Physics 8.321, Fall 2021

Homework #8

Due **Friday, December 3** by 8:00 PM.

1. [Sakurai and Napolitano Problem 28, Chapter 2 (page 154)]

Consider an electron confined to the interior of a hollow cylindrical shell whose axis coincides with the z-axis. The wave function is required to vanish on the inner and outer walls, $\rho = \rho_a, \rho_b$ and also at the top and bottom, $z = 0, L$.

(**a**) Find the energy eigenfunctions. (Do not bother with normalization.) Show that the energy eigenvalues are given by

$$E_{lmn} = \left(\frac{\hbar^2}{2m_e}\right)\left[k_{mn}^2 + \left(\frac{l\pi}{L}\right)^2\right] \quad (l = 1, 2, 3, \ldots, m = 0, 1, 2, \ldots),$$

where k_{mn} is the nth root of the transcendental equation

$$J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) - N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a) = 0\,,$$

where J_m, N_m are Bessel functions of the first and second kind (N_m is also sometimes called a Neumann function and can also be denoted Y_m); Bessel functions with integer values of m are also known as *cylindrical harmonics* in analogy with spherical harmonics.

(**b**) Repeat the same problem when there is a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ for $0 < \rho < \rho_a$. Note that the energy eigenvalues are influenced by the magnetic field even though the electron never "touches" the magnetic field.

(**c**) Compare, in particular, the ground state of the $B = 0$ problem with that of the $B \neq 0$ problem. Show that if we require the ground-state energy to be unchanged in the presence of B, we obtain "flux quantization"

$$\pi\rho_a^2 B = \frac{2\pi N\hbar c}{e}, \quad (N = 0, \pm1, \pm2, \ldots)\,.$$

Solution:

(a) In cylindrical coordinate (ρ, ϕ, z):

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

Write the wavefunction as

$$\psi(\rho, \phi, z) = R(\rho)\Phi(\phi)Z(z)$$

The Schrödinger equation is

$$-\frac{\hbar^2}{2m_e}\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\right) = E\psi$$

Divide both sides by ψ:

$$\Rightarrow \underbrace{\frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)}_{\text{function of }\rho\text{ only}} + \frac{1}{\Phi}\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \underbrace{\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}}_{\text{function of }z\text{ only}} = \underbrace{-\frac{2m_e E}{\hbar^2}}_{\text{independent of any variables}}$$

So we can set

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -k_z^2 \Rightarrow Z = A\sin(k_z z) + B\cos(k_z z)$$

Since $Z(z=0) = Z(z=L) = 0$, $Z(z) = \sin\left(\frac{l\pi}{L}z\right)$, $l = 1, 2, \ldots$

$$\text{Set} \quad \frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} = -m^2 \Rightarrow \Phi = e^{\pm im\phi}$$

Since $\Phi(0) = \Phi(2\pi) \Rightarrow m = 0, 1, \ldots$

$$\Rightarrow \frac{1}{R}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) - \frac{m^2}{\rho^2} - k_z^2 = -\frac{2m_e E}{\hbar^2}$$

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \left(\underbrace{\frac{2m_e E}{\hbar^2} - k_z^2}_{\equiv k^2} - \frac{m^2}{\rho^2}\right) R = 0$$

We know the solution to the above equation is Bessel Function: $J_m(k\rho)$ and $N_m(k\rho)$:

$$R(\rho) = c_1 J_m(k\rho) + c_2 N_m(k\rho)$$

Imposing the boundary condition: $R(\rho = \rho_a) = R(\rho = \rho_b) = 0$, we get

$$J_m(k\rho_a)N_m(k\rho_b) - J_m(k\rho_b)N_m(k\rho_a) = 0$$

We index the solution as k_{mn}, where m means it belongs to $m-$th order Bessel function, n means it is $n-$th root. The energy eigenfunctions therefore are

$$\psi_{lmn}(\rho, \phi, z) = \left(c_1 J_m(k_{mn}\rho) + c_2 N_m(k_{mn}\rho)\right) e^{\pm im\phi} \sin\left(\frac{l\pi}{L}z\right)$$

and energy eigenvalues are

$$E_{lmn} = \frac{\hbar^2}{2m_e}\left(k_{mn}^2 + \left(\frac{l\pi}{L}\right)^2\right)$$

(b,c) In cylindrical coordinate:

$$\nabla \times \mathbf{A} = \mathbf{e}_\rho\left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right) + \mathbf{e}_\phi\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right) + \mathbf{e}_z\frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right)$$

$$\nabla\chi = \mathbf{e}_\rho\frac{\partial\chi}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial\chi}{\partial\phi} + \mathbf{e}_z\frac{\partial\chi}{\partial z}$$

Now we seek a representation of the vector potential $\mathbf{A}$ such that $\nabla \times \mathbf{A} =$ magnetic field holds *for all regions.* ($B\mathbf{e}_z$, when $\rho \leq \rho_a$, and 0 when $\rho \geq \rho_a$) We assume $\mathbf{A}$ has the form: $\mathbf{A} = A_\phi(\rho)\mathbf{e}_\phi$ then

$$A_\phi(\rho) = \begin{cases} \frac{B\rho}{2}, & \text{if } \rho \leq \rho_a; \\ \frac{B\rho_a^2}{2\rho}, & \text{if } \rho_a \leq \rho \leq \rho_b. \end{cases}$$

