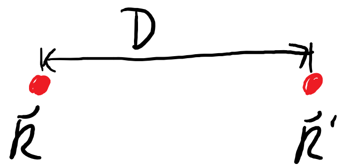
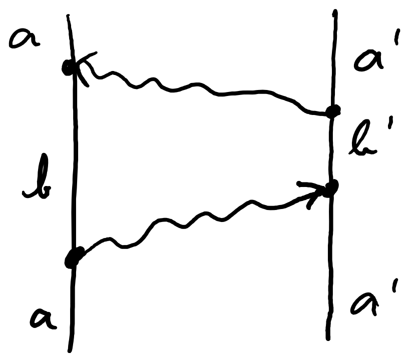
Interaction by photon exchange

— Van-der-Waals interaction



$$H_I' = -\vec{d}\cdot\vec{E}_\perp(\vec{R}) - \vec{d}\,'\cdot\vec{E}_\perp(\vec{R}')$$



conservation of energy
$\Rightarrow$ first process coupling the two atoms is a fourth order process

$\Rightarrow$ exchange of pairs of photons between the two atoms.
All intermediate states off-resonant.

Few low-frequency modes ($k^2 dk$ small, $\langle\cdot\rangle \propto \sqrt{\omega}$)
large wave-vectors interfere $e^{i\vec{k}\cdot\vec{R}}$, $e^{i\vec{k}\cdot\vec{R}'}$

$\Rightarrow$ dominant $k \sim \dfrac{1}{D}$.

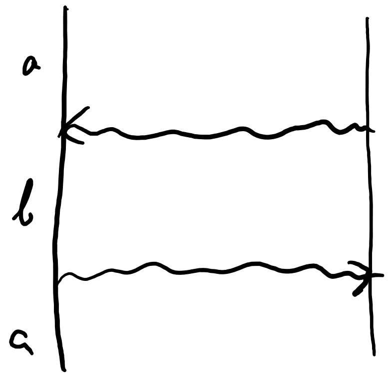
Two limits:

$D \ll \lambda_{ab} = \dfrac{\hbar c}{|E_a - E_b|}$: $\hbar\omega \gg \hbar\omega_0$.

photons exist for very short time.

$\Rightarrow$ horizontal photon lines

$D \ll \lambda_{ab}$



defect $\approx \hbar\omega$

It's like both atoms excited simultaneously.

effective Hamiltonian δV with

$$\langle b, b' | \delta V | a, a' \rangle =$$

$$= -\sum_{k\vec{\varepsilon}} \frac{1}{\hbar\omega} \langle b, b', 0 | H_I | b, a'; k\vec{\varepsilon} \rangle \langle b, a'; k\vec{\varepsilon} | H_I | a, a', 0 \rangle$$

$$= -\sum_{k\vec{\varepsilon}} \frac{1}{2\varepsilon_0 L^3} (\vec{d}\cdot\vec{\varepsilon})(\vec{d}'\cdot\vec{\varepsilon})\, e^{ik(R-R')} + h.c.$$

$$= -\frac{1}{\varepsilon_0} \sum_{ij} d_i d_j\, \delta^{\perp}_{ij}(R-R')$$

$$\delta_{ij}(R-R') = \frac{-\delta_{ij} + 3n_i n_j}{4\pi D^3} \qquad \sum_{ij} \varepsilon_i \varepsilon_j = \delta_{ij} - \frac{k_i k_j}{k^2}$$

