Name: **Huan Q. Bui**
Course: **8.422 - AMO II**
Problem set: **#4**
Due: Friday, Mar 10, 2022
References:

1. Squeezing Hamiltonian.

The dimensionless squeezing Hamiltonian is

$$\mathcal{H} = \frac{\hbar\omega}{2}(\tilde{p}^2 - \tilde{x}^2).$$

With $p = (\tilde{p} - \tilde{x})/\sqrt{2}$ and $x = (\tilde{p} + \tilde{x})/\sqrt{2}$, we have $[x, p] = \tilde{x}, \tilde{p} = i$ and the Hamiltonian becomes

$$\mathcal{H} = \hbar\omega x p$$

plus an offset which we ignore.

(a) Let $\psi_0(x)$ be given. In the spirit on the question, we will simply plug $\psi_0(x(t))$ into the Schrödinger equation as a test solution and see what happens:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\psi_0(x(t)) &= \hbar\omega x p \psi_0(x(t)) \\ i\psi_0'(x(t))x'(t) &= -i\omega x(t) t \psi_0'(x(t)) \\ x'(t) &= -\omega x(t), \end{aligned}$$

where we have used $p = -i\partial_x$. We see that we could arrive this same differential equation for $x(t)$ irrespective of the functional form of $\psi_0(x)$, so we may conclude that $\psi(x,t) = \psi_0(x(t))$. The equation for $x(t)$ is:

$$x(t) = x_0 e^{-\omega t}.$$

(b) Let $\tilde{\psi}_0(p)$ be given. Before calculating more, let us modify our Hamiltonian so that the operators are in a "good" order. By adding a constant term $-i\hbar\omega$ to H (just like what we did in the beginning) to get

$$H \longrightarrow \hbar\omega x p - i\hbar\omega = \hbar\omega p x,$$

without changing the dynamics. With $x = i\partial/\partial p$ in momentum space we calculate, similar to Part (a):

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\tilde{\psi}_0(p(t)) &= i\hbar\omega p(t)\frac{\partial}{\partial p}\tilde{\psi}_0(p(t)) \\ \tilde{\psi}_0'(p(t))p'(t) &= \omega p(t)\tilde{\psi}_0'(p(t)) \\ p'(t) &= \omega p(t). \end{aligned}$$

From here, we find $p(t)$:

$$p(t) = p_0 e^{\omega t}.$$

(c) Suppose $\psi_0(x) = e^{-x^2/2}/\pi^{1/4}$, which is the wavefunction of the vacuum in the x-representation. This wavefunction has width $1/\sqrt{2}$. At a time t, space is rescaled by a factor $e^{-\omega t}$ and acquires an extra factor of $e^{-\omega t/2}$ in order to maintain normalization. In any case, because $x \longrightarrow x e^{-\omega t}$, the new wavefunction is

$$\psi_0(x) = \frac{e^{-\omega t/2}}{\pi^{1/4}}\exp\left[-\frac{1}{2}x^2 e^{-2\omega t}\right].$$

We can just read off from here that the new width Δx becomes $e^{\omega t}$ times the original width of $1/\sqrt{2}$.

(d) By a similar argument to that in Part (c), the new wavefunction in momentum space acquires an extra factor for $e^{\omega t/2}$ to maintain normalization. Because $p \to pe^{\omega t}$, we see that the new width Δp is now $e^{-\omega t}$ times the original width of $1/\sqrt{2}$.

(e) From Parts (c) and (d), we see quite easily that $\Delta x \Delta p = e^{\omega t - \omega t}/2 = 1/2$, which is the same as the uncertain at $t = 0$, as expected.

