QUANTUM FIELD THEORY

Sep 13, 2020

Preface. These notes come from Prof. Patton's QFT notes, Peskin-Schröder QFT, and "Quantum field theory & condensed matter" – Shankar.

Enjoy!

Conventions

- $\hbar = c = 1$
- $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$
- $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1\ -1\ -1\ -1)$
- $p^2 = p_\mu p^\mu = E^2 - |p|^2 = mc^2 = m$
- $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$
- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$
- $\partial_\mu = \dfrac{\partial}{\partial x^\mu} = \left(\dfrac{\partial}{\partial x^0}, \nabla\right)$
- $\varepsilon^{0123} = +1 \ ; \ \varepsilon_{0123} = -1$

  $\varepsilon^{1230} = -1$
- $E = i\dfrac{\partial}{\partial x^0} \ , \ \vec{p} = -i\vec{\nabla}$
- $p^\mu = i\partial^\mu$

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $$\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \dfrac{d}{dx}\theta(x)$

- n-dimensional Dirac δ-fn:

$$\int d^n x\, \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(x)$$

$$\tilde{f}(x) = \int d^4k\, e^{ik\cdot x} f(x)$$

- $$\int d^4x\, e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k)$$

- EM $\Phi = \dfrac{Q}{4\pi r}$ ← Coulomb potential

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ ; \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\Phi, \vec{A}) \ ; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of classical Field Theory

☒ Lagrangian Field Theory:

$$S = \int \underline{L}\, dt = \int \mathcal{L}(\phi, \partial_\mu\phi)\, d^4x$$

Lag-density

$$\left(\underline{L} = \mathcal{L}\, d^3x\right)$$

(Lagrangian)

Principle of least action:

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \right\}$$

(E-L eqn)

$$= \int d^4x \left\{ \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right] \right\}$$

$$\Rightarrow \boxed{0 = \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi}}$$

FTC → term vanishes @ boundary

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

Ex $\mathcal{L} = \phi^2 : \quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi \, ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow 2\phi = 0$

$$\left.\begin{aligned} &\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) \\ &\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = 0 \, ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2\partial^\mu \phi \end{aligned}\right\} \Rightarrow \partial^\mu \phi = 0.$$

Ex Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

- $\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$
- $\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi .$

$$E\text{-}L \Rightarrow \quad -m^2 \phi = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$\rightarrow \boxed{(\Box + m^2)\phi = 0}$$

Klein-Gordon Eqn.

relativistic particle of mass m.

Ex $\phi = e^{-ip\cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

Noether's Thm

Thm

$$\boxed{\text{For every continuous symmetry, there exists a conserved current } j^\mu \text{ which implies a local conservation law}}$$

$$\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j} \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

Ex Conservation of charge ...

$$Q = \int j^0 \, d^3x$$

(gauss' law)

$$\Rightarrow \frac{\partial Q}{\partial t} = \int \frac{dj^0}{dt} d^3x = -\int \vec{\nabla}\cdot\vec{j} \, d^3x$$

$$= -\oint \vec{J} \, d^2s$$

Idea Consider continuous transf. → infinitesimally... (local)

$$(\star) \qquad \phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$$

small

$(\star)$ is a symmetry if EOM invariant under $(\star)$.

$\Leftrightarrow$ S is invariant.

$\Rightarrow$ $\mathcal{L}$ must be invariant, up to $\alpha\partial_\mu J^\mu(x)$, for some J^μ.

Let us compare this expectation for $\Delta\mathcal{L}$ to the result obtained by varying the fields ...

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\mu(\alpha\Delta\phi)$$

$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right\}\right)}_{0}\Delta\phi$$

$$\Rightarrow \quad \Delta\mathcal{L} = \partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right\}$$

So $\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi$ is the desired J^μ.

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu.$$

Ex massless KG field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)$$

Consider transformation:

$$\phi(x) \longrightarrow \phi(x) + \alpha \quad , \quad \Delta\phi = 1$$

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi = \partial^\mu\phi.$$

Check $\partial_\mu j^\mu = \partial_\mu\partial^\mu\phi = \Box\phi = 0$ since

$(m^2 + \Box)\phi = 0$ & $m = 0$

Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi.$$

again, EOM is

$$(m^2 + \Box)\phi = 0.$$

Symmetry: $\phi \to e^{i\alpha}\phi$.

For infinitesimal transf we have:

$$\alpha \Delta \phi = i\alpha\phi \qquad \text{(Taylor expand)}$$
$$\alpha \Delta \phi^\dagger = -i\alpha \phi^\dagger$$

One can check that

$$j^\mu = i\left[(\partial^\mu \phi^\dagger)\phi - \phi^\dagger(\partial^\mu \phi)\right]$$

is the conserved current.

$\boxed{\text{Ex}}$

Space-time translation:

Consider infi. translation:

$$x^\mu \to x^\mu - a^\mu$$

in field transform this becomes:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Lagrangian is a scaler $\Rightarrow$ must transform the same way:

$$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Compare this to the eqn:

$$\mathcal{L} \longrightarrow \mathcal{L} + a\,\partial_\mu J^\mu ,$$

we have

$$J^\mu = \delta^\mu_\nu \mathcal{L}$$

$\Rightarrow$ apply thm, we find:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}(\partial_\mu \phi) - \delta^\nu_\mu \mathcal{L}$$

Make μ explicit...

$$\boxed{T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\,\partial_\mu \phi - \delta^\nu_\mu \mathcal{L}}$$

STRESS-ENERGY TENSOR. (or Energy-Momentum tensor)

<u>Conserved charge</u> is the Hamiltonian...

$$H = \int T^{00} d^3x = \int \mathcal{H}\, d^3x \quad (\text{time-translation})$$

momentum $\longrightarrow$ $$p^i = \int T^{0i} d^3x = -\int \pi\, \partial_i \phi\, d^3x \quad (\text{spatial translation})$$

Ex Klein-Gordon field again...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2.$$

$$T^{00} = \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial^0\phi - g^{00}\mathcal{L} = \ldots$$

$$= \frac{1}{2}(\partial_0\phi)(\partial^0\phi) - \frac{1}{2}(\partial_i\phi)(\partial^i\phi) + \frac{1}{2}m^2\phi^2$$

So

$$H = \int T^{00}d^3x = \int\left\{\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2\right\}d^3x$$

note: all terms are positive ... (sum of squares)

→ can't fall into arbitrary negative energy.

$$\boxed{\text{THE KLEIN-GORDON FIELD} \infty \text{ HARMONIC OSCILLATORS}}$$

promote: ϕ, π to operators & impose suitable commutation relations...

Recall ...

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0.$$

Harmonic oscillator: $H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2 q^2$

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$

- creation: $a^{\dagger} = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$

- $a^{\dagger}a = \frac{1}{\omega}H - \frac{1}{2} \Leftrightarrow H = \omega\left(a^{\dagger}a + \frac{1}{2}\right)$

  $\uparrow$

  # operator...

- $|0\rangle$, $a|0\rangle = 0$.

  $|n\rangle = \frac{1}{\sqrt{n!}}\underbrace{a^{\dagger}\ldots a^{\dagger}}_{n}|0\rangle$

- $[a^{\dagger}a, a] = -a$

- $[a^{\dagger}a, a^{\dagger}] = a^{\dagger}$

- $[H, a] = -\omega a$; $[H, a^{\dagger}] = \omega a^{\dagger}$

  a lowers by ω

  $a^{\dagger}$ raises by ω

- $Ha^{\dagger}|n\rangle = (E_n + \omega)a^{\dagger}|n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

For continuum system ... commutation relations become ..

$$\boxed{\begin{aligned} [\phi(x), \pi(y)] &= i\delta^{(3)}(x-y) \\ [\phi(x), \phi(y)] &= [\pi(x), \pi(y)] = 0. \end{aligned}}$$

Next, the Hamiltonian is now also an operator ...
To find spec(H), Fourier transf $\phi(x)$

$$\rightarrow \quad \phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \, e^{ip\cdot x} \phi(\vec{p},t)$$

Recall KG eqn: $(m^2 + \Box)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) \right\} \phi(\vec{p},t) = 0$$

$\Rightarrow$ This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}.$$

Now, $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2 \qquad (m=1)$

$\rightarrow$ know spectrum! $\quad (n+\frac{1}{2})\omega.$

$$\phi = \frac{1}{\sqrt{2\omega}}(a^\dagger + a) \quad ; \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger)$$

$$[a, a^\dagger] = 1.$$

Since it's more convenient to work in position space

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^+ e^{-ip\cdot x} \right) \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^+ e^{ipx} \right) \end{aligned}}$$

note

$$\begin{cases} a_p \text{ goes with } e^{+ip\cdot x} \\ a_p^+ \text{ goes with } e^{-ip\cdot x}. \end{cases}$$

- Easy to show that

$$\begin{cases} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \end{cases}$$

- Can re-arrange...

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p + a_{-p}^+) e^{ip\cdot x}. \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} (a_p - a_{-p}^+) e^{+ip\cdot x}. \end{cases}$$

→ get commutation relation between a_p:

$$\boxed{[a_p ; a_{p'}^+] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,')}$$

Now, can check that

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{w_{p'}}{w_p}} \times e^{i(p\cdot x + p'\cdot x')} \times$$

$$\left([a^\dagger_{-p}, a_{p'}] - [a_p, a^\dagger_{-p'}] \right)$$

$$= i\delta^{(3)}(x - x') \quad \checkmark$$

- Now, can express Hamiltonian in terms of ladder ops

Recall that

$$H = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} \right\}$$

KG field, last time

$$= \int d^3x \left\{ \frac{1}{2}(\partial_0 \phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \right)$$

So ...

$$H = \int d^3x \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

with $\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a^\dagger_{-p}) e^{ip\cdot x}$ we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p + a^\dagger_{-p}) e^{ip\cdot x}$$

results in $\delta(p-p')$

$\Rightarrow p = p'$

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\, \overbrace{e^{i(p+p')\cdot x}} \Big\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4}(a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'})$$

$$+ \frac{-p\cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}}(a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) \Big\}$$

Some $\delta^{(3)}$ will appear...

$$= \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)$$

So

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)}$$

→ can evaluate commutators...

$$\boxed{[H, a_p^\dagger] = \omega_p a_p^\dagger \; ; \quad [H, a_p] = -\omega a_p}$$

With H, can find momentum operator...

KG field → from $P^i = \int d^3x\, T^{0i} = -\int \pi \partial_i \phi\, d^3x$, we get

$$\boxed{\begin{aligned} \vec{P} &= -\int d^3x\, \pi(x) \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a_p^\dagger a_p . \end{aligned}}$$

$a_p^\dagger$ creates momentum $\vec{p}$ & energy $\underset{E_p > 0}{\omega_p} = \sqrt{|p|^2 + m^2}$.

Excitation: $a_p^\dagger a_q^\dagger \ldots |0\rangle \equiv$ "particles".

↳ <u>each excitation $a^\dagger_p$ is a particle.</u>

$\Rightarrow$ get particle statistics...

Consider 2-particle state... $a_p^\dagger a_q^\dagger |0\rangle$.

Since $[a_p^\dagger, a_q^\dagger] = 0$, we have

$$\boxed{a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle}$$

$\rightarrow$ Klein Gordon particles follow Bose-Einstein stats.

- <u>Normalization</u> $\langle 0|0\rangle = 1$.

$$|p\rangle \propto a_p^\dagger |0\rangle .$$

This normalization $\longrightarrow$ $\langle q|p\rangle = (2\pi)^3 \delta^{(3)}(q-p)$ is <u>NOT Lorentz inv</u>

Pf Under a Lorentz boost $\rightarrow$ $p_3' = \gamma(p_3 + \beta E)$

$$E' = \gamma(E + \beta p_3)$$

using the identity $\rightarrow$ $\delta(f(x) - f(x_0)) = \dfrac{1}{|f'(x_0)|}\delta(x - x_0)$

we compute: $\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \cdot \left(\dfrac{dp_3'}{dp_3}\right)$

$\underbrace{\delta(p_1-q_1)\delta(p_2-q_2)}_{\text{same}}\underbrace{\delta(p_3-q_3)}_{\text{boosted}}$

$$= \delta^{(3)}(p'-q') \cdot \gamma\left(1 + \beta \, dE/dp_3\right)$$

$$= \delta^{(3)}(p'-q') \frac{\gamma}{E}(E + \beta p_3)$$

$$= \delta^{(3)}(p'-q')(E'/E)$$

$\boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\left(\frac{E'}{E}\right)}$ $\rightarrow 1 \Leftrightarrow E = E_p = E'$

So

For normalization to work $\rightarrow$ use $\underset{}{E_p} \overset{w_p}{\parallel}$, not E.

$\rightarrow$ define: $\boxed{|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$

So that $\boxed{\langle p|q\rangle = 2E_p\,(2\pi)^3\,\delta^{(3)}(p-q)}$

Completeness relation...

$$\boxed{1 = \int \frac{d^3p}{(2\pi)^3}\, |p\rangle \frac{1}{2E_p} \langle p|}$$

1 particle

Interpret $\phi(x)|0\rangle$ .... we know that $a_p^\dagger$ creates momentum p & energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|p\rangle, \qquad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

Pf. By defn...

annihilates.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\, e^{-ip\cdot x}|0\rangle \qquad \square$$

$\rightarrow$ $\phi(x)|0\rangle$ is a lin. superposition of single-particle states

~~Note HW1, HW2 easy~~

that have well-defn momentum.

When nonrelativistic $\rightarrow E_p \simeq$ Constant!

$\Rightarrow$ [$\phi(x)$ acting on the vacuum, "creates a particle at position x".]

$\hookrightarrow$ <u>Confirm this</u> by computing.

$$\langle 0|\,\phi(x)\,|p\rangle = \langle 0|\int\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_{p'}}}\left(a_{p'}e^{ip'\cdot x} + a_{p'}^{\dagger}e^{-ip'\cdot x}\right)\sqrt{2E_p}\,a_p^{\dagger}|0\rangle$$

$$\boxed{\langle 0|\,\phi(x)\,|p\rangle = e^{ip\cdot x}}$$

$\hookrightarrow$ <u>Interpretation</u>: position-space representation of the single-particle wfn of the state $|p\rangle$, just like ...

$$\boxed{\langle x|p\rangle \propto e^{ip\cdot x}} \text{ in QM!}$$

$\langle 0|\phi(x) \sim \langle x| \ldots$ (don't take this literally, ofc).

<u>Note</u> HW1, HW2 are easy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time → we quantized KG field in the Schrödinger picture.

→ Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$\mathcal{U}(t) = e^{-iHt}$ is the time evolution.

$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle$. → State evolve in time

→ In the Heisenberg picture,... Operators evolve in time

$$\mathcal{O}(t) = \mathcal{U}^\dagger(t)\,\mathcal{O}(0)\,\mathcal{U}(t).$$

So that

$$\langle\psi_1|\mathcal{O}(t)|\psi_1\rangle = \langle\psi_1(t)|\mathcal{O}|\psi_1(t)\rangle$$

↓ Heisenberg ↓ Schrödinger.

→ make the operators ϕ & π time-dependent like this

$$\boxed{\begin{aligned}\phi(x) \to \phi(x,t) &= e^{iHt}\phi(x)e^{-iHt}\\ \pi(x) \to \pi(x,t) &= e^{iHt}\pi(x)e^{-iHt}\end{aligned}}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

$$= \int d^3x' \left(i\delta^{(3)}(x-x')\,\pi'(x,t) \right)$$

$(\phi \leftrightarrow \phi)$
$\Rightarrow$ only nontrivial term is 1st.

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t}\phi(x,t) = \pi(x,t)}$$

<u>and</u>

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')(-\nabla^2 + m^2)\,\phi(x',t) \right)$$

(integrate by parts here)

$$= -i(-\nabla^2 + m^2)\,\phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\pi(x,t) = (m^2 - \nabla^2)\,\phi(x,t)}$$

Combining these 2 results we get...

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) = (\nabla^2 - m^2)\,\phi(x,t)}$$

rearranging this gives

$$\boxed{(\Box + m^2)\,\phi(x,t) = 0} \longrightarrow$$ just the KG eqn...

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

<u>Recall</u>: $H a_p = a_p (H - E_p)$ → from comm. rule.

$$\Longrightarrow \left(\begin{array}{c}\text{proof by}\\ \text{induction}\end{array}\right) \quad \boxed{H^n a_p = a_p (H - E_p)^n}$$

Similarly, $\boxed{H^n a_p^\dagger = a_p^\dagger (H + E_p)^n}$

→ So, we have

$$\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}}$$ → from above...

and

$$\boxed{e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{+iE_p t}}$$

→ Now ... we want to write $\phi(x,t)$ in terms of these operators. (since $\phi(x)$, $\pi(x)$ is a combo of a & $a^\dagger$)

- We know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before...

$$\phi(x) = \phi(x,0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

Substitute this into $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$ we find ...

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-iE_pt} e^{ip\cdot x} + a_p^\dagger e^{iE_pt} e^{-ip\cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow p\cdot x = E_p \cdot x^0 - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p} = E_pt - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p}$$

So...

$$\boxed{\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right\}\Bigg|_{p^0=E_p}}$$

Similarly, we can find

$$\boxed{\pi(x,t) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{E_p}{2}} \left(a_p e^{-i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right)\Bigg|_{p^0=E_p}}$$

from defn or from $\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$.

Note we can also do everything but starting from P and not H. But we won't worry about that.

Causality Note that causality is broken ~~then~~ without the presence of fields.

$\rightarrow$ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture...

...the amplitude for a particle to propagate from y to x is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

"Two-point correlation function"

Let $\boxed{\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0|\text{ in terms of } a_p \text{ \& } a_p^\dagger \ldots |0\rangle$$

$$= \langle 0|\, a_p\, a_q^\dagger\, |0\rangle =$$

$$= (2\pi)^3\,\delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,e^{-ip\cdot(x-y)}$$

More explicitly...

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

@ $p^0 = E_p$
$p^{0\prime} = E_{p'}$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\,\frac{1}{\sqrt{2E_p}}\,\frac{1}{\sqrt{2E_{p'}}}\,\langle 0|\, e^{-ip\cdot x}\, e^{+ip'\cdot y}\, a_p a_{p'}^\dagger\, |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\left(\frac{1}{\sqrt{2E_p}}\,\frac{1}{\sqrt{2E_{p'}}}\right) e^{-ip\cdot x + ip'\cdot y}\,(2\pi)^3\,\delta^{(3)}(p-p')$$

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\cdot e^{-ip\cdot(x-y)} \qquad \square.$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$...

