Electromagnetic Theory
Problem Set #4 (8.311)

[Due Wed. March 9, 2022]

Problem 3.13. A stand-alone charge Q is distributed, in some way, in the volume of a body made of a uniform linear dielectric with a dielectric constant κ. Calculate the polarization charge Q_{ef} residing on the surface of the body, provided that it is surrounded by free space.

Solution: Let us apply the *macroscopic* Gauss law (see Eq. (3.34) of the lecture notes) to two closed surfaces – one (S_{in}) immediately inside the body's surface and one (S_{out}) immediately outside it, so that the stand-alone charge inside them is the same (Q):

$$\oint_{S_{in}} D_n d^2r = Q, \qquad \oint_{S_{out}} D_n d^2r = Q. \tag{*}$$

The inner surface is inside the dielectric, so that at all its points $\mathbf{D} = \varepsilon\mathbf{E}$, i.e. $D_n = \varepsilon E_n$ (where $\varepsilon \equiv \kappa\varepsilon_0$ is dielectric's permittivity), while at the outer surface $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $D_n = \varepsilon_0 E_n$. Plugging these relations into Eqs. (*), we get

$$\oint_{S_{in}} E_n d^2r = \frac{Q}{\varepsilon}, \qquad \oint_{S_{out}} E_n d^2r = \frac{Q}{\varepsilon_0}. \tag{**}$$

But according to the "microscopic" Gauss law (see Eq. (1.16) of the lecture notes), applied to the slim volume between two surfaces (which, by the problem's conditions, does not contain any stand-alone surface charge, but may have a surface polarization charge Q_{ef}),

$$\oint_{S_{out}} E_n d^2r - \oint_{S_{in}} E_n d^2r = \frac{Q_{ef}}{\varepsilon_0}.$$

Plugging in the integrals from Eqs. (**), we get

$$\frac{Q_{ef}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} - \frac{Q}{\varepsilon}, \qquad \text{i.e. } Q_{ef} = Q\left(1 - \frac{\varepsilon_0}{\varepsilon}\right) \equiv Q\left(1 - \frac{1}{\kappa}\right).$$

Note that this result does not describe the part of the polarization ("effective") charge $\rho_{ef} = -\nabla\cdot\mathbf{P}$ that may be distributed in the volume of the body. (This part vanishes only if the polarization inside the body is uniform: $\mathbf{P}$ = const.)

<u>Problem 3.15</u>. A point charge q is located at distance $r >> R$ from the center of a uniform sphere of radius R, made of a uniform linear dielectric. In the first nonzero approximation in small parameter R/r, calculate the interaction force, and the energy of interaction between the sphere and the charge.

Solution: The point charge's field $E = q/4\pi\varepsilon_0 r^2$, and at the sphere's location, is nearly uniform on the scale of its radius $R << r$. From the problem solved in Sec. 3.4 of the lecture notes, we know that such uniform field induces in a dielectric sphere a dipole moment of magnitude $p = 4\pi\varepsilon_0 ER^3(\kappa - 1)/(\kappa + 2)$, directed along the initial electric field – see Eq. (3.64). Hence we can use the general formula for the radial component of the dipole field, $E_r = 2p\cos\theta/4\pi\varepsilon_0 r^3$ (see the first form of Eq. (3.13) of the lecture notes), with $\theta = 0$, to calculate the magnitude of the interaction force:

$$F = qE_r = q\frac{2}{4\pi\varepsilon_0 r^3}p = q\frac{2}{4\pi\varepsilon_0 r^3}4\pi\varepsilon_0 R^3\frac{\kappa-1}{\kappa+2}E = q\frac{2}{r^3}R^3\frac{\kappa-1}{\kappa+2}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = \frac{1}{4\pi\varepsilon_0}2\frac{\kappa-1}{\kappa+2}\frac{q^2R^3}{r^5}.$$

