8.422 Pset 4 Solution, 2023

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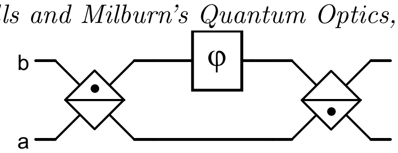
This solution benefits a great deal from the work of the previous TA (Joshua Ramette in 2021, and possibly many more in earlier years).

1 Problem 1 Better Phase Measurements with Squeezed Vacuum

This problem was adapted from Walls and Milburn's Quantum Optics, section 8.3.2



First, let's follow the mode operators through the interferometer, labelling the modes from left to right as *in*, *mid* and *out*.

$$b_{mid} = \frac{a_{in}+b_{in}}{\sqrt{2}}e^{i\phi} \qquad a_{mid} = \frac{b_{in}-a_{in}}{\sqrt{2}} \tag{1}$$

$$\begin{aligned} b_{out} = \frac{b_{mid}-a_{mid}}{\sqrt{2}} &= \frac{a_{in}}{2}\left(e^{i\phi}+1\right)+\frac{b_{in}}{2}\left(e^{i\phi}-1\right) && (2)\\ &= e^{-i\phi/2}\left(a_{in}\cos\frac{\phi}{2}+ib_{in}\sin\frac{\phi}{2}\right) && (3)\\ a_{out} = \frac{a_{mid}+b_{mid}}{\sqrt{2}} &= \frac{a_{in}}{2}\left(e^{i\phi}-1\right)+\frac{b_{in}}{2}\left(e^{i\phi}+1\right) && (4)\\ &= e^{-i\phi/2}\left(ia_{in}\sin\frac{\phi}{2}+b_{in}\cos\frac{\phi}{2}\right) && (5)\end{aligned}$$

Note depending on your convention for each beam splitter, you can get a phase, or you may swap the place of the cos and sin at your output mode.

1.1 (a)

Calculate the output signal $\langle M\rangle = \langle b_{out}^\dagger b_{out} - a_{out}^\dagger a_{out}\rangle$ and $\langle \Delta M^2\rangle$ using a coherent state in mode a and vacuum in mode b. Calculate the Signal-to-Noise Ratio (SNR) for this measurement, $\langle M\rangle/\sqrt{\langle\Delta M^2\rangle}$.

To find $\langle M \rangle$ and $\langle \Delta M^2 \rangle$, first calculate $\langle N_{out,b} \rangle$ and $\langle N_{out,a} \rangle$

$$\begin{aligned}
\langle N_{out,b} \rangle = \langle b^\dagger_{out} b_{out} \rangle &= \langle a^\dagger_{in} a_{in} \rangle \cos^2 \frac{\phi}{2} + \langle b^\dagger_{in} b_{in} \rangle \sin^2 \frac{\phi}{2} umber \\
&\quad + i \left(\langle a^\dagger_{in} b_{in} \rangle - \langle a_{in} b^\dagger_{in} \rangle \right) \sin \frac{\phi}{2} \cos \frac{\phi}{2} && (6) \\
\langle N_{out,a} \rangle = \langle a^\dagger_{out} a_{out} \rangle &= \langle a^\dagger_{in} a_{in} \rangle \sin^2 \frac{\phi}{2} + \langle b^\dagger_{in} b_{in} \rangle \cos^2 \frac{\phi}{2} umber \\
&\quad - i \left(\langle a^\dagger_{in} b_{in} \rangle - \langle a_{in} b^\dagger_{in} \rangle \right) \sin \frac{\phi}{2} \cos \frac{\phi}{2} && (7) \\
\langle M \rangle = \langle N_{out,b} \rangle - \langle N_{out,a} \rangle &= \left(\cos^2 \frac{\phi}{2} - \sin^2 \frac{\phi}{2} \right) \left(\langle a^\dagger_{in} a_{in} \rangle - \langle b^\dagger_{in} b_{in} \rangle \right) umber \\
&\quad + 2i \left(\langle a^\dagger_{in} b_{in} \rangle - \langle a_{in} b^\dagger_{in} \rangle \right) \cos \frac{\phi}{2} && (8) \\
&= \cos\phi \left(\langle N_{in,a} \rangle - \langle N_{in,b} \rangle \right) umber \\
&\quad + i \sin\phi \left(\langle a^\dagger_{in} \rangle \langle b_{in} \rangle - \langle a_{in} \rangle \langle b^\dagger_{in} \rangle \right) && (9) \\
&= \boxed{\cos\phi \left(|\alpha|^2 \right)} && (10)
\end{aligned}$$

