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Course: **8.421 - AMO I**
Problem set: **#9**
Due: Friday, April 15, 2022.

1. Transition Lifetimes and Blackbody Radiation.

(a) (i) The rate of *absorption* is given by the product of Einstein's B coefficient and the average number of photons per mode $\langle n\rangle_{\omega_0}$ where ω_0 is angular frequency associated with the (dominant) transition with $\lambda_0 = 590$ nm.

$$W = B\langle n\rangle\hbar\omega_0\rho(\omega_0) = B\langle n\rangle\hbar\omega_0\frac{\omega_0^2}{\pi^2c^3} = 1/60\text{ s}^{-1}.$$

where $\rho(\omega_0)$ is the density of states. From lecture, we know that the Einstein's B coefficient can be written in terms of the (known) Einstein's $A = \Gamma_0 = 1/\tau$ coefficient:

$$B = \frac{\pi^2c^3}{\hbar\omega_0^3}A = \frac{\pi^2c^3}{\hbar\omega_0^3}\frac{1}{\tau}$$

With this, we can plug in the numbers to find

$$\langle n\rangle = W\tau \approx 2.67\times10^{-10}.$$

(ii) For blackbody radiation, we have

$$\langle n\rangle = \frac{1}{e^{\hbar\omega_0/k_BT}-1} \implies T \approx 1100\text{ K}.$$

We see that in order for the absorption rate to reach 1 photon per minute, the blackbody temperature has to be $\sim$ 1100 K, which is way above room temperature (of course the higher the temperature, the higher the absorption rate, and vice versa). As a result, there is no need to shield the vacuum system from room temperature radiation or col the vacuum system to cryogenic temperatures (as expected).

(b) Here we estimate the lifetime of hydrogen in the $F = 1$ hyperfine level of the $1S$ state.

(i) $F = 1 \to F = 0$ is a magnetic dipole transition.

(ii) To estimate the lifetime of the $F = 1$ state, we may assume that the (magnetic dipole) transition matrix element is μ_B. From here, we work entirely in the CGS unit system to find

$$\Gamma_0 = \frac{4\omega_0^3\mu_B^2}{3\hbar c^3} \approx 2.91\times10^{-15}\text{ s}^{-1}$$

where the numerical values for the fundamental constants can be found on Wikipedia. The lifetime is

$$\tau = \frac{1}{\Gamma_0} \approx 1.1\times10^7\text{ years}.$$

(c) Now we look at a hydrogen BEC in the $F = 1$ state.

(i) To find the average number of photons per mode from blackbody radiation at the 21 cm line at 300 K and 4 K, we simply calculate

$$\langle n\rangle = \frac{1}{e^{\hbar\omega/k_BT}-1}$$

at the corresponding temperatures and angular frequency. The answers are

$$T = 300\text{ K},\qquad \langle n\rangle \approx 4375$$
$$T = 4\text{ K},\qquad \langle n\rangle \approx 57.84.$$

(ii) Similar to what we did before (but reversed), we can find the transition rates at $T = 300$ K and $T = 4$ K. We will also need the lifetime $\tau \approx 1.1 \times 10^7$ years for this calculation.

$$T = 300\text{ K}, \qquad W = \langle n \rangle/\tau \approx 1.27 \times 10^{-11}\text{ s}^{-1}$$
$$T = 4\text{ K}, \qquad W = \langle n \rangle/\tau \approx 1.68 \times 10^{-13}\text{ s}^{-1}$$

(iii) It is clear that we should not be concerned about blackbody radiation from the environment limiting our experiment with hydrogen in the $F = 1$ state if we need a trapping times on the order of seconds/minutes.

(d) The lifetime is much longer for the case of a magnetic dipole transition in hydrogen where there are more photons per mode mainly because the lifetime τ scales as $1/\omega_0^3$. The ratio between the sodium wavelength of 590 nm versus the 21 cm of hydrogen is $\sim 10^{-6}$. This gives a reduction factor of 10^{-18}. On top of this, there is also another factor of α^2 reduction when replacing the electric with magnetic dipole matrix element.

