# THE QUANTUM ISING CHAIN FOR BEGINNERS

Sep 28, 2020

## 1 The Jordan-Wigner transformation

There are techniques to deal with large assemblies of bosons & fermions. But not with spin systems.

→ need a way to "map" the hard problem to easy!

→ Consider single spin-$\frac{1}{2}$ ⇒ 3 components of spin operator:

- $\sigma^x$, $\sigma^y$, $\sigma^z$. Hilbert space is $\{|\uparrow\rangle, |\downarrow\rangle\}$

- Eigenstates: $\begin{cases} \sigma^z|\uparrow\rangle = |\uparrow\rangle \\ \sigma^z|\downarrow\rangle = -|\downarrow\rangle \end{cases}$

- Commutation relation (from angular momentum $J^i \leftrightarrow \sigma^i$)

index

$$\boxed{[\sigma_j^i, \sigma_{j'}^{i'}] = 0} \quad , \quad \boxed{[\sigma_j^x, \sigma_j^y] = 2i\sigma_j^z}$$

site

cyclic

same site, obey usual comm. relation (can be written with $\varepsilon^{ijk}$ ...)

- Define $\sigma^{\pm} = \dfrac{\sigma_j^x \pm i\sigma_j^y}{2}$

gives $\boxed{\sigma^+|\downarrow\rangle = |\uparrow\rangle \quad , \quad \sigma^-|\uparrow\rangle = |\downarrow\rangle}$

& $\boxed{\{\sigma_j^+, \sigma_j^-\} = \mathbb{1}}$ → anticommutator, typical of rules for fermions

→ Should we describe spins w/ bosons or fermions?

→ Let's start w/ bosons... (hard...)

Suppose have single boson $\hat{b}^\dagger$ with associated vacuum state $|0\rangle$ s.t. $\hat{b}|0\rangle = |0\rangle$

then because $[\hat{b}, \hat{b}^\dagger] = 1$, can have

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{b}^\dagger)^n|0\rangle \qquad n = 0, 1, \ldots, \infty$$

→ Now, since we want only spin-$\frac{1}{2}$ ⇒ truncate Hilbert space so that $(\hat{b}^\dagger)^2|0\rangle = 0$ ⇒ get sth like Hilbert space of single spin-$\frac{1}{2}$.

(!) this kind of truncation is called "hard-core boson"

Now, how do we relate $\hat{b}^\dagger$, $\hat{b}$, to the Pauli matrices?

→ Observe that if we identify $\begin{cases} |0\rangle \leftrightarrow |\uparrow\rangle \\ |1\rangle \leftrightarrow |\downarrow\rangle \end{cases}$

then $\hat{b}^\dagger|0\rangle = |1\rangle \longleftrightarrow |\downarrow\rangle = \hat{\sigma}^-|\uparrow\rangle$

and so on...

so
$$\boxed{\begin{cases} \hat{\sigma}_j^+ \equiv \hat{b}_j \\ \hat{\sigma}_j^- \equiv \hat{b}_j^\dagger \\ \hat{\sigma}_j^z \equiv 1 - 2\hat{b}_j^\dagger\hat{b}_j \end{cases} \rightarrow \begin{cases} \hat{\sigma}_j^x = \hat{b}_j^\dagger + \hat{b}_j \\ \hat{\sigma}_j^y = i(\hat{b}_j^\dagger - \hat{b}_j) \\ \hat{\sigma}_j^z = 1 - 2\hat{b}_j^\dagger\hat{b}_j \end{cases}}$$

• Note that $[\hat{b}_j, \hat{b}_i] = 0 = [\hat{b}_j^\dagger, \hat{b}_i^\dagger]$ $(j \neq i)$

(like how $\vec{\sigma}$ commutes @ different sites)

→ But $\hat{b}_j^\dagger, \hat{b}_j$ are not ordinary bosonic operators.

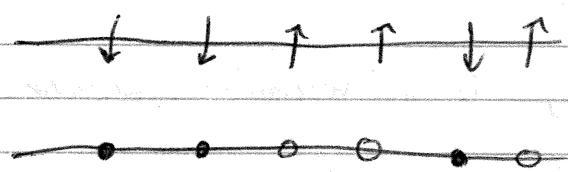
• Also ~~Note~~ that b/c of the truncation, $(\hat{b}_j^\dagger)^2 |0\rangle = 0$, and that

$$\{\hat{b}_j, \hat{b}_j^\dagger\} = \mathbb{1}.$$

$\Rightarrow$ $\boxed{\text{At most one boson is allowed at one site}}$

Now, if we pay close attention... the hard-core boson representation is calling out "spinless fermions" $\hat{c}_j^\dagger$

↳ Why? b/c the absence of double occupancy is actually enforced by the Pauli Exclusion Princip & that the anti-commutation rule comes for free!



h.c. bosons

spinless fermions

$$= \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_5^\dagger |0\rangle = \hat{c}_1^\dagger \hat{c}_2^\dagger \hat{c}_5^\dagger |0\rangle$$

There's a difficulty, however. The mapping $\hat{\sigma}_j^- \to \hat{b}_j^\dagger$ can be done in any dimension.

But writing $\hat{b}_j^\dagger$ in terms of $\hat{c}_j^\dagger$ is only useful in 1D.

b/c there's a natural ordering of the sites!

What is this mapping $\hat{b}_j \longrightarrow \hat{c}_j$ ?

$\rightarrow$ the Jordan - Wigner transformation!

# operator

$$\boxed{\hat{b}_j = \hat{K}_j \hat{c}_j = \hat{c}_j \hat{K}_j \quad \text{where} \quad \hat{K}_j = e^{i\pi \sum_{j'=1}^{j-1} \hat{n}_{j'}} = \prod_{j'=1}^{j-1} (1 - 2\hat{n}_{j'})}$$

where we've introduced $\hat{K}_j$ the "non-local" string operator

$\rightarrow$ now, $\hat{K}_j$ is just a sign! $\hat{K}_j = \pm 1$.

$\rightarrow$ Intuitively, $\boxed{\hat{K}_j \text{ counts the parity of \# of fermions } \underline{\text{before}} \text{ site } j.}$

Now, $\hat{K}_j = e^{i\pi \sum_{j'=1}^{j-1} \hat{n}_{j'}}$, b/c $\hat{K}_j = \pm 1$

$\rightarrow$ $\boxed{\hat{K}_j = \hat{K}_j^{+} = \hat{K}_j^{-1} \text{ , and } \hat{K}_j^{2} = 1.}$

Why? $\hat{n}_j = \hat{c}_j^{+} \hat{c}_j$ is a number operator.

$\rightarrow$ we will now show that if we take $c_j$ to be the fermionic operators with the anti-comm. rel.

$$\boxed{\{ \hat{c}_j , \hat{c}_{j'}^{+} \} = \delta_{j,j'} \quad \& \quad \{ \hat{c}_j , \hat{c}_{j'} \} = \{ \hat{c}_j^{+}, \hat{c}_{j'}^{+} \} = 0}$$

then the expected properties of the $\hat{b}_j$, $\hat{b}_j^{+}$ ! will follow...

<u>Same-site property:</u> (anti-commutation relation)

$$\{\hat{b}_j, \hat{b}_j^{\dagger}\} = 1$$

$$\{\hat{b}_j, \hat{b}_j\} = \{\hat{b}_j^{\dagger}, \hat{b}_j^{\dagger}\} = 0$$

<u>Different-site property:</u> (commutation relation)

$$[\hat{b}_j, \hat{b}_{j'}^{\dagger}] = 0$$

$$[\hat{b}_j, \hat{b}_{j'}] = 0 \qquad [\hat{b}_j^{\dagger}, \hat{b}_{j'}^{\dagger}] = 0$$

@ different sites, always commute.

i.e. that $\hat{b}_j^{(\dagger)}$'s are hard-core bosons.

→ To show the same-site property, just use the fact that

$$\hat{b}_j^{\dagger}\hat{b}_j = (\hat{c}_j^{\dagger}\underbrace{\hat{k}_j^{\dagger})(\hat{k}_j}_{\hat{k}_j^2}\hat{c}_j) = \hat{c}_j^{\dagger}(1)\hat{c}_j = \hat{c}_j^{\dagger}\hat{c}_j$$

⇒ $\hat{b}_j$'s follow the same anti-comm. relation as $\hat{c}_j$

Similarly — $\hat{b}_j\hat{b}_j^{\dagger} = \hat{c}_j\hat{c}_j^{\dagger}$.