Notice $A_\phi(\rho)$ is continuous at $\rho = \rho_a$. Instead of solving

$$\frac{1}{2m_e}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2 \psi = E\psi$$

in cylindrical coordinates, we make use of "gauge principle". We seek a function χ such that $\mathbf{A} = \nabla\chi$ when $\rho_a \leq \rho \leq \rho_b$. Then, the new wavefunction is $\tilde{\psi} = e^{-\frac{ie}{\hbar c}\chi}\psi$ (Note: the charge q is $q = -e < 0$ for electron.) Use the formula given at the beginning, it is easy to see that the choice

$$\chi = \frac{B}{2}\rho_a^2\phi$$

would do.

$$\Rightarrow \tilde{\psi} = e^{-\frac{ie}{\hbar c}\chi}\psi = \left(c_1 J_m(k_{mn}\rho) + c_2 N_m(k_{mn}\rho)\right) e^{\pm i m' \phi} \sin\left(\frac{l\pi}{L}z\right)$$

which

$$m' = m \pm \frac{e}{\hbar c}\frac{B}{2}\rho_a^2 = m \pm \frac{\Phi_B}{\left(\frac{hc}{e}\right)}$$

where $\Phi_B = \pi\rho_a^2 \times B$ is the magnetic flux and hc/e is called flux quantum. Apart from the phase, it is important that the quantum numbers also change. Notice that the wavefunction still has to be single-valued in ϕ, so we require m' instead of m to be integer. The corresponding energies are

$$E_{lmn} = \frac{\hbar^2}{2m_e}\left(k_{mn}^2 + \left(\frac{l\pi}{L}\right)^2\right) \quad l = 1, 2, ..., m' = 0, 1, ...$$

It depends on magnetic field B.

- The ground states are the same when m is also an integer. Therefore, for $N = 0, \pm 1, ...$

$$\frac{e}{\hbar c}\frac{B}{2}\rho_a^2 = N \Rightarrow \pi\rho_a^2 B = \frac{2\pi\hbar c}{e}N$$

or

$$\Phi_B = N\frac{hc}{e}$$

2. [Sakurai and Napolitano Problem 39, Chapter 2 (page 155)]

An electron moves in the presence of a uniform magnetic field in the z-direction ($\mathbf{B} = B\hat{\mathbf{z}}$).

(a) Evaluate

$$[\Pi_x, \Pi_y],$$

where

$$\Pi_x = p_x - eA_x/c, \quad \Pi_y = p_y - eA_y/c.$$

(b) By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB|\hbar}{mc}\right)(n + 1/2) \ ,$$

where $\hbar k$ is the continuous eigenvalue of the p_z operator and n is a nonnegative integer including zero.

Solution:

(a)

$$[\Pi_x, \Pi_y] = -\frac{e}{c}\left([p_x, A_y] + [A_x, p_y]\right) = \frac{i\hbar e}{c}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = \frac{i\hbar e}{c}\left(\nabla \times \mathbf{A}\right)_z = \frac{i\hbar e}{c}B$$

(b) This problem implicitly assumes a gauge such that $A_z = 0$, $\Rightarrow \Pi_z = p_z$. The Hamiltonian

$$\begin{aligned} H &= \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2 + \Pi_z^2\right) \\ &= \frac{1}{2m}p_z^2 + \frac{1}{2m}\left(\Pi_x^2 + \Pi_y^2\right) \end{aligned}$$

Now compare

$$[x, p] = i\hbar \Leftrightarrow [\Pi_x, \Pi_y] = \frac{i\hbar e}{c}B$$

If we define $Y = c\Pi_x/eB$, we have $[Y, \Pi_y] = i\hbar$ and

$$H = \frac{1}{2m}\Pi_y^2 + \frac{1}{2}m\omega_c^2 Y^2 + \frac{1}{2m}p_z^2$$

where the cyclotron frequency is $\omega_c = |eB|/mc$. The first two terms are exactly the Hamiltonian of 1D harmonic oscillator, which has energy $\hbar\omega_c(n + 1/2)$. Note that Π_x, Π_y are chosen by the above gauge to be functions of x, y only, so H is independent of z, $\Rightarrow [p_z, H] = 0 \Rightarrow$ eigenvalues is the sum of eigenvalues of each terms.

$$\Rightarrow \boxed{E = \frac{\hbar^2 k^2}{2m} + \hbar\omega_c\left(n + \frac{1}{2}\right)}$$

3. Write the wave functions $\psi_{k,n}$ for the states associated with the eigenvalues $E_{k,n}$ computed in problem 2, working in the following three gauges:

(a) $\mathbf{A} = (-yB, 0, 0)$

(b) $\mathbf{A} = (-yB/2, xB/2, 0)$

(c) $\mathbf{A} = (0, xB, 0)$

Solution:

(a) $\mathbf{A} = (-yB, 0, 0)$

$$\Rightarrow H = \frac{1}{2m}\left((P_x + \frac{qB}{c}y)^2 + P_y^2 + P_z^2\right)$$

