Practice Handout For The Final Exam MA353; Term: S19

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These Problems, TYC's and Theorems are likely to pop-up on the Final Exam tickets.

1 Underlying Space Resolution For Normal Operators

Theorem 1.1 ☞ **Kernel Of The Adjoint**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. Then

$$Nullspace\left(\mathcal{L}^*\right) = Range\left(\mathcal{L}\right)^\perp \ .$$

Proof. Consider the following string of equivalences.

$$\begin{aligned} Y \in Range\left(\mathcal{L}\right)^\perp &\iff \langle \mathcal{L}(X), Y\rangle = 0, \quad \text{for all} \quad X \in \boldsymbol{V} \\ &\iff \langle X, \mathcal{L}^*(Y)\rangle = 0, \quad \text{for all} \quad X \in \boldsymbol{V} \\ &\iff \mathcal{L}^*(Y) = O \\ &\iff Y \in Nullspace\left(\mathcal{L}^*\right) \ . \end{aligned}$$

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Test Your Comprehension 1.2 ☞ **Range Of The Adjoint**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. Then

$$Range\left(\mathcal{L}^*\right) = Nullspace\left(\mathcal{L}\right)^{\perp} \ .$$

Theorem 1.3 ☞ **Kernels Of Normal Operators And Adjoints**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a <u>normal</u> linear operator on $\boldsymbol{V}$. Then

$$Nullspace\left(\mathcal{L}^*\right) = Nullspace\left(\mathcal{L}\right) \ .$$

Proof. Consider the following string of equivalences.

$$\begin{aligned}
Y \in Nullspace\left(\mathcal{L}\right) &\Longleftrightarrow \left\|\mathcal{L}(Y)\right\|^2 = 0 \\
&\Longleftrightarrow \langle \mathcal{L}(Y), \mathcal{L}(Y)\rangle = 0 \\
&\Longleftrightarrow \langle \mathcal{L}^*\mathcal{L}(Y), Y\rangle = 0 \\
&\Longleftrightarrow \langle \mathcal{L}\mathcal{L}^*(Y), Y\rangle = 0 \\
&\Longleftrightarrow \langle \mathcal{L}^*(Y), \mathcal{L}^*(Y)\rangle = 0 \\
&\Longleftrightarrow \left\|\mathcal{L}^*(Y)\right\|^2 = 0 \\
&\Longleftrightarrow Y \in Nullspace\left(\mathcal{L}^*\right) \ .
\end{aligned}$$

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Test Your Comprehension 1.4 ☞ **Underlying Space Resolution For Normal Operators**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a <u>normal</u> linear operator on $\boldsymbol{V}$. Then

$$\boldsymbol{V} = Range\left(\mathcal{L}\right) \oplus Nullspace\left(\mathcal{L}\right) \ .$$

2 Isometries and Unitaries

Problem 1 ☞ Shuttling Inner Products

1. Suppose that $\left(\boldsymbol{V}, \langle\ ,\ \rangle\right)$ is an fdips with an orthonormal basis Γ. Let $\mathcal{A} : \mathbb{C}^n \longrightarrow \boldsymbol{V}$ be the atrix corresponding to Γ. Prove that

$$\langle V, W\rangle = \mathcal{A}^{-1}(V) \,\bar{\bullet}\, \mathcal{A}^{-1}(W)\ ,$$

   for every $V, W \in \boldsymbol{V}$.

2. Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are fdips with ortho-bases Γ and Ω respectively, and $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ is a linear function. Prove that $\left[\mathcal{L}^*\right]_{\Gamma\leftarrow\Omega}$ is the conjugate transpose of $[\mathcal{L}]_{\Omega\leftarrow\Gamma}$.

Problem 2 ☞ Isometries

Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are inner product spaces, and $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ is a linear function.

1. Argue that the following claims are equivalent.

   (a) $\mathcal{L}$ is an **isometry**; i.e. $\left\|\mathcal{L}(v)\right\|_{\boldsymbol{W}} = \|v\|_{\boldsymbol{V}}$, for all $v \in \boldsymbol{V}$.

