Spring, 2021

Physics 312: Physics of Fluids
Assignment #5 (Solutions)

Background Reading

Friday, Mar. 12: Tritton 5.4,
Kundu & Cohen 4.1 - 4.3

Monday, Mar. 15: Kundu & Cohen 4.7

Wednesday, Mar. 17: Tritton 5.6, 5.7,
Kundu & Cohen 4.10, 4.11

Informal Written Reflection

Due: Thursday, March 18 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, March 19 (in class)

1. In class, we used an arbitrary *fixed* volume to derive the continuum expression of mass conservation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0.$$

We could instead have derived this expression by considering an arbitrary *material* volume. Show that this alternative derivation leads to the same result. (This is Kundu and Cohen, Chapter 4, Problem 2.)

(Hint: You'll use an integral theorem strategy like the one we used in class except, since you're now dealing with a material volume instead of a fixed volume, the theorem takes a slightly different form. The version you'll need is called the *Reynolds transport theorem* in Kundu and Cohen, Chapter 4...)

Solution:

The total mass in a material volume $\mathcal{V}$ is conserved:

$$\frac{D}{Dt}\int_{\mathcal{V}} \rho \, \mathrm{d}\mathcal{V} = 0.$$

Applying the Reynolds transport theorem to this volume,

$$\begin{aligned} \frac{D}{Dt}\int_{\mathcal{V}} \rho \, \mathrm{d}\mathcal{V} &= \int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \, \mathrm{d}\mathcal{V} + \int_{\mathcal{A}} \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}\mathcal{A} \\ &= \int_{\mathcal{V}} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] \mathrm{d}\mathcal{V} = 0. \end{aligned}$$

Since this is true for any material volume, the quantity in brackets must vanish (and that gives us the continuity equation).

2. In class, we used an arbitrary *material* volume to derive the continuum expression of momentum conservation,

$$\rho \frac{D}{Dt} u_i = \rho g_i + \frac{\partial}{\partial x_j} \tau_{ij}.$$

We could instead have derived this expression by considering an arbitrary *fixed* volume...

   (a) The i-th component of total momentum of the fluid contained in a fixed volume V is given by

$$M_i = \int_V \rho u_i \, \mathrm{d}V.$$

   The i-th component of total force of the fluid contained in this volume is given by

$$F_i = \int_V \left[\rho g_i + \frac{\partial}{\partial x_j} \tau_{ij} \right] \mathrm{d}V.$$

The *momentum principle* (for a fixed volume) relates these two expressions in the following way:

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t} M_i + \int_A \rho u_i u_j n_j \, \mathrm{d}A.$$

Can you interpret the expression $\rho u_j n_j \, \mathrm{d}A$ (appearing in the last term) as a flux? Use your answer to develop a physical interpretation of the momentum principle. Explain your ideas carefully.

(b) Use the momentum principle to rederive our main result (this is Kundu and Cohen, Chapter 4, Problem 4):

$$\rho \frac{D}{Dt} u_i = \rho g_i + \frac{\partial}{\partial x_j} \tau_{ij}.$$

(Hint: You'll need the continuity equation and a theorem telling you how to take a time derivative of an integral over a fixed volume.)

Solution:

(a) $\rho u_j n_j \, \mathrm{d}A$ represents the mass flux through the area element $\mathrm{d}A$. $\rho u_i u_j n_j \, \mathrm{d}A$ represents the i-th component of the momentum flux through this element. Thus, the momentum principle states that the total force on a fixed volume of fluid must equal the rate of change of the momentum within the volume plus the total outward momentum flux through the boundary of the volume.

(b) The rate of change of M_i is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} M_i = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho u_i \, \mathrm{d}V &= \int_V \frac{\partial}{\partial t}(\rho u_i) \, \mathrm{d}V \\
&= \int_V \left[u_i \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_i}{\partial t} \right] \mathrm{d}V.
\end{aligned}$$

The momentum flux term can be rewritten as

$$\begin{aligned}
\int_A \rho u_i u_j n_j \, \mathrm{d}A &= \int_V \frac{\partial}{\partial x_j} (\rho u_i u_j) \, \mathrm{d}V \\
&= \int_V \left[u_i \frac{\partial}{\partial x_j}(\rho u_j) + \rho u_j \frac{\partial u_i}{\partial x_j} \right] \mathrm{d}V
\end{aligned}$$

Then, plugging these expressions into the right side of the momentum principle gives us

$$\int_V \left[u_i \left(\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) \right) + \rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) \right] \mathrm{d}V$$

The first group of parenthesis contains the continuity equation and, therefore, equals zero. The second group of parenthesis is simply the rate of change of u_i following the flow. Set this integral equal to the F_i integral and note that the fixed volume V is arbitrary to complete the problem.

3. The solution to the one-dimensional diffusion equation,

$$\frac{\partial}{\partial t}\phi(x,t) = k\frac{\partial^2}{\partial x^2}\phi(x,t),$$

can be written as a *convolution*:

$$\phi(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\,\phi(y,0)\,\mathrm{d}y,$$

where $\phi(y,0)$ is an initial condition for $\phi(x,t)$ (y is just a dummy variable) and the convolution kernel G has the form of a Gaussian,

$$G(x,t) = \frac{1}{\sqrt{4\pi kt}}\exp(-\frac{x^2}{4kt}).$$

Note that $\phi(x,t) = G(x,t)$ if the initial condition is a delta function ($\phi(y,0) = \delta(y)$). How does $G(x,t)$ change shape as t increases? How does the integral of $G(x,t)$ over all x vary with time?...Use your answers to these questions to interpret the behavior of solutions to the one-dimensional diffusion equation.

Solution:

(i) The standard form of a Gaussian function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{x^2}{2\sigma^2}),$$

where σ is the standard deviation of $f(x)$ about its mean $(x = 0)$. Note that, in the case of $G(x,t)$ given above, $\sigma(t) = \sqrt{2kt}$. Thus, as t increases, the "width" of the curve increases.

(ii) The integral of $G(x,t)$ over all x equals 1 for any choice of t. Thus, the integral of $\phi(x,t)$ over all x is a conserved quantity:

$$\begin{aligned}\int_{-\infty}^{\infty} \phi(x,t)\,\mathrm{d}x &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} G(x-y,t)\,\phi(y,0)\,\mathrm{d}y \right) \mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} G(x-y,t)\,\mathrm{d}x \right) \phi(y,0)\,\mathrm{d}y \qquad (1) \\ &= \int_{-\infty}^{\infty} \phi(y,0)\,\mathrm{d}y.\end{aligned}$$

As t increases and the curve spreads out, it flattens to preserve the area under it.

(iii) This behavior matches what we expect from a diffusion problem: an initially concentrated distribution of ϕ gradually spreads out, without any creation or destruction, until it evenly fills any space available to it. If the initial concentration were more interesting, we would still expect this behavior and, indeed, this is the meaning of the convolution integral.