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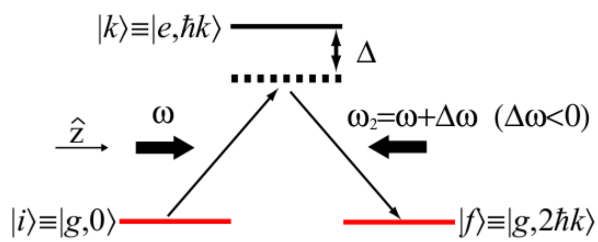
Course: **8.421 - AMO I**
Problem set: **#12**
Due: Monday, May 9, 2022.

I'd like to apologize to the grader in advance. This is possibly the most hand-wavy and *sus* pset I've done so far in AMO 1. I'm unsure on some of my answers while some are definitely wrong. Also, some of the "correct" answers are not obtained in a rigorous way, which bothers me quite a lot. I think this is a combination of the instructions not being clear, me not understanding what the instructions are getting at, and the fact that I have other things to take care of as this is the end of the semester. In any case, I tried my best to give sensible responses.

1. Bragg Scattering. Consider he energy level diagram below where the states of the atom can be written as $|\text{internal, external}\rangle = |\text{internal}\rangle \otimes |\text{external}\rangle$. Here the internal state is either $|g\rangle$ or $|e\rangle$, and the momentum state is $|\boldsymbol{p}\rangle$. If two counter-propagating lasers are tuned as indicated, recoil momentum will be transferred to the atoms by redistributing photons between the beams. We want to look at this "Bragg scattering" in two ways:

- by describing it as a two-photon stimulated Raman process
- by considering the mechanical effect of the AC Stark shift potential seen by an atom



1. **Two-Photon Stimulated Raman Process.**
   (a) By conservation of energy, we have
   $$\hbar\omega = \frac{(2\hbar k)^2}{2m} + \hbar(\omega + \Delta\omega),$$
   by comparing the energy of $|i\rangle$ and $|f\rangle$ to $|k\rangle$. From here, we find
   $$\Delta\omega = -4\frac{\hbar k^2}{2m} = -4\omega_r$$
   where $\omega_r = \hbar k^2/2m$ is the recoil frequency.
   (b) Assume that the beams are counter-propagating along the z-axis and have the same polarization (which we may assume to be x), then
   $$\begin{aligned} E_1 &= E_0\cos(kz - \omega_1 t) \\ E_2 &= E_0\cos(-kz - (\omega_1 + \Delta\omega)t). \end{aligned}$$

Since the electric fields E_j only couple the state $\left|j\right\rangle$ to the intermediate state $\left|k\right\rangle$[1], the interaction Hamiltonian is

$$\begin{aligned}\mathcal{H}_{\text{int}} = &-\frac{1}{2}\left\langle g\right|\hat{e}\cdot\vec{d}\left|e\right\rangle E_0\left(\left\langle 0\right|e^{ikz}\left|\hbar k\right\rangle e^{-i\omega_1 t}\left|g\right\rangle\left\langle k\right| + h.c.\right)\\ &-\frac{1}{2}\left\langle e\right|\hat{e}\cdot\vec{d}\left|g\right\rangle\left(\left\langle \hbar k\right|e^{-ikz}\left|2\hbar k\right\rangle e^{-i(\omega_1+\Delta\omega)t}\left|g\right\rangle\left\langle k\right| + h.c.\right).\end{aligned}$$

Define two relevant Rabi frequencies:

$$\begin{aligned}\Omega_1 &= \frac{E_0}{\hbar}\left\langle g\right|\hat{e}\cdot\vec{d}\left|e\right\rangle\left\langle 0\right|e^{ikz}\left|\hbar k\right\rangle\\ \Omega_2 &= \frac{E_0}{\hbar}\left\langle e\right|\hat{e}\cdot\vec{d}\left|g\right\rangle\left\langle \hbar k\right|e^{-ikz}\left|2\hbar k\right\rangle.\end{aligned}$$

I thiiiink this is correct?

(c) The wavefunction for a particle with definite momentum $p = \hbar k$ in a 1D box of length L in position space is

$$\psi(z) = \frac{1}{\sqrt{L}}e^{ipz/\hbar}e^{iEt/\hbar} = \frac{1}{\sqrt{L}}e^{ipz/\hbar}e^{i(p^2/2m)t/\hbar} = \frac{1}{\sqrt{L}}e^{ikz}e^{i\hbar k^2 t/2m}.$$

(d) Here we calculate the two-photon Rabi frequency for the Raman process shown in the figure. Assuming that $\left|i\right\rangle, \left|k\right\rangle, \left|f\right\rangle$ gave external wavefunctions of the form we wrote down the with the appropriate momenta. The *full* Rabi frequency for $\vec{E}_1$ is

$$\begin{aligned}\Omega_1 &= \frac{E_0 D_{ge}}{\hbar}\left[\frac{e^{i\hbar k^2 t/2m}(e^{2ikL}-1)}{kL}\right].\\ \Omega_2 &= \frac{E_0 D_{eg}}{\hbar}e^{3i\hbar k^2 t/2m}.\end{aligned}$$

The two-photon Rabi frequency is therefore

$$\Omega_{R2} = \frac{\Omega_1^*\Omega_2}{2\Delta} = \frac{iE_0^2 D_{eg}^2(e^{2ikL}-1)}{4\hbar^2\Delta kL}e^{ik(-2Lm+\hbar kt)/m}.$$

I don't think any of these calculations are correct. Does the Rabi frequency include the external part or not? The papers I've read that describe Bragg scattering/stimulated Raman processes don't seem to include the external part into the Rabi frequency.

