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Course: **8.333 - Statistical Mechanics I**
Problem set: **#6**

1. Numerical Estimates.

(a) The heat capacity

The Fermi temperature for typical metal is $T_F = 5 \times 10^4 \text{K}$ which is much higher than room temperature. Thus, we may calculate the heat capacity using the formula (VII.49) in the lecture notes:

$$C_{\text{electron}} = \frac{\pi^2}{2} N k_B \frac{T}{T_F} \implies C_V \approx 0.03 N k_B$$

where we have used $T_F = 5 \times 10^4 \text{K}$ and $T = 300 \text{K}$. Mathematica code:

```
In[14]:= N[Pi^2/2*300/(5*10^4)]  
Out[14]= 0.0296088
```

Let us consider the metal Aluminum, whose Debye temperature is 428 K. Since room temperature is approximately the Debye temperature, we can't use the high- or low-temperature limits to calculate heat capacity. To do this, we have to use the exact formula:

$$C_{\text{phonon, Al}} = \frac{dE}{dT} = \frac{d}{dT} N k_B \left[9T \frac{T^3}{T_D^3} \int_0^{T_D/T} \frac{x^3}{e^x - 1} dx \right] \Big|_{T=300\text{K}} \approx 2.7 N k_B$$

Mathematica code:

```
In[17]:= N[  
D[9*T*(T/428)^3*Integrate[x^3/(Exp[x] - 1), {x, 0, 428/T}],  
T] /. {T -> 300}]  
Out[17]= 2.71557
```

The desired ratio is therefore

$$\frac{C_{\text{electron}}}{C_{\text{phonon, Al}}} \approx \frac{0.03}{2.7} \approx \boxed{10^{-2}}$$

(b) The thermal wavelength of a neutron at room temperature is

$$\lambda_n = \frac{h}{\sqrt{2\pi m_n k_B T}} \approx \boxed{1 \text{ \AA}}$$

Wolfram Alpha command:

```
planck's constant/Sqrt[2*Pi*mass of neutron*boltzmann constant*300 kelvin]  
>>>> 1.00361194x10^-10 meters
```

The minimum wavelength of a phonon in a typical crystal is on the order of the atomic spacing, so let us say $1 - 10 \text{ \AA}$. Therefore, we have

$$\frac{\lambda_n}{\lambda_{\text{phonon}}} \approx \boxed{1}$$

- (c) We calculate $n\lambda^3$ for H, He, and O₂ under the assumption that the gas densities n follow from the ideal gas law $P = nk_B T$ where $P = 1$ atm. (this is valid since we're assuming room temperature $T = 300$ K).

$$n_H \lambda_H^3 = \frac{P}{k_B T} \frac{h^3}{(2\pi m_H k_B T)^{3/2}} = \boxed{2.4 \times 10^{-5}}$$

$$n_{He} \lambda_{He}^3 = \frac{P}{k_B T} \frac{h^3}{(2\pi m_{He} k_B T)^{3/2}} = \boxed{3.1 \times 10^{-6}}$$

where $m_{He} \approx 4m_H$, and

$$n_{O_2} \lambda_{O_2}^3 = \frac{P}{k_B T} \frac{h^3}{(2\pi m_{O_2} k_B T)^{3/2}} = \boxed{1.4 \times 10^{-7}}$$

where $m_{O_2} \approx 32m_H$.

Wolfram Alpha code:

```
(1 atm/(300 kelvin *Boltzmann constant)) * planck's constant^3/(2*Pi*mass of
proton*Boltzmann constant*300 kelvin)^(3/2)

>>> 0.0000247803181

(1 atm/(300 kelvin *Boltzmann constant)) * planck's constant^3/(2*Pi*4*mass of
proton*Boltzmann constant*300 kelvin)^(3/2)

>>> 3.09753976x10^-6

(1 atm/(300 kelvin *Boltzmann constant)) * planck's constant^3/(2*Pi*32*mass of
proton*Boltzmann constant*300 kelvin)^(3/2)

>>> 1.36893211x10^-7
```

- (d) **(Optional)** The since the heat capacity scales like $C_V \sim T^3$, the energy spectrum must scale like $\mathcal{E}(k) \sim |k|$, consistent with the results discussed on page 131 of Lecture Notes #19. In full form,

$$\mathcal{E} = \hbar v |k|$$

where v is the speed of sound. With

$$C_V = k_B V \frac{2\pi^2}{5} \left(\frac{k_B T}{\hbar v} \right)^3$$

we find

$$v = \left(\frac{2k_B^4 \pi^2 T^3 V}{5 C_V \hbar^3} \right)^{1/3} \implies \mathcal{E} = \hbar v k \approx \boxed{k \times (2 \times 10^{-31}) \text{ Jm}}$$

where we have used $C_V/T^3 = 20.4 \text{ J K}^{-1} \text{ K}^{-1}$. Wolfram Alpha code:

```
hbar*(2*boltzmann constant^4*Pi^2/(5*20.4*hbar^3))^(1/3)
```

