Problem Set 5

Due: Friday 5pm, Mar 11, via Canvas upload or in envelope outside 26-255
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1 Magnetic field of a magnetic dipole (5 pts.)

In lecture I showed that the magnetic field of an infinitesimal current loop (infinitesimal radius r) with fixed magnetic moment contains a δ-function piece: The field at the center is $B \propto I/r \propto \mu/r^3$ and the integral of the field over the volume of the current loop $\int d^3r\, \boldsymbol{B} \propto \boldsymbol{\mu}$ was a constant just given by the dipole moment. So $\boldsymbol{B} \propto \boldsymbol{\mu}\,\delta(\boldsymbol{r})$. A sphere of uniform magnetization also has a constant magnetic field in the center, as it has surface currents, can thus be modeled like many current loops stacked together (a rotating shell of charge). The constant magnetic field inside again gives the δ-function piece upon shrinking to an infinitesimal sphere.

Are those just some special cases? No. We will prove the δ-function piece as a generic portion of the magnetic field created by a magnetic dipole.

It is a mathematical identity. Essentially every E&M equation involving δ-functions can be derived from it:

$$\boxed{\partial_i \partial_j \left(\frac{1}{r}\right) = -\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2}\right) = -\partial_i \left(\frac{x_j}{r^3}\right) = \frac{3\hat{\boldsymbol{r}}_i \hat{\boldsymbol{r}}_j - \delta_{ij}}{r^3} - \frac{4\pi}{3}\,\delta_{ij}\,\delta^3(\boldsymbol{r})} \tag{1}$$

To make sure the definitions are clear, $\boldsymbol{r} \equiv x_i \hat{\boldsymbol{e}}_i \equiv x\hat{\boldsymbol{e}}_x + y\hat{\boldsymbol{e}}_y + z\hat{\boldsymbol{e}}_z$, $r \equiv |\boldsymbol{r}|$, $\hat{\boldsymbol{r}} \equiv \boldsymbol{r}/r$, $\hat{r}_i \equiv x_i/r$, and $\partial_i \equiv \partial/\partial x_i$.

Note on grading: this problem is checked for completion of each part. Each part has the following number of points allocated: part a) 1, b) 2.5, c) 1.5.

a) Show that the famous identity

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(\boldsymbol{r})$$

follows immediately from Eq. 1.

In Cartesian coordinates

$$\nabla^2 \left(\frac{1}{r}\right) = \partial_j \partial_j \left(\frac{1}{r}\right)$$

so we can use the identity Eq. 1 to write

$$\nabla^2 \left(\frac{1}{r}\right) = \frac{3\hat{\boldsymbol{r}}_j \hat{\boldsymbol{r}}_j - \delta_{jj}}{r^3} - \frac{4\pi}{3}\,\delta_{jj}\,\delta^3(\boldsymbol{r})$$

Then we use $\delta_{jj} = 3$ and $\hat{\boldsymbol{r}}_j \hat{\boldsymbol{r}}_j = 1$, so

$$\nabla^2 \left(\frac{1}{r}\right) = -4\pi\delta^3(\boldsymbol{r}) \ .$$

b) Using the identity Eq. 1, show that the vector potential for a magnetic dipole,

$$\boldsymbol{A}_{\mathrm{dip}}(\boldsymbol{r}) = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2} \ ,$$

can be used to find immediately that the magnetic field of a magnetic dipole is given by

$$\boldsymbol{B}_{\mathrm{dip}}(\boldsymbol{r}) = \boldsymbol{\nabla} \times \boldsymbol{A} = \frac{3(\boldsymbol{m} \cdot \hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^3} + \frac{8\pi}{3}\boldsymbol{m}\,\delta^3(\boldsymbol{r}).$$

This therefore contains the δ-piece, all important for the 21 cm line of hydrogen.

