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Course: **8.309 - Classical Mechanics III**
Problem set: **#2**

1. Canonical Transformations.

(a) Consider the following two generating functions:

$$\boxed{F_2(q,P) = q_i P_i \qquad \text{and} \qquad F_3(p,Q) = -p_i Q_i}$$

For F_2:

$$p_i = \frac{\partial F_2}{\partial q_i} = P_i, \qquad Q_i = \frac{\partial F_2}{\partial P_i} = q_i$$

as desired.

And for F_3:

$$q_i = -\frac{\partial F_3}{\partial p_i} = Q_i, \qquad P_i = -\frac{\partial F_3}{\partial Q_i} = p_i$$

as desired.

(b) $Q = p/t, P = -qt$. We can choose the following generating function $\boxed{F_1(q,Q,t) = Qqt}$. Let's check that it works:

$$p = \frac{\partial F_1}{\partial q} = Qt \implies Q = \frac{p}{t}, \qquad P = -\frac{\partial F_1}{\partial Q} = qt.$$

(c) We want $F_1 = F_1(q,Q,t)$, so

$$P = -\frac{\partial F_1}{\partial Q} = q^m p^n \implies F = -Qq^m p^n + g(q).$$

We also want

$$p = \frac{\partial F_1}{\partial q} = -Qmq^{m-1}p^n + g'(q) = Q^{1/l}q^{-k/l}.$$

This equality forces $g(q) = 0$, and so $m = -1, l = 1, n = 0, k = 2$. Therefore,

$$\boxed{Q = q^2 p \qquad P = q^{-1}}$$

(d) Under the gauge transformation $\vec{A} \to \vec{A}' = \vec{A} + \nabla f(\vec{x},t)$, $\phi \to \phi' = \phi - \partial_t f(\vec{x},t)$, we have

$$\mathcal{H}' = \frac{1}{2m}(\vec{P} - q\vec{A}')^2 + q\phi' = \frac{1}{2m}(\vec{P} - q\vec{A} - q\nabla f(\vec{X},t))^2 + q\phi(\vec{X},t) - q\frac{\partial}{\partial t}f(\vec{X},t).$$

Since the $(\vec{p} - q\vec{A})$ term is unchanged, we must have

$$(\vec{P} - q\vec{A}')^2 = (\vec{P} - q\vec{A} - q\nabla f(\vec{X},t))^2 = (\vec{p} - q\vec{A})^2.$$

which gives us one of the transformation:

$$\vec{P} = \vec{p} + q\nabla f(\vec{X},t) \implies \vec{p} = \vec{P} - q\nabla f(\vec{X},t)$$

The other transformation is simply $\vec{x} = \vec{X}$, and this gives

$$\mathcal{H}'(\vec{X}, \vec{P}, t) = \mathcal{H}(\vec{X}, \vec{P}, t) - q\frac{\partial}{\partial t} f(\vec{X}, t).$$

There is more than one way to show that the transformation

$$\vec{x} = \vec{X} \qquad \vec{p} = \vec{P} - q\boldsymbol{\nabla} f(\vec{X}, t)$$

is canonical, but we will do this by checking that the fundamental Poissons brackets are correct:

$$\begin{aligned}
\{X_i, X_j\}_{\vec{x},\vec{p}} &= \sum_k \frac{\partial X_i}{\partial x_k}\frac{\partial X_j}{\partial p_k} - \frac{\partial X_i}{\partial p_k}\frac{\partial X_j}{\partial x_k} = \sum_k \delta_{ik}\delta_{jk} - \delta_{ik}\delta_{jk} = 0 \\
\{P_i, P_j\}_{\vec{x},\vec{p}} &= \sum_k \frac{\partial P_i}{\partial x_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial P_i}{\partial p_k}\frac{\partial P_j}{\partial x_k} = 0 \\
\{X_i, P_j\}_{\vec{x},\vec{p}} &= \sum_k \frac{\partial X_i}{\partial x_k}\frac{\partial P_j}{\partial p_k} - \frac{\partial X_i}{\partial p_k}\frac{\partial P_j}{\partial x_k} = \sum_k \delta_{ik}\delta_{jk} = \delta_{ij}.
\end{aligned}$$

