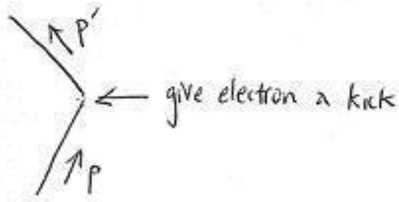
Chapter 6: Radiative corrections

Soft Brehmsstrahlung – low frequency radiation when electron undergoes sudden acceleration

Classical picture:

At $\vec{x}=0$ and time $t=0$ an electron is given a momentum kick



We will get the radiation from Maxwell's equation once we know the current density as a function of space + time.

For a particle at rest $j^\mu = \underset{\substack{\uparrow \\ \text{electric} \\ \text{charge}}}{e} \cdot (\text{particle density}, \vec{0})$

$$= (1,0,0,0)\cdot e\, \underset{\substack{\nwarrow \\ \text{for particle at origin}}}{\delta^{(3)}(\vec{x})}$$

(In classical picture we can specify x + p)

We can write this as

$$j^\mu(x) = \int dt' \; (1,0,0,0)^\mu \, e \, \delta^{(4)}(x - y(t'))$$

where $y^\mu(t') = (t', 0, 0, 0)$
is the worldline
of the particle

In the general case, for particle with worldline
$$y^\mu(\tau)$$

it is not difficult to guess (using Lorentz covariance as a guide)

$$j^\mu(x) = e \int d\tau \; \frac{dy^\mu(\tau)}{d\tau} \, \delta^{(4)}(x - y(\tau))$$

For a given time t, the $\delta(t - y(\tau))$ picks the τ such that $y^0(\tau) = t$. At that propertime we have $\delta^{(3)}(\vec{x} - \vec{y}(\tau))$ and the correct four-velocity $\frac{dy^\mu(\tau)}{d\tau}$.

We can check that $j^\mu(x)$ is a conserved current.
Let $f(x)$ be any function such that $f(x) \to 0$ as $x^\mu \to \infty$.

Then
$$\int d^4x \; f(x) \, \partial_\mu j^\mu(x) = \int d^4x \, f(x) \, e \int d\tau \, \frac{dy^\mu(\tau)}{d\tau} \, \partial^x_\mu \delta^{(4)}(x - y(\tau))$$

$$= -e \int d\tau \; \frac{dy^\mu(\tau)}{d\tau} \, \frac{\partial f(x)}{\partial x^\mu}\bigg|_{x = y(\tau)}$$

$$= -e \int d\tau \; \frac{d f(y^\mu(\tau))}{d\tau} = -e \cdot f(y^\mu(\tau))\bigg|_{\tau=-\infty}^{\tau=\infty} = 0$$

For our particle that gets a kick at $\vec{x}=0$, $\tau=0$, we patch together two worldlines ...

$$y^\mu(\tau) = \begin{cases} \frac{p^\mu}{m}\tau & \text{for } \tau<0 \quad \substack{\text{particle with}\\ \text{(momentum } p^\mu)} \\ \frac{p'^\mu}{m}\tau & \text{for } \tau>0 \quad \substack{\text{particle with}\\ \text{(momentum } p'^\mu)} \end{cases}$$

So then $$j^\mu(x) = e\int_0^\infty d\tau \frac{p'^\mu}{m} \delta^{(4)}(x-\tfrac{p'}{m}\tau) + e\int_{-\infty}^0 d\tau \frac{p^\mu}{m}\delta^{(4)}(x-\tfrac{p}{m}\tau)$$

The Fourier transform is

$$\tilde{j}^\mu(k) = \int d^4x\, e^{ik\cdot x} j^\mu(x)$$

$$= e\int_0^\infty d\tau\, \frac{p'^\mu}{m} e^{i(k\cdot\frac{p'}{m}+i\varepsilon)\tau} + e\int_{-\infty}^0 \frac{p^\mu}{m} e^{i(\frac{k\cdot p}{m}-i\varepsilon)\tau}$$

to help integrals converge at infinity

$$(\varepsilon\to 0^+) \qquad = ie\left(\frac{p'^\mu}{k\cdot p'+i\varepsilon} - \frac{p^\mu}{k\cdot p-i\varepsilon}\right)$$

From Maxwell's equations,

(in Lorentz gauge $\partial_\mu A^\mu = 0$)

we have $\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = j^\nu$

$$\Rightarrow \partial_\mu\partial^\mu A^\nu = j^\nu$$

So $$-k^2 \tilde{A}^\mu(k) = \tilde{j}^\mu(k)$$

and $$\tilde{A}^\mu(k) = -\frac{1}{k^2}\tilde{j}^\mu(k)$$

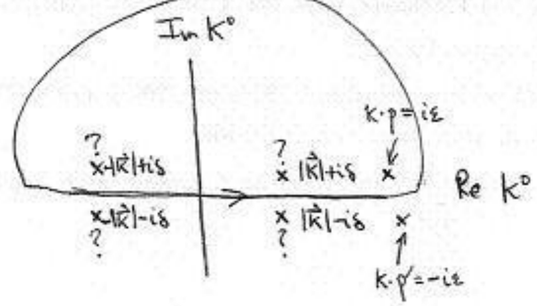
$$= -\frac{ie}{k^2}\left(\frac{p'^\mu}{k\cdot p' + i\varepsilon} - \frac{p^\mu}{k\cdot p - i\varepsilon}\right)$$

$$A^\mu(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ik\cdot x}\, \frac{(-i e)}{\underset{(k^0)^2-\vec{k}^2}{\overset{\shortparallel}{k^2}}} \left(\frac{p'^\mu}{k\cdot p' + i\varepsilon} - \frac{p^\mu}{k\cdot p - i\varepsilon} \right)$$

Let us now try to figure out from physical intuition how the poles from $\frac{1}{k^2}$ should be placed.