(e) If $\mathcal{H}'$ is the perturbation due to E_1, E_2, and if we treat the system as an effective two-level system, then we can simply identify $\hbar\left\langle i\right|\mathcal{H}'\left|f\right\rangle$ with the two-photon Rabi frequency $\Omega_{2\text{R}}$ found above. There might a factor of 4 missing or something. I'm also not sure what I'm doing here is correct.

2. **AC Stark Shift**

(a) Here we calculate the AC Stark shift $U(z,t)$ of an atom in the ground state $\left|g\right\rangle$ due to the total electric field $E_1 + E_2$. By (my potentially faulty) intuition, the Stark shift of $\left|g\right\rangle$ due to the two electric fields is the sum of the individual AC Stark shifts:

$$U(z,t) = -\frac{E_0^2 D_{eg}^2}{4\hbar^2\Delta}\left(\cos^2(kz) + \cos^2(-kz)\right) = -\frac{E_0^2 D_{eg}^2}{2\hbar^2\Delta}\cos^2(kz)$$

(b) From this we can calculate the coupling $\left|i\right\rangle U(z,t)\left|f\right\rangle$ due to the mechanical potential presented by the AC Stark shift. For this we only focus on the external part of the wavefunction:

[1] I just realized that the figure uses k for both the internal state and momentum... which could be little confusing

2. Spontaneous Two-Photon Emission

1. **The Göppert-Mayer formula.** From second order perturbation theory, the amplitude for state $|b\rangle$ after excitation for a time t is:

$$a_b^{[2]} = \frac{1}{4\hbar^2} \sum_f \left\{ \frac{H_{bf,2} H_{fa,1}}{\omega_1 - \omega_{fa}} \frac{e^{i(\omega_{ba}-\omega_1-\omega_2)} - 1}{\omega_{ba} - \omega_1 - \omega_2} + \frac{H_{bf,1} H_{fa,2}}{\omega_2 - \omega_{fa}} \frac{e^{i(\omega_{ba}-\omega_1-\omega_2)} - 1}{\omega_{ba} - \omega_1 - \omega_2} \right\}.$$

(i) The excitation rate $\Gamma_{ab}(\omega_1)$ when one beam as frequency ω_1 can be calculated using the Fermi's golden rule

$$\begin{aligned}
\Gamma_{ab}(\omega_1) &= \frac{2\pi}{\hbar} \left| a_b^{[2]} \right|^2 \delta(\omega_{ba} - (\omega_1 + \omega_2)) \\
&= \frac{\pi E_1^2 E_2^2 e^4}{8\hbar^4} \delta(\omega_{ba} - (\omega_1 + \omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a| \boldsymbol{r} \cdot \boldsymbol{e}_1 \left| f \right\rangle \left\langle f \right| \boldsymbol{r} \cdot \boldsymbol{e}_2 |b\rangle}{\omega_1 - \omega_{fa}} + \frac{\langle a| \boldsymbol{r} \cdot \boldsymbol{e}_2 \left| f \right\rangle \left\langle f \right| \boldsymbol{r} \cdot \boldsymbol{e}_1 |b\rangle}{\omega_2 - \omega_{fa}} \right\} \right|^2.
\end{aligned}$$

(ii) Since we have the constraint:

$$\omega_{ba} = \omega_b - \omega_a = \omega_1 + \omega_2,$$

we must have that

$$\omega_1 - \omega_{fa} = \omega_1 - \omega_f + \omega_a = \omega_b - \omega_2 - \omega_f = -(\omega_2 + \omega_{fb})$$

and similarly

$$\omega_2 - \omega_{fa} = -(\omega_1 + \omega_{fb}).$$

With these,

$$\begin{aligned}
\Gamma_{ab}(\omega_1) &= \frac{\pi E_1^2 E_2^2 e^4}{8\hbar^4} \delta(\omega_{ba} - (\omega_1 + \omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a| \boldsymbol{r} \cdot \boldsymbol{e}_1 \left| f \right\rangle \left\langle f \right| \boldsymbol{r} \cdot \boldsymbol{e}_2 |b\rangle}{\omega_2 + \omega_{fb}} + \frac{\langle a| \boldsymbol{r} \cdot \boldsymbol{e}_2 \left| f \right\rangle \left\langle f \right| \boldsymbol{r} \cdot \boldsymbol{e}_1 |b\rangle}{\omega_1 + \omega_{fb}} \right\} \right|^2,
\end{aligned}$$

as desired.