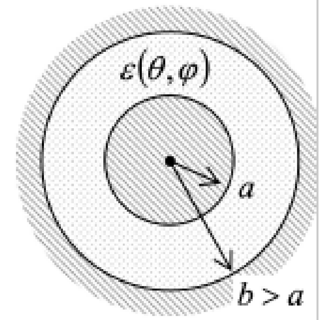
Integrating the result from ∞ to r, we get the potential energy of the charge-sphere interaction (attraction, for any sign of q):

$$U = -\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\frac{\kappa-1}{\kappa+2}\frac{q^2R^3}{r^4}.$$

Another way to obtain the last result is to use Eq. (3.15b) of the lecture notes for the interaction energy of a dipole with an external electric field, accounting for the fact that in our case the dipole moment is induced by the external field itself. In that approach, the interaction force's magnitude may be obtained at the second step, by calculating the (minus) gradient of U, i.e. its derivative over r.

Finally, the same results may be obtained by the proper generalization of the image charge method. (My strong recommendation to the reader is to explore this way as well.)

Problem 3.17. A spherical capacitor (see the figure on the right) is filled with a linear dielectric whose permittivity ε depends on spherical angles θ and φ, but not on the distance r from the system's center. Give an explicit expression for its capacitance C.



Solution: Let us prove that the boundary problem for the field distribution inside the capacitor is satisfied by the following radially-directed fields:

$$\mathbf{E} = \mathbf{n}_r E(r), \qquad \mathbf{D} = \mathbf{n}_r \varepsilon(\theta, \varphi) E(r),$$

Since the curl of such a vector **E** equals zero,[70] the first macroscopic Maxwell equation for **E**, given by Eq. (3.36) of the lecture notes, is satisfied. On the other hand, the macroscopic Maxwell equation (3.32) for **D** is satisfied if $\nabla \cdot \mathbf{D} = 0$. For the selected form of **D**, this equation is reduced to[71]

$$\frac{d}{dr}\left[r^2 E(r)\right] = 0.$$

Its straightforward integration yields the same result,

$$E(r) = \frac{c_1}{r^2},$$

as for the capacitor without the dielectric (i.e. with $\varepsilon = \varepsilon_0 = \text{const}$), which was analyzed in Sec. 2.3(iii) of the lecture notes, so that the boundary conditions for the corresponding electrostatic potential,

$$\phi(r) = \phi(a) - \int_a^r E(r)dr = \phi(a) + \frac{c_1}{r} + c_2,$$

may be satisfied by the same choice of the constants $c_{1,2}$ just as in that case.

Hence, our proof is accomplished, and now we may use Eq. (2.53) to write the expression for the electric field on the surface of any conductor – say, the inner one:

$$E(a) = \frac{V}{a^2}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}.$$

Only at this stage, the difference with the dielectric-free case cuts in, because the areal density of stand-alone charges on the surface of the conductor has to be calculated using the integral of Eq. (3.54) rather than of Eq. (2.3):

$$\sigma_a = D(a) = \varepsilon(\theta, \varphi) E(a) = \varepsilon(\theta, \varphi) \frac{V}{a^2}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1},$$

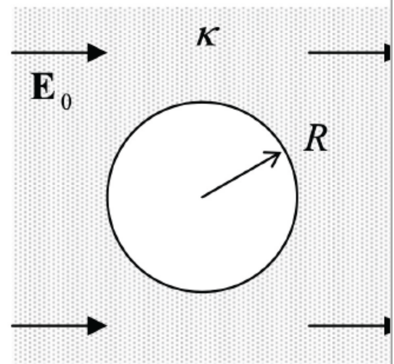
so that the mutual capacitance is

$$C \equiv \frac{Q_a}{V} = \frac{1}{V}\int_{r=a} \sigma_a d^2 r = \frac{a^2}{V}\int_{r=a} \sigma_a d\Omega = \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \int_{4\pi} \varepsilon(\theta, \varphi) d\Omega \equiv \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\varphi\, \varepsilon(\theta, \varphi).$$

For the particular case $\varepsilon = \text{const}$, this double integral is equal to $4\pi\varepsilon$, and we immediately get an evident generalization of the (easy) Problem 2.8(i):

$$C = 4\pi\varepsilon\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \equiv 4\pi\varepsilon \frac{ab}{b-a}.$$

Problem 3.19. A uniform electric field $\mathbf{E}_0$ has been created (by distant external sources) inside a uniform linear dielectric. Find the change of the electric field, created by carving out a cavity in the shape of a round cylinder of radius R, with its axis normal to the external field – see the figure on the right.



