ULTRACOLD FERMI GAS

- A Quick Guide -

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October 19, 2022

Preface

Greetings,

This is my personal “cheatsheet” for BEC1 experiment with strongly interacting Fermi gas (ironic, I know, but there’s a lot of history behind this irony). All information relevant to what I do in the lab plus what I find interesting is accumulated here.

Enjoy!

Contents

Chapter 1

Ideal Fermi Gas

1.1 Thermodynamics of a non-relativistic gas

For a non-relativistic gas of particles with spin s in three dimensions with $\mathcal{E}(\vec{k}) = \hbar^2 k^2/2m$, we have

$$\begin{aligned} \beta P_\eta &= \frac{g}{\lambda^3} f^\eta_{5/2}(z) \\ n_\eta &= \frac{g}{\lambda^3} f^\eta_{3/2}(z) \\ \epsilon_\eta &= \frac{3}{2} P_\eta \end{aligned}$$

with $g = 2s+1$ is the spin degeneracy factor, z is the fugacity parameter $e^{\beta\mu}$, and $\eta = -1$ for fermions and $\eta = 1$ for bosons, and

$$f^\eta_m(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{dx\, x^{m-1}}{z^{-1} e^x - \eta}.$$

The three equations above completely describe the thermodynamics of ideal quantum gases as a function of z. To rewrite these equations in terms of the density n_η, we have to express $f^\eta_m(z)$ in terms of n_η. To do this, we have to expand $f^\eta_m(z)$ in terms of z in the high temperature limit. The details of this procedure can be found in Chapter 7 of [2].

One of the main corollaries of the expansion is that the natural dimensionless expansion parameter

$$\frac{n_\eta \lambda^3}{g}$$

appears. Here, $\lambda = \sqrt{h^2/2\pi m k_B T}$ is the de Broglie wavelength. For example,

$$P_\eta = n_\eta k_B T \left[1 - \frac{\eta}{2^{5/2}} \left(\frac{n_\eta \lambda^3}{g} \right) + \dots \right]$$

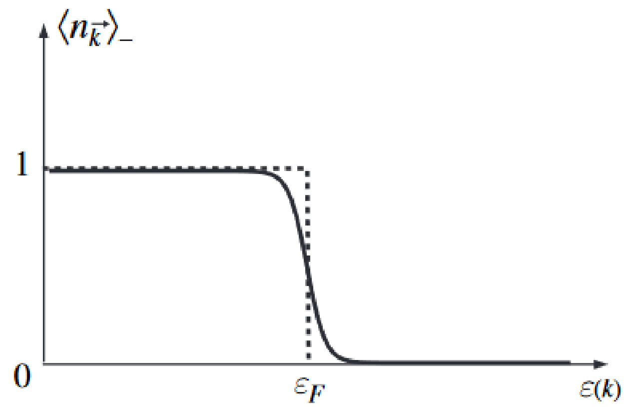
We see that quantum mechanical effects become important in the regime where $n_\eta \lambda^3 \geq g \sim 1$. This is called the **quantum degenerate limit**. Fermi and Boses gases behave very differently in this limit of high density and low temperature. In this document, we will focus mostly on fermions.

1.2 Thermodynamics of an ideal fermi gas

In the quantum degeneracy limit, fermions obey Fermi-Dirac statistics. The average occupation number as a function of momentum (or energy) is

$$\langle n(\vec{k}) \rangle = \frac{1}{e^{(\mathcal{E}(\vec{k}) - \mu)/k_B T} + 1} \xrightarrow{T \to 0} \begin{cases} 1 & \text{if } \mathcal{E} < \mu \\ 0 & \text{if } \mathcal{E} \geq \mu \end{cases}$$

Here μ is the chemical potential and $\mathcal{E}(\vec{k})$ is the energy. We see that μ is a limiting value for $\mathcal{E}(\vec{k})$ (at zero temperature), and call it the **Fermi energy** $\mathcal{E}_F$. At $T \to 0$, all one-particle states of energy less than $\mathcal{E}_F$ are occupied, forming a **Fermi sea**.