where we have used the fact that $\langle N_{in,a} \rangle = |\alpha|^2$, $\langle N_{in,b} \rangle = 0$, and that $\langle b_{in} \rangle = \langle b^\dagger_{in} \rangle = 0$. Note that we can commute a_{in} and b_{in} because the inputs are not entangled. Note depending on your convention of beam splitter, this result may differ by a sign. However, it does not affect the qualitative discussion below. In real experiment, it depends on how the phase is shifted when light goes through the coatings in your beam splitter.

Next we need $\langle \Delta M^2 \rangle = \langle M^2 \rangle - \langle M \rangle^2$. Substituting the expressions for $a_{out}, a^\dagger_{out}$ and $b_{out}, b^\dagger_{out}$ into $\langle M^2 \rangle = \langle (b^\dagger_{out} b_{out} - a^\dagger_{out} a_{out})^2 \rangle$ gives a general expression which we will use here and later in the problem. In 3 (a) this is relatively simple because mode b_{in} is in the vacuum state and most terms disappear. In 3 (c) this will be a little more difficult since there are more terms in the case of the squeezed vacuum, but by remembering that the squeezed vacuum state entering mode b_{in} contains only even photon number states, after expanding M^2 you can ignore all terms with odd combinations of b_{in} and/or $b^\dagger_{in}$. After ignoring these odd combinations the expression looks like the following, which is still general for both vacuum and squeezed vacuum in mode b:

$$\begin{aligned}
\langle M^2 \rangle &= \cos^2\phi \left[\langle N^2_{in,a} \rangle - 2 \langle N_{in,a} \rangle \langle N_{in,b} \rangle + \langle N^2_{in,b} \rangle \right] umber \\
&\quad - \sin^2\phi \left[\langle a^{\dagger 2}_{in} \rangle \langle b^2_{in} \rangle - \langle a^\dagger_{in} a_{in} \rangle \langle b_{in} b^\dagger_{in} \rangle - \langle a_{in} a^\dagger_{in} \rangle \langle b^\dagger_{in} b_{in} \rangle + \langle a^2_{in} \rangle \langle b^{\dagger 2}_{in} \rangle \right] && (11)
\end{aligned}$$

The, evaluating this for the case where the coherent state is in mode a and the vacuum in mode b gives:

$$\begin{aligned}
\langle M^2 \rangle &= \cos^2\phi \left[\langle N^2_{in,a} \rangle - 2 \langle N_{in,a} \rangle \langle N_{in,b} \rangle + \langle N^2_{in,b} \rangle \right] - \sin^2\phi \left[\langle N_{in,a} \rangle \right] && (12) \\
\langle \Delta M^2 \rangle &= \cos^2\phi \left[\langle \Delta N^2_{in,b} \rangle + \langle \Delta N^2_{in,a} \rangle \right] - \sin^2\phi \left[\langle N_{in,a} \rangle \right] && (13) \\
&= \cos^2\phi |\alpha|^2 + \sin^2\phi |\alpha|^2 = \boxed{|\alpha|^2} && (14)
\end{aligned}$$

Using these results, the SNR is then:

$$\boxed{\frac{\langle M \rangle}{\sqrt{\langle \Delta M^2 \rangle}} = |\alpha| \cos\phi} \qquad (15)$$

1.2 (b)

Find ϕ_{min} for (a), i.e. the smallest ϕ for which the SNR is 1.