Since $[P_x, H] = [P_z, H] = 0$, we can choose the eigenfunctions as

$$\psi(x,y,z) = e^{\frac{i}{\hbar}(p_x x + p_z z)}\chi(y) \qquad -\infty < p_x, p_z < \infty$$

Plug into $H\psi = E\psi$, we find $\chi(y)$ has to satisfy

$$\frac{1}{2m}\left((p_x + \frac{qB}{c}y)^2 - \hbar^2\frac{\partial^2}{\partial y^2} + p_z^2\right)\chi(y) = E\chi(y)$$

Let $cp_x/qB \equiv -y_0$

$$\Rightarrow -\frac{\hbar^2}{2m}\chi''(y) + \frac{1}{2}m(\frac{qB}{mc})^2(y-y_0)^2\chi(y) = (E - \frac{p_z^2}{2m})\chi(y)$$

which is a one-dimensional SHO, oscillating around y_0, with frequency $\omega_c = |qB|/mc$. So the total eigenenergy is

$$E = \frac{p_z^2}{2m} + (n + \frac{1}{2})\hbar\omega_c$$

the wavefunction is

$$\begin{aligned}\psi(x,y,z) &\propto e^{\frac{i}{\hbar}(p_x x + p_z z)} \times \text{displaced SHO in } y \text{ direction} \\ &= e^{\frac{i}{\hbar}(p_x x + p_z z)} \times e^{-\frac{(y-y_0)^2}{2l_B^2}} H_n(\frac{y-y_0}{l_B})\end{aligned}$$

where $l_B = \sqrt{\frac{\hbar}{m\omega_c}} = \sqrt{\frac{\hbar c}{|eB|}}$ is the length scale called the magnetic length. $l_B \approx 26$ nm when B=1T for electron.

(b) There are different approaches to this problem with various difficulties.
Method 1:
Notice that the gauge in this problem is related to that in last problem by

$$(-\frac{yB}{2}, \frac{xB}{2}, 0) = (-yB, 0, 0) + \nabla\left(\frac{xyB}{2}\right)$$

Then the new wavefunction is

$$\psi(x,y,z) \propto e^{\frac{iq}{\hbar c}\frac{xyB}{2}} e^{\frac{i}{\hbar}(p_x x + p_z z)} \times e^{-\frac{(y-y_0)^2}{2l_B^2}} H_n(\frac{y-y_0}{l_B})$$

Notice that each Landau level has a large degeneracy. The above for different p_x is mathematically a correct basis for Landau level n, but this is physically not insightful. In particular, it does not respect the rotational symmetry in symmetric gauge.
Method 2:

In cylindrical coordinate, the gauge is $A_\theta = \frac{1}{2}Br$, $A_r = A_z = 0$. Then $\Pi_\theta = -i\hbar\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{qBr}{2c}$ and the Schrodinger equation becomes

$$(-\frac{\hbar^2}{2m}(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}) + i\frac{\hbar qB}{2mc}\frac{\partial}{\partial\theta} + \frac{q^2B^2r^2}{8mc^2} + \frac{\partial^2}{\partial z^2})\psi(r,\theta,z) = E\psi(r,\theta,z)$$

Set $\psi(r,\theta,z)$ as

$$\psi(r,\theta,z) = e^{ik_z z}e^{-il\theta}\chi(r),$$

As above we require l to be integer. For simplicity we let $qB > 0$. We then find $\chi(r)$ satisfies

$$(-\frac{\hbar^2}{2m}(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}) + \frac{\hbar^2 l^2}{2mr^2} + \frac{1}{8}m\omega_c^2 r^2)\chi(r) = (E - \frac{1}{2}l\hbar\omega_c - \frac{\hbar^2k_z^2}{2m})\chi(r)$$

The left hand side is the Hamiltonian for a two-dimensional isotropic harmonic oscillator, and has energy $(n_r + \frac{1}{2}|l| + \frac{1}{2})\hbar\omega_c$, $n_r = 0, 1, 2\ldots$. So the eigenenergy and eigenfunction are

$$E_{n_r,k_z,l} = (n_r + \frac{1}{2}l + \frac{1}{2}|l| + \frac{1}{2})\hbar\omega_c + \frac{\hbar^2k_z^2}{2m}$$

$$\psi_{n_r,k_z,l}(r,\theta,z) = e^{ik_z z}e^{-il\theta}e^{-\frac{r^2}{4l_B^2}}r^{|l|}L_n^{(|l|)}(\frac{r^2}{2l_B^2})$$

where $L_n^{(|l|)}$ are associated Laguerre polynomials.
Method 3:
As usual the same problem can also be treated with ladder operators. This method is closely parallel to Problem 7. Recall

$$H = \frac{1}{2m}\left[\left(-i\hbar\frac{\partial}{\partial x} + \frac{qBy}{2c}\right)^2 + \left(-i\hbar\frac{\partial}{\partial y} - \frac{qBx}{2c}\right)^2\right]$$