Now, to show the different-site property ... ~~use the~~

↳ Consider $[\hat{b}_{j_1}, \hat{b}_{j_2}]$, assuming $j_2 > j_1$.

By the JW transform, we find that

$$\boxed{\hat{b}_{j_2}^\dagger \hat{b}_{j_1}} = \hat{c}_{j_2}^\dagger \hat{k}_{j_2}^\dagger \hat{k}_{j_1} \hat{c}_{j_1}$$

$$= \hat{c}_{j_2}^\dagger \left\{ e^{-i\pi \sum_{j'=1}^{j_2-1} \hat{n}_{j'}} \right\} \left\{ e^{i\pi \sum_{j'=1}^{j_1-1} \hat{n}_{j'}} \right\} \hat{c}_{j_1}$$

$$= \hat{c}_{j_2}^\dagger \, e^{-i\pi \sum_{j'=j_1}^{j_2-1} \hat{n}_{j'}} \hat{c}_{j_1}$$

$$= e^{-i\pi \sum_{j'=j_1}^{j_2-1} \hat{n}_{j'}} \hat{c}_{j_2}^\dagger \hat{c}_{j_1}$$

$K_{j_1} \leftrightarrow c_{j_2}$

$$= -\exp\left\{-i\pi \sum_{j'=j_1}^{j_2-1} \hat{n}_{j'}\right\} \hat{c}_{j_1} \hat{c}_{j_2}^\dagger$$

← by anti comm. relation

$$= \boxed{+\hat{c}_{j_1} \exp\left\{-i\pi \sum_{j'=j_1}^{j_2-1} \hat{n}_{j'}\right\} \hat{c}_{j_2}^\dagger}$$

where the last eq comes from the fact that

$$-\exp\left\{-i\pi \sum_{j'=j_1}^{j_2-1} \hat{n}_{j'}\right\} \hat{c}_{j_1} \hat{c}_{j_2}^\dagger$$

annihilates site $j_1 \Rightarrow \hat{n}_{j_1} = 0$

whereas

$$\hat{c}_{j_1} \exp\left\{-i\pi \sum_{j'=j_1}^{j_2-1} \hat{n}_{j'}\right\} \hat{c}_{j_2}^\dagger$$

has $\hat{n}_{j_1} = 1$ since there is no $\hat{c}_{j_1}$ present.

→ differ by $(-)$

Similarly, can show that

$$\boxed{\hat{b}_{j_1} \hat{b}_{j_2}^\dagger = \hat{c}_{j_1} \exp\left\{-i\pi \sum_{j'=j_1}^{j_2-1} \hat{n}_{j'}\right\} \hat{c}_{j_2}^\dagger}$$

With these, ... can check that $\boxed{\{\hat{b}_{j_1}, \hat{b}_{j_2}^{\dagger}\} = 0}$

→ all other different-site relations are proven similarly.

<u>Facts</u>

$$\prod_{j'=1}^{j-1} (1 - 2\hat{n}_{j'}) \prod_{j'=1}^{j} (1 - 2\hat{n}_{j'}) = 1 - 2\hat{n}_j$$

since $(1 - 2\hat{n}_{j'})^2 = 1$ & terms with different $j'$'s commute.

<u>note</u> true b/c $\hat{n}_j$ can only be 0 or 1.

With this relation, we get ...

- $\hat{b}_j^{\dagger} \hat{b}_j = \hat{c}_j^{\dagger} \hat{c}_j$
- $\hat{b}_j^{\dagger} \hat{b}_{j+1}^{\dagger} = \hat{c}_j^{\dagger} (1 - 2\hat{n}_j) c_{j+1}^{\dagger} = \hat{c}_j^{\dagger} \hat{c}_{j+1}^{\dagger}$
- $\hat{b}_j^{\dagger} \hat{b}_{j+1} = \hat{c}_j^{\dagger} (1 - 2\hat{n}_j) \hat{c}_{j+1} = \hat{c}_j^{\dagger} \hat{c}_{j+1}$
- $\hat{b}_j \hat{b}_{j+1} = \hat{c}_j (1 - 2\hat{n}_j) \hat{c}_{j+1} = \hat{c}_j [1 - 2(1 - \hat{c}_j \hat{c}_j^{\dagger})] \hat{c}_{j+1}$
  $= -\hat{c}_j \hat{c}_{j+1}$
- $\hat{b}_j \hat{b}_{j+1}^{\dagger} = \hat{c}_j (1 - 2\hat{n}_j) \hat{c}_{j+1} = \hat{c}_j [1 - 2(1 - \hat{c}_j \hat{c}_j^{\dagger})] \hat{c}_{j+1}^{\dagger}$
  $= -\hat{c}_j \hat{c}_{j+1}^{\dagger}$

To summarize, the JW transformation is given by

$$\begin{cases} \hat{\sigma}_j^x = K_j(\hat{c}_j^+ + \hat{c}_j) = \hat{b}_j^+ + \hat{b}_j \\ \hat{\sigma}_j^y = K_j i(\hat{c}_j^+ - \hat{c}_j) = i(\hat{b}_j^+ - \hat{b}_j) \\ \hat{\sigma}_j^z = 1 - 2\hat{n}_j = 1 - 2\hat{c}_j^+ c_j = 1 - 2b_j^+ b_j \end{cases}$$

where

$$\hat{K}_j = \prod_{j'=1}^{j-1} (1 - 2\hat{n}_{j'})$$

Under this map, spin operators become local ferm. op.

$$\hat{\sigma}_j^z = 1 - 2\hat{n}_j = (\hat{c}_j^+ + \hat{c}_j)(\hat{c}_j^+ - \hat{c}_j)$$

$$\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x = [\hat{c}_j^+ \hat{c}_{j+1}^+ + \hat{c}_j^+ \hat{c}_{j+1} + h.c.]$$

$$\hat{\sigma}_j^y \hat{\sigma}_{j+1}^y = -[\hat{c}_j^+ \hat{c}_{j+1}^+ - \hat{c}_j^+ \hat{c}_{j+1} + h.c.]$$

<u>Note</u> a longitudinal field term involving a single $\hat{\sigma}_j^x$ <u>cannot</u> be translated into a simple <u>local</u> fermionic operator.

Oct 6, 2020 <u>Note</u> Boundary conditions are very important.

Often assumes periodic boundary conditions → (PBC)

i.e. model is defined on a <u>ring geometry</u>

where we understand that $\hat{\sigma}_0^\alpha \equiv \hat{\sigma}_L^\alpha$, $\hat{\sigma}_{L+1}^\alpha = \hat{\sigma}_1^\alpha$

This implies the same PBC for hard-core bosons

Hence, e.g. $\hat{b}_L^\dagger \hat{b}_{L+1} \equiv \hat{b}_L^\dagger \hat{b}_1$.

Oct 7, 2020 → But things can go wrong for fermions when we look at

$$\hat{b}_L^\dagger \hat{b}_1 = e^{i\pi \sum_{j=1}^{L-1} \hat{n}_j} \hat{c}_L^\dagger \hat{c}_1$$

→ 1 due to $\hat{c}_L^\dagger$

$$= e^{i\pi \sum_{j=1}^{L-1} \hat{n}_j + \hat{n}_L} \cdot (-1)\, \hat{c}_L^\dagger \hat{c}_1$$

$$= -e^{i\pi \sum_{j=1}^{L} \hat{n}_j} \hat{c}_L^\dagger \hat{c}_1$$

$$\Rightarrow \boxed{\hat{b}_L^\dagger \hat{b}_1 = -e^{i\pi \hat{N}} \hat{c}_L^\dagger \hat{c}_1}$$

"fermion parity"

where $\hat{N} = \sum_{j=1}^{L} \hat{n}_j = \sum_{j=1}^{L} \hat{c}_j^\dagger \hat{c}_j = \sum_{j=1}^{L} \hat{b}_j^\dagger \hat{b}_j$

is the total # of particles.