When the SNR=1 and expanding around the highest resolution point where $\phi \approx \frac{\pi}{2}$, we have

$$\cos\left(\phi - \frac{\pi}{2}\right) \approx \boxed{\phi_{min} = \frac{1}{|\alpha|}} \qquad (16)$$

1.3 (c)

Repeat (a-b) using squeezed vacuum in port b ($|a\rangle = |\alpha\rangle$ and $|b\rangle = S(\epsilon)|0\rangle$). What degree of squeezing do you need to make $\phi_{min,\ squeezed} = \frac{1}{2}\phi_{min,\ coherent}$? Find the intermediate average photon number, $\langle L\rangle = \langle b^\dagger_{mid} b_{mid} + a^\dagger_{mid} a_{mid}\rangle$. Does $\langle L\rangle$ depend on squeezing? (Note that (a,b) are special cases of (c) where $\epsilon = 0$).

We again can use Eq. 11 which we derived in full generality only assuming an even number of photons sent into mode b to reduce the number of terms (which is the case for squeezed vacuum):

$$\begin{aligned}\langle M^2\rangle &= \cos^2\phi\left[\langle N^2_{in,a}\rangle - 2\langle N_{in,a}\rangle\langle N_{in,b}\rangle + \langle N^2_{in,b}\rangle\right] umber \\ &\quad - \sin^2\phi\left[\langle a^{\dagger 2}_{in}\rangle\langle b^2_{in}\rangle - \langle a^\dagger_{in} a_{in}\rangle\langle b_{in} b^\dagger_{in}\rangle - \langle a_{in} a^\dagger_{in}\rangle\langle b^\dagger_{in} b_{in}\rangle + \langle a^2_{in}\rangle\langle b^{\dagger 2}_{in}\rangle\right] \qquad (17)\end{aligned}$$

We now just have to evaluate various expectations of raising and lowering operators of modes a and b using a coherent state and squeezed vacuum. This can be done using our result from problem 1 (b) where we have an expression in the Heisenberg picture for the squeezed raising and lower operators $b_{in}, b^\dagger_{in}$ in terms of the unsqueezed operators $b_0, b^\dagger_0$:

$$\begin{aligned} b_{in} &= b_0\cosh\epsilon + b^\dagger_0 \sinh\epsilon umber \\ b^\dagger_{in} &= b_0\sinh\epsilon + b^\dagger_0\cosh\epsilon umber \end{aligned} \qquad (18)$$

The expectations then come out as (you should have already calculated these in problems 2(a) and 2(c)):

$$\begin{aligned} \langle b^\dagger_{in} b_{in}\rangle &= \sinh^2\epsilon && (19) \\ \langle b_{in} b^\dagger_{in}\rangle &= \cosh^2\epsilon umber \\ \langle b^\dagger_{in} b^\dagger_{in}\rangle &= \langle b_{in} b_{in}\rangle = \cosh\epsilon\sinh\epsilon umber \\ \langle N^2_{in,b}\rangle &= \langle b^\dagger_{in} b_{in} b^\dagger_{in} b_{in}\rangle = \sinh^4\epsilon + \frac{1}{4}(\cosh 4\epsilon - 1) umber \\ \langle N^2_{in,a}\rangle &= \langle a^\dagger_{in} a_{in} a^\dagger_{in} a_{in}\rangle = |\alpha|^2(|\alpha|^2+1) && (20) \\ & && (21) \end{aligned}$$

For the squeezed vacuum input we then have

$$\boxed{\langle M\rangle = \cos\phi\left(|\alpha|^2 - \sinh^2\epsilon\right)}$$

. Note that this gives us the result we got in part (i) for $\epsilon = 0$, as it should.

