MATRIX ANALYSIS

A Quick Guide

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Preface

Greetings,

Matrix Analysis: A Quick Guide to is compiled based on my MA353: Matrix Analysis notes with professor Leo Livshits. The sections are based on a number of resources: *Linear Algebra Done Right* by Axler, *A Second Course in Linear Algebra* by Horn and Garcia, *Matrices and Linear Transformations* by Cullen, *Matrices: Methods and Applications* by Barnett, *Problems and Theorems in Linear Algebra* by Prasolov, *Matrix Operations* by Richard Bronson, and professor Leo Livshits' own textbook (in the making). Prerequisites: some prior exposure to a first course in linear algebra.

The development of this text will come in layers. The first layer, one that I am working on during the course of S'19 MA353, will be an overview of the key topics listed in the table of contents. As the semester progresses, I will be constantly updating the existing notes, as well as adding prof. Livshits' problems and my solutions to the problems. The second layer will come after the course is over, when concepts will have hopefully "come together."

I will decide how much narrative I should put into the text as text is developed over the semester. I'm thinking that I will only add detailed explanations wherever I find fit or necessary for my own studies. I will most likely keep the text as condensed as I can.

Enjoy!

Contents

1 List of Special Matrices & Their Properties

1. **Hermitian/Self-adjoint**: $H = H^\dagger$. A Hermitian matrix is matrix that is equal to its own conjugate transpose:

$$H \text{ is Hermitian} \iff H_{ij} = \bar{H}_{ji}$$

Properties 1.1.

(a) H is Hermitian $\iff \langle w, Hv \rangle = \langle Hw, v \rangle$, where $\langle , \rangle$ denotes the inner product.

(b) H is Hermitian $\iff \langle v, Hv \rangle \in \mathbb{R}$.

(c) H is Hermitian $\iff$ it is *unitarily diagonalizable* with *real eigenvalues.*

Unitary: $U^*U = UU^* = I = U^\dagger U = UU^\dagger$. The real analogue of a unitary matrix is an orthogonal matrix. The following list contain the properties of U:

(a) U preserves the inner product:

$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$

(b) U is normal: it commutes with $U^* = U^\dagger$.

(c) U is diagonalizable:

$$U = VDV^*,$$

where D is diagonal and unitary, and V is unitary.

(d) $|\det(U)| = 1$ (hence the real analogue to U is an orthogonal matrix)

(e) Its eigenspaces are orthogonal.

(f) U can be written as

$$U = e^{iH},$$

where H is a Hermitian matrix.

(g) Any square matrix with unit Euclidean norm is the average of two unitary matrices.

2. **Cofactor**: ?

2 List of Operations

1. **Conjugate transpose** is what its name suggests.

2. **Classical adjoint/Adjugate/adjunct** of a square matrix is the transpose of its cofactor matrix.

3 List of Algorithms

4 Complex Numbers

4.1 A different point of view

We often think of complex numbers as

$$a + ib$$

where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. While there is nothing "bad" about this way of thinking - in fact thinking of complex numbers as $a+ib$ allows us to very quickly and intuitively do arithmetics operations on them - a "matrix representation" of complex numbers can give us some insights on "what we actually do" when we perform complex arithmetics.

Let us think of

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

as a different representation of the same object - the same complex number "$a + ib$." Note that it does not make sense to say the matrix representation **equals** the complex number itself. But we shall see that a lot of the properties of complex numbers are carried into this matrix representation under interesting matricial properties.

$$\boxed{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \sim a + ib}$$

First, let us break the the matrix down:

$$a + ib = a \times 1 + i \times b \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aI + b\mathcal{I}.$$

Right away, we can make some "mental connections" between the representations:

$$I \sim 1$$
$$\mathcal{I} \sim i.$$

Now, we know that complex number multiplications commute:

$$(a + ib)(c + id) = (c + id)(a + ib).$$

Matrix multiplications are not commutative. So, we might wonder whether commutativity holds under the this new representation of complex numbers. Well, the answer is yes. We can readily verify that

$$(aI + b\mathcal{I})(cI + b\mathcal{I}) = (cI + b\mathcal{I})(aI + b\mathcal{I}).$$

How about additions? Let's check:

$$(a+ib)+(c+id)=(a+c)+i(b+d)\sim\begin{pmatrix} a+c & -(b+d) \\ (b+d) & a+c \end{pmatrix}=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}+\begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Ah! Additions work. So, the new representation of complex numbers seems to be working flawlessly. However, we have yet to gain any interesting insights into the connections between the representations. To do that, we have to look into changing the form of the matrix. First, let's see what conjugation does:

$$(a+ib)^*=a-ib\sim\begin{pmatrix} a & b \\ -b & a \end{pmatrix}=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^\top$$

