What can we say about the polarization sum

$$\sum_{\substack{\text{polarizations}\\ i}} \varepsilon_i^{*\mu}(k)\,\varepsilon_i^{\nu}(k) \;?$$

Let's try to find out by an example. Let $k^\mu = (k,0,0,k)$. Then the two physical transverse polarizations are usually taken as

$$\varepsilon_1^\mu = (0,1,0,0)\ ,\quad \varepsilon_2^\mu = (0,0,1,0)$$

So $$\sum_{\substack{\text{polarizations}\\ i}} \varepsilon_i^{*\mu}\varepsilon_i^{\nu} = \begin{array}{c} \xrightarrow{\quad\nu\quad} \\ \mu\downarrow\begin{bmatrix} 0&0&0&0\\ 0&1&0&0\\ 0&0&1&0\\ 0&0&0&0 \end{bmatrix} \end{array}.$$ But this

is a bit ugly... let's try to find a more familiar form. Note that the polarizations enter the calculation in the form $\sum_i |\varepsilon^*_{i\mu}(k)\mathcal{M}^\mu(k)|^2$ which is

$$\sum_i \varepsilon_i^{*\mu}(k)\varepsilon_i^{\nu}(k)\,\mathcal{M}_\mu(k)\mathcal{M}^*_\nu(k) = |\mathcal{M}_1(k)|^2 + |\mathcal{M}_2(k)|^2$$

But we know from the Ward identity that

$$k_\mu \mathcal{M}^\mu(k) = 0.$$

So $$\mathcal{M}^0(k) - \mathcal{M}^3(k) = 0 \Rightarrow \mathcal{M}^0(k) = \mathcal{M}^3(k).$$

So we could in fact take

$$\sum_i \epsilon_i^{*\mu}(k)\,\epsilon_i^{\nu}(k) = -g^{\mu\nu}$$

and <u>still</u> get

$$\sum_i \epsilon_i^{*\mu}(k)\,\epsilon_i^{\nu}(k)\,\mathcal{M}_\mu(k)\,\mathcal{M}_\nu^*(k) = -|\mathcal{M}_0|^2 + |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + |\mathcal{M}_3|^2$$
$$= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 .$$

So while it's not an equality, we can always take

$$\sum_i \epsilon_i^{*\mu}(k)\,\epsilon_i^{\nu}(k) = -g^{\mu\nu}.$$

Let's go back to Compton scattering.

We found

$$i\mathcal{M} = -ie^2\,\epsilon_\mu^*(k')\,\epsilon_\nu(k)\,\bar{u}(p')\left[\frac{\gamma^\mu \not{k}\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not{k}'\gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right]u(p)$$

We now square + sum over spins and divide by # of initial spins (= 2 electron × 2 photon = 4)

from photon polarization sums

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{e^4}{4} g_{\mu\rho} g_{\sigma\nu}$$

$$\cdot \operatorname{tr}\left\{ (\not p' + m)\left[\frac{\gamma^\mu \not k \gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{\gamma^\nu \not k' \gamma^\mu - 2\gamma^\nu p^\mu}{2p\cdot k'}\right](\not p + m) \left[\frac{\gamma^\sigma \not k \gamma^\rho + 2\gamma^\rho p^\sigma}{2p\cdot k} + \frac{\gamma^\rho \not k' \gamma^\sigma - 2\gamma^\sigma p^\rho}{2p\cdot k'}\right]\right\}$$

This is a lot of work and we don't do all details here. In the numerator get terms like

$$\operatorname{tr}\left[\not p' \gamma^\mu \not k \gamma^\nu \not p \gamma_\nu \not k \gamma_\mu\right]$$

We use fact that $\gamma^\nu \not p \gamma_\nu = -2\not p$ and so we have

$$-2\operatorname{tr}\left[\not p' \gamma^\mu \not k \not p \not k \gamma_\mu\right]$$

Now use fact that $\gamma^\mu \not k \not p \not k \gamma_\mu = -2 \not k \not p \not k$ (order is reversed ... $\gamma^\mu \not a \not b \not c \gamma_\mu = -2 \not c \not b \not a$)

and so we have

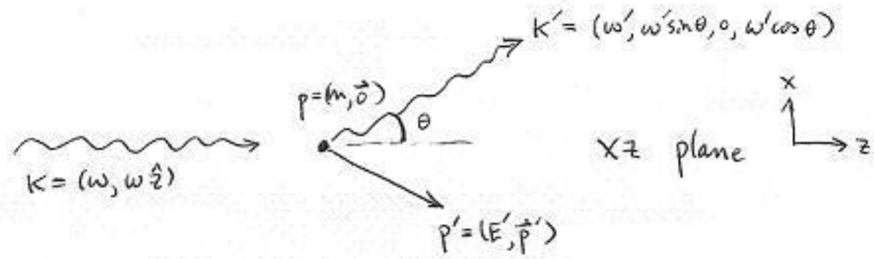
$$+4\operatorname{tr}\left[\not p' \not k \not p \not k\right] = 4\cdot 4\left[(p'\cdot k)(p\cdot k) - (p'\cdot p)\underset{0}{\underset{\shortparallel}{(k\cdot k)}} + (p'\cdot k)(p\cdot k)\right]$$

$$= 32\,(p'\cdot k)(p\cdot k)$$

In the end you get (not difficult ... just rather long)

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = 2e^4\left[\frac{p\cdot k'}{p\cdot k}+\frac{p\cdot k}{p'\cdot k} + 2m^2\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)+m^4\left(\frac{1}{p\cdot k}-\frac{1}{p\cdot k'}\right)\right]$$

For Compton scattering the usual frame is the rest frame of the initial electron.