So, the transformation is indeed canonical. To find $F_2(\vec{x}, \vec{P}, t)$, we require that

$$p_i = \frac{\partial F_2}{\partial x_i} = P_i - q\frac{\partial}{\partial x_j} f(\vec{x}, t) \qquad \text{and} \qquad X_i = \frac{\partial F_2}{\partial P_i} = x_i$$

where we have used $\vec{x} = \vec{X}$ to bring the independent variables to $\vec{x}$ and $\vec{P}$. The first equation implies

$$F_2(\vec{x}, \vec{P}, t) = x_i P_i - f(\vec{x}, t) + g(\vec{P}).$$

With this and the second equation we have

$$X_i = x_i - \frac{\partial}{\partial P_i} g(\vec{P}).$$

We can just set $g(\vec{P}) = 0$. With these, we may write

$$\boxed{F_2(\vec{x}, \vec{P}, t) = x_i P_i - q f(\vec{x}, t) = \vec{x} \cdot \vec{P} - q f(\vec{x}, t)}$$

2. Harmonic Oscillator.

(a)

$$\begin{aligned}
\{Q, Q\}_{\vec{q},\vec{p}} &= \frac{\partial Q}{\partial q}\frac{\partial Q}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial Q}{\partial p} = 0 \\
\{P, P\}_{\vec{q},\vec{p}} &= \frac{\partial P}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial p}\frac{\partial P}{\partial q} = 0 \\
\{Q, P\}_{\vec{q},\vec{p}} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} = \frac{ia}{2ia} - \frac{-ia}{2ia} = 1.
\end{aligned}$$

Thus, the given transformation is indeed canonical.

(b) From the given transformations, we could solve for q, p in terms of P, Q:

$$q = \frac{1}{2ia}(Q - 2iaP) \qquad \text{and} \qquad p = \frac{1}{2}(Q + 2iaP).$$

The linear harmonic oscillator Hamiltonian can thus be written as

$$\begin{aligned}
\mathcal{H} &= \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2} \\
&= \frac{1}{2m}\left[\frac{1}{2}(Q + 2iaP)\right]^2 + \frac{m\omega^2}{2}\left[\frac{1}{2ia}(Q - 2iaP)\right]^2 \\
&= \frac{1}{8m}\left[(Q + 2iaP)^2 - \frac{m^2\omega^2}{a^2}(Q - 2iaP)^2\right].
\end{aligned}$$

An obvious choice for a would be $\boxed{a = m\omega}$, which gives

$$\mathcal{H} = \frac{1}{8m}\left[(Q + 2im\omega P)^2 - (Q - 2im\omega P)^2\right] = \frac{1}{8m} 8im\omega QP = i\omega QP.$$

It is clear that the canonical transformation is not explicitly time-dependent, therefore the generating function is not time-dependent. As a result, $\boxed{K = H = i\omega QP}$. Now we solve for the equations of motion:

$$\begin{aligned}
\dot{Q} &= \frac{\partial K}{\partial P} = i\omega Q \implies Q(t) = Q_0 e^{i\omega t} \\
\dot{P} &= -\frac{\partial K}{\partial Q} = -i\omega P \implies P(t) = P_0 e^{-i\omega t},
\end{aligned}$$

where

$$\begin{aligned}
Q_0 &= p(0) + im\omega q(0) \\
P_0 &= \frac{p(0) - im\omega q(0)}{2im\omega}.
\end{aligned}$$

From $Q(t), P(t)$, we can invert to solve for $q(t), p(t)$:

$$\begin{aligned}
\boxed{p(t)} &= \frac{1}{2}\left(Q(t) + 2im\omega P(t)\right) \\
&= \frac{1}{2}\left((p(0) + im\omega q(0))e^{i\omega t} - 2im\omega \frac{p(0) - im\omega q(0)}{2im\omega} e^{-i\omega t}\right) \\
&= \frac{1}{2}\left((p(0) + im\omega q(0))(\cos\omega t + i\sin\omega t) - (p(0) - im\omega q(0))(\cos\omega t - i\sin\omega t)\right) \\
&= \boxed{p(0)\cos\omega t - m\omega q(0)\sin\omega t}
\end{aligned}$$

and

$$\begin{aligned}
\boxed{q(t)} &= \frac{1}{2im\omega}(Q(t) - 2im\omega P(t)) \\
&= \frac{1}{2ia}((p(0) + im\omega q(0))e^{i\omega t} - 2im\omega \frac{p(0) - im\omega q(0)}{2im\omega} e^{-i\omega t}) \\
&= \frac{1}{2ia}\left((p(0) + im\omega q(0))(\cos\omega t + i\sin\omega t) - (p(0) - im\omega q(0))(\cos\omega t - i\sin\omega t)\right) \\
&= \boxed{q(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t}
\end{aligned}$$