(f) Here we rewrite the Hamiltonian in proper symmetrized form:

$$H = \frac{\hbar\omega}{2}(xp + px).$$

With $a = (x + ip)/\sqrt{2}$ and $a^\dagger = (x - ip)/\sqrt{2}$, we find

$$\begin{aligned} H &= \frac{\hbar\omega}{2}\left[\frac{a + a^\dagger}{\sqrt{2}}\frac{a - a^\dagger}{\sqrt{2}i} + \frac{a - a^\dagger}{\sqrt{2}i}\frac{a + a^\dagger}{\sqrt{2}}\right] \\ &= \frac{\hbar\omega}{4i}(a^2 - aa^\dagger + a^\dagger a - a^{\dagger 2} + a^2 + aa^\dagger - a^\dagger a - a^{\dagger 2}) \\ &= \frac{\hbar\omega}{2i}(a^2 - a^{\dagger 2}). \end{aligned}$$

We find the familiar Hamiltonian that appeared in lecture.

2. Disentangling the Squeezing Operator.

In this problem we show that

$$e^{\frac{r}{2}(a^{\dagger 2} - a^2)} |0\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh r)^n |2n\rangle .$$

To do this, we follow the parts below.

(a) We first calculate the following commutators:

$$\begin{aligned} [a^2, a^{\dagger 2}] &= aaa^\dagger a^\dagger - a^\dagger a^\dagger aa \\ &= a(1 + a^\dagger a)a^\dagger - a^\dagger(aa^\dagger - 1)a \\ &= 1 + a^\dagger a + (1 + a^\dagger a)(1 + a^\dagger a) - a^\dagger aa^\dagger a + a^\dagger a \\ &= 4a^\dagger a + 2. \end{aligned}$$

$$\begin{aligned} [a^2, a^\dagger a] &= aaa^\dagger a - a^\dagger aaa \\ &= a(1 + a^\dagger a)a - (aa^\dagger - 1)aa \\ &= 2a^2. \end{aligned}$$

$$\begin{aligned} [a^{\dagger 2}, a^\dagger a] &= a^\dagger a^\dagger a^\dagger a - a^\dagger aa^\dagger a^\dagger \\ &= a^\dagger a^\dagger (aa^\dagger - 1) - a^\dagger(1 + a^\dagger a)a^\dagger \\ &= -2a^{\dagger 2}. \end{aligned}$$

From these, we conclude that the Lie algebra of operators $\{a^2, a^{\dagger 2}, a^\dagger a + 1/2\}$ is closed under commutation. This means we must be able to write

$$e^{\frac{r}{2}(a^{\dagger 2} - a^2)} = e^{\frac{u}{2}a^{\dagger 2}} e^{t(^\dagger a + 1/2)} e^{\frac{v}{2}a^2}.$$

Our job now is to find the numbers u, t, v, which are functions of r. To do this, we find any other Lie algebra whose three operators obey the same commutation relations which allows us to more easily find u, t, v. It turns out that Pauli matrices work.

(b) Consider the replacement:

$$a^\dagger a + \frac{1}{2} \longrightarrow \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$a^2 \longrightarrow -\sigma_- = -\sigma_x + i\sigma_y = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}$$
$$a^{\dagger 2} \longrightarrow \sigma_+ = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

Let us check that the commutation relations above still hold.

$$[a^2, a^{\dagger 2}] \longrightarrow [-\sigma_-, \sigma_+] = [-\sigma_x, i\sigma_y] + [i\sigma_y, \sigma_x] = 4\sigma_z \longleftarrow 4a^\dagger a + \frac{1}{2} \quad \checkmark$$

$$[a^2, a^\dagger a] \longrightarrow [-\sigma_-, \sigma_z - 1/2] = [-\sigma_-, \sigma_z] = 2i\sigma_y - 2\sigma_x = 2(-\sigma_-) \longleftarrow 2a^2 \quad \checkmark$$

$$[a^{\dagger 2}, a^\dagger a] = [\sigma_+, \sigma_z - 1/2] = [\sigma_+, \sigma_z] = -2i\sigma_y - 2\sigma_x = -2\sigma_+ \longleftarrow -2a^{\dagger 2} \quad \checkmark$$

