Continuity: Exercises 4.11, 14, 17, 18, 20, 21, 22, 23, Baby Rudin

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4.11 *Proof.* Let $f : X \to Y$ be a uniformly continuous map. Let a Cauchy sequence $\{x_n\} \subset X$ be given. To prove: $\{f(x_n)\}$ is Cauchy in Y. Let ϵ be given. We want to show that for sufficiently large m, n, $d_Y(f(x_n) - f(x_m)) < \epsilon$. Now, by uniform continuity of f, this holds whenever $d_X(x_n, x_m) < \delta$ for some $\delta > 0$. By the Cauchy-ness of $\{x_n\}$, this holds for any $\delta > 0$, provided sufficiently large m, n (which we assumed). So the claim is proven.

We want to use this to prove the following statement in Exercise 13: for E a dense subset of X and f a uniformly continuous *real* function defined on E, that f has a continuous extension from E to X. To do this, let $E \subset X$ be given. E is dense in X. $f : E \to \mathbb{R}$ is a uniformly continuous function. E is dense in X so for every $x \in X \setminus E$, there is a sequence $\{x_n\} \subset E$ such that $x_n \to x$. From the proof above we know $\{f(x_n)\}$ is Cauchy in $f(E) \subset \mathbb{R}$ and so $f(x_n) \to \mathfrak{f} \in \mathbb{R}$.

We define the continuous extension as follows:

$$g(x) = \begin{cases} f(x), & x \in E \\ \lim_{n\to\infty} f(x_n), & x \in X \setminus E, \{x_n\} \subset E \text{ s.t. } x_n \to x. \end{cases}$$

We claim that this is well-defined. To check this, we want to make sure $f(x_n)$ and $f(y_n)$ converge to the same value, provided the sequences $\{x_n\}$ and $\{y_n\}$ converge to the same value. For $\{x_n\}, \{y_n\} \subset E$ such that $x_n, y_n \to x \in X \setminus E$, we want to show $f(x_n), f(y_n) \to f(x)$. Let $\epsilon > 0$ be given, there exists $\delta > 0$ for which $\left|f(x) - f(y)\right| < \epsilon$ whenever $d_X(x, y) < \delta$. For sufficiently large n, $d_X(x_n, y_n) \leq d_X(x_n, x) + d(x, x) + d(x, y_n) < \delta$, which implies $\left|f(x_n) - f(y_n)\right| < \epsilon$. And so $f(x_n), f(y_n) \to f(x)$.

Finally we want to show $g(x)$ is continuous on X. To do this, we consider a sequence $\{p_n\}$ in X that converges to some p in X. For every $p_n \in X$ there is some $q_n \in E$ such that $d_X(p_n q_n) < d_X(p_n, p)$ (because E is dense in X) and $\left|g(p_n) = g(q_n)\right| < 1/n$. It follows that $d_X(q_n, p) \leq d_X(q_n, p_n) + d_X(p, p_n) < 2d_X(p_n, p) \to 0$ which means $q_n \to p$ as well. Now, because $\{q_n\} \subset E$ converges to $p \in X$, we have that $g(q_n) \to g(p)$. We want to show $g(p_n) \to f(p)$. Well, $\left|g(p) - g(p_n)\right| \leq \left|g(p) - g(q_n)\right| + \left|g(q_n) - g(p_n)\right| < \left|g(p) - g(q_n)\right| + 1/n$. This goes to zero as $n \to \infty$. So, $g(p_n) \to g(p)$ as desired. So, g is continuous in X. □

4.14 *Proof.* Let f be a continuous mapping from I into I where $I = [0, 1]$ is the closed unit interval. We want to show $f(x) = x$ for at least one $x \in I$. Consider the function $g(x) = f(x) - x$. g is continuous because f and id are continuous functions. $x, f(x) \in [0, 1]$, and so $g(0) = f(0) - 0 \geq 0$ and $g(1) = f(1) - 1 \leq 0$. If $g(0) = 0$ or $g(1) = 1$ then we have $f(1) = 1$ or $f(0) = 0$. Else, since g is continuous, $g(1) < 0 < g(0)$ implies that there is some $x \in [0, 1]$ such that $g(x) = f(x) - x = 0$ (Theorem 4.23, aka IVT). □

4.17 *Proof.* Let f be a real function defined on (a, b). We want to show that the set of points at which f has a simple discontinuity is at most countable.