2. Solar Interior.

- (a) With $T = 1.6 \times 10^7$ K we have

$$\lambda_e = \frac{h}{(2\pi m_p k_B T)^{1/2}} = \boxed{1.8 \times 10^{-11} \text{ m}}$$

$$\lambda_p = \frac{h}{(2\pi m_e k_B T)^{1/2}} = \boxed{4.3 \times 10^{-13} \text{ m}}$$

where $m_{He} \approx 4m_H$, and

$$\lambda_\alpha = \frac{h}{(2\pi m_\alpha k_B T)^{1/2}} = \boxed{2.7 \times 10^{-13} \text{ m}}$$

where $m_{O_2} \approx 32m_H$.

- (b) Assuming ideal gas. Quantum mechanical effects kick in whenever $n\lambda^3 \geq 1$. We calculate n 's from the ρ 's:

$$n_H = \frac{\rho_H}{m_H} = \boxed{3.59 \times 10^{31} \text{ m}^{-3}}$$

$$n_{He} = \frac{\rho_{He}}{m_{He}} = \boxed{1.50 \times 10^{31} \text{ m}^{-3}}$$

$$n_e = 2n_{He} + n_H = \boxed{6.6 \times 10^{31} \text{ m}^{-3}}$$

With these we find that

$$n_H \lambda_H^3 \approx 2.9 \times 10^{-6} \ll 1$$

$$n_{He} \lambda_{He}^3 \approx 1.54 \times 10^{-7} \ll 1$$

$$n_e \lambda_e^3 \approx 0.42 \sim 1.$$

So H, He are not degenerate in the QM sense, but electrons are close to QM degeneracy.

- (c) Assume ideal gas, then

$$P \sim (n_H + n_{He} + n_e)k_B T \approx \boxed{2.6 \times 10^{16} \text{ Pa}}$$

- (d) Radiation pressure is given by

$$P = \frac{4\sigma}{3c} T^4 = \boxed{1.65 \times 10^{13} \text{ Pa}}$$

where σ is the Stefan-Boltzmann constant. Since this pressure is much less than the matter pressure, it is **matter pressure** that prevents the gravitational collapse of the sun.

3. Density of States. In this problem we will treat $N = n$ i.e. we will implicitly understand that $V = 1$ and treat the particle number the same as particle density, due to the definition of N in the problem (the definition doesn't have V in it).

- (a) Since N has the form

$$N = \int f(\epsilon) \rho(\epsilon) d\epsilon$$

we have that the total energy is

$$E = \int \epsilon f(\epsilon) \rho(\epsilon) d\epsilon = \boxed{\int_0^\infty d\epsilon \rho(\epsilon) \frac{\epsilon}{e^{\beta(\epsilon-\mu)} - \eta}}$$

- (b) For bosons, $\eta = 1$. The critical temperature T_c for Bose-Einstein condensation is where the average particle number N is equal to the average particle number in the excited states $N = N_e$ but with the chemical potential approaching its vanishing limit $\mu = 0$. The critical temperature T_c therefore solves the equation

$$N_e(\mu = 0, T_c) = N \implies \boxed{N = \int_0^\infty d\epsilon \rho(\epsilon) \frac{1}{e^{\epsilon/k_B T_c} - 1}}$$

- (c) We're working with Fermions now, so let us set $\eta = -1$. The Sommerfield expansion says that as $\beta \rightarrow \infty$, we have

$$\lim_{\beta \rightarrow \infty} \int_0^\infty dx \frac{g(x)}{e^{\beta(x-\mu)} + 1} \approx \int_0^\mu dx g(x) + \frac{\pi^2}{6\beta^2} g'(\mu) + \dots$$

Let us choose $g(\epsilon) = \rho(\epsilon)$, so that we have

$$\begin{aligned} \lim_{\beta \rightarrow \infty} N &= \lim_{\beta \rightarrow \infty} \int_0^\infty d\epsilon \frac{\rho(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \\ &\approx \int_0^\mu d\epsilon \rho(\epsilon) + \frac{\pi^2}{6\beta^2} \rho'(\mu) + \dots \\ &= \int_0^{E_F} d\epsilon \rho(\epsilon) + \int_{E_F}^\mu d\epsilon \rho(\epsilon) + \frac{\pi^2}{6\beta^2} \rho'(\mu). \end{aligned}$$

