Matrix Theory in a 2-Qubit Entangler

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Matrix Analysis

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Presentation layout

Qubits & Quantum Gates

Qubit: A quantum system with two measurable physical states $|0\rangle$ and $|1\rangle$,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Before measurement,

$$|\psi\rangle = a\,|0\rangle + b\,|1\rangle \in \mathbb{C}^2, \quad |a|^2 + |b|^2 = 1.$$

Physically,

$$P(|\psi\rangle \to |0\rangle) = |a|^2 \quad P(|\psi\rangle \to |1\rangle) = |b|^2.$$

Quantum gate: a unitary transformation on $|\psi\rangle$.

Qubits & Quantum Gates

Multiple Qubits: States of k qubits is a vector in $\otimes^k \mathbb{C}^2$ with basis vectors

$$|x_1 \dots x_k\rangle = |x_1\rangle \otimes \cdots \otimes |x_k\rangle\,, \qquad x_i \in \{0,1\}.$$

"$\otimes$": Kronecker product. If $\mathcal{A} \in \mathbb{M}_{m\times n}$ and $\mathcal{B} \in \mathbb{M}_{p\times q}$, then

$$\mathcal{A} \otimes \mathcal{B} = \begin{bmatrix} a_{11}\mathcal{B} & \dots & a_{1n}\mathcal{B} \\ \vdots & \ddots & \vdots \\ a_{m1}\mathcal{B} & \dots & a_{mn}\mathcal{B} \end{bmatrix}.$$

Kronecker Products

Example: Representing the classical numbers "1" and "0" with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$0_2 \equiv |00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top, |11\rangle = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

In fact, $|00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis for $\otimes^2 \mathbb{C}^2$, the 2-qubit system.

Kronecker Products

Doesn't care where scalar goes...

$$(\alpha\mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\alpha\mathcal{B}) = \alpha(\mathcal{A} \otimes \mathcal{B})$$

Associative:

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$$

Left-distributive:

$$\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = \mathcal{A} \otimes \mathcal{B} + \mathcal{A} \otimes \mathcal{C}$$

Right-distributive:

$$(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes \mathcal{B} + \mathcal{B} \otimes \mathcal{C}$$

Tensor Products

hello

Quantum Gates

Quantum Gates: Represented by unitary matrices $\rightarrow$ Reversible. Act on spaces of one or many qubits. Example:

$$Hadamard : H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad CNOT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Measurements: Irreversible $\rightarrow$ Not quantum gates.

Multi-qubit systems

- Representing a multi-qubit state as many single-qubit states?
- Representing a multi-qubit gate as many single-qubit gates?

Recipe

What do we need to entangle two qubits?

- Tensor products
- Hadamard gate
- CNOT gate
- Measure

Tensor Products

The *tensor product* of $\mathbf{V} = \mathbb{C}^{\Sigma_1}$ and $\mathbf{W} = \mathbb{C}^{\Sigma_2}$ is

$$\mathbf{V} \otimes \mathbf{W} = \mathbb{C}^{\Sigma_1 \times \Sigma_2}.$$

Elementary tensors span $\mathbf{V} \otimes \mathbf{W}$. For $|v\rangle \in \mathbf{V}$ and $|w\rangle \in \mathbf{W}$,

$$|v\rangle \otimes |w\rangle \equiv |v\rangle\,|w\rangle \equiv |vw\rangle \in \mathbf{V} \otimes \mathbf{W}.$$

<u>Example</u>: Representing the classical number "1" with two qubits:

$$1_2 \equiv |01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Tensor Products (cont.)

span$(|00\rangle, |01\rangle, |10\rangle, |11\rangle) = \mathbf{V} \otimes \mathbf{W}$, where

$$|00\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top, |10\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top, |11\rangle = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

Linear independence $\rightarrow$ $(|00\rangle, |01\rangle, |10\rangle, |11\rangle)$ form a computational basis.

A *generic state*: For $|a_{00}|^2 + |a_{01}|^2 + |a_{10}|^2 + |a_{11}|^2 = 1$,

$$|\psi\rangle = a_{00}|00\rangle + a_{01}|01\rangle + a_{10}|10\rangle + a_{11}|11\rangle.$$

Tensor Products (cont.)

