Chapter 1

Introduction

1.1 A Brief History of Atomic Physics

The history of atomic physics is well demonstrated through the BEC 1 cold atom experiment. The experiment creates a gas of ultracold atoms, with both bosons and fermions.

1.1.1 Denegerate Gases

Ultracold is between $1\,\mu$K and $1\,$nK. At these temperatures, atoms move at approximately $1\,$mm/s. In addition, particles behave as waves, with the de Broglie wavelength given by

$$\lambda = \frac{h}{mv}, \tag{1.1}$$

where h is Planck's constant, m is the mass, and v is the velocity. Heisenberg expressed this in terms of uncertainty. Temperature is the uncertainty of velocity (squared), so

$$\Delta x \approx \frac{\hbar}{\Delta p} \approx \frac{\hbar}{\sqrt{T}}. \tag{1.2}$$

Once our atoms are overlapping, we need to consider quantum statistical mechanics. Overlapping occurs when $\Delta x \approx n^{-1/3}$. Here, bosons and fermions behave differently.

Fermions have half-integer spin. They experience Pauli blocking, with enough fermions forming a Fermi sea. There is no phase transition at low temperatures.

Bosons have interger spin. They can share quantum states (and tend to do so). At low temperatures, Bosons undergo a phase transition, forming a Bose-Einstein condensate(BEC). This occurs at roughly the temperature where their wavefunctions begin to overlap. Note that this temperature is much higher than the temperature where no other states are available.

By combining the earlier equations, we can find the degeneracy temperature is

$$k_B T_{\text{Degeneracy}} \approx \frac{\hbar^2}{m} n^{2/3} \approx E_F. \tag{1.3}$$

This is approximately the Fermi energy. Note the dependence on density: electrons in a metal are dense enough that they are degenerate at room temperature.

1.1.2 Cooling to Degeneracy

There are two methods for cooling. Laser cooling can typically cool atoms to less than $\sim 1\,\mathrm{mK}$. Laser cooling works by using momentum transfer between atoms and absorbed photons. Atoms re-emit in a random direction, which sets a minimum temperature reachable by this method. Evaporative cooling allows further cooling to $\sim 10\,\mathrm{nK}$. Hot atoms are allowed to escape, while cold atoms remain, causing the average temperature to decrease. Evaporation typically takes place in a magnetic trap. Because magnetic fields maxima can not occur in free space, this requires atoms be in states that have the highest energy at magnetic field minima. We can use this to make evaporation more efficient: we can flip the spin of hot atoms (using RF fields) and cause them to quickly escape the trap. This was invented at MIT, intiially with hydrogen. The technique was then used to make a BEC of sodium.

Laser Cooling Techniques

Bill Phillips developed the Zeeman slower: a technique for cooling a beam of atoms that keeps them resonant with a laser by changing a magnetic field to compensate for the changing Doppler shift.

Steve Chu developed the optical molasses technique for holding atoms, consisting of six counter-propogating beams with a magnetic field. Claude Cohen-Tannoudji explained why this was more effective than expected: the counter-propogating beams form an optical lattice, and atoms may be more likely to make transitions from more excited states, decaying to lower energy states. This decreases the temperature below what one would expect.

Eric Cornell, Wolfgang Ketterle, and Carl Wieman won the Nobel Prize in 2001 for forming the first BECs.

1.2 Overview

Unlike AMO 1, this course will have more advanced mathematics, in order to pursue second quantization.

Chapter 2

2.1

An atom is a quantum system, with few levels. Photons are generally non-relativistic, meaning in the visible, IR, microware, or RF spectrum. This allows us to avoid thinking about virtual electrons and similar.

We would like to understand certain phenomenological results, such as the refractive index (which is actually complex) and the absorption coefficient.

2.2 Lorentz Model

In the Lorentz model, we consider an electron on a spring connected to the nucleus. This forms a dipole. Note that here we actually have infinite levels.

2.2.1 Undamped

Without damping, the equation of motion is

$$m\frac{d^2\vec{r}}{dt^2} = -m\omega_0^2\vec{r}, \tag{2.1}$$

where ω_0 is the resonance frequency, motivated by the quantum harmonic oscillator.

We use the dipole approximation for a driving field, which is justified because optical wavelengths are much larger than an atom. The field is

$$\vec{E}(t) = \vec{E}\cos(\omega t). \tag{2.2}$$

We then have

$$m\frac{d^2\vec{r}}{dt^2} = -m\omega_0^2\vec{r} + q\vec{E}\cos(\omega t). \tag{2.3}$$

Defining the dipole $\vec{d} = q\vec{r}$,

$$\frac{d^2\vec{d}}{dt^2} + \omega_0^2\vec{d} = \frac{q^2}{m}\vec{E}\cos(\omega t). \tag{2.4}$$

It is often better to use complex notation:

$$\vec{E}(\vec{r},t) = \vec{E}(\vec{r})\cos(\omega t + \phi) = \vec{E}(\vec{r})\frac{e^{-i\phi}}{2}e^{-i\omega t} + \vec{E}(\vec{r})\frac{e^{i\phi}}{2}e^{i\omega t} = \vec{E}^{+}(\vec{r})e^{-i\omega t} + \vec{E}^{-}e^{i\omega t}. \tag{2.5}$$

$\vec{E}^+$ is for positive frquency, $\vec{E}^-$ is for the negative frequency. The electric field must be real, so it can be written as

$$\vec{E}(t) = \vec{E}^+ e^{-i\omega t} + \text{c.c.} = 2\text{Re}\left[\vec{E}^+ e^{-i\omega t}\right]. \tag{2.6}$$

Be careful, as $\vec{E}^+ = \vec{E}/2$. $\vec{E}^+$ and $\vec{E}^-$ and associated notation will eventually correspond to photon creation and annihilation operators for quantum fields.

We can then write

$$\frac{d^2\vec{d}^+}{dt^2} + \omega_0^2 \vec{d}^+ = \frac{q^2}{m}\vec{E}^+ e^{-i\omega t}. \tag{2.7}$$

The real dipole is then $\vec{d} = 2\text{Re}\left[\vec{d}^+\right]$. Note that although we have not explicitly included damping, we assume transients will disappear. Let the electric field have polarization $\vec{\epsilon}$. Then $\vec{E}^+ = \vec{\epsilon} E_0^+$. We guess the solution is

$$\vec{d}^+ = \vec{\epsilon} d_0^+ e^{-i\omega t}. \tag{2.8}$$

Our equation becomes

$$-\omega^2 d_0^+ + \omega_0^2 d_0^+ = \frac{q^2 E_0^+}{m}, \tag{2.9}$$

from which we find

$$d_0^+ = \frac{q^2 E_0^+/m}{\omega_0^2 - \omega^2} \approx \frac{q^2 E_0^+/m}{2\omega\left(\omega_0 - \omega\right)}. \tag{2.10}$$

Let the polarizability α be given by

$$\vec{d}^+ = \alpha(\omega)\vec{E}^+. \tag{2.11}$$

The units of this are the units of dipole over electric field, which ultimately becomes a volume (in CGS units!):

$$[\alpha] = \left[\frac{d}{E}\right] = \left[\frac{qr^3}{q}\right] = \left[r^3\right]. \tag{2.12}$$

Solving for α,

$$\alpha(\omega) = \frac{q^2/m}{\omega_0^2 - \omega^2}. \tag{2.13}$$

Some limits: at DC, this is positive, as d should be in phase with E. At DC,

$$\alpha_{\text{DC}} = \frac{q^2}{m\omega_0^2} \sim \frac{q^2}{mc}\lambda^2 = r_e \lambda^2. \tag{2.14}$$

Where r_e is the classical radius of the electron, given by $mc^2 = q^2/r_e$.

For Bohr hydrogen, we would expect $\alpha_H \propto a_0^3$, where a_0 is the Bohr radius. $r_e = \frac{q^2}{mc^2}$ and $a_0 = \frac{\hbar^2}{me^2}$. The fine structure constant, $\alpha_{137} = \frac{e^2}{\hbar c} \approx \frac{1}{137}$ will appear here as well.

$$\frac{a_0}{r_e} = \frac{\hbar^2}{me^2}\frac{mc^2}{e^2} = \left(\frac{\hbar c}{e^2}\right)^2 = \alpha_{137}^{-2}. \tag{2.15}$$

Therefore, the Bohr radius is about 10000x larger than the classical electron size. The Bohr radius is related to the Rydberg

$$\text{Ry} \sim \hbar\omega \sim \frac{e^2}{a_0} \sim \frac{\hbar^2}{ma_0^2}. \tag{2.16}$$

Note that $\hbar\omega = \frac{hc}{\lambda}$. Therefore,

$$\lambda \sim \frac{cma_0^2}{h} \tag{2.17}$$

Note that $\frac{h}{mc}$ is the Compton wavelengthThis is the energy at which photons can create electrons. So,

$$\frac{\alpha_{\mathrm{DC}}}{\alpha_H} = \frac{r_e\lambda^2}{a_0^3} \sim \frac{\alpha_{137}^2 a_0 \left(\frac{a_0}{\alpha_{137}}\right)^2}{a_0^3} = 1. \tag{2.18}$$

This implies this classical model is not too unreasonable in practice.

Given N atoms per unit volume, polarization density is

$$\vec{P}^+ = N\vec{d}^+ = N\alpha(\omega)\vec{E}^+ = \vec{\epsilon}\frac{Nq^2/m}{\omega_0^2 - \omega^2}\vec{E}^+ e^{-i\omega t}. \tag{2.19}$$

The susceptibility χ is defined by

$$\vec{P} = \epsilon_0\chi\vec{E}. \tag{2.20}$$

χ is given by

$$\chi(\omega) = \frac{N}{\epsilon_0}\alpha(\omega) = \frac{Nq^2/m\epsilon_0}{\omega_0^2 - \omega^2}. \tag{2.21}$$

2.2.2 Damped

Let there also be damping γ. The Q of an oscillator is ω_0/γ. This could be due to radiation due to acceleration or collisions.

An Aside on the Bohr Model

What prevents electrons from undergoing radiative damping into the nucleus?

The Heisenberg uncertainty principle requires $\Delta x\Delta p \sim \hbar$. The energy of an electron in a hydrogen atom is

$$\frac{\Delta p^2}{2m} - \frac{e^2}{\Delta x} \sim \frac{\hbar}{m\Delta x^2} - \frac{e^2}{\Delta x} \implies \frac{\hbar}{m\Delta x^2} \sim \frac{e^2}{\Delta x} \implies \Delta x \sim \frac{\hbar^2}{me^2} = a_0 \approx 0.5 \times 10^{-10}\,\mathrm{m}. \tag{2.22}$$

$$m\frac{d^2\vec{r}}{dt^2} + m\omega_0^2\vec{r} + m\gamma\frac{d\vec{r}}{dt} = q\vec{E}\cos(\omega t). \tag{2.23}$$

In terms of the real dipole,

$$\frac{d^2\vec{d}^+}{dt^2} + \omega_0\vec{d}^+ + \gamma\frac{d\vec{d}^+}{dt} = \frac{q^2}{m}\vec{\epsilon}\vec{E}^+ e^{-i\omega t}. \tag{2.24}$$

Making the same ansatz and solving the associated algebraic equation, we find

$$d_0^+ = \frac{q^2E_0^+}{m}\frac{1}{\omega_0^2 - \omega^2 - i\gamma\omega} \equiv \left|d_0^+\right| e^{i\delta}. \tag{2.25}$$

The peak is no longer at ω_0, there is a slight offset due to γ. The phase shift δ is $\pi/2$ at resonance, with it going to 0 at DC and π at high frequencies.

$$\delta = \arctan\left(\frac{\gamma\omega}{\omega_0^2 - \omega^2}\right). \tag{2.26}$$

This is a phase lag for a real dipole. The magnitude (as a function of ω) is proportional to

$$\frac{1}{\omega_0 - \omega^2 - i\gamma\omega} = \frac{\omega_0^2 - \omega^2 + i\gamma\omega}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2} = \frac{1}{\omega_0^2}\left(u + iv\right). \tag{2.27}$$

For the real dipole, we then have

$$\vec{d} = \left[\frac{q^2 E_0}{m\omega_0^2}\left(u + iv\right)e^{-i\omega t}\right] = d_{\mathrm{DC}}\left(u\cos\left(\omega t\right) + v\sin\left(\omega t\right)\right). \tag{2.28}$$

Therefore,

$$\vec{d} = \frac{q^2 E_0}{m}\left\{\frac{\omega_0^2 - \omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2}\cos\left(\omega t\right) + \frac{\gamma^2\omega^2}{\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2}\sin\left(\omega t\right)\right\}. \tag{2.29}$$

This does not peak on resonance, but time averaged power does:

$$P(t) = \vec{F}\cdot\dot{\vec{r}} = \vec{E}\cdot\dot{\vec{d}} = E_0\cos\left(\omega t\right)d_{\mathrm{DC}}\left[-\omega u\sin\left(\omega t\right) + \omega v\cos\left(\omega t\right)\right] = d_{\mathrm{DC}}E_0\omega\left(-\frac{1}{2}u\sin\left(2\omega t\right) + v\cos^2\left(\omega t\right)\right). \tag{2.30}$$

We find that the in phase response (u) averages to zero, but the out of phase response does not. The average is

$$\langle P\rangle = \frac{1}{2}d_{\mathrm{DC}}E_0\omega v. \tag{2.31}$$

Near resonance, we approximate

$$\omega_0^2 - \omega^2 = \left(\omega_0 + \omega\right)\left(\omega_0 - \omega\right) = 2\omega_0\left(\omega_0 - \omega\right). \tag{2.32}$$

Observables

From the susceptibility, we can find the index of refraction. In a material, we replae $\vec{E}$ with $\vec{D} = \epsilon_0\vec{E} + \vec{P} = \epsilon\vec{E}$. The dielectric constant ϵ is related to the susceptibility by

$$\epsilon = \epsilon_0\left(1 + \chi\right). \tag{2.33}$$

The index of refraction determines the phase velocity in the material, with

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \implies n \approx \sqrt{\frac{\epsilon}{\epsilon_0}} = \sqrt{1+\chi} \approx 1 + \frac{\chi}{2}. \tag{2.34}$$

The first approximation assumes μ is unchanged; the second approximation assume χ is small. Therefore,

$$\bar{n}(\omega) = 1 + \frac{Nq^2}{2m\epsilon_0\omega_0^2}\left(u + iv\right). \tag{2.35}$$

For a plane wave,

$$E(z) = E_0 e^{ikz} = E_0 e^{ink_0 z} = E_0 e^{i\mathrm{Re}(n)k_0 z} e^{-\mathrm{Im}(n)k_0 z}. \tag{2.36}$$

The phas eindex is then

$$n(\omega) = \mathrm{Re}\left(\bar{n}(\omega)\right) = 1 + \frac{Nq^2}{2m\epsilon_0\omega_0^2} u = 1 + \frac{1}{2}\chi_{\mathrm{DC}} u. \tag{2.37}$$

The intensity lost per unit distance is described by an absorption coefficient a, given by

$$\frac{dI}{dz} = -aI \implies I(z) = I_0 e^{-az}. \tag{2.38}$$

This is then

$$a(\omega) = 2k_0 \mathrm{Im}\left(\bar{n}(\omega)\right) = \frac{1}{c}\chi_{\mathrm{DC}}\omega v. \tag{2.39}$$

The DC susceptibility is

$$\chi_{\mathrm{DC}} = \frac{N}{\epsilon}\alpha_{\mathrm{DC}} = \frac{Nq^2}{m\epsilon_0\omega_0^2}. \tag{2.40}$$

Absorption is proportional to dissipated power, which makes sense. The absorption coefficient is also related to the scattering cross section by

$$a = \sigma N. \tag{2.41}$$

The scattering rate is given by the probability of scattering in a given distance with a given cross section per unit time:

$$N\sigma * v\, dt/dt = N\sigma v. \tag{2.42}$$

On resonance, the classical scattering cross section is

$$\sigma_{\mathrm{classical}} = \frac{q^2}{m\epsilon_0 c\gamma}. \tag{2.43}$$

Smaller γ leads to more efficient scattering on resonance. The (true) quantum mechanical scattering cross section is

$$\sigma_{\mathrm{quantum,isotropic}} = \frac{\lambda_0^2}{2\pi}. \tag{2.44}$$

This relates to a unitary limit. No matter the properties of the atom, the scattering probability can not exceed one. Oscillator strength is defined as the ratio of the classical cross section to the quantum cross section.

The real and imaginary part of the susceptibility obey Kramers-Kronig relations

$$\mathrm{Re}\left[\chi(\omega)\right] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Im}\left[\chi(\omega')\right]}{\omega' - \omega}\, d\omega' \tag{2.45}$$

$$\mathrm{Im}\left[\chi(\omega)\right] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\left[\chi(\omega')\right]}{\omega' - \omega}\, d\omega' \tag{2.46}$$

$$\tag{2.47}$$

This comes from complex analysis. $\mathcal{P}$ denotes the Cauchy principal part. This applies for causal drives. This implies there is no dispersion without absorption, and no absorption without dispersion.

2.3 Einstein Model (1916)

Getting closer to photons...

Einstein considered a gas of atoms in thermal equilibrium with some light, in a thermal distribution. Consider an atom with two energy levels, $|a\rangle$ (lower) and $|b\rangle$ (higher). Let N_a atoms be in state a and N_b atoms be in state b, with $N_a + N_b = 1$. For atoms to be in thermal equilibrium with light, there must be some absorption and emission processes. Einstein included spontaneous emission in order for oscillating charges to lose energy. Depending on the phase between the atom and the light field, energy can be transferred either way.

The rate equation for b is

$$\frac{dN_b}{dt} = -\frac{dN_a}{dt} = -AN_b + B\left(N_a - N_b\right)u\left(\nu = \frac{\omega_0}{2\pi}\right). \tag{2.48}$$

A describes spontaneous emission, B describes stimulated emission and absorption. A is a relaxation, so it should correspond to γ. In equilibrium, $\frac{dN_b}{dt} = 0$. Therefore,

$$AN_b = B\left(N_a - N_b\right)u(\nu) = B\left(1 - 2N_b\right)u(\nu). \tag{2.49}$$

This implies

$$N_b = \frac{B}{\gamma + 2Bu}u. \tag{2.50}$$

At small u, this is linear in u. At large u, N_b asymptotes to $1/2$. The power exchanged with the light field has the sign of $N_b - N_a$, which implies the possibility of lasers.

In thermal equilibrium, Boltzmann's law gives

$$\frac{N_b}{N_a} = e^{-(E_b - E_a)/k_B T} = e^{-h\nu/k_B T}. \tag{2.51}$$

Therefore,

$$Ae^{-h\nu/k_B T} = B\left(1 - e^{-h\nu/k_B T}\right)u(\nu). \tag{2.52}$$

This gives us the Planck spectrum

$$u(\nu) = \frac{A}{B}\frac{1}{e^{h\nu/k_B T} - 1}. \tag{2.53}$$

This works well to describe thermal radiation fields, and also quite well for population in laser media. Note that no photons are yet needed.

2.4 Maxwell's Equations

Maxwell's equations are

$$\boldsymbol{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{2.54}$$

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \tag{2.55}$$

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{2.56}$$

$$\boldsymbol{\nabla} \times \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} + \mu_0 \vec{j}. \tag{2.57}$$

Our goal is to transform these to a harmonic potential, so we can easily quantize. In classical electromagnetism, these fields couple to charges via the Lorentz force. For $v \ll c$,

$$m_\alpha \ddot{\vec{r}}_\alpha = q_\alpha \left(\vec{E} + \vec{v}_\alpha \times \vec{B}\right). \tag{2.58}$$

2.4.1 Conserved Quantities

We have local conservation of charge

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla} \cdot \vec{j} = 0. \tag{2.59}$$

This is implied by the first and last Maxwell's equations. The charge density and current density are defined by

$$\rho(\vec{r}, t) = \sum_\alpha q_\alpha \delta\left(\vec{r} - \vec{r}_\alpha(t)\right) \text{ and} \tag{2.60}$$

$$\vec{j}(\vec{r}, t) = \sum_\alpha q_\alpha \vec{v}_\alpha(t) \delta\left(\vec{r} - \vec{r}_\alpha(t)\right). \tag{2.61}$$

The total energy of particles and fields is given by

$$H = \sum_\alpha \frac{1}{2} m_\alpha \vec{v}_\alpha^2 + \frac{\epsilon_0}{2}\int d^3\vec{r}\left[\vec{E}^2\left(\vec{r}, t\right) + \vec{B}^2\left(\vec{r}, t\right)\right] \tag{2.62}$$

(ignoring some ϵ_0 and μ_0 factors). The momentum is

$$\vec{P} = \sum_\alpha m_\alpha \vec{v}_\alpha + \epsilon_0 \int d^3\vec{r}\vec{E} \times \vec{B}. \tag{2.63}$$

Recall the Poynting vector, which appears here. The angular momentum is

$$\vec{J} = \sum_\alpha m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha + \epsilon_0 \int d^3\vec{r}\,\vec{r} \times \left(\vec{E} \times \vec{B}\right). \tag{2.64}$$

2.4.2 Potentials

For Maxwell's equations in free space, we can use the vector and scalar potentials

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \implies \vec{B} = \boldsymbol{\nabla} \times \vec{A} \tag{2.65}$$

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \implies \boldsymbol{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\left(\boldsymbol{\nabla} \times \vec{A}\right) = 0 \implies \boldsymbol{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0 \implies \vec{E} = -\boldsymbol{\nabla} U - \frac{\partial \vec{A}}{\partial t}. \tag{2.66}$$

Plugging these into Maxwell's equations, the second and third are automatically satisfied. The other two become

$$\nabla^2 U = -\frac{\rho}{\epsilon_0} - \boldsymbol{\nabla} \cdot \frac{\partial \vec{A}}{\partial t} \text{ and} \tag{2.67}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = \mu_0\vec{j} - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial U}{\partial t}\right). \tag{2.68}$$

We also define the d'Alembertian

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \tag{2.69}$$

Note that $\vec{A}(\vec{r}, t_0)$ and $\frac{\partial \vec{A}}{\partial t}(\vec{r}, t_0)$ $\forall r$ are sufficient to fully define the fields at time t_0.

Gauge Invariance

The vector potential and scalar potential are not uniquely defined. They can be modified by

$$\vec{A} \to \vec{A}' = \vec{A} + \boldsymbol{\nabla} f, \quad U \to U' = U - \frac{\partial}{\partial t} f. \tag{2.70}$$

The Lorentz gauge is $\boldsymbol{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial U}{\partial t} = 0$, or (in relativistic notation), $\partial_\mu A^\mu = 0$. This is best for relativistic theories that treat time and space the same. Then Maxwell's equations become

$$\Box U = \frac{\rho}{\epsilon_0} \text{ and } \Box\vec{A} = \mu_0\vec{j}, \text{ or } \partial_\nu\partial^\mu A^\mu = \mu_0 j^\mu, \tag{2.71}$$

where

$$\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla^2\right), \quad A^\mu = \left(\frac{1}{c}U, \vec{A}\right), \text{ and } j^\mu = \left(c\rho, \vec{j}\right). \tag{2.72}$$

We will usually be dealing with the non-relativistic case. Note that we must ensure that our momenta remain well below the Compton momentum,

$$k \ll k_c = \frac{2\pi}{\lambda_c}, \quad \lambda_c = \frac{\hbar}{mc}. \tag{2.73}$$

At this momentum, the energy is sufficient to produce new electrons (and other particles). For our non-relativistic particles, it is often easiest to use the Coulomb gauge: $\boldsymbol{\nabla}\cdot\vec{A} = 0$. Then Maxwell's equations become

$$\nabla^2 U = -\frac{\rho}{\epsilon_0} \text{ and } \Box\vec{A} = \mu_0\vec{j} - \frac{1}{c^2}\boldsymbol{\nabla}\frac{\partial}{\partial t}U. \tag{2.74}$$

Note that $U(\vec{r}, t)$ is completely specified by $\rho(\vec{r}, t)$. This is clearly non-relativistic, as changes to U violate locality.

2.4.3 Electrodynamics in Reciprocal Space

It is useful to convert differential equations into algebraic equations. We will use a transformation to reciprocal space (a Fourier transform) to do this.

$$\vec{\mathcal{E}}(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{r}\,\vec{E}(\vec{r}, t)e^{-i\vec{k}\cdot\vec{r}} \tag{2.75}$$

$$\vec{E}(\vec{k}, t) = \frac{1}{(2\pi)^{3/2}}\int d^3\vec{r}\,\vec{\mathcal{E}}(\vec{r}, t)e^{i\vec{k}\cdot\vec{r}}. \tag{2.76}$$

We will define similar pairs for $\vec{B} \leftrightarrow \vec{\mathcal{B}}$, $\vec{A} \leftrightarrow \vec{\mathcal{A}}$, $U \leftrightarrow \mathcal{U}$.

The field equations are "local" in reciprocal space:

$$i\vec{k} \cdot \vec{\mathcal{E}} = \frac{\rho}{\epsilon_0}, \tag{2.77}$$

$$i\vec{k} \cdot \vec{\mathcal{B}} = 0, \tag{2.78}$$

$$i\vec{k} \times \vec{\mathcal{E}} = -\dot{\vec{\mathcal{B}}}, \text{ and} \tag{2.79}$$

$$i\vec{k} \times \vec{\mathcal{B}} = \frac{1}{c^2}\dot{\vec{\mathcal{E}}} + \mu_0 \vec{j}. \tag{2.80}$$

While we still have time derivatives, the spatial derivatives have been eliminated. Note that we can already see that $\vec{\mathcal{B}}$ must be orthogonal to the direction of propagation.

The continuity equation becomes

$$i\vec{k} \cdot \vec{j} + \dot{\rho} = 0. \tag{2.81}$$

The relationships between fields and potentials become

$$\vec{\mathcal{B}} = i\vec{k} \times \vec{\mathcal{A}} \text{ and} \tag{2.82}$$

$$\vec{\mathcal{E}} = -\dot{\vec{\mathcal{A}}} - i\vec{k}\mathcal{U}. \tag{2.83}$$

Gauge transformations are now

$$\vec{\mathcal{A}} \to \vec{\mathcal{A}}' = \vec{\mathcal{A}} + i\vec{k}f \text{ and} \tag{2.84}$$

$$\mathcal{U} \to \mathcal{U}' = \mathcal{U} - \dot{f}. \tag{2.85}$$

The equations for the potentials become

$$k^2\mathcal{U} = \frac{\rho}{\epsilon_0} + i\vec{k} \cdot \dot{\vec{\mathcal{A}}} \text{ and} \tag{2.86}$$

$$\frac{1}{c^2}\ddot{\vec{\mathcal{A}}} + k^2\vec{\mathcal{A}} = \mu_0\vec{j} - i\vec{k}\left(i\vec{k} \cdot \vec{\mathcal{A}} + \frac{1}{c^2}\dot{\mathcal{U}}\right). \tag{2.87}$$

2.4.4 Longitudinal and transverse fields

We now define the longitudinal and transverse fields. Longitudinal fields have no curl:

$$\boldsymbol{\nabla} \times \vec{V}_{\parallel}(\vec{r}) = 0 \text{ or } i\vec{k} \times \vec{\mathcal{V}}_{\parallel}(\vec{k}) = 0. \tag{2.88}$$

Transverse fields have no divergence:

$$\boldsymbol{\nabla} \cdot \vec{V}_{\perp}(\vec{r}) = 0 \text{ or } i\vec{k} \cdot \vec{\mathcal{V}}_{\perp}(\vec{k}) = 0. \tag{2.89}$$

In reciprocal space, the decompositive is simply

$$\vec{\mathcal{V}}(\vec{k}) = \vec{\mathcal{V}}_{\parallel}(\vec{k}) + \vec{\mathcal{V}}_{\perp}(\vec{k}), \tag{2.90}$$

where

$$\vec{\mathcal{V}}_{\parallel}(\vec{k}) = \vec{\kappa}\left(\vec{\kappa} \cdot \vec{\mathcal{V}}(\vec{k})\right) \tag{2.91}$$

$$\mathcal{V}_\perp(\vec{k}) = \vec{\mathcal{V}}(\vec{k}) - \vec{\mathcal{V}}_\parallel(\vec{k}) = \left(\vec{\kappa} \times \vec{\mathcal{V}}\right) \times \vec{\kappa}. \tag{2.92}$$

where $\vec{\kappa}$ is a unit vector in the direction of $\vec{k}$.

From Maxwell's first equation,

$$\vec{\mathcal{B}}_\parallel = \vec{\kappa}\left(\vec{\kappa} \cdot \vec{B}\right) = \vec{0} = \vec{B}_\parallel. \tag{2.93}$$

The magnetic field is thus purely transverse. Looking at the second equation,

$$\vec{\mathcal{E}}_\parallel(\vec{k}) = -\frac{i}{\epsilon_0}\rho(\vec{k})\frac{\vec{k}}{k^2}. \tag{2.94}$$

Since this is a product of two functions of $\vec{k}$, the real space version is given by a convolution. The Fourier transform of $-\frac{i}{\epsilon_0}\frac{\vec{k}}{k^2}$ is

$$-\frac{i}{\epsilon_0}\frac{\vec{k}}{k^2} \leftrightarrow \frac{(2\pi)^{3/2}}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}. \tag{2.95}$$

Taking the convolution,

$$\vec{E}_\parallel(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\rho(\vec{r}',t)\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3} = \frac{1}{4\pi\epsilon_0}\sum_\alpha q_\alpha \frac{\vec{r}-\vec{r}_\alpha(t)}{|\vec{r}-\vec{r}_\alpha(t)|^3}. \tag{2.96}$$

This is the typical instantaneous Coulomb field of ρ. It turns out to be true regardless of the gauge, but an instantaneous component of $\vec{E}_\perp$ will cancel this effect.

Note that from $\vec{\mathcal{E}} = -\dot{\vec{\mathcal{A}}} - i\vec{k}\mathcal{U}$,

$$\vec{\mathcal{E}}_\perp = -\dot{\vec{\mathcal{A}}}_\perp \implies \vec{E}_\perp = -\dot{\vec{A}}_\perp \tag{2.97}$$

$$\vec{\mathcal{E}}_\parallel = -\dot{\vec{\mathcal{A}}}_\parallel - i\vec{k}\mathcal{U} \implies \vec{E}_\parallel = -\dot{\vec{A}}_\parallel - \boldsymbol{\nabla}U. \tag{2.98}$$

In the Coulomb gauge, $\boldsymbol{\nabla}\cdot\vec{A} = 0 \implies i\vec{k}\cdot\vec{\mathcal{A}} = 0 \implies \vec{\mathcal{A}}_\parallel = 0$. Therefore,

$$\vec{E}_\perp = -\dot{\vec{A}}_\perp \text{ and } \vec{E}_\parallel = -\boldsymbol{\nabla}U. \tag{2.99}$$

Thus, the Coulomb gauge potential is given by

$$U = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}'\frac{\rho(\vec{r}',t)}{|\vec{r}-\vec{r}'|}. \tag{2.100}$$

Gauge transformation can not change $\vec{A}_\perp$. This implies that $\vec{A}_\perp$ is more real.

In every gauge,

$$\vec{\mathcal{E}}_\perp = -\dot{\vec{\mathcal{A}}}_\perp \text{ and} \tag{2.101}$$

$$\vec{\mathcal{B}} = \vec{\mathcal{B}}_\perp = i\vec{k}\times\vec{\mathcal{A}}_\perp. \tag{2.102}$$

$\vec{A}_\perp$ has only two components, which hints at the existance of only two polarizations of light.

The longitudinal part of the last Maxwell's equation is

$$\dot{\vec{\mathcal{E}}}_\parallel + \frac{1}{\epsilon_0}\vec{j}_\parallel = 0 \implies i\vec{k}\cdot\dot{\vec{\mathcal{E}}}_\parallel + \frac{1}{\epsilon}i\vec{k}\cdot\vec{j}_\parallel = i\vec{k}\cdot\dot{\vec{\mathcal{E}}} + \frac{1}{\epsilon}i\vec{k}\cdot\vec{j} = \frac{\dot{\rho}}{\epsilon_0} + \frac{i\vec{k}\cdot\vec{j}}{\epsilon_0} = 0 \implies \dot{\rho} + \boldsymbol{\nabla}\cdot\vec{j} = 0, \tag{2.103}$$

which is simply the charge continuity equation.

Energy

We will now decompose the energy into longitudinal and transverse components.

From the Parseval-Plancherel theorem,

$$\frac{\epsilon_0}{2}\int d^3\vec{r}\vec{E}\cdot\vec{E}=\frac{\epsilon_0}{2}\int d^3\vec{k}\vec{\mathcal{E}}^*\cdot\vec{\mathcal{E}}. \tag{2.104}$$

Next, we decompose into longitudinal and transverse components. Note that $\vec{\mathcal{E}}_\perp\cdot\vec{\mathcal{E}}_\parallel=0$. Therefore,

$$\frac{\epsilon_0}{2}\int d^3\vec{r}\vec{E}^2=\frac{\epsilon_0}{2}\int d^3\vec{k}\left|\vec{\mathcal{E}}_\parallel(\vec{k})\right|^2+\frac{\epsilon_0}{2}\int d^3\vec{k}\left|\vec{\mathcal{E}}_\perp(\vec{k})\right|^2. \tag{2.105}$$

The longitudinal energy is thus

$$H_{\text{long}}=\frac{\epsilon_0}{2}\int d^3\vec{k}\left|\vec{\mathcal{E}}_\parallel(\vec{k})\right|^2=\frac{\epsilon_0}{2}\int d^3\vec{r}\vec{E}_\parallel(\vec{r}). \tag{2.106}$$

The transverse energy is

$$H_{\text{trans}}=\frac{\epsilon_0}{2}\int d^3\vec{k}\left(\left|\vec{\mathcal{E}}_\perp(\vec{k})\right|^2+c^2\left|\vec{\mathcal{B}}(\vec{k})\right|^2\right)=\frac{\epsilon_0}{2}\int d^3\vec{r}\left(\vec{E}_\perp^2(\vec{r})+c^2\vec{B}^2(\vec{r})\right). \tag{2.107}$$

H_{long} is the Coulomb energy:

$$H_{\text{long}}=\frac{1}{2\epsilon_0}\int d^3k\rho^*(\vec{k})\frac{\rho(\vec{k})}{k^2}=\frac{1}{8\pi\epsilon_0}\iint d^3rd^3r'\frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|}. \tag{2.108}$$

For a system of charges,

$$\rho(\vec{r})=\sum_\alpha q_\alpha\delta\left(\vec{r}-\vec{r}_\alpha(t)\right)\implies\rho(\vec{k})=\sum_\alpha\frac{q_\alpha}{(2\pi)^{3/2}}e^{-i\vec{k}\cdot\vec{r}_\alpha}. \tag{2.109}$$

Therefore,

$$H_{\text{long}}=V_{\text{Coulomb}}=\sum_\alpha\frac{q_\alpha^2}{2\epsilon_0\left(2\pi\right)^3}\int d^3k\frac{1}{k^2}+\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{2\epsilon_0\left(2\pi\right)^3}\int d^3k\frac{e^{-i\vec{k}(\vec{r}_\alpha-\vec{r}_\beta)}}{k^2}=\sum_\alpha\epsilon^\alpha_{\text{Coulomb}}+\frac{1}{8\pi\epsilon_0}\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\vec{r}_\alpha-\vec{r}_\beta|}. \tag{2.110}$$

The first term is the Coulomb self energy of the particle.

$$\epsilon^\alpha_{\text{Coulomb}}=\frac{q_\alpha^2}{2\epsilon_0\left(2\pi\right)^3}\int d^3k\frac{1}{k^2}=\frac{q_\alpha^2}{4\pi^2\epsilon_0}k_c. \tag{2.111}$$

It is infinite without some cutoff, which we choose to be the Compton momentum k_c. The second term is typical Coulomb interaction.

The full Hamiltonian is thus

$$H=\sum_\alpha\frac{1}{2}m_\alpha^2\dot{\vec{r}}_\alpha^2+V_{\text{Coulomb}}+H_{\text{trans}}. \tag{2.112}$$

Momentum

We again use the longitudinal transverse decomposition.

$$\vec{P}_{\text{long}} = \epsilon_0 \int d^3\vec{r}\vec{E}_{\parallel} \times \vec{B}(\vec{r}) = \epsilon_0 \int d^3k\vec{\mathcal{E}}^*_{\parallel}(\vec{k}) \times \vec{\mathcal{B}}(\vec{k}) \tag{2.113}$$

$$\vec{P}_{\text{trans}} = \epsilon_0 \int d^3\vec{r}\vec{E}_{\perp} \times \vec{B}(\vec{r}) = \epsilon_0 \int d^3k\vec{\mathcal{E}}^*_{\perp}(\vec{k}) \times \vec{\mathcal{B}}(\vec{k}). \tag{2.114}$$

We plug in $\vec{\mathcal{E}}_{\parallel} = -\frac{i}{\epsilon_0}\rho(\vec{k})\frac{\vec{k}}{k^2}$ and $\vec{\mathcal{B}} = i\vec{k} \times \vec{\mathcal{A}}$. Using these,

$$\vec{P}_{\text{long}} = \int d^3\vec{k}\rho^* \frac{\vec{k}}{k^2} \times \left(i\vec{k} \times \vec{\mathcal{A}}\right) = \int d^3\vec{k}\rho^* \left[\vec{\mathcal{A}} - \vec{\kappa}\left(\vec{\kappa} \cdot \vec{\mathcal{A}}\right)\right] = \int d^3\vec{k}\rho^* \vec{\mathcal{A}}_{\perp} = \int d^3r\rho\vec{A}_{\perp} = \sum_{\alpha} q_\alpha \vec{A}_{\perp}(\vec{r}_\alpha). \tag{2.115}$$

This momentum is at the position of each particle, so we can think of it as an additional momentum each particle gains. The total momentum is then

$$\vec{P} = \sum_{\alpha} \left(m_\alpha \dot{\vec{r}}_\alpha + q_\alpha \vec{A}_{\perp}(\vec{r}_\alpha)\right) + \vec{P}_{\text{trans}} \tag{2.116}$$

The terms are known as the mechanical momentum, the longitudinal field momentum, and the transverse field momentum. We then define

$$\vec{p}_\alpha = m_\alpha \dot{\vec{r}}_\alpha + q_\alpha \vec{A}_{\perp}(\vec{r}_\alpha), \tag{2.117}$$

which allows us to write

$$\vec{P} = \sum_{\alpha} \vec{p}_\alpha + \vec{P}_{\text{trans}}. \tag{2.118}$$

$\vec{p}_\alpha$ is the canonical/generalized momentum in the Coulomb gauge, so this definition and subsequent substitution is appropriate.