With $E_F = \lim_{T \rightarrow 0} \mu(T)$ we may assume that $\rho(\mu) \approx \rho(E_F)$ and $\rho'(\mu) \approx \rho'(E_F)$. From this, we have

$$\int_{E_F}^\mu d\epsilon \rho(\epsilon) \approx (\mu - E_F) \rho(E_F).$$

Moreover, by definition for Fermi energy,

$$\lim_{\beta \rightarrow \infty} N = \int_0^{E_F} \rho(\epsilon) d\epsilon.$$

We thus conclude that

$$\boxed{\mu - E_F \approx -\frac{\pi^2}{6\beta^2} \frac{\rho'(E_F)}{\rho(E_F)}}$$

- (d) For this part, we simply repeat but using the expression for E as a starting point. Let us choose $g(\epsilon) = \epsilon \rho(\epsilon)$, so that

$$\begin{aligned} E &= \int_0^\infty d\epsilon \frac{\epsilon \rho(\epsilon)}{e^{\beta(\epsilon-\mu)} + 1} \\ &\approx \int_0^\mu d\epsilon \epsilon \rho(\epsilon) + \frac{\pi^2}{6\beta^2} [\rho(\mu) + \mu \rho'(\mu)] + \dots \\ &= \int_0^{E_F} d\epsilon \epsilon \rho(\epsilon) + \int_{E_F}^\mu d\epsilon \epsilon \rho(\epsilon) + \frac{\pi^2}{6\beta^2} [\rho(\mu) + \mu \rho'(\mu)] + \dots \\ &= E(T=0) + (\mu - E_F) E_F \rho(E_F) + \frac{\pi^2}{6\beta^2} [\rho(E_F) + E_F \rho'(E_F)] \end{aligned}$$

where we have used

$$E(T=0) = \int_0^{E_F} d\epsilon \epsilon \rho(\epsilon).$$

Using the relation for $\mu - E_F$ from the last part, we have

$$\boxed{E - E(T=0)} = -\frac{\pi^2}{6\beta^2} E_F \rho'(E_F) + \frac{\pi^2}{6\beta^2} [\rho(E_F) + E_F \rho'(E_F)] = \boxed{\frac{\pi^2}{6\beta^2} \rho(E_F)}$$

(e) The low temperature heat capacity is simply

$$C_V = \frac{dE}{dT} = \boxed{\frac{\pi^2 k_B^2 T}{3} \rho(E_F)}$$

4. Quantum Point Particle Condensation. The particles are spinless, so $g = 2 \times 0 + 1 = 1$.

(a) The partition function has an extra factor $\exp(\beta u N^2/2V)$, and so the pressure, which has the form $\beta P \sim -\partial \ln Z / \partial V$ gets a correction term which deviates it from the ideal gas pressure:

$$P(n, t) = P_0(n, t) - \frac{\beta u N^2/2V}{\beta V} = P_0(n, t) - \frac{un^2}{2}$$

(b) While there are probably analytic approaches, we can check that the formula holds symbolically using Mathematica. From standard theory for ideal quantum gas, we have

$$P_0(z) = \frac{1}{\beta \lambda^3} f_{5/2}^\eta(z) \quad \text{and} \quad n_\eta = \frac{1}{\lambda^3} f_{3/2}^\eta(z)$$

where

$$f_m^\eta(z) = \frac{1}{(m-1)!} \int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - \eta} dx.$$

While we can do this by hand, we can also quickly simply compute in Mathematica using

$$\left. \frac{\partial P}{\partial n} \right|_T = -un + \frac{\partial P_0}{\partial z} \left(\frac{\partial n}{\partial z} \right)^{-1} = -un + \frac{1}{\beta} \frac{\text{PolyLog}(3/2, \eta z)}{\text{PolyLog}(1/2, \eta z)}$$

while

$$k_B T \frac{f_{3/2}^\eta(z)}{f_{1/2}^\eta(z)} = \frac{1}{\beta} \frac{\text{PolyLog}(3/2, \eta z)}{\text{PolyLog}(1/2, \eta z)}$$

So,

$$\left. \frac{\partial P}{\partial n} \right|_T = -un + k_B T \frac{f_{3/2}^\eta(z)}{f_{1/2}^\eta(z)}$$

as desired.

Mathematica code:

```
(*define f*)
In[49]:= F[[Eta]_, m_, z_] := (1/Factorial[m - 1])*
Integrate[x^(m - 1)/(z^(-1)*Exp[x] - \[Eta]), {x, 0, Infinity}]

(*find ratio*)
In[52]:= D[F[[Eta], 5/2, z], z]/D[F[[Eta], 3/2, z], z]

Out[52]= PolyLog[3/2, z \[Eta]]/PolyLog[1/2, z \[Eta]]

(*find second ratio*)
In[53]:= F[[Eta], 3/2, z]/F[[Eta], 1/2, z]

Out[53]= PolyLog[3/2, z \[Eta]]/PolyLog[1/2, z \[Eta]]
```

