Homework 4 8.370/18.435/2.111 Due 10/12

1. Consider a quantum gate on two qubits that applies a σ_x to the second qubit if the first qubit is $|0\rangle$ and a σ_z to the second qubit if the first qubit is $|1\rangle$.

   (a) Write down a 4×4 matrix for the action of this gate.

   Solution:

   The action of this gate is

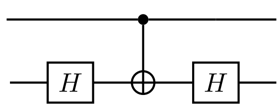
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

   The top left corner gives the action when the first qubit is $|0\rangle$ and the bottom right corner when it is $|1\rangle$.

   (b) Show how to build a quantum circuit for this gate using CNOT gates and one-qubit gates.

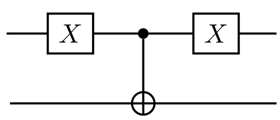
   Solution:

   What we will do is first apply the σ_z if the first qubit is 1, and then apply the σ_x if the first qubit is 0. For the first, recall that the circuit

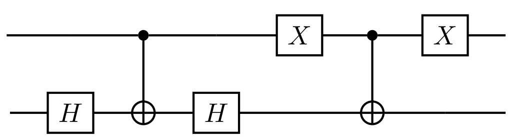


   performs a controlled Z. If the first qubit is $|0\rangle$, the second qubit has two H gates applied, and their product is I. If the first qubit is $|1\rangle$, the second qubit has $HXH = Z$ applied.

   Now, we need to figure out how to build a CNOT circuit which applies the X operation to the second qubit when the first qubit is $|0\rangle$ rather than $|1\rangle$. The solution to this is to change $|0\rangle$ to $|1\rangle$ on the first qubit, and then change it back after the CNOT gate. This gives the quantum circuit:



   Thus, putting everything together, we get the quantum circuit:



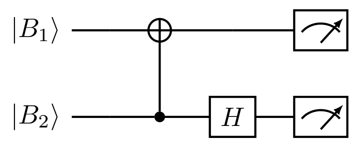
2. Suppose that we have three parties, Alice, Bob, and Charlie. Alice and Bob share a pair of qubits in the Bell state

$$\frac{1}{\sqrt{2}}\big(|00\rangle + |11\rangle \big)$$

   Bob and Charlie also share a pair of qubits in that Bell state. (So between all three of them, they hold two Bell states consisting of four qubits total.)

   Alice, Bob, and Charlie would like to perform some operations so that Alice and Charlie end up with a pair of qubits in this Bell state. Can they? If so, how? (They are allowed to measure any of their original qubits that they want.)

Solution: Here we apply quantum teleportation. Denote the EPR pair between Alice and Bob as $|A\rangle, |B_1\rangle$, and the EPR pair between Bob and Charlie as $|B_2\rangle, |C\rangle$. We wish to teleport $|B_2\rangle$ to Alice, at the cost of the first EPR pair. Consider the following quantum circuit.



Now Bob can send the measurement result from $|B_1\rangle, |B_2\rangle$ to Alice, and Alice will apply different gates on $|A\rangle$. If $B_1B_2 = 00$, we apply I; 01, apply X; 10, apply Z; 11, apply ZX. One can check that after these operations, the resulting qubits $|A\rangle, |C\rangle$ forms an EPR pair.

3. Alice has two classical bits, a_1 and a_2. Bob has two classical bits, b_1 and b_2. They would like to send Charlie two classical bits which are functions of their bits, namely $c_1 = a_1 \oplus b_1$ and $c_2 = a_2 \oplus b_2$, where $\oplus$ is the XOR operation. However, they have only very limited resources to do this. Alice and Charlie share a pair of entangled bits in the state

$$\frac{1}{\sqrt{2}}\big(\,|00\rangle + |11\rangle\,\big)$$

They are allowed to send one qubit from Alice to Bob, and one qubit from Bob to Charlie (and none of them are allowed to send any classical bits). Can they accomplish their task? If so, how?

Solution: Their resources are limited enough that there seems to be only one thing to do: Alice encodes her two bits using superdense coding and sends her half of the EPR pair to Bob. Bob then alters them by applying a unitary transform and sends the qubit on to Charlie. Charlie then decodes using superdense coding. So the question is: can they find an encoding scheme so that Bob can apply a unitary transform that does this. In fact, he can. (There are other very similar situations where he couldn't; for example, if Alice and Bob had two numbers modulo 4, and they wanted Bob to get the sum.)

