Physics 8.321, Fall 2021
Homework #7

Due **Friday, November 19** by 8:00 PM.

1. Use the WKB approximation to calculate the spectrum of energies for a particle in the following 1D potentials (in units $\hbar = m = 1$):

(a) Harmonic oscillator potential $V(x) = \frac{1}{2}x^2$.

(b) Box potential $V(x) = 0$ for $0 \le x \le L$, $V(x) = \infty$ otherwise.

(c) $V(x) = |kx^\alpha|$, with $\alpha, k > 0$.

(d) Compare with the exact values for the ground state and first excited state in each case (set $k = 1/4, \alpha = 4$ in case (c), and compare with the results $E_0 = 0.4208, E_1 = 1.5079$ from problem set 5). In which case does the WKB approximation do the worst? Why?

Answer: Recall WKB energy quantization hypothesis

$$\oint p\,dq = (n + \frac{1}{2})h = (n + \frac{1}{2})2\pi\hbar = (n + \frac{1}{2})2\pi$$

where $p = \sqrt{2(E - V)}$. If you integrate only *half* period, the condition is

$$\int_{x_1}^{x_2} p\,dq = (n + \frac{1}{2})\pi\hbar = (n + \frac{1}{2})\pi$$

where x_1, x_2 are turning points.

(a) For $V(x) = \frac{1}{2}x^2$, turning points are $-x_1 = x_2 = \sqrt{2E}$, so we have

$$\int_{x_1}^{x_2} p\,dq = \int_{-\sqrt{2E}}^{\sqrt{2E}} \sqrt{2(E - \frac{1}{2}x^2)}\,dx = E\pi = (n + \frac{1}{2})\pi \Rightarrow \boxed{E = (n + \frac{1}{2})}$$

We are lucky.

(b) Turning points are $x_1 = 0$ and $x_2 = L$, so we have

$$\int_0^L \sqrt{2E}\,dx = \sqrt{2E}L = (n + \frac{1}{2})\pi \Rightarrow \boxed{E_n = \frac{(n + \frac{1}{2})^2\pi^2}{2L^2}}$$

The exact answer is $E_n = \frac{n^2\pi^2}{2L^2}$ Doesn't mean we are not lucky. See below for the discussion about WKB applied to box potential:

Physically, the infinite barrier is the limiting case of a 'large' barrier. As long as it is not infinite, no matter how large the barrier is, we always have the $\pi/4$ from connecting two wavefunctions at different regions, and thus WKB gives an energy different from the exact energy.

On the other hand, if you view the 'infinite' barrier as a mathematical property, that is, there is an abrupt change of the box potential from 0 to ∞ at the boundary $x = 0$, and $x = L$. The $V = \infty$ region now is inaccessible to the particle, and there is no wavefunctions at different regioins to connect, only the boundary condition $\psi(x = 0) = \psi(x = L) = 0$ to be satisfied. The particle in the box then NEVER experiences the abrupt change of the potential. If you take this viewpoint, then WKB can give you exact energy for box potential: Assume for a 1-D potential $V(x)$, the turning points are a (left), and b (right). Then for $V(x) < E$, and away from turning points, the wave function at $x > a$ is

$$\psi(x) = \frac{C}{\sqrt{p}} \sin\left(\frac{1}{\hbar}\int_a^x p\,dx + \alpha\right)$$

where C is the normalization constant, and α is determined by matching to the exact solution at turning point, and is found to be $\frac{\pi}{4}$.
The same, the wave function at $x < b$ is

$$\psi(x) = \frac{C'}{\sqrt{p}} \sin\left(\frac{1}{\hbar}\int_x^b p\,dx + \beta\right)$$

where $\beta = \frac{\pi}{4}$ too.
By demanding the two function to be the same, the quantities in sin must sum to integral π,

$$\frac{1}{\hbar}\int_a^x p\,dx + \alpha + \frac{1}{\hbar}\int_x^b p\,dx + \beta = \frac{1}{\hbar}\int_a^b p\,dx + \frac{\pi}{2} \equiv (n+1)\pi \quad n = 0, 1, 2, \ldots$$