Starting from the vector potential of a magnetic dipole,

$$\boldsymbol{A}_{\mathrm{dip}}(\boldsymbol{r}) = \frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2},$$

we find using index notation that

$$\begin{aligned} B_{\mathrm{dip},i} &= \epsilon_{ijk}\partial_j A_k = \epsilon_{ijk}\partial_j \left(\frac{\boldsymbol{m} \times \hat{\boldsymbol{r}}}{r^2}\right)_k = \epsilon_{ijk}\partial_j \left(\frac{\epsilon_{k\ell m} m_\ell \hat{\boldsymbol{r}}_m}{r^2}\right) \\ &= m_\ell \epsilon_{kij}\epsilon_{k\ell m}\partial_j \left(\frac{\hat{\boldsymbol{r}}_m}{r^2}\right) . \end{aligned}$$

Now we use the identity

$$\epsilon_{kij}\epsilon_{k\ell m} = \delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}$$

as well as the identity Eq. 1 to give

$$\begin{aligned} B_{\mathrm{dip},i} &= m_\ell(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \left[\frac{\delta_{jm} - 3\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_m}{r^3} + \frac{4\pi}{3}\,\delta_{jm}\,\delta^3(\boldsymbol{r})\right] \\ &= (m_i\delta_{jm} - m_j\delta_{im}) \left[\frac{\delta_{jm} - 3\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_m}{r^3} + \frac{4\pi}{3}\,\delta_{jm}\,\delta^3(\boldsymbol{r})\right] . \end{aligned}$$

Before we continue, note that

$$\delta_{jm}\frac{\delta_{jm} - 3\hat{\boldsymbol{r}}_j\hat{\boldsymbol{r}}_m}{r^3} = \frac{3-3}{r^3} = 0$$

Thus,

$$\begin{aligned}
B_{\mathrm{dip},i} &= \left[-\frac{m_i - 3(m_j \hat{\boldsymbol{r}}_j)\hat{\boldsymbol{r}}_i}{r^3} + \frac{4\pi}{3}\left(3m_i - m_i\right)\delta^3(\boldsymbol{r})\right] \\
&= \frac{3(m_j \hat{\boldsymbol{r}}_j)\hat{\boldsymbol{r}}_i - m_i}{r^3} + \frac{8\pi}{3} m_i \delta^3(\boldsymbol{r}).
\end{aligned}$$

In vector notation, this becomes

$$\boldsymbol{B}_{\mathrm{dip}}(\boldsymbol{r}) = \frac{3(\boldsymbol{m}\cdot\hat{\boldsymbol{r}})\,\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^3} + \frac{8\pi}{3}\boldsymbol{m}\,\delta^3(\boldsymbol{r}).$$

c) Let's now prove the identity. First, show that it is valid for $r \neq 0$. Then, verify the δ-function piece by integrating over a small sphere of radius ϵ about the origin. You will need to use the identity:

$$\int_V \boldsymbol{\nabla}\psi \,\mathrm{d}^3x = \int_S \psi \,\mathrm{d}\boldsymbol{a}$$

where S is the surface bounding the volume V, and $\mathrm{d}\boldsymbol{a}$ the vector denoting the infinitesimal surface element $\mathrm{d}\boldsymbol{a} = \mathrm{d}a\,\hat{\boldsymbol{n}}$, where $\mathrm{d}a$ is the infinitesimal area of the surface element and $\hat{\boldsymbol{n}}$ is the outward normal vector. This allows to convert

$$\int_{r<\epsilon} \partial_i \left(\frac{\hat{r}_j}{r^2}\right) \mathrm{d}^3x$$

into a surface integral on a sphere of radius ϵ (so here $\psi = \frac{\hat{r}_j}{r^2}$). Evaluate the surface integral to show that the δ-function term in Eq. 1 is correct.

Note: You may be concerned that the integral in Eq. 1 is ill-defined, since $\hat{r}_j/r^2$ is not differentiable at $r = 0$. Mathematicians give a rigorous meaning to the equation Eq. 1 by defining both sides as distributions, which means that they are defined by the result of multiplying them by an arbitrary test function $\phi(\boldsymbol{r})$ (assumed smooth and falling off rapidly at $r \to \infty$) and then integrating both sides over all space. The derivative $\partial_i\left(\frac{\hat{r}_j}{r^2}\right)$ is then given an unambigious meaning by defining the derivative of a distribution by integration by parts: all the derivatives are applied to the test function $\phi(\boldsymbol{r})$. So $\int \phi(\boldsymbol{r})\partial_i\left(\frac{\hat{r}_j}{r^2}\right)\mathrm{d}^3x = -\int \partial_i\phi(\boldsymbol{r})\left(\frac{\hat{r}_j}{r^2}\right)\mathrm{d}^3x$, and this is well-defined as the measure $\mathrm{d}^3x = r^2\mathrm{d}r\sin\theta\mathrm{d}\theta\mathrm{d}\phi$ cancels the r^2 in the denominator.

