1.4 Position, momentum, and translation

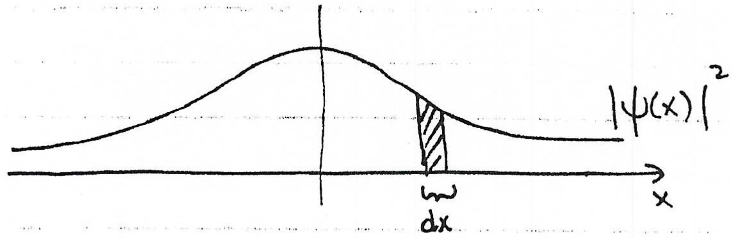
Until now, all explicit examples involved finite-dimensional matrices.

Generalize to continuous degrees of freedom

Want to describe particle in 3D by wavefunction $\psi(x,y,z)$

Simplify to 1D : $\psi(x)$

Want $|\psi(x)|^2 dx$ = probability particle is in region dx



Natural Hilbert space: $\mathcal{L}^2(\mathbb{R})$:
Square integrable functions $\int_{-\infty}^{\infty} |\psi(x)|^2 < \infty$.

[To precisely define, need Lebesgue measure, ...]

Can do QM in this framework.
$\mathcal{L}^2(\mathbb{R})$ is a separable Hilbert space.

Typical observables on $\mathcal{L}^{(2)}(\mathbb{R})$:

$P_{[a,b]}$ projection on interval $[a,b]$

$$(P_{[a,b]} f)(x) = \begin{cases} f(x), & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

$P_{[a,b]}$ are good operators, and diagonalize into ON basis w/ evalues 1 or ϕ.

QM breaks down at some scale $> 10^{-33}$ cm, but can use $P_{[a,b]}$ for any current expts.

<u>Problem</u> : This approach, while mathematically correct, introduces unnecessary complications from physical point of view.

Most operators of interest <u>cannot</u> be diagonalized in $\mathcal{H}$.

Ex.

	position	x	
	momentum	$p = -i\hbar\, \partial/\partial x$	
	Energy	$H = p^2/2m + V(x)$	(for example)

estates of x $\Longrightarrow$  support at a point, $|\psi|^2 = 0$ except at x.

" " p $\Rightarrow$  support everywhere, $\int |\psi|^2 = \infty$

<u>Solution</u> (Dirac) : Ignore this problem. Treat all these operators as acceptable and include their eigenvectors formally, <u>even if not in $\mathcal{H}$.</u>

["Quote from Von Neumann]

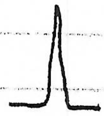
Dirac's approach:

Replace discrete basis $|a_i\rangle$ with continuous basis $|\xi\rangle$, ξ in continuous domain (like $(-\infty, \infty)$).

$$A|a_i\rangle = a_i|a_i\rangle \implies \Xi|\xi\rangle = \xi|\xi\rangle$$

$$\langle a_i|a_j\rangle = \delta_{ij} \implies \langle\xi|\xi'\rangle = \delta(\xi-\xi')$$

$$\sum_i |a_i\rangle\langle a_i| = \mathbb{1} \implies \int_D d\xi\, |\xi\rangle\langle\xi| = \mathbb{1}.$$

[Brief review of Dirac δ function:  "distribution"

$$\delta(\xi) = 0, \quad \text{when } \xi \neq 0$$

$$\int_{-a}^{a} \delta(\xi)\, d\xi = 1, \quad \forall\, a > 0.$$

Property: $$\int_{-\infty}^{\infty} \delta(\xi) f(\xi)\, d\xi = f(0)$$ for smooth f.

Can realize $\delta(\xi)$ as a limit of smooth functions

e.g. $$\delta(\xi) = \lim_{a\to\infty} \sqrt{\frac{a}{\pi}}\, e^{-a\xi^2}$$]

$$= \lim_{\varepsilon\to 0} \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2} = \frac{1}{2\pi}\int dk\, e^{ikx}$$

Generally, can allow operators with partly continuous & partly discrete spectrum. (e.g. H-1 bound + free states)

<u>Example</u>: "position basis".

$$\boxed{X|x'\rangle = x'|x'\rangle}$$

[Notation: X is always operator, $x', x'', \ldots$ are eigenvalues]

$$\boxed{\langle x'|x''\rangle = \delta(x'-x''), \quad \int_{-\infty}^{\infty} dx' |x'\rangle\langle x'| = \mathbb{1}.}$$

$|x'\rangle$ is <u>not in $\mathcal{H}$</u>, but can still treat as state for most operations; justifiable in terms of appropriate limits.

Note: x is not measurable to arbitrary precision experimentally. (can [illegible] $\ell \sim 10^{-35}$ cm) So not really an observable, but a convenient formal tool.

Working with x, p:

Write

$$|\psi\rangle = \int dx' \, |x'\rangle\langle x'|\psi\rangle$$

$$= \int dx' \, \psi(x') \, |x'\rangle$$

so $\psi(x') = \langle x'|\psi\rangle$.

The probability that $a \leq x \leq b$ is given by

$$\int_a^b dx' \, |\psi(x')|^2 = \int_a^b dx' \, \langle\psi|x'\rangle\langle x'|\psi\rangle.$$

$$= \langle\psi|P_{[a,b]}|\psi\rangle.$$

<u>Momentum operator</u>

$$\boxed{p = -i\hbar \frac{\partial}{\partial x}}$$

<u>p is generator of translation</u>

$$e^{iap/\hbar} f(x) = e^{a\partial/\partial x} f(x) = f(x+a).$$

Commutation relation $\boxed{[x, p] = i\hbar}$

[recall $[A, B] = \mathbb{1}$ imp. for finite dim]

[Related to $\{x, p\} = 1$ through classical - quantum correspondence, foundation of "matrix mechanics" of Bohr, Jordan, etc... (more later)]

From general uncertainty relation,

$$\boxed{\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle \geq \hbar^2/4.}$$

So wavefunctions can't be localized in x and in p.

