Gaussian Beams and Lasers

- Gaussian Beams Introduction
- Matrix Method
- Equivalent Ray tracing
- Example calculation
- Laser Basics

ECE 5616
Curtis

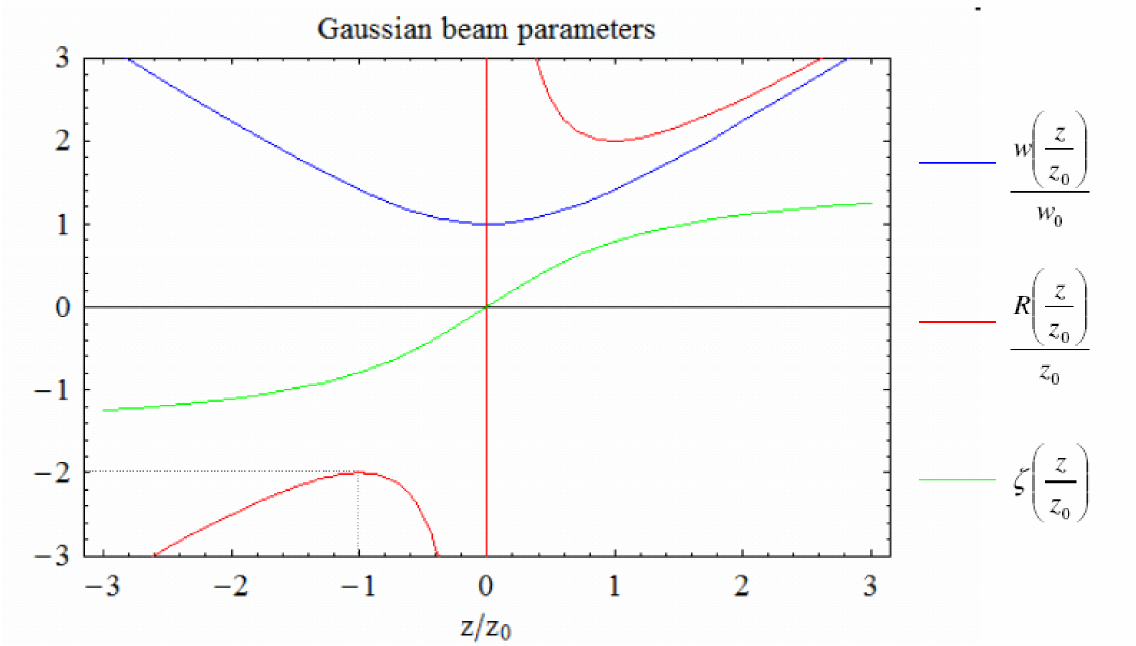
Gaussian Beams

Solution of scalar paraxial wave equation (Helmholtz equation) is a Gaussian beam, given by:

$$E(\vec{r}) = A_0 \frac{w_0}{w(z)} e^{-\frac{\rho^2}{w^2(z)} - jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)} \qquad \text{where}$$

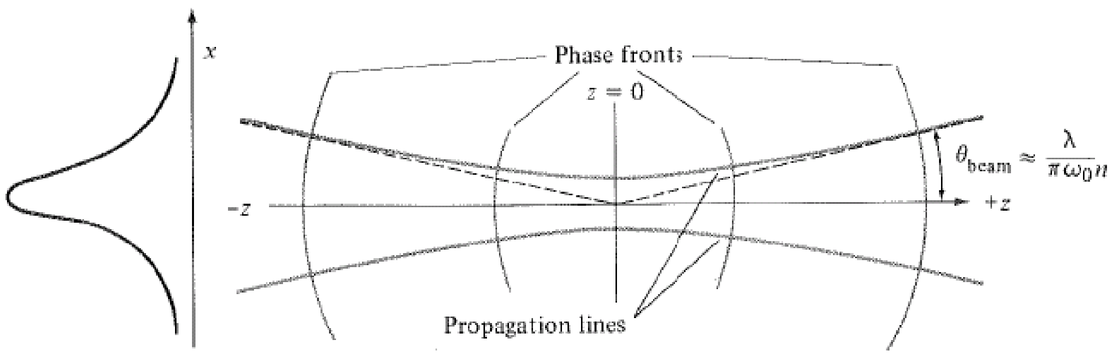
Size	*Radius of curvature*	*Gouy phase*
$w(z) = w_0\sqrt{1+\left(\frac{z}{z_0}\right)^2}$	$R(z) = z\left[1+\left(\frac{z_0}{z}\right)^2\right]$	$\zeta(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$

Note that R(z) does not obey ray tracing sign convention. Unfortunately there's no particularly good way to fix this.



Gaussian Beams

Main points



Gaussian beam can be completely described once you know two things

1. w_o beam waist, which is the point where the field is down 1/e compared to on axis, wavelength
2. Z=0, location of beam waist

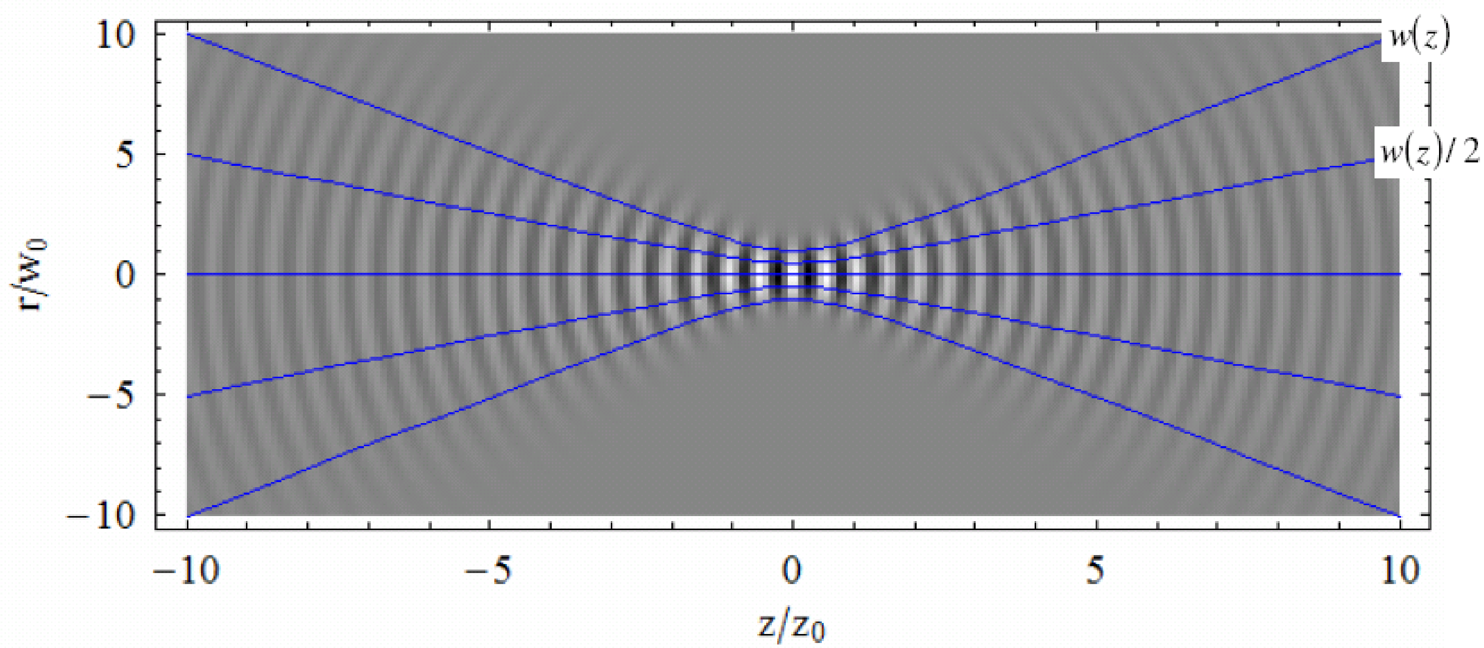
$$\theta_{beam} = \tan^{-1}\left(\frac{\lambda}{\pi\omega_0 n}\right) \simeq \frac{\lambda}{\pi\omega_0 n} \qquad \text{for } \theta_{beam} \ll \pi$$

Half apex angle for far field of aperture w_o, about 86% of beam power is contained within this cone

Gaussian Beams

Detailed view, Rayleigh distance

Real part of E vs. radius and z



At $z = z_0$,

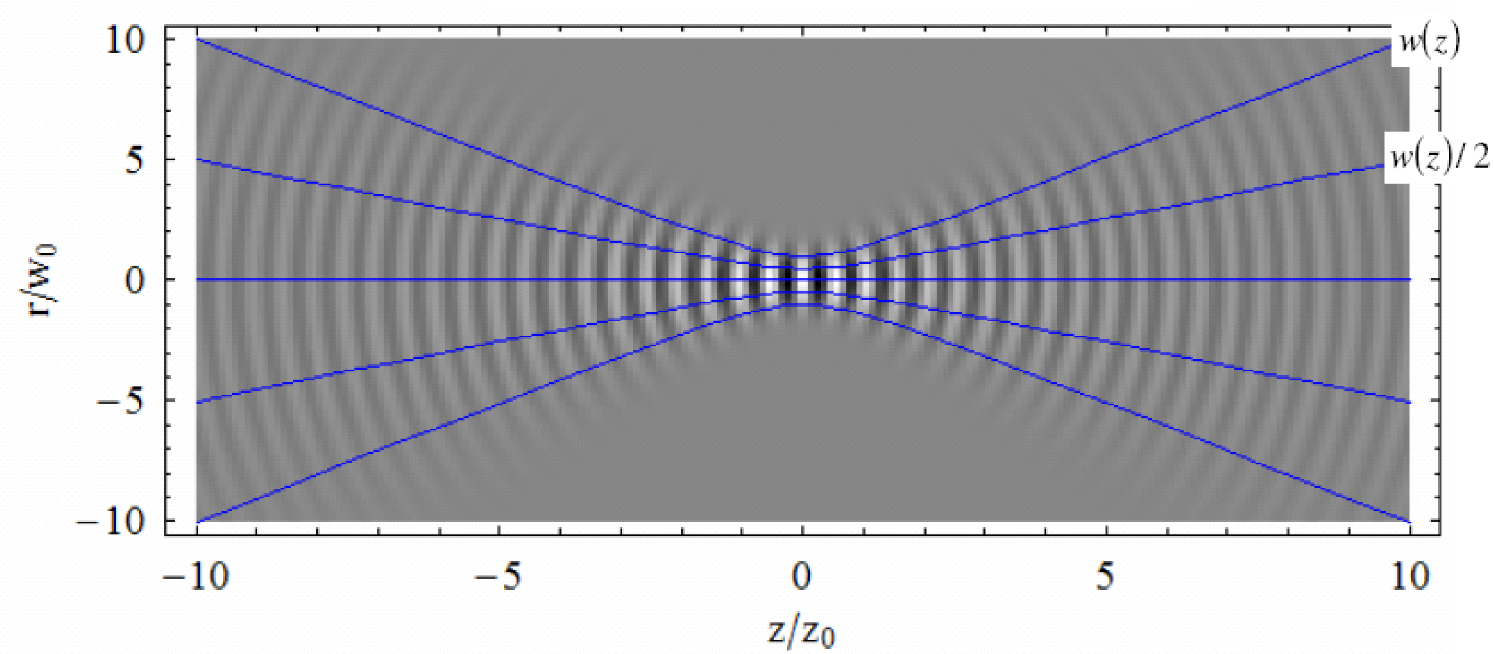
$$I(0,z_0) = I(0,0)/2 \qquad w(z_0) = \sqrt{2}w_0 \qquad R(z_0) = 2z_0 \text{ (min value)}$$

Measure of the convergence of the beam, smaller z_o stronger convergence.
The phase on the beam axis is retarded by $\pi/4$ relative to plane wave.