   (b) $\left\langle\mathcal{L}(v), \mathcal{L}(z)\right\rangle_{\boldsymbol{W}} = \langle v, z\rangle_{\boldsymbol{V}}$, for all $v, z \in \boldsymbol{V}$.

   (c) $\mathcal{L}^*\mathcal{L} = \mathcal{I}_{\boldsymbol{V}}$.

   (d) $\left\|\mathcal{L}(v)\right\|_{\boldsymbol{W}} = 1$, for every unit vector $v \in \boldsymbol{V}$.

2. Argue that the following claims are equivalent.

   (a) $\mathcal{L}$ is a scalar multiple of an isometry.

   (b) $\left\|\mathcal{L}(v)\right\|_{\boldsymbol{W}} = \left\|\mathcal{L}(z)\right\|_{\boldsymbol{W}}$, for any unit vectors $v, z \in \boldsymbol{V}$.

   (c) $\mathcal{L}$ **preserves orthogonality**; i.e.

   if $\langle v, z\rangle_{\boldsymbol{V}} = 0$ then $\left\langle\mathcal{L}(v), \mathcal{L}(z)\right\rangle_{\boldsymbol{W}} = 0$.

   Hint: If v and z are orthonormal, then $v - z$ and $v + z$ are orthogonal.

3. Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are fdips with orthonormal bases Γ and Ω respectively. Argue that $\mathcal{L}$ is an isometry if and only if the columns of $\left[\mathcal{L}\right]_{\Omega\leftarrow\Gamma}$ are orthonormal.

Terminology 2.1

Invertible isometric operators $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ are said to be **unitary**.

Problem 3 ☞ **Unitaries**

Suppose that $\boldsymbol{V}$ is an inner product space, and $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ is a linear function.

1. Argue that the following claims are equivalent.
   (a) $\mathcal{L}$ is unitary.
   (b) $\mathcal{L}$ is invertible, and $\mathcal{L}^{-1} = \mathcal{L}^{*}$.
2. Argue that the following are equivalent in the case that $\boldsymbol{V}$ is fdips, and Γ is an orthonormal basis of $\boldsymbol{V}$.
   (a) $\mathcal{L}$ is unitary.
   (b) The columns of $\left[\mathcal{L}\right]_{\Gamma\leftarrow\Gamma}$ form an orthonormal basis (of the appropriate $\mathbb{C}^n$).

Problem 4 ☞ **Unitary Equivalence For Matrices**

Let $\boldsymbol{V}$ be an n-dimensional inner product space.

1. Suppose that Γ and Ω are two orthonormal bases of $\boldsymbol{V}$. Argue that $\left[\mathcal{I}\right]_{\Omega\leftarrow\Gamma}$ is a unitary matrix.
2. Argue that for each unitary matrix $\mathcal{U}$ in $\mathbb{M}_n$ there exist orthonormal bases Γ and Ω of $\boldsymbol{V}$ such that $\mathcal{U} = \left[\mathcal{I}\right]_{\Omega\leftarrow\Gamma}$.
3. Suppose that $\mathcal{A}, \mathcal{B} \in \mathbb{M}_n$. Argue that the following claims are equivalent.
   (a) There exists a linear function $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$ and orthonormal bases Γ and Ω of $\boldsymbol{V}$ such that
   $$\mathcal{A} = \left[\mathcal{L}\right]_{\Gamma\leftarrow\Gamma} \quad \text{and} \quad \mathcal{B} = \left[\mathcal{L}\right]_{\Omega\leftarrow\Omega} .$$
   (b) There exists a unitary matrix $\mathcal{U}$ in $\mathbb{M}_n$ such that
   $$\mathcal{B} = \mathcal{U}^{-1}\mathcal{A}\mathcal{U} .$$