Momentum basis

Construct a basis of states with

$$\boxed{p|p'\rangle = p'|p'\rangle.}$$

know $\quad -i\,\partial/\partial x' \langle x'|p'\rangle = p' \langle x'|p'\rangle$

so $\quad \langle x'|p'\rangle = N e^{ip'x'/\hbar}$

want $\quad \boxed{\langle p'|p''\rangle = \delta(p' - p'')}$

so

$$\int dx' \langle p'|x'\rangle\langle x'|p''\rangle = \int dx' |N|^2 e^{ix'(p''-p')/\hbar}$$

$$= |N|^2 \cdot 2\pi\hbar \cdot \delta(p' - p'')$$

so $\quad |N|^2 = \dfrac{1}{2\pi\hbar},$

$$\langle x'|p'\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ip'x'/\hbar}.$$

Completeness for momentum states

$$\int dp' \, |p'\rangle\langle p'|$$
$$= \int dx' \, dx'' \, dp' \, |x'\rangle\langle x'|p'\rangle\langle p'|x''\rangle\langle x''|$$
$$= \int dx' \, dx'' \, dp' \, |x'\rangle \frac{1}{2\pi\hbar} e^{ip'(x''-x')/\hbar} \langle x''|$$
$$= \int dx' \, |x'\rangle\langle x'| \quad = \mathbb{1}.$$

Fourier transforms

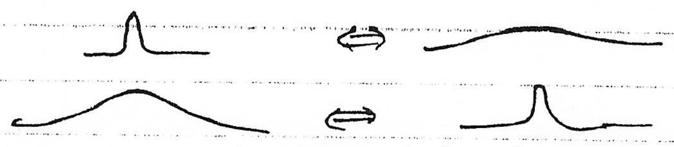
$$|\psi\rangle = \int dp' \, |p'\rangle\langle p'|\psi\rangle$$
$$= \int dp' \, \phi(p') \, |p'\rangle$$

$$\phi(p') = \langle p'|\psi\rangle$$
$$= \int dx' \, \langle p'|x'\rangle\langle x'|\psi\rangle$$
$$= \frac{1}{\sqrt{2\pi\hbar}} \int dx' \, e^{-ip'x'/\hbar} \, \psi(x')$$

similarly

$$\psi(x') = \frac{1}{\sqrt{2\pi\hbar}} \int dp' \, e^{ip'x'/\hbar} \, \phi(p')$$

Uncertainty principle $\langle \Delta x^2 \rangle \langle \Delta p^2 \rangle \geq \hbar^2/4$ relates width of wavefunction $\psi(x')$ and Fourier transform $\phi(p')$



[Example: inequality saturated for Gaussian e^{-x^2/d^2}]

Generalize to 3D

$$\mathcal{H} = \mathcal{H}^{(x)} \otimes \mathcal{H}^{(y)} \otimes \mathcal{H}^{(z)}$$

$$|\psi\rangle = \int dx\, dy\, dz \quad \psi(x,y,z)\ |x,y,z\rangle$$

$|x,y,z\rangle = |x\rangle \otimes |y\rangle \otimes |z\rangle$ is basis for $\mathcal{H}$

Translation group

$$T(\vec{a})\ |\vec{x}\rangle = |\vec{x}+\vec{a}\rangle, \qquad \vec{a} \in \mathbb{R}^3.$$

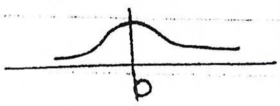
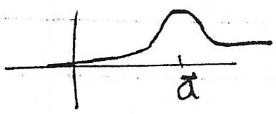
$\mathbb{R}^3$ forms a group under addition $\vec{a}+\vec{b}$.
(closed, associative, identity ϕ, inverses $-\vec{a}$).

A representation of a group G on $\mathcal{H}$ is a map R from G to linear operators on $\mathcal{H}$ so that $R(\text{Identity}) = \mathbb{1}$, $R(ab) = R(a)R(b)$.

T is a unitary representation of the 3D translation group on $\mathcal{H}$.

$$T^{\dagger}(\vec{a}) = T^{-1}(\vec{a}) = T(-\vec{a})$$
$$T(\vec{a}+\vec{b}) = T(\vec{a})T(\vec{b})$$
$$T(0) = \mathbb{1}$$

Realization: $T(\vec{a}) = e^{-i\vec{a}\cdot\vec{p}/\hbar}$. $\quad [p_i,p_j]=0 \Rightarrow T(\vec{a})T(\vec{b}) = T(\vec{b})T(\vec{a})$

Active picture :  $\Rightarrow$ 

Passive picture : $\psi(\vec{x}) \Longrightarrow \psi(\vec{x}-\vec{a})$
(peak at $\vec{a}$)

$$T(\vec{a}) \int \psi(\vec{x}')\, |\vec{x}'\rangle\, d\vec{x}' = \int \psi(\vec{x}')\, |\vec{x}'+\vec{a}\rangle\, d\vec{x}'$$
$$= \int \psi(\vec{x}''-\vec{a})\, |\vec{x}''\rangle\, d\vec{x}''$$