} = \frac{\kappa_2 x_2}{R^2 + x_2^2 - 2Rx_2\cos\phi} .$$

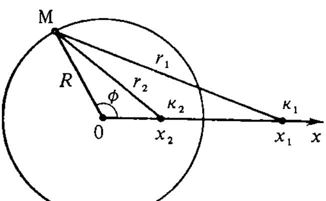


Figure 2.49.1.

On rearranging and equating terms with and without $\cos\phi$, we find that when $\kappa_1 = \kappa_2$, any cylindrical surface whose axis is parallel to the charged filaments and lies in the plane of the filaments will be an equipotential surface, provided its radius satisfies the condition $R^2 = x_1 x_2$. When $x_1 = 0$, the solution $\kappa_2 = 0$ will also exist. This corresponds to cylindrical equipotential surfaces in the field due to one of the filaments.

2.50 Referring to figure 2.50.1, the radius R of the required sphere and the position of its centre are respectively given by

$$R^2 = z_1 z_2 , \qquad \frac{z_1}{z_2} = \frac{q_1^2}{q_2^2} .$$

The potential on the surface of this sphere is zero.

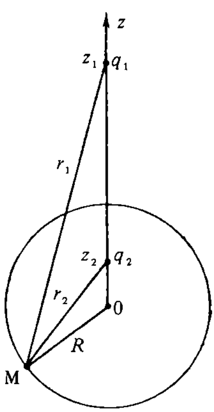


Figure 2.50.1.

2.51

$$\nabla^2\varphi = q\nabla^2\frac{\exp(-\alpha r)}{r} = q\nabla^2\frac{1}{r} + q\nabla^2\frac{\exp(-\alpha r) - 1}{r}$$

$$= 4\pi q\delta(r) + \frac{q}{r}\frac{\partial^2}{\partial r^2}\left[r\frac{\exp(-\alpha r) - 1}{r}\right] = -4\pi q\delta(r) + \frac{q\alpha^2\exp(-\alpha r)}{r} .$$

Thus, we have a point charge q at the origin and a spherical symmetric volume distribution with a density

$$\rho = -\frac{q\alpha^2\exp(-\alpha r)}{4\pi r} , \qquad \int\rho\, d^3r = -q .$$

2.52 The point charge e_0 at the origin is surrounded by a volume charge with a density

$$\rho(r) = -\frac{e_0}{\pi a^3}\exp\left(-\frac{2r}{a}\right) .$$

This is the charge distribution in the hydrogen atom (see problem 2.15).

2.53

$$U = \int \frac{e_0}{r}\rho(r)\,\mathrm{d}^3r = -\frac{e_0^2}{\pi a^3}\int_0^\infty r\exp\left(-\frac{2r}{a}\right)4\pi\,\mathrm{d}r = -\frac{e_0^2}{a}\,.$$

2.54

$$U = \frac{5e_0^2}{4a}\,.$$

2.55

$$U = \frac{q_1q_2}{a}\,, \qquad F = \frac{q_1q_2}{a^2}\,.$$

2.56

$$R = \frac{32\pi\gamma}{E_0^2}\,.$$

2.57

$$U = \int_{l_1}\int_{l_2}\frac{\kappa_1\kappa_2\,\mathrm{d}l_1\,\mathrm{d}l_2}{r_{12}} = \frac{q_1q_2}{4\pi^2ab}\int_0^{2\pi}\int_0^{2\pi}\frac{ab\,\mathrm{d}l_1\,\mathrm{d}l_2}{[c^2+a^2+b^2-2ab\cos(\phi_1-\phi_2)]^{1/2}}\,,$$

where the integration is carried out over all elements $\mathrm{d}l_1$, $\mathrm{d}l_2$ of both rings and ϕ_1, ϕ_2 are the angles which define the position of the elements. Integrating with respect to ϕ_2 and substituting $\phi_1 = \pi-2\phi$ we have

$$U = \frac{q_1q_2k}{\pi(ab)^{1/2}}K(k)\,,$$

where

$$k = \frac{2(ab)^{1/2}}{[c^2+(a+b)^2]^{1/2}} \quad \text{and} \quad K(k) = \int_0^{\frac{1}{2}\pi}\frac{\mathrm{d}\phi}{(1-k^2\sin^2\phi)^{1/2}}\,.$$

The function $K(k)$ is the complete elliptical integral of the first kind.

In evaluating the force

$$F = -\frac{\partial U}{\partial c} = -\frac{\partial U}{\partial k}\frac{\partial k}{\partial c}$$

it is necessary to use the formula [see Gradshteyn and Ryzhik, 1965, formula (8.112)]

$$2k^2\frac{\mathrm{d}K(k)}{\mathrm{d}(k^2)} = \frac{E(k)}{1-k^2} - K(k)\,,$$

where $E(k) = \int_0^{\frac{1}{2}\pi}(1-k^2\sin^2\phi)^{1/2}\,\mathrm{d}\phi$ is the complete elliptical integral of the second kind. Finally,

$$F = \frac{q_1q_2ck^3}{4\pi(ab)^{3/2}}\frac{E(k)}{1-k^2}\,.$$

2.58

$$F = -\frac{3qr(p \cdot r)}{r^5} + \frac{qp}{r^3}, \qquad N = \frac{q[p \wedge r]}{r^3}.$$

2.59

$$U = \frac{p_1 p_2}{r^3}(\sin\vartheta_1 \sin\vartheta_2 \cos\varphi - 2\cos\vartheta_1 \cos\vartheta_2),$$

where ϑ_1 is the angle between $\boldsymbol{r}$ and $\boldsymbol{p}_1$, ϑ_2 is the angle between $\boldsymbol{r}$ and $\boldsymbol{p}_2$, φ is the angle between the planes $(\boldsymbol{r}, \boldsymbol{p}_1)$ and $(\boldsymbol{r}, \boldsymbol{p}_2)$, and

$$F = \frac{3p_1 p_2}{r^4}(\sin\vartheta_1 \sin\vartheta_2 \cos\varphi - 2\cos\vartheta_1 \cos\vartheta_2).$$

The force reaches a maximum when $\vartheta_1 = \vartheta_2 = \varphi = 0$, i.e. when the dipoles are parallel.

2.60

$$U_{21} = \int \rho(r')\varphi_1(r')\, d^3r' = \sum_{l,m} \left(\frac{4\pi}{2l+1}\right)^{1/2} a_{lm} \int r'^l Y_{lm}(\vartheta', \phi')\, d^3r'$$

$$= \sum_{l,m} a_{lm} Q^*_{lm}.$$

3

Electrostatics of conductors and dielectrics

a Basic concepts and methods of electrostatics

3.1

$$\varphi_1 = \varphi_2 = \frac{2}{\epsilon_1+\epsilon_2}\frac{q}{r}\,, \qquad D_1 = \frac{2\epsilon_1}{\epsilon_1+\epsilon_2}\frac{q\mathbf{r}}{r^3}\,, \qquad D_2 = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2}\frac{q\mathbf{r}}{r^3}\,.$$

3.2

$$\varphi_1 = \varphi_2 = \varphi_3 = \frac{2\pi}{\epsilon_1\alpha_1+\epsilon_2\alpha_2+\epsilon_3\alpha_3}\frac{q}{r}\,, \qquad D_i = \frac{2\pi\epsilon_i}{\epsilon_1\alpha_1+\epsilon_2\alpha_2+\epsilon_3\alpha_3}\frac{q\mathbf{r}}{r^3}\,.$$

3.3 The boundary conditions (φ = constant on the surface of the conductor and $\varphi = 0$ when $r \to \infty$) may be satisfied by a potential of the form $\varphi = C/r$ where the constant C is determined from the conditions

$$\oint_S D_n\, \mathrm{d}^2S = 4\pi q\,, \qquad C = \frac{2q}{\epsilon_1+\epsilon_2}\,.$$

Hence we have

$$\varphi = \frac{2q}{\epsilon_1+\epsilon_2}\frac{1}{r}\,,$$

and the surface distributions are

$$\sigma_1 = \frac{q\epsilon_1}{2\pi a^2(\epsilon_1+\epsilon_2)}\,, \qquad \sigma_2 = \frac{q\epsilon_2}{2\pi a^2(\epsilon_1+\epsilon_2)}\,,$$

$$\sigma_{1\mathrm{b}} = \frac{q(\epsilon_1-1)}{2\pi a^2(\epsilon_1+\epsilon_2)}\,, \qquad \sigma_{2\mathrm{b}} = \frac{q(\epsilon_2-1)}{2\pi a^2(\epsilon_1+\epsilon_2)}\,.$$

3.4

$$C = \left[\frac{(\epsilon-1)\Omega}{4\pi}+1\right]\frac{ab}{b-a}\,.$$

3.5

$$C = \left[\frac{1}{\epsilon_1}\left(\frac{1}{a}-\frac{1}{c}\right)+\frac{1}{\epsilon_2}\left(\frac{1}{c}-\frac{1}{b}\right)\right]^{-1}$$

The bound charges are located where the dielectric is nonuniform, i.e. on the spheres with radii a, b, c:

$$\sigma_{a,\mathrm{b}} = -\frac{q}{4\pi a^2}\frac{\epsilon_1-1}{\epsilon_1}\,, \qquad \sigma_{b,\mathrm{b}} = \frac{q}{4\pi b^2}\frac{\epsilon_2-1}{\epsilon_2}\,,$$

$$\sigma_{c,\mathrm{b}} = \frac{q}{4\pi c^2}\left(\frac{1}{\epsilon_2}-\frac{1}{\epsilon_1}\right),$$

where q is the charge on the inner electrode of the capacitor. The total bound charge on the capacitor is zero.

3.7 The capacitance is

$$C = \frac{\epsilon_0 S}{4\pi a \ln 2} \, .$$

The surface density of bound charges is given by

$$\sigma_b = -\sigma\left(1 - \frac{1}{\epsilon_0}\right), \quad \text{when} \quad x = 0 \, ,$$

$$\sigma_b = \sigma\left(1 - \frac{1}{2\epsilon_0}\right), \quad \text{when} \quad x = a \, .$$

The volume density is given by

$$\rho_b = \frac{\sigma a}{\epsilon_0 (x+a)^2} \, ,$$

where σ is the charge at $x = 0$.

3.8

(a) $f_0 = \dfrac{E_0^2}{8\pi} = \dfrac{V^2}{8\pi d^2}$;

(b) $f = \dfrac{D^2}{8\pi\epsilon} = \dfrac{1}{\epsilon} f_0$ (liquid dielectric),

$f = \dfrac{D^2}{8\pi} = f_0$ (solid dielectric);

(c) $f = \dfrac{\epsilon E^2}{8\pi} = \epsilon f_0$ (liquid dielectric),

$f = \dfrac{(\epsilon E)^2}{8\pi} = \epsilon^2 f_0$ (solid dielectric).

3.9

(a) $F = \dfrac{(\epsilon - 1) b h_2 V^2}{8\pi\epsilon h_1 [h_1 - h_2(\epsilon - 1)/\epsilon]}$;

(b) $F = \dfrac{2\pi(\epsilon - 1) h_1 h_2 q^2 [h_1 - h_2(\epsilon - 1)/\epsilon]}{b\{a[h_1 - h_2(\epsilon - 1)/\epsilon] + (\epsilon - 1) h_2 x/\epsilon\}}$.

3.10 Consider the pressures at the points A and B in the liquid (figure 3.10.1). At B the pressure is equal to the atmospheric pressure p_{atm}. The pressure at A may be found either from equation (3.a.25), according to which

$$p_A = p_{atm} + \frac{E^2}{8\pi} \frac{\tau \partial \epsilon}{\partial \tau}$$

(where $p_{\text{atm}} = p_0$, $E = V/d$), or by noting the fact that p_{A} differs from the pressure at the surface of the liquid in the capacitor by the hydrostatic pressure rgh so that

$$p_{\text{A}} = rgh + \tau \frac{E^2}{8\pi}\frac{\partial \epsilon}{\partial \tau} - \frac{\epsilon - 1}{8\pi} E^2 + p_{\text{atm}}$$

[cf equation (3.a.23)]. On comparing the two expressions we find that

$$h = \frac{\epsilon - 1}{8\pi g \tau} E^2 .$$

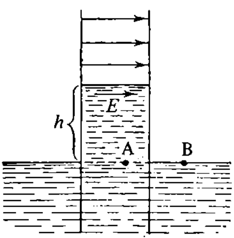


Figure 3.10.1.

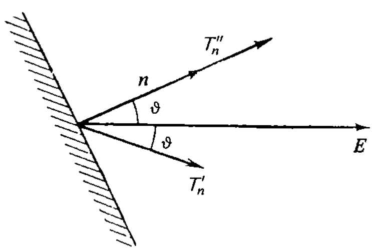


Figure 3.11.1.

3.11 The Maxwell stress tensor T_n' is oriented so that the electric field $\boldsymbol{E}$ bisects the angle between $\boldsymbol{n}$ and T_n' (figure 3.11.1)

$$|T_n'| = w = \frac{\epsilon E^2}{8\pi}$$

for any orientation of the area. The strictional stress is given by

$$T_n'' = \frac{E^2 \tau n}{8\pi}\frac{\partial \epsilon}{\partial \tau}$$

and is always directed along the normal $\boldsymbol{n}$ ('negative pressure').

3.12 (a) Consider the cylindrical coordinate system illustrated in figure 3.12.1a. In the xy-plane the field is radial and is given by

$$E = \frac{2qr}{\epsilon(r^2 + \frac{1}{4}a^2)^{3/2}} .$$

The force F acting on one of the charges, for example the left-hand charge, is obtained by adding the stresses applied to the elements d^2S of this plane on the right-hand side

$$T_z\, \mathrm{d}^2S = -\frac{\epsilon}{8\pi} E^2\, \mathrm{d}^2S = -\frac{\epsilon q^2}{2\pi}\frac{r^2}{(r^2 + \frac{1}{4}a^2)^3 \epsilon^2}\, \mathrm{d}^2S ,$$

if we use the Maxwell stress tensor. Hence

$$F_z = \int T_z \, d^2S = -\frac{1}{2\pi}\epsilon q^2 \int_0^\infty \frac{r^2 2\pi r \, dr}{\epsilon^2 (r^2 + \frac{1}{4}a^2)^3} = -\frac{q^2}{\epsilon a^2} \, .$$

This is in fact the force acting between two charges in a uniform dielectric. However, if this calculation is carried out with the total stress tensor, then the force becomes $F_z + \Delta F_z$ where

$$\Delta F_z = \frac{q^2 \tau}{\epsilon^2 a^2} \frac{\partial \epsilon}{\partial \tau} \, .$$

The correction ΔF_z is due to the strictional term. However, the theory which takes into account the electrostrictional stresses must also allow for the fact that the liquid will be drawn into the field, and for the associated increase in the hydrostatic pressure, which is given by

$$\Delta p = \frac{E^2 \tau}{8\pi} \frac{\partial \epsilon}{\partial \tau}$$

[equation (3.a.25)]. The resulting hydrostatic force is

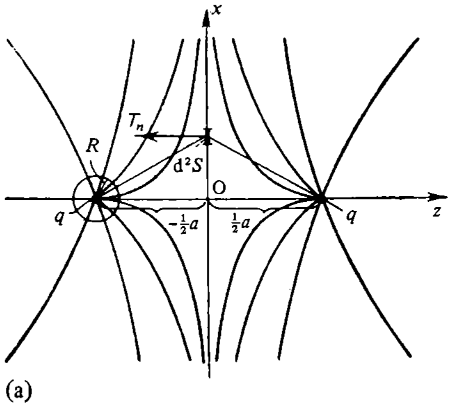
$$\Delta F_{z\mathrm{h}} = -\frac{q^2 \tau}{\epsilon^2 a^2} \frac{\partial \epsilon}{\partial \tau} = -\Delta F_z \, .$$

The total force between the charges is thus given by

$$F_z + \Delta F_z + \Delta F_{z\mathrm{h}} = -\frac{q^2}{\epsilon a^2}$$

and is, therefore, equal to the force which is obtained without allowance for the strictional forces. Consequently it is the total force due to both electrical and mechanical effects.

(b) The same results are obtained by considering the stresses on the surface of a small sphere of radius R whose centre coincides with the



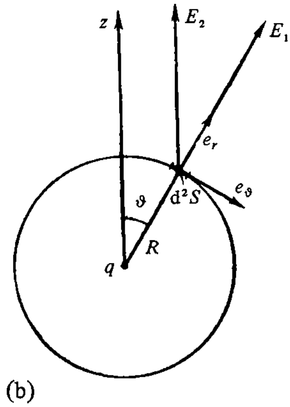


Figure 3.12.1.

point charge q on which the force is acting (figure 3.12.1b). Introduce spherical polars and consider the Maxwell stress

$$T_n' = \frac{\epsilon}{4\pi}(E_r\boldsymbol{E} - \tfrac{1}{2}E^2\boldsymbol{e}_r) ,$$

where $\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2$ and the field due to the charge on which the force is acting is

$$\boldsymbol{E}_1 = \frac{q}{\epsilon R^2}\boldsymbol{e}_r ,$$

while the field due to the second charge is

$$\boldsymbol{E}_2 = \frac{q}{\epsilon a^2}(\boldsymbol{e}_\vartheta \sin\vartheta - \boldsymbol{e}_r \cos\vartheta)$$

and may be looked upon as uniform, since the distance between the charges is $a \gg R$. Hence adding the stresses on the surface of the sphere we find that

$$\boldsymbol{F} = \int T_n' \, \mathrm{d}^2S = \frac{q}{\epsilon a^2}\boldsymbol{e}_z .$$

The inclusion of strictional stresses does not modify this result owing to hydrostatic compensation.

3.13

$$\varphi_0 = \left(\frac{8mg}{\epsilon - 1}\right)^{1/2} ,$$

where g is the acceleration due to gravity.

3.14

When $z \geqslant 0$, $\qquad \varphi = \varphi_1 = \dfrac{q}{\epsilon_1 r_1} + \dfrac{(\epsilon_1 - \epsilon_2)}{\epsilon_1(\epsilon_1 + \epsilon_2)}\dfrac{q}{r_2}$.

When $z \leqslant 0$, $\qquad \varphi = \varphi_2 = \dfrac{2}{\epsilon_1 + \epsilon_2}\dfrac{q}{r_1}$.

3.15

$$\sigma_\mathrm{b} = \frac{1}{4\pi}\left(\frac{\partial\varphi_2}{\partial z} - \frac{\partial\varphi_1}{\partial z}\right)\bigg|_{z=0} = \frac{qa}{2\pi r^3}\frac{\epsilon_1 - \epsilon_2}{\epsilon_1(\epsilon_1 + \epsilon_2)} ,$$

where

$$r = (x^2 + y^2 + a^2)^{1/2} = r_1|_{z=0} = r_2|_{z=0} .$$

When $\epsilon_2 \to \infty$, the problem reduces to that of a point charge q placed in the dielectric ϵ_1 just outside the plane surface of the conductor. The surface density is then

$$\sigma_\mathrm{b} \to -\frac{qa}{2\pi r^3 \epsilon_1} .$$

This limiting density is in fact the sum of the density of bound charges on the surface of the dielectric and the density of free charges on the surface of the conductor.

3.16

$$F = \frac{q^2}{4a^2}\frac{\epsilon_1 - \epsilon_2}{\epsilon_1(\epsilon_1 + \epsilon_2)} .$$

When $\epsilon_1 > \epsilon_2$, the charge is repelled from the boundary between the dielectrics, and when $\epsilon_1 < \epsilon_2$ the charge is attracted. A charge which is at first in the medium with the higher permittivity is repelled and moves away from the boundary. A charge which is at first in the medium with the lower permittivity is attracted towards the boundary, passes through it, and having reached the second medium is repelled by it and moves to infinity. All these results will hold provided that frictional effects experienced by the charge while it is moving in the media are neglected.

The above expression for the force F may be obtained in various ways:

(a) by considering the interaction between two point charges q' and q'';

(b) by calculating the force on the point charge which is due to bound charges on the separation boundary;

(c) with the aid of the Maxwell stress tensor.

In the latter case it is convenient to discuss stresses on the separation boundary, or on the surface of a small sphere surrounding the charge.

3.17

$$F_1 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1(\epsilon_1 + \epsilon_2)}\frac{q_1^2}{4a^2} + \frac{q_1 q_2}{2(\epsilon_1 + \epsilon_2)a^2} ,$$

$$F_2 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2(\epsilon_1 + \epsilon_2)}\frac{q_2^2}{4a^2} + \frac{q_1 q_2}{2(\epsilon_1 + \epsilon_2)a^2} .$$

The fact that the forces acting on q_1 and q_2 are not equal is due to the fact that the two charges do not by themselves form a closed mechanical system: there are also bound charges on the separation boundary between the two dielectrics. The vector sum of the forces on the boundary and on the charges q_1 and q_2 is equal to zero.

3.18 If it is assumed that in the metal $\varphi = 0$, then in the dielectric

$$\varphi = \frac{q}{\epsilon r_1} - \frac{q}{\epsilon r_2}$$

(see figure 3.14.1: charge q at A, charge $-q$ at B; $\epsilon_1 = \epsilon$, $\epsilon_2 = \infty$). The term $-q/\epsilon r_2$ is due to the charge induced in the conductor and the bound charges on the dielectric. It is equivalent to the potential of a point charge $-q/\epsilon$ at the point $z = -a$. The charge $-q/\epsilon$ is called the image of the charge q/ϵ in the plane $z = 0$ (the factor $1/\epsilon$ represents the effect of the dielectric).

$$\sigma = -\frac{qa}{2\pi r^3}, \qquad F = -\frac{q^2}{4a^2\epsilon},$$

where r is the position vector in the plane $z = 0$.

3.19 The field between the conducting planes is due to the charge systems shown in figure 3.19.1.

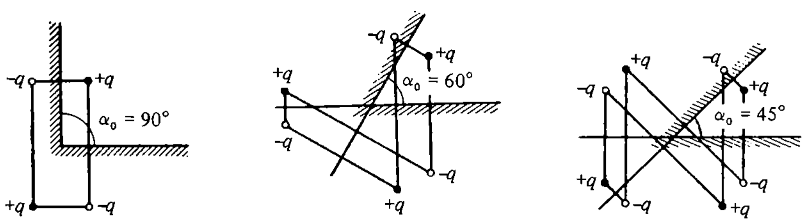


Figure 3.19.1.

3.20 Suppose that the dipole is located at the point $(0, 0, z)$. If the components of the dipole moment $\boldsymbol{p}$ along the x-, y-, and z-axes are $(p \sin\phi, 0, p\cos\phi)$ then the components of its image $\boldsymbol{p}'$ along these axes will be $(-p\sin\phi, 0, p\cos\phi)$.

$$U = \frac{(\boldsymbol{p}\cdot\boldsymbol{p}')r^2 - 3(\boldsymbol{p}\cdot\boldsymbol{r})(\boldsymbol{p}'\cdot\boldsymbol{r})}{2\epsilon r^5} = -\frac{p^2}{16z^3\epsilon}(1+\cos^2\phi),$$

$$F_z = -\frac{3p^2}{16z^4\epsilon}(1+\cos^2\phi), \qquad N_\phi = -\frac{p^2\sin^2\phi}{16z^3\epsilon}.$$

The factor $\frac{1}{2}$ in the expression for U is due to the fact that the field $\boldsymbol{E}'$ due to the dipole moment $\boldsymbol{p}'$ is proportional to $\boldsymbol{p}$. When $\boldsymbol{p}$ is increased by $\mathrm{d}\boldsymbol{p}$ at constant orientation, the interaction energy increases by

$$\mathrm{d}U = -(\boldsymbol{E}'\cdot\mathrm{d}\boldsymbol{p})$$

and hence

$$U = \int_0^p \mathrm{d}U = -\tfrac{1}{2}(\boldsymbol{E}'\cdot\boldsymbol{p})$$

(cf the solution to problem 3.37).

The dipole is attracted to the plane whatever its orientation. The couple N tends to align the dipole with the z-axis, either in the positive or the negative direction ($\phi = 0, \pi$). The couple will also vanish when $\phi = \frac{1}{2}\pi$, but this equilibrium position is unstable.

3.21 Consider a set of polar coordinates with the origin at the centre of the sphere and the z-axis parallel to $\boldsymbol{E}_0$. The potential may be sought in the form of a series in Legendre polynomials (cf the solution of problem 3.25).

The final result is

$$\varphi_1 = -\frac{3\epsilon_2}{\epsilon_1+2\epsilon_2}E_0 r\cos\vartheta\,, \quad \text{when} \quad r<a\,,$$

$$\varphi_2 = -E_0 r\cos\vartheta + \frac{\epsilon_1-\epsilon_2}{\epsilon_1+2\epsilon_2}E_0 a^3\frac{\cos\vartheta}{r^2}\,, \quad \text{when} \quad r>a\,.$$

The electric field inside the sphere is uniform and is given by

$$E_1 = \frac{3\epsilon_2}{\epsilon_1+2\epsilon_2}E_0 \quad \begin{cases} > E_0, & \text{when} \quad \epsilon_2 > \epsilon_1\,, \\ < E_0, & \text{when} \quad \epsilon_2 < \epsilon_1\,. \end{cases}$$

Outside the sphere the resultant field consists of the external uniform field E_0 and the field due to an electric dipole of moment

$$p = E_0 a^3 \frac{\epsilon_1-\epsilon_2}{\epsilon_1+2\epsilon_2}\,.$$

This secondary field is due to bound charges on the surface of the dielectric sphere with densities

$$\sigma_b = \frac{3}{4\pi}\frac{\epsilon_1-\epsilon_2}{\epsilon_1+2\epsilon_2}E_0\cos\vartheta\,, \qquad \rho_b = 0\,.$$

It is easy to understand the reason for such a charge distribution by considering each small element of the polarised dielectric as an elementary dipole.

3.22 For the dielectric with unaltered polarisation

$$\Delta\boldsymbol{E} = -\tfrac{4}{3}\pi\boldsymbol{P}$$

(see problem 2.36). For a normal dielectric

$$\Delta\boldsymbol{E} = -\frac{12\pi\epsilon}{(2\epsilon+1)(\epsilon-1)}\boldsymbol{P}\,.$$

3.23

$$\varphi = -(\boldsymbol{E}_0\cdot\boldsymbol{r}) + \frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3} \qquad (r \geqslant R)\,,$$

where $\boldsymbol{p} = R^3\boldsymbol{E}_0$, R^3 is the polarisability of the sphere, and

$$\sigma = \frac{3\epsilon_0}{4\pi}E_0\cos\vartheta\,.$$

3.24 The force F on one of the charges may be found by multiplying q_1 by the field strength due to the second charge, q_2, in the cavity around q_1. Since the cavity is small, the field inside it is uniform and equal to

$$\frac{3\epsilon E_0}{2\epsilon+1} = \frac{3q}{(2\epsilon+1)a^2}\,,$$

where $E_0 = q/\epsilon a^2$ is the uniform field in the vicinity of the cavity. Hence,

$$F = \frac{3q^2}{(2\epsilon + 1)a^2} \ .$$

This force is different from the force between charges in a uniform liquid dielectric with the same permittivity (see problem 3.12). If, by analogy with problem 3.12, it were required to find the force on the plane of symmetry, then a calculation involving only the Maxwell stresses would yield

$$F_1 = \frac{q^2}{\epsilon a^2} \ .$$

This differs both from the force F on the charge itself and from the total electric stress force (the strictional term is ignored in view of its complicated form for a solid). An equal force will act on any region of the dielectric surrounding the cavity containing the charge. One part of this force, namely

$$\frac{3q^2}{(2\epsilon + 1)a^2}$$

acts on the point charge q and the other part, namely,

$$-\frac{(2\epsilon - 1)q^2}{(2\epsilon + 1)a^2\epsilon}$$

acts on the bound charges on the surface of the cavity.

3.25 Let the origin of spherical polar coordinates be at the centre of the sphere (figure 3.25.1) and consider the potential in the form

$$\varphi(r, \vartheta, \phi) = \frac{q}{\epsilon r_1} + \sum_{l,m}\left(a_{lm}r^l + \frac{b_{lm}}{r^{l+1}}\right)P_{lm}(\cos\vartheta)\exp(im\phi) \ , \tag{3.25.1}$$

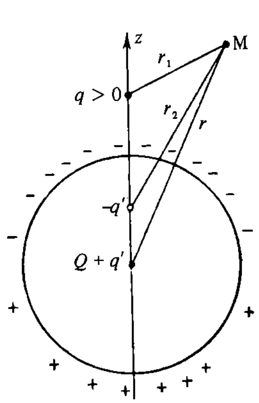


Figure 3.25.1.

where r_1 is the distance between q_1 and the point of observation. The series in equation (3.25.1) represents the field due to charges induced on the sphere. This field should vanish at infinity, and therefore $a_{lm} = 0$. Because of the symmetry of the problem, the potential is independent of the angle ϕ, and hence terms with $m \neq 0$ will also vanish. The remaining constants, $b_l \equiv b_{l0}$, will be determined from the boundary conditions. In case (a) the potential of the sphere is

$$\varphi(R, \vartheta) = V = \text{constant} .$$

Using the expansion for q/r_1 given in problem 2.28 we have

$$\varphi(R, \vartheta) = \sum_{l=0}^{\infty} \left(\frac{qR^l}{\epsilon a^{l+1}} + \frac{b_l}{R^{l+1}}\right) P_l(\cos\vartheta) = V ,$$

and hence

$$b_l = -\frac{qR^{2l+1}}{\epsilon a^{l+1}} \quad \text{when} \quad l \neq 0 , \qquad b_0 = VR - \frac{Rq}{\epsilon a} ,$$

so that the potential outside the sphere is given by

$$\varphi(r, \vartheta) = \frac{q}{\epsilon r_1} + \frac{VR}{r} - \frac{qR}{\epsilon a} \sum_{l=0}^{\infty} \left(\frac{R^2}{a}\right)^l \frac{P_l(\cos\vartheta)}{r^{l+1}} . \tag{3.25.2}$$

The density of charges induced on the surface of the sphere can now be found and is given by

$$\sigma(R, \vartheta) = -\frac{\epsilon}{4\pi} \frac{\partial \varphi}{\partial r}\bigg|_{r=R} = \frac{\epsilon V}{4\pi R} - \frac{q}{4\pi} \sum_{l=0}^{\infty} (2l+1) \frac{R^{l-1}}{a^{l+1}} P_l(\cos\vartheta) .$$

In case (b) the potential V is unknown and should be expressed in terms of the charge Q of the sphere. Clearly,

$$Q = 2\pi \int \sigma(R, \vartheta) R^2 \sin\vartheta \, d\vartheta = \epsilon VR - \frac{qR}{a} ,$$

and hence

$$V = \frac{Q}{\epsilon R} + \frac{q}{\epsilon a} .$$

Using the solution of problem 2.28, equation (3.25.2) may be rewritten in the form

$$\varphi = \frac{q}{\epsilon r_1} + \frac{Q + q'}{\epsilon r} - \frac{q'}{\epsilon r_2} ,$$

where

$$q' = q\frac{R}{a} , \qquad r_2 = (r^2 + a'^2 - 2a'r\cos\vartheta)^{1/2} , \qquad a' = \frac{R^2}{a} .$$

Thus, in the region $r > a$, the potential due to the sphere and the point charge is equivalent to the potential due to four point charges lying along the axis of symmetry, namely, a charge q at a distance a from the origin and its three images, i.e. charges Q and $q' = qR/a$ at the origin and a charge $-q'$ at the point $a' = R^2/a$. The charge $-q'$ represents the effect of charges induced on the surface of the sphere which is nearest to q (figure 3.25.1). The sign of these charges is clearly opposite to that of q. The charge $+q'$ represents the effect of charges on the distant side of the sphere and is, of course, of the same sign as q. If the sphere as a whole is neutral, then the term including Q will vanish. When the sphere is earthed ($V = 0$) the potential is given by

$$\varphi = \frac{q}{\epsilon r_1} - \frac{q'}{\epsilon r_2} .$$

3.26

$$\varphi(M) = \frac{q}{\epsilon r_1} - \frac{q'}{\epsilon r_2} + V ,$$

where (see figure 3.26.1)

$$q' = q\frac{R}{a} . \qquad a' = \frac{R^2}{a} .$$

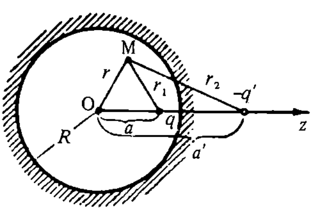


Figure 3.26.1.

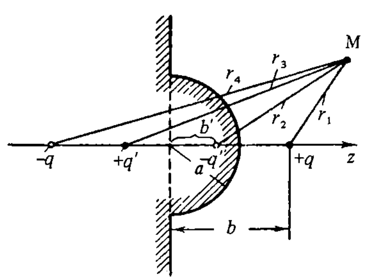


Figure 3.27.1.

3.27

$$\varphi(M) = \frac{q}{r_1} - \frac{q'}{r_2} + \frac{q'}{r_3} - \frac{q}{r_4} ,$$

where

$$q' = \frac{qa}{b} \qquad b' = \frac{a^2}{b}$$

(see figure 3.27.1). The charge on the boss is given by

$$Q = -q\left[1 - \frac{b^2 - a^2}{b(a^2 + b^2)^{1/2}}\right] .$$

3.28 Outside the sphere

$$\varphi \equiv \varphi_1 = \frac{q}{\epsilon_1 r} ;$$

in the conductor

$$\varphi \equiv \varphi_3 = \frac{q}{\epsilon_1 R_1} ;$$

in the cavity

$$\varphi \equiv \varphi_2 = \frac{q}{\epsilon_2 r_1} - \frac{q'}{\epsilon_2 r_2} + \frac{q}{\epsilon_1 R_1}$$

(see figure 3.28.1) where

$$q' = \frac{qR_2}{a} , \qquad a' = \frac{R_2^2}{a} .$$

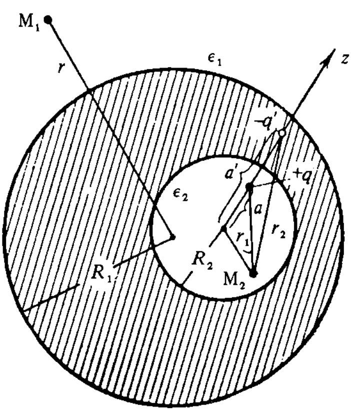


Figure 3.28.1.

3.29

(a) $$\varphi_1(r, \vartheta) = q\sum_{l=0}^{\infty} \frac{2l+1}{l\epsilon_1 + (l+1)\epsilon_2} \frac{r^l}{a^{l+1}} P_l(\cos\vartheta) , \quad \text{when} \quad r \leqslant R ,$$

$$\varphi_2(r, \vartheta) = \frac{q}{\epsilon_2 r_1} + q\frac{\epsilon_2 - \epsilon_1}{\epsilon_2} \sum_{l=0}^{\infty} \frac{l}{l\epsilon_1 + (l+1)\epsilon_2} \frac{R^{2l+1}}{a^{l+1}} \frac{P_l(\cos\vartheta)}{r^{l+1}} , \quad \text{when } r \geqslant R ,$$

where r_1 is the distance between the point charge q and the point of observation. In this case the potential cannot be derived from a simple set of images. As $\epsilon_1 \to \infty$ we get the result of problem 3.25.

(b) $$F = -q^2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_2} \sum_{l=0}^{\infty} \frac{l(l+1)}{l\epsilon_1 + (l+1)\epsilon_2} \frac{R^{2l+1}}{a^{2l+3}} ;$$

$$F = \frac{q^2}{\epsilon_2 (2a)^2} - 2q^2 \frac{\epsilon_1 - \epsilon_2}{\epsilon_2} \sum_{n=0}^{\infty} \frac{2n(2n+1)}{2n\epsilon_1 + (2n+1)\epsilon_2} \frac{R^{4n+1}}{a^{4n+3}} .$$

The repulsive charge between charges of the same sign is weakened by the polarisation of the dielectric, when $\epsilon_1 > \epsilon_2$, and is strengthened, when $\epsilon_1 < \epsilon_2$.

3.30

$$\varphi_1 = \frac{q}{\epsilon_1 r_1} + q\frac{\epsilon_1 - \epsilon_2}{\epsilon_1}\sum_{l=0}^{\infty}\frac{l+1}{\epsilon_1 l + \epsilon_2(l+1)}\frac{a^l r^l}{R^{2l+1}}P_l(\cos\vartheta)\,, \quad \text{when } r \leqslant R\,,$$

$$\varphi_2 = q\sum_{l=0}^{\infty}\frac{2l+1}{\epsilon_1 l + \epsilon_2(l+1)}\frac{a^l}{r^{l+1}}P_l(\cos\vartheta)\,, \quad \text{when} \quad r \geqslant R\,,$$

where r_1 is the distance between the point of observation and the charge q. When $a = 0$,

$$\varphi_1 = \frac{q}{\epsilon_1 r} + \left(1 - \frac{\epsilon_2}{\epsilon_1}\right)\frac{q}{\epsilon_2 R}\,, \qquad \varphi_2 = \frac{q}{\epsilon_2 r}\,.$$

3.31 Let the surfaces of the inner and outer spheres be denoted by S_1 and S_2, respectively, and suppose that the potential of the inner sphere is zero. It is convenient to solve the problem in terms of polar coordinates with the polar axis joining the centres of the spheres and the origin at the centre of the inner sphere. The equation of the surface S_1 is then $r = a$. In order to obtain the equation for S_2 we note that in the triangle OO′A (figure 3.31.1):

$$\frac{1}{b} = \frac{1}{(R^2 + c^2 - 2cR\cos\vartheta)^{1/2}}\,. \tag{3.31.1}$$

On expanding, and neglecting terms of order higher than c, we find from equation (3.31.1) that the equation of S_2 is

$$R(\vartheta) = b + cP_1(\cos\vartheta)\,, \tag{3.31.2}$$

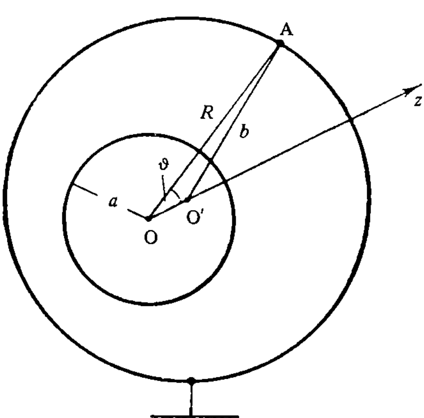


Figure 3.31.1.

where

$$P_1(\cos\vartheta) = \cos\vartheta .$$

The term $cP_1(\cos\vartheta) = c\cos\vartheta$ in equation (3.31.2) describes the departure from spherical symmetry and vanishes when $c \to 0$. It is natural to seek the potential in the form of an expansion in spherical harmonics (see appendix 2) and to retain only the first two terms. The second term, which represents the departure from spherical symmetry, should be proportional to c. Thus,

$$\varphi(r, \vartheta) = \left(A_1 + \frac{B_1}{r}\right) + c\left(A_2 r + \frac{B_2}{r^2}\right)\cos\vartheta ,$$

where the A_i and B_i are determined from the boundary conditions

$$\varphi|_{S_1} = \text{constant} , \qquad \varphi|_{S_2} = 0 , \qquad \oint_{S_1} \frac{\partial\varphi}{\partial n}\mathrm{d}S_1 = -4\pi q .$$

The final result is

$$\varphi = q\left(\frac{1}{r} - \frac{1}{b}\right) + \frac{qc}{b^3 - a^3}\left(r - \frac{a^3}{r^2}\right)\cos\vartheta ,$$

and hence the density of charge on the inner sphere is given by

$$\sigma = \frac{q}{4\pi a^2} - \frac{3qc}{4\pi(b^3 - a^3)}\cos\vartheta ;$$

and the force on the inner sphere is given by

$$F = -\frac{qc}{b^3 - a^3} .$$

3.32

$$\Delta C = \frac{a^2 b^2 c^2}{(b-a)^2(b^3 - a^3)} .$$

3.33 When the charge q is increased by $\mathrm{d}q$, the energy U of its interaction with the sphere increases by $\mathrm{d}U = \varphi'\,\mathrm{d}q$, where φ' is the potential due to charges induced on the sphere. However, this potential is proportional to q, $\varphi' = \text{constant} \times q$, and hence

$$U = \int_0^q \mathrm{d}U = \tfrac{1}{2}\text{const.}q^2 = \tfrac{1}{2}\varphi' q . \qquad (3.33.1)$$

If the magnitude of φ' were independent of q, then the interaction energy would be larger by a factor of two ($U = \varphi' q$). Using equation (3.33.1) and the results of problem 3.25, we find that

$$U = -\frac{q^2 R}{2\epsilon(a^2 - R^2)} ,$$

and hence

$$F = -\frac{q^2 aR}{\epsilon(a^2-R^2)^2} .$$

3.34

$$U = \frac{Qq}{\epsilon a} - \frac{q^2R^3}{2a^2\epsilon(a^2-R^2)} ; \qquad F = \frac{Qq}{\epsilon a^2} - \frac{q^2R^3(2a^2-R^2)}{\epsilon a^3(a^2-R^2)^2} .$$

When the two charges are of the same sign $Qq > 0$ and the force of interaction may vanish. When q is sufficiently large, or a is very small, the force may even be negative (attraction).

3.35 The test charge q should be small compared with the charges on other conductors and dielectrics, and should not be too near to any irregularities in the medium, e.g. boundaries of conductors and dielectrics, so that the effects due to charges induced by the test body should be small. For example, in measuring the electric field due to a charged conducting sphere, it is necessary for the force due to the electrical image to be small compared with qQ/a^2, where Q is the charge on the sphere and a is the distance between the test charge and the centre of the sphere. This leads to the condition (cf the solution of the preceding problem)

$$\left|\frac{Q}{q}\right| \gg 2\frac{(2a/R-1)^2}{(a/R)(a/R-1)^2} .$$

This condition will be satisfied provided a/R is not too small and q/Q is not too large.

3.36 The image of the electric dipole

$$p = p(e_x \sin\alpha + e_z \cos\alpha)$$

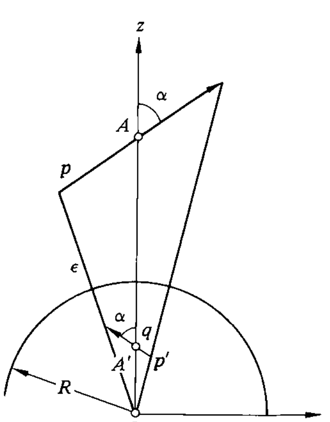


Figure 3.36.1.

in the earthed sphere consists of a point charge

$$q = \frac{pR}{r^2}\cos\alpha$$

and a dipole

$$\boldsymbol{p}' = p\left(\frac{R}{r}\right)^3(-\boldsymbol{e}_x \sin\alpha + \boldsymbol{e}_z \cos\alpha)$$

at the point A' (figure 3.36.1) which is at a distance $r' = R^2/r$ from the centre of the sphere.

$$U = -\frac{p^2 R(r^2\cos^2\alpha + R^2)}{2\epsilon(r^2 - R^2)^3},$$

$$F = -\frac{p^2 Rr}{\epsilon(r^2 - R^2)^4}[(2r^2 + R^2)\cos^2\alpha + 3R^2],$$

$$N = -\frac{p^2 Rr^2 \sin 2\alpha}{2\epsilon(r^2 - R^2)^3}.$$

In the limit as $r \to R$, we find on substituting $r = R + z$, $R \to \infty$, $z =$ constant that the results are identical with those of problem 3.20 (dipole near a conducting plane).

3.37

$$\sigma = -\frac{3p}{4\pi R^3}\cos\vartheta,$$

where ϑ is the angle between $\boldsymbol{p}$ and the line joining the centre and the point of observation. Induced charges give rise to a uniform field

$$E' = \frac{p}{R^3}$$

in the cavity.

3.38 The forces acting on the irregularity may be obtained by differentiating the quantity

$$U' = \sum_{l,m} a_{lm} Q_{lm}^*$$

with Q_{lm}^* held constant. The quantity U' differs from the true energy of interaction of the irregularity with the external field; the latter is defined as the work which must be done in order to produce a potential φ in the presence of the irregularity [cf equation (3.a.16)]. It must, however, be remembered that the moments Q_{lm} depend on the external field. In particular, if the irregularity is in the form of an uncharged conductor or dielectric, then the true energy of interaction of the irregularity with the external field is given by

$$U = \tfrac{1}{2} \sum_{l,\, m} a_{lm} Q^*_{lm} \ . \tag{3.38.1}$$

The factor $\frac{1}{2}$ may be obtained as in the solution of problem 3.33 by recalling that the Q_{lm} are proportional to the a_{lm}. In the determination of the generalised forces with the aid of equation (3.38.1), which involves differentiation with respect to the generalised coordinates, both Q_{lm} and a_{lm} must be looked upon as variable quantities.

3.39 $U_0 = q\varphi_0 - (\boldsymbol{p} \cdot \boldsymbol{E}_0)$ and

$$\varphi_1 = \varphi_0 - (\boldsymbol{r} \cdot \boldsymbol{E}_0) , \qquad \varphi_2 = \frac{q}{\epsilon r} + \frac{(\boldsymbol{p} \cdot \boldsymbol{r})}{\epsilon r^3} ,$$

$$\boldsymbol{F} = q\boldsymbol{E}_0 + (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}_0 , \qquad \boldsymbol{N} = [\boldsymbol{p} \wedge \boldsymbol{E}_0]$$

(the couple is evaluated about the origin).

3.41 The body tends to occupy a position in which its potential energy $U = \frac{1}{2}(\boldsymbol{p} \cdot \boldsymbol{E})$ is a minimum. Suppose that the coordinate axes lie along the principal axes of the tensor β_{ik} so that

$$U = -\tfrac{1}{2}[\beta^{(x)} E_x^2 + \beta^{(y)} E_y^2 + \beta^{(z)} E_z^2] \ .$$

It is clear that if $\beta^{(x)} \geqslant \beta^{(y)} \geqslant \beta^{(z)} > 0$, then U will be a minimum when $\boldsymbol{E}$ is parallel to the x-axis. When $\beta^{(x)} \leqslant \beta^{(y)} \leqslant \beta^{(z)} < 0$, the minimum occurs when $\boldsymbol{E}$ is parallel to the z-axis.

3.42 The axis of the rod and the plane of the disc tend to become parallel to the field when $\epsilon_1 > \epsilon_2$, and perpendicular to the field when $\epsilon_1 < \epsilon_2$.

3.43

$$F = \frac{\epsilon_2 - \epsilon_1}{\epsilon_2} q^2 \sum_{l=0}^{\infty} \frac{l(l+1)}{l\epsilon_1 + (l+1)\epsilon_2} \frac{R^{2l+1}}{a^{2l+3}} \ .$$

Attraction occurs for $\epsilon_2 < \epsilon_1$ and repulsion for $\epsilon_2 > \epsilon_1$. In the case of a conducting sphere $\epsilon_1 \to \infty$. On summing the geometrical progression we find that the interaction energy is

$$U = -\frac{q^2 R}{2\epsilon_2 (R^2 - a^2)} ,$$

and hence

$$F = -\frac{q^2 a R}{\epsilon_2 (a^2 - R^2)^2}$$

(cf problem 3.33).

Consider now the calculation of the force with the aid of equation (3.a.16). In the integral

$$U' = \frac{1}{8\pi}\int_{V'}(\epsilon_2-\epsilon_1)(\boldsymbol{E}\cdot\boldsymbol{E}_1)\,\mathrm{d}^3\boldsymbol{r}'$$

the volume V' is bounded by a sphere S which is infinitely close to the surface of the dielectric sphere and lies wholly inside the latter. The integral differs from the potential energy of interaction, U, between the point charge and the sphere by an infinitesimal amount. Instead of the total electric field $\boldsymbol{E}$ and the field of the point charge $\boldsymbol{E}_1$ in the uniform dielectric ϵ_2, let us use the corresponding potentials φ and φ_1 and take the constant quantity $\epsilon_2-\epsilon_1$ outside the integral, so that

$$U' = \frac{\epsilon_2-\epsilon_1}{8\pi}\int_{V'}(\nabla\varphi\cdot\nabla\varphi_1)\,\mathrm{d}^3\boldsymbol{r}'\,.$$

Using Green's formula

$$\int(\nabla\varphi\cdot\nabla\varphi_1)\,\mathrm{d}^3\boldsymbol{r} = \oint_S \varphi\frac{\partial\varphi_1}{\partial n}\,\mathrm{d}^2S+\int\varphi\nabla^2\varphi_1\,\mathrm{d}^3\boldsymbol{r}\,,$$

and the fact that inside the sphere $\nabla^2\varphi_1 = 0$, we find the following expression for U:

$$U = \frac{\epsilon_2-\epsilon_1}{\epsilon_2}q^2\sum_{l=0}^{\infty}\frac{l}{l\epsilon_1+(l+1)\epsilon_2}\frac{R^{2l+1}}{a^{2l+3}}\,.$$

This expression is identical with that obtained from equation (3.38.1) of problem 3.38. Hence the formula for F turns out to be identical with that given above.

3.44

$$C_{\text{mut}} = \frac{\kappa}{\varphi_1-\varphi_2} = \tfrac{1}{2}\left(\operatorname{arcosh}\frac{a^2-R_1^2-R_2^2}{2R_1R_2}\right)^{-1}$$

3.45

$$\sigma = \pm\frac{|E|}{4\pi} = \pm\frac{4b^2\kappa^2}{\pi[(x^2-y^2-b^2)^2+4x^2y^2]}\,,$$

where

$$b = \frac{2R^4}{a^2(a^2-4R^2)^{1/2}}\,.$$

The origin is at the centre of the straight line joining the axes of the cylinders and chosen as the x-axis.

3.46

$$C = \tfrac{1}{2}\left(\operatorname{arcosh}\frac{R_1^2+R_2^2-a^2}{2R_1R_2}\right)^{-1}\,.$$

3.47 If the x-, y-, and z-axes are parallel to the principal axes of the tensor ϵ_{ik}, then

$$\varphi(x, y, z) = \frac{e'}{r'} = \frac{e}{(\epsilon^{(x)}\epsilon^{(y)}\epsilon^{(z)})^{1/2}}\left(\frac{x^2}{\epsilon^{(x)}}+\frac{y^2}{\epsilon^{(y)}}+\frac{z^2}{\epsilon^{(z)}}\right)^{-1/2} \tag{3.47.1}$$

For an arbitrary orientation of the coordinate system, equation (3.47.1) may be written in the form

$$\varphi(r) = \frac{e}{(|\epsilon_{ik}|\epsilon_{ik}^{-1}x_i x_k)^{1/2}},$$

where $|\epsilon_{ik}|$ is the determinant of the tensor ϵ_{ik}.

3.48

$$\boldsymbol{E} = \boldsymbol{E}_0 - \frac{(\epsilon_{ik}-\delta_{ik})n_i E_{0k}}{\epsilon_{lm}n_l n_m}\boldsymbol{n}.$$

3.49

$$C = \frac{S\epsilon^{(z)}}{4\pi d},$$

where z is measured along the normal to the plates of the capacitor.

3.50 If the x- and z-axes lie in the plane $(\boldsymbol{E}_0, \boldsymbol{n})$ and the z-axis is parallel to $\boldsymbol{n}$, then

$$\tan\vartheta = \frac{E_x}{E_z} = \frac{\epsilon_{zz}\tan\vartheta_0}{1-\epsilon_{zx}\tan\vartheta_0},$$

where $\tan\vartheta_0 = E_{0x}/E_{0z}$. A given line of force in the dielectric will then remain in the plane $(\boldsymbol{E}_0, \boldsymbol{n})$.

b Coefficients of potential and capacitance

3.52 Suppose that the charge on the first conductor is q_1 and the charge on the outer surface of the second conductor is q' (the charge on the inner surface of the second conductor is $-q_1$; this follows from the electrostatic Gauss theorem). Equation (3.b.3) then assumes the form

$$\left.\begin{aligned} q_1 &= c_{11}V_1 + c_{12}V_2, \\ -q_1+q' &= c_{12}V_1 + c_{22}V_2, \end{aligned}\right\} \tag{3.52.1}$$

and hence

$$q' = (c_{11}+c_{12})V_1 + (c_{12}+c_{22})V_2. \tag{3.52.2}$$

Once q' is specified the field in the external space is defined, including the potential V_2 of the second conductor. Equation (3.52.2) should therefore hold for all values of V_1 at given q' and V_2, which can only be true if

$$c_{11}+c_{12} = 0. \tag{3.52.3}$$

The first of the equations in (3.52.1) then takes the form:

$$q_1 = c_{11}(V_1 - V_2)\,. \tag{3.52.4}$$

It follows from equations (3.52.2), (3.52.3) and (3.52.4) that

$$C = c_{11} = -c_{12} = -c_{21}\,, \qquad C' = c_{12} + c_{22}\,.$$

3.53

$$s_{11} = \frac{c_{22}}{c_{11}c_{22} - c_{12}^2}\,, \qquad s_{22} = \frac{c_{11}}{c_{11}c_{22} - c_{12}^2}\,,$$

$$s_{12} = s_{21} = -\frac{c_{12}}{c_{11}c_{22} - c_{12}^2}\,.$$

3.55

$$C = \frac{c_{11}c_{22} - c_{12}^2}{c_{11} + c_{22} + 2c_{12}}\,.$$

3.56

$$q_1 = \frac{s_{11} - 2s_{12} + s_{13}}{s_{11} - s_{12}}\frac{q}{8}\,, \qquad q_2 = \frac{q}{2}\,,$$

$$q_3 = \frac{q}{4}\,, \qquad q_4 = \frac{s_{11} - s_{13}}{s_{11} - s_{12}}\frac{q}{8}\,.$$

3.57

$$q_1 = -\frac{2a}{b}q\,, \qquad q_2 = -\frac{a}{b}q\,, \qquad q_3 = \frac{3a^2}{b^2}q\,.$$

3.58

$$F = \frac{\epsilon C_1 C_2(\epsilon r V_1 - C_2 V_2)(\epsilon r V_2 - C_1 V_1)}{(\epsilon^2 r^2 - C_1 C_2)^2}\,.$$

3.59

$$q_1 = q\frac{V_0 - V_P}{V_1 - V_0}\,, \qquad q_0 = q\frac{V_1 - V_P}{V_0 - V_1}\,.$$

3.61 The self-capacitance of the joined conductors is given by

$$c_{00} = c_{11} + c_{22} + 2c_{12}\,.$$

The mutual capacitance between the joined conductors and the ith conductor of the system is

$$c_{0i} = c_{1i} + c_{2i}\,.$$

3.62 The reduction in the energy is

$$\Delta W = \frac{(q - q')^2}{4}\frac{r - b}{rb}\,.$$

3.63 To within terms of the order of $1/r$ the force F is

$$F = -\frac{bC^2q^2}{r^3[C+ab(b-a)^{-1}]^2} \, .$$

3.64 When the sphere and the conductor touch, they assume the same potential

$$V_1 = qs_{11}+(Q-q)s_{12} = qs_{12}+(Q-q)s_{22} = V_2 \, ,$$

and hence

$$\frac{s_{11}-s_{12}}{s_{22}-s_{12}} = \frac{Q}{q}-1 \, , \tag{3.64.1}$$

where s_{ik} are the coefficients of potential and subscripts 1 and 2 refer to the sphere and the conductor respectively. Let q_k be the charge on the conductor after the kth contact. Since the potentials are equal on contact, it follows that

$$q_k s_{11}+(Q+q_{k-1}-q_k)s_{12} = q_k s_{12}+(Q-q_k+q_{k-1})s_{22} \, .$$

Hence using equation (3.64.1) we obtain the recurrence relation

$$q_k = q+\frac{q}{Q}q_{k-1} \, . \tag{3.64.2}$$

Successive application of equation (3.64.2) yields, as $k \to \infty$,

$$q = \lim_{k\to\infty} q_k = q\left[1+\frac{q}{Q}+\left(\frac{q}{Q}\right)^2+\left(\frac{q}{Q}\right)^3+\ldots\right] = \frac{qQ}{Q-q} \, .$$

c Special methods of electrostatics

3.65 The Laplace equation assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(R_\xi\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right) = 0 \, , \qquad R_\xi = [(\xi+a^2)(\xi+b^2)(\xi+c^2)]^{1/2} \, .$$

This equation must be integrated subject to the boundary conditions: φ = constant when $\xi = 0$ (surface of the ellipsoid), and $\varphi \to 0$ when $\xi \to \infty$. Since $r = (x^2+y^2+z^2)^{1/2} \to \infty$ when $\xi \to r^2$ we have

$$\varphi(\xi) = \frac{q}{2\epsilon}\int_\xi^\infty\frac{\mathrm{d}\xi}{R_\xi} \, , \qquad \frac{1}{C} = \frac{1}{2\epsilon}\int_0^\infty\frac{\mathrm{d}\xi}{R_\xi} \, ,$$

and hence

$$\sigma = -\frac{\epsilon}{4\pi}\frac{\partial\varphi}{\partial n}\bigg|_{\xi=0} = -\frac{\epsilon}{4\pi}\left(\frac{1}{h_1}\frac{\partial\varphi}{\partial\xi}\right)_{\xi=0} = \frac{q}{4\pi abc}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-1/2}$$

The charge densities at the ends of the semiaxes are proportional to the lengths of the semiaxes, so that

$$\sigma_a:\sigma_b:\sigma_c = a:b:c \, .$$

3.66 When $a = b > c$ (oblate ellipsoid):

$$\varphi = \frac{q}{\epsilon(a^2-c^2)^{1/2}} \arctan\left(\frac{a^2-c^2}{\xi+c^2}\right)^{1/2}, \qquad C = \frac{\epsilon(a^2-c^2)^{1/2}}{\arccos(c/a)} .$$

In particular, when $c = 0$ (disc), $C = 2\epsilon a/\pi$. When $a > b = c$ (prolate ellipsoid)

$$\varphi = \frac{q}{2\epsilon(a^2-b^2)^{1/2}} \ln\frac{(\xi+a^2)^{1/2}+(a^2-b^2)^{1/2}}{(\xi+a^2)^{1/2}-(a^2-b^2)^{1/2}} ,$$

$$C = \frac{\epsilon(a^2-b^2)^{1/2}}{\ln\{[a+(a^2-b^2)^{1/2}]/b\}} .$$

In particular, when $b \ll a$ (a rod)

$$C = \frac{\epsilon a}{\ln(2a/b)} .$$

3.67 To start with, let us suppose that the ellipsoid is uncharged and therefore $q = 0$. If the external uniform field E_0 is parallel to the x-axis, then

$$\varphi_0 = -E_0 x = \mp E_0 \left[\frac{(\xi+a^2)(\eta+a^2)(\zeta+a^2)}{(b^2-a^2)(c^2-a^2)}\right]^{1/2}$$

The negative sign corresponds to $x > 0$ and the positive sign to $x < 0$. Both φ_0 and the potential φ' due to charges induced on the ellipsoid satisfy the Laplace equation. Substituting $\varphi' = \varphi_0 F(\xi)$ into the Laplace equation we obtain the following differential equation for the unknown function $F(\xi)$:

$$\frac{d^2F}{d\xi^2} + \frac{dF}{d\xi} \ln[R_\xi(\xi+a^2)] = 0 .$$

This equation can easily be integrated. The solution satisfying the boundary conditions is

$$\varphi|_{q=0} = \varphi_0\left\{1 - \left[\int_\xi^\infty \frac{d\xi}{(\xi+a^2)R_\xi}\right] \Big/ \left[\int_0^\infty \frac{d\xi}{(\xi+a^2)R_\xi}\right]\right\} .$$

If the ellipsoid has a total charge q, then the solution satisfying the conditions

$$\varphi|_{\xi=0} = \text{constant} \quad \text{and} \quad -\oint_S \frac{\partial\varphi}{\partial n} d^2S = 4\pi q$$

(S is a closed surface surrounding the ellipsoid) may be obtained from the superposition principle (see problem 3.65):

$$\varphi|_q = \varphi_{q=0} + \tfrac{1}{2}q \int_\xi^\infty \frac{d\xi}{R_\xi} .$$

3.68 The potential is of the same form as in the preceding problem. The integrals entering into the expression for the potential may be expressed in terms of elementary functions whenever the ellipsoid has rotational symmetry. The final result is

$$\varphi = -E_0 x \left\{1 - \left[\ln\frac{(1+\xi/a^2)^{1/2}+e}{(1+\xi/a^2)^{1/2}-e} - \frac{2e}{(1+\xi/a^2)^{1/2}}\right] \Big/ \left(\ln\frac{1+e}{1-e} - 2e\right)\right\},$$

where a and b are the semiaxes, $e = (1-b^2/a^2)^{1/2}$ is the eccentricity of the ellipsoid, and the x-axis is perpendicular to the plane,

$$x = \frac{a}{e}\left(1+\frac{\xi}{a^2}\right)^{1/2}\left(1+\frac{\zeta}{a^2}\right)^{1/2}$$

(see problem 1.66). The field strength reaches a maximum at the apex of the ellipsoid:

$$\frac{E_{\max}}{E_0} = -\frac{1}{E_0 h_\xi}\frac{\partial\varphi}{\partial\xi}\bigg|_{\xi,\,0,\,\zeta=-b^2} = \frac{2e^3(1-e^2)^{-1}}{\ln[(1+e)/(1-e)]-2e} = \frac{1}{n^{(x)}},$$

where $n^{(x)}$ is a depolarisation coefficient (see problem 3.70). For a sphere, $e = 0$ and $E_{\max}/E_0 = 3$. In the case of a long rod (lightning conductor):

$$\frac{E_{\max}}{E_0} = \frac{a^2}{b^2}\left(\ln\frac{2a}{b} - 1\right)^{-1}, \qquad a \gg b,$$

and therefore a discharge is more likely to occur at the end of the rod rather than at any other point on it.

3.69 The field at an arbitrary distance from the ellipsoid may be obtained by superimposing three fields of the form established in problem 3.67 (the field $\boldsymbol{E}_0$ will be resolved along the principal axes of the ellipsoid). At large distances from the ellipsoid

$$\varphi = \varphi_0 + \frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3},$$

$$p_x = \beta^{(x)}E_x, \qquad p_y = \beta^{(y)}E_y, \qquad p_z = \beta^{(z)}E_z.$$

The principal values of the polarisability tensor for the ellipsoid are

$$\beta^{(x)} = \frac{abc}{3n^{(x)}}, \qquad \beta^{(y)} = \frac{abc}{3n^{(y)}}, \qquad \beta^{(z)} = \frac{abc}{3n^{(z)}}.$$

3.70

$$n^{(x)} = \frac{1-e^2}{2e^2}\left(\ln\frac{1+e}{1-e} - 2e\right) \leqslant \tfrac{1}{3},$$

$$n^{(y)} = n^{(x)} = \frac{1-n^{(x)}}{2},$$

where $e = (1-b^2/a^2)^{1/2}$ is the eccentricity of the ellipsoid. When $e \to 1$ (a rod),

$$n^{(x)} = 0\,, \qquad n^{(y)} = n^{(z)} = \tfrac{1}{2}\,.$$

When $e \ll 1$ (nearly spherical shape),

$$n^{(x)} = \tfrac{1}{3} - \tfrac{2}{15}e^2\,, \qquad n^{(y)} = n^{(z)} = \tfrac{1}{3} + \tfrac{1}{15}e^2\,.$$

3.71

$$n^{(z)} = \frac{1+e^2}{e^3}(e - \arctan e) \geqslant \tfrac{1}{3}\,,$$

$$n^{(x)} = n^{(y)} = \frac{1-n^{(z)}}{2}\,, \qquad e = \left[\left(\frac{a}{c}\right)^2 - 1\right]^{1/2}$$

For a disc

$$n^{(z)} = 1\,, \qquad n^{(x)} = n^{(y)} = 0\,.$$

3.72 $\varphi = \varphi_x + \varphi_y + \varphi_z$. Inside the ellipsoid,

$$\varphi_x = \varphi_{1x} = -E_0 x\left(1 + \frac{\epsilon_1 - \epsilon_2}{\epsilon_2}n^{(x)}\right)^{-1}$$

Outside the ellipsoid,

$$\varphi_x = \varphi_{2x} = -E_0 x\left\{1 - \frac{abc(\epsilon_1-\epsilon_2)}{2[\epsilon_2 + (\epsilon_1-\epsilon_2)n^{(x)}]}\int_\xi^\infty \frac{d\xi}{(\xi+a^2)R_\xi}\right\}\,,$$

where

$$n^{(x)} = \tfrac{1}{2}abc\int_0^\infty \frac{d\xi}{(\xi+a^2)R_\xi}\,.$$

The potentials φ_y and φ_z are given by analogous expressions with x replaced by y and z, and a by b and c, respectively. Inside the ellipsoid the field is uniform and is given by

$$\boldsymbol{E}_1 = \frac{E_{0x}\boldsymbol{e}_x}{1+[(\epsilon_1-\epsilon_2)/\epsilon_2]n^{(x)}} + \frac{E_{0y}\boldsymbol{e}_y}{1+[(\epsilon_1-\epsilon_2)/\epsilon_2]n^{(y)}} + \frac{E_{0z}\boldsymbol{e}_z}{1+[(\epsilon_1-\epsilon_2)/\epsilon_2]n^{(z)}}\,.$$

At large distances from the ellipsoid

$$\varphi_2 = -(\boldsymbol{E}_0\cdot\boldsymbol{r}) + \frac{(\boldsymbol{p}\cdot\boldsymbol{r})}{r^3}\,,$$

where

$$p_x = \beta^{(x)}E_x\,, \qquad \beta^{(x)} = \frac{abc}{3[\epsilon_2/(\epsilon_1-\epsilon_2)+n^{(x)}]}\,,$$

and so on.

3.73 Using equation (3.a.16) we have

$$U = \frac{abc(\epsilon_2 - \epsilon_1)E_0^2\{2[\epsilon_2 + n(\epsilon_1 - \epsilon_2)]\sin^2\vartheta + [\epsilon_1 + \epsilon_2 + n(\epsilon_2 - \epsilon_1)]\cos^2\vartheta\}}{6[\epsilon_2 + \epsilon_1 + n(\epsilon_2 - \epsilon_1)][\epsilon_2 + n(\epsilon_1 - \epsilon_2)]},$$

$$N = -\frac{\partial U}{\partial \vartheta} = \frac{abc(\epsilon_2 - \epsilon_1)^2 E_0^2 (3n-1)\sin 2\vartheta}{6[\epsilon_2 + \epsilon_1 + n(\epsilon_2 - \epsilon_1)][\epsilon_2 + n(\epsilon_1 - \epsilon_2)]},$$

where ϑ is the angle between the axis of symmetry and the field $\boldsymbol{E}_0$, and n is the depolarisation coefficient with respect to the axis of symmetry of the ellipsoid (see, for example, the solution of the preceding problem). It is clear from the latter formula that the external field tends to turn over the axes of symmetry of prolate ($n < \frac{1}{3}$) and oblate ($n > \frac{1}{3}$) ellipsoids into positions, respectively, parallel and perpendicular to the field. For a conducting ellipsoid $\epsilon_1 \to \infty$, and

$$N = \frac{abc(3n-1)E_0^2 \sin 2\vartheta}{6n(1-n)}.$$

3.74 The potential energy of a charged drop in the form of an ellipsoid of revolution with an eccentricity $e = (1 - b^2/a^2)^{1/2}$ and volume equal to that of a sphere of radius R and charge q may be written down in the form

$$U(e) = \frac{q^2}{2C} + \alpha S = \frac{q^2(1-e^2)^{1/3}}{4Re}\ln\frac{1+e}{1-e} + 2\pi R^2\gamma\left[(1-e^2)^{1/3} + \frac{\arcsin e}{e(1-e^2)^{1/6}}\right],$$

where we have used the expression for the capacitance of a prolate ellipsoid of revolution given in the solution of problem 3.66.

In order to investigate the stability of a charged spherical drop, consider the function $U(e)$ for small values of e. The expansion for U up to terms of order e^4 is

$$U(e) = \frac{q^2}{2R} + 4\pi R^2\gamma + \frac{e^4}{45}\left(8\pi R^2\gamma - \frac{q^2}{2R}\right).$$

It is clear from this formula that if the charge on the drop is such that $q < q_c = (16\pi R^3\gamma)^{1/2}$, then for small deformations the drop will tend to return to the spherical state and will therefore be stable. When $q > q_c$, the drop will be unstable because the deformation will continue to increase. The process ends with the division of the unstable drop into two or more parts. It is easy to verify that when the charged drop divides into two equal spherical drops the energy is reduced by a factor $2^{2/3}$. The process finally ends with the formation of stable drops. This is clear from the expression for the critical charge. As the dimensions of the drop are reduced, the critical charge decreases in proportion to the square root of the volume, whereas the total charge on the drop decreases in proportion to the volume. It follows that the stability conditions are satisfied for sufficiently small drop radii.

3.75

$$\varphi = -\frac{E_0 z}{\pi}\left(\arctan\frac{a}{\xi^{1/2}} - \frac{a}{\xi^{1/2}}\right) = -\frac{E_0}{\pi}(-\eta)^{1/2}\left(\frac{\xi^{1/2}}{a}\arctan\frac{a}{\xi^{1/2}} - 1\right) ,$$

where $\xi^{1/2}$ should be taken with the positive sign for $z > 0$ and the negative sign for $z < 0$. At large distances from the aperture $\xi \approx r^2$ and the field becomes

$$\varphi \approx \frac{E_0 a^3 z}{3\pi r^3} , \qquad \text{when} \quad z > 0 .$$

This is the field of an electric dipole lying along the z-axis and having a moment $p = E_0 a^3/3\pi$. Hence it is clear that the lines of force passing through the aperture close on the other side of the metal screen.

3.76

$$\sigma = -\frac{E_0}{4\pi^2}\left[\pi - \arcsin\frac{a}{r_1} + \frac{a}{(r_1^2 - a^2)^{1/2}}\right] , \qquad \text{when} \quad z = -0 ,$$

$$\sigma = -\frac{E_0}{4\pi^2}\left[\frac{a}{(r_1^2 - a^2)^{1/2}} - \arcsin\frac{a}{r_1}\right] , \qquad \text{when} \quad z = +0 ,$$

where $r_1 = (\xi + a^2)^{1/2}$ is the distance between the centre of the aperture and the point of observation on the plane.

3.77 The equation to be solved is $\nabla^2\varphi = -4\pi q \delta(\boldsymbol{r} - \boldsymbol{r}_0)$. In cylindrical coordinates the δ-function is given by

$$\delta(\boldsymbol{r} - \boldsymbol{r}_0) = \frac{1}{r_0}\delta(r - r_0)\delta(\phi - \gamma)\delta(z) .$$

The Fourier component of the potential $\varphi(r, \phi, z)$

$$\varphi_k(r, \phi) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \varphi(r, \phi, z)\cos kz \, \mathrm{d}k$$

satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi_k}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi_k}{\partial\phi^2} - k^2\varphi_k = -\frac{4q}{r_0}\delta(r - r_0)\delta(\phi - \gamma) \qquad (3.77.1)$$

and the boundary conditions (see figure 3.77.1)

$$\varphi_k(r, 0) = \varphi_k(r, \beta) = 0 , \qquad (3.77.2)$$

$$\varphi_k(\infty, \phi) = 0 . \qquad (3.77.3)$$

Consider the homogeneous equation corresponding to equation (3.77.1). The particular solutions satisfying equation (3.77.2) are the products $R_n(r)\sin(n\pi\phi/\beta)$ $(n = 1, 2, 3, ...)$ where $R_n(r)$ is equal either to $I_{n\pi/\beta}(kr)$ or to $K_{n\pi/\beta}(kr)$ (apart from a constant factor). The solution of the nonhomogeneous equation (3.77.1) will be sought in the form of a superposition of these particular solutions:

$$\varphi_k = \begin{cases} \sum\limits_{n=1}^{\infty} A_n I_{n\pi/\beta}(kr) \sin\dfrac{n\pi\phi}{\beta}\,, & \text{when} \quad r < a\,, \\ \sum\limits_{n=1}^{\infty} B_n K_{n\pi/\beta}(kr) \sin\dfrac{n\pi\phi}{\beta}\,, & \text{when} \quad r > a\,. \end{cases} \tag{3.77.4}$$

This solution satisfies equation (3.77.3) and is bounded at $r = 0$ (see appendix 3). In order to determine the constants A_n and B_n we shall first use the fact that the potential is continuous at $r = r_0$. This gives

$$\frac{B_n}{A_n} = \frac{I_{n\pi/\beta}(kr_0)}{K_{n\pi/\beta}(kr_0)}\,.$$

Second, we shall require that the potential given by equations (3.77.4) should satisfy equation (3.77.1). Substitute equation (3.77.4) into equation (3.77.1), multiply both sides by $\sin(m\pi\phi/\beta)$ $(m = 1, 2, 3, \ldots)$ and integrate over ϕ between 0 and β. Since the functions $\sin(n\pi\phi/\beta)$ are orthogonal in the above range, we have

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_m}{\mathrm{d}r}\right) - \left(k^2 + \frac{m^2\pi^2}{\beta^2 r^2}\right)R_m = -\frac{8q}{\beta r_0}\delta(r - r_0)\sin\frac{m\pi\gamma}{\beta}\,, \tag{3.77.5}$$

where

$$R_m(r) = \begin{cases} A_m I_{m\pi/\beta}(kr)\,, & \text{when} \quad r < a\,, \\ B_m K_{m\pi/\beta}(kr)\,, & \text{when} \quad r > a\,. \end{cases}$$

The function $R_m(r)$ is continuous at $r = r_0$, but its first derivative with respect to r has a discontinuity at that point and changes by

$$b \equiv R'_m(r_0 + 0) - R'_m(r_0 - 0) = kB_m K'_{m\pi/\beta}(kr_0) - kA_m I'_{m\pi/\beta}(kr_0)\,.$$

The second derivative is $R''_m(r) = b\delta(r - r_0)$.

Substituting this expression into equation (3.77.5) and rejecting terms which are bounded at $r = r_0$ we obtain the second equation for A_n, B_n:

$$kB_n K'_{n\pi/\beta}(kr_0) - kA_n I'_{n\pi/\beta}(kr_0) = -\frac{8q}{\beta r_0}\sin\frac{n\pi\gamma}{\beta}\,.$$

In simplifying the expressions for A_n and B_n it is useful to employ the relation

$$K_\nu(x)I'_\nu(x) - K'_\nu(x)I_\nu(x) = \frac{1}{x}\,.$$

3.79

$$\varphi(r, \phi, z) = \frac{2q}{\pi}\left\{\frac{1}{R_0}\arctan\left[\frac{\cosh\frac{1}{2}\eta + \cos\frac{1}{2}(\phi - \gamma)}{\cosh\frac{1}{2}\eta - \cos\frac{1}{2}(\phi - \gamma)}\right]^{1/2}\right.$$
$$\left. - \frac{1}{R'_0}\arctan\left[\frac{\cosh\frac{1}{2}\eta + \cos\frac{1}{2}(\phi + \gamma)}{\cosh\frac{1}{2}\eta - \cos\frac{1}{2}(\phi + \gamma)}\right]^{1/2}\right\}\,,$$

where

$$R_0 = [r_0^2+r^2+z^2-2rr_0\cos(\gamma-\phi)]^{1/2} = [2rr_0]^{1/2}[\cosh\eta-\cos(\gamma-\phi)]^{1/2},$$

$$R_0' = [r_0^2+r^2+z^2-2rr_0\cos(\gamma+\phi)]^{1/2} = [2rr_0]^{1/2}[\cosh\eta-\cos(\gamma+\phi)]^{1/2}.$$

3.80 $\sigma = \text{const}\, r^{(\pi/\beta-1)}$, where r is the distance from the line of intersection. In the special case of two intersecting surfaces in the field of a point charge (see problem 3.77), we have

$$-\frac{q\sqrt{\pi}r_0^{\pi/\beta}\sin(\pi\gamma/\beta)}{\beta^2(r_0^2+z^2)^{\pi/\beta+1/2}}\frac{\Gamma(\pi/\beta+\frac{1}{2})}{\Gamma(\pi/\beta+1)} = \text{constant}.$$

From this it is clear that $\sigma\to 0$ as $r\to 0$ and $\beta<\pi$; $\sigma\to\infty$ as $r\to 0$ and $\beta>\pi$. In the special case where the charge lies on the line of intersection, $\sigma \propto r^{-1/2}$.

3.81 Suppose that the charge q is at the origin and the z-axis is perpendicular to the surface of the plate. The equations of the front and rear surfaces will then be $z = a$ and $z = a+c$ respectively. The potential will be sought in the form

$$\left.\begin{aligned}
\varphi_1 &= q\int_0^\infty J_0(kr_1)\exp(-k|z|)\,dk+\int_0^\infty A_1(k)J_0(kr_1)\exp(kz)\,dk, \\
&\qquad\qquad -\infty<z<a, \\
\varphi_2 &= \int_0^\infty B_1(k)J_0(kr_1)\exp(-kz)\,dk+\int_0^\infty B_2(k)J_0(kr_1)\exp(kz)\,dk, \\
&\qquad\qquad a<z<b, \\
\varphi_3 &= \int_0^\infty A_2(k)J_0(kr_1)\exp(-kz)\,dk, \qquad b<z<\infty, \\
&\qquad\qquad \text{where}\quad b=a+c.
\end{aligned}\right\} \tag{3.81.1}$$

The boundary conditions on the surfaces of the plate are equivalent to four algebraic equations for the coefficients A_1, A_2, B_1, and B_2. On solving these equations we have

$$\begin{aligned}
A_1 &= q\beta\frac{\exp(-2kb)-\exp(-2ka)}{1-\beta^2\exp(-2kc)}, & A_2 &= q\frac{1-\beta^2}{1-\beta^2\exp(-2kc)}, \\
B_1 &= \frac{q(1-\beta)}{1-\beta^2\exp(-2kc)}, & B_2 &= \frac{q\beta(1-\beta)\exp(-2kb)}{1-\beta^2\exp(-2kc)},
\end{aligned} \tag{3.81.2}$$

where

$$\beta = \frac{\epsilon-1}{\epsilon+1}, \qquad b = a+c.$$

Equations (3.81.2) and (3.81.1) together give the solution of our problem. At large distances from the plate ($z>0$), the field is of the form

$$\varphi(r_1,z) \approx \frac{q}{(r_1^2+z^2)^{1/2}}+\frac{pz}{(r_1^2+z^2)^{3/2}},$$

where

$$r_1 = (x^2+y^2)^{1/2}, \qquad p = -\frac{(\epsilon-1)^2}{2\epsilon}cq.$$

3.82

$$\varphi(\mathrm{M}) = \frac{2q}{\epsilon+1}\int_0^\infty \frac{\sinh k(a-|z|)}{\cosh ka} J_0(kr_1)\,dk,$$

where $r_1 = (x^2+y^2)^{1/2}$ (figure 3.82.1). As $(z^2+r_1^2)^{1/2} \to 0$ (near the charge),

$$\varphi \to \frac{2q}{(\epsilon+1)(r_1^2+z^2)^{1/2}}$$

(cf problem 3.1). The potential φ may be written in the form

$$\varphi = \frac{2q}{\epsilon+1}\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{[r_1^2+(z-2an)^2]^{1/2}}.$$

The corresponding system of images is shown in figure 3.87.1.

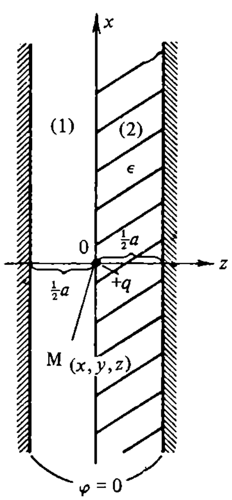


Figure 3.82.1.

3.83 The use of bispherical coordinates will ensure that the surfaces of the inner and outer electrodes will have the equations $\xi = \xi_1$ and $\xi = \xi_2$ respectively. To achieve this the z-axis must be drawn through the centres of the spheres as shown in figure 3.83.1. The coordinates of the centres will then be $z_1 = a\coth\xi_1$ and $z_2 = a\coth\xi_2$, where a is the parameter of the bispherical coordinates. The relations between the radii of the electrodes and the quantities a, ξ_1, ξ_2 are

$$a = a_1\sinh\xi_1, \qquad a = a_2\sinh\xi_2, \qquad b = z_2-z_1 = a(\coth\xi_2-\coth\xi_1),$$

and hence

$$\cosh\xi_1 = \frac{a_2^2 - a_1^2 - b^2}{2a_1 b}, \qquad \cosh\xi_2 = \frac{a_2^2 + b^2 - a_1^2}{2a_2 b}.$$

In the space between the electrodes the function ψ satisfies the equation

$$\frac{\partial^2\psi}{\partial\xi^2} + \frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial\psi}{\partial\eta}\right) + \frac{1}{\sin^2\eta}\frac{\partial^2\psi}{\partial\phi^2} - \tfrac{1}{4}\psi = 0. \tag{3.83.1}$$

Separating the variables in equation (3.83.1) and recalling the fact that ψ is independent of the azimuthal angle ϕ, we find the following special solution which is bounded at $\eta = 0, \pi$:

$$\psi_l(\xi, \eta) = [A_l \cosh(l+\tfrac{1}{2})\xi + B_l \sinh(l+\tfrac{1}{2})\xi]P_l(\cos\eta),$$

where $l = 0, 1, 2, 3, \ldots$.

The function ψ will be sought in the form of the series

$$\psi(\xi, \eta) = \sum_{l=0}^{\infty} \psi_l(\xi, \eta).$$

The coefficients A_l and B_l will be determined from the boundary conditions

$$\psi(\xi_2, \eta) = 0, \qquad \psi(\xi_1, \eta) = V(2\cosh\xi_1 - 2\cos\eta)^{-1/2}$$
$$= V\sum_{l=0}^{\infty} \exp[-(l+\tfrac{1}{2})\xi_1]P_l(\cos\eta).$$

The final result is

$$\varphi(\xi, \eta) = V(2\cosh\xi - 2\cos\eta)^{1/2} \times \sum_{l=0}^{\infty} \frac{\exp[-(l+\tfrac{1}{2})\xi_1]\sinh(l+\tfrac{1}{2})(\xi-\xi_2)}{\sinh(l+\tfrac{1}{2})(\xi_1-\xi_2)} P_l(\cos\eta). \tag{3.83.2}$$

The capacitance is given by

$$C = \frac{q_1}{V} = \frac{1}{4\pi V}\int_0^{\pi}\int_0^{2\pi} \frac{1}{h_\xi}\frac{\partial\varphi}{\partial\xi}h_\eta h_\phi\bigg|_{\xi=\xi_1} d\eta\, d\phi.$$

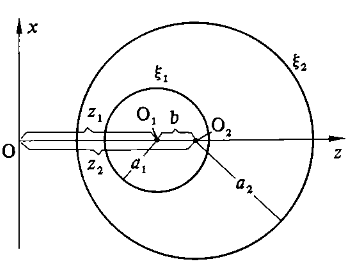


Figure 3.83.1.

Substituting equation (3.83.2) into this expression, and remembering that Legendre polynomials form an orthogonal set, we have

$$C = \frac{a_1}{2} + a_1 \sinh \xi_1 \sum_{l=0}^{\infty} \exp[-(2l+1)\xi_1] \coth[(l+\tfrac{1}{2})(\xi_2 - \xi_2)] .$$

3.84

$$C = \frac{a_1 a_2}{a_2 - a_1} + \frac{a_1^2 a_2^2 b^2}{(a_2 - a_1)^2 (a_2^3 - a_1^3)} .$$

3.85

$$c_{11} = \tfrac{1}{2} a_1 + a_1 \sinh \xi_1 \sum_{l=0}^{\infty} \exp[-(l+\tfrac{1}{2})\xi_1] \coth[(l+\tfrac{1}{2})(\xi_1 + \xi_2)] ,$$

$$c_{22} = \tfrac{1}{2} a_2 + a_2 \sinh \xi_2 \sum_{l=0}^{\infty} \exp[-(l+\tfrac{1}{2})\xi_2] \coth[(l+\tfrac{1}{2})(\xi_1 + \xi_2)] ,$$

$$c_{12} = -a_1 \sinh \xi_1 \sum_{l=0}^{\infty} \frac{\exp[-(l+\frac{1}{2})(\xi_1 + \xi_2)]}{\sinh[(l+\frac{1}{2})(\xi_1 + \xi_2)]} ,$$

where

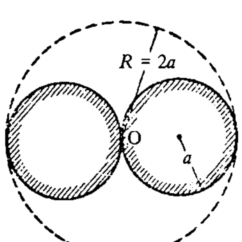
$$\cosh \xi_1 = \frac{b^2 + a_1^2 - a_2^2}{2ba_1} , \qquad \cosh \xi_2 = \frac{b^2 - a_1^2 + a_2^2}{2ba_2} .$$

The surfaces of the first and second conductor are given by $\xi = -\xi_1$ and $\xi = \xi_2$, respectively, where $a_1 \sinh \xi_1 = a_2 \sinh \xi_2$.

3.86 $c_{11} = a_1(1 + mn + mn^3 + m^2 n^2)$,
$c_{12} = -a_1 n(1 + mn)$,
$c_{22} = a_2(1 + mn + m^3 n + m^2 n^2)$,

where $m = a_1/b$ and $n = a_2/b$.

3.87 Suppose that the potential of the spheres is zero and that the potential at infinity is $-V$. Inversion of the system with respect to a sphere of radius $R = 2a$, whose centre lies at the point of contact of the conducting spheres, yields a plane parallel capacitor with the earthed electrodes at a distance $2R$ from each other. (In figure 3.87.1 the spheres of inversion are shown by the dashed curves.) The inner regions of the



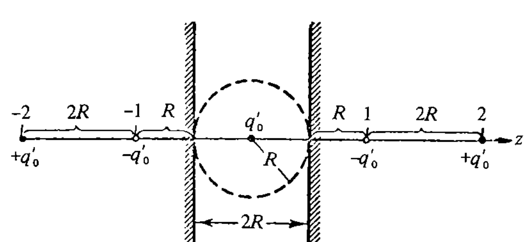


Figure 3.87.1.

spheres are then found to correspond to the outer regions of the capacitor. The centre of inversion in the capacitor corresponds to an infinitely distant point of the original system at a potential V. A point charge $q'_0 = -RV$ is therefore located at the centre of inversion. In the inverted system the field may be looked upon as being due to the following infinite system of images (see problem 3.82; $\epsilon = 1$): point charges $(-1)^n q'_0$ which lie at the points $z'_n = 2Rn$ on the z'-axis. The latter passes through the centre of inversion at right angles to the capacitor plates. Since we are interested in the capacitance the total charge of the original system must now be determined:

$$q = 2\sum_{n=1}^{\infty} q_n = 2\sum_{n=1}^{\infty} \frac{q'_n R}{z'_n} = q'_0 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -q'_0 \ln 2 = RV \ln 2 .$$

In summing up the series, we have made use of the well-known expansion for $\ln 2$. Hence, the capacitance is given by

$$C = \frac{q}{V} = 2a \ln 2 .$$

In order to determine the potential from equations (3.c.1) and (3.c.2), consider a cylindrical coordinate system with the z-axis lying along the axis of symmetry of the system and the origin at the point of contact of the spheres. In these coordinates

$$z' = \frac{R^2 z}{r^2}, \qquad r'_1 = \frac{R^2 r_1}{r^2}, \qquad r^2 = r_1^2 + z^2$$

and the potential is given by

$$\varphi(r) = \frac{q}{C} - \frac{R^2 q}{Cr} \int_0^{\infty} \frac{\sinh\{k(R - R^2|z|/r^2)\}}{\cosh kR} J_0\left(\frac{kR^2 r_1}{r^2}\right) \mathrm{d}k .$$

The term q/C is added in order to ensure that $\varphi(r)$ will vanish as $r \to \infty$.

3.89 The angle β at which the spherical surfaces intersect is given by $\beta = 2\pi - |\xi_2 - \xi_1|$ if ξ_1 and ξ_2 are of the same sign and by $\beta = 2\pi - |\xi_1 + \xi_2|$ if ξ_1 and ξ_2 are of different sign. If we choose the centre of inversion O on the line of intersection of the spheres, and let the radius of inversion be equal to $2a$, then the result of inversion will be a set of two planes intersecting at an angle β with the line of intersection (z'-axis) and perpendicular to the plane of symmetry ($\phi = 0, \pi$) of the conductor under investigation. Figure 3.89.1 illustrates the case $\xi_1 > 0$, $\xi_2 < 0$. After inversion a charge $q'_0 = -2aV$ appears at O. It can easily be shown that $\gamma = \xi_1$ if γ is measured from the plane into which the spherical surface $\xi = \xi_1$ is transformed. Upon inversion, the surfaces $\xi =$ constant become the half-planes $\phi' =$ constant, where

$$\xi = \begin{cases} \gamma - \phi' & 0 \leqslant \phi' \leqslant \pi + \gamma , \\ \gamma - \phi' + 2\pi & \pi + \gamma < \phi' < \beta \qquad (\text{when } \beta > \pi + \gamma) . \end{cases} \tag{3.89.1}$$

The distances r and r' can be expressed in terms of the coordinates ρ, ξ of the point of observation M:

$$r = \frac{2a\exp(\frac{1}{2}\rho)}{[2(\cosh\rho - \cos\xi)]^{1/2}}, \qquad r' = 2a\exp(-\rho). \tag{3.89.2}$$

The above equations are obtained by using the relations between Cartesian and toroidal coordinates given in problem 1.68, and considering the congruent triangles OO′M′ and OO′M. From the expression for the potential of a wedge given in problem 3.78, and also equations (3.89.1) and (3.89.2), the following expression for the capacitance is found:

$$C = \frac{q}{V} = \lim_{\substack{r\to\infty \\ (\text{or } \rho\to 0, \\ \xi\to 0)}} \frac{r(\varphi+V)}{V} = \frac{a}{\pi}\int_0^\infty \frac{d\zeta}{\sinh\frac{1}{2}\zeta}\left[\frac{\pi}{\beta}\frac{\sinh(\pi\zeta/\beta)}{\cosh(\pi\zeta/\beta)-\cos(2\pi\gamma/\beta)} - \frac{\pi}{\beta}\frac{\sinh(\pi\zeta/\beta)}{\cosh(\pi\zeta/\beta)-1} + \frac{\sinh\zeta}{\cosh\zeta - 1}\right].$$

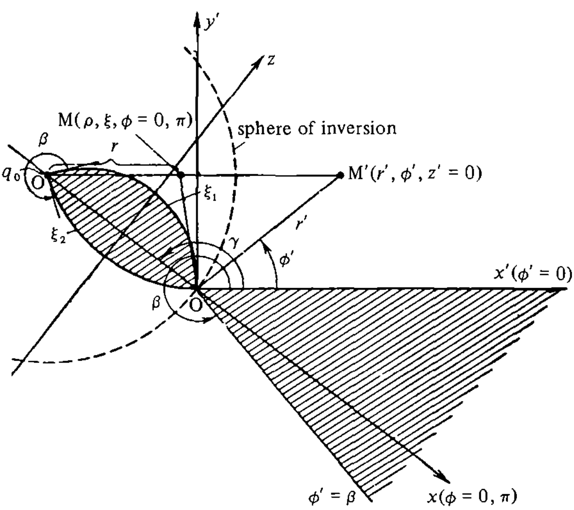


Figure 3.89.1.

3.90

(a) $C = \dfrac{R}{\pi}(\sin\theta + \theta)$,

(b) $C = 2R\left(1 - \dfrac{1}{\sqrt{3}}\right) \approx \frac{11}{13}R$,

where the integral from the solution of the preceding problem is evaluated with the aid of the substitution $x = \exp(-\frac{1}{2}\zeta)$.

3.91

$$C = \frac{a}{2}\left(5 - \frac{4}{\sqrt{3}}\right).$$

Steady currents

4.1
$$\frac{J}{J_2} = \frac{V_1 - V_2}{V_2 - \mathscr{E}} .$$

4.2 The resistance of the galvanometer coil should be equal to the external resistance R.

4.3 $R = \frac{3}{2}r$, when $n = 2$,
$R = \frac{13}{7}r$, when $n = 3$,
$R = \frac{47}{22}r$, when $n = 4$.

The number of circulating currents which must be used may be reduced by making use of the symmetry of the circuit. For example, for $n = 3$, the number of circulating currents may be reduced to three.

4.4 Consider the circulating currents indicated in figure 4.4.1. Kirchhoff's equation for the circuit $\mathrm{B}_k \mathrm{A}_k \mathrm{A}_{k+1} \mathrm{B}_{k+1}$ is

$$J_{k-1} + J_{k+1} = \left(2 + \frac{R}{r}\right) J_k . \tag{4.4.1}$$

This second order linear difference equation has two linearly independent solutions, namely, $\exp(k\alpha)$ and $\exp(-k\alpha)$, where[1]

$$\sinh\tfrac{1}{2}\alpha = \tfrac{1}{2}\left(\frac{R}{r}\right)^{1/2} \tag{4.4.2}$$

The general solution of equation (4.4.1) is of the form

$$J_k = A' \exp(k\alpha) + B' \exp(-k\alpha) .$$

It will be convenient to rearrange the terms and rewrite equation (4.4.1) in the form

$$J_k = A \cosh[(\beta - k)\alpha] , \tag{4.4.3}$$

where A and β are constants. These constants can be determined from the boundary conditions at the ends of the line. The Kirchhoff equation for the last cell is

$$J_n(R + R_a + r) - J_{n-1}r = 0 . \tag{4.4.4}$$

Substituting expression (4.4.3) into equation (4.4.4) and using equation (4.4.2), we obtain, after some rearrangement, the following expression

[1] In deriving this equation and some of those which follow, it is useful to bear in mind that the formulae in hyperbolic trigonometry are obtained from those in ordinary trigonometry by the substitutions

$$\cos\alpha \rightarrow \cosh\alpha , \qquad \sin\alpha \rightarrow \mathrm{i}\sinh\alpha .$$

for β:

$$\tanh\beta\alpha = \frac{R_a \cosh n\alpha + (Rr)^{1/2} \sinh(n+\frac{1}{2})\alpha}{R_a \sinh n\alpha + (Rr)^{1/2} \cosh(n+\frac{1}{2})\alpha} . \tag{4.4.5}$$

The constant A may be determined from Kirchhoff's equation for the first cell:

$$J_0(R+R_i+r) - J_1 r = \mathcal{E} . \tag{4.4.6}$$

After rearrangement equation (4.4.6) yields

$$A = \frac{\mathcal{E}}{R_i \cosh\beta\alpha + (Rr)^{1/2} \sinh(\beta+\frac{1}{2})\alpha} .$$

The final expression for the current in the section $A_k A_{k+1}$ is

$$J_k = \frac{\mathcal{E} \cosh(\beta - k)\alpha}{R_i \cosh\beta\alpha + (Rr)^{1/2} \sinh(\beta+\frac{1}{2})\alpha} . \tag{4.4.7}$$

The constants α and β in this relation are given by equations (4.4.2) and (4.4.5).

For dry insulation when $r \to \infty$, $\alpha \to 0$, equation (4.4.7) assumes the expected form

$$J_k = \frac{\mathcal{E}}{R_i + R_a + (n+1)R} . \tag{4.4.8}$$

The ratio of the e.m.f.s ensuring the same current through the load for dry and poor insulation can be obtained from equations (4.4.7) and (4.4.8), and is given by

$$\frac{\mathcal{E}}{\mathcal{E}_0} = \frac{R_i \cosh\beta\alpha + (Rr)^{1/2} \sinh(\beta+\frac{1}{2})\alpha}{[R_i + R_a + (n+1)R] \cosh(\beta - n + \frac{1}{2})\alpha} .$$

When the load resistance is zero ($R_a = 0$), we have from equation (4.4.5) the simple result

$$\beta = n + \tfrac{1}{2} .$$

4.5 Let $J(x)$ and $\varphi(x)$ be the current and potential of the central conductor (relative to ground) at the point x. It follows that

$$\varphi(x) = -\rho' \frac{dJ}{dx} , \qquad J = -\rho \frac{d\varphi}{dx} ,$$

and

$$\frac{d^2 J}{dx^2} = \frac{\rho}{\rho'} J .$$

4.6

$$J(x) = \frac{\mathcal{E} \cosh s(x - x_0)}{R_i \cosh s x_0 + (\rho\rho')^{1/2} \sinh s x_0} , \tag{4.6.1}$$

where $s = (\rho/\rho')^{1/2}$. The constant x_0 can be determined from

$$\tanh s(x_0 - a) = \frac{R_a}{(\rho\rho')^{1/2}} . \tag{4.6.2}$$

When $R_i = R_a = 0$, we have

$$J(x) = \frac{\mathcal{E} \cosh s(x-a)}{(\rho\rho')^{1/2} \sinh sa} .$$

In the absence of leakage $\rho' \to \infty$, $x_0 \to a$, $s \to 0$, and the current along the cable is

$$J_0 = \frac{\mathcal{E}}{R_i + \rho a + R_a} .$$

The following substitutions must be made when using equation (4.4.7) of the solution of problem 4.4:

$$R = \rho\, \mathrm{d}x , \qquad r = \frac{\rho'}{\mathrm{d}x} , \qquad k = \frac{x}{\mathrm{d}x} , \qquad n = \frac{a}{\mathrm{d}x} .$$

It then follows from equation (4.4.2) of the solution of problem 4.4 that $\alpha = s\,\mathrm{d}x$. The quantity β in that solution is related to x_0 by $\beta = x_0/\mathrm{d}x$ so that $\beta\alpha = x_0 s$. Substitution of these expressions into equations (4.4.5) and (4.4.7) of the solution of problem 4.4 leads to equations (4.6.1) and (4.6.2) given above.

4.7

$$E_1 = \frac{\gamma_2 V}{\gamma_1 h_2 + \gamma_2 h_1} , \qquad D_1 = \frac{\epsilon_1 \gamma_2 V}{\gamma_1 h_2 + \gamma_2 h_1} ,$$

$$E_2 = \frac{\gamma_1 V}{\gamma_1 h_2 + \gamma_2 h_1} , \qquad D_2 = \frac{\epsilon_2 \gamma_1 V}{\gamma_1 h_2 + \gamma_2 h_1} ,$$

$$j_1 = j_2 = \frac{\gamma_1 \gamma_2 V}{\gamma_1 h_2 + \gamma_2 h_1} .$$

At the boundary of separation between the plates

$$\sigma_b = \frac{E_2 - E_1}{4\pi} - \sigma = \frac{\gamma_2(\epsilon_1 - 1) - \gamma_1(\epsilon_2 - 1)}{4\pi(\gamma_1 h_2 + \gamma_2 h_1)} V ,$$

$$\sigma = \frac{D_2 - D_1}{4\pi} = \frac{(\epsilon_2 \gamma_1 - \epsilon_1 \gamma_2) V}{4\pi(\gamma_1 h_2 + \gamma_2 h_1)} .$$

The quantity V is positive, if the first plate is in contact with the positively charged electrode.

At the boundary between the electrode and the first plate

$$\sigma = \frac{D_1}{4\pi} , \qquad \sigma_b = \frac{E_1 - D_1}{4\pi} .$$

At the boundary between the electrode and the second plate

$$\sigma = -\frac{D_2}{4\pi}, \qquad \sigma_b = -\frac{E_2 - D_2}{4\pi}.$$

4.8

$$\frac{\tan\beta_1}{\tan\beta_2} = \frac{\gamma_1}{\gamma_2},$$

where β_1 and β_2 are, respectively, the angles between the current flow-lines and the normal to the separation boundary in the first and second media.

4.9

$$\varphi = \begin{cases} \dfrac{Jz}{\pi a^2 \gamma}, & \text{when} \quad 0 \leqslant r \leqslant a, \\ \dfrac{Jz \ln(r/b)}{\pi a^2 \gamma \ln(a/b)}, & \text{when} \quad a < r \leqslant b, \\ 0, & \text{when} \quad b < r. \end{cases}$$

It is clear from this formula that the electric field in the space between the conductors is not parallel to the z-axis. The absence of a finite radial component of the electric field E_r shows that the surface charges on the cylindrical surfaces of the conductors are given by

$$\sigma_1 = \frac{\epsilon E_r}{4\pi}\bigg|_{r=a} = \frac{\epsilon Jz}{4\pi^2 a^3 \gamma \ln(a/b)},$$

$$\sigma_2 = \frac{\epsilon E_r}{4\pi}\bigg|_{r=b} = -\frac{\epsilon Jz}{4\pi^2 a^2 b \gamma \ln(a/b)}.$$

When $z = 0$, the surface densities σ_1 and σ_2 become equal to zero. The position of the section on which $\sigma_1 = \sigma_2 = 0$ is indeterminate. This section may be displaced if an additional constant charge is placed on the conductor. The relation between the two charges $q_1 = 2\pi a \sigma_1$ and $q_2 = 2\pi b \sigma_2 = -q_1$ per unit length of the conductor and envelope (at the same value of z) and the potential difference between them

$$V = \int_a^b E_r \, \mathrm{d}r = -\frac{Jz}{a^2 \gamma}$$

is

$$\frac{q_1}{V} = \frac{1}{2\ln(b/a)} = \text{constant}.$$

The ratio q_1/V is equal to the capacitance per unit length in the corresponding electrostatic problem (cylindrical capacitor). The magnetic field is, of course, of the same form as for an infinitely long straight conductor carrying a current J. This is explained by the fact that the current density in the infinitely thick envelope vanishes and as a result the reverse current does not produce a magnetic field.

4.10 $E_0 = -k(\gamma_2 l_1 + \gamma_1 l_2)\mathcal{E}_0$, $\quad E_1 = k\gamma_2\mathcal{E}_0$, $\quad E_2 = k\gamma_1\mathcal{E}_0$,

where $k = \gamma_0/l_0(\gamma_0\gamma_1 l_2 + \gamma_0\gamma_2 l_1 + \gamma_1\gamma_2 l_0)$ and $\mathcal{E}_0 = E_e l_0$ is the e.m.f. of the source. Inside the source the electric field acts in the opposite direction to the direction of flow of the current ($E_0 < 0$).

The charges responsible for this electric field are located on the separation boundaries between conductors of different conductivities, and may be determined with the aid of the proper boundary conditions. For example, the charge on the boundary 01 is given by

$$q_{01} = \frac{r^2}{4}(E_1 - E_0)\,.$$

4.11 Consider, for example, the energy flux through the surface of conductor 0. The magnetic field near the surface is the same as the field due to an infinitely long straight conductor ($H = 2J/cr$). The Poynting vector

$$S = \frac{c}{4\pi}[E_0 \wedge H]$$

can be easily shown to be parallel to the outward normal to the surface of the conductor (E_0 is the field strength in conductor 0 and its direction is opposite to the direction of the current; see problem 4.10). The energy flux through the surface of this conductor is therefore equal to

$$2\pi r l_0 s = JV\,,$$

where $V = E_0 l_0$ is the potential difference between the ends of the conductor. The quantity JV is the difference between the work done by the e.m.f. and the Joule losses per unit time in the source. The former is equal to $\mathcal{E}_0 J$ where $\mathcal{E}_0 = E_e l_0$.

Per unit time an amount of energy JV leaves the source, flows in the surrounding space (mainly outside the conductors), enters conductors 1 and 2 through their surface, and is converted into Joule heat. The total amount of energy entering conductors 1 and 2 per unit time is JV_1 and JV_2 respectively. This may be easily proved with the aid of the Poynting vector, as indicated above.

4.12 The resistance is given by

$$R = \int_1^2 \frac{dl}{S\gamma}\,,$$

where dl is parallel to the normal to the equipotential surface of area S. The limits 1 and 2 represent the bounding surfaces of the conductor.

4.13

(a) $R = \dfrac{1}{4\pi\gamma}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$;

(b) $R = \dfrac{1}{4\pi\gamma_1}\left(\dfrac{1}{a} - \dfrac{1}{c}\right) + \dfrac{1}{4\pi\gamma_2}\left(\dfrac{1}{c} - \dfrac{1}{b}\right)$;

(c) $R = \dfrac{1}{2\pi l\gamma}\ln\dfrac{b}{a}$.

4.14

$$R = \frac{1}{2\pi\gamma_2}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{2\pi\gamma_1}\frac{1}{b} .$$

4.15

$$J_k = \frac{4\pi\gamma}{\epsilon}q_k , \qquad R_{ik} = \frac{\epsilon}{4\pi\gamma}s_{ik} .$$

4.16

$$C = \frac{\epsilon}{4\pi\gamma R} .$$

4.17

$$Q = \sum_{i,k} R_{ik} J_i J_k .$$

4.18

$$R = \frac{\epsilon}{4\pi\gamma}(s_{11} - 2s_{12} + s_{22}) = \frac{\epsilon}{4\pi\gamma}\frac{c_{11} + 2c_{12} + c_{22}}{c_{12}^2 - c_{11}c_{22}} .$$

4.19

$$R = \frac{V_1 - V_2}{J} = R_1 + R_2 - \frac{1}{\pi\gamma l} \approx R_1 + R_2 ,$$

where $R_1 = 1/2\pi\gamma a_1$ and $R_2 = 1/2\pi\gamma a_2$ are the resistances of the joined conductors (cf problem 4.14).

4.20 Let $e_0 = (1 - b^2/a^2)^{1/2}$ be the eccentricity of the ellipsoid of revolution, where b/a is the ratio of the semiminor to semimajor axes. It follows that for an oblate ellipsoid

$$R = \frac{1}{\gamma}\left(\frac{4}{3\pi^2 V}\right)^{1/3}\frac{(1 - e_0^2)^{1/6}}{e_0}\arccos(1 - e_0^2)^{1/2}$$

while for a prolate ellipsoid

$$R = \frac{1}{\gamma}\frac{(1 - e_0^2)^{1/3}}{(6\pi^2 V)^{1/3} e_0}\ln\frac{1 + e_0}{1 - e_0} .$$

For a given volume V, the most efficient shape is either a very elongated or a very flattened conductor.

4.21 The current density in the space between the electrodes

$$j = \rho v \tag{4.21.1}$$

is independent of x [$v(x)$ is the particle velocity at a given point x]. The relation between the potential $\varphi(x)$ and the velocity is

$$v = \left(-\frac{2e\varphi}{m}\right)^{1/2} \tag{4.21.2}$$

($\varphi = 0$ when $x = 0$).

It follows from equations (4.21.1) and (4.21.2) that $\rho = j(-m/2e\varphi)^{1/2}$, since the Poisson equation is

$$\frac{d^2\varphi}{dx^2} = -4\pi j\left(-\frac{m}{2e\varphi}\right)^{1/2} \tag{4.21.3}$$

Integration of equation (4.21.3) subject to the boundary conditions $d\varphi/dx\,|_{x=0} = 0$ and $\varphi\,|_{x=a} = \varphi_0$ yields

$$j = \frac{1}{9\pi a^2}\left(\frac{2\,|e\,|}{m}\right)^{1/2} |\varphi_0|^{3/2}\,,$$

which is the so-called three-halves law (Child–Langmuir law).

5

Magnetostatics

5.1

$$H_r = H_z = 0\,, \qquad H_\phi = \begin{cases} \dfrac{2Jr}{ca^2}\,, & \text{when} \quad r < a\,, \\ \dfrac{2J}{cr}\,, & \text{when} \quad a \leqslant r \leqslant b\,, \\ 0\,, & \text{when} \quad r > b\,. \end{cases}$$

5.2 Let the z-axis lie along the axis of the cylinder so that the Cartesian components of the vector potential A satisfy the following equation:

$$\nabla^2 A_x = 0\,, \qquad \nabla^2 A_y = 0\,, \qquad \nabla^2 A_z = -\frac{4\pi\mu_0}{c} j_z\,,$$

where $j_z = 0$ when $r > a$ and $j_z = J/\pi a^2$ when $r \leqslant a$.

Since the current J does not enter into the equations for A_x and A_y, these components may be put equal to zero. A_z will then depend only on the distance r from the z-axis. If we integrate the equation for A_z and use the continuity conditions for A_z and H_ϕ on the boundary $r = a$, and also the condition that H must be finite at $r = 0$, we have for $r < a$,

$$A_z = C - \frac{\mu_0 J}{c}\left(\frac{r}{a}\right)^2, \qquad B_\phi = \frac{2\mu_0 J}{ca^2} r\,, \qquad H_\phi = \frac{2J}{ca^2} r\,.$$

When $r > a$

$$A_z = C - \frac{J}{c}\left(\mu_0 + 2\mu \ln\frac{r}{a}\right), \qquad B_\phi = \frac{2\mu J}{cr}\,, \qquad H_\phi = \frac{2J}{cr}\,,$$

where C is an arbitrary constant.

5.3 When $r < a$

$$A_z = C_1\,, \qquad \boldsymbol{B} = 0\,,$$

and when $a \leqslant r \leqslant b$

$$A_z = \frac{2\mu_0 J a^2}{c(b^2 - a^2)}\left(\ln\frac{r}{a} - \frac{r^2}{2a^2}\right) + C_2\,, \qquad B_\phi = \frac{2\mu_0 J}{c(b^2 - a^2)}\left(r - \frac{a^2}{r}\right).$$

When $r > b$

$$A_z = \frac{2\mu J}{c} \ln\frac{b}{r} + C_3\,, \qquad B_\phi = \frac{2\mu J}{cr}\,.$$

The remaining components of A and $\boldsymbol{B}$ are all equal to zero. Any two of the constants entering into the expression for A_z may be expressed in terms of the third constant by using the continuity of the vector potential at the boundaries.

5.4

$$H_x = \frac{2J}{ca}\left(\arctan\frac{a+2x}{2y}+\arctan\frac{a-2x}{2y}\right),$$

$$H_y = \frac{J}{ca}\ln\frac{(x-a/2)^2+y^2}{(x+a/2)^2+y^2}, \qquad H_z = 0 .$$

The y-axis is perpendicular to the strip and passes through its axis.

5.5 The repulsive force on each of the plates is given by

$$f = \frac{4J^2}{c^2a^2}\left(a\arctan\frac{a}{b} - \tfrac{1}{2}b\ln\frac{a^2+b^2}{b^2}\right).$$

5.6

$$A_z = \frac{2J}{c}\ln\frac{r_2}{r_1} = \frac{J}{c}\ln\frac{(a+x)^2+y^2}{(a-x)^2+y^2},$$

$$H_x = \frac{\partial A_z}{\partial y} = -\frac{8J}{c}\frac{axy}{r_1^2r_2^2},$$

$$H_y = -\frac{\partial A_z}{\partial x} = -\frac{2J}{c}\left(\frac{a-x}{r_1^2}+\frac{a+x}{r_2^2}\right),$$

where the position of the two currents J and $-J$ in the plane perpendicular to them is defined by the coordinates $(a, 0)$ and $(-a, 0)$, respectively; r_1 and r_2 are the distances from the points $(a, 0)$ and $(-a, 0)$ to the point of observation.

5.7 (a) in the space between the planes $H = 4\pi i/c$, in the remaining space $H = 0$; (b) in the space between the planes $H = 0$, in the remaining space $H = 4\pi i/c$. The magnetic field is perpendicular to the currents and parallel to the planes in both cases.

5.8

$$H_y = \frac{2Jd}{c(b^2-a^2)}, \qquad H_x = H_z = 0,$$

where the y-axis is perpendicular to the plane containing the axes of the cylinders.

5.9 In the cylindrical system of coordinates whose z-axis is perpendicular to the plane of the ring and passes through its centre

$$A_\phi = \frac{2\mu J}{c}\left(\frac{a}{r}\right)^{1/2}\left[\left(\frac{2}{k}-k\right)K(k)-\frac{2}{k}E(k)\right], \qquad A_z = A_r = 0,$$

where $k^2 = 4ar/[(a+r)^2+z^2]$ and $K(k)$ and $E(k)$ are the complete elliptic integrals.

The components of the magnetic field are

$$H_r = \frac{2Jz}{cr[(a+r)^2+z^2]^{1/2}}\left[-K(k)+\frac{a^2+r^2+z^2}{(a-r)^2+z^2}E(k)\right],$$

$$H_z = \frac{2J}{c[(a+r)^2+z^2]^{1/2}}\left[K(k)+\frac{a^2-r^2-z^2}{(a-r)^2+z^2}E(k)\right], \qquad H_\phi = 0 .$$

On the axis of the ring ($r = 0$) these expressions become

$$H_r = 0, \qquad H_z = \frac{2\pi a^2 J}{c(a^2+z^2)^{3/2}} .$$

5.10 The flux of magnetic induction remains constant along any such tube. Hence the equation of the surface of the tube is

$$N = \int_S (B \cdot \mathrm{d}S) = f(r,z) = \text{constant},$$

where the surface of integration S has a circular cross-section of radius r in the plane perpendicular to the axis of symmetry (the centre of the circle lies on the axis of symmetry). Since A_ϕ is independent of ϕ, Stokes' theorem gives

$$\int (B \cdot \mathrm{d}^2S) = \oint (A \cdot \mathrm{d}l) = 2\pi r A_\phi(r,z) = \text{constant}.$$

The lines of intersection of these surfaces with the planes ϕ = constant are the required lines of magnetic induction.

5.11 The components of the magnetic field are given by

$$H_z = -\frac{\partial\psi}{\partial z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}H^{(2n)}(z)(\tfrac{1}{2}r)^{2n} = H(z)-\tfrac{1}{4}r^2H''(z)+\dots,$$

$$H_r = -\frac{\partial\psi}{\partial r} = \sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!n!}H^{(2n-1)}(z)(\tfrac{1}{2}r)^{2n-1} = -\tfrac{1}{2}rH'(z)+\dots,$$

$$H_\phi = 0 .$$

The vector potential can be expressed in terms of the magnetic field with the aid of Stokes' theorem and the relation $H = \mathrm{curl}A$. The result is

$$A_\phi(r,z) = \frac{1}{r}\int_0^r H_z r\,\mathrm{d}r = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(n+1)!}H^{(2n)}(z)(\tfrac{1}{2}r)^{2n+1} = \tfrac{1}{2}rH(r)-\dots .$$

5.12

$$H_z = \frac{2\pi nJ}{c}(\cos\theta_1+\cos\theta_2),$$

where (see figure 5.12.1):

$$\cos\theta_1 = \frac{h-z}{[a^2+(h-z)^2]^{1/2}}, \qquad \cos\theta_2 = \frac{z}{(a^2+z^2)^{1/2}} .$$

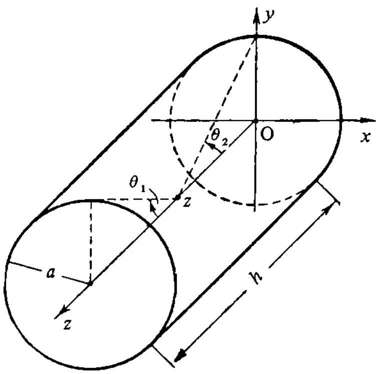


Figure 5.12.1.

