The fluctuation-dissipation theorem

The fluctuation-dissipation theorem says that when there is a process that dissipates energy, turning it into heat (e.g. friction), there is a reverse process related to thermal fluctuations.

A general relation between linear response (susceptibility) and a pair correlation function (dynamical structure factor)

General proof (Callen, Welton)
Examples:
Brownian random walks, Einstein theory
Johnson-Nyquist noise, classical and quantum

Weak probes and Kubo susceptibility (reminder)

- The magic of the linear response: probing the system in its ground state through a non-equilibrium process
- The notion of linear susceptibility χ_{ji}: response of an observable O_j to the perturbation O_i
- System Hamiltonian $\mathcal{H} = \mathcal{H}_0 + O_i f_i(t)$; Susceptibility

$$\chi_{ji} = \frac{\text{"response"}}{\text{"force"}\ f_i}, \ \ \langle O_j(t)\rangle = \int_{-\infty}^{\infty} dt' \chi_{ji}(t-t') f_i(t')$$

- Relate χ_{ji} to the properties of the system in equilibrium:
$\chi_{ji}(t-t') = \frac{i}{\hbar}\Theta(t-t')\langle G|[O_j(t), O_i(t')]|G\rangle$
- At $T > 0$ replace $\langle G|...|G\rangle \rightarrow \frac{1}{Z}\sum_\alpha e^{-\beta E_\alpha}\langle\alpha|...|\alpha\rangle$

Examples (reminder)

Perturb system by driving it out of equilibrium, then measure an observable O_i. E.g. particle density (compressibility), current (conductivity), or magnetization (χ_{spin}):

System Hamiltonian with a perturbation describing a weak probe.

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}'(t), \quad \mathcal{H}'(t) = \sum_j O_j A_j(t)$$

E.g. for O_i being particle density:

$$\mathcal{H}' = \int \hat{\rho}(x,t) U(x,t) d^3x$$

with $\hat{\rho}(x,t) = \sum_i \delta(x - x_i(t))$ in 1st quantization and $\hat{\rho}(x,t) = \psi^\dagger(x)\psi(x)$ in 2nd quantization. Or, a magnetic coupling

$$\mathcal{H}' = -\int \hat{m}_z(x,t) H_z(x,t) d^3x, \quad \hat{m}_z = \mu(\hat{\rho}_\uparrow - \hat{\rho}_\downarrow)$$

Or, electric current coupled to the EM vector potential

$$\mathcal{H}' = -\int \frac{1}{c}\mathrm{j}(x,t)\mathrm{A}(x,t) d^3x, \quad \mathrm{E} = -\frac{1}{c}\partial \mathrm{A}/\partial t$$

Linear response theory: derive the Kubo formula

A system driven out of equilibrium, $H = H_0 + H'$, $H' = \sum_i O_i A_i(t)$,

$$\langle O_j(t)\rangle_{n.e.} = \langle O_j(t)\rangle + \int d\tau \chi_{ji}(t-\tau) A_i((\tau) + ... \qquad (2)$$

Steps to derive the response function χ_{ji}: 1) Express χ_{ji} through the thermal equilibrium state in distant past $A_j(\tau) \to 0$ as $\tau \to -\infty$:

$$\langle O_j(t)\rangle_{n.e.} = Z_0^{-1} \sum_\alpha e^{-\beta\epsilon_\alpha} \langle\alpha| U^\dagger(-\infty, t) O_j U(-\infty, t)|\alpha\rangle, \quad Z_0 = \mathrm{Tr} e^{-\beta H_0}$$

2) Express the evolution operator as time-ordered power series $U(t_0, t) = U_0(t-t_0) \times \mathrm{Texp}\left(-i\int_{t_0}^t dt' H_I'(t')\right)$. Here we are working in the "interaction representation" $H_I' = U_0^\dagger(t-t_0) H' U_0(t-t_0)$, $U_0(t-t_0) = \exp(-iH_0 t/\hbar)$. (the choice of the initial time t_0 is completely arbitrary)

3) Next, expand in H_I' as $U(t_0, t) = U_0(t-t_0)\left[1 - i\int_{t_0}^t dt' H_I'(t') + ...\right]$.

At 1st order in H_I' this gives (a result independent of the time t_0)

$$\chi_{ji}(t-t') = \Theta(t-t') \frac{i}{\hbar} \langle [O_j(t), O_i(t')]\rangle \qquad (3)$$

The Heaviside function is a direct consequence of causality – that is, an applied field can impact the future dynamics but not the past dynamics.