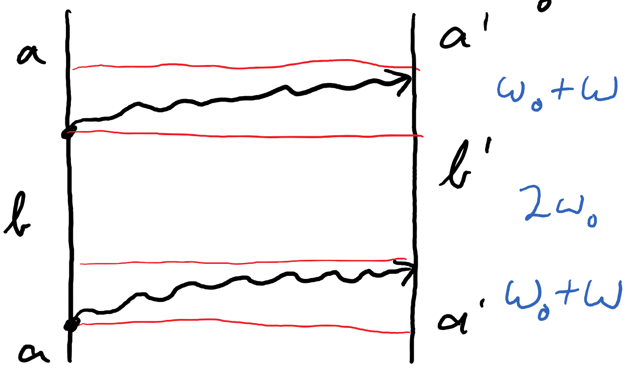
$\Rightarrow$ dipole-dipole interaction! $|a, a'\rangle \to |b, b'\rangle$

$$\Delta E = \sum_{bb'} \frac{\langle a a' | \delta V | b b' \rangle \langle b b' | \delta V | a a' \rangle}{E_a + E_{a'} - E_b - E_{b'}}$$

$$= -\frac{C_6}{D^6}$$

short distance $D \ll \lambda_{ab}$

$\omega \gg \omega_0$

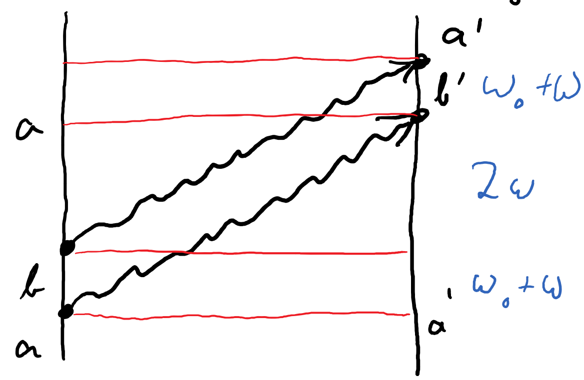


energy defects

$$\frac{1}{\omega^2\omega_0}$$

large distance $D \gg \lambda_{ab}$

$\omega \ll \omega_0$



energy defects

$$\frac{1}{\omega_0^2\omega}$$

So for $D \gg \lambda_{ab}$ we get one more factor of ω

$$\omega \propto \frac{1}{\lambda} \propto \frac{1}{D}$$

$$\frac{1}{D^6} \xrightarrow[\text{long range}]{} \frac{1}{D^7}$$

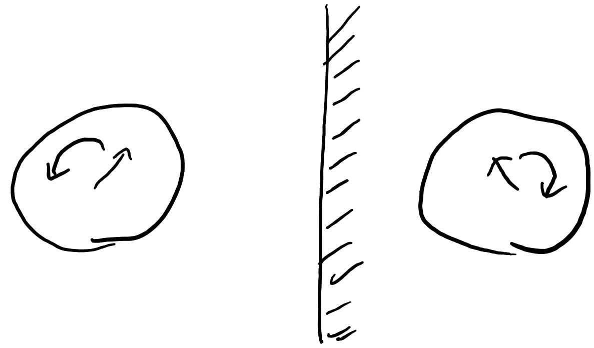
instantaneous $\longrightarrow$ retarded potentials

Long-range potentials

	$r < \lambda_{ab} = 137 a_0$	$r > \lambda_{ab} = 137 a_0$
atom-atom	$\frac{e^2}{a_0} \frac{a_0^6}{r^6}$	$\frac{\hbar c \; a_0^6}{r^7}$
atom-wall	$\frac{(e a_0)^2}{z^3}$	$\frac{\hbar c}{z^4} a_0^3$
wall-wall	$\frac{\hbar c}{z^4}$	

Atom - Wall

$z \ll \lambda_{ab} \approx \frac{a_0}{\alpha} \approx 137\, a_0$:



