8.(3)09 Classical Mechanics III

Assignment 8: Solutions

November 7, 2021

1. Flow Geometries [12 points for 8.09 who do all 3 parts for 4 points each as shown, 6 points for 8.309 who do any 2 out of 3 parts for 3 points each]

Recall that $\vec{v} = \vec{\nabla}\phi$, where the gradient is given in cylindrical coordinates as $\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}$. In all parts ϕ is independent of z, and hence so is $\vec{v}$; the flow is two-dimensional.

(a) [4 points] In this case $\vec{v} = 2Cr[\hat{r}\cos(2\theta) - \hat{\theta}\sin(2\theta)]$, with no $\hat{z}$ component. The set of points where $\vec{v} = 0$ are the points for which $r = 0$, i.e. the z-axis (we may call this the stagnation line). We require the flow to always be parallel to the bounding surfaces. For this case (and cases (b) and (c)) this can be assured by picking surfaces at constant θ at which $\hat{\theta} \cdot \vec{v} = 0$. Then the velocity is entirely in the $\hat{r}$ direction, parallel to the surface.

For case (a), we can take bounding surfaces $\theta = 0$ and $\theta = \pi/2$, corresponding to the $(+x)z$-halfplane and $(+y)z$-halfplane. (The flow exists in the region $0 \leq \theta \leq \pi/2$.)

At $\theta = 0$ (the halfplane defined by $x > 0,\ y = 0$), $\vec{v} = 2Cr\hat{r} = 2Cx\hat{x}$.

At $\theta = \pi/2$ (the halfplane defined by $y > 0,\ x = 0$), $\vec{v} = -2Cr\hat{r} = -2Cy\hat{y}$.

(b) [4 points] In this case $\vec{v} = 4Cr^3[\hat{r}\cos(4\theta) - \hat{\theta}\sin(4\theta)]$. Again the stagnation point is the z-axis. We can take bounding surfaces $\theta = 0$ and $\theta = \pi/4$ (at these surfaces $\sin(4\theta) = 0$, so $\vec{v} \parallel \hat{r}$).

At $\theta = 0$, $\vec{v} = 4Cr^3\hat{r}$ and the flow goes in the direction of increasing $\hat{r}$.

At $\theta = \pi/4$, $\vec{v} = -4Cr^3\hat{r}$ and the flow goes in the direction of decreasing $\hat{r}$.

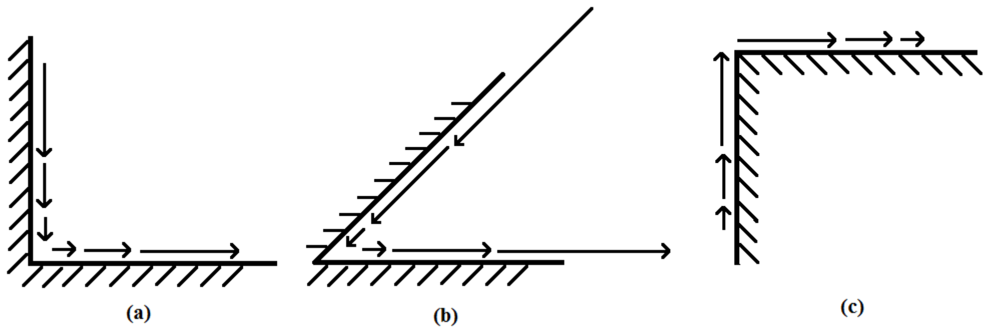
(c) [4 points] In this case $\vec{v} = \frac{2}{3}Cr^{-1/3}[\hat{r}\cos(\frac{2\theta}{3}) - \hat{\theta}\sin(\frac{2\theta}{3})]$. There are actually no stagnation points at finite r (since at $r = 0$ the velocity blows up). We can take bounding surfaces $\theta = 0$ and $\theta = \frac{3\pi}{2}$, since at these surfaces the flow is parallel to the $\hat{r}$ direction. The flow exists in the region $0 \leq \theta \leq \frac{3\pi}{2}$; this describes a possible flow around the outside of a right angle corner (as opposed to (a), describing a flow in the inside of a right angle corner).

At $\theta = 0$, $\vec{v} = \frac{2}{3}\frac{C}{r^{1/3}}\hat{r} = \frac{2}{3}\frac{C}{r^{1/3}}\hat{x}$.