Not every $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|c\rangle, |d\rangle \in \mathbb{C}^2$ such that

$$\begin{aligned} |c\rangle \otimes |d\rangle = |\beta_{00}\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \\ &= \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \end{aligned}$$

Examples: Bell states

$$|\beta_{10}\rangle = \frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle$$
$$|\beta_{01}\rangle = \frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle$$
$$|\beta_{11}\rangle = \frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle$$

Tensor Products (cont.)

For operators: $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$, $\mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}\,|v\rangle) \otimes (\mathcal{B}\,|w\rangle).$$

Not all $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ can be written as $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$.

Example:

$$CNOT_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$SWAP \neq S_1 \otimes S_2$

Consider the 2-qubit $SWAP$ map:

$$SWAP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W}).$$

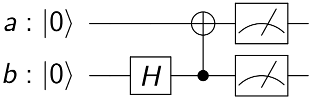
Observe:

$$SWAP(|0\rangle \otimes |1\rangle) = |1\rangle \otimes |0\rangle .$$

Suppose for $S_1 \in \mathcal{L}(\mathbf{V}), S_2 \in \mathcal{L}(\mathbf{W})$

$$SWAP = S_1 \otimes S_2$$

Example: 2-Qubit Entanglement Circuit



$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{cases} |00\rangle \to |00\rangle \\ |10\rangle \to |10\rangle \\ |01\rangle \to |11\rangle \\ |11\rangle \to |01\rangle \end{cases}$$

Example: Entanglement (cont.)

Notice:

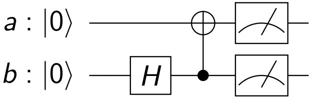
$$(I\,|0\rangle) \otimes (H_b\,|0\rangle) = (I \otimes H_b)(|0\rangle \otimes |0\rangle)$$

$$\begin{bmatrix}1\\0\end{bmatrix} \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} & \mathcal{O}\\ \mathcal{O} & \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}^\top = \begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0\end{bmatrix}^\top$$

$\rightarrow$ Possible to write H as $I \otimes H_b$. Not possible for $CNOT_b$.

Example: Entanglement (cont.)



$$C_b(I \otimes H)\left(\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \begin{bmatrix}1\\0\end{bmatrix}_b\right) = C_b\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_b\right)$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |1\rangle$$

$\rightarrow$ **Entangled**

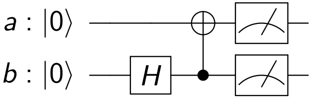
Tensor Products (cont.)

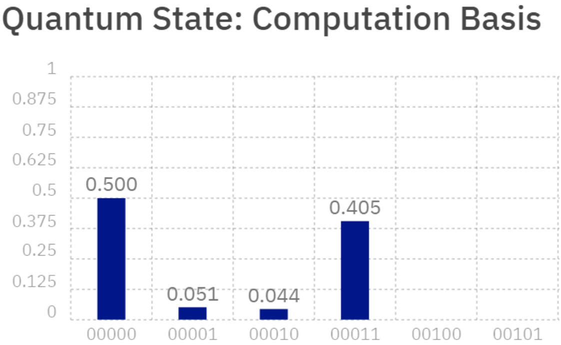
Other properties:

- Bilinear: linear in both arguments.
- Associative
- Distributive
- Not commutative
- $(\mathcal{A} \otimes \mathcal{B})^\dagger = \mathcal{A}^\dagger \otimes \mathcal{B}^\dagger$.
- $\text{Tr}(\mathcal{A} \otimes \mathcal{B}) = \text{Tr}(\mathcal{A}) \cdot \text{Tr}(\mathcal{B})$.
- $\det(\mathcal{A} \otimes \mathcal{B}) = (\det(\mathcal{A}))^m \cdot \det(\mathcal{B})^n$, where $m = \text{size}(\mathcal{A}), n = \text{size}(\mathcal{B})$.

Simulation on IBM-Q

Entanglement circuit, revisited





Recap

- Representing multi-qubit systems with tensor products.
- Representing multi-qubit operators with tensor products.
- Not all multi-qubit states are a tensor product of 1-qubit states.
- Not all multi-qubit gates are a tensor product of 1-qubit gates.
- Entanglement on IBM-Q.

References

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