33. There is yet another bijection that lets us prove that a set of size n has 2^n subsets. Namely, for each subset S of $[n] = \{1, 2, \ldots, n\}$, define a function (traditionally denoted by χ_S) as follows.

$$\chi_S(i) = \begin{cases} 1 \text{ if } i \in S \\ 0 \text{ if } i \notin S \end{cases}$$

The function χ_S is called the *characteristic function* of S. Notice that the characteristic function is a function from $[n]$ to $\{0, 1\}$.

(a) For practice, consider the function $\chi_{\{1,3\}}$ for the subset $\{1, 3\}$ of the set $\{1, 2, 3, 4\}$. What are

i. $\chi_{\{1,3\}}(1)$?
Solution: 1 ▮

ii. $\chi_{\{1,3\}}(2)$?
Solution: 0 ▮

iii. $\chi_{\{1,3\}}(3)$?
Solution: 1 ▮

iv. $\chi_{\{1,3\}}(4)$?
Solution: 0 ▮

(b) We define a function f from the set of subsets of $[n] = \{1, 2, \ldots, n\}$ to the set of functions from $[n]$ to $\{0, 1\}$ by $f(S) = \chi_S$. Explain why f is a bijection.

Solution: Suppose S and T are subsets of $[n]$. If $i \in S$ but $i \notin T$, then $\chi_S(i) = 1$ but $\chi_T(i) = 0$. Thus if $S \neq T$, then $\chi_S \neq \chi_T$. Therefore f is one-to-one. Given a function g from $[n]$ to $\{0, 1\}$, let $S = \{i | g(i) = 1\}$. Then by definition, $g = \chi_S = f(S)$. Therefore f is onto, so it is a bijection. ▮

(c) Why does the fact that f is a bijection prove that $[n]$ has 2^n subsets?

Solution: We have seen that there are 2^n functions from $[n]$ to the two-element set $\{0, 1\}$, and we have just described a bijection between the set of all such functions and the subsets of $[n]$. ▮

➔ •43. In how many ways may we string n distinct beads on a necklace without a clasp? (Perhaps we make the necklace by stringing the beads on a string, and then carefully gluing the two ends of the string together so that the joint can't be seen. Assume someone can pick up the necklace, move it around in space and put it back down, giving an apparently different way of stringing the beads that is equivalent to the first.)

Solution: We can obtain a permutation of the beads by cutting the necklace and stretching it out in a straight line. We can partition the permutations according to which necklace they come from in this process. Two permutations are in the same block if we get one either by circularly permuting the other and/or by reversing the other (this corresponds to flipping the necklace over in space). Thus each necklace corresponds to $2n$ permutations so by the quotient principle we have $n!/2n = (n-1)!/2$ ways to string n distinct beads on a necklace. ▮

➔ ·45. (This becomes especially relevant in Chapter 6, though it makes an important point here.) In how many ways may we attach two identical red beads and two identical blue beads to the corners of a square (with one bead per corner) free to move around in (three-dimensional) space?

Solution: Two ways; either the red beads are side-by-side or diagonally opposite. If we think about partitioning lists of 2 Rs and 2 Bs so that two are in the same block if we get one from the other by moving the square, we get two blocks, $\{RRBB, BRRB, BBRR, RBBR\}$ and $\{RBRB, BRBR\}$. This is an example of a problem with a good deal of symmetry in which the blocks of the relevant partition have different sizes. ∎

Supp ·3. Write down a list of all 16 zero-one sequences of length four starting with 0000 in such a way that each entry differs from the previous one by changing just one digit. This is called a Gray Code. That is, a *Gray Code* for 0-1 sequences of length n is a list of the sequences so that each entry differs from the previous one in exactly one place. Can you describe how to get a Gray Code for 0-1 sequences of length five from the one you found for sequences of length 4? Can you describe how to prove that there is a Gray code for sequences of length n?

Solution: (One of many) 0000, 0001, 0011, 0010, 0110, 0111, 0101, 0100, 1100, 1101, 1111, 1110, 1010, 1011, 1001, 1000. To get a code for sequences of length 5, put a zero at the end of each of the sequences we have. Follow that revised sequence by 10001, and write the remainder of the original sequence in reverse order with a 1 at the end of each term. (Don't reverse the individual length four sequences, just the sequence of sequences!) If we do this with a Gray Code for sequences of length n, we get a Gray code for sequences of length $n+1$. Thus we can get a Gray code for sequences of any length we wish. In the terminology of Chapter 2, we just described the inductive step of an inductive proof that Gray Codes exist for sequences of any length. ∎

Supp ➔4. Use the idea of a Gray Code from Problem 3 to prove bijectively that the number of even-sized subsets of an n-element set equals the number of odd-sized subsets of an n-element set.

Solution: Each sequence in the Gray Code is the characteristic function of a set, and the number of elements of the set is the number of ones in the sequence. Since each sequence differs in just one place from the preceding one, the sequences alternate between having an even number of ones and an odd number of ones. Since the first sequence is all zeros and there are 2^n sequences, the last one has an odd number of zeros. Thus the map that takes each sequence except the last to the next one, and takes the last to the first is a bijection between the characteristic functions of sets with an even number of elements and sets with an odd number of elements. ∎