For simplicity we let $qB > 0$. Define

$$a = \sqrt{\frac{c}{2\hbar qB}}\left[\left(-i\hbar\frac{\partial}{\partial x} + \frac{qBy}{2c}\right) + i\left(-i\hbar\frac{\partial}{\partial y} - \frac{qBx}{2c}\right)\right]$$

$$a^\dagger = \sqrt{\frac{c}{2\hbar qB}}\left[\left(-i\hbar\frac{\partial}{\partial x} + \frac{qBy}{2c}\right) - i\left(-i\hbar\frac{\partial}{\partial y} - \frac{qBx}{2c}\right)\right]$$

It is straightforward to check that $\left[a, a^\dagger\right] = 1$ and $H = \hbar\omega_c\left(a^\dagger a + 1/2\right)$, so it seems to be the same as a 1D harmonic oscillator. However, to consider degeneracies, we need to define another set of ladder operators that commute with H. Now define

$$b = \sqrt{\frac{c}{2\hbar qB}}\left[\left(-i\hbar\frac{\partial}{\partial x} - \frac{qBy}{2c}\right) - i\left(-i\hbar\frac{\partial}{\partial y} + \frac{qBx}{2c}\right)\right]$$

$$b^\dagger = \sqrt{\frac{c}{2\hbar qB}}\left[\left(-i\hbar\frac{\partial}{\partial x} - \frac{qBy}{2c}\right) + i\left(-i\hbar\frac{\partial}{\partial y} + \frac{qBx}{2c}\right)\right]$$

It is easy to check that $\left[b, b^\dagger\right] = 1$ and $a, a^\dagger$ commute with $b, b^\dagger$. Therefore, the eigenstates are $|n, l\rangle \propto a^{\dagger n} b^{\dagger l} |0, 0\rangle$ with energy $\hbar\omega_c (n + 1/2)$. The state $|0, 0\rangle$ is defined by $a|0, 0\rangle = b|0, 0\rangle = 0$, or

$$-i\hbar\frac{\partial\psi_{00}}{\partial x} - i\frac{qBx}{2c}\psi_{00} = -i\hbar\frac{\partial\psi_{00}}{\partial y} - i\frac{qBy}{2c}\psi_{00} = 0$$

$$\psi_{00} = e^{-\left(x^2+y^2\right)/4l_B^2} \Rightarrow \psi_{nl} = \left[\left(-i\hbar\frac{\partial}{\partial x} + \frac{qBy}{2c}\right) - i\left(-i\hbar\frac{\partial}{\partial y} - \frac{qBx}{2c}\right)\right]^n (x + iy)^l\, e^{-\left(x^2+y^2\right)/4l_B^2}$$

See Problem 7's solution on how to get the closed form.

It is interesting to note the very different character of the solution in parts (a) and (b). In both cases there is an infinite degeneracy at each energy level; this degeneracy is labeled by a continuous parameter in the gauge (a) and a discrete parameter in the gauge (b), however. Of course, it is possible to go back and forth between the two bases of eigenstates at a given energy level by a gauge transformation. This is a useful exercise which relates a continuous set of states (the momentum basis) to a discrete set of states (the Hermite polynomials multiplied by a Gaussian) as two sets of basis vectors for the same countably-infinite dimensional Hilbert space.

(c) Exactly as in Part(a), but $x \leftrightarrow y$, and the equilibrium point x_0 has different sign as y_0 in Part(a).

4. Consider a charged particle in crossed electric and magnetic fields

$$\mathbf{B} = (0, 0, B), \qquad \mathbf{E} = (E, 0, 0)$$

Solve the eigenvalue problem in one of the three gauges of the previous problem.
Solution:
It is simpler to work in Landau gauge: take $\mathbf{A} = (0, Bx, 0)$, $\phi = -Ex$. The Hamiltonian is

$$H = \frac{1}{2m}\left[P_x^2 + \left(P_y - \frac{qB}{c}x\right)^2 + P_z^2\right] - qEx$$

Since $[H, P_y] = [H, P_z] = 0$, we take the wavefunction to have the form

$$\psi(x, y, z) = e^{ik_y y + ik_z z}\chi(x)$$

Then $\chi(x)$ satisfies

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_c^2\left(x - (\frac{c\hbar k_y}{qB} + \frac{qE}{m\omega_c^2})\right)^2\right)\chi(x) = (E - \frac{\hbar k_z^2}{2m} + \frac{c\hbar k_y E}{B} + \frac{mc^2E^2}{2B^2})\chi(x)$$

Here we have completed the square. Again, the left hand side is a displaced SHO. So

$$E_{n,k_y,k_z} = (n + \frac{1}{2})\hbar\omega_c + \frac{\hbar k_z^2}{2m} - \frac{c\hbar k_y E}{B} - \frac{mc^2E^2}{2B^2}$$

$$\begin{aligned}\psi(x, y, z) &= e^{ik_y y + ik_z z} \times \text{displaced SHO in } x\text{−direction} \\ &= e^{ik_y y + ik_z z} \times e^{-\frac{(x-x_0)^2}{2l_B^2}} H_n(\frac{x - x_0}{l_B})\end{aligned}$$

where

$$x_0 = \frac{c\hbar k_y}{qB} + \frac{qE}{m\omega_c^2}$$

Notice that the energy can be written as

$$E_{n,k_y,k_z} = (n + \frac{1}{2})\hbar\omega_c + \frac{\hbar k_z^2}{2m} - qEx_0 + \frac{1}{2}mv_d^2$$

where $v_d = cE/B$ is the drift velocity. We should have been able to guess at the form of answer from the beginning.

