Classical Mechanics III (8.09 & 8.309) Fall 2021
Assignment 1

Massachusetts Institute of Technology
Physics Department   *Due Mon. September 20, 2021*
September 10, 2021   *6:00pm*

Announcements

The first week is dedicated primarily to review, discussing and expanding upon pre-requisite material (seen for example in Classical Mechanics II).

- On this problem set, **8.09 students** should do problems 1 to 5, and **8.309 students** should do problems 1 to 4 and 6. There is also a 7th problem that is not for credit, but which everyone should read and consider since the result will be used later in the course.

- All problem set solutions will be turned in electronically. We encourage people to submit PDF files, created either by taking pictures or scanning handwritten solutions, or by using LaTeX to create solutions (example files to get started with that are available on the course webpage).

- After you have made an attempt at the problems, we encourage you to learn more by discussion with your fellow students about places where you are stuck. Please list the names of the students you have talked with on your problem set. The solutions you submit must be your own work and NOT a copy of another student's work.

Reading Assignment for the first two weeks of the course

- Lagrangian and Hamiltonian classical mechanics, symmetries and conservation laws. Use of generalized coordinates to handle constraints to motion on surfaces or curves. Read the following parts of **Goldstein** to review and extend concepts you have already seen in Classical Mechanics II: pgs. 34-45, 55-56, 61-63, 334-342, then 343-347 and 353-356 (ie. specific parts of sections 2.1-2.3, 2.6, 2.7, 8.1, 8.2, 8.5.) If you are feeling rusty with Clasical Mechanics II material you might want to review sections 1-10, and 40 of **Landau & Lifshitz**.

- On problem set #2 we will cover: D'Alembert's principle and systems with holonomic and non-holonomic constraints, and the use of Lagrange multipliers to determine forces of constraint. If you want to read ahead the corresponding reading is **Goldstein**: sections 1.3-1.6, 2.4, 2.5, 8.5.

Problem Set 1

These five problems are intended to help you review and generalize results you have seen in an early classical mechanics course.

1. **Two Particles in a Gravitational Field** [12 points, everyone]

   Consider two masses m_1 and m_2 moving in three dimensions which are attracted to each other gravitationally, and are also acted on by a uniform gravitational field with acceleration strength g in the $+\hat{x}$ direction.

   (a) What is the Lagrangian? Use center of mass coordinates and the vector distance $\vec{r}$ between the two particles. This Lagrangian naturally splits itself into the sum of two parts, explain physically why this makes sense.

   (b) Consider the part of the Lagrangian depending on $\vec{r}$ and determine the Lagrangian and compute the Euler-Lagrange equations of motion using spherical coordinates.

   (c) Using the Lagrangian in (b), what is the corresponding Hamiltonian? What are the Hamilton equations of motion?

2. **Double Pendulum in a Plane with Gravity** [20 points, everyone]

   Since the double pendulum is our mascot on the course homepage, we'd best find the equations that govern its motion.
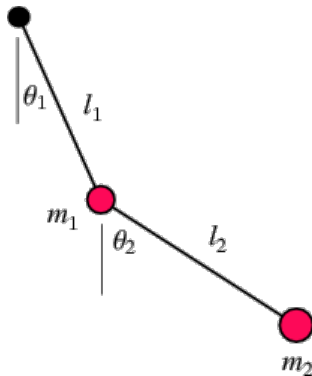
   a) Obtain the Lagrangian and equations of motion for the double pendulum illustrated here. The point at the top is fixed and the oscillations take place in a two-dimensional plane. The length of the pendula are ℓ_1 and ℓ_2, and the rigid bars of the pendula can be taken to be massless. The balls on the ends have mass m_1 and m_2.

   b) Now take $m_1 = m_2 = m$ and find the Hamiltonian for this system, and the Hamilton equations of motion.

   [If you wish, you can make use of results derived in lecture. You may also use Mathematica but be sure to write out intermediate steps in your solution set.]

3. **Point Mass on a Hoop** [12 points, everyone]

   Do Goldstein Ch.2 Problem #18. Use spherical coordinates (r, θ, ϕ) with the origin at the center of the hoop.

4. **Spring System on a Plane** [10 points, everyone]

A massless spring has an unstretched length b and spring constant k, and is used to connect two particles of mass m_1 and m_2. The system rests on a frictionless table and may oscillate and rotate.

(a) What is the Lagrangian? Write it with two-dimensional Cartesian coordinates.

(b) Setup a suitable set of generalized coordinates to better account for the symmetries of this system. What are the Lagrangian and equations of motion in these variables?

(c) Identify as many conserved generalized momenta as you can that are associated to cyclic coordinates in the Lagrangian from part b). If you think you are missing some, try to improve your answer to b). Show that there is a solution that rotates but does not oscillate, and discuss what happens to this solution for an increased rate of rotation. (A closed form solution is not necessary. A graphical solution will suffice.)

5. **Jerky Mechanics** [6 points, 8.09 ONLY]

Consider an extension of classical mechanics where the equation of motion involves a triple time derivative, $\dddot{x} = f(x, \dot{x}, \ddot{x}, t)$. Lets use Hamilton's principle to derive the corresponding Euler-Lagrange equations. Start with a Lagrangian of the form $L(q_i, \dot{q}_i, \ddot{q}_i, t)$ for n generalized coordinates q_i, and make use of Hamilton's principle for paths $q_i(t)$ that have zero variation of both q_i and $\dot{q}_i$ at the end points. Show that

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial q_i} = 0$$

for each $i = 1, \ldots, n$.

6. **Routhian Mechanics** [6 points, 8.309 ONLY]

Another version of Classical Mechanics makes use of what is known as the Routhian R. The Routhian can be defined by starting with a Lagrangian L for n generalized coordinates and performing the following Legendre transformation:

$$R(q_1, \ldots, q_n, p_1, \ldots, p_s, \dot{q}_{s+1}, \ldots, \dot{q}_n, t) = \left(\sum_{k=1}^{s} p_k \dot{q}_k\right) - L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t),$$

where $1 < s < n$. Starting with this definition, derive equations of motion that are expressed in terms of partial derivatives of R for all of the variables. You should find that R is essentially part way between the Lagrangian and Hamiltonian formalisms.

[Aside: For reasons that should be evident from your solution, if you had a problem where you know the solution is easier in terms of H for some variables, but is easier in terms of L for other variables, then you might choose to use R instead of H or L.]

7.* **Extra Problem: Equivalent Lagrangians** [not for credit]

Some of you have seen this problem already in Classical Mechanics II. If it does not sound familiar then I suggest you try it. (It will not be graded but solutions will be provided, and we will make use of this fact later in the course.)

Let $L(q, \dot{q}, t)$ be the Lagrangian for a particle with coordinate q, which satisfies the Euler-Lagrange equations. Show that the Lagrangian

$$L' = L + \frac{dF(q, t)}{dt}$$

also satisfies the Euler-Lagrange equations where F is an arbitrary differentiable function.