and this gives

$$\int_a^b p\,dx = (n + \frac{1}{2})\pi\hbar$$

For box potential, the particle cannot access to the classically forbidden region, so the wave function is

$$\psi(x) \begin{cases} \frac{C}{\sqrt{p}} \sin\left(\frac{1}{\hbar}\int_0^x p\,dx\right), & x > 0; \\ \frac{C'}{\sqrt{p}} \sin\left(\frac{1}{\hbar}\int_x^L p\,dx\right), & x < L. \end{cases}$$

This gives

$$\frac{1}{\hbar}\int_0^L p\,dx = n\pi$$

which gives you the exact energy.

(c) $V(x) = kx^\alpha$. (We will only express the integral at $x > 0$ region.) The turning point is $x_0 = (\frac{E}{k})^{1/\alpha}$, therefore

$$(n + \frac{1}{2})\pi\hbar = 2\int_0^{x_0} \sqrt{2(E - kx^\alpha)} = 2\sqrt{2}\sqrt{E}(\frac{E}{k})^{\frac{1}{\alpha}}\,{}_2F_1(-\frac{1}{2}, \frac{1}{\alpha}, 1 + \frac{1}{\alpha}, 1)$$

where ${}_2F_1$ is the famous Hypergeometric function:

$${}_2F_1(a, b, c; z) \equiv \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \times \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$$

From this we have

$$
{}_2F_1(-\frac{1}{2},\frac{1}{\alpha},1+\frac{1}{\alpha},1) = \frac{\Gamma(1+\frac{1}{\alpha})}{\Gamma(\frac{1}{\alpha})\Gamma(1)} \underbrace{\int_0^1 t^{\frac{1}{\alpha}-1}(1-t)^{\frac{1}{2}}\,dt}_{=\frac{\sqrt{\pi}\Gamma(\frac{1}{\alpha})}{2\Gamma(\frac{3}{2}+\frac{1}{\alpha})}}
$$

Furthermore, since $\Gamma(z+1) = z\Gamma(z)$, we can write

$$
\begin{aligned}
\Gamma(1+\frac{1}{\alpha}) &= \frac{1}{\alpha}\Gamma(\frac{1}{\alpha}) \\
\Gamma(\frac{3}{2}+\frac{1}{\alpha}) &= \frac{\alpha+2}{2\alpha}\Gamma(\frac{1}{2}+\frac{1}{\alpha})
\end{aligned}
$$

Finally, we have

$$
(n+\frac{1}{2})\pi\hbar = \sqrt{E}E^{1/\alpha} \times \underbrace{2\sqrt{2}(\frac{1}{k})^{\frac{1}{\alpha}}\sqrt{\pi}\frac{\Gamma(\frac{1}{\alpha})}{(\alpha+2)\Gamma(\frac{1}{2}+\frac{1}{\alpha})}}_{\equiv C}
$$

yielding

$$
\boxed{E_n = \left(\frac{(n+\frac{1}{2})\pi\hbar}{C}\right)^{\frac{2\alpha}{\alpha+2}}}
$$

The integral

$$
\int_0^{\left(\frac{E}{k}\right)^{1/\alpha}} \sqrt{E-kx^\alpha}\,dx
$$

can also be done in a more elementary way. Set

$$
\begin{aligned}
kx^\alpha = E\sin^2\theta \Rightarrow dx &= \frac{2}{\alpha}\left(\frac{E}{k}\right)^{\frac{1}{\alpha}}\cos\theta(\sin\theta)^{2/\alpha-1}\,d\theta, \\
\sqrt{E-kx^\alpha} &= \sqrt{E}\sqrt{1-\sin^2\theta} = \sqrt{E}\cos\theta
\end{aligned}
$$

