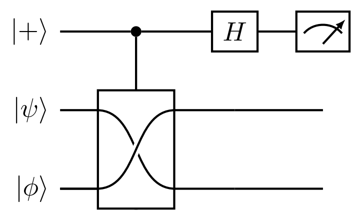
Homework 9   8.370/18.435/2.111   Due 12/07

1. The SWAP test.

   This is a test for figuring out whether two pure quantum states $|\phi\rangle$ and $|\psi\rangle$ are the same.

   Suppose we have the quantum circuit:



   where the gate is a controlled SWAP (CSWAP), a unitary gate that operates as follows:

$$\begin{aligned}&\text{CSWAP}\,|0\rangle\,|\phi\rangle\,|\psi\rangle = |0\rangle\,|\phi\rangle\,|\psi\rangle,\\ &\text{CSWAP}\,|1\rangle\,|\phi\rangle\,|\psi\rangle = |1\rangle\,|\psi\rangle\,|\phi\rangle,\end{aligned}$$

   (a) If $|\phi\rangle = |\psi\rangle$, what is the probability that we observe $|0\rangle$ on the top wire?

   (b) If $\langle\phi|\psi\rangle = 0$, what is the probability that we observe $|0\rangle$ on the top wire?

   (c) Now, even though it wasn't designed to be used this way, suppose that we apply the SWAP test with the inputs being two identical density matrices,

$$\rho_1 = \rho_2 = \sqrt{p}\,|0\rangle\langle 0| + \sqrt{1-p}\,|1\rangle\langle 1|\,.$$

   What is the probability that we observe $|0\rangle$ on the top wire?

2. Suppose k and ℓ are two odd numbers. There is a generalization of the nine-qubit code to a $k\ell$-qubit code that has the following codewords:

$$\begin{aligned}|0\rangle_L &= \frac{1}{2^{\ell/2}}\Big(|\underbrace{000\ldots0}_{k}\rangle + |\underbrace{111\ldots1}_{k}\rangle\Big)^{\otimes\ell},\\ |1\rangle_L &= \frac{1}{2^{\ell/2}}\Big(|\underbrace{000\ldots0}_{k}\rangle - |\underbrace{111\ldots1}_{k}\rangle\Big)^{\otimes\ell}.\end{aligned}$$

   How many bit errors can this code correct? How many phase errors can this code correct? How many syndrome bits do you need to measure to correct the bit errors? How many syndrome bits do you need to measure to correct the phase errors? (For the last two questions, I am asking how many bits are in the syndrome you compute, not how many bits you need to XOR to find each bit of the syndrome.)

3. Suppose I encode a qubit $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ with the 7-qubit Hamming code, with code C_1 having generator matrix G and C_2 having generator matrix H, as given in class:

$$G = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad H = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

The encoded state is stored in a noisy memory, and two errors occur (this is one more error than the code is designed to correct). When you decode the state with errors, what one-qubit state will you obtain if:

(a) there is a σ_x error on qubit 3 and a σ_z error on qubit 6,

(b) there is a σ_x error on qubit 3 and a σ_y error on qubit 6.

4. There is a CSS code with distance 2 (which thus can detect one error, but not correct any) with the following codes C_1 and C_2:

$$C_1 = \{0000, 0011, 0101, 1001, 0110, 1010, 1100, 1111\}, \qquad C_2 = \{0000, 1111\}.$$

(a) Write down the four codewords of this code.

For the next three questions, you don't have to list what happens to all four codewords if you can give a systematic description of the effects of the operations.

(b) Suppose you apply a σ_x to qubits 1 and 2 of this code (assume the qubits are labeled 1, 2, 3, 4). Does this operation take codewords to states in the code? What states does it take the four codewords to?

(c) Suppose you apply a σ_z to qubits 1 and 3 of this code (assume the qubits are labeled 1, 2, 3, 4). Does this operation take codewords to codewords? What states does it take the four codewords to?

(d) Suppose you apply H to all four qubits. Does this operation take the four codewords to states in the code? What states does it take the four codewords to?