(c) With

$$\frac{r}{2}\left(a^{\dagger 2} - a^2\right) = \frac{r}{2}(\sigma_+ + \sigma_-) = r\sigma_x = \begin{pmatrix} 0 & r \\ r & 0 \end{pmatrix},$$

we find

$$e^{\frac{r}{2}\left(a^{\dagger 2} - a^2\right)} = \begin{pmatrix} \cosh r & \sinh r \\ \sinh r & \cosh r \end{pmatrix}.$$

Mathematica code:

```
In[7]:= MatrixExp[r*PauliMatrix[1]] // FullSimplify
Out[7]= {{Cosh[r], Sinh[r]}, {Sinh[r], Cosh[r]}}
```

(d) Similarly, we have

$$U = e^{\frac{u}{2}a^{\dagger 2}} = \exp\left[\frac{u}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\right] = \exp\left[\begin{pmatrix} 0 & u \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

$$T = e^{t\left(a^\dagger a + 1/2\right)} = e^{t\sigma_z} = \exp\left[\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix}\right] = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$V = e^{\frac{v}{2}a^2} = \exp\left[\frac{v}{2}\begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix}\right] = \exp\left[\begin{pmatrix} 0 & 0 \\ -v & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}.$$

With these,

$$UTV = \begin{pmatrix} e^t - e^{-t}uv & e^{-t}u \\ -e^{-t}v & e^{-t} \end{pmatrix}.$$

Mathematica code:

```
In[14]:= U = MatrixExp[{{0, u}, {0, 0}}]
Out[14]= {{1, u}, {0, 1}}

In[15]:= T = MatrixExp[t*PauliMatrix[3]]
Out[15]= {{E^t, 0}, {0, E^-t}}

In[16]:= V = MatrixExp[{{0, 0}, {-v, 0}}]
Out[16]= {{1, 0}, {-v, 1}}

In[18]:= U . T . V // FullSimplify
Out[18]= {{E^t - E^-t u v, E^-t u}, {-E^-t v, E^-t}}
```

(e) Comparing the results of Parts (c) and (d) we easily find that

$$t = -\ln\cosh r, \qquad u = -v = e^t \sinh r = \tanh r.$$

so we have

$$e^{\frac{r}{2}\left(a^{\dagger 2}-a^2\right)} = e^{\frac{\tanh r}{2}a^{\dagger 2}} e^{-\ln\cosh r\left(a^\dagger a+1/2\right)} e^{-\frac{\tanh r}{2}a^2} = \frac{1}{\sqrt{\cosh r}} e^{\frac{\tanh r}{2}a^{\dagger 2}} e^{-\ln\cosh r(a^\dagger a)} e^{-\frac{\tanh r}{2}a^2}$$

(f) Applying the operator above to $|0\rangle$, we realize that the two right most operators act on $|0\rangle$ as the identity, so we end up with

$$\begin{aligned} e^{\frac{r}{2}\left(a^{\dagger 2}-a^2\right)} |0\rangle &= \frac{1}{\sqrt{\cosh r}} e^{\frac{\tanh r}{2}a^{\dagger 2}} |0\rangle \\ &= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{2^n n!} a^{\dagger 2n} |0\rangle \\ &= \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh r)^n |2n\rangle , \end{aligned}$$

as desired.

3. Generation of Squeezed States by Two-Photon Interactions.

Consider a mode $(\vec{k}, \vec{\epsilon})$ with wavevector $\vec{k}$ and polarization $\vec{\epsilon}$ of the EM field with frequency ω whose Hamiltonian is given by

$$H = \hbar\omega a^\dagger a + i\hbar\Lambda \left[(a^\dagger)^2 e^{-2i\omega t} - a^2 e^{2i\omega t}\right].$$

The first term is the energy of the mode of the free field. The second term describes a two-photon interaction process such as parametric amplification (a classical wave of frequency 2ω generating two photons with frequency ω). Λ is a real quantity characterizing the strength of the interaction. In this problem, we will show that this Hamiltonian produces squeezed vacuum and explore how it acts on coherent states.