3. Poisson Brackets and Conserved Quantities. Given

$$\mathcal{H} = q_1 p_1 - q_2 p_2 + aq_1^2 + bq_2^2$$

where a, b are constants. To show that $u_1 = (p_1 + aq_1)/q_2$ and $u_2 = q_1 q_2$ are constants of motion we check that $du_1/dt = 0 = du_2/dt$. Since $\partial u_1/\partial t = \partial u_2/\partial t = 0$, it suffices to check that $\{u_1, \mathcal{H}\} = \{u_2, \mathcal{H}\} = 0$:

$$\begin{aligned}
\{u_1, \mathcal{H}\} &= \left(\frac{\partial u_1}{\partial q_1}\frac{\partial \mathcal{H}}{\partial p_1} - \frac{\partial u_1}{\partial p_1}\frac{\partial \mathcal{H}}{\partial q_1}\right) + \left(\frac{\partial u_1}{\partial q_2}\frac{\partial \mathcal{H}}{\partial p_2} - \frac{\partial u_1}{\partial p_2}\frac{\partial \mathcal{H}}{\partial q_2}\right) \\
&= \left[\frac{a}{q_2}q_1 - \frac{1}{q_2}(p_1 + 2aq_1)\right] + \left[-\frac{(p_1 + aq_1)}{q_2^2}(-q_2) - 0\right] \\
&= \frac{1}{q_2}\left[aq_1 - p_1 - 2aq_1 + p_1 + aq_1\right] \\
&= 0 \qquad \checkmark
\end{aligned}$$

$$\begin{aligned}
\{u_2, \mathcal{H}\} &= \left(\frac{\partial u_2}{\partial q_1}\frac{\partial \mathcal{H}}{\partial p_1} - \frac{\partial u_2}{\partial p_1}\frac{\partial \mathcal{H}}{\partial q_1}\right) + \left(\frac{\partial u_2}{\partial q_2}\frac{\partial \mathcal{H}}{\partial p_2} - \frac{\partial u_2}{\partial p_2}\frac{\partial \mathcal{H}}{\partial q_2}\right) \\
&= \left[q_2 q_1 - 0\right] + \left[q_1(-q_2)\right] \\
&= 0. \qquad \checkmark
\end{aligned}$$

4. Angular Momentum and the Laplace-Runge-Lenz vector.

(a) To avoid unnecessary repeated indices we will replace i, j, k by a, b, c when needed:

$$\boxed{\{x_i, L_j\}} = \{x_i, \epsilon_{jka}x_k p_a\} = \sum_l \frac{\partial x_i}{\partial x_l}\frac{\partial \epsilon_{jka}x_k p_a}{\partial p_l} - \frac{\partial x_i}{\partial p_l}\frac{\partial \epsilon_{jka}x_k p_a}{\partial x_l} = \sum_l \delta_{il}\epsilon_{jki}x_k\delta_{al} = \epsilon_{jki}x_k = \boxed{\epsilon_{ijk}x_k}$$

$$\boxed{\{p_i, L_j\}} = \{p_i, \epsilon_{jka}x_k p_a\} = \sum_l \frac{\partial p_i}{\partial x_l}\frac{\partial \epsilon_{jka}x_k p_a}{\partial p_l} - \frac{\partial p_i}{\partial p_l}\frac{\partial \epsilon_{jka}x_k p_a}{\partial x_l} = \sum_l -\delta_{il}\epsilon_{jka}p_a\delta_{kl} = -\epsilon_{jia}p_a = \boxed{\epsilon_{ijk}p_k}$$

