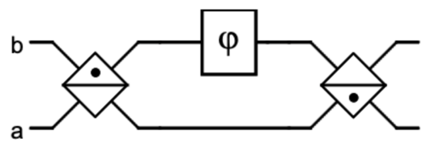
Name: **Huan Q. Bui**
Course: **8.422 - AMO II**
Problem set: **#5**
Due: Friday, Mar 17, 2022
References:

1. Better Phase Measurement with Squeezed Vacuum.

One application of squeezed light is to measure phase shifts with better precision than can be achieved with the same number of photons in a coherent state. In this problem, we employ a Mach-Zehnder interferometer with coherent and squeezed input states. This interferometer uses two 50/50 beamsplitters and has a phase shift in one path of ϕ and has output ports of b_{out} and a_{out}.



(a) Suppose the input on port a is a coherent state $|\alpha\rangle$ and the input on port b is the vacuum state $|0\rangle$. Here we calculate the output signal $\langle M\rangle = \langle b^\dagger_{\text{out}} b_{\text{out}} - a^\dagger_{\text{out}} a_{\text{out}}\rangle$, its variance $\langle \Delta M^2\rangle = \langle \Delta(b^\dagger_{\text{output}} b_{\text{output}} - a^\dagger_{\text{output}} a_{\text{output}})^2$, and the signal-to-noise ratio $\langle M\rangle / \sqrt{\langle \Delta M^2\rangle}$.

Let's first compute the output ports in terms of the input ports. After the first beamsplitter, we have

$$b_1 = BbB^\dagger = \frac{b-a}{\sqrt{2}} \qquad \text{and} \qquad a_1 = BaB^\dagger = \frac{b+a}{\sqrt{2}}.$$

Remembering that the beamsplitter with the dot facing up is B, we calculate the output after it:

$$B\,|\alpha\rangle_a\,|0\rangle_b = e^{-|\alpha|^2/2} B e^{\alpha a^\dagger} B^\dagger\,|0\rangle_a\,|0\rangle_b = e^{-|\alpha|^2/2} \exp\left(\alpha \frac{a^\dagger + b^\dagger}{\sqrt{2}}\right)|0\rangle\,|0\rangle = \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{a_1} \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{b_1}.$$

After the phase shift in the b mode we have

$$\left|\frac{\alpha}{\sqrt{2}}\right\rangle_{a_1} \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{b_1} \to \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{a_1} \left|\frac{\alpha e^{i\varphi}}{\sqrt{2}}\right\rangle_{b_1}.$$

Finally, we apply $B^\dagger$ to this state:

$$\begin{aligned} B^\dagger \left|\frac{\alpha}{\sqrt{2}}\right\rangle_{a_1} \left|\frac{\alpha e^{i\varphi}}{\sqrt{2}}\right\rangle_{b_1} &= e^{-|\alpha|^2/2} B^\dagger e^{\alpha a^\dagger/\sqrt{2}} e^{\alpha e^{i\varphi} b^\dagger/\sqrt{2}} B B^\dagger\,|0\rangle\,|0\rangle \\ &= e^{-|\alpha|^2/2} \exp\left(\frac{\alpha}{2}(a^\dagger - b^\dagger) + \frac{\alpha e^{i\varphi}}{2}(a^\dagger + b^\dagger)\right)|0\rangle\,|0\rangle \\ &= e^{-|\alpha|^2/2} \exp\left(a^\dagger \frac{\alpha(1+e^{i\varphi})}{2} + b^\dagger \frac{\alpha(-1+e^{i\varphi})}{2}\right)|0\rangle\,|0\rangle \\ &= \left|\frac{\alpha(1+e^{i\varphi})}{2}\right\rangle_{a_{\text{out}}} \left|\frac{\alpha(-1+e^{i\varphi})}{2}\right\rangle_{b_{\text{out}}}. \end{aligned}$$

With this we can compute the signal:

$$\langle M\rangle = \langle b^\dagger_{\text{out}} b_{\text{out}} - a^\dagger_{\text{out}} a_{\text{out}}\rangle = \langle b^\dagger_{\text{out}} b_{\text{out}}\rangle - \langle a^\dagger_{\text{out}} a_{\text{out}}\rangle = \frac{|\alpha|^2}{4}\left|-1+e^{i\varphi}\right|^2 - \frac{|\alpha|^2}{4}\left|1+e^{i\varphi}\right|^2 = -|\alpha|^2\cos\varphi$$

