8.(3)09 Section 6

October 15, 2021

1 Infinitesimal generators

Recall that $G(q, P, t)$ is a generating function for transformations specified by $F_2(q, P, t) = qP + \epsilon G(q, P, t)$ where ϵ is an infinitesimal quantity. The transformations of the phase space coordinates are given by

$$\delta q_i = \epsilon \frac{\partial G}{\partial p_i}\,, \quad \delta p_i = \epsilon \frac{\partial G}{\partial q_i} \tag{1}$$

(a)

Show that $G(q, P, t) = p_i$ generates translations in q_i and leaves the other qs alone

(b)

Show that $G(q, P, t) = L_z = xp_y - yp_x$ generates rotations around the z axis.

2 Canonical transformations

(a)

Determine if the following is a canonical transformation:

$$Q = \arctan \frac{\alpha q}{p}\,, \quad P = \frac{\alpha q^2}{2}\left(1 + \frac{p^2}{\alpha^2 q^2}\right) \tag{2}$$

(b)

Determine if the following is a canonical transformation:

$$Q = \ln(1 + \sqrt{q}\cos p)\,, \quad P = 2(1 + \sqrt{q}\cos p)\sqrt{q}\sin p \tag{3}$$

Hint: consider the generating function $F_3 = -(e^Q - 1)^2 \tan p$

(c)

Determine if the following is a canonical transformation:

$$Q = \ln(1 + \cos q \sin p)\,, \quad P = \ln(1 + \sin q \cos p). \tag{4}$$

3 1D SHO constant of motion

Determine if $u = \ln(p + im\omega q) - i\omega t$ is a constant of motion in the 1D simple harmonic oscillator.

4 Hamilton-Jacobi for a free particle in 1D

Consider a free particle moving in one dimension.

(a)

Write down and solve the Hamilton-Jacobi equation for Hamilton's principle function S for this system.

(b)

Write down and solve the Hamilton-Jacobi equation for Hamilton's characteristic function W for this system.

5 Hamilton-Jacobi for a free particle in 3D

Consider a free particle moving in three dimensions. Write down and solve the Hamilton-Jacobi equation for Hamilton's characteristic function W for this system.

6 Hamilton-Jacobi for the two body problem

Consider two fixed unequal point masses m_1 and m_2 a distance $2a$ apart in three dimensions, and a third particle of mass m free to move. It will be helpful to align the z axis to lie on the line between the particles.

(a)

Write down the Hamiltonian for this system in cylindrical coordinates (r, ϕ, z) for particle m.

(b)

Write down the Hamilton-Jacobi equation for Hamilton's characteristic function W and determine if it is separable.

(c)

Consider the coordinate transform to ellipsoidal polar coordinates (u, v, ϕ): $r = a \sinh v \sin u$, $z = a \cosh v \cos u, \phi = \phi$. Write the Hamiltonian in these coordinates.

(d)

Write down the Hamilton-Jacobi equation for Hamilton's characteristic function W and separate it.