Recall the the four operations that the sender makes in superdense coding are id, σ_x, σ_y, and σ_z. Recall further than $\sigma_y = -i\sigma_x\sigma_z$. Alice can encode $(0,0)$ by the identity, $(1,0)$ by σ_x, $(0,1)$ by σ_z, and $(1,1)$ by σ_y. However, since global phases don't affect quantum states, she could just as easily encode $(1,1)$ by $\sigma_x\sigma_z$ or $\sigma_z\sigma_x$.

One way to do it is encode $a_1 = 1$ by applying σ_x to Alice's half of the EPR pair and $a_2 = 1$ by applying σ_z to it. Now, Bob wants to change the EPR pair so that it encodes $a_1 \oplus b_1$ and $a_2 \oplus b_2$. He can do this by applying σ_x to the qubit he receives from Alice if $b_1 = 1$ and applying σ_z if $b_2 = 1$. Charlie can then decode the two bits by doing a Bell basis measurement on the qubit he receives and his half of the EPR pair.

4. Recall that on an earlier homework, we considered a scenario where a challenger gave you an unknown quantum state which was either $|0\rangle$, $|+\rangle$, $|1\rangle$, $|-\rangle$, and challenged you to clone it. You give them back two qubits, and they measure both of them to check whether they were both the original state. You succeed if both the measurement results agree with the original state. There was a strategy that succeeded with probability $\frac{5}{8}$. In this problem, you will analyze a batter strategy, that succeeds with a larger probability.

Suppose we have a mapping B that embeds a single qubit into a subspace of a three-qubit system that behaves as follows (and takes superpositions of the inputs to superpositions of the outputs, as you would expect):

$$B|0\rangle = \frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|101\rangle$$
$$B|1\rangle = \frac{\sqrt{2}}{\sqrt{3}}|111\rangle + \frac{1}{\sqrt{6}}|100\rangle + \frac{1}{\sqrt{6}}|010\rangle$$

You take the qubit they give you and apply B, then give them the first two qubits of the result. What is the probability that you pass the chalenger's test?

In fact, this is the optimal cloning strategy ... you cannot succeed with a higher probability.

Solution: Suppose that they give us $|0\rangle$. To compute the probability that we pass their test, we need to compute

$$\begin{aligned} {}_{12}\langle 00|\, B\,|0\rangle &= {}_{12}\langle 00| \left(\frac{\sqrt{2}}{\sqrt{3}}|000\rangle_{123} + \frac{1}{\sqrt{6}}|011\rangle_{123} + \frac{1}{\sqrt{6}}|101\rangle_{123} \right) \\ &= \frac{\sqrt{2}}{\sqrt{3}}|0\rangle_3 \end{aligned}$$

Squaring the length of this vector gives you the probability that the see $|00\rangle$, so you pass their test with probability $\frac{2}{3}$.

The case with $|1\rangle$ is entirely symmetric, since you're just replacing 0s with 1s in the quantum state. So in this case, you also pass with probability $\frac{2}{3}$.

Now, let's suppose they give you $|+\rangle$. You give them the first qubits of the state $B|+\rangle$, which is

$$\frac{1}{\sqrt{3}}|000\rangle + \frac{1}{2\sqrt{3}}|011\rangle + \frac{1}{2\sqrt{3}}|101\rangle + \frac{1}{\sqrt{3}}|111\rangle + \frac{1}{2\sqrt{3}}|100\rangle + \frac{1}{2\sqrt{3}}|010\rangle$$

Taking the inner product of

$${}_{12}\langle{+}{+}| = \frac{1}{2}\Big({}_{12}\langle 00| + {}_{12}\langle 01| + {}_{12}\langle 10| + {}_{12}\langle 11| \Big)$$

with this state gives

$$\frac{1}{\sqrt{3}}|0\rangle_3 + \frac{1}{\sqrt{3}}|1\rangle_3,$$

which also has length $\frac{\sqrt{2}}{\sqrt{3}}$, so this probability is also $\frac{2}{3}$.