1. Suppose that $x-y = (t, 0, 0, 0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \, \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t}$$

(timelike)

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\underset{t \to \infty}{=} e^{-imt} \cdot \longrightarrow \text{dominated by region where } p \approx 0.$$

2. Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

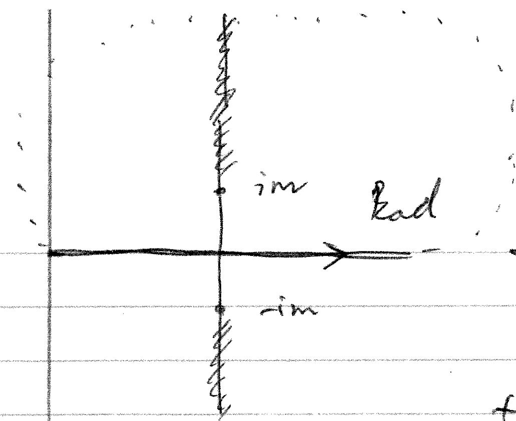
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \, \frac{p^2}{2E_p} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

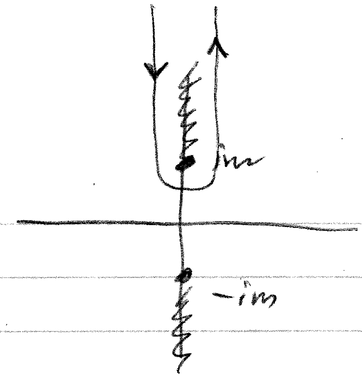
$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \, \frac{p e^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ $\pm im$ (singularity)...

→ must change contour... & Wick rotate



$\longrightarrow$ push contour + Wick rotate ...



To get

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \, \frac{\rho e^{-\rho r}}{\sqrt{\rho^2 - m^2}}$$

$\rho = -ip$ (Wick rotate)

$$\Rightarrow \quad \boxed{\mathcal{D}(x-y) \underset{r\to\infty}{=} \frac{e^{-mr}}{r}} \longrightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^\mu (x-y)_\mu = -(\vec{x}-\vec{y})^2 < 0$ is spacelike, cannot have causality between x & y.

$$\mathcal{D}(x-y) \neq 0 \Rightarrow \ ??? \text{ paradox?}$$

$\rightarrow$ <u>NO!</u> To discuss causality, we should ask not whether particles can propagate over spacelike intervals ...

... but whether a "<u>measurement</u>" ~~at~~ at one point can affect a measurement at another point whose separation from the first is spacelike -

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement @ x, call this $\Phi(x)$. & a local measurement @ y, called $\Phi'(y)$

So long as $[\Phi(x), \Phi'(y)] = 0$, the 2 measurements don't affect one another.

~ measure the field ϕ @ x & @ y,

If $[\phi(x), \phi(y)] = 0$ ~~when~~ $(x-y)^2 < 0$ then we're good

$$\cancel{\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x}[a_p, a^\dagger_{p'}]e^{ip'\cdot y} + e^{ip\cdot x}[a^\dagger_p, a_{p'}]e^{-ip'\cdot y} \right\}}$$

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \times \left\{ \left[(a_p e^{-ip\cdot x} + a^\dagger_p e^{ip\cdot x}), (a_{p'} e^{-ip'\cdot y} + a^\dagger_{p'} e^{ip'\cdot y}) \right] \right.$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6\, 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x}\underbrace{[a_p, a^\dagger_{p'}]}_{(2\pi)^3 \delta^3(p-p')} e^{ip'\cdot y} + e^{ip\cdot x}\underbrace{[a^\dagger_p, a_{p'}]}_{-(2\pi)^3 \delta^3(p-p')} e^{-ip'\cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p}\right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

So $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$.

Since $D(y-x)$ is Lorentz invariant, can take

$y-x \to x-y$. (possible b/c $(x-y)^2 < 0$)

Note that when $(x-y)^2 > 0 \to$ there's no continuous transf that takes $y-x \to x-y$

$\to$ so this is only possible because $(x-y)^2 < 0$ (spacelike).

So $D(y-x) = D(x-y)$

So, $[\phi(x), \phi(y)] = 0$ ✓

The Klein-Gordon Propagator

Let's look at $[\phi(x), \phi(y)]$ in more details ...

$[\phi(x), \phi(y)]$ is just a number

$\Rightarrow$ can write $[\phi(x), \phi(y)] \equiv \langle 0| [\phi(x), \phi(y)] |0\rangle$

$$\to \langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right\}$$

(assuming $x^0 > y^0$)

Poles @ $p^2 = m^2$ ⇓ $p^0 = \pm E_p$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_p} - \frac{1}{2E_p} e^{-ip(x-y)} \Big|_{p^0 = -E_p} \right.$$

(27)

The Klein-Gordon Propagator

→ Before looking at this ... need to look at Green's Function & Contour Integrals ...

→ Consider function

$$\frac{1}{(p_0^0 - E_p)(p_0^0 + E_p)} \qquad E_p = \sqrt{|p|^2 + m^2}.$$

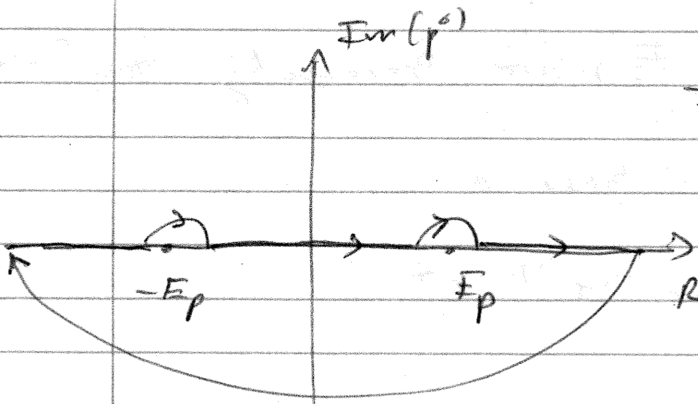
→ Poles at $p_0^0 = \pm E_p$.

Look at FT:

$$\boxed{f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}}$$

→ How to integrate this?

If $t > 0$ →



→ $f(t) = (-2\pi i) \sum \mathrm{Res}(f)$

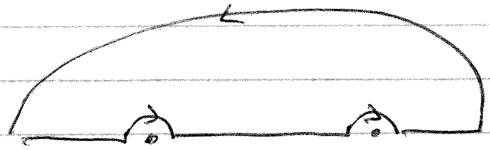
$$= (-2\pi i) \left\{ \frac{e^{iE_p t}}{2\pi(-2E_p)} + \frac{e^{-iE_p t}}{2\pi(2E_p)} \right\}$$

↑ close contour ...
but clockwise → reverse sign

→

$$\boxed{f(t) = \frac{-i}{2E_p}\left(e^{+iE_p t} - e^{-iE_p t} \right)} \qquad (t > 0)$$

If $t<0$ close contour above poles



$\rightarrow f(t) = 0.$

$\rightarrow$ So, altogether, we have...

$$f(t) = \int \frac{1}{2\pi} \frac{dp^0}{(p^0-E_p)(p^0+E_p)} e^{-ip^0 t}$$

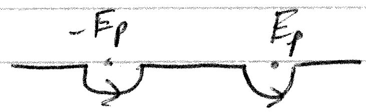
$$= \Theta(t)\left(\frac{-i}{2E_p}\right)\left(e^{-iE_p t} - e^{+iE_p t}\right)$$

where $\Theta(t)$ is the Heaviside step fn...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

$\longrightarrow$ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



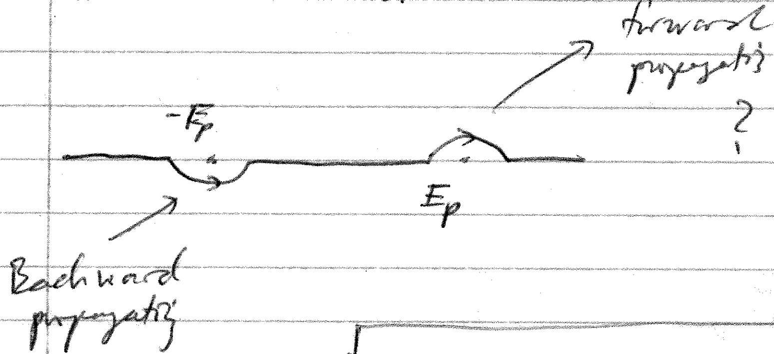
, then we'll get

$t > 0 \rightarrow f(t) = 0$

$t < 0 \rightarrow f(t) = \Theta(-t)\frac{i}{2E_p}\left(e^{-iE_p t} - e^{iE_p t}\right)$

$\rightarrow$ Advanced / Backward Propagating Green's fn.

☒ What about



$$\longrightarrow \boxed{f(t) = \Theta(t)\left(\frac{-i}{2E_p}\right)e^{-iE_pt} + \Theta(-t)\left(\frac{-i}{2E_p}\right)e^{+iE_pt}}$$

↑ Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$

... Consider this quantity...

$$\langle 0|\,[\phi(x),\phi(y)]\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\underbrace{\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{ \frac{1}{2E_p} e^{-ip(x-y)} + \frac{1}{-2E_p} e^{-ip(x-y)} \right\}$$

pole @ $p_0 = E_p$ pole @ $p_0 = -E_p$

(4D integral) $\longrightarrow$

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3} \underbrace{\int \frac{dp^0}{2\pi i}\,\frac{-1}{p^2-m^2}\,e^{-ip(x-y)}}_{f(t)\text{ before, where}}$$

$$(p^0 - E_p)(p^0 + E_p) = p^{0\,2} - |p|^2 - m^2 = p^2 - m^2 .$$

Now, call ...

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle}$$

Then

$$(\Box+m^2)D_R(x-y) = \Box D_R(x-y) + m^2 D_R(x-y)$$

$$= (\Box\theta(x^0-y^0))\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ 2(\partial_\mu\theta(x^0-y^0))\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ \theta(x^0-y^0)\underbrace{(\Box+m^2)\langle 0|[\phi(x),\phi(y)]|0\rangle}_{\cancel{\text{constant}}}$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$+ 2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0$$

needs verification (easy)

$$= \delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$= -i\delta^{(4)}(x-y)$$

$$-i\delta^{(3)}(x-y)$$

So

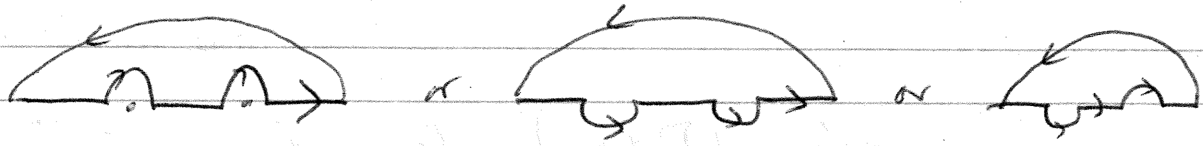
$$\boxed{(\Box+m^2)D_R(x-y) = -i\delta^{(4)}(x-y)}$$

$\longrightarrow$ $D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0$ @ $x^0 < y^0$

$\Rightarrow$ $D_R(x-y) \equiv$ "Retarded" Green's fn

Now ... As we have seen, there are many ways to take the contour...

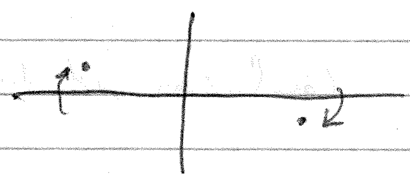


or ... or ... etc

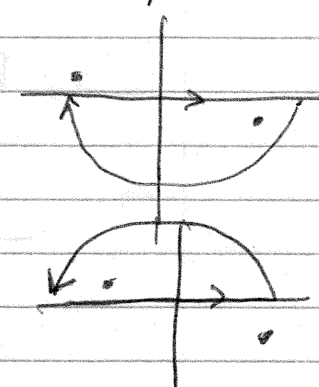
$\longrightarrow$ Use the <u>Feynman prescription</u> instead

$$\boxed{D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}}$$

$\rightarrow$ Convenient! B/c now poles are $p^0 = \pm(E_p - i\varepsilon)$



$\longrightarrow$ when $x^0 > y^0 \rightarrow$ take contour below



when $x^0 < y^0 \rightarrow$ take contour above.

$\rightarrow$ get same expression but with $x \leftrightarrow y$.

So,

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$\equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle$$

"time-ordering" symbol.

<u>Time-ordering symbol</u> $\Rightarrow$ instructs us to place the operators that follows in order with the latest to the left.

→ apply $(\Box + m^2)$ to last line, get D_F is Green's fn of Klein-Gordon operator.

→ $D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator...

→ propagation amplitude

→ But we can't much calculation at this point just yet.

→ B/c we've only looked at the <u>free</u> KG theory

→ field eqn in this case is linear : there are no <u>interactions</u>...

→ this theory is too simple to make any predictions...

→ need perturbation ....

One kind of interaction that ~~eh~~ can also be solved ...

Particle Creation by a classical Source

Consider the source $j(x)$

Recall... free field: $(\Box + m^2)\phi = 0$

now... $(\Box + m^2)\phi = j(x)$ fixed fn in space time.

$j(x)$ is nonzero only for a finite time interval

The associated Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2} m^2 \phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is easiest to solve...

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

With a source...

$$\phi(x) = \phi_0(x) + i\int d^4y \, D_R(x-y)\, j(y)$$

$\vdots$

We won't worry about this for now...

Some problems & Insights

① Classical EM (no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

ⓐ Identify $\begin{cases} E^i = -F^{0i} \\ \varepsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \; ; \; \vec{\nabla} \cdot \vec{E} = 0 \quad (\nu = 0)$$

$$-\partial_t \mathcal{E} + \vec{\nabla} \times \vec{E} = 0 \quad (\nu = i)$$

② Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi\right)$$

Derive E-L eqn:

$$\boxed{\begin{aligned} i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ &= 0 \\ -i\partial_t \phi - \frac{1}{2m} \nabla^2 \phi &= 0 \end{aligned}}$$

now... write $\phi \to e^{-i\theta}\phi$, $\phi^+ \to e^{i\theta}\phi^+$

$$\sim \phi - i\theta\phi \qquad \sim \phi^+ + i\theta\phi^+$$

$$\to \Delta\phi \sim -i\theta \qquad \Delta\phi^+ \sim i\phi^+$$

So that

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^+)}\Delta\phi^+$$

conserved current.

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \to \ldots \text{compute} \ldots$$

→ can get Hamiltonian...

there's a Ham. formula in book, but we won't worry abt this.

$$\mathcal{H} = \int d^3x \left(\pi^+\pi + (\nabla\phi)^+(\nabla\phi) + m^2\phi^+\phi\right)$$

(3) If we take $(x-y)^2 = -r^2$ can implicitly write

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2 \to D(x-y)$ can be written ~~as~~ in terms of Bessels Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation... $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu}$

→ what happens to $\phi(x)$ under Λ?

we require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \to \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform ... $\partial_\mu \phi(x) \to \partial_\mu(\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu(\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu \phi(x))^2 \to g^{\mu\nu}(\partial_\mu \phi'(x))(\partial_\nu \phi'(x))$$

$$= g^{\mu\nu}[\partial_\mu(\phi(\Lambda^{-1}x))][\partial_\nu(\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu}\{(\Lambda^{-1})^\rho{}_\mu \partial_\rho \phi\}\{(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \phi\}(\Lambda^{-1}x)$$

$$= (\partial\phi)^2(\Lambda^{-1}x) \leftarrow = g^{\rho\sigma}\partial_\rho\phi\, \partial_\sigma\phi\, (\Lambda^{-1}x) \to \text{evaluated}$$

Recall

$$\boxed{\begin{aligned} \phi(x) &\to \phi(\Lambda^{-1}x) \\ \partial_\mu \phi(x) &\to (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x) \\ (\partial_\mu \phi(x))^2 &\to \cancel{g^{\rho\sigma}} \; (\partial_\mu \phi)^2 (\Lambda^{-1}x) \end{aligned}}$$

So it is clear that

$$\boxed{\mathcal{L} \longrightarrow \mathcal{L}(\Lambda^{-1}x)}$$

Lagrangian is Lorentz-invariant.

→ The action $S = \int d^4x \, \mathcal{L}$ is also Lorentz inv.

→ also clear that EOM is also Lorentz inv.

$$(\Box + m^2)\,\phi'(x) = \left((\Lambda^{-1})^\mu{}_\nu \partial_\mu (\Lambda^{-1})^{\sigma\mu} \partial_\sigma + m^2\right)\phi(\Lambda^{-1}x)$$

$$= (\partial^\sigma \partial_\sigma + m^2)\,\phi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Sep 20, 2020

→ How do we find Lorentz-invariant theories, in general?

→ for simplicity, restrict attention to lin. transf

→ $\phi_a = v \in \mathbb{C}^n$, → matrix giving Lorentz transform Λ.

→ $\boxed{\Phi_a(x) \Rightarrow \underbrace{M_{ab}(\Lambda)}_{n\times n} \Phi_b(\Lambda^{-1}x)}$

→ the most general nonlinear transf can be built out of linear ones ⇒ suffices to consider M only.

↳ for short, write $\Phi \longmapsto M(\Lambda)\Phi$.

→ what are the possible allowed $M(\Lambda)$?

Obs $\{\Lambda\}$ form a group $M(\Lambda'')M(\Lambda) = M(\Lambda')$ → $\Lambda''\Lambda = \Lambda'$

→ the correspondence between Λ & M must be preserved under multiplication.

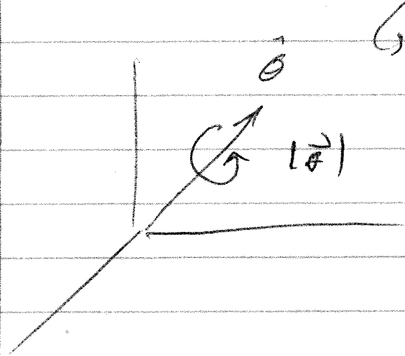
$\{\Lambda\}$ Lorentz group $\longmapsto \{M(\Lambda)\} \to$ n-dim representation of the Lorentz group

↳ $\boxed{?}$ what are the finite-dim matrix reps of the Lorentz group?

Ex in QM... spin $\frac{1}{2}$ → $\{M\}$ are the 2x2 unitary matrices with determinant 1.

→ $\boxed{\mathcal{U} = e^{-i\theta^i\sigma^i/2}}$ → $\sum_i \theta^i\sigma^i/2$

3 arbitrary parameters & Pauli matrices.



$\mathcal{U}(\vec\theta) = e^{-i\vec\theta\cdot\vec\sigma/2}$

→ In the case for arbitrary spin representations...

$$\mathcal{U}(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i \sum_l \epsilon^{jkl} J^l$$

→ check that this works for spin 1/2:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i\sum_l \epsilon^{jkl} \frac{\sigma^l}{2} \checkmark$$

→ for spinless particles... $\Psi(\vec{x})$ can be decomposed into orbital angular momentum states $J = 0, 1, 2, \ldots$
(no intrinsic spin ⇒ $J = L$)

- $\vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$
- $J^j = i \sum_l \epsilon^{jkl} x^k \nabla^l$
- $\nabla^l = -\nabla_l = -\dfrac{\partial}{\partial x^l}$

But the cross product is special to 3D case.

→ write operator as antisymmetric tensor...

$$\boxed{J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)}$$ → reproduces the cross product.