Next,

$$\begin{aligned}
\{L_i, L_b\} &= \{\epsilon_{ijk}x_j p_k, \epsilon_{bca}x_c p_a\} \\
&= \sum_l \frac{\partial \epsilon_{ijk}x_j p_k}{\partial x_l}\frac{\partial \epsilon_{bca}x_c p_a}{\partial p_l} - \frac{\partial \epsilon_{ijk}x_j p_k}{\partial p_l}\frac{\partial \epsilon_{bca}x_c p_a}{\partial x_l} \\
&= \sum_l \delta_{jl}\epsilon_{ijk}p_k\delta_{al}\epsilon_{bca}x_c - \delta_{kl}\epsilon_{ijk}x_j\delta_{cl}\epsilon_{bca}p_a \\
&= \epsilon_{ijk}\epsilon_{bcj}p_k x_c - \epsilon_{ijk}\epsilon_{bka}x_j p_a \\
&= -\epsilon_{ikj}\epsilon_{bcj}p_k x_c + \epsilon_{ijk}\epsilon_{bak}x_j p_a \\
&= -(\delta_{ib}\delta_{kc} - \delta_{ic}\delta_{kb})p_k x_c + (\delta_{ib}\delta_{ja} - \delta_{ia}\delta_{jb})x_j p_a \\
&= -\delta_{ib}p_k x_k + x_i p_b + \delta_{ib}x_j p_j - x_b p_i \\
&= x_i p_b - x_b p_i.
\end{aligned}$$

Putting back $b = j$ we get the final result

$$\boxed{\{L_i, L_j\}} = x_i p_j - x_j p_i = (\delta_{iu}\delta_{jv} - \delta_{iv}\delta_{ju})x_u p_v = \epsilon_{ijk}\epsilon_{uvk}x_u p_v = \epsilon_{ijk}\epsilon_{kuv}x_u p_v = \boxed{\epsilon_{ijk}L_k}$$

To calculate $\{L_i \vec{L}^2\}$, we have to be more explicit with our index notation. To be explicit we will say $L_a L_a = L_a^2 \neq \vec{L}^2$ and use the "product rule" for Poisson brackets $\{uv, w\} = u\{v, w\} + \{u, w\}v$.

$$\{L_i, \vec{L}^2\} = \{L_i, L_i L_i + L_j L_j + L_k L_k\} = \{L_i, L_i L_i\} + \{L_i, L_j L_j\} + \{L_i, L_k L_k\}$$

since $\{L_i, L_i\} = 0$, the first term on the RHS is zero, which leaves us with

$$\begin{aligned}
\boxed{\{L_i, \vec{L}^2\}} &= -\{L_j L_j, L_i\} - \{L_k L_k, L_i\} \\
&= -L_j\{L_j, L_i\} - \{L_j, L_i\}L_j - L_k\{L_k, L_i\} + \{L_k, L_i\}L_k \\
&= 2L_j\epsilon_{ijk}L_k + 2L_k\epsilon_{ika}L_a \\
&= 2\epsilon_{ijk}L_jL_k - 2\epsilon_{iak}L_aL_k \\
&= \boxed{0}
\end{aligned}$$

(b) Since $\vec{A} = \vec{p} \times \vec{L} - \mu k\vec{r}/r$, to show that $\vec{A}$ is conserved we check that $\{\vec{A}, \mathcal{H}\} = 0$, i.e., $\{A_i, \mathcal{H}\} = 0$. To do this, we must first write everything in terms of components and computer their derivatives

$$\begin{aligned}
\vec{A} = \vec{p} \times \vec{L} - \frac{\mu k\vec{r}}{r} \implies A_i &= \epsilon_{ijk}p_jL_k - \frac{\mu k\vec{r}_i}{r} \\
&= \epsilon_{ijk}\epsilon_{kab}p_jr_ap_b - \frac{\mu kr_i}{r} \\
&= \epsilon_{ijk}\epsilon_{abk}p_jr_ap_b - \frac{\mu kr_i}{r} \\
&= (\delta_{ia}\delta_{jb} - \delta_{ib}\delta_{ja})p_jr_ap_b - \frac{\mu kt_i}{r} \\
&= r_ip_jp_j - p_jr_jp_i - \frac{\mu kr_i}{r} \\
&= r_i\vec{p}^2 - p_jr_jp_i - \frac{\mu kr_i}{r}
\end{aligned}$$

With this,

$$\begin{aligned}
\frac{\partial A_i}{\partial r_l} &= \delta_{il}\vec{p}^2 - p_jp_i\delta_{jl} - \mu k\left(\delta_{il}\frac{1}{r} - \frac{r_i}{r^2}\frac{\partial r}{\partial r_l}\right) \\
&= \delta_{il}\vec{p}^2 - p_lp_i - \mu k\left(\frac{\delta_{il}}{r} - \frac{r_ir_l}{r^3}\right) \\
&= \delta_{il}\vec{p}^2 - p_lp_i - \frac{\mu k}{r}\left(\delta_{il} - \frac{r_ir_l}{r^2}\right)
\end{aligned}$$

where we have used the fact that

$$\frac{\partial r}{\partial r_l} = \frac{\partial\sqrt{r_lr_l}}{\partial r_l} = \frac{1}{\sqrt{r_lr_l}}2\frac{1}{2}r_l = \frac{r_l}{r}.$$