(iii) The terms in the sum are simply the amplitudes of different possible decay paths via an arbitrary intermediate state $\left| f \right\rangle$. Each summand has two terms due to the fact that the two beams can switch roles.

(iv) Now we obtain the expression of $A(\omega_1)$. To do this, let us first write E_1, E_2 in terms of the photon number raising and lowering operators:

$$E_j = \sqrt{\frac{\hbar \omega_j}{2V\epsilon_0}} (a - a^\dagger).$$

With this, we find

$$\langle n| E_j^2 |n\rangle = \frac{\hbar \omega_j}{V\epsilon_0} \left(n_{\omega_j} + \frac{1}{2} \right).$$

We can take this as E_j^2. Now we replace n_j with the densities of modes at frequency ω_j:

$$n_{\omega_j} \longrightarrow \frac{2\omega_j^3}{\pi c^3}$$

We may ignore the $1/2$ to get

$$\begin{aligned}
\Gamma_{ab}(\omega_1) &= \frac{\pi E_1^2 E_2^2 e^4}{8\hbar^4}\delta(\omega_{ba} - (\omega_1 + \omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \,|f\rangle \langle f|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \,|b\rangle}{\omega_2 + \omega_{fb}} + \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \,|f\rangle \langle f|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \,|b\rangle}{\omega_1 + \omega_{fb}} \right\} \right|^2 \\
&= \frac{e^4 \omega_1^3 \omega_2^3}{2\pi\hbar^2 c^6}\delta(\omega_{ba} - (\omega_1 + \omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \,|f\rangle \langle f|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \,|b\rangle}{\omega_2 + \omega_{fb}} + \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \,|f\rangle \langle f|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \,|b\rangle}{\omega_1 + \omega_{fb}} \right\} \right|^2
\end{aligned}$$

From here, we find that

$$A(\omega_1)\, d\omega_1 = \frac{e^4 \omega_1^3 \omega_2^3}{2\pi\hbar^2 c^6} \left\langle \left| \sum_f \left\{ \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \,|f\rangle \langle f|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \,|b\rangle}{\omega_2 + \omega_{fb}} + \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \,|f\rangle \langle f|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \,|b\rangle}{\omega_1 + \omega_{fb}} \right\} \right|^2 \right\rangle_{\text{avg}} d\omega_1$$

which is off by some numerical factor from the answer given in the problem.

Remark: I'm definitely doing something weird here. Why is the problem talking about how to convert absorption rate into spontaneous emission rate one way and suggests to do it a different way? I guess they give the same answer. I think my brain is not working right now.

2. $2S$ **natural lifetime.**

   (i) Here we use $2P$ as the only intermediate state and assume $\omega_{2S-2P} = \omega_{fb} = 0$. Moreover, we use the Bohr radius a_0 for both relevant matrix elements of z. With these, we find

$$\begin{aligned}
\frac{1}{\tau} &= A_\tau \\
&= \frac{1}{2}\int_0^{\omega_{ba}} A(\omega_1)\, d\omega_1 \\
&= \frac{1}{2}\frac{8e^4}{3\pi\hbar^2 c^6}\int_0^{\omega_{ba}} \omega_1^3(\omega_{ba} - \omega_1)^3 \left| a_0^2 \left(\frac{1}{\omega_1} + \frac{1}{\omega_{ba} - \omega_1} \right) \right|^2 d\omega_1 \\
&= \frac{1}{2}\frac{8e^4}{3\pi\hbar^2 c^6}\frac{a_0^4 \omega_{ba}^5}{6},
\end{aligned}$$

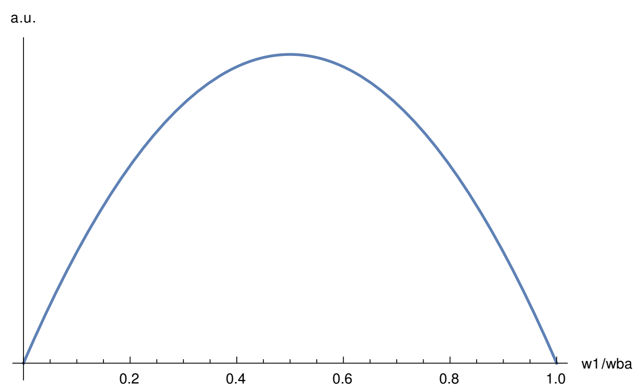
   where we have used the formula provided by Breit and Teller (1940) provided at the end of Part 1.

   (ii) To actually get a sensible lifetime out of this, we have to convert the formula above back to SI units by including a factor of $1/(4\pi\epsilon_0)^2$:

$$\tau = \left(\frac{1}{(4\pi\epsilon_0)^2}\frac{1}{2}\frac{8e^4}{3\pi\hbar^2 c^6}\frac{a_0^4 \omega_{ba}^5}{6} \right)^{-1} \approx 0.095 \text{ s}.$$

   Given that the actual natural lifetime of $2S$ is 0.122 s, this result is quite remarkable.

   (iii) Here we plot $A(\omega_1)$ over its range:


a.u.
0.2
0.4
0.6
0.8
1.0
w1/wba