Next we compute the variance in the signal:

$$\begin{aligned}
\langle \Delta M^2 \rangle &= \langle M^2 \rangle - \langle M \rangle^2 \\
&= \langle (b_{\text{out}}^\dagger b_{\text{out}} - a_{\text{out}}^\dagger a_{\text{out}})^2 \rangle - |\alpha|^4 \cos^2\varphi \\
&= \langle b_{\text{out}}^\dagger b_{\text{out}} b_{\text{out}}^\dagger b_{\text{out}} + a_{\text{out}}^\dagger a_{\text{out}} a_{\text{out}}^\dagger a_{\text{out}} - 2 b_{\text{out}}^\dagger b_{\text{out}} a_{\text{out}}^\dagger a_{\text{out}} \rangle - |\alpha|^4 \cos^2\varphi \\
&= \langle n_{b,\text{out}}^2 \rangle + \langle n_{a,\text{out}}^2 \rangle - 2\langle n_{b,\text{out}} \rangle \langle n_{a,\text{out}} \rangle - |\alpha|^4 \cos^2\varphi \\
&= \left| \frac{\alpha(-1+e^{i\varphi})}{2} \right|^4 + \left| \frac{\alpha(-1+e^{i\varphi})}{2} \right|^2 + \left| \frac{\alpha(1+e^{i\varphi})}{2} \right|^4 + \left| \frac{\alpha(1+e^{i\varphi})}{2} \right|^2 \\
&\quad - 2\left| \frac{\alpha(-1+e^{i\varphi})}{2} \right|^2 \left| \frac{\alpha(1+e^{i\varphi})}{2} \right|^2 - |\alpha|^4 \cos^2\varphi \\
&= \frac{|\alpha|^4}{16}(16\cos^2\varphi) - |\alpha|^4\cos^2\varphi + \frac{4|\alpha|^2}{4} \\
&= |\alpha|^2.
\end{aligned}$$

With these, the sign-to-noise ratio in the signal is

$$\text{SNR} = \left| \frac{\langle M \rangle}{\sqrt{\langle \Delta M^2 \rangle}} \right| = \frac{|\alpha|^2 |\cos\varphi|}{|\alpha|} = |\alpha \cos\varphi|.$$

(b) The minimal detectable phase is the phase ϕ_{min} for which SNR = 1. Assuming we have a strong coherent state then $\cos\varphi$ is small. This means we want to expand $\cos\varphi$ near $\pi/2$.

$$1 = |\alpha| \cos\varphi_{\text{min}} \approx \pm|\alpha| \left(\varphi_{\text{min}} - \frac{\pi}{2} \right) \implies \varphi_{\text{min}} \approx \pm\frac{1}{|\alpha|} + \frac{\pi}{2}.$$

(c) Now we repeat the calculation but with the squeezed vacuum $S(r)|0\rangle$ entering port b instead. In view of the last problem set where we decomposed the squeezing operator into

$$S(r) = e^{\frac{r}{2}(b^{\dagger 2} - b^2)} = e^{\frac{u}{2} b^{\dagger 2}} e^{t(b^\dagger b + 1/2)} e^{\frac{v}{2} a^2}$$

with $u = -v = \tanh r$ and $t = -\ln\cosh r$, we find that

$$B|\alpha\rangle_a S(r)|0\rangle_b = \frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh r}} B e^{\alpha a^\dagger} e^{\frac{\tanh r}{2} b^{\dagger 2}} |0\rangle_a |0\rangle_b .$$

Note here that the terms with t, v act as identities on the vacuum state, so only the u term remains. Under the beamsplitter transformation $B \cdot B^\dagger$, we have

$$\frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh r}} B|\alpha\rangle_a S(r)|0\rangle_b \to \frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh r}} e^{\frac{\alpha}{\sqrt{2}}(a^\dagger + b^\dagger)} e^{\frac{\tanh r}{4}(b^\dagger - a^\dagger)^2} |00\rangle$$

We notice that the

(d)

2. Hanbury-Brown and Twiss Experiment with Atoms.

This problem illustrates the coherence and collimation requirements for performing a Hanbury Brown and Twiss (HBT) experiment with atoms. If a free particle starts at point A at time $t = 0$ with amplitude ψ_A then the amplitude at another point 1 and time $t = \tau$ is proportional to $\psi_A e^{i(\vec{k}\cdot\vec{r}_{A1} - \omega\tau)}$.