$\Rightarrow$ correlated dipoles

$$\Rightarrow \quad V_{a-w} = \frac{(e a_0)^2}{z^3}$$

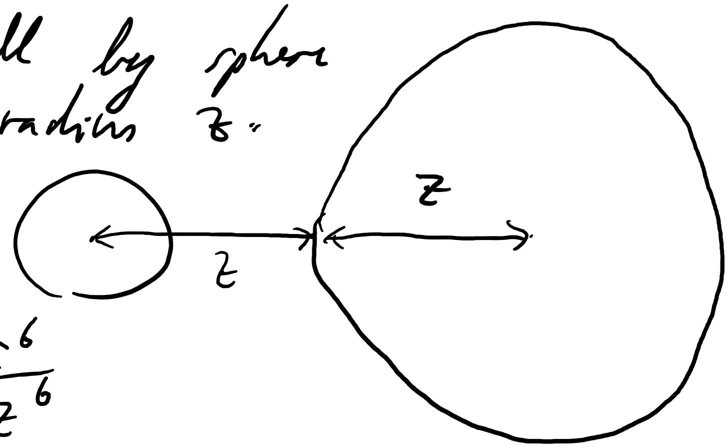
$z \gg \lambda_{ab} = 137\, a_0$

Use Spruch's formula

$$V(r) = \frac{\hbar}{c^5 r} \int_0^{c/r} d\omega \; \alpha_1(\omega)\, \alpha_2(\omega)\, \omega^5$$

but trick: replace wall by sphere of radius z.

$\alpha_{sphere} \propto z^3$



$$\Rightarrow V_{a-w} = \frac{\hbar}{c^5 z}\, a_0^3 z^3 \, \frac{c^6}{z^6}$$

$$= \frac{\hbar c}{z^4}\, a_0^3$$

Other way:

Polarizable system in presence of background field:

energy $\alpha(\omega)|E_0|^2(\omega,\vec{x}) = \alpha(\omega)\, u(\omega,\vec{x})$

$u(\omega,\vec{x})$ = energy density $= \frac{\hbar\omega}{V}$ for vacuum.

$$\Rightarrow \quad \Sigma = \int_0^\infty d\omega\, N(\omega)\, u(\omega,\vec{x})\, \alpha(\omega)$$

$$= \frac{\hbar}{c^3} \int_0^\infty d\omega\, \alpha(\omega)\, \omega^3 \quad = \text{infinite}$$

Now: atom at distance z from an ideal wall

$\Rightarrow$ fluctuation with $\omega \gg \frac{c}{z}$ not affected by presence of the wall.

fluctuation with $\omega \ll \frac{c}{z}$ greatly affected.

$\Rightarrow$ While self-energy $\Sigma(z)$ is also infinite

$\Sigma - \Sigma(z)$ is finite

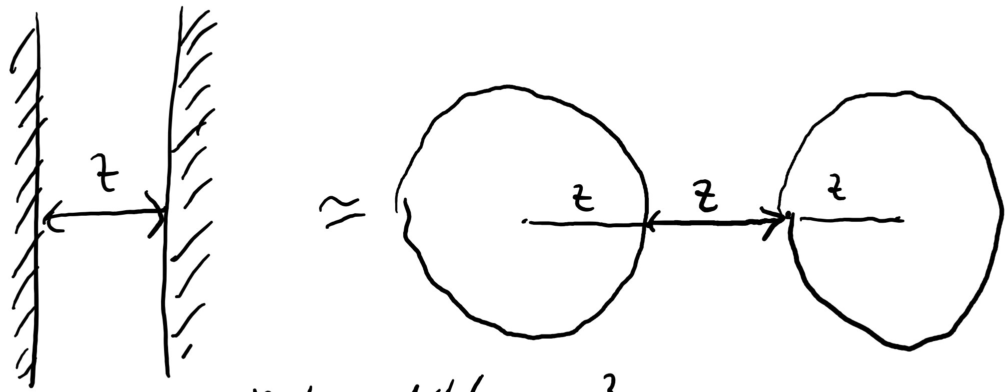
contributions $\omega \gg \frac{c}{z}$ cancel