5. Compute the spherical harmonics $Y_2^m(\theta, \phi)$ explicitly. Express these both as functions of θ, ϕ and of $z = \cos\theta, x = \sin\theta\cos\phi, y = \sin\theta\sin\phi$. Compute $\sum_m |Y_2^m|^2$.

Solution:
Begin with recalling in position space we have

$$L_\pm = \frac{\hbar}{i} e^{\pm i\phi} \left(\pm i \frac{\partial}{\partial\theta} - \cot\theta \frac{\partial}{\partial\phi} \right),$$

and

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}.$$

Then from the definition of spherical harmonics

$$L_z Y_l^m(\theta, \phi) = m\hbar Y_l^m(\theta, \phi),$$

we get

$$Y_l^m(\theta, \phi) \propto e^{im\phi} f(\theta).$$

Similarly, from

$$L_+ Y_l^l(\theta, \phi) = 0,$$

we easily obtain the full expression

$$Y_l^l(\theta, \phi) = c_l e^{il\phi} \sin^l\theta,$$

here c_l is the normalization factor, which can be determined by demanding

$$\int_0^{2\pi} d\phi \int_{-1}^{+1} d(\cos\theta) Y_{l'}^{m'^*} Y_l^m = \delta_{ll'}\delta_{mm'}.$$

This yields, after performing the integration (using Mathematica),

$$c_l = \frac{(-1)^l}{2^l l!}\sqrt{\frac{(2l+1)(2l)!}{4\pi}}.$$

Now we can apply L_- repeatedly to generate the lower spherical harmonics. Recall after each applications we would have

$$L_- Y_l^m = C_l^m Y_l^{m-1}$$

where

$$C_l^m = \sqrt{l(l+1) - m(m-1)},$$

is the appropriate normalization factor to keep the spherical harmonics normalized properly. Note that we only need to lower until we hit Y_l^0, then we can use the fact $Y_l^{-m} = (-1)^m (Y_l^m)^*$ to get spherical harmonics for $m < 0$. This fact can be easily seen from the definition of spherical harmonics.

In the case of $l = 2$ this procedure yields, using the Cartesian coordinates

$$\begin{aligned}
Y_2^2(\theta,\phi) &= \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{i2\phi} = \sqrt{\frac{15}{32\pi}} (x+iy)^2, \\
Y_2^1(\theta,\phi) &= -\sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta\, e^{i\phi} = -\sqrt{\frac{15}{8\pi}} z(x+iy), \\
Y_2^0(\theta,\phi) &= \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) = \sqrt{\frac{5}{16\pi}} (3z^2 - 1), \\
Y_2^{-1}(\theta,\phi) &= \sqrt{\frac{15}{8\pi}} \sin\theta\cos\theta\, e^{-i\phi} = \sqrt{\frac{15}{8\pi}} z(x-iy), \\
Y_2^{-2}(\theta,\phi) &= \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{-i2\phi} = \sqrt{\frac{15}{32\pi}} (x-iy)^2.
\end{aligned}$$

From this it is a matter of simple algebra to see

$$\boxed{\sum_{m=-2}^{m=2} |Y_2^m|^2 = \frac{5}{4\pi}.}$$

6. Using separation of variables, show that the eigenstates of the Hamiltonian for a spherically symmetric potential $V(\mathbf{r})$ may be written in the form

$$\Psi_{E,l,m} = R_{El}(r) Y_l^m(\theta,\phi)$$

where $R_{El}(r) = \frac{1}{r} u_{El}(r)$ and u_{El} satisfies

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \right] u_{El}(r) = E u_{El}(r)\,.$$

Solution:
The Schrödinger equation

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V(r) \right)\psi = E\psi,$$

takes the following form in spherical coordinate:

$$-\frac{\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) + \frac{L^2}{2mr^2}\psi + V(r)\psi = E\psi,$$

where

$$L^2 = -\hbar^2 \left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (\sin\theta \frac{\partial}{\partial\theta}) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right).$$

Since $V(r)$ is spherically symmetric, (H, L^2, L_z) is a complete set of commuting observables. Therefore we can take the wavefunctions as

$$\psi_{Elm}(r,\theta,\phi) = R_{El}(r) Y_l^m(\theta,\phi)$$

where $l = 0, 1, 2\ldots$, $m = -l, -l+1, \ldots, l-1, l$ and as usual spherical harmonics satisfy

$$L^2 Y_l^m(\theta,\phi) = \hbar^2 l(l+1) Y_l^m(\theta,\phi). \tag{1}$$

In addition, the form of the equation, $\frac{\partial^2}{\partial r^2}(r\psi)$, suggests us to the following transformation

$$R_{El}(r) \equiv \frac{u_{El}(r)}{r},$$

for which $u_{El}(r)$ satisfies

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r) \right) u_{El}(r) = E u_{El}(r),$$

after we put spherical harmonics and separate it. Notice that the radial equation is independent of the quantum number m($\leftarrow$ NOT the mass m), so usually the energy is $(2l+1)-$fold degenerate.