Calculating $\langle \Delta M^2\rangle$ from this point just involves plugging in our coherent state and squeezed vacuum expectation values:

$$
\begin{aligned}
\langle M^2\rangle &= \cos^2\phi\left[\langle N_{in,a}^2\rangle - 2\langle N_{in,a}\rangle\langle N_{in,b}\rangle + \langle N_{in,b}^2\rangle\right] umber \\
&\quad - \sin^2\phi\left[\langle a_{in}^{\dagger 2}\rangle\langle b_{in}^2\rangle - \langle a_{in}^\dagger a_{in}\rangle\langle b_{in} b_{in}^\dagger\rangle - \langle a_{in} a_{in}^\dagger\rangle\langle b_{in}^\dagger b_{in}\rangle + \langle a_{in}^2\rangle\langle b_{in}^{\dagger 2}\rangle\right] umber \\
&= \cos^2\phi\left[|\alpha|^2(|\alpha|^2+1) - 2|\alpha|^2\sinh^2\epsilon + \sinh^4\epsilon + \frac{1}{4}(\cosh 4\epsilon - 1)\right] umber \\
&\quad - \sin^2\phi\left[\alpha^{*2}\cosh\epsilon\sinh\epsilon - |\alpha|^2\cosh^2\epsilon - (|\alpha|^2+1)\sinh^2\epsilon + \alpha^2\cosh\epsilon\sinh\epsilon\right] umber \\
\langle \Delta M^2\rangle &= \cos^2\phi\left[|\alpha|^2(|\alpha|^2+1) - 2|\alpha|^2\sinh^2\epsilon + \sinh^4\epsilon + \frac{1}{4}(\cosh 4\epsilon - 1)\right] umber \\
&\quad - \sin^2\phi\left[\alpha^{*2}\cosh\epsilon\sinh\epsilon - |\alpha|^2\cosh^2\epsilon - (|\alpha|^2+1)\sinh^2\epsilon + \alpha^2\cosh\epsilon\sinh\epsilon\right] umber \\
&\quad - \cos^2\phi[|\alpha|^4 - 2|\alpha|^2\sinh^2\epsilon + \sinh^4\epsilon] umber \\
&= \cos^2\phi\left[|\alpha|^2 + \frac{1}{4}(\cosh 4\epsilon - 1)\right] umber \\
&\quad - \sin^2\phi\left[(\alpha^{*2}+\alpha^2)\cosh\epsilon\sinh\epsilon - |\alpha|^2\cosh^2\epsilon - (|\alpha|^2+1)\sinh^2\epsilon\right] umber \\
&= \cos^2\phi\left[|\alpha|^2 + \frac{1}{4}(\cosh 4\epsilon - 1)\right] umber \\
&\quad - \sin^2\phi\left[\alpha^2(2\cosh\epsilon\sinh\epsilon - \cosh^2\epsilon - \sinh^2\epsilon) - \sinh^2\epsilon\right] umber \\
&= \cos^2\phi\left[|\alpha|^2 + \frac{1}{4}(\cosh 4\epsilon - 1)\right] umber \\
&\quad + \sin^2\phi\left[\alpha^2 e^{-2\epsilon} + \sinh^2\epsilon\right] umber
\end{aligned}
$$

Where we have assumed α real and used $2\cosh\epsilon\sinh\epsilon - \cosh^2\epsilon - \sinh^2\epsilon = -e^{-2\epsilon}$.
The SNR is then

$$
\frac{\langle M\rangle}{\sqrt{\langle \Delta M^2\rangle}} = \frac{\cos\phi\left(\alpha^2 - \sinh^2\epsilon\right)}{\sqrt{\cos^2\phi\left[\alpha^2 + \frac{1}{4}(\cosh 4\epsilon - 1)\right] + \sin^2\phi\left[\alpha^2 e^{-2\epsilon} + \sinh^2\epsilon\right] umber}}
$$

