

(1)

QUANTUM FIELD THEORY

Sept 13, 2020

Before: These notes come from Prof. Paltin's QFT notes, Peskin-Schroeder QFT, and "Quantum Field Theory & Condensed Matter" - Shankar.

Enjoy!

4

Conventions

$$t = c = 1$$

$$[\text{length}] = [\text{time}] = [\text{energy}] = [\text{mass}]$$

$$g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1 - 1 - 1 - 1)$$

$$p^2 = p_\mu p^\mu = E^2 - |\vec{p}|^2 = mc^2 = m$$

$$x_\mu = g_{\mu\nu} x^\nu = (x^0, \vec{x})$$

$$x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \vec{\nabla} \right)$$

$$\epsilon^{0123} = +1 ; \quad \epsilon_{0123} = -1$$

$$\epsilon^{1230} = -1$$

$$E = i \frac{\partial}{\partial x^0}, \quad \vec{p} \doteq \vec{\nabla}$$

$$p^\mu = i \partial^\mu$$

(2)

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx} \theta(x)$

- n -dimensional Dirac δ -fn:

$$\int d^n x \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot x} \tilde{f}(k)$$

$$\tilde{f}(k) = \int d^4 x e^{ik \cdot x} f(x)$$

- $\int d^4 x e^{ik \cdot x} = (2\pi)^4 \delta^{(4)}(k)$

- EM $\Phi = \frac{Q}{4\pi r} \leftarrow$ Coulomb potential

(3)

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi hc} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0, \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\vec{E}, \vec{A}), \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of classical Field Theory

- Lagrangian Field Theory:

$$S = \int \underline{L} dt = \int \underline{L}(\phi, \partial_\mu \phi) d^4x \quad \left(\underline{L} = L d^4x \right)$$

Principle of least action:

$$0 = \delta S \quad (\text{Lagrangian})$$

$$(\text{E-L eqn}) = \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi + \frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \underline{L}}{\partial \phi} \delta \phi - \partial_\mu \left(\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right) \delta \phi + \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \delta \phi \right] \right\}$$

$$\Rightarrow 0 = \partial_\mu \left[\frac{\partial \underline{L}}{\partial (\partial_\mu \phi)} \right] - \frac{\partial \underline{L}}{\partial \phi}$$

FTC \rightarrow term vanishes
at boundary

(4)

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_m \left\{ \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} \right\}$$

Ex $\mathcal{L} = \phi^2 : \quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi ; \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 0 \Rightarrow 2\phi = 0$

$$\begin{aligned} \mathcal{L} &= (\partial_m \phi) (\partial^m \phi) \\ \Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} &= 0 ; \quad \frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = 2\partial^m \phi \end{aligned} \quad \left. \right\} \Rightarrow \partial^m \phi = 0,$$

Ex Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2} (\partial_m \phi) (\partial^m \phi) - \frac{1}{2} m^2 \phi^2$$

$$\therefore \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial (\partial_m \phi)} = \partial^m \phi .$$

relativistic particle
of mass m .

$$E-L \Rightarrow -m^2 \phi = \partial_m \partial^m \phi = \square \phi$$

$$\rightarrow \boxed{(\square + m^2) \phi = 0}$$

(Klein - Gordon Eqn.)

$$\text{Ex } \phi = e^{-ip \cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$$

(5)

Noether's Thm

Thm

For every continuous symmetry, there exists a conserved current j^μ which implies a local conservation law

$$\partial_\nu j^\nu = -\vec{\nabla} \cdot \vec{j} \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

Ex Conservation of charge -- (gauss' law)

$$\begin{aligned} Q &= \int j^0 d^3x \\ \Rightarrow \frac{\partial Q}{\partial t} &= \int \frac{d j^0}{dt} d^3x = - \int \vec{\nabla} \cdot \vec{j} d^3x \\ &= - \oint \vec{j} d^2s \end{aligned}$$

Idea Consider continuous transf. \rightarrow infinitesimally (local)

$$(\star) \quad \phi(x) \rightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$$

↑
small

(\star) is a symmetry if EOM invariant under (\star).

$\Rightarrow S$ is invariant.

$\Rightarrow L$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$,
for some J^μ .

(6)

Let us compare this expectation for ΔL to the result obtained by varying the fields ...

$$\begin{aligned}\alpha \Delta L &= \frac{\partial L}{\partial \phi} (\alpha \Delta \phi) + \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \right) \partial_\mu (\alpha \Delta \phi) \\ &= \alpha \partial_\mu \left(\frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right) + \alpha \underbrace{\left(\frac{\partial L}{\partial \phi} - \partial_\mu \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \right\} \right) \Delta \phi}_{0}\end{aligned}$$

$$\Rightarrow \Delta L = \partial_\mu \left\{ \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi \right\}$$

So $\frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi$ is the desired J^μ .

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi - J^\mu$$

Ex massless KG field

$$L = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)$$

Consider transformation:

$$\phi(x) \rightarrow \phi(x) + \alpha, \quad \Delta \phi = 1$$

$$j^\mu = \frac{\partial L}{\partial (\partial_\mu \phi)} \Delta \phi = \partial^\mu \phi.$$

Check $\partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = \square \phi = 0$ since
 $(m^2 + \nabla^2) \phi = 0 \quad \uparrow$

(7)

Ex Complex KG field

$$\mathcal{L} = (\partial_\mu \phi^+) (\partial^\mu \phi) - m^2 \phi^+ \phi.$$

again, EOM \Rightarrow

$$(m^2 + \Box) \phi = 0.$$

Symmetry: $\phi \rightarrow e^{i\alpha} \phi$.

For infinitesimal transf we have:

$$\begin{aligned}\alpha D\phi &= i\alpha \phi && \text{(Taylor expand)} \\ \alpha D\phi^+ &= -i\alpha \phi^+\end{aligned}$$

One can check that

$$j^\mu = i [(\partial^\mu \phi^+) \phi - \phi^+ (\partial^\mu \phi)]$$

\Rightarrow the conserved current.

[Ex]

Space-time translation:

Consider inf. translation:

$$x^\mu \rightarrow x^\mu - a^\mu$$

\hookrightarrow in field transforms this becomes:

$$\phi(x) \rightarrow \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

(8)

Lagrangian is a scalar \Rightarrow must transform the same way:

$$L \rightarrow L + a^{\mu} \partial_{\mu} L = L + a^{\nu} \partial_{\mu} (s_{\nu}^{\mu} L)$$

Compare this to the eqn:

$$L \rightarrow L + a \partial_{\mu} J^{\mu}$$

we have

$$J^{\mu} = \cancel{s}_{\nu}^{\mu} L$$

\Rightarrow apply this, we find:

$$J^{\mu} = \frac{\partial L}{\partial (\partial_{\mu} \phi)} (\partial_{\mu} \phi) - s_{\nu}^{\mu} L$$

value μ explicit...

$$\boxed{T_{\mu}^{\nu} = \frac{\partial L}{\partial (\partial_{\nu} \phi)} \partial_{\mu} \phi - \delta_{\mu}^{\nu} L}$$

\hookrightarrow STRESS-ENERGY TENSOR, (or Energy-momentum tensor)

Conserved charge \Rightarrow the Hamiltonian

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad (\text{time-translation})$$

$$\text{momentum} \rightarrow p^i = \int T^{0i} d^3x = - \int \mathcal{H} \partial_i \phi d^3x \quad (\text{spatial translation})$$

(9)

Ex Klein-Gordon field again

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2.$$

$$T^{00} = \frac{\partial F}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} = \dots$$

$$= \frac{1}{2} (\partial_0 \phi) (\partial^0 \phi) - \frac{1}{2} (\partial_i \phi) (\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

so

$$H = \int T^{00} d^3x = \int \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial t} \right)^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} d^3x$$

(note: all terms are positive ... (sum of squares))

→ can't fall into arbitrary negative energy

THE KLEIN-GORDON FIELD as HARMONIC OSCILLATOR

promote: ϕ, π to operators \Rightarrow impose suitable commutation relations

Reall ...

$$[q_i, p_j] = i \hbar \delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0.$$

$$\text{Harmonic oscillator: } H = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 q^2$$

(10)

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}} \left(g\sqrt{m\omega} + ip \frac{1}{\sqrt{m\omega}} \right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}} \left(g\sqrt{m\omega} - ip \frac{1}{\sqrt{m\omega}} \right)$

- $a^\dagger a = \frac{1}{\omega} H - \frac{1}{2} \quad (\Rightarrow \quad H = \omega(a^\dagger a + \frac{1}{2})$



operator...

- $|0\rangle, a|0\rangle = 0.$

$$|n\rangle = \frac{1}{\sqrt{n!}} \underbrace{a^\dagger \dots a^\dagger}_{n} |0\rangle$$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -\omega a ; [H, a^\dagger] = \omega a^\dagger$

a lowers by ω

a^\dagger raises by ω

- $H a^\dagger |n\rangle = (E_n + \omega) a^\dagger |n\rangle$

- $H |0\rangle = \frac{1}{2} \omega |0\rangle \rightarrow E_0 = \frac{1}{2} \omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

(11)

For continuous system ... commutation relations become ..

$$[\phi(x), \pi(y)] = i\delta^{(1)}(x-y)$$

$$[\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0.$$

Next, the Hamiltonian is now also an operator...
To find $\text{spec}(H)$, Fourier transf $\phi(x)$

$$\rightarrow \phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p} \cdot \vec{x}} \phi(\vec{p}, t)$$

Recall KG eqn: $(m^2 + D)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|\vec{p}|^2 + m^2) \right\} \phi(\vec{p}, t) = 0$$

\rightarrow This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2}$$

$$\text{Now, } H_{SHO} = \frac{1}{2} \vec{p}^2 + \frac{1}{2} \omega^2 \phi^2 \quad (m=1)$$

\rightarrow know spectrum! $(n + \frac{1}{2})\omega$.

$$\phi = \frac{1}{\sqrt{2\omega}} (at + a) ; \vec{p} = -i\sqrt{\frac{\omega}{2}} (a - at)$$

$$[a, a^\dagger] = 1.$$

(12)

Since it's more convenient to work in position space

$$\boxed{\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p e^{ip \cdot x} + a_p^\dagger e^{-ip \cdot x})}$$

$$\boxed{\pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p e^{ip \cdot x} - a_p^\dagger e^{-ip \cdot x})}$$

Note

$$\left\{ \begin{array}{l} a_p \text{ goes with } e^{+ip \cdot x} \\ a_p^\dagger \text{ goes with } e^{-ip \cdot x}. \end{array} \right.$$

9 Easy to show that

$$\left\{ \begin{array}{l} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}). \end{array} \right.$$

* Can re-arrange...

$$\left\{ \begin{array}{l} \phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} (a_p + a_{-p}^\dagger) e^{ip \cdot x} \\ \pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{-ip \cdot x}. \end{array} \right.$$

→ set commutation relation between a_p :

$$\boxed{[a_p; a_{p'}^\dagger] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')}$$

(13)

Now, can check that

$$\begin{aligned} [\phi(x), \pi(x')] &= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{w_p}{w_{p'}}} x \cdot e^{ip \cdot x} (p \cdot x + p' \cdot x') \\ &\quad \left([a_{-p}^\dagger, a_p] - [a_p, a_{-p}^\dagger] \right) \\ &= i \delta^{(3)}(x - x') \quad \checkmark \end{aligned}$$

• Now, can express Hamiltonian in terms of ladder ops

recall that

KG field, but
done

$$H = \int d^3 x \left\{ \frac{\partial f}{\partial (\partial_0 \phi)} \partial^0 \phi - g^{00} \partial^0 f \right\}$$

$$= \int d^3 x \left\{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial f}{\partial (\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial f}{\partial \dot{\phi}} \right)$$

so ...

$$H = \int d^3 x \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi) \cdot (\nabla \phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

$$\text{with } \pi(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} (a_p - a_{-p}^\dagger) e^{ip \cdot x}$$

we get

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2 w_p}} (a_p + a_{-p}^\dagger) e^{-ip \cdot x}$$

(14)

$$H = \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(p+p') \cdot x} \left\{ -\frac{\sqrt{w_p w_{p'}}}{4} (a_p - a_{-p}^\dagger) (a_{p'} + a_{-p'}^\dagger) \right. \\ \left. + \frac{-p \cdot p' + m^2}{4\sqrt{w_p w_{p'}}} (a_p + a_{-p}^\dagger) (a_{p'} + a_{-p'}^\dagger) \right\}$$

results in $C(p-p')$
 $\Rightarrow p = p'$

Some $S^{(3)}$
will appear...

$$= \int \frac{d^3p}{(2\pi)^3} w_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

Σ

$$H = \int \frac{d^3p}{(2\pi)^3} w_p (a_p^\dagger a_p + \frac{1}{2} [a_p, a_p^\dagger])$$

→ can evaluate commutators...

$$[H, a_p^\dagger] = w_p a_p^\dagger; [H, a_p] = -w_p a_p$$

With H , can find momentum operator...

kG field \rightarrow from $p^i = \int d^3x T^{0i} = - \int \nabla \phi_i \cdot \vec{p} d^3x$, we get

$$\begin{aligned} \vec{P} &= - \int d^3x \vec{\nabla} \phi(x) \cdot \vec{p}(x) \\ &= \int \frac{d^3p}{(2\pi)^3} \vec{p} a_p^\dagger a_p. \end{aligned}$$

$E_p \xrightarrow{\parallel 0}$

a_p^\dagger creates momentum \vec{p} & energy $w_p = \sqrt{|\vec{p}|^2 + m^2}$.

Excitation: $a_p^\dagger a_q^\dagger \dots |0\rangle$ = "particles".

↳ such excitation at p is a particle.

\Rightarrow set particle statistics --

Consider 2-particle state $a_p^+ a_q^+ |0\rangle$.

Since $[a_p^+, a_q^+] = 0$, we have

$$a_p^+ a_q^+ |0\rangle = a_q^+ a_p^+ |0\rangle$$

\Rightarrow Klein Gordon particles follow Bose-Einstein state.

* Normalization $\langle 0|0 \rangle = 1$.

$$\langle p | \propto a_p^+ |0\rangle$$

This $\rightarrow \langle q | p \rangle = (2\pi)^3 \delta^{(3)}(q-p) \rightarrow$ NOT Lorentz inv

PF Under a Lorentz boost $\rightarrow p'_3 = \gamma(p_3 + \beta E)$

using the identity $E' = \gamma(E + \beta p_3)$

$$\rightarrow \delta(f(x) - f(x_0)) = \frac{1}{|f'(x_0)|} \delta(n - n_0)$$

$$\text{we can write: } \delta^{(3)}(p-q) = \delta^6(p'-q') \cdot \left(\frac{dp'_3}{dp_3} \right)$$

$$\begin{aligned} & \underbrace{\delta(p_1-q_1)}_{\text{same}} \underbrace{\delta(p_2-q_2)}_{\text{boosted}} \underbrace{\delta(p_3-q_3)}_{\text{boosted}} \\ &= \delta^{(3)}(p'-q') \cdot \gamma \left(1 + \beta \frac{dE}{dp_3} \right) \\ &= \delta^{(3)}(p'-q') \frac{\gamma}{E} (E + \beta p_3) \\ &= \delta^{(3)}(p'-q') \left(\frac{E'}{E} \right) \end{aligned}$$

(16)

$$\text{So } \boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \left(\frac{E'}{E}\right)} \rightarrow 1 \Leftrightarrow E = E_p = E'$$

For normalization to work \rightarrow use E_p , not E .

\rightarrow define: $|p\rangle = \sqrt{2E_p} a_p^\dagger |0\rangle$

to find $\boxed{\langle p|q\rangle = 2E_p (2\pi)^3 \delta^{(3)}(p-q)}$

Completeness relation ...

1 particle $\rightarrow \boxed{1 = \int \frac{d^3 p}{(2\pi)^3} |p\rangle \frac{1}{2E_p} \langle p|}$

RS Interpret $\phi(x)|0\rangle \dots$ we know that a_p^\dagger creates momentum p energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle , \quad a_p^\dagger |0\rangle = \frac{1}{\sqrt{2E_p}} |p\rangle$$

Pf. By defn ...

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ipx} + a_p^\dagger e^{-ipx})$$

$$\rightarrow \phi(x)|0\rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ipx} |p\rangle \quad \square$$

$\rightarrow \phi(x)|0\rangle$ is a lin. superposition of single-particle states

Note Ch 8 & 9

(17)

Hint here: well-defn momentum.

When nonrelativistic $\rightarrow E_p \approx \text{constant}!$

\Rightarrow $\phi(x)$ acting on the vacuum, "creates a particle at position x ".

\hookrightarrow Confirm this by computing -

$$\langle 0 | \phi(x) | p \rangle = \langle 0 | \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p^\dagger e^{ip \cdot x} + a_p^- e^{-ip \cdot x}) \sqrt{2E_p} a_p | p \rangle$$

$$\boxed{\langle 0 | \phi(x) | p \rangle = e^{ip \cdot x}}$$

\hookrightarrow Interpretation: position-space representation of the single-particle wf_n of the state $|p\rangle$, just like

$$\boxed{\langle n | p \rangle \propto e^{ip \cdot x}} \text{ in QM!}$$

$\langle 0 | \phi(x) | \sim x | \dots$ (don't take this literally, ofc).

Note Hw1, Hw2 are copy, so we'll skip for now.

ep 14, 2020

THE KLEIN-GORDON FIELD IN SPACETIME

Last time \rightarrow we quantized KG field in the Schrödinger picture.

\rightarrow Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$U(t) = e^{-iHt}$ is the time evolution.

$$|\psi(t)\rangle = e^{-iHt} |\psi(0)\rangle \xrightarrow{\text{state evolves in time}}$$

\rightarrow In the Heisenberg picture, ... Operators evolve in time

$$\theta(t) = U^\dagger(t) \theta(0) U(t).$$

to treat

$$\langle \psi_1 | \theta(t) | \psi_2 \rangle = \langle \psi_1(t) | \theta(t) | \psi_2(t) \rangle$$

\downarrow

Heisenberg

\downarrow

Schrödinger.

\rightarrow make the operators ϕ, π time-dependent like this

$$\phi(x) \rightarrow \phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$$

$$\pi(x) \rightarrow \pi(x, t) = e^{iHt} \pi(x) e^{-iHt}$$

Recall Heisenberg eqn of motion $i\frac{\partial}{\partial t} \theta = [\theta, H]$

(19)

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$\frac{i}{\partial t} \phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\} \right]$$

$$(\phi \leftrightarrow \phi) \quad = \int d^3x' \left(i\delta^{(3)}(x-x') \pi'(x,t) \right)$$

\rightarrow only continual term is 1^{st} .

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t} \phi(x,t) = \pi(x,t)}$$

and

$$\frac{i}{\partial t} \pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\pi)^2 + \frac{1}{2}m^2\phi^2 \right\} \right]$$

$$= \int d^3x' \left(-i\delta^{(3)}(x-x') (-\nabla^2 + m^2) \phi(x',t) \right)$$

(integrate by parts here)

$$= -i(-\nabla^2 + m^2) \phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t} \pi(x,t) = (m^2 - \nabla^2) \phi(x,t)}$$

Combining these 2 results we get ...

$$\boxed{\frac{\partial^2}{\partial t^2} \phi(x,t) = (\nabla^2 - m^2) \phi(x,t)}$$

\hookrightarrow rearranging this gives

$$\boxed{(\nabla^2 + m^2) \phi(x,t) = 0} \rightarrow \text{just the KG eqn...}$$

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

Recall: $H_{\text{ap}} = a_p^{\dagger} (H - E_p) \rightarrow$ from comm. rule -

\Rightarrow (proof by induction)

$$H^n a_p = a_p (H - E_p)^n$$

Similarly,

$$H^n a_p^{\dagger} = a_p^{\dagger} (H + E_p)^n$$

\rightarrow So, we have

$$e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t} \rightarrow \text{from above...}$$

and

$$e^{iHt} a_p^{\dagger} e^{-iHt} = a_p^{\dagger} e^{+iE_p t}$$

\rightarrow Now -- we want to write $\phi(x, t)$ in terms of these operators. (since $\phi(x)$ is a comb of a & a^{\dagger})

$\pi(x)$
we know that $\phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before ...

$$\phi(x) = \phi(x, 0) = \int \frac{dp}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ip \cdot x} + a_p^{\dagger} e^{-ip \cdot x})$$

substitute this into $\phi(x, t) = e^{iHt} \phi(x) e^{-iHt}$ we find

(21)

$$\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-ip \cdot x} + a_p^+ e^{ip \cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow p \cdot x = E_p \cdot x^0 - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p} = E_p t - \vec{p} \cdot \vec{x} \Big|_{p^0 = E_p}$$

So,

$$\boxed{\phi(x, t) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p} \cdot \vec{x}} + a_p^+ e^{+i\vec{p} \cdot \vec{x}} \right\} \Big|_{p^0 = E_p}}$$

Similarly, we can find

$$\boxed{\Pi(x, t) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} (a_p e^{-i\vec{p} \cdot \vec{x}} - a_p^+ e^{+i\vec{p} \cdot \vec{x}}) \Big|_{p^0 = E_p}}$$

from defn or from $\Pi(x, t) = \frac{\partial}{\partial t} \phi(x, t)$.

Note we can also do everything, but starting from P and not H . But we won't worry about that.



Causality Note that causality is broken when without the presence of fields.

→ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture --

-- the amplitude for a particle to propagate from $y \rightarrow x$ is given by

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle$$

"Two-point correlation function"

Let $\boxed{D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle}$

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \langle 0 | \text{in terms of } a_p^+ a_q^- | 0 \rangle$$

$$= \langle 0 | a_p^+ a_q^- | 0 \rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip \cdot (x-y)}$$

More explicitly --

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$\begin{cases} \textcircled{2} p' = \vec{p} \\ p'_0 = E_{p'} \end{cases}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | e^{-ipx} e^{ip'y} a_p^+ a_{p'}^- | 0 \rangle$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(\frac{1}{\sqrt{2E_p}} \right) \left(\frac{1}{\sqrt{2E_{p'}}} \right) e^{-ip \cdot x - ip' \cdot y} (2\pi)^3 \delta^{(3)}(p-p')$$

$$D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip \cdot (x-y)} \quad \square$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$.

(1) Suppose that $x-y = (t, \vec{v}, 0, 0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}t}$$

$$(\text{timelike}) = \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2} e^{-iEt}$$

$$= e^{-imt} \xrightarrow{\text{dominated by region above}} \text{dominated by region above}$$

$t - i\omega$

$p \approx 0 -$

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

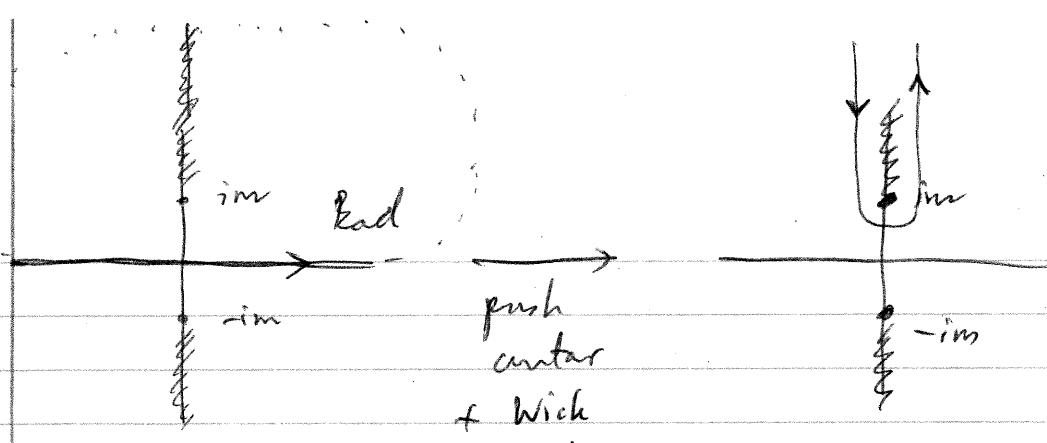
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2Ep} e^{i\vec{p} \cdot \vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \frac{p^2}{2Ep} \frac{e^{ipr} - e^{-ipr}}{i pr}$$

$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{pe^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ tim (singularity) ...

→ must change contour... \rightarrow which rotate



To get

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r} \int_m^\infty d\rho \frac{\rho e^{-i\rho r}}{\sqrt{\rho^2 - m^2}} \quad (\text{Wick rotate})$$

$$\Rightarrow \boxed{\mathcal{D}(x-y) = \frac{e^{-mr}}{r}} \rightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^m (x-y)_m = -(\vec{x}-\vec{y})^2 < 0$
is spacelike, cannot have causality between
 $x-y$.

$\mathcal{D}(x-y) \neq 0 \Rightarrow ??? \text{ paradox?}$

$\rightarrow \underline{\text{No!}}$ To discuss causality, we should ask not whether particles can propagate over spacelike intervals --

-- but whether a "measurement" at one point can affect a measurement at another point where separation from the first is spacelike --

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement $\phi(x)$, call this $\phi(x)$. or a local measurement $\phi(y)$, called $\phi(y)$

So long as $[\phi(x), \phi(y)] = 0$, the 2 measurements don't affect one another.

→ measure the field $\phi @ x + @ y$,

If $[\phi(x), \phi(y)] = 0$ when $(x-y)^2 < 0$ then we've good

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} \right. \\ \left. + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\}$$

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \times \left\{ (a_p e^{-ip \cdot x} + a_p^\dagger e^{ip \cdot x}), \right. \\ \left. (a_p^\dagger e^{-ip' \cdot y} + a_p e^{ip' \cdot y}) \right\}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6 2\sqrt{E_p E_{p'}}} \left\{ e^{-ip \cdot x} [a_p, a_{p'}^\dagger] e^{ip' \cdot y} \right. \\ \left. + e^{ip \cdot x} [a_p^\dagger, a_{p'}] e^{-ip' \cdot y} \right\} \\ (2\pi)^3 \delta^3(p - p') \quad -(2\pi)^3 \delta^3(p - p')$$

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2E_p} \right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

$$\text{So } [\phi(x), \phi(y)] = D(x-y) - D(y-x).$$

Since $D(y-x)$ is Lorentz invariant, can take

$$y-x \rightarrow x-y. \text{ (possible b/c } (x-y)^2 < 0)$$

Note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

\rightarrow so this is why possible because $(x-y)^2 < 0$
(negative).

$$\text{So } D(y-x) = D(x-y)$$

~~$$\text{So, } [\phi(x), \phi(y)] = 0 \quad \checkmark$$~~

~~24~~

~~The Klein-Gordon Propagator~~

Let's look at $[\phi(x), \phi(y)]$ in more details..

$[\phi(x), \phi(y)]$ is just a number

~~\Rightarrow can write $[\phi(x), \phi(y)] \equiv \langle 0 | [\phi(x), \phi(y)] | 0 \rangle$~~

~~$$\rightarrow \langle 0 | [\phi(x), \phi(y)] | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip \cdot (x-y)} - e^{ip \cdot (x-y)} \right\}$$~~

$$\Delta \text{ Poles}$$

$$E_p^2 = m^2$$

$$p^0 = \pm E_p$$

(assuming $x^0 > y^0$)

$$= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip \cdot (x-y)} \right|_{p^0=E_p} - \frac{1}{2E_p} e^{-ip \cdot (x-y)} \right\}_{p^0=-E_p}$$

$= -E_0$

The Klein-Gordon Propagator

→ Before looking at this -- need to look at Green's Function & Contour Integrals --

→ Consider function

$$\frac{1}{(p_0^0 - E_p)(p_0^0 + E_p)} \quad E_p = \sqrt{|p|^2 + m^2}.$$

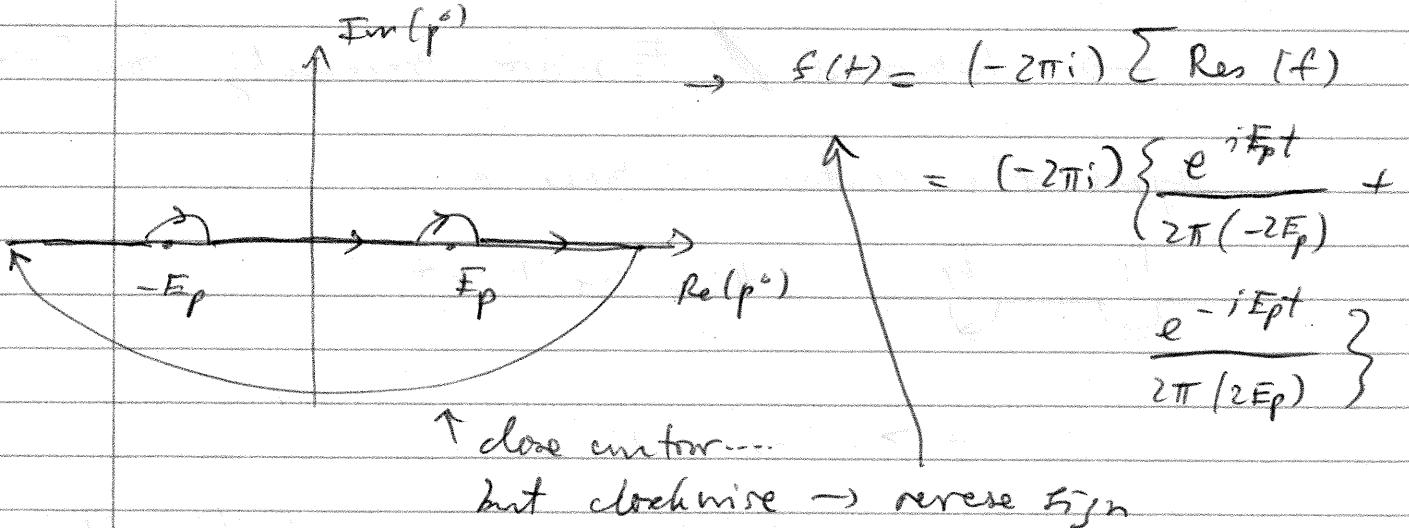
→ Poles at $p_0^0 = \pm E_p$.

Lab at FT:

$$f(t) = \int_{-\infty}^{\infty} \frac{dp^0}{2\pi i} \frac{e^{-ip^0 t}}{(p^0 - E_p)(p^0 + E_p)}$$

→ How to integrate this?

If $t > 0 \rightarrow$ ~~crosses poles~~



$$\rightarrow f(t) = \frac{-i}{2E_p} (e^{+iE_p t} - e^{-iE_p t}) \quad (t > 0)$$

If $t < 0$ close contours above poles



$$\rightarrow f(t) = 0.$$

→ So, altogether, we have ...

$$\begin{aligned} f(t) &= \int \frac{1}{2\pi} \frac{dp^0}{(p_0^0 E_p)(p^0 + E_p)} e^{-ip^0 t} \\ &= \Theta(t) \left(\frac{-i}{2E_p} \right) (e^{-iE_p t} - e^{+iE_p t}) \end{aligned}$$

where $\Theta(t)$ is the Heaviside Step fn ...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

→ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as

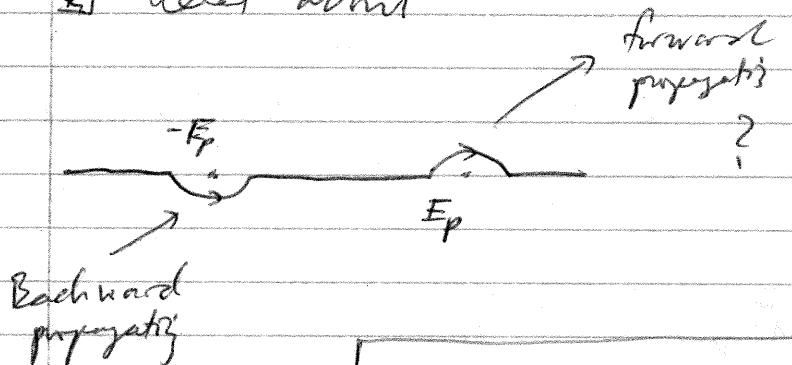


$$t > 0 \rightarrow f(t) = 0$$

$$t < 0 \rightarrow f(t) = \Theta(-t) \frac{i}{2E_p} (e^{-iE_p t} - e^{+iE_p t})$$

→ Advanced / Backward Propagating Green's fn.

What about



$$\rightarrow \boxed{f(t) = \Theta(+)(-\frac{i}{2E_p}) e^{-iE_p t} + \Theta(-)(-\frac{i}{2E_p}) e^{+iE_p t}}$$

Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$

Consider this quantity...

$$\begin{aligned} \langle 0 | [\phi(x), \phi(y)] | 0 \rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) \\ &= \int \frac{d^3 p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip(x-y)} + \frac{1}{-2E_p} e^{-ip(x-y)} \right\} \\ &\quad \uparrow \quad \downarrow \\ &\quad \text{pole} \quad \text{pole @} \\ &\quad @ p_0 = E_p \quad p_0 = -E_p \end{aligned}$$

$$\begin{aligned} \text{(41) integral} \rightarrow &= \int \frac{d^3 p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \underbrace{\frac{-i}{p^0 - m^2}}_{f(+)} e^{-ip(x-y)} \\ &\quad \underbrace{\qquad\qquad\qquad}_{\text{f(+)} \text{ before, where}} \end{aligned}$$

$$(p^0 - E_p)(p^0 + E_p) = p^{0^2} - |p|^2 - m^2 = p^2 - m^2$$

Now, call ...

$$D_R(x-y) = \theta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle$$

Then

$$\begin{aligned}
 \rightarrow (\square + m^2) D_R(x-y) &= \square D_R(x-y) + m^2 D_R(x-y) \\
 &= (\square \theta(x^0 - y^0)) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + 2(\partial_\mu \theta(x^0 - y^0)) \partial^\mu \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &\quad + \theta(x^0 - y^0) (\square + m^2) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \\
 &= -\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \quad \text{cancel} \\
 &\quad + 2\delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle + 0 \\
 &= \delta(x^0 - y^0) \langle 0 | [\bar{\psi}(x), \psi(y)] | 0 \rangle \quad \text{milds} \\
 &= -i \delta^{(4)}(x-y) \quad \text{renormalization} \\
 &\quad \downarrow \quad \text{(easy)} \\
 &\quad -i \delta^{(3)}(x-y)
 \end{aligned}$$

So

$$(\square + m^2) D_R(x-y) = -i \delta^{(4)}(x-y)$$

$\rightarrow D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

Since $D_R(x-y) = 0 @ x^0 < y^0$

$\Rightarrow D_R(x-y) \equiv \text{"Retarded" Green's fn}$

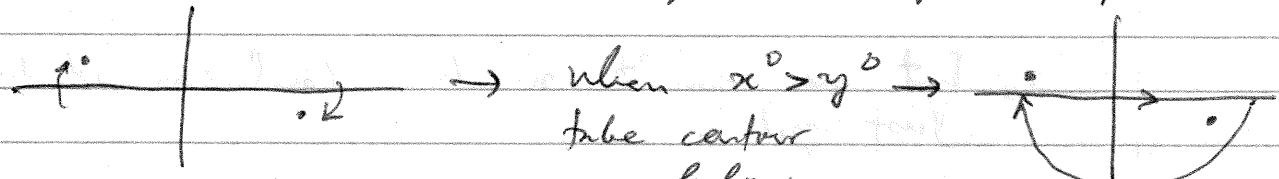
Now ... As we have seen, there are many ways to take the contour ...



→ Use the Feynman prescription instead

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)}$$

Convenient! B/c now poles are $p^0 = \pm(E_p - i\epsilon)$



when $x^0 < y^0 \rightarrow$
take contour above.

→ get same expression
but with $x \leftrightarrow y$.

So,

$$D_F(x-y) = \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases}$$

$$= \Theta(x^0 - y^0) \langle 0 | \phi(x) \phi(y) | 0 \rangle + \Theta(y^0 - x^0) \langle 0 | \phi(y) \phi(x) | 0 \rangle$$

$$= \langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle$$

"time-ordering" symbol -

Time-ordering symbol \Rightarrow instructs us to place the operators & heat follows in order with the latest to the left.

\rightarrow apply $(D + m^2)$ to last line, set D_F is Green's fn of Klein-Gordon Operator,

$$() \quad \overbrace{\hspace{10em}}^{\text{---}}$$

$D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator--

\hookrightarrow propagation amplitude

\rightarrow But we can't much calculation at this point just yet.

\rightarrow B/c we've only looked at the free K-G theory

\rightarrow Field eqn in this case is linear : there are no interactions--

\rightarrow this theory is too simple to make any predictions--

\rightarrow need perturbation --

One kind of interaction it's can also be solved



Particle Creation by a classical Source

Consider the source $j(x)$

Result... free field: $(D + m^2)\phi = 0$

→ now... $(D + m^2)\phi = j(x)$ Field ϕ is
 ↗ space time.

$j(x)$ is nonzero only for a finite time interval

The associated lagrangian is

$$L = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is
 enough to solve

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ip_i x} + a_p^+ e^{ip_i x})$$

With a source...

$$\phi(x) = \phi_0(x) + i \int d^3y D_R(x-y)j(y)$$

We won't worry about this for now...