We can then write the total energy as

$$H = \sum_{\alpha} \frac{1}{2m_\alpha}\left[\vec{p}_\alpha - q_\alpha \vec{A}_{\perp}(\vec{r}_\alpha)\right]^2 + V_{\text{Coulomb}} + H_{\text{trans}} \tag{2.119}$$

Angular Momentum

The total angular momentum is

$$\vec{J} = \sum_{\alpha} \vec{r}_\alpha \times \vec{p}_\alpha + \epsilon_0 \int d^3\vec{r}\vec{r} \times \left(\vec{E}_{\perp} \times \vec{B}\right). \tag{2.120}$$

2.4.5 Normal Variables

Normal variables mean orthogonal, uncoupled variables. "Normal" in the sense of perpendicular.

Recall that for transverse fields, the last two Maxwell's equations become

$$\dot{\vec{\mathcal{B}}} = -i\vec{k} \times \vec{\mathcal{E}}_{\perp} \implies \vec{k} \times \dot{\vec{\mathcal{B}}} = ik^2\vec{\mathcal{E}}_{\perp} \tag{2.121}$$

$$\dot{\vec{\mathcal{E}}} = ic^2\vec{k}\times\vec{\mathcal{B}} - \frac{1}{\epsilon_0}\vec{j}_\perp(\vec{k},t). \tag{2.122}$$

These are coupled equations, but in reciprocal space can be simplified algebraically.

$$\vec{\mathcal{B}} = i\vec{k}\times\vec{\mathcal{A}} = i\vec{k}\times\vec{\mathcal{A}}_\perp \implies \vec{k}\times\vec{\mathcal{B}} = -ik^2\vec{\mathcal{A}}_\perp. \tag{2.123}$$

Thus, for $\omega = ck$,

$$\dot{\vec{\mathcal{A}}}_\perp = -\vec{\mathcal{E}}_\perp \text{ and } \dot{\vec{\mathcal{E}}}_\perp = \omega^2\vec{\mathcal{A}}_\perp - \frac{1}{\epsilon_0}\vec{j}_\perp(\vec{k},t). \tag{2.124}$$

This is analogous to a driven harmonic oscillator with

$$\dot{\vec{x}} = \frac{\vec{p}}{m} \text{ and} \frac{\dot{\vec{p}}}{m} = -\omega_0^2\vec{x} + \frac{q}{m}\vec{E}(t). \tag{2.125}$$

To uncouple these equations, we use the same procedure as for a harmonic oscillator. We define the normal variables (dropping to 1D for convenience)

$$\alpha = \mathcal{N}\left(x + i\frac{p}{m\omega_0}\right) \text{ and } \beta = \mathcal{N}\left(x - i\frac{p}{m\omega_0}\right) = \alpha^* \tag{2.126}$$

where $\mathcal{N}$ is a normalization factor. The equations of motion are then

$$\dot{\alpha} = \mathcal{N}\left(\dot{x} + i\frac{\dot{p}}{m\omega_0}\right) = \mathcal{N}\left(\frac{p}{m} - i\omega_0 x + i\frac{qE}{m\omega_0}\right) = -i\omega_0\alpha + i\frac{q\mathcal{N}}{m\omega_0}E. \tag{2.127}$$

We can quantize this:

$$\alpha \to a, \quad \beta \to a^\perp, \tag{2.128}$$

in which case we require

$$\left[a, a^\perp\right] = -\mathcal{N}^2\frac{i}{m\omega_0}2\left[x,p\right] = \mathcal{N}^2\frac{2\hbar}{m\omega_0} = 1. \tag{2.129}$$

From this, we determine the normalization to be

$$\mathcal{N} = \sqrt{\frac{m\omega_0}{2\hbar}}. \tag{2.130}$$

Following the analogy backwards,

$$\vec{\alpha}(\vec{k},t) = \mathcal{N}(k)\left(\vec{\mathcal{A}}_\perp - i\frac{\vec{\mathcal{E}}_\perp}{\omega_0}\right) \text{ and } \vec{\alpha}^*(\vec{k},t) = \mathcal{N}(k)\,() \tag{2.131}$$

Therefore,

$$\alpha^*(\vec{k},t) = -\frac{i}{ck}\mathcal{N}(\vec{k})\left(c\vec{\kappa}\times\vec{\mathcal{B}}^*(\vec{k}) - \vec{\mathcal{E}}^*_\perp(\vec{k})\right) = \ldots = +\vec{\beta}(-\vec{k},t). \tag{2.132}$$

The time evolution is

$$\dot{\vec{\alpha}}(\vec{k},t) + i\omega_0\vec{\alpha}(\vec{k},t) = \frac{i\mathcal{N}(\vec{k})}{\epsilon_0\omega}\vec{j}_\perp(\vec{k},t). \tag{2.133}$$

We can get the electric and magnetic fields from this, and verify that it is equivalent to Maxwell's equations.

$$\vec{\mathcal{E}}_\perp(\vec{k},t) = \frac{i}{2}\frac{\omega_0}{\mathcal{N}(k)}\left(\vec{\alpha}\vec{k},t) - \vec{\alpha}^*(-\vec{k},t)\right) \tag{2.134}$$

$$\vec{\mathcal{B}}(\vec{k},t) = \frac{i}{2}\frac{k}{\mathcal{N}(k)}\vec{\kappa}\times\left(\vec{\alpha}\vec{k},t) + \vec{\alpha}^*(\vec{k},t)\right). \tag{2.135}$$

When $\vec{j}_\perp = 0$, then we get the usual free evolution

$$\vec{\alpha}(\vec{k},t) = \vec{\alpha}(\vec{k},0)e^{-i\omega_0 t}. \tag{2.136}$$

In this case, $\vec{\alpha}$ are normal.

When $\vec{j}_\perp(\vec{k},t)$ is from an external source, that is, it is independent of $\vec{\alpha}(\vec{k},t)$, then each $\vec{\alpha}(\vec{k},t)$ evolves according to the Fourier coefficients of $\vec{j}_\perp(\vec{k},t)$. However, in many cases, $\vec{j}_\perp(\vec{k},t)$ is due to particles interacting with the field, which provides a coupling betwee then different values of $\vec{k}$ and t.

$\vec{\alpha}$ is composed of $\vec{\mathcal{E}})_\perp$ and $\vec{\mathcal{B}}$, so it is transverse. Thus, for each $\vec{k}$, $\vec{\alpha}$ can be expanded in terms of two unit vectors. We often denote these $\vec{\epsilon}$ and $\vec{\epsilon}'$, which are normalized and orthogonal with each other and $\vec{\kappa}$. Therefore,

$$\vec{\alpha}(\vec{k},t) = \vec{\epsilon}\alpha_\epsilon(\vec{k},t) + \vec{\epsilon}'\alpha'_\epsilon(\vec{k},t) = \sum_\epsilon \vec{\epsilon}\alpha_\epsilon(\vec{k},t) \text{ where } \alpha_\epsilon(\vec{k},t) = \vec{\epsilon}\cdot\vec{\alpha}(\vec{k},t). \tag{2.137}$$

The equation of motion is then

$$\dot{\alpha}_\epsilon + i\omega\alpha_\epsilon = \frac{i\mathcal{N}(k)}{\epsilon_0\omega}\vec{\epsilon}\cdot\vec{j}. \tag{2.138}$$

The (transverse) energy can be expressed in terms of α_ϵ as

$$H_{\text{trans}} = \frac{\epsilon_0}{2}\int d^3\vec{r}\left(\vec{E}_\perp^2(\vec{r}) + c^2\vec{B}^2(\vec{r})\right) \tag{2.139}$$

$$= \frac{\epsilon_0}{2}\int d^3\vec{k}\left(\vec{\mathcal{E}}_\perp^*(\vec{r})\vec{\mathcal{E}}_\perp(\vec{r}) + c^2\vec{\mathcal{B}}^*(\vec{r})\vec{\mathcal{B}}(\vec{r})\right) \tag{2.140}$$

$$= \epsilon_0\int d^3\vec{k}\frac{\omega_0^2}{4\mathcal{N}(k)}\left(\vec{\alpha}^*(\vec{k})\cdot\vec{\alpha}(\vec{k}) + \vec{\alpha}^*(-\vec{k})\cdot\vec{\alpha}(-\vec{k})\right). \tag{2.141}$$

When quantizing, we want this to be

$$\epsilon_0\int d^3\vec{k}\sum_\epsilon\frac{\hbar\omega}{2}\left(a_\epsilon^*(\vec{k},t)a_\epsilon(\vec{k},t) + a_\epsilon(\vec{k},t)a_\epsilon^*(\vec{k},t)\right). \tag{2.142}$$

This sets the normalization to

$$\mathcal{N}^2 = \frac{\epsilon_0\omega^2}{2\hbar\omega} = \frac{\epsilon_0\omega}{2\hbar} \implies \mathcal{N} = \sqrt{\frac{\epsilon_0\omega}{2\hbar}}. \tag{2.143}$$

This also implies the desired commutation relation,

$$\left[a(\vec{k}), a^\dagger(\vec{k}')\right] = \delta\left(\vec{k} - \vec{k}'\right). \tag{2.144}$$

The momentum becomes

$$\vec{P}_{\text{trans}} = \int d^3\vec{k} \sum_\epsilon \frac{\hbar\vec{k}}{2} \left(\alpha_\epsilon^* \alpha_\epsilon + \alpha_\epsilon \alpha \epsilon^*\right). \tag{2.145}$$

The fields can now be written

$$\vec{E}_\perp(\vec{r},t) = u \int d^3\vec{k} \sum_\epsilon \mathcal{E}_\omega \left(\alpha_\epsilon(\vec{k},t)\vec{\epsilon} e^{i\vec{k}\ cdot\vec{r}} - \alpha_\epsilon^*(\vec{k},t)\vec{\epsilon} e^{-i\vec{k}\ cdot\vec{r}}\right) \tag{2.146}$$

$$\vec{B}(\vec{r},t) = u \int d^3\vec{k} \sum_\epsilon \mathcal{B}_\omega \left(\alpha_\epsilon(\vec{k},t)\vec{\kappa}\times\vec{\epsilon} e^{i\vec{k}\ cdot\vec{r}} - \alpha_\epsilon^*(\vec{k},t)\vec{\kappa}\times\vec{\epsilon} e^{-i\vec{k}\ cdot\vec{r}}\right) \tag{2.147}$$

where

$$\mathcal{E}_\omega = \sqrt{\frac{\hbar\omega}{2\epsilon_0 (2\pi)^3}} \text{ and } \mathcal{B}_\omega = \frac{\mathcal{E}_\omega}{c}. \tag{2.148}$$

For free fields,

$$\vec{j}_\perp = 0 \implies \alpha_\epsilon(\vec{k},t) = \alpha_\epsilon(\vec{k}) e^{-i\omega t} \tag{2.149}$$

so

$$\vec{E}_\perp(\vec{r},t) = \int d^3\vec{k} \sum_\epsilon \mathcal{E}_\omega \left(\alpha_\epsilon(\vec{k})\vec{\epsilon} e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)} + \text{c.c.}\right). \tag{2.150}$$

For the complex electric field notation,

$$\vec{E}_\perp^+(\vec{r},t) = \int d^3\vec{k} \sum_\epsilon \mathcal{E}_\omega \alpha_\epsilon(\vec{k})\vec{\epsilon} e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}. \tag{2.151}$$

2.5 Quantization of the Electromagnetic Field

Now, let us actually quantize.

2.5.1 Boxes

Working in infinite space is difficult, so we will typically start with a box, then take infinite limits later. Consider a cubic box with side length L, which then has volume $V = L^3$. We will assume periodic boundary conditions. Momentum is thus quantized, with

$$k_{x,y,z} = \frac{2\pi}{L} n_{x,y,z}. \tag{2.152}$$

For periodic boundary conditions, $n_{x,y,z} \in \mathbb{Z}^3$. We then write our normal variables as

$$\alpha_\epsilon(\vec{k},t) \to \alpha_{\vec{k},\epsilon}(t) \text{ or } \alpha_i \text{ with } \left(\vec{k}_i, \vec{\epsilon}_i\right). \tag{2.153}$$

Integrals become sums according to

$$\int d^3\vec{k} \sum_\epsilon f\left(\vec{k},\vec{\epsilon}\right) \leftrightarrow \left(\frac{2\pi}{L}\right)^3 \sum_i f\left(\vec{k}_i,\vec{\epsilon}_i\right). \tag{2.154}$$

In our new notation, the energy of a single field mode is written

$$H_i = \frac{\epsilon_0}{2} \frac{(2\pi)^3}{V} \left(|\mathcal{E}_i|^2 + \omega_i^2 |\mathcal{A}_i|^2\right). \tag{2.155}$$

2.5.2 Quantization

For quantization, we map

$$\mathcal{A}_i \to \hat{\mathcal{A}}_i \text{ and } \mathcal{E}_i \to \hat{\mathcal{E}}_i. \tag{2.156}$$

For the harmonic oscillator, $[\hat{x}, \hat{p}] = i\hbar$. Using our analogy from earlier, this implies the commutation relation

$$\left[\hat{\mathcal{A}}_i, \hat{\mathcal{E}}_i\right] = -\frac{V}{(2\pi)^3 \epsilon_0} i\hbar. \tag{2.157}$$

We have

$$\left[\hat{a}, \hat{a}^\dagger\right] = 1 \text{ for } \mathcal{N} = \sqrt{\frac{\epsilon_0 \omega_i}{2\hbar} \frac{(2\pi)^3}{V}}. \tag{2.158}$$

For each mode, the Hamiltonian is

$$\hat{H}_i = \frac{\hbar\omega_i}{2}\left(\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger\right). \tag{2.159}$$

The full Hamiltonian is then

$$\hat{H} = \sum_i \hbar\omega_i \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}\right), \tag{2.160}$$

using the commutation relation to simplify slightly.

The momentum is

$$\hat{P} = \sum_i \hbar\vec{k}_i \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}\right). \tag{2.161}$$

Since the average momentum is zero ($\sum_i \vec{k}_i = 0$), this simplifies to

$$\hat{P} = \sum_i \hbar\vec{k}_i \hat{a}_i^\dagger \hat{a}_i. \tag{2.162}$$

The electric field operator can be written

$$\hat{\vec{E}}_\perp(\vec{r}) = i\sum_i \mathcal{E}_i \left(\vec{\epsilon}_i \hat{a}_i e^{i\vec{k}_i\cdot\vec{r}} - \vec{\epsilon}_i \hat{a}_i^\dagger e^{-i\vec{k}_i\cdot\vec{r}}\right) \text{ with } \mathcal{E}_i = \sqrt{\frac{\hbar\omega_i}{2\epsilon_0 V}}. \tag{2.163}$$

Note that

$$\left[\hat{\vec{E}}_\perp(\vec{r}), \hat{\vec{E}}_\perp(\vec{r}')\right] \neq 0, \tag{2.164}$$

which implies there are no eigenstates of $\hat{\vec{E}}_\perp(\vec{r})$ for all $\vec{r}$.

Henceforth, we will stop putting hats on operators.

2.5.3 Combining with the Particles

Combining with the particles again, the total Hamiltonian is

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left(\vec{p}_\alpha - q_\alpha \vec{A}_\perp(\vec{r}_\alpha)\right)^2 + \sum_\alpha \left(-g_\alpha \frac{q_\alpha}{2m_\alpha}\right)\vec{S}_\alpha \cdot \vec{B}(\vec{r}_\alpha) + V_{\text{Coulomb}} + H_R. \tag{2.165}$$

The second term is new, this is a manually inserted spin interaction with the magnetic field. g_α is a g-factor (2 for electrons). Recall

$$V_{\text{Coulomb}} = \sum_\alpha \epsilon^\alpha_{\text{Coulomb}} + \frac{1}{8\pi\epsilon_0} \sum_{\alpha\neq\beta} \frac{q_\alpha q_\beta}{|\vec{r}_\alpha - \vec{r}_\beta|} \text{ with } \epsilon^\alpha_{\text{Coulomb}} = \frac{q_\alpha}{4\pi^2\epsilon_0} k_c. \tag{2.166}$$

The field Hamiltonian is

$$H_R = \sum_i \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2} \right). \tag{2.167}$$

We can also divide the Hamiltonian by interaction:

$$H_I = H_{I1} + H_{I1}^S + H_{I2}, \tag{2.168}$$

where

$$H_{I1} = \sum_\alpha \frac{q_\alpha}{m_\alpha} \vec{p}_\alpha \cdot \vec{A}_\perp(\vec{r}_\alpha), \tag{2.169}$$

$$H_{I1}^S = -\sum_\alpha g_\alpha \frac{q_\alpha}{2m_\alpha} \vec{S}_\alpha \cdot \vec{B}(\vec{r}_\alpha), \tag{2.170}$$

and

$$H_{I2} = \ldots. \tag{2.171}$$

2.5.4 State Space

What is the Hilbert space that this new Hamiltonian acts in?

It can be written as a tensor product of the particle space and the field space:

$$(\text{Particles}) \otimes (\text{Fields}), \tag{2.172}$$

where the particles space is the tensor product of a space for each particle. The fields space is the space of number states for each field mode.

$$H_R |\{n_i\}\rangle = \left(\sum_i \left(n_i + \frac{1}{2} \right) \hbar\omega_i \right) |\{n_i\}\rangle. \tag{2.173}$$

$$P_R |\{n_i\}\rangle = \left(\sum_i n_i \hbar \vec{k}_i \right) |\{n_i\}\rangle. \tag{2.174}$$

$|\{n_i\}\rangle$ denotes a state with n_1 photons in mode 1, n_i photons in mode i, etc. Recall

$$a_i |n_i\rangle = \sqrt{n_i} |n_i - 1\rangle \text{ for } n_i \neq 0 \tag{2.175}$$

$$a_i^\dagger |n_i\rangle = \sqrt{n_i + 1} |n_i + 1\rangle \tag{2.176}$$

$$a_i |0\rangle = 0. \tag{2.177}$$

2.5.5 The Vacuum

$|0\rangle$ is the vacuum.

$$H\,|0\rangle = \sum_i \frac{1}{2}\hbar\omega_i\,|0\rangle = E_v\,|0\rangle\,. \tag{2.178}$$

Note that

$$\langle 0|H|0\rangle = 0 \tag{2.179}$$

since $a_i\,|0\rangle = 0$ and $\langle 0|\,a_i^\dagger = 0$.

However, the vacuum is not exactly nothing. $\langle 0|E^2|0\rangle$ contains $aa^\dagger$, and $\langle 0|aa^\dagger|0\rangle = 1$. Thus,

$$\langle 0|E^2|0\rangle = \sum_i |\mathcal{E}_i|^2 = \sum_i \frac{\hbar\omega_i}{2\epsilon_0 V}. \tag{2.180}$$

Therefore, the uncertainty of the vacuum energy $\Delta\vec{E}$ is not zero. We have fluctuations of the vacuum. This is implied by the Heisenberg uncertainty relation. This leads to a few phenomena, one of which is a vacuum Stark Effect. Since $\Delta E \propto E^2$, we will get a vacuum Stark Shift. This is the Lamb Shift.

2.5.6 Dipole Interaction

How do we get a dipole interaction?

Make a long wavelength approximation: $\lambda \gg a_0$. Then the Hamiltonian can be written

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left(\vec{p}_\alpha - q_\alpha \vec{A}_\perp(\vec{0})\right)^2 + V_{\text{Coulomb}} + \sum_i \hbar\omega_i\left(a_i^\dagger a_i + \frac{1}{2}\right). \tag{2.181}$$

We define a unitary transformation

$$T = \exp\left(-\frac{i}{\hbar}\vec{d}\cdot\vec{A}_\perp(\vec{0})\right) = \exp\left(\sum_i\left(\lambda_i^* a_i - \lambda_i a_i^\dagger\right)\right) \tag{2.182}$$

where λ_i are dipole field interaction strengths, given by

$$\lambda_i = \frac{i}{\sqrt{2\epsilon_0\hbar\omega_i V}}\vec{\mathcal{E}}_i\cdot\vec{d}. \tag{2.183}$$

These use the classical definition of the dipole, $\vec{d}$. This transformation translates the fields and the particle momenta, since $\vec{d} = \sum_\alpha q_\alpha \vec{r}_\alpha$.

First, note some conjugation relations

$$T\vec{r}_\alpha T^\dagger = 0, \tag{2.184}$$

$$T\vec{p}_\alpha T^\dagger = \vec{p}_\alpha + q_\alpha\vec{A}(\vec{0}), \tag{2.185}$$

$$Ta_iT^\dagger = a_i + \lambda_i, \text{ and} \tag{2.186}$$

$$Ta_i^\dagger T^\dagger = a_i^\dagger + \lambda_i^*. \tag{2.187}$$

Applying the transformation to the Hamiltonian,

$$H' = THT^\dagger = \sum_\alpha \frac{p_\alpha^2}{2m_\alpha} + V_{\text{Coulomb}} + \mathcal{E}_{\text{dipole}} + \sum_i \hbar\omega_i \left(a_i^\dagger a_i + \frac{1}{2}\right) - \vec{d}\cdot i \sum_i \mathcal{E}_i \left(a_i \vec{\epsilon}_i - a_i^\dagger \vec{\epsilon}_i\right). \quad (2.188)$$

The term

$$i \sum_i \mathcal{E}_i \left(a_i \vec{\epsilon}_i - a_i^\dagger \vec{\epsilon}_i\right) \quad (2.189)$$

is just the electric field. The $\mathcal{E}_{\text{dipole}}$ term is the dipole self-energy

$$\mathcal{E}_{\text{dipole}} = \sum_i \frac{1}{2\epsilon_0 V} \left(\vec{\mathcal{E}}_i \cdot \vec{d}\right)^2. \quad (2.190)$$

The velocity operator for particles under this transformation is

$$\vec{v}'_\alpha = T\vec{v}_\alpha T^\dagger = \frac{p_\alpha}{m}, \quad (2.191)$$

which is the mechanical momentum over mass.

The fields transform as

$$\vec{A}'_\perp = T\vec{A}_\perp T^\dagger = \vec{A}_\perp \quad (2.192)$$

$$\vec{E}'_\perp = T\vec{E}_\perp T^\dagger = \vec{E}_\perp - \frac{1}{\epsilon_0}\vec{P}_\perp, \quad (2.193)$$

where $\vec{P}_\perp$ is the polarization density operator, which can be shown to be

$$\vec{P}_\perp = \sum_i \frac{\vec{\epsilon}_i \left(\vec{\epsilon}_i \cdot \vec{d}\right)}{V} e^{i\vec{k}_i \cdot \vec{r}}. \quad (2.194)$$

This is analogous to the polarization of a material in classical electromagnetism. The electric field becomes (approximately) the displacement field (typically denoted $\vec{D}$) under this transformation.

Now, let us consider the transformation of the displacement field:

$$\vec{D}'_\perp = \epsilon_0 \vec{E}'_\perp + \vec{P}'_\perp = \vec{E}_\perp = i \sum_j \mathcal{E}_j \left(a_j \vec{\epsilon}_j e^{i\vec{k}_j \cdot \vec{r}} - a_j^\perp \vec{\epsilon}_j e^{-i\vec{k}_j \cdot \vec{r}}\right). \quad (2.195)$$

The dipole Hamiltonian is then

$$H'_I = -\vec{d} \cdot \frac{\vec{D}'(0)}{\epsilon_0} = -\vec{d} \cdot \vec{E}_\perp(0). \quad (2.196)$$

The transformed Hamiltonian is purely linear, which is nice.

2.6 Quantum States of Light

2.6.1 Review and Notation Change

Recall that we had an analogy between the electromagnetic field and a harmonic oscillator. Here we change Fourier transform convention, which eliminates some factors of 2π:

Our fields are

$$\vec{E}_\perp(\vec{r}) = i\sum_j \mathcal{E}_j\vec{\epsilon}_j\left(a_j e^{i\vec{k}_j\cdot\vec{r}} - a_j^\dagger e^{-i\vec{k}_j\cdot\vec{r}}\right) \tag{2.197}$$

$$\vec{V}(\vec{r}) = i\sum_j \mathcal{E}_j\frac{\vec{k}_j\times\vec{\epsilon}_j}{\omega_j}\left(a_j e^{i\vec{k}_j\cdot\vec{r}} - a_j^\dagger e^{-i\vec{k}_j\cdot\vec{r}}\right) \tag{2.198}$$

$$\vec{A}_\perp(\vec{r}) = i\sum_j \frac{\mathcal{E}_j}{\omega_j}\vec{\epsilon}_j\left(a_j e^{i\vec{k}_j\cdot\vec{r}} - +a_j^\dagger e^{-i\vec{k}_j\cdot\vec{r}}\right) \tag{2.199}$$

where $\mathcal{E}_j = \sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V}}$. The normalization constant satisfies $\epsilon_0\mathcal{E}^2 V = \frac{1}{2}\hbar\omega$, which makes sense as the amplitude of a single photon.

There is an uncertainty relation between $\vec{E}_\perp$ and $\vec{A}$:

$$\Delta\mathcal{A}_{\perp,j}\Delta\mathcal{E}_{\perp,j} \geq \frac{\hbar V}{2\epsilon_0}. \tag{2.200}$$

The first order vacuum expectation values are all zero. The fluctuations are

$$\left(\Delta\vec{E}_\perp\right)^2 = \sum_j \frac{\hbar\omega_j}{2\epsilon_0 V} \tag{2.201}$$

$$\tag{2.202}$$

2.6.2 Number States

Suppose we have $n_i = 0\forall i \neq j$, and $n_j \neq 0$. Let $|n_j\rangle = |\{n_i\}\rangle$. Note that the other states still exist, they simply have no excitations. The first order expectation values of this state are

$$\langle n_j|\vec{E}_\perp|n_j\rangle = \langle n_j|\vec{B}|n_j\rangle = 0, \tag{2.203}$$

as $\vec{E}_\perp$ and $\vec{B}$ contain raising and lowering operators, which will lead to the final inner product being zero. The fluctuations are

$$c^2\Delta B^2 = \Delta E^2 = \langle n_j|E_\perp^2|n_j\rangle = (2n_j+1)\,\mathcal{E}_j^2, \tag{2.204}$$

$2n_j + 1$ times the vacuum value. This seems quite uncertain. However, like a harmonic oscillator, number states are not well localized in $\vec{E}_\perp$-$\vec{A}_\perp$ phase space. Number states form a annulus in $\vec{E}_\perp$-$\vec{A}_\perp$ phase space: the magnitude of the electric field is well defined, but phase is not.

2.6.3 Coherent States

Another frequently encountered class of states are the coherent states. Coherent states are also known as quasi-classical states or Glauber states.

A classical electric field can be written

$$\vec{E}_{\text{classical}} = \sum_j \mathcal{E}_j\vec{\epsilon}_j\left(\alpha_j e^{i\vec{k}\cdot\vec{r}} + \text{c.c.}\right). \tag{2.205}$$

We would like to find a quantum field that is similar. Considering the quantum electric field,

$$\vec{E}_{\text{quantum}} = \sum_j \mathcal{E}_j \vec{\epsilon} \left(a_j e^{i\vec{k}\cdot\vec{r}} + \text{c.c.} \right), \tag{2.206}$$

this would require a state that is an eigenstate of a_j. Note that since $\left[a_j, a_j^\dagger\right] = 1 \neq 0$, this state will not be an eigenstate of $a_j^\dagger$, but perhaps we can find something close enough.

The Eigenstate of a

We define a coherent state as an eigenstate of the annihilation operator a:

$$a \left|\alpha\right\rangle = \alpha \left|\alpha\right\rangle. \tag{2.207}$$

We would like to write this in terms of number states

$$\left|\alpha\right\rangle = \sum_{n=0}^{\infty} c_n \left|n\right\rangle \tag{2.208}$$

for some constants c_n.

$$\alpha \left|\alpha\right\rangle = a \left|\alpha\right\rangle \tag{2.209}$$

$$\implies \alpha \sum_{n=0}^{\infty} c_n \left|n\right\rangle = a \sum_{n=0}^{\infty} c_n \left|n\right\rangle = \sum_{n=0}^{\infty} c_n a \left|n\right\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n} \left|n-1\right\rangle = \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} \left|n\right\rangle \tag{2.210}$$

$$\implies c_{n+1}\sqrt{n+1} = \alpha c_n \tag{2.211}$$

$$\implies c_n = \frac{\alpha}{\sqrt{n}} c_{n-1} = \frac{\alpha^n}{\sqrt{n!}} c_0. \tag{2.212}$$

We then define c_0 in terms of a normalization:

$$\langle\alpha|\alpha\rangle = 1 = |c_0|^2 \sum_{n,m} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!m!}} \langle n|m\rangle = |c_0|^2 \sum_n \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2} \implies c_0 = e^{-|\alpha|^2/2}. \tag{2.213}$$

So, coherent states are

$$\left|\alpha\right\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left|n\right\rangle. \tag{2.214}$$

Note that this is not an eigenstate of $a^\dagger$:

$$a^\dagger \left|\alpha\right\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \sqrt{n+1} \left|n+1\right\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{n\alpha^{n-1}}{\sqrt{n!}} \left|n\right\rangle. \tag{2.215}$$

This is the derivative of $\left|\alpha\right\rangle$ with respect to α, except for the normalization.

Time Evolution

Let us consider the time evolution of a coherent state. Suppose at $t = 0$ we have the state $|\alpha\rangle$. Then this evolves according to

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-iHt/\hbar}|n\rangle \tag{2.216}$$

$$= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i(n+1/2)\omega t}|n\rangle \tag{2.217}$$

$$= e^{-|\alpha|^2/2}e^{-i\omega t/2}\sum_{n=0}^{\infty}\frac{\left(\alpha e^{-i\omega t}\right)^n}{\sqrt{n!}} = e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle. \tag{2.218}$$

Thus, it is simply another coherent state. This is the same evolution we would expect for a classical harmonic oscillator: the system exchanges energy between the "position" and "momentum" as the phase rotates.

Distribution Properties

What is the probability of measuring the system in state $|n\rangle$ (that is, having the energy $\left(n+\frac{1}{2}\right)\hbar\omega$)? This is

$$P(n) = |c_n|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}. \tag{2.219}$$

This is a Poisson distribution. The mean photon number is

$$\langle n\rangle = barn = \langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2. \tag{2.220}$$

We can also calculate this from the moments of the Poisson distribution. This means we can also write our distribution as

$$P(n) = e^{-\bar{n}}\frac{\bar{n}^n}{n!}. \tag{2.221}$$

Recall that the width of a Poisson distribution is given by the square root of the expected value. For large $\bar{n}$, the distribution of number state expected values will look like a Gaussian. We now compute

$$\Delta n^2 = \langle n^2\rangle - \langle n\rangle^2 = \langle\alpha|a^\dagger a a^\dagger a|\alpha\rangle - |\alpha|^4 = \langle\alpha|a^\dagger\left(a^\dagger a+1\right)a|\alpha\rangle - |\alpha|^4 = |\alpha|^4 + \bar{n} - |\alpha|^4 = \bar{n}, \tag{2.222}$$

which is exactly what we expect for a Poisson distribution.

Now we reintroduce the electric field. The expected value is

$$\langle\Psi(t)|\vec{E}_\perp|\Psi(t)\rangle = \sum_j i\mathcal{E}_j\vec{\epsilon}_j\left[\alpha e^{i\left(\vec{k}_j\cdot\vec{r}-\omega t\right)} - \alpha^* e^{-i\left(\vec{k}_j\cdot\vec{r}-\omega t\right)}\right]. \tag{2.223}$$

This resembles the classical field nicely. However, there is some non-classical width:

$$\langle\Psi(t)|E_\perp^2|\Psi(t)\rangle = \Big(\langle\Psi(t)|\vec{E}_\perp|\Psi(t)\rangle\Big)^2 + \mathcal{E}_j^2. \tag{2.224}$$

This implies $\Delta E_\perp^2 = \mathcal{E}_j^2$. This is small, the same as that of the vacuum, but it is non-zero.

An overcomplete, non-orthgonal basis

A downside to the coherent states is that they are not orthogonal. We find

$$|\langle\beta|\alpha\rangle|^2 = \left| e^{-|\alpha|^2/2} e^{-|\beta|^2/2} \sum_{m,n} \frac{\alpha^n \beta^{*m}}{\sqrt{n!m!}} \langle n|m\rangle \right|^2 \tag{2.225}$$

$$= e^{-|\alpha|^2} e^{-\beta|^2} \left| \sum_n \frac{(\alpha\beta^*)^n}{n!} \right|^2 = e^{-|\alpha|^2} e^{-\beta|^2} \left| e^{\alpha\beta^*} \right|^2 = e^{-|\alpha-\beta|^2}. \tag{2.226}$$

So, they are not orthogonal, but they overlap less with distance.

Coherent states are still a complete (in fact overcomplete) basis of the Hilbert space. One can show this by finding

$$\frac{1}{\pi} \int d(\operatorname{Re}\alpha)\ d(\operatorname{Im}\alpha)\, |\alpha\rangle\langle\alpha| = \mathbb{I}. \tag{2.227}$$

Calculating,

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \int \frac{d^2\alpha}{\pi} \left[e^{-|\alpha|^2/2} \sum_{m=0}^{\infty} \frac{\alpha^{*m}}{\sqrt{m!}} |m\rangle \right] \left[e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle n| \right] \tag{2.228}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} \frac{\alpha^{*m}}{\sqrt{m!}} \frac{\alpha^n}{\sqrt{n!}} |m\rangle\langle n| \tag{2.229}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|m\rangle\langle n|}{\pi\sqrt{m!n!}} \int_0^{\infty} d|\alpha| \int_0^{2\pi} d\arg\alpha \ \ |\alpha|^{m+n+1} e^{i(n-m)\arg\alpha} e^{-|\alpha|^2} \tag{2.230}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|m\rangle\langle n|}{\pi\sqrt{m!n!}} \int_0^{\infty} d|\alpha||\alpha|^{m+n+1} e^{-|\alpha|^2} \int_0^{2\pi} d\arg\alpha \ \ e^{i(n-m)\arg\alpha} \tag{2.231}$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{|m\rangle\langle n|}{\pi\sqrt{m!n!}} \int_0^{\infty} d|\alpha||\alpha|^{m+n+1} e^{-|\alpha|^2} 2\pi\delta_{n,m} \tag{2.232}$$

$$= \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} 2 \int_0^{\infty} d|\alpha||\alpha|^{2n+1} e^{-|\alpha|^2} \tag{2.233}$$

$$= \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} n! = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{I}. \tag{2.234}$$

The first step follows from the form of coherent states. The second step is a simple rearrangement. In the third step, we convert to polar coordinates for the integration, introducing an additional factor of $|\alpha|$ from the Jacobian. In the fourth step, we rearrange again. In the fifth step, we evaluate the azimuthal integral. This integral is recognizable as a form of the Kronecker delta function, which is one of the best delta functions. The sixth step uses the Kronecker delta function to simplify the radial integral. In the seventh step, we evaluate the radial integral. Finally, we cancel various terms to find the anticipated identity operator.

We can attempt to write a number state in terms of coherent states:

$$|n\rangle = \frac{1}{\pi} \int d^2\alpha\ |\alpha\rangle \langle\alpha|n\rangle \tag{2.235}$$

$$= \frac{1}{\pi} \int d^2\alpha \, e^{-|\alpha|^2/2} \frac{(\alpha^*)^n}{\sqrt{n!}} |\alpha\rangle \,. \tag{2.236}$$

This will be peaked for $|\alpha| = \sqrt{n}$. Note that the weight is independent of $\arg \alpha$, which makes sense, as the phase of number states is not well defined. We can also write a coherent state in terms of other coherent states:

$$|\beta\rangle = \frac{1}{\pi} \int d^2\alpha \, |\alpha\rangle \langle\alpha|\beta\rangle \tag{2.237}$$

$$= \frac{1}{\pi} \sum_{n=0}^{\infty} \int d^2\alpha \, e^{-\left(|\alpha|^2+|\beta|^2\right)/2+\alpha^*\beta} |\alpha\rangle \,. \tag{2.238}$$

The over-completeness implies that for any state $|\psi\rangle$,

$$|\psi\rangle = \int d^2\alpha \, c_\alpha |\alpha\rangle \;\text{ with } c_\alpha = \frac{1}{\pi} \langle\alpha|\psi\rangle \,. \tag{2.239}$$

However, this decomposition is not unique, as $\langle\alpha|\alpha'\rangle \neq 0$.

Coherent States as Displaced Vacuum

The displacement operator is

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}. \tag{2.240}$$

Then a coherent state can be written (no proof here, but it can be proven)

$$|\alpha\rangle = D(\alpha) |0\rangle \,. \tag{2.241}$$

For real α, the displacement operator becomes

$$D(\alpha) = e^{\alpha\left(a^\dagger - a\right)}, \tag{2.242}$$

which is a position translation operator for a position-momentum harmonic oscillator, so this makes sense.

2.6.4 Statistical Properties of Quantum Light

Phase Space

Let us first consider a classical harmonic oscillator in which $m\omega = \hbar = 1$. Then the creation and annihilation operators can be written

$$\alpha^\dagger = \frac{x + ip}{\sqrt{2}} \text{ and } \alpha = \frac{x + ip}{\sqrt{2}}. \tag{2.243}$$

The inverse of these gives

$$x = \frac{\alpha + \alpha^\dagger}{\sqrt{2}} \text{ and } p = \frac{\alpha - \alpha^\dagger}{\sqrt{2}i}. \tag{2.244}$$

We can also write these $p = \sqrt{2}\,\mathrm{Im}\,\alpha$ and $x = \sqrt{2}\,\mathrm{Re}\,\alpha$. The x-p plane is then an α plane. This is our phase space. Classically, any point in this plane represents a state of the system, which will follow a fixed trajectory. Quantum mechanically, uncertainty requires that we not have a single point, but rather a larger region.

