Matrix Theory in a 2-Qubit Entangler

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Matrix Analysis

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Presentation layout

Quantum Bits - Qubits

Qubits:

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle\,,$$

where $|a|^2 + |b|^2 = 1$.

Measurement: Probabilistic

$$P(|\psi\rangle \to |0\rangle) = |a|^2 \qquad P(|\psi\rangle \to |1\rangle) = |b|^2$$

Entanglement

Consider a quantum state $|\psi\rangle$. It is not always possible to find $|x\rangle$ and $|y\rangle$ such that $|\psi\rangle = |x\rangle \otimes |y\rangle$.

Recipe

What do we need to entangle two qubits?

- Tensor products
- Hadamard gate
- CNOT gate
- Measure

Tensor Products

The *tensor product* of $\mathbf{V} = \mathbb{C}^{\Sigma_1}$ and $\mathbf{W} = \mathbb{C}^{\Sigma_2}$ is

$$\mathbf{V} \otimes \mathbf{W} = \mathbb{C}^{\Sigma_1 \times \Sigma_2}.$$

Elementary tensors span $\mathbf{V} \otimes \mathbf{W}$. For $|v\rangle \in \mathbf{V}$ and $|w\rangle \in \mathbf{W}$,

$$|v\rangle \otimes |w\rangle \equiv |v\rangle\,|w\rangle \equiv |vw\rangle \in \mathbf{V} \otimes \mathbf{W}.$$

Example: Representing the classical number "1" with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Tensor Products (cont.)

$\text{span}(|00\rangle, |01\rangle, |10\rangle, |11\rangle) = \mathbf{V} \otimes \mathbf{W}$, where

$$|00\rangle = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^\top, |10\rangle = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}^\top, |11\rangle = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\top.$$

Linear independence $\rightarrow$ $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ form a computational basis.

A *generic state*: For $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$,

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.$$

Tensor Products (cont.)

Not every $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|c\rangle, |d\rangle$ such that

$$|c\rangle \otimes |d\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \rightarrow \textbf{Entangled}.$$

Tensor Products (cont.)

Bilinearity:

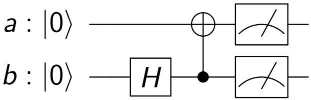
$$|a\rangle \otimes (\alpha |v\rangle + \beta |w\rangle) = \alpha |av\rangle + \beta |aw\rangle$$
$$(\alpha |v\rangle + \beta |w\rangle) \otimes |b\rangle = \alpha |vb\rangle + \beta |wb\rangle$$

Of operators: $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$, $\mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A} |v\rangle) \otimes (\mathcal{B} |w\rangle).$$

But not all $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ can be written as $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$ $\rightarrow$ **Entangled**.

Example: Entanglement



$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ Unitary Operations $\equiv$ Quantum Gates

Example: Entanglement (cont.)

Notice:

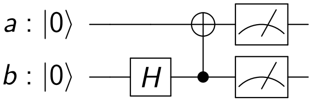
$$(I \otimes H_b)(|0\rangle \otimes |0\rangle) = (I\,|0\rangle) \otimes (H_b\,|0\rangle)$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \mathcal{O} \\ \mathcal{O} & \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}^\top = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}^\top$$

$\rightarrow$ Possible to write H as $I \otimes H_b$.

Example: Entanglement (cont.)



$$C_b(I \otimes H)\left(\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \begin{bmatrix}1\\0\end{bmatrix}_b\right) = C_b\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_b\right)$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle \rightarrow \textbf{Entangled}$$

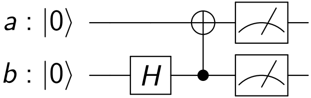
Tensor Products (cont.)

Other properties:

- Associative
- Distributive
- Not commutative
- $(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$.
- $\text{Tr}(\mathcal{A} \otimes \mathcal{B}) = \text{Tr}(\mathcal{A}) \cdot \text{Tr}(\mathcal{B})$.
- $\det(\mathcal{A} \otimes \mathcal{B}) = (\det(\mathcal{A}))^m \cdot \det(\mathcal{B})^n$, where m is the dimension of $\mathcal{A}$ and n of $\mathcal{B}$.

Simulation on IBM-Q

Entanglement circuit, revisited



Recap

What did we learn on the show tonight, Craig?

Q-circuit user guide [EF04]

quantum addition of classical numbers [CC16]

Mike and Ike [NC02]

Handbook of Linear Algebra [Hog07]

addition on quantum computer [Dra00]

QFT quick math [Bac]

Matrix analysis (where I read about unitary matrices) [HJ90]

References

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- Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge university press, 1990.
- Leslie Hogben, *Handbook of linear algebra*, Chapman and Hall/CRC, 2007.
- Michael A Nielsen and Isaac Chuang, *Quantum computation and quantum information*, 2002.