Definition 2.2 ☞ **Unitary Similarity/Equivalence**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ and $\mathcal{M}$ are linear operators on $\boldsymbol{V}$. We say that $\mathcal{L}$ and $\mathcal{M}$ are **unitarily similar (a.k.a. unitarily equivalent)** if there exists a unitary $\mathcal{U}$ on $\boldsymbol{V}$ such that

$$\mathcal{M} = \mathcal{U}^{-1}\mathcal{L}\mathcal{U} = \mathcal{U}^{*}\mathcal{L}\mathcal{U}\,.$$

Problem 5 ☞ **Properties Of Unitary Similarity/Equivalence**

1. Argue that unitary equivalence is an equivalence relation on $\mathfrak{L}(\boldsymbol{V},\boldsymbol{V})$.

2. Argue that unitary equivalence preserves normality, "unitary-ness" and "self-adjoint-ness".

3 Spectral Theorem

Fact 3.1 ☞ **Half of The Spectral Theorem**

Suppose that $\boldsymbol{V}$ is an n-dimensional inner product space, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. In class we have shown that if $\mathcal{L}$ is normal then $\boldsymbol{V}$ has an orthonormal basis comprised of the eigenvectors of $\mathcal{L}$.

Problem 6 ☞ **Spectral Theorem For Normal Matrices**

1. Argue that the following claims are equivalent for a square matrix $\mathcal{A}$.
   (a) $\mathcal{A}$ is normal.

   (b) $\mathcal{A}$ is unitarily equivalent to a diagonal matrix.

2. Argue that the following claims are equivalent for a square matrix $\mathcal{B}$.
   (a) $\mathcal{B}$ is unitary.

   (b) $\mathcal{B}$ is unitarily equivalent to a diagonal matrix with diagonal entries of modulus 1.

3. Argue that the following claims are equivalent for a square matrix $\mathcal{C}$.
   (a) $\mathcal{C}$ is self-adjoint.
   (b) $\mathcal{C}$ is unitarily equivalent to a diagonal matrix with real entries.

Problem 7 ☞ **Spectral Theorem For Normal Linear Operators**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. Prove each of the following claims.

1. $\mathcal{L}$ is normal if and only if $\boldsymbol{V}$ has an orthonormal basis comprised of the eigenvectors of $\mathcal{L}$.
2. $\mathcal{L}$ is unitary if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ have modulus 1.
3. $\mathcal{L}$ is self-adjoint if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ are real.

4 Spectral Resolution

Fact 4.1 ☞ **Diagonalizability And Resolutions Of Identity**

Suppose that $\boldsymbol{V}$ is a finite-dimensional vector space, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. In class we have shown that $\mathcal{L}$ is diagonalizable exactly when $\boldsymbol{V}$ has a basis comprised of the eigenvectors of $\mathcal{L}$, and from that we had deduced the $\mathcal{L}$ is diagonalizable exactly when there exist idempotents $\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_k$ which resolve the identity on $\boldsymbol{V}$, such that

$$\mathcal{L} = \lambda_1 \mathcal{E}_1 + \lambda_2 \mathcal{E}_2 + \cdots + \lambda_k \mathcal{E}_k \ ,$$

for some *distinct* λ_i. (This is **a spectral resolution of** $\mathcal{L}$.)

In that case we showed that

$$\sigma_{\mathbb{C}}(\mathcal{L}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \ ,$$

and that the range of $\mathcal{E}_i$ is the eigenspace of $\mathcal{L}$ corresponding to λ_i, and the kernel of $\mathcal{E}_i$ is the direct sum of the remaining eigenspaces. This uniquely specifies the $\mathcal{E}_i$'s.

Test Your Comprehension 4.2

Argue that O is the only normal nilpotent operator on an fdips $\boldsymbol{V}$.

Terminology 4.3 ☞ **Atomic Spectral Idempotents**

Under the set-up of fact 4.1, the $\mathcal{E}_i$'s are said to be **the atomic spectral idempotents of $\mathcal{L}$.**

Problem 8 ☞ **Diagonalizability And Linear Combinations Of Mutually Annihilating Idempotents**

Suppose that $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_p$ are idempotent linear operators on a vector space $\boldsymbol{V}$, such that

$$\mathcal{G}_i \mathcal{G}_j = O \, ,$$

whenever $i \neq j$. In this case we say that the $\mathcal{G}_i$'s are **mutually annihilating idempotents**.

