CLASSICAL FIELD THEORY

A Quick Guide

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Preface

Greetings,

Classical Field Theory, A Quick Guide to is compiled based on my independent study PH492: Topics in Classical Field Theory notes with professor Robert Bluhm. Sean Carroll's *Spacetime and Geometry: An Introduction to General Relativity*, along with other resources, serves as the main guiding text.

This text is a continuation of *General Relativity and Cosmology, A Quick Guide to*. Familiarity with classical mechanics, linear algebra, vector calculus, and especially general relativity is expected. I will not be covering a review of general relativity, but instead will jump directly into an introduction to field theory and the Lagrangian formulation of general relativity and Einstein equations.

Enjoy!

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1 Introduction to Classical Field Theory

1.1 Overview of Lagrangian Formulation of Classical Mechanics

1.2 Lagrangian Formulation in Field Theory

Proposition 1.1. All fundamental physics obeys least action principles.

The action S is defined as

$$S = \int_a^b \mathcal{L}\, dt.$$

where $\mathcal{L}$ is called the Lagrangian.

Refer for Farlow's *Partial Differential Equation*, page 353, for detailed explanation of Lagrange's calculus of variations.

I will derive the Euler-Lagrange equation(s) here, but we are not going to use it in the following subsection for the introduction to field theory for now.

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0.$$

1.3 Field Theory: A Mechanical Example

In this subsection we take a look at how the Lagrangian formulation of classical mechanics can give rise to Newton's second law of motion. In mechanics, the Lagrangian often takes the form:

$$\mathcal{L} = K - V, \tag{1.1}$$

where K is the kinetic energy, and V is the potential energy. Let us consider a simple example where

$$K = \frac{1}{2} m\dot{x}^2$$
$$V = V(x).$$

Variations on the Lagrangian gives

$$\begin{aligned}
\delta\mathcal{L} &= \delta\left(\frac{1}{2}m\dot{x}^2 - V(x)\right) \\
&= m\dot{x}\delta\dot{x} - \frac{dV}{dx}\delta x \\
&= m\dot{x}\dot{\delta x} - \frac{dV}{dx}\delta x \\
&= m\left(-\ddot{x}\delta x + \frac{d}{dt}\dot{x}\delta x\right) - \frac{dV}{dx}\delta x \\
&= -m\ddot{x}\delta x - m\frac{d}{dt}\dot{x}\delta x - \frac{dV}{dx}\delta x.
\end{aligned}$$

It follows that the variations on the action gives

$$\delta S = \int_a^b \delta L\, dt = -\int_a^b \left(m\ddot{x} + \frac{dV}{dx}\right)\delta x\, dt.$$

The principle of least action requires $\delta S = 0$ for all δx. Therefore it follows that

$$m\ddot{x} + \frac{dV}{dx} = 0,$$

which is simply Newton's second law of motion in disguise.

Before we move on, we should note that in order for the Lagrangian formulation to work in electromagnetism or in general relativity, we need to promote the Lagrangian to its relativistic version where the Lagrangian is given by

$$L = \int_a^b \mathcal{L}\, d^3x.$$

$\mathcal{L}$ is called the Lagrangian density, but we can colloquially refer to it as "the Lagrangian." The relativistic action hence takes the form

$$S = \int \mathcal{L}\, d^4x,$$

where d^4x implies integrating over all spacetime.

1.4 Introduction to Fields

In field theory, most physical objects are described as "fields." Let us dive into the first two fields that we are more or less familiar with: scalar fields and vector fields.

