·82. Prove that $R(m,n) \leq R(m-1,n) + R(m,n-1)$.

Solution: If there are $R(m-1,n) + R(m,n-1)$ people in a room, choose one person, say person P. By the generalized pigeonhole principle, there are either $R(m-1,n)$ people with whom P is acquainted or $R(m,n-1)$ people with whom person P is unacquainted. In the first case, among the people with whom person P is acquainted, there are either n mutual strangers, in which case we are done, or there are $m-1$ people with whom person P is acquainted. These $m-1$ people and person P form m people who are mutually acquainted, and so we have m mutual acquaintances. On the other hand, if P is unacquainted with $R(m,n-1)$ people, then among these people, there are either m mutually acquainted people, in which case we are done, or among these people there are $m-1$ mutually unacquainted people, and these $m-1$ people together with P make m mutual strangers. Thus in every case, if there are $R(m-1,n) + R(m,n-1)$ people in a room, there are either at least m mutual acquaintances or at least n mutual strangers. Therefore $R(m,n) \leq R(m-1,n) + R(m,n-1)$. ∎

→ 83. (a) What does the equation in Problem 82 tell us about $R(4,4)$?

Solution: $R(4,4) \le R(3,4) + R(4,3) = 9 + 9 = 18$. ▮

$*$(b) Consider 17 people arranged in a circle such that each person is acquainted with the first, second, fourth, and eighth person to the right and the first, second, fourth, and eighth person to the left. Can you find a set of four mutual acquaintances? Can you find a set of four mutual strangers?

Solution: You cannot find either. If there were a set of four mutual acquaintances, you could assume by symmetry that it includes person 1, and two people from among those one, two, four, and eight places to the right. Thus you can assume your set of four acquaintances contains person 1 and two from among persons 2, 3, 5, and 9. However, persons 2 and 5, 2 and 9 and 3 and 9 are not acquainted. Thus three of the mutually acquainted people are either persons 1, 2, and 3, persons 1, 5, and 9 or persons 1, 3, and 5. However, person 5 is not acquainted with the person one, two, or eight places to the left of person 1, so if person 5 is in the set of mutual acquaintances, then person 14 must be as well. However, person 3 and person 9 are not acquainted with person 14. Thus our set must contain persons 1, 2, and 3. However, person 3 is not acquainted with the person one, two, four, or eight persons to the left of person 1, so there is no set of four mutual acquaintances. A similar argument holds for nonacquaintances. ▮

(c) What is $R(4,4)$?

Solution: 18. ▮

84. (Optional) Prove the inequality of Problem 81 by induction on $m+n$.

Solution: We want to prove that if $m \ge 2$ and $n \ge 2$, then when there are $\binom{m+n-2}{m-1}$ people in a room, there are either m mutual acquaintances or n mutual strangers. If $m+n=4$, then $m=2$ and $n=2$, and if there are $\binom{2+2-2}{1}=2$ people in a room, there are either two who know each other or two who don't.

Now assume that when $m+n=k-1$, it is the case that if there are $\binom{m+n-2}{m-1}$ people in a room, then there are either m mutual acquaintances or n mutual strangers. Suppose that $m'+n'=k$ and there are $\binom{m'+n'-2}{m'-1}$ people in a room. If $n'=2$, then we know that with $\binom{m'+2-2}{m'-1}=m'$ people in a room, there are either m' mutual acquaintances or two mutual strangers, and similarly if $m'=2$ there are either two mutual acquaintances or n mutual strangers among $\binom{m'+n'-2}{m-1}$ people. Thus we may assume that both m' and n' are greater than two. Since $\binom{m'+n'-2}{m-1} = \binom{(m'-1)+n-2}{m'-2} + \binom{m+(n-1)-2}{m-1}$, if there are $\binom{m'+n'-2}{m'-1}$ people in a room, then a given person, say person P, is either acquainted with $\binom{(m'-1)+n'-2}{m'-1}$ of them (call this case 1) or is a stranger with $\binom{m'+(n-1)-2}{m-1}$ of them (call this case 2). Notice that $m'-1+n'=k-1$ and $m'+n'-1=k-1$. Thus in case 1, our inductive hypothesis tells us that either $m'-1$ of person P's acquaintances are mutually acquainted, in which case they and person P form m' mutual acquaintances, or n' of P's acquaintances are mutual strangers, in which case we have n' mutual strangers. Similarly in case 2 we have either m' mutual acquaintances or n' mutual strangers. Thus by the principle of mathematical induction, for all values of $m+n$ greater than or equal to 4, if we have $\binom{m+n-2}{m-1}$ people in a room, then we have either m mutual acquaintances or n mutual strangers, so that $R(m,n)$ exists and is no more than $\binom{m+n-2}{m-1}$. ▮