Guassian Beams

Detailed view, at beam center and far away

Real part of E vs. radius and z



Near beam center ($z \ll z_o$)

Beam intensity is ~ uniform across wavefront and Guay phase is zero

In other words it is like a plane wave.

Far from beam waist ($z \gg z_o$)

Wave is ~ like a spherical wave (paraxially)

Since $R(z) \approx z$ and $w(z) \approx w_o z/z_o$, but with extra phase of $\zeta(z) = \pi/2$.

Gaussian Beams

Conversion formulas

Any constant times w(z) is a ray path. Note that the rays are converging and diverging spherical wave *except* near the focus where they <u>bend</u>. Ergo rays do <u>not</u> always travel in straight lines – the region near the focus violates the slowly-varying envelope approximation.

Conversion formulas

$$w_0 = \sqrt{\frac{\lambda}{\pi} z_0} = \frac{\lambda}{\pi}\frac{1}{\theta_0} \qquad \theta_0 = \frac{\lambda}{\pi}\frac{1}{w_0} = \sqrt{\frac{\lambda}{\pi}\frac{1}{z_0}} \qquad z_0 = \frac{\lambda}{\pi}\theta_0^{-2} = \frac{\pi}{\lambda} w_0^2 = \frac{w_0}{\theta_0}$$

Gaussian Beam Parameter q(z)

The complete Gaussian beam expression normalized to intensity

$$E(\vec{r}) = A_0 \left(\frac{1}{\sqrt{\pi} w_0} \right)^{D/2} \frac{w_0}{w(z)} e^{-\frac{\rho^2}{w^2(z)} - jkz - jk\frac{\rho^2}{2R(z)} + j\zeta(z)}$$

D is number of transverse dimensions $=1,2$

Define the complex radius of curvature

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\frac{\lambda}{\pi\, w^2(z)}$$

Gaussian Beam Parameter q(z)

What is q(z)?

$$q(z) = \frac{1}{\dfrac{1}{z\left[1+\left(\dfrac{z_0}{z}\right)^2\right]} - j\dfrac{1}{z_0\left[1+\left(\dfrac{z}{z_0}\right)^2\right]}}$$

$$= \left[\frac{z - j\,z_0}{z^2 + z_0^2}\right]^{-1}$$

$$= z + j\,z_0$$

$$\arg(q) = \tan^{-1}\frac{z_0}{z}$$

$$\frac{|q|}{z_0} = \sqrt{1+\left(z/z_0\right)^2} = \frac{w(z)}{w_0}$$

Can now write Gaussian above as

$$E(\vec{r}) = j\,A_0\left(\frac{1}{\sqrt{\pi}\,w_0}\right)^{D/2}\frac{z_0}{q(z)}e^{-jk\frac{\rho^2}{2q(z)} - jkz}$$

Note that phase of j/q(z) is ζ.

How does q change with transfer and refraction ?

Free space:

$$q_1 = z_1 + j\, z_0$$

Start with expression for q(z)

$$q_2 = z_2 + j\, z_0 = q_1 + (z_2 - z_1)$$

so $q_2 = q_1 + \Delta z$

Thin lens

$$\frac{1}{q(z)} = \frac{1}{R(z)} - j\,\frac{\lambda}{\pi\, w^2(z)}$$

Start with expression for 1/q(z)

$$\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$$

Thin lens equation expressed as change in curvature of wave
NOTE HOW GAUSSIAN BEAM SIGN CONVENTION HAS CHANGED THE SIGN

$$\frac{1}{q'} = \frac{1}{q} - \frac{1}{f}$$

Apply to 1/q

$$q' = \frac{q}{-q/f + 1}$$

Solve for q

ABCD Matrix for Guassians

Remember the ABCD matrices for thin lens refraction and free-space transfer

$$\mathrm{R}_k = \begin{bmatrix} 1 & 0 \\ -\phi_k & 1 \end{bmatrix} \qquad \mathrm{T}_k = \begin{bmatrix} 1 & t_k' \\ 0 & 1 \end{bmatrix}$$

and define the evolution equation for q

$$q' = \frac{Aq + B}{Cq + D}$$

ABCD Matrix for Guassians

Check for free space:

$$q' = \frac{1q + t_k'}{0q + 1} = q + t_k'$$

Check for thin lens:

$$q' = \frac{1q + 0}{-\phi_k q + 1} = \frac{q}{-q/f + 1}$$

Which says, rather remarkably, that we can model the propagation of a Gaussian beam through a paraxial optical system using ray matrices.

Matrix Method

Consider next the propagation of a Gaussian beam through two lenslike media that are adjacent to each other. The ray matrix describing the first one is (A_1, B_1, C_1, D_1) while that of the second one is (A_2, B_2, C_2, D_2). Taking the input beam parameter as q_1 and the output beam parameter as q_3, we have

$$q_2 = \frac{A_1 q_1 + B_1}{C_1 q_1 + D_1}$$

for the beam parameter at the output of medium 1 and

$$q_3 = \frac{A_2 q_2 + B_2}{C_2 q_2 + D_2}$$

and after combining the last two equations,

$$q_3 = \frac{A_T q_1 + B_T}{C_T q_1 + D_T}$$

where

$$\begin{vmatrix} A_T & B_T \\ C_T & D_T \end{vmatrix} = \begin{vmatrix} A_2 & B_2 \\ C_2 & D_2 \end{vmatrix} \begin{vmatrix} A_1 & B_1 \\ C_1 & D_1 \end{vmatrix}$$

So can use them like before for multiple elements

Amnon Yariv, Optical Electronics

Example

Gaussian beam focusing

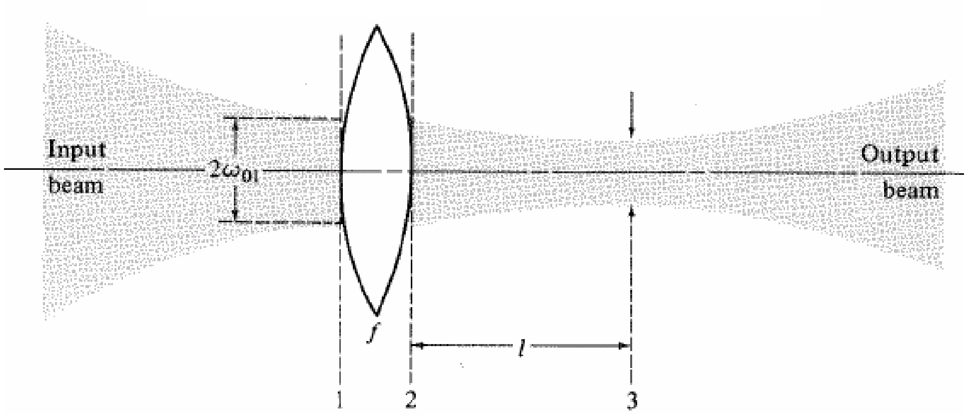
At the input plane 1 $\omega = \omega_{01}$, $R_1 = \infty$ so that

$$\frac{1}{q_1} = \frac{1}{R_1} - i\frac{\lambda}{\pi\omega_{01}^2 n} = -i\frac{\lambda}{\pi\omega_{01}^2 n}$$

$$\frac{1}{q_2} = \frac{1}{q_1} - \frac{1}{f} = -\frac{1}{f} - i\frac{\lambda}{\pi\omega_{01}^2 n}$$

$$q_2 = \frac{1}{-1/f - i(\lambda/\pi\omega_{01}^2 n)} = \frac{-a+ib}{a^2+b^2}$$

$$a \equiv \frac{1}{f} \qquad b \equiv \frac{\lambda}{\pi\omega_{01}^2 n}$$



Example

Gaussian beam focusing

$$q_3 = q_2 + l = \frac{-a}{a^2+b^2} + \frac{ib}{a^2+b^2} + l$$

$$\frac{1}{q_3} = \frac{1}{R_3} - i\frac{\lambda}{\pi\omega_3^2 n}$$

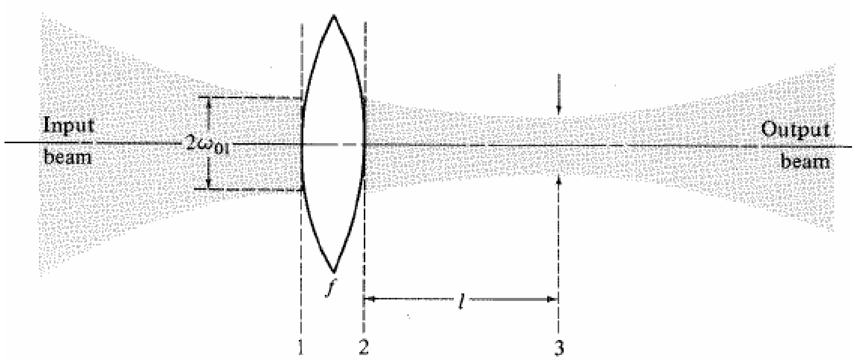
$$= \frac{[-a/(a^2+b^2)+l] - ib/(a^2+b^2)}{[-a/(a^2+b^2)+l]^2 + b^2/(a^2+b^2)^2}$$

Since plane 3 is, according to the statement of the problem, to correspond to the output beam waist, $R_3 = \infty$. Using this fact in the last equation leads to

$$l = \frac{a}{a^2+b^2} = \frac{f}{1+(f/\pi\omega_{01}^2 n/\lambda)^2} = \frac{f}{1+(f/z_{01})^2}$$

as the location of the new waist, and to

$$\frac{\omega_3}{\omega_{01}} = \frac{f\lambda/\pi\omega_{01}^2 n}{\sqrt{1+(f\lambda/\pi\omega_{01}^2 n)^2}} = \frac{f/z_{01}}{\sqrt{1+(f/z_{01})^2}}$$



Example

Gaussian beam in lens waveguide/resonator

$$\begin{vmatrix} A_T & B_T \\ C_T & D_T \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}^S$$

where (A, B, C, D) is the matrix for propagation through a single two-lens, unit cell $(\Delta s = 1)$ We can use a well-known formula for the sth power of a matrix with a unity determinant (unimodular) to obtain

$$A_T = \frac{A \sin(s\theta) - \sin[(s-1)\theta]}{\sin\theta}$$

$$B_T = \frac{B \sin(s\theta)}{\sin\theta}$$

$$C_T = \frac{C \sin(s\theta)}{\sin\theta}$$

$$D_T = \frac{D \sin(s\theta) - \sin[(s-1)\theta]}{\sin\theta}$$

where $$\cos\theta = \tfrac{1}{2}(A + D) = \left(1 - \frac{d}{f_2} - \frac{d}{f_1} + \frac{d^2}{2 f_1 f_2}\right)$$

For mirrors
F=2/R

$$q_{s+1} = \frac{\{A \sin(s\theta) - \sin[(s-1)\theta]\} q_1 + B \sin(s\theta)}{C \sin(s\theta) q_1 + D \sin(s\theta) - \sin[(s-1)\theta]}$$

Example

Gaussian beam in lens waveguide/resonator

For Gaussian beam confinement θ must be real. This yields the condition below for stable beam confinement or resonance…