Coherent State Basis Expansion

We can expand an arbitrary operator in a coherent state basis:

$$O = \frac{1}{\pi}\int d^2\alpha O\,|\alpha\rangle\langle\alpha| = \frac{1}{\pi^2}\int d^2\alpha \int d^2\beta\,|\beta\rangle\langle\beta|\,O\,|\alpha\rangle\langle\alpha|\,. \tag{2.245}$$

We have

$$\langle\beta|O|\alpha\rangle = \sum_{m,n}\langle\beta|m\rangle\ \langle m|O|n\rangle\,\langle n|\alpha\rangle = e^{-\left(|\alpha|^2+|\beta|^2\right)/2}\sum_{m,n}\langle m|O|n\rangle\,\frac{\left(\beta^*\right)^m\alpha^n}{\sqrt{m!n!}}. \tag{2.246}$$

We can obtain the trace of O in the number basis, and we want to show this works for the coherent state basis too:

$$\mathrm{Tr}\,O = \sum_n\langle n|O|n\rangle = \frac{1}{\pi}\int d^2\alpha\sum_n\langle n|O|\alpha\rangle\,\langle\alpha|n\rangle \tag{2.247}$$

$$= \frac{1}{\pi}\int d^2\alpha\sum_n\langle\alpha|n\rangle\ \langle n|O|\alpha\rangle = \frac{1}{\pi}\int d^2\alpha\,\langle\alpha|O\rangle\,\alpha. \tag{2.248}$$

For the density matrix,

$$\rho = \frac{1}{\pi^2}\int d^2\alpha\int d^2\beta\ |\beta\rangle\ \langle\beta|\rho|\alpha\rangle\,\langle\alpha|\,. \tag{2.249}$$

The trace of a density matrix is one, so

$$\mathrm{Tr}\,\rho = \frac{1}{\pi}\int d^2\alpha\ \langle\alpha|\rho|\alpha\rangle = 1. \tag{2.250}$$

We can express the density matrix using the Glauber P-distribution, which is not quite a probability distribution:

$$\rho = \int d^2\alpha\,P(\alpha)\,|\alpha\rangle\langle\alpha|\,. \tag{2.251}$$

$P(\alpha)$ has some properties like a probability distribution:

$$\int d^2\alpha\,P(\alpha) = \int d^2\alpha\,P(\alpha)\sum_n\langle n|\alpha\rangle\,\langle\alpha|n\rangle = \sum_n\langle n|\rho|n\rangle = 1. \tag{2.252}$$

The diagonal elements of the density matrix in the coherent state basis are

$$\langle\beta|\rho|\beta\rangle = \int d^2\alpha\,P(\alpha)\,\langle\beta|\alpha\rangle\,\langle\alpha|\beta\rangle = \int d^2\alpha\,P(\alpha)e^{-|\alpha-\beta|^2}. \tag{2.253}$$

$P(\alpha)$ can be used to calculate expectation values

$$\langle O\rangle = \mathrm{Tr}\,(\rho O) = \sum_n\langle n|\int d^2\alpha\,P(\alpha)\,|\alpha\rangle\ \langle\alpha|O|n\rangle \tag{2.254}$$

$$= \sum_n\int d^2\alpha\,P(\alpha)\sum_n\langle n|\alpha\rangle\ \langle\alpha|O|n\rangle = \int d^2\alpha\,P(\alpha)\,\langle\alpha|O|\alpha\rangle\,. \tag{2.255}$$

To use this, write O in terms of $a^\dagger$ and a, apply normal ordering, and then perform the integral.

Although $P(\alpha)$ integrates to unity, it can become singular or negative. So it is not a probability density.

Q Distributions

To get closer to a probability distribution, we consider the Q distribution.

This is an approximation of the probability distribution given by

$$Q_\rho(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha\rangle = \int d^2\alpha'\, P(\alpha') e^{-|\alpha-\alpha'|^2}. \tag{2.256}$$

Note that this is always positive, which is nice. If $P(\alpha')$ varies slowly on the scale of the Gaussian $e^{-|\alpha-\alpha'|^2}$, we can approximate this as

$$Q_\rho(\alpha) \approx P(\alpha) \int d^2\alpha\, e^{-|\alpha-\alpha'|^2} = \pi P(\alpha). \tag{2.257}$$

Example: The Vacuum

Suppose $\rho = |0\rangle\langle 0|$. Then

$$Q_{|0\rangle}(\alpha) = \frac{1}{\pi}|\langle\alpha|0\rangle|^2 = \frac{1}{\pi}e^{-|\alpha|^2}. \tag{2.258}$$

Plotting this on the α plane (phase space), we see a Gaussian centered at the origin with unity width.

Example: Coherent State

Suppose $\rho = |\beta\rangle\langle\beta|$. Then

$$Q_\beta(\alpha) = \frac{1}{\pi}|\langle\alpha|\beta\rangle|^2 = \frac{1}{\pi}e^{-|\alpha-\beta|^2}. \tag{2.259}$$

Plotting this on the α plane (phase space), we see a Gaussian centered at β, again with unity width. This is a displaced vacuum state.

Example: Number State

Suppose $\rho = |n\rangle\langle n|$. $P(\alpha)$ would likely be unpleasant. The Q distribution is

$$Q_n(\alpha) = \frac{1}{\pi}|\langle n|\alpha\rangle|^2 = \frac{1}{\pi}e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}. \tag{2.260}$$

For large n, a 1D slice will be approach a Gaussian. From this,

$$Q_n(\alpha) \approx \frac{1}{\pi}\frac{1}{\sqrt{2\pi|\alpha|^2}}e^{-\left(n-|\alpha|^2\right)^2/2|\alpha|^2} \tag{2.261}$$

$$\approx \frac{1}{\pi}\frac{1}{\sqrt{2\pi n}}e^{-\left(n-|\alpha|^2\right)^2/2n} \tag{2.262}$$

$$= \frac{1}{\pi}\frac{1}{\sqrt{2\pi n}}e^{-\left(\sqrt{n}-|\alpha|\right)^2\left(\sqrt{n}+\alpha\right)^2/2n} \tag{2.263}$$

$$\approx \frac{1}{\pi}\frac{1}{\sqrt{2\pi n}}e^{-2\left(\sqrt{n}-|\alpha|\right)^2}. \tag{2.264}$$

Looking at this in 2D, we have a ring of radius $\sqrt{n}$ and thickness 1. The ring nature is clear from the independence of the expression from $\arg\alpha$. Note that the width is mostly a consequence of how the Q distribution works, a true number state is better defined.

Example: Thermal State

Consider the thermal state

$$\rho = \frac{e^{-\beta H}}{Z} = \sum_n \frac{e^{-\beta n\hbar\omega}}{Z} |n\rangle\langle n| \text{ where } Z = \sum_n e^{-n\beta\hbar\omega} = \frac{1}{1-e^{-\beta\hbar\omega}}. \tag{2.265}$$

Here we neglect the ground state energy. Then

$$\rho = \sum_n \rho_n |n\rangle\langle n| \text{ where } \rho_n = e^{-\beta n\hbar\omega}\left(1-e^{-\beta\hbar\omega}\right). \tag{2.266}$$

This is the distribution for light from black bodies, such an incandescent lamps. Incandescent lamps are not cool.

The mean excitation is

$$\langle n\rangle = \bar{n} = \mathrm{Tr}\left(\rho a^\dagger a\right) = \sum_n n\rho_n = -\frac{\partial}{\partial\left(\beta\hbar\omega\right)}\sum_n \rho_n = -\frac{\partial}{\partial\left(\beta\hbar\omega\right)}Z \tag{2.267}$$

$$= \frac{e^{-\beta\hbar\omega}}{\left(1-e^{-\beta\hbar\omega}\right)^2}\left(1-e^{-\beta\hbar\omega}\right) = \frac{e^{-\beta\hbar\omega}}{1-e^{-\beta\hbar\omega}} = \frac{1}{e^{\beta\hbar\omega}-1}. \tag{2.268}$$

Note that for $\hbar\omega \ll k_B T$, this approaches $\frac{k_B T}{\hbar\omega}$, while for $\hbar\omega \gg k_B T$ approaches $e^{-\beta\hbar\omega}$. We thus see lots of low energy photons, and few high energy photons.

One can show that

$$p_n = \frac{\bar{n}^n}{\left(1+\bar{n}\right)^{n+1}}. \tag{2.269}$$

The most probable n is zero.

The variance in the photon number is

$$\Delta n^2 = \left\langle n^2\right\rangle - \langle n\rangle^2. \tag{2.270}$$

We let $p_n = x^2\left(1-x\right)$, and note that

$$n\left(n-1\right)p_n = \left(1-x\right)x^2\frac{d^2}{dx^2}x^n \tag{2.271}$$

$$\implies \sum_n n\left(n-1\right)p_n = \sum_n \left(1-x\right)x^2\frac{d^2}{dx^2}x^n = x^2\left(1-x\right)\frac{d^2}{dx^2}\frac{1}{1-x} \tag{2.272}$$

$$= \frac{2x^2}{\left(1-x\right)^2} = \frac{2}{\left(1/x-1\right)^2} = 2\bar{n}^2. \tag{2.273}$$

Therefore,

$$\left\langle n^2\right\rangle - \langle n\rangle = 2\bar{n}^2 \implies \delta n^2 = \left\langle n^2\right\rangle - \langle n\rangle^2 = 2\bar{n}^2 + \bar{n} - \bar{n}^2 = \bar{n}^2 + \bar{n}. \tag{2.274}$$

This is distinctly wider than a Poisson distribution, which makes sense for a thermal distribution.

Finally, the Q distribution.

$$Q_{\mathrm{th}}(\alpha) = \frac{1}{\pi}\langle\alpha|\rho_{\mathrm{th}}|\alpha\rangle = \frac{1}{\pi}\sum_n p_n \left|\langle\alpha|n\rangle\right|^2 \tag{2.275}$$

$$= \frac{1}{\pi}\frac{1}{1+\bar{n}} e^{-|\alpha|^2/(1+\bar{n})}. \tag{2.276}$$

This is a Gaussian centered at the origin with a (much) wider width $\sqrt{1+\bar{n}}$ than the vacuum state.

Now, let us examine the time dependence of a coherent state in the Q distribution picture. Recall

$$e^{-iHt/\hbar}\left|\alpha\right\rangle = e^{-i\omega t/2}\left|\alpha e^{-i\omega t}\right\rangle. \tag{2.277}$$

On the $\mathrm{Re}\,\alpha$-$\mathrm{Im}\,\alpha$ plane, we have a Gaussian blob that circles clockwise.

2.7 Correlations

2.7.1 Measurements

How are photons detected?

Photons are detected by absorption. The electric field operator is

$$\vec{E}(\vec{r},t) = \sum_j \mathcal{E}\vec{\epsilon}_j \left(a_j e^{-i\vec{k}\cdot\vec{r}-i\omega_j t} + a_j^\dagger e^{i\vec{k}\cdot\vec{r}+i\omega_j t}\right). \tag{2.278}$$

Note that this contains both creation and annihilation operators. Detectors measure $\vec{E}^{(+)}$ (the positive frequency prat), since they always absorb (annihilate) photons. The final state $|f\rangle$ does not matter to us, so (for a good photodetector) we sum over final states:

$$P = \sum_f \left|\left\langle f|\vec{E}^{(+)}|i\right\rangle\right|^2 = \sum_f \left\langle i|\vec{E}^{(-)}|f\right\rangle \left\langle f|\vec{E}^{(+)}|i\right\rangle = \left\langle i|\vec{E}^{(-)}\vec{E}^{(+)}|i\right\rangle. \tag{2.279}$$

This is a normal ordered operator, and thus easy to measure with the P distribution. For emission of a photon, this is reverse, so that is related to the Q distribution.

When we absorb two photons (at different places and times), we find something more complicated. The transition is

$$\left\langle f|\vec{E}^{(+)}\left(\vec{r}',t'\right)\vec{E}^{(+)}\left(\vec{r},t\right)|i\right\rangle. \tag{2.280}$$

The probability is

$$P = \sum_f \left|\left\langle f|\vec{E}^{(+)}(\vec{r}',t')\vec{E}^{(+)}(\vec{r},t)|i\right\rangle\right|^2 = \left\langle i|\vec{E}^{(-)}(\vec{r},t)\vec{E}^{(-)}(\vec{r}',t')\vec{E}^{(+)}(\vec{r}',t')\vec{E}^{(+)}(\vec{r},t)|i\right\rangle \tag{2.281}$$

For a mixed state, we must consider the density matrix:

$$\rho = \sum_i p_i \left|i\right\rangle\!\left\langle i\right|. \tag{2.282}$$

$$P = \mathrm{Tr}\left(\rho\vec{E}^{(-)}(\vec{r},t)\vec{E}^{(+)}(\vec{r},t)\right). \tag{2.283}$$

For two photons:

$$P = \mathrm{Tr}\left(\rho\vec{E}^{(-)}(\vec{r},t)\vec{E}^{(-)}(\vec{r}',t')\vec{E}^{(+)}(\vec{r}',t')\vec{E}^{(+)}(\vec{r},t)\right). \tag{2.284}$$

2.7.2 Correlation Fucntions

The first order correlation function is more general, allowing $\vec{E}^{(-)}$ and $\vec{E}^{(+)}$ at different points:

$$G^{(1)}\left(\vec{r}t,\vec{r}'t\right)=\mathrm{Tr}\left(\rho\vec{E}^{(-)}(\vec{r},t)\vec{E}^{(+)}(\vec{r}',t')\right). \tag{2.285}$$

For second order:

$$G^{(1)}\left(\vec{r}_1t_1\vec{r}_2t_2,\vec{r}_3t_3\vec{r}_4t_4\right)= \tag{2.286}$$

The normalized verions are

$$g^{(1)}()=\frac{G^{(1)}\left(\vec{r}t,\vec{r}'t'\right)}{\sqrt{G^{(1)}\left(\vec{r}t,\vec{r}t\right)G^{(1)}\left(\vec{r}'t',\vec{r}'t'\right)}} \tag{2.287}$$

$$g^{(2)}\left(\vec{r}_1t_1,\vec{r}_2t_2,\vec{r}_3t_3,\vec{r}_4t_4\right)=\frac{G^{(2)}\left(\vec{r}_1t_1,\vec{r}_2t_2,\vec{r}_3t_3,\vec{r}_4t_4\right)}{\sqrt{\prod_{j=1}^{4}G^{(2)}\left(\vec{r}_jt_j,\vec{r}_jt_j\right)}}. \tag{2.288}$$

2.7.3 Uncertainty of the Coherent State

Recall that a coherent state $|\alpha\rangle$ evolves as $\left|\alpha e^{-i\omega t}\right\rangle$. The Q distribution is a Gaussian blob that rotates about the origin of the α plane. Recall that we can also write $|\alpha\rangle=D(\alpha)\,|0\rangle$.

What would we measure if we measured the electric field of a coherent state? We know that

$$\vec{E}\propto\langle\alpha|a-a^\dagger|\alpha\rangle\,. \tag{2.289}$$

This is the imaginary component of α. Since the imaginary component oscillates in time, we expect to see the electric field oscillating sinusoidally. There will also be noised due to the uncertainty in the imaginary component of α. For conjugate variables P and Q (with commutator $i\hbar$), the Heisenberg uncertainty relation demands $\Delta P\Delta Q=\hbar/2$. For a position-momentum harmonic oscillator and the electromagnetic field:

$$P=\frac{i}{\sqrt{2}}\left(a^\dagger-a\right)\sqrt{\hbar m\omega}=i\sqrt{\frac{\hbar\epsilon_0\omega}{2V}}\left(a^\dagger-a\right)\text{ and} \tag{2.290}$$

$$Q=\sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger+a\right)=\sqrt{\frac{\hbar V}{2\epsilon_0\omega}}\left(a^\dagger+a\right). \tag{2.291}$$

Using this, we can find the noise of the coherent state. For P,

$$\langle\alpha|P|\alpha\rangle=iP_0\left(\alpha^*-\alpha\right), \tag{2.292}$$

$$\langle\alpha|P^2|\alpha\rangle=-P_0^2\,\langle\alpha|\left(a^\dagger-a\right)\left(a^\dagger-a\right)|\alpha\rangle \tag{2.293}$$

$$=-P_0^2\,\langle\alpha|(\alpha^*-a)\left(a^\dagger-\alpha\right)|\alpha\rangle \tag{2.294}$$

$$=-P_0^2\,\langle\alpha|-\alpha^*\alpha+\alpha^2+\alpha^{*2}-aa^\dagger|\alpha\rangle \tag{2.295}$$

$$=-P_0^2\,\langle\alpha|-\alpha^*\alpha+\alpha^2+\alpha^{*2}-a^\dagger a-1|\alpha\rangle \tag{2.296}$$

$$=-P_0^2\,\langle\alpha|-\alpha^*\alpha+\alpha^2+\alpha^{*2}-\alpha^*\alpha-1|\alpha\rangle \tag{2.297}$$

$$=\left(\,\langle\alpha|P|\alpha\rangle\right)^2+P_0^2. \tag{2.298}$$

For Q,

$$\langle\alpha|Q|\alpha\rangle = Q_0\left(\alpha^* - \alpha\right), \tag{2.299}$$

$$\langle\alpha|Q^2|\alpha\rangle = \left(\langle\alpha|Q|\alpha\rangle\right)^2 + Q_0^2. \tag{2.300}$$

Therefore, $\Delta P^2 = P_0^2$ and $\Delta Q^2 = Q_0^2$, so

$$\Delta P \Delta Q = P_0 Q_0 = \frac{\hbar}{2}, \tag{2.301}$$

which shows that coherent states are minimum uncertainty states.

2.7.4 Non-Classical Light

For classical light, the correlation function $g^{(2)}$ is

$$g_{\text{cl}}^{(2)}(\tau) = \frac{\langle I(t) I\left(t+\tau\right)\rangle}{\langle I\rangle^2}. \tag{2.302}$$

This is always greater than or equal to 1. See the problem set. So, non-classical light may have $g^{(2)} < 1$.

In quantum mechanics, the probability of absorbing two photons (from a single mode) is

$$\sum_j \left|\langle j|aa|i\rangle\right|^2 = \langle i|a^\dagger a^\dagger aa|i\rangle\,. \tag{2.303}$$

This implies the quantum correlation function $g^{(2)}$ is

$$g^{(2)} = \frac{\left\langle a^\dagger(\tau)a^\dagger(\tau)aa\right\rangle}{\left\langle a^\dagger a\right\rangle^2} = \frac{\left\langle n^2\right\rangle - \langle n\rangle}{\langle n\rangle^2}. \tag{2.304}$$

Since there is a negative $\langle n\rangle$ in there, it can be be less than 1.

Fano Factors and the Non-Quantum Nature of the Number State

The Fano factor

$$F = \frac{\left\langle n^2\right\rangle - \langle n\rangle^2}{\langle n\rangle} - 1 \tag{2.305}$$

describes how non-classical some light is.

Example: Poisson Fano

For a Poisson distribution, $\Delta n^2 = \langle n\rangle$, which implies $\left\langle n^2\right\rangle - \langle n\rangle^2 = \langle n\rangle$. Therefore,

$$\frac{\left\langle n^2\right\rangle - \langle n\rangle^2}{\langle n\rangle} = \frac{\langle n\rangle}{\langle n\rangle} = 1 \implies F = 0. \tag{2.306}$$

This implies that the Poisson distribution (which describes coherent states) are classical (barely).

So far, we have found

State Type	$\langle n^2 \rangle$	F	$g^{(2)}(0)$
Thermal	$2\bar{n}^2 + \bar{n}$	$\bar{n}$	2
Coherent	$\bar{n}^2 + \bar{n}$	0	1
$\|n\rangle$	$\bar{n}^2$	-1	$1 - \frac{1}{n}$

We thus see that number states are quantum in nature.

How do we create a number state? A coherent state is not close at all, as it has a distribution of number states. Recall the Q distribution of a number state. We found that it forms a ring in α space. Experimentally, there are a few options:

- Place an atom (or ion) in a cavity. The cavity enhances the probability of emitting one photon into the cavity mode, so if we excite the atom with a π pulse, then within Γ^{-1}, the atom should emit a photon into the cavity mode.

- The heralded photon method. Many atoms are placed in a cavity in a ground state $|1\rangle$, with there being an excited state and a second low energy state $|2\rangle$. The atoms are pumped into the excited state, and a single photon detector detects the photons emitted by the $|e\rangle \to |2\rangle$ relaxation. Once the single photon detector detects a photon, there will be one atom in the $|2\rangle$ state (though which atom is not known, leading to a superposition). For N atoms, the superposition state is

$$\frac{|211\ldots\rangle + |121\ldots\rangle + \ldots}{\sqrt{N}}. \tag{2.307}$$

  This gives a superradiant enhancement (N times stronger cavity coupling than a single atom) of emission into a single mode. Then, pumping is applied from $|2\rangle$ to $|e\rangle$. With the superradiant enhancement, a single $|e\rangle \to |1\rangle$ photon will be emitted with a higher probability of being in the cavity mode.

Experimental Results

The suppressed $g^{(2)}(0)$ has been measured.

The earliest experiment was the Hanbury-Brown and Twiss experiment. Light from stars was passed through a beamsplitter, with a detector placed on each output branch of the beamsplitter. Note that a vacuum state enters the beamsplitter input port without incoming light. Then, coincidences are detected between the two detectors. A number state will be detected on one detector, and not the other. Coherent states will be detected with Poisson noise. Thermal states will appear bunched: the signals on the two detectors will be correlated. Hanbury-Brown and Twiss measured light from stars, which is thermal, so they saw "bunching": intensity at the detectors correlated.

Later expeirments (e.g. Kimble et al.) used the same technique with non-classical light, observing anti-bunching: a weaker correlation with $\tau = 0$ than at later times.

The group of Haroche used Rydberg atoms as cavity state detectors to probe cavity occupation, observing quantum jumps in the cavity electric field. The Rydberg atoms gain phase proportional to the number to photons in the cavity. Since there is no photon absorption, many atoms can be used. This actually leads to number states being created, as the phase is more accurately determined.

2.7.5 Squeezed Light

Inverted Harmonic Oscillator

It is possible to redistributed uncertainty between ΔP and ΔQ. Consider a harmonic oscillator,

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{2.308}$$

and an inverted harmonic oscillator,

$$H = \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2. \tag{2.309}$$

The harmonic oscillator can be re-written as, defining reduced variables $\tilde{p} = \frac{p}{\sqrt{m\omega}}$ and $\tilde{x} = \sqrt{m\omega}x$,

$$H = \frac{\omega}{2}\left(\frac{p^2}{m\omega} + m\omega x^2\right) = \frac{\omega}{2}\left(\tilde{p}^2 + \tilde{x}^2\right). \tag{2.310}$$

In phase space, the system rotates about the origin. Hamilton's equations are motion are

$$\dot{\tilde{x}} = \frac{\partial H}{\partial \tilde{p}} = \omega\tilde{p} \text{ and } \dot{\tilde{p}} = -\frac{\partial H}{\partial \tilde{x}} = -\omega\tilde{x}. \tag{2.311}$$

We introduce complex coordinats $z = \tilde{x} + i\tilde{p}$, which implies

$$\dot{\tilde{z}} = \omega\tilde{p} - i\omega\tilde{x} = -i\omega\left(\tilde{x} + i\tilde{p}\right) = -i\omega z \implies \tilde{z} = e^{-i\omega t}z. \tag{2.312}$$

We thus see the expected circular motion (in phase space).

The inverted harmonic oscillator can also be re-written, using the same coordinates:

$$H = \frac{\omega}{2}\left(\tilde{p}^2 - \tilde{x}^2\right). \tag{2.313}$$

This gives us hyperbolic trajectories in phase space. This results in squeezing in phase space: a distribution will approach the line $\tilde{x} = \tilde{p}$. Hamilton's equations of motion are

$$\dot{\tilde{x}} = \omega\tilde{p} \text{ and } \dot{\tilde{p}} = \omega\tilde{x}. \tag{2.314}$$

We define new coordinates aligned with the squeezed and anti-squeezed axes:

$$x' = \frac{\tilde{x} + \tilde{p}}{\sqrt{2}} \text{ and } p' = \frac{\tilde{p} - \tilde{x}}{\sqrt{2}}. \tag{2.315}$$

Then the equations of motion become

$$\dot{x}' = \omega x' \implies x' = e^{\omega t}x'_0 \text{ and} \tag{2.316}$$

$$\dot{p}' = -\omega p' \implies p' = e^{-\omega t}p'_0. \tag{2.317}$$

The real space $\tilde{x}$ coordinate is

$$\tilde{x}(t_= (x' - p')/\sqrt{2} = \cosh(\omega t)\,\tilde{x}(0) + \sinh(\omega t)\,\tilde{p}(0). \tag{2.318}$$

Now consider the quantum case. We change our reduced variables to be dimensionless:

$$\tilde{p} = \frac{p}{\sqrt{m\hbar\omega}} \text{ and } \tilde{x} = \sqrt{\frac{m\omega}{\hbar}}x. \tag{2.319}$$

We can use the creation and annihilation operators. For the harmonic oscillator,

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right). \tag{2.320}$$

For the inverted harmonic oscillator, we find

$$H = -\frac{\hbar\omega}{2}\left(a^2 + \left(a^\dagger\right)^2\right). \tag{2.321}$$

In rotated coordinates, the harmonic oscillator Hamiltonian is

$$H = \frac{\hbar\omega}{2}\left(p'^2 + x'^2\right) \tag{2.322}$$

and the inverted harmonic oscillator Harmonic is

$$H = \frac{\hbar\omega}{2}\left(x'p' + p'x'\right). \tag{2.323}$$

With creation and annihilation operators

$$b = \frac{x' + ip'}{\sqrt{2}} \text{ and } b^\dagger = \frac{x' - ip'}{\sqrt{2}}, \tag{2.324}$$

the harmonic oscillator and inverted harmonic oscillator are (respectively)

$$H = \frac{\hbar\omega}{2}\left(b^\dagger b + bb^\dagger\right) \text{ and} \tag{2.325}$$

We can generalize this to arbitrary rotations. With

$$x' = \tilde{x}\cos\frac{\varphi}{2} + \tilde{p}\sin\frac{\varphi}{2} \text{ and } p' = -\tilde{x}\sin\frac{\varphi}{2} + \tilde{p}\cos\frac{\varphi}{2}, \tag{2.326}$$

the Hamiltonian for the inverted harmonic oscillator becomes

$$H = \frac{\hbar\omega}{2}\left(\cos\varphi\left(p'^2 - x'^2\right) + \sin\varphi\left(x'p' + p'x'\right)\right). \tag{2.327}$$

For corresponding creation and annihilation operators ($b = \frac{x'+ip'}{\sqrt{2}}$ and $b^\dagger = \frac{x'-ip'}{\sqrt{2}}$),

$$H = -\frac{\hbar\omega}{2}\left(b^2 e^{i\varphi} + \left(b^\dagger\right)^2 e^{-i\varphi}\right) = \frac{1}{2}\left(zb^2 + z^*\left(b^\dagger\right)^2\right) \text{ where } z = -\hbar\omega e^{i\varphi}. \tag{2.328}$$

Classical Squeezing of a Harmonic Oscillator

Light is analogous to a harmonic oscillator, so how do we squeeze that?

Consider a classical harmonic oscillator. The potential is

$$V(x) = \frac{1}{2}m\omega^2 x^2 \implies m\ddot{x} = -m\omega^2 x. \tag{2.329}$$

The solution is

$$x(t) = c_0 \cos(\omega t) + s_0 \sin(\omega t). \tag{2.330}$$

Now, add a small parametric drive

$$V(x) = \frac{1}{2}m\omega^2 x^2 (1 + \epsilon \sin(2\omega t)). \tag{2.331}$$

The force is then

$$F = -V'(x) = -m\omega^2 x(t)(1 + \epsilon \sin(2\omega t)) = m\ddot{x}(t). \tag{2.332}$$

Suppose ϵ is small, such that things will not change much in a single period. Make the ansatz $c_0 \to c_0(t)$ and $s_0 \to s_0(t)$, with $c_0(t)$ and $s_0(t)$ slowly varying. With this ansatz,

$$\dot{x}(t) = -c_0\omega sin(\omega t) + s_0\omega\cos(\omega t) + \dot{c}_0\cos(\omega t) + \dot{s}_0\sin(\omega t) \tag{2.333}$$

$$\ddot{x}(t) = -\omega^2 x(t) - 2\omega\dot{c}_0\sin(\omega t) + 2\omega\dot{s}_0\cos(\omega t) + (\ddot{c}\ldots, \quad \ddot{s}_0\ldots \ll \omega\dot{c}_0, \quad \omega\dot{s}_0). \tag{2.334}$$

Therefore,

$$-m\omega^2 x(t)(1 + \epsilon\sin(2\omega t)) = -\omega^2 x(t) - 2\omega\dot{c}_0\sin(\omega t) + 2\omega\dot{s}_0\cos(\omega t) \tag{2.335}$$

$$-m\omega^2 x(t)\epsilon\sin(2\omega t) = -2\omega\dot{c}_0\sin(\omega t) + 2\omega\dot{s}_0\cos(\omega t). \tag{2.336}$$

We note that, plugging in the unperturbed sollution for $x(t)$,

$$x(t)\sin(2\omega t) = (c_0\cos(\omega t) + s_0\sin(\omega t))\sin(2\omega t) \tag{2.337}$$

$$= c_0\cos(\omega t)\sin(2\omega t) + s_0\sin(\omega t)\sin(2\omega t) \tag{2.338}$$

$$= \frac{c_0}{2}(\sin(\omega t) + \sin(3\omega t)) + \frac{s_0}{2}(\cos(\omega t) - \cos(3\omega t)) \tag{2.339}$$

$$\approx \frac{c_0}{2}\sin(\omega t) + \frac{s_0}{2}\cos(\omega t). \tag{2.340}$$

So,

$$-2\omega\dot{c}_0\sin(\omega t) + 2\omega\dot{s}_0\cos(\omega t) = -\omega^2\frac{\epsilon}{2}(c_0\sin(\omega t) + s_0\cos(\omega t)) \tag{2.341}$$

which implies (using orthogonality of sin and cos),

$$\dot{c}_0 = \frac{\omega\epsilon}{4}c_0 \text{ and } \dot{s}_0 = -\frac{\omega\epsilon}{4}s_0. \tag{2.342}$$

Therefore,

$$c_0(t) = e^{\epsilon\omega t/4} \text{ and } s_0(t) = e^{-\epsilon\omega t/4}. \tag{2.343}$$

This is squeezing, along the s axis.

Quantum Squeezing

Based on the classical case, we expect that we want a parametric drive at twice the frequency. To get this, we need non-linear optics to create a 2ω drive. An example is an optical parametric oscillator, such as a KDP crystal. In such a material, a 2ω photon can create two ω photons. Let the creation and annihilation operators for 2ω be $b^\dagger$ and b and the creation and annihilation opreators for ω be $a^\dagger$ and a. Then the Hamiltonian is

$$H = aab^\dagger + a^\dagger a^\dagger b. \tag{2.344}$$

For a strong coherent state,

$$b\left|\beta\right\rangle = \beta\left|\beta\right\rangle \text{ and } b^dagger\left|\beta\right\rangle = \beta^*\left|\beta\right\rangle. \tag{2.345}$$

This works because the field is very strong. Choosing $\beta = \frac{r}{2}e^{i\varphi}$, then Hamiltonian is

$$H = \frac{r}{2}\left[a^2 e^{i\varphi} + \left(a^\dagger\right)^2 e^{-i\varphi}\right], \tag{2.346}$$

which is the squeezing Hamiltonian.

The squeezing operator is the operator for time evolution under this Hamiltonian. For $\varphi = \pi/2$, we find

$$S(r) = \exp\left[\frac{r}{2}\left(a^2 - \left(a^\dagger\right)^2\right)\right]. \tag{2.347}$$

Let $z = r\exp\left(\varphi - \pi/2\right)$. Then

$$S(z) = \exp\left[\frac{1}{2}\left(za^2 - z^*\left(a^\dagger\right)^2\right)\right]. \tag{2.348}$$

We can also write $S(z) = e^{iA}$ with

$$A = -\frac{i}{2}\left(za^2 - z^*\left(a^\dagger\right)^2\right). \tag{2.349}$$

Since A is Hermitian, $S(z)$ is unitary. It is similar to the displacement operator $D(\alpha) = \exp\left(\alpha a^\dagger - \alpha^* a\right)$, but it is not linear. We can think of $S(z)$ as translating each point α in phase space by an amount $-\frac{z^*}{2}\alpha^*$. Consider $z = 2\epsilon$ is real. Then α is translated by $-\epsilon\alpha^*$, or

$$x + iy \to x\left(1 - \epsilon\right) + iy\left(1 + \epsilon\right). \tag{2.350}$$

This is damping of x and amplifying of y, which is squeezing.

Now let us examine the effects on operators

$$\hat{x} = \frac{a^\dagger + a}{\sqrt{2}} \text{ and } \hat{p} = i\frac{a^\dagger - a}{\sqrt{2}}. \tag{2.351}$$

Since $[\hat{x}, \hat{p}] = i$, $\Delta x \Delta p \geq \frac{1}{2}$. In a coherent state,

$$\Delta x^2 = \frac{1}{2} \text{ and } \Delta p^2 = \frac{1}{2}. \tag{2.352}$$

Now consider the squeezed state. Define

$$t = S(z)aS^\dagger(z) \tag{2.353}$$
$$= e^{iA}ae^{-iA} \tag{2.354}$$
$$= a + [iA, a] + \frac{1}{2!}[iA, [iA,]] + \dots \tag{2.355}$$
$$= a + z^*a^\dagger + \frac{1}{2!}|z|^2a + \frac{1}{3!}|z|^2z^*a^\dagger + \dots \tag{2.356}$$
$$= a\cosh(r) + a^\dagger e^{i\varphi}\sinh(r). \tag{2.357}$$

Furthermore,

$$t^\dagger = a^\dagger\cosh(r) + ae^{-i\varphi}\sinh(r). \tag{2.358}$$

Let

$$t = \mu^*a - \nu a^\dagger. \tag{2.359}$$

Note that

$$\left[t, t^\dagger\right] = 1, \tag{2.360}$$

so we have the same algebra as for creation and annihilation operators.

We can introduce squeezed states as eigenstates of t:

$$t\left|\tau\right\rangle_t = \tau\left|\tau\right\rangle_t \tag{2.361}$$

We can also find a squeezed vacuum

$$\left|\tau\right\rangle_t = D(\tau)\left|0\right\rangle_t \tag{2.362}$$
$$D(\tau) = \exp\left(\tau t^\dagger - \tau^* t\right) \tag{2.363}$$
$$t\left|0\right\rangle_t = 0. \tag{2.364}$$

The squeezed vacuum is thus

$$\left|0\right\rangle_t = S(z)\left|0\right\rangle. \tag{2.365}$$

We can verify that this emerges squeezing the vacuum

$$t\left|0\right\rangle_t = tS(z)\left|0\right\rangle = S(z)aS^\dagger(z)S(z)\left|0\right\rangle = S(z)a\left|0\right\rangle = 0. \tag{2.366}$$

We can also define squeezed quadratures:

$$\hat{x}_t = \frac{t + t^\dagger}{\sqrt{2}} \text{ and } \hat{p}_t = i\frac{t^\dagger - t}{\sqrt{2}}. \tag{2.367}$$

$$\hat{x}_t = \frac{1}{\sqrt{2}}S(z)\left(a^\dagger a\right)S^\dagger(z) \tag{2.368}$$

For $\varphi = 0$,

$$\hat{x}_t = \hat{x}e^r \text{ and } \hat{p}_t = \hat{p}e^{-r}. \tag{2.369}$$

Therefore,

$${}_t\left\langle\tau|\hat{x}_t|\tau\right\rangle_t = \frac{1}{\sqrt{2}}\left\langle\tau|t + t^\dagger|\tau\right\rangle = \dots. \tag{2.370}$$

For $\hat{x}$ and $\hat{p}$,

$$\langle\tau|\hat{x}|\tau\rangle_t = e^{-r}\langle\tau|\hat{x}_t|\tau\rangle_t \text{ and } \langle\tau|\hat{x}^2|\tau\rangle_t = e^{-2r}\langle\tau|\hat{x}_t^2|\tau\rangle_t \tag{2.371}$$

$$\langle\tau|\hat{p}|\tau\rangle_t = e^{r}\langle\tau|\hat{p}_t|\tau\rangle_t \text{ and } \langle\tau|\hat{p}^2|\tau\rangle_t = e^{2r}\langle\tau|\hat{p}_t^2|\tau\rangle_t \tag{2.372}$$

So

$$\Delta x^2 = e^{-2r}\langle\tau|\hat{x}^2|\tau\rangle_t - \left(e^{-r}\langle\tau|\hat{x}|\tau\rangle_t\right)^2 = \frac{1}{2}e^{-2r} \tag{2.373}$$

$$\Delta p^2 = e^{2r}\langle\tau|\hat{p}^2|\tau\rangle_t - \left(e^{r}\langle\tau|\hat{p}|\tau\rangle_t\right)^2 = \frac{1}{2}e^{2r} \tag{2.374}$$

$$\tag{2.375}$$

We still have $\Delta x\Delta p = \frac{1}{2}$, but $\Delta p \gg \frac{1}{\sqrt{2}}$ and $\Delta x \ll \frac{1}{\sqrt{2}}$.

The electric field of a squeezed state is

$$\hat{E}(r) = i\mathcal{E}_l\left[ae^{i\vec{k}\cdot\vec{r}} - a^\dagger e^{-i\vec{k}\cdot\vec{r}}\right] \tag{2.376}$$

$$= i\mathcal{E}_l\left[\left(a - a^\dagger\right)\cos\left(\vec{k}\cdot\vec{r}\right) + i\left(a + a^\dagger\right)\sin\left(\vec{k}\cdot\vec{r}\right)\right] \tag{2.377}$$

$$= -\sqrt{2}\mathcal{E}_l\left[\hat{p}\cos\left(\vec{k}\cdot\vec{r}\right) + \hat{x}\sin\left(\vec{k}\cdot\vec{r}\right)\right] \tag{2.378}$$

$$= -\sqrt{2}\mathcal{E}_l\left[\hat{p}_t e^{r}\cos\left(\vec{k}\cdot\vec{r}\right) + \hat{x}e^{-r}\sin\left(\vec{k}\cdot\vec{r}\right)\right]. \tag{2.379}$$

Then

$$\langle\tau|\hat{E}|\tau\rangle_t = -\mathcal{E}_l\mathcal{E}_l\left[i\left(\tau^* - \tau\right)e^{r}\cos\left(\vec{k}\cdot\vec{r}\right) + \left(\tau^* + \tau\right)e^{-r}\sin\left(\vec{k}\cdot\vec{r}\right)\right]. \tag{2.380}$$

Mathematically, this is equivalent to a coherent state with $\alpha = \tau\cosh r - \tau^*\sinh r$. So,

$$\langle\tau|\hat{E}^2|\tau\rangle_t = 2\mathcal{E}_l^2\langle\tau|\hat{p}_t^2 e^{2r}\cos^2\left(\vec{k}\cdot\vec{r}\right) + \hat{x}_t^2 e^{-2r}\sin^2\left(\vec{k}\cdot\vec{r}\right) + \left(\hat{p}_t\hat{x}_t + \hat{x}_t\hat{p}_t\right)\sin\left(\vec{k}\cdot\vec{r}\right)\cos\left(\vec{k}\cdot\vec{r}\right)|\tau\rangle_t \quad = 2\mathcal{E}_l^2\ldots \tag{2.381}$$

The photon number is

$$\langle\tau|N|\tau\rangle_t = \langle\tau|a^\dagger t|\tau\rangle_t. \tag{2.382}$$

Using

$$a = t\cosh r - t^\dagger\sinh r \text{ and } a^\dagger = t^\dagger\cosh r - t\sinh r, \tag{2.383}$$

we find

$$N = a^\dagger a = t^\dagger t\cosh^2 r + tt^\dagger\sinh^2 r - \left(t^2 + \left(t^\dagger\right)^2\right)\sinh r\cosh r = t^\dagger t\left(\cosh^2 r + \sinh^2 r\right) - \left(t^2 + \left(t^\dagger\right)^2\right)\sinh r\cosh r + \sinh^2 r. \tag{2.384}$$

This implies

$$\langle\tau|N|\tau\rangle_t = |\tau|^2\left(\cosh^2 r + \sinh^2 r\right) - \left(\tau^2 + \tau^{*2}\right)\sinh r\cosh r + \sinh^2 r \tag{2.385}$$

$$= \left|\tau\cosh r - \tau^*\sinh r\right|^2 + \sinh^2 r \tag{2.386}$$

$$= |\alpha|^2 + \sinh^2 r. \tag{2.387}$$

The average photon number is larger than that of a coherent state. Note that

$$\langle 0|N|0\rangle_t = \sinh^2 r \neq 0. \tag{2.388}$$

The fluctuations in the photon number are

$$\Delta N^2 \approx |\alpha|^2 e^{-2r} \approx \langle N\rangle e^{-2r}, \tag{2.389}$$

which allows ΔN to be reduced (at the expense of phase noise).