1.4.1 Scalar Fields

A scalar field can be used to describe particles of spin 0. A scalar field has only one component, or one degree of freedom, making it the "simplest case" of the fields we will discuss. Let us now consider a moving field in one dimension, which has the form

$$\phi(s) \sim e^{-i\mathbf{k}\cdot\mathbf{x}},$$

where

$$\mathbf{k} = K^\mu = (K^0, \vec{K})$$
$$\mathbf{x} = X^\mu = (X^0, \vec{X}).$$

Remember that K^μ is the wavenumber vector, and X^μ is the position vector. Also recall that the metric is Minkowskian at this point of consideration (we are still in flat spacetime. General curved spacetime will come later):

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Doing the inner product of X^μ and K^μ gives

$$\phi(x) = e^{-iK^0 t + i\vec{k}\cdot\vec{x}}.$$

We shall choose "natural units" such that $\hbar = c = 1$. This gives

$$\phi(x) = e^{-i\omega t} e^{i\vec{k}\cdot\vec{x}}.$$

Now, particles obey the following Einstein mass-energy equivalence:

$$E^2 = m^2 + \vec{p}^2.$$

But because of our choice of units, $E = c\hbar K^0 = K^0$, and $\vec{p} = \hbar\vec{k} = \vec{k}$. This gives

$$\left(K^0\right)^2 - \vec{k}^2 = m^2$$
$$K^\mu K_\mu = m^2.$$

So, massive particles obey $K^\mu K_\mu = m^2$, while massless particles obey $K^\mu K_\mu = 0$.

Now, we might wonder how we know that the scalar field has the above form. The answer is derived from, you guessed it, the Lagrangian for a scalar field. Let us consider a single scalar field in classical mechanics where

$$\text{Kinetic energy: } K = \frac{1}{2}\dot{\phi}^2$$
$$\text{Gradient energy: } G = \frac{1}{2}\left(\nabla\phi\right)^2$$
$$\text{Potential energy: } P = V(\phi).$$

Note: I haven't found a satisfactory explanation to what a "gradient energy" is. I'll come back to this term later.

We currently have three terms, but we would like our Lagrangian density to have the form $\mathcal{L} = K - V$. So, let us combine the kinetic energy and gradient energy terms into one:

$$K' = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\nabla\phi\right)^2.$$

We shall verify that

$$K' = -\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\nabla\phi\right)^2.$$

This turns out to be quite straightforward:

$$\begin{aligned}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) &= \eta^{\mu\nu}\left(\partial_\mu\phi\right)\left(\partial_\nu\phi\right)\\ &= \left(\partial_0\phi\right)^2 - \left(\partial_j\phi\right)^2\\ &= \dot{\phi}^2 - \left(\nabla\phi\right)^2.\end{aligned}$$

So, a good choice of Lagrangian for our scalar field would be

$$\mathcal{L} \sim K' - V = -\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - V(\phi).$$

In order for the action to be extremized, i.e. $\delta S = 0$, we require that $\delta\mathcal{L} = 0$ for any $\delta\phi$. Varying $\mathcal{L}$ with respect to ϕ gives

$$\begin{aligned}\delta\mathcal{L} &= \delta\left(-\frac{1}{2}\left(\partial_\mu\phi\right)\left(\partial^\mu\phi\right) - V(\phi)\right)\\ &= -\frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu\phi\,\partial^\mu\delta\phi\right) - \frac{dV(\phi)}{d\phi}\delta\phi\\ &= -\partial_\mu\delta\phi\,\partial^\mu\phi - \frac{dV(\phi)}{d\phi}\delta\phi.\end{aligned}$$

Now, integration by parts tells us that

$$\partial_\mu\left(\partial^\mu\phi\,\delta\phi\right) = \partial^\mu\partial_\mu\phi\,\delta\phi + \partial_\mu\delta\phi\,\partial^\mu\phi.$$

So,

$$\partial_\mu\delta\phi\,\partial^\mu\phi = \partial_\mu\left(\partial^\mu\phi\,\delta\phi\right) - \partial^\mu\partial_\mu\phi\,\delta\phi.$$