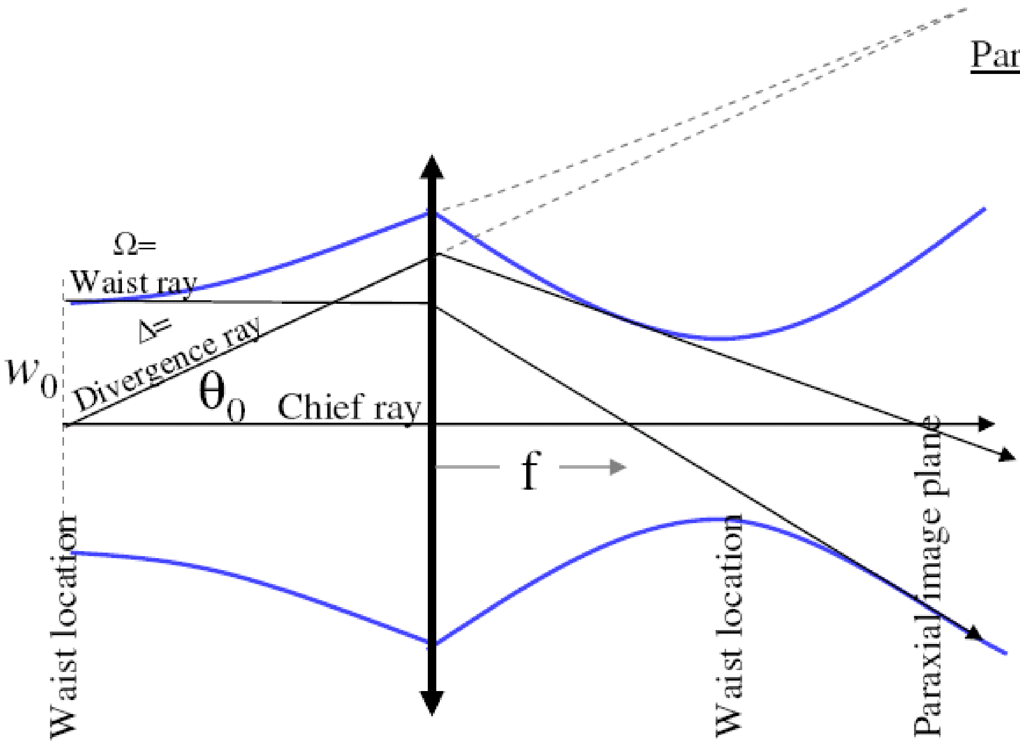
$$0 \le \left(1 - \frac{d}{2f_1}\right)\left(1 - \frac{d}{2f_2}\right) \le 1$$

For mirrors
F=2/R

This is the same condition as we derived for RAYS !!!

Representation of Gaussian Beams by complex rays (1)

Define the following three rays. Note their suggestive names and relationship to the Gaussian beam.



Paraxial ray trajectory form

$$\Omega(z) = w_0$$

$$\Delta(z) = z\,\theta_0$$

ABCD vector form

$$\Omega_0 = \begin{bmatrix} w_0 \\ 0 \end{bmatrix}$$

$$\Delta_0 = \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix}$$

Representation of Gaussian Beams by complex rays (1')

Define the complex ray trajectory

$$\Gamma(z) = \Delta(z) + j\Omega(z)$$

This is Greynolds' definition and yields the proper form of q. Arnaud's definition yields q*.

You can then show that this ray contains $q(z)$

$$\frac{\Gamma(z)}{d\Gamma/dz} = \frac{y_\Delta + j\,y_\Omega}{u_\Delta + j\,u_\Omega}$$

Ray heights over ray slopes

$$= \frac{z\theta_0 + jw_0}{\theta_0} = z + j\,z_0 = q(z)$$

E.g. at z=0

J. Arnaud, Applied Optics, V24, N4, p. 538, 15 Feb 1985
A. W. Greynolds, SPIE V 560, p. 33, 1985
M&M A2.5

Representation of Gaussian Beams by complex rays (2)

First we note:

$$\left(y_\Omega u_\Delta - y_\Delta u_\Omega\right)\frac{n}{n'} = \frac{\lambda'}{\pi}$$

Lagrange invariant (N_{spots}=1)

By brute for tracing of the rays, we can find the following Gaussian parameters based on the two rays at that point:

$$\theta_0 = \sqrt{u_\Delta^2 + u_\Omega^2}$$

Which gives all other beam parameters

$$w(z) = \sqrt{y_\Omega^2(z) + y_\Delta^2(z)}$$

1/e field radius *at this z*

Representation of Gaussian Beams by complex rays (2')

We could use these two and the expressions for the Gaussian beam parameters to generate the complete Gaussian, but this would be a bit tedious. A more elegant way is to use the complex ray formalism:

$$q(z) = \frac{\Gamma(z)}{d\Gamma/dz}$$

$$= \frac{y_\Delta + zu_\Delta + j(y_\Omega + zu_\Omega)}{u_\Delta + ju_\Omega} \qquad \text{At plane } z \neq 0$$

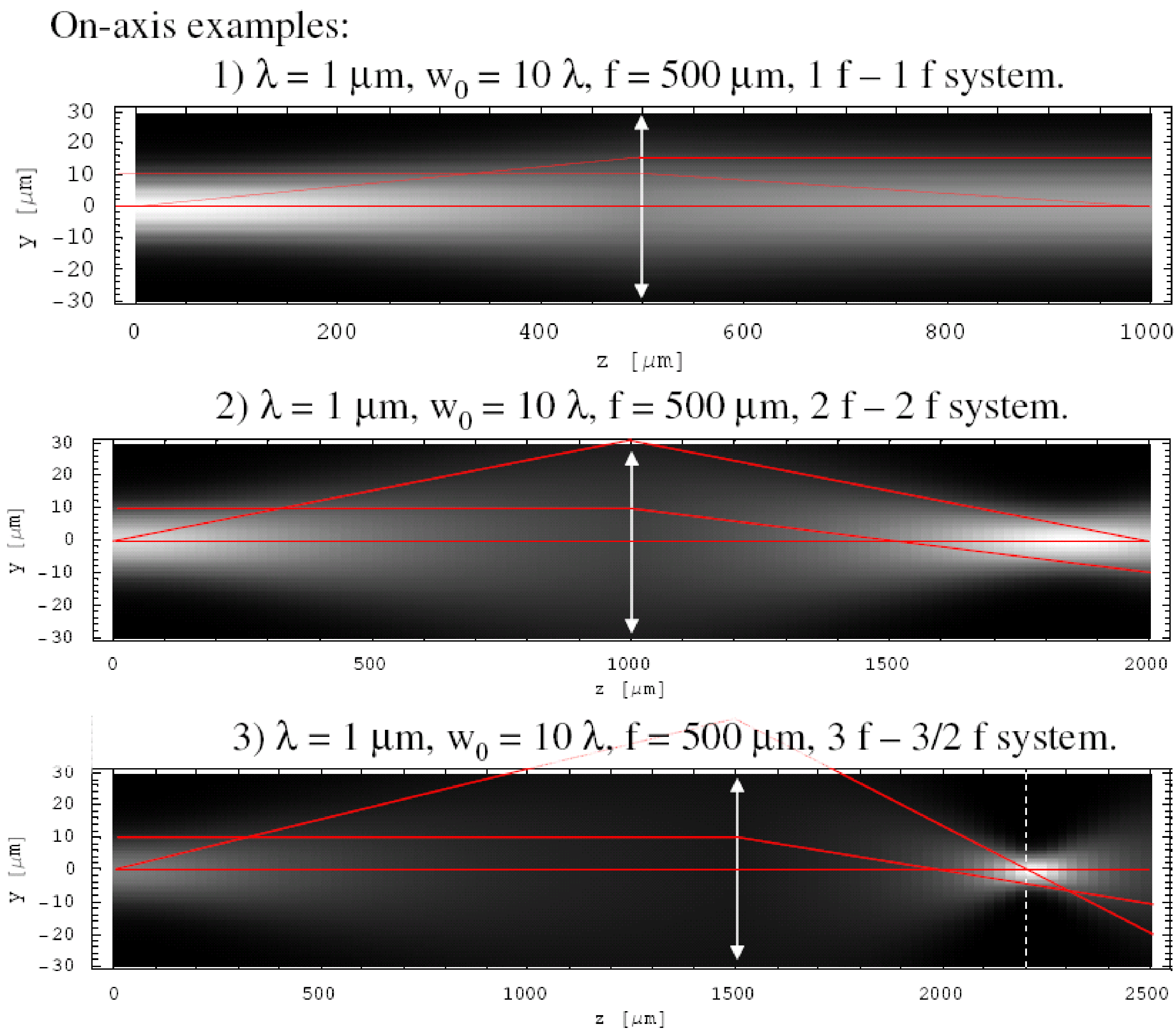
Which, apart from the on-axis phase k_0S gives the full Gaussian beam at this plane z.

R. Herloski, S. Marshall, R. Antos, Applied Optics, V22, N8, p 1168, 15 Apr 1983

Representation of Guassian Beams by complex rays (3)

Notes

- In (1), second waist is at Fourier plane, as expected.
- In (2), second waist occurs before image plane, as expected.
- In (3), as distance to lens increases, waist moves to paraxial image plane

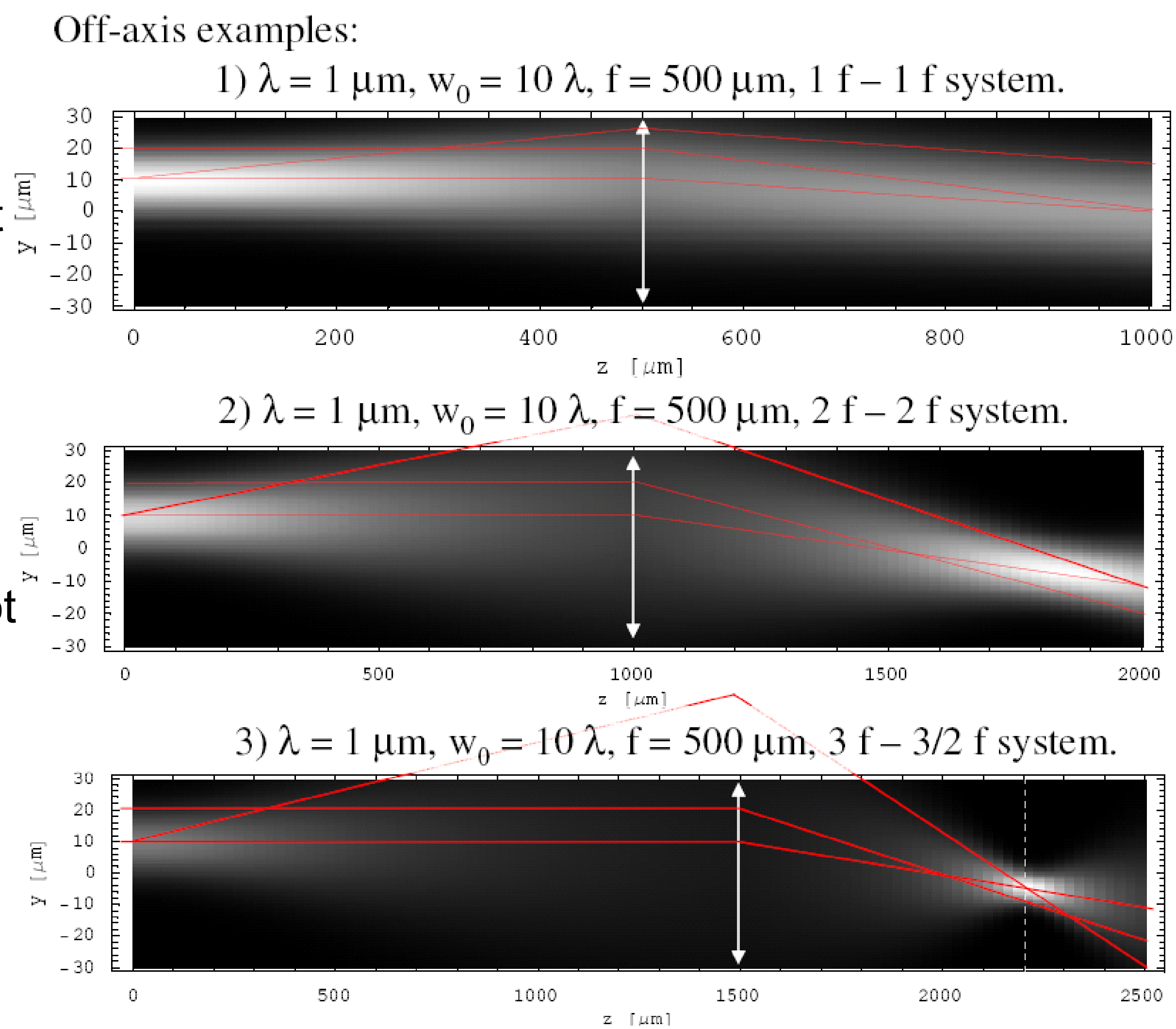


Representation of Guassian Beams by complex rays (4)