We also need to find $\partial A_i/\partial p_l$:

$$\frac{\partial A_i}{\partial p_l} = 2r_ip_j\delta_{jl} - r_j\delta_{jl}p_i - p_jr_j\delta_{il} = 2r_ip_l - r_lp_i - p_jr_j\delta_{il}.$$

The next step is to do the same for $\mathcal{H} = \vec{p}^2/2\mu - k/r$:

$$\frac{\partial\mathcal{H}}{\partial r_l} = -k\frac{\partial}{\partial r_l}\frac{1}{r} = \frac{k}{r^2}\frac{\partial r}{\partial r_l} = \frac{kr_l}{r^3}.$$

$$\frac{\partial\mathcal{H}}{\partial p_l} = \frac{1}{2\mu}\frac{\partial p_ip_i}{\partial p_l} = \frac{1}{\mu}p_i\delta_{il} = \frac{p_l}{\mu}.$$

With these, we are ready:

$$
\begin{aligned}
\{A_i, \mathcal{H}\} &= \sum_l \frac{\partial A_i}{\partial r_l}\frac{\partial \mathcal{H}}{\partial p_l} - \frac{\partial A_i}{\partial p_l}\frac{\partial \mathcal{H}}{\partial r_l} \\
&= \sum_l \left(\delta_{il}\vec{p}^2 - p_l p_i - \frac{\mu k}{r}\left(\delta_{il} - \frac{r_i r_l}{r^2}\right)\right)\frac{p_l}{\mu} - (2r_i p_l - r_l p_i - p_j r_j \delta_{il})\frac{k r_l}{r^3} \\
&= \cancel{\frac{1}{\mu}p_i\vec{p}^2} - \cancel{\frac{1}{\mu}p_i\vec{p}^2} - \frac{k}{r}\left(p_i - \frac{r_i r_l p_l}{r^2}\right) - \frac{k}{r^3}(2r_i p_l r_l - r_l p_i r_l - p_j r_j r_i) \\
&= -p_i\frac{k}{r} + r_i\frac{k\vec{r}\cdot\vec{p}}{r^3} - r_i\frac{2k\vec{r}\cdot\vec{p}}{r^3} + p_i\frac{k}{r} + r_i\frac{k\vec{r}\cdot\vec{p}}{r^3} \\
&= \left(-p_i\frac{k}{r} + p_i\frac{k}{r}\right) + \left(r_i\frac{k\vec{r}\cdot\vec{p}}{r^3} - r_i\frac{2k\vec{r}\cdot\vec{p}}{r^3} + r_i\frac{k\vec{r}\cdot\vec{p}}{r^3}\right) \\
&= 0.
\end{aligned}
$$

So, the Laplace-Runge-Lenz vector is conserved, as desired.

(c) We first find what $\vec{L}^2$ is in terms of $\vec{x}, \vec{p}$:

$$
\begin{aligned}
\vec{L}^2 &= L_i L_i \\
&= \epsilon_{ijk} L r_j p_k \epsilon_{ibc} r_b p_c = \epsilon_{jki}\epsilon_{bci} r_j r_b p_k p_c \\
&= (\delta_{jb}\delta_{kc} - \delta_{jc}\delta_{kb}) r_j r_b p_k p_c \\
&= r_j r_j p_k p_k - r_j p_j r_k p_k \\
&= \vec{r}^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2
\end{aligned}
$$

With this, we expand the LHS and simplify to get $\vec{A}^2$:

$$
\begin{aligned}
\mu^2 k^2 + 2\mu\mathcal{H}\vec{L}^2 &= \mu^2 k^2 + 2\mu\left(\frac{\vec{p}^2}{2\mu} - \frac{k}{r}\right)\left[\vec{r}^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2\right] \\
&= \mu^2 k^2 + \left(\vec{p}^2 - \frac{2\mu k}{r}\right)\left[\vec{r}^2\vec{p}^2 - (\vec{r}\cdot\vec{p})^2\right] \\
&= \mu^2 k^2 + \vec{p}^2\vec{r}^2\vec{p}^2 - \vec{p}^2(\vec{r}\cdot\vec{p})^2 - \frac{2\mu k}{r}\vec{r}^2\vec{p}^2 + \frac{2\mu k}{r}(\vec{r}\cdot\vec{p})^2 \\
&= \vec{r}^2\left(\vec{p}^2\vec{p}^2 - \frac{2\mu k}{r}\vec{p}^2 + \frac{\mu^2 k^2}{r^2}\right) - 2(\vec{r}\cdot\vec{p})^2\left(\vec{p}^2 - \frac{\mu k}{r}\right) + (\vec{r}\cdot\vec{p})^2\vec{p}^2 \\
&= (r_i r_i)\left(\vec{p} - \frac{\mu k}{r}\right)^2 - 2(r_i p_i)(\vec{r}\cdot\vec{p})\left(\vec{p}^2 - \frac{\mu k}{r}\right) + (\vec{r}\cdot\vec{p})p_i p_i \\
&= \underbrace{\left[r_i\left(\vec{p} - \frac{\mu k}{r}\right) - (\vec{r}\cdot\vec{p})p_i\right]^2}_{A_i} \\
&= \vec{A}^2 \quad \checkmark
\end{aligned}
$$

where we have used the previously derived result:

$$
A_i = r_i\vec{p}^2 - p_j r_j p_i - \frac{k\mu r_i}{r} = r_i\left(\vec{p}^2 - \frac{\mu k}{r}\right) - (\vec{r}\cdot\vec{p})p_i.
$$

5. An Exponential Potential.

(a) $\mathcal{H} = p^2 + e^x$. We want $K = P^2 = H = p^2 + e^x$. Since $p > 0$, we have $p = \sqrt{P^2 - e^x}$. With this,

$$p = \frac{\partial F_2}{\partial x} = \sqrt{P^2 - e^x} \implies F_2(x, P) = 2\sqrt{P^2 - e^x} - 2P\tanh^{-1}\left(\frac{\sqrt{P^2 - e^x}}{P}\right) + g(P).$$

Since we don't have any requirements from $X = X(x, P)$, we may as well set $g(P) = 0$, so that

$$\boxed{F_2(x, P) = 2\sqrt{P^2 - e^x} - 2P\tanh^{-1}\left(\frac{\sqrt{P^2 - e^x}}{P}\right)}$$

Mathematica code:

```
F2 = Integrate[Sqrt[P^2 - E^x], x]

>>>> 2 Sqrt[-E^x + P^2] - 2 P ArcTanh[Sqrt[-E^x + P^2]/P]
```

(b) The transformation equations can be obtained from F_2. First, assuming $P > 0$ we have that

$$\boxed{P(x, p) = \sqrt{p^2 + e^x}}$$

Next,

$$X(x, p) = \frac{\partial F_2}{\partial P} = \frac{\partial}{\partial P}\left[2\sqrt{P^2 - e^x} - 2P\text{arctanh}\left(\frac{\sqrt{P^2 - e^x}}{P}\right)\right] = -2\text{arccoth}\left(\frac{P}{\sqrt{P^2 - e^x}}\right)$$

Substituting in the result for P we find

$$\boxed{X(x, p)} = -2\text{arccoth}\left(\frac{\sqrt{p^2 + e^x}}{\sqrt{p^2 + e^x - e^x}}\right) = \boxed{-2\text{arccoth}\left(\frac{\sqrt{p^2 + e^x}}{p}\right)}$$

Mathematica code:

```
D[F2, P] // FullSimplify

>>>> -2 ArcCoth[P/Sqrt[-E^x + P^2]]
```

Assuming $P < 0$ gives us another set of transformation equations:

$$\boxed{P(x, p) = -\sqrt{p^2 + e^x}} \qquad \boxed{X(x, p) = -2\text{arccoth}\left(\frac{-\sqrt{p^2 + e^x}}{p}\right)}$$

(c) To determine $x(t), p(t)$ we may solve for the Hamilton EOMs for K first:

$$\begin{aligned}
\dot{P} = -\frac{\partial K}{\partial X} = 0 &\implies P = \pm\sqrt{p^2 + e^x} = \text{constant} = \pm P_0 := \pm\sqrt{p(0)^2 + e^{x(0)}} \\
\dot{X} = \frac{\partial K}{\partial P} = 2P &\implies X = 2Pt + X_0 \\
&\implies -2\text{arccoth}\left(\frac{\pm\sqrt{p^2 + e^x}}{p}\right) = -2\text{arccoth}\left(\frac{\pm P_0}{p(t)}\right) = \pm 2P_0 t + C \\
&\implies p(t) = \mp P_0 \tanh\left(\frac{X_0}{2} \pm tP_0\right)
\end{aligned}$$