Ah, so conjugation to a complex number in the traditional representation is the same as transposition in the matrix representations. What about the amplitude square? Let us call

$$M=\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We have

$$(a+ib)(a-ib)\sim\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}=MM^\top=(a^2+b^2)I=\det(M)I$$

Interesting. But observe that if $\det(M)\neq 0$

$$\frac{1}{\det(M)}MM^\top=I.$$

This tells us that

$$M^\top=M^{-1},$$

where M^{-1} is the inverse of M, and, not surprisingly, it corresponds to the reciprocal to the complex number $a+ib$. We can readily show that

$$M^{-1}\sim(a+ib)^{-1}=\frac{1}{a^2+b^2}(a-ib).$$

Remember that we can also think of a complex number as a column vector:

$$c+id\sim\begin{pmatrix} c \\ d \end{pmatrix}.$$

Let us look back at complex number multiplication under matrix representation:

$$(a+ib)(c+id)=(ac-bd)+i(bc+ad)\sim\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix}=\begin{pmatrix} ac-bd \\ bc+ad \end{pmatrix}.$$

Multiplication actually works in this "mixed" way of representing complex numbers as well. Now, observe that what we just did was performing a linear transformation on a vector in $\mathbb{R}^2$. It is always interesting to look at the geometrical interpretation of this transformation. To do this, let us call N the "normalized" version of M:

$$N = \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We immediately recognize that N is an orthogonal matrix. This means N is an orthogonal transformation (length preserving). Now, it is reasonable to define

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$$
$$\sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

We can write N as

$$N = \begin{pmatrix} \cos\theta & -\sin\theta \sin\theta & \cos\theta \end{pmatrix},$$

which is a physicists' favorite matrix: the rotation by θ. So, let us write M in terms of N:

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \sqrt{a^2+b^2} N = \sqrt{a^2+b^2} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We can interpret M as a rotation by θ, followed by a scaling by $\sqrt{(a^2+b^2)}$. But what $\sqrt{a^2+b^2}$ exactly is just the "length" or the "amplitude" of the complex number $a+ib$, if we think of it as an arrow in a plane.

4.2 Relevant properties and definitions

1. The *modulus* of $z = a+ib$ is the "amplitude" of z, denoted by $|z| = \sqrt{a^2+b^2} = z\bar{z}$.
2. The modulus is *multiplicative*, i.e.

$$|wz| = |w||z|.$$

3. Triangle inequality:

$$|z+w| \leq |z| + |w|.$$

   We can readily show this geometrically, or algebraically.

4. The *argument* of $z = a+ib$ is θ, where

$$\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right), \text{ if } a > 0 \\ \frac{\pi}{2} + k2\pi, k \in \mathbb{R} \text{ if } a = 0, b > 0 \\ -\frac{\pi}{2} + k2\pi, k \in \mathbb{R} \text{ if } a = 0, b < 0 \\ \text{Undefined if } a = b = 0. \end{cases}$$

5. The *conjugate* of $a+ib$ is $a-ib$. Conjugation is *additive* and *multiplicative*, i.e.

$$\begin{aligned} \bar{z + w} &= \bar{z} + \bar{w} \\ \bar{wz} &= \bar{w}\bar{z}. \end{aligned}$$

   Note that we can also show the multiplicative property with the matrix representation as well:

$$\bar{wz} \sim (WZ)^\top = Z^\top W^\top \sim \bar{z}\bar{w} = \bar{w}\bar{z}.$$

6. Euler's identity, generalized to de Moivre's formula:

$$z^n = r^n e^{in\theta}.$$

5 Vector Spaces & Linear Functions

5.1 Review of Linear Spaces and Subspaces

Properties 5.1. of linear spaces:

1. Commutativity and associativity of addition
2. Existence of an additively neutral element (null element). Zero multiples of elements give the null element: $0 \cdot V = \mathbf{0}$
3. Every element has an (unique) additively antipodal element
4. Scalar multiplication distributes over addition
5. Multiplicative identity:
6. $ab \cdot V = a \cdot (bV)$
7. $(a+b)V = aV + bV$

W is a subspace of V if

1. $S \subseteq V$
2. S is non-empty
3. S is closed under addition and scalar multiplication

Properties 5.2. that are interesting/important/maybe-not-so-obvious:

1. If S is a subspace of V and $S \neq V$ then S is a proper subspace of V.
2. If X in a subspace of Y and Y is a subspace of Z, then X is a subspace of W.
3. Non-trivial linear (non-singleton) spaces are infinite.

