8.512 Recitation 7

- Pset 3 due today
- Pset 4 posted, due 04/04
- Spring break, no class next week

Today: 1) Ferromagnetic Heisenberg Model
2) Semi-classical (Bloch) spin wave theory
3) Holstein-Primakoff transformation
4) Magnon dispersion and thermodynamics

Ref: "Theory of Magnetism" Ch. 8
by C. Timm

1) Heisenberg Model

$$H = \underbrace{-\frac{1}{2}}_{\text{double counting}} \sum_{ij} J_{ij} \vec{S}_i \cdot \vec{S}_j = -\frac{1}{2} \sum_{ij} J_{ij} \left(\frac{S_i^+ S_j^- + S_i^- S_j^+}{2} + S_i^z S_j^z \right)$$

where $J_{ij} = J_{ji} > 0$, $S_i^{\pm} = S_i^x \pm i S_i^y$

$[\vec{S}, H] = 0 \Rightarrow$ consider simultaneous eigenstates of H, $\vec{S} \cdot \vec{S}$, S^z

- GS is a ferromagnet w/ all spins aligned and max spin S

$$|\psi_0\rangle = |S, S, \ldots, S\rangle \qquad (S^z \text{ basis})$$
$$E_0 = -\frac{1}{2} \sum_{ij} J_{ij} S^2$$

Spontaneosly breaks rotational symmetry $\langle \vec{S} \rangle \neq 0$

- For simplicity, consider a Bravais lattice $\{\vec{R}_i\}$ w/ J_{ij} only depending on distance $J_{ij} = J(\vec{R}_i - \vec{R}_j) = J(\Delta R)$

- Consider low-energy excitations above GS. Naively, we may choose

$$|\psi_1\rangle = |S-1, S, \ldots, S\rangle$$

But this has large excitation energy

$$\Delta E_1 = -\sum_{j\neq 1} J_{ij}\, S(S-1) + \sum_{j\neq 1} J_{ij}\, S^2 = S \sum_{j\neq 1} J_{ij}$$

2) To find lower-energy excitations, consider spin dynamics

$$\hbar \frac{d\vec{S}_i}{dt} = i[H, \vec{S}_i] = \sum_j J_{ij}\, \vec{S}_i \times \vec{S}_j$$

Further assume small deviations from GS $\Rightarrow S_i^z \simeq S$

$$\begin{cases} \hbar \dfrac{dS_i^x}{dt} = \sum_j J_{ij}(S_i^y S_j^z - S_i^z S_j^y) \simeq S \sum_j J_{ij}(S_i^y - S_j^y) \\ \hbar \dfrac{dS_i^y}{dt} = \sum_j J_{ij}(S_i^z S_j^x - S_i^x S_j^z) \simeq S \sum_j J_{ij}(S_j^x - S_i^x) \\ \hbar \dfrac{dS_i^z}{dt} = \sum_j J_{ij}(S_i^x S_j^y - S_i^y S_j^x) \simeq 0 \end{cases}$$

$$\Rightarrow \hbar \frac{dS_i^\pm}{dt} = \pm i S \sum_j J_{ij}(S_j^\pm - S_i^\pm) \qquad \text{(decoupled ODEs)}$$

- Fourier transform $\quad S_i^- = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{R}_i} S_{\vec{k}}^-$

$$\hbar \frac{dS_{\vec{k}}^-}{dt} = iS \frac{1}{N} \sum_{ij} e^{-i\vec{k}\cdot\vec{R}_i} J_{ij} \sum_{k'} \left(e^{i\vec{k}'\cdot\vec{R}_i} - e^{i\vec{k}'\cdot\vec{R}_j}\right) S_{k'}^-$$

$$= iS \frac{1}{N} \sum_{i,\Delta R} J_{\Delta R} \sum_{k'} e^{-i(\vec{k}-\vec{k}')\cdot\vec{R}_i} \left(1 - e^{-i\vec{k}'\cdot\Delta\vec{R}}\right) S_{k'}^-$$

$$= iS \sum_{\Delta R} J_{\Delta R} \left(1 - e^{-i\vec{k}\cdot\Delta\vec{R}}\right) S_k^- = iS\left(J(0) - J(\vec{k})\right) S_k^-$$

$$= iS \sum_{\Delta R} J_{\Delta R} \left(1 - e \quad\quad \right) S_K = iS\left(J(0) - J(K)\right) S_K$$

where $\Delta\vec{R} = \vec{R}_i - \vec{R}_j$, $J_{\Delta R} = J_{ij}$, $J(\vec{K}) = \sum_{\Delta R} J_{\Delta R}\, e^{-i\vec{K}\cdot\Delta\vec{R}}$

$\Rightarrow$ exp. solutions: $S_K^-(t) = M_K\, e^{i\omega_K t}$

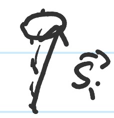
$$S_i^-(t) = \frac{1}{\sqrt{N}} \sum_K M_K\, e^{i(\vec{K}\cdot\vec{R}_i + \omega_K t)}$$

$$\hbar\omega_K = \left(J(0) - J(\vec{K})\right) S$$

in original variables:
$$\begin{cases} S_i^x(t) = \frac{1}{\sqrt{N}} \sum_K M_K \cos(\vec{K}\cdot\vec{R}_i + \omega_K t) \\ S_i^y(t) = \frac{1}{\sqrt{N}} \sum_K M_K \sin(\vec{K}\cdot\vec{R}_i + \omega_K t) \\ S_i^z(t) = S \end{cases}$$

plane waves!

