Name: **Huan Q. Bui**
Course: **8.370 - QC**
Problem set: **#1**
Due: Wednesday, Sep 21, 2022.

1. Useful properties of unitary matrices

(a) Consider a d-dimension quantum space with orthonormal bases $\{|1\rangle, |2\rangle, \dots, |d\rangle\}$ and $\{|v_1\rangle, |v_2\rangle, \dots, |v_d\rangle\}$. We shall construct a unitary matrix U for which $U|j\rangle = |v_j\rangle$. To this end, we use the standard basis $\{|e_1\rangle, |e_2\rangle, \dots, |e_d\rangle\}$ as an intermediate basis. The matrix that transforms $|e_j\rangle$ to $|j\rangle$ is simply one whose jth-column has the components of $|j\rangle$ in the standard basis:

$$|j\rangle = U_A |e_j\rangle \;\; \forall j = 1, 2, \dots, d \quad \text{if} \quad U_A = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ |1\rangle & |2\rangle & \dots & |d\rangle \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

Similarly,

$$|v_j\rangle = U_B |e_j\rangle \;\; \forall j = 1, 2, \dots, d \quad \text{if} \quad U_B = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ |v_1\rangle & |v_2\rangle & \dots & |v_d\rangle \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

Since the provided bases are orthonormal, it is clear by definition of U_A and U_B that $U_A^\dagger U_A = U_B^\dagger U_B = \mathbb{I}$, so both U_A and U_B are unitary. Our desired matrix U is then given by

$$U = U_B U_A^\dagger = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ |v_1\rangle & |v_2\rangle & \dots & |v_d\rangle \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix} \begin{pmatrix} \leftarrow & \langle 1| & \rightarrow \\ \leftarrow & \langle 2| & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & \langle d| & \rightarrow \end{pmatrix},$$

which is also unitary since $U^\dagger U = U_A U_B^\dagger U_B U_A^\dagger = U_A U_A^\dagger = \mathbb{I}$. It is clear that $U|j\rangle = |v_j\rangle$, but to see explicitly, suppose we apply U to $|1\rangle$. The application of $U_A^\dagger$ returns the column vector $|e_1\rangle = (1\ 0\ 0\ \dots)^\top$. The subsequent application of U_B therefore returns its first column, which is $|v_1\rangle$, as desired.

(b) Let an orthonormal basis $\{|v_1\rangle, |v_2\rangle, \dots, |v_d\rangle\}$ be given. In the standard basis $\{|e_k\rangle\}$, we may write

$$|v_i\rangle = \sum_{k=1}^{d} (v_i)_k |e_k\rangle,$$

so that

$$\sum_{i=1}^{d} |v_i\rangle\langle v_i| = \sum_{i=1}^{d} \left[\sum_{k=1}^{d} (v_i)_k |e_k\rangle\right] \left[\sum_{l=1}^{d} (v_i)_l^* \langle e_l|\right].$$

Using the fact that $|e_m\rangle\langle e_n| = 0$ if $m \neq n$ and $|e_m\rangle\langle e_m| = \Pi_m$ we have

$$\sum_{i=1}^{d} |v_i\rangle\langle v_i| = \sum_{i=1}^{d}\sum_{k=1}^{d} |(v_i)_k|^2 \Pi_k = \sum_{k=1}^{d}\sum_{i=1}^{d} |(v_i)_k|^2 \Pi_k = \sum_{k=1}^{d} \Pi_k = \mathbb{I},$$

where we have used the fact that the given basis is orthonormal in the third equality and resolution of identity with standard projections in the last equality.

2. Angle between quantum states and angle between associated points on the Bloch sphere

(a) The point $p_i = (x_i, y_i, z_i)$ on the Bloch sphere is associated with the quantum state of a qubit $|v_i\rangle$ where

$$|v_i\rangle\langle v_i| - |\bar{v}_i\rangle\langle\bar{v}_i| = x_i\sigma_x + y_i\sigma_y + z_i\sigma_z$$

where $|\bar{v}_i\rangle$ is orthogonal to $|v_i\rangle$. Because the quantum system is 2-dimensional and $|v_i\rangle \perp |\bar{v}_i\rangle$, we have that $\{|v_i\rangle, |\bar{v}_i\rangle\}$ is an orthonormal basis. This implies

$$|v_i\rangle\langle v_i| + |\bar{v}_i\rangle\langle\bar{v}_i| = \mathbb{I}.$$

Combine this with the equation above, we find that

$$|v_i\rangle\langle v_i| = \frac{\mathbb{I} + x_i\sigma_x + y_i\sigma_y + z_i\sigma_z}{2} = \frac{\mathbb{I} + \vec{p}_i\cdot\vec{\sigma}}{2}.$$

(b) Using the fact that

$$|\langle v_1|v_2\rangle|^2 = \langle v_1|v_2\rangle\langle v_2|v_1\rangle = \mathrm{Tr}\left(|v_1\rangle\langle v_1|v_2\rangle\langle v_2|\right),$$

which can be proved using the cyclic property of the trace, we find that

$$|\langle v_1|v_2\rangle|^2 = \mathrm{Tr}\left(\frac{\mathbb{I}+\vec{p}_1\cdot\vec{\sigma}}{2}\frac{\mathbb{I}+\vec{p}_2\cdot\vec{\sigma}}{2}\right) = \frac{1+\vec{p}_1\cdot\vec{p}_2}{2}. \qquad \text{(using Mathematica)}$$

Let θ denote the angle between $|v_1\rangle$ and $|v_2\rangle$ and θ' denote the angle between $\vec{p}_1$ and $\vec{p}_2$, then

$$\theta = \arccos|\langle v_1|v_2\rangle| = \arccos\left(\sqrt{\frac{1+\cos\theta'}{2}}\right) = \arccos\left(\left|\cos\frac{\theta'}{2}\right|\right) \to \frac{\theta'}{2}$$

If we ignore a possible minus sign due to relative orientation, the angle θ between quantum states is **half** the angle between associated points on the Bloch sphere. This makes sense, as *orthogonal* quantum states occupy opposite poles on the Bloch sphere.

Mathematica code:

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In[7]:= Id = {{1, 0}, {0, 1}};

In[8]:= \[Sigma]x = PauliMatrix[1];

In[9]:= \[Sigma]y = PauliMatrix[2];

In[10]:= \[Sigma]z = PauliMatrix[3];

In[11]:= \[Sigma] = {\[Sigma]x, \[Sigma]y, \[Sigma]z};

In[15]:= p1 = {x1, y1, z1};

In[16]:= p2 = {x2, y2, z2};

In[32]:= M = (Id + Dot[p1, \[Sigma]]) . (Id + Dot[p2, \[Sigma]])/4;

In[29]:= Tr[M] // Simplify

Out[29]= 1/2 (1 + x1 x2 + y1 y2 + z1 z2)
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3. von Neumann measurement We have

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1+i}{\sqrt{3}}|1\rangle.$$

Suppose we make a von Neumann measurement in the basis

$$\left\{\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right\}$$

4. Qutrit

(a)

(b)

5. Perfect polarizing filter

(a)

(b)

6.

(a)

(b)

(c)