Fourier representation of susceptibility (reminder)

Since the unperturbed Hamiltonian is time-independent it is clear that the linear response is diagonal in frequency. Namely, if the system is perturbed at a frequency ω, the linear response will be at frequency ω as well:

$$\langle O_j(\omega)\rangle = \sum_i \chi_{ji}(\omega) A_i(\omega)$$

where $\langle O_j(\omega)\rangle$, $\chi_{ji}(\omega)$ and $A_i(\omega)$ are the Fourier transforms of $\langle O_j(t)\rangle$, $\chi_{ji}(t)$ and $A_i(t)$:

$$A_i(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} A_i(t)dt, \quad \chi_{ji}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t}\chi_{ji}(t)dt$$

☺**:** Mathematically speaking, this is perfectly natural since under FT a convolution in Eq.(2) turns into a product.
☺**:** We will use both positive and negative frequencies!
☺**:** Since $f(\omega) = f^*(-\omega)$ for FT of a real-valued function, the real and imaginary parts of $\chi_{ji}(\omega) = \chi'_{ji}(\omega) + i\chi''_{ji}(\omega)$ are even and odd in ω, respectively.

The arrow of time and Fourier transform (reminder)

Because of causality, $\chi(\tau < 0) = 0$ in Eq.(2). The Fourier transform

$$\chi_{ij}(z) = \int_{-\infty}^{\infty} dt e^{izt} \chi_{ij}(t) = \int_{0}^{\infty} dt e^{izt} \chi_{ij}(t)$$

is therefore analytic in the upper half plane of complex frequency, $\mathrm{Im}\, z > 0$. This analyticity property is a nontrivial, but extremely useful mathematical consequence of the arrow of time.

Many constraints on the ω dependence, both the obvious ones and the surprising ones.

To illustrate the connection between causality and the analytic properties under Fourier transform consider $\chi(t) = \Theta(t) A e^{-\gamma t}$. This is a memory function with the memory loss rate $\gamma > 0$.

In this case we have $O_j(t) = \int_{-\infty}^{t} dt' A e^{-\gamma(t-t')} f_i(t')$. The Fourier transform

$$\chi(z) = \int_{0}^{\infty} dt A e^{izt - \gamma t} = \frac{A}{\gamma - iz}. \tag{4}$$

This expression has a pole at $z = -i\gamma$ in the lower halfplane $\mathrm{Im} z < 0$, and is analytic at $\mathrm{Im} z > 0$.

Analytic functions? See excellent 18.04 notes, or a summary at the end

Sanity check: harmonic oscillator susceptibility (reminder)

Hamiltonian $H = \frac{p^2}{2m} + \frac{m\omega_0^2}{2}x^2 - exE(t)$. Find Kubo polarizability?
Dipole moment $d = \langle O_j \rangle = ex$, the "force" E couples to $O_i = -d$.
Dynamic polarizabity, defined as $\langle d(t) \rangle = \int_{-\infty}^{t} \chi(t-t')E(t')dt'$, equals

$$\chi_{Kubo}(t-t') = -\frac{i}{\hbar}\langle G|[d(t), d(t')]|G\rangle$$

Quantum harmonic oscillator evolution is identical to the classical one.
Therefore $x(t) = x(t')\cos\omega_0(t-t') + \frac{p(t')}{m\omega_0}\sin\omega_0(t-t')$. Plugging it in the Kubo formula and combining with the equal-time commutators $[x(t'), x(t')] = 0$, $[x(t'), p(t')] = i\hbar$ gives a result (!!!) identical to the classical oscillator polarizability response

$$\chi_{Kubo}(t-t') = -\frac{ie^2}{\hbar} i\hbar \frac{1}{m\omega_0}\sin\omega_0(t-t') = \frac{e^2}{m\omega_0}\sin\omega_0(t-t')$$

Fourier transform (infinitesimal damping η added to control convergence)

$$\chi(\omega) = \int_0^\infty dt e^{i\omega t - \eta t}\chi(t) = \frac{e^2}{2im\omega_0}\left(\frac{1}{\eta - i(\omega+\omega_0)} - \frac{1}{\eta - i(\omega-\omega_0)}\right)$$

$$= \frac{e^2}{m(\omega_0^2 - (\omega+i\eta)^2)}, \quad \text{complex poles}: \omega_{1,2} = \pm\omega_0 - i\eta, \ \mathrm{Im}\,\omega_{1,2} < 0$$