Therefore, variations on $\mathcal{L}$ is:

$$\delta\mathcal{L} = -\left[\partial_\mu\left(\partial^\mu\phi\,\delta\phi\right) - \partial^\mu\partial_\mu\phi\,\delta\phi\right] - \frac{dV(\phi)}{d\phi}\delta\phi.$$

It follows that the action is

$$S = \int_a^b \delta\mathcal{L}\,d^4x = \int_a^b\left\{-\left[\partial_\mu\left(\partial^\mu\phi\,\delta\phi\right) - \partial^\mu\partial_\mu\phi\,\delta\phi\right] - \frac{dV(\phi)}{d\phi}\delta\phi\right\}\,d^4x.$$

The total derivative term $\partial_\mu \left(\partial^\mu \phi \, \delta\phi\right)$ vanishes as we require the variations $\delta\phi = 0$ at a and b. This leaves us with

$$S = \int_a^b \left\{ \partial^\mu \partial_\mu \phi - \frac{dV(\phi)}{d\phi} \right\} \delta\phi \, d^4x.$$

We require that this equality hold for any variation $\delta\phi$. So it must be true that

$$\partial^\mu \partial_\mu \phi - \frac{dV(\phi)}{d\phi} = 0.$$

We introduce a new operator, the **d'Alembertian**:

$$\Box \equiv \partial^\mu \partial_\mu \equiv \partial_\nu \partial^\mu \equiv \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2.$$

The requirement we just derived now becomes the **Klein-Gordon equation**:

$$\Box\phi - \frac{dV}{d\phi} = 0.$$

Remember that we are working with Lagrangian for a scalar field. It can easily be shown the connection between the Klein-Gordon equations and Newton's second law of motion, by separating the temporal and spatial derivatives from the d'Alembertian and rewriting a few things:

$$\Box\phi - \frac{dV}{d\phi} = \ddot{\phi} - \vec{\nabla}^2\phi - \frac{dV(\phi)}{d\phi} = 0.$$

We can see the time second derivative on the field ϕ and the ϕ-derivative on the potential field resemble "acceleration" and "force" in Newton's second law.

Let us return to our original question of why a scalar field has the form $\phi \sim e^{-i\mathbf{k}\cdot\mathbf{x}}$. From our derivation of the Klein-Gordon equation, we observe that a scalar field ϕ must be a solution to the Klein-Gordon equation. Now, we verify that

$$\phi = e^{-i\mathbf{k}\cdot\mathbf{x}}$$

is a solution to the KG equation. We simply unpack the d'Alembertian and attack the derivatives step-by-step. The first derivative is

$$\begin{aligned}
\partial_\mu \phi &= \partial_\mu \left(e^{-i\mathbf{k}\cdot\mathbf{x}} \right) \\
&= -i\partial_\mu \left(\mathbf{k}\cdot\mathbf{x}\right) e^{-i\mathbf{k}\cdot\mathbf{x}} \\
&= -i\partial_\mu \left(K_\nu X^\nu\right) e^{-iK_\alpha X^\alpha} \\
&= -iK_\nu \, \partial_\mu X^\nu \, \phi \\
&= -iK_\nu \delta^\nu_\mu \phi \\
&= -iK_\mu \phi
\end{aligned}$$

Next, we attack the second derivative:

$$\begin{aligned}\partial^\mu \partial_\mu \phi &= \eta^{\mu\nu}\partial_\nu \partial_\mu \phi \\ &= \eta^{\mu\nu}\partial_\nu \left(-iK_\mu \phi\right) \\ &= -iK_\mu \eta^{\mu\nu}\left(-iK_\nu \phi\right) \\ &= (-i)^2 K^\mu K_\mu \phi.\end{aligned}$$

If $K^\mu K_\mu = m^2$ (as we have shown before), then

$$\Box\phi + m^2\phi = (-m^2 + m^2)\phi = 0,$$

which satisfies the Klein-Gordon equation. So, as long as $K^\mu K_\mu = m^2$ is satisfied, ϕ of the given form is a solution to the KG equation and is a legitimate scalar field.