Similarly, one finds that

$$\hat{b}_L^\dagger \hat{b}_1^\dagger = e^{i\pi \sum_{j=1}^{L-1} \hat{n}_j} \hat{c}_L^\dagger \hat{c}_1^\dagger$$

$$\Rightarrow \boxed{\hat{b}_L^\dagger \hat{b}_1^\dagger = -e^{i\pi \hat{N}} \hat{c}_L^\dagger \hat{c}_1^\dagger}$$

→ This shows that boundary conditions are affected by the <u>fermion parity</u>:

$$e^{i\pi \hat{N}} = (-1)^{\hat{N}}$$

(ABC)

In particular, PBC → anti-periodic when $\hat{N}$ is <u>even</u>

when PBC is open (OBC) ⇒ no problem.

TFIM

## (2) The transverse-field Ising model : fermionic formulation

**Info** There is a class of 1D spin systems in which a ~~fer~~-fermionic re-formulation can be useful.

→ most noteworthy is the XXZ chain (Heisenberg)

$$\hat{H}_{XXZ} = \sum_j \left( J_j^{\perp} (\hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y) + J_j^{zz} \hat{\sigma}_j^z \hat{\sigma}_{j+1}^z \right) - \sum_j h_j \hat{\sigma}_j^z$$

The corresponding fermionic formulation reads

$$\hat{H}_{XXZ} \to \sum_j \left( 2J_j^{\perp} \{ \hat{c}_j^{\dagger} \hat{c}_{j+1} + h.c. \} + J_j^{zz} (2\hat{n}_j - 1)(2\hat{n}_{j+1} - 1) \right) + \sum_j h_j (2\hat{n}_j - 1)$$

single-site

→ shows that the fermions interact at <u>nearest-neighbors</u>, due to the $J_j^{zz}$ term.

//

→ now look at 1D models where the fermionic Hamiltonian can be diagonalized exactly since it is quadratic in the fermions...

→ e.g. XY model & TFIM

After a rotation in spin space, can write the Hamiltonian ⟶ (allowing for non-uniform, possibly random couplings) as follows:

$$\boxed{\hat{H} = -\sum_{j=1}^{L} \left( J_j^x \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + J_j^y \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right) - \sum_{j=1}^{L} h_j \hat{\sigma}_j^z}$$

can be chosen to be iid from $\mathcal{U}(0,1)$

For system size $L<\infty$, with <u>OBC</u>, then the sum only actually runs from $1 \to L-1$, with $J_L^{x,y}=0$

But if we have PBC $\Rightarrow$ sum runs from $1 \to L$, and we assume that

$$\hat{\sigma}_{L+1}^\alpha = \hat{\sigma}_1^\alpha .$$

→ when $J_j^y = 0$, we have the TFIM:

$$\hat{H}_{IS} = -\sum_{j=1}^{L} J_j^x \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x - \sum_{j=1}^{L} h_j \hat{\sigma}_j^z .$$

→ when $J_j^y = J_j^x$, we have the isotropic XY model

$$\hat{H}_{XY\,iso} = -\sum_{j=1}^{L} J_j^{x,y} \left\{ \hat{\sigma}_j^x \hat{\sigma}_{j+1}^x + \hat{\sigma}_j^y \hat{\sigma}_{j+1}^y \right\} - \sum_{j=1}^{L} h_j \hat{\sigma}_j^z$$

Now, let's write $\hat{H}$ in terms of hard-core bosons.

$$\boxed{\hat{H} \to -\sum_{j=1}^{L} \left( J_j^+ \hat{b}_j^\dagger \hat{b}_{j+1} + J_j^- \hat{b}_j^\dagger \hat{b}_{j+1}^\dagger + \text{h.c.} \right) + \sum_{j=1}^{L} h_j (2\hat{n}_j - 1)}$$

where $J_j^\pm = J_j^x \pm J_j^y$

Next, we switch from hard core bosons to spinless fermions.

Now, since
$$\hat{b}_j^{+}\hat{b}_{j+1} = \hat{c}_j^{+}\hat{c}_{j+1}$$
$$\hat{b}_j^{+}\hat{b}_{j+1}^{+} = \hat{c}_j^{+}\hat{c}_{j+1}$$
$$\hat{b}_j^{+}\hat{b}_j = \hat{n}_j = \hat{n}_j = \hat{c}_j^{+}\hat{c}_j$$
none depends on the string operator $\hat{k}_j$

→ the Hamiltonian in the fermionic picture is identical.

<u>Remark:</u> { In the fermionic context the pair creation & annihilation terms are characteristic of the BCS theory of superconductivity }

⟹ <u>tricky point</u> is just the boundary conditions

→ If we use <u>OBC</u> → first sum runs over $1 \to L-1$, and there is no term involving the site $L+1$

$$\to \hat{H}_{OBC} = -\sum_{j=1}^{L-1}\left(J_j^{+}\hat{c}_j^{+}\hat{c}_{j+1} + J_j^{-}\hat{c}_j^{+}\hat{c}_{j+1}^{+} + h.c.\right) + \sum_{j=1}^{L} h_j\,(2\hat{n}_j - 1)$$

notice the range of the summations

If we use PBC, terms like $\hat{b}^{+}\hat{b}$ can close @ $L+1$:
$$\hat{b}_L^{+}\hat{b}_{L+1} \equiv \hat{b}_L^{+}\hat{b}_1 = -(-1)^{\hat{N}}\hat{c}_L^{+}\hat{c}_1$$
$$\hat{b}_L^{+}\hat{b}_{L+1}^{+} \equiv \hat{b}_L^{+}\hat{b}_1^{+} = -(-1)^{\hat{N}}\hat{c}_L^{+}\hat{c}_1^{+}$$
→ just as we showed before...

$$\Rightarrow \boxed{\hat{H}_{PBC} = \hat{H}_{OBC} + (-1)^{\hat{N}}\left\{J_L^{+}\hat{c}_L^{+}\hat{c}_1 + J_L^{-}\hat{c}_L^{+}\hat{c}_1^{+} + h.c.\right\}}$$

**Info** Notice that the number of fermions $\hat{N}$ is not conserved by the Hamiltonian in the PBC:

$$\hat{N} = \sum_{j=1}^{L} \hat{c}_j^{\dagger} \hat{c}_j \not\longleftrightarrow \hat{\mathcal{H}}_{PBC} \quad \text{(can check this)}$$

<u>But</u> the fermionic parity $(-1)^{\hat{N}} = e^{i\pi \hat{N}}$ is a constant of motion since

$$e^{i\pi\hat{N}} = \pm 1 \longleftrightarrow \hat{\mathcal{H}}_{PBC} \checkmark$$

→ From the fermionic perspective, it is as if we have anti-boundary condition (ABC)

<u>whence</u> $\{\ \hat{c}_{L+1} = -\hat{c}_1$ if $\hat{N}$ even

or <u>PBC</u> if $\{\ \hat{c}_{L+1} = \hat{c}_1$ if $\hat{N}$ odd

→ This symmetry can also be seen from the nearest-neighbor $\hat{\sigma}^x_j \hat{\sigma}^x_{j+1}$ & $\hat{\sigma}^y_j \hat{\sigma}^y_{j+1}$

→ these can only flip <u>pairs</u> of spins.

→ <u>Parity</u> of the overall magnetization along $z$ is <u>unchanged</u>.

$$\boxed{\hat{\mathcal{P}} = \prod_{j=1}^{L} \hat{\sigma}^z_j = \prod_{j=1}^{L} (1 - 2\hat{n}_j)}$$

→ parity operator.

→ <u>Hermitian</u>

Here is a $\mathbb{Z}_2$-symmetry which the system breaks in the ordered ferromagnetic phase

<u>Remark:</u> $\hat{\mathcal{P}}$ flips all the $\hat{\sigma}^x_j$ & $\hat{\sigma}^y_j$:

i.e. $\boxed{\hat{\mathcal{P}} \hat{\sigma}_j^{x,y} \hat{\mathcal{P}} = -\hat{\sigma}_j^{x,y}}$ (parity transform)

→ $\hat{\mathcal{P}} \hat{\mathcal{H}} \hat{\mathcal{P}} = \hat{\mathcal{H}}$.

