Topic: PDEs and Euler-Lagrange Equations.

Plan: I will first discuss two motivating examples of optimization problems that use calculus of variations: showing that a straight line gives the shortest distance between two points (in Euclidean geometry), and the famous Brachistochrone problem. These examples will lead up to the idea of a common **functional** of the form

$$J[\phi_i] = \int_a^b \mathcal{L}(t, \phi_i, \dot{\phi}_i)\, dt$$

and how the **Euler-Lagrange equations**

$$\frac{\partial \mathcal{L}}{\partial \phi_i} = \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i}\right),$$

where $\mathcal{L}$ is called the **Lagrangian**, arise when finding the function $\bar{\phi}_i$ that minimizes $J[\phi_i]$. I will then show we can:

1. Solve PDE's by formulating them as minimization problems. For example, the solution to the Dirichlet problem

$$(*) \begin{cases} \nabla^2 u = 0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

   minimizes

$$J[\phi] = \frac{1}{2}\int_\Omega |\nabla \phi|^2\, dx$$

   where

$$\mathcal{L} = \frac{1}{2}|\nabla \phi|^2.$$

2. Recognize the Euler-Lagrange equations as PDE's whose solutions minimize $J[\phi_i]$. I will discuss *Hamilton's Principle of Least Action* and how "equations on motion" including Newton's second law of motion, the wave equation, etc. can be derived from this principle.