review lecture notes
(you should read egs. we covered) 1

Midterm Review 8.09

Hamilton's Principle

$$\delta \int_1^2 dt\ L(q, \dot{q}, t) = 0$$

$$L = T - V$$

Kinetic energy (T); ptnl energy (V) (~~forces from~~)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

(no constraints yet q_j indep.)

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

D'Alembert's Principle

constraint forces do no virtual work

$$\sum_i (\dot{\vec{p}}_i - \vec{F}_i)\cdot \delta\vec{r}_i = 0$$

$\delta\vec{r}_i$: virtual displacements [inf, satisfy constraints, fixed t]

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = R_j$$

R_j: generalized forces not from V

eg. friction force $\vec{f}_i = -h_i(v_i)\frac{\vec{v}_i}{v_i}$, $\vec{v}_i = \dot{\vec{r}}_i$

$$R_j = -\frac{\partial F}{\partial \dot{q}_j}, \quad F = \sum_i \int_0^{v_i} dv_i'\ h_i(v_i')$$

L with holonomic Constraints

$$f_\alpha(q_1, .., q_n, t) = 0$$

- If we don't care about constraint forces use K eqtns $f_\alpha = 0$ to find $n-k$ indep coords q_j
  $$d/dt\ \partial L/\partial \dot{q}_j - \partial L/\partial q_j = 0 \qquad n-k \text{ eqtns}$$
- If we do care, then use Lagrange mult. λ_α for those constraints

$$\left.\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} &= \underbrace{\sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j}}_{\text{Generalized Forces of Constraint}} \\ f_\alpha(q, t) &= 0 \end{aligned}\right\} \quad n+k \text{ eqtns for } n+k \text{ vars}$$

L with Semi-holonomic Constraints

$$g_\beta = \sum_j a_{\beta j}(q,t)\,\dot{q}_j + a_{\beta t}(q,t) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot{q}_j}$$

eg: constraints to surfaces

eg: rolling constraints

Hamilton Eqtns

$$H(q,p,t) = \dot{q}_i p_i - L(q,\dot{q},t)$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

[2n 1st order eqtns]

[q, p equal footing]

Transformations

- Any Point Trnsfm $Q_i = Q_i(q,t)$ gives $L'(Q,\dot{Q},t) = L(q,\dot{q},t)$
  with $\frac{d}{dt}\frac{\partial L'}{\partial \dot{Q}_i} - \frac{\partial L'}{\partial Q_i} = 0$ (same form)

- Transformation is canonical $Q_i = Q_i(q,p,t)$, $P_i = P_i(q,p,t)$
  if there is a $K(Q,P,t)$ so that

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

call (q_i, p_i), (Q_i, P_i) canonical coordinates

Cyclic Coords

no q_1 in L or H $\Rightarrow$ $\dot{p}_1 = 0$, p_1 = constant

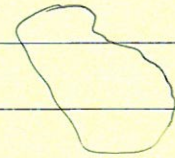
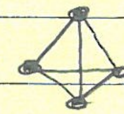
eg. no force on CM coord $\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i = \frac{1}{M}\int dV \rho \vec{r}$

conj. mom. $\vec{P}$ = const.

eg. no ϕ dep., p_ϕ = ang. mom = const.

eg. no t dep. $\frac{dH}{dt} = \frac{\partial H}{\partial t} = 0$, H = const

may or may not have $H = E = T + V$

Rigid Bodies continuous  discrete 

6 coords: 3 translations

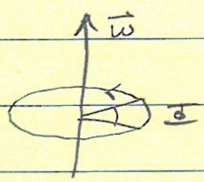
3 rotations (ϕ, θ, ψ) Euler Angles

↑ about fixed pt

$$\vec{r}\,' = U_\psi U_\theta U_\phi \vec{r} = U\vec{r}$$

$$U_\phi = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad U^T U = \mathbb{1} \quad \text{orthogonal}$$