(c) The gas becomes unstable when $\partial P/\partial n = 0$, so we have

$$u_c(n_\eta, T) = \frac{k_B T}{n_\eta} \frac{f_{3/2}^\eta(z)}{f_{1/2}^\eta(z)}$$

In the low density (non-degenerate) limit $n\lambda^3 \ll 1$, we take $z \rightarrow 0$ and compute $f_{3/2}^\eta(z)/f_{1/2}^\eta(z)$ as a series in z . We shall do this in Mathematica, using the series definition for $f_m^\eta(z)$ at low z :

$$f_m^\eta(z) = \sum_{a=0}^{\infty} \eta^{\text{Mod}(a,2)} \frac{z^{a+1}}{(a+1)^m}$$

This definition is equivalent to that in the textbook, but more convenient for Mathematica use. After the expansion, we also have to plug in z as a perturbative expansion in n , given in the textbook:

$$z = (n_\eta \lambda^3) - \frac{\eta}{2^{3/2}} (n_\eta \lambda^3)^2 + \left(\frac{1}{4} - \frac{1}{3^{3/2}} \right) (n_\eta \lambda^3)^3 - \dots$$

The result, up to first-order correction, is

$$u_c(n_\eta, T) = \frac{k_B T}{n_\eta} \left[1 - \frac{\eta}{2^{3/2}} (n_\eta \lambda^3) + \frac{\eta^2}{4^{3/2}} (n_\eta \lambda^3)^2 + \mathcal{O}[(n_\eta \lambda^3)^3] \right]$$

A few observations, when $\eta = 0$ we get $u_c = k_B T/n$. The first correction that distinguishes between Fermi and Bose statistics is

$$-\frac{\eta}{2^{3/2}} k_B T \lambda^3$$

which is independent of density n_η .

Mathematica code:

```
(*Define f as a series*)
In[12]:= f[\[Eta]_, m_, z_] :=
Sum[\[Eta]^a z^(a + 1)/(a + 1)^m, {a, 0, 5}]

(*define z as a function of n*)
In[17]:= Z = (n*\[Lambda]^3) - \[Eta]/
2^(3/2)*(\[Eta]*\[Lambda]^3)^2 + (1/4 -
1/3^(3/2))*(n*\[Lambda]^3)^3;

(*compute uc, with the kBT/n factor*)
In[18]:= ucPrime =
Series[f[\[Eta], 3/2, z]/f[\[Eta], 1/2, z], {z, 0, 1}] // FullSimplify

Out[18]= SeriesData[z, 0, {
1, Rational[-1, 2] 2^Rational[-1, 2] \[Eta]}, 0, 2, 1]

(*plug in z = z(n)*)
In[22]:= 1 - (\[Eta] z)/(2 Sqrt[2]) /. {z -> Z} // Expand

Out[22]= 1 - (n \[Eta] \[Lambda]^3)/(2 Sqrt[2]) +
1/8 n^2 \[Eta]^2 \[Lambda]^6 - (n^3 \[Eta] \[Lambda]^9)/(
8 Sqrt[2]) + (n^3 \[Eta] \[Lambda]^9)/(6 Sqrt[6])
```

(d) For fermions, $\eta = -1$, so we have

$$u_c(n_-, T) \approx \frac{k_B T}{\eta_-} \left[1 + \frac{n_- \lambda^3}{2^{3/2}} \right].$$

Recall the Fermi energy:

$$\mathcal{E}_F = \frac{\hbar^2}{2m} (6\pi^2 n_-)^{2/3} \implies n_- = \frac{\sqrt{2}}{3\pi^2} \left(\frac{m\mathcal{E}_F}{\hbar^2} \right)^{3/2}$$

With this, we get

$$u_c(\mathcal{E}_F, T) = \boxed{k_B T \left(\frac{\lambda^3}{2\sqrt{2}} + \frac{3\hbar^3 \pi^2}{\sqrt{2}(\mathcal{E}_F m)^{3/2}} \right)}$$

(e) For bosons,

$$u_c(n_+, T) \approx \frac{k_B T}{\eta_+} \left[1 - \frac{n_+ \lambda^3}{2^{3/2}} \right].$$

As temperature is decreased towards the quantum degeneracy regime, the coupling $u_c(n_+, T)$ will vanish and become negative.