5.13 We use the vector potential method to solve this problem. The density of the surface current which is produced as a result of the rotation of the sphere is

$$\boldsymbol{i} = \boldsymbol{e}_\phi \frac{e\omega}{4\pi a} \sin\vartheta ,$$

where the polar axis lies along the vector $\boldsymbol{\omega}$. The vector potential at all points outside the surface of the sphere satisfies the Laplace equation. In view of the symmetry of the system, the vector potential may be chosen so that A_ϕ is the only finite component. A_ϕ will then be independent of ϕ. It follows that the equation for the vector potential is

$$\nabla^2 A_\phi - \frac{1}{r^2 \sin^2\vartheta} A_\phi = 0 \tag{5.13.1}$$

(see the solution of problem 1.47). Because the current density is proportional to $\sin\vartheta$, it is natural to look for the solution of equation (5.13.1) in the form

$$A_\phi(r, \vartheta) = F(r) \sin\vartheta . \tag{5.13.2}$$

It will be seen below that $F(r)$ can be chosen so that it will satisfy both the above equation and the boundary conditions, which justifies the choice (5.13.2). We note that the vector potential (5.13.2) satisfies the condition $\mathrm{div}\boldsymbol{A} = 0$, which is necessary if equation (5.13.1) is to hold. A_ϕ and $\boldsymbol{H} = \mathrm{curl}\boldsymbol{A}$ can be obtained by determining $F(r)$ with the aid of equation (5.13.1) and the boundary conditions. The magnetic field inside the sphere ($r < a$) is

$$\boldsymbol{H} = \frac{2e}{3c}\boldsymbol{\omega} .$$

The field outside the sphere ($r > a$) is

$$H = \frac{3r(m \cdot r)}{r^5} - \frac{m}{r^3},$$

where $m = (ea^2/3c)\omega$ is the magnetic moment of the system.

5.14 At all points where $j = 0$ it may be assumed that $H = -\mathrm{grad}\,\psi$. The equation $\mathrm{curl}H = 0$ is then satisfied for all ψ and the equation $\mathrm{div}H = 0$ yields $\nabla^2\psi = 0$. The latter equation may be solved subject to the additional condition

$$\int_l (H \cdot \mathrm{d}l) = \frac{4\pi}{c}J,$$

where l is a contour drawn round the current J. Consider the cylindrical coordinates r, ϕ, z and suppose that the solution is of the form $\psi = \psi(\phi)$. The final result is

$$\psi = -\frac{2J}{c}\phi, \qquad H_\phi = \frac{2J}{cr}, \qquad H_r = H_z = 0.$$

5.15 (a) In order to ensure that the scalar potential ψ of the magnetic field is a single-valued function, let us choose a surface S (figure 5.15.1) drawn on the linear current and assume that the change in ψ on passing through this surface is given by

$$\psi(2) - \psi(1) = \frac{4\pi}{c}J. \tag{5.15.1}$$

The points 1 and 2 lie infinitely near to each other while still remaining on opposite sides of the surface. The direction of the line drawn from 1 to 2 forms a right-handed system with the direction of the current (see figure 5.15.1). The solution of the Laplace equation may be written in

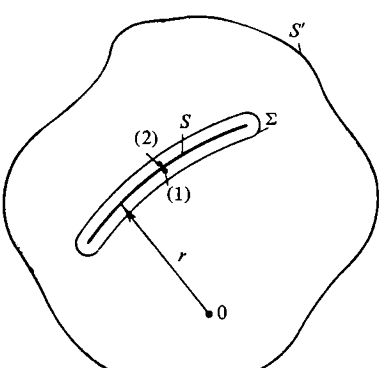


Figure 5.15.1.

the form

$$\psi = \frac{1}{4\pi}\oint\left[\frac{1}{r}\frac{\partial\psi}{\partial n} - \psi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\right] d^2S\,, \tag{5.15.2}$$

where the integration may be carried out over an infinitely distant closed surface S' and over all closed surfaces Σ_i at a finite distance from the origin, within which ψ and $\partial\psi/\partial n$ have discontinuities. In the example under consideration, the integral over the infinitely distant surface will vanish because the field source (linear current) has finite dimensions. There are no surfaces across which the normal derivative $\partial\psi/\partial n = -H_n$ is discontinuous since H_n is a continuous function. It follows that the integral in equation (5.15.2) should be taken over a single surface Σ surrounding S.

Let us shrink Σ until it coincides with S. Since $1/r$, $\partial\psi/\partial n$, and $\partial(1/r)/\partial n$ are all continuous across S, equation (5.15.2) will assume the form

$$\psi = -\frac{1}{4\pi}\int[\psi(1) - \psi(2)]\frac{\partial}{\partial n}\left(\frac{1}{r}\right) d^2S\,,$$

where the integration is now carried out over the open surface S.

Using equation (5.15.1) we have

$$\psi = \frac{J}{c}\int\frac{\partial}{\partial n}\left(\frac{1}{r}\right) d^2S = -\frac{J}{c}\int\frac{(\boldsymbol{r}\cdot d^2\boldsymbol{S})}{r^3}\ . \tag{5.15.3}$$

The integral $\int(\boldsymbol{r}\cdot d^2\boldsymbol{S})/r^3$ represents the solid angle Ω subtended by the linear current at the point of observation, and hence equation (5.15.3) may be rewritten in the form

$$\psi = -\frac{J}{c}\Omega\,.$$

Ω is positive if the radius vector $\boldsymbol{r}$ drawn from the point of observation to any point on the surface S, and the direction of the current form a right-handed system.

(b) Transform the line integral into a surface integral using the result of problem 1.55. Thus,

$$\boldsymbol{A} = \frac{J}{c}\int[d^2\boldsymbol{S}\wedge\nabla(1/r)] = \frac{J}{c}\int[\nabla_{\mathrm{M}}(1/r)\wedge d^2\boldsymbol{S}]\,,$$

where ∇_{M} represents differentiation with respect to the coordinates of the point of observation M. Using the relation $\boldsymbol{H} = \mathrm{curl}\boldsymbol{A}$ we have

$$\boldsymbol{H} = \frac{J}{c}\int(d^2\boldsymbol{S}\cdot\nabla_{\mathrm{M}})\nabla_{\mathrm{M}}(1/r) = \frac{J}{c}\nabla_{\mathrm{M}}\int(d^2\boldsymbol{S}\cdot\nabla_{\mathrm{M}}(1/r))\,. \tag{5.15.4}$$

[In carrying out the transformation, use was made of the result $\nabla^2(1/r) = 0$, and it was assumed that the point $r = 0$ did not lie on the surface of integration.] Comparing equation (5.15.4) with $H = -\text{grad}\,\psi$ we have

$$\psi = -\frac{J}{c}\int(\mathrm{d}^2S \cdot \nabla_{\mathrm{M}}(1/r)) = -\frac{J}{c}\int\frac{(r \cdot \mathrm{d}^2S)}{r^3} = -\frac{J}{c}\Omega\,,$$

5.16 $F = 0\,,\qquad N = [m \wedge H]\,,$

where

$$m = \frac{J}{c}\int n \cdot \mathrm{d}^2S$$

is the magnetic moment of the current-carrying conductor.

5.17

$$U = \frac{(m_1 \cdot m_2)}{r^3}\ \frac{3(m_1 \cdot r)(m_2 \cdot r)}{r^5}\,,$$

$$F_2 = -F_1 = \frac{3}{r^5}[(m_1 \cdot r)m_2 + (m_2 \cdot r)m_1 + (m_1 \cdot m_2)r] - \frac{15}{r^7}(m_1 \cdot r)(m_2 \cdot r)r$$

where r is the position vector of the second loop relative to the first, F_1 and F_2 are the forces on the two loops, and the couples acting on them are given by

$$N_1 = \frac{3(m_2 \cdot r)[m_1 \wedge r]}{r^5} + \frac{[m_2 \wedge m_1]}{r^3}\,,$$

$$N_2 = \frac{3(m_1 \cdot r)[m_2 \wedge r]}{r^5} + \frac{[m_1 \wedge m_2]}{r^3}\,.$$

It should be noted that $N_1 \neq -N_2$ and

$$N_1 + N_2 + [r \wedge F_2] = 0\,.$$

If the magnetic moments are parallel ($m_1 = m_1 n$, $m_2 = m_2 n$, $r = r r_0$, where n and r_0 are unit vectors) then

$$F_2 = \frac{3m_1 m_2[2n\cos\vartheta - r_0(5\cos^2\vartheta - 1)]}{r^4}\,,$$

where ϑ is the angle between n and r_0.

5.19 The potential function of the current J_2 in the field of J_1 is

$$u_{21} = \frac{2J_1J_2}{c^2}\ln a + \text{constant}\,,$$

where a is the distance between the currents. The force per unit length of the second current is given by

$$f = -\frac{\partial u_{21}}{\partial a} = -\frac{2J_1J_2}{c^2 a}\,.$$

When the currents are of the same sign they attract each other.

5.20 The force F and the couple N can be obtained by differentiating the potential function

$$U(r,\alpha) = -\frac{J_1 J_2 a}{c^2} \ln \frac{4r^2 + a^2 + 4ar\cos\alpha}{4r^2 + a^2 - 4ar\cos\alpha} \, .$$

5.21

$$N = \frac{4JJ'a}{c}(\sin\varphi - \varphi\cos\varphi) \, .$$

5.22

$$\mathcal{L} = \tfrac{1}{2}\mu_0 + 2\mu\ln\frac{b}{a} \, .$$

5.23

$$\mathcal{L} = 2\mu\ln\frac{b}{a} \, .$$

5.24 $L_{12} = 4\pi[b-(b^2-a^2)^{1/2}]$;

$$F = \frac{J_1 J_2}{c^2}\frac{\partial L_{21}}{\partial b} = \frac{4\pi J_1 J_2}{c^2}\left[1 - \frac{b}{(b^2-a^2)^{1/2}}\right] .$$

5.25 It is convenient to use equation (5.0.23). As in the solution of problem 2.21, we have

$$L_{12} = 4\pi(ab)^{1/2}\left[\left(\frac{2}{k}-k\right)K(k) - \frac{2}{k}E(k)\right] ,$$

where

$$K(k) = \int_0^{\frac{1}{2}\pi} \frac{d\psi}{(1-k^2\sin^2\psi)^{1/2}} \, , \qquad E(k) = \int_0^{\frac{1}{2}\pi} (1-k^2\sin^2\psi)^{1/2}\,d\psi \, ,$$

$$k^2 = \frac{4ab}{(a+b)^2 + l^2} \, .$$

When $l \gg a, b$ the parameter k is small so that

$$k^2 \approx \frac{4ab}{l^2} \, , \qquad k \approx \frac{2(ab)^{1/2}}{l} \ll 1 \, .$$

It follows that the following approximate formulae for E and K can be used:

$$K(k) = \tfrac{1}{2}\pi(1 + \tfrac{1}{4}k^2 + \tfrac{9}{64}k^4) \, , \qquad E(k) = \tfrac{1}{2}\pi(1 - \tfrac{1}{4}k^2 - \tfrac{3}{64}k^4) \, .$$

By retaining in the expression for L_{12} only the terms proportional to k^3, we get the first nonvanishing approximation for L_{12}, which is

$$L_{12} = \frac{2\pi^2 a^2 b^2}{l^3} \, .$$

The latter result can also be obtained from the equation $L_{12} = c\Phi_{12}/J_1$ by regarding the two rings as magnetic dipoles.

When

$$a \approx b \gg l\,, \qquad k \approx 1\,, \qquad K(k) \approx \ln\frac{4}{(1-k^2)^{1/2}}\,, \qquad E(k) \approx 1\,,$$

$$L_{12} = 4\pi a\left\{\ln\frac{8a}{[l^2+(a-b)^2]^{1/2}} - 2\right\}.$$

5.26 In the notation of the preceding problem

$$F = \frac{4\pi J_1 J_2}{c^2}\frac{l}{[(a+b)^2+l^2]^{1/2}}\left[-K(k)+\frac{a^2+b^2+l^2}{(a+b)^2+l^2}E(k)\right].$$

5.27 $\mathcal{L} = 4\pi n^2 S$.

For a solenoid having a large though finite length h, the total self-inductance (neglecting edge effects) is $L = 4\pi n^2 Sh$.

5.28 The magnetic energy is given by

$$W = \frac{1}{2c^2}\int\frac{(i_1 \cdot i_2)}{R}\,\mathrm{d}^2S_1\,\mathrm{d}^2S_2\,,$$

where d^2S_1 and d^2S_2 are elements of the surface of the solenoid, R is the distance between them, i ($i_1 = i_2 = i = nJ$) is the density of the surface current, by which the actual current flowing in the solenoid may be replaced, and n is the number of turns per unit length. The integral can conveniently be evaluated in terms of cylindrical coordinates:

$$W = \frac{\pi n^2 a^2 J^2}{c^2}\int_0^h \mathrm{d}z_1\int_0^h \mathrm{d}z_2\oint\frac{\cos\phi\,\mathrm{d}\phi}{[(z_1-z_2)^2+4a^2\sin^2\frac{1}{2}\phi]^{1/2}}$$
$$= \frac{2\pi^2 a^2 n^2 J^2 h}{c^2}\left(1-\frac{8a}{3\pi h}\right),$$

where terms of the order of $(a/h)^2$ or higher are neglected. Hence

$$L = 4\pi^2 a^2 n^2 h\left(1-\frac{8a}{3\pi h}\right).$$

When $a/h \ll 1$, this result will be identical with that obtained in the preceding solution:

$$L = 4\pi^2 a^2 n^2 h = 4\pi n^2 Sh\,.$$

5.29 For a circular cross section

$$L = 4\pi N^2[b-(b^2-a^2)^{1/2}]\,.$$

The self-inductance per unit length of an infinite solenoid $\mathcal{L} = L/(2\pi b)$ may be obtained by letting $b \to \infty$ at a constant number of turns per unit length $n = N/(2\pi b)$ (cf problem 5.27):

$$\mathcal{L} = 4\pi^2 n^2 a^2 = 4\pi n^2 S .$$

For a rectangular cross section

$$L = 2N^2 h \ln\left(\frac{2b+a}{2b-a}\right) .$$

When $b \gg a$, we have $\mathcal{L} = 4\pi n^2 S$ as before.

If the current flows directly along the shell of the torus, the self-inductance is diminished by a factor N^2 as compared to the self-inductance of a torus with a wire wound around it. Hence we have for a torus of circular cross section

$$L = 4\pi[b-(b^2-a^2)^{1/2}] ,$$

and for a torus with a rectangular cross section

$$L = 2h \ln\left(\frac{2b+a}{2b-a}\right) .$$

5.30 The magnetic energy per unit length of the system can be calculated with the aid of equation (5.0.16). The vector potential for a straight conductor was obtained in the solution of problem 5.2. For conductor 1 (figure 5.30.1) this potential can be written in the form

$$A_{1z} = C-\frac{Jr_1^2}{ca^2} , \qquad \text{when} \quad r_1 < a , \tag{5.30.1}$$

$$A_{1z} = C-\frac{J}{c}\left(1+2\ln\frac{r_1}{a}\right) , \qquad \text{when} \quad r_1 > a .$$

The vector potential due to conductor 2 is obtained by making the following substitutions in equation (5.30.1): $J \to -J$, $a \to b$, $r_1 \to r_2$.

The magnetic energy is given by

$$W = \frac{J}{2\pi ca^2}\int_1 (A_{1z}+A_{2z})\,\mathrm{d}^2S_1 - \frac{J}{2\pi cb^2}\int_2 (A_{1z}+A_{2z})\,\mathrm{d}^2S_2 .$$

Recalling the relation between the inductance and magnetic energy, we find that

$$\mathcal{L} = 1+2\ln\frac{h^2}{ab} .$$

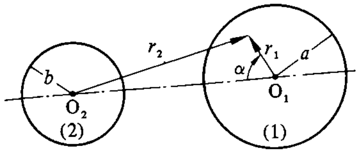


Figure 5.30.1.

5.31 The total magnetic energy of the current flowing in the conductor consists of two parts:

$$W = W_1 + W_2\,, \tag{5.31.1}$$

where

$$W_1 = \frac{\mu_0}{8\pi}\int H_1^2\,\mathrm{d}^3r$$

is the total energy stored inside the conductor (the integration is carried out over the volume of the conductor), and

$$W_2 = \frac{\mu}{8\pi}\int H_2^2\,\mathrm{d}^3r$$

is the energy stored in the remaining space. Let us suppose that we may introduce a parameter r_0 which has the dimensions of length and satisfies the condition

$$a \ll r_0 \ll R\,, \tag{5.31.2}$$

where a is the radius of the conductor and R is the radius of curvature of the axial line of the conductor (which in general will not be a constant). It follows that at distances smaller than r_0 the magnetic field can be considered as identical with that due to an infinitely long straight conductor. In particular, inside the conductor

$$H_1 = \frac{2Jr}{ca^2}$$

(cf problem 5.2). This may be used to find the 'internal' energy W_1, which is given by

$$W_1 = \frac{\mu_0 l J^2}{4c^2}\,. \tag{5.31.3}$$

In order to determine the 'external' energy W_2, consider a surface S on an arbitrary contour lying on the surface of the conductor and introduce a scalar potential ψ. This potential will exhibit a discontinuity across S which is given by

$$\psi_+ - \psi_- = \frac{4\pi}{c}J\,. \tag{5.31.4}$$

The integral in the expression for W_2 may be transformed as follows:

$$\int(\boldsymbol{B}\cdot\boldsymbol{H})\,\mathrm{d}^3r = -\int(\boldsymbol{B}\cdot\operatorname{grad}\psi)\,\mathrm{d}^3r = -\int\operatorname{div}(\psi\boldsymbol{B})\,\mathrm{d}^3r = -\oint\psi B_n\,\mathrm{d}^2S\,,$$

where the subscript 2 has been omitted and $\operatorname{div}\boldsymbol{B} = 0$. In the final integral the integration should be carried out over both sides of the surface S and the surface of the conductor S' (see figure 5.31.1, which shows a cross section of the conductor). The integral over an infinitely distant surface will be zero in view of the finite dimensions of the conductor. Thus

$$W_2 = -\frac{1}{8\pi}\int_{S'} \psi B_n \, d^2S + \frac{1}{8\pi}\int_S \psi_+ B_n \, d^2S - \frac{1}{8\pi}\int_S \psi_- B_n \, d^2S .$$

The first of these integrals is zero because in view of equation (5.31.2) the magnetic field on S' is identical with the field due to a straight conductor and has, therefore, only a tangential component. The remaining integrals may be transformed with the aid of equation (5.31.4) and the fact that B_n is continuous. Thus

$$W_2 = \frac{J}{2c}\int_S B_n \, d^2S . \tag{5.31.5}$$

At large distances from the conductor ($r > r_0$) the magnetic field is independent of the current distribution across the conductor and hence it may be assumed that the current flows along the axis. At small distances ($a \leqslant r < r_0$) the field is identical with the field due to an infinitely long circular cylinder, and again it may be considered that the current flows along the axis. Thus, the integral (5.31.5) represents the flux of magnetic induction due to a current flowing along the axis of the conductor through a surface drawn on a closed contour lying on the surface of the conductor. Using equation (5.0.22) we have

$$W_2 = \frac{J^2}{2c^2}L' . \tag{5.31.6}$$

Finally, from equations (5.31.1), (5.31.3) and (5.31.6) and the relation between self-inductance and the magnetic energy, we obtain the required formula for the self-inductance:

$$L = \tfrac{1}{2}\mu_0 l + L' .$$

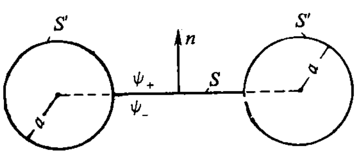


Figure 5.31.1.

5.32 Using the result of the preceding problem we have

$$L' = 4\pi\mu b\left(\ln\frac{8b}{a} - 2\right) ,$$

where μ is the magnetic permeability of the medium in which the conductor is located. The total self-inductance is

$$L = 4\pi b\left(\mu \ln\frac{8b}{a} - 2\mu + \tfrac{1}{4}\mu_0\right) ,$$

or

$$L = 4\pi b\left(\ln\frac{8b}{a} - \frac{7}{4}\right),$$

when $\mu_0 = \mu = 1$.

5.33

$$L_{12} = 2l - 2(a^2+l^2)^{1/2} + 2a\ln\frac{a+(a^2+l^2)^{1/2}}{l}.$$

5.34 Using the result of the preceding problem we have

$$L_{12} = 8\left[l - 2(a^2+l^2)^{1/2} + (2a^2+l^2)^{1/2} + a\ln\frac{a+(a^2+l^2)^{1/2}}{l} - a\ln\frac{a+(2a^2+l^2)^{1/2}}{(a^2+l^2)^{1/2}}\right],$$

$$F = \frac{8J_1J_2}{c^2}\left[\frac{a^2+2l^2}{l(a^2+l^2)^{1/2}} - \frac{l(2a^2+l^2)^{1/2}}{a^2+l^2} - 1\right].$$

5.35

$$L = 2\mu_0 b + 8\mu b\left[\ln\frac{2b}{a(1+\sqrt{2})} + \sqrt{2} - 2\right].$$

5.36 Using equation (5.0.13) in the integration with respect to the angles, and the relation $\overline{n_i n_k} = \frac{1}{3}\delta_{ik}$ (see problem 1.32), we have:
for a uniform volume distribution of charge

$$m = \frac{ea^2}{5c}\omega;$$

for a uniform surface distribution of charge

$$m = \frac{ea^2}{3c}\omega.$$

For a sphere whose radius is equal to the classical radius of the electron ($2\cdot 8\times 10^{-15}$ m) and whose magnetic moment is equal to the magnetic moment of the electron as measured experimentally ($0\cdot 9\times 10^{-24}$ J T^{-1}), it turns out that the linear velocity at the equator of the sphere is $v = a\omega \approx 10^{11}$ m s^{-1} and is therefore greater than the velocity of light in vacuo. This indicates that classical ideas cannot be used to describe electron spin (Akhiezer and Berestetskii, 1965).

5.38 For the secondary field, $\mathrm{curl}\boldsymbol{H}' = 0$, i.e. it may be described by a scalar potential defined by $\boldsymbol{H}' = -\mathrm{grad}\,\psi$. Hence the equation for the potential is the same as in electrostatics in the case of a nonuniform medium, i.e.

$$\mathrm{div}(\mu\,\mathrm{grad}\,\psi) = -4\pi\rho_m,$$

where

$$\rho_m = -\frac{1}{4\pi}(\boldsymbol{H}_0 \cdot \operatorname{grad}\mu) ,$$

represents the density of magnetic charges. On the boundary of separation between two media the tangential components must satisfy the conditions

$$H'_{1\tau} = H'_{2\tau} , \quad \text{or} \quad \frac{\partial \psi_1}{\partial \tau} = \frac{\partial \psi_2}{\partial \tau}$$

whereas the normal components must satisfy the conditions

$$\mu_2 H'_{2n} - \mu_1 H'_{1n} = (\mu_1 - \mu_2) H_{0n} , \quad \text{or} \quad \mu_1 \frac{\partial \psi_1}{\partial n} - \mu_2 \frac{\partial \psi_2}{\partial n} = 4\pi\sigma_m .$$

In the above expression

$$\sigma_m = \frac{1}{4\pi}(\mu_1 - \mu_2) H_{0n}$$

is the surface density of magnetic charge. We note that the expression for σ_m may be obtained from the volume density of charge by means of the following formula

$$\sigma_m = \lim_{h \to 0} \rho_m h .$$

Let us replace the separation boundary by a thin layer of thickness h. The vector $\operatorname{grad}\mu$ will then lie along the normal to the layer and will be equal to $(\mu_2 - \mu_1)/h$, and hence

$$\rho_m = -\frac{1}{4\pi h}(\mu_2 - \mu_1) H_{0n} , \qquad \sigma_m = \lim_{h \to 0} \rho_m h = \frac{1}{4\pi}(\mu_1 - \mu_2) H_{0n} .$$

5.39

$$\boldsymbol{H}_1 = \frac{\mu_2}{\mu_1 + \mu_2} \boldsymbol{H}_0 , \qquad \boldsymbol{H}_2 = \frac{\mu_1}{\mu_1 + \mu_2} \boldsymbol{H}_0 ,$$

where $\boldsymbol{H}_0$ is the field due to the circuit in vacuo and $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ are the fields in the media with permeabilities μ_1, μ_2.

5.40 The magnetic field in medium 1 is the same as the field produced in vacuo by two straight currents

$$J_1 = \mu J \quad \text{and} \quad J_2 = \frac{\mu_1(\mu_2 - \mu_1)}{\mu_1 + \mu_2} J ,$$

where J_1 flows along the same conductor as the original current J, and J_2 flows along a conductor which is a mirror image of the first conductor with respect to the separation boundary between the two media. The magnetic field in medium 2 is identical with the field produced in vacuo by a current

$$J_1 = \frac{2\mu_1\mu_2}{\mu_1+\mu_2} J$$

which flows along the same conductor as the original current J.

5.41 The field vectors satisfy the relations $\mathrm{curl}\boldsymbol{H} = 0 = \mathrm{div}\boldsymbol{B}$ in all space, and hence it is possible to introduce a scalar potential ψ which is defined by $\boldsymbol{H} = -\mathrm{grad}\,\psi$ and satisfies the Laplace equation. This reduces the magnetostatic problem to the corresponding electrostatic problem. The solution inside the sphere is (see problem 3.21)

$$\boldsymbol{H}_1 = \frac{3}{\mu+2}\boldsymbol{H}_0 .$$

Outside the sphere the solution is

$$\boldsymbol{H}_2 = \boldsymbol{H}_0 + \boldsymbol{H}_\mathrm{d} ,$$

where $\boldsymbol{H}_\mathrm{d}$ is the field due to a magnetic dipole whose moment is given by

$$\boldsymbol{m} = \frac{\mu-1}{\mu+2} a^3 \boldsymbol{H}_0 .$$

Since the field inside the sphere is uniform, the magnetisation is constant and is given by

$$\boldsymbol{M} = \frac{3\boldsymbol{m}}{4\pi a^3} = \frac{3(\mu-1)}{4\pi(\mu+2)}\boldsymbol{H}_0 .$$

The equivalent volume current density is therefore zero, i.e.

$$j_\mathrm{mol} = c\,\mathrm{curl}\boldsymbol{M} = 0 .$$

The surface current density is given by

$$i_\mathrm{mol} = c[\boldsymbol{n} \wedge (\boldsymbol{M}_2 - \boldsymbol{M}_1)] .$$

This expression may be obtained from equation (5.0.3) by taking it to the limit (cf the derivation of the boundary condition for the tangential component H_r from Maxwell's equations). Substituting $\boldsymbol{M}_2 = 0$ and $\boldsymbol{M}_1 = \boldsymbol{M}$ we have

$$i_\mathrm{mol} = \boldsymbol{e}_\phi \frac{3c(\mu-1)}{4\pi(\mu+2)} H_0 \sin\vartheta .$$

It is interesting to note that the surface current may be obtained by rotating a sphere with a uniform surface charge about one of its diameters (see problem 5.13).

5.42 Consider a coordinate system whose axes lie along the principal axes of the permeability tensor. The field components inside the sphere will then be equal to

$$\frac{3}{\mu^{(k)}+2}H_{0k}\,,$$

where H_0 is the external field. Inside the sphere $H_2 = H_0 + H_d$ where H_d is the field due to a magnetic dipole having a moment m with

$$m_k = \frac{\mu^{(k)}-1}{\mu^{(k)}+2}a^3 H_{0k}\,.$$

The couple acting on the sphere is $N = [m \wedge H_0]$.

5.43

$$H = \left[1 - \frac{1-(a/b)^2}{(\mu_1+\mu_2)^2/(\mu_1-\mu_2)^2-(a/b)^2}\right]H_0\,.$$

When $\mu_1 \gg \mu_2$ the field in the cavity is considerably reduced (magnetic screening).

5.44

$$H = \left[1 - \frac{1-(a/b)^3}{\frac{1}{2}(\mu_1+2\mu_2)(2\mu_1+\mu_2)/(\mu_1-\mu_2)^2-(a/b)^3}\right]H_0\,.$$

When $\mu_1 \gg \mu_2$ the field is considerably reduced ($H \ll H_0$).

5.45 The magnetic field is given by $H = \operatorname{curl} A$ where

$$A_z = -\frac{\mu_1 J}{4\pi a^2}r^2 + \frac{2\mu_1}{\mu_1+\mu_2}B_0 r\sin\phi\,, \qquad \text{when} \quad r < a\,,$$

$$A_z = \frac{\mu_2 J}{2\pi}\ln\frac{a}{r} + \left[1+\left(\frac{\mu_1-\mu_2}{\mu_1+\mu_2}\right)\frac{a^2}{r^2}\right]B_0 r\sin\phi\,, \qquad \text{when} \quad r > a\,.$$

The z-axis lies along the axis of the cylinder; the remaining components of A are zero.

5.48

$$f = \frac{2J^2a^2(\mu-1)}{c^2b(b^2-a^2)(\mu+1)}\,.$$

5.49

$$f = \frac{2J^2b(\mu-1)}{c^2(a^2-b^2)(\mu+1)}\,.$$

5.50

$$H_i = \frac{1}{\mu_i}\left(\frac{2\pi\mu_1\mu_2\mu_3}{\mu_1\mu_2\alpha_3+\mu_2\mu_3\alpha_1+\mu_1\mu_3\alpha_2}\right)H_0\,,$$

where H_0 is the field produced by the current in vacuo.

5.51 Outside the sphere the magnetic induction B and the magnetic field H are related by the usual formula $B_2 = \mu_2 H_2$. According to equation (5.0.27), the magnetic induction inside the sphere is given by

$$B_1 = \mu_1 H_1 + 4\pi M_0 ,$$

where M_0 is the constant magnetisation. Using the scalar potential as in problem 5.41, we have

$$\psi_1 = -(H_1 \cdot r) , \qquad \psi_2 = \frac{(m \cdot r)}{r^3} ,$$

where

$$H_1 = -\frac{4\pi M_0}{2\mu_2 + \mu_1} , \qquad m = \frac{4\pi a^3 M_0}{2\mu_2 + \mu_1} .$$

Thus the field inside the sphere is uniform and the field outside the sphere is the same as that due to a magnetic dipole of moment m.

5.52 The field inside the cylinder is

$$H_1 = -\frac{4\pi M_0}{\mu_2 + \mu_1} .$$

The field outside the cylinder is

$$H_2 = \frac{2r(m \cdot r)}{r^4} - \frac{m}{r^2} ,$$

where M_0 is the constant magnetisation and

$$m = \frac{4\pi a^2 M_0}{\mu_2 + \mu_1} .$$

5.53 The field inside the sphere is

$$H_1 = \frac{3}{\mu + 2} H_0 - \frac{4\pi M_0}{\mu + 2} .$$

Outside the sphere the field is

$$H_2 = H_0 + \frac{3r(m \cdot r)}{r^5} - \frac{m}{r^3} , \tag{5.53.1}$$

where

$$m = \frac{4\pi a^3 M_0}{\mu + 2} + \frac{\mu - 1}{\mu + 2} a^3 H_0 .$$

Since the external field is uniform, it follows that the resultant force on the sphere is zero. However, the sphere will experience a couple if the directions of M_0 and H_0 are different. This couple may be computed with the aid of the magnetic field stress tensor. The couple is given by

$$N_i = \oint_S e_{ikl} x_k T_{lm} \, \mathrm{d}^2 S_m , \tag{5.53.2}$$

where T_{lm} is the stress tensor given by equation (5.0.26) and e_{ikl} is the skew-symmetric unit tensor. The integration is carried out over the surface of the magnet. Substituting equation (5.0.26) into equation (5.53.2) we have (in vector notation)

$$N = \frac{1}{4\pi}\oint [r \wedge H_2](H_2 \cdot \mathrm{d}^2S) - \frac{1}{8\pi}\oint H_2^2[r \wedge \mathrm{d}^2S] . \tag{5.53.3}$$

Since the centre of this sphere is taken as the origin, it follows that r and d^2S are parallel and the second integral in equation (5.53.3) is equal to zero. In order to evaluate the first integral let

$$\mathrm{d}^2S = n\,\mathrm{d}^2S = na^2\,\mathrm{d}^2\omega , \qquad r = an ,$$

and substitute for H_2 from equation (5.53.1). This gives

$$N = \frac{1}{4\pi}\int \{a^3[n \wedge H_0] + [m \wedge n]\}\left[(H_0 \cdot n) + \frac{2}{a^3}(m \cdot n)\right] \mathrm{d}^2\omega .$$

The components of the couple are therefore given by

$$N_i = a^3 e_{ikl} H_{0l} H_{0m} \overline{n_k n_m} + 2e_{ikl} H_{0l} m_s \overline{n_k n_s} + e_{ikl} m_k H_{0m} \overline{n_l n_m} + \frac{2}{a^3} e_{ikl} m_k m_s \overline{n_l n_s} . \tag{5.53.4}$$

Using the relation $\overline{n_k n_m} = \frac{1}{3}\delta_{km}$ (see problem 1.32) we find that two of the four terms on the right-hand side of equation (5.53.4) are equal to zero, and the remaining terms lead to the following expression for the couple

$$N = [m \wedge H_0] .$$

Finally, when m is expressed in terms of the constant magnetisation we have

$$N = \frac{4\pi a^3}{\mu + 2}[M_0 \wedge H_0] .$$

As can be seen from this formula, the induced part of the magnetic moment $[(\mu - 1)/(\mu + 2)]a^3 H_0$ does not contribute to the resultant couple.

5.54

$$F = \frac{3m^2}{16a^4}\frac{\mu - 1}{\mu + 2}(1 + \cos^2\theta) , \qquad N = \frac{m^2}{8a^3}\frac{\mu - 1}{\mu + 2}\sin\theta \cos\theta ,$$

where a is the distance of the magnet from the plane and θ is the angle between m and the normal to the plane. When $\mu \gg 1$ (soft iron in a weak magnetic field), the result becomes identical to that for an electric dipole in the vicinity of a metal surface (see problem 3.20).

5.55 The required parameters may be obtained by replacing the electrical quantities by the corresponding magnetic quantities in the solution of problem 3.73. In particular, for arbitrary coordinate axes, the internal field $\boldsymbol{H}_1$ in the ellipsoid is of the form

$$H_{1k} = H_{0k} - 4\pi N_{kl} M_l \,,$$

where $\boldsymbol{M}$ is the magnetisation and the N_{kl} are the demagnetisation coefficients. The principal values of the demagnetisation tensor were denoted by $n^{(i)}$ in problem 3.69 and were referred to as the depolarisation coefficients.

5.56 The formula obtained in the preceding solution will also hold for an anisotropic magnetic material. A further relation connecting $\boldsymbol{M}$ and $\boldsymbol{H}_1$ is

$$H_{1k} + 4\pi M_k = \mu_{kl} H_{1l} \,.$$

Combining these formulae we have

$$H_{0k} = b_{km} H_{1m} \,,$$

where

$$b_{km} = \delta_{km} - N_{km} + N_{kl}\mu_{lm} \,,$$

and hence

$$H_{1k} = b_{km}^{-1} H_{0m} \,,$$

where b_{km}^{-1} are the components of the reciprocal tensor. They may be determined with the aid of the formulae obtained in problem 1.11.

Consider a special case. Suppose that the coordinate axes lie along the principal axes of the ellipsoid. If the tensor μ_{ik} referred to these axes has the diagonal form

$$\mu_{ik} = \begin{pmatrix} \mu^{(x)} & 0 & 0 \\ 0 & \mu^{(y)} & 0 \\ 0 & 0 & \mu^{(z)} \end{pmatrix} ,$$

then the tensor b_{ik} and its reciprocal b_{ik}^{-1} will also be diagonal:

$$b_{ik}^{-1} = \begin{pmatrix} [1+N^{(x)}(\mu^{(x)}-1)]^{-1} & 0 & 0 \\ 0 & [1+N^{(y)}(\mu^{(y)}-1)]^{-1} & 0 \\ 0 & 0 & [1+N^{(z)}(\mu^{(z)}-1)]^{-1} \end{pmatrix} .$$

Electrical and magnetic properties of matter

a Polarisation of matter in a constant field

6.1 $\beta = \frac{3}{4}a_0^3$.

If the charge of the electron is distributed uniformly within a sphere of radius a_0 then $\beta = a_0^3$. The model considered in this problem is very approximate and yields only the order of magnitude. Accurate quantum mechanical calculations show that for hydrogen $\beta = \frac{9}{2}a_0^3$.

6.2 It is clear from the symmetry of the molecule that one of the principal axes of the polarisability tensor will lie along the axis of the molecule, while the two other axes may be chosen arbitrarily in the plane perpendicular to the axis of the molecule. It follows that of the three principal values of the polarisability tensor, only two will be distinct. In order to determine these principal values, it is necessary to consider the following situations separately.

(a) The external field is parallel to the axis of the molecule. It is clear that the induced dipole moment of each of the atoms will be parallel to the external field. The dipole moments are given by

$$\boldsymbol{p}' = \beta'(\boldsymbol{E}+\boldsymbol{E}') , \qquad \boldsymbol{p}'' = \beta''(\boldsymbol{E}+\boldsymbol{E}'') , \tag{6.2.1}$$

where $\boldsymbol{E}$ is the external field and $\boldsymbol{E}'$ and $\boldsymbol{E}''$ are the additional fields produced at the centre of each atom by the other atom. The fields $\boldsymbol{E}'$ and $\boldsymbol{E}''$ may be expressed in terms of the dipole moments of the corresponding atoms by means of the formula for the field strength due to a dipole having a moment $\boldsymbol{p}$ and the fact that all the vectors are parallel to the axis of the molecule. If $\boldsymbol{p}'$ and $\boldsymbol{p}''$ are determined from equation (6.2.1), using the relation $\boldsymbol{p} = \boldsymbol{p}'+\boldsymbol{p}'' = \beta^{(1)}\boldsymbol{E}$, we have,

$$\beta^{(1)} = \left[\frac{1}{\beta'} - \frac{2(a^3+2\beta')}{a^3(a^3+2\beta'')}\right]^{-1} + \left[\frac{1}{\beta''} - \frac{2(a^3+2\beta'')}{a^3(a^3+2\beta')}\right]^{-1}$$

(b) The external field is perpendicular to the axis of the molecule. Using a similar method, we have

$$\beta^{(2)} = \beta^{(3)} = \left[\frac{1}{\beta'} + \frac{a^3-\beta'}{a^3(a^3-\beta'')}\right]^{-1} + \left[\frac{1}{\beta''} + \frac{a^3-\beta''}{a^3(a^3-\beta')}\right]^{-1}$$

When $\beta' = \beta''$ the above expressions assume the simpler form

$$\beta^{(1)} = \frac{2\beta'}{1-2\beta'/a^3} , \qquad \beta^{(2)} = \frac{2\beta'}{1+\beta'/a^3} .$$

The average polarisability is given by

$$\bar{\beta} = \tfrac{1}{3}(\beta^{(1)}+2\beta^{(2)}) = \tfrac{2}{3}\beta'\left[\frac{1}{1-2\beta'/a^3} + \frac{2}{1+\beta'/a^3}\right] .$$

6.5 (a) The dielectric as a whole will be anisotropic. The principal values of the polarisability tensor are [see equation (6.a.4)]

$$\alpha^{(i)} = \frac{N\beta^{(i)}}{1-\frac{4}{3}\pi N\beta^{(i)}} \,.$$

(b) When the molecules are randomly oriented there will be no preferred macroscopic directions other than the direction of the external field. Hence, the average dipole moment of a molecule $\bar{\boldsymbol{p}}$ will be proportional to the field E acting on the molecule:

$$\bar{\boldsymbol{p}} = \beta E \,.$$

On the other hand,

$$\bar{p}_i = \overline{\beta_{ik} E_k} = \bar{\beta}_{ik} E_k \,,$$

where the averaging process is carried out over a macroscopically small volume. It follows from the last two expressions that

$$\beta = \bar{\beta}_{11} = \bar{\beta}_{22} = \bar{\beta}_{33} \,, \qquad \bar{\beta}_{ik} = 0 \,, \qquad \text{when} \quad i \neq k \,,$$

and hence

$$\beta = \tfrac{1}{3}\overline{(\beta_{11}+\beta_{22}+\beta_{33})} \,.$$

However, as the sum of the diagonal components of a tensor is an invariant and is equal to the sum of the principal values $\beta^{(1)}+\beta^{(2)}+\beta^{(3)}$ (see problem 1.9), we have

$$\beta = \tfrac{1}{3}(\beta^{(1)}+\beta^{(2)}+\beta^{(3)}) \,.$$

The polarisation coefficient α of the dielectric is related to β by the usual formula (6.a.4′).

6.6 If the axis of the molecule is at an angle θ to the direction of the external field, $\boldsymbol{E}_0$, then the energy of the molecule is given by

$$W = -\tfrac{1}{2}(\boldsymbol{p}\cdot\boldsymbol{E}_0) = -\tfrac{1}{2}(\beta_1\cos^2\theta+\beta_2\sin^2\theta)E_0^2 \,.$$

The number of particles per unit volume whose axes lie at the angle θ to $\boldsymbol{E}_0$ is given by the Boltzmann formula (6.a.6). In the normalisation condition (6.a.7), the quantity N should then represent the number of particles per unit volume. The polarisation vector is given by $\boldsymbol{P} = N\bar{\boldsymbol{p}}$ where $\bar{\boldsymbol{p}}$ is the dipole moment of a molecule averaged over the Boltzmann distribution. Since, in the absence of the field, the molecules are randomly oriented, $\bar{\boldsymbol{p}}$ will be parallel to the direction of the external field.

In view of the above considerations, $\bar{p}$ may be calculated from the formula

$$\bar{p} = \frac{1}{N}\int p_{\parallel}\,\mathrm{d}N = \frac{E_0\displaystyle\int_0^{\pi}\exp(-W(\theta)/kT)(\beta_1\cos^2\theta+\beta_2\sin^2\theta)\sin\theta\,\mathrm{d}\theta}{\displaystyle\int_0^{\pi}\exp(-W(\theta)/kT)\sin\theta\,\mathrm{d}\theta} \,,$$

where $p_\parallel$ is the component of the dipole moment of the molecule which is parallel to the field. Since by definition the field is weak, it is sufficient to retain only those terms which are linear in $a = (\beta_1 - \beta_2)E_0^2/2kT \ll 1$. Moreover, $P = N\overline{p} = \alpha E_0$, and hence

$$\alpha = N\beta_2 + \tfrac{1}{3}N(\beta_1 - \beta_2)\left[1 + \tfrac{2}{15}\frac{(\beta_1 - \beta_2)E_0^2}{kT}\right].$$

As can be seen from this formula, the relation between P and E_0 is nonlinear and α is a function of E_0. Let us estimate the magnitude of the correction term at room temperatures (T = 300 K). Assuming that $\beta_1 - \beta_2$ is of the order of 10^{-30} m^3, we have $kT/(\beta_1 - \beta_2) \approx 10^6$. Thus, the correction term is small provided $E_0 \ll 10^5$ V m^{-1}. Neglecting the correction term, we have the same expression for α as before:

$$\alpha = \tfrac{1}{3}N(\beta_1 + 2\beta_2)$$

(see problem 6.5).

6.9 The additional potential due to the quadrupole polarisation of the dielectric is given by

$$\varphi = \tfrac{1}{2}\int \frac{\partial^2(1/R)}{\partial x_i\, \partial x_k} Q_{ik}\, \mathrm{d}^3 r\,, \tag{6.9.1}$$

where R is the distance of the point of observation from the volume element $\mathrm{d}^3 r$ and the integration is carried out over the volume of the dielectric. On the other hand, the volume and surface charges will in general give rise to a potential

$$\varphi = \int \frac{\rho'}{R}\mathrm{d}^3 r + \int \frac{\sigma'}{R}\mathrm{d}^2 S + \int (\tau' \cdot \nabla(1/R))\, \mathrm{d}^2 S\,, \tag{6.9.2}$$

where ρ' is the volume density of charges, σ' is the surface density of charges, and τ' is the strength of the double layer. Comparison of equation (6.9.1) with equation (6.9.2) yields

$$\rho' = \tfrac{1}{2}\frac{\partial^2 Q_{ik}}{\partial x_i\, \partial x_k}\,, \qquad \sigma' = -\tfrac{1}{2}\frac{\partial Q_{in}}{\partial x_i}\,, \qquad \tau_k' = \tfrac{1}{2}Q_{ki} n_i\,. \tag{6.9.3}$$

Thus, the quadrupole polarisation is equivalent to a volume density ρ' inside the dielectric, a surface density σ', and an electric double layer of strength τ' on the surface of the dielectric. Since the volume and surface densities in the dielectric are related to the polarisation vector by the formulae $\rho' = -\mathrm{div}\boldsymbol{P}'$, $\sigma' = P_n'$, it follows from equation (6.9.3) that the quadrupole polarisation is equivalent to an additional dipole polarisation given by

$$P_k' = -\tfrac{1}{2}\frac{\partial Q_{ik}}{\partial x_i}$$

and a double layer of strength τ_k'.

The formulae given by equation (6.9.3) can also be obtained from a consideration of the energy of the dielectric which is caused by the quadrupole polarisation.

6.10 $\epsilon = \frac{1}{4}[1+3x+3(1+\frac{2}{3}x+x^2)^{1/2}]$,

where $x = 4\pi N\beta$. The polarisability β for polar substances in weak fields is given by $\beta = p^2/(3kT)$, where p is the molecular dipole moment, k is the Boltzmann constant, and T is the temperature. When $x \ll 1$ the difference between the field acting on the molecule and the average field is very small and

$$\epsilon = 1+x = 1+4\pi N\beta .$$

6.11 The total magnetic susceptibility is equal to the sum of the paramagnetic and diamagnetic susceptibilities

$$\chi = \frac{N\boldsymbol{m}^2}{3kT} - \frac{Ne^2}{6mc^2}\overline{r^2} . \tag{6.11.1}$$

This formula includes the magnetic moment of a rotator $\boldsymbol{m}$, which may be calculated as follows. In general,

$$\boldsymbol{m} = \frac{e}{2mc}\boldsymbol{K} , \tag{6.11.2}$$

where $\boldsymbol{K}$ is the angular momentum of the particle. In the case of the rotator, $\boldsymbol{K}$ is related to the kinetic energy by the formula

$$W_k = \frac{K^2}{2ma^2} .$$

It follows that the average of the square of the angular momentum is given by

$$\overline{K^2} = 2ma^2\overline{W}_k . \tag{6.11.3}$$

However, the average kinetic energy $\overline{W}_k$ may be found from the theorem of equipartition of energy. Since the rotator has two degrees of freedom, $\overline{W}_k = kT$. Substituting equations (6.11.2) and (6.11.3) into (6.11.1) we find that $\chi = 0$. This result is in accordance with the general theorem which states that the total magnetic moment of a body obeying classical statistics is zero. A finite magnetic moment will be obtained only when it is assumed that there are discrete electronic orbits in atoms. This assumption lies outside the framework of classical theories. [For details see, e.g. the book by Levich (1970).]

6.12 The concentration of ions N and electrons n is given by the Boltzmann formula (6.a.6):

$$N = N_0 \exp\left(-\frac{Ze\varphi}{kT}\right) , \qquad n = n_0 \exp\left(\frac{e\varphi}{kT}\right) , \tag{6.12.1}$$

where $\varphi(x, y, z)$ is the electrostatic potential. The preexponential factors are chosen so that as $T \to \infty$, when the interaction between the particles becomes negligible, N and n become equal to N_0 and n_0 respectively. Using equation (6.12.1), the charge density is given by

$$\rho = ZeN - en = e\left[ZN_0 \exp\left(-\frac{Ze\varphi}{kT}\right) - n_0 \exp\left(\frac{e\varphi}{kT}\right)\right] .$$

The potential φ can be determined by solving the Poisson equation

$$\nabla^2\varphi = -4\pi\rho = -4\pi e\left[ZN_0 \exp\left(-\frac{Ze\varphi}{kT}\right) - n_0 \exp\left(\frac{e\varphi}{kT}\right)\right] . \tag{6.12.2}$$

In order to solve this equation, we shall assume that the interaction energy is small in comparison with thermal energy, so that

$$\left|\frac{Ze\varphi}{kT}\right| \ll 1 , \qquad \left|\frac{e\varphi}{kT}\right| \ll 1 .$$

Expanding the exponentials into series, and retaining only those terms which are linear in φ, we have

$$\rho = -\frac{\kappa^2}{4\pi}\varphi , \qquad \kappa^2 = \frac{4\pi e^2(Z^2N_0 + n_0)}{kT} ,$$

where the gas is assumed to be electrically neutral, so that $ZN_0 = n_0$. Hence, equation (6.12.2) may be written in the form

$$\nabla^2\varphi = \kappa^2\varphi . \tag{6.12.3}$$

The potential φ can only be a function of the distance r to the ion under consideration. The spherically symmetric solution of equation (6.12.3) is of the form

$$\varphi = C_1\frac{\exp(-\kappa r)}{r} + C_2\frac{\exp(\kappa r)}{r} .$$

$C_2 = 0$ because the potential must be finite at infinity. C_1 is determined from the condition that when $r \ll \kappa^{-1}$ the potential should become identical with the Coulomb potential of the ion under consideration

$$\varphi|_{r \ll 1/\kappa} = \frac{Ze}{r} = \frac{C_1}{r} , \qquad C_1 = Ze .$$

Thus, the ion is surrounded by a cloud of electrons and other ions whose density decreases exponentially and whose mean radius κ^{-1} decreases with decreasing temperature.

The above method of calculating the potential is due to Debye and Hückel and has been used in their theory of strong electrolytes. The constant κ^{-1} is known as the Debye radius.

6.13 The electric induction inside the plate is given by

$$D(x) = E_0 \frac{\cosh \kappa x}{\cosh \kappa h} ,$$

where $\kappa = (4\pi e^2 n_0/\epsilon kT)^{1/2}$. When $\kappa h \gg 1$, near the surfaces $x = \pm h$, we have

$$D(x) = E_0 \exp[-\kappa(h - |x|)] ,$$

and hence it follows that when $|x - h| \gg \kappa^{-1}$, $D(x) = 0$, i.e. the field penetrates into the conductor to a depth κ^{-1}. A charge

$$\rho = \frac{1}{4\pi}\frac{\partial D}{\partial x} = \pm \frac{\kappa E_0}{4\pi} \exp[-\kappa(h - |x|)]$$

is concentrated in a layer of this thickness. The density of the 'surface' charge, which is considered in the macroscopic theory, is obtained by integrating ρ. At $x = h$ we have

$$\sigma = \int \rho \, \mathrm{d}x = -\frac{\kappa E_0}{4\pi} \int_0^\infty \exp(-\kappa x') \, \mathrm{d}x' = \frac{E_0}{4\pi} ,$$

which is the same as the usual boundary condition at the surface of a conductor.

6.14

$$\varphi = \varphi_0 \frac{\sinh \kappa x}{\sinh \kappa h} , \qquad \kappa = \left(\frac{8\pi e^2 n_0}{\epsilon kT}\right)^{1/2}$$

The magnitude of κ^2 is found to be twice as large as in the preceding problem, since there are two kinds of moving ions.

b Polarisation of matter in a variable field

6.15 $\epsilon = 1 + 4\pi N a^3$; $\qquad \mu = 1 - 2\pi N a^3 < 1$.

This type of dielectric is diamagnetic. The permittivity ϵ and the permeability μ are independent of frequency, since it is assumed that the spheres are perfect conductors. In order that the artificial dielectric should behave as a continuous medium, the wavelength must be much greater than both the average distance between the spheres and the radius of each sphere:

$$\lambda \gg l , \qquad \lambda \gg a ,$$

where l is the average distance between the spheres. The difference between the effective field and the average field can only be neglected when the polarisability of the medium is small, i.e. when $4\pi N a^3 \ll 1$.

6.16 The equation of motion for an electron is of the form

$$m\ddot{r} + \eta \dot{r} = eE_0 \exp(-\mathrm{i}\omega t) . \qquad (6.16.1)$$

The special solution of this equation corresponding to forced oscillations is of the form

$$r = -\frac{eE_0 \exp(-\mathrm{i}\omega t)}{m(\omega^2 + \mathrm{i}\gamma\omega)} ,$$

where $\gamma = \eta/m$.

The dipole moment per unit volume is obtained by multiplying r by the charge of the electron e and the number of particles per unit volume N. The polarisability of the medium $\alpha(\omega)$ and the permittivity $\epsilon(\omega)$ can then be determined from

$$\epsilon(\omega) = 1 + 4\pi\alpha(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2 + \mathrm{i}\gamma\omega} , \tag{6.16.2}$$

where

$$\omega_\mathrm{p}^2 = \frac{4\pi e^2 N}{m} .$$

The relation between the resistivity ρ and the coefficient η may be found with the aid of equation (6.16.1) and Ohm's law:

$$\rho \equiv \frac{1}{\sigma} = \frac{\eta}{Ne^2} . \tag{6.16.3}$$

This result can also be obtained by comparing the permittivity (6.16.2) with the complex permittivity [equation (8.a.8)] expressed in terms of the conductivity:

$$\epsilon(\omega) = \epsilon' + \mathrm{i}\frac{4\pi\sigma}{\omega} . \tag{6.16.4}$$

Separating the real and imaginary parts in equation (6.16.2) we find that

$$\epsilon' = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2 + \gamma^2} , \qquad \sigma = \frac{e^2 N\gamma}{m(\omega^2 + \gamma^2)} . \tag{6.16.5}$$

It follows from equation (6.16.5) that ϵ' and σ are functions of frequency. When $\omega \ll \gamma$ they assume their static values

$$\epsilon' = 1 - \frac{\omega_\mathrm{p}^2}{\gamma} < 1 , \qquad \sigma = \frac{e^2 N}{m\gamma} .$$

It follows from equations (6.16.4) and (6.16.5) that the complex permittivity of a conducting medium becomes infinite at low frequencies ($\omega \to 0$). At high frequencies it is of the form

$$\epsilon(\omega) = \epsilon'(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2} .$$

This dependence of the permittivity on frequency at high frequencies will also hold for dielectrics.

Let us estimate the order of magnitude of $\gamma = \eta/m$ for copper for which the static conductivity is $\sigma = 5 \times 10^{17}\ \mathrm{s}^{-1}$. Equation (6.16.3) gives

$$\gamma = \frac{Ne^2}{\sigma m} = \frac{N_0 e^2 d}{\sigma m A} ,$$

where $N_0 \approx 6 \times 10^{23}$ (Avogadro's number), $A \approx 6{\cdot}35 \times 10^{-2}\ \mathrm{kg\ mol^{-1}}$ (atomic weight), and $d \approx 8{\cdot}9 \times 10^3\ \mathrm{kg\ m^{-3}}$ (density of copper). It follows that $\gamma \approx 10^{14}\ \mathrm{s}^{-1}$. We recall that in the visible region of the spectrum the frequencies are of the order of $10^{15}\ \mathrm{s}^{-1}$.

Thus, it may be concluded that in this example the conductivity remains equal to the static value right up to frequencies lying in the infrared region of the spectrum. However, it must be remembered that at high frequencies, when the mean free path of electrons becomes comparable with the depth of penetration of the field into the metal, spatial nonuniformity of the field becomes important and the macroscopic permittivity ϵ loses its meaning.

The results obtained in this problem can be applied, within a limited range, to a metal and also to a semiconductor or an ionised gas (plasma), provided the motion of the positive ions can be neglected. Calculations of the permittivity of a plasma, taking into account the motion of the positive ions, are given in the solution of problem 6.25.

6.17 Since the molecules of the dielectric are not spherically symmetric, the external field $\boldsymbol{E}_0$ will partially orient them and the dielectric as a whole will become anisotropic. Since $\mathcal{E} \ll E_0$, the orienting effect of the alternating field may be neglected. Since the external electric field $\boldsymbol{E}_0$ is responsible for the anisotropy, one of the principal axes of the permittivity tensor will lie in the direction of this field, while the other two principal axes will be perpendicular to $\boldsymbol{E}_0$.

Let β'_{ik} represent the components of the polarisability of the molecule along these axes (i, $k = 1$ correspond to the axis parallel to $\boldsymbol{E}_0$). The components β'_{ik} are given by the usual formula

$$\beta'_{ik} = \alpha_{il}\alpha_{km}\beta_{lm} = (\beta - \beta')\alpha_{i1}\alpha_{k1} + \beta'\delta_{ik} ,$$

where α_{il} are the cosines of the angles between the axis of symmetry of the molecule and the principal axes of the permittivity tensor ($\alpha_{il}\alpha_{kl} = \delta_{ik}$ in view of the orthogonality of the matrix α_{ik}). In order to calculate the permittivity tensor per unit volume of the dielectric, it is necessary to find the statistical average of the quantities β'_{ik} with the aid of the Boltzmann formula, i.e. it is necessary to average the product $\alpha_{i1}\alpha_{k1}$.

The quantities α_{1i} can be expressed in terms of the polar angles of the axis of symmetry of the molecule in the dashed system ϑ, ϕ:

$$\alpha_{11} = \cos\vartheta , \qquad \alpha_{12} = \sin\vartheta\cos\phi , \qquad \alpha_{13} = \sin\vartheta\sin\phi .$$

Using the same averaging procedure as in problem 6.6, we have, to within terms which are linear in $a = (\beta_0 - \beta_0')E_0^2/(2kT)$:

$$\overline{\alpha_{11}^2} = \tfrac{1}{3}(1+\tfrac{4}{15}a)\,, \qquad \overline{\alpha_{12}^2} = \overline{\alpha_{13}^2} = \tfrac{1}{3}(1-\tfrac{2}{15}a)\,,$$

$$\overline{\alpha_{i1}\alpha_{k1}} = 0\,, \qquad \text{when} \quad i \neq k\,,$$

where β_0 and β_0' are the static values of the polarisability tensor. Hence,

$$\overline{\beta_{11}'} = \tfrac{1}{3}(\beta-\beta')(1+\tfrac{4}{15}a)+\beta'\,,$$

$$\overline{\beta_{22}'} = \overline{\beta_{33}'} = \tfrac{1}{3}(\beta-\beta')(1-\tfrac{2}{15}a)+\beta'\,.$$

Neglecting the difference between the field acting on the molecule and the average field, we obtain the principal values of the permittivity tensor:

$$\epsilon^{(1)} = 1+4\pi N\overline{\beta_{11}'}\,, \qquad \epsilon^{(2)} = \epsilon^{(3)} = 1+4\pi N\overline{\beta_{22}'}\,.$$

This result shows that in a strong constant electric field, the dielectric will become anisotropic with respect to high frequency oscillations (for example, visible radiation). The appearance of anisotropy under the action of an electric field is known as the Kerr effect. This effect is a very rapid one: the relaxation time is of the order of 10^{-10} s. It is determined by the time necessary to establish statistical equilibrium in the dielectric. The Kerr effect is widely used in technology for high frequency modulation of the intensity of light.

6.18 Assuming that the parameter $pE_0/(kT) = a$ is small, we have, to within terms of the order of a^2,

$$\overline{\beta_{11}'} = \tfrac{1}{3}(\beta-\beta')(1+\tfrac{2}{15}a^2)+\beta'\,,$$

$$\overline{\beta_{22}'} = \overline{\beta_{33}'} = \tfrac{1}{3}(\beta-\beta')(1-\tfrac{1}{15}a^2)+\beta'\,,$$

$$\epsilon^{(1)} = 1+4\pi N\overline{\beta_{11}'}\,, \qquad \epsilon^{(2)} = \epsilon^{(3)} = 1+4\pi N\overline{\beta_{22}'}\,.$$

The symbols have the same meaning as in the preceding problem.

6.19 Let the field amplitude E increase by $\mathrm{d}E$ ($\mathrm{d}E_x$, $\mathrm{d}E_y$, $\mathrm{d}E_z$). The work done on a molecule is then given by

$$\mathrm{d}A = \tfrac{1}{2}\,\mathrm{Re}(\boldsymbol{p}\cdot \mathrm{d}E^*) = \tfrac{1}{4}[(\boldsymbol{p}\cdot \mathrm{d}E^*)+(\boldsymbol{p}^*\cdot \mathrm{d}E)]\,,$$

where $p_i = \beta_{ik}E_k$ is the component of the dipole moment of the system. Since there is no absorption of energy, this work is used entirely to increase the average potential energy of the molecule in the external field:

$$\mathrm{d}A = \mathrm{d}\overline{W}\,.$$

Hence, $\mathrm{d}A$ should be a perfect differential of a function of the field amplitude. $\mathrm{d}\overline{W}$ may be rewritten in the form

$$\mathrm{d}\overline{W} = \tfrac{1}{4}\sum_{i,k}(\beta_{ik}E_k\,\mathrm{d}E_i^* + \beta_{ki}^*E_i^*\,\mathrm{d}E_k)\,.$$

It is clear that this quantity is a perfect differential, provided $\beta_{ik} = \beta^*_{ki}$. When this is so we have

$$d\overline{W} = \tfrac{1}{4}\sum_{i,k} \beta_{ik}(\mathcal{E}_k \, d\mathcal{E}^*_i + \mathcal{E}^*_i \, d\mathcal{E}_k) = \tfrac{1}{4}\sum_{i,k} \beta_{ik} \, d(\mathcal{E}^*_i \mathcal{E}_k) = d(\tfrac{1}{4}\boldsymbol{p} \cdot \mathcal{E}^*) ,$$

or

$$\overline{W} = \tfrac{1}{4}(\boldsymbol{p} \cdot \mathcal{E}^*) .$$

Similarly, it can be shown that the magnetic polarisability tensor for a system in which there is no dissipation of energy is Hermitian.

6.21 The equation of motion for an atomic electron which is bound to the nucleus by an elastic force is

$$\ddot{\boldsymbol{r}} + \omega_0^2 \boldsymbol{r} = \frac{e}{m}\left\{\boldsymbol{E}_0 \exp(-i\omega t) + \left[\frac{\boldsymbol{v}}{c} \wedge \boldsymbol{H}_0\right]\right\} ,$$

where ω_0 is the frequency of the eigenoscillations. On solving this equation by the method of successive approximations we obtain the following approximate expression:

$$\boldsymbol{r} = \frac{e\boldsymbol{E}}{m(\omega_0^2 - \omega^2)} - i\frac{e\omega}{m^2 c(\omega_0^2 - \omega^2)^2}[\boldsymbol{E} \wedge \boldsymbol{H}_0] .$$

In order to obtain the polarisability tensor for the atom we shall use the representation of the vector product involving the skew-symmetric tensor e_{ikl} (see problem 1.26). This gives

$$\beta_{ik} = \frac{e^2}{m(\omega_0^2 - \omega^2)}\delta_{ik} - i\frac{e^2 \omega H_{0l}}{m^2 c(\omega_0^2 - \omega^2)^2} e_{ikl} .$$

This tensor is Hermitian, in accordance with the general result established in problem 6.19. The gyration vector (see problem 6.20) is now of the form

$$\boldsymbol{g} = -\frac{e^2 \omega}{m^2 c(\omega_0^2 - \omega^2)^2}\boldsymbol{H}_0 = -\frac{2e\omega}{m(\omega_0^2 - \omega^2)^2}\boldsymbol{\omega}_{\mathrm{L}} ,$$

where $\boldsymbol{\omega}_{\mathrm{L}} = e\boldsymbol{H}_0/(2mc)$ is a vector with its length equal to the Larmor frequency.

6.22

$$\epsilon_{ik} = \begin{pmatrix} \tfrac{1}{2}(\epsilon^+ + \epsilon^-) & \tfrac{1}{2}i(\epsilon^+ - \epsilon^-) & 0 \\ -\tfrac{1}{2}i(\epsilon^+ - \epsilon^-) & \tfrac{1}{2}(\epsilon^+ + \epsilon^-) & 0 \\ 0 & 0 & \epsilon^0 \end{pmatrix} ,$$

where

$$\epsilon^{\pm} = 1-\frac{\omega_{\mathrm{p}}^2}{\omega(\omega \pm 2\omega_{\mathrm{L}})-\omega_0^2}\,, \qquad \epsilon^0 = 1-\frac{\omega_{\mathrm{p}}^2}{\omega^2-\omega_0^2}\,, \qquad \omega_{\mathrm{p}}^2 = \frac{4\pi e^2 N}{m}\,.$$

The magnitude of the gyration vector is given by

$$g = \tfrac{1}{2}(\epsilon^{+}-\epsilon^{-})$$

and its direction is parallel to the z-axis. The result established in the solution of the preceding problem can be obtained from the exact solution above by assuming that $2\omega_{\mathrm{L}}\,\omega \ll |\omega_0^2-\omega^2|$.

6.23 The tensor ϵ_{ik} is of the same form as in the preceding problem. However, its components $\epsilon^{\pm}$ and ϵ^0 are given by the following expressions

$$\epsilon^{\pm} = 1-\frac{\omega_{\mathrm{p}}^2}{\omega^2+\omega(\mathrm{i}\gamma \pm 2\omega_{\mathrm{L}})}\,, \qquad \epsilon^0 = 1-\frac{\omega_{\mathrm{p}}^2}{\omega^2+\mathrm{i}\omega\gamma}\,,$$

$$\gamma = \frac{\eta}{m}\,, \qquad \omega_{\mathrm{L}} = -\frac{eH_0}{2mc} > 0\,, \qquad \omega_{\mathrm{p}}^2 = \frac{4\pi e^2 N}{m}\,.$$

Owing to the presence of 'friction' ($\eta \neq 0$) in the electron gas there is dissipation of energy and the tensor ϵ_{ik} is not Hermitian.

6.24 $\boldsymbol{j} = \sigma \boldsymbol{E} + [\boldsymbol{E} \wedge \boldsymbol{a}]$,

where

$$\sigma = \frac{e^2 N}{m\gamma}\,, \qquad \boldsymbol{a} = \frac{e^3 N}{m^2\gamma^2 c}\boldsymbol{H}_0\,, \qquad \gamma = \frac{\eta}{m}\,.$$

The magnetic field gives rise to a current whose direction is perpendicular to the electric field (Hall current). The reciprocal relation (in the same approximation) is

$$\boldsymbol{E} = \frac{1}{\sigma}\boldsymbol{j} + R[\boldsymbol{j} \wedge \boldsymbol{H}_0]\,,$$

where $R = (ceN)^{-1}$ is the Hall constant. The electrical conductivity tensor is

$$\sigma_{ik} = \sigma\delta_{ik} - e_{ikl}a_l\,.$$

6.25 Let m, $\dot{\boldsymbol{r}}$, and $-e$ be the mass, velocity, and charge of an electron, and let the corresponding quantities for an ion be M, $\dot{\boldsymbol{R}}$, and $+e$. The equations of motion are

$$\left.\begin{aligned} m\ddot{\boldsymbol{r}} &= -e\boldsymbol{E}_0\exp(-\mathrm{i}\omega t) - \frac{e}{c}[\dot{\boldsymbol{r}} \wedge \boldsymbol{H}_0] - m\gamma(\dot{\boldsymbol{r}}-\dot{\boldsymbol{R}})\,, \\ M\ddot{\boldsymbol{R}} &= e\boldsymbol{E}_0\exp(-\mathrm{i}\omega t) + \frac{e}{c}[\dot{\boldsymbol{R}} \wedge \boldsymbol{H}_0] - m\gamma(\dot{\boldsymbol{R}}-\dot{\boldsymbol{r}})\,, \end{aligned}\right\} \qquad (6.25.1)$$

where $\boldsymbol{H}_0$ is the constant uniform magnetic field and $m\gamma$ is the 'friction' coefficient. The frictional force is proportional to the relative velocity of the electrons and ions, i.e. to the differences $\dot{\boldsymbol{r}}-\dot{\boldsymbol{R}}$ and $\dot{\boldsymbol{R}}-\dot{\boldsymbol{r}}$ for electrons and ions respectively. The electric field $\boldsymbol{E}$ is a harmonic function of time, so that $\boldsymbol{E} = \boldsymbol{E}_0 \exp(-\mathrm{i}\omega t)$.

The solution of equations (6.25.1) will be sought in the form

$$\boldsymbol{r} = \boldsymbol{r}_0 \exp(-\mathrm{i}\omega t)\,, \qquad \boldsymbol{R} = \boldsymbol{R}_0 \exp(-\mathrm{i}\omega t)\,. \tag{6.25.2}$$

Let the z-axis be in the direction of $\boldsymbol{H}_0$ and introduce the following cyclic components of the vectors $\boldsymbol{r}_0$ and $\boldsymbol{R}_0$:

$$r_{0\pm 1} = \mp\frac{1}{\sqrt{2}}(r_{0x} \pm \mathrm{i}r_{0y})\,, \qquad R_{0\pm 1} = \mp\frac{1}{\sqrt{2}}(R_{0x} \pm \mathrm{i}R_{0y})\,.$$

Substituting equation (6.25.2) into equations (6.25.1) we have, after addition,

$$-\mathrm{i}\omega(m\boldsymbol{r}_0+M\boldsymbol{R}_0) = \frac{e}{c}[(\boldsymbol{R}_0-\boldsymbol{r}_0)\wedge \boldsymbol{H}_0]\,.$$

The left-hand side of the latter equation may be rewritten in the form

$$-\mathrm{i}\omega[(M+m)\boldsymbol{R}_0+m(\boldsymbol{r}_0-\boldsymbol{R}_0)]\,.$$

Neglecting m in comparison with M, we have

$$\omega R_{0\pm 1} = \left(\pm\Omega_H+\omega\frac{m}{M}\right)s_{\pm 1}\,, \tag{6.25.3}$$

where

$$\Omega_H = \frac{eH_0}{Mc} \qquad \text{and} \qquad \boldsymbol{s} = \boldsymbol{R}_0-\boldsymbol{r}_0\,.$$

Let us now divide the first of the equations in (6.25.1) by m, and the second by M, and subtract one from the other. Neglecting eE/M in comparison with eE/m, $m\gamma s/M$ in comparison with γs, and $e[\dot{\boldsymbol{R}} \wedge \boldsymbol{H}_0]/(Mc)$ in comparison with $e[\dot{\boldsymbol{r}} \wedge \boldsymbol{H}_0]/(mc)$, we have

$$\left.\begin{aligned} (-\mathrm{i}\omega+\gamma\mp\mathrm{i}\omega_H)s_{\pm 1} \mp \mathrm{i}\omega_H R_{0\pm 1} &= \frac{e}{m}E_{0\pm 1}\,, \\ -\omega^2 s_z &= \frac{eE_{0z}}{m}+\mathrm{i}\omega\gamma s_z\,, \end{aligned}\right\} \tag{6.25.4}$$

where $\omega_H = eH_0/mc$. Equations (6.25.3) and (6.25.4) can then be used to determine $\boldsymbol{s}$.

The polarisation vector $\boldsymbol{P}$ may be calculated from the formula $\boldsymbol{P} = Ne\boldsymbol{s}\exp(-\mathrm{i}\omega t)$, where N is the number of ions per unit volume, which is equal to the number of electrons per unit volume.

The components of the permittivity tensor are

$$\epsilon^{\pm} = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma \pm \omega_H - \omega_H \Omega_H/\omega)} ,$$

$$\epsilon^{(z)} = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)} .$$

The component $\epsilon^{(z)}$ is of the same form as the scalar permittivity in the absence of a magnetic field, which was obtained in problem 6.16. The permittivity becomes infinite when $\omega \to 0$. The components $\epsilon^{\pm}$ will contain the extra term $\omega_H \Omega_H$ in the denominator when the motion of the ions is taken into account. It may be neglected when $\Omega_H/\omega \ll 1$, i.e. at high frequencies ω. However, this term becomes important at low frequencies. As $\omega \to 0$ it is responsible for the fact that $\epsilon^{\pm}$ remains finite:

$$\epsilon^{\pm} = 1 + \frac{\omega_p^2}{\omega_H \Omega_H} .$$

This means that very low frequency waves may exist in a plasma (magnetohydrodynamic waves). The theory of propagation of electromagnetic waves in a plasma, taking the oscillation of the positive ions into account, is considered in problem 8.47.

6.26 In the system of coordinates in which the x_3-axis is parallel to the special direction, the tensor T_{ik} should be of the form

$$T_{ik} = \begin{pmatrix} T & T_a & 0 \\ -T_a & T & 0 \\ 0 & 0 & T_{\|} \end{pmatrix} .$$

This is in agreement with the results obtained in problems 6.22, 6.23, and so on.

6.28 Since the field is switched on at $t = 0$, it follows from the principle of causality that $\boldsymbol{P}(t) = 0$ when $t < 0$. Denoting the permittivity by $\alpha = \alpha' + i\alpha''$, we have

$$\boldsymbol{P}(t) = \int_{-\infty}^{\infty} \alpha(\omega') \boldsymbol{E}(\omega') \exp(-i\omega t)\, d\omega' = \frac{\boldsymbol{E}_0}{2\pi} \int_{-\infty}^{\infty} \alpha(\omega') \exp(-i\omega' t)\, d\omega' , \tag{6.28.1}$$

where $\boldsymbol{E}(\omega')$ is the Fourier component of the field $\boldsymbol{E}(t) = \boldsymbol{E}_0 \delta(t)$. Next, multiply equation (6.28.1) by $\exp(i\omega t)$ and integrate with respect to t between $-\infty$ and 0. Since $\boldsymbol{P}(t) = 0$ for $t < 0$, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega' \alpha(\omega') \int_{-\infty}^{0} \exp[-i(\omega' - \omega)t]\, dt = 0 .$$

By using equation (A1.17), and separating the real and imaginary parts, we have

$$\alpha'(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha''(\omega')}{\omega'-\omega}\mathrm{d}\omega' , \qquad \alpha''(\omega) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha'(\omega')\,\mathrm{d}\omega'}{\omega'-\omega} .$$

The Kramers–Kronig relation follows immediately.

6.29

$$\epsilon'(\omega) = 1+\frac{\epsilon_0-1}{1+\omega^2\tau^2} .$$

6.31

$$\mathrm{curl}\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} , \qquad \mathrm{curl}\boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{D}'}{\partial t}+\frac{4\pi}{c}\boldsymbol{j} ,$$

$$\mathrm{div}\boldsymbol{D}' = 4\pi\rho , \qquad \mathrm{div}\boldsymbol{B} = 0 .$$

c Ferromagnetic resonance

6.32 $M_x = A\sin(\omega_0 t+\alpha)$, $\quad M_y = A\cos(\omega_0 t+\alpha)$, $\quad M_z = C$,

where $\omega_0 = \gamma H_0$, α is the initial phase, A and C are constants related by the condition $M^2 = M_0^2$, i.e. $A^2+C^2 = M_0^2$, where M_0 is the saturation magnetisation. The motion of the magnetisation vector takes the form of the usual Larmor precession.

6.33 The solution of the equation

$$\frac{\mathrm{d}\boldsymbol{M}}{\mathrm{d}t} = -\gamma[\boldsymbol{M}\wedge\boldsymbol{H}_0]+\omega_r(\chi_0\boldsymbol{H}_0-\boldsymbol{M}) \tag{6.33.1}$$

will be sought in the form

$$M_x = m_x\exp(-\mathrm{i}\omega t) , \qquad M_y = m_y\exp(-\mathrm{i}\omega t) ,$$
$$M_z = M_0+m_z\exp(-\mathrm{i}\omega t) ,$$

where ω is the unknown frequency and the z-axis is parallel to $\boldsymbol{H}_0$.

On taking the components of equation (6.33.1) along the coordinate axes, and substituting for $\boldsymbol{M}$, we obtain a system of algebraic equations. The condition that these equations should be consistent is

$$\omega_0^2-(\omega+\mathrm{i}\omega_r)^2 = 0 .$$

The frequency ω turns out to be complex:

$$\omega = \omega_0-\mathrm{i}\omega_r .$$

The presence of losses gives rise to damped motion. The components m_x and m_y exhibit a phase difference of $\frac{1}{2}\pi$. The vector $\boldsymbol{M}$ executes a damped precession about $\boldsymbol{H}_0$.

6.34 Let the z-axis be parallel to $\boldsymbol{H}$. The resultant magnetic field will then have the components $h_x\exp(-\mathrm{i}\omega t)$, $h_y\exp(-\mathrm{i}\omega t)$, and $H_0+h_z\exp(-\mathrm{i}\omega t)$.

The solution of the Landau–Lifshitz equation (6.c.2) will be sought in the form

$$M_x = m_x \exp(-\mathrm{i}\omega t)\,, \qquad M_y = m_y \exp(-\mathrm{i}\omega t)\,,$$
$$M_z = M_0 + m_z \exp(-\mathrm{i}\omega t)\,, \tag{6.34.1}$$

where M_0 is the saturation magnetisation. This form of the solution corresponds to the assumption that the Larmor precession has been damped out and the oscillations are maintained by the high frequency (forcing) field. Hence, the quantities m_x, m_y, m_z should be regarded as small (of the order of h or smaller). Substituting equation (6.34.1) into the Landau–Lifshitz equation, and rejecting terms which are quadratic in h and m, we obtain the components of $\boldsymbol{m}$:

$$m_x = \chi_0 \frac{\omega_0^2}{\omega_0^2-\omega^2} h_x - \chi_0 \frac{\mathrm{i}\omega\omega_0}{\omega_0^2-\omega^2} h_y\,,$$

$$m_y = \chi_0 \frac{\mathrm{i}\omega\omega_0}{\omega_0^2-\omega^2} h_x + \chi_0 \frac{\omega_0^2}{\omega_0^2-\omega^2} h_y\,, \qquad m_z = 0\,.$$

It is clear from these formulae that the behaviour of m_x and m_y as functions of ω for fixed $\omega_0 = \gamma H_0$ or as functions of H_0 for given ω has a resonance character; ferromagnetic resonance will occur at $\omega = \omega_0$, i.e. m_x and m_y become infinite.

The infinite magnitude of $\boldsymbol{m}$ at resonance is due to the approximate method used in the solution of the Landau–Lifshitz equation. The exact solution (see problem 6.36) should ensure that $|\boldsymbol{M}|$ remains constant, since it follows from the Landau–Lifshitz equation that $M^2 =$ constant. When the problem is solved by the method of successive approximations, taking losses into account, $\boldsymbol{M}$ will also remain finite.

6.35

$$\chi_{ik} = \begin{pmatrix} \chi_\perp & -\mathrm{i}\chi_a & 0 \\ \mathrm{i}\chi_a & \chi_\perp & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\chi_\perp = \chi_0 \frac{\omega_0^2}{\omega_0^2-\omega^2}\,, \qquad \chi_a = \chi_0 \frac{\omega\omega_0}{\omega_0^2-\omega^2}\,,$$

$$\mu_{ik} = \begin{pmatrix} \mu_\perp & -\mathrm{i}\mu_a & 0 \\ \mathrm{i}\mu_a & \mu_\perp & 0 \\ 0 & 0 & \mu_\parallel \end{pmatrix},$$

and

$$\mu_\perp = 1 + 4\pi\chi_\perp\,, \qquad \mu_a = 4\pi\chi_a\,, \qquad \mu_\parallel = 1\,.$$

As can be seen from the above formulae, χ_{ik} and μ_{ik} are Hermitian tensors ($\mu_{ik} = \mu_{ki}^*$). This means that the medium is gyrotropic and there are no losses.

Figure 6.35.1 shows the components μ_{ik} as functions of frequency for $H_{0\,\mathrm{res}} \approx 2{\cdot}7 \times 10^5$ A m^{-1}.

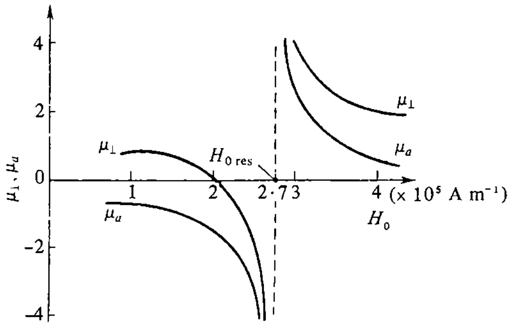


Figure 6.35.1.

6.36

$$M_x = \frac{\omega_1}{\Delta\omega} C \cos\omega t\,, \qquad M_y = -\frac{\omega_1}{\Delta\omega} \sin\omega t\,, \qquad M_z = C\,,$$

where $\Delta\omega = \omega_0 - \omega$, $\omega_0 = \gamma H_0$, $\omega_1 = \gamma h$. The constant C may be determined from the condition $M_x^2 + M_y^2 + M_z^2 = M_0^2$, which follows from the Landau-Lifshitz equation:

$$C = \frac{|\Delta\omega|}{\Omega} M_0\,,$$

where $\Omega = (\Delta\omega^2 + \omega_1^2)^{1/2}$.

The expression for C includes the modulus $|\Delta\omega|$ since $M_z > 0$. The components of M are given by

$$\left.\begin{aligned} M_x &= \pm\frac{\omega_1}{\Omega} M_0 \cos\omega t = \chi h_x\,, \\ M_y &= \pm\frac{\omega_1}{\Omega} M_0 \sin\omega t = \chi h_y\,, \qquad M_z = \frac{|\Delta\omega|}{\Omega} M_0\,. \end{aligned}\right\} \tag{6.36.1}$$

The $\pm$ sign corresponds to the sign of $\Delta\omega$. It follows from these equations that the relation between M and h is nonlinear: the coefficient of proportionality χ is a function of h:

$$\chi = \pm\frac{\gamma M_0}{(\Delta\omega^2 + \omega_1^2)^{1/2}}\,.$$

The precession angle ϑ (the angle between M and H_0) is given by

$$\sin\vartheta = \frac{M_\perp}{M_0} = \frac{\omega_1}{\Omega}\,,$$

where $M_\perp = (M_x^2+M_y^2)^{1/2}$. At the ferromagnetic resonance $\Delta\omega = 0$ and equations (6.36.1) yield

$$M_x = \pm M_0 \cos\omega t\,, \qquad M_y = \pm M_0 \sin\omega t\,, \qquad M_z = 0\,.$$

The vector $\boldsymbol{M}$ will then rotate with a frequency ω in the plane perpendicular to $\boldsymbol{H}_0$, and its components will remain finite.

6.37 $\boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{m}\exp(-\mathrm{i}\omega t)$,

where $\boldsymbol{M}_0$ lies in the direction of $\boldsymbol{H}_0$ and the components of $\boldsymbol{m}$ are given by

$$m_x = \chi_0 \frac{\Omega^2 - \mathrm{i}\omega\omega_r}{\Omega^2-\omega^2-2\mathrm{i}\omega\omega_r} h_x - \mathrm{i}\chi_0 \frac{\omega\omega_0}{\Omega^2-\omega^2-2\mathrm{i}\omega\omega_r} h_y\,,$$

$$m_y = \mathrm{i}\chi_0 \frac{\omega\omega_0}{\Omega^2-\omega^2-2\mathrm{i}\omega\omega_r} h_x + \chi_0 \frac{\Omega^2 - \mathrm{i}\omega\omega_r}{\Omega^2-\omega^2-2\mathrm{i}\omega\omega_r} h_y\,,$$

$$m_z = \chi_0 \frac{\omega_r}{\omega_r - \mathrm{i}\omega} h_z\,,$$

$$\Omega = (\omega_0^2 + \omega_r^2)^{1/2}\,, \qquad \omega_0 = \gamma H_0\,.$$

As can be seen from these formulae, the presence of losses ($\omega_r \neq 0$) will ensure that the amplitude of $\boldsymbol{m}$ will remain finite at resonance.

6.38

$$\mu_{ik} = \begin{pmatrix} \mu_\perp & -\mathrm{i}\mu_a & 0 \\ \mathrm{i}\mu_a & \mu_\perp & 0 \\ 0 & 0 & \mu_\parallel \end{pmatrix},$$

$$\mu_\perp = \mu_\perp' + \mathrm{i}\mu_\perp''\,, \qquad \mu_a = \mu_a' + \mathrm{i}\mu_a''\,,$$

$$\mu_\perp' = 1 + 4\pi\chi_0 \frac{\Omega^2(\Omega^2-\omega^2) + 2\omega^2\omega_r^2}{(\Omega^2-\omega^2)^2 + 4\omega^2\omega_r^2}\,,$$

$$\mu_\perp'' = 4\pi\chi_0 \frac{\omega\omega_r(\Omega^2+\omega^2)}{(\Omega^2-\omega^2)^2 + 4\omega^2\omega_r^2}\,,$$

$$\mu_a' = 4\pi\chi_0 \frac{\omega\omega_0(\Omega^2-\omega^2)}{(\Omega^2-\omega^2)^2 + 4\omega^2\omega_r^2}\,,$$

$$\mu_a'' = \frac{\omega^2\omega_0\omega_r}{(\Omega^2-\omega^2)^2 + 4\omega^2\omega_r^2}\,,$$

where

$$\Omega = (\omega_0^2 + \omega_r^2)^{1/2}\,, \qquad \omega_0 = \gamma H_0\,,$$

$$\mu_\parallel = 1 + 4\pi\chi_0 \frac{\omega_r}{\omega_r - \mathrm{i}\omega}\,,$$

$$H_{0\,\mathrm{res}} \approx 2{\cdot}7 \times 10^5\ \mathrm{A\,m^{-1}}\,.$$

Figure 6.38.1 shows the dependence of $\mu'_\perp$ and $\mu''_\perp$ on the constant field H_0. The dependence of μ'_a and μ''_a on H_0 is similar.

The imaginary parts of $\mu''_\perp$ and μ''_a have a maximum at $H_0 = H_{0\,\mathrm{res}} \approx \omega/\gamma$, while the real parts of $\mu'_\perp$, μ'_a reach extremal values at $H_0 \approx (\omega \pm \omega_r)/\gamma$.

The curves shown in figure 6.38.1 are similar to the dispersion curves for $\epsilon(\omega)$ (see figure 6.b.1).

The imaginary parts of the components $\mu''_\perp$, μ''_a, and $\mu''_\parallel$ determine the dissipation of electromagnetic energy. They are zero at $\omega_r = 0$.

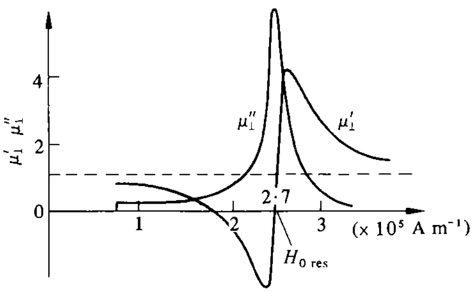


Figure 6.38.1.

6.39

$$\Delta H_0 = \frac{\omega_r}{\gamma} \, .$$

6.40 Let the coordinate axes lie along the principal axes of the ellipsoid, with the z-axis lying along the field H_0. In that system of coordinates the tensor N_{ik} will be diagonal, and hence the components of the Landau–Lifshitz equation along the coordinate axes will be

$$\left.\begin{aligned} \dot{M}_x &= -\gamma[H_0 + 4\pi(N^{(y)} - N^{(z)})M_z]M_y \, , \\ \dot{M}_y &= \gamma[H_0 + 4\pi(N^{(x)} - N^{(z)})M_z]M_x \, , \\ \dot{M}_z &= -4\pi\gamma(N^{(x)} - N^{(y)})M_x M_y \, . \end{aligned}\right\} \tag{6.40.1}$$

The equations are therefore nonlinear. Assuming that the departure of the vector $\boldsymbol{M}$ from the equilibrium position (the direction of the z-axis) is small, the solution will be sought in the form

$$\boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{m}\exp(-\mathrm{i}\omega t) \, , \tag{6.40.2}$$

where the vector $\boldsymbol{M}_0$ is parallel to the z-axis. The system of equations given by (6.40.1) may be linearised by substituting equation (6.40.2) into it and then neglecting terms containing m^2. Equating the determinant of the system to zero, we have

$$\omega^2 \equiv \omega_k^2 = \gamma^2[H_0 + 4\pi(N^{(x)} - N^{(z)})M_0][H_0 + 4\pi(N^{(y)} - N^{(z)})M_0] \, .$$

6.41 $\omega = \omega_k + \mathrm{i}\omega_r[1+\frac{1}{2}\chi_0(N^{(x)}+N^{(y)})]$,

$$\chi_0 = \frac{M_0}{H_0 - N^{(z)}M_0}\ .$$

The value of ω_k is given in the solution of the preceding problem.

6.42

$$\chi_{ik} = \begin{pmatrix} \chi_1 & -\mathrm{i}\chi_a & 0 \\ \mathrm{i}\chi_a & \chi_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the z-axis lies along $\boldsymbol{H}_0$, and

$$\chi_1 = \frac{1}{\Delta}\{\gamma^2 M_0[H_0 + (N^{(y)} - N^{(z)})M_0] - \mathrm{i}\chi_0\omega\omega_r\}\,,$$

$$\chi_2 = \frac{1}{\Delta}\{\gamma^2 M_0[H_0 + (N^{(x)} - N^{(z)})M_0] - \mathrm{i}\chi_0\omega\omega_r\}\,,$$

where

$$\Delta = (\omega_k^2 - \omega^2) - \mathrm{i}\omega\omega_r[2+\chi_0(N^{(x)}+N^{(y)})]\,,$$

$$\chi_0 = \frac{M_0}{H_0 - N^{(z)}M_0}\,, \qquad \chi_a = -\frac{1}{\Delta}\gamma\omega M_0\,.$$

Since the demagnetising factors enter into the expressions for the components of the tensor χ_{ik}, the position of the resonance and the width of the resonance line will depend on the shape of the body.

6.43 The equations of motion for the magnetisation vectors $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ are of the form

$$\frac{\mathrm{d}\boldsymbol{M}_1}{\mathrm{d}t} = -\gamma[\boldsymbol{M}_1 \wedge (\boldsymbol{H}_0 - \lambda\boldsymbol{M}_2)]\,, \qquad \frac{\mathrm{d}\boldsymbol{M}_2}{\mathrm{d}t} = -\gamma[\boldsymbol{M}_2 \wedge (\boldsymbol{H}_0 - \lambda\boldsymbol{M}_1)]\,.$$

The solution will be sought in the form

$$\boldsymbol{M}_1 = \boldsymbol{M}_{10} + \boldsymbol{m}_1\exp(-\mathrm{i}\omega t)\,, \qquad \boldsymbol{M}_2 = \boldsymbol{M}_{20} + \boldsymbol{m}_2\exp(-\mathrm{i}\omega t)\,,$$

where $\boldsymbol{M}_{10}$ and $\boldsymbol{M}_{20}$ are the equilibrium values of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ respectively.

It is convenient to use the cyclic components

$$M_{j\pm} = M_{jx} \pm \mathrm{i}M_{jy} \qquad (j = 1, 2)\,.$$

The precession eigenfrequencies are given by

$$\omega_{01} = \gamma H_0\,, \qquad \omega_{02} = \gamma\lambda|M_{10} - M_{20}|\,.$$

These formulae will hold provided $\lambda|M_{10} - M_{20}| \gg H_0$. The frequency ω_{01} is the same as in the case of a ferromagnetic without sublattices. The frequency ω_{02} depends on the molecular field and is usually much greater than ω_{01}.

d Superconductivity

6.44 $j_n = 0\,, \qquad \operatorname{div} j_s = 0\,, \qquad E = 0\,,$

$$\operatorname{curl}\Lambda j_s = -\frac{1}{c}H\,, \qquad \operatorname{curl}H = \frac{4\pi}{c}j_s\,, \qquad \operatorname{div}H = 0\,.$$

Eliminating j_s or H from these equations we get

$$\nabla^2 j_s = \frac{1}{\delta^2}j_s\,, \qquad \nabla^2 H = \frac{1}{\delta^2}H\,,$$

where $\delta = (\Lambda c^2/4\pi)^{1/2}$ characterizes the depth to which the magnetic field penetrates into the superconductor (or the thickness of the layer in which the superconducting current is concentrated).

6.45

$$H_x = H_z = 0\,, \qquad H_y = H_0 \exp\left(-\frac{x}{\delta}\right)\,,$$

$$j_x = j_y = 0\,, \qquad j_z = -\frac{c}{4\pi}\frac{\partial H_y}{\partial x} = \frac{cH_0}{4\pi\delta}\exp\left(-\frac{x}{\delta}\right)\,.$$

6.46

$$F_x = -\frac{1}{c}\int_0^\infty j_z H_y \, dx = \frac{H_0^2}{8\pi}\,.$$

The force F_x tends to expel the superconductor from the field. This is the way the diamagnetism of the superconductor manifests itself.

6.47

$$H_x = H_z = 0\,, \qquad H_y = H_0\frac{\cosh(x/\delta)}{\cosh(a/\delta)}\,,$$

$$\overline{M}_y = \frac{1}{2a}\frac{1}{2c}\int_{-a}^{a}[r \wedge j_s]_y \, dx + \frac{1}{8\pi a}\int_{-a}^{a} x\frac{\partial H_y}{\partial x}\, dx = \frac{1}{8\pi a}\int_{-a}^{a}(H_{0y} - H_0)\, dx$$

$$= -\frac{H_0}{4\pi}\left(1 - \frac{\delta}{a}\tanh\frac{a}{\delta}\right)\,.$$

$\overline{M}_y$ has the opposite sign from the field (diamagnetism). When $\delta \ll a$ the magnetic moment $\overline{M}_y \approx -H_0/4\pi$. This corresponds to a magnetic susceptibility $\chi = -\frac{1}{4}\pi$ and a permeability $\mu = 1 + 4\pi\chi = 0$.

6.48

$$H_z = H_0\frac{I_0(r/\delta)}{I_0(a/\delta)}\,,$$

$$\overline{M}_z = \frac{1}{2\pi a^2}\int_0^a (H_z - H_0) r \, dr = -\frac{H_0}{4\pi}\left[1 - 2\frac{\delta}{a}\frac{I_1(a/\delta)}{I_0(a/\delta)}\right]\,,$$

where I_0 and I_1 are modified Bessel functions.

6.49 Outside the sphere

$$H_r = \left(H_0 + \frac{2m}{r^3}\right)\cos\vartheta\,, \qquad H_\vartheta = \left(-H_0 + \frac{m}{r^3}\right)\sin\vartheta\,,$$

where m is a constant, the physical meaning of which is that of a magnetic moment.

Inside the sphere

$$j_\phi = f(r)\sin\vartheta\,, \qquad j_r = j_\vartheta = 0\,.$$

The function $j_\phi(r, \vartheta)$ satisfies the equation

$$\nabla^2 j_\phi - \frac{1}{r^2}\sin^2\vartheta\, j_\phi = 0$$

(see the answer to problem 1.47), whence

$$j_\phi(r, \vartheta) = \frac{cA}{4\pi r^2}\left(\sinh\frac{r}{\delta} - \frac{r}{\delta}\cosh\frac{r}{\delta}\right)\,.$$

Here A is an integration constant. The components H_r and H_ϑ of the magnetic field inside the sphere can be expressed in terms of $j_\phi(r, \vartheta)$:

$$H_r = \frac{2\delta^2 A}{r^3}\left(\sinh\frac{r}{\delta} - \frac{r}{\delta}\cosh\frac{r}{\delta}\right)\cos\vartheta\,,$$

$$H_\vartheta = \frac{\delta^2 A}{r^3}\left[\left(1 + \frac{r^2}{\delta^2}\right)\sinh\frac{r}{\delta} - \frac{r}{\delta}\cosh\frac{r}{\delta}\right]\sin\vartheta\,.$$

The constants m and A are determined from the condition that H_r and H_ϑ must be continuous at $r = a$:

$$m = -\tfrac{1}{2}H_0 a^3\left(1 - 3\frac{\delta}{a}\coth\frac{a}{\delta} + 3\frac{\delta^2}{a^2}\right)\,, \qquad A = -\tfrac{3}{2}\,\frac{H_0 a}{\sinh(a/\delta)}\,.$$

When $\delta \ll a$, we get $m = -\tfrac{1}{2}H_0 a^3$ (cf the answer to problem 5.41 with $\mu = 0$), $A = 0$. When $\delta \gg a$, we have $m = -H_0 a^5/30\delta^2$.

6.50

$$j_r = j_\phi = 0\,, \qquad j_z = \frac{J}{2\pi a\delta}\left[\frac{I_0(r/\delta)}{I_1(r/\delta)}\right]\,,$$

$$H_r = H_z = 0\,, \qquad H_\phi = \begin{cases} \dfrac{J}{2\pi ca}\dfrac{I_1(r/\delta)}{I_1(a/\delta)}\,, & \text{when } r < a\,, \\[2ex] \dfrac{J}{2\pi cr}\,, & \text{when } r > a\,; \end{cases}$$

I_0 and I_1 are modified Bessel functions.

6.51 We integrate the Maxwell equation $\operatorname{curl}\boldsymbol{E} = -(1/c)(\partial \boldsymbol{H}/\partial t)$ with $\boldsymbol{E} = \Lambda \partial \boldsymbol{j}_s/\partial t$, over an arbitrary closed contour $\mathfrak{L}$ which passes inside the superconductor and goes round the opening. Applying Stokes' theorem gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_S H_n \, \mathrm{d}^2S + \oint_{\mathfrak{L}} \Lambda(j_s \cdot \mathrm{d}\mathfrak{L})\right] = 0 \, ,$$

where S is a surface based on the contour $\mathfrak{L}$. If the contour $\mathfrak{L}$ lies completely outside the limits of the layer of thickness $\approx \delta$ adjoining the surface of the superconductor, along it $j_s = 0$, and we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int H_n \, \mathrm{d}^2S = 0 \, .$$

6.52

$$J' = -\frac{cH_0S\cos\vartheta}{L} + J \, .$$

6.53

$$J = \frac{c\Phi_0}{L} \, .$$

Quasi-stationary electromagnetic fields

a Quasi-stationary phenomena in linear conductors

7.1

$$J(t) = \frac{\pi a^2 \omega H_0}{c[R^2+(\omega L/c^2)^2]^{1/2}} \sin(\omega t - \varphi) ,$$

where $\tan\varphi = \omega L/(c^2 R)$,

$$N(t) = -\frac{\omega(\pi a^2 H_0)^2}{c^2[R^2+(\omega L/c^2)^2]^{1/2}} \sin\omega t \sin(\omega t - \varphi) ,$$

$$\overline{P} = \frac{\omega^2}{2c^2}\frac{(\pi a^2 H_0)^2 R}{[R^2+(\omega L/c^2)^2]} = \tfrac{1}{2} J_0^2 R .$$

In these expressions L is the inductance of the ring (see problem 5.32), R is its resistance, and J_0 is the amplitude of the current in the ring. The origin of time is chosen so that at $t = 0$ the plane of the loop is perpendicular to H_0.

7.2

$$\overline{N} = \frac{\omega}{2c^2}\frac{(SH_0)^2 R}{R^2+[\omega L/c^2 - 1/(\omega C)]^2} .$$

7.3 The average generalised force which tends to increase the generalised coordinate q_i is equal to

$$-\frac{J_0^2}{2c^6}\frac{\omega^2 L L_{12}}{R^2+(\omega L/c^2)^2}\frac{\partial L_{12}}{\partial q_i} ,$$

where R and L are the resistance and inductance, respectively, of the second circuit and L_{12} is the coefficient of mutual inductance.

7.4

$$\overline{F} = -\frac{\omega^2 L L_{12} |\mathcal{E}_0|^2}{2c^6\{[R^2+\omega^2(L_{12}^2 - L^2)/c^4]^2 + 4\omega^2 L^2 R^2/c^4\}}\frac{\partial L_{12}}{\partial q_i} .$$

7.5

$$\omega_{1,2}^2 = \frac{c^2[(L_1+L_2)C + L_1 C_1 + L_2 C_2]}{2L_1 L_2(C_1 C_2 + CC_1 + CC_2)} \pm \frac{c^2\{[L_1(C+C_1) - L_2(C+C_2)]^2 + 4L_1 L_2 C^2\}^{1/2}}{2L_1 L_2(C_1 C_2 + CC_1 + CC_2)} .$$

When there is no coupling between the circuits, i.e. when $C = 0$, the frequencies ω_1 and ω_2 become equal to $c/(L_1 C_1)^{1/2}$ and $c/(L_2 C_2)^{1/2}$, which corresponds to independent oscillations in each of the circuits. For very tight coupling ($C \gg C_1, C_2$) there is only a single frequency $\omega = c/(L'C')^{1/2}$, where $L' = L_1 L_2/(L_1+L_2)$, $C' = C_1 + C_2$. This corresponds to oscillations in a single circuit in which the capacitances C_1, C_2 and inductances L_1, L_2 are connected in parallel.

7.6

$$\omega_{1,2}^2 = \tfrac{1}{2}c^2\left(\frac{1}{LC_1}+\frac{1}{LC_2}+\frac{1}{L_1C_1}+\frac{1}{L_2C_2}\right)$$
$$\pm\tfrac{1}{2}c^2\left\{\left[\frac{1}{C_1}\left(\frac{1}{L}+\frac{1}{L_1}\right)-\frac{1}{C_2}\left(\frac{1}{L}+\frac{1}{L_2}\right)\right]^2+\frac{4}{L^2C_1C_2}\right\}^{1/2}$$

7.7

$$\omega_{1,2}^2 = c^2\frac{L_1C_1+L_2C_2\pm[(L_1C_1-L_2C_2)^2+4C_1C_2L_{12}^2]^{1/2}}{2C_1C_2(L_1L_2-L_{12}^2)}\ .$$

7.8 When the system of equations for the currents is set up and the determinant of the system is equated to zero, we have after some rearrangement the following fourth-order equation

$$\omega^4+\mathrm{i}\omega^3\left(\frac{1}{\tau_1}+\frac{1}{\tau_2}\right)-\omega^2(\omega_1^2+\omega_2^2)-\mathrm{i}\omega\left(\frac{\omega_1^2}{\tau_2}+\frac{\omega_2^2}{\tau_1}\right)+\omega_1^2\omega_2^2 = 0\ , \qquad (7.8.1)$$

where

$$\omega_1 = \frac{c}{(L_1C_1)^{1/2}}\ , \qquad \omega_2 = \frac{c}{(L_2C_2)^{1/2}}\ , \qquad \tau_1 = RC_1\ , \qquad \tau_2 = RC_2.$$

The coefficients of this equation are complex and hence the frequency ω will also be complex, so that $\omega = \omega'+\mathrm{i}\omega''$. In the zero-order approximation the terms including τ_1 and τ_2 in equation (7.8.1) may be neglected, so that the equation will assume the simpler form

$$\omega^4-\omega^2(\omega_1^2+\omega_2^2)+\omega_1^2\omega_2^2 = 0\ . \qquad (7.8.2)$$

This equation has the following solutions: $\omega_1^{(0)} = \omega_1$ and $\omega_2^{(0)} = \omega_2$. Thus, in this approximation, $\omega'' = 0$ and there is no dissipation of energy (since it was assumed that R is infinite). The oscillations in each circuit take place independently. In the next approximation $\omega = \omega^{(0)}+\Delta\omega'+\mathrm{i}\omega''$, where ω'' and $\Delta\omega'$ are at least of the order of τ^{-1}. Accordingly we shall neglect higher-order terms. If ω is substituted into equation (7.8.1), and equation (7.8.2) is used, we have, since the real and imaginary parts are separately zero,

$$\Delta\omega' = 0\ , \qquad \omega_1'' = -\frac{1}{2\tau_2}\ , \qquad \omega_2'' = -\frac{1}{2\tau_1}\ .$$

The correction to ω' which contains R will only appear in the next approximation.

7.9

$$J_1 = \frac{\mathscr{E}}{Z_1[1+\omega^2 L_{12}^2/(c^4 Z_1 Z_2)]}\,, \qquad J_2 = \frac{\mathrm{i}\omega L_{12}}{c^2 Z_2} J_1\,;$$

$$Z_1 = R_1 + \mathrm{i}\left(\frac{1}{\omega C_1} - \frac{\omega L_1}{c^2}\right), \qquad Z_2 = R_2 + \mathrm{i}\left(\frac{1}{\omega C_2} - \frac{\omega L_2}{c^2}\right);$$

$$J_{1\,\max} = \frac{\mathscr{E}}{R}\,, \qquad \text{when} \quad \omega = \frac{c}{\{L_1 C_1[1 - L_{12}^2/(L_1 L_2)]\}^{1/2}}\,.$$

7.10

$$Z = \frac{R - \mathrm{i}\omega L/c^2}{1 - \omega^2/\omega_1^2 - \mathrm{i}\omega RC}\,,$$

where $\omega_1 = c(LC)^{-1/2}$ is the eigen frequency of the oscillations in the circuit. The impedance Z becomes infinite when $R = 0$ and $\omega = \omega_1$. This property of the two-terminal network is used in electronics (rejector circuit).

7.11

$$C = C_0\,, \qquad L = L_0\,, \qquad R = \frac{\gamma L_0}{c^2}\,,$$

where

$$L_0 = \frac{c^2}{\omega_p^2 C_0}\,.$$

7.12

$$Q = \tfrac{1}{2}\mathrm{Re}(UJ^*) = \tfrac{1}{2}|J|^2\,\mathrm{Re}\left(\frac{1}{Z}\right) = \tfrac{1}{2}\frac{\gamma\omega_p^2}{\omega^2+\gamma^2} C_0 |U_0|^2\,,$$

$$\overline{W} = \tfrac{1}{4}\left(1 + \frac{\omega_p^2}{\omega^2+\gamma^2}\right) C_0 |U_0|^2\,.$$

7.13

$$C = C_0\,, \qquad L = \frac{c^2}{\omega_p^2 C_0}\,, \qquad C_1 = \frac{\omega_p^2}{\omega_0^2} C_0\,, \qquad R = \frac{\gamma L}{c^2} = \frac{\gamma}{\omega_p^2 C_0}\,.$$

7.14

$$Q = \tfrac{1}{2}\frac{\omega^2 \omega_p^2 \gamma}{(\omega^2 - \omega_0^2)^2 + \omega^2\gamma^2} C_0 |U_0|^2\,,$$

$$\overline{W} = \tfrac{1}{4}\left[1 + \frac{\omega_p^2(\omega^2+\omega_0^2)}{(\omega^2-\omega_0^2)^2 + \omega^2\gamma^2}\right] C_0 |U_0|^2\,.$$

7.15 Let the currents flowing through the inductance, capacitance, and the battery be denoted by J_1, J_2, and J_3. According to Kirchhoff's laws

$$J_1 + J_2 + J_3 = 0\,, \qquad \frac{L}{c^2}\dot{J}_1 = \frac{q(t)}{C} = \mathscr{E}(t) + J_3 R\,, \tag{7.15.1}$$

where $q(t)$ is the charge on the capacitor plate, which is related to J_2 by the formula $J_2 = \dot{q}$, and

$$\mathcal{E}(t) = \begin{cases} 0\,, & \text{when} \quad t < 0\,, \\ \mathcal{E}\,, & \text{when} \quad t > 0\,. \end{cases}$$

From equation (7.15.1) we obtain a second-order equation for the current J_1. The corresponding characteristic equation has the following roots

$$x = -\frac{1}{2RC} \pm \left[\left(\frac{1}{2RC}\right)^2 - \omega_0^2\right]^{1/2}, \qquad \omega_0^2 = \frac{c^2}{LC}\,.$$

The following three cases are possible, depending on the relation between R, L, and C:

(a) $\omega_0 > \dfrac{1}{2RC}$.

Using the method of variation of arbitrary constants due to Lagrange (Smirnov, 1964), we have

$$J_1(t) = \frac{\mathcal{E}}{R}\left[1 - \exp\left(-\frac{t}{2RC}\right)\left(\frac{\sin\omega t}{2\omega RC} + \cos\omega t\right)\right]\,,$$

where $\omega = [\omega_0^2 - (2RC)^{-2}]^{1/2}$.

(b) $\omega_0 < \dfrac{1}{2RC}$.

Here

$$J_1(t) = \frac{\mathcal{E}}{R}\left[1 - \exp\left(-\frac{t}{2RC}\right)\left(\frac{\sinh\Omega t}{2\Omega RC} + \cosh\Omega t\right)\right]\,,$$

where $\Omega = [(4R^2C^2)^{-1} - \omega_0^2]^{1/2}$.

(c) $\omega_0 = \dfrac{1}{2RC}$:

$$J_1(t) = \frac{\mathcal{E}}{R}\left[1 - \left(1 + \frac{t}{2RC}\right)\exp\left(-\frac{t}{2RC}\right)\right]\,.$$

Under the conditions (b) and (c) the transients are completely aperiodic and there are no oscillations.

7.16

$$U_2(t) = \begin{cases} 0\,, & \text{when} \quad t < 0\,, \\ U_0 \exp\left(-\dfrac{t}{RC}\right), & \text{when} \quad 0 < t < T\,, \\ U_0\left\{\exp\left(-\dfrac{t}{RC}\right) - \exp\left[-\dfrac{(t-T)}{RC}\right]\right. \,, & \text{when} \quad t > T\,. \end{cases}$$

7.17

$$U_2(t) = \begin{cases} 0\,, & \text{when} \quad t < 0\,, \\ U_0 \exp\left(-\dfrac{Rc^2 t}{L}\right), & \text{when} \quad 0 < t < T\,, \\ U_0 \left\{\exp\left(-\dfrac{Rc^2 t}{L}\right) - \exp\left[-\dfrac{Rc^2(t-T)}{L}\right]\right\}, & \text{when} \quad t > T. \end{cases}$$

7.18 The form of the pulse should be

$$U_1(t) = \begin{cases} 0\,, & \text{when} \quad t < -T\,, \\ hE_0\left(1+\dfrac{t}{T}+\dfrac{RC}{T}\right), & \text{when} \quad -T < t < 0\,, \\ hE_0\left(1-\dfrac{t}{T}\right), & \text{when} \quad 0 < t < T\,, \\ 0\,, & \text{when} \quad t > T\,. \end{cases}$$

The origin of time is chosen so that the field between the plates of the capacitor is a maximum at $t = 0$.

7.19

$$J(t) = \frac{\mathcal{E}_0}{[R^2 + (\omega L/c^2)^2]^{1/2}}\left[\cos(\omega t + \varphi_0 - \varphi) - \exp\left(-\frac{Rc^2 t}{L}\right)\cos(\varphi_0 - \varphi)\right],$$

where $\tan\varphi = \omega L/(c^2 R)$. The transient is absent when $\tan\varphi_0 = -Rc^2/(\omega L)$. This condition has a simple interpretation: the stationary value of the current should be zero at $t = 0$.

7.20 For a simple harmonic dependence of the currents on time, the Kirchhoff equation for the nth section is of the form

$$-\frac{\omega L}{c^2} J_n + \frac{1}{\omega C}(2J_n - J_{n-1} - J_{n+1}) = 0\,. \tag{7.20.1}$$

Equation (7.20.1) is a linear difference equation with the independent variable n assuming integral values. It has two linearly independent solutions, namely $\sin\kappa n$ and $\cos\kappa n$ (cf problem 4.4); the frequencies of the eigenoscillations can be expressed in terms of the parameter κ:

$$\omega^2 = 4\omega_0^2 \sin^2\tfrac{1}{2}\kappa\,, \qquad \omega_0 = \frac{c}{(LC)^{1/2}}\,. \tag{7.20.2}$$

Using the boundary conditions $J_0 = J_N = 0$, we have

$$J_n = A\sin\kappa n\,, \qquad \kappa = \frac{\pi r}{N}\,, \tag{7.20.3}$$

where r is an integer, $r = 1, 2, \ldots$. When $r = 0$, the current in the circuit is zero. However, owing to the periodicity of the factor $\sin\frac{1}{2}\kappa$ which enters into equation (7.20.2), the number of eigenfrequencies of the system will be finite. To obtain the frequency spectrum it is sufficient to vary r within the range $1 \leqslant r \leqslant N$. The parameter κ will then vary in the range $0 < \kappa \leqslant \pi$, and to each κ there will be one eigenfrequency. The total number of such frequencies will be N, as should be the case for a system of N coupled circuits. They will lie in the range $0 < \omega \leqslant 2\omega_0$.

In order to determine the significance of κ introduce the coordinate $y_n = an$, where n refers to the nth section and a is the 'length' of a section. Equation (7.20.3) can then be written in the form

$$\mathcal{J}_n(t) = \mathcal{J}_0 \sin k y_n \exp(-\mathrm{i}\omega_k t) , \tag{7.20.4}$$

where the time factor has been included and $k = \kappa/a$.

Equation (7.20.4) represents a superposition of two waves travelling in opposite directions. The quantity k plays the role of a 'wave vector' for the oscillations which propagate along the line from different discrete sections. The phase and group velocities of these waves are given by the formulae

$$v_{\mathrm{ph}} = \frac{\omega}{k} , \qquad v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} .$$

Since ω is a nonlinear function of k the phase and group velocities are different, i.e. dispersion takes place. From equation (7.20.2) we have

$$v_{\mathrm{ph}} = \frac{2\omega_0}{k} \sin\tfrac{1}{2}ka , \qquad v_{\mathrm{g}} = \omega_0 a \cos\tfrac{1}{2}ka .$$

The quantity $2\pi/k$ may be interpreted as the 'wavelength' in a discrete chain. For long waves ($\lambda \gg a$) we have $ka \ll 1$ and hence $v_{\mathrm{ph}} = v_{\mathrm{g}} = \omega_0 a$, that is, the phase and group velocities are independent of k and there is no dispersion. Figure 7.20.2 shows plots of v_{g} and ω as functions of k.

The electrical oscillations of the system considered in this problem are analogous to the mechanical oscillations of a linear atomic chain, which

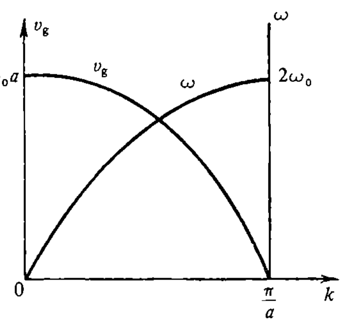


Figure 7.20.2.

may be used as a one-dimensional model of a crystal. The inductance L is then analogous to the mass of an atom, while $1/C$ is analogous to the elastic constant[1].