→86. Suppose we have two numbers n and m. We consider all possible ways to color the edges of the complete graph K_m with two colors, say red and blue. For each coloring, we look at each n-element subset N of the vertex set M of K_m. Then N together with the edges of K_m connecting vertices in N forms a complete graph on n vertices. This graph, which we denote by K_N, has its edges colored by the original coloring of the edges of K_m.

(a) Why is it that, if there is no subset $N \subseteq M$ so that all the edges of K_N are colored the same color for any coloring of the edges of K_m, then $R(n,n) > m$?

Solution: Another way to say there is no such subset is to say that it is not possible to find a K_n all of whose edges are red or a K_n all of whose edges are blue. This means that $R(n,n) > n$. ▮

(b) To apply the probabilistic method, we are going to compute the average, over all colorings of K_m, of the number of sets $N \subseteq M$ with $|N| = n$ such that K_N *does* have all its edges the same color. Explain why it is that if the average is less than 1, then for some coloring there is no set N such that K_N has all its edges colored the same color. Why does this mean that $R(n,n) > m$?

Solution: If the average of n nonnegative integers is less than one, they cannot all be one or more, so one has to be zero. Thus in this context there must be some coloring that has no set N so that K_N has all its edges colored the same color. ▮

(c) We call a K_N *monochromatic* for a coloring c of K_m if the color $c(e)$ assigned to edge e is the same for every edge e of K_N. Let us define mono(c,N) to be 1 if N is monochromatic for c and to be 0 otherwise. Find a formula for the average number of monochromatic K_Ns over all colorings of K_m that involves a double sum first over all edge colorings c of K_m and then over all n-element subsets $N \subseteq M$ of mono(c,N).

Solution:

$$\frac{1}{2^{\binom{m}{2}}} \sum_{c:c \text{ is a coloring of } K_m} \; \sum_{N:N\subseteq M,\ |N|=n} \mathrm{mono}(c,N).$$

▮

(d) Show that your formula for the average reduces to $2\binom{m}{n} \cdot 2^{-\binom{n}{2}}$

Solution:

$$\begin{aligned}
&\frac{1}{2^{\binom{m}{2}}} \sum_{c:c \text{ is a coloring of } K_m} \; \sum_{N:N\subseteq M,\ |N|=n} \mathrm{mono}(c,N) \\
=\ &\frac{1}{2^{\binom{m}{2}}} \sum_{N:N\subseteq M,\ |N|=n} \; \sum_{c:c \text{ is a coloring of } K_m} \mathrm{mono}(c,N) \\
=\ &2^{-\binom{m}{2}} \sum_{N:N\subseteq M,\ |N|=n} 2\cdot 2^{\binom{m}{2}-\binom{n}{2}} \\
=\ &2\binom{m}{n}2^{-\binom{n}{2}}
\end{aligned}$$

▮

(e) Explain why $R(n,n) > m$ if $\binom{m}{n} \le 2^{\binom{n}{2}-1}$.

Solution: $R(n,n) > m$ if the average above is less than 1. Thus $R(n,n) > m$ if $2\binom{m}{2}2^{-\binom{n}{2}} < 1$, which is equivalent to $\binom{m}{n} < 2^{\binom{n}{2}-1}$. ▮

*(f) Explain why $R(n,n) > \sqrt[n]{n!2^{\binom{n}{2}-1}}$.