The poles $\omega_{1,2}$ reside in the lower halfplane of complex ω. This agrees

Express $\chi_{ji}(\omega)$ through microscopic quantities (reminder)

Use the eigenstates of H_0, $\epsilon_\alpha|\alpha\rangle = H_0|\alpha\rangle$, and identity decomposition $1 = \sum_\alpha |\alpha\rangle\langle\alpha|$ to bring $\chi_{ji}(\omega) = \int dt e^{i\omega t} \frac{i}{\hbar}\langle[O_j(t), O_i(0)]\rangle$ to the form

$$\chi_{ji}(\omega) = \frac{i}{\hbar Z_0}\sum_{\alpha\beta} e^{-\beta\epsilon_\alpha}\langle\alpha|O_j|\beta\rangle\langle\beta|O_i|\alpha\rangle \int_0^\infty e^{i(\epsilon_\alpha-\epsilon_\beta)t}e^{i\omega t}e^{-\delta t}dt$$

$$-\frac{i}{\hbar Z_0}\sum_{\alpha\beta} e^{-\beta\epsilon_\beta}\langle\beta|O_j|\alpha\rangle\langle\alpha|O_i|\beta\rangle \int_0^\infty e^{i(\epsilon_\alpha-\epsilon_\beta)t}e^{i\omega t}e^{-\delta t}dt$$

$$= \frac{1}{Z_0}\sum_{\alpha\beta}\langle\beta|O_j|\alpha\rangle\langle\alpha|O_i|\beta\rangle \frac{e^{-\beta\epsilon_\beta} - e^{-\beta\epsilon_\alpha}}{\omega - (\epsilon_\beta - \epsilon_\alpha) + i\delta} \tag{5}$$

We swapped α and β in 2nd term, and added a factor $e^{-\delta t}$ to assure convergence.
This result is completely general (no approximations made!). We will use it later to derive the fluctuation-dissipation theorem and sum rules.

Eq.(5) is an explicit expression that would be useful if we knew the many-body eigenstates and the respective matrix elements. Usually we don't, so we will seek other ways to evaluate the response functions.

Symmetry Properties (reminder)

Since O_j are Hermitian operators, it follows that (check!)

$$\chi_{ji}(\omega) = -\chi_{ij}(-\omega) = -[\chi_{ji}(-\omega)]^* = [\chi_{ij}(\omega)]^*$$

is a Hermitian matrix. Decomposing into a sum of the real and imaginary parts

$$\chi_{ji}(\omega) = \chi'_{ji}(\omega) + i\chi''_{ji}(\omega)$$

and setting $j = i$ we see that $\chi''_{jj}(\omega)$ is real and an odd function of ω. Likewise, $\chi'_{jj}(\omega)$ is a real, even function of ω.
Other symmetry properties of χ_{ji} can be derived from symmetries of H. For instance, if H is time-reversal invariant, and if $f_i(t) \to \epsilon_i f_i(-t)$ under time-reversal, then it is easy to see that

$$\chi_{ji}(\omega) = -\epsilon_i\epsilon_j\chi_{ij}(-\omega) = \epsilon_i\epsilon_j\chi_{ji}(\omega)$$

The reason we need to include the "signature," ϵ_i, is that some operators that we are interested in, such as a position or an electric potential, are even under time-reversal, $\epsilon_i = 1$, while others, such as the current or the magnetic field, are odd, $\epsilon_i = -1$.

Identification of $\chi''_{ij}(\omega)$ with dissipation (reminder)

Consider the rate at which power is absorbed from a generic external field

$$P(t) = \frac{d\langle H\rangle}{dt} = \sum_j \frac{\partial\langle H\rangle}{\partial A_j}\dot{A}_j = \sum_j \langle O_j(t)\rangle_{n.e.}\dot{A}_j$$

$$= \int d\tau \chi_{ji}(\tau) A_i(t-\tau)\dot{A}_j(t) = \iint \frac{d\omega d\nu}{(2\pi)^2} e^{i(\omega-\nu)t}\chi_{ji}(\omega)A_i(\omega)i\nu A_j(-\nu)$$