1. Argue that $\mathcal{G}_1 + \mathcal{G}_2 + \cdots + \mathcal{G}_p$ is an idempotent and infer that

$$\alpha_1 \mathcal{G}_1 + \alpha_1 \mathcal{G}_2 + \cdots + \alpha_1 \mathcal{G}_p$$

   is a diagonalizable operator.

2. Argue that a linear operator on $\boldsymbol{V}$ is diagonalizable exactly when it is a linear combination of some mutually annihilating idempotent operators.

Fact 4.4 ☞ **Polynomial Interpolation**

Given *distinct* complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$, and any complex numbers $\beta_1, \beta_2, \ldots, \beta_m$, with $m \in \mathbb{N}$, there exists exactly one polynomial p of degree at most $m-1$, such that

$$p(\alpha_i) = \beta_i \, , \quad \text{for all} \quad i \, .$$

Problem 9 ☞ **Atomic Spectral Idempotents Are Polynomials In $\boldsymbol{L}$**

Suppose that $\boldsymbol{V}$ is a finite-dimensional vector space, and $\mathcal{L}$ is a diagonalizable linear operator on $\boldsymbol{V}$ with

$$\sigma_{\mathbb{C}}(\mathcal{L}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \ .$$

Argue that each corresponding atomic spectral idempotent $\mathcal{E}_i$ of $\boldsymbol{L}$ can be expressed as $p_i(\boldsymbol{L})$, for the unique polynomial p_i of degree at most $k-1$ which maps λ_i to 1, and all other eigenvalues of $\boldsymbol{L}$ to zero.

Problem 10 ☞ **Uniqueness of Spectral Resolution**

Suppose that $\boldsymbol{V}$ is a finite-dimensional vector space, and $\mathcal{L}$ is a diagonalizable linear operator on $\boldsymbol{V}$ with

$$\sigma_{\mathbb{C}}(\mathcal{L}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \ .$$

1. Suppose that

   $$\mathcal{L} = \gamma_1\mathcal{F}_1 + \gamma_2\mathcal{F}_2 + \cdots + \gamma_m\mathcal{F}_m \ ,$$

   for some idempotents $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ which resolve the identity on $\boldsymbol{V}$, and some *distinct* complex numbers $\gamma_1, \gamma_2, \ldots, \gamma_m$.
   Argue that $m = k$, that

   $$\sigma_{\mathbb{C}}(\mathcal{L}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} = \{\gamma_1, \gamma_2, \ldots, \gamma_k\}$$

   and that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ are the atomic spectral idempotents of $\mathcal{L}$.

2. Explain how one discerns the atomic spectral idempotents and the eigenspaces of $\mathcal{L}$ from any $\mathcal{L}$-eigenbasis of $\boldsymbol{V}$. Justify your claims.

Problem 11 ☞ **Spectral Resolutions Of Normal Operators**

Suppose that $\boldsymbol{V}$ is an fdips, that $\mathcal{L}$ is a normal linear operator on $\boldsymbol{V}$ with

$$\sigma_{\mathbb{C}}(\mathcal{L}) = \{\lambda_1, \lambda_2, \ldots, \lambda_k\} \ .$$

1. Argue that every atomic spectral idempotent of $\mathcal{L}$ is normal, and conclude that every atomic spectral idempotent of $\mathcal{L}$ is a non-

negative (i.e. positive semi-definite) operator.

Non-negative idempotents are said to be **ortho-projections** or **projections**, for short.

2. Argue that
$$\boldsymbol{V} = \boldsymbol{E}_{\lambda_1} \oplus \boldsymbol{E}_{\lambda_2} \oplus \cdots \oplus \boldsymbol{E}_{\lambda_k} \ ,$$
where $\boldsymbol{E}_\lambda$ is the eigenspace of $\mathcal{L}$ corresponding to the eigenvalue λ.