7. Consider a particle in the 2D potential $(m = 1)$

$$V(x,y) = \frac{1}{2}\omega^2(x^2 + y^2).$$

Use raising and lowering operators $a_x^\dagger, a_y^\dagger, a_x, a_y$ to compute the spectrum and degeneracies of the Hamiltonian. For each value of the energy, what eigenvalues of J_z are possible? Are H, J_z a complete set of commuting observables? Write the states at the lowest 3 energy levels that are simultaneous eigenstates of these observables.

Solution:

We have a Hamiltonian that can be described as that for two independent oscillators with the same frequency

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}\omega^2 y^2.$$

For each coordinate, we can define the raising and lowering operators as follows:

$$a = \sqrt{\frac{\omega}{2\hbar}} \left(Q + i\frac{P}{\omega} \right).$$

$$a^\dagger = \sqrt{\frac{\omega}{2\hbar}} \left(Q - i\frac{P}{\omega} \right).$$

Their inverse is also useful to keep in mind

$$
\begin{aligned}
Q &= \sqrt{\frac{\hbar}{2\omega}} \left(a^\dagger + a\right), \\
P &= i\sqrt{\frac{\hbar\omega}{2}} \left(a^\dagger - a\right).
\end{aligned}
$$

The Hamiltonian we get after this change of variables is as usual

$$H = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + 1),$$

So we see we have two decoupled harmonic oscillators. From this, the energy eigenstates are $|n_x, n_y\rangle \sim (a_x^\dagger)^{n_x}(a_y^\dagger)^{n_y}|0\rangle$ and the energy spectrum is

$$\boxed{E = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega = (n_x + n_y + 1)\hbar\omega.}$$

where $n_x, n_y = 0, 1, 2, \ldots$. Note that when $n_x + n_y = n$, n_x can be $0, 1, 2, \ldots, n$, so the degeneracy for each level of energy is $\boxed{n+1}$.

Now we can define the angular momentum operator in z-direction as

$$
\begin{aligned}
L_z &\equiv xp_y - yp_x \\
&= \boxed{\frac{\hbar}{i}(a_x^\dagger a_y - a_x a_y^\dagger).}
\end{aligned}
$$

Note that operators belonging to different coordinates commute, since oscillators are non-interacting.

Observe that $\boxed{[H, L_z] = 0}$ from the rotational symmetry of the problem. But note that $|n_x, n_y\rangle$ are not eigenstate of L_z so they are not simultaneous eigenstates of H and L_z. So now we have three routes to proceed to understand values L_z takes and whether or not H and L_z form a *complete* set of commuting observables:

- The easiest way to understand the states is to note that the states at energy level $n = n_x + n_y$ can be written as

$$\psi_{n_x, n-n_x} \propto H_{n_x}\left(\sqrt{\frac{\omega}{\hbar}}x\right)H_{n_y}\left(\sqrt{\frac{\omega}{\hbar}}y\right)e^{-\omega\left(x^2+y^2\right)/2\hbar},$$

  where $H_n(x)$ is the Hermite polynomial with leading term of order x^n. Recall that $H_n(x)$ has a constant term when x is even, and is proportional to x when x is odd. Writing $L_z = -i\hbar\partial/\partial\phi$ we see that linear combinations of these wavefunctions have the form $e^{im\phi}f(r)$ with $r = x^2 + y^2, e^{i\phi} = x + iy$ and $m = n, n-2, \ldots, -n$. Since m has $(n+1)$ possible values, H, L_z form a complete set of commuting observables. We work this out explicitly below in the cases $n = 0, 1, 2$, but first give some more abstract arguments for the complete set of commuting observables.

- Define

$$
\begin{aligned}
a_d &= \frac{1}{\sqrt{2}}(a_x - ia_y), \\
a_d^\dagger &= \frac{1}{\sqrt{2}}(a_x^\dagger + ia_y^\dagger), \\
a_g &= \frac{1}{\sqrt{2}}(a_x + ia_y). \\
a_g^\dagger &= \frac{1}{\sqrt{2}}(a_x^\dagger - ia_y^\dagger).
\end{aligned}
$$

It's easy to check these four operators have the same commutator relations as the original four. The Hamiltonian and angular momentum operator in z-direction takes the form:

$$
\begin{aligned}
H &= \hbar\omega(a_d^\dagger a_d + a_g^\dagger a_g + 1), \\
L_z &= \hbar(a_d^\dagger a_d - a_g^\dagger a_g).
\end{aligned}
$$

So the states $|n_d, n_g\rangle \sim (a_d^\dagger)^{n_d}(a_g^\dagger)^{n_g}|0\rangle$ have energy $E = (n_d + n_g + 1)\hbar\omega$. Furthermore, it's also easy to see that $|n_d, n_g\rangle$ are eigenstates of L_z with eigenvalues $L_z = \hbar(n_d - n_g)$. Therefore, $|n_d, n_g\rangle$ states are simultaneous eigenstates of H, and L_z.