then

$$
\begin{aligned}
\int_0^{\left(\frac{E}{k}\right)^{1/\alpha}} \sqrt{E-kx^\alpha}\,dx &= \sqrt{E}\frac{2}{\alpha}\left(\frac{E}{k}\right)^{\frac{1}{\alpha}}\int_0^{\pi/2}\cos^2\theta(\sin\theta)^{2/\alpha-1}\,d\theta \\
&= \sqrt{E}\frac{2}{\alpha}\left(\frac{E}{k}\right)^{\frac{1}{\alpha}}\frac{\sqrt{\pi}\Gamma(\frac{1}{\alpha})}{4\Gamma(\frac{3}{2}+\frac{1}{\alpha})}
\end{aligned}
$$

(d) For $k = 1/4$, $\alpha = 4$, we get $\boxed{E_0 = 0.344127}$, $\boxed{E_1 = 1.48895}$. The values from Problem set 5 is $E_0 = 0.4208$, and $E_1 = 1.5079$.

- If you keep the mass m, then the energy has the form

$$
E_n = A(\alpha)(n+\frac{1}{2})^{2\alpha/(\alpha+2)} \times k^{2/(\alpha+2)} \times \left(\frac{\hbar^2}{2m}\right)^{\alpha/(\alpha+2)}
$$

where $A(\alpha)$ is a dimensionless function of α.

– If you use "naive" energy quantization condition $(n+\frac{1}{2})\pi$ for all cases, the box potential deviates the most. However, if you take the second viewpoint as discussed above, then actually you get exact values in part (a), and (b), while a not-too-bad estimate for part (c).
– When you set $\alpha = 1$, then you get an estimate for part (b) in Problem (3).

$$E_n = \left(\frac{3\pi}{4}(n+\frac{1}{2})\right)^{2/3} \times \left(\frac{\lambda\hbar^2}{2m}\right)^{1/3}$$

– A careful use of WKB also gives exact energy for hydrogen atom problem.
– Now you have more confidence on WKB; it gives exact energy spectrum for SHO, hydrogen atom, and box potential (when second viewpoint is taken).

2. [Sakurai and Napolitano Problem 32, Chapter 2 (page 154)]

Define the partition function as

$$Z = \int d^3x' K(\mathbf{x}', t; \mathbf{x}', 0)|_{\beta = it/\hbar}\,,$$

as in (2.6.20)-(2.6.22). Show that the ground-state energy is obtained by taking

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta},\ (\beta \to \infty)\,.$$

Illustrate this for a particle in a one-dimensional box, using the spectrum of energies for that system.

Answer:

$$Z = \sum_{n=0} e^{-\beta E_n}$$

$$\begin{aligned}
\Rightarrow -\frac{1}{Z}\frac{\partial Z}{\partial \beta} &= \frac{\sum_{n=0} E_n e^{-\beta E_n}}{\sum_{n=0} e^{-\beta E_n}} \\
&= \frac{E_0 e^{-\beta E_0} + \sum_{n\geq 1} E_n e^{-\beta E_n}}{e^{-\beta E_0} + \sum_{n\geq 1} e^{-\beta E_n}} \\
&= \frac{E_0 + \sum_{n\geq 1} E_n e^{-\beta(E_n - E_0)}}{1 + \sum_{n\geq 1} e^{-\beta(E_n - E_0)}} \quad \text{Both denominator and numerator } \times e^{\beta E_0}
\end{aligned}$$

Since E_0 is the ground energy, and in one dimension, the ground state energy is not degenerate, so $E_n - E_0 > 0$, $\forall\, n \geq 1$. $\Rightarrow e^{-\beta(E_n - E_0)} \to 0$, when $\beta \to \infty$. Therefore,

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta} \xrightarrow{\beta\to\infty} E_0$$

Put $E_n = \frac{\pi^2\hbar^2}{2mL^2}n^2$, $n = 1, 2....$ Notice this time the index n starts from 1, and re-do the above.

$$\boxed{-\frac{1}{Z}\frac{\partial Z}{\partial \beta} \xrightarrow{\beta\to\infty} \frac{\pi^2\hbar^2}{2mL^2}}$$

3. [Sakurai and Napolitano Problem 34, Chapter 2 (page 155)]

(a) Write down an expression for the action for a classical solution of the simple harmonic oscillator from $x(0) = x$ to $x(t') = x'$.