Solution: $\binom{m}{n} < 2^{\binom{n}{2}-1}$ is the same as $\frac{m^{\underline{n}}}{n!} < 2^{\binom{n}{2}-1}$. And since $m^{\underline{n}} < m^n$, the inequality $\frac{m^{\underline{n}}}{n!} < 2^{\binom{n}{2}-1}$ holds if the inequality $\frac{m^n}{n!} \leq 2^{\binom{n}{2}-1}$ holds. And this last inequality holds if $m \leq \sqrt[n]{n!2^{\binom{n}{2}-1}}$ holds. Thus $R(n,n) > m$ for any m such that $m \leq \sqrt[n]{n!2^{\binom{n}{2}-1}}$, which implies that $R(n,n) > \sqrt[n]{n!2^{\binom{n}{2}-1}}$. ∎

(g) By using Stirling's formula, show that if n is large enough, then $R(n,n) > \sqrt{2^n} = \sqrt{2}^n$. (Here large enough means large enough for Stirling's formula to be reasonably accurate.)

Solution: Using Stirling's approximation to $n!$ we get

$$R(n,n) > \sqrt[n]{\frac{n^n}{e^n}\sqrt{2\pi n}2^{\frac{n^2-n-2}{2}}} = \frac{n}{e}2^{\frac{n^2-n-2}{2n}}\sqrt[2n]{2\pi n} > 2^{n/2} = \sqrt{2}^n.$$

∎

89. Show that there is only one solution to Recurrence 2.1 that satisfies $s_0 = 1$.

Solution: We prove by induction on n that there is one and only one value s_n that satisfies both $s_n = 2s_{n-1}$ for $n > 0$ and $s_0 = 1$. First, there is clearly one and only one value s_0 that satisfies $s_0 = 1$. Now assume that $k > 0$ and there is one and only one value s_{k-1} that satisfies the two equations. Then $s_k = 2s_{k-1}$ is the one and only one value that satisfies the two equations. Therefore by the principle of mathematical induction, for all nonnegative integers n there is one and only one value s_n that satisfies the equations $s_0 = 1$ and $s_k = 2s_{k-1}$ for all $k > 0$. (Note that since we were making a statement about s_n for all nonnegative integers n it was not appropriate to use n as the dummy variable in the recursive equation $s_k = 2s_{k-1}$.) ∎

90. A first-order recurrence relation is one which expresses a_n in terms of a_{n-1} and other functions of n, but which does not include any of the terms a_i for $i < n-1$ in the equation.

(a) Which of the recurrences 2.1 through 2.6 are first order recurrences?

Solution: The recurrences 2.1, 2.2, 2.3, and 2.4 are all examples of first order recurrences. The recurrences 2.5 and 2.6 are not. ∎

(b) Show that there is one and only one sequence a_n that is defined for every nonnegative integer n, satisfies a given first order recurrence, and satisfies $a_0 = a$ for some fixed constant a.

Solution: A first order recurrence will give a_n in terms of a_{n-1}, that is, there will be a function f such that $a_n = f(a_{n-1})$ for all $n > 0$. We prove by induction that there is one and only one solution to a first order recurrence that satisfies $a_0 = a$ for some fixed constant a. First, there is one and only one value for a_0. Now suppose that when $n = k-1$, there is one and only one value possible for a_{n-1}. Then a_k is uniquely determined by $a_k = f(a_{k-1})$. Thus the truth of the statement that a_{k-1} is uniquely determined by the equations $a_0 = a$ and $a_n = f(a_{n-1})$ implies the truth of the statement that a_k is determined uniquely by the equations $a_0 = a$ and $a_n = f(a_{n-1})$. Therefore by the principle of mathematical induction, a_k is uniquely determined by the equations $a_0 = a$ and $a_n = f(a_{n-1})$ for all nonnegative integers k. ∎

95. A person who is earning \$50,000 per year gets a raise of \$3000 a year for n years in a row. Find a recurrence for the amount a_n of money the person earns over $n+1$ years. What is the total amount of money that the person earns over a period of $n+1$ years? (In $n+1$ years, there are n raises.)

