<u>Def</u>: A matrix Lie group is a closed subgroup $G \leq GL(n,\mathbb{C})$ for some $n \in \mathbb{N}$ (closed w.r.t. the topology induced from $M_n(\mathbb{C})$) → using operator norm

$$\|A\| = \sup\left\{ \frac{\|Ax\|}{\|x\|} \,\middle|\, x \in \mathbb{C}^n \setminus \{0\} \right\}$$

<u>examples</u>

- $GL(n,\mathbb{C})$ general linear group over $\mathbb{C}$
- $SL(n,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid \det A = 1\}$ special linear group over $\mathbb{C}$
- $GL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{C}) \mid \bar{X} - X = 0\}$ general linear group over $\mathbb{R}$
- $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}$ special linear group over $\mathbb{R}$
- $O(n) = \{A \in GL(n,\mathbb{R}) \mid A^t A = \mathbb{1}\}$ orthogonal group
- $SO(n) = \{A \in O(n) \mid \det A = 1\}$ special orthogonal group
- $U(n) = \{A \in GL(n,\mathbb{C}) \mid A^* A = \mathbb{1}\}$ unitary group
- $SU(n) = \{A \in U(n) \mid \det A = 1\}$ special unitary group

] real Lie groups despite having complex matrices!

Def: A real or complex Lie algebra is a vector space V over $\mathbb{R}$ or $\mathbb{C}$ with an operation $[\cdot,\cdot]: V \times V \to V$ (Lie bracket) satisfying

- $[\alpha x + \beta y, z] = \alpha [x,z] + \beta [y,z]$
- $[z, \alpha x + \beta y] = \alpha [z,x] + \beta [z,y]$
- $[y,x] = -[x,y]$
- $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$

Def: Let $G \leq GL(n,\mathbb{C})$ be a MLG. The associated Lie algebra is the set $Lie(G) = \{X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in G \ \forall \varepsilon \in \mathbb{R}\}$ with Lie bracket
$[X,Y] = XY - YX$

↳ matrix exponential $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$

Note: $Lie(G)$ is usually denoted by $\mathfrak{g}$

proof can be found in "extra" folder on google drive