For an ideal Fermi gas,

$$\mathcal{E}(\vec{k}) = \frac{\hbar^2 k^2}{2m} \implies \mathcal{E}_F := \frac{\hbar^2 k_F^2}{2m}$$

where k_F is the **Fermi wavenumber**. The Fermi wavenumber k_F can be computed from the number density as:

$$n = \int_{k<k_F} g \times \frac{d^3\vec{k}}{(2\pi)^3} = g \int_{k<k_F} 4\pi k^2 \frac{dk}{(2\pi)^3} = \frac{g}{6\pi^2} k_F^3.$$

So,

$$\boxed{k_F = \left(\frac{6\pi^2 n}{g}\right)^{1/3} \implies \mathcal{E}_F = \frac{\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3}}$$

To extract thermodynamics properties of the degenerate Fermi gas in three dimensions (untrapped), we consider the high-z limit of $f_m^-(z)$. Again, Chapter 7 of [2] shows the details of the various expansions. The result is the **Sommerfeld expansion**:

$$\lim_{z\to\infty} f_m^-(z) = \frac{(\ln z)^m}{m!}\left[1 + \frac{\pi^2}{6}\frac{m(m-1)}{(\ln z)^2} + \frac{7\pi^4}{360}\frac{m(m-1)(m-2)(m-3)}{(\ln z)^4} + \dots\right]$$

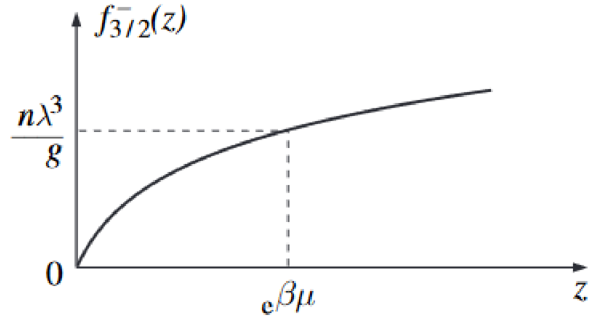
From the n_η equation in the previous section, we find that in the degenerate limit, the density and chemical potential are related by

$$\frac{n\lambda^3}{g} = f_{3/2}^-(z) = \frac{(\ln z)^{3/2}}{(3/2)!}\left[1 + \frac{\pi^2}{6}\frac{3}{2}\frac{1}{2}(\ln z)^{-2} + \dots\right] \gg 1$$

To lowest order, we have

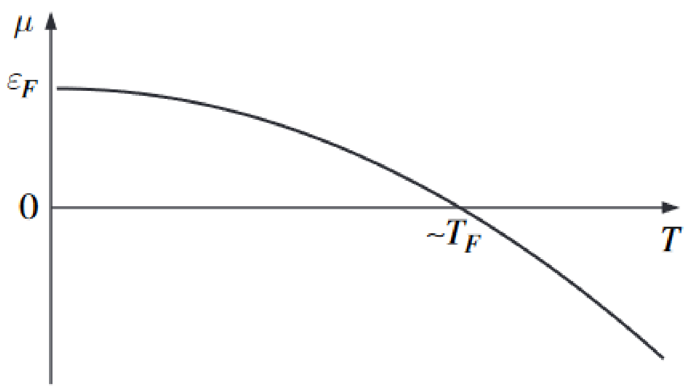
$$\lim_{T\to 0} \ln z = \left(\frac{3}{4\sqrt{\pi}}\frac{n\lambda^3}{g}\right)^{2/3} = \frac{\beta\hbar^2}{2m}\left(\frac{6\pi^2 n}{g}\right)^{2/3} = \beta\mathcal{E}_F \implies \lim_{T\to 0}\mu = \mathcal{E}_F$$

which is what we found earlier by looking at the mean occupation number for fermions.



At finite but small temperatures, we have corrections:

$$\ln z \approx \beta \mathcal{E}_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right]$$



We see that the chemical potential $\mu = k_B T \ln z$ is positive at low temperature and negative at high temperature. It changes sign at $T \sim T_F = \mathcal{E}_F / k_B$.

