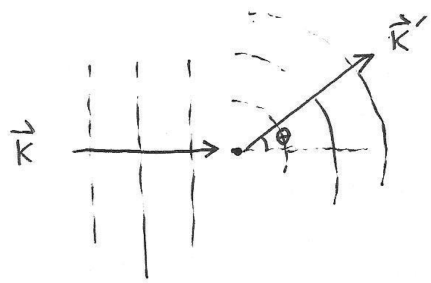
PY 711

In non-relativistic quantum mechanics, rotational invariance simplifies scattering problems for



$$|\vec{k}'| = |\vec{k}| = k$$

elastic scattering

central potentials $V(r)$. At very large distances,
$r = |\vec{r}|$

$$\psi(\vec{r}) = \underbrace{e^{i\vec{k}\cdot\vec{r}}}_{\text{incoming plane wave}} + \underbrace{f(\vec{k}',\vec{k})\,\frac{e^{ikr}}{r}}_{\text{outgoing spherical wave}}$$

Scattering amplitude

$$f(\vec{k}',\vec{k}) = \sum_{L=0}^{\infty} f_L(k)\,\underbrace{P_L(\cos\theta)}_{\text{Legendre polynomials}}$$

Decompose according to angular momentum and solve for each L (possibly coupled if the particles have intrinsic spin).

So then why does relativity make things more complicated? After all we are imposing a larger symmetry group ... Lorentz invariance.

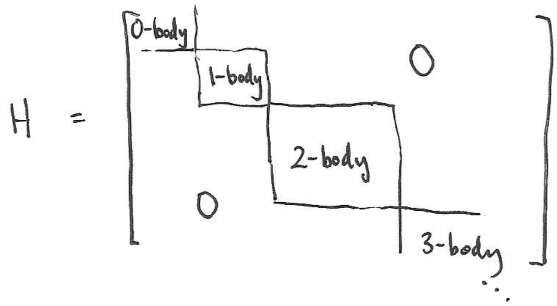
Answer: Particles can appear out of the vacuum.

At high energies we can have the reaction

$$\underset{\text{proton}}{p} + p \longrightarrow p + p + p + \underset{\text{antiproton}}{\bar{p}}$$

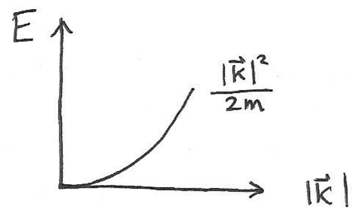
The solution of a high energy scattering problem will involve many particles.

In nonrelativistic physics the Hamiltonian is block diagonal in the number of particles

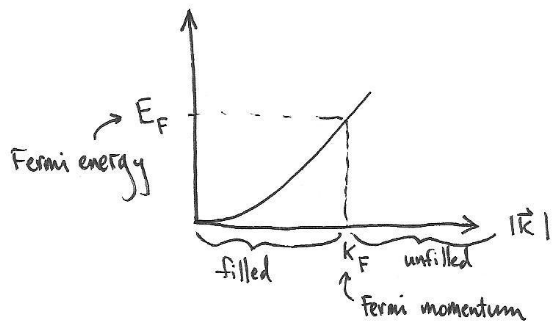


But in relativistic physics there are terms in the Hamiltonian connecting different n-body blocks.

A similar thing happens in nonrelativistic many-body systems. Consider a nonrelativistic electron without interactions. The kinetic energy as a function of momentum is



Suppose we have a lot of electrons.
The lowest energy states are filled with electrons.



$$E_F = \frac{k_F^2}{2m}$$

Assume that with interactions turned on, this picture is still qualitatively correct.

For $|\vec{k}| > k_F$ we can add an extra electron. This is a "particle" excitation. Sometimes the term is "quasiparticle" when there are interactions.

For $|\vec{k}| < k_F$ we can remove an electron from the "Fermi sea." This is called a "hole" excitation or "quasihole" in the presence of interactions.

We can now have a nonrelativistic scattering process particle + particle $\longrightarrow$ particle + particle + particle + hole

Looks similar to our relativistic proton scattering. Many of the tools in field theory apply to both relativistic systems and nonrelativistic many-body systems.

Relativistic conventions and notation

"Natural" units

$$\hbar = c = 1$$

(later we also set $k_B = 1$)

So $[\text{length}] = \cancel{c} \cdot [\text{time}] = [\text{time}]$

$[\text{energy}] = \cancel{c}\,[\text{momentum}] = \cancel{c^2}\,[\text{mass}]$

Everything can be written as MeV or kg or any unit of your choice raised to some power.