So that $J^3 = J^{12}$, etc.

→ generate to 4D: → 6 operators that generate { 3 boosts, 3 rotations, of the Lorentz group.

$$\boxed{J^{\mu\nu} = +i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

→ Spatial Rotations: $J^{jk} = i(x^j\partial^k - x^k\partial^j)$

→ Lorentz boosts along x^j axis: $J^{0j} = i(x^0\partial^j - x^j\partial^0)$

→ Now, want to get commutation rules...
→ compute the commutators of differential ops to get

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\sigma}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)}$$

Ex 3 rotations:

$$J^{12} = -J^{21}$$
$$J^{23} = -J^{32}$$
$$J^{13} = -J^{31}$$

3 boosts

$$J^{01} = -J^{10}$$
$$J^{02} = -J^{20}$$
$$J^{03} = -J^{30}$$

→ 6 total generating matrices...

Ex Consider the 4x4 matrices $(J^{\mu\nu})_{\alpha\beta}$ where μ,ν label which of the 6 matrices, while α,β label the component / matrix element...

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^{\mu}_{\ \alpha}\delta^{\nu}_{\ \beta} - \delta^{\mu}_{\ \beta}\delta^{\nu}_{\ \alpha}\right)}$$

Can verify that $(J^{\mu\nu})_{\alpha\beta}$ satisfies the comm. relation...

→ These are matrices that act on ordinary Lorentz 4-vectors...
to see this...

→ Look at elements of the Lorentz group...

$$\mathcal{U}(\omega_{\mu\nu}) = \text{Exp}\left[\frac{-i}{2} \omega_{\mu\nu} J^{\mu\nu} \right]$$

infinitesimally →
$$\simeq \mathbb{I} + \frac{-i}{2} \omega_{\mu\nu} J^{\mu\nu}$$
$$\simeq \delta^{\alpha}_{\beta} + \frac{-i}{2} \omega_{\mu\nu} (J^{\mu\nu})^{\alpha}{}_{\beta}$$

So, infinitesimally...

$$V^{\alpha} \longrightarrow \delta^{\alpha}_{\beta} + \frac{-i}{2} \omega_{\mu\nu} (J^{\mu\nu})^{\alpha}{}_{\beta} V^{\beta}$$

→ $\omega_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles...

V_{α}, V_{β} → 4-vectors...

<u>Ex 1</u> When $\omega_{12} = -\omega_{21} = \theta$, $\omega_{\mu\nu} = 0$ else, we get

$$[V^{\mu}] \longrightarrow \begin{array}{c c} & \begin{array}{cccc} t & x & y & z \end{array} \\ \begin{array}{c} t \\ x \\ y \\ z \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} [V^{\mu}]$$

→ Infinitesimal $\boxed{\text{ROTATION on } xy \text{ plane.}}$

<u>Ex 2</u> When $\omega_{01} = -\omega_{10} = \beta \Rightarrow$ get
$\omega_{\mu\nu} = 0$ else

$$[V^{\mu}] \xrightarrow{\sim} \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} [V^{\mu}] \rightarrow \boxed{\text{BOOST along } x}$$

$\boxed{\text{THE DIRAC EQUATION}}$

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to ~~develop~~ formalism for finding all other ~~formalisms~~ representations...
(Problem 3.1)

→ focus on spin $\frac{1}{2}$ systems...

→ In this case, use Dirac's trick ~~due to~~ -

Suppose we had a set of 4 $n \times n$ matrices γ^μ satisfying:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} * I}$$

then we could write down an n-dim representation of the Lorentz algebra...

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

These matrices satisfy the commutation relation...

$$\rightarrow [S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

→ Verify that this trick works in 3D Euclidean space:

in which case, $\boxed{\gamma^j = i\sigma^j}$ → $\boxed{\{\gamma^i, \gamma^j\} = -2\delta^{ij}}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\sum_k \varepsilon^{ijk}\sigma^k = J^l$$

Which is what we saw before $\Rightarrow$ angular momentum.

$$\left\{\begin{aligned} J^3 &= S^{12} = \tfrac{1}{2}\sigma^3 \\ J^2 &= S^{31} = \tfrac{1}{2}\sigma^2 \\ J^1 &= S^{23} = \tfrac{1}{2}\sigma^1 \end{aligned}\right\}$$

$\rightarrow$ now, want $S^{\mu\nu}$ for 4D Minkowski space...

$\rightarrow$ matrices γ^μ must be at least 4×4.

$\rightarrow$ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2\times 2} \\ \mathbb{I}_{2\times 2} & 0 \end{pmatrix} \;;\; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

"Weyl" / "Chiral" representation.

$\rightarrow$ In this case, the boost & rotation generators are ...

Boosts $\rightarrow$ $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

(sum) sum

Rotations $\rightarrow$ $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \sum_k \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \sum_k \frac{1}{2}\varepsilon^{ijk}\Sigma^k$

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at *all* transformations under which the ~~Lagrangian~~ Action S is "invariant"

→ In particular, we want S to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\begin{cases} \phi(x) \longrightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x^{\mu}) \rightarrow \phi(\Lambda^{-1\,\mu}{}_{\nu}\, x^{\nu}). \end{cases}$$

→ (check that S is invariant)

→ But this is very simple ... ⇒ There are *many* more transforms that leave S Lorentz invariant.

→ How do we find all of them?

⇒ For simplicity, we'll just restrict ourselves to *linear* ~~combinations~~ transformations

→ look at transformations of the form.

$$\phi_a(x) \longrightarrow \sum_b M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \longrightarrow M(\Lambda)\,\phi}$$

These matrices M must be "nice" in the sense that M must obey...

$$\boxed{\phi \to M(\Lambda'')M(\Lambda)\phi = M(\Lambda''\Lambda)\phi}$$

This says that $\{M\}$ (the collection of M's) must be a <u>representation</u> of the Lorentz group.

<u>What??</u> So... recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

$$\to \quad \boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

<u>A representation</u> π of a group G is a function satisfying the property

$$\underbrace{\pi(g_1)}_{\in G}\ \underbrace{\pi(g_2)}_{\in G} = \underbrace{\pi(g_1 g_2)}_{\in G}$$

With this, it is clear that

$$\boxed{\{\Lambda\}\ \text{Lorentz group} \implies \{M\}\ \text{is a representation of}\ \{\Lambda\}}$$

So... what are these M's?

→ <u>Ex</u> Rotation group for spin $\frac{1}{2}$ particles

For spin-$\frac{1}{2}$, the most important nontrivial representation is the 2D representation.

↳ these are unitary matrices with det = 1
(2x2)

$\Rightarrow$ In general: $\mathcal{U} = e^{-i\vec{\sigma}\cdot\vec{\theta}/2}$

$\vec{\sigma} \rightarrow$ Pauli matrices
$\vec{\theta} \rightarrow$ angle.

For infinitesimal rotations, we can write

$$\mathcal{U} = 1 - \frac{i}{\hbar}\vec{J}\cdot\vec{\theta} \equiv 1 - \vec{J}\cdot\vec{\theta}$$

$\{\mathcal{U}\}$ form a Lie-algebra of the L-group.

$\vec{J}$ here are the "generator" of the Lie algebra

when $\{\mathcal{U}\}$ is a representation of the rotational group, we identify

$$\vec{J} \Leftrightarrow \frac{\vec{\sigma}}{2}$$

$\rightarrow$ $\vec{J}$ as the quantum angular momentum operator

$\rightarrow$ satisfies the commutation relation

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

like the generators of SO(3), namely the Pauli matrices...

→ finite rotations are formed by matrix exp.

$$R = \exp\left[-i\theta^i \sigma^i\right]$$

Sep 27, 2020

Back to present problem...

to get generator of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

(3D) $\vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$

(4D) $\boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)}$$

→ any matrices that are ~~re~~ represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)$$

→ by symmetry, μ, ν ~~table~~ label which of the six matrices we want;

→ α, β label components.

The Dirac Eqn

What are the representations of the Lorentz group?

, especially for spin-$\frac{1}{2}$?

Dirac's trick: if we have a set of 4 $n \times n$ matrices γ^μ which satisfies:

Dirac algebra →
$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} * I_{n\times n}}$$

Then the n-dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4}\left[\gamma^\mu, \gamma^\nu\right]}$$

→ In other words, $S^{\mu\nu}$ satisfies ...

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho}\right)$$

(*) Note that this trick works ~~ns~~ in any dim.

e.g. take $\gamma^j = i\sigma^j$ so that $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

$$\Rightarrow S^{ij} = \frac{1}{2}\varepsilon^{ijk}\sigma^k \rightarrow \text{just as before.}$$

2D representation of the rotation group.

Spin $\frac{1}{2}$: $J^3 = J^{12} = \frac{1}{2}\sigma^3$; $J^2 = \frac{1}{2}\sigma^2 = S^{31}$, $J^1 = S^{23} = \frac{1}{2}\sigma^1$

One such representation for the Dirac algebra is

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4\times 4} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4\times 4}}$$

Weyl / chiral representation.

get

Boosts $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

Hermitian Defn

not ~~[illegible]~~, but Ψ is also classical field, not a wfn

All 4-component field Ψ that transforms under boosts & rotations according to is called a Dirac spinor

S^{ij} are Hermitian
S^{0i} are anti-Hermitian

fine b/c Ψ is a classical field, not a ~~k~~ wfn.

//

Now, what is the field eqn for ψ?

→ try $(\Box + m^2)\psi = 0$ ← KG field eqn.

But this obviously works because the representation are block-diagonal ...

→ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~the~~ at transformation of γ matrices

In an expression we can think of ...

$$[\cdots]\,\Lambda_{\frac{1}{2}}^{-1}\,[4\times 4]\,\Lambda_{\frac{1}{2}}\,[\vdots]$$

$\frac{1}{2}$: for spin $\frac{1}{2}$

where $\Lambda_{\frac{1}{2}} = \exp\left\{\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right\}$

$$\simeq 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\Rightarrow [\gamma^\mu] \to [\Lambda_{\frac{1}{2}}^{-1}][\gamma^\mu][\Lambda_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$= \ldots$ (some terms of higher order cancelled ...)

$$= \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}\underbrace{[\gamma^\mu, S^{\alpha\beta}]}_{?}$$

where a quick computation shows that

$$\boxed{[\gamma^\mu, S^{\alpha\beta}] = (\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma^\nu}$$

where

$$\mathcal{J}^{\alpha\beta} = +i\,(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu)$$

So...
$$\boxed{\gamma^\mu \longrightarrow \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma^\nu} = \Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}}$$

→ γ^μ transforms like 4-vectors ...!

⇒ γ^μ are invariant under simultaneous rotations of their <u>vectors</u> & <u>spinor</u> indices...

→ can treat "μ" on γ^μ as a vector index!

→ can dot γ^μ into ∂_μ to form a Lorentz-inv. differential operator...

<u>Dirac eqn</u>

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}$$

Check that this is Lorentz-inv:

Let $\Psi(x) \longrightarrow \Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$ then

$$i\gamma^\mu\partial_\mu\Psi \to (i\gamma^\mu\Lambda^{+1}_{\frac{1}{2}})(\partial_\mu(\Psi(\Lambda^{-1}x)))$$

$$= i\Lambda_{\frac{1}{2}}(\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}) \cdot (\Lambda^{-1})^\alpha{}_\mu(\partial_\alpha\Psi)(\Lambda^{-1}x)$$

~~same~~ Lorentz transform

$$= i\Lambda_{\frac{1}{2}} (\Lambda)^{\mu}{}_{\nu} \gamma^{\nu} \cdot (\Lambda^{-1})^{\alpha}{}_{\mu} \, (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \underbrace{(\Lambda)^{\mu}{}_{\nu}(\Lambda)^{\alpha}{}_{\mu}}_{\delta^{\alpha}_{\nu}} (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \partial_\mu \Psi(\Lambda^{-1}x)$$

$$\Rightarrow \quad i\gamma^{\mu}\partial_\mu \Psi(x) \to \Lambda_{\frac{1}{2}} i\gamma^{\mu} \Psi(\Lambda^{-1}x)$$

$\to$ transforms the same way as $\Psi(\Lambda^{-1}x)$.

Cleaner way:

Let $[i\gamma^{\mu}\partial_\mu - m]\Psi(x) \to [i\gamma^{\mu}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m]\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$

$$= \Lambda_{\frac{1}{2}} \underbrace{\Lambda^{-1}_{\frac{1}{2}} [i\gamma^{\mu}\Lambda^{-1\,\nu}{}_{\mu}\partial_\nu - m] \Lambda_{\frac{1}{2}}} \Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}} \{ i\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m\} \Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}} \{ i\gamma^{\nu}\partial_\nu - m\} \Psi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i\gamma^{\mu}\partial_\mu - m)\Psi$$

$$\to \quad 0 = (-i\gamma^{\mu}\partial_\mu - m)(+i\gamma^{\nu}\partial_\nu - m)\Psi$$

$$= (\gamma^{\mu}\gamma^{\nu}\partial_\nu\partial_\mu + m^2)\Psi \quad = \ldots$$

$$= \left(\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2 \right) \psi$$

$$= \left[\frac{1}{2} \left(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \right) \partial_\mu \partial_\nu + m^2 \right] \psi$$

$$= \left[\frac{1}{2} \{ \gamma^\mu, \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right] \psi$$

KG eqn.

$$= \left(g^{\mu\nu} \partial_\mu \partial_\nu + m^2 \right) \psi = \left(\Box + m^2 \right) \psi = 0$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger \psi$ doesn't work b/c under a boost,

$$\psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \Lambda_{\frac{1}{2}} \psi \neq \psi^\dagger \psi \quad \text{since } \Lambda_{\frac{1}{2}} = \exp\{-i\omega_{\mu\nu} S^{\mu\nu}\}$$

not unitary ... since not all $S^{\mu\nu}$ are Herm.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, get

$$\bar{\psi} \rightarrow \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \gamma^0 \cong \psi^\dagger \left(1 + \frac{i}{2} \omega_{\mu\nu} (S^{\mu\nu})^\dagger \right) \gamma^0$$

when ~~[illegible]~~ $\mu \neq 0$ & $\nu \neq 0$, $(S^{\mu\nu})^\dagger = (S^{\mu\nu})$

$$\& \; (S^{\mu\nu} \leftrightarrow \gamma^0)$$

when $\mu = 0$ or $\nu = 0$, $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$

$\Rightarrow$ $S^{\mu\nu}$ anti-commutes w/ γ^0.

$$\rightarrow \quad \bar{\Psi} \rightarrow \Psi^\dagger \left(1 + \frac{i}{2} w_{\mu\nu} (S^{\mu\nu})^\dagger\right) \gamma^0$$

$$= \underbrace{\Psi^\dagger \gamma^0}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right)$$

$$= \bar{\Psi}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right) = \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \quad \text{as desired.}$$

$$\rightarrow \quad \boxed{\bar{\Psi} \rightarrow \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1}}$$

and so $\boxed{\bar{\Psi}\Psi = \Psi^\dagger \gamma^0 \Psi}$ is a Lorentz scalar.

Similarly, can show that

$\boxed{\bar{\Psi}\gamma^\mu \Psi}$ is a Lorentz vector.

$\Rightarrow$ the correct Lorentz-inv Dirac Lagrangian is

$$\boxed{\mathcal{L}_{\text{Dirac}} = \bar{\Psi}(i\gamma^\mu \partial_\mu - m)\Psi}$$

$$\begin{cases} \text{E-L eqn for } \bar{\Psi} \text{ gives } (i\gamma^\mu \partial_\mu - m)\Psi = 0 \\ \text{E-L eqn for } \Psi \text{ gives } -i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} = 0 \end{cases}$$

WEYL SPINOR

Recall that

$$\boxed{\begin{aligned} S^{0i} &= \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \end{aligned}}$$

Since block-diagonal $\Rightarrow$ Dirac representation of the Lorentz group is reducible.

$\rightarrow$ Can form 2-D representations by considering each block separately...

$\rightarrow$ write $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\begin{matrix} \rightarrow \text{Left-handed} \\ \rightarrow \text{RH} \end{matrix}$ Weyl spinors

Under infinitesimal boost $\vec{\beta}$ & rotation $\vec{\theta}$, these transform as

$$\boxed{\begin{aligned} \psi_L &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_L \\ \psi_R &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_R . \end{aligned}}$$

Recall that $\left(\tanh(\vec{\beta}) = \frac{|v|}{c}\right)$.

$\rightarrow$ Transf of ψ_R is equiv to transf of ψ_L^*

By noting that

$$\psi_L^* \longrightarrow \left(1 + i\tilde{\theta}\cdot\frac{\vec{\sigma}}{2}^* - \tilde{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

noting that $\underset{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}{\sigma^2}\vec{\sigma}^* = -\vec{\sigma}\,\underset{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}{\sigma^2}$ $\quad (\sigma^2 = \sigma^y)$

we find.

$$\underset{\sim\sim}{\sigma^2\psi_L^*} \longrightarrow \sigma^2\left[1 + i\tilde{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \tilde{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right]\psi_L^*$$

$$= \underbrace{\left[1 - i\tilde{\theta}\cdot\frac{\tilde{\sigma}}{2} + \tilde{\beta}\cdot\frac{\vec{\sigma}}{2}\right]}_{\text{like } \psi_R \text{ transform.}}\underset{\sim\sim}{\sigma^2\psi_L^*}$$

So $\sigma^2\psi_L^*$ transform like ψ_R...

With $\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$, the Dirac eqn has form.

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0 \iff \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0}$$

When $m = 0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left.\begin{cases} i(\partial_0 - \tilde{\sigma}\cdot\vec{\nabla})\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0 \end{cases}\right\} \longrightarrow \text{Weyl eqns.}$$

$\longrightarrow$ important for neutrinos & weak force studies...

For convenience let us define ...

$$\boxed{\sigma^\mu \equiv (1, \vec{\sigma}) \quad ; \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})}$$

so that

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$

With this, can simplify notation .. Dirac eqn becomes ...

$i\sigma\cdot\partial \to i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})$

$$\boxed{\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0}$$

$i\bar{\sigma}\cdot\partial \to i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})$

& the Weyl eqns become:

$$\boxed{\begin{aligned} (i\bar{\sigma}\cdot\partial)\psi_L &= 0 \\ (i\sigma\cdot\partial)\psi_R &= 0 \end{aligned}}$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p)\, e^{-ip\cdot x} \quad , \quad p^2 = m^2$$

so that $p^0 = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}$.

Look only solutions with positive frequency ... that is
$E_p = p^0 > 0$ ....

ψ solves Dirac eq^n $\Rightarrow$ $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$\Leftrightarrow$ $(i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$

$\Rightarrow$ $\boxed{(i\gamma^\mu p_\mu - m)\, u(p) = 0}$

go to rest frame $\Rightarrow$ $p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $\Lambda_{\frac{1}{2}}$.