(a) The equation of motion for $a(t)$ in the Heisenberg picture is

$$\frac{d}{dt} a_H = \frac{i}{\hbar}[H_H, a_H],$$

where the Hamiltonian in the Heisenberg picture is simply the Schrödinger equation but in terms of a_H and $a_H^\dagger$:

$$H_H = \hbar\omega a_H^\dagger a_H + i\hbar\Lambda \left[(a_H^\dagger)^2 e^{-2i\omega t} - a_H^2 e^{2i\omega t}\right].$$

Now we compute the commutators:

$$[a^\dagger a, a] = -[a, a^\dagger a] = -[a, a^\dagger]a - a^\dagger[a, a] = -a.$$

$$[(a^\dagger)^2, a] = -[a, (a^\dagger)^2] = -[a, a^\dagger]a^\dagger - a^\dagger[a, a^\dagger] = -2a^\dagger$$

$$[a^2, a] = 0$$

From these, we find the equations of motion for a and $a^\dagger$

$$\begin{aligned}\dot{a} &= -i\omega a + 2\Lambda a^\dagger e^{-2i\omega t}\\ \dot{a}^\dagger &= i\omega a^\dagger + 2\Lambda a e^{2i\omega t}\end{aligned}$$

In matrix form:

$$\begin{pmatrix}\dot{a}\\ \dot{a}^\dagger\end{pmatrix} = \begin{pmatrix}-i\omega & 2\Lambda e^{-2\omega t}\\ 2\Lambda e^{2i\omega t} & i\omega\end{pmatrix}\begin{pmatrix}a\\ a^\dagger\end{pmatrix}.$$

To solve this, we consider the ansatz $a(t) = b(t)e^{-i\omega t}$. After we plug in this substitution, the system of equations above becomes:

$$\dot{b}(t)e^{-i\omega t} = 2\Lambda b^\dagger(t)e^{-2i\omega t + i\omega t} \implies \dot{b}(t) = 2\Lambda b^\dagger(t) \implies \dot{b}^\dagger(t) = 2\Lambda b(t).$$

From here, we find the equations of motion for $b(t)$ and $b^\dagger(t)$:

$$\ddot{b}(t) = 4\Lambda^2 b(t) \qquad \text{and} \qquad \ddot{b}^\dagger(t) = 4\Lambda^2 b^\dagger(t).$$

(b) The contribution of the mode $(\vec{k}, \vec{\epsilon})$ to the electric field is

$$\vec{E}(\vec{r}, t) = i\mathcal{E}_\omega \vec{\epsilon}\left[a(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger(t)e^{-i\vec{k}\cdot\vec{r}}\right]$$

where $a(t) = b(t)e^{-i\omega t}$. Consider the quantities

$$b_P(t) = \frac{b(t) + b^\dagger(t)}{2} \qquad \text{and} \qquad b_Q(t) = \frac{b(t) - b^\dagger(t)}{2i}.$$

We now show that they represent physically two quadrature components of the field. To this end, we simply plug in expression for $a(t)$ in terms of $b(t)$ and expand the electric field in terms of trigonometric functions:

$$\begin{aligned}\vec{E}(\vec{r}, t) &= i\mathcal{E}_\omega \vec{\epsilon}\left[b(t)e^{i(\vec{k}\cdot\vec{r} - \omega t)} - b^\dagger(t)e^{-i(\vec{k}\cdot\vec{r} - \omega t)}\right]\\ &= i\mathcal{E}_\omega \vec{\epsilon}\left[[b(t) - b^\dagger(t)]\cos\left(\vec{k}\cdot\vec{r} - \omega t\right) + i[b(t) + b^\dagger(t)]\sin\left(\vec{k}\cdot\vec{r} - \omega t\right)\right]\\ &= -2\mathcal{E}_\omega \vec{\epsilon}\left[b_Q(t)\cos\left(\vec{k}\cdot\vec{r} - \omega t\right) + b_P(t)\sin\left(\vec{k}\cdot\vec{r} - \omega t\right)\right].\end{aligned}$$

