Due: **Monday, February 28 by 5pm**
Reading: 8.512 classnotes Superfluidity and Superconductivity; Girvin and Yang, Chap 19 "Superconductivity: basic phenomena and phenomenological theories"
Michael Cross' Lectures 6 and 14 on the BKT transition and Ginzburg-Landau theory.
Tinkham Chaps 1, 2 and 1st section of Chap 3; (note active links)

Vortices in superfluids

1. Vortex energy and stability.

Weakly interacting Bose gas is described by the Hamiltonian

$$H = \int d^3x \left[\psi^\dagger \left(-\frac{\nabla^2}{2m} \right) \psi - \mu \psi^\dagger \psi + \frac{g}{2} \left(\psi^\dagger \psi \right)^2 \right], \tag{1}$$

where $g > 0$ is the coupling strength, and μ is the chemical potential. Replacing the quantum fields ψ, $\psi^\dagger$ by classical fields, $\psi = \sqrt{n} e^{i\theta}$, $\psi^* = \sqrt{n} e^{-i\theta}$, we can write the system energy as $E = \int d^3x \left[\frac{1}{2} mn\mathbf{v}_s^2 + \frac{\hbar^2}{8mn} (\nabla n)^2 - \mu n + \frac{g}{2} n^2 \right]$, where $\mathbf{v}_s = \frac{\hbar}{m} \nabla\theta$. Analyze the long-wavelength limit of this expression (i.e. the behavior at distances much larger than the healing length $\xi = \hbar/(mgn)^{1/2}$). Argue that away from zeros of ψ, at leading order in gradients of θ and n, the energy takes the form

$$E = \int d^3x \left[\frac{m}{2} n\mathbf{v}_s^2(x) + \frac{g}{2} \left(n(x) - n_0 \right)^2 \right] \tag{2}$$

where $n_0 = \mu/g$ is the equilibrium superfluid density. Setting to zero $\delta E/\delta\theta$ gives the incompressible flow condition $\nabla \cdot \mathbf{v}_s = 0$, which is compatible with the constant density condition enforced by $\delta E/\delta n = 0$ (when taken at leading order in the gradients of θ and n). Since $\mathbf{v}_s = \frac{\hbar}{m} \nabla\theta$, the flow has zero vorticity away from singularities (vortices), $\nabla \times \mathbf{v}_s = 0$, and quantized circulation $\oint \mathbf{v}_s \cdot d\mathbf{x} = 2\pi p \frac{\hbar}{m}$ due to the vortices (here p is an integer).

(a) [10 pts] Consider a single-quantum vortex in a cylinder of radius R and height d, positioned at the cylinder axis. Find the vortex energy $E_v = \int d^3x \frac{1}{2} mn\mathbf{v}_s^2(x)$. Assume that R and d are much larger than the healing length ξ which sets the diameter of vortex core.

(b) [10 pts] Consider a bucket of helium having radius R and height d, rotating at an angular frequency ω. Comparing the energies for the system with and without a vortex, find the range of rotation frequencies for which the vortex-free state is stable and determine the lowest value of ω at which the first vortex enters the rotating bucket. Take into account that the energy in a rotating frame is given by $E'_v = E_v - M\omega$, where M is the mechanical angular momentum.

2. Berezinskii-Kosterlitz-Thouless transition

(a) [10 pts] In very thin superconducting and superfluid films the superfluid state can be destroyed with increasing temperature by the proliferation of vortices induced by thermal fluctuations. To study this effect, consider the full free energy of a single vortex $F = E - TS$ in a cylindrical sample of radius R and a small height d, defined by adding to the energy of a vortex $E = \int d^2x \frac{1}{2} \rho_s \mathbf{v}_s^2$ derived in class [see Question 1] the contribution due to the vortex configurational entropy.

Calculate the entropy using the formula $S = k_\mathrm{B} \ln \Omega$, where Ω is the number of non-overlapping places to put the vortex core of radius $\sim \xi$ inside the circular cross-section of the cylinder. Compare the entropic contribution to F and the vortex energy E. From this, find the critical temperature T_{BKT} above which the total free energy of the vortex becomes negative at large R. Above critical temperature, $T > T_{\mathrm{BKT}}$, it becomes favorable for vortices to enter the sample, and superfluidity to be destroyed, even if the superfluid is not rotating.

(b) [10 pts] Very thin superfluid helium films, with thickness d in the nanometer range, can be prepared and studied. In such films the transition from a normal fluid to a superfluid occurs at temperatures below the value T_λ in ^{4}He bulk. Experiments show that as the ^{4}He film thickness decreases the transition temperature also decreases, varying linearly with the superfluid density with a universal prefactor, as shown in the figure. Explain these observations. Use no more than 100 words to answer this question.

