Problem Set 3

Due: Friday 5pm, March 3rd via Canvas upload or in the envelope in front of 26-255
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Office hour: TBA on Canvas, likely Wednesday.

1 Classical Coherence of Light

Consider a classical light field. The classical expressions for first-order and second-order coherence are

$$\begin{aligned} g^{(1)}(\tau) &= \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle} \\ g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} &= \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle^2} \end{aligned}$$

where the $\langle\rangle$ denotes a statistical averaging over many measurements. We may implement it as the time-average $\langle f(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{d}t\, f(t)$, but you will not need this explicit definition.

a) Prove that $|g^{(1)}(\tau)| \leq 1$.
Hint: $\langle f, g\rangle \equiv \langle f^*(t)g(t)\rangle$ defines a scalar product, for which we have Cauchy's inequality: $|\langle f, g\rangle|^2 \leq \langle f, f\rangle \cdot \langle g, g\rangle$.

b) Prove that for zero time-delay, the second order coherence obeys the inequality $g^{(2)}(0) \geq 1$.

Hint: Consider that $\langle (I(t) - \langle I(t)\rangle)^2\rangle \geq 0$.

This implies that light in a number state, which has $g^{(2)}(0) < 1$, has no classical analog.

c) Using a similar argument, show that:

$$g^{(2)}(\tau) \leq g^{(2)}(0)$$

This implies that anti-bunched light, with $g^{(2)}(\tau) \geq g^{(2)}(0)$, has no classical analog.

d) Consider chaotic classical light generated by an ensemble of ν atoms. The total electric field can be expressed as $E(t) = \sum_{i=1}^{\nu} E_i(t)$, where the phases of the E_i are random. Show that when ν is large,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2 \tag{1}$$

Hint: Note that statistical averages in which the random electric field phases do not cancel are zero. Also note that fields emitted by different atoms are uncorrelated.

2 Quantum Coherence of Light

Consider light in a single mode of the radiation field. The quantum mechanical expressions for first-order and second-order coherence are

$$
\begin{aligned}
g^{(1)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2) &= \frac{\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^+(\mathbf{r}_2 t_2)\rangle}{[\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^+(\mathbf{r}_1 t_1)\rangle\langle \hat{E}^-(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_2 t_2)\rangle]^{1/2}} \\
g^{(2)}(\mathbf{r}_1 t_1, \mathbf{r}_2 t_2; \mathbf{r}_2 t_2, \mathbf{r}_1 t_1) &= \frac{\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^-(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_1 t_1)\rangle}{\langle \hat{E}^-(\mathbf{r}_1 t_1)\hat{E}^+(\mathbf{r}_1 t_1)\rangle\langle \hat{E}^-(\mathbf{r}_2 t_2)\hat{E}^+(\mathbf{r}_2 t_2)\rangle}
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{E}^+(\mathbf{r}t) &= i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} a\, e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})} \\
\hat{E}^-(\mathbf{r}t) &= -i\left(\frac{\hbar\omega}{2\epsilon_0 V}\right)^{1/2} a^\dagger\, e^{+i(\omega t - \mathbf{k}\cdot\mathbf{r})}
\end{aligned}
$$

(we consider here a field along some fixed polarization).

a) Using these expressions, show that the second order coherence may be written as

$$
g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} = \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2}
$$

b) Using the expression you just derived, show that, for light in a number state $|n\rangle$ where $n > 2$, $|g^{(1)}| = 1$ and $g^{(2)} = 1 - 1/n$, independent of space-time separation. What is $g^{(1)}$ and $g^{(2)}$ for $n = 0$ and $n = 1$?

c) Show that, for light in a coherent state $|\alpha\rangle$, $|g^{(1)}| = 1$ and $|g^{(2)}| = 1$.

d) Show that, for chaotic light with density matrix

$$
\hat{\rho} = (1 - e^{-\hbar\omega/k_B T})\sum_n e^{-n\hbar\omega/k_B T}|n\rangle\langle n|
$$

$|g^{(1)}| = 1$ and $g^{(2)} = 2$.

Note that the values of the first and second-order coherence functions in part (c) satisfy the classical relation you derived in problem 1 for chaotic light (equation 1). It can be proved that equation 1 holds quantum mechanically for multi-mode chaotic light.

e) Compute $g^{(2)}(\tau)$ for the following states, as a function of α:

$$
\begin{aligned}
|\psi_+\rangle &= \frac{|\alpha\rangle + |-\alpha\rangle}{\sqrt{2}\sqrt{1 + e^{-2|\alpha|^2}}} \\
|\psi_-\rangle &= \frac{|\alpha\rangle - |-\alpha\rangle}{\sqrt{2}\sqrt{1 - e^{-2|\alpha|^2}}}
\end{aligned}
$$

Do either of these two states show non-classical second-order coherence? Why (or why not)?

