Problem Set 2

Due: Friday 5pm, February 24th via Canvas upload or in the envelope in front of 26-255
TA: Eric Wolf
Email: eawolf@mit.edu
Office hour: TBA on Canvas, most likely Wednesday
Reading: For the coherent state parts, what will help a lot is reading Weissbluth's chapter, and Glauber's article, on Canvas/Reading, and also under Lectures&Notes. Glauber's original paper is an absolute classic.

1 When the mechanical momentum is not the canonical momentum

In this problem we will see that the motion of neutral atoms in a rotating frame can be described as the motion of a charged particle experiencing a scalar potential and an effective magnetic field. Let's consider free motion in the x-y plane. The transformation from the lab frame to a frame rotating at angular frequency Ω about the z-axis is

$$\tilde{x}(t) = x\cos(\Omega t) + y\sin(\Omega t) \quad (1)$$
$$\tilde{y}(t) = -x\sin(\Omega t) + y\cos(\Omega t) \quad (2)$$

(a) Write the kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ of a particle of mass m in terms of the coordinates and velocities in the rotating frame, $\tilde{x}$, $\tilde{y}$, $\dot{\tilde{x}}$ and $\dot{\tilde{y}}$.

(b) The Lagrangian $\mathcal{L}(\tilde{x}, \tilde{y}, \dot{\tilde{x}}, \dot{\tilde{y}}, t)$ is just the kinetic energy you found above. Find the canonical momentum $\tilde{p}_x = \frac{\partial \mathcal{L}}{\partial \dot{\tilde{x}}}$ and $\tilde{p}_y = \frac{\partial \mathcal{L}}{\partial \dot{\tilde{y}}}$.

(c) The poisson brackets are defined as $\{f, g\} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}$, where $x_1 \equiv \tilde{x}$, $x_2 \equiv \tilde{y}$, $p_1 = \tilde{p}_x$, $p_2 = \tilde{p}_y$. From that definition, $\{\tilde{x}, \tilde{p}_x\} = 1$ naturally, as well as $\{\tilde{p}_x, \tilde{p}_x\} = 0$, $\{\tilde{p}_x, \tilde{p}_y\} = 0$ etc. Show however that $\{m\dot{\tilde{x}}, m\dot{\tilde{y}}\} \neq 0$. *Note: In the quantum mechanical description of the same problem we will accordingly have that the velocity operators for different directions do not commute.*

(d) Obtain the Hamiltonian from the Legendre transformation $H = \sum_i \dot{x}_i p_i - \mathcal{L}$. This replaces the dependence of $\mathcal{L}$ on $\dot{\tilde{x}}$ and $\dot{\tilde{y}}$ by the dependence of H on $\tilde{p}_x$ and $\tilde{p}_y$. Rewrite the Hamiltonian in terms of a vector potential $\vec{A}$ (assuming a fictitious charge q for the particles) and an effective potential $V(\tilde{x}, \tilde{y})$. What is the effective magnetic field $\vec{B} = \nabla \times \vec{A}$ and how do we call the effective potential V?

(e) Completing the square, rewrite the Hamiltonian like $H = \frac{\tilde{p}_x^2}{2m} + \frac{\tilde{p}_y^2}{2m} + W$ and give the expression for the operator W in terms of $\tilde{x}$, $\tilde{y}$, $\tilde{p}_x$ and $\tilde{p}_y$. Can you explain in words why all that is needed to describe motion of the particle in a rotating frame is adding that operator W?

(f) From Hamilton's equations $\dot{\tilde{x}} = \frac{\partial H}{\partial \tilde{p}_x}$ etc. derive the equation of motion for the particle in the rotating frame, identifying well-known effective forces that you will find.

2 Quantum description of a charged particle in a uniform magnetic field - Landau levels

The Hamiltonian for a charged particle of charge $q > 0$ moving freely in the x-y plane in a uniform magnetic field $\vec{B} = B\hat{z}$ pointing along the z-axis is

$$H = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2$$

Let us ignore motion along z. We will use the symmetric gauge $\vec{A} = -\frac{1}{2}\vec{r} \times \vec{B} = \left(-\frac{yB}{2}, \frac{xB}{2}, 0\right)$.

(a) Obtain the classical equations of motion ($m\ddot{x} = \ldots$, $m\ddot{y} = \ldots$), using e.g. Hamilton's equations or directly the Lorentz force. To solve, introduce $z = x + iy$, find an equation for $\ddot{z} = \ldots$, solve for $z(t)$ and thus obtain $x(t)$ and $y(t)$. You will find cyclotron motion of frequency ω_c that you should find. Call x_0 and y_0 the center of the orbit you will find.

(b) Quantum solution: Complete the square (yes, this is similar to problem 1e), and find that the Hamiltonian is that of a standard two-dimensional oscillator $H_{\text{h.o.}}$ with additional coupling to the angular momentum $L_z = xp_y - yp_x$.