Some problems & Insights

(1) Classical EM (no sources) follow from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\text{where } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \epsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \quad \boxed{(\vec{\nabla} \cdot \vec{E} = 0) \quad (\nu = 0)}$$

\downarrow

$$\boxed{-\partial_t \vec{E} + \vec{\nabla} \times \vec{E} = 0} \quad (\nu = i)$$

(2) Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi^- - m^2 \phi^+ \phi^- \right)$$

Derive E-L eqn:

$$\boxed{i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ = 0}$$

$$\boxed{-i\partial_t \phi^- - \frac{1}{2m} \nabla^2 \phi^- = 0}$$

Now... write $\phi \rightarrow e^{-i\theta} \phi$, $\phi^+ \rightarrow e^{i\theta} \phi^+$

$$\begin{aligned} &\sim \phi - i\theta \phi \\ &\rightarrow \Delta \phi \sim -i\theta \end{aligned}$$

$$\begin{aligned} &\sim \phi^+ + i\theta \phi^+ \\ &\Delta \phi^+ \sim i\theta \end{aligned}$$

So that

$$j^{\mu} = \frac{\partial f}{\partial (\partial_{\mu} \phi)} \Delta \phi + \frac{\partial f}{\partial (\partial_{\mu} \phi^+)} \Delta \phi^+$$

↑
conserved current -

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial f}{\partial (\partial_x \phi)} \rightarrow \dots \text{conjugate\dots}$$

→ can get Hamiltonian → there's a formula in book,
but we worry abt this.

3) If we take $(x-y)^2 = -r^2 \rightarrow$ can implicitly write

$$\langle 0 | \phi(x) \phi(y) | 0 \rangle = D(x-y) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2 \rightarrow D(x-y)$ can be written in terms of Bessel Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation...

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu$$

→ What happens to $\phi(x)$ under Λ ?

We require that $\phi'(x') = \phi(x)$

$$\rightarrow \boxed{\phi(x) \rightarrow \phi'(x) = \phi(\Lambda^{-1}x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ What about $\partial_\mu \phi(x)$?

Under transform -- $\partial_\mu \phi(x) \rightarrow \partial_\mu (\phi(\Lambda^{-1}x))$

$$\rightarrow \boxed{\partial_\mu (\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\rho_\mu (\partial_\rho \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho_\mu (\Lambda^{-1})^\sigma_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\rightarrow (\partial_\mu \phi(x))^2 \rightarrow g^{\mu\nu} (\partial_\mu \phi'(x)) (\partial_\nu \phi'(x))$$

$$= g^{\mu\nu} [\partial_\mu (\phi(\Lambda^{-1}x))] [\partial_\nu (\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu} \{ (\Lambda^{-1})^\rho_\mu \partial_\rho \phi \} \{ (\Lambda^{-1})^\sigma_\nu \partial_\sigma \phi \} (\Lambda^{-1}x)$$

$$= (\partial_\mu \phi)^2 (\Lambda^{-1}x) \leftarrow = g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi (\Lambda^{-1}x) \rightarrow \text{evaluated}$$

(37)

Exercise

$$\phi(x) \rightarrow \phi(\tilde{x})$$

$$\partial_\mu \phi(x) \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)(\tilde{x})$$

$$(\partial_\mu \phi(x))^2 \rightarrow (\tilde{x})^\mu (\partial_\mu \phi)^2 (\tilde{x})^\nu$$

So it is clear that

$$L \rightarrow L(\tilde{x})$$

↑

Lagrangian is Lorentz-invariant.

→ The action $S = \int d^4x L$ is also Lorentz inv.

→ also clear that EOM is also Lorentz inv.

$$\begin{aligned} (\square + m^2) \phi(x) &= (\tilde{x})^\mu \partial_\mu (\tilde{x})^\nu \partial_\nu + m^2 \phi(\tilde{x}) \\ &= (\partial^\mu \partial_\mu + m^2) \phi(\tilde{x}) \\ &= 0 \quad \checkmark \end{aligned}$$

20

Lep 10, 2020

→ How do we find Lorentz-invariant theories, in general?

→ For simplicity, restrict attention to lin. transf

→ $\phi_a = \phi \in \mathbb{C}^n$, → matrix giving Lorentz transf in A .

$$\rightarrow \boxed{\Phi_a(x) \rightarrow M_{ab}(A) \Phi_b(\tilde{x})}$$

$n \times n$

The

→ most general nonlinear laws can be built
out of linear ones \Rightarrow suffices to consider M
only.

↳ for short, write $\Phi \mapsto M(\mathbf{A})\Phi$.

→ What are the possible allowed M(D)?

Q12 $\{M(\Delta)\}$ form a group $M(\Delta') M(\Delta) = M(\Delta')$
 $\Delta' \Delta = \Delta'$

→ the correspondence between A & M must be preserved under multiplication.

$\{1\}$ Lorentz group $\rightarrow \{M(1)\} \rightarrow$ n-dim representation of the Lorentz group

↳ ? What are the finite-dim matrix repr of the Lorentz group?

Ex in $\otimes M$ -- spin $\frac{1}{2} \rightarrow \{\mathbf{M}\}$ we have 2×2 unitary matrices with determinant 1.

$$U = e^{-i\theta^i \sigma^i/2} \rightarrow \sum_i \theta^i \sigma^i/2$$

\vec{O}

3 arbitrary parameters
8 pauli matrices.

$$u(\vec{r}) = e^{-i\vec{\theta} \cdot \vec{r}/c}$$

→ In the case for arbitrary spin representations...

$$U(\vec{\theta}) = e^{-i \vec{\theta} \cdot \vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i \sum_l \epsilon^{ijk} J^l$$

→ Check that this works for spin $\frac{1}{2}$:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2} \right] = i \sum_l \epsilon^{jkl} \frac{\sigma^l}{2} \quad \checkmark$$

→ for spinless particles... $\psi(\vec{x})$ can be decomposed into orbital angular momentum states. $J=0, 1, 2, \dots$
(no intrinsic spin $\Rightarrow J=L$)

$$\bullet \quad \vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i \vec{\nabla})$$

$$\bullet \quad J^j = i \sum_l \epsilon^{jkl} x^k \nabla^l$$

$$\bullet \quad \nabla^l = -\partial_x^l = -\frac{\partial}{\partial x^l}$$

But the cross product is special to 3D case.

→ write operators in antisymmetric tensor...

$$\boxed{J^{ij} = -i(x^i \partial^j - x^j \partial^i)} \rightarrow \text{represents free cross product.}$$

so that $J^3 = J^{12}$, etc.

→ generate to 4D: → 6 operators that generate 3 boosts, 3 rotations,

$$\boxed{J^{\mu\nu} = +i(x^\mu \partial^\nu - x^\nu \partial^\mu)} \quad \text{of the Lorentz group.}$$

$\{ \rightarrow$ Spatial Rotations: $J^{ik} = i(x^i \partial^k - x^k \partial^i)$

\rightarrow Lorentz boosts along x^j axis: $J^{ij} = i(x^i \partial^j - x^j \partial^i)$

\rightarrow Now, want to get commutation rules.

\rightarrow compute the commutators of differential ops

to get

$$[J^{mn}, J^{pq}] = i(g^{rp}J^{m\sigma} - g^{n\sigma}J^{m\rho} - g^{n\sigma}J^{mp} + g^{m\sigma}J^{np})$$

$$\left. \begin{array}{l} \text{Ex 1 rotations: } J^{12} = -J^{21} \\ J^{23} = -J^{32} \\ J^{13} = -J^{31} \end{array} \right\} \Rightarrow 6 \text{ total metrics...}$$

$$\left. \begin{array}{l} \text{3 boostr} \\ J^{01} = -J^{10} \\ J^{02} = -J^{20} \\ J^{03} = -J^{30} \end{array} \right\}$$

Ex Consider the 4×4 matrix $(J^{mn})_{\alpha\beta}$ where μ, ν label which of the 6 metrics, while α, β label the component/matrix element...

$$(J^{mn})_{\alpha\beta} = i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu)$$

\hookrightarrow can verify that $(J^{mn})_{\alpha\beta}$ satisfies the comm. relation...

\rightarrow These are matrices that act on ordinary Lorentz 4-vectors...

to see this...

→ Look at elements of the Lorentz group

$$U(w_{\mu\nu}) = \exp \left[-i \frac{\omega_{\mu\nu}}{2} J^{\mu\nu} \right]$$

infinitesimally $\rightarrow \sim I + \frac{-i}{2} w_{\mu\nu} J^{\mu\nu}$

$$\sim \delta_p^\alpha + \frac{-i}{2} w_{\mu\nu} (J^{\mu\nu})_p^\alpha$$

So, infinitesimally...

$$V^\alpha \rightarrow \delta_p^\alpha + \frac{-i}{2} w_{\mu\nu} (J^{\mu\nu})_p^\alpha V^\beta$$

$w_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles.

$V_\alpha, V_\beta \rightarrow 4$ -vectors..

Ex 1 When $w_{12} = -w_{21} = \theta$, $w_{\mu\nu} = 0$ else, we get

$$[V^\mu] \rightarrow \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} [V^\mu]$$

→ Infinitesimal ROTATION on xy plane.

Ex 2 when $w_{01} = -w_{10} = \beta \Rightarrow$ get
 $w_{\mu\nu} = 0$ else

$$[V^\mu] \rightarrow \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} [V^\mu] \rightarrow \boxed{\text{BOOST along } x}$$

THE DIRAC EQUATION

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to develop formalism for finding all other ~~formalisms~~ representations...
(problem 3.1)

→ focus on spin $\frac{1}{2}$ systems...

→ In this case, use Dirac's trick due to -

Suppose we had a set of 4 $n \times n$ matrices γ^{μ} satisfying:

$$\{\gamma^{\mu}, \gamma^{\nu}\} = \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}\mathbf{I}$$

Then we could write down an n -dim representation of the Lorentz algebra...

$$S^{\mu\nu} = \frac{i}{4} [\gamma^{\mu}, \gamma^{\nu}]$$

These matrices satisfy the commutation relation...

$$\rightarrow [S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho})$$

→ Verify that this trick works in 3D Euclidean space

in which case, $\gamma^0 = \gamma^5$ → $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \sum_k i \epsilon^{ijk} \sigma^k = J^i$$

Which is what we saw before as angular momentum.

$$\left\{ J^1 = S^{12} = \frac{1}{2} \sigma^3 \right\}$$

$$\left\{ J^2 = S^{31} = \frac{1}{2} \sigma^2 \right\}$$

$$\left\{ J^3 = S^{23} = \frac{1}{2} \sigma^1 \right\}$$

→ now, want S^{mn} for 4D Minkowski space...

→ Matrices γ^m must next be at least 4×4 .

→ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2 \times 2} \\ \mathbb{I}_{2 \times 2} & 0 \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

"Weyl" / "Chiral" representations.

→ In this case, the boost + rotation generators are ..

Boots
in

$$S^{0i} = \frac{i}{4} [\gamma^0, \gamma^i] = -\frac{i}{2} \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

Rotations
in

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \sum_k \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} = \sum_k \frac{1}{2} \epsilon^{ijk} \sum_l \sigma^l$$

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at all transformations under which the ~~law~~^{Action's} ~~gives~~ ~~it~~ is invariant

→ In particular, we want \mathcal{S} to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\left\{ \begin{array}{l} \phi(x) \rightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x^\mu) \rightarrow \phi(\Lambda^{\mu\nu} x^\nu). \end{array} \right. \rightarrow \begin{array}{l} \text{check that} \\ \mathcal{S} \text{ is invariant} \end{array}$$

→ But this is very simple ... ⇒ There are many more transformations that leave \mathcal{S} Lorentz invariant.

→ How do we find all of them?

→ For simplicity, we'll just restrict ourselves to linear combinations of transformations

→ Look at transformations of the form

$$\phi_a(x) \rightarrow \sum_b M_{ab}(\Lambda) \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \rightarrow M(\Lambda) \phi}$$

These matrices M must be "nice" in the sense that M must obey...

Ex

$$\boxed{\phi \rightarrow M(\Lambda') M(\Lambda) \phi = M(\Lambda' \Lambda) \phi}$$

This says that $\{M\}$ (the collection of M 's) must be a representation of the Lorentz group.

What?? So, recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

 \rightarrow

$$\boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

of a group to

A representation Π is a function π satisfying the property

$$\pi(g_1) \pi(g_2) = \pi(g_1 g_2)$$

 \uparrow \uparrow \uparrow g_1 $\in G$ $\in G$

With this, it is clear that

$$\boxed{\{\Lambda\} \text{ Lorentz group} \Rightarrow \{M\} \text{ is a representation of } \{\Lambda\}}$$

So... what are these M ?

 \rightarrow Ex

Rotation group for spin $1/2$ particles

For spin - $\frac{1}{2}$, the most important nontrivial representation is the 2D representation:

→ These are unitary matrices with $\det = 1$
 (2×2)

$$\Rightarrow \text{In general: } U = e^{-i \vec{\sigma} \cdot \vec{\theta}/2}$$

$\vec{\sigma}$ → Pauli matrices
 $\vec{\theta}$ → angle.

For infinitesimal rotations, we can write

$$U = I - i \frac{\vec{\sigma}}{\hbar} \cdot \vec{\theta} = I - \vec{\tau} \cdot \vec{\theta}$$

{U} form a Lie-algebra of the L-group.

$\vec{\tau}$ here are the "generators" of the Lie algebra

when {U} is a representation of the rotational group, we identify

$$\vec{\tau} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{\tau}$ is the quantum angular momentum operator

→ satisfies the commutation relation

$$[\vec{\tau}^i, \vec{\tau}^j] = i \epsilon^{ijk} \vec{\tau}^k$$

like the generators of $SO(3)$, namely the Pauli matrices -

→ finite rotations are formed by matrix exp.

$$R = \exp\left[-i\theta^i \hat{J}^i\right]$$

\longleftarrow \rightarrow ~~Angular momentum~~

Sep 27, 2020

Back to present problem...

to get generator of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

$$(3D) \quad \vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\sigma})$$

$$(4D) \quad \boxed{J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\rho} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho})}$$

→ any matrices that are to represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu)}$$

→ by symmetry, μ, ν take label which of the six matrices we want;

→ α, β label components.

The Dirac Eqn.

What are the representations of the Lorentz group?
especially for spin- $\frac{1}{2}$?

Dirac's trick if we have a set of $4 \times n \times n$ matrices γ^μ which satisfies:

Dirac algebra

$$\rightarrow \boxed{\{ \gamma^\mu, \gamma^\nu \} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\gamma^{\mu\nu} \star I_{n \times n}}$$

Then the n -dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4} \{ \gamma^\mu, \gamma^\nu \}}$$

\rightarrow In other words, $S^{\mu\nu}$ satisfies:-

$$[\gamma^{\mu\nu}, \gamma^{\rho\sigma}] = i(g^{\mu\rho} S^{\nu\sigma} - g^{\nu\rho} S^{\mu\sigma} - g^{\mu\sigma} S^{\nu\rho} + g^{\nu\sigma} S^{\mu\rho})$$

* Note that this trick works also in any dim.

e.g. take $\gamma^0 = i\sigma^3$ so that $\{ \gamma^i, \gamma^j \} = -2\delta^{ij}$

$$\Rightarrow \boxed{S^{ij} = \frac{1}{2} \epsilon^{ijk} S^k} \rightarrow \text{just as before.}$$

2D representation of the rotation group.

$$\text{Spin } \frac{1}{2}: J^1 = J^{12} = \frac{1}{2}\sigma^3; J^2 = \frac{1}{2}\sigma^2 = S^{21}; J^3 = S^{23} = \frac{1}{2}\sigma^1$$

One such representation for the Dirac algebra is

$$\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4 \times 4}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4 \times 4}$$

Weyl / chiral representation.

get

$$\text{Boosts } S^{0j} = \frac{i}{4} [\gamma^0, \gamma^j] = \frac{-i}{2} \begin{pmatrix} \sigma^j & 0 \\ 0 & -\sigma^j \end{pmatrix}$$

Rotations

$$S^{ij} = \frac{i}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} 0 & \sigma^k \\ 0 & 0 \end{pmatrix} = \frac{1}{2} \epsilon^{ijk} \epsilon^{lk}$$

Hermitian Def'n

not rotation
but Ψ is also

classical

field, not a
wfn

All 4-component field Ψ that transforms under
boosts + rotations according to \rightarrow is called
a Dirac spinor

S^{ij} are Hermitian

S^{0i} are anti-Hermitian

∴ fine b/c Ψ is a classical field, not a wfn.

Now, what is the field eqn for ψ ?

→ try $(\Box + m^2)\psi = 0 \leftarrow \text{KG field eqn.}$

But this obviously works because the representations are block-diagonal...

→ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~at~~ at transformation of δ matrices

In an expression we can think of...

$$[\dots] \Delta_{\frac{1}{2}} [4 \times 4] \Delta_{\frac{1}{2}} [\cdot, \cdot] \xrightarrow{\frac{1}{2} \text{ for spin } \frac{1}{2}}$$

where $\Delta_{\frac{1}{2}} = \exp \left\{ \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right\}$

$$\simeq 1 - \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu}$$

$$\Rightarrow [\gamma^1] \rightarrow [\Delta_{\frac{1}{2}}] [\gamma^1] [\Delta_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right) \gamma^1 \left(1 - \frac{i}{2} \omega_{\mu\nu} S^{\mu\nu} \right)$$

$$= \dots \quad (\text{some terms of higher order cancelled...})$$

$$= \gamma^1 - \frac{i}{2} \omega_{\alpha\beta} \underbrace{[\gamma^1, S^{\alpha\beta}]}_{?}$$

above a quick computation shows that

$$[\gamma^\mu, \gamma^\nu] = (\gamma^{\mu\nu})_\nu \gamma^\nu$$

where

$$\gamma^{\mu\nu} = i(g^{\mu\alpha}\gamma_\nu - g^{\nu\alpha}\gamma_\nu)$$

So ...

$$\boxed{\gamma^\mu \rightarrow \gamma^\mu - \frac{i}{2} \omega_{\alpha\beta} (\gamma^{\alpha\beta})_\nu \gamma^\nu = \tilde{1}_{\frac{1}{2}} \gamma^\mu \tilde{1}_{\frac{1}{2}}}$$

$\rightarrow \gamma^\mu$ transforms like 4-vectors ... !

$\Rightarrow \gamma^\mu$ are invariant under simultaneous rotations of
their vectors & spinor indices.

I can treat " μ " or γ^μ as a vector index!

\rightarrow can dot γ^μ into ∂_μ to form a Lorentz-

inv. differential operator ...

Dine eqn

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

check that this is Lorentz-inv:

Lit $\psi(x) \rightarrow \tilde{1}_{\frac{1}{2}} + (\tilde{1}'x)$ then

$$i\gamma^\mu \partial_\mu \psi \rightarrow (i\gamma^\mu \tilde{1}_{\frac{1}{2}}) \partial_\mu (\psi(\tilde{1}'x))$$

$$= i\tilde{1}_{\frac{1}{2}} (\tilde{1}' \gamma^\mu \tilde{1}_{\frac{1}{2}}) \cdot (\tilde{1}')^\mu \partial_\mu (\psi(\tilde{1}'x))$$

some Lorentz transform

$$\begin{aligned}
 &= i \Delta_{\frac{1}{2}} (\Delta)^{\mu}_{\nu} \gamma^{\nu} \cdot (\Delta)_{\mu}^{\alpha} (\partial_2 \psi)(\Delta' x) \\
 &= i \Delta_{\frac{1}{2}} \gamma^{\mu} \underbrace{(\Delta)^{\mu}_{\nu} (\Delta)_{\nu}^{\alpha}}_{\delta^{\alpha}_{\nu}} (\partial_2 \psi)(\Delta' x) \\
 &= i \Delta_{\frac{1}{2}} \gamma^{\mu} \partial_{\mu} \psi(\Delta' x)
 \end{aligned}$$

$$\Rightarrow i \gamma^{\mu} \partial_{\mu} \psi(x) \rightarrow \Delta_{\frac{1}{2}} i \gamma^{\mu} \psi(\Delta' x)$$

→ transforms the same way as $\psi(\Delta' x)$

Cleaner way:

$$\begin{aligned}
 \text{Let } [i \gamma^{\mu} \partial_{\mu} - m] \psi(x) &\rightarrow [\overbrace{i \gamma^{\mu} (\Delta')^{\nu}_{\mu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \psi(\Delta' x)] \Delta_{\frac{1}{2}} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \Delta_{\frac{1}{2}}^{-1} [\overbrace{i \gamma^{\mu} (\Delta')^{\nu}_{\mu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \psi(\Delta' x)] \Delta_{\frac{1}{2}} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \left\{ i \gamma^{\mu} \overbrace{(\Delta')^{\nu}_{\mu} \partial_{\nu} - m}^{\Delta_{\frac{1}{2}}} \right\} \psi(\Delta' x) \\
 &= \Delta_{\frac{1}{2}} \left\{ i \gamma^{\nu} \partial_{\nu} - m \right\} \psi(\Delta' x) \\
 &= 0 \quad \checkmark
 \end{aligned}$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i \gamma^{\mu} \partial_{\mu} - m) \psi$$

$$\begin{aligned}
 \rightarrow 0 &= (-i \gamma^{\mu} \partial_{\mu} - m) (+i \gamma^{\nu} \partial_{\nu} - m) \psi \\
 &= (\gamma^{\mu} \gamma^{\nu} \partial_{\nu} \partial_{\mu} + m^2) \psi = ...
 \end{aligned}$$

$$\begin{aligned}
 &= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2) \psi \\
 &= \left[\frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu + m^2 \right] \psi \\
 &= \left[\frac{1}{2} \{ \gamma^\mu \gamma^\nu \} \partial_\mu \partial_\nu + m^2 \right] \psi \quad \xrightarrow{\text{LG eqn.}} \\
 &= (g^{\mu\nu} \partial_\mu \partial_\nu + m^2) \psi = (D + m^2) \psi = 0
 \end{aligned}$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger \psi$ doesn't work b/c under a boost,

$$\psi^\dagger \frac{1}{\sqrt{2}} \Delta_{\frac{1}{2}} \psi \neq \psi^\dagger \psi \text{ since } \frac{1}{\sqrt{2}} = \exp \left\{ -i \omega \gamma^\mu S^\mu \right\}$$

not unitary ... since not all $S^{\mu\nu}$ are Herms.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger \gamma^0}$$

Then under infinitesimal transform, set

$$\bar{\psi} \rightarrow \bar{\psi} \frac{1}{\sqrt{2}} \gamma^0 \simeq \bar{\psi} \left(1 + i \frac{1}{2} \omega_{\mu\nu} (S^{\mu\nu})^+ \right) \gamma^0$$

when ~~assume~~ $\omega_0 \neq 0 \Rightarrow \omega \neq 0$, $(S^{\mu\nu})^+ = (S^{\mu\nu})^-$

$$i (S^{\mu\nu} \leftrightarrow \gamma^0)$$

When $\mu=0$ or $\nu=0$, $(S^{\mu\nu})^+ = -S_{\mu\nu}^\mu$

$S^{\mu\nu}$ anti-commutes w/ γ^0 .

$$\rightarrow \bar{\psi} \rightarrow \psi^+ \left(1 + \frac{i}{2} \gamma_\mu \nu (S^{\mu\nu})^+ \right) \gamma^0$$

$$= \underbrace{\psi^+}_{\gamma^0} \gamma^0 \left(1 + \frac{i}{2} \gamma_\mu \nu S^{\mu\nu} \right)$$

$$= \bar{\psi} \left(1 + \frac{i}{2} \gamma_\mu \nu S^{\mu\nu} \right) = \bar{\psi} \gamma_1^{-1} \text{ as desired.}$$

$$\rightarrow \boxed{\bar{\psi} \rightarrow \bar{\psi} \gamma_1^{-1}}$$

and so $\boxed{\bar{\psi} \psi = \psi^+ \gamma^0 \psi}$ is a Lorentz scalar.

Similarly, can show that

$$\boxed{\bar{\psi} \gamma^\mu \psi}$$
 is a Lorentz vector.

\rightarrow the correct Lorentz-invariant Dirac Lagrangian is

$$\boxed{L_{\text{Dirac}} = \bar{\psi} (\gamma^\mu \partial_\mu - m) \psi}$$

{-L eqn for $\bar{\psi}$ gives $(\gamma^\mu \partial_\mu - m) \psi = 0$

{-L eqn for ψ gives $-i \partial_\mu \bar{\psi} \gamma^\mu - m \bar{\psi} = 0$

WEYL SPINOR

Recall that

$$\begin{aligned} S^{0j} &= \frac{-i}{2} \begin{pmatrix} \sigma^i & \alpha \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & \alpha \\ 0 & \sigma^k \end{pmatrix} \end{aligned}$$

Since block-diagonal \Rightarrow Dirac representation of the Lorentz group is reducible.

\rightarrow Can form 2-D representations by considering each block separately.

$$\rightarrow \text{write } \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \xrightarrow{\text{left-handed Weyl spinors}}$$

Under infinitesimal boost $\vec{\beta}$ + rotation $\vec{\theta}$, these transform as

$$\begin{aligned} \psi_L &\rightarrow \left(1 - i \vec{\theta} \cdot \vec{\sigma}/2 - i \vec{\beta} \cdot \vec{\sigma}/2 \right) \psi_L \\ \psi_R &\rightarrow \left(1 - i \vec{\theta} \cdot \vec{\sigma}/2 + i \vec{\beta} \cdot \vec{\sigma}/2 \right) \psi_R \end{aligned}$$

Recall that $(\tanh(\vec{\beta}) = \frac{1+i}{i})$.

\rightarrow Transform of ψ_R is equiv to trans of ψ_L^\dagger

By writing that

$$\psi_L^* \rightarrow \left(1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}^*}{2} - \vec{\beta} \cdot \frac{\vec{\sigma}^*}{2} \right) \psi_L^*$$

noting that $\vec{\sigma}^* \vec{\sigma}^* = -\vec{\sigma} \vec{\sigma}$ ($\vec{\sigma}^2 = \vec{\sigma}^2$)

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

we find.

$$\vec{\sigma}^2 \psi_L^* \rightarrow \vec{\sigma}^2 \left[1 + i\vec{\sigma} \cdot \frac{\vec{\sigma}}{2} - \vec{\beta} \cdot \frac{\vec{\sigma}^*}{2} \right] \psi_L^*$$

$$= \left[1 - i\vec{\sigma} \cdot \frac{\vec{\sigma}}{2} + \vec{\beta} \cdot \frac{\vec{\sigma}}{2} \right] \psi_L^*$$

like ψ_R transform.

$\underline{\text{So }} \vec{\sigma}^2 \psi_L^*$ transform like ψ_R ..

With $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$, the Dirac eqn has form.

$$(i\vec{\sigma}^m \partial_m - m) \Psi = 0 \Leftrightarrow \begin{pmatrix} -m & i(\vec{\sigma}_0 + \vec{\sigma} \cdot \vec{\beta}) \\ i(\vec{\sigma}_0 - \vec{\sigma} \cdot \vec{\beta}) & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

When $m=0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left\{ \begin{array}{l} i(\vec{\sigma}_0 - \vec{\sigma} \cdot \vec{\beta}) \psi_L = 0 \\ i(\vec{\sigma}_0 + \vec{\sigma} \cdot \vec{\beta}) \psi_R = 0 \end{array} \right\} \rightarrow \underline{\text{Welfl eqns.}}$$

\rightarrow important for neutrinos & weak force studies..

For convenience let us define -

$$\sigma^{\mu} = (1, \vec{\sigma}) ; \quad \bar{\sigma}^{\mu} = (1, -\vec{\sigma})$$

So that $\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix}$ $\sigma^{\mu} = (1, \vec{\sigma}, \vec{\sigma}^2, \vec{\sigma}^3)$

With this, can simply rotation. Dirac eqn becomes -

$$\begin{pmatrix} -m & i\vec{\sigma} \cdot \vec{\alpha} \\ i\vec{\sigma} \cdot \vec{\alpha} & m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

$$i(\vec{\alpha} + \vec{\sigma} \cdot \vec{\nabla}) \psi_L = 0$$

$$i(\vec{\alpha} - \vec{\sigma} \cdot \vec{\nabla}) \psi_R = 0$$

∴ the Weyl eqns become :

$$(i\vec{\sigma} \cdot \vec{\alpha}) \psi_L = 0$$

$$(i\vec{\sigma} \cdot \vec{\alpha}) \psi_R = 0$$

A hint

$$p^* = \sqrt{p^2 + m^2} = E_p$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p) e^{-ip \cdot x} , \quad p^2 = m^2$$

Look only solutions with positive frequency ... that is
 $E_p = p^0 > 0 \dots$

Ψ solves Dirac eqn $\rightarrow (i\gamma^\mu \partial_\mu - m)\Psi = 0$

$$\rightarrow (i\gamma^\mu \partial_\mu - m) u(p) e^{-ip \cdot x} = 0$$

$$\rightarrow \boxed{(i\gamma^\mu p_\mu - m) u(p) = 0}$$

Get rest frame $\Rightarrow p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $A_{1/2}$.

In rest frame, we have

$$(i\gamma^\mu p_\mu - m) u(p) \rightarrow (m\gamma^0 - m) u(p_0) = m(\gamma^0 - 1) u(p_0) = 0$$

$$\Rightarrow m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$$

$$\rightarrow \boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} \xrightarrow{\text{two-component spinor}}}$$

just a factor ξ with norm. constraint.

$$\xi^\dagger \xi = 1,$$

$\cancel{\alpha}$

What are those ξ ?

Look at rotation generators ...

$$\boxed{s^{ij} = \frac{1}{2} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

$$\text{In particular, } S^2 = S'^2 = \frac{1}{2} \begin{pmatrix} 6^2 & 0 \\ 0 & 0^2 \end{pmatrix}$$

$$\text{So if } \left\{ \begin{array}{l} S = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ S = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$$

$$\text{Now, we're in rest frame, so } p' = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now, boost to frame where particle has velocity ...

$$\vec{v} = v \cdot \hat{z} \cdot \circ \quad \text{Let } \tanh(\eta) = \frac{v}{c}.$$

↗ "rapidity"

$$\text{Then } \begin{pmatrix} E \\ p^3 \end{pmatrix} = p' = \begin{pmatrix} \cosh \eta & 0 & 0 & \sinh \eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \eta & 0 & 0 & \cosh \eta \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m \cosh \eta \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (\text{minh})$$

(infinitesimal $\frac{1}{2}$)

$\frac{1}{2} \rightarrow$ just the Lorentz transform.

$$\rightarrow \text{In this frame, } \left\{ \begin{array}{l} E = m \cosh \eta \\ p^3 = m \sinh \eta \end{array} \right.$$

Now, apply the same boost to $\alpha(p)$...

$$\begin{aligned} \alpha(p) &= \frac{1}{2} \sqrt{m} \begin{pmatrix} s \\ s \end{pmatrix} \rightarrow \left(\frac{1}{2} \right) = \exp \left(\frac{-i}{2} \omega_{\mu\nu} S^{\mu\nu} \right) \\ &= \exp \left(\frac{-i}{2} \gamma \begin{pmatrix} 0^3 & 0 \\ 0 & -0^3 \end{pmatrix} \right) \sqrt{m} \begin{pmatrix} s \\ s \end{pmatrix} \\ &\quad \text{~} \uparrow i \cdot (0^3 - s) \end{aligned}$$

So, infinitesimally -

$$\exp \left\{ \frac{-i}{2} \eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix} \right\} \approx \begin{pmatrix} \cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3 & 0 \\ 0 & \cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3 \end{pmatrix}$$

So Rest

$$u(p) \approx \sqrt{m} \begin{pmatrix} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) \xi \\ (\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) \xi \end{pmatrix} \quad \text{---}$$

Simplify ... note that

$$\begin{aligned} (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3)^2 &= \dots \\ &= \cosh \eta - \sinh \eta \sigma^3 \\ &= \frac{E}{m} - \frac{P^3}{m} \sigma^3 = \frac{p \cdot \sigma}{m} \\ &= \frac{p^\mu \sigma^\mu}{m} \quad \text{where } \sigma^\mu = (1, \vec{\sigma}) \end{aligned}$$

So ... $\left\{ (\cosh \frac{\eta}{2} - \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \sigma}{m}}$

$\& \quad (\cosh \frac{\eta}{2} + \sinh \frac{\eta}{2} \sigma^3) = \sqrt{\frac{p \cdot \bar{\sigma}}{m}}$

So -
$$\boxed{u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \xi \end{pmatrix}} \rightarrow \text{current = valid for any arbitrary direction of } p.$$

Fact
$$\left((p \cdot \sigma) (p \cdot \bar{\sigma}) = (p^\mu)^2 - \vec{p}^2 = p^2 = m^2 \right)$$

(61)

Now, back to example

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow p \cdot \sigma = \dots = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$$

$$\sqrt{p \cdot \sigma} = \begin{pmatrix} \sqrt{E - p^3} & 0 \\ 0 & \sqrt{E + p^3} \end{pmatrix}$$

and

$$\sqrt{p \cdot \bar{\sigma}} = \begin{pmatrix} \sqrt{E + p^3} & 0 \\ 0 & \sqrt{E - p^3} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E - p^3} (1) \\ \sqrt{E + p^3} (0) \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then (spin $-\frac{1}{2}$)

$$v(p) = \begin{pmatrix} \sqrt{E + p^3} (0) \\ \sqrt{E - p^3} (1) \end{pmatrix}$$

In the massless limit, $E \rightarrow p^3$ ($E^2 = \sqrt{mc^2 + (p^3)^2}$)

$$\Rightarrow \boxed{u(p) = \begin{pmatrix} (0) \\ \sqrt{2E} (1) \end{pmatrix} \text{ spin } \frac{1}{2}}$$

$$\boxed{v(p) = \begin{pmatrix} \sqrt{2E} (0) \\ (0) \end{pmatrix} \text{ spin } -\frac{1}{2}}$$

These states: $u(p) = \sqrt{2E} \begin{pmatrix} 0 \\ i \\ 0 \end{pmatrix}$, $u(p) = \sqrt{2E} \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}$ are eigenstates of the helicity operator

$$\boxed{h = \vec{p} \cdot \vec{S} = \sum_i \frac{1}{2} p_i^i \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}} = \frac{1}{2} \vec{p} \cdot \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

When $\{ h = \frac{1}{2} \Rightarrow \text{call Right-handed}$

$\{ h = -\frac{1}{2} \Rightarrow \text{call Left-handed}$

Note: Dirac helicity is frame-dependent... (for massive particle). — since can boost so that momentum is in the opposite direction,

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\left\{ \begin{array}{l} i(\partial_0 - \vec{\sigma} \cdot \vec{\nabla}) \psi_L = i(\vec{\sigma} \cdot \vec{\partial}) \psi_L = 0 \\ i(\partial_0 + \vec{\sigma} \cdot \vec{\nabla}) \psi_R = i(\vec{\sigma} \cdot \vec{\partial}) \psi_R = 0 \end{array} \right.$$

Plug $\psi = u(p) e^{-ip \cdot x} \sim$, $\partial_0 \rightarrow -iE$

$$\vec{\nabla} \rightarrow i\vec{p}$$

↓, with $m=0$, $\tilde{p} = E\vec{p}$.

$$\Rightarrow h = \frac{-1}{2}$$

$$\Rightarrow \text{get } \left\{ (E + E\vec{p} \cdot \vec{\sigma}) \psi_L = 0 \Rightarrow (E)(1+2h) \psi_L = 0 \right.$$

$$\left. (E - E\vec{p} \cdot \vec{\sigma}) \psi_R = 0 \Rightarrow (E)(1-2h) \psi_R = 0 \right. \Rightarrow h = \frac{1}{2}$$

$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}$, as expected

#

Recap... $\psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 > 0) \rightarrow \text{positive frequency}$

 $\Rightarrow u(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix} \rightarrow \text{spinor.}$

when $\psi(x) = u(p) e^{-ip \cdot x} \quad (p^0 < 0) \rightarrow \text{negative frequency}$

$$\Rightarrow (u(p)) = \dots \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ -\sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix}$$

#

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = (\xi^+ \sqrt{p \cdot \sigma} \xi^+ \sqrt{p \cdot \bar{\sigma}}) \cdot \begin{pmatrix} \sqrt{p \cdot \sigma} \xi \\ \sqrt{p \cdot \bar{\sigma}} \bar{\xi} \end{pmatrix}$$

$$= \xi^+ \left[(p \cdot \sigma) + (p \cdot \bar{\sigma}) \right] \xi$$

$$\Rightarrow u^\dagger u = 2E_p \xi^+ \xi \quad \xrightarrow{\text{depends on } p!}$$

\sim ~~also~~ $u^\dagger u$ is not a Lorentz-inv scalar.
just like $\psi^\dagger \psi$.