2.7.6 Beamsplitters and Interferometry

We measure squeezing by beating our squeezed light with a local oscillator, using a beamsplitter.

Beamsplitter

Consider two electric fields incident on a semi-reflective mirror, $E_1^{(+)}$ and $E_2^{(+)}$. Using momentum and energy conservation, the outgoing light is constrained. One option that works is

$$E_3^{(+)} = \frac{1}{\sqrt{2}}\left(E_1^{(+)} + E_2^{(+)}\right) \text{ and } E_4^{(+)} = \frac{1}{\sqrt{2}}\left(E_1^{(+)} - E_2^{(+)}\right). \tag{2.390}$$

In general, we must have

$$\left|E_1^{(+)}\right|^2 + \left|E_2^{(+)}\right|^2 = \left|E_3^{(+)}\right|^2 + \left|E_4^{(+)}\right|^2. \tag{2.391}$$

The system must also satisfy time reversal. If $E_0 \to tE_0$ and rE_0, then rE_0 and tE_0 must only produce E_0 and the other arm must cancel. Therefore, $r^2+t^2 = 1$ (reflecting twice plus transmitting twice). We also need $trE_0+r'tE_0 = 0$, which implies $r' = -r$: reflection from the opposite direction should be negated.

It is easier to model a quantum beamsplitter in the Heisenberg picture. Our operators transform as

$$\hat{E}_3^{(+)} = \frac{1}{\sqrt{2}}\left(\hat{E}_1^{(+)} + \hat{E}_2^{(+)}\right) \text{ and } \hat{E}_4^{(+)} = \frac{1}{\sqrt{2}}\left(\hat{E}_1^{(+)} - \hat{E}_2^{(+)}\right). \tag{2.392}$$

The reation and annihilation operators transform as

$$a_3 = \frac{1}{\sqrt{2}}\left(a_1 + a_2\right) \text{ and } a_3 = \frac{1}{\sqrt{2}}\left(a_1 - a_2\right). \tag{2.393}$$

This transformation is canonical, so

$$\left[a_3, a_3^\dagger\right] = \left[a_4, a_4^\dagger\right] = 1. \tag{2.394}$$

Homodyne

Homodyne detection uses light of the same frequency (e.g. the laser picked off before the squeezing) as the local oscillator. Let the state of the local oscillator be $|\psi_2\rangle$, with operator b, and the squeezed light be $|\psi_1\rangle$, with operator a. We can detect amplitude (or photon number) at each output as currents, i_3 and i_4. We will take the difference between these two amplitudes. Assume $|\psi_2\rangle = |\beta\rangle$

is a coherent state (with fairly large amplitude, which generally implies $b^\dagger |\beta\rangle \approx \beta |\beta\rangle$). Since the frequencies are the same, $\mathcal{E}_l$ is the same throughout. The difference in amplitudes is

$$\bar{d} = \langle\psi|E^{(-)}(\vec{r}_3)E^{(+)}(\vec{r}_3)|\psi\rangle - \langle\psi|E^{(-)}(\vec{r}_4)E^{(+)}(\vec{r}_4)|\psi\rangle \tag{2.395}$$

where

$$E^{(+)}(\vec{r}_3) = i\mathcal{E}_l e^{i\vec{k}_3\cdot\vec{r}_3}a_0 \text{ and } E^{(+)}(\vec{r}_4) = i\mathcal{E}_l e^{i\vec{k}_4\cdot\vec{r}_4}b_0, \tag{2.396}$$

with

$$a_0 = \frac{a+b}{\sqrt{2}} \text{ and } b_0 = \frac{a-b}{\sqrt{2}}. \tag{2.397}$$

Our state is

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle = |\psi_1\rangle \otimes |\beta\rangle \tag{2.398}$$

which implies

$$\bar{d} = -\mathcal{E}_l^2 \langle\psi|a_0^\dagger a_0 - b_0^\dagger b_0|\psi\rangle = -\mathcal{E}_l^2 \langle\psi|a^\dagger b - b^\dagger a|\psi\rangle . \tag{2.399}$$

For a coherent state,

$$\langle\beta|b|\beta\rangle = \beta e^{-i\omega t} = |\beta|e^{i\varphi}e^{-i\omega t}. \tag{2.400}$$

If $|\psi_1\rangle$ describes a single mode of the field,

$$\langle\psi_1(t)|a|\psi_1(t)\rangle = e^{-i\omega t}\langle\psi_1(0)|a|\psi_1(0)\rangle . \tag{2.401}$$

This is the same frequency as for the b component, so the result is time-independent. We thus find

$$\bar{d} = -\mathcal{E}_l|\beta|\langle\psi_1(0)|\left(e^{-i\varphi}a + e^{i\varphi}a^\dagger\right)|\psi_1(0)\rangle . \tag{2.402}$$

By changing the local oscillator phase φ, we can change the quadrature measurement. The quadratures of the electric field are

$$\hat{E}_Q = \mathcal{E}_l\left(a + a^\dagger\right) = \mathcal{E}_l\sqrt{\frac{2}{\hbar}}\hat{x} \tag{2.403}$$

$$\hat{E}_P = -i\mathcal{E}_l\left(a - a^\dagger\right) = \mathcal{E}_l\sqrt{\frac{2}{\hbar}}\hat{p}. \tag{2.404}$$

Then

$$\bar{d} = -|\beta|\mathcal{E}_l\left(\cos\varphi\langle\psi_1(0)|\hat{E}_Q|\psi_1(0)\rangle + \sin\varphi\langle\psi_1(0)|\hat{E}_P|\psi_1(0)\rangle\right). \tag{2.405}$$

So far, this works for classical or quantum states. Th noise is related to $\overline{(i_3 - i_4)^2} = \bar{i_3^2} + \bar{i_4}^2 - 2\overline{i_3 i_4}$, and ultimately one could find

$$\bar{d^2} = \mathcal{E}_l^4\left(|\beta|^2\langle\psi_1(0)|\left(e^{-i\varphi}a + e^{i\varphi}a^\dagger\right)^2|\psi_1(0)\rangle + \langle\psi_1(0)|a^\dagger a|\psi_1(0)\rangle\right). \tag{2.406}$$

As before, we can adjust φ to measure in each quadrature.

Next, suppose $|t|^2 = 0.99$. Then we have an unbalanced homodyne. We then have

$$a_0 = ta + \sqrt{1-t^2}b \approx \left(a + \sqrt{1-t^2}\beta\right) = D\left(\sqrt{1-t^2}\beta\right)aD\left(\sqrt{t-t^2}\right)^\dagger. \tag{2.407}$$

This is a technique to implement the displacement operator.

Suppose we want to make an attenuator: $|\alpha\rangle \to |t\alpha\rangle$ for $|t|^2 < 1$. This can not shrink the entire state, as that would violate uncertainty. The beamsplitter in this configuration actually mixes in vacuum, leading to $a_0 = ta + rb$. This creates sufficient noise to increase the uncertainty to the required level. It turns out

$$\langle n_0 \rangle = \langle \alpha, 0 | (ta + rb)^\dagger (ta + rb) | \alpha, 0 \rangle = |t|^2 |\alpha|^2 \tag{2.408}$$

and

$$\left\langle n_0^2 \right\rangle = \langle \alpha, 0 | (ta + rb)^\dagger (ta + rb) (ta + rb)^\dagger (ta + rb) | \alpha, 0 \rangle = |t|^4 |\alpha|^4 + |t|^2 |\alpha|^2 \tag{2.409}$$

using that the transformation is canonical:

$$\left[ta + rb, t^* a^\dagger + r^* b^\dagger \right] = |t|^2 + |r|^2 = 1. \tag{2.410}$$

Thus,

$$\Delta n_0^2 = \left\langle n_0^2 \right\rangle - \langle n_0 \rangle^2 = |t|^2 |\alpha|^2 = \langle n_0 \rangle, \tag{2.411}$$

which is what we would expect.

Number States

Now, suppose we have single photons entering a beamsplitter. For a general beamsplitter,

$$H = i\theta \left(ab^\dagger - a^\dagger b \right) \implies B = e^{iH} = \exp\left(\theta \left(ab^\dagger - a^\dagger b \right) \right). \tag{2.412}$$

H coherently transfers a photon from a to b and vice-versa. This is similar to a spin flip, but there are an infinite number of levels. Using this analogy, we introduce

$$s_+ = a^\dagger b, \quad s_- = ab^\dagger, \quad s_z = a^\dagger a - b^\dagger b, \quad s_x = ab^\dagger + a^\dagger b, \text{ and } s_y = i \left(ab^\dagger - a^\dagger b \right). \tag{2.413}$$

Note that the commutation relations are correct for angular momentum: the algebra is correct. Thus, the beamsplitter is the operator for rotation about the y axis:

$$B = \exp\left(\theta s_y \right). \tag{2.414}$$

We can compute

$$BaB^\dagger = a \cos\theta + b \sin\theta = a' \text{ and } BbB^\dagger = -a \sin\theta + b \cos\theta = b'. \tag{2.415}$$

Note that these are orthogonal. For the special case $\theta = \frac{\pi}{4}$, we have

$$b' = \frac{b - a}{\sqrt{2}} \text{ and } a' = \frac{b + a}{\sqrt{2}}. \tag{2.416}$$

We can also write this as a matrix

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \tag{2.417}$$

A phase shifter is done by the operator $e^{i\varphi}$ applied to one mode (but not the other). Now consider the space of single photons, $|10\rangle = b^\dagger |00\rangle$ and $|01\rangle = a^\dagger |00\rangle$. Applying the beamsplitter:

$$B |10\rangle = Bb^\dagger B^\dagger B |0\rangle = -\sin\theta |01\rangle + \cos\theta |10\rangle. \tag{2.418}$$

Similarly,

$$B\left|01\right\rangle = \cos\theta\left|01\right\rangle + \sin\theta\left|10\right\rangle. \tag{2.419}$$

B conserves the photon number. Consider now the state with one photon in each input port

$$B\left|11\right\rangle = -\sqrt{2}\sin\theta\cos\theta\left|02\right\rangle + \sqrt{2}\sin\theta\cos\theta\left|20\right\rangle + \left(\cos^2\theta - \sin^2\theta\right)\left|11\right\rangle. \tag{2.420}$$

For a balanced beamsplitter, both photons go into the same output port. Consider just the subspace $\{\left|01\right\rangle, \left|10\right\rangle\}$. It is a two level system. It can be proven that any state $\left|\psi\right\rangle$ can be created from $\left|01\right\rangle$ using beamsplitters and phase shifters.

Proof. Let

$$\left|\psi\right\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{2.421}$$

The beamsplitter does

$$B_\theta\left|\psi\right\rangle = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{2.422}$$

The phase shifter does

$$e^{-i\varphi/2}\begin{pmatrix}e^{i\varphi/2} & 0\\ 0 & e^{-i\varphi/2}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}. \tag{2.423}$$

Thus, $B(\theta) = R_y\left(-2\theta\right)$ and $e^{-i\varphi} = R_z\left(-\varphi\right)$. Using Euler angles, any unitary can be written as

$$U = e^{i\alpha}R_z(\beta)R_y(\gamma)R_z(\delta). \tag{2.424}$$

□

Mach-Zehnder Interferometer

With two beamsplitters, one can make an interferometer. Consider a pair of beamsplitters with the output ports of the first ($3'$ and $4'$) redirected into the input ports of the second: a Mach-Zehnder interferometer. Let the output 3 of the first beamsplitter reflect off of mirror M_3 and the output 4 of the first beamsplitter reflect off of mirror M_4. We again have a detector at each output (3 and 4) of the second beamsplitter. The detectors measure the difference $\overline{S_{\text{in}}M_3S_{\text{out}}} - \overline{S_{\text{in}}M_4S_{\text{out}}}$.

The electric fields are

$$\hat{E}_3^{(+)} = \frac{1}{\sqrt{2}}\left(-\hat{E}_{3'}^{(+)}e^{ikL_{3'}} + \hat{E}_{4'}^{(+)}e^{ikL_{4'}}\right) \tag{2.425}$$

$$\hat{E}_4^{(+)} = \frac{1}{\sqrt{2}}\left(\hat{E}_{3'}^{(+)}e^{ikL_{3'}} + \hat{E}_{4'}^{(+)}e^{ikL_{4'}}\right) \tag{2.426}$$

$$\ldots \tag{2.427}$$

$$\hat{E}_3^{(+)} = e^{ik\bar{L}'}\left(-i\sin\left(\frac{k\delta L}{2}\right)\hat{E}_1^{(+)} - \cos\left(\frac{k\delta L}{2}\right)\hat{E}_1^{(+)}\right)\hat{E}_4^{(+)} \quad = e^{ik\bar{L}'}\left(\cos\left(\frac{k\delta L}{2}\right)\hat{E}_1^{(+)} + i\sin\left(\frac{k\delta L}{2}\right)\hat{E}_1^{(+)}\right) \tag{2.428}$$

In quantum terms, assuming the beamsplitters are identical,

$$\left|\text{out}\right\rangle = B^\dagger PB\left|\text{in}\right\rangle = R_y\left(\frac{\pi}{2}\right)R_z\left(-\varphi\right)R_y\left(-\frac{\pi}{2}\right)\left|\text{in}\right\rangle = R_x\left(-\varphi\right)\left|\text{in}\right\rangle \tag{2.429}$$

where P is the phase shift operator and $\varphi = k\delta L$, δL being the path length difference. This is fairly similar to a Ramsey spectrometer. The intensity difference can thus be used to determine the phase between the two paths.

Suppose

$$|\text{in}\rangle = |\psi_1\rangle \otimes |0\rangle . \tag{2.430}$$

Then the output signal is

$$w_3(r_3, t) = s \left|\left|\hat{E}_3^{(+)}(r_3)\,|\text{in}\rangle\right|\right|^2 = s \sin^2\left(\frac{k\delta L}{2}\right) \left|\left|\hat{E}_1^{(+)}(r_3)\,|\psi_1\rangle\right|\right|^2 . \tag{2.431}$$

This only depends on the path difference of δL, the dependence of $\bar{L}'$ is gone. Consider the case $|\psi_1(t)\rangle = \left|\alpha e^{-i\omega t}\right\rangle$. Then

$$w_3 = w_1 \sin^2\left(\frac{k\delta L}{2}\right) \text{ with } w_1 = s\mathcal{E}^2|\alpha|^2. \tag{2.432}$$

We get a modulation of the signal at the incoming power.

Now suppose $|\psi_1\rangle = |1\rangle$. Then

$$w_3 = w_1 \sin^2\left(\frac{k\delta L}{2}\right) \text{ with } w_1 = s\mathcal{E}^2 \text{ and} \tag{2.433}$$

$$w_4 = w_1 \cos^2\left(\frac{k\delta L}{2}\right). \tag{2.434}$$

We thus see (even) a single photon showing interference.

If we placed detectors immediately after the first beamsplitter, a single photon state would show no coincidences (simultaneous detections) between both detectors. To be mathematical, $a^2\,|1\rangle = 0$.

Two Photons

Consider the case where we have two single photon states incident on each of a single beamsplitter's input ports. Then the output field is

$$\hat{E}_3^{(+)}\,|\psi\rangle \propto \frac{a-b}{\sqrt{2}}\,|1\rangle \otimes |1\rangle = \frac{|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle}{\sqrt{2}}. \tag{2.435}$$

So, the detected field is

$$\left|\left|\hat{E}_3^{(+)}\,|\psi\rangle\right|\right|^2 = \mathcal{E}^2\left(\frac{1}{2} + \frac{1}{2}\right). \tag{2.436}$$

The same result applies for port 4. However, the cross-correlation is zero. TODO If the photons have the same mode at the input, they always exit through the same output port. This is known as quantum coalescnece or Hong-Ou-Mandel interference. The paths where both photons reflect or both photons transmit are forbidden due to the bosonic nature of photons. The paths should be added, but the reflection path has a negative due to the mirror back surface having negative reflection coefficient.

This has been observed using photons emitted from pairs of atoms: in a beam, between trapped atoms. Tis can also be done with electrons, but since electrons are fermions, the sign is reversed, electrons are more often detected on opposite ports. The Hong-Ou-Mandel effect has also been shown with pairs of (bosonic) atoms in tweezers. A related effect can be shown by atoms undergoing walks in 1D lattices. Fermions can also be seen to spatially avoid correlations.

Non-Linear Mach-Zehnder Interferometer

Suppose that instead of polarization $\vec{P} = \epsilon_0 \chi \vec{E}$, we consider the non-linear polarization

$$\vec{P} = \epsilon_0 \left(\chi' \vec{E} + \chi^2 E^2 + \chi^3 E^3 + \ldots \right). \tag{2.437}$$

The E^3 term is called the Kerr effect. It can be considered an intensity dependent index of refraction. The Hamiltonian takes the form

$$H_{xpm} = -\chi a^\dagger a b^\dagger b, \tag{2.438}$$

where b, $b^\dagger$ correspond to the intensity of another field. For a crystal of length L, we have

$$K = \exp\left[-iHL\right] = \exp\left[i\chi L a^\dagger a b^\dagger b\right]. \tag{2.439}$$

We can choose $\chi L = \pi$, which means that for one photon in a and one photon in b, we get a π phase shift:

$$K\left|0,0\right\rangle = \left|0,0\right\rangle, \quad K\left|1,0\right\rangle = \left|1,0\right\rangle, \quad K\left|0,1\right\rangle = \left|0,1\right\rangle, \quad K\left|1,1\right\rangle = -\left|1,1\right\rangle. \tag{2.440}$$

This is a bit like a controlled swap (in the language of quantum information).

In terms of beamsplitter operators,

$$\left|\text{out}\right\rangle = B_{ab}^\dagger K_{bc} B_{ab} \left|\text{in}\right\rangle. \tag{2.441}$$

Entangled States of Light

Consider now a situation where we have four input modes, a, b, c, and d. a and b are combined on a beamsplitter, while c and d are combined on a separate beamsplitter. The output c' of the c-d beamsplitter is combined with output b' of the a-b beamsplitter through a Kerr element.

For $a = c = \left|1\right\rangle$, $b = d = \left|0\right\rangle$, the state after the beamsplitters is

$$\left|\phi_1\right\rangle = \left(\left|01\right\rangle + \left|10\right\rangle\right)\left(\left|01\right\rangle + \left|10\right\rangle\right) = \left|0101\right\rangle + \left|0110\right\rangle + \left|1001\right\rangle + \left|1010\right\rangle. \tag{2.442}$$

After the Kerr element, we have

$$\left|\phi_2\right\rangle = \left|0101\right\rangle - \left|0110\right\rangle + \left|1001\right\rangle + \left|1010\right\rangle. \tag{2.443}$$

Then after the second beamsplitters,

$$\left|\phi_3\right\rangle = -\left|0110\right\rangle - \left|0110\right\rangle + \left|1001\right\rangle + \left|1001\right\rangle = \frac{\left|1001\right\rangle + \left|011\right\rangle}{\sqrt{2}}. \tag{2.444}$$

This is an entangled state.

Chapter 3

Entanglement

Quantum entanglement generates correlations between systems stronger than classical correlations.

A bi-partite $|\psi_A B\rangle$ of a composite system $A + B$ is entangled if and only if there do not exist stats $|\psi\rangle_A$ and $|\psi\rangle_B$ such that $|\psi_{AB}\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$.

3.1 Einstein-Podolsky-Rosen Pair

Consider two particles that have interacted for some time and have a (perfectly) known separation. Now measure (exactly) the velocity p_1 of the first particle. Then the velocity of the second must be $p_2 = -p_1$. This implies that measuring the first particle leads to information of the second particle being known. One could also measure the position x_1 of the first particle, and then know x_2.

This seems like a problem: measuring the first particle changes the "reality" of the second particle. However, since the separation is known, the particles are entangled, which resolves this.

3.1.1 Bell's Inequalities

Bell's inequalities provided a theorem linking certain measureable inequalities to the possibility of a local probabilistic theory.

Consider two spin-1/2 particles prepared in a state with zero magnetic moment (entangled). Then the initial state is

$$|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2 . \tag{3.1}$$

Now separate the particles, and measure the spin of the first particle. Despite the distance, the spin of the second particle will be instantaneously known.

This leads to some inequalities that one can measure. Experiments were eventually performed to measure these inequalities. Photon polarization was used (rather than spins). It was eventually found that the inequalities were violated.

Let Alice and Bob each have one photon from the state $|\psi\rangle$. Let Alice measure the photon in random orthogonal basis Q, R and Bob measure the photon in random orthogonal basis S, T. Consider

$$QS + RS + RT - QT = (Q + R) S + (R - Q) T = \pm 2, \tag{3.2}$$

since $Q+R=0$ or ± 2(as they are orthogonal, and ± 1 each) and $R-Q$ the opposite. Let $p\left(q,r,s,t\right)$ be the probability that before measurement, the initial state has $Q=q$, $R=r$, $S=s$,and $T=t$. Then

$$\langle QS+RS+RT-QT\rangle =\sum_{qrst}p\left(q,r,s,t\right)\left(qs+rs+rt-qt\right)\leq \sum_{qrst}2p\left(q,r,s,t\right)=2. \tag{3.3}$$

This implies

$$\langle QS\rangle +\langle RS\rangle +\langle RT\rangle -\langle QT\rangle \leq 2. \tag{3.4}$$

Quantum mechanics violates this inequality. Let Alice measure $Q=|0\rangle\langle 0|-|1\rangle\langle 1|$ and $R=|0\rangle\langle 1|+|1\rangle\langle 0|$ and Bob measure $S=\frac{-|0\rangle\langle 0|-|0\rangle\langle 1|-|1\rangle\langle 0|+|1\rangle\langle 1|}{\sqrt{2}}$ and $T=\frac{|0\rangle\langle 0|-|0\rangle\langle 1|-|1\rangle\langle 0|-|1\rangle\langle 1|}{\sqrt{2}}$. With initial state $|\psi_{EPR}\rangle =\frac{|HV\rangle +|VH\rangle}{\sqrt{2}}$. Then

$$\langle QS\rangle =\langle RS\rangle =\langle RT\rangle =-\langle QT\rangle =\frac{1}{\sqrt{2}}, \tag{3.5}$$

so

$$\langle QS\rangle +\langle RS\rangle +\langle RT\rangle -\langle QT\rangle =2\sqrt{2}>2. \tag{3.6}$$

Chapter 4

Lecture 11: Atoms and Cavities

4.1 Jaynes-Cummings Hamiltonian

4.1.1 Derivation from QED Hamiltonian

The Hamiltonian for a signgle electron charge interacting with a single mode of the electromagnetic field is

$$H = \frac{1}{2m}\left[\vec{p} - e\vec{A}\left(\vec{r},t\right)\right]^2 - e\phi\left(\vec{r},t\right) + V(r) + H_{\text{field}}. \tag{4.1}$$

The free field Hamiltonian is $H_{\text{field}} = \hbar\omega a^\dagger a$. We will typically use the Coulomb guage.

For a plane wave, the atom holds the electron at a constant position relative to the field. Therefore, $\vec{A}\left(\vec{r},t\right) \sim \vec{A}\left(\vec{r}_0,t\right)$ where $\vec{r}_0$ is the position of the atom.

We then transform into a moving frame using

$$\exp\left[\frac{ie}{\hbar}\left(\vec{A}\left(\vec{r}_0,t\right)\cdot\vec{r}\right)\right]. \tag{4.2}$$

This operator is unitary, and translates the momentum by $e\vec{A}\left(\vec{r}_0,t\right)$, canceling the time dependent component of the Hamiltonian.

Then the equation of motion is

$$i\hbar\partial_t\left|\phi\right\rangle = \left[\left(\frac{p^2}{2m} + V(r) + H_{\text{field}}\right) - e\vec{r}\cdot\vec{E}\right]\left|\phi\right\rangle. \tag{4.3}$$

Now suppose we have H_I in the case of a two-level atom, eigenstates $|e\rangle$ and $|g\rangle$. Typically, $\langle r\rangle = 0$ for both eigenstates. For superpositions, it may be non-zero. Therefore, we can let

$$\vec{r} = d\hat{x}\left(|g\rangle\langle e| + |e\rangle\langle g|\right). \tag{4.4}$$

In terns of the Pauli matrices,

$$\vec{r} = \hat{x}d\sigma_x. \tag{4.5}$$

Suppose the electric field is applied along the $\hat{x}$ direction, so

$$\vec{E} = E\hat{x}\left(a + a^\dagger\right). \tag{4.6}$$

Then the interaction Hamiltonian is

$$H_I = dE\left(|g\rangle\langle e| + |e\rangle\langle g|\right)\left(a + a^\dagger\right). \tag{4.7}$$

This has four terms, which can create or destroy photons while moving from $|e\rangle$ to $ketg$ or $|g\rangle$ to $|e\rangle$. Two of these terms are co-rotating, while two are counter-rotating.

We thus make the rotating wave approximation and consider only the two near resonant terms:

$$H_I = dE\left(a\,|g\rangle\langle e| + a^\dagger\,|e\rangle\langle g|\right) = g\left[a^\dagger\sigma^- + a\sigma^+\right], \tag{4.8}$$

where $\sigma^- = |g\rangle\langle e|$ and $\sigma^+ = |e\rangle\langle g|$. This is the Jaynes-Cummings interaction Hamiltonian.

4.1.2 Classical Control of a Spin

Suppose our electromagnetic field is a strong coherent state $|\alpha\rangle$ with $\alpha \gg 1$. Therefore, we can approximate $a^\dagger|\alpha\rangle \sim a^*|\alpha\rangle$. Then the Hamiltonian can be written

$$H = \frac{\omega_0}{2}\sigma_z + g_0\alpha_0\left[e^{i\omega t}\sigma^- + e^{-i\omega t}\sigma^+\right], \tag{4.9}$$

where the first term is the free Hamiltonian of the atom (letting the transition frequency be ω_0 and the field has frequency ω. We also define $g = g_0\alpha_0$. Then we can rewrite

$$H = \frac{\omega_0}{2}\sigma_z + g\left(\sigma_x\cos(\omega t) + \sigma_y\sin(\omega t)\right). \tag{4.10}$$

We define $|\phi(t)\rangle = e^{i\omega t\sigma_z/2}|\chi(t)\rangle$, so we can write the Schrödinger equation as

$$i\partial_t|\phi(t)\rangle = \left[e^{i\omega\sigma_z t/2}He^{-i\omega\sigma_z t/2} - \frac{\omega}{2}\sigma_z\right]|\phi(t)\rangle. \tag{4.11}$$

Using

$$e^{i\omega\sigma_z t/2}\sigma_x e^{-i\omega\sigma_z t/2} = \sigma_x\cos(\omega t) - \sigma_y\sin(\omega t), \tag{4.12}$$

this becomes

$$i\partial_t|\phi(t)\rangle = \left[\frac{\omega_0-\omega}{2} + g\sigma_x\right]|\phi(t)\rangle. \tag{4.13}$$

The solution is

$$|\phi(t)\rangle = \exp\left[i\left(\frac{\omega_0-\omega}{2} + g\sigma_x\right)t\right]|\phi(0)\rangle. \tag{4.14}$$

This corresponds to rotation of the state about an axis

$$\hat{n} = \frac{\hat{z} + \frac{2g}{\omega_0-\omega}\hat{x}}{\sqrt{1+\left(\frac{2g}{\omega_0-\omega}\right)^2}} \tag{4.15}$$

by an angle

$$|\vec{n}| = t\sqrt{\left(\frac{\omega_0-\omega}{2}\right)^2 + g^2}. \tag{4.16}$$

On resonance, we get rotation about the $\hat{x}$ axis $e^{igt\sigma_x}$. For large δ, the spin mostly rotates by $e^{igt\sigma_z}$, but there is an additional correction proportional to g^2/δ, which is the AC Stark shift.

4.1.3 Quantum Control

Now suppose $|\alpha\rangle$ is no longer weak. We consider the Jaynes-Cummings Hamiltonian again

$$H = \frac{\omega_0}{2}\sigma_z + \omega a^\dagger a + g\left[a^\dagger\sigma^- + a\sigma^+\right]. \tag{4.17}$$

This Hamiltonian is solvable.

Zero Detuning Case

Consider $\omega - \omega_0$. Then the Hamiltonian becomes

$$H = g\left[a^\dagger\sigma^- + a\sigma^+\right]. \tag{4.18}$$

We note that

$$\sigma^+\sigma^- = |e\rangle\langle e| \text{ and } \sigma^-\sigma^+ = |g\rangle\langle g|. \tag{4.19}$$

Therefore,

$$\left[a^\dagger\sigma^- + a\sigma^+\right]^{2k} = \left(aa^\dagger\right)^k |e\rangle\langle e| + \left(a^\dagger a\right)^k |g\rangle\langle g| \text{ and} \tag{4.20}$$

$$\left[a^\dagger\sigma^- + a\sigma^+\right]^{2k+1} = \left(aa^\dagger\right)^k a\,|e\rangle\langle g| + a^\dagger\left(a^\dagger a\right)^k |g\rangle\langle e|. \tag{4.21}$$

Letting $n = a^\dagger a$, we find the time evolution operator U:

$$U = e^{-it} \tag{4.22}$$

$$= \sum_k \frac{(-iHt)^k}{k!} \tag{4.23}$$

$$= \cos\left(gt\sqrt{n+1}\right)|e\rangle\langle e| + \cos\left(gt\sqrt{n}\right)|g\rangle\langle g| - i\frac{\sin\left(gt\sqrt{n+1}\right)}{\sqrt{n+1}}a\,|e\rangle\,g - ia^\dagger\frac{\sin\left(gt\sqrt{n+1}\right)}{\sqrt{n+1}}|g\rangle\langle e|. \tag{4.24}$$

An arbitrary state of the atom and field

$$|\psi\rangle = \sum_n c_n\,|e,n\rangle + d_n\,|g\rangle\,n \tag{4.25}$$

can then be time evolved.

4.1.4 General Time Evolution

Consider our previous solution in the case where the initial state is $|e\rangle$. The time evolution is

$$U(t)\sum_n c_n^0\,|e,n\rangle = \sum_n c_n^0\left[\cos\left(gt\sqrt{n+1}\right)|e,n\rangle - i\sin\left(gt\sqrt{n+1}\right)|g,n+1\rangle\right] = \sum_n\left[c_n(t)\,|e,n\rangle + d_{n+1}(t)\,|g,n+1\rangle\right]. \tag{4.26}$$

Let the polarization be

$$P(t) = \sum_n |c_n(t)|^2 - |d_n(t)|^2. \tag{4.27}$$

When we define the generalized Rabi frequency $\Omega_n^2 = \delta^2 + 4g^2(n+1)$, one can show

$$P(t) = \sum_n c_n^2\left[\frac{\delta^2 + 4g^2(n+1)\cos.(\Omega_n t)}{\Omega_n^2}\right]. \tag{4.28}$$

Vacuum Rabi Oscillations

Suppose we start with no photons, such that only $c_0 = 1$. Then

$$P(t) = \frac{\delta^2 + 4g^2 \cos(\Omega t)}{\Omega_0^2}. \tag{4.29}$$

The atom oscillates from the ground excited state to the ground state and back. These oscillations have been observed.

At time $t = \pi/2g$, the system is in a number operator eigenstate:

$$a^\dagger a |\psi\rangle = -ia^\dagger a |g\rangle |1\rangle = |\psi\rangle . \tag{4.30}$$

This is not like a coherent state.

Note that the ability to absorb the photon and return to the excited state makes this process quite different from spontaneous emission. A very good cavity is required in order to suppress spontaneous emission.

Consider the case where we have a weak coherent state. At zero detuning, the polarization as a function of time is

$$P(t) = \sum_n c_n^2 \cos(\Omega_n t) . \tag{4.31}$$

As the cos's dephase, this will go to half. For a strong coherent state, the photon number distribution is peaked at $|\alpha|^2$, so oscillations can occur for a while, but eventually decay. For a weak coherent state, the decay is faster. However, there can eventually be a revival in the oscillation when the oscillations are back in phase after time α/g. Consider the phase of the nth number state:

$$\phi_n(t) = g\sqrt{n+1}t \approx g\sqrt{\bar{n} + \delta n}t \approx gt\sqrt{n}\left(1 + \frac{\delta n}{2\bar{n}}\right) = gt\sqrt{\bar{n}}\left(\frac{\bar{n} + n}{2\bar{n}}\right). \tag{4.32}$$

To first order, the phase is linear in n. For sufficiently long time, a small δn will lead to dephasing. Photon number fluctuations become significant when $\frac{gt\Delta n}{\sqrt{\bar{n}}}$ is near 1:

$$\tau_{\text{collapse}} \sim \frac{\sqrt{\bar{n}}}{\delta n g} \sim \frac{1}{g}. \tag{4.33}$$

Revivals occur when neighboring states differ by π in phase:

$$\tau_{\text{revival}} \sim \pi\frac{\sqrt{\bar{n}}}{g} \sim \frac{2\pi|\alpha|}{g}. \tag{4.34}$$

Collapse and revival is common in non-linear isolated quantum systems.

The state ceases to be a coherent state after some time. We can write the sin as

$$\sin(\phi_n) = \frac{1}{2}\left(e^{i\phi_n} - e^{-i\phi_n}\right) \approx e^{ig\sqrt{\bar{n}}t/2}e^{ignt/2\sqrt{\bar{n}}} - e^{-ig\sqrt{\bar{n}}t/2}e^{-ignt/2\sqrt{\bar{n}}}. \tag{4.35}$$

The ground state contribution to the wavefunction can then be written at arbitrary times as

$$\sum_n e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} i\sin\left(gt\sqrt{n+1}\right)|g\rangle\,|n+1\rangle \approx \frac{1}{2}|g\rangle \otimes \sum_n e^{-|\alpha|^2/2}\left(e^{ig\sqrt{\bar{n}}t/2}\frac{\left(\alpha e^{igt/2\sqrt{\bar{n}}}\right)^n}{\sqrt{n!}} - e^{-ig\sqrt{\bar{n}}t/2}\frac{\left(\alpha e^{-igt/2\sqrt{\bar{n}}}\right)^n}{\sqrt{n!}}\right) \tag{4.36}$$

$$= \frac{1}{2}|g\rangle \otimes \left(e^{ig\sqrt{\bar{n}}t/2}\left|\alpha e^{igt/2\sqrt{\bar{n}}}\right\rangle - e^{-ig\sqrt{\bar{n}}t/2}\left|\alpha e^{-igt/2\sqrt{\bar{n}}}\right\rangle\right) \tag{4.37}$$

There is a similar form for the excited state. This is known as a cat state.

4.2 Transition amplitudes and S-matrix

Consider an initial state $|\psi_i\rangle$ and final state $|\psi_f\rangle$. The transition probability is

$$\langle\psi_f|U(t_f,t_i)|\psi_i\rangle\,. \tag{4.38}$$

For a perturbative calculation, we divide the Hamiltonian into $H = H_0 + V$. We then write states as sums of eigenstates $|\phi_n\rangle$ of H_0, then consider amplitudes between the eigenstates:

$$\langle\phi_m|U(t_f,t_i)|\phi_n\rangle\,. \tag{4.39}$$

Suppose the perturbation V is applied at time t_i (long) before the collision and removed at time t_f after the collision, so there is some time where H_0 eigenstates are correct. This is done by multiplying it by an envelop function $\lambda(t)$. This allows the eigenstates of H_0 to be good states as $t_i \to -\infty$ and $t_f \to +\infty$. The S-matrix is the matrix of our amplitudes.

4.2.1 Interaction Picture

In the Interaction Picutre, we transform waveforms as

$$\left|\tilde{\psi}(t)\right\rangle = e^{iH_0t/\hbar}\,|\psi(t)\rangle \tag{4.40}$$

and operators as

$$\tilde{A}(t) = e^{iH_0t/\hbar}Ae^{-iH_0t/\hbar}. \tag{4.41}$$

The Schrödinger equation becomes

$$i\hbar\frac{d}{dt}\left|\tilde{\psi}(t)\right\rangle = -H_0\left|\tilde{\psi}(t)\right\rangle + e^{iH_0t/\hbar}\left(H_0+V\right)|\psi(t)\rangle = \tilde{V}(t)\left|\tilde{\psi}(t)\right\rangle, \tag{4.42}$$

which is often simpler to solve.