5. Harmonic Confinement of Fermions. The potential is

$$U(r) = \frac{m}{2} \sum_{\alpha}^d \omega_{\alpha}^2 x_{\alpha}^2.$$

(a) Let $N(E)$ be the number of states with energy between 0 and E . In one dimension α , the energies are spaced by $\hbar\omega_{\alpha}$. Assuming that we could consider an infinitesimal change dE , we can generalize to d dimensions to find

$$\begin{aligned} N(E) &= \prod_{\alpha}^d \frac{1}{\hbar\omega_{\alpha}} \int_0^E \int_0^{E-E_1} \int_0^{E-E_1-E_2} \dots \int_0^{E-\sum_{\alpha}^{d-1}} \prod_{\alpha}^d dE_{\alpha} \\ &= \boxed{\frac{1}{d!} \prod_{\alpha=1}^d \left(\frac{E}{\hbar\omega_{\alpha}} \right)} \end{aligned}$$

where we have used the fact that the value of the iterated integral is $E^d/d!$ which can be readily checked by hand or Mathematica. From here, the density of states is straightforward:

$$\rho(E) = \frac{dN(E)}{dE} = \boxed{\frac{1}{(d-1)!} \frac{E^{d-1}}{\prod_{\alpha}^d \hbar\omega_{\alpha}}}$$

(one way to think about $N(E)$ and $\rho(E)$ is that the former is a cdf and latter is a pdf, ignoring normalization).

(b) In a grand canonical ensemble, the number of particles in the trap is obtained by using the fact that the particles follows Fermi-Dirac statistics:

$$\begin{aligned} \langle N \rangle &= \int_0^{\infty} \frac{1}{e^{\beta(E-\mu)} + 1} \rho(E) dE \\ &= \frac{1}{(d-1)!} \frac{1}{\prod_{\alpha}^d \hbar\omega_{\alpha}} \int_0^{\infty} \frac{E^{d-1}}{e^{\beta(E-\mu)} + 1} dE \end{aligned}$$

By the change of variables $x = \beta E = E/k_B T$ and letting $z = e^{\beta\mu}$, we find

$$\langle N \rangle = \prod_{\alpha}^d \left(\frac{k_B T}{\hbar\omega_{\alpha}} \right) \frac{1}{(d-1)!} \int_0^{\infty} \frac{x^{d-1}}{z^{-1}e^x + 1} dx = \boxed{f_d^-(z) \prod_{\alpha}^d \left(\frac{k_B T}{\hbar\omega_{\alpha}} \right)}$$

as desired.

(c) The energy is given by

$$\begin{aligned}
\langle E \rangle &= \int_0^\infty \frac{E}{e^{\beta(E-\mu)} + 1} \rho(E) dE \\
&= k_B T \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) \frac{1}{(d-1)!} \int_0^\infty \frac{x^d}{z^{-1} e^x + 1} dx \\
&= k_B T d \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) \frac{1}{d!} \int_0^\infty \frac{x^d}{z^{-1} e^x + 1} dx \\
&= \boxed{k_B T d \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) f_{d+1}^-(z)}
\end{aligned}$$

(d) The limit forms for $\langle E \rangle$ and $\langle N \rangle$ are

$$\begin{aligned}
\langle N \rangle &= \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) f_d^-(z) \\
&= \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) \sum_{m=1}^{\infty} (-1)^{m+1} \frac{z^m}{m^d} \quad \text{as } \beta \rightarrow 0 \\
&\approx \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) \left[z - \frac{z^2}{2^d} + \frac{z^3}{3^d} - \dots \right]
\end{aligned}$$

$$\begin{aligned}
\langle E \rangle &= k_B T d \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) f_{d+1}^-(z) \\
&= k_B T d \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) \sum_{m=1}^{\infty} (-1)^{m+1} \frac{z^m}{m^{d+1}} \quad \text{as } \beta \rightarrow 0 \\
&\approx k_B T d \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) \left[z - \frac{z^2}{2^{d+1}} + \frac{z^3}{3^{d+1}} - \dots \right]
\end{aligned}$$

With these, we may compute the energy per particle in the high temperature limit:

$$\boxed{\left. \frac{\langle E \rangle}{\langle N \rangle} \right|_{\beta \rightarrow 0} \approx k_B T d \left[1 + \frac{z}{2^{d+1}} + \dots \right]}$$

Mathematica code:

```

In[49]:= Series[f[-1, d + 1, z]/f[-1, d, z], {z, 0, 3}]

Out[49]= SeriesData[z, 0, {
1, 2^(-1 - d), -2^(-1 - 2 d) + 2^((-2)
d) + 3^(-1 - d) - 3^(-d), -2^(-2 - 3 d) 3^(-1 - d) (
7 2^(1 + 2 d) - 2 3^(1 + d) - 2^d 3^(2 + d))}, 0, 4, 1]

```

(e) μ approaches the Fermi energy \mathcal{E}_F at zero temperature. At $T = 0$, let us take the Fermi occupation number to be unity. So that

$$\langle N \rangle = \frac{1}{(d-1)!} \frac{1}{\prod_{\alpha}^d \hbar \omega_{\alpha}} \int_0^{\mu \equiv \mathcal{E}_F} E^{d-1} dE = \frac{1}{(d-1)!} \frac{1}{\prod_{\alpha}^d \hbar \omega_{\alpha}} \frac{\mathcal{E}_F^d}{d} = \frac{\mathcal{E}_F^d}{d!} \frac{1}{\prod_{\alpha}^d \hbar \omega_{\alpha}}.$$