For $r \neq 0$, the calculation is straightforward. We first note that

$$\partial_i r = \partial_i (x_j x_j)^{1/2} = (x_k x_k)^{-1/2} x_j \partial_i x_j = \frac{x_j \delta_{ij}}{r} = \frac{x_i}{r} = \hat{\boldsymbol{r}}_i$$

which should be familiar by now. Then

$$\partial_i \left(\frac{x_j}{r^3} \right) = \frac{\delta_{ij}}{r^3} - 3\frac{x_j}{r^4}\partial_i r = \frac{\delta_{ij} - 3\hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j}{r^3} \quad (\text{for } r \neq 0) \ .$$

To find the δ-function, we integrate $\partial_i(\hat{\boldsymbol{r}}_j/r^2)$ over a small sphere $r < \epsilon$ about the origin. The contribution from $\partial_i \left(\frac{x_j}{r^3}\right)$ vanishes when integrated over all angles, but we will nonetheless find a contribution by integrating over the origin, which is encoded in the δ-function. In Griffiths' Problem 1.61(a), p. 56, we showed that

$$\int_{\mathcal{V}} (\boldsymbol{\nabla} T)\, \mathrm{d}^3x = \int_S T\, \mathrm{d}\boldsymbol{a}$$

where S is the surface bounding the volume $\mathcal{V}$. (We proved this identity in Problem 7(a) of Problem Set 1.) In index notation, this becomes

$$\int_{\mathcal{V}} \partial_i T\, \mathrm{d}^3x = \int_S T\, \mathrm{d}a_i$$

where $\mathrm{d}a_i$ is the i'th component of the surface area element vector $\mathrm{d}\boldsymbol{a} = \mathrm{d}a$, where $\mathrm{d}a$ is the infinitesimal area of the surface element and is the outward normal vector. For integration over a sphere, $\mathrm{d}\boldsymbol{a} = r^2 \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi\, \hat{\boldsymbol{r}}$. Now we let $T = (\hat{\boldsymbol{r}}_j/r^2)$, and we have

$$\int_{r<\epsilon} \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3x = \int_{r=\epsilon} \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \hat{\boldsymbol{r}}_i\ \mathrm{d}a = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi$$

The last integral can be evaluated by brute force, using

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}}$$

but it can be found easily if one recognizes that it must be rotationally invariant, and that the only rotationally invariant tensor is δ_{ij}. Thus it must have the form

$$\int \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi = A\,\delta_{ij}$$

where we can determine A most easily by multiplying both sides by δ_{ij} and summing over the repeated indices. This gives $4\pi = 3A$, so $A = 4\pi/3$ and

$$\int_{r<\epsilon} \partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \mathrm{d}^3x = \int \hat{\boldsymbol{r}}_i\hat{\boldsymbol{r}}_j \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi = \frac{4\pi}{3}\delta_{ij}$$

Since this result is obtained for arbitrarily small ϵ, we conclude that

$$\partial_i \left(\frac{\hat{\boldsymbol{r}}_j}{r^2} \right) \quad \text{must contain a term} \quad \frac{4\pi}{3}\delta_{ij}\delta^3(\boldsymbol{r}) \ .$$

2 Atoms in magnetic fields: the Breit-Rabi formula (10 pts.)

The Hamiltonian for an atom in a magnetic field along $\hat{z}$ may be written

$$H = ah\,\boldsymbol{I}\cdot\boldsymbol{J} + (g_J\mu_B m_J - g_I\mu_B m_I)\,B_z$$

Note on grading: this problem is checked for completion of each part. Each part has the following number of points allocated: part a) 3, b) 2, c) 1, d) 1, e) 2, and f) 1.

a) Restrict attention to the case $J = 1/2$, but arbitrary I. Show that the energies of states are given by the Breit-Rabi formula

$$E_m^{\pm} = -\frac{ah}{4} - m g_I \mu_B B_z \pm \frac{ahF^+}{2}\sqrt{1 + \frac{2mx}{F^+} + x^2}$$

The parameter x is given by $x = (g_I + g_J)\,\mu_B B_z/(ahF^+)$, where $F^+ = I + 1/2$. m is the z component of the total angular momentum (remember: $m = m_F = m_J + m_I$ is always a good quantum number, at any magnetic field).

