8.09/8.309, Classical Mechanics III, Fall 2021
MIDTERM
Tuesday November 2, 7:30-9:30pm
You have 120 minutes.

Answer all problems in the white books provided. Write YOUR NAME on EACH book you use.

There are four problems, totalling 100 points. You should do all four. The problems are worth 10, 27, 29, 34 points.

None of the problems requires extensive algebra. If you find yourself lost in a calculational thicket, stop and think.

No books, notes, or calculators allowed.

Some potentially useful information

- Euler-Lagrange equations for generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j}\ , \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot q_j}$$

constraints: holonomic $f_\alpha(q,t) = 0$ or semiholonomic $g_\beta = \sum_j a_{\beta j}(q,t)\dot q_j + a_{\beta t}(q,t) = 0$

- Generalized forces: $d/dt(\partial L/\partial \dot q_j) - \partial L/\partial q_j = R_j$

Friction forces: $\vec f_i = -h(v_i)\vec v_i/v_i$, $\vec v_i = \dot{\vec r}_i$ gives $R_j = -\partial \mathcal{F}/\partial \dot q_j\ ,\ \ \mathcal{F} = \sum_i \int_0^{v_i} dv_i'\, h(v_i')$

- Hamilton's equations for canonical variables (q_j, p_j): $\quad \dot q_j = \dfrac{\partial H}{\partial p_j}\ , \quad \dot p_j = -\dfrac{\partial H}{\partial q_j}$

- Hamiltonian for a Lagrangian quadratic in velocities

$$L = L_0(q,t) + \dot{\vec q}^{\,T}\cdot \vec a + \frac{1}{2}\dot{\vec q}^{\,T}\cdot \hat T \cdot \dot{\vec q} \quad \Rightarrow \quad H = \frac{1}{2}(\vec p - \vec a)^T \cdot \hat T^{-1}\cdot(\vec p - \vec a) - L_0(q,t)$$

- The Moment of Inertia Tensor and its relations:

$$I_{ab} = \int dV\, \rho(\vec r)[\vec r^{\,2}\delta_{ab} - r_a r_b] \qquad \text{or} \qquad I^{ab} = \sum_i m_i[\delta^{ab}\vec r_i^{\,2} - r_i^a r_i^b]$$

$$I_{ab}^{(Q)} = M(\delta_{ab}\,\vec R^2 - R_a R_b) + I_{ab}^{(\mathrm{CM})}\ , \qquad \hat I' = \hat U\,\hat I\,\hat U^T$$

- Euler's Equations:

$$\begin{aligned} I_1\dot\omega_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1 \\ I_2\dot\omega_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2 \\ I_3\dot\omega_3 - (I_1 - I_2)\omega_1\omega_2 &= \tau_3 \end{aligned}$$

- Vibrations: $L = \frac{1}{2}\dot{\vec\eta}^{\,T}\cdot \hat T \cdot \dot{\vec\eta} - \frac{1}{2}\vec\eta^{\,T}\cdot \hat V \cdot \vec\eta$ has Normal modes $\vec\eta^{\,(k)} = \vec a^{\,(k)}\exp(-i\omega^{(k)}t)$

$$\det(\hat V - \omega^2 \hat T) = 0\ , \qquad \left(\hat V - [\omega^{(k)}]^2\,\hat T\right)\cdot \vec a^{\,(k)} = 0\ , \qquad \vec\eta = \mathrm{Re}\sum_k C_k\,\vec\eta^{\,(k)}$$

- Generating functions for Canonical Transformations: $K = H + \partial F_i/\partial t$ and

$$F_1(q,Q,t):\quad p_i = \frac{\partial F_1}{\partial q_i}\ ,\ P_i = -\frac{\partial F_1}{\partial Q_i}\ , \qquad F_2(q,P,t):\quad p_i = \frac{\partial F_2}{\partial q_i}\ ,\ Q_i = \frac{\partial F_2}{\partial P_i}$$

- Poisson Brackets: $\quad [u,v]_{q,p} = \displaystyle\sum_j \left[\frac{\partial u}{\partial q_j}\frac{\partial v}{\partial p_j} - \frac{\partial u}{\partial p_j}\frac{\partial v}{\partial q_j}\right]\ , \qquad \frac{du}{dt} = [u,H] + \frac{\partial u}{\partial t}$