So we see that $b_P(t)$ and $b_Q(t)$ represent the two quadrature components of the electric field, as desired. Since $b_P(t)$ and $b_Q(t)$ are linear superpositions of $b(t)$ and $b^\dagger(t)$, $b_Q(t)$ and $b_P(t)$ also solve the similar differential equation as $b(t)$ and $b^\dagger(t)$:

$$\ddot{b}_P(t) = 4\Lambda^2 b_P(t) \qquad \text{and} \qquad \ddot{b}_Q(t) = 4\Lambda^2 b_Q(t),$$

where $b_P(t)$ and $b_Q(t)$ are uncoupled, unlike $b(t)$ and $b^\dagger(t)$. Let us now solve for $b_P(t), b_Q(t), b(t), b^\dagger(t)$ in terms of $b(0)$ and $b^\dagger(0)$. From its differential equation we find the general form for $b(t)$ and $b^\dagger(t)$:

$$\begin{aligned}b(t) &= c_1 e^{2\Lambda t} + c_2 e^{-2\Lambda t} \implies b(0) = c_1 + c_2\\ b^\dagger(t) &= c_1 e^{2\Lambda t} - c_2 e^{-2\Lambda t} \implies b^\dagger(0) = c_1 - c_2.\end{aligned}$$

From $b_P(0)$ and $b_Q(0)$ we can solve for c_1, c_2:

$$c_1 = b_P(0) \qquad \text{and} \qquad c_2 = ib_Q(0).$$

And we're done:

$$\begin{aligned}
b(t) &= b_P(0)e^{2\Lambda t} + ib_Q(0)e^{-2\Lambda t} = \frac{b(0)+b^\dagger(0)}{2}e^{2\Lambda t} + \frac{b(0)-b^\dagger(0)}{2}e^{-2\Lambda t} \\
b^\dagger(t) &= b_P(0)e^{2\Lambda t} - ib_Q(0)e^{-2\Lambda t} = \frac{b(0)+b^\dagger(0)}{2}e^{2\Lambda t} - \frac{b(0)-b^\dagger(0)}{2}e^{-2\Lambda t} \\
b_P(t) &= b_P(0)e^{2\Lambda t} \\
b_Q(t) &= b_Q(0)e^{-2\Lambda t}.
\end{aligned}$$

(c) Assume that at time $t = 0$ the electromagnetic field is in the vacuum state. Then at time t:

$$\begin{aligned}
\langle N \rangle_t &= \langle 0 | a^\dagger(t) a(t) | 0 \rangle \\
&= \langle 0 | b^\dagger(t) b(t) | 0 \rangle \\
&= \langle 0 | \left(\frac{b+b^\dagger}{2} e^{2\Lambda t} - \frac{b-b^\dagger}{2} e^{-2\Lambda t} \right) \left(\frac{b+b^\dagger}{2} e^{2\Lambda t} + \frac{b-b^\dagger}{2} e^{-2\Lambda t} \right) | 0 \rangle \\
&= \langle 0 | \left(\frac{a+a^\dagger}{2} e^{2\Lambda t} - \frac{a-a^\dagger}{2} e^{-2\Lambda t} \right) \left(\frac{a+a^\dagger}{2} e^{2\Lambda t} + \frac{a-a^\dagger}{2} e^{-2\Lambda t} \right) | 0 \rangle \\
&= \frac{e^{4\Lambda t}}{4} + \frac{e^{-4\Lambda t}}{4} - \frac{1}{2} \\
&= \frac{1}{2}(\cosh(4\Lambda t) - 1).
\end{aligned}$$

Here we have used $a = a(0) = b(0)$ and $a^\dagger = a^\dagger(0) = b^\dagger(0)$. We notice that the average number of photon increases exponentially as a function of t.