7.21

$$\Delta r = \frac{2N}{\pi}\frac{\Delta\omega}{(4\omega_0^2-\omega^2)^{1/2}} .$$

7.22 Let the currents through the section with self-inductances L_1 and L_2 be J and J'. The Kirchhoff equation will then be of the form

$$-\frac{\omega L_1}{c^2}J_n + \frac{1}{\omega C}(2J_n - J'_n - J'_{n-1}) = 0 ,$$

$$-\frac{\omega L_2}{c^2}J'_n + \frac{1}{\omega C}(2J'_n - J_n - J_{n+1}) = 0 .$$

Substituting $\omega_1 = c(L_1C)^{-1/2}$ and $\omega_2 = c(L_2C)^{-1/2}$ we have

$$\left.\begin{aligned}(2\omega_1^2-\omega^2)J_n &= \omega_1^2(J'_n + J'_{n-1}) ,\\ (2\omega_2^2-\omega^2)J'_n &= \omega_2^2(J_n + J_{n+1}) .\end{aligned}\right\} \qquad (7.22.1)$$

The solution of this system of equations will be sought in the form

$$J_n = A\exp(\mathrm{i}\kappa n) , \qquad J'_n = B\exp(\mathrm{i}\kappa n) ,$$

where A, B, and κ are constants. Substitution into equation (7.22.1) yields

$$\begin{aligned}A(2\omega_1^2-\omega^2) &= B\omega_1^2[1+\exp(-\mathrm{i}\kappa)] ,\\ B(2\omega_2^2-\omega^2) &= A\omega_2^2[1+\exp(-\mathrm{i}\kappa)] .\end{aligned} \qquad (7.22.2)$$

Equating the determinant of this system to zero, we have the following relation between the frequency ω and κ:

$$\omega^2 = \omega_1^2+\omega_2^2 \pm [(\omega_1^2+\omega_2^2)^2 - 4\omega_1^2\omega_2^2\sin^2\tfrac{1}{2}\kappa]^{1/2} . \qquad (7.22.3)$$

To obtain the complete spectrum of oscillations, the quantity κ must be varied between 0 and π. As in problem 7.20, the values of κ will be found from the boundary conditions.

The most important difference between the present arrangement and that involving identical sections is the fact that there are now two frequencies to each value of κ, which can be seen from equation (7.22.3). There are, therefore, two branches of oscillations. Let us denote the frequencies of these oscillations by ω_+ and ω_- where the subscripts + and − correspond to the same signs in equation (7.22.3). Figure 7.22.2 shows the frequencies as functions of κ. The oscillations corresponding to ω_- are analogous to those in the case of identical sections. In

[1] For details about the oscillations of atomic chains see, e.g. books by Born and Huang (1954) or Wannier (1959). The analogy between electrical and mechanical oscillations is discussed in the book by Brillouin (1946).

particular, for small κ (long waves) we have

$$\omega_- = \frac{\omega_1\omega_2}{[2(\omega_1^2+\omega_2^2)]^{1/2}}\kappa\,,$$

i.e. there is no dispersion. For the ω_+ branch, the dispersion relation for small κ is

$$\omega_+ = a + b\kappa^2\,.$$

As $\kappa \to 0$ the phase velocity becomes infinite while the group velocity tends to zero.

To investigate the character of the oscillations in both branches, consider the ratio of the current amplitudes in adjacent sections for very long ($\kappa \ll 1$) and very short ($\kappa \sim \pi$) waves. By using equation (7.22.2), we have for $\kappa \ll 1$ for the ω_- branch:

$$\left(\frac{A}{B}\right)_- \approx 1\,,$$

and for the ω_+ branch:

$$\left(\frac{A}{B}\right)_+ \approx -\frac{\omega_1^2}{\omega_2^2} = -\frac{L_2}{L_1}\,.$$

For the oscillations corresponding to the ω_- branch the amplitudes are equal and the oscillations are in phase. In the other case, the oscillations in adjacent sections are in antiphase while the amplitudes are inversely proportional to the inductances. When $\kappa = \pi$,

$$\omega_+ = 2^{1/2}\omega_1\,, \qquad \omega_- = 2^{1/2}\omega_2\,.$$

On passing to the limit $\kappa \to \pi$ in equation (7.22.2), we have

$$\left(\frac{B}{A}\right)_+ \to 0\,, \qquad \left(\frac{A}{B}\right)_- \to 0\,.$$

Thus in the limit $\kappa = \pi$, oscillations with frequency $\omega_+ = c(2/L_1C)^{1/2}$ will take place only in the sections with inductances L_1, whereas

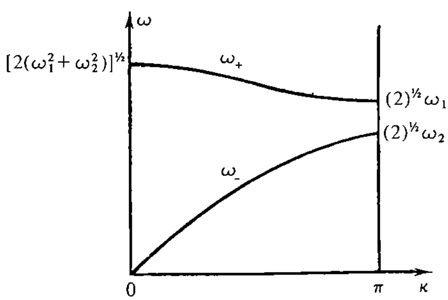


Figure 7.22.2.

oscillations with the frequency $\omega_- = c(2/L_2C)^{1/2}$ will take place in sections with inductances L_2.

The oscillations considered in this problem are an analogue of acoustic and optical oscillations in a linear atomic chain consisting of two types of atoms with different masses (see the references in the footnote to problem 7.20).

7.23 $J_n = Aq_1^n + Bq_2^n$,

where q_1 and q_2 are the roots of the equation

$$q^2 - \left(2 + \frac{Z_1}{Z_2}\right)q + 1 = 0 \,. \tag{7.23.1}$$

The constants A and B may be determined from the boundary conditions $J_N = 0$, $(J_0 - J_1)Z_2 = U_1$. The second condition means that a voltage U_1 is applied across the points $a'b'$ (see figure 7.23.1). Since $q_1q_2 = 1$, which follows from equation (7.23.1), we have

$$U_2 = J_{N-1}Z_2 = U_1 \frac{q_2 - q_1}{(1-q_1)q_2^N - (1-q_2)q_1^N} \,.$$

7.24 The transmission coefficient K can be determined from the results obtained in the preceding solution:

$$K = \frac{q_1 - q_2}{(1-q_2)q_1^N - (1-q_1)q_2^N} \,.$$

The denominator of this expression includes q_1^N and q_2^N. Since $q_1q_2 = 1$, there are two possible cases, namely (a) $|q_1| = |q_2| = 1$ and (b) $|q_1| > 1$, $|q_2| < 1$. In the first case the moduli of q_1^N and q_2^N will be equal to unity and K will also be of the order of unity. In the second case, when $N \gg 1$, $|q_1^N| \gg 1$, $|q_2^N| \ll 1$ and hence

$$K = \frac{q_1 - q_2}{(1-q_2)q_1^N} \ll 1 \,.$$

The frequency ranges for the cases (a) and (b) may be determined from equation (7.23.1) of problem 7.23, from which it follows that

$$q_{1,2} = 1 + \frac{Z_1}{2Z_2} \pm \left[\left(1 + \frac{Z_1}{2Z_2}\right)^2 - 1\right]^{1/2}$$

If the expression in square brackets is negative, then q_1 and q_2 are conjugate roots whose moduli are equal to unity, and this corresponds to $|q_1| = |q_2| = 1$. When this expression is positive, q_1 and q_2 are real and different and this corresponds to $|q_1| > 1$, $|q_2| < 1$. The range of Z_1 and Z_2 for the first of the two cases can be found by equating to zero the expression in square brackets, and the result is

$$-4 \leqslant \frac{Z_1}{Z_2} \leqslant 0 \,.$$

This corresponds to values of ω^2 lying between $c^2/(L_1C_1)$ and $c^2(4C_1+C_2)/[C_1C_2(4L_2+L_1)]$.

7.25 Consider the nth closed section of the long artificial transmission line (figure 7.25.1). It may be regarded as the equivalent circuit for a section of a line with distributed parameters having a length a, inductance ΔL, and capacitance ΔC. For an arbitrary time-dependence of the current in the line, the Kirchhoff equation for this section will be of the form

$$-\frac{1}{c^2}\Delta L\frac{\partial J_n}{\partial t}+\frac{q_{n-1,n}}{\Delta C}-\frac{q_{n+1,n}}{\Delta C}=0\,, \tag{7.25.1}$$

where $q_{n-1,n}$ and $q_{n+1,n}$ are the charges on the upper plates of the two capacitors. From the derivative of equation (7.25.1) with respect to time and the relations $\dot{q}_{n-1,n}=-J_n+J_{n-1}$ and $\dot{q}_{n,n+1}=J_n-J_{n+1}$, we have

$$\frac{1}{c^2}\Delta L\frac{\partial^2 J_n}{\partial t^2}+\frac{1}{\Delta C}(2J_n-J_{n-1}-J_{n+1})=0\,. \tag{7.25.2}$$

It is now necessary to transform from the variable n to z, which is the distance along the line with distributed parameters. We shall, therefore, substitute

$$J_n(t)=J(z,t)\,,\qquad J_{n-1}(t)=J(z-a,t)\,,\qquad J_{n+1}(t)=J(z+a,t)$$

and evaluate the differences

$$J_n-J_{n-1}\approx\frac{\partial J}{\partial z}a-\tfrac{1}{2}\frac{\partial^2 J}{\partial z^2}a^2\,,\qquad J_n-J_{n+1}\approx-\frac{\partial J}{\partial z}a-\tfrac{1}{2}\frac{\partial^2 J}{\partial z^2}a^2\,.$$

If these differences are substituted into equation (7.25.2), together with $L=\Delta L/a$ and $C=\Delta C/a$, which are the inductance and capacitance per unit length, we have

$$\frac{L}{c^2}\frac{\partial^2 J}{\partial t^2}=\frac{1}{C}\frac{\partial^2 J}{\partial z^2}\,. \tag{7.25.3}$$

This is the differential equation for a long lossless line. A real long line will always exhibit losses associated with the finite resistance of the leads and imperfect insulation.

Figure 7.25.2 shows the equivalent circuit when the insulation is not assumed to be perfect. The equation for a long line of this kind

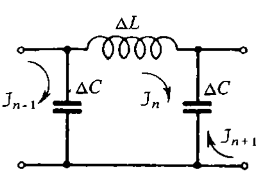


Figure 7.25.1.

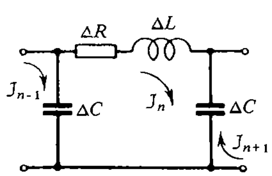


Figure 7.25.2.

(the telegraph equation) can be obtained in a similar way and is

$$\frac{L}{c^2}\frac{\partial^2 J}{\partial t^2}+R\frac{\partial J}{\partial t} = \frac{1}{C}\frac{\partial^2 J}{\partial z^2} ,$$

where R is the active resistance of the leads per unit length.

7.26 From the solution of equation (7.25.3) obtained in the preceding solution, we have $\omega = vk$, where $v = c(LC)^{-1/2}$ is the velocity of propagation of waves in a long line, $k = \pi r/l$, $r = 1, 2, 3, \ldots$, and L and C are, respectively, the inductance and capacitance per unit length. In contrast to a line with lumped parameters, here the number of eigen-oscillations is infinite. This is due to the fact that a long line has an infinite number of degrees of freedom while with lumped parameters the number of degrees of freedom, N, was finite. The absence of dispersion is characteristic of a perfect long line.

7.27 We shall use Ohm's law in the differential form $\boldsymbol{j} = \sigma(\boldsymbol{E}+\boldsymbol{E}_e)$, where $\boldsymbol{E}_e$ is the external field strength. The field strengths can be expressed in terms of the potentials:

$$\boldsymbol{E} = -\nabla\varphi-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} , \qquad \boldsymbol{E}_e = \frac{\boldsymbol{j}}{\sigma}+\nabla\varphi+\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} .$$

Assuming that the conductor is thin, we can integrate both parts of the latter equation along the circuit:

$$\oint(\boldsymbol{E}_e \cdot \mathrm{d}\boldsymbol{l}) = \oint\left(\frac{\boldsymbol{j}}{\sigma}\cdot \mathrm{d}\boldsymbol{l}\right)+\oint(\nabla\varphi\cdot \mathrm{d}\boldsymbol{l})+\frac{1}{c}\oint\left(\frac{\partial \boldsymbol{A}}{\partial t}\cdot \mathrm{d}\boldsymbol{l}\right) . \qquad (7.27.1)$$

The integral on the left-hand side of equation (7.27.1) is equal to the external e.m.f. $\mathcal{E}_e$ which is applied to the circuit. The integral

$$\oint\left(\frac{\boldsymbol{j}}{\sigma}\cdot \mathrm{d}\boldsymbol{l}\right) = JR$$

represents the Joule losses per unit time, while

$$\oint(\nabla\varphi\cdot \mathrm{d}\boldsymbol{l}) = \oint \mathrm{d}\varphi = 0 .$$

The last integral may be transformed as follows. If

$$\boldsymbol{A} = \frac{1}{c}\oint\frac{J}{r}\left(t-\frac{r}{c}\right)\mathrm{d}\boldsymbol{l}' ,$$

$$J\left(t-\frac{r}{c}\right) = J_0\exp\left[-\mathrm{i}\omega\left(t-\frac{r}{c}\right)\right] , \qquad \frac{\partial \boldsymbol{A}}{\partial t} = -\mathrm{i}\omega \boldsymbol{A} ,$$

is substituted into equation (7.27.1), and the real and imaginary parts are separated, we have

$$\mathcal{E}_e(t) = J(t)\left\{\left[R+\frac{\omega}{c^2}\oint\oint\frac{1}{r}\sin\frac{\omega r}{c}(\mathrm{d}\boldsymbol{l}\cdot \mathrm{d}\boldsymbol{l}')\right]-\frac{\mathrm{i}\omega}{c^2}\oint\oint\frac{1}{r}\cos\frac{\omega r}{c}(\mathrm{d}\boldsymbol{l}\cdot \mathrm{d}\boldsymbol{l}')\right\} .$$

The expression in braces represent the complex impedance of the circuit. The effective resistance is equal to $R+R_r(\omega)$ where

$$R_r(\omega) = \frac{\omega}{c^2}\oint\oint\frac{1}{r}\sin\frac{\omega r}{c}(\mathbf{d}\boldsymbol{l}\cdot\mathbf{d}\boldsymbol{l}')\,.$$

The resistance R is associated with losses giving rise to the heating of the conductor, while $R_r(\omega)$ represents the energy lost by radiation and is referred to as the radiation resistance (see the next problem).

The reactance is equal to $-\mathrm{i}\omega L(\omega)/c^2$ where

$$L(\omega) = \oint\oint\frac{1}{r}\cos\frac{\omega r}{c}(\mathbf{d}\boldsymbol{l}\cdot\mathbf{d}\boldsymbol{l}')$$

is the inductance and is a function of frequency.

Consider the limit where $c/\omega = \lambda/2\pi \gg l$, where the quantity l is the linear dimension of the circuit. In the region of integration, $\omega r/c \ll 1$. When the cosine is expanded into a series, and third- and higher-order terms are neglected, we have

$$L(\omega) \approx \oint\oint\frac{1}{r}(\mathbf{d}\boldsymbol{l}\cdot\mathbf{d}\boldsymbol{l}') - \tfrac{1}{2}\left(\frac{\omega}{c}\right)^2\oint\oint r(\mathbf{d}\boldsymbol{l}\cdot\mathbf{d}\boldsymbol{l}')\,.$$

The first term in this equation is independent of frequency and represents the ordinary inductance, while the second term gives a correction which is significant at high frequencies. In practice, the self-inductance should be calculated from equation (5.0.18), since the integral $\oint\oint(\mathbf{d}\boldsymbol{l}\cdot\mathbf{d}\boldsymbol{l}')/r$ is divergent. This divergence is due to the fact that the conductor is assumed to be infinitely thin (linear).

In the expansion for the sine in the expression for $R_r(\omega)$, the third-order term must also be included since the integral of the first (linear) term is zero. The radiation resistance is then given by

$$R_r(\omega) = -\frac{\omega^4}{6c^5}\oint\oint r^2(\mathbf{d}\boldsymbol{l}\cdot\mathbf{d}\boldsymbol{l}')\,.$$

7.28

$$L(\omega) = L+\frac{64\pi^4a^4}{3\lambda^3}\,. \qquad R_r(\omega) = \frac{2\pi^2}{3c}\left(\frac{2\pi a}{\lambda}\right)^4\,.$$

The current-carrying ring acts as a magnetic dipole. The energy emitted per unit time is equal to $2\ddot{\boldsymbol{m}}^2/(3c^3)$ where $\boldsymbol{m}$ is the magnetic dipole moment. The coefficient of proportionality between the emitted energy and the mean square current is equal to $2\pi^2a^2\omega^4/(3c^5)$ and is the same as $R_r(\omega)$.

b Eddy currents and skin effect

7.29

$$H(x) = H_0\left[\frac{\sinh^2(x/\delta)+\cos^2(x/\delta)}{\sinh^2(h/\delta)+\cos^2(h/\delta)}\right]^{1/2},$$

$$H_0 = \frac{4\pi}{c}J_0 n\,; \qquad \delta = \frac{c}{(2\pi\mu\sigma\omega)^{1/2}}\,.$$

When $\delta \ll h$, the magnetic field is $H(x) = H_0 \exp[-(h-|x|)/\delta]$. When $\delta \gg h$, the field is $H(x) = H_0$ (cf problem 5.7).

7.30 Since the system is symmetric with respect to the axis of the cylinder and the primary magnetic field H_0 is uniform, it is clear that the eddy currents induced in the cylinder will follow circular paths which lie in planes perpendicular to the axis. These currents will produce the same magnetic field as that due to a large number of separate coaxial solenoids. However, the field caused by the solenoid is zero outside the system; inside it will be parallel to the axis of the solenoid. Thus, the total magnetic field outside the cylinder is equal to H_0 while inside the cylinder it is given by the first equation in (7.b.4), which in view of the axial symmetry is of the form

$$\frac{d^2H}{dr^2}+\frac{1}{r}\frac{dH}{dr}+k^2H = 0\,,$$

where

$$k = \frac{1+i}{\delta}\,, \qquad H = H_z(r)\,, \qquad H_\phi = H_r = 0\,.$$

The boundary condition is $H(a) = H_0$.

The solution which is finite at $r = 0$ and which satisfies this boundary condition can be expressed in terms of the zero-order Bessel function:

$$H = H_0\frac{J_0(kr)}{J_0(ka)}\,.$$

Outside the cylinder $H = H_0$, when $a \leqslant r \leqslant b$, and $H = 0$, when $r > b$. The current density and the electric field inside the cylinder may be calculated from equation (7.b.3):

$$j = j_\phi = \sigma E_\phi = \frac{kc\,J_1(kr)}{4\pi\,J_0(ka)}H_0\,, \qquad E_r = E_z = 0\,.$$

To determine the electric field outside the cylinder we shall use the Maxwell equation for curl$E$ in the integral form

$$\oint E_l\,dl = \frac{i\omega}{c}\int B_n\,d^2S\,.$$

Inside the cylinder there is only a single component of the electric field (E_ϕ). It follows from the boundary condition on the surface of the cylinder and from the symmetry of the system that outside the cylinder the field E will also have the single component E_ϕ which is a function of r only. If the contour of integration is taken to be circular, the line integral turns out to be $2\pi r E_\phi$. The surface integral may be evaluated with the aid of equation (A3.12). The final result is

$$E_\phi = \frac{kcH_0 J_1(ka)}{4\pi\sigma\, J_0(ka)}\frac{a}{r} + \frac{\mathrm{i}\omega H_0}{2cr}(r^2 - a^2)\,, \qquad a \leqslant r \leqslant b\,,$$

$$E_\phi = \frac{kcH_0 J_1(ka)}{4\pi\sigma\, J_0(ka)}\frac{a}{r} + \frac{\mathrm{i}\omega H_0}{2cr}(b^2 - a^2)\,, \qquad r > b\,.$$

In the absence of the cylinder, i.e. when $a = 0$, the field is given by

$$E_\phi = \frac{\mathrm{i}\omega H_0 r}{2c} \quad (r < b)\,, \qquad E_\phi = \frac{\mathrm{i}\omega H_0 b^2}{2cr} \quad (r > b)\,.$$

Thus, the additional magnetic field caused by the presence of the cylinder is equal to zero for $r > a$, even though the additional electric field is finite. This is because of the fact that the exact equation $\mathrm{curl}\boldsymbol{H} = (\partial \boldsymbol{D}/\partial t)/c$, which holds outside the conductor, was replaced by the approximate equation $\mathrm{curl}\boldsymbol{H} = 0$ (the displacement current is neglected in the quasi-stationary approximation). In the rigorous solution the additional magnetic field outside the conductor will also be finite (see problem 8.54 where the diffraction of a plane wave by a conducting cylinder is considered).

7.31 At low frequencies ($|ka| \ll 1$ or $\delta \gg a$)

$$j = \mathrm{i}\frac{cH_0}{4\pi}\frac{r}{\delta^2} = \frac{\mathrm{i}\sigma\omega H_0}{2c}r\,,$$

and hence the current density is a linear function of r and is proportional to the frequency.

At high frequencies ($|ka| \gg 1$ or $\delta \ll a$) it is possible to use the asymptotic expression for the Bessel function and the result is

$$j = (\mathrm{i}-1)\frac{cH_0}{4\pi\delta}\left(\frac{a}{r}\right)^{1/2} \exp\left[-(1+\mathrm{i})\frac{(a-r)}{\delta}\right].$$

When $a - r \gg \delta$, the current density becomes infinitesimal. Thus at high frequencies the current is largely concentrated in a thin surface layer.

7.32

$$Q = -\frac{\pi a^2 n^2 J_0^2}{\sigma}\mathrm{Re}\left[\frac{kJ_1(ka)}{J_0(ka)}\right], \qquad k^2 = \mathrm{i}\frac{4\pi\mu\sigma\omega}{c^2}\,.$$

When $|ka| \ll 1$ (low frequencies)

$$Q = \frac{\pi n^2 J_0^2}{16\sigma}\left(\frac{a}{\delta}\right)^4 = \pi^3\sigma\left(\frac{a^2\mu n\omega J_0}{c^2}\right)^2.$$

When $|ka| \gg 1$ (high frequencies)

$$Q = \frac{\pi n^2 J_0^2}{\sigma}\left(\frac{a}{\delta}\right) = \frac{an^2 J_0^2}{c}\left(\frac{2\pi^3\mu\omega}{\sigma}\right)^{1/2}$$

The energy dissipation at low and high frequencies is proportional to ω^2 and $\omega^{1/2}$, respectively.

7.33

$$\beta = \beta' + \mathrm{i}\beta'' = -\tfrac{1}{4}a^2\left[1 - \frac{2}{ka}\frac{J_1(ka)}{J_0(ka)}\right], \qquad k^2 = \mathrm{i}\frac{4\pi\sigma\omega}{c^2}.$$

When $|ka| \gg 1$ (high frequencies)

$$\beta' = -\tfrac{1}{4}a^2\left[1 - \frac{c}{a(2\pi\sigma\omega)^{1/2}}\right], \qquad \beta'' = \frac{ca}{4(2\pi\sigma\omega)^{1/2}}.$$

It follows that at high frequencies $\beta'' \to 0$, i.e. the losses are reduced as a result of the absence of the field in the conductor. When $|ka| \ll 1$ (low frequencies)

$$\beta' = -\frac{\pi^2 a^6\sigma^2\omega^2}{12c^4}, \qquad \beta'' = \frac{\pi a^4\sigma\omega}{8c^2}.$$

Thus $\beta \to 0$ as $\omega \to 0$; this is related to the fact that $\mu = 1$, i.e. the static magnetic polarisability is zero.

7.34 The magnetic moment due to the induced currents will, in view of the symmetry of the system, be parallel to the external magnetic field. Hence, the total magnetic field $\boldsymbol{H}_2$ outside the cylinder will be of the form

$$\boldsymbol{H}_2(\boldsymbol{r}) = \frac{4\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^4} - \frac{2\boldsymbol{m}}{r^2} + \boldsymbol{H}_0,$$

where $\boldsymbol{m}$ is the unknown magnetic moment per unit length of the cylinder, which is parallel to $\boldsymbol{H}_0$, and $\boldsymbol{r}$ is the position vector in the plane perpendicular to the axis of the cylinder. The vector potential corresponding to $\boldsymbol{H}_2$ is

$$\boldsymbol{A}_2 = \frac{2[\boldsymbol{m}\wedge\boldsymbol{r}]}{r^2} + [\boldsymbol{H}_0\wedge\boldsymbol{r}].$$

The components of $\boldsymbol{A}_2$ are

$$A_{2z} \equiv A_2 = \left(\frac{2m}{r} + H_0 r\right)\sin\phi, \qquad A_{2r} = A_{2\phi} = 0. \tag{7.34.1}$$

The angle ϕ is measured from the direction of $\boldsymbol{H}_0$.

Thus, the vector potential in the external space has only a longitudinal component which is proportional to $\sin\phi$. The continuity conditions for the field components at the boundary may be satisfied if the vector potential in the internal region is sought in an analogous form:

$$A_{1z} \equiv A_1 = F(r)\sin\phi, \qquad A_{1r} \equiv A_{1\phi} = 0. \tag{7.34.2}$$

The electric field $\boldsymbol{E}$ will in general depend on both A and φ. As usual, we shall impose the additional condition

$$\operatorname{div}\boldsymbol{A} + \frac{\epsilon}{c}\frac{\partial\varphi}{\partial t} = 0 .$$

Since $\operatorname{div}\boldsymbol{A} = 0$, which follows from equations (7.34.1) and (7.34.2), we have $\partial\varphi/\partial t = -\mathrm{i}\omega\varphi = 0$ so that

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{A}}{\partial t} = \frac{\mathrm{i}\omega\boldsymbol{A}}{c} .$$

It follows that A will satisfy the same equation as the electric field [see equation (7.b.4)]. The solution of this equation which is bounded at $r = 0$ is a Bessel function

$$F(r) = CJ_1(kr) , \qquad A_1 = CJ_1(kr)\sin\phi . \tag{7.34.3}$$

The constants C and m in equations (7.34.1) and (7.34.3) may be determined from the condition that the internal ($\boldsymbol{H}_1$) and external ($\boldsymbol{H}_2$) fields must be equal at $r = a$ (surface of the cylinder). Using equation (A3.9) we have

$$C = \frac{2H_0}{kJ_0(ka)} , \qquad m = -\tfrac{1}{2}a^2H_0\left[1-\frac{2}{ka}\frac{J_1(ka)}{J_0(ka)}\right] . \tag{7.34.4}$$

It follows from the expression for m that the transverse magnetic polarisability of the cylinder,

$$\beta = -\tfrac{1}{2}a^2\left[1-\frac{2}{ka}\frac{J_1(ka)}{J_0(ka)}\right] ,$$

is larger by a factor of two as compared with its longitudinal polarisability (see problem 7.33). The components of the magnetic field inside the cylinder can be determined from equations (7.34.3) and (7.34.4) and are given by

$$H_{1r} = \frac{1}{r}\frac{\partial A_1}{\partial\phi} = 2H_0\frac{J_1(kr)}{krJ_0(ka)}\cos\phi , \qquad H_{1z} = 0 ,$$

$$H_{1\phi} = -\frac{\partial A_1}{\partial r} = -2H_0\frac{J_1'(kr)}{J_0(ka)}\sin\phi .$$

Finally, the current density is given by $j = (c/4\pi)\operatorname{curl}\boldsymbol{H}$, and hence

$$j_z = -\frac{cH_0}{2\pi}\frac{J_1(kr)}{J_0(ka)}\sin\phi , \qquad j_\phi = j_r = 0 .$$

It is clear from this formula that the currents in the two halves of the cylinder, $0 \leqslant \phi \leqslant \pi$ and $\pi \leqslant \phi \leqslant 2\pi$, will always flow in opposite directions. The total current through the cross section of the cylinder is zero. The radial dependence of the current is the same as for a cylinder placed in a longitudinal field (see problem 7.31). It should be noted, however, that for a longitudinal field the currents flow over circles in planes perpendicular to the axis of the cylinder, whereas for a transverse field the currents flow along the axis of the cylinder.

7.35 The average energy dissipation per unit length of the cylinder is most simply calculated with the aid of equation (7.b.9) by considering the flux of energy through the lateral surface of the cylinder. From the results of the preceding problem we have

$$Q = -\frac{ac^2H_0^2}{8\pi\sigma}\mathrm{Re}\left[\frac{kJ_1(ka)}{J_0(ka)}\right].$$

The same result can be obtained with the aid of equation (7.b.8) where the integration of the product of Bessel functions can be carried out with the aid of equation (A3.13).

7.36 The electric and magnetic fields inside the cylinder may be found as in problem 7.34 for a linearly polarised external field:

$$H_r = -\frac{2H_0J_1(kr)}{krJ_0(ka)}\exp(\mathrm{i}\phi), \qquad H_\phi = -2\mathrm{i}H_0\frac{J_1'(kr)}{J_0(ka)}\exp(\mathrm{i}\phi),$$

$$j_z = \frac{\mathrm{i}ckH_0}{2\pi}\frac{J_1(kr)}{J_0(ka)}\exp(\mathrm{i}\phi).$$

The force per unit volume of the cylinder is given by

$$f = \frac{1}{c}[j \wedge H],$$

where it is assumed that $\mu = 1$ inside the cylinder. The radial component of this force gives rise to a radial pressure, whereas the azimuthal component is responsible for a couple. Since j and H are complex quantities, the average value of the azimuthal component of the force is

$$\overline{f_\phi} = \frac{1}{2c}\mathrm{Re}(j_zH_r^*). \tag{7.36.1}$$

The couple per unit length of the cylinder may be found by multiplying the average force given by equation (7.36.1) by r and integrating over the cross section of the cylinder. The integral may be evaluated with the aid of equation (A3.13). The result is

$$\overline{N} = -\frac{aH_0^2}{|k|^2}\mathrm{Re}\left[k\frac{J_1(ka)}{J_0(ka)}\right]. \tag{7.36.2}$$

The same result may be obtained in another way. The moment of the forces may be expressed in terms of the magnetic moment of the system

$$N(t) = [\boldsymbol{m}(t) \wedge \boldsymbol{H}_0(t)].$$

Writing $\overline{N_z} = \overline{N}$ in terms of the complex amplitudes of $\boldsymbol{H}_0$ and $\boldsymbol{m}$, and $\boldsymbol{m}$ in terms of the transverse magnetic polarisability of the cylinder (see problem 7.34), we arrive again at equation (7.36.2).

At low frequencies, equation (7.36.2) yields

$$\bar{N} = \frac{a^4 H_0^2}{4\delta^2} = \frac{\pi\sigma\omega}{4c^2} H_0^2 a^4 ,$$

whereas at high frequencies,

$$\bar{N} = \tfrac{1}{2} a\delta H_0^2 = \frac{ca}{(8\pi\sigma\omega)^{1/2}} H_0^2 .$$

It is clear from these formulae that the couple is zero both at very low and very high frequencies. If the field is linearly polarised, the average couple is zero (formally this follows from the fact that the integral with respect to ϕ in the formula for N is zero; see problem 7.34 in which j and $\boldsymbol{H}$ are found for this situation). The couple is produced by a 'rotating' field. The induction motor is based on the phenomenon discussed in this problem.

7.37 Together with a fixed system of coordinates with the z-axis along the axis of the cylinder and the x-axis along the external magnetic field $\boldsymbol{H}_0$, we consider a system of coordinates ξ, η, z which rotates with the cylinder and whose z-axis also lies along the axis of the cylinder. In this system the external magnetic field may be written in the form

$$\boldsymbol{H}_0(t) = (\boldsymbol{H}_{01} - \mathrm{i}\boldsymbol{H}_{02}) \exp(-\mathrm{i}\omega t) ,$$

where $\boldsymbol{H}_{01}$ and $\boldsymbol{H}_{02}$ are constant vectors of equal length H_0 and are parallel to the ξ and η axes. A field of this kind was discussed in the preceding problem. The couple, which will tend to retard the cylinder, is given by

$$\bar{N} = -\frac{aH_0^2}{|k|^2} \mathrm{Re}\left[k \frac{J_1(ka)}{J_0(ka)}\right] .$$

7.38 It was shown in the solution of problem 7.30 that the currents induced in the cylinder as a result of changes in the external longitudinal field do not give rise to an additional magnetic field outside the cylinder. Inside the cylinder, on the other hand, the field due to these currents is longitudinal and is a function of r only. This field satisfies the equation

$$\frac{\partial^2 H}{\partial r^2} + \frac{1}{r}\frac{\partial H}{\partial r} - \frac{4\pi\mu\sigma}{c^2}\frac{\partial H}{\partial t} = 0 . \tag{7.38.1}$$

It is clear that the magnetic field inside the cylinder will eventually be damped out. Hence, the special solution of equation (7.38.1) will be sought in the form $F(r)\exp(-\gamma t)$ where $\gamma > 0$ is a constant. The function $F(r)$ satisfies the Bessel equation

$$F''(r) + \frac{1}{r}F'(r) + k^2 F(r) = 0 , \tag{7.38.2}$$

where $k^2 = 4\pi\mu\sigma\gamma/c^2$.

The solution of equation (7.38.2) which is finite at $r = 0$ is of the form $F(r) = CJ_0(kr)$. Since the external field H_0 is switched off, and the additional field caused by the eddy currents outside the cylinder is zero, the condition at the boundary is $H|_{r=a} = 0$, i.e.

$$J_0(ka) = 0 .$$

It follows that $k_m a = \beta_m$, $m = 1, 2, \ldots$, where the quantities β_m are the zeros of J_0. The possible values of γ are

$$\gamma_m = \frac{c^2\beta_m^2}{4\pi\mu\sigma a^2} . \tag{7.38.3}$$

The general solution of equation (7.38.1) which corresponds to the boundary value problem under consideration is

$$H(r, t) = \sum_m C_m J_0(k_m r)\exp(-\gamma_m t) ,$$

where the C_m may be determined from the initial condition

$$H(r, 0) = \sum_m C_m J_0(k_m r) .$$

By using the orthogonality condition

$$\int_0^1 xJ_0(k_m x)J_0(k_n x)\,dx = \tfrac{1}{2}[J_0'(k_m)]^2\delta_{mn} ,$$

we have

$$C_m = \frac{2}{a^2[J_0'(k_m a)]^2}\int_0^a H(r, 0)J_0(k_m r)r\,dr .$$

Initially, the field $H(r, 0)$ is equal to the external field H_0, since the constant magnetic field is unaffected by the presence of an infinite cylinder whose axis is parallel to it. From equations (A3.12) and (A3.9) we have

$$C_m = \frac{2H_0}{(k_m a)J_1(k_m a)} .$$

The rate of attenuation of the field will be determined by the smallest of the γ_m, i.e. by γ_1. The latter quantity may be obtained by substituting the smallest root of the Bessel function ($\beta_1 \approx 2{\cdot}4$) into equation (7.38.3).

7.39 In the zero-order approximation (in frequency) the magnetic field inside the sphere is given by

$$\boldsymbol{H} = \frac{3}{\mu+2}\boldsymbol{H}_0 , \tag{7.39.1}$$

which was obtained in problem 5.41. The electric field inside the sphere will be zero in this approximation [see equation (7.b.3)], since a constant magnetic field does not give rise to an electric field. In order to determine the electric field in the next (linear in frequency) approximation we shall use equation (7.b.2) in an integral form.

It is clear from the symmetry of the system that the currents in the sphere will flow in circular paths in planes perpendicular to $\boldsymbol{H}_0$ and the electric field will be in the same direction. In the spherical system of coordinates with the z-axis parallel to $\boldsymbol{H}_0$ we have

$$E = \frac{\mathrm{i}\omega H}{2c} r \sin\vartheta , \qquad j = \sigma E ,$$

where H is given by equation (7.39.1). The heat Q liberated in the sphere is obtained by integrating $q = \frac{1}{2}\sigma |E|^2$ over the volume of the sphere. The result is

$$Q = \frac{3\pi a^5 \sigma \omega^2 H_0}{5c^2(\mu+2)^2} .$$

7.40 The magnetic field outside the sphere is

$$\boldsymbol{H} = \boldsymbol{H}_0 + \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} ,$$

where $\boldsymbol{m} = -\frac{1}{2}a^3\boldsymbol{H}_0$ and $\beta = -\frac{1}{2}a^3$ is the magnetic polarisability of the sphere for a strong skin effect. Inside the sphere

$$H_\vartheta = -\tfrac{3}{2}H_0 \exp\left[-(1-\mathrm{i})\frac{z}{\delta}\right] \sin\vartheta , \qquad H_r = H_\phi = 0 ,$$

where z is measured from the surface along the inward normal to the conductor, and the polar axis of the spherical system is parallel to $\boldsymbol{H}_0$;

$$Q = \frac{3a^2 c}{8}\left(\frac{\mu\omega}{2\pi\sigma}\right)^{1/2} H_0^2 .$$

7.41 For a strong skin effect the field inside the ellipsoid is zero, whereas the field outside the ellipsoid satisfies the equations $\operatorname{curl}\boldsymbol{H} = 0$, $\operatorname{div}\boldsymbol{H} = 0$ and the boundary conditions $H_n|_S = 0$, $\boldsymbol{H}|_{r\to\infty} \to \boldsymbol{H}_0$ where $\boldsymbol{H}_0$ is the external field and S represents the surface of the ellipsoid.

Compare now this problem with the case of a dielectric ellipsoid with $\epsilon = 0$ placed in a uniform electric field. The electric field outside the ellipsoid will satisfy the equations

$$\operatorname{curl}\boldsymbol{E} = 0 , \qquad \operatorname{div}\boldsymbol{E} = 0 , \tag{7.41.1}$$

and the boundary conditions

$$E_n|_S = \epsilon E_{n\,\mathrm{int}}|_S = 0 , \qquad \boldsymbol{E}|_{r\to\infty} \to \boldsymbol{E}_0 . \tag{7.41.2}$$

The conditions for the tangential components of $\boldsymbol{E}$ need not be considered since equations (7.41.1) and (7.41.2) uniquely define $\boldsymbol{E}$ in the external region.

We see that the problem of a conducting ellipsoid for a strong skin effect is identical with the problem of a dielectric ellipsoid for which $\epsilon = 0$. The magnetic polarisabilities in the direction of the principal axes of the ellipsoid are obtained by substituting $\epsilon_1 = 0$ into the formulae given in the solution of problem 3.72:

$$\beta^{(i)} = -\frac{V}{4\pi(1-n^{(i)})} ,$$

where $n^{(i)}$ is the depolarisation coefficient and V is the volume of the ellipsoid.

For a very prolate ellipsoid of revolution with semiaxes a and $b \gg a$ (a rod), we have (see problem 3.70)

$$\beta_\perp = -\tfrac{2}{3}a^2 b , \qquad \beta_\parallel = -\tfrac{1}{3}a^2 b .$$

For a very oblate ellipsoid ($b \ll a$; a disc)

$$\beta_\perp = -\frac{2a^3}{3\pi} , \qquad \beta_\parallel = -\tfrac{1}{3}a^2 b \to 0 , \qquad \text{as} \quad b \to 0 .$$

7.42 In view of the axial symmetry of the system, both the distribution of eddy currents in the sphere and the electric field will exhibit axial symmetry. Hence, the electric field will have only a single component, which cannot be a function of ϕ, so that $E_\phi = f(r, \vartheta)$.

The solution of equation (7.b.4) for the resultant electric field $\boldsymbol{E}$ will be sought in the form

$$E_\phi = F(r)\sin\vartheta , \qquad E_r = E_\vartheta = 0 .$$

Using the expression for the Laplacian of a vector in spherical coordinates (see problem 1.47), we obtain the equation for $F(r)$, and this can be reduced to the Bessel equation by substituting $F(r) = r^{-1/2}\chi(r)$. The solution of this equation which is bounded at $r = 0$ is

$$\chi(r) = AJ_{3/2}(kr) .$$

The magnetic field inside the sphere may be determined from equation (7.b.2). The magnetic field in the external region is the sum of the external field $\boldsymbol{H}_0$ and the magnetic field due to the dipole $\boldsymbol{m}$ whose direction is parallel to that of $\boldsymbol{H}_0$. Thus,

$$\boldsymbol{H}_2 = \boldsymbol{H}_0 + \frac{3\boldsymbol{r}(\boldsymbol{m}\cdot\boldsymbol{r})}{r^5} - \frac{\boldsymbol{m}}{r^3} .$$

The constants A and m can be determined from the boundary conditions for $\boldsymbol{H}$ at the surface of the sphere. If the Bessel functions of half-integral order are expressed in terms of trigonometric functions, then

$$m = -\tfrac{1}{2}a^3\left(1 - \frac{3}{k^2a^2} + \frac{3}{ka}\cot ka\right)H_0 , \qquad A = \frac{3ia\omega}{c\sin ka}\left(\frac{\pi}{8k}\right)^{1/2} H_0 .$$

7.43

$$Q = -\tfrac{3}{8}a\delta^2 H_0^2 \left\{1 - \frac{a}{\delta}\left[\frac{\sinh(2a/\delta) + \sin(2a/\delta)}{\cosh(2a/\delta) - \cos(2a/\delta)}\right]\right\} .$$

7.44

$$R = \frac{l}{ca}\left(\frac{\omega}{\pi\sigma}\right)^{1/2} \mathrm{Re}\left[(1+\mathrm{i})\frac{J_0(ka)}{J_1(ka)}\right],$$

where $kc = (1+\mathrm{i})(2\pi\sigma\omega)^{1/2}$.

When $|ka| \ll 1$ (low frequencies)

$$R = R_0\left[1 + \tfrac{1}{12}\left(\frac{\pi\sigma\omega a^2}{c^2}\right)^2\right],$$

where $R_0 = l/\pi\sigma a^2$ is the DC resistance.

When $|ka| \gg 1$ (high frequencies)

$$R = \frac{l}{2\pi\sigma a\delta} = \frac{l}{ca}\left(\frac{\omega}{2\pi\sigma}\right)^{1/2}$$

It follows from the latter formula that the effective cross section of a conductor for a strong skin effect is equal to $2\pi a\delta$.

7.45

$$R = \frac{\omega\delta_2}{2ac^2}\left\{\frac{(\delta_1^2 - \delta_2^2)\sin(2h/\delta_2) - (\delta_1^2 + \delta_2^2)\sinh(2h/\delta_2) + 2\delta_1\delta_2\cos(2h/\delta_2)}{[\delta_1\sin(h/\delta_2) + \delta_2\cos(h/\delta_2)]^2 + (\delta_1^2 + \delta_2^2)\sinh^2(h/\delta_2)}\right\}$$

where

$$\delta_1 = \frac{c}{(2\pi\sigma_1\omega)^{1/2}}, \qquad \delta_2 = \frac{c}{(2\pi\sigma_2\omega)^{1/2}} .$$

7.46

$$H' = \frac{2H_0}{ka\sinh kh + 2\cosh kh},$$

where $k = (1+\mathrm{i})/\delta$.

When $|kh| \ll 1$ (low frequencies), $H' = H_0$ and hence the presence of the cylindrical shell has no effect on the magnitude of the field, When $|kh| \gg 1$ (high frequencies), we have

$$\sinh kh \approx -\mathrm{i}\cosh kh \approx \tfrac{1}{2}\mathrm{i}\exp\left[(1-\mathrm{i})\frac{h}{\delta}\right] .$$

Since $a \gg \delta$, it follows that

$$H' = -(1+\mathrm{i})\frac{\delta}{2a}\exp\left[-(1-\mathrm{i})\frac{h}{\delta}\right]H_0 , \qquad H_0 \gg |H'| .$$

The strong reduction in the field is due to the fact that the currents induced in the shell give rise to an additional field in the cavity which acts in the opposite direction.

7.47

$$j = \frac{2iJ_0\mu\sigma\omega}{c^2ka}\frac{\cos k(h-x)}{\sin kh},$$

where x is measured from the surface along the inward radial direction;

$$R = \frac{1}{2\pi c\delta\sigma}\left\{\frac{\sinh(2h/\delta)+\sin(2h/\delta)}{2[\sinh^2(h/\delta)+\sin^2(h/\delta)]}\right\}.$$

When $\delta \ll h$, the hollow and solid conductors have the same resistance.

7.48 Consider the cylindrical system of coordinates illustrated in figure 7.48.1. For a weak skin effect, the component of the magnetic field which is tangential to the wall of the tube at the surface S of this wall must satisfy the condition

$$H_{2\tau} - H_{1\tau} = \frac{4\pi}{c}i, \tag{7.48.1}$$

where $i = \sigma hE = \zeta E$ is the surface current and ζ is the surface conductance.

The electric field will clearly have a single component only (z component) and should be continuous across S:

$$E_1 = E_2 = E.$$

The remaining steps of the solution are very similar to those in the solution of problem 3.31. To within terms of the order of l/a, the equation of the boundary may be written in the form

$$r = a + l\cos\phi.$$

The vector potential, whose direction is parallel to that of the current, will be sought in the form

$$A_1 = -\frac{2J}{c}\ln\frac{r}{a} + C_1 r\cos\phi + C,$$

$$A_2 = -\frac{2J'}{c}\ln\frac{r}{a} + \frac{B_1}{r}\cos\phi,$$

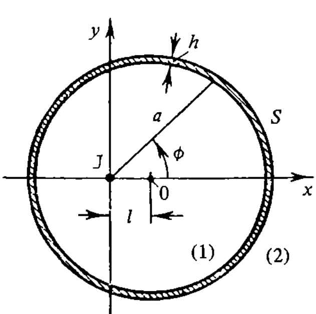


Figure 7.48.1.

where C_1 and B_1 are functions of the time and are of the first order with respect to l/a, and J' is of zero order with respect to l/a.

For a weak skin effect ($h \ll \delta$) the vector potential satisfies the condition

$$A_1 = A_2\,, \qquad \text{for} \quad r = a + l\cos\phi\,.$$

Hence when terms of the order of $(l/a)^2$ are neglected

$$B_1 = a^2C_1 + \frac{2(J'-J)l}{c}\,, \qquad C = 0\,.$$

The quantity H_τ in the boundary condition (7.48.1) may be replaced by H_ϕ. It can be easily verified that this will give rise to an error of the order of $(l/a)^2$. Since

$$H_\phi = -\frac{\partial A}{\partial r}\,, \qquad i = \zeta E = -\frac{\zeta}{c}\frac{\partial A}{\partial t}\,,$$

we have on S

$$\frac{\partial A_1}{\partial r} - \frac{\partial A_2}{\partial r} = -\frac{4\pi\zeta}{c^2}\frac{\partial A_1}{\partial t}\,,$$

or, to within terms of the order of l/a,

$$\frac{2(J'-J)}{ca} + 2C_1\cos\phi = \frac{4\pi\zeta}{c^2}\left[\frac{2}{ca}\frac{\mathrm{d}(Jl)}{\mathrm{d}t} + a\frac{\mathrm{d}C_1}{\mathrm{d}t}\right]\cos\phi\,.$$

It follows at once that $J = J'$, which is due to the fact that the skin effect was assumed to be weak. The differential equation for C_1 is

$$\frac{\mathrm{d}C_1}{\mathrm{d}t} + \rho C_1 = \frac{2}{a^2c}\frac{\mathrm{d}(Jl)}{\mathrm{d}t}\,. \tag{7.48.2}$$

The parameter $\rho = c^2/(2\pi a\zeta)$ is equal to the resistance per unit length of the tube in electromagnetic units.

The solution of equation (7.48.2) may easily be obtained by the method of variation of arbitrary constants. It is of the form

$$C_1 = \frac{2}{ca^2}\int_{-\infty}^{t}\exp[\rho(\tau - t)]\frac{\mathrm{d}}{\mathrm{d}\tau}[J(\tau)l(\tau)]\,\mathrm{d}\tau\,,$$

where it has been assumed that the current is zero as $t \to -\infty$.

The force f per unit length of the current J may be calculated from the formula

$$f_x = -\frac{1}{c}JH_y'\,,$$

where H_y' is the magnetic field along the straight line parallel to the current J due to the current flowing in the tube. The vector potential for this field is

$$A' = C_1 r \cos\phi = C_1 y \ ,$$

and hence

$$H'_y = -\frac{\partial A'}{\partial y} = -C_1 \ .$$

Finally,

$$f_x = \frac{2J(t)}{c^2a^2}\int_{-\infty}^{t} \exp[\rho(\tau - t)]\frac{\mathrm{d}}{\mathrm{d}\tau}[J(\tau)l(\tau)]\,\mathrm{d}\tau \ .$$

Consider some special cases. When J = constant,

$$f_x = \frac{2J^2}{c^2a^2}\int_{-\infty}^{t} \exp[\rho(\tau - t)]\dot{l}(\tau)\,\mathrm{d}\tau \ .$$

When the current departs from the axis of the cylinder ($\dot{l} > 0$) there will be a force tending to prevent this departure. For a slow motion ($\ddot{l} \ll \rho\dot{l}$), integration by parts will yield

$$f_x = \frac{2J^2}{c^2a^2}\left(\frac{\dot{l}}{\rho} - \frac{\ddot{l}}{\rho^2} + \ldots\right) \ .$$

In particular, for a uniform displacement $l = vt$ the retarding force will be

$$f_x = \frac{2J^2v}{c^2a^2\rho} \ .$$

7.49

$$f_x = \frac{2J^2(t)l(t)}{c^2a^2} \ .$$

Propagation of electromagnetic waves

a Plane waves in a homogeneous medium. Reflection and refraction. Wavepackets

8.1 The amplitude of the first wave is $\boldsymbol{E}_1 = a\boldsymbol{e}_x$, the amplitude of the second wave is $\boldsymbol{E}_2 = b\exp(\mathrm{i}\chi)\boldsymbol{e}_y$; a and b are real. The resultant amplitude is

$$\boldsymbol{E}_0 = \boldsymbol{E}_1+\boldsymbol{E}_2 = a\boldsymbol{e}_x + b\exp(\mathrm{i}\chi)\boldsymbol{e}_y .$$

In order to discuss the polarisation it is convenient to shift the origin from which the phases are measured so that the vibrations taking place along the mutually perpendicular directions differ in phase by $\frac{1}{2}\pi$. Consider a new amplitude given by $\boldsymbol{E}_0' = \boldsymbol{E}_0\exp(\mathrm{i}\alpha) = \boldsymbol{E}' - \mathrm{i}\boldsymbol{E}''$, where the vectors $\boldsymbol{E}'$ and $\boldsymbol{E}''$ are real and $(\boldsymbol{E}'\cdot\boldsymbol{E}'') = 0$ (figure 8.1.1). Hence

$$\left.\begin{aligned}\boldsymbol{E}' &= a\boldsymbol{e}_x\cos\alpha + b\boldsymbol{e}_y\cos(\alpha-\chi)\\ \boldsymbol{E}'' &= a\boldsymbol{e}_x\sin\alpha + b\boldsymbol{e}_y\sin(\alpha-\chi)\end{aligned}\right\} . \tag{8.1.1}$$

The phase difference α may be obtained from the condition $(\boldsymbol{E}'\cdot\boldsymbol{E}'') = 0$:

$$a^2\cos\alpha\sin\alpha + b^2\sin(\alpha-\chi)\cos(\alpha-\chi) = 0 ,$$

so that

$$\tan 2\alpha = \frac{b^2\sin 2\chi}{a^2+b^2\cos 2\chi} . \tag{8.1.2}$$

Having determined α from equation (8.1.2), we can substitute it into equation (8.1.1) and determine $\boldsymbol{E}'$ and $\boldsymbol{E}''$. Let us now introduce new axes $x' \parallel \boldsymbol{E}'$ and $y' \parallel \boldsymbol{E}''$ in the xy plane. The components along these axes are then given by

$$\begin{aligned}E_{x'} &= E'\cos[(\boldsymbol{k}\cdot\boldsymbol{r}) - \omega t + \alpha] ,\\ E_{y'} &= E''\sin[(\boldsymbol{k}\cdot\boldsymbol{r}) - \omega t + \alpha] .\end{aligned}$$

It is clear that $E_{x'}^2/E'^2 + E_{y'}^2/E''^2 = 1$, i.e. the end point of the vector $\boldsymbol{E}$ describes an ellipse.

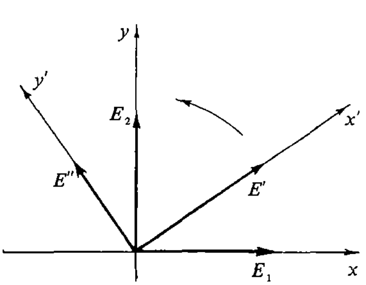
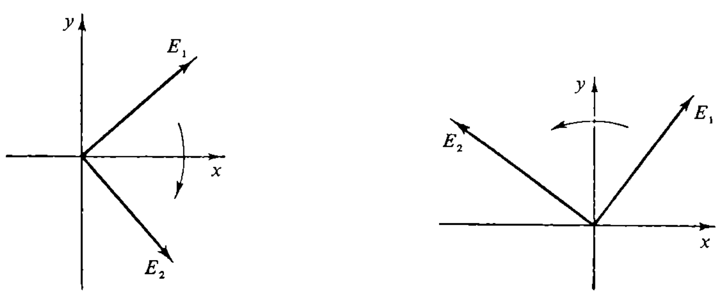


Figure 8.1.1.

In general $\boldsymbol{E}', \boldsymbol{E}'' \neq 0$. The vibrations along the x'-axis lead the vibrations along the y'-axis by $\frac{1}{2}\pi$. If the x', y', z system forms a right-handed set (figure 8.1.1), then for an observer looking along the negative direction of the z-axis, the vector $\boldsymbol{E}$ will rotate in the anticlockwise direction. The polarisation is then said to be elliptical and laevorotatory. If the x'-, y'-, z-axes form a left-handed system, then $\boldsymbol{E}$ will rotate in a clockwise direction and the elliptical polarisation will be dextrorotatory. Circular and linear polarisation corresponds to $\boldsymbol{E}' = \boldsymbol{E}''$ and $\boldsymbol{E}' = 0$ or $\boldsymbol{E}'' = 0$, respectively.

8.2 When $\chi = 0$, the polarisation is linear and the plane of polarisation contains the bisector of the angle between the x- and y-axes. When $\chi = \pi$, the polarisation is also linear and the plane of polarisation passes through the bisector of the angle between the x- and y-axes. When $\chi = \frac{1}{2}\pi$, the polarisation is circular and dextrorotatory (figure 8.2.1a). When $\chi = -\frac{1}{2}\pi$, the polarisation is circular and laevorotatory (figure 8.2.1b). In the remaining cases the polarisation is elliptical and dextrorotatory when $0 < \chi < \pi$ $[\cos\frac{1}{2}\chi > 0,\ \sin\frac{1}{2}\chi > 0]$, with axes as shown in figure 8.2.1a and laevorotatory when $-\pi < \chi < 0$ (figure 8.2.1b).



(a) (b)

Figure 8.2.1.

8.3 When $a = b$ the polarisation is linear. When $a > b$ the polarisation is elliptical and dextrorotatory. When $a < b$ the polarisation is elliptical and laevorotatory. Circular polarisation occurs only when $b = 0$ (dextrorotatory) or $a = 0$ (laevorotatory).

8.4

$$P = [1 - 4|I_{ik}|\{\mathrm{Tr}(I_{ik})\}^{-2}]^{1/2},$$

where $|I_{ik}|$ is the determinant of the tensor I_{ik}. When $|I_{ik}| = 0$, the degree of polarisation $P = 1$.

8.6 We introduce rectangular axes $x' \parallel \boldsymbol{a}$ and $y' \parallel \boldsymbol{b}$. In terms of these axes the complex amplitude of the field is

$$\boldsymbol{E}_0 = a\boldsymbol{e}_{x'} \pm \mathrm{i}b\boldsymbol{e}_{y'},$$

where the plus sign corresponds to right-hand and the minus sign to left-hand elliptical polarisation. The phase has been chosen to be zero for the x'-component of the field. The intensity is $I = a^2+b^2$. If we now express the unit vectors $\boldsymbol{e}_{x'}$, $\boldsymbol{e}_{y'}$, in terms of $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ we get for the components I_{ik}:

$$\begin{aligned} I_{11} &= a^2\cos^2\vartheta + b^2\sin^2\vartheta\,,\\ I_{22} &= a^2\sin^2\vartheta + b^2\cos^2\vartheta\,,\\ I_{12} &= (a^2-b^2)\sin\vartheta\cos\vartheta \mp 2\mathrm{i}ab = I_{21}^*\,. \end{aligned}$$

The upper sign corresponds to right-hand and the lower sign to left-hand elliptical polarisation. When $b = 0$ the polarisation is linear and the tensor I_{ik} becomes

$$I_{ik} = I\begin{pmatrix} \cos^2\vartheta & \sin\vartheta\cos\vartheta \\ \sin\vartheta\cos\vartheta & \sin^2\vartheta \end{pmatrix}.$$

When $a = b = (\frac{1}{2}I)^{1/2}$ the polarisation is circular and

$$I_{ik} = \tfrac{1}{2}I\begin{pmatrix} 1 & \mp\mathrm{i} \\ \pm\mathrm{i} & 1 \end{pmatrix}.$$

8.7 The amplitude of the resultant wave is

$$\boldsymbol{E} = \boldsymbol{E}_1+\boldsymbol{E}_2 = E[\boldsymbol{e}^{(1)}+\boldsymbol{e}^{(2)}\exp(\mathrm{i}\alpha)]\,,$$

where α is a randomly varying phase difference and $|E|^2 = I$. By definition, the components of the polarisation tensor are given by [see equation (8.a.14)]

$$I_{ik} = \overline{E_i E_k^*} = \overline{I[e^{(1)}+e^{(2)}\exp(\mathrm{i}\alpha)]_i[e^{(1)}+e^{(2)}\exp(-\mathrm{i}\alpha)]_k}\,.$$

Averaging with respect to time yields $\overline{\exp(\pm\mathrm{i}\alpha)} = 0$ and hence the polarisation tensor becomes

$$I_{ik} = I\begin{pmatrix} 1+\cos^2\vartheta & \sin\vartheta\cos\vartheta \\ \sin\vartheta\cos\vartheta & 1-\cos^2\vartheta \end{pmatrix}.$$

Hence, using the result from problem 8.4 we get

$$P = |\cos\vartheta|\,.$$

We can obtain the same result by diagonalising the tensor I_{ik}. Equating the determinant of the set of equations (8.a.16) to zero we find that $I_1 = 1+|\cos\vartheta|$, $I_2 = 1-|\cos\vartheta|$. Hence we get again

$$P = \frac{I_1-I_2}{I_1+I_2} = |\cos\vartheta|\,.$$

The basis vectors are $\boldsymbol{e}_1 = (\cos\frac{1}{2}\vartheta, \sin\frac{1}{2}\vartheta)$ and $\boldsymbol{e}_2 = (-\sin\frac{1}{2}\vartheta, \cos\frac{1}{2}\vartheta)$. They are real in the case considered.

The resulting wave consists of an unpolarised part of intensity $I(1-|\cos\vartheta|)$ and a linearly polarised part along the direction of

$e_1 = (\cos\frac{1}{2}\vartheta, \sin\frac{1}{2}\vartheta)$ with an intensity $I|\cos\vartheta|$:

$$(I_{ik}) = I(1-|\cos\vartheta|)(\delta_{ik})+I|\cos\vartheta|\begin{pmatrix}\cos^2\frac{1}{2}\vartheta & \sin\frac{1}{2}\vartheta\cos\frac{1}{2}\vartheta\\ \sin\frac{1}{2}\vartheta\cos\frac{1}{2}\vartheta & \sin^2\frac{1}{2}\vartheta\end{pmatrix}.$$

The resulting wave is completely polarised (but not monochromatic) when $\vartheta = 0$. When $\vartheta = \frac{1}{2}\pi$ it is completely unpolarised.

8.8 The polarisation tensor is given by

$$I_{ik} = \begin{pmatrix}I_1+\frac{1}{2}I_2 & \frac{1}{2}I_2\\ \frac{1}{2}I_2 & \frac{1}{2}I_2\end{pmatrix},$$

where the x_1-axis lies along the direction of polarisation of the first wave. The degree of polarisation is

$$P = \frac{2(I_1^2+I_2^2)^{1/2}}{I_1+I_2+(I_1^2+I_2^2)^{1/2}}$$

The resulting wave consists of an unpolarised wave of intensity $\frac{1}{2}(I_1+I_2)(1-P)$ and a linearly polarised wave. The angle between the direction of the linear polarisation and the direction of the polarisation of the first wave is

$$\vartheta = \arctan\frac{2I_2(I_1^2+I_2^2)^{1/2}}{I_1(I_1+I_2)+(3I_1+2I_2)(I_1^2+I_2^2)^{1/2}}$$

8.9

$$\rho = \frac{1-\xi}{1+\xi}.$$

When $\xi = 0$ the wave is unpolarised; when $\xi = 1$ it is completely polarised. Substituting $\xi_i = \xi\eta_i$, where $\eta_1^2+\eta_2^2+\eta_3^2 = 1$, we have

$$I_{ik} = \frac{1}{2}I(1-\xi)\delta_{ik}+\frac{1}{2}I\xi\left(1+\sum_{l=1}^{3}\eta_l\tau_{ik}^{(l)}\right).$$

The first term in this expression corresponds to a completely unpolarised state, and the second term corresponds to a completely polarised state. In case (a), $\eta_3 = 1$, $\eta_1 = \eta_2 = 0$.

On comparing

$$I''_{ik} = I\xi\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

with $I_{ik} = In_in_k^*$, we see that in this case $n_1 = 1$, $n_2 = 0$, i.e. the tensor I''_{ik} describes a wave which is linearly polarised along the direction of the x-axis (the waves propagating along the z-axis).

Similarly, it is easy to show that in case (b) $\eta_1 = 1$, $\eta_2 = \eta_3 = 0$ and the wave is linearly polarised along the direction at 45° to the x-axis, whereas in case (c) $\eta_2 = 1$, $\eta_1 = \eta_3 = 0$ and the wave is circularly polarised.

8.10 Since the vector $\boldsymbol{E}$ is linearly polarised, the amplitude $\boldsymbol{E}_0$ may be assumed to be real. Since $\operatorname{div}\boldsymbol{E} = 0$, we have $(\boldsymbol{k}' \cdot \boldsymbol{E}_0) = 0 = (\boldsymbol{k}'' \cdot \boldsymbol{E}_0)$, i.e. $\boldsymbol{E}_0$ is perpendicular to the $\boldsymbol{k}'$, $\boldsymbol{k}''$ plane. Since $\operatorname{curl}\boldsymbol{E} = -(\mu/c)\partial\boldsymbol{H}/\partial t$, it follows that

$$\frac{\mu\omega}{c}\boldsymbol{H}' = [\boldsymbol{k}' \wedge \boldsymbol{E}_0], \qquad \frac{\mu\omega}{c}\boldsymbol{H}'' = [\boldsymbol{k}'' \wedge \boldsymbol{E}_0],$$

i.e. $\boldsymbol{H}'$ and $\boldsymbol{H}''$ are perpendicular to $\boldsymbol{E}_0$ and $\boldsymbol{H}' \perp \boldsymbol{k}'$, $\boldsymbol{H}'' \perp \boldsymbol{k}''$. The end point of the vector $\boldsymbol{H}$ describes an ellipse in the $\boldsymbol{k}'$, $\boldsymbol{k}''$ plane (figure 8.10.1).

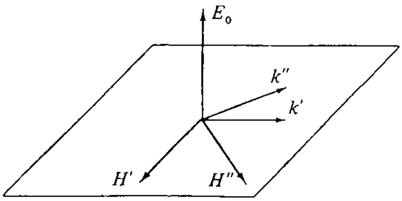


Figure 8.10.1.

8.11 Both waves will be elliptically polarised. One of the principal axes of the polarisation ellipse lies in the plane of incidence while the other is perpendicular to it. The semiaxes are given by the following formulae:

(1) reflected wave:

$$E_{\parallel} = \frac{\tan(\theta_0 - \theta_2)}{\tan(\theta_0 + \theta_2)} E_0, \qquad E_{\perp} = \frac{\sin(\theta_2 - \theta_0)}{\sin(\theta_2 + \theta_0)} E_0;$$

(2) refracted wave:

$$E_{\parallel} = \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_0 + \theta_2)\cos(\theta_0 - \theta_2)} E_0, \qquad E_{\perp} = \frac{2\cos\theta_0 \sin\theta_2}{\sin(\theta_0 + \theta_2)} E_0,$$

where θ_0 is the angle of incidence, θ_2 is the angle of refraction, and E_0 is the absolute magnitude of the amplitude of the incident wave. When $\theta_0 = \frac{1}{2}\pi - \theta_2$ (Brewster's angle), the reflected wave is linearly polarised.

8.12 Unpolarised (natural) light may be looked upon as an incoherent superposition of two waves of 'complementary' polarisation and equal intensity. The incident beam can therefore be regarded as consisting of two incoherent components one of which ($E_{\parallel}$) is polarised in the plane of incidence and the other ($E_{\perp}$) is polarised in the perpendicular plane. The intensities of these waves are equal, so that $I_{\parallel} = I_{\perp} = I$.

After reflection the two components will remain incoherent. Fresnel's formulae then give

$$I_{ik}^{(1)} = I\frac{\sin^2(\theta_0 - \theta_2)}{\sin^2(\theta_0 + \theta_2)}\left[e_i^{\perp} e_k^{\perp} + \frac{\cos^2(\theta_0 + \theta_2)}{\cos^2(\theta_0 - \theta_2)} e_i^{\parallel} e_k^{\parallel}\right],$$

$$\rho_1 = \frac{\cos^2(\theta_0 + \theta_2)}{\cos^2(\theta_0 - \theta_2)} < 1,$$

where $e^{\perp}$ and $e^{\parallel}$ are unit vectors in the direction of the polarisation of the transverse and longitudinal components. These vectors lie in the plane perpendicular to the direction of propagation of the reflected light. The degree of depolarisation of the incident light is 1. The light becomes polarised after reflection.

Similarly, for refracted light we have

$$I_{ik}^{(2)} = \frac{4I\cos^2\theta_0 \sin^2\theta_2}{\sin^2(\theta_0+\theta_2)} \left[e_i^{\perp} e_k^{\perp} + \frac{e_i^{\parallel} e_k^{\parallel}}{\cos^2(\theta_0-\theta_2)}\right],$$

$$\rho_2 = \cos^2(\theta_0-\theta_2) < 1 .$$

8.13

$$R = \frac{(\epsilon_1-\epsilon_2)^2}{2(\epsilon_1+\epsilon_2)}, \qquad \rho_1 = 0, \qquad \rho_2 = \frac{4\epsilon_1\epsilon_2}{(\epsilon_1+\epsilon_2)^2},$$

where ϵ_1 and ϵ_2 are the permittivities of the two dielectrics.

8.14

$$E_{\perp 1} = (-1+2\zeta\cos\theta_0)E_{\perp 0}, \qquad E_{\parallel 1} = \left(1-\frac{2\zeta}{\cos\theta_0}\right)E_{\parallel 0},$$

$$E_{\perp 2} = 2\zeta\cos\theta_0 E_{\perp 0}, \qquad E_{\parallel 2} = 2\zeta E_{\parallel 0} .$$

The expressions for $E_{\parallel 1}$ and $E_{\parallel 2}$ will hold only if the glancing angle is such that $\varphi_0 = \frac{1}{2}\pi - \theta_0 \gg |\zeta|$. The following formulae will hold for $\varphi_0 \ll 1$:

$$E_{\parallel 1} = \frac{\varphi_0-\zeta}{\varphi_0+\zeta}E_{\parallel 0}, \qquad E_{\parallel 2} = \frac{\varphi_0\zeta}{\varphi_0+\zeta}E_{\parallel 0},$$

where $|\zeta|$ and φ_0 are arbitrary.

8.15 $R_{\perp} = 1-4\zeta'\cos\theta_0$.

$R_{\perp}$ is very nearly equal to unity for all angles of incidence and reaches a minimum at $\theta_0 = 0$ (normal incidence). Moreover,

$$R_{\parallel} = 1-\frac{4\zeta'}{\cos\theta_0} \quad \text{when} \quad \varphi_0 = \tfrac{1}{2}\pi-\theta_0 \gg 4\zeta',$$

$$R_{\parallel} = \frac{(\varphi_0-\zeta')^2+\zeta''^2}{(\varphi_0+\zeta')^2+\zeta''^2} \quad \text{when} \quad \varphi_0 \ll 1 .$$

The angle φ_0 at which $R_{\parallel}$ is a minimum may be found from the condition $\partial R_{\parallel}/\partial\varphi_0 = 0$ and the result is

$$\varphi_0 = \Phi_0 = |\zeta|, \qquad R_{\parallel} = \frac{|\zeta|-\zeta'}{|\zeta|+\zeta'} .$$

The angle Φ_0 is an analogue of the Brewster angle since the magnitude of $R_{\parallel}$ is a minimum at $\varphi_0 = \Phi_0$ (when the wave is incident on the boundary of the dielectric at the Brewster angle, the coefficient $R_{\parallel}$ is also a minimum and is in fact equal to zero).

8.16 The polarisation of the reflected wave depends on the phase difference between the longitudinal and the transverse components. By using the results of the two preceding problems, we find that

$$E_{\perp 1} \approx -E_{\perp 0} = \exp(i\delta_\perp)E_{\perp 0}\,, \qquad \delta_\perp = \pi\,;$$

$$E_{\parallel 1} = \left[\frac{|\zeta|-\zeta'}{|\zeta|+\zeta'}\right]^{1/2} \exp(i\delta_\parallel)E_{\parallel 0}\,, \qquad \tan\delta_\parallel = -\frac{2\Phi_0\zeta''}{\Phi_0^2-|\zeta|^2} \to \infty\,,$$

i.e. $\delta_\parallel = \frac{1}{2}\pi$.

Thus, the phase difference is $\delta = \delta_\perp - \delta_\parallel = \frac{1}{2}\pi$. In general, the reflected wave is elliptically polarised and one of the axes of the ellipse lies in the plane of incidence. When $|E_{\parallel 1}| = |E_{\perp 1}|$ the polarisation is circular. When $E_{\parallel 0} = 0$ or $E_{\perp 0} = 0$ the polarisation is linear.

8.17 Fresnel's formulae give

$$n' = \frac{\sin\theta_0 \tan\theta_0 \cos 2\rho}{1+\sin^2 2\rho\cos\delta}\,, \qquad n'' = \frac{\sin\theta_0\tan\theta_0\sin 2\rho\sin\delta}{1+\sin 2\rho\cos\delta}\,.$$

8.18

$$R = \frac{(\sqrt{\epsilon}-\sqrt{\epsilon'})^2}{(\sqrt{\epsilon}+\sqrt{\epsilon'})^2} + \frac{4}{(\sqrt{\epsilon}+\sqrt{\epsilon'})^4}\left(\frac{\epsilon'}{\epsilon}\right)^{1/2}\left(\frac{2\pi\sigma}{\omega}\right)^2,$$

where ϵ is the permittivity of the medium from which the light emerges, and ϵ' is the real part of the permittivity of the conducting medium.

8.19 The phase difference between $E_{\perp 1}$, E_0 and $E_{\parallel 1}$, E_0 may be determined from the Fresnel formulae:

$$\tan\tfrac{1}{2}\delta_\perp = \frac{(\sin^2\theta_0-n^2)^{1/2}}{\cos\theta_0}\,, \qquad \tan\tfrac{1}{2}\delta_\parallel = \frac{(\sin^2\theta_0-n^2)^{1/2}}{n^2\cos\theta_0}\,. \tag{8.19.1}$$

Since $\delta_\perp \neq \delta_\parallel$, the wave is elliptically polarised.

The polarisation will become circular when the following conditions are satisfied

$$\delta = \delta_\parallel - \delta_\perp = \tfrac{1}{2}\pi\,; \qquad E_{\parallel 0} = E_{\perp 0}\,.$$

The second of these conditions requires that the incident wave should be polarised in the plane making an angle of $\frac{1}{4}\pi$ with the plane of incidence. Let us consider whether the first of the above two conditions can be satisfied. From equation (8.19.1) we have

$$\tan\tfrac{1}{2}\delta = \frac{\cos\theta_0(\sin^2\theta_0-n^2)^{1/2}}{\sin^2\theta_0}\,.$$

It follows that $\delta = 0$ when $\theta_0 = \arcsin n$ and $\theta_0 = \frac{1}{2}\pi$. The maximum value of δ occurs between these two points. It is easy to show that $\tan\frac{1}{2}\delta_{max} = (1-n^2)/(2n)$. In order that $\tan\frac{1}{2}\delta$ should be equal to unity ($\delta = \frac{1}{2}\pi$), the following inequalities must be satisfied: $1-n^2 \geqslant 2n$, $n \leqslant 0{\cdot}414$.

8.20 When the vector $\boldsymbol{E}_0$ is normal to the plane of incidence, the transverse and longitudinal components of Poynting's vector are given by

$$\left.\begin{aligned} S_\perp &= \frac{c^2 k''}{8\pi\omega} E_0^2 \exp(-2k''z)\sin 2(k'x-\omega t)\,, \\ S_\parallel &= \frac{c^2 k''}{8\pi\omega} E_0^2 \exp(-2k''z)[1-\cos 2(k'x-\omega t)]\,, \end{aligned}\right\} \tag{8.20.1}$$

where the z-axis is normal to the boundary of the media and the x-axis is the line of intersection of the plane of incidence and the separation boundary,

$$k' = k_2 \sin\theta_0\,, \qquad k'' = k_2(\sin^2\theta_0 - n^2)^{1/2}\,,$$

where $k_2 = \omega n_2/c$ is the wavevector in the second medium and θ_0 is the angle of incidence.

It is clear from equation (8.20.1) that the frequency of the vibrations in the direction perpendicular to the boundary is 2ω. The time average of the energy flux entering the second medium is zero. The average magnitude of $S_\parallel$ is not zero, so that there is a finite flow of energy along the separation boundary.

The equation for the Poynting vector lines in the second medium is

$$z = \frac{1}{k''}\ln\frac{|\sin k'x|}{C}\,,$$

where C is a constant of integration. Figure 8.20.1 illustrates the form of these lines. In the first medium the S lines have a more complicated form.

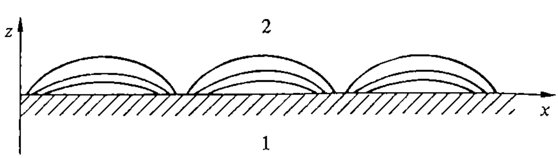


Figure 8.20.1.

8.21 We find from the Fresnel formulae equations (8.a.19) and (8.a.20) that as $\theta_0 \to \frac{1}{2}\pi$ the amplitude of the penetrating wave $E_1 \to \theta$ and the amplitude of the reflected wave $E_2 \to -E_0$. This means that a plane monochromatic wave cannot propagate along the dividing boundary of two dielectrics.

8.22 In this case the law of refraction is

$$k_1 \sin\theta_0 = k_2 \sin\theta_2\,, \qquad k_1 = \frac{\omega}{c}(\epsilon_1)^{1/2}\,,$$

$$k_2 = \frac{\omega}{c}\left(\epsilon_2' + \mathrm{i}\frac{4\pi\sigma_2}{\omega}\right)^{1/2} = k_2' + \mathrm{i}k_2''\,,$$

where $\sin\theta_2$ and $\cos\theta_2$ are complex quantities. Let $\cos\theta_2 = \rho\exp(i\alpha)$, where ρ and α are real constants which are functions of θ_0 and of the electrical constants of the medium. The parameters ρ and α can be determined from

$$\rho^2\cos 2\alpha = 1-\frac{k_1^2}{|k_2|^2}\sin^2\theta_0\,, \qquad \rho^2\sin 2\alpha = \frac{2k_1^2 k_2' k_2''}{|k_2|^4}\sin^2\theta_0\,.$$

The wave in the second conducting medium 2 is

$$\boldsymbol{E}_2(\boldsymbol{r}, t) = \boldsymbol{E}_2\exp\{i[(k_2\boldsymbol{e}_2\cdot\boldsymbol{r})-\omega t]\}\,.$$

If the real and imaginary parts of the product $(k_2\boldsymbol{e}_2\cdot\boldsymbol{r})$ are separated, we have

$$(k_2\boldsymbol{e}_2\cdot\boldsymbol{r}) = (k_2'+ik_2'')(x\sin\theta_2+z\cos\theta_2) = izp(\theta_0)+xk_1\sin\theta_0+zq(\theta_0)\,,$$

where

$$p(\theta_0) = \rho(k_2'\sin\alpha+k_2''\cos\alpha)\,, \qquad q(\theta_0) = \rho(k_2'\cos\alpha-k_2''\sin\alpha)\,.$$

Thus

$$\boldsymbol{E}_2(\boldsymbol{r}, t) = \boldsymbol{E}_2\exp(-pz)\exp[i(xk_1\sin\theta_0+zq-\omega t)]\,.$$

It is clear that the directions of propagation and damping are not the same. The planes of constant amplitude z = constant are parallel to the surface of the conductor. The planes of constant phase are given by

$$xk_1\sin\theta_0+zq(\theta_0) = \text{constant}\,,$$

from which it follows that the vector $\boldsymbol{k}_2'$ which defines the direction of propagation of the wave makes an angle $\psi = \arctan[k_1\sin\theta_0/q(\theta_0)]$ with the z-axis (figure 8.22.1). The phase velocity in the conducting medium is a function of the angle of incidence and is given by

$$v_{\text{ph}} = \frac{\omega}{[q^2(\theta_0)+k_1^2\sin^2\theta_0]^{1/2}}\,.$$

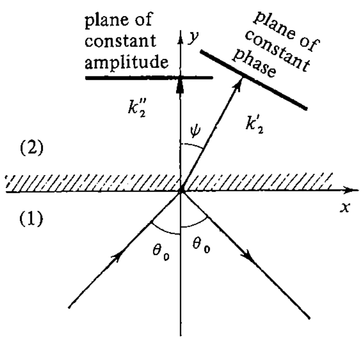


Figure 8.22.1.

8.23 In order to determine the reflection coefficient for the plane layer it is necessary to find the relation between the amplitudes of the reflected and incident waves. This relation may be obtained by two methods.

In the first method it is determined from the boundary conditions. Since the tangential components of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ are continuous across the boundaries $z = 0$ and $z = a$, and since in front of the layer there are waves propagating in both directions while behind the layer there is only the transmitted wave propagating in the positive direction of the z-axis, we have from the boundary conditions

$$E_1 = \frac{\alpha_{12}+\alpha_{23}\exp(-2ik_2a)}{1+\alpha_{12}\alpha_{23}\exp(-2ik_2a)}E_0\,, \tag{8.23.1}$$

where E_1 and E_0 are the amplitudes of the reflected and incident waves respectively, and

$$\alpha_{12} = \frac{1-n_{12}}{1+n_{12}}\,, \qquad \alpha_{23} = \frac{1-n_{23}}{1+n_{23}}\,, \qquad n_{ik} = \left(\frac{\epsilon_k}{\epsilon_i}\right)^{1/2}, \qquad k_2 = \frac{\omega}{c}(\epsilon_2)^{1/2}\,.$$

The second method of solution is based on a consideration of multiple reflections from the separation boundaries. If we use the Fresnel formulae at normal incidence, we find that the amplitude of the wave which has been reflected from the boundary $z = 0$ only once is $E_0' = \alpha_{12}E_0$. The amplitude of the wave which has entered the layer is $E_\alpha' = \beta_{12}E_0$ where

$$\beta_{12} = \frac{2}{1+n_{12}}\,.$$

The amplitude of the wave passing from the layer into the region $z < 0$ after single reflection at the $z = a$ plane is

$$E_1' = \beta_{21}\beta_{12}\alpha_{23}E_0\exp(-2ik_2a)\,.$$

The amplitude of the wave entering the region $z < 0$ after s reflections at $z = a$ is

$$E_s' = \beta_{21}\beta_{12}\alpha_{23}\exp(-2ik_2a)[\alpha_{21}\alpha_{23}\exp(-2ik_3a)]^{s-1}\,.$$

The total amplitude E_1 of the wave reflected from the plane layer is equal to the sum of all the E_s:

$$E_1 = \sum_{s=0}^{\infty} E_s' = \alpha_{12}E_0+\beta_{21}\beta_{12}\alpha_{23}\exp(-2ik_2a)\sum_{s=1}^{\infty}[\alpha_{21}\alpha_{23}\exp(-2ik_2a)]^{s-1}\,.$$

By using the formula for the sum of an infinite geometric progression, this expression can be shown to be identical with equation (8.23.1).

The reflection coefficient is defined by $R = |E_1|^2/|E_0|^2$. The minimum of R may be found in the usual way. The reflection coefficient is a minimum if the thickness of the layer satisfies the condition

$$a = a_n = \tfrac{1}{4}n\lambda_2\,, \qquad n = 1, 2, 3, \ldots\,,$$

where λ_2 is the wavelength inside the layer.

Consider the minimum thickness of the layer $a = \frac{1}{4}\lambda_2$. When R is equated to zero, we find that the condition for the absence of reflection is

$$\epsilon_2 = (\epsilon_1\epsilon_3)^{1/2} .$$

8.24 The equation satisfied by the electric field is [see equation (8.a.12)]

$$\frac{d^2E}{dz^2}+\frac{\omega^2}{c^2}\left(\epsilon-\frac{\Delta\epsilon}{\epsilon^{z/a}+1}\right)E = 0 . \tag{8.24.1}$$

It is required to find the solution of this problem which is bounded for all z and satisfies the appropriate conditions as $z \to \pm\infty$. When $z \to -\infty$, the solution should be

$$E(z) \to A\exp(ik_0z)+B\exp(-ik_0z) , \tag{8.24.2}$$

where $k_0 = \omega/c$. This is a superposition of two waves, namely, the incident and the reflected waves. When $z \to +\infty$, there should only be the transmitted wave, so that

$$E(z) \to C\exp(ikz) , \tag{8.24.3}$$

where $k = \omega\epsilon^{1/2}/c$.

Let us substitute $\xi = -\exp(-z/a)$ into equation (8.24.1). The new variable is such that $-\infty \leqslant \xi \leqslant 0$ when z varies from $-\infty$ to $+\infty$. If we use the substitution $E(\xi) = \xi^{-ika}\psi(\xi)$ we obtain the following equation for the new unknown function $\psi(\xi)$:

$$\xi(1-\xi)\frac{d^2\psi}{d\xi^2}+(1-2ika)(1-\xi)\frac{d\psi}{d\xi}+\kappa^2a^2\psi = 0 , \tag{8.24.4}$$

where $\kappa^2 = \omega^2\Delta\epsilon/c^2$. This is the hypergeometric equation.

It follows from equation (8.24.3) that the function $\psi(\xi)$ should tend to a constant limit when $\xi \to 0$. The hypergeometric function which is a solution of equation (8.24.4) and satisfies the above conditions is given by

$$F(\alpha, \beta, \gamma, z) = 1+\frac{\alpha\beta}{\gamma(1!)}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)(2!)}z^2+\dots .$$

The solution of equation (8.24.4) may therefore be written in the form

$$\psi = CF\left[-i(k+k_0)a,\ -i(k-k_0)a,\ 1-2ika,\ -\exp\left(-\frac{z}{a}\right)\right] .$$

In order to find the form of the function ψ when $\xi \to -\infty$, we shall use the asymptotic expression [see Gradshteyn and Ryzhik (1965) equation (7.232,2)]

$$F(\alpha, \beta, \gamma, \xi) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-\xi)^{-\alpha}+\frac{\Gamma(\gamma)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-\xi)^{-\beta} .$$

An examination of this equation will show that the condition given by equation (8.24.2) is satisfied.

The reflection coefficient is therefore given by

$$R = \left|\frac{B}{A}\right|^2 = \left|\frac{\Gamma(2ik_0a)\Gamma[1-i(k+k_0)a]\Gamma[-i(k+k_0)a]}{\Gamma(-2ik_0a)\Gamma[1-i(k-k_0)a]\Gamma[-i(k-k_0)a]}\right|^2 .$$

This equation may be simplified if we recall that

$$\left|\frac{\Gamma(2ik_0a)}{\Gamma(-2ik_0a)}\right| = \left|\frac{\Gamma(2ik_0a)}{\Gamma^*(2ik_0a)}\right| = 1 \qquad \text{and} \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z} .$$

The final expression for R is therefore

$$R = \frac{\sinh^2\pi a(k-k_0)}{\sinh^2\pi a(k+k_0)} .$$

For small a ($ka \ll 1$), the expression for the reflection coefficient becomes identical with the well-known expression which holds for a step change in ϵ:

$$R = \frac{(k-k_0)^2}{(k+k_0)^2} .$$

R is a monotonically decreasing function of a. For large ka, the reduction is exponential:

$$R = \exp(-4\pi k_0 a) , \qquad ka \gg 1 .$$

8.25 For normal incidence on a nonhomogeneous layer, the electric field is a function of z only and satisfies the equation

$$\frac{d^2E}{dz^2} + \frac{\omega^2}{c^2}\epsilon(\omega, z)E = 0 . \tag{8.25.1}$$

Let $z_1 = m\omega^2/(4\pi e^2 N_0)$ so that $\epsilon = 1 - z/z_1$. On substituting $\xi = (\omega^2/c^2 z_1)^{1/3}(z_1 - z)$ into equation (8.25.1) we have[(1)]

$$\frac{d^2E}{d\xi^2} + \xi E = 0 . \tag{8.25.2}$$

The simplest way to obtain the solution of equation (8.25.2) is to expand $E(\xi)$ into a Fourier integral:

$$E(\xi) = \int_{-\infty}^{\infty} E(u)\exp(i\xi u)\, du , \qquad E(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(\xi)\exp(-i\xi u)\, d\xi .$$

Substitution into equation (8.25.2) yields the first-order differential equation

$$\frac{dE(u)}{du} + iu^2E(u) = 0 .$$

The solution of this equation is

$$E(u) = A'\exp(-\tfrac{1}{3}iu^3) ,$$

(1) A similar equation is used in quantum mechanics to describe the motion of a particle in a uniform field.

whence

$$E(\xi) = A'\int_{-\infty}^{\infty} \exp[-i(\tfrac{1}{3}u^3 - \xi u)]\,du\,.$$

By rewriting $\exp -i(\frac{1}{3}u^3 - \xi u)$ in its trigonometric form and noting that the integral of $\sin(\frac{1}{3}u^3 - \xi u)$ is zero, we have

$$E(\xi) = \frac{A}{\sqrt{\pi}}\int_0^{\infty} \cos(\tfrac{1}{3}u^3 - \xi u)\,du\,.$$

The function

$$\Phi(\xi) = \frac{1}{\sqrt{\pi}}\int_0^{\infty} \cos(\tfrac{1}{3}u^3 + \xi u)\,du$$

is known as Airy's function. Airy's function may be expressed in terms of Bessel functions of order $\frac{1}{3}$. Hence, finally,

$$E(\xi) = A\Phi(-\xi)\,.$$

The constant A should be determined from the boundary conditions.

Consider now the behaviour of $E(\xi)$ at large $|\xi|$. By using the asymptotic form of $\Phi(\xi)$ we have, for large positive ξ,

$$E(\xi) = \frac{A}{\xi^{1/4}}\sin(\tfrac{2}{3}\xi^{3/2} + \tfrac{1}{4}\pi)\,,$$

so that the field is oscillatory in character.

For large negative ξ

$$E(\xi) = \frac{A}{2|\xi|^{1/4}}\exp(-\tfrac{2}{3}|\xi|^{3/4})\,,$$

so that the field falls off exponentially. The reason for this is that negative ξ corresponds to negative values of the permittivity ϵ. When $\epsilon < 0$, the wavevector $k = \omega\epsilon^{1/2}/c$ is purely imaginary, i.e. absorption takes place. However, this absorption is not a process in which the electromagnetic energy is converted into heat. In fact, the absorption gives rise to a reflection of the wave from a layer with negative ϵ. This is so because the permittivity is real, and therefore there are no losses.

8.26 $\Psi(x, 0) = A(x, 0)\exp(ik_0 x)\,,$

where $A(x, 0) = a_0\sqrt{\pi}\Delta k\exp(-\frac{1}{4}x^2\Delta k^2)$. The amplitude $A(x, 0)$ of the wavepacket is Gaussian. It becomes negligibly small when $|x\Delta k| \gg 1$. It follows that the width of the packet in ordinary space is related to its 'width' in k-space by $\Delta k\Delta x \approx 1$. This is a universal relation which holds for electromagnetic or any other wavepackets. It is of particular importance for probability waves in quantum mechanics, since it leads to the Heisenberg relation for the space and momentum coordinates of fundamental particles.

8.27 $\Psi(0, t) = A(0, t)\exp(-i\omega_0 t)$,

where $A(0, t) = a_0\sqrt{\pi}\Delta\omega\exp(-\frac{1}{4}t^2\Delta\omega^2)$, and $\Delta t\Delta\omega \approx 1$.

8.28

$$\Delta x_{\min} = \frac{\lambda}{2\pi\sin\theta} ,$$

where θ is the angle of the cone formed by the rays drawn from the objective of the microscope to the object under consideration.

8.29 The wavepulse sent out by a radar set has a width Δx which is related to the transverse spread $k_\perp$ of the wavevectors by $(\Delta x)k_\perp \geqslant 1$. On the other hand, it is clear that $\Delta x/l \approx k_\perp/k$. Hence the uncertainty in the position of the object is $\Delta x \geqslant (l\lambda)^{1/2}$.

8.30 The wavepacket is described by

$$\Psi(\boldsymbol{r}, t) = 4\pi a_0\left(\frac{\pi q^3}{2\rho^3}\right)^{1/2} J_{1/2}(\rho q)\exp\{i[(\boldsymbol{k}_0\cdot\boldsymbol{r}) - \omega_0 t]\} ,$$

where $J_{1/2}(x) = (2/\pi x)^{1/2}[\sin x/(x-\cos x)]$ is a Bessel function and $\rho = |\boldsymbol{r}-\boldsymbol{v}_g t|$. The group velocity $\boldsymbol{v}_g = \partial\omega/\partial\boldsymbol{k}$ is a vector with components $\partial\omega/\partial k_x$, $\partial\omega/\partial k_y$, $\partial\omega/\partial k_z$. The amplitude of the wavepacket is now appreciably different from zero only in the (spherically symmetric) region $\rho q \leqslant 1$.

It is clear from the expression for $\Psi(\boldsymbol{r}, t)$ that the shape of the wavepacket is independent of time. This is due to the linear dispersion law which is strictly true only for waves travelling in a vacuum. When higher terms in the expansion of ω as a function of k are taken into account, the packet will spread out and its form will no longer be time independent. The packet as a whole moves with the group velocity $\boldsymbol{v}_g$.

8.31 Writing the function $\omega(k)$ in the form

$$\omega = \omega_0 + v_g(k-k_0) + \beta(k-k_0)^2 ,$$

we have

$$\Psi(x, t) = a_0\left(\frac{\pi}{\alpha+i\beta t}\right)^{1/2}\exp\left[-\frac{(x-v_g t)^2}{4(\alpha+i\beta t)} + i(k_0 x - \omega_0 t)\right] .$$

It is convenient to consider the square of the modulus of this expression:

$$|A(x, t)|^2 = \frac{\pi a_0^2}{(\alpha^2+\beta^2 t^2)^{1/2}}\exp\left[-\frac{\alpha(x-v_g t)^2}{2(\alpha^2+\beta^2 t^2)}\right],$$

which determines the intensity of the wave. It is clear from this expression that the intensity of the wave as a function of x at given t is a Gaussian curve whose width l is given by

$$l = \left[\frac{2(\alpha^2+\beta^2 t^2)}{\alpha}\right]^{1/2} .$$

Thus the width increases with time, but the height of the packet decreases, owing to the presence of the factor $(\alpha^2+\beta^2 t^2)^{-1/2}$.

The wavepacket spreads out symmetrically both in the $t = +\infty$ and $t = -\infty$ directions. This spread is not associated with the absorption of energy because k is real. The absence of dissipation is clear from the fact that the integral $\int_{-\infty}^{+\infty} |A(x, t)|^2 \, dx = (\pi/2\alpha)^{1/2} a_0^2$ is independent of time, i.e. the 'total intensity' is conserved. The reason for the spread of the wavepacket is that the phase velocities $v_{ph} = \omega/k$ of the various plane waves entering into the superposition which represents the wavepacket are different; owing to dispersion, the ratio ω/k is a function of k.

8.32 When $\omega \ll \omega_0$

$$v_{ph} = \frac{c}{\sqrt{\epsilon_0}}\left(1 - \frac{\omega_p^2\omega^2}{2\epsilon_0\omega_0^4}\right) < c\,, \qquad v_g = \frac{c}{\sqrt{\epsilon_0}}\left(1 - \frac{3\omega_p^2\omega^2}{2\epsilon_0\omega_0^4}\right) < c\,,$$

where $\epsilon_0 = \epsilon(0)$. When $\omega \gg \omega_0$

$$v_{ph} = c\left(1 + \frac{\omega_p^2}{2\omega^2}\right) > c\,, \qquad v_g = c\left(1 - \frac{\omega_p^2}{2\omega^2}\right) < c\,.$$

In the latter case $v_{ph} v_g \approx c^2$. Near the resonance frequency $(\omega \approx \omega_0)$, the concept of group velocity loses its significance.

8.33 It follows from the solution of problems 8.30 and 8.31 that the function describing the wavepacket is

$$E(x, t) = E_0(x, t) \exp[i(k_0 x - \omega_0 t)]\,.$$

The amplitude $E_0(x, t)$ varies much more slowly than the function $\exp[i(k_0 x - \omega_0 t)]$ (the periods of these functions are in the ratio of $\Delta k/k_0$). If the change in E_0 as compared with $\exp[i(k_0 x - \omega_0 t)]$ is neglected, we find from Maxwell's equations that

$$H(x, t) = \frac{k_0 c}{\omega_0 \mu} E(x, t)$$
$$= \left(\frac{\epsilon}{\mu}\right)^{1/2} E_0(x, t) \exp[i(k_0 x - \omega_0 t)]\,.$$

The energy flux averaged over the period $2\pi/\omega_0$ of the high frequency component is given by

$$\bar{\gamma}(x, t) = \frac{c}{8\pi} |\mathrm{Re}[E \wedge H^*]| = \frac{c}{8\pi}\left(\frac{\epsilon}{\mu}\right)^{1/2} E_0(x, t) E_0^*(x, t)\,.$$

Since $\bar{\gamma} = v\bar{w}$, the energy transport velocity is

$$v = \frac{c}{(d/d\omega)[\omega(\epsilon\mu)^{1/2}]} = \frac{d\omega}{dk} = v_g\,.$$

b Plane waves in anisotropic and gyrotropic media

8.34

$$\cos\alpha = \frac{(\epsilon_\parallel - \epsilon_\perp)\sin\theta\cos\theta}{(\epsilon_\parallel^2\cos^2\theta + \epsilon_\perp^2\sin^2\theta)^{1/2}}\,, \qquad \tan\vartheta = \frac{\epsilon_\perp}{\epsilon_\parallel}\tan\theta\,.$$

8.35 The boundary conditions for the field vectors will be satisfied at all points on the boundary of separation when the tangential components of the wavevectors of the incident, reflected, and both refracted waves, are equal at the boundary of separation. For the ordinary wave this gives

$$k_0\sin\theta_0 = k_1\sin\theta_2'\,, \qquad \frac{\sin\theta_2'}{\sin\theta_0} = (\epsilon_\perp\mu)^{1/2}\,.$$

The ray direction (the direction of the Poynting vector) in the ordinary wave is the same as the direction of the wavevector, and is therefore at an angle θ_2' to the normal to the boundary. In the case of the extraordinary wave

$$k_0\sin\theta_0 = k_2\sin\theta_2'' = k_0\sin\theta_2''\left(\frac{\epsilon_\perp\epsilon_\parallel\mu}{\epsilon_\perp\sin^2\theta_2'' + \epsilon_\parallel\cos^2\theta_2''}\right)^{1/2}$$

[see equation (8.b.2)]. Hence,

$$\sin^2\theta_2'' = \frac{\epsilon_\parallel\sin^2\theta_0}{\epsilon_\perp\epsilon_\parallel\mu + (\epsilon_\parallel - \epsilon_\perp)\sin^2\theta_0}\,.$$

The angle ϑ'' between the ray and the optic axis (which is parallel to the normal to the separation boundary) is, in accordance with the result of the preceding problem, given by

$$\tan\vartheta'' = \frac{\epsilon_\perp}{\epsilon_\parallel}\tan\theta_2'' = \frac{(\epsilon_\perp)^{1/2}\sin\theta_0}{[\epsilon_\parallel(\epsilon_\parallel\mu - \sin^2\theta_0)]^{1/2}}\,.$$

The angle of reflection from the crystal is equal to the angle of incidence, just as for an isotropic medium.

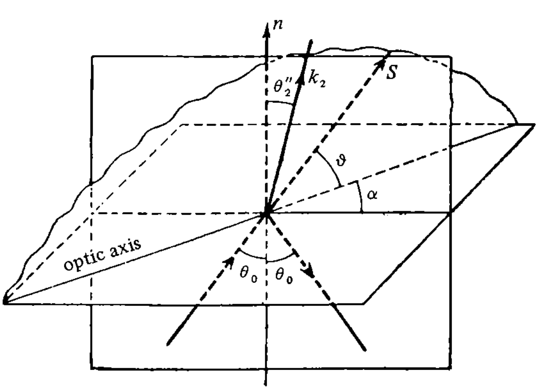


Figure 8.36.1.

8.36 The ordinary ray lies in the plane of incidence, and makes an angle θ_2' with the normal to the surface, where

$$\sin\theta_2' = (\epsilon_\perp \mu)^{1/2} \sin\theta_0 .$$

The wavevector $\boldsymbol{k}_2$ of the extraordinary wave lies in the plane of incidence and makes an angle θ_2'' with the normal, which is given by

$$\sin^2\theta_2'' = \frac{\epsilon_\perp \sin^2\theta_0}{\epsilon_\perp \epsilon_\parallel \mu + (\epsilon_\perp - \epsilon_\parallel)\sin^2\theta_0 \cos^2\alpha} .$$

The ray direction in the extraordinary wave does not lie in the plane of incidence. The ray direction lies in the same plane as $\boldsymbol{k}_2$ and the optic axis, and makes an angle ϑ with the latter, where (figure 8.36.1)

$$\tan\vartheta = \frac{[\epsilon_\perp^2 \epsilon_\parallel \mu + \epsilon_\perp(\epsilon_\perp - \epsilon_\parallel)\sin^2\theta_0 \cos^2\alpha]^{1/2}}{\epsilon_\parallel \sin\theta_0 \cos\alpha} .$$

8.37 Substitution of the plane wave expressions for $\boldsymbol{E}$ and $\boldsymbol{H}$ into Maxwell's equations (8.a.1) to (8.a.4) produces the equation for the amplitudes and the wavevectors of waves which may propagate in the given medium:

$$[\boldsymbol{k} \wedge [\boldsymbol{k} \wedge \boldsymbol{H}_0]] = -\frac{\omega^2 \epsilon}{c^2} \hat{\mu} \boldsymbol{H}_0 . \qquad (8.37.1)$$

Let θ be the angle between the wavevector $\boldsymbol{k}$ and the z-axis, and consider the components of (8.37.1) along the coordinate axes.

A biquadratic equation in k is obtained by equating the determinant of the system to zero. The solution of this equation is

$$k_{1,2}^2 = \frac{\omega^2 \epsilon \mu_\parallel}{2c^2} \, \frac{\mu \sin^2\theta + (2\mu_\perp/\mu_\parallel) \pm [\mu^2 \sin^4\theta + (2\mu_a/\mu_\parallel)^2 \cos^2\theta]^{1/2}}{(\mu_\perp/\mu_\parallel - 1)\sin^2\theta + 1} , \qquad (8.37.2)$$

where

$$\mu = \frac{\mu_\perp^2 - \mu_a^2 - \mu_\perp \mu_\parallel}{\mu_\parallel^2} .$$

Two waves can propagate in each direction with different phase velocities $v_{1,2} = \omega/k_{1,2}$. These velocities are functions of the angle θ. There are no directions for which the two phase velocities are equal, since the expression in square brackets in equation (8.37.2) does not vanish for any value of θ. With $\mu_a = 0$ in equation (8.37.2), the latter will give the phase velocities of waves which can propagate in a nongyrotropic but magnetically anisotropic crystal:

$$k_1^2 = \frac{\omega^2}{c^2} \epsilon \mu_\perp , \qquad k_2^2 = \frac{\omega^2}{c^2} \left(\frac{\epsilon \mu_\perp \epsilon_\parallel}{\mu_\parallel \cos^2\theta + \mu_\perp \sin^2\theta} \right) .$$

The first of these waves (ordinary wave) has a velocity $v_1 = c/(\epsilon\mu_\perp)^{1/2}$ which is independent of the direction of propagation. The velocity of the second wave (extraordinary wave) is a function of the angle between the axis of symmetry of the crystal and the direction of propagation. The two velocities are equal, and the two waves become indistinguishable, when the direction of propagation is along the axis of symmetry ($\theta = 0$).

8.38 Two waves can propagate in any direction with phase velocities $v_{1,2} = \omega/k_{1,2}$, where $k_{1,2}$ is given by equation (8.37.2), in which the magnetic quantities should be replaced (for the purposes of this problem) by the corresponding electrical quantities.

8.39 A plane wave propagating in the direction of a constant magnetic field will split into two waves with opposite circular polarisations and different phase velocities $v_\pm = c[\epsilon(\mu_\perp \pm \mu_a]^{-1/2}$. When the propagation takes place at right angles to the constant magnetic field, one of the waves [velocity $v = c/(\epsilon\mu_\parallel)^{1/2}$] will be purely transverse ($\boldsymbol{E} \perp \boldsymbol{k}$, $\boldsymbol{H} \perp \boldsymbol{k}$). It is analogous to the waves which propagate in an isotropic medium with scalar parameters ϵ and $\mu = \mu_\parallel$. In the case of the second wave {velocity $v = c[\mu_\perp/\epsilon(\mu_\perp^2 - \mu_a^2)]^{1/2}$} the vector $\boldsymbol{E}$ will be parallel to the constant magnetic field, and the vector $\boldsymbol{H}$ will have a component in the direction of propagation. Thus, a wave with an arbitrary polarisation will split into two linearly polarised waves.

All the results obtained in this problem remain valid when ϵ is a Hermitian tensor and μ is a scalar. All that is required is that the magnetic quantities should be replaced by the corresponding electrical quantities, and vice versa.

8.40 It was shown in the solution of the preceding problem that two waves with different phase velocities and opposite circular polarisations can propagate in the direction of a magnetic field. It follows that a wave whose polarisation is not circular will split into two circularly polarised waves. Since the phase velocities of the two waves are different, the phase difference between them will vary from point to point, and hence the polarisation of the resultant wave will be different at different points.

A detailed calculation will show that the polarisation of the resultant wave will remain linear, but the plane of polarisation will rotate through an angle $\chi = \frac{1}{2}(k_+ - k_-)z$. This is known as the Faraday effect. The quantities k_+ and k_- are the wavevectors of the two circularly polarised waves, and may be found from the solutions of problems 8.39 and 6.22. For a weak magnetic field

$$\chi = VHz\,,$$

where V is the Verdet constant. When the atoms of the material may be looked upon as harmonic oscillators, the constant V is given by

$$V = \frac{2\pi e^3 N}{nm^2 c}\,\frac{\omega^2}{(\omega^2 - \omega_0^2)^2}\,,$$

where $n = \epsilon^{1/2}$ is the refractive index in the absence of the magnetic field.

8.41 It follows from symmetry considerations that the wavevectors of the reflected and transmitted waves should be perpendicular to the separation boundary. Both waves will be circularly polarised in the same direction as the incident wave. The amplitude of the reflected wave is

$$H_1 = \frac{(\epsilon)^{1/2} - (\mu_\perp \pm \mu_a)^{1/2}}{(\epsilon)^{1/2} + (\mu_\perp \pm \mu_a)^{1/2}} H_0 ,$$

where H_0 is the amplitude of the incident wave, ϵ is the permittivity, and $\mu_\perp$ and μ_a are the components of the permeability tensor of the ferrite (see problem 8.37). The amplitude of the transmitted wave is given by

$$H_2 = \frac{2(\epsilon)^{1/2}}{(\epsilon)^{1/2} + (\mu \pm \mu_a)^{1/2}} H_0 ,$$

where the signs + and − correspond to waves with right- and left-handed circular polarisations.

8.42 The wavevectors of the reflected and transmitted waves are perpendicular to the separation boundary. The reflected wave is elliptically polarised with semiaxes

$$H_1' = H_0 \frac{\epsilon - (\mu^2 - \mu_a^2)^{1/2}}{[(\epsilon)^{1/2} + (\mu + \mu_a)^{1/2}][(\epsilon)^{1/2} + (\mu - \mu_a)^{1/2}]} ,$$

$$H_1'' = H_0 \frac{[\epsilon(\mu - \mu_a)]^{1/2} - [\epsilon(\mu + \mu_a)]^{1/2}}{[(\epsilon)^{1/2} + (\mu + \mu_a)^{1/2}][(\epsilon)^{1/2} + (\mu - \mu_a)^{1/2}]} .$$

The direction of H_1' is the same as the direction of polarisation of the vector $\boldsymbol{H}$ of the incident wave. The transmitted wave splits into two waves with amplitudes

$$H_2' = \frac{H_0(\epsilon)^{1/2}}{(\epsilon)^{1/2} + (\mu + \mu_a)^{1/2}} , \qquad H_2'' = \frac{H_0(\epsilon)^{1/2}}{(\epsilon)^{1/2} + (\mu - \mu_a)^{1/2}} ,$$

which are circularly polarised in opposite directions. Their velocities of propagation are different (see solution of problem 8.39).

8.43 If the wavelength is much smaller than the radius of the discs and smaller than the distance between neighbouring discs, then the artificial dielectric may be looked upon as a continuous medium. The electric field of the incident wave is parallel to the planes of the discs. Hence, in the absence of the external magnetic field, $\boldsymbol{H}_0$, the polarisability of the dielectric will be $\alpha = N\beta_e$, where $\beta_e = 4a^3/3\pi$ is the longitudinal (relative to the plane of the disc) electrical polarisability of a disc, and N is the number of discs per unit volume.

The longitudinal magnetic polarisability of a disc β_m is zero (see problem 7.41) and hence the magnetic susceptibility of the dielectric for the given direction of the magnetic field is also zero.

The presence of the external magnetic field $\boldsymbol{H}_0$ gives rise to the Hall effect. Thus, conduction electrons which make up the current in each disc will be deflected under the action of the field $\boldsymbol{H}_0$ and will give rise to an additional electric field $\boldsymbol{E}_\mathrm{H}$ which should balance the deflecting effect of the magnetic field. This gives rise to an additional electric moment in each disc which in turn produces a change in the polarisation vector in the medium and in the electric induction. To calculate the change in the induction, it is convenient to consider the total density of the polarisation current $\partial \boldsymbol{P}/\partial t$ in the dielectric, rather than the current in an isolated disc.

On the first approximation in H_0, the field $\boldsymbol{E}_\mathrm{H}$ due to the Hall effect is given by

$$\boldsymbol{E}_\mathrm{H} = R[\boldsymbol{H}_0 \wedge \boldsymbol{j}] = R\left[\boldsymbol{H}_0 \wedge \frac{\partial \boldsymbol{P}}{\partial t}\right],$$

where R is the Hall constant and $\boldsymbol{P} = \alpha \boldsymbol{E}$ is the polarisation on the zero-order approximation. Owing to the presence of the field $\boldsymbol{E}_\mathrm{H}$, the polarisation vector will increase by

$$\Delta \boldsymbol{P} = \alpha \boldsymbol{E}_\mathrm{H} = \alpha^2 R\left[\boldsymbol{H}_0 \wedge \frac{\partial \boldsymbol{E}}{\partial t}\right],$$

and hence the induction $\boldsymbol{D}$ will be given by

$$\boldsymbol{D} = \boldsymbol{E} + 4\pi(\boldsymbol{P} + \Delta \boldsymbol{P}) = \epsilon \boldsymbol{E} + 4\pi\alpha^2 R\left[\boldsymbol{H}_0 \wedge \frac{\partial \boldsymbol{E}}{\partial t}\right], \qquad (8.43.1)$$

where $\epsilon = 1 + 4\pi N\beta_\mathrm{e}$ is the permittivity in the absence of the external magnetic field.

When the electric field $\boldsymbol{E}$ is a simple harmonic function of time, equation (8.43.1) will give the following relation between $\boldsymbol{D}$ and $\boldsymbol{E}$

$$\boldsymbol{D} = \epsilon \boldsymbol{E} + \mathrm{i}[\boldsymbol{E} \wedge \boldsymbol{g}],$$

where $\boldsymbol{g} = 4\pi\alpha^2 \omega R \boldsymbol{H}_0$ is the gyration vector [see equation (8.b.4)]. Thus the medium will be gyrotropic. It follows from the results of problem 8.39 that two waves can propagate in the direction of vector $\boldsymbol{g}$. The two waves are circularly polarised in opposite directions, and have different phase velocities $v_\pm = \omega/k_\pm$. On calculating $k_\pm$ in the usual way we find that

$$k_\pm^2 = \frac{\omega^2}{c^2}(\epsilon \pm g).$$

8.44 A wave for which the electric vector is parallel to the conductors is reflected from the lattice as from a continuous metal plane. A wave for which the electric vector is at right angles to the conductors will propagate as in free space since it does not excite any currents in the lattice.

8.45 The solution of the Maxwell equations will be sought in the form of plane waves. The amplitude $\boldsymbol{E}_0$ of such waves must satisfy the following equations

$$[k \wedge E_0] = \frac{\omega}{c} H_0 , \qquad [k \wedge H_0] = -\frac{\omega}{c} \epsilon(\omega) E_0 .$$

However, for a longitudinal electric field $[k \wedge E_0] = 0$, and hence $H_0 = 0$ and $\epsilon(\omega) E_0 = 0$.

It follows from the latter result that a longitudinal electric field will exist provided

$$\epsilon(\omega) = 0 .$$

The frequencies ω_α of the longitudinal oscillations are determined by this equation and are, as a rule complex, $\omega_\alpha = \tilde{\omega}_\alpha - i\gamma_\alpha$. This means that oscillations, once they are excited, will be damped. If the condition $\gamma_\alpha \ll \tilde{\omega}_\alpha$ is satisfied, the damping during one period of the oscillations is small. Such oscillations will be long-lived.

In the case of a plasma with dielectric permittivity

$$\epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)}$$

(see problem 6.16) the frequency of the longitudinal oscillations is

$$\omega_0 = (\omega_p^2 + \tfrac{1}{4}\gamma^2)^{1/2} - \tfrac{1}{2} i\gamma .$$

As $\gamma \to 0$ this frequency becomes the plasma frequency:

$$\omega_0 = \omega_p = \left(\frac{4\pi e^2 N}{m}\right)^{1/2}$$

According to this formula, the frequency ω is independent of the wavevector, and hence the group velocity of longitudinal plasma waves is zero. However, this result holds only on the first approximation, and arises because the spatial nonuniformity of the electric field is not taken into account. Longitudinal plasma waves consist of oscillations of the electron cloud relative to the ion cloud (the ions are assumed to be fixed in the present approximation).

8.46 $E(x, z, t) = E_0 \exp[-\alpha|x| + i(kz - \omega t)]$,

where the frequency ω is determined from the condition $\epsilon(\omega) = -1$: $\omega = \omega_p/\sqrt{2}$.

The damping coefficient α can be expressed in terms of the wavevector k:

$$\alpha = k\left[1 - \tfrac{1}{2}\left(\frac{\omega_p}{kc}\right)^2\right]^{1/2} .$$

In the case of a slow wave $\alpha \approx k$. The wavevector k can have any magnitude. The amplitude E_0 has components

$$E_{0y} = 0 , \qquad E_{0x} = \pm \frac{ik}{\alpha} E_{0z} \approx \pm i E_{0z} ,$$

where the plus (minus) sign corresponds to the region $x > 0$ $(x < 0)$. The polarisation is thus nearly circular and the vector $\boldsymbol{E}$ rotates in the xz-plane. The amplitude of the magnetic field $\boldsymbol{H}_0$ $(0, H_{0y}, 0)$ is small compared to that of E_0: $H_{0y} = E_{0z}\omega/(kc) \ll E_{0z}$, which is characteristic for plasma oscillations. The wave considered is called a plasma surface wave.

8.47 It follows from the solution of problem 8.39 that two waves with opposite circular polarisations can propagate in the direction of the magnetic field. The wavevectors of these waves are given by (see problem 6.25)

$$\frac{k^2c^2}{\omega^2} = \epsilon_\pm = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma \mp \omega_H - \omega_H\Omega_H/\omega)} \, .$$

When $\Omega_H \ll \omega$ the effect of the motion of the positive ions is very small and may be neglected. In the opposite case, namely $\Omega_H \gg \omega$ and $\gamma\omega \ll \omega_H\Omega_H$, the positive ions play the leading role in the process, and

$$\frac{k^2c^2}{\omega^2} = 1 + \frac{\omega_p^2}{\omega_H\Omega_H} = 1 + \frac{4\pi NMc^2}{H_0^2} \, .$$

Both waves propagate with the same phase velocity v_{ph}, which is equal to their group velocity v_g:

$$v_g = v_{\text{ph}} = \frac{\omega}{k} = \frac{c}{(1 + 4\pi NMc^2/H_0^2)^{1/2}} \tag{8.47.1}$$

or

$$v_g = v_{\text{ph}} = \frac{H_0}{(4\pi NM)^{1/2}} = \frac{H_0}{(4\pi\tau)^{1/2}} \, . \tag{8.47.2}$$

The latter result holds provided that the second term in the denominator of equation (8.47.1) is $\gg 1$. The quantity $\tau = NM$ is the density of the gas (the electron mass can clearly be neglected). If the motion of the positive ions is not taken into account, then instead of the finite constant velocity given by equation (8.47.2), we would have a zero velocity for $\omega \to 0$, and the corresponding waves would not be able to propagate. Thus, mechanical oscillations of a gas and the oscillations of the electromagnetic field are in this case very closely related. Waves propagating with the velocity given by equation (8.47.2) are known as magnetohydrodynamic waves. They are of particular importance in astrophysical and other processes.

8.48 The linearised equation relating the amplitudes of the high-frequency components of the magnetisation ($\boldsymbol{m}_0$) and the magnetic field ($\boldsymbol{h}_0$) can be deduced from equations (6.c.2) and (6.c.3) and is

$$i\omega\boldsymbol{m}_0 = -\gamma[\boldsymbol{M}_0 \wedge \boldsymbol{h}_0] - \gamma[\boldsymbol{m}_0 \wedge \boldsymbol{H}_0] + \gamma q k^2[\boldsymbol{M}_0 \wedge \boldsymbol{m}_0] \, , \tag{8.48.1}$$

where $\boldsymbol{M}_0$ is the saturation magnetisation and is parallel to the magnetic field $\boldsymbol{H}_0$. If the z-axis is chosen to be parallel to $\boldsymbol{H}_0$ ($z = x_3$), then the components of the tensor μ_{ik} are [see equation (8.48.1)]:

$$\mu_{11} = \mu_{22} = 1+\frac{\omega_M(\omega_0+\alpha k^2)}{(\omega_0+\alpha k^2)-\omega^2}, \tag{8.48.2}$$

$$\mu_{12} = -\mu_{21} = -\mathrm{i}\frac{\omega\omega_M}{(\omega_0+\alpha k^2)^2-\omega^2}, \qquad \mu_{33} = 1,$$

where

$$\omega_0 = \gamma H_0, \qquad \omega_M = 4\pi\gamma M_0, \qquad \alpha = q\gamma M_0.$$

The remaining components of μ_{ik} are zero.