General Displacement with one point fixed is instantaneous rotation about some axis



defines angular velocity $\vec{\omega}(t)$

$$|\vec{\omega}| = \frac{d\Phi}{dt}$$

Inertial & Body Axes

$$\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{body} + \vec{\omega}\times$$

$$\left(\frac{d\vec{r}}{dt}\right)_{inertial} = \left(\frac{d\vec{r}}{dt}\right)_{body} + \vec{\omega}\times\vec{r}\,,$$

2nd $\frac{d}{dt}$ derivative gave Coriolis, Centrifugal Fictitious Forces

$$T = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2}\vec{\omega}^T \cdot \hat{I}^{(cm)} \cdot \vec{\omega}$$ for rotation about cm point

$$\vec{L} = \hat{I}\cdot\vec{\omega}\,, \qquad T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$$ for rotation about any fixed point

$$\hat{I}_{ab} = \sum_i m_i \left(\delta_{ab}\vec{r}_i^{\,2} - r_i^a r_i^b\right)$$

or

$$\hat{I}_{ab} = \int_{Vol} dV\, \rho(\vec{r}) \left[\delta_{ab}\vec{r}^{\,2} - r_a r_b\right]$$

} depend on fixed "origin" & axes choice

Translate Origin : $\hat{I}_{ab}^{(Q)} = \hat{I}_{ab}^{(cm)} + M\left(\delta^{ab}\vec{R}^2 - R^a R^b\right)$

Rotate Axes : $\hat{I}' = U \hat{I} U^T$

<u>Principal Axes</u> [given $\hat{I}$ for some origin & axes]

$\hat{I}\vec{\xi} = \lambda\vec{\xi}$ $\qquad \det(\hat{I} - \lambda \mathbb{1}) = 0$ $\qquad$ 3 λ's

3 $\vec{\xi}$'s

3 $\vec{\xi}$'s give principal axes , λ's are principal moments of inertia I_1, I_2, I_3

For these axes [same origin] $\quad \hat{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$

<u>Euler Equations</u> use principal body axes to describe rotation of rigid body about fixed pt.

$I_1 \dot{\omega}_1 - (I_2 - I_3)\,\omega_2 \omega_3 = \tau_1$

etc

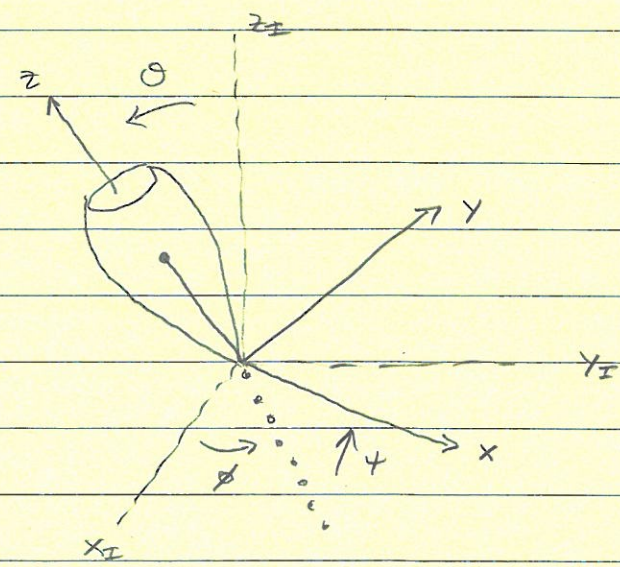
components in body frame

eg. Symmetric Top

$\dot{\theta}$ nutation

$\dot{\phi}$ precession

$\dot{\psi}$ rotation about z



<u>Vibrations</u> expand about min. of ptnl.

$$q_i = q_{0i} + \eta_i$$

$$V = V(q_0) + \frac{1}{2} V_{ij}(q_0)\,\eta_i \eta_j + \cdots$$

$$T = \frac{1}{2} T_{ij}(q_0)\,\dot\eta_i \dot\eta_j + \cdots$$

$$L = \frac{1}{2} \dot{\vec\eta}^T \hat T \dot{\vec\eta} - \frac{1}{2} \vec\eta^T \hat V \vec\eta$$