In rest frame, we have

$(i\gamma^\mu p_\mu - m)\, u(p) \Rightarrow (m\gamma^0 - m)\, u(p_0) = m(\gamma^0 - 1)\, u(p_0) = 0$

$\Rightarrow$ $m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$

$\Rightarrow$ $\boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}}$ $\rightarrow$ two-component spinor

just a factor, with norm. constraint.

$\xi^\dagger \xi = 1$.

<u>What are these ξ?</u>

Look at <u>rotation</u> generators...

$$\boxed{S^{ij} = \frac{1}{2}\varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

In particular, $S^z = S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}$

So if $\left\{ \begin{array}{l} \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$

Now, we're in rest frame, so $p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Now, boost to frame where particle has velocity...

$\vec{v} = v\cdot\vec{z}$. Let $\underbrace{\tanh(\eta)}_{\text{"rapidity"}} = \frac{v}{c}$.

Then

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = p^\mu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh(\eta) \end{pmatrix}\begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ 0 \\ 0 \\ m\sinh\eta \end{pmatrix}$$

(infinitesimal $\Lambda_{\frac{1}{2}}$)

$\Lambda_{\frac{1}{2}} \rightarrow$ just the Lorentz transform.

$\rightarrow$ In this frame, $\boxed{\begin{array}{l} E = m\cosh\eta \\ p^3 = m\sinh\eta \end{array}}$

Now, apply the same boost to $u(p)$ ...

$$u(p) = \Lambda_{\frac{1}{2}} \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad \rightarrow (\Lambda_{\frac{1}{2}}) = \exp\left(\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

$$= \exp\left(\frac{-1}{2}\eta\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right)\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

S^{03} ... $i/(\sigma^3 \quad \sigma^3)$

So, infitesimally...

$$\exp\left\{ \frac{-1}{2}\eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right\} \simeq \begin{pmatrix} \cosh \eta/2 - \sinh \eta/2\, \sigma^3 & 0 \\ 0 & \cosh \eta/2 + \sinh \frac{\eta}{2}\sigma^3 \end{pmatrix}$$

So that

$$u(p) \simeq \sqrt{m} \begin{pmatrix} (\cosh \eta/2 - \sinh \eta/2\, \sigma^3)\xi \\ (\cosh \eta/2 + \sinh \eta/2\, \sigma^3)\xi \end{pmatrix} \ldots$$

Simplify ... note that

$$\left(\cosh \eta/2 - \sinh \eta/2\, \sigma^3\right)^2 = \ldots$$

$$= \cosh \eta - \sinh \eta\, \sigma^3 \quad p_\mu \sigma^\mu / m$$

$$= E/m - p^3/m\, \sigma^3 = \frac{p\cdot\sigma}{m}$$

$$= \frac{p_\mu \sigma^\mu}{m} \quad \text{where } \sigma^\mu = (1, \vec{\sigma})$$

So...

$$(\cosh \eta/2 - \sinh \eta/2\, \sigma^3) = \sqrt{\frac{p\cdot\sigma}{m}}$$

&

$$(\cosh \eta/2 + \sinh \eta/2\, \sigma^3) = \sqrt{\frac{p\cdot\bar{\sigma}}{m}}$$

So...

$$\boxed{u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}} \longrightarrow \text{compact \& valid for any arbitrary direction of } \vec{p}.$$

Fact $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0)^2 - \vec{p}^{\,2} = p^2 = m^2.$

Now, back to example...

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow \quad p\cdot\sigma = \cdots = \begin{pmatrix} E-p^3 & 0 \\ 0 & E+p^3 \end{pmatrix}$$

so

$$\sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}$$

and

$$\sqrt{p\cdot\bar\sigma} = \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}$$

pick $\xi = \begin{pmatrix}1\\0\end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix}0\\1\end{pmatrix}$ then (spin $\frac{-1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}$$

In the massless limit, $E \to p^3$ $\left(E^2 = \sqrt{m^2c^4 + (p^3)^2}\right)$

$$\rightarrow \boxed{\begin{array}{ll} u(p) = \begin{pmatrix} \begin{pmatrix}0\\0\end{pmatrix} \\ \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} & \text{spin } \frac{1}{2} \\ u(p) = \begin{pmatrix} \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} \end{pmatrix} & \text{spin } \frac{-1}{2} \end{array}}$$

These states: $u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $u(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are eigenstates of the <u>helicity</u> operator

$$\boxed{h \equiv \hat{p}\cdot\vec{S} = \sum_i \frac{1}{2}\hat{p}^i \begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}} = \frac{1}{2}\hat{p}\cdot\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

When $\begin{cases} h = \frac{1}{2} \Rightarrow \text{call Right-handed} \\ h = \frac{-1}{2} \Rightarrow \text{call Left-handed}\end{cases}$

<u>Note</u>: Dirac helicity is frame-dependent... (for massive particle). — Since can boost so that momentum is in the opposite direction.

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = i(\bar{\sigma}\cdot\partial)\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = i(\sigma\cdot\partial)\psi_R = 0 \end{cases}$$

Plug $\psi = u(p)e^{-ip\cdot x}$ in, $\partial_0 \longrightarrow -iE$

$\vec{\nabla} \longrightarrow i\vec{p}$.

So, with $m = 0$, $\vec{p} = E\hat{p}$.

$$\Rightarrow \text{get} \begin{cases} (E + E\hat{p}\cdot\sigma)\psi_L = 0 \Rightarrow (E)(1+2h)\psi_L = 0 \xrightarrow{} h = \frac{-1}{2} \\ (E - E\hat{p}\cdot\sigma)\psi_R = 0 \Rightarrow (E)(1-2h)\psi_R = 0 \longrightarrow h = \frac{1}{2} \end{cases}$$

$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}$, as expected

//

Recap... $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 > 0)$ → positive frequency

$\rightarrow$ $u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$ → spinor.

when $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 < 0)$ → negative frequency

$\Rightarrow$ $(u(p)) = \dots = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$

//

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = \left(\xi^\dagger\sqrt{p\cdot\sigma},\ \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)\cdot\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger\Big[(p\cdot\sigma) + (p\cdot\bar\sigma)\Big]\xi$$

→ depends on p!

$\Rightarrow$ $u^\dagger u = 2E_p\, \xi^\dagger\xi$ ~~dlll~~

→ ~~dll~~ $u^\dagger u$ is <u>not</u> a Lorentz-inv scalar. just like $\psi^\dagger\psi$.

$\Rightarrow$ to make one such Lorentz-inv scalar, define

$\boxed{\bar u(p) = u^\dagger(p)\gamma^0}$ $\Longrightarrow$ $\boxed{\bar u u = 2m\,\xi^\dagger\xi}$ Lorentz-inv (indep of $\vec p$)

So, with ~~then~~ $\bar{u}u = u^\dagger \gamma^0 u = 2m \underbrace{\xi^\dagger \xi}_{1} = 2m$

→ convenient to choose ONB spinors. ξ^1 & ξ^2.

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\;\xi^s \end{pmatrix} \qquad s = 1,2}$$

Normalize:

$$\bar{u}^r(p)\, u^s(p) = 2m\, \delta^{rs} \Leftrightarrow u^{r\dagger}(p)\, u^s(p) = 2E_p\, \delta^{rs}$$

For the negative - freq solns, we get

$$\bar{v}^r(p)\, v^s(p) = -2m\, \delta^{rs} \Leftrightarrow v^{r\dagger}(p)\, v^s(p) = +2E_p\, \delta^{rs}$$

and

v, u are orthogonal to each other ...

$$\bar{u}^r(p)\, v^s(p) = \bar{v}^r(p)\, u^s(p) = 0$$

Finally, talk about $\boxed{\text{spin sums}}$

→ useful when evaluating Feynman diagrams.

→ when we need to sum all spin-$\frac{1}{2}$ polarization

Since $\{\xi^s\}$ form an ONB, ...

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}},\ \xi^{s\dagger}\sqrt{p\cdot\sigma} \right)$$

$$= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \cdot \left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar{\sigma}} \right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

"completeness" $\longrightarrow$

$$= \begin{pmatrix} p\cdot\sigma & m \\ m & p\cdot\bar{\sigma} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix}$$

$$\left(\begin{aligned} \sqrt{p\cdot\bar{\sigma}}\,\sqrt{p\cdot\sigma} &= \cancel{\tfrac{1}{2}\left((p\cdot\bar{\sigma})+(p\cdot\sigma)\right)^2 - p\cdot\sigma - p\cdot\bar{\sigma}} = m \\ &= \sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} = \sqrt{\left((p^0,\vec{p})\cdot(1,-\vec{\sigma})\right)\left((p^0,\vec{p})\cdot(1,\vec{\sigma})\right)} \\ &= \sqrt{(p^0)^2 - p^2} = m. \end{aligned} \right)$$

So,

$$\boxed{\begin{aligned} \sum_{s=1,2} u^s(p)\,\bar{u}^s(p) &= \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar{\sigma} & m \end{pmatrix} = \underbrace{p\cdot\gamma}_{\not{p}} + m\,I \\ \sum_{s=1,2} v^s(p)\,\bar{v}^s(p) &= \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} = \underbrace{p\cdot\gamma}_{\not{p}} - m\,I \end{aligned}}$$

Feynman's slash notation

→ The combo $\gamma \cdot p$ occur so often that Feynman introduced the notation:

$$\boxed{\not{p} \equiv \gamma^\mu p_\mu = p_\mu \gamma^\mu}$$

~~Gamma Matrices & Dirac Field Bilinears~~

<u>Exercise</u> Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L. The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

$$m = \text{Majorana mass.}$$

(a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.

(b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

(a) In general, infinitesimal Lorentz transforms on ψ_L has the form

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

→ Rotation has the form:

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

$$\Rightarrow \sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L(x) \quad \text{(notes)}$$

Lorentz transform:

$$\psi_L(x) \to \Lambda_{\frac{1}{2}}\psi_L(\Lambda^{-1}x)$$

$$\partial_\mu\psi_L(x) \longrightarrow (\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

→ put these together...

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(\Lambda^{-1}x)$$

$$\sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

$$\rightarrow \quad -im\sigma^2\psi_L^*(x) \longrightarrow -im\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

Next, $i\bar{\sigma}\cdot\partial\psi_L(x) = i\bar{\sigma}^\mu\partial_\mu\psi_L(x)$

$$\Rightarrow i\bar{\sigma}^\mu(\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= i\bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)(\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

use trick: multiply:

$$1 \equiv \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right) \quad \text{(rot × inv\_rot)}$$

$$\Rightarrow \rightarrow i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\overbrace{\left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)}$$

$$\times\ (\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_{L}(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\boxed{?}\,(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_{L}(\Lambda^{-1}x)$$

what is $\boxed{?}$

$$\rightarrow \boxed{?} = \left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\vec{\sigma}/2\right)$$

$$\cong \bar{\sigma}^{\mu} + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\bar{\sigma}^{\mu} - i\vec{\theta}\cdot\bar{\sigma}^{\mu}\frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^{\mu} - \frac{i}{2}\vec{\theta}\left[\bar{\sigma}^{\mu},\vec{\sigma}\right]$$

can show that

$$= \bar{\sigma}^{\mu} - i\theta\left[J_{4}^{ij}\right]^{\mu}_{\ \nu}\bar{\sigma}^{\nu}$$

$$\downarrow$$

$$i\left(g^{i\mu}\delta^{j}_{\nu} - g^{j\mu}\delta^{i}_{\nu}\right)$$

$$\Rightarrow \boxed{?} = (\Lambda_{4})^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu} \longrightarrow$$ $\bar{\sigma}^{\mu}$ transforms like 4-vector

$$\Rightarrow i\bar{\sigma}\cdot\partial\Psi_{L}(x) \Rightarrow i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Lambda^{\mu}_{\ \nu}\bar{\sigma}^{\nu}(\Lambda^{-1})^{\alpha}_{\ \mu}\partial_{\alpha}\Psi_{L}(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\delta^{\alpha}_{\nu}\bar{\sigma}^{\nu}\partial_{\alpha}\Psi_{L}(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\nu}\partial_{\nu}\Psi_{L}(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}\cdot\partial\Psi_{L}(\Lambda^{-1}x)$$

$\rightarrow \quad i\bar{\sigma}\cdot\partial\psi_L(x) - im\,\sigma^2\psi_L^*(x) = 0$

$\rightarrow$ leads to infinitesimal rotation...

$$\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\left\{\underbrace{i\bar{\sigma}\cdot\partial\psi_L(\Lambda^{-1}x) - im\,\sigma^2\psi_L^*(\Lambda^{-1}x)}_{=0}\right\} = 0$$

$\Rightarrow$ done! So Majorana eqn is invariant under infinitesimal rotations...

(b) <u>Boosts</u> (proceed in a similar way...)

<u>Key</u>

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= \bar{\sigma}^\mu - \frac{1}{2}\vec{\beta}\left\{\bar{\sigma}^\mu, \vec{\sigma}\right\}$$

$$\rightarrow \quad \bar{\sigma}^\mu \longrightarrow \bar{\sigma}^\mu - i\vec{\beta}\left[\mathcal{J}_4^{0i}\right]^\mu{}_\nu\,\bar{\sigma}^\nu$$

$$\rightarrow \quad \left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = (\Lambda)^\mu{}_\nu\,\bar{\sigma}^\nu \quad \text{as before}$$

so plug this back into the original eqn.

Sep 28, 2020 $\boxed{\text{Dirac Matrices \& Dirac Field Bilinears}}$

Oct 2, 2020 Recall that $\bar{\psi}\psi$ is Lorentz scalar...

Recall that $\bar{\psi}\gamma^\mu\psi$ is also a 4-vector.

$\Rightarrow$ $\boxed{?}$ Consider $\bar{\psi}\Gamma\psi$, where Γ is any 4×4

→ can we decompose into terms that have definite transformation properties under the Lorentz group?

↳ Γ can be written as combo of 16-element basis defined by

$$
\begin{array}{lll}
1: & \mathbb{1} & \to 1 \\
4: & \gamma^\mu & \to 4C2 \\
6: & \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} & \to 4C3 \\
4: & \gamma^{\mu\nu\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^{\sigma]} & \to 4C2 \\
1: & \gamma^{\mu\nu\sigma\rho} = \gamma^{[\mu}\gamma^\nu\gamma^\sigma\gamma^{\rho]} & \to 4C2
\end{array}
$$

16 total.

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisym tensor of successively higher rank.

→ Introduce

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$= \frac{-i}{4!}\varepsilon^{\mu\nu\sigma\rho}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho$$

0123 → 1

1023 → −1

↳ totally anti-symmetric

Note that → $\boxed{(\gamma^5)^2 = \mathbb{1}}$

$$\rightarrow \boxed{\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^3)^\dagger \ldots (\gamma^0)^\dagger \\ &= +i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5\end{aligned}}$$

also

$$\boxed{\begin{aligned}\{\gamma^5, \gamma^\mu\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \quad (-1) \\ &= 0\end{aligned}}$$

and thus

$$\boxed{[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}\{\gamma^\mu, \gamma^\nu\}] = 0}$$

⇒ Eigenstates of γ^5 with different eigv don't mix under Lorentz transform.

→ In basis, can write..

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{array}{l} \rightarrow \text{ for } \psi_L \text{ (left-hd)} \\ \rightarrow \text{ for } \psi_R \text{ (right-hd)} \end{array}$$

→ a Dirac spinor with only L/R component is an eigenstate of γ^5 with eigv $(-1)/(1)$.

With γ^5, can rewrite the table of 4×4 matrices as

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}\{\gamma^\mu, \gamma^\nu\}$	tensor	6
$\gamma^\mu\gamma^5$	pseudo vector	4
γ^5	pseudo scalar	1
		16

pseudo-vector/scaler is due to the fact that they transform like vector/scaler, BUT with an additional under Lorentz transf. → sign change under parity-transf.

Ex Parity transf: $\vec{x} \to -\vec{x}$

$$\hookrightarrow (x^0, x^i) \to (x^0, -x^i)$$

If instead $(x^0; \vec{x}) \to -(x^0; -\vec{x}) = (-x^0; \vec{x})$ under parity, we call this a pseudo-vector

→ pseudo vect/scaler flips sign under parity transf.

→ From vector & pseudo-vector we can form 2 currents out of Dirac field bilinears...

$$\boxed{\begin{aligned} j^\mu(x) &= \bar{\Psi}(x)\gamma^\mu\Psi(x) \to \text{vector current} \\ j^{\mu 5}(x) &= \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) \to \text{pseudo vector current} \end{aligned}}$$

Assume that Ψ satisfies Dirac eqn... $\bar{\Psi} = \Psi^\dagger\gamma^0$

$$\to i\not{\partial}\Psi = m\Psi \quad \& \quad -i\not{\partial}\bar{\Psi} = m\bar{\Psi} \quad (\text{since } \bar{\gamma}^\mu = \gamma^\mu)$$

→ compute div of these currents...

$$\partial_\mu j^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu(\partial_\mu\Psi)$$

$$= im\bar{\Psi}\Psi + \bar{\Psi}(-im\Psi) = 0$$

$$\to \boxed{\partial_\mu j^\mu = 0}$$

→ j^μ is always <u>conserved</u> if $\psi(x)$ satisfies Dirac eqn

→ It is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha}\psi(x)$$

<u>Similarly</u>

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= (\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi + (-1)\bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi$$

$$= im\bar{\psi}\gamma^5\psi + (-1)(-i)m\bar{\psi}\gamma^5\psi$$

→ $\boxed{\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\gamma}$ ⇝ "axial vector current

→ if $m=0$ then $\partial_\mu j^{\mu 5}$ is conserved.

→ When $m=0$, $j^{\mu 5}$ is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha\gamma^5}\psi(x).$$

(we worry about ~~that~~ the rest of this section in ~~Weyl~~ Peskin's ...)

QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi .$$

→ Canonical momentum conjugate to ψ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0 \psi)} = i\bar{\psi}\gamma^0 = i\psi^\dagger .$$

→ Hamiltonian is given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\psi)}\partial_0\psi - \mathcal{L} = \cancel{i\bar{\psi}\gamma^0\partial_0\psi} - \cancel{i\bar{\psi}\gamma^0\partial_0\psi}$$

$$- i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi$$

$$= -i\bar{\psi}\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi$$

Thus, $\boxed{H = \int \mathcal{H} d^3x = \int d^3x\, \bar{\psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\psi}$

→ now let's figure out the commutators to make this a quantum field theory...

→ $\boxed{\text{HOW } \underline{\text{NOT}} \text{ TO QUANTIZE THE DIRAC FIELD}}$

This won't work!

<u>Guess</u> $$[\psi_a(\vec{x}), i\psi_b^\dagger(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

spinor components $(a, b = 1, 2, 3, 4)$

i.e. $$[\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

In matrix notation...

$$[\psi_a(\vec{x}), \psi_a^\dagger(\vec{y})] = \mathbb{1}_{4\times4}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix} \qquad [\,-\;-\;-\;-\,]$$

Also guess $$[\psi_a(\vec{x}), \psi_b(\vec{y})] = 0$$

$$= [\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})] = 0$$

Note that

$$[\psi(\vec{x}), \bar{\psi}(\vec{y})] = [\psi(x), \psi^\dagger(y)\gamma^0]$$

$$= [\psi(x), \psi^\dagger(y)]\gamma^0 = \gamma^0\delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote...