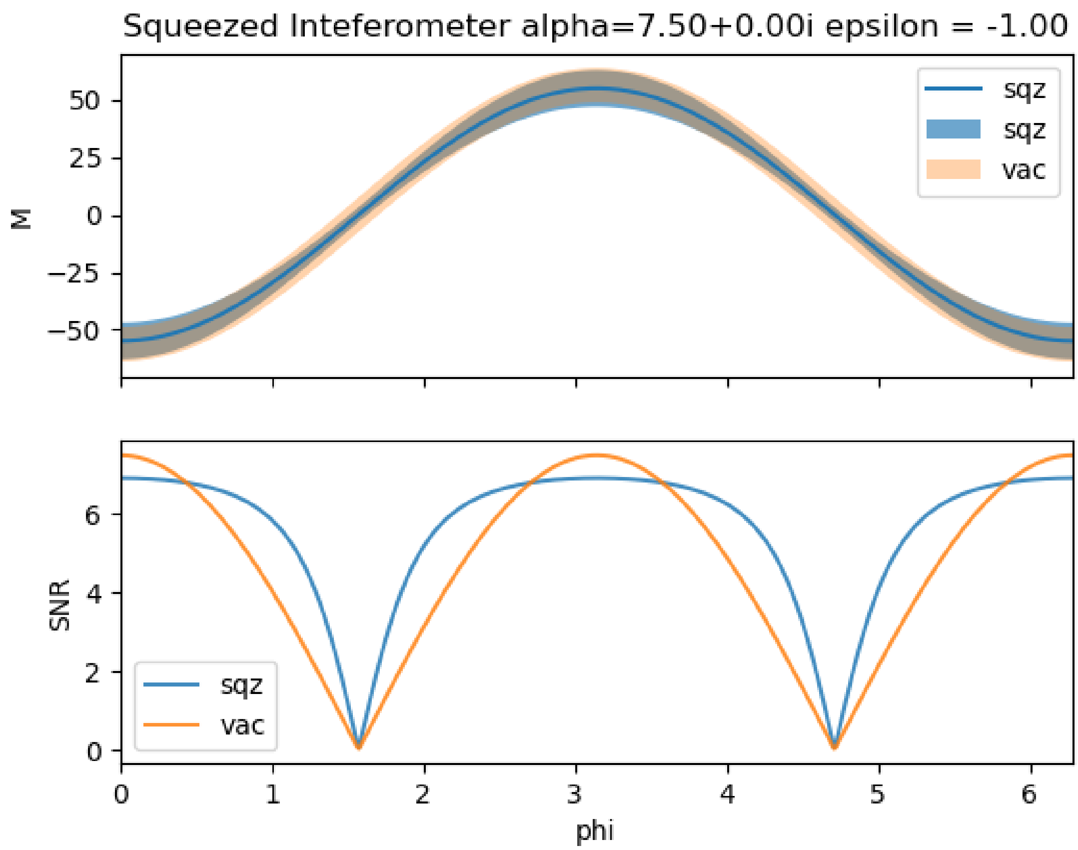
Further assume moderate squeezing and a strong coherent beam (i.e. assuming terms with α dominate terms with just ϵ, so that $\alpha^2 \gg \sinh^2\epsilon$ and $\alpha^2 e^{-2\epsilon} \gg \sinh^2\epsilon$) and again expand the denominator around the most sensitive angle $\phi \approx \frac{\pi}{2}$ so that $\cos\phi \approx \phi$ and $\sin\phi \approx 1$. For ϕ very close to $\pi/2$ we then have $\cos^2\phi \approx 0$ and the SNR is then approximately:

$$
\boxed{\frac{\langle M\rangle}{\sqrt{\langle \Delta M^2\rangle}} = \alpha e^{\epsilon}\cos\phi} \tag{22}
$$

and the angle at which the SNR = 1 is then $\boxed{\phi_{min} = \frac{1}{e^{\epsilon}\alpha}}$.