From this, we can invert to find $x(t)$:

$$e^{x(t)} = P^2 - p^2 = P_0^2 - P_0^2\tanh^2\left(\frac{X_0}{2} \pm tP_0\right) \implies x(t) = \ln\left[P_0^2 - P_0^2\tanh^2\left(\frac{X_0}{2} \pm tP_0\right)\right]$$

where P_0 is already defined above and $X_0 = -2\text{arccoth}(\pm P_0/p(0))$. We also answer whether the signs matter i.e., whether there is a "preferred" set of transformation equations. Suppose that $P > 0$, then

$$p(t) = -P_0\tanh\left(\frac{X_0}{2} + tP_0\right).$$

At $t = 0$, we have $p(0) = -P_0\tanh(X_0/2)$ where $X_0 = -2\text{arccoth}(P_0/p(0)) < 0$, and so $p(t) > 0$. ✓

On the other hand if we pick the $P < 0$ transformation then

$$p(t) = P_0\tanh\left(\frac{X_0}{2} - tP_0\right).$$

At $t = 0$, we have $p(0) = P_0\tanh(X_0/2)$ where $X_0 = -2\text{arccoth}(-P_0/p(0)) > 0$, and so $p(t) > 0$. ✓

Therefore, either transformation works, so we may as well choose the positive transformation can obtain the following solution:

$$\boxed{p(t) = -P_0\tanh\left(\frac{X_0}{2} + tP_0\right) \qquad \text{and} \qquad x(t) = \ln\left[P_0^2 - P_0^2\tanh^2\left(\frac{X_0}{2} + tP_0\right)\right]}$$

where

$$X_0 = -2\text{arccoth}\left(\frac{\sqrt{p(0) + e^{x(0)}}}{p(0)}\right) \qquad \text{and} \qquad P_0 = \sqrt{p(0)^2 + e^{x(0)}}$$

Mathematica code:

```
(* Positive solution *)
Solve[-2 ArcCoth[Z/p[t]] == 2*Z*t + C, p[t]] // FullSimplify

p[t] -> -Z Tanh[C/2 + tZ]

(* Negative solution *)
Solve[-2 ArcCoth[-Z/p[t]] == -2*Z*t + C, p[t]] // FullSimplify

p[t] -> Z Tanh[C/2 - tZ]
```

6. Projectile with Hamilton-Jacobi.

Our (Hamiltonian) coordinates will be p_x, p_y, x, y. The Hamiltonian is simply the energy:

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + mgy = E$$

The Hamilton-Jacobi equation says

$$\mathcal{H}\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}, t\right) + \frac{\partial S}{\partial t} = 0$$

which in our case becomes

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2\right] + mgy + \frac{\partial S}{\partial t} = 0 \implies \frac{1}{2m}\left[\left(\frac{\partial S_{xy}}{\partial x}\right)^2 + \left(\frac{\partial S_{xy}}{\partial y}\right)^2\right] + mgy = \alpha_1 = -\frac{\partial S_t}{\partial t}$$

where we have used the fact that $p_i = \partial S/\partial x_i$ and that S is separable in spatial and time coordinates: $S = S_{xy} + S_t$ where S_{xy} has no t-dependence and S_t has no spatial dependence. From here we find that

$$S_t = -\alpha_1 t.$$

We now solve the spatial part:

$$\frac{1}{2m}\left[\left(\frac{\partial S_{xy}}{\partial x}\right)^2 + \left(\frac{\partial S_{xy}}{\partial y}\right)^2\right] + mgy = \alpha_1 \implies \frac{1}{2m}\left(\frac{\partial S_{xy}}{\partial x}\right)^2 = \alpha_1 - \frac{1}{2m}\left(\frac{\partial S_{xy}}{\partial y}\right)^2 - mgy = \alpha_2$$

where α is constant. By separability we may as well write $S_{xy} = S_x + S_y$ where S_x has no y-dependence and S_y has no x-dependence. From here we find that

$$\frac{\partial S_x}{\partial x} = \pm\sqrt{2m\alpha_2} \implies S_x = \pm\int \sqrt{2m\alpha_2}\,dx$$

and

$$\frac{\partial S_y}{\partial y} = \pm\sqrt{2m(\alpha_1 - \alpha_2 - mgy)} \implies S_y = \pm\int \sqrt{2m(\alpha_1 - \alpha_2 - mgy)}\,dy.$$