5.2 Review of Linear Maps

Consider linear spaces V and W and elements $v \in V$ and $w \in W$ and scalars $\alpha, \beta \in \mathbb{R}$, a function $F : V \to W$ is a linear map if

$$F[\alpha v + \beta w] = \alpha F[v] + \beta F[w].$$

Properties 5.3.

1. $F[\mathbf{0}_V] = \mathbf{0}_W$
2. $G[w] = (\alpha \cdot F)[w] = \alpha \cdot F[w]$
3. Given $F : V \to W$ and $G : V \to W$, $H[v] = F[v] + G[v] = (F+G)[v]$ is call the sum of the functions F and G.

4. Linear combinations of linear maps are linear.
5. Compositions of linear maps are linear.
6. Compositions distributes over linear combinations of linear maps.
7. Inverses of linear functions (if they exist) are linear.
8. Inverse of a bijective linear function is a bijective linear function

5.3 Review of Kernels and Images

Definition 5.1. Let $F : V \to W$ be given. The kernel of F is defined as

$$\ker(F) = \{v \in V | F[v] = \mathbf{0}_W\}.$$

Properties 5.4. Let $F : V \to W$ a linear map be given. Also, consider a linear map G such that $F \circ G$ is defined

1. F is null $\iff \ker(F) = V \iff \mathrm{Im}(F) = \mathbf{0}_W$
2. $\ker(F)$ is a subspace of V
3. $\mathrm{Im}(F)$ is a subspace of W
4. F is injective $\iff \ker(F) = \mathbf{0}_V$
5. $\ker(F) \subseteq \ker(F \circ G)$.
6. F injective $\implies \ker(F) = \ker(F \circ G)$

Definition 5.2. Let $F : V \to W$ be given. The image of F is defined as

$$\mathrm{Im}(F) = \{w \in W | \exists v \in V, F[v] = w\}.$$

5.4 Atrices

Definition 5.3. Atrix functions: Let $V_1, \ldots, V_m \in \mathbf{V}$ be given. Consider $f : \mathbb{R}^m \to \mathbf{V}$ be defined by

$$f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \sum_{i=1}^{m} a_i V_i.$$

We denote f by

$$\begin{pmatrix} V_1 & V_2 & \ldots & V_m \end{pmatrix}.$$

We refer to the V_i's as the **columns** of f, even though there doesn't have to be any columns. Basically, f is simply a function that takes in an ordered list of coefficients and returns a linear combination of V_i with the respective coefficients. A matrix is a special atrix. Not every atrix is a matrix.

Properties 5.5. of atrices

1. The V_i's - the columns of an atrix - are the images of the standard basis tuples.

2. $\mathbf{e}_j \in \ker(V_1 \dots V_m) \iff V_j = \mathbf{0}_V$, where $\mathbf{e}_j$ denotes a standard basis tuple with a 1 at the j^{th} position. To put in words, a kernel of an atrix contains a standard basis if and only if one of its columns in a null element.

3. $\mathrm{Im}(f) \equiv \mathrm{Im}(V_1 \dots V_m) = \mathrm{span}(V_1 \dots V_m)$

4. B is a null atrix $\iff \ker(B) = \mathbb{R}^m \iff \mathrm{Im}(B) = \mathbf{0}_V \iff V_j = \mathbf{0}_V \forall j = 1, 2, \dots, m$.

5. f is a linear function $\mathbb{R}^m \to V \iff f$ is an atrix function $\mathbb{R}^m \to V$.

6. An atrix A is bijective/invertible, then its inverse A^{-1} is a linear function, but is an atrix only if A is a matrix.

7. Linear combinations of atrices are atrices:

$$\begin{aligned} &\alpha \cdot \begin{pmatrix} V_1 & \cdots & V_m \end{pmatrix} + \beta \cdot \begin{pmatrix} W_1 & \dots & W_m \end{pmatrix} \\ &= \begin{pmatrix} \alpha V_1 + \beta W_1 & \dots & \alpha V_m + \beta W_m \end{pmatrix} \end{aligned}$$

8. Compositions of two atrices are NOT defined unless the atrix going first is a matrix. Consider $F : \mathbb{R}^m \to W$ and $G : \mathbb{R}^n \to T$. $F \circ G$ is only defined if $T = \mathbb{R}^m$. This make G an $n \times m$ matrix. It follows that the atrix $F \circ G$ has the form

$$\begin{pmatrix} f(g_1) & f(g_2) & \dots & f(g_m) \end{pmatrix}.$$

9. Consider $F : \mathbb{R}^m \to V$ and $G : \mathbb{R}^k \to V$. $\mathrm{Im}(F) \subseteq \mathrm{Im}(G) \iff F = G \circ C$, with $C \in \mathbb{M}_{k \times m}$, i.e. C is an $k \times m$ matrix.