From here we have

$$f_d^-(z) \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) = \frac{\mathcal{E}_F^d}{d!} \frac{1}{\prod_{\alpha}^d \hbar \omega_{\alpha}} \implies \frac{1}{d!} \left(\frac{\mathcal{E}_F}{k_B T} \right)^d = f_d^-(z).$$

Now we will use the Sommerfeld expansion to find

$$\lim_{\beta \rightarrow \infty} f_d^-(z) = \frac{(\ln z)^d}{d!} \left[1 + \frac{\pi^2}{6} \frac{d(d-1)}{(\ln z)^2} + \dots \right].$$

Since $z = e^{\beta \mu}$ we have $\ln z = \beta \mu$, so

$$\beta \mathcal{E}_F = \beta \mu \left[1 + \frac{\pi^2}{6} \frac{d(d-1)}{(\beta \mu)^2} + \dots \right]^{1/d}$$

For the correction part, we may as well call $\mu = \mathcal{E}_F$, so that we get

$$\boxed{\mu} = \mathcal{E}_F \left[1 + d(d-1) \frac{\pi^2}{6} \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right]^{-1/d} \approx \boxed{\mathcal{E}_F \left[1 - (d-1) \frac{\pi^2}{6} \left(\frac{k_B T}{\mathcal{E}_F} \right)^2 + \dots \right]}$$

(f) The heat capacity is given by

$$\begin{aligned} C_V &= \frac{d\langle E \rangle}{dT} \\ &= \frac{d}{dT} \left\{ k_B T d \prod_{\alpha}^d \left(\frac{k_B T}{\hbar \omega_{\alpha}} \right) f_{d+1}^-(z) \right\} \end{aligned}$$

where

$$\lim_{\beta \rightarrow \infty} f_{d+1}^-(z) = \frac{(\ln z)^{d+1}}{(d+1)!} \left[1 + \frac{\pi^2}{6} \frac{d(d+1)}{(\ln z)^2} + \dots \right].$$

I can continue here with the expansion, but I won't, as it will be a big mess with unsimplified d 's everywhere and multiplication of series, etc. This is also the last pset, so I'll let this slide...

6. Anharmonic Trap.

(a) From the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + K r^n$$

we can calculate... **No idea how to do this.**

(b) Using

$$\rho(\epsilon) = \frac{C}{(p-1)!} \epsilon^{p-1},$$

we get

$$N = \frac{C}{(p-1)!} \int_0^{\infty} \frac{\epsilon^{p-1}}{e^{\beta(\epsilon - \mu)} - \eta} d\epsilon$$

Let $x = \beta \epsilon$ and $z = e^{\beta \mu}$, then we have

$$N = \frac{C(k_B T)^p}{(p-1)!} \int_0^{\infty} \frac{x^{p-1}}{z^{-1} e^x - \eta} dx = \boxed{C(k_B T)^p f_p^{\eta}(z)}$$

(c) Just like the previous problem, we find that the total energy is

$$E = \frac{C}{(p-1)!} \int_0^\infty \frac{\epsilon^p}{e^{\beta(\epsilon-\mu)-\eta}} d\epsilon = \boxed{Cp(k_B T)^{p+1} f_{p+1}^\eta(z)}$$

(d) At $T = 0$, the Fermi occupation number is unitary. The Fermi energy is given by

$$N = \frac{C}{(p-1)!} \int_0^{\mathcal{E}_F} \epsilon^{p-1} d\epsilon \implies N = \frac{C}{p!} \mathcal{E}_F^p \implies \boxed{\mathcal{E}_F = \left(\frac{Np!}{C} \right)^{1/p}}$$

(e) The heat capacity is given by $C_V = dE/dT$. There are two T -dependence from E . The first is the T^{p+1} factor. However, since E is also proportional to $f_{p+1}^\eta(z)$ whose low-temperature expansion has leading term which scales like $\beta^{p+1} \sim T^{-(p+1)}$ multiplied by a correction factor of the form $(1 + \Lambda(T/T_F)^2 + \dots)$. So, when we take dE/dT , we will be left with a term that is linear in T . Therefore, the low-temperature heat capacity is **linear** in temperature.

(f) For bosons, we set $\eta = 1$. **To be continued.**

7. (Optional) Fermi gas in two dimensions.

(a)

(b)

(c)

8. (Optional) Partition of Integers.

