8.512 Theory of Solids II HW # 1 Due: 2/14/2022

Due: **Monday, Feb 14 by 5pm (submit your work on canvas)**
Reading: 8.512 classnotes, Michael Cross' lecture notes. Also, textbook "Statistical Mechanics" by Pathria covers some relevant introductory statmech topics; luckily, MIT library has an electronic version of this text. (note active links)

Superfluidity

1. Collective excitations in a superfluid ^{4}He. Critical velocity of a superflow.

(a) [10 pts] What are phonons, rotons, and maxons, and what does their dispersion curve have to do with superfluidity? Research textbooks/web to find out. Use no more than 100 words for your explanation.

(b) [10 pts] We can expect that superfluidity breaks down if the phase gradient of the condensate wavefunction $\psi(\mathbf{r}) = \sqrt{n_s}e^{i\theta(\mathbf{r})}$ is too large. For liquid helium, a reasonable guess would be that this occurs when the phase varies by 2π over $1\mathring{A}$, comparable to the interatomic distance. What is the "critical velocity"? Estimate by order of magnitude the critical velocity value for superfluid helium.

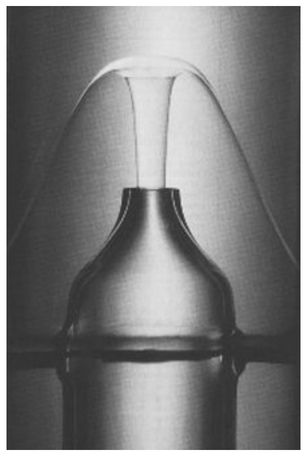

FIGURE 1. The fountain effect in superfluid helium.

2. The fountain effect [10 pts]

The fountain effect is a striking phenomenon observed in superfluid helium when a tube having a porous plug at the bottom end and a funnel at the top end is dipped in it halfway and illuminated with a light torch. This creates a frictionless fountain which lasts indefinitely (as long as helium remains in a superfluid state). Watch video showing the frictionless fountain effect at 1:30-1:44

Research textbooks/web and give a brief explanation of the effect in terms of the presence of two components in BEC at $T > 0$, the condensate and a gas of thermally excited quasiparticles. Use no more than 100 words for your explanation.

3. Quasiparticles in a superfluid as collective excitations.

We argued in class that key aspects of BEC can be understood from the the many-body Hamiltonian

$$H = \int d^3x \left[-\frac{1}{2m}\psi^\dagger \nabla^2 \psi - \mu \psi^\dagger \psi + \frac{g}{2}\psi^\dagger\psi^\dagger\psi\psi \right], \quad \psi(x) = \sum_{\mathbf{k}} \frac{e^{i\mathbf{kx}}}{\sqrt{V}} a_{\mathbf{k}}, \tag{1}$$

where $g > 0$ is the coupling strength, and μ is the chemical potential.

(a) [10 pts] Starting from Heisenberg evolution equation for ψ and $\psi^\dagger$, $\partial_t \psi = \frac{i}{\hbar}[H, \psi]$, $\partial_t \psi^\dagger = \frac{i}{\hbar}[H, \psi^\dagger]$, and using the commutator algebra,

$$[\psi(x), \psi(x')] = [\psi^\dagger(x), \psi^\dagger(x')] = 0, \quad [\psi(x), \psi^\dagger(x')] = \delta(x - x').$$

derive the Gross-Pitaevsky (GP) equations

$$(2) \qquad i\partial_t \phi = \left(-\frac{1}{2m}\nabla^2 - \mu\right)\phi + g\phi^2\phi^*, \quad -i\partial_t \phi^* = \left(-\frac{1}{2m}\nabla^2 - \mu\right)\phi^* + g\phi^{*2}\phi.$$

Here $\phi(\mathbf{x}, t)$ and $\phi^*(\mathbf{x}, t)$ are classical (c-number) fields associated with the operators ψ, $\psi^\dagger$.

Analyze stationary solutions of the GP equations. Show that the ground state, rather than being unique, is in fact infinitely degenerate, with the degeneracy parameterized by different values of the phase: $\phi_0 = |\phi_0|e^{i\theta}$, $0 < \theta < 2\pi$. This is pictured in Fig.2.

(b) [10 pts] Consider the dynamics of excitations about the ground state. Linearize the GP equations about the ground state found in part a) via $\phi(x) \approx \phi_0 + \delta\phi$, and derive coupled linearized equations of motion

$$(3) \quad i\partial_t \delta\phi = \left(-\frac{1}{2m}\nabla^2 + gn_0\right)\delta\phi + gn_0\delta\phi^*, \quad -i\partial_t \delta\phi^* = \left(-\frac{1}{2m}\nabla^2 + gn_0\right)\delta\phi^* + gn_0\delta\phi$$

where $n_0 = |\phi_0|^2$ is particle density found in part a) and, without loss of generality, we set $\theta = 0$. Construct plane wave solutions of these equations, in which $\delta\phi(x, t) \propto e^{-i\omega t + i\mathbf{kx}}$, $\delta\phi^*(x, t) \propto e^{-i\omega t + i\mathbf{kx}}$. Find the dispersion relation $\omega(k)$ and compare it to that for noninteracting particles.

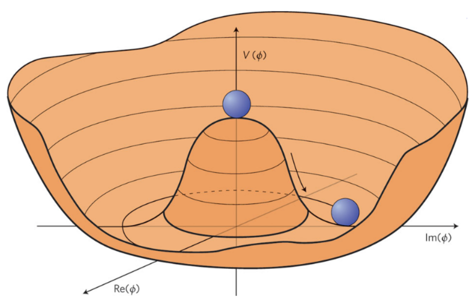

FIGURE 2. Spontaneous symmetry breaking in BEC described by the Mexican hat potential $-\mu|\phi|^2 + \frac{1}{2}g|\phi|^4$.

Identical quantum particles. Creation and annihilation operators

Reading: 8.512 classnotes on 2nd quantization; Marder Appendix C on 2nd quantization; Cross' notes on 2nd quantization (note active links)

4. Properties of a and $a^\dagger$

a) [10pts] The basic commutation relations for boson annihilation and creation operators are:

$$(4) \qquad [a_i, a_j^\dagger]_- = \delta_{ij}, \quad [a_i, a_j]_- = [a_i^\dagger, a_j^\dagger]_- = 0$$

where $[A, B]_- = AB - BA$. From this definition, show that the normalized eigenstates $|n\rangle$ of the particle number operators $\hat{n}_i = a_i^\dagger a_i$ have the properties:

$$(5) \qquad a_i^\dagger a_i |n_i\rangle = n_i |n_i\rangle, \quad n_i = 0, 1, 2, \ldots$$

(6) $$a_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle, \quad a_i^\dagger|n_i\rangle = \sqrt{n_i + 1}|n_i + 1\rangle$$

[Hint: 1) Evaluate the commutators between a, $a^\dagger$ and $\hat{n} = a^\dagger a$, and show that the operators a and $a^\dagger$ map the eigenstates $|n\rangle$ of $\hat{n}$ to the eigenstates $|n-1\rangle$ and $|n+1\rangle$, respectively. 2) From the identity $\langle\psi|\psi\rangle \geq 0$ for $|\psi\rangle = a|n\rangle$ argue that the eigenvalues of $\hat{n}$ take non-negative values. Combine 1) and 2) to show that n_i are nonnegative integers and obtain the $\sqrt{n_i}$, $\sqrt{n_i+1}$ matrix elements. This is discused in more detail in Cross' notes, see Reading section.]

b) [10pts] Fermion annihilation and creation operators are defined by the anticommutation relations:

(7) $$[a_i, a_j^\dagger]_+ = \delta_{ij}, \quad [a_i, a_j]_+ = [a_i^\dagger, a_j^\dagger]_+ = 0$$

where $[A, B]_+ = AB + BA$. Show that:

(8) $$a_i^\dagger a_i|n_i\rangle = n_i|n_i\rangle, \quad n_i = 0, 1$$

(9) $$a_i|n_i\rangle = \sqrt{n_i}|n_i - 1\rangle, \quad a_i^\dagger|n_i\rangle = \sqrt{1 - n_i}|n_i + 1\rangle$$

In other words, $a|1\rangle = |0\rangle$, $a|0\rangle = 0$, $a^\dagger|0\rangle = |1\rangle$, $a^\dagger|1\rangle = 0$, where 0 is a null vector and $|0\rangle$ is the $n = 0$ eigenstate of $n = a^\dagger a$.

5. Canonical transformations

a) [10pts] Operators a_i, $a_i^\dagger$ and $\tilde{a}_j$, $\tilde{a}_j^\dagger$ are annihilation and creation operators for a particle in states $\{\phi_i\}$ and $\{\tilde{\phi}_j\}$ that form two different orthonormal complete sets in the single-particle Hilbert space. Indicate the relations between these operators.

b) [10pts] Solve the one-dimensional tight-binding model

(10) $$H = -t\sum_x a_x^\dagger a_{x+1} + a_{x+1}^\dagger a_x, \quad x = 0, \pm1, \pm2...$$

with periodic boundary conditions, $x + N \equiv x$, by the canonical transformation

(11) $$a_x = \frac{1}{\sqrt{N}}\sum_k e^{ikx}a_k, \quad a_x^\dagger = \frac{1}{\sqrt{N}}\sum_k e^{-ikx}a_k^\dagger$$

That is, write H in terms of the a_k operators and show that it becomes diagonal.

6. An attempt at Bogoliubov transformations [10pts] Does the transformation

$$a' = \alpha a + \beta a^\dagger, \quad a'^\dagger = \alpha a^\dagger + \beta a,$$

where α and β are real numbers, preserve the canonical commutation relations? Why or why not? In other words, can one do a canonical transformation that preserves the commutator algebra for just one boson?