Using $I_\pm = I_x \pm iI_y$ and $J_\pm = J_x \pm iJ_y$, we can rewrite the Hamiltonian as

$$H = \frac{ah}{\hbar^2}\left(\frac{I_+J_- + I_-J_+}{2} + I_zJ_z\right) + \left(g_J\frac{J_z}{\hbar} - g_I\frac{I_z}{\hbar}\right)\mu_0 B_z$$

This manifestly commutes with F_z (note that the action of each term leaves $m_I + m_J$ unchanged). Since $[H, F_z] = 0$, m_F is a good quantum number; i.e. states of different m_F are not mixed. Therefore, it is legitimate to solve the problem within a subspace of fixed $m_F = m$.

When $J = 1/2$, this subspace is spanned by $\{\left|m_I = m - \frac{1}{2}, m_J = \frac{1}{2}\right\rangle, \left|m_I = m + \frac{1}{2}, m_J = -\frac{1}{2}\right\rangle\}$. If we describe the Hamiltonian in this basis, it is expressed as the following 2×2 matrix.

$$\begin{pmatrix} ah\left(m-\frac{1}{2}\right)\frac{1}{2} + \left(\frac{g_J}{2} - g_I\left(m-\frac{1}{2}\right)\right)\mu_0 B_z & \frac{ah}{2}\sqrt{I(I+1) - \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)}\sqrt{\frac{1}{2}\frac{3}{2} + \frac{1}{2}\frac{1}{2}} \\ \frac{ah}{2}\sqrt{I(I+1) - \left(m-\frac{1}{2}\right)\left(m+\frac{1}{2}\right)}\sqrt{\frac{1}{2}\frac{3}{2} + \frac{1}{2}\frac{1}{2}} & ah\left(m+\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(-\frac{g_J}{2} - g_I\left(m+\frac{1}{2}\right)\right)\mu_0 B_z \end{pmatrix}$$

Introducing $x = (g_I + g_J)\,\mu_0 B_z/ahF^+$ with $F^+ = I + 1/2$, this can be simplified to

$$-\frac{ah}{4} - g_I m \mu_0 B_z + \frac{ah}{2}\begin{pmatrix} m + F^+x & \sqrt{F^{+2} - m^2} \\ \sqrt{F^{+2} - m^2} & -m - F^+x \end{pmatrix}$$

The eigenvalues of the matrix in the last term are

$$\lambda = \pm F^+\sqrt{1 + \frac{2mx}{F^+} + x^2}$$

So the eigenenergies are

$$E_m^{\pm} = -\frac{ah}{4} - m g_I \mu_0 B_z \pm \frac{ahF^+}{2}\sqrt{1 + \frac{2mx}{F^+} + x^2} \tag{2}$$

This analysis does not cover the stretched states $m = \pm F^+$ where $m_I = \pm I$, $m_J = \pm 1/2$ and the sign is the same in both. These states are actually much simpler to deal with, since they have no near-degenerate partner to mix with: the energy is just the expectation value of the Hamiltonian in the state. The result is

$$E_{\pm F^+} = -\frac{ah}{4} \mp F^+ g_I \mu_0 B_z + \frac{ahF^+}{2}(1 \pm x)$$

so that equation 2 with the positive sign in front of the radical is valid also for the stretched states.

It is worth noting that the two states with the same m have different energy shifts. Also, we can check that the "center of mass" is zero, i.e. that the average shift over all the states vanishes.

$$\begin{aligned}
\sum_{-F^++1}^{F^+-1} E_m^+ + E_m^- &= -\frac{ah}{4} \times 2 \times (2F^+ - 1) = -ahF^+ + \frac{ah}{2} \\
E_{+F^+} &= -\frac{ah}{4} - F^+ g_I \mu_0 B_z + \frac{ahF^+}{2}(1 + x) \\
E_{-F^+} &= -\frac{ah}{4} + F^+ g_I \mu_0 B_z + \frac{ahF^+}{2}(1 - x) \\
E_{+F^+} + E_{-F^+} &= -\frac{ah}{2} + ahF^+
\end{aligned}$$

b) For the case of $I = 3/2$, make a clear sketch of energies vs x. Take advantage of the non-crossing rule (levels of the same m do not cross). Be sure to extend your figure to very high field ($1/x \ll g_I/g_J$). Label the lines with quantum numbers at low and high fields and indicate m. You may use the values of ^{87}Rb, which has $I = 3/2$, $g_I = 0.0009954$, and $g_J = 2.002331$.