Next we compute,

$$\begin{aligned}
\langle 0 | b_P(t) | 0 \rangle &= \frac{e^{2\Lambda t}}{2} \langle 0 | a + a^\dagger | 0 \rangle = 0 \\
\langle 0 | b_Q(t) | 0 \rangle &= \frac{e^{-2\Lambda t}}{2} \langle 0 | a - a^\dagger | 0 \rangle = 0 \\
\langle 0 | b_P(t)^2 | 0 \rangle &= \frac{e^{4\Lambda t}}{4} \langle 0 | (a + a^\dagger)^2 | 0 \rangle = \frac{e^{4\Lambda t}}{4} \\
\langle 0 | b_Q(t)^2 | 0 \rangle &= -\frac{e^{-4\Lambda t}}{4} \langle 0 | (a - a^\dagger)^2 | 0 \rangle = \frac{e^{-4\Lambda t}}{4}.
\end{aligned}$$

With these we find the dispersions:

$$\Delta b_P(t) = \frac{e^{2\Lambda t}}{2} \qquad \text{and} \qquad \Delta b_Q(t) = \frac{e^{-2\Lambda t}}{2}.$$

We clearly see that there is squeezing: one quadrature grows while the other decays exponentially. However, $\Delta b_P(t)\Delta b_Q(t) = 1/2$ for all times t, as it should be due to Heisenberg.

(d) In class, we defined the squeezed vacuum with parameter $z = re^{i\phi}$ as

$$|0_z\rangle = S(z)|0\rangle = \exp\left[\frac{1}{2}z^* a^2 - \frac{1}{2}z a^{\dagger 2}\right]|0\rangle = \frac{1}{\sqrt{\cosh r}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh r)^n e^{i2n\phi} |2n\rangle.$$

We will now show that the two-photon interaction Hamiltonian above produces this state and this operator when we apply it for a time t_0.

Since the free field part of the Hamiltonian $\hbar\omega a^\dagger a$ annihilates the vacuum state $|0\rangle$, we can simply ignore it. This leaves us with only the two-photon part. The time evolution operator is then given by

$$\begin{aligned} e^{iHt_0/\hbar} &= \exp\left[\Lambda\left(a(t_0)^2 e^{2i\omega t_0} - a^\dagger(t_0)^2 e^{-2i\omega t_0}\right)\right] \\ &= \exp\left[\Lambda e^{2i\omega t_0} a(t_0)^2 - \Lambda e^{-2i\omega t_0} a^\dagger(t_0)^2\right] \\ &= \exp\left[\Lambda b(t_0)^2 - \Lambda b^\dagger(t_0)^2\right] \\ &= \exp\left[\Lambda b(t_0)^2 - \Lambda b^\dagger(t_0)^2\right]. \end{aligned}$$

Now we compute $b(t_0)^2 - b^\dagger(t_0)$:

$$\begin{aligned} b(t_0)^2 - b^\dagger(t_0)^2 &= \frac{1}{4}\left[(b+b^\dagger)e^{2\Lambda t_0} + (b-b^\dagger)e^{-2\Lambda t_0}\right]^2 - \frac{1}{4}\left[(b+b^\dagger)e^{2\Lambda t_0} - (b-b^\dagger)e^{-2\Lambda t_0}\right]^2 \\ &= \frac{1}{4}\left[2(b+b^\dagger)(b-b^\dagger) + 2(b-b^\dagger)(b+b^\dagger)\right] \\ &= \frac{1}{2}\left[bb - bb^\dagger + b^\dagger b - b^\dagger b^\dagger + b^2 + bb^\dagger - b^\dagger b - b^\dagger b^\dagger\right] \\ &= b^2 - b^{\dagger 2}. \end{aligned}$$

So, we have

$$e^{iHt_0/\hbar} = \exp\left[\Lambda b^2 - \Lambda b^{\dagger 2}\right].$$

By identifying this with the squeezing operator $S(z) = \exp\left[z^* b^2/2 - z b^{\dagger 2}/2\right]$, we find that $z = 2\Lambda$. We have derived the resulting state in Problem 2, so there is no need to do that again here. We notice that there is no dependence on t_0.

(e) In order to plot the follow Q distributions we must calculate the amplitudes.