\Rightarrow to make one such Lorentz-inv scalar, define

$$\bar{u}(p) = u^\dagger(p) \gamma^0$$



$$\bar{u}u = 2m \xi^+ \xi \quad \begin{matrix} \text{Lorentz-inv} \\ (\text{indep of } \vec{p}) \end{matrix}$$

$$\text{L}, \text{ wish after } \bar{u}n = u^r \gamma^0 n = 2m \xi^+ \xi^- = 2m$$

→ convenient to choose ONB spinors, ξ^1, ξ^2 .

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} & \xi^s \\ \sqrt{p \cdot \bar{\sigma}} & \xi^s \end{pmatrix} \quad s=1,2}$$

Normalize:

$$\boxed{\bar{u}^r(p) u^s(p) = 2m \delta^{rs} \Leftrightarrow u^{r\dagger}(p) u^s(p) = 2E_p \delta^{rs}}$$

For the negative-freq solns, we get

$$\boxed{\bar{v}^r(p) v^s(p) = -2m \delta^{rs} \Leftrightarrow v^{r\dagger}(p) v^s(p) = +2E_p \delta^{rs}}$$

and

v, u are orthogonal to each other...

$$\boxed{\bar{u}^r(p) v^s(p) = \bar{v}^r(p) u^s(p) = 0}$$

†

Finally, talk about spin sumrs

→ useful when evaluating Feynman diagrams.

→ when we need to sum all spin- $\frac{1}{2}$ polarizations

Since $\{\xi^s\}$ form an ONB,

$$\sum_{s=1,2} \xi^s \xi^{s*} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using this, we find that

$$\begin{aligned} \sum_{s=1,2} n^s(p) \bar{n}^s(p) &= \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \sigma} \xi^s} \right) \cdot \left(\xi^{s*} \sqrt{p \cdot \sigma}, \xi^{s*} \sqrt{p \cdot \sigma} \right) \\ &= \sum_s \left(\frac{\sqrt{p \cdot \sigma} \xi^s}{\sqrt{p \cdot \sigma} \xi^s} \right) \cdot \left(\xi^{s*} \sqrt{p \cdot \sigma}, \xi^{s*} \sqrt{p \cdot \sigma} \right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \xrightarrow{\text{"completeness"}} &= \begin{pmatrix} p \cdot \sigma & m \\ m & p \cdot \sigma \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix} \end{aligned}$$

$$\begin{aligned} \sqrt{p \cdot \sigma} \sqrt{p \cdot \sigma} &= \sqrt{(p \cdot \sigma + p \cdot \sigma - p \cdot \sigma + p \cdot \sigma) m m} \\ &= \sqrt{(p \cdot \sigma)(p \cdot \sigma)} = \sqrt{((p^0, \vec{p}) \cdot (1, -\vec{\sigma})) ((p^0, \vec{p}) \cdot (1, \vec{\sigma}^2))} \\ &= \sqrt{(p^0)^2 - p^2} = m. \end{aligned}$$

$\frac{p}{\cancel{p}}$

$$\boxed{\sum_{s=1,2} n^s(p) \bar{n}^s(p) = \begin{pmatrix} m & p \cdot \sigma \\ p \cdot \sigma & m \end{pmatrix} = p \cdot \gamma + m I} \quad \begin{array}{l} \text{Feyn-} \\ \text{man's} \\ \text{slash} \\ \text{notation} \end{array}$$

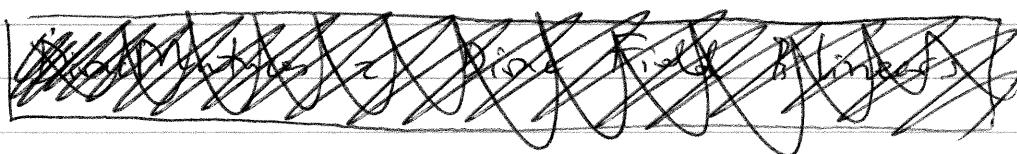
$\frac{p}{\cancel{p}}$

$$\boxed{\sum_{s=1,2} n^s(p) \bar{n}^s(p) = \begin{pmatrix} -m & p \cdot \sigma \\ p \cdot \sigma & -m \end{pmatrix} = p \cdot \gamma - m I}$$

→ The combos $\partial \cdot p$ occur so often that Feynman introduced the notation:

$$\not{p} = \partial^\mu p_\mu = p_\mu \partial^\mu$$

#

Exercise

Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L .
The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

m = Majorana mass.

- (a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.
- (b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

a) In general, infinitesimal Lorentz transform on $\Psi_L(x)$ has the form

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} + \tilde{\rho} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

→ Rotation has the form:

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(x)$$

$$\Rightarrow \sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L(x) \quad (\text{notes})$$

Lorentz transformed:

$$\Psi_L(x) \rightarrow \Lambda_{\frac{1}{2}} \Psi_L(\Lambda^{-1}x)$$

$$\partial_\mu \Psi_L(x) \rightarrow (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x)$$

→ put these together ...

$$\Psi_L(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \Psi_L(\Lambda^{-1}x)$$

$$\sigma^2 \Psi_L^*(x) \rightarrow \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(\Lambda^{-1}x)$$

$$\Rightarrow -im \sigma^2 \Psi_L^*(x) \rightarrow -im \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \sigma^2 \Psi_L^*(\Lambda^{-1}x)$$

$$\text{Next, } i\vec{\sigma} \cdot \partial \Psi_L(x) = i\vec{\sigma}^\mu \partial_\mu \Psi_L(x)$$

$$\Rightarrow i\vec{\sigma}^\mu (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x) \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right)$$

$$= i\vec{\sigma}^\mu \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) (\Lambda^{-1})^\alpha_\mu \partial_\alpha \Psi_L(\Lambda^{-1}x)$$

we find: multiply:

$$1 = \left(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \left(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}\right) \quad (\text{rot} \times \text{inv.-rot})$$

$$\Rightarrow \rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})(1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\mu (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \\ \times (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \boxed{?} (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

Want is $\boxed{?}$

$$\rightarrow \boxed{?} = (1 + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\mu (1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2})$$

$$\approx \bar{\sigma}^\mu + i\vec{\theta} \cdot \frac{\vec{\sigma}}{2} \bar{\sigma}^\mu - i\vec{\theta} \cdot \vec{\sigma} \frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^\mu - \frac{i}{2} \vec{\theta} [\bar{\sigma}^\mu, \frac{\vec{\sigma}}{2}]$$

\downarrow
 \downarrow can show want

$$= \bar{\sigma}^\mu - i\vec{\theta} [J_\mu^{\alpha\beta}] \bar{\sigma}^\nu$$

\downarrow

$$i(g^{\mu\nu} \delta_\nu^\alpha - g^{\mu\nu} \delta_\nu^\alpha)$$

$$\Rightarrow \boxed{?} = (\Delta_4)^\mu_\nu \bar{\sigma}^\nu \rightarrow \bar{\sigma}^\mu transforms like 4-vector$$

$$\Rightarrow i\vec{\theta} \cdot \partial \Psi_L(x) \Rightarrow i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \Delta_\nu^\mu \bar{\sigma}^\nu (\Delta')^\alpha_\mu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \delta_\nu^\alpha \bar{\sigma}^\nu \partial_\alpha \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^\nu \partial_\nu \Psi_L(\Delta' x)$$

$$= i(1 - i\vec{\theta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma} \cdot \partial \Psi_L(\Delta' x)$$

✓

$$\Rightarrow i\bar{\sigma} \cdot \partial \psi_c(x) - im \bar{\sigma}^2 \psi_c^*(x) = 0$$

\rightarrow due to infinitesimal rotations ...

$$(1 - i\tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \underbrace{\{ i\bar{\sigma} \cdot \partial \psi_c(\tilde{x}) - im \bar{\sigma}^2 \psi_c^*(\tilde{x}) \}}_{=0} = 0$$

\Rightarrow done! So Majorana eqn is invariant under infinitesimal rotations.

\rightarrow

① Bosons (proceed in a similar way ...)

Key

$$(1 - \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \tilde{\beta} \cdot \frac{\vec{\sigma}}{2})$$

$$= \bar{\sigma}^M - \frac{1}{2} \tilde{\beta} \{ \bar{\sigma}^M, \bar{\sigma} \}$$

$$\Rightarrow \bar{\sigma}^M \rightarrow \bar{\sigma}^M - i\tilde{\beta} [\bar{\sigma}^M]_r \bar{\sigma}^r$$

$$\Rightarrow (1 - \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) \bar{\sigma}^M (1 + \tilde{\beta} \cdot \frac{\vec{\sigma}}{2}) = (1)_r \bar{\sigma}^r \text{ as before}$$

so plug this back into the original eqn.

\cancel{x}

Sep 28, 2020

Dirac Matrices & Dirac Field Bilinears

Oct 2, 2020 Recall that $\bar{\psi}\psi$ is Lorentz scalar...

Recall that $\bar{\psi}\gamma^\mu\psi$ is also a 4-vector.

$\Rightarrow \boxed{?}$ Consider $\bar{\psi}\tilde{\Gamma}\psi$, where $\tilde{\Gamma}$ is any 4×4

\rightarrow can we decompose $\tilde{\Gamma}$ into terms that have definite transformation properties under the Lorentz group?

$\hookrightarrow \tilde{\Gamma}$ can be written as combo of 16-element basis defined by

$$\left. \begin{array}{lll}
 1: & \mathbb{1} & \rightarrow 1 \\
 4: & \gamma^\mu & \rightarrow 4C2 \\
 6: & \gamma^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \equiv \gamma^{\mu\nu\rho\sigma} & \rightarrow 4C3 \\
 4: & \gamma^{\mu\nu\rho} = \gamma^{\mu\rho}\gamma^{\nu\rho} & \rightarrow 4C2 \\
 1: & \gamma^{\mu\nu\rho\sigma} = \gamma^{\mu\rho}\gamma^{\nu\sigma} & \rightarrow 4C2
 \end{array} \right\}$$

16 total.

\rightarrow all are anti-symmetric products.

\rightarrow Each set of matrices transform as an antisymmetric tensor of successively higher ranks.

\rightarrow Introduct

$$\bar{\psi}^s = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$= -\frac{i}{4!} \epsilon^{\mu\nu\rho\sigma} \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma$$

$$\begin{aligned}
 0123 &\rightarrow 1 \\
 7023 &\rightarrow -1
 \end{aligned}$$

\hookrightarrow totally anti-symmetric

$$\text{Note that } \rightarrow \boxed{(8^5)^2 = 11}$$

$$\rightarrow \overline{(Y^s)^+} = -i(Y^?)^+ \dots = (Y^e)^+$$

$$= + \gamma^2 \gamma^2 \gamma^1 \gamma^0 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \gamma^5$$

also

$$\{g^s, g^m\} = i g^0 g^1 g^2 g^3 g^m + i g^m g^0 g^1 g^2 g^3 \\ = 0$$

and Hens

$$[\gamma^5, \gamma^{\mu\nu}] = [\gamma^5, \frac{i}{4} \{ \gamma^\mu, \gamma^\nu \}] = 0$$

\Rightarrow Eigenstates of γ^5 with different eigenvalues don't mix under Lorentz-transf.m.

→ In basis, can write

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \text{for } \Psi_L \text{ (left-hd)} \\ \rightarrow \text{for } \Psi_R \text{ (right-hd),}$$

\rightarrow a Dirac spinor with only L/R component is an eigenstate of γ^5 with eig $\sqrt{(-1)/(1)}$.

With δ^5 , can rewrite the table of 4×4 matrices as

$\frac{1}{2}$	scalar	1
γ^m	vector	4
$\delta^{MV} = \frac{1}{2}\{\gamma^m, \gamma^n\}$	tensor	6
$\gamma^M \gamma^S$	pseudo vector	4
γ^S	pseudo scalar	<u>1</u>
		16

pseudo-vector/scalar is due to the fact that they transform like vector/scalar, BUT with an additional under Lorentz transf \rightarrow in charge under parity-transf.

Ex Parity transf: $\vec{x} \rightarrow -\vec{x}$

$$\hookrightarrow (x^0, x^i) \rightarrow (x^0, -x^i)$$

If instead $(x^0, \vec{x}) \rightarrow -(\vec{x}, -x^0) = (-x^0, \vec{x})$
under parity, we call this a pseudo-vector

\rightarrow pseudo vector/scalar flips sign under parity transf.

\rightarrow From vector + pseudo-vector we can form 2 currents out of Dirac field bilinears -

$j^\mu(x) = \bar{\psi}(x) \gamma^\mu \psi(x) \rightarrow$ vector current
$j^{\mu 5}(x) = \bar{\psi}(x) \gamma^\mu \gamma^5 \psi(x) \rightarrow$ pseudo vector current

Assume that ψ satisfies Dirac eqn.. $\bar{\psi} = \psi^\dagger \gamma^0$

$$\rightarrow i \not{D} \psi = m \psi \quad \rightarrow i \not{D} \bar{\psi} = m \bar{\psi} \quad (\text{Given } \not{D} = \not{\partial}^0,$$

\rightarrow compute div of these currents -

$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi}) \not{\partial}^\mu \psi + \bar{\psi} \not{\partial}^\mu (\partial_\mu \psi)$$

$$= i m \bar{\psi} \psi + \bar{\psi} (-i m \psi) = 0$$

$$\rightarrow \boxed{\partial_\mu j^\mu = 0}$$

$\rightarrow j^m$ is always conserved if $\psi(x)$ satisfies
Dirac eqn

\rightarrow It is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{i\alpha} \psi(x)$$

Similarity

$$\begin{aligned}\partial_m j^{ms} &= (\partial_m \bar{\psi}) \gamma^m \gamma^5 \psi + \cancel{\bar{\psi} \gamma^m \gamma^5 \partial_m \psi} \\ &= (\partial_m \bar{\psi}) \gamma^m \gamma^5 \psi + (-1) \bar{\psi} \gamma^5 \gamma^m \cancel{\partial_m \psi} \\ &= \text{im } \bar{\psi} \gamma^5 \psi + (-1)(-i) m \bar{\psi} \gamma^5 \psi\end{aligned}$$

$\rightarrow \boxed{\partial_m j^{ms} = 2 \text{im } \bar{\psi} \gamma^5 \psi} \rightsquigarrow$ axial vector current

\rightarrow if $m=0$ then $\partial_m j^{ms}$ is conserved.

\rightarrow When $m=0$, j^m is the Noether current for the symmetry

$$\psi(x) \rightarrow e^{i\alpha \gamma^5} \psi(x)$$

(we worry about the rest of this section in ~~Wojciech~~ Pashkin's ...)

-4

QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\psi} (\gamma \not{d} - m) \psi = \bar{\psi} (\gamma^0 \partial_0 + \vec{\gamma} \cdot \vec{\partial}) \psi - m \bar{\psi} \psi .$$

→ Canonical momentum conjugate to ψ is

$$\frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} = \gamma^0 \bar{\psi} \gamma^0 = \gamma^0 \bar{\psi} \gamma^0 = i \psi^+ .$$

→ Hamiltonian is given by

$$\begin{aligned} \mathcal{H} &= \frac{\partial \mathcal{L}}{\partial (\partial_0 \psi)} \partial_0 \psi - \mathcal{L} = i \bar{\psi} \gamma^0 \partial_0 \psi - i \bar{\psi} \gamma^0 \partial_0 \psi \\ &\quad - i \bar{\psi} \vec{\gamma} \cdot \vec{\partial} \psi + m \bar{\psi} \psi \\ &= -i \bar{\psi} \vec{\gamma} \cdot \vec{\partial} \psi + m \bar{\psi} \psi \end{aligned}$$

Thus,

$$\mathcal{H} = \int \mathcal{H} d^3x = \int d^3x \bar{\psi} (-i \vec{\gamma} \cdot \vec{\partial} + m) \psi$$

→ now let's figure out the commutators to make this a quantum field theory...

→ DO NOT QUANTIZE THE DIRAC FIELD

This won't work!

Guess $\left[\psi_a(\vec{x}), i\psi_b^+(\vec{y}) \right] = i\delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}$

↑ spin ↑
components $(a, b = 1, 2, 3, 4)$

i.e.

$$\left[\psi_a(\vec{x}), \psi_b^+(\vec{y}) \right] = \delta^{(3)}(\vec{x}-\vec{y}) \delta_{ab}.$$

In matrix notation ...

$$\left[\psi_a(\vec{x}), \psi_b^+(\vec{y}) \right] = \mathbf{1}_{4 \times 4} \delta^{(3)}(\vec{x}-\vec{y})$$

↓ ↓
[:] [---]

Also guess $\left[\psi_a(\vec{x}), \psi_b(\vec{y}) \right] = 0$

$$\left[\psi_a^+(\vec{x}), \psi_b^+(\vec{y}) \right] = 0$$

No heat

$$\left[\psi(\vec{x}), \psi(\vec{y}) \right] = \left[\psi(\vec{x}), \psi^+(\vec{y}) \right] \gamma^0$$

$$= \left[\psi(\vec{x}), \psi^+(\vec{y}) \right] \gamma^0 = \gamma^0 \delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote -

(real) field $\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \vec{a}_p + \vec{a}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}$. (FT)

For complex field \rightarrow we get

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \vec{a}_p + \vec{b}_{-p}^\dagger \right\} e^{i\vec{p}\cdot\vec{x}}.$$

In the case of Dirac field, need spin degrees of freedom.

Try --

$$\Psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{+i\vec{p} \cdot \vec{x}}$$

↑
Spin degrees of freedom

Former components: $\Psi(\vec{x}) = u(p) e^{i\vec{p} \cdot \vec{x}}$

$$2 \quad \Psi^+(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ \hat{a}_p^r u^r(p) + \hat{b}_{-\vec{p}}^r v^r(-\vec{p}) \right\} e^{-i\vec{p} \cdot \vec{x}}$$

Recall about u, v also solves Dirac eqn in the reverse
heat (in momentum space --)

$$p^m \delta_m u^r(p) = mu^r(p) \quad p^m \delta_m v^r(p) = -mv^r(p)$$

We can by the commutators --

$$[\hat{a}_p^r, \hat{a}_{p'}^{s\dagger}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{b}_p^r, \hat{b}_{p'}^{s\dagger}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\hat{a}_p^r, \hat{b}_{p'}^{s\dagger}] = 0$$

The rest are all zero --

We find heat \rightarrow as desired --

$$[\Psi_a(\vec{x}), \Psi_b(\vec{y})] = 0 = [\Psi_a^+(\vec{x}), \Psi_b^+(\vec{y})]$$

77

We also find that

$$\{\Psi_a(\vec{x}), \Psi_b^+(\vec{y})\} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian ...

$$H = \int d^3x \left[-i\vec{\nabla} \cdot \vec{\psi} + m\vec{\psi}\vec{\psi} \right]$$

$$= \int d^3x \left\{ \psi^0 \underbrace{\left[-i\vec{\nabla} \cdot \vec{\psi} + m \right]}_{\text{just const}} \psi \right\}$$

just const

$$\text{Now, with } p^m \partial_\mu u^r(p) = mu^r(p)$$

$$\rightarrow (\vec{p} \cdot \vec{\nabla} + m) u^r(p) = p^0 \delta^0 u^r(p) = E_p \delta^0 u^r(p)$$

$$\text{Similarly, SIC } p^m \partial_\mu v^r(p) = -mv^r(p)$$

$$(\vec{p} \cdot \vec{\nabla} + m) v^r(p) = -E_p \delta^0 v^r(p).$$

So ...

$$\rightarrow [-i\vec{\nabla} \cdot \vec{\psi} + m] \psi = [-i\vec{\nabla} \cdot \vec{\psi} + m] \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} [a_p^r u_p^r + b_p^r v_p^r] e^{ip \cdot \vec{x}}$$

$$= \psi^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ E_p a_p^r u_p^r(p) - E_p b_p^r v_p^r(p) \right\} e^{ip \cdot \vec{x}}$$

So ...

$$H = \int d^3x \left\{ \psi^+ \psi^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \dots \right\} e^{ip \cdot \vec{x}} \right\}$$

play in ...

$$\rightarrow H = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} E_p \left\{ a_p^r a_p^r - b_{+p}^r b_{+p}^{r+} \right\}$$

\downarrow
 $b_{+p}^{r+} b_{+p}^r + \text{const}$

!

→ By creating more and more particles with b_{+p}^r , we can lower the energy indefinitely

→ This is bad...

→ So we should use Fermi-Dirac statistics instead → anti-commutators instead of commutators...

Requirement.

$$\left\{ a_p^r, a_q^{s+} \right\} = \left\{ b_{+p}^r, b_{+q}^{s+} \right\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p} - \vec{q})$$

↑
 no longer harmonic! ↗ all other
 anti-commutators
 are zero...

When this is true, we find that

$$\left\{ \psi_a(\vec{x}), \psi_b^+(\vec{y}) \right\} = S^{(3)}(\vec{x} - \vec{y}) \delta_{ab}$$

$$\left\{ \psi_a(\vec{x}), \psi_b(\vec{y}) \right\} = \left\{ \psi_a^+(\vec{x}), \psi_b^+(\vec{y}) \right\} = 0$$

where we're using

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p^r u_r(\vec{p}) + b_{-p}^{r+} v_r(-\vec{p}) \right] e^{i\vec{p} \cdot \vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p (\hat{a}_p^{rt} \hat{a}_p^r - \hat{b}_{-p}^r \hat{b}_{-p}^{rt}) - \hat{b}_{-p}^{rt} \hat{b}_p^r + \text{const}$$

$$\Rightarrow \mathcal{H} = \boxed{\int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ \hat{a}_p^{rt} \hat{a}_p^r + \hat{b}_{-p}^r \hat{b}_{-p}^{rt} \right\}}$$

now good, b/c E is bold below...

→ also can compute

$$\tilde{P} = \int \frac{d^3 p}{(2\pi)^3} \sum_{r=1,2} \tilde{p} (\hat{a}_p^{rt} \hat{a}_p^r + \hat{b}_{-p}^{rt} \hat{b}_{-p}^r)$$

To avoid sign confusion, we will usually write

$$\Psi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(\hat{a}_p^r u_r^r(\vec{p}) e^{+i\vec{p} \cdot \vec{x}} + \hat{b}_{-p}^{rt} v_r^r(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} \right)$$

As a Heisenberg field,

$$\Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(\hat{a}_p^r u_r^r(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} + \hat{b}_{-p}^{rt} v_r^r(\vec{p}) e^{+i\vec{p} \cdot \vec{x}} \right)$$

where:

- | | | |
|---|---------------------|------------------------------|
| { | \hat{a}_p^r | : annihilates particles |
| | \hat{a}_p^{rt} | : creates particles |
| | \hat{b}_p^r | : annihilates anti-particles |
| | \hat{b}_{-p}^{rt} | : creates anti-particles. |

Vacuum state as $|0\rangle$ where

$$\begin{cases} \hat{a}_p^\dagger |0\rangle = 0 \\ \hat{b}_p^\dagger |0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ conserved norm:

$$|\vec{p}, s\rangle = \sqrt{2E_p} \hat{a}_p^\dagger |0\rangle$$

so that

$$|\vec{p}, s\rangle |\vec{q}, r\rangle = \sqrt{2E_p} \sqrt{2E_q} \delta^{(3)}(\vec{p} - \vec{q}) \delta^{rs}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lorentz}} \psi'(x) = \gamma \frac{1}{\sqrt{2}} \psi(\gamma^{-1} x)$$

recall that with $\omega_{12} = -\omega_{21} = \theta$

$$\begin{cases} \omega_{12} = -\omega_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2} [\gamma^\mu, \gamma^\nu] \end{cases} \Rightarrow \exp\left\{-i\omega_{\mu\nu} \gamma^\nu \frac{\gamma^\mu}{2}\right\} = 1 - i\vec{\theta} \cdot \vec{\gamma}$$

$$= 1 - i\vec{\theta} \cdot \vec{\gamma}$$

$$\rightarrow \text{and } \psi(\gamma^{-1} x) \approx [1 - \vec{\theta} \cdot \vec{\gamma}] \psi(x)$$

$$\vec{\gamma} = \vec{x} \times (-i\vec{\nabla})$$

so we'd $\psi \rightarrow \psi + S\psi$ where

$$S\psi = \psi' - \psi = \left(\frac{i}{2}\vec{\theta} \cdot \vec{\gamma}\right)\psi(x) - \vec{\theta} \cdot (\vec{x} \times \vec{\nabla})\psi(x)$$

By Noether's Thm,

$$\vec{J}_{\text{total}} \text{ (total spin)} = \int \frac{d^3x}{2} \left[\bar{\psi}^\dagger (-i\vec{\gamma} \cdot \vec{\nabla}) \psi + \frac{1}{2} \bar{\psi}^\dagger \vec{\Sigma} \psi \right].$$

~~to~~

We won't worry about the rest of this section about propagators

\rightarrow we'll come back to them later when looking at Feynman diagrams.

~~to~~

DISCRETE SYMMETRIES OF THE DIRAC THEORY

Basically, we have

Parity — Time Reversal

Charge
Conjugation

~~to~~

Recall that we before, we looked at implementation of continuous Lorentz transform -

\rightarrow found that $\gamma_1 \in$ Lorentz group

$\exists U(1)$ unitary for which

$$U(1) \psi(x) U(1)^\dagger = \gamma_2' \psi(\gamma_1 x).$$

\rightarrow Now, we'll look about discrete symmetries on the Dirac field.

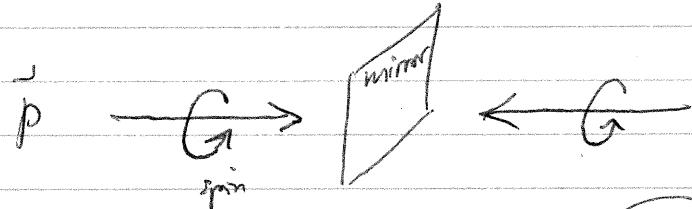
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant:

→ e.g. { time-reversal },
{ parity }.

[Parity] (P) : flips direction of spatial vectors

$$P: (t, \vec{x}) \rightarrow (t, -\vec{x})$$

↔ mirror sym → change the handedness.



→ Note momentum flip sign, but spin is unchanged.

[Time-reversal]

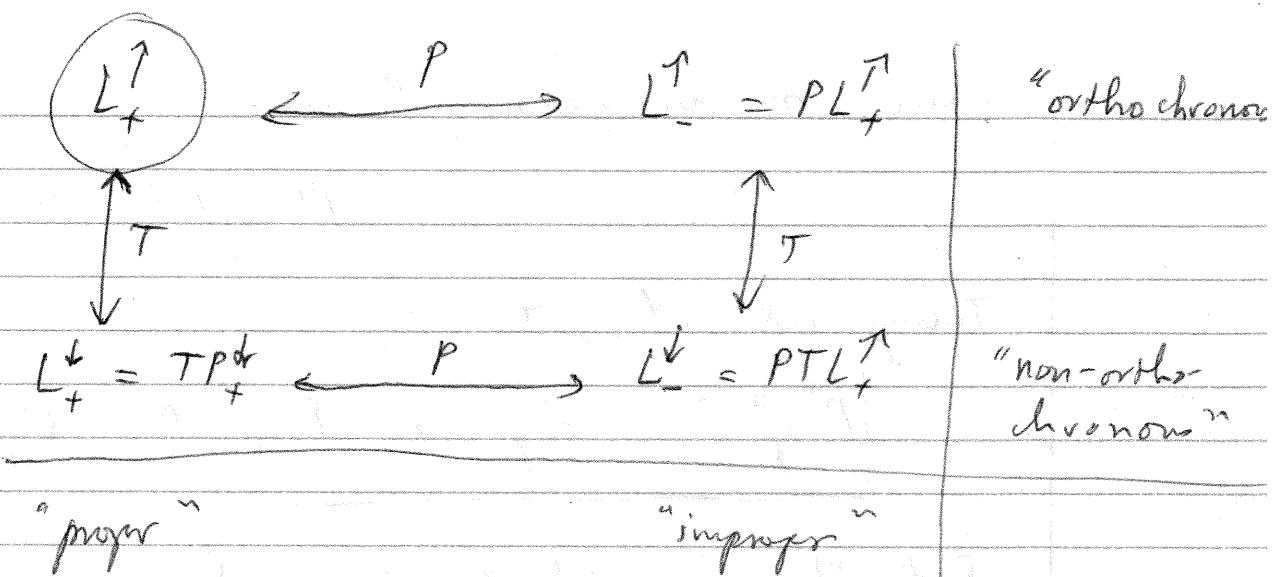
$$T: (t, \vec{x}) \rightarrow (-t, \vec{x})$$

P,T don't belong to the "proper" Lorentz group L_+

→ the full Lorentz group breaks into 4 disjoint subsets ...

(L)

(03)



charge conjugation \rightarrow intercharge particles & anti-particles.

\hookrightarrow non-space-time.

Let's look at Parity.

Note that because $P: (t, \vec{x}) \rightarrow (t, -\vec{x})$

\rightarrow momentum flips sign

but not spin! \rightarrow what is P ? As an operator?

$$\xrightarrow{\text{---}} \xrightarrow{\text{---}} \xleftarrow{\text{---}} \xleftarrow{\text{---}}$$

As an operator on creation/annihilation ops, we want

$$P^\dagger a_{\vec{p}}^s P = a_{\vec{p}}^s \quad \& \quad P^\dagger b_{\vec{p}}^s P = b_{-\vec{p}}^s$$

where, as discussed, P must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, set

$$\boxed{P^\dagger \tilde{a}_p^s P = a_{-\vec{p}}^{s\dagger} \quad P^\dagger \tilde{b}_p^s P = b_{-\vec{p}}^{s\dagger}}$$

But there might be too restrictive --- we can get better constraints by requiring that:

$$\boxed{P^\dagger \tilde{a}_p^s P = \eta_a a_{-\vec{p}}^{s\dagger} \quad P^\dagger \tilde{b}_p^s P = \eta_b b_{-\vec{p}}^{s\dagger}}$$

as long as $\eta_a^2 = (\eta_b)^2 = 1$ are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel:

$$\left\{ \begin{array}{l} P^\dagger \tilde{a}_p^s \tilde{a}_{\vec{p}}^{s\dagger} P = a_{-\vec{p}}^{s\dagger} a_{-\vec{p}}^s \\ P^\dagger \tilde{b}_p^s \tilde{b}_{\vec{p}}^{s\dagger} P = b_{-\vec{p}}^{s\dagger} b_{-\vec{p}}^s \end{array} \right.$$

With this, let's ~~see~~ implement parity condition on $\psi(x)$

$$\rightarrow P^\dagger \psi P = ? \quad \left(\begin{array}{l} \text{to find out what these} \\ \eta_a + \eta_b \text{ must be...} \end{array} \right)$$

$$P^t \chi(x) P = \int \frac{1}{(2\pi)^3} \frac{1}{\sqrt{2\tilde{p}}} \sum_{s=1,2} (\gamma_a^s a_{-\vec{p}}^s u^s(p) e^{-i\tilde{p} \cdot \vec{x}} + \gamma_b^s b_{-\vec{p}}^s v^s(\vec{p}) e^{i\tilde{p} \cdot \vec{x}})$$

Define $\begin{cases} \tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}) \end{cases}$

Note that

$$u^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ \sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} \quad \text{where } \sigma = (1, \vec{\sigma}) \quad \bar{\sigma} = (1, -\vec{\sigma})$$

$$\begin{aligned} &= \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} u^s(-\tilde{p}) \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} \xi^s \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u^s(-\tilde{p}) \end{aligned}$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\tilde{p})}$$

and

$$v^s(p) = \begin{pmatrix} \sqrt{p \cdot \sigma} \xi^s \\ -\sqrt{p \cdot \bar{\sigma}} \xi^s \end{pmatrix} = \dots = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} v^s(-\tilde{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(-\tilde{p})}$$

With these, we find that

$$\tilde{p} \cdot \tilde{x} = p \cdot x$$

(86)

$$P^+ \bar{\psi}(x) P = 8^0 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \left(\gamma_a \frac{a^s}{-p} u^s(-p) e^{-ip \cdot \tilde{x}} + \gamma_b^* \frac{b^s}{-p} v^s(-p) e^{ip \cdot \tilde{x}} \right)$$

Now, notice that if $\gamma_a = \gamma_b^*$ then it's "nice":

$$(\gamma_a = \gamma_b^*) \Rightarrow P \bar{\psi}(x) P = \gamma_a 8^0 \bar{\psi}(\tilde{x}) \quad \rightarrow P_{\text{transf}} \text{ in final form}$$

\rightarrow sufficient to choose $\gamma_a = 1 = -\gamma_b^*$

relative sign between fermions - antifermions --

-4

Now, useful to know how various Dirac field bilinears transform under parity ...

Recall ... 5 of them:

$$\bar{\psi} \psi, \bar{\psi} \gamma^\mu \psi, ; \bar{\psi} [\gamma^\mu, \gamma^\nu] \psi$$

$$\bar{\psi} \gamma^\mu \gamma^\nu \psi, ; \bar{\psi} \gamma^\nu \psi.$$

\rightarrow find these, first compute: $P \bar{\psi}(x) P$ --

$$P^+ \bar{\psi}(x) P = P^+ \bar{\psi}^+(x) \gamma^0 P \stackrel{\curvearrowright}{=} (P^+ \bar{\psi} P)^+ \gamma^0 \quad (\gamma^0 = \gamma^0)$$

$$\rightarrow = \gamma_a^* (\gamma^0 \bar{\psi}(\tilde{x}))^+ \gamma^0 = \gamma_a^* \bar{\psi}^+(\tilde{x}) \gamma^0 \gamma^0$$

$$\rightarrow \boxed{P^+ \bar{\psi} P = \gamma_a^* \bar{\psi}(\tilde{x}) \gamma^0}$$

With this --

$$\begin{aligned}
 p^\dagger \bar{\psi} \psi p &= \underbrace{p^\dagger \bar{\psi}(x) p}_{(x)(x)} \underbrace{p^\dagger \psi(x) p}_{\text{II}} \\
 &= \gamma_a^\dagger \bar{\psi}(\tilde{x}) \gamma^0 \gamma_a \gamma^0 \psi(\tilde{x}) \\
 &= |\gamma_a|^2 \bar{\psi}(\tilde{x}) \psi(\tilde{x})
 \end{aligned}$$

scalar

$$p^\dagger \bar{\psi} \psi p(x) = \bar{\psi} \psi(\tilde{x}). \quad (\text{scalar})$$

scalar.

can also show --

$$\begin{aligned}
 p^\dagger \bar{\psi}(x) \underbrace{\gamma^\mu}_{\text{(vector)} \atop \text{fields}} \psi p &= \bar{\psi} \gamma^0 \gamma^\mu \gamma^0 \psi(\tilde{x}) \\
 &= \begin{cases} + \bar{\psi} \gamma^\mu \psi(\tilde{x}) & \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi(\tilde{x}) & \mu = 1, 2, 3 \end{cases}
 \end{aligned}$$

$$p^\dagger (i \cancel{\partial} \gamma^5 \psi) p = i \bar{\psi}(\tilde{x}) \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) = -i \cancel{\partial} \gamma^5 \psi(\tilde{x})$$

↑
pseudo
scalar
(-)

~~$$\begin{aligned}
 &\cancel{\partial} \gamma^5 \gamma^0 \gamma^5 \gamma^0 \psi(\tilde{x}) \\
 &+ \cancel{\partial} \gamma^5 \gamma^0 \gamma^5 \psi(\tilde{x}) \\
 &- \cancel{\partial} \gamma^5 \gamma^0 \gamma^5 \psi(\tilde{x}) \\
 &\mu = 0, 1, 2, 3
 \end{aligned}$$~~

$$p^\dagger \bar{\psi} \gamma^\mu \gamma^5 \psi p = \bar{\psi}(\tilde{x}) \gamma^0 \gamma^\mu \gamma^5 \gamma^0 \psi(\tilde{x})$$

↑
pseudo
vector.
(-)

$$\begin{aligned}
 &= \begin{cases} - \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) & \mu = 0 \\ + \bar{\psi} \gamma^\mu \gamma^5 \psi(\tilde{x}) & \mu = 1, 2, 3 \end{cases}
 \end{aligned}$$

Note The relative sign: $-\gamma_a = \gamma_b^*$ is important.

for the relationship between fermion - anti - fermi

Consider ~~and~~ fermion - anti fermion state...

$$\begin{aligned}
 & a_p^{st} b_q^{st} |0\rangle \xrightarrow{P} P(a_p^{st} b_q^{st} |0\rangle) \\
 &= P^+ (a_p^{st} b_q^{st}) P |0\rangle \\
 &= \underbrace{P^+ a_p^{st} P P^+ b_q^{st} P}_{\gamma_a} |0\rangle \\
 &= (\gamma_a) a_{-p}^{st} \gamma_b b_{-q}^{st} |0\rangle \\
 &= -(\gamma_b \gamma_b^*) a_{-p}^{st} b_{-q}^{st} |0\rangle \\
 &= -a_{-p}^{st} b_{-q}^{st} |0\rangle
 \end{aligned}$$

→ a state containing a fermion-antifermion pair gets an (-1) under parity transformation.

extra

—

[TIME REVERSAL].

if T is unitary $\Rightarrow [T, H] = 0$

$$\rightarrow T^+ e^{iHt} T = e^{iHt + T^+ T} = e^{iHt}$$

→ no good...