At $\theta = \frac{3\pi}{2}$, $\vec{v} = -\frac{2}{3}\frac{C}{r^{1/3}}\hat{r} = \frac{2}{3}\frac{C}{r^{1/3}}\hat{y}$.

Note the potential flow formalism isn't really valid at $r \approx 0$, since $\vec{v}$ blows up; in fact there is a vortex at $r = 0$.

Plots of these flows are shown below.



2. The Stream Function [8.309 ONLY, 6 points]

(a) [2 points] ψ is constant, so $\mathrm{d}\psi = 0$. We have

$$0 = \mathrm{d}\psi = \frac{\partial \psi}{\partial \mathrm{x}}\mathrm{dx} + \frac{\partial \psi}{\partial \mathrm{y}}\mathrm{dy} = -\mathrm{v_y dx} + \mathrm{v_x dy}.$$

This implies that

$$\frac{dy}{dx} = \frac{v_y}{v_x},$$

which is the streamline equation confined to the x-y plane, i.e. the path is tangent to the instantaneous velocity.

(b) [4 points] Irrotational flow means that $\nabla \times \vec{v} = 0$. Taking the $\hat{z}$ component of the curl, we have

$$\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{z} = 0,$$

which means

$$-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} = 0.$$

This is equivalent to $\nabla^2 \psi = 0$, which is Laplace's Equation.

In polar coordinates we have the stream function $\psi(r, \theta)$ and velocity potential $\phi(r, \theta)$. We can convert the cartesian definitions to polar coordinates to find that the velocity components are

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \; v_\theta = -\frac{\partial \psi}{\partial r} \text{ (opposite signs are ok)}$$

Also we know that $\vec{v} = \nabla \phi$ so

$$v_r = \frac{\partial \phi}{\partial r}, \; v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}.$$

Since $\psi = \frac{2}{3} r^{3/2} \sin(\frac{3}{2}\theta)$,

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = r^{1/2}\cos(\frac{3}{2}\theta)$$

and

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = -r^{1/2}\sin(\frac{3}{2}\theta).$$

Therefore $\phi = \frac{2}{3}r^{3/2}\cos(\frac{3}{2}\theta)$ is the solution.

Checking Laplace's Equation we find (not required):

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{1}{2}r^{-1/2}\sin(\frac{3}{2}\theta) + r^{-1/2}\sin(\frac{3}{2}\theta) - \frac{3}{2}r^{-1/2}\sin(\frac{3}{2}\theta) = 0$$

3. Ideal Fluid Flow around a Cylinder [16 points]

(a) [6 points] We need to find a velocity potential ϕ satisfying $\nabla^2\phi = 0$, for which we can derive the velocity $\vec{v} = \vec{\nabla}\phi$. Infinitely far away from the cylinder we must have $\vec{v} \to \vec{u}$, so let us set

$$\vec{v} = \vec{u} + \vec{v}\,'; \qquad \vec{v}\,' = \vec{\nabla}\phi'$$

where

$$\nabla^2\phi' = 0; \qquad \vec{\nabla}\phi' \to 0 \text{ at } r \to \infty.$$

Moreover, by continuity we require that, over any closed surface S,

$$\oint_S \vec{v}\cdot d\hat{n} = \oint_S \vec{\nabla}\phi'\cdot d\hat{n} = 0.$$

(This does not follow automatically from $\vec{\nabla}\cdot\vec{v} = 0$, since the surface S can also enclose the cylinder.)

By symmetry, the velocity and the potential must be independent of z; hence the flow is two dimensional. Ignoring the z-direction, the Laplacian is

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Now recall the 2d point source solution $\phi' = A\ln r$. This satisfies Laplace's equation, but fails to satisfy continuity: $\oint_S \vec{\nabla}\phi'\cdot d\hat{n} \neq 0$. However, following the example of flow past a sphere done in class, we guess that we can take the derivative of the solution with respect to any spatial direction:

$$\phi' = \vec{A}\cdot\vec{\nabla}(\ln r) = \frac{\vec{A}\cdot\hat{r}}{r}.$$

(Note that the solution in class directly presupposes $\vec{A} = A\vec{u}$; we won't make this assumption here.)