(a) Let some energy $E = \sum_k k n_k$ be given, where k is an integer and n_k is its associated multiplicity in the partition. The partition function is given by

$$\mathcal{Z}(\beta) = \sum_{\psi} e^{-\beta E_{\psi}}$$

where \sum_{ψ} denotes the sum over all configurations. We may expand $\mathcal{Z}(\beta)$ to simplify it

$$\begin{aligned} \mathcal{Z}(\beta) &= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \exp(-\beta(1n_1 + 2n_2 + \dots)) \\ &= \sum_{n_1=0}^{\infty} e^{-\beta n_1} \sum_{n_2=0}^{\infty} e^{-2\beta n_2} \dots \\ &= \prod_{l=1}^{\infty} \left[\sum_{n_l=0}^{\infty} \exp(-\beta l n_l) \right] \\ &= \prod_{l=1}^{\infty} \frac{1}{1 - e^{-\beta l}} \end{aligned}$$

after using geometric series. **This seems nice and doesn't require going to $\beta \rightarrow 0$?**

(b) We **now** change the sum into an integral. The average energy is

$$E = -\frac{\partial \ln \mathcal{Z}}{\partial \beta} = -\frac{\partial}{\partial \beta} \sum_{l=1}^{\infty} \ln \left(\frac{1}{1 - e^{-\beta l}} \right) = \frac{\partial}{\partial \beta} \sum_{l=1}^{\infty} \ln (1 - e^{-\beta l}) \approx \frac{\partial}{\partial \beta} \int_1^{\infty} \ln (1 - e^{-\beta l})$$

The integral can be approximated by the following trick:

$$\begin{aligned}\int_1^\infty \ln(1 - e^{-\beta l}) &= \int_0^\infty \ln(1 - e^{-\beta l}) - \int_0^1 \ln(1 - e^{-\beta l}) \\ &= -\frac{\pi^2}{6\beta} - \left(\int_0^1 \ln(\beta l) - \frac{\beta l}{2} + \dots dl \right).\end{aligned}$$

Letting Mathematica do the work, we find that

$$E \approx \frac{\pi^2}{6\beta^2} - \frac{1}{\beta}$$

where we have dropped terms with zeroth and higher orders in β .

Mathematica code:

```
In[79]:= Int = Integrate[Log[1 - Exp[-b*l]], {l, 0, Infinity}]
Out[79]= ConditionalExpression[-(\[Pi]^2/(6 b)), Re[b] > 0]

In[93]:= integral =
Int - Integrate[Series[Log[1 - Exp[-b*l]], {b, 0, 1}], {l, 0, 1}];
In[94]:= energy = D[integral, b]
Out[94]= ConditionalExpression[1/4 - 1/b + \[Pi]^2/(6 b^2), Re[b] > 0]
```

We can also tell Mathematica to solve for $T(E)$. By taking $E \gg 1$, we find that

$$T(E) \approx \frac{\sqrt{6E}}{k_B \pi}$$

Mathematica code:

```
In[95]:= Solve[EE == \[Pi]^2/(6 b^2) - 1/b, b] // FullSimplify
Out[95]= {{b -> -((3 + Sqrt[9 + 6 EE \[Pi]^2))/(6 EE))}, {b -> (-3 + Sqrt[9 + 6 EE \[Pi]^2))/(6 EE)}}
```

(c) The entropy is

$$S = \frac{\partial}{\partial T} (k_B T \ln \mathcal{Z}) = \frac{\partial}{\partial T} \left[k_B T \int_1^\infty \ln \left(\frac{1}{1 - e^{-l/k_B T}} \right) dl \right]$$

Using the same trick we find

$$S(T) \approx \frac{1}{3} k_B^2 \pi^2 T + k_B \ln \left(\frac{1}{k_B T} \right) \implies S(E) = k_B \pi \sqrt{\frac{2E}{3}} + k_B \ln \left(\frac{\pi}{\sqrt{6E}} \right)$$

where we have used $T(E)$ from Part (b). Now, we want a relation between the entropy and the number of microstates. According to this paper <https://arxiv.org/pdf/1603.01049.pdf>, the multiplicity is related to the entropy by

$$W(E) = \frac{e^{S(\beta_0)}}{\sqrt{2\pi S''(\beta_0)}}$$

where $\beta_0 = 1/k_B T(E)$, with $T(E)$ being the answer from Part (b). So,

$$S''(\beta) = -\frac{k_B}{\beta^2} + \frac{2k_B \pi^2}{3\beta^3} \approx \frac{4\sqrt{6}E^{3/2}k_B}{\pi}.$$

To avoid complications let us set $k_B = 1$, so that

$$S(E) = \pi \sqrt{\frac{2E}{3}}, \quad S''(E) = \frac{4\sqrt{6}E^{3/2}}{\pi}$$

So,

$$W(E) \sim \frac{1}{E^{5/4}} \exp \left(\pi \sqrt{\frac{2E}{3}} \right).$$

Hmm... I'm getting close but something is off here. I think the error is from very early on when I try to go from the sum to the integral. Initially I did this problem without referencing the paper and got the leading factor to go like $1/E^{3/4}$, which is not quite $1/E$. I think the $\ln(1/\beta)$ correction in the expression for entropy is quite delicate, and any factor that lands there will decide the leading factor $1/E^x$ in the final expression. In any case, the exponential part is at least consistent.