2. Saturation Intensity.

(a) We first manipulate the Einstein A coefficient so that it is written in terms of the oscillator strength f, the fine structure constant α and the transition frequency ω. The oscillator strength is given by

$$f_{21} = \frac{2m\omega_{21}}{3\hbar}\frac{1}{2J_1+1}\sum_{m_1,m_2}|\langle J_1 m_1|\vec{r}|J_2 m_2\rangle|^2 = \frac{2m\omega_{21}}{3\hbar}\frac{S_{12}}{2J_1+1}.$$

where S_{12} is the line strength, while

$$A_{12} = \frac{4\omega^3 e^2}{3\hbar c^3}\sum_{m_2}|\langle 1m_1|\vec{r}|2m_2\rangle|^2 = \frac{4\omega^3 e^2}{3\hbar c^3}\frac{S_{12}}{2J_1+1}.$$

From here we have that

$$A_{12} = \frac{4e^2\omega_{21}^3}{3\hbar c^3}\frac{3\hbar f_{21}}{2m\omega_{21}} = \frac{2e^2\omega_{21}^2 f_{21}}{mc^3} = \frac{2\hbar\omega_{21}^2 f_{21}}{mc^2}\frac{e^2}{\hbar c} = \frac{2\alpha\hbar\omega_{21}^2 f_{21}}{mc^2}.$$

Assuming $f_{21} = 1$ we may use this formula to estimate the lifetime of sodium. Plugging in the numerical values for the constants above (in CGS units), we find

$$A \approx 1.917 \times 10^8\text{ s}^{-1} \implies \tau = \frac{1}{A} \approx 5.2\text{ ns}.$$

More precisely, if we call f the *absorption* oscillator strength, then we actually have

$$A = \frac{2\alpha\hbar\omega^2}{mc^2}\frac{2J_1+1}{2J_2+1}f.$$

Assuming that we're working the D lines of sodium, we will take $J_1 = 1/2$ and $J_2 = 1/2$ and $3/2$. From Steck's, we find that $f = 0.64$ for the $3^2S_{1/2} \to 3^2P_{3/2}$ transition and $f = 0.32$ for the $3^2S_{1/2} \to 3^2P_{1/2}$ transition. Plugging in the numbers we find that

$$\tau_{1/2\to 3/2} \approx \tau_{1/2\to 3/2} \approx 16\text{ ns},$$

as expected.

(b) The saturation intensity for the principal transition in sodium, with $\sigma_0 = 3\lambda^2/2\pi$, is:

$$I_{\text{sat}} = \frac{\hbar\omega A}{2\sigma_0} \approx 63.3\text{ W/m}^2.$$

where we are using A from the previous part and ignoring fine and hyperfine structure.

3. Saturation of Atomic Transitions.

(a) Here we consider a two-state atom with $R_{ge} = R_{eg} = R$ the stimulated absorption/emission rate and $A = \Gamma$ the spontaneous emission rate. Define the saturation parameter s as $s = 2R/\Gamma$. In equilibrium, we have

$$\Gamma N_b + RN_b - RN_a = 0 \implies \frac{N_b}{N_a} = \frac{R}{R+\Gamma} = \frac{s\Gamma/2}{s\Gamma/2+\Gamma} = \frac{s}{s+2}.$$

(b) The equilibrium spontaneous emission rate per atom AN_b can be expressed in terms of Γ and s and the total density $N = N_a + N_b = 1$. From part (a), we have

$$\frac{N_b}{N} = \frac{N_b}{N_b + N_a} = \frac{N_b/N_a}{N_b/N_a + 1} = \frac{s}{s+(s+2)} = \frac{1}{2}\frac{s}{s+1} = N_b.$$