The velocity is given by

$$\begin{aligned}
\vec{v}' = \vec{\nabla}\phi' &= \vec{\nabla}\left(\frac{\vec{A}\cdot\hat{r}}{r}\right) \\
&= \vec{\nabla}\left(\frac{A_x x + A_y y}{x^2+y^2}\right), \qquad \text{since } \vec{A}\cdot\hat{r} = A_x\frac{x}{\sqrt{x^2+y^2}} + A_y\frac{y}{\sqrt{x^2+y^2}} \\
&= \frac{A_x\hat{x} + A_y\hat{y}}{x^2+y^2} - 2\frac{(A_x x + A_y y)(x\hat{x}+y\hat{y})}{(x^2+y^2)^2} \\
&= \frac{\vec{A} - 2(\vec{A}\cdot\hat{r})\hat{r}}{r^2}.
\end{aligned}$$

Written alternatively,

$$\vec{v}' = \frac{\vec{A}}{r^2} - \frac{2(\vec{A}\cdot\vec{r})\vec{r}}{r^4}.$$

This solution automatically satisfies Laplace's equation, since $\nabla^2\phi' = \vec{A}\cdot\vec{\nabla}(\nabla^2 \ln r) = 0$. It also satisfies continuity: if we take the closed surface S to be a cylinder of radius r and length L, then (the contribution from the caps cancel out)

$$\oint_S \vec{v}\cdot d\hat{n} = L\int_0^{2\pi} -\frac{\vec{A}\cdot\hat{r}}{r^2} r\,d\theta = 0.$$

The full velocity is $\vec{v} = \vec{v}' + \vec{u} = \vec{u} + \frac{\vec{A}-2(\vec{A}\cdot\hat{r})\hat{r}}{r^2}$. We now impose the boundary condition that $\hat{r}\cdot\vec{v} = 0$ at $r = R$, or

$$\vec{u}\cdot\hat{r} - \frac{\vec{A}\cdot\hat{r}}{R^2} = 0.$$

Since this holds for all $\hat{r}$ (which can vary over the cylinder), we must have that $\vec{A} = R^2\vec{u}$. The velocity profile is given by

$$\vec{v} = \left(1 + \frac{R^2}{r^2}\right)\vec{u} - \frac{2R^2(\vec{u}\cdot\hat{r})}{r^2}\hat{r}.$$

(b) [4 points] We want to apply Bernoulli's equation

$$\frac{\rho u^2}{2} + p_0 = \frac{\rho v^2}{2} + p$$

where p_0 is the pressure at $r \to \infty$. Squaring the velocity from part (a), we have

$$v^2 = \left(1+\frac{R^2}{r^2}\right)^2 u^2 + \frac{4R^4(\vec{u}\cdot\hat{r})^2}{r^4} - \frac{4R^2(\vec{u}\cdot\hat{r})^2}{r^2}\left(1+\frac{R^2}{r^2}\right).$$

We only need the result at $r = R$; using $\vec{u}\cdot\hat{r} = u\cos\theta$,

$$\begin{aligned}
v^2|_{r=R} &= 4u^2 + 4u^2\cos^2\theta - 8u^2\cos^2\theta \\
&= 4u^2(1-\cos^2\theta)
\end{aligned}$$

and since $u^2 - v^2|_{r=R} = u^2(4\cos^2\theta - 3)$, we get

$$p = p_0 + \frac{1}{2}\rho u^2(4\cos^2\theta - 3)$$

on the cylinder.

(c) [6 points] Expanding $\vec{v}$ in Cartesian components, we have

$$\begin{aligned}\vec{v} &= \left(1 + \frac{R^2}{r^2}\right) u\hat{x} - \frac{2R^2ux}{r^4}(x\hat{x} + y\hat{y}) \\ &= \left(1 + \frac{R^2}{r^2} - \frac{2R^2x^2}{r^4}\right) u\hat{x} - \frac{2uR^2xy}{r^4}\hat{y}\end{aligned}$$

and so

$$v_x = u\left(1 + \frac{R^2}{r^2} - \frac{2R^2x^2}{r^4}\right), \qquad v_y = -\frac{2uR^2xy}{r^4}.$$