Typically, we are not interested in the rapidly oscillating pieces of P but only in its time average. We thus integrate over t [to enforce approximate δ-function in frequency through $\int dt e^{i(\omega-\nu)t} = 2\pi\delta(\omega-\nu)$]:

$$\int dt P(t) = \int \frac{d\nu}{2\pi}\chi_{ji}(\nu)A_i(\nu)i\nu A_j(-\nu)$$

Since $A_j(t)$ is real, $A_j(-\omega) = A_j^*(\omega)$. Using $\chi_{ji}(-\nu) = \chi_{ji}^*(\nu)$ we have

$$\int dt P(t) = \int \frac{d\nu}{2\pi}\nu\chi''_{ji}(\nu)A_i(\nu)A_j^*(\nu)$$

Since dissipated power is always positive (2nd law of thermodynamics), it follows that $\nu\chi''_{ii}(\nu) \geq 0$. (Which agrees with the symmetry properties, see above)

This result checks with the harmonic oscillator response found above:

$\chi''(\omega) \sim \delta(\omega-\omega_0) - \delta(\omega+\omega_0))$ (odd in ω and positive for $\omega > 0$).

The Fluctuation-Dissipation Theorem

Equilibrium fluctuations are defined as a correlation function

$$S_{ij}(t) = \langle (O_i(t) - \langle O_i \rangle)(O_j(0) - \langle O_j \rangle) \rangle$$

In solids, spatial and temporal correlations (of fluctuations) of particle density, magnetization, etc, are measured in the scattering experiments. Or, by noise measurements of current fluctuations in electric circuits, etc.

Relate S to χ? It follows from the cyclic property of the trace that

$$S_{ij}(t) = \sum_{\alpha\gamma} e^{-\beta\epsilon_\alpha} \langle \alpha | \delta O_i(t) | \gamma \rangle \langle \gamma | \delta O_j(0) | \alpha \rangle = \sum_{\alpha\gamma} e^{-\beta\epsilon_\alpha + it(\epsilon_\alpha - \epsilon_\gamma)}$$

$$\times \langle \alpha | \delta O_i(0) | \gamma \rangle \langle \gamma | \delta O_j(0) | \alpha \rangle = [\alpha \leftrightarrow \gamma, i \leftrightarrow j] = S_{ji}(-t - i\beta).$$

Therefore, the Fourier transform $S(\omega) = \int_{-\infty}^{\infty} S(t) e^{i\omega t} dt$ obeys an interesting identity $S_{ij}(\omega) = S_{ji}(-\omega) e^{\beta\omega}$ known as the detailed balance relation.

We can express $S_{ij}(\omega)$ through the many-body eigenstates of H_0 as

$$S_{ji}(\omega) = \int dt \tfrac{1}{Z_0} \textstyle\sum_{\alpha\beta} e^{-\beta\epsilon_\alpha} \langle \alpha | O_j | \beta \rangle \langle \beta | O_i | \alpha \rangle e^{i(\omega - \epsilon_\beta + \epsilon_\alpha)t}$$

$$= \frac{1}{Z_0} \sum_{\alpha\beta} e^{-\beta\epsilon_\alpha} \langle \alpha | O_j | \beta \rangle \langle \beta | O_i | \alpha \rangle 2\pi \delta(\omega - \epsilon_\beta + \epsilon_\alpha). \qquad (6)$$

Comparing Eqs. (6) and (5), we find an amazing fluctuation-dissipation relation

$$\chi''_{ji}(\omega) = S_{ji}(\omega) \frac{1}{2\hbar} (1 - e^{-\beta\hbar\omega})$$

Fluctuation-Dissipation Theorem (discussion)

A relation between two quantities that have quite different meaning:
* χ'' is the rate of dissipation of the nonequilibrium state after system is driven out of equilibrium by $A(t)$
* S describes fluctuations in pristine equilibrium state

The equivalence of χ'' and S permits us to predict a large host of non-equilibrium phenomena based exclusively on the knowledge of the equilibrium ensemble. Indeed, since we have identified $\chi''(\omega)$ with dissipation, we can think of it as being some measure of the "density of states" for excitations with energy $\hbar\omega$ (aka the spectral function).
In this way we can obtain an intuitive understanding of the equilibrium fluctuations of the system as related to the thermal occupation of a set of harmonic oscillator modes according to

$$S_{ij}(\omega) = 2\left\{\theta(\omega)\left[1+\bar{n}(\omega)\right] - \theta(-\omega)\bar{n}(|\omega|)\right\}\chi''_{ij}(\omega)$$

where $\bar{n}(\omega) = 1/(e^{\beta\omega}-1)$ is the Bose occupancy factor with chemical potential $\mu = 0$ describes "spontaneous" and "stimulated" processes.
The limit $\beta\omega \ll 1$ gives the classical version of the fluctuation-dissipation theorem,