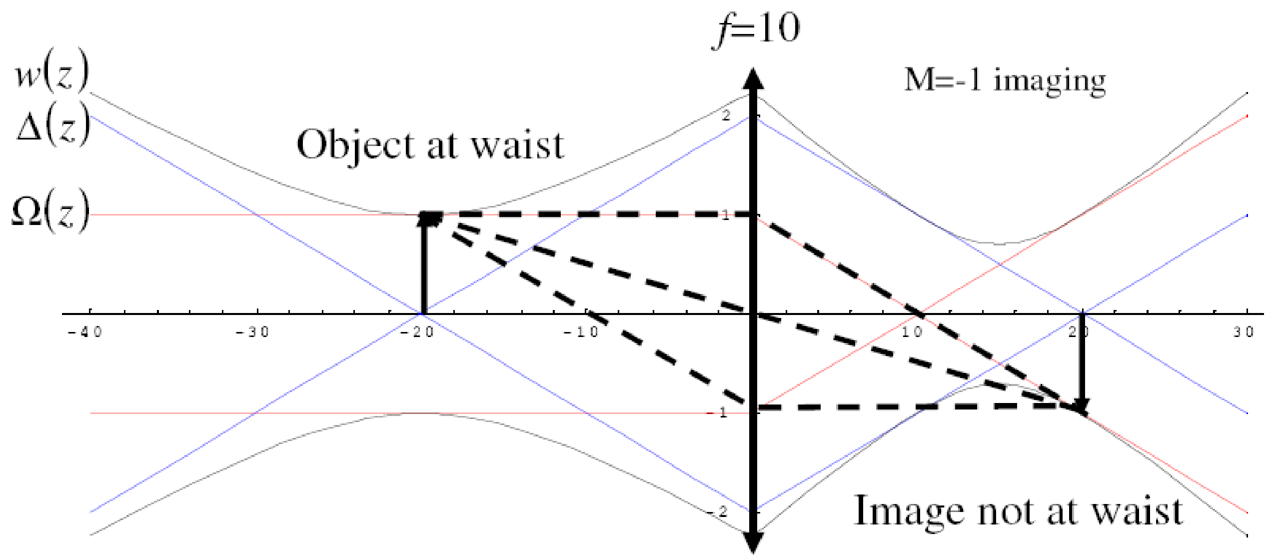
Notes
- In (1), waist is centered at zero (as expected of FT geometry)

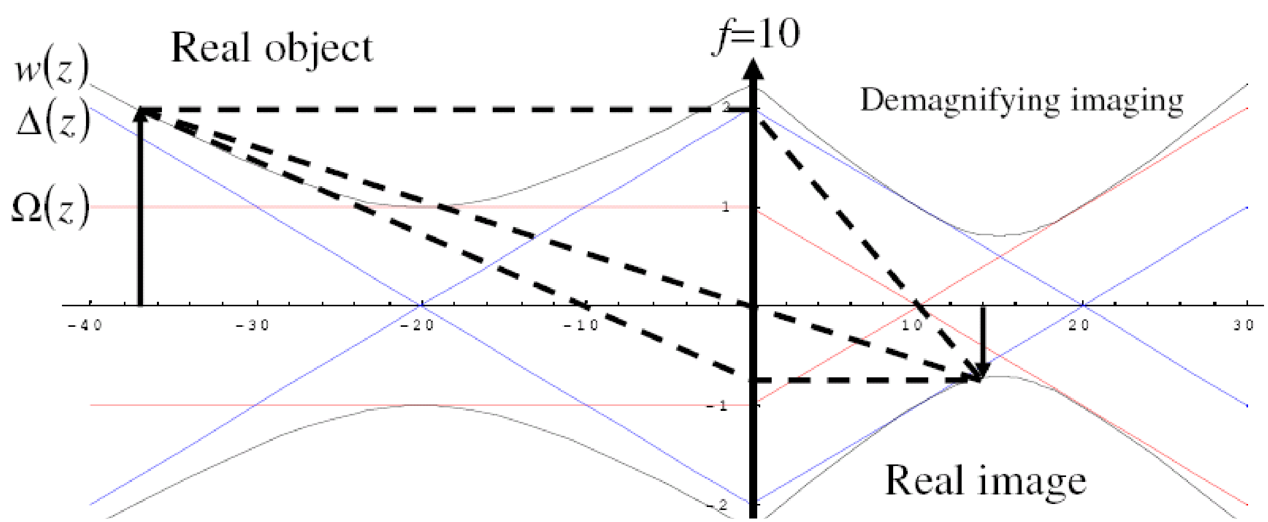
- In (2), image is at -10 μm, expected from M=-1.

- This type of problem is not possible with the ABCD formalism.



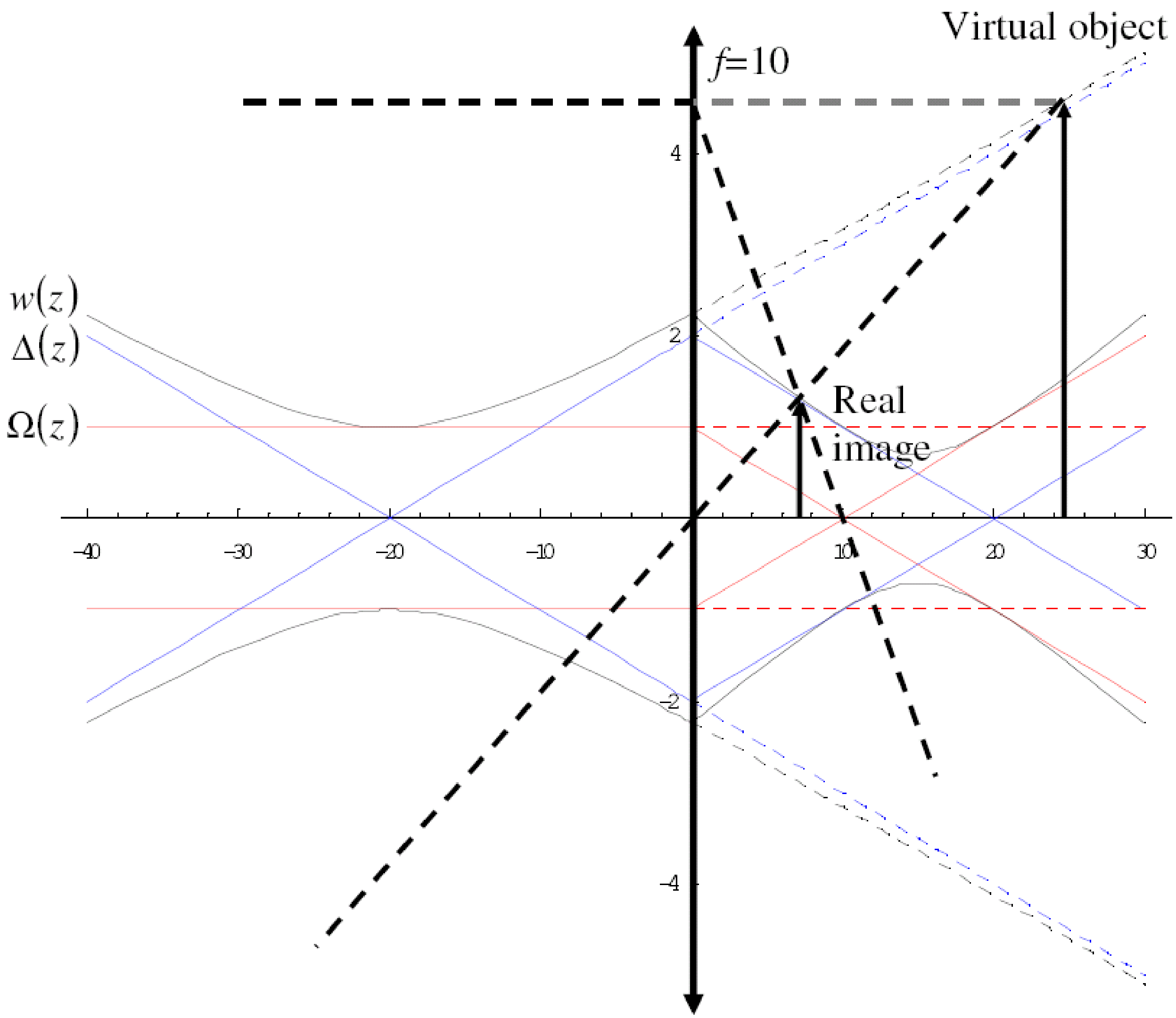
Do Guassian Beams Obey Paraxial Imaging ? (1/3)