As can be seen from equation (8.48.2), the magnetic permeability is now a function of both frequency and the wavevector. This is due to the fact that the magnetisation at each point depends on the magnitude of the magnetic field not only at that particular point but also at the neighbouring points (this is because of the term $q\nabla^2\boldsymbol{M}$ in the expression for $\boldsymbol{H}_{\text{eff}}$). The dependence of the permittivity or the magnetic permeability on the wavevector is called spatial dispersion. The dependence of μ on $\boldsymbol{k}$ is significant only for very nonuniform fields (short wavelengths).

8.49 Consider the plane wave solutions of the Maxwell equations and the equation of motion of the magnetisation vector (6.c.2):

$$\boldsymbol{E} = \boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})-\omega t], \qquad \boldsymbol{H} = \boldsymbol{H}_0+\boldsymbol{h}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})-\omega t],$$

$$\boldsymbol{M} = \boldsymbol{M}_0+\boldsymbol{m}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})-\omega t].$$

The amplitudes of the fields and of the magnetisation satisfy the following system of equations

$$c[\boldsymbol{k}\wedge\boldsymbol{h}_0] = -\omega\epsilon\boldsymbol{E}_0, \qquad c[\boldsymbol{k}\wedge\boldsymbol{E}_0] = \omega(\boldsymbol{h}_0+4\pi\boldsymbol{m}_0),$$

$$(\boldsymbol{k}\cdot[\boldsymbol{h}_0+4\pi\boldsymbol{m}_0]) = 0, \tag{8.49.1}$$

$$\mathrm{i}\omega\boldsymbol{m}_0 = -\gamma[\boldsymbol{M}_0\wedge\boldsymbol{h}_0]-\gamma[\boldsymbol{m}_0\wedge\boldsymbol{H}_0]+\gamma qk^2[\boldsymbol{M}_0\wedge\boldsymbol{m}_0]. \tag{8.49.2}$$

On eliminating $\boldsymbol{E}_0$ and $\boldsymbol{h}_0$ from equations (8.49.1) and (8.49.2) and substituting

$$u = \frac{\omega_M}{\Omega}, \qquad x = \frac{\omega}{\Omega}, \qquad \xi = \frac{ck}{\Omega\sqrt{\epsilon}}, \qquad \Omega = \omega_0+\omega_1+\omega_M,$$

$$\omega_0 = \gamma H_0, \qquad \omega_1 = \gamma qk^2M_0, \qquad \omega_M = 4\pi\gamma M_0,$$

we have

$$\mathrm{i}x\boldsymbol{m}_0 = \frac{u}{x^2-\xi^2}\{x^2[\boldsymbol{e}_z\wedge\boldsymbol{m}_0]+\xi^2(\boldsymbol{n}\cdot\boldsymbol{m}_0)[\boldsymbol{e}_z\wedge\boldsymbol{n}]\}+(1-u)[\boldsymbol{e}_z\wedge\boldsymbol{m}_0], \tag{8.49.3}$$

where $\boldsymbol{n} = \boldsymbol{k}/k$, $\boldsymbol{e}_z$ is a unit vector in the direction of $\boldsymbol{H}_0$, and $\boldsymbol{M}_0$ is parallel to $\boldsymbol{H}_0$.

Take the x-axis in the $\boldsymbol{n}$, $\boldsymbol{e}_z$ plane and denote the angle between $\boldsymbol{e}_z$ and $\boldsymbol{n}$ by θ. The following system of linear equations is then obtained from equation (8.49.3):

$$\mathrm{i}x m_{0x} + \left(1+\frac{u\xi^2}{x^2-\xi^2}\right) m_{0y} = 0\,,$$

$$\left(1+\frac{u\xi^2}{x^2-\xi^2}\cos^2\theta\right) m_{0x} - \mathrm{i}x m_{0y} = 0\,.$$

The condition that these equations should be consistent yields the required dispersion relation:

$$\left(1+\frac{u\xi^2}{x^2-\xi^2}\right)\left(1+\frac{u\xi^2}{x^2-\xi^2}\cos^2\theta\right) - x^2 = 0\,.$$

This equation is of the third degree in ω^2 ($\omega^2 = \Omega^2 x^2$; Ω is independent of ω) and hence three types of wave with different dispersion relations can propagate in the medium under consideration. Two of these dispersion relations were investigated in the solution of problem 8.37, where it was assumed that $\omega_1 = 0$. They correspond to ordinary electromagnetic waves propagating in a gyrotropic medium. To investigate the third type of wave, let us suppose that $\omega^2\epsilon/c^2k^2 \ll 1$ ($x^2 \ll \xi^2$). If x^2 is neglected in comparison with ξ^2 in the denominator of equation (8.49.3), we have the third dispersion relation

$$\omega^2 = (\omega_0+\omega_1)(\omega_0+\omega_1+\omega_M\sin^2\theta)\,, \tag{8.49.4}$$

where $\omega_1 = q\gamma k^2 M_0$. Since $\omega^2\epsilon \ll c^2k^2$ we find, assuming that ω_0, ω_1, and ω_M are comparable in magnitude, that the dispersion law (8.49.4) will hold provided $\xi^2 \gg 1$.

Consider now the relative magnitudes of E_0 and h_0 for waves with the dispersion relation (8.49.4). Using the Maxwell equation (8.49.1) and the condition $\omega^2\epsilon/c^2k^2 \ll 1$ we have

$$\boldsymbol{E}_0 \approx \frac{4\pi\omega}{ck^2}[\boldsymbol{k}\wedge\boldsymbol{m}]\,; \qquad \boldsymbol{h}_0 \approx 4\pi\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{m})\,.$$

Thus, $E_0 \ll h_0$. The waves under consideration are purely magnetic oscillations of the magnetisation vector, for which the electric field does not appear. They are known as spin waves, and determine many magnetic, thermal, and electrical properties of ferromagnetics.

8.50 Let the y-axis be perpendicular to the surface of the metal (positive in the inward direction) and let the z-axis lie along the magnetic field. Since the impedance ζ is independent of the angle of incidence of the wave, we shall consider normal incidence only. Solution of Maxwell's equations and the definition of surface impedance yield

$$\zeta_{xx} = (1-\mathrm{i})\left(\frac{\omega\mu_\parallel}{8\pi\sigma}\right)^{1/2}, \qquad \zeta_{zz} = (1-\mathrm{i})\left(\frac{\omega\mu}{8\pi\sigma_3}\right)^{1/2}, \qquad \zeta_{xz} = \zeta_{zx} = 0\,,$$

where

$$\sigma = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1}, \qquad \mu = \frac{\mu_\perp^2 - \mu_a^2}{\mu_\perp}.$$

The dependence of ζ_{zz} on frequency has a resonance character (see problem 6.35, in which the components of μ_{ik} are calculated). The component ζ_{xx} does not exhibit resonance properties, since $\mu_\parallel = 1$.

8.51

$$\zeta_\pm = \pm\frac{E_{\pm 1}}{h_{\pm 1}} = -(1-\mathrm{i})\left(\frac{\omega\mu^\pm}{8\pi\sigma^\pm}\right)^{1/2},$$

where $\mu^\pm = \mu_\perp \pm \mu_a$, $\sigma^\pm = \sigma_1 \pm \sigma_2$; $E_{\pm 1}$ and $h_{\pm 1}$ are the cyclic components of $\boldsymbol{E}$ and $\boldsymbol{h}$ [$h_{\pm 1} = \mp(h_x \pm \mathrm{i}h_y)/\sqrt{2}$].

c Scattering of electromagnetic waves by macroscopic bodies. Diffraction

8.52 It is convenient to introduce cylindrical coordinates with the z-axis lying along the axis of the cylinder, and the angle α measured from the direction of the wavevector $\boldsymbol{k}$ of the incident wave. In view of the symmetry of the problem, the field vectors are independent of z, and the only finite components are E_z, H_r, and H_ϕ. In what follows we shall omit the time factor $\exp(-\mathrm{i}\omega t)$ and will use the wave equation (8.a.6) for $\boldsymbol{E}$ and the Maxwell equation (8.a.1). The first of these equations will give E_z, and the second will give H_r and H_ϕ in terms of E_z:

$$H_r = \frac{1}{\mathrm{i}kr}\frac{\partial E_z}{\partial \phi}, \qquad H_\phi = -\frac{1}{\mathrm{i}k}\frac{\partial E_z}{\partial r}. \tag{8.52.1}$$

The secondary field $E' = E - E_0$, which is due to the presence of the cylinder, satisfies the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E'}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E'}{\partial \phi^2} + k^2 E' = 0. \tag{8.52.2}$$

Let $E' = R(r)\Phi(\phi)$ and separate the variables in equation (8.52.2):

$$\frac{\mathrm{d}^2 R_m}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R_m}{\mathrm{d}r} + \left(k^2 - \frac{m^2}{r^2}\right)R_m = 0, \tag{8.52.3}$$

$$\frac{\mathrm{d}^2\Phi_m}{\mathrm{d}\phi^2} + m^2\Phi_m = 0, \tag{8.52.4}$$

where m^2 is the separation constant. The general solution of equation (8.52.2) is then the sum of the special solutions over all the allowed values of m:

$$E'(r, \phi) = \sum_m \Phi_m(\phi)R_m(r).$$

In order to obtain the solution of the Bessel equation [equation (8.52.3)] in a form convenient for our purposes, consider the boundary condition for $r \to \infty$. Since E' represents the secondary field, which is due to the currents induced in the cylinder, it follows that when $r \to \infty$ this field will take the form of cylindrical waves, travelling in the outward direction. This means that E' should, in this region, be of the form

$$E' = E_0 f(\phi) \frac{\exp(\mathrm{i}kr)}{r^{1/2}} . \tag{8.52.5}$$

The latter condition will be satisfied if the Hankel function $H_m^{(1)}(kr)$ is taken as the solution of equation (8.52.2). For large r this function is of the form (see appendix 3)

$$H_m^{(1)}(kr) = \left(\frac{2}{\pi kr}\right)^{1/2} \exp[\mathrm{i}(kr - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi)] \qquad (kr \gg 1) .$$

The second linearly independent solution will contain a term of the form constant $\times\, r^{-1/2}\exp(-\mathrm{i}kr)$ which represents a cylindrical wave flowing in the inward direction. This wave cannot appear under the conditions of the present problem. Hence, the solution of equation (8.52.3) may be written in the form $R_m(r) = H_m^{(1)}(kr)$. Equation (8.52.4) has the following solution:

$$\Phi_m(\phi) = A_m \exp(\mathrm{i}m\phi) + B_m \exp(-\mathrm{i}m\phi) .$$

Since the field remains unaltered when ϕ changes by 2π, the number m should be an integer. If it is considered that m may assume both positive and negative values, then the final solution will be of the form

$$E'(r, \phi) = E_0' \sum_{m=-\infty}^{\infty} A_m H_m^{(1)}(kr) \exp(\mathrm{i}m\phi) . \tag{8.52.6}$$

At large distances equation (8.52.6) will become identical to equation (8.52.5) and

$$f(\phi) = \left(\frac{2}{\pi k}\right)^{1/2} \sum_m A_m \exp[\mathrm{i}(m\phi - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi)] .$$

The coefficients A_m in the series given by equation (8.52.6) may be determined from the boundary condition for the surface of the cylinder. Since the cylinder is a perfect conductor, it follows that

$$E' + E_0 = 0 \qquad \text{when} \quad r = a$$

or

$$\exp(\mathrm{i}ka\cos\phi) + \sum_{m=-\infty}^{\infty} A_m H_m^{(1)}(ka) \exp(\mathrm{i}m\phi) = 0 .$$

Since the functions $\exp(\mathrm{i}m\phi)$ are orthogonal, we have

$$\int_0^{2\pi} \exp[\mathrm{i}(ka\cos\phi - m'\phi)]\, \mathrm{d}\phi + 2\pi A_{m'} H_{m'}^{(1)}(ka) = 0 ,$$

and hence, by equation (A3.11), we have

$$A_m = \frac{\mathrm{i}^m J_m(ka)}{H_m^{(1)}(ka)} .$$

The resultant electric field is therefore given by

$$E(r, \alpha) = E'_0 \exp(\mathrm{i}kr\cos\phi) - E'_0 \sum_m \frac{\mathrm{i}^m J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(kr) \exp(\mathrm{i}m\phi) .$$

The components of the magnetic field can be determined with the aid of equation (8.52.1) and are given by

$$H_r = -E'_0 \sin\phi \exp(\mathrm{i}kr\cos\phi) - E'_0 \sum_m \frac{\mathrm{i}^m m J_m(ka)}{H_m^{(1)}(ka)} \frac{H_m^{(1)}(kr)}{kr} \exp(\mathrm{i}m\phi) ,$$

$$H_\phi = -E'_0 \cos\phi \exp(\mathrm{i}kr\cos\phi) + E'_0 \sum_m \frac{\mathrm{i}^{m-1} J_m(ka)}{H_m^{(1)}(ka)} \frac{\mathrm{d}H_m^{(1)}(kr)}{\mathrm{d}(kr)} \exp(\mathrm{i}m\phi) .$$

The secondary electric field is transverse in all space; the secondary magnetic field becomes transverse at large distances from the cylinder $kr \gg 1$ (wave zone), when the longitudinal component H_r vanishes owing to the presence of the factor kr in the denominator.

The surface current density can be determined from the boundary condition for the tangential component of $\boldsymbol{H}$:

$$j(\phi) = j_z(\phi) = \frac{c}{4\pi} H_\phi(a, \phi) .$$

Finally, the total current is given by

$$J = -\tfrac{1}{2}\mathrm{i}caE'_0 \left[J_1(ka) - \frac{J_0(ka) H_1^{(1)}(ka)}{H_0^{(1)}(ka)} \right] .$$

8.53 Since the field is two-dimensional, the quantity $\mathrm{d}I$ in the general equation (8.c.1) $\mathrm{d}\sigma_s = \mathrm{d}I/\overline{\gamma}_0$ must be looked upon as the intensity of secondary waves per unit length within an angle $\mathrm{d}\phi$, so that $\mathrm{d}I = \overline{\gamma r}\,\mathrm{d}\phi$. The differential scattering cross section will then have the dimensions of length. With the use of the results of problem 8.52, we have
8.52, we have

$$\mathrm{d}\sigma_s = |f(\phi)|^2 \mathrm{d}\phi ,$$

where

$$f(\phi) = \left(\frac{2}{\pi k}\right)^{1/2} \sum_m \mathrm{i}^m \frac{J_m(ka)}{H_m^{(1)}(ka)} \exp[\mathrm{i}(m\phi - \tfrac{1}{2}m\pi - \tfrac{1}{4}\pi)] . \tag{8.53.1}$$

In general, the latter result is very complicated. However, it becomes much simpler when $ka \ll 1$. It is then sufficient to retain only the term with $m = 0$, which yields an isotropic distribution of the secondary radiation:

$$d\sigma_s = \frac{\pi\, d\phi}{2k \ln^2(ka)} = \frac{\lambda}{4 \ln^2(ka)} d\phi\, .$$

The total cross section is obtained by integrating equation (8.53.1) over ϕ. Owing to the orthogonality of the functions $\exp(im\phi)$, we have

$$\sigma_s = \frac{4}{k} \sum_{m=-\infty}^{\infty} \left| \frac{J_m(ka)}{H_m^{(1)}(ka)} \right|^2 . \tag{8.53.2}$$

When $ka \ll 1$, equation (8.53.2) assumes the simple form

$$\sigma_s = \frac{\pi\lambda}{2 \ln^2(ka)}\, .$$

8.54

$$H_z = H_0 \left[\exp(ikr\cos\phi) - \sum_{m=-\infty}^{\infty} i^m \frac{J'_m(ka)}{H_m^{(1)'}(ka)} H_m^{(1)}(kr) \exp(im\phi) \right] ,$$

$$E_r = H_0 \left[\sin\phi \exp(ikr\cos\phi) + \frac{1}{kr} \sum_{m=-\infty}^{\infty} i^m \frac{mJ'_m(ka)}{H_m^{(1)'}(ka)} H_m^{(1)}(kr) \exp(im\phi) \right] ,$$

$$E_\phi = H_0 \left[\cos\phi \exp(ikr\cos\phi) + \sum_{m=-\infty}^{\infty} i^{m+1} \frac{J'_m(ka)}{H_m^{(1)'}(ka)} H_m^{(1)'}(kr) \exp(im\phi) \right] ,$$

where ϕ is measured from the direction of $\boldsymbol{k}$, and the axis of the cylindrical system of coordinates lies along the axis of the cylinder.

$$d\sigma_s(\phi) = \tfrac{1}{8}\pi(ka)^3 a(1 - 2\cos\phi)^2\, d\phi\, ,$$

$$\sigma_s = \tfrac{3}{4}\pi^2 k^3 a^4\, .$$

8.55 $d\sigma'_s = \cos^2\varphi\, d\sigma_\parallel + \sin^2\varphi\, d\sigma_\perp$,

$$d\sigma''_s = \tfrac{1}{2}(d\sigma_\parallel + d\sigma_\perp)\, .$$

8.56 The unpolarised wave can be looked upon as a superposition of two incoherent components of equal intensity. The electric vector $\boldsymbol{E}$ of one of the two components can be taken to lie along the axis of the cylinder, and the electric vector of the other component is then perpendicular to the axis. The scattering cross sections for the first and second components were obtained in problems 8.53 and 8.54. The degree of depolarisation ρ is given by the ratio of the intensities of the scattered waves:

$$\rho = \frac{d\sigma_\perp}{d\sigma_\parallel} = \tfrac{1}{4}(ka)^4 \ln^2(ka)(1 - 2\cos\phi)^2\, .$$

Since $ka \ll 1$, it follows that ρ is very small, i.e. the scattered waves are almost completely polarised for all scattering angles. When $\cos\phi = 0{\cdot}5$, i.e. when $\phi = 60°$, the depolarisation coefficient is zero.

8.57

$$H_z = H_0 \left[\exp(\mathrm{i}kr\cos\phi) + \sum_{m=-\infty}^{\infty} \mathrm{i}^m \frac{J'_m(ka) - \mathrm{i}\zeta J_m(ka)}{\mathrm{i}\zeta H_m^{(1)}(ka) - H_m^{(1)'}(ka)} H_m^{(1)}(kr)\, \exp(\mathrm{i}m\phi)\right],$$

where ζ is the surface impedance of the metal,

$$H_\phi = H_r = 0\,, \qquad \boldsymbol{E} = \frac{\mathrm{i}}{k}\operatorname{curl}\boldsymbol{H}\,.$$

8.58

$$Q = \tfrac{1}{4}ac\zeta' H_0^2 \sum_m \left|\frac{J'_m N_m - J_m N'_m}{\mathrm{i}\zeta H_m^{(1)} - H_m^{(1)'}}\right|^2,$$

where ζ' is the real part of the surface impedance. The cylindrical functions J_m, N_m, and $H_m^{(1)}$ (see appendix 3) and their derivatives are taken at the point ka. The absorption cross section is given by

$$\sigma_{\mathrm{a}} = \frac{Q_0}{\gamma_0} = 2\pi a\zeta' \sum_m \left|\frac{J'_m N_m - J_m N'_m}{\mathrm{i}\zeta H_m^{(1)} - H_m^{(1)'}}\right|^2.$$

When $ka \ll 1$, i.e. when $\lambda \gg a$, the field in the neighbourhood of the cylinder is quasi-stationary (conducting cylinder in a longitudinal quasi-stationary magnetic field; see problem 7.30). Hence, if we express ζ' in terms of the conductivity σ with the aid of equations (8.a.9) and (8.a.11), we have the following expression for Q:

$$Q = \tfrac{1}{8}acH_0^2 \left(\frac{\mu\omega}{2\pi\sigma}\right)^{1/2},$$

which is the same as that found in the solution of problem 7.32 for a strong skin effect.

8.59 When $r > a$

$$E_z = E_0 \left[\exp(\mathrm{i}kr\cos\phi) + \sum_{m=-\infty}^{\infty} \mathrm{i}^m \frac{\zeta J'_m(ka)J_m(k'a) - J_m(ka)J'_m(k'a)}{H_m^{(1)}(ka)J'_m(k'a) - \zeta H_m^{(1)'}(ka)J_m(k'a)} H_m^{(1)}(kr)\exp(\mathrm{i}m\phi)\right],$$

and when $r < a$

$$E_z = E_0\zeta \sum_{m=-\infty}^{\infty} \mathrm{i}^m \frac{J'_m(k'a)H_m^{(1)}(ka) - J'_m(k'a)H_m^{(1)'}(ka)}{J'_m(k'a)H_m^{(1)}(ka) - \zeta J_m(k'a)H_m^{(1)}(ka)} J_m(k'r)\exp(\mathrm{i}m\phi)\,,$$

where E_0 is the amplitude of the incident wave, $\zeta = (\mu/\epsilon)^{1/2}$, $k = \omega/c$, $k' = \omega(\epsilon\mu)^{1/2}/c$, and the remaining components of $\boldsymbol{E}$ are zero. The field $\boldsymbol{H}$ may be calculated from the formula

$$\boldsymbol{H} = \frac{c}{\mathrm{i}\omega\mu}\operatorname{curl}\boldsymbol{E}\,.$$

8.60 The dipole moments of the sphere may be written in the form

$$\boldsymbol{p} = \beta_e \boldsymbol{E}_0 \exp(-i\omega t) , \qquad \boldsymbol{m} = \beta_m \boldsymbol{H}_0 \exp(-i\omega t) ,$$

where β_e and β_m are the electric and magnetic polarisabilities of the sphere, which in general are complex.

The components of the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ of the scattered wave may be obtained from equations (12.a.17) and (12.a.20):

$$H_\phi = E_\theta = \frac{\omega^2 E_0}{c^2 r}(\beta_e \cos\theta + \beta_m)\cos\phi ,$$

$$H_\theta = -E_\phi = \frac{\omega^2 E_0}{c^2 r}(\beta_e + \beta_m)\sin\phi .$$

The angles θ and ϕ define the direction of scattering as shown in figure 8.60.1.

The differential scattering cross section can be determined from equation (8.c.1) and is given by

$$\frac{d\sigma_s(\theta, a)}{d\Omega} = \frac{\omega^4}{c^4}[|\beta_e|^2(\cos^2\theta \cos^2\phi + \sin^2\phi) + |\beta_m|^2(\cos^2\theta \sin^2\phi + \cos^2\phi) + (\beta_e\beta_m^* + \beta_e^*\beta_m)\cos\theta] .$$

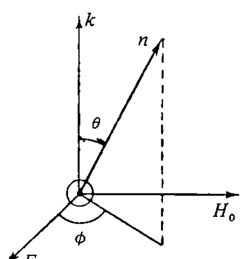


Figure 8.60.1.

8.61

$$d\sigma_s(\theta) = \tfrac{1}{2}[d\sigma_s(\theta, \phi) + d\sigma_s(\theta, \phi + \tfrac{1}{2}\pi)$$

$$= \frac{\omega^4}{2c^4}[(|\beta_e|^2 + |\beta_m|^2)(1 + \cos^2\theta) + 2(\beta_e\beta_m^* + \beta_e^*\beta_m)\cos\theta]\, d\Omega ,$$

$$\sigma_s = \frac{8\pi\omega^4}{3c^4}(|\beta_e|^2 + |\beta_m|^2) .$$

In order to determine the degree of depolarisation of the scattered radiation, it is necessary to find the principal directions of the polarisation tensor. This can easily be done in the present example because of the symmetry of the problem. For given $\boldsymbol{k}$ and $\boldsymbol{n}$ (cf figure 8.60.1), the special directions for $\boldsymbol{E}_0$ will be the direction of the normal to the plane of scattering and the direction lying in the plane of scattering and perpendicular to $\boldsymbol{k}$.

The differential cross sections for these two directions were obtained in the solution of the preceding problem. The degree of depolarisation ρ is given by the ratio of the smaller to the larger of these two quantities. When $|\beta_m| < |\beta_e|$,

$$\rho = \frac{d\sigma_s(\theta, 0)}{d\sigma_s(\theta, \frac{1}{2}\pi)} = \left|\frac{\beta_m + \beta_e \cos\theta}{\beta_m \cos\theta + \beta_e}\right|^2 .$$

8.62 For a dielectric sphere

$$d\sigma_{sd} = \frac{\omega^4 a^6}{2c^4}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2 (1 + \cos^2\theta)\, d\Omega ,$$

$$\sigma_{sd} = \frac{8\pi\omega^4 a^6}{3c^4}\left(\frac{\epsilon - 1}{\epsilon + 2}\right)^2 ; \qquad \rho_d = \cos^2\theta .$$

For a perfectly conducting sphere

$$d\sigma_{s\,cond} = \frac{\omega^4 a^6}{8c^4}[5(1 + \cos^2\theta) - 8\cos\theta]\, d\Omega ,$$

$$\sigma_{s\,cond} = \frac{10\pi\omega^4 a^6}{3c^4} . \qquad \rho_{cond} = \left(\frac{1 - 2\cos\theta}{2 - \cos\theta}\right)^2$$

It is clear from the formula for $d\sigma_{sd}$ that the scattering cross section of a dielectric sphere is symmetric with respect to the forward ($\theta = 0$) and the backward ($\theta = \pi$) directions. Moreover, $d\sigma_{sd}(0)/d\sigma_{sd}(\pi) = 1$. The scattering cross section for a conducting sphere is much more anisotropic and asymmetric: $d\sigma_{s\,cond}(0)/d\sigma_{s\,cond}(\pi) = \frac{1}{9}$. Light scattered by a dielectric sphere through an angle $\theta = \frac{1}{2}\pi$ will be completely polarised. Scattering by a perfectly conducting sphere results in complete polarisation when $\cos\theta = 0{\cdot}5$ ($\theta = 60°$).

The above formulae for the dielectric sphere are rigorous provided it is possible to neglect effects associated with the finite velocity of propagation of the electromagnetic wave inside the sphere, i.e. when the wavelength inside the sphere is large compared with its radius. For a perfectly conducting sphere, the wave will not propagate inside the sphere, and it is sufficient for the condition $a \ll \lambda$ to be satisfied, where λ is the wavelength in the medium surrounding the sphere.

8.63 Just as in problem 8.60 it is necessary to consider the emission of radiation by the induced electric and magnetic dipoles $\boldsymbol{p}$ and $\boldsymbol{m}$. Consider the system of coordinates illustrated in figure 8.63.1. The vector $\boldsymbol{k}$ of the primary wave lies in the xz plane. Consider two cases of polarisation of the incident wave, namely, (a) $\boldsymbol{E}_0$ lying in the plane of incidence, i.e. the xz plane, and (b) $\boldsymbol{E}_0$ normal to the plane of incidence. For (a) the component of the electric field which is parallel to the plane of the disc is given by $E_{0\parallel} = E_0 \sin\phi$, and the perpendicular component is given by $E_{0\perp} = E_0 \cos\phi$. In this approximation ($a \ll \lambda$), the electric moment $\boldsymbol{p}$

may be calculated on the assumption that it is the same as the static moment of a conducting disc in a uniform electric field.

According to the solutions of problems 3.69 and 3.71, the longitudinal polarisability of the disc is $\beta_{e\parallel} = 4a^3/3\pi$, and the transverse polarisability is $\beta_{e\perp} = 0$. Hence,

$$p_x = \beta_{e\parallel} E_{0x} = -\frac{4a^3}{3\pi} E_0 \sin\phi ,$$

$$p_y = p_z = 0 .$$

The magnetic field will only have a longitudinal component. However, the longitudinal magnetic polarisability of the disc is zero (see problem 7.41), and hence $\boldsymbol{m} = 0$.

The differential scattering cross section is given by

$$\mathrm{d}\sigma_s = \frac{16a^6\omega^4}{9\pi^2c^4} \sin^2\phi (1 - \sin^2\vartheta \cos^2\varphi)\, \mathrm{d}\Omega , \tag{8.63.1}$$

and the total scattering cross section is given by

$$\sigma_s = \frac{128a^6\omega^4}{27\pi c^4} \sin^2\phi . \tag{8.63.2}$$

For condition (b) we have

$$p_y = \frac{4a^3}{3\pi} E_0 , \qquad p_x = p_z = 0 ,$$

$$m_z = \frac{2a^3}{3\pi} E_0 \cos\phi , \qquad m_x = m_y = 0,$$

$$\mathrm{d}\sigma_s = \frac{16a^6\omega^4}{9\pi^2c^4} [1 + \sin^2\vartheta(\tfrac{1}{4}\cos^2\phi - \sin^2\varphi) + \sin\vartheta \cos\phi \cos\varphi]\, \mathrm{d}\Omega , \tag{8.63.3}$$

$$\sigma_s = \frac{128a^6\omega^4}{27\pi c^4} (1 + \tfrac{1}{4}\cos^2\phi) ,$$

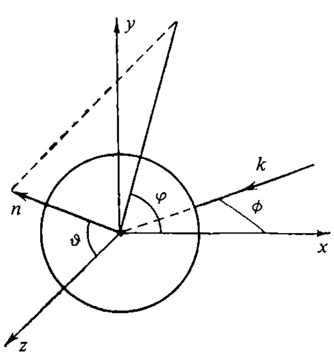


Figure 8.63.1.

For an unpolarised wave, we have from equations (8.63.1), (8.63.2), and (8.63.3)

$$\begin{aligned} d\sigma_s &= \frac{8a^6\omega^4}{9\pi^2c^4}[1+\sin^2\vartheta(1-\tfrac{1}{4}\cos^2\phi-\cos^2\phi\cos^2\varphi) \\ &\qquad\qquad +\sin^2\phi+\sin\vartheta\cos\phi\cos\varphi]\,d\Omega\ , \\ \sigma_s &= \frac{128a^6\omega^4}{27\pi c^4}(1-\tfrac{3}{8}\cos\phi)\ . \end{aligned}$$

8.64

$$d\sigma_s = \frac{a^4h^2\omega^4(\epsilon-1)^2}{18c^4\epsilon^2}(1+\cos^2\vartheta)\,d\Omega\ ,$$

where ϑ is the scattering angle, and hence

$$\sigma_s = \frac{8\pi a^4h^2\omega^4(\epsilon-1)^2}{27c^4\epsilon^2}\ .$$

8.65 Consider the coordinate system shown in figure 8.65.1. The vector k of the primary wave lies in the xz plane. The cylinder may be approximately represented by an elongated ellipsoid of revolution with semiaxes a and h. It follows from the solutions of problems 3.69, 3.70, and 7.41 that the longitudinal component of the electrical polarisability of a very elongated ellipsoid of revolution is larger than its transverse electric and magnetic polarisability by a factor of roughly h/a. Hence, this scattering cross section will be very dependent on whether there is a finite longitudinal component of the electric field in the incident wave. If this component is appreciable, then the secondary radiation will be due

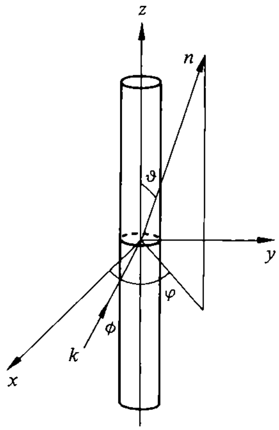


Figure 8.65.1.

to the z-component of the electric dipole moment. The remaining components of the electric moment, and also the magnetic moment, may be neglected. Assuming that $\boldsymbol{E}_0$ lies in the xz plane, we have

$$d\sigma_s = \frac{\omega^4 h^6}{9c^4 \ln^2(h/a)} \sin^2\phi \sin^2\vartheta \, d\Omega ,$$

$$\sigma_s = \frac{8\pi\omega^4 h^6}{27c^4 \ln^2(h/a)} \sin^2\phi .$$

If the longitudinal component of $\boldsymbol{E}_0$ is zero, then the scattering is due to the transverse components of the electric moment and the magnetic moment, which have the same order of magnitude. Then,

$$d\sigma_s = \frac{a^4 h^2 \omega^4}{9c^4}[(1+2n_x \sin\phi)^2 + 3\cos^2\phi + n_z^2(4-\sin^2\phi) + 8n_z \cos\phi + 2n_x n_z \sin 2\phi] \, d\Omega ,$$

$$\sigma_s = \frac{40\pi a^4 h^2 \omega^4}{27c^4}(1+\tfrac{3}{5}\cos^2\phi) ,$$

where n_i $(i = x, y, z)$ are the components of the unit vector in the direction of scattering. The scattering cross section for an unpolarised wave is given by

$$\frac{d\sigma_s}{d\Omega} = \frac{\omega^4 h^6}{18c^4 \ln^2(h/a)} \sin^2\phi \sin^2\vartheta ,$$

$$\sigma_s = \frac{4\pi\omega^4 h^6}{27c^4 \ln^2(h/a)} \sin^2\phi .$$

8.66 When the vector $\boldsymbol{E}_0$ is polarised in the xz plane (figure 8.65.1),

$$d\sigma_{s\parallel} = \frac{4\omega^4 a^4 h^2}{9c^4}\left(\frac{\epsilon-1}{\epsilon+1}\right)^2 [(1-n_x^2)\cos^2\phi + \tfrac{1}{4}(\epsilon+1)^2(1-n_z^2)\sin^2\phi - \tfrac{1}{2}(\epsilon+1)n_x n_z \sin 2\phi] \, d\Omega .$$

When the vector $\boldsymbol{E}_0$ is polarised at right angles to the xz plane

$$d\sigma_{s\perp} = \frac{4\omega^4 a^4 h^2}{9c^4}\left(\frac{\epsilon-1}{\epsilon+1}\right)^2 (1-\sin^2\vartheta \sin^2\varphi) \, d\Omega .$$

8.67 The total intensity of the electric field at a general point in space may be written in the form

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0(\boldsymbol{r}, t) + \boldsymbol{E}'(\boldsymbol{r}, t) ,$$

where

$$\boldsymbol{E}_0(\boldsymbol{r}, t) = \boldsymbol{E}_0 \exp[i(\boldsymbol{k} \circ \boldsymbol{r}) - i\omega t]$$

represents the incident wave, and $\boldsymbol{E}'(\boldsymbol{r}, t)$ represents the scattered (secondary) radiation. At each point inside the sphere the polarisation vector $\boldsymbol{P}(\boldsymbol{r}, t)$ is proportional to $\boldsymbol{E}$ and approximately to $\boldsymbol{E}_0$ since the scattered field is much smaller than the incident field ($E' \ll |E_0|$) when $(\epsilon - 1)/4\pi \ll 1$ [(2)].

The scattered field E' can be expressed in terms of the Hertz vector [see equation (12.a.13)]

$$Z(r, t) = \int \frac{P(r', t) \exp[i(kR - \omega t)]}{R} d^3 r'$$

through the formula

$$E' = \operatorname{curl}\operatorname{curl} Z - 4\pi P = \operatorname{curl}\operatorname{curl} E_0 \frac{\exp(ikr)}{r} \int \exp(i[k - kn] \cdot r') \, d^3 r' .$$

The difference $k - kn$ represents the change in the wavevector on scattering. Let this change be denoted by q where $q = 2k \sin\frac{1}{2}\theta$ and θ is the angle of scattering. In order to evaluate the integral, take the polar axis in the direction of q, when

$$\int \exp[i(q \cdot r')] r'^2 \, dr' \, d\Omega = 4\pi \frac{\sin qa - qa \cos qa}{q^3} . \tag{8.67.1}$$

In evaluating the double curl in equation (8.67.1) we retain only terms proportional to $1/r$, so that

$$\operatorname{curl}\operatorname{curl} E_0 \frac{\exp(ikr)}{r} = k^2 [n \wedge [E_0 \wedge n]] \frac{\exp(ikr)}{r} .$$

Finally, the expression for the scattered field is

$$E' = \tfrac{1}{3} \frac{\omega^2 a^3}{c^2} \frac{\epsilon - 1}{3} [n \wedge [E_0 \wedge n]] \varphi(qa) \frac{\exp(ikr)}{r} , \tag{8.67.2}$$

where

$$\varphi(qa) = \frac{3(\sin qa - qa \cos qa)}{(qa)^3} = 3 \left[\frac{\pi}{(qa)^3}\right]^{1/2} J_{3/2}(qa) .$$

Compare the expression given by equation (8.67.2) with the expression which holds for small a (see problem 8.62). If equation (8.67.2) is taken in the limit $qa \ll 1$,

$$E' = \frac{\omega^2 a^3}{c^2} \frac{\epsilon - 1}{3} [n \wedge [E_0 \wedge n]] \frac{\exp(ikr)}{r} , \tag{8.67.3}$$

since $\varphi(qa) \approx 1$ when $qa \ll 1$. On the other hand, if E' is evaluated from the formula

$$E' = [n \wedge [\ddot{p} \wedge n]] \frac{\exp(ikr)}{c^2 r} ,$$

(2) We note that the method involving expansions in powers of a small parameter, which is used in this preceding problem, is analogous to the Born approximation in quantum mechanics. The latter method is widely used in the theory of scattering of particles by quantum mechanical systems.

where

$$p = \frac{\epsilon - 1}{\epsilon + 2} a^3 E_0$$

is the static dipole moment of the sphere, we have

$$E' = \frac{\omega^2 a^3}{c^2} \frac{\epsilon - 1}{\epsilon + 2} [n \wedge [E_0 \wedge n]] \frac{\exp(ikr)}{r} \ . \tag{8.67.4}$$

It is clear that the factor $\frac{1}{3}$ has been replaced by $(\epsilon + 2)^{-1}$. However, this difference between equation (8.67.3) and (8.67.4) is not really significant because equation (8.67.3) is valid only to within a factor $(\epsilon - 1)$.

The differential scattering cross section is

$$\frac{d\sigma_s(\theta, \phi)}{d\Omega} = \frac{\omega^4 a^6 (\epsilon - 1)^2}{9c^4} \varphi^2(qa)(\sin^2\phi + \cos^2\phi \cos^2\theta)$$

where the angles θ and ϕ are defined in figure 8.60.1.

This cross section differs from the scattering cross section for a small dielectric sphere (see solution to problem 8.62) by the presence of the new factor $\varphi^2(qa)$ and the fact that the term $(\epsilon + 2)^2$ in the denominator has been replaced by 9. The new factor, $\varphi^2(qa)$, represents the interference between secondary waves from different elements of the sphere. The degree of depolarisation of the scattered light will therefore be the same as for a small dielectric sphere:

$$\rho = \cos^2\theta \ .$$

Averaging over the polarisations yields

$$\frac{d\sigma_s(\theta)}{d\Omega} = \frac{\omega^4 a^6 (\epsilon - 1)^2}{18c^4} \varphi^2(qa)(1 + \cos^2\theta) \ .$$

Finally, consider a very large sphere ($ka \gg 1$). If the angles are such that $qa \gg 1$, then $\varphi(qa) \to 0$, and the cross section for this range of angles is very small. It follows from the explicit expression for q that $qa \gg 1$ is equivalent to the condition $\theta \gg (ka)^{-1}$. Thus, if the sphere is large, the scattering is mainly in the forward direction, in the angular range $\theta \lesssim (ka)^{-1}$.

8.68 When $ka \gg 1$, the function $\varphi^2(qa)$, which enters into the expression for the differential cross section (see preceding problem), is appreciably different from zero only within the angular range $\theta \lesssim (ka)^{-1}$. In this angular range, the factor $(1 + \cos^2\theta)$ may be regarded as constant and equal to 2. Hence,

$$\sigma_s = \frac{2\pi\omega^4 a^6 (\epsilon - 1)^2}{9c^4} \int_0^\pi \varphi^2(qa) \sin\theta \, d\theta \ .$$

Substitute $y = qa = 2ka\sin\frac{1}{2}\theta$. In the limit $ka \gg 1$

$$\sigma_s = \frac{\pi\omega^2 a^4(\epsilon-1)^2}{18c^2} .$$

For a small sphere ($ka \ll 1$) we can replace $\epsilon+2$ by 3 (see solution of problem 8.62), so that

$$\sigma_s = \frac{8\pi\omega^4 a^6(\epsilon-1)^2}{27c^4} .$$

As can be seen from these results, the dependence on the frequency and on the radius of the sphere is different in the two extremes. Thus, the frequency dependence is of the form $\sim\omega^4$ and $\sim\omega^2$, while the dependence on the radius is of the form $\sim a^6$ and $\sim a^4$.

8.69 Consider the result

$$\sigma_a = -\frac{1}{E_0^2}\,\mathrm{Re}\int([\boldsymbol{E}\wedge\boldsymbol{H}^*]\cdot\boldsymbol{n})r^2\,\mathrm{d}\Omega , \tag{8.69.1}$$

where $\boldsymbol{n} = \boldsymbol{r}/r$ and σ_a is the absorption cross section. The integration is carried out over the surface of a large sphere surrounding the scatterer. Equation (8.69.1) expresses the fact that the scattering cross section is proportional to the energy flux through the surface of the sphere in the inward direction. If we substitute the expression for $\boldsymbol{E}$ (as given in the problem) into equation (8.69.1) and the following expression for $\boldsymbol{H}$

$$\boldsymbol{H} = E_0\left\{[\boldsymbol{n}_0\wedge\boldsymbol{e}]\exp(\mathrm{i}kz)+[\boldsymbol{n}\wedge\boldsymbol{F}(\boldsymbol{n})]\frac{\exp(\mathrm{i}kr)}{r}\right\}$$

and remember that

$$(\boldsymbol{n}\cdot\boldsymbol{F}(\boldsymbol{n})) = 0 ,$$

we have

$$\begin{aligned}\frac{1}{E_0^2}\,\mathrm{Re}\,([\boldsymbol{E}\wedge\boldsymbol{H}^*]\cdot\boldsymbol{n}) = {}& (\boldsymbol{n}_0\cdot\boldsymbol{n})+\frac{|\boldsymbol{F}|^2}{r^2}+\tfrac{1}{2}[(\boldsymbol{e}\cdot\boldsymbol{F})+(\boldsymbol{n}_0\cdot\boldsymbol{n})(\boldsymbol{e}\cdot\boldsymbol{F}) \\ & -(\boldsymbol{e}\cdot\boldsymbol{n})(\boldsymbol{n}_0\cdot\boldsymbol{F})]\frac{\exp[\mathrm{i}k(r-z)]}{r} \\ & +\tfrac{1}{2}[(\boldsymbol{e}^*\cdot\boldsymbol{F}^*)+(\boldsymbol{n}_0\cdot\boldsymbol{n})(\boldsymbol{e}^*\cdot\boldsymbol{F}^*) \\ & -(\boldsymbol{e}^*\cdot\boldsymbol{n})(\boldsymbol{n}_0\cdot\boldsymbol{F}^*)]\frac{\exp[-\mathrm{i}k(r-z)]}{r} .\end{aligned}$$

The integral of the first term over the angles is zero, and the integral of the second term is equal to the total scattering cross section σ_s. The integrals of the remaining components may be transformed by integrating by parts:

$$\frac{1}{r}\int (\boldsymbol{n}_0\cdot\boldsymbol{n})(\boldsymbol{e}\cdot\boldsymbol{F})\exp[\mathrm{i}k(r-z)]r^2\,\mathrm{d}\Omega$$

$$=\frac{1}{\mathrm{i}k}\int_0^{2\pi}\mathrm{d}\varphi\Bigg[\{(\boldsymbol{n}_0\cdot\boldsymbol{n})(\boldsymbol{e}\cdot\boldsymbol{F})\exp[\mathrm{i}kr(1-\cos\vartheta)]\}\big|_{\vartheta=0}^{\vartheta=\pi}$$

$$-\int_0^{\pi}\exp[\mathrm{i}kr(1-\cos\vartheta)]\frac{\partial}{\partial\cos\vartheta}(\boldsymbol{n}_0\cdot\boldsymbol{n})(\boldsymbol{e}\cdot\boldsymbol{F})\,\mathrm{d}\cos\vartheta\Bigg].$$

Repeated integration by parts in the last integral will give terms that are proportional to $1/r$, and may therefore be neglected. Moreover, the term with the oscillating factor $\exp(2\mathrm{i}kr)$ can also be neglected, since it will not contribute to the total energy flux. In order to show this, we recall that strictly monochromatic waves do not, in fact, exist. In reality, any 'monochromatic' wave is really a superposition of harmonics whose frequencies lie within a more or less narrow interval $\Delta\omega$. Since r is large, the result of averaging the factor $\exp(2\mathrm{i}kr)$ over any such interval will be zero. It follows that

$$\frac{1}{r}\int (\boldsymbol{n}_0\cdot\boldsymbol{n})(\boldsymbol{e}\cdot\boldsymbol{F})\exp[\mathrm{i}k(r-z)]r^2\,\mathrm{d}\Omega=\frac{2\pi\mathrm{i}}{k}(\boldsymbol{e}\cdot\boldsymbol{F}(\boldsymbol{n}_0)).$$

The integrals of the remaining components may be evaluated in a similar way. Terms containing the factors $(\boldsymbol{e}\cdot\boldsymbol{n})$ and $(\boldsymbol{e}^*\cdot\boldsymbol{n})$ will not contribute to the final result in view of the fact that $(\boldsymbol{e}\cdot\boldsymbol{n}_0)=0$. If we substitute the evaluated integrals into equation (8.69.1), we have the following final expression

$$\sigma_\mathrm{t}=\frac{4\pi}{k}\operatorname{Im}(\boldsymbol{e}\cdot\boldsymbol{F}(\boldsymbol{n}_0)).$$

The latter result, the so-called optical theorem, admits of a simple physical interpretation. The total cross section is a measure of the attenuation of the primary wave. This attenuation is due to the interference between the incident wave and that part of the scattered wave which has the same polarisation and direction of propagation as the incident wave. It follows that the total cross section depends on the forward scattering amplitude.

8.70 The scattered wave is due to the electric and magnetic dipole moments which are induced by the incident wave. The scattering amplitude $\boldsymbol{F}(\boldsymbol{n})$ (see preceding problem) can be determined from equations (12.a.17) and (12.a.20).

The final result is

$$\sigma_\mathrm{a}=\frac{4\pi\omega}{c}(\beta_\mathrm{e}''+\beta_\mathrm{m}'').$$

8.71 $\sigma_\mathrm{a}=6\pi b^2\zeta$.

8.72 The force is in the direction of the wavevector of the incident wave, and is of the form

$$\overline{F} = \frac{\overline{\gamma_0}}{c}\left\{\sigma_a + \int (1-\cos\vartheta)\left[\frac{d\sigma_s(\vartheta,\varphi)}{d\Omega}\right] d\Omega\right\},$$

where $\overline{\gamma_0}$ is the average energy flux density in the incident wave, and the integration is carried out over the entire solid angle.

8.73 For a perfectly conducting sphere

$$\overline{F} = \frac{43a^6\omega^4}{96c^4}E_0^2 .$$

For a dielectric sphere

$$\overline{F} = \frac{a^6\omega^4}{3c^4}\left(\frac{\epsilon-1}{\epsilon+2}\right)^2 E_0^2 .$$

8.74 The diffraction formula (8.c.4) can be used. The plane in which the screen is located can be taken as the plane of integration. It follows that on the surface of integration

$$u = A\frac{\exp(ikR_1)}{R_1}, \qquad d^2S_n = 2\pi r\,dr\cos(R_1, z) = 2\pi\frac{z_1 r}{R_1}\,dr ,$$

where A = constant. By substituting these expressions into equation (8.c.4), and transforming to the new variable $\rho = R+R_1$, we have

$$u_P(z) = -ikAz_1\int_a^\infty \frac{\exp[ik(R+R_1)]}{RR_1^2} r\,dr$$

$$= -ikAz_1\int_{\rho_0}^\infty \frac{\exp(ik\rho)}{\rho R_1(\rho)}\,d\rho . \qquad (8.74.1)$$

where

$$\rho_0 = (a^2+z^2)^{1/2} + (a^2+z_1^2)^{1/2} .$$

Integration by parts may be used to rewrite equation (8.74.1) in the form of a series with increasing negative powers of $k\rho$. Since $\lambda \ll a$, the higher-order terms are negligible, and only the first term need be retained. Hence,

$$u_P(z) = \frac{u_0 z_1 \exp[ik(a^2+z^2)^{1/2}]}{\rho_0},$$

where $u_0 = A(a^2+z_1^2)^{-1/2}\exp[ik(a^2+z_1^2)^{1/2}]$ is the amplitude of the incident wave at the surface of the screen. Since the intensity is proportional to the square of the amplitude, we have

$$I(z) = I_0\frac{z_1^2}{[(a^2+z_1^2)^{1/2}+(a^2+z^2)^{1/2}]^2} .$$

At a point which is symmetrical with respect to the screen ($z_1 = z$) the intensity is given by

$$I(z) = \tfrac{1}{4}\frac{I_0 z^2}{a^2+z^2} \; .$$

It follows that a bright axial spot will appear for points which are not too close to the screen. This result, which is in conflict with the hypothesis of the rectilinear propagation of light, was first theoretically predicted by Poisson in 1818. He used it as an argument against Fresnel's theory of diffraction and the wave theory of light as a whole. However, experiments carried out by Arago and Fresnel confirmed the presence of the bright spot which appears as the result of the symmetry of the screen. Waves from the zone which surrounds the periphery of the disc arrive at the axial point in phase. It is clear that all points lying along the axial line will have this property, and at such points the intensity will be much greater than at neighbouring points which do not lie on the z-axis.

8.75 If we use the Babinet principle [see equation (8.c.6)], we have for $z = z_1 \gg a$

$$I = I_0 \sin^2\frac{ka^2}{2z} \; ,$$

where I_0 is the intensity of the primary wave at the aperture.

8.76 When $z \gg a$,

$$I = 4I_0 \sin^2\frac{ka^2}{4z} \; .$$

The intensity along the axis of the circular aperture oscillates, decreasing to zero for $z \to \infty$. The reduction in the intensity is due to the fact that the parallel incident beam becomes divergent after diffraction at the aperture.

8.77 Using equation (8.c.5) we have, in the case of Fraunhofer diffraction,

$$\mathrm{d}I = I_0 \frac{[aJ_1(ak\alpha) - bJ_1(bk\alpha)]^2}{\alpha^2}\,\mathrm{d}\Omega \; ,$$

where α is the angle of diffraction, and I_0 is the intensity of the incident light. For a circular aperture

$$\mathrm{d}I = I_0' \frac{J_1^2(ak\alpha)}{\pi\alpha^2}\,\mathrm{d}\Omega \; ,$$

where $I_0' \approx \pi a^2 |u_0|^2$ is the total intensity of the radiation incident on the aperture.

8.78 The diffracted wave is given by

$$u_{\mathrm{P}} = \frac{u_0 \exp(\mathrm{i}kR_0)}{2\pi\mathrm{i}R_0}\int \exp[\mathrm{i}(\boldsymbol{q}_{\parallel}\cdot\boldsymbol{r})]\,\mathrm{d}^2 S_n \; ,$$

where $\boldsymbol{k}'-\boldsymbol{k} = \boldsymbol{q}$; $\boldsymbol{q}_\parallel$ and $\boldsymbol{q}_\perp$ are respectively the components of $\boldsymbol{q}$ in the plane of the screen and in the perpendicular direction.

It is convenient to use polar coordinates with the origin at the centre of the aperture and the polar axis in the direction of $\boldsymbol{q}_\parallel$. This yields

$$u_{\mathrm{P}} = \frac{u_0 \exp(\mathrm{i}kR_0)k\cos\theta}{2\pi\mathrm{i}R_0}\int \exp(-\mathrm{i}q_\parallel r\cos\varphi)r\,\mathrm{d}r\,\mathrm{d}\varphi\,,$$

where θ is the angle of incidence. Using equations (A3.11), (A3.13), and (A3.9) we have

$$\mathrm{d}I = |u_{\mathrm{P}}|^2R_0^2\,\mathrm{d}\Omega = I_0\frac{J_1^2(q_\parallel a)}{\pi q_\parallel^2}\,\mathrm{d}\Omega\,,$$

where $I_0 \sim |u_0|^2\pi a^2\cos\theta$ is the total intensity of the light incident on the aperture.

Assuming that the angle of diffraction α, i.e. the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$, is small, we have

$$q_\parallel = k\alpha(1-\sin^2\theta\cos^2\alpha')^{1/2}\,, \qquad \mathrm{d}I = I_0\frac{J_1^2(ka\alpha[1-\sin^2\theta\cos^2\alpha']^{1/2})}{\pi\alpha^2(1-\sin^2\theta\cos^2\alpha')}\,\mathrm{d}\Omega\,,$$

where α' is the azimuthal angle between $\boldsymbol{q}$ and the plane of incidence. The latter formula does not hold at glancing angles ($\theta \approx \frac{1}{2}\pi$).

8.79 Application of Kirchhoff's formula in vector form (8.c.7) yields the following expressions for the radiated field

$$E_\vartheta = H_\phi = -\mathrm{i}kabE_0\frac{\exp(\mathrm{i}kR)}{\pi R}\left(\frac{\sin k_x'a}{k_x'a}\right)\left(\frac{\sin k_y'b}{k_y'b}\right)(1+\cos\vartheta)\sin\phi\,,$$

$$E_\phi = -H_\vartheta = -\mathrm{i}kabE_0\frac{\exp(\mathrm{i}kR)}{\pi R}\left(\frac{\sin k_x'a}{k_x'a}\right)\left(\frac{\sin k_y'b}{k_y'b}\right)(1+\cos\vartheta)\cos\phi\,,$$

where ϑ and ϕ are the polar angles measured from an axis perpendicular to the plane of the aperture; $k_x' = k\sin\vartheta\cos\phi$ and $k_y' = k\sin\vartheta\sin\phi$ are the components of the wavevector of the diffracted wave.

The angular distribution of the radiation is given by

$$\mathrm{d}I = I_0\frac{abk^2}{4\pi^2}\left(\frac{\sin k_x'a}{k_x'a}\right)^2\left(\frac{\sin k_y'b}{k_y'b}\right)^2(1+\cos\vartheta)^2\,\mathrm{d}\Omega\,,$$

where $I_0 = (cab/2\pi)E_0^2$ is the intensity of the incident wave.

8.80 Let the x-, y-, and z-axes lie in the direction of the vectors $\boldsymbol{E}_0$, $\boldsymbol{H}_0$, and $\boldsymbol{k}$ respectively. The radiated field will then be given by

$$E_\vartheta = H_\phi = -\tfrac{1}{2}\mathrm{i}ka^2E_0\frac{\exp(\mathrm{i}kR)}{R}\left[\frac{J_1(ka\sin\vartheta)}{ka\sin\vartheta}\right](1+\cos\vartheta)\cos\phi\,,$$

$$E_\phi = -H_\vartheta = \tfrac{1}{2}\mathrm{i}ka^2E_0\frac{\exp(\mathrm{i}kR)}{R}\left[\frac{J_1(ka\sin\vartheta)}{ka\sin\vartheta}\right](1+\cos\vartheta)\sin\phi\,,$$

$$\mathrm{d}I = \tfrac{1}{4}I_0\left[\frac{J_1(ka\sin\vartheta)}{\sin\vartheta}\right]^2(1+\cos\vartheta)^2\,\mathrm{d}\Omega\,,$$

where $I_0 = (ca^2/8)E_0^2$ is the intensity of the incident wave. When $\vartheta \ll 1$, we have

$$dI = I_0 \frac{J_1^2(ka\vartheta)}{\pi\vartheta^2}\,d\Omega\,.$$

This result was obtained in the solution of problem 8.77 with the aid of the scalar diffraction formula.

d Coherence and interference

8.83

$$\Delta\Omega = \frac{l_\perp^2}{R^2} \sim \frac{c^2}{\nu^2 L^2} = \left(\frac{\lambda}{L}\right)^2$$

The solid angle of the coherence is independent of the distance R from the source.

8.84

$$\Delta\lambda \approx 3{\cdot}52 \times 10^{-12}\ \text{m}; \qquad l_\perp \sim \frac{\lambda R}{L} = 5{\cdot}4 \times 10^{-5}\ \text{m};$$

$$l_\parallel \sim \frac{\lambda^2}{\Delta\lambda} = 7{\cdot}1 \times 10^{-2}\ \text{m}; \qquad \Delta\Omega \sim 1{\cdot}3 \times 10^{-31}\ \text{sr};$$

$$\Delta V = l_\perp^2 l_\parallel \sim 2{\cdot}1 \times 10^{-10}\ \text{m}^3.$$

8.85 $R = 9{\cdot}46 \times 10^{18}$ km, i.e. larger by a factor $6{\cdot}3 \times 10^5$ than the Earth–Sun distance. Hence it follows that $l_\perp \approx 3{\cdot}4 \times 10$ m, i.e. larger by a factor $6{\cdot}3 \times 10^5$ than $l_\perp$ in the preceding problem. As far as $l_\parallel \approx \lambda^2/\Delta\lambda \approx 7{\cdot}1 \times 10^{-2}$ m and $\Delta\Omega \approx 1{\cdot}3 \times 10^{-31}$ sr are concerned, they retain the same values as in the preceding problem. The coherence volume $\Delta V \approx 8{\cdot}3 \times 10\ \text{m}^3$ is larger by a factor 4×10^{11} than the coherence volume for the solar radiation on the Earth. The increase in the degree of coherence of light while it propagates is a characteristic feature. This refers only to transverse coherence.

8.86 $l_\parallel \sim \lambda^2/\Delta\lambda \approx 3 \times 10^6$ m. As a cone of light with an opening angle $\Delta\vartheta \sim \lambda/D = 10^{-5}$ emerges from the laser the coherence volume close to the laser has the shape of a cone with its vertex at the laser.

$$l_\perp = \begin{cases} D = 5 \times 10^{-2}\ \text{m at the laser,} \\ \dfrac{2 l_\parallel \lambda}{D} \approx 60\ \text{m at the base of the coherence cone.} \end{cases}$$

$$\Delta V = \tfrac{1}{3}\pi(\tfrac{1}{2}l_\perp)^2 l_\parallel \approx 2{\cdot}8 \times 10^9\ \text{m}^3\,.$$

8.87

$$\delta = \left[\exp\left(\frac{2\pi\hbar\nu}{kT}\right) - 1\right]^{-1} = \left[\exp\left(\frac{2\pi\hbar c}{\lambda kT}\right) - 1\right]^{-1};$$

with
$\delta \approx \lambda kT/2\pi\hbar c \approx 200$ for $\lambda = 10^{-2}$ m, $T = 273$ K;
$\delta = \exp(-100) \approx 10^{-43}$ for $\lambda = 5 \times 10^{-7}$ m, $T = 273$ K,
$\delta = [\exp(2\cdot 73) - 1]^{-1} \approx 0\cdot 07$ for $\lambda = 5 \times 10^{-7}$ m, $T = 10000$ K.

8.88 $\delta = 5 \times 10^{18}$, $T = 1\cdot 4 \times 10^{23}$ K.

8.89 $\Gamma(\tau) = \frac{1}{2}\int_0^\pi I(\omega)(1 + \cos\omega\tau)\,\mathrm{d}\omega.$

8.90

$$\Gamma(\tau) = I\frac{\sin\Delta\omega\tau}{2\pi\tau}\cos\omega_0\tau\,.$$

8.91 The path difference for the light from one of the independent emitters which is at the point (x', y') is $s_1 - s_2 \approx (xx' + yy')/R$ (see figure 8.91.1) if we take into account that the transverse size of the source is much larger than $D = (x^2 + y^2)^{1/2}$. The field at the points $\boldsymbol{r}_1(0, 0)$ and $\boldsymbol{r}_2(x, y)$ is produced by all the emitters in the source:

$$u(\boldsymbol{r}_1, t) = \sum_i u_i(t)\,, \qquad u(\boldsymbol{r}_2, t) = \sum_i u_i(t)\exp\left[-\frac{ik(xx_i' + yy_i')}{R}\right],$$

where $u_i(t)$ is the intensity of the field of the ith emitter at the first aperture at time t. The correlation function is

$$\Gamma(\boldsymbol{r}_1, \boldsymbol{r}_2, 0) = \overline{u^*(\boldsymbol{r}_1, t)u(\boldsymbol{r}_2, t)} = \sum_i \overline{u_i^*(t)u_i(t)}\exp[-ik(xx_i' + yy_i')/a]$$
$$+ \sum_{i\neq j}\overline{u_i^*(t)u_j(t)\exp[-ik(xx_j' + yy_j')/R]}\,.$$

The second term in Γ arises from the incoherence of the independent emitters. The first term is the averaged intensity of the radiation from the separate emitters where the path difference $s_1 - s_2$ is taken into account. Changing from a sum to an integration we get

$$\gamma(x, y) = \iint I(x', y')\exp\left[-\frac{ik(xx' + yy')}{R}\right]\mathrm{d}x'\,\mathrm{d}y' \bigg/ \iint I(x', y')\,\mathrm{d}x'\,\mathrm{d}y'\,,$$

where the integration is over the transverse cross section of the source.

8.92

(a) $B(D) = |\gamma(D, 0)| = \cos\dfrac{\pi D\alpha}{\lambda}$; (b) $B(D) = \dfrac{2\lambda}{\pi D}J_1\left(\dfrac{\pi\alpha D}{\lambda}\right)$.

8.93

(a) $\rho = \dfrac{R}{\alpha} = \dfrac{2DR}{\lambda} = 1\cdot 5 \times 10^6$ km ;

(b) $d = \dfrac{R}{\alpha} = \dfrac{DR}{1\cdot 22\lambda} = 6\cdot 2 \times 10^8$ km ;

the diameter of Betelgeuze is approximately 450 times the diameter of the Sun and hence larger than the diameter not only of the Earth's orbit, but also even that of Mars!

8.94 A plane wave $u_1 = A_1 \exp[\mathrm{i}(\boldsymbol{k}_1 \cdot \boldsymbol{r})] = |A_1| \exp[\mathrm{i}(\boldsymbol{k}_1 \cdot \boldsymbol{r}) + \alpha_1]$ with phase α_1 and amplitude A_1, which both vary randomly such that $\overline{A_1} = 0$, while $\overline{|A_1|^2}$ has a constant nonvanishing value, comes from the first source. A wave $u_2 = A_2 \exp[\mathrm{i}(\boldsymbol{k}_2 \cdot \boldsymbol{r})]$ with similar properties comes from the second source. Both these waves enter the photoelements P_1 and P_2. The nonaveraged signal from the photoelement P_1 would be proportional to

$$I(\boldsymbol{r}_1, t) = |u_1(\boldsymbol{r}_1, t) + u_2(\boldsymbol{r}_1, t)|^2 = |A_1|^2 + |A_2|^2 + A_1 A_2^* \exp\{\mathrm{i}([\boldsymbol{k}_1 - \boldsymbol{k}_2] \cdot \boldsymbol{r})\} + A_1^* A_2 \exp\{-\mathrm{i}([\boldsymbol{k}_1 - \boldsymbol{k}_2] \cdot \boldsymbol{r})\}. \tag{8.94.1}$$

The signal (8.94.1) is subject to random fluctuations owing to the fluctuations of the phases of A_1 and A_2 at frequencies much lower than the frequency of the waves u_1, u_2 coming from the sources. These fluctuations are, nevertheless, not registered and an averaged intensity is observed. When only one of the detectors is included the averaged intensity is

$$\overline{I(\boldsymbol{r}_1, t)} = \overline{|A_1|^2} + \overline{|A_2|^2} = \overline{I(\boldsymbol{r}_2, t)},$$

which is independent of $\boldsymbol{k}_1 - \boldsymbol{k}_2$ (the phases of A_1 and A_2 fluctuate independently so that $\overline{A_1 A_2^*} = \overline{A_1^* A_2} = 0$).

Now let the signals from the photoelements P_1 and P_2 from the beginning enter the multiplier in which the intensities $I(\boldsymbol{r}_1, t)$ and $I(\boldsymbol{r}_2, t)$ are multiplied before registration. The signal observed as a result will be proportional to

$$\overline{I(\boldsymbol{r}_1, t) I(\boldsymbol{r}_2, t)} = (\overline{|A_1|^2} + \overline{|A_2|^2})^2 + 2\overline{|A_1|^2}\,\overline{|A_2|^2} \cos\{([\boldsymbol{k}_1 - \boldsymbol{k}_2] \cdot [\boldsymbol{r}_1 - \boldsymbol{r}_2])\}.$$

It depends on $\boldsymbol{k}_1 - \boldsymbol{k}_2$ and, hence, on the angular distance between separated sources. If the distance $\boldsymbol{r}_1 - \boldsymbol{r}_2$ between the detectors is changed and the weakening and strengthening of the signal are observed, one can determine this angular distance.

8.95

$$\Delta\varphi = \frac{2\pi(n-1)x}{\lambda},$$

where the coordinate x is reckoned from the refracting edge perpendicular to it.

If by any means whatever there is a phase shift given by $\Delta\varphi \propto x$ in the xy-plane, such a plane will turn the front of a plane wave in the direction of increasing x, i.e. it will operate just like a prism.

8.96 The phase shift at a distance x from the axis of the lens is, in the case of a converging lens,

$$\Delta\varphi = -\frac{\pi x^2}{\lambda f},$$

where f is the focal distance determined by the equation

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1}+\frac{1}{R_2}\right).$$

In the case of a diverging lens

$$\Delta\varphi = +\frac{\pi x^2}{\lambda f}.$$

8.97 The intensity distribution of the light on the photographic plate is

$$I(x) = |A_1 \exp(ikx\vartheta_1)+A_2 \exp(ikx\vartheta_2)|^2 = I_1+I_2+2(I_1I_2)^{1/2}\cos(kx\vartheta),$$

where $\vartheta_2 = \vartheta+\vartheta_1$, $k = 2\pi/\lambda$, $I_1 = |A_1|^2$, $I_2 = |A_2|^2$ and the coordinate x is measured along the photographic plate as shown in figure 8.97.1. The blackening distribution on the developed photographic plate is determined by the intensity distribution $I(x)$. The transmissivity $T(x)$ is proportional to $[I(x)]^{-\frac{1}{2}\gamma}$, where γ is the gamma of the photographic emulsion, and is a periodic function of x with period λ/ϑ. It can be written in the form $T(x) = a+b\cos(kx\vartheta)$ (a and b are constants) if we retain only the two lowest harmonics. We can consider the developed plate to be a diffraction grating that splits the incident plane wave into plane beams with propagation directions θ which are determined by the relation $(\lambda/\vartheta)\sin\theta = n\lambda$, $n = 0, \pm1, \pm2, \ldots$. The main beams are the central, zeroth order one and the two first order beams in the directions $\theta = \pm\vartheta$. We note that these three basic beams can be obtained by multiplying the incident wave $A_0 \exp(ikz)$ by the transmissivity $T(x)$. We then get a wave field behind the photographic plate of the form

$$A_0 a \exp(ikz)+\tfrac{1}{2}A_0 b \exp[ik(z+x\vartheta)]+\tfrac{1}{2}A_0 b \exp[ik(z-x\vartheta)],$$

where the first term describes the undeflected central beam, the second the first order beam deflected by $+\vartheta$, and the third the first order beam deflected by $-\vartheta$.

8.98 The reference field at the plate has the form

$$u_1 = A_0 \exp(-i\beta x), \qquad \beta = \frac{2\pi(n-1)\alpha}{\lambda}.$$

We shall omit here, and in what follows, the common factor $\exp(ikz-i\omega t)$. The field diffracted by the aperture is:

$$u_2 = A(x)\exp\left(\frac{i\pi x^2}{\lambda f}\right).$$

The total field is

$$u(x) = u_1 + u_2 \ ,$$

and the intensity

$$I(x) = |u(x)|^2 = A_0^2 + A^2(x) + 2A_0 A(x)\cos\left(\beta x + \frac{\pi x^2}{\lambda f}\right).$$

The intensity distribution contains some information about the phase of the diffracted wave thanks only to the presence of the reference beam.

8.99 The transmissivity $T(x)$ of the developed photographic emulsion is

$$T(x) \propto [I(x)]^{-\frac{1}{2}\gamma} = A_0^{-\gamma}\left[1 + \frac{A^2(x)}{A_0^2} + 2\frac{A(x)}{A_0}\cos\left(\beta x + \frac{\pi x^2}{\lambda f}\right)\right]^{-\frac{1}{2}\gamma}$$

$$\approx A_0^{-\gamma-2}\left[A_0^2 - \tfrac{1}{2}\gamma A^2(x) - \gamma A_0 A(x)\cos\left(\beta x + \frac{\pi x^2}{\lambda f}\right)\right] ,$$

if we use the condition $A_0 \gg A(x)$. We can rewrite the last equation in the form

$$T(x) \propto 2A_0^2 - \gamma A^2(x) - \gamma A_0 A(x)\exp\left[\mathrm{i}\left(\beta x + \frac{\pi x^2}{\lambda f}\right)\right]$$
$$- \gamma A_0 A(x)\exp\left[-\mathrm{i}\left(\beta x + \frac{\pi x^2}{\lambda f}\right)\right] . \qquad (8.99.1)$$

This is called Gabor's hologram equation.

When the hologram is illuminated by a plane monochromatic light wave $A_0\exp[\mathrm{i}(kz - \omega t)]$ a wave field appears behind the hologram which is the result of diffraction by the hologram. This field can be obtained simply

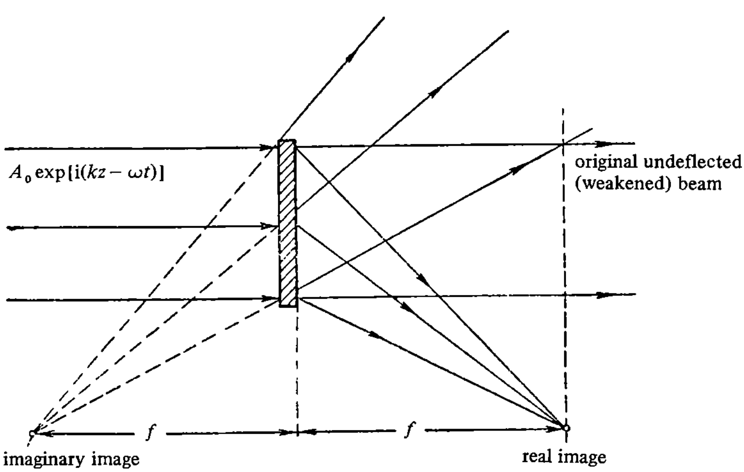


Figure 8.99.1.

(see problem 8.97) by multiplying the original wave field $A'_0\exp[i(kz-\omega t)]$ by the transmissivity $T(x)$ given by Gabor's formula (8.99.1). We then get a field of the form

$$\begin{aligned} u \sim\ & [2A_0^2-\gamma A^2(x)]\exp(\mathrm{i}kz-\mathrm{i}\omega t) \\ & -\gamma A_0 A(x)\exp(\mathrm{i}kz-\mathrm{i}\omega t)\exp\left(\mathrm{i}\beta x-\mathrm{i}\frac{\pi x^2}{\lambda f}\right) \\ & -\gamma A_0 A(x)\exp(\mathrm{i}kz-\mathrm{i}\omega t)\exp\left(-\mathrm{i}\beta x-\mathrm{i}\frac{\pi x^2}{\lambda f}\right). \end{aligned} \tag{8.99.2}$$

The first term in equation (8.99.2) corresponds to the nonuniform diffraction [due to $A^2(x)$] weakening of the incident wave. The angle of diffraction is small since $A(x)$ is a smoothly varying function as compared to the exponentials. The second term acts as a combination of a prism which deflects the beams upwards and a diverging lens with a focusing length f (see problems 8.95 and 8.96). The third term behaves as a combination of a prism which deflects the beams downwards and a converging lens. Finally, when a plane monochromatic wave is transmitted through the hologram the original wave fronts are established (figure 8.99.1): the plane wave and the spherical front from the aperture. The latter is reconstructed twice: in the form of a wave from the real image and a wave from the imaginary image.

8.100

$$\begin{aligned} \exp(\mathrm{i}k'z)T(x) \propto\ & \left[2A_0^2-2\gamma A^2\left(1+\cos\frac{4\pi Dx}{\lambda f}\right)\right]\exp(\mathrm{i}k'z) \\ & -\gamma A_0 A\left\{\exp\left[\frac{\mathrm{i}\pi(x-D)^2}{\lambda f}\right]+\exp\left[\frac{\mathrm{i}\pi(x+D)^2}{\lambda f}\right]\right\}\exp(\mathrm{i}\beta x+\mathrm{i}k'z) \\ & -\gamma A_0 A\left\{\exp\left[-\frac{\mathrm{i}\pi(x-D)^2}{\lambda f}\right]+\exp\left[-\frac{\mathrm{i}\pi(x+D)^2}{\lambda f}\right]\right\}\exp(-\mathrm{i}\beta x+\mathrm{i}k'z). \end{aligned}$$

The second and third terms, as in the preceding problem, describe the field deflected upwards and downwards and focused in two pairs of points. However, the focusing lengths of the corresponding diverging and converging lenses are different, namely

$$f' = \frac{\lambda}{\lambda'}f.$$

The linear magnification is given by the formula

$$\frac{2\Delta}{2D} = \frac{p+q}{p} = \frac{\lambda'}{\lambda}\frac{q}{f},$$

where

$$\frac{1}{p}+\frac{1}{q} = \frac{1}{f'} = \frac{\lambda'}{\lambda f},$$

p is the distance of the source of wavelength λ' from the hologram and q the distance of the image from the hologram (figure 8.100.1). To achieve amplification we must use a wavelength $\lambda' > \lambda$, for the reconstruction, and place the source at a finite distance p from the hologram.

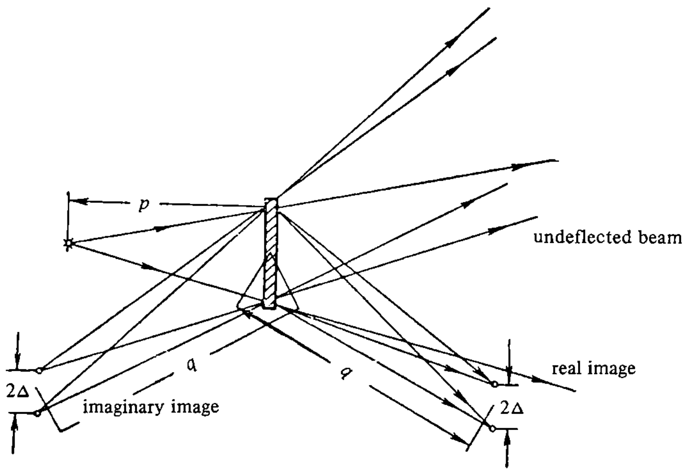


Figure 8.100.1.

8.101 The intensity distribution on the hologram can be transferred without important distortions if the spatial period of the diffraction picture is larger than d,

$$\frac{1}{|\beta + \pi x/\lambda f|} \geqslant d$$

(see the solution of problem 8.98). The maximum size of the hologram in the x-direction, $2x_{\max} \approx 2\lambda f/d$, is restricted by this condition. This size plays the role of the lens diameter in the Rayleigh theory of resolving power (cf problem 8.28). If we apply the Rayleigh criterion for the minimum size s of an object which can be resolved we get

$$s \approx \frac{\lambda}{2\vartheta} \approx \frac{\lambda f}{2x_{\max}} \approx \tfrac{1}{2}d\,.$$

Here ϑ is half the opening angle of the cone of light going from the hologram to the image.

e X-ray diffraction

8.102 It is necessary that the inequality $\omega \gg \omega_{\mathrm{at}}$ is satisfied. However, this is not sufficient. Let us first of all consider the case when the coherence length l is large compared with the dimensions L of the body. For sufficiently small scattering angles $\vartheta < \lambda/L$ the product $qL \ll 1$

and the exponentials in formulae (8.e.2) or (8.e.4) for the cross sections are close to unity: $\int n \exp[\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{r})] \, \mathrm{d}^3\boldsymbol{r} = NZ$. If the wavelength $\lambda \gtrsim L$, this is satisfied for any angle. We then get, e.g. from equation (8.e.2)

$$\mathrm{d}\sigma = r_0^2 N^2 Z^2 \sin^2\theta \, \mathrm{d}\Omega \,. \tag{8.102.1}$$

This formula corresponds to the coherent Thomson scattering by all NZ charges in the body. If, however, to take an example, the coherence length is less than the interatomic distance but larger than the size of an atom, when $\vartheta < \lambda/l$ one can add coherently only the contributions from the Z electrons of an atom, and in equation (8.102.1) one must replace N^2Z^2 by NZ^2. For large values of the angle the magnitude of the cross section decreases steeply because of the fast oscillating factor $\exp[\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{r})]$ in the integrand.

8.103 We can write the electron density in the gas as a sum of terms referring to separate atoms, $n(\boldsymbol{r}) = \sum_{a=1}^{N} n_{\mathrm{at}}(\boldsymbol{r}-\boldsymbol{R}_a)$, where a characterises the instantaneous position of the ath atom. We have then

$$\begin{aligned}\left|\int n(\boldsymbol{r}) \exp[\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{r})] \, \mathrm{d}^3\boldsymbol{r}\right|^2 &= \left|\sum_a \exp[\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{R}_a)] \int n_{\mathrm{at}}(\boldsymbol{r}') \exp[\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{r}')] \, \mathrm{d}^3\boldsymbol{r}'\right|^2 \\ &= |F_{\mathrm{at}}(\boldsymbol{q})|^2 \left|\sum_a \exp[\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{R}_a)]\right|^2 ,\end{aligned} \tag{8.103.1}$$

where $\boldsymbol{r}' = \boldsymbol{r}-\boldsymbol{R}_a$, and $F_{\mathrm{at}}(\boldsymbol{q})$ is the atomic formfactor (8.e.6). The averaging in equation (8.103.1) must be over all positions $\boldsymbol{R}_a$. Since the atoms in a gas are distributed randomly we have

$$\overline{\left|\sum_a \exp[\mathrm{i}(\boldsymbol{q} \cdot \boldsymbol{R}_a)]\right|^2} = N \,.$$

Altogether, for unpolarised radiation we have

$$\mathrm{d}\sigma = \tfrac{1}{2} r_0^2 (1+\cos^2\vartheta) |F_{\mathrm{at}}(\boldsymbol{q})|^2 \, \mathrm{d}\Omega \,. \tag{8.103.2}$$

For the density $n_{\mathrm{at}}(\boldsymbol{r})$ given in the problem the evaluation of the formfactor is elementary and we get

$$F_{\mathrm{at}}(q) = \frac{8\pi}{a(a^{-2}+q^2)^2} \,.$$

Finally

$$\mathrm{d}\sigma(\vartheta) = 32\pi^2 \frac{r_0^2}{a^2} n_{0\,\mathrm{at}} N \frac{1+\cos^2\vartheta}{[a^{-2}+(4\pi/\lambda)^2 \sin^2\frac{1}{2}\vartheta]^4} \, \mathrm{d}\Omega \,.$$

From the experimentally observed cross section [equation (8.103.2)] we can obtain the absolute magnitude of the formfactor. To find the electron distribution, in general, we also need to know the phase of the formfactor.

8.104

$$\mathrm{d}\sigma = \tfrac{1}{2}Nr_0^2(1+\cos^2\vartheta)|F_{\mathrm{at}}(q)|^2 2\left(1+\frac{\sin qR}{qR}\right)\mathrm{d}\Omega\,.$$

The cross section differs from the cross section for scattering by isolated atoms by the structure factor $2(1+\sin qR/qR)$, which depends on the relative position of the atoms in a molecule.

8.105

$$\overline{\mathrm{d}\sigma} = Nr_0^2(1+\cos^2\vartheta)|F_{\mathrm{at}}(q)|^2\frac{1}{b\sqrt{\pi}}\int_{-\infty}^{\infty}\left[1+\frac{\sin q(R_0+x)}{q(R_0+x)}\right]\exp\left(-\frac{x^2}{b^2}\right)\mathrm{d}x\,.$$

The relative magnitude of $1/q$ and b is important. When $q \gg 1/b$ the fast oscillating term with $\sin[q(R_0+x)]$ will vanish. The thermal motion annihilates the structure effect under those conditions. When $q \ll 1/b$ the structure factor has the same form $1+(\sin qR_0)/qR_0$ as for the case of nonmoving nuclei.

8.107 We take the x-, y-, and z-axes along the edges L_1, L_2, and L_3 of the single crystal.

$$\begin{aligned}\int n(\boldsymbol{r})\exp[\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{r})]\,\mathrm{d}^3r &= F_{\mathrm{at}}(\boldsymbol{q})\sum_{\boldsymbol{R}}\exp\mathrm{i}(\boldsymbol{q}\cdot\boldsymbol{R})\\ &= F_{\mathrm{at}}(\boldsymbol{q})\left[\sum_{n_1=0}^{N_1}\exp(\mathrm{i}q_x a n_1)\right]\left[\sum_{n_2=0}^{N_2}\exp(\mathrm{i}q_y a n_2)\right]\left[\sum_{n_3=0}^{N_3}\exp(\mathrm{i}q_z a n_3)\right]\\ &= F_{\mathrm{at}}(\boldsymbol{q})\left[\frac{1-\exp(\mathrm{i}q_x aN_1)}{1-\exp(\mathrm{i}q_x a)}\right]\left[\frac{1-\exp(\mathrm{i}q_y aN_2)}{1-\exp(\mathrm{i}q_y a)}\right]\left[\frac{1-\exp(\mathrm{i}q_z aN_3)}{1-\exp(\mathrm{i}q_z a)}\right],\end{aligned}$$

where $N_1 = L_1/a$, $N_2 = L_2/a$, and $N_3 = L_3/a$ are the number of elementary cells along the edges L_1, L_2, and L_3; clearly $N = N_1N_2N_3$. Using equation (8.e.4) we get

$$\begin{aligned}\mathrm{d}\sigma &= \tfrac{1}{2}r_0^2(1+\cos^2\vartheta)|F_{\mathrm{at}}(\boldsymbol{q})|^2\\ &\times\left[\frac{\sin^2(\frac{1}{2}q_x aN_1)}{\sin^2(\frac{1}{2}q_x a)}\right]\left[\frac{\sin^2(\frac{1}{2}q_y aN_2)}{\sin^2(\frac{1}{2}q_y a)}\right]\left[\frac{\sin^2(\frac{1}{2}q_z aN_3)}{\sin^2(\frac{1}{2}q_z a)}\right]\mathrm{d}\Omega\,.\end{aligned}\tag{8.107.1}$$

The positions of the main maxima can be obtained from the condition that the denominators vanish, whence it follows that $q_x = 2\pi m_x/a$, $q_y = 2\pi m_y/a$, $q_z = 2\pi m_z/a$, where m_x, m_y, and m_z are integers. These equations are the von Laue equations written down in the components, since the components of $\boldsymbol{g}$ can be expressed in terms of the equations $\boldsymbol{g} = (m_x/a, m_y/a, m_z/a)$. In the maxima the cross section is

$$\mathrm{d}\sigma = \tfrac{1}{2}r_0^2(1+\cos^2\vartheta)|F_{\mathrm{at}}(2\pi\boldsymbol{g})|^2\frac{(L_1L_2L_3)^2}{a^6}\mathrm{d}\Omega\,.$$

It is proportional to the square of the volume of the crystal. The results of problems 8.107 to 8.111 are valid only if the single crystal is completely within the coherence volume (see section d of chapter 8).

8.108

$$d\sigma = \tfrac{1}{2}r_0^2(1+\cos^2\vartheta)|F_{at}(q)|^2$$
$$\times \frac{\sin^2(\frac{1}{2}q_z aN_3)}{4\sin^2(\frac{1}{2}q_y a)\sin^2(\frac{1}{2}q_z a)}\left[\left\{\frac{\sin(\frac{1}{2}q_x aN_1)}{\sin(\frac{1}{2}q_x a)} - \frac{\sin[\frac{1}{2}(q_x+q_y)aN_1]}{\sin[\frac{1}{2}(q_x+q_y)a]}\right\}^2\right.$$
$$\left. + 4\sin^2(\tfrac{1}{2}q_y aN_1)\frac{\sin(\frac{1}{2}q_x aN_1)\sin[\frac{1}{2}(q_x+q_y)aN_1]}{\sin(\frac{1}{2}q_x a)\sin[\frac{1}{2}(q_x+q_y)a]}\right] d\Omega ,$$

where $N_1 = L_1/a$, $N_3 = L_3/a$. The positions of the main maxima are expressed in terms of the von Laue condition: $\boldsymbol{q} = 2\pi\boldsymbol{g}$, where $\boldsymbol{g} = (m_x/a, m_y/a, m_z/a)$. In the maxima the cross section is

$$d\sigma = \tfrac{1}{2}r_0^2(1+\cos^2\vartheta)|F_{at}(2\pi\boldsymbol{g})|^2\frac{(L_1^2L_3)^2}{4a^6}\,d\Omega .$$

The angle ϑ is connected with $\boldsymbol{q} = 2\pi\boldsymbol{g}$ through equation (8.e.3).

8.109 When $k \gg 1/a$ the diffraction picture is concentrated in the small angle region since, according to equation (8.e.3) and the von Laue equation, $k\vartheta = 2\pi g \sim 1/a$ and $\vartheta \sim 1/ak \ll 1$; in that case $q \ll k$.

If we write $\boldsymbol{\kappa} = \boldsymbol{q} - 2\pi\boldsymbol{g}$, in the region of the diffraction spot close to a given main maximum the magnitude of κ is given by $\kappa \ll 2\pi g \ll k$. If we take the square of the equation

$$\boldsymbol{k} = \boldsymbol{k}_0 + 2\pi\boldsymbol{g} + \boldsymbol{\kappa}$$

and note that $k^2 = k_0^2$ and

$$(\boldsymbol{g}\cdot\boldsymbol{k}_0) = -\pi g^2 , \tag{8.109.1}$$

we then find that $([\boldsymbol{k}_0 + 2\pi\boldsymbol{g}]\cdot\boldsymbol{\kappa}) + \kappa^2 = 0$, whence it is clear that when $\kappa \ll g$ we have $\boldsymbol{\kappa} \perp \boldsymbol{k}_0 + 2\pi\boldsymbol{g}$, i.e. the extra vector $\boldsymbol{\kappa}$ is at right angles to the wavevector corresponding to scattering in the direction of the main maximum. The equation $([\boldsymbol{k}_0 + 2\pi\boldsymbol{g}]\cdot\boldsymbol{\kappa}) = 0$ can be written in the form $\kappa_z \approx -2\pi[g_x/k_0)\kappa_x + (g_y/k_0)\kappa_y]$, from which it is clear that $|\kappa_z| \ll |\kappa_x|, |\kappa_y|$. With the aid of equation (8.109.1) we find that the ratio

$$\frac{\sin^2(\frac{1}{2}q_z aN_3)}{\sin^2(\frac{1}{2}q_z a)} = \frac{\sin^2(\frac{1}{2}\kappa_z aN_3)}{\sin^2(\frac{1}{2}\kappa_z a)}$$

is an appreciably flatter function of κ_z than the first two ratios, and it can be replaced by its value N_3^2 in the maximum ($\kappa_z = 0$). The cross section becomes ($\vartheta \ll 1$)

$$d\sigma = 4r_0^2|F_{at}(2\pi\boldsymbol{g})|^2N_3^2\,\frac{\sin^2(\frac{1}{2}\kappa_x aN_1)}{(\frac{1}{2}\kappa_x a)^2}\,\frac{\sin^2(\frac{1}{2}\kappa_y aN_2)}{(\frac{1}{2}\kappa_y a)^2}\,d\Omega ,$$

from which it is clear that the angular width of the main maximum is of the order of magnitude of $(kaN_1)^{-1}$ and $(kaN_2)^{-1}$ in the x- and y-directions respectively. If we write the element of solid angle in the form $d\Omega = d\kappa_x\, d\kappa_y/k^2$ and integrate over κ_x and κ_y from $-\infty$ to $+\infty$ we get

$$\sigma = 4r_0^2|F_{at}(2\pi g)|^2\left(\frac{\pi}{ak}\right)^2 N_3^2 N_1 N_2 \, .$$

The cross section depends in different ways on the longitudinal and transverse dimensions. If these are approximately the same, the total cross section is proportional to $V^{4/3}$, where V is the volume of the body, and the angular width is proportional to $(V^{4/3}/V^2)^{1/2} = 1/V^{1/3}$.

8.110

$$d\sigma = 32r_0^2(1+\cos^2\vartheta)|F_{at}(2\pi g)|^2 \times \frac{\sin^2(\frac{1}{2}\kappa_x L_x)}{\kappa_x^2}\frac{\sin^2(\frac{1}{2}\kappa_y L_y)}{\kappa_y^2}\frac{\sin^2(\frac{1}{2}\kappa_z L_z)}{\kappa_z^2}\, d\Omega \, ,$$

where

$$\kappa_x k_{gx} + \kappa_y k_{gy} + \kappa_z k_{gz} = 0 \, , \qquad \boldsymbol{k}_g = \boldsymbol{k}_0 + 2\pi \boldsymbol{g} \, .$$

8.111

$$d\sigma = 8\pi r_0^2(1+\cos^2\vartheta)|F_{at}(2\pi g)|^2 \frac{(\sin\kappa R - \kappa R\cos\kappa R)}{\kappa^6}\, d\Omega \, .$$

9

Electromagnetic oscillations in finite bodies

9.1 For E-waves

$$E_z = E_0 \sin(\kappa_1 x)\sin(\kappa_2 y)\,,$$

where

$$\kappa_1 = \frac{n_1\pi}{a}\,,\qquad \kappa_2 = \frac{n_2\pi}{b}\,,\qquad n_1, n_2 = 1, 2, \ldots\,,$$

and the origin is chosen to lie at the corner of the rectangular cross section with lengths a and b along the x- and y-axes respectively.

For H-waves

$$H_z = H_0\cos(\kappa_1 x)\cos(\kappa_2 y)$$

with the same κ_1, κ_2, but now one of the n_1 and n_2 can take the value zero. It follows that in transverse directions the field is of the standing wave type. The relation between the propagation constant k and ω is of the form

$$k^2 = \frac{\omega^2}{c^2} - \pi^2\left(\frac{n_1^2}{a^2}+\frac{n_2^2}{b^2}\right).$$

The transverse field components may be expressed in terms of E_z, H_z with the aid of Maxwell's equations.

9.2 For E-waves

$$\alpha = \frac{2\zeta'\omega}{ck\kappa^2 ab}(\kappa_1^2 b + \kappa_2^2 a)\,,$$

where

$$\kappa^2 = \kappa_1^2 + \kappa_2^2\,,\qquad \zeta' = \operatorname{Re}\zeta\,.$$

For H_{n0} waves

$$\alpha = \frac{\zeta'\omega}{ckab}\; a + \frac{2\kappa^2 b}{k^2+\kappa^2}\quad .$$

For $H_{n_1 n_2}$ $(n_1,\, n_2 \neq 0)$:

$$\alpha = \frac{2c\kappa^2\zeta'}{\omega kab}\left[a + b + \frac{k^2}{\kappa^4}(\kappa_1^2 a + \kappa_2^2 b)\right]\,.$$

The notation is the same as in the preceding problem.

9.3 E-waves.

(a) Even solutions $[E_x(x) = E_x(-x),\ H_y(x) = H_y(-x),\ E_z(x) = -E_z(-x)]$: when $x > a$

$$E_z = A\exp(-sx)\,,\qquad E_x = \frac{ik}{s}A\exp(-sx)\,,\qquad H_y = \frac{i\omega}{sc}A\exp(-sx)\,;$$

when $-a \leqslant x \leqslant a$

$$E_z = B \sin \kappa x\,, \qquad E_x = \frac{\mathrm{i}k}{\kappa} B \cos \kappa x\,, \qquad H_y = \frac{\mathrm{i}\omega\epsilon}{\kappa c} B \cos \kappa x\,;$$

when $x < -a$

$$E_x = -A \exp(sx)\,, \qquad E_x = \frac{\mathrm{i}k}{s} A \exp(sx)\,, \qquad H_y = \frac{\mathrm{i}\omega}{sc} A \exp(sx)$$

where $A = B \exp(sa) \sin \kappa a$; the remaining components of E and H are zero. The parameters κ and s may be determined from the following equations

$$(\kappa a)^2 + (sa)^2 = \frac{\omega^2 a^2}{c^2}(\epsilon\mu - 1)\,,$$

$$sa = \frac{1}{\epsilon} \kappa a \tan \kappa a\,. \tag{9.3.1}$$

These equations can easily be solved graphically. The possible values of κ and s correspond to the points of intersection of the curves defined by equation (9.3.1) with the circle of radius $r = (\omega a/c)(\epsilon\mu - 1)^{1/2}$ (figure 9.3.1). For given ω, a, ϵ, and μ there are a finite number of points of intersection, i.e. a finite number of types of wave, for which the field distribution is described by the above equations. In particular, when $r < \pi$, there is a single E_{00} wave.

Consider now the propagation constant

$$k = \left(\frac{\omega^2 \epsilon\mu}{c^2} - \kappa^2\right)^{1/2} = \left(\frac{\omega^2}{c^2} + s^2\right)^{1/2} \tag{9.3.2}$$

for given parameters of the dielectric layer and given types of wave. It is clear from figure 9.3.1 that for frequencies near the limiting frequency at which the given type of wave appears, the parameter s approaches zero, while k tends to ω/c. At such frequencies the wave has the same propagation constant as it has in vacuo, and the field penetrates to large distances in the layer. The parameter s increases with increasing ω

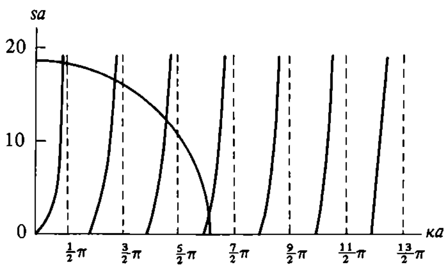


Figure 9.3.1.

while κ remains bounded. At the same time, k tends to $(\omega/c)(\epsilon\mu)^{1/2}$, i.e. to a value which corresponds to a wave propagating in an infinite dielectric medium with parameters ϵ, μ. For sufficiently large ω, and therefore large s, the field is almost entirely concentrated within the dielectric layer.

(b) Odd solutions $[E_x(x) = -E_x(-x),\ H_y(x) = -H_y(-x),\ E_z(x) = E_z(-x)]$:

when $x > a$

$$E_z = A\exp(-sx)\,, \qquad E_x = \frac{ik}{s}A\exp(-sx)\,, \qquad H_y = \frac{i\omega}{sc}A\exp(-sx)\,;$$

when $-a \leqslant x \leqslant a$

$$E_z = B\cos\kappa x\,, \qquad E_x = -\frac{ik}{\kappa}B\sin\kappa x\,, \qquad H_y = -\frac{i\omega\epsilon}{\kappa c}B\sin\kappa x\,;$$

when $x < -a$

$$E_z = A\exp(sx)\,, \qquad E_x = -\frac{ik}{s}A\exp(sx)\,, \qquad H_y = -\frac{i\omega}{sc}A\exp(sx)\,,$$

where $A = B\exp(sa)\cos\kappa a$; the remaining components of E and H are zero. The parameters s and κ may be determined from the equations

$$(\kappa a)^2 + (sa)^2 = \frac{\omega^2 a^2}{c^2}(\epsilon\mu - 1)\,, \qquad sa = -\frac{1}{\epsilon}\kappa a\cot\kappa a\,.$$

The propagation constant k is related to κ and s by equation (9.3.2).

It is easily shown by a graphical method that odd electric waves cannot exist when $r < \frac{1}{2}\pi$. The remaining properties are the same as for even waves.

H-waves may be analysed in a similar way.

9.4 Even *E*-waves and odd *H*-waves can propagate along the layer, and their characteristics (propagation constant, field configuration in the region $x > 0$, and so on) are the same as in the preceding problem.

9.5 *E*-waves.

In order to determine these waves, it is necessary to solve the following equation for the longitudinal component of the electric field:

$$\frac{\partial^2 E_z}{\partial r^2} + \frac{1}{r}\frac{\partial E_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 E_z}{\partial\phi^2} + \kappa^2 E_z = 0\,.$$

This equation can be integrated by the method of separation of variables. Special solutions are of the form

$$E_z(r, \phi) = J_m(\kappa r)\sin(m\phi + \psi_m)\,,$$

where J_m is the Bessel function of order m and ψ_m is an arbitrary constant. In order that the field should return to its original value when ϕ changes by 2π, the quantity m must be an integer ($m = 0, 1, 2, \ldots$).

The transverse components of the electric and magnetic fields can be expressed in terms of E_z with the aid of Maxwell's equations:

$$E_r = \frac{ik}{\kappa}J'_m(\kappa r)\sin(m\phi + \psi_m)\,, \qquad E_\phi = \frac{imk}{\kappa^2 r}J_m(\kappa r)\cos(m\phi + \psi_m)\,,$$

$$H_r = -\frac{im\omega}{\kappa^2 cr}J_m(\kappa r)\cos(m\phi + \psi_m)\,, \qquad H_\phi = \frac{i\omega}{\kappa c}J'_m(\kappa r)\sin(m\phi + \psi_m)\,.$$

The possible values of κ are determined by the boundary conditions at the waveguide wall:

$$E_r|_{r=a} = 0\,, \qquad E_\phi|_{r=a} = 0\,.$$

This yields $\kappa_{mn}a = \alpha_{mn}$ where α_{mn} is the nth root of the Bessel function: $J_m(\alpha_{mn}) = 0$, $n = 1, 2, 3, \ldots$.

Thus, the above waves are characterised by two subscripts, namely m and n. When $m = 0$ the field exhibits rotation symmetry with respect to the z-axis. For an ideal waveguide, the phases ψ_m are determined by the conditions of excitation. In reality, however, they are very dependent on defects in the waveguide walls (departures from circular cross section, longitudinal cracks, and so on).

The wave propagation along the waveguide is possible when $k = [(\omega/c)^2 - \kappa^2]^{1/2}$ is real. Hence a wave characterised by m, n will propagate in the waveguide, provided its frequency is such that

$$\omega^2 \geqslant \frac{c^2\alpha_{mn}^2}{a^2}\,.$$

The minimum possible frequency for a (0, 1) wave is

$$\omega_0 = \frac{c\alpha_{01}}{a} \approx 2{\cdot}4\frac{c}{a}\,.$$

The corresponding wavelength is

$$\lambda_0 = \frac{2\pi c}{\omega_0} \approx 2{\cdot}6a\,,$$

which is of the order of the radius of the waveguide.

The H-waves are

$$H_z = J_m(\kappa r)\sin(m\phi + \psi_m) \qquad (m = 0, 1, 2, \ldots)\,.$$

The propagation constant is given by

$$k^2 = \frac{\omega^2}{c^2} - \frac{\beta_{mn}^2}{a^2} \qquad (n = 1, 2, \ldots)\,,$$

where β_{mn} is the nth root of the equation $J'_m(\beta_{mn}) = 0$. The smallest root is $\beta_{11} \approx 1{\cdot}8$, and this corresponds to a limiting frequency $\omega_0 \approx 1{\cdot}8c/a$ and a limiting wavelength $\lambda_0 = 2\pi c/\omega_0 \approx 3{\cdot}5a$.

The limiting frequency is lower for H-waves than for E-waves. When the frequency lies within the limits $\omega_{0,E} > \omega > \omega_{0,H}$ then only H_{11} waves will propagate.

9.6 E-waves:

$$\alpha = \frac{\omega\zeta'}{cak} .$$

H-waves of the (m, n) type:

$$\alpha = \frac{c\zeta'\kappa^2}{\omega ak}\left[1+\frac{m^2\omega^2}{c^2\kappa^2(a^2\kappa^2-m^2)}\right],$$

where $\zeta' = \mathrm{Re}\,\zeta$.

9.7 The wavevector k and the frequency ω of the waves in the waveguide are related by

$$\frac{\omega^2}{c^2} = k^2+\kappa^2 ,$$

where κ is a constant which depends on the type of waves and the transverse dimensions of the waveguide. Hence

$$v_{\mathrm{ph}} = \frac{\omega}{k} = \frac{c}{[1-(\lambda/\lambda_0)^2]^{1/2}} , \qquad v_{\mathrm{g}} = \frac{\mathrm{d}\omega}{\mathrm{d}k} = c[1-(\lambda/\lambda_0)^2]^{1/2} ,$$

where λ_0 is the limiting wavelength.

It is clear from the above formulae that $v_{\mathrm{ph}} > c$, $v_{\mathrm{g}} < c$, and $v_{\mathrm{ph}}v_{\mathrm{g}} = c^2$. This result holds for a waveguide in vacuo (the dielectric properties of air may be neglected for the range of phenomena under consideration).

If the waveguide is filled with a dielectric and the dispersion of ϵ and μ may be neglected, then all the above formulae remain valid, provided c is replaced by $v = c/(\epsilon\mu)^{1/2}$. Hence, for this type of waveguide $v_{\mathrm{ph}} = c/(\epsilon\mu)^{1/2}[1-(\lambda/\lambda_0)^2]^{1/2}$ may be less than c and the wave is 'slowed down' (see problem 9.13).

9.8 $H_z = \frac{1}{2}H_0\{\exp[\mathrm{i}(\kappa_1 x+kz)]+\exp[\mathrm{i}(-\kappa_1 x+kz)]\}\exp(-\mathrm{i}\omega t)$.

The directions of propagation of the two plane waves into which H_{10} is resolved are at an angle θ to the axis of the waveguide (figure 9.8.1),

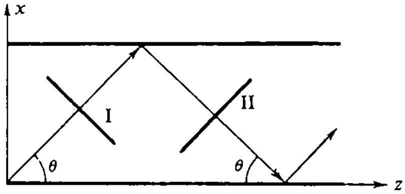


Figure 9.8.1.

where

$$\cos\theta = \frac{k}{(k^2+\kappa_1^2)^{1/2}} = \left[1-\left(\frac{\lambda}{\lambda_0}\right)^2\right]^{1/2}.$$

The phase plane I moves with a velocity c in the direction making an angle θ with the z-axis. However, its velocity along the axis of the waveguide will be larger:

$$v = \frac{c}{\cos\theta} = \frac{c}{[1-(\lambda/\lambda_0)^2]^{1/2}} = v_{\mathrm{ph}}\ .$$

This is, in fact, the phase velocity of the wave in the waveguide.

The group velocity is equal to the rate of transport of energy. However, for a plane wave propagating in a vacuum, the energy is transported with a velocity c in the direction of propagation of the wave. Each component plane wave of $H_{1\,0}$ will undergo multiple reflections from the walls of the waveguide, and will therefore travel over a zigzag path. The resultant velocity along the axis of the waveguide will then be

$$v = c\cos\theta = c\left[1-\left(\frac{\lambda}{\lambda_0}\right)^2\right]^{1/2},$$

which is equal to the group velocity v_g.

9.9

$$H_\phi = E_r = \frac{A}{r}\exp\left[\mathrm{i}\omega\left(\frac{z}{c}-t\right)\right], \tag{9.9.1}$$

where A is a constant, and the remaining components are all equal to zero.

The energy flux is given by

$$\bar{\gamma} = \frac{A^2c}{4}\ln\frac{b}{a}\ .$$

For a single perfect conductor, the field outside the conductor is given by equation (9.9.1) and the total energy flux through the z = constant plane is infinite ($\gamma \to \infty$ when $b \to \infty$). It follows that this type of wave cannot be maintained by a source of finite power, and therefore this case is of no physical significance.

9.10 E-waves:

$$E_z = [A_{mn}J_m(\kappa_{mn}r)+B_{mn}N_m(\kappa_{mn}r)]\sin(m\phi+\psi_m)\,, \quad m = 0, 1, 2, \ldots,$$

where κ_{mn} is the nth root of the equation

$$J_m(\kappa a)N_m(\kappa b)-J_m(\kappa b)N_m(\kappa a) = 0\ .$$

In this expression N_m and J_m are the cylindrical functions (see appendix 3) and A_{mn} and B_{mn} are constants which are related by

$$A_{mn}J_m(\kappa_{mn}a)+B_{mn}N_m(\kappa_{mn}a) = 0\ .$$

H-waves:

$$H_z = [C_{mn}J_m(\kappa_{mn}r)+D_{mn}N_m(\kappa_{mn}r)]\sin(m\phi+\psi_m), \quad m = 0, 1, 2, \dots,$$

where κ_{mn} is the nth root of the equation

$$J'_m(\kappa a)N'_m(\kappa b)-N'_m(\kappa a)J'_m(\kappa b) = 0,$$

and C_{mn} and D_{mn} are related by

$$C_{mn}J'_m(\kappa_{mn}a)+D_{mn}N'_m(\kappa_{mn}a) = 0.$$

The remaining components of the electric and magnetic fields may be expressed in terms of E_z and H_z by means of Maxwell's equations.

9.11

$$\alpha = \frac{\zeta'(a+b)}{2ab\ln(b/a)},$$

where $\zeta' = \mathrm{Re}\,\zeta$.

9.12 If the field is symmetric with respect to the axis of the conductor, then the longitudinal component E_z will satisfy the following differential equation:

$$\frac{d^2E_z}{dr^2}+\frac{1}{r}\frac{dE_z}{dr}+\kappa^2E_z = 0. \tag{9.12.1}$$

Since the conductor has a finite conductivity the parameters k and κ will be complex. The sign of κ will be defined so that $\mathrm{Im}\,\kappa = \kappa'' > 0$.

The general solution of equation (9.12.1) may be written in the form

$$E_z(r) = A'H_0^{(1)}(\kappa r)+B'H_0^{(2)}(\kappa r),$$

where $H_0^{(1)}$ and $H_0^{(2)}$ are Hankel functions. It follows from the asymptotic behaviour of these functions (see appendix 3) and the condition $\mathrm{Im}\,\kappa > 0$ that $B' = 0$ since otherwise the field would diverge at infinity. The remaining components of E and H can be expressed in terms of E_z with the aid of Maxwell's equations:

$$E_z = A'H_0^{(1)}(\kappa r), \qquad E_r = \frac{ik}{\kappa}A'H_1^{(1)}(\kappa r),$$

$$H_\phi = \frac{i\omega}{\kappa c}A'H_1^{(1)}(\kappa r). \tag{9.12.2}$$

For sufficiently large κr, the functions $H_0^{(1)}$ and $H_1^{(1)}$ are proportional to $(\kappa r)^{-1/2}\exp(-\kappa'' r)$, and therefore the electromagnetic field is damped out exponentially at large distances from the conductor. The maximum concentration of the field exists near the conductor, and hence the wave is a surface type wave.

The boundary condition

$$E_z = \zeta H_\phi$$

at the surface of the conductor leads to the following characteristic equation for κ:

$$\kappa a \frac{H_0^{(1)}(\kappa a)}{H_1^{(1)}(\kappa a)} = \mathrm{i}\zeta \frac{\omega}{c} a ,$$

where ζ is the surface impedance of the metal. For a good conductor $|\zeta| \ll 1$, and hence the latter equation can only be satisfied for small κa. If we use the approximate formulae for $H_0^{(1)}$ and $H_1^{(1)}$ (appendix 3) we have

$$(\kappa a)^2 \ln\left(\frac{\gamma \kappa a}{2\mathrm{i}}\right) = \mathrm{i}\zeta \frac{\omega}{c} a , \qquad \ln\gamma = 0{\cdot}5772 . \tag{9.12.3}$$

This transcendental equation cannot be solved by a graphical method because it involves complex quantities. It was solved by Sommerfeld, who used an iteration method based on the fact that $\ln \kappa a$ varies much more slowly than κa. Let $(\gamma \kappa a/2\mathrm{i})^2 = u$, $-\mathrm{i}\gamma^2 \omega \zeta a/2c = v$. Equation (9.12.3) can then be written in the form

$$u \ln u = v .$$

If an approximate solution u_n (nth order approximation) is known, then a more accurate value u_{n+1} [$(n+1)$st approximation] may be obtained from the formula

$$u_{n+1} \ln u_n = v .$$

In the zero-order approximation $u_0 = v$, and hence

$$u_1 = \frac{v}{\ln v} , \qquad u_2 = \frac{v}{\ln(v/\ln v)} , \qquad u_3 = \frac{v}{\ln[v/\ln(v/\ln v)]} , \dots .$$

For decimeter waves ($\lambda = 2\pi c/\omega = 3 \times 10^{-1}$ m), which propagate along a copper conductor of radius equal to 1 mm (the conductivity of copper is $5{\cdot}2 \times 10^{17}$ s^{-1}), the above method and equations (8.a.9) to (8.a.11) yield

$$u \approx (4{\cdot}2 + 4{\cdot}5\mathrm{i}) \times 10^{-8} ,$$

and hence

$$k = \frac{\omega}{c}[1 + (6{\cdot}0 + 6{\cdot}4\mathrm{i}) \times 10^{-5}] .$$

The phase velocity is given by

$$v_{\mathrm{ph}} = \frac{\omega}{\mathrm{Re}\, k} = (1 - 6 \times 10^{-5})c < c ,$$

and hence the wave is slightly slowed down.

This result can be understood from the following considerations. For a perfect conductor, the transverse electromagnetic wave has phase velocity c

and the field inside the conductor is zero. In the case of a finite conductivity a proportion of the energy will propagate within the conductor, and since the velocity of propagation in the metal is appreciably less than c, it follows that the electromagnetic wave will, on the average, be slowed down. Moreover, damping will appear.

Consider the nature of the field in the limiting case $\zeta \to 0$ (perfect conductor). It follows from equation (9.12.3) that $\kappa \to 0$, $k \to \omega/c$. Using the expressions for $H_0^{(1)}$ and $H_1^{(1)}$ for small arguments of these functions, we have from equation (9.12.2)

$$E_z = \lim_{\kappa \to 0} \frac{2iA'}{\pi} \ln\left(\frac{\gamma\kappa r}{2i}\right), \qquad E_r = \lim_{\kappa \to 0} \frac{2kA'}{\pi\kappa^2}\frac{1}{r},$$

$$H_\phi = \lim_{\kappa \to 0} \frac{2kA'}{\pi\kappa^2}\frac{1}{r}.$$

Since the field components cannot become infinite, it must be assumed that A' is proportional to κ^2. Let $A' = A\kappa^2/k$, so that

$$E_r = H_\phi = \frac{A}{r}, \qquad E_z = 0.$$

This is a purely transverse electromagnetic wave propagating with a velocity c.

9.13 The components of the electromagnetic field in the waveguide are given by the following expressions:

when $r \leqslant a$

$$E_z = E_0 J_0(\kappa_1 r), \qquad E_r = -i\frac{k}{\kappa_1}E_0 J_1(\kappa_1 r), \qquad H_\phi = -i\frac{\omega}{c\kappa_1}E_0 J_1(\kappa_1 r);$$

when $a \leqslant r \leqslant b$

$$E_z = AJ_0(\kappa_2 r) + BN_0(\kappa_2 r), \qquad E_r = -i\frac{k}{\kappa_2}[AJ_1(\kappa_2 r) + BN_1(\kappa_2 r)],$$

$$H_\phi = -i\frac{\epsilon\omega}{c\kappa_2}[AJ_1(\kappa_2 r) + BN_1(\kappa_2 r)],$$

where

$$\kappa_1 = \left(\frac{\omega^2}{c^2} - k^2\right)^{1/2}, \qquad \kappa_2 = \left(\frac{\epsilon\omega^2}{c^2} - k^2\right)^{1/2},$$

and E_0, A, and B are constants.

The boundary conditions may be written in the form

$$E_z|_{r=b} = 0, \qquad E_z|_{r=a-0} = E_z|_{r=a+0}, \qquad H_\phi|_{r=a-0} = H_\phi|_{r=a+0},$$

and the boundary condition for E_ϕ will be satisfied automatically. When the constants A, B, and E_0 are eliminated, we have the following transcendental equation relating k and ω:

$$\frac{\epsilon\kappa_1}{\kappa_2}\frac{J_0(\kappa_1 a)}{J_1(\kappa_1 a)} = \frac{J_0(\kappa_2 a)N_0(\kappa_2 b)-N_0(\kappa_2 a)J_0(\kappa_2 b)}{J_1(\kappa_2 a)N_0(\kappa_2 b)-N_1(\kappa_2 a)J_0(\kappa_2 b)} \,. \tag{9.13.1}$$

This equation can be considerably simplified when $a \ll b$. Consider the wave which has the maximum k. If the waveguide were fully filled with a dielectric ($a = 0$), then the corresponding value of κ_2 would be $\kappa_{02} = \alpha_{01}/b$ where $\alpha_{01} = 2{\cdot}4$, $J_0(\alpha_{01}) = 0$ (see problem 9.5).

We shall seek a solution which is not very different from κ_{02}:

$$\kappa_2 = \kappa_{02}+\kappa_2' = \frac{\alpha_{01}}{b}+\frac{\Delta\alpha}{b}\,,$$

where the order of magnitude of $\Delta\alpha$ is not lower than that of a/b. Assuming that $\alpha_{01}(a/b) \ll 1$, and using the approximate formulae for J_0, N_0, J_1, and N_1 (appendix 3), we have instead of equation (9.13.1)

$$\epsilon\left[(\kappa_2 a)^2 N_0(\kappa_2 b)+\frac{2}{\pi}J_0(\kappa_2 b)\right] = (\kappa_2 a)^2\left[N_0(\kappa_2 b)+\frac{2}{\pi}\ln\frac{2}{\gamma\kappa_2 a}J_0(\kappa_2 b)\right].$$

If we substitute

$$N_0(\kappa_2 b) = N_0(\alpha_{01}+\Delta\alpha) \approx N_0(\alpha_{01})\,, \qquad J_0(\kappa_2 b) = -J_1(\alpha_{10})\Delta\alpha\,,$$

and neglect the small logarithmic term, we have

$$\Delta\alpha = \left(1-\frac{1}{\epsilon}\right)\frac{\pi\alpha_{01}^2}{4}\frac{N_0(\alpha_{01})}{J_1(\alpha_{01})}\left(\frac{a}{b}\right)^2 .$$

The phase velocity is then given by

$$v_{\text{ph}} = \frac{\omega}{k} = \frac{\omega}{[\epsilon\omega^2/c^2-(\alpha_{01}^2+2\alpha_{01}\Delta\alpha)/b^2]^{1/2}}\,.$$

By substituting $\omega_0 = \alpha_{01}(c/b) \approx 2{\cdot}4(c/b)$ (minimum frequency for a waveguide not containing a dielectric) and using tabulated values for $N_0(\alpha_{01})$ and $J_1(\alpha_{01})$ we have

$$v_{\text{ph}} = c\left\{\epsilon-\left(\frac{\omega_0}{\omega}\right)^2\left[1+3{\cdot}7\left(1-\frac{1}{\epsilon}\right)\frac{a^2}{b^2}\right]\right\}^{-1/2} \tag{9.13.2}$$

When the waveguide is completely filled with the dielectric ($a = 0$), the phase velocity is given by

$$v_{\text{ph}} = c\left[\epsilon-\left(\frac{\omega_0}{\omega}\right)^2\right]^{-1/2}.$$

The limiting frequency of a partially filled waveguide is given by

$$\omega_1 = \frac{\omega_0}{(\epsilon)^{1/2}}\left[1+1{\cdot}85\left(1-\frac{1}{\epsilon}\right)\frac{a^2}{b^2}\right]$$

and lies between the limiting frequencies of an empty and fully filled waveguide:

$$\frac{\omega_0}{(\epsilon)^{1/2}} < \omega_1 < \omega_0 \, .$$

The phase velocity given by equation (9.13.2) becomes smaller than c at frequencies

$$\omega > \frac{\omega_0}{(\epsilon-1)^{1/2}}\left[1 + 1{\cdot}85\left(1 - \frac{1}{\epsilon}\right)\frac{a^2}{b^2}\right].$$

Thus, a partially or fully-filled waveguide is a retarding system in which the phase velocity of electromagnetic waves may be less than c. An important property of slow waves is the fact that they can effectively interact with charged particle beams. This interaction may be used both for the generation and the amplification of ultra-high frequency electromagnetic oscillations (backward wave oscillator, magnetron) and for the acceleration of particles (linear accelerator).

