# Noether's theorem for a scaler field in flat space

Note: Einsten's implicit sum notation is used here.

**Def:** The smooth one-parameter subgroup of transformations

$$\tilde{x}^\mu = X^\mu_\varepsilon(x) \qquad \tilde{\phi}(\tilde{x}) = F_\varepsilon(\phi(x))$$

is an infinitesimal symmetry of the action $S[\phi] = \int_{U\subseteq\mathbb{R}^n} d^n x\, \mathcal{L}(\phi(x), \partial\phi(x), x)$

if $\delta S = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{S}[\tilde{\phi}] = 0$ for all $\phi(x)$ $\left( \tilde{S}[\tilde{\phi}] = \int_{X_\varepsilon(U)} d^n\tilde{x}\, \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x}), \tilde{x}) \right)$

derivatives w.r.t. $\tilde{x}$

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**Theorem:** if $\tilde{x}^\mu = X^\mu_\varepsilon(x)$, $\tilde{\phi}(\tilde{x}) = F_\varepsilon(\phi(x))$ is an infinitesimal symmetry

of $S[\phi] = \int_{U\subseteq\mathbb{R}^n} d^n x\, \mathcal{L}(\phi(x), \partial\phi(x), x)$ for all (nice) $U \subseteq \mathbb{R}^n$

integration over $U$ should be defined

then $\partial_\mu \left[ \mathcal{L}\,\delta x^\mu + \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} (\delta\phi - \phi_{,\nu}\,\delta x^\nu) \right] = 0$ when $\phi$ satisfies the

notation for $\partial_\mu\phi$

Euler-Lagrange eqs. $\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial\phi_{,\mu}} = 0.$ $\left( \delta x^\mu = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} X^\mu_\varepsilon(x)\ ,\ \delta\phi(\phi) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} F_\varepsilon(\phi) \right)$

conserved current

**Note:** in Minkowski space this is a continuity equation $\partial_\mu j^\mu = 0$

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**proof** Very similar to the particle case. Let's set $D_{\varepsilon=0} = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}$

Note that

$D_{\varepsilon=0}(A(\varepsilon)B(\varepsilon))$
$\quad \|$
$(D_{\varepsilon=0}A(\varepsilon))B(0) + A(0)(D_{\varepsilon=0}B(\varepsilon))$

By assumption $\delta S = D_{\varepsilon=0}\tilde{S}[\tilde{\phi}] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \tilde{S}[\tilde{\phi}] = 0.$

$$\tilde{S}[\tilde{\phi}] = \int_{X_\varepsilon(U)} d^n\tilde{x}\, \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x}), \tilde{x})$$

$$= \int_U d^n x\, |\det J_\varepsilon|\, \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x}), \tilde{x})$$

where $J_\varepsilon$ is the Jacobian matrix, $(J_\varepsilon)^\mu{}_\nu = \frac{\partial X^\mu_\varepsilon}{\partial x^\nu} = \frac{\partial\tilde{x}^\mu}{\partial x^\nu}$

Since everything is smooth, $(J_\varepsilon^{-1})^\mu{}_\nu = \frac{\partial x^\mu}{\partial\tilde{x}^\nu}$

Some information that we need: Let $A$ be an invertible matrix.

- $\frac{d}{d\varepsilon} \det(A) = \det(A)\, \mathrm{tr}\left(A^{-1} \frac{dA}{d\varepsilon}\right)$ (Jacobi's formula)
- $\frac{d}{d\varepsilon} A^{-1} = - A^{-1} \frac{dA}{d\varepsilon} A^{-1}$

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Now:

- $D_{\varepsilon=0} (J_\varepsilon)^\mu{}_\nu = D_{\varepsilon=0} \frac{\partial X^\mu_\varepsilon}{\partial x^\nu} = \partial_\nu \delta x^\mu$
- $D_{\varepsilon=0} \det J_\varepsilon = \det(\underbrace{J_0}_{\mathbb{1}})\, \mathrm{tr}(\underbrace{J_0^{-1}}_{\mathbb{1}} D_{\varepsilon=0} J_\varepsilon) = \mathrm{tr}(D_{\varepsilon=0} J_\varepsilon) = \partial_\mu \delta x^\mu$
- $D_{\varepsilon=0} |\det J_\varepsilon| = \frac{|\det J_0|}{\det J_0} D_{\varepsilon=0} \det J_\varepsilon = \partial_\mu \delta x^\mu$
- $D_{\varepsilon=0} J_\varepsilon^{-1} = - J_0^{-1} (D_{\varepsilon=0} J_\varepsilon) J_0^{-1} = - D_{\varepsilon=0} J_\varepsilon$
- $D_{\varepsilon=0} \tilde{\phi}(\tilde{x}) = D_{\varepsilon=0} F_\varepsilon(\phi(x)) = \delta\phi(\phi(x))$
- $\frac{\partial \tilde{\phi}}{\partial \tilde{x}^\mu} = \frac{\partial}{\partial \tilde{x}^\mu} F_\varepsilon(\phi(x)) = \frac{dF_\varepsilon}{d\phi}(\phi(x))\, \partial_\nu \phi(x) \underbrace{\frac{\partial x^\nu}{\partial \tilde{x}^\mu}}_{(J_\varepsilon^{-1})^\nu{}_\mu}$