- Classically $\vec{S}_i(t)$ precesses on a cone around $\hat{z}$



Ex: 1D chain w/ N.N. interactions J, lattice spacing a

$$J(K) = 2J\cos(Ka)$$
$$\hbar\omega_K = 2JS\left(1 - \cos(Ka)\right) \approx JS(Ka)^2$$
at $Ka \ll 1$

$\boxed{\omega_K \sim K^2}$ is generic for FM excitations

Similarly

$\boxed{\omega_K \sim K}$ is generic for AFM excitations

- Note that Heisenberg eq. after Fourier trans. becomes

$$\hbar\omega_K S_K^- = [H, S_K^-]$$

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$\Rightarrow H\,(S_k^-|\psi_0\rangle) = (\hbar\omega_k S_k^- + S_k^- H)|\psi_0\rangle = (\hbar\omega_k + E_0)(S_k^-|\psi_0\rangle)$

$\Rightarrow S_k^-|\psi_0\rangle$ is an eigenstate of H w/ $E = E_0 + \hbar\omega_k$

S_k^- creates one spin wave "quantum", called magnon. with energy $\hbar\omega_k$

spin wave $\rightarrow$ magnon
E&M wave $\rightarrow$ photon

- The $K=0$ magnon costs no energy and corresponds to a Goldstone mode!
- This semi-classical approx. assumes non-interacting magnons.

3) Holstein–Primakoff transformation:
 a) introduce a bosonic mode a_i for every spin
 b) constrain $n_i = a_i^\dagger a_i \leq 2S$

$$\begin{cases} S_i^- = a_i^\dagger \sqrt{2S - n_i} \\ S_i^+ = \sqrt{2S - n_i}\, a_i \\ S_i^z = S - n_i \end{cases}$$

Canonical commutation relations preserved ✓
norm preserved $\vec{S}_i \cdot \vec{S}_i = S(S+1)$ ✓

H can now be expanded as $H = O(S^2) + O(S^1) + O(S^0) + \ldots$

- At low temperatures $k_B T \ll J$, the # of excitations above GS is small $\langle n_i \rangle \ll S$. This also holds in the classical limit $S \rightarrow \infty$. To leading order $O(S^1)$

$$\begin{cases} S_i^- = a_i^\dagger \sqrt{2S} \\ S_i^+ = a_i \sqrt{2S} \\ S_i^z = S - a_i^\dagger a_i \end{cases}$$

$$H = \underset{O(S^2)}{E_0} + S \sum_{ij} J_{ij} \left(a_i^\dagger a_i + a_j^\dagger a_j - a_i^\dagger a_j - a_j^\dagger a_i \right) + O(S^0)$$

H can be diagonalized by Fourier transform

$$a_i = \frac{1}{\sqrt{N}} \sum_k e^{i\vec{k}\cdot\vec{R}} a_k$$

$$\Rightarrow H = E_0 + S \sum_k \left(J(0) - J(\vec{k}) \right) a_k^\dagger a_k + O(S^0)$$

Same dispersion relation as before! In 1D

$$H = E_0 + \sum_k \hbar\omega_k a_k^\dagger a_k + \ldots$$

$$\boxed{\hbar\omega_k = 2JS(1 - \cos(ka))}$$

- This makes it more clear how magnon excitations emerge
- The next $O(S^0)$ term gives magnon-magnon interactions

$$\hbar\omega_k \approx JS(ka)^2 \left[1 - \frac{1}{S} \cdot \frac{\zeta(\frac{5}{2})}{32} \cdot \frac{1}{\pi^{3/2} (\beta J S)^{5/2}} \right]$$

4) Magnons are bosons with $\langle n_k \rangle = \dfrac{1}{e^{\beta\hbar\omega_k} - 1}$

$$\langle n \rangle = \sum_k \langle n_k \rangle = V \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta\hbar\omega_k} - 1}$$

Each magnon reduces total spin by 1:

$$\underset{\substack{\uparrow \\ \text{magnetization}}}{M} = \frac{g\mu_B (NS - \langle n \rangle)}{V} = \underbrace{\frac{g\mu_B NS}{V}}_{M_{sat}\ (\text{max magnetisation})} - g\mu_B \int \frac{d^d k}{(2\pi)^d} \frac{1}{e^{\beta\hbar\omega_k} - 1}$$

• In $d=3$ integral converges

$$M = M_{sat} - g\mu_B \cdot \frac{\zeta(3/2)}{8} \cdot \frac{1}{(\pi \beta J S a^2)^{3/2}}$$

Magnetization reduction $\sim T^{3/2}$ (Bloch's $T^{3/2}$ law)

Similarly $g(E) \sim \sqrt{E}$ and $C \sim T^{3/2}$

• In $d=1,2$ the integral diverges! Infinitely many magnons will be excited and magnetic order will be destroyed in $d \leq 2$ at $T > 0$.

(Mermin – Wagner theorem)

• For AFM in $d=3$:

$$\omega_k \sim k$$
$$g(E) \sim E^2$$
$$C \sim T^3$$