$g_{\mu\nu}$ metric tensor

Greek indices include time and space

$$\mu = \underset{\text{time}}{0}, \underbrace{1,2,3}_{\text{space}}$$

Regular indices are meant to indicate only space

$$i = 1, 2, 3$$

As a matrix $$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

So $g_{00} = +1$, $g_{11} = g_{22} = g_{33} = -1$. All others are zero.

$$\underset{\text{four-vector}}{x^\mu} = (x^0, x^1, x^2, x^3) = (x^0, \vec{x})$$

the textbook uses bold font for this

For two four-vectors a^μ and b^μ we can define the Lorentz scalar product or contraction

$$a \cdot b = a^\mu b^\nu g_{\mu\nu} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$$

This quantity is Lorentz invariant. It is convenient to define a lower index object

$$x_\mu = \sum_\nu g_{\mu\nu} x^\nu = g_{\mu 0} x^0 + g_{\mu 1} x^1 + g_{\mu 2} x^2 + g_{\mu 3} x^3$$

$$= g_{\mu\nu} x^\nu$$ repeated index means that you sum over this index

$$= (x^0, -\vec{x})$$

Note then that

$$a \cdot b = a^\mu b_\mu = a_\mu b^\mu$$

Consider the energy-momentum four-vector

$$p^\mu = (E, \vec{p})$$

$p^0 = E$

Then $\quad p \cdot p = E^2 - \vec{p}^{\,2} = m^2.$

We use the shorthand a^2 for the scalar product of a four-vector with itself, $a \cdot a$. So $p^2 = m^2$.

Classical Field Theory

Action $S = \int \underset{\substack{\uparrow \\ \text{Lagrangian}}}{L}\, dt$

$$= \int \underset{\substack{\uparrow \\ \text{Lagrange density}}}{\mathcal{L}}\, d^3x\, dt = \int \mathcal{L}\, d^4x$$

Let ϕ be a real-valued function of spacetime. Let $\mathcal{L}$ be a function of ϕ and derivatives of ϕ, which we denote $\partial_\mu \phi$.

Suppose we are given the initial configuration

$$\phi(t_i, \vec{x}) = f_{initial}(\vec{x})$$

and the final configuration

$$\phi(t_f, \vec{x}) = f_{final}(\vec{x}).$$

We get ϕ for times between t_i and t_f by extremizing S. For any small perturbation

$$\phi = \phi_{\text{stationary}} + \delta\phi ,$$

vanishes at boundaries

we should have $\delta S = 0$. So we replace $\phi \to \phi + \delta\phi$ and collect the $\delta\phi$ terms...

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right]$$

Since ∂_μ is a linear operator

$$\delta(\partial_\mu \phi) = \partial_\mu(\phi + \delta\phi) - \partial_\mu \phi$$
$$= \partial_\mu(\delta\phi)$$

So $\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu(\delta\phi)} \right]$

[integrate by parts
$u\,dv = d(uv) - v\,du$]

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \underbrace{\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right]} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \delta\phi \right]$$

This can be converted into a surface integral over the boundary, where $\delta\phi$ vanishes

So we are left with

$$0 = \delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right]\delta\phi$$

Since $\delta\phi$ is arbitrary for points inside the boundary, we have

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0$$

These are the Euler-Lagrange equations of motion for the classical field ϕ.

Examples on how to use this...

Suppose $\mathcal{L} = \phi^2$. Then $\frac{\partial \mathcal{L}}{\partial \phi} = 2\phi$, $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0$.

Suppose $\mathcal{L} = \partial_\mu \phi \, \partial^\mu \phi$. Then $\frac{\partial \mathcal{L}}{\partial \phi} = 0$.

Maybe $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}$ is not so obvious.

It might be helpful to write

$$\mathcal{L} = \partial_\mu \phi_1 \partial^\mu \phi_2$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)}$$

and then set $\phi_1 = \phi_2 = \phi$ in the end.

When computing $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_1)}$ we get $\partial^\mu \phi_2$.

When computing $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)}$ it is useful to write $\mathcal{L}$ as $\mathcal{L} = \partial^\mu \phi_1 \partial_\mu \phi_2$. Then clearly $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_2)} = \partial^\mu \phi_1$.

So $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi_1 + \partial^\mu \phi_2 = 2 \cdot \partial^\mu \phi$.

Example: Klein-Gordon field

Let $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$

Euler-Lagrange equations give

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi, \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = 0$$

$$\uparrow \qquad\qquad \uparrow$$

$$-m^2\phi - \partial_\mu(\partial^\mu \phi) = 0$$

$$\Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0$$

this is the Klein-Gordon equation

$$\partial_\mu \partial^\mu = \left(\frac{\partial}{\partial x^0}\right)^2 - \left(\frac{\partial}{\partial x^1}\right)^2 - \left(\frac{\partial}{\partial x^2}\right)^2 - \left(\frac{\partial}{\partial x^3}\right)^2$$

We can find solutions by setting

$$\phi(x) = e^{-ip\cdot x}$$

$$\partial_\mu \partial^\mu (e^{-ip\cdot x}) = (-ip_\mu)(-ip^\mu) = -p^2$$

The Klein-Gordon equation gives $p^2 = m^2$.
We conclude that this describes a relativistic particle with no interactions and mass m.