2.5. Quantization of the e.m. field

Quantize in a cubic box of side length L, volume $V = L^3$, with periodic boundary conditions: $k_{x,y,z} = \frac{2\pi}{L} n_{x,y,z}$

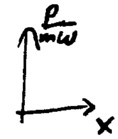
$$\alpha_\varepsilon(\vec{k}, t) \to \alpha_{\vec{k}_\varepsilon}(t) \quad \text{or simply} \quad \alpha_i \qquad (i = (\vec{k}_i, \vec{\varepsilon}_i))$$

Correspondence: $\int d^3k \sum_\varepsilon f(\vec{k}, \vec{\varepsilon}) \leftrightarrow \left(\frac{2\pi}{L}\right)^3 \sum_i f(\vec{k}_i, \vec{\varepsilon}_i)$

2.5.1 Analogy with harmonic oscillator

One field mode		Harmonic Oscillator
$\dot{A}_i = -\mathcal{E}_i$	$A_i \hat{=} x$	$\dot{x} = \frac{p}{m}$
$\dot{\mathcal{E}}_i = \omega_i^2 A_i$	$\mathcal{E}_i \hat{=} -\frac{p}{m}$	$\frac{\dot{p}}{m} = -\omega^2 x$
$H_i = \frac{\varepsilon_0}{2} \frac{(2\pi)^3}{V} \left(\lvert\mathcal{E}_i\rvert^2 + \omega_i^2 \lvert A_i\rvert^2\right)$	$\varepsilon_0 \frac{(2\pi)^3}{V} \hat{=} m$	$H = \frac{m}{2}\left(\left(\frac{p}{m}\right)^2 + \omega^2 x^2\right)$
$\alpha_i = N_i \left(A_i + \frac{i}{\omega_i} \mathcal{E}_i\right)$		$\alpha = N\left(x + i\frac{p}{m\omega}\right)$
$\frac{d\alpha_i}{dt} = -i\omega_i \alpha_i$		$\frac{d\alpha}{dt} = -i\omega\alpha$

α point in phase space



2.5.2 Commutation relations

One field mode

$$A_i \to \hat{A}_i$$
$$\mathcal{E}_i \to \hat{\mathcal{E}}_i$$
$$[\hat{A}_i, \hat{\mathcal{E}}_i] \hat{=} [\hat{x}, -\frac{\hat{p}}{m}] = -i\frac{\hbar}{m} \hat{=} -\frac{V}{(2\pi)^3}\frac{i\hbar}{\varepsilon_0}$$

$\hat{a}_i$ annihilation operator associated to α_i

$[\hat{a}_i, \hat{a}_i^+] = 1$ for

$$N = \sqrt{\frac{\varepsilon_0 \omega_i}{2\hbar} \frac{(2\pi)^3}{V}}$$

Harmonic Oscillator

$$x \to \hat{x}$$
$$p \to \hat{p}$$
$$[\hat{x}, \hat{p}] = i\hbar$$

$\hat{a}$ annihilation operator associated to α

$[\hat{a}, \hat{a}^+] = 1$ for choice

$$N = \sqrt{\frac{m\omega}{2\hbar}}$$

2.5.3 Physical operators

One field mode

$$H_i = \frac{\hbar\omega_i}{2}\left(\alpha_i^*\alpha_i + \alpha_i\alpha_i^*\right) \quad \left(= \hbar\omega_i|\alpha_i|^2\right)$$

$$\hat{H}_i = \frac{\hbar\omega_i}{2}\left(\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger\right)$$

$$\boxed{\hat{H} = \sum_i \hbar\omega_i\left(\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\right)}$$

Harmonic oscillator

$$H = \frac{\hbar\omega}{2}\left(\alpha^*\alpha + \alpha\alpha^*\right)$$

$$\hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

$$\vec{P} = \sum_i \frac{\hbar\vec{k}_i}{2}\left(\alpha_i^*\alpha_i + \alpha_i\alpha_i^*\right)$$

$$\hat{\vec{P}} = \sum_i \frac{\hbar\vec{k}_i}{2}\left(\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger\right) = \sum_i \hbar\vec{k}_i\left(\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\right)$$

but $\sum_i \vec{k}_i = 0$ so

$$\boxed{\hat{\vec{P}} = \sum_i \hbar\vec{k}_i\,\hat{a}_i^\dagger a_i}$$

$$\hat{\vec{E}}(\vec{r}) = i\sum_i \mathcal{E}_i\left(\vec{\varepsilon}_i\,\hat{a}_i\,e^{i\vec{k}_i\cdot\vec{r}} - \vec{\varepsilon}_i\,\hat{a}_i^\dagger e^{-i\vec{k}_i\cdot\vec{r}}\right)$$

with $\mathcal{E}_i = \sqrt{\dfrac{\hbar\omega_i}{2\varepsilon_0 V}}$

Comment: Within Lagrangian formalism, one sees that the momentum conjugate to $\mathcal{A}_{\perp\varepsilon}$ is $\Pi_\varepsilon = \varepsilon_0\dot{\mathcal{A}}_{\perp\varepsilon} = -\varepsilon_0\mathcal{E}_{\perp\varepsilon}$

The canonical commutation relations are then

$[\mathcal{A}_\varepsilon(\vec{k}), \Pi_{\varepsilon'}^\dagger(\vec{k}')] = i\hbar\,\delta_{\varepsilon\varepsilon'}\,\delta(\vec{k}-\vec{k}')$. This agrees with