What if $T^+ T = -H$? or $[T, H] = 0$?

But this \Rightarrow no good either since implies that H is unbounded ...

\rightarrow Assume this ...

"Time-reversal is conjugate-linear/anti-linear"

Assume:

T is unitary

$$T^* T = c^* \quad (c \in \mathbb{C})$$

$$[T, H] = 0$$

With those

$$T^* e^{-iHt} T = e^{-iHt} \quad \checkmark$$

\rightarrow Time-reversal:

momentum

\downarrow

spin

are reversed

\rightarrow like watching a movie played back-wards

$$G \xrightarrow{\quad} T \xrightarrow{\quad} \leftarrow \int$$

Flipping momentum is easy.

What abt flipping spinor? We know that

In some basis --

$$\xi(\uparrow) = \begin{pmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi} \sin \theta/2 \\ \cos \theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s=1, 2$ & define

reversed
spin

$$\xi^{-s} = -i\sigma^2 (\xi^s)^{\dagger}.$$

→ This is the flipped spinor → σ^z

It is clear that

$$\begin{aligned} \xi^{-s} &= -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow), \xi(\downarrow))^{\dagger} \\ &= (\xi(\downarrow), -\xi(\uparrow))^{\dagger} \end{aligned}$$

where $\xi^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{-1\dagger}$

$$\xi^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{-2\dagger}$$

→ This is convenient since our time reversal op. involves complex conjugation --

→ Can show: $\boxed{i\vec{\sigma}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{s\dagger} \\ \sqrt{\tilde{p} \cdot \vec{\sigma}} (-i\sigma^2) \xi^{s\dagger} \end{pmatrix}}$

So if we use the identity ...

$$\{\sqrt{\tilde{p} \cdot \sigma} \gamma^2 = \sigma^2 \sqrt{\tilde{p} \cdot \sigma^2}\}$$

(prove using $\sigma^2 \bar{\sigma}^2 = -\bar{\sigma}^2 \sigma^2$)

then we get

$$\begin{aligned} u^{-s}(\tilde{p}) &= \begin{pmatrix} \sqrt{\tilde{p} \cdot \sigma} (-i\sigma^2) \xi^{s\pm} \\ \sqrt{\tilde{p} \cdot \bar{\sigma}} (-i\sigma^2) \xi^{s\mp} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2) \sqrt{\tilde{p} \cdot \sigma^2} \xi^{s\pm} \\ (-i\sigma^2) \sqrt{\tilde{p} \cdot \bar{\sigma}^2} \xi^{s\mp} \end{pmatrix} \\ &= (-i) \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} [u^s(p)]^\star = -\gamma^1 \gamma^3 [u^s(p)]^\star \end{aligned}$$

\uparrow
 $\sigma^2 = \sigma^2$

element-wise
cmplx conjugation

similarly,

$$v^{-s}(\tilde{p}) = -\gamma^1 \gamma^3 [\vartheta^s(p)]^\star$$

in this relation, v^{-s} contains

$$\xi^{(-s)} = -\xi^s$$

a 360° flip
introduces
a $(-)$ sign.

~~introduces~~
~~effect~~

Now we can define time reversal operation on the creation - annihilation operators ---

shores
can't
cross here -->

$$T^+ a_{\vec{p}}^s T = \bar{a}_{-\vec{p}}^s \quad ; \quad T^+ b_{\vec{p}}^s T = \bar{b}_{-\vec{p}}^s \quad \begin{array}{l} \text{flip spin} \\ \text{flip} \end{array}$$

where $\left\{ \begin{array}{l} \bar{a}_{-\vec{p}}^s = (a_{-\vec{p}}^\downarrow, -a_{-\vec{p}}^\uparrow) \\ \bar{b}_{-\vec{p}}^s = (b_{-\vec{p}}^\downarrow, -b_{-\vec{p}}^\uparrow) \end{array} \right\}$ just like what we did with $\left\{ \begin{array}{l} s^s = (s(\uparrow), -s(\downarrow)) \end{array} \right\}$

if $\left\{ \begin{array}{l} a_{\vec{p}}^s = (a_{\vec{p}}^\uparrow, a_{\vec{p}}^\downarrow) \\ b_{\vec{p}}^s = (b_{\vec{p}}^\uparrow, b_{\vec{p}}^\downarrow) \end{array} \right\}$ analogous to what we did before ...

With this, let's evaluate $T^\dagger \Psi(x) T$:

$$\begin{aligned} T^\dagger \Psi(x) T &= \int \frac{d^3 p}{(2\pi)^3} \sum_{s=1,2} T^+ (a_{\vec{p}}^s u_s^s(p) e^{-ip \cdot x} + b_{\vec{p}}^{s+} v_s^s(p) e^{+ip \cdot x}) T \\ &= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^2 \left\{ \bar{a}_{-\vec{p}}^s [u_s^s(p)]^* e^{-ip \cdot x} \right. \\ &\quad \left. + \bar{b}_{-\vec{p}}^{s+} [v_s^s(p)]^* e^{+ip \cdot x} \right\} \end{aligned}$$

where under T , $= \gamma^1 \gamma^2 \Psi(x_T)$, $x_T = (-t, \vec{x})$

$$\left\{ \begin{array}{l} a_{\vec{p}}^s \xrightarrow{T} \bar{a}_{-\vec{p}}^s \end{array} \right.$$

$$\text{and this } \bar{a}_{\vec{p}}^s = \gamma_0 \gamma_1 \gamma_2 \bar{a}_{-\vec{p}}^s$$

$$\rightarrow \bar{a}_{\vec{p}}^s \bar{a}_{-\vec{p}}^{s+} = \bar{a}_{-\vec{p}}^s \bar{a}_{\vec{p}}^{s+}$$

$$\bullet T^\dagger e^{-ip \cdot x} T = \mathbb{1} e^{+ip \cdot x}; T^+ a_{\vec{p}}^s T = [u_{\vec{p}}^s]^*$$

note sign here
choose ↑
93

Becare $\{\tilde{u}^s(p)\}^* = \gamma_1 \gamma_3 u^{-s}(\tilde{p})$, we have

$$T^+ \psi(x) T = \gamma' \gamma^3 \int \frac{d^2 \tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^2 \left\{ a_{\tilde{p}}^{-s} \tilde{u}^s(\tilde{p}) e^{i\tilde{p}(t_1, \tilde{x})} \right.$$

$$\left. + b_{\tilde{p}}^{-s} \tilde{v}^{-s}(\tilde{p}) e^{-i\tilde{p}(t_1, \tilde{x})} \right\}$$

$$= \gamma' \gamma^3 \psi(-t, x)$$

$$= -\tilde{p}(-t, \tilde{x})$$

$\Rightarrow \boxed{T^+ \psi(x, t) T = \gamma' \gamma^3 \psi(x, -t)}$

Next, can check the action of T on bilinears...

$$T^+ \bar{\psi} T = T^+ \psi^+ \gamma^0 T = T^+ \psi^+ T \gamma^0 \xrightarrow{\text{real}}$$

$$= (\gamma' \gamma^3 \psi(x_T))^+ \gamma^0 = \psi^+(x_T) \gamma^3 \gamma^1 \gamma^0$$

$$= \psi^+(x_T) \gamma^3 \gamma^1 \gamma^0$$

$$= + \psi^+(x_T) \gamma^0 \gamma^3 \gamma^1$$

$\Rightarrow \boxed{T^+ \bar{\psi} T = -\bar{\psi}(x_T) \gamma^1 \gamma^3}$

with this, can compute the rest---

Scalar $\boxed{T \bar{\psi} \psi T = \underbrace{\bar{\psi}(-\gamma' \gamma^3)}_{(x)} \underbrace{(\gamma' \gamma^3)}_{11} \psi(x_T) = \bar{\psi}(x_T) \psi(x_T)}$

Pseudoscalar \rightarrow set (-)

\Rightarrow "pseudo"

$$\boxed{T i \bar{\psi} \gamma^5 \psi T = -i \bar{\psi}(-\gamma' \gamma^3) (\gamma' \gamma^3) \psi(x_T) = -i \bar{\psi}(x_T) \gamma^5 \psi(x_T)}$$

Vector

$$\boxed{T^+ \bar{\psi} \gamma^\mu \psi T = \bar{\psi} (-\gamma^1 \gamma^3) (\gamma^\mu)^T (\gamma^1 \gamma^3) \psi}$$

(x)

$$= \begin{cases} + \bar{\psi} \gamma^\mu \psi (x_T) & \mu = 0 \\ - \bar{\psi} \gamma^\mu \psi (x_T) & \mu = 1, 2, 3 \end{cases}$$

This makes sense... Recall that $\bar{\psi} \gamma^0 \psi$ is the charge density

↳ $\bar{\psi} \gamma^0 \psi$ should be the same under T -

as we saw: $T^+ \bar{\psi} \gamma^0 \psi T = \bar{\psi} \gamma^0 \psi$.

but current density (time-dy) must reverse sign

$$\rightarrow T^+ \bar{\psi} \gamma^5 \psi T = - \bar{\psi} \gamma^5 \psi \quad \checkmark.$$

→

Charge Conjugation - Matter-anti-matter flip

{ anti-particles \rightarrow particles are swapped.

{ spin + momentum are the same.

Let $\left\{ \begin{array}{l} C^\dagger a_p^+ C = b_p^- \\ C^\dagger b_p^- C = a_p^+ \end{array} \right\} \rightarrow$ ignore phases...

How should C act on $\psi(x)$?

First, look at relation ...

$$(v^s(p))^{\pm} = \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}} & (-i\gamma^2) \xi^{s\pm} \\ \sqrt{p\cdot\bar{\sigma}} & (-i\gamma^2) \xi^{s\pm} \end{pmatrix}^{\pm} = \begin{pmatrix} -i\gamma^2 \sqrt{p\cdot\bar{\sigma}} \xi^{s\pm} \\ i\gamma^2 \sqrt{p\cdot\bar{\sigma}} \xi^{s\pm} \end{pmatrix}^{\pm}$$

$$= \begin{pmatrix} 0 & -i\gamma^2 \\ -i\gamma^2 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}} \xi^s \\ \sqrt{p\cdot\bar{\sigma}} \xi^s \end{pmatrix} = \cancel{\text{both}}$$

→ set

$$\boxed{u^s(p) = -i\gamma^2 (v^s(p))^{\pm}}$$

$$\boxed{v^s(p) = -i\gamma^2 (u^s(p))^{\pm}}$$

$$\rightarrow C^+ \psi(x) C = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^2 \left\{ -i\gamma^2 b_p^s (v^s(p))^* e^{-ip \cdot x} - i\gamma^2 a_p^{s\pm} (u^s(p))^{\pm} e^{ip \cdot x} \right\}$$

$$= -i\gamma^2 \psi^*(x) = -i\gamma^2 (\psi^+)^T = -i(\bar{\psi} \gamma^0 \gamma^2)^T$$

$$\Rightarrow \boxed{C^+ \psi(x) C = -i(\bar{\psi} \gamma^0 \gamma^2)^T} \rightarrow C \text{ is a unitary op.}$$

On bilinears ... first, find $\bar{\psi} = (\psi^+)^+ \gamma^0 = \psi^0$

$$\boxed{C^+ \bar{\psi} \psi^0 C = C^+ \psi^+ \gamma^0 C = \underbrace{C^+ \psi^+}_{\psi^0} \gamma^0 = -i \psi^T \gamma^0 \gamma^0}$$

$$= (-i \gamma^2 \psi)^T \gamma^0 = (-i \gamma^0 \gamma^2 \psi)^T$$

Next ...

$$C^+ \bar{\psi} \psi C = (-i(\bar{\psi} \gamma^0 \gamma^2 \psi)^T) (-i(\bar{\psi} \gamma^0 \gamma^2 \psi)^T) = \dots =$$

$$= -[(-i \bar{\psi} \gamma^0 \gamma^2)(-i \bar{\psi} \gamma^0 \gamma^2)]^T = +\bar{\psi} \gamma^0 \gamma^2 \gamma^0 \gamma^2 \psi$$

$$= +\bar{\psi} \gamma^0 \gamma^2 \gamma^2 \gamma^0 \psi = +\bar{\psi} \psi$$

(P)

$$\text{So } \boxed{C^\dagger \bar{\gamma}^4 C = \bar{\gamma}^\dagger \gamma} \rightarrow \text{reduces}$$

vector

$$\boxed{C_i^\dagger \bar{\gamma}^i \gamma^5 C = i (-; \gamma^0 \gamma^2 \gamma)^T \gamma^5 (-; \bar{\gamma}^0 \bar{\gamma}^2)^T = ; \bar{\gamma}^i \gamma^5}$$

pseudo-scalar

$$\boxed{C^\dagger \bar{\gamma}^m \gamma^5 C = - \bar{\gamma}^m \gamma^5}$$

pseudo scalar

$$\boxed{C^\dagger \bar{\gamma}^m \gamma^5 C = + \bar{\gamma}^m \gamma^5}$$

(I'll skip the derivations... to save time)

Summary

	$\bar{\gamma} \gamma$	$i \bar{\gamma} \gamma^5 \gamma$	$\bar{\gamma} \gamma^m \gamma$	$\bar{\gamma} \gamma^m \gamma^5 \gamma$	$\bar{\gamma} \gamma^m \gamma^5 \gamma$	∂_μ
P	+1	-1	$(-1)^m$	$-(-1)^m$	$(-1)^m (-1)^v$	$(-1)^m$
T	+1	-1	$(-1)^m$	$(-1)^m$	$-(-1)^m (-1)^v$	$-(-1)^m$
C	+1	+1	-1	+1	-1	+1
CPT	+1	+1	-1	-1	+1	-1

Notice that

$$L = \bar{\gamma} (i \gamma^\mu \partial_\mu - m) \gamma \text{ is invariant under } C, P, T \text{ separately}$$

→ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

Problem 5

↳ (to be continued...)

Invariance under CPT is required for any Lorentz invariant local Hermitian op.

Correlation functions for Dirac fields

$\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle \rightarrow$ Dirac propagation amplitudes
 ↓ ↑
 only "a" only "a"
 term contributes term contributes

Recall -

$$\rightarrow \bar{\psi}_A(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ a_A^S u_A^S(p) e^{-ip \cdot x} + b_A^{S+} v_A^S(p) e^{-ip \cdot x} \right\}$$

$$\rightarrow \bar{\psi}_B(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{S=1,2} \left\{ b_B^S \bar{v}_B^S(p) e^{-ip \cdot x} + a_B^{S+} \bar{u}_B^S(p) e^{ip \cdot x} \right\}$$

where $\{a_A^S, a_B^{S+}\} = \{b_A^S, b_B^{S+}\} = (2\pi)^3 \delta^{(3)}(p-q)/8$

$$\rightarrow \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_S u_A^S(p) \bar{v}_B^S(p)}_{AB} e^{-ip(x-y)}$$

$$= (i\gamma_x - m) \underbrace{\int \frac{d^3 p}{(2\pi)^3 / 2E_p}}_{AB} e^{-ip(x-y)}$$

$$(p+m)_{AB}$$

$$\boxed{\langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle = (i\gamma_x + m)_{AB} D(x-y)}$$

$$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \underbrace{\varphi_A^s(p) \varphi_B^s(p)}_{(\varphi-m)_{AB}} e^{-ip(x-y)}$$

↑ ↑
 6 terms 6 terms
 contribute contribute

$$= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{1}{2E_p} \right) (\varphi-m)_{AB} e^{-ip(x-y)}$$

$\langle 0 | \bar{\psi}_B(y) \psi_A(x) | 0 \rangle = - (i\partial_x + m)_{AB} \mathcal{D}(y-x)$

Feynman Propagator

$$S_f^{AB}(x-y) = \begin{cases} \langle 0 | \bar{\psi}_A(x) \bar{\psi}_B(y) | 0 \rangle & \text{for } x^0 > y^0 \\ -\langle 0 | \bar{\psi}_B(y) \bar{\psi}_A(x) | 0 \rangle & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0 | T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \} | 0 \rangle$$

↑
--- fine-ordering ---

where $T \{ \bar{\psi}_A(x), \bar{\psi}_B(y) \}$

$$= \theta(x^0 - y^0) \bar{\psi}_A(x) \bar{\psi}_B(y)$$

$$= \theta(y^0 - x^0) \bar{\psi}_B(y) \bar{\psi}_A(x)$$

minus sign for Fermions

(99)

let's check the calculations.

$$\langle 0 | \psi_A(x) \bar{\psi}_B(y) | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0 | \left\{ \sum_s a_{Ap}^s \bar{u}_A^s(p) e^{-ipx} + b_{Ap}^{s\dagger} u_A^s(p) e^{-ipx} \right\}$$

$$\times \left\{ \sum_s b_{Bp}^{s\dagger} \bar{u}_B^s(p') e^{-ip'y} + a_{Bp}^s u_B^s(p') e^{-ip'y} \right\} | 0 \rangle$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 p'}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \langle 0 | \sum_s a_{Ap}^s \bar{u}_A^s(p) \sum_s a_{Bp}^{s\dagger} \bar{u}_B^s(p') e^{-i(p-x-p'y)}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^6} \frac{1}{2} \frac{1}{\sqrt{E_p E_{p'}}} \sum_s \langle 0 | \bar{u}_A^s(p) \bar{u}_B^s(p') | 0 \rangle (2\pi)^3 \delta^{(3)}(p-p') e^{i(p-x-p'y)}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \underbrace{\sum_s \bar{u}_A^s(p) \bar{u}_B^s(p)}_{(p+m)_{AB}} e^{-ip(x-y)}$$

$$(p+m)_{AB} = (\gamma^\mu p_\mu + m)_{AB} \begin{pmatrix} \text{spin sum} \\ \text{relations} \end{pmatrix}$$

$$= (p+m)_{AB} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip(x-y)} \quad \checkmark$$

Similarly, we can get the other relation too...

-g

Oct 5, 2020

(1) Recall Dirac bispinor field --

$$\psi(\vec{x}) = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(\vec{p}) \right] e^{i\vec{p} \cdot \vec{x}}$$

$$\text{Use } \{a_{\vec{p}}^r, a_{\vec{p}'}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{p}'}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(\vec{p}-\vec{p}')}$$

and all other anti-comm = 0, derive the following:

$$\{\psi_a(\vec{x}), \psi_b^+(\vec{y})\} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y})$$

(2) The momentum operator is the Noether charge associated with spatial translation.

$$\vec{P} = -i \int d^3x \psi^+(\vec{x}) \vec{\nabla} \psi(\vec{x})$$

Show Keert

$$\vec{P} = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \vec{p} \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r \right)$$

Oct 10, 2020

(1) Well...

$$\{\psi_a(x), \psi_b^+(y)\}$$

$$= \psi_a(x) \psi_b^+(y) + \psi_b^+(y) \psi_a(x)$$

~~$$\frac{1}{2} \int \frac{d^3 p_1 d^3 p_2}{(2\pi)^6} \frac{1}{2\sqrt{E_{p_1} E_{p_2}}} e^{i(\vec{p}_1 \vec{x} + \vec{p}_2 \vec{y})}$$~~

To keep things clean --

$\psi_a(x)\psi_b^+(y) + \psi_b(x)\psi_a^+(y)$ which involves the factor ...

$$\begin{aligned}
 & a \sum_{r=1}^2 \sum_{s=1}^2 \left[\bar{a}_{p_a}^{1r} u^r(p_a) + \bar{b}_{-p_a}^{1s} v^s(-p_a) \right] \left[\bar{a}_{p_b}^{1s} u^r(p_b) + \bar{b}_{-p_b}^{1t} v^t(-p_b) \right] \\
 & + \sum_{r=1}^2 \sum_{s=1}^2 \left[\left(\bar{a}_{p_b}^{1s} u^r(p_b) \right)^+ + \left(\bar{b}_{-p_b}^{1t} v^s(-p_b) \right)^+ \right] \left[\bar{a}_{p_a}^{1r} u^r(p_a) + \bar{b}_{-p_a}^{1s} v^s(-p_a) \right] \\
 & = \sum_{r,s=1}^2 \left\{ \bar{a}_{p_a}^{1r}, \bar{a}_{p_b}^{1s} \right\} u^r(p_a) u^s(p_b) + \left\{ \bar{b}_{p_a}^{1r}, \bar{b}_{p_b}^{1s} \right\} v^r(p_a) v^s(p_b) \\
 & = \sum_{r,s=1}^2 (i\pi)^r \delta^{rs} \delta(p_a - p_b) \left\{ u^r(p_a) u^s(p_b) + v^r(p_a) v^s(p_b) \right\}
 \end{aligned}$$

$$\Rightarrow \{\psi_a(x), \psi_b^+(y)\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^2 \left\{ u^r(p) u^r(p) + v^r(p) v^r(-p) \right\}$$

Now, we want to convert $u^r \rightarrow \bar{u}^r$

\rightarrow need γ^0 . In particular, recall that $\gamma\gamma=0$
and $\overline{u^r(p)}\gamma^0 = \bar{u}^r(p)$ \Rightarrow we have

$$\{\psi_a(x), \psi_b^+(y)\} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^2 \left\{ u_p^r \bar{u}_p^r \gamma^0 + v_p^r \bar{v}_p^r \gamma^0 \right\}$$

$$= \left(\frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \sum_{r=1}^2 \left\{ u_p^r \bar{u}_p^r + \frac{v_p^r \bar{v}_p^r}{-p-p} \right\} \gamma^0 \right) \text{ (spin sum)}$$

$$= \left(\frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left(p_+ \gamma + m \gamma + p_- \gamma - m \gamma \right) \gamma^0 \right) \delta_{ab}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} [2\cancel{\langle \vec{p} \cdot \vec{r} \rangle} \delta^3]$$

recall that
only $\vec{p} \rightarrow -\vec{p}$ (102)

$$\text{Rather } = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}(\vec{x}-\vec{y})} \left\{ E_p \cancel{\langle \vec{x} - \vec{p}, \vec{x} + \vec{p} \rangle} + E_p \cancel{\langle \vec{y} - \vec{p}, \vec{y} + \vec{p} \rangle} \right\} \delta^3$$

$$= \int \frac{d^3 p}{(2\pi)^3} e^{i\vec{p}(\vec{x}-\vec{y})} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}).$$

$$\boxed{\delta_{ab} \{ \psi_a(x), \psi_b^\dagger(y) \} = \delta_{ab} \delta^{(3)}(x-y)}$$

4

$$(2) \text{ Let } \vec{p} = -i \int d^3 x \vec{x} \psi^\dagger(\vec{x}) \vec{\nabla} \psi(\vec{x}).$$

then must

$$p = (p^+, \vec{p})$$

$$\vec{p} = \sum_{i=1}^2 \int \frac{d^3 p}{(2\pi)^3} \vec{p} \left(\frac{a^+}{p^+} a^r_p + b^+_p b^r_p \right).$$

$$i\vec{p} \cdot \vec{x} = \cancel{a^+ a^r} + i\vec{b}^+ \vec{x}$$

Well,

$$\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} e^{i\vec{p} \cdot \vec{x}} \left\{ a^r_p u^r_p + b^r_p v^r_p \right\}.$$

$$\delta \psi(x) = \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (i\vec{p})$$

$$(2) \quad \Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a_p^s u_p^s e^{-ip_x} + b_p^{s\dagger} v_p^s e^{ip_x} \right)$$

$$\rightarrow \nabla \Psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s (i\vec{p}) \left\{ a_p^s u_p^s e^{-ip_x} - b_p^{s\dagger} v_p^s e^{ip_x} \right\}$$

$$\hat{\Psi}(x) = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \sum_{r=1}^{\infty} \left\{ b_q^r v_q^r (q) e^{-iq_x} + a_q^{r\dagger} u_q^r (q) e^{iq_x} \right\}$$

$$\sim \int d^3 x (i) \nabla \hat{\Psi} D \Psi$$

$$= \int d^3 x \int \frac{d^3 q}{(2\pi)^3} \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \frac{1}{\sqrt{E_p E_q}} e^{ix(p-q)} \hat{p}_x \times \sum_s \sum_r$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2} \frac{1}{E_p} \hat{p}_x \left\{ a \sum_{s,r=1}^2 \left(a_p^{s\dagger} a_p^s \hat{v}_p^r u_p^r - b_p^{r\dagger} b_p^r \hat{v}_p^s u_p^s \right) \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{2E_p}{2E_p} \hat{p}_x \left(a_p^{r\dagger} a_p^r - b_p^{r\dagger} b_p^r \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \hat{p}_x \left(a_p^{r\dagger} a_p^r - b_p^{r\dagger} b_p^r \right)$$

Finally ... $\{ b_p^r, b_p^{r\dagger} \} = (2\pi)^3 \delta^{rr} (\hat{p} - \hat{p})$

$$\Rightarrow -b_p^r b_p^{r\dagger} = b_p^{r\dagger} b_p^r - (2\pi)^3 \delta^{rr} \delta(\hat{p} - \hat{p})$$

$$\Rightarrow \hat{p}_x = \int d^3 x \Psi^+(x) (-i\partial_x) \Psi(x) \quad \rightarrow \text{momentum op}$$

$$\boxed{\hat{p}_x = \int \frac{d^3 p}{(2\pi)^3} \hat{p}_x \left(a_p^{r\dagger} a_p^r + b_p^{r\dagger} b_p^r \right)}$$

More problems

① Let \mathcal{U} be the following unitary op:

$$\mathcal{U} = \exp \left\{ -i\frac{\pi}{2} \sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r\dagger} - b_p^{r\dagger}) \right\}$$

Investigate the effect of \mathcal{U} on $a_p^r - b_p^{r\dagger}$. I.e.
compute $\mathcal{U}^\dagger a_p^r \mathcal{U} = n^r b_p^{r\dagger}$.

What type of transform does \mathcal{U} produce?
~~-x~~ $\xrightarrow{+X}$

Well...

$$\mathcal{U}^\dagger a_p^r \mathcal{U} = \exp \left\{ +i\frac{\pi}{2} \left(\sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r\dagger} - b_p^{r\dagger}) \right) \right\}$$

$\xleftarrow{+X} \exp \left\{ i\frac{\pi}{2} \sum_{r=1}^2 \int \frac{dp}{(2\pi)^3} (a_p^{rt} - b_p^{rt}) (a_p^{r\dagger} - b_p^{r\dagger}) \right\}$

\Rightarrow no good way to do this except for powers,

Recall that $e^X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$, $e^{-X} = \sum_{n=0}^{\infty} \frac{1}{n!} (-X)^n$

$$\begin{aligned} \mathcal{U}^\dagger a_p^r \mathcal{U} &\simeq \left(\sum_{n=0}^{\infty} \frac{(X^r)^n}{n!} \right) a_p^r \left(\sum_{m=0}^{\infty} \frac{(X^{r\dagger})^m}{m!} \right) = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} (X^r)^n (X^{r\dagger})^m \\ &= (X^r) a_p^r X^{r\dagger} \end{aligned}$$

but note that \mathcal{U} unitary iff X hermitian.

$$\rightarrow X^\dagger = X \rightarrow \mathcal{U}^\dagger a_p^r \mathcal{U} = \sum_{n,m=0}^{\infty} \frac{1}{n! m!} X^n a_p^r X^m.$$

Weilil theorem:

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!} [X, [X, Y]] + \frac{1}{3!} [X, [X, [X, Y]]]$$

2 Identity relating comm & anti-comm: ---

$$[AB, C] = ABC - CAB$$

$$= ABC + ACB - ACB - CAB$$

$$= A\{B, C\} - \{A, C\}B.$$

We will need to compute $\{X, g_p^r\}$

$$U_{\frac{q}{p}}^T U = g_p^r + [X, g_p^r] + \frac{1}{2!} [X, [X, g_p^r]] + \dots$$

→ need to compute

$$[X, g_p^r] = X g_p^r - g_p^r X = \dots = ?$$

$$= \left(\frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left\{ (a_p^{st} - b_p^{st}) (a_p^s - b_p^s) g_p^r \right. \right. \\ \left. \left. - \left\{ g_p^r (a_p^{st} - b_p^{st}) (a_p^s - b_p^s) \right\} \right\} \right)$$

$$= \frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left[\left(a_p^{st} - b_p^{st} \right) \left\{ a_p^s - b_p^s \right\} g_p^r \right. \\ \left. - \left\{ a_p^{st} - b_p^{st}, a_p^s \right\} (a_p^r - b_p^r) \right]$$

$$= \frac{i\pi}{2} \sum_{s=1}^2 \int \frac{dp}{(2\pi)^3} \left[- (2\pi)^3 s^{rs} \delta^{(3)}(\vec{p} - \vec{q}) (a_p^r - b_p^r) \right]$$

$$= -\frac{i\pi}{2} (a_q^r - b_q^r)$$

Next turn,

$$\begin{aligned} [X, [X, a_q^r]] &= [X, -\frac{i\pi}{2}(a_q^r - b_q^r)] \\ &= \frac{-i\pi}{2}[X, a_q^r] + \frac{i\pi}{2}[X, b_q^r] = \dots \\ &= 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r) \end{aligned}$$

$$So \dots e^X a_p^r e^{-X} = a_p^r - \frac{i\pi}{2}(a_q^r - b_q^r) + 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r)$$

what's next?

+ ?

each step $\rightarrow +2\left(\frac{i\pi}{2}\right)$ = alt (+) sign.

$$\begin{aligned} \rightarrow u^\dagger a_p^r u &= a_p^r - \frac{i\pi}{2}(a_q^r - b_q^r) + \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2(a_q^r - b_q^r) \\ &\quad - \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3(a_q^r - b_q^r) + \frac{1}{4!} \left(\frac{i\pi}{2}\right)^4 6(-) \end{aligned}$$

$$= a_p^r \left\{ 1 - \frac{i\pi}{2} + \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2 - \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3 + \frac{6}{4!} \left(\frac{i\pi}{2}\right)^4 + \dots \right\}$$

~~check this~~

$$\begin{aligned} &+ b_p^r \left\{ \frac{i\pi}{2} - \frac{1}{2!} 2\left(\frac{i\pi}{2}\right)^2 + \frac{1}{3!} 4\left(\frac{i\pi}{2}\right)^3 - \frac{6}{4!} \left(\frac{i\pi}{2}\right)^4 + \dots \right\} \\ &= a_p^r \left\{ 1 - \frac{1}{2} \cdot 2 \right\} + b_p^r \cdot \left\{ \frac{1}{2} - 2 \right\} \end{aligned}$$

$$= b_p^r \Rightarrow \boxed{u^\dagger a_p^r u^* = b_p^r}$$

$\rightarrow u$ corresponds to charge conjugation!

(2) Using Dirac annihilation creation ops, construct P for which

$$P^+ a_p^r P = a_{-p}^r \quad P^+ b_p^r P = b_{-p}^r.$$

Last time, we find that \rightarrow target

$$[X, a_p^r] = -\left(\frac{i\pi}{2}\right) (a_p^r - b_p^r)$$

\rightarrow we want X for which

$$\{ [X, a_p^r] = -\left(\frac{i\pi}{2}\right) (a_p^r - a_{-p}^r) .$$

$$[X, b_p^r] = -\left(\frac{i\pi}{2}\right) (b_p^r + b_{-p}^r) .$$

\rightarrow fit

$$P = \exp \left\{ -\frac{i\pi}{2} \sum_{r=1}^2 \int \frac{d^3 p}{(2\pi)^3} \left\{ a_p^{\dagger} (a_p^r - b_{-p}^r) + b_p^{\dagger} (b_p^r + b_{-p}^r) \right\} \right\}$$



check this, like last time

\rightarrow should work! \square

Olv a_p^r only int. w/ 1st term $a^{\dagger}()a = 0$

$$a^{\dagger}() = \delta^{--}(\omega) \quad \checkmark$$

Same with b_p^r \checkmark

Interacting Fields = Feynman Diagrams

To get better description of the real world, need to include interactions in the theory.

To preserve causality, new terms may involve products of fields at the same spacetime point!

↳ $\phi^4(x)$ ✓, but not $\phi(x)\phi(y)$.

$$\rightarrow H_{\text{int}} = \int d^3x \, H_{\text{int}}[\phi(x)] = - \int d^3x \, \partial_{\mu} \phi \partial^{\mu} \phi$$

→ insist that H_{int} is a func of the fields, not of their derivatives.

→ Common ex in perturb physics = CMT:

$$\boxed{L = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4}$$

Note

— $\Pi(x)$ is still $\partial_\mu \phi(x)$ since there are no new terms involving $\partial_\mu \phi$ interaction.

λ : dimensionless "coupling constant".

→ In general, adding interactions preserves invariance.

However, no matter what the true physics looks like at high momenta or short distances, the low momentum / long distance physics is well-approximated by an "effective" FT.

with "renormalizable" interactions.

→ these interactions where coupling constant are has dimensions $\boxed{d > 0}$

$[\text{Mass}]^d$ where $d > 0$.

$$\text{Ex} \quad -\frac{1}{2} m^2 \phi^2 \sim \frac{2}{4!} \phi^4 \text{ same dim}$$

→ $\lambda \sim [\text{Mass}]^0 \rightarrow \text{renormalizable.}$

But $-\frac{\lambda_6}{6!} \phi^6 \rightarrow \underline{\text{not}} \text{ renormalizable.}$

(since $\lambda \sim [\text{Mass}]^{-2}$)

Perturbation Expansion

$$\text{Let } H = H_0 + f_{\text{int}} \rightsquigarrow = \int d^3 p \frac{1}{4!} \phi^4(x)$$

$$\uparrow$$

KG, free

→ we will generate power series in λ .

At any t_0 , we can write

$$\phi(t_0, \vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{i\vec{p} \cdot \vec{x}} + a_p^\dagger e^{-i\vec{p} \cdot \vec{x}} \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \tilde{a}_p e^{i\vec{p} \cdot \vec{x}} + \tilde{a}_p^\dagger e^{-i\vec{p} \cdot \vec{x}} \right\}$$

$$a_p^\dagger$$

where we've let
 a_p^\dagger absorb e^{iEt_0}

The Heisenberg field is then given by:

$$\rightarrow \boxed{\phi(t, \vec{x}) = e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)}}$$

If there's no interaction then we have

$$\rightarrow \boxed{\phi_{\text{free}}(t, \vec{x}) = e^{iH_0(t-t_0)} \phi_{\text{free}}(t_0, \vec{x}) e^{-iH_0(t-t_0)}}$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p^* e^{-ip \cdot x} + a_p^+ e^{ip \cdot x} \right\} \Big|_{\substack{x_0 = t-t_0 \\ p^0 = E_p}}$$

Define this to be $\phi_I(t, \vec{x})$, the interaction picture field

The interaction picture field = Heisenberg field when $\lambda = 0$.

Now, look at Heisenberg field ..

$$\begin{aligned} \phi(t, \vec{x}) &= e^{iH(t-t_0)} \phi(t_0, \vec{x}) e^{-iH(t-t_0)} \\ &= \underbrace{e^{iH(t-t_0)} e^{-iH_0(t-t_0)}}_{\phi_I(t, \vec{x})} e^{+iH_0(t-t_0)} e^{-iH(t-t_0)} \\ &= U^+(t, t_0) \phi_I(t, \vec{x}) U(t, t_0) \end{aligned}$$

\rightarrow Evolve the operator on $\phi_I(t, \vec{x})$

Time evolution operator

OR

Evolve the state by $U(t, t_0) \rightarrow U|\phi\rangle ..$

→ now, we want to express $U(t, t_0)$ entirely in ϕ_I

To do this, note that $U(t, t_0)$ solves SE:

$$\begin{aligned} \frac{\partial}{\partial t} U(t, t_0) &= e^{iH_0(t-t_0)} (H - H_0) e^{-iH_0(t-t_0)} \\ &= e^{iH_0(t-t_0)} \text{ Hint } e^{-iH_0(t-t_0)} \\ &= \underbrace{e^{iH_0(t-t_0)} \text{ Hint } e^{-iH_0(t-t_0)}}_{e^{iH_0(t-t_0)} e^{-iH_0(t-t_0)}} e^{iH_0(t-t_0)} e^{-iH_0(t-t_0)} \\ &= H_I(t) U(t, t_0) \end{aligned}$$

where

$$H_I(t) = e^{iH_0(t-t_0)} \text{ Hint } e^{-iH_0(t-t_0)}$$

$$= \int d^3x \frac{\partial}{4!} \phi^4 \quad [3]$$

$$= \int d^3x e^{iH_0(t-t_0)} \overbrace{\frac{\partial}{4!} \phi^4}^{\rightarrow} \overbrace{e^{-iH_0(t-t_0)}}^{\leftarrow}$$

$$= \int d^3x \frac{\partial}{4!} \phi^4 \quad \checkmark$$

→ this is the Hamiltonian in the interaction picture.

So since U solves the SE:

$$\frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0),$$

U must look like

$$U(t, t_0) \sim \exp \{-iH_I t\}$$

More carefully, we actually have that

$$U(t, t_0) = T \exp \left\{ -i \int_{t_0}^t dt' H_I(t') \right\}$$

Lyman's formula

time-ordering symbol.