9.14 The boundary conditions at an anisotropically conducting plane are

$$E_{1x} = E_{2x} = 0 \, , \qquad H_{1x} = H_{2x} \, , \qquad E_{1z} = E_{2z} \, .$$

The index 1 corresponds to the region $y > 0$, and the index 2 to the region $y < 0$. The first two equations occur as a consequence of the perfect conductivity of the strips, and the last two express that there is no current in the direction at right angles to the strips. Moreover, $E_y = E_z = 0$ for $x = \pm a$ and all field components must be bounded as $y \to \pm\infty$.

If we solve the Maxwell equations with these boundary conditions we find

$$E_{1x} = 0 \, , \qquad E_{1y} = -B\exp(-\beta y)\cos\alpha x \, , \qquad E_{1z} = \mathrm{i}B\frac{\beta}{k}\exp(-\beta y)\cos\alpha x \, ,$$

$$H_{1x} = B\left(\frac{k_0\epsilon}{k} - \frac{\alpha^2}{k_0 k}\right)\exp(-\beta y)\cos\alpha x \, ,$$

$$H_{1y} = B\frac{\alpha\beta}{kk_0}\exp(-\beta y)\sin\alpha x \, , \qquad H_{1z} = -\mathrm{i}B\frac{\alpha}{k_0}\exp(-\beta y)\sin\alpha x \, ,$$

where B is a constant and $k_0 = \omega/c$,

$$\alpha = \alpha_m = \tfrac{1}{2}(2m+1)\pi \, ,$$

$$\beta = \beta_m = (k^2 - k_0^2\epsilon + \alpha_m^2)^{1/2} \, , \qquad m = 0, 1, 2, \ldots \, .$$

The wavevector can be expressed in terms of ω through the formula

$$k = k_m = \left[\frac{(k_0^2\epsilon/\alpha_m^2 - 1)(1 - k_0^2/\alpha_m^2)}{1 - (\epsilon+1)k_0^2/\alpha_m^2}\right]^{1/2} .$$

For a given m the wave can propagate provided its frequency ω lies within the range

$$\frac{1}{(\epsilon)^{1/2}} \leqslant \frac{\omega}{c\alpha_m} \leqslant \left(\frac{2}{\epsilon+1}\right)^{1/2};$$

k varies then from 0 to ∞.

If $\epsilon = 1$ (there is no dielectric) the system becomes a resonator: oscillations with discrete frequencies $\omega_m = c\alpha_m$ are possible in it. When $\epsilon > 1$ the setup considered is a retarding system. The group and phase velocities in it are less than the velocity of light.

9.15 E-waves cannot exist in this instance. The H-waves are given by

$$H_z = \frac{iE_0 c}{\omega\mu}\left(\kappa\cos\kappa x - k\frac{\mu_a}{\mu_\perp}\sin\kappa x\right), \qquad H_x = \frac{E_0 c}{\omega\mu}\left(k\sin\kappa x - \kappa\frac{\mu_a}{\mu_\perp}\cos\kappa x\right),$$

$$E_z = E_0\sin\kappa x, \qquad E_x = E_y = H_y = 0,$$

where

$$\kappa = \frac{n\pi}{a}, \qquad k = \left[\frac{\omega^2\epsilon_\parallel\mu}{c^2} - \left(\frac{n\pi}{a}\right)^2\right]^{1/2}, \qquad n = 1, 2, 3, \ldots,$$

$$\mu = \mu_\perp - \frac{\mu_a^2}{\mu_\perp}.$$

The limiting frequency is $\omega_0^{(n)} = c\kappa_n/(\epsilon_\parallel\mu)$.

It follows from the above formulae for H_z and H_x that the configuration of the magnetic field for this type of wave depends on the sign of k, i.e. on the direction of propagation of the wave, and on the sign of μ_a, i.e. on the direction of the constant magnetic field. This effect is associated with the gyrotropy of the medium filling the waveguide.

9.16 Maxwell's equations for the complex conjugate amplitudes E_0^*, H_0^* are of the form

$$\operatorname{curl} E_0^* - ik_0[e_z \wedge E_0^*] = -\frac{i\omega}{c}H_0^*, \qquad \operatorname{curl} H_0^* - ik_0[e_z \wedge H_0^*] = \frac{i\omega}{c}E_0^*.$$

The amplitudes E, H satisfy the equations

$$\operatorname{curl} E + ik[e_z \wedge E] = \frac{i\omega}{c}\hat{\mu}'H, \qquad \operatorname{curl} H + ik[e_z \wedge H] = -\frac{i\omega}{c}\hat{\epsilon}'E,$$

where $\hat{\mu}'H$, $\hat{\epsilon}'E$ are vectors with components $\mu'_{ik}H_k$, $\epsilon'_{ik}E_k$; $\mu'_{ik} = \epsilon'_{ik} = \delta_{ik}$ outside the region filled by the dielectric. Inside the region, $\mu'_{ik} = \mu_{ik}$, $\epsilon'_{ik} = \epsilon_{ik}$.

It follows from the equations for $\operatorname{curl} E_0^*$ and $\operatorname{curl} H$ that

$$H\operatorname{curl} E_0^* - E_0^*\operatorname{curl} H + i(k-k_0)[e_z \wedge E_0^*]H = -i\omega[(H\cdot H_0^*) - \hat{\epsilon}'(E\cdot E_0^*)]. \tag{9.16.1}$$

Integrate both parts of this equation over the cross section of the waveguide S. The first two terms may be transformed as follows:

$$\int_S (H \operatorname{curl} E_0^* - E_0^* \operatorname{curl} H)\, d^2S = \frac{1}{l}\int_V \operatorname{div}[E_0^* \wedge H]\, d^3r .$$

The integral on the right-hand side of this expression is evaluated over the volume bounded by the walls of the waveguide and two cross sections at a distance l from each other (the integration is independent of z).

Next, from Gauss' theorem, we have

$$\int \operatorname{div}[E_0^* \wedge H]\, d^3r = \int ([E_0^* \wedge H] \cdot n)\, d^2S = \int ([n \wedge E_0^*] \cdot H)\, d^2S .$$

On the wall of the waveguide $[n \wedge E_0^*] = 0$ in view of the boundary condition $E_{0\tau} = 0$, while the integrals over the cross sections have opposite signs and cancel out. Hence,

$$\int ([n \wedge E_0^*] \cdot H)\, d^2S = 0$$

and equation (9.16.1) gives

$$(k-k_0)\int_S ([E_0^* \wedge H] \cdot e_z)\, d^2S = -\omega\left[\int_S (H \cdot H_0^*)\, d^2S - \int_S (E \cdot E_0^*)\, d^2S - \int_{\Delta S} \Delta\hat{\epsilon}(E \cdot E_0^*)\, d^2S\right] \tag{9.16.2}$$

where $\Delta\hat{\epsilon} = \hat{\epsilon} - \hat{1}$ and ΔS is the cross-sectional area of the region occupied by the dielectric.

Similarly, if we use the equations for $\operatorname{curl} E$ and $\operatorname{curl} H_0^*$ we have

$$(k-k_0)\int_S ([E \wedge H_0^*] \cdot e_z)\, d^2S = \frac{\omega}{c}\left[\int_S (H \cdot H_0^*)\, d^2S - \int_S (E \cdot E_0^*)\, d^2S + \int_{\Delta S} \Delta\hat{\mu}(H \cdot H_0^*)\, d^2S\right], \tag{9.16.3}$$

where $\Delta\hat{\mu} = \hat{\mu} - \hat{1}$.

By combining equations (9.16.2) and (9.16.3) we obtain the formula given in the problem. This formula represents the exact relation between the change in the propagation constant and the field amplitudes. However, in the majority of situations the exact solution of the problem of a waveguide partially filled with a dielectric cannot be obtained. The amplitudes of the disturbed field E and H can only be determined approximately, provided that the transverse dimensions of the region occupied by the dielectric are small. When this is so, the formula for the change in the propagation constant may be used to calculate the value of Δk, which is an important characteristic of the wave propagating in the waveguide. Examples of calculations based on this method are given in the solution of problems 9.17 to 9.19.

9.17 For a thin plate, the amplitudes of the disturbed fields may be approximately expressed in terms of the undisturbed amplitudes, which for an H_{10} wave are of the form

$$H_{0z} = H_0 \cos\frac{\pi x}{a}\ , \qquad H_{0x} = -\frac{ik_0 a}{\pi} H_0 \sin\frac{\pi x}{a}\ ,$$

$$E_{0y} = \frac{i\omega a}{\pi c} H_0 \sin\frac{\pi x}{a}\ , \qquad E_{0x} = E_{0z} = H_{0y} = 0\ .$$

These expressions may be obtained from the solutions of problem 9.1. Let us now neglect the change in the field amplitudes outside the volume occupied by the plate, and also the variation in the fields within the plate. This is equivalent to neglecting terms of the order of d^2 or higher. The boundary conditions which must be satisfied on the surface of the plate are

$$E_y = E_{0y}\ , \qquad H_z = H_{0z}\ , \qquad \mu_\perp H_x - i\mu_a H_y = H_{0x}\ , \qquad H_y = H_{0y} = 0\ ,$$

where the undisturbed amplitudes on the right-hand side of the equations should be taken at $x = x_1$. These equations define the amplitudes of the disturbed field in the plate.

The integral in the numerator of the expression for Δk (see the preceding problem) is equal to the product of the integrand and the cross-sectional area bd of the plate, since the field is independent of y and the dependence on x is neglected.

The undisturbed amplitudes may be substituted into the integral in the denominator. The final result is

$$\Delta k = \frac{d}{k_0 a}\left\{\left[\frac{(\epsilon-1)\omega^2}{c^2} + \left(1-\frac{1}{\mu_\perp}\right)k_0^2\right]\sin^2\frac{\pi x_1}{a} + (\mu_\parallel - 1)\left(\frac{\pi}{a}\right)^2\cos^2\frac{\pi x_1}{a}\right\}\ .$$

Since $\mu_\perp$ depends on the magnitude of the constant magnetising field H_0 (see problem 6.35), it follows that Δk will also depend on this field. A change in H_0 gives rise to a change in the phase of the wave. Devices which are based on these phenomena are widely used in electronics for phase transformation.

9.18

$$\Delta k = \frac{\omega d}{4\pi c}\frac{b-a}{ab\ln(b/a)}\left(\epsilon - \frac{1}{\mu_\perp}\right)\ .$$

9.19

$$\text{(a)}\ \Delta k = \frac{\omega d}{4\pi c}\frac{b-a}{ab\ln(b/a)}\left(\epsilon - \frac{1}{\mu_\parallel}\right)\ ; \qquad \text{(b)}\ \Delta k = \frac{\omega d}{4\pi c}\frac{b-a}{ab\ln(b/a)}\left(\epsilon - \frac{1}{\mu_\perp}\right)\ .$$

In (a) the change in the propagation constant Δk is practically independent of the magnitude of the constant magnetic field H_0 since $\mu_\parallel \approx 1$ (see problem 6.35). This is explained by the fact that inside the plate the high-frequency magnetic field has the same direction as the constant field, and does not support the precession of the magnetisation vector $\boldsymbol{M}$.

In (b) the high-frequency magnetic field inside the plate is perpendicular to the constant field, $\mu_\perp$ depends on H_0, and this dependence is of the resonance type.

9.20 If we integrate equation (9.0.1) with the boundary conditions (9.0.2) we find

$$\left.\begin{aligned} E_x &= A_1 \cos(k_1 x)\sin(k_2 y)\sin(k_3 z)\,, \\ E_y &= A_2 \sin(k_1 x)\cos(k_2 y)\sin(k_3 z)\,, \\ E_z &= A_3 \sin(k_1 x)\sin(k_2 y)\cos(k_3 z)\,, \end{aligned}\right\} \tag{9.20.1}$$

where the A_i are constants and

$$k_1 = \frac{n_1 \pi}{a}\,, \qquad k_2 = \frac{n_2 \pi}{b}\,, \qquad k_3 = \frac{n_3 \pi}{h}\,, \qquad \omega^2 = c^2(k_1^2 + k_2^2 + k_3^2)\,,$$

$n_1, n_2, n_3 = 0, 1, 2, \ldots$, and we have dropped the time factor $\exp(-\mathrm{i}\omega t)$.

One can use the Maxwell equations to express the vector $\boldsymbol{H}$ in terms of $\boldsymbol{E}$.

The equation $\mathrm{div}\boldsymbol{E} = 0$ leads to the condition that the waves are transverse, $(\boldsymbol{A}\cdot\boldsymbol{k}) = 0$, where the vector $\boldsymbol{A} = (A_1, A_2, A_3)$. Hence it follows that oscillations for given $k_x, k_y, k_z \neq 0$ are two-fold degenerate since the vector $\boldsymbol{A}$ can be chosen in the plane perpendicular to $\boldsymbol{k}$ in two independent and arbitrary ways. For each such $\boldsymbol{k}$ we put:

$$\boldsymbol{A}_{k\sigma} = A\boldsymbol{e}_{k\sigma}\,, \qquad \sigma = 1, 2\,,$$

where $\boldsymbol{e}_{k\sigma}$ is a unit vector such that $(\boldsymbol{e}_{k1}\cdot\boldsymbol{e}_{k2}) = 0$ and $(\boldsymbol{e}_{k\sigma}\cdot\boldsymbol{k}) = 0$, and the constant $A = (32\pi/V)^{1/2}$, where $V = abh$ is the volume of the resonator.

The eigenfunctions are now all mutually orthogonal and normalised:

$$\int (\boldsymbol{E}_{\nu'}\cdot\boldsymbol{E}_\nu)\,\mathrm{d}^3\boldsymbol{r} = 4\pi\delta_{\nu\nu'}\,.$$

By direct integration of equations (9.20.1) one can easily verify that this relation holds. The indices ν, ν' stand for sets of four numbers: n_1, n_2, n_3, and σ.

If one of the components of $\boldsymbol{k}$ vanishes, there is no degeneracy since in that case there is only one constant in the solution (9.20.1).

9.21

$$\Delta N = \frac{V}{\pi^2 c^3}\omega^2 \Delta\omega\,.$$

9.22 The E-waves are

$$E_z = \mathrm{E}_0 J_m(\kappa r)\sin(m\phi + \psi_m)\cos kz\,\exp(-\mathrm{i}\omega t)\,, \qquad H_z = 0\,,$$

$$E_r = -\frac{k}{\kappa}\mathrm{E}_0 J_m'(\kappa r)\sin(m\phi + \psi_m)\sin kz\,\exp(-\mathrm{i}\omega t)\,,$$

$$E_\phi = -\frac{mk}{\kappa^2 r} E_0 J_m(\kappa r)\cos(m\phi + \psi_m)\sin kz \exp(-i\omega t)\,,$$

$$H_r = -\frac{im\omega}{\kappa^2 cr} E_0 J_m(\kappa r)\cos(m\phi + \psi_m)\cos kz \exp(-i\omega t)\,,$$

$$H_\phi = \frac{i\omega}{\kappa c} E_0 J_m'(\kappa r)\sin(m\phi + \psi_m)\cos kz \exp(-i\omega t)\,,$$

where $k = l\pi/h$, $l = 0, 1, 2, \ldots$, $\kappa_{mn} = \alpha_{mn}/a$, and α_{mn} are the roots of the equation

$$J_m(\alpha_{mn}) = 0\,, \qquad \omega^2 = c^2(\kappa_{mn}^2 + k^2)\,.$$

The H-waves are

$$H_z = H_0 J_m(\kappa r)\sin(m\phi + \psi_m)\sin kz \exp(-i\omega t)\,,$$

with $k = l\pi/h$, $l = 1, 2, \ldots$; the value $l = 0$ is impossible; $\kappa_{mn} = \beta_{mn}/a$ where β_{mn} is a root of the equation $J_m'(\beta_{mn}) = 0$; $\omega^2 = c^2(\kappa_{mn}^2 + k^2)$. The remaining field components may be expressed in terms of H_z with the aid of Maxwell's equations.

When $m \neq 0$, both E- and H-waves are in general doubly degenerate, since to each eigenfrequency there correspond two eigenfunctions, for example,

$$H_z = H_0 J_m(\kappa r)\sin m\phi \sin kz \exp(-i\omega t)\,,$$

and

$$H_z = H_0 J_m(\kappa r)\cos m\phi \sin kz \exp(-i\omega t)\,.$$

9.23 In the quasi-stationary approximation we can consider the given system as an oscillating contour consisting of a capacitor of capacitance $C = R^2/4d$ and an inductive spool which has a self-inductance given by $L = 4\pi b[\ln(8b/a) - \frac{7}{4}]$. (See problem 5.32 for the calculation of the self-inductance of a given loop.) By making use of Thomson's formula (7.a.3) we find

$$\omega_0 = \frac{c}{(\pi b/d)[\ln(8b/a) - \frac{7}{4}]^{1/2}}\,.$$

The quasi-stationary approximation is applicable provided $\lambda_0 = 2\pi c/\omega_0$ is much larger than the dimensions of the system, i.e. $\lambda_0 \gg R, b$.

9.24

$$\omega_0 = \frac{c}{a}\left(\frac{d}{\pi b}\right)^{1/2}$$

9.25 In the quasi-stationary approximation ($\lambda_0 = 2\pi c/\omega_0 \gg a, b$) we assume that the electrical field is completely concentrated between the plates of the capacitor, while the magnetic field is inside the toroidal cavity. Under those assumptions the resonator is equivalent to a normal oscillating contour consisting of a capacitance and an inductance. The capacitance of the capacitor is $C = (b-a)^2/4d$, and the self-inductance of the torus $L = 4\pi[b-(b^2-a^2)^{1/2}]$ (see problem 5.29). The eigenfrequency is

$$\omega_0 = \frac{c}{b-a}\left\{\frac{d}{\pi[b-(b^2-a^2)^{1/2}]}\right\}^{1/2}$$

Higher types of oscillations of the resonator considered here cannot be evaluated in the quasi-stationary approximation, since they do not satisfy the condition $\lambda \gg a, b$.

9.26

$$\omega_0 = \frac{2c}{2b-a}\left\{\frac{d}{2\pi h \ln[(2b+a)/(2b-a)]}\right\}^{1/2}$$

9.27 In the coaxial waveguide, which is shorted at one end (at $z = 0$) by the perfectly conducting diaphragm, a standing transverse wave can be realised with field strengths:

$$E_r = \frac{A}{r}\sin\frac{\omega z}{c}\exp(-i\omega t), \qquad H_\phi = -\frac{iA}{r}\cos\frac{\omega z}{c}\exp(-i\omega t). \tag{9.27.1}$$

In any plane at right angles to the axis of the waveguide the electrical field distribution is the same as in a cylindrical capacitor and one may assume that it is produced by a potential difference

$$\Delta\varphi = A\ln\frac{b}{a}\sin\frac{\omega z}{c}$$

between the central rod and the outside surface.

This potential difference must be equal to the field strength on the plates of the capacitor which is formed by the end face of the rod and the upper cover of the resonator:

$$\Delta\varphi|_{z=h} = \frac{q}{C}. \tag{9.27.2}$$

Here $C = a^2/(4d)$ is the capacitance of the capacitor, and q the charge on one of the plates which can be expressed in terms of the current J which flows along the rod (or the current in the outside surface which has the same magnitude and the opposite direction)

$$J = -i\omega q.$$

If we evaluate the current strength from the known magnetic field (9.27.1) and substitute it, as well as the potential difference, into equation (9.27.2), we find a transcendental equation which determines the eigenfrequency:

$$\cot\frac{\omega h}{c} = \frac{2\pi a^2\omega}{cd}\ln\frac{b}{a}.$$

One can easily solve this equation graphically. When $\omega h/c \ll 1$ (which means that $\lambda \gg 2\pi h$—the quasi-stationary approximation) we find

$$\omega = \frac{c}{[(a^2/4d)2h\ln b/a]^{1/2}} = \frac{c}{(LC)^{1/2}} ,$$

where L is the self-inductance of a section of length h of a coaxial line. In this approximation one can evaluate only one—the lowest—eigenfrequency (cf the solutions of the preceding four problems).

When $d = 0$ (a coaxial waveguide section which is shorted at both ends) we have

$$\omega_m = \frac{\pi c}{h} m , \qquad m = 1, 2, \ldots .$$

This means that an integral number of half-waves must fit along the length of the resonator: $h = \frac{1}{2} m\lambda_m$.

9.28 The field in the resonator is described by the Maxwell equations (8.a.1) and (8.a.2) with $\boldsymbol{B} = \boldsymbol{H}$ and $\boldsymbol{D} = \boldsymbol{E}$. If we take the scalar product of the first one with $\boldsymbol{H}_\nu$ and of the second one with $\boldsymbol{E}_\nu$ and integrate over the whole of the volume of the resonator we get:

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int(\boldsymbol{H}\cdot\boldsymbol{H}_\nu)\,\mathrm{d}^3\boldsymbol{r} &= -c\int(\boldsymbol{H}_\nu\cdot\operatorname{curl}\boldsymbol{E})\,\mathrm{d}^3\boldsymbol{r} , \\ \frac{\mathrm{d}}{\mathrm{d}t}\int(\boldsymbol{E}\cdot\boldsymbol{E}_\nu)\,\mathrm{d}^3\boldsymbol{r} &= c\int(\boldsymbol{E}_\nu\cdot\operatorname{curl}\boldsymbol{H})\,\mathrm{d}^3\boldsymbol{r} . \end{aligned}\right\} \tag{9.28.1}$$

If we assume that the eigenfunctions $\boldsymbol{E}_\nu$, $\boldsymbol{H}_\nu$ are orthonormal, according to equation (9.0.3), we can evaluate the integrals on the left-hand sides of equations (9.28.1):

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int(\boldsymbol{H}\cdot\boldsymbol{H}_\nu)\,\mathrm{d}^3\boldsymbol{r} &= 4\pi\dot{p}_\nu , \\ \frac{\mathrm{d}}{\mathrm{d}t}\int(\boldsymbol{E}\cdot\boldsymbol{E}_\nu)\,\mathrm{d}^3\boldsymbol{r} &= 4\pi\dot{q}_\nu . \end{aligned}\right\} \tag{9.28.2}$$

The eigenfunctions $\boldsymbol{E}_\nu$, $\boldsymbol{H}_\nu$ satisfy the equations

$$\left.\begin{aligned} \operatorname{curl}\boldsymbol{E}_\nu &= \mathrm{i}k_\nu\boldsymbol{H}_\nu , & \operatorname{curl}\boldsymbol{H}_\nu &= -\mathrm{i}k_\nu\boldsymbol{E}_\nu , \\ \operatorname{curl}\operatorname{curl}\boldsymbol{E}_\nu &= k_\nu^2\boldsymbol{E}_\nu , & \operatorname{curl}\operatorname{curl}\boldsymbol{H}_\nu &= k_\nu^2\boldsymbol{H}_\nu , \end{aligned}\right\} \tag{9.28.3}$$

where the $k_\nu(k_1, k_2, k_3)$ are the corresponding eigenwavevectors (they were evaluated in problems 9.20 and 9.22). We can use equations (9.28.3) to transform the integrals on the right-hand sides of equations (9.28.1),

$$\begin{aligned} \operatorname{div}[\boldsymbol{E}\wedge\operatorname{curl}\boldsymbol{E}_\nu] &= (\operatorname{curl}\boldsymbol{E}_\nu\cdot\operatorname{curl}\boldsymbol{E}) - (\boldsymbol{E}\cdot\operatorname{curl}\operatorname{curl}\boldsymbol{E}_\nu) \\ &= \mathrm{i}k_\nu(\boldsymbol{H}_\nu\cdot\operatorname{curl}\boldsymbol{E}) - k_\nu^2(\boldsymbol{E}_\nu\cdot\boldsymbol{E}) , \end{aligned}$$

whence

$$\int (\boldsymbol{H}_\nu \cdot \operatorname{curl} \boldsymbol{E})\, \mathrm{d}^3 r = -\mathrm{i}k_\nu \int (\boldsymbol{E}_\nu \cdot \boldsymbol{E})\, \mathrm{d}^3 r + \frac{1}{\mathrm{i}k_\nu} \int \operatorname{div} [\boldsymbol{E} \wedge \operatorname{curl} \boldsymbol{E}_\nu]\, \mathrm{d}^3 r$$
$$= -4\pi \mathrm{i} k_\nu q_\nu + \oint (\boldsymbol{H}_\nu \cdot [\boldsymbol{n} \wedge \boldsymbol{E}])\, \mathrm{d}^2 S\,, \tag{9.28.4}$$

where the last integral is taken over the inner surface of the resonator and $\boldsymbol{n}$ is a unit vector directed into the conductor along the normal. However, the field at the wall of the resonator satisfies condition (8.a.10), which can be written in the form

$$\zeta \boldsymbol{H}_\tau = [\boldsymbol{n} \wedge \boldsymbol{E}]\,. \tag{9.28.5}$$

The eigenfunction $\boldsymbol{H}_\nu$ of a resonator with perfect conductivity has only a tangential component on the wall, so that when substituting equation (9.28.5) into the integral in equation (9.28.4) we can replace $\boldsymbol{H}_\tau$ by $\boldsymbol{H}$. As a result, on combining equations (9.28.1) to (9.28.5) we get the equation

$$\dot{p}_\nu - \mathrm{i}\omega_\nu q_\nu = -\frac{c\zeta}{4\pi} \oint (\boldsymbol{H}_\nu \cdot \boldsymbol{H})\, \mathrm{d}^2 S\,. \tag{9.28.6}$$

The second equation can be reduced similarly:

$$\dot{q}_\nu - \mathrm{i}\omega_\nu p_\nu = 0\,.$$

We study the effect of the finite conductivity of the walls on the νth type of oscillations of the perfect resonator. The perturbed field $\boldsymbol{H}$ must go over into the unperturbed field as $\zeta \to 0$, i.e. in the sum

$$\boldsymbol{H} = \sum_{\nu'} p_{\nu'} \boldsymbol{H}_{\nu'}$$

only the term with $\nu' = \nu$ must remain. Hence, the amplitudes $p_{\nu'}$ with $\nu' \neq \nu$ must be proportional to ζ and substituting them into equation (9.28.6) gives terms of order ζ^2 or higher. If we neglect such terms and replace $\boldsymbol{H}$ in equation (9.28.6) by $p_\nu \boldsymbol{H}_\nu$, we get an equation of the form

$$\dot{p}_\nu - \mathrm{i}\omega_\nu q_\nu = -p_\nu \frac{c\zeta}{4\pi} \oint H_\nu^2\, \mathrm{d}^2 S\,. \tag{9.28.7}$$

If we use equation (9.28.7) to eliminate one of the variables (p_ν), the equation we get for the other one

$$\ddot{q}_\nu + \left(\frac{c\zeta}{4\pi} \oint H_\nu^2\, \mathrm{d}^2 S \right) \dot{q}_\nu + \omega_\nu^2 q_\nu = 0\,. \tag{9.28.8}$$

The quantity within the brackets is complex. Equation (9.28.8) thus describes a harmonic oscillator on which a ‘friction force’

$$-\left(\frac{c\zeta'}{4\pi} \oint H_\nu^2\, \mathrm{d}^2 S \right) \dot{q}_\nu$$

acts, where ζ' is the real part of the surface impedance.

On solving the equation we find a complex correction $\Delta\omega_\nu - i\gamma_\nu$ to the eigenfrequency of the perfect resonator. The losses lead to a damping of the eigenoscillations, with a damping rate

$$\gamma_\nu = \frac{c\zeta'}{8\pi}\oint H_\nu^2\, d^2S ,$$

and to a shift in the eigenfrequencies by an amount

$$\Delta\omega_\nu = \frac{c\zeta''}{8\pi}\oint H_\nu^2\, d^2S ,$$

where ζ'' is the imaginary part of the surface impedance; the changed eigenfrequency is thus $\widetilde{\omega}_\nu = \omega_\nu + \Delta\omega_\nu$.

The connection between the Q of the resonator and the damping rate is given by formula (9.0.4).

9.29

$$Q_\nu = \frac{\omega_\nu a}{4c\zeta'} = \left(\frac{\pi a^2 \sigma \omega_\nu}{2c^2}\right)^{1/2}$$

The system loses its resonance properties at sufficiently high frequencies when the distance between neighbouring eigenfrequencies becomes comparable with the width of the resonance curve determined by the damping rate $\gamma_\nu = \omega_\nu/2Q_\nu$. At high frequencies it follows from the results of problem 9.21 that the distance between neighbouring eigenfrequencies is

$$\frac{\Delta\omega}{\Delta N} = \frac{\pi^2 c^3}{a^3}\frac{1}{\omega^2} .$$

On equating this quantity to the damping rate γ we find the frequency region for which the system has resonance properties:

$$\omega \leqslant 10^9 \sigma^{1/5} a^{4/5} .$$

For $a \approx 10^{-2}$ m and $\sigma = 10^{17}\ s^{-1}$ we have: $\omega \lesssim 3 \times 10^{12}\ s^{-1}$.

9.30 If we expand $\boldsymbol{E}$ and $\boldsymbol{H}$ in terms of the eigenfunctions of a perfect resonator, as was done in problem 9.28, we get for the amplitudes p_ν and q_ν the set of equations:

$$\dot{p}_\nu - i\omega q_\nu + 2i\sum_{\nu'} \Delta\Omega_{\nu'} j_{\nu'} = 0 , \tag{9.30.1}$$

$$\dot{q}_\nu - i\omega_\nu p_\nu = -\frac{1}{c} j_\nu \exp(-i\omega t) , \tag{9.30.2}$$

where $\Delta\Omega_\nu = \Delta\omega_\nu - i\gamma_\nu$ is the complex shift in the eigenfrequencies and

$$j_\nu = \int (\boldsymbol{j} \cdot \boldsymbol{E}_\nu)\, d^3\boldsymbol{r} .$$

We look for the solution of equations (9.30.1) and (9.30.2) in the form

$$p_\nu = p_\nu^0 \exp(-i\omega t), \qquad q_\nu = q_\nu^0 \exp(-i\omega t).$$

Eliminating the q_ν^0 we get

$$p_\nu^0(\omega^2 - 2\omega\Delta\Omega_\nu - \omega_\nu^2) = \frac{i\omega_\nu}{c} j_\nu + 2\omega \sum_{\nu'}{}' \Delta\Omega_{\nu'} p_{\nu'}^0 . \tag{9.30.3}$$

The prime at the summation sign indicates that the term with $\nu' = \nu$ is absent (it has been taken to the other side).

We solve the equation (9.30.3) by the method of successive approximations. In the zeroth approximation we drop the sum on the right-hand side and obtain

$$p_\nu^0 = \frac{i\omega_\nu j_\nu}{c(\omega^2 - 2\omega\Delta\Omega_\nu - \omega_\nu^2)} . \tag{9.30.4}$$

In the next approximation we get a correction to equation (9.30.4) which is equal to

$$\frac{2\omega}{\omega^2 - 2\omega\Delta\Omega_\nu - \omega_\nu^2} \sum_{\nu'}{}' \Delta\Omega_{\nu'} p_{\nu'}^0 .$$

It is small if ω is close to ω_ν while all other eigenfrequencies $\omega_{\nu'}$ satisfy the condition $|\omega - \omega_{\nu'}| \gg |\Delta\Omega_{\nu'}|$.

We express the denominator in equation (9.30.4) in terms of the quality factor Q_ν and the changed eigenfrequency $\tilde{\omega}_\nu = \omega_\nu + \Delta\omega_\nu$. We have

$$\omega\Delta\Omega_\nu = \omega\Delta\omega_\nu - i\omega\gamma_\nu \approx \omega_\nu\Delta\omega_\nu - \frac{i\omega\tilde{\omega}_\nu}{2Q_\nu} ,$$

which is valid near resonance ($|\omega - \omega_\nu| \ll \omega$). Hence

$$p_\nu^0 \approx \frac{i\omega_\nu j_\nu}{c(\omega^2 - \tilde{\omega}_\nu^2 + i\omega\tilde{\omega}_\nu/Q_\nu)} , \qquad q_\nu^0 \approx -\frac{i\omega j_\nu}{c(\omega^2 - \tilde{\omega}_\nu^2 + i\omega\tilde{\omega}_\nu/Q_\nu)} .$$

The frequency dependence of the field amplitudes has a resonance character. For a given j the field at resonance is larger the higher the Q of the resonator:

$$q_{\nu\,\mathrm{res}}^0 = p_{\nu\,\mathrm{res}}^0 = \frac{j_\nu Q_\nu}{c\omega_\nu} .$$

It follows also from the formulae obtained here that one must place the conductor with the current in the antinode of the electrical field $\boldsymbol{E}_\nu$ and orient it along $\boldsymbol{E}_\nu$. In that case the quantities j_ν, and hence p_ν^0, q_ν^0, will have their maximum values.

9.31 If the wave field of energy W, which fills the resonator, is reflected once from a mirror, the energy loss is $W(1-R)$. After a time dt the energy loss is

$$dW = -W(1-R)\frac{c\,dt}{L} ,$$

where $c\,dt/L$ is the number of reflections. From the definition (9.0.4) of the quality factor we have

$$Q_1 = \frac{\omega W}{-dW/dt} = \frac{\omega L}{c(1-R)} ,$$

where ω is the frequency of the oscillations considered.

The radiation through the side surface is connected with the fact that a beam of light with a limited extent in the transverse direction cannot be rigorously directed. It necessarily has a transverse wavevector component $\Delta k_\perp$ which can be estimated from the condition $\Delta k_\perp D \approx 1$ (see problem 8.26). This leads to the fact that light rays propagating from the one mirror to the other form a slightly diverging beam with opening angle

$$2\theta = \frac{2\Delta k_\perp}{k} = \frac{2c}{D\omega} .$$

Part of the beam does not fall on the second mirror (figure 9.31.2) and the energy loss at one reflection is $WL\theta/D$. After a time dt the losses are

$$dW = -W\frac{L\theta}{D}\frac{c\,dt}{L} = -W\frac{c^2}{D^2\omega}dt .$$

The Q due to this radiation is

$$Q_2 = \frac{D^2\omega^2}{c^2} .$$

If the losses in the mirrors and due to the radiation are small, they are additive. The total Q is then given by the formula

$$\frac{1}{Q} = \frac{1}{Q_1} + \frac{1}{Q_2} .$$

For the given parameters we have:

$$Q_1 \approx 4 \times 10^5 ; \qquad Q_2 \approx 4 \times 10^8 \gg Q_1 ; \qquad Q \approx Q_1 \approx 4 \times 10^5 .$$

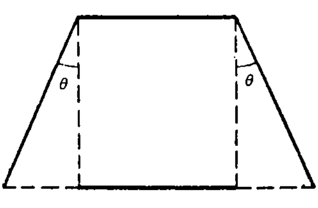


Figure 9.31.2.

9.32 If the beam of light originally propagates along the normal to the plane of one of the mirrors, after the nth reflection the angle between the normal and the beam will be equal to $n\beta$ (figure 9.32.1). After the nth passage between the mirrors the beam is displaced by a distance $n\beta L$; the number of reflections N before the beam leaves the resonator can be estimated from the relation

$$\sum_{n=1}^{N} n\beta L \approx D\,.$$

For $N \gg 1$ we get $N = (2D/\beta L)^{1/2}$, which corresponds to a damping time for the eigenoscillations that is given by

$$\tau = N\frac{L}{c} = \frac{1}{c}\left(\frac{2DL}{\beta}\right)^{1/2}$$

This time can be identified with the reciprocal of the damping rate γ:

$$\gamma = \frac{1}{\tau} = c\left(\frac{\beta}{2DL}\right)^{1/2}$$

The Q due to the mirrors not being parallel is

$$Q_3 = \frac{\omega}{2\gamma} = \frac{\omega}{2c}\left(\frac{2DL}{\beta}\right)^{1/2}$$

In order that the fact that the mirrors are not parallel may not appreciably change the Q of the resonator the condition $Q_3 \leqslant Q$ must be satisfied, where Q is the Q of the resonator with parallel mirrors. Hence

$$\beta \lessdot \frac{\omega^2}{2c^2}\frac{DL}{Q^2}\,.$$

For the parameters given in the preceding problem this leads to

$$\beta \leqslant 0{\cdot}0012\,.$$

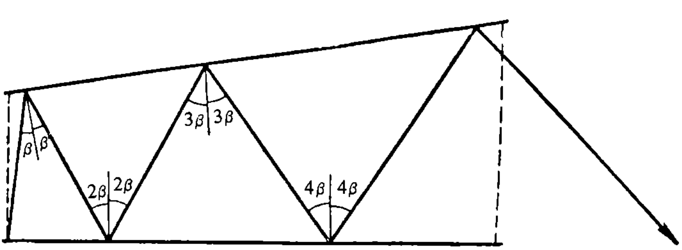


Figure 9.32.1.

9.33 (a) The angle ϑ takes on discrete values given by the formula

$$\frac{L}{\cos\vartheta} = \tfrac{1}{2}n\lambda\,,$$

where $n \gg 1$ is a positive integer. If, for a given λ, the value $\vartheta = 0$ is possible, which corresponds to $n = n_0$ $(L = \frac{1}{2}n_0\lambda)$, the discrete values of the angle $\vartheta_k \ll 1$ are given by the formula

$$\vartheta_k = \left(\frac{k\lambda}{L}\right)^{1/2}$$

(b) We found in problem 9.31 the quality factor Q_1 due to losses in the mirrors. The quality factor Q_2 due to losses through radiation is, as to order of magnitude,

$$Q_2 = \frac{\omega D}{c\vartheta}, \quad \text{when} \quad \vartheta > \theta ; \qquad Q_2 = \frac{\omega D}{c\theta} = \frac{D^2\omega^2}{c^2}, \quad \text{when} \quad \vartheta < \theta , \tag{9.33.1}$$

where θ is the diffraction angle, determined in problem 9.31.

If $Q_1 < Q_{2\,\max}$, the total Q of the resonator, for those kinds of oscillations for which $Q_2(\vartheta) > Q_1$, will be practically the same and close to Q_1. If $Q_1 > Q_{2\,\max}$, Q will basically be determined by the magnitude of Q_2, in accordance with equation (9.33.1).

Special theory of relativity

a Lorentz transformations

10.1

$$x - x_0 = \frac{x' - x'_0 + V(t' - t'_0)}{(1-\beta^2)^{1/2}}, \qquad y - y_0 = y' - y'_0,$$

$$z - z_0 = z' - z'_0, \qquad t - t_0 = \frac{t' - t'_0 + (V/c^2)(x' - x'_0)}{(1-\beta^2)^{1/2}}.$$

10.4 The coordinates of clocks indicating equal times $t = t'$ in S and S' are

$$x = \frac{c^2}{V}\left(1 - \frac{1}{\gamma}\right)t, \qquad x' = -\frac{c^2}{V}\left(1 - \frac{1}{\gamma}\right)t.$$

It is clear from these formulae that the point at which $t = t'$ executes a uniform motion in each of the systems. In the system in which this point is at rest, S and S' move in opposite directions with equal velocities $V_0 = (c^2/V)(1 - 1/\gamma)$, where V_0 is the relativistic half of the velocity V in the sense that the relativistic addition of the two velocities V_0 yields V.

10.5 The length of one period in the frame S' is $T' = 2l/c$; in the frame S the time T_1 it takes the 'spot' to move along the rod in the direction of the relative velocity V can be evaluated from the formula

$$T_1 = \frac{1}{c}\left[l\left(1 - \frac{V^2}{c^2}\right)^{1/2} + VT_1\right],$$

whereas the time T_2 for the motion in the opposite direction is obtained by replacing V by $-V$. For the ratio of T' to $T = T_1 + T_2$ we find

$$\frac{T'}{T} = \left(1 - \frac{V^2}{c^2}\right)^{1/2},$$

whence follows equation (10.a.7).

10.7 (a) No. 12 h 00 min can be indicated simultaneously by two clocks in one of the systems and by only one clock in the other system.

(b) The readings of spatially coincident clocks are independent of the choice of the reference frame, so that

$$t_{A'} = 12\text{ h }00\text{ min} + \frac{l_0}{V} = 13\text{ h }00\text{ min};$$

$$t_A = 12\text{ h }00\text{ min} + l_0\frac{(1 - V^2/c^2)^{1/2}}{V} = 12\text{ h }36\text{ min}.$$

In view of the relativity of simultaneity, the readings of the clocks B and B' will depend on the choice of the reference frame.

For an observer on the 'platform' (figure 10.7.1a)

$$t_{B'} = 12\,\text{h}\;21\cdot6\,\text{min}\,, \qquad t_B = t_A = 12\,\text{h}\;36\,\text{min}\,.$$

For an observer on the 'train' (figure 10.7.1b)

$$t_{B'} = t_{A'} = 13\,\text{h}\;00\,\text{min}\,, \qquad t_B = 13\,\text{h}\;14\cdot4\,\text{min}\,.$$

(c) For an observer on the 'platform'

$$t_A = 13\,\text{h}\;00\,\text{min} = t_B\,, \quad t_{B'} = 12\,\text{h}\;36\,\text{min}\,, \quad t_{A'} = 13\,\text{h}\;14\cdot4\,\text{min}\,.$$

For an observer on the 'train'

$$t_A = 12\,\text{h}\;21\cdot6\,\text{min}\,, \quad t_{A'} = t_{B'} = 12\,\text{h}\;36\,\text{min}\,, \quad t_B = 13\,\text{h}\;00\,\text{min}\,.$$

The clocks whose readings are compared with the readings of the two clocks in the other reference frames will always lag behind.

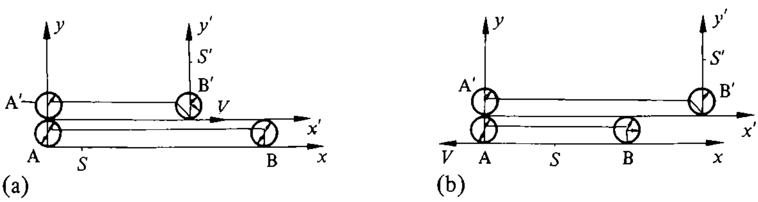


Figure 10.7.1.

10.8 According to the clocks on the earth $\Delta t = 8$ years. In estimating the necessary supplies, a time interval of $\Delta t_0 = 0\cdot01\Delta t \approx 1$ month must be taken (according to clocks on the rocket);

$$T = mc^2(\gamma - 1) \equiv 2\cdot5 \times 10^{16}\ \text{kWh}\,.$$

This amount of energy exceeds the annual world output of electrical energy by a factor of about 10^4.

10.9

$$v = \frac{2l_0\Delta t}{(\Delta t)^2 + l_0^2/c^2}\,.$$

According to the observer moving with the first ruler (figure 10.9.1a), the left-hand ends will pass each other first. For an observer at rest relative to the second ruler (figure 10.9.1b), the right-hand ends will meet first. According to an observer relative to whom the two rulers move with equal speeds, the ends will coincide simultaneously.

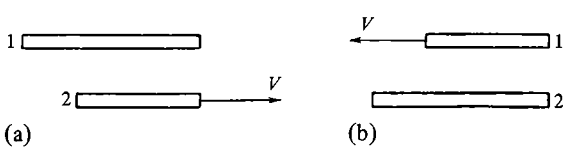


Figure 10.9.1.

10.10 Consider the transverse and longitudinal components of the position vector $\boldsymbol{r}$:

$$\boldsymbol{r}_{\parallel} = \boldsymbol{V}\frac{(\boldsymbol{r}\cdot\boldsymbol{V})}{V^2}, \qquad \boldsymbol{r}'_{\parallel} = \boldsymbol{V}\frac{(\boldsymbol{r}'\cdot\boldsymbol{V})}{V^2};$$

$$\boldsymbol{r}_{\perp} = \boldsymbol{r}-\boldsymbol{r}_{\parallel}, \qquad \boldsymbol{r}'_{\perp} = \boldsymbol{r}'-\boldsymbol{r}'_{\parallel}.$$

On applying the Lorentz transformation (10.a.1) to $\boldsymbol{r}_{\parallel}$ and $\boldsymbol{r}_{\perp}$ we have

$$\boldsymbol{r}_{\parallel} = \gamma(\boldsymbol{r}'_{\parallel}+\boldsymbol{V}t'), \qquad \boldsymbol{r}_{\perp} = \boldsymbol{r}'_{\perp}.$$

Finally,

$$\boldsymbol{r} = \gamma(\boldsymbol{r}'+\boldsymbol{V}t')+(\gamma-1)\frac{[[\boldsymbol{r}'\wedge\boldsymbol{V}]\wedge\boldsymbol{V}]}{V^2},$$

$$t = \gamma\left[t'+\frac{(\boldsymbol{r}'\cdot\boldsymbol{V})}{c^2}\right].$$

10.11

$$\boldsymbol{A} = \gamma\left(\boldsymbol{A}'+\frac{\boldsymbol{V}}{c}A'_0\right)+(\gamma-1)\frac{[[\boldsymbol{A}'\wedge\boldsymbol{V}]\wedge\boldsymbol{V}]}{V^2},$$

$$A_0 = \gamma\left[A'_0+\frac{(\boldsymbol{A}'\cdot\boldsymbol{V})}{c^2}\right].$$

10.12

$$\boldsymbol{v} = \boldsymbol{v}_{\parallel}+\boldsymbol{v}_{\perp} = \frac{\boldsymbol{v}'+\boldsymbol{V}+(\gamma-1)(\boldsymbol{V}/V^2)[(\boldsymbol{v}'\cdot\boldsymbol{V})+V^2]}{\gamma[1+(\boldsymbol{v}'\cdot\boldsymbol{V})/c^2]},$$

where $\boldsymbol{v}$ and $\boldsymbol{v}'$ are the velocities in the systems S and S'. It is also possible to differentiate with respect to time the radius vector $\boldsymbol{r}$ taken as a function of $\boldsymbol{r}'$ and t' as given by the formula obtained in problem 10.10.

10.16 The angle of the Thomas precession is determined by the relation

$$\varphi = -\arccos\frac{v^2(1-V^2/c^2)^{1/2}+V^2(1-v^2/c^2)^{1/2}}{V^2+v^2-V^2v^2/c^2}.$$

In the case where $v, V \ll c$ the angle $\varphi \approx 0$. As $v \to c$, the angle $\varphi \to -\arccos(1-V^2/c^2)^{1/2}$; if then also $V \to c$, we have $\varphi \to \frac{1}{2}\pi$.

10.17

$$l = l_0\frac{1-v^2/c^2}{1+v^2/c^2}.$$

10.18 (a) $V = 2\times 0{\cdot}9c = 1{\cdot}8c$; (b) $V = 0{\cdot}994c$.

10.19 The relative velocity of the two particles in the system in which one of them is at rest is $V = 2v/(1+v^2/c^2)$. Hence,

$$\mathscr{E} = \frac{mc^2}{(1-V^2/c^2)^{1/2}} = mc^2\left[2\left(\frac{\mathscr{E}_0}{mc^2}\right)^2-1\right].$$

In the ultrarelativistic case $\mathscr{E}_0 \gg mc^2$, and hence $\mathscr{E} = 2\mathscr{E}_0^2/mc^2$. Suppose that the particles under acceleration are electrons ($mc^2 = 0{\cdot}5$ MeV). Then for an energy $\mathscr{E}_0 = 50$ MeV, the gain in the power supplied by the accelerator is a factor of 200: $\mathscr{E} = 10000$ MeV.

10.20 This problem is similar to problem 10.12 in that it can be solved in two ways. The result is

$$\dot{\boldsymbol{v}} = \frac{1}{\gamma^2 s^2}\dot{\boldsymbol{v}}' - \frac{(\gamma-1)(\dot{\boldsymbol{v}}' \cdot \boldsymbol{V})\boldsymbol{V}}{\gamma^3 s^3 V^2} - \frac{(\dot{\boldsymbol{v}}' \cdot \boldsymbol{V})\boldsymbol{v}'}{\gamma^2 s^3 c^2} ,$$

where

$$s = 1 + \frac{(\boldsymbol{v}' \cdot \boldsymbol{V})}{c^2} .$$

It is clear from these formulae that if the particle moves with a constant acceleration $\dot{\boldsymbol{v}}'$ in one system, then the acceleration $\dot{\boldsymbol{v}}$ in another reference system will in general be a function of time (because the variable velocity $\boldsymbol{v}'$ enters into the transformation formulae).

10.21

$$w_i^2 = -\gamma^6\left\{\dot{\boldsymbol{v}}^2 - \left[\dot{\boldsymbol{v}} \wedge \frac{\boldsymbol{v}}{c}\right]^2\right\} = -\gamma^4\left[\dot{v}^2 + \gamma^2 v^2 \frac{\dot{v}^2}{c^2}\right] < 0 ,$$

i.e. the 4-dimensional acceleration is a space-like vector.

10.22 Let S' be the system in which the particle is instantaneously at rest. In accordance with the solution of problem 10.20

$$\dot{\boldsymbol{v}}' = \gamma^2\left[\dot{\boldsymbol{v}} + \frac{\gamma-1}{v^2}(\dot{\boldsymbol{v}} \cdot \boldsymbol{v})\boldsymbol{v}\right] .$$

Hence, the square of the acceleration is given by

$$\dot{\boldsymbol{v}}'^2 = \gamma^4\left[\dot{v}^2 + \frac{\gamma^2(\dot{\boldsymbol{v}} \cdot \boldsymbol{v})^2}{c^2}\right] = \gamma^6\left\{\dot{\boldsymbol{v}}^2 - \left[\dot{\boldsymbol{v}} \wedge \frac{\boldsymbol{v}}{c}\right]^2\right\} . \qquad (10.22.1)$$

If the velocity of the particle varies in magnitude only, then $\dot{\boldsymbol{v}} \parallel \boldsymbol{v}$ and

$$\dot{\boldsymbol{v}}' = \gamma^3 \dot{\boldsymbol{v}} .$$

If the velocity of the particle varies only in direction, then $\boldsymbol{v} \perp \dot{\boldsymbol{v}}$ and $(\boldsymbol{v} \cdot \dot{\boldsymbol{v}}) = 0$, so that

$$\dot{\boldsymbol{v}}' = \gamma^2 \dot{\boldsymbol{v}} .$$

Equation (10.22.1) can also be obtained by another simpler method which involves the use of the expression for the square of the 4-dimensional acceleration which was found in the solution to the preceding problem. The quantity w_i^2 is a 4-invariant. This means that the evaluation of w_i^2 should yield the same result in S as in S'. Since the velocity of the particle in S' is $\boldsymbol{v}' = 0$, this leads directly to equation (10.22.1).

10.23

$$v(t) = \frac{wt + v_0/[1-(v_0/c)^2]^{1/2}}{\{1+[(wt)^2/c^2+(v_0^2/c^2)/(1-v_0^2/c^2)]^2\}^{1/2}} ,$$

$$x(t) = \frac{c^2}{w}\left\{\left[1+\left(\frac{wt}{c}+\frac{v_0/c}{1-v_0^2/c^2}\right)^2\right]^{1/2} - \frac{v_0/c}{1-v_0^2/c^2}\right\} + x_0 .$$

In the ultrarelativistic limit:

$$v(t) \approx c , \qquad x(t) \approx ct - \frac{c^2}{w} .$$

In the nonrelativistic limit:

$$v(t) = v_0 + wt , \qquad x(t) = x_0 + v_0 t + \tfrac{1}{2} w t^2 .$$

10.24 According to clocks in the stationary frame

$$T = \frac{1}{|\dot{v}|}\int_0^V \frac{dv}{(1-v^2/c^2)^{1/2}} = \frac{v}{|\dot{v}|(1-v^2/c^2)^{1/2}} = 47\cdot 5 \text{ years} ,$$

while according to the clocks at rest in the rocket

$$\tau = \frac{c}{2|\dot{v}|}\ln\left|\frac{1+v/c}{1-v/c}\right| = 2\cdot 5 \text{ years} .$$

10.25 Equation (10.25.1) describes a Lorentz transformation with a small relative velocity $\Delta \boldsymbol{v}$ and a rotation over an angle $\Delta\varphi = |\Delta\boldsymbol{\varphi}|$, where the axis of rotation goes through the origin, parallel to the vector $\Delta\boldsymbol{\varphi}$. As $\Delta\boldsymbol{v}$ and $\Delta\boldsymbol{\varphi}$ are small, these transformations can be performed in any order. The instantaneously comoving frame is thus a rotating one. This rotation is a purely kinematic relativistic effect and is the so-called Thomas precession (cf problem 10.16).

When $v \ll c$ equation (10.25.2) becomes

$$\Delta\boldsymbol{v} \approx \delta\boldsymbol{v} , \qquad \Delta\boldsymbol{\varphi} = \frac{1}{2c^2}[\delta\boldsymbol{v} \wedge \boldsymbol{v}] .$$

In that limit one can consider the quantity

$$\boldsymbol{\omega}_{\mathrm{T}} = \frac{\Delta\boldsymbol{\varphi}}{\delta t} = \frac{1}{2c^2}[\dot{\boldsymbol{v}} \wedge \boldsymbol{v}]$$

to be the angular velocity of the Thomas precession of the instantaneously comoving frame relative to the laboratory frame S.

10.26 In S

$$\cos\alpha = \frac{(\boldsymbol{v}_1 \cdot \boldsymbol{v}_2)}{|\boldsymbol{v}_1||\boldsymbol{v}_2|} ,$$

while in S'

$$\cos\alpha' = \frac{([\boldsymbol{v}_1 - \boldsymbol{V}]\cdot[\boldsymbol{v}_2 - \boldsymbol{V}]) - (1/c^2)([\boldsymbol{v}_1 \wedge \boldsymbol{V}]\cdot[\boldsymbol{v}_2 \wedge \boldsymbol{V}])}{\{(\boldsymbol{v}_1 - \boldsymbol{V})^2 - (1/c^2)[\boldsymbol{v}_1 \wedge \boldsymbol{V}]^2\}^{1/2}\{(\boldsymbol{v}_2 - \boldsymbol{V})^2 - (1/c^2)[\boldsymbol{v}_2 \wedge \boldsymbol{V}]^2\}^{1/2}} .$$

10.27 In S' the angle tends to zero. In order to show this, let $V = V_0 c$, where $|V_0| = 1$. If we use the formula for $\cos\alpha'$ derived in the preceding problem, and the result

$$([a \wedge b] \cdot [a_1 \wedge b_1]) = (a \cdot a_1)(b \cdot b_1) - (a \cdot b_1)(a_1 \cdot b) ,$$

we have

$$\cos\alpha' = \frac{c^2 - (v_1 \cdot V) - (v_2 \cdot V) + (1/c^2)(v_1 \cdot V)(v_2 \cdot V)}{\{[c - (v_1 \cdot V)/c]^2 [c - (v_2 \cdot V)/c]^2\}^{1/2}} = 1 ,$$

and hence $\alpha' = 0$. This contraction of the angular distribution is a specifically relativistic effect which appears in many phenomena.

10.28 The determination of the angle of aberration may be reduced to the determination of the angle α_1 between the direction of the ray AC (figure 10.28.1) and the direction of the velocity of the Earth, v, in its first position, and the angle α_2 between the ray direction BC and the direction of the velocity of the earth, v', in its second position (after six months). The angle of aberration δ is then given by $\delta = (\pi - \alpha_2) - \alpha_1$. The angles α_1 and α_2 may be computed from equation (10.a.15) by expressing them in terms of the angle ϑ which is observed in the reference frame in which the Sun is at rest, and is equal to the angle between the light ray OC and the velocity of the Earth:

$$\tan(\pi - \alpha_1) = \frac{1}{\gamma} \frac{\sin\vartheta}{\cos\vartheta - \beta} , \qquad \tan(\pi - \alpha_2) = -\frac{1}{\gamma} \frac{\sin\vartheta}{\cos\vartheta + \beta} ,$$

where $\beta = v/c$ and $\gamma = (1 - \beta^2)^{-1/2}$. Hence,

$$\tan\tfrac{1}{2}\delta = \left(\frac{1 - \cos\delta}{1 + \cos\delta}\right)^{1/2} = \beta\gamma \sin\vartheta .$$

We note that the three angles in figure 10.28.1 refer to different reference frames, and the drawing itself is conventional (for example,

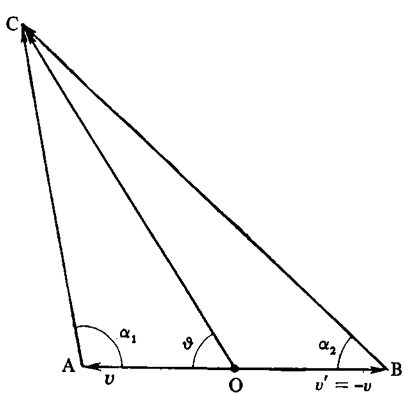


Figure 10.28.1.

AC = CO = CB = c). It is clear from these results that the angle of aberration δ depends only on the relative velocity v of the Earth and the Sun, and is independent of the velocity of the solar system relative to the star.

10.29 If the position of the Earth on the orbit is defined by the azimuth angle φ and $\boldsymbol{a} = (0, a_\vartheta, a_\phi)$ is a vector drawn from the point (ϑ, ϕ) on the celestial sphere to the point of the apparent position of the star on this sphere, then

$$a_\vartheta = -\beta\cos\vartheta\sin(\phi-\varphi)\,, \qquad a_\phi = -\beta\cos(\phi-\varphi)\,.$$

It is clear that the apparent position of the star on the celestial sphere will, in the course of a year, describe an ellipse with semi-axes $\beta\cos\vartheta$ and β.

10.30 The beam, which in S lies in the solid angle $\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\phi$, will be observed in S' in the solid angle $\mathrm{d}\Omega' = \sin\vartheta'\,\mathrm{d}\vartheta'\,\mathrm{d}\phi'$. Since $\phi = \phi'$ and $\cos\vartheta' = (\cos\vartheta-\beta)/(1-\beta\cos\vartheta)$ it follows that

$$\mathrm{d}\Omega' = \sin\vartheta'\,\mathrm{d}\vartheta'\,\mathrm{d}\phi' = \frac{1-\beta^2}{(1-\beta\cos\vartheta)^2}\,\mathrm{d}\Omega\,,$$

where, of course,

$$\int \mathrm{d}\Omega' = \int \mathrm{d}\Omega = 4\pi\,.$$

10.31

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega'} = \frac{N_0\,\mathrm{d}\Omega}{4\pi\,\mathrm{d}\Omega'} = \frac{N_0}{4\pi}\frac{1-\beta^2}{(1-\beta\cos\vartheta')^2}\,,$$

where N_0 is the total number of visible stars.

10.32

$$\omega = \gamma\omega'\left[1+\frac{(\boldsymbol{n}'\cdot\boldsymbol{V})}{c}\right] \quad \text{or} \quad \omega = \frac{\omega'}{\gamma[1-(\boldsymbol{n}\cdot\boldsymbol{V})/c]}\,,$$

$$\boldsymbol{k} = \gamma\left(\boldsymbol{k}'+\frac{\boldsymbol{V}\omega'}{c^2}\right)+(\gamma-1)\left[[\boldsymbol{k}'\wedge\boldsymbol{V}]\wedge\frac{\boldsymbol{V}}{V^2}\right],$$

where

$$\boldsymbol{n} = \frac{\boldsymbol{k}}{k}\,, \qquad \boldsymbol{n}' = \frac{\boldsymbol{k}'}{k'}\,, \qquad k = \frac{\omega}{c}\,.$$

10.33 If ω_0 is the frequency in the frame in which the source is at rest, and V is the velocity of the source relative to the detector, then the detector will record the (lower) frequency $\omega = \omega_0(1-V^2/c^2)^{1/2}$ (this is the so-called red shift). The angle α between the ray and the direction of motion of the source in the frame in which it is at rest is given by

$$\cos\alpha = -\frac{V}{c} .$$

The angle α is close to $\frac{1}{2}\pi$ only when $V \ll c$. As $V \to c$, $\alpha \to \pi$.

10.34

(a) $\lambda = \lambda_0\left(\dfrac{1-V/c}{1+V/c}\right)^{1/2}$; (b) $\lambda = \lambda_0\left(\dfrac{1+V/c}{1-V/c}\right)^{1/2}$

10.35

$$\omega = \omega_0\frac{(1-\beta^2)^{1/2}}{1-\beta\cos\theta} , \qquad J = J_0\frac{(1-\beta^2)^{3/2}}{(1-\beta\cos\theta)^2} .$$

The frequencies are the same, that is $\omega = \omega_0$, when $\theta = \theta_0$ where $\cos\theta_0 = [1-(1-\beta^2)^{1/2}]/\beta$; in that case $J = J_0(1-\beta^2)^{1/2}$. The intensities are the same, $J = J_0$, in the case in which $\theta = \theta_1 < \theta_0$, where $\cos\theta_1 = [1-(1-\beta^2)^{3/4}]/\beta$. When the light source is far from the observer, while approaching him, so that $\theta < \theta_0$, the frequency $\omega > \omega_0$ because of the Doppler effect ('violet' shift). If $\theta < \theta_1$ the intensity J will also be larger than J_0: a moving source looks brighter than a fixed one. The intensity is a maximum for $\theta = 0$ and equals $J_{max} = J_0(1+\beta)^{3/2}/(1-\beta)^{1/2}$. When $\theta > \theta_0$ the frequency $\omega < \omega_0$ and the observer sees a 'red' shift; the light intensity is now less than for a fixed source. These effects are particularly noticeable when $V \approx c$ when

$$\omega_{max} = \omega_0\left(\frac{1+\beta}{1-\beta}\right)^{1/2} \gg \omega_0 \quad \text{and} \quad J_{max} = J_0\frac{(1+\beta)^{3/2}}{(1-\beta)^{1/2}} \gg J_0 ,$$

while the angle

$$\theta_0 \approx \sqrt{2}(1-\beta)^{1/4} \ll 1 ,$$

so that the reddening of the light starts when the source is still far from the observer, but approaching him. This begins to occur at a distance $l \approx d/\theta_0$.

The number of photons emitted per unit laboratory time in the range of angles $0 < \theta < \theta_0$ is

$$N_1 = J_0(1-\beta^2)\int_0^{\theta_0}\frac{2\pi\sin\theta\, d\theta}{(1-\beta\cos\theta)^2} = 2\pi J_0(1-\beta^2)^{1/2}\frac{1+\beta-(1-\beta^2)^{1/2}}{\beta}$$

$$= 2\pi J_0(1-\beta^2)^{1/2}(1+\cos\theta_0) ,$$

and in the range of angles $\theta_0 < \theta < \pi$

$$N_2 = 2\pi J_0(1-\beta^2)^{1/2}\frac{(1-\beta^2)^{1/2}-1+\beta}{\beta} = 2\pi J_0(1-\beta^2)^{1/2}(1-\cos\theta_0) .$$

Clearly $N_1+N_2 = 4\pi J_0(1-\beta^2)^{1/2}$ corresponds to the total number of photons emitted per unit time in all directions. N_1 and N_2 have the same value when $\beta \ll 1$ and $\cos\theta_0 \approx 0$. If, however, β approaches unity, N_1 becomes much larger than N_2. In the ultrarelativistic case the overwhelming majority of the photons are thus emitted within a narrow angle $\theta < \theta_0$ and are then subject to a violet shift.

10.36 By using the solution of the preceding problem we get

$$I = J\hbar\omega = I_0\frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^3},$$

where $I_0 = J_0\hbar\omega_0$ is the isotropically distributed light intensity in the rest frame of the source. The total light flux

$$\Phi = \int_{(4\pi)} I\,\mathrm{d}\Omega = 2\pi I_0(1-\beta^2)^2\int_0^\pi \frac{\sin\theta\,\mathrm{d}\theta}{(1-\beta^2\cos^2\theta)^3} = 4\pi I_0 = \Phi_0$$

is the same in the rest frame of the source as in the laboratory frame (compare this with the result of problem 12.46).

10.37 Consider the system S' in which the mirror is at rest, and the laboratory system S. Let α_1' and α_2' be the angles between the wavevectors k_1' and k_2' of the incident and reflected waves and the direction of the velocity V of the mirror (figure 10.37.1). Let the frequencies before and after reflection be denoted by ω_1' and ω_2'. The analogous quantities in S will be denoted by the same symbols but without primes. According to the laws of reflection, in S' we have $\omega_1' = \omega_2' = \omega'$ and $\alpha_2' = \pi - \alpha_1'$. Hence $\cos\alpha_2' = -\cos\alpha_1'$. We can now express ω' in terms of ω, and $\cos\alpha'$ in terms of $\cos\alpha$, with the aid of equations (10.a.4) and (10.a.14). Solution of the resulting equations for ω_2 and $\cos\alpha_2$ yields

$$\cos\alpha_2 = -\frac{(1+\beta^2)\cos\alpha_1 - 2\beta}{1-2\beta\cos\alpha_1+\beta^2}, \qquad \omega_2 = \omega_1\frac{1-2\beta\cos\alpha_1+\beta^2}{1-\beta^2}.$$

When $\beta \to 1$, then for normal incidence on a receding mirror $\omega_2 \to 0$, while for normal incidence on an approaching mirror $\omega_2 \to \infty$.

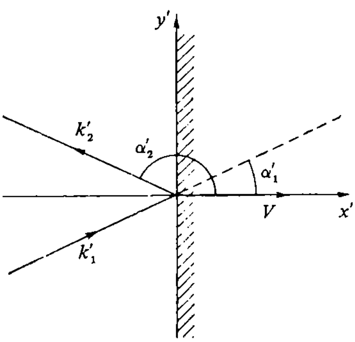


Figure 10.37.1.

10.38 $\omega_1 = \omega_2$. The angle of incidence is equal to the angle of reflection.

10.39 The image is formed by light quanta which reach the photographic plate simultaneously. However, these quanta were emitted by points of a moving body and, in general, were not emitted simultaneously. This follows as a consequence from the fact that the distances from different points of the body to the photographic plate are not the same and also from the fact that events which are simultaneous in one frame of reference are not simultaneous in another frame. The image of a moving object will therefore not be the same as that of a fixed object.

The quanta which are emitted simultaneously from different points of the edge A′B′ in the system S' (the cube) also reach the photographic plate at the same time. The length of the image AB will be the same as in the case of a fixed cube and will be determined solely by the contraction caused by the distance to the object and the focal distance of the camera. We take this length as 1.

For a fixed cube the image of the edge E′F′ would overlap that of A′B′ (in the limiting case of a sufficiently small solid angle when all light rays would be parallel). In the case of a moving cube, quanta from the edge E′F′ reach the photographic plate at the same time as those from the edge A′B′, if the first were emitted earlier by a time interval $\Delta t = l_0/c$ (in the frame S). Before that time interval the edge E′F′ occupied the position $E_1'F_1'$ and before the edge A′B′ emitted light it traversed a distance Vl_0/c. Therefore the edge EF will not be covered by the edge AB and the images of the edges A′E′ and B′F′ will have length $V/c = \beta$ and not zero, as for a fixed cube, and the whole face A′B′F′E′ will be photographed in the shape of the rectangle ABFE (figure 10.39.2a) with side ratio $1:\beta$.

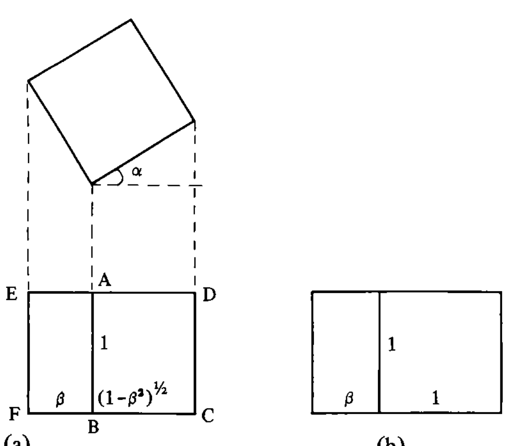


Figure 10.39.2.

The quanta forming the images of the edges A′B′ and C′D′ are emitted simultaneously by the cube in the frame S. It follows from the Lorentz transformation (10.a.1) that in the system S' the quanta from the edge C′D′ must be emitted earlier than those from the edge A′B′ by a time interval $\Delta t' = \gamma V l/c^2$, where l is the length of the edges B′C′ and A′D′ in the frame S. We may assume that in the system S', at points which are at a distance $\Delta x' = l_0$ from one another, two events occur, at times which are separated by $\Delta t'$. The distance between them in the system S is determined by using equation (10.a.1):

$$l \equiv \Delta x = \gamma(\Delta x' - V\Delta t') ,$$

whence, by substituting for $\Delta x'$ and $\Delta t'$, we find the length of the edges BC and AD in the system S: $l = l_0(1-\beta^2)^{1/2}$. They were subject to the usual Lorentz contraction. If we take the contraction in the camera into account, we find the lengths of their images to be $(1-\beta^2)^{1/2}$.

In figure 10.39.2a we show the picture of the image of the cube. It is interesting to note that the same image would be produced by a fixed cube turned over an angle $\alpha = \arcsin(V/c)$ relative to the direction of V. The apparent form of the object in this case is not subject to a distortion due to the Lorentz contraction—the object was only 'turned' over an angle α. It turns out (see the next few problems) that this result is true for any object and any angle between the velocity and the direction of the line of sight. It is merely necessary that the object is seen under a small solid angle.

If the Galilean transformations were valid, the edges A′D′ and B′C′ would not be subject to a Lorentz contraction and the image would be the one shown in figure 10.39.2b. The back (with respect to the direction of motion) face of the cube would be photographed, as before. The apparent form of a moving object would thus be subject to a distortion.

10.40

(a) $l = l_0|(1-\beta^2)^{1/2}\cos\alpha' - \beta\sin\alpha'|, \qquad \beta = \dfrac{V}{c}.$

The value $\alpha'_{\max}$ for which the function $|(1-\beta^2)^{1/2}\cos\alpha' - \beta\sin\alpha'|$ has a maximum is determined by the condition $\tan\alpha'_{\max} = -\beta/(1-\beta^2)^{1/2}$. In that case $l = l_0$: the largest length l is thus equal to l_0. The image is in this case equivalent to the image of a fixed rod which is oriented parallel to the photographic plate. The rod is 'turned' over an angle $\pi - \alpha'_{\max}$.

(b) $\alpha' = \arctan\left[\dfrac{(1-\beta^2)^{1/2}}{\beta}\right];$

in this case the image is obtained as if the rod were fixed and oriented at right angles to the photographic plate.

(c) If two observers, fixed in the system S, simultaneously make notches in the xy-plane in the points M and N which the end points of the rod pass at that moment, the section MN obtained by them will make an angle α with the x-axis, where

$$\alpha = \arctan\left[\frac{\tan\alpha'}{(1-\beta^2)^{1/2}}\right].$$

10.41 The image will have the shape of a circle. The hemisphere shown shaded in figure 10.41.1 is photographed. It is bounded by the plane A′B′ which makes an angle

$$\alpha' = \arctan\frac{\beta}{(1-\beta^2)^{1/2}}$$

with the direction of V (in the frame of the sphere). In spite of one's natural intuitive ideas, a moving sphere does not appear to an observer as an ellipsoid, flattened in the direction of the velocity. The Lorentz contraction is invisible! This does not, of course, mean that it is not present.

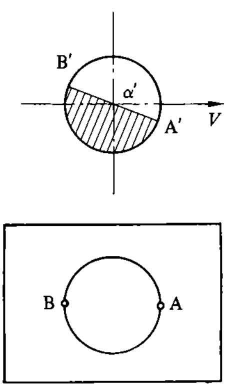


Figure 10.41.1.

10.42 The apparent positions of the cube are shown in figure 10.42.2. When $V/c < \cos\alpha$ we see the leading face A′D′ and the lower face A′B′. If in the optical system of the camera there is no contraction of the dimensions of the object, we have

$$\mathrm{AB} = l_0(1-\beta^2)^{1/2}\frac{\sin\alpha}{1-\beta\cos\alpha}, \qquad \mathrm{AD} = l_0\frac{\cos\alpha-\beta}{1-\beta\cos\alpha}.$$

Using these formulae we can find the angle ϑ over which the cube is turned:

$$\vartheta = \tfrac{1}{2}\pi-\alpha-\theta, \quad \text{where} \quad \tan\theta = \frac{\cos\alpha-\beta}{\sin\alpha(1-\beta^2)^{1/2}}.$$

When $V/c = \cos\alpha$, we have $\vartheta = \frac{1}{2}\pi - \alpha$ and only the lower face A′B′ can be seen. When $V/c > \cos\alpha$ the lower and trailing faces are seen,

$$\vartheta = \tfrac{1}{2}\pi - \alpha + \arctan\frac{\beta - \cos\alpha}{(1-\beta^2\sin\alpha)^{1/2}} .$$

Finally, when $V/c \to 1$, only the trailing face can be seen; the lower face is subject to a Lorentz contraction and vanishes, and $\vartheta = \pi - \alpha$.

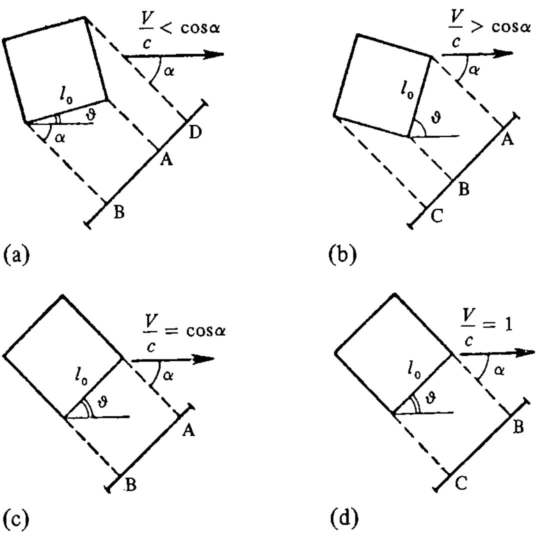


Figure 10.42.2.

10.43 Consider a plane wave of frequency ω' and wavevector $\boldsymbol{k}'$ $(k'\cos\alpha', k'\sin\alpha', 0)$, $\boldsymbol{k}' \perp \mathrm{O}z$, propagating in a frame S' fixed in the medium. The phase velocity $v' = c/n = \omega'/k'$ in the system S' is independent of angle α' which is determined by the direction of propagation of the wave. The field components are proportional to $\exp(-\mathrm{i}k_i'x_i')$, where $k_i' = (\omega'/c, \boldsymbol{k}')$. Since the phase $k_i x_i = k_i' x_i'$ is invariant under a Lorentz transformation, k_i is a 4-vector (wave 4-vector). By using equations (10.a.4) and (10.a.14) we can find the components k_i in the frame S, with respect to which the medium moves with a velocity $\boldsymbol{V} \parallel \mathrm{O}x$, and hence

$$\omega = \gamma\omega'(1+\beta n\cos\alpha') , \qquad (10.43.1)$$

$$\tan\alpha = \frac{\sin\alpha'}{\gamma(\cos\alpha' + \beta/n)} ,$$

$$v = \frac{\omega}{k} = c\frac{1+\beta n\cos\alpha'}{[(n\cos\alpha'+\beta)^2 + n^2\sin^2\alpha'(1-\beta^2)]^{1/2}} , \qquad (10.43.2)$$

where $\beta = V/c$ and $\gamma = (1-\beta^2)^{-1/2}$. It is clear from equation (10.43.2) that the phase velocity in a moving medium depends on the direction of propagation. Thus, a particular form of anisotropy appears in a moving medium.

10.44 The required velocity may be found from equation (10.43.2) of the preceding problem ($\alpha' = 0$):

$$v = c\frac{1+\beta n(\lambda')}{n(\lambda')+\beta} \approx \frac{c}{n(\lambda')} + V\left[1 - \frac{1}{n^2(\lambda')}\right],$$

where $\lambda' = 2\pi c/\omega'$ and ω' is the frequency observed in the frame S' in which the medium is at rest. Using equation (10.43.1) of the preceding solution, we have, to within terms of the first order in V/c,

$$\frac{\lambda'}{\lambda} = \frac{\omega}{\omega'} = 1 + \frac{nV}{c},$$

whence,

$$\frac{c}{n(\lambda')} = \frac{c}{n(\lambda)} - \frac{c}{n^2}\frac{\mathrm{d}n}{\mathrm{d}\lambda}\lambda\frac{nV}{c},$$

and finally,

$$v = \frac{c}{n(\lambda)} + V\left[1 - \frac{1}{n^2(\lambda)} - \frac{\lambda}{n(\lambda)}\frac{\mathrm{d}n(\lambda)}{\mathrm{d}\lambda}\right].$$

b Four-dimensional vectors and tensors

10.48 One three-dimensional tensor of rank 2 [$A_{\alpha\beta}$ ($\alpha, \beta = 1, 2, 3$)], two three-dimensional vectors [$A_{0\alpha}$ and $A_{\alpha 0}$ ($\alpha = 1, 2, 3$)], and a three-dimensional scalar (A_{00}).

10.49 The skew-symmetric 4-tensor A_{ik} may be written in the form

$$A_{ik} = \begin{pmatrix} 0 & -B_1 & -B_2 & -B_3 \\ B_1 & 0 & A_3 & -A_2 \\ B_2 & -A_3 & 0 & A_1 \\ B_3 & A_2 & -A_1 & 0 \end{pmatrix},$$

where $\boldsymbol{A} = (A_1, A_2, A_3)$ and $\boldsymbol{B} = (B_1, B_2, B_3)$ are three-dimensional vectors, or more precisely, $\boldsymbol{B}$ is a polar vector and $\boldsymbol{A}$ an axial vector.

10.53 The invariant quantity

$$\mathrm{d}\varphi = \frac{\partial\varphi}{\partial x_0}\mathrm{d}x_0 + \frac{\partial\varphi}{\partial x_1}\mathrm{d}x_1 + \frac{\partial\varphi}{\partial x_2}\mathrm{d}x_2 + \frac{\partial\varphi}{\partial x_3}\mathrm{d}x_3$$

has the same form in all inertial systems of reference; since the $\mathrm{d}x_i$ ($i = 0, 1, 2, 3$) are the components of a 4-vector, the set of quantities

$$\nabla_i\varphi = \left(\frac{\partial\varphi}{\partial x_0}, -\frac{\partial\varphi}{\partial x_1}, -\frac{\partial\varphi}{\partial x_2}, -\frac{\partial\varphi}{\partial x_3}\right)$$

is also a 4-vector. The four-dimensional gradient operator, defined by the relation

$$\nabla_i = \left(\frac{\partial}{\partial x_0}, -\nabla\right),$$

where ∇ is the three-dimensional gradient operator, thus transforms as a 4-vector.

10.54

$$T_{ik} = \nabla_k A_i, \qquad \nabla_k = \left(\frac{\partial}{\partial x_0}, -\frac{\partial}{\partial x_1}, -\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_3}\right).$$

The four-dimensional divergence is

$$\nabla_i A_i = \frac{\partial A_0}{\partial x_0} + \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \text{invariant}.$$

10.55 (a) A scalar; (b) a 4-vector.

10.56 The condition for A_i and B_i to be parallel may be written in the form

$$\frac{\alpha_{0i}A_0}{\alpha_{0i}B_0} = \frac{-\alpha_{1i}A_1}{-\alpha_{1i}B_1} = \frac{-\alpha_{2i}A_2}{-\alpha_{2i}B_2} = \frac{-\alpha_{3i}A_3}{-\alpha_{3i}B_3},$$

where the numerator and denominator of each fraction is multiplied by the same number. Hence, using the properties of equal fractions, we have

$$\frac{A_i}{B_i} = \frac{\alpha_{0i}A_0 - \alpha_{1i}A_1 - \alpha_{2i}A_2 - \alpha_{3i}A_3}{\alpha_{0i}B_0 - \alpha_{1i}B_1 - \alpha_{2i}B_2 - \alpha_{3i}B_3} = \frac{A_i'}{B_i'}.$$

10.57 There are four different components. Except for the sign, they are identical with the components of the vector $A_i = \frac{1}{6} e_{iklm} A_{klm}$ and hence,

$$A_0 = A_{123} = -A_{213} = \ldots, \qquad A_1 = -A_{230} = A_{320} = \ldots,$$
$$A_2 = -A_{310} = A_{130} = \ldots, \qquad A_3 = -A_{120} = A_{210} = \ldots.$$

The remaining components A_{ikl} are all zero (they have at least two equal indices). It follows that the nonzero components of A_{ikl} transform on four-dimensional rotations and reflections as the components of a four-dimensional pseudovector.

10.59 If $x_i = \alpha_{ik} x_k'$, then the matrix $\hat{\alpha}$ is of the form

$$\hat{\alpha} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ \sinh\alpha & -\cosh\alpha & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

10.60 The required matrix $\hat{g}$ may be written in the form of the product of three matrices:

$$\hat{g} = \hat{g}(\vartheta, \phi)\hat{g}(\alpha)\hat{g}^{-1}(\vartheta, \phi).$$

The matrix

$$\hat{g}(\vartheta, \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\vartheta\cos\phi & \sin\phi & -\sin\vartheta\cos\phi \\ 0 & -\cos\vartheta\sin\phi & -\cos\phi & -\sin\vartheta\sin\phi \\ 0 & \sin\vartheta & 0 & -\cos\vartheta \end{pmatrix}$$

describes spatial rotation of the reference frame (figure 10.60.1).

$$x_i = \sum_k g_{ik}(\vartheta, \phi) x_k'' .$$

The matrix

$$\hat{g}(\alpha) = \begin{pmatrix} \cosh\alpha & 0 & 0 & -\sinh\alpha \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sinh\alpha & 0 & 0 & -\cosh\alpha \end{pmatrix}.$$

corresponds to the transition from S''' to S'', where the former moves along the x_3''-axis with the velocity $V = c\tanh\alpha$ [see the special form of the Lorentz transformation (10.a.1)]. Finally, the matrix $\hat{g}^{-1}(\vartheta, \phi)$ describes a rotation as a result of which the frame S' transforms into S''' (figure 10.60.1), and is identical with the transpose of $\hat{g}(\vartheta, \phi)$. On multiplying the matrices, we have

$$\hat{g} =$$

$$\begin{pmatrix} \cosh\alpha & -\omega_1\sinh\alpha & -\omega_2\sinh\alpha & -\omega_3\sinh\alpha \\ \omega_1\sinh\alpha & \omega_1^2(1-\cosh\alpha)-1 & \omega_1\omega_2(1-\cosh\alpha) & \omega_1\omega_3(1-\cosh\alpha) \\ \omega_2\sinh\alpha & \omega_1\omega_2(1-\cosh\alpha) & \omega_2^2(1-\cosh\alpha)-1 & \omega_2\omega_3(1-\cosh\alpha) \\ \omega_3\sinh\alpha & \omega_1\omega_3(1-\cosh\alpha) & \omega_2\omega_3(1-\cosh\alpha) & \omega_3^2(1-\cosh\alpha)-1 \end{pmatrix},$$

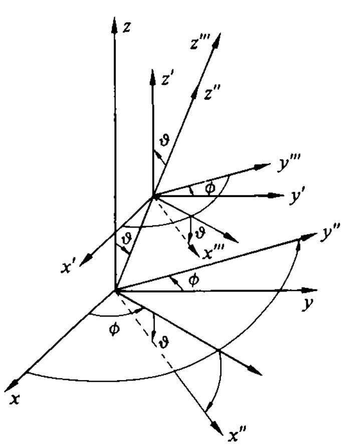


Figure 10.60.1.

where

$$\omega_1 = \sin\vartheta\cos\phi\,, \qquad \omega_2 = \sin\vartheta\sin\phi\,, \qquad \omega_3 = \cos\vartheta\,.$$

c Relativistic electrodynamics

10.61 In vacuo

$$\boldsymbol{E} = \gamma\left(\boldsymbol{E}' - \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{H}'\right]\right) - (\gamma - 1)\boldsymbol{V}\frac{(\boldsymbol{V}\cdot\boldsymbol{E}')}{V^2}\;;$$

$$\boldsymbol{H} = \gamma\left(\boldsymbol{H}' + \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{E}'\right]\right) - (\gamma - 1)\boldsymbol{V}\frac{(\boldsymbol{V}\cdot\boldsymbol{H}')}{V^2}\,.$$

In a medium

$$\boldsymbol{P} = \gamma\left(\boldsymbol{P}' + \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{M}'\right]\right) - (\gamma - 1)\boldsymbol{V}\frac{(\boldsymbol{V}\cdot\boldsymbol{P}')}{V^2}\;;$$

$$\boldsymbol{M} = \gamma\left(\boldsymbol{M}' - \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{P}'\right]\right) - (\gamma - 1)\boldsymbol{V}\frac{(\boldsymbol{V}\cdot\boldsymbol{M}')}{V^2}\,.$$

The transformation formulae for the pairs of vectors $\boldsymbol{E}$, $\boldsymbol{B}$ and $\boldsymbol{D}$, $\boldsymbol{H}$ are analogous to the transformation formulae for $\boldsymbol{E}$, $\boldsymbol{H}$ in vacuo.

10.62 The problem has an infinite number of solutions. If the system S' (moving with a velocity $\boldsymbol{V}$) in which $\boldsymbol{E}'$ is parallel to $\boldsymbol{H}'$ has been found, then in any other reference frame, which moves relative to S' along this common direction, the vectors $\boldsymbol{E}$ and $\boldsymbol{H}$ will again be parallel [this follows from equation (10.c.3)]. In view of this, let us try to find the reference frame S' which moves at right angles to the plane $\boldsymbol{E}$, $\boldsymbol{H}$. From the condition for the vectors $\boldsymbol{E}'$ and $\boldsymbol{H}'$ to be parallel, i.e. $[\boldsymbol{E}' \wedge \boldsymbol{H}'] = 0$, and the transformation formulae of the previous problem, we have

$$\frac{\boldsymbol{V}}{c} = [\boldsymbol{E} \wedge \boldsymbol{H}]\frac{E^2 + H^2 - [(E^2 - H^2)^2 + 4(\boldsymbol{E}\cdot\boldsymbol{H})^2]^{1/2}}{2[\boldsymbol{E} \wedge \boldsymbol{H}]^2}\,.$$

Using the field invariants we have

$$E'^2 = \tfrac{1}{2}\{E^2 - H^2 + [(E^2 - H^2)^2 + 4(\boldsymbol{E}\cdot\boldsymbol{H})^2]^{1/2}\}\,,$$
$$H'^2 = \tfrac{1}{2}\{H^2 - E^2 + [(E^2 - H^2)^2 + 4(\boldsymbol{E}\cdot\boldsymbol{H})^2]^{1/2}\}\,.$$

10.63 When $E > H$, a reference frame should exist in which $H' = 0$ and $E' = (E^2 - H^2)^{1/2}$. When $E < H$, there exists a reference frame in which $E' = 0$ and $H' = (H^2 - E^2)^{1/2}$.

When $E > H$ we have

$$\boldsymbol{V} = c\frac{[\boldsymbol{E} \wedge \boldsymbol{H}]}{E^2}\,, \qquad \boldsymbol{E}' = \frac{\boldsymbol{E}}{E}(E^2 - H^2)^{1/2}\,.$$

In any frame S'' moving in the direction of $\boldsymbol{E}'$ with an arbitrary velocity, the magnetic field will also be zero.

When $E < H$

$$V = c\frac{[H \wedge E]}{H^2}, \qquad H' = \frac{H}{H}(H^2 - E^2)^{1/2}.$$

10.64 When $\kappa < J/c$ and the reference frame moves with velocity $V = c^2\kappa/J$ parallel to the axis of the cylinder and to $[E \wedge H]$, the electric field E' is zero and the magnetic field is given by

$$H' = \frac{2J}{cr}\left(1 - \frac{c^2\kappa^2}{J^2}\right)^{1/2}$$

When $\kappa > J/c$ and the reference frame moves with velocity $V = J/\kappa$ parallel to the axis of the cylinder and to $[E \wedge H]$, the magnetic field H' is zero and

$$E' = \frac{2\kappa}{r}\left(1 - \frac{J^2}{c^2\kappa^2}\right)^{1/2}$$

When $\kappa = J/c$, there is no reference frame in which there is only an electric or only a magnetic field. As can be seen from the above formulae, when $\kappa \to J/c$, the velocity of such a reference frame tends to the velocity of light, and the magnitude of the two fields tend to zero.

10.65 (a) At a fixed time ($\mathrm{d}t = 0$) we get the equations $[\mathbf{d}\boldsymbol{r} \wedge \boldsymbol{H}] = 0$, $(\boldsymbol{E} \cdot \mathbf{d}\boldsymbol{r}) = 0$. The first of these shows that $\mathbf{d}\boldsymbol{r}$ is parallel to $\boldsymbol{H}$, i.e. $\mathbf{d}\boldsymbol{r}$ is an element of length of a magnetic field line. The set (10.65.2) can be written in the form $F_{ik}\,\mathrm{d}x_k = 0$, whence follows its relativistic invariance. Here F_{ik} is the field tensor and $\mathrm{d}x_k$ the increment of the coordinates.

(b) The compatibility condition for the set is $(\boldsymbol{E} \cdot \boldsymbol{H}) = 0$. It is relativistically invariant and shows that one can only introduce relativistically invariant magnetic field lines when the electrical and magnetic fields are at right angles to one another.

(c) The integrability conditions of the set of equations has the form

$$\left[\boldsymbol{H} \wedge \left(\mathrm{curl}\boldsymbol{E} + \frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t}\right)\right] - \boldsymbol{E}\,\mathrm{div}\boldsymbol{H} = 0,$$

or, in covariant form,

$$F_{ik}\,e_{klmn}\frac{\partial F_{mn}}{\partial x_l} = 0,$$

and this is always satisfied, by virtue of the Maxwell equations.

(d) Writing the set (10.65.2) in the form $(\boldsymbol{E} \perp \boldsymbol{H})$

$$\mathbf{d}\boldsymbol{r} = \frac{\boldsymbol{H}(\boldsymbol{H} \cdot \mathrm{d}\boldsymbol{r})}{H^2} + c\frac{[\boldsymbol{E} \wedge \boldsymbol{H}]}{H^2}\mathrm{d}t,$$

we immediately verify that the statement under (d) is correct.

10.66 If we write the equations given in the problem in three-dimensional form, they become

$$[\mathrm{d}\boldsymbol{r} \wedge \boldsymbol{E}] - c\boldsymbol{H}\,\mathrm{d}t = 0\,, \qquad (\boldsymbol{H}\cdot \mathrm{d}\boldsymbol{r}) = 0$$

whence it follows that at any fixed time ($\mathrm{d}t = 0$) the condition $[\mathrm{d}\boldsymbol{r} \wedge \boldsymbol{E}] = \boldsymbol{0}$ that the increment $\mathrm{d}\boldsymbol{r}$ and the electrical field $\boldsymbol{E}$ are parallel, is satisfied. The compatibility condition is $(\boldsymbol{E}\cdot\boldsymbol{H}) = 0$, and the equations are integrable when

$$\left[\boldsymbol{E} \wedge \left(\operatorname{curl}\boldsymbol{H} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}\right)\right] + \boldsymbol{H}\operatorname{div}\boldsymbol{E} = \boldsymbol{0}\,.$$

This last equation imposes the following condition on the charge and current distributions:

$$[\boldsymbol{E} \wedge \boldsymbol{j}] + c\boldsymbol{H}\rho = 0\,.$$

If the conditions given here are not satisfied it is impossible to introduce invariant electrical field lines. The field lines move at right angles to their direction with a velocity

$$\boldsymbol{u} = -c\frac{[\boldsymbol{E} \wedge \boldsymbol{H}]}{E^2}\,.$$

10.68

$$\varphi = \frac{e}{R^*}\,, \qquad \boldsymbol{A} = e\frac{\boldsymbol{V}}{cR^*}\,,$$

$$\boldsymbol{E} = \frac{e\boldsymbol{R}}{\gamma^2 R^{*3}} = \frac{e\boldsymbol{R}(1 - V^2/c^2)}{R^3[1-(V^2/c^2)\sin^2\vartheta]^{3/2}}\,, \qquad \boldsymbol{H} = \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{E}\right],$$

where $R^* = [(x - Vt)^2 + (1-\beta^2)(y^2+z^2)]^{1/2}$, $(Vt, 0, 0)$ are the coordinates of the moving charge at time t, $\boldsymbol{R} = (x - Vt, y, z)$ is the position vector of the charge relative to the point of observation at time t, and ϑ is the angle between $\boldsymbol{R}$ and $\boldsymbol{V}$.

10.69

$$\boldsymbol{G} = \frac{e^2\gamma}{6\pi c^2 R}\boldsymbol{V}\,.$$

10.70 It follows from the formulae derived in the preceding solution that along the line of motion of the charge ($\vartheta = 0, \pi$) the field E is smaller than the Coulomb field $E_{\mathrm{Coul}} = e/R^2$ by a factor $1 - V^2/c^2$, while in the perpendicular direction ($\vartheta = \frac{1}{2}\pi$) the field E is greater by a factor $(1 - V^2/c^2)^{-1/2}$. When $V \sim c$ the field is large only within the angular range $\Delta\vartheta \sim (1 - V^2/c^2)^{1/2}$ near the equatorial plane.

The condition $E_{\parallel} = E'_{\parallel}$ refers to the same points in the 4-space. However, if in the rest system of the charge, a point A lies on the x-axis at a distance R from the charge, then in the laboratory system the same point will lie at a distance $R(1-\beta^2)^{1/2}$. If we compare the values of $E_{\parallel}$

at the point $R(1-\beta^2)^{1/2}$ and $E'_\parallel$ at the point R, we have

$$E_\parallel = \frac{eR(1-\beta^2)^{1/2}(1-\beta^2)}{[R(1-\beta^2)^{1/2}]^3} = \frac{e}{R^2} = E'_\parallel \,,$$

as was to be expected.

10.71

$$\varphi = \frac{(\boldsymbol{p}_0 \cdot \boldsymbol{r}^*)}{\gamma r^{*3}} \,, \qquad \boldsymbol{A} = \frac{\boldsymbol{V}}{c}\varphi \,,$$

$$\boldsymbol{E} = \frac{3\boldsymbol{R}(\boldsymbol{p}_0 \cdot \boldsymbol{r}^*) - \boldsymbol{p}_0 r^{*2}}{\gamma^2 r^{*5}} \,, \qquad \boldsymbol{H} = \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{E}\right] ,$$

where $\boldsymbol{R} = (x-Vt, y, z)$, $\boldsymbol{r}^* = (x-Vt, y/\gamma, z/\gamma)$; the dipole moves along the x-axis and its position vector at time t is $\boldsymbol{V}t$.

10.72

$$\boldsymbol{p} = \boldsymbol{p}' + \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{m}\right] - (\gamma-1)\boldsymbol{V}\frac{(\boldsymbol{V} \cdot \boldsymbol{p}')}{\gamma V^2} \,,$$

$$\boldsymbol{m} = \boldsymbol{m}' - \left[\frac{\boldsymbol{V}}{c} \wedge \boldsymbol{p}'\right] - (\gamma-1)\boldsymbol{V}\frac{(\boldsymbol{V} \cdot \boldsymbol{m}')}{\gamma V^2} \,,$$

where $\boldsymbol{p}'$ and $\boldsymbol{m}'$ are the dipole moments in the rest system.

10.73 If we use the formulae for the transformation of the four-dimensional current density, we find that the sides 2 and 4 of the rectangle shown in figure 10.73.1 are uncharged, whereas the sides 1 and 3 carry charges $q_1 = -q_3 = -(V/c)(J'a/c)$ where J' is the current in the system S' in which the loop is at rest. Hence (or from the result of the preceding problem), it follows that the electric dipole moment of the loop as observed in S' is $p = q_3 b = (V/c)m'$, where $m' = J'ab/c$ is the magnetic moment of the loop as observed in S'.

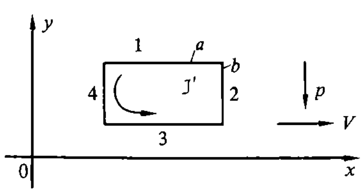


Figure 10.73.1.

10.74 Let u_i be the four-dimensional velocity of the medium. Consider the 4-invariant [see equation (10.c.15)]

$$-f_i u_i = \gamma(\boldsymbol{f} \cdot \boldsymbol{V}) - \gamma[Q + (\boldsymbol{f} \cdot \boldsymbol{V})] = -\gamma Q = \text{invariant} \,.$$

If Q_0 is the amount of heat released per unit volume of the medium per unit time in the system in which the medium is at rest, then $Q = Q_0(1-\beta^2)^{1/2}$.

10.75

$$w = \gamma^2\left(w' + \frac{2\beta}{c}S'_x + \beta^2 T'_{xx}\right), \qquad S_x = \gamma^2[(1+\beta^2)S'_x + Vw' + VT'_{xx}],$$

$$S_y = \gamma(S'_y + VT'_{xy}), \qquad S_z = \gamma(S'_z + VT'_{xz}),$$

$$T_{xx} = \gamma^2\left(T'_{xx} + \frac{2\beta}{c}S'_x + \beta^2 w'\right), \qquad T_{yy} = T'_{yy}, \qquad T_{yz} = T'_{yz},$$

$$T_{zz} = T'_{zz}, \qquad T_{xy} = \gamma\left(T'_{xy} + \frac{\beta}{c}S'_y\right), \qquad T_{xz} = \gamma\left(T'_{xz} + \frac{\beta}{c}S'_z\right).$$

10.76 $T_{ii} = 0$.

10.77 The momentum and energy of the field in the volume V at time $t = x_0/c$ are given by the integrals $\int T_{0\alpha}\,\mathrm{d}^3S$ and $\int T_{00}\,\mathrm{d}^3S$ respectively, where d^3S is an element of the hypersurface x_4 = constant (it is evident that $\mathrm{d}^3S = \mathrm{d}^3\boldsymbol{r}$). The momentum and energy of the field at the time $t' = x'_0/c$ is given by analogous integrals. Consider an arbitrary auxiliary constant 4-vector a_i and evaluate the sum $T_{0i}a_i$. Next, consider the 4-volume Ω which is bounded by a cylindrical hypersurface S, whose generator is parallel to the x_0-axis, and the two hyperplanes x_0 = constant and x'_0 = constant (figure 10.77.1).

Applying the four-dimensional form of Gauss' theorem to the surface integral of the function $T_{0i}a_i$ over this hypersurface, we have

$$\oint T_{0i}a_i\,\mathrm{d}^3S = \int_\Omega \frac{\partial T_{0i}}{\partial x_i}a_i\,\mathrm{d}^4\Omega = 0,$$

since $\partial T_{0i}/\partial x_i = 0$ in the absence of charges. On the cylindrical hypersurface $T_{0i} = 0$, since the field is zero on the boundary of the volume V. Hence (taking into account the direction of the normal), we have

$$a_i \int T_{0i}\,\mathrm{d}^3\boldsymbol{r} = a'_i \int T'_{0i}\,\mathrm{d}^3\boldsymbol{r}'.$$

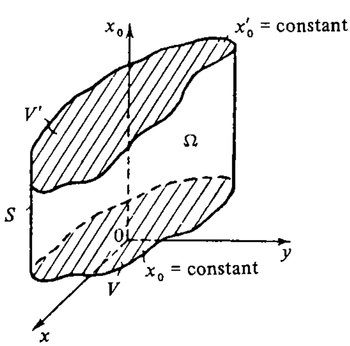


Figure 10.77.1.

In other words, the quantity $a_i \int T_{0i}\, d^3r$ is invariant with respect to the Lorentz transformation, and hence $\int T_{0i}\, d^3r$ should be a 4-vector (cf problems 10.55 and 1.4).

10.78 Consider the change in K_{ik} during a time dt, where K_{ik} is a functional of the space-like hypersurface t = constant. It will be necessary to compare the values of K_{ik} at two hyperplanes t = constant and $t+dt$ = constant. Since the field at infinity is zero, the difference between integrals over these hyperplanes may be transformed into an integral over a closed hypersurface S which is formed by adding an infinitely distant lateral hyperplane to the above two hyperplanes. The resulting integral may be transformed in accordance with Gauss' theorem:

$$\oint_S A_{ikl}\, d^3S_l = \int_\Omega \frac{\partial A_{ikl}}{\partial x_l}\, d^4\Omega ,$$

where Ω is the volume enclosed by the closed hypersurface S. The integrand on the right-hand side may be transformed as follows:

$$\frac{\partial A_{ikl}}{\partial x_l} = \frac{\partial}{\partial x_l}(x_i T_{kl} - x_k T_{il}) = T_{ki} - T_{ik} + x_i \frac{\partial T_{kl}}{\partial x_l} - x_k \frac{\partial T_{il}}{\partial x_l} ,$$

where $T_{ik} = T_{ki}$, owing to the symmetry of the stress 4-tensor.

Consider now the integral

$$\int x_i \frac{\partial T_{kl}}{\partial x_l}\, d^4\Omega = -\frac{1}{c}\int x_i F_{kl} j_l\, d^4\Omega .$$

Since we are concerned with a system of point particles, we have

$$\int x_i F_{kl} j_l\, d^3r = \sum e x_i F_{kl} \frac{dx_l}{dt} ,$$

where the right-hand side of this expression involves the coordinates of the particles and functions of them at time t. From the equations of motion of the particles,

$$\frac{e}{c} F_{kl} \frac{dx_l}{dt} = \frac{dp_k}{dt} .$$

The integral $\int x_k (\partial T_{il}/\partial x_l)\, d^4\Omega$ may be considered in a similar way. Thus, the integral with respect to Ω becomes equal to $-\sum(x_i\, dp_k/dt - x_k\, dp_i/dt)\,dt$ and is cancelled by an equal sum over the particles. Hence,

$$\frac{dK_{ik}}{dt} = 0 \quad \text{and} \qquad K_{ik} = \text{constant} .$$

10.79 The total angular momentum of the particles and the field within the volume is

$$K_{\alpha\beta}(t) = \sum k_{\alpha\beta} - \frac{1}{c}\int_t (x_\alpha T_{\beta\gamma} - x_\beta T_{\alpha\gamma})\, d^3S_\gamma ,$$

where $k_{\alpha\beta} = x_\alpha p_\beta - x_\beta p_\alpha$ is the angular momentum of one of the particles, and the integral is evaluated over the part of the hyperplane $t =$ constant whose projection on to three-dimensional space is V. $K_{\alpha\beta}(t+\mathrm{d}t)$ may be written down in a similar way. The angular momentum lost by the system in a time $\mathrm{d}t$ is

$$-\mathrm{d}K_{\alpha\beta} = K_{\alpha\beta}(t) - K_{\alpha\beta}(t+\mathrm{d}t) = -\sum \mathrm{d}k_{\alpha\beta} + \frac{1}{c}\int_{t+\mathrm{d}t} \ldots - \frac{1}{c}\int_t \ldots .$$

The difference between integrals over near hyperplanes may be rewritten in another form, since

$$\int_t + \int_{t+\mathrm{d}t} + \int_{S_{\text{side}}} = \oint$$

over a closed cylindrical hypersurface (see figure 10.79.1) whose generators are parallel to the time axis(1). As in the previous solution, it can be shown that $\oint$ is cancelled by $-\Sigma\,\mathrm{d}k_{\alpha\beta}$, and hence

$$-\mathrm{d}K_{\alpha\beta} = \frac{1}{c}\int_{S_{\text{side}}} (x_\alpha T_{\beta\gamma} - x_\beta T_{\alpha\gamma})\,\mathrm{d}^3S_\gamma \,.$$

The elements of the hypersurface S_{side} are normal to the t-axis and may be written in the form $\mathrm{d}S_\gamma = c\,\mathrm{d}t\,n_\gamma\,\mathrm{d}^2f$, where d^2f is an element of the ordinary surface enclosing the volume V and $\boldsymbol{n}$ is a unit normal to this element. Hence, the rate of loss of angular momentum of the system is

$$-\frac{\mathrm{d}K_{\alpha\beta}}{\mathrm{d}t} = \int (-x_\alpha T_{\beta\gamma} + x_\beta T_{\alpha\gamma}) n_\gamma\,\mathrm{d}^2f\,. \qquad (10.79.1)$$

Consider now the tensor $R_{\alpha\beta\gamma} = x_\beta T_{\alpha\gamma} - x_\alpha T_{\beta\gamma}$, which is skew-symmetric with respect to the subscripts α, β. This tensor may be interpreted as the angular momentum flux density, as is evident from equation (10.79.1).

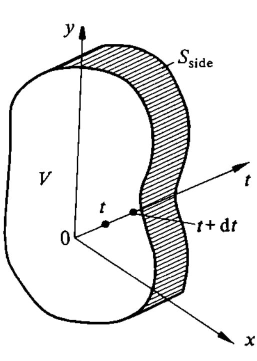


Figure 10.79.1.

(1) It must be remembered that such diagrams are purely conventional.

The component $R_{\alpha\beta\gamma}$ is equal to the $\alpha\beta$ component of the total angular momentum $K_{\alpha\beta}$ which flows per unit time through a unit area at right angles to the x_γ-axis. In a way similar to that in which $K_{\alpha\beta}$ may be replaced by the pseudovector $\boldsymbol{K}$, it is possible to introduce a pseudovector equivalent of $R_{\alpha\beta\gamma}n_\gamma$. Equation (10.79.1) will then be of the form

$$-\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}t} = \int \boldsymbol{R}\,\mathrm{d}^2f\,, \tag{10.79.2}$$

$$\boldsymbol{R} = \frac{E^2+H^2}{8\pi}[\boldsymbol{r}\wedge\boldsymbol{n}] - \frac{1}{4\pi}\{[\boldsymbol{r}\wedge\boldsymbol{E}](\boldsymbol{n}\cdot\boldsymbol{E}) + [\boldsymbol{r}\wedge\boldsymbol{H}](\boldsymbol{n}\cdot\boldsymbol{H})\}\,. \tag{10.79.3}$$

In deriving equation (10.79.3) use was made of the expression given by equation (10.c.7) for the components $T_{\alpha\beta}$.

11

Relativistic mechanics

a Energy and momentum

11.1

$$p = \frac{1}{c}[T(T+2mc^2)]^{1/2} .$$

11.2

$$v = \frac{cp}{(p^2+m^2c^2)^{1/2}} .$$

11.3

$$\beta = \frac{v}{c} = \left[1-\left(\frac{\mathcal{E}_0}{\mathcal{E}}\right)^2\right]^{1/2} ,$$

where $\mathcal{E}_0 = mc^2$. In the nonrelativistic case $\beta \approx (2T/\mathcal{E}_0)^{1/2}$; in the ultrarelativistic case $\beta = 1-\frac{1}{2}(\mathcal{E}_0/\mathcal{E})^2$.

11.4

(a) $T = \frac{1}{2}mv^2+\frac{3}{8}m\frac{v^2}{c^2}+ \ldots ,$

(b) $T = \frac{p^2}{2m}-\frac{1}{8}\frac{p^4}{m^3c^2}+ \ldots .$

11.5

$$v = \left[\frac{2eV}{m}\frac{1+eV/2mc^2}{(1+eV/mc^2)^2}\right]^{1/2}$$

In particular, when $eV \ll mc^2$

$$v = \left(\frac{2eV}{m}\right)^{1/2}\left(1-\frac{3}{4}\frac{eV}{mc^2}\right) \ll c .$$

When $eV \gg mc^2$

$$v = c\left[1-\frac{1}{2}\left(\frac{mc^2}{eV}\right)^2\right] \approx c .$$

11.6 (a) $v = 0{\cdot}0342c$; (b) $v = 0{\cdot}9999985c$; (c) $0{\cdot}81c$; (d) $0{\cdot}9956c$.

11.7

$$F = \frac{J}{ce}[T(T+2mc^2)]^{1/2} , \qquad W = \frac{J}{e}T .$$

11.8

$$p = \frac{2mv^2N}{1-v^2/c^2} .$$

The pressure in the system attached to the body and in the system attached to the gas is the same. This may be verified either by direct calculation of the pressure in each of the systems, or by applying the Lorentz transformation to the 4-force [see equation (11.b.3)].

11.9 The length of the nth tube is given by

$$L_n = \frac{v_n}{2\nu} = \frac{c}{2\nu}\left[1-\left(\frac{mc^2}{nV_e e+mc^2}\right)^2\right]^{1/2},$$

where v_n is the particle velocity in the nth tube. At the beginning of the acceleration $mc^2 \gg neV_e$ and $L_n \approx (2eV_e n/m)^{1/2}/2\nu$. In the ultrarelativistic limit, $T_n \gg mc^2$, $v \approx c$, and $L_n \approx c/2\nu$.

The total length of the accelerator is given by

$$L = \sum_n L_n \approx \frac{c}{2\nu}\int_0^N \left[1-\left(\frac{mc^2}{nV_e e+mc^2}\right)^2\right]^{1/2} \mathrm{d}n$$

$$\approx \frac{c}{2\nu e V_e}\left\{[(NeV_e+mc^2)^2-m^2c^4]^{1/2}-mc^2\arccos\left(\frac{mc^2}{NeV_e+mc^2}\right)\right\}.$$

11.10 The ratio of the intensities is given by

$$\frac{I_h}{I_0} = \exp\left(\frac{h}{v\tau}\right) \approx \exp\left(\frac{h}{\tau_0 c}\frac{m_\mu c^2}{\mathcal{E}}\right) = 2\cdot 5,$$

where $\tau = \tau_0[1-(v^2/c^2)]^{-1/2}$ is the life-time of a μ-meson moving with a velocity v. In the absence of time dilatation, the intensity ratio is given by (assuming $v = c$)

$$\frac{I_h'}{I_0'} \approx \exp\left(\frac{h}{\tau_0 c}\right) \approx 94\cdot 4.$$

The first result ($I_h/I_0 \approx 2\cdot 5$) is in agreement with observation and therefore constitutes a direct experimental proof of the existence of a relativistic slowing down of clocks.

11.11

$$\tan\vartheta = \frac{1}{\gamma}\frac{p'\sin\vartheta'}{p'\cos\vartheta'+V\mathcal{E}'/c^2} = \frac{1}{\gamma}\frac{\sin\vartheta'}{\cos\vartheta'+V/v'},$$

where

$$\gamma = \frac{1}{(1-V^2/c^2)^{1/2}}, \qquad \mathcal{E} = \gamma(\mathcal{E}'+p'V\cos\vartheta'),$$

and p, p' are the momenta of the particle in S and S' respectively.

The approximate formula given in the question may be used provided $\cos\frac{1}{2}\vartheta' \gg |1-V/v'|^{1/2}$, where $v' = p'c^2/\mathcal{E}'$ is the velocity of the particle in S'. The energy in the ultrarelativistic limit is $\mathcal{E} \approx pc \approx 2\gamma\mathcal{E}'\cos^2\frac{1}{2}\vartheta'$.

11.12 Consider the $\mathrm{d}N$ particles moving within the solid angle $\mathrm{d}\Omega'$ in S'. In S these $\mathrm{d}N$ particles will move within a solid angle $\mathrm{d}\Omega = \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\phi$, generated by the velocity vectors of these particles in S. The angular distribution in S, which is described by the function $F(\vartheta, \phi)$, is then given by

$$F(\vartheta, \phi)\,\mathrm{d}\Omega = F'(\vartheta', \phi')\,\mathrm{d}\Omega' = \mathrm{d}W = \frac{\mathrm{d}N}{N} .$$

The angle ϑ' expressed as a function of ϑ is given by the formula

$$\cos^2\vartheta = \frac{1}{1+\tan^2\vartheta} = \frac{(\cos\vartheta' + V/v')^2}{(\cos\vartheta' + V/v')^2 + (1/\gamma^2)\sin^2\vartheta'} ,$$

which follows from the solution of problem 11.11 ($v' = p'c^2/\mathcal{E}'$ is the velocity of the particles in S'). Since $\phi = \phi'$, we have

$$F(\vartheta, \phi) = F'[\vartheta'(\vartheta), \phi]\frac{\gamma^2[(\cos\vartheta' + V/v')^2 + (1/\gamma^2)\sin^2\vartheta']^{3/2}}{1 + (V/v')\cos\vartheta'} .$$

For ultrarelativistic particles $v' = c$, and the angular distribution in S assumes the simpler form (cf problem 10.30)

$$F(\vartheta, \phi) = F'[\vartheta'(\vartheta), \phi]\frac{[1 + (V/c)\cos\vartheta']^2}{1 - V^2/c^2} .$$

We note that particles moving at different angles ϑ in S have different energies, in spite of the fact that in S' the energies are the same.

11.13 The distribution function f is an invariant quantity. This means that when we change to another frame of reference S':

$$f'(\boldsymbol{r}', \boldsymbol{p}', t') = f(\boldsymbol{r}, \boldsymbol{p}, t) ,$$

where on the right-hand side of the equation we must use equation (10.a.4) to express $\boldsymbol{r}$, $\boldsymbol{p}$, and t in terms of the primed quantities.

