Recall quantization of quantum system...

$$[q, p] = qp - pq = i$$

Harmonic oscillator

$$H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2$$

Define ladder operators

$$a \text{ (annihilation)} = \frac{q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}}{\sqrt{2}}$$

$$a^\dagger \text{ (creation)} = \frac{q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}}{\sqrt{2}}$$

Then $a^\dagger a = \frac{1}{2} m\omega q^2 + \frac{1}{2m\omega} p^2 - \frac{i}{2}\underbrace{(pq - qp)}_{-i}$

$$= \frac{1}{\omega} H - \frac{1}{2}$$

So $H = \omega(a^\dagger a + \frac{1}{2})$.

Call the ground state (lowest energy state) $|0\rangle$.

It satisfies

$$a|0\rangle = 0$$

We write all the other normalized energy eigenstates as

$$|n\rangle = \frac{1}{\sqrt{n!}} \underbrace{a^\dagger a^\dagger \cdots a^\dagger}_{n} |0\rangle$$

Since $[a, a^\dagger] = -\frac{i}{2}(\underbrace{qp - pq}_{i}) + \frac{i}{2}(\underbrace{pq - qp}_{-i})$

$$= 1,$$

$$[a^\dagger a, a] = a^\dagger a a - a a^\dagger a = [a^\dagger, a]\, a = -a$$

$$[a^\dagger a, a^\dagger] = a^\dagger a a^\dagger - a^\dagger a^\dagger a = a^\dagger [a, a^\dagger] = a^\dagger$$

Therefore $[H, a] = -\omega a$

$$[H, a^\dagger] = \omega a^\dagger$$

So a lowers the energy by ω
$a^\dagger$ raises the energy by ω ...

$$H a |n\rangle = a H |n\rangle + (-\omega a)|n\rangle$$

$$= E_n a|n\rangle + (-\omega)\, a|n\rangle$$

$$= (E_n - \omega)\, a|n\rangle$$

$$H a^\dagger |n\rangle = (E_n + \omega) a^\dagger |n\rangle$$

In ordu that the ladder of energy states does not extend to negative infinity, we imposed

$$a|0\rangle = 0.$$

Therefore

$$H|0\rangle = \omega(a^\dagger a + \tfrac{1}{2})|0\rangle$$
$$= \tfrac{1}{2}\omega|0\rangle$$

And more generally

$$E_n = (n + \tfrac{1}{2})\omega$$

We now try to quantize our field theory. Since we start with free (non-interacting) fields we already know what the energy eigenstates are...

The ground state consists of the vacuum $|0\rangle$. Next are the one-particle states with momentum $\vec{p}$. We label these as $|\vec{p}\rangle_{NR}$. The NR is for non-relativistic normalization. This is the usual normalization

$$_{NR}\langle \vec{p} \,|\, \vec{p}\,' \rangle_{NR} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,')$$

In this course we use the following conventions for Fourier transforms...

$$\tilde{f}(p) = \int_{-\infty}^{\infty} dx \; e^{ipx} f(x)$$

$$f(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi} \; e^{-ipx} \tilde{f}(p)$$

The integration is always $\frac{1}{2\pi}$ for each dp. This results in a factor of 2π for each

$\delta(p)$. So you will see

$$\frac{d^3\vec{p}}{(2\pi)^3} \quad \text{or} \quad \frac{d^4 p}{(2\pi)^4}$$

$$(2\pi)^3\, \delta^3(\vec{p}) \quad \text{or} \quad (2\pi)^4\, \delta^4(p)$$

Consider the case with one free bosonic field. For each momentum $\vec{p}$ we can have zero, one, two, ... particles each with momentum $\vec{p}$. The energy associated with each particle with momentum $\vec{p}$ is

$$\omega_{\vec{p}} = \sqrt{\vec{p}^{\,2} + m^2}$$

(later we use the notation $E_{\vec{p}}$ to denote the same quantity)

This looks a lot like our harmonic

oscillator... actually a bunch of them, one for each momentum $\vec{p}$. There is no coupling between the different "momentum $\vec{p}$" oscillators.

Let $a^\dagger_{\vec{p}}$ be the creation operator for particles of momentum $\vec{p}$ so that

$$|\vec{p}\rangle_{NR} = a^\dagger_{\vec{p}} |0\rangle$$

↖ vacuum

Let the Hermitian conjugate be $a_{\vec{p}}$, the annihilation operator. As before we have

$$a_{\vec{p}} |0\rangle = 0.$$

How about the commutation relations? Clearly we should have

$$[a_{\vec{p}}, a_{\vec{p}'}] = 0 = [a^\dagger_{\vec{p}}, a^\dagger_{\vec{p}'}]$$

We also expect $[a_{\vec{p}}, a^\dagger_{\vec{p}'}] \propto \delta^{(3)}(\vec{p}-\vec{p}')$,

but what is the correct normalization?

Since we choose the usual vacuum state normalization, $\langle 0|0\rangle = 1$, it follows that

$$
\begin{aligned}
[a_{\vec{p}}, a^\dagger_{\vec{p}'}] &= \langle 0|[a_{\vec{p}}, a^\dagger_{\vec{p}'}]|0\rangle \\
&= \underbrace{\langle 0|a_{\vec{p}}}_{{}_{NR}\langle\vec{p}|}\underbrace{a^\dagger_{\vec{p}'}|0\rangle}_{|\vec{p}'\rangle_{NR}} - \langle 0|a^\dagger_{\vec{p}'}\underbrace{a_{\vec{p}}|0\rangle}_{0} \\
&= {}_{NR}\langle\vec{p}|\vec{p}'\rangle_{NR} = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')
\end{aligned}
$$

So $[a_{\vec{p}}, a^\dagger_{\vec{p}'}] = (2\pi)^3\delta^{(3)}(\vec{p}-\vec{p}')$.

