Matrix Theory in a 2-Qubit Entangler

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Matrix Analysis

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Presentation layout

1. Quantum Entanglement
2. Matrix Theory
3. Simulation on IBM-Q
4. Recap

Quantum Bits - Qubits

Qubits:

$$|\psi\rangle = a|0\rangle + b|1\rangle,$$

where $|a|^2 + |b|^2 = 1$.

Measurement: Probabilistic

$$P(|\psi\rangle \to |0\rangle) = |a|^2 \quad P(|\psi\rangle \to |1\rangle) = |b|^2$$

Entanglement

When qubits “coordinate”:

Recipe

What do we need to entangle two qubits?

- Tensor products
- Hadamard gate
- CNOT gate
- Measure

Tensor Products

The *tensor product* of $\mathbf{V} = \mathbb{C}^{\Sigma_1}$ and $\mathbf{W} = \mathbb{C}^{\Sigma_2}$ is

$$\mathbf{V} \otimes \mathbf{W} = \mathbb{C}^{\Sigma_1 \times \Sigma_2}.$$

Elementary tensors span $\mathbf{V} \otimes \mathbf{W}$. For $|v\rangle \in \mathbf{V}$ and $|w\rangle \in \mathbf{W}$,

$$|v\rangle \otimes |w\rangle \equiv |v\rangle \, |w\rangle \equiv |vw\rangle \in \mathbf{V} \otimes \mathbf{W}.$$

Example: Representing the classical number "1" with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Tensor Products (cont.)

span$(|00\rangle, |01\rangle, |10\rangle, |11\rangle) = \mathbf{V} \otimes \mathbf{W}$, where

$$|00\rangle = \begin{bmatrix}1 & 0 & 0 & 0\end{bmatrix}^\top, |10\rangle = \begin{bmatrix}0 & 0 & 1 & 0\end{bmatrix}^\top, |11\rangle = \begin{bmatrix}0 & 0 & 0 & 1\end{bmatrix}^\top.$$

A *generic state*: For $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$,

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.$$

Not every $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|c\rangle, |d\rangle$ such that

$$|c\rangle \otimes |d\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}^\top \rightarrow \textbf{Entangled}.$$

Tensor Products (cont.)

Bilinearity:

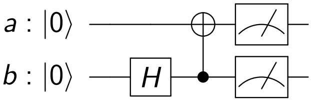
$$|a\rangle \otimes (\alpha\,|v\rangle + \beta\,|w\rangle) = \alpha\,|av\rangle + \beta\,|aw\rangle$$
$$(\alpha\,|v\rangle + \beta\,|w\rangle) \otimes |b\rangle = \alpha\,|vb\rangle + \beta\,|wb\rangle$$

Of operators: $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$, $\mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}\,|v\rangle) \otimes (\mathcal{B}\,|w\rangle).$$

But not all $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ can be written as $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$ $\rightarrow$ **Entangled**.

Example: Entanglement

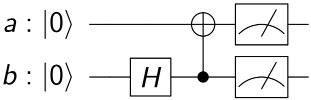


$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$\rightarrow$ Unitary Operations

Example: Entanglement (cont.)



$$C_b(I \otimes H)\left(\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \begin{bmatrix}1\\0\end{bmatrix}_b\right) = C_b\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_b\right)$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle \rightarrow \textbf{Entangled}$$

Tensor Products (cont.)

Other properties:

- Associative
- Distributive
- Not commutative
- $(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$.
- $\mathrm{Tr}(\mathcal{A} \otimes \mathcal{B}) = \mathrm{Tr}(\mathcal{A}) \cdot \mathrm{Tr}(\mathcal{B})$.
- $\det(\mathcal{A} \otimes \mathcal{B}) = (\det(\mathcal{A}))^m \cdot \det(\mathcal{B})^n$, where m is the dimension of $\mathcal{A}$ and n of $\mathcal{B}$.

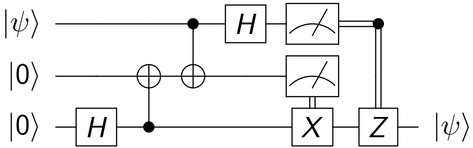
Unitary Operations

- Quantum Fourier Transform
- Control-phase
-

Unitary Operations: QFT

Simulation on IBM-Q

A sample quantum circuit.



Recap

What did we learn on the show tonight, Craig?
Q-circuit user guide [EF04]
quantum addition of classical numbers [CC16]
Mike and Ike [NC02]
Handbook of Linear Algebra [Hog07]
addition on quantum computer [Dra00]
QFT quick math [Bac]
Matrix analysis (where I read about unitary matrices) [HJ90]

References

- Dave Bacon, *The quantum fourier transform and jordan's algorithm*.
- AV Cherkas and SA Chivilikhin, *Quantum adder of classical numbers*, Journal of Physics: Conference Series, vol. 735, IOP Publishing, 2016, p. 012083.
- Thomas G Draper, *Addition on a quantum computer.*, arXiv preprint quant-ph/0008033 (2000).
- Bryan Eastin and Steven T Flammia, *Q-circuit tutorial*, arXiv preprint quant-ph/0406003 (2004).
- Roger A. Horn and Charles R. Johnson, *Matrix analysis*, Cambridge university press, 1990.
- Leslie Hogben, *Handbook of linear algebra*, Chapman and Hall/CRC, 2007.
- Michael A Nielsen and Isaac Chuang, *Quantum computation and quantum information*, 2002.