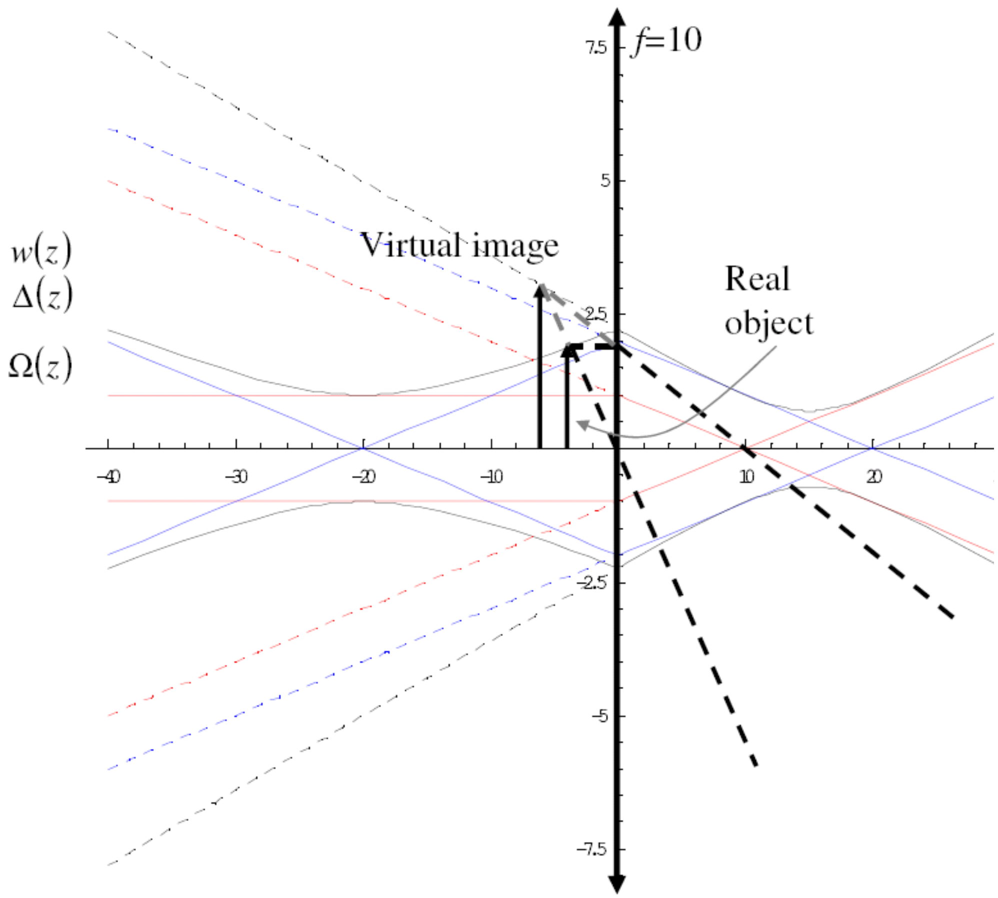
Answer: Yes. The image is also a Gaussian E field distribution in amplitude and any point on the object down from the peak by some value, say 1/e for the point $w(z)$, will image to the point on the image down from the peak by the same value. Shown above only for real objects conjugate to real images ($-t$ f).

Do Guassian Beams Obey Paraxial Imaging ? (2/3)



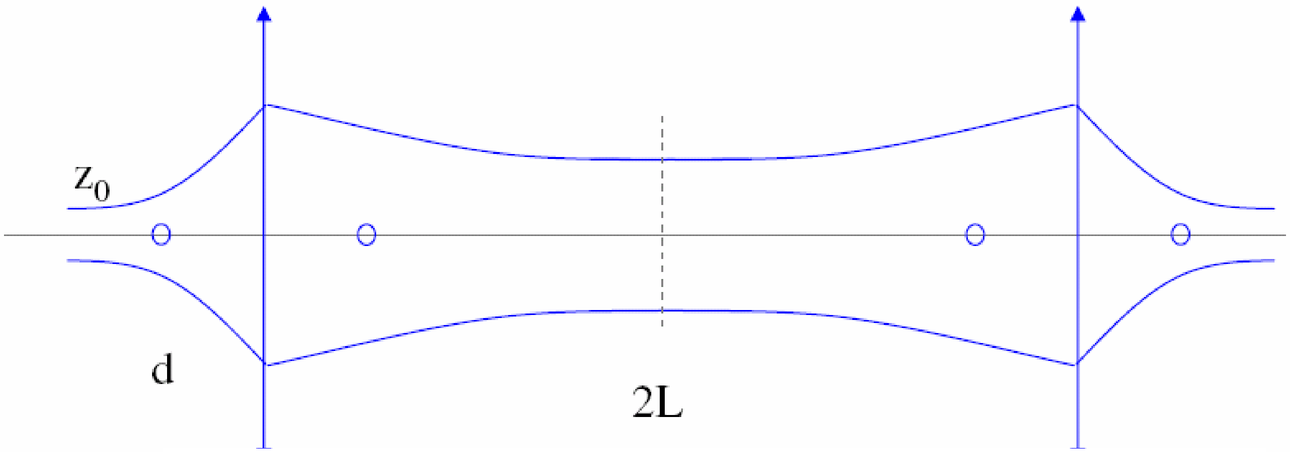
Works for virtual objects as well…

Do Guassian Beams Obey Paraxial Imaging ? (3/3)



Conclusion: All parts of the object Gaussian image correctly to the appropriate parts of the image Gaussian including both real and virtual objects and images.
Corollary: If you apply paraxial imaging to the object Gaussian over all z, you generate the image Gaussian over all z. Gaussian beams obey paraxial imaging exactly.

Example: Collimation Lens



$$M = T(L)R(\phi)T(d)$$ ABCD from start to center

$$= \begin{bmatrix} 1-L\phi & L+d(1-L\phi) \\ -\phi & 1-d\phi \end{bmatrix}$$

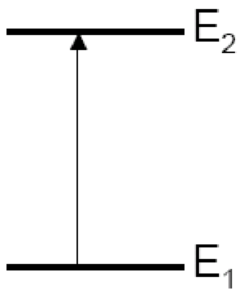
$$q(L) = \frac{L+d(1-L\phi)+jz_0(1-L\phi)}{1-d\phi-j\phi\, z_0}$$ q at center starting with $q = j\, z_0$

$$\mathrm{Re}[q(L')] = 0 \Rightarrow L' = \frac{\phi\, z_0^2 - d(1-d\phi)}{\phi^2 z_0^2 + (1-d\phi)^2}$$ Where is waist?

$$q(L') = j\frac{z_0}{\phi^2 z_0^2 + (1-d\phi)^2} = j\, z_{0-NEW}$$ What is new Rayleigh range?

Laser Basics

Absorption and Emission



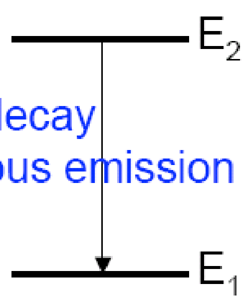
An atom/ion in a material, such as glass, ca
absorb light,

$$h\nu = E_2 - E_1$$

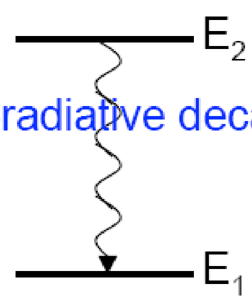
ν is the frequency of light (photon)

After absorption, the material does not stay in the excited state indefinitely, but it will go back to the ground state either by

emitting light or giving off heat



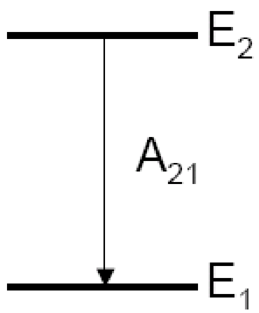
radiative decay
spontaneous emission



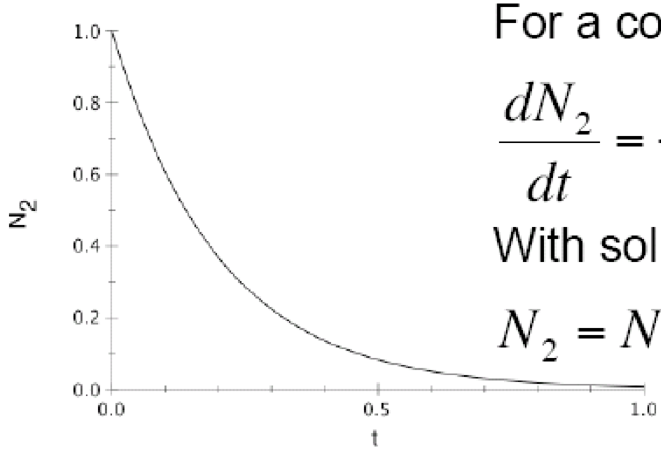
non-radiative decay

Laser basics

Spontaneous Emission and Lifetime



There is a certain probability (A_{21}) for the atom to decay radiatively



For a collection of N_2 atoms in the excited state:

$$\frac{dN_2}{dt} = -A_{21}N_2$$

With solution:

$$N_2 = N_{20}e^{-A_{21}t} = N_{20}e^{-t/\tau}$$

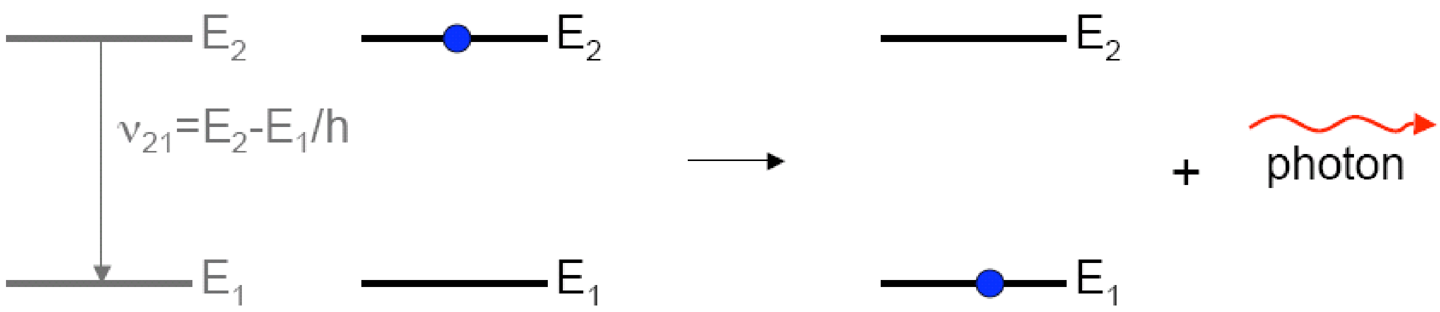
τ is the lifetime

Including non-radiative decay:

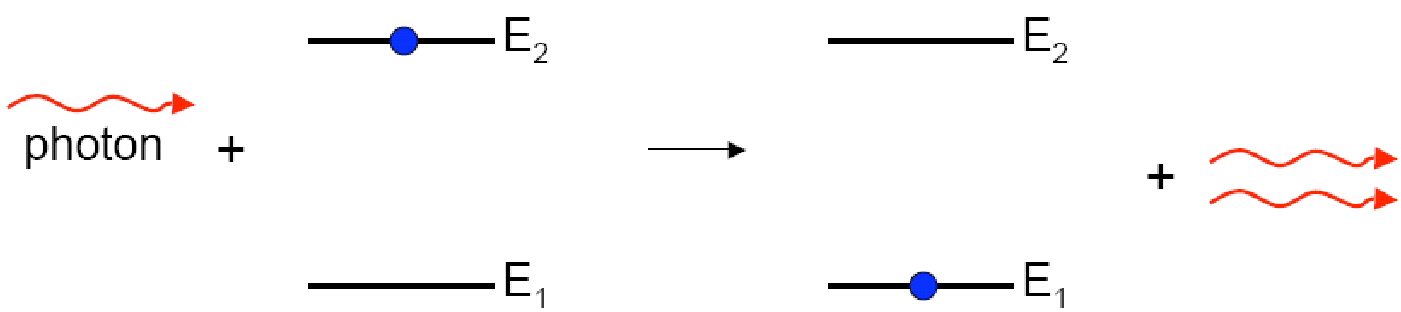
$A_{tot} = A_{21} + A_{nr}$ $\longrightarrow$ $\tau = 1/A_{tot}$

Laser basics

Spontaneous and Stimulated Emission



spontaneous emission



stimulated emission → light amplification!