So, we find

$$\Gamma N_b = \frac{1}{2}\frac{\Gamma s}{s+1}.$$

Recalling the optical Bloch equations, we know that

$$N_b = \frac{\Omega^2/\Gamma^2}{1+2\Omega^2/\Gamma^2}.$$

Thus we may identify s with $2\Omega^2/\Gamma^2$. The scattering cross section has the form

$$\sigma = \frac{\sigma_0}{1+I/I_{\text{sat}}} = \frac{\sigma_0}{1+2\Omega^2/\Gamma^2}.$$

where we have ignored the (small) detuning term and used the definition of I_{sat}. We can immediately see that $\sigma(s)$ bleaches out as $\sigma(s=0)/(1+s)$, as desired.

(c) For $s = 1$, the energy density $\langle w \rangle_{\text{SAT}}$ per unit frequency is given by

$$\langle w \rangle = \frac{R}{B},$$

where B is the Einstein B coefficient. With $s = 1$, we find that

$$s = 1 = \frac{2R}{\Gamma} \implies R = \frac{\Gamma}{2} = B\langle w \rangle \implies \langle w \rangle = \frac{\Gamma}{2B}.$$

$\langle w \rangle$ is independent of the atomic dipole matrix element since both the $A = \Gamma$ and B Einstein coefficients are proportional to the matrix element (modulus) squared.

(d) With

$$\frac{\Gamma}{B} = \frac{8\pi\hbar}{\lambda^3} = \frac{\hbar\omega^3}{\pi^2 c^3},$$

we find

$$\langle w \rangle_{\text{SAT}} = \frac{\hbar\omega^3}{2\pi^2 c^3}.$$

The mean occupation number n per photon mode for $s = 1$ is given by

$$\langle n \rangle = \frac{B\langle w \rangle_{\text{SAT}}}{\Gamma} = \frac{B}{\Gamma}\frac{\Gamma}{2B} = \frac{1}{2}.$$

(e) We have laser light of intensity I_0 and Lorentzian lineshape centered at the atomic transition frequency ω_0 with FWHM $\Gamma' \gg \Gamma$. The energy density of this beam per frequency interval at ω_0 is

$$S(\omega_0) = \frac{I(\omega_0)}{c} = I_0 \frac{1}{\pi c} \frac{\Gamma'/2}{(\omega_0 - \omega_0)^2 + (\Gamma'/2)^2} = \frac{2I_0}{\pi c \Gamma'}.$$

We now want I_s such that

$$s = 1 = \frac{2R}{\Gamma} = \frac{2B\langle w \rangle}{\Gamma} = 2S(\omega_0)\frac{B}{\Gamma} = \frac{4I_s}{\pi c \Gamma'} \frac{\pi^2 c^3}{\hbar \omega^3} = \frac{4\pi c^2}{\Gamma' \hbar \omega^3} I_s \implies I_s = \frac{\Gamma' \hbar \omega^3}{4\pi c^2}.$$

(f) The stimulated broadband absorption rate is given by

$$R = B\langle w \rangle = B\frac{2I_0}{\pi c \Gamma'} = \frac{2I_0}{\pi c \Gamma'} \frac{\Gamma \pi^2 c^3}{\hbar \omega^3} = \frac{2\pi \Gamma c^2 I_0}{\hbar \omega^3 \Gamma'} = \frac{\omega_R^2}{\Gamma'},$$

as desired. Here, we have used the result seen on the last problem set:

$$\omega_R^2 = \frac{2\pi \Gamma c^2 I_0}{\hbar \omega^3}.$$

For $s = 1$, we have $I_0 = I_s$, so

$$\omega_R^2 = \frac{2\pi \Gamma c^2}{\hbar \omega^3} \frac{\Gamma' \hbar \omega^3}{4\pi c^2} = \frac{\Gamma \Gamma'}{2}.$$

(g) If we set $\Gamma' = \Gamma$, then we get exactly the saturation intensity of a monochromatic laser beam and the Rabi frequency at saturation. This is not surprising since when setting $\Gamma' = \Gamma$ we are no longer in the broadband regime.