Table I. Amplitude of the various Fourier components for two tones on a Boehm flute blown with different embouchure averaged over several trials.

	A_1 (440 Hz)		A_2 (880 Hz)	
Order of the harmonic	Relaxed circular embouchure	Tensed elongated embouchure	Relaxed circular embouchure	Tensed elongated embouchure
1 (fundamental)	1.00	1.00	1.00	1.00
2	0.55	0.36	0.17	0.25
3	0.27	0.23	0.02	0.02
4	0.08	0.06	0.06	0.08
5	0.07	0.03	0.02	0.05
6	0.02	0.01	0.02	0.02
7	0.02	0.02	0.01	0.00
Musical impression:	Rich	Pure	Bottom	Thin

harmonics. The blowing mechanism which excites the higher harmonics in the first register, namely, the rounded lips, causes apparently the opposite effect in the second register.

The results for the top octave were quite varied, which is not surprising since the high notes on the flute are the most difficult to control. There was no inner consistency among the individual trials for either particular tone quality. The velocity and pressure of the air stream when obtaining a note of the high register are considerably greater than in the lower two registers and much more sensitive to the slightest changes in embouchure and blowing technique. Musical skill is necessary to be able always to produce the quality desired in this register.

Figure 1 shows a typical oscillograph picture of A_2 obtained with the two different embouchure techniques.

Table I shows the amplitudes of the various harmonics normalized with respect to the fundamental. Since the ear does not distinguish phase, that information, though obtained from the Fourier program, was not analyzed.

No correction was made for the frequency dependence of the microphone, since we were more interested in relative harmonic content than absolute values. The large amplitude of the overtones was somewhat surprising to us, at first, since textbooks refer to a flute as a relatively pure sine wave generator,[1,2] but this is true, as we observed, only at sound levels below those used in actual music making. The full sound of the flute is relatively rich in harmonics,[3] and the relative intensities of these can be controlled to an extent by embouchure techniques, as we have demonstrated.

In conclusion, we would like to remark that the project described provides a strong stimulus for a musically interested student to learn about Fourier synthesis and the associated experimental computational methods. It also demonstrated in the case studied a correlation between the musical and mathematical descriptions of a sound.

[1]J. G. Roederer, *Introduction to the Physics and Psychophysics of Music* (English Universities, London, 1973), p. 28.

[2]C. A. Taylor, *The Physics of Musical Sounds* (English Universities, London, 1965), p. 161.

[3]F. S. Crawford, Am. J. Phys. **42**, 282 (1974).

Prehistory of the "Runge–Lenz" vector

Herbert Goldstein
Division of Nuclear Science and Engineering
Columbia University
New York, New York 10027

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Heintz's exposition[1] of the "Runge–Lenz" vector perhaps provides an occasion to remark on the earlier history of this vector invariant of the Kepler problem, and to appeal for further information about the various eponymic changes it has undergone. When Lenz[2] made use of this vector to calculate the energy levels of perturbed Kepler motions on the basis of the old quantum theory, he described it as "little known," and referred to a then popular text by Runge[3] on vector analysis. In his discussion of the vector invariant, Runge makes no claim for originality. The vector appears in a section intended to illustrate procedures for differentiating and integrating vectors by application to central force motion. After proving the invariance of the angular momentum vector Runge goes on briefly to show that, if the central force varies inversely as the square of the distance, another constant vector can be obtained from the equation of motion. In turn, from this constant vector the orbit equation is then derived. No suggestion is made in the text that Runge was the first to discover this vector constant, any more than the invariance of the angular momentum vector. Pauli, in his pioneer 1926 paper[4] on the derivation of the levels of the hydrogen atom on the basis of the new matrix mechanics, starts again from the vector invariant, which he notes as "previously utilized by Lenz." Nevertheless, the vector has entered almost all subsequent physics literature as the "Runge–Lenz vector."

The earliest appearance of the constants of motion making up this vector invariant that I know of is in Vol. 1 of Laplace's *Traité de mécanique celeste,* which appeared in

1799.[5] Laplace's physical reasoning is clear, explicit, and reasonably complete. It deserves to be described in some detail. Laplace collects seven integrals of the motion for the reduced Kepler problem, not explicitly dependent on time. The first three of these, in modern notation, are respectively L_z/μ, $-L_y/\mu$, and L_x/μ, where μ is the reduced mass and $\mathbf{L}$ is the total angular momentum. The next three, again in modern notation, are the cartesian components of $\mathbf{A}/\mu^2$, where $\mathbf{A}$ is the Runge–Lenz vector in the form

$$\mathbf{A}=\mathbf{p}\times\mathbf{L}-\mu k\mathbf{r}/r.$$

The final constant is a function of the total energy in the form of a, the semimajor axis of the ellipse for negative energy orbits. Laplace then notes that there can only be five independent first integrals that are not explicit functions of time. After showing that two of the constants can indeed be expressed in terms of the rest, he remarks: "While these integrals are insufficient to determine x,y,z as functions of time, they do determine the nature of the orbit." Laplace then proves that the first three constants—the conserved components of the angular momentum—indicate the motion is in a plane. Finally, he performs the equivalent of taking the dot product of $\mathbf{r}$ with $\mathbf{A}$, obtaining the orbit equation as a conical section.