$\omega \ll \frac{c}{z}$ roughly comparable

$$\Rightarrow \quad V_{a\text{-wall}}(z) = \frac{\hbar}{c^3} \int_0^{c/z} d\omega\, \alpha(\omega)\, \omega^3$$

$$= \frac{\hbar c}{z^4}\, a_0^3$$

Wall-wall: Trick: replace walls by spheres of radius z



Polarizability z^3

Need $\frac{\text{Force}}{\text{unit area}} = \frac{1}{z^2}\frac{\partial V}{\partial z} \approx \frac{V}{z^3}$

$$V = \frac{\hbar}{c^5 z} z^6 \int_0^{c/z} d\omega\, \omega^5$$

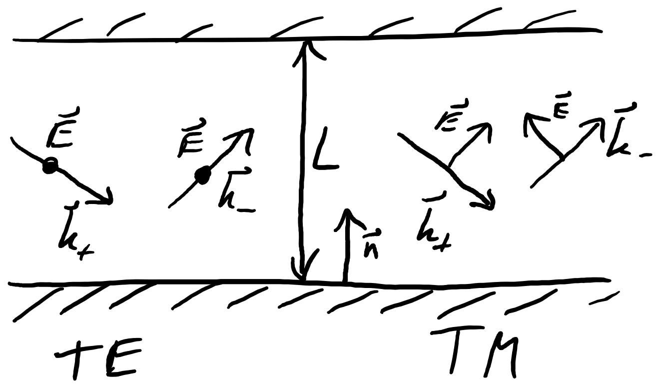
$$= \frac{\hbar c}{z}$$

So $\frac{F}{A} = \frac{\hbar c}{z^4}$ Casimir 1948

Casimir Effect – Full calculation

see Serge Haroche, Les Houches Summer School Lectures 1990

Cavity



TE and TM modes exist in cavity.

superpositions of plane waves with

$$\vec{k}_\pm = \mp l \vec{n} + k \vec{\varphi}$$

$\vec{n}$ – normal to mirror
$\vec{\varphi}$ – parallel

$$\frac{\omega^2}{c^2} = l^2 + k^2$$

Total electric field along mirrors and magnetic field normal to surfaces must vanish at $z=0$ and $z=L$.

$\Rightarrow$ l is quantized

$$l = \frac{m\pi}{L}.$$

$$\Rightarrow \quad \omega^2 = m^2\omega_0^2 + k^2c^2$$

with $\omega_0 = \frac{c\pi}{L}$

Field distributions

$$\vec{\alpha}^{E}_{m,k,p}(z,\vec{\rho}) = \sqrt{\frac{2}{V}}\,\sin\left(\frac{m\pi z}{L}\right) e^{i\vec{k}\cdot\vec{\rho}}\;\hat{\rho}\times\hat{n}$$

$$\vec{\alpha}^{M}_{m,k,p}(z,\rho) = \sqrt{\frac{\beta_m}{V}}\left[\frac{ck}{\omega}\cos\left(\frac{m\pi z}{L}\right)\hat{n} - i\,\frac{m\omega_0}{\omega}\sin\left(\frac{m\pi z}{L}\right)\hat{\rho}\right] e^{i\vec{k}\cdot\vec{\rho}}$$

$$\beta_m = 1 \text{ if } m=0$$
$$= 2 \text{ if } m>0$$

$V = L a^2$, a arbitrary side length.

$m=0$ modes have no spatial variation along z.

$\Rightarrow$ since transverse electric field must vanish at $z=0$ and $z=L$, it must vanish everywhere

$\Rightarrow$ No TE mode with $m=0$.

TM with $m=0$ has $\sqrt{\frac{1}{V}}$ normalization

all others (for which $\langle \sin^2\rangle = \langle \cos^2\rangle = \frac{1}{2}$) have $\sqrt{\frac{2}{V}}$.

