Lecture 5 - Quantum States of Light

Review from last lecture

- Analogy with harmonic oscillator:

$$
\begin{array}{c|c|c}
\text{One e.m. field mode} & \text{Correspondence} & \text{Harmonic Oscillator} \\
\dot{\mathcal{A}}_i = -\mathcal{E}_i & \mathcal{A}_i \hat{=} x & \dot{x} = \dfrac{p}{m} \\
\dot{\mathcal{E}}_i = \omega_i^2 \mathcal{A}_i & \mathcal{E}_i \hat{=} -\dfrac{p}{m} & \dfrac{\dot{p}}{m} = -\omega^2 x \\
H_i = \frac{\epsilon_0}{2} \frac{1}{V} \left(|\mathcal{E}_i|^2 + \omega_i^2 |\mathcal{A}_i|^2 \right) & \epsilon_0 \frac{1}{V} \hat{=} m & H = \frac{m}{2} \left(\left(\frac{p}{m}\right)^2 + \omega^2 x^2 \right) \\
\alpha_i = \mathcal{N}_i \left(\mathcal{A}_i - \dfrac{i}{\omega_i} \mathcal{E}_i \right) & & \alpha = \mathcal{N} \left(x + i \dfrac{p}{m\omega} \right) \\
\dfrac{d\alpha_i}{dt} = -i\omega_i \alpha_i & & \dfrac{d\alpha}{dt} = -i\omega\alpha
\end{array}
$$

- Quantization and Commutation relations:

$$
\begin{array}{c|c}
\text{One e.m. field mode} & \text{Harmonic Oscillator} \\
\mathcal{A}_i \to \hat{\mathcal{A}}_i & x \to \hat{x} \\
\mathcal{E}_i \to \hat{\mathcal{E}}_i & p \to \hat{p} \\
\left[\hat{\mathcal{A}}_i, \hat{\mathcal{E}}_i \right] = -\frac{V}{\epsilon_0} i\hbar & [\hat{x}, \hat{p}] = i\hbar \\
\hat{a}_i \text{ annihilation operator} & \hat{a} \text{ annihilation operator} \\
\text{associated to } \alpha_i & \text{associated to } \alpha \\
\left[\hat{a}_i, \hat{a}_i^\dagger \right] = 1 \text{ for} & \left[\hat{a}, \hat{a}^\dagger \right] = 1 \text{ for} \\
\mathcal{N} = \sqrt{\dfrac{\epsilon_0 \omega_i}{2\hbar} \dfrac{1}{V}} & \mathcal{N} = \sqrt{\dfrac{m\omega}{2\hbar}}
\end{array}
$$

- Fields:

$$\vec{E}_\perp(\vec{r}) = i\sum_j \mathcal{E}_j \vec{\epsilon}_j \left(a_j e^{i\vec{k}_j\vec{r}} - a_j^\dagger e^{-i\vec{k}_j\vec{r}}\right)$$

$$\vec{B}(\vec{r}) = i\sum_j \mathcal{E}_j \frac{\vec{k}_j \times \vec{\epsilon}_j}{\omega_j} \left(a_j e^{i\vec{k}_j\cdot\vec{r}} - a_j^\dagger e^{-i\vec{k}_j\cdot\vec{r}}\right)$$

$$\vec{A}_\perp(\vec{r}) = \sum_j \frac{\mathcal{E}_j}{\omega_j}\vec{\epsilon}_j \left(a_j e^{i\vec{k}_j\cdot\vec{r}} + a_j^\dagger e^{-i\vec{k}_j\cdot\vec{r}}\right)$$

where $\mathcal{E}_j = \sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V}}$ (careful: This pure number should not to be confused with the complete Fourier transform of the classical field mode $\vec{\mathcal{E}}_\perp$, which is a function of $\vec{k}$.)

- Uncertainty relation:

  Just like $\Delta x \Delta p \geq \frac{\hbar}{2}$ we have, using our correspondence $\mathcal{A}_j \equiv x$ and $\mathcal{E}_j \equiv -p/m$ as well as $\epsilon_0/V \equiv m$:

  $$\Delta\mathcal{A}_{\perp,j}\Delta\mathcal{E}_{\perp,j} \geq \frac{\hbar}{2}\frac{V}{\epsilon_0}$$

  Quick reminder on units: $\vec{\mathcal{E}}_\perp(\vec{k}_j) = \int \mathrm{d}^3 r\, \vec{E}_\perp(\vec{r}) e^{-i\vec{k}_j\cdot\vec{r}}$ has units of $[EV]$. Also, $\dot{\vec{A}}$ has units of an electric field, so $[\mathcal{A}] = [EVt]$. So $[\epsilon_0\mathcal{A}\mathcal{E}] = \left[\epsilon_0 E^2 V^2 t\right]$. $\epsilon_0 E^2$ is an energy density, so $[\epsilon_0\mathcal{A}\mathcal{E}] = [\text{Energy}\cdot\text{time}\cdot\text{Volume}] = [\hbar V]$.

- Vacuum properties:

  We have $\langle 0|\,\vec{E}_\perp(\vec{r})\,|0\rangle = \langle 0|\,\vec{B}(\vec{r})\,|0\rangle = \langle 0|\,\vec{A}(\vec{r})\,|0\rangle = 0$

  but since $\langle 0|\, a_j a_k^\dagger\,|0\rangle = \delta_{jk}$ we find the

- Vacuum fluctuations:

  $$\left(\Delta\vec{E}_\perp\right)^2 = \langle 0|\,(\vec{E}_\perp(\vec{r}))^2\,|0\rangle = \sum_j \mathcal{E}_j^2 = \sum_j \frac{\hbar\omega_j}{2\epsilon_0 V}$$

  $$\left(\Delta\vec{B}_\perp\right)^2 = \langle 0|\,(\vec{B}(\vec{r}))^2\,|0\rangle = \sum_j \frac{1}{c^2}\mathcal{E}_j^2 = \sum_j \frac{\hbar\omega_j}{2\epsilon_0 c^2 V}$$

$$\left(\Delta\vec{A}_\perp\right)^2 = \langle 0|\left(\vec{A}_\perp(\vec{r})\right)^2|0\rangle = \sum_j \frac{1}{\omega_j^2}\mathcal{E}_j^2 = \sum_j \frac{\hbar}{2\epsilon_0 V\omega_j}$$

- Field values for number states:

$$|n_1 = 0, \ldots, n_{j-1} = 0, n_j, n_{j+1} = 0, \ldots\rangle \equiv |n_j\rangle$$

$$\langle n_j|\vec{E}_\perp(\vec{r})|n_j\rangle = \langle n_j|\vec{B}(\vec{r})|n_j\rangle = 0$$

$$c^2\Delta B^2 = \Delta E_\perp^2 = \langle n_j|\vec{E}_\perp(\vec{r})^2|n_j\rangle = (2n_j+1)\mathcal{E}_j^2$$

So this is $(2n_j + 1)$ times the $\Delta E_\perp^2$ in vacuum.

Coherent States (Quasi-classical states, Glauber states)

I want a state in which the E-field and B-field are as close to a classical state as possible.

$$\vec{E}_{\text{classical}} = \sum_j \mathcal{E}_j\vec{\epsilon}\left(\alpha_j e^{i\vec{k}_j\cdot\vec{r}} + \text{c.c.}\right)$$

$$\vec{E}_{\text{quantum}} = \sum_j \mathcal{E}_j\vec{\epsilon}\left(a_j e^{i\vec{k}_j\cdot\vec{r}} + \text{c.c.}\right)$$

This leads us to define the coherent state $|\alpha\rangle$ as an eigenstate of the annihilation operator with eigenvalue α_j:

$$\boxed{a_j|\alpha_j\rangle = \alpha_j|\alpha_j\rangle}$$

(for one mode, say mode j - let's drop the index j in the following to unclutter).

Expand over number states:

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$$

$$|\alpha\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n}\, |n-1\rangle = \sum_{n=0}^{\infty} c_{n+1}\sqrt{n+1}\, |n\rangle$$
$$\stackrel{!}{=} \alpha \sum_{n=0}^{\infty} c_n\, |n\rangle$$

$$\Rightarrow c_{n+1}\sqrt{n+1} = \alpha c_n \qquad \text{or} \qquad c_n = \frac{\alpha}{\sqrt{n}} c_{n-1}$$

$$\Rightarrow c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$$

$$\Rightarrow |\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle$$

Normalization:

$$\langle\alpha|\alpha\rangle = 1 = |c_0|^2 \sum_{\substack{m=0\\n=0}}^{\infty} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!m!}} \langle m|n\rangle = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2\, e^{|\alpha|^2}$$

$$\Rightarrow c_0 = e^{-\frac{|\alpha|^2}{2}}$$

up to a phase factor. So we finally have found

$$\boxed{|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, |n\rangle}$$

We may immediately note that $|\alpha\rangle$ is *not* an eigenstate of the creation operator:

$$a^\dagger\, |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1}\, |n+1\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=1}^{\infty} \frac{n\alpha^{n-1}}{\sqrt{n!}}\, |n\rangle$$

which is clearly not proportional to $|\alpha\rangle$. In fact, one sometimes works with an alternative normalization where $||\alpha\rangle = e^{\frac{|\alpha|^2}{2}}\, |\alpha\rangle$, which allows one to right simply

$$a^\dagger \left|\left|\alpha\right\rangle\right. = \frac{\partial}{\partial\alpha} \left|\left|\alpha\right\rangle\right.$$

(simply from $\frac{\partial}{\partial\alpha}\alpha^n = n\alpha^{n-1}$)

- Time evolution:

  Let's start at $t = 0$ with $|\alpha\rangle$. At time t we find:

$$\begin{aligned}|\psi(t)\rangle &= e^{-iHt/\hbar}\,|\alpha\rangle \\ &= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-(n+\frac{1}{2})\omega t}\,|n\rangle \\ &= e^{-i\frac{\omega t}{2}} e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}\,|n\rangle \\ &= e^{-i\frac{\omega t}{2}}\,\left|\alpha e^{-i\omega t}\right\rangle\end{aligned}$$

  Probability of finding the value $(n+\frac{1}{2})\hbar\omega$ for the energy (or the value n for the photon number) is time independent:

$$P(n) = |c_n|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$$

  Mean photon number: $\bar{n} = \left\langle \hat{N} \right\rangle = \langle\alpha|\, a^\dagger a\, |\alpha\rangle = \alpha^*\alpha = |\alpha|^2$

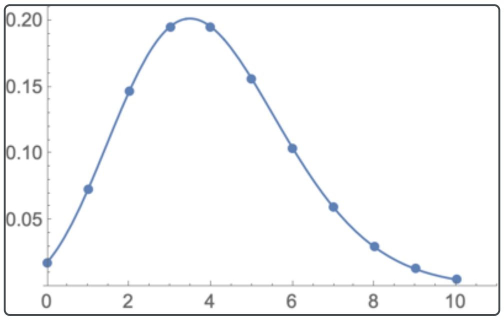
  Equivalently we can calculate:

$$\left\langle \hat{N} \right\rangle = \sum_n nP(n) = \sum_n n e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = |\alpha|^2 \sum_n e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} = |\alpha|^2$$

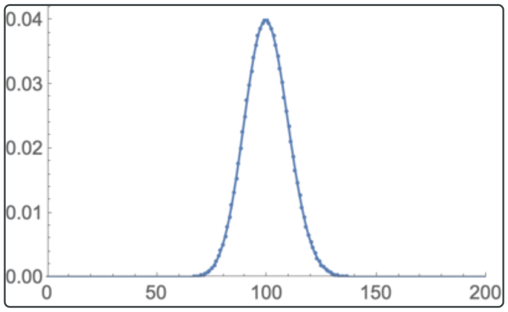
  So $\bar{n} = |\alpha|^2$ and

$$\boxed{P(n) = e^{-\bar{n}} \frac{\bar{n}^n}{n!}} \qquad \text{Poisson Distribution}$$

$P(n)$ for $\bar{n} = 4$:



$P(n)$ for $\bar{n} = 100$



- Standard deviation:

$$\begin{aligned}\Delta n^2 = (\Delta N)^2 = \left\langle N^2 \right\rangle - \langle N \rangle^2 &= \langle \alpha | \, a^\dagger a a^\dagger a \, | \alpha \rangle - |\alpha|^4 \\ &= \langle \alpha | \, a^\dagger (a^\dagger a + 1) a \, | \alpha \rangle - |\alpha|^4 \\ &= |\alpha|^4 + \langle N \rangle - |\alpha|^4 \\ &= \langle N \rangle = \bar{n}\end{aligned}$$

So $\boxed{\Delta n^2 = \bar{n}}$ as it should be for the Poisson distribution.

- Electric field: Use $|\psi(t)\rangle = e^{-i\frac{\omega t}{2}} \left| \alpha e^{-i\omega t} \right\rangle$ and find (again, we omit the index j):

$$\langle \psi(t) | \, \vec{E}_\perp(\vec{r}) \, | \psi(t) \rangle = i \mathcal{E} \vec{\epsilon} \left\{ \alpha e^{i(\vec{k} \cdot \vec{r} - \omega t)} - \alpha^* e^{-i(\vec{k} \cdot \vec{r} - \omega t)} \right\}$$

(which is just the classical field)

$$\begin{aligned}\langle \psi(t) | \, \vec{E}_\perp(\vec{r})^2 \, | \psi(t) \rangle &= \langle \psi(t) | \, (\ldots a \cdots - a^\dagger \ldots)(\ldots a \cdots - \ldots a^\dagger \ldots) \, | \psi(t) \rangle \\ &= \left(\langle \psi(t) | \, \vec{E}_\perp(\vec{r}) \, | \psi(t) \rangle \right)^2 + \mathcal{E}^2\end{aligned}$$

Therefore

$$\Rightarrow \boxed{\Delta \vec{E}_\perp^2 = \mathcal{E}^2}$$

which is the same fluctuation as in the vacuum state!

- Are coherent states orthonormal? Let's find out:

$$
\begin{aligned}
|\langle\beta|\alpha\rangle|^2 &= \left| e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \sum_{m,n} \frac{\alpha^n \beta^{*m}}{\sqrt{n!}\sqrt{m!}} \langle m|n\rangle \right|^2 \\
&= e^{-|\alpha|^2} e^{-|\beta|^2} \left| e^{\alpha\beta^*} \right| \\
&= e^{-|\alpha-\beta|^2}
\end{aligned}
$$

They are not orthogonal!

But they do obey the closure relation:

$$
\frac{1}{\pi} \int \mathrm{d}(\operatorname{Re}\alpha)\mathrm{d}(\operatorname{Im}\alpha)\, |\alpha\rangle\langle\alpha| = 1
$$

Equivalently we can write

$$
\frac{1}{\pi} \int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| = 1
$$

Proof:

$$
\frac{1}{\pi} \int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| = \frac{1}{\pi} \sum_{m,n} \frac{|m\rangle\langle n|}{\sqrt{m!n!}} \int \mathrm{d}^2\alpha\, e^{-|\alpha|^2} \alpha^m (\alpha^*)^n
$$

$$
\begin{aligned}
\int \mathrm{d}^2\alpha\, e^{-|\alpha|^2} \alpha^m (\alpha^*)^n &= \int_0^\infty \mathrm{d}r\, e^{-r^2} r^{m+n+1} \int_0^{2\pi} e^{i(m-n)\theta} \mathrm{d}\theta \\
&= 2\pi \int_0^\infty \mathrm{d}r\, e^{-r^2} r^{m+n+1} \delta_{mn} \\
&= 2\pi\delta_{mn} \int_0^\infty \mathrm{d}r\, e^{-r^2} r^{2n+1} \\
&= 2\pi\delta_{mn} \frac{1}{2} \int_0^\infty \mathrm{d}u\, e^{-u} u^n \\
&= \pi\, n!\, \delta_{mn}
\end{aligned}
$$

So

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| = \frac{1}{\pi}\sum_{m,n}\frac{|m\rangle\langle n|}{\sqrt{m!n!}}\pi\, n!\,\delta_{mn} = \sum_n |n\rangle\langle n| = 1$$

Application to number state:

$$|n\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha|n\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, e^{-\frac{|\alpha|^2}{2}}\frac{(\alpha^*)^n}{\sqrt{n!}}|\alpha\rangle$$

The weight of the state $|\alpha\rangle$ can be seen to be peaked at $|\alpha| = \sqrt{n}$, which is independent of the argument of α.

Application to a coherent state:

$$|\beta\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha|\beta\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, e^{-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\alpha^*\beta}|\alpha\rangle$$

So a coherent state can be written as a superposition of coherent states! The basis is said to be overcomplete. Still, we have

$$\langle\beta|\beta\rangle = \frac{1}{\pi}\int \mathrm{d}^2\alpha\, e^{-|\alpha-\beta|^2} = 1$$

as it should be.

Decomposition: $|\psi\rangle = \int \mathrm{d}^2\alpha\, c_\alpha\, |\alpha\rangle$ with $c_\alpha = \frac{1}{\pi}\langle\alpha|\psi\rangle$

This decomposition is not unique as

$\langle\alpha|\alpha'\rangle \neq 0$, but $|\langle\alpha|\alpha'\rangle|^2 = e^{-\left|\alpha-\alpha'\right|^2} \to 0$

if α and α' are widely separated.

- We say that the basis of coherent states $|\alpha\rangle$ is overcomplete.