Note that given $n = n_d + n_g$, and $m\hbar = (n_d - n_g)\hbar$ uniquely determines n_d, and n_g, so we can write these states as $|n, m\rangle$ for $n = 0, 1, 2, \ldots$ and $m = n, n-2, \ldots, -n$ as well. In this case, states have the energy $E = (n+1)\hbar\omega$, and at this energy level the angular momenta can be $L_z = n\hbar$, $(n-2)\hbar$, $(n-4)\hbar$, $\ldots$, $-n\hbar$. The degeneracy is $(n+1)$, as before. From this we see H and L_z are obviously a complete set of commuting observables.

- Alternatively, we can define a new set of angular momentum operators

$$
\begin{aligned}
J_+ &= \hbar a_x^\dagger a_y, \\
J_- &= \hbar a_y^\dagger a_x, \\
J_z &= \frac{\hbar}{2}(a_x^\dagger a_x - a_y^\dagger a_y).
\end{aligned}
$$

From this we can show

$$
\begin{aligned}
[J_z, J_\pm] &= \pm\hbar J_\pm, \\
[J_+, J_-] &= 2\hbar J_z,
\end{aligned}
$$

and construct

$$
\begin{aligned}
J_x &= \frac{1}{2}(J_+ + J_-) = \frac{\hbar}{2}(a_x^\dagger a_y + a_y^\dagger a_x), \\
J_y &= \frac{1}{2i}(J_+ - J_-) = \frac{\hbar}{2i}(a_x^\dagger a_y - a_y^\dagger a_x).
\end{aligned}
$$

Then the energy is related to

$$
J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = \hbar^2\frac{n}{2}(\frac{n}{2} + 1) \equiv \hbar^2 j(j+1),
$$

and the eigenvalues of each J_x, J_y, or J_z are from $-\frac{n}{2}\hbar$ to $\frac{n}{2}\hbar$. Notice that our L_z is $2 \times J_y$, so the spectrum is from $-n\hbar$ to $n\hbar$, consistent with the results above.

- Now we are going to write the states at the lowest 3 energy levels that are simultaneous eigenstates of H and L_z, using our conventions for the states from above and the wavefunctions in the position space. In order to do that start with the lowest level for which we have $n_x = 0 = n_y$ and $E = \hbar\omega$.Note that there is only one state at this level and in position space it is given by
$$|n_x = 0, n_y = 0\rangle \sim e^{-\frac{\omega}{2\hbar}x^2} e^{-\frac{\omega}{2\hbar}y^2} = e^{-\frac{\omega}{2\hbar}r^2},$$
Since there is no angular dependence on this it is easy to this is also an eigenstate of L_z with eigenvalue 0. Therefore we can write
$$|n = 0, m = 0\rangle = |n_x = 0, n_y = 0\rangle \sim e^{-\frac{\omega}{2\hbar}r^2}.$$
Now focus on the second-lowest level, whose energy is given by $E = 2\hbar\omega$. There are two states at this level, whose expressions in position space are given by
$$\begin{aligned} |n_x = 1, n_y = 0\rangle &\sim x e^{-\frac{\omega}{2\hbar}r^2}, \\ |n_x = 0, n_y = 1\rangle &\sim y e^{-\frac{\omega}{2\hbar}r^2}. \end{aligned}$$
We will assume they are multiplied by the some numerical factor, which is going to be unimportant. Now note that these are not eigenstates of L_z, but we can form eigenstates of L_z by taking the following linear combinations
$$\begin{aligned} |n = 1, m = 1\rangle &= \frac{1}{\sqrt{2}} \left(|n_x = 1, n_y = 0\rangle + i|n_x = 0, n_y = 1\rangle\right) \sim e^{i\phi} e^{-\frac{\omega}{2\hbar}r^2}, \\ |n = 1, m = -1\rangle &= \frac{1}{\sqrt{2}} \left(|n_x = 1, n_y = 0\rangle - i|n_x = 0, n_y = 1\rangle\right) \sim e^{-i\phi} e^{-\frac{\omega}{2\hbar}r^2}. \end{aligned}$$
as one can easily check. Note that we used $e^{i\phi} = x + iy$ as above. Finally, for the third-lowest level, whose energy is given by $E = 3\hbar\omega$, we have 3 states and they are (with proper pre-factors; whole expressions are not needed, but their relative factors are important)
$$\begin{aligned} |n_x = 1, n_y = 1\rangle &\sim \sqrt{2} xy e^{-\frac{\omega}{2\hbar}r^2}, \\ |n_x = 2, n_y = 0\rangle &\sim \left(x^2 - \frac{\hbar}{\omega}\right) e^{-\frac{\omega}{2\hbar}r^2}, \\ |n_x = 0, n_y = 2\rangle &\sim \left(y^2 - \frac{\hbar}{\omega}\right) e^{-\frac{\omega}{2\hbar}r^2}, \end{aligned}$$
Again, these are not eigenstates of L_z, but we can form eigenstates of L_z by taking the following linear combinations
$$\begin{aligned} |n = 2, m = 2\rangle &= \frac{1}{2}\Big(|n_x = 2, n_y = 0\rangle + i\sqrt{2}|n_x = 1, n_y = 1\rangle - |n_x = 2, n_y = 0\rangle\Big) \\ &\sim e^{2i\phi} e^{-\frac{\omega}{2\hbar}r^2}, \\ |n = 2, m = 0\rangle &= \frac{1}{\sqrt{2}} \left(|n_x = 2, n_y = 0\rangle + |n_x = 0, n_y = 2\rangle\right) \sim \left(r^2 - \frac{2\hbar}{\omega}\right) e^{-\frac{\omega}{2\hbar}r^2}, \\ |n = 2, m = -2\rangle &= \frac{1}{2}\Big(|n_x = 2, n_y = 0\rangle - i\sqrt{2}|n_x = 1, n_y = 1\rangle - |n_x = 2, n_y = 0\rangle\Big) \\ &\sim e^{-2i\phi} e^{-\frac{\omega}{2\hbar}r^2}, \end{aligned}$$