(b) Construct $\langle x_n, t_n | x_{n-1}, t_{n-1}\rangle$ for a simple harmonic oscillator using Feynman's prescription for $t_n - t_{n-1} = \Delta t$ small. Show that the leading terms in this propagator can can be expressed in the form

$$K(x', x, \Delta t) = K_{\text{free}}(x', x, \Delta t)\left[c_0(x, x') + c_1(x, x')\Delta t + \mathcal{O}((\Delta t)^2)\right],$$

and find $c_0(x, x')$ and $c_1(x, x')$. Further, show that this matches with the leading terms in a term-by-term expansion of the propagator written in terms of the time development operator in the Hamiltonian picture.

Answer:

(a) Writing the classical solution as

$$x_{\text{cl}}(t) \equiv A\sin(\omega t + \phi)$$

The "boundary" conditions are

$$\begin{aligned} x_{\text{cl}}(0) &\equiv x = A\sin\phi \\ x_{\text{cl}}(t) &\equiv x' = A\sin(\omega t + \phi) \end{aligned}$$

The action is

$$\begin{aligned} S[x_{\text{cl}}(t)] &= \int_0^t dt' \left(\frac{m}{2}\dot{x}^2(t') - \frac{m\omega^2}{2}x^2(t')\right) \\ &= \frac{1}{2}mA^2\omega^2 \int_0^t dt' \left(\cos^2(\omega t' + \phi) - \sin^2(\omega t' + \phi)\right) \\ &= \frac{1}{2}mA^2\omega^2 \int_0^t dt' \cos(2\omega t' + 2\phi) \\ &= \frac{m\omega}{4}A^2\left(\sin(2\omega t + 2\phi) - \sin 2\phi\right) \end{aligned}$$

Now we have to express A, and ϕ by x, and x'. First, we have

$$\frac{x}{x'} = \frac{A\sin\phi}{A\sin(\omega t + \phi)} = \frac{\sin\phi}{\sin\omega t\cos\phi + \cos\omega t\sin\phi}$$

Solving for $\cos\phi$:

$$A\cos\phi = \frac{x'}{\sin\omega t} - \frac{x\cos\omega t}{\sin\omega t}$$

Thus

$$\begin{aligned} A^2\sin 2\phi &= 2\underbrace{A\sin\phi}_{x}\times A\cos\phi \\ &= \frac{2x}{\sin\omega t}(x' - x\cos\omega t) \end{aligned}$$

$$
\begin{aligned}
A^2\sin(2\omega t+2\phi) &= 2A\sin(\omega t+\phi)\times A\cos(\omega t+\phi)\\
&= 2x'A(\cos\omega t\cos\phi-\sin\omega t\sin\phi)\\
&= 2x'\left(\cos\omega t\left(\frac{x'}{\sin\omega t}-\frac{x\cos\omega t}{\sin\omega t}\right)-x\sin\omega t\right)\\
&= \frac{2x'}{\sin\omega t}(x'\cos\omega t-x)
\end{aligned}
$$

We then get

$$
\boxed{S[x_{\text{cl}}(t)]=\frac{m\omega}{2\sin\omega t}\left((x^2+x'^2)\cos\omega t-2xx'\right)}
$$

For starting time t_0, replace t above by $t-t_0$.