⇒ So the Hamiltonian (in the spin pic) is invariant

Now, let us focus on diagonalizing the Hamiltonian:

→ Define projectors on the subspaces with even & odd # of parti

$$\boxed{\begin{aligned}\hat{P}_{even} &= \frac{1}{2}\left(\mathbb{1} + e^{i\pi\hat{N}}\right) = \hat{P}_0 \qquad (-1)^{\hat{N}} \\ &\& \\ \hat{P}_{odd} &= \frac{1}{2}\left(\mathbb{1} - e^{i\pi\hat{N}}\right) = \hat{P}_1 \qquad (-1)^{\hat{N}}\end{aligned}}$$

With these, can define 2 fermionic Ham's acting on the $2^{L-1}$-dim even/odd parity subspaces of the full Hilbert space:

$$\boxed{\hat{H}_0 = \hat{P}_0 \hat{H}_{PBC} \hat{P}_0 \quad \& \quad \hat{H}_1 = \hat{P}_1 \hat{H}_{PBC} \hat{P}_1}$$

so that

$$\hat{H}_{PBC} = \underbrace{\begin{pmatrix} \hat{H}_0 & 0 \\ 0 & \hat{H}_1 \end{pmatrix}}_{2^{L-1}} \rightarrow 2^{L-1}$$

Now, observe that if we write a $2^L$ fermionic Hamilt. of the form

$$\hat{\mathbb{H}}_{p=0,1} = -\sum_{j=1}^{L-1}\left(J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1} + h.c.\right)$$

$$+ (-1)^p \left(J_L^+ \hat{c}_L^\dagger \hat{c}_1^\dagger + J_L^- \hat{c}_L^\dagger \hat{c}_1 + h.c.\right)$$

$$+ \sum_{j=1}^{L} h_j \left(2\hat{n}_j - 1\right)$$

then we have that:

→ $\boxed{\text{When } p=1}$

$$\left\{ \hat{H}_{p=1} = -\sum_{j=1}^{L} \left( J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1} + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger \right) + \sum_{j=1}^{L} h_j (2\hat{n}_j - 1) \right\}$$
$$+h.c.$$

= a <u>legitimate PBC-fermionic Hamiltonian.</u>

$$(\hat{c}_{L+1} = \hat{c}_1)$$

→ $\boxed{\text{When } p=0}$

$$\left\{ \hat{H}_{p=0} = -\sum_{j=1}^{L} \left( J_j^+ \hat{c}_j^\dagger \hat{c}_{j+1} + J_j^- \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger \right) + \left( J_L^+ \hat{c}_L^\dagger \hat{c}_1 + J_L^- \hat{c}_L^\dagger \hat{c}_1^\dagger \right) \right.$$
$$+h.c. \qquad\qquad +h.c.$$
$$\left. + \sum_{j=1}^{L} h_j (2\hat{n}_j - 1) \right\}$$

= a ABC - fermionic Hamiltonian, where we identify $\hat{c}_{L+1} = -\hat{c}_1$.

→ But, since $p = 0, 1 \neq \hat{N}$ in general since $\hat{N} = \sum_{j=1}^{L} \hat{c}_j^\dagger \hat{c}_j$

→ $H_{p=0,1}$ are not exactly the PBC-spin Hamiltonian form.

However, it is true that

$$\left\{ \begin{aligned} &\hat{H}_0 = \hat{P}_0 \hat{H}_{PBC} \hat{P}_0 = \hat{P}_0 \hat{H}_0 \hat{P}_0 = \hat{H}_0 \hat{P}_0 \hat{P}_0 = \hat{H}_0 \hat{P}_0 . \\ \text{and } \quad &\hat{H}_1 = \hat{P}_1 H_{PBC} \hat{P}_1 = \hat{P}_1 \hat{H}_1 \hat{P}_1 = \hat{H}_1 \hat{P}_1 \hat{P}_1 = \hat{H}_1 \hat{P}_1 \end{aligned} \right\}$$

- sanity check:

$$\hat{P}_0 \hat{P}_0 = \frac{1}{4} \left( \mathbb{1} + e^{i\pi\hat{N}} \right) \left( 1 + e^{i\pi\hat{N}} \right) = \frac{1}{4} \left( \underbrace{\mathbb{1} + 2 e^{i\pi\hat{N}} + \cancel{e^{i\pi 2\hat{N}}}}_{2\cdot\mathbb{1}} \right)$$

$$= \frac{1}{2} \left( \mathbb{1} + e^{i\pi\hat{N}} \right) = \hat{P}_0$$

→ similarly for $\hat{P}_1$.

Further, show that $\vec{P}_0 \leftrightarrow \vec{\mathbb{H}}_0$:

This is easy b/c $\vec{\mathbb{H}}_{p=0,1}$ conserves fermionic parity ($e^{i\pi \vec{N}}$), just like $H_{PBC}$.

$\rightarrow \ \vec{\mathbb{H}}_{0,1} \Leftrightarrow \vec{P}_{0,1}$.

---

$\triangle!$ $\hat{H}_{0,1} \neq \hat{\mathbb{H}}_{0,1}$ - $\vec{H}_{0,1}$ acts on $2^{L-1}$-dim vector space,

$\hat{\mathbb{H}}_{0,1}$ acts on $2^{L}$-dim vector space.

$\vec{H}_{0,1}$ are blocks with $2^{L-1}$ eigenvalues.

$\vec{\mathbb{H}}_{0,1}$ lives in the full Hilbert space.

Note In the OBC case, since we don't have to worry about what happens when $j = L+1$,

there is no distinction between $\vec{\mathbb{H}}_0$ & $\vec{\mathbb{H}}_1$.

$\rightarrow \ \vec{\mathbb{H}}_0 = \vec{\mathbb{H}}_1 = \mathbb{H}$.

In the OBC case, can just set $\vec{H}_{OBC} = \vec{\mathbb{H}}$ & work with a single fermionic Hamiltonian.

---

## 3 The Uniform Ising model

To put the formalism into context, recall how the numerics show that some states of the ~~unif~~ (non) uniform Ising model can ~~have~~ obey $\Pi X = 1$ or $\Pi X = -1$,

↳ This is exactly what we mean by partitioning $\hat{H}$ into $\hat{H}_0$ & $\hat{H}_1$ where $\hat{H}_0$ ~~tests~~ ~~refers~~ states with $\Pi X = 1$ & $\hat{H}_1$ with $\Pi X = -1$.

→ In any case, let's look at the uniform case where

$J_j^x = J^x$ ; $J_j^y = J^y$ ; $h_j = h$ → uniform!

→ Customary to parameterize

$$\begin{cases} J^x = J(1+x)/2 \\ J^y = J(1+x)/2 \end{cases}$$

So that

$$J^{\pm} = J^x \pm J^y = J \text{ and } xJ \text{, respectively.}$$

With these, the Ham looks like ...

$$\boxed{\begin{aligned} \hat{H}_{OBC} = & -J \sum_{j=1}^{L-1} \left( \hat{c}_j^\dagger \hat{c}_{j+1} + x \hat{c}_j^\dagger \hat{c}_{j+1}^\dagger + h.c. \right) \\ & + h \sum_{j=1}^{L} \left( 2 \hat{c}_j^\dagger \hat{c}_j - 1 \right) \end{aligned}}$$

(remember that OBC is "nice")

and

$$\boxed{\hat{H}_{PBC} = \hat{H}_{OBC} + (-1)^{\hat{N}} J \left( \hat{c}_L^\dagger \hat{c}_1 + x \hat{c}_L^\dagger \hat{c}_1^\dagger + h.c. \right)}$$

Now, let us assume that $L$ is even. This is not a big restriction & is useful.

/!\ Should not confound $L$ even/odd with even/odd parity of states!

Recall that in the spin-PBC, if

$$\begin{cases} \hat{N} \text{ odd} \Rightarrow \hat{c}_{L+1} \equiv \hat{c}_1 , \\ \hat{N} \text{ even} \Rightarrow \hat{c}_{L+1} \equiv -\hat{c}_1 . \end{cases}$$

The Hamiltonian conserves $e^{i\pi\hat{N}}$ → need to consider ~~the~~ both cases where $\hat{N}$ = odd/even when diagonalizing $\hat{H}$.