And for $|-\rangle$, the inner product of

$${}_{12}\langle{-}{-}| = \frac{1}{2}\Big({}_{12}\langle 00| - {}_{12}\langle 01| - {}_{12}\langle 10| + {}_{12}\langle 11| \Big)$$

with

$$\frac{1}{\sqrt{3}}|000\rangle + \frac{1}{2\sqrt{3}}|011\rangle + \frac{1}{2\sqrt{3}}|101\rangle - \frac{1}{\sqrt{3}}|111\rangle - \frac{1}{2\sqrt{3}}|100\rangle - \frac{1}{2\sqrt{3}}|010\rangle$$

is

$$\frac{1}{\sqrt{3}}|0\rangle_3 + \frac{1}{\sqrt{3}}|1\rangle_3,$$

also giving probability $\frac{2}{3}$.

You can show that this strategy in fact will clone any state the challenger gives you with probability $\frac{2}{3}$.

5. You can implement the strategy in problem 2 by taking the qubit $|t\rangle$ they gave you, appending some two-qubit state $|\psi\rangle$, and applying the following circuit with four CNOT gates:

   pset4fig.pdf

   Find the state $|\psi\rangle$ and show that this circuit gives the desired results.

   Solution:

   The easy way to do this is to work backwards. Start with

$$B|0\rangle = \frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|101\rangle$$

   Now apply the last gate, $\mathrm{CNOT}(2,1)$ (which is its own inverse). We end up with the state

$$\mathrm{CNOT}(2,1)\Big(\frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|101\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|111\rangle + \frac{1}{\sqrt{6}}|101\rangle$$

   Then continuing, we have

$$\mathrm{CNOT}(3,1)\Big(\frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|111\rangle + \frac{1}{\sqrt{6}}|101\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|001\rangle$$

   and

$$\mathrm{CNOT}(1,3)\Big(\frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|001\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|001\rangle$$

   and finally we reach the initial state:

$$\mathrm{CNOT}(1,3)\Big(\frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|001\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}|000\rangle + \frac{1}{\sqrt{6}}|011\rangle + \frac{1}{\sqrt{6}}|001\rangle$$

One can do the same thing for

$$B\left|1\right\rangle = \frac{\sqrt{2}}{\sqrt{3}}\left|111\right\rangle + \frac{1}{\sqrt{6}}\left|100\right\rangle + \frac{1}{\sqrt{6}}\left|010\right\rangle$$

We have

$$\mathrm{CNOT}(2,1)\Big(\frac{\sqrt{2}}{\sqrt{3}}\left|111\right\rangle + \frac{1}{\sqrt{6}}\left|100\right\rangle + \frac{1}{\sqrt{6}}\left|010\right\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}\left|011\right\rangle + \frac{1}{\sqrt{6}}\left|100\right\rangle + \frac{1}{\sqrt{6}}\left|110\right\rangle$$

and

$$\mathrm{CNOT}(3,1)\Big(\frac{\sqrt{2}}{\sqrt{3}}\left|011\right\rangle + \frac{1}{\sqrt{6}}\left|100\right\rangle + \frac{1}{\sqrt{6}}\left|110\right\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}\left|111\right\rangle + \frac{1}{\sqrt{6}}\left|100\right\rangle + \frac{1}{\sqrt{6}}\left|110\right\rangle$$

and

$$\mathrm{CNOT}(1,3)\Big(\frac{\sqrt{2}}{\sqrt{3}}\left|111\right\rangle + \frac{1}{\sqrt{6}}\left|100\right\rangle + \frac{1}{\sqrt{6}}\left|110\right\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}\left|110\right\rangle + \frac{1}{\sqrt{6}}\left|101\right\rangle + \frac{1}{\sqrt{6}}\left|111\right\rangle$$

and finally we reach the initial state:

$$\mathrm{CNOT}(1,2)\Big(\frac{\sqrt{2}}{\sqrt{3}}\left|110\right\rangle + \frac{1}{\sqrt{6}}\left|101\right\rangle + \frac{1}{\sqrt{6}}\left|111\right\rangle\Big) = \frac{\sqrt{2}}{\sqrt{3}}\left|100\right\rangle + \frac{1}{\sqrt{6}}\left|111\right\rangle + \frac{1}{\sqrt{6}}\left|101\right\rangle$$

Comparing the two initial states, we see that:

$$\left|\psi\right\rangle = \frac{\sqrt{2}}{\sqrt{3}}\left|00\right\rangle + \frac{1}{\sqrt{6}}\left|11\right\rangle + \frac{1}{\sqrt{6}}\left|01\right\rangle$$