(b) When $t-t_0=\Delta t\to 0$,

$$
\begin{aligned}
S[x_{\text{cl}}(t)] &\to \frac{m\omega}{2\omega\Delta t\left(1-\frac{(\omega\Delta t)^2}{6}\right)}\left((x_n^2+x_{n-1}^2)(1-\frac{\omega^2\Delta t^2}{2})-2x_nx_{n-1}\right)\\
&= \frac{m}{2\Delta t}(x_n-x_{n-1})^2-\frac{m\omega^2}{6}(x_n^2+x_{n-1}^2+x_nx_{n-1})\Delta t
\end{aligned}
$$

$$
\begin{aligned}
\langle x_n,t_n|x_{n-1},t_{n-1}\rangle &\to \sqrt{\frac{m}{2\pi i\hbar\Delta t}}e^{\frac{i}{\hbar}\left(\frac{m}{2\Delta t}(x_n-x_{n-1})^2-\frac{m\omega^2}{6}(x_n^2+x_{n-1}^2+x_nx_{n-1})\Delta t\right)}\\
&\to \sqrt{\frac{m}{2\pi i\hbar\Delta t}}e^{\frac{i}{\hbar}\left(\frac{m}{2\Delta t}(x_n-x_{n-1})^2\right)}e^{-\frac{im\omega^2}{6\hbar}(x_n^2+x_{n-1}^2+x_nx_{n-1})\Delta t}\\
&\to K_{free}(x',x,\Delta t)\left(1-\frac{im\omega^2}{6\hbar}(x_n^2+x_{n-1}^2+x_nx_{n-1})\Delta t\right)\\
&\to \delta(x_n-x_{n-1})\left(1-\frac{im\omega^2}{2\hbar}(x_n^2)\Delta t\right)
\end{aligned}
$$

where

$$
\sqrt{\frac{1}{\pi z}}e^{-\frac{x^2}{z}}\longrightarrow\delta(x)\quad\text{when } z\to 0
$$

is used. $z=\frac{2i\hbar\Delta t}{m}$ in our case. This identity shows that $K_{free}(x',x,\Delta t)\to\delta(x_n-x_{n-1})$ as $\Delta t\to 0$, so using the δ function we may set $x_n=x_{n-1}$ as in the final line. The expansion coefficients are then

$$
\begin{aligned}
c_0(x,x') &= 1\\
c_1(x,x') &= -\frac{im\omega^2}{6\hbar}(x_n^2+x_{n-1}^2+x_nx_{n-1}),
\end{aligned}
$$

where any expression equivalent to c_1 above when setting $x_n=x_{n-1}$ is allowed.

4. (Optional, extra credit)

In this problem we use the semiclassical WKB-type approximation to gain some insight into the rate of nuclear fusion reactions between a pair of protons in the sun. At very close distances, nuclear forces between a pair of protons will overcome the Coulomb repulsion between their common positive charges and the nuclei can fuse into a deuteron, giving an initial stage towards the production of helium that releases energy and provides the sun's power. The Coulomb barrier for two protons to reach this distance classically requires energies of $V_C \sim 0.2$ MeV (1 MeV $\cong 1.6 \times 10^{-13}$ J). At the temperature of the sun, no protons have this much energy. So quantum tunneling is needed to drive the sun's nuclear reactions.

(a) Use the semiclassical approximation to show that the *tunneling probability* (per unit time) for a particle with energy E to pass through a classically forbidden barrier described by a potential $V(x)$ is given by

$$P_T(E) \sim \exp\left(-\frac{2}{\hbar}\int_{x_1(E)}^{x_2(E)} dx \sqrt{2m(V(x)-E)}\right)$$

where x_1, x_2 are the classical turning points.

(b) Using $m = M_p/2$ (the reduced mass in the two-proton) system and the Coulomb potential describing the repulsion between two protons, estimate the tunneling rate of a proton of energy E to penetrate the Coulomb barrier by reaching a distance $r \sim 0$ from another proton.

(c) Use the Boltzmann distribution $P_B(E) = e^{-E/k_BT}/Z$ giving the probability that a quantum system at temperature T has energy E to derive the Maxwell-Boltzmann distribution for the probability per unit energy that a particle in a hot 3D gas has a given energy (assuming energy can be taken as a continuous variable)

$$\frac{dP_B}{dE} = \left(\frac{2\pi}{(\pi k_B T)^{3/2}}\right)\sqrt{E}e^{-E/k_BT}\,.$$

(d) The temperature in the sun is roughly 1.5×10^7 K. Combine the Maxwell-Boltzmann distribution with the tunneling probability to derive the Gamow distribution

$$dP_{\text{Gamow}} = P_T \times dP_B$$

describing the window where nuclear fusion is possible. Use this to estimate the fraction of protons that can get close enough for a fusion reaction.