Recalling that streamlines are given by $\frac{dy}{dx} = \frac{v_y}{v_x}$ (the z-component can be ignored here since the flow is 2-dimensional), we have

$$\begin{aligned}\frac{dy}{dx} &= -\frac{2R^2xy}{(r^4 + R^2r^2 - 2R^2x^2)} \\ &= -\frac{2R^2xy}{[(x^2+y^2)^2 + R^2(x^2+y^2) - 2R^2x^2]}.\end{aligned}$$

We can verify that the given curve is a streamline: taking the x-derivatives of both sides of

$$x^2 + y^2 = \frac{R^2}{1 - K/y}$$

we get

$$\begin{aligned}2x + 2y\frac{dy}{dx} &= -\frac{KR^2}{y^2(1-K/y)^2}\frac{dy}{dx} \\ &= -\frac{Kr^4}{y^2R^2}\frac{dy}{dx} \qquad \text{(plugging curve equation in)} \\ &= -\frac{r^4}{yR^2}\left(1 - \frac{R^2}{r^2}\right)\frac{dy}{dx} \qquad \text{(eliminating } K\text{)}\end{aligned}$$

which gives after multiplication by $-yR^2$

$$-2xyR^2 = \frac{dy}{dx}[2y^2R^2 + r^4 - R^2r^2] = \frac{dy}{dx}[r^4 + R^2r^2 - 2R^2x^2]$$

which is exactly our differential equation, thus proving the curve is a streamline.

Aside: This differential equation can actually be solved explicitly, if one uses cylindrical coordinates. Specifically, since $0 = d\vec{r} \times \vec{v} = (v_\theta dr - rv_r d\theta)\hat{z} + \cdots$, we have

$$\frac{dr}{d\theta} = \frac{rv_r}{v_\theta}.$$

Now using $\vec{u} = u\hat{x} = u(\hat{r}\cos\theta - \hat{\theta}\sin\theta)$, we get $\vec{u}\cdot\hat{r} = u\cos\theta$ and $\vec{u}\cdot\hat{\theta} = -u\sin\theta$, which leads to

$$v_r = \hat{r}\cdot\vec{v} = \left(1+\frac{R^2}{r^2}\right)\vec{u}\cdot\hat{r} - \frac{2R^2(\vec{u}\cdot\hat{r})}{r^2} = \left(1-\frac{R^2}{r^2}\right)u\cos\theta$$

$$v_\theta = \hat{\theta}\cdot\vec{v} = \left(1+\frac{R^2}{r^2}\right)\vec{u}\cdot\hat{\theta} = -\left(1+\frac{R^2}{r^2}\right)u\sin\theta.$$

Thus

$$\frac{dr}{d\theta} = r\frac{v_r}{v_\theta} = -\frac{r(r^2-R^2)}{r^2+R^2}\cot\theta$$

$$\frac{(r^2+R^2)}{r(r^2-R^2)}dr = -d\theta\cot\theta$$

which can be integrated to give

$$\ln\left(\frac{r^2-R^2}{r}\right) = -\ln\sin\theta + \ln K$$

where $\ln K$ is an arbitrary constant. Thus

$$\frac{r^2-R^2}{r} = \frac{K}{\sin\theta} = \frac{K}{y/r}$$

which after rearranging gives

$$r^2 = \frac{R^2}{1-K/y}$$

as the general equation obeyed by streamlines.

4. A Spherical Sound Wave [10 points]

(a) [3 points] The wave equation is

$$\frac{\partial^2 p}{\partial t^2} = c_s^2\,\nabla^2 p$$

where in spherical coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

We can verify that the wave equation holds for $p = \frac{1}{r}f(r-c_st)$:

$$\begin{aligned}
\nabla^2 p &= \frac{1}{r^2}\frac{\partial}{\partial r}r^2\left[-\frac{1}{r^2}f(r-c_st) + \frac{1}{r}\frac{\partial f}{\partial r}(r-c_st)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left[-f + r\frac{\partial f}{\partial r}\right] \\
&= \frac{1}{r^2}\left[-\frac{\partial f}{\partial r} + \frac{\partial f}{\partial r} + r\frac{\partial^2 f}{\partial r^2}\right] \\
&= \frac{1}{r}\frac{\partial^2 f}{\partial r^2} = \frac{1}{r}f''(r-c_st)
\end{aligned}$$

$$\frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial t}\left[-c_s \frac{1}{r} f'(r - c_s t)\right] = c_s^2 \frac{1}{r} f''(r - c_s t)$$

and hence the wave equation holds: $\frac{\partial^2 p}{\partial t^2} = c_s^2 \nabla^2 p$.