11.14 Let n_1 and n_2 be the numbers of scattered and scattering particles per unit volume, and consider the scattering process in the system S. The total number of particles $\mathrm{d}N$ scattered into a solid angle $\mathrm{d}\Omega$ in a time t by target particles lying within a volume V is given by $\mathrm{d}N = \mathrm{d}\sigma_{12}J_{12}n_2Vt$, where σ_{12} is the cross section and $J_{12} = n_1v_1$. In the S' system, the corresponding expression is $\mathrm{d}N = \mathrm{d}\sigma'_{12}J'_{12}n'_2V't'$, where $J'_{12} = n'_1|\boldsymbol{v}'_1 - \boldsymbol{v}'_2|$. In the latter system $\mathrm{d}N$ represents the number of particles scattered into the solid angle $\mathrm{d}\Omega'$ which corresponds to $\mathrm{d}\Omega$. Thus,

$$\mathrm{d}\sigma_{12}n_1n_2v_1Vt = \mathrm{d}\sigma'_{12}n'_1n'_2|\boldsymbol{v}'_1 - \boldsymbol{v}'_2|V't' .$$

Like the four-dimensional electrical current density $(\rho c, \rho\boldsymbol{v})$ the quantity $(nc, n\boldsymbol{v})$ is a 4-vector. It follows that

$$n_1n_2 = n'_1n'_2\left[1 - \frac{(\boldsymbol{v}'_1 \cdot \boldsymbol{v}'_2)}{c^2}\right] , \qquad (11.14.1)$$

since the scalar product of 4-vectors is an invariant quantity. In view of equation (11.14.1) and the fact that the 4 volume is an invariant quantity, we have $Vt = V't'$, and hence finally

$$d\sigma'_{12} = d\sigma_{12}\frac{v_1[1-(\boldsymbol{v}'_1\cdot\boldsymbol{v}'_2)/c^2]}{|\boldsymbol{v}'_1-\boldsymbol{v}'_2|} \,. \tag{11.14.2}$$

In the special case, where $\boldsymbol{v}'_1$ is parallel to $\boldsymbol{v}'_2$,

$$\boldsymbol{v}_1 = \frac{\boldsymbol{v}'_1-\boldsymbol{v}'_2}{1-(\boldsymbol{v}'_1\cdot\boldsymbol{v}'_2)/c^2}$$

(see problem 10.12), and it follows from equation (11.14.2) that the cross section is an invariant quantity, i.e.

$$d\sigma_{12} = d\sigma'_{12}\,.$$

This case occurs, for example, in the transformation from the laboratory system to the centre of mass system. We note that if the flux is defined by $J_{12} = n_1\widetilde{v}$ where $\widetilde{v} = v_1[1-(\boldsymbol{v}'_1\cdot\boldsymbol{v}'_2)/c^2]$, then the cross section is invariant under any Lorentz transformation.

11.15

$$dW = \frac{d\Omega}{4\pi\gamma^2(1-\beta\cos\vartheta)^2}\,, \qquad \int_{4\pi} dW = 1\,,$$

where $\beta = v/c$.

11.16

$$f = \frac{1+\beta}{1-\beta}\,,$$

and hence

$$\mathcal{E} = mc^2\frac{(1+f)}{2f^{1/2}}\,,$$

where m is the rest mass of the π^0-meson.

11.18 Since the momentum of the photon is $p = \mathcal{E}/c$, it follows that (cf problem 11.11)

$$\mathcal{E} = \frac{\mathcal{E}'}{\gamma(1-\beta\cos\vartheta)}\,, \qquad \mathcal{E}' = \tfrac{1}{2}mc^2\,, \qquad \beta = \frac{v}{c}\,.$$

From a comparison of the resulting expression

$$d\mathcal{E} = -\frac{\mathcal{E}'d(1-\beta\cos\vartheta)}{\gamma(1-\beta\cos\vartheta)^2}$$

with the angular distribution of γ-rays found in the answer to problem 11.11, the following probability distribution is found for the energies of photons produced in the disintegration

$$dW(\mathcal{E}) = \frac{|d\mathcal{E}|}{\mathcal{E}_{max}-\mathcal{E}_{min}}\,,$$

where $\mathscr{E}_{min} = \mathscr{E}'(1-\beta)^{1/2}/(1+\beta)^{1/2}$ is the minimum energy of the γ-rays (at $\vartheta = \pi$) and $\mathscr{E}_{max} = \mathscr{E}'(1+\beta)^{1/2}/(1-\beta)^{1/2}$ is the maximum energy (at $\vartheta = 0$). Hence, it is evident that the spectrum of the γ-rays produced as a result of the disintegration is rectangular in the laboratory system, i.e. all energies between $\mathscr{E}_{min}$ and $\mathscr{E}_{max}$ are equally probable.

11.19

$$m = \frac{2(\mathscr{E}_1\mathscr{E}_2)^{1/2}}{c^2} .$$

11.20 $m^2 = m_1^2+m_2^2+2\{[(p_1^2+m_1^2c^2)(p_2^2+m_2^2c^2)]^{1/2}-p_1p_2\cos\vartheta\}$;
$c = 1$; $\quad m_\pi = 139\cdot58$ MeV.

11.21 $m_1^2 = m^2+m_2^2-2\{[(p^2+m^2c^2)(p_2^2+m_2^2c^2)]^{1/2}-pp_2\cos\vartheta_2\}$,
$c = 1$.

11.22

$$m^2 = \mathscr{E}^2-c^2p^2 = m_1^2+m_2^2+\frac{2m_1m_2}{(1-v^2/c^2)^{1/2}} ,$$

$$V = \frac{p}{\mathscr{E}} = \frac{m_1v}{m_1+m_2(1-v^2/c^2)^{1/2}} , \qquad c = 1 .$$

11.23

$$p_\mu = \frac{(m_\pi^2-m_\mu^2)c}{2m_\pi} , \qquad d = \tfrac{1}{2}c\tau\left(\frac{m_\pi}{m_\mu}-\frac{m_\mu}{m_\pi}\right) .$$

11.24

$$T_1 = \mathscr{E}_1-m_1c^2 = (m_0-m_1-m_2)(m_0-m_1+m_2)\frac{c^2}{2m_0} ,$$

$$T_2 = \mathscr{E}_2-m_2c^2 = (m_0-m_1-m_2)(m_0+m_1-m_2)\frac{c^2}{2m_0} ;$$

(a) $\dfrac{T_\alpha}{T_n} = 58\cdot5$; (b) $\dfrac{T_\nu}{T_\mu} = 7\cdot27$; (c) $\dfrac{T_\gamma}{T_n} = \dfrac{2mc^2}{\Delta\mathscr{E}}$,

where m is the rest mass of the disintegrating nucleus, $\Delta\mathscr{E}$ is the energy of its excitation, while $mc^2 \gg \Delta\mathscr{E}$.

From the general formulae for T_1 and T_2, and also from the examples given here, it is clear that the lighter particle carries away most of the energy.

11.25

$$m_1 = m\frac{(1-v_1^2/c^2)^{1/2}}{1+v_1/v_2} . \tag{11.25.1}$$

Using equation (11.25.1) we find the following differential equation for the mass of the rocket:

$$m\frac{dv}{dm}+w\left(1-\frac{v^2}{c^2}\right) = 0 ,$$

whence

$$\frac{v}{c} = \frac{1-(m/m_0)^{2w/c}}{1+(m/m_0)^{2w/c}} .$$

11.26 The equation of motion is now

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma v = f .$$

We find for the mass:

$$m = m_0 \exp\left(-\frac{\chi}{k}\right) , \qquad \sinh\chi = \frac{ft}{c} .$$

11.27

$$Q_a = T_b\left[1+\frac{T_b+2m_b}{m_a+(T_b^2+2T_b m_b+m_a^2)^{1/2}}\right] ;$$

$$Q_{\Sigma^+} = 109{\cdot}6\ \mathrm{MeV} ; \qquad M_{\Sigma^+} = 1188{\cdot}7\ \mathrm{MeV} \qquad (\Sigma^+ \to \mathrm{n}+\pi^+) ;$$

$$Q_{\Sigma^+} = 116{\cdot}1\ \mathrm{MeV} ; \qquad M_{\Sigma^+} = 1189{\cdot}3\ \mathrm{MeV} \qquad (\Sigma^+ \to \mathrm{p}+\pi^0) .$$

The two values for M_{Σ^+} agree well with one another.

11.28

$$\omega = \frac{\Delta\mathcal{E}}{\hbar}\left(1-\frac{\Delta\mathcal{E}}{2mc^2}\right) .$$

The energy, $\hbar\omega$, carried away by the quantum is less than $\Delta\mathcal{E}$ by an amount $(\Delta\mathcal{E})^2/(2mc^2)$, which is carried away by the nucleus in its recoil. If the nucleus is rigidly bound to the crystalline lattice, it does not receive any energy (since its mass $M \gg m$ is very large) and the quantum carries away the whole of the energy $\hbar\omega = \Delta\mathcal{E}$.

11.29 (a) The energy conservation law determines the bounds of the equilateral triangle ABC (figure 11.29.1a), the height of which, BO, is equal to the decay energy $Q = m-m_1-m_2-m_3$ $(c = 1)$. The distance of the point D from the base AC equals T_1 by construction and the distances from D to AB and BC can easily be calculated and turn out to be equal to T_2 and T_3, respectively.

(b) The magnitudes of the momenta, when the masses of all particles are given, are determined by giving two of the energies, e.g. T_1 and T_2 (as $T_3 = Q-T_1-T_2$), or x and y, which are linear combinations of them. The momenta of the particles which are produced in the decay are the sides of a triangle ($\boldsymbol{p}_1+\boldsymbol{p}_2+\boldsymbol{p}_3 = 0$ in the rest frame of the decaying particle). The angles of the triangle characterise the relative directions in which the particles fly off and can be found once we know p_1, p_2, and p_3.

(c) The limits of the allowed region are determined by the conditions

$$p_1+p_2 \geqslant p_3\,, \qquad -p_3 \leqslant p_1-p_2 \leqslant p_3\,.$$

These conditions lead to the shaded region in figure 11.29.1b. At the top the region is bounded by the line $y = (m-m_1)^2/2m$, and below by the hyperbolae $x = \pm[\frac{1}{3}(y^2+2m_1y)]^{1/2}$.

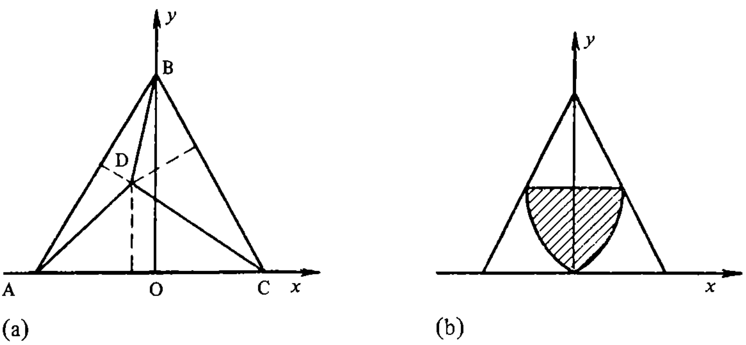


Figure 11.29.1.

11.30 The Dalitz plot has the form shown in figure 11.29.1b.

(a) $T_{1\,\max} \approx T_{2\,\max} \approx T_{3\,\max} \approx 69{\cdot}8$ MeV .

(b) $T_{1\,\max} = \dfrac{(m-m_1)^2}{2m} \approx 127$ MeV ,

$$T_{2\,\max} = T_{3\,\max} = \frac{(m^2-m_1^2)}{2m} \approx 228 \text{ MeV}\,.$$

The maximum momenta of all three particles are the same.

11.31 The Dalitz plot is given in figure 11.31.1 under the assumption that $Q \ll m_\pi$.

$$\text{OB} = Q\,, \qquad R = \tfrac{1}{3}Q\,, \qquad T_{\max} = \tfrac{2}{3}Q \approx 50 \text{ MeV}\,.$$

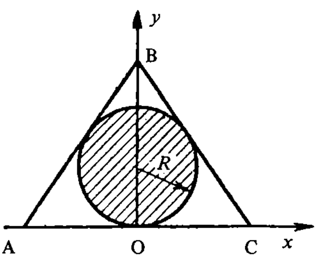


Figure 11.31.1.

11.32 The Dalitz plot is given in figure 11.32.1,

$$\mathrm{OB} = Q\,, \qquad T_{\max} \approx 210\ \mathrm{MeV}\,.$$

The inner curve is given by the equation

$$x = \pm\left\{\frac{(2m_\pi y + y^2)[(m_\omega - m_\pi)^2 - 4m_\pi^2 - 2m_\omega y]}{3[(m_\omega - m_\pi)^2 - 2m_\omega y]}\right\}^{1/2}.$$

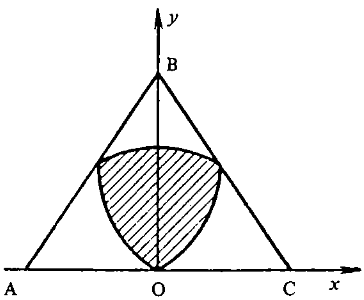


Figure 11.32.1.

11.33 The δ-function of the 4-vector must be understood to be the product of four δ-functions of its components:

$$\delta(p_i - p_{i1} - p_{i2} - p_{i3}) = \delta(\boldsymbol{p} - \boldsymbol{p}_1 - \boldsymbol{p}_2 - \boldsymbol{p}_3)\delta(\mathcal{E} - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3)\,. \quad (11.33.1)$$

By integrating over $\mathrm{d}^3\boldsymbol{p}_3$ and using equation (11.33.1) we get the relation

$$\Gamma = \int \frac{\mathrm{d}^3\boldsymbol{p}_1\,\mathrm{d}^3\boldsymbol{p}_2}{\mathcal{E}_1\mathcal{E}_2\mathcal{E}_3}\delta([p_1^2 + p_2^2 + m_3^2 + 2p_1p_2\cos\vartheta]^{1/2} - \mathcal{E}_3)\,, \quad (11.33.2)$$

where $\mathcal{E}_3 = m - \mathcal{E}_1 - \mathcal{E}_2$ and ϑ is the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$.

We write $\mathrm{d}^3\boldsymbol{p}_2$ in the form $\mathrm{d}^3\boldsymbol{p}_2 = p_2^2\,\mathrm{d}p_2\,\mathrm{d}\Omega_2$, where $\mathrm{d}\Omega_2$ is an element of solid angle. If we take the direction of $\boldsymbol{p}_1$ as polar axis we have $\mathrm{d}\Omega_2 = 2\pi\sin\vartheta\,\mathrm{d}\vartheta$. Moreover, it follows from equation (11.a.3) that $p_2\,\mathrm{d}p_2 = \mathcal{E}_2\,\mathrm{d}\mathcal{E}_2$. Using equation (A1.18) we can transform the δ-function in equation (11.33.2):

$$\delta([p_1^2 + p_2^2 + m_3^2 + 2p_1p_2\cos\vartheta]^{1/2} - \mathcal{E}_3)$$
$$= 2\mathcal{E}_3\delta(2p_1p_2\cos\vartheta + p_1^2 + p_2^2 + m_3^2 - \mathcal{E}_3^2)\,. \quad (11.33.3)$$

Since $-1 \leqslant \cos\vartheta \leqslant 1$, the integral in equation (11.33.2) will be nonvanishing only if the following inequalities hold:

$$p_1 + p_2 \geqslant p_3\,, \qquad p_1 - p_2 \leqslant p_3\,, \qquad p_1 - p_2 \geqslant -p_3\,;$$

these are just the inequalities which determine the boundaries of the allowed region in the Dalitz plot.

Using equations (11.33.3) and (A1.15) and integrating over ϑ we get

$$\Gamma = \pi\int\frac{d^3\boldsymbol{p}_1\, d\mathscr{E}_2}{\mathscr{E}_1 p_1} = 4\pi^2\int d\mathscr{E}_1\, d\mathscr{E}_2\,.$$

We now integrate over the variables

$$x = \frac{T_2 - T_3}{\sqrt{3}} = \frac{\mathscr{E}_1 + 2\mathscr{E}_2 + m_3 - m_2 - m}{\sqrt{3}}\,, \qquad y = T_1 = \mathscr{E}_1 - m_1\,,$$

which were used to construct the Dalitz plots. If we transform the element $d\mathscr{E}_1\, d\mathscr{E}_2$ we find

$$\Gamma = 2\pi^2\sqrt{3}\int dx\, dy\,,$$

where the domain of integration is bounded by the inner curves in the plot (see figures 11.29.1b to 11.32.1).

The last formula shows that the phase element $d\Gamma = 2\pi^2\sqrt{3}\, dx\, dy$ is proportional to the element of area in the Dalitz plot. The energies T_1, T_2, and T_3 of the particles that are formed in the decay can be measured experimentally and plotted in the Dalitz plot. The density of points is then proportional to the quantity ρ (see the statement of this problem) which can thus be found from experimental data.

11.34 Consider the energy-momentum 4-vector p_i of the system of particles. It is conserved, i.e. the corresponding components of the energy-momentum before and after the reaction are equal. At the threshold kinetic energy, T_0, the particles produced in the reaction are at rest in the centre of mass frame (we note that in the laboratory frame the particles cannot be at rest for the threshold energy T_0 since this would violate the law of conservation of momentum). The total 4-momentum vector of the system before the reaction is (in the laboratory system)

$$p_i^{(0)} = \left(\frac{\mathscr{E}_0}{c} + m_1 c, \boldsymbol{p}_0\right)\,,$$

where $\mathscr{E}_0$ is the total energy and $\boldsymbol{p}_0$ is the total momentum at the threshold.

After the reaction, the 4-momentum in the centre of mass system is $p_i' = (Mc, 0)$. In view of the invariance of the square of a 4-vector, and the law of conservation of 4-momentum, we have $p_i^{(0)2} = p_i'^2$. Explicitly,

$$M^2c^2 = \frac{\mathscr{E}_0^2}{c^2} + 2m_1\mathscr{E}_0 + m_1^2c^2 - p_0^2\,,$$

and hence,

$$T_0 = \frac{c^2}{2m_1}(M - m_1 - m)(M + m_1 + m)\,.$$

11.35

(a) $T_0 = 288$ MeV, (b) $T_0 = 160$ MeV,

(c) $T_0 = 763$ MeV, (d) $T_0 = 2\frac{m_p}{m}(m + 2m_p)c^2$,

where m_p is the rest mass of the proton. In the special case of a collision with a proton $m = m_p$ and hence $T_0 = 6m_pc^2 = 5{\cdot}63$ GeV. The approximate formula for the threshold energy is

$$T_0 = \frac{2(A+2)}{A}m_pc^2 .$$

For large A this may be replaced by $T_0 \approx 2m_pc^2$.

11.36

$$T_0 = \left(1 + \frac{m}{M}\right)\Delta\mathcal{E} .$$

In case (a) we have $\Delta\mathcal{E} = T_0 = 2{\cdot}18$ MeV $(m = 0)$, in accordance with the above approximate formula. When the exact formula is used (see problem 11.34), the result is larger by $|Q|^2/2Mc^2 \approx 0{\cdot}0012$ MeV, where $Q = -(M - m_1 - m)c^2$. In case (b) the approximate formula yields $T_0 = 2|Q| = 7{\cdot}96$ MeV, and the difference between this result and the exact value is $0{\cdot}003$ MeV.

11.37 The equation for the reaction is

$$\gamma + (\text{particle}) \to e^+ + e^- + (\text{particle}) .$$

The threshold energy may be found from the general formula (see problem 11.34)

$$T_0 = \hbar\omega_0 = \frac{c^2}{2m_1}(m_1 + 2m - m_1)(m_1 + 2m + m_1) = 2mc^2\left(1 + \frac{m}{m_1}\right),$$

where m is the rest mass of the electron (or positron).

When there is no particle, so that $m_1 \to 0$, the threshold energy T_0 tends to infinity, which means that the reaction is impossible. The latter result may also be obtained by showing that the relation $k_i = p_{+i} + p_{-i}$ cannot be satisfied, where k_i, p_{+i}, and p_{-i} are the 4-momenta of the photon, positron, and electron, respectively. Indeed, taking the square of both sides of this equation we have

$$k_i^2 = (\mathcal{E}_+ + \mathcal{E}_-)^2 - (\boldsymbol{p}_+ + \boldsymbol{p}_-)^2 .$$

However, $k_i^2 = 0$, while the invariant quantity on the right-hand side does not vanish for any values of $\boldsymbol{p}_+$, $\boldsymbol{p}_-$. This is clear, if we change to the system of reference in which $\boldsymbol{p}_+ + \boldsymbol{p}_- = 0$.

11.39

$$v = \frac{c(\mathcal{E}^2 - m_1^2c^4)^{1/2}}{\mathcal{E} + m_2c^2} .$$

11.40 According to the conservation of 4-momentum

$$p_{1i}^{(0)}+p_{2i}^{(0)} = p_{1i}+p_{2i}\,.$$

To determine the angle of scattering of the first particle, take p_{1i} to the left-hand side of the equation, and square both sides, so that

$$p_{1i}^{(0)2}+p_{2i}^{(0)2}+p_{1i}^2+2p_{1i}^{(0)}p_{2i}^{(0)}-2p_{1i}^{(0)}p_{1i}-2p_{2i}^{(0)}p_{1i} = p_{2i}^2\,. \tag{11.40.1}$$

In view of equation (11.a.7)

$$p_{1i}^{(0)2} = p_{1i}^2 = m_1^2c^2\,,\quad p_{2i}^{(0)2} = p_{2i}^2 = m_2^2c^2\,.$$

The scalar products are transformed as follows ($\boldsymbol{p}_2^{(0)} = 0$):

$$-p_{1i}^{(0)}p_{2i}^{(0)} = (\boldsymbol{p}_1^{(0)}\cdot\boldsymbol{p}_2^{(0)})-\frac{1}{c^2}\mathcal{E}_1^{(0)}\mathcal{E}_2^{(0)} = -\mathcal{E}_0m_2\,,\quad p_{2i}^{(0)}p_{1i} = m_2\mathcal{E}_1\,,$$

$$-p_{1i}^{(0)}p_{1i} = (\boldsymbol{p}_1^{(0)}\cdot\boldsymbol{p}_1)-\frac{1}{c^2}\mathcal{E}_1^{(0)}\mathcal{E}_1 = p_0p_1\cos\vartheta_1-\frac{\mathcal{E}_0\mathcal{E}_1}{c^2}\,,$$

where $p_0 = (\mathcal{E}_0^2-m_1^2c^4)^{1/2}/c^2$. Substituting these expressions into equation (11.40.1), it is found that

$$\cos\vartheta_1 = \frac{\mathcal{E}_1(\mathcal{E}_0+m_2c^2)-\mathcal{E}_0m_2c^2-m_1^2c^4}{c^2p_0p_1}\,.$$

Similarly,

$$\cos\vartheta_2 = \frac{(\mathcal{E}_0+m_2c^2)(\mathcal{E}_2-m_2c^2)}{c^2p_0p_2}\,.$$

11.41

$$\mathcal{E}_1 = m_1c^2\frac{(\gamma_0+m_2/m_1)(1+\gamma_0m_2/m_1)\pm\cos\vartheta_1(\gamma_0^2-1)(m_2^2/m_1^2-\sin^2\vartheta_1)^{1/2}}{(\gamma_0+m_2/m_1)^2-(\gamma_0^2-1)\cos^2\vartheta_1} \tag{11.41.1}$$

$$\mathcal{E}_2 = \frac{(\gamma_0+m_2/m_1)^2+(\gamma_0^2-1)\cos^2\vartheta_2}{(\gamma_0+m_2/m_1)^2-(\gamma_0^2-1)\cos^2\vartheta_2}m_2c^2\,,$$

where $\gamma_0 = \mathcal{E}_0/m_1c^2$.

It is evident from these formulae that when $m_1 > m_2$ the scattering angle must be such that $\vartheta_1 < \arcsin(m_2/m_1)^{1/2}$ and the positive sign should be taken in equation (11.41.1). To each value of ϑ_1 there are two values of $\mathcal{E}_1$.

When $m_1 = m_2$, the scattering angle ϑ_1 will not exceed $\frac{1}{2}\pi$, and to each value of ϑ_1 there is only one value of the energy which corresponds to the positive sign in equation (11.41.1). The negative sign would correspond to $\mathcal{E}_1 = m_1c^2$ whatever the scattering angle, and this is unphysical.

When $m_1 < m_2$ all scattering angles are possible, and to each value of ϑ_1 there is one value of $\mathcal{E}_1$. The positive sign must be taken in equation (11.41.1) when $0 < \vartheta_1 < \frac{1}{2}\pi$, and the negative sign when $\frac{1}{2}\pi < \vartheta_1 < \pi$. With this choice of signs, the scattering of incident particles through large angles will correspond to large energy losses, as expected.

11.42

$$\mathscr{E} \approx \frac{\mathscr{E}_0}{1+(\mathscr{E}_0/Mc^2)(1-\cos\vartheta)} .$$

11.43

$$\mathscr{E} \approx \frac{\mathscr{E}_0-\Delta E}{1+(\mathscr{E}_0/Mc^2)(1-\cos\vartheta)} .$$

11.44

$$T_1 = \frac{T_0\cos^2\vartheta_1}{1+\frac{1}{2}(T_0/mc^2)\sin^2\vartheta_1} .$$

11.45

$$T_1 = T_0\left(\frac{m_1}{m_1+m_2}\right)^2\left\{1+\left(\frac{m_2}{m_1}\right)^2-2\sin^2\vartheta_1 \pm 2\cos\vartheta_1\left[\left(\frac{m_2}{m_1}\right)^2-\sin^2\vartheta_1\right]^{1/2}\right\};$$

$$T_2 = \frac{4m_1m_2}{(m_1+m_2)^2}T_0\cos^2\vartheta_2 .$$

The signs should be chosen as in the solution to problem 11.41.

11.46 The angle $\chi = \vartheta_1+\vartheta_2$ between the directions of motion of the particles is given by

$$\tan\chi = \frac{(v_1'+v_2')(1-V^2/c^2)^{1/2}\sin\vartheta'}{(V^2/c^2)v_1'\sin^2\vartheta'+(V-v_1')(1-\cos\vartheta')}$$

(cf problem 10.26). When $m_1 = m_2$, we have $v_1' = v_2' = V$ and

$$\tan\chi = \frac{2c^2(1-V^2/c^2)^{1/2}}{V^2\sin\vartheta'} .$$

In this case, $\chi < 90°$. In the nonrelativistic limit $\chi \to 90°$.

11.47 Proceeding as in the solution of problem 11.40, we find that

$$\omega = \frac{\omega_0(\mathscr{E}_0/c-p_0\cos\vartheta_0)}{\mathscr{E}_0/c-p_0\cos\vartheta_1+(\hbar\omega_0/c)(1-\cos\vartheta)} ,$$

where ϑ is the angle between the directions of motion of the primary and secondary photons, and ϑ_1 is the angle between the initial direction of motion of the electron and the direction of motion of the photon after scattering. If the electron is initially at rest, then

$$\omega = \frac{\omega_0}{1+(\hbar\omega_0/mc^2)(1-\cos\vartheta)} .$$

11.48 The energy of the scattered quantum is a maximum when $\vartheta_0 = \vartheta = \pi$, $\vartheta_1 = 0$, i.e. for a head-on collision in which the quantum is scattered backwards. In that case

$$\hbar\omega \approx \hbar\omega_0\frac{2\mathscr{E}_0}{(mc^2)^2/2\mathscr{E}_0+2\hbar\omega_0} . \qquad (11.48.1)$$

It is clear from equation (11.48.1) that the quantum is considerably 'hardened' in the ultrarelativistic case, $\hbar\omega \gg \hbar\omega_0$. We note two particular cases. When $\hbar\omega_0 \ll (mc^2)^2/\mathcal{E}_0$, equation (11.48.1) gives:

$$\mathcal{E}_0 \gg \hbar\omega \approx 4\hbar\omega_0\left(\frac{\mathcal{E}_0}{mc^2}\right)^2 \gg \hbar\omega_0.$$

If, on the other hand, $\hbar\omega_0 \gg (mc^2)^2/\mathcal{E}_0$, $\hbar\omega$ approaches $\mathcal{E}_0$.

11.49

$$\mathcal{E}-\mathcal{E}_0 = \hbar\omega_0\frac{p_0c(\cos\vartheta_0-\cos\vartheta_1)+\hbar\omega_0(1-\cos\vartheta)}{\mathcal{E}_0-p_0\cos\vartheta_1+\hbar\omega_0(1-\cos\vartheta)}\ .$$

The designation of the angles is the same as in problem 11.47. An electron which is initially at rest always increases its energy when it collides with a photon:

$$\mathcal{E}-mc^2 = \frac{(\hbar\omega_0)^2(1-\cos\vartheta)}{mc^2+\hbar\omega_0(1-\cos\vartheta)}\ .$$

If before the scattering the electron had a momentum $p_0 \gg \hbar\omega/c$, its energy increases in the scattering, when $\vartheta_0 < \vartheta_1$, and decreases in the opposite case. The maximum acceleration of the electron occurs when $\vartheta_0 = 0$, $\vartheta = \vartheta_2 = \pi$. In that case

$$\mathcal{E}-\mathcal{E}_0 = 2\hbar\omega_0\frac{p_0c+\hbar\omega_0}{\mathcal{E}_0+p_0c+2\hbar\omega_0}\ .$$

In the case where the electron is nonrelativistic, but $p_0c \gg \hbar\omega$, we have $\mathcal{E}-\mathcal{E}_0 = 2\hbar\omega_0(v_0/c) \ll \hbar\omega_0$. If, instead, the electron is ultrarelativistic, $\mathcal{E}-\mathcal{E}_0 \approx \hbar\omega_0$ and the conditions for accelerating the electron are an optimum.

11.50 $s = 4(m^2+q^2)\,, \qquad t = -2q^2(1-\cos\vartheta)\,, \qquad u = -2q^2(1+\cos\vartheta)\,.$

11.51

$$\mathcal{E}_a = \frac{(s-m_a^2-m_b^2)}{2m_b}\,, \qquad p_a = \frac{[\lambda(s, m_a^2, m_b^2)]^{1/2}}{2m_b}\,;$$

$$\mathcal{E}_a' = \frac{(s+m_a^2-m_b^2)}{2\sqrt{s}}\,, \qquad p' = \frac{[\lambda(s, m_a^2, m_b^2)]^{1/2}}{2\sqrt{s}}\,, \qquad \mathcal{E}_b' = \frac{(s-m_a^2+m_b^2)}{2\sqrt{s}}\,,$$

where

$$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz\ .$$

Since in the centre-of-mass frame $p_a = -p_b$, the meaning of the quantity s is that it is the square of the total energy in that frame of reference:

$$s = (\mathcal{E}_a'+\mathcal{E}_b')^2 = (\mathcal{E}_c'+\mathcal{E}_d')^2\ .$$

11.52

$$\mathcal{E}_c = \frac{(m_b^2 + m_c^2 - u)}{2m_b}, \qquad \mathcal{E}_d = \frac{(m_b^2 + m_d^2 - t)}{2m_b}; \qquad c = 1 .$$

11.53

$$\cos\theta = \frac{(s - m_a^2 - m_b^2)(m_b^2 + m_c^2 - u) + 2m_b^2(t - m_a^2 - m_c^2)}{[\lambda(s, m_a^2, m_b^2)\lambda(u, m_b^2, m_c^2)]^{1/2}}$$

$$\cos\theta' = \frac{s^2 + s(2t - m_a^2 - m_b^2 - m_c^2 - m_d^2) + (m_a^2 - m_b^2)(m_c^2 - m_d^2)}{[\lambda(s, m_a^2, m_b^2)\lambda(s, m_c^2, m_d^2)]^{1/2}} .$$

Here $c = 1$ and the quantity λ was defined in the solution to problem 11.51.

11.54 The meaning of the quantity $s = (\mathcal{E}'_\pi + \mathcal{E}'_p)^2$ is that it is the square of the total energy of the two particles in the centre-of-mass frame; it is therefore always positive. Its minimum value $s_{\min} = (m+M)^2$ corresponds to the case where the pion, of mass m, and the proton, of mass M, are at rest in the centre-of-mass frame. Therefore,

$$(m+M)^2 \leqslant s \leqslant \infty.$$

The cosine of the scattering angle θ' in the centre-of-mass system is connected with s and t through the relation

$$\cos\theta' = \frac{s^2 + s(2t - 2M^2 - m^2) + M^2(M^2 - m^2)}{(s - M^2)[s^2 - 2s(M^2 + m^2) + (M^2 - m^2)^2]^{1/2}} . \qquad (11.54.1)$$

Since $-1 \leqslant \cos\theta' \leqslant 1$, we find, by substituting equation (11.54.1) for $\cos\theta'$ into these inequalities, the values of t which are allowed for a given value of s.

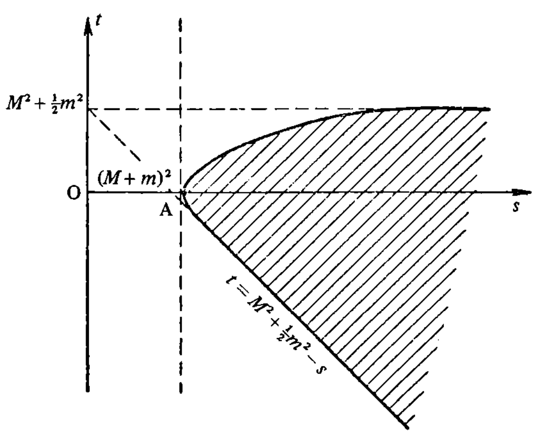


Figure 11.54.1.

The physical region is shaded in figure 11.54.1. The threshold for the reaction corresponds to the point A where

$$s_A = (M+m)^2, \qquad t_A = -m^2M/(M+m).$$

$$T_0 = m+\frac{m^2}{2M}, \qquad T_\pi = \frac{m^3}{2M(M+m)}.$$

11.55 The required regions are shown in figure 11.55.1.

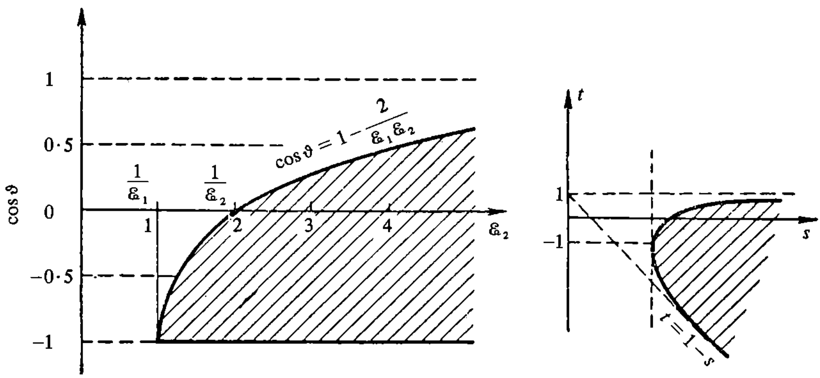
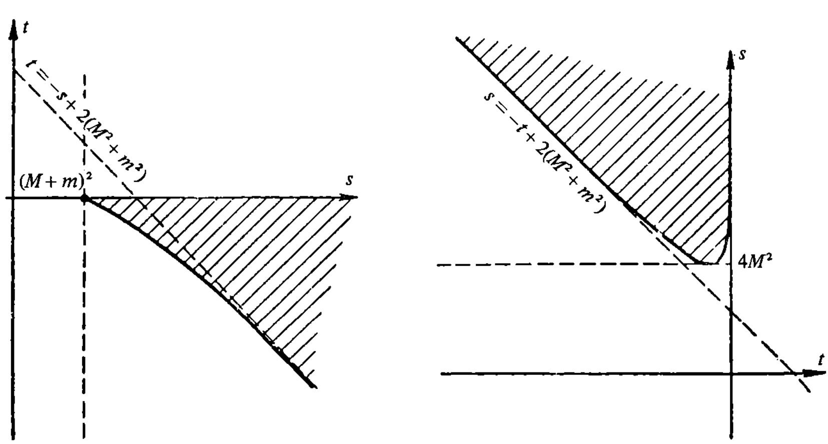


Figure 11.55.1.

11.56 The allowed regions for the first two processes are shown in figure 11.56.1a and for the third process in figure 11.56.1b.

One can construct a single kinematic diagram for all three processes by considering them as three possible channels for a single reaction in which two nucleons and two pions participate. The initial and final states of the



(a) (b)

Figure 11.56.1.

pions and nucleons in the channels considered differ in energies, momenta, and charges (and also a few other characteristics which are studied in quantum theory).

In order to construct this diagram (figure 11.56.2) we draw three lines on which $s = 0$, $t = 0$, and $u = 0$, in such a way that they intersect and form an equilateral triangle with height $h = s+t+u = m_a^2+m_b^2+m_c^2+m_d^2$ ($c = 1$). The values $s = s_0 =$ constant correspond to a straight line, parallel to the $s = 0$ axis and at a distance $|s_0|$ from it. This line must be drawn at the side where the triangle is, if $s_0 > 0$ and at the other side, if $s_0 < 0$. The lines $t =$ constant and $u =$ constant are constructed in a similar way.

As a result we have constructed an oblique-angled coordinate system in the plane, such that each point in the plane corresponds to three (positive or negative) numbers s, t, and u. The sum of these three numbers satisfies the necessary relation [equation (11.a.14)]. To verify this we consider an arbitrary point D and draw from it perpendiculars onto the sides AB, BC, and AC, or their extensions. Since

area of ABC = area of ABD − (area of BCD + area of ACD) ,

we have

$$\mathrm{DM} - \mathrm{DN} - \mathrm{DK} = h = m_a^2+m_b^2+m_c^2+m_d^2 .$$

However, $-\mathrm{DN} = s$, $-\mathrm{DK} = t$, $\mathrm{DM} = u$, whence follows equation (11.a.14).

It is convenient for us to change the definitions of s, t, and u somewhat as compared to equation (11.a.13). Let

$$s = (p_{ai}+p_{bi})^2 , \qquad t = (p_{ai}+p_{ci})^2 , \qquad u = (p_{ai}+p_{di})^2 ,$$

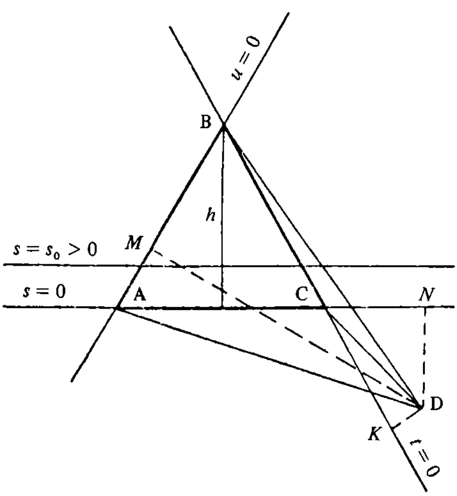


Figure 11.56.2.

where we put $p_i = (-\mathscr{E}, -\boldsymbol{p})$ for particles which disappear as a result of the reaction, and $p_i = (\mathscr{E}, \boldsymbol{p})$ for particles produced in the reaction. This rule for the signs corresponds to the fact that $\sum_a p_{ai} = 0$, as in the case of a decay. We indicate the pions by the indices a and b and the nucleons by c and d. For the channel (c) we then have

$$p_{ai} = (-\mathscr{E}_a, -\boldsymbol{p}_a), \qquad p_{bi} = (-\mathscr{E}_b, -\boldsymbol{p}_b), \qquad p_{ci} = (\mathscr{E}_c, \boldsymbol{p}_c),$$
$$p_{di} = (\mathscr{E}_d, \boldsymbol{p}_d);$$
$$s = (\mathscr{E}_a' + \mathscr{E}_b')^2 = (\mathscr{E}_c' + \mathscr{E}_d')^2 \geqslant 4M^2;$$

the allowable values of t are obtained from the condition $|\cos\theta'| \leqslant 1$.

The boundary of the physical region is given by the equation

$$s = -t - \frac{(M^2 - m^2)^2}{t} + 2(M^2 + m^2) \geqslant 4M^2; \tag{11.56.1}$$

it is a hyperbola with asymptotes $t = 0$ and $u = 0$ (figure 11.56.3).

In the case of channel (a) we get

$$p_{ai} = (-\mathscr{E}_a, -\boldsymbol{p}_a), \qquad p_{ci} = (-\mathscr{E}_c, -\boldsymbol{p}_c), \qquad p_{bi} = (\mathscr{E}_b, \boldsymbol{p}_b),$$
$$p_{di} = (\mathscr{E}_d, \boldsymbol{p}_d).$$

The physical region is bounded by the line $s = 0$ and the hyperbola

$$s = -t - \frac{(M^2 - m^2)^2}{t} + 2(M^2 + m^2), \qquad t \geqslant (M + m)^2,$$

which is the second branch of the hyperbola given by equation (11.56.1).

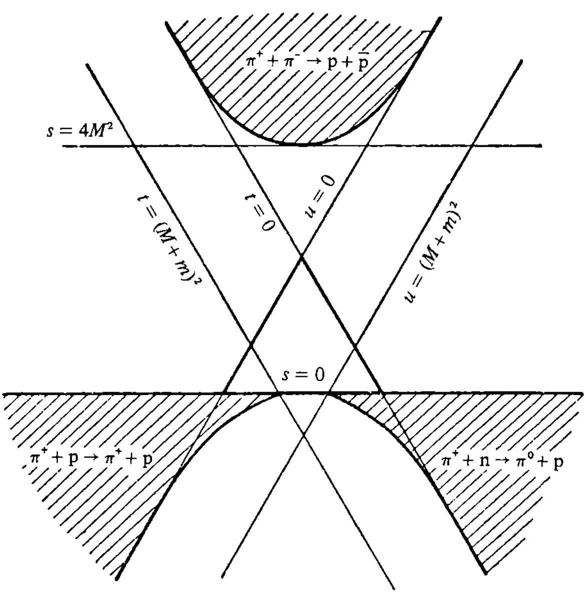


Figure 11.56.3.

One can similarly construct the physical region for the channel (b). From what we have said it is clear that the diagram which we have obtained is very similar to the Dalitz plot for three-particle decay (see problem 11.29).

The similarity is caused by the fact that in both cases four particles participate whose 4-momenta satisfy the condition $p_{ai}+p_{bi}+p_{ci}+p_{di} = 0$, by virtue of the momentum conservation law. Taking into account that for given masses of all particles, we have $m_a^2 = p_{ai}^2$, and so on, and we can therefore construct only two independent invariants from the 4-momenta, for instance, $s = (p_{ai}+p_{bi})^2$ and $t = (p_{ai}+p_{ci})^2$. Thus we need a two-dimensional space (the kinematic plane) to depict such processes.

11.58 When a particle moving in a medium with a 4-momentum p_{0i} emits a photon with a 4-momentum $k_i = (\hbar\omega/c, \hbar\omega n/c)$, then the laws of conservation of energy and momentum may be described by the four-dimensional equation

$$p_{0i} = p_i + k_i \,,$$

where p_i is the 4-momentum of the particle after the emission of the photon. Put k_i on the left-hand side of this equation and square both sides. After simple rearrangement it is found that

$$\cos\vartheta = \frac{1}{n\beta}\left[1+\frac{\pi\Lambda}{n\lambda}(n^2-1)(1-\beta^2)^{1/2}\right] \,, \qquad (11.58.1)$$

where $\Lambda = \hbar/mc$ is the Compton wavelength of the particle, $\lambda = 2\pi c/\omega n$ is the wavelength of the photon, and $\beta = v/c$. The second term, which is of the order of Λ/λ, is usually very small. This term represents quantum corrections (Λ is proportional to $\hbar$). When it is neglected, the relation given by equation (11.58.1) reduces to the classical condition for the emission of Cherenkov radiation, which is

$$\cos\vartheta = \frac{1}{n\beta} \,.$$

11.60 Let p_{0i} and p_i represent the 4-momenta of the particle before and after the emission and k_i the '4-momentum' of the photon. The law of conservation of energy and momentum may then be written in the form

$$p_{0i} - k_i = p_i \,.$$

On squaring both sides of this equation, and neglecting terms containing $\hbar^2$, it is found that

$$(m^2 - m_0^2)c^2 - 2(\mathbf{p}\cdot\mathbf{k}) + \frac{2\mathcal{E}_0 k}{c} = 0 \,,$$

where m_0 is the rest mass of the excited particle and m is the rest mass of the particle in the ground state.

Since $c^2(m_0 - m)(m_0 + m) \approx 2\hbar\omega_0 m$ we have

$$n(\omega)\beta\cos\vartheta = 1 - \frac{\omega_0}{\omega}(1-\beta^2)^{1/2}\,, \tag{11.60.1}$$

where $\beta = v/c$. When $\omega_0 \to 0$, equation (11.60.1) becomes

$$n(\omega)\beta\cos\vartheta = 1\,,$$

which is the condition for the emission of Cherenkov radiation. It follows that this radiation is not associated in any way with a change in the internal state of the particle.

When $\omega_0 \neq 0$, equation (11.60.1) is conveniently rewritten in the form

$$\omega = \frac{\omega_0(1-\beta^2)^{1/2}}{1 - n(\omega)\beta\cos\vartheta}\,. \tag{11.60.2}$$

which describes the Doppler effect in a refracting medium (cf problem 10.43). It may be used when $n(\omega)\beta\cos\vartheta < 1$ and differs from the corresponding formula for the Doppler effect in a vacuum only by the presence of $n(\omega)$ in the denominator. When $\beta \ll 1$ no fundamentally new effects arise, but when $\beta \approx 1$ and there is dispersion in the medium, the phenomenon becomes more complicated.

In general, equation (11.60.2) is a nonlinear equation in ω (n is a function of ω!), and there may be more than one solution. Instead of a single shifted line, as in the usual Doppler effect, there are then several lines in the laboratory frame (composite Doppler effect).

11.61 Proceeding as in the solution of the preceding problem, the following results are obtained.

Emission at a frequency ω, which is accompanied by excitation of the particle, may occur when the velocity of the particle $v = \beta c$ exceeds the threshold value $c/n(\omega)\cos\vartheta$ where ϑ is the angle between the direction of the velocity v and the direction of the momentum of the photon. The energy necessary for this to occur is taken at the expense of the kinetic energy. This type of emission is observed at a fixed value of ω only in a certain range of acute angles ϑ within the Cherenkov cone whose surface is defined by $n\beta\cos\vartheta = 1$. The observed frequency ω is then given by

$$\omega = \frac{\omega_0(1-\beta^2)^{1/2}}{n(\omega)\beta\cos\vartheta - 1}\,, \qquad n(\omega)\beta\cos\vartheta > 1\,,$$

which is an equation in ω as in the preceding problem. In general, this equation has several solutions (composite superrelativistic Doppler effect).

11.62 We denote the angle between the initial electron momentum $\boldsymbol{p}_0$ and the direction of propagation of the soft quantum by ϑ_1 and the angle between $\boldsymbol{p}_0$ and the direction of propagation of the hard quantum by ϑ_2.

From the law of conservation of 4-momentum (cf problem 11.58), and by making the assumption that $\hbar\omega_1 \ll \mathcal{E}_0$, $\hbar\omega_0 \ll \mathcal{E}_0$, we find

$$\cos\vartheta_2 = \frac{c}{v_0 n(\omega_1)} + \frac{\hbar\omega_2}{\hbar\omega_1}\frac{[1-(v_0/c)\cos\vartheta_1]}{(v_0/c)n(\omega_1)} . \tag{11.62.1}$$

From this it is clear that the hard Cherenkov quantum propagates inside the Cherenkov cone corresponding to the soft Cherenkov quantum of frequency ω_1. The opening angle of this cone is, in the accuracy to which we are working, determined by the condition $\cos\vartheta_1 = c/v_0 n(\omega_1)$. As in the case of the usual Cherenkov emission, for the occurrence of hard Cherenkov emission it is necessary that the inequality $v_0 > c/n(\omega_1)$ is satisfied. This is possible only if $n(\omega_1) > 1$. Hence one of the quanta must be sufficiently soft. Solving equation (11.62.1) for $\hbar\omega_2$ we get

$$\hbar\omega_2 = \hbar\omega_1 \frac{[n(\omega_1)v_0/c]\cos\vartheta_1 - 1}{1-(v_0/c)\cos\vartheta_2} .$$

The maximum value of the energy $\hbar\omega_2$ is reached for $\vartheta_1 = \vartheta_2 = 0$:

$$(\hbar\omega_2)_{\max} = \hbar\omega_1 \frac{n(\omega_1)v_0/c - 1}{1 - v_0/c} .$$

11.63

$$\hbar\omega_2 = \frac{2\hbar\omega_1[n(\omega_1)\cos\vartheta_1 - 1]}{(mc^2/\mathcal{E}_0)^2 + 2(\hbar\omega_1/\mathcal{E}_0)[n(\omega_1)\cos\vartheta_1 - 1] + \vartheta_2^2} .$$

The maximum value of $\hbar\omega_2$ is reached for $\vartheta_1 = \vartheta_2 = 0$. Particular cases: when $\mathcal{E}_0 \ll (mc^2)^2/\hbar\omega_1$, we have

$$(\hbar\omega_2)_{\max} \approx 2\hbar\omega_1 \left(\frac{\mathcal{E}_0}{mc^2}\right)^2 [n(\omega_1) - 1] ;$$

when $\mathcal{E}_0 \gg (mc^2)^2/\hbar\omega_1$, we have

$$(\hbar\omega_2)_{\max} \approx \mathcal{E}_0 .$$

It is clear from the last equation that a hard Cherenkov quantum can take away the larger part of the initial energy of an ultrarelativistic electron.

11.64 The scattering angle takes on discrete values, determined by the equation

$$\sin\tfrac{1}{2}\vartheta = \frac{n\pi\hbar}{a p_0} ,$$

with

$$\frac{1}{a} = \frac{n_1}{a_1} + \frac{n_2}{a_2} + \frac{n_3}{a_3} ,$$

where the n_i are integers.

11.66 When $\hbar\omega \ll \mathcal{E}_0$, we have

$$\hbar\omega = \frac{(qc)^2/2\mathcal{E}_0}{(mc^2/\mathcal{E}_0)^2 + \vartheta^2 - 2(qc/\mathcal{E}_0)} \; .$$

The energy $\hbar\omega$ of a bremsstrahlung quantum takes on discrete values for fixed values of the angle ϑ, since the momentum transferred, $\boldsymbol{q} = 2\pi\hbar\boldsymbol{g}$, is discrete.

11.67

$$\nu' = \nu\gamma\left(1 + \frac{v}{c}\cos\psi\right) \; .$$

$$\sin\tfrac{1}{2}(\psi' - \psi) = \tanh\tfrac{1}{2}\alpha \sin\tfrac{1}{2}(\psi' + \psi) \, , \qquad \tanh\alpha = \frac{v}{c} \; .$$

b The motion of charged particles in an electromagnetic field

11.68

$$\frac{m}{(1 - v^2/c^2)^{1/2}}\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + \frac{m v \boldsymbol{v}}{c^2(1 - v^2/c^2)^{3/2}}\frac{\mathrm{d}v}{\mathrm{d}t} = \boldsymbol{F} \; ;$$

(a) $\dfrac{m}{(1 - v^2/c^2)^{3/2}}\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F}$, (b) $\dfrac{m}{(1 - v^2/c^2)^{1/2}}\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F}$, $\boldsymbol{v} \perp \boldsymbol{F}$;

(c) $m\dfrac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F}$.

The quantities $m[1 - (v^2/c^2)]^{-3/2}$ and $m[1 - (v^2/c^2)]^{-1/2}$ are sometimes referred to as the longitudinal and transverse masses, respectively.

11.69

$$\boldsymbol{F} = \frac{1}{\gamma}\boldsymbol{F}' + \left(1 - \frac{1}{\gamma}\right)\frac{(\boldsymbol{v}\cdot\boldsymbol{F}')\boldsymbol{v}}{v^2} \; ,$$

$$\boldsymbol{F}' = \gamma\boldsymbol{F} - (\gamma - 1)\frac{(\boldsymbol{v}\cdot\boldsymbol{F})\boldsymbol{v}}{v^2} \; ,$$

where $\gamma = [1 - (v^2/c^2)]^{-1/2}$.

11.70

$$F = \gamma^2\frac{mv^2}{R} \; .$$

11.72

$$\psi(\alpha) = -\frac{2\kappa(1 - \beta^2)}{[(1 - \beta^2)\cos^2\alpha + \sin^2\alpha]^{1/2}}\ln r \, ,$$

where $\beta = v/c$ and r is the distance of the point of observation from the conductor.

11.73

$$F = \frac{2\epsilon\kappa}{\gamma r} \; .$$

The problem may be solved in different ways: (a) the electromagnetic force on the moving point charge due to the linear charge and current can be computed directly (the Lorentz contraction must be taken into account!), (b) the force can be evaluated in the reference frame in which the magnetic field is zero, with subsequent application of the transformation formulae for the 4-force, and (c) one can use the convection potential ψ obtained in the preceding problem and defined by $F = -e\,\mathrm{grad}\,\psi$.

11.74

$$F = e\left(1-\frac{v^2}{c^2}\right)\frac{2J_r}{vr},$$

where r is the distance of the electron from the axis of the beam. The current through a circle of radius r is given by

$$J_r = \frac{2\pi v}{(1-\beta^2)^{1/2}}\int_0^r \rho(r)r\,\mathrm{d}r$$

and the electron velocity is given by (see preceding problem)

$$v = \left(1+\frac{eV}{mc^2}\right)^{-1}\left(1+\frac{eV}{2mc^2}\right)\left(\frac{2eV}{m}\right)^{1/2}.$$

At the surface an electron experiences a force $F = e(1-v^2/c^2)(2J/va)$ where a is the radius of the beam.

11.75 The acceleration of an outer electron is normal to the axis of the beam and to the electron velocity, and hence in the laboratory system we have (see solutions to problems 11.68 and 11.74)

$$\dot{v}_n = \frac{(1-v^2/c^2)^{1/2}}{m}F = \frac{2eJ}{mav}\left(1-\frac{v^2}{c^2}\right)^{3/2}.$$

The broadening of the beam is given by

$$\Delta a = \frac{\dot{v}_n t^2}{2} = \frac{\dot{v}_n L^2}{2v^2}.$$

Since $\Delta a \ll L$, we have $\dot{v}_n L/v \ll v$ or $\dot{v}_n t \ll v < c$. The broadening Δa can therefore be calculated from a nonrelativistic formula.

The same result may be obtained for Δa by considering the broadening of the beam in the reference frame moving together with the electrons. In this frame the electrons experience only an electrostatic force.

11.76 Let the direction of the x-axis be parallel to $\boldsymbol{E}$. The differential equations of motion in the four-dimensional form are

$$\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} = \frac{|e|E}{mc}\frac{\mathrm{d}(ct)}{\mathrm{d}\tau}, \qquad \frac{\mathrm{d}^2y}{\mathrm{d}\tau^2} = 0, \qquad \frac{\mathrm{d}^2z}{\mathrm{d}\tau^2} = 0, \qquad \frac{\mathrm{d}^2(ct)}{\mathrm{d}\tau^2} = \frac{|e|E}{mc}\frac{\mathrm{d}x}{\mathrm{d}\tau}.$$

Integration of these equations subject to the initial conditions

$$x = y = z = ct = 0\,, \qquad \frac{dx}{d\tau} = \frac{p_{0x}}{m}\,, \qquad \frac{dy}{d\tau} = \frac{p_{0y}}{m}\,,$$

$$\frac{dz}{d\tau} = 0\,, \qquad c\frac{dt}{d\tau} = \frac{\mathcal{E}_0}{mc} \quad \text{at} \quad \tau = 0\,, \qquad \text{where} \quad \mathcal{E}_0 = (c^2p_0^2 + m^2c^4)^{1/2}\,,$$

leads to the following equations for the particle trajectory in the four-dimensional space:

$$x = \frac{\mathcal{E}_0}{|e|E}\left(\cosh\frac{|e|E\tau}{mc} - 1\right) + \frac{cp_{0x}}{|e|E}\sinh\frac{|e|E\tau}{mc}\,,$$

$$y = \frac{p_{0y}\tau}{m}\,, \qquad z = 0\,,$$

$$ct = \frac{\mathcal{E}_0}{|e|E}\sinh\frac{|e|E\tau}{mc} + \frac{cp_{0x}}{|e|E}\left(\cosh\frac{|e|E\tau}{mc} - 1\right)\,.$$

From the last equation we have

$$\tau = \frac{mc}{|e|E}\ln\frac{p_{0x} + |e|Et + [(p_{0x} + |e|Et)^2 + m^2c^2 + p_{0y}^2]^{1/2}}{p_{0x} + \mathcal{E}_0/c}\,.$$

Using this expression, and eliminating the sinh and cosh terms from the first and last equations, we obtain the following equations for the trajectory in the three-dimensional form

$$x(t) = \frac{c}{|e|E}\left\{[(p_{0x} + |e|Et)^2 + m^2c^2 + p_{0y}^2]^{1/2} - \frac{\mathcal{E}_0}{c}\right\};$$

$$y(t) = \frac{cp_{0y}}{|e|E}\ln\frac{p_{0x} + |e|Et + [(p_{0x} + |e|Et)^2 + m^2c^2 + p_{0y}^2]^{1/2}}{p_{0x} + \mathcal{E}_0/c}\,;$$

$$z(t) = 0\,.$$

When $p_0 \ll mc$ and $t \ll mc/|e|E$, the motion is nonrelativistic. Then the expressions for x, y, and z go over into the usual nonrelativistic expressions

$$x(t) = \frac{p_{0x}}{m}t + \frac{|e|E}{2m}t^2\,; \qquad y(t) = \frac{p_{0y}}{m}t\,.$$

After a sufficiently long interval of time ($t \gg mc/|e|E$), the velocity of the particle will approach the velocity of light (even when it is initially small). Then

$$x(t) = ct - \frac{mc^2}{|e|E}\,, \qquad y(t) = \frac{cp_{0y}}{|e|E}\ln\left(\frac{2|e|Et}{mc}\right)$$

and the motion becomes uniform (with velocity c). The functions $x(t)$ and $y(t)$ are illustrated in figures 11.76.1a and 11.76.1b respectively. The motion for which $p_{0y} = 0$ (see figure 11.76.1a) is usually referred to as hyperbolic.

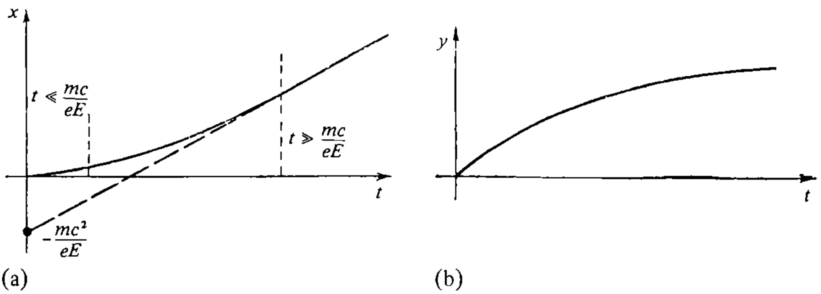


Figure 11.76.1.

11.77 The particle trajectory is given by

$$x = \frac{\mathcal{E}_0}{|e|E}\left[\cosh\left(\frac{|e|E}{cp_{0y}}y\right) - 1\right] + \frac{cp_{0x}}{|e|E}\sinh\left(\frac{|e|E}{cp_{0y}}y\right) .$$

In the nonrelativistic limit $\mathcal{E}_0 = mc^2$, $p_0 \ll mc$, and $|e|Ey/cp_{0y} \ll 1$. The latter result follows from the fact that $|e|E\tau$, which is the momentum given to the particle, should be small in comparison with mc in the nonrelativistic approximation. Thus,

$$x = \frac{m|e|Ey^2}{2p_{0y}^2} + \frac{p_{0x}}{p_{0y}}y .$$

11.78

$$l = \frac{\mathcal{E} - mc^2}{eE} .$$

11.79 Let the z-axis be parallel to H and consider the differential equations of motion in the four-dimensional form[1].

$$\frac{d^2x}{d\tau^2} = \omega_1\frac{dy}{d\tau} , \qquad \frac{d^2y}{d\tau^2} = -\omega_1\frac{dx}{d\tau} , \qquad \frac{d^2z}{d\tau^2} = 0 , \qquad \frac{d^2t}{d\tau^2} = 0 ,$$

where $\omega_1 = eH/mc$.

The first two equations may be conveniently written in the form $d^2u/d\tau^2 + i\omega_1 du/d\tau = 0$, where $u = x + iy$. It follows from the latter equation that

$$ct = \frac{\mathcal{E}_0}{mc}\tau , \qquad \mathcal{E}_0 = c(p_0^2 + m^2c^2)^{1/2} , \qquad \mathcal{E} = mc^2\frac{dt}{d\tau} = \mathcal{E}_0 .$$

(1) It is possible to start from the three-dimensional equation $d\boldsymbol{p}/dt = e[\boldsymbol{v} \wedge \boldsymbol{H}]/c$ by making the substitution $\boldsymbol{p} = \mathcal{E}\boldsymbol{v}/c^2$ and using the fact that $\mathcal{E}$ = constant (the magnetic field does no work on the particle).

The energy of the particle is independent of time, since the forces due to the magnetic field do no work. By integrating the equations for u and z, separating the real and imaginary parts of u, and expressing the proper time τ in terms of t, it is found that

$$\left.\begin{aligned} x &= R_1\cos(\omega_2 t+\alpha)+\frac{cp_{0y}}{eH}+x_0\,, \\ y &= -R_1\sin(\omega_2 t+\alpha)-\frac{cp_{0x}}{eH}+y_0\,, \\ z &= v_{0z}t\,. \end{aligned}\right\} \tag{11.79.1}$$

It is evident from equation (11.79.1) that the particle will travel along a helix. The radius of the helix is $R = |R_1|$ where $R_1 = p_{0\perp}c/eH$ and $p_{0\perp} = (p_{0x}^2+p_{0y}^2)^{1/2}$. The associated frequency is given by $\omega = |\omega_2|$ where $\omega_2 = eHc/\mathcal{E}$ (the sign of the charge may be negative). The pitch of the helix is given by

$$\frac{2\pi|v_{0z}|}{\omega} = \frac{2\pi\mathcal{E}|v_{0z}|}{|e|Hc}\,,$$

where $v_{0z} = p_{0z}c^2/\mathcal{E}$.

It is clear that $R = v_{0\perp}/\omega$ where $v_{0\perp} = p_{0\perp}c^2/\mathcal{E}$ is the velocity component perpendicular to the field. At low particle velocities $\mathcal{E} = mc^2$, and

$$R = \frac{mcv_{0\perp}}{|e|H}\,, \qquad \omega = \frac{|e|H}{mc}\,.$$

The angle α is given by

$$\sin\alpha = -\frac{p_{0x}}{p_{0\perp}}\,, \qquad \cos\alpha = -\frac{p_{0y}}{p_{0\perp}}\,.$$

11.80

$$x = a\sin\omega t+\frac{cE_y}{H}t\,, \qquad y = a(\cos\omega t-1)\,, \qquad z = \frac{eE_z}{2m}t^2+v_{0z}t\,,$$

where

$$a = \frac{1}{\omega}\left(v_{0x}-\frac{cE_y}{H}\right).$$

The motion in the direction of the z-axis takes the form of uniform acceleration under the action of the z component of the electric field. In the x, y plane, the orbit is circular, of radius a, and the centre of the circle moves in the direction perpendicular to the $(\boldsymbol{E}, \boldsymbol{H})$ plane. The drift velocity is $v_{dr} = cE_y/H$. The possible projections of the trajectories on the x, y plane are shown in figure 11.80.1. The curves shown in figures 11.80.1a, 11.80.1c, 11.80.1e, and 11.80.1g are trochoids, while those in figures 11.80.1b and 11.80.1f are cycloids. The motion is nonrelativistic when $v_0 \ll c$, $E_y/H \ll 1$, and the time t is not too large, i.e.

$$t \ll \frac{mc}{eE_z} = \frac{H}{\omega E_z} .$$

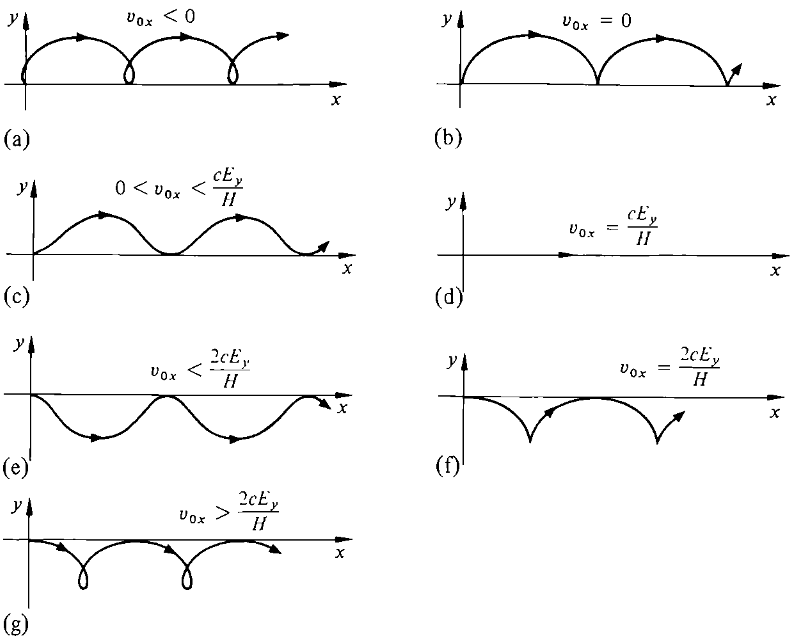


Figure 11.80.1.

11.81

$$x = \frac{p_{0x}c}{eH}\sin\kappa H\tau + \frac{p_{0y}c}{eH}(\cos\kappa H\tau - 1) ,$$

$$y = \frac{p_{0x}c}{eH}(\cos\kappa H\tau - 1) + \frac{p_{0y}c}{eH}\sin\kappa H\tau ,$$

$$z = \frac{\mathcal{E}_0}{eH}(\cosh\kappa E\tau - 1) + \frac{p_{0z}c}{eH}\sinh\kappa E\tau ,$$

$$ct = \frac{p_{0z}c}{eE}(\cosh\kappa E\tau - 1) + \frac{\mathcal{E}_0}{eE}\sinh\kappa E\tau ,$$

where $\kappa = e/mc$.

11.82 (a) Let the electric field $\boldsymbol{E}$ be parallel to the y-axis, and the magnetic field $\boldsymbol{H}$ be parallel to the z-axis (in the S system). At $t = 0$ the particle is at the point $x = y = z = 0$ and has a momentum $\boldsymbol{p}_0$. The motion is different when $E > H$ and when $H > E$. In the first case, $(\boldsymbol{E} \cdot \boldsymbol{H}) = 0$, $E^2 - H^2 > 0$, and hence there exists a frame S' in which the magnetic field is zero. It is clear from the Lorentz transformations for the field that S' should move relative to S with velocity $V = cH/E$ along the x-axis (see problem 10.61). The required equations of motion for the particle in S may be obtained from the equations of motion in a uniform electric field E' with the aid of the Lorentz transformation $x = (x' + Vt')/(1 - V^2/c^2)^{1/2}$, and so on. In this method E', p'_{0x}, p'_{0y}, p'_{0z} should be expressed in terms of the corresponding quantities without primes. The final result is

$$\left.\begin{aligned}
x &= \frac{E(cp_{0x}E - \mathcal{E}_0 H)}{mc(E^2 - H^2)}\tau + \frac{H(\mathcal{E}_0 E - cp_{0x}H)}{e(E^2 - H^2)^{3/2}} \sinh \kappa(E^2 - H^2)^{1/2}\tau \\
&\quad + \frac{cp_{0y}H}{e(E^2 - H^2)}[\cosh \kappa(E^2 - H^2)^{1/2}\tau - 1] , \\
y &= \frac{\mathcal{E}_0 E - cp_{0x}H}{e(E^2 - H^2)}[\cosh \kappa(E^2 - H^2)^{1/2}\tau - 1] \\
&\quad + \frac{p_{0y}c}{e(E^2 - H^2)^{1/2}} \sinh \kappa(E^2 - H^2)^{1/2}\tau , \qquad z = \frac{p_{0z}}{m}\tau , \\
ct &= \frac{H(cp_{0x}E - \mathcal{E}_0 H)}{mc(E^2 - H^2)}\tau + \frac{E(\mathcal{E}_0 E - cp_{0x}H)}{e(E^2 - H^2)^{1/2}} \sinh \kappa(E^2 - H^2)^{1/2}\tau \\
&\quad + \frac{cp_{0y}E}{e(E^2 - H^2)}[\cosh \kappa(E^2 - H^2)^{1/2}\tau - 1] , \\
&\text{where} \\
\kappa &= \frac{e}{mc} .
\end{aligned}\right\} \qquad (11.82.1)$$

When $H > E$, the transformation to the system in which there is only a magnetic field leads to the same result as that given by equations (11.82.1), except that E is replaced by H. In carrying out this replacement, it must be remembered that $\sinh \mathrm{i}\alpha = \mathrm{i} \sin \alpha$ and $\cosh \mathrm{i}\alpha = \cos \alpha$.

The solution for $E = H$ may be obtained from the above formulae by letting $E \to H$. The result is

$$x = \frac{\kappa^2 H^2}{6m}\left(\frac{\mathcal{E}_0}{c} - p_{0x}\right)\tau^3 + \frac{cp_{0y}\kappa^2 H}{2e}\tau^2 + \frac{p_{0x}}{m}\tau ,$$

and so on. The solution for limit (b) is similar to the solutions used in problems 11.76 and 11.79.

11.83

$$T = mc^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau} - 1\right)$$

and hence, for example, in the system considered in problem 11.81, we have

$$T = \mathcal{E}_0 \cosh \kappa E\tau + cp_{0z} \sinh \kappa E\tau - mc^2 .$$

11.84 Using the result obtained in problem 10.61, and evaluating V/c to within terms of the first order in E/H, we find that $V/c = E_y/H$. The solution is similar to that for problem 11.82. In all the calculations, second and higher orders of E_y/H, E_z/H, and v_0/c should be neglected. The final result is

$$\left.\begin{aligned} x &= a\sin\omega t + \frac{v_{0y}}{\omega}(\cos\omega t - 1) + c\frac{E_y}{H}t\,, \\ y &= a(\cos\omega t - 1) + \frac{v_{0y}}{\omega}\sin\omega t\,, \\ z &= \frac{eE_z t^2}{2m} + v_{0z}t\,, \end{aligned}\right\} \tag{11.84.1}$$

where

$$a = \frac{v_{0x} - cE_y/H}{\omega} \quad \text{and} \quad \omega = \frac{eH}{mc}\,.$$

At $t = 0$ the particle is at the point $x = y = z = 0$. The formulae given by equations (11.84.1) contain the solution of problem 11.80 as a special case.

11.85 Let the x-axis be parallel to the direction of propagation of the plane wave. The wave field can then be described by two functions of t', for example, $E_y(t')$ and $E_z(t')$:

$$\boldsymbol{E} = [0, E_y(t'), E_z(t')]\,, \qquad \boldsymbol{H} = [0, -E_z(t'), E_y(t')]\,.$$

Using equation (11.b.8) we find that $t' = \tau$ and hence the parametric equations of motion are

$$x(\tau) = \frac{1}{2m^2c}\int_0^\tau p_\perp^2\,\mathrm{d}\tau\,, \qquad y(\tau) = \frac{1}{m}\int_0^\tau p_y\,\mathrm{d}\tau\,,$$

$$z(\tau) = \frac{1}{m}\int_0^\tau p_z\,\mathrm{d}\tau\,, \qquad t(\tau) = \tau + \frac{1}{2m^2c^2}\int_0^\tau p_\perp^2\,\mathrm{d}\tau\,,$$

where

$$\boldsymbol{p}_\perp = e\int_0^\tau \boldsymbol{E}(t')\,\mathrm{d}t' = \boldsymbol{e}_y p_y + \boldsymbol{e}_z p_z$$

is the component of the momentum of the particle in the $\boldsymbol{E}$, $\boldsymbol{H}$ plane.

11.86 The particle moves in the x,z-plane, and its equation of motion is $\gamma\dot{z} = c(\gamma - 1)$. The orbit is given by the equations:

$$x = b(1-\cos\theta)\,, \qquad z = \tfrac{1}{8}kb^2(\sin 2\theta - 2\theta)\,, \qquad b = \frac{ea}{mk^2c^2}\,.$$

11.87 The coordinates of the particle are given by

$$x = x_0 \cos\omega t\,, \qquad y = y_0 \cosh\omega t\,, \qquad z = vt\,,$$

where $\omega^2 = 2ek/m$.

It follows from the form of $x(t)$ and $y(t)$ that this type of lens may be used to produce a charged particle beam in the form of a flat ribbon.

11.88

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\dot{r}}{(1-v^2/c^2)^{1/2}}\right] = \frac{mr\dot{\phi}^2}{(1-v^2/c^2)^{1/2}} + eE_r + \frac{e}{c}(-H_\phi \dot{z} + H_z r\dot{\phi})\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{mr^2\dot{\phi}}{(1-v^2/c^2)^{1/2}}\right] = e\left[E_\phi + \frac{1}{c}(H\dot{z} - H_z\dot{r})\right]r\,,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\dot{z}}{(1-v^2/c^2)^{1/2}}\right] = e\left[E_z + \frac{1}{c}(H_\phi\dot{r} - H_r r\dot{\phi})\right]\,.$$

The first and last of these equations have the form of the usual Newtonian laws of motion, except that they contain the variable mass $m/(1-v^2/c^2)^{1/2}$. Moreover, the right-hand side of the first equation contains the term $mr\dot{\phi}^2/(1-v^2/c^2)^{1/2}$ which is independent of the form of the electromagnetic forces (centripetal force). The second equation gives the time derivative of the angular momentum of the particle about the z-axis in terms of the z component of the moment of the Lorentz force.

11.89 When $H = 0$, the electron trajectories are rectilinear. As the magnetic field increases, the trajectories become more and more curved in the plane perpendicular to the axis. Consider the cylindrical coordinates r, ϕ, z, where the z-axis lies along the axis of the cylinder. The electrons will not reach the anode if at $r = b$ their velocity is parallel to the surface of the anode, i.e. when $\dot{r}|_{r=b} = 0$. Moreover, $\dot{\phi}|_{r=b} = v_{\max}/b$. Let us use the second of the equations given in the solution to the preceding problem, which in the present situation assumes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{mr^2\dot{\phi}}{(1-v^2/c^2)^{1/2}}\right] = -\frac{e}{c}H(r)r\frac{\mathrm{d}r}{\mathrm{d}t}\,. \tag{11.89.1}$$

Integrating equation (11.89.1) along the particle trajectory between $r = a$ and $r = b$, we have

$$\left.\frac{mr^2\dot{\phi}}{(1-v^2/c^2)^{1/2}}\right|_{r=a}^{r=b} = -\frac{e}{2\pi c}\int_a^b 2\pi Hr\,\mathrm{d}r = -\frac{e\Phi}{2\pi c}\,.$$

Hence

$$\Phi_{\text{crit}} = \frac{2\pi cb}{|e|}p_{\max} = 2\pi cb\left[\frac{2mV}{|e|}\left(1 + \frac{|e|V}{2mc^2}\right)\right]^{1/2}\,, \tag{11.89.2}$$

where we have used the result of problem 11.1 and the fact that $T_{\max} = |e|V$.

When the potential difference is small, so that $|e|V \ll mc^2$, which is equivalent to saying that $v \ll c$, equation (11.89.2) assumes the simpler form

$$\Phi_{\text{crit}} = 2\pi cb\left(\frac{2mV}{|e|}\right)^{1/2}$$

11.90 The potential difference V should be larger than

$$V_{\text{crit}} = \left(\frac{4J^2}{c^2}\ln^2\frac{b}{a}+\frac{m^2c^4}{e^2}\right)^{1/2} - \frac{mc^2}{|e|}\ .$$

When $|e|V \ll mc^2$ (nonrelativistic electrons), we have

$$V_{\text{crit}} = \frac{2J^2|e|}{mc^4}\ln^2\frac{b}{a}\ .$$

11.91

$$b = a\exp\left(\frac{p_0c^2}{J|e|}\right),$$

where $p_0 = mv_0/(1-v_0^2/c^2)^{1/2}$.

11.93 Consider the cylindrical coordinates r, ϕ whose origin coincides with the charge Ze and whose polar axis is parallel to the angular momentum of the particle. The motion will then take place in the $z = 0$ plane and r will be the distance between the charges $-e$ and Ze. The first two equations in the solution to problem 11.88 will then be

$$\left.\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{m\dot{r}}{(1-v^2/c^2)^{1/2}}\right] = \frac{mr\dot{\phi}^2}{(1-v^2/c^2)^{1/2}} - \frac{Ze^2}{r^2}\ ,\\ &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{mr^2\dot{\phi}}{(1-v^2/c^2)^{1/2}}\right] = 0\ . \end{aligned}\right\} \tag{11.93.1}$$

It follows from the second equation that the angular momentum is an integral of the motion, so that

$$\frac{mr^2\dot{\phi}}{(1-v^2/c^2)^{1/2}} = K = \text{constant}\ . \tag{11.93.2}$$

The second integral of the motion is the total energy of the system:

$$\frac{mc^2}{(1-v^2/c^2)^{1/2}} - \frac{Ze^2}{r} = \mathcal{E} = \text{constant}\ . \tag{11.93.3}$$

It follows from equation (11.93.3) that there are two main types of trajectories. At large values of r the total energy is $\mathcal{E} = mc^2+T$ (T is the kinetic energy), since as $r \to \infty$, the potential energy $Ze^2/r \to 0$. Since $T \geqslant 0$, it follows that when $E < mc^2$ the particle cannot reach large distances from the attracting centre, and its trajectory will lie in a finite region (finite motion). When $E > mc^2$, there will be trajectories extending to infinity (infinite motion).

Let us now find the differential equation for the particle trajectory. It follows from equation (11.93.3) that

$$\frac{d}{dt} = \frac{K(1-v^2/c^2)^{1/2}}{mr^2}\frac{d}{d\phi} \,. \tag{11.93.4}$$

Substituting equations (11.93.3) and (11.93.4) into the first equation in (11.93.1) we obtain the required differential equation:

$$\frac{d^2}{d\phi^2}\left(\frac{1}{r}\right) + (1-\rho^2)\frac{1}{r} = \frac{Ze^2\mathcal{E}}{K^2c^2} \,, \tag{11.93.5}$$

where $\rho = Ze^2/Kc$.

When $\rho \neq 1$, the integration of this equation yields

$$r = \frac{p}{1+\epsilon\cos(1-\rho^2)^{1/2}\phi} \,, \qquad p = \frac{K^2c^2 - Z^2e^4}{Ze^2\mathcal{E}} \,. \tag{11.93.6}$$

where ϵ is an integration constant. The second integration constant may be eliminated by a suitable choice of the origin of the angle ϕ, and the quantity ϵ may be expressed in terms of $\mathcal{E}$ and K. The trajectories are symmetric with respect to the x-axis ($\phi = 0$).

Consider the case $\rho < 1$ in greater detail. As can be seen from equation (11.93.6), the particle will not approach the centre to distances smaller than $r_{\text{min}} = p/(1+\epsilon)$ if it is assumed that $\epsilon > 0$. In equation (11.93.6) the origin for the angle ϕ is chosen so that $r = r_{\text{min}}$ when $\phi = 0$. The particle can therefore reach the distance r_{min} from the centre several times. At all such points $\dot{r} = 0$ and the velocity is perpendicular to the radius vector $\boldsymbol{r}$. Hence,

$$K = \frac{mvr_{\text{min}}}{(1-v^2/c^2)^{1/2}} \,.$$

By eliminating v from this expression and from

$$\mathcal{E} = \frac{mc^2}{(1-v^2/c^2)^{1/2}} - \frac{Ze^2}{r_{\text{min}}}$$

and expressing r_{min} in terms of ϵ it is found that

$$\epsilon = \frac{1}{\rho}\left[1 - \frac{m^2c^4}{\mathcal{E}^2}(1-\rho^2)\right]^{1/2} \tag{11.93.7}$$

It follows from equation (11.93.7) that $\epsilon < 1$ when $\mathcal{E} < mc^2$. The motion is then finite, and the trajectory is quasi-ellipsoidal (figure 11.93.1). It takes the form of a rosette lying between circles of radii $p/(1+\epsilon)$ and $p/(1-\epsilon)$. The rosette may be obtained by rotating the nonrelativistic elliptical trajectory in its own plane. A complete oscillation of r from the minimum value $r_{\text{min}} = p/(1+\epsilon)$ (pericentre) to the maximum value $r_{\text{max}} = p/(1-\epsilon)$ (apocentre) and back to a new minimum, occurs when ϕ increases by $2\pi/(1-\rho^2)^{1/2}$. The pericentre of the orbit is thus rotated through an angle $2\pi[(1-\rho^2)^{-1/2}-1]$ in each period of r. If $(1-\rho^2)^{1/2}$ is a rational number, then the trajectory will close after a finite number of revolutions.

When $\mathcal{E} > mc^2$, $\epsilon > 1$. The motion is infinite and the trajectory is quasi-hyperbolic (figure 11.93.2). It has two branches extending to infinity at $\phi = \pm\phi_0$, where $\phi_0 = \arccos(-1/\epsilon)/(1-\rho^2)^{1/2}$. A particle approaching the charge Ze along one of these branches may execute a number of revolutions about the charge before it leaves it to escape to infinity along the other branch.

The case $\mathcal{E} = mc^2$ corresponds to $\epsilon = 1$. Here, the motion is again infinite and the trajectory is quasi-parabolic.

As $\rho \to 1$, the above trajectories become elliptical ($\epsilon < 1$), hyperbolic ($\epsilon > 1$), and parabolic ($\epsilon = 1$), and correspond to the usual nonrelativistic Keplerian solutions. This is inevitable, since $v/c \ll 1$ when $\rho \ll 1$[(2)].

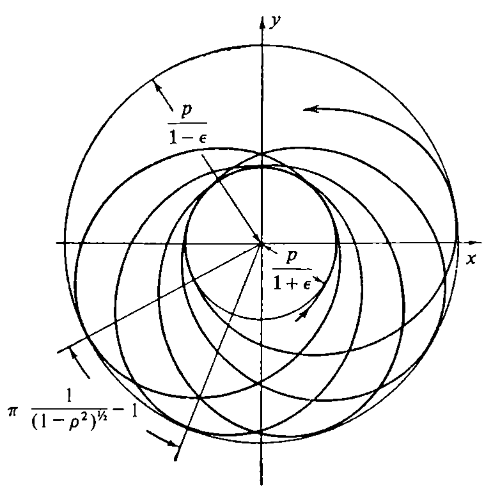


Figure 11.93.1.

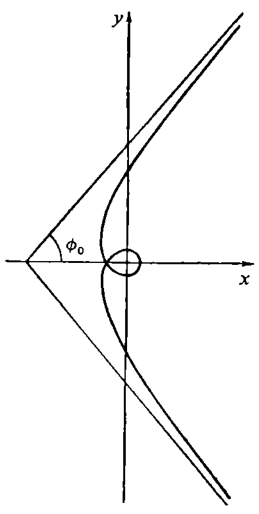


Figure 11.93.2.

11.94 When $\rho > 1$, the solution of equation (11.93.5) in the preceding problem can conveniently be rewritten as

$$r = \frac{p_1}{-1+\epsilon_1\cosh(\rho^2-1)^{1/2}\phi}, \tag{11.94.1}$$

(2) In the nonrelativistic limit

$$\rho = \frac{Ze^2}{Kc} \sim \frac{Ze^2}{rmvc} \sim \frac{|U|}{mvc}.$$

According to the virial theorem $|U| = 2T \approx mv^2$, so that $\rho \approx v/c \ll 1$.

where

$$p_1 = \frac{-K^2c^2 + Z^2e^4}{Ze^2\mathcal{E}} ,$$

$$\epsilon_1 = \left[\frac{1}{\rho^2} + \frac{m^2c^4}{\mathcal{E}^2}\left(1 - \frac{1}{\rho^2}\right)\right]^{1/2}$$

The trajectories described by equation (11.94.1) become spirals which pass through the origin as $\phi \to \pm\infty$. The particle passes through the centre of force (in the nonrelativistic limit this is only possible if $K = 0, \rho = \infty$). When $\mathcal{E} > mc^2$ we have $\epsilon_1 < 1$ and the trajectory has two branches which reach out to infinity at $\phi = \pm\phi_0$, where $\phi_0 = \operatorname{arcosh}(1/\epsilon_1)/(\rho^2 - 1)^{1/2}$ (figure 11.94.1). When $\mathcal{E} < mc^2$ we have $\epsilon_1 > 1$ and the trajectory is as shown in figure 11.94.2.

When $\rho = 1$, the solution given by equation (11.94.1) cannot be used, and the differential equation for the trajectory must be integrated again. The result of this integration is

$$r = \frac{2Ze^2\mathcal{E}}{\mathcal{E}^2(\phi^2 - 1) + m^2c^4} .$$

The trajectory is then also a spiral passing through the centre when $\phi \to \pm\infty$, although the centre is approached much more slowly than in the case $\rho > 1$. The general nature of the trajectory is similar to that illustrated in figures 11.94.1 and 11.94.2.

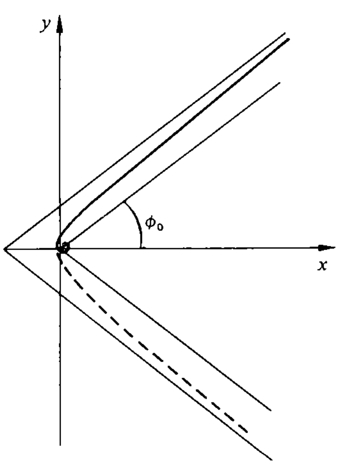


Figure 11.94.1.

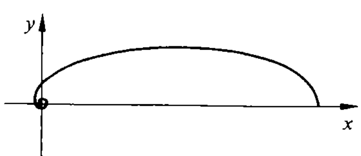


Figure 11.94.2.

11.95 When $Ze^2/Kc < 1$,

$$r = \frac{p}{-1 + \epsilon\cos\phi(1 - Z^2e^4/K^2c^2)^{1/2}} ,$$

where

$$p = -\frac{Z^2e^4 - K^2c^2}{Ze^2\mathcal{E}}, \qquad \epsilon = \frac{c}{Ze^2\mathcal{E}}[K^2\mathcal{E}^2 - m^2c^2(K^2c^2 - Z^2e^4)]^{1/2} > 1 .$$

The trajectory is quasi-hyperbolic (figure 11.95.1). Its two branches reach infinity at $\phi = \pm\phi_0$, where

$$\phi_0 = \frac{1}{(1 - Z^2e^4/K^2c^2)^{1/2}} \arccos\frac{1}{\epsilon} .$$

When $Ze^2/Kc \ll 1$, the particle moves on a hyperbola. This corresponds to nonrelativistic motion with $v \ll c$ (see footnote to the solution of problem 11.93).

When $Ze^2/Kc > 1$,

$$r = -\frac{p}{1 - \epsilon\cosh\phi(Z^2e^4/K^2c^2 - 1)^{1/2}},$$

where $\epsilon < 1$. The general character of the trajectory is similar to that in the previous example.

The two branches reach infinity at

$$\phi = \pm\frac{1}{(Z^2e^4/K^2c^2 - 1)^{1/2}} \operatorname{arcosh}\frac{1}{\epsilon} .$$

When $Ze^2/Kc = 1$,

$$r = \frac{2Ze^2\mathcal{E}}{\mathcal{E}^2(1 - \phi^2) - m^2c^4} .$$

The two branches of the trajectory reach infinity when

$$\phi = \frac{(\mathcal{E}^2 - m^2c^4)^{1/2}}{\mathcal{E}} .$$

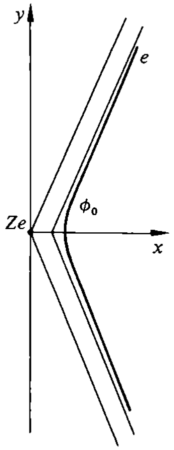


Figure 11.95.1.

11.97 When $ee' < 0$ (attraction)

$$r = \frac{a|\epsilon^2 - 1|}{1 + \epsilon\cos\phi},$$

where

$$a = \left|\frac{ee'}{2\mathcal{E}}\right|, \qquad \epsilon = \left(1 + \frac{2\mathcal{E}K^2}{\mu e^2 e'^2}\right)^{1/2}, \qquad \mu = \frac{m_1 m_2}{m_1 + m_2},$$

$K = \mu r^2\dot{\phi}$ is the angular momentum, $\mathcal{E} = ee'/r + \frac{1}{2}\mu v^2$ is the total energy of the particle, and r, ϕ are the polar coordinates. The particle trajectory is a conic section: it is an ellipse ($\epsilon < 1$) when $\mathcal{E} < 0$, a hyperbola with the charge e' at one of its foci ($\epsilon > 1$) when $\mathcal{E} > 0$, and a parabola ($\epsilon = 1$) when $\mathcal{E} = 0$.

When $ee' > 0$ (repulsion)

$$r = \frac{a(\epsilon^2 - 1)}{-1 + \epsilon\cos\phi}.$$

In this case $\mathcal{E} > 0$, the eccentricity $\epsilon > 1$, and the trajectory is a hyperbola with the charge e' at the external focus.

11.98 The differential scattering cross section may be calculated from the formula

$$\sigma(\theta) = \frac{s\,\mathrm{d}s}{\sin\theta\,\mathrm{d}\theta}, \tag{11.98.1}$$

where θ is the angle of scattering corresponding to a given value of the impact parameter s. The relation between s and θ may be found from the

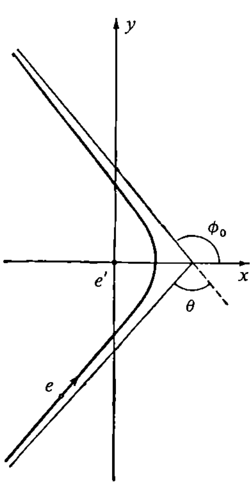


Figure 11.98.1.

equation for the trajectory of the particle (see preceding problem). In the case of attraction ($ee' < 0$), $\cos\phi > -\epsilon^{-1}$. The angle ϕ varies between $-\phi_0$ and ϕ_0 (figure 11.98.1) as the particle traverses its orbit ($\cos\phi_0 = -\epsilon^{-1}$).

The scattering angle θ is the complement of the angle between the asymptotes of the hyperbolic trajectory. It follows from figure 11.98.1 that $\frac{1}{2}\theta = -\frac{1}{2}\pi + \phi_0$ and hence

$$\cot^2\tfrac{1}{2}\theta = \operatorname{cosec}^2\tfrac{1}{2}\theta - 1 = \sec^2\phi_0 - 1 = \epsilon^2 - 1 = \frac{2\mathcal{E}K^2}{me^2e'^2}.$$

The angular momentum is $K = mv_0 s$, so that

$$s^2 = \frac{e^2e'^2}{m^2v_0^4}\cot^2\tfrac{1}{2}\theta .$$

Differentiation and substitution into equation (11.98.1) yields

$$\sigma(\theta) = \left(\frac{ee'}{2mv_0^2}\right)^2 \operatorname{cosec}^4\tfrac{1}{2}\theta .$$

This is the well-known Rutherford formula. The same result is obtained when $ee' > 0$.

11.99 For $ee' < 0$ (attraction),

$$\theta = \left[\frac{2cK}{(c^2K^2 - Z^2e^4)^{1/2}} - 1\right]\pi - \frac{2cK}{(c^2K^2 - Z^2e^4)^{1/2}}\arctan\frac{v_0(c^2K^2 - Z^2e^4)^{1/2}}{cZe^2},$$

where v_0 is the velocity of the charge when $r \to \infty$.

When $ee' > 0$ (repulsion)

$$\theta = \pi - \frac{2cK}{(c^2K^2 - Z^2e^4)^{1/2}}\arctan\frac{v_0(c^2K^2 - Z^2e^4)^{1/2}}{cZe^2} .$$

11.100 Large impact parameters s correspond to small angles of scattering. Hence, by putting $K = p_0 s$, where p_0 is the momentum at $r \to \infty$, it is possible to find the required dependence of the angle θ on s by letting $s \to \infty$ (evidently, $K > |ee'|/c$) in the general formulae derived in the preceding problem. On passing to the limit, the same result is obtained both for $ee' < 0$ and $ee' > 0$, namely,

$$\theta = \pi - 2\arctan\frac{v_0p_0s}{|ee'|} = \frac{2Ze^2}{v_0p_0s} \ll 1 ,$$

and hence $s = 2|ee'|/v_0p_0\theta$ and

$$\sigma(\theta) = \frac{s\,\mathrm{d}s}{\theta\,\mathrm{d}\theta} = 4\left(\frac{ee'}{v_0p_0}\right)^2\frac{1}{\theta^4} .$$

11.101

$$x = vt = \tfrac{2}{3}\left[\frac{(\epsilon+1)}{(\epsilon-1)}\frac{ma^3}{e^2}\right]^{1/2} .$$

11.102 The accelerating electric field is given by

$$E_\phi = \frac{1}{2\pi rc}\frac{\mathrm{d}\Phi}{\mathrm{d}t} ,$$

where r is the radius of the orbit of the electron, Φ is the magnetic flux intercepted by the orbit, and ϕ is the azimuth angle of the electron.

When the electron travels along the orbit through a distance $r\,\mathrm{d}\phi$, the work done by the field E_ϕ is

$$\delta A = E_\phi r\,\mathrm{d}\phi . \qquad (11.102.1)$$

The acceleration of the electron takes place on an orbit of constant radius $r = cp/eH_0$ (see problem 11.79), where H_0 is the magnetic field at the orbit. This field is perpendicular to the plane of the orbit and increases with time. From the condition $\mathrm{d}r = 0$ we find that

$$\mathrm{d}p = \frac{p}{H_0}\mathrm{d}H_0 . \qquad (11.102.2)$$

The electron energy $\mathcal{E} = c(p^2+m^2c^2)^{1/2}$ is increased by

$$\mathrm{d}\mathcal{E} = \frac{c^2p\,\mathrm{d}p}{\mathcal{E}} = \frac{c^2p^2\,\mathrm{d}H_0}{\mathcal{E}H_0} , \qquad (11.102.3)$$

in which equation (11.102.2) has been used. It follows that

$$\delta A = \mathrm{d}\mathcal{E} . \qquad (11.102.4)$$

Substituting equations (11.102.1) and (11.102.3) into equation (11.102.4), and using the equation $c^2p/\mathcal{E} = v = r\,\mathrm{d}\phi/\mathrm{d}t$, we have, after integration,

$$\Phi = 2\Phi_0 ,$$

where $\Phi_0 = \pi r^2 H_0$.

The latter result is the required 2:1 rule.

11.103 The energy of interaction U between the two charged particles is given by equation (11.b.8) into which the charge e_1 of one of the particles and the retarded potentials φ_2, A_2 for the other particle should be substituted. From the explanations put forward in problem 12.36, we have

$$\varphi_2 = \frac{e_2}{R}+\frac{e_2}{2c^2}\frac{\partial^2 R}{\partial t^2} , \qquad A_2 = \frac{e_2 v_2}{cR} ,$$

where R is the distance between the particles. By writing χ in the form

$$\chi = \frac{e_2}{2c}\frac{\partial R}{\partial t} ,$$

and carrying out a gauge transformation of the potentials, it is found that the new potentials are

$$\varphi_2' = \varphi_2 - \frac{1}{c}\frac{\partial\chi}{\partial t} = \frac{e_2}{R}, \qquad A_2' = A_2 + \operatorname{grad}\chi = \frac{e_2[v_2 + (n\cdot v_2)n]}{2cR},$$

where $n = R/R$. Hence, the energy of interaction is given by the Breit formula

$$U = e_1\varphi_2 - \frac{e_1}{c}(v_1\cdot A_2) = \frac{e_1e_2}{R}\left\{1 - \frac{1}{2c^2}[(v_1\cdot v_2) + (v_1\cdot n)(v_2\cdot n)]\right\}.$$

This formula gives an approximate representation of the fact that the force acting on one of the two interacting charged particles at a distance R from one another is determined by the previous position and state of motion of the other charge. Energy and momentum are communicated by the charges to the field and are transported by the field from one charge to the other in the time interval R/c. The particles and the field form a single system, and hence an exact description of the motion of a system of interacting particles cannot be given without considering the degrees of freedom of the field.

11.104

$$L = \frac{m_1v_1^2}{2} + \frac{m_1v_1^4}{8c^2} + \frac{m_2v_2^2}{2} + \frac{m_2v_2^4}{8c^2} + \frac{e_1e_2}{R} - \frac{e_1e_2}{2c^2R}[(v_1\cdot v_2) + (v_1\cdot n)(v_2\cdot n)]$$

11.105 The magnetic moment of the particle precesses about the direction of the magnetic field with an angular frequency $\omega = -\chi H$.

11.106 According to equation (10.c.3), in the instantaneously comoving frame there is a magnetic field

$$H' = -\frac{1}{c}[v \wedge E],$$

where E is the electrical field in the fixed frame, and where we have assumed that $v \ll c$. In the comoving frame the spin angular momentum changes according to the law

$$\left(\frac{ds}{dt}\right)_{\text{comoving}} = [m \wedge H'].$$

If we use the equations given in the statement of the problem we find

$$\left(\frac{ds}{dt}\right)_{\text{fixed}} = \left[m \wedge \left(H' - \frac{mc}{e}\omega_T\right)\right]$$

On comparing this equation with equation (6.c.1) we see that, in the case considered, H_{eff} has the form

$$H_{\text{eff}} = H' - \frac{mc}{e}\omega_T.$$

However,

$$\dot{v} = \frac{e}{m}E\,, \qquad E = -\frac{d\varphi}{dr}\frac{r}{r}\,, \quad \text{and} \quad H_{\text{eff}} = -\frac{1}{2mc}\frac{1}{r}\frac{d\varphi}{dr}l\,,$$

where l is the orbital angular momentum of the particle. The energy of the interaction between the magnetic moment and the effective field has the usual form

$$U = -(m \cdot H_{\text{eff}})\,,$$

and if we take the derivative of this expression with respect to the angle which gives the orientation of m we can find the generalised force acting upon the magnetic moment. Finally we get

$$U = \frac{e}{2m^2c^2}\frac{1}{r}\frac{d\varphi}{dr}(l \cdot s)\,.$$

This expression is used in the quantum theory of atoms and is called the spin-orbit interaction energy.

11.107 The interaction energy arises only if we take the Thomas precession into account, and it has the form

$$U = -\frac{1}{2m^2c^2}\frac{1}{r}\frac{dV}{dr}(l \cdot s)\,. \tag{11.107.1}$$

The situation considered in this problem is approximately realised in atomic nuclei. On the nucleons in a nucleus there act strong nonelectrical (nuclear) forces and relatively weak electrostatic forces which can be neglected. The spin-orbit interaction energy is thus given by equation (11.107.1), where V is the potential of the nuclear forces. When calculating the nuclear energy levels it is very important to take the spin-orbit interaction into account.

11.108 A reflection will occur for an antiparallel orientation of the magnetic moment and the field, provided the glancing angle α is such that $\sin\alpha \leqslant (m_0H/T)^{1/2}$.

11.109 The motion of the neutron along the conductor is uniform. The motion in the plane perpendicular to the conductor occurs in the potential $U = \pm 2m_0J/cr$. It follows that the projections of the neutron trajectories on this plane are similar to the trajectories of the uniform motion of two charges e and e' which interact in accordance with Coulomb's law (see problem 11.97), provided ee' is replaced by $\pm 2m_0J/c$ and $\mathcal{E} = \frac{1}{2}m\dot{r}^2 + K^2/2mr^2 + U(r)$ is regarded as the energy of the transverse motion ($K = mr^2\dot{\phi}$ is the angular momentum). In particular, when $\mathcal{E} < 0$ the neutron executes a finite motion about the conductor.

11.110

$$l(\phi) = \frac{m_0J}{cmv_0^2\sin^2\frac{1}{2}\phi}\,.$$

Emission of electromagnetic waves

a The Hertz vector and the multipole expansion

12.3

$$\nabla^2\varphi = -4\pi\rho\,, \qquad \nabla^2 A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \frac{1}{c}\,\mathrm{grad}\frac{\partial\varphi}{\partial t} - \frac{4\pi}{c}j\,.$$

12.6 We first of all note that the Coulomb gauge, in contrast to the Lorentz gauge, is not a Lorentz invariant concept.

Let A'' and φ'' be the potentials obtained from A and φ through the infinitesimal Lorentz transformation which transforms S into S'. In that case

$$A'' = A + \beta\varphi\,, \qquad \varphi'' = \varphi + (\beta \cdot A)\,, \tag{12.6.1}$$

where $\beta = v/c$ $(\beta = |\beta|)$. Moreover,

$$\nabla' \equiv \nabla - \frac{\beta}{c}\frac{\partial}{\partial t}\,. \tag{12.6.2}$$

Equations (12.6.1), (12.6.2), and $\mathrm{div}A = 0$ lead, to first order in β, to the result

$$(\nabla' \cdot A'') + \left(\beta \cdot \left[\frac{1}{c}\frac{\partial A''}{\partial t'} - \nabla'\varphi\right]\right) = 0\,, \tag{12.6.3}$$

which shows that, indeed, the transformed potentials do not correspond to the Coulomb gauge.

We now look for a gauge transformation

$$A'' = A' + \nabla'\zeta\,, \qquad \varphi'' = \varphi' - \frac{1}{c}\frac{\partial\zeta}{\partial t'}\,, \tag{12.6.4}$$

such that

$$(\nabla' \cdot A') = 0\,.$$

On substituting equation (12.6.4) into (12.6.3), we see that if ζ satisfies the inhomogeneous differential equation

$$\nabla'^2\zeta + \frac{2}{c}(\beta \cdot \nabla')\frac{\partial\zeta}{\partial t'} = \left(\beta \cdot \left[\nabla'\varphi' - \frac{1}{c}\frac{\partial A'}{\partial t'}\right]\right), \tag{12.6.5}$$

A' corresponds to the Coulomb gauge in S'.

12.7 Clearly ζ will be of the form

$$\zeta = \zeta_0 \exp[\mathrm{i}(k \cdot r) - \mathrm{i}\omega t]\,,$$

and by using equation (12.6.5) from the solution to the preceding problem we can find ζ_0. Finally, we find for the required vector potential the expression

$$A' = A - (\hat{n}\beta \cdot A)\,,$$

where $\hat{n} = k/k$.

12.9 The angular momentum flux density is given by

$$R = \frac{[n \wedge \ddot{p}](n \cdot \dot{p})}{2\pi c^3 r^2}\,.$$

In evaluating the quantity $-\mathrm{d}K/\mathrm{d}t = \int R r^2\,\mathrm{d}\Omega$ it is convenient to use the formula $\overline{n_i n_k} = \frac{1}{3}\delta_{ik}$ (see chapter 1). The result is

$$-\frac{\mathrm{d}K(t)}{\mathrm{d}t} = \frac{2}{3c^2}[\dot{p} \wedge \ddot{p}]\Big|_{t' = t - r/c}\,.$$

12.10 The magnetic lines of force are in the form of circles whose planes are perpendicular to the z-axis, and whose centres lie on this axis. The electric lines of force are described by the following equations:

$$C_1 = \sin^2\vartheta\left[\frac{1}{r}\cos(kr - \omega t) + k\sin(kr - \omega t)\right], \qquad C_2 = \phi\,,$$

where C_1 and C_2 are constants.

12.11

$$H = \frac{1}{c}\frac{\partial\,\mathrm{curl}Z}{\partial t}$$

$$= ea\left[e_\vartheta\left(-\mathrm{i}\frac{\omega^2}{c^2 r} + \frac{\omega}{cr^2}\right) + e_\phi\left(\frac{\omega^2}{c^2 r} + \mathrm{i}\frac{\omega}{cr^2}\right)\cos\vartheta\right]\exp[\mathrm{i}(kr - \omega t + \phi)]\,,$$

$$E = \mathrm{curl\,curl}Z = ea\left[e_r\left(-\mathrm{i}\frac{\omega}{cr^2} + \frac{1}{r^3}\right)2\sin\vartheta + e_\vartheta\left(\frac{\omega^2}{c^2 r} + \mathrm{i}\frac{\omega}{cr^2} - \frac{1}{r^3}\right)\cos\vartheta\right.$$
$$\left. + e_\phi\left(\mathrm{i}\frac{\omega^2}{c^2 r} - \frac{\omega}{cr^2} - \frac{\mathrm{i}}{r^3}\right)\right]\exp[\mathrm{i}(kr - \omega t + \phi)]\,.$$

In the wave zone $r \gg \lambda = 2\pi c/\omega$ and the expressions for E and H assume the simpler form:

$$H = ea\frac{\omega^2}{c^2 r}(-\mathrm{i}e_\vartheta + e_\phi\cos\vartheta)\exp[\mathrm{i}(kr - \omega t + \phi)]\,,$$

$$E = ea\frac{\omega^2}{c^2 r}(e_\vartheta\cos\vartheta + \mathrm{i}e_\phi)\exp[\mathrm{i}(kr - \omega t + \phi)] = [H \wedge n]\,.$$

When the emission takes place into the upper hemisphere ($\cos\vartheta > 0$), the polarisation is in general elliptical and left handed (it becomes circular when $\vartheta = 0$). When the radiation is emitted into the lower hemisphere ($\cos\vartheta < 0$), the elliptical polarisation is right handed, and the special case of circular polarisation occurs at $\vartheta = \pi$. Waves emitted in the equatorial plane are linearly polarised. The angular distribution and the total intensity are respectively given by

$$\frac{\overline{dI}}{d\Omega} = (\overline{\gamma} \cdot nr^2) = \frac{e^2\omega^4 a^2}{8\pi c^3}(1+\cos^2\vartheta)\,, \qquad \overline{I} = \frac{2\omega^4 e^2 a^2}{3c^3}\,.$$

The above situation obtains, for example, in the case of a charge moving in a magnetic field.

12.12 $p = m = 0\,, \qquad Q \neq 0\,,$

$$H = \frac{1}{c}[\dot{A} \wedge n]$$

$$= -\frac{4ea^2\omega^3}{c^3 r}\sin\vartheta[e_\vartheta \cos(2\omega t' - 2\phi) + e_\phi \cos\vartheta \sin(2\omega t' - 2\phi)]\,.$$

The frequency of the oscillations in the charge and current distributions, and therefore the frequency of the field, is equal to twice the orbital frequency ω. In general, the polarisation is elliptical, and becomes circular as $\vartheta \to 0, \pi$ and linear when $\vartheta = \frac{1}{2}\pi$.

$$\frac{\overline{dI}}{d\Omega} = \frac{2e^2a^4\omega^6}{\pi c^5}\sin^2\vartheta(1+\cos^2\vartheta)\,, \qquad \overline{I} = \frac{32}{5}\frac{e^2a^4\omega^6}{c^5}\,.$$

When one of the charges is removed, the intensity increases by a factor of the order of $(\lambda/a)^2$, i.e. very considerably, since the condition $a/\lambda \ll 1$ is satisfied.

12.13 If the angle between the position vectors of the charges is equal to $\pi - \varphi$, then

$$\varphi = (\tfrac{12}{5})^{1/2}\frac{a\omega}{c}\,.$$

12.14 Let the dipole moment of the oscillator which leads in phase lie along the x-axis, and let the xy-plane contain the dipole moments of both oscillators. The magnetic field is then given by

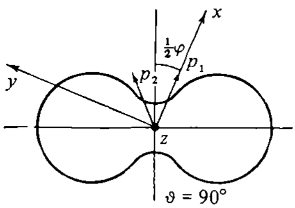
$$H(r,t) = H\exp(-i\omega t') = \frac{\omega^2 p}{c^2 r}\{e_\vartheta[\sin\phi + i\sin(\phi-\varphi)]$$

$$+ e_\phi[\cos\phi + i\cos(\phi-\varphi)]\cos\vartheta\}\exp(-i\omega t')\,,$$

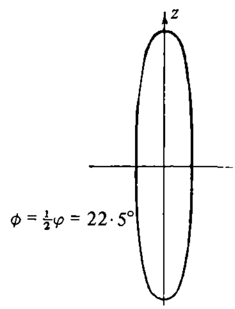
$$\frac{\overline{dI}}{d\Omega} = \frac{p^2\omega^4}{8\pi c^3}\{2 - [\cos^2\phi + \cos^2(\phi-\varphi)]\sin^2\vartheta\}\,, \qquad \overline{I} = \frac{2p^2\omega^4}{3c^3}\,,$$

where the unit vector n in the direction of propagation of the wave is defined by the polar angles $\vartheta = 0, \pi$.

The angular distribution is a maximum in the directions $\vartheta = 0$ and $\vartheta = \pi$, which are perpendicular to the dipole moments. The distribution is illustrated in figure 12.14.1 for $\varphi = 45°$. Figure 12.14.1a shows the angular (ϕ-) distribution in the plane $\vartheta = 90°$, and the angular (ϑ-) distribution in the plane $\phi = \frac{1}{2}\varphi = 22\cdot 5°$ is shown in figure 12.14.1b.



(a)



(b)

Figure 12.14.1.

12.15 When the origin of the phase is displaced by γ, the new amplitude becomes $\boldsymbol{H}\exp(-\mathrm{i}\gamma) = \boldsymbol{H}_1 - \mathrm{i}\boldsymbol{H}_2$. Since $(\boldsymbol{H}_1 \cdot \boldsymbol{H}_2) = 0$, we have

$$\tan 2\gamma = 2\frac{\sin\phi\sin(\phi-\varphi)+\cos\phi\cos(\phi-\varphi)\cos^2\vartheta}{\sin^2\phi-\sin^2(\phi-\varphi)+[\cos^2\phi-\cos^2(\phi-\varphi)]\cos^2\vartheta}\ . \tag{12.15.1}$$

$\cos\gamma$ and $\sin\gamma$ can now be determined from equation (12.15.1), and hence $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ may be found as functions of ϑ, ϕ, and φ.

When $\vartheta = 90°$, the polarisation is linear and the plane of polarisation is perpendicular to the xy-plane. When $\vartheta = 0, \pi$, the polarisation is elliptical and the ratio of semi-axes of the ellipse is equal to $\tan\frac{1}{2}\varphi$. In particular, the polarisation is circular when $\varphi = \frac{1}{2}\pi$ and $\vartheta = 0, \pi$. The values $\phi = \frac{1}{2}\varphi$, $\frac{1}{2}(\varphi \pm \pi)$, and $\frac{1}{2}\varphi + \pi$ are equally simple to discuss. In all these cases the polarisation is in general elliptical. When $\phi = \frac{1}{2}\varphi, \frac{1}{2}\varphi+\pi$, the polarisation is circular in the directions defined by $\tan\frac{1}{2}\varphi = |\cos\vartheta|$. When $\phi = \frac{1}{2}(\varphi \pm \pi)$, the directions in which the polarisation is circular are defined by $\cot\frac{1}{2}\varphi = |\cos\vartheta|$.

12.16

$$\bar{\boldsymbol{\gamma}} = \frac{e^2a^2\omega^4}{8\pi c^3r^2}(1+\cos^2\vartheta)\boldsymbol{e}_r + \frac{e^2a^2\omega^3}{4\pi c^2r^3}\sin\vartheta\boldsymbol{e}_\phi\ ,$$

$$\boldsymbol{N} = \frac{2}{3}\frac{e^2a^2\omega^3}{c^3}\boldsymbol{e}_z\ .$$

The latter result may be obtained by recalling that the rate of loss of angular momentum $\mathrm{d}\boldsymbol{K}/\mathrm{d}t = -(2/3c^3)[\dot{\boldsymbol{p}} \wedge \boldsymbol{p}]$ (see problem 12.9) is equal to the couple $\boldsymbol{N}$ on the screen. Alternatively, the couple may be computed directly from the formula

$$\boldsymbol{N} = \frac{1}{c}\int_{r \gg a}[\boldsymbol{r} \wedge \bar{\boldsymbol{\gamma}}r^2]\,\mathrm{d}\Omega\ .$$

12.17

$$H = \frac{m\omega^2 \sin\varphi}{c^2 r}(e_\vartheta \cos\vartheta + \mathrm{i}e_\phi)\exp[\mathrm{i}(kr - \omega t + \phi)] ,$$

$$E = \frac{m\omega^2 \sin\varphi}{c^2 r}(-e_\phi \cos\vartheta + \mathrm{i}e_\vartheta)\exp[\mathrm{i}(kr - \omega t + \phi)] ,$$

where

$$m = \frac{4\pi}{3}a^3 M ,$$

$$\frac{\mathrm{d}\bar{I}}{\mathrm{d}\Omega} = \frac{m^2\omega^4 \sin^2\varphi}{8\pi c^3}(1 + \cos^2\vartheta) , \qquad \bar{I} = \frac{2m^2\omega^4 \sin^2\varphi}{3c^3} .$$

12.18

$$\frac{\mathrm{d}\bar{I}}{\mathrm{d}\Omega} = \frac{9}{800\pi}\frac{\omega^6 q^2 R_0^4 a^2}{c^5}\sin^2\vartheta\cos^2\vartheta , \qquad \bar{I} = \frac{3}{500}\frac{\omega^6 q^2 R_0^4 a^2}{c^5} .$$

12.19

$$E = \frac{qr}{r^3} , \qquad H = 0 .$$

12.20 The Hertz vector $Z(r, t)$ may be resolved into monochromatic components. Using the expansion given by equation (A3.20) of appendix 3, we have

$$Z_p(r, t) = \frac{p(t')}{r} ,$$

where $t' = t - (r/c)$ and

$$Z_Q(r, t) = \frac{1}{2r^2}Q(t') + \frac{1}{2rc}\dot{Q}(t') , \tag{12.20.1}$$

$$Z_m(r, t) = \frac{[m(t') \wedge n]}{r} + \frac{c}{r^2}\left[\left(\int m(t')\,\mathrm{d}t'\right) \wedge n\right] . \tag{12.20.2}$$

These formulae will hold for $r \gg a$, where a is a linear dimension of the system. The arbitrary constant which arises as a result of the integration in equation (12.20.2) will have no effect on the magnitude of the field strength.

