Problem Set 5

Due: Friday 5pm, Mar 11, via Canvas upload or in envelope outside 26-255
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1 Magnetic field of a magnetic dipole (5 pts.)

In lecture I showed that the magnetic field of an infinitesimal current loop (infinitesimal radius r) with fixed magnetic moment contains a δ-function piece: The field at the center is $B \propto I/r \propto \mu/r^3$ and the integral of the field over the volume of the current loop $\int d^3r\, \boldsymbol{B} \propto \boldsymbol{\mu}$ was a constant just given by the dipole moment. So $\boldsymbol{B} \propto \boldsymbol{\mu}\, \delta(\boldsymbol{r})$. A sphere of uniform magnetization also has a constant magnetic field in the center, as it has surface currents, can thus be modeled like many current loops stacked together (a rotating shell of charge). The constant magnetic field inside again gives the δ-function piece upon shrinking to an infinitesimal sphere.

Are those just some special cases? No. We will prove the δ-function piece as a generic portion of the magnetic field created by a magnetic dipole.

It is a mathematical identity. Essentially every E&M equation involving δ-functions can be derived from it:

$$\boxed{\partial_i \partial_j \left(\frac{1}{r}\right) = -\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i \left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\, \delta_{ij}\, \delta^3(\boldsymbol{r})} \tag{1}$$

To make sure the definitions are clear, $\boldsymbol{r} \equiv x_i \hat{\boldsymbol{e}}_i \equiv x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z$, $r \equiv |\boldsymbol{r}|$, $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$, $\hat{r}_i \equiv x_i/r$, and $\partial_i \equiv \partial/\partial x_i$.

a) Show that the famous identity

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(\boldsymbol{r})$$

follows immediately from Eq. 1.

b) Using the identity Eq. 1, show that the vector potential for a magnetic dipole,

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{r}) = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2}\ ,$$

can be used to find immediately that the magnetic field of a magnetic dipole is given by

$$\boldsymbol{B}_{\text{dip}}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^3} + \frac{8\pi}{3}\boldsymbol{m}\,\delta^3(\boldsymbol{r}).$$

This therefore contains the δ-piece, all important for the 21 cm line of hydrogen.

c) Let's now prove the identity. First, show that it is valid for $r \neq 0$. Then, verify the δ-function piece by integrating over a small sphere of radius ϵ about the origin. You will need to use the identity:

$$\int_V \boldsymbol{\nabla}\psi \, \mathrm{d}^3x = \int_S \psi \, \mathrm{d}\boldsymbol{a}$$

where S is the surface bounding the volume V, and $\mathrm{d}\boldsymbol{a}$ the vector denoting the infinitesimal surface element $\mathrm{d}\boldsymbol{a} = \mathrm{d}a\,\hat{\boldsymbol{n}}$, where $\mathrm{d}a$ is the infinitesimal area of the surface element and $\hat{\boldsymbol{n}}$ is the outward normal vector. This allows to convert

$$\int_{r<\epsilon} \partial_i \left(\frac{\hat{r}_j}{r^2}\right) \mathrm{d}^3x$$

into a surface integral on a sphere of radius ϵ (so here $\psi = \frac{\hat{r}_j}{r^2}$). Evaluate the surface integral to show that the δ-function term in Eq. 1 is correct.

Note: You may be concerned that the integral in Eq. 1 is ill-defined, since $\hat{r}_j/r^2$ is not differentiable at $r = 0$. Mathematicians give a rigorous meaning to the equation Eq. 1 by defining both sides as distributions, which means that they are defined by the result of multiplying them by an arbitrary test function $\phi(\boldsymbol{r})$ (assumed smooth and falling off rapidly at $r \to \infty$) and then integrating both sides over all space. The derivative $\partial_i \left(\frac{\hat{r}_j}{r^2}\right)$ is then given an unambigious meaning by defining the derivative of a distribution by integration by parts: all the derivatives are applied to the test function $\phi(\boldsymbol{r})$. So $\int \phi(\boldsymbol{r})\partial_i \left(\frac{\hat{r}_j}{r^2}\right) \mathrm{d}^3x = -\int \partial_i\phi(\boldsymbol{r}) \left(\frac{\hat{r}_j}{r^2}\right) \mathrm{d}^3x$, and this is well-defined as the measure $\mathrm{d}^3x = r^2\mathrm{d}r \sin\theta \mathrm{d}\theta \mathrm{d}\phi$ cancels the r^2 in the denominator.