→ need to introduce $\hat{H}_0$ & $\hat{H}_1$:

$$\boxed{\hat{H}_{p=0,1} = -J\sum_{j=1}^{L} \left( \hat{c}_j^{\dagger} \hat{c}_{j+1} + x\, \hat{c}_j^{\dagger} \hat{c}_{j+1}^{\dagger} + h.c. \right) + h \sum_{j=1}^{L} \left( 2\hat{n}_j - 1 \right)}$$

where $p=0$ goes with even parity

$p = 1$ " odd.

and ~~[illegible] that~~ we're assuming that

$$\boxed{\hat{c}_{L+1} \equiv (-1)^{p+1}\, \hat{c}_1}$$

compact way to write dependence on $p$

→ This $\hat{H}_{p=0,1}$ might look "wrong" but it isn't!

Now, to proceed, we will look at fermionic ops in "momentum" space:

$$\begin{cases} \hat{c}_k = \dfrac{e^{-i\phi}}{\sqrt{L}} \displaystyle\sum_{j=1}^{L} e^{-ikj}\, \hat{c}_j & \text{(FT)} \\ \hat{c}_j = \dfrac{e^{i\phi}}{\sqrt{L}} \displaystyle\sum_{k} e^{+ijk}\, \hat{c}_k & \text{(inv FT)} \end{cases}$$

- We'll use $e^{i\phi}$ to correct phase later... (only useful math term)
- $k$ depends on $p$!

$\rightarrow$ For $p=1$, $\hat{c}_{L+1} = \hat{c}_1$

$$\Rightarrow \hat{c}_{L+1} = \frac{e^{i\phi}}{\sqrt{L}} \sum_k e^{i(L+1)k}\, \hat{c}_k$$

$$\parallel$$

$$\hat{c}_1 = \frac{e^{i\phi}}{\sqrt{L}} \sum_k e^{ik}\, \hat{c}_k .$$

$$\rightarrow e^{ikL} = 1$$

standard PBC choices for $k$

$$\rightarrow \boxed{p = 1 \Rightarrow \mathcal{K}_{p=1} = \left\{ k = \frac{2\pi n}{L} \;;\; n = -\frac{L}{2}+1, \ldots, 0, \ldots \frac{L}{2} \right\}}$$

Similarly,

$$\boxed{p = 0 \Rightarrow \mathcal{K}_{p=0} = \left\{ k = \pm \frac{(2n-1)\pi}{L} \;;\; n = 1, \ldots, \frac{L}{2} \right\}}$$

Now, let us try to express $\hat{H}_{p=0,1}$ in terms of the $\hat{c}_k$'s

↳ $\hat{H}_{p=0,1}$ in "momentum" space...

$$\hat{H}_{p=0,1} = -J \sum_{j=1}^{L} \left( \hat{c}_j^{\dagger} \hat{c}_{j+1} + x \hat{c}_j^{\dagger} \hat{c}_{j+1}^{\dagger} + h.c. \right) + h \sum_{j=1}^{L} (2\hat{n}_j - 1)$$

where remember that $\hat{c}_{L+1} = (-1)^{p+1} \hat{c}_1$.

- With $\hat{c}_j$ written in $\hat{c}_k$'s, ~~[illegible]~~ we need some useful identities first before writing $\hat{H}_p$ in $k$:

(1) $$\frac{1}{L} \sum_{j=1}^{L} e^{-i(k-k')j} = \delta_{kk'}$$

(2) $$\sum_k 2\cos(k)\, \hat{c}_k^{\dagger} \hat{c}_k = \sum_k \cos(k) \left( \hat{c}_k^{\dagger} \hat{c}_k - \hat{c}_{-k} \hat{c}_{-k}^{\dagger} \right)$$

where we used the anti-comm relation and

(3) $$\sum_k \cos(k) = 0$$, and

(4) $$\sum_k (2\hat{c}_k^{\dagger} \hat{c}_k - 1) = \sum_k \left( \hat{c}_k^{\dagger} \hat{c}_k - \hat{c}_{-k} \hat{c}_{-k}^{\dagger} \right)$$

→ With these, we find that:

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$$\hat{H}_p = -J \sum_{j=1}^{L} \left( \hat{c}_j^{\dagger} \hat{c}_{j+1} + x \hat{c}_j^{\dagger} \hat{c}_{j+1}^{\dagger} + h.c. \right) + h \sum_{j=1}^{L} (2\hat{n}_j - 1)$$

where $\hat{c}_{L+1} = (-1)^{p+1} \hat{c}_1$

in terms of $\hat{c}_k, \hat{c}_k^{\dagger}$ →

$$= \boxed{ -J \sum_{k}^{K_p} \left[ 2\cos k\, \hat{c}_k^{\dagger} \hat{c}_k + x \left( \underline{e^{-2i\phi}}\, e^{ik} \hat{c}_k^{\dagger} \hat{c}_{-k}^{\dagger} + h.c. \right) \right] + h \sum_{k}^{K_p} (2\hat{c}_k^{\dagger} \hat{c}_k - 1) }$$

Let's verify this ...

$$\hat{c}_j^{\dagger} \hat{c}_{j+1} = \left( \frac{e^{-i\phi}}{\sqrt{L}} \sum_k e^{-ijk} \hat{c}_k^{\dagger} \right) \left( \frac{e^{i\phi}}{\sqrt{L}} \sum_{k'} e^{+i(j+1)k'} \hat{c}_{k'} \right)$$

$$= \frac{1}{L} \sum_{k,k'} e^{-i(kj - k'(j+1))} \hat{c}_k^{\dagger} \hat{c}_{k'}$$

$$= \frac{1}{L} \sum_{k,k'} e^{-i(k-k')j} e^{ik'} \hat{c}_k^{\dagger} \hat{c}_{k'}$$

so
$$\sum_{j=1}^{L} \hat{c}_j^{\dagger} \hat{c}_{j+1} = \frac{1}{L} \sum_{j=1}^{L} \sum_{k,k'} e^{-i(k-k')j} e^{ik'} \hat{c}_k^{\dagger} \hat{c}_{k'}$$

$$= \frac{1}{L} \sum_{k,k'} \sum_{j=1}^{L} e^{-i(k-k')j} e^{ik'} \hat{c}_k^{\dagger} \hat{c}_{k'}$$

now look at $k \in K_{p=0}$ & $K_{p=1}$ $\dashrightarrow$ $2\cos k$.
$$= \sum_{k,k'} \delta_{kk'} e^{ik'} \hat{c}_k^{\dagger} \hat{c}_{k'} \rightsquigarrow \text{not quite ..}$$

since $p = 0, 1$ gives different answers ..

Rather ...

$$\sum_{j=1}^{L} \hat{c}_j^{\dagger} \hat{c}_{j+1} = \sum_{j=1}^{L-1} \hat{c}_j^{\dagger} \hat{c}_{j+1} + (-1)^{p+1} \hat{c}_L^{\dagger} \hat{c}_1$$

$$= \frac{1}{L} \sum_{k,k'} \sum_{j=1}^{L-1} e^{-i(k-k')j} e^{ik'} \hat{c}_k^{\dagger} \hat{c}_k + \cancel{(-1)^{p+1}}$$

$$+ \frac{(-1)^{p+1}}{L} \sum_{k,k'} e^{-i(k-k')L} e^{ik'} \hat{c}_k^{\dagger} \hat{c}_{k'}$$

$$e^{-iLk} e^{iLk'} e^{ik'} \hat{c}_k^{\dagger} \hat{c}_{k'}$$

some trig identity comes in here,

Basically, when $k \in K_{p=0}$ or $K_{p=1}$

$$= \dots$$

$$= \sum_{k}^{K_p} 2\cos(k)\, \hat{c}_k^{\dagger} \hat{c}_{k'} \checkmark$$

2cos k works ✓

That's good enough of a check -- we'll just take the answer as it is ---

→ Now, notice that the coupling of $-k$ & $k$ in the anomalous pair-creation term:

$$\left[e^{-2i\phi} e^{ik} \hat{c}_k^{\dagger} \hat{c}_{-k}^{\dagger}\right]$$

with the exceptions for $p=1$ (PBC), $k=0, \pi$.