Solution: By Problem 94 we saw that if b_n is the salary in year n, then $b_n = 50,000 + 3000n$. If a_n is the total amount earned over the period of from year 0 through the end of year n, a period of $n+1$ years, then $a_n = a_{n-1} + b_n = a_{n-1} + 50,000 + 3000n$. Further, $a_n = \sum_{i=0}^{n} b_i = \sum_{i=0}^{n} 50,000 + 3000i = 50,000(n+1) + 3000\sum_{i=0}^{n} i = 50,000(n+1) + 1500(n(n+1)$. ▮

96. An *arithmetic series* is a sequence s_n equal to the sum of the terms a_0 through a_n of an arithmetic progression. Find a recurrence for the sum s_n of an arithmetic progression with initial value a_0 and common difference c (using the language of Problem 94). Find a formula for general term s_n of an arithmetic series.

Solution: $s_n = \sum_{i=0}^{n} a_0 + ci = (n+1)a_0 + c\sum_{i=0}^{n} i = (n+1)a_0 + cn(n+1)/2$. ▮

➔·102. The sum of the degrees of the vertices of a (finite) graph is related in a natural way to the number of edges.

(a) What is the relationship?

Solution: The sum of the degrees of the vertices is twice the number of edges. ▮

(b) Find a proof that what you say is correct that uses induction on the number of edges.

Solution: If a graph has no edges, then the sum of the degrees of the vertices is 0, which is twice the number of edges. Now suppose that whenever a graph has $n-1$ edges, the sum of the degrees of the vertices is twice the number of edges. Let G be a graph with n edges, and delete an edge from G to get G'. The the sum of the degrees of G' is $2(n-1)$, and adding the edge back into G' to get G either increases the degrees of exactly two vertices by one each or increases the degree of one vertex by 2. Thus the sum of the degrees of the vertices of G is $2n$, which is twice the number of edges. Thus by the principle of mathematical induction, for all nonnegative integers n, if a graph has n edges, then the sum of the degrees of the vertices is twice the number of edges. ▮

(c) Find a proof that what you say is correct which uses induction on the number of vertices.

(d) Find a proof that what you say is correct that does not use induction.

Solution: The sum of the degrees of the vertices is the sum over all edges of the number of times that edge touches a vertex, which is twice the number of edges. ▮

102. (c) Solution.

If there are no vertices, then there are no edges, so the sum of the degrees and the number of edges are 0.
Assume that whenever a graph has $n-1$ vertices, $\sum_{i=1}^{n-1} d_i = 2E$, where d_i is the degree of the i^{th} vertex and E is the number of edges.
Let G be a graph with n vertices and delete a vertex along with all edges connected to it to get G'. Let d_i' be the degrees of the vertices in G', with $i=1,2,\ldots,n-1$, and let E' be the number of edges in G'. Then $\sum_{i=1}^{n-1} d_i' = 2E'$.
Now add the vertex and edges you removed back in. Suppose the degree of this vertex is d_n. If there are ℓ loops on this vertex (ie. edges that begin and end on it) then there are $d_n - 2\ell$ edges that connect this vertex to vertices in G', increasing the sum of the degrees of the vertices of G' by $d_n - 2\ell$. So the sum of the degrees of G is $2E' + d_n + (d_n - 2\ell)$, and the number of edges in G that are not in G' is $\ell + (d_n - 2\ell) = d_n - \ell$. So the sum of the degrees of G is $2E' + 2(d_n - \ell) = 2(E' + d_n - \ell) =$ twice the number of edges in G.

103. What can you say about the number of vertices of odd degree in a graph?

Solution: The number of vertices of odd degree must be even, because otherwise the sum of the degrees of the vertices would be odd. ▮

4. (a) Give a recurrence for the number of ways to divide $2n$ people into sets of two for tennis games. (Don't worry about who serves first.)

Solution: $t_{2n} = (2n-1)t_{2n-2}$. ▮

(b) Give a recurrence for the number of ways to divide $2n$ people into sets of two for tennis games and to determine who serves first.)

Solution: $t_{2n} = 2(2n-1)t_{2n-2}$. ▮

(a) Given $2n$ people, pick one of them. There are $2n-1$ people she could be paired with. The remaining $2n-2$ people can be paired in t_{2n-2} ways.
(b) There are 2 ways to determine who serves first in the first pair described in (a).