The full solution is therefore

$$S = S_x + S_y + S_t = \pm\int \sqrt{2m\alpha_2}\,dx \pm \int \sqrt{2m(\alpha_1 - \alpha_2 - mgy)}\,dy - \alpha_1 t.$$

From here, we solve for the constants β_1, β_2:

$$\begin{aligned}
\beta_1 &= \frac{\partial S}{\partial \alpha_1} = -t \pm \int \frac{m}{\sqrt{2m(\alpha_1 - \alpha_2 - mgy)}}\,dy \\
\beta_2 &= \frac{\partial S}{\partial \alpha_2} = \pm\int \frac{m}{\sqrt{2m\alpha_2}}\,dx \mp \int \frac{m}{\sqrt{2m(\alpha_1 - \alpha_2 - gmy)}}\,dy = \pm\int \frac{m}{\sqrt{2m\alpha_2}}\,dx \mp (\beta_1 + t)
\end{aligned}$$

Solving the first equation for y we find

$$t + \beta_1 = \pm\frac{1}{mg}\sqrt{2m(\alpha_1 - \alpha_2 - mgy)} \implies y(t) = \frac{\alpha_1 - \alpha_2}{mg} - \frac{1}{2}g(t + \beta_1)^2.$$

Inverting the first equation we find

$$x(t) = \pm\frac{\sqrt{2m\alpha_2}}{m}\left[\beta_2 \pm (\beta_1 + t)\right].$$

It remains to solve for $\alpha_1, \alpha_2, \beta_1, \beta_2$. We know that $\alpha_1 = E$, the total energy, which is just the initial kinetic energy (as the object was launched from the ground), so $E = \alpha_1 = mv_0^2/2$. At $t = 0$, we have

$$x(0) = \pm\frac{\sqrt{2m\alpha_2}}{m}[\beta_2 \pm \beta_1] = y(0) = \frac{mv_0^2/2 - \alpha_2}{mg} - \frac{1}{2}g\beta_1^2 = 0.$$

So $\beta_2 = \mp\beta_1$. Since $x(t) > 0$, we may take the positive solution:

$$x(t) = \frac{\sqrt{2m\alpha_2}}{m}t \implies \dot{x}(0) = v_0\cos\theta = \frac{\sqrt{2m\alpha_2}}{m} \implies \alpha_2 = \frac{mv_0^2\cos^2\theta}{2}.$$

We then solve for β_1:

$$\frac{mv_0^2/2 - mv_0^2\cos^2\theta/2}{mg} = \frac{1}{2}g\beta_1^2 \implies \beta_1 = \pm\sqrt{\frac{v_0^2\sin^2\theta}{g^2}}.$$

In order for $\dot{y}(0) > 0$, $\beta_2 < 0$. Thus, we take the negative solution for β_2:

$$\dot{y}(0) = -g\beta_1 > 0 \implies \beta_1 < 0 \implies \beta_1 = -\frac{v_0 \sin\theta}{g}$$

Putting everything back we get the equations for $x(t)$ and $y(t)$:

$$\boxed{x(t) = v_0 t \cos\theta} \quad \checkmark \qquad \boxed{y(t) = \frac{v_0^2 \sin^2\theta}{2g} - \frac{1}{2}g\left(t - \frac{v_0 \sin\theta}{g}\right)^2 = v_0 t \sin\theta - \frac{1}{2}gt^2} \quad \checkmark$$

From here, it is easy to find the momentum coordinates:

$$\begin{aligned} p_x &= \frac{\partial S}{\partial x} = \sqrt{2m\alpha_2} = mv_0 \cos\theta \quad \checkmark \\ p_y &= \frac{\partial S}{\partial y} = \sqrt{2m(\alpha_1 - \alpha_2 - mgy)} = m(v_0 \sin\theta - gt) \quad \checkmark \end{aligned}$$

Finally, we want to solve for the equation of the trajectory, i.e. we want to solve for $y(x)$. To do this, we solve for t in terms of x:

$$t(x) = \frac{x}{v_0 \cos\theta}$$

and plug into the equation for $y(t)$:

$$\boxed{y(x) = \frac{v_0 \sin\theta x}{v_0 \cos\theta} - \frac{1}{2}g\left(\frac{x^2}{v_0^2 \cos\theta^2}\right) = x\tan\theta - \frac{x^2}{2g^2 v_0^2 \cos^2\theta}}$$