12.21 The field of the magnetic dipole is

$$E_m(r, t) = -\frac{1}{c}\dot{A}_m = \frac{[n \wedge \ddot{m}(t')]}{c^2 r} + \frac{[n \wedge \dot{m}(t')]}{cr^2} ,$$

$$H_m(r, t) = \mathrm{curl}A_m = \frac{3n(m \cdot n) - m}{r^3} + \frac{3n(\dot{m} \cdot n) - \dot{m}}{cr^2} + \frac{[n \wedge [n \wedge \ddot{m}]]}{c^2 r} .$$

The field of the electric dipole may be obtained from that due to the magnetic dipole by the substitution

$$\boldsymbol{m} \to \boldsymbol{p}\,, \qquad \boldsymbol{H}_m \to \boldsymbol{E}_e\,, \qquad \boldsymbol{E}_m \to -\boldsymbol{H}_e\,.$$

12.22

$$\frac{dI}{d\Omega} = \frac{\omega_0^4}{4\pi c^3}[p^2(1-\sin^2\vartheta\cos^2\phi)+m^2\sin^2\vartheta+mp\sin\vartheta\sin\phi]\,;$$

$$I = \frac{2\omega_0^4}{3c^3}(p^2+m^2)\,.$$

Here we have used a system of coordinates with the x-axis along $\boldsymbol{p}$ and the z-axis along $\boldsymbol{m}$. The dipole moments in both cases have the values

$$p = p_0\cos\omega_0 t\,, \qquad m = m_0\cos\omega_0 t\,,$$

where $p_0 = q_0 d$, $m_0 = \pi R^2 q_0\omega_0/c$, q_0 is the maximum charge of one of the capacitor plates, which is determined by the conditions under which the system is excited, d is the width of the gap, and R is the radius of the ring in case (a) and of the cylindrical shell in case (b).

Averaging the intensity of the radiation over a period of the oscillations we get

$$\frac{\overline{dI}}{d\Omega} = \frac{\omega_0^4}{8\pi c^3}\{p_0^2(1-\sin^2\vartheta\cos^2\phi)+m_0^2\sin^2\vartheta\}\,, \qquad \bar{I} = \frac{\omega_0^4}{3c^3}(p_0^2+m_0^2)\,.$$

12.23 The dipole moments of the system are zero, and the electric quadrupole moment has the single nonzero component Q_{zz} (if the z-axis is parallel to $\boldsymbol{p}_0$).

Hence, the vector $\boldsymbol{Q}$ is parallel to the z-axis and is equal to $\boldsymbol{Q}(t') = Q_0\cos\vartheta\cos\omega t'\boldsymbol{e}_z$ (when the origin of time is suitably chosen), where $Q_0 = 2p_0 a$.

It is convenient to carry out the calculation in the complex form using expression (12.20.1) of the solution of problem 12.20, and taking the components of $\boldsymbol{Z}$ along the axes of the spherical system of coordinates. Separating out the real part, we have the following final result:

$$H_\phi = \tfrac{1}{4}Q_0\sin 2\vartheta\left[\left(\frac{k^3}{r}-\frac{3k}{r^3}\right)\sin(\omega t-kr)-\frac{3k^2}{r^2}\cos(\omega t-kr)\right],$$

$$E_r = \tfrac{1}{2}Q_0(3\cos^2\vartheta-1)\left[\left(\frac{3}{r^4}-\frac{k^2}{r^2}\right)\cos(\omega t-kr)-\frac{3k}{r^3}\sin(\omega t-kr)\right],$$

$$E_\vartheta = \tfrac{1}{4}Q_0\sin 2\vartheta\left[\left(\frac{6}{r^4}-\frac{3k^2}{r^2}\right)\cos(\omega t-kr)+\left(\frac{k^3}{r}-\frac{6k}{r^3}\right)\sin(\omega t-kr)\right],$$

$$\frac{\overline{dI}}{d\Omega} = \frac{Q_0^4\omega^6}{32\pi c^5}\sin^2\vartheta\cos^2\vartheta\,, \qquad \bar{I} = \frac{Q_0^2\omega^6}{60c^5}\,,$$

where $Q_0 = 2p_0 a$.

12.24 Consider the coordinate system shown in figure 12.24.1. The current distribution in the antenna is given by

$$J = J_0 \sin k(\xi + \tfrac{1}{2}l)\exp(-\mathrm{i}\omega t)\ ,$$

where

$$k = \frac{\omega}{c} = \frac{m\pi}{l}\ .$$

According to equation (12.a.9) the electric dipole moment per unit length of the antenna is $P = (\mathrm{i}/\omega)J$. The element $\mathrm{d}\xi$ of the antenna may be looked upon as an electric dipole oscillator with a moment $\mathrm{d}p = P\,\mathrm{d}\xi$. Since $\mathrm{d}\xi \ll \lambda$, it follows that the magnetic field at the point A due to this element may be calculated from equations (12.a.17) and (12.a.20):

$$\mathrm{d}H_0(r_0, t) = -\frac{\omega^2}{c^2 r} e_\phi \sin\vartheta P\left(t - \frac{r}{c}\right)\mathrm{d}\xi\ ,$$

where

$$r = r_0 - \xi\cos\vartheta\ .$$

Since we are only interested in the field in the wave zone, the quantity $\sin\vartheta/r$, which is a slowly varying function in the region $r \gg l$, may be taken out from under the integral sign. Thus,

$$H_r = H_\vartheta = 0\ ,$$

$$H_\phi = -\frac{\mathrm{i}\omega\sin\vartheta}{c^2 r_0} J_0 \exp[\mathrm{i}(kr_0 - \omega t)]\int_{-\frac{1}{2}l}^{\frac{1}{2}l}\exp(\mathrm{i}k\xi\cos\vartheta)\sin m\pi\left(\frac{\xi}{l}+\frac{1}{2}\right)\mathrm{d}\xi\ .$$

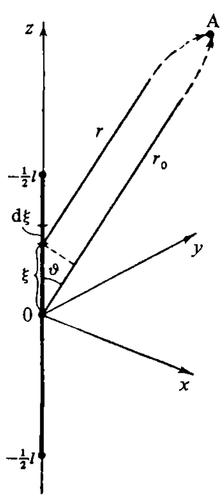


Figure 12.24.1.

The angular distribution may be found by evaluating the above integral, since $\overline{\mathrm{d}I/\mathrm{d}\Omega} = (c/4\pi)\overline{H_\phi^2}r_0^2$.

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \begin{cases} \dfrac{J_0^2}{2\pi c}\dfrac{\cos^2(\frac{1}{2}m\pi\cos\vartheta)}{\sin^2\vartheta}, & m \text{ odd;} \\[2ex] \dfrac{J_0^2}{2\pi c}\dfrac{\sin^2(\frac{1}{2}m\pi\cos\vartheta)}{\sin^2\vartheta}, & m \text{ even.} \end{cases}$$

The angular distribution is illustrated by the polar diagrams in figure 12.24.2. The dashed curves show the current distribution along the antenna; the continuous curves represent the angular distribution of the emitted radiation.

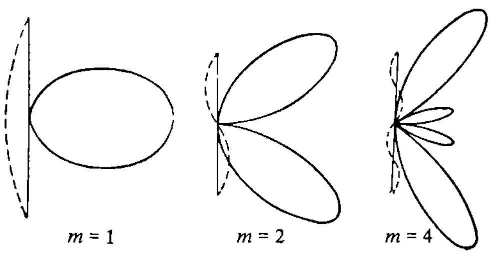


Figure 12.24.2.

12.25

$$\overline{I} = \frac{J_0^2}{2c}[\ln(2\pi m) + C - \mathrm{Ci}(2\pi m)]\,,$$

$$R = 2\frac{\overline{I}}{J_0^2} = \frac{1}{c}[\ln(2\pi m) + C - \mathrm{Ci}(2\pi m)]\,.$$

12.26

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{J_0^2}{2\pi c}\frac{\sin^2\vartheta\sin^2[\frac{1}{2}kl(1-\cos\vartheta)]}{(1-\cos\vartheta)^2}\,,$$

$$\overline{I} = \frac{J_0^2}{c}\left[C - 1 + \ln\frac{4\pi l}{\lambda} - \mathrm{Ci}\left(\frac{4\pi l}{\lambda}\right) + \frac{\sin(4\pi l/\lambda)}{4\pi l/\lambda}\right],$$

where $\lambda = 2\pi/k$ is the wavelength of the emitted wave, and ϑ is the polar angle measured from the ξ-axis.

It is easy to show that a travelling wave emits more intensely than a standing wave with the same values of l, λ, and J_0.

12.27 If the distance r of the point of observation A (r_0, ϑ, ϕ) (figure 12.27.1) from the loop is large $(r \gg a)$, then the position vectors $\boldsymbol{r}$ of all the elements of the ring, $\mathrm{d}\boldsymbol{l}$, may be regarded as parallel and

$$r = r_0 - a\cos\varphi = r_0 - a\sin\vartheta\cos(\phi' - \phi)$$

(see problem 1.1). An element $\mathbf{d}l$ has an electric dipole moment $\mathbf{d}p = P\,\mathrm{d}l = (\mathrm{i}/\omega)J\,\mathrm{d}l$, where P is the electric dipole moment per unit length of the wire which gives rise to a magnetic field at A of magnitude [see equation (12.a.20)]

$$
\begin{aligned}
\mathrm{d}H(r_0, t) &= -\frac{\omega^2}{c^2}\frac{[\mathrm{d}p(t') \wedge n]}{r} \\
&= -\mathrm{i}\frac{\omega a}{c^2}\frac{J_0}{r_0}\exp[-\mathrm{i}\omega t + \mathrm{i}kr_0 - \mathrm{i}ak\sin\vartheta\cos(\phi' - \phi)]\sin n\phi' \\
&\qquad \times [\cos(\phi' - \phi)e_\vartheta + \cos\vartheta\sin(\phi' - \phi)e_\phi]\,\mathrm{d}\phi' .
\end{aligned}
$$

In the denominator of the latter expression quantities of the order of a have been neglected in comparison with r_0. This cannot be done in the exponential, since ak is not small in general and has an important effect on the phase. The field may be found by integration:

$$
\begin{aligned}
H_\vartheta = &-\frac{\mathrm{i}\omega a}{c^2}\frac{J_0}{r_0}\exp[\mathrm{i}(kr_0 - \omega t)] \\
&\times \int_{-\pi}^{\pi} \cos(\phi' - \phi)\sin n\phi' \exp[-\mathrm{i}ka\sin\vartheta\cos(\phi' - \phi)]\,\mathrm{d}\phi' .
\end{aligned}
$$

The expression for H_ϕ may be obtained from that for H_ϑ by replacing $\cos(\phi' - \phi)$ by $\sin(\phi' - \phi)$ in the preexponential factor in the integration.

Substituting $\beta = \phi' - \phi$ we have

$$
\begin{aligned}
H_\vartheta = &-\frac{\mathrm{i}\omega a}{c^2}\frac{J_0}{r_0}\exp[\mathrm{i}(kr_0 - \omega t)] \\
&\times \left[\cos n\phi \int_{-\pi}^{\pi} \cos\beta\sin n\beta\exp(-\mathrm{i}ka\sin\vartheta\cos\beta)\,\mathrm{d}\beta \right. \\
&\left. + \sin n\phi \int_{-\pi}^{\pi} \cos\beta\cos n\beta\exp(-\mathrm{i}ka\sin\vartheta\cos\beta)\,\mathrm{d}\beta \right] .
\end{aligned}
$$

The first of the integrals in the brackets will vanish because the integrand is odd. The second integral has an even integrand so that its limits may

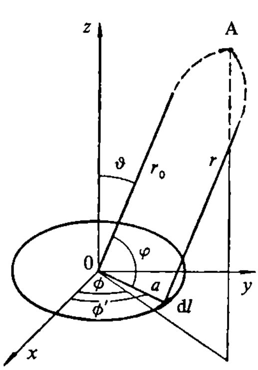


Figure 12.27.1.

be transformed to 0, π and the integral itself may be expressed in terms of the derivative of a Bessel function (see appendix 3). Thus,

$$H_\vartheta(r_0, t) = -E_\phi = \frac{2\pi\omega a}{c^2}\frac{J_0}{r_0}\exp[\mathrm{i}(kr_0 - \omega t - \tfrac{1}{2}n\pi)]\sin n\phi J_n'(ka\sin\vartheta) \,.$$

Since $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$, similar calculations will yield

$$H_\phi(r_0, t) = E_\vartheta = \frac{2\pi\omega a n J_0 \exp[\mathrm{i}(kr - \omega t - \tfrac{1}{2}n\pi)]}{c^2 r_0}\cos n\phi \frac{J_n(ka\sin\vartheta)}{ka\tan\vartheta} \,.$$

12.28

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{\omega^4 p_0^2}{2\pi c^3}(1 - \sin^2\vartheta\cos^2\phi)\cos^2\left(\frac{\pi}{2}\cos^2\tfrac{1}{2}\vartheta\right) \,,$$

where ϑ, ϕ are the polar angles characterising the direction of emission (see the polar diagrams in figure 12.28.1). The oscillator which leads in phase is in the upper position along the z-axis.

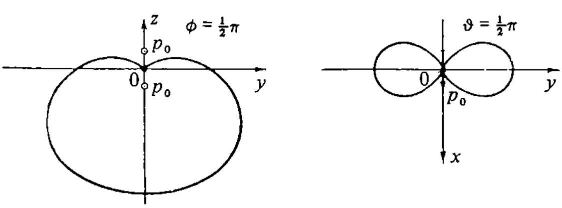


Figure 12.28.1.

12.29 Since $j = \rho v = \rho\,\mathrm{d}r/\mathrm{d}t$, it follows that $(j_x, j_y, j_z) \to (-j_x, -j_y, +j_z)$ where the reflected currents are calculated at the reflected points. $j_x(r) = -j_x'(r')$, and so on.

Similarly, from the usual definitions and equations (12.a.1) and (12.a.2) in the Cartesian form, we have

$$(p_x, p_y, p_z) \to (-p_x, -p_y, p_z)\,, \qquad (Q_x, Q_y, Q_z) \to (-Q_x, -Q_y, -Q_z)\,,$$
$$(m_x, m_y, m_z) \to (m_x, m_y, -m_z)\,, \qquad (E_x, E_y, E_z) \to (-E_x, -E_y, E_z)\,,$$
$$(H_x, H_y, H_z) \to (H_x, H_y, -H_z)\,.$$

12.30 The boundary conditions $H_n = 0$ and $E_\tau = 0$ are satisfied at the surface ($z = 0$) of the conductor: this is a direct consequence of the results of the preceding problem. In the special case of the electric dipole oscillator, the electromagnetic field in the half-space $z > 0$ is the same as the field due to an electric dipole oscillator of moment $\boldsymbol{p} = 2e_z f(t)\sin\varphi_0$. The field vanishes when $\varphi_0 = 0$ (dipole parallel to the plane) and reaches a maximum when $\varphi_0 = \frac{1}{2}\pi$ (dipole perpendicular to the plane). The total energy emitted into the half-space $z > 0$ for $\varphi_0 = \frac{1}{2}\pi$ is greater by a factor of 4 than the energy emitted by this oscillator when it is at a large distance from the conducting plane.

12.31

$$E_\vartheta = H_\phi = \frac{\omega^3 p_0 a}{2c^3 r}\cos 2\vartheta \cos\phi \cos\omega t' ,$$

$$E_\phi = -H_\vartheta = \frac{\omega^3 p_0 a}{2c^3 r}\cos\vartheta \sin\phi \cos\omega t' ,$$

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{p_0^2 a^2 \omega^6}{32\pi c^6}(\cos^2 2\vartheta \cos^2\phi + \cos^2\vartheta \sin^2\phi) .$$

12.32

$$H_r = 0 , \qquad H_\vartheta = -\frac{ik}{\sin\vartheta}\frac{\partial u}{\partial\phi} , \qquad H_\phi = \mathrm{i}k\frac{\partial u}{\partial\vartheta} ,$$

$$E_r = k^2 ur + \frac{\partial^2(ur)}{\partial r^2} , \qquad E_\vartheta = \frac{1}{r}\frac{\partial^2(ru)}{\partial r\partial\vartheta} , \qquad E_\phi = \frac{1}{r\sin\vartheta}\frac{\partial^2(ru)}{\partial r\partial\phi} .$$

12.34

$$u = \frac{p_0}{b}\frac{\exp(\mathrm{i}kR)}{R}$$
$$-\mathrm{i}kp_0\sum_{l=0}^{\infty}(2l+1)\frac{h_l^{(1)}(kb)}{b}\left.\frac{\mathrm{d}[rj_l(kr)]/\mathrm{d}r}{\mathrm{d}[rh_l^{(1)}(kr)]/\mathrm{d}r}\right|_{r=a} h_l^{(1)}(kr)P_l(\cos\vartheta) .$$

The fields $\boldsymbol{E}$ and $\boldsymbol{H}$ are given by the formulae obtained in the solution of problem 12.32. In order to determine the angular distribution it is necessary to use the asymptotic expressions for the spherical Hankel functions [equation (A3.19)]. This gives

$$E_\phi = H_\vartheta = 0 , \qquad H_\phi = \mathrm{i}k\frac{\partial u}{\partial\vartheta} = F(\vartheta)\frac{\exp(\mathrm{i}kr)}{r} = E_\vartheta ,$$

where

$$F(\vartheta) = \frac{p_0 k^2}{b}\sum_{l=0}^{\infty}\frac{2l+1}{\mathrm{i}^{l-1}}\left.\frac{j_l(kb)\,\mathrm{d}[rh_l^{(1)}(kr)]/\mathrm{d}r - h_l^{(1)}(kb)\,\mathrm{d}[rj_l(kr)]/\mathrm{d}r}{\mathrm{d}[rh_l^{(1)}(kr)]/\mathrm{d}r}\right|_{r=a}$$
$$\times\frac{\mathrm{d}P_l(\cos\vartheta)}{\mathrm{d}\vartheta} ,$$

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{c}{8\pi}|H_\phi|^2 r^2 = \frac{c}{8\pi}|F(\vartheta)|^2 .$$

b The electromagnetic field of a moving point charge

12.35 The potential φ for the field of the particle is given by

$$\varphi(\boldsymbol{r}, t) = \int\frac{\rho(\boldsymbol{r}', t-R/c)}{R}\mathrm{d}^3\boldsymbol{r}' = e\int\frac{\delta[\boldsymbol{r}'-\boldsymbol{r}_0(t-R/c)]}{R}\mathrm{d}^3\boldsymbol{r}' , \qquad (12.35.1)$$

where $R = |\boldsymbol{r}-\boldsymbol{r}'|$. This integral may be evaluated with the aid of the formula $\int f(\boldsymbol{R}_1)\delta(\boldsymbol{R}_1)\,\mathrm{d}^3\boldsymbol{R}_1 = f(0)$ (see appendix 1).

Let $R_1 = r' - r_0(t - R/c)$. The Jacobian of the transformation is

$$\left|\frac{\partial R_1}{\partial r'}\right| = 1 - \frac{(v \cdot R)}{cR} .$$

In terms of the new variables, the integral in equation (12.35.1) becomes

$$\varphi(r, t) = e\int \frac{\delta(R_1)\,d^3R_1}{R - (v \cdot R)/c} = \left.\frac{e}{R - (v \cdot R)/c}\right|_{R_1 = 0} .$$

The condition $R_1 = 0$ means that on the right-hand side of this formula all the quantities are referred to the point $r' = r_0(t - R/c)$ at which the charge was located at the retarded time $t' = t - R/c$. The vector potential may be calculated in a similar way.

12.36

$$\varphi(r, t) = \int \frac{\rho(r', t - R/c)}{R}\,d^3r' = \sum_{n=0}^{\infty} \frac{(-1)^n}{c^n n!}\frac{\partial^n}{\partial t^n}\int R^{n-1}\rho(r', t)\,d^3r'$$

$$= e\sum_{n=0}^{\infty} \frac{(-1)^n}{c^n n!}\frac{d^n R_0^{n-1}}{dt^n} ,$$

where $R_0 = |r - r_0(t)|$ and

$$A(r, t) = e\sum_{n=0}^{\infty} \frac{(-1)^n}{c^{n+1} n!}\frac{d^n[v(t)R_0^{n-1}]}{dt^n} .$$

All the quantities on the right- and left-hand sides of these equations are to be taken at the same instants of time. The retarded interaction is thus formally reduced to the instantaneous interaction. The above expansions may be used for a sufficiently slow ($v \ll c$) and continuous motion (acceleration and all its derivatives finite), provided R is not too large.

12.39 For small v/c, equation (12.b.3) becomes

$$E = \left.\frac{er}{r^3} + 3\frac{er(r \cdot v)}{cr^4} - \frac{ev}{cr^2} + \frac{e[r \wedge [r \wedge \dot{v}]]}{c^2 r^3}\right|_{t' = t - r/c} ,$$

$$H = \left.\frac{e[v \wedge r]}{cr^3} + \frac{e[\dot{v} \wedge r]}{c^2 r^2}\right|_{t' = t - r/c} ,$$

where r is the distance between the point of observation and any point in the region in which the motion of the charge takes place.

The first three terms in the expression for E and the first term in the expression for H are proportional to r^{-2} and are the leading terms at relatively small distances from the charge (quasi-stationary zone). The electric field in this zone is essentially the Coulomb field $E = er/r^3$, while the magnetic field is given by the Biot–Savart law $H = e[v \wedge r]/cr^3$. At large distances from the charge (in the wave zone), the leading terms are the last terms, which are proportional to r^{-1}. These terms represent the radiation field and are of the form

$$E = \frac{e[n \wedge [n \wedge \dot{v}]]}{c^2 r} , \qquad H = \frac{e[\dot{v} \wedge n]}{c^2 r} ,$$

where $n = r/r$. The position of the boundary between the quasi-stationary and the wave zones is defined by the condition

$$\frac{e/r_b^2}{e|\dot{v}|/c^2 r_b} \sim 1 ,$$

and hence

$$r_b \approx a\left(\frac{c^2}{v^2}\right) ,$$

since $|\dot{v}| \sim v^2/a$, where a is of the same order of magnitude as the linear dimensions of the region in which the motion of the charge takes place.

12.40

$$\frac{dI}{d\Omega} = \frac{e^2}{4\pi c^3}[\dot{v} \wedge n]^2 , \qquad I = \frac{2e^2}{3c^3}\dot{v}^2 ,$$

where $n = r/r$.

12.41 (a) The portion of energy $-d\mathscr{E}$ emitted by the particle into a solid angle $d\Omega$ in a time dt' will travel past the point of observation in a time dt. Hence,

$$-\frac{d\mathscr{E}}{dt'\,d\Omega} = \frac{dI}{d\Omega}\frac{dt}{dt'} .$$

Since $t = t' + R/c$ and $\partial R/\partial t' = -(n \cdot v)$ where $n = R/R$, we have

$$dt = dt'\left[1 - \frac{(n \cdot v)}{c}\right] ,$$

and hence, finally,

$$-\frac{d\mathscr{E}}{dt'\,d\Omega} = \left[1 - \frac{(n \cdot v)}{c}\right]\frac{dI}{d\Omega} .$$

(b) The energy emitted by the charge in a time interval dt' is localised between two spheres. The first sphere has its centre at the point O at which the charge was located at time t', while the second sphere has its centre at the point O$'$ where the charge was located at time $t' + dt'$ (figure 12.41.1). The radius of the first sphere is $R + c\,dt'$ and the radius of the second sphere is R. Consider the volume element

$$d^3r = d^2S\,dR = R^2\,d\Omega[c - (n \cdot v)]\,dt' .$$

The electromagnetic energy residing in this volume is

$$\mathrm{d}W = \frac{E^2}{4\pi}\mathrm{d}^3r = \frac{cE^2}{4\pi}\left[1-\frac{(\boldsymbol{n}\cdot\boldsymbol{v})}{c}\right]R^2\,\mathrm{d}\Omega\,\mathrm{d}t'\,.$$

Hence, the rate of loss of energy

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'\,\mathrm{d}\Omega} = \frac{\mathrm{d}W}{\mathrm{d}t'\,\mathrm{d}\Omega}$$

is given by the expression quoted above.

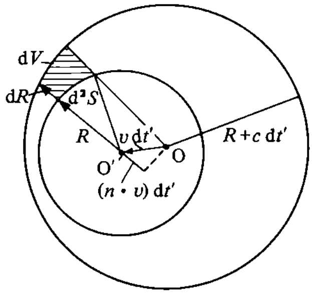


Figure 12.41.1.

12.44

(a) $$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = \frac{2e^2}{3c^3}\gamma^6\left\{\dot{v}^2 - \left[\dot{\boldsymbol{v}}\wedge\frac{\boldsymbol{v}}{c}\right]^2\right\}\,;$$

(b) $$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'} = \frac{2e^4}{3m^2c^3}\frac{\{\boldsymbol{E}+[\boldsymbol{v}\wedge\boldsymbol{H}]/c\}^2-(\boldsymbol{E}\cdot\boldsymbol{v})^2/c^2}{1-v^2/c^2}\,.$$

12.45

$$-\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t'} = -\frac{\boldsymbol{v}}{c^2}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'}\,,$$

where $\boldsymbol{v}$ is the velocity of the particle at time t'.

12.46 Consider the rate of loss of energy by the particle in two reference frames, namely, the frame S_0 in which the particle is instantaneously at rest, and the laboratory frame S in which the particle has a velocity $\boldsymbol{v}$. In S_0 the emission is of the electric dipole type, and hence there is no loss of momentum in S_0. This follows from the central symmetry of the angular distribution of the emitted radiation in this reference frame (or from the result of the preceding problem).

Consider now the amount of energy $-\mathrm{d}\mathcal{E}_0$ emitted by the particle in a time $\mathrm{d}t_0' = \mathrm{d}\tau$ in the frame S_0. The energy loss observed in S in a time interval $\mathrm{d}t' = \mathrm{d}\tau/(1-v^2/c^2)^{1/2}$ is then $-\mathrm{d}\mathcal{E} = -\mathrm{d}\mathcal{E}_0/(1-v^2/c^2)^{1/2}$.

Hence the rate of loss of energy is given by

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t'} = -\frac{\mathrm{d}\mathscr{E}_0/(1-v^2/c^2)^{1/2}}{\mathrm{d}\tau/(1-v^2/c^2)^{1/2}} = -\frac{\mathrm{d}\mathscr{E}_0}{\mathrm{d}\tau} \,.$$

This result is independent of v, which means that the rate of loss of energy integrated over all directions is the same in all reference frames.

The total radiated intensity at a time t is given by the integral of the normal component of the Poynting vector over a sphere of radius R and centred at the point occupied by the particle at the retarded time $t' = t - R/c$. In contrast to the invariant quantity $-\mathrm{d}\mathscr{E}/\mathrm{d}t'$, the total radiated intensity does not exhibit simple relativistic transformation properties.

12.47

$$\frac{\mathrm{d}I(t)}{\mathrm{d}\Omega} = \frac{c}{4\pi}E^2R^2 = \frac{e^2\dot{v}^2\sin^2\vartheta}{4\pi c^3(1-\beta\cos\vartheta)^6} \,,$$

where ϑ is the angle between the velocity $\boldsymbol{v}$ and the direction of emission $\boldsymbol{n}$, and $\beta = v/c$. The polar diagram of the radiation is shown in figure 12.47.1. At low velocities v, the intensity emitted in the forward and backward directions is the same. Emission in the forward direction predominates when v becomes comparable with c. Maximum emission is observed at an angle ϑ_0 which is given by

$$\cos\vartheta_0 = \frac{1}{4\beta}[(1+12\beta^2)^{1/2} - 1] \,.$$

When $\beta \to 0$, $\vartheta_0 \to \frac{1}{2}\pi$; when $\beta \to 1$, $\vartheta_0 \to 0$. In the ultrarelativistic limit the emission is thus mainly in directions at small angles to the particle velocity. Putting $\vartheta \ll 1$ we can write $\mathrm{d}I/\mathrm{d}\Omega$ in the form

$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2\dot{v}^2\vartheta^2}{2\pi c^3[(mc^2/\mathscr{E})^2 + \vartheta^2]^6} \,.$$

It is clear from this formula that an ultrarelativistic particle emits mainly within a cone of opening angle $\psi = mc^2/\mathscr{E}$.

The total radiation intensity is

$$I = \int \frac{\mathrm{d}I}{\mathrm{d}\Omega}\,\mathrm{d}\Omega = \frac{2e^2\dot{v}^2}{3c^3}\,\frac{1+\frac{1}{5}\beta^2}{(1-\beta^2)^4} \,.$$

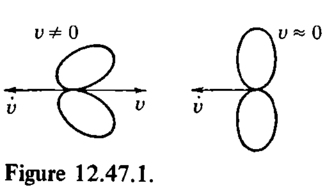


Figure 12.47.1.

The total rate of loss of energy is

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t'} = \frac{2e^2}{3c^3}\frac{\dot{v}^2}{(1-\beta^2)^3} \; .$$

12.48 The total bremsstrahlung emitted in the direction $\mathrm{d}\Omega$ over the whole period when the particle passes by is given by

$$\frac{\mathrm{d}\Delta W}{\mathrm{d}\Omega} = \int \frac{\mathrm{d}I}{\mathrm{d}\Omega}\,\mathrm{d}t = \int \left(-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}\Omega\,\mathrm{d}t'}\right)\mathrm{d}t'$$

$$= \frac{e^2 v_0^2}{16\pi c^3 \tau}\frac{\sin^2\vartheta}{\cos\vartheta}\left\{\frac{1}{[1-(v_0/c)\cos\vartheta]^4} - 1\right\} ,$$

where ϑ is the angle between the direction of motion of the particle and the direction of emission of the radiation $\boldsymbol{n}$.

The observed duration of the pulse depends on the angle ϑ between the velocity of the particle and the direction of emission:

$$\Delta t = \tau\left(1 - \frac{v_0}{2c}\cos\vartheta\right) .$$

12.49

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t'} = \frac{2e^4 H^2 p^2}{3m^4 c^5} \; .$$

12.50

$$-\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t'} = \frac{2e^4 H^2 \sin^2\theta}{3m^2 c(1-\beta^2)} \; .$$

When $\theta \gg (1-v^2/c^2)^{1/2}$ a fixed observer, far from the electron, registers separate pulses of radiation emitted at those moments when the electron velocity is directed at the observer (within a cone with opening angle

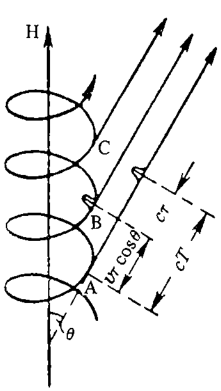


Figure 12.50.1.

$\psi \approx (1-v^2/c^2)^{1/2}$, see problem 12.47). The time between the pulses is (figure 12.50.1)

$$\tau = T\left(1 - \frac{v_\parallel \cos\theta}{c}\right) \approx T\sin^2\theta \ ,$$

where $T = 2\pi\mathcal{E}/ecH$ is the period of the cyclotron rotation, $\mathcal{E}$ is the particle energy, and $v_\parallel = v\cos\theta$ is the component of the velocity along the direction of the field. Thus, owing to the translational motion of an electron with velocity $v_\parallel$ the radiation emitted over a time T passes through a fixed sphere after a time τ. Hence

$$I = -\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t'}\frac{T}{\tau} = \frac{2e^4H^2}{3m^2c(1-v^2/c^2)} \ .$$

When $\theta \lesssim \psi \ll 1$, we have

$$I = \frac{2e^4H^2}{3m^2c(1-v^2/c^2)}\frac{\theta^2}{\frac{1}{2}(mc^2/\mathcal{E})^2 + \theta^2} \ .$$

12.51

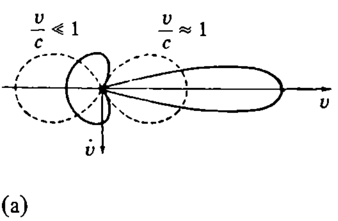
$$\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \frac{e^2|\dot{v}|^2}{4\pi c^3}\frac{(1-\beta\cos\vartheta)^2 - (1-\beta^2)\sin^2\vartheta\cos^2\phi}{(1-\beta\cos\vartheta)^6} \ ,$$

where $\beta = v/c$.

Suppose that the polar axis is in the direction of the velocity and the azimuth angle ϕ is measured from the direction of the acceleration. The angular distribution of the emitted radiation is shown in figure 12.51.1. No radiation is emitted in directions defined by

$$\gamma\left(1 - \frac{v}{c}\cos\vartheta\right) = \sin\vartheta|\cos\phi| \ .$$

In particular, when $\phi = 0, \pi$ (figure 12.51.1a) no radiation is emitted at $\vartheta = \cos^{-1}(v/c)$. When $\phi = \frac{1}{2}\pi, \frac{3}{2}\pi$ (figure 12.51.1b) the intensity is finite for all ϑ.



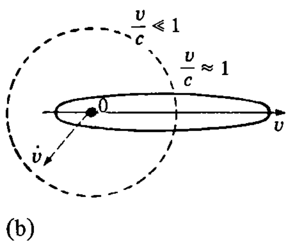


Figure 12.51.1.

12.52

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = -\frac{\overline{\mathrm{d}\mathscr{E}}}{\mathrm{d}\Omega\,\mathrm{d}t'} = \frac{e^4H^2\beta^2}{8\pi^2m^2c^3}(1-\beta^2)\int_0^{2\pi}\frac{(1-\beta^2)\cos^2\vartheta+(\beta-\sin\vartheta\cos\phi)^2}{(1-\beta\sin\vartheta\cos\phi)^5}\,\mathrm{d}\phi$$

$$= \frac{e^4H^2\beta^2(1-\beta^2)}{8\pi m^2c^3}\,\frac{1+\cos^2\vartheta-\frac{1}{4}\beta^2(1+3\beta^2)\sin^4\vartheta}{(1-\beta^2\sin^2\vartheta)^{7/2}},$$

where $\beta = v/c$.

The origin of the azimuth angle is chosen so that the direction of the vector $\boldsymbol{n}$ is characterised by the polar angles $\vartheta, \frac{1}{2}\pi$. In the ultrarelativistic case, the emission of radiation is confined to the neighbourhood of the plane of the orbit in the angular range $\Delta\vartheta \approx (1-\beta^2)^{1/2}$.

12.53

$$\left.\begin{aligned} A_{n\vartheta} &= \frac{e\beta\exp(ikR_0)}{2\pi R_0}\cos\vartheta\int_0^{2\pi}\cos\phi'\exp[\mathrm{i}(n\phi'-n\beta\sin\vartheta\sin\phi')]\,\mathrm{d}\phi',\\ A_{n\phi} &= \frac{e\beta\exp(ikR_0)}{2\pi R_0}\int_0^{2\pi}\sin\phi'\exp[\mathrm{i}(n\phi'-n\beta\sin\vartheta\sin\phi')]\,\mathrm{d}\phi',\end{aligned}\right\}$$

where $\boldsymbol{k} = (\omega/c)\boldsymbol{n}$, the origin of coordinates is at the centre of the orbit, the z-axis is perpendicular to the plane of the orbit, the direction of $\boldsymbol{k}$ is characterised by the polar angles $\vartheta, \frac{1}{2}\pi$, and R_0 is the distance from the centre of the orbit to the point of observation. Hence

$$\left.\begin{aligned} H_{n\phi} &= \mathrm{i}\frac{\omega}{c}nA_{n\vartheta} \approx \mathrm{i}\frac{\beta en\exp(ikR_0)}{aR_0}\cot\vartheta J_n(n\beta\sin\vartheta),\\ H_{n\vartheta} &= -\mathrm{i}\frac{\omega}{c}nA_{n\phi} = \frac{e\beta^2n\exp(ikR_0)}{aR_0}J'_n(n\beta\sin\vartheta).\end{aligned}\right\}\qquad(12.53.1)$$

The polarisation of the emitted radiation is, in general, elliptical. The ratio of the semi-axes of the ellipse is equal to

$$\beta\tan\vartheta\left[\frac{J'_n(n\beta\sin\vartheta)}{J_n(n\beta\sin\vartheta)}\right].$$

The sense in which the ellipse is traversed is determined by the sign of this ratio. When $\vartheta = 0$ and $\vartheta = \frac{1}{2}\pi$, the polarisation is, respectively, circular and linear. For sufficiently large n and β, the polarisation is linear in those directions for which J'_n/J_n is zero or has a pole.

12.54 The presence of higher harmonics in the spectrum of the radiated field is explained by the fact that the time taken by the field between equivalent points on the orbit is finite, and is, in general, comparable to the orbital period (provided the velocity of the charge is comparable to the velocity of light, c). Hence, the time taken by the field emitted during one-half of an orbital period to travel past the point of observation, when the particle approaches this point, is smaller than the corresponding time during the next half-period. Since the coordinates of the particle are not simple harmonic functions of time, it follows that the field is a complicated periodic function of time which can be represented as a superposition of Fourier components.

It is to be expected that when $\beta \to 0$ the higher harmonics will vanish. In fact, when $x \approx 0$, $n > 0$ we have (see appendix 3)

$$J_n(x) \approx \frac{x^n}{2^n n!} , \qquad J'_n(x) \approx \frac{x^{n-1}}{2^n (n-1)!} .$$

It is clear from these equations that when $\beta \to 0$, the important harmonics are those with $|n| = 1$. Thus (cf solution of problem 12.11),

$$H_\phi = H_{1\phi} + H_{-1\phi} = -\frac{e\beta^2}{a}\frac{\cos\vartheta \sin kR_0}{R_0} ,$$

$$H_\vartheta = H_{1\vartheta} + H_{-1\vartheta} = \frac{e\beta^2}{a}\frac{\cos kR_0}{R_0} .$$

12.55

$$\frac{\mathrm{d}I_n}{\mathrm{d}\Omega} = \frac{c}{2\pi}|H_n|^2 R_0^2 = \frac{cn^2e^2\beta^2}{2\pi a^2}[\cot^2\vartheta J_n^2(n\beta\sin\vartheta) + \beta^2 J_n'^2(n\beta\sin\vartheta)] .$$

If the circular motion occurs under the action of a constant uniform magnetic field H, then

$$a = \frac{mc^2\beta}{eH(1-\beta^2)^{1/2}} .$$

12.56 The nth harmonic of the field emitted by a single charge was obtained in the solution of problem 12.53 [equation (12.53.1)]. The expressions for the harmonics due to a number of charges will clearly differ from each other only in the initial phases. Let ψ_l be the phase difference between the field due to the lth electron relative to the electron which we shall agree to call electron number 1, so that the final field (in real form) is

$$H_{n\vartheta} = \frac{e\beta^2 n}{aR_0} J'_n(n\beta\sin\vartheta) \sum_{l=1}^{N} \cos n\left(\omega t - \frac{\omega R_0}{c} + \psi_l\right) .$$

There is an analogous expression for $H_{n\phi}$. The average intensity over the period $T = 2\pi/\omega$ is

$$\mathrm{d}I_{nN} = \frac{c}{4\pi}\frac{1}{T}\int_0^T (H_{n\vartheta}^2 + H_{n\phi}^2)\,\mathrm{d}t R_0^2\,\mathrm{d}\Omega = S_N\,\mathrm{d}I_n ,$$

where $\mathrm{d}I_n$ is the intensity due to a single electron (see preceding problem) and S_N is a factor which represents interference effects between the fields emitted by the electrons (the so-called coherence factor):

$$S_N = N + \sum_{\substack{l,l'=1\\(l\neq l')}}^{N} \cos n(\psi_l - \psi_{l'}) .$$

Consider three special cases:

(a) electrons distributed at random along the orbit:

$$\sum \cos n(\psi_l - \psi_{l'}) = 0 ;$$

(b) uniform distribution of electrons along the orbit

$$\psi_l = \frac{2\pi}{N}(l-1)$$

and

$$\begin{aligned} S_N &= N\sum_{l=2}^{N} \cos 2\pi(l-1)\frac{n}{N} \\ &= \tfrac{1}{2}N\left\{\sum_{l=1}^{N} \exp\left[2\pi \mathrm{i}(l-1)\frac{n}{N}\right] + \sum_{l=1}^{N} \exp\left[-2\pi \mathrm{i}(l-1)\frac{n}{N}\right]\right\} \\ &= N(-1)^n \frac{\sin n\pi}{\tan(n\pi/N)} = \begin{cases} 0, & \text{if } \dfrac{n}{N} \text{ is not an integer,} \\ N^2, & \text{if } \dfrac{n}{N} \text{ is an integer.} \end{cases} \end{aligned}$$

(c) if the electrons form a bunch, then all the differences $\psi_l - \psi_{l'}$ are small, provided n is not too large, so that the linear dimensions of the bunch are small compared with the corresponding wavelength. All the $\cos n(\psi_l - \psi_{l'})$ in the expression for S_N can then be replaced by 1. Hence, $S_N = N^2$. The factor S_N decreases with increasing n. The actual value of S_N depends on the precise distribution of the electrons in the bunch, and cannot be given in a general form.

12.57 Let the origin be at the centre of inertia of the system of charges. The electric dipole moment of the system is then given by

$$\boldsymbol{p} = e_1\boldsymbol{r}_1 + e_2\boldsymbol{r}_2 = \mu\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\boldsymbol{r} ,$$

where $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and $\mu = m_1 m_2/(m_1+m_2)$.

Since the ratios e/m are different, it follows that $\boldsymbol{p} \neq 0$ and the system will radiate as an electric dipole ($v/c \ll 1$). The instantaneous intensity is

$$I(t) = \frac{2\ddot{\boldsymbol{p}}^2}{3c^3} = \frac{2\mu^2}{3c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 \ddot{\boldsymbol{r}}^2(t') .$$

Since the equation of motion of the charges is $\mu\ddot{\boldsymbol{r}} = e_1 e_2 \boldsymbol{r}/r^3$,

$$I = \frac{2e_1^2e_2^2}{3c^3}\frac{(e_1/m_1 - e_2/m_2)^2}{r^4} .$$

To calculate the time average of the intensity, $\bar{I} = T^{-1}\int_0^T I\,\mathrm{d}t'$, we can replace integration with respect to t' by integration with respect to the angle ϕ by substituting $\mathrm{d}t' = (\mu r^2/K)\,\mathrm{d}\phi$ (K is the angular momentum) and using the equation of the trajectory. The result is

$$\bar{I} = \frac{2^{3/2}}{3c^3}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\frac{\mu^{5/2}|e_1e_2|^3|\mathscr{E}|^{3/2}}{K^5}\left(3-\frac{2|\mathscr{E}|K^2}{\mu e_1^2e_2^2}\right).$$

12.58

$$\overline{\frac{\mathrm{d}K}{\mathrm{d}t}} = -\frac{2^{7/2}\mu^{3/2}|\mathscr{E}|^{3/2}}{3c^3}\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\frac{K}{K^3}\ .$$

12.59 If we proceed as in the solution of problem 12.57, we can write the second derivative of the dipole moment as

$$\ddot{p} = \left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)\frac{e_1e_2 r}{r^3}\ . \tag{12.59.1}$$

The evaluation of A presents no difficulties. To determine B it is necessary to know $\ddot{p}_z$, i.e. the component of $\ddot{p}$ in the direction of the initial motion of the scattered particles as a function of the coordinates r, ϕ (the polar coordinates in the plane of the relative motion of the particles). It must be remembered that in the equation for the trajectory of relative motion $[-1+\epsilon\cos\phi = a(\epsilon^2-1)/r]$, the angle ϕ is measured from the axis of symmetry (z'-axis) of the trajectory. Thus, $y' = r\sin\phi$, $z' = r\cos\phi$. The angle between the z- and z'-axes is $\pi-\phi_0$ ($\cos\phi_0 = 1/\epsilon$), and hence

$$z = -z'\cos\phi_0 - y'\sin\phi_0 = -\frac{r}{\epsilon}[\cos\phi+(\epsilon^2-1)^{1/2}\sin\phi]\ .$$

By using equation (12.59.1), and recalling that $\sin\phi$ is an odd function, we have

$$\int_0^\infty\int_{-\infty}^{+\infty}\ddot{p}_z^2\,\mathrm{d}t\,s\,\mathrm{d}s = e_1^2e_2^2\left(\frac{e_z}{m_1}-\frac{e_2}{m_2}\right)^2\int_0^\infty\int_{-\infty}^{+\infty}\frac{\cos^2\phi+(\epsilon^2-1)\sin^2\phi}{\epsilon^2r^4}\,\mathrm{d}t\,s\,\mathrm{d}s\ .$$

The equation for the trajectory may be used to express $\cos^2\phi$ and $\sin^2\phi$ in terms of r and ϵ. On substituting $\epsilon^2 = u$, $s\,\mathrm{d}s = \tfrac{1}{2}a^2\,\mathrm{d}u$ we find that the above integral can be transformed to

$$\frac{a}{v_0}\int_{2a}^\infty\frac{\mathrm{d}r}{r^3}\int_1^{(r/a-1)^2}\left[-\frac{a^2}{r^2}u+\left(4\frac{a^2}{r^2}-2\frac{a}{r}+1\right)+\left(-5\frac{a^2}{r^2}+6\frac{a}{r}-2\right)\frac{1}{u}\right.$$
$$\left.+2\left(\frac{a}{r}-1\right)^2\frac{1}{u^2}\right]\frac{\mathrm{d}u}{[(r/a-1)^2-u]^{1/2}}\ .$$

The evaluation of the integral with respect to u leads to a logarithmic term which may be transformed by integration by parts. To evaluate the integral with respect to r it is convenient to substitute $x = 2a/r$ which converts this integral into a sum of a number of B-functions which are defined by

$$B(k, l) = \int_0^1 x^{k-1}(1-x)^{l-1}\,dx = \frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}\,.$$

The final result is

$$A = \frac{8\pi}{9}e_1e_2\left(\frac{e_1}{m_1}-\frac{e_2}{m_2}\right)^2\mu v_0\,, \qquad B = 0\,.$$

12.60 In the approximation considered here v = constant and the trajectory of the particle is a straight line. Suppose that the motion of the particle takes place in the xz-plane in the direction parallel to the z-axis. In terms of these coordinates.

$$\begin{aligned} &\boldsymbol{n} = (n_x, n_y, n_z), \quad \text{where} \quad n_x = \sin\vartheta\cos\phi, \quad n_y = \sin\vartheta\sin\phi,\\ &n_z = \cos\vartheta, \quad \boldsymbol{r} = (s, 0, vt'), \quad r = (s^2+v^2t'^2)^{1/2},\\ &\boldsymbol{v} = (0, 0, v). \end{aligned}$$

Since $\boldsymbol{v} = c^2\boldsymbol{p}/\mathscr{E}$ where $\mathscr{E} = mc^2/(1-\beta^2)^{1/2}$ and $\beta = v/c$, we have

$$\dot{\boldsymbol{v}} = \frac{c^2\dot{\boldsymbol{p}}}{\mathscr{E}} - \frac{c^2\boldsymbol{p}\dot{\mathscr{E}}}{\mathscr{E}^2}\,.$$

Moreover, from the equation of motion of the particle we have $\dot{\boldsymbol{p}} = e_1e_2\boldsymbol{r}/r^3$, and the law of conservation of energy requires that $\mathscr{E} + e_1e_2/r$ = constant. When the last equation is differentiated with respect to t' we have

$$\dot{\mathscr{E}} = \frac{e_1e_2\dot{r}}{r^2} = \frac{e_1e_2(\boldsymbol{r}\cdot\boldsymbol{v})}{r^3}\,,$$

so that

$$\dot{\boldsymbol{v}} = \frac{e_1e_2c^2}{\mathscr{E}}\left[\boldsymbol{r} - \frac{\boldsymbol{p}(\boldsymbol{r}\cdot\boldsymbol{v})}{\mathscr{E}}\right] = \frac{e_1e_2c^2}{\mathscr{E}r^3}[s\boldsymbol{e}_x + vt'(1-\beta^2)\boldsymbol{e}_z]\,.$$

Substituting these expressions into equation (12.b.4) we have

$$\frac{d\Delta W_n}{d\Omega} = \frac{e_1^4e_2^2c^4}{4\pi c^3\mathscr{E}^2(1-\beta n_z)^5}\left\{s^2[(1-\beta n_z)^2 - n_x^2(1-\beta^2)]\int_{-\infty}^{+\infty}\frac{dt'}{(s^2+v^2t'^2)}\right.$$
$$\left. + c^2\beta^2(1-\beta^2)^2(1-n_z)^2\int_{-\infty}^{\infty}\frac{t'^2\,dt'}{(s^2+v^2t'^2)^3}\right\}\,.$$

Integration yields

$$\frac{d\Delta W_n}{d\Omega} = \frac{e_1^4e_2^2(1-\beta^2)}{32m^2c^3s^3v(1-\beta n_z)^5}[4-3n_x^2-n_z^2-6\beta n_z+\beta^2(-2+3n_x^2+5n_z^2)$$
$$+\beta^4(1-n_z^2)]\,. \tag{12.60.1}$$

In the nonrelativistic limit $\beta \to 0$ and

$$\frac{d\Delta W_n}{d\Omega} = \frac{e_1^4 e_2^2}{32 m^2 c^3 s^3 v}(4 - 3n_x^2 - n_z^2) .$$

In the ultrarelativistic limit $\beta \approx 1$ and

$$\frac{d\Delta W_n}{d\Omega} = \frac{3 e_1^4 e_2^2 (1-\beta)}{2^9 m^2 c^4 s^3 \sin^4 \frac{1}{2}\vartheta} .$$

When $\vartheta \leqslant (1-\beta)^{1/2}$, the latter formula becomes incorrect, and the exact expression (12.60.1) must be employed.

12.61

$$\Delta W = \frac{\pi e_1^4 e_2^2}{12 m^2 c^3 s^3 v}\frac{4-\beta^2}{1-\beta^2}, \qquad \Delta \boldsymbol{p} = \frac{\boldsymbol{v}\Delta W}{c^2} .$$

12.62

$$\frac{d\Delta W_\omega}{d\omega} = \frac{8 e_1^4 e_2^2 \omega^2 c}{3\pi v^4}\left[K_1^2\left(\frac{\omega s}{v}\right) + K_0^2\left(\frac{\omega s}{v}\right)\right] .$$

12.63 Equation (12.b.11) will hold for all frequencies ω since the collision time $\tau = 0$. For scattering by a perfectly hard sphere, the angle of incidence equals the angle of scattering, and hence $|\boldsymbol{v}_2 - \boldsymbol{v}_1|^2 = 2v \sin\frac{1}{2}\vartheta$, where ϑ is the angle of scattering which is related to the impact parameter s by $s = a \sin\frac{1}{2}\vartheta$ when $s \leqslant a$. When $s > a$, the particle is not scattered. Hence,

$$d\kappa_\omega = \frac{2e^2}{3\pi c^3} 4v^2 \int_0^a \sin^2\tfrac{1}{2}\vartheta \, 2\pi s \, ds \, d\omega = \frac{4e^2 a^2 v^2}{3c^3} d\omega .$$

The above differential emission is independent of frequency. Hence, the total emission is

$$\kappa = \int_0^\infty d\kappa_\omega = \infty .$$

This is due to the fact that the sphere was assumed to be perfectly hard. In fact, perfectly hard spheres do not exist, $\tau \neq 0$, and for large values of ω the above expression for $d\kappa_\omega$ is invalid.

12.64 Equation (12.b.8) for the differential emission may be written in the form

$$\frac{d\kappa_n}{d\Omega} = 2\pi \int_0^\infty \int_{-\infty}^\infty \overline{\frac{dI}{d\Omega}} \, dt s \, ds . \tag{12.64.1}$$

The intensity is given by $dI/d\Omega = (c/4\pi)H^2 r^2$, where $\boldsymbol{H} = [\boldsymbol{A} \wedge \boldsymbol{n}]/c$. In equation (12.64.1), the averaging of the radiated intensity should be carried out over all directions in the plane perpendicular to the direction of the incident beam. It is convenient to write the vector product in the expression for $\boldsymbol{H}$ in the form $H_\alpha = e_{\alpha\beta\gamma}\dot{A}_\beta n_\gamma / c$ where $e_{\alpha\beta\gamma}$ is the skew-symmetric unit pseudotensor (cf problems 1.24 and 1.26), and the summation is carried out over the repeated indices. The components of the vector potential A_β may be expressed in terms of the components of the quadrupole moment $Q_{\beta\epsilon}$ which are given by equation (12.a.19):

$$A_\beta = \frac{1}{2c^2 r}\ddot{Q}_{\beta\epsilon} n_\epsilon \; .$$

Thus,

$$H_\alpha = \frac{1}{2c^3 r} e_{\alpha\beta\gamma} \dddot{Q}_{\beta\epsilon} n_\gamma n_\epsilon$$

and

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{1}{16\pi c^5} \dddot{Q}_{\beta\epsilon} \dddot{Q}_{\beta'\epsilon'} e_{\alpha\beta\gamma} e_{\alpha\beta'\gamma'} \overline{n_\gamma n_\epsilon n_{\gamma'} n_{\epsilon'}} \; .$$

Consider a polar system of coordinates with the polar axis in the direction of the incident beam, and the particle e_2, m_2 at the origin. The average should be taken for a fixed value of the component $n_z \equiv n_3 = \cos\vartheta$, where ϑ represents the direction in which the radiation is emitted. It is easy to show that

$$\left.\begin{aligned} \overline{n_i n_k} &= \tfrac{1}{2}\delta_{ik}(1-n_3^2)\,, \\ \overline{n_i n_k n_l n_m} &= \tfrac{1}{8}(\delta_{ik}\delta_{lm} + \delta_{il}\delta_{km} + \delta_{im}\delta_{kl})(1-n_3^2)^2\,, \\ \overline{n_i} &= \overline{n_i n_k n_l} = 0\,, \end{aligned}\right\} \tag{12.64.2}$$

where the subscripts i, k, l assume the values 1, 2.

Using equation (12.64.2) and the equation

$$e_{\alpha\beta\gamma} e_{\alpha\beta'\gamma'} = \delta_{\beta\beta'}\delta_{\gamma\gamma'} - \delta_{\beta\gamma'}\delta_{\gamma\beta'}\,,$$

we have

$$\begin{aligned} \frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{1}{16\pi c^5}\{ & (\dddot{Q}_{\beta 3}^2 - \dddot{Q}_{33}^2)\cos^4\vartheta + \tfrac{1}{2}(\dddot{Q}_{\beta\beta'}^2 - 3\dddot{Q}_{\beta 3}^2 + 6\dddot{Q}_{33}^2 - 2\dddot{Q}_{33}\dddot{Q}_{\beta\beta}) \\ & \times \sin^2\vartheta\cos^2\vartheta + \tfrac{1}{8}[2\dddot{Q}_{\beta\beta'}^2 - (\dddot{Q}_{\beta\beta})^2 - 3\dddot{Q}_{33}^2 + 2\dddot{Q}_{33}\dddot{Q}_{\beta\beta}]\sin^4\vartheta\}\,. \end{aligned} \tag{12.64.3}$$

Substituting equation (12.64.3) into equation (12.64.1) we have finally

$$\frac{\mathrm{d}\kappa_n}{\mathrm{d}\Omega} = A + BP_2(\cos\vartheta) + CP_4(\cos\vartheta)\,, \tag{12.64.4}$$

where P_2 and P_4 are the Legendre polynomials (see appendix 2) and

$$\left.\begin{aligned} A &= \frac{1}{120c^5}\int_{-\infty}^{\infty}\int_0^{\infty}[3\dddot{Q}_{\beta\beta'}^2 - (\dddot{Q}_{\beta\beta})^2]s\,\mathrm{d}s\,\mathrm{d}t\,, \\ B &= \frac{1}{168c^5}\int_{-\infty}^{\infty}\int_0^{\infty}[-3\dddot{Q}_{\beta\beta'}^2 + 2(\dddot{Q}_{\beta\beta})^2 + 9\dddot{Q}_{\beta 3}^2 - 6\dddot{Q}_{33}\dddot{Q}_{\beta\beta}]s\,\mathrm{d}s\,\mathrm{d}t\,, \end{aligned}\right\} \tag{12.64.5}$$

Equations (12.64.5) continued over

$$\left.\begin{aligned} C = \frac{1}{280c^5}\int_{-\infty}^{\infty}\int_0^{\infty}[-2\dddot{Q}^2_{\beta\beta'} + 2\dddot{Q}^2_{\beta 3} - (\dddot{Q}_{\beta\beta})^2 \\ -35\dddot{Q}^2_{33} + 10\dddot{Q}_{33}\dddot{Q}_{\beta\beta}]s\,\mathrm{d}s\,\mathrm{d}t\,. \end{aligned}\right\} \tag{12.64.5}$$

12.65 The total emission is

$$\kappa = \int \frac{\mathrm{d}\kappa_n}{\mathrm{d}\Omega}\,\mathrm{d}\Omega\,.$$

Using equations (12.64.4) and (12.64.5) of the preceding solution, we have (see appendix 2):

$$\kappa = 4\pi A = \frac{\pi}{30c^5}\int_{-\infty}^{\infty}\int_0^{\infty}[3\dddot{Q}^2_{\alpha\beta} - \dddot{Q}^2_{\beta\beta}]s\,\mathrm{d}s\,\mathrm{d}t\,. \tag{12.65.1}$$

Let x_α be the Cartesian components of the relative position vector of the particles and $v_\alpha = \dot{x}_\alpha$ the Cartesian components of the corresponding velocities. Hence, bearing in mind the equation of relative motion of the particles, we have

$$\ddot{x}_\alpha = \frac{2e^2x_\alpha}{mr^3}\,, \qquad \dddot{x}_\alpha = \frac{2e^2}{m}\frac{(rx_\alpha - 3x_\alpha v_r)}{r^6}\,,$$

where $v_r = \dot{r}$. Substituting these expressions into equation (12.65.1) we have

$$\kappa = \frac{4\pi e^6}{15m^2c^5}\int_{-\infty}^{\infty}\int_0^{\infty}\frac{v^2 + 11v_\phi^2}{r^4}s\,\mathrm{d}s\,\mathrm{d}t\,, \tag{12.65.2}$$

where $v^2 = v_\phi^2 + v_r^2$.

In view of the conservation of energy and angular momentum, $v^2 = v_0^2 - 4e^2/mr$ and $v_\phi = v_0 s/r$. In order to evaluate the integral in equation (12.65.2) it is convenient to substitute $\mathrm{d}t = \mathrm{d}r/v_r = \mathrm{d}r/(v^2 - v_\phi^2)^{1/2}$. The final result is

$$\kappa = \frac{4\pi}{9}\frac{e^4v_0^3}{mc^5}\,.$$

c Interaction of charged particles with radiation

12.66 The momentum of the field is

$$\boldsymbol{G} = \int \boldsymbol{g}\,\mathrm{d}^3\boldsymbol{r}\,,$$

where $\boldsymbol{g} = [\boldsymbol{E} \wedge \boldsymbol{H}]/4\pi c$ and the integral is evaluated over all space. The magnetic field of the moving particle is $\boldsymbol{H} = [\boldsymbol{v} \wedge \boldsymbol{E}]/c$, since in the rest system of the particle (S') the magnetic field is zero. Hence,

$$\boldsymbol{g} = \frac{1}{4\pi c^2}[\boldsymbol{v}E^2 - \boldsymbol{E}(\boldsymbol{v}\cdot\boldsymbol{E})]\,.$$

Using equation (12.b.3) we have

$$E_x = E'_x\,, \qquad E_y = \frac{E'_y}{(1-\beta^2)^{1/2}}\,, \qquad E_z = \frac{E'_z}{(1-\beta^2)^{1/2}}\,,$$

where the x-axis is parallel to $\boldsymbol{v}$. Moreover, in view of the Lorentz contraction, the relation between the volume elements is $\mathrm{d}^3\boldsymbol{r} = \mathrm{d}^3\boldsymbol{r}'(1-\beta^2)^{1/2}$ and hence,

$$\boldsymbol{G} = \frac{\boldsymbol{v}}{4\pi c^2(1-\beta^2)^{1/2}}\int (E_y'^2+E_z'^2)\,\mathrm{d}^3\boldsymbol{r}' = \frac{\boldsymbol{v}}{4\pi c^2(1-\beta^2)^{1/2}}\frac{2}{3}\int E'^2\,\mathrm{d}^3\boldsymbol{r}'\,. \quad (12.66.1)$$

The last transformation is a consequence of the spherical symmetry of the field in the S' system.

If it is assumed that the rest mass of the particle is of purely electromagnetic origin, i.e. it is the mass of its electric field and is given by Einstein's relation $W' = m_0c^2$, then

$$m_0 = \frac{1}{c^2}\frac{1}{8\pi}\int E'^2\,\mathrm{d}^3\boldsymbol{r}'\,. \quad (12.66.2)$$

The momentum of the field should then be equal to $m_0\boldsymbol{v}/(1-\beta^2)^{1/2}$, although it follows from equation (12.66.1) that this cannot be so[(1)]. The momentum of the field depends on the velocity $\boldsymbol{v}$ just as for a particle, i.e.

$$\boldsymbol{G} = \frac{m_0'\boldsymbol{v}}{(1-\beta^2)^{1/2}}\,.$$

However, the 'mass' is $m_0' = \frac{4}{3}m_0 \neq m_0$, i.e. it is not equal to the rest mass of the particle m_0 as given by equation (12.66.2).

The presence of the coefficient $\frac{4}{3}$ in the expression for $\boldsymbol{G}$ shows that the energy and momentum of the electromagnetic field of the particle do not form a 4-vector and cannot be identified with the energy and momentum of the particle.

It should be noted that the electromagnetic mass defined by equation (12.66.2) is infinite for a point particle.

12.67

$$W_m = \frac{1}{8\pi}\int H^2\,\mathrm{d}^3\boldsymbol{r} = \frac{1}{2}\frac{m_0'v^2}{(1-\beta^2)^{1/2}}\,,$$

where m_0' is defined in the preceding problem.

The total electromagnetic energy of the particle is

$$W = \frac{1}{8\pi}\int (E^2+H^2)\,\mathrm{d}^3\boldsymbol{r} = m_0c^2\left[\frac{1}{(1-\beta^2)^{1/2}} - \frac{1}{4}(1-\beta^2)^{1/2}\right]$$

which is different from the velocity dependence $m_0c^2/(1-\beta^2)^{1/2}$ of the energy of a particle (cf preceding problem).

(1) On this assumption the energy of the field should be equal to $m_0c^2/(1-\beta^2)^{1/2}$; it will be shown in the next problem that this is not correct.

12.68 We shall neglect terms of the order of v/c and higher, and consider the effect of an element de_1 on another element de_2. The Coulomb part of the electric field is spherically symmetric and does not contribute to the self-force. The quasi-stationary magnetic field cannot contribute to it either. Thus, it is sufficient to consider only that part of the electric field strength $\mathrm{d}\boldsymbol{E}$ of the element de_1 which depends on the acceleration. The force acting on the element de_2 is

$$\mathrm{d}\boldsymbol{F} = -\mathrm{d}e_2\,\mathrm{d}\boldsymbol{E} = \frac{\mathrm{d}e_1\,\mathrm{d}e_2}{c^2 r}[\dot{\boldsymbol{v}} - \boldsymbol{r}_0(\boldsymbol{r}_0 \cdot \dot{\boldsymbol{v}})] ,$$

where $\boldsymbol{r}_0 = \boldsymbol{r}/r$, and $\boldsymbol{r}$ is the position vector of de_2 relative to de_1. The total force acting on the particle is

$$\boldsymbol{F} = \int \mathrm{d}\boldsymbol{F} = -\tfrac{4}{3}\frac{W_0}{c^2}\dot{\boldsymbol{v}} ,$$

where $W_0 = \frac{1}{2}\int \mathrm{d}e_1\,\mathrm{d}e_2/r$ is the energy of the electromagnetic field of the particle at rest. The factor $\frac{4}{3}$ is obtained as a result of integration over the directions $\boldsymbol{r}_0$. If the rest mass of the particle is defined by $m_0' = 4W_0/3c^2$ (see problem 12.66) then the total self-force is given by

$$\boldsymbol{F} = -m_0'\dot{\boldsymbol{v}} .$$

Thus, when retarded effects are neglected, this force becomes identical with the inertial force.

12.69 The force acting on an element of charge de_2 due to an element de_1 is determined by the acceleration $\dot{\boldsymbol{v}}$ of the latter at the time t':

$$\mathrm{d}\boldsymbol{F}(t) = -\frac{\mathrm{d}e_1\,\mathrm{d}e_2}{c^2 r}[\dot{\boldsymbol{v}} - \boldsymbol{r}_0(\boldsymbol{r}_0 \cdot \dot{\boldsymbol{v}})]\,|_{t' = t - r/c} .$$

Expanding the acceleration $\dot{\boldsymbol{v}}$ in powers of $|t' - t| = r/c$ we have

$$\dot{\boldsymbol{v}}(t') = \dot{\boldsymbol{v}}(t) + (t' - t)\ddot{\boldsymbol{v}}(t) = \dot{\boldsymbol{v}}(t) - \frac{r}{c}\ddot{\boldsymbol{v}}(t) .$$

The required force is found by integrating with respect to e_1 and e_2 (see preceding solution):

$$\boldsymbol{F} = -m_0'\dot{\boldsymbol{v}} + \tfrac{2}{3}\frac{e^2}{c^3}\ddot{\boldsymbol{v}} .$$

The second term on the right-hand side is the force of radiative friction. It is independent of the structure of the particle and retains its form as the size of the particle tends to zero. The proper energy W_0, and therefore the electromagnetic mass m_0, become infinite in the limit of a point particle. The terms which have been neglected are of the order of $(t' - t)^n$, where $n \geqslant 2$, and are proportional to r_0^{n-1} and tend to zero in the limit of a point particle (r_0 is the radius of the particle).

12.70

$$T = \frac{m^2c^3a_0^3}{4e^4} = 3{\cdot}2 \times 10^{-13} \text{ s} .$$

The approximations made in the solution will be valid if the energy loss per period is small in comparison with the total energy of the electron, i.e. $\tau|d\mathscr{E}/dt| \ll |\mathscr{E}|$, and hence $ac/v \gg r_0 = e^2/mc^2$, where r_0 is the classical radius of the electron. This condition ceases to be valid at distances of the order of 10^{-15} m, when classical electrodynamics breaks down.

The very short lifetime obtained from the above calculations clearly shows that classical ideas about the motion of an electron in an atom (definite orbits and so on) are incorrect. These (and other) fundamental difficulties of classical physics have been removed by quantum mechanics.

12.71

$$\mathscr{E}(t) = mc^2 \coth\left(\frac{2e^4H^2}{3m^3c^5}t + \tfrac{1}{2}\ln\frac{\mathscr{E}_0 + mc^2}{\mathscr{E}_0 - mc^2}\right) .$$

When $t \to \infty$ we have $\mathscr{E}(t) \to mc^2$, i.e. the particle comes to rest. The radius of the orbit can be expressed in terms of $\mathscr{E}(t)$:

$$r(t) = \frac{cp}{eH} = \frac{1}{eH}[\mathscr{E}^2(t) - m^2c^4]^{1/2} .$$

When $t \to \infty$ we have $r(t) \to 0$, i.e. the particle moves on a closing spiral.

12.72

$$\mathscr{E}_{cr} = mc^2\left(\frac{3a^2\omega}{2cr_0}\right)^{1/3}$$

where $r_0 = e^2/mc^2$.

12.73 The equation of motion (including the radiative friction term) is

$$\ddot{r} + \omega_0^2 r = \tfrac{2}{3}\frac{e^2}{mc^3}\dddot{r} . \tag{12.73.1}$$

The characteristic equation corresponding to equation (12.73.1) is

$$k^2 + \omega_0^2 = \tfrac{2}{3}\frac{e^2}{mc^3}k^3 . \tag{12.73.2}$$

Since the radiative friction force is small in comparison with the quasi-elastic force, equation (12.73.2) can be solved by successive approximations. In the zero-order approximation the right-hand side is neglected and $k \approx k_0 = \pm i\omega_0$. In the first-order approximation k_0 may be substituted for k in the right-hand side of equation (12.73.2) so that $k \approx k_1 = \pm i\omega_0 - \frac{1}{2}\gamma$ where

$$\gamma = \tfrac{2}{3}\frac{e^2\omega_0^2}{mc^3} .$$

Consider the solution corresponding to the negative sign:

$$r = r_0 \exp(-\tfrac{1}{2}\gamma t)\exp(-i\omega_0 t) \qquad (t > 0)\,.$$

This solution holds when $\gamma \ll \omega_0$ and represents damped oscillations.

The energy of the oscillator falls off as the square of the modulus of its amplitude:

$$W = W_0 \exp(-\gamma t)\,.$$

The quantity $1/\gamma$ can therefore be called the lifetime of the excited state of the oscillator.

The electric field strength of the radiation is proportional to r so that

$$E = \int_{-\infty}^{+\infty} E_\omega \exp(-i\omega t)\,d\omega = \begin{cases} E_0 \exp(-i\omega_0 t)\exp(-\tfrac{1}{2}\gamma t)\,, & t > 0\,; \\ 0 \quad , & t < 0\,; \end{cases}$$

and

$$E_\omega = \frac{E}{2\pi}\int_0^\infty \exp[-(\tfrac{1}{2}\gamma + i\omega_0)t + i\omega t]\,dt = \frac{E_0}{2\pi[i(\omega-\omega_0)-\tfrac{1}{2}\gamma]}\,.$$

Hence, the spectral distribution of the emitted intensity is

$$\frac{dI_\omega}{d\omega} = \frac{I_0\gamma}{2\pi}\frac{1}{(\omega-\omega_0)^2+\tfrac{1}{4}\gamma^2}\,, \qquad (12.73.3)$$

where $I_0 = \int_{-\infty}^{+\infty} dI_\omega$ is the total intensity. The spectral distribution (12.73.3) is of the characteristic resonance form (figure 12.73.1).

The width of the spectral line is characterised by the quantity $\Delta\omega = \gamma$. The natural width is very small [in terms of wavelengths it is $\Delta\lambda = \Delta(2\pi c/\omega) = 4\pi r_0/3 = 1\cdot 17 \times 10^{-14}$ m].

If it is assumed that the radiation is emitted in discrete portions (this assumption does, of course, take us outside the framework of classical electrodynamics), then the uncertainty in the energy of the photons, $\Delta\mathcal{E} = \hbar\Delta\omega = \hbar\gamma$, is related to the lifetime of the excited state, $\tau = 1/\gamma$, by

$$(\Delta\mathcal{E})\tau = \hbar\,.$$

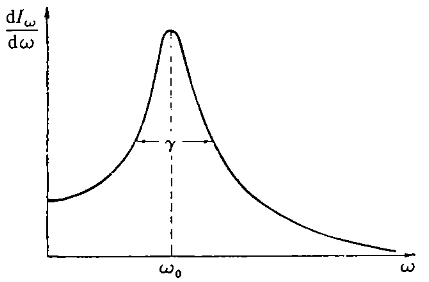


Figure 12.73.1.

This is a special case of the very general quantum mechanical uncertainty relation for energy and time.

12.74

$$\frac{dI_\omega}{d\omega} = I_0 \exp\left[-\left(\frac{\omega-\omega_0}{\gamma_D}\right)^2\right],$$

where $\gamma_D = (2kT\omega_0^2/mc^2)^{1/2}$ is the Doppler line width and I_0 is the intensity at $\omega = \omega_0$. The line width is a function of temperature and may be used as a measure of the temperature of a gas.

12.75

$$\frac{dI_\omega}{d\omega} = \frac{I\Gamma}{2\pi}\frac{1}{(\omega-\omega_0)^2+\frac{1}{4}\Gamma^2},$$

where $I = \int_{-\infty}^{+\infty} dI_\omega$.

12.76 If the wave is polarised along the x-axis then

$$x_\omega = \frac{eE_{x\omega}}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma},$$

where $\gamma = \frac{2}{3}(e^2\omega_0^2/mc^3)$.

The energy absorbed by the oscillating electron is[(2)]

$$\Delta W = \int_{-\infty}^{+\infty} eE_x(t)\dot{x}(t)\,dt = \frac{2\pi e^2}{m}\int_0^\infty |E_{x\omega}|^2 \frac{2\omega^2\gamma}{(\omega_0^2-\omega^2)^2+\omega^2\gamma^2}d\omega$$

since $(\dot{x})_\omega = -i\omega x_\omega$. The integrand in the latter expression represents the spectral distribution of the absorption. It follows that γ is a measure of the width of the absorption line, just as in the case of emission. Since, by definition, the width of the spectral distribution is large compared with the natural line width γ, it follows that

$$\Delta W = \frac{2\pi e^2}{m}|E_{x\omega_0}|^2 2\omega_0^2\gamma \int_{-\omega_0}^{\infty}\frac{d\xi}{(2\omega_0\xi)^2+\omega_0^2\gamma^2},$$

where $\xi = \omega-\omega_0$. The lower limit may be replaced by $-\infty$ since $\gamma \ll \omega_0$. The result of the integration is

$$\Delta W = \frac{2\pi^2 e^2}{m}|E_{x\omega_0}|^2 = 2\pi^2 r_0 c S_{\omega_0},$$

where $r_0 = e^2/mc^2$ is the classical radius of the electron. We see that the result is independent of γ. The dependence on frequency is

(2) It is easy to show that

$$\int_{-\infty}^{+\infty} A(t)B(t)\,dt = 2\pi\int_0^\infty (A_\omega B_\omega^* + A_\omega^* B_\omega)\,d\omega.$$

indirect: the quantity ΔW is proportional to the spectral density S_{ω_0} at the resonant frequency ω_0 of the oscillator. It is clear from the above derivation that the same result would be obtained for an unpolarised wave group incident on an anisotropic oscillator. The density S_ω would then be the sum of the intensities of all the polarised waves of frequency ω entering into this group.

12.77 (a) $\Delta W = 2\pi^2 r_0 c S_{\omega_0} \cos^2\vartheta$;
(b) $\Delta W = \pi^2 r_0 c S_{\omega_0} \sin^2\theta$;
(c) $\Delta W = \frac{4}{3}\pi^2 r_0 c S_{\omega_0}$.

12.78 The equation of motion of the harmonic oscillator is

$$\ddot{r} + \omega_0^2 r = \frac{2e^2}{3mc^3}\dddot{r} + \frac{e}{m}E_0 \exp(-\mathrm{i}\omega t) , \qquad (12.78.1)$$

where it is assumed that the electric field is uniform in the region occupied by the oscillator, and the effects of the magnetic force, which are effects of the order of v/c, can be neglected.

The solution of equation (12.78.1) which corresponds to forced oscillations is

$$r = \frac{e}{m}\frac{E}{\omega_0^2 - \omega^2 - \mathrm{i}\omega\gamma} .$$

Hence the time average of the intensity scattered in a given direction is given by

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{1}{4\pi c^3}\overline{|e[\ddot{r} \wedge n]|^2} = \frac{cE_0^2 r_0^2}{8\pi}\frac{\omega^4 \sin^2\vartheta}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} ,$$

where ϑ is the angle between the direction of propagation of the scattered radiation, n, and the direction of polarisation of the incident wave. The time average of the energy flux density in the incident wave is $\overline{\gamma}_0 = cE_0^2/8\pi$. The differential scattering cross section is given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{\overline{\gamma}_0}\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = r_0^2 \frac{\omega^4 \sin^2\vartheta}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} .$$

The total scattering cross section is therefore given by

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = \frac{8\pi}{3}r_0^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2} .$$

For a strongly bound electron, when $\omega_0 \gg \omega$,

$$\sigma = \frac{8\pi}{3}\frac{r_0^2\omega^4}{\omega_0^4} .$$

A characteristic feature of this relation is the frequency-dependence: $\sigma \propto \omega^4$.

For a weakly bound electron and small radiative friction $\gamma \approx 0$, $\omega \approx 0$, and $\sigma = 8\pi r_0^2/3$.

12.79

$$H = -\frac{Ae^2}{mc^2 r}(e_\phi \cos\vartheta - \mathrm{i}e_\vartheta)\exp[-\mathrm{i}(\omega t' - \phi)] ,$$

where ϑ, ϕ are the polar angles of the direction of propagation of the scattered wave (the incident wave is assumed to propagate in the direction of the z-axis) and A is the amplitude of the incident wave.

It is clear from the expression for $\boldsymbol{H}$ that the scattered wave will, in general, be elliptically polarised. The waves scattered in the forward and backward directions will be circularly polarised, while the waves scattered into the xy-plane will be linearly polarised. The differential and total scattering cross sections are given by

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \tfrac{1}{2}r_0^2(1+\cos^2\vartheta) , \qquad \sigma = \frac{8\pi}{3}r_0^2 .$$

12.80 $\rho = \cos^2\vartheta$.

12.81 For a linearly polarised wave

$$\mathrm{d}\sigma_{\mathrm{pol}} = r_0^2\frac{(1-\beta^2)(1-\beta)^2}{(1-\beta\cos\vartheta)^6}[(1-\beta\cos\vartheta)^2-(1-\beta^2)\sin^2\vartheta\cos^2\phi] ,$$

where ϑ, ϕ are the polar angles of the direction of propagation of the scattered wave, the z-axis is parallel to the velocity $\boldsymbol{v}$ of the charge, $\beta = v/c$, and the azimuth angle ϕ is measured from the direction of the vector $\boldsymbol{E}$ in the incident wave.

For an unpolarised wave

$$\mathrm{d}\sigma_{\mathrm{unpol}} = r_0^2\frac{(1-\beta^2)(1-\beta)^2}{(1-\beta\cos\vartheta)^6}[\tfrac{1}{2}(1+\beta^2)(1+\cos^2\vartheta)-2\beta\cos\vartheta] .$$

12.82 Consider the motion of the oscillator in a magnetic field $\boldsymbol{H}$ parallel to z and suppose that

$$\omega_0 \gg \frac{eH}{2mc} = \omega_{\mathrm{L}} .$$

Following the method employed in the solution of problem 11.79, we have

$$\boldsymbol{r} = A_1(\boldsymbol{e}_x+\mathrm{i}\boldsymbol{e}_y)\exp[-\mathrm{i}(\omega_0-\omega_{\mathrm{L}})t]+A_2(\boldsymbol{e}_x-\mathrm{i}\boldsymbol{e}_y)\exp[-\mathrm{i}(\omega_0+\omega_{\mathrm{L}})t]$$
$$+A_3\boldsymbol{e}_z\exp(-\mathrm{i}\omega t) ,$$

where A_1, A_2, and A_3 are the constants of integration which can be determined from the initial conditions.

It is clear from the expression for $\boldsymbol{r}$ that when the oscillator is placed in the magnetic field it becomes anisotropic and the frequency of its oscillations splits into three frequencies ω_0 and $\omega_0 \pm \omega_L$. When the radiation is observed in an arbitrary direction, the polarisation of each of the monochromatic components is in general found to be elliptical. In particular, two spectral lines are observed along the z-axis (along the field $\boldsymbol{H}$). These lines are circularly polarised in opposite directions. All three monochromatic components are observed in the direction perpendicular to the field and are linearly polarised. In this situation, the electric vector of the undisplaced spectral line is parallel to the magnetic field, while the electric vectors of the two displaced lines are perpendicular to this direction.

d Expansion of an electromagnetic field in terms of plane waves

12.84

$$\boldsymbol{E}_\omega(\boldsymbol{r}) = -\mathrm{grad}\,\varphi_\omega(\boldsymbol{r}) + \mathrm{i}\frac{\omega}{c}\boldsymbol{A}_\omega(\boldsymbol{r}) , \qquad \boldsymbol{H}_\omega(\boldsymbol{r}) = \mathrm{curl}\boldsymbol{A}_\omega(\boldsymbol{r}) ,$$

$$\boldsymbol{E}_{\boldsymbol{k}}(t) = -\mathrm{i}\boldsymbol{k}\varphi_{\boldsymbol{k}}(t) - \frac{1}{c}\dot{\boldsymbol{A}}_{\boldsymbol{k}} , \qquad \boldsymbol{H}_{\boldsymbol{k}}(t) = \mathrm{i}[\boldsymbol{k} \wedge \boldsymbol{A}_{\boldsymbol{k}}(t)] ,$$

$$\boldsymbol{E}_{\boldsymbol{k}\omega} = -\mathrm{i}\boldsymbol{k}\varphi_{\boldsymbol{k}\omega} + \mathrm{i}\frac{\omega}{c}\boldsymbol{A}_{\boldsymbol{k}\omega} , \qquad \boldsymbol{H}_{\boldsymbol{k}\omega} = \mathrm{i}[\boldsymbol{k} \wedge \boldsymbol{A}_{\boldsymbol{k}\omega}] .$$

12.85

(a) $$\mathrm{curl}\boldsymbol{E}_\omega = \frac{\mathrm{i}\omega\mu}{c}\boldsymbol{H}_\omega , \qquad \mathrm{div}\,\epsilon\boldsymbol{E}_\omega = 4\pi\rho_\omega ,$$

$$\mathrm{curl}\boldsymbol{H}_\omega = -\frac{\mathrm{i}\omega\epsilon}{c}\boldsymbol{E}_\omega + \frac{4\pi}{c}\boldsymbol{j}_\omega , \qquad \mathrm{div}\,\mu\boldsymbol{H}_\omega = 0 ;$$

(b) $$\mathrm{i}[\boldsymbol{k} \wedge \boldsymbol{E}_{\boldsymbol{k}}] = -\frac{1}{c}\dot{\boldsymbol{B}}_{\boldsymbol{k}} , \qquad \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{D}_{\boldsymbol{k}}) = 4\pi\rho_{\boldsymbol{k}} ,$$

$$\mathrm{i}[\boldsymbol{k} \wedge \boldsymbol{H}_{\boldsymbol{k}}] = \frac{1}{c}\dot{\boldsymbol{D}}_{\boldsymbol{k}} + \frac{4\pi}{c}\boldsymbol{j}_{\boldsymbol{k}} , \qquad (\boldsymbol{k} \cdot \boldsymbol{B}_{\boldsymbol{k}}) = 0 ;$$

(c) $$[\boldsymbol{k} \wedge \boldsymbol{E}_{\boldsymbol{k}\omega}] = \frac{\omega\mu}{c}\boldsymbol{H}_{\boldsymbol{k}\omega} , \qquad \mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{E}_{\boldsymbol{k}\omega}) = \frac{4\pi\rho_{\boldsymbol{k}\omega}}{\epsilon} ,$$

$$\mathrm{i}[\boldsymbol{k} \wedge \boldsymbol{H}_{\boldsymbol{k}\omega}] = -\frac{\mathrm{i}\omega\epsilon}{c}\boldsymbol{E}_{\boldsymbol{k}\omega} + \frac{4\pi}{c}\boldsymbol{j}_{\boldsymbol{k}\omega} , \qquad (\boldsymbol{k} \cdot \boldsymbol{H}_{\boldsymbol{k}\omega}) = 0 .$$

12.86

(a) $$\nabla^2\varphi_\omega + \frac{\epsilon\mu\omega^2}{c^2}\varphi_\omega = -\frac{4\pi\rho_\omega}{\epsilon} , \qquad \nabla^2\boldsymbol{A}_\omega + \frac{\epsilon\mu\omega^2}{c^2}\boldsymbol{A}_\omega = -\frac{4\pi\mu\boldsymbol{j}_\omega}{c} ,$$

$$\mathrm{div}\boldsymbol{A}_\omega - \frac{\mathrm{i}\omega\epsilon\mu}{c}\varphi_\omega = 0 ;$$

(b) $\epsilon\mu\ddot{\varphi}_k + k^2c^2\varphi_k = \dfrac{4\pi c^2\rho_k}{\epsilon}$, $\qquad \epsilon\mu\ddot{A}_k + k^2c^2A_k = 4\pi c\mu j_k$,

$-\mathrm{i}c(k\cdot A_k)+\epsilon\mu\dot{\varphi}_k = 0$;

(c) $\left(k^2-\dfrac{\epsilon\mu\omega^2}{c^2}\right)\varphi_{k\omega} = \dfrac{4\pi}{\epsilon}\rho_{k\omega}$, $\qquad \left(k^2-\dfrac{\epsilon\mu\omega^2}{c^2}\right)A_{k\omega} = \dfrac{4\pi\mu}{c}j_{k\omega}$,

$(k\cdot A_{k\omega})-\dfrac{\omega\epsilon\mu}{c}\varphi_{k\omega} = 0$.

12.87 Consider equation (12.d.2′). Integration with respect to the angles yields

$$\varphi_k = \frac{1}{(2\pi)^3}\int\varphi(r)\exp[-\mathrm{i}(k\cdot r)]\,\mathrm{d}^3r = \frac{e}{2\pi^2k}\int_0^\infty \sin kr\,\mathrm{d}r\,.$$

The last integral does not, in general, have a definite value, since the quantity

$$I_N = \int_0^N \sin kr\,\mathrm{d}r = \frac{1-\cos kN}{k}$$

does not tend to any particular limit as $N\to\infty$. It is easy to see, however, that the indeterminate term containing $\cos kN$ does not contribute to the potential $\varphi(r)$ when I_N is substituted into equation (12.d.2) with $N\to\infty$.

This follows from the fact that as $N\to\infty$, owing to rapid oscillations,

$$\int_0^\infty \frac{\cos kN}{k}\exp[\mathrm{i}(k\cdot r)]\,\mathrm{d}^3k \to 0\,.$$

Thus, in effect,

$$\varphi_k = \frac{e}{2\pi^2k^2}\,.$$

We note that the result $I = \lim\limits_{N\to\infty} I_N = 1/k$ may be obtained for example, if I is defined as the limit of $\displaystyle\int_0^\infty \exp(-br)\sin kr\,\mathrm{d}r$ as $b\to 0$.

The same result can also be obtained in another way. The application of the Laplace operator ∇^2 to both sides of the equation

$$\varphi(r) = \int\varphi_k\exp[\mathrm{i}(k\cdot r)]\,\mathrm{d}^3k$$

yields

$$(\nabla^2\varphi)_k = -k^2\varphi_k\,.$$

On the other hand, we may obtain the expression for the Fourier component $(\nabla^2\varphi)_k = -e/2\pi^2$ from a Fourier expansion of both sides of

Poisson's equation $\nabla^2\varphi = -4\pi e\delta(\boldsymbol{r})$. The required result is obtained by equating these two expressions for $(\nabla^2\varphi)_{\boldsymbol{k}}$.

12.88

$$\boldsymbol{E}_k = -\mathrm{i}\boldsymbol{k}\varphi_k = -\frac{\mathrm{i}e\boldsymbol{k}}{2\pi^2k^2} \, .$$

12.89 Since the volume density is $\rho(\boldsymbol{r}, t) = e\delta(\boldsymbol{r}-\boldsymbol{v}t)$ it follows that

$$\rho_{k\omega} = \frac{e}{(2\pi)^4}\int\!\!\int\delta(\boldsymbol{r}-\boldsymbol{v}t)\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})+\mathrm{i}\omega t]\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}t$$

$$= \frac{e}{(2\pi)^4}\int_{-\infty}^{+\infty}\exp\{\mathrm{i}[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]t\}\,\mathrm{d}t = \frac{e}{8\pi^3}\delta[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega] \, .$$

Hence, using the results of the preceding solution, we have

$$\varphi_{k\omega} = \frac{e}{2\pi^2}\frac{\delta[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]}{k^2-\omega^2/c^2} \, ,$$

and similarly

$$\boldsymbol{A}_{k\omega} = \frac{e\boldsymbol{v}}{2\pi^2c}\frac{\delta[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]}{k^2-\omega^2/c^2} \, .$$

Using the expressions for the component of the field strength (see solution of problem 12.84) we have

$$\boldsymbol{E}_{k\omega} = \mathrm{i}\frac{e}{2\pi^2}\frac{\delta[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]}{k^2-\omega^2/c^2}\left(-\boldsymbol{k}+\frac{\boldsymbol{v}\omega}{c^2}\right) ,$$

$$\boldsymbol{H}_{k\omega} = \mathrm{i}[\boldsymbol{k}\wedge\boldsymbol{A}_{k\omega}]$$

$$= \mathrm{i}\frac{e}{2\pi^2c}[\boldsymbol{k}\wedge\boldsymbol{v}]\frac{\delta[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]}{k^2-\omega^2/c^2} \, .$$

All the field components contain the factor $\delta[(\boldsymbol{k}\cdot\boldsymbol{v})-\omega]$ which is due to the dispersion relation $\omega = (\boldsymbol{k}\cdot\boldsymbol{v})$. Hence, all the Fourier expansions of the electromagnetic field in this case are, in fact, three rather than four-dimensional. For example, the potential is given by

$$\varphi = \int_{(k)}\int_{-\infty}^{\infty}\frac{e}{2\pi^2}\frac{\delta[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})]}{k^2-\omega^2/c^2}\exp\{\mathrm{i}[(\boldsymbol{k}\cdot\boldsymbol{r})-\omega t]\}\,\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}\omega$$

$$= \int\varphi_{\boldsymbol{k}}(t)\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})]\,\mathrm{d}^3\boldsymbol{k} \, ,$$

where

$$\varphi_{\boldsymbol{k}}(t) = \frac{e}{2\pi^2}\frac{\exp[-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{v})t]}{k^2-\omega^2/c^2} \, .$$

12.91 Consider the scalar potential. According to equations (c) of the solution of problem 12.86 ($\epsilon = \mu = 1$),

$$\varphi_{k\omega} = \frac{4\pi\rho_{k\omega}}{k^2 - \omega^2/c^2} .$$

The Fourier component of the charge density is given by

$$\begin{aligned}\rho_{k\omega} &= -\frac{1}{(2\pi)^4}\int (\boldsymbol{p}\cdot\operatorname{grad}\delta(\boldsymbol{r}-\boldsymbol{v}t))\exp\{-\mathrm{i}[(\boldsymbol{k}\cdot\boldsymbol{r})-\omega t]\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}t \\ &= \frac{1}{(2\pi)^4}\int (\boldsymbol{p}\cdot\operatorname{grad}\exp\{-\mathrm{i}[(\boldsymbol{k}\cdot\boldsymbol{r})-\omega t]\})\,\delta(\boldsymbol{r}-\boldsymbol{v}t)\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}t \\ &= -\mathrm{i}\frac{(\boldsymbol{p}\cdot\boldsymbol{k})}{(2\pi)^3}\delta[\omega-(\boldsymbol{k}\cdot\boldsymbol{v})] .\end{aligned}$$

The dispersion relation $\omega = (\boldsymbol{k}\cdot\boldsymbol{v})$ is identical with that for the field of a uniformly moving point charge (see problem 12.89). Using the method given in the solution of the preceding problem we have

$$\varphi(\boldsymbol{r}, t) = -(\boldsymbol{p}\cdot\operatorname{grad})\frac{1}{r^*} = \frac{(\boldsymbol{p}\cdot\boldsymbol{r}_0)}{r^{*3}} ,$$

where

$$\boldsymbol{r}_0 = \left(x - vt,\ \frac{y}{\gamma^2},\ \frac{z}{\gamma^2}\right), \qquad r^* = \left[(x-vt)^2 + \frac{1}{\gamma^2}(y^2+z^2)\right]^{1/2}$$

A similar calculation of the vector potential yields

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{[\boldsymbol{m}\wedge\boldsymbol{r}^*]}{r^{*3}} + \frac{\boldsymbol{v}(\boldsymbol{p}\cdot\boldsymbol{r}_0)}{cr^{*3}} .$$

12.92

(a) $\boldsymbol{A} = \dfrac{[\boldsymbol{m}_0\wedge\boldsymbol{r}^*]}{\gamma r^{*3}}, \qquad \varphi = 0 ;$

(b) $\boldsymbol{A} = \dfrac{[\boldsymbol{m}_0\wedge\boldsymbol{r}^*]}{r^{*3}}, \qquad \varphi = \dfrac{(\boldsymbol{v}\cdot\boldsymbol{A})}{c} .$

12.95 Let us expand all the vectors in Maxwell's equations into irrotational and solenoidal parts[3] (or longitudinal and transverse; see preceding problem):

$$\boldsymbol{E} = \boldsymbol{E}_\parallel + \boldsymbol{E}_\perp , \qquad \boldsymbol{j} = \boldsymbol{j}_\parallel + \boldsymbol{j}_\perp , \qquad \boldsymbol{H} = \boldsymbol{H}_\perp , \qquad \boldsymbol{H}_\parallel = 0 .$$

[3] The expansion of the electromagnetic field into longitudinal and transverse parts is used in one of the variants of quantum electrodynamics. In this expansion the transverse part of the field is quantised while the longitudinal part is not quantised: photons correspond to the former part.

Equating the transverse parts of the vectors, we have from Maxwell's equations,

$$\operatorname{curl}\boldsymbol{E}_\perp = -\frac{1}{c}\dot{\boldsymbol{H}}, \qquad \operatorname{curl}\boldsymbol{H} = \frac{1}{c}\dot{\boldsymbol{E}}_\perp + \frac{4\pi}{c}\boldsymbol{j}_\perp,$$

$$\operatorname{div}\boldsymbol{E}_\perp = 0, \qquad \operatorname{div}\boldsymbol{H} = 0.$$

The longitudinal (irrotational) part of the electric field is given by

$$\operatorname{div}\boldsymbol{E}_\parallel(\boldsymbol{r}, t) = 4\pi\rho(\boldsymbol{r}, t), \qquad \operatorname{curl}\boldsymbol{E}_\parallel(\boldsymbol{r}, t) = 0,$$

which resemble the equations of electrostatics. The time enters into them as a parameter. It follows that $\boldsymbol{E}_\parallel$ is the Coulomb field.

12.96 According to the solution of problem 12.86(b), we have

$$\ddot{q}_{k\lambda} + \omega_k^2 q_{k\lambda} = 0,$$

where $\omega_k = kc$.

This is the equation of a linear harmonic oscillator. Its general solution is

$$q_{k\lambda}(t) = a_{k\lambda}\exp(-\mathrm{i}\omega t) + b_{k\lambda}\exp(\mathrm{i}\omega t).$$

The coefficients $a_{k\lambda}$ and $b_{k\lambda}$ are related by

$$\boldsymbol{e}_{k\lambda}a_{k\lambda} = \boldsymbol{e}^*_{-k\lambda}b^*_{-k\lambda}, \qquad \boldsymbol{e}_{k\lambda}b_{k\lambda} = \boldsymbol{e}^*_{-k\lambda}a^*_{-k\lambda},$$

which follow from the fact that $A(\boldsymbol{r}, t) = A^*(\boldsymbol{r}, t)$.

If the unit vectors which describe the main states of polarisation of waves with opposite wavevectors, $\boldsymbol{k}$ and $-\boldsymbol{k}$, are such that

$$\boldsymbol{e}_{k\lambda} = \boldsymbol{e}^*_{-k\lambda},$$

then

$$\left.\begin{aligned} &a_{k\lambda} = b^*_{-k\lambda}, \qquad b_{k\lambda} = a^*_{-k\lambda}, \\ &q_{k\lambda}(t) = a_{k\lambda}\exp(-\mathrm{i}\omega t) + a^*_{-k\lambda}\exp(\mathrm{i}\omega t). \end{aligned}\right\} \tag{12.96.1}$$

The fields can be expressed in terms of the coordinates $q_{k\lambda}(t)$:

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial A}{\partial t} = -\frac{1}{\pi\sqrt{2}}\int \boldsymbol{e}_{k\lambda}q_{k\lambda}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})]\,\mathrm{d}^3k,$$

$$\boldsymbol{H} = \operatorname{curl}A = \frac{\mathrm{i}c}{\pi\sqrt{2}}\int [\boldsymbol{k}\wedge\boldsymbol{e}_{k\lambda}]q_{k\lambda}\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})]\,\mathrm{d}^3k.$$

Consider the energy of the electromagnetic field

$$W = \frac{1}{8\pi}\int (E^2 + H^2)\,\mathrm{d}^3r.$$

Since $\boldsymbol{E}$, $\boldsymbol{H}$ are real, we have

$$\frac{1}{8\pi}\int E^2\,\mathrm{d}^3\boldsymbol{r} = \frac{1}{8\pi}\int (\boldsymbol{E}\cdot\boldsymbol{E}^*)\,\mathrm{d}^3\boldsymbol{r}$$

$$= \frac{1}{2\pi^2}\frac{1}{8\pi}\int\!\!\int\!\!\int \sum_{\lambda\lambda'} [(\boldsymbol{e}_{k\lambda}\cdot\boldsymbol{e}_{k'\lambda'})\dot{q}_{k\lambda}\dot{q}^*_{k'\lambda'} \times \exp\{\mathrm{i}([\boldsymbol{k}-\boldsymbol{k}']\cdot\boldsymbol{r})\}]\,\mathrm{d}^3\boldsymbol{r}\,\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'$$

$$= \tfrac{1}{2}\int\sum_{\lambda} \dot{q}_{k\lambda}\dot{q}^*_{k\lambda}\,\mathrm{d}^3\boldsymbol{k}\,,$$

where we have used equation (A1.15) of appendix 1 and the orthogonality of the polarisation unit vectors belonging to the same $\boldsymbol{k}$ but different $\lambda[(\boldsymbol{e}_{k1}\cdot\boldsymbol{e}^*_{k2}) = 0]$. The energy of the magnetic field may be calculated in a similar way. The total energy of the electromagnetic field is thus given by

$$W = \tfrac{1}{2}\int\sum_{\lambda} (\dot{q}_{k\lambda}\dot{q}^*_{k\lambda} + \omega_k^2 q_{k\lambda}q^*_{k\lambda})\,\mathrm{d}^3\boldsymbol{k}\,. \tag{12.96.2}$$

It consists of the energies of the separate field oscillators

$$W_{k\lambda} = \tfrac{1}{2}(\dot{q}_{k\lambda}\dot{q}^*_{k\lambda} + \omega_k^2 q_{k\lambda}q^*_{k\lambda})\,.$$

The field energy (12.96.2) may be expressed directly in terms of the coefficients $a_{k\lambda}$ with the aid of equation (12.96.1):

$$W = 2\int\sum_{\lambda} \omega_k^2 a_{k\lambda}a^*_{k\lambda}\,\mathrm{d}^3\boldsymbol{k}\,. \tag{12.96.3}$$

Similarly, the momentum of the field may be shown to be

$$\boldsymbol{G} = \frac{1}{4\pi c}\int [\boldsymbol{E}\wedge\boldsymbol{H}]\,\mathrm{d}^3\boldsymbol{r} = \frac{1}{8\pi c}\int\{[\boldsymbol{E}\wedge\boldsymbol{H}^*]+[\boldsymbol{E}^*\wedge\boldsymbol{H}]\}\,\mathrm{d}^3\boldsymbol{r}$$

$$= \frac{\mathrm{i}}{2}\int\sum_{\lambda}\boldsymbol{k}(\dot{q}_{k\lambda}q^*_{k\lambda} - \dot{q}^*_{k\lambda}q_{k\lambda})\,\mathrm{d}^3\boldsymbol{k}\,.$$

The oscillator coordinates $q_{k\lambda}$ are analogous to the coordinates describing the normal oscillations of a mechanical system. The main difference is that the field is a system with an infinite number of degrees of freedom. This analogy may be employed in the application of formal methods of quantum mechanics to the solution of problems in quantum electrodynamics.

12.97

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{c}{\pi\sqrt{2}}\int\sum_{\lambda}\boldsymbol{e}_{k\lambda}\left[Q_{k\lambda}(t)\cos(\boldsymbol{k}\cdot\boldsymbol{r}) - \frac{1}{\omega}\dot{Q}_{k\lambda}(t)\sin(\boldsymbol{k}\cdot\boldsymbol{r})\right]\mathrm{d}^3\boldsymbol{k}\,,$$

$$\boldsymbol{E}(\boldsymbol{r},t) = -\frac{1}{\pi\sqrt{2}}\int\sum_{\lambda}\boldsymbol{e}_{k\lambda}[\dot{Q}_{k\lambda}\cos(\boldsymbol{k}\cdot\boldsymbol{r}) + \omega Q_{k\lambda}\sin(\boldsymbol{k}\cdot\boldsymbol{r})]\,\mathrm{d}^3\boldsymbol{k}\,,$$

$$H(r, t) = -\frac{1}{\pi\sqrt{2}}\int\sum_{\lambda}[k \wedge e_{k\lambda}]\left[Q_{k\lambda}\sin(k\cdot r)+\frac{1}{\omega}\dot{Q}_{k\lambda}\cos(k\cdot r)\right]\mathrm{d}^3k\,.$$

In deriving the expression for $E(r, t)$ we have made use of the fact that

$$\ddot{Q}_{k\lambda}+\omega_k^2 Q_{k\lambda} = 0\,.$$

The expression for the field energy can be derived most simply from equation (12.96.3) of the preceding solution by expressing the coefficients $a_{k\lambda}$ and $a^*_{k\lambda}$ in terms of $Q_{k\lambda}$ and $\dot{Q}_{k\lambda}$:

$$a_{k\lambda} = \tfrac{1}{2}Q_{k\lambda}\exp(\mathrm{i}\omega t)+\frac{\mathrm{i}}{2\omega}\dot{Q}_{k\lambda}\exp(\mathrm{i}\omega t)\,,$$

$$a^*_{k\lambda} = \tfrac{1}{2}Q_{k\lambda}\exp(-\mathrm{i}\omega t)-\frac{\mathrm{i}}{2\omega}\dot{Q}_{k\lambda}\exp(-\mathrm{i}\omega t)\,.$$

Hence,

$$W = \tfrac{1}{2}\sum_{\lambda}\int(\dot{Q}^2_{k\lambda}+\omega_k^2 Q^2_{k\lambda})\,\mathrm{d}^3k\,.$$

It is evident from the latter equation that the energy of a free electromagnetic field may be written as a sum of the energies of field oscillators, which is similar in form to the corresponding expression for a mechanical oscillating system:

$$W = \int\sum_{\lambda}W_{k\lambda}\,\mathrm{d}^3k\,,$$

where

$$W_{k\lambda} = \tfrac{1}{2}(\dot{Q}^2_{k\lambda}+\omega_k^2 Q^2_{k\lambda})\,.$$

Evaluation of the field momentum G yields

$$G = \int\sum_{\lambda}W_{k\lambda}\frac{k}{kc}\,\mathrm{d}^3k = \frac{1}{4\pi c}\int[E\wedge H]\,\mathrm{d}^3r\,.$$

The relation between the momentum of a single oscillator, $G_{k\lambda}$, and its energy is

$$G_{k\lambda} = \frac{kW_{k\lambda}}{kc}\,.$$

A similar equation relates the energy and the momentum of particles moving with the velocity of light in the direction k (photons!).

12.98 If the equations given in the solution of problem 12.86(b) are multiplied by $e^*_{k\lambda}$ then it is found that the transverse component of the potential $A_k(t)$ is given by

$$\ddot{q}_{k\lambda}(t)+\omega_k^2 q_{k\lambda}(t) = F_{k\lambda}(t)\,, \tag{12.98.1}$$

where

$$F_{k\lambda}(t) = \frac{ec}{2\pi^2}(e^*_{k\lambda} \cdot v(t)) \exp[-i(k \cdot r_0(t))] ,$$

and $r_0(t)$ is the position vector of the particle and v is its velocity at time t. In the nonrelativistic limit

$$m\ddot{r}_0 = F + eE(r_0) , \tag{12.98.2}$$

where m is the mass of the particle, F is that part of the force on the particle which is of nonelectromagnetic origin, and

$$E(r_0) = -\frac{1}{\pi\sqrt{2}}\int e_{k\lambda}\dot{q}_{k\lambda} \exp[i(k \cdot r_0)] \, d^3k$$

is the field strength at the point at which the particle is located. The force on the particle which is due to the magnetic field is neglected, since it is assumed that $v \ll c$. Equation (12.98.1) is, in fact, the equation of forced oscillations of the oscillator under the action of the external force $F_{k\lambda}(t)$. The motion of the particle and of the electromagnetic field which interact with each other is described by the system of equations (12.98.1) and (12.98.2).

12.99 The change in the energy of one oscillator is

$$\frac{dW_{k\lambda}}{dt} = \tfrac{1}{2}(F_{k\lambda}\dot{q}^*_{k\lambda} + F^*_{k\lambda}\dot{q}_{k\lambda}) .$$

The rate of change of the field energy is

$$\frac{dW}{dt} = \tfrac{1}{2}\int \sum_{\lambda} (F_{k\lambda}\dot{q}^*_{k\lambda} + F^*_{k\lambda}\dot{q}_{k\lambda}) \, d^3k .$$

12.100 The force $F_{k\lambda}(t)$ is given by

$$F_{k\lambda}(t) = b_{k\lambda} \cos\omega_0 t ,$$

where

$$b_{k\lambda} = \frac{e}{\pi\sqrt{2}}(v_0 \cdot e_{k\lambda}) , \qquad v_0 = \omega_0 r_0$$

(for the sake of simplicity the discussion is confined to linearly polarised field oscillators, so that the unit vectors $e_{k\lambda}$ are real). Integration of equation (12.98.1) yields

$$q_{k\lambda} = \frac{b_{k\lambda}}{\omega_k^2 - \omega_0^2}(\cos\omega_0 t - \cos\omega_k t) ,$$

if the oscillators are not excited at $t = 0$. Substituting this expression for $q_{k\lambda}$ into the expression for the rate of change of the field energy $dW_{k\lambda}/dt$

found in the solution of the preceding problem, we have

$$\frac{\mathrm{d}W_{k\lambda}}{\mathrm{d}t} = \frac{b_{k\lambda}^2}{\omega_k^2 - \omega_0^2}(\omega_k \cos\omega_0 t \sin\omega_k t - \omega_0 \cos\omega_0 t \sin\omega_0 t) .$$

Integration of this expression with respect to t between 0 and t yields the amount of energy transmitted by the particle to the field oscillator $(\boldsymbol{k}, \lambda)$ in a time t:

$$W_{k\lambda} = \int_0^t \frac{\mathrm{d}W_{k\lambda}}{\mathrm{d}t}\mathrm{d}t = \frac{b_{k\lambda}^2}{\omega_k^2 - \omega_0^2}\left[\tfrac{1}{2}\omega_k \frac{1-\cos(\omega_k+\omega_0)t}{\omega_k+\omega_0} + \tfrac{1}{2}\omega_k \frac{1-\cos(\omega_k-\omega_0)t}{\omega_k-\omega_0} - \tfrac{1}{4}\omega_0 \frac{1-\cos 2\omega_0 t}{\omega_0}\right] .$$

When $\omega_k = \omega_0$ and as $t \to \infty$ the second term in brackets is very large in comparison with the first and third terms. The excitation of the oscillators is therefore found to exhibit a resonance property: the oscillators whose frequency is close to the frequency of the exciting force $F_{k\lambda}$ are excited first. We shall therefore retain only the resonance term, and sum the energies received by the field oscillators whose frequencies are near ω_0, with $\boldsymbol{k}$ lying inside a solid angle $\mathrm{d}\Omega$ and the polarisation unit vector $\boldsymbol{e}_{k1}(\boldsymbol{e}_{k2})$ fixed in direction:

$$\mathrm{d}W = \sum_{k,\lambda} W_{k\lambda} = \frac{\mathrm{d}\Omega}{2c^3}\int_{\omega_0-\delta}^{\omega_0+\delta} \sum_\lambda \frac{\omega_k^3 b_{k\lambda}^2}{\omega_k+\omega_0}\frac{1-\cos(\omega_k-\omega_0)t}{(\omega_k-\omega_0)^2}\mathrm{d}\omega_k .$$

The integrand in the latter expression has a sharp maximum at $\omega_k = \omega_0$. This maximum is particularly narrow for large t. For sufficiently large t the slowly varying factor $\sum_\lambda \omega_k^3 b_{k\lambda}^2/(\omega_k+\omega_0)$ can be taken outside the integral sign and ω_k may be replaced by ω_0. In the remaining integral δ may be allowed to tend to infinity so that (see appendix 1)

$$\int_{-\infty}^{\infty} \frac{1-\cos\alpha t}{\alpha^2}\mathrm{d}\alpha = \pi t , \qquad t \to \infty .$$

The final result is therefore

$$\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \frac{\pi(b_{k1}^2 + b_{k2}^2)\omega_0^2}{2c^3}t .$$

Hence, the intensity emitted in a given direction is given by the following well-known formula

$$\frac{\overline{\mathrm{d}I}}{\mathrm{d}\Omega} = \frac{1}{t}\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \frac{e^2\omega_0^2\overline{v^2}\sin^2\vartheta}{4\pi c^3} ,$$

where $\overline{v^2} = \frac{1}{2}v_0^2$ is the mean square velocity of the oscillating particle and ϑ is the angle between $\boldsymbol{v}_0$ and $\boldsymbol{k}$. In deriving the latter formula we have used the following relation, which can easily be proved:

$$(\boldsymbol{v}_0 \cdot \boldsymbol{e}_{k1})^2 + (\boldsymbol{v}_0 \cdot \boldsymbol{e}_{k2})^2 = v_0^2 \sin^2\vartheta .$$

Integration with respect to the angles yields the total intensity:

$$\bar{I} = \frac{2e^2\omega_0^2\overline{v^2}}{3} .$$

12.102 Consider the approximate solution of equations (12.98.1) and (12.98.2) in the solution of problem 12.98. Neglect the reaction of the radiation, and substitute the field strength $\boldsymbol{E} = \boldsymbol{E}_0 \cos\omega t$ of the incident wave into equation (12.98.2). The solution corresponding to forced oscillations is

$$\boldsymbol{r}(t) = \frac{e}{m}\boldsymbol{E}_0 \frac{\cos\omega t}{\omega_0^2 - \omega^2} . \qquad (12.102.1)$$

The motion of the particle under the action of the incident wave will excite radiation field oscillators in accordance with equation (12.98.1) of problem 12.98, in which the force $F_{k\lambda}$ should be expressed in terms of $\boldsymbol{r}(t)$ so that

$$F_{k\lambda} = \frac{e^2\omega}{m\pi\sqrt{2}} \frac{(\boldsymbol{e}_{k\lambda} \cdot \boldsymbol{E}_0)}{\omega^2 - \omega_0^2} \sin\omega t .$$

The polarisation unit vectors are chosen to be real. The solution of equation (12.98.1) of problem 12.98, subject to the initial condition $q_{k\lambda}(0) = 0$, is

$$q_{k\lambda}(t) = \frac{e^2}{m\pi\sqrt{2}} \frac{\omega(\boldsymbol{E}_0 \cdot \boldsymbol{e}_{k\lambda})}{(\omega_k^2 - \omega^2)(\omega^2 - \omega_0^2)} (\sin\omega t - \sin\omega_k t) .$$

Proceeding as in the solution of problem 12.100, we obtain the following expression for the intensity in the direction $\boldsymbol{k}$, corresponding to polarisation characterised by the unit vector $\boldsymbol{e}_{k\lambda}$:

$$\frac{\mathrm{d}I_{k\lambda}}{\mathrm{d}\Omega} = \frac{1}{t}\frac{\mathrm{d}W_{k\lambda}}{\mathrm{d}\Omega} = \frac{e^4}{8\pi m^2 c^3} \frac{\omega^4(\boldsymbol{E}_0 \cdot \boldsymbol{e}_{k\lambda})^2}{(\omega^2 - \omega_0^2)^2} .$$

It follows from equation (12.102.1) that the scattered radiation is linearly polarised in the plane containing $\boldsymbol{E}_0$ and $\boldsymbol{k}$. If ϑ is the angle between $\boldsymbol{E}_0$ and $\boldsymbol{k}$, then

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{8\pi}{cE_0^2}\frac{\mathrm{d}I}{\mathrm{d}\Omega} = \left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2} \sin^2\vartheta ,$$

which is in agreement with the result of problem 12.78. Integration with respect to the angles yields the following expression for the total scattering cross section:

$$\sigma = \frac{8\pi}{3}\left(\frac{e^2}{mc^2}\right)^2 \frac{\omega^4}{(\omega_0^2 - \omega^2)^2} .$$

13

The radiation emitted during the interaction of charged particles with matter

13.1 By expressing the field vectors in the form of the Fourier integral over the space and time coordinates

$$E(R, t) = \int \mathsf{E}(k, \omega) \exp\{i[(k \cdot R) - \omega t]\} d^3k \, d\omega \ldots ,$$

we obtain the following system of algebraic equations for the Fourier amplitudes from Maxwell's equations:

$$\left.\begin{aligned} &\kappa[n \wedge \mathsf{E}(k, \omega)] = \mathsf{H}(k, \omega) , \\ &\kappa[n \wedge \mathsf{H}(k, \omega)] = -\epsilon(\omega)\mathsf{E}(k, \omega) - i\frac{ev}{2\pi^2\omega^2}\delta\left[\frac{\kappa}{c}(n \cdot v) - 1\right] , \\ &\kappa\epsilon(\omega)(n \cdot \mathsf{E}(k, \omega)) = -i\frac{ec}{2\pi^2\omega^2}\delta\left[\frac{\kappa}{c}(n \cdot v) - 1\right] , \\ &\kappa(n \cdot \mathsf{H}(k, \omega)) = 0 , \end{aligned}\right\} \qquad (13.1.1)$$

where $\mathsf{H}(k, \omega)$ is the Fourier amplitude of the magnetic field, $k = (\omega/c)\kappa n$, κ is a parameter which can be expressed in terms of ω and k, and n is a unit vector. In deriving equations (13.1.1) it must be remembered that the Fourier amplitude of the function $\delta(R - vt)$ is equal to $\delta[(k \cdot v) - \omega]/8\pi^3$ and that $\delta(\alpha x) = \delta(x)/|\alpha|$. The system (13.1.1) yields

$$\left.\begin{aligned} &\mathsf{E}(k, \omega) = -\frac{iec}{2\pi^2\omega^2}\frac{\kappa n - (v/c)\epsilon}{\epsilon(\kappa^2 - \epsilon)}\delta\left[\frac{\kappa}{c}(n \cdot v) - 1\right] , \\ &\mathsf{H}(k, \omega) = \frac{ie\kappa}{2\pi^2\omega^2}\frac{[n \wedge v]}{\kappa^2 - \epsilon}\delta\left[\frac{\kappa}{c}(n \cdot v) - 1\right] . \end{aligned}\right\} \qquad (13.1.2)$$

The fields may be determined by the inverse Fourier transformation. Consider $E_z(R, t)$ first. It follows from equations (13.1.2) that

$$\mathsf{E}_z(k, \omega) = -\frac{iec}{2\pi^2\omega^2}\frac{\kappa\cos\theta - \beta\epsilon}{\epsilon(\kappa^2 - \epsilon)}\delta(\beta\kappa\cos\theta - 1) ,$$

and hence,

$$\begin{aligned} E_z(R, t) = &-\frac{ie}{2\pi^2c^2}\int_{-\infty}^{\infty} \omega \, d\omega \exp(-i\omega t) \int_0^{\infty} \kappa^2 \, d\kappa \\ &\times \iint \left[\frac{\kappa\cos\theta - \beta\epsilon}{\epsilon(\kappa^2 - \epsilon)} \exp\left\{ i\frac{\omega}{c}\kappa[r\sin\theta\cos(\Phi - \varphi) - z\cos\theta] \right\} \right. \\ &\left. \times \, \delta(\beta\kappa\cos\theta - 1)\sin\theta \right] d\theta \, d\Phi , \end{aligned} \qquad (13.1.3)$$

where r is the component of R in the xy plane, φ is the angle between r and the x-axis, $\beta = v/c$, and θ and Φ are the polar angles of n.

The integral with respect to Φ may be expressed in terms of the Bessel function $J_0[(\omega/c)\kappa r \sin\theta]$ [see equation (A3.11) of appendix 3].