The first type of of simple discontinuity is where $f(x-) < f(x+)$. Let E be the set on which $f(x-) < f(x+)$. With each point $x \in E$, we associate a triple (p, q, r) of rational numbers such that

1. $f(x-) < p < f(x+)$
2. $a < q < t < x \implies f(t) < p$
3. $x < t < r < b \implies f(t) > p$

The first item is always possible be done because $\mathbb{Q}$ is dense in $\mathbb{R}$. The second item is possible because when $f(x-)$ exists, let $\epsilon = p - f(x-) > 0$ be given, there is a $\delta > 0$ such that whenever $x - t < \delta$, $f(t) - f(x-) < \epsilon = p - f(x-)$, which implies $f(t) < p$. Now, we can always find a rational $q \in (x - \delta, x)$ such that for all $q < t < x$, $f(t) < p$. The third item follows from a similar argument.

Next we want to show the association is unique. Suppose we can also assign the same (p, q, r) to $y \neq x$:

1. $f(y-) < p < f(y+)$
2. $a < q < t < y \implies f(t) < p$
3. $y < t < r < b \implies f(t) > p$

We want to get to a contradiction. WLOG, assume $y < x$, then there is a number $y < s < x$, we have

1. From y: $y < s < r < b \implies f(t) > p$
2. From x: $a < q < s < x \implies f(t) < p$

which is a contradiction, since they cannot hold simultaneously. Thus, this association is unique. And because $\mathbb{Q}^3$ is still countable, there are countable such unique associations, and thus there must be at most countable such simple discontinuities.

The simple discontinuity of type $f(x-) > f(x+)$ can be dealt with in a similar manner. So, let's consider the third type where $f(x-) = f(x+) = y$. For this type, the number p in the association is no longer necessary, so we consider the following association with just two rational numbers (q, r) where:

1. $a < q < t < x \implies \left|f(t) - z\right| > \left|f(x) - z\right|$
2. $x < t < r < b \implies \left|f(t) - z\right| > \left|f(x) - z\right|$

Let's show this association is unique. Suppose $x < y$, then if we can have the same association for both x, y then we must have

1. $a < q < y < x \implies \left|f(x) - z\right| > \left|f(x) - z\right|$
2. $x < y < r < b \implies \left|f(y) - z\right| > \left|f(x) - z\right|$

which is a contradiction. So, the association is unique and thus the simple discontinuities of this type is at most countable. □

4.18 *Proof.* Let the function f defined on $\mathbb{R}$ be given by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/n, & x = m/n \end{cases}$$

where x in the second case is rational, with m, n are integers with no nontrivial common divisor and $n > 0$. When $x = 0$, we take $n = 1$. We want to show that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Let $x_0 \in \mathbb{R}$ be given. We claim that $\lim_{x \to x_0} f(x) = 0$. Let $\epsilon > 0$ be given. Take $q_0 \in \mathbb{N}$ such that $1/q_0 < \epsilon$. Now, for any interval $(x_0 - x', x_0 + x')$ for any $\infty > x' > 0$, there are finitely rationals p/q with denominator $q \in (0, q_0]$. And so we can always find a $\delta > 0$ such that any rational p/q in the interval $(x_0 - \delta, x_0 + \delta)$ has denominator $q > q_0$. Consider this δ, then if $x \in (x_0 - \delta, x_0 + \delta)$ is irrational then of course $f(x) = 0$, else if x is rational then $f(x) = f(p/q) = 1/q < 1/q_0$, which means $\left|f(x) - 0\right| < 1/q_0 < \epsilon$ for any $x \in (x_0 - \delta, x_0 + \delta)$. So, $\lim_{x \to x_0} f(x) = 0$ for all $x_0 \in \mathbb{R}$.