$I = 3/2$ gives $F^+ = 2$.

$$\begin{aligned}
\frac{E_m^{\pm}}{2ah} &= -\frac{1}{8} - \frac{g_I}{g_I + g_J} m x \pm \frac{1}{2}\sqrt{1 + mx + x^2} \\
\frac{E_2}{2ah} &= -\frac{1}{8} - \frac{g_I}{g_I + g_J} 2x + \frac{1}{2}(1 + x) \\
\frac{E_{-2}}{2ah} &= -\frac{1}{8} - \frac{g_I}{g_I + g_J}(-2)x + \frac{1}{2}(1 - x)
\end{aligned}$$

These energy levels are shown in figure (1). There are 8 lines in the figure, but it is not possible to resolve them at the scale of the figure. Let us consider specific ranges of x values.

Since $g_I/g_J \ll 10^{-3}$, we can take $x \sim g_J\mu_0 B_z/2ah$. Thus x is a good measure of the ratio of the effect of the hyperfine Hamiltonian to that of the Zeeman Hamiltonian caused by the electron spin. In other words, x represents the degree of coupling between $\left|m_I = m - \frac{1}{2}, m_J = \frac{1}{2}\right\rangle$ and $\left|m_I = m + \frac{1}{2}, m_J = -\frac{1}{2}\right\rangle$ in the $\{m_F = m\}$ subspace.

(a) $x \ll 1$, figure (2)
Here the levels are mainly determined by the hyperfine interaction term and the Zeeman term lifts its degeneracy. Levels split proportional to the g-factors, which you will calculate later on.

(b) $x \sim 1$, figure (2)
Both terms compete and show interesting phenomena like field-independent transitions, which is the topic of the next few questions.

(c) $x > 1$, figures (3-1) and (3-2)
Now the Hamiltonian starts to look like

$$H \sim m_J g_J \mu_0 m_J B_z + ahm_J m_I$$

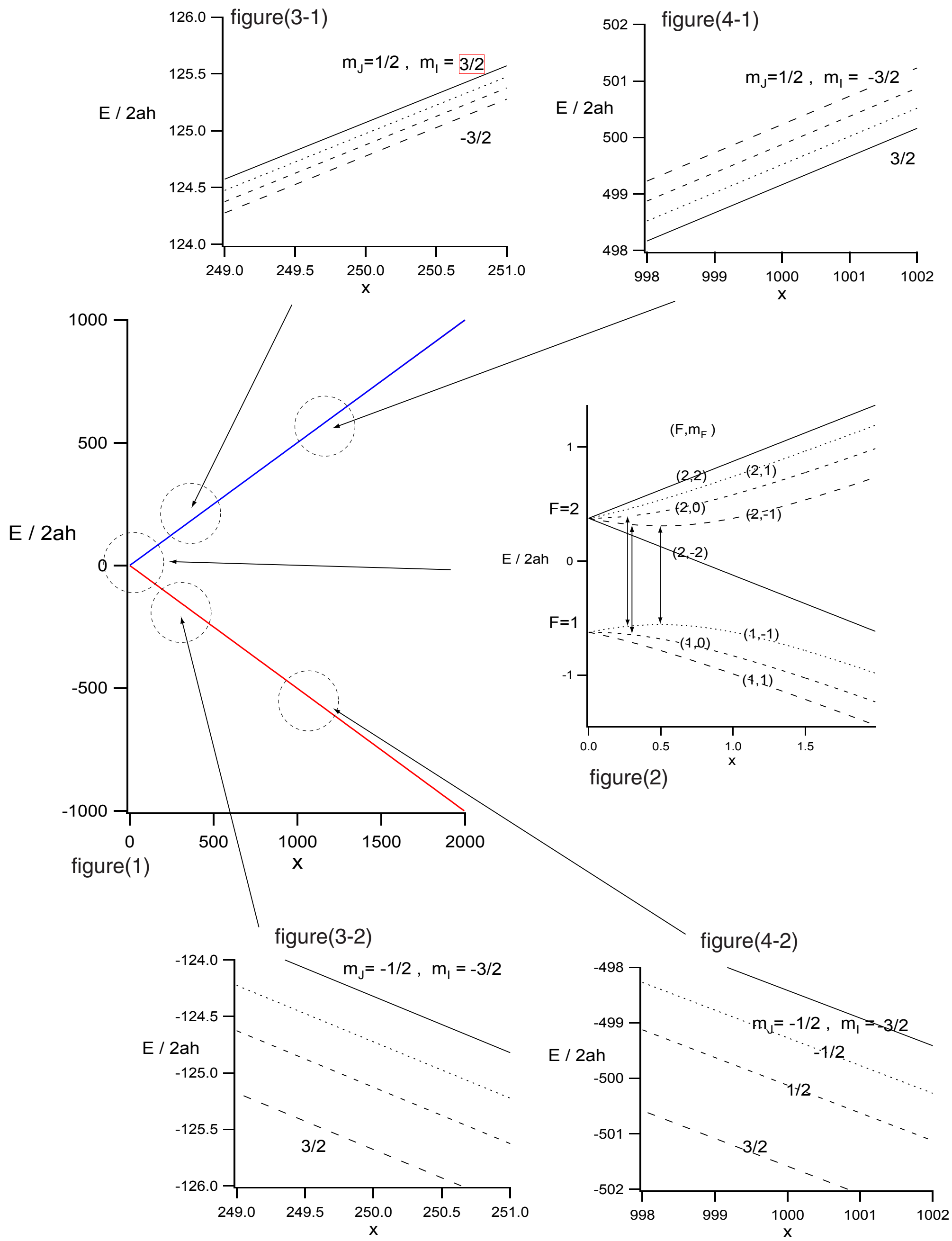
So the electron spin dominates the spectrum $(g_J\mu_0 m_J B_z)$, but its degeneracy is lifted by $(ahm_J m_I)$.