Problem 12 ☞ **Another Form Of The Spectral Theorem For Normal Operators**

Suppose that $\boldsymbol{V}$ is an fdips, that $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. Argue that the following claims are equivalent.

1. $\mathcal{L}$ is normal.

2. $\mathcal{L}$ is a linear combination of ortho-projections that resolve the identity on $\boldsymbol{V}$.

3. $\mathcal{L}$ is a linear combination of mutually annihilating projections.

Problem 13 ☞ **Commuting With A Normal**

Suppose that $\boldsymbol{V}$ is an fdips, that $\mathcal{L}$ is a normal linear operator on $\boldsymbol{V}$. Argue that the following claims are equivalent for an operator $\mathcal{M}$ on $\boldsymbol{V}$.

Hint: Problem 9 can be handy here.

1. $\mathcal{M}$ commutes with $\mathcal{L}$.

2. $\mathcal{M}$ commutes with every atomic spectral projection of $\mathcal{L}$.

5 Non-Negative Operators

Test Your Comprehension 5.1

Argue that an operator $\mathcal{L}$ on a vector space $\boldsymbol{V}$ is self-adjoint exactly when

$$\langle \mathcal{L}(X), X\rangle \in \mathbb{R}, \quad \text{for all} \quad X \in \boldsymbol{V}.$$

Problem 14 ☞ **Another Polarization Identity**

Suppose that $\mathcal{L}$ is a linear operator on an inner product space $\boldsymbol{V}$.

1. Argue that

$$4\langle \mathcal{L}(X), Y\rangle = \langle \mathcal{L}(X+Y), X+Y\rangle - \langle \mathcal{L}(X-Y), X-Y\rangle$$

$$+\, i\langle \mathcal{L}(X+iY), X+iY\rangle - i\langle \mathcal{L}(X-iY), X-iY\rangle \,.$$

2. Infer that

$$\langle \mathcal{L}(Z), Z\rangle = 0, \quad \text{for all} \quad Z \in \boldsymbol{V} \qquad \Longleftrightarrow \qquad \mathcal{L} = O\,,$$

and that for an operator $\mathcal{M}$ on $\boldsymbol{V}$,

$$\langle \mathcal{L}(Z), Z\rangle = \langle \mathcal{M}(Z), Z\rangle, \quad \text{for all} \quad Z \in \boldsymbol{V} \qquad \Longleftrightarrow \qquad \mathcal{L} = \mathcal{M}\,.$$

Terminology 5.2

Suppose that $\boldsymbol{V}$ is an inner product space, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. We say that $\mathcal{L}$ is **non-negative (a.k.a. positive semi-definite)** if

$$\langle \mathcal{L}(X), X\rangle \geq 0, \quad \text{for all} \quad X \in \boldsymbol{V}.$$

We say that $\mathcal{L}$ is **positive (a.k.a. positive definite)** if

$$\langle \mathcal{L}(X), X\rangle > 0, \quad \text{for all non-zero } X \in \boldsymbol{V}.$$

Note that every non-negative operator is self-adjoint, by TYC 5.1.

Test Your Comprehension 5.3

Argue that a sum of two self-adjoint operators (when defined) is also self-adjoint, and the sum of two non-negative operators (when defined) is non-negative. Is the sum of two positive operators (when defined) necessarily positive?

Test Your Comprehension 5.4

Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are inner product spaces, and $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ is a linear function. Argue that $\mathcal{L}^*\mathcal{L}$ and $\mathcal{L}\mathcal{L}^*$ are non-negative linear operators.

Test Your Comprehension 5.5

Suppose that $\mathcal{L}$ and $\mathcal{M}$ are linear operators on an inner product space $\boldsymbol{V}$. Verify each of the following claims.

1. If $\mathcal{L}$ is self-adjoint, so is $\mathcal{M}^*\mathcal{L}\mathcal{M}$.

2. If $\mathcal{L}$ is non-negative, so is $\mathcal{M}^*\mathcal{L}\mathcal{M}$.

Problem 15 ☞ **Non-Negativity (And Positivity) Conditions I**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. Prove each of the following claims.