Let us solve for ω':

$$p'^2 = m^2 \quad \text{and so}$$

$$m^2 = (p')^2 = (p+k-k')^2 = \underbrace{p^2}_{m^2} + \underbrace{\cancel{k^2}}_{=0} + \underbrace{\cancel{k'^2}}_{=0} + 2p\cdot(k-k') - 2k\cdot k'$$

$$= m^2 + 2p\cdot(k-k') - 2k\cdot k' = m^2 + 2m(\omega-\omega') - 2\omega\omega'(1-\cos\theta)$$

So
$$2m(\omega-\omega') - 2\omega\omega'(1-\cos\theta) = 0$$

$$\Rightarrow \frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{m}(1-\cos\theta)$$

$$\Rightarrow \omega' = \frac{1}{\frac{1}{m}(1-\cos\theta)+\frac{1}{\omega}} = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}$$

We now have to work out the phase space integrals (we are not in the center of mass frame)

We have

$$\int \frac{d^3\vec{k}'}{(2\pi)^3 2\omega'} \frac{d^3\vec{p}'}{(2\pi)^3 2E'} (2\pi)^4 \delta^{(4)}(k+p'-k-p)$$

$$\begin{pmatrix}\text{use } \delta^{(3)} \text{ to} \\ \text{kill } d^3\vec{p}'\end{pmatrix} = \int \frac{d^3\vec{k}'}{(2\pi)^3 2\omega'} \frac{1}{2E'} (2\pi)\, \delta(\omega'+E'-\omega-m) \quad \text{where } \vec{p}' = \vec{k}+\vec{p}-\vec{k}'$$

$$\boxed{E' = \sqrt{(\vec{k}+\vec{p}-\vec{k}')^2+m^2} = \sqrt{(\vec{k}-\vec{k}')^2+m^2} = \sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2}}$$

$$= \int \frac{d^3\vec{k}'}{(2\pi)^3 2\omega' 2E'}\, 2\pi\, \delta(\omega' + \overbrace{\sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2}}^{E'} - \omega - m)$$

$$\left(\text{using } \delta(f(x)) = \frac{\delta(x-x_0)}{|f'(x_0)|}\right)$$

$$= \int \frac{\omega'^2 d\omega'\, d\Omega}{(2\pi)^3\, 4\omega' E'}\, 2\pi\, \frac{\delta\left(\omega' - \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}\right)}{\left|1 + \frac{\omega'-\omega\cos\theta}{E'}\right|}$$

$$E' = \sqrt{\omega^2+\omega'^2-2\omega\omega'\cos\theta+m^2}$$

$$= \frac{1}{8\pi}\int_{-1}^{1} \frac{d\cos\theta \cdot \omega'}{\left|1+\frac{\omega'-\omega\cos\theta}{E'}\right| \cdot E'} = \frac{1}{8\pi}\int_{-1}^{1} \frac{d\cos\theta \cdot \omega'}{|E'+\omega'-\omega\cos\theta|}$$

But $E'+\omega' =$ total energy $= \omega + m$ (initial energy)

So we have $\displaystyle \frac{1}{8\pi}\int_{-1}^{1} \frac{d\cos\theta\ \omega'}{|\omega+m-\omega\cos\theta|} = \frac{\omega'}{8\pi}\int \frac{d\cos\theta}{m+\omega(1-\cos\theta)}$

The Velocity of the electron is zero, $|V_A - V_B| = 1$.

$$V_A = 0, \quad V_B = 1$$

So $$\frac{d\sigma}{d\cos\theta} = \frac{1}{2\omega}\frac{1}{2m}\frac{\omega'}{8\pi(m+\omega(1-\cos\theta))} \times \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

Since $\frac{1}{m+\omega(1-\cos\theta)} = \frac{\omega'}{m\omega}$, we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\left[2e^4\left(\frac{p\cdot k'}{p\cdot k} + \frac{p\cdot k}{p\cdot k'} + 2m^2\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right) + m^4\left(\frac{1}{p\cdot k} - \frac{1}{p\cdot k'}\right)^2\right)\right]$$

Using $p\cdot k = m\omega$ and $p\cdot k' = m\omega'$, we have

$$\frac{d\sigma}{d\cos\theta} = \frac{1}{32\pi}\frac{\omega'^2}{m^2\omega^2}\left[2e^4\left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + 2m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) + m^2\left(\frac{1}{\omega} - \frac{1}{\omega'}\right)^2\right)\right]$$

Clever way to write this...

use $\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}$ and so

$$m\left(\frac{1}{\omega} - \frac{1}{\omega'}\right) = m\left(\frac{1}{\omega} - \frac{1+\frac{\omega}{m}(1-\cos\theta)}{\omega}\right)$$
$$= -(1-\cos\theta)$$

So $$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} + \underbrace{\left[-2(1-\cos\theta) + (1-\cos\theta)^2\right]}_{-\sin^2\theta}\right]$$

$$= \frac{\pi\alpha^2}{m^2}\left(\frac{\omega'}{\omega}\right)^2\left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\theta\right]$$

where $\omega' = \frac{\omega}{1+\frac{\omega}{m}(1-\cos\theta)}$

Klein-Nishina formula

As $\omega \to 0$, $\frac{\omega'}{\omega} \to \frac{1}{1+0} = 1$. So $\frac{d\sigma}{d\cos\theta} \to \frac{\pi\alpha^2}{m^2}(1+\cos^2\theta)$

and $\sigma_{total} \to \frac{8\pi\alpha^2}{3m^2}$. These are the Thomson cross-sections for classical E+M scattering off an electron.