Transition Amplitudes in the Interaction Picutre

We now have

$$\left|\tilde{\psi}(t_f)\right\rangle = \tilde{U}(t_f,t_i)\left|\tilde{\psi}(t_i)\right\rangle. \tag{4.43}$$

The time evolution operator in the interaction picture is

$$\tilde{U}(t_f, t_i) = e^{iH_0 t_f/\hbar} U(t_f, t_i) e^{-iH_0 t_i/\hbar}. \tag{4.44}$$

Since

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = (H_0 + V) |\psi(t)\rangle , \tag{4.45}$$

we have

$$U(t_f, t_i) = U_0(t_f, t_i) + \frac{1}{i\hbar} \int_{t_i}^{t_f} dt\, U_0(t_f, t) V U(t, t_i) \tag{4.46}$$

where

$$U_0(t_f, t_i) = e^{-iH_0(t_f - t_i)/\hbar}. \tag{4.47}$$

The Schrödinger equation gives us

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = (H_0 + V) |\psi(t)\rangle \tag{4.48}$$

$$i\hbar \frac{d}{dt} U(t, t_i) |\psi(t_i)\rangle = (H_0 + V) U(t, t_i) |\psi(t_i)\rangle \text{ so} \tag{4.49}$$

$$i\hbar \frac{d}{dt} U(t, t_i) = (H_0 + V) U(t, t_i), \tag{4.50}$$

so this U ought to satisfy that. Clearly, $U(t_f, t_i)$ satisfies $U(t_i, t_i) = \mathbb{I}$. We calculate

$$i\hbar \frac{d}{dt_f} U(t_f, t_i) = i\hbar \frac{d}{dt_f} \left[U_0(t_f, t_i) + \frac{1}{i\hbar} \int_{t_i}^{t_f} dt\, U_0(t_f, t) V U(t, t_i) \right] \tag{4.51}$$

$$= i\hbar U_0(t_f, t_i) + \int_{t_i}^{t_f} dt \left(\frac{d}{dt_f} U_0(t_f, t_i) \right) V U(t, t_i) + U_0(t_f, t_i) V U(t_f, t_i) \tag{4.52}$$

$$= H_0 U_0 + H_0 \int_{t_i}^{t_f} dt U_0(t_f, t_i) V U(t, t_i) + V U(t_f, t_i) \tag{4.53}$$

$$= H_0 \left(U_0 + \int_{t_i}^{t_f} dt U_0(t_f, t_i) V U(t, t_i) \right) + V U(t_f, t_i) \tag{4.54}$$

$$= (H_0 + V) U(t_f, t_i). \tag{4.55}$$

So,

$$i\hbar \frac{d}{dt_f} U(t_f, t_i) = (H_0 + V) U(t_f, t_i). \tag{4.56}$$

We can then make a perturbative expansion

$$U^{(0)} = U_0 \tag{4.57}$$

$$U^{(1)} = U_0 + U_0 V U \tag{4.58}$$

$$U^{(2)} = U_0 + U_0 V U_0 + U_0 V U_0 V U \tag{4.59}$$

$$U^{(3)} = \ldots. \tag{4.60}$$

$$U(t_f, t_i) = U_0(t_f, t_i) + \sum_{n=1}^{\infty} U^{(n)}(t_f, t_i). \tag{4.61}$$

This sum does not actually converge, but that is fine. We find

$$U^{(n)}(t_f,t_i)=\left(\frac{1}{i\hbar}\right)^n\int_{t_f\geq\tau_n\ldots\tau_2\geq\tau_1\geq t_i}d\tau_n\ldots d\tau_2\,d\tau_1\times e^{-iH_0(t_f-\tau_n)/\hbar}V\ldots Ve^{-iH_0(\tau_2-\tau_1)/\hbar}Ve^{-iH_0(\tau_1-t_i)/\hbar}. \tag{4.62}$$

This is a series of free interactions with instantaneous interactions occuring at various times. In the interaction picture,

$$\tilde{U}^{(n)}(t_f,t_i)=\left(\frac{1}{i\hbar}\right)^n\int_{t_f\geq\tau_n\ldots\tau_2\geq\tau_1\geq t_i}d\tau_n\tilde{V}(\tau_n)\ldots\tilde{V}(\tau_2)\tilde{V}(\tau_1) \tag{4.63, 4.64}$$

4.2.2 Scattering Matrix

The scattering matrix is a matrix with elements given by

$$S_{fi}=\langle\varphi_f|\tilde{U}(t_f,t_i)|\varphi_i\rangle=\delta_{fi}+\sum_{n=1}^{\infty}S_{fi}^{(n)},\ \text{where}\ S_{fi}^{(n)}=\langle\varphi_f|\tilde{U}^{(n)}(t_f,t_i)|\varphi_i\rangle\,. \tag{4.65}$$

First Order

The first order (for time independent V)is

$$S_{fi}^{(1)}=\frac{1}{i\hbar}\int_{t_i}^{t_f}d\tau_1\,V_{fi}e^{i(E_f-E_i)\tau_1/\hbar} \tag{4.66}$$

$$=\frac{V_{fi}}{i\hbar}\int_{-T/2}^{T/2}dt\,e^{i(E_f-E_i)t/\hbar} \tag{4.67}$$

$$=-2\pi iV_{fi}\delta^{(T)}\left(E_f-E_i\right), \tag{4.68}$$

where $t_i=-T/2$ and $t_f=+T/2$. $\delta^{(T)}$ would be an exact delta function as $T\to\infty$, but for finite times is a sinc function:

$$\delta^{(T)}\left(E_f-E_i\right)=\frac{1}{2\pi}\int_{-T/2}^{T/2}\frac{d\tau_1}{\hbar}e^{i(E_f-E_i)\tau_1/\hbar}=\frac{1}{\pi}\frac{\sin\left(\frac{E_f-E_i}{\hbar}\frac{T}{2}\right)}{E_f-E_i}. \tag{4.69}$$

The first zero occurs at approximately $1/T$, and the height is $T/2\pi\hbar$. This does tend to a Dirac delta function (one of the best delta functions) for infinite T. This gives us energy conservation over long times.

Second Order

Now let us consider 2nd order:

$$S_{fi}^{(2)}=\left(\frac{1}{i\hbar}\right)^2\int_{T/2\geq\tau_2\geq\tau_1\geq-T/2}d\tau_1\,d\tau_2\sum_k\langle\varphi_f|V|\varphi_k\rangle\,\langle\varphi_k|V|\varphi_i\rangle\,e^{i(E_f-E_k)\tau_2/\hbar}e^{i(E_k-E_i)\tau_1/\hbar}. \tag{4.70}$$

We want to get rid of time ordering, which we accomplish using the Heaviside function $\theta(\tau_2 - \tau_1)$. Then we use that

$$e^{-iE_k(\tau_2-\tau_1)/\hbar}\theta(\tau_2-\tau_1) = \lim_{\eta\to 0^+} -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, \frac{e^{-iE(\tau_2-\tau_1)/\hbar}}{E - E_k + i\eta}. \tag{4.71}$$

This is a manifestation of the Cauchy integral formula. The function $\frac{e^{-iE(\tau_2-\tau_1)/\hbar}}{E-E_k+i\eta}$ has a pole at $E = E_k - i\eta$. When $\tau_2 < \tau_1$, the argument of the exponential goes to zero for infinite imaginary E, so we can do the integral by looping to positive imaginary E. Because there is no pole in the upper imaginary plane, this goes to zero, which is what we expect for $\theta(\tau_2 - \tau_1)$. When $\tau_2 > \tau_1$, the loop must go to negative imaginary E, and encircles the pole once, giving us the $2\pi i e^{-iE_k(\tau_2-\tau_1)/\hbar}$ residue. The value of the integral is then $2\pi i$, so we find the Heaviside function.

Returning to the second order calculation,

$$S_{fi}^{(2)} = \left(\frac{1}{i\hbar}\right)^2 \left(-\frac{1}{2\pi i}\right) \int_{-T/2}^{T/2} d\tau_2 \int_{-T/2}^{T/2} d\tau_1 \int_{-\infty}^{\infty} dE\, e^{i(E_j-E)\tau_2/\hbar} e^{-i(E_i-E)\tau_1/\hbar} W_{fi}(E), \tag{4.72}$$

where

$$W_{fi}(E) = \lim_{\eta\to 0^+} \sum_k \frac{V_{jk}V_{ki}}{E - E_k + i\eta}. \tag{4.73}$$

The time integrals give us $2\pi\hbar\delta^{(T)}(E_f - E)$ and $2\pi\hbar\delta^{(T)}(E_i - E)$. Therefore, E_i, E_f, and E must be the same to within $\hbar/T$. We assume that $W_{fi}(E)$ varies slowly over the interval $\hbar/T$, making the expression

$$S_{fi}^{(2)} = -\frac{1}{2\pi i}\frac{4\pi^2\hbar^2}{(i\hbar)^2}\left[\lim_{\eta\to 0^+}\sum_k \frac{V_{fk}V_{fi}}{E_i - E_k + i\eta}\right] \int_{-\infty}^{\infty} dE\, \delta^{(T)}(E - E_i)\, \delta^{(T)}(E - E_f) \tag{4.74}$$

$$= -\frac{1}{2\pi i}\frac{4\pi^2\hbar^2}{(i\hbar)^2}\left[\lim_{\eta\to 0^+}\sum_k \frac{V_{fk}V_{ki}}{E_i - E_k + i\eta}\right] \delta^{(T)}(E_i - E_f). \tag{4.75}$$

Note that when we have resonant intermediate states (e.g. $\exists k$ s.t. $E_k \sim E_i$), the perturbative approach may fail. Simplifying and summarizing,

$$S_{fi} = \delta_{fi} - 2\pi i \delta^{(T)}(E_f - E_i)\left[V_{fi} + \lim_{\eta\to 0^+}\sum_k \frac{V_{fk}V_{ki}}{E_i - E_k + i\eta} + O(V^3)\right]. \tag{4.76}$$

The infinite series in square brackets is known as the T matrix. This can be written

$$T_{fi} = \langle\varphi_f|V|\varphi_i\rangle + \langle\varphi_f|V\frac{1}{E_i - H + i\eta}V|\varphi_i\rangle. \tag{4.77}$$

It turns out that in general,

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{B}(B - A)\frac{1}{A}. \tag{4.78}$$

So, for $A = E_i - H + i\eta$ and $B = E_i - H_0 + i\eta$,

$$\frac{1}{E_i - H + i\eta} = \frac{1}{E_i - H_0 + i\eta} + \frac{1}{E_i - H_0 + i\eta}V\frac{1}{E_i - H + i\eta}. \tag{4.79}$$

These inverse A and B terms are Green's functions.

Chapter 5

Lecture 13: Scattering Matrix Processes

We expect to find things like Rabi oscillations, Raman transitions, etc.

5.1 Transition Probabilities

The probability of transitioning from an initial state $|\varphi_i\rangle$ to a final state $|\varphi_f\rangle$ is

$$P_{fi}(T) = |S_{fi}|^2 = 4\pi^2 \left[\delta^{(T)}\left(E_i - E_f\right)\right]^2 \times \left| V_{fi} + \lim_{\eta \to 0^+} \sum_k \frac{V_{fk} V_{ki}}{E_i - E_k + i\eta} + \ldots \right|^2 . \tag{5.1}$$

The squared approximate delta funciton is

$$\left[\delta^{(T)}\left(E_f - E_i\right)\right]^2 = \frac{1}{\pi^2} \frac{\sin^2\left(\left(E_f - E_i\right)\frac{T}{2\hbar}\right)}{\left(E_f - E_i\right)^2}. \tag{5.2}$$

We expect this to resemble a Rabi oscillation. It does have the oscillator component. For a continuum, we find

$$\int_{-\infty}^{\infty} dE_f \left[\delta^{(T)}\left(E_f - E_i\right)\right]^2 = \frac{T}{2\pi\hbar} \propto T. \tag{5.3}$$

The probability grows linearly in time. For two discrete states,

$$P_{fi}(T) = \frac{4\left|V_{fi}\right|^2}{\left(E_f - E_i\right)^2} \sin^2\left(\frac{\left(E_f - E_i\right)T}{2\hbar}\right). \tag{5.4}$$

The exact (Rabi) formula is

$$P_{fi}(T) = \frac{4\left|V_{fi}\right|^2}{\left(E_f - E_i\right)^2 + 4\left|V_{fi}\right|^2} \sin^2\left(\frac{\sqrt{\left(E_f - E_i\right)^2 + \left|V_{fi}\right|^2}T}{2\hbar}\right). \tag{5.5}$$

These are different, but to first order, the Rabi oscillation formula becomes the same (since the squares of V_{fi} disappear). As we approach resonance, the higher order terms become more important. If we neglect these terms anyway and consider the resonant case (which makes $E_f - E_i$ small, and sin roughly linear), we find

$$P_{fi}(T) = |V_{fi}|^2 \frac{T^2}{\hbar^2} \propto T^2. \tag{5.6}$$

Chapter 6

Lecture 12 and 13

6.1 S-matrix for Atom-Photon Interactions

6.1.1 Details of the Hamiltonian

The Hamiltonian is

$$H = H_A + H_R + H_I, \tag{6.1}$$

where H_A describes atoms, H_R describes radiation ($a^\dagger a$ for each mode), and H_I describes interactions. In the long wavelength approximation, the interaction is

$$H_I = -\frac{q}{m}\vec{p}\cdot\vec{A}(\vec{r}_0) + \frac{q^2}{2m}\vec{A}^2(\vec{r}_0). \tag{6.2}$$

Alternatively, the Goppert-Meyer transformation gives us the dipole Hamiltonian

$$H_I' = -\vec{d}\cdot\vec{E}(\vec{r}_0). \tag{6.3}$$

Note that these are all operators (though hats are not written).

We can decompose the two terms in the Hamiltonian (in the first form) as

$$H_{I1} = \frac{q}{m}\sum_l \sqrt{\frac{\hbar}{2\epsilon_0\omega V}}\vec{p}\cdot\vec{\epsilon}_l\left(a_l + a_l^\dagger\right) \text{ and} \tag{6.4}$$

$$H_{I2} = \frac{q^2}{2m}\frac{\hbar}{2\epsilon_0 V}\sum_{jl}\frac{\vec{\epsilon}_j\cdot\vec{\epsilon}_l}{\sqrt{\omega_l\omega_j}}\times\left(a_j a_l^\dagger + a_j^\dagger a_l + a_j a_l + a_j^\dagger a_l^\dagger\right). \tag{6.5}$$

We take $\vec{r}_0$ to be zero here for simplicity.

When spin is included, there is an additional spin term:

$$H_S = g\mu_B\vec{S}\cdot\vec{B}(\vec{r}_0). \tag{6.6}$$

6.1.2 H_{I1} Processes

What does H_{I1} do to the unperturbed states?

H_{I1} is linear in the fields. For a two-level (levels $|a\rangle$ and $|b\rangle$) atoms, the eigenstates of $H_0 = H_A + H_R$ are

$$|a, n_1, \ldots, n_j, \ldots\rangle \text{ and } |b, n_1, \ldots, n_j, \ldots\rangle . \tag{6.7}$$

Absorption

H_{I1} allows for absorption:

$$|\psi_i\rangle = |a, n_j\rangle \to |\psi_f\rangle = |b, n_j = n_j - 1\rangle . \tag{6.8}$$

a_j provides this:

$$a_j |n_j\rangle = \sqrt{n_j}\, |n_j - 1\rangle . \tag{6.9}$$

So,

$$\Omega_1 = \langle b, n_j - 1|H_{I1}|a, n_j\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2\epsilon_0\omega_j V}}\, \langle b|\vec{p}\cdot\vec{\epsilon}_j|a\rangle \sqrt{n_j}. \tag{6.10}$$

The $\sqrt{n_j}$ makes this proportional to the laser power. The transition probability is proportional to intensity:

$$P_{i\to f}(T) \propto |\langle b, n_j - 1|H_{I1}|a, n_j\rangle|^2 \propto n_j \propto \text{ intensity}. \tag{6.11}$$

If we consider time dependence,

$$P_{i\to f}(T) = \frac{\Omega_1^2}{\Omega_1^2 + \delta^2} \sin^2\left(\sqrt{\Omega_1^2 + \delta^2}\frac{T}{2}\right) \tag{6.12}$$

where $\delta = \omega_i - \frac{E_b - E_a}{\hbar}$.

We think the probability should become largest when energy is conserved, which requires

$$E_a + n_j\hbar\omega_j = E_b + (n_j - 1)\,\hbar\omega_j \implies E_b = E_a + \hbar\omega_j. \tag{6.13}$$

Emission

We can also get emission.

$$a_j^\dagger |n_j\rangle = \sqrt{n_j + 1}\, |n_j + 1\rangle , \tag{6.14}$$

which gives us the matrix element

$$\langle a, n_{j+1}|H_{I1}|b, n_j\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2\epsilon_0 V\omega_j}} \times \langle a|\vec{p}\cdot\vec{\epsilon}_j|b\rangle \sqrt{n_j + 1}. \tag{6.15}$$

We again expect energy conservation (which will be shown shortly), which implies

$$E_a = E_b - \hbar\omega_j. \tag{6.16}$$

Note that a large n_j increases the probability of emission occurring. This is stimulated emission. For $n_j = 0$, spontaneous emission is still possible. The matrix element for this is

$$\langle a, 1|H_{I1}|b, 0\rangle = -\frac{q}{m}\sqrt{\frac{\hbar}{2\epsilon_0 V\omega_j}} \times \langle a|\vec{p}\cdot\vec{\epsilon}_j|b\rangle . \tag{6.17}$$

We would like to determine $\langle a|\vec{p}\cdot\vec{\epsilon}_j|b\rangle$. We use $-i\frac{m}{\hbar}[\vec{r}, H_A] = \vec{p}$, from the quantum equivalent of Hamilton's equations. Then

$$\langle a|\vec{p}\cdot\vec{\epsilon}_j|b\rangle = -i\frac{m}{\hbar}\langle a|[\vec{r}, H_A]|b\rangle\cdot\vec{\epsilon}_j = -i\frac{m}{\hbar}(E_b - E_a)\,\langle a|\vec{r}\cdot\vec{\epsilon}_j|b\rangle = -i\frac{m\omega_0}{q}\langle a|\vec{d}\cdot\vec{\epsilon}_j|b\rangle . \tag{6.18}$$

6.1.3 H_{I2} Processes

H_{I2} describes elastic scattering: a change in a photon with no change in the atomic state. Note that the long wavelength approximation is significant here.

Four processes are possible:

- $a_j^\dagger a_l^\dagger$: two photons appear (off resonant, unlikely)
- $a_j a_l$: two photons disappear (off resonant, unlikely)
- $a_j^\dagger a_l$ and $a_j a_l^\dagger$: one photon destroyed, one new photon created. To conserve energy, need $\omega_j = \omega_l$.

The amplitude for the $a_j^\dagger a_l$ term is

$$\langle a, n_j = 1, n_l = n_l - 1 | H_{I2} | a, n_j = 0, n_l = n_l \rangle = \frac{q^2}{m} \frac{\hbar}{2\epsilon_0 V} \frac{\vec{\epsilon}_j \cdot \vec{\epsilon}_l}{\omega_j} \sqrt{n_l}, \tag{6.19}$$

using $\omega_j = \omega_l$.

Chapter 7

Lecture 13: Transition into Continuum

7.1 Fermi's Golden Rule

Consider a final state with energy near E_f with other parameters β after time T. The probability density of transitioning to a state near E_f is

$$dP\left(E_f,\beta,T\right)=\int_{E\in\delta E_f}dE\int_{\beta\in\delta\beta_f}d\beta\,\rho\left(E,\beta\right)\left|\left\langle E,\beta|\tilde{U}(T)|\varphi_i\right\rangle\right|^2. \tag{7.1}$$

The matrix element is what was determined in the previous two chapters. Using this,

$$dP\left(E_f,\beta,T\right)=\int dE\,d\beta 4\pi^2\left|V\left(E,\beta_i,\varphi_i\right)\right|^2\left[\delta^{(T)}\left(E-E_i\right)\right]^2. \tag{7.2}$$

One of the delta functions evaluates approximately to $\frac{T}{2\pi\hbar}$, so

$$dP\left(E_f,\beta,T\right)=\int dE\,d\beta 4\pi^2\rho\left(E,\beta\right)\left|V\left(E,\beta,\varphi_i\right)\right|^2\frac{T}{2\pi\hbar}\delta^{(T)}\left(E-E_i\right). \tag{7.3}$$

Since this grows linearly, what is more interesting is the transition probability per unit time, which is the transition rate:

$$\delta w\left(E_f,\beta_f\right)=\frac{1}{T}\delta P\left(E_f,\beta_f,T\right). \tag{7.4}$$

So,

$$\delta w=\frac{2\pi}{\hbar}\int dE\,d\beta\rho\left(E,\beta\right)\left|V\left(E,\beta,\varphi_i\right)\right|^2\delta^{(T)}\left(E-E_i\right). \tag{7.5}$$

We can do the energy integral. We then divide by β to find

$$\frac{dw}{d\beta}=\frac{2\pi}{\hbar}\left|V\left(E,\beta,\varphi_i\right)\right|^2\rho\left(E_f=E_i,\beta_f\right). \tag{7.6}$$

This is Fermi's Golden Rule.

7.2 Scattering Cross Section

To determine the scattering cross section, we simply divide the rate per angle by the incoming photon flux. The incoming photon flux is

$$\phi_i = c\frac{n_i}{L^3}. \tag{7.7}$$

The rate to go into an angle Ω is then

$$\frac{dw}{d\Omega} = \frac{d\Gamma}{d\Omega} = c\frac{n_i}{L^3}\frac{d\sigma}{d\Omega} = \phi_i\frac{d\sigma}{d\Omega}, \tag{7.8}$$

which gives us the scattering cross section $\frac{d\sigma}{d\Omega}$.

7.3 Spontaneous Emission

What happens when we have an atom in an excited state transitioning into being in the ground state with a single photon?

7.3.1 Background Math

To first order, our scattering matrix is

$$S_{fi} = \delta_{fi} - 2\pi i\delta^{(T)}\left(E_f - E_i\right)V_{fi}. \tag{7.9}$$

Then our transition probability is

$$P_{fi} = \left|2\pi i\delta^{(T)}\left(E_f - E_i\right)V_{fi}\right|^2 = T\frac{2\pi}{\hbar}\left|V_{fi}\right|^2\delta^{(T)}\left(E_f - E_i\right). \tag{7.10}$$

On resonance, we get another factor of T:

$$P_{fi} = T^2\frac{\left|V_{fi}\right|^2}{\hbar^2}. \tag{7.11}$$

This is the beginning of an oscillation.

The probability of remaining in the initial state must be the $1-$ the sum of the probabilities of transitioning to other states:

$$P_{ii} = 1 - \sum_{f\neq i}P_{fi} = 1 - T\frac{2\pi}{\hbar}\sum_{f\neq i}\left|V_{fi}\right|^2\delta^{(T)}\left(E_f - E_i\right) = 1 - \Gamma T. \tag{7.12}$$

This is linear in T. In reality, this is exponential decay for a short time.

To second order

$$S_{fi} = \delta_{fi} - 2\pi i\delta^{(T)}\left(E_f - E_i\right)\left[V_{fi} + \sum_k\frac{V_{ik}V_{kf}}{E_i - E_k + i\eta}\right]. \tag{7.13}$$

Then the probability of not transitioning is

$$P_{ii}(T) = \left| 1 - 2\pi i \delta^{(T)}(0) \left[V_{ii} + \sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} \right] \right|^2 \tag{7.14}$$

$$= \left| 1 - 2\pi i \frac{T}{2\pi\hbar} \sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} \right] \tag{7.15}$$

$$= \left| 1 - \frac{iT}{\hbar} \sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} \right]. \tag{7.16}$$

The states k are truly a coninuum, so

$$|V_{ik}|^2 \to |V(E)|^2 \text{ and } \sum_k \to \int dE \int d\beta\, \rho\,(E,\beta)\,. \tag{7.17}$$

Then

$$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} \to \int dE\, d\beta\, \rho\,(E,\beta)\, \frac{|V(E)|^2}{E_i - E + i\eta}. \tag{7.18}$$

It turns out that

$$\frac{1}{x + i\eta} = \frac{1}{x + i\eta} \frac{x - i\eta}{x - i\eta} = \frac{x}{x^2 + \eta^2} - \frac{i\eta}{x^2 + \eta^2}. \tag{7.19}$$

These are Lorentzians. The integral is

$$\lim_{\eta\to 0} \int dx\, f(x) \frac{x}{x^2 + \eta^2} \approx \lim_{\eta to 0} \left[\int_{-\infty}^{-\eta} dx + \int_{\eta}^{\infty} dx \right] \frac{f(x)}{x} + \int_{-\eta}^{\eta} dx\, f(x) \frac{x}{x^2 + eta^2} \tag{7.20}$$

$$= P \int_{-\infty}^{\infty} dx \frac{f(x)}{x} + f(0) \int_{-\eta}^{\eta} dx \frac{x}{x^2 + \eta^2} \tag{7.21}$$

$$= P \int_{-\infty}^{\infty} dx \frac{f(x)}{x} \tag{7.22}$$

using that $f(0)$ times an odd function is zero when integrated. The principal integral can be approximated as $\pi f(0)$:

$$\lim_{\eta\to 0} \int dx\, f(x) \frac{\eta}{x^2 + \eta^2} \approx f(0) \int_{-\infty}^{\infty} dx \frac{\eta}{x^2 + \eta^2}. \tag{7.23}$$

$$\lim_{\eta\to 0} -\frac{i\eta}{x^2 + \eta^2} = -i\pi\delta(x). \tag{7.24}$$

So,

$$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} = \Delta - i\pi \int d\beta\, \rho\,(E,\beta)\, |V\,(E_i,\beta)|^2 = \Delta - i\frac{\hbar\Gamma}{2} \tag{7.25}$$

where

$$\Delta = P \int_{-\infty}^{\infty} dx \frac{f(x)}{x}. \tag{7.26}$$

Then

$$P_{ii} = \left|1 - iT\frac{1}{\hbar}\left(\Delta - i\hbar\frac{\Gamma}{2}\right)\right|^2 = \left|1 - T\frac{\Gamma}{2} - iT\frac{\Delta}{\hbar}\right|^2 = 1 - \Gamma T + \dots. \tag{7.27}$$

In the non-perturbative case, one finds

$$S_{ii} = e^{-\Gamma T/2}e^{-iDeltaT/\hbar}, \tag{7.28}$$

which corresponds to a decay and a level shift. Then

$$P_{ii} = |S_{ii}|^2 = e^{-\Gamma T}. \tag{7.29}$$

7.3.2 Spontaneous Emission for Atoms

Our initial state is an excited state $|b\rangle$. The final atomic state is a lower state $|a\rangle$, with $E_b - E_a = \hbar\omega_0$. An emitted photon can have wavevector

$$\vec{k}_j = \frac{2\pi}{L}\left(n_x\vec{e}_x + n_y\vec{e}_y + n_z\vec{e}_z\right) \tag{7.30}$$

which implies frequency $\omega_j = ck_j$.

Fermi's Golden Rule gives us

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar}\left|\left\langle a, \vec{k}\vec{\epsilon}\middle| H_{I1}\middle| b, 0\right\rangle\right|^2 \rho\left(\theta, \varphi, \hbar\omega = \hbar\omega_0\right). \tag{7.31}$$

The matrix element is

$$\left\langle a, \vec{k}\vec{\epsilon}\middle| H_{I1}\middle| b, 0\right\rangle = \left\langle a, \vec{k}\vec{\epsilon}\middle| -\vec{d}\vec{E}\middle| b, 0\right\rangle = \langle a|\hat{\vec{d}}|b\rangle\left\langle \vec{k}\vec{\epsilon}\middle| -i\sum_l \mathcal{E}_l\vec{\epsilon}_l\left(a_l - a_l^\dagger\right)\middle| b, 0\right\rangle. \tag{7.32}$$

Let $d\vec{e}_z = \langle a|\hat{\vec{d}}|b\rangle$. The only light amplitude that matters is $\mathcal{E}_k$, and the only polarization is $\vec{\epsilon}$ (since we assumed an output polarization). Therefore,

$$\left\langle a, \vec{k}\vec{\epsilon}\middle| H_{I1}\middle| b, 0\right\rangle = id\vec{e}_z\cdot\mathcal{E}_k\vec{\epsilon}. = i\sqrt{\frac{\hbar\omega_0}{2\epsilon_0 V}}d\left(\vec{\epsilon}\cdot\vec{e}_z\right). \tag{7.33}$$

The density of states is given by the number of photons in a certain phase space region. We are interested in the number of photons in a volume of k space.

$$dN = \rho\, dE\, d\Omega = \left(\frac{L}{2\pi}\right)^3 d^3k = \left(\frac{L}{2\pi}\right)^3 k^2\, dk\, d\Omega = \left(\frac{L}{2\pi}\right)^3 k^2\frac{dE}{\hbar c}\, d\Omega \implies \rho = \left(\frac{L}{2\pi}\right)^3\frac{k^2}{\hbar c} = \left(\frac{L}{2\pi}\right)^3\frac{E^2}{(\hbar c)^3}, \tag{7.34}$$

using that $E = \hbar ck \implies dE = \hbar c\, dk$. Therefore,

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar}\frac{\hbar\omega_0}{2\epsilon_0 V}d^2\left(\vec{\epsilon}\cdot\vec{e}_z\right)^2\frac{V}{(2\pi)^3}\frac{E^2}{(\hbar c)^3} = \frac{2\pi}{\hbar}\frac{\hbar\omega_0}{2\epsilon_0}d^2\left(\vec{\epsilon}\cdot\vec{e}_z\right)^2\frac{1}{(2\pi)^3}\frac{E^2}{(\hbar c)^3}. \tag{7.35}$$

Fortunately, the volume factors cancel. Making further cancellations and combinations (noting that $E = \hbar\omega_0$),

$$\frac{d\Gamma}{d\Omega} = \frac{1}{8\pi^2\epsilon_0}\frac{\omega_0^3}{\hbar c^3}d^2\left(\vec{\epsilon}\cdot\vec{e}_z\right)^2. \tag{7.36}$$

Note the scaling with ω_0^3: higher energy transitions decay much faster. This is due to both the matrix element growing with ω_0 and the density of states growing with ω_0^2 (the surface of a sphere).

We also need to sum over all directions and polarizations. There are two polarizations $\vec{\epsilon}$ and $\vec{\epsilon}'$, which are orthogonal to each other and to $\vec{k}/|\vec{k}|$. Therefore,

$$1 = (\vec{\epsilon}\cdot\vec{e}_z)^2+(\vec{\epsilon}'\cdot\vec{e}_z)^2+\left(\frac{\vec{k}}{|\vec{k}|}\cdot\vec{e}_z\right)^2 = \left(1-\cos^2\theta\right)+\cos^2\theta \implies (\vec{\epsilon}\cdot\vec{e}_z)^2+(\vec{\epsilon}'\cdot\vec{e}_z)^2 = \sin^2\theta. \quad (7.37)$$

Since

$$\int d\Omega\,\sin^2\theta = 2\pi\int_{-1}^{1} d(\cos\theta)\,\sin^2\theta = 2\pi\int_{-1}^{1} du\,\left(1-u^2\right) = 2\pi\left(2-\frac{2}{3}\right) = \frac{8\pi}{3}, \quad (7.38)$$

we find the total rate of emission is

$$\Gamma = \frac{d^2\omega_0^3}{3\pi\epsilon_0\hbar c^3}. \quad (7.39)$$

The dipole is $d = qz_{ab}$, where z_{ab} is on the order of the Bohr radius. We recall the fine structure constant is

$$\alpha = \frac{q^2}{4\pi\epsilon_0\hbar c}. \quad (7.40)$$

Using these,

$$\Gamma = \frac{4}{3}\alpha\frac{\omega_0^3 z_{ab}^2}{c^2}. \quad (7.41)$$

We would like to determine the Q factor of the transition. This is given by

$$\frac{1}{Q} = \frac{\Gamma}{\omega_0} = \frac{4}{3}\alpha\frac{\omega_0^2 z_{ab}^2}{c^2}. \quad (7.42)$$

What is the order of magnitude here? We can approximate

$$z_{ab} = a_0 = \frac{\hbar^2}{me^2} = \frac{1}{\alpha}\frac{\hbar}{mc} \quad (7.43)$$

and

$$\hbar\omega_0 = \alpha^2 mc^2, \quad (7.44)$$

since there must be units of energy, but the energy must be sub-relativistic. Then

$$\frac{1}{Q}\sim\alpha^3 \implies Q\sim 10^6. \quad (7.45)$$

7.3.3 Wigner-Weisshopf: Exponential Decay

Consider our decay from an initial state to a continuum again. Let the initial state be $|i\rangle$ and the final state be $|k\rangle$ with $E_k = k\epsilon$, where ϵ is the spacing between continuum states. Let the coupling be w. From Fermi's Golden Rule,

$$\Gamma = \frac{2\pi}{\hbar}|w|^2\cdot\frac{1}{\epsilon}. \quad (7.46)$$

The wave function can be written as (choosing the initial state energy to be zero)

$$|\psi(t)\rangle = \gamma_i(t)\,|i\rangle + \sum_{k=-\infty}^{\infty} \gamma_k(t) e^{-i\epsilon k t/\hbar}\,|k\rangle\,. \tag{7.47}$$

Note that since off-resonant processes are supressed, having the sum run from $-\infty$ to ∞ is only a small deviation from reality.

The Hamiltonian can be written

$$H = \epsilon_i\,|i\rangle\langle i| + \sum_{k=-\infty}^{\infty} k\epsilon\,|k\rangle\langle k| + w\sum_{k=-\infty}^{\infty}\left(|k\rangle\langle i| + |i\rangle\langle k|\right) = \sum_{k=-\infty}^{\infty} k\epsilon\,|k\rangle\langle k| + w\sum_{k=-\infty}^{\infty}\left(|k\rangle\langle i| + |i\rangle\langle k|\right), \tag{7.48}$$

where the second step follows from choosing the initial state energy to be zero. From the Schrödinger equation

$$i\hbar\frac{d}{dt}\,|\psi(t)\rangle = H\,|\psi(t)\rangle \tag{7.49}$$

$$\implies i\hbar\dot{\gamma}_i(t)\,|i\rangle + i\bar{\sum}_{k=-\infty}^{\infty}\dot{\gamma}_k(t)e^{-i\epsilon k t/\hbar}\,|k\rangle + i\hbar\sum_{k=-\infty}^{\infty}\epsilon k\gamma_k(t)e^{-i\epsilon k t/\hbar}\,|k\rangle \tag{7.50}$$

$$= w\gamma_i\sum_{k=-\infty}^{\infty}|k\rangle + \sum_{k=-\infty}^{\infty}\gamma_k(t)e^{-i\epsilon k t/\hbar}\epsilon k\,|k\rangle + w\sum_{k=-\infty}^{\infty}\gamma_k(t)e^{-i\epsilon k t/\hbar}\,|i\rangle\,. \tag{7.51}$$

Projecting onto $|i\rangle$ and $|k\rangle$, we find

$$i\hbar\frac{d}{dt}\gamma_i(t) = w\sum_{k=-\infty}^{\infty}\gamma_k(t)e^{-ik\epsilon t/\hbar} \tag{7.52}$$

$$i\hbar\frac{d}{dt}\gamma_k(t) = we^{ik\epsilon t/\hbar}\gamma_i(t). \tag{7.53}$$

Initially, $\gamma_k(0) = 0$, so the solution to the second equation is

$$\gamma_k(t) = \frac{w}{i\hbar}\int_0^t dt'\,\gamma_i(t')e^{ik\epsilon t'/\hbar}, \tag{7.54}$$

which implies the solution to the first equation is

$$\frac{d}{dt}\gamma_i(t) = -\frac{\Gamma}{2\pi\hbar}\int_0^t dt'\,\gamma_i(t')\left[\sum_{k=-\infty}^{\infty}\epsilon e^{ik\epsilon\left(t'-t\right)/\hbar}\right]. \tag{7.55}$$

We approximate the spectrum of final states as a continuum

$$\sum_{k=-\infty}^{\infty}\epsilon e^{ik\epsilon\left(t'-t\right)/\hbar} \approx \int_{-\infty}^{\infty} dE\,e^{iE\left(t'-t\right)/\hbar} = 2\pi\hbar\delta\left(t'-t\right). \tag{7.56}$$

Then

$$\frac{d}{dt}\gamma_i(t) = -\Gamma\int_{-t}^{0} d\tau\,\delta(\tau)\gamma_i\left(t+\tau\right) = -\frac{\Gamma}{2}\gamma_i(t) \implies \gamma_i(t) = e^{-\Gamma t/2}. \tag{7.57}$$

The probability of remaining in the initial state is thus

$$P_{ii}(t) = e^{-\Gamma t}. \tag{7.58}$$

Chapter 8

Lecture 14 Transition into Continuum Continued, Scattering, and Atom-Atom Interactions

8.1 Spontaneous Emission

8.1.1 Wigner-Weisshopf Notes

Recall that within the S-matrix element for remaining in the initial state, we found

$$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} = \Delta - i\hbar\frac{\Gamma}{2} \tag{8.1}$$

for a continuum of k. That is, a continuum coupling leads to shift and decay. We do not see the shift in the Wigner-Weisshopf derivation because it assumes an equal number of higher and lower energy states, so all shifts cancel. Note that the exponential decay actually requires a Lorentzian lineshape, which implies symmetry. Therefore, decays will often have a slight deviation.

8.1.2 Scattering

Suppose we have initial state consisting of a ground state atom and a laser

$$|\varphi_i\rangle = \left|a, N\vec{k}_0, \vec{\epsilon}_0\right\rangle \tag{8.2}$$

coupled with couplig V_{if} to state consisting of an excited state atom and a laser

$$|\varphi_f\rangle = \left|b, (N-1)\,\vec{k}_0, \vec{\epsilon}_0\right\rangle \tag{8.3}$$

and this state in turn coupled with coupled V_{fg} to the state consisting of a ground state atom, a laser, and a spontaneously emitted photon

$$|\varphi_g\rangle = \left|a, (N-1)\,\vec{k}_0\vec{\epsilon}_0, \vec{k}\vec{\epsilon}\right\rangle. \tag{8.4}$$

If $V_{fg} = 0$, we get Rabi oscillations with $\Omega \propto dE \propto d\sqrt{N}$. If $V_{if} = 0$, we get exponential decay from $|\varphi_f\rangle$ to $|\varphi_g\rangle$.

Suppose both couplings are non-zero.

First, consider the case $\Omega \gg \Gamma$. We would expect to get damped Rabi oscillations with the damping on timescale $1/\Gamma$.

Next, let us consider $\Gamma \gg \Omega$. The coherent Rabi oscillations will not occur. $|\varphi_i\rangle$ will decay at a rate we can determine from Fermi's Golden Rule. Due to the coupling V_{fg}, $|\varphi_f\rangle$ acts as a continuum of width Γ, so

$$\Gamma_i = \frac{2\pi}{\hbar}\left|V_{if}\right|^2 \frac{1}{\Gamma}. \tag{8.5}$$

8.2 Scattering

We can represent scattering in Feynman diagrams. Consider the case of an atom scattering a single photon. There are three possible diagrams.

In the first, an atom absorbs a photon, is in an excited state, then emits a photon. Call this process α.

In the second, an atom emits a photon, is in an excited state, then absorbs a photon. Call this process β.

Finally, the atom could emit and absorb a photon at the same time. Call this process γ.

For initial state a, final state a', intial photon ω, and final photon ω', all of these diagrams require

$$E_{a'} + \hbar\omega' = E_a + \hbar\omega. \tag{8.6}$$

When $a' = a$, energy conservation demains $\omega' = \omega$. This is elastic scattering. Otherwise, scattering is inelastic.

8.2.1 Scattering Amplitudes

To determine scattering amplitudes, we use the T-matrix. For the α process

$$\tau_{ij}^{\alpha} = \sum_b \lim_{\eta\to 0^+} \frac{\left\langle a', \vec{k}'\vec{\epsilon}'\middle|H_{I1}\middle|b, 0\right\rangle \left\langle b, 0\middle|H_{I1}\middle|a, \vec{k}\vec{\epsilon}\right\rangle}{E_a + \hbar\omega - E_b + i\eta}. \tag{8.7}$$

For the β process

$$\tau_{ij}^{\beta} = \sum_b \lim_{\eta\to 0^+} \frac{\left\langle a', \vec{k}'\vec{\epsilon}'\middle|H_{I1}\middle|b, \vec{k}\vec{\epsilon}\vec{k}'\vec{\epsilon}'\right\rangle \left\langle b, \vec{k}\vec{\epsilon}\vec{k}'\vec{\epsilon}'\middle|H_{I1}\middle|a, \vec{k}\vec{\epsilon}\right\rangle}{E_a - \hbar\omega - E_b + i\eta}. \tag{8.8}$$

For the γ process,

$$\tau_{ij}^{\gamma} = \left\langle a', \vec{k}'\vec{\epsilon}'\middle|H_{E2}\middle|a, \vec{k}\vec{\epsilon}'\right\rangle. \tag{8.9}$$

8.2.2 Overview of Processes

Rayleigh Scattering

Suppose we have a high energy transition, but only low energy photons,

$$\hbar\omega \ll |E_b - E_a| \,. \tag{8.10}$$

One will find $\sigma \propto \omega^4$. Each photon (initial and scattered) contributes a factor of ω, while the density of states contributes a factor of ω^2.