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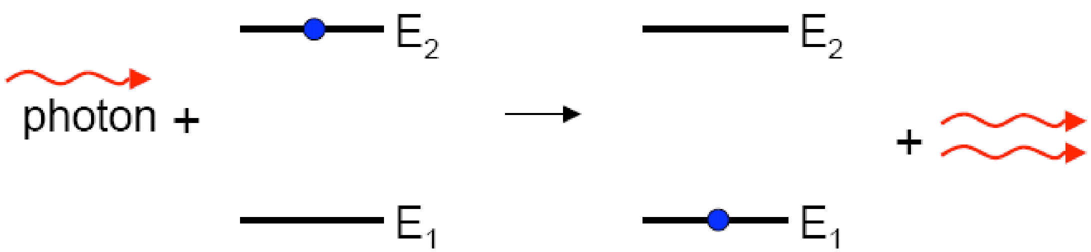
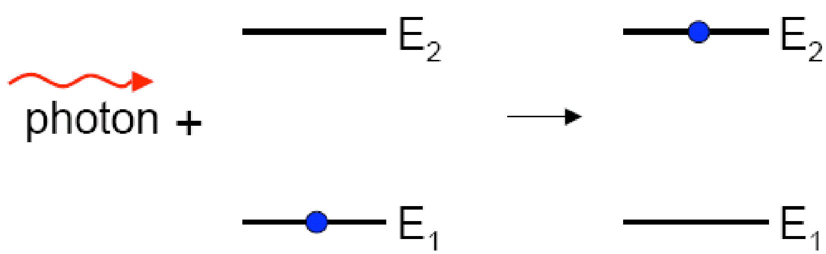
Laser basics

Stimulated Emission versus Absorption



stimulated emission

But we also have



absorption

To have net amplification of light (gain) we need $N_2 > N_1$

We need population inversion

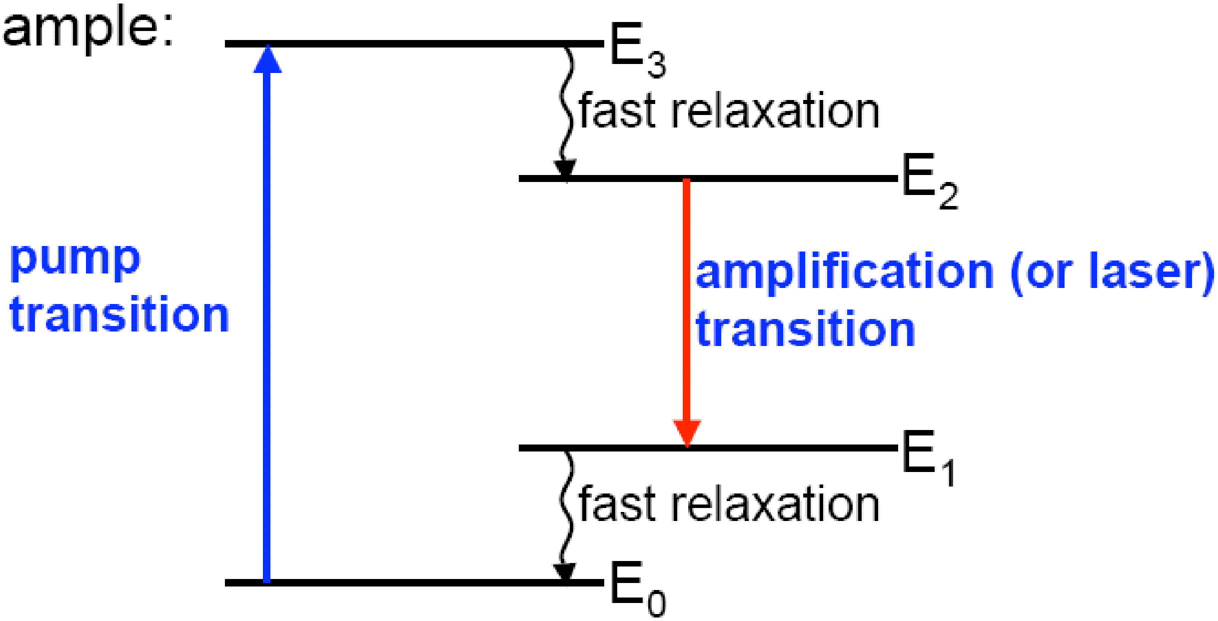
Laser basics

Population Inversion

If there are only 2 levels inversion is not possible

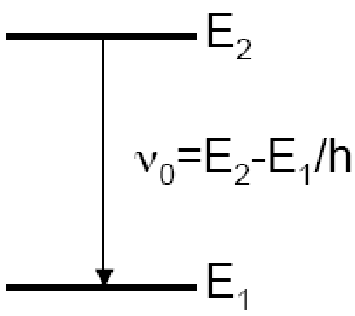
But if we have >3 levels inversion can be obtained

Example:



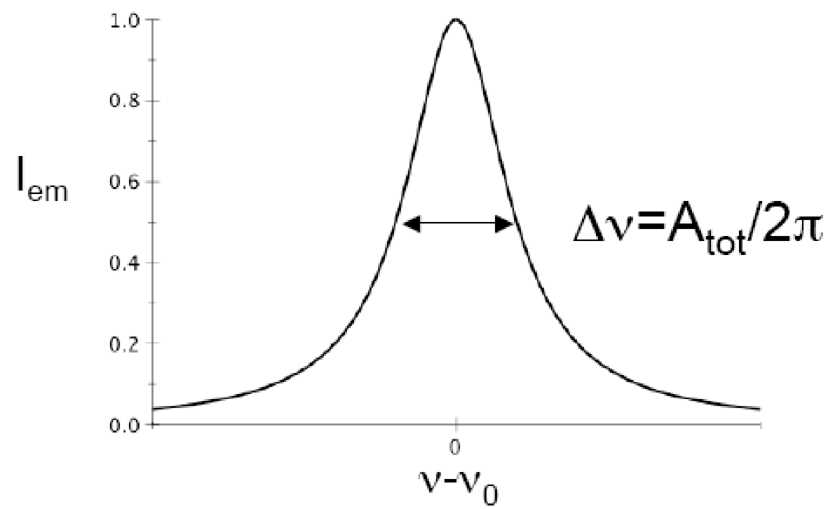
Line Broadening:

Homogeneous



The finite lifetime of the excited state leads to a broadening of the emission linewidth:

$$I(\nu) = I_0 \frac{A_{tot}/4\pi^2}{(\nu-\nu_0)^2 + (A_{tot}/4\pi)^2}$$



The lineshape is Lorentzian and the same for all atoms

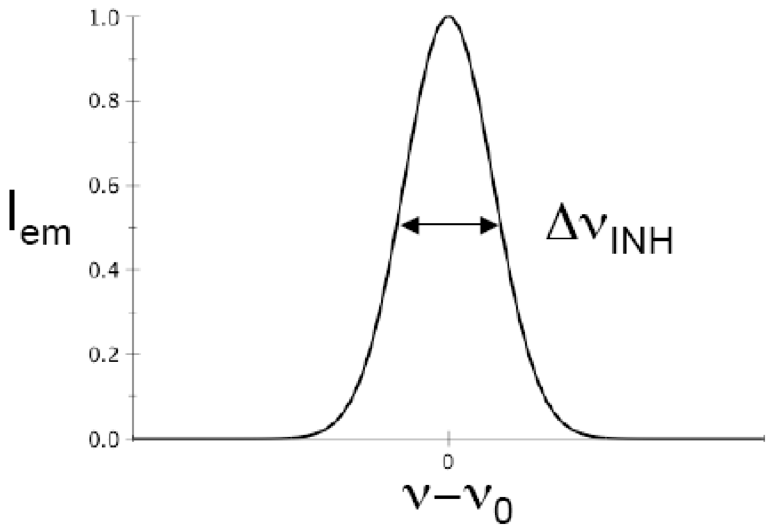
homogeneous broadening

Line Broadening:

Inhomogeneous

There is also a broadening that results from the fact that not all atoms have the same surroundings (glass!) $\longrightarrow$ different atoms have slightly different transition frequencies (gas) different velocities
The spread in frequencies is characterized by $\Delta\nu_{INH}$

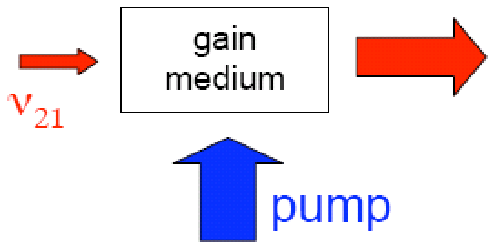
$$I(\nu) = I_0 \frac{2(\ln 2)^{1/2}}{\pi^{1/2}\Delta\nu_{INH}} \exp\left\{-\left[\frac{4(\ln 2)(\nu-\nu_0)^2}{(\Delta\nu_{INH})^2}\right]\right\}$$



The resulting lineshape is Gaussian   inhomogeneous broadening

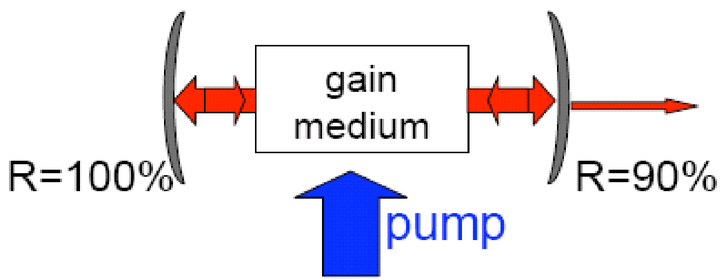
Laser basics

Amplification and Lasing



amplifier

Because of stimulated process, amplified light has direction and phase of incoming signal



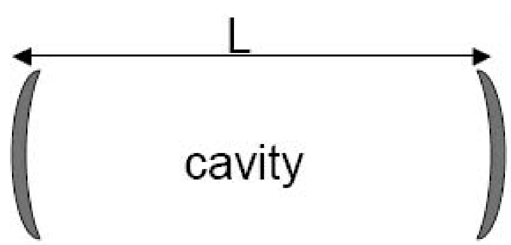
laser =
amplifier + optical cavity

laser light has the following properties:

- highly directional
- highly monochromatic
- highly coherent

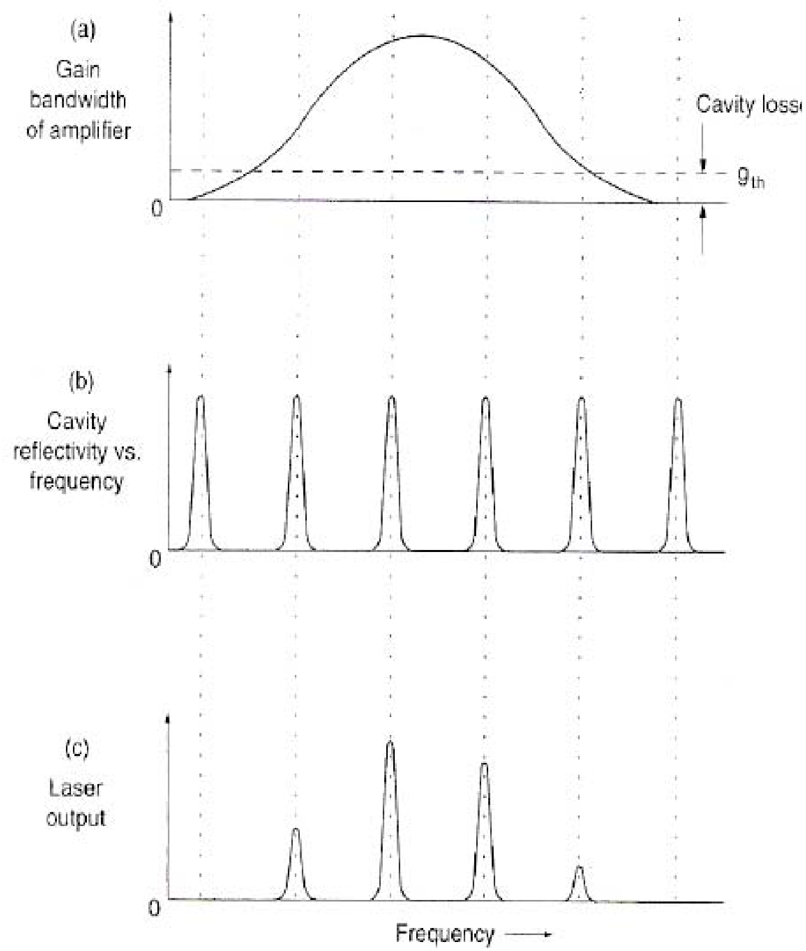
Laser basics

Cavity Modes

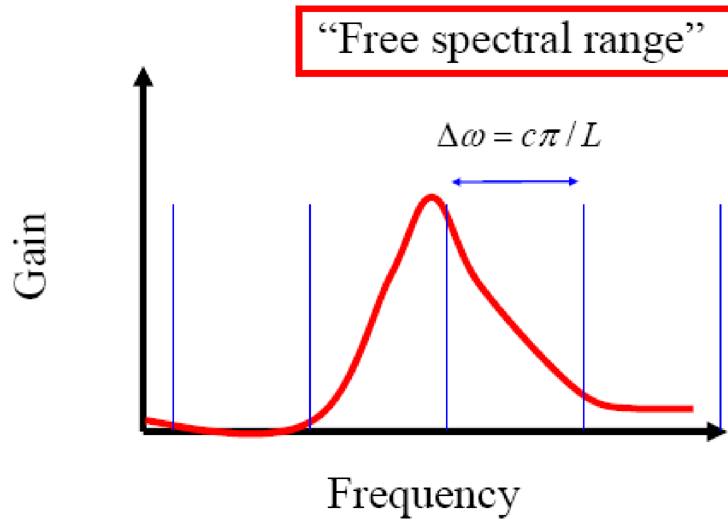


axial mode frequencies $\nu = nc/2L$

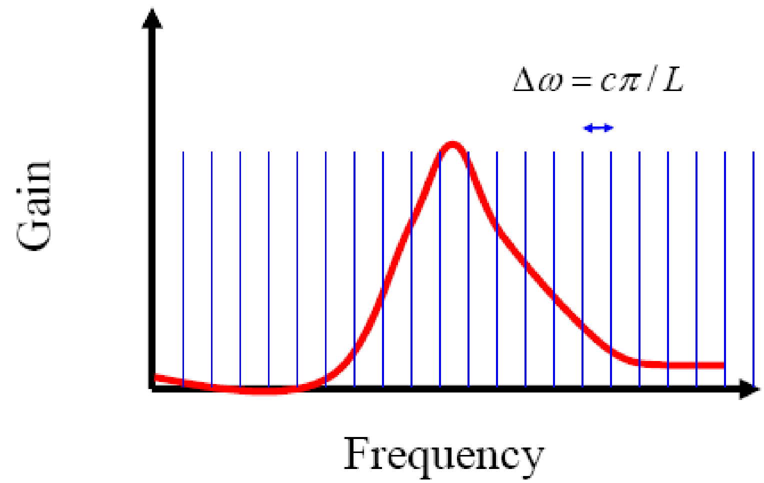
axial mode separation $\Delta\nu = c/2L$



Single versus Multimode



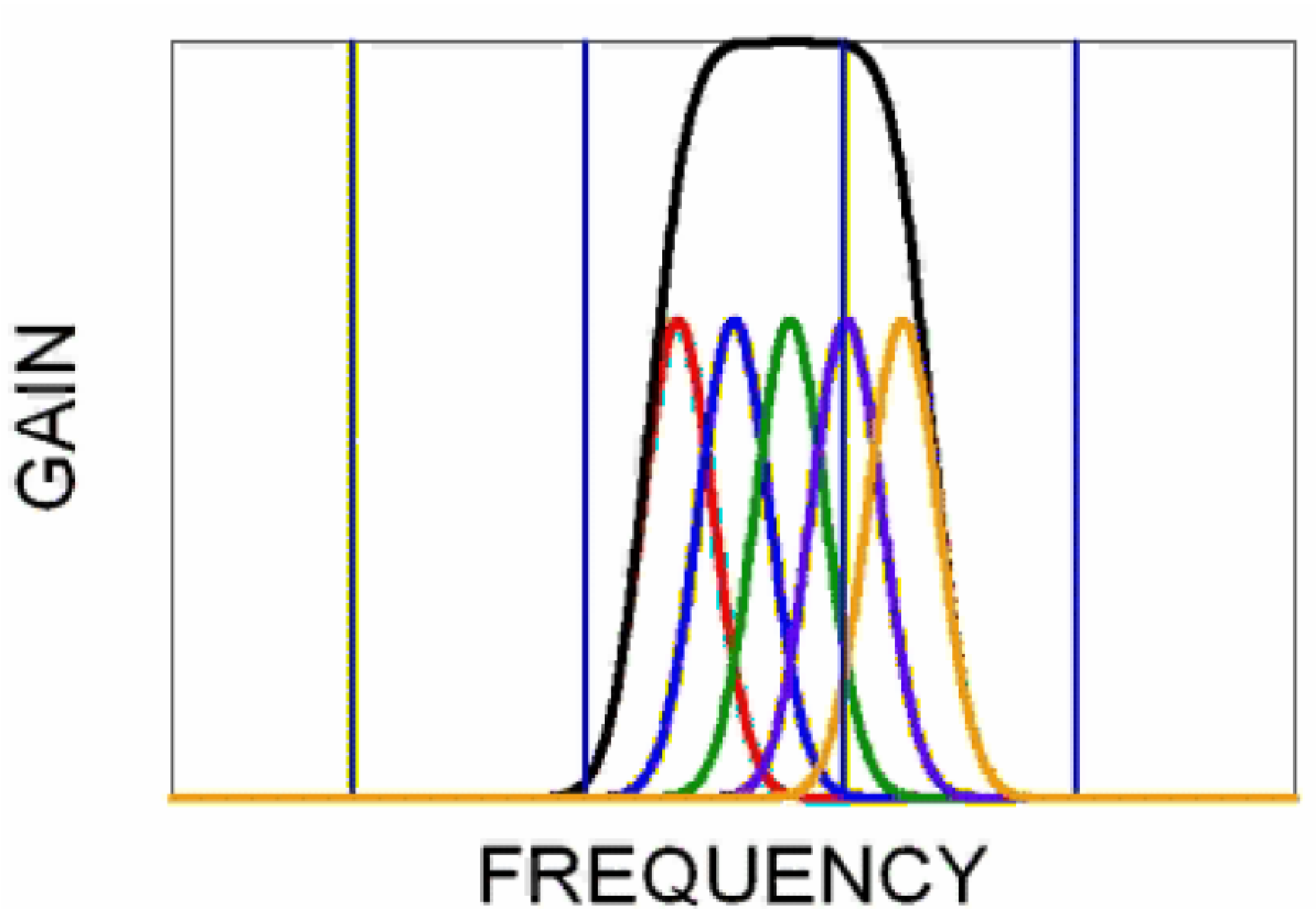
Single Mode



Multimode Mode

Spectral hole burning

Five species of atom shown. (For example, five atoms in gas with different velocities.)
Black is total response. Blue lines are cavity modes.



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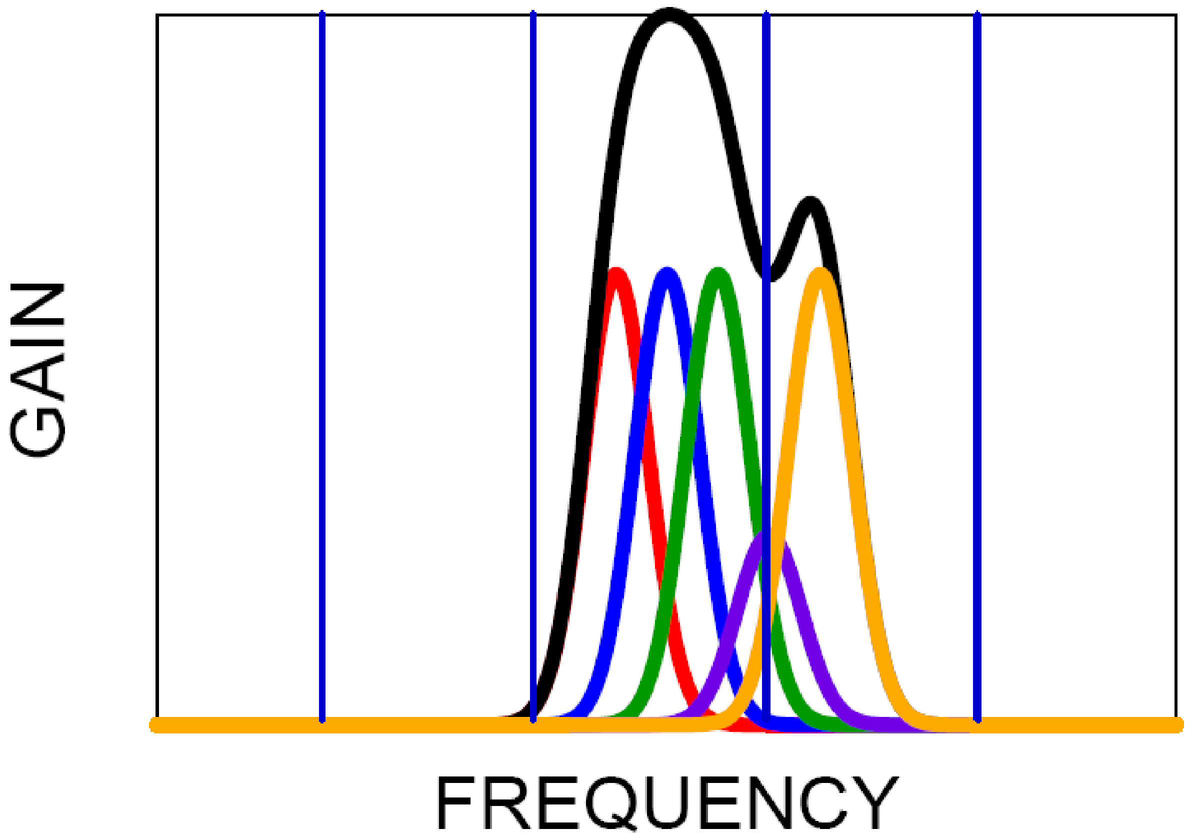
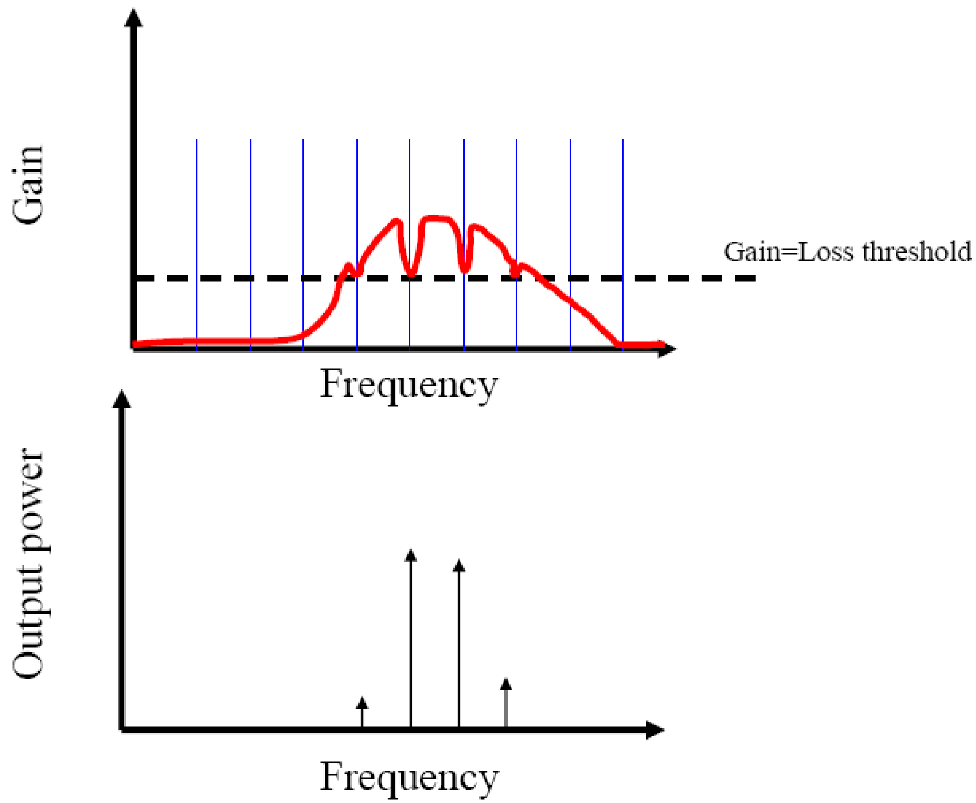