Normal Modes: $\vec\eta^{(k)} = \vec a^{(k)} e^{-i\omega^{(k)} t}$

$$\hat V \vec a = \lambda \hat T \vec a$$

$\det(\hat V - \lambda \hat T) = 0$ gives $\lambda^{(k)} = [\omega^{(k)}]^2$'s

$(\hat V - (\omega^{(k)})^2 \hat T) \cdot \vec a^{(k)} = 0$ gives $\vec a^{(k)}$'s

norm choice $\vec a^{(l)T} \hat T \vec a^{(k)} = \delta^{lk}$

General Solution is superposition $\vec\eta = \mathrm{Re} \sum_k \underbrace{C_k}_{\text{complex}} \vec\eta^{(k)}$

Normal Coords $\vec\eta = A \vec\xi$, $A = \begin{pmatrix} \uparrow & & \uparrow \\ \vec a^{(1)} & \cdots & \vec a^{(n)} \\ \downarrow & & \downarrow \end{pmatrix}$

$$L = \frac{1}{2} \sum_i \dot\xi_i^2 - \frac{1}{2} \sum_i (\omega^{(i)})^2 \xi_i^2$$

$$\ddot\xi_i + (\omega^{(i)})^2 \xi_i = 0$$

<u>Generating Functions for Canonical Transfm</u>

$Q_i = Q_i(q,p,t)$
$P_i = P_i(q,p,t)$ canonical,

consider inversions

$\begin{cases} q_i = q_i(Q,P,t) \\ p_i = p_i(Q,P,t) \end{cases}$

or $\begin{cases} p_i = p_i(q,Q,t) \\ P_i = P_i(q,Q,t) \end{cases}$ etc

$$p_i \dot q_i - H = P_i \dot Q_i - K + \frac{dF}{dt}$$

$F = F_1(q, Q, t)$ generates canonical transfm through

$$p_i = \frac{\partial F_1}{\partial q_i} = p_i(q, Q, t)$$

$$K = H + \frac{\partial F_1}{\partial t}$$

$$P_i = -\frac{\partial F_1}{\partial Q_i} = P_i(q, Q, t)$$

$F = F_2(q, P, t) - Q_i P_i$ gives $p_i = \frac{\partial F_2}{\partial q_i}$, $Q_i = \frac{\partial F_2}{\partial P_i}$

$$K = H + \partial F_2/\partial t$$

Poisson Brackets

$$[u, v]_{q,p} = \sum_i \left(\frac{\partial u}{\partial q_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial q_i} \right)$$

- Fundamental Brackets $\quad [q_j, q_k]_{q,p} = 0 = [p_j, p_k]_{q,p}$

  $$[q_j, p_k]_{q,p} = \delta_{jk}$$

- transfm $Q = Q(q, p, t)$, $P = P(q, p, t)$ canonical iff

  $$[Q_j, Q_k]_{q,p} = 0$$
  $$[P_j, P_k]_{q,p} = 0$$
  $$[Q_j, P_k]_{q,p} = \delta_{jk}$$

- for canonical vars poisson brackets are same

  $$[u, v]_{q,p} = [u, v]_{Q,P}$$

- Eqtns of Motion $\quad \frac{du}{dt} = [u, H]_{q,p} + \frac{\partial u}{\partial t}$ any $u(q, p, t)$

  H for p, q

- Conserved $\quad \frac{du}{dt} = 0 \iff [u, H]_{q,p} + \frac{\partial u}{\partial t} = 0$

eg. Infinitesimal Canonical Transfm

Hamilton - Jacobi

Hamilton's Principal function $S(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n, t)$

gen. function which transforms to

$$K = 0 = H\left(q_1, \ldots, q_n, \frac{\partial S}{\partial q_1}, \ldots, \frac{\partial S}{\partial q_n}, t\right) + \frac{\partial S}{\partial t} = 0$$