- Relations for Hamilton's Principle function, $S = S(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n,t)$

$$K = 0\ , \qquad P_i = \alpha_i\ , \qquad Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i}\ , \qquad p_i = \frac{\partial S}{\partial q_i}$$

- Relations for Hamilton's Characteristic function, $W = W(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n)$

$$K = H = \alpha_1\ , \qquad P_i = \alpha_i\ , \qquad \beta_1 + t = \frac{\partial W}{\partial \alpha_1}\ , \qquad \beta_{i>1} = \frac{\partial W}{\partial \alpha_i}\ , \qquad p_i = \frac{\partial W}{\partial q_i}$$

- Action Angle Variables: $\quad J = \displaystyle\oint p\,dq\ , \qquad w = \frac{\partial W(q,J)}{\partial J}\ , \qquad \dot w = \frac{\partial H(J)}{\partial J} = \nu(J)$

1. **Semi-short answer problems** [10 points]

These problems require less algebra, and a correct answer with no work shown will receive full credit. Your answers should be short.

(a) [2 points] Given three principal moments of inertia $\frac{2}{3}I$, $\frac{7}{3}I$, and $\frac{1}{3}I$ about the x, y, and z axes respectively, about which axis is a torque free rotation unstable to a small perturbation? No calculations are required.

(b) [3 points] For a mass m in a frame rotating with angular velocity $\vec{w}$ we have

$$m\vec{a}_r = \vec{F} - m\vec{w} \times (\vec{w} \times \vec{r}) - 2m\vec{w} \times \vec{v}_r - m\dot{\vec{w}} \times \vec{r}$$

where the terms are various fictitious forces. If a fast train goes round a circular curve, and a person on the train slowly walks to the back of the train, then what is the direction of the Coriolis force that they feel?

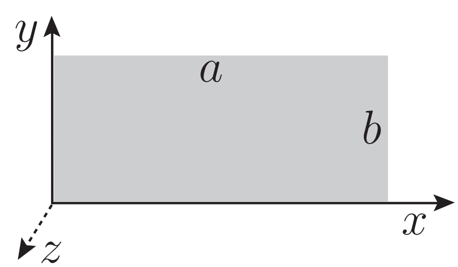
(c) [5 points] Consider the generating function $F = -Q_1P_1 + R(q_1, P_1) + G(q_2, Q_2)$, where R, G are some functions. What are the partial derivative equations that will determine the relations between the new and old canonical coordinates?

2. **Rotations of a Thin Rectangle** [27 points]

Consider a thin rectangle with sides a and b, total mass m, and uniform surface density. [Full marks can be obtained in each part below without doing the previous part.]

(a) [7 points] Show that the components of the moment of inertia tensor $\hat{I}$ about the origin for the (x, y, z) axes have the form

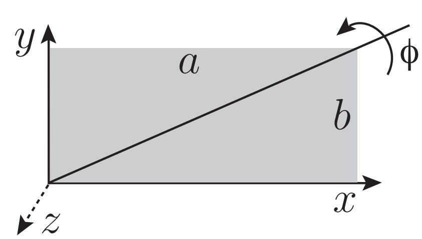
$$\hat{I} = \begin{pmatrix} A & C & 0 \\ C & B & 0 \\ 0 & 0 & D \end{pmatrix},$$



and find the values for A, B, C and D.

(b) [10 points] For the same origin, a rotated set of axes (x', y', z) makes the moment of inertia tensor diagonal. Find a formula for the angle θ that the x'-axis makes with respect to the x-axis. Write your answer in terms of A, B, C, and D without substituting values from part (a).

(c) [10 points] Suppose that the thin rectangle undergoes rotations about the diagonal axis of the rectangle by an angle $\phi(t)$ and has translational velocity $v(t)$ of this diagonal axis in the $(\hat{x}+\hat{y}+\hat{z})/\sqrt{3}$ direction. What is the kinetic energy of this system in terms of m, a, b, $v(t)$ and $