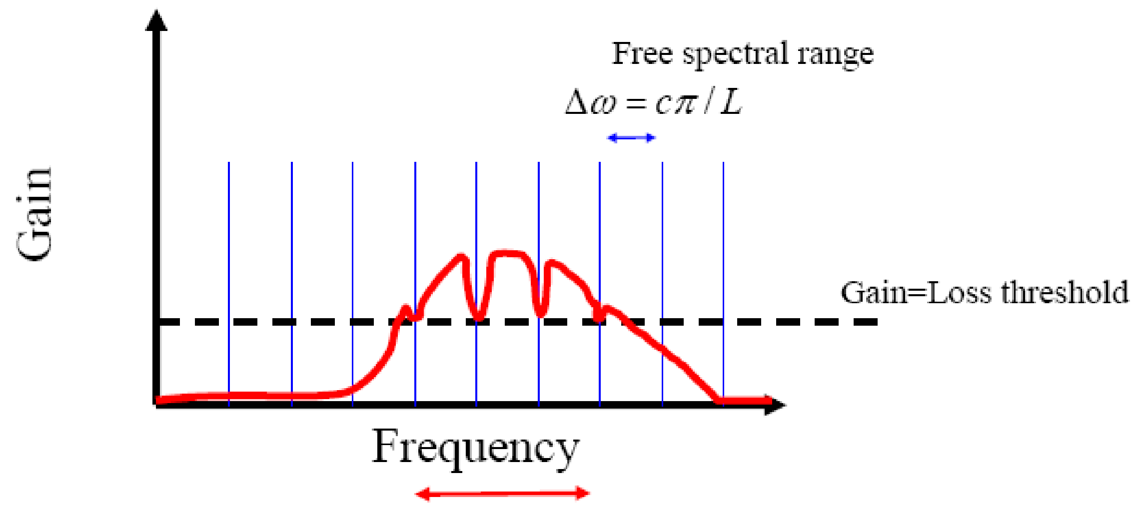
This will be a single mode laser. What if broadening > free spectral range?

Multimode/Inhomogeneous

Inhomogeneously broadened:



Multimode Lasers

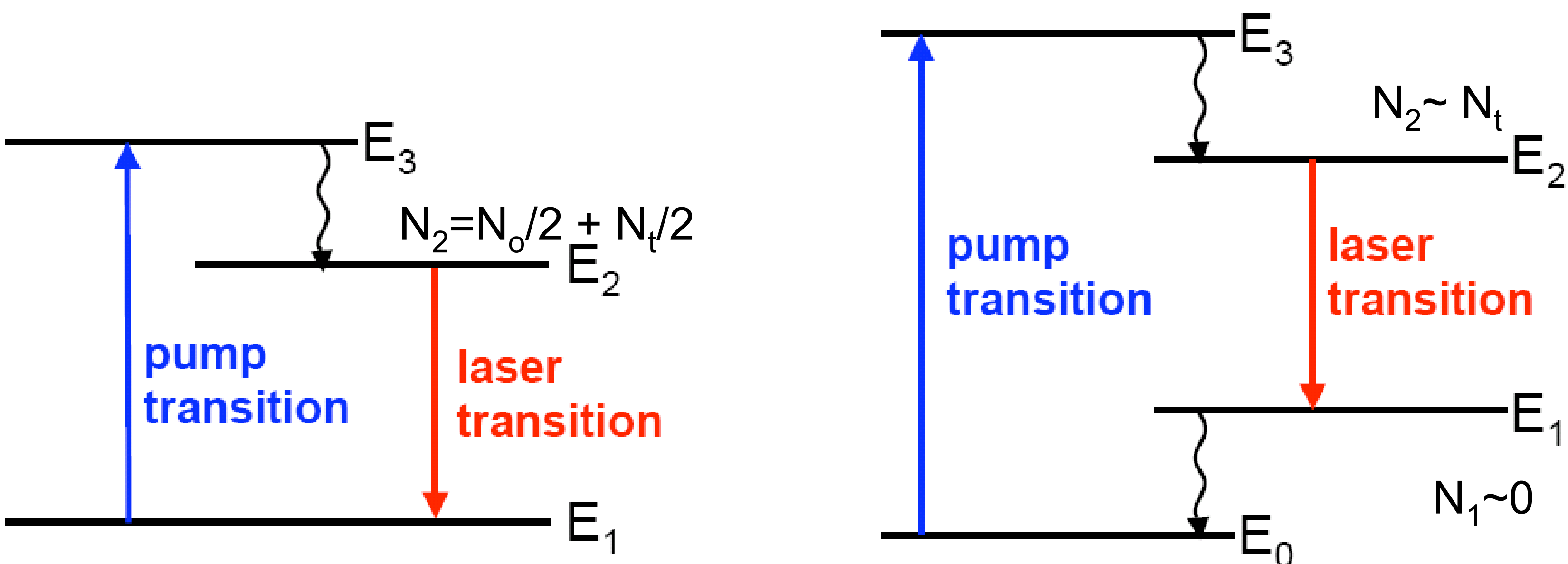


Examples:

HeNe: Doppler (inhomogeneous) broadened to 1.5 GHz
Free spectral range for 0.1 meters = 0.15 GHz
So *ten modes oscillate.*
Low pressure CO_2 laser doppler broadened to 100 MHz
If length of cavity is 0.1 meters, FSR = 0.15 GHz
So *one mode oscillates* i.e. *single mode.*
Nd-YAG homogeneous broadened to 140 GHz
Since it's homogeneous, it will tend to single mode, even though it's got 100s of allowed modes in its (unsaturated) gain curve.

Laser basics

3 level vs 4 level systems



3-level system example : Er^{3+}

$N_1=N_o/2 - N_t/2$

4-level system example : Nd^{3+}

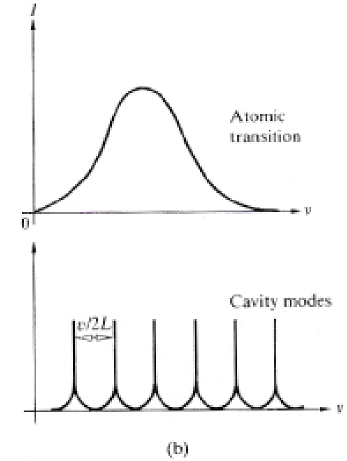
In 3-level system more than 50% of level 1 needs to be pumped, so it is harder to obtain inversion:
Pump and laser transition share a level

$$\frac{(N_2)_{3level}}{(N_2)_{4level}} \approx \frac{N_o}{2N_t}$$

Can be large ~100

Laser basics

Some simple laser equations



note: cw lasers!

$$e^{2g_{th}L} = \frac{1}{R^2}$$

gain =loss and ν = nc/2L

$g_{th} = \sigma\Delta N_{th}$ where σ is the emission cross-section (m^2).
R = mirror reflectivity
(Add in α which is passive loss in medium)

The lasing threshold is achieved when the pump rate (proportional to pump power) is high enough to obtain ΔN_{th}.
If the pump rate is increased further the steady state laser intensity (power/area), I_{ss}, grows according to

$$I_{ss} = (P/P_{th} - 1)I_{sat}$$

Here P is the pump power, P_{th} the pump power needed to reach threshold and I_{sat} the saturation intensity (a fixed parameters for a given laser transition)

Threshold Inversion

Minimum Power Required

Gain equals losses and then solve for ΔN

$$N_t = \frac{8\pi n^2 t_{spont}}{g(\upsilon)\lambda^2}(\alpha - \frac{1}{l}\ln r_1 r_2)$$

$$\frac{1}{g(\upsilon_o)} \approx \Delta\upsilon$$

Linewidth

Also 1/lw will give you minimum pulse length

The cavity decay time (t_c) assuming α=0, r_1 ~ r_2 is approximately

$$t_c \approx \frac{nl}{c(1-R)}$$

The threshold population is many times written using t_c as

$$N_t = \frac{8\pi n^3 \upsilon^2 t_{spont}}{c^3 t_c g(\upsilon)}$$

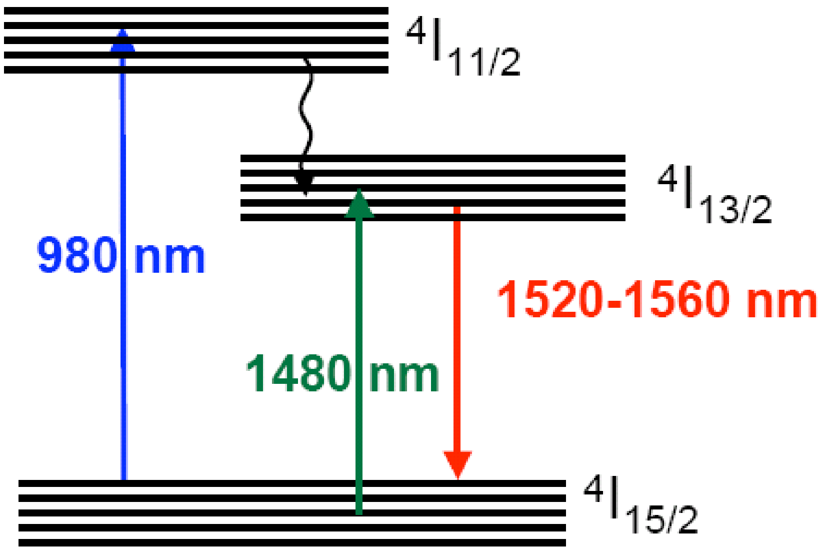
Power needed to do this is given by

$$(P_s)_{4level} = \frac{N_t h\upsilon V}{t_2}$$

Example

Erbium Doped Fiber Amplifier: EDFA

Glass fibers: long interaction lengths, compact and robust



Energy levels of Er^{3+}
Pumping bands @ 980 or 1480 nm

Schematic diagram of EDFA