PY 711 Fall 2010
Homework 9: Due Tuesday, November 2

1. In this problem we consider a universe with one time dimension and either three, two, or one spatial dimensions. The interactions are described by a Lagrange density involving three real scalar fields,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi_X)(\partial^\mu \phi_X) - \frac{1}{2}m_X^2\phi_X^2 + \frac{1}{2}(\partial_\mu \phi_Y)(\partial^\mu \phi_Y) - \frac{1}{2}m_Y^2\phi_Y^2 \\ &\quad + \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 - \lambda\Phi\phi_X\phi_Y, \end{aligned} \tag{1}$$

where $M > m_X + m_Y$.

   (a) (5 points) Consider the case when the number of spatial dimensions is three. Calculate the total decay rate of the Φ particle in its center of mass frame to lowest non-vanishing order in λ.

   (b) (5 points) Consider the case when the number of spatial dimensions is two. Determine how the relevant formulas derived in class would change in two spatial dimensions, and calculate the total decay rate of the Φ particle in its center of mass frame to lowest non-vanishing order in λ.

   (c) (5 points) Consider the case when the number of spatial dimensions is one. Again determine how the relevant formulas derived in class would change in one spatial dimension, and calculate the total decay rate of the Φ particle in its center of mass frame to lowest non-vanishing order in λ.

KELLY PATTON
HW #9

14/15

1. IN THIS PROBLEM WE CONSIDER A UNIVERSE WITH ONE TIME DIMENSION AND EITHER THREE, TWO, OR ONE SPATIAL DIMENSIONS. THE INTERACTIONS ARE DESCRIBED BY A LAGRANGE DENSITY INVOLVING THREE REAL SCALAR FIELDS,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_x)(\partial^\mu \phi_x) - \frac{1}{2} m_x^2 \phi_x^2 + \frac{1}{2}(\partial_\mu \phi_y)(\partial^\mu \phi_y) - \frac{1}{2} m_y^2 \phi_y^2$$

$$+ \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2} M^2 \Phi^2 - \lambda \Phi \phi_x \phi_y,$$

WHERE $M > m_x + m_y$.

a. CONSIDER THE CASE WHEN THE NUMBER OF SPATIAL DIMENSIONS IS THREE. CALCULATE THE TOTAL DECAY RATE OF THE Φ PARTICLE IN ITS CENTER OF MASS FRAME TO LOWEST NON-VANISHING ORDER IN λ.

From class, for two particle final state

$$d\Gamma = \frac{1}{2E^I} \frac{d^3 p_x}{(2\pi)^3 2E_x} \frac{d^3 p_y}{(2\pi)^3 2E_y} |\mathcal{M}|^2 (2\pi)^4 \delta^4 (k^F_{tot} - k^I_{tot})$$

We defined

$$d\Pi_2 = \frac{d^3 p_x}{(2\pi)^3 2E_x} \frac{d^3 p_y}{(2\pi)^3 2E_y} (2\pi)^4 \delta^4 (k^F_{tot} - k^I_{tot})$$

$$= \frac{d^3 p_x}{(2\pi)^3 2E_x} \frac{d^3 p_y}{(2\pi)^3 2E_y} (2\pi)^4 \delta(E_x + E_y - E_{CM}) \underbrace{\delta^3(p_x + p_y)}_{|\vec{p}_x| = |\vec{p}_y| = p}$$

(center of mass frame)

$$= \frac{d^3 p}{(2\pi)^3 (2E_x)(2E_y)} (2\pi)\, \delta(E_x + E_y - M)$$

($E_{CM} = M$ in center of mass frame)

$$E_x = \sqrt{p^2 + m_x^2}$$

$$E_y = \sqrt{p^2 + m_y^2}$$

$$\delta(f(x)) = \frac{\delta(x - x_0)}{f'(x_0)}$$

$$\delta\left(\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2} - M\right) = \frac{\delta(p - p_0)}{\left(\frac{p}{E_x} + \frac{p}{E_y}\right)}$$

a. CONTINUED

$$d\Pi_2 = \frac{p^2 d\Omega}{16\pi^2 (2E_x)(2E_y)} \frac{1}{\left(\frac{p}{E_x} + \frac{p}{E_y}\right)}$$

(p so that $E_x + E_y - M = 0$)
(integral over dp done)

$$d\Pi_2 = \frac{d\Omega}{16\pi^2} \frac{p}{M}$$

Now, what is $|\mathcal{M}|^2$?