Why T ? Why ordering? \Rightarrow B/c $H(t_1) \not\rightarrow H(t_2)$ when $t_1 \neq t_2$.

" T " puts the latest operators on the left.

hence $i \partial_t U(t, t_0) = \underline{\underline{H_I(t)}} U(t, t_0)$.

As a power series in λ :

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt' H_I(t') + \frac{(-i)^2}{2!} \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 T \{ H_1(t_1) H_2(t_2) \} + \dots$$

$$+ \frac{(-i)^3}{3!} \int_{t_0}^t \int_{t_0}^t \int_{t_0}^t dt_1 dt_2 dt_3 T \{ H_1(t_1) H_2(t_2) H_3(t_3) \} + \dots$$

Note "the time-ordering of the exponential is just ~~the time-ordering~~ the Taylor series of the terms time-ordered ...".

\rightarrow Now, we want to generalise $U(t, t_0)$ to $U(t, t')$

↑
referendum

This generalization is natural -

$$U(t, t') = T \left\{ \exp \left[-i \int_{t'}^t dt'' H_I(t'') \right] \right\} \quad (t \geq t')$$

Then we see that b/c both t, t' are variables -- we find :

$$i\partial_t U(t, t') = H_I(t) U(t, t')$$

$$i\partial_{t'} U(t, t') = -U(t, t') H_I(t').$$

and thus --

$$U(t, t') = e^{iH_0(t-t')} e^{-iH(t-t')} e^{-iH_0(t-t')}$$

so U is unitary.

Further, for $t_1 \geq t_2 \geq t_3$,

$$U(t_1, t_2) U(t_2, t_3) = U(t_1, t_3)$$

$$U(t_1, t_3) [U(t_2, t_3)]^\dagger = U(t_1, t_2)$$

Now, let $|0\rangle$ be gnd state of H_0

$|S\rangle$ be gnd state of H

$|n\rangle$ be gnd label all $|E_n\rangle$ of H .

then, $(E_0 = \langle \psi_0 | H | \psi_0 \rangle)$

$$\langle e^{-iHT} | \psi_0 \rangle = e^{-iE_0 T} |\psi_0\rangle \langle \psi_0 | \psi_0 \rangle + \sum_{n \neq 0} e^{-iE_n T} |n\rangle \langle n | \psi_0 \rangle$$

Assume $H_0 |0\rangle = 0$ in consider $T \rightarrow \infty$ limit.

↑

Then $e^{-iE_n T}$ dies slowest for $n=0$, and so --

$$T \rightarrow \infty (1 - i\varepsilon)$$

$$\rightarrow e^{iHT} |\psi_0\rangle \rightarrow e^{-iE_0 T} |\psi_0\rangle \langle \psi_0 | \psi_0 \rangle \quad \text{assume } \langle \psi_0 | \psi_0 \rangle \neq 0$$

So

$$|\psi_0\rangle = \lim_{T \rightarrow \infty (1 - i\varepsilon)} (e^{-iE_0 T} \langle \psi_0 | \psi_0 \rangle)^{-1} e^{-iHT} |\psi_0\rangle$$

Now, since T large, we can shift it by a small constant --

$$\begin{aligned} |\psi_0\rangle &= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0(T+t_0)} \langle \psi_0 | \psi_0 \rangle \right\}^{-1} e^{-iH(T+t_0)} |\psi_0\rangle \\ &= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0(t_0 - (-T))} \langle \psi_0 | \psi_0 \rangle \right\}^{-1} e^{-iH(t_0 - (-T))} |\psi_0\rangle \\ &= \lim_{T \rightarrow \infty (1 - i\varepsilon)} \left\{ e^{-iE_0(t_0 - (-T))} \langle \psi_0 | \psi_0 \rangle \right\}^{-1} e^{-iH(t_0 - (-T))} e^{-iH_0(-T-t_0)} |\psi_0\rangle \\ &= |\psi_0\rangle \text{ since } H_0 |\psi_0\rangle = 0. \end{aligned}$$

$$\Rightarrow |\psi_0\rangle = \lim_{T \rightarrow \infty (1 - i\varepsilon)} \frac{U(t_0, -T) |\psi_0\rangle}{e^{-iE_0(t_0 - (-T))} \langle \psi_0 | \psi_0 \rangle}$$

Similarly,

$$\langle \sigma | = \lim_{T \rightarrow \infty (1-i\epsilon)} e^{-iE_0(T-t_0)}$$

$$\langle 0 | u(T, t_0)$$

$$\langle 0 | \sigma \rangle$$

So, putting these together gives a correlation function -

For $x^0 > y^0 > t_0$, we have

$$\rightarrow \langle \sigma | \phi(x) \phi(y) | \sigma \rangle$$

$$= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, t_0) (u(x^0, t_0))^+ \phi_I(x) u(x^0, t_0)}{e^{-iE_0 t + (T-t_0)} \langle 0 | \sigma \rangle}$$

$$\times (u(y^0, t_0))^+ \phi_I(y) u(y^0, t_0) \times$$

$$u(t_0 - T)$$

$$e^{-iE_0 t + (T+t_0)} \langle \sigma | 0 \rangle$$

$$= \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, x^0) \phi_I(x) u(x^0, y^0) \phi_I(y) u(y^0, -T) | 0 \rangle}{| \langle 0 | \sigma \rangle |^2 e^{-iE_0 (2T)}}$$

awkward ...

so divide the whole thing by $1 = \langle \sigma | \sigma \rangle$

$$1 = \langle \sigma | \sigma \rangle = \frac{\langle 0 | u(T, t_0) u(t_0, -T) | 0 \rangle}{| \langle 0 | \sigma \rangle |^2 e^{-iE_0 (2T)}} m(T, -T)$$

To get (for $x^0 > y^0$)

$$\langle \sigma | \phi(x) \phi(y) | \sigma \rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \frac{\langle 0 | u(T, x^0) \phi_I(x) u(x^0, y^0) \phi_I(y) u(y^0, -T) | 0 \rangle}{\langle 0 | u(T, -T) | 0 \rangle}$$

So, we have shown, by replacing U^* with Dyson's formula (w/ time-ordering)

$$\boxed{\langle 0 | T \{ \phi(x) \phi(y) \} | 0 \rangle} = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{\langle 0 | T \{ \phi_I(x) \phi_I(y) \} \exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} | 0 \rangle}{\langle 0 | T \exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} | 0 \rangle}$$

So, looks like the term

$$\exp \left\{ -i \int_{-T}^T dt H_I(t) \right\} \text{ can be treated & can be found, so } \swarrow$$

Wick's theorem

→ So, we have reduced the problem of calculating correlation functions to evaluating

$$\boxed{\langle 0 | T \{ \phi_I(x_1) \phi_I(x_2) \dots \phi_I(x_n) \} | 0 \rangle}$$

→ This is the vacuum exp-value of time-ordered products of finite number of field operators.

$n=2 \rightarrow$ get Feynman operator.

$n>2 \rightarrow$ can use ~~for~~ brute force, but there are also ways to simplify calculations.

Now, we study

$$\boxed{\langle 0 | T \{ \phi_I(x) \phi_I(y) \} | 0 \rangle}$$

Recall that

$$\phi_I(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \left\{ a_p^- e^{-ip \cdot x} + a_p^+ e^{+ip \cdot x} \right\}$$

Oct 11,
2020

$$\text{Call } \phi_I^+(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} a_p^- e^{-ip \cdot x}$$

$$\text{and } \phi_I^-(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} a_p^+ e^{+ip \cdot x}$$

which is useful b/c

$$\phi_I^+(x)|0\rangle = 0, \quad \langle 0|\phi_I^-(x) = 0.$$

only has annihilation ops

only has creation ops

→ For $x^> y^>$, $x^> y^<$

$$\begin{aligned} \Gamma \{ \phi_I^+(x) \phi_I^-(y) \} &= \phi_I^+(x) \phi_I^+(y) + \phi_I^-(x) \phi_I^-(y) \\ &\quad + \phi_I^+(x) \phi_I^-(y) + \phi_I^-(x) \phi_I^+(y) \end{aligned}$$

$$= \phi_I^+(x) \phi_I^+(y) + \phi_I^-(y) \phi_I^+(x) + \phi_I^-(x) \phi_I^+(y) + \phi_I^-(x) \phi_I^-(y)$$

$$+ [\phi_I^+(x), \phi_I^-(y)]$$

every of these terms has the form $a_p^+ a_p^+ a_k^- a_k^-$

i.e. creation ops always on the left.

→ "Normal order" → less vanishing vacuum expectation value

What can we say about the commutator?

It's just a number, there's no creation/annihilation op's in it!

$$\begin{aligned} [\phi_I^+(x), \phi_I^-(y)] &= \langle 0 | [\sum \phi_I^+(x), \phi_I^-(y)] | 0 \rangle \\ &= \langle 0 | \phi_I^+(x) \phi_I^-(y) | 0 \rangle = \langle 0 | \phi_I^-(y) \phi_I^+(x) | 0 \rangle. \end{aligned}$$

With this, we can write

$$\begin{aligned} T\{\phi_I^+(x) \phi_I^-(y)\} &= \overbrace{\phi_I^+(x) \phi_I^+(y)} + \overbrace{\phi_I^-(x) \phi_I^+(y)} + \overbrace{\phi_I^-(y) \phi_I^+(x)} \\ &\quad + \overbrace{\phi_I^-(x) \phi_I^-(y)} + \langle 0 | \phi_I^-(x) \phi_I^-(y) | 0 \rangle \end{aligned}$$

Now, define the normal ordering symbol "N"

s.t. N takes the string of at's and rearranges them so that at's are on the left

$$\text{ex. } \left\{ \begin{array}{l} N(a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger) = a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger \\ N(a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger) = a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger \cdot \text{ ordering for } a_{\vec{p}}, a_{\vec{q}}^\dagger \\ N(a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger a_{\vec{p}}^\dagger) = a_{\vec{p}}^\dagger a_{\vec{q}}^\dagger a_{\vec{p}}^\dagger \text{ doesn't matter} \end{array} \right. \text{ since they commute.}$$

\Rightarrow Note N is not a well-defined mathematical operation

$$\text{e.g. } N(\sum a_{\vec{p}, \vec{q}}^\dagger) \neq N((2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}))$$

\rightarrow it is only a lexicographic convention.

Now, let us consider general x^0, y^0 , then

$$T\{\phi_I(x)\phi_I(y)\} = N\{\phi_I(x), \phi_I(y)\}$$

$$+ \begin{cases} [\phi_I^+(x), \phi_I^-(y)] & \text{for } x^0 > y^0 \\ (\phi_I^+(y), \phi_I^-(x)) & \text{for } x^0 < y^0 \end{cases}$$

→ Let us define the continuation of $\phi_I(x), \phi_I(y)$ as

$$\phi_I^+(x)\phi_I^-(y) = \begin{cases} \sum \phi_I^+(x), \phi_I^-(y) & x^0 > y^0 \\ \sum \phi_I^+(y), \phi_I^-(x) & x^0 < y^0 \end{cases}$$

Then notice that, from our previous derivation,

$$\boxed{\phi_I^+(x)\phi_I^-(y)} = \begin{cases} \langle 0 | T\{\phi_I^+(x)\phi_I^-(y)\} | 0 \rangle & x^0 > y^0 \\ \langle 0 | \phi_I^-(y)\phi_I^+(x) | 0 \rangle & y^0 > x^0. \end{cases}$$

So,

$$\left. \begin{aligned} \phi_I^+(x)\phi_I^-(y) &= \langle 0 | T\{\phi_I^+(x)\phi_I^-(y)\} | 0 \rangle \\ &= D_F(x-y) \quad \sim \text{Feynman propagator} \\ &= \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)} \end{aligned} \right\}$$

With this, we have that

(20)

→ carry out the I^{th} subscript

$$T \{ \phi(x) \phi(y) \} = N \left\{ \phi(x) \phi(y) + \underbrace{\phi(x) \phi(y)}_{\phi(y) \phi(x)} \right\}$$

$$\rightarrow T \{ \phi(x) \phi(y) \} = N \{ \phi(x) \phi(y) \} + \text{"contraction"}$$

In fact, the generalization of this is called
Wick's Theorem

$$T \{ \phi(x_1) \phi(x_2) \dots \phi(x_m) \}$$

$$= N \{ \phi(x_1) \phi(x_2) \dots \phi(x_m) + \text{all possible contractions} \}$$

$$\text{Ex } T \{ \phi_1 \phi_2 \phi_3 \phi_4 \}$$

$$(\phi_n = \phi(x_n))$$

$$= N \{ \phi_1 \phi_2 \phi_3 \phi_4 +$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4}$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4}$$

$$+ \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} + \overbrace{\phi_1 \phi_2 \phi_3 \phi_4}^{\phi_1 \phi_2 \phi_3 \phi_4} \}$$

What does $N \{ \phi_1 \phi_2 \phi_3 \phi_4 \}$ mean?

$$N \{ \phi_1 \phi_2 \phi_3 \phi_4 \} = \overbrace{\phi_1 \phi_2}^{\phi_1 \phi_2} N \{ \phi_3 \phi_4 \}$$

$$= D_f(x_1 - x_2) N \{ \phi_3 \phi_4 \}.$$

Proof \rightarrow prove by induction. $n=2$ is good
(Feynman)

\rightarrow assume this holds for $n-1$.

Let $W(\phi_1 \dots \phi_n) = N\{\phi_1 \phi_2 \dots \phi_n + \text{all possible contractions}\}$

To prove $W(\phi_1 \dots \phi_n) = T\{\phi_1 \phi_2 \dots \phi_n\}$.

W/l/o/j: let $x^0 \geq x_1^0 \geq \dots \geq x_n^0$.

Then $T\{\phi_1 \dots \phi_n\} = \phi_1 T\{\phi_2 \dots \phi_n\}$ since
 $\phi_1 \in W(\phi_1 \dots \phi_n)$ done.

$$\text{So } T\{\phi_1 \dots \phi_n\} = \underbrace{\phi_1^+ W(\phi_2 \dots \phi_n)}_X + \underbrace{W(\phi_2 \dots \phi_n) \phi_1^+}_Y + [\phi_1^+, W]$$

Let $X = \phi_1^+ W + W \phi_1^+$; $Y = [\phi_1^+, W]$.

$X+Y$ are normal ordered: X contains all contractions in $W(\phi_1 \dots \phi_n)$ which don't contact ϕ_1 with anything.

Y contains all contractions in $W(\phi_1 \dots \phi_n)$ which contracts ϕ_1 with something.

$$\text{So } T(\phi_1 \phi_2 \dots \phi_n) = W(\phi_1 \dots \phi_n).$$

(we won't worry too much abt this proof.)

\rightarrow the main idea is the theorem itself). \square

In any case, we have another way to explicitly write out the result of Wick's Theorem:

$$T\{\phi_1, \phi_2, \dots, \phi_n\} = N \left\{ \exp \left[\frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^n \phi_i \phi_j \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j} \right] \phi_1 \dots \phi_n \right\}$$

↑
we'll see why later on.

~~it~~

Feynman Diagrams

Wick's Theorem allows us to write

$$\langle 0 | T\{\phi_1, \dots, \phi_n\} | 0 \rangle$$

in terms of sums and products of Feynman propagators.

→ Now, we will develop ~~the~~ diagrammatic expressions.

Recall that

$$T\{\phi_1, \phi_2, \phi_3, \phi_4\} = N \left\{ \phi_1 \phi_2 \phi_3 \phi_4 + \text{all possible contractions} \right\}$$

But the only contribution to

$\langle 0 | T\{\phi_1, \phi_2, \phi_3, \phi_4\} | 0 \rangle$ is when all the ϕ 's are contracted -

↳ This b/c whenever things are in normal order, the exp value vanishes - $\rightarrow N(\bar{\phi}_1 \phi_2 \phi_3 \phi_4) = \bar{\phi}_1 \phi_2 N(1, \phi_3, \phi_4)$

→ to "escape" from normal order, ϕ 's have to be contracted

This means that

$$\{T\{\phi_1 \phi_2 \phi_3 \phi_4\}\} = \overbrace{\phi_1 \phi_2 \phi_3 \phi_4} + \underbrace{\phi_1 \phi_2 \phi_3 \phi_4}_{\text{L}} + \phi_1 \phi_2 \phi_3 \phi_4$$

→ can write this as Feynman diagrams...

$$T\{\phi_1 \phi_2 \phi_3 \phi_4\} = \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ | \quad | \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ | \quad | \\ 3 \quad 4 \end{array} + \begin{array}{c} 1 \quad 2 \\ \text{---} \quad \text{---} \\ | \quad | \\ 3 \quad 4 \end{array}$$

↳ Interpretation

Particles are created at 2 spacetime points, each propagates to one of the other points, then get annihilated.

→ total amplitude of the process is the sum of the diagrams.

Well... what about something like...

$$\langle 0 | T\{\phi(x) \phi(y)\} \exp\left\{-i \int_{-\infty}^{\infty} dt H_I(t)\right\} \rangle | 0 \rangle ?$$

Well... as a power series in λ , the lowest order term is

$$\langle 0 | T\{\phi(x) \phi(y)\} | 0 \rangle = D_\phi(x-y) \cdot \xrightarrow{x} \xrightarrow{y}$$

$$\text{1st order } \langle 0 | T\{\phi(x) \phi(y)\} (-i) \left(\int_{-\infty}^{\infty} dt H_I(t) \right) | 0 \rangle$$

$$= \langle 0 | T\{\phi(x) \phi(y) (-i) \int d^4 z \bar{\phi}_q(z) \phi^q(z)\} | 0 \rangle$$

(124)

$$\begin{aligned}
 &= -\frac{i\gamma}{4!} \int d^4 z \langle 0 | T \{ \phi(x) \phi(y) \phi(z) \phi(\tau) \phi(1\tau) \phi(2) \} | 0 \rangle \\
 &= -\frac{i\gamma}{4!} \int d^4 z \left\{ \phi(x) \phi(y) \cdot \left\{ \phi(z) \phi(\tau) \phi(+) \phi(+) \phi(1\tau) + \phi_z \phi_z \phi_2 \phi_2 \right. \right. \\
 &\quad \left. \left. + \phi_z \phi_+ \phi_+ \phi_2 \right\} \right. \\
 &\quad \left. + \phi(x) \phi(y) \phi(1z) \phi(2) \phi(1z) \phi(2) \right\} \\
 &\qquad \qquad \qquad \xrightarrow{\text{12 terms, but are identical}}
 \end{aligned}$$

$$= \begin{array}{c} x \\ \text{---} \\ y \end{array} + \begin{array}{c} x \\ \swarrow \quad \searrow \\ y \quad z \end{array} \xrightarrow{\text{1 propagator}} \int d^4 z \mathcal{D}_F(x-z) \mathcal{D}_F(y-z)$$

$$\int d^4 z \mathcal{D}_F(x-y) \mathcal{D}_F(z-z) \mathcal{D}_F(1\tau z)$$

↑
12 of these.

→ each contraction \mathcal{D}_F is a line.

each quantum point is a dot.

→ but need to distinguish "external" and "internal" points.

↓ ↓
x, y z

Each internal point is associated w/ a factor of $-i\gamma \int d^4 z$, with combinatorial factor...

How do we count these combinatorial factors?

Oct, 19,
2020

well.. Each H_I has 4 ϕ_I 's: $\phi(z_1)\phi(z_2)\phi(\bar{z}_1)\phi(\bar{z}_2)$

→ interchanging free subtraction "end" will give the same amplitude.

→ so for each H_I we expect a factor of 4!

→ cancels out the $\frac{1}{4!}$ in $\frac{1}{4!}\phi^4$.

In a diagram with more than one power of H_I

We can exchange all the interaction ends of one H_I with interaction ends of the other H_I .

↳ Since we integrate over all $z_1, z_2 \rightarrow$ these gives the same amplitude.

⇒ For a diagram with n "internal vertices",

(i.e. # of H_I 's), we get a factor of $n!$ This cancels the $\frac{1}{n!}$ factor from Taylor series expansion

of $\exp\{-is^2 H_I(4)dt\}$.

~~→ Can be a small subtlety - but is~~

For example... consider the \mathcal{T} term:

(

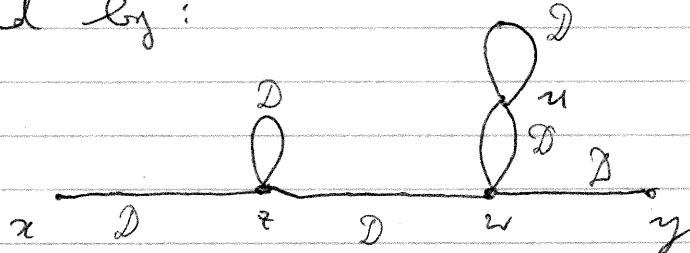
$$\begin{aligned}
 & \langle 0 | \phi(x) \phi(y) \frac{1}{3!} \left(\frac{-i\gamma}{4!} \right)^3 \int d^4 z d^4 w d^4 u \delta^{(4)}(x-z) \delta^{(4)}(y-w) \delta^{(4)}(z-u) \delta^{(4)}(w-y) \\
 & = \frac{1}{3!} \left(\frac{-i\gamma}{4!} \right)^3 \int d^4 z d^4 w d^4 u D_F(x-z) D_F(z-u) D_F(w-u) D_F(w-y)
 \end{aligned}$$

The number of contractions that give this same expression is

$$3! \times (4 \cdot 3) \times (4 \cdot 3 \cdot 2 \cdot 1) \times (9 \cdot 3) \times (1/2)$$

↑ ↑ ↑ ↑ ↑
 interchange placement placement ... interchange
 vertices of contraction of contraction for w of $w-u$
 into \bullet into w into u vertex contraction

Represented by:



Now... there is a subtlety here with all of this, and that's symmetry factors.



Symmetry factors

→ Best to consider the simplest diagram with the most general problem.

→ with $\frac{3!}{3!} \phi^3$ theory --

At 2nd order in λ , $\langle 0 | T \{ \exp \int_{-\infty}^{\infty} H_1(t) dt \} | 0 \rangle$

gives 5th like

$$\left(\langle 0 | \phi(x) \phi(x) \phi(x) \phi(y) \phi(y) | 0 \rangle \right) d^4x d^4y$$

↳ Feynman diagram is



$$D_F(x-y) D_F(x-y) D_F(x-y).$$

Naively we expect $2!$ from interchanging x and y .

and $3! \times 3! = 36$ from interchanging the ϕ 's at x and ϕ 's at y .

→ Expect 72. But are actually only 6:

$$\overbrace{\phi_x^2 \phi_y^2}^1 \rightarrow \left\{ \begin{array}{c} \overbrace{\phi_x \phi_x \phi_x \phi_y \phi_y}^1 \\ \overbrace{\phi_x \phi_x \phi_x \phi_y \phi_y}^1 \end{array} \right\} \text{ total } = 6.$$

Similarly for $\phi_x^1 - \phi_y^2$ ($\times 2$)

$\phi_x^1 - \phi_y^1$ ($\times 2$)

\rightarrow we have overshot by 12, because

$$\cancel{\phi_x \phi_y \phi_z \phi_{\bar{x}} \phi_{\bar{y}} \phi_{\bar{z}}} = \cancel{\phi_x \phi_{\bar{x}} \phi_y \phi_{\bar{y}} \phi_z \phi_{\bar{z}}}$$

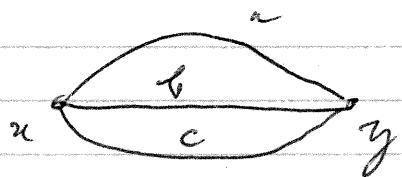
what about graphs to?

(1) simultaneously swap $x^1 = x^2$ and $y^1 = y^2$

$$\cancel{\phi_x \phi_{\bar{x}} \phi_y \phi_{\bar{y}}} = \phi_x \phi_{\bar{x}} \phi_x \phi_{\bar{y}} \phi_{\bar{y}}$$

(2) exchanging all $x^i = y^j$'s don't do anything either ...

We can see what's going on by looking at the vertices = propagators ...



$a \leftrightarrow b$ don't
 $\downarrow \uparrow$ change
the diagram

\Rightarrow diagram has a permutation symmetry

$x \leftrightarrow y$ also don't
change the diagram

This has $2! \times 3! = 12$ elements; which is ten times we overshot with ...

This is called the symmetry factor 5.

The number of diagrams or terms

$$\frac{1}{5} (3!) (3!) (2!) = 6.$$

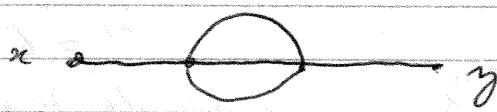
Some examples



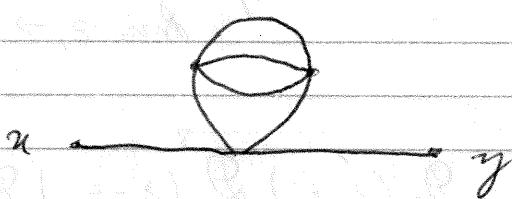
$$s = 2 \quad (x-y)$$



$$s = 2 \cdot 2 \cdot 2 = 8$$



$$s = 3! = 6$$



$$s = 3! \cdot 2 = 12$$

Now, we are ready to state the Feynman rules for position space...

built out this rule lets us find

$$\langle 0 | T \{ \phi(x) \phi(y) \} \exp \left\{ -i \int_{-\infty}^{\infty} p_4(t) dt \right\} | 0 \rangle$$

= (sum all possible diagrams with)
the external points

where each diagram is built out of
propagators
vertices
external pts

Feynman Rules for ϕ^4 theory

① For each propagator $x \xrightarrow{\quad} y = D_F(x-y)$

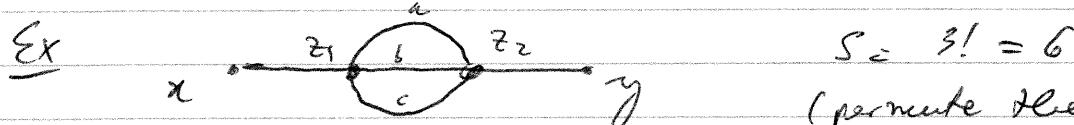
② For each vertex

$$\times z = (-i\gamma) \int d^4 z$$

③ For each external point $x = 1$

④ Divide by symmetry factor.

$\frac{1}{4!}$
no here



(permute the 3 propagators
from $z_1 \rightarrow z_2$)

Amplitude:

$$\left(\frac{-i\gamma}{1}\right)^2 \cdot \frac{1}{6} \cdot \int d^4 z_1 d^4 z_2 D_F(x-z_1) D_F^3(z_1-z_2) D_F(z_2-y)$$

Interpretation

- Each of the vertex factor $(-i\gamma)$ is the amplitude for the emission and/or absorption of particles at a vertex.

- The integral $\int d^4 z$ instructs us to sum over all points where their process can occur.

→ This is the principle of superposition!

- $\int d^4 z$ is addition of amplitudes

↳ Feynman rules tell us to multiply the amplitudes for each

independent part of the process.

*

Now, in most calculations, it is simpler to express the Feynman rules in terms of momentum.

→ We want momentum-space Feynman diagrams

To do this, we write $D_F(x-y)$ in Fourier space

$$D_F(x-y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \frac{i}{p^2 - m^2 + i\epsilon}$$

Now, we present this in a diagram by assigning a 4-momentum p to each propagator.

When 4 lines meet at a vertex, we get

$$\begin{array}{c} p_4 \\ \diagup \\ p_1 \\ \diagdown \\ p_3 \end{array} \quad \rightarrow \quad \int d^4 z e^{-ip_1 z} e^{-ip_2 z} e^{-ip_3 z} e^{+ip_4 z} \\ = (2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 - p_4)$$

i.e. momentum is conserved at each vertex.

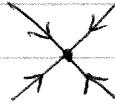
The delta functions can be used to perform integrals for the propagators...

→ From here we get momentum-space Feynman rules.

6

Momentum-space Feynman rules

① Each propagator $\xrightarrow{p} = \frac{i}{p^2 - m^2 + i\epsilon}$

② Each vertex  $= -i\gamma$

③ Each external point  $= e^{-ip \cdot x}$

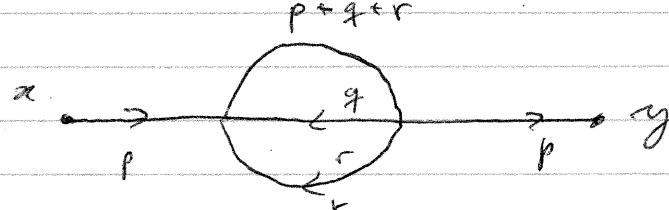
④ Imose momentum conservation at each vertex

⑤ Integrate over each ~~so~~ undetermined momenta

$$\int \frac{d^4 p}{(2\pi)^4}$$

⑥ Divide by symmetry factor.

EV



$$\begin{aligned}
 &= \left(\frac{-i\gamma}{2}\right)^2 \frac{1}{6} \cdot \int \left\{ \left(\frac{i}{p^2 - m^2 + i\epsilon} \right)^2 \frac{i}{(q+p+r)^2 - m^2 + i\epsilon} \frac{i}{q^2 - m^2 + i\epsilon} \right. \\
 &\quad \left. \cdot \frac{i}{r^2 - m^2 + i\epsilon} \right\} \frac{d^4 q}{(2\pi)^4} \frac{d^4 r}{(2\pi)^4} .
 \end{aligned}$$

There is one subtlety, however...

Consider diagrams without external vertices.

↳ Here are diagrams from the form:

$$\langle 0 | T \{ \exp (-i \int_{-\infty}^{\infty} H_0(t) dt) \} | 0 \rangle$$

→ These are called "vacuum diagrams".

At order β^2 we have ...



β^2



and



and

$(\infty_x \infty_y) \rightarrow$ disconnected diagram,

⇒ There is $S=2$ for $(\infty_x \infty_y)$

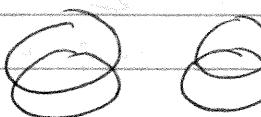
In general, a vacuum diagram has connected subdiagrams V_i which appear n_i times

v

v_1

v_2

v_3



copies

→

$$n_1 = 2$$



$$n_3 = 1$$

$$n_2 = 3$$

The amplitude for the total diagram is the product:

$$\boxed{\prod_i \left(\frac{1}{n_i!} V_i^{n_i} \right)}$$

Now, the sum over all connected diagrams can be written as

$$\sum_{\text{all possible connected pieces}} \sum_{\text{all connected pieces}} (\text{value of connected piece}) \times \left\{ \prod_i \frac{1}{n_i!} (v_i)^{n_i} \right\}$$

The sum of the unconnected pieces factors out, giving

$$= \left(\sum_{\text{connected}} \right) \times \sum_{\{\text{un}\}} \left(\prod_i \frac{1}{n_i!} (v_i)^{n_i} \right)$$

sum all the values of the connected pieces

Now...

$$\sum_{\{\text{un}\}} \left(\prod_i \frac{1}{n_i!} (v_i)^{n_i} \right)$$

$$= \prod_i \sum_{\{n_i\}} \frac{1}{n_i!} (v_i)^{n_i}$$

$$= \prod_i \exp(v_i)$$

$$= \exp\left(\sum_i v_i\right)$$

$$\rightarrow \boxed{\sum \text{all diagrams} = \sum \text{connected} \times \exp(\sum \text{disconnected})}$$

Now, recall that the sum 'll vacuum diagrams is just going to be

$$\exp\left(\sum_i V_i\right).$$

$$\Rightarrow \langle 0 | T \left\{ \exp \left(-i \int_{-\infty}^{\infty} H_I(t) dt \right) \right\} | 0 \rangle = \exp\left(\sum_i V_i\right)$$

and as we have agreed --

$$\begin{aligned} \langle 0 | T \left\{ \phi_I(x) \phi_I(y) \exp \left\{ -i \int_{-\infty}^{\infty} H_I(t) dt \right\} \right\} | 0 \rangle \\ = (\text{connected}) \times \exp\left(\sum_i V_i\right). \end{aligned}$$

\Rightarrow And so we have that

$$\langle S_2 | T \{ \phi(x) \phi(y) \} | S_2 \rangle = \lim \frac{\langle 0 | T \{ (\phi(x) \phi(y)) \text{ (perturb)} \} | 0 \rangle}{\langle 0 | T \{ \exp(-i \int H_I(t) dt) \} | 0 \rangle}$$

= \sum all connected diagrams with 2 external pts.

More generally --

$$\frac{\langle 0 | T \{ \phi(x_1) \dots \phi(x_n) \exp \left(-i \int_{-\infty}^{\infty} H_I(t) dt \right) \} | 0 \rangle}{\langle 0 | T \{ \exp \left(-i \int_{-\infty}^{\infty} H_I(t) dt \right) \} | 0 \rangle}$$

= \sum connected diagrams with
end points x_1, \dots, x_n

Cross section & the S-matrix

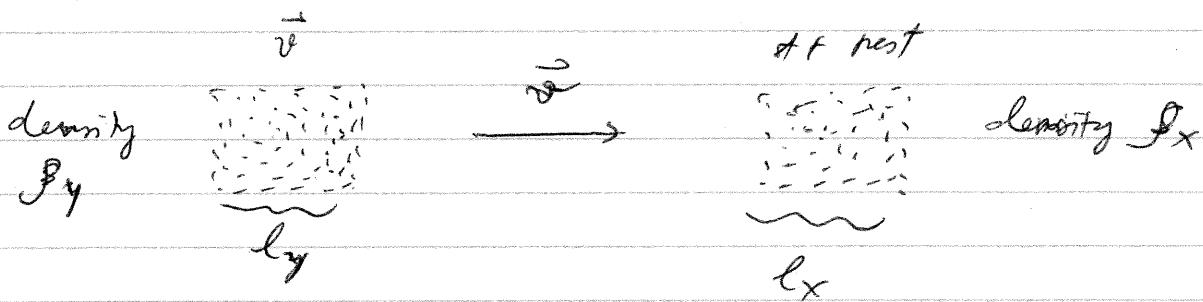
Now let us derive a formula for computing the n-point correlation function...

→ Next task is to compute quantities that can be measured

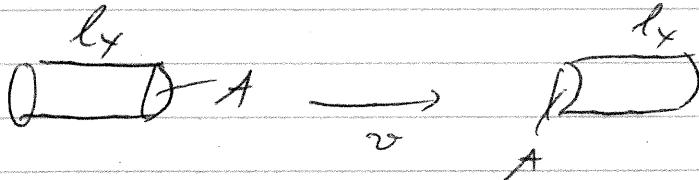
↳ cross section & decay rates.

The cross section:

Consider collision of 2 beams of particles with relatively well-defined momenta.



$\left\{ \begin{array}{l} l_y, l_x \text{ are observed from rest} \\ f_y, f_x \text{ are densities at rest. Let } A \text{ be the} \\ \text{cross sectional area of overlap.} \end{array} \right.$



Total # of particles... $N_x = f_x l_x A$

$$N_y = f_y l_y A$$

→ Total # of scatterings is proportional to $N_x N_y$.

Let total number of scatterings be

$$N_x \cdot N_y = \left(\frac{\sigma}{A} \right)$$

↑ probability one particular X particle & Y collide.

Call σ the effective area or "cross section" of the scattering process.

Let $N_x = 1$, then

$$\sigma = \frac{\text{total # scatterings}}{p_y \cdot l_y}$$

For small time interval ...

$$\sigma = \frac{\text{total # scatterings} / \Delta t}{p_y \cdot (l_y / \Delta t)}$$

scattering rate
particle flux.

The differential cross section is the portion of σ in which the final particle momenta lie inside some window of momenta.

↪ write this as

$$\frac{d\sigma}{dp_1 \cdots dp_n}, \text{ so - tent}$$

$\uparrow \quad \uparrow$
final particle momenta.

$$\int \frac{d\sigma}{d^3 p_1 \cdots d^3 p_n} \cdot d^3 \vec{p}_1 \cdots d^3 \vec{p}_n = \int d\sigma = \sigma.$$

Now, if there are only 2 free particles then
there are only two free parameters ...

Why? two spatial momenta \rightarrow 6 degrees

4-momentum conservation \rightarrow 4 ~~parameters~~
constraints

\Rightarrow can take these two degrees to be
orient. angles $\theta = \phi$

\rightarrow Then we can measure $\boxed{\frac{d\sigma}{d\Omega}(\theta, \phi)}$

where $d\Omega$ is the solid angle differential

$$d\Omega = d\cos\theta d\phi$$

"Differential cross section" refers to $\frac{d\sigma}{d\Omega}$

Let's look at an example ... Consider a periodic
box with length L in all orders.

Spatial momentum mode are now discrete:

$$\vec{k} = \frac{2\pi}{L} (n_x, n_y, n_z) . \quad n_i \in \mathbb{Z} .$$

Have comm. relation:

$$[\hat{a}_{\vec{k}}, \hat{a}_{\vec{k}'}^\dagger] = \delta_{\vec{k}, \vec{k}'} V \quad \rightarrow V = V' \text{ (volume)}$$

$\Rightarrow V \rightarrow \infty$

$$\delta_{\vec{k}, \vec{k}'} V = \iiint_{\text{box}} dx_1 dx_2 dx_3 e^{i(\vec{k}-\vec{k}') \cdot \vec{x}}$$

$$\rightarrow \iiint_{\text{box}} dx_1 dx_2 dx_3 e^{i(\vec{k}-\vec{k}') \cdot \vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}-\vec{k}').$$

In this box ...