(d) $x > g_J/g_I$, figures (4-1) and (4-2)
Now even the nuclear magnetic moment's Zeeman energy is larger than the hyperfine interaction, and the Hamiltonian becomes

$$H \sim g_J\mu_0 m_J B_z - g_I\mu_0 m_I B_z$$

In other words, the energy cost of anti-aligning the nuclear magnetic moment to the field can overpower the energy reduction of aligning the nuclear and electron magnetic moments. So although the electron spin still dominates the spectrum, the way the degeneracy lifted is different than in the previous case. For example, $|m_J = 1/2, m_I = 3/2\rangle$ has the highest energy of all the eight levels in figure (3). This is no longer true if $x \gg g_J/g_I$; then $|m_J = 1/2, m_I = -3/2\rangle$ has the highest energy. The order of the four $m_J = 1/2$ lines are reversed.

c) There are some values of magnetic field x where the resonance frequency ΔE for the (magnetic) dipole transition (selection rules $\Delta m = 0, \pm 1$) is first-order field independent (i.e., the leading order term in ΔE is of $O((\Delta x)^2)$). Show those values for magnetic field and corresponding transitions on your figure from part b).



figure(3-1)
E / 2ah
m_J=1/2 , m_I = 3/2
-3/2
x
figure(4-1)
E / 2ah
m_J=1/2 , m_I = -3/2
3/2
x
E / 2ah
x
figure(1)
(F,m_F)
(2,2)
(2,1)
(2,0)
(2,-1)
(2,-2)
F=2
F=1
(1,-1)
(1,0)
(1,1)
E / 2ah
x
figure(2)
figure(3-2)
m_J= -1/2 , m_I = -3/2
E / 2ah
3/2
x
figure(4-2)
m_J= -1/2 , m_I = -3/2
-1/2
1/2
-3/2
E / 2ah
x

As magnetic field is increased from zero, the energy of $|F = 2, m_F = 0\rangle$ does not initially vary (since $m = 0$) and then starts to increase as mixing with $|F = 1, m_F = 0\rangle$ becomes important. The energy of $|F = 0, m_F = -1\rangle$ first goes up and then decreases. So the energy difference of those two levels first decreases, goes through a *minimum* (at $x \sim 0.3$), and then starts to increase. If the frequency of the transition is measured at the field corresponding to this minimum, the change of the transition frequency with magnetic field fluctuations is of order $\Delta E = O((\Delta x)^2)$ instead of $O(\Delta x)$. This is called a "first order field independent transition". From figure (b), we can find four of those transitions at low ($x \ll 1$) field.

$$
\begin{array}{cc}
|1,0\rangle \leftrightarrow |2,0\rangle & x \approx 0 \\
|1,-1\rangle \leftrightarrow |2,-1\rangle & x \approx 1/2 \\
|1,0\rangle \leftrightarrow |2,-1\rangle & x \approx 2-\sqrt{3} \\
|1,-1\rangle \leftrightarrow |2,0\rangle & x \approx 2-\sqrt{3}
\end{array}
$$

The approximate values for x are found by solving for the critical point of the energy difference, taking $g_I/(g_I + g_J) \approx 0$.

In addition, there is a first order field independent transition for $|1,-1\rangle \leftrightarrow |2,-2\rangle$ at high field. One can see it by observing the following: at low field (small x), the slope of the energy curve of $|1,-1\rangle$ is greater than the slope of the energy curve of $|2,-2\rangle$. At high field, the slopes of the two energy curves are almost the same, but due to the contribution from m_I, the slope of $|1,-1\rangle$ becomes slightly smaller than that of $|2,-2\rangle$. Therefore, we expect also a change in the sign of the derivative of the energy difference at high field. One can find the approximate value for x by expanding the energy difference of $|1,-1\rangle \leftrightarrow |2,-2\rangle$ for large x, and solving for the critical point. One then obtains $x \approx \frac{1}{4}\sqrt{\frac{3(g_I+g_J)}{g_I}}$.

d) Which of the transitions you just found connect states that can be confined in a magnetic trap, i.e. a field configuration with a local minimum in the magnitude of the magnetic field? Note that Maxwell's equations forbid a static local maximum of the magnetic field in free space.

For a state to be magnetically trappable, its energy must be an *increasing* function of magnetic field, so that it will be confined in a local minimum of the field. Of the four transitions found in the preceding exercise, only $\boxed{|1,-1\rangle \leftrightarrow |2,0\rangle}$ is between states both of which are trappable *at the field for which the transition is first-order field-insensitive.*

e) The first-order insensitive transition(s) between trappable states you have just found occur(s) for states with a relatively weak dipole moment (dependence of state energy on magnetic field) and at large offset fields. This makes for weak magnetic traps: it is hard to generate large field gradients (needed for tight trapping, particularly of weak dipoles) at a large offset field. The transition $|F=1, m_F=-1\rangle \leftrightarrow |F=2, m_F=1\rangle$ in ^{87}Rb is not an allowed magnetic dipole transition ($\Delta m = 2$), but it can be driven as a two-photon transition using one of the $m=0$ sublevels as a virtual intermediate state. Like the examples you have looked at in the preceding questions, this transition is first-order insensitive to field fluctuations at some properly-chosen offset field. However, in this case the correct offset field is small, and the two states have large ($\mu_B/2$) magnetic dipole moments. This makes it valuable for precision spectroscopy of trapped atomic samples and for efforts to make compact neutral-atom clocks. For examples, see G.K. Campbell *et al.*, Science *313*:649–652 (2006), D.S. Hall *et al.*, PRL **81**:1543 (1998), D.M. Harber *et al.*, PRA **66**:053616 (2002). Calculate the magnetic field for which the two-photon transition is first-order field-insensitive. In ^{87}Rb, $I = 3/2$, $2a = 6.835\,\text{GHz}$, $g_I = 0.0009954$, and $g_J = 2.002331$ in the ground ($5^2S_{1/2}$) state. Using $F^+ = I + 1/2 = 2$ and $\mu_0 B_z / ahF^+ = x/(g_I + g_J)$:

$$\frac{E_1^+ - E_{-1}^-}{2ah} = -2\frac{g_I}{g_I + g_J}x + \frac{1}{2}\left(\sqrt{1+x+x^2} + \sqrt{1-x+x^2}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{E_1^+ - E_{-1}^-}{2ah}\right) = -2\frac{g_I}{g_I + g_J} + \frac{1}{4}\left(\frac{1+2x}{\sqrt{1+x+x^2}} - \frac{1-2x}{\sqrt{1-x+x^2}}\right)$$

We seek the field for which this derivative vanishes. Now the solution we are looking for will be for small x. We can see this either by noting that $x=0$ is a solution if we neglect the first term that accounts for the nuclear magnetic moment (it is a small correction since $g_I/g_J \sim 5\times 10^{-4}$), or by considering that $x = 1 \Leftrightarrow B_z = ahF^+/\mu_0(g_I + g_J) = 2.44G$, which is a very large field by the standards of an atomic physics laboratory. We therefore make a small-x expansion:

$$\frac{E_1^+ - E_{-1}^-}{2ah} = -2\frac{g_I}{g_I + g_J}x + 1 + \frac{3}{8}x^2 + O(x^3)$$

$$\frac{\partial}{\partial x}\left(\frac{E_1^+ - E_{-1}^-}{2ah}\right) = -2\frac{g_I}{g_I + g_J} + \frac{3}{4}x + O(x^2)$$

This vanishes for $x = 8g_I/3(g_I + g_J) = 1.325 \times 10^{-3}$, so the field at which the transition is first-order field-insensitive is $\boxed{B_z = 3.23G}$.

f) ^{87}Rb atoms magnetically trapped in the $|F=1, m=-1\rangle$ state at $1\,\mu$K temperature are distributed over a magnetic field range of about 30 mG. Calculate the inhomogeneous width of the $|F=1, m=-1\rangle \leftrightarrow |F=2, m=1\rangle$ transition at zero magnetic field and at the magnetic field found in the previous question.

At zero field, the inhomogeneous width is given by

$$\begin{aligned}
\Delta\left(\frac{E_1^+ - E_{-1}^-}{h}\right) &= \left|\frac{\partial}{\partial x}\left(\frac{E_1^+ - E_{-1}^-}{h}\right)_{x=0}\right|\Delta x \\
&= 2a \times 2\frac{g_I}{g_I + g_J}\Delta x \\
&\approx 6.835 \times 10^{-3}\frac{30G}{2.44G} \approx \boxed{84}
\end{aligned}$$

While near the field-independent point it is given by

$$\begin{aligned}
\Delta\left(\frac{E_1^+ - E_{-1}^-}{h}\right) &= \left|\frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{E_1^+ - E_{-1}^-}{h}\right)_{x=1.325\times 10^{-3}}\right|(\Delta x)^2 \\
&= 2a \times \frac{1}{2}\frac{3}{4}(\Delta x)^2 \\
&\approx 6.835 \times \frac{3}{8}\left(\frac{30G}{2.44G}\right)^2 \approx \boxed{0.4}
\end{aligned}$$

A modest adjustment of the bias field can thus narrow up the microwave transition by two orders of magnitude or, more realistically, reveal the next-largest line-broadening mechanism.

3 Atomic g factors (5 pts.)

Find g factors for the following states of Na ($I = 3/2$):

$$\begin{array}{ll}
{}^2P_{1/2} & F = 1, 2 \\
{}^2P_{3/2} & F = 0, 1, 2, 3 \\
{}^2S_{1/2} & F = 1, 2
\end{array}$$

Can you find the g factors for the states of maximum angular momentum (so-called stretched states ${}^2P_{3/2}, F = 3$ and ${}^2S_{1/2}, F = 2$) without resorting to the formula for the g factor derived in class?

Note on grading: this problem is out of 5. 4 point is for correct values of g, and 1 point is for getting the g factor for ${}^2S_{1/2}$, $F = 2$ and ${}^2P_{3/2}$, $F = 3$ states from physical picture, without using formula.

As derived in the class, the general formula for the Landé g factor is

$$g = \left(1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}\right)\left(\frac{F(F+1) + J(J+1) - I(I+1)}{2F(F+1)}\right)$$

By plugging in numbers, we obtain

$$
\begin{aligned}
{}^2P_{1/2} &: \begin{cases} -1/6 & F=1 \\ 1/6 & F=2 \end{cases} \\
{}^2P_{3/2} &: 2/3 \quad F=1,2,3 \\
{}^2S_{1/2} &: \begin{cases} -1/2 & F=1 \\ 1/2 & F=2 \end{cases}
\end{aligned}
$$

The formula gives $g = 2/3$ for ${}^2P_{3/2}, F = 0$ but this is meaningless since the $F = 0$ state has no magnetic moment.

For the stretched states the formula is unnecessary: all the angular momenta are then aligned with each other and their magnetic moments just add. Thus for ${}^2S_{1/2}, F = 2$ we have a total magnetic dipole μ_0 from the electron spin and a total angular momentum $F = 2$, so the g factor is $1/2$; while for ${}^2P_{3/2}, F = 3$ we have a total magnetic dipole $2\mu_0$ (one each from electron spin and orbital angular momentum) and a total angular momentum $F = 3$ for a g factor of $2/3$.