8.422 AMO II - Quantization of the E&M Field

February 8th, 2023

Classical Electrodynamics

- Vector and Scalar Potentials solve "free" Maxwell's equations

$$
\begin{aligned}
\vec{\nabla} \cdot \vec{B} = 0 \quad &\Leftrightarrow \quad \vec{B} = \vec{\nabla} \times \vec{A} \\
\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \quad &\Leftrightarrow \quad \vec{E} = -\vec{\nabla} U - \frac{\partial}{\partial t}\vec{A}
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- Gauge invariance

$$\begin{aligned} \vec{A} \quad &\rightarrow \quad \vec{A}' = \vec{A} + \vec{\nabla} f \\ U \quad &\rightarrow \quad U' = U - \frac{\partial}{\partial t} f \end{aligned}$$

For non-relativistic particles, use Coulomb gauge: $\vec{\nabla} \cdot \vec{A} = 0$

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- Maxwell's equations are now

$$\begin{aligned} \Delta U \quad &= \quad -\frac{1}{\epsilon_0}\rho \\ \Box \vec{A} \quad &= \quad \mu_0 \vec{j} - \frac{1}{c^2}\vec{\nabla}\frac{\partial}{\partial t} U \end{aligned}$$

E&M in reciprocal space

$$\begin{aligned} \vec{\mathcal{E}}(\vec{k},t) &= \int \mathrm{d}^3 r \vec{E}(\vec{r},t) e^{-i\vec{k}\cdot\vec{r}} \\ \vec{E}(\vec{r},t) &= \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \vec{\mathcal{E}}(\vec{k},t) e^{i\vec{k}\cdot\vec{r}} \end{aligned}$$

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$\Rightarrow$ Magnetic field is transverse!

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- Gauge invariance

$$\vec{\mathcal{A}} \quad \rightarrow \quad \vec{\mathcal{A}}' = \vec{\mathcal{A}} + \vec{k} f$$

$$\mathcal{U} \quad \rightarrow \quad \mathcal{U}' = \mathcal{U} - \dot{f}$$

For non-relativistic particles, use Coulomb gauge: $\vec{k}\cdot\vec{\mathcal{A}} = 0$

$\Rightarrow$ Longitudinal electric field $\vec{\mathcal{E}}_{||} \equiv \vec{\kappa}(\vec{\kappa}\cdot\vec{\mathcal{E}}) = -i\vec{k}\mathcal{U}$

$\Rightarrow$ Transverse electric field $\vec{\mathcal{E}}_{\perp} = -\dot{\vec{\mathcal{A}}} = -\dot{\vec{\mathcal{A}}}_{\perp}$