Recall that $K_{p=1} = \left\{ k = \frac{2\pi n}{L}, n = -\frac{L}{2}+1, \ldots, 0, \ldots \frac{L}{2} \right\}$

when $k = 0, \pi$, $\underline{e^{ik} = \pm 1}$ → no separate $-k$ partner

since h.c. cancels ---

⇒ Useful to manipulate the (normal) number-conserving terms to rewrite the Hamiltonian ...

We use the fact that

anti-comm relation

$$\sum_k 2\cos(k)\, \hat{c}_k^{\dagger} \hat{c}_k = \sum_k \cos(k)\, 2\hat{c}_k^{\dagger} \hat{c}_k$$

$$= \sum_k \cos(k) \left[\hat{c}_k^{\dagger} \hat{c}_k - \hat{c}_{-k} \hat{c}_{-k}^{\dagger}\right]$$

and by defn

to write the Ham as:

$$\boxed{\hat{H}_p = \sum_p^{K_p} \left\{ (h - J\cos k)(\hat{c}_k^{\dagger} \hat{c}_k - \hat{c}_{-k} \hat{c}_{-k}^{\dagger}) - \gamma J (e^{-2i\phi} e^{ik} \hat{c}_k^{\dagger} \hat{c}_{-k}^{\dagger} + h.c.) \right\}}$$

Might be a typo in the document.

$$= -2J\hat{n}_0 + 2J\hat{n}_\pi + h(2\hat{n}_0 - 1) + h(2\hat{n}_\pi - 1)$$

$$= -2J(\hat{n}_0 - \hat{n}_\pi) + 2h(\hat{n}_0 + \hat{n}_\pi - 1)$$

The remaining $p=1$ terms and all terms for $p=0$, come into pairs $(k,-k)$.

→ Define the positive $k$ values as follows...

leaving out $k = 0, \pi$

$$\boxed{\begin{aligned} \mathcal{K}^+_{p=1} &= \left\{ k = \frac{2n\pi}{L},\ n = 1, 2, \ldots, \frac{L}{2} - 1 \right\} \\ \mathcal{K}^+_{p=0} &= \left\{ k = \frac{(2n-1)\pi}{L},\ n = 1, 2, \ldots, \frac{L}{2} \right\} \end{aligned}}$$

With this, we can write the Hamiltonian as...

$$\boxed{\hat{H}_0 = \sum_{k \in K^+_{p=0}} \hat{H}_k \quad \& \quad \hat{H}_1 = \sum_{k \in K^+_{p=1}} \hat{H}_k + \hat{H}_{k=0,\pi}}$$

OBC PBC

where

$$\hat{H}_k = 2(h - J\cos k)(\hat{c}^\dagger_k \hat{c}_k - \hat{c}^\dagger_{-k}\hat{c}_{-k}) - 2 \times J(\sin k)(ie^{-2i\phi}\hat{c}^\dagger_k \hat{c}^\dagger_{-k} - ie^{2i\phi}\hat{c}_{-k}\hat{c}_k)$$

each is $4\times4$.

factor of 2 due to the $(-k,k)$ symmetry...

and

$$\hat{H}_{k=0,\pi} = -2J(\hat{n}_0 - \hat{n}_\pi) + 2h(\hat{n}_0 + \hat{n}_\pi - 1)$$

this is a "$4\times4$" matrix

where we have used the fact that

$$\sum_k (2\hat{c}_k^\dagger \hat{c}_k - 1) = \sum_k (\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k}^{*} \hat{c}_{-k}^\dagger)$$

to manipulate the transverse field term & write

$$\hat{H}_p = -J\sum_k^{K_p} (2\cos k\, \hat{c}_k^\dagger \hat{c}_k) + h\sum_k^{K_p} (2\hat{c}_k^\dagger \hat{c}_k - 1) - J\sum_k^{K_p} \chi\left(e^{-2i\phi} e^{ik} \hat{c}_k^\dagger \hat{c}_{-k}^\dagger + h.c.\right).$$

$$= \sum_k^{K_p} \left\{ (h - J\cos k)(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k}^{*} \hat{c}_{-k}^\dagger) - \chi J\left(e^{-2i\phi} e^{ik} \hat{c}_k^\dagger \hat{c}_{-k}^\dagger + h.c.\right) \right\}.$$

now, notice that when $p = 1$ & $k = 0$ or $\pi$, we can take out the two terms with these indices and write

$$\hat{H}_{k=0,\pi} = -2J(\underbrace{\hat{n}_0}_{\hat{c}_0^\dagger \hat{c}_0} - \underbrace{\hat{n}_\pi}_{\hat{c}_\pi^\dagger \hat{c}_\pi}) + 2h(\underbrace{\hat{n}_0}_{\hat{c}_0^\dagger \hat{c}_0} + \underbrace{\hat{n}_\pi}_{\hat{c}_\pi^\dagger \hat{c}_\pi} - 1)$$

where does this come from? well... look at terms with $p = 1$, $k = 0, \pi$:

$$\cancel{(-J+h)(\hat{c}_0^\dagger \hat{c}_0 - \hat{c}_{-0}^\dagger \hat{c}_{-0}) + (h+J)(\hat{c}_\pi^\dagger \hat{c}_\pi - \hat{c}_{-\pi}^\dagger \hat{c}_{-\pi})}$$

$$\hat{H} \quad -J(2\hat{c}_0^\dagger \hat{c}_0) - J(-2)(\hat{c}_\pi^\dagger \hat{c}_\pi) + h(2\hat{c}_0^\dagger \hat{c}_0 - 1) + h(2\hat{c}_\pi^\dagger \hat{c}_\pi - 1)$$

$$-J\chi\{\underbrace{e^{-i\phi} \hat{c}_0^\dagger \hat{c}_0^\dagger + h.c.\}}_{0} - J\chi\{\underbrace{e^{-i\phi} \hat{c}_\pi^\dagger \hat{c}_\pi^\dagger (-1) + h.c.\}}_{0}$$

Now, a closer look at the Hamiltonian $\hat{H}_k$:

$$\hat{H}_k = 2(h - J\cos k)\left(\hat{c}_k^\dagger \hat{c}_k - \hat{c}_{-k}\hat{c}_{-k}^\dagger\right)$$
$$- 2i\chi J(\sin k)\left\{ e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger - e^{2i\phi}\hat{c}_{-k}\hat{c}_k \right\}$$

Look at the collection (basis): $\boxed{\left\{ \hat{c}_k^\dagger \hat{c}_{-k}^\dagger |0\rangle, |0\rangle, \hat{c}_k^\dagger|0\rangle, \hat{c}_{-k}^\dagger|0\rangle \right\}}$

(1) $$\hat{H}_k \underbrace{\hat{c}_k^\dagger \hat{c}_{-k}^\dagger |0\rangle} = 2(h - J\cos k)\left(\hat{c}_k^\dagger \hat{c}_k - \cancel{\hat{c}_{-k}\hat{c}_{-k}^\dagger}\right)\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle$$
$$- J2i\chi\sin k\left\{ \cancel{e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger} - e^{+2i\phi}\hat{c}_{-k}\hat{c}_k\right\}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle$$
$$= 2(h - J\cos k)\underbrace{\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle} + 2i\chi J(\sin k)e^{i2\phi}|0\rangle$$

(2) $$\hat{H}_k|0\rangle = 2(h - J\cos k)\left(\cancel{\hat{c}_k^\dagger \hat{c}_k} - \hat{c}_{-k}\hat{c}_{-k}^\dagger\right)|0\rangle$$
$$- J2i\chi(\sin k)\left\{ e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger - \cancel{e^{2i\phi}\hat{c}_{-k}\hat{c}_k}\right\}|0\rangle$$
$$= -2(h - J\cos k)\underbrace{\cancel{\hat{c}_{-k}\hat{c}_{-k}^\dagger}}_{1}|0\rangle - 2i\chi J\sin k\, e^{-2i\phi}\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle$$

(3) $\hat{H}_k \hat{c}_k^\dagger|0\rangle = \cancel{2(h - J\cos k)}\ 0$

(4) $\hat{H}_k \hat{c}_{-k}^\dagger|0\rangle = 0$

So we see that in the subspace $\{ \underset{A}{\hat{c}_k^\dagger \hat{c}_{-k}^\dagger|0\rangle}, \underset{B}{|0\rangle}, \underset{C}{\hat{c}_k^\dagger|0\rangle}, \underset{D}{\hat{c}_{-k}^\dagger|0\rangle} \}$,

$$\hat{H}_k = \begin{array}{c|c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \left(\begin{array}{cc|cc} 2(h - J\cos k) & -2i\chi J\sin k\, e^{-2i\phi} & 0 & 0 \\ 2i\chi J\sin k\, e^{2i\phi} & -2(h - J\cos k) & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

## Check of dimensions

- Recall that both $\hat{H}_{p=0,1}$ have $2^L$ eigenvalues.