(b) [4 points] The velocity is given by the equation

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{\rho_a} \vec{\nabla} p$$

where ρ_a is the air density. We can compute:

$$\begin{aligned}
\frac{\partial \vec{v}}{\partial t} &= -\frac{1}{\rho_a} \hat{r} \frac{\partial}{\partial r} p \\
&= -\frac{\hat{r}}{\rho_a}\left[-\frac{1}{r^2} f(r - c_s t) + \frac{1}{r} f'(r - c_s t)\right]
\end{aligned}$$

Now setting $g(x) = \int^x dx' f(x')$, we have $g(r - c_s t) = -c_s \int^t dt' f(r - c_s t')$, and hence

$$\begin{aligned}
\vec{v} &= -\int^t dt' \frac{\hat{r}}{\rho_a}\left[-\frac{1}{r^2} f(r - c_s t') + \frac{1}{r} f'(r - c_s t')\right] \\
&= \frac{\hat{r}}{\rho_a c_s}\left[-\frac{1}{r^2} g(r - c_s t) + \frac{1}{r} g'(r - c_s t)\right].
\end{aligned}$$

where of course $g' = f$. (Note that the definition of g already carries an arbitrary constant.)

(c) [3 points] We again evaluate both sides of the wave equation:

$$\begin{aligned}
\nabla^2 \vec{v} &= \hat{r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial v_r}{\partial r}\right) - \frac{2}{r^2} v_r\right] \\
&= \frac{\hat{r}}{\rho_a c_s}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{2g}{r} - 2g' + r g''\right) - \left(-\frac{2g}{r^4} + \frac{2g'}{r^3}\right)\right] \\
&= \frac{\hat{r}}{\rho_a c_s}\left[\frac{1}{r^2}\left(-\frac{2g}{r^2} + \frac{2g'}{r} - g'' + r g'''\right) - \left(-\frac{2g}{r^4} + \frac{2g'}{r^3}\right)\right] \\
&= \frac{\hat{r}}{\rho_a c_s}\left[-\frac{g''(r - c_s t)}{r^2} + \frac{g'''(r - c_s t)}{r}\right]
\end{aligned}$$

and

$$\frac{\partial^2}{\partial t^2} \vec{v} = \frac{\hat{r}}{\rho_a c_s}\left[-\frac{1}{r^2} c_s^2 g''(r - c_s t) + \frac{1}{r} c_s^2 g'''(r - c_s t)\right]$$

and hence

$$\frac{\partial^2 \vec{v}}{\partial t^2} = c_s^2 \, \nabla^2 \vec{v}$$

as expected.

5. Viscous Fluid Velocity between Coaxial Cylinders [10 points]

Since the flow is steady, $\frac{\partial \vec{v}}{\partial t} = 0$ and $\frac{\partial p}{\partial t} = 0$, and the Navier-Stokes equation becomes

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{1}{\rho}\vec{\nabla}p + \nu \nabla^2 \vec{v}$$

By symmetry, $\vec{v}$ depends only on r, and $v_\theta = 0$. Moreover by continuity the surface integral of $\vec{v}$ over a cylinder with radius r should vanish, and hence $v_r = 0$ as well. Therefore we can set

$$\vec{v} = v(r)\hat{z}.$$

Also by symmetry, p is independent of θ; by assumption p is independent of z as well, so $p = p(r)$. The r-component of the Navier-Stokes equation gives $\frac{\partial p}{\partial r} = 0$, so p is actually constant.

The z-component of the Navier-Stokes equation thus simply reduces to

$$\nabla^2 v(r) = 0.$$

We can take as a solution $v = b \ln r + C$. We now match the boundary conditions using the no-slip condition:

$$v(r = R_1) = 0 = b \ln R_1 + C \ \Rightarrow C = -b \ln R_1$$

$$v(r = R_2) = u = b \ln(R_2/R_1) \ \Rightarrow b = \frac{u}{\ln(R_2/R_1)}$$

so the velocity profile of the liquid is

$$\vec{v} = u \frac{\ln(r/R_1)}{\ln(R_2/R_1)} \hat{z}.$$

We also want the friction force per unit length. The friction per unit area is given by the viscous stress tensor σ'_{ik}, of which the components in cylindrical coordinates is given in p. 48 of Landau. The only nonvanishing component is

$$\sigma'_{rz} = \eta \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) = \frac{\eta u}{r \ln(R_2/R_1)}.$$

There is a friction force acting in the z-direction per unit area perpendicular to $\hat{r}$. To get the friction per unit length, we multiply by $2\pi r$:

$$\frac{\text{friction}}{\text{length}} = \frac{2\pi \eta u}{\ln(R_2/R_1)}$$

Since the result is independent of r, this holds for both cylinders (except the forces are of course in opposite directions).