Following what we did for $\frac{\lambda}{4!}\phi^4$ theory, we should get

$$(-i\lambda) = i\mathcal{M}. \Rightarrow |\mathcal{M}|^2 = \lambda^2$$

$$\Gamma = \int d\Gamma = \int d\Pi_2 \frac{1}{2E_{CM}} |\mathcal{M}|^2$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p}{M} \frac{1}{2M} \lambda^2$$

$$= \frac{4\pi p \lambda^2}{32\pi^2 M^2}$$

$$\boxed{\Gamma = \frac{\lambda^2}{8\pi} \frac{p}{M^2}}$$

b. CONSIDER THE CASE WHEN THE NUMBER OF SPATIAL DIMENSIONS IS TWO. DETERMINE HOW THE RELEVANT FORMULAS DERIVED IN CLASS WOULD CHANGE IN TWO SPATIAL DIMENSIONS, AND CALCULATE THE TOTAL DECAY RATE OF THE Φ PARTICLE IN ITS CENTER OF MASS FRAME TO LOWEST NON-VANISHING ORDER IN λ.

Dropping one spatial dimension will make the formula for $d\Gamma$

$$d\Gamma = \frac{d^2p_x}{(2\pi)^2(2E_x)}\frac{d^3p_y}{(2\pi)^2(2E_y)}(2\pi)^3\,\delta(E_x+E_y-M)\,\delta^2(\vec{p}_x+\vec{p}_y)\,\frac{1}{2M}|\mathcal{M}|^2$$

Where the terms $\frac{d^3p}{(2\pi)^3} \rightarrow \frac{d^2p}{(2\pi)^2}$ lose a factor of $\frac{1}{2\pi}$ for the lost dimension, the four dimensional $\delta^4(k^F_{tot}-k^I_{tot})(2\pi)^4$ also loses a factor of (2π) and becomes three dimensional $\rightarrow$

$$(2\pi)^3\,\underbrace{\delta(E_x+E_y-M)}_{\text{time}}\,\underbrace{\delta^2(\vec{p}_x+\vec{p}_y)}_{\text{2 spatial dimensions}}$$

So now

$$d\Pi_2 = \frac{d^2p_x}{(2\pi)^2(2E_x)}\frac{d^2p_y}{(2\pi)^2(2E_y)}(2\pi)^3\,\delta(E_x+E_y-M)\,\underbrace{\delta^2(\vec{p}_x+\vec{p}_y)}_{|\vec{p}_x|=|\vec{p}_y|=p}$$

$$= \frac{p\,dp\,d\theta}{(2\pi)^2(2E_x)(2E_y)}(2\pi)\,\delta(E_x+E_y-M)$$

Use the same trick for the δ-function and integrate over p.

$$d\Pi_2 = \frac{d\theta}{2\pi(2E_x)(2E_y)}\frac{1}{\left(\frac{p}{E_x}+\frac{p}{E_y}\right)}$$

$$= \frac{d\theta}{8\pi M}$$

$|\mathcal{M}|^2 = \lambda^2$ still, no change in the diagram

$$\Gamma = \int\frac{d\theta}{8\pi M}\frac{1}{2M}\lambda^2$$

$$\boxed{\Gamma = \frac{\lambda^2}{8M^2}}$$

C. CONSIDER THE CASE WHEN THE NUMBER OF SPATIAL DIMENSIONS IS ONE. AGAIN DETERMINE HOW THE RELEVANT FORMULAS DERIVED IN CLASS WOULD CHANGE IN ONE SPATIAL DIMENSION, AND CALCULATE THE TOTAL DECAY RATE OF THE Φ PARTICLE IN ITS CENTER OF MASS FRAME TO LOWEST NON-VANISHING ORDER IN λ.

To go to one dimension, drop another factor of $\frac{1}{(2\pi)}$ in the integration term and a factor of (2π) from the δ-function.

two directions

$$d\Gamma = \frac{dp_x}{(2\pi)(2E_x)} \frac{dp_y}{(2\pi)(2E_y)} (2\pi)^2 \delta(E_x + E_y - M)\, \delta(p_x + p_y) \frac{1}{2M} |\mathcal{M}|^2$$

So

$$\Gamma = \int \frac{dp_x}{(2\pi)(2E_x)} \frac{dp_y}{(2\pi)(2E_y)} (2\pi)^2 \delta(E_x + E_y - M)\, \delta(p_x + p_y) \frac{\lambda^2}{2M}$$

$$= \int \frac{dp}{4E_xE_y} \delta(E_x + E_y - M) \frac{\lambda^2}{2M}$$

$$= \int \frac{dp}{4} \frac{\lambda^2}{2M} \frac{1}{\sqrt{p^2+m_x^2}\sqrt{p^2+m_y^2}} \delta\left(\sqrt{p^2+m_x^2} + \sqrt{p^2+m_y^2} - M\right)$$