10. Consider an atrix $A : \mathbb{R}^m \to V$. $A = (V_1 \dots V_m)$. A is NOT injective.
$\iff \exists$ a non-trivial linear combination of the columns of A that gives $\mathbf{0}_V$
$\iff A = [\mathbf{0}_v]$ or $\exists j | V_j$ is linear combination of other columns of A
$\iff$ The first column of A is $\mathbf{0}_V$ or $\exists j | V_j$ is a linear combination some of $V_i, i < j$.

11. If atrix $A : \mathbb{R} \to V$ has a single column then it is injective if the column is not $\mathbf{0}_V$.

Properties 5.6. of elementary column operations for atrices. Elementary operations on the columns of F can be expressed as a composition of F and an appropriate elementary matrix E, $F \circ E$.

1. Swapping i^{th} and j^{th} columns: $F \circ E^{[i] \leftrightarrow [j]}$.

2. Scaling the j^{th} column by α: $F \circ E^{\alpha \cdot [j]}$.

3. Adjust the j^{th} column by adding to it $\alpha \times i^{th}$ column: $F \circ E^{[i] \stackrel{+}{\leftarrow} \alpha \cdot [j]}$.

4. Elementary column operations do not change the 'jectivity nor image of F.

5. If a column of F is a linear combination of some of the other columns then elementary column operations can turn it into $\mathbf{0}_V$.

6. Removing/Inserting null columns or columns that are linear combinations of other columns does not change the image of F.

7. Given atrix A, it is possible to eliminate (or not) columns of A to end up with an atrix B with $\text{Im}(B) = \text{Im}(A)$.

8. If B is obtained from insertion of columns into atrix A, then $\text{Im}(B) = \text{Im}(A) \iff$ the insert columns $\in \text{Im}(A)$.

9. Inserting columns to a surjective A does not destroy surjectivity of A. (A is already having extra or just enough columns)

10. A surjective $\iff$ A is obtained by inserting columns (or not) into an invertible atrix $\iff$ deleting some columns of A (or not) gives an invertible matrix.

11. If A is injective, then column deletion does not destroy injectivity. (A is already "lacking" or having just enough columns)

12. The new atrix obtained from inserting columns from $\text{Im}(A)$ into A is injective $\iff$ A is injective.

5.5 Linear Independence, Span, and Bases

5.5.1 Linear Independence

$$\begin{aligned}
& X_1 \dots X_m \text{ are linearly independent} \\
\iff & F = [X_1 \dots X_m] \text{ injective} \\
\iff & \textstyle\sum a_i X_i = 0 \iff a_i = 0 \forall i \\
\iff & \mathbf{0}_V \notin \{X_i\} \text{ and none are linear combinations of some of the others.} \\
\iff & X_1 \neq \mathbf{0}_V \text{ and } X_j \text{ is not a linear combination of any of } X_i\text{'s for } i < j.
\end{aligned}$$

Properties 5.7.

1. The singleton list is linearly independent if its entry is not the null element

2. Sublists of a linearly independent list are linearly independent

3. List operations cannot create/destroy linearly independence.

4. If a linear map $L : X \to W$ injective, then $X_1 \dots X_m$ linearly independent $\iff$ $L(X_1) \dots L(X_m)$ linearly independent.

5.5.2 Span

Properties 5.8.

1. Spans are subspaces. $\text{span}(X_1 \dots X_m), X_j \in V$ is a subspace of V.
2. $\text{span}(X_1 \dots X_j) \subseteq \text{span}(X_1 \dots X_k)$ if $j \leq k$.
3. Adding elements to a list that spans V produces a list that spans V.
4. The following list operations do not change the span of V: removing the null/linearly dependent element, inserting a linearly dependent element, scaling element(s), adding (multiples) of an element to another element.
5. It is possible to reduce a list that spans to a list that spans AND have linearly independent elements.
6. Consider $A : V \to W$. If X_i span W then $A(X_i)$ span $\text{Im}(A)$. $i = 1, 2, \dots, m$.
7. If $A : V \to W$ invertible, then X_i span $V \iff A(X_i)$ span W, $i = 1, 2, \dots, m$.

5.5.3 Bases

Properties 5.9.