1. $\mathcal{L}$ is non-negative if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ are non-negative.

2. $\mathcal{L}$ is positive if and only if $\mathcal{L}$ is normal, and all eigenvalues of $\mathcal{L}$ are positive.

3. $\mathcal{L}$ is non-negative if and only if $\mathcal{L}$ is self-adjoint, and all eigenvalues of $\mathcal{L}$ are non-negative.

4. $\mathcal{L}$ is positive if and only if $\mathcal{L}$ is non-negative and invertible.

Hint: Spectral decomposition and functional calculus can be handy here.

Problem 16 ☞ **Non-Negativity Conditions II**

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a linear operator on $\boldsymbol{V}$. Prove that the following claims are equivalent.

1. $\mathcal{L}$ is non-negative.
2. $\mathcal{L} = \mathcal{P}^2$ for some non-negative operator $\mathcal{P}$ on $\boldsymbol{V}$.*
3. $\mathcal{L} = \mathcal{M}^2$ for some self-adjoint operator $\mathcal{M}$ on $\boldsymbol{V}$.
4. $\mathcal{L} = \mathcal{T}^*\mathcal{T}$ for some operator $\mathcal{T}$ on $\boldsymbol{V}$.
5. $\mathcal{L} = \mathcal{K}\mathcal{K}^*$ for some operator $\mathcal{K}$ on $\boldsymbol{V}$.

*Such a $\mathcal{P}$ is said to be **a non-negative square root of** $\mathcal{L}$.

Problem 17

Suppose that $\boldsymbol{V}$ is an fdips, and $\mathcal{L}$ is a non-negative linear operator on $\boldsymbol{V}$. Argue that for any $X \in \boldsymbol{V}$ the following claims are equivalent.

1. $\mathcal{L}(X) = O$.
2. $\langle \mathcal{L}(X), X \rangle = 0$.

Problem 18 ☞ **Non-Negativity Conditions For Matrices**

1. Argue that the following claims are equivalent for a square matrix $\mathcal{P}$.
   (a) $\mathcal{P}$ is non-negative.
   (b) $\mathcal{P}$ is unitarily equivalent to a diagonal matrix with non-negative diagonal entries.
2. Argue that the following claims are equivalent for a square matrix $\mathcal{P}$.
   (a) $\mathcal{P}$ is positive.
   (b) $\mathcal{P}$ is unitarily equivalent to a diagonal matrix with positive diagonal entries.

Theorem 5.6 ☞ **Uniqueness Of Non-Negative Square Roots**

For each non-negative linear operator $\mathcal{L}$ on an fdips $\boldsymbol{V}$ there is a unique non-negative operator $\mathcal{R}$ on $\boldsymbol{V}$ such that

$$\mathcal{R}^2 = \mathcal{L} \; .$$

This $\mathcal{R}$ is said to be **THE square root of** $\mathcal{L}$ and it is denoted by $\sqrt{\mathcal{L}}$.

Proof. Let us begin by writing down a spectral resolution of $\mathcal{L}$:

$$\mathcal{L} = \lambda_1 \mathcal{E}_1 + \cdots + \lambda_k \mathcal{E}_k \; .$$

Keep in mind that all eigenvalues of $\mathcal{L}$ are non-negative. Then, as we have seen in class,

$$\mathcal{R} := \sqrt{\lambda_1}\,\mathcal{E}_1 + \cdots + \sqrt{\lambda_k}\,\mathcal{E}_k$$

defines a non-negative (see TYC 5.3) linear operator $\mathcal{R}$ that squares to $\mathcal{L}$.

Suppose that $\mathcal{P}$ is another non-negative linear operator on $\boldsymbol{V}$ that squares to $\mathcal{L}$. If we can demonstrate that $\mathcal{P} = \mathcal{R}$, the proof will be complete.

Note that

$$\mathcal{P}\mathcal{L} = \mathcal{P}^3 = \mathcal{L}\mathcal{P} \; ,$$

so that $\mathcal{P}$ commutes with $\mathcal{L}$.