(real field) $$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\vec{a}_p + \vec{a}^{\,\dagger}_{-p}\right\} e^{i\vec{p}\cdot\vec{x}}. \quad \text{(FT)}$$

For <u>complex</u> field, we get

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\hat{a}_p + \vec{b}^{\,\dagger}_{-p}\right\} e^{+i\vec{p}\cdot\vec{x}}.$$

In the case of Dirac field, need spin degree of freedom...

Try ...

$$\Psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{r} u^r(\vec{p}) + b_{-\vec{p}}^{r} v^r(-\vec{p}) \right\} e^{+i\vec{p}\cdot\vec{x}}$$

Spin degrees of freedom

Fourier components : $\Psi(x) = u(p) e^{ip}$

$$\Psi^{+}(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}}^{r+} u^{r+}(\vec{p}) + b_{-\vec{p}}^{r+} v^{r+}(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

Recall that u, v also solves Dirac eqn in the sense that (in momentum space...)

$$\boxed{p^\mu \gamma_\mu u^r(\vec{p}) = m u^r(p) \quad \& \quad p^\mu \gamma_\mu v^r(\vec{p}) = -m v^r(p)}$$

We can try the commutators ...

$$[a_{\vec{p}}^{r}, a_{\vec{p}'}^{s+}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[b_{\vec{p}}^{r}, b_{\vec{p}'}^{s}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[a_{\vec{p}}^{r}, a_{\vec{p}'}^{s}] = 0$$

$\vdots$

the rest are all zero...

We find that

as desired...

$$[\Psi_a(\vec{x}), \Psi_b(\vec{y})] = 0 = [\Psi_a^{+}(\vec{x}), \Psi_b^{+}(\vec{y})]$$

We also find that

$$[\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})] = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian...

$$\mathcal{H} = \int d^3x\,\left[-i\bar{\psi}\,\vec{\gamma}\cdot\vec{\nabla}\psi + m\bar{\psi}\psi\right]$$

$$= \int d^3x\,\left\{\psi^\dagger\gamma^0\left[-i\vec{\gamma}\cdot\vec{\nabla}+m\right]\psi\right\}$$

Now, with $p^\mu\gamma_\mu u^r(p) = m u^r(p)$

$$\longrightarrow (\vec{p}\cdot\vec{\gamma}+m)\,u^r(p) = p^0\gamma^0 u^r(p) = E_p\gamma^0 u^r(p)$$

just coefs

Similarly, b/c $p^\mu\gamma_\mu v^{(r)}(p) = -m v^{(r)}(p)$

$$\longrightarrow (\vec{p}\cdot\vec{\gamma}+m)\,v^r(p) = -E_p\gamma^0 v^{(r)}(p).$$

So....

$$\to\ [-i\vec{\gamma}\cdot\vec{\nabla}+m]\psi = [-i\vec{\gamma}\cdot\vec{\nabla}+m]\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left[a_p^r u_p^r + b_{-p}^{r\dagger} v^r\right]e^{i\vec{p}\cdot\vec{x}}$$

$$= \gamma^0\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{E_p a_p^r u^r(\vec{p}) - E_{p} b_{-p}^{r\dagger} v^r(\vec{p})\right\}e^{i\vec{p}\cdot\vec{x}}$$

So....

$$\mathcal{H} = \int d^3x\,\left\{\psi^\dagger\gamma^0\gamma^0\sum_{r=1,2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\cdots\right\}e^{i\vec{p}\cdot\vec{x}}\right\}$$

plug in

$$\rightarrow \mathcal{H} = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} E_p \left\{ a_p^{r\dagger} a_p^r - \underbrace{b_{+p}^r b_{+p}^{r\dagger}}_{=\, b_{+p}^{r\dagger} b_{+\vec{p}}^r + \cancel{\text{const}}} \right\}$$

$\boxed{!}$

→ By creating more and more particles with $b^\dagger$, we can lower the energy indefinitely

→ This is bad...

→ So we should use Fermi-Dirac statistics instead → anti-commutators instead of commutators...

→ Require that

$$\boxed{\{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

no longer bosonic! & all other anti-commutators are zero...

When this is true, we find that

$$\boxed{\begin{aligned} \{\Psi_a(\vec{x}), \Psi_b^\dagger(\vec{y})\} &= \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab} \\ \{\Psi_a(\vec{x}), \Psi_b(\vec{y})\} &= \{\Psi_a^\dagger(x), \Psi_b^\dagger(y)\} = 0 \end{aligned}}$$

where we're using

$$\Psi(x) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_{\vec{p}} \left(a_p^{r\dagger} a_{\vec{p}}^r - \underbrace{b_{-\vec{p}}^r b_{-\vec{p}}^{r\dagger}}_{-b_{-p}^{r\dagger} b_{-p}^r + \cancel{\text{const}}} \right)$$

$$\Rightarrow \boxed{\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{-\vec{p}}^{r\dagger} b_{-\vec{p}}^r \right\}}$$

now good, b/c E is bdd below...

→ also can compute...

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{+\vec{p}}^{r\dagger} b_{+\vec{p}}^r \right)}$$

To avoid sign confusion, we will usually write...

$$\boxed{\Psi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_{\vec{p}}^r u^r(\vec{p}) e^{+i\vec{p}\cdot\vec{x}} + b_{\vec{p}}^{r\dagger} v^r(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right)}$$

As a Heisenberg field,

$$\boxed{\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u^r(\vec{p}) e^{-ip\cdot x} + b_p^{r\dagger} v^r(\vec{p}) e^{ip\cdot x} \right)}$$

where:

$$\begin{cases} a_p^r & : \text{annihilates particles} \\ a_p^{r\dagger} & : \text{creates particles} \\ b_p^r & : \text{annihilates anti-particles} \\ b_p^{r\dagger} & : \text{creates anti-particles.} \end{cases}$$

Vacuum state as $|0\rangle$ where

$$\begin{cases} a_p^s|0\rangle = 0 \\ b_p^s|0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ covariant norm:

$$\boxed{|\vec{p},s\rangle = \sqrt{2E_p}\; a_p^{s\dagger}|0\rangle}$$

so that

$$\boxed{\langle \vec{p},s|\vec{q},r\rangle = \sqrt{2E_p}\sqrt{2E_{\vec{q}}}\;\delta^{(3)}(\vec{p}-\vec{q})\,\delta^{rs}}$$

Now, look at Lorentz transform ...

$$\Psi(x) \xrightarrow{\text{Lortz}} \Psi'(x) = \Lambda_{\frac{1}{2}}\Psi(\Lambda_4^{-1}x)$$

recall that with $\rightsquigarrow \exp\{-i\omega_{\mu\nu}S^{\mu\nu}/2\} \simeq \mathbb{1} - \frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}$

$$\begin{cases} \omega_{12} = -\omega_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} \end{cases}$$

$= \mathbb{1} - \frac{i}{2}\vec{\theta}\cdot\big(\vec{\sigma}$

$\rightarrow$ and $\Psi(\Lambda_4^{-1}x) \simeq [1 - \vec{\theta}\cdot\vec{J}]\Psi(x)$

$$\vec{J} = \vec{x}\times(-i\vec{\nabla})$$

So ~~Which~~ $\Psi \to \Psi + \delta\Psi$ where

$$\delta\Psi = \Psi' - \Psi = \left(\frac{-i}{2}\vec{\theta}\cdot\vec{\Sigma}\right)\Psi(x) - \vec{\theta}\cdot(\vec{x}\times\vec{\nabla})\,\Psi(x)$$

By Noether's thm,

$$\vec{J}_{total}\ (\text{total spin}) = \int \frac{d^3x}{2}\left[\psi^\dagger(-i\vec{x}\times\vec{\nabla})\psi + \frac{1}{2}\psi^\dagger\vec{\Sigma}\psi\right].$$

We won't worry abt the rest of this section about propagators

↳ we'll come back to them later when looking at Feynman diagrams.

$$\boxed{\text{DISCRETE SYMMETRIES OF THE DIRAC THEORY}}$$

Basically, we have

Parity —— Time Reversal

Charge conjugation

Recall that before, we looked at implementation of <u>continuous</u> Lorentz transform –

→ found that $\forall \Lambda \in$ Lorentz group

$\exists\, \mathcal{U}(\Lambda)$ unitary for which

$$\mathcal{U}(\Lambda)\psi(x)\mathcal{U}^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\psi(\Lambda x).$$

→ Now, we'll talk about <u>Discrete</u> symmetries on the Dirac field.

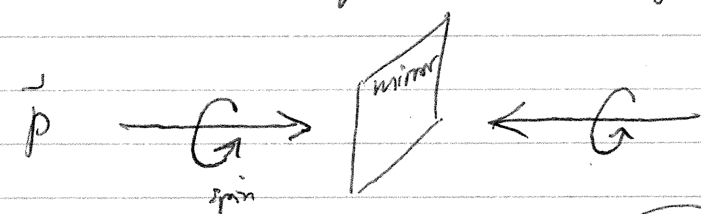
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant –

$\rightarrow$ e.g. $\left\{ \begin{array}{c} \text{time-reversal} \\ \text{parity} \end{array} \right\}$.

$\boxed{\text{Parity}}$ (P) : flips direction of spatial vectors

$$\boxed{P : (t, \vec{x}) \longrightarrow (t, -\vec{x})}.$$

$\rightsquigarrow$ mirror sym $\rightarrow$ change the handedness.



$\rightsquigarrow$ Note: momentum flips sign, but spin is unchanged.

$\boxed{\text{Time-reversal}}$

$$\boxed{T : (t, \vec{x}) \rightarrow (-t, \vec{x})}$$

P, T don't belong to the "proper" Lorentz group $L_+^\uparrow$

$\rightarrow$ the full Lorentz group breaks into 4 disconnected subsets ...

$\hookrightarrow$

$$\begin{tikzcd} L_+^{\uparrow} \arrow[r, leftrightarrow, "P"] \arrow[d, leftrightarrow, "T"] & L_-^{\uparrow} = PL_+^{\uparrow} \arrow[d, leftrightarrow, "T"] \\ L_+^{\downarrow} = TL_+^{\uparrow} \arrow[r, leftrightarrow, "P"] & L_-^{\downarrow} = PTL_+^{\uparrow} \end{tikzcd}$$

"orthochronous"

"non-orthochronous"

"proper" "improper"

$\boxed{\text{charge conjugation}}$ → interchange particles & anti-particles.

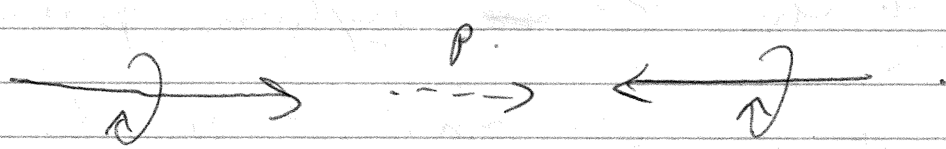
↳ non-space-time.

Let's look at $\boxed{\text{Parity}}$.

Note that because $P: (t, \vec{x}) \to (t, -\vec{x})$

→ momentum flips sign

but not spin! ⇒ what is P? As an operator?



As an operator on creation / annihilation ops, we want

$$\boxed{P^\dagger a_{\vec{p}}^s P = a_{-\vec{p}}^s \quad \& \quad P^\dagger b_{\vec{p}}^s P = b_{-\vec{p}}^s}$$

where, as discussed, P must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, get

$$\boxed{P^\dagger a^{s\dagger}_{\vec p} P = a^{s\dagger}_{-\vec p} \quad \& \quad P^\dagger b^{s\dagger}_{\vec p} P = b^{s\dagger}_{-\vec p}}$$

But these might be too restrictive ... we can get better constraints by requiring that

$$P^\dagger a^s_{\vec p} P = \eta_a a^s_{-\vec p} \quad \& \quad P^\dagger b^s_{\vec p} P = \eta_b b^s_{-\vec p}$$

as long as $|\eta_a|^2 = |\eta_b|^2 = \mathbb{1}$ are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel:

$$\left\{ \begin{aligned} P^\dagger a^{s'\dagger}_{\vec p} a^s_{\vec p} P &= a^{s'\dagger}_{-\vec p} a^s_{-\vec p} \\ P^\dagger b^{s'\dagger}_{\vec p} b^{s'}_{\vec p} P &= b^{s\dagger}_{-\vec p} b^s_{-\vec p}. \end{aligned} \right\}$$

With this, let's implement parity condition on $\Psi(x)$

(to find out what these η_a, η_b must be ...)

$$\rightarrow P^\dagger \Psi P = ?$$

$$P^{\dagger}\psi(x)P = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\Big(\eta_a\, a^s_{-\vec p}\, u^s(p)\, e^{-i\tilde p\cdot\tilde x} + \eta_b^{*}\, b^{s\dagger}_{-\vec p}\, v^s(\vec p)\, e^{i\tilde p\cdot\tilde x}\Big)$$

<u>Define</u>

$$\begin{cases}\tilde p = (E_p, -\vec p) \\ \tilde x = (t, -\vec x).\end{cases}$$

<u>Note that</u>

$$u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix} \qquad \text{where } \sigma = (\mathbb{1}, \vec\sigma), \quad \bar\sigma = (\mathbb{1}, -\vec\sigma)$$

$$= \begin{pmatrix}\sqrt{\tilde p\cdot\bar\sigma}\,\xi^s \\ \sqrt{\tilde p\cdot\sigma}\,\xi^s\end{pmatrix}$$

$$= \begin{pmatrix}0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}\underbrace{\begin{pmatrix}\sqrt{\tilde p\cdot\sigma}\,\xi^s \\ \sqrt{\tilde p\cdot\bar\sigma}\,\xi^s\end{pmatrix}}_{u^s(-\vec p)} = \underbrace{\begin{pmatrix}0 & \mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}}_{\gamma^0} u^s(-\vec p)$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\vec p)}$$

and

$$v^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ -\sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix} = \cdots = \begin{pmatrix}0 & -\mathbb{1} \\ -\mathbb{1} & 0\end{pmatrix} v^s(-\vec p)$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(p)}$$

With these, we find that ...

$\tilde{p}\cdot\tilde{x} = p\cdot x$

$$P^\dagger \psi(x) P = \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(\eta_a a^s_{-p}\, u^s(-p) e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^* b^s_{-p}\, v^s(-p) e^{i\tilde{p}\cdot\tilde{x}} \Big)$$

now, notice that if $\eta_a = \eta_b^*$ then it's "nice":

$$\boxed{(\eta_a = \eta_b^*) \Rightarrow P\psi^\dagger(x) P = \eta_a \gamma^0 \psi(\tilde{x})}$$ → P-transf in final form

→ suffices to choose $\eta_a = 1 = -\eta_b$.

relative sign between fermions & antifermions ...

Now, useful to know how various Dirac field bilinears transform under parity ...

<u>Recall ... 5 of them:</u>

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad i\bar{\psi}[\gamma^\mu, \gamma^\nu]\psi$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi, \quad i\bar{\psi}\gamma^5\psi.$$

to find these, first compute: $P\bar{\psi}(x)P$ ...

$$P^\dagger\bar{\psi}(x)P = P^\dagger\psi^\dagger(x)\gamma^0 P = (P^\dagger\psi P)^\dagger\gamma^0 \qquad (\gamma^{0\dagger} = \gamma^0)$$

$$= \eta_a^* (\gamma^0\psi(\tilde{x}))^\dagger \gamma^0 = \eta_a^* \psi^\dagger(\tilde{x})\gamma^0\gamma^0$$

$$\rightarrow \boxed{P^\dagger\bar{\psi}P = \eta_a^* \bar{\psi}(\tilde{x})\gamma^0}$$

With this ---

$$P^\dagger \underset{(x)}{\bar\Psi} \underset{(x)}{\Psi} P = \underbrace{P^\dagger \bar\Psi(x) P}\overset{\mathbb{1}}{\widetilde{P^\dagger}} \Psi(x) P .$$

$$= \eta_a^* \bar\Psi(\tilde x)\gamma^0 \eta_a \gamma^0 \Psi(\tilde x)$$

$$= |\eta_a|^2 \bar\Psi(\tilde x)\Psi(\tilde x)$$

Scalar $\boxed{P^\dagger \bar\Psi \Psi P(x) = \bar\Psi\Psi(\tilde x)}$ (scalar) bilinear.

can also show ...

$$\boxed{\begin{aligned} P^\dagger \underbrace{\bar\Psi(x) \gamma^\mu}\Psi P &= \bar\Psi\gamma^0\gamma^\mu\gamma^0\Psi(\tilde x) \\ \text{(vector)} \quad &= \begin{cases} + \bar\Psi\gamma^\mu\Psi(\tilde x) & \mu = 0 \\ - \bar\Psi\gamma^\mu\Psi(\tilde x) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

$$\boxed{P^\dagger (i\bar\Psi\gamma^5\Psi) P = i\bar\Psi(\tilde x)\gamma^0\gamma^5\gamma^0\Psi(\tilde x) = -i\bar\Psi\gamma^5\Psi(\tilde x)}$$

pseudo scalar ↳ (−)

$$\boxed{\begin{aligned} P^\dagger \bar\Psi\gamma^\mu\gamma^5\Psi(x) P &= \bar\Psi(\tilde x)\gamma^0\gamma^\mu\gamma^5\gamma^0\Psi(\tilde x) \\ &= \begin{cases} - \bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 0 \\ + \bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

pseudo vector.

(−)

<u>Note</u> the relative sign: $\boxed{-\eta_a = \eta_b^*}$ is important.
for the relationship between fermions - anti-fermi

<u>Consider</u> ~~and~~ fermion - antifermion state...

$$a_p^{s\dagger} b_q^{s\dagger}|0\rangle \xrightarrow{P} P\left(a_p^{s\dagger} b_q^{s\dagger}|0\rangle\right)$$

$$= P^\dagger\left(a_p^{s\dagger} b_q^{s\dagger}\right)P|0\rangle$$

$$= \underbrace{P^\dagger a_p^{s\dagger} P}P^\dagger b_q^{s\dagger} P|0\rangle$$

$$= \eta_a\, a_{-p}^{s\dagger}\, \eta_b\, b_{-q}^{s\dagger}|0\rangle$$

$$= -(\eta_b \eta_b^*)\, a_{-p}^{s\dagger}\, b_{-q}^{s\dagger}|0\rangle$$

$$= -a_{-p}^{s\dagger} b_{-q}^{s\dagger}|0\rangle$$

$\rightarrow$ a state containing a fermion - antifermion pair gets an extra (-1) under parity transform.