2 Atoms in magnetic fields: the Breit-Rabi formula (10 pts.)

The Hamiltonian for an atom in a magnetic field along $\hat{z}$ may be written

$$H = ah\,\boldsymbol{I}\cdot\boldsymbol{J} + (g_J\mu_B m_J - g_I\mu_B m_I)\,B_z$$

a) Restrict attention to the case $J = 1/2$, but arbitrary I. Show that the energies of states are given by the Breit-Rabi formula

$$E_m^{\pm} = -\frac{ah}{4} - m g_I \mu_B B_z \pm \frac{ahF^+}{2}\sqrt{1 + \frac{2mx}{F^+} + x^2}$$

The parameter x is given by $x = (g_I + g_J)\,\mu_B B_z/(ahF^+)$, where $F^+ = I + 1/2$. m is the z component of the total angular momentum (remember: $m = m_F = m_J + m_I$ is always a good quantum number, at any magnetic field).

b) For the case of $I = 3/2$, make a clear sketch of energies vs x. Take advantage of the non-crossing rule (levels of the same m do not cross). Be sure to extend your figure to very high field ($1/x \ll g_I/g_J$). Label the lines with quantum numbers at low and high fields and indicate m. You may use the values of ^{87}Rb, which has $I = 3/2$, $g_I = 0.0009954$, and $g_J = 2.002331$.

c) There are some values of magnetic field x where the resonance frequency ΔE for the (magnetic) dipole transition (selection rules $\Delta m = 0, \pm 1$) is first-order field independent (i.e., the leading order term in ΔE is of $O((\Delta x)^2)$). Show those values for magnetic field and corresponding transitions on your figure from part b).

d) Which of the transitions you just found connect states that can be confined in a magnetic trap, i.e. a field configuration with a local minimum in the magnitude of the magnetic field? Note that Maxwell's equations forbid a static local maximum of the magnetic field in free space.

e) The first-order insensitive transition(s) between trappable states you have just found occur(s) for states with a relatively weak dipole moment (dependence of state energy on magnetic field) and at large offset fields. This makes for weak magnetic traps: it is hard to generate large field gradients (needed for tight trapping, particularly of weak dipoles) at a large offset field. The transition $|F = 1, m_F = -1\rangle \leftrightarrow |F = 2, m_F = 1\rangle$ in ^{87}Rb is not an allowed magnetic dipole transition ($\Delta m = 2$), but it can be driven as a two-photon transition using one of the $m = 0$ sublevels as a virtual intermediate state. Like the examples you have looked at in the preceding questions, this transition is first-order insensitive to field fluctuations at some properly-chosen offset field. However, in this case the correct offset field is small, and the two states have large ($\mu_B/2$) magnetic dipole moments. This makes it valuable for precision spectroscopy of trapped atomic samples and for efforts to make compact neutral-atom clocks. For examples, see G.K. Campbell *et al.*, Science *313*:649–652 (2006), D.S. Hall *et al.*, PRL **81**:1543 (1998), D.M. Harber *et al.*, PRA **66**:053616 (2002). Calculate the magnetic field for which the two-photon transition is first-order field-insensitive. In ^{87}Rb, $I = 3/2$, $2a = 6.835\,\text{GHz}$, $g_I = 0.0009954$, and $g_J = 2.002331$ in the ground ($5^2S_{1/2}$) state.

f) ^{87}Rb atoms magnetically trapped in the $|F = 1, m = -1\rangle$ state at $1\,\mu$K temperature are distributed over a magnetic field range of about 30 mG. Calculate the inhomogeneous width of the $|F = 1, m = -1\rangle \leftrightarrow |F = 2, m = 1\rangle$ transition at zero magnetic field and at the magnetic field found in the previous question.

3 Atomic g factors (5 pts.)

Find g factors for the following states of Na ($I = 3/2$):

$$\begin{array}{lr} {}^2P_{1/2} & F = 1, 2 \\ {}^2P_{3/2} & F = 0, 1, 2, 3 \\ {}^2S_{1/2} & F = 1, 2 \end{array}$$

Can you find the g factors for the states of maximum angular momentum (so-called stretched states ${}^2P_{3/2}, F = 3$ and ${}^2S_{1/2}, F = 2$) without resorting to the formula for the g factor derived in class?