$\dot{P}_i = 0 \;,\; \dot{Q}_i = 0$ ← time-dependent H-J eqtn

$P_i = \alpha_i = \text{constant}$

$$\left.\begin{aligned} p_i &= \frac{\partial S}{\partial q_i} \\ Q_i &= \beta_i = \partial S/\partial \alpha_i = \text{constant} \end{aligned}\right\} \text{solve to get} \quad \begin{aligned} q_j &= q_j(\alpha, \beta, t) \\ p_j &= p_j(\alpha, \beta, t) \end{aligned}$$

Hamilton's Characteristic function $W(q_1, \ldots, q_n, \alpha_1, \ldots, \alpha_n)$

$$H\left(q_1, \ldots, q_n, \frac{\partial W}{\partial q_1}, \ldots, \frac{\partial W}{\partial q_n}\right) = \alpha_1$$

H-J eqtn time indep.

$P_i = \alpha_i$

$p_i = \dfrac{\partial W}{\partial q_i}$ W is generating function

$K = \alpha_1$

$\beta_1 = \dfrac{\partial W}{\partial \alpha_1} - t$ $\quad \dot{Q}_1 = \dfrac{\partial K}{\partial \alpha_1} = 1 \;,\; Q_1 = \beta_1 + t$

$\beta_{i>1} = \dfrac{\partial W}{\partial \alpha_i}$ $\quad \dot{Q}_{i>1} = 0 \quad Q_{i>1} = \beta_i$

Solve by Separating Variables

$$W = W_1(q_1, \alpha) + W_2(q_2, \alpha) + \cdots + W_n(q_n, \alpha)$$

Cyclic Coords: q_1 cyclic $\Rightarrow$ $W_1(q_1, \alpha) = \underbrace{\gamma}\, q_1$

$p_1 = \gamma = \text{constant}$

$(= \alpha_2 \text{ say})$

eg. Harmonic Osc.

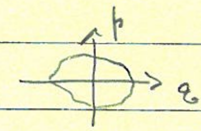
eg. Kepler Problem $\qquad \alpha_1 = E \quad , \quad \alpha_\theta = l = |\vec{L}|$

$\alpha_\phi = (\vec{L} \text{ about } \hat{z})$

Action-Angle Variables

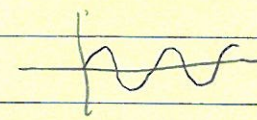
1-dim $\qquad H(q,p) = \alpha_1 \quad \Rightarrow \quad p = p(q, \alpha_1)$

Oscillation $\qquad p, q$ periodic



Rotation $\qquad p$ periodic

q unbounded



Action Variable $\qquad J = \oint p\, dq = J(\alpha) = \text{const.}$

over one cycle

consider $\quad W = W(q, J)$

Angle Variable $\qquad w = \dfrac{\partial W}{\partial J} \quad , \quad \dot{w} = \dfrac{\partial H(J)}{\partial J} = \nu$

= constant

= frequency

$\nu = \dfrac{1}{\tau} \quad , \quad \tau$ period

Many Vars $\qquad (q_i, p_i)$ oscillate or rotate, completely separable

$$J_i \equiv \oint p_i\, dq_i = J_i(\alpha)$$

$$w_i = \frac{\partial W}{\partial J_i} = w_i(q_1, \ldots, q_n, \alpha_1, \ldots \alpha_n)$$

$$\dot{w}_i = \frac{\partial H}{\partial J_i} = \nu_i(J_1, \ldots, J_n) \qquad \text{frequencies}$$

eg. Kepler $\qquad E = \dfrac{-2\pi^2 k^2 m}{(J_r + J_\theta + J_\phi)^2} \qquad$ so $\quad \nu_\theta = \nu_\phi = \nu_r$