Solution: Introducing the usual polar coordinates, we can use the general solution (2.112) of the Laplace equation, and our experience with using it for the problem shown in Fig. 2.15 of the lecture notes, as the guidance for looking for the electrostatic potential ϕ in the following form:

$$\phi\Big|_{\rho\leq R} = a_1\rho\cos\varphi, \qquad \phi\Big|_{\rho\geq R} = \left(-E_0\rho + \frac{b_1}{\rho}\right)\cos\varphi,$$

where the coefficient a_1 has the sense of the uniform field inside the cavity (with the minus sign). Using the boundary conditions of continuity of ϕ and $\varepsilon\partial\phi/\partial n$ (and hence $\kappa\partial\phi/\partial\rho$) on the cavity surface ($\rho = R$), we get the following system of two linear equations,

$$a_1 R = -E_0 R + \frac{b_1}{R}, \qquad a_1 = \kappa\left(-E_0 - \frac{b_1}{R^2}\right),$$

for two unknown coefficients, a_1 and b_1. Solving the system, we get:

$$a_1 = -\frac{2\kappa}{\kappa+1}E_0, \quad b_1 = -\frac{\kappa-1}{\kappa+1}E_0 R^2.$$

As a result, the electrostatic potential distribution may be represented as

$$\phi\Big|_{\rho<R} = -\frac{2\kappa}{\kappa+1}E_0\rho\cos\varphi,$$

$$\phi\Big|_{\rho>R} = -E_0\left(\rho + \frac{\kappa-1}{\kappa+1}\frac{R^2}{\rho}\right)\cos\varphi = -E_0 x\left(1 + \frac{\kappa-1}{\kappa+1}\frac{R^2}{x^2+y^2}\right),$$

where x is the Cartesian coordinate along the initial field.

As a (necessary for us all :-) sanity check, at $\kappa = 1$ (uniform space with no dielectric), the potential distribution is the same at both $\rho > R$ and $\rho < R$:

$$\phi = \phi_0 = -E_0\rho\cos\theta,$$

and corresponds to the uniform field $E_0\mathbf{n}_x$. In the general case, $\kappa \neq 1$, the electric field, $\mathbf{E} = -\nabla\phi$, is:

$$\mathbf{E}\Big|_{\rho<R} = \frac{2\kappa}{\kappa+1}E_0\mathbf{n}_x, \qquad \mathbf{E}\Big|_{\rho>R} = E_0\mathbf{n}_x + \frac{\kappa-1}{\kappa+1}E_0\left[\mathbf{n}_x\frac{R^2(y^2-x^2)}{(x^2+y^2)^2} - \mathbf{n}_y\frac{2R^2xy}{(x^2+y^2)^2}\right].$$

From here, the electric field change from its original value $E_0\mathbf{n}_x$ is:

$$\Delta\mathbf{E}\Big|_{\rho<R} = \left(\frac{2\kappa}{\kappa+1} - 1\right)E_0\mathbf{n}_x \equiv \frac{\kappa-1}{\kappa+1}E_0\mathbf{n}_x, \qquad \Delta\mathbf{E}\Big|_{\rho>R} = \frac{\kappa-1}{\kappa+1}E_0\left[\mathbf{n}_x\frac{R^2(y^2-x^2)}{(x^2+y^2)^2} - \mathbf{n}_y\frac{2R^2xy}{(x^2+y^2)^2}\right].$$

It is curious that in the limit $\kappa \to \infty$, the internal electric field *increases* by exactly $E_0\mathbf{n}_x$, i.e. by 100% of its initial value, while $\mathbf{D}$ *drops* dramatically (by a factor of κ) as a result of the cavity cut. The reader is invited to interpret this fact in the light of the thin-gap "experiments" discussed in Sec. 3.4 of the lecture notes – see Fig. 3.9.