With this, if x_0 is irrational then $\lim_{x \to x_0} f(x) = 0 = f(x_0)$, so f is continuous there. If x_0 is rational, then $\lim_{x \to x_0} f(x) = 0$ but $f(x) \neq 0$, which means f has a simple discontinuity there. □

4.20 *Proof.* If E is a nonempty subset of a metric space X, define the distance from $x \in X$ to E by $\rho_E(x) = \inf_{z \in E} d(x, z)$.

1. $\rho_E(x) = 0 \iff x \in \bar{E}$. Suppose $x \in \bar{E}$, then $x \in E \cup E'$. If $x \in E$ then obviously $\rho_E(x) = d(x, x) = 0$. If x is a limit point of E then for every $\epsilon > 0$ there is some $q \in E$ such that $d(x, q) < \epsilon$. This means $\rho_E(x) = 0$ as well. Suppose $\rho_E(x) = 0$. If $x \notin \bar{E} = E \cup E'$ then there exists $\epsilon > 0$ such that $\mathcal{N}_\epsilon(x)$ does not contain any point in E, which means $d(x, z) \geq \epsilon$ for every $z \in E$. This is clearly a contradiction.

2. Prove that ρ_E is a uniformly continuous function on X, by showing that $\left|\rho_E(x) - \rho_E(y)\right| \leq d(x, y)$ for all $x, y \in X$. Let $x, y \in X$ be given. Let $z \in E$ be given, then $\rho_E(x) \leq d(x, y) + d(y, z) \leq$. This holds for all z, so $\rho_E(x) \leq d(x, y) + \rho_E(y)$. And so, $\left|\rho_E(x) - \rho_E(y)\right| \leq d(x, y)$. Thus, ρ_E is a uniformly continuous function on X because for any $\epsilon > 0$, there is a $\delta = \epsilon$ such that for any $x, y \in X$, whenever $d(x, y) < \delta = \epsilon$, $\left|\rho_E(x) - \rho_E(y)\right| \leq d(x, y) < \delta = \epsilon$.

□

4.21 *Proof.* Suppose K and F are disjoint sets in a metric space X and K is compact, F closed. We want to show that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$. Well, from problem 20, $\rho_F(x) = 0 \iff x \in F$ since F is closed. Also, from problem 20, we have that $d(p, q) \leq \left|\rho_F(p) - \rho_F(q)\right| = \left|\rho_F(p)\right|$. Now, ρ_F is a (uniformly) continuous function on the compact set K, so by Theorem 4.16 there is a point p_0 such that $\rho_F(p_0) = \inf_{t \in K} \rho_F(t)$. And so we have $d(p, q) \geq \left|\rho_F(p)\right| \geq \left|\rho_F(p_0)\right|$. So, if we let $\delta = \left|\rho_F(p_0)\right|/2$ then clearly, $d(x, y) > \delta$.

Suppose the "compactness" is dropped. Consider $X = \mathbb{R}$, $K = \mathbb{N}$ and $F = \{n + 1/2^n : n \in \mathbb{N}\}$. Then obviously K, F are closed and disjoint, but some large elements on both sets can get arbitrarily close to each other, i.e., $d(n, n + 1/2^n) \to 0$ as $n \to \infty$. $\square$

4.22 *Proof.* Let disjoint nonempty closed sets A, B be given and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)}, \quad p \in X.$$

Obviously, $0 \le f(p) \le 1$ for all p since it is a ratio of a nonnegative number to a larger positive number (which we know is positive because $A \cap B = \emptyset$). $\rho_A(p) = 0 \iff x \in \bar{A} = A$ (problem 20), so $f(p) = 0 \iff p \in A$. The same argument goes for $p \in B$, except that $p \in B \iff f(p) = \rho_A(p)/\rho_A(p) = 1$. Note that because $A \cap B = \emptyset$, this ratio is defined. We now want to show f is continuous on X. This is easy because it just follows from the fact that both ρ_A and ρ_B are continuous on X.