Let $\quad E_F = \sqrt{p^2+m_x^2} + \sqrt{p^2+m_y^2}$

$$dE_F = \frac{E_F p}{\sqrt{p^2+m_x^2}\sqrt{p^2+m_y^2}}\, dp$$

$$\Gamma = \frac{\lambda^2}{8M} \int \frac{dE_F}{E_F\, p} \delta(E_F - M)$$

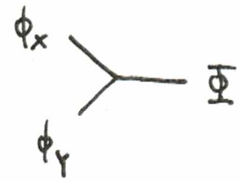
$$\boxed{\Gamma = \frac{\lambda^2}{8M^2 p}} \times 2 \qquad \left(p \text{ such that } \sqrt{p^2+m_x^2} + \sqrt{p^2+m_y^2} = M\right)$$

PY 711 Solutions #9

1. In d spatial dimensions the amplitude is

$$\langle \text{final} | S - 1 | \text{initial} \rangle = i\mathcal{M}\,(2\pi)\,\delta(E^F_{tot} - E^I_{tot})\,(2\pi)^d\,\delta^{(d)}(\vec{k}^F_{tot} - \vec{k}^I_{tot}).$$

To lowest non-trivial order the relevant diagram is



$i\mathcal{M} = -i\lambda$

In the center of mass frame we have $\vec{k}^I_{tot} = 0$, $E^I_{tot} = M$.

The decay rate is $\Gamma = \frac{\text{probability}}{\text{time}} = \frac{1}{2M}\int d\Pi_2\, |\mathcal{M}|^2$,

where
$$\int d\Pi_2 = \int \frac{d^d\vec{k}_x}{(2\pi)^d 2E_x}\frac{d^d\vec{k}_Y}{(2\pi)^d 2E_Y}(2\pi)\,\delta(E^F_{tot} - M)\,(2\pi)^d\,\delta^{(d)}(\vec{k}^F_{tot})$$

$$\left[\begin{array}{l} E_x = \sqrt{\vec{k}_x^2 + m_x^2}, \\ E_Y = \sqrt{\vec{k}_Y^2 + m_Y^2} \end{array}\right.$$

Integrating $\vec{k}_Y$ using the $\delta^{(d)}(\vec{k}^F_{tot})$ gives

$$\int d\Pi_2 = \int \frac{d^d\vec{k}_x}{(2\pi)^d (2E_x)(2E_Y)}(2\pi)\,\delta(E^F_{tot} - M)$$

$$= \int d\Omega \int \frac{dk_x\, k_x^{d-1}}{(2\pi)^d (2E_x)(2E_Y)}(2\pi)\,\delta(E^F_{tot} - M)$$

$$\left[\begin{array}{l} \vec{k}_Y = -\vec{k}_x, \quad E^F_{tot} = E_x + E_Y, \\ k_x = |\vec{k}_x|, \quad k_Y = |\vec{k}_Y| = k_x \end{array}\right.$$

Since $\frac{dE_x}{dk_x} = \frac{k_x}{E_x}$ and $\left.\frac{dE_Y}{dk_Y}\right|_{k_Y = k_x} = \frac{k_x}{E_Y}$,

$$\int d\Pi_2 = \int d\Omega\, \frac{k_x^{d-1}(2\pi)}{(2\pi)^d \cdot 4}\,\frac{1}{E_x E_Y\left(\frac{k_x}{E_x} + \frac{k_x}{E_Y}\right)} = \int d\Omega\, \frac{k_x^{d-2}}{(2\pi)^{d-1} \cdot 4M}$$

$[k_x$ solves $E^F_{tot} = E_x + E_Y = M$

The value of k_X which sets $E^F_{tot} = E_X + E_Y = M$ is

$$k_X = \frac{1}{2M}\sqrt{(M-m_X-m_Y)(M-m_X+m_Y)(M+m_X-m_Y)(M+m_X+m_Y)}$$

For general dimension d, we find

$$\Gamma = \int d\Omega \cdot \frac{\lambda^2}{8M^2} \cdot \frac{k_X^{d-2}}{(2\pi)^{d-1}}$$

a) For $d=3$, $\Gamma = 4\pi \cdot \frac{\lambda^2}{8M^2} \cdot \frac{k_X^1}{(2\pi)^2} = \frac{\lambda^2 k_X}{8\pi M^2}$

b) For $d=2$, $\Gamma = 2\pi \cdot \frac{\lambda^2}{8M^2} \cdot \frac{k_X^0}{(2\pi)^1} = \frac{\lambda^2}{8M^2}$

c) For $d=1$, $\Gamma = 2 \cdot \frac{\lambda^2}{8M^2} \cdot \frac{k_X^{-1}}{(2\pi)^0} = \frac{\lambda^2}{4k_X M^2}$