1. A list is a basis of V if the elements are linearly independent and they span V.
2. A singleton linear space has no basis.
3. Re-ordering the elements of a basis gives another basis.
4. $\{X_1 \dots X_m\}$ is a basis of V if $(X_1 \dots X_m)$ is injective AND $\text{Im}(X_1 \dots X_m) = V$, i.e. $(X_1 \dots X_m)$ invertible.
5. If $\{X_i\}$ is a basis of V and $\{Y_i\}$ is a list of elements in W, then there exists a unique linear function $L : V \to W$ satisfying

$$L[X_i] = Y_i$$

6. If $A : V \to W$ bijective, then $\{V_i\}$ forms a basis of V and $\{A(X_i)\}$ forms a basis of W.
7. Elementary operations on bases give bases.

5.6 Linear Bijections and Isomorphisms

Definition 5.4. Let linear spaces V, W be given. V is **isomorphic** to W if $\exists F : V \to W$ bijective. We say $V \sim W$.

Properties 5.10.

1. A non-zero scalar multiple of an isomorphism is an isomorphism.
2. A composition of isomorphism is an isomorphism.
3. "Isomorphism" behaves like an equivalence relation:
   (a) Reflexivity: $V \sim V$.
   (b) Symmetry: if $V \sim W$ then $W \sim V$.
   (c) Transitivity: if $V \sim W$ and $W \sim Z$ then $V \sim Z$.
4. Consider $F : V \to W$ an isomorphism.
   (a) Isomorphisms preserve linear independence. V_i's are linearly independent in $V \iff F(V_i)$'s are linearly independent in W.
   (b) isomorphisms preserve spanning. V_i's span $V \iff F(V_i)$'s span W.
   (c) Isomorphisms preserve bases. $\{V_i\}$ is a basis of $V \iff F\{(V_i)\}$ is a basis of W.
5. If $V \sim W$ then $\dim(V) = \dim(W)$ (finite or infinite).
6. If a linear map $A : V \to W$ is given and $A(X_i)$'s are linearly independent, then A_i's are linearly independent.
7. If a linear map $A : V \to W$ is injective and $\{X_i\}$ is a basis of V then $\{A(X_i)\}$ is a basis of $\mathrm{Im}(A)$

5.7 Rank-Nullity Theorem

5.8 Coordinatization & matricial representation of linear functions

6 Products of vector spaces & Sums of subspaces

6.1 Direct Sums

Definition 6.1. Let U_j, $j = 1, 2, \ldots m$ are subspaces of V. $\sum_1^m U_j$ is a *direct sum* if each $u \in \sum U_j$ can be written in only one way as $u = \sum_1^m u_j$. The direct sum $\sum_i^m U_j$ is denoted as $U_1 \oplus \cdots \oplus U_m$.

Properties 6.1.

1. Condition for direct sum: If all U_j are subspaces of V, then $\sum_1^m U_j$ is a direct sum $\iff$ the only way to write 0 as $\sum_1^m u_j$, where $u_j \in U_j$ is to take $u_j = 0$ for all j.

2. If U, W are subspaces of V and $U \bigcap W = \{0\}$ then $U + W$ is a direct sum.

6.2 Products and Quotients of Vector Spaces

6.3 Products of Vector Spaces

Definition 6.2. Product of vectors spaces

$$V_1 \times \cdots \times V_m = \{(v_1, \ldots, v_m) : v_j \in V_j, j = 1, 2, \ldots, m\}.$$

Definition 6.3. Addition on $V_1 \times \cdots \times V_m$:

$$(u_1, \ldots, u_m) + (v_1, \ldots, v_m) = (u_1 + v_1, \ldots, v_m + u_m).$$

Definition 6.4. Scalar multiplication on $V_1 \times \cdots \times V_m$:

$$\lambda(v_1, \ldots, v_m) = (\lambda v_1, \ldots, \lambda v_m).$$

Properties 6.2.

1. Product of vectors spaces is a vector space.

   V_j are vectors spaces over $\mathbb{F} \implies V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.

2. Dimension of a product is the sum of dimensions:

$$\dim(V_1 \times \cdots \times V_m) = \sum_1^m \dim(V_j)$$

6.4 Products & Direct Sums

Properties 6.3.

1. Let $U_1, \ldots, U_m$ be subspaces of V. Define a linear map $\Gamma : U_1 \times \cdots \times U_m \to U_1 + \cdots + U_m$ by:

$$\Gamma(u_1, \ldots, u_m) = \sum_1^m u_j.$$

   $U_1 + \cdots + U_m$ is a direct sum $\iff \Gamma$ is injective.

2. Let U_j be finite-dimensional and are subspaces of V.

$$U_1 \oplus \cdots \oplus U_m \iff \dim(U_1 + \cdots + U_m) = \sum_1^m \dim(U_j)$$

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