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 Course: **8.370 - QC**
 Problem set: **#6**
 Due: Wednesday, Nov 2, 2022
 Collaborators/References:

1. Pauli group, Clifford group

(a) The Clifford group is

$$C \equiv \{\text{unitaries } U : U g U^\dagger \in \mathcal{P} \forall g \in \mathcal{P}\}$$

where \mathcal{P} is the Pauli group. Suppose $A, B \in C$. Then for any $g \in \mathcal{P}$:

$$ABg(AB)^\dagger = A \underbrace{(BgB^\dagger)}_{\in \mathcal{P}} A^\dagger = Ag'A^\dagger \in \mathcal{P}.$$

So $AB \in C$. Moreover, if $A \in C$, then for any $g \in \mathcal{P}$

$$A^\dagger g A = (Ag^\dagger A^\dagger)^\dagger = (AgA^\dagger)^\dagger \in \mathcal{P}.$$

So every element of C has an inverse. The identity element is simply the identity matrix. Associativity is inherited from associativity of matrix multiplication. So, C is a group.

(b) The Pauli group for 1 qubit is given by

$$\mathcal{P}_1 = \{\pm I, \pm iI, \pm X, \pm iX, \pm Y, \pm iY, \pm Z, \pm iZ\}.$$

The Pauli group for n qubits is simply generated, via the tensor product, by the operators in \mathcal{G}_1 , which are generated by the Pauli matrices X, Y, Z . H is unitary. So, it remains to check that H is in C_1 . Using the fact that $H^\dagger = H^{-1} = H$, we can simply check that $HXH, HYH, HZH \in \mathcal{P}_2$:

$$HXH = Z \in \mathcal{P}_1$$

$$HZH = X \in \mathcal{P}_1$$

$$HYH = -Y \in \mathcal{P}_1$$

And we're done!

(c) To check that the $CNOT$ gate is in C , we check that it is in C_2 . $CNOT$ is unitary, so we check that $CNOT g CNOT^\dagger$ is in \mathcal{P}_2 for all generators g of \mathcal{P}_2 . The generators of \mathcal{P}_2 are once again X, Y, Z tensored with the identity matrix either on the first or second qubit. This means there are six cases:

$$CNOT X \otimes I CNOT^\dagger = X \otimes X \in \mathcal{P}_2$$

$$CNOT Y \otimes I CNOT^\dagger = Y \otimes X \in \mathcal{P}_2$$

$$CNOT Z \otimes I CNOT^\dagger = Z \otimes I \in \mathcal{P}_2$$

$$CNOT I \otimes X CNOT^\dagger = I \otimes X \in \mathcal{P}_2$$

$$CNOT I \otimes Y CNOT^\dagger = Z \otimes Y \in \mathcal{P}_2$$

$$CNOT I \otimes Z CNOT^\dagger = Z \otimes Z \in \mathcal{P}_2 \quad \checkmark$$

Here, the $CNOT$ gate in consideration is one where the first qubit is the control. However, since the other $CNOT$ gate only differs on on which qubits it uses as control and target, we only need to check one of the two $CNOT$ s.

(d) We want to check that $T \notin C_1$. Consider $g = X \in \mathcal{P}_1$.

$$T X T^\dagger = \begin{pmatrix} 0 & e^{-i\pi/4} \\ e^{i\pi/4} & 0 \end{pmatrix} \notin \mathcal{P}_1.$$

So T is not in the Clifford group.

2. Gotta erase workbits!

3. Simon's algorithm

4. Partial transpose

(a) Suppose M is separable, i.e.,

$$M = \sum_i \lambda_i |v_i\rangle \langle v_i| \otimes |w_i\rangle \langle w_i|$$

where λ_i 's are positive. Then the partial transpose of M according to the definition in the problem is

$$pt(M) = \sum_i \lambda_i (|v_i\rangle \langle v_i|)^\top \otimes |w_i\rangle \langle w_i|.$$

Since $\Pi_i = |v_i\rangle \langle v_i|$ are orthogonal projections, the matrices $(|v_i\rangle \langle v_i|)^\top$ are also orthogonal projections. We may very well consider the transposition as a unitary change of basis in the first qubit and write

$$pt(M) = \sum_i \lambda_i |v'_i\rangle \langle v'_i| \otimes |w_i\rangle \langle w_i|.$$

It is clear that the spectrum of $pt(M)$ is exactly the same as that of M in this case, so $pt(M)$ must also be positive. However, we could also be explicit: Let $x = \sum_{i,j} c_{ij} |v'_i\rangle |w_j\rangle$. Then

$$\langle x | pt(M) | x \rangle = \sum_i \lambda_i |c_{ii}|^2 \geq 0.$$

And we're done.

(b) The density matrix for $|\psi\rangle = (1/\sqrt{2})(|00\rangle + |11\rangle)$ is

$$\rho = |\psi\rangle \langle \psi| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

From here we find

$$pt(\rho) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The eigenvalues of this matrix are $-1/2, 1/2, 1/2, 1/2$, so $pt(\rho)$ is not non-negative which implies that ρ is not separable in view of Part (a).

5. Teleporting a qutrit directly

Instead of embedding the qutrit in a set of qubits of higher dimensions, we can teleport qutrits directly. Let $\omega = e^{2\pi i/3}$, the cube root of unity. Define

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The analogue of the EPR pair is the state

$$|EPR_3\rangle = \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)$$

and the analogue of the Pauli matrices are the nine matrices $P^a T^b$, with $0 \leq a, b < 3$. We teleport the qutrit as follows:

Alice has some qutrit in state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle$. She now measures her two qutrits (the qutrit in state $|\psi\rangle$ and her half of the EPR_3 pair) in the EPR-pair basis which consists of nine states in the form

$$(\mathcal{I} \otimes P^a T^b)(|00\rangle + |11\rangle + |22\rangle).$$

Alice will then send her measurement result (one of the nine possible ones) to Bob, and Bob will apply one of nine unitaries to his qutrit to obtain ψ . In particular, if Alice sees a state associated with $P^a T^b$, then Bob applies $P^a T^b$ to his qutrit. Below we will show explicitly.

Here we show the nine-element basis for Alice's measurement, the resulting state on Bob's qutrit, and Bob's corrective unitary for each case:

$$\begin{aligned} (1/\sqrt{3})(\mathcal{I} \otimes P^0 T^0)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})\mathcal{I}(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^0 T^0 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^1 T^0)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})P^2(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^1 T^0 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^2 T^0)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})P^1(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^2 T^0 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^0 T^1)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})T^2(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^0 T^1 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^1 T^1)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})P^2 T^2(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^1 T^1 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^2 T^1)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})P T^2(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^2 T^1 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^0 T^2)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})T(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^0 T^2 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^1 T^2)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})P^2 T(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^1 T^2 \\ (1/\sqrt{3})(\mathcal{I} \otimes P^2 T^2)(|00\rangle + |11\rangle + |22\rangle) &\rightarrow |B\rangle = (1/\sqrt{3})P T(\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle) \rightarrow P^2 T^2. \end{aligned}$$