1.4.2 Vector Fields: An Electromagnetic Example

Vector fields describe particles of spin 1 such as photons. Unlike scalar fields ϕ where there is only one degree of freedom, a vector field is represented by A_μ with $\mu = 0, 1, 2, 3$, hence having 4 degrees of freedom. Electromagnetism is a field theory where the relevant field is a vector field, A_μ, called the vector potential.

$$A_\mu = (A_0, \vec{A}).$$

The first component of the vector potential, A_0 is the electrostatic potential V where $\vec{E} = -\vec{\nabla}V$. The other spatial components of A_μ, forming $\vec{A}$, form the vector potential from which the magnetic field and full electric field is derived:

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times \vec{A} \\ \vec{E} &= -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\end{aligned}$$

Let us consider the (cleverly chosen) Lagrangian density for electromagnetism:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu,$$

where $j^\mu = (\rho, \vec{J})$ is a combination of the charge density and current density. The electromagnetic field strength tensor is given by:

$$\begin{aligned}F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\ &= \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & -B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & B^2 & B^1 & 0 \end{pmatrix}.\end{aligned}$$

With this definition, we can also have an equivalent definition:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Recall the cyclic identity (this can be readily verified - we in fact have covered this in the GR notes):

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0.$$

We can easily show that this identity yields two of four Maxwell's equations:

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0. \end{aligned}$$

The remaining Maxwell equations come from varying the action and minimizing the action: $\delta S = 0$ with respect to the vector potential A_μ. Similar to what we have done before, we want to vary the Lagrangian. Now, the E&M Lagrangian has two terms. The term involving the vector potential is simple:

$$\delta \left(j^\mu A_\mu \right) = j^\mu \, \delta A_\mu$$

true for all δA_μ, so if the field strength tensor is zero, then $j^\mu = 0$. The term involving the field strength tensor is a little more complicated, but certainly doable:

$$\begin{aligned} \delta \left(\frac{-1}{4} F^{\mu\nu} F_{\mu\nu} \right) &= \frac{-1}{4} \, \delta \left[\left(\partial^\mu A^\nu - \partial^\nu A^\nu \right) \left(\partial_\mu A_\nu - \partial_\nu A_\mu \right) \right] \\ &= \frac{-1}{2} \, \delta \left(\partial^\mu A^\nu \, \partial_\mu A_\nu - \partial_\mu A_\nu \, \partial^\mu A^\nu \right) \\ &= \frac{-1}{2} \left(\partial^\mu \, \delta A^\nu \, \partial_\mu A_\nu + \partial^\mu A^\nu \, \partial_\mu \, \delta A_\nu - \partial_\mu \, \delta A_\nu \, \partial^\mu A^\nu - \partial_\mu A_\nu \, \partial^\mu \, \delta A^\nu \right). \end{aligned}$$

Raising and lowering indices gives

$$\begin{aligned} \delta \left(\frac{-1}{4} F^{\mu\nu} F_{\mu\nu} \right) &= \frac{-1}{2} \left(\partial^\mu \, \delta A^\nu \, \partial_\mu A_\nu + \partial_\nu A_\mu \, \partial^\nu \, \delta A^\mu - \partial_\mu \, \delta A_\nu \, \partial^\mu A^\nu - \partial^\nu A^\mu \, \partial_\nu \, \delta A_\mu \right) \\ &= \partial^\nu \, \delta A^\mu \, \partial_\nu A_\mu - \partial_\mu \, \delta A_\nu \, \partial^\mu A^\nu. \end{aligned}$$