Therefore $\mathcal{P}$ commutes with all atomic spectral projections of $\mathcal{L}$, and so commutes with $\mathcal{R}$ (Problem 13).

Since $\mathcal{P}$ and $\mathcal{R}$ are self-adjoint, so is $\mathcal{P}-\mathcal{R}$, and since the only normal nilpotent operator on $\boldsymbol{V}$ is the zero operator (TYC 4.2), our task shall be complete if we can show that $(\mathcal{P}-\mathcal{R})^2 = \mathcal{O}$.

Note that

$$(\mathcal{P}-\mathcal{R})^2 = \mathcal{P}^2 - \mathcal{P}\mathcal{R} - \mathcal{R}\mathcal{P} + \mathcal{R}^2$$

$$= 2\mathcal{L} - 2\mathcal{P}\mathcal{R} \; .$$

Hence we have reduced the task to showing that $\mathcal{L} = \mathcal{P}\mathcal{R}$.

Let us continue by observing that the following equality holds.

$$O = \mathcal{L} - \mathcal{L} = \mathcal{P}^2 - \mathcal{R}^2 = \big(\mathcal{P} + \mathcal{R}\big)\big(\mathcal{P} - \mathcal{R}\big) \ .$$

In particular, for any $X \in \boldsymbol{V}$, letting $Y = \big(\mathcal{P} - \mathcal{R}\big)(X)$, we get

$$\big(\mathcal{P} + \mathcal{R}\big)(Y) = O,$$

and consequently

$$0 = \langle \big(\mathcal{P} + \mathcal{R}\big)(Y), Y \rangle = \langle \mathcal{P}(Y), Y \rangle + \langle \mathcal{R}(Y), Y \rangle \ .$$

Since $\mathcal{P}$ and $\mathcal{R}$ are non-negative operators, $\langle \mathcal{P}(Y), Y \rangle$ and $\langle \mathcal{R}(Y), Y \rangle$ are non-negative, and hence both have to be zero.

In particular, $\mathcal{P}(Y) = O$ by Problem 17. This demonstrates that

$$\mathcal{P}\big(\mathcal{P} - \mathcal{R}\big)(X) = O, \quad \text{for all} \quad X \in \boldsymbol{V} \ ,$$

and so

$$\mathcal{P}\big(\mathcal{P} - \mathcal{R}\big) = \mathcal{O} \ ,$$

which reduces to the desired equality

$$\mathcal{L} - \mathcal{P}\mathcal{R} = \mathcal{P}^2 - \mathcal{P}\mathcal{R} = \mathcal{O} \ . \qquad \blacksquare$$

Problem 19 ☞ **Properties of $\sqrt{\mathcal{L}^*\mathcal{L}}$ and $\sqrt{\mathcal{L}\mathcal{L}^*}$**

Suppose that $\boldsymbol{V}$ and $\boldsymbol{W}$ are fdips, and $\mathcal{L} : \boldsymbol{V} \longrightarrow \boldsymbol{W}$ is a linear function. Argue that

1. $\left\| \sqrt{\mathcal{L}^*\mathcal{L}}(X) \right\| = \left\| \mathcal{L}(X) \right\|$, for all $X \in \boldsymbol{V}$ and
   $\left\| \sqrt{\mathcal{L}\mathcal{L}^*}(Y) \right\| = \left\| \mathcal{L}^*(Y) \right\|$, for all $Y \in \boldsymbol{W}$.

2. $Nullspace\left(\sqrt{\mathcal{L}^*\mathcal{L}}\right) = Nullspace\left(\mathcal{L}\right)$ and
   $Nullspace\left(\sqrt{\mathcal{L}\mathcal{L}^*}\right) = Nullspace\left(\mathcal{L}^*\right)$.

3. $Range\left(\sqrt{\mathcal{L}^*\mathcal{L}}\right) = Range\left(\mathcal{L}^*\right)$ and
   $Range\left(\sqrt{\mathcal{L}\mathcal{L}^*}\right) = Range\left(\mathcal{L}\right)$.