6. Viscous Shedding with Dimensional Analysis [6 points]

(a) [3 points] Letting ℓ, t be units of length and time, respectively, we have

$$[f] = t^{-1}, \qquad [D] = [L] = \ell, \qquad [V] = \ell t^{-1}$$

$$[\nu] = \frac{[V]t^{-1}}{[V]\ell^{-2}} = \ell^2 t^{-1}$$

(the last equation comes from noticing that in the Navier-Stokes equation $\frac{\partial \vec{v}}{\partial t}$ and $\nu \nabla^2 \vec{v}$ have the same units). Since there are 5 quantities and 2 nontrivial dimensions, we expect to be able to form $5 - 2 = 3$ independent dimensionless quantities, which can be taken to be

$$\alpha = \frac{L}{D}, \quad R = \frac{VD}{\nu}, \quad \frac{fD}{V}$$

(there are other possibilities, e.g. the D's in the latter two can be replaced by L). Therefore we can write the last variable as a function of the first two:

$$\frac{fD}{V} = \phi\Big(\frac{VD}{\nu}, \frac{L}{D}\Big)$$

$$f = \frac{V}{D}\phi\Big(\frac{VD}{\nu}, \frac{L}{D}\Big)$$

where ϕ is an arbitrary dimensionless function.

(b) [1 point] If we shrink D and L to $D' = \frac{D}{2}$ and $L' = \frac{L}{2}$, α is fixed automatically, but to keep R fixed we need $V' = 2V$, i.e. the wind velocity needs to be doubled.

(c) [2 points] Now if the two chimneys have the same dimensionless parameters α and R, then

$$f' = \frac{V'}{D'}\phi(\frac{V'D'}{\nu}, \frac{L'}{D'}) = 4\frac{V}{D}\phi(\frac{VD}{\nu}, \frac{L}{D}) = 4f$$

i.e. the frequency of vortex shedding is four times as large.

7. Golf Ball Drag with Dimensional Analysis [6 points]

(a) [3 points] Letting m, ℓ, t be units of mass, length, and time respectively. Then

$$[F_D] = m\ell t^{-2} \qquad [V] = \ell t^{-1} \qquad [D] = \ell \qquad [\omega] = t^{-1}$$

$$[\rho] = m\ell^{-3} \qquad \left[\frac{\eta}{\rho}\right] = \ell^2 t^{-1}, \qquad [c_s] = \ell t^{-1}.$$

We have 7 quantities and 3 nontrivial dimensions, so we can form $7 - 3 = 4$ independent dimensionless constants. We can take the numbers to be

$$M = \frac{V}{c_s}, \quad R = \frac{\rho V D}{\eta}, \quad \frac{D\omega}{V}, \quad \frac{F_D}{\rho V^2 D^2}$$

so we can take the last as a dimensionless function of the first three, i.e.

$$F_D = \rho V^2 D^2 \phi\left(\frac{V}{c_s}, \frac{D\omega}{V}, \frac{\rho V D}{\eta}\right).$$

(b) [3 points] In this case ϕ cannot possibly depend on the Mach number $\frac{V}{c_s}$, and so

$$F_D = \rho V^2 D^2 \phi\left(\frac{D\omega}{V}, \frac{\rho V D}{\eta}\right)$$

With this assumption, if we replace V by $V' = 2V$ then to keep the dimensionless constants we need

$$\frac{\rho V' D'}{\eta} = \frac{\rho V D}{\eta} \Rightarrow D' = \frac{D}{2}$$

$$\frac{D'\omega'}{V'} = \frac{D\omega}{V} \Rightarrow \omega' = 4\omega.$$

In this case actually, $F'_D = F_D$, i.e. the drag force is unchanged.