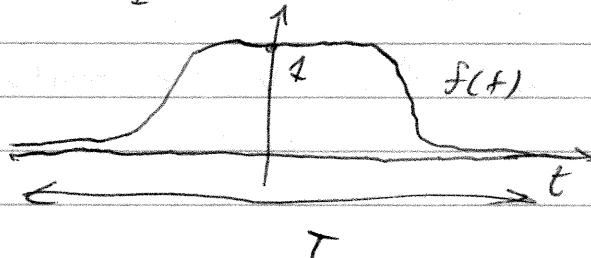
$$\begin{aligned} \phi(x) &= \sum_{\vec{k}} \frac{(2\pi/L)^3}{(2\pi)^3 \sqrt{2E_{\vec{k}}}} (\hat{a}_{\vec{k}} e^{-ik \cdot x} + \hat{a}_{\vec{k}}^\dagger e^{ik \cdot x}) \\ &= \sum_{\vec{k}} \frac{1}{\sqrt{2E_{\vec{k}}}} (\hat{a}_{\vec{k}} e^{-ik \cdot x} + \hat{a}_{\vec{k}}^\dagger e^{ik \cdot x}) \end{aligned}$$

Oct 25, 2020

Now imagine starting with free field theory ...

at some early time then switching on the interactions slowly, and then slowly switching off the interactions ...

i.e. $H_I(t) \rightarrow H_I(t) f(t)$ where $f(t)$ looks like



such that

such that $\int_{-\infty}^{\infty} f(t) dt = T$, $\int_{-\infty}^{\infty} (f(t))^2 dt = T$

$$\text{Let } S = T \exp \left\{ -i \int_{-\infty}^{\infty} dt H_I(t) f(t) \right\}$$

Define the S matrix as

$$\langle \text{final} | S | \text{initial} \rangle$$

where $|\text{initial}\rangle$ is a free particle state with momentum \vec{k}_I^I + energy E_I^I

and $|\text{final}\rangle$ is a free particle state with momentum \vec{k}_F^F + energy E_F^F

Now look at $S - I$:

$$\langle \text{final} | S - I | \text{initial} \rangle$$

* as $T \rightarrow \infty$, $V \rightarrow \infty$ we can write this amplitude as

$$\langle \text{final} | S - I | \text{initial} \rangle = i \cdot N \cdot (2\pi)^4 \delta(E_{tot}^F - E_{tot}^I) \prod_j \delta^{(3)}(\vec{k}_{tot}^F - \vec{k}_{tot}^I)$$

fn of the momenta

* For finite T and finite V we instead have

$\langle \text{Final } | S-I | \text{initial} \rangle$

$$= i M \int_{-\infty}^{\infty} f(t) e^{i(E_{\text{tot}}^F - E_{\text{tot}}^I)t} \cdot \delta_{k_{\text{tot}}^F, k_{\text{tot}}^I} \cdot V$$

So,

$$|\langle \text{Final } | S-I | \text{initial} \rangle|^2 = |M|^2 \cdot \delta_{k_{\text{tot}}^F, k_{\text{tot}}^I} \cdot V^2$$

$$\cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) f(t') e^{i(E_{\text{tot}}^F - E_{\text{tot}}^I)(t-t')} dt dt'$$

as $T \rightarrow \infty$, this is some constant times $\delta(E_{\text{tot}}^F - E_{\text{tot}}^I)$

What is this constant?

To get this \rightarrow integrate w.r.t E_{tot}^F .

$$\rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} dE_{\text{tot}}^F e^{-i(E_{\text{tot}}^F - E_{\text{tot}}^I)(t-t')} \right\} f(t) f(t') dt dt'$$

$2\pi \delta(t-t')$

$$= 2\pi \int_{-\infty}^{\infty} f^2(t) dt = 2\pi \cdot T.$$

So the constant is $2\pi \cdot T$ and so

$$|\langle \text{Final } | S-I | \text{initial} \rangle|^2 = |M|^2 (2\pi) \delta(E_{\text{tot}}^F - E_{\text{tot}}^I) \delta_{k_{\text{tot}}^F, k_{\text{tot}}^I} V^2 T$$

Now, we have been using relative normalizations

$$\left\{ \begin{array}{l} \langle \text{initial} | \text{initial} \rangle = \prod_i (2E_i^I \cdot V) \\ \qquad \qquad \qquad \text{becomes} \\ \qquad \qquad \qquad (2\pi)^3 \delta^{(3)}(0) \text{ as } V \rightarrow \infty \end{array} \right.$$

$$\langle \text{final} | \text{initial} \rangle = \prod_i (2E_i^F \cdot V)$$

To get transition probability per unit time ---

$$\frac{\text{probability}}{\text{time}} = \frac{1}{T} \frac{|\langle \text{final} | S - I | \text{initial} \rangle|^2}{\langle \text{final} | \text{final} \rangle \langle \text{initial} | \text{initial} \rangle}$$

$$= \frac{|M|^2 (2\pi)^3 \delta(E_{\text{tot}}^F - E_{\text{tot}}^I) \delta(\vec{k}_{\text{tot}}^F, \vec{k}_{\text{tot}}^I) \cdot V^2}{\prod_i (2E_i^F \cdot V) \prod_i (2E_i^I \cdot V)}$$

$$\text{As } V \rightarrow \infty, \quad \delta(\vec{k}_{\text{tot}}^F, \vec{k}_{\text{tot}}^I) \cdot V \rightarrow (2\pi)^3 \delta^{(3)}(\vec{k}_{\text{tot}}^F - \vec{k}_{\text{tot}}^I)$$

If we sum over final states in some window then we have

$$\sum_{\vec{k}_1^F, \dots, \vec{k}_{n_F}^F} \frac{1}{(2E_1^F \cdot V)} \dots \frac{1}{(2E_{n_F}^F \cdot V)} \frac{|M|^2 (2\pi)^3 \delta(\vec{k}_{\text{tot}}^F - \vec{k}_{\text{tot}}^I) \cdot V}{(2E_1^I \cdot V) \dots (2E_{n_I}^I \cdot V)}$$

$$\text{As } V \rightarrow \infty, \quad \frac{d^3 \vec{k}_1^F}{(2\pi)^3 2E_1^F} \dots \frac{d^3 \vec{k}_{n_F}^F}{(2\pi)^3 2E_{n_F}^F} \quad \left\{ \begin{array}{l} n_I = \# \text{ initial particles} \\ n_F = \# \text{ final particles} \end{array} \right.$$

Consider single particle decay ... ($n_F = 2$)

The total decay rate is $\Gamma = \int dP$ where

$$dP = \frac{1}{2E^F} \left(\prod_{i=1}^{n_F} \frac{d^3 \vec{k}_i^F}{(2\pi)^3 2E_i^F} \right) |M|^2 (2\pi)^4 \delta^{(4)}(\vec{k}_{tot}^F - \vec{k}_{tot}^I)$$

For a 2-particle initial state, the cross section is given by

$$\sigma = \frac{\text{probability}}{\text{time} \cdot \text{flux density}}$$

Flux density = relative velocity between beam and target
 \times density of incoming beam in lab frame.

We have normalized probability for one incoming beam particle \rightarrow density = $1/V$, and flux

$$\text{flux} = \frac{1/\vec{v}_A - \vec{v}_B}{V} \quad \vec{v}_A, \vec{v}_B = \text{velocity of particles in lab frame}$$

$$\therefore d\sigma = \left(\prod_{i=1}^{n_F} \frac{d^3 \vec{k}_i^F}{(2\pi)^3 2E_i^F} \right) (2\pi)^4 \delta^{(4)}(\vec{k}_{tot}^F - \vec{k}_{tot}^I) \frac{|M|^2}{2E_A E_B |\vec{v}_A - \vec{v}_B|}$$

call this $d\Gamma_{n_F}$

Now consider special case: 2 final particles ($\eta_F = 2$)
in COM frame

$$\int d\Omega_2 = \int \frac{dp_1 \, d\Omega}{(2\pi)^3 2E_1 2E_2} (2\pi) \delta(E_{\text{com}} - E_1 - E_2)$$

where final particle energies are $E_1 = \sqrt{(\vec{p}_1)^2 + m_1^2}$, $E_2 = \sqrt{(\vec{p}_2)^2 + m_2^2}$

~~$$= \int d\Omega_2 = \int \frac{d\Omega}{16\pi^2 E_1 E_2}$$~~

$$\Rightarrow \int d\Omega_2 = \int \frac{d\Omega}{16\pi^2} \int_0^\infty \frac{p_1^2}{\sqrt{(\vec{p}_1)^2 + m_1^2}} \delta(-E_{\text{com}} + \sqrt{(\vec{p}_1)^2 + m_1^2} + \sqrt{(\vec{p}_2)^2 + m_2^2}) dp_1$$

Recall that $\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}$

$$\frac{dE_1}{dp_1} = \frac{d\sqrt{(\vec{p}_1)^2 + m_1^2}}{dp_1} = \frac{1}{2} \frac{2p_1}{\sqrt{(\vec{p}_1)^2 + m_1^2}} = \frac{p_1}{E_1}$$

Likewise $\frac{dE_2}{dp_2} = \dots = \frac{p_2}{E_2}$.

$$\therefore \int d\Omega_2 = \int \frac{d\Omega}{16\pi^2} \frac{p_1^2}{E_1 E_2 \left(\frac{p_1}{E_1} + \frac{p_2}{E_2} \right)} \Bigg|_{p_1 \text{ s.t. } E_{\text{com}} = E_1 + E_2}$$

$$= \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_1 + E_2} = \int \frac{d\Omega}{16\pi^2} \frac{p_1}{E_{\text{com}}}$$

So for 2 particles \rightarrow 2 particles ...

$$\left(\frac{dc}{dr} \right)_{\text{com}} = \frac{10^4 F_{\text{inel}} \cdot (M)^2}{2E_A E_B (\vec{r}_A - \vec{r}_B) \cdot 16\pi^2 F_{\text{com}}} \quad (F_{\text{com}} = F_A + F_B)$$

Now, it is conventional to define the T-matrix as

$$S = 1 + iT$$

$$\text{Claim } \langle \vec{p}_1^F, \dots, \vec{p}_{n_F}^F | i^* T | \vec{p}_A, \vec{p}_B \rangle$$

$$= \left(\lim_{t \rightarrow \infty} \langle \vec{p}_1^F, \dots, \vec{p}_{n_F}^F | T \exp \left[i \int_{-\infty}^{\infty} dt' H(t') \right] | \vec{p}_A, \vec{p}_B \rangle \right)_{\text{free}} *$$

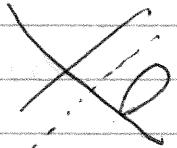
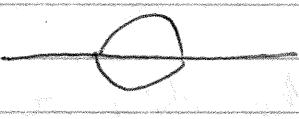
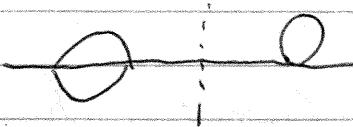
where $*$ = connected diagrams only
+ "asymptotic" diagrams only

"asymptotic" \equiv diagrams can't be broken into
disconnected pieces by cutting one
internal line
(i.e. 1 particle irreducible)

ex

(not asymptotic)

(asymptotic)



The claim won't be proven now, but the idea is similar as before...

$$|s\rangle = \lim_{T \rightarrow \infty (1-i\epsilon)} \left(e^{-iE_0 T} |s\rangle_0 \right)^\dagger e^{-iHT} |0\rangle$$

and we would like something similar...

$$\langle \tilde{p}_1 \dots \tilde{p}_n \rangle \propto \lim_{T \rightarrow \infty (1-i\epsilon)} e^{-iHT} \langle \tilde{p}_1, \dots, \tilde{p}_n \rangle_{\text{free}}$$

For now, we'll just take the claim as true...

→ Note that

$$\begin{aligned} \phi_I^+(x) |\vec{p}\rangle_{\text{free}} &= \underbrace{\int \frac{d^3 k}{(2\pi)^3 \sqrt{2E_k}}}_{\text{relativistic}} \underbrace{a_k^+}_{\text{normalization}} e^{-ik \cdot x} \underbrace{\sqrt{2E_p}}_{\text{free}} \underbrace{a_p^+}_{\text{free}} |0\rangle \\ &= e^{-ip \cdot x} |0\rangle \end{aligned}$$

We can think of taking the commutator of $\phi_I^+(x)$ with the a_p^+ from $|\vec{p}\rangle_{\text{free}}$

→ suggest the relation

$$\boxed{\phi_I^+(x) |\vec{p}\rangle_{\text{free}} = \cancel{\text{commute}} e^{-ip \cdot x}}$$

Now drop the "free" subscript - really...

$$\langle \tilde{p} | \phi_I^+(x) = \cancel{\text{commute}} e^{+ip \cdot x} \langle 0 | \quad \text{define}$$

$$\boxed{\langle \tilde{p} | \phi(x) = e^{+ip \cdot x}}$$

→ Set Feynman Rules in position space with external lines

Propagator: $\frac{x-y}{D_F(x-y)}$

Internal vertex: $\cancel{X}_z \quad (-i\lambda) \int d^4 z$

Each external line $\cancel{\ell} \quad e^{ip \cdot x}$

Divide by symmetry factor S .

Feynman rules in momentum space with external line

Propagator: $\frac{i}{p^2 - m^2 + i\epsilon}$

Internal vertex: $\cancel{X} \quad -i\lambda = \text{momentum conservation}$

External line

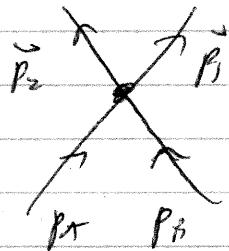
$\cancel{\ell} \quad \text{no extra factor (just 1)}$

Integrate over all ¹ conserved momenta & divide by S .

\cancel{q}

S₁ $\langle \tilde{p}_1, \tilde{p}_2 | i\tau | \tilde{p}_A, \tilde{p}_B \rangle$ at lowest order...

well...



Feynman amplitude:

$$iM = -i\lambda$$

S₂

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{(\vec{p}^{\text{final}} / M)^2}{2E_A E_B |\vec{v}_A - \vec{v}_B| 16\pi^2 E_{\text{cm}}}$$

Let $p = |\vec{p}^{\text{final}}| = |\vec{p}_A| = |\vec{p}_B|$ all same since masses are all same

$$E_{\text{cm}} = 2E_A = 2\sqrt{p^2 + m^2}$$

$$|\vec{v}_A - \vec{v}_B| = 2|\vec{v}_A| = \frac{2|\vec{p}_A|}{E_A} = \frac{2p}{E_A}$$

$$\left(\frac{d\sigma}{d\Omega} \right)_{\text{cm}} = \frac{\pi^2 p}{\frac{2}{E_A} (2E_A)(2E_B) 16\pi^2 E_{\text{cm}}} = \frac{\pi^2}{64\pi^2 E_{\text{cm}}^2}$$

This is spherically symmetric, so

$$\sigma_{\text{tot}} = (4\pi) \frac{\pi^2}{64\pi^2 E_{\text{cm}}^2} \cdot \frac{1}{2} \rightarrow \text{particles in final state are identical so need a } 1/2 \text{ factor...}$$

$$\sigma_{\text{tot}} = \frac{\pi^2}{32\pi^2 E_{\text{cm}}^2} \rightarrow \text{our first QFT cross section!}$$

Feynman Rules for Fermions

Oct 26
2020

So far we've discussed only the ϕ^4 theory --

→ need to generalize results to theories containing fermions.

→ need to generalize defn. of time ordering
2 normal ordering symbols to include fermions --

Recall --

$$T\{\psi_a(x) \overline{\psi_b(y)}\} = \begin{cases} \psi_a(x) \overline{\psi_b(y)} & x^0 > y^0 \\ -\overline{\psi_b(y)} \psi_a(x) & x^0 < y^0 \end{cases}$$

The Feynman propagator, under this defn is

$$\begin{aligned} S_F(x-y) &= \int \frac{d^4 p}{(2\pi)^4} \frac{i(p+m)}{p^2 - m^2 + i\epsilon} e^{-ip \cdot (x-y)} \\ &= \langle 0 | T\{\psi_a(x) \overline{\psi_b(y)}\} | 0 \rangle \end{aligned}$$

(Recall that $p = \gamma^\mu p_\mu = \vec{p} \cdot \vec{\gamma}$)

Generalize of T to more than two fermion fields --

$$T\{\psi_1 \psi_2 \psi_3 \psi_4\} = \begin{cases} \psi_1 \psi_2 \psi_3 \psi_4 & \text{if } x_1^0 > x_2^0 > x_3^0 > x_4^0 \\ -\psi_2 \psi_1 \psi_3 \psi_4 & \text{if } x_2^0 > x_1^0 > x_3^0 > x_4^0 \\ -\psi_3 \psi_2 \psi_1 \psi_4 & \text{if } x_3^0 > x_2^0 > x_1^0 > x_4^0 \\ \vdots & \end{cases}$$

Rule: $x(-1)$ if odd # permutations.

$x(+1)$ if even # permutations.

Similarly, for normal ordering symbol..

$$N \{ a_{p_1}^{\dagger} a_{p_2}^{\dagger} a_{p_3}^{\dagger} a_{p_4}^{\dagger} \} = (-1)^{a_{p_4}^{\dagger} a_{p_1}^{\dagger} a_{p_2}^{\dagger} a_{p_3}^{\dagger}}$$

$x(-1)$ if odd permutation of fields

$x(+1)$ if even

With these, can generalize to get Wick's theorem..

1st case: 2 Dirac fields $T \{ \psi_a(x) \overline{\psi_b(y)} \}$

$$T \{ \psi_a(x) \overline{\psi_b(y)} \} = N \{ \psi_a(x) \overline{\psi_b(y)} \} + \overbrace{\psi_a(x) \overline{\psi_b(y)}}^1$$

$$\text{where } \overbrace{\psi_a(x) \overline{\psi_b(y)}}^1 = \begin{cases} \{ \psi_a^+(x), \overline{\psi_b^-}(y) \} & x^0 > y^0 \\ -\{ \overline{\psi_b^-}(y), \psi_a^+(x) \} & x^0 < y^0 \end{cases}$$

$$= \langle 0 | T \{ \psi_a(x) \overline{\psi_b(y)} \} | 0 \rangle$$

$$= \overbrace{S_F(x-y)}$$

$$= -\overline{\psi_b(y)} \psi_a(x)$$

where recall that

$\psi^+, \bar{\psi}^+$ are the positive frequency part of $\psi, \bar{\psi}$
 \rightarrow i.e. part with annihilation operators.

$\psi^-, \bar{\psi}^-$... "negative" creation operators.

Also we note that:

$$\boxed{\psi_a(x) \bar{\psi}_b(y) = \overline{\psi_a(x)} \overline{\bar{\psi}_b(y)} = 0}$$

\rightarrow Just as we proved Wick's thm for boson, we can show the same for fermions

$$T\{\psi_1 \bar{\psi}_2 \psi_3 \bar{\psi}_4 \dots\} = N[\psi_1 \bar{\psi}_2 \psi_3 \bar{\psi}_4 + \text{all possible combinations}]$$

where we note that an expression such as

$$N[\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4] = - \bar{\psi}_3 \bar{\psi}_4 N[\psi_2 \psi_4]$$

gets a minus sign since the $\bar{\psi}_3$ must loop over the ψ_2

Helpful hint for any fully contracted quantity, count the number of times the combination lines cross-over tells you if the # of perm. is odd/even...

Ex $\boxed{\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6}$ \rightarrow even

$\boxed{\psi_1 \psi_2 \bar{\psi}_3 \bar{\psi}_4 \bar{\psi}_5 \bar{\psi}_6}$ \rightarrow odd.

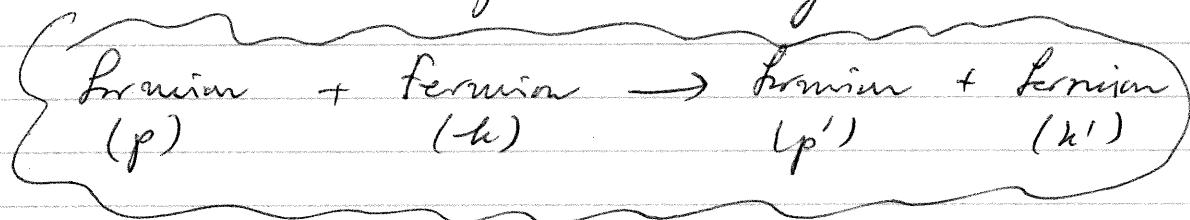
Yukawa Theory

Now we consider the simplest theory with fermions.

$$H_{\text{Yukawa}} = H_{\text{Dirac}} + H_{\text{plain-Gordon}} + \int d^3x g \bar{\psi} \gamma^\mu \phi$$

A simplified model of QED. We will carefully work out the rules of calculations for Yukawa theory before going to QED.

We will consider two-particle scattering reaction:



The leading contribution comes from the H_F^2 term of the S-matrix:

$$\langle p', k' | T \left\{ \frac{1}{2!} (-ig) \int d^3x \bar{\psi}_I \gamma^\mu \phi_I (-ig) \int d^3y \bar{\psi}_I \gamma^\mu \phi_I \right\} | p, k \rangle$$

Now we Wick's theorem to reduce this to N-product of contractions -> can act on unrenormalized fields

Represent this as the contraction:

$$\begin{aligned} \text{at } & \text{vertices} \\ Y_I(x) | \tilde{p}, s \rangle &= \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \sum_s \left(\tilde{a}_{\tilde{p}'}^{(s)} u^{(s)}(\tilde{p}') e^{-ip' \cdot x} \sqrt{2E_{\tilde{p}'}} \right) \left(\tilde{a}_{\tilde{p}}^{(+)}) | 0 \rangle \right) \\ &= e^{-ip \cdot x} u^s(p) | 0 \rangle \end{aligned}$$

Fermion state with momentum \tilde{p} , spin s

Define:

$$\text{So } \boxed{\langle \vec{p}, s | \psi_{\pm}(x) | \vec{p}, s \rangle = e^{-i\vec{p} \cdot x} u^{\pm}(\vec{p})}$$

$$\text{Similarly, } \boxed{\langle \vec{k}, s | \bar{\psi}_{\pm}(x) | \vec{k}, s \rangle = e^{-i\vec{k} \cdot x} \bar{u}^{\pm}(\vec{k})}$$

$$\left\{ \begin{array}{l} \boxed{\langle \vec{p}, s | \bar{\psi}_{\mp}(x) = e^{+i\vec{p} \cdot x} \bar{u}^{\mp}(\vec{p})} \\ \boxed{\langle \vec{k}, s | \psi_{\mp}(x) = e^{+i\vec{k} \cdot x} u^{\mp}(\vec{k})} \end{array} \right.$$

So, typically, a contribution to the matrix element is
the interaction ...

$$\langle \vec{p}', \vec{k}' | \frac{1}{2!} (\text{fig}) \int d^4q \bar{\psi} \gamma^4 \phi(-iq) \int d^4q' \bar{\psi} \gamma^4 \phi(\vec{p}, \vec{k})$$

Up to a (-) sign, the value of this quantity is

$$\begin{aligned} J = & (-iq)^2 \int \frac{d^4q}{(2\pi)^4} \delta^4(q^2 - m_\phi^2) (p' - p + q) \\ & \times (2\pi)^4 \delta^{(4)}(k' - k - q) \bar{u}(p') u(p) \bar{u}(k') u(k) \\ & \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ & \quad \phi - \phi \quad \quad \quad \vec{p}' \leftrightarrow \vec{q} \quad q_p \quad \vec{k}' - \vec{q} \quad \vec{q} - \vec{k} \end{aligned}$$

Int.

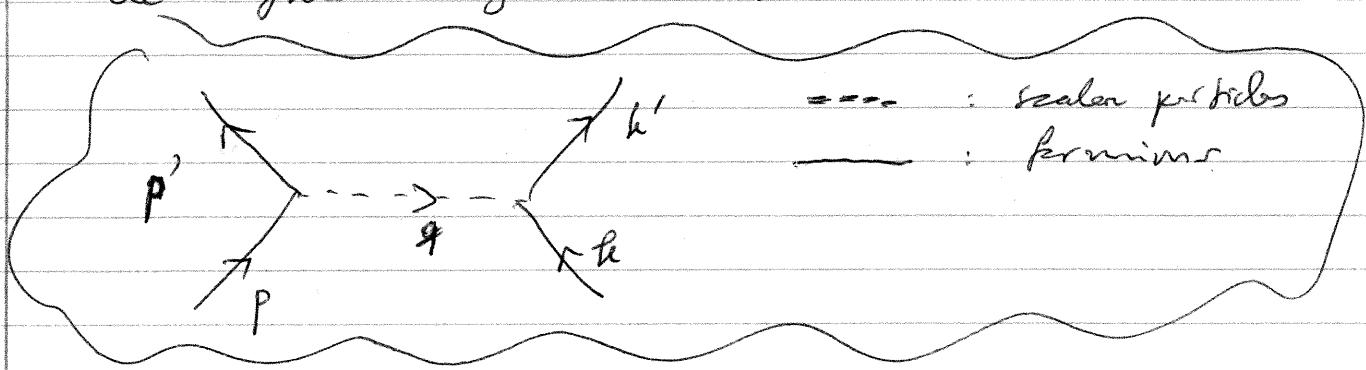
over all
intermediate
momenta.

where $q = p - p' = k' - k$.

→ Upon using the δ functions, $J = iM (2\pi)^4 \delta^{(4)}(\sum p)$

$$\text{where } M = \frac{-i q^2}{q^2 - m_\phi^2} \bar{u}(p') u(p) \bar{u}(k') u(k)$$

The Feynman diagram for this is ...



Feynman rules for fermions in momentum space

① Propagators: $\overline{\phi(x)\phi(y)} = \frac{i}{q^2 - m_\phi^2 + i\epsilon}$

$$\overline{\psi(x)\bar{\psi}(y)} = \frac{i(p+m)}{p^2 - m^2 + i\epsilon}$$

② Vertices = $(-ig)$

③ External leg contractions:
 a) $\overline{\phi(q)} = \frac{1}{q} = 1$

b) $\overline{\psi}\phi = \frac{1}{q} = 1$

c) $\overline{\psi(p,s)} = u^s(p)$ d) $\overline{\langle p,s | \bar{\psi}} = \overline{u^s(p)}$
(fermions)

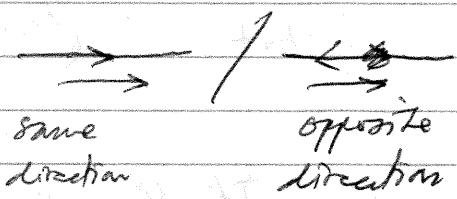
e) $\overline{\psi(k,s)} = \overline{v^s(k)}$ f) $\langle k,s | \bar{\psi} = \overline{v^s(k)}$
(antifermion)

④ Momentum conservation at each vertex

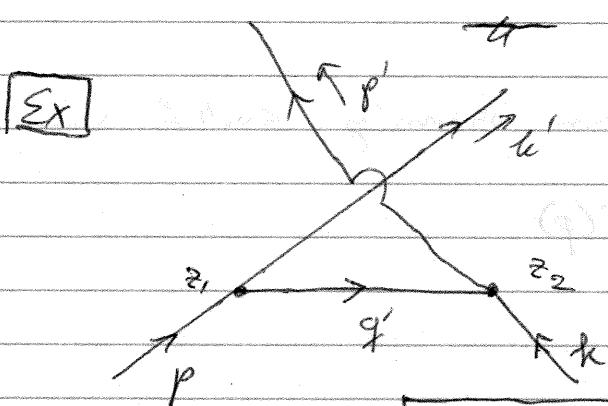
- 5) Integrate over intermediate momenta
6) Figure out sign of diagram

Note { initial & final state have momentum pointing in
out

external particle / antiparticle (\Rightarrow)



Relevant example: There are 2 cross-cross, so (+1).



$$M = \int d^4z_1 d^4z_2 \langle k', p' | \bar{\psi}_1 \psi_1 \phi_1 \bar{\psi}_2 \psi_2 \phi_2 | \tilde{p}, \tilde{k} \rangle$$

cross-cross = 3 \Rightarrow (-1)

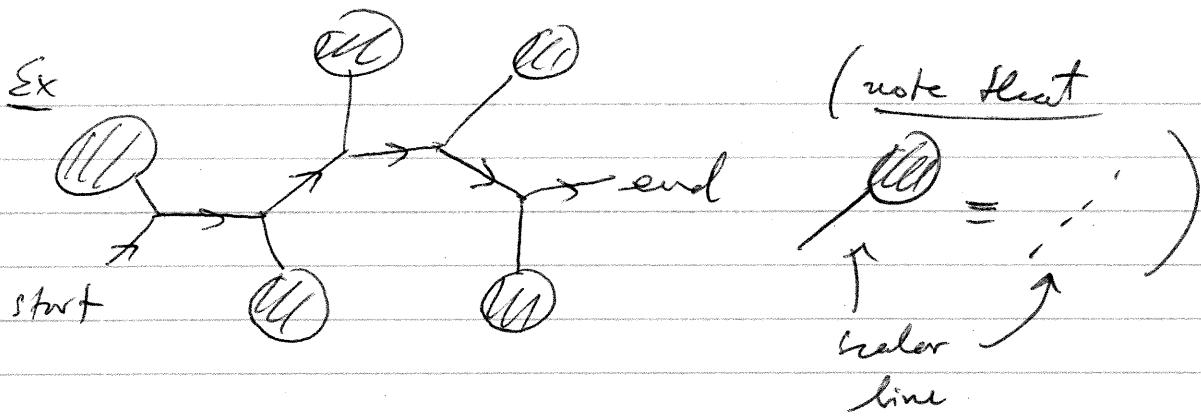
$$\rightarrow M = (-ig)^2 (-1) \cdot \frac{i}{q'^2 - m_\phi^2 + i\epsilon} (\bar{u}(k') u(p)) (\bar{u}(p) u(k))$$

↑ ↑ ↑ ↑ ↑ ↑
2 vertices 3 c.c. ↓ ↓ ↓ ↓
 ↓ ↓ ↓ ↓
 φ-φ φ-φ φ-φ φ-φ

$$\langle u' |\bar{\psi} \psi | p' \rangle \langle p' |\bar{\psi} \psi | k' \rangle$$

where $q' = p - k'$.

{ } Tips for each fermion line that doesn't close into loop, follow the particle number arrow to the end



If the end is an outgoing fermion write down as

$$\rightarrow \not{p} \bar{u}(p)$$

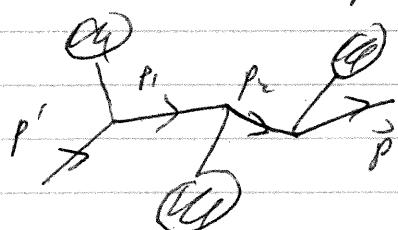
(1)

If the end is an incoming antifermion --

$$\not{p} \bar{v}(p)$$

(1)

Write down fermion propagators you encounter as you follow the particle number arrow backwards



$$\bar{u}(p) \frac{i(p_2 + m)}{p_2^2 - m^2 + i\epsilon} \cdot \frac{i(p_3 + m)}{p_3^2 - m^2 + i\epsilon} \dots$$

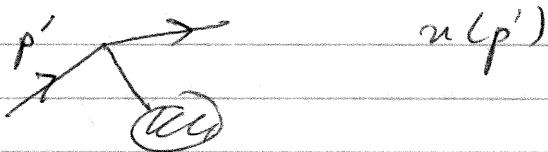
Note if

$$\not{p} \rightarrow \frac{i(p + m)}{p^2 - m^2 + i\epsilon}$$

if

$$\not{p} \rightarrow \frac{i(-p + m)}{p^2 - m^2 + i\epsilon}$$

If the start is an incoming fermion...



If -- incoming antifermion



If the fermion lines form a closed loop

$$\begin{aligned}
 &= \overline{\psi} \psi \overline{\psi} \psi \overline{\psi} \psi \\
 &= (-1) \text{tr} \left\{ \overline{\psi} \gamma^5 \psi \overline{\psi} \gamma^5 \psi \overline{\psi} \gamma^5 \psi \right\} \\
 &= (-1) \text{tr} \left\{ S_F S_F S_F S_F \right\}
 \end{aligned}$$

→ a closed loop always gives a factor of (-1) & the trace of a product of Dirac matrices.

"trace" because we sum over the spinor indices.

The Yukawa Potential

Consider non-relativistic scattering of 2 different fermions
 \rightarrow interact via exchange of a scalar particle.

Ignore $\mathcal{O}(\vec{p}/m^2)$ corrections, momenta are

$$\left\{ \begin{array}{l} p = (m, \vec{p}) , \quad k = (m, \vec{k}) \\ p' = (m, \vec{p}') , \quad k' = (m, \vec{k}') \end{array} \right.$$

$$(p' - p)^2 = -|\vec{p}' - \vec{p}|^2 + \mathcal{O}(p^4)$$

$$= (m - m) - (\vec{p}' - \vec{p})^2 + \mathcal{O}(\dots)$$

$$u^s(p) = \sqrt{m} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}, \text{ etc.}$$

where $\xi^s \xi^{s'} = \delta_{ss'}$

Spin products: $\left\{ \begin{array}{l} \bar{u}^s(p') u^s(p) = 2m \xi^s \xi^s = 2m s^{ss'} \\ \bar{u}^r(k) u^r(k) = 2m \xi^r \xi^r = 2m s^{rr'} \end{array} \right.$

\rightarrow the spin of each particle is conserved.

\rightarrow amplitude: $iM = \frac{i\theta^2}{(\vec{p}' - \vec{p})^2 + m^2} (\bar{u}^s(p') u^s(p)) (\bar{u}^r(k) u^r(k))$

$$iM = \frac{\delta g^2}{|\vec{p}' - \vec{p}|^2 + m^2} \frac{2m s^{rr'} 2m s^{ss'}}{2m s^{rr'} 2m s^{ss'}}$$

$$\text{Okay so } iM = \frac{i\bar{g}^2}{(\vec{p}' - \vec{p})^2 + m_q^2} 2m \delta^{rr'} 2m \delta^{ss'}$$

Compared with the Born approximation to the scattering amplitude in nonrelativistic scattering ~~interaction~~
~~with~~ ΩM , in form of the potential $V(x)$:

$$\langle p' | iT | p \rangle = -i \tilde{V}(\vec{q}) (2\pi) \delta(E_{\vec{p}'} - E_{\vec{p}})$$

where $\vec{q} = \vec{p}' - \vec{p}$.
where does this come from?