Raman Scattering

In Raman scattering, $a' \neq a$. This can be used to determine vibrational states of molecules. When $E_{a'} > E_a$, the process is Stokes. When $E_{a'} < E_a$, the process is anti-Stokes.

Thompson Scattering

In Thompson scattering, $\hbar\omega \gg E_I$, where E_I is the ionization energy. The atom is not ionized, and $\hbar\omega' = \hbar\omega$. The γ process dominates here, as the denominator for the α and β processes is quite large.

8.3 Atom-Atom Interactions

We will begin with the Van-der-Waals and Casimir forces. These are purely quantum.

8.3.1

Two electrons interact via a Coulomb repulsion:

$$V_{ee} = \frac{e^2}{r}. \tag{8.11}$$

For an electron and an atom, the electric field from the electron induces a dipole moment in the atom leading to an energy

$$-\vec{d}\cdot\vec{E} = -\left(\alpha\vec{E}\right)\cdot\vec{E} = -\alpha\left|\vec{E}\right|^2. \tag{8.12}$$

Since $|\vec{E}| = e/r^2$,

$$V_{ae} = -\alpha\frac{e^2}{r^4}. \tag{8.13}$$

This is a clear, classical story.

For an atom and an atom, the classical answer is zero, as there is no electric field to induce a dipole. Quantum mechanically, we find (in second order)

$$V_{aa} = -\frac{\left(\frac{\vec{d}\cdot\vec{d}'}{r^3}\right)^2}{\Delta E}. \tag{8.14}$$

Although $\vec{d}$ and $\vec{d'}$ are zero, quantum mechanically their squares are non-zero due to noise/zero point fluctuations. We can estimate $\Delta E \sim e^2/a_0$ as being the ground state to first excited state (nearest state with opposite parity) energy. $|\vec{d}| \sim ea_0$. Then

$$V_{aa} \sim -\frac{\left(\frac{(ea_0)^2}{r^3}\right)^2}{e^2/a_0} \sim -\frac{e^2}{a_0} \cdot \frac{a_0^6}{r^6}. \tag{8.15}$$

This tends to be larger for larger atoms (which is actually relevant, due to the a_0^6 scaling).

Recall from electromagnetism that there is an interaction between an electron and a conducting plane:

$$V_{e,\ \text{wall}} = -\frac{e^2}{z}. \tag{8.16}$$

This emerges from considering an image charge. For an atom, there will be a perfectly correlated image charge for the atom. This leads to an attraction

$$V = -\frac{\left|\vec{d}\right|^2}{z^3} \sim -\frac{(ea_0)^2}{z^3}. \tag{8.17}$$

8.3.2 A General Argument

Conisder two polarizable systems. Suppose we have an external electric field $\vec{E}_0$ which induces dipole moments $\vec{d_1}, \vec{d_2} = \alpha\vec{E}_0$. Then there is an interaction

$$V(x_1, x_2, t) = \int_0^\infty d\omega\, N(\omega)\vec{d_1} \cdot \vec{E}_{2\to1}(\omega, t). \tag{8.18}$$

The density of states is

$$N(\omega)\, d\omega = 2V\, d^3k = 2\omega^2\, d\omega \sin\theta\, d\theta\, d\phi\, \frac{V}{c^3}. \tag{8.19}$$

The electric field is

$$\vec{E}_{2\to1}(\omega, t) = \frac{\vec{d_2} - 3\vec{d}_{2z}\hat{z}}{r^3} + \frac{\dot{\vec{d}}_2 - 3\dot{d}_{2z}\hat{z}}{cr^2} + \frac{\ddot{\vec{d}}_2 - \ddot{d}_{2z}\hat{z}}{c^2 r}. \tag{8.20}$$

The first two terms are te familiar $1 - 3\cos^2\theta$. The last term is a radiation term, proportional to $\alpha_2\vec{E}_0\frac{\omega^2}{c^2 r}$. We also add a cutoff frequency determined by the separation between the atoms, c/r. Therefore,

$$V(\vec{x}_1, \vec{x}_2) = \frac{V}{c^5 r}\int_0^{c/r} \alpha_1(\omega)\alpha_2(\omega)\omega^2 \left|E_0(\omega, \vec{x}_1)\right|^2 \omega^2\, d\omega. \tag{8.21}$$

We choose E_0 to be due to vacuum fluctuations, so

$$\left|E_0(\omega, \vec{x}_1)\right|^2 \sim \frac{\hbar\omega}{V}. \tag{8.22}$$

Then

$$V(\vec{x}_1, \vec{x}_2) = \frac{\hbar}{c^5 r}\int_0^{c/r} \alpha_1(\omega)\alpha_2(\omega)\omega^5\, d\omega. \tag{8.23}$$

Let our $\alpha(\omega)$ be given by the electron on a spring model. Then

$$\alpha(\omega) = \frac{e^2/m}{\omega_0^2 - \omega^2}. \tag{8.24}$$

We note that c/r is very low frequency. Therefore, for atoms $\alpha(\omega) \sim 1/\omega^2$.

For electron-electron interactions, $\omega_0 = 0$, so we keep $\alpha = e^2/m\omega^2$. Therefore,

$$V_{ee} \sim \frac{\hbar e^4}{c^3 m^2 r^3}. \tag{8.25}$$

For atom-electron interactions

$$V_{ae} \sim \frac{\hbar e^2 \alpha}{mcr^5}, \tag{8.26}$$

where α is the atom polarizability.

For atom-atom interactions, $\alpha(\omega) \sim 1/\omega_0^2$. Then

$$V_{aa} \sim \frac{\hbar c \alpha_1 \alpha_2}{r^7}. \tag{8.27}$$

This applies at longer distances than the Van-der-Waals interaction.

8.3.3 Interaction by Photon Exchange

A more rigorous argument.

Our interaction Hamiltonian is

$$H_I' = -\vec{d} \cdot \vec{E}_\perp(\vec{R}) - \vec{d}' \cdot \vec{E}_\perp(\vec{R}'). \tag{8.28}$$

The diagram for the interaction is that one atom emits a photon and transitions to the excited state, then the second atom absorbs this photon and transitions to the exicted state. This is off resonant, so the second atom needs to emit and transition to the ground state, and the first atom then absorbs this phton and returns to the ground state.

Chapter 9

Lecture 15 Atom-Atom Interactions

9.1 Van-der-Waals revisited

Consider two atoms in the ground state $|a\rangle$, with an excited state $|b\rangle$. The Hamiltonian is

$$H_I' = -\vec{d}\cdot\vec{E}_\perp(\vec{R}) - \vec{d}'\cdot\vec{E}_\perp(\vec{R}'). \tag{9.1}$$

The Feynman diagram has the first atom emit a photon, which the second atom absorbs, then the second atom emit a photon, which the first atom absorbs. This leaves both atoms in the ground state and conserves energy. Note that the intermediate state is far off resonance.

Low frequency modes will not have a strong contribution, due to a small density of states and the matrix element (which is raised to the fourth) scaling with $\sqrt{\omega}$. High frequencies will also not contribute too much, due to too many interferences. Thus, $k \sim 1/D$, where D is the separation, will be most significant.

9.1.1 Short Distance Limit

We can also consider this wavelength relative to the exictation wavelength. When $D \ll \lambda_{ab} = \frac{\hbar c}{|E_a - E_b|}$, we have $\hbar\omega \gg \hbar\omega_{ab}$. Since the distance is small, the photons will not exist for long. The atoms appear to be excited simultaneously.

We can write an effective interaction matrix element

$$\langle b,b'|\delta V|a,a'\rangle = -\sum_{\vec{k}\vec{\epsilon}} \frac{1}{\hbar\omega} \left\langle b,b',0\middle|H_I\middle|b,a',\vec{k}\vec{\epsilon}\right\rangle \left\langle b,a',\vec{k}\vec{\epsilon}\middle|H_I\middle|a,a',0\right\rangle \tag{9.2}$$

$$= -\sum_{\vec{k}\vec{\epsilon}} \frac{1}{2\epsilon_0 V} \left(\vec{d}\cdot\vec{\epsilon}\right)\left(\vec{d}\cdot\vec{\epsilon}\right) e^{i\vec{k}\cdot(\vec{r}-\vec{r}')} + \text{h.c.} \tag{9.3}$$

$$= -\frac{1}{\epsilon_0}\sum_{ij} d_i d_j \delta_{ij}^\perp(\vec{r}-\vec{r}'). \tag{9.4}$$

We approximate the denominator as $\hbar\omega$, since the photons are higher energy. We use that

$$\sum_{ij} \epsilon_i \epsilon_k + \frac{k_i k_j}{k^2} = \delta_{ij} \implies \sum_{ij} \epsilon_i \epsilon_k = \delta_{ij} - \frac{k_i k_j}{k^2} \tag{9.5}$$

and

$$\delta_{ij}^{\perp}(\vec{r} - \vec{r}') = \frac{-\delta_{ij} + 3u_i u_j}{4\pi D^3}, \text{ where } u_i = \frac{r_i}{|r|}. \tag{9.6}$$

Evaluating these in the matrix element, we find the typical dipole-dipole interaction.

Using this interaction, the energy shift (in second order) is

$$\Delta E = \sum_{b,b'} \frac{\langle a, a'|\delta V|b, b'\rangle \ \langle b, b'|\delta V|a, a'\rangle}{E_a + E'_a - E_b - E'_b} = -\frac{C_6}{D^6}. \tag{9.7}$$

C_6 is defined as the coefficient for the separation.

9.1.2 Large Distance Limit

Now our photons will be around for a while, while the atoms are not excited for long. One atom must emit two photons, and then the second atom absorbs both photons. The photon wavelength is much larger, so $\omega \ll \omega_{ab}$. The energy defect is now twice the transition energy, since this is larger, $2\hbar\omega_{ab}$. Since $\omega \approx 1/\lambda \approx 1/D$, and our denominator has fewer ω factors, the potential becomes $1/D^7$. This can also be considered due to propagation time (i.e. the potential is now the retarded potential).

9.2 Overview

For atom-atom interactions at short distances, the energy shift is roughly

$$\frac{e^2}{a_0}\frac{a_0^6}{r^6}. \tag{9.8}$$

For atom-atom interactions at long distances, the energy shift is roughly

$$\frac{\hbar c a_0^6}{r^7}. \tag{9.9}$$

For an atom interacting with a wall at short distances, we have

$$\frac{(ea_0)^2}{z^3}. \tag{9.10}$$

For an atom interacting with a wall at long distances, we have

$$\frac{\hbar c}{z^4} a_0^3. \tag{9.11}$$

Finally, for a wall interacting with a wall, we have

$$\frac{\hbar c}{z^4}. \tag{9.12}$$

9.3 Atom-Wall Interactions Revisited

9.3.1 Short Distances

Suppose $z \ll \lambda_{ab} \sim \frac{a_0}{\alpha} \approx 137a_0$, then we can consider this to be the interaction of correlated dipoles, which gives

$$V_{a-w} = \frac{(ea_0)^2}{z^3}. \tag{9.13}$$

9.3.2 Long Distances

Now suppose $z \gg \lambda_{ab} \approx 137a_0$. Then we can use Sprunch's formula

$$V(r) = \frac{\hbar}{c^5 r}\int_0^{c/r} d\omega\, \alpha_1(\omega)\alpha_2(\omega)\omega^5. \tag{9.14}$$

This is a general interaction for two polarizable things. We consider an atom to be subject to some perturbing field that leads to it developing a dipole moment, then consider the field from this dipole moment acting on the second atom. The ω^5 emerges from an electric field being generated by an accelerating charge giving two factors of ω (the second derivative of a dipole), the density of states giving two factors of ω (the surface of a sphere), and a factor of ω for the vacuum energy. The cutoff is motivated by high frequency photons dephasing.

We replace the wall by a sphere of radius z. The polarizability is then z^3. This implies a potential of

$$V_{a-w} \sim \frac{\hbar}{c^5 z} a_0^3 z^3 \frac{c^6}{z^6} = \frac{\hbar c}{z^4} a_0^3. \tag{9.15}$$

9.3.3 Another way

We have a polarizable system in a background field, which has energy

$$\alpha(\omega)|E_0|^2(\omega, \vec{x}) = \alpha(\omega)u(\omega, \vec{x}), \tag{9.16}$$

where u is the energy density. For vacuum, this is $\hbar\omega/V$. Then the energy is

$$\mathcal{E} = \int_0^\infty d\omega\, N(\omega)u(\omega, \vec{x})\alpha(\omega) = \frac{\hbar}{c^3}\int_0^\infty d\omega\, \alpha(\omega)\omega^3. \tag{9.17}$$

This will be infinite, but we want to consider a difference.

Now suppose we have anm atom at distance z from the wall. Fluctuations with $\omega \gg c/z$ will not be affected by the presence of the wall. Fluctuations with $\omega \ll c/z$ will be greatly affected by the presence of the wall. Since the wall does not effect the high frequency component, the difference $\mathcal{E} - \mathcal{E}(z)$ will be finite. Therefore, we compute

$$V_{a-w}(z) = \frac{\hbar}{c^3}\int_0^{c/z} d\omega\, \alpha(\omega)\omega^3 \approx \frac{\hbar}{c^3}\int_0^{c/z} d\omega\, a_0^3\omega^3 = \frac{\hbar}{c^3}a_0^3\frac{c^4}{z^4} = \frac{\hbar c}{z^4}a_0^3. \tag{9.18}$$

9.4 Wall-Wall Interactions

9.4.1 Spherical Walls

Using the wall to sphere replacement, we can approximate the polarizability as z^3. The force per unit area is

$$\frac{1}{z^2}\frac{\partial V}{\partial z} \approx \frac{V}{z^3}. \tag{9.19}$$

$$V = \frac{\hbar}{c^5 z} z^6 \int_0^{c/z} d\omega \, omega^5 = \frac{\hbar c}{z}. \tag{9.20}$$

Putting these together,

$$\frac{F}{A} = \frac{\hbar c}{z^4} \tag{9.21}$$

9.4.2

Consider two planar mirrors seprated by some distance L along $\hat{z}$, with $z = 0$ at one mirror and $z = L$ at the other. This forms a cavity, which supports two types of modes. One set of modes has the electric field transverse to the mirror separation (TE modes), the other set has the magnetic field transverse to the mirror separation (TM modes).

We can also divide modes into modes propagating along $+\hat{z}$, labelled $\vec{k}_+$ and modes propagating along $-\hat{z}$, labelled $\vec{k}_-$. Let φ be the angle of $\vec{k}$ about $\hat{z}$. Then we can write

$$\vec{k}_\pm = \mp l\hat{k} + k\vec{\varphi}, \tag{9.22}$$

where $\vec{\varphi}$ is parallel to the mirrors. The frequency of the light is

$$\frac{\omega^2}{c^2} = l^2 + k^2. \tag{9.23}$$

If our mirrors are perfectly conductive, we must no electric field along the surface. Since the mirrors must not have surface currents, the magnetic field can not be normal to the surface. This gives us a quantization condition for l:

$$l = \frac{m\pi}{L}, \quad m \in \mathbb{Z}. \tag{9.24}$$

Thus,

$$\omega^2 = m^2\omega_0^2 + k^2c^2, \text{ where } \omega_0 = \frac{c\pi}{L} \text{ is the free spectral range.} \tag{9.25}$$

We can now write the fields in terms of $\vec{\alpha}$ for TE and TM modes:

$$\vec{\alpha}^E_{m,k,\varphi}(z,\vec{\rho}) = \sqrt{\frac{2}{V}} \sin\left(\frac{m\pi z}{L}\right) e^{i\vec{k}\cdot\vec{\rho}} \vec{\varphi} \times \hat{z} \text{ and} \tag{9.26}$$

$$\vec{\alpha}^M_{m,k,\varphi}(z,\vec{\rho}) = \sqrt{\frac{\beta m}{V}} \left[\frac{ck}{\omega}\cos\left(\frac{m\pi z}{L}\right)\vec{n} - \frac{im\omega_0}{\omega}\sin\left(\frac{m\pi z}{L}\right)\vec{\varphi}\right] e^{i\vec{k}\cdot\vec{\rho}}. \tag{9.27}$$

βm is 1 for $m = 0$ and 2 for $m > 0$. For $m = 0$, the sin is zero, implying there is no sin mode. The cos is always constant non-zero, which does not vanish at the mirrors and averages to 1, rather

than 1/2 that various non-zeor cos arguments average to. The vector potentials associated with the normal modes are

$$\vec{A}^E(z,\rho) = \sum_{m,k,\varphi} \left\{ \sqrt{\frac{\hbar}{2\epsilon_0 \omega}} \vec{\alpha}^E_{mk\varphi} a^E_{mk\varphi} + \text{h.c.} \right\} \text{ and} \tag{9.28}$$

$$\vec{A}^M(z,\rho) = \sum_{m,k,\varphi} \left\{ \sqrt{\frac{\hbar}{2\epsilon_0 \omega}} \vec{\alpha}^M_{mk\varphi} a^M_{mk\varphi} + \text{h.c.} \right\}. \tag{9.29}$$

The total vector potential is $\vec{A} = \vec{A}^E + \vec{A}^M$. This is a quantization of the electric field in constrained space.

To find the ground/vacuum state energy, we simply need to count the modes. In order to count modes, we add a transverse box with (very large) side length a, such that $V = La^2$. We apply cyclic boundary conditions, which quantizes k to

$$k_x, k_y = \frac{2\pi}{a} n_x, \frac{2\pi}{a} n_y. \tag{9.30}$$

The number of modes for a given m with k between k and $k + dk$ is

$$d^2k \frac{a^2}{(2\pi)^2} = \frac{a^2}{2\pi} k\, dk = \frac{a^2}{2\pi c^2} \omega\, d\omega, \tag{9.31}$$

using the dispersion relation $\frac{\omega^2}{c^2} = l^2 + k^2$, with l fixed. A given ω can be obtained for $m = 0$ to $\left\lfloor \frac{\omega}{\omega_0} \right\rfloor$, where $\lfloor x \rfloor$ (floor) denotes the last integer smaller than x. For $m > 0$, we have TE and TM modes. $m = 0$ only has a TM mode. Then the density of states is

$$\rho^{\text{cav}}(\omega) = \frac{a^2 \omega}{2\pi c^2} \left[1 + 2 \left\lfloor \frac{\omega}{\omega_0} \right\rfloor \right] = \frac{V \omega \omega_0}{2\pi^2 c^3} \left[1 + 2 \sum_{m=1}^{\infty} \Theta\left(\frac{\omega}{\omega_0} - m \right) \right], \tag{9.32}$$

where Θ is the Heaviside step function. In free space, $L \to \infty$, which implies $\omega_0 \to 0$. Then Θ is always equal to 1, and we find

$$\rho^{\text{free}}(\omega) \to \frac{V \omega^2}{\pi^2 c^3}. \tag{9.33}$$

Using this density of states, the ground state energy is

$$W(L) = \sum_{\text{modes}} \frac{\hbar\omega}{2} = \int_0^\infty d\omega\, \frac{\hbar\omega}{2} \rho^{\text{cav}}(\omega) = \frac{a^2 \hbar}{4\pi^2 c^2} \left[I_0 + 2 \sum_{m=1}^{\infty} I_m \right]. \tag{9.34}$$

We then wish to evaluate

$$I_m = \int_{m\omega_0}^\infty d\omega\, \omega^2. \tag{9.35}$$

We add a high frequency cutoff, assuming such photons will be high enough frequency that the mirrors (the electrons in the mirrors) will no longer be able to respond. This is given by $e^{-\lambda\omega/c}$, with $\lambda \ll L$. Then

$$I_m = \int_{m\omega_0}^\infty d\omega\, \omega^2 e^{-\lambda\omega/c} = c^2 \frac{\partial^2}{\partial \lambda^2} \int_{m\omega_0}^\infty d\omega\, e^{-\lambda\omega/c} = c^3 \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{\lambda} e^{-m\pi\lambda/L} \right] \tag{9.36}$$

using $\omega_0 = \pi c/L$. We then have a simple geometric series, so

$$\sum_{m=1}^{\infty} c^3 \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{\lambda} e^{-m\pi\lambda/L} \right] = c^3 \frac{\partial^2}{\partial \lambda^2} \left[\sum_{m=1}^{\infty} \frac{e^{-m\pi\lambda/L}}{\lambda} \right] = \frac{c^3 \pi}{L} \frac{\partial^2}{\partial \lambda^2} \left\{ \frac{L}{\pi\lambda} \frac{1}{e^{\pi\lambda/L} - 1} \right\}. \tag{9.37}$$

Expanding this for small λ/L (since the cutoff wavelength should be much less than the distance between the mirrors),

$$\frac{L}{\pi\lambda} \frac{1}{e^{\pi\lambda/L} - 1} = \frac{L^2}{(\pi\lambda)^2} - \frac{L}{2\pi\lambda} + \frac{1}{12} - \frac{1}{720} \left(\frac{\pi\lambda}{L} \right)^2 + \dots. \tag{9.38}$$

Taking the second derivative of this, the energy is

$$W(L) = \frac{a^2\hbar}{4\pi c^2} I_0 + \frac{a^2 \hbar c}{2} \left[\frac{6L}{\pi^2 \lambda^4} - \frac{1}{\pi\lambda^3} - \frac{2\pi^2}{720 L^3} + \dots \right]. \tag{9.39}$$

Now, let us embed the cavity (one of the mirrors, anyway) inside a larger cavity. Then we can look at the force on the mirror inside the cavity. Let the distance from the first mirror to the mirror of the larger cavity be $L_0 \gg L$. The sum of the energies of the inner and outer cavity is then

$$W_T(L) = \frac{a^2\hbar}{2\pi c^2} I_0 + \frac{a^2 \hbar c}{2} \left[\frac{6L_0}{\pi^2 \lambda^4} - \frac{2}{\pi\lambda^3} - \frac{2\pi^2}{720 L^3} + \dots \right]. \tag{9.40}$$

To determine the force on the mirrors in the smaller cavity, we take the derivative (difference) of the potential with displacement:

$$W_T(L') - W_T(L) = -\frac{a^2 \pi^2 \hbar c}{720} \left(\frac{1}{L'^3 - L^3} \right). \tag{9.41}$$

The potential energy is

$$U(L) = -\frac{\pi^2 \hbar c}{720} \frac{a^2}{L^3}. \tag{9.42}$$

The corresponding pressure is

$$P_{\rm vac} = \frac{1}{a^2} \frac{\partial U}{\partial L} = \frac{\pi^2 \hbar c}{240} \frac{1}{L^4}. \tag{9.43}$$

This is approximately $10^{-3}\,\mathrm{Pa}$ for $L = 1\,\mu\mathrm{m}$.

Chapter 10

Lecture 16: Continuum Coupling

Our equation for a photon interacting on resonance with an atom still has an infinity in the denominator, which is problematic.

10.1 Discrete to Continuum Coupling

Consider a discrete state $|\varphi\rangle$ coupled to a continuum of states. Let the coupling be v (Hermitian) and the states be $|k\rangle$ with energy spacing δ, as in the Wigner-Weisshopf setup. Let $E_k = k\delta$ (relative to the energy of $|\varphi\rangle$). Our matrix elements are

$$\langle\varphi|V|k\rangle = v = \langle k|V|\varphi\rangle \text{ and } \langle\varphi|V|\varphi\rangle = \langle k|V|k\rangle = 0. \tag{10.1}$$

Our Hamiltonian is $H = H_0 + V$. Let the eigenstates of H be $|\psi_\mu\rangle$. We want to solve

$$H|\psi_\mu\rangle = E_\mu|\psi_\mu\rangle. \tag{10.2}$$

Similar to the Wigner-Weisshopf method, we project onto $|k\rangle$ and $|\varphi\rangle$. Doing this,

$$\langle k|H|\psi_\mu\rangle = E_k\langle k|\psi_\mu\rangle + v\langle\varphi|\psi_\mu\rangle = E_\mu\langle k|\psi_\mu\rangle \tag{10.3}$$

$$\langle\varphi|H|\psi_\mu\rangle = E_0\langle\varphi|\psi_\mu\rangle + \sum_k v\langle k|\psi_\mu\rangle = E_0\langle\varphi|\psi_\mu\rangle + \sum_k v\langle k|\psi_\mu\rangle = E_\mu\langle\varphi|\psi_\mu\rangle, \tag{10.4}$$

using that $E_0 = 0$. From the first equation, we can write

$$\langle k|\psi_\mu\rangle = v\frac{\langle\varphi|\psi_\mu\rangle}{E_\mu - E_k}. \tag{10.5}$$

It is unlikely that the eigenstates are unshifted, so the denominator is probably non-zero. Plugging this into the second equation, we find

$$\sum_k \frac{v^2}{E_\mu - E_k}\langle\varphi|\psi_\mu\rangle = E_\mu\langle\varphi|\psi_\mu\rangle \implies E_\mu = \sum_k \frac{v^2}{E_\mu - E_k}. \tag{10.6}$$

This is a nice eigenvalue equation.

We also ought to have normalization, which implies

$$\sum_k |\langle k|\psi_\mu\rangle|^2 + |\langle\varphi|\psi_\mu\rangle|^2 = 1. \tag{10.7}$$

Using this and our first projection equation, we have

$$\langle\varphi|\psi_\mu\rangle = \sqrt{1 + \sum_k \left(\frac{v}{E_\mu - E_k}\right)^2}. \tag{10.8}$$

In addition, this gives us

$$\langle k|\psi_\mu\rangle = \frac{\frac{v}{E_\mu - E_k}}{\sqrt{1 + \sum_k \left(\frac{v}{E_\mu - E_k}\right)^2}}. \tag{10.9}$$

The eigenvalue equation holds in general, and can be solved for our evenly spaced states. Define $z \equiv E_\mu/\delta$. Then our sum effectively becomes

$$\sum_k \frac{1}{z-k}. \tag{10.10}$$

This is holomorphic except at integer z (technically meromorphic). Such functions are uniquely defined by their singularities and residues. It turns out that this sum satisfies

$$\sum_k \frac{1}{z-k} = \frac{\pi}{\tan(\pi z)}. \tag{10.11}$$

Using a similar process for our state sum,

$$\sum_k \frac{1}{(z-k)^2} = \frac{\pi^2}{\sin^2(\pi z)}. \tag{10.12}$$

Computing our actual sum,

$$\sum_k \frac{v^2}{E_\mu - E_k} = \frac{v^2\pi}{\delta\tan(\pi E_\mu/\delta)} = E_\mu. \tag{10.13}$$

This is numerically solvable, but not analytically solvable. We can also write this in a form inspired by Fermi's Golden Rule:

$$\frac{1}{\tan(\pi E_\mu/\delta)} = \frac{2E_\mu}{\hbar\Gamma} \text{ for } \Gamma = \frac{2\pi}{\hbar}v^2\frac{1}{\delta}. \tag{10.14}$$

We define a "mixing angle"

$$\phi_\mu = \tan^{-1}\left(\frac{\hbar\Gamma}{2E_\mu}\right). \tag{10.15}$$

In terms of this, the eigenvalue equation becomes

$$\frac{E_\mu}{\delta} = m + \frac{\phi_\mu}{\pi}. \tag{10.16}$$

Now let us turn to the energy denominator.

$$1+v^2\sum_k\left(\frac{1}{E_\mu-E_k}\right)^2 = 1+\frac{v^2}{\delta^2}\sum_k\frac{1}{(E_\mu/\delta-k)^2} \tag{10.17}$$

$$= 1+\frac{\pi^2v^2}{\delta^2}\left(1+\frac{1}{\tan^2(\pi E_\mu/\delta)}\right) \tag{10.18}$$

$$= 1+\frac{\pi^2v^2}{\delta^2}+\frac{E_\mu^2}{v^2} \tag{10.19}$$

$$= \frac{1}{v^2}\left(v^2+\left(\frac{\hbar\Gamma}{2}\right)^2+E_\mu^2\right). \tag{10.20}$$

Then

$$\langle\varphi|\psi_\mu\rangle = \frac{v}{\sqrt{v^2+(\hbar\Gamma/2)^2+E_\mu^2}}. \tag{10.21}$$

This should be reminiscent of common denominators: Rabi frequency (v^2), decay rate (Γ^2), and energy (E_μ^2).

10.1.1 Dissolving in Continuum

Now suppose we can tune the energy of the original state. Consider an energy interval $[E, E+dE]$ with $\hbar\Gamma \gg dE \gg \delta$. We want to determine how much of state $|\varphi\rangle$ is now in that energy range. That is, for $E < E_\mu < E+dE$,

$$dN_\varphi = \sum|\langle\varphi|\psi_\mu\rangle|^2 \approx \frac{dE}{\delta}|\langle\varphi|\psi_\mu\rangle|^2. \tag{10.22}$$

Then

$$\frac{dN_\varphi}{dE} = \frac{1}{\delta}|\langle\varphi|\psi_\mu\rangle|^2 = \frac{v^2}{\delta}\frac{1}{v^2+(\hbar\Gamma/2)^2+E_\mu^2} = \frac{\frac{\hbar\Gamma}{2\pi}}{v^2+(\hbar\Gamma/2)^2+E_\mu^2}. \tag{10.23}$$

Taking the energy spacing to zero and keeping Γ (v^2/δ) fixed, we find

$$\frac{dN_\varphi}{dE} = \frac{\hbar\Gamma/2\pi}{(\hbar\Gamma/2)^2+E^2}. \tag{10.24}$$

The original state is now "dissolved" over a width in the continuum.

10.1.2 Discrete State Coupled to Dissolved State

Suppose now we have another discrete state $|\chi\rangle$ coupled to the original $|\varphi\rangle$. Note that (after the coupling is switched on) these are states of both atoms and light. Let the interaction coupling $|\chi\rangle$ and $|\varphi\rangle$ be

$$\langle\varphi|W|\chi\rangle = w. \tag{10.25}$$

Also, let $\langle k|W|\chi\rangle = 0$. Since $|\chi\rangle$ is now coupled to the continuum (indirectly), it will acquire a linewidth and finite lifetime. To determine this coupling, we consider

$$\langle\psi_\mu|W|\chi\rangle = \langle\psi_\mu|\varphi\rangle\,\langle\varphi|W|\chi\rangle = \langle\psi_\mu|\varphi\rangle\,w. \tag{10.26}$$

The lifetime of $|\chi\rangle$ is then (from Fermi's Golden Rule)

$$\Gamma_\chi = \frac{2\pi}{\hbar} \left| \langle \psi_\mu | W | \chi \rangle \right|^2 \frac{1}{\delta} = \frac{2\pi}{\hbar} w^2 \left| \langle \psi_\mu | \varphi \rangle \right|^2 \frac{1}{\delta} = w^2 \frac{\Gamma}{\left(\hbar\Gamma/2\right)^2 + E_\chi^2}. \tag{10.27}$$

For the resonant coupling case, when $E_\chi = E_\varphi = 0$,

$$\Gamma_\chi = \frac{4w^2}{\hbar^2 \Gamma}. \tag{10.28}$$

This is the quantum Zeno effect: when Γ is strong, the excited state decays quickly. This makes it easy to determine if there is probability amplitude in the excited state. Due to this constant "measurement", there is little coherent evolution into the excited state. It thus takes $|\chi\rangle$ longer to decay.

Chapter 11

Lecture 17: Continuum Coupling Continued, Non-Perturbative Transition Amplitudes

11.1 Resonant Scattering Through a Discrete Level

Let $|\chi_i\rangle$ and $|\chi_f\rangle$ have the same energy E_χ. For example,

$$|\chi_i\rangle = \left|a, \vec{k}\vec{\epsilon}\right\rangle \text{ and } |\chi_f\rangle = \left|a, \vec{k}'\vec{\epsilon}'\right\rangle. \tag{11.1}$$

Suppose these states are coupled to a discrete state $|\varphi\rangle$ (e.g. an atom in the excited state, having absorbed a photon). Let the coupling coefficients be

$$\langle\varphi|W|\chi_i\rangle = w_i, \quad \langle\varphi|W|\chi_f\rangle = w_j, \quad \langle\chi_{i,j}|W|\chi_{i,j}\rangle = 0, \text{ and } \langle k|W|\chi_{i,j}\rangle = 0. \tag{11.2}$$

To lowest order, we usethe T matrix. We consider the first order result:

$$T_{fi} = \lim_{\eta\to 0^+} \langle\chi_f| W \frac{1}{E_\chi - H_0 + i\eta} W |\chi_i\rangle \approx \lim_{\eta\to 0^+} \frac{\langle\chi_f|W|\varphi\rangle \langle\varphi|W|\chi_i\rangle}{E_\chi - E_\varphi + i\eta}. \tag{11.3}$$

For $|\varphi\rangle$ resonant with $|\chi_{i,j}\rangle$, this becomes problematic. Therefore, we must consider the exact result:

$$T_{fi} = \lim_{\eta\to 0^+} \langle\chi_f|W|\varphi\rangle \langle\varphi| \frac{1}{E_\chi - H + i\eta} |\varphi\rangle \langle\varphi|W|\chi_i\rangle. \tag{11.4}$$

We can do this for the situation we considered since last chapter. We use the closure relation to determine

$$\langle\varphi|\frac{1}{E_\chi - H + i\eta}|\varphi\rangle = \sum_\mu \frac{|\langle\varphi|\psi_\mu\rangle|^2}{E_\chi - E_\mu + i\eta}. \tag{11.5}$$

Using the results from the last chapter,

$$T_{fi} = \lim_{\eta\to 0} w_i w_j^* \delta \sum_\mu \frac{\hbar\Gamma/2\pi}{(E_\chi - E_\mu + i\eta)\left(E_\mu^2 + (\hbar\Gamma/2)^2 + v^2\right)}. \tag{11.6}$$

Taking $\delta \to 0$, $\delta \sum_\mu \to \int dE$. Then

$$T_{fi} = \lim_{\eta \to 0^+} w_i w_j^* \int_{-\infty}^{\infty} dE \frac{\hbar\Gamma/2\pi}{(E_\chi - E + i\eta)\left(E^2 + (\hbar\Gamma/2)^2\right)} = w_j^* w_i \frac{1}{E_\chi + i\hbar\Gamma/2}. \tag{11.7}$$

There is now no divergence: T_{fi} has a resonance at $E_\chi = 0$, with width Γ.

11.1.1 Scattering of Light by an Atom

Consider an atom in the ground state with a photon: $\left|a, \vec{k}\vec{\epsilon}\right\rangle$. It will be coupled to the excited state with no photon: $|b, 0\rangle$. This state couples to a continuum, decaying to some state $\left|a, \vec{k}'\vec{\epsilon}'\right\rangle$. Then the T matrix element is

$$T_{a\vec{k}'\vec{\epsilon}', a\vec{k}\vec{\epsilon}} = \left\langle a, \vec{k}'\vec{\epsilon}' \middle| H_{I1} \frac{1}{E_a + \hbar\omega - H + i\eta} H_{I1} \middle| a, \vec{k}\vec{\epsilon} \right\rangle. \tag{11.8}$$

The intermediate state $|b, 0\rangle$ is dissolved in the continuum, so the denominator is not problematic. The sum becomes

$$\sum_\mu \frac{1}{x - E_\mu + i\eta} \to \int dE P\left(\frac{1}{x - E} + i\pi\delta(x - E)\right) \tag{11.9}$$

We can simply replace E_b with $E_b - i\hbar\Gamma/2$. For example, for a time evolution

$$e^{-iE_b t/\hbar} \to e^{-iE_b t/\hbar} e^{-\Gamma t/2}; \tag{11.10}$$

we see exponential decay. The T matrix element is

$$T_{fi} = \frac{\left\langle a, \vec{k}'\vec{\epsilon}' \middle| H_{I1} \middle| b, 0 \right\rangle \left\langle b, 0 \middle| H_{I1} \middle| a, \vec{k}\vec{\epsilon} \right\rangle}{\hbar\omega + E_a - E_b + i\hbar\Gamma/2}. \tag{11.11}$$

This calculation is non-perturbative:

$$\frac{1}{\omega - \omega_0 + i\Gamma/2} = \sum_{n=0}^{\infty} (-1)^n \frac{(i\Gamma/2)^n}{(\omega - \omega_0)^{n+1}}. \tag{11.12}$$

11.1.2 Resonance Fluorescence

Incident monochromatic light at ω is scattered. The frequency ω does not change; energy must be conserved. The intensity of the scattering varies with the detuning according to the line width. Incident light with a uniform specturm leads to scattered light with a Lorentzian spectrum of width Γ centered at $\omega = \omega_0$.

With higher intensity, multi-photon processes are possible, which may lead to different output frequencies. For example, if two photons with energy ω are absorbed, photons with energy ω' and ω'' may be emitted. This process could occur in order of ω absorbed, ω' emitted, ω absorbed, ω'' emitted.

11.2 Fano Resonances

Consider again our case of $|\chi\rangle$ coupled to $|\varphi\rangle$ (at energy 0), with $|\varphi\rangle$ coupled to a continuum with spacing $|k\rangle$. Fano resonances occur when $|\chi\rangle$ is also coupled to the continuum of $|k\rangle$s. Let the coupling of $|\varphi\rangle$ to the continuum be v, the coupling of $|\chi\rangle$ to $|\varphi\rangle$ be w, and the coupling of $|\chi\rangle$ to the continuum be w'. Since there are two paths from $|\chi\rangle$ to the continuum, interference is likely. Mathematically, we let $\langle\chi|w|\varphi\rangle = w$ and $\langle\chi|w|k\rangle = w'$.

We want to dtermine the probability of exciting $|\psi_\mu\rangle$ from $|\chi\rangle$.