(e) The mass of the sun is 1.989×10^{30} kg, Earth's distance from the sun is $R_{\text{orbit}} \cong 1.5 \times 10^8$ km, and the insolation at the top of Earth's atmosphere is $I \cong 1366$ W/m^2. One nuclear fusion of four protons to a single helium nucleus, which involves two *pp* fusion events, releases roughly 26.7 MeV. Estimate the rate at which *pp* fusion events occur in the sun. The large discrepancy between this rate and that you computed in (c) results from the fact that this process involves the weak nuclear interactions, and are suppressed by a very small factor since these interactions are so weak (coupling constant $\sim 10^{-10}$).

Answer:

(a) Recall that the wavefunction decreases exponentially in a classically forbidden region from one classical turning point x_1 to another x_2 according to (assuming $x_2 > x_1$ throughout the problem)

$$\psi(x_2) = \psi(x_1) \exp\left(-\frac{1}{\hbar}\int_{x_1(E)}^{x_2(E)} dx \sqrt{2m(V(x) - E)}\right) ,$$

by the WKB approximation. Squaring both sides and interpreting the ratio $|\psi(x_2)|^2/|\psi(x_1)|^2$ as the tunneling probability, we immediately see that the tunneling probability per unit time $P_T(E)$ will scale as

$$P_T(E) \sim \exp\left(-\frac{2}{\hbar}\int_{x_1(E)}^{x_2(E)} dx \sqrt{2m(V(x) - E)}\right) .$$

Note that it makes sense to say that $|\psi(x_2)|^2/|\psi(x_1)|^2$ is a tunneling probability, since it is the ratio of probability densities on the two sides of the classically forbidden region and it is higher when tunneling happens from x_1 to x_2, just as in for free particle transmission/scattering probability. Normalization is left arbitrary.

(b) The Coulomb potential describing the repulsion between two protons is given by

$$V(r) = \frac{e^2}{4\pi\epsilon_0 r}.$$

From this, the classical turning point for a given energy E would be

$$r_c = \frac{e^2}{4\pi\epsilon_0 E}$$

With this we can evaluate the following integral (using reduced mass of two protons $m = M_p/2$)

$$\int_0^{r_c} dr \sqrt{2m(V(r) - E)} = \int_0^{r_c} dr \sqrt{M_p\left(\frac{e^2}{4\pi\epsilon_0 r} - E\right)} = \frac{e^2}{4\pi\epsilon_0}\sqrt{\frac{M_p}{E}}\int_0^1 dx \sqrt{\frac{1}{x} - 1} = \frac{e^2}{8\epsilon_0}\sqrt{\frac{M_p}{E}}$$

Above we defined $x = (4\pi\epsilon_0 r)/e^2$ and evaluated the integral with $\int_0^1 dx\sqrt{1/x - 1} = \pi/2$. From this we get the tunneling rate

$$P_T(E) \sim \exp\left(-\frac{e^2}{4\epsilon_0\hbar}\sqrt{\frac{M_p}{E}}\right) .$$

using the result from part a. Note that problem instructs us to take one of the turning points as $r = 0$, which we did while we are evaluating the integral above already. This can be justified by remembering there would be additional nuclear forces that will modify the Coulomb potential at these scales, and we expect another turning point at (or very close to) $r = 0$. This approximation is valid for energies well below the actual potential barrier of $V_C \sim 0.2$ MeV.