(c) Introduce annihilation operators

$$\begin{aligned} a_x &= \frac{1}{\sqrt{2}}\left(\frac{x}{l_B} + i\frac{p_x l_B}{\hbar}\right) \\ a_y &= \frac{1}{\sqrt{2}}\left(\frac{y}{l_B} + i\frac{p_y l_B}{\hbar}\right) \end{aligned}$$

with $\left[a_x, a_x^\dagger\right] = 1$, $\left[a_y, a_y^\dagger\right] = 1$, and other commutators zero.
Find the choice of l_B that allows to write the harmonic oscillator part of H in the form

$$H_{\text{h.o.}} = \frac{\hbar\omega_c}{2}\left(\hat{n}_x + \hat{n}_y + 1\right)$$

and find the expression of L_z in terms of a_x, a_y, $a_x^\dagger$, $a_y^\dagger$.

(d) We still need to bring L_z, the angular momentum, into a better form. Operators for motion along x only, or along y only, are not a good idea as these motions do not respect the symmetry of circular motion about z. A much better idea is to introduce annihilation operators for *left-handed* and *right-handed* circular motion about z:

$$\begin{aligned} a &= \frac{1}{\sqrt{2}}\left(a_x + ia_y\right) \\ b &= \frac{1}{\sqrt{2}}\left(a_x - ia_y\right) \end{aligned}$$

Find an expression of L_z in terms of $\hat{n}_a = a^\dagger a$ and $\hat{n}_b = b^\dagger b$.

(e) Combine the results for $H_{\text{h.o.}}$ and L_z to write a simple equation for H in terms of some of the operators introduced. What are the eigenenergies? These are called Landau levels. There is a vast degeneracy within each Landau level. Physically, where does this degeneracy come from? *Hint: You can see the origin for the degeneracy already in your classical solution.*

(f) Express the observables x, y, v_x, v_y (velocities) and the center of orbit variables x_0 and y_0 (also called guiding center coordinates) in terms of a, b, $a^\dagger$, $b^\dagger$. One can also introduce the cyclotron coordinates $\xi = -v_y/\omega_c$ and $\eta = v_x/\omega_c$. As a check, you should find that

$$x = x_0 + \xi \tag{3}$$

$$y = y_0 + \eta \tag{4}$$

(g) Compute the commutator of the center of orbit operators $[x_0, y_0]$. Find an uncertainty relation $\Delta x_0 \Delta y_0 \geq \dots$ for how well one can simultaneously measure the guiding center x_0 and y_0 coordinates. Motion of the guiding centers of cyclotron orbits is thus motion in non-commutative geometry. An analogous commutator and uncertainty relation exists between the cyclotron coordinates ξ and η. What is $[x, y]$?

Note: The following problem you can do without having done any of the above.

(h) Let's put the idea of non-commutative geometry to the test. We will place ourselves in the lowest Landau level, the ground state of cyclotron motion, and we start with the particle in the "vacuum" $|0\rangle$ of the guiding center motion: $b\,|0\rangle = 0$. That is, the particle is localized (within the limits of Heisenberg's uncertainty between x_0 and y_0) at the origin $\langle 0|x_0|0\rangle = \langle 0|y_0|0\rangle = 0$. We now want to apply a force along x_0, i.e. we switch on a Hamiltonian

$$H_F = -F x_0 = -\frac{1}{2} F\, l_B (b + b^\dagger)$$

Calculate $|\Psi(t)\rangle = e^{-iH_F t/\hbar}\,|0\rangle$ in terms of coherent states $|\beta\rangle$. These are eigenstates of b, i.e. $b\,|\beta\rangle = \beta\,|\beta\rangle$.
Find $\langle x_0\rangle(t) = \langle\Psi(t)|x_0|\Psi(t)\rangle$ and $\langle y_0\rangle(t) = \langle\Psi(t)|y_0|\Psi(t)\rangle$.

Hint: After reading Weissbluth section 4.6 (see Canvas/Files/Reading or Lectures and Notes) you may realize that the time-evolution operator $\exp(-iH_F t/\hbar)$ *is nothing else than the displacement operator. To learn more, see the next problem.*

For a direct application of the ideas of Problems 1 and 2 using a Bose-Einstein condensate rapidly rotating in a harmonic trap, see Richard Fletcher et al., Science 372,1318 (2021) , "Geometric Squeezing into the Lowest Landau Level.".

3 Properties of the coherent state $|\alpha\rangle$

(a) Compute the overlap of two coherent states $\langle\alpha|\beta\rangle$, for arbitrary complex α and β.

(b) Prove that the coherent states form an over-complete basis, that is

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \mathbb{1}\,. \tag{5}$$

(c) The BCH Lemma states that $e^{A+B} = e^A e^B e^{-[A,B]/2}$ for operators A and B satisfying $[A, B] = c$, where c is a complex number. Use this lemma to prove that the displacement operator $D(\alpha)$ defined by $D(\alpha)|0\rangle = |\alpha\rangle$ may be written as $D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right]$.

