

MA439: Functional Analysis
Tychonoff Spaces: Exercises 1-6 on p.36, Ben Mathes

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Exercise 1 (Ex 1, p.36). Let \mathcal{X} be a topological space. Prove that if d is a continuous pseudometric, then the sets $\{y \in \mathcal{X} : d(x, y) > \delta\}$ are open, where $x \in \mathcal{X}$ and $\delta \in \mathbb{R}$.

Proof. Let $O = \{y \in \mathcal{X} : d(x, y) > \delta\}$. We want to show that each $y \in O$ is an interior point of O . Let $y \in O$ be given, then $d(x, y) > \delta$. This means that $d(x, y) \geq \delta + \epsilon$ for some $\epsilon > 0$. d is a continuous pseudometric, so every d -ball is an open subset of \mathcal{X} . In particular, $B_d(y, \epsilon/2)$ is an open subset of \mathcal{X} . By the triangle inequality, for any $z \in B_d(y, \epsilon/2)$, $z \in O$. Thus, $B_d(y, \epsilon/2) \subseteq O$. So, O is open as desired. \square

Exercise 2 (Ex 2, p.36). Let \mathcal{X} be a topological space. Prove that d is a continuous pseudometric on \mathcal{X} if and only if the function $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$.

Proof. (\implies) Suppose that d is a continuous pseudometric on \mathcal{X} . Let $\epsilon > 0$ and $x \in \mathcal{X}$. f_x^d is continuous at $y \in \mathcal{X}$ if and only if for every $\epsilon > 0 \exists f(y) \in G \subseteq \mathcal{X}$ open for which $|f_x^d(y) - f_x^d(y')| < \epsilon$ whenever $y' \in G$. Note that $|f_x^d(y) - f_x^d(y')| = |d(x, y) - d(x, y')| \leq d(y, y')$. So, we just take $G = B_d(y, \epsilon)$.

(\impliedby) Let d be a pseudometric and suppose that $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$. We want to show that every d -ball is open in \mathcal{X} . To this end, let $x \in \mathcal{X}$ and $\delta > 0$ be given and consider $B_d(x, \delta) = \{y \in \mathcal{X} : d(x, y) < \delta\} = \{y \in \mathcal{X} : f_x^d(y) < \delta\} = \{y \in \mathcal{X} : f_x^d(y) \in (-\delta, \delta)\}$ which is open by continuity of f_x^d . So we're done. \square

Exercise 3 (Ex 3, p.36). Let \mathcal{X} be a Tychonoff space whose topology is generated by the family of pseudometrics \mathcal{G} . Prove that the topology on \mathcal{X} is the same as the weak topology induced by the family of functions f_x^d where $x \in \mathcal{X}$, $d \in \mathcal{G}$.¹

Proof. One inclusion is trivial. It remains to show the other inclusion. Tychonoff: for every closed set $F \subseteq \mathcal{X}$ and every $x \in F$, there exists a continuous function $f : \mathcal{X} \rightarrow \mathbb{R}$ for which $f[F] = \{0\}$ and $f(x) = 1$. From \mathcal{G} , use balls as a subbase and build the topology from those balls. Alternatively, we can build the functions $\{f_x^d : x \in \mathcal{X}, d \in \mathcal{G}\}$ and build the (open-ball) topology by taking inverse images of open sets. From the previous exercise, we have that weak topology $\implies f_x^d$ are all continuous, which implies that all balls are open relative to the weak topology, which implies that the new (open ball) topology is contained in the weak topology. \square

Exercise 4 (Ex 4, p.36). Assume \mathcal{X} is a Tychonoff space with generating family \mathcal{G} . If E is a subset of \mathcal{X} , let \mathcal{G}_E denote the set of restrictions of elements of \mathcal{G} to E . Prove that the resulting Tychonoff Topology on E generated by the family \mathcal{G}_E is the same as the topological **subspace topology** that E inherits from the topology on \mathcal{X} .

Proof. get base from finite intersection of balls. G open iff for every $x \in G$ there exist finitely many $d_1, \dots, d_k \in \mathcal{G}$ and $\epsilon_1, \dots, \epsilon_k > 0$ such that $\cap_{i=1}^k B_{d_i}(x, \epsilon_i) \subseteq G$. \square

¹completely regular \equiv Tychonoff

Exercise 5 (Ex 5, p.36). *Give an example of a continuous pseudometric on $(0,1)$ that is not the restriction of a continuous pseudometric on \mathbb{R} to $(0,1)$.*

Proof. blah □

Exercise 6 (Ex 6, p.36). *Prove that a bounded continuous pseudometric on $(0,1)$ is the restriction of a continuous pseudometric on \mathbb{R} to $(0,1)$. (?CHECK?)*

Proof. blah □

Exercise 7 (Ex 7, p.36). *Proof.* □

Exercise 8 (Ex 8, p.36). *Proof.* □

Exercise 9 (Ex 9, p.36). *Proof.* □

Exercise 10 (Ex 10, p.36). *Proof.* □

Exercise 11 (Ex 11, p.36). *Proof.* □

Exercise 12 (Ex 12, p.36). *Proof.* □

Exercise 13 (Ex 14, p.36). *Proof.* □