(a) **Correlation function.** Assume we have a particle at A with amplitude ψ_A and a particle at B with amplitude ψ_B. Then the joint probability P of finding one particle at A and one particle at B is

$$P = \left|\psi_A e^{i\phi_{A1}}\psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}}\psi_B e^{i\phi_{B1}}\right|^2$$

and is proportional to the second-order coherence function $g^{(2)}(1,2)$. The $\pm$ is for bosons/fermions. Here, $\phi_{A1} = \vec{k}_A \cdot r_{A1} - \omega\tau$, etc. Here we want to calculate P as a function of $\vec{r}_{21}$, the vector from point 2 to point 1 on the detector. To do this, we simply put:

$$\vec{r}_{B1} = \vec{r}_{B2} + \vec{r}_{21} \qquad \text{and} \qquad \vec{r}_{A2} = \vec{r}_{A1} - \vec{r}_{21}.$$

From these, we find

$$\begin{aligned} P &= \left|\underbrace{\psi_A\psi_B e^{-2i\omega\tau}e^{i\vec{k}_A\cdot\vec{r}_{A1}}e^{i\vec{k}_B\cdot\vec{r}_{B2}}}_{C} \pm \underbrace{\psi_A\psi_B e^{-2i\omega\tau}e^{i\vec{k}_A\cdot\vec{r}_{A1}}e^{i\vec{k}_B\cdot\vec{r}_{B2}}}_{C}\, e^{-i\vec{k}_A\cdot\vec{r}_{21}}e^{i\vec{k}_B\cdot\vec{r}_{21}}\right|^2 \\ &= \left|C(1 \pm e^{-i\vec{k}_A\cdot\vec{r}_{21}}e^{i\vec{k}_B\cdot\vec{r}_{21}})\right|^2 \\ &= |\psi_A\psi_B|^2\left|1 \pm e^{i(\vec{k}_B-\vec{k}_A)\cdot\vec{r}_{21}}\right|^2. \end{aligned}$$

(b) **Transverse collimation.** Assume that we are given a source with transverse dimension W and detector with transverse dimension w where $\vec{r}_{21} \le w$. The distance between the source and the detector d is much greater than all other distances. The transverse component of the phase factor from part (a) can be written as $\phi_t = (\vec{k}_A - \vec{k}_B)_t \cdot (\vec{r}_{21})_t$. Assume that the signal at the detector is mainly due to atoms with wavevectors distributed around $\vec{k}_0$.

As seen from the detector, the angular size of the source is given by W/d. In order to see second-order correlation effects, the de Broglie wavelength of the particles, after taking into account the angular size of the target due to being at a distance d away from the detector, must be much larger than the detector size. As a result, we must have that

$$w \ll \frac{\lambda_{dB}}{W/d} \implies Ww \ll d\lambda_{dB},$$

as desired. This is related to the transverse collimation requirement.

Now we consider a ^{6}Li MOT at 500 μK. The de Broglie wavelength is

$$\lambda_{dB} = \frac{h}{p} = \frac{h}{\sqrt{2\pi m_{^6\mathrm{Li}} k_B T}} \approx 3.18 \times 10^{-8}\ \mathrm{m} = 31.8\ \mathrm{nm}.$$

Assuming the MOT and detector have equal size, i.e., $W \approx w$, then an upper bound on the magnitude of $W \approx w$ is simply

$$\sqrt{d\lambda_{dB}} \approx 0.000056\ \mathrm{m} = 56\ \mu\mathrm{m}$$

where we have used $d = 10$ cm.

(c) **Longitudinal collimation**

(d) **Phase-space volume enhancement**