Vector potential $\vec{A}(z,\vec{\rho}) = \vec{A}^{E}(z,\vec{\rho}) + \vec{A}^{M}(z,\vec{\rho})$

$$\vec{A}^{E}(z,\vec{\rho}) = \sum_{mkp}\left\{\sqrt{\frac{\hbar}{2\varepsilon_0\omega}}\;\vec{\alpha}^{E}_{mkp}(z,\vec{\rho})\,a^{E}_{mkp} + h.c.\right\}$$

$$\vec{A}^{M}(z,\vec{\rho}) = \sum_{mkp}\left\{\sqrt{\frac{\hbar}{2\varepsilon_0\omega}}\;\vec{\alpha}^{M}_{mkp}(z,\vec{\rho})\,a^{M}_{mkp} + h.c.\right\}$$

Counting modes:

cyclic boundary conditions $\Rightarrow$ k_x, k_y quantized in units of $\frac{2\pi}{a}$.

# of modes for given m between k and $k+dk$ is

$$d^2k \cdot \frac{a^2}{(2\pi)^2} = \frac{a^2}{2\pi} k\,dk \underset{\frac{\omega^2}{c^2} = l^2 + k^2}{\uparrow=} \frac{a^2}{2\pi c^2}\omega\, d\omega$$

A given ω can be obtained for $m = 0$ to $\mathrm{Int}\left(\frac{\omega}{\omega_0}\right)$

For $m > 0$ we have TE and TM mode

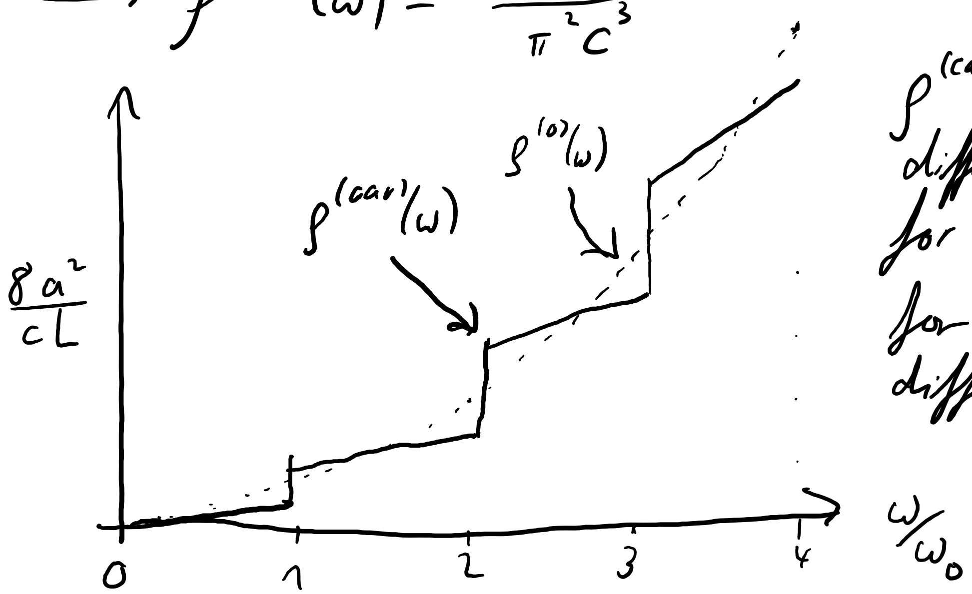
$m = 0$ one TM mode.

$$\Rightarrow \rho^{(cav)}(\omega) = \frac{a^2\omega}{2\pi c^2}\left[1 + 2\,\mathrm{Int}\left(\frac{\omega}{\omega_0}\right)\right]$$

$$= \frac{V\omega\omega_0}{2\pi^2 c^3}\left[1 + 2\sum_{m=1}^{\infty}\Theta\left(\frac{\omega}{\omega_0} - m\right)\right]$$

In free space $L \to \infty$, $\omega_0 \to 0$

$$\longrightarrow \rho^{(0)}(\omega) = \frac{V\omega^2}{\pi^2 c^3}$$



$\rho^{(cav)}$ and $\rho^{(0)}$ differ significantly for $\omega \lesssim$ few ω_0.