- $Q_1(\alpha)$ is for the squeezed vacuum $S(z)|0\rangle$:

$$Q_1(\alpha) = |\langle\alpha| S(z)|0\rangle|^2 = \left|\frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh|z|}}\sum_{n=0}^{\infty}\frac{(\alpha^*)^{2n}}{2^n n!}(\tanh|z|)^n\right|^2.$$

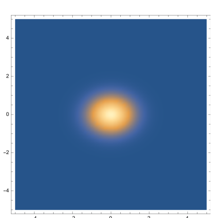

Figure 1: $z = 0.2$

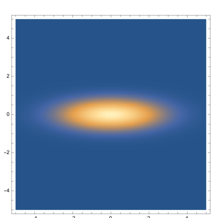

Figure 2: $z = 1.2$

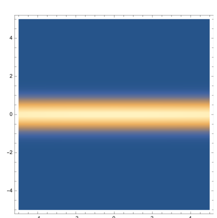

Figure 3: $z = 4.0$

- Displaced squeezed state $Q_2(\alpha)$:

$$Q_2(\alpha) = |\langle\alpha| D(\beta)S(z)|0\rangle|^2 = \left|\frac{e^{-|\alpha-\beta|^2/2}}{\sqrt{\cosh|z|}}\sum_{n=0}^{\infty}\frac{(\alpha^*-\beta^*)^{2n}}{2^n n!}(\tanh|z|)^n\right|^2.$$

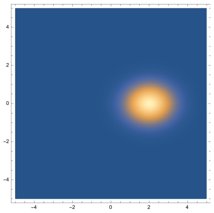

Figure 4: $z = 0.2$

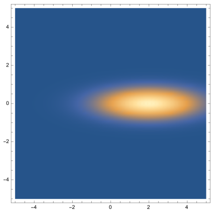

Figure 5: $z = 1.2$

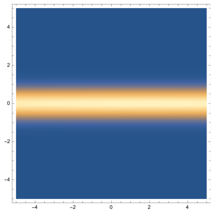

Figure 6: $z = 4.0$

where we have used $\langle\alpha| D(\beta) = \langle\alpha| D^{\dagger}(-\beta) = \langle\alpha - \beta|$. For this and the next part of the problem let us choose $\beta = 2$.

- A squeezed coherent state $Q_3(\alpha)$:

$$
\begin{aligned}
Q_3(\alpha) &= |\langle\alpha| S(z)D(\beta)|0\rangle|^2 = |\langle\alpha| D(\beta_-)S(z)|0\rangle|^2 \\
&= |\langle\alpha| D(\beta\cosh z - \beta^* \sinh z)S(z)|0\rangle|^2 \\
&= \left| \frac{e^{-|\alpha-\beta\cosh z+\beta^*\sinh z|^2/2}}{\sqrt{\cosh|z|}} \sum_{n=0}^{\infty} \frac{(\alpha^* - \beta^*\cosh^* z + \beta\sinh^* z)^{2n}}{2^n n!} (\tanh|z|)^n \right|^2 .
\end{aligned}
$$

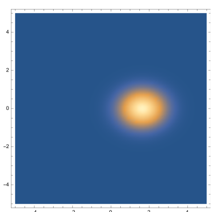

Figure 7: $z = 0.2$

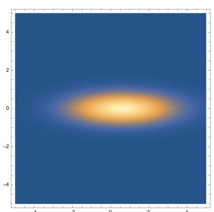

Figure 8: $z = 1.2$

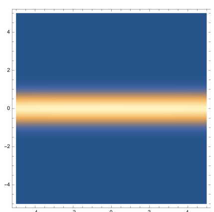

Figure 9: $z = 4.0$

Looking at Q_2 and Q_3, we see that they behave very similarly, except for the rate at which the peak of the Q-distribution gets displaced as a function of z. This makes sense, as this rate is determined by the hyperbolic trigonometric functions of z for Q_3. What's interesting is that the squeezed coherent state is NOT the same as the displaced squeezed vacuum.

By thinking about how the time evolution of the Q-distribution reflects on the measurement of the electric field, we can see that the two-photon interaction generates phase squeezing.