Comparing this with the result we had in (ii), we find we need a squeezing parameter $\epsilon = \ln(2)$ to increase our sensativity by a factor of 2 (so that $\phi_{min,squeezed} = \frac{1}{2}\phi_{min,coherent}$).

The following figure demonstrates how the SNR differs between sending a squeezed state vs a vaccum state through port b.



1.4 (d)

Find the minimal differential length change and the minimal detectable strain. Calculate these for 5W of 1064nm light, 4 km path length and coherent vacuum input in mode b, as well as for a 6dB squeezed vacuum in mode b for a measurement time t.

The minimum differential length is

$$\boxed{dL = \lambda\phi_{min}/(2\pi)}$$

and the minimal detectable strain is

$$\boxed{\frac{dL}{L} = \lambda\phi_{min}/(2\pi L)}$$

. For our coherent laser beam, we have $|\alpha|^2 = \bar{N} = \frac{5\mathrm{W}\ t\lambda}{hc} \approx 2.68 \times 10^{19}t$.

For a coherent state, putting in our numbers gives

$$\boxed{\phi_{min,coherent} = \frac{1}{|\alpha|} \approx 0.2 \times 10^{-9}/\sqrt{t}}$$

,

$$\boxed{dL_{coherent} = \frac{\lambda\phi_{min}}{2\pi} \approx 33\mathrm{am}/\sqrt{t}}$$

(attometers), and

$$\boxed{\frac{dL_{coherent}}{L} \approx 8.2 \times 10^{-21}/\sqrt{t}}$$

.

For a 6dB squeezed state, all of the above answers are multiplied by a factor of $e^r \approx \frac{1}{2}$ where we take $6 \approx -8.7\epsilon$ from problem 2b (choosing negative ϵ to pick the squeezing quadtrature that increases our sensativity):

$$\boxed{\phi_{min,squeezed} \approx 0.1 \times 10^{-9}/\sqrt{t},\ dL_{squeezed} \approx 17\text{am}/\sqrt{t},\text{ and } \frac{dL_{squeezed}}{L} \approx 4.1 \times 10^{-21}/\sqrt{t}}$$

.

2 Problem 2: Hanbury Brown and Twiss Experiment with Atoms

2.1 (a) Correlation function

This part is fairly straightforward substitution,

$$\begin{aligned} P &= \left|\psi_A e^{i\phi_{A1}}\psi_B e^{i\phi_{B2}} \pm \psi_A e^{i\phi_{A2}}\psi_B e^{i\phi_{B1}}\right|^2 \\ &= |\psi_A\psi_B|^2 \left|e^{i(\mathbf{k}_A\cdot\mathbf{r}_{A1}-\omega\tau+\mathbf{k}_B\cdot\mathbf{r}_{B2}-\omega\tau)} \pm e^{i(\mathbf{k}_A\cdot\mathbf{r}_{A2}-\omega\tau+\mathbf{k}_B\cdot\mathbf{r}_{B1}-\omega\tau)}\right|^2 \\ &= 2|\psi_A\psi_B|^2 \left[1 \pm \cos(\mathbf{k}_A\cdot(\mathbf{r}_{A2}-\mathbf{r}_{A1}) + \mathbf{k}_B\cdot(\mathbf{r}_{A1}-\mathbf{r}_{A2}))\right] \\ &= 2|\psi_A\psi_B|^2 \left[1 \pm \cos((\mathbf{k}_A-\mathbf{k}_B)\cdot\mathbf{r}_{21})\right] \end{aligned} \tag{23}$$

2.2 (b) Transverse Collimation

What does it mean to "see a second-order correlation effect"? As discussed in class, the second order correlation function is a measure of the probability of detecting *two* particles (e.g. atoms or photons) within a certain distance/time of each other. Due to the Pauli exclusion principle, it should be relatively *less* likely to detect two fermions 'close' to each other in space or time, and due to bosonic enhancement, relatively *more* likely to detect two bosons 'close' to each other. We can see this explicitly occurring in Equation (23). As long as the distance between points of detection, $\mathbf{r}_{21}$ is small enough that $(\mathbf{k}_A-\mathbf{k}_B)\cdot\mathbf{r}_{21} \equiv \phi_t \ll 2\pi \Longrightarrow \cos \simeq 1$ it indicates an increased probability of detecting two bosons and a decreased probability of detecting two fermions.