(d) Let the electric field operator be $E_x = i\mathcal{E}\left(a\, e^{ikz} - a^\dagger\, e^{-ikz}\right)$, where $\mathcal{E} = \sqrt{\hbar\omega/2\epsilon_0 V}$ is the electric field amplitude for one photon inside the cavity volume V. For a freely evolving coherent state $|\alpha\rangle = |\alpha(t)\rangle$, compute the average electric field $\langle E_x\rangle = \langle\alpha|E_x|\alpha\rangle$ and the root-mean-square deviation of the electric field $\sqrt{\langle \Delta E_x^2\rangle} = \sqrt{\langle\alpha|E_x^2|\alpha\rangle - |\langle E_x\rangle|^2}$. Why is $\sqrt{\langle \Delta E_x^2\rangle}$ independent of time and field strength $|\alpha|$? Why is the result the same as for the vacuum state $\alpha = 0$?

4 Pseudo-probability distribution plots

Pseudo-probability distributions such as the $Q(\alpha)$ function provide useful and insightful ways to depict quantum states of light. In this problem, we explore some important states and their depictions. Due to the technical challenge of analytically calculating $Q(\alpha)$, it is satisfactory to give numerical answers for some parts of this problem. In particular, the $Q(\alpha)$ function is defined as

$$Q_\rho(\alpha) \equiv \langle\alpha|\rho|\alpha\rangle\,, \tag{6}$$

and it is readily computed numerically by using the fact that any pure state $\rho = |\psi\rangle\langle\psi|$ can be represented using

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle\,, \tag{7}$$

and

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\,. \tag{8}$$

Compute and plot the following:

(i)
$$Q_1(\alpha) = |\langle\alpha|\psi_1\rangle|^2\,, \tag{9}$$

for
$$|\psi_1\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|12\rangle\,, \tag{10}$$

where $|12\rangle$ is the twelve photon number eigenstate. Plot for various values of ϕ and θ. Is this a minimum uncertainty state?

(ii)
$$Q_2(\alpha) = |\langle\alpha|\psi_2\rangle|^2\,, \tag{11}$$

for
$$|\psi_2\rangle = \frac{|\beta\rangle + |-\beta\rangle}{\sqrt{2}}\,, \tag{12}$$

say, with $\beta = 3$. Try plotting this on a logarithmic scale, and you should see interference fringes around the origin. What is that due to?

(iii)
$$Q_3(\alpha) = |\langle\alpha|\psi_3\rangle|^2\,, \tag{13}$$

for
$$|\psi_3\rangle = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{ik\phi}|k\rangle\,, \tag{14}$$

where $|k\rangle$ is a Fock state of k photons, say with $N = 10$ and $\phi = \pi/4$. How would you interpret the physical meaning of this state? What happens as $N \to \infty$?

(iv)

$$Q_4(\alpha) = |\langle\alpha|0_\epsilon\rangle|^2 \,, \tag{15}$$

where $|0_\epsilon\rangle = S(\epsilon)|0\rangle$ is called the squeezed vacuum with parameter ϵ. You may use

$$S(\epsilon)|0\rangle = \frac{1}{\sqrt{\cosh\epsilon}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh\epsilon)^n |2n\rangle \,. \tag{16}$$

Compare plots made with $\epsilon = 0.2$, 1.2, and 4, for example.

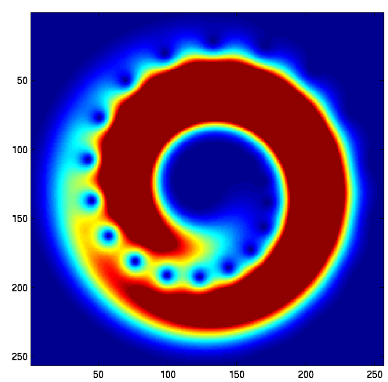
(v)

$$Q_5(\alpha) = |\langle\alpha|e^{iH_{\text{kerr}}t}|\beta\rangle|^2 \,, \tag{17}$$

where the Kerr effect Hamiltonian is

$$H_{\text{kerr}} = \xi a^\dagger a(a^\dagger a - 1) = \xi n(n-1) \,. \tag{18}$$

This is easily numerically computed by using the number basis representation of the coherent state $|\beta\rangle$. Take $\beta = 4$, and $\xi = \pi/128$, for example, and generate a sequence of plots as a function of time. At what time does the initial coherent state evolve to become two superposed coherent states? When does it return to its original state? The Kerr-type nonlinearity is important both in nonlinear optics, and in interacting cold atomic gases. It produces interesting and useful squeezed states, such as that depicted in this sample $Q(\alpha)$ plot:



For fun: Guess what the homepage logo is showing.