Physics 8.321, Fall 2021
Homework #8

Due **Friday, December 3** by 8:00 PM.

1. [Sakurai and Napolitano Problem 28, Chapter 2 (page 154)]

Consider an electron confined to the interior of a hollow cylindrical shell whose axis coincides with the z-axis. The wave function is required to vanish on the inner and outer walls, $\rho = \rho_a, \rho_b$ and also at the top and bottom, $z = 0, L$.

(**a**) Find the energy eigenfunctions. (Do not bother with normalization.) Show that the energy eigenvalues are given by

$$E_{lmn} = \left(\frac{\hbar^2}{2m_e}\right)\left[k_{mn}^2 + \left(\frac{l\pi}{L}\right)^2\right] \quad (l = 1, 2, 3, \ldots, m = 0, 1, 2, \ldots),$$

where k_{mn} is the nth root of the transcendental equation

$$J_m(k_{mn}\rho_b)N_m(k_{mn}\rho_a) - N_m(k_{mn}\rho_b)J_m(k_{mn}\rho_a) = 0\,,$$

where J_m, N_m are Bessel functions of the first and second kind (N_m is also sometimes called a Neumann function and can also be denoted Y_m); Bessel functions with integer values of m are also known as *cylindrical harmonics* in analogy with spherical harmonics.

(**b**) Repeat the same problem when there is a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ for $0 < \rho < \rho_a$. Note that the energy eigenvalues are influenced by the magnetic field even though the electron never "touches" the magnetic field.

(**c**) Compare, in particular, the ground state of the $B = 0$ problem with that of the $B \neq 0$ problem. Show that if we require the ground-state energy to be unchanged in the presence of B, we obtain "flux quantization"

$$\pi\rho_a^2 B = \frac{2\pi N\hbar c}{e}, \quad (N = 0, \pm 1, \pm 2, \ldots)\,.$$

2. [Sakurai and Napolitano Problem 39, Chapter 2 (page 155)]

An electron moves in the presence of a uniform magnetic field in the z-direction ($\mathbf{B} = B\hat{\mathbf{z}}$).

(a) Evaluate

$$[\Pi_x, \Pi_y]\,,$$

where

$$\Pi_x = p_x - eA_x/c, \quad \Pi_y = p_y - eA_y/c\,.$$

(b) By comparing the Hamiltonian and the commutation relation obtained in (a) with those of the one-dimensional oscillator problem, show how we can immediately write the energy eigenvalues as

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \left(\frac{|eB|\hbar}{mc}\right)(n + 1/2) \ ,$$

where $\hbar k$ is the continuous eigenvalue of the p_z operator and n is a nonnegative integer including zero.

3. Write the wave functions $\psi_{k,n}$ for the states associated with the eigenvalues $E_{k,n}$ computed in problem 2, working in the following three gauges:

(a) $\mathbf{A} = (-yB, 0, 0)$

(b) $\mathbf{A} = (-yB/2, xB/2, 0)$

(c) $\mathbf{A} = (0, xB, 0)$

4. Consider a charged particle in crossed electric and magnetic fields

$$\mathbf{B} = (0, 0, B), \qquad \mathbf{E} = (E, 0, 0)$$

Solve the eigenvalue problem in one of the three gauges of the previous problem.

5. Compute the spherical harmonics $Y_2^m(\theta, \phi)$ explicitly. Express these both as functions of θ, ϕ and of $z = \cos\theta, x = \sin\theta\cos\phi, y = \sin\theta\sin\phi$. Compute $\sum_m |Y_2^m|^2$.

6. Using separation of variables, show that the eigenstates of the Hamiltonian for a spherically symmetric potential $V(\mathbf{r})$ may be written in the form

$$\Psi_{E,l,m} = R_{El}(r) Y_l^m(\theta, \phi)$$

where $R_{El}(r) = \frac{1}{r} u_{El}(r)$ and u_{El} satisfies

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{\hbar^2 l(l+1)}{2mr^2} + V(r)\right] u_{El}(r) = E u_{El}(r) \, .$$

7. Consider a particle in the 2D potential $(m = 1)$

$$V(x, y) = \frac{1}{2}\omega^2(x^2 + y^2) \, .$$

Use raising and lowering operators $a_x^\dagger, a_y^\dagger, a_x, a_y$ to compute the spectrum and degeneracies of the Hamiltonian. For each value of the energy, what eigenvalues of J_z are possible? Are H, J_z a complete set of commuting observables? Write the states at the lowest 3 energy levels that are simultaneous eigenstates of these observables.