Many modern authors of monographs on celestial mechanics acknowledge Laplace's description of these constants of the motion. To mention a few, Tisserand in Vol. 1 of his magistral treatise[6] follows Laplace's treatment closely (with due acknowledgement). Stumpff[7] describes the vector invariant peculiar to the Kepler problem as the Laplace vector (which seems to be the preferable designation) and uses it extensively. Wintner[8] obtains the orbit equation in the reduced two-body problem via the Laplace vector, and in a historical note remarks that this procedure is due to Laplace and "seems to be quite forgotten, although it was discovered by Jacobi also (*Werke*, Bd. 4, 1842, p. 282)." What complicates the situation is that a number of authors ascribe the discovery to Hamilton. Thus Hill[9] speaks of the vector as "first noticed by Hamilton." Milne[10] refers to it as Hamilton's integral and describes its derivation as "a brilliant artifice due to Hamilton." No reference to Hamilton's published works is given. I have been unable to track it down further, and would appreciate any indication where it may be found in Hamilton's papers. Also, is there any reference to those constants of motion forming the Laplace vector earlier than Laplace, or was he really the original discoverer? A cursory search through Book I of Newton's Principia has not revealed any suggestive trace relating to these constants of the motion, but a more careful examination by a Newton scholar would be highly desirable.

Two specific comments on Heintz's paper[1] may be added. The vector $\mathbf{L}\times\mathbf{A}$ does *not* lie along the line of nodes. The line of nodes is traditionally the intersection of the orbital plane with the xy plane and may be at any angle to the line of the periapsis, whereas $\mathbf{L}\times\mathbf{A}$, by Heintz's definition, must be perpendicular to the line of the periapsis. To the very useful list of references compiled by Heintz may be added the thesis by Decoster, which appeared as a French report,[11] and which contains a bibliography of 207 items through 1969 relating to the invariants of the simple problems of two and three degrees of freedom.

[1]W. H. Heintz, Am. J. Phys. **42**, 1078 (1974).

[2]W. Lenz, Z. Phys. **24**, 197 (1924).

[3]C. Runge, *Vektoranalysis* (Hirzel, Leipzig, 1919), Vol. 1.

[4]W. Pauli, Z. Phys. **36**, 336 (1926) [translated in *Sources of Quantum Mechanics*, edited by B. L. Van der Waerden (Dover, New York, 1968), p. 387 ff].

[5]P. S. de Laplace, *Traité de mécanique celeste* [Paris, An VII. (1798—1799)], Tome I, Premiere Partie, Livre II, p. 165 ff; also *Oeuvres* (Gauthier-Villars, Paris, 1843), Tome I, p. 187 ff.

[6]F. Tisserand, *Traité de mécanique celeste* (Gauthier-Villars, Paris, 1889), Tome I, p. 95 ff.

[7]K. Stumpff, *Himmelsmechanik* (VEB Deutscher Verlag der Wissenschaften, Berlin, 1959), Band I, p. 92.

[8]A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton University, Princeton, NJ, 1941), pp. 241 and 422.

[9]R. Hill, *Principles of Dynamics* (Pergamon, Oxford, 1964), p. 25, footnote.

[10]E. A. Milne, *Vectorial Mechanics* (Interscience, New York, 1948), p. 237.

[11]A. Decoster, *Algebre et groupe de Lie d'invariance de l'atome d'hydrogene non relativiste*, CEA - R - 4112 (1970) [available from ERDA Technical Information Center, Oak Ridge, TN].

Repeatable equivalent examinations for premedical students*

John Lohr
Fusion Research Center
University of Texas at Austin
Austin, Texas 78712

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A recent paper by Golden, Fuller, and Jensen[1] discussed the use of repeatable equivalent examinations in several different sorts of physics courses. The great pressure under which premedical students labor in quest of high grades suggests that the system may be particularly applicable to courses in which these students constitute an appreciable fraction of the total.

I have now used this examination format in two introductory courses totaling about 140 students. The distribution of interests was 30% premedical, 33% health-related sciences such as pharmacy and predental, and a variety of others, mostly sciences. In each course four 1-h examinations and a final were scheduled. The first presentation of each exam was during class, while the second occurred about a week after the first in an evening. The first set of grades was posted within about three days of the examination. Attendance at either presentation or both was completely optional, with the proviso that no make-up examinations would be given. The two examinations were