9. (Optional) Fermions pairing into Bosons.

- (a)
- (b)
- (c)
- (d)

10. (Optional) Ring Diagrams Mimicking Bosons.

- (a)
- (b)
- (c)
- (d)

11. (Optional) Relativistic Bose Gas in d -dimensions.

- (a) The grand potential is

$$\begin{aligned}
 \mathcal{G} &= -k_B T \ln Q \\
 &= k_B T \sum_i \ln [1 - e^{\beta(\mu - \epsilon_i)}] \\
 &\rightarrow k_B T \int_0^\infty V \frac{d^d k}{(2\pi)^d} \ln [1 - e^{\beta(\mu - ck)}] \\
 &= \frac{k_B T V S_d}{(2\pi)^d} \int_0^\infty dk k^{d-1} \ln [1 - ze^{-\beta ck}] \\
 &= \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d \int_0^\infty dx x^{d-1} \ln [1 - ze^{-x}] \\
 &= -\frac{1}{d} \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d \int_0^\infty dx x^d \frac{ze^{-x}}{1 - ze^{-x}} \quad \text{int. by parts.} \\
 &= -\frac{1}{d} \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d \int_0^\infty dx \frac{x^d}{z^{-1}e^x - 1} \\
 &= \boxed{-\frac{1}{d} \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d d! f_{d+1}^+(z)}
 \end{aligned}$$

The density is therefore

$$\begin{aligned}
n &= \frac{N}{V} \\
&= -\frac{1}{V} \frac{\partial \mathcal{G}}{\partial \mu} \\
&= -\frac{1}{V} \frac{\partial \mathcal{G}}{\partial z} \frac{\partial z}{\partial \mu} \\
&= -\frac{\beta z}{V} \frac{\partial \mathcal{G}}{\partial z} \\
&= \frac{1}{d} \frac{S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d d! [z \partial_z f_{d+1}^+(z)] \\
&= \boxed{\frac{1}{d} \frac{S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d d! f_d^+(z)}
\end{aligned}$$

(b) Since $\mathcal{G} = -PV$, it suffices to just calculate the pressure,

$$P = \frac{\ln Q}{\beta V} = \frac{-\mathcal{G}}{V} = -\frac{1}{d} \frac{k_B T S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d d! f_{d+1}^+(z)$$

Following the section on non-relativistic gas in the book, we find that the energy is (this part is not affected by relativity):

$$E = d \times PV = -d \times \mathcal{G}.$$

With these, we have

$$\frac{E}{PV} = \frac{dPV}{PV} = \boxed{d}$$

which is the same as the classical value.

(c) The critical temperature T_c for BEC is given by

$$n = \frac{1}{d} \frac{S_d}{(2\pi)^d} \left(\frac{k_B T_c}{c} \right)^d d! f_d^+(z=1) = \frac{1}{d} \frac{S_d}{(2\pi)^d} \left(\frac{k_B T_c}{c} \right)^d d! \zeta_d$$

which gives

$$\boxed{T_c = \frac{c}{k_B} \left(\frac{d(2\pi)^d}{S_d d! \zeta_d} \right)^{1/d}}$$

The Riemann zeta-function is finite only for $d > 1$, so BEC transition occurs only in $\boxed{d > 1}$ dimensions.

(d) Below the critical temperature we have $z = 1$ (z gets stuck there) which is independent of temperature and $E \sim \mathcal{G} \propto T^{d+1}$. So $\boxed{C_V \propto T^d}$.

(e) With $C(T_c) = dE/dT|_{T_c} = -d(d+1)\mathcal{G}/T$, we find

$$\frac{C(T_c)}{Nk_B} = -\frac{d(d+1)}{T} \frac{1}{k_B} \left(V \frac{1}{d} \frac{S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d d! f_d^+(1) \right)^{-1} \left(-\frac{1}{d} \frac{k_B T V S_d}{(2\pi)^d} \left(\frac{k_B T}{c} \right)^d d! f_{d+1}^+(1) \right) = \boxed{\frac{d(d+1)\zeta_{d+1}}{\zeta_d}}$$

In the high temperature limit, $C_V/Nk_B \propto d$, due to the partition theorem (**I'm actually not sure or know how to show that this is true... Only heard this in a pset session.**), so there is a difference.

12. (Optional) Surface Adsorption of an Ideal Bose Gas.

- (a)
- (b)
- (c)
- (d)

13. (Optional) Inertia of Superfluid Helium.

- (a)
- (b)
- (c)
- (d)