We can again integrate by parts on the two terms similar to the following steps

$$\begin{aligned} \partial^\nu \, \delta A^\mu \, \partial_\nu A_\mu &= \partial^\nu (\partial_\mu A_\nu \, \delta A^\mu) - (\partial^\nu \partial_\mu A_\nu) \delta A^\mu = \partial_\mu (\partial^\nu A^\mu \, \delta A_\nu) - (\partial^\nu \partial_\mu A_\nu) \delta A^\mu \\ \partial_\mu \, \delta A_\nu \partial^\mu A^\nu &= \partial_\mu (\partial^\mu A^\nu \, \delta A_\nu) - \partial_\mu (\partial^\mu A^\nu) \, \delta A_\nu = \partial_\mu (\partial^\mu A^\nu \, \delta A_\nu) - \partial_\mu (\partial^\mu A^\nu) \, \delta A_\nu. \end{aligned}$$

and eliminate the total derivative from the action integral. Assuming that the term with the current density and vector potential is zero, we are eventually (after lowering/raising the indices correctly, of course) left with the requirement

$$\partial_\mu (\partial^\mu A^\nu) \, \delta A_\nu - (\partial^\nu \partial_\mu A_\nu) \delta A^\mu \equiv \left(\Box A^\mu - \partial^\mu \partial^\nu A_\nu \right) \, \delta A_\mu = 0$$

for all δA_μ, which forces the following identity:

$$\Box A^\mu - \partial^\nu \partial^\mu A_\nu = \Box A^\mu - \partial_\nu \partial^\mu A^\nu = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = \partial_\nu F^{\mu\nu} = 0.$$

Now, with the current density and vector potential terms, we get the requirement

$$\partial_\nu F^{\mu\nu} = j^\mu.$$

This identity gives the remaining two Maxwell's equations.

We can look at photons as an example. Photons do not carry a current/charge, so $j^\mu = 0$. Therefore the equation of motion can be derived from just

$$\partial_\nu F^{\mu\nu} = 0.$$

Now, we have an interesting problem to think about: We know that photons can have 2 independent transverse polarizations, i.e. there are 2 massless modes for photons. However, A^μ has 4 degrees of freedom, not 2. So why does our theory require more than 2 degrees of freedom to describe a physical quantity that only has 2 degrees of freedom? The answer to this is that there are 2 degrees of freedom in A_μ that don't matter. The first is the A_0 factor - the electrostatic potential. Why A_0 does not matter in describing photons can be illustrated if we look at the case where $\mu = 0$:

$$\begin{aligned}\Box A_0 - \partial_0\, \partial^\nu A_\nu &= \partial^0 \partial_0 A_0 + \partial^j \partial_j A_0 - \partial_0 \partial^0 A_0 - \partial_0 \partial^j A_j \\ &= \partial^j \partial_j A_0 - \partial_0 \partial^j A_j \\ &= 0.\end{aligned}$$

We see that A_0 is not a propagating mode, or the **ghost** mode, or the **auxiliary** mode. This is actually a good thing in our theory. In fact, the Lagrangian is actually chosen such that the time second derivative vanishes.

Now that we have sort of explained why one degree of freedom of A_μ does not matter. What about the other one that shouldn't matter? The short answer to this is the keyword **gauge symmetry** in the theory. If we look back at how the field strength tensor is defined:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_n u A_\mu,$$

and attempt to transform (**gauge transform**)

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \Lambda(x),$$

then we observe that

$$\begin{aligned}F'_{\mu\nu} &= \partial_\mu(A_\nu + \partial_\nu \Lambda(x)) - \partial_\nu(A_\mu + \partial_\mu \Lambda(x)) \\ &= \partial_\mu A_\nu - \partial_n u A_\mu \\ &= F^{\mu\nu},\end{aligned}$$

i.e. there is a way to choose $\Lambda(x)$ such that we eliminate one A_μ mode, leaving just 4-1-1=2 modes.

Note: This is a very hand-wavy-type of explanation. For a better, more detailed, and more satisfactory explanation, please refer to Sean Carroll's book, p. 40. I will come back to this section later when my understanding of classical field theory develops, and hopefully provide a better explanation here.