(Make sure you agree with the normalization given)

3 The quantum beamsplitter

Let the beamsplitter operator B, acting with angle θ on modes a and b, be defined by

$$B = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right] . \tag{2}$$

a) We can show that B conserves the total photon number and leaves coherent states as coherent states. Prove that B leaves $n_a + n_b = a^\dagger a + b^\dagger b$ unchanged. Also prove that $B^\dagger B = I$.

b) Let $|\alpha\rangle$ be a coherent state. Compute $B|0\rangle_b|\alpha\rangle_a$, and show that the output is a tensor product of coherent states for all θ. Your result should be consistent with the intuition that the beamsplitter has well defined transmission and reflection coefficients; give these as a function of θ.

c) There is close connection between the Lie group $SU(2)$ and the algebra of two coupled harmonic oscillators, which is useful for understanding B. Let's define

$$s_z = a^\dagger a - b^\dagger b \qquad\qquad s_+ = a^\dagger b \qquad\qquad s_- = ab^\dagger , \tag{3}$$

and let $s_\pm = (s_x \pm is_y)/2$. What is $B(\theta)$ in spin space? What is $a^\dagger a + b^\dagger b$ in spin space? Show that s_x, s_y, and s_z have the same commutation relations as the Pauli matrices. This relationship also explains why $a^\dagger a + b^\dagger b$ is invariant; it is the Casimir operator of the algebra.

d) How does a beamsplitter transform an input photon-number eigenstate? Let

$$B(\theta) = \exp\left[\theta\left(-a^\dagger b + ab^\dagger\right)\right] , \tag{4}$$

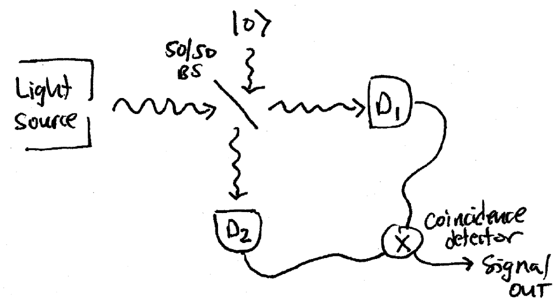
and $B = B(\pi/4)$ be a 50/50 beamsplitter, such that

$$BaB^\dagger = \frac{a+b}{\sqrt{2}} \qquad \text{and} \qquad BbB^\dagger = \frac{-a+b}{\sqrt{2}} . \tag{5}$$

Compute $B|0\rangle|n\rangle$, where the first label is mode b, and the second label is mode a. Note that the result is *not* $|n/2\rangle|n/2\rangle$, because $|n\rangle$ is a photon number eigenstate, and not a coherent state. What photon number states have the largest amplitude? How sharp is the distribution for $n = 10$, and $n = 100$, or as a function of n, if a general solution exists? Hint: use the binomial expansion on $(a^\dagger + b^\dagger)^n$.

4 The Hanbury-Brown Twiss experiment and $g^{(2)}(\tau)$

The second-order coherence function $g^{(2)}(\tau)$ is often measured in the laboratory using an experiment first developed by Hanbury-Brown and Twiss in the 1950's, for studying the light from distant stars. This experiment involves mixing light from the input source with the vacuum, $|0\rangle$, on a 50/50 beamsplitter, and measuring the intensity-intensity correlation function at the output using two detectors and a coincidence circuit:



This problem examines how this experiment measures $g^{(2)}(\tau)$, and what results are obtained for different input states of light.

Let a, $a^\dagger$, and b, $b^\dagger$ be the raising and lowering operators for the two modes of light input to the beamsplitter, and let the unitary transformation performed by the beamsplitter be defined by

$$a_1 = UaU^\dagger = \frac{a+b}{\sqrt{2}} \tag{6}$$

$$b_1 = UbU^\dagger = \frac{a-b}{\sqrt{2}}\,. \tag{7}$$

For light input in state $|\psi\rangle$, you are given that the output of the coincidence circuit is a voltage

$$V_\psi = V_0\langle\psi, 0|\, a_1^\dagger a_1 b_1^\dagger b_1 \,|\psi, 0\rangle\,, \tag{8}$$

where V_0 is some proportionality constant, and $|\psi, 0\rangle$ denotes a state with $|\psi\rangle$ in mode "a" and $|0\rangle$ in mode "b". In other words, the voltage is the average of the product of the two detected photon signals. Show that V_ψ gives a measure of $g^{(2)}(\tau)$,

$$g^{(2)}(\tau) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} \tag{9}$$

up to an additive offset and normalization.