When $t = x^0 < 0$, the electron momentum is still p^μ and no kick has been delivered. So the term containing p'^μ cannot make any contribution to $A^\mu(x)$ for $x^0 < 0$.

When $x^0 < 0$ we continue the contour in the upper half plane for the k^0 integral.



We have the possibility of poles at $-|\vec{k}| \pm i\delta$ and $|\vec{k}| \pm i\delta$. But if we have a pole at $-|\vec{k}|+i\delta$ or $|\vec{k}|+i\delta$ we get a contribution that depends on p'^μ. So both poles must be in the lower half plane.

So for $x^0 < 0$, we get the residue at $k \cdot p = +i\varepsilon$,

$$A^\mu(x) = \int \frac{d^3\vec{k}}{(2\pi)^4} e^{i\vec{k}\cdot\vec{x}} e^{-i\frac{\vec{k}\cdot\vec{p}}{p^0}t} (2\pi i)\frac{ie}{k^2}\frac{p^\mu}{p^0} \quad \left(k^0 = \frac{\vec{k}\cdot\vec{p}}{p^0}\right)$$

This looks a bit unfamiliar. It may look more familiar in the rest frame when $p^0 = m$, $\vec{p} = 0$. Then

$$A^\mu(x) = e \int \frac{d^3\vec{k}}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \frac{(1,0,0,0)}{|\vec{k}|^2}$$

We have done this integral... it gives the Coulomb potential in the $\mu = 0$ component and zero for the other components.

What about after scattering at $x^0 = 0$? For $x^0 > 0$ we have three poles that produce residues. The residue at $k \cdot p' = -i\varepsilon$ is completely analogous to the integral we just did for $x^0 < 0$ and the $k \cdot p = +i\varepsilon$ pole. We get the field due to an electron with momentum p'^μ.

The interesting Brehmsstrahlung radiation comes from the other two poles ... at $k^0 = |\vec{k}| - i\delta$ and $k^0 = -|\vec{k}| - i\delta$.

Their residues give

$$A^\mu_{rad}(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \left[\overbrace{\frac{-e}{2|\vec{k}|} e^{+i\vec{k}\cdot\vec{x}} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\bigg|_{k^0=|\vec{k}|}}^{\text{residue at } k^0=|\vec{k}|-i\delta} e^{-i|\vec{k}|t} + \underbrace{\frac{e}{2|\vec{k}|} e^{+i\vec{k}\cdot\vec{x}} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\bigg|_{k^0=-|\vec{k}|}}_{\text{residue at } k^0=-|\vec{k}|-i\delta} e^{i|\vec{k}|t} \right]$$

We can write the second term as

$$\int \frac{d^3\vec{k}}{(2\pi)^3} \frac{-e}{2|\vec{k}|} e^{-i\vec{k}\cdot\vec{x}} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\bigg|_{k^0=|\vec{k}|} e^{i|\vec{k}|t}$$

by taking the dummy variable $\vec{k} \to -\vec{k}$. Note that this the complex conjugate of the first term.

So $$A^\mu_{rad}(x) = \mathrm{Re} \int \frac{d^3\vec{k}}{(2\pi)^3} \mathcal{A}^\mu(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, e^{-i|\vec{k}|t}$$

where $$\mathcal{A}^\mu(\vec{k}) = -\frac{e}{|\vec{k}|} \left(\frac{p'^\mu}{k\cdot p'} - \frac{p^\mu}{k\cdot p} \right)\bigg|_{k^0=|\vec{k}|}$$

By definition $$E^i(x) = -F^{0i} = -\partial_0 A^i - \partial_i A^0 = -\partial_0 \vec{A} - \vec{\nabla} A^0$$
$$B^i(x) = -\tfrac{1}{2}\varepsilon^{ijk} F^{jk} = \vec{\nabla} \times \vec{A}$$

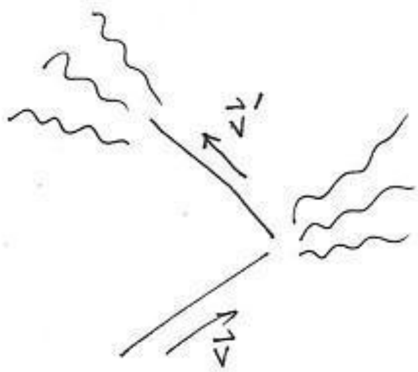
Let us choose a frame whue the electron initial and final energies are the same...

$$p^0 = p'^0 = E$$

Let $k^\mu = (|\vec{k}|, \vec{k})$, $p^\mu = (E, E\vec{v})$, $p'^\mu = (E, E\vec{v}')$

Then
$$\frac{1}{k\cdot p'} = \frac{1}{E|\vec{k}|\,(1-\hat{k}\cdot\vec{v}')}$$
$$\frac{1}{k\cdot p} = \frac{1}{E|\vec{k}|\,(1-\hat{k}\cdot\vec{v})}$$

Radiation is peaked when $\hat{k}$ points in the direction of $\vec{v}$ or $\vec{v}'$.



Note also that $k_\mu a^\mu = 0$.