$$\rightarrow D_{\varepsilon=0} \frac{\partial \tilde{\phi}}{\partial \tilde{x}^\mu} = \overbrace{\left(\frac{d}{d\phi}\delta\phi\right) \partial_\nu \phi}^{\partial_\nu \delta\phi} \underbrace{(J_0^{-1})^\nu{}_\mu}_{\delta^\nu_\mu} - \underbrace{\frac{dF_0}{d\phi}}_{\rightarrow 1}(\phi(x))\, \partial_\nu \phi\, \partial_\mu \delta x^\nu$$

$$= \partial_\mu \delta\phi - \phi_{,\nu} \partial_\mu \delta x^\nu$$

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$$0 = \delta S = \int_U d^n x \left\{ (D_{\varepsilon=0} |\det J_\varepsilon|)\, \overset{\varepsilon=0}{\mathcal{L}}(\phi(x), \partial\phi(x), x) + |\det J_0|\, D_{\varepsilon=0} \mathcal{L}(\tilde{\phi}(\tilde{x}), \tilde{\partial}\tilde{\phi}(\tilde{x}), \tilde{x}) \right\}$$

$$= \int_U d^n x \left\{ \mathcal{L}\, \partial_\mu \delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\partial_\mu \delta\phi - \phi_{,\nu} \partial_\mu \delta x^\nu) \right\}$$

$\rightarrow$ use this to check if $\delta S = 0$ for the given transformation!

Here is where we use the EL eqs.

- $\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \partial_\mu \delta\phi = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi\right) - \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\right) \delta\phi = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \delta\phi\right) - \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi$

$$\bullet\ \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \partial_\mu \delta x^\nu = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \delta x^\nu \right) - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \right) \delta x^\nu$$

$$= \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \delta x^\nu \right) - \left( \partial_\mu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \right) \phi_{,\nu} \delta x^\nu - \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\mu\nu} \delta x^\nu$$

$$= \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \delta x^\nu \right) - \frac{\partial \mathcal{L}}{\partial \phi} \phi_{,\mu} \delta x^\mu - \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} \phi_{,\nu\mu} \delta x^\mu}_{\text{swapped dummy indices}}$$

Putting everything together:

$$0 = \delta S = \int_U d^n x \left\{ \partial_\mu \left[ \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta \phi - \phi_{,\nu} \delta x^\nu) \right] + \mathcal{L} \partial_\mu \delta x^\mu + \underbrace{\frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \phi_{,\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} \phi_{,\nu\mu} \delta x^\mu}_{\delta x^\mu \partial_\mu \mathcal{L}(\phi(x), \partial\phi(x), x)} \right\}$$

$$= \int_U d^n x\, \partial_\mu \left[ \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta \phi - \phi_{,\nu} \delta x^\nu) \right]$$

<u>Since this holds for all $U$, it must be</u> $\partial_\mu \left[ \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\delta \phi - \phi_{,\nu} \delta x^\nu) \right] = 0$

<u>Example:</u> If $\mathcal{L}$ does not depend on $x$ then

$X^\mu_\varepsilon(x) = x^\mu + \varepsilon a^\mu$ $(a \in \mathbb{R}^n)$, $F_\varepsilon = id$ is a symmetry of $S[\phi]$ for every $U \subseteq \mathbb{R}^n$.

In fact $\delta x^\mu = a^\mu$ $\delta \phi = 0$

$$\delta S = \int_U d^n x \left[ \mathcal{L} \underbrace{\partial_\mu \delta x^\mu}_{=0} + \overset{=0}{\frac{\partial \mathcal{L}}{\partial x^\mu}} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi} \overset{=0}{\delta \phi} + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\overset{=0}{\partial_\mu \delta \phi} - \phi_{,\nu} \partial_\mu \delta x^\nu) \right] = 0$$

which means that

$$0 = \partial_\mu \left[ \mathcal{L} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} (\cancel{\delta \phi} - \phi_{,\nu} \delta x^\nu) \right]$$

$$= \partial_\mu \left[ a^\mu \mathcal{L} - a^\nu \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \right] = a^\nu \partial_\mu \left[ \delta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} \right]$$

since $a^\nu$ is arbitrary, this tells us that the energy-momentum tensor

$T^\mu{}_\nu = \frac{\partial \mathcal{L}}{\partial \phi_{,\mu}} \phi_{,\nu} - \delta^\mu{}_\nu \mathcal{L}$ satisfies $\partial_\mu T^\mu{}_\nu = 0$