$[\mathcal{A}_i, \mathcal{E}_j] = -\frac{V}{(2\pi)^3}\frac{i\hbar}{\varepsilon_0}\delta_{ij}$ as $1 = \int d^3k\,\delta(\vec{k}-\vec{k}') \leftrightarrow \frac{(2\pi)^3}{V}\sum_k \frac{V}{(2\pi)^3}\delta_{kk'}$

or $\delta(\vec{k}-\vec{k}') \leftrightarrow \frac{V}{(2\pi)^3}\delta_{kk'}$

2.6 Total hamiltonian & Momentum

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left(\vec{p}_\alpha - q_\alpha \vec{A}_\perp(\vec{r}_\alpha)\right)^2 + \underbrace{\sum_\alpha \left(-g_\alpha \frac{q_\alpha}{2m_\alpha}\right)\vec{S}_\alpha \cdot \vec{B}(\vec{r}_\alpha)}_{\text{from Dirac equation}}$$

$$+ V_{coul} + H_R .$$

$$V_{coul} = \frac{\varepsilon_0}{2}\int d^3r\, \vec{E}_\parallel^{\,2}(\vec{r}) = \frac{\varepsilon_0}{2}\int d^3k \left|\mathcal{E}_\parallel(\vec{k})\right|^2$$

$$= \frac{1}{2\varepsilon_0}\int d^3k \frac{\rho^*(\vec{k})\rho(\vec{k})}{k^2} = \sum_\alpha \varepsilon^\alpha_{coul} + \frac{1}{8\pi\varepsilon_0}\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\vec{r}_\alpha - \vec{r}_\beta|}$$

$$\varepsilon^\alpha_{coul} = \frac{q_\alpha^2}{2\varepsilon_0}\int \frac{d^3k}{(2\pi)^3}\frac{1}{k^2} = \frac{q_\alpha^2}{4\varepsilon_0\pi^2}k_c \qquad \text{using cut-off } k_c$$

$$H_R = \frac{\varepsilon_0}{2}\int d^3r\left[\vec{E}_\perp^{\,2} + c^2\vec{B}^2\right]$$

$$= \sum_i \hbar\omega_i\left(\hat{a}_i^+\hat{a}_i + \frac{1}{2}\right)$$

Momentum:

$$\vec{P} = \sum_\alpha \vec{p}_\alpha + \vec{P}_R$$

$$\vec{P}_R = \sum_i \hbar\vec{k}_i\, \hat{a}_i^+ a_i$$

$$H = H_p + H_R + H_I$$

$$H_p = \sum_\alpha \frac{\vec{p}_\alpha^{\,2}}{2m} + V_{coul} \qquad \text{particle hamiltonian}$$

$$H_I = H_{I1} + H_{I2} + H^S_{I1}$$

$$H_{I1} = -\sum_\alpha \frac{q_\alpha}{m_\alpha}\vec{p}_\alpha \cdot \vec{A}_\perp(\vec{r}_\alpha)$$

$$H^S_{I1} = -\sum_\alpha g_\alpha \frac{q_\alpha}{2m_\alpha}\vec{S}_\alpha \cdot \vec{B}(\vec{r}_\alpha)$$

$$H_{I2} = \sum_\alpha \frac{q_\alpha^2}{2m_\alpha}\vec{A}_\perp^{\,2}(\vec{r}_\alpha)$$

2.7 State Space

$$\mathcal{E} = \mathcal{E}_{particles} \otimes \mathcal{E}_{Radiation}$$

$$\mathcal{E}_{particles} = \ldots \otimes \mathcal{E}_\alpha \otimes \ldots \qquad \mathcal{E}_\alpha \text{ space for particle } \alpha$$

$$\mathcal{E}_{radiation} = \ldots \otimes \mathcal{E}_i \otimes \ldots \qquad \mathcal{E}_i \text{ space for mode } i$$

Orthonormal basis for $\mathcal{E}_i$ is $\{|n_i\rangle\}$ of energy eigenstates of oscillator at i. Writing $|\{n_i\}\rangle$ for $|n_1\rangle \ldots |n_i\rangle \ldots$:

$$H_R \, |\{n_i\}\rangle = \Big[\sum_i (n_i + \tfrac{1}{2}) \hbar\omega_i\Big] \, |\{n_i\}\rangle$$

$$\vec{P}_R \, |\{n_i\}\rangle = \Big(\sum_i n_i \hbar \vec{k}_i\Big) \, |\{n_i\}\rangle$$

$$a_i |n_i\rangle \underset{n_i \neq 0}{=} \sqrt{n_i}\, |n_i - 1\rangle$$

$$a_i^+ |n_i\rangle = \sqrt{n_i + 1}\, |n_i + 1\rangle$$

$$a_i \, |0\rangle = 0$$

$|\{n_i\}\rangle$ – state of the field containing n_1 photons in mode 1, n_j photons in mode j, etc.

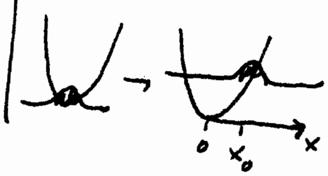
Vacuum: $|0\rangle \qquad (n_1 = 0, \ldots, n_j = 0, \ldots)$

property: $a_i \, |0\rangle = 0 \quad \forall i$

Coherent states: $\quad |\alpha_i\rangle = T^+(\alpha_i)\, |0\rangle$

$$T(\alpha_i) = e^{\alpha_i^* a_i - \alpha_i a_i^+}$$

analogy h.osc.: $e^{-i\hat{p}\frac{x_0}{\hbar}}$



$$T(\alpha_i)\, a_i \, T^+(\alpha_i) = a_i + \alpha_i$$

State $|\alpha_i\rangle$ is eigenstate of the annihilation operator a_i with eigen value α_i:

$$a_i \, |\alpha_i\rangle = \alpha_i \, |\alpha_i\rangle$$

$$|\alpha_i\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n_j = 0}^{\infty} \frac{(\alpha_j)^{n_j}}{\sqrt{n_j!}} \, |n_j\rangle$$