- We can also construct the eigenstates $|n_d, n_g\rangle$ in coordinate space in general. Using the relationship between $a_d, a_d^\dagger, a_g, a_g^\dagger$ and $a_x, a_x^\dagger, a_y, a_y^\dagger$, we see that

$$
\begin{aligned}
|n_d, n_g\rangle &= \frac{1}{\sqrt{n_d!\, n_g!}}(a_d^\dagger)^{n_d}(a_g^\dagger)^{n_g}|0,0\rangle \\
&= \sum_{k=0}^{n_d}\sum_{j=0}^{n_g}(-1)^j i^{j+k}\frac{\sqrt{n_d!\, n_g!}}{j!\, k!\,(n_d-k)!\,(n_g-j)!}(a_x^\dagger)^{n_d+n_g-k-j}(a_y^\dagger)^{j+k}|0,0\rangle \\
\Rightarrow \langle x, y|n_d, n_g\rangle &= \sum_{k,j}\frac{(-1)^j i^{j+k}\sqrt{n_d!\, n_g!}}{2^{(n_d+n_g)/2}\sqrt{\pi}\, j!\, k!\,(n_d-k)!\,(n_g-j)!}H_{n_d+n_g-k-j}(x)H_{j+k}(y)e^{-(x^2+y^2)/2}
\end{aligned}
$$

in units where $\hbar = \omega = m = 1$.

Here is an alternative way to construct the eigenstates. Please refer to the Problem 2 of Problem Set 5. From Problem Set 5, we know

$$
\frac{1}{\pi^{1/4}}e^{-\frac{1}{2}x^2+\sqrt{2}xa^\dagger-\frac{1}{2}(a^\dagger)^2}|0\rangle = |x\rangle
$$

we can furthermore write it as

$$
\begin{aligned}
|x\rangle &= \sum_{n=0}^{\infty} f_n(x)\frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle \\
&= \sum_{n=0}^{\infty} f_n(x)|n\rangle
\end{aligned}
$$

Project the above equation to $\langle n'|$, we know that $f_n(x) = H_n(x)$. Exactly the same idea can be applied to 2-dimensional case:

$$
\frac{1}{\pi^{1/2}}e^{-\frac{1}{2}(x^2+y^2)+\sqrt{2}(xa_x^\dagger+ya_y^\dagger)-\frac{1}{2}((a_x^\dagger)^2+(a_y^\dagger)^2)}|0\rangle\otimes|0\rangle = |x\rangle\otimes|y\rangle
$$

Writing $x = r\cos\phi$, $y = r\sin\phi$, and use $a_d^\dagger$, $a_g^\dagger$ defined previously:

$$
\frac{1}{\pi^{1/2}}e^{-\frac{1}{2}(x^2+y^2)+\sqrt{2}(xa_x^\dagger+ya_y^\dagger)-\frac{1}{2}((a_x^\dagger)^2+(a_y^\dagger)^2)} = \frac{1}{\pi^{1/2}}e^{-a_d^\dagger a_g^\dagger+a_d^\dagger z+a_g^\dagger z^*}e^{-\frac{1}{2}|z|^2}
$$

where $z = re^{i\phi}$.

$$
\frac{1}{\pi^{1/2}}e^{-a_d^\dagger a_g^\dagger+a_d^\dagger z+a_g^\dagger z^*}e^{-\frac{1}{2}|z|^2}|0\rangle\otimes|0\rangle = \sum_{n_d,n_g=0}^{\infty}\psi_{n_d,n_g}(r,\phi)\frac{(a_d^\dagger)^{n_d}}{\sqrt{n_d!}}\frac{(a_g^\dagger)^{n_g}}{\sqrt{n_g!}}|0\rangle\otimes|0\rangle
$$

$\psi_{n_d,n_g}(r,\phi)$ can be found from Mathematical Formula Handbook. We further set $n_d+n_g = n$, and $n_d - n_g = m$, then

$$
\psi_{n,m}(r,\phi) = \frac{1}{\pi^{1/2}}(-1)^{\frac{n-m}{2}}\sqrt{\frac{(\frac{n-m}{2})!}{(\frac{n+m}{2})!}}L^m_{(n-m/2)}(r^2)\times(e^{im\phi}r^m e^{-\frac{1}{2}r^2})
$$

Here, L's are associated Laguerre polynomials like Problem 3(b). However note that m can be either positive or negative above. One can check these expression matches for the first three levels above.