The integral with respect to θ takes the form

$$\int_0^{\pi} f(\theta)\delta(\beta\kappa\cos\theta - 1)\sin\theta\,\mathrm{d}\theta = \frac{1}{\beta\kappa}\int_{-\beta\kappa}^{\beta\kappa}\varphi(y)\delta(y-1)\,\mathrm{d}y\,. \tag{13.1.4}$$

This integral has a nonzero value only when $\beta\kappa \geqslant 1$ and hence the lower limit of κ is equal to $1/\beta$. In equation (13.1.3) this is automatically allowed for because of the presence of the δ-function. However, after integration with respect to y, the δ-function will vanish and it will be necessary to take into account explicitly the lower limit of integration. Integration of equation (13.1.4) with respect to y yields

$$\frac{1}{\beta\kappa}\varphi(1) = \frac{1}{\beta\kappa}f(\theta)\bigg|_{\cos\theta = 1/\beta\kappa}\,. \tag{13.1.5}$$

Now substitute equation (13.1.5) into (13.1.3), and change the variable so that $x = (\kappa^2 - \beta^{-2})^{1/2}$. Since κ varies between $1/\beta$ and infinity, the limit of x will be 0 and infinity. $E_z(\boldsymbol{R}, t)$ will then be of the form

$$E_z(\boldsymbol{R}, t) = \frac{\mathrm{i}e}{\pi c^2}\int_{-\infty}^{\infty}\omega\,\mathrm{d}\omega\exp\left[\mathrm{i}\omega\left(\frac{z}{v} - t\right)\right]\left(1 - \frac{1}{\beta^2\epsilon}\right)\int_0^{\infty}\frac{J_0[(\omega/c)rx]x\,\mathrm{d}x}{x^2 + 1/\beta^2 - \epsilon}\,.$$

Equation (A3.16) of appendix 3 may be used to complete the integration with respect to x:

$$E_z(\boldsymbol{R}, t) = \frac{\mathrm{i}e}{\pi c^2}\int_{-\infty}^{\infty}\left(1 - \frac{1}{\beta^2\epsilon}\right)K_0(sr)\exp\left[\mathrm{i}\omega\left(\frac{z}{v} - t\right)\right]\omega\,\mathrm{d}\omega\,, \tag{13.1.6}$$

where $s^2 = (\omega^2/v^2) - (\omega^2/c^2)\epsilon(\omega)$. The sign of s should be chosen so that $\mathrm{Re}\,s > 0$ since otherwise the integral with respect to ω will diverge. The integration with respect to ω in equation (13.1.6) can only be completed by specifying the precise form of the function $\epsilon(\omega)$.

In order to evaluate $E_x(\boldsymbol{R}, t)$, let us again begin with integration with respect to Φ.

Integration with respect to θ can be carried out with the aid of the δ-function, and the subsequent integration with respect to $x = (\kappa^2 - \beta^{-2})^{1/2}$ may be completed with the aid of the formula

$$\int_0^{\infty}\frac{J_1(xr)x^2\,\mathrm{d}x}{x^2 + k^2} = kK_1(kr)\,,$$

which is obtained in appendix 3 [equation (A3.16)] after differentiation with respect to r ($J_0' = -J_1$, $K_0' = -K_1$).

The result is

$$E_x(\boldsymbol{R}, t) = \cos\varphi\frac{e}{\pi v}\int_{-\infty}^{\infty}\frac{s}{\epsilon}K_1(sr)\exp\left[\mathrm{i}\omega\left(\frac{z}{v} - t\right)\right]\mathrm{d}\omega\,.$$

$E_y(\boldsymbol{R}, t)$ and $\boldsymbol{H}(\boldsymbol{R}, t)$ may be determined in a similar way. E_y differs from E_x in that $\cos\varphi$ is replaced by $\sin\varphi$. Hence, in cylindrical coordinates

$$E_r(\boldsymbol{R}, t) = \frac{e}{\pi v}\int_{-\infty}^{\infty}\frac{s}{\epsilon}K_1(sr)\exp\left[\mathrm{i}\omega\left(\frac{z}{v}-t\right)\right]\mathrm{d}\omega\,, \qquad E_\varphi = 0\,. \tag{13.1.7}$$

The result for $\boldsymbol{H}$ is

$$H_\varphi(\boldsymbol{R}, t) = \frac{e}{\pi c}\int_{-\infty}^{\infty}sK_1(sr)\exp\left[\mathrm{i}\omega\left(\frac{z}{v}-t\right)\right]\mathrm{d}\omega\,, \qquad H_z = H_r = 0\,. \tag{13.1.8}$$

It follows from equations (13.1.6) to (13.1.8) that the electromagnetic field is axially symmetric.

The above formulae will hold only in the region $r \gg a$, where a is of the order of the interatomic distance. When $r \ll a$, it is necessary to take into account the spatial dispersion of the permittivity.

13.2 It follows from equations (13.1.6) to (13.1.8) of the preceding solution that the monochromatic components of the fields, $\boldsymbol{E}_\omega(\boldsymbol{R}, t)$ and $\boldsymbol{H}_\omega(\boldsymbol{R}, t)$, are

$$E_{\omega z}(\boldsymbol{R}, t) = \frac{\mathrm{i}e\omega}{\pi c^2}\left(1-\frac{1}{\beta^2\epsilon}\right)K_0(sr)\exp\left[\mathrm{i}\omega\left(\frac{z}{v}-t\right)\right]\ldots\,, \tag{13.2.1}$$

where

$$s^2 = \frac{\omega^2}{v^2}-\frac{\omega^2}{c^2}\epsilon(\omega)\,, \qquad \mathrm{Re}\,s > 0\,, \tag{13.2.2}$$

and K_n are the modified Bessel functions.

In the wave zone $|sr| \gg 1$ and hence the asymptotic expression given by equation (A3.8) of appendix 3 may be used:

$$K_n(sr) = \left(\frac{\pi}{2sr}\right)^{1/2}\exp(-sr)\,. \tag{13.2.3}$$

It follows from equation (13.2.2) that when $\epsilon(\omega)$ is real s will also be real provided $\beta^{-2} > \epsilon(\omega)$ or $\beta n(\omega) < 1$ [$n(\omega)$ is the refractive index at frequency ω]. When $\beta n(\omega) > 1$, the quantity s will be purely imaginary.

For real s [$s > 0$ in view of equation (13.2.2)], the field will decay exponentially in the wave zone, and there will be no emission of radiation. For a purely imaginary s, the field amplitude in the wave zone will vary as $r^{-1/2}$, which corresponds to cylindrical waves. We shall now show that these waves will diverge, i.e. radiation will, in fact, be emitted.

Suppose that

$$s = \pm\frac{\omega}{c}\left[\frac{1}{\beta^2}-\epsilon(\omega)\right]^{1/2} = \pm\mathrm{i}\frac{\omega}{c}n(\beta^2n^2-1)^{1/2} \tag{13.2.4}$$

and consider the sign in front of the square root. The lossless dielectric under consideration is the limiting case of a weakly absorbing dielectric with a complex refractive index $n = n'+in''$. In order that the imaginary part of the refractive index, n'', should, in fact, represent the absorption of energy (so that the amplitude of the corresponding wave will decrease rather than increase), it is necessary to satisfy the condition $n'' > 0$ when $\omega > 0$ and $n'' < 0$ when $\omega < 0$. Assuming that n'' is very small, we may write

$$[\beta^2(n'+in'')^2-1]^{1/2} \approx (\beta^2 n'^2-1)^{1/2}\left(1+i\frac{\beta n'n''}{\beta^2 n'^2-1}\right) .$$

It follows that the condition $\operatorname{Re} s > 0$ will be satisfied provided the negative sign is taken in equation (13.2.4). Letting $n'' \to 0$, we have

$$s = -i\frac{\omega n}{c}(\beta^2 n^2-1)^{1/2} .$$

The negative sign corresponds to diverging waves, since the exponential factor in equation (13.2.1) will be of the form

$$\exp\{i[(\boldsymbol{k}\cdot\boldsymbol{R})-\omega t]\} = \exp\{i[k(z\cos\theta+r\sin\theta)-\omega t]\},$$

where $k = \omega n/c$, $\cos\theta = (\beta n)^{-1}$, $\sin\theta = [1-(\beta n)^{-2}]^{1/2}$, $k\cos\theta = k_z = k_\parallel$, and $k\sin\theta = k_\perp$ (the components of the wavevector). Thus, when the condition $\beta n(\omega) > 1$ is satisfied, a particle moving in the dielectric with a constant velocity $v = \beta c$ will emit electromagnetic waves of frequency ω (Cherenkov radiation).

The condition $\beta n > 1$ means that the velocity of the particle should exceed the phase velocity of the wave of frequency ω in the given medium. It follows from the expression for the wavevector $\boldsymbol{k}$ that the radiation will be emitted at an angle θ to the velocity of the particle, where

$$\cos\theta = \frac{1}{\beta n(\omega)} .$$

This characteristic directional property of the radiation is a consequence of the coherence of the waves emitted by the particle at the various points along its trajectory (see problem 13.4).

The phase velocity of Cherenkov waves

$$v_{\text{ph}} = \frac{\omega}{c} = \frac{c}{n}$$

is the same as for all the transverse electromagnetic waves. The polarisation of the radiation can easily be determined from equation (13.2.1). Thus, the vector $\boldsymbol{H}$ is perpendicular to the plane containing the trajectory of the particle and the wavevector $\boldsymbol{k}$, while the vector $\boldsymbol{E}$ lies in this plane and is perpendicular to $\boldsymbol{k}$ in the wave zone. The fact that $\boldsymbol{k}$ and $\boldsymbol{E}$ are perpendicular may be verified by evaluating the scalar product $(\boldsymbol{k}\cdot\boldsymbol{E}_\omega)$. The total energy per unit path length of the Cherenkov radiation, w_{Ch}, is equal to the time integral of the flux of the Poynting vector through an infinitely distant cylindrical surface of unit length, surrounding the trajectory of the particle:

$$w_{\mathrm{Ch}} = 2\pi r\int_{-\infty}^{\infty}\frac{c}{4\pi}[E \wedge H]_r\,\mathrm{d}t = -\frac{cr}{2}\int_{-\infty}^{\infty}H_\varphi E_z\,\mathrm{d}t\,.$$

This equation may be rewritten with the aid of the formula given in the solution to problem 12.76:

$$w_{\mathrm{Ch}} = -2\pi cr\,\mathrm{Re}\int\limits_{\beta n(\omega)>1} H^*_{\omega\varphi}E_{\omega z}\,\mathrm{d}\omega\,, \qquad (13.2.5)$$

where the monochromatic components $H_{\omega\varphi}$, $E_{\omega z}$ should be taken in the wave zone, and the integration should be carried out over the frequency region in which the radiation condition $\beta n(\omega) > 1$ is satisfied. From equations (13.2.1) to (13.2.3) we have, finally,

$$w_{\mathrm{Ch}} = \frac{e^2}{c^2}\int\limits_{\beta n(\omega)>1}\left(1-\frac{c^2}{v^2n^2}\right)\omega\,\mathrm{d}\omega\,.$$

13.3

$$w_{\mathrm{Ch}} = \frac{e^2\omega_0^2}{2v^2}(\beta^2-1)+\frac{e^2\omega_0^2}{2v^2}(\epsilon_0-1)\ln\frac{\epsilon_0}{\epsilon_0-1}\,.$$

Under the conditions given in the problem, $w_{\mathrm{Ch}} \approx 5\times10^5$ eV m^{-1}. The radiation is largely concentrated in the angular range $\theta_0 \leqslant \theta \leqslant \frac{1}{2}\pi$ where $\beta^2\epsilon_0\cos^2\theta_0 = 1$.

13.4 Each point of the trajectory may be regarded as a source of elementary excitation propagating in the form of a spherical wave with a velocity $v_{\mathrm{ph}} = c/n$ (figure 13.4.1). The resultant wave front will be the envelope of the elementary spherical waves. The normal to the wave front will be at an angle θ to the trajectory. It follows from the figure that $\cos\theta = 1/\beta n$.

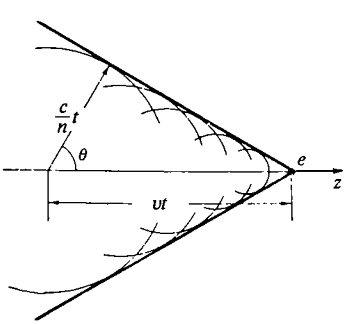


Figure 13.4.1.

13.5 The field due to a uniformly moving charged particle is a result of the superposition of plane waves with frequencies $\omega = (\boldsymbol{k} \cdot \boldsymbol{v})$, where $\boldsymbol{v}$ is the velocity of the particle and $\boldsymbol{k}$ is the wavevector (see problem 12.89). In an infinite dielectric, the possible oscillations have frequencies $\omega = kc/n$, where n is the refractive index of the medium. It follows from the resonance condition

$$\frac{kc}{n} = (\boldsymbol{k} \cdot \boldsymbol{v}) = kv\cos\theta$$

that $\cos\theta = c/vn$. Since $\cos\theta \leqslant 1$, it follows that $vn/c \geqslant 1$, which is the condition for the existence of the Cherenkov radiation.

13.6

$$\tau = \frac{l}{v}\tan^2\theta\ , \qquad I = w_{\mathrm{Ch}}\, v \cot^2\theta\ ,$$

where $\cos\theta = c/nv$, while w_{Ch} is the Cherenkov radiation energy per unit length, which was evaluated in problem 13.2.

13.8 When $\beta n < 1$, i.e. when $v < v_{\mathrm{ph}}$,

$$\varphi = \frac{e}{\epsilon[(z-vt)^2 + r^2(1-\beta^2 n^2)]^{1/2}}\ .$$

This expression may be obtained by the method used in the solution of problem 12.90.

When $\beta n > 1$, this method cannot be used, since the integrand will then have a pole at $k^2 = \epsilon\mu(\boldsymbol{k} \cdot \boldsymbol{v})^2/c^2$.

In terms of cylindrical coordinates in $\boldsymbol{k}$-space,

$$\varphi(\boldsymbol{R}, t) = \frac{e}{2\pi^2\epsilon}\int\frac{\exp[\mathrm{i}k_z(z-vt) + \mathrm{i}k_\perp r\cos\phi]}{k_\perp^2 - k_z^2(\beta^2 n^2 - 1)}k_\perp\, \mathrm{d}k_\perp\, \mathrm{d}k_z\, \mathrm{d}\phi\ .$$

The integral with respect to k_z may be evaluated by the method of residues. The denominator will have zeros at $k_z = \pm k_\perp/(\beta^2 n^2 - 1)^{1/2}$. Suppose that n has a small imaginary part $n'' > 0$ when $k_z > 0$ and $n'' < 0$ when $k_z < 0$ (see the analogous analysis in problem 13.2); here, the sign of ω is the same as the sign of k_z, since $\omega = (\boldsymbol{k} \cdot \boldsymbol{v})$. Hence, both zeros will be displaced into the lower half-plane of the complex variable k_z. When $z > vt$, the contour of integration may be completed by an arc of infinite radius in the upper half-plane (on which the integrand will be zero). Since the denominator has no zeros in the upper half-plane, the integral with respect to k_z will then vanish. When $z < vt$, the contour of integration will be closed in the lower half-plane. Both poles will then contribute to the integral and the result will be

$$\int_{-\infty}^{\infty}\frac{\exp[\mathrm{i}k_z(z-vt)]}{k_\perp^2 - k_z^2(\beta^2 n^2 - 1)}\mathrm{d}k_z = -\frac{2\pi}{k_\perp(\beta^2 n^2 - 1)^{1/2}}\sin\frac{k_\perp(z-vt)}{(\beta^2 n^2 - 1)^{1/2}}\ .$$

The integral over ϕ may be expressed in terms of the Bessel function $J_0(k_\perp r)$ [see equation (A3.11)]. Thus, when $\beta n > 1$

$$\varphi(R, t) = \begin{cases} \dfrac{2e}{\epsilon[(z-vt)^2 - r^2(\beta^2 n^2 - 1)]^{1/2}}, & \text{when } z < vt - r(\beta^2 n^2 - 1)^{1/2}, \\ 0, & \text{in the remaining space}. \end{cases} \tag{13.8.1}$$

The vector potential A may be obtained by multiplying φ by $\epsilon\mu v/c$.

Equation (13.8.1) shows that when the Cherenkov condition $\beta n > 1$ is satisfied, the field is discontinuous. It exists only within a cone whose surface is described by the equation

$$z - vt + r(\beta^2 n^2 - 1)^{1/2} = 0. \tag{13.8.2}$$

The normal to the surface of the cone is at an angle $\theta = \arccos(1/\beta n)$ to the direction of motion of the particle. It follows from equation (13.8.2) that the conical wave propagates along the z-axis with the velocity of the particle.

Electromagnetic waves are not the only waves which may have the above structure. For example, discontinuous acoustic waves of the above type may be excited by a projectile moving in air with a velocity greater than the velocity of sound (ballistic shock wave). Similar waves may be produced on the surface of water by a fast-moving vessel.

13.9 The Cherenkov radiation will be emitted provided $\beta n > 1$ where $n(\omega) = [\epsilon(\omega)\mu(\omega)]^{1/2}$; the vector potential is of the form

$$A_x = \frac{\mathrm{i}J}{c}\int \frac{\exp\{\mathrm{i}(\omega/v)[y - vt + (\beta^2 n^2 - 1)^{1/2}|z|]\}}{(\beta^2 n^2 - 1)^{1/2}} \frac{\mu(\omega)\,\mathrm{d}\omega}{\omega};$$

$$w_{\mathrm{Ch}} = \frac{2J^2}{c^2 v} \int\limits_{\beta n > 1} \frac{\mu(\omega)\,\mathrm{d}\omega}{(\beta^2 n^2 - 1)^{1/2}}.$$

The retarding force may be calculated from the formula $F = [j \wedge B]/c$ where B should be taken at the point $z = 0$, $y = vt$. The force acts in the negative direction of the y-axis, and its absolute magnitude equals the energy loss per unit path length: $F_y = -w_{\mathrm{Ch}}$. This result is a direct consequence of the law of conservation of energy.

13.10

$$w_{\mathrm{Ch}} = \frac{2e^2}{c^2} \int\limits_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right)\left(1 \pm \cos\frac{\omega l}{v}\right)\omega\,\mathrm{d}\omega.$$

The positive and negative signs correspond respectively to the two cases quoted in the problem. The spectral density of the radiation emitted by two identical charges differs from the spectral density of the radiation emitted by a single charge by a factor of $2[1 + \cos(\omega l/v)]$. Hence, the intensity of the harmonics with frequencies $\omega = (2\pi v/l)n$ $(n = 0, 1, 2, ...)$ will increase by a factor of 4, while harmonics with frequencies $\omega = (\pi v/l)(2n+1)$ will vanish. The reverse situation will occur when the charges differ in sign.

In order to obtain the energy loss for a point dipole lying along the direction of motion, the quantity $1-\cos(\omega l/v)$ should be expanded into a series. This yields

$$w_{\mathrm{Ch}} = \frac{p^2}{c^2 v^2} \int\limits_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) \omega^3 \,\mathrm{d}\omega ,$$

where p is the electric moment of the dipole in the laboratory system.

13.11

$$w_{\mathrm{Ch}} = \frac{p^2}{c^2 v^2} \int\limits_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) [\cos^2\phi + \tfrac{1}{2}\sin^2\phi(\beta^2 n^2 - 1)]\omega^3 \,\mathrm{d}\omega ,$$

where $n = \epsilon^{1/2}$ and p is the electric dipole moment in the laboratory system.

13.12

$$w_{\mathrm{Ch}} = \frac{m^2}{c^2 v^2} \int\limits_{\beta n > 1} \left(1 - \frac{1}{\beta^2 n^2}\right) n^2 \omega^3 \,\mathrm{d}\omega .$$

13.13 The energy loss per unit path length is given by the time integral of the energy flux through a cylindrical surface of unit length and radius a, which surrounds the trajectory of the particle. The losses may be calculated with the aid of equation (13.2.5) in the solution of problem 13.2, in which the fields should be taken at $r = a$ and the integration should be carried out over all frequencies between 0 and infinity. Using the expressions for the field components obtained in the solution of problem 13.1, we have

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}l} = \frac{2e^2\omega_0^2}{\pi v^2} \mathrm{Re}\, \mathrm{i} \int_0^\infty \left(\frac{1-x^2}{\epsilon_0 - x^2} - \beta^2\right) s^* a K_1(s^* a) K_0(sa) x \,\mathrm{d}x , \qquad (13.13.1)$$

where $x = \omega/\omega_0$, $\epsilon(0) = \epsilon_0 = 1 + \omega_p^2/\omega_0^2$ is the static permittivity, and

$$s^2 = \frac{\omega_0^2}{c^2}\left(\frac{1}{\beta^2} - 1\right) \frac{b - x^2}{1 - x^2} x^2 , \qquad b = \frac{c^2 - \epsilon_0 v^2}{c^2 - v^2} . \qquad (13.13.2)$$

It follows from equation (13.13.1) that the imaginary part of the integral is the only one to contribute to the losses. The functions K_0 and K_1 are real for a real argument, and hence the imaginary part of the integral will be determined by the range of values of x for which s is complex. As can be seen from equation (13.13.2), this range will depend on the sign and magnitude of the parameter b. When $b > 0$ (this means that $v < c/\epsilon_0^{1/2}$), s will be purely imaginary for x in the range $(b^{1/2}, 1)$, and real outside this interval. When $b < 0$ (this corresponds to $v > c/\epsilon_0^{1/2}$), s will be imaginary for $0 \leqslant x \leqslant 1$ and real for $x > 1$.

In addition to the above intervals of x, certain individual points at which the denominator $\epsilon_0 - x^2$ is zero will contribute to the imaginary part of the integral. At these points $x = \pm\epsilon_0^{1/2}$, and since the integration in equation (13.13.1) is carried out for $x > 0$, it is sufficient to consider the single pole at $x = \epsilon_0^{1/2} > 1$. When losses are neglected this pole will lie on the real axis. It will be displaced into the lower half-plane of the complex variable, ω, when the losses are taken into account[1]. This can easily be seen from the explicit expression for $\epsilon(\omega)$ [see equation (6.b.5)]. The integral may be evaluated either by introducing a damping parameter which tends to zero after the integration, or by slightly deforming the path of integration and by-passing the pole by a circle of infinitely small radius in the upper half-plane. Denoting integration over this semicircle by $\frown$, we have

$$\int_{\frown} \frac{1-x^2}{\epsilon_0 - x^2} s^* a K_1(s^* a) K_0(sa) x \, \mathrm{d}x = \tfrac{1}{2}\mathrm{i}(1-\epsilon_0)\frac{\omega_0 a(\epsilon_0)^{1/2}}{v} \times K_0\left[\frac{\omega_0 a(\epsilon_0)^{1/2}}{v}\right] K_1\left[\frac{\omega_0 a(\epsilon_0)^{1/2}}{v}\right]. \quad (13.13.3)$$

Next, consider the integral over the region in which s is purely imaginary. We note that for purely imaginary argument the cylindrical functions K_0 and K_1 are related by

$$s^* a K_1(s^* a) K_0(sa) - sa K_1(sa) K_0(s^* a) = \tfrac{1}{2}\pi\mathrm{i} ,$$

which follows from the properties of the Wronskian of the system of solutions of the Bessel equation. Hence,

$$\mathrm{Re}\,\mathrm{i} \int_{s^2<0} \left(\frac{1-x^2}{\epsilon_0 - x^2} - \beta^2\right) s^* a K_1(s^* a) K_0(sa) x \, \mathrm{d}x = -\tfrac{1}{2}\pi \int_{s^2<0} \left(\frac{1-x^2}{\epsilon_0 - x^2} - \beta^2\right) x \, \mathrm{d}x . \quad (13.13.4)$$

The latter integral may be evaluated by elementary methods. The limits of integration are chosen as indicated above.

Substituting equations (13.13.3) and (13.13.4) into (13.13.1), we have for $v < c/\epsilon_0^{1/2}$,

$$-\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}l} = \frac{2\pi e^4 N}{m v^2}\left\{\frac{2a\omega_0(\epsilon_0)^{1/2}}{v} K_0\left[\frac{a\omega_0(\epsilon_0)^{1/2}}{v}\right] K_1\left[\frac{a\omega_0(\epsilon_0)^{1/2}}{v}\right] - \beta^2 - \ln(1-\beta^2)\right\} \quad (13.13.5)$$

[1] This is in accordance with the general theorem according to which $\epsilon(\omega)$ has no zeros in the upper half-plane.

and for $v > c/\epsilon_0^{1/2}$,

$$-\frac{d\mathcal{E}}{dl} = \frac{2\pi e^4 N}{mv^2}\left\{\frac{2a\omega_0(\epsilon_0)^{1/2}}{v}K_0\left[\frac{a\omega_0(\epsilon_0)^{1/2}}{v}\right]K_1\left[\frac{a\omega_0(\epsilon_0)^{1/2}}{v}\right] - \frac{1-\beta^2}{\epsilon_0 - 1} + \ln\frac{\epsilon_0}{\epsilon_0 - 1}\right\}. \tag{13.13.6}$$

The part of total losses which does not vanish when $a \to \infty$ [terms which do not contain a in equations (13.13.5) and (13.13.6)] will represent the energy loss by the emission of transverse waves (Cherenkov effect):

$$-\left(\frac{d\mathcal{E}}{dl}\right)_{\text{Ch}} \equiv w_{\text{Ch}} = \frac{e^2\omega_p^2}{2v^2}[-\beta^2 - \ln(1-\beta^2)] \qquad v < \frac{c}{(\epsilon_0)^{1/2}},$$

$$-\left(\frac{d\mathcal{E}}{dl}\right)_{\text{Ch}} \equiv w_{\text{Ch}} = \frac{e^2\omega_p^2}{2v^2}\left(-\frac{1-\beta^2}{\epsilon_0 - 1} + \ln\frac{\epsilon_0}{\epsilon_0 - 1}\right) \qquad v > \frac{c}{(\epsilon_0)^{1/2}}.$$

The latter expression was obtained in the solution of problem 13.3.

The terms with K_0, K_1 in equations (13.13.5) and (13.13.6), which depend on a, enter as a result of integration round the pole $\omega \equiv \Omega = (\omega_0^2 + \omega_p^2)^{1/2}$ at which ϵ becomes equal to zero. Longitudinal oscillations are excited at these frequencies (see problem 8.45) and hence the expression

$$-\left(\frac{d\mathcal{E}}{dl}\right)_{\text{tot}} = \frac{e^2\omega_p^2\Omega a}{v^3}K_0\left(\frac{\Omega a}{v}\right)K_1\left(\frac{\Omega a}{v}\right) \tag{13.13.7}$$

describes the losses on excitation of longitudinal oscillations (polarisation losses). When $\Omega a/v \ll 1$, equation (13.13.7) assumes the simple form [see equation (A3.6)]

$$-\left(\frac{d\mathcal{E}}{dl}\right)_{\text{tot}} = \frac{e^2\omega_p^2}{v^2}\ln\frac{v}{\Omega a}.$$

When $\Omega a/v \gg 1$, the quantity $-(d\mathcal{E}/dl)_{\text{tot}}$ becomes very small [it is proportional to $\exp(-\Omega a/v)$]. This shows that the effect of the polarisation of the medium is small at low velocities.

The above macroscopic method of calculating the losses is due to Fermi.

13.14

$$-\frac{d\mathcal{E}}{dl} = \frac{e^2\omega_p^3 a}{v^3}K_0\left(\frac{\omega_p a}{v}\right)K_1\left(\frac{\omega_p a}{v}\right). \tag{13.14.1}$$

When $\omega_p a/v \ll 1$, which occurs when the velocity of the particle is sufficiently large, one can use the approximate formula (A3.6) for K_n. Equation (13.14.1) will then become

$$-\frac{d\mathcal{E}}{dl} = \frac{e^2\omega_p^2}{v^2}\ln\frac{2v}{\gamma\omega_p a}. \tag{13.14.2}$$

It follows from equations (13.13.1) and (13.13.2) that the losses experienced by the particle will be very dependent on ω_p [the frequency of longitudinal plasma oscillations (see problem 8.45)].

Cherenkov radiation is not emitted by a plasma, since the condition $\beta^2\epsilon \geqslant 1$ is not satisfied, in view of the fact that $\epsilon(\omega) < 1$ at all frequencies (however, Cherenkov emission will become possible if the plasma is placed in a magnetic field).

In a quantum mechanical interpretation, the excitation of plasma waves is equivalent to the appearance of certain discrete elementary excitations (these are the quasi-particles known as plasmons). The energy of each plasmon is equal to $\hbar\omega_p$ where $\hbar = 1{\cdot}05 \times 10^{-34}$ J s (Dirac's constant; $\hbar = h/2\pi$, where h is Planck's constant). For metals, the energy $\hbar\omega_p$ lies between 5 and 30 eV. Thus, when plasma oscillations are excited, the particle will lose energy in discrete steps. Studies of these discrete energies yield valuable information about the properties of solids.

13.15 Consider the expansion of the current density (figure 3.15.1)

$$j = j_z = \begin{cases} -ev\delta(z-vt)\delta(x)\delta(y)\,, & z \geqslant 0\,, \\ -ev\delta(z+vt)\delta(x)\delta(y)\,, & z \leqslant 0\,, \end{cases}$$

$(-\infty < t \leqslant 0)$, in the Fourier integral with respect to the time:

$$j = \int j_\omega \exp(-\mathrm{i}\omega t)\,\mathrm{d}t\,, \qquad j_\omega = \begin{cases} -\dfrac{e}{2\pi}\exp\left(-\mathrm{i}\dfrac{\omega}{v}z\right)\delta(x)\delta(y)\,, & z \geqslant 0\,, \\ -\dfrac{e}{2\pi}\exp\left(\mathrm{i}\dfrac{\omega}{v}z\right)\delta(x)\delta(y)\,, & z \leqslant 0\,. \end{cases} \tag{13.15.1}$$

Next, introduce the polarisation vector (12.a.9)

$$P_\omega = -\frac{j_\omega}{\mathrm{i}\omega}\,, \tag{13.15.2}$$

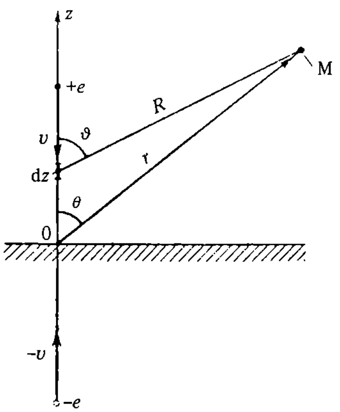


Figure 13.15.1.

and suppose that $\boldsymbol{P}_\omega$ lies along the z-axis.

Equations (13.15.1) and (13.15.2) show that the charge and current densities due to the moving particle can be represented by a set of harmonic oscillators whose spatial distribution is of the form

$$P_\omega = \begin{cases} -\dfrac{\mathrm{i}e}{2\pi\omega}\exp\left(-\mathrm{i}\dfrac{\omega}{v}z\right)\delta(x)\delta(y)\,, & z \geqslant 0\,, \\ -\dfrac{\mathrm{i}e}{2\pi\omega}\exp\left(\mathrm{i}\dfrac{\omega}{v}z\right)\delta(x)\delta(y)\,, & z \leqslant 0\,. \end{cases} \tag{13.15.3}$$

The presence of $\delta(x)\delta(y)$ in equation (13.15.3) shows that the oscillators are, in fact, distributed along the line of motion of the charge.

The oscillators lying along the element dz will produce at a point M in the wave zone a magnetic field (see figure 13.15.1)

$$\mathrm{d}\boldsymbol{H}_\omega = -\frac{\omega^2\exp(\mathrm{i}kR)}{c^2R^2}[\boldsymbol{P}_\omega \wedge \boldsymbol{R}]\,\mathrm{d}z = -\frac{\omega^2\exp(\mathrm{i}kR)}{c^2R}P_\omega \sin\vartheta\, \boldsymbol{e}_\phi\,\mathrm{d}z\,. \tag{13.15.4}$$

After integrating equation (13.15.4) with respect to z the following expression for the total field is obtained:

$$H_{\omega\phi} = \frac{\mathrm{i}e\omega}{2\pi c^2}\left\{\int_{-\infty}^{0}\frac{\exp[\mathrm{i}(\omega z/v + kR)]\sin\vartheta}{R}\mathrm{d}z + \int_0^{\infty}\frac{\exp[-\mathrm{i}(\omega z/v - kR)]\sin\vartheta}{R}\mathrm{d}z\right\}.$$

In the latter expression the integrands are the products of oscillating and decreasing functions, and therefore the main contribution to the integral will be that due to the region near $z = 0$. This is owing to the fact that the emission takes place at the boundary between the vacuum and the metal. To evaluate the integrals approximately, let $R = r - z\cos\theta$ in the exponentials. Expressing $\sin\vartheta$ in terms of R we have

$$H_{\omega\phi} = \frac{\mathrm{i}e\omega\exp(\mathrm{i}kr)r\sin\theta}{2\pi c^2}\left\{\int_{-\infty}^{0}\frac{\exp[\mathrm{i}(\omega/v)(1-\beta\cos\theta)z]}{R^2}\mathrm{d}z + \int_0^{\infty}\frac{\exp[-\mathrm{i}(\omega/v)(1+\beta\cos\theta)z]}{R^2}\mathrm{d}z\right\}.$$

These integrals can be expanded into series in powers of R^{-1} after integrating by parts. Retaining only terms up to $1/R$, we have

$$H_\phi = E_\theta = \frac{e\omega}{2\pi c^2}\left[\frac{1}{(\omega/v)(1-\beta\cos\theta)} + \frac{1}{(\omega/v)(1+\beta\cos\theta)}\right]\frac{\sin\theta\exp(\mathrm{i}kr)}{r}\,. \tag{13.15.5}$$

The second term in this expression represents the radiation field due to the sudden deceleration of the charge, while the first term represents the radiation due to the image.

The intensity of the radiation of frequency ω which is emitted into a solid angle $\mathrm{d}\Omega$ is given by

$$\mathrm{d}I(\omega, \theta) = c|E(\omega, \theta)|^2 r^2 \mathrm{d}\Omega = \frac{e^2 v^2}{\pi^2 c^3} \frac{\sin^2\theta \,\mathrm{d}\Omega}{(1-\beta^2\cos^2\theta)^2} \, . \tag{13.15.6}$$

In the nonrelativistic limit ($\beta \ll 1$) equation (13.15.6) gives the dipole radiation

$$\mathrm{d}I(\omega, \theta) = \frac{e^2 v^2}{\pi^2 c^3} \sin^2\theta \,\mathrm{d}\Omega \, , \tag{13.15.7}$$

whose intensity is proportional to the square of the velocity of the particle. We note that the intensity of the radiation is independent of the mass of the particle.

The integrals of equations (13.15.6) and (13.15.7) with respect to ω, which represent the angular distribution of the total radiation, will be found to diverge. This is due to the fact that the metal is regarded as a perfect conductor. In reality, the metal cannot be looked upon as a perfect conductor even in the infrared region, so that at high frequencies both equations (13.15.6) and (13.15.7) will not hold.

The spectral distribution of the total radiation may be obtained by integrating equation (13.15.6) over the upper hemisphere:

$$I(\omega) = \frac{4e^2 v^2}{3\pi c^3} \left(\frac{3}{8} \frac{\beta^2 - 1}{\beta^3} \ln \frac{1+\beta}{1-\beta} - \frac{3}{4\beta^2} \right) \, . \tag{13.15.8}$$

In the ultrarelativistic limit, when the total energy, $\mathcal{E}$, of the particle is much smaller than the rest energy, mc^2, equation (13.15.8) may be simplified to read

$$I(\omega) = \frac{2e^2}{\pi c} \ln \frac{\mathcal{E}}{mc^2} \, .$$

The intensity of the radiation increases logarithmically with the energy.

In the nonrelativistic limit the expression in brackets in equation (13.15.8) becomes equal to unity and

$$I(\omega) = \frac{4e^2 v^2}{3\pi c^3} \, .$$

13.16 The Fourier component of the polarisation vector is

$$P_\omega = -\frac{\mathrm{i}e}{2\pi\omega} \exp\left(-\mathrm{i}\frac{\omega}{v} z\right) \delta(x)\delta(y) \, .$$

To begin with, consider the field at the point A due to oscillators lying in the region $z > 0$ (figure 13.16.1). It is sufficient to consider the oscillators lying near $z = 0$, since they will be largely responsible for the radiated field (see preceding problem).

Consider an oscillator $\boldsymbol{p}_{\mathrm{B}}$ on the z-axis near $z = 0$ (point B) and an oscillator $\boldsymbol{p}_{\mathrm{A}}$ at the point A at which the field is to be calculated. Suppose they are equal in absolute magnitude and parallel to the z-axis, and that the distance between them is large compared with the wavelength. The oscillator $\boldsymbol{p}_{\mathrm{B}}$ will produce a field at A whose amplitude $\boldsymbol{E}_+$ will be at an angle to the z-axis which is approximately equal to $\frac{1}{2}\pi - \theta$ (figure 13.16.1). The waves from A and B will arrive by two routes, namely, directly and after reflection from the boundary of the dielectric. In the figure, the corresponding amplitudes are denoted by $\boldsymbol{E}'$ and $\boldsymbol{E}''$. These two amplitudes are at angles $\frac{1}{2}\pi - \vartheta' \approx \frac{1}{2}\pi - \theta$ to the z-axis. Hence, according to the reciprocity theorem, $\boldsymbol{E}_+ = \boldsymbol{E}' + \boldsymbol{E}''$ or, since in the wave zone $\boldsymbol{H} = [\boldsymbol{n} \wedge \boldsymbol{E}]$, we have $\boldsymbol{H}_+ = -\boldsymbol{H}' - \boldsymbol{H}''$ (all the three vectors $\boldsymbol{H}_+$, $\boldsymbol{H}'$, and $\boldsymbol{H}''$ are perpendicular to the AOz plane).

The wave emitted by A and reaching B directly will give rise to a field

$$\mathrm{d}H' = \frac{\omega^2 \exp(\mathrm{i}kR)}{c^2 R} P_\omega \sin\theta \, \mathrm{d}z \, .$$

The amplitude of the reflected wave may be determined with the aid of the Fresnel formulae, since the distance AC is large and the wave emitted at A may be looked upon as a plane wave near the point C. Using equation (8.a.20), and remembering that there is a phase change and that $\vartheta' \approx \theta$, we have

$$\mathrm{d}H'' = \frac{\omega^2 f \exp(\mathrm{i}kR')}{c^2 R'} P_\omega \sin\theta \, \mathrm{d}z \, ,$$

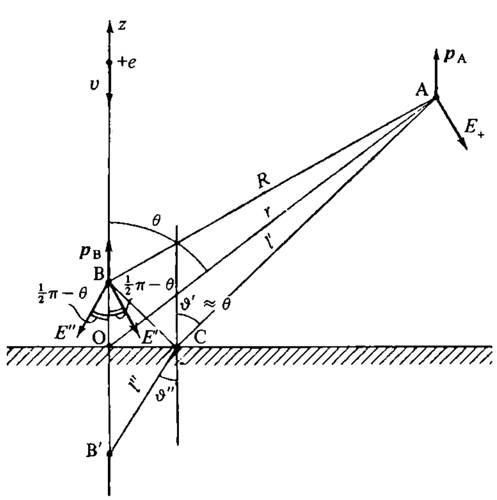


Figure 13.16.1.

where

$$f = \frac{\epsilon\cos\theta - (\epsilon - \sin^2\theta)^{1/2}}{\epsilon\cos\theta + (\epsilon - \sin^2\theta)^{1/2}}\,, \qquad R' = \mathrm{ACB}\,.$$

The field H_+, which is produced at A by all the oscillators lying in the region $z > 0$, may be obtained by integrating the sum $-(\mathrm{d}H' + \mathrm{d}H'')$ with respect to z between 0 and ∞. The integration may be carried out precisely as in the preceding problem. The result is

$$H_+ = \frac{ev}{2\pi c^2}\left(\frac{1}{1+\beta\cos\theta} + \frac{f}{1-\beta\cos\theta}\right)\frac{\sin\theta\exp(\mathrm{i}kr)}{r}\,.$$

This formula may easily be interpreted by comparison with the analogous formula (13.15.5) in the solution to the preceding problem. The first term represents the field of the particle moving in the vacuum and suddenly coming to rest at the point $z = 0$. The second term represents the field due to the image $(-ef)$ which moves in the dielectric towards the particle and also comes to rest at $z = 0$. In contrast to a perfect conductor, the strength of the image is reduced by a factor f and its magnitude depends on the frequency ω of the particular harmonic [through $\epsilon(\omega)$] and on the position of the point of observation (through the angle θ).

The field H_- due to the dipoles at $z < 0$ can be determined in a similar way. The wave leaving A will arrive at B after refraction at the separation boundary. Using the Fresnel formulae again, we have

$$\mathrm{d}H_- = -\frac{\omega^2}{\epsilon c^2 R''}(1+f)P_\omega \sin\vartheta' \exp(\mathrm{i}\varphi)\,\mathrm{d}z\,, \tag{13.16.1}$$

where $R'' = l' + l''$ is the length of the broken line ACB$'$ (see figure 13.16.1). The phase φ is retarded, so that

$$\varphi = \frac{\omega}{c}l' + \frac{\omega}{c}(\epsilon)^{1/2} l''\,.$$

When $|z| \ll r$ $(z < 0)$, we have $l' = r + z\tan\vartheta''\sin\theta$, $l'' = -z/\cos\vartheta''$. Since the law of refraction is $\sin\vartheta'' = \epsilon^{-1/2}\sin\vartheta'$ we have, on substituting θ for ϑ'

$$\varphi = \frac{\omega}{c}r - \frac{\omega}{c}z(\epsilon - \sin^2\theta)^{1/2}\,.$$

The field due to the dipoles at $z < 0$ may be obtained by integrating equation (13.16.1) between $-\infty$ and 0, and the result is

$$H_- = -\frac{ev}{2\pi\epsilon c^2}(1+f)\frac{1}{1-\beta(\epsilon - \sin^2\theta)^{1/2}}\frac{\sin\theta\exp(\mathrm{i}kr)}{r}\,.$$

The total field at A is equal to $H_+ + H_-$. The intensity of the radiation of frequency ω which is emitted into a solid angle $\mathrm{d}\Omega$ is

$$\mathrm{d}I(\omega, \theta) = \frac{e^2 v^2}{4\pi^2 c^3} A^2(\omega, \theta) \sin^2\theta \, \mathrm{d}\Omega ,$$

where

$$A(\omega, \theta) = \frac{2\beta \cos\theta}{1 - \beta \cos^2\theta} + (1+f)\left\{ \frac{1}{\epsilon[1 - \beta(\epsilon - \sin^2\theta)^{1/2}]} - \frac{1}{1 - \beta\cos\theta} \right\} .$$

The quantity A depends on the frequency through $\epsilon(\omega)$.

In the nonrelativistic limit $\beta \ll 1$, and hence

$$\mathrm{d}I(\omega, \theta) = \frac{e^2 v^2}{\pi^2 c^3} \frac{(\epsilon - 1)^2 \sin^2\theta \cos^2\theta}{[\epsilon \cos\theta + (\epsilon - \sin^2\theta)^{1/2}]^2} \mathrm{d}\Omega .$$

Plasma physics

a The motion of separate particles in a plasma

14.3

$$dw(\vartheta) = \tfrac{1}{2}(n)^{1/2}\frac{\sin\vartheta\, d\vartheta}{[1+(n-1)\cos^2\vartheta]^{3/2}}\,, \qquad \overline{\mathscr{E}^2} = \frac{2n+1}{3}\mathscr{E}_0^2 - \frac{2(n-1)}{3}m^2c^4\,.$$

In the nonrelativistic case for the average kinetic energy $\overline{T}$ in the final state we have

$$\overline{T} = \frac{2n+1}{3}T_0\,, \qquad T_0 = \frac{p_0^2}{2m}\,.$$

14.4 To evaluate $\boldsymbol{v}_d$ we must find the additional particle velocity caused by the presence of a field gradient ∇H and averaged over the cyclotron rotation period. We write down the equation of motion for the transverse particle velocity $\boldsymbol{v}_\perp$:

$$\frac{d\boldsymbol{v}_\perp}{dt} = \frac{eH}{mc}[\boldsymbol{v}_\perp \wedge \boldsymbol{h}]\,. \tag{14.4.1}$$

Here $\boldsymbol{h}$ is the unit vector in the direction of the magnetic field. $H(\boldsymbol{r})$ occurs in this equation, that is, the value of the field at the position of the particle. For this quantity we write

$$H(\boldsymbol{r}) = H(\boldsymbol{R}) + (\boldsymbol{r}\cdot\nabla H)\,, \tag{14.4.2}$$

where $\boldsymbol{R}$ is the position of the guiding centre, and $\boldsymbol{r}$ is the position of the particle when reckoned from the guiding centre as origin. To a first approximation we may assume that the guiding centre is not displaced in a transverse direction during a single rotation of the particle. Substituting equation (14.4.2) into (14.4.1) we get an equation of motion of the form

$$\frac{d\boldsymbol{v}_\perp}{dt} = [\boldsymbol{v}_\perp \wedge \boldsymbol{\Omega}]\left[1 + \frac{(\boldsymbol{r}\cdot\nabla H)}{H}\right], \tag{14.4.3}$$

where $\boldsymbol{\Omega} = eH(\boldsymbol{R})\boldsymbol{h}/mc$.

We split $\boldsymbol{v}_\perp$ into two components: the velocity $\boldsymbol{v}_0 = d\boldsymbol{r}_0/dt$ in a uniform field and a small correction $\boldsymbol{v}_1$:

$$\boldsymbol{v}_\perp = \boldsymbol{v}_0 + \boldsymbol{v}_1\,.$$

In the correction term in equation (14.4.3) we can replace the quantities $\boldsymbol{v}_\perp$ and $\boldsymbol{r}$ by $\boldsymbol{v}_0$ and $\boldsymbol{r}_0$. Bearing in mind that

$$\frac{d\boldsymbol{v}_0}{dt} = [\boldsymbol{v}_0 \wedge \boldsymbol{\Omega}]\,, \tag{14.4.4}$$

we get the following equation for $\boldsymbol{v}_1$:

$$\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t} = [\{\boldsymbol{v}_1 + \boldsymbol{v}_0(\boldsymbol{r}_0 \cdot \nabla H)\} \wedge \boldsymbol{\Omega}] .$$

We take the average over a rotation period of the particle of both sides of this equation. When averaging the derivative $\mathrm{d}\boldsymbol{v}_1/\mathrm{d}t$ we get

$$\overline{\frac{\mathrm{d}\boldsymbol{v}_1}{\mathrm{d}t}} = \frac{\boldsymbol{v}_1(t+T) - \boldsymbol{v}_1(t)}{T} \approx 0$$

up to terms in first order in the small quantity ∇H. If we average the right-hand side we find

$$\boldsymbol{v}_\mathrm{d} = \overline{\boldsymbol{v}_1} = -\overline{\boldsymbol{v}_0(\boldsymbol{r}_0 \cdot \nabla H)} .$$

The quantities $\boldsymbol{v}_0$ and $\boldsymbol{r}_0$ correspond to the motion of the particle in a uniform field and can be obtained from equation (14.4.4). We can choose them such that

$$\boldsymbol{r}_0 = R_\perp(\boldsymbol{e}_1 \sin\Omega t + \boldsymbol{e}_2 \cos\Omega t) , \qquad \boldsymbol{v}_0 = v_\perp[\boldsymbol{r}_0 \wedge \boldsymbol{h}] ,$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors orthogonal to $\boldsymbol{h}$ and to one another. If we now carry out the averaging we get the required expression for $\boldsymbol{v}_\mathrm{d}$.

14.5 The quantity $\gamma\mu$ is the adiabatic invariant for a relativistic particle, where $\gamma = (1 - v^2/c^2)^{-1/2}$ and $\mu = p_\perp v_\perp/2H$ is the magnetic moment. If the kinetic energy of the particle is conserved, we have $\gamma =$ constant and $\mu =$ constant. The latter relation is satisfied for a nonrelativistic particle, for which $\gamma \approx 1$, even when its energy is not conserved.

14.6 $F = -(\boldsymbol{\mu} \cdot \nabla H)$,

where $\boldsymbol{\mu} = p_\perp v_\perp \boldsymbol{h}/2H$ is the magnetic moment produced by the rotation of the particle. This expression is the same as the right-hand side of equation (14.a.2), if we put $\boldsymbol{E} = 0$ in it, since it follows from the Maxwell equation $\operatorname{div}\boldsymbol{H} = 0$ that $H \operatorname{div}\boldsymbol{h} = -(\boldsymbol{h} \cdot \nabla H)$.

14.7

$$\sin\vartheta > \left(\frac{H}{H_\mathrm{m}}\right)^{1/2}$$

14.8

$$R = 1 - \frac{H}{H_\mathrm{m}} .$$

14.9

$$r = r_0\left(\frac{H_0}{H}\right)^{1/2} ,$$

where r_0 is the distance of the guiding centre from the axis of the trap in the field H_0, and r is the same distance after the field has been changed to H. Increasing the field causes the plasma to be compressed towards the axis of the trap.

14.10 The guiding centre will go over to the field line

$$r = l, \qquad \phi = \frac{2cq}{Hv_{\parallel}l^2}.$$

14.11 The guiding centre of a proton moves uniformly along a circle of radius $r = 2r_*$ which lies in the equatorial plane with an angular velocity

$$\omega_d = \frac{3c\mathscr{E}}{e\mu}r - \frac{3GmM}{e\mu},$$

where G is the gravitational constant; $R \approx 226$ km, $T \approx 14{\cdot}9$ s.

14.12 (a) If we evaluate the products $[\boldsymbol{h} \wedge \nabla H]$ and $[\boldsymbol{h} \wedge (\boldsymbol{h} \cdot \nabla)\boldsymbol{h}]$ for the field of a magnetic dipole, we find from equation (14.a.1) that motion across the magnetic field lines reduces to azimuthal drift for which the distance from the Earth's centre and the latitudinal angle remain unchanged. Moreover, the guiding centre moves along the line of force, the equation of which is

$$r = r_0 \cos^2\lambda, \tag{14.12.1}$$

where r_0 is the distance in the equatorial plane of the line of force to the centre. As we neglect the gravitational field the particle energy remains constant.

On using the well-known expressions for the field strength of a magnetic dipole, and also equations (14.12.1), (14.a.1), and (14.a.5) we find the angular velocity of the azimuthal drift:

$$\omega_d = \frac{(v_d)_\phi}{r} = -\frac{3cpvr_0\sin^2\phi}{2e\mu}\frac{1+\sin^2\lambda}{\cos^3\lambda(3\sin^2\lambda+1)} - \frac{cpvr_0}{e\mu}\frac{\cos^3\lambda(3\sin^2\lambda-1)}{(3\sin^2\lambda+1)},$$

where p and v are the proton momentum and velocity.

(b) By using equation (14.a.5) we find the condition determining $\lambda_m > 0$:

$$\frac{\cos^6\lambda_m}{(3\sin^2\lambda_m+1)^{1/2}} = \sin^2\alpha.$$

The particles move in the region $-\lambda_m \leqslant \lambda \leqslant \lambda_m$.

(c) The proton will reach the Earth's surface provided

$$r_0\cos^2\lambda_m \leqslant r_*,$$

where r_* is the radius of the Earth's sphere.

14.13 Through an area $d\sigma = s\,ds\,d\phi$ of the plane perpendicular to the direction of motion of the particle, $nv\,d\sigma$ particles will pass per unit time.

They transfer to the fixed particle a momentum equal to

$$m\Delta v_z nv \, d\sigma \,, \tag{14.13.1}$$

where Δv_z is the change in the z-component of the velocity of a single particle when it is scattered by the fixed particle.

The required force which equals the total momentum transferred per unit time is obtained by integrating equation (14.13.1) over the total cross section of the particle beam. We must express Δv_z in terms of the impact parameter s. Since the collisions are elastic we have

$$\Delta v_z = -2v \sin^2 \tfrac{1}{2}\theta \,, \tag{14.13.2}$$

where θ is the scattering angle. We found in problem 11.98 the connection between θ and s:

$$s^2 = \frac{e^2 e'^2}{m^2 v^4} \cot^2 \tfrac{1}{2}\theta \,. \tag{14.13.3}$$

After substituting equations (14.13.2) and (14.13.3) into (14.13.1) and integrating over ϕ we get an expression for the force:

$$\boldsymbol{F} = \frac{4\pi}{m} e^2 e'^2 n \Lambda \frac{\boldsymbol{v}}{v^3} \,,$$

where

$$\Lambda = \ln \left(s_\mathrm{m} \frac{mv^2}{ee'} \right) .$$

As $s_\mathrm{m} \to \infty$, which corresponds to an unbounded beam, the quantity Λ diverges. This result is explained by the long-range nature of the Coulomb forces, as a result of which even particles that fly past at however large a distance interact with the fixed particle. In actual fact in a plasma any charge is screened by charges of the opposite sign so that with a given particle only those particles interact which fly past it at a distance which does not exceed the screening radius. The screening radius [Debye radius (cf problem 14.31) or Debye–Hückel radius] was evaluated for a static, equilibrium plasma in problem 6.12 (see also problem 14.31):

$$s_\mathrm{m} = \frac{kT}{4\pi(e'^2 N + e^2 n)} \,,$$

where e and e' are the electronic and ionic charges, and n and N their densities.

The quantity Λ is called the Coulomb logarithm. Neglecting the weak v-dependence of Λ, we may assume that $\Lambda =$ constant, with a value of the order 10.

14.14

$$\overline{\boldsymbol{F}}(\boldsymbol{v}) = -\frac{4\pi}{\mu} e^2 e'^2 \Lambda \int \frac{\boldsymbol{v} - \boldsymbol{v}'}{|\boldsymbol{v} - \boldsymbol{v}'|^3} f(\boldsymbol{v}') \, d^3 \boldsymbol{v}' \,, \tag{14.14.1}$$

where $\mu = mm'/(m+m')$ is the reduced mass.

It is useful to bear in mind the following electrostatic analogue: we can write expression (14.14.1) in the form of an electrical force $\boldsymbol{F} = q\boldsymbol{E}$, acting upon a charge $q = -4\pi e^2 e'^2 \Lambda/\mu$ due to an 'electrostatic field'

$$\boldsymbol{E}(\boldsymbol{v}) = -\mathrm{grad}_v \varphi(\boldsymbol{v}) \,,$$

where

$$\varphi(\boldsymbol{v}) = \int \frac{f(\boldsymbol{v}')\,\mathrm{d}^3\boldsymbol{v}'}{|\boldsymbol{v}-\boldsymbol{v}'|}$$

is the 'electrostatic potential', satisfying the Poisson equation

$$\nabla_v^2 \varphi(\boldsymbol{v}) = -4\pi f(\boldsymbol{v}) \,.$$

14.15 The energy of the test particle is not changed when it collides with fixed, infinitely heavy particles. The change in its mean momentum is described by the equation

$$\frac{\mathrm{d}\overline{\boldsymbol{p}}}{\mathrm{d}t} = \overline{\boldsymbol{F}} \,, \tag{14.15.1}$$

where $\overline{\boldsymbol{F}}$ is the average force. It is convenient to evaluate it using the electrostatic analogue shown in the preceding problem. The velocity distribution of the particles in the medium is described by the function $f(\boldsymbol{v}) = n\delta(\boldsymbol{v})$. Therefore $\varphi(\boldsymbol{v}) = n/v$, $\boldsymbol{E} = n\boldsymbol{v}/v^3$,

$$\overline{\boldsymbol{F}} = -\frac{4\pi}{m} e^2 e'^2 \Lambda \frac{\boldsymbol{v}}{v^3} \,.$$

$\overline{\boldsymbol{F}}$ has the nature of a 'friction force' tending to diminish the directed velocity of the particles. However, this friction is smaller the larger the particle velocity ($\overline{F} \propto 1/v^2$, 'decreasing friction').

Integrating equation (14.15.1) we find

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 \exp\left(-\frac{t}{\tau}\right)$$

where $\tau = mv^3/4\pi e^2 e'^2 n\Lambda$ is a characteristic time of loss of directed particle velocity.

14.16

$$\overline{\boldsymbol{F}} = \begin{cases} 0\,, & \text{when } v < v_0\,, \\ 4\pi e^2 e'^2 \Lambda \left(\dfrac{1}{m} + \dfrac{1}{m'}\right) \dfrac{n\boldsymbol{v}}{v^3}\,, & \text{when } v > v_0\,. \end{cases}$$

14.17

$$\overline{F} = \begin{cases} -4\pi e^2 e'^2 \Lambda n \left(\dfrac{1}{m} + \dfrac{1}{m'}\right) \dfrac{\boldsymbol{v}_0}{v_0} , & \text{when } (\boldsymbol{v} \cdot \boldsymbol{v}_0) > v_0^2 , \\ 4\pi e^2 e'^2 \Lambda n \left(\dfrac{1}{m} + \dfrac{1}{m'}\right) \dfrac{\boldsymbol{v}_0}{v_0} , & \text{when } (\boldsymbol{v} \cdot \boldsymbol{v}_0) < v_0^2 . \end{cases}$$

14.18 A friction force

$$\overline{\boldsymbol{F}} = -\frac{4\pi e^4 n \Lambda}{m} \frac{\boldsymbol{v}}{v^3} \tag{14.18.1}$$

acts upon an electron moving with velocity v in a medium of fixed equally charged ions (see the solution of problem 14.15). We note that the velocity dependence of the force F can also be obtained from the following considerations. The friction force is the loss of momentum of a particle per unit time owing to collisions. If the average time between collisions is τ, and in each collision a particle loses momentum of the order of its total momentum mv (which means that as the result of a collision the electron is deflected over a large angle), we have

$$F \approx \frac{mv}{\tau} . \tag{14.18.2}$$

In such a collision the electron approaches the ion at a distance at which its kinetic energy is of the order of its potential energy:

$$\tfrac{1}{2} mv^2 \approx \frac{e^2}{r} . \tag{14.18.3}$$

This approximate equality enables us to estimate the collision cross section:

$$\sigma \approx \pi r^2 \approx \frac{4\pi e^4}{m^2 v^4} \tag{14.18.4}$$

and the average time between collisions

$$\tau \approx \frac{1}{n\sigma v} \approx \frac{m^2 v^3}{4\pi n e^4} . \tag{14.18.5}$$

Substituting τ into equation (14.18.2) we find $F \approx 4\pi n e^4 / m v^2$, or, taking into account the retarding nature of the force,

$$\boldsymbol{F} \approx -\frac{4\pi n e^4 \boldsymbol{v}}{m v^3} ,$$

which differs from equation (14.18.1) by the absence of the Coulomb logarithm Λ. This is natural: when using the estimates (14.18.2) to (14.18.5) we neglected distant collisions with small momentum transfers, and their contributions lead to the Coulomb logarithm.

Let us now average equation (14.18.1) over the possible electron velocities. To do this we put

$$\boldsymbol{v} = \boldsymbol{u}+\boldsymbol{v}_{\text{th}} ,$$

where $\boldsymbol{v}_{\text{th}}$ is the thermal velocity and $\boldsymbol{u}$ the velocity acquired under the influence of the electrical field $\boldsymbol{E}$. If $u \ll v_{\text{th}}$ we can put $v^3 \approx v_{\text{th}}^3$ in the denominator of equation (14.18.1). It is, however, impossible to neglect u as compared to v_{th} in the enumerator since we get $\overline{v}_{\text{th}} = 0$ when we average over the directions of the thermal velocity. As a result we get

$$\overline{\boldsymbol{F}} = \frac{4\pi n e^4 \Lambda}{m v_{\text{th}}^3} \boldsymbol{u} , \tag{14.18.6}$$

where now v_{th} is a quantity of the order of magnitude of the mean thermal velocity. In the case of a Maxwell distribution we have $v_{\text{th}}^2 = 3kT/m$. For $u \ll v_{\text{th}}$ we thus get $F \propto u$.

When $u \gg v_{\text{th}}$, we put $v \approx u$ and obtain

$$\overline{F} \approx \frac{4\pi n e^4 \Lambda}{m u^2} , \tag{14.18.7}$$

i.e. $\overline{F} \propto 1/u^2$. The maximum of $\overline{F}$ will clearly correspond to a value of $u \sim v_{\text{th}}$; then both equations (14.18.6) and (14.18.7) give the same value

$$\overline{F} \approx \frac{4\pi n e^4 \Lambda}{m v_{\text{th}}^2} . \tag{14.18.8}$$

The function $\overline{F}(u)$ is sketched in figure 14.18.1.

If the field in the plasma $E < \overline{F}_{\text{max}}/e = E_{\text{D}}$, the braking force for a value of u satisfying the equation $\overline{F}(u) = eE$ will exceed the accelerating force eE and the electrons cannot be further accelerated. This is the range of fields E for which the normal Ohm law holds. In the case $E > E_{\text{D}}$ the accelerating force is larger than the braking force and the electrons have a possibility of being accelerated without limit. (In actual fact the electron gas as a whole is not accelerated because of collective

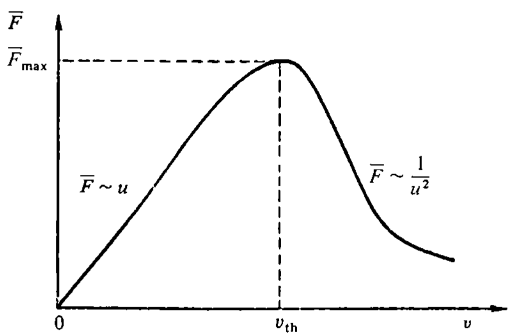


Figure 14.18.1.

effects and its resistance may even increase.) This is called the effect of 'run-away electrons'.

If we substitute $v_{\rm th}^2 = 3kT/m$ into equation (14.18.8) we get

$$E_{\rm D} = \tfrac{1}{3}\frac{e\Lambda}{D^2}\,, \qquad D^2 = \frac{kT}{4\pi ne^2}\,.$$

An exact calculation for the same case gives (Leontovich, 1965, volume 1)

$$E_{\rm D} = 0{\cdot}214\frac{e\Lambda}{D^2}\,.$$

Our order-of-magnitude estimate gave a result which was close to the exact value. The critical field value $E = E_{\rm D}$ is called the Dreicer field.

b Collective motions in a plasma

14.19 It is obvious to assume that the velocity of the liquid motion is along the z-axis and only depends on the transverse coordinate x. Since a moving conducting liquid carries along its magnetic lines of force, during the motion there must arise a longitudinal magnetic field component $H_z(x)$. We therefore look for the unknown functions $\boldsymbol{v}$ and $\boldsymbol{H}$ in the form $\boldsymbol{v}\ (0, 0, v(x))$ and $\boldsymbol{H}\ (H_0, 0, H_z(x))$; equations (14.b.3) and (14.b.4) are then identically satisfied. Equations (14.b.1) and (14.b.2) take the form

$$\frac{\mathrm{d}v}{\mathrm{d}x}+\frac{c^2}{4\pi\sigma H_0}\frac{\mathrm{d}^2H_z}{\mathrm{d}x^2} = 0\,, \tag{14.19.1}$$

$$\frac{\mathrm{d}^2v}{\mathrm{d}x^2}+\frac{H_0}{4\pi\eta}\frac{\mathrm{d}H_z}{\mathrm{d}x} = \frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}z}\left(p+\frac{H^2}{8\pi}\right), \tag{14.19.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(p+\frac{H^2}{8\pi}\right) = 0\,.$$

From the last equation it follows that $p+H^2/8\pi$ depends on z only. However,

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p+\frac{H^2}{8\pi}\right) = \frac{\mathrm{d}p}{\mathrm{d}z} = \text{constant}\,,$$

since $H^2/8\pi$ is independent of z. Equations (14.19.1) and (14.19.2) are thus ordinary linear differential equations to determine the unknown functions $v(x)$ and $H_z(x)$. Eliminating $\mathrm{d}H_z/\mathrm{d}x$ we get the following equation for $u = \mathrm{d}v/\mathrm{d}x$:

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\frac{1}{x_0^2}u = 0\,, \qquad x_0 = \frac{c}{H_0}\left(\frac{\eta}{\sigma}\right)^{\frac{1}{2}},$$

whence we find

$$v = x_0\left[A\exp\left(\frac{x}{x_0}\right)-B\exp\left(-\frac{x}{x_0}\right)\right]+C\,. \tag{14.19.3}$$

The boundary conditions have the form $v(\pm a) = 0$, since a viscous liquid sticks at the walls. Moreover, from symmetry considerations it follows that $v(x) = v(-x)$. From the boundary conditions and equation (14.19.3) we find

$$v(x) = v_0 \frac{\cosh(a/x_0) - \cosh(x/x_0)}{\cosh(a/x_0) - 1}, \tag{14.19.4}$$

where v_0 is a new constant which has the meaning of the velocity at the middle plane $x = 0$; we can express v_0 in terms of the pressure gradient:

$$v_0 = -\frac{ax_0}{\eta} \frac{\cosh(a/x_0) - 1}{\sinh(a/x_0)} \frac{\mathrm{d}p}{\mathrm{d}z}.$$

The magnetic field is determined from equations (14.19.2) and (14.19.4) and the boundary conditions $H_z(\pm a) = 0$:

$$H_z(x) = -\frac{4\pi\eta}{c}(\sigma\eta)^{1/2} v_0 \frac{(x/a)\sinh(a/x_0) - \sinh(x/x_0)}{\cosh(a/x_0) - 1}.$$

The ratio $a/x_0 = M$ is called the Hartmann number. When $M \ll 1$, we have

$$v_0 = \frac{a^2}{\eta}\frac{\mathrm{d}p}{\mathrm{d}z}, \qquad v(x) = v_0\left(1 - \frac{x^2}{a^2}\right), \tag{14.19.5}$$

as in ordinary hydrodynamics. The magnetic field $H_z = 0$ to first order in the Hartmann number. The longitudinal field H_z only appears in the higher approximations.

In the opposite limiting case, $M \gg 1$, we get

$$v_0 = -\frac{a^2}{\eta M}\frac{\mathrm{d}p}{\mathrm{d}z}; \qquad v(x) = v_0\left\{1 - \exp\left[-\frac{(a - |x|)}{x_0}\right]\right\}. \tag{14.19.6}$$

A comparison of equations (14.19.5) and (14.19.6) shows that the mean velocity of the liquid decreases with increasing H_0, while the velocity profile becomes flatter in the main part of the flow, but varies steeply in a layer of thickness x_0 at the walls. The longitudinal magnetic field in this limit has the form

$$H_z(x) = \frac{4\pi a^2(\sigma\eta)^{1/2}}{cM}\frac{\mathrm{d}p}{\mathrm{d}z}\left[\frac{x}{a} - \sinh\left(\frac{x}{x_0}\right)\exp\left(-\frac{a}{x_0}\right)\right].$$

One can see from this formula that the field decreases with increasing Hartmann number. The largest value of H_z occurs for $M \approx 1$.

The current density in the moving liquid can be evaluated from the Maxwell equation $j = (c/4\pi)\,\mathrm{curl}\boldsymbol{H}$. Only the y-component of the current is different from zero:

$$j_y(x) = \frac{c}{H_0}\frac{\mathrm{d}p}{\mathrm{d}z}\left[1 - \frac{a}{x}\frac{\cosh(x/x_0)}{\sinh(a/x_0)}\right].$$

The magnetic field H_z produced by it vanishes everywhere outside the region occupied by the liquid. There remains only the transverse field H_0.

14.20

$$v(x) = v_0\left[\frac{\sinh(x/x_0)}{\sinh(a/x_0)}\right] .$$

The current density is

$$j_y(x) = \frac{c\eta v_0}{H_0 x_0^2}\left[\frac{\sinh(x/x_0)}{\sinh(a/x_0)}\right] .$$

This current produces a magnetic field

$$H_z(x) = \frac{4\pi\eta v_0}{H_0 x_0}\frac{\cosh(a/x_0) - \cosh(x/x_0)}{\sinh(a/x_0)} ,$$

which vanishes for $|x| \geqslant a$.

14.21 The magnetic field has one component

$$H_\phi \equiv H(r) = \frac{4\pi}{cr}\int_0^r rj(r)\,\mathrm{d}r .$$

If we integrate the equation of motion (14.b.1) with the boundary condition $p|_{r \geqslant a} = 0$ we obtain

$$p(r) = \frac{1}{8\pi}\int_r^a \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2H^2)\,\mathrm{d}r , \tag{14.21.1}$$

where $H = (4\pi/cr)\int_0^r rj(r)\,\mathrm{d}r$ for $r < a$, and $H = 2J/cr$ for $r > a$.

In order to connect the current strength J with T and N we put $p = n(r)kT$, where k is Boltzmann's constant, and integrate both sides of equation (14.21.1) over the cross-sectional area of the plasma column. We get

$$J = 2c(NkT)^{1/2} .$$

For $T \approx 10^8$ K and $N \approx 10^{21}$ m^{-3} (values which are characteristic for thermonuclear investigations) we have

$$J = 7{\cdot}5 \times 10^4 \text{ A} .$$

14.22 The current must flow in a thin surface layer. In that case the pressure inside the column will be constant:

$$p = \frac{J^2}{2\pi c^2 a^2} .$$

14.23 By taking the r-component of equation (14.b.6) and putting $\boldsymbol{v} = v\boldsymbol{r}/r$, v = constant, we get the following equation to determine H_r:

$$\frac{\partial H_r}{\partial t} = -\frac{2v}{r}H_r - v\frac{\partial H_r}{\partial r} .$$

The solution of this equation can be expressed in terms of an arbitrary function of the arguments $r-vt$, ϑ, and ϕ:

$$H_r(r, \vartheta, \phi) = \frac{1}{r^2}F(r-vt, \vartheta, \phi) . \tag{14.23.1}$$

The boundary condition has the form

$$H_r|_{r=a} = H_{0r}(\vartheta, \phi+\Omega t) = \frac{1}{a^2}F(a-vt, \vartheta, \phi)$$

(the argument has been written $\phi+\Omega t$ since we want to go over to the fixed system of coordinates). Thus we have

$$F(a-vt, \vartheta, \phi) = a^2 H_{0r}(\vartheta, \phi+\Omega t) .$$

Hence equation (14.23.1) can be written in the form

$$H_r(r, \vartheta, \phi, t) = \frac{a^2}{r^2}H_{0r}\left[\vartheta, \phi-\frac{(r-a)\Omega}{v}+\Omega t\right] . \tag{14.23.2}$$

Similarly we get

$$H_\vartheta = \frac{a}{r}H_{0\vartheta}\left[\vartheta, \phi-\frac{(r-a)\Omega}{v}+\Omega t\right] , \quad H_\phi = \frac{a}{r}H_{0\phi}\left[\vartheta, \phi-\frac{(r-a)\Omega}{v}+\Omega t\right]$$

We obtain the following connection between the components of the vector $\boldsymbol{H}_0$ from the equation $\operatorname{div}\boldsymbol{H} = 0$:

$$-\frac{a\Omega}{v}\frac{\partial H_{0r}}{\partial \phi}\sin\vartheta+\frac{\partial}{\partial\vartheta}(H_{0\vartheta}\sin\vartheta)+\frac{\partial H_{0\phi}}{\partial\phi} = 0 .$$

For $H_{0\vartheta} = 0$ we find $H_\vartheta = 0$ and

$$H_{0\phi} = \frac{a\Omega}{v}H_{0r}\sin\vartheta+f(\vartheta) ;$$

if we put $f(\vartheta) = 0$ we find

$$H_\phi(r, \vartheta, \phi, t) = \frac{a^2\Omega}{vr}H_{0r}\left[\vartheta, \phi-\frac{(r-a)\Omega}{v}+\Omega t\right]\sin\vartheta . \tag{14.23.3}$$

Parker has used the model considered here as a model to describe the interplanetary magnetic field produced by solar plasma currents (the solar wind). In Parker's model of the interplanetary magnetic field $H_\vartheta = 0$ and H_r and H_ϕ are given by equations (14.23.2) and (14.23.3). Measurements of the interplanetary magnetic field performed by satellites and rockets show that the average magnetic field in the neighbourhood of the Earth's orbit is satisfactorily described by Parker's model.

14.24 The lines of force have the form of an Archimedean spiral:

$$r = \frac{v}{\Omega}(\phi-\phi_0) , \qquad \phi_0 = \text{constant} ,$$

$$\theta = \arctan\frac{r_0\Omega}{v} \approx 56^\circ\,; \qquad H \approx 3{\cdot}6 \times 10^{-3}\ \mathrm{A\,m^{-1}}\,.$$

14.25

$$\epsilon_\perp = 1 + \frac{4\pi c^2 \rho}{H^2}$$

where ρ is the plasma density. The value of $\epsilon_\perp$ found here is obtained from the results of problem 6.25 in the limit as $\omega \to 0$.

14.26

$$\omega = \omega_p = \left(\frac{4\pi n e^2}{m}\right)^{1/2},$$

where m is the electron mass.

14.27 When $\omega < \omega_p$ we have $R = 1$,

$$\boldsymbol{E} = \frac{2iq}{k+iq}\boldsymbol{E}_0 \exp(-qz - i\omega t)\,,$$

where $q = (\omega/c)[(\omega_p^2/\omega^2) - 1]^{1/2}$, $k = \omega/c$, and $\boldsymbol{E}_0$ is the amplitude of the incident wave. The penetration depth is

$$\delta = \frac{1}{q} = \frac{c}{(\omega_p^2 - \omega^2)^{1/2}}\,;$$

$\delta \approx c/\omega_p$ when $\omega \ll \omega_p$. The damping of the wave is caused not by the dissipation of energy, but by the appearance of currents in the plasma which produce a field of the opposite sign.

When $\omega > \omega_p$

$$R = \left(\frac{k-q}{k+q}\right)^2, \qquad \boldsymbol{E} = \frac{2q}{k+q}\boldsymbol{E}_0 \exp(iqz - i\omega t)\,,$$

where $q = (\omega^2 - \omega_p^2)^{1/2}/c$; the wave propagates in the plasma without being damped.

14.28 The position vector of a particle can be written in the form

$$\boldsymbol{R}(t) = \boldsymbol{R}_0 + \boldsymbol{v}_0 t + \boldsymbol{R}_1(t)\,,$$

where $\boldsymbol{v}_0$ is the particle velocity when there is no field present (its thermal velocity); $\boldsymbol{R}_0$ is the position vector at $t = 0$; and $\boldsymbol{R}_1(t)$ is the extra term caused by the action of the electrical field of the plane wave (we assume that the particles are nonrelativistic and neglect the magnetic field).

The quantity $\boldsymbol{R}_1$ satisfies the equation

$$m\ddot{\boldsymbol{R}}_1 = e\boldsymbol{E}_0 \exp\{i[(\boldsymbol{k}\cdot\boldsymbol{R}_0) + (\boldsymbol{k}\cdot\boldsymbol{v}_0)t + (\boldsymbol{k}\cdot\boldsymbol{R}_1) - \omega t]\}\,. \tag{14.28.1}$$

In the index of the exponential we can neglect the term $(\boldsymbol{k}\cdot\boldsymbol{R}_1)$, by assuming that the inequality $kR_1 \ll 1$ holds. In this approximation, which is the one linear in $\boldsymbol{E}_0$, the solution of equation (14.28.1), which corresponds to forced oscillations, has the form

$$\boldsymbol{R}_1(t) = -\frac{e}{m}\frac{\boldsymbol{E}_0 \exp[\![\mathrm{i}\{(\boldsymbol{k}\cdot\boldsymbol{R}_0)-[\omega-(\boldsymbol{k}\cdot\boldsymbol{v}_0)]t\}]\!]}{[\omega-(\boldsymbol{k}\cdot\boldsymbol{v}_0)]^2}\,. \tag{14.28.2}$$

The particle velocity can be written in the form[1]

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 + \frac{\mathrm{i}e}{m}\frac{\boldsymbol{E}_0}{[\omega-(\boldsymbol{k}\cdot\boldsymbol{v}_0)]}\exp[\![\mathrm{i}\{(\boldsymbol{k}\cdot\boldsymbol{R}_0)-[\omega-(\boldsymbol{k}\cdot\boldsymbol{v}_0)]t\}]\!]\,. \tag{14.28.3}$$

The current produced by a single particle with initial position and velocity equal to $\boldsymbol{R}_0$ and $\boldsymbol{v}_0$, respectively, can be written as

$$\boldsymbol{j}_1(\boldsymbol{r}, t) = e\boldsymbol{v}(t)\delta[\boldsymbol{r}-\boldsymbol{R}(t)]\,, \tag{14.28.4}$$

where $\boldsymbol{v}(t)$ is the velocity at the point $\boldsymbol{r} = \boldsymbol{R}(t)$. To evaluate the total current $\boldsymbol{j}(\boldsymbol{r}, t)$ we must multiply equation (14.28.4) by the number of particles with initial velocity $\boldsymbol{v}_0$ in the volume element $\mathrm{d}^3\boldsymbol{R}_0$ and integrate over all possible values of $\boldsymbol{R}_0$ and $\boldsymbol{v}_0$:

$$\boldsymbol{j}(\boldsymbol{r}, t) = en\int \boldsymbol{v}(t)\delta[\boldsymbol{r}-\boldsymbol{R}(t)]f(\boldsymbol{v}_0)\,\mathrm{d}^3\boldsymbol{v}_0\,\mathrm{d}^3\boldsymbol{R}_0\,.$$

First of all we integrate over $\boldsymbol{R}_0$. The argument of the δ-function depends in a complicated way on $\boldsymbol{R}_0$, so that we change to a new integration variable $\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{v}_0 t + \boldsymbol{R}_1(\boldsymbol{R}_0, t)$. If we evaluate the Jacobian of the transformation up to terms linear in $\boldsymbol{E}_0$ we get

$$\mathrm{d}^3\boldsymbol{R}_0 = \frac{\partial(R_{0x}, R_{0y}, R_{0z})}{\partial(R_x, R_y, R_z)}\mathrm{d}^3\boldsymbol{R} \approx [1-\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{R}_1)]\,\mathrm{d}^3\boldsymbol{R}\,. \tag{14.28.5}$$

After this the integration over $\boldsymbol{R}$ is easy and can be performed with the use of a formula such as (A1.4). By substituting expression (14.28.5) in the integrand and again neglecting terms $(\boldsymbol{k}\cdot\boldsymbol{R}_1)$ in the index of the exponentials, we get

$$\boldsymbol{j}(\boldsymbol{r}, t) = en\int\left\{\boldsymbol{v}_0 + \frac{\mathrm{i}e\boldsymbol{E}}{m[\omega-(\boldsymbol{k}\cdot\boldsymbol{v}_0)]} + \frac{\mathrm{i}e\boldsymbol{v}_0(\boldsymbol{k}\cdot\boldsymbol{E})}{m[\omega-(\boldsymbol{k}\cdot\boldsymbol{v}_0)]^2}\right\}f(\boldsymbol{v}_0)\,\mathrm{d}^3\boldsymbol{v}_0\,, \tag{14.28.6}$$

where $\boldsymbol{E} = \boldsymbol{E}_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r})-\mathrm{i}\omega t]$. The point $(\boldsymbol{k}\cdot\boldsymbol{v}_0) = \omega$ is not a singularity of the integrand since we have assumed that $f(\boldsymbol{v}_0) = 0$ for $(\boldsymbol{k}\cdot\boldsymbol{v}_0) = \omega$. We can thus expand in the ratio $v_0/v_{\mathrm{ph}} = kv_0/\omega$, with the assumption that the characteristic velocities are small compared to the

[1] The divergence of equations (14.28.2) and (14.28.3) when $(\boldsymbol{k}\cdot\boldsymbol{v}_0) = \omega$ is connected with the fact that 'resonance' particles have not been taken into account properly; 'resonance' particles are those with velocities that satisfy the condition $(\boldsymbol{k}\cdot\boldsymbol{v}_0) = \omega$. In order to remove this difficulty we assume that there are no particles in the plasma with such velocities, i.e. we exclude from our considerations the range of velocities which satisfy the inequality $|\boldsymbol{v}-\boldsymbol{v}_0| \ll v_0$.

phase velocity. This enables us to write equation (14.28.6) in the form

$$j(r, t) = \int \left\{ v_0 + \frac{ieE_0}{m\omega}\left[1+\frac{(k\cdot v_0)}{\omega}+\frac{(k\cdot v_0)^2}{\omega^2}\right] + \frac{ie(k\cdot E)v_0}{m\omega^2}\left[1+\frac{2(k\cdot v_0)}{\omega}\right]\right\} f(v_0)\,\mathrm{d}^3v_0 \,.$$

If we assume that $f(v_0)$ is independent of the angles, we get

$$j(r, t) = \frac{ine^2}{m\omega}\left[\left(1+\frac{k^2\overline{v_\parallel^2}}{\omega^2}\right)E_\perp + \left(1+\frac{3k^2\overline{v_\parallel^2}}{\omega^2}\right)E_\parallel\right] . \qquad (14.28.7)$$

Here

$$E_\parallel = \frac{(k\cdot E)k}{k^2}\,, \qquad E_\perp = E - E_\parallel\,, \qquad \overline{v_\parallel^2} = 2\pi\int v_\parallel^2 f(v) v_\perp \,\mathrm{d}v_\perp\,\mathrm{d}v_\parallel \,.$$

In the case of a Maxwell distribution $\overline{v_\parallel^2} = T/2m$.

From equation (14.28.7) we find the conductivity tensor:

$$\sigma_{\alpha\beta} = \frac{ine^2}{m\omega}\left[\left(1+\frac{k^2\overline{v_\parallel^2}}{\omega^2}\right)\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right) + \left(1+\frac{3k^2\overline{v_\parallel^2}}{\omega^2}\right)\frac{k_\alpha k_\beta}{k^2}\right] .$$

It is purely imaginary, and corresponds to the absence of energy dissipation. If we now use the formula

$$\epsilon_{\alpha\beta} = \delta_{\alpha\beta} + \frac{4\pi i}{\omega}\sigma_{\alpha\beta}$$

to evaluate the dielectric permittivity tensor, we find

$$\epsilon_{\alpha\beta}(\omega, k) = \epsilon_\perp\left(\delta_{\alpha\beta}-\frac{k_\alpha k_\beta}{k^2}\right) + \epsilon_\parallel\frac{k_\alpha k_\beta}{k^2}\,,$$

where

$$\epsilon_\perp = 1 - \frac{\omega_p^2}{\omega^2}\left(1+\frac{k^2\overline{v_\parallel^2}}{\omega^2}\right), \qquad \epsilon_\parallel = 1 - \frac{\omega_p^2}{\omega^2}\left(1+\frac{3k^2\overline{v_\parallel^2}}{\omega^2}\right), \qquad \omega_p^2 = \frac{4\pi ne^2}{m} .$$

When this result is compared with that obtained when there is no thermal motion ($\overline{v_\parallel^2} = 0$) we see that an important new effect appears: the wavevector dependence of ϵ—spatial dispersion. At the same time the dielectric permittivity becomes a tensorial quantity. The k-dependence of ϵ is explained by the fact that the current in any given point is produced by particles coming from neighbouring regions where the field is not the same as in that point. The spatial inhomogeneity of the field together with the thermal motion of the particles leads to the spatial dispersion of the dielectric permittivity.

In this problem we have not sufficiently correctly considered the behaviour of the resonance particles [for which $(k\cdot v_0) = \omega$]; this has led to the loss of a small imaginary part of $\epsilon_\parallel$ which describes the transfer of energy from the wave to the particles (Landau damping which exists even in a collisionless plasma; see problem 14.38).

14.29

$$v_{\text{ph}} = \frac{\omega_{\text{p}}}{k} + \frac{3k\overline{v_\parallel^2}}{2\omega_{\text{p}}} \approx \frac{\omega_{\text{p}}}{k}\,; \qquad v_{\text{g}} = \frac{3k\overline{v_\parallel^2}}{\omega_{\text{p}}}\,.$$

When there is no thermal motion $v_{\text{g}} = 0$. The plasma oscillations propagate therefore as the result of the transport of electromagnetic energy by the particles.

14.30 (a) $\rho(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, 0)\cos\omega_{\text{p}}t$;

$$\text{(b)}\quad \rho(x, t) = \operatorname{Re}\left\{\rho_0 x_0^2 \frac{x}{(x_0^2 + 4\mathrm{i}\beta t)^{3/2}} \exp\left[-\frac{x^2}{(x_0^2 + 4\mathrm{i}\beta t)} - \mathrm{i}\omega_{\text{p}}t\right]\right\},$$

where $\beta = 3\overline{v_\parallel^2}/2\omega_{\text{p}}$. In the case (b) we see not only the plasma oscillations of the charge density with frequency ω_{p}, but also its relaxation due to the thermal motion of the particles: $\rho(x, t) = 0$ as $t \to \infty$.

14.31 The function $\varphi(\boldsymbol{r})$ must satisfy Poisson's equation

$$\nabla^2\varphi = 4\pi e n(\boldsymbol{r}) - 4\pi e n_0 - 4\pi q\delta(\boldsymbol{r})\,. \tag{14.31.1}$$

At equilibrium $n(\boldsymbol{r})$ is related to $\varphi(\boldsymbol{r})$ through Boltzmann's formula

$$n(\boldsymbol{r}) = A\exp\left(\frac{e\varphi}{kT}\right), \tag{14.31.2}$$

where k is Boltzmann's constant and where A is a normalisation constant which follows from the condition

$$\int n(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r} = n_0\,.$$

At infinity, where the influence of q can be neglected and where we can put $\varphi(\infty) = 0$, we can determine A and we find that $A = n_0$.

If equation (14.31.2) is substituted into (14.31.1) and the assumption that $e\varphi/kT \ll 1$ is used so that the exponential can be expanded and only the term which is linear in φ is retained we find

$$\nabla^2\varphi - \kappa^2\varphi = -4\pi q\delta(\boldsymbol{r})\,,$$

where κ satisfies the equation

$$\kappa^2 = 4\pi n_0\frac{e^2}{kT}\,. \tag{14.31.3}$$

Since the system has spherical symmetry, we find the following solution of equation (14.31.3):

$$\varphi(r) = \frac{q}{r}\exp(-\kappa r)\,.$$

We note that this is a screened potential with the so-called *Debye radius* r_D as the screening distance, which is given by the expression

$$r_D = \frac{1}{\kappa} = \left(\frac{kT}{4\pi n_0 e^2}\right)^{1/2} \tag{14.31.4}$$

14.32 The derivation in the preceding solution is valid only if the system can really be treated as a quasi-continuum, which means that the change in $n(r)$ or $\varphi(r)$ is small over distances d that are of the order of the mean interparticle distance, that is, for d satisfying the relation

$$d \sim n^{-1/3} .$$

This means that the following condition must be satisfied:

$$r_D \gg d \sim n^{-1/3} .$$

By using equation (14.31.4) we see that this condition can be written in the form

$$kT \gg \frac{e^2}{d} \sim e^2 n_0^{1/3} , \tag{14.32.1}$$

which expresses the fact that the derivation is valid, provided the average kinetic energy of an electron is large compared to its average potential energy. A plasma for which condition (14.32.1) is satisfied is called a *hot dilute plasma*, and the approximation used in that case is called the Debye approximation.

14.33 By using the Boltzmann formula [equation (14.31.2) of problem 14.31] and the approximation $e\varphi/kT \ll 1$ [compare condition (14.32.1) from the preceding problem] we find

$$n(r) = n_0 + \frac{q\kappa^2}{4\pi e}\varphi(r) , \tag{14.33.1}$$

when the additional infinitesimal point charge is present.

14.34 With the use of equation (14.33.1) we find

$$n_{\text{exc}} = \int [n(r) - n_0]\, \mathrm{d}^3 r = \frac{q\kappa^2}{e}\int_0^\infty \exp(-\kappa r)\, \mathrm{d}r = \frac{q}{e} .$$

14.35

$$E_{\text{int}} = \int \frac{qe[n(r) - n_0]}{r}\, \mathrm{d}^3 r = \frac{q^2}{r_D} .$$

14.36

$$E_D = \tfrac{1}{2}\int \mathrm{d}^3 r \int \mathrm{d}^3 r' \frac{e^2[n(r) - n_0][n(r') - n_0]}{|\boldsymbol{r} - \boldsymbol{r}'|} .$$

If we use equation (A2.13) we can evaluate E_D:

$$E_D = \tfrac{1}{2}q^2\kappa^4\left[\int_0^\infty dr\exp(-\kappa r)\int_0^r r'\,dr'\exp(-\kappa r') + \int_0^\infty r\,dr\exp(-\kappa r)\int_r^\infty dr'\exp(-\kappa r')\right] = \frac{q^2}{4r_D}\,.$$

14.37

$$n_D \approx \tfrac{4}{3}\pi n_0 r_D^3 = \frac{1}{3(4\pi)^{1/2}}\left(\frac{kT}{e^2 n_0^{1/3}}\right)^{3/2}$$

We see from equation (14.32.1) of the solution to problem 14.32 that when the Debye approximation is valid n_D is a very large number.

14.38 The electric field corresponds to a potential φ in which the electrons will move. Consider this potential in a frame moving with the phase velocity ω/k of the wave. In this frame electrons with velocities close to ω/k may be trapped in the field of the wave. If φ_0 $(= E_0/k)$ is the amplitude of the potential, electrons with velocities between $\omega/k - v_c$ and $\omega/k + v_c$ will have insufficient kinetic energy in the moving frame to pass over the potential barrier in the sinusoidal potential, if v_c satisfies the relation $\tfrac{1}{2}mv_c^2 = e\varphi_0$. This means that electrons with velocities between $\omega/k - v_c$ and ω/k will be accelerated in order to keep up with the wave while those with velocities between ω/k and $\omega/k + v_c$ will be decelerated. Since in a Maxwell distribution there are more electrons in the first velocity range than in the second, on average the wave will give up energy to the electrons, and thus be damped. The damping rate γ will be proportional to the number of electrons with velocity in the neighbourhood of ω/k. In the case of a Maxwell distribution we thus have (we use k_B for the Boltzmann constant to avoid confusion with the wavenumber k)

$$\gamma \propto \exp\left[-\frac{m(\omega/k)^2}{2k_B T}\right]. \tag{14.38.1}$$

In the case where the wave is a Langmuir wave, so that the frequency is given by equation (14.b.8), we get from equation (14.38.1)

$$\gamma \propto \exp\left(-\frac{m\omega_p^2}{2k^2 k_B T}\right) = \exp\left(-\frac{1}{2k^2 r_D^2}\right), \tag{14.38.2}$$

where we have used equation (14.31.4) of problem 14.31 for the Debye radius r_D (we note in passing that if we introduce a thermal velocity v_T by the relation $v_T^2 = k_B T$, we can write $r_D = v_T/\omega_p$).

We note from equation (14.38.2) that Landau damping only becomes significant for wavenumbers of the order of $1/r_D$ whereas for smaller wavenumbers Landau damping is negligibly small.

appendices

Appendix 1 [1]

The δ-function

Dirac's δ-function is defined by[2]

$$\delta(x-a) = \begin{cases} 0 & x \neq a\,, \\ \infty & x = a\,, \end{cases} \tag{A1.1}$$

$$\int_{\Delta} \delta(x-a)\,\mathrm{d}x = 1\,. \tag{A1.2}$$

The integration in equation (A1.2) is carried out over an arbitrary interval Δ which includes the point $x = a$.

The δ-function has the following properties

$$\delta(x) = \delta(-x)\,, \tag{A1.3}$$

$$\int_{\Delta} f(x)\delta(x-a)\,\mathrm{d}x = f(a)\,, \tag{A1.4}$$

$$\delta(\alpha x) = \frac{1}{|\alpha|}\delta(x)\,, \tag{A1.5}$$

where $f(x)$ is a continuous function. The three-dimensional δ-function is defined in an analogous fashion:

$$\delta(\boldsymbol{r}-\boldsymbol{a}) = \delta(x-a_x)\delta(y-a_y)\delta(z-a_z) = \begin{cases} 0\,, & \text{if } \boldsymbol{r} \neq \boldsymbol{a}\,, \\ \infty\,, & \text{if } \boldsymbol{r} = \boldsymbol{a}\,; \end{cases} \tag{A1.6}$$

$$\int_{V} f(\boldsymbol{r})\delta(\boldsymbol{r}-\boldsymbol{a})\,\mathrm{d}^3\boldsymbol{r} = \begin{cases} f(\boldsymbol{a})\,, & \text{if } \boldsymbol{a} \text{ is inside the volume } V\,, \\ 0\,, & \text{if } \boldsymbol{a} \text{ is outside the volume } V\,, \end{cases} \tag{A1.7}$$

where $f(\boldsymbol{r})$ is a continuous function.

The δ-function may be of use to describe the charge distribution of a point particle in space. The volume density of this distribution is given by

$$\rho(\boldsymbol{r}) = e\delta(\boldsymbol{r}-\boldsymbol{a})\,, \tag{A1.8}$$

where e is the charge and $\boldsymbol{a}$ is the position vector of the particle.

It is also possible to define a derivative of the δ-function:

$$\int_{\Delta} f(x)\frac{\partial\delta(x-a)}{\partial x}\,\mathrm{d}x = -\frac{\partial f(a)}{\partial a}\,, \tag{A1.9}$$

[1] For details of the topics discussed in the appendices see, for instance, Gradshteyn and Ryzhik (1965), Jahnke and Emde (1945), Whittaker and Watson (1952), Morse and Feshbach (1953), or Abramovitz and Stegun (1965).

[2] A mathematically correct definition of the δ-function requires a generalisation of the usual concept of a function; the δ-function belongs to the class of singular generalised functions.

where the integration is carried out by parts. Higher-order derivatives are defined in a similar fashion:

$$\int_{\Delta} f(x)\delta^{(n)}(x-a)\,\mathrm{d}x = (-1)^n f^n(a)\,. \tag{A1.10}$$

The δ-function may also be looked upon as the derivative of a function which exhibits a finite discontinuous change b at the point $x = a$. Thus, if $f(a+0)-f(a-0) \equiv b$ then

$$\frac{\partial f}{\partial x} = b\delta(x-a) + \text{bounded function}\,. \tag{A1.11}$$

A convenient representation of the δ-function and its derivatives may be obtained by considering a graph of the continuous function $\delta(x-a, \alpha)$ which is illustrated in figure A1.1. The parameter α represents the width of the interval within which $\delta(x-a, \alpha)$ is different from zero. The δ-function and its derivatives are then defined as the following limits:

$$\delta(x-a) = \lim_{\alpha \to 0} \delta(x-a, \alpha)\,, \qquad \frac{\partial\delta(x-a)}{\partial x} = \lim_{\alpha \to 0} \frac{\partial\delta(x-a, \alpha)}{\partial x}\,,$$

Many nonsingular functions, that depend on a parameter, show the properties of a δ-function for certain limits of the parameter. For

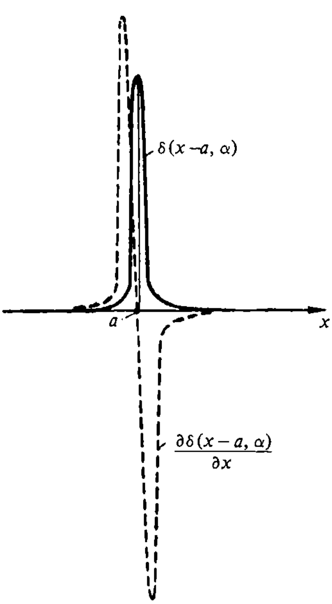


Figure A1.1.

instance,

$$\delta(x) = \lim_{\alpha \to 0} \frac{1}{\pi} \frac{\alpha}{\alpha^2 + x^2} , \tag{A1.12}$$

$$\delta(x) = \lim_{k \to \infty} \frac{1}{\pi} \frac{\sin kx}{x} , \tag{A1.13}$$

$$\delta(x) = \lim_{k \to \infty} \frac{1}{\pi} \frac{\sin^2 kx}{kx^2} . \tag{A1.14}$$

For integer k and n

$$\lim_{k \to \infty} \frac{1}{\pi} \frac{\sin 2kx}{\sin x} = \sum_{n=-\infty}^{+\infty} \delta(x - n\pi) . \tag{A1.14$'$}$$

One can easily check that any of the representations (A1.12) to (A1.14′) satisfy equations (A1.1) to (A1.5) and also the definition (A1.9) of the derivative of the δ-function. In evaluating integrals such as $\int f(x)\delta(x-a)\,\mathrm{d}x$ with the aid of equations (A1.12) to (A1.14′) it should be remembered that the limit as $\alpha \to 0$ or as $k \to \infty$ should be taken after the integration. For instance, using (A1.12) we have:

$$\int f(x)\delta(x-a)\,\mathrm{d}x = \lim_{\alpha \to 0} \int f(x)\delta(x-a, \alpha)\,\mathrm{d}x .$$

A consideration of the Fourier integral, or use of the representation (A1.13), leads to a further useful representation of the δ-function:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(\mathrm{i}kx)\,\mathrm{d}k = \frac{1}{\pi} \int_0^{\infty} \cos kx\,\mathrm{d}k . \tag{A1.15}$$

The two generalised functions $\delta_+(x)$ and $\delta_-(x)$ are similar to the δ-function. They are defined by a relation which is similar to that given by equation (A1.15):

$$\delta_\pm(x) = \frac{1}{2\pi} \int_0^{\infty} \exp(\pm \mathrm{i}kx)\,\mathrm{d}k . \tag{A1.16}$$

The generalised functions are related to the δ-function:

$$\delta_\pm(x) = \tfrac{1}{2}\delta(x) \pm \frac{\mathrm{i}}{2\pi} P \frac{1}{x} , \tag{A1.17}$$

so that $\delta(x) = \delta_+(x) + \delta_-(x)$. The symbol P in equation (A1.17) represents the principal value of the integral:

$$\begin{aligned}\int_{a_1}^{a_2} f(x)\delta_\pm(x-a)\,\mathrm{d}x &= \tfrac{1}{2} f(a) \pm \frac{\mathrm{i}}{2\pi} P \int_{a_1}^{a_2} \frac{f(x)}{x-a}\,\mathrm{d}x \\ &= \tfrac{1}{2} f(a) \pm \frac{\mathrm{i}}{2\pi} \lim_{\epsilon \to 0} \left[\int_{a_1}^{a-\epsilon} \frac{f(x)}{x-a}\,\mathrm{d}x + \int_{a+\epsilon}^{a_2} \frac{f(x)}{x-a}\,\mathrm{d}x \right] ,\end{aligned}$$

where $a_1 < a < a_2$, $\epsilon > 0$. The symbol $\displaystyle⨍_a^b$ is frequently used instead of $P\displaystyle\int_a^b$.

If the argument of the δ-function is a single-valued function of the independent variable x, we have

$$\delta[\varphi(x)] = \sum_i \frac{1}{|\varphi'(\alpha_i)|}\delta(x-\alpha_i)\,, \tag{A1.18}$$

where the α_i are the roots of the equation $\varphi(x) = 0$.

Spherical Legendre functions

The spherical function of order l, m is defined by

$$Y_{lm}(\vartheta, \phi) = \delta_m \left[\frac{(2l+1)}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}\right]^{1/2} P_{lm}(\cos\vartheta)\exp(im\phi)\,, \tag{A2.1}$$

where ϑ and ϕ are the polar angles and the integers l, m are such that $l \geqslant 0$, $-l \leqslant m \leqslant l$ and $\delta_m = (-1)^m$ when $m \geqslant 0$, and $\delta_m = 1$ when $m < 0$. In the above definition $P_{lm}(\cos\vartheta)$ represents the associated Legendre polynomial

$$P_{lm}(x) = (1-x^2)^{\frac{1}{2}|m|}\frac{\mathrm{d}^{|m|}P_l(x)}{\mathrm{d}x^{|m|}}\,, \tag{A2.2}$$

where $P_l(x)$ is the ordinary Legendre polynomial which is identical with $P_{lm}(x)$ when $m = 0$ since

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l(x^2-1)^l}{\mathrm{d}x^l} = P_{l0}(x)\,. \tag{A2.3}$$

The associated Legendre polynomials are the solutions of the following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^2-1)\frac{\mathrm{d}P_{lm}(x)}{\mathrm{d}x}\right] + \left[l(l+1) - \frac{m^2}{x^2-1}\right]P_{lm}(x) = 0\,. \tag{A2.4}$$

The following formulae are useful in practice:

$$\left.\begin{aligned}
Y_{lm}(0,0) &= \left(\frac{2l+1}{4\pi}\right)^{1/2}\delta_{m0}\,,\\
Y_{lm}(\vartheta,\phi) &= (-1)^l Y_{lm}(\pi-\vartheta, \pi+\phi)\,,\\
P_l(1) &= 1\,,\\
P_{2n}(0) &= (-1)^n\frac{(2n-1)!!}{(2n)!!}\,,\\
P_{2n+1}(0) &= 0\,;
\end{aligned}\right\} \tag{A2.5}$$

$$\left.\begin{aligned}
(l+1)P_{l+1}(x) &= (2l+1)xP_l(x) - lP_{l-1}(x)\,,\\
(x^2-1)\frac{\mathrm{d}P_l(x)}{\mathrm{d}x} &= l[xP_l(x) - P_{l-1}(x)]\,,
\end{aligned}\right\} \tag{A2.6}$$

where the symbol $n!!$ represents the product of all the successive integers having the same parity as n between 2 and n when n is even, and between 1 and n when n is odd. The spherical functions with $l = 0, 1, 2$ are of the form

$$\left.\begin{aligned}
Y_{00} &= \frac{1}{(4\pi)^{1/2}}\,, \\
Y_{10} &= \left(\frac{3}{4\pi}\right)^{1/2} \cos\vartheta\,, \\
Y_{1,\pm 1} &= \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\vartheta \exp(\pm \mathrm{i}\phi)\,, \\
Y_{20} &= \left(\frac{5}{4\pi}\right)^{1/2} \frac{3\cos^2\vartheta - 1}{2}\,, \\
Y_{2,\pm 1} &= \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\vartheta\cos\vartheta \exp(\pm \mathrm{i}\phi)\,, \\
Y_{2,\pm 2} &= \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\vartheta \exp(\pm 2\mathrm{i}\phi)\,.
\end{aligned}\right\} \tag{A2.7}$$

The spherical functions form a complete orthonormal set of functions on the surface of a sphere. This means that

$$\int Y_{lm}^*(\vartheta, \phi) Y_{l'm'}(\vartheta, \phi)\, \mathrm{d}\Omega = \delta_{ll'}\delta_{mm'}\,, \tag{A2.8}$$

where $\mathrm{d}\Omega = \sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\phi$ is an element of solid angle. It also means that an arbitrary function of ϑ and ϕ which has an integrable square may be expanded into a series in terms of the spherical functions, i.e.

$$f(\vartheta, \phi) = \sum_{l,m} a_{lm} Y_{lm}(\vartheta, \phi)\,. \tag{A2.9}$$

The coefficients a_{lm} are given by

$$a_{lm} = \int Y_{lm}^*(\vartheta, \phi) f(\vartheta, \phi)\, \mathrm{d}\Omega\,. \tag{A2.10}$$

Functions of the form $r^{-l-1} Y_{lm}(\vartheta, \phi)$ and $r^l Y_{lm}(\vartheta, \phi)$ are called spherical harmonics. It is easily shown with the aid of equation (A2.4) that spherical harmonics are special solutions of the Laplace equation at all points other than $r = 0$:

$$\nabla^2 \left[\frac{Y_{lm}(\vartheta, \phi)}{r^{l+1}}\right] = 0\,, \qquad \nabla^2[r^l Y_{lm}(\vartheta, \phi)] = 0\,. \tag{A2.11}$$

The superposition of spherical harmonics with arbitrary coefficients

$$\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} (a_{lm} r^{-l-1} + b_{lm} r^l) Y_{lm}(\vartheta, \phi)\,, \tag{A2.12}$$

is also a solution of the Laplace equation. If $\boldsymbol{r}(r, \vartheta, \phi)$ and $\boldsymbol{r}'(r', \vartheta', \phi')$ are the position vectors of any two points in space which are such that $r > r'$

(see figure 2.0.2) then

$$\frac{1}{R} = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{(r^2 - 2rr'\cos\gamma + r'^2)^{1/2}} = \sum_{l=0}^{\infty} \frac{r'^l}{r^{l+1}} P_l(\cos\gamma) . \tag{A2.13}$$

The function R^{-1} is referred to as the generating function for the Legendre polynomials. The following addition theorem holds for spherical functions:

$$P_l(\cos\gamma) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}(\vartheta, \phi) Y_{lm}^*(\vartheta', \phi') . \tag{A2.14}$$

The angles ϑ, ϕ and ϑ', ϕ' enter symmetrically into equation (A2.14). Substitution of equation (A2.14) into (A2.13) yields the following expansion:

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{r'^l}{(2l+1)r^{l+1}} Y_{lm}(\vartheta, \phi) Y_{lm}^*(\vartheta', \phi') . \tag{A2.15}$$

It follows from equation (A2.13) that

$$\frac{1}{(\cosh\xi - \cos\eta)^{1/2}} = \sqrt{2} \sum_{l=0}^{\infty} \exp[-(l+\tfrac{1}{2})|\xi|] P_l(\cos\eta) , \tag{A2.16}$$

where $r'/r = \exp(-|\xi|)$.

Cylindrical functions

The cylindrical functions $Z_p(kx)$ are the solutions of the Bessel equation

$$\frac{d^2Z}{dx^2}+\frac{1}{x}\frac{dZ}{dx}+\left(k^2-\frac{p^2}{x^2}\right)Z = 0\,. \tag{A3.1}$$

The solution which is bounded at $x = 0$ for $p \geqslant 0$ is known as the cylindrical function of the first kind (or the Bessel function of the first kind). It is given by

$$J_p(x) = \frac{x^p}{2^p}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2^{2k}k!\Gamma(p+k+1)}\,. \tag{A3.2}$$

Since the square of p enters into equation (A3.1), it follows that J_{-p} is also a solution of the above equation. Any linear combination of J_p and J_{-p} will also be a solution. The cylindrical function of the second kind (Neumann function) is defined by

$$N_p(x) = \frac{J_p(x)\cos p\pi - J_{-p}(x)}{\sin p\pi}\,. \tag{A3.3}$$

This function is occasionally called a Weber function and is denoted by $Y_p(x)$. Cylindrical functions of the third kind (Hankel functions) which are defined by

$$\left.\begin{aligned} H_p^{(1)}(x) &= J_p(x)+\mathrm{i}N_p(x)\,,\\ H_p^{(2)}(x) &= J_p(x)-\mathrm{i}N_p(x)\,,\end{aligned}\right\} \tag{A3.4}$$

are also frequently employed.

Any linear combination of any two linearly independent cylindrical functions will be a general solution of the Bessel equation. Such linearly independent functions will, in particular, be the Bessel functions $J_p(x)$ and $J_{-p}(x)$ when p is not an integer. In the case when $p = n$, where n is an integer, the functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent since $J_{-n}(x) = (-1)^nJ_n(x)$. When this is so, the general solution will be a linear combination of, say, J_n and N_n.

Cylindrical functions of a purely imaginary argument are known as modified Bessel functions. When n is an integer they are defined by

$$I_n(x) = \mathrm{i}^{-n}J_n(\mathrm{i}x)\,, \qquad K_n(x) = \tfrac{1}{2}\pi\mathrm{i}^{n+1}H_n^{(1)}(\mathrm{i}x)\,. \tag{A3.5}$$

The function $K_n(x)$ is referred to as the Macdonald function.

In physical problems it is frequently necessary to have approximate expressions for the cylindrical functions of small and large values of the argument. When $|x| \ll 1$, we have

$$\left.\begin{aligned} &J_p(x) \approx \frac{x^p}{2^p\Gamma(p+1)}\,, \quad J_n(x)\approx\frac{x^n}{2^nn!}\,, \quad I_n(x)\approx\frac{x^n}{2^nn!}\,,\\ &J_0(x)\approx 1-\tfrac{1}{4}x^2\,, \qquad I_0(x)\approx 1+\tfrac{1}{4}x^2\,,\end{aligned}\right\} \tag{A3.6}$$

Equations (A3.6) continued over

$$\left.\begin{aligned} N_n(x) &\approx -\frac{2^n(n-1)!}{\pi x^n}\,, \qquad K_n(x) \approx \frac{2^{n-1}(n-1)!}{x^n}\,, \\ N_0(x) &\approx \frac{2}{\pi}\ln\frac{\gamma x}{2}\,, \qquad K_0(x) \approx \ln\frac{2}{\gamma x}\,, \end{aligned}\right\} \tag{A3.6}$$

where $n \geqslant 1$, $\ln\gamma = 0{\cdot}5772$.

The expressions for the Hankel functions for $|x| \ll 1$ may be obtained from equation (A3.6) with the aid of equation (A3.4). In particular,

$$H_0^{(1,2)}(x) \approx 1 \pm \frac{2\mathrm{i}}{\pi}\ln\frac{\gamma x}{2} = \pm\frac{2\mathrm{i}}{\pi}\ln\left(-\frac{\gamma x}{\pm 2\mathrm{i}}\right). \tag{A3.7}$$

Asymptotic expressions for the cylindrical functions ($|x| \ll 1$) are

$$\left.\begin{aligned} J_p(x) &\approx \left(\frac{2}{\pi x}\right)^{1/2}\cos(x-\tfrac{1}{2}p\pi-\tfrac{1}{4}\pi)\,, \\ N_p(x) &\approx \left(\frac{2}{\pi x}\right)^{1/2}\sin(x-\tfrac{1}{2}p\pi-\tfrac{1}{4}\pi)\,, \\ H_p^{(1,2)}(x) &\approx \left(\frac{2}{\pi x}\right)^{1/2}\exp[\pm\mathrm{i}(x-\tfrac{1}{2}p\pi-\tfrac{1}{4}\pi)]\,, \\ I_n(x) &\approx \frac{\exp x}{(2\pi x)^{1/2}}\,, \qquad K_n(x) \approx \left(\frac{\pi}{2x}\right)^{1/2}\exp(-x)\,. \end{aligned}\right\} \tag{A3.8}$$

The following relationships may be shown to hold:

$$\left.\begin{aligned} Z_{p-1}(x)+Z_{p+1}(x) &= \frac{2p}{x}Z_p(x)\,, \\ Z_{p-1}(x)-Z_{p+1}(x) &= 2Z_p'(x)\,, \qquad Z_0'(x) = -Z_1(x)\,; \end{aligned}\right\} \tag{A3.9}$$

$$\left.\begin{aligned} I_{p-1}(x)-I_{p+1}(x) &= \frac{2p}{x}I_p(x)\,, \qquad I_{p-1}(x)-I_{p+1}(x) = 2I_p'(x)\,, \\ K_{p-1}(x)-K_{p+1}(x) &= -\frac{2p}{x}K_p(x)\,, \\ K_{p-1}(x)+K_{p+1}(x) &= -2K_p'(x)\,. \end{aligned}\right\} \tag{A3.10}$$

where Z_p represents J_p, N_p, or $H_p^{(1,2)}$.

The Bessel functions may also be written in integral form:

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi}\int_\alpha^{\alpha+2\pi}\exp[\mathrm{i}(x\sin\varphi-n\varphi)]\,\mathrm{d}\varphi \\ &= \frac{(-\mathrm{i})^n}{2\pi}\int_\alpha^{\alpha+2\pi}\exp[\mathrm{i}(x\cos\varphi-n\varphi)]\,\mathrm{d}\varphi\,, \end{aligned} \tag{A3.11}$$

where α is a real number.

Integrals involving Bessel functions may be evaluated with the aid of equations (A3.9) and (A3.10). In particular,

$$\int x^p Z_p(x)\,\mathrm{d}x = x^p Z_{p-1}(x)\,, \qquad \int x^{-p} Z_p(x)\,\mathrm{d}x = -x^{-p} Z_{p+1}(x)\,, \qquad (A3.12)$$

$$\int_0^1 x J_p(\alpha x) J_p(\beta x)\,\mathrm{d}x = \frac{\alpha J_p'(\alpha) J_p(\beta) - \beta J_p'(\beta) J_p(\alpha)}{\beta^2 - \alpha^2}\,, \qquad (A3.13)$$

The following integral formulae are also employed in the present collection ($\mathrm{Re}\,k > 0$):

$$(\rho^2 + z^2)^{-1/2} = \int_0^\infty \exp(-k|z|) J_0(k\rho)\,\mathrm{d}k\,, \qquad (A3.14)$$

$$\int_0^\infty \frac{\cos px\,\mathrm{d}x}{(q^2+x^2)^{s+1}} = \sqrt{\pi}\left(\frac{p}{2q}\right)^{s+\frac{1}{2}} \frac{K_{s+\frac{1}{2}}(pq)}{\Gamma(s+1)}\,, \qquad (A3.15)$$

$$\int_0^\infty \frac{J_0(xr) x\,\mathrm{d}x}{x^2 + k^2} = k K_0(kr)\,. \qquad (A3.16)$$

Spherical Bessel functions of the first kind and spherical Hankel functions of the first and second kind [denoted by the common symbol $z_l(x)$] are defined by

$$j_l(x) = \left(\frac{\pi}{2x}\right)^{1/2} J_{l+\frac{1}{2}}(x)\,, \qquad h_l^{(1,2)}(x) = \left(\frac{\pi}{2x}\right)^{1/2} H_{l+\frac{1}{2}}^{(1,2)}(x)\,. \qquad (A3.17)$$

For small x,

$$j_l(x) = \frac{x^l}{(2l+1)!!}\,, \qquad h_l^{(1,2)}(x) \text{ diverges as } x^{-l-1}\,. \qquad (A3.18)$$

For large x,

$$j_l(x) = \frac{1}{x}\cos[x - \tfrac{1}{2}(l+1)\pi]\,,$$
$$h_l^{(1,2)}(x) = \frac{1}{x}\exp\{\pm \mathrm{i}[x - \tfrac{1}{2}(l+1)\pi]\}\,. \qquad (A3.19)$$

When $R = |\boldsymbol{r} - \boldsymbol{r}'|$ (see figure 2.0.2) then

$$\frac{\exp(\mathrm{i}kR)}{R} = 4\pi \mathrm{i}k \sum_{l,\,m} j_l(kr') h_l^{(1)}(kr) Y_{lm}(\vartheta, \phi) Y_{lm}^*(\vartheta', \phi') \quad (r > r')\,. \qquad (A3.20)$$

The functions $z_l(kr) Y_{lm}(\vartheta, \phi)$ are special solutions of the wave equation for monochromatic waves: $\nabla^2\varphi + k^2\varphi = 0$. Any linear combination of these functions is also a solution.

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