Recall

$$\langle\varphi|\psi_\mu\rangle = \frac{v}{\sqrt{v^2 + (\hbar\Gamma/2)^2 + E_\mu^2}} \text{ and} \tag{11.13}$$

$$\langle k|\psi_\mu\rangle = v\frac{\langle\varphi|\psi_\mu\rangle}{E_\mu - E_k}. \tag{11.14}$$

We need to calculate $\langle\psi_\mu|W|\chi\rangle$. There are two terms: the indirect coupling through $|\varphi\rangle$ or the direct coupling. This gives us

$$\langle\psi_\mu|W|\chi\rangle = \frac{1}{\sqrt{v^2 + (\hbar\Gamma/2)^2 + E_\mu^2}}\left[\langle\varphi|W|\chi\rangle\, v + \sum_k \langle k|W|\chi\rangle\, \frac{v^2}{E_\mu - E_k}\right]. \tag{11.15}$$

Recall that we also found

$$\sum_k \frac{v^2}{E_\mu - E_k} = E_\mu. \tag{11.16}$$

Using this, and the independence of v on k, we have

$$\langle\psi_\mu|W|\chi\rangle = \frac{wv + w'E_\mu}{\sqrt{v^2 + (\hbar\Gamma/2)^2 + E_\mu^2}}. \tag{11.17}$$

Note that since the sign of E_μ can vary, this can have some zeros. We write the energy in units of linewidth: $\mathcal{E}_\mu = E_\mu/(\hbar\Gamma/2)$. We also define

$$q = \frac{\delta}{\pi v}\frac{w}{w'} \text{ and } \xi = \frac{4v^2}{\hbar^2\Gamma^2} = \frac{2}{\pi}\frac{\delta}{\hbar\Gamma}. \tag{11.18}$$

q is the ratio of the coupling to $|\varphi\rangle$ to the coupling to $|k\rangle$. $1/\xi$ is the number of continuum states within Γ. Using these, we can write

$$\frac{|\langle\psi_\mu|W|\chi\rangle|^2}{w^2} = \frac{|q + \mathcal{E}_\mu|^2}{1 + \mathcal{E}_\mu^2 + \xi}. \tag{11.19}$$

When $q = 0$, there is no coupling on resonance, since the coupling with $|\varphi\rangle$ has shifted the states. When $q \gg 1$, the original coupling through $|\varphi\rangle$ is dominant, so the coupling has a width γ. For $q \sim 2$, there is some zero where it is impossible to couple to a shifted state, as well as some of the original peak from coupling through $|\varphi\rangle$.

11.3 Transition Amplitudes, Non-Perturbative

Now let us find a non-perturbative way for calculating transition amplitudes.

Consider Schrödinger's equation for the time evolution operator:

$$i\hbar\frac{d}{dt}U(t,t') = (H_0 + V)\,U(t,t') \tag{11.20}$$

with $U(t',t') = \mathbb{I}$. We previously proved the integral equation

$$U(t,t') = U_0(t,t') + \frac{1}{i\hbar}\int_{t'}^{t} dt_1 U_0(t,t_1)VU(t_1,t'), \tag{11.21}$$

where

$$U_0(t,t') = \exp\left(-iH_0\left(t-t'\right)/\hbar\right). \tag{11.22}$$

We can not exponentiate H directly, as it could be time dependent. Maybe bad things happen over time. Our integral equation is like a convolution, which suggests it could be simplified by a Fourier transform. However, the limits are wrong for a convolution. We will take this approach anyway and fix it.

11.3.1 Green's Functions

We define

$$K_+(t,t') = U(t,t')\Theta\left(t-t'\right) \text{ and } K_{0+}(t,t') = U_0(t,t')\Theta\left(t-t'\right). \tag{11.23}$$

We can also perform the change

$$\int_{t'}^{t} dt_1 \to \int_{-\infty}^{\infty} dt_1 \Theta\left(t-t_1\right)\Theta\left(t_1-t'\right). \tag{11.24}$$

Using these, we can write

$$K_+(t,t') = K_{0+}(t,t') + \frac{1}{i\hbar}\int_{-\infty}^{\infty} dt_1 K_{0+}(t,t_1)VK_+(t_1,t'). \tag{11.25}$$

U solved the Schrödinger equation; one can show that K_+ solves

$$\left(i\hbar\frac{d}{dt} - H\right)K_+(t,t') = \hbar\delta\left(t-t'\right). \tag{11.26}$$

This makes K_+ the (retarded) Green's function for the Hamiltonian. The advanced Green's function is K_-, given by

$$K_-(t,t') = -U(t,t')\Theta\left(t'-t\right). \tag{11.27}$$

Since δ is symmetric, K_- obeys the same equation as K_+.

11.3.2 Application of the Convolution Theorem

We here assume that time translation symmetry applies, so $K_+(t,t') = K_+(\tau)$.

$$K_+(\tau) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iE\tau/\hbar} G_+(E). \tag{11.28}$$

$$G_+(E) = \frac{1}{i\hbar}\int_{-\infty}^{\infty} d\tau\, e^{iE\tau/\hbar} K_+(\tau). \tag{11.29}$$

Now we assume H is time-independent and write

$$K_+(\tau) = e^{-iH\tau/\hbar}\Theta(\tau). \tag{11.30}$$

Then

$$G_+(E) = \frac{1}{i\hbar}\int_0^{\infty} d\tau\, e^{i(E-H)\tau/\hbar} = \lim_{\eta\to 0^+}\frac{1}{i\hbar}\int_0^{\infty} d\tau\, e^{i(E-H+i\eta)\tau/\hbar} = \lim_{\eta\to 0^+}\frac{1}{E-H+i\eta}, \tag{11.31}$$

where in the second step we add $i\eta$ for force convergence to occur at ∞, which preserves causality. For G_-, there is a similar formula:

$$G_-(E) = \lim_{\eta\to 0^+}\frac{1}{E-H-i\eta}. \tag{11.32}$$

Now we can apply the convolution theorem[1]:

$$G_+(E) = G_{0+}(E) + G_{0+}(E)VG_+(E). \tag{11.33}$$

The G_+ function is a special case of the resolvent. The resolvent is some function $G(z) = \frac{1}{z-H}$ for complex z. $G_\pm$ can be written in terms of the resolvent:

$$G_\pm(E) = \lim_{\eta\to 0^+} G\left(E \pm i\eta\right). \tag{11.34}$$

We ultimately want to determine $U(\tau)$. We can write

$$U(\tau) = K_+(\tau) - K_-(\tau) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iE\tau/\hbar}\left(G_-(E) - G_+(E)\right) = \frac{1}{2\pi i}\int_{C_+ + C_-} dz\, e^{-iz\tau/\hbar} G(z), \tag{11.35}$$

where in the first step we use $\Theta(\tau) + \Theta(-\tau) = 1$, the second step uses our Fourier transform, and the third step uses the resolvant and defines paths C_+ and C_-. C_+ is a path from $(+\infty, \eta i)$ to $(-\infty, \eta i)$ and C_- is a path from $(-\infty, -\eta i)$ to $(+\infty, -\eta i)$ for some small η. The singularities of G are all on the real axis, since H is Hermitian and has real eigenvalues. Singularities could occur at discrete or continuous eigenvalues.

The resolvant formula can also be derived from

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{B}\left(B - A\right)\frac{1}{A}: \tag{11.36}$$

$$G(z) = G_0(z) + G_0(z)VG(z) \tag{11.37}$$

for $A = z - H$ and $B = z - H_0$.

[1] Also known as the "Fourier trick"

Chapter 12

Lecture 18: Non-perturbative Transition Amplitudes Continue

12.1 Transition Amplitudes, Non-Perturbative

Now we wish to obtain some useful results. Starting with

$$G(z) = G_0(z) + G_0(z)VG(z), \tag{12.1}$$

we can perform a series expansion:

$$G(z) + G_0 + G_0VG_0 + G_0VG_0VG_0 + \dots. \tag{12.2}$$

This resembles a geometric series, suggesting we could write

$$G(z) = \frac{1}{1 - G_0(z)V}G_0(z). \tag{12.3}$$

This is hard for matrices, but somewhat easier to calculate for matrix elements. Let $|\varphi_i\rangle$ be eigenstates of H_0 and consider

$$G_{kl}(z) = \langle\varphi_k|G(z)|\varphi_l\rangle \text{ and } V_{ij} = \langle\varphi_i|V|\varphi_j\rangle\,. \tag{12.4}$$

Then we can write

$$G_{kl}(z) = \frac{1}{z - E_k}\delta_{kl} + \frac{1}{z - E_k}V_{kl}\frac{1}{z - E_l} + \sum_i \frac{1}{z - E_k}V_{ki}\frac{1}{z - E_i}V_{il}\frac{1}{z - E_l} + \dots. \tag{12.5}$$

This equation is somewhat interpretable. $G(z)$ is a propagator, telling us the probability of transitioning from $|\varphi_k\rangle$ to $|\varphi_l\rangle$. The first term is simply the probability of already being in that state. The second term describes a single-step transition: detunings make the process less likely and a single coupling is required. The third term describes a two step transition.

12.1.1 Resumming

For some classes of diagrams, we can formally resum them.

Suppose $|\varphi_b\rangle$ is a discrete state of H_0 with energy E_b. We want to determine the probability of remaining in that state. The original propagator is

$$G_{0b}(z) = \langle\varphi_b|G_0|\varphi_b\rangle = \frac{1}{z - E_b}, \tag{12.6}$$

which varies rapidly when $z \simeq E_b$. We then regroup terms in the expansion. Consider first only the terms where $\frac{1}{z-E_b}$ occurs twice. Possible terms include being in state b, having one interaction, and remaining in state b and being in state b, having one interaction, entering an intermediate state i (not b), having another interaction, and returning to state b. We define the level shift operator for b as the sum of terms that have initial and final factors of $\frac{1}{z-E_b}$ (that is, the system begins and ends in state b). Mathematically,

$$R_b(z) = V_{bb} + \sum_{i\neq b} V_{bi}\frac{1}{z - E_i}V_{ib} + \sum_{i\neq b, j\neq b} V_{bi}\frac{1}{z - E_i}V_{ij}\frac{1}{z - E_j}V_{jb} + \dots. \tag{12.7}$$

We can also consider terms that have $\frac{1}{z-E_b}$ three times. This can be weitten in terms of the level shift operator again: the state begins in b, goes through level shift processes, re-enters state b, goes through level shift processes again, and finally ends in state b. These third order processes contribute

$$\frac{1}{(z - E_b)^2}\left[R_b(z)\right]^2. \tag{12.8}$$

This allows us to write the propagator as

$$G_b(z) = \sum_{n=1}^{\infty}\frac{\left[R_b(z)\right]^{n-1}}{(z - E_b)^n} = \frac{1}{z - E_b}\sum_{n=0}^{\infty}\left(\frac{R_b(z)}{z - E_b}\right)^n = \frac{1}{z - E_b - R_b(z)}. \tag{12.9}$$

This explains why the level shift operator is called the "level shift operator." Imaginary components of $R_b(z)$ will lead to decays.

Isolated states with small transitions

If $|\varphi_b\rangle$ is well-isolated and V is small, then

$$R_b(z) = V_{bb} + \sum_{i\neq b} V_{bi}\frac{1}{z - E_i}V_{bi} \tag{12.10}$$

is a reasonable approximation. This allows us to treat this perturbatively, but only making certain assumptions (that may be more generally applicable).

Energy independent level shift

Sometimes we can neglect the energy dependence of $R_b(E)$, in particular, if we only want to consider $z \simeq E_b$. Then we can write

$$G_b\left(E \pm i\eta\right) = \frac{1}{E \pm i\eta - E_b - R_b\left(E \pm i\eta\right)} \approx \frac{1}{E \pm i\eta - E_b - R_b\left(E_b \pm i\eta\right)}. \tag{12.11}$$

This works as R_b is generally smooth in the neighborhood of E_b, as it contains no factors of $1/\left(z - E_b\right)$.

12.1.2 Physics Examples

Atomic Decay

Suppose $|\varphi_b\rangle = |b,0\rangle$, where $|b\rangle$ is an excited atomic state. We want to find $G_b(z) = \langle b,0|G(z)|b,0\rangle$ in order to deduce the time evolution of the state, $U_b(\tau) = \langle b,0|U(\tau)|b,0\rangle$. We know that

$$G_b(z) = \frac{1}{z - E_b - R_b(z)}. \tag{12.12}$$

There will be a non-zero factor $\langle b_0|H_{I2}|b_0\rangle$, which we will simply consider part of H_0. The interesting components are (to second order)

$$R_b(z) = \sum_a \sum_{\vec{k},\vec{\epsilon}} \frac{\left\langle b,0\middle|H_{I1}\middle|a,\vec{k}\vec{\epsilon}\right\rangle \left\langle a,\vec{k}\vec{\epsilon}\middle|H_{I1}\middle|b,0\right\rangle}{z - E_a - \hbar\omega}. \tag{12.13}$$

This takes into account various processes where the atom emits a photon, then reabsorbs it (returning to state $|b\rangle$, possibly repeating this many times. There are other processes involving emitting multiple photons before reabsorbing any, but these are farther off resonant or correspond to more complicated atomic state structures.

So, we equate

$$R_b\left(E \pm i\eta\right) = \sum_a \sum_{\vec{k},\vec{\epsilon}} \frac{\left|\left\langle b,0\middle|H_{I1}\middle|a,\vec{k}\vec{\epsilon}\right\rangle\right|^2}{E \pm i\eta - E_a - \hbar\omega} = \hbar\Delta_b(E) \mp i\hbar\frac{\Gamma_b(E)}{2}. \tag{12.14}$$

To solve for $\Delta_b(E)$ and $\Gamma_b(E)$, we use

$$\frac{1}{x \pm i\eta} = \mathcal{P}\left(\frac{1}{x}\right) \mp i\pi\delta(x), \tag{12.15}$$

where $\mathcal{P}$ denotes the principal part. Therefore,

$$\Delta_b(E) = \frac{1}{\hbar}\mathcal{P}\sum_a \sum_{\vec{k},\vec{\epsilon}} \frac{\left|\left\langle b,0\middle|H_{I1}\middle|a,\vec{k}\vec{\epsilon}\right\rangle\right|^2}{E - E_a - \hbar\omega} \tag{12.16}$$

and

$$\Gamma_b(E) = \frac{2\pi}{\hbar}\sum_a \sum_{\vec{k},\vec{\epsilon}} \left|\left\langle b,0\middle|H_{I1}\middle|a,\vec{k}\vec{\epsilon}\right\rangle\right|^2 \delta\left(E - E_a - \hbar\omega\right). \tag{12.17}$$

Note that the expression for Γ_b resembles Fermi's Golden Rule, though it is more general, being energy dependent. Therefore,

$$G_b\left(E \pm i\eta\right) = \frac{1}{E \pm i\eta - E_b - \hbar\Delta_b(E) \pm i\frac{\hbar}{2}\Gamma_b(E)}. \tag{12.18}$$

To see exponential decay, we approximate Δ_b and Γ_b for $E \simeq E_b$. Then

$$\Delta_b(E) = \Delta_b(E_b) = \frac{1}{\hbar}\mathcal{P}\sum_a \sum_{\vec{k},\vec{\epsilon}} \frac{\left|\left\langle b,0\middle|H_{I1}\middle|a,\vec{k}\vec{\epsilon}\right\rangle\right|^2}{E_b - E_a - \hbar\omega} \tag{12.19}$$

$$\Gamma_b(E) = \Gamma_b(E_b) = \frac{2\pi}{\hbar} \sum_a \sum_{\vec{k},\vec{\epsilon}} \left| \left\langle b, 0 \middle| H_{I1} \middle| a, \vec{k}\vec{\epsilon} \right\rangle \right|^2 \delta\left(E_b - E_a - \hbar\omega\right). \tag{12.20}$$

which implies

$$G_b\left(E \pm i\eta\right) \approx \frac{1}{E \pm i\eta - E_b - \hbar\Delta_b(E_b) \pm i\frac{\hbar}{2}\Gamma_b(E_b)}. \tag{12.21}$$

The time evolution is then

$$U_b(\tau) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iE\tau/\hbar}\left(G_+(E) - G_-(E)\right) \tag{12.22}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iE\tau/\hbar}\left(\frac{1}{E + i\eta - E_b - \hbar\Delta_b + i\hbar\Gamma_b/2} - \frac{1}{E - i\eta - E_b - \hbar\Delta_b - i\hbar\Gamma_b/2}\right) \tag{12.23}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, e^{-iE\tau/\hbar}\frac{2i\left(\hbar\Gamma_b/2 + \eta\right)}{\left(E - E_b - \hbar\Delta_b\right)^2 + \left(\eta + \hbar\Gamma_b/2\right)^2}. \tag{12.24}$$

Evaluating this (along with the $\eta \to 0$ limit),

$$U_b(\tau) = e^{-i(E_b + \hbar\Delta_b)\tau/\hbar} e^{-\Gamma_b|\tau|/2}. \tag{12.25}$$

This is not entirely trivial to calculate, and requires adding the advanced and retarded Green's functions to get a Lorentzian.

A Transition

We can also compute

$$\left\langle a, \vec{k}\vec{\epsilon} \middle| U(\tau) \middle| b, 0 \right\rangle. \tag{12.26}$$

It can be fairly directly shown that

$$\left\langle a, \vec{k}\vec{\epsilon} \middle| U(\tau) \middle| b, 0 \right\rangle = \frac{1}{\hbar}\frac{\left\langle a, \vec{k}\vec{\epsilon} \middle| H_I \middle| b, 0 \right\rangle}{\omega - \tilde{\omega}_{ba} + i\Gamma_b/2} e^{-i\left(\tilde{E}_a + \hbar\omega\right)\tau/\hbar} \tag{12.27}$$

where

$$\tilde{\omega}_{ba} = \frac{\tilde{E}_b - \tilde{E}_a}{\hbar}, \quad \tilde{E}_b = E_b + \hbar\Delta_b, \text{ and } \tilde{E}_a = E_a + \hbar\Delta_a. \tag{12.28}$$

Chapter 13

Lecture 19: Master Equation

How to handle light in a reservoir of thermal light. We will need to use a density matrix, then trace over the reservoir. This will allow us to derive the optical Bloch equations.

13.1 Review of Einstein's Rate Equations

For a population of atoms immersed in blackbody radiation, we get the coupled differential equations

$$\frac{dN_g}{dt} = A_{eg}N_e + u(\omega_{eg})\left[B_{eg}N_e - B_{ge}B_g\right] \text{ and} \tag{13.1}$$

$$\frac{dN_e}{dt} = -A_{eg}N_e + u(\omega_{eg})\left[B_{eg}N_g - B_{ge}B_e\right]. \tag{13.2}$$

13.2 Setup

So far, we have dealt with systems that contain interacting light and atoms. However, this is too precise for many situations. Often, we do not care about the light (e.g. for laser cooling), so we can trace out those parts of the system. This means that we must move beyond the Schrödinger equation and use density matrices.

13.2.1 Density Matrix Review

Recall that for a pure state, the density matrix is written

$$\rho = |\psi\rangle\langle\psi|. \tag{13.3}$$

For a two level system,

$$|0\rangle \to \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, \quad |1\rangle \to \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}, \text{ and } \frac{|0\rangle + |1\rangle}{\sqrt{2}} \to \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}. \tag{13.4}$$

The density matrix also allows statistical (not quantum) mixtures of states to be represented. For example,

$$\frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \tag{13.5}$$

is one way of representing a 50%-50% mixture of states $|0\rangle$ and $|1\rangle$. However, this is not unique.

$$\rho = \frac{1}{4}\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \tag{13.6}$$

is either

$$\rho = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| \tag{13.7}$$

or

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b| \tag{13.8}$$

for

$$|a\rangle = \frac{\sqrt{3}|0\rangle + |1\rangle}{2} \text{ and } |b\rangle = \frac{\sqrt{3}|0\rangle - |1\rangle}{2}. \tag{13.9}$$

The eigenvalues of ρ are probabilities, so they should be real and positive, as well as sum to 1. Recall that $\mathrm{Tr}(\rho) = 1$ and if and only if a state is a pure state, $\mathrm{Tr}\left(\rho^2\right) = 1$.

Evolution of the Density Matrix

The time evolution of the density matrix is given by

$$\rho = |\psi\rangle\langle\psi| \implies \dot{\rho} = \left|\dot{\psi}\right\rangle\left\langle\psi\right| + \left|\psi\right\rangle\left\langle\dot{\psi}\right| = -\frac{i}{\hbar}\left[H, \rho\right]. \tag{13.10}$$

Now consider, for example, the Jaynes-Cummings Hamiltonian:

$$H = \frac{\hbar\omega_0}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) + \Omega\left(|g\rangle\langle e| + |e\rangle\langle g|\right) = \frac{\hbar\omega_0}{2}\sigma_z + \Omega\sigma_x. \tag{13.11}$$

For a density matrix

$$\rho = \begin{pmatrix} \rho_{ee} & \rho_{eg} \\ \rho_{ge} & \rho_{gg} \end{pmatrix}, \tag{13.12}$$

the time evolution is thus

$$\dot{\rho} = \begin{bmatrix} i\Omega\left(\rho_{eg} - \rho_{ge}\right) & i\omega_0\rho_{ge} - i\Omega\left(\rho_{ee} - \rho_{gg}\right) \\ -i\omega_0\rho_{eg} + i\Omega\left(\rho_{ee} - \rho_{gg}\right) & i\Omega\left(\rho_{eg} - \rho_{ge}\right) \end{bmatrix}. \tag{13.13}$$

This is simply a rotation on the Bloch sphere about an axis

$$\vec{n} = \frac{\hbar\omega_0}{2}\hat{z} + \Omega\hat{x}. \tag{13.14}$$

To see this, we write the density matrix as a Bloch vector

$$\rho = \frac{1}{2}\left(\mathbb{I} + \vec{r}\cdot\vec{\sigma}\right) \tag{13.15}$$

for a real vectory $\vec{r}$ and the vector of Pauli matrices $\vec{\sigma}$.

13.2.2 Approximations

We combine four ideas for this

1. Density matrix evolution: hopefully we can find $\dot{\rho} = \mathcal{L}(\rho)$ for the "superoperator" $\mathcal{L}$, the Liouvillian. Such equations for the density matrix are called master equations.

2. Partial trace: we start with a density matrix for the system and the environment (light), then trace out the environment so we only have to worry about the system. For ρ being the total density matrix of the system and the environment,

$$\rho_{\text{sys}} = \text{Tr}_{\text{env}}(\rho) = \sum_{\text{env}} \langle \text{env}|\rho|\text{env}\rangle \,. \tag{13.16}$$

3. Assumptions about the environment: the environment is a reservoir (the electromagnetic field). We assume this reservoir is large and unchanging, which is the Born approximation. We also assume that correlation times τ_c within the reservoir are short, which is the Markov approximation.

4. We will generally want to find equations for times $\Delta t \gg \tau_c$ but still short compared to the atomic relation time $T_r \gg \Delta t$. This requires that these timescales be relatively different.

13.2.3

So, we want to find

$$\frac{\Delta \rho_{\text{sys}}}{\Delta t} = M \rho_{\text{sys}} \tag{13.17}$$

for some matrix M.

13.3 Beamsplitter Model from Ike

13.3.1 Single Beamsplitter

Consider a state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ incident on one port of a beamsplitter. On the other port, let the incident state be $|0\rangle$. Let one output be simply discarded. What will the other output be?

The probability that there is a single photon that is discarded is $p_1 = \beta^2 \sin^2\theta$, in which case the output is $|0\rangle$. To determine the other output, we consider the full output:

$$\begin{aligned} |\phi\rangle &= e^{i\theta\left(a^\dagger b + b^\dagger a\right)} \left[|\psi\rangle \otimes |0\rangle\right] && (13.18)\\ &= \alpha\,|00\rangle + \beta\left(\cos\theta\,|10\rangle + \sin\theta\,|01\rangle\right) && (13.19)\\ &= \left[\alpha\,|0\rangle + \beta\cos\theta\,|1\rangle\right) \otimes |0\rangle + \beta\sin\theta\,|0\rangle \otimes |1\rangle \,. && (13.20) \end{aligned}$$

Therefore, if a photon is not detected, the output is

$$|\psi_0\rangle = \frac{\alpha\,|0\rangle + \beta\cos\theta\,|1\rangle}{\sqrt{\alpha^2 + \beta^2\cos^2\theta}} \tag{13.21}$$

with probability p_1.

Single Beamsplitter in terms of Density Matrices

Now let us use density matrices to evaluate this.

$$\rho_{\text{in}} = |\psi\rangle\langle\psi| = \begin{pmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{pmatrix}. \tag{13.22}$$

Then the output is

$$\rho_{\text{out}} = p_1 |\psi_1\rangle\langle\psi_1| + p_0 |\psi_0\rangle\langle\psi_0| = \begin{pmatrix} \alpha^2 + \beta^2 \sin^2\theta & \alpha\beta\cos\theta \\ \alpha\beta\cos\theta & \beta^2\cos^2\theta \end{pmatrix}. \tag{13.23}$$

Note that $\rho \neq |\chi\rangle\langle\chi|$ for any pure state $|\chi\rangle$. We now have a mixed state (due to discarding some information).

The change in ρ in the beamsplitter is

$$\Delta\rho = \rho_{\text{out}} - \rho_{\text{in}} = \begin{pmatrix} -\beta^2(\cos 2\theta - 1)/2 & \alpha\beta(\cos\theta - 1) \\ \alpha\beta(\cos\theta - 1) & \beta^2(\cos 2\theta - 1)/2 \end{pmatrix} \tag{13.24}$$

13.3.2 Chaining Beamsplitters

Now suppose we pass the state through multiple beamsplitters, one after the other. Using the Born approximation, these beamsplitters can be identical, with identical input and output states. We choose a small angle for each beamsplitter:

$$\theta = \sqrt{\Gamma\Delta t/2}. \tag{13.25}$$

We can thus write (using the Markov approximation) a coarse-grained evolution

$$\frac{\Delta\rho}{\Delta t} = \begin{pmatrix} \dot\rho_{00} & \dot\rho_{01} \\ \dot\rho_{10} & \dot\rho_{11} \end{pmatrix} = -\Gamma \begin{pmatrix} -\beta^2 & \alpha\beta/2 \\ \alpha\beta/2 & \beta^2 \end{pmatrix}. \tag{13.26}$$

We see some fairly simple results from this. The populations change according to

$$\frac{d}{dt}\rho_{00} = +\Gamma\rho_{11} \tag{13.27}$$

$$\frac{d}{dt}\rho_{11} = -\Gamma\rho_{11}. \tag{13.28}$$

This looks exactly like decay of an atom coupled to the vacuum. The coherences (off-diagonals) behave according to

$$\frac{d}{dt}\rho_{01} = -\frac{\Gamma}{2}\rho_{01} \tag{13.29}$$

$$\frac{d}{dt}\rho_{10} = -\frac{\Gamma}{2}\rho_{10}. \tag{13.30}$$

This is also a familiar result.

For true atom-photon interactions, the coherences are slightly more complicated:

$$\frac{d}{dt}\rho_{01} = -\left(i\Delta + \frac{\Gamma}{2}\right)\rho_{01}. \tag{13.31}$$

In the full derivation of the Master Equation, we will find

$$\Gamma = \frac{2\pi}{\hbar}\sum_{\vec{k}\vec{\epsilon}} \left|\left\langle g, \vec{k}\vec{\epsilon}\middle| V \middle| e, 0\right\rangle\right|^2 \delta\left(\hbar\omega - \hbar\omega_{eg}\right). \tag{13.32}$$

13.4 Derivation (with More Math)

Our Hamiltonian is

$$H = H_A + H_R + V \tag{13.33}$$

where H_A describes atoms, H_R describes the light reservoir, and V describes interactions. The density matrix of the full system is

$$\frac{d}{dt}\rho(t) = \frac{1}{i\hbar}\left[H, \rho(t)\right]. \tag{13.34}$$

We go into the interaction picture, where

$$\frac{d}{dt}\tilde{\rho}(t) = \frac{1}{i\hbar}\left[\tilde{V}(t), \tilde{\rho}(t)\right]. \tag{13.35}$$

Recall that to do this,

$$\tilde{\rho}(t) = e^{i(H_A+H_R)t/\hbar}\rho(t)e^{-i(H_A+H_R)t/\hbar}. \tag{13.36}$$

We can then integrate the density matrix evolution equation to find the change over a short time Δt:

$$\tilde{\rho}\left(t+\Delta t\right) = \tilde{\rho}(t) + \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \left[\tilde{V}(t'), \tilde{\rho}(t')\right] \tag{13.37}$$

Plugging in the result for the density matrix another time and taking the difference, we find

$$\Delta\tilde{\rho}(t) = \tilde{\rho}\left(t+\Delta t\right) - \tilde{\rho}(t) \tag{13.38}$$

$$= \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \left[\tilde{V}(t'), \tilde{\rho}(t)\right] + \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \left[\tilde{V}(t'), \left[\tilde{V}(t''), \tilde{\rho}(t'')\right]\right]. \tag{13.39}$$

We then take the trace over the reservoir to get the atom density matrix: $\tilde{\rho}_A(t) = \mathrm{Tr}_R\,\tilde{\rho}(t)$[1]. Then

$$\Delta\tilde{\rho}_A(t) = \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \,\mathrm{Tr}_R\left[\tilde{V}(t'), \tilde{\rho}(t)\right] + \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \,\mathrm{Tr}_R\left[\tilde{V}(t'), \left[\tilde{V}(t''), \tilde{\rho}(t'')\right]\right]. \tag{13.40}$$

We can also consider the density matrix of the reservoir: $\tilde{\rho}_R(t) = \mathrm{Tr}_A\,\tilde{\rho}(t)$. We assume that it is not affected by the coupling to the atom and it is stationary, which means

$$\tilde{\rho}_R(t) \simeq \tilde{\rho}_R(0) = \rho_R. \tag{13.41}$$

Therefore, we can consider eigenstates of H_R to be good states. Let the eigenstates be $|\mu\rangle$ with $H_R\,|\mu\rangle = E_\mu\,|\mu\rangle$. Then we can write

$$\rho_R = \sum_\mu p_\mu\,|\mu\rangle\langle\mu|\,. \tag{13.42}$$

For example, if the reservoir is in thermal equilibrium at T, $p_\mu = \frac{1}{Z}e^{-E_\mu/k_BT}$ for $Z = \sum_\mu e^{-E_\mu/k_BT}$.

[1] API here uses $\tilde{\sigma}$. This seems needlessly easy to confuse with Pauli matrices, so I will simply attach a subscript to ρ to denote a subsystem

We assume we may write the interaction as

$$V = -AR \tag{13.43}$$

for an observable A acting on the atom state and an observable R acting on the radiation state. In the interaction picture

$$\tilde{V} = -\tilde{A}\tilde{R} \text{ where } \tilde{A} = e^{iH_A t/\hbar} A e^{-iH_A t/\hbar} \text{ and } \tilde{R} = e^{iH_R t/\hbar} R e^{-iH_R t/\hbar}. \tag{13.44}$$

We further assume that the average of R acting on the reservoir state is zero:

$$\langle R \rangle_R = \mathrm{Tr}_R \left(\rho_R R\right) = \mathrm{Tr}_R \left(\rho_R \tilde{R}(t)\right) = 0. \tag{13.45}$$

This then implies

$$\mathrm{Tr}_R \left[\rho_R \tilde{V}(t)\right] = \tilde{A}(t) \,\mathrm{Tr}\left[\rho_R \tilde{R}(t)\right] = 0. \tag{13.46}$$

We also define the two-time average (a correlation function of the reservoir):

$$g\left(t', t''\right) = \mathrm{Tr}\left(\rho_R \tilde{R}(t')\tilde{R}(t'')\right). \tag{13.47}$$

This is where the Markov approximation will apply: $g\left(t', t''\right)$ must be zero for $|t' - t''|$ not small. $\mathrm{Re} g\left(t', t''\right)$ is the correlation function for dynamics of fluctuations of R. $\mathrm{Im} g\left(t, t''\right)$ is the linear susceptibility. We note that

$$g\left(t', t''\right) = \mathrm{Tr}\left[\rho_R \tilde{R}(t')\tilde{R}(t'')\right] \tag{13.48}$$

$$= \mathrm{Tr}\left[\rho_R e^{iH_R t'/\hbar} R e^{-iH_R\left(t'-t''\right)/\hbar} R e^{-iH_R t''/\hbar}\right] \tag{13.49}$$

$$= \mathrm{Tr}\left[\rho_R e^{iH_R\left(t'-t''\right)/\hbar} R e^{-iH_R\left(t'-t''\right)/\hbar} R\right] \tag{13.50}$$

$$\text{so} \tag{13.51}$$

$$g(\tau) = \mathrm{Tr}\left[\rho_R \tilde{R}(\tau)\tilde{R}(0)\right]. \tag{13.52}$$

Using these assumptions,

$$\Delta\tilde{\rho}_A(t) = \frac{1}{i\hbar}\int_t^{t+\Delta t} dt' \,\mathrm{Tr}_R\left[\tilde{V}(t'), \tilde{\rho}(t)\right] + \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \,\mathrm{Tr}_R\left[\tilde{V}(t'), \left[\tilde{V}(t''), \tilde{\rho}(t'')\right]\right] \tag{13.53}$$

$$= \left(\frac{1}{i\hbar}\right)^2 \int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \,\mathrm{Tr}_R\left[\tilde{V}(t'), \left[\tilde{V}(t''), \tilde{\rho}(t'')\right]\right]. \tag{13.54}$$

The second equality follows from the average of R being zero. For small V and $\Delta t \ll T_R$, then $\tilde{\rho}$ evolves slowly, so $\tilde{\rho}(t'') \simeq \tilde{\rho}(t)$. We also assume that $\tilde{\rho}(t) \simeq \tilde{\rho}_A(t) \otimes \rho_R$. This amouts to neglecting correlations between the atom and reservoir, which is valid for a reservoir much larger than the atom. It will be valid if $\tau_c \ll \Delta t \ll T_R$.

Applying this additional set of assumptions,

$$\frac{\Delta\tilde{\rho}_A}{\Delta t} = -\frac{1}{\hbar^2}\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \int_t^{t'} dt'' \;\mathrm{Tr}_R\left[\tilde{V}(t'), \left[\tilde{V}(t''), \tilde{\rho}_A(t) \otimes \rho_R\right]\right]. \tag{13.55}$$

$\Delta\tilde{\rho}_A/\Delta t$ depends only on $\tilde{\rho}_A(t)$, which is nice, as it depends only on the present state of the system.

Since $\tilde{V} = -\tilde{A}\tilde{R}$, $\mathrm{Tr}_R\left(\rho_R\tilde{R}(t')\tilde{R}(t'')\right) = g\left(t', t''\right)$, which will go to zero for $|t' - t''| \gg \tau_c$. We can use the form of the reservoir density matrix to write

$$g(\tau) = \mathrm{Tr}\sum_\mu \left[p_\mu \left|\mu\right\rangle\!\left\langle\mu\right| \tilde{R}(\tau)\tilde{R}(0)\right] \tag{13.56}$$

$$= \mathrm{Tr}\sum_\mu \left[p_\mu \left\langle\mu|\tilde{R}(\tau)\tilde{R}(0)|\mu\right\rangle\right] \tag{13.57}$$

$$= \mathrm{Tr}\sum_{\mu\nu} \left[p_\mu \left\langle\mu|\tilde{R}(\tau)|\nu\right\rangle \left\langle\nu|\tilde{R}(0)|\mu\right\rangle\right] \tag{13.58}$$

$$= \sum_{\mu\nu} p_\mu \left|R_{\mu\nu}\right|^2 e^{i\omega_{\mu\nu}\tau}. \tag{13.59}$$

Note that $g(\tau)$ is a sum over rapidly oscillting terms. This explains the short correlation time τ_c. Using this,

$$\frac{\Delta\tilde{\rho}_A(t)}{\Delta t} = -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \left\{g(\tau)\left[\tilde{A}(t')\tilde{A}\left(t'-\tau\right)\tilde{\rho}_A(t) - \tilde{A}\left(t'-\tau\right)\tilde{\rho}_A(t)\tilde{A}(t')\right]\right. \tag{13.60}$$

$$\left. + g(-\tau)\left[\tilde{\rho}_A(t)\tilde{A}\left(t'-\tau\right)\tilde{A}(t') - \tilde{A}(t')\tilde{\rho}_A(t)\tilde{A}\left(t'-\tau\right)\right]\right\}. \tag{13.61}$$

Now we introduce a basis of states for the atomic states. Let these be denoted by letters, with

$$H_A\left|a\right\rangle = E_a\left|a\right\rangle. \tag{13.62}$$

Then we can find

$$\frac{\Delta\tilde{\rho}_{A,ab}(t)}{\Delta t} = \sum_{cd}\gamma_{abcd}\tilde{\rho}_{A,cd}(t) \tag{13.63}$$

with

$$\gamma_{abcd} = -\frac{1}{\hbar^2}\int_0^\infty d\tau \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \left\{g(\tau)\left[\delta_{bd}\sum_n \tilde{A}_{an}(t')\tilde{A}_{nc}\left(t'-\tau\right) - \tilde{A}_{ac}\left(t'-\tau\right)\tilde{A}_{db}(t')\right]\right. \tag{13.64}$$

$$\left. g(-\tau)\left[\delta_{ac}\sum_n \tilde{A}_{dn}\left(t'-\tau\right)\tilde{A}_{nb}(t') - \tilde{A}_{ac}(t')\tilde{A}_{db}\left(t'-\tau\right)\right]\right\}. \tag{13.65}$$

The dependence on t' comes from $\tilde{A}(t')$, which has time dependence from Bohr frequency of the atom: $\tilde{A}_{an}(t') \sim \exp\left(i\omega_{an}t'\right)$. These integrals can be fairly easily evaluated. The integral over t' can then be evaluated as

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} dt'\, e^{i(\omega_{ab}-\omega_{cd})t'} = e^{i(\omega_{ab}-\omega_{cd})t} e^{i(\omega_{ab}-\omega_{cd})\Delta t/2}\mathrm{sinc}\left(\left(\omega_{ab}-\omega_{cd}\right)\Delta t/2\right). \tag{13.66}$$

So, for $|\omega_{ab} - \omega_{cd}| \gg 1\Delta t$, we can approximate the sinc function as small and neglect these terms. This is the secular approximation one can make. Making this approximation, we find

$$\frac{\Delta\tilde{\rho}_{A,ab}(t)}{\Delta t} = \sum_{cd} \mathcal{R}_{abcd}\tilde{\rho}_{A,cd}(t) \tag{13.67}$$

where the sum is over states such that $|\omega_{ab} - \omega_{cd}| \ll 1/\Delta t$ and

$$\mathcal{R}_{abcd} = -\frac{1}{\hbar^2}\int_0^\infty d\tau \left\{ g(\tau)\left[\delta_{bd}\sum_n A_{an}A_{nc}e^{i\omega_{cn}\tau} - A_{ac}A_{db}e^{i\omega_{ca}\tau}\right]\right. \tag{13.68}$$

$$\left. g(-\tau)\left[\delta_{ac}\sum_n A_{dn}A_{nb}e^{i\omega_{nd}\tau} - A_{ac}A_{db}e^{i\omega_{bd}\tau}\right]\right\}. \tag{13.69}$$

Finally, we convert back to the Schrödinger picture using

$$\rho_{A,ab}(t) = e^{-i\omega_{ab}t}\tilde{\rho}_{ab}(t). \tag{13.70}$$

Therefore,

$$\frac{d\rho_{A,ab}(t)}{dt} = -i\omega_{ab}\rho_{ab}(t) + e^{-i\omega_{ab}t}\frac{d\tilde{\rho}_{A,ab}(t)}{dt} = -i\omega_{ab}\rho_{ab}(t) + \sum_{cd}\mathcal{R}_{abcd}\rho_{A,cd}(t). \tag{13.71}$$

Our first term is free evolution. The second term is the result of interacting with the radiation reservoir. The system of equations is linear with time-independent coefficients, so it is solvable.