TIME REVERSAL

if T is unitary $\rightarrow$ $[T, H] = 0$

$\rightarrow$ $T^\dagger e^{iHt} T = e^{iHt} T^\dagger T = e^{iHt}$

$\rightarrow$ no good...

what if $T^\dagger H T = -H$? or $\{T, H\} = 0$?

But this is no good either since implies that
H is unbounded ...

→ Assume this ...

$\boxed{\text{"Time-reversal is conjugate-linear / anti-linear"}}$

Assume:

T is unitary

$$T^\dagger c T = c^* \qquad (c \in \mathbb{C}).$$

$$[T, H] = 0$$

With these ...

$$T^\dagger e^{-iHt} T = e^{iHt} \quad ✓$$

→ time reversal: momentum & spin are reversed

↳ like watching a movie played backwards

Flipping momentum is easy.

what abt flipping spinor s? We know that

In some basis ...

$$\xi(\uparrow) = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s = 1, 2$ & define

reversed spin

$$\boxed{\xi^{-s} = -i\sigma^2 (\xi^s)^*.}$$

σ^y

This is the flipped spinor ...

It is clear that

$$\xi^{-s} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow)^*, \xi(\downarrow)^*)$$

$$= (\xi(\downarrow), -\xi(\uparrow))^*$$

where
$$\begin{cases} \xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{1*} \\ \xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{2*} \end{cases}$$

→ This is convenient since our time reversal op. involves complex conjugation ...

→ Can show:
$$\boxed{u^{-s}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\,\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\,\xi^{s*} \end{pmatrix}}$$

So if we use the identity ...

$$\sqrt{\tilde{p}\cdot\sigma}\,\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}$$

(prove using $\sigma^2\vec{\sigma} = -\vec{\sigma}^*\sigma^2$

then we get

$$u^{-s}(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\xi^{s*} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2)\sqrt{\tilde{p}\cdot\sigma^*}\,\xi^{s*} \\ (-i\sigma^2)\sqrt{\tilde{p}\cdot\bar{\sigma}^*}\,\xi^{s*} \end{pmatrix}$$

$$= (-i)\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(p)]^* = -\gamma^1\gamma^3[u^s(p)]^*$$

$\sigma^2 = \sigma^y$

$$\Rightarrow \boxed{u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^*}$$

element-wise complex conjugation

Similarly, ...

$$\boxed{v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^*}$$

in this relation, v^{-s} contains $\boxed{\xi^{-(-s)} = -\xi^s}$

a 360° flip introduces a $(-)$ sign.

~~[illegible]~~

Now we can define time reversal operation on the creation - annihilation operators...

spins don't matter here... $\rightarrow$ $\boxed{T^\dagger a^s_{\vec p} T = a^{-s}_{-\vec p} \quad \text{e} \quad T^\dagger b^s_{\vec p} T = b^{-s}_{-\vec p}}$ $\rightarrow$ flip spin, $\rightarrow$ flip momentum

where we now define

$$a^{-s}_{-\vec p} = (a^{\downarrow}_{-\vec p}, -a^{\uparrow}_{-\vec p})$$
$$b^{-s}_{-\vec p} = (b^{\downarrow}_{-\vec p}, -b^{\uparrow}_{-\vec p})$$

just like what we did with $\xi^{-s} = (\xi(\downarrow), -\xi(\uparrow))$

if

$$a^s_{\vec p} = (a^{\uparrow}_{\vec p}, a^{\downarrow}_{\vec p})$$
$$b^s_{\vec p} = (b^{\uparrow}_{\vec p}, b^{\downarrow}_{\vec p})$$

analogous to what we did before...

With this, let's evaluate $T^\dagger \Psi(x) T$:

$$\rightarrow \quad T^\dagger \Psi(x) T = \int \frac{d^3 p}{(2\pi)^3} \underbrace{\sum_{s=1,2}}_{\frac{1}{\sqrt{2E_p}}} T^\dagger \left(a^s_{\vec p} u^s(p) e^{-ip\cdot x} + b^{s\dagger}_p v^s(p) e^{+ip\cdot x} \right) T$$

$$= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^{2} \left\{ a^{-s}_{-\vec p} [u^s(p)]^* e^{ip\cdot x} + b^{-s\,\dagger}_{-p} [v^s(p)]^* e^{-ip\cdot x} \right]$$

where under T, $\qquad = \gamma^1 \gamma^3 \Psi(x_T), \; x_T = (-t, \vec x)$

- $a^s_p \xrightarrow{T} a^{-p}_{-p}$
- ~~$u^s(p)$~~ ~~$u^{-s}(\tilde p) = -\gamma^1\gamma^3 [u(p)]^*$~~
- $\rightarrow$ ~~$\gamma^1\gamma^3 u^{-s}(\tilde p) = [u^s(p)]^*$~~
- $T^\dagger e^{-ip\cdot x} T = e^{+ip\cdot x}$; $T^\dagger u^s_p T = [u^s_p]^*$

note sign change here

Because $\{u^s(p)\}^* = \gamma_1\gamma_3 u^{-s}(\tilde{p})$, we have

$$T^\dagger \psi(x) T = \gamma^1\gamma^3 \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^{2} \left\{ a^{-s}_{\tilde{p}} u^{-s}(\tilde{p}) e^{i\tilde{p}(t,-\vec{x})} + b^{-s\dagger}_{\tilde{p}} v^{-s}(\tilde{p}) e^{-i\tilde{p}(t,-\vec{x})} \right\}$$

$$= \gamma^1\gamma^3 \psi(-t, x) \qquad = -\tilde{p}(-t,\vec{x})$$

$$\Rightarrow \boxed{T^\dagger \psi(x,t) T = \gamma^1\gamma^3 \psi(x,-t)}$$

Next, can check the action of T on bilinears...

$$T^\dagger \bar{\psi} T = T^\dagger \psi^\dagger \gamma^0 T = T^\dagger \psi^\dagger T \gamma^0 \quad \text{real}$$

$$= (\gamma^1\gamma^3\psi(x_T))^\dagger \gamma^0 = \psi^\dagger(x_T)\gamma^{3\dagger}\gamma^{1\dagger}\gamma^0$$

$$\qquad\qquad -\gamma^3 \quad -\gamma^1$$

$$= \psi^\dagger(x_T)\gamma^3\gamma^1\gamma^0$$

$$= +\psi^\dagger(x_T)\gamma^0\gamma^3\gamma^1$$

$$\Rightarrow \boxed{T^\dagger \bar{\psi} T = -\bar{\psi}(x_T)\gamma^1\gamma^3}$$

with this, can compute the rest...

Scalar $\boxed{T\bar{\psi}\psi T_{(x)} = \bar{\psi}\underbrace{(-\gamma^1\gamma^3)(\gamma^1\gamma^3)}_{\mathbb{1}}\psi(x_T) = \bar{\psi}(x_T)\psi(x_T)}$

Pseudoscalar $\to$ get $(-)$

"pseudo"

$$\boxed{T^\dagger i\bar{\psi}\gamma^5\psi T = -i\bar{\psi}(-\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3)\psi(x_T) = -i\bar{\psi}(x_T)\gamma^5\psi(x_T)}$$

Vector

$$\boxed{\begin{aligned} T^\dagger \bar{\Psi}\gamma^\mu \Psi T &= \bar{\Psi}(-\gamma^1\gamma^3)(\gamma^\mu)^* (\gamma^1\gamma^3)\Psi \\ (x) \quad & \\ &= \begin{cases} +\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu = 0 \\ -\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu = 1,2,3. \end{cases} \end{aligned}}$$

This makes sense ... Recall that $\bar{\Psi}\gamma^0\Psi$ is the charge density

↳ $\bar{\Psi}\gamma^0\Psi$ should be the same under T.

as we saw: $T^\dagger\bar{\Psi}\gamma^0\Psi T = \bar{\Psi}\gamma^0\Psi$.

but current density (time-dep) must reverse si[gn]

$\rightarrow \quad T^\dagger\bar{\Psi}\gamma^i\Psi T = -\bar{\Psi}\gamma^i\Psi$ ✓.

Charge Conjugation - Matter-anti-matter flip

↳ anti particles & particles are swapped.
spin & momentum are the same.

Let $\left\{ \begin{aligned} C^\dagger a_p^s C &= b_p^s \\ C^\dagger b_p^s C &= a_p^s . \end{aligned} \right\} \rightarrow$ ignore phases...

How should C act on $\Psi(x)$?

First, look at relation...

$$(v^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\,(-i\sigma^2)\,\xi^{s*} \\ \sqrt{p\cdot\bar\sigma}\,(-i\sigma^2)\,\xi^{s*} \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma^*}\,\xi^{s*} \\ i\sigma^2\sqrt{p\cdot\sigma^*}\,\xi^{*} \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} =$$

$\rightarrow$ set

$$\boxed{\begin{aligned} u^s(p) &= -i\gamma^2 (v^s(p))^* \\ v^s(p) &= -i\gamma^2 (u^s(p))^* \end{aligned}}$$

$$\rightarrow \quad C^\dagger \psi(x) C = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_p}} \sum_{s=1}^{2} \Big\{ -i\gamma^2 b_p^s (v^s(p))^* e^{-ip\cdot x} - i\gamma^2 a_p^{s\dagger} (u^s(p))^* e^{ip\cdot x} \Big)$$

$$= -i\gamma^2 \psi^*(x) = -i\gamma^2 (\psi^\dagger)^T = -i(\bar\psi\gamma^0\gamma^2)^T$$

$$\Rightarrow \boxed{C^\dagger \psi(x) C = -i(\bar\psi\gamma^0\gamma^2)^T} \quad \rightarrow \; C \text{ is a unitary op.}$$

On bilinears... first, find $\quad \bar\psi^\dagger = (\psi^\dagger)^\dagger\gamma^0 = \psi\gamma^0$

$$\boxed{\begin{aligned} C^\dagger\bar\psi(x)C &= C^\dagger\psi^\dagger\gamma^0 C = \underbrace{C^\dagger\psi^\dagger C}\gamma^0 = -i\psi^T\gamma^2\gamma^0 \\ &= (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T \end{aligned}}$$

Next ....

$$C^\dagger\bar\psi\psi C = \left(-i(\gamma^0\gamma^2\psi)^T\right)\left(-i(\bar\psi\gamma^0\gamma^2)^T\right) = \cdots\cdots =$$

$$= -\left[(-i\bar\psi\gamma^0\gamma^2)(-i\gamma^0\gamma^2\psi)\right]^T = +\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi$$

$$= + \bar\psi\gamma^0\gamma^2\gamma^2\gamma^0\psi = +\bar\psi\psi$$

So... $\boxed{C^\dagger \bar{\Psi}\Psi C = \bar{\Psi}\Psi}$ $\longrightarrow$ scalar

vector $\boxed{C^\dagger i\bar{\Psi}\gamma^5\Psi C = i(-i\gamma^0\gamma^2\Psi)^T\gamma^5(-i\bar{\Psi}\gamma^0\gamma^2)^T = i\bar{\Psi}\gamma^5\Psi}$

pseudo-scalar $C^\dagger\bar{\Psi}\gamma^\mu\Psi C = -\bar{\Psi}\gamma^\mu\Psi$

pseudo vector. $C^\dagger\bar{\Psi}\gamma^\mu\gamma^5\Psi C = +\bar{\Psi}\gamma^\mu\gamma^5\Psi$

(I'll skip the derivations... to save time)

<u>Summary</u>

	$\bar{\Psi}\Psi$	$i\bar{\Psi}\gamma^5\Psi$	$\bar{\Psi}\gamma^\mu\Psi$	$\bar{\Psi}\gamma^\mu\gamma^5\Psi$	$\bar{\Psi}\sigma^{\mu\nu}\Psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

notice that

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi \text{ is invariant}$$

under C, P, T separately

$\rightarrow$ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

$\rightarrow$ Invariance under CPT is required for any Lorentz invariant local Hermitian op.

$\boxed{\text{Problem 5}}$

$\hookrightarrow$ (to be continued...)

$\boxed{\text{Correlation functions for Dirac fields}}$

$$\langle 0|\underbrace{\psi_a(x)}_{\substack{\text{only "}a\text{"}\\ \text{term contributes}}}\underbrace{\bar{\psi}_b(y)}_{\substack{\text{only "}a^{\dagger}\text{"}\\ \text{term contributes}}}|0\rangle \rightsquigarrow \text{Dirac propagation amplitudes...}$$

Recall...

$$\to \psi_A(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{a^s_{Ap}u^s_A(p)e^{-ip\cdot x} + \cancel{b^{s\dagger}_{Ap}v^s_A(p)e^{-ip\cdot x}}\right\}$$

$\langle 0|$

$$\to \bar{\psi}_B(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{\cancel{b^s_{Bp}\bar{v}^s_B(p)e^{-ip\cdot x}} + a^{s\dagger}_{Ap}\bar{u}^s_B(p)e^{ip\cdot x}\right\}$$

$|0\rangle$

where $\{a^r_p, a^{s\dagger}_q\} = \{b^r_p, b^{s\dagger}_q\} = (2\pi)^3\delta^{(3)}(p-q)\delta^{rs}$

$$\to \langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\bar{u}^s_B(p)}_{(\not{p}+m)_{AB}}e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{AB}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

$$\boxed{\langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = (i\not{\partial}_x + m)_{AB}\,\mathcal{D}(x-y)}$$

- - -

$$\langle 0|\bar{\Psi}_B(y)\Psi_A(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s v_A^s(p)\bar{v}_B^s(p)}_{(\not{p}-m)_{AB}} e^{-ip(x-y)}$$

$\bar{\Psi}_B(y)$: b terms contribute

$\Psi_A(x)$: $b^\dagger$ terms contribute

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_p}\right)(\not{p}-m)_{AB}\, e^{-ip(x-y)}$$

$$\boxed{\langle 0|\bar{\Psi}_B(y)\Psi_A(x)|0\rangle = -(i\not{\partial}_x+m)_{AB}\,\mathcal{D}(y-x)}$$

Feynman Propagator

$$S_F^{AB}(x-y) = \begin{cases} \langle 0|\Psi_A(x)\bar{\Psi}_B(y)|0\rangle & \text{for } x^0>y^0 \\ -\langle 0|\bar{\Psi}_B(y)\Psi_A(x)|0\rangle & \text{for } x^0<y^0 \end{cases}$$

$$= \langle 0|T\{\Psi_A(x),\bar{\Psi}_B(y)\}|0\rangle$$

... time-ordering ...

where $T\{\Psi_A(x),\bar{\Psi}_B(y)\}$

$$= \theta(x^0-y^0)\,\Psi_A(x)\bar{\Psi}_B(y)$$
$$-\,\theta(y^0-x^0)\,\bar{\Psi}_B(y)\Psi_A(x)$$

"minus sign for Fermions"

Let's check the calculations...

$$\langle 0|\psi_A(x)\bar{\psi}_B(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_{p'}}}\langle 0|\left\{\sum_s a^s_{A\vec{p}}\, u^s_A(p)e^{-ip\cdot x} + \cancel{b^{s\dagger}_{A\vec{p}}\, v^s_A(p)e^{-ip\cdot x}}\right\}$$

$$\times\left\{\cancel{\sum_s b^s_{B\vec{p}'}\,\bar{v}^s_B(p')e^{-ip'\cdot y}} + a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{ip'\cdot y}\right\}|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\langle 0|\sum_s a^s_{A\vec{p}}\,u^s_A(p)\sum_s a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{-i(p\cdot x - p'\cdot y)}|0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\sum_s \overset{\text{“}1\text{”}}{\langle 0|}u^s_A(p)\bar{u}^s_B(p')|0\rangle(2\pi)^3\delta^{(3)}(p-p')$$

$$\times e^{i(p\cdot x - p'\cdot y)}$$

$$= \int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\bar{u}^s_B(p)}\, e^{-ip(x-y)}$$

$$(\not{p}+m)_{AB} = (\gamma^\mu p_\mu + m)_{AB} \quad \left(\frac{\text{spin sum}}{\text{relations}}\right)$$

$$= (i\not{\partial}_x + m)_{AB}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}\ \checkmark$$

Similarly, we can get the other relation too...

Oct 5, 2020

(1) Recall Dirac bispinor field...

$$\psi(x) = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a^r_{\vec{p}} u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}} v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Use $\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{p}'} \} = \{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{p}'} \} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}')$

and all other anti-comm $= 0$, derive the following:

$$\{ \psi_a(\vec{x}), \psi_b^\dagger(\vec{y}) \} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y})$$

(2) The momentum operator is the Noether charge associated with spatial translation...

$$\vec{P} = -i \int d^3\vec{x}\, \psi^\dagger(x) \vec{\nabla} \psi(\vec{x})$$

Show that

$$\vec{P} = \sum_{r=1}^{2} \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} + b^{r\dagger}_{\vec{p}} b^r_{\vec{p}} \right)$$

Oct 10, 2020

(1) Well...

$$\{ \psi_a(x), \psi_b^\dagger(y) \}$$

$$= \psi_a(x) \psi_b^\dagger(y) + \psi_b^\dagger(y) \psi_a(x)$$

$$\cancel{\sum_{r=1}^{2} \int \frac{d^3p_a\, d^3p_b}{(2\pi)^6\, 2\sqrt{E_{p_a} E_{p_b}}} e^{i(\vec{p}_a\cdot\vec{x} - \vec{p}_b\cdot\vec{y})} \left[a^r \right.}$$

To keep things clean...

$\Psi_a(x)\,\Psi_b^\dagger(y) + \Psi_b^\dagger(x)\Psi_a(x)$ which involves the factor...

$$= \sum_{r=1}^{2}\sum_{s=1}^{2}\left[\hat{a}^r_{p_a} u^r(p_a) + \hat{b}^{r\dagger}_{-p_a} v^r(-p_a)\right]\left[(\hat{a}^s_{p_b} u^s(p_b))^\dagger + (\hat{b}^{s\dagger}_{-p_b} v^s(-p_b))^\dagger\right]$$

$$+ \sum_{r=1}^{2}\sum_{s=1}^{2}\left[(\hat{a}^s_{p_b} u^s(p_b))^\dagger + (\hat{b}^{s\dagger}_{-p_b} v^s(-p_b))^\dagger\right]\left[\hat{a}^r_{p_a} u^r(p_a) + \hat{b}^{r\dagger}_{-p_a} v^r(-p_a)\right]$$

$$= \sum_{r,s=1}^{2}\left\{ \{\hat{a}^r_{p_a}, \hat{a}^{s\dagger}_{p_b}\} u^r(p_a) u^{s\dagger}(p_b) + \{\hat{b}^r_{p_a}, \hat{b}^{\dagger s}_{p_b}\} v^r(p_a) v^s(p_b)\right\}$$

$$= \sum_{r,s=1}^{2} (2\pi)^3 \delta^{r,s}\,\delta^{(3)}(p_a - p_b)\left\{ u^r(p_a) u^{s\dagger}(p_b) + v^r(p_a) v^{s\dagger}(-p_b)\right\}$$

$\Rightarrow \{\Psi_a(x), \Psi_b^\dagger(y)\}$

$$= \int \frac{d^3 p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^{2}\left\{ u^r(p) u^{r\dagger}(p) + v^r(-p) v^{r\dagger}(-p)\right\}$$

now, we want to convert $u^{s\dagger} \rightarrow \bar{u}$

$\rightarrow$ need γ^0. In particular, recall that $\gamma^0\gamma^0 = \mathbb{1}$

from page 63 $\Rightarrow$ and $\boxed{u^{s\dagger}(p)\gamma^0 = \bar{u}^s(p)}$, we have

$$\{\Psi_a(x), \Psi_b^\dagger(y)\} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^{2}\left\{ u^r_p u^{r\dagger}_p \gamma^0 + v^r_{-\vec{p}} v^{r\dagger}_{-\vec{p}}\gamma^0\right\}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^{2}\left\{ u^r_p \bar{u}^r_p + v^r_{-\vec{p}} \bar{v}^r_{-\vec{p}}\right\}\gamma^0 \quad \left(\text{spin sum}\right)$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left(\underset{E_0\gamma^0 - \vec{p}\cdot\vec{\gamma}}{p_+\cdot\gamma} + \cancel{mI} + \underset{E_0\gamma^0 + \vec{p}\cdot\vec{\gamma}}{p_-\cdot\gamma} - \cancel{mI}\right)\gamma^0 \Big|_{\delta_{a,b}}$$

recall that only $\vec{p} \to -\vec{p}$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \cancel{[2 p\cdot\gamma]}\gamma^0$$

Rather
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left\{ E_p\cancel{\gamma^0} - \cancel{\vec{p}\cdot\vec{\gamma}} + E_p\cancel{\gamma^0} + \cancel{\vec{p}\cdot\vec{\gamma}} \right\}\cancel{\gamma^0}_{ab}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y}).$$

So
$$\boxed{\{ \psi_a(x), \psi_b^\dagger(y) \} = \delta_{ab}\,\delta^{(3)}(x-y)}$$

~~(2) Let~~
$$\cancel{\vec{P} = -i\int d^3\vec{x}\; \psi^\dagger(\vec{x})\,\vec{\nabla}\psi(\vec{x}).}$$

~~Show that~~

$p = (p^0, \vec{p})$

$$\cancel{\vec{P} = \sum_{r=1}^{2}\int \frac{d^3p}{(2\pi)^3}\,\vec{p}\left(a_{\vec{p}}^{r\dagger}a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger}b_{\vec{p}}^r\right).}$$

$ip\cdot x = \cancel{\ldots} + i\vec{p}\cdot\vec{x}$

~~Well.~~
$$\cancel{\psi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}} e^{ip\cdot x}\sum_{r=1}^{2}\left\{ a_{\vec{p}}^r u_{\vec{p}}^r + b_{-\vec{p}}^{r\dagger} v_{-\vec{p}}^r \right\}.}$$

~~So~~

$$\cancel{\nabla\psi(x) = \sum_{r=1}^{2}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}(i\vec{p})}$$

(2) $\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a^s_{\vec{p}} u^s_p e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s_p e^{ip\cdot x} \right)$

$\rightarrow \quad \nabla\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (i\vec{p}) \left\{ a^s_{\vec{p}} u^s_p e^{-ip\cdot x} - b^{s\dagger}_{\vec{p}} v^s_p e^{ip\cdot x} \right\}$

$\Psi^\dagger(x) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{r=1}^{2} \left\{ b^r_q v^{\dagger r}(q) e^{-iqx} + a^{r\dagger}_q u^{\dagger r}_q e^{iqx} \right\}$

$\rightarrow \quad \int d^3x (-i) \Psi^\dagger \nabla \Psi$

$\delta^{(3)}(p-q)$

$= \int d^3x \int \frac{d^3q}{(2\pi)^3} \frac{d^3p}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_q}} e^{ix(p-q)} \vec{p} \times \sum_s \times \sum_r$

$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \frac{1}{E_p} \vec{p} \left\{ \sum_{s,r=1}^{2} \left(a^{r\dagger}_{\vec{p}} a^s_{\vec{p}} u^{\dagger r}_p u^s_p - b^r_{\vec{p}} b^{s\dagger}_{\vec{p}} v^{\dagger r}_p v^s_p \right) \right\}$

$2E_p \delta^{rs}$

other uv terms $= 0$

$= \int \frac{d^3p}{(2\pi)^3} \frac{2E_p}{2E_p} \vec{p} \left(a^{r\dagger}_p a^r_p - b^r_p b^{r\dagger}_p \right)$

$= \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{r\dagger}_p a^r_p - b^r_p b^{r\dagger}_p \right)$

ignore...

Finally ... $\{ b^r_p, b^{r\dagger}_p \} = (2\pi)^3 \delta^{rr}(\vec{p}-\vec{p})$

$\Rightarrow \quad -b^r_p b^{r\dagger}_p = b^{r\dagger}_p b^r_p - (2\pi)^3 \delta^{rr} \delta(\vec{p}-\vec{p})$

momentum op

$\Rightarrow \quad \vec{P} = \int d^3x \, \Psi^\dagger(x)(-i\nabla)\Psi(x)$

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p} \left(a^{r\dagger}_p a^r_p + b^{r\dagger}_p b^r_p \right)}$$

✓

More problems

(1) Let $\mathcal{U}$ be the following unitary op:

$$\mathcal{U} = \exp\left\{\frac{-i\pi}{2}\sum_{r=1}^{2}\int\frac{d^3\vec p}{(2\pi)^3}\left(a_{\vec p}^{r\dagger} - b_{\vec p}^{r\dagger}\right)\left(a_{\vec p}^{r} - b_{\vec p}^{r}\right)\right\}$$

investigate the effect of $\mathcal{U}$ on $a_{\vec p}^{r}$ & $b_{\vec p}^{r}$. I.e. compute

$$\mathcal{U}^\dagger a_{\vec p}^{r}\,\mathcal{U} \quad \& \quad \mathcal{U}^\dagger b_{\vec p}^{r}\,\mathcal{U}.$$

What type of transform does $\mathcal{U}$ produce?

—//—

Well...

$$\mathcal{U}^\dagger a_{\vec p}^{r}\,\mathcal{U} = \exp\Big\{\underbrace{+\frac{i\pi}{2}\Big(\sum_{r=1}^{2}\int\frac{d^3 p}{(2\pi)^3}\left(a_{\vec p}^{r\dagger} - b_{\vec p}^{r\dagger}\right)\left(a_{\vec p}^{r} - b_{\vec p}^{r}\right)\Big)^\dagger}_{X^\dagger}\Big\}$$

$$\cdot\, a_{\vec p}^{r} \,\cdot\, \exp\Big\{\underbrace{\frac{i\pi}{2}\sum_{r=1}^{2}\int\frac{d^3 p}{(2\pi)^3}\left(a_{\vec p}^{r\dagger} - b_{\vec p}^{r\dagger}\right)\left(a_{\vec p}^{r} - b_{\vec p}^{r}\right)}_{X}\Big\}$$

$\Rightarrow$ no good way to do this except for power series,

Recall that $e^{x} = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$, $e^{-x} = \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n$

So
$$\mathcal{U}^\dagger a_{\vec p}^{r}\,\mathcal{U} = \left(\sum_{n=0}^{\infty}\frac{(X^\dagger)^n}{n!}\right) a_{\vec p}^{r}\left(\sum_{m=0}^{\infty}\frac{(X)^m}{m!}\right) = \sum_{n,m=0}^{\infty}\frac{1}{n!\,m!}\,(X^\dagger)^n a_{\vec p}^{r} X^m$$

but notice that $\mathcal{U}$ unitary iff X hermitian.

$$\rightarrow X^\dagger = X \rightarrow \mathcal{U}^\dagger a_{\vec p}^{r}\,\mathcal{U} = \sum_{n,m=0}^{\infty}\frac{1}{n!\,m!}\,X^n a_{\vec p}^{r} X^m.$$

Useful theorem:

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots$$

& Identity relating comm & anti-comm:

$$[AB,C] = ABC - CAB$$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B,C\} - \{A,C\}B.$$

We ~~get~~ will need to compute ~~X~~, a_p^s

$$\mathcal{U}^\dagger a_{\vec{q}}^r \mathcal{U} = a_{\vec{q}}^r + [X, a_{\vec{q}}^r] + \frac{1}{2!}[X,[X,a_{\vec{q}}^r]] + \cdots$$

$\rightarrow$ need to compute

$$[X, a_{\vec{q}}^r] = X a_{\vec{q}}^r - a_{\vec{q}}^r X = \cdots = ?$$

$$= \left(+\frac{i\pi}{2}\sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\left\{(a_{\vec{p}}^{s\dagger} - b_{\vec{p}}^{s\dagger})(a_p^s - b_p^s)\, a_{\vec{q}}^r - a_{\vec{q}}^r (a_{\vec{p}}^{s\dagger} - b_{\vec{p}}^{s\dagger})(a_p^s - b_p^s)\right\}\right)$$

$$= \frac{i\pi}{2}\sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\left[(a_p^{s\dagger} - b_p^{s\dagger})\underbrace{\{a_{\vec{p}}^s - b_{\vec{p}}^s, a_{\vec{q}}^r\}}_{0} - \{a_p^{s\dagger} - b_p^{s\dagger}, a_{\vec{q}}^r\}(a_p^s - b_p^s)\right]$$

$$= \frac{i\pi}{2}\sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\left[-(2\pi)^3\delta^{rs}\delta^{(3)}(\vec{p}-\vec{q})(a_{\vec{p}}^s - b_{\vec{p}}^s)\right]$$

$$= -\frac{i\pi}{2}(a_{\vec{q}}^r - b_{\vec{q}}^r)$$

Next term,

$$[X,[X,a_q^r]] = [X, \tfrac{-i\pi}{2}(a_q^r - b_q^r)]$$

$$= \tfrac{-i\pi}{2}[X, a_q^r] + \tfrac{i\pi}{2}[X, b_q^r] = \ldots$$

$$= 2\left(\tfrac{i\pi}{2}\right)^2 (a_q^r - b_q^r)$$

So... $e^X a_p^r e^{-X} = a_q^r - \frac{i\pi}{2}(a_q^r - b_q^r) + 2\left(\frac{i\pi}{2}\right)^2 (a_q^r - b_q^r)$

what's next? $+ \, ?$

each step $\rightarrow$ $\times 2\left(\frac{i\pi}{2}\right)$ & alt $(-)$ sign.

$$\rightarrow \quad \mathcal{U}^\dagger a_p^r \,\mathcal{U} = a_p^r - \frac{i\pi}{2}(a_p^r - b_p^r) + \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 (a_p^r - b_p^r),$$

$$- \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 (a_p^r - b_p^r) + \frac{1}{4!}\left(\frac{i\pi}{2}\right)^4 6\,(\ldots) + \ldots$$

$$= a_p^r \left\{ 1 - \frac{i\pi}{2} + \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 - \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 + \frac{6}{4}\left(\frac{i\pi}{2}\right)^4 + \ldots \right\}$$

check this!

$$+ \; b_p^r \left\{ \frac{i\pi}{2} - \frac{1}{2!}2\left(\frac{i\pi}{2}\right)^2 + \frac{1}{3!}4\left(\frac{i\pi}{2}\right)^3 - \frac{6}{4}\left(\frac{i\pi}{2}\right)^4 + \ldots \right\}$$

$$= a_p^r \left\{ 1 - \tfrac{1}{2}\cdot 2 \right\} + b_p^r \cdot \left\{ \tfrac{1}{2}\cdot 2 \right\}$$

$$= b_p^r \quad \rightarrow \quad \boxed{\mathcal{U}^\dagger a_p^r\, \mathcal{U} = b_p^r}$$

$\rightarrow$ <u>$\mathcal{U}$ corresponds to charge conjugation!</u>

2) using Dirac annihilation & creation ops, construct P for which

$$P^\dagger a^r_{\vec p} P = a^r_{-\vec p} \quad \& \quad P^\dagger b^r_{\vec p} P = b^r_{-\vec p}.$$

Last time, we find that

$$[X, a^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(a^r_{\vec p} - b^r_{\vec p}\right) \rightarrow \text{target}$$

→ we want X for which

$$\begin{cases} [X, a^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(a^r_{\vec p} - a^r_{-\vec p}\right). \\ [X, b^r_{\vec p}] = -\left(\frac{i\pi}{2}\right)\left(b^r_{\vec p} + b^r_{-\vec p}\right). \end{cases}$$

→ Let

$$P = \exp\left\{\frac{i\pi}{2}\sum_{r=1}^{2}\int\frac{d^3p}{(2\pi)^3}\left\{a^{r\dagger}_{\vec p}\left(a^r_{\vec p} - a^r_{-\vec p}\right) + b^{r\dagger}_{\vec p}\left(b^r_{\vec p} + b^r_{-\vec p}\right)\right\}\right\}$$

check this, like last time

→ should work! :)

Obv a^r_p only int. w/ 1st term $a^\dagger(\,)a = 0$

$aa^\dagger(\,) = \delta^{\cdots}(\sim)$ ✓

Same with b^r_p ✓.

Interacting Fields & Feynman Diagrams

To get better description of the real world, need to include interactions in the theory.

To preserve causality, new terms may involve only products of fields at the same spacetime point!

↳ $\phi^4(x)$ ✓, but not $\phi(x)\phi(y)$.

→ $H_{int} = \int d^3x \, \mathcal{H}_{int}[\phi(x)] = -\int d^3x \, \mathcal{L}_{int}[\phi(x)]$

→ insist that H_{int} is a fn only of the fields, not of their derivatives.

→ Common ex in particle physics & CMT:

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4}$$

note

$\pi(x)$ is still $\partial_0 \phi_0(x)$ since there are not any interaction terms involving $\partial_0 \phi$

λ: dimensionless "coupling constant".

→ In general, adding interactions produces divergences

However, no matter what the true physics looks like at high momenta or short distances, the low momentum / long distance physics is well-approximated by an "effective" FT.

with "renormalizable" interactions.

→ those interactions where coupling constant are has dimensions $\boxed{d > 0}$

$[\text{Mass}]^d$ where $d > 0$.

Ex $\quad -\frac{1}{2} m^2 \phi^2 \quad \frac{\lambda}{4!} \phi^4$ same dim

$\rightarrow \quad \lambda \sim [\text{mass}]^0 \rightarrow$ Renormalizable.

But $\quad -\frac{\lambda_6}{6!} \phi^6 \rightarrow$ <u>not</u> renormalizable.

since $\lambda \sim [\text{mass}]^{-2}$.

$\boxed{\text{Perturbation Expansion}}$

Let $\quad H = H_0 + H_{int} \rightsquigarrow = \int d^3x \, \frac{\lambda}{4!} \phi^4(x)$

$\uparrow$

KG, free

→ we will generate power series in λ.

At any t_0, we can write

$$\phi(t_0, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec{p}} \, e^{ip\cdot x} + a^\dagger_{\vec{p}} \, e^{-ip\cdot x} \right\}$$

$$\equiv \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \underset{\substack{\parallel \\ a_{\vec{p}}}}{\tilde{a}_{\vec{p}}} \, e^{i\vec{p}\cdot\vec{x}} + \tilde{a}^\dagger_{p} \, e^{-i\vec{p}\cdot\vec{x}} \right\}$$

where we've let $a_{\vec{p}}$ absorb $e^{\pm i E t_0}$

The Heisenberg field is then given by:

$$\rightarrow \boxed{\phi(t,\vec{x}) = e^{iH(t-t_0)}\,\phi(t_0,x)\,e^{-iH(t-t_0)}}$$

If there's no interaction then we have

$$\rightarrow \boxed{\begin{aligned}\phi_{free}(t,\vec{x}) &= e^{iH_0(t-t_0)}\,\phi(t_0,x)\,e^{-iH_0(t-t_0)} \\ &= \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{a_p e^{-ip\cdot x} + a_p^\dagger e^{+ip\cdot x}\right\}\end{aligned}}\Bigg|_{\substack{x^0 = t-t_0 \\ p^0 = E_{\vec{p}}}}$$

Define this to be $\phi_I(t,\vec{x})$, the interaction picture field

the interaction picture field $\equiv$ Heisenberg field when $\lambda = 0$.

Now, look at Heisenberg field...

$$\begin{aligned}\phi(t,\vec{x}) &= e^{iH(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH(t-t_0)} \\ &= \underbrace{e^{iH(t-t_0)}e^{-iH_0(t-t_0)}}\,\phi_I(t,\vec{x})\,e^{+iH_0(t-t_0)}e^{-iH(t-t_0)} \\ &= \mathcal{U}^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,\mathcal{U}(t,t_0)\end{aligned}$$

time evolution operator

→ Evolve the operator as $\phi_I(t,\vec{x})$

OR

Evolve the states by $\mathcal{U}(t,t_0) \rightsquigarrow \mathcal{U}|\phi\rangle$ ...

→ now, we want to express $\mathcal{U}(t,t_0)$ entirely in ϕ_I

To do this, note that $\mathcal{U}(t,t_0)$ solves SE:

$$i\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = e^{iH_0(t-t_0)}(H-H_0)e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)} H_{int}\, e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}}\, e^{iH_0(t-t_0)} e^{-iH(t-t_0)}$$

$$= H_I(t)\,\mathcal{U}(t,t_0)$$

where

$$H_I(t) = e^{iH_0(t-t_0)} H_{int}\, e^{-iH_0(t-t_0)}$$

$$= \int d^3x\, \frac{\lambda}{4!}\phi_I^4 \;\; \boxed{?}$$

$$= \int d^3x\, e^{iH_0(t-t_0)}\frac{\lambda}{4!}\phi^4\, e^{-iH_0(t-t_0)}$$

$$= \int d^3x\, \frac{\lambda}{4!}\phi_I^4 \quad \checkmark$$

→ this is the Hamiltonian in the interaction picture.

So since $\mathcal{U}$ solves the SE:

$$i\partial_t\,\mathcal{U}(t,t_0) = H_I(t)\,\mathcal{U}(t,t_0),$$

$\mathcal{U}$ must look like

$$\boxed{\mathcal{U}(t,t_0) \sim \exp\{-iH_I t\}}$$

More carefully, we actually have that

$$\boxed{\mathcal{U}(t,t_0) = \underset{\substack{\uparrow \\ \text{time-ordering symbol}}}{T} \exp\left\{ -i\int_{t_0}^{t} dt'\, H_I(t') \right\}}$$ Dyson's formula

Why T? Why ordering? $\Rightarrow$ B/c $H(t_1) \not\leftrightarrow H(t_2)$ when $t_1 \neq t_2$.

"T" puts the latest operators on the left.

hence $i\partial_t \mathcal{U}(t,t_0) = \underline{\underline{H_I(t)}}\, \mathcal{U}(t,t_0)$.

As a pwr series in λ:

$$\mathcal{U}(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt'\, H_I(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2\, T\{H_1(t_1) H_2(t_2)\}$$
$$+ \frac{(-i)^3}{3!}\int_{t_0}^{t}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 dt_3\, T\{H_1(t_1) H_2(t_2) H_3(t_3)\} + \cdots$$

Need "the time-ordering of the exponential is just ~~the time-ordering~~ the Taylor series of the terms time-ordered..".

$\rightarrow$ Now, we want to generalize $\mathcal{U}(t, \underset{\substack{\uparrow\\ \text{reference time}}}{t_0})$ to $\mathcal{U}(t,t')$

This generalization is natural...

$$\boxed{\mathcal{U}(t,t') \equiv T\left\{\exp\left[-i\int_{t'}^{t} dt''\,\mathcal{H}_I(t'')\right]\right\} \quad (t \geq t')}$$

Then we see that b/c both t, t' are variables... we find:

$$\boxed{\begin{aligned} i\partial_t\, \mathcal{U}(t,t') &= \mathcal{H}_I(t)\,\mathcal{U}(t,t') \\ i\partial_{t'}\, \mathcal{U}(t,t') &= -\,\mathcal{U}(t,t')\,\mathcal{H}_I(t'). \end{aligned}}$$

and thus...

$$\boxed{\mathcal{U}(t,t') = e^{iH_0(t-t_0^*)}\, e^{-iH(t-t')}\, e^{-iH_0(t-t_0^*)}}$$

So $\mathcal{U}$ is unitary.

Further, for $t_1 \geq t_2 \geq t_3$,

$$\begin{aligned} \mathcal{U}(t_1,t_2)\,\mathcal{U}(t_2,t_3) &= \mathcal{U}(t_1,t_3) \\ \mathcal{U}(t_1,t_3)\,[\mathcal{U}(t_2,t_3)]^{\dagger} &= \mathcal{U}(t_1,t_2) \end{aligned}$$

Now, let $|0\rangle$ ~~be~~ be gnd state of $\mathcal{H}_0$

$|\Omega\rangle$ be gnd state of $\mathcal{H}$

$|n\rangle$ be ~~[illegible]~~ label all $|E_n\rangle$ of $\mathcal{H}$.

Then, $(E_0 = \langle\Omega|H|\Omega\rangle)$

$$e^{-iHT}|0\rangle = e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

Assume $H_0|0\rangle = 0$ & consider $T\to\infty$ limit.

$\updownarrow$

$T\to\infty(1-i\varepsilon)$

Then e^{-iE_nT} dies slowest for $n=0$, and so...

$$\to\ e^{-iHT}|0\rangle \to e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle$$

assume $\langle\Omega|0\rangle\neq 0$

So

$$\boxed{|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle}$$

Now, since T large, we can shift it by a small constant...

$$|\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right\}^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right\}^{-1} \underbrace{e^{-iH(t_0-(-T))}}_{\mathcal{U}(t_0,-T)}|0\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \left\{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right\}^{-1} e^{-iH(t_0-(-T))}\underbrace{e^{-iH_0(-T-t_0)}|0\rangle}_{=|0\rangle \text{ since } H_0|0\rangle=0}$$

$$\to\ |\Omega\rangle = \lim_{T\to\infty(1-i\varepsilon)} \frac{\mathcal{U}(t_0,-T)|0\rangle}{e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle}$$

Similarly,

$$\langle\Omega| = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle}$$

So, putting these together gives a correlation function...

For $x^0 > y^0 > t_0$, we have

$$\rightarrow \langle\Omega|\phi(x)\phi(y)|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle} \overbrace{\left(\mathcal{U}(x^0;t_0)\right)^{\dagger}\phi_I(x)\,\mathcal{U}(x^0;t_0)}^{\phi(x)}$$

$$\times\left(\mathcal{U}(y^0;t_0)\right)^{\dagger}\phi_I(y)\,\mathcal{U}(y^0;t_0)\times$$

$$\frac{\mathcal{U}(t_0,-T)}{e^{-iE_0(T+t_0)}\langle\Omega|0\rangle}$$

$$= \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,x^0)\phi_I(x)\,\mathcal{U}(x^0;y^0)\phi_I(y)\,\mathcal{U}(y^0;-T)|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}}$$

awkward...

so divide the whole thing by $1 = \langle\Omega|\Omega\rangle$

$$1 = \langle\Omega|\Omega\rangle = \frac{\langle 0|\overbrace{\mathcal{U}(T,t_0)\,\mathcal{U}(t_0,-T)}^{\mathcal{U}(T,-T)}|0\rangle}{|\langle 0|\Omega\rangle|^2 e^{-iE_0(2T)}}.$$

to get (for $x^0 > y^0$)

$$\boxed{\langle\Omega|\phi(x)\phi(y)|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\mathcal{U}(T,x^0)\phi_I(x)\,\mathcal{U}(x^0;y^0)\phi_I(y)\,\mathcal{U}(y^0;-T)|0\rangle}{\langle 0|\mathcal{U}(T,-T)|0\rangle}}$$

So, we have that, by replacing U's with Dyson's formula (w/ time-ordering) ...

$$\boxed{\langle\Omega| T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0| T\{\phi_I(x)\phi_I(y)\exp\{-i\int_{-T}^{T} dt\, H_I(t)\}\}|0\rangle}{\langle 0| T\exp\{-i\int_{-T}^{T} dt\, H_I(t)\}|0\rangle}}$$

So, looks like the term

$$\exp\left\{-i\int_{-T}^{T} dt\, H_I(t)\right\}$$ ~~can~~ ~~expanded~~ & can be found, so

Wick's theorem

→ So, we have reduced the problem of calc. correlation functions to evaluating

$$\boxed{\langle 0| T\{\phi_I(x_1)\phi_I(x_2)\cdots\phi_I(x_n)\}|0\rangle}$$

→ This is the vacuum exp-value of time-ordered products of finite number of field operators.

$n = 2 \Rightarrow$ get Feynman operator.

$n > 2 \rightarrow$ can use brute force, but there are also ways to simplify calculations.

Now, we study $\boxed{\langle 0| T\{\phi_I(x)\phi_I(y)\}|0\rangle}$

Recall that

Oct 11, 2020

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_{\vec p}\, e^{-ip\cdot x} + a^\dagger_{\vec p}\, e^{+ip\cdot x} \right\}$$

Call
$$\phi_I^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_{\vec p}\, e^{-ip\cdot x}$$

and
$$\phi_I^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a^\dagger_{\vec p}\, e^{+ip\cdot x}$$

which is ~~useful~~ b/c

$$\phi_I^+(x)|0\rangle = 0, \qquad \langle 0|\phi_I^-(x) = 0.$$

only has annihilation ops — only has creation ops.

→ For $x^0 > y^0$, $x^0 > y^0$

$$T\{\phi_I(x)\phi_I(y)\} = \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + \phi_I^+(x)\phi_I^-(y) + \phi_I^-(x)\phi_I^+(y)$$

$$= \phi_I^+(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^-(y) + [\phi_I^+(x), \phi_I^-(y)]$$

every of these terms has the form $a^\dagger_{\vec p} a^\dagger_{\vec q} a_{\vec k} a_{\vec k}$

i.e. creation ops always on the left.

→ "Normal order" → has vanishing vacuum exp value.

What can we say about the commutator?

[just a number, there's no creation/annihilation op. in it!

$$[\phi_I^+(x), \phi_I^-(y)] = \langle 0|[\phi_I^+(x), \phi_I^-(y)]|0\rangle$$

$$= \langle 0|\phi_I^+(x)\phi_I^-(y)|0\rangle = \langle 0|\phi_I(x)\phi_I(y)|0\rangle .$$

With these, we can write

$$\boxed{T\{\phi_I(x)\phi_I(y)\} = \phi_I^+(x)\phi_I^+(y) + \phi_I^-(x)\phi_I^+(y) + \phi_I^-(y)\phi_I^+(x) + \phi_I^-(x)\phi_I^-(y) + \langle 0|\phi_I(x)\phi_I(y)|0\rangle}$$

Now, <u>define</u> the "normal ordering symbol" N

s.t. N takes the string a. $a^\dagger$'s and rearranges them so that $a^\dagger$'s are on the left

Ex.

$$\left\{\begin{aligned} &N(a^\dagger_{\vec p}\, a_{\vec p}) = a^\dagger_{\vec p}\, a_{\vec p} \\ &N(a_{\vec p}\, a^\dagger_{\vec p}) = a^\dagger_{\vec p}\, a_{\vec p} . \\ &N(a_{\vec p}\, a_{\vec q}\, a^\dagger_{\vec p}) = a^\dagger_{\vec p}\, a_{\vec p}\, a_{\vec q} \end{aligned}\right.$$

ordering for $a_{\vec p}$, $a_{\vec q}$ doesn't matter since they commute.

→ <u>Note</u> N is not a well-defined mathematical operation

e.g. $N([a_{\vec p}, a^\dagger_{\vec q}]) \neq N((2\pi)^3 \delta^{(3)}(\vec p - \vec q))$

→ it is only a lexicographic convention.

Now, let us consider general x^0, y^0, then

$$T\{\phi_I(x)\,\phi_I(y)\} = N\{\phi_I(x)\,\phi_I(y)\} + \begin{cases} [\phi_I^+(x),\, \phi_I^-(y)] & \text{for } x^0 > y^0 \\ [\phi_I^+(y),\, \phi_I^-(x)] & \text{for } x^0 < y^0 \end{cases}$$

→ Let us define the contraction of $\phi_I(x)$, $\phi_I(y)$ as

$$\overbrace{\phi_I(x)\,\phi_I(y)} = \begin{cases} [\phi_I^+(x),\, \phi_I^-(y)] & x^0 \geq y^0 \\ [\phi_I^+(y),\, \phi_I^-(x)] & x^0 < y^0 \end{cases}$$

then notice that, from our previous observation,

$$\overbrace{\phi_I(x)\,\phi_I(y)} = \begin{cases} \langle 0|\, \phi_I(x)\phi_I(y)\,|0\rangle & x^0 \geq y^0 \\ \langle 0|\, \phi_I(y)\,\phi_I(x)\,|0\rangle & y^0 > x^0. \end{cases}$$

So,

Feynman propagator

$$\begin{aligned} \overbrace{\phi_I(x)\phi_I(y)} &= \langle 0|\, T\{\phi_I(x)\,\phi_I(y)\}\,|0\rangle \\ &= \mathcal{D}_F(x-y) \rightsquigarrow \text{Feynman propagator.} \\ &= \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2-m^2+i\epsilon}\, e^{-ip(x-y)} \end{aligned}$$

With these, we have that

→ leaving out the I's subscripts

$$T\{\phi(x)\phi(y)\} = \mathcal{N}\{\phi(x)\phi(y) + \overbrace{\phi(x)\phi(y)}\}$$

$$\overbrace{\phi(x)\phi(y)} = \overbrace{\phi(y)\phi(x)}$$

$$\to \quad T\{\phi(x)\phi(y)\} = \mathcal{N}\{\phi(x)\phi(y)\} + \text{"contraction"}.$$

In fact, the generalization of this is called

<u>Wick's Theorem</u>

$$\boxed{\begin{aligned} &T\{\phi(x_1)\phi(x_2)\dots\phi(x_m)\} \\ &= N\{\phi(x_1)\phi(x_2)\dots\phi(x_m) + \text{all possible contractions}\} \end{aligned}}$$

<u>Ex</u> $T\{\phi_1\phi_2\phi_3\phi_4\}$ $\qquad (\phi_n = \phi(x_n))$

$$= N\{\phi_1\phi_2\phi_3\phi_4 +$$

$$+ \overbrace{\phi_1\phi_2}\phi_3\phi_4 + \phi_1\overbrace{\phi_2\phi_3}\phi_4 + \phi_1\phi_2\overbrace{\phi_3\phi_4}$$

$$\overbrace{\phi_1\phi_2\phi_3\phi_4} + \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \phi_1\overbrace{\phi_1\phi_3\phi_4} + \overbrace{\phi_1\phi_2}\overbrace{\phi_3\phi_4}$$

$$+ \overbrace{\phi_1\phi_2\phi_3}\phi_4 + \underbrace{\phi_1\overbrace{\phi_2\phi_3}\phi_4}\}$$

what does $\mathcal{N}\{\overbrace{\phi_1\phi_2}\phi_3\phi_4\}$ mean?

$$N\{\overbrace{\phi_1\phi_2}\phi_3\phi_4\} = \overbrace{\phi_1\phi_2}\,\mathcal{N}\{\phi_3\phi_4\}$$

$$= \mathcal{D}_F(x_1 - x_2)\,\mathcal{N}\{\phi_3\phi_4\}.$$

$\cancel{4}$

Proof → prove by induction. $n=2$ is good (Feynman)

→ assume this holds for $n-1$.

Let $W(\phi_1 \ldots \phi_n) = N\{\phi_1 \phi_2 \cdots \phi_n + \text{all possible contr.}\}$

To prove $W(\phi_1 \ldots \phi_n) = T\{\phi_1 \phi_2 \ldots \phi_n\}$.

W/l/o/g: let $x_1^0 \geq x_2^0 \geq \ldots \geq x_n^0$.

Then $T\{\phi_1 \ldots \phi_n\} = \phi_1 T\{\phi_2 \ldots \phi_n\}$

$$= \phi_1 W(\phi_2 \ldots \phi_n) \quad \text{since } n-1 \text{ true.}$$

So $T\{\phi_1 \ldots \phi_n\} = \underbrace{\phi_1^- W(\phi_2 \ldots \phi_n) + W(\phi_2 \ldots \phi_n)\phi_1^+}_{X} + \underbrace{[\phi_1^+, W]}_{Y}$

Let $X \equiv \phi_1^- W + W\phi_1^+$; $Y \equiv [\phi_1^+, W]$.

$X+Y$ are normal ordered: X contains all contractions in $W(\phi_1 \ldots \phi_n)$ which does not contract ϕ_1 w/ anything.

Y contains all contractions in $W(\phi_1 \ldots \phi_n)$ which contracts ϕ_1 with something.

So $T(\phi_1 \phi_2 \ldots \phi_n) = W(\phi_1 \ldots \phi_n)$.

(we won't worry too much abt this proof.

→ the main idea is the theorem itself).

□

In any case, we have another way to explicitly write out the result of Wick's Thm:

$$T\{\phi_1\phi_2\cdots\phi_n\} = N\left\{\exp\left[\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n} \phi_i\phi_j \frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}\right]\phi_1\cdots\phi_n\right\}$$

we'll see why later on.

Feynman Diagrams

Wick's Thm allows ~~us~~ to write

$$\langle 0|T\{\phi_1\cdots\phi_n\}|0\rangle$$

in terms of sums and products of Feynman propagators.

→ Now, we will develop ~~a~~ diagrammatic expressions.

Recall that

$$T\{\phi_1\phi_2\phi_3\phi_4\} = N\{\phi_1\phi_2\phi_3\phi_4 + \text{all possible contractions}\}$$

But the only contribution to

$\langle 0|T\{\phi_1\phi_2\phi_3\phi_4\}|0\rangle$ is when <u>all</u> the ϕ's are contracted.

This b/c whenever things are in normal order, the exp value vanishes. → $N(\phi_1\phi_2\phi_3\phi_4) = \phi_1\phi_2 N(\phi_3\phi_4)$

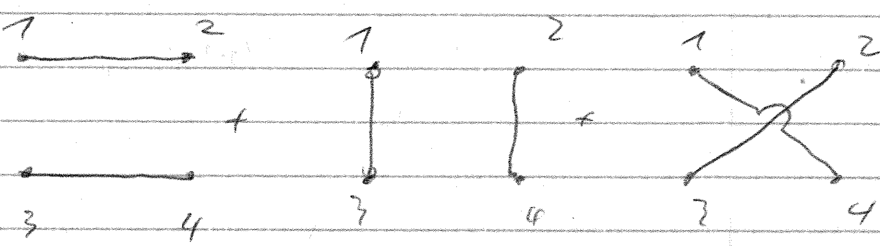
→ to "escape" from normal order, ϕ's have to be contracted

This means that

$$\boxed{T\{\phi_1\phi_2\phi_3\phi_4\} = \overline{\phi_1\phi_2}\,\overline{\phi_3\phi_4} + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4}$$

→ Can write this as Feynman diagrams...

$T\{\phi_1\phi_2\phi_3\phi_4\} =$



↳ $\boxed{\text{Interpretation}}$

Particles are created at 2 spacetime points, each propagates to one of the other points, then get annihilated.

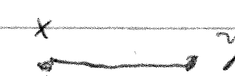
→ total amplitude of the process is the sum of the diagrams.

Well... what about something like...

$$\langle 0|\, T\left\{\phi(x)\,\phi(y)\,\exp\left\{-i\int_{-\infty}^{\infty} dt\, H_I(t)\right\}\right\} |0\rangle \;?$$

- Well... as a pwr series in λ, the lowest order term is

$$\langle 0|\, T\{\phi(x)\phi(y)\}|0\rangle = D_F(x-y).$$



- $\underline{1^{st}\text{ order}}$ $\langle 0|\, T\{\phi(x)\phi(y)\,(-i)\left(\int_{-\infty}^{\infty} dt\, H_I(t)\right)_{dt} |0\rangle$

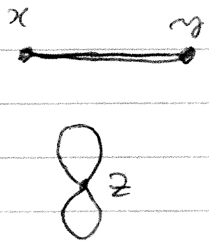
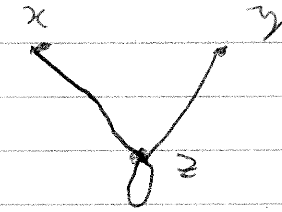
$$= \langle 0|\, T\{\phi(x)\,\phi(y)\,(-i)\int d^4z\, \tfrac{\lambda}{4!}\,\phi^4(z)\}\,|0\rangle$$

$$= \dots \frac{-i\lambda}{4!}\int d^4z \,\langle 0|T\{\phi(x)\phi(y)\overbrace{\phi(z)\phi(z)\phi(z)\phi(z)}^{\phi^4(z)}\}|0\rangle$$

$$= \frac{-i\lambda}{4!}\int d^4z \Big\{ \phi(x)\phi(y)\cdot\{\phi(z)\phi(z)\phi(z)\phi(z) + \phi_z\phi_z\phi_z\phi_z + \phi_z\phi_z\phi_z\phi_z\}$$

$$+ \phi(x)\phi(y)\phi(z)\phi(z)\phi(z)\phi(z) \Big\}$$

→ 12 terms that are identical

=  +  ⟶ propagators

γ   γ

$\int d^4z \, D_F(x-z) D_F(z-y) D_F(z-z)$

↑

12 of these.

$\int d^4z \, D_F(x-y) D_F(z-z) D_F(z-z)$

↳ each contraction D_F is a line.

each spacetime point is a dot.

→ but need to distinguish "external" and "internal" points.

↓ x, y ↓ z

Each internal point is associated w/ a factor of $-i\lambda\int d^4z$, with combinatorial factor...

How do we count these combinatorial factors?

Oct,
2020