When we perform an actual experiment, unless $\phi_t \ll 2\pi$ for all the different wavevector pairs coming from the source and all the different possible detected positions, the cosine term will average to zero and we will see no difference in probability for bosons versus fermions. The maximum difference in transverse wavevector, for particles coming opposite edges of the cloud but arriving at nearly the same point on the detector, is $(\mathbf{k}_A-\mathbf{k}_B)_{max} \simeq k_0\frac{W}{d}$. The maximum separation between detected particles is given by the width of the detector $(\mathbf{r}_{21})_{max} \simeq w$. So to ensure that $\phi_t \ll 2\pi$, we must have: $k_0\frac{W}{d}w \ll 2\pi$,or

$$Ww \ll \lambda_{dB} d \text{ where } \lambda_{dB} = \frac{2\pi}{k_0}.$$

Another way to approach the problem is to determine under what conditions the two particles being detected lie within a single phase space cell, (i.e. $\delta x\delta p \ll h$) which, roughly speaking, means that they are detected in the same quantum state. Since two fermions cannot occupy the same quantum state, we would expect to see second order correlation effects in this case. Substituting $\mathbf{r}_{21}$ for δx and $\hbar(\mathbf{k}_A-\mathbf{k}_B)$ for δp, it is clear that we will arrive at the same result given above.

The deBroglie wavelength at 500 μK is $\lambda_{dB} = h/\sqrt{2\pi m k_B T} = 32$ nm. Given $d = 10$cm, this means $W, w \ll 56\mu$m.

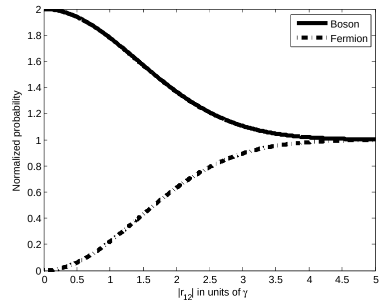
2.3 (c) Longitudinal Collimation

(i) Average P over the given wavevector distribution, normalizing appropriately:

$$\langle P\rangle = 2|\psi_A\psi_B|^2 \frac{\int_{-\infty}^{\infty}(1\pm\cos((\mathbf{k}_A-\mathbf{k}_B)\cdot\mathbf{r}_{21}))\, e^{-|\mathbf{k}_A-\mathbf{k}_B|^2\gamma^2} d(\mathbf{k}_A-\mathbf{k}_B)}{\int_{-\infty}^{\infty} e^{-|\mathbf{k}_A-\mathbf{k}_B|^2\gamma^2} d(\mathbf{k}_A-\mathbf{k}_B)}.$$

We could proceed by integrating separately over the transverse and longitudinal wavevector components: $d(\mathbf{k}_A-\mathbf{k}_B) = d(\mathbf{k}_A-\mathbf{k}_B)_t d(\mathbf{k}_A-\mathbf{k}_B)_l$, but we can avoid some work by using the condition we derived above $\phi_t \equiv (\mathbf{k}_A-\mathbf{k}_B)_t\cdot(\mathbf{r}_{21})_t \ll 2\pi$. This means we can neglect any transverse contribution to the cosine, leaving numerator and denominator with the same ϕ_t dependence, and therefore cancels. The remaining (longitudinal) integral is straightforward:

$$\langle P\rangle = 2|\psi_A\psi_B|^2 \left[1 \pm \frac{\int_{-\infty}^{\infty}\cos(\delta k_l(\mathbf{r}_{21})_l)\, e^{-\delta k_l^2\gamma^2} d(\delta k_l)}{\int_{-\infty}^{\infty} e^{-\delta k_l^2\gamma^2} d(\delta k_l)}\right] = 2|\psi_A\psi_B|^2 \left[1 \pm e^{-(\mathbf{r}_{12})_l^2/4\gamma^2}\right]$$



Spatial correlation effects can be seen for $(\mathbf{r}_{12})_l \leq 2\gamma$.