(cf. Griffiths Chapter 21 on the Born approximation)

→ Check out Born's approximation to scattering amplitude in nonrelativistic quantum mechanics

Nov 1, 2020

so we find that

$$\text{Amplitude} = iM = \frac{i\bar{g}^2}{(\vec{p}' - \vec{p})^2 + m_q^2} 2m \delta^{rr'} \delta^{ss'} \rightarrow \text{FT}\{V(r)\}$$

$$2 \quad \langle p' | iT | p \rangle = -i \tilde{V}(\vec{q}) (2\pi) \delta(E_{\vec{p}'} - E_{\vec{p}})$$

$$(\vec{q} = \vec{p}' - \vec{p})$$

→ for Yukawa interaction.

$$\boxed{V(\vec{q}) = -\frac{g^2}{|\vec{q}|^2 + m_q^2}}$$

With $\tilde{V}(q)$, can take inv FT to get V ...

$$V(\vec{r}) = \int \frac{d^3 q}{(2\pi)^3} \frac{-g^2}{|q|^2 + m_q^2} e^{i\vec{q} \cdot \vec{r}}$$

$$= \frac{-g^2}{(4\pi)^3} \int_0^\infty \frac{1}{1} \frac{q^2}{iqr} \frac{e^{iqr} - e^{-iqr}}{iqr} \frac{1}{q^2 + m_q^2}$$

$$= \frac{-g^2}{4\pi^2 ir} \int_{-\infty}^\infty dq \frac{q e^{iqr}}{q^2 + m_q^2}$$

evaluate w/ CIF

$$\Rightarrow V(r) = \boxed{\frac{-g^2}{4\pi} \frac{e^{-m_q r}}{r}} \rightsquigarrow \text{attractive Yukawa potential}$$

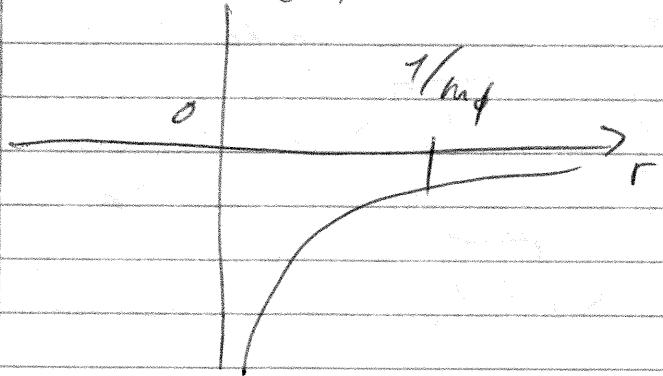
Range $1/m_q = \frac{t}{m_q c}$, the Compton wavelength of the decayed baryon. (Yukawa predicted the mass of the pion from this)

short range

\rightarrow

short-range

$V(r)$



Review of QUANTUM ELECTRODYNAMICS

To get to QED, we go from Yukawa theory by
replace the scalar field particle ϕ with
a vector particle A_μ

→ replace Yukawa Hamilton with

$$H_{\text{int}} = \int d^3x e \bar{\psi} \gamma^\mu \psi A_\mu.$$

→ Vector particle has spin 1

Recall that fermions come with spinor indices
 $u^c(p)$

→ Photon has a polarization vector $\epsilon^\mu(p)$

Feynman Rules (momentum space) → (derivation)

Photon propagator: $\mu \xrightarrow{q} \nu \xrightarrow{q^2 + i\epsilon} \frac{-ig_{\mu\nu}}{q^2 + i\epsilon}$

New vertex $\left. \begin{array}{c} a \\ b \end{array} \right\} \xrightarrow{\text{fermion}} \left. \begin{array}{c} \mu \\ \nu \end{array} \right\} \xrightarrow{\text{Lorentz index}} -i \gamma^\mu_{ab}$

External photon lines $A_\mu |\vec{p}, \varepsilon\rangle \leftarrow \begin{array}{c} \mu \\ \nu \end{array} = \epsilon_\mu(p)$

$\langle \vec{p}, \varepsilon | A_\mu \rangle \leftarrow \begin{array}{c} \mu \\ \nu \end{array} = \epsilon_\mu^\nu(p)$

To justify these rules... recall that in Lorentz gauge we have

$$\partial_\mu A^\nu = 0, \text{ in which case}$$

$$\partial_\mu F^{\mu\nu} = 0 \Rightarrow \boxed{\partial_\mu \partial^\mu A^\nu = 0.}$$

So, such A^ν ($\nu = 0, 1, 2, 3$) satisfies the keq with zero mass...

→ In \vec{p} space... solutions are plane waves...

$$\left\{ \epsilon_\nu(p) e^{-ip \cdot x}, \text{ where } p^2 = 0 \text{ (massless)} \right.$$

$\epsilon_\nu(p) = 4\text{-vector}$

In Fourier components...

$$A_\mu(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=0}^3 (a_p^r \epsilon_m^r(p) e^{-ip \cdot x} + a_p^{r\dagger} \epsilon_m^r(p) e^{ip \cdot x})$$

$$\text{where } p^0 = E_p = \sqrt{\vec{p}^2}$$

where $r = 0, 1, 2, 3 \rightarrow$ basis for polarization vectors.

But notice that photons in the real world only have 2 polarizations...

→ More on this later...

11

The Coulomb potential

Consider nonrelativistic scattering calculation of the previous section ...

→ Look at scattering of 2 fermions of the same mass ...

→ Leading-order contribution is ... new vertices

$$iM = \frac{\not{p} + \not{k}}{\not{p} - \not{k}} = (-ie)^2 \overline{u}(p') \gamma^5 u(p) \frac{-ig_{\mu\nu}}{(\not{p}' - \not{p})^2} \times \overbrace{\overline{u}(k') \gamma^\nu u(k)}^{\text{pion line}}$$

In nonrelativistic limit ...

$$\overline{u}^s(p') \gamma^5 u(p) = [\xi^{s+} \xi^{s+}] \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}_{xm}$$

$$= 0 \quad \forall i = 1, 2, 3$$

and

$$\overline{u}^s(p') \gamma^0 u(p) = (\xi^{s+} \xi^{s+}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \xi^s \\ \xi^s \end{pmatrix}_{xm}$$

$$= 2m \delta^{ss'}$$

So ... almost like the Yukawa case ...

$$iM \approx \frac{-ie^2}{-(\not{p} - \not{p}')^2} (2m)^2 g_{00} \delta^{ss'} \delta^{rr'}$$

Same as Yukawa except the $(-)$ sign $\approx m_p = 0$.

\rightarrow this potential is repulsive! \rightarrow the potential is of the Coulomb potential...

$$V(r) = \frac{e^2}{4\pi r} = \frac{\alpha}{r}$$

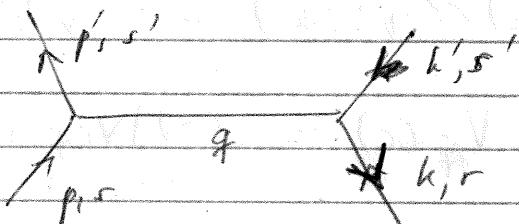
$$\alpha = \frac{e^2}{4\pi} \sim \frac{1}{137}$$

(fine structure constant)

What about fermion-anti-fermion subtrig?

For Yukawa = QED?

{Yukawa:}



$$|\tilde{p}, s; \tilde{k}, r\rangle = (\sqrt{2\varepsilon_p} \sqrt{2\varepsilon_k} a_p^{s\dagger} b_k^{r\dagger}) |0\rangle$$

$$\langle \tilde{p}', s'; \tilde{k}', r' | = \langle 0 | \sqrt{2\varepsilon_{p'}} \sqrt{2\varepsilon_{k'}} b_{k'}^{r'} a_{p'}^{s'}$$

and so...

$$\langle \tilde{k}, r'; \tilde{p}', s' | \underbrace{\gamma^\mu \gamma^\nu}_{\text{cross}} \underbrace{\gamma^\rho \gamma^\sigma}_{\text{cross}} | \tilde{p}, s, \tilde{k}, r \rangle \quad (3 \text{ cross-cross})$$

For fermions...

$$\bar{u}^{s'}(p') u^s(p) = 2m \delta^{ss'}$$

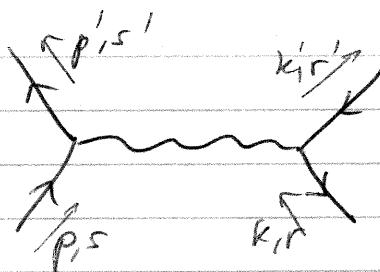
anti fermion

$$\bar{v}^r(k) v^{r'}(k') = \begin{pmatrix} \bar{s}^r & -\bar{s}^{r'} \\ \bar{s}^{r'} & \bar{s}^r \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \bar{\xi}^{r'} \\ \bar{\xi}^r \end{pmatrix} x_m$$

$$= -2m \delta^{rr'}$$

$$\text{so } V_{\bar{f}f}(r) = (-1)(-1) V_{ff}(r) = V_{ff}(r) \quad (\text{attractive})$$

QED



→ has an overall (-1) sign

For fermion ... set $\bar{n}^s(p') \delta^{\mu} n^s(p) \rightarrow \bar{n}^s(p') \delta^{\mu} n^s(p)$
 $\approx 2m \delta^{ss}$

For antifermion ...

$$\bar{v}^r(k) \delta^{\mu} v^r(k') \rightarrow \bar{v}^r(k) r^{\mu} v^r(k') = 2m \delta^{rr}$$

~~$\delta V_{ff}(r) = (-1)V_{ff}(r) = -V_{ff}(r)$ (attractive)~~

~~4~~

<u>Exchange particle</u>	<u>ff or $\bar{f}\bar{f}$</u>	<u>ff</u>
scalar (Yukawa)	attractive	attractive
weak (QED)	repulsive	attractive
tensor (gravity)	attractive	attractive

~~4~~

Some problems...

(167)

Problem

(normal ordering - applications in QM & coherent states...)

$$\rightarrow \text{both QM & QMHO: } H = \frac{1}{2}(p^2 + q^2)$$

Since $[p, q] = -i$ is usual...

$$a = \frac{1}{\sqrt{2}}(q + ip)$$

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip)$$

Let $|Y_0\rangle$ be the ground state of H with normalization
 $\langle Y_0 | Y_0 \rangle = 1.$ For any $z \in \mathbb{C}$, we define the coherent state

$$|z\rangle = ce^{z a^\dagger} |Y_0\rangle$$

where c is such that $\langle z | z \rangle = 1.$ (a) What is $c?$

$$\text{Well } \langle z | z \rangle = 1 = \langle Y_0 | e^{z^* a} c^* c e^{z a^\dagger} | Y_0 \rangle$$

$$\text{Recall } = |c|^2 \langle Y_0 | e^{z^* a} e^{z a^\dagger} | Y_0 \rangle$$

$$\begin{aligned} (a^\dagger)^n |Y_0\rangle &= \sqrt{n!} |Y_n\rangle \\ &= |c|^2 \sum_{m,n} \langle Y_0 | \frac{(z^* a)^n}{n!} \frac{(z a^\dagger)^m}{m!} | Y_0 \rangle \\ &= |c|^2 \sum_{m,n} (z^*)^n (z)^m \frac{1}{\sqrt{n! m!}} \langle Y_n | Y_m \rangle \\ &= \sum_n \frac{|z|^2}{n!} |c|^2 \\ &= e^{|z|^2} |c|^2 \end{aligned}$$

$$\Rightarrow c = \exp \left\{ -|z|^2 / 2 \right\}$$

(b) What is $\langle z_1 | z_2 \rangle$?

Answer (easy to get)

$$\langle z_1 | z_2 \rangle = \exp \left\{ -\frac{|z_1|^2 - |z_2|^2}{2} \right\} \exp [z_1^* z_2]$$

(c) Show that $|z\rangle$ is eig. state of a - b d.eigv

\hookrightarrow plug in & use commutator rule to

$$[a, (a^\dagger)^n] = n(a^\dagger)^{n-1} \text{ to get}$$

$$\langle a | z \rangle = z | z \rangle$$

(d) $\forall n, m \in \mathbb{Z}^+$, determine the normal-ordered expectation value

$$\langle z | : \phi^m z^n : | z \rangle$$

$$\rightarrow \text{use } p = \frac{-i}{\sqrt{2}}(a - a^\dagger)$$

$$q = \frac{i}{\sqrt{2}}(a + a^\dagger)$$

\hookrightarrow then expand the normal ordering in binomial exp.

$$:p^m q^n: = \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} (-a^\dagger)^k (a^\dagger)^l a^{n-k} a^{m-l}$$

$$= \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^n \sum_{l=0}^m \binom{n}{k} \binom{m}{l} (a^\dagger)^{k+l} a^{n+m-k-l} (-1)^k$$

So ... with $a/7 = z/7 > z \quad \langle z | a^k = (z/z)^k$

$$\begin{aligned} \langle z | p^m q^n | \bar{p} \bar{q} \rangle &= \frac{(-i)^n}{2^{(n+m)/2}} \sum_{l=0}^n \sum_{k=0}^m \binom{n}{k} \binom{m}{k} \langle z | (a^+)^{k+l} a^{n+m-k-l} | z \rangle \\ &= \underbrace{\dots}_{\sim} (z^{\pm})^{n+k} \underbrace{(z^{\pm})^m}_{(-1)^k} \\ &= \frac{(-i)^n}{2^{(n+m)/2}} (z - z^2)^n (z + z^2)^m \\ &= [\sqrt{z} \operatorname{Im}(z)]^n [\sqrt{z} \operatorname{Re}(z)]^m \end{aligned}$$

~~for d=1, 2, 3~~

More problems

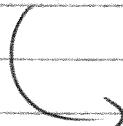
- (1) Consider a universe with 1 dim in \mathbb{R}^d $d=1, 2, 3$ for space.

$$\begin{aligned} L &= \frac{1}{2} (\partial_m \phi_x) (\partial^m \phi_x) - \frac{1}{2} m_x^2 \phi_x^2 + \frac{1}{2} (\partial_m \phi_y) (\partial^m \phi_y) - \frac{1}{2} m_y^2 \phi_y^2 \\ &\quad + \frac{1}{2} (\partial_m \Phi) (\partial^m \Phi) - \frac{1}{2} M^2 \Phi^2 - 2 \Phi \phi_x \phi_y \end{aligned}$$

where $M > m_x + m_y$.

- (a) $d=3$: Find decay rate of the Φ particle into com
frame to the lowest non-vanishing order in λ .

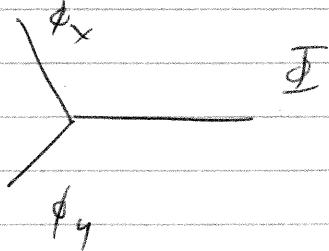
Do the same for $d=2, 1$.



Amplitude is given by

$$\langle \text{final} / s - 1 | \text{initial} \rangle = iM(2\pi) \delta(E_{\text{tot}}^F - E_{\text{tot}}^I) \\ (2\pi)^4 \delta^4(\vec{k}_{\text{tot}}^F - \vec{k}_{\text{tot}}^I)$$

To lowest non-trivial order--



$$\text{In com, } \vec{k}_{\text{tot}}^F = \mathbf{0}; \quad E_{\text{tot}}^F = M$$

$$\text{Decay rate: } P = \frac{\text{probability}}{\text{time}} = \frac{1}{2M} \int d\Omega_2 |M|^2$$

where

$$\int d\Omega_2 = \int \frac{d^4 \vec{k}_x}{(2\pi)^4 2E_x} \frac{d^4 \vec{k}_y}{(2\pi)^4 2E_y} (2\pi) \delta(E_{\text{tot}}^F - M) (2\pi)^4 \delta^{(4)}(\vec{k}_{\text{tot}}^F)$$

$$= \int \frac{d^4 \vec{k}_x}{(2\pi)^4 (2E_x)(2E_y)} \frac{k_x^{d-1}}{(2\pi)^4} (2\pi) \delta(E_{\text{tot}}^F - M)$$

$$\text{where } E_i = \sqrt{\vec{k}_i^2 + m_i^2} \quad ; \quad \frac{dE_i}{dk_i} = \frac{k_i}{E_i}$$

$$\rightarrow \int d\Omega_2 = \int d\Omega_2 \frac{k_x^{d-1} (2\pi)}{(2\pi)^4 \cdot 4} \frac{1}{E_x E_y \left(\frac{k_x}{E_x} + \frac{k_y}{E_y} \right)} = \int d\Omega_2 \frac{k_x^{d-2}}{(2\pi)^{d-1} 4M}$$

The value of k_x for which $E_{\text{tot}}^F = E_x + E_y = M$ is

$$E_{\text{tot}}^F = \frac{1}{2M} \sqrt{(M - m_x - m_y)(M - m_x + m_y)(M + m_x - m_y)} \\ \times (M + m_x + m_y)$$

$$\text{So } P = \left(\int d\Omega \right) \cdot \frac{\pi^2}{8M} \frac{a_x^{d-2}}{(2\pi)^{d-1}}$$

$$\text{when } d=3, \quad P = \frac{\pi^2 k_x}{8\pi M^2}$$

$$d=2, \quad P = \frac{\pi^2}{8M^2}$$

$$d=1, \quad P = \frac{\pi^2}{4k_x M^2}$$

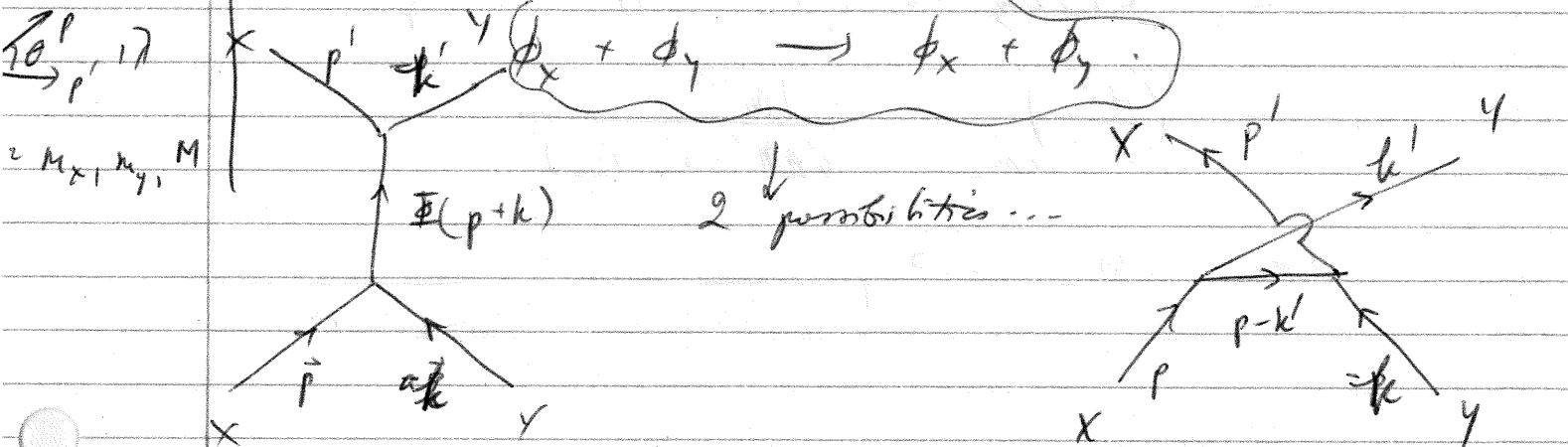
More problems. Consider the same Lagrangian density ...

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_x)(\partial^\mu \phi_x) - \frac{1}{2} m_x^2 \phi_x^2 + \frac{1}{2} (\partial_\mu \phi_y)(\partial^\mu \phi_y) - \frac{1}{2} m_y^2 \phi_y^2$$

$$\text{Find } \Sigma + \frac{1}{2} (\partial_\mu \mathcal{E})(\partial^\mu \mathcal{E}) - \frac{1}{2} m^2 \mathcal{E}^2 - 2 \mathcal{E} \phi_x \phi_y.$$

Σ as a function of $p = |\vec{p}|$

Consider $d=3$. In com frame consider elastic scattering of $\phi_x + \phi_y \rightarrow \phi_x + \phi_y$ to lowest non vanishing order in λ .



$$iM_1 = (-i\lambda)^2 \frac{i}{(p+k)^2 - M^2 + i\epsilon}$$

$$iM_2 = (-i\lambda)^2 \frac{i}{(p-k')^2 - M^2 + i\epsilon}$$

$$\rightarrow iM = iM_2 + iM_4 = -i\lambda^2 \left[\frac{1}{(\vec{p}-\vec{k})^2 - M^2 + i\varepsilon} + \frac{1}{(\vec{p}+\vec{k})^2 - M^2 + i\varepsilon} \right]$$

$$\text{In com ... } p = (E_x, \vec{p}) \quad k = (E_y, -\vec{p})$$

$$p' = (E_x, \vec{p}') \quad k' = (E_y, -\vec{p}')$$

$$|\vec{p}| = |\vec{p}'|$$

$$\text{where } E_{ci} = \sqrt{\vec{p}_i^2 + m_i^2}, \quad E_{cm} = E_x + E_y.$$

We have a formula that says ...

$$\left(\frac{d\sigma}{d\Omega} \right)_{cm} = \frac{|\vec{p}| |M|^2}{2E_x 2E_y |\vec{v}_x - \vec{v}_y| (16\pi^2) E_{cm}}$$

$$\text{where } v_x = \vec{p}/E_x \rightarrow v_y = -\vec{p}/E_y$$

$$\rightarrow |\vec{v}_x - \vec{v}_y| = |\vec{p}| \left(\frac{1}{E_x} + \frac{1}{E_y} \right)$$

$$\rightarrow 2E_x 2E_y |\vec{v}_x - \vec{v}_y| = 4|\vec{p}| (E_x + E_y)$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega} \right)_{cm} = \frac{|M|^2}{64\pi^2 (E_x + E_y)^2}$$

$$\text{now... } iM = -i\lambda^2 \left(\frac{1}{(\vec{p}-\vec{k})^2 - M^2 + i\varepsilon} + \frac{1}{(\vec{p}+\vec{k})^2 - M^2 + i\varepsilon} \right)$$

Using ...

$$\begin{aligned}
 (p-h')^2 &= (E_x - E_y)^2 - (\vec{p} + \vec{p}')^2 \\
 &= (E_x - E_y)^2 - (2\vec{p}^2 + 2\vec{p}'^2 \cos\theta) \\
 &= (E_x - E_y)^2 - 2\vec{p}^2 (1 + \cos\theta)
 \end{aligned}$$

2

$$(p+h)^2 = (E_x + E_y)^2$$

we get

$$|M|^2 = \lambda^4 \left\{ \frac{1}{(E_x - E_y)^2 - 2\vec{p}^2(1+\cos\theta) - h^2} + \frac{1}{(E_x + E_y)^2 - h^2} \right\}$$

So

$$\frac{(d\sigma)}{d\Omega} = \frac{\lambda^4}{64\pi^2 (E_x + E_y)^2} \left[\frac{1}{(\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2})^2} \right]^2$$

so, explicitly ...

$$\left\{ \frac{(d\sigma)}{d\Omega} \right\}_{cm} = \frac{\lambda^4}{64\pi^2} \left[\frac{1}{(\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2})^2} \right]^2 \times \frac{1}{(\sqrt{p^2 + m_x^2} - \sqrt{p^2 + m_y^2})^2 - 2\vec{p}^2(1+\cos\theta) - h^2} + \frac{1}{(\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2} - h)^2}$$

4

10v 8,
2020

Trace Condition of γ^{μ}

Show that in Weyl's representation

$$\gamma^\mu = \begin{pmatrix} \alpha & \beta^\mu \\ \bar{\beta}^\mu & \bar{\alpha} \end{pmatrix}$$

$$\Rightarrow \boxed{\text{Tr } \gamma^\mu = 0}$$

$$\text{Tr } \gamma^\mu \gamma^\nu = \frac{1}{2} \text{Tr} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu)$$

$$= \frac{1}{2} \text{Tr} (2g^{\mu\nu} \cdot \mathbb{1}_{4 \times 4}) = \boxed{4g^{\mu\nu} = \text{Tr} \gamma^\mu \gamma^\nu}$$

$$\begin{aligned} \text{Tr} (\underbrace{\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma}_{\text{R}}) &= \text{Tr} [2g^{\mu\nu} \gamma^\rho \gamma^\sigma - \underbrace{\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma}_{\text{L}}] \\ &= \text{Tr} (2g^{\mu\nu} \gamma^\rho \gamma^\sigma - \gamma^\nu (2g^{\mu\rho}) \gamma^\sigma + \underbrace{\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma}_{\text{R}}) \\ &= \text{Tr} (2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + \gamma^\mu \gamma^\rho (2g^{\nu\sigma}) \\ &\quad - \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\mu) \end{aligned}$$

$$\Rightarrow \boxed{2 \text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = \text{Tr} (2g^{\mu\nu} \gamma^\rho \gamma^\sigma - 2g^{\mu\rho} \gamma^\nu \gamma^\sigma + 2g^{\nu\sigma} \gamma^\mu \gamma^\rho)}$$

$$\Rightarrow \boxed{\text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\nu\sigma} g^{\mu\rho}}$$

Next... $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3$

$$\text{Tr } \gamma^5 = \text{Tr } \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \text{Tr } \gamma^5 \gamma^0 \gamma^1 \gamma^2 = -\text{Tr } \gamma^5 \Rightarrow \boxed{\text{Tr } \gamma^5 = 0}$$

Also, $\text{Tr} \left(\underbrace{\gamma^1 \gamma^2 \dots \gamma^n}_{n \text{ odd}} \gamma^5 \right) = 0$

Since by moving γ^5 to the front, we get a factor of $(-1)^n$. if n odd then $(-1) \Rightarrow$ cancel to get 0

Similarly ...

$$\text{Tr} (\gamma^1 \dots \gamma^n) = 0 \quad \text{if } n \text{ odd}$$

More results

- $\text{Tr} (1_{\alpha \bar{\alpha}}) = 4$

- $\text{Tr} (\gamma^1 \dots \gamma^n) = 0, n \text{ odd}$

- $\text{Tr} (\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$

- $\text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4g^{\mu\nu} g^{\rho\sigma} - 4g^{\mu\rho} g^{\nu\sigma} + 4g^{\mu\sigma} g^{\nu\rho}$

- $\text{Tr} (\gamma^5) = 0$

- $\text{Tr} \left(\underbrace{\gamma^1 \dots \gamma^n}_{\text{odd}} \gamma^5 \right) = 0$

- $\text{Tr} (\gamma^\mu \gamma^\nu \gamma^\sigma) = 0$

- $\text{Tr} (\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^\tau) = -4i \epsilon^{\mu\nu\rho\sigma\tau}$ completely antisymmetric

More useful contraction identities

$$\begin{aligned} \gamma_\mu \gamma^\mu &= \gamma^\nu \gamma^\mu g_{\mu\nu} = \frac{1}{2} (\gamma^{\mu\nu} + \gamma^{\nu\mu}) g_{\mu\nu} \\ &= \frac{1}{2} 2g^{\mu\nu} g_{\mu\nu} = 4. \end{aligned}$$

$$\gamma_\mu \gamma^\nu \gamma^\mu = \delta_\mu^\nu (2g^{\nu\mu} - \gamma^\mu \gamma^\nu) = +2\delta^\nu - 4\delta^\nu = -2\delta^\nu$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4g^{\nu\rho}$$

$$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$$

With this we're ready & look at chapter 5

Chapter 5. Elementary Processes of QED

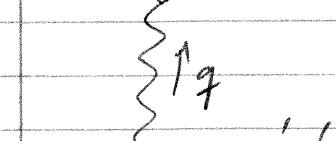
$$e^+ e^- \rightarrow \mu^+ \mu^-$$

↑

one of the most simple processes in QED

$\xrightarrow{\gamma}$, we will calculate the regularized cross-section

$$iM = (1-ie)^2 (\bar{v}^s(p') \gamma^\mu v^s(p)) (\bar{n}^r(k) \gamma^\nu n^r(k')).$$



$$\times \left(\frac{-ig_{\mu\nu}}{q^2 + i\varepsilon} \right)$$



$$= \frac{ie^2}{(p+p')^2} (\bar{v}^s(p') \gamma^\mu v^s(p)) (\bar{n}^r(k) \gamma_\mu n^r(k'))$$

27

→ need $|M|^2$. notice that

$$(\bar{v}^m v)^* = \overline{(\bar{v}^n u)} = \bar{u} \bar{v}^n v = \bar{u} v^m v,$$

recall that $\bar{M} = \gamma^0 M^+ \gamma^0$

$$\text{and so } \bar{\gamma}^0 = \gamma^0 \gamma^{0+} \gamma^0 = \gamma^0$$

$$\bar{\gamma}^i = \gamma^0 \gamma^{i+} \gamma^0 = \gamma^i$$

$$\Rightarrow |M|^2 = \frac{e^4}{((p+p')^2)^2} \times (\bar{v}^{s'}(p') \bar{v}^m u^s(p)) (\bar{u}^r(k) v_n v^r(k')) \\ \times (\bar{u}^s(p) \bar{v}^{s'} v^{s'}(p')) (\bar{v}^{r'}(k') v_n v^r(k))$$

$$= \frac{e^4}{((p+p')^2)^2} \left\{ \bar{v}^{s'}(p') \bar{v}^m u^s(p) \bar{u}^s(p) \bar{v}^{s'}(p') \right\}$$

$$\times \left\{ \bar{u}^r(k) v_n \bar{v}^{s'}(k') v^{s'}(k') \bar{v}^{r'}(k') v^r(k) \right\}$$

This simplifies when we throw away the spin info.

$$\rightarrow \text{Want to compute } \frac{1}{2} \sum_s \frac{1}{2} \sum_{s'} \sum_{r r'} |M(s, s' \rightarrow r, r')|^2$$

$$\text{Recall completeness: } \sum_s u^s(p) \bar{u}^s(p) = p + m$$

$$\sum_s v^s(p) \bar{v}^s(p) = p - m$$

With this, we find that

(77)

$$\sum_{ss'rr'} \bar{v}^s(p') \gamma^m u^s(p) \bar{u}^s(p) \gamma^r v^{s'}(p')$$

$$= \bar{v}(p') \gamma^m u^s(p) \bar{u}^s(p) \gamma^r v^{s'}(p')$$

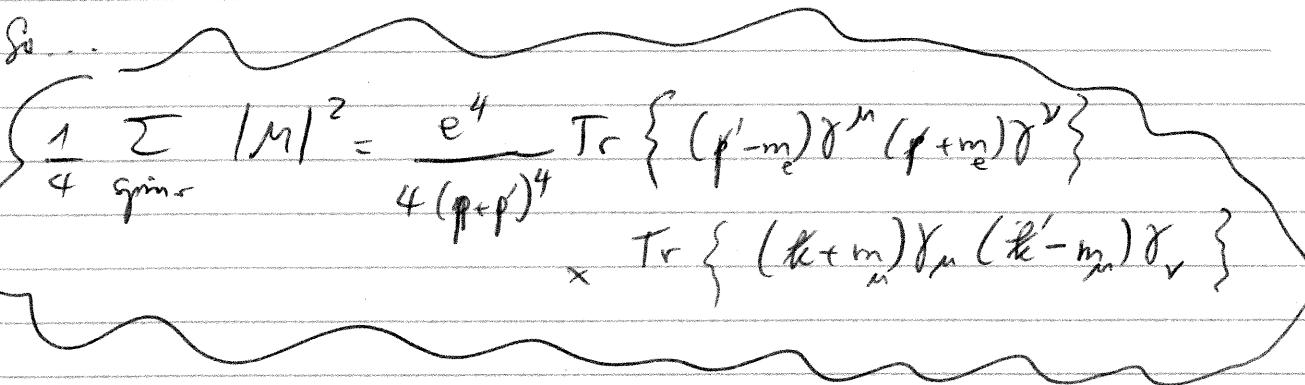
= ℓ_p ?

$$= \sum_{ss'} \bar{v}_a^s(p') \underbrace{\gamma^m}_{ab} \underbrace{u_b^s(p)}_{da} \underbrace{\bar{u}_c^s(p)}_{cd} \underbrace{\gamma^r}_{cd} \underbrace{v_d^{s'}(p')}$$

$$= (p'-m)_a \gamma^m_{ab} (p+m)_{bd} \gamma^r_{cd}$$

$$= \text{Tr} \{ (p'-m) \gamma^m (p+m) \gamma^r \}$$

So ...



$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{e^4}{4(p+p')^4} \text{Tr} \{ (p'-m_e) \gamma^m (p+m_e) \gamma^r \} \times \text{Tr} \{ (k+m_n) \gamma_m (k'-m_n) \gamma_r \}$$

Now, look at each term ...

$$\rightarrow \text{Tr} \{ \gamma^m (p+m_e) \gamma^r (p-m_e) \}$$

$$= \text{Tr} \{ \gamma^\mu p^\nu \gamma^\lambda p^\beta \} - \text{Tr} \{ \gamma^\mu \gamma^\nu \} m_e^2$$

\uparrow \uparrow
 $p_\alpha \delta^\alpha$ $p_\beta \delta^\beta$

$$= 4 p^\mu p^\nu - 4 g^{\mu\nu} p \cdot p' + 4 p^\nu p^\mu - 4 g^{\mu\nu} m_e^2$$

$$= 4 [p^\mu p^\nu + p^\nu p^\mu - g^{\mu\nu} (p \cdot p' + m_e^2)]$$

Similarly, ...

$$\begin{aligned} & \text{Tr} \left\{ \delta_{\mu}(k' - m_{\mu}) \delta_{\nu}(k + m_{\mu}) \right\} \\ &= 4 \left[k'_\mu k_\nu + k'_\nu k_\mu - g_{\mu\nu} (k' \cdot k + m_{\mu}^2) \right] \end{aligned}$$

So ...

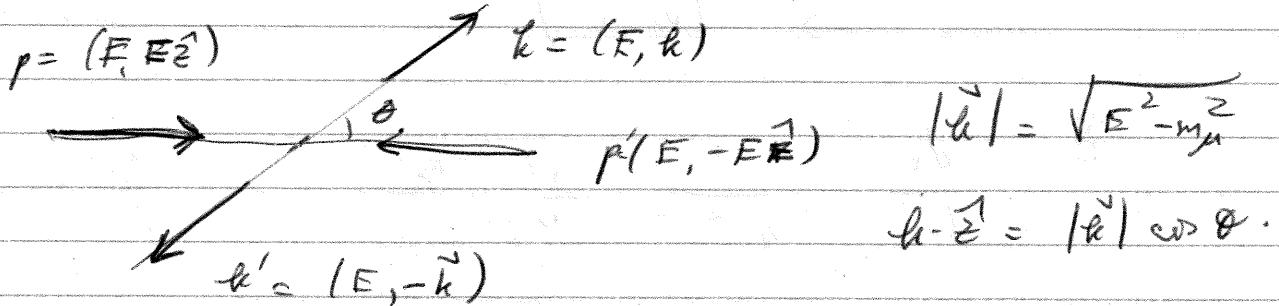
$$\sum_{\text{spins}} \frac{1}{4} |M|^2 = \frac{4e^4}{q^4} \left\{ \begin{array}{l} 2(p \cdot k)(p' \cdot k') + 2(p \cdot k')(p' \cdot k) \\ - 2(k \cdot k')(p \cdot p' + m_e^2) \\ + 2(p \cdot p')(k' \cdot k + m_{\mu}^2) \\ + 4(k' \cdot k + m_{\mu}^2)(p' \cdot p + m_e^2) \end{array} \right\}$$

When energy is sufficiently high, can neglect
 ~~m_e~~ ... $m_e \rightarrow$ set $m_e = 0$...

Get ...

$$\boxed{\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{4e^4}{q^4} \left\{ 2(p \cdot k')(p \cdot k) + 2(p \cdot k')(p' \cdot k) + 2m_{\mu}^2(p \cdot p') \right\}}$$

To get a more explicit formula \rightarrow go to com frame



$$q^2 = (p + p')^2 = 2E^2 - \theta = 4E^2$$

$$p \cdot p' = \frac{q^2}{2} = 2E^2$$

$$p \cdot k = E^2 - E(\vec{k}) \cos \theta \quad p \cdot k' = E^2 + E(\vec{k}') \cos \theta$$

$$p' \cdot k' = E^2 - E(\vec{k}') \cos \theta \quad p' \cdot k = E^2 + E(\vec{k}) \cos \theta$$

So

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{e^4}{16E^4} \left\{ E^2 (E - (\vec{k}/\cos \theta))^2 + E^2 (E + (\vec{k}'/\cos \theta))^2 + 2m_n^2 E^2 \right\}$$

$$= \frac{e^4}{E^4} \left\{ E^2 / (E^2 + (\vec{k}/\cos \theta)^2) + m_n^2 E^2 \right\}$$

$(E^2 - m_n^2)$

$$\Rightarrow \left\{ \frac{1}{4} \sum_{\text{spins}} |M|^2 = e^4 \left\{ 1 + \frac{m_n^2}{E^2} + \left(1 - \frac{m_n^2}{E^2}\right) \cos^2 \theta \right\} \right\}$$

Now... for 2 body scattering into a 2-body system

$$\frac{d\sigma}{d\Omega} = \frac{1}{4F_A F_B (v_A - v_B)} \frac{1}{16\pi^2 E_{cm}} \times \frac{1}{4} \sum_{\text{spins}} |M|^2$$

where $m_e \approx 0$ and so $\vec{p} = \vec{e}^+$ are nearly at the speed of light

$$\rightarrow |v_A - v_B| \approx 2c = 2 \rightarrow = 2E_A = 2E_B = 2E$$

So

$$\frac{d\sigma}{d\Omega} = \frac{1}{4E^2 \cdot 2} \cdot \frac{\sqrt{E^2 - m_n^2}}{16\pi^2 E_{cm}} \cdot \left\{ 1 + \frac{m_n^2}{E^2} + \left(1 - \frac{m_n^2}{E^2}\right) \cos^2 \theta \right\}$$

$$= \frac{e^4}{256\pi^2 E^2} \sqrt{1 - \frac{m_n^2}{E^2}} \left\{ 1 + \frac{m_n^2}{E^2} + \left(1 - \frac{m_n^2}{E^2}\right) \cos^2 \theta \right\}$$

Integrate over $d\Omega$ gives ...

$$\sigma_{\text{tot}} = \frac{e^4 \cdot 2\pi}{288 T^2 E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ 2 \left(1 + \frac{m_\mu^2}{E^2} \right) + \frac{2}{3} \left(1 - \frac{m_\mu^2}{E^2} \right) \right\}$$

$$= \frac{e^4}{64\pi E^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left\{ \frac{4}{3} \left(1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right) \right\}$$

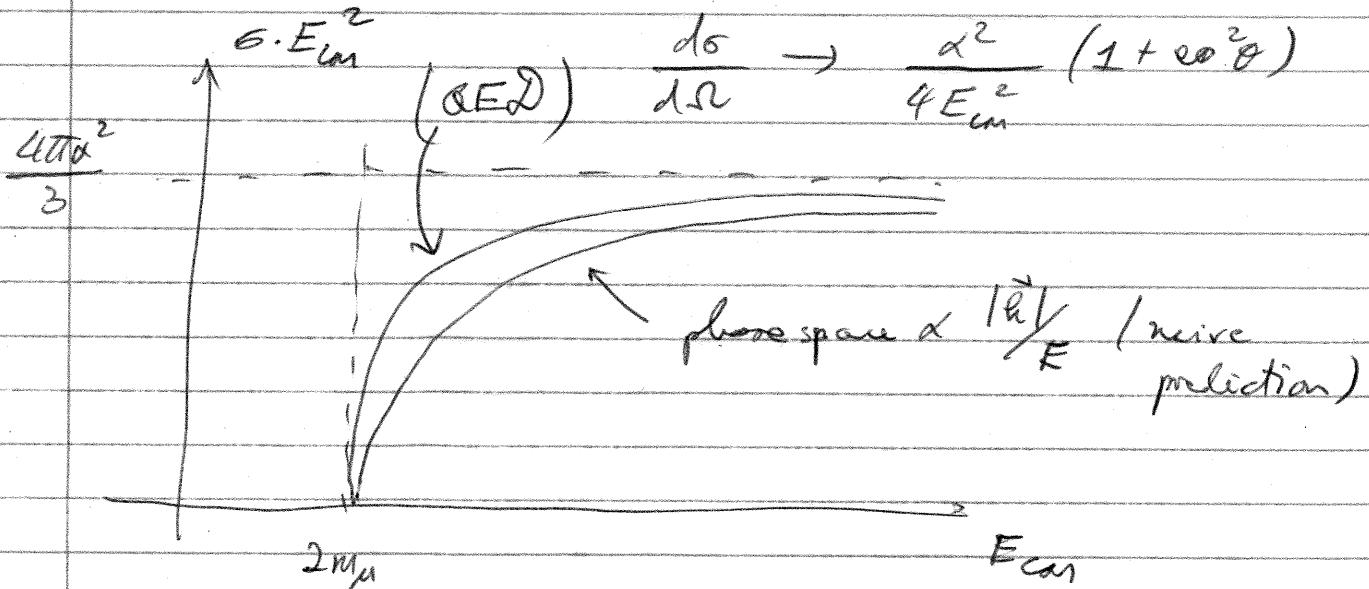
Conventionally, we write this as ...

$$\sigma_{\text{tot}} = \frac{4\pi \alpha^2}{3 E_{\text{cm}}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right)$$

where $\alpha = e^2 / 4\pi$

When $E_{\text{cm}} < 2m_\mu \rightarrow$ cross section is zero
there won't enough energy to produce
a muon + anti-muon pair.

$$\text{As } E_{\text{cm}} \rightarrow \infty \rightarrow \sigma_{\text{tot}} \rightarrow \frac{40\alpha^2}{3^2 c^2}$$



Summary of method

- ① Draw Feynman diagram for the process
- ② Use Feynman rules to calculate M
- ③ Calculate $|M|^2$, averaged or sum over all spins using the completeness relation.
- ④ Evaluate traces using the trace theorem, collect terms & simplify as much as possible.
- ⑤ Rest-frame. Draw a picture of kinematics. Express all 4-momentum vectors in terms of variables such as $E = \theta$.
- ⑥ Plug $|M|^2$ into cross section formula & integrate over all phase space variables to find σ_{tot} .

~~24~~

Production of Quarks - Anti-quark pairs

An important cross section is

$$e^+ e^- \rightarrow \text{hadrons}$$

↑

any number of strongly interacting
particles.

$\text{QCD} \rightarrow$ all hadrons are composed of three fermions
called "quarks" (which come in many "flavors")

↳ each quark has mass

electric charge
 additional quantum number
 (color)

↳ simplest process in QCD is

$$e^+ e^- \rightarrow q\bar{q} \quad (\text{quark-antiquark pair})$$

To modify the $e^+ e^- \rightarrow \mu^+ \mu^-$ to get here, we
need 3 things:

- Replace the massless e with $Q/\epsilon l \rightarrow$ quark
charge
- Color each quark 3 times, one for each color
- Include strong interaction within the produced
quark + anti-quark

First 2 charges are easy: masses + charge are known.

u, c, t quarks: $Q = 2/3$

d, s, b quarks: $Q = -1/3$

→ finally since $M \propto \text{charge}^2 \rightarrow \text{interest } Q^2$.

→ counting: multiply everything by 3.

Finally, in high energy limit, strong interaction can be neglected (Nobel 2004)

→ σ_{tot} for $e^+ e^- \rightarrow g\bar{g}$ looks like ...

$$\sigma_{\text{tot}} \rightarrow R = \frac{4\pi r^2}{8E_{\text{cm}}^2} \cdot (3Q^2) = (3Q^2)R$$

$$\frac{86.8 \text{ nbarn}}{(E_{\text{cm}} \text{ in GeV})^2}$$

→ Asymptotically,

$$\sigma(e^+e^- \rightarrow \text{hadrons}) \xrightarrow[E \rightarrow \infty]{} 3 \cdot (2; Q^2) R$$

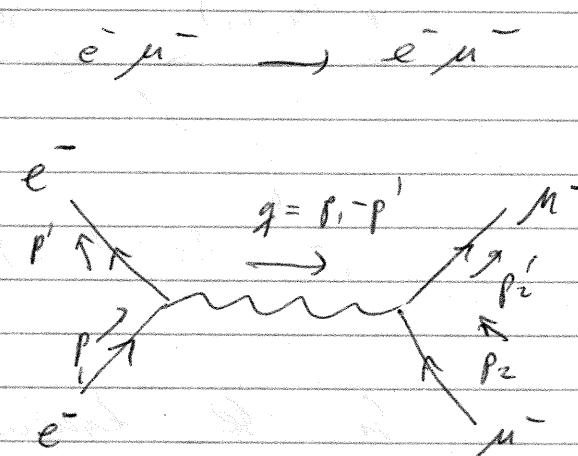
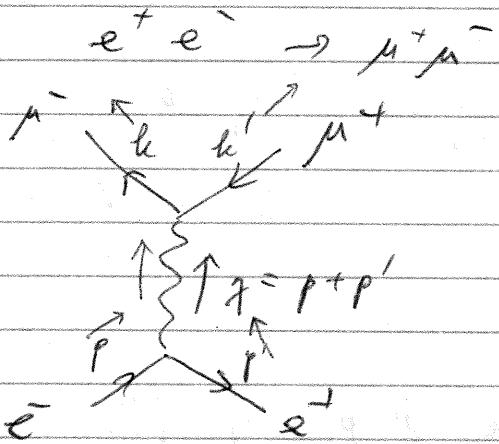
(there's more to this... but we won't be noisy about this too much)

(all + lot of experimental details.)

Crossing symmetry

$$e^+ e^- \rightarrow \mu^+ \mu^-$$

Consider a related process: $e^- \bar{\mu}^- \rightarrow e^+ \mu^+$



$$\sum_{\text{spins}} \frac{1}{4} |M|^2 = \frac{8e^4}{q^4} \left\{ (p \cdot h)(p' \cdot u') + (p \cdot u')(p \cdot h) + m_\mu^2 (p \cdot p') \right\}$$

$$iM = \frac{(ie)}{q^2} (\bar{u}(p_1) \gamma^\mu u(p_1)) \times (\bar{u}(p'_2) \gamma^\nu u(p'_2))$$

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{q^2} \left[(p_1 \cdot p'_2)(p'_1 \cdot p_2) + (p_1 \cdot p_2)(p'_1 \cdot p'_2) - m_\mu^2 (p_1 \cdot p'_1) \right]$$

Compare...

$$p \longleftrightarrow p_1$$

$$h \longleftrightarrow p'_1$$

$$p' \longleftrightarrow -p'$$

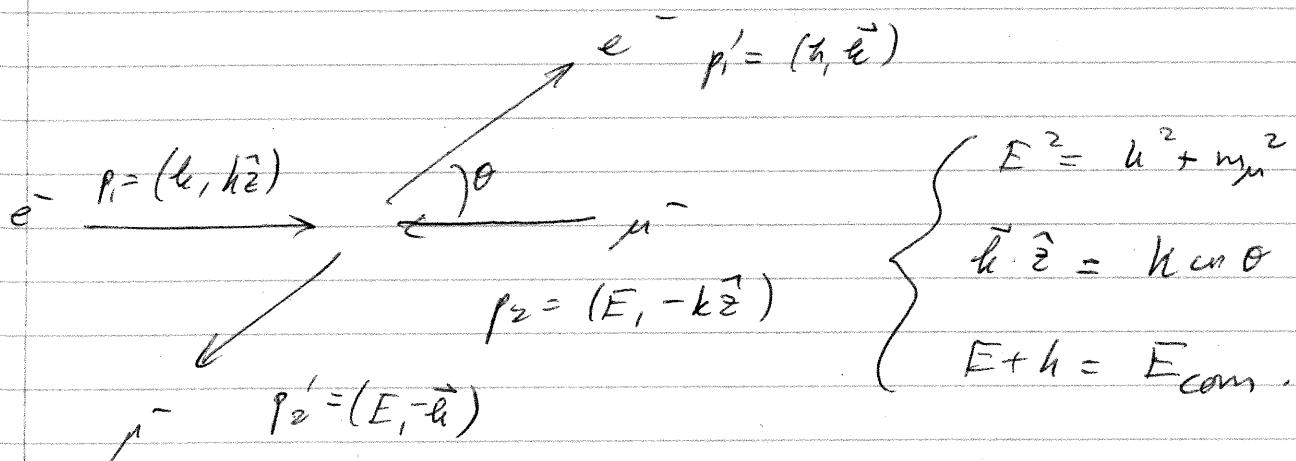
$$h' \longleftrightarrow -p'_2$$

neutrino \rightarrow antineutrino

particle \rightarrow antiparticle

neutrinos \rightarrow -neutrinos

The kinematics here will be completely different.



So let's evaluate the $p_1 \cdot p_2$, $p_1' \cdot p_2'$ --- combos etc.

$$\frac{1}{4} \sum_{\text{symm}} |p_1|^2 = \frac{2e^4}{a^2(1-\cos\theta)^2} \left\{ (E+h)^2 + (E+h \cos\theta)^2 - m_\mu^2 (1-\cos\theta) \right\}$$

$$\left(\frac{d\sigma}{ds} \right)_{\text{cm}} = \frac{|p_1|^2}{64\pi^2 (E+h)^2} \quad (\text{which is obtained from the 1-massless particle limit of the 2-body } -2 \text{ body } \frac{d\sigma}{ds})$$

$$\left(\frac{d\sigma}{ds} \right)_{\text{cm}} = \frac{\alpha^2}{2h^2 (E+h)^2 (1-\alpha\theta)^2} \left\{ (E+h)^2 + (E+h \cos\theta)^2 - m_\mu^2 (1-\cos\theta) \right\}$$

$$\text{where } h = \sqrt{E^2 - m_\mu^2}.$$

When $E \rightarrow \infty$, can set $m_\mu = 0$, and

$$\left(\frac{d\sigma}{ds} \right)_{\text{cm}} \sim \frac{\alpha^2}{2E_{\text{cm}}^2 (1-\cos\theta)^2} (4 + (1+\alpha\theta)^2)$$

Note that $\left(\frac{d\sigma}{ds} - \frac{1}{s^4} \right)$ (singular as $\theta \rightarrow 0$)

This is because of the photon propagator being nearly on shell ($q^2 \approx 0$).

The same result can be seen in non-relativistic scattering (Rutherford scattering)

↳ Direct cross section is due to the first term Coulomb force has a range.

General Crossing Symmetry

→ The general section is an example of crossing symmetry ...

→ For a scalar particle ...

$$\mu(\phi q) + X \rightarrow Y = \mu(X \rightarrow Y + \bar{\phi} q)$$

↳ flip sign of q and you get the new amplitude.

→ For fermion spinors, there is an additional minus sign for the regularized open sums since

$$\sum_{\text{spins}} \bar{u}(p) u(q) = \not{p} + m \quad \text{while}$$

$$\sum_{\text{spins}} v(-p) \bar{v}(q) = -\not{p} - m = -(\not{p} + m)$$

→ $\times (-1)$ for each flipped fermion as well ...

Mandelstam variables. (Convenient for crossing symmetry)

2 body \rightarrow 2 body scattering...

Diagram showing two particles with momenta p and p' . The Mandelstam variables are defined as:

$$\left. \begin{array}{l} s = (p+p')^2 = (t+u)^2 \\ t = (t-p)^2 = (t'-p')^2 \\ u = (t'-p)^2 = (u'-p')^2 \end{array} \right\}$$

(Mandelstam variables)

With this, we can write ...

Feynman diagram for $e^+e^- \rightarrow \mu^+\mu^-$ via annihilation. The incoming particles are e^- and e^+ , and the outgoing particles are μ^- and μ^+ . The invariant mass is given by:

$$M = \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{s^2} \left\{ \left(\frac{t}{2}\right)^2 + \left(\frac{u}{2}\right)^2 \right\}$$

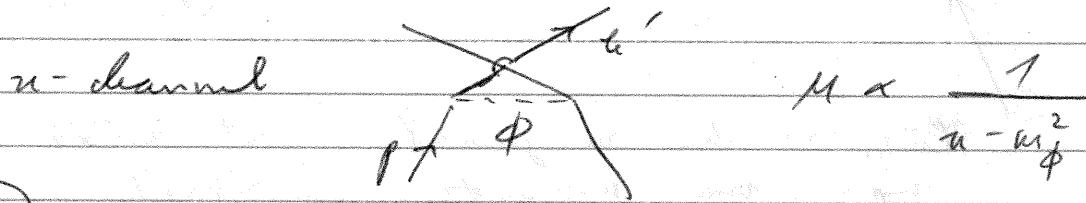
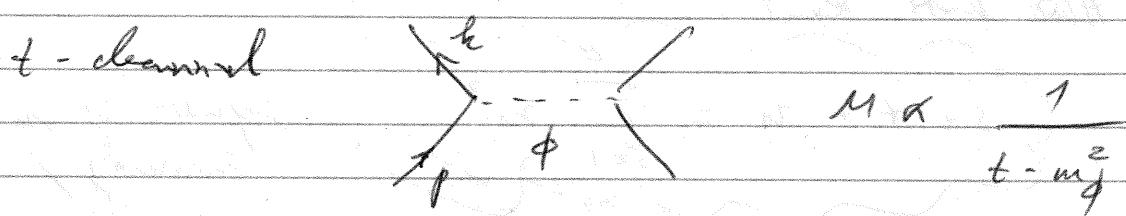
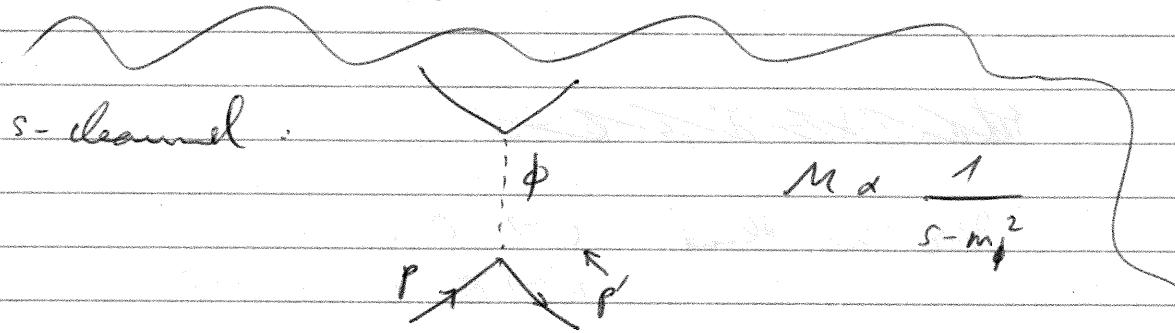
and

Feynman diagram for $e^- \mu^+ \rightarrow e^- \mu^-$ via annihilation. The incoming particles are e^- and μ^+ , and the outgoing particles are e^- and μ^- . The invariant mass is given by:

$$M = \frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{8e^4}{t^2} \left\{ \left(\frac{s}{2}\right)^2 + \left(\frac{u}{2}\right)^2 \right\}$$

When a 2 body - 2 body diagram contains only one virtual particle ...

↳ conventionally described that particle as being in extra "channels"



To get better feel ... let's look at what happens in com frame ...

$$\begin{array}{ccc}
 p: (E, \vec{p}) & \xrightarrow{\phi} & h = (E, \vec{p}) \\
 & \searrow & \swarrow \\
 & p' = (E, -\vec{p}) & \\
 & \nearrow & \downarrow \\
 v' = (E, -\vec{p}) & &
 \end{array}$$

$$\left\{ \begin{array}{l} s = (p+p')^2 = (h'+h)^2 = (2E)^2 = E_{cm}^2 \\ t = (h-p)^2 = -p^2 \sin^2 \theta - p^2 (\cos \theta - 1)^2 = -p^2 (1 - \cos \theta) \\ u = (h'-p)^2 = -p^2 \sin^2 \theta - p^2 (\cos \theta + 1)^2 = -2p^2 (1 + \cos \theta) \end{array} \right.$$

~~Other terms are zero~~

We see that $t \rightarrow 0$ as $\theta \rightarrow 0$
 $u \rightarrow 0$ or $\theta \rightarrow \pi$

Also note that ...

$$s+t+u = \sum_{i=1}^4 m_i^2$$

square of mass of incoming / outgoing particles

This is due in general -- just add anything up to momentum conservation.

$$(p+p'-h'-h)^2 = 0.$$

~~t~~

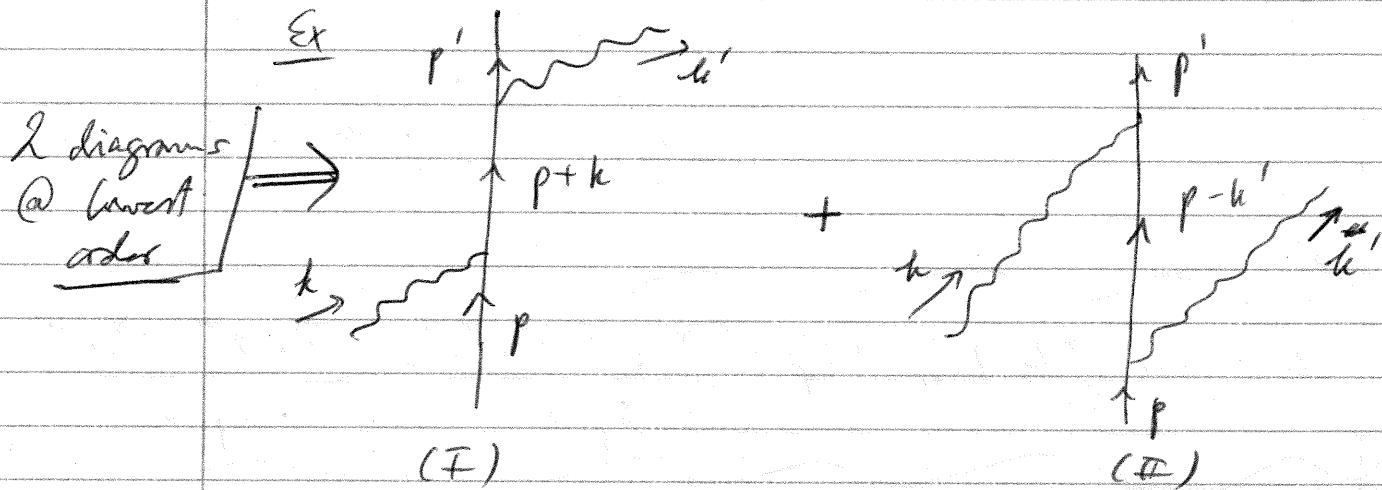
Coupton scattering $e^\gamma \rightarrow e^\gamma$

\rightarrow X-ray + R-ray or electrons...

We will calculate σ_{tot} to lowest order in α .

(we will be using all the techniques we learned so far)

+ how to deal with external photons.



Let $\epsilon_\nu(h)$ + $\epsilon_\nu(h')$ be the photon polarization vectors

$$iM = (I) + (II)$$

$$= \bar{u}(p') (-ie\gamma^\mu) \epsilon_\mu^\pm(h') \frac{i(p+k+m)}{(p+k)^2 - m^2} (-ie\gamma^\nu) \epsilon_\nu(h) u(p)$$

$$+ \bar{u}(p') (-ie\gamma^\nu) \epsilon_\nu(h) \frac{i(p-k+m)}{(p-k)^2 - m^2} (-ie\gamma^\mu) \epsilon_\mu^\pm(h') u(p)$$

$$= -ie^2 \epsilon_\mu^\pm(h') \epsilon_\nu(h) \bar{u}(p') \left[\frac{\gamma^\mu (p+k+m) \gamma^\nu}{(p+k)^2 - m^2} + \frac{\gamma^\nu (p-k+m) \gamma^\mu}{(p-k)^2 - m^2} \right] u(p)$$

$$\text{now note that } p^2 = m^2 = 0, \quad k^2 = 0$$

$$\rightarrow (p+k)^2 - m^2 = 2p \cdot k$$

$$(p-k)^2 - m^2 = -2p \cdot k'$$

and finally ...

$$\begin{aligned}
 (\rho + m) \gamma^\nu u(p) &= (2\rho^\nu - \gamma^\nu p^\mu + \gamma^\nu m) u(p) \\
 &= 2\rho^\nu u(p) - \underbrace{\gamma^\nu (p-m) u(p)}_{\text{o.b.s dep of } u(p)} \\
 &= 2\rho^\nu u(p)
 \end{aligned}$$

So...

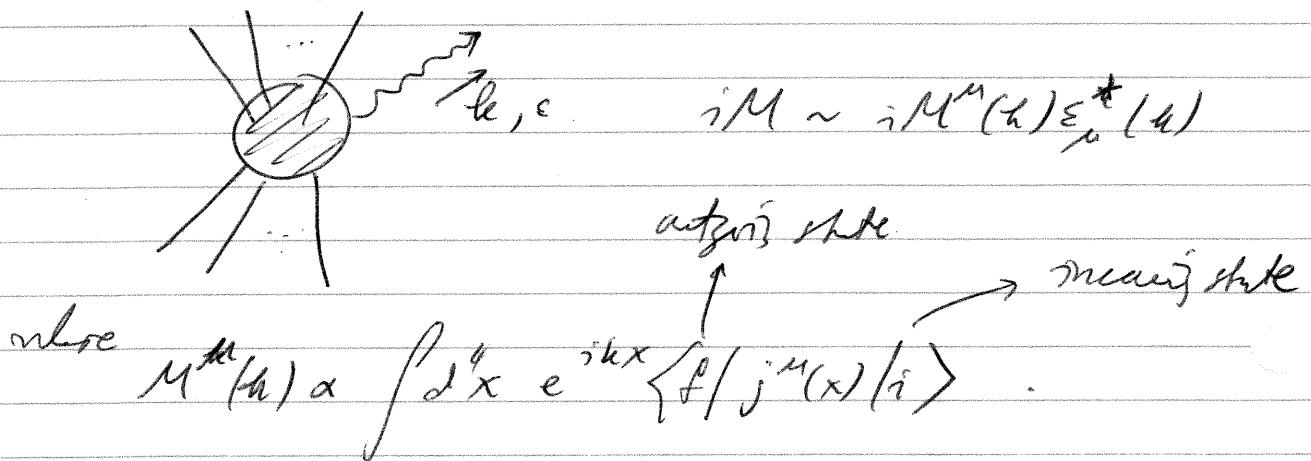
$$iM = -ie^2 \epsilon_\mu^{*\frac{1}{2}}(k') \epsilon_\nu(k) \bar{u}(p') \left\{ \frac{\gamma^\mu k' \gamma^\nu + 2\gamma^\mu p^\nu}{2p \cdot k} + \frac{-\gamma^\nu k' \gamma^\mu + 2\gamma^\mu p^\nu}{-2p \cdot k'} \right\} u(p)$$

Next, ... Photon Polarization + Ward Identity

Interaction for QED has the form

$$e \int d^4x j^\mu A_\mu \quad \text{where } j^\mu \text{ is the conserved electric charge current.}$$

Consider a process with outgoing photon with momentum k & polarization ϵ .



(4)

Now, since $\partial_\mu j^\mu(x) = 0 \dots$

$$\begin{aligned}
 k_\mu M^\mu(k) &\propto \int d^4x e^{ik\cdot x} \langle h(j^\mu(x))/i \rangle \\
 &= -i \int d^4x \partial_\mu (e^{ik\cdot x}) \langle f | j^\mu(x) | i \rangle \\
 &= i \int d^4x e^{ik\cdot x} \langle f | \partial_\mu j^\mu(x) | i \rangle \\
 &= 0.
 \end{aligned}$$

→ This is an example of the Ward identity in QED.

→ When you replace the polarization vector by my external photon (momentum or outgoing) by the momentum k_μ , the amplitude vanishes...

$$[k_\mu M^\mu(k) = 0]$$

More on this later...

→ This is essentially a statement of current conservation.

↳ which is a consequence of gauge symmetry of QED.

mm

What can we say about the polarization sum

$$\sum_i \epsilon_i^{\mu}(\boldsymbol{k}) \epsilon_i^{\nu}(\boldsymbol{k}) ?$$

polarizations

↳ Consider an example -- Let $\boldsymbol{k} = (k, 0, 0, k)$

→ 2 physical polarizations are usually taken as

$$\epsilon_1^{\mu} = (0, 1, 0, 0)$$

$$\epsilon_2^{\mu} = (0, 0, 1, 0) \quad \checkmark$$

$$\rightarrow \sum_i \epsilon_i^{\mu} \epsilon_i^{\nu} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix} / \infty$$

But this is ugly -- → can find a more
simpler form --

Nov 9,
2020

Note that polarization enters calculation in the form

$$\sum_i |\epsilon_i^{\mu}(\boldsymbol{k}) M^{\mu}(\boldsymbol{k})|^2$$

which is

$$\sum_i \epsilon_i^{\mu}(\boldsymbol{k}) \epsilon_i^{\nu}(\boldsymbol{k}) M_{\mu}(\boldsymbol{k}) M_{\nu}^{\dagger}(\boldsymbol{k})$$

$$= |M_1(\boldsymbol{k})|^2 + |M_2(\boldsymbol{k})|^2$$

$$= |M_1(\boldsymbol{k})|^2 + |M_2(\boldsymbol{k})|^2 + |M_3(\boldsymbol{k})|^2 - |M_0(\boldsymbol{k})|^2$$

since $k^{\mu} : (k, 0, 0, k)$

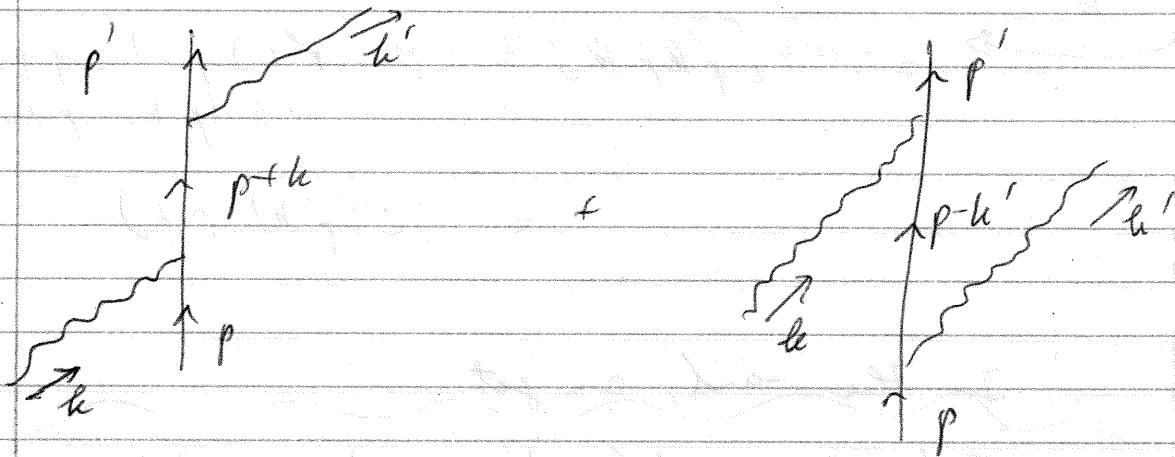
$$= -g_{\mu\nu} M^{\mu}(\boldsymbol{k}) M^{\nu \dagger}(\boldsymbol{k}) - k^{\mu} M_{\mu} = 0 \quad (\text{Ward identity})$$

So we can take $\left\{ \sum_i \epsilon_i^{\mu\nu}(h) \epsilon_i^\nu(h) = -g^{\mu\nu} \right.$

even though this is not really an equality...

Now back to Compton scattering...

\hookrightarrow Lee Klein - Nishina Formula



Recall that we found

$$iM = -ie^2 \bar{\epsilon}_\mu^\mu(h) \epsilon_\nu(h) \bar{u}(p') \left\{ \frac{\gamma^\mu h^\nu + 2\gamma^\nu p^\mu}{2p \cdot h} + \frac{\gamma^\nu h^\mu - 2\gamma^\mu p^\nu}{2p \cdot h'} \right\}$$

\hookrightarrow now, we square, sum & divide by # of initial spins (2 electrons \times 2 photons = 4)

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = \frac{e^4}{4} \underbrace{\gamma_{\mu\nu} \gamma^{\mu\nu}}_{\text{from polarization sums}} \times (p+m) \left\{ \frac{\gamma^\mu h^\nu + 2\gamma^\nu p^\mu}{2 \cdot p \cdot h} + \frac{\gamma^\nu h^\mu - 2\gamma^\mu p^\nu}{2 \cdot p \cdot h'} \right\}$$

$$\times (p+m) \times \left\{ \frac{\gamma^\mu h^\nu + 2\gamma^\nu p^\mu}{2 \cdot p \cdot h} + \frac{\gamma^\nu h^\mu - 2\gamma^\mu p^\nu}{2 \cdot p \cdot h'} \right\}$$

There's a lot of work & we don't do all details here ...

$$\text{Use: } \partial_r (\gamma^\mu K \gamma^\nu \partial_\nu K \gamma_\mu) = -2\kappa (\gamma^\mu K \rho K \gamma_\mu)$$

$$\text{since } \gamma^\mu \partial_\mu = -2\rho$$

$$\text{Next: } \gamma^\mu K \rho K \gamma_\mu = -2K \rho K$$

so ...

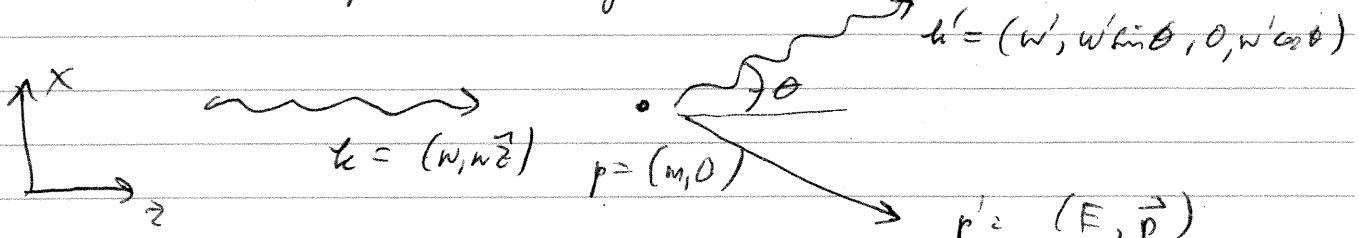
$$\begin{aligned} \rightarrow & +4\kappa [\gamma^\mu K \rho K] = 4 \cdot 4 \left((\rho \cdot K)(p \cdot h) - (\rho \cdot p)(h \cdot K) \right. \\ & \quad \left. + (\rho \cdot K)(p \cdot h) \right) \\ & = 32 (\rho \cdot h)(p \cdot h) \end{aligned}$$

In the end, we get

$$\frac{1}{4} \sum_{\text{spins}} |M|^2 = 2e^4 \left\{ \frac{p \cdot h'}{p \cdot K} + \frac{p \cdot h}{p' \cdot K} + 2m^2 \left(\frac{1}{p \cdot K} - \frac{1}{p \cdot h'} \right) \right.$$

$$\left. + m^4 \left(\frac{1}{p \cdot h} - \frac{1}{p \cdot h'} \right)^2 \right\}$$

Now, from Compton scattering, the frame is --



$$\begin{aligned} p'^2 &= m^2 \\ \Rightarrow m^2 &= (p')^2 = (p + k - k')^2 = p^2 + k'^2 + k'^2 + 2p \cdot (k - k') \\ &= m^2 + 2p \cdot (k - k') - 2k \cdot k' \end{aligned}$$

$$= m^2 + 2m(w-w') - 2mw'(1-\cos\theta)$$

$$\text{So } 2m(w-w') \approx 2ww'(1-\cos\theta) = 0$$

$$\Rightarrow \left\{ \begin{array}{l} w' = \frac{1}{1 + \frac{w(1-\cos\theta)}{m}} \\ \frac{1}{m}(1-\cos\theta) + \frac{1}{w} = 1 + \frac{w(1-\cos\theta)}{m} \end{array} \right.$$

Cauchy's formula for the shift in the electron wavelength -

→ Next, have to work out the phase space integrals

$$\int dT_2 = \int \frac{d^3 h'}{(2\pi)^3 2w'} \frac{1}{(2\pi)^3 2E'} (2\pi)^4 \delta^{(4)}(k' + p' - k - p)$$

$$\rightarrow \int \frac{d^3 h'}{(2\pi)^3 2w' 2E'} \frac{1}{(2\pi)^3 2w' 2E'} (2\pi) \delta(w' + E' - w - m) \quad (\vec{p} = \vec{k} + \vec{p}' - \vec{k}')$$

$$= \int \frac{d^3 h'}{(2\pi)^3 2w' 2E'} (2\pi) \delta\left(w' + \sqrt{w'^2 + w^2 - 2ww'\cos\theta + m^2} - w - m\right)$$

$$\text{now we } P(f(x)) = \frac{\delta(x-x_0)}{|f'(x_0)|}$$

$$= \int \frac{w'^2 dw' d\Omega}{(2\pi)^3 4w'E'} (2\pi) \frac{\delta(w' - \sqrt{1 + \frac{w^2}{m}(1-\cos\theta)})}{\left|1 + \frac{w' - w\cos\theta}{E'}\right|} \rightarrow \sqrt{\dots}$$

$$= \frac{1}{4\pi} \int_{-1}^1 \frac{dw' \cos\theta \cdot w'}{\left|1 + \frac{w' - w\cos\theta}{E'}\right| \cdot E'} \rightarrow \frac{1}{8\pi} \int_{-1}^1 \frac{dw' \cos\theta \cdot w'}{(E' + w' - w\cos\theta)}$$

But $E' + w' = \text{total energy} = w + m$ (initial energy)

So, we have...

$$\frac{1}{8\pi} \int_{-1}^1 \frac{dw \theta w'}{|w+m-w\cos\theta|} = \frac{w'}{8\pi} \int_0^\pi \frac{d\cos\theta}{m+w(1-\cos\theta)}$$

Now, $v_{\text{electron}} = 0$, so $|v_A - v_B| = 1$

$$\text{So } \frac{d\sigma}{d\cos\theta} = \frac{1}{2w} \frac{1}{2m} \frac{w'}{8\pi(m+w(1-\cos\theta))} \frac{1}{4 \text{ barns}} |M|^2$$

Since $\frac{1}{m+w(1-\cos\theta)} = \frac{w'}{mw}$, we have

$$\frac{d\sigma}{d\cos\theta} \geq \frac{1}{32\pi} \frac{w'^2}{m^2 w^2} \left\{ 2e^4 \left(\frac{p \cdot h'}{p \cdot h} + \frac{p \cdot h}{p \cdot h'} + 2m \left(\frac{1}{p \cdot h} - \frac{1}{p \cdot h'} \right) + m^2 \left(\frac{1}{p \cdot h} - \frac{1}{p \cdot h'} \right)^2 \right) \right\}$$

Using $p \cdot h = mw$ $\Rightarrow p \cdot h' = m \cdot w'$

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi} \frac{w'^2}{m^2 w^2} \left\{ 2e^4 \left(\frac{w'}{w} + \frac{w}{w'} + 2m \left(\frac{1}{w} - \frac{1}{w'} \right) + m^2 \left(\frac{1}{w} - \frac{1}{w'} \right)^2 \right) \right\}$$

$$= \underbrace{\frac{\pi \alpha^2}{m^2} \left(\frac{w'}{w} \right)^2}_{\nearrow} \left\{ \frac{w''}{w} + \frac{w}{w'} - 6m^2 \theta^2 \right\}$$

Klein-Nishina formula (1929)

Thomson cross

When $w \rightarrow 0$, $w'/w \rightarrow 1$

section for

$$\frac{d\sigma}{d\cos\theta} \rightarrow \frac{\pi \alpha^2}{m^2} (1 + w^2 \theta^2), \quad \sigma_{\text{tot}} = \frac{8\pi \alpha^2}{3m^2}$$

scattering of
classical EM
by free e^-

Radiative Corrections

Soft Bremsstrahlung \rightarrow low freq radiation when
 e^- under goes sudden acceleration