$$S_{ij}(\omega) = \left[\frac{2k_B T}{\omega}\right]\chi''_{ij}(\omega).$$

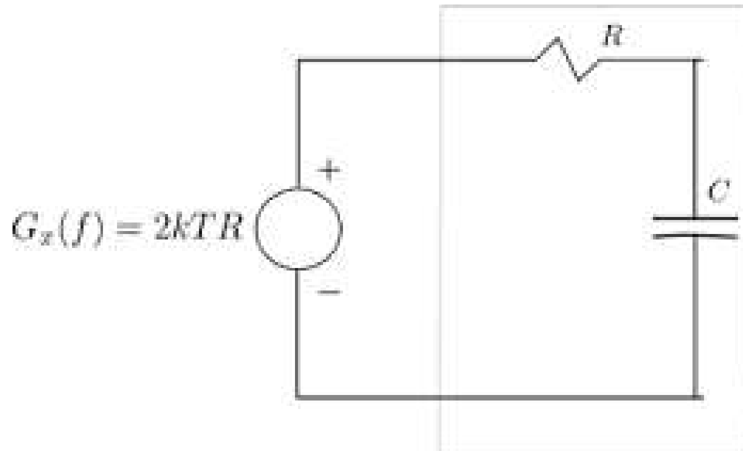
Fluctuation-Dissipation Theorem: two examples

The fluctuation-dissipation theorem says that when there is a process that dissipates energy, turning it into heat (e.g. friction), there is a reverse process related to thermal fluctuations.
One textbook example is the Brownian random-walk motion. A particle that is being kicked and dragged.
Langevin dynamics: $\mathrm{v} = \mu f(x) + \delta \mathrm{v}(t)$, where μ is mobility, $f(x) = -\nabla U(x)$ is external force, and velocity fluctuations $\delta \mathrm{v}(t)$ define diffusion constant $D = \int_0^\infty \frac{1}{3} \langle \delta \mathrm{v}(t) \delta \mathrm{v}(0) \rangle dt$.
Brownian motion converts heat energy into kinetic energy—the reverse of drag. Due to fluctuation-dissipation theorem, the mobility μ and diffusivity D are related by $D = k_B T \mu$.
In other words, the fluctuation of the particle at rest has the same origin as the dissipative frictional force one must do work against, if one tries to perturb the system in a particular direction [Einstein (1905)].

The relation $D = k_B T \mu$ is found by demanding that Boltzmann distribution $e^{-\beta U(x)}$ is a steady state of the Fokker-Planck equation $\partial_t p = D \nabla^2 p - \nabla(\mu \mathrm{f} p)$.

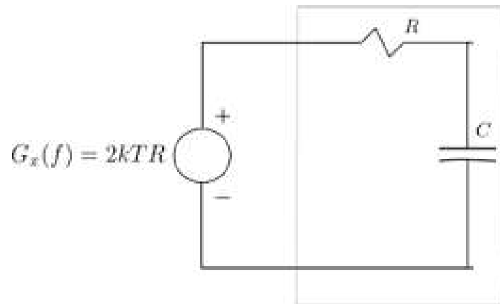
Fluctuation-Dissipation Theorem: two examples

Another example is the Johnson noise in an electrical resistance arising due to its inner thermal fluctuations.



The Kirchhoff-Langevin dynamics: $IR = V + \delta V(t)$, where V is external voltage and $\delta V(t)$ is Johnson noise intrinsic to the resistor.
Noise spectral power $\langle \delta V(\omega)\delta V(-\omega)\rangle_{\hbar\omega \ll k_B T} = 2Rk_B T$ (Nyquist, 1928).
E.g. an RC circuit in thermal equilibrium, $\frac{1}{C}Q(t) = -R\dot{Q}(t) + \delta V(t)$.
Fluctuations δV are due to a small and rapidly-fluctuating current caused by the thermal fluctuations of the electrons and atoms in the resistor.
Johnson noise converts heat energy into electrical energy—the reverse of power dissipation by a resistor.