What should the Hamiltonian look like?

We guess, in analogy with the harmonic oscillator, Single oscillator

$$H = \omega(a^\dagger a + \tfrac{1}{2})$$
$$= \omega(a^\dagger a + \tfrac{1}{2}[a, a^\dagger]) = \tfrac{1}{2}\omega(a^\dagger a + a a^\dagger)$$

Free field theory

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_{\vec{p}} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \tfrac{1}{2} \underbrace{[a_{\vec{p}}, a^\dagger_{\vec{p}}]}_{\text{Kind of strange...}} \right)$$

$\frac{1}{2\pi}$ for each dp

Kind of strange...

$$= \frac{(2\pi)^3}{2}\, \delta^{(3)}(\vec{p} - \vec{p})$$
$$= \frac{(2\pi)^3}{2}\, \delta^{(3)}(\vec{0})$$

We get the correct ladder structure

$$[H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}}\, a^\dagger_{\vec{p}}$$
$$[H, a_{\vec{p}}] = -\omega_{\vec{p}}\, a_{\vec{p}}$$

The $[a_{\vec{p}}, a^\dagger_{\vec{p}}]$ in the definition of H is kind of strange. It is there for each momentum $\vec{p}$ and it diverges.

This is troubling for combing general relativity and quantum mechanics. But if we ignore general relativity and gravity effects, the energy divergence is harmless. The vacuum energy is divergent, but the energy separation any excitation (1 particle, 2 particles, etc.) is finite in this free field example.

We note that $a^\dagger_{\vec{p}} a_{\vec{p}}$ (when integrated with $\frac{d^3\vec{p}}{(2\pi)^3}$ gives the number of particles with momentum $\vec{p}$. So it is easy to guess that the momentum operator should be

$$\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$$

If we were only interested in free bosonic fields, this is the whole story. But we want to study interacting fields. To do this we first reconstruct the p's and q's associated with the annihilation and creation operators.

Back to the single oscillator...

$$q = \sqrt{\frac{1}{2m\omega}}\,(a + a^\dagger)$$
$$p = -i\sqrt{\frac{m\omega}{2}}\,(a - a^\dagger)$$

Actually it is more convenient to absorb the factors of $\sqrt{m}$ in the definitions of q and p:

$$q = \sqrt{\frac{1}{2\omega}}\,(a + a^\dagger)$$
$$p = -i\sqrt{\frac{\omega}{2}}\,(a - a^\dagger)$$

Then our original Hamiltonian is

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2$$

Let us define $\phi_{\vec{p}}$ as the analog of q

$\Pi_{\vec{p}}$ as the analog of p

Then we have

$$\phi_{\vec{p}} = \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}} + a^{\dagger}_{\vec{p}} \right)$$
$$\Pi_{\vec{p}} = -i \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}} - a^{\dagger}_{\vec{p}} \right)$$

Actually it is more convenient to work in position space. We want to Fourier transform back to coordinate space variable $\vec{x}$. Should we use $e^{-i\vec{p}\cdot\vec{x}}$ or $e^{+i\vec{p}\cdot\vec{x}}$?

Answer: one of each. We want

$$\langle 0 | \phi(\vec{x}) | \vec{p} \rangle_{NR} \propto e^{i\vec{p}\cdot\vec{x}}$$
$$_{NR}\langle \vec{p} | \phi(\vec{x}) | 0 \rangle \propto e^{-i\vec{p}\cdot\vec{x}}$$

since this is kind of like $\langle \vec{x} | \vec{p} \rangle$

and $\langle \vec{p} | \vec{x} \rangle$ in single-particle quantum mechanics. So we use $e^{i\vec{p}\cdot\vec{x}}$ for the $a_{\vec{p}}$ term and $e^{-i\vec{p}\cdot\vec{x}}$ for the $a_{\vec{p}}^\dagger$ term.

This gives

$$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

$$\pi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

We can now show that (homework)

$$[\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})]$$

and $$[\phi(\vec{x}), \pi(\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y})$$

We previously derived for the free field Hamiltonian

$$H = \int d^3\vec{x} \, \underbrace{\left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \overset{1}{\overset{\shortparallel}{g^{00}}} \mathcal{L} \right]}_{= \frac{1}{2}(\partial^0\phi)^2 + \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \frac{1}{2}m^2\phi^2}$$

where $\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi$

In order to quantize this field theory it is usual to define canonical conjugate

$$\Pi(\vec{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi(\vec{x})$$

(just like $p = \frac{\partial L}{\partial \dot{q}}$)

We can then write (replacing $\partial^0\phi$ with Π)

$$H = \int d^3\vec{x} \, \underset{\substack{\uparrow \\ \text{Hamiltonian} \\ \text{density}}}{\mathcal{H}} = \int d^3\vec{x} \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\vec{\nabla}\phi)(\vec{\nabla}\phi) + \frac{1}{2}m^2\phi^2 \right]$$

Similarly we get

$$P^i = \int d^3\vec{x} \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^i \phi \right] = \int d^3\vec{x}\, \Pi\, \partial^i \phi$$

momentum in i direction

$$\Rightarrow \vec{P} = - \int d^3\vec{x}\, \Pi\, \vec{\nabla} \phi$$

minus since $\vec{\nabla} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ is a lower index object

We can show that using our expressions for $\phi(\vec{x})$ and $\Pi(\vec{x})$,

$$H = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \omega_{\vec{p}} \left(a^\dagger_{\vec{p}} a_{\vec{p}} + \frac{1}{2} [a_{\vec{p}}, a^\dagger_{\vec{p}}] \right)$$

$$\vec{P} = \int \frac{d^3\vec{p}}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}},$$

as we found before.