for $\omega \gg \omega_0$, difference is negligible

(construct by adding mode by mode)

Casimir effect: The variation of $\rho^{(cav)}(\omega)$ with L leads to a variation of the total field vacuum energy with L

$\Rightarrow$ force that pulls mirrors together

$$W(L) = \sum_{\text{modes}} \frac{\hbar\omega}{2} = \int_0^\infty d\omega \, \frac{\hbar\omega}{2} \rho^{(cav)}(\omega)$$

$$= \frac{a^2\hbar}{4\pi c^2}\left[I_0 + 2\sum_{m=1}^{\infty} I_m \right]$$

with $I_m = \int_{m\omega_0}^{\infty} d\omega \, \omega^2$

$W(L)$ diverges. I_0 divergence doesn't depend on L.
for I_m $(m \neq 0)$ introduce converging term $e^{-\lambda\omega/c}$.

$$I_m = \int_{m\pi c/L}^{\infty} \omega^2 e^{-\lambda\omega/c} d\omega$$

$$= c^2 \frac{\partial^2}{\partial\lambda^2} \int_{m\pi c/L}^{\infty} e^{-\lambda\omega/c} d\omega$$

$$= c^3 \frac{\partial^2}{\partial\lambda^2}\left[- \frac{e^{-m\pi\lambda/L}}{\lambda} \right]$$

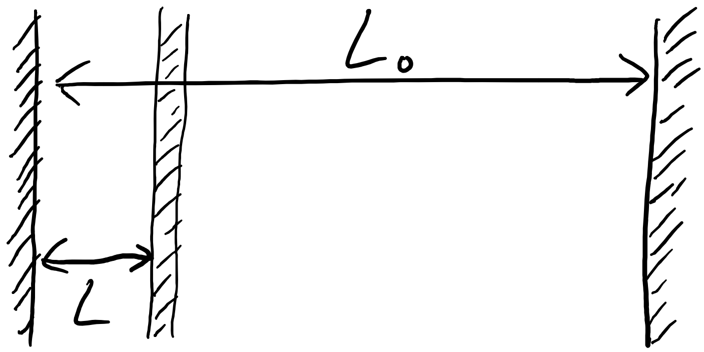
$$\sum_{m=1}^{\infty} I_m = - c^3 \frac{\partial^2}{\partial\lambda^2}\left\{ \sum_{m=1}^{\infty} \frac{e^{-m\pi\lambda/L}}{\lambda} \right\}$$

$$= \frac{c^3\pi}{L} \frac{\partial^2}{\partial\lambda^2}\left\{ \frac{L}{\pi\lambda} \frac{1}{e^{\frac{\pi\lambda}{L}} - 1} \right\}$$

$$\frac{L}{\pi\lambda}\frac{1}{e^{\pi\lambda/L}-1} = \frac{L^2}{(\pi\lambda)^2} - \frac{L}{2\pi\lambda} + \frac{1}{12} - \frac{1}{720}\left(\frac{\pi\lambda}{L}\right)^2 + \dots$$

$$W(L) = \frac{a^2\hbar}{4\pi c^2} I_0 + \frac{a^2\hbar c}{2}\left[\frac{6L}{\pi^2\lambda^4} - \frac{1}{\pi\lambda^3} - \frac{2\pi^2}{720L^3} + \dots\right]$$

Trick: Embed mirror in between larger gap:



Total energy

$$W_T(L) = \frac{a^2\hbar}{2\pi c^2} I_0 + \frac{a^2\hbar c}{2}\left[\frac{6L_0}{\pi^2\lambda^4} - \frac{2}{\pi\lambda^3} - \frac{2\pi^2}{720L^3} + \dots\right]$$

(neglect $\frac{1}{(L_0-L)^3} \ll \frac{1}{L^3}$)