A direct derivation of Johnson noise (Nyquist thm)



Consider an RC circuit made of a capacitor C and resistor R in series with a Johnson noise source $\delta V(t)$:

$$\frac{Q}{C} + IR = \delta V(t)$$

In Fourier harmonics, this reads $Q_\omega\left(\frac{1}{C} + i\omega R\right) = \delta V_\omega$. The spectrum of charge fluctuations on the capacitor

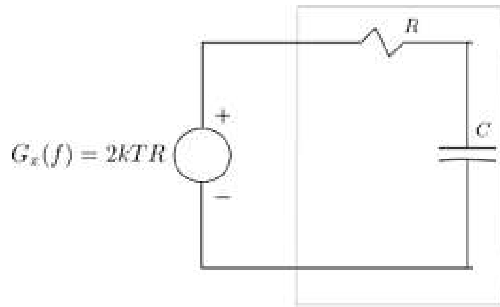
$$\langle Q_{-\omega} Q_\omega \rangle = \frac{\langle \delta V_{-\omega} \delta V_\omega \rangle}{\frac{1}{C^2} + \omega^2 R^2}$$

Demand that capacitor's thermal energy equals $k_B T$:

$$E_c = \left\langle \frac{Q^2}{2C} \right\rangle = \int \frac{d\omega}{2\pi} \frac{\langle \delta V_{-\omega} \delta V_\omega \rangle}{2C(\frac{1}{C^2} + \omega^2 R^2)} = \frac{1}{2R} \langle \delta V_{-\omega} \delta V_\omega \rangle = k_B T.$$

Which instantly gives $\langle V_{-\omega} V_\omega \rangle_{\hbar\omega \ll k_B T} = 2Rk_B T$. (Independent of C!)

Johnson noise and Fluctuation-Dissipation Theorem



"Canonical susceptibility" for a resistor $RI(t) = V(t)$?
Hamiltonian $H' = V(t)\Delta Q$, where $\Delta Q(t) = \int_{-\infty}^{t} I(t')dt'$ is the charge transfered through the resistor. Therefore, we want a relation of the form $\Delta Q_\omega = \chi(\omega)V_\omega$.
Comparing this to $R\Delta\dot{Q} = V$ gives $\Delta Q_\omega = \frac{1}{i\omega R}V_\omega$ and $\chi(\omega) = \frac{1}{i\omega R}$.
The FDT relation $S(\omega) = \hbar \coth \frac{\hbar\omega}{2T}\chi''(\omega)$ gives charge and current fluctuations

$$\langle \Delta Q_{-\omega}\Delta Q_\omega \rangle = \hbar \coth \frac{\hbar\omega}{2T}\frac{1}{\omega R}, \quad \langle I_{-\omega} I_\omega \rangle = \hbar\omega \coth \frac{\hbar\omega}{2T}\frac{1}{R}$$

Noisy resistor is usually modeled as a noiseless resistor in series with the Johnson voltage noise source $\delta V_\omega = RI_\omega$ with the spectrum

$$\langle \delta V_{-\omega}\delta V_\omega \rangle = \hbar\omega \coth \frac{\hbar\omega}{2T}R = 2\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right)R$$

Analogy with the Brownian random walks

Identify current with particle velocity, and capacitor charge — with particle displacement: $I \to v$, $-Q \to x$. Then $\frac{Q}{C} + IR = \delta V(t)$ turns into an equation for a Brownian particle moving due to thermal kicks and being dragged by an external force of a harmonic oscillator

$$v = \mu f(x) + \delta v, \quad \delta v = \frac{\delta V}{R}, \quad \mu = \frac{1}{R}, \quad f(x) = -\partial_x \frac{x^2}{2C} = -\frac{x}{C}$$

with the drag coefficient $\mu = \frac{1}{R}$.
This can be described by a diffusion equation for the probability distribution $p(x,t)$ of particle displacement (aka the Fokker-Planck equation): $\partial_t p(x,t) = D\partial_x^2 p(x,t) - \partial_x(\mu f(x) p(x,t))$. Derivation: $\partial_t p(x,t) = -\partial_x j$, $j = -D\partial_x p(x,t) + \mu f(x) p(x,t)$.
Now, demand that the equilibrium distribution $p(x) \sim e^{-\beta U(x)}$, $U(x) = \frac{x^2}{2C}$ describes the steady state. [Independent of $\mu = 1/R$!]
This predicts a relation between particle diffusivity and drag coefficient

$$D = k_B T \mu.$$

Combining it with the Einstein formula for particle diffusivity

$D = \int_0^\infty dt \langle \delta v(t) \delta v(0) \rangle = \frac{1}{2} \langle \delta v_{-\omega} \delta v_\omega \rangle$ gives the particle velocity noise spectrum $\langle \delta v_{-\omega} \delta v_\omega \rangle = 2\mu k_B T$. Going back to the electric quantities $\delta v = \delta I = \frac{1}{R}\delta V$, $\mu = \frac{1}{R}$ yields the Johnson noise spectrum $2Rk_B T$.