Since there are $\frac{L}{2}$ such terms for $H_k$, we get a dimension of

$$\left(\tfrac{L}{2}\right)\ \underbrace{4}_{\substack{(4\times 4) \\ H_k}}{}^{\overbrace{L/2}^{(L/2)\ H_k\text{'s}}} = 2^L \checkmark$$

↳ each $H_k$ has 4 eigs.

- notice that $\hat{H}_{k=0,\pi}$ also works in a 4-dim subspace:

$$\{ |0\rangle,\ \hat{c}_0^{\dagger}\hat{c}_{\pi}^{\dagger}|0\rangle,\ \hat{c}_0^{\dagger}|0\rangle,\ \hat{c}_{\pi}^{\dagger}|0\rangle \}.$$

and there are $\frac{L}{2}-1$ wave vectors in $K_1^+$

$\Rightarrow$ again a total dimension for $\hat{H}_1$ of

$$\underbrace{4^{L/2-1}}_{\substack{\left(\frac{L}{2}-1\right) \text{ of } H_k \\ \text{where } k\in K^+_{p=1}}} \cdot \underbrace{4}_{\substack{\text{just} \\ \hat{H}_{k=0,\pi}}} = 2^L \quad \checkmark$$

- Finally, recall that the correct eigs are obtained from the block Hamiltonians $\hat{H}_{p=0,1} = P\hat{H}P$ which have $2^{L-1}$ eigs, those with even $(p=0)$ & odd $(p=1)$ fermion parity.

→ Now, we want to further simplify things...

Look at

$$H_k = \begin{pmatrix} 2(h - J\cos k) & -2i\chi J \sin k\, e^{-2i\phi} & 0 & 0 \\ 2i\chi J \sin k\, e^{2i\phi} & -2h(h - J\cos h) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}$$

in the basis $\{ c_k^{\dagger} c_{-k}^{\dagger}|0\rangle, |0\rangle, c_k^{\dagger}|0\rangle, c_{-k}^{\dagger}|0\rangle \}$.

→ We want to isolate just the nontrivial block of this matrix, so that

$$\boxed{H_k = \begin{pmatrix} \mathcal{H}_k & 0 \\ 0 & 0 \end{pmatrix}_{4\times 4} \quad \text{where } \mathcal{H}_k \text{ is } 2\times 2.}$$

⇒ Need a Bogoliubov transformation.

To this end, define a 2-component spinor:

$$\boxed{\Psi_k = \begin{pmatrix} c_k \\ c_{-k}^{\dagger} \end{pmatrix} \quad \& \quad \Psi_k^{\dagger} = \left( c_k^{\dagger}, c_{-k} \right)}$$

with the anti-commutation relation...

$$\boxed{\{ \Psi_{k,\alpha}, \Psi_{k',\alpha'}^{\dagger} \} = \delta_{\alpha,\alpha'} \delta_{k,k'}}$$

($\alpha$ denotes the component of $\Psi_k$ : $\alpha = 1, 2$)

With this, can rewrite $H_k$ $(4\times 4)$ as

$$\hat{H}_k = \sum_{\alpha,\alpha'} \underbrace{\hat{\Psi}^\dagger_{k\alpha}}_{\substack{4\times 1 \\ \text{or } (2)\times 2}} \underbrace{(\mathcal{H}_k)_{\alpha\alpha'}}_{(2\times 2) \text{ in } (2\times 2)} \hat{\Psi}_{k\alpha'}$$

$$= \left(\hat{c}^\dagger_k,\ \hat{c}_{-k}\right) \underbrace{\begin{pmatrix} 2(h - J\cos k) & -2i\chi J \sin k\, e^{-2i\phi} \\ 2i\chi J \sin k\, e^{2i\phi} & -2(h - J\cos k) \end{pmatrix}}_{(\mathcal{H}_k)_{2\times 2}} \begin{pmatrix} \hat{c}_k \\ \hat{c}^\dagger_{-k} \end{pmatrix}$$

So
$$\boxed{\hat{H}_k = \sum_{\alpha,\alpha'} \hat{\Psi}_{k\alpha} (\mathcal{H}_k)_{\alpha\alpha'} \hat{\Psi}_{k\alpha'}}$$

$$\left(\boxed{2},\ \boxed{2}\right) \begin{pmatrix} \boxed{2} & \boxed{2} \\ \boxed{2} & \boxed{2} \end{pmatrix} \begin{pmatrix} \boxed{2} \\ \boxed{2} \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{2} & 0 \\ 0 & 0 \end{pmatrix}$$

Now, look at this new matrix

$$\boxed{\mathcal{H}_k = \begin{pmatrix} 2(h - J\cos k) & -2i\chi J \sin k\, e^{-2i\phi} \\ 2i\chi J \sin k\, e^{2i\phi} & -2(h - J\cos k) \end{pmatrix}}$$

This can be expressed in terms of new pseudo-spin Pauli matrices $\hat{\tau}^{x,y,z}$ as

$$\boxed{\mathcal{H}_k = R_k \cdot \vec{\tau}}$$

where

$$R_k = 2\left(-\chi J \sin 2\phi \sin k,\ \chi J \cos 2\phi \sin k,\ (h - J\cos k)\right)^T$$

"effective magnetic field".

Let's verify this ...

$$R_k \cdot \vec{\tau} = 2\begin{pmatrix} -\chi J \sin 2\phi \sin k \\ \chi J \cos 2\phi \sin k \\ h - J\cos k \end{pmatrix} \cdot \hat{\tau} \qquad \begin{pmatrix} \hat{\sigma}^x \\ \hat{\sigma}^y \\ \hat{\sigma}^z \end{pmatrix}$$

$$= 2(-\chi J \sin 2\phi \sin k)\hat{\sigma}^x + 2\chi J \cos 2\phi \sin k\, \hat{\sigma}^y + (h - J\cos k)\hat{\sigma}^z$$

$$= 2\begin{pmatrix} 0 & -\chi J \sin 2\phi \sin k \\ -\chi J \sin 2\phi \sin k & 0 \end{pmatrix} + 2\begin{pmatrix} h - J\cos k & 0 \\ 0 & -h + J\cos k \end{pmatrix}$$

$$+ 2\begin{pmatrix} 0 & -i\chi J \cos 2\phi \sin k \\ +i\chi J \cos 2\phi \sin k & 0 \end{pmatrix}$$

$$= 2\begin{pmatrix} h - J\cos k & -i\chi J \sin k\, e^{-2i\phi} \\ +i\chi J \sin k\, e^{+2i\phi} & -h + J\cos k \end{pmatrix}$$

✓

**Info** | We can now see the role of the arbitrary phase ... $\phi$

For $\phi = 0$, $\mathbf{R}_k$ lives in the $y$-$z$ plane

$\phi = \pi/4$, $\mathbf{R}_k$ lives in the $x$-$z$ plane

$\Rightarrow \mathcal{H}_k$ is real.