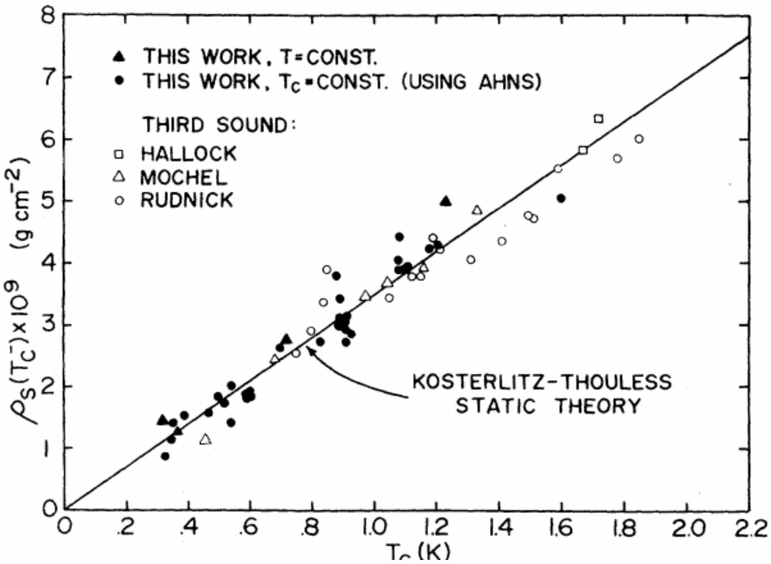

FIGURE 1. The dependence of the transition temperature in ultra-thin films of ^{4}He on the film thickness (from D. J. Bishop and J. D. Reppy, Phys. Rev. Lett., 40, 1727, 1978)

Superconductivity: basic phenomena

3. London equations and magnetic field penetration in a superconductor

a) [10 pts] In physics, phenomenology is the way of reasoning when direct calculation is impossible. Restate the phenomenological reasoning that allowed London brothers to arrive at London equations

$$\frac{\partial \mathbf{j}_s}{\partial t} = \frac{n_s e^2}{m}\mathbf{E}, \quad \nabla \times \mathbf{j}_s = -\frac{n_s e^2}{mc}\mathbf{B} \tag{3}$$

Use no more than 100 words in answering this question.

b) [10 pts] Show that London equations predict that superconductors expel magnetic field (Meissner effect). The mechanism of the Meissner effect is that supercurrents induced in a narrow surface layer exactly cancel the B field within the superconductor material.

Use London equations to derive an equation for B inside the superconductor, $\nabla\times(\nabla\times\mathbf{B}) = \frac{1}{\lambda_L^2}\mathbf{B}$, where $\lambda_L = \left(\frac{mc^2}{4\pi n_s e^2}\right)^{1/2}$ is London penetration length (in a typical superconductor $\lambda_L \lesssim 10$nm). Solve this equation for an infinite superconducting plate of finite thickness $2d$. Assume that a static magnetic field of magnitude B_0 is applied parallel to the plate. Find both the magnetic field and the supercurrent inside the plate. Sketch the magnetic field and supercurrent for $2d = 0.1\lambda_L$, λ_L and $10\lambda_L$.

4. Ginzburg-Landau theory

The Ginzburg-Landau (GL) theory is a symmetry-based approach, formulated in terms of a free energy for a complex valued function $\psi(r)$, the order parameter representing a collective wavefunction of superconducting electrons:

$$F = \int d^3r \left[\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \gamma\left| \left(i\hbar\nabla + \frac{2e}{c}\mathbf{A} \right) \psi \right|^2 + \frac{\mathbf{B}^2}{8\pi} \right], \quad \beta, \gamma > 0 \tag{4}$$

The GL approach is valid for temperatures T near T_c. The quadratic term changes sign at $T = T_c$, such that $\alpha(T) = a(T - T_c)$, $a > 0$.

a) [10 pts] First, analyze *symmetry breaking.* Consider superconductor in the absence of a B field. Use the variational principle $\frac{\delta F}{\delta \psi} = 0$ to show that the order parameter satisfies

$$-\gamma\nabla^2\psi + \alpha\psi + \beta|\psi|^2\psi = 0 \tag{5}$$

Show that ψ vanishes at $T > T_c$, and is described by a particle in a Mexican hat potential at $T < T_c$ such that $\psi = \psi_0 e^{i\theta}$, with the order parameter phase taking any value $0 < \theta < 2\pi$.

b) [10 pts] Next, use GL theory to obtain *London equations.* Write magnetic field as $\mathbf{B} = \nabla \times \mathbf{A}$ and use the variational principle $\frac{\delta F}{\delta \mathbf{A}} = 0$ to derive Ampère's law for supercurrent

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j}_s, \quad \mathbf{j}_s = -\eta(\mathbf{A} - \frac{\hbar c}{2e}\nabla\theta) \tag{6}$$

Find the constant η in terms of the superfluid density $n_s = |\psi|^2$ and show that Eqs.(6) are equivalent to London equations. London penetration length λ_L therefore provides a direct measure of the superfluid density.

c) [10 pts] Use the result of part **b)** to show that magnetic flux trapped in a superconducting current-carrying ring is *quantized.* Consider a superconducting ring which is threaded by a magnetic field and carries a persistent current. Show that the order parameter phase winds by a multiple of 2π. Argue that, so long as the thickness of the ring is much greater than λ_L, the magnetic flux trapped inside the ring is quantized in the units of (superconducting) flux quantum $\Phi_0 = hc/2e$.

[Be careful: while magnetic field is expelled from the superconducting ring bulk, the vector potential is generally nonzero but has zero curl, $\nabla \times \mathbf{A} = 0$.]