Einstein theory and Fluctuation-Dissipation Thm

Since the external force f couples to particle displacement, $\mathcal{H}' = -\mathrm{f}\mathrm{x}(t)$, the appropriate susceptibility is $\mathrm{x}_\omega = \chi(\omega)\mathrm{f}_\omega$, where we introduced Fourier harmonics $\mathrm{x}_\omega = \int dte^{i\omega t}\mathrm{x}(t)$, $\mathrm{f}_\omega = \int dte^{i\omega t}\mathrm{f}(t)$, etc. Comparing this to the equation of motion $\mathrm{v} = \mu\mathrm{f}$ gives

$$\chi(\omega) = \frac{i\mu}{\omega}.$$

The FDT relation (taken in the classical limit) is $S(\omega) = \frac{2k_B T}{\omega}\chi''(\omega)$, which predicts the displacement fluctuations spectral density

$$\langle \delta \mathrm{x}_\omega \delta \mathrm{x}_{-\omega} \rangle = \frac{2k_B T}{\omega^2}\mu.$$

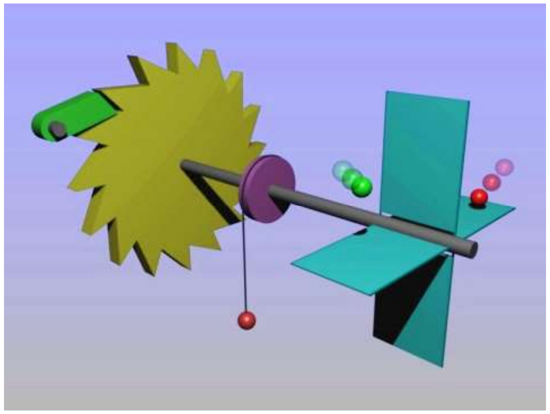
Multiplying both sides by ω and using the relation $\mathrm{v}_\omega = -i\omega \mathrm{x}_\omega$ gives velocity fluctuation spectrum

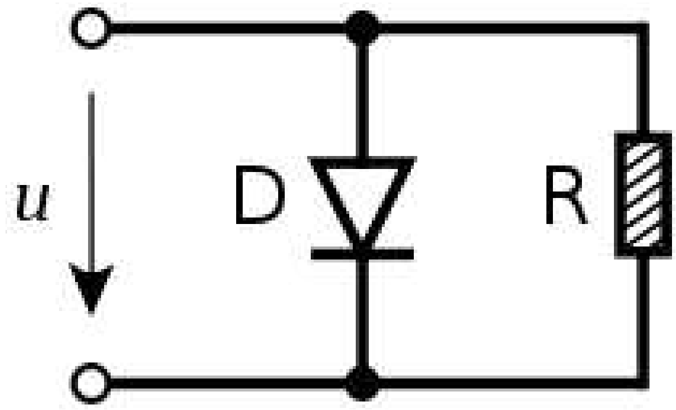
$$\langle \delta v_{x,\omega} \delta v_{x,-\omega} \rangle = 2k_B T\mu,$$

where $v_{x,\omega}$ are Fourier harmonics of the particle velocity x component, $v_{x,\omega} = \int dte^{i\omega t}v_x(t)$. Einstein relation then follows after expressing the diffusion constant through $\langle \delta v_{x,\omega} \delta v_{x,-\omega} \rangle$ in the limit of small ω.

Feynman's ratchet and pawl paradox

Can thermal bath do work?





Thermal noise can convert heat into mechanical work provided that the system is not in thermal equilibrium with the environment. Here temperatures of two susbsystems (ratchet and pawl) have to be imbalanced. No mechanical work can be produced if the ratchet and pawl temperatures are equal. A simplest heat engine, efficiency $\sim \Delta T$. The spinning direction is clockwise if $\Delta T > 0$ and counterclockwise if $\Delta T < 0$. This topic, in part because of the insightful discussion in Feynman lectures, has triggered a lot of theor and exper activity.

Symmetrized and non-symmetrized fluctuations

For symmetrized correlation function $\tilde{S}_{ij}(t) = \frac{1}{2}\langle O_i(t)O_j(0) + O_j(0)O_i(t)\rangle$ the Fluctuation-Dissipation theorem reads:

$$\tilde{S}_{ij}(\omega) = \frac{1}{2} i\hbar\left(\chi^*_{ji} - \chi_{ij}\right) \coth\frac{\hbar\omega}{2T}$$

For a single variable $O_i(t) = x(t)$ this gives

$$\langle x(\omega)x(-\omega)\rangle = \hbar\chi''(\omega) \coth\frac{\hbar\omega}{2T}.$$

The Johnson-Nyquist noise spectral function (symmetrized)

$$\langle \delta V(\omega)\delta V(-\omega)\rangle = \hbar\omega R \coth\frac{\hbar\omega}{2T}$$

at low frequency $\omega \to 0$ giving the textbook result $\langle \delta V^2\rangle = 2RT$

Both symmetrized and non-symmetrized fluctuations are measurable (yet by very different techniques). Which one is more 'natural', i.e. more easily measurable? It depends: e.g. scattering experiments access non-symmetrized fluctuations, whereas for Brownian motion or Johnson noise it is the symmetrized fluctuations that are being measured.

Johnson Noise Thermometry

JNT is a well established experimental technique that finds wide applications in temperature scale metrology and in the development of reliable thermometers for harsh environments.
Also, there've been much interest in the extensions of JNT and the Fluctuation-Dissipation Theorem to out-of-equilibrium systems. Here is one example from the literature.
Low-frequency current noise in a voltage-biased tunnel junction

$$S_I = \langle \delta I^2 \rangle = R^{-1}\left[T_n + \frac{eV}{2k_B}\coth\left(\frac{eV}{2k_B T}\right)\right]$$

can be used for high-precision temperature measurement [Spietz et al. Science 300, 1929 (2003)]

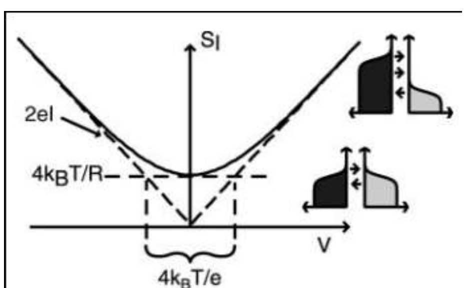


Fig. 1. Theoretical plot of current spectral density of a tunnel junction (Eq. 3) as a function of dc bias voltage. The diagonal dashed lines indicate the shot noise limit, and the horizontal dashed line indicates the Johnson noise limit. The voltage span of the intersection of these limits is $4k_BT/e$ and is indicated by vertical dashed lines. The bottom inset depicts the occupancies of the states in the electrodes in the equilibrium case, and the top inset depicts the out-of-equilibrium case where $eV \gg k_BT$.

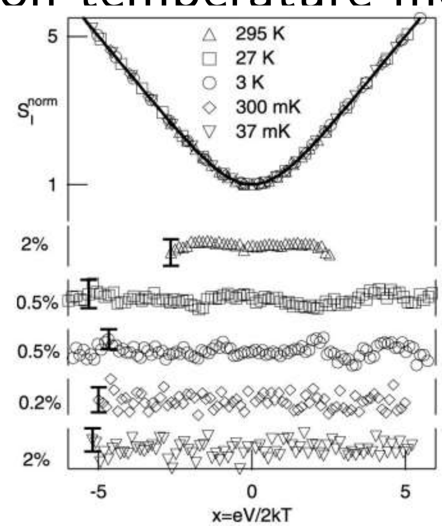


Fig. 3. Normalized junction noise plotted versus normalized voltage at various temperatures. Noise power is normalized to the zero bias (Johnson) noise, and bias voltage is scaled relative to temperature. In these units, the data follow the universal function xcoth(x), depicted by the solid line. The residuals have the indicated fractional standard deviations and are shown below. This plot shows that the "gas law" for the junction noise is obeyed over four decades in temperature, with a significant systematic effect at the room temperature. Error bars indicate stated approximate SD of residuals.

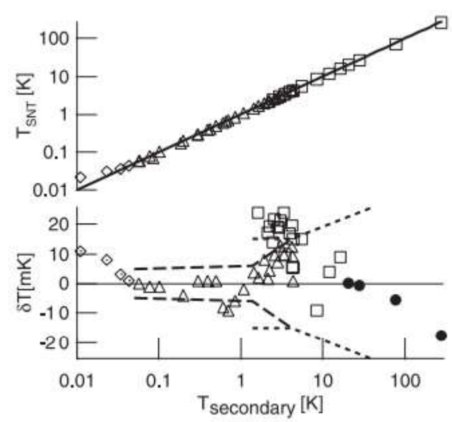


Fig. 4. Comparison of temperature as measured by the SNT (T_{SNT}) to temperature as measured by secondary thermometers ($T_{secondary}$). Temperature is displayed on logarithmic axes, and the solid line indicates the line $T_{SNT} = T_{second}$.