Now, one can diagonalize $\mathcal{H}_k$ and find the eigs:

$$\boxed{\varepsilon_{k\pm} = \pm \varepsilon_k} \quad \text{with}$$

$$\varepsilon_k = |R_k| = 2J\sqrt{\left(\cos k - \frac{h}{J}\right)^2 + \chi^2 \sin^2 k}$$

<u>Now, fix $\phi = 0$.</u> So that

$$\mathbf{R}_k = (0,\ 2\chi J \sin k,\ h - J\cos k)^T \equiv (0,\ y_k,\ z_k)^T$$

$\rightarrow$ For the positive energy eigenvectors ... we have ...

$$\mathcal{H}_k \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} 0 \\ y_k \\ z_k \end{pmatrix} \cdot \begin{pmatrix} \hat{\sigma}^x \\ \hat{\sigma}^y \\ \hat{\sigma}^z \end{pmatrix} = y_k \hat{\sigma}^y + z_k \hat{\sigma}^z = \varepsilon_k \begin{pmatrix} u_k \\ v_k \end{pmatrix}$$

$$\rightarrow \boxed{\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{\sqrt{2\varepsilon_k(\varepsilon_k + z_k)}} \begin{pmatrix} \varepsilon_k + z_k \\ i y_k \end{pmatrix}}$$

more explicitly ...

$$\begin{cases} z_k u_k - i y_k v_k = \varepsilon_k u_k \\ i y_k u_k - z_k v_k = \varepsilon_k v_k \end{cases}$$

gives

eigv 1
$$\begin{pmatrix} u_{k+} \\ v_{k+} \end{pmatrix} \equiv \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \frac{1}{\sqrt{2\varepsilon_k(\varepsilon_k + z_k)}} \begin{pmatrix} \varepsilon_k + z_k \\ i y_k \end{pmatrix}$$

For the negative energy eigenvectors ... $\varepsilon_{k-} = -\varepsilon_k$, we have

$$\begin{cases} z_k(-v_k^*) - i y_k u_k^* = -\varepsilon_k(-v_k^*) = \varepsilon_{k-}(-v_k^*) \\ i y_k(-v_k^*) - z_k u_k^* = -\varepsilon_k(u_k^*) = \varepsilon_{k-}(u_k^*) \end{cases}$$

to get

eigv 2
$$\begin{pmatrix} u_{k-} \\ v_{k-} \end{pmatrix} \equiv \begin{pmatrix} -v_k^* \\ u_k^* \end{pmatrix} = \frac{1}{\sqrt{2\varepsilon_k(\varepsilon_k + z_k)}} \begin{pmatrix} i y_k \\ \varepsilon_k + z_k \end{pmatrix}$$

So, the unitary $U_k$ that diagonalizes $\mathcal{H}_k$ is

$$U_k = \begin{pmatrix} u_k & -v_k^* \\ v_k & u_k^* \end{pmatrix}$$

$$\hookrightarrow \boxed{U_k^\dagger \mathcal{H}_k U_k = \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix}}$$

Define a new fermion two-component operator:

$$\boxed{\hat{\Phi}_k = \mathcal{U}_k^\dagger \hat{\Psi}_k = \begin{pmatrix} u_k^* & +v_k^* \\ -v_k & u_k^* \end{pmatrix}\begin{pmatrix} \hat{c}_k \\ \hat{c}_{-k}^\dagger \end{pmatrix} \equiv \begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}_{-k}^\dagger \end{pmatrix}}$$

where

$$\left\{ \begin{aligned} \hat{\gamma}_k &= u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^\dagger \\ \hat{\gamma}_{-k}^\dagger &= -v_k \hat{c}_k + u_k^* \hat{c}_{-k}^\dagger \end{aligned} \right\}$$

~~Verify~~

~~Claim~~ $\cancel{\boxed{\hat{\gamma}_k \text{ is indeed a fermion}}}$

~~Proof~~ $\cancel{\{\hat{\gamma}_k, \hat{\gamma}_k^\dagger\} = \{u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^{*}, v_k \hat{c}_k + u_k \hat{c}_{-k}^\dagger\}}$

$\cancel{= |u_k|^2 \{\hat{c}_k, \hat{c}_{+k}^\dagger\} + |v_k|^2 \{\hat{c}_{-k}^\dagger, \hat{c}_{-k}\}}$

Why can we define $\hat{\gamma}_k^\dagger$ this way?

Say $\hat{\gamma}_k = u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^\dagger$

$\rightarrow \hat{\gamma}_k^\dagger = u_k \hat{c}_k^\dagger + v_k \hat{c}_{-k}$

$\rightarrow \hat{\gamma}_{-k}^\dagger = u_{-k} \hat{c}_{-k}^\dagger + v_{-k} \hat{c}_{+k}$

But $u_{-k} = u_k$ & $v_{-k} = -v_k$ (eigvalues)

$\Rightarrow \hat{\gamma}_{-k}^\dagger = u_k \hat{c}_{-k}^\dagger - v_k \hat{c}_k$ ✓

So the defn above makes sense!

Claim $\boxed{\hat{\gamma}_k \text{ is a fermion}}$

Pf
$$\{\hat{\gamma}_k, \hat{\gamma}_k^\dagger\} = \{u_k^* \hat{c}_k + v_k^* \hat{c}_{-k}^\dagger, \; u_k \hat{c}_k^\dagger + v_k \hat{c}_{-k}\}$$
$$= |u_k|^2 \underbrace{\{\hat{c}_k, \hat{c}_k^\dagger\}}_{1} + |v_k|^2 \underbrace{\{\hat{c}_{-k}^\dagger, \hat{c}_{-k}\}}_{1}$$
$$= |u_k|^2 + |v_k|^2 = 1.$$
$\square$.

With this new definition, we can write $\hat{H}_k$ (4x4) as

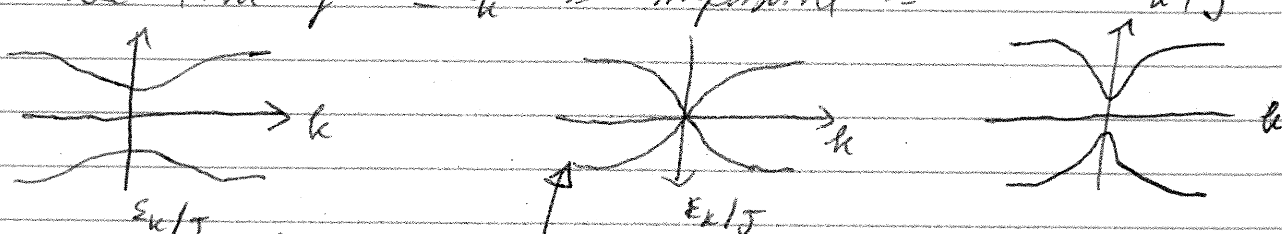
$$\hat{H}_k = \hat{\Psi}_k^\dagger \mathcal{H}_k \hat{\Psi}_k$$
$$= \underbrace{\hat{\Psi}_k^\dagger \mathcal{U}_k}\, \underbrace{\mathcal{U}_k^\dagger \mathcal{H}_k \mathcal{U}_k}\, \mathcal{U}_k^\dagger \hat{\Psi}_k$$
$$= \hat{\Phi}_k^\dagger \begin{pmatrix} \varepsilon_k & 0 \\ 0 & -\varepsilon_k \end{pmatrix} \hat{\Phi}_k = \varepsilon_k \hat{\Phi}_k^\dagger \hat{\sigma}^z \hat{\Phi}_k$$

$(\hat{\gamma}_k^\dagger, \hat{\gamma}_{-k})$ $\quad \begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}_{-k}^\dagger \end{pmatrix}$

$$= \varepsilon_k \left( \hat{\gamma}_k^\dagger \hat{\gamma}_k - \hat{\gamma}_{-k} \hat{\gamma}_{-k}^\dagger \right)$$
$$= \varepsilon_k \left( \hat{\gamma}_k^\dagger \hat{\gamma}_k + \hat{\gamma}_{-k}^\dagger \hat{\gamma}_{-k} - 1 \right)$$
anti-comm relation.

$$\Rightarrow \boxed{\hat{H}_k = \varepsilon_k \left( \hat{\gamma}_k^\dagger \hat{\gamma}_k + \hat{\gamma}_{-k}^\dagger \hat{\gamma}_{-k} - 1 \right)}$$

The form of $\pm \varepsilon_k$ is important ...




(ferromagnetic) $\quad$ (critical) $\quad$ (paramagnetic)

GAPLESS

- Critical point  Gapless linear spectrum.
- Ferro vs Paramagnetic : indistinguishable, But topology is distinctly different
  → we'll see this later.

3.1. Ground state & Excited states of the Ising model