This establishes a converse of Exercise 3: Every closed set $A \subset X$ is $Z(f)$ for some continuous real f on X. Setting $V = f^{-1}([0, 1/2))$ and $W = f^{-1}((1/2, 1])$. We want to show V, W are open and disjoint.

f is a continuous function $X \to [0, 1]$. By Theorem 4.8, because $[0, 1/2)$ and $(1/2, 1]$ are open sets in $[0, 1]$, V, W must be open in X. Further, $f(A) = \{0\} \subset [0, 1/2)$ and $f(B) = \{1\} \subset (1/2, 1]$, so $A \subset V$ and $B \subset W$. $\square$

4.23 *Proof.* A real-valued function f defined in (a, b) is *convex* if

$$f(\lambda x + (1 - \lambda)y) \le \lambda f(x) + (1 - \lambda) f(y)$$

whenever $x, y \in (a, b)$, $0 < \lambda < 1$. We first want to show that every convex function is continuous. Next, we want to show that every increasing convex function of a convex function is convex. Finally, if f is convex in (a, b) and if $a < s < t < u < b$, we want to show that

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(s)}{u - s} \le \frac{f(u) - f(t)}{u - t}$$

We will prove the last item first. Let $s, t, u \in (a, b)$ such that $s < t < u$. Then we can put

$$t = \frac{t - s}{u - s} u + \frac{u - t}{u - s} s.$$

Obviously $\frac{t-s}{u-s} + \frac{u-t}{u-s} = 1$ and both are greater than 0. f is convex, so

$$f(t) = f\left(\frac{t - s}{u - s} u + \frac{u - t}{u - s} s\right) \le \frac{t - s}{u - s} f(u) + \frac{u - t}{u - s} f(s) = \frac{t - s}{u - s} f(u) + \left[1 - \frac{t - s}{u - s}\right] f(s)$$

After some **nontrivial** rearranging (too much LaTeX-ing here so I'll skip — sorry) we get

$$\frac{f(t) - f(s)}{t - s} \le \frac{f(u) - f(s)}{u - s} \le \frac{f(u) - f(t)}{u - t}.$$

Now we prove that f is continuous. Let $\epsilon > 0$ be given. For any $x > y \in [x_1, x_2]$, there are also x_0, x_3 such that $x_0 < x_1 < x_2 < x_3$. By the inequalities we have

$$\frac{f(x) - f(y)}{x - y} \le \frac{f(x_3) - f(y)}{x_3 - y} \le \frac{f(x_2) - f(y)}{x_2 - y}$$

and

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} \leq \frac{f(x) - f(x_0)}{x - x_0} \leq \frac{f(x) - f(y)}{x - y}.$$

And so,

$$\left|f(x) - f(y)\right| \leq \left|x - y\right| \max\left\{\frac{\left|f(x_3) - f(x_2)\right|}{|x_3 - x_2|}, \frac{\left|f(x_1) - f(x_0)\right|}{|x_1 - x_0|}\right\} \equiv C\left|x - y\right|.$$

Let $\delta = \min\{\epsilon/C, \frac{x_2 - x_1}{2}\}$, then we have

$$\left|f(x) - f(y)\right| \leq C\frac{\epsilon}{C} = \epsilon.$$

So f is continuous on (a, b).

Finally we want to show that every increasing convex function of a convex function is convex. Let $h(x) = g(f(x))$ where g is an increasing convex function and h is a convex function. For $x, y \in (a, b)$ and $\lambda \in (0, 1)$, we have that

$$\begin{aligned} h(\lambda x + (1 - \lambda)y) &= g(f(\lambda x + (1 - \lambda)y)) \\ &\leq g(\lambda f(x) + (1 - \lambda)f(y)) \\ &\leq \lambda g(f(x)) + (1 - \lambda)g(f(y)) \\ &= \lambda h(x) + (1 - \lambda)h(y). \end{aligned}$$

So we're done. □