Using a similar approach, one finds the low-temperature expansion for pressure:

$$\beta P = \beta P_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right], \qquad \boxed{P_F = \frac{2}{5} n \mathcal{E}_F}$$

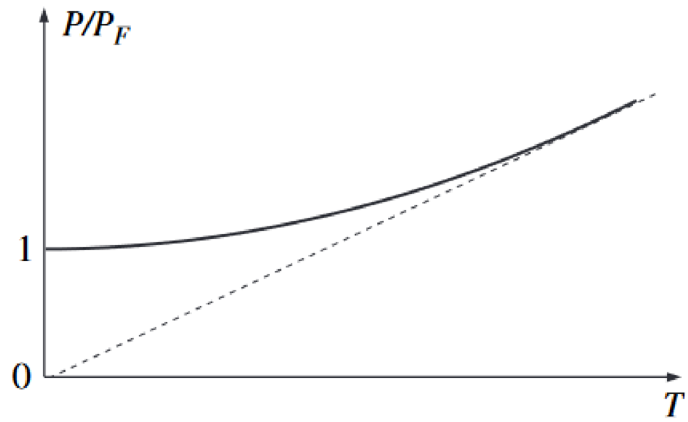
Here P_F is the **fermi pressure**. Note that the fermi gas at zero temperature has nonzero pressure and internal energy, unlike its classical counterpart.

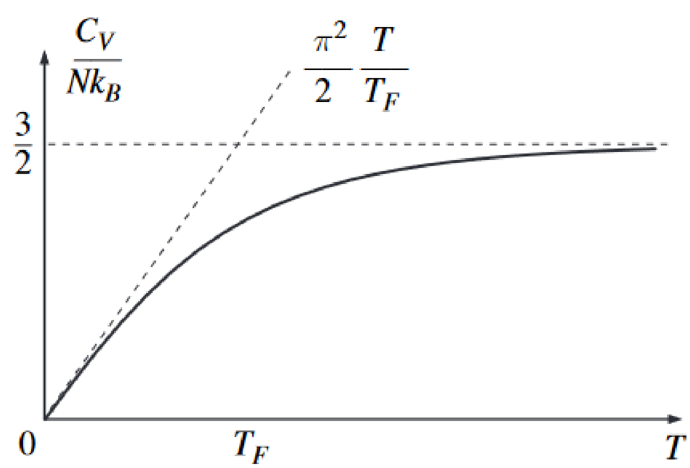
The internal energy (density) is obtained from the third equation in the previous section:

$$\boxed{\epsilon = \frac{E}{V} = \frac{3}{2} P = \frac{3}{5} n k_B T_F \left[1 + \frac{5}{12} \pi^2 \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right]}$$

From here, one finds the low-temperature heat capacity:

$$C_V = \frac{dE}{dT} = \frac{\pi^2}{2} N k_B \left(\frac{T}{T_F} \right) + \mathcal{O} \left(\frac{T}{T_F} \right)^3$$





Note: the linear scaling of C_V as $T \to 0$ is a general feature of a fermi gas and is valid in all dimensions. The physical interpretation is as follows: At small temperatures, all single-particle states are occupied. As a result, only particles within a distance of about $k_B T$ of the fermi energy can be thermally excited. This represents only a small fraction (T/T_F) of the fermions. Each excited particle gains $k_B T$, leading to a change in E of $k_B T N(T/T_F)$. So $C_V \propto dE/dT \sim N k_B T/T_F$ which is linear in T. This conclusion is also **valid for interacting fermi gases**.

1.3 Ideal Fermi gas in a harmonic trap

See next chapter. This section is more naturally treated in conjunction with a number of related topics.

Chapter 2

Interacting Fermi Gas

2.1 Hyperfine Structure

2.2 Collisional Properties

2.3 Cooling and Trapping techniques

2.4 RF Spectroscopy

2.5 Feshbach resonance

This section is adapted from [1] and [3].

2.5.1 Scattering theory

2.5.2 Application: Spherical Well Model

2.5.3 Application: Coupled Square Well Model

2.5.4 An intuitive picture

2.6 Analysis of density distributions

Chapter 3

Strongly Interacting Fermi Gas

Bibliography

[1] S. L. Campbell. Feshbach resonances in ultracold gases.

[2] M. Kardar. *Statistical physics of particles*. Cambridge University Press, 2007.

[3] W. Ketterle and M. W. Zwierlein. Making, probing and understanding ultracold fermi gases. *La Rivista del Nuovo Cimento*, 31(5):247–422, 2008.