(c) From the Boltzmann distribution we can immediately write the differential probability $dP_B(E)$ that a quantum system at temperature T has energy E as follows

$$dP_B(E) = \frac{1}{Z} e^{-\frac{E}{k_B T}} \rho(E) dE.$$

Here $\rho(E)$ is called the *density of states*, for which $\rho(E)dE$ counts the number of states between E and $E + dE$. Then the meaning of the expression above is clear: basically it is the probability times the number of states with energy E that has that probability. Our job now is to find the $\rho(E)$.

In order to do this remember that for non-relativistic particle we have the relation

$$E = \frac{p^2}{2m} \implies p = p_x^2 + p_y^2 + p_z^2 \sim \sqrt{E}.$$

This implies that we can alternatively count states with constant momentum p, but p_x, p_y, p_z are arbitrary as long as they satisfy the constraint $p^2 = p_x^2 + p_y^2 + p_z^2$. Now observe that the differential $dp_x dp_y dp_z$ is proportional to the number of states with momenta (p_x, p_y, p_z). Since direction is not important, we should integrate angular directions in momentum space to find the measure number of states with constant momentum p is proportional to. This is easy to do, which gives $p^2 dp$. Then we simply observe that

$$p^2 dp \sim \sqrt{E} dE,$$

and this will be clearly proportional to number of states with energy E. So we find $\rho(E) \sim \sqrt{E}$ from this quick argument. So we obtain

$$\frac{dP}{dE} = N \sqrt{E} e^{-E/k_B T}$$

Essentially the factor of $\sqrt{E}$ comes from integrating over the directions of motion and N is some E-independent normalization constant (combining Z and proper normalization of density of states $\rho(E)$) which we still need to determine. But this is easy to do, just observe we should have

$$\int_0^\infty dE \frac{dP}{dE} = 1$$

in order to have probabilities to add up to 1. Then we should evaluate it and find

$$N \int_0^\infty dE \sqrt{E} e^{-E/k_B T} = 1 \implies N \frac{\sqrt{\pi (k_B T)^3}}{2} = 1 \implies N = \frac{2\pi}{(\pi k_B T)^{3/2}}$$

and this gives the answer

$$\frac{dP_B}{dE} = \left(\frac{2\pi}{(\pi k_B T)^{3/2}} \right) \sqrt{E} e^{-E/k_B T}.$$

(d) The Gamow distribution would be

$$dP_{\text{Gamow}} = P_T \times dP_B = \left(\frac{2\pi}{(\pi k_B T)^{3/2}} \right) \sqrt{E} \exp \left(-\frac{E}{k_B T} - \pi\alpha \sqrt{\frac{M_p c^2}{E}} \right) dE$$

combining our answers above and using fine-structure constant $\alpha = e^2/(4\pi\epsilon_0\hbar c)$. Remember that this is the probability for a proton with energy E at temperature T to tunnel near another proton, so in order to estimate the fraction of protons that can get close enough for a fusion reaction, we ought to integrate this from $E = 0$ to $E = \infty$ (Technically to some $E \ll V_C \sim 0.2$ MeV by our approximation, but since the probabilities are dying off exponentially we can approximate it with infinity). If you do this on Mathematica you would find this fraction is roughly 10^{-5} (per unit time).

(e) Luminosity is simply $L \cong 4\pi R_{\text{orbit}}^2 I \cong 3.86 \times 10^{26}$ W and we have 26.7 MeV $\cong 4.27 \times 10^{-12}$ J. So the number of reactions per second is roughly 9×10^{37} events/s by dividing these two quantities. Mass of the proton is 1.67×10^{-27} kg, so the sun contains roughly 1.2×10^{57} protons, so roughly 3×10^{-19} of the protons in the sun fuse each second (giving a lifetime of roughly 10^{10} years until 10% of the hydrogen is used up, after which the hydrogen-burning phase will be completed, compatible with the sun's age of a bit over 4 billion years). The discrepancy between this slow rate and the computation in (d) (with taking the unit time as $\mathcal{O}(1)$ seconds) of more than 10 orders of magnitude comes from the fact that they are suppressed by requiring weak nuclear interactions.