For a different configuration L' we get $W_T(L')$.

$$W_T(L') - W_T(L) = -\frac{a^2\pi^2\hbar c}{720}\left(\frac{1}{L'^3} - \frac{1}{L^3}\right)$$

$$\Rightarrow \quad U(L) = -\frac{\pi^2\hbar c}{720}\frac{a^2}{L^3}$$

Pressure $P_{vac} = \frac{1}{a^2}\frac{\partial U}{\partial L} = \frac{\pi^2\hbar c}{240}\frac{1}{L^4}$

$P_{vac} = 10^{-3}\,Pa$ for $L = 1\,mm$. $\hat{=}$ one electron per L^2

$= 10^{-5}\,mbar$

Physical justification for λ:
Plasma frequency!
Above ω_{pl} mirrors are transparent.

Two ways to calculate Casimir effect:

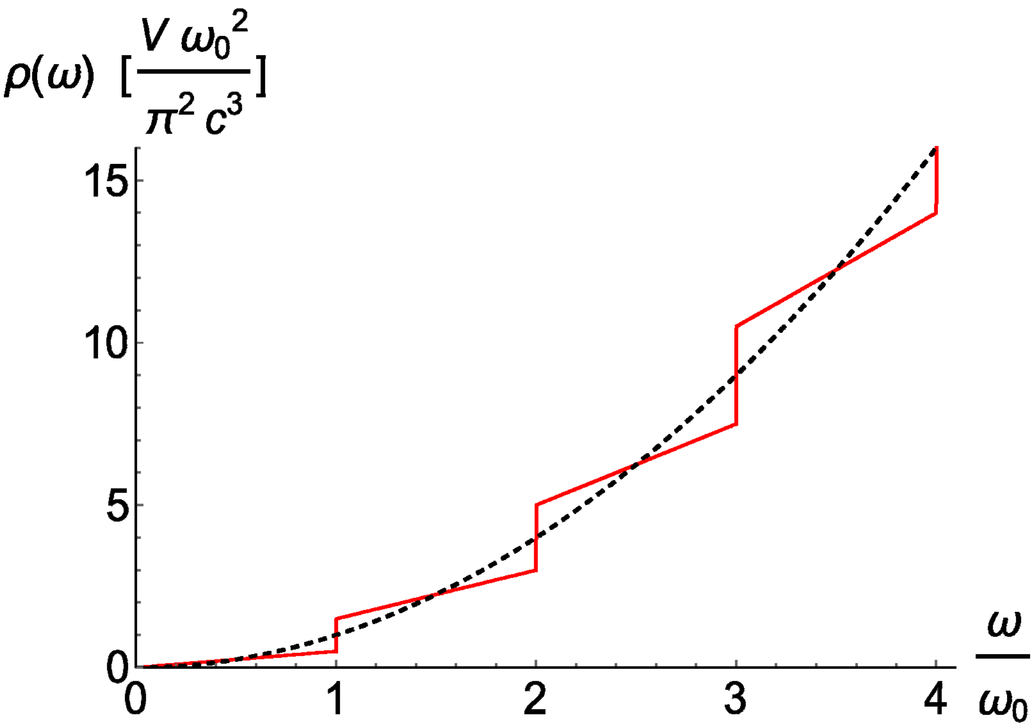
$$\Delta E = \sum \frac{1}{2}\hbar\omega - \sum \frac{1}{2}\hbar\omega_0$$

or S-matrix approach

$$\langle 0 | U(t) | 0 \rangle \sim |\langle \psi | 0 \rangle|^2 \, e^{-i \Delta E t/\hbar}$$

→ no reference to vacuum energy needed.
See Jaffe.

Plot of the cavity density of states (red) and the free space density of states



Zoom in onto the region from $\omega = 0$ to $\omega = \omega_0$:

It is nice to see how here the cavity density of states is clearly falling behind the one of free space: