It is remarkable that there is no known formula for $P(k, n)$, nor is there one for $P(k)$. This section is devoted to developing methods for computing values of $P(n, k)$ and finding properties of $P(n, k)$ that we can prove even without knowing a formula. Some future sections will attempt to develop other methods.

We have seen that the number of partitions of k into n parts is equal to the number of ways to distribute k identical objects to n recipients so that each receives at least one. If we relax the condition that each recipient receives at least one, then we see that the number of distributions of k identical objects to n recipients is $\sum_{i=1}^{n} P(k, i)$ because if some recipients receive nothing, it does not matter which recipients these are. This completes rows 7 and 8 of our table of distribution problems. The completed table is shown in Figure 3.2. Every entry in that table tells us how to count something. There are quite a few theorems that you have proved which are summarized by Table 3.2. It would be worthwhile to try to write them all down! The methods we used to complete Figure 3.2 are extensions of the basic counting principles we learned in Chapter 1. The remaining chapters of this book develop more sophisticated kinds of tools that let us solve more sophisticated kinds of counting problems.

3.3.4 Partitions into distinct parts

Often $Q(k, n)$ is used to denote the number of partitions of k into distinct parts, that is, parts that are different from each other.

172. Show that
$$Q(k, n) \le \frac{1}{n!}\binom{k-1}{n-1}.$$

Solution: The number of compositions of k into n parts is $\binom{k-1}{n-1}$. Thus the number of compositions of k into n distinct parts is less than $\binom{k-1}{n-1}$. Divide the compositions of k into n distinct parts into blocks with two compositions in the same block if one is a rearrangement of the other. Because the parts are distinct, each block has $n!$ members. Further, there is a bijection between the blocks of this partition and the partitions of k into n distinct parts. Since the number of compositions of k into n distinct parts is less than $\binom{k-1}{n-1}$, the number of partitions of k into n distinct parts is less than $\frac{1}{n!}\binom{k-1}{n-1}$. ∎

173. Show that the number of partitions of seven into three parts equals the number of partitions of 10 into three distinct parts.

relationship.

Solution: The number of partitions of k into n parts is equal to the number of partitions of $k+\binom{n}{2}$ into n distinct parts. The bijection from partitions of k with n parts to partitions of $k+\binom{n}{2}$ with n distinct parts that proves this is the one that takes a partition $\lambda_n\lambda_{n-1}\cdots\lambda_1$ of k with $\lambda_i > \lambda_{i+1}$ and adds $i-1$ to λ_i to get λ_i'. Then λ' is a partition into distinct parts, and the number it partitions is $k+1+2+\cdots+n-1 = k+\binom{n}{2}$. The proof that it is a bijection is the fact that subtracting $n-i$ from the ith part of a partition of k into distinct parts yields a partition of k, because part $i+j$ is at least j smaller than part i. ∎

·175. Find a recurrence that expresses $Q(k,n)$ as a sum of $Q(k-n,m)$ for appropriate values of m.

Solution: Suppose λ is a partition of k into n distinct parts. Either 1 is one of those parts or not. Thus if we subtract 1 from each part, we either get a partition of $k-n$ into $n-1$ parts or a partition of $k-n$ into n parts. If λ and λ' are different partitions of k into n distinct parts, they go to different partitions. Each partition of $k-n$ into $n-1$ parts or n parts can be gotten in this way from a corresponding partition of k into n parts. Thus we have a bijective correspondence and $Q(k,n) = Q(k-n,n-1) + Q(k-n,n)$. ∎

∗176. Show that the number of partitions of k into distinct parts equals the number of partitions of k into odd parts.

Solution: We start by giving a function from the set of partitions of k to the set of partitions of k with (only) odd parts. Clearly such a function cannot be one to one. Then we show that when restricted to the partitions with distinct parts it is one-to-one and onto by constructing an inverse. Given a partition $\lambda_1^{i_1}\lambda_2^{i_2}\cdots\lambda_n^{i_n}$, write $\lambda_i = \gamma_i 2^{k_i}$, where γ_i is odd. (Thus 2^{k_i} is the highest power of 2 that is a factor of λ_i, so it is 1 if λ_i is odd.). It is possible that $\gamma_i = \gamma_j$, for example if $\lambda_i = 36$ and $\lambda_j = 18$, then $\gamma_i = \gamma_j = 9$. We construct a new partition π whose parts are the numbers γ_j as follows: Given an odd number p, let the multiplicity $m(p)$ of p in π be $\sum_{j:\gamma_j=m} 2^{k_j}$. Thus $\sum_{p:m(p)\neq 0} m(p)p = k$. Therefore, π is a partition of k whose parts are all odd.

Now consider a partition π of k whose parts are all odd. Let $\pi = \pi_1^{r_1}\pi_2^{r_2}\cdots\pi_t^{r_t}$, with $\pi_i > \pi_{i+1}$. (In terms of the multiplicity function m, $m(\pi_i) = r_i$, and $\sum_{i=1}^{t} r_i\pi_i = k$.) We are going to write the binary

(e) In how many ways can they distribute all the trees if they are identical and anyone may receive any number of trees?

Solution: $\binom{r+s-1}{r}$ ▮

(f) In how many ways can all the trees be distributed and planted if the trees are distinct, any family can get any number, and a family must plant its trees in an evenly spaced row along the road?

Solution: $s^{\overline{r}} = (r+s-1)^{\underline{r}}$ ▮

(g) Answer the question in Part 2f assuming that every family must get a tree.

Solution: $r!\binom{r-1}{s-1}$ ▮

(h) Answer the question in Part 2e assuming that each family must get at least one tree.

Solution: $\binom{r-1}{s-1}$ ▮

3. In how many ways can n identical chemistry books, r identical mathematics books, s identical physics books, and t identical astronomy books be arranged on three bookshelves? (Assume there is no limit on the number of books per shelf.)

Solution: $\frac{(n+r+s+t+2)!}{n!r!s!t!2!}$ ▮

➔4. One formula for the Lah numbers is

$$L(k,n) = \binom{k}{n}(k-1)^{\underline{k-n}}$$

Find a proof that explains this product.

Solution: First choose the n elements which will be the first member of the part they lie in. (This, in effect, labels the n parts.) Then assign the remaining $k-n$ elements to their parts by making an ordered function of $n-k$ objects to n recipients in $(n+(k-n)-1)^{k-n} = (k-1)^{k-n}$ ways. ▮

5. What is the number of partitions of n into two parts?

Solution: $n/2$ if n is even and $(n-1)/2$ if n is odd, equivalently, $\lfloor n/2 \rfloor$. ▮

6. What is the number of partitions of k into $k-2$ parts?

Solution: A partition of k into $k-2$ parts will have either one part of size 3 and $k-3$ parts of size 1, or two parts of size 2 and $k-4$ parts of size 1. Thus the number of partitions of k into $k-2$ parts is

$$\begin{aligned}\binom{k}{3}+\binom{k}{2}\binom{k-2}{2}/2 &= k(k-1)(n-2)/6+k(k-1)(k-2)(k-3)/8 \\ &= k(k-1)(k-2)(1/6+(k-3)/8) \\ &= k(k-1)(k-2)(3k-5)/24.\end{aligned}$$

∎

7. Show that the number of partitions of k into n parts of size at most m equals the number of partitions of $mn-k$ into no more than n parts of size at most $m-1$.

   Solution: If we take the complement of the Young diagram of a partition of k into n parts of size at most m in an rectangle with n rows and m columns, the number we partition will be $mn-k$, and we will have no more than n parts, each of size at most $m-1$. And if we take the complement of a partition of this second kind in the same rectangle, we will get a partition of the first kind. ∎

8. Show that the number of partitions of k into parts of size at most m is equal to the number of partitions of $k+m$ into m parts.

   Solution: Given the first kind of partition, take the conjugate (giving a partition of k into at most m parts), add one to each part, and then add enough parts of size 1 to get a total of m parts. It is straightforward that this process can be reversed. ∎

9. You can say something pretty specific about self-conjugate partitions of k into distinct parts. Figure out what it is and prove it. With that, you should be able to find a relationship between these partitions and partitions whose parts are consecutive integers, starting with 1. What is that relationship?

   Solution: In a self-conjugate partition, the number of parts is the size of the largest part. If these parts are distinct, this means that each number between 1 and the largest part appears once as a part. That is, the parts are a list of consecutive integers, starting with 1. ∎

10. What is $s(k,1)$?

    Solution: Since $s(k,1)$ is the coefficient of x^1 in

    $$x^{\underline{k}} = x(x-1)(x-2)\cdot(x-(k-1)),$$