13.5 Physical Content

13.5.1 Populations

The populations are $\rho_{A,aa}$. These have no free evolution (since $\omega_{aa} = 0$). Assuming that no coherence has a low free evolution frequency, the secular approximation implies populations couple only to other populations via

$$\frac{d\rho_{A,aa}}{dt} = \sum_c \mathcal{R}_{aacc}\rho_{A,cc}. \tag{13.72}$$

Then we can expand the $\mathcal{R}$ element for $a \neq c$ to find

$$\mathcal{R}_{aacc} = \frac{1}{\hbar^2}\int_{-\infty}^{\infty} d\tau\, g(\tau)\,|A_{ac}|^2\, e^{i\omega_{ca}t} \tag{13.73}$$

$$= \frac{1}{\hbar^2}\sum_\mu p_\mu \sum_\nu \int_{-\infty}^{\infty} d\tau\ |R_{\mu\nu}|^2\,|A_{ac}|^2\, e^{i(\omega_{\mu\nu}+\omega_{ca})t}. \tag{13.74}$$

We then set $R_{aacc} = \Gamma_{c\to a}$, which implies

$$\Gamma_{c\to a} = \frac{2\pi}{\hbar}\sum_\mu p_\mu \sum_\nu |\langle \nu, a|V|\mu, c\rangle|^2\,\delta\left(E_\mu + E_c - E_\nu - E_a\right). \tag{13.75}$$

Fermi's Golden Rule! By examining the expression for $\mathcal{R}_{abcd}$, we find that

$$\mathcal{R}_{aaaa} = -\sum_{n\neq a} \Gamma_{a\to n}, \tag{13.76}$$

which also just makes sense. A combined equation for the evolution of populations is thus

$$\frac{d\rho_{A,aa}}{dt} = \sum_{c\neq a} \left(\rho_{A,cc}\Gamma_{c\to a} - \rho_{A,aa}\Gamma_{a\to c}\right). \tag{13.77}$$

This gives a detailed balance steady state:

$$\rho^{\text{st}}_{A,aa}\Gamma_{a\to c} = \rho^{\text{st}}_{A,cc}\Gamma_{c\to a}. \tag{13.78}$$

For a reservoir in thermal equilibrium,

$$e^{-E_a/k_BT}\Gamma_{a\to c} = e^{-E_c/k_BT}\Gamma_{c\to a}. \tag{13.79}$$

Non-degenerate Coherences

When the Bohr frequency of a coherence is non-degenerate (over the secular range), its equation can be simplified to

$$\frac{d}{dt}\rho_{A,ab} = -i\omega_{ab}\rho_{A,ab} + \mathcal{R}_{abab}\rho_{A,ab}. \tag{13.80}$$

We thus compute

$$\mathcal{R}_{abab} = -\frac{1}{\hbar}\int_0^\infty d\tau \left\{ g(\tau)\left[\sum_n |A_{an}|^2 e^{i\omega_{an}\tau} - A_{aa}A_{bb}\right] + g(-\tau)\left[\sum_n |A_{bn}|^2 e^{i\omega_{bn}\tau} - A_{aa}A_{bb}\right]\right\}. \tag{13.81}$$

Following this calculation further, we can write

$$\mathcal{R}_{abab} = -\Gamma_{ab} - i\Delta_{ab} \tag{13.82}$$

where $\Delta_{ab} = \Delta_a - \Delta_b$ with

$$\Delta_a = \frac{1}{\hbar}\mathcal{P}\sum_\mu p_\mu \sum_\nu \sum_n \frac{|\langle \nu, n|V|\mu, a\rangle|^2}{E_\mu + E_a - E_\nu - E_n} \tag{13.83}$$

and $\Gamma_{ab} = \Gamma^{\text{nonad.}}_{ab} + \Gamma^{\text{ad.}}_{ab}$. Δ_a is the Lamb shift of state $|a\rangle$. The Γ_{ab} terms are further defined as

$$\Gamma^{\text{nonad.}}_{ab} = \frac{1}{2}\left(\sum_{n\neq a}\Gamma_{a\to n} + \sum_{n\neq b}\Gamma_{b\to n}\right) \text{ and} \tag{13.84}$$

$$\Gamma^{\text{ad.}}_{ab} = \frac{2\pi}{\hbar}\sum_\mu p_\mu \sum_\nu \delta\left(E_\mu - E_\nu\right)\left(\frac{1}{2}|\langle \nu, a|V|\mu, a\rangle|^2 + \frac{1}{2}|\langle \nu, b|V|\mu, b\rangle|^2 - \text{Re}\left[\langle \mu, a|V|\nu, a\rangle\ \langle \nu, b|V|\mu, b\rangle\right]\right). \tag{13.85}$$

We see a shift in the frequency of the coherence and a damping. The damping is due both to population decay (the nonadiabatic component) and processes which perturb the phase of one of the states without changing it (the adiabatic component).

Degenerate Coherences

Finally, there is the case of a degeneracy between ab and cd. Then

$$\mathcal{R}_{abcd} = \frac{2\pi}{\hbar} \sum_{\mu} p_{\mu} \sum_{\nu} \langle \nu, a|V|\mu, c\rangle \, \langle \mu, d|V|\nu, b\rangle \, \delta \left(E_{\mu} + E_c - E_{\nu} - E_a\right). \tag{13.86}$$

This is not particularly interpretable.

Chapter 14

Lecture 20: Optical Bloch Equations

14.1 Recap of the Master Equation

Consider the Hamiltonian

$$H = H_A + H_R + V. \tag{14.1}$$

In the interaction picture, a density matrix $\tilde{\rho}$ evolves according to

$$i\hbar\dot{\tilde{\rho}} = [V, \rho], \tag{14.2}$$

which implies

$$i\hbar\tilde{\rho}(t) = \int_0^t \left[\tilde{V}(t'), \tilde{\rho}(t')\right] dt'. \tag{14.3}$$

We then make some assumptions to simplify this to

$$\dot{\tilde{\rho}} = -\frac{1}{\hbar^2}\int_0^t \left[\tilde{V}(t), \left[\tilde{V}(t'), \tilde{\rho}(t')\right]\right] dt'. \tag{14.4}$$

We then assume we can separate the density matrix as

$$\tilde{\rho}(t) \simeq \tilde{\rho}_A(t) \otimes \tilde{\rho}_R(0) = \tilde{\rho}_A(t) \otimes \rho_R. \tag{14.5}$$

Then

$$\dot{\tilde{\rho}}_A = -\frac{1}{\hbar^2}\,\mathrm{Tr}_R \int_0^t \left[\tilde{V}(t), \left[\tilde{V}(t'), \tilde{\rho}_A(t') \otimes \rho_R\right]\right] dt'. \tag{14.6}$$

We then write the interaction as

$$V = -AR. \tag{14.7}$$

This applies in many cases, such as the dipole interaction.

14.1.1 Two-level example

Suppose in particular that

$$V = \frac{\hbar\Omega}{2}\left(b^\dagger |g\rangle\langle e| + b\,|e\rangle\langle g|\right). \tag{14.8}$$

Define (for notational simplicity) $\hat{V} = b^\dagger |g\rangle\langle e| + b\,|e\rangle\langle g|$. We recall the definition

$$\sigma_\pm = \frac{1}{2}\left(\sigma_x \pm i\sigma_y\right). \tag{14.9}$$

Then we can write $\sigma_- = |g\rangle\langle e|$ and $\sigma_+ = |e\rangle\langle g|$.

Let the radiation be purely in the vacuum state. Then we compute

$$\left[\hat{V}, \tilde{\rho}_A \otimes |0\rangle\langle 0|\right] = \left[b^\dagger\sigma_- + b\sigma_+, \tilde{\rho}_A \otimes |0\rangle\langle 0|\right] \tag{14.10}$$

$$= \sigma_-\tilde{\rho}_A \otimes |1\rangle\langle 0| - \tilde{\rho}_A\sigma_+ \otimes |0\rangle\langle 1|. \tag{14.11}$$

We ultimately want the double commutator:

$$\left[\hat{V}, \left[\hat{V}, \tilde{\rho}_A \otimes |0\rangle\langle 0|\right]\right] = \sigma_+\sigma_-\tilde{\rho}_A \otimes |0\rangle\langle 0| - 2\sigma_-\tilde{\rho}_A\sigma_+ \otimes |1\rangle\langle 1| + \tilde{\rho}_A\sigma_+\sigma_- \otimes |0\rangle\langle 0| + \text{non-EM diagonal modes}. \tag{14.12}$$

Since we take the trace over the radiation modes, we can ignore any radiation modes that are not diagonal. Therefore, we find

$$\dot{\tilde{\rho}}_A = -\frac{\Gamma}{2}\left[\sigma_+\sigma_-\tilde{\rho}_A - 2\sigma_-\tilde{\rho}_A\sigma_+ + \tilde{\rho}_A\sigma_+\sigma_-\right]. \tag{14.13}$$

This is the Master equation for a two level atom coupled to a single EM mode. This is known as the Lindblad form. This form is useful and common; it ensures that the density matrix remains valid.

Plugging in a density matrix

Consider

$$\tilde{\rho}_A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{14.14}$$

Then

$$\dot{\tilde{\rho}}_A = -\frac{\Gamma}{2}\begin{bmatrix} 2a & b \\ c & -2d \end{bmatrix} = -\frac{\Gamma}{2}\begin{bmatrix} 2\rho_{ee} & \rho_{eg} \\ \rho_{ge} & -2\rho_{ee} \end{bmatrix}. \tag{14.15}$$

Note that the excited state population decays at rate Γ, while the coherences are decaying slower, at $\Gamma/2$.

14.2 Optical Bloch Equations

Suppose that in addition to our decay (from the Master Equation) we have a drive as well. If the drive rate is sufficiently different from the vacuum decay, we can simply add them. This gives us the Optical Bloch Equations:

$$\dot{\rho}_{ee} = i\Omega\left(\rho_{eg} - \rho_{ge}\right) - \Gamma\rho_{ee} \tag{14.16}$$

$$\dot{\rho}_{ge} = i\omega_0\rho_{ge} - i\Omega\left(\rho_{ee} - \rho_{gg}\right) - \frac{\Gamma}{2}\rho_{ge} \tag{14.17}$$

$$\rho_{gg} = 1 - \rho_{ee} \tag{14.18}$$

$$\rho_{eg} = \rho_{ge}^*. \tag{14.19}$$

One can derive this more carefully as well. The derivation is complicated, as the resolvent method does not work (too many intermediate states) and the master equation method does not work (drive has a very long coherence time, violating Markov approximation).

14.2.1 Setup

The Hamiltonian for this scenario is

$$H = H_A + H_R - \vec{d}\cdot\left(\vec{E}_L\left(\vec{0},t\right) + \vec{E}_\perp\left(\vec{0}\right)\right). \tag{14.20}$$

The drive is described by

$$\vec{E}_L\left(\vec{0},t\right) = \vec{\mathcal{E}}_0\cos\left(\omega_L t\right). \tag{14.21}$$

$\vec{E}_\perp$ is the vacuum. So,

$$i\hbar\dot{\rho}_A = \left[H_A - \vec{d}\cdot\vec{\mathcal{E}}_0\cos\left(\omega_L t\right), \rho_A\right] \tag{14.22}$$

With excited atomic state $|b\rangle$ and ground state $|a\rangle$, the Optical Bloch Equations become

$$\dot{\rho}_{A,bb} = i\Omega\cos\left(\omega_L t\right)\left(\rho_{A,ba} - \rho_{A,ab}\right) - \Gamma\rho_{A,bb} \tag{14.23}$$

$$\dot{\rho}_{A,aa} = -i\Omega\cos\left(\omega_L t\right)\left(\rho_{A,ba} - \rho_{A,ab}\right) + \Gamma\rho_{A,bb} \tag{14.24}$$

$$\dot{\rho}_{A,ab} = i\omega_0\rho_{A,ab} - i\Omega\cos\left(\omega_L t\right)\left(\rho_{A,bb} - \rho_{A,aa}\right) - \frac{\Gamma}{2}\rho_{A,ab} \tag{14.25}$$

$$\dot{\rho}_{A,ba} = -i\omega_0\rho_{A,ba} + i\Omega\cos\left(\omega_L t\right)\left(\rho_{A,bb} - \rho_{A,aa}\right) - \frac{\Gamma}{2}\rho_{A,ba}, \tag{14.26}$$

where we define the Rabi frequency

$$\hbar\Omega = -\vec{d}_{ab}\cdot\vec{\mathcal{E}}_0 \tag{14.27}$$

with

$$\vec{d}_{ab} = \langle a|\vec{d}|b\rangle, \tag{14.28}$$

which we take to be real. This is valid if the Rabi frequency Ω is much smaller than vacuum fluctuations, $\Omega \ll 1/\tau_c$. This is required for the Markov approximation.

14.2.2 Rotating Wave Approximation and Rotating Frame

The dipole operator can be written

$$\vec{d} = \vec{d}_{ab}\left(|b\rangle\langle a| + |a\rangle\langle b|\right) = \vec{d}_+ + \vec{d}_-. \tag{14.29}$$

We can also define $S_+ = |b\rangle\langle a|$ and $S_- = |a\rangle\langle b|$, which allows us to write $\vec{d}_\pm = \vec{d}_{ab}S_\pm$. Therefore,

$$-\vec{d}\cdot\vec{\mathcal{E}}_0\cos\left(\omega_L t\right) = \frac{1}{2}\hbar\Omega\left[S_+e^{-i\omega_L t} + S_-e^{i\omega_L t} + S_-e^{-i\omega_L t} + S_+e^{i\omega_L t}\right] \tag{14.30}$$

The first term is absorption of a photon and excitation of the atom. The second term is emission of a photon and de-excitation of the atom. These two terms are near-resonant. The next two terms are off-resonant. $S_- e^{-i\omega_L t}$ is emission of a photon and de-excitation of the atom. The final term is emission of a photon and excitation of the atom. We now ignore the off-resonant terms.

We then define a new density matrix rotating with the laser:

$$\hat{\rho}_{A,ba} = \rho_{A,ba} e^{i\omega_L t}, \tag{14.31}$$

$$\hat{\rho}_{A,ab} = \rho_{A,ab} e^{-i\omega_L t}, \tag{14.32}$$

$$\hat{\rho}_{A,aa} = \rho_{A,aa}, \text{ and } \hat{\rho}_{A,bb} = \rho_{A,bb}. \tag{14.33}$$

The new Optical Bloch Equations are thus

$$\dot{\hat{\rho}}_{A,bb} = i\frac{\Omega}{2}\left(\hat{\rho}_{A,ba} - \hat{\rho}_{A,ab}\right) - \Gamma\hat{\rho}_{A,bb} \tag{14.34}$$

$$\dot{\hat{\rho}}_{A,aa} = -i\frac{\Omega}{2}\left(\hat{\rho}_{A,ba} - \hat{\rho}_{A,ab}\right) + \Gamma\hat{\rho}_{A,bb} \tag{14.35}$$

$$\dot{\hat{\rho}}_{A,ab} = -i\delta_L\hat{\rho}_{A,ab} - i\frac{\Omega}{2}\left(\hat{\rho}_{A,bb} - \hat{\rho}_{A,aa}\right) - \frac{\Gamma}{2}\hat{\rho}_{A,ab} \tag{14.36}$$

$$\dot{\hat{\rho}}_{A,ba} = i\delta_L\hat{\rho}_{A,ba} + i\frac{\Omega}{2}\left(\hat{\rho}_{A,bb} - \hat{\rho}_{A,aa}\right) - \frac{\Gamma}{2}\hat{\rho}_{A,ba} \tag{14.37}$$

where $\delta_L = \omega_L - \omega_0$. We can already see that

$$\frac{d}{dt}\left[\hat{\rho}_{A,aa} + \hat{\rho}_{A,bb}\right] = 0. \tag{14.38}$$

Note that this neglects things such as the Lamb shift.

14.2.3 Bloch Vectors

We introduce rotating versions of the $S_\pm$ operators (and an S_z operator):

$$\hat{S}_+ = e^{-i\omega_L t} S_+ = e^{-i\omega_L t} |b\rangle\langle a| \tag{14.39}$$

$$\hat{S}_- = e^{i\omega_L t} S_- = e^{i\omega_L t} |a\rangle\langle b| \tag{14.40}$$

$$\hat{S}_z = \frac{1}{2}\left(|b\rangle\langle b| - |a\rangle\langle a|\right). \tag{14.41}$$

Taking averages, we find

$$\left\langle \hat{S}_+ \right\rangle = \mathrm{Tr}\left(\rho_A S_+ e^{-i\omega_L t}\right) = \rho_{A,ab} e^{-i\omega_L t} = \hat{\rho}_{A,ab} \tag{14.42}$$

$$\left\langle \hat{S}_- \right\rangle = \hat{\rho}_{A,ba} \tag{14.43}$$

$$\left\langle \hat{S}_z \right\rangle = \mathrm{Tr}\left(\rho_A \frac{1}{2}\left(|b\rangle\langle b| - |a\rangle\langle a|\right)\right) = \frac{1}{2}\left(\hat{\rho}_{A,bb} - \hat{\rho}_{A,aa}\right). \tag{14.44}$$

The expectation value of the evolution is

$$\left\langle \dot{\hat{S}}_+ \right\rangle = -\left(i\delta_L + \frac{\Gamma}{2}\right)\left\langle \hat{S}_+ \right\rangle - i\Omega\left\langle S_z \right\rangle \tag{14.45}$$

with a similar equation for $\dot{\hat{S}}_-$ and

$$\left\langle \dot{\hat{S}}_z \right\rangle = i\frac{\Omega}{2}\left(\left\langle \hat{S}_- \right\rangle - \left\langle \hat{S}_+ \right\rangle\right) - \Gamma\left(\langle S_z \rangle + \frac{1}{2}\right). \tag{14.46}$$

We then introduce the Bloch Vector

$$u = \frac{1}{2}\left(\hat{\rho}_{A,ab} + \hat{\rho}_{A,ba}\right) \tag{14.47}$$

$$v = \frac{1}{2i}\left(\hat{\rho}_{A,ab} - \hat{\rho}_{A,ba}\right) \tag{14.48}$$

$$w = \frac{1}{2}\left(\hat{\rho}_{A,bb} - \hat{\rho}_{A,aa}\right). \tag{14.49}$$

The phases are chosen such that the drive is aligned with the u component. In the spin-model, the drive acts as a magnetic field along this axis. In this form, the Optical Bloch Equations describe the evolution of the Bloch Vector

$$\dot{u} = \delta_L v - \frac{\Gamma}{2}u \tag{14.50}$$

$$\dot{v} = -\delta_L u - \Omega w - \frac{\Gamma}{2}v \tag{14.51}$$

$$\dot{w} = \Omega v - \Gamma w - \frac{\Gamma}{2}. \tag{14.52}$$

Note that the drive does not appear in the first equation as there is no precession if the spin is aligned with the driving field.

Dipole Moment from Bloch Vector

In terms of the Bloch vector, the dipole moment of the atom is

$$\langle \vec{d} \rangle = 2\vec{d_{ab}}\left(u\cos\left(\omega_L t\right) - v\sin\left(\omega_L t\right)\right). \tag{14.53}$$

Note that here we can see that u is the component in phase with the drive and v is the quadrature component.

We have two time constants in this equation, $T_1 = 1/\Gamma$ and $T_2 = 2/\Gamma$ for spontaneous emission.

14.3 Solutions of the Optical Bloch Equations

First, we write the Optical Bloch Equations as a matrix

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -\Gamma/2 & \delta_L & 0 \\ -\delta_L & -\Gamma/2 & -\Omega \\ 0 & \Omega & -\Gamma \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\Gamma/2 \end{pmatrix}. \tag{14.54}$$

14.3.1 On Resonance

There is a resonance at $\omega_L - \omega_0$ (where $\delta_L = 0$).

Weak Drive

First, consider the case $\Omega \to 0$. The solutions will be superpositions of exponentials $e^{-r_\lambda t}$. Two rates will be $r_{1,2} = \Gamma/2$ (the decay of the coherences) and the final rate will be $r_3 = \Gamma$. This is purely damped, with no oscillations.

Strong Drive

Suppose then $\Omega \gg \Gamma$. Then we ought to have Rabi oscillations. When the Bloch vector is along v, the damping rate is $\Gamma/2$. When the vector is along w, the damping rate is Γ. When Ω is large, we expect the average damping rate to be signifcant. This is $3\Gamma/4$. During the math, it turns out $r_{1,2} = \pm i\Omega + \frac{3}{4}\Gamma$. The third solution is just $r_3 = \Gamma/2$, corresponding to motion along u.

Response of the Dipole Moment

Recall

$$\langle \vec{d} \rangle = 2\vec{d}_{ab} \left(u \cos(\omega_L t) - v \sin(\omega_L t) \right). \tag{14.55}$$

Since we have an oscillating dipole, there ought to be emission. What is the spectrum of this emission? This will be the spectrum of the Bloch vector, as this describes the oscillating dipole.

In the Rabi oscillating case ($\Omega \gg \Gamma$), there will be three peaks: at ω_L and $\omega_L \pm \Omega$. This makes sense, as the Bloch vector is modulated by the drive. The width of the ω_L line will be Γ. The width of the $\omega_L \pm \Omega$ sidebands will be $3\Gamma/2$. This is the Mollow triplet.

14.3.2 Off Resonance

Now we consider the case $|\delta_L| > |\Omega|$. Then we have an effective field that will be approximately along w. We will thus have precession about w at frequency δ_L damped as $\Gamma/2$ and purely damped motion along w, damped at Γ. Thus, the solutions are $r_{1,2} = \pm i\delta_L + \Gamma/2$ and $r_3 = \Gamma$. The spectrum still has three peaks. The center peak at ω_L has width 2Γ, while the side peaks at $\omega_L \pm \delta_L$ have width Γ.

Chapter 15

Lecture 21: Around the Bloch Again and Quantum Monte Carlo

15.1 Quantum Monte Carlo

Many transitions occur randomly, which can be observed for single atoms. This suggests that a probabilistic approach might be viable.

Consider an atom with excited state $|e\rangle$ and ground state $|g\rangle$. Let the initial state be $|\psi\rangle$. We start with the (non-Hermitian) Hamiltonian

$$H = H_0 - i\frac{\Gamma}{2}|e\rangle\langle e|\,.$$

We then apply a procedure:

1. We compute the decay probability $dp = \Gamma dt\,|\langle e|\psi\rangle|^2$.
2. Take a uniformly distributed random number $0 \leq \epsilon \leq 1$.
3. When $\epsilon < dp$, we set $|\psi\rangle = |g\rangle$.
4. If $\epsilon \geq dp$, then set $|\psi\rangle = \left[e^{-iHdt}/\sqrt{1-dp}\right]|\psi\rangle$ (normal evolution).
5. We then repeat.

This is similar to the beamsplitter model introduced previously.

If $H_0 = 0$, then for an initial state $|\psi\rangle = \alpha|g\rangle + \beta|e\rangle$, the step 4 evolution gives

$$\frac{\alpha|g\rangle + \beta e^{-\Gamma dt/2}|e\rangle}{\sqrt{|\alpha|^2 + |\beta|^2 e^{-\Gamma dt}}} \to |\psi\rangle\,.$$

For a two-level system, this approach turns out to be equivalent to the optical Bloch equations.

15.1.1 General Open Quantum Systems

This can also be done for general open quantum systems, with more complex Lindblad operators.

15.2 Around the Bloch Again

Recall the Bloch vector form of the optical Bloch equations:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -\Gamma/2 & \delta_L & 0 \\ \delta_L & -\Gamma/2 & -\Omega_1 \\ 0 & -\Omega_1 & -\Gamma \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -\Gamma/2 \end{pmatrix}.$$

15.2.1 Steady-State Solutions

The steady-state solutions are

$$\begin{aligned} u_{\mathrm{st}} &= \frac{\Omega_1}{2} \frac{\delta_L}{\delta_L^2 + \Gamma^2/4 + \Omega_1^2/2} \\ v_{\mathrm{st}} &= \frac{\Omega_1}{2} \frac{\Gamma/2}{\delta_L^2 + \Gamma^2/4 + \Omega_1^2/2} \\ w_{\mathrm{st}} + \frac{1}{2} = rho_{A,bb}^{\mathrm{st}} &= \frac{\Omega_1^2}{4} \frac{1}{\delta_L^2 + \Gamma^2/4 + \Omega_1^2/2}. \end{aligned}$$

Recall u describes the in-phase response of the Bloch vector to the drive, v describes the out of phase response of the Bloch vector to the drive, and w describes populations.

The denominator is a Lorentzian centered at $\delta_L = 0$, with a FWHM given by power broadening: the natural linewidth plus an Ω_1 term.

The population w in the excited state increases with the intensity. The out of phase v response describes the energy (similar to a harmonic oscillator). It has a resonance lineshape. Note that out of phase response emerges from damping. The in phase response has a dispersive lineshape. These are similar to what you get for a driven harmonic oscillator.

Saturation

At low intensity, u and v are proportional to Ω_1. At high intensity, a saturation effect takes over and they are proportional to $1/\Omega_1$. The population is proportional to Ω_1^2 at low intensity, and saturates to $1/2$ at high intensity. The saturation parameter is

$$s = \frac{\Omega_1^2/2}{\delta_L^2 + \Gamma^2/4}.$$

Note that the required intensity to saturate increases off resonance. In terms of this parameter,

$$u_{\mathrm{st}} = \frac{\delta_L}{\Omega_1} \frac{s}{1+s}, \quad v_{\mathrm{st}} = \frac{\Gamma}{2\Omega_1} \frac{s}{1+s}, \text{ and } \rho_{A,bb}^{\mathrm{st}} = \frac{1}{2} \frac{s}{1+s}.$$

15.3 Light Emitted by Atom

Consider an atom emitting light that is detected by a photodetector. The electric field at the detector is

$$E\left(\vec{r}_D, t\right) = \eta d\left(t - \vec{r}_D/c\right),$$

where d is the dipole of the atom and η is some proportionality constant. Note that we use the proper time-delayed form. Then we separate

$$E^{\pm} = \eta e^{\mp i\omega_c(t-\vec{r}_D/c)} S_{\pm}\left(t-\vec{r}_D/c\right),$$

as the detector only detects E^{+}. Using

$$d = d_{ab}\left(S_- e^{-i\omega_L t} + S_+ e^{+i\omega t}\right).$$

The average intensity is then (as derived previously)

$$\langle I(t)\rangle = \left\langle E^{(-)}\left(\vec{r}_D,t\right) E^{(+)}\left(\vec{r}_D,t\right)\right\rangle.$$

The spectral density is given by the Fourier transform

$$I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{-i\omega\tau}\left\langle E^{(-)}\left(\vec{r}_D,t+\tau\right) E^{(+)}\left(\vec{r}_D,t\right)\right\rangle.$$

Plugging in our expressions for $E^{\pm}$,

$$\langle I(t)\rangle = \eta^2\left\langle S_+\left(t-\vec{r}_D/c\right) S_-\left(t-\vec{r}_D/c\right)\right\rangle,$$

which implies

$$I(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\tau e^{i(\omega_1-\omega)\tau}\left\langle S_+\left(t+\tau-\vec{r}_D/c\right) S_-\left(t-\vec{r}_D/c\right)\right\rangle.$$

The total intensity is

$$\langle I(t)\rangle = \eta^2\rho_{A,bb}\left(t-\vec{r}_D/c\right),$$

which is proportional to the excited state probability.

We then split up $S_{\pm}$ into an average and fluctuations:

$$S_{\pm}\left(t-\vec{r}_D/c\right) = \langle S_{\pm}\left(t-\vec{r}_D/c\right)\rangle + \delta S_{\pm}\left(t-\vec{r}_D/c\right).$$

The average ofthe fluctuations is zero. Then $\langle S_+(t)\rangle$ and $\langle S_+(t)S_-(t)\rangle$ do not depend on time. So

$$\langle I\rangle = \eta^2\langle S_+\rangle\langle S_-\rangle + \eta^2\left\langle \delta S_+\delta S_-\right\rangle.$$

The first term is the mean dipole, also known as the coherent part. It results in the same field one would get from driving a classical dipole. The second term is the fluctuations of the dipole, which is incoherent.

The coherent part is

$$\frac{1}{\eta^2}\langle I_{\rm coh}\rangle = |\langle S_+\rangle|^2 = |u_{\rm st}+iv_{\rm st}|^2 = \frac{1}{2}\frac{s}{(1+s)^2}.$$

The incoherent part is

$$\frac{1}{\eta^2}\langle I_{\rm incoh}\rangle = \langle S_+S_-\rangle - |\langle S_+\rangle|^2 = \rho^{\rm st}_{A,bb} - \langle u_{\rm st}+iv_{\rm st}|^2 = \frac{1}{2}\frac{s^2}{(1+s)^2}.$$

The coherent part contributes a δ-function in frequency, while the incoherent part contributes the broader Mollow triplet.

Looking at the spectral density:

$$I(\omega) = I_{\text{coh}}(\omega) + I_{\text{incoh}}(\omega).$$

$$I_{\text{coh}}(\omega) = \frac{\eta^2}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i(\omega_L-\omega)\tau}\left|\langle S_+\rangle\right|^2 = \propto \delta\left(\omega_L-\omega\right).$$

This is monochromatic, given by the mean dipole oscillating in the forced regime. There will be radiation emitted at ω_L.

$$I_{\text{incoh}}(\omega) = \frac{\eta^2}{2\pi}\int_{-\infty}^{\infty} d\tau\, e^{i(\omega_L-\omega)\tau}\left\langle \delta S_+\left(t+\tau\right)\delta S_-(t)\right\rangle.$$

This is more complex. It turns out that $\langle \delta S_+\left(t+\tau\right)\delta S_-(t)\rangle$ obeys the same equation as sometime else. The equation is the optical Bloch equations:

$$\frac{d}{dt}\left\langle \delta S_+\left(t\right)\delta S_-(0)\right\rangle = \sum_{q'} B_{qq'}\left\langle \delta S_{q'}\left(t\right)\delta S_-(0)\right\rangle,$$

where $B_{qq'}$ is the Bloch matrix for the Bloch vector. We thus see that the spectrum is the same as the solutions of the Bloch equations.

15.4 Light Forces: an Application of the Optical Bloch Equations

15.4.1 Work Done by Driving Field

Work is defined by

$$dW = q\mathcal{E}_0\cos\left(\omega_L t\right)\cdot dr,$$

where dr is the distance travelled. So, in a unit time,

$$\frac{dW}{dt} = q\mathcal{E}_0\cos\left(\omega_L t\right)\frac{dr}{dt} = \mathcal{E}_0\cos\left(\omega_L t\right)\langle\dot{\vec{d}}\rangle.$$

The average work is then

$$\overline{\frac{dW}{dt}} = 2\vec{d}_{ab}\mathcal{E}_0\omega_L\left[\overline{\cos^2\left(\omega_L t\right)}v + \overline{\sin\left(\omega_L t\right)\cos\left(\omega_L t\right)}u\right] = \hbar\Omega\omega_L v,$$

using in the first step the equation for the evolution of the dipole we found previously. Note that since we take a derivative, the v component is proportional to cos, while u is proportional to sin. We can also write this in terms of the average number of absorbed photons per unit time:

$$\left\langle\frac{dN}{dt}\right\rangle = \frac{1}{\hbar\omega_L}\overline{\frac{dW}{dt}} = \Omega v.$$

We can also look at the w optical Bloch equation:

$$w = \frac{1}{2}\left(\rho_{A,bb} - \rho_{A,aa}\right) = \rho_{A,bb} - \frac{1}{2}$$

and

$$\dot{\rho}_{A,bb} = \Omega v - \Gamma \rho_{A,bb} = \left\langle \frac{dN}{dt} \right\rangle - \Gamma \rho_{A,bb}.$$

Then in steady state,

$$\left\langle \frac{dN}{dt} \right\rangle = \Gamma \rho_{A,bb}.$$

So, the atom absorbs a photon for each time it decays.

15.4.2 Derivation

External Degrees of Freedom

Let us add the moment of the atom to the Hamiltonian again:

$$H = \frac{\vec{p}^2}{2M} + H_A + H_R - \vec{d}\cdot\left(\vec{E}_l(\vec{R},t) + \vec{E}_\perp(\vec{R})\right),$$

where again $\vec{E}_l$ is the laser field and $\vec{E}_\perp$ is the vacuum field. Then the Heisenberg equations of motion give

$$\dot{\vec{R}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{M} \text{ and}$$
$$\dot{\vec{p}} = M\ddot{\vec{R}} = -\frac{\partial H}{\partial \vec{R}} = \sum_j d_j \boldsymbol{\nabla}_R \left[E_{l,j}(\vec{R},t) + E_{\perp,j}(\vec{R})\right].$$

Let the average (classical) position of the atom be $\vec{r} \equiv \langle \vec{R} \rangle$. We also assume the de Broglie wavelength $\frac{\hbar}{Mv} = \lambda_{\text{dB}} \ll \lambda_l$ so that we do not have to worry about the phase of the light varying over the atom. We further assume fluctuations of the electromagnetic vacuum are uniform (the vacuum does not have spatial correlations), so $\boldsymbol{\nabla}_R E_\perp(\vec{R}) = 0$. Then

$$M\ddot{\vec{r}} = \sum_j \langle d_j \rangle \boldsymbol{\nabla} E_{lj}(\vec{r},t).$$

Timescales

Now we assume that the internal timescale is much faster than the external timescale. This allows us to use the optical Bloch equations to determine $\langle d_j \rangle$ without considering the external motion. Mathematically,

$$T_{\text{int}} = \frac{1}{\Gamma}, \frac{1}{\Omega} \text{ and } vT_{\text{int}} \sim \frac{v}{\Gamma} \ll \lambda.$$

The timescale of external motion is given by the recoil energy:

$$T_{\text{ext}} = \frac{\hbar}{E_{\text{rec}}} \text{ where } E_{\text{rec}} = \frac{\hbar^2 k^2}{2M}.$$

Typically, $\hbar\Gamma \gg E_{\text{rec}}$, which implies $T_{\text{int}} \ll T_{\text{ext}}$. For Na, a typical alkali atom, $\hbar\Gamma = 400E_{\text{rec}}$, so this holds.

Evaluating

Now let us define the light field

$$\vec{E}_l(\vec{r},t) = \vec{\epsilon}\mathcal{E}_0(\vec{r})\cos\left(\omega_L t + \phi(\vec{r})\right).$$

We will choose $\phi(\vec{0}) = 0$. Then the gradient of component j is

$$\boldsymbol{\nabla} E_{lj}|_{\vec{r}=0} = \epsilon_j\left[\cos\left(\omega_L t\right)\boldsymbol{\nabla}\mathcal{E}_0 - \sin\left(\omega_L t\right)\mathcal{E}_0\boldsymbol{\nabla}\phi\right]$$

The dipole is

$$\langle d_j\rangle = 2\left(\vec{d}_{ab}\right)_j\left[u_{\rm st}\cos\left(\omega_L t\right) - v_{\rm st}\sin\left(\omega_L t\right)\right].$$

Combining these, the force is

$$\mathcal{F} = \sum_j \langle d_j\rangle\boldsymbol{\nabla}E_{lj} = \left(\vec{\epsilon}\cdot\vec{d}_{ab}\right)\left[u_{\rm st}\boldsymbol{\nabla}\mathcal{E}_0 + v_{\rm st}\mathcal{E}_0\boldsymbol{\nabla}\phi\right].$$

Forces

We see there are two forces. One in phase (reactive) and one out of phase (dissipative):

$$\vec{F}_{\rm reactive} = \left(\vec{\epsilon}\cdot\vec{d}_{ab}\right)u_{\rm st}\boldsymbol{\nabla}\mathcal{E}_0 \text{ and}$$
$$\vec{F}_{\rm diss} = \left(\vec{\epsilon}\cdot\vec{d}_{ab}\right)v_{\rm st}\mathcal{E}_0\boldsymbol{\nabla}\phi.$$

Introducing the Rabi frequency $\Omega = -\vec{d}_{ab}\cdot\vec{\epsilon}\frac{\mathcal{E}_0}{\hbar}$,

$$\vec{F}_{\rm reactive} = -\hbar\Omega u_{\rm st}\vec{\alpha} \text{ where } \vec{\alpha} = \frac{\boldsymbol{\nabla}\Omega}{\Omega} \text{ and}$$
$$\vec{F}_{\rm diss} = -\hbar\Omega v_{\rm st}\vec{\beta} \text{ where } \vec{\beta} = \boldsymbol{\nabla}\phi.$$

15.4.3 Radiation Pressure

Suppose our light is a plane wave:

$$\vec{E}_l(\vec{r},t) = \vec{\epsilon}\mathcal{E}_0\cos\left(\omega t - \vec{k}_L\cdot\vec{r}\right).$$

Then $\phi(\vec{r}) = -\vec{k}_L\cot\vec{r}$, which implies $\vec{\beta} = -\vec{k}_L$. The dissipative force is then radiation pressure:

$$\vec{F}_{\rm diss} = \Omega v_{\rm st}\hbar\vec{k}_L = \left\langle\frac{dN}{dt}\right\rangle_{\rm st}\hbar\vec{k}_L = \Gamma\rho^{\rm st}_{A,bb}\hbar\vec{k}_L = \hbar k_L\frac{\Gamma}{2}\frac{\Omega_1^2/2}{\left(\omega_L-\omega\right)^2+\Gamma^2/4+\Omega_1^2/2}.$$

Note that stimulated emission would be in the same direction as the laser, so it would not result in a net force. Therefore, the dissipative (or spontaneous) force is needed for radiation pressure. The force is a Lorentzian centered at the resonance frequency. For low intensity, it is proportional to intensity, but it will saturate at high intensity.

The acceleration from this force is $a = F/M \sim 10^6\,\mathrm{m/s^2} = 10^5 g$, where g is the gravitational constant near the surface of Earth.

We want to determine the total external interaction time T_{ext}. The final velocity is given by a Doppler shift. This velocity is

$$aT_{\text{ext}} = \frac{\hbar k_L}{M}\Gamma T_{\text{ext}} = v.$$

$kv \approx \Gamma$ is when the Doppler shift will become large. This implies

$$T_{\text{ext}} = \frac{M}{\hbar k^2} = \frac{\hbar}{E_{\text{rec}}}.$$

15.4.4 Dipole Force

Suppose our light is an optical lattice:

$$\vec{E}_l = \vec{\epsilon}\mathcal{E}_0 \cos(k_L z)\cos(\omega t).$$

The phase is a constant zero, but the amplitude has a sinusoidal variation. The reactive force is proportional to the in-phase response:

$$F_{\text{reactive}} \propto u_{\text{st}}.$$

This means that energy is not exchanged with the light field (overall). What actually happens is light is redistributed between the two counter-propagating light beams. The atom absorbs from one beam and (stimulated) emits into the other, depending on where it is on the standing wave.

The force will be proportional to the detuning.