(ii) The second half of this part is just geometry. Because we have a pulsed source, all the atoms are localized within a longitudinal distance L at time $t = 0$. For two atoms in different parts of the cloud to reach the detector at the same later time $t = \tau$, it must be the case that the one starting further away had a larger enough velocity to 'catch-up' to the one starting closer. The largest difference in velocity between two particles arriving simultaneously at the detector will occur when one particle is from the "front" of the cloud, $v_{min} = d/\tau$, while the other is from the "back", $v_{max} = (d + L)/\tau$. Thus the difference in velocity, Δv, between any two particles detected at the same time must be $\leq L/\tau$. Using the relation $\hbar k = mv$, we find:

$$(\mathbf{k}_A - \mathbf{k}_B)_l \leq \frac{mL}{\hbar\tau} = \frac{mvL}{\hbar d},$$

written in terms of the "average" velocity $v = \frac{d}{\tau}$.

By the same arguments as given in part (b) above, in order to see second order correlation effects, we must have $\phi_l \equiv (\mathbf{k}_A - \mathbf{k}_B)_l(\mathbf{r}_{21})_l \ll 2\pi$. Since we have assumed that our detector has no longitudinal extent, $(\mathbf{r}_{21})_l = 0$ and this condition is trivially satisfied. In other words, all of the atoms detected at a particular time are guaranteed to be in the same longitudinal phase space cell. If, however, our detector has some finite response time, t_r, then we can attribute to it an 'effective' length $(\mathbf{r}_{21})_l = vt_r$. Now, in order to see second order correlations, we must have $(\mathbf{k}_A - \mathbf{k}_B)_l\,(\mathbf{r}_{21})_l \leq \frac{mvL}{\hbar d} vt_r \ll 2\pi$, or,

$$t_r \ll \frac{hd}{mLv^2} = 0.12 \text{ ms}$$

2.4 (d) Phase-Space Volume Enhancement

The initial phase space cell volume is $\delta x \delta y \delta z = \lambda_{dB}^3$.

In each dimension, we know that the uncertainty in position and momentum corresponding to a single phase space cell is given by $\delta k \delta r \simeq 2\pi$.

From part (b), in each of the two transverse dimensions we have $(\mathbf{k}_A - \mathbf{k}_B)_t = \delta k_t \simeq k_0 W/d$. Thus:

$$\delta x_t = \frac{2\pi}{\delta k_t} = \frac{d}{W}\frac{2\pi}{k_0} = \frac{d}{W}\lambda_{dB}.$$

From part (c), in the longitudinal direction we have $(\mathbf{k}_A - \mathbf{k}_B)_l = \delta k_l \simeq \frac{mvL}{\hbar d} = k_0 L/d$. Thus:

$$\delta x_l = \frac{2\pi}{\delta k_l} = \frac{d}{L}\frac{2\pi}{k_0} = \frac{d}{L}\lambda_{dB}.$$

Our new phase space cell volume (after expansion of the cloud) is $\delta x_t^2 \delta x_l = \frac{d^3}{W^2 L}\lambda_{dB}^3$. So the phase space volume has increased by $\frac{d^3}{W^2 L} \approx 10^{12}$ for $d = 10$cm and $L \approx W \approx 10\mu$m.

The phase space density of the Lithium MOT is given by $n\lambda_{DB}^3$. Using the given numbers, the phase space density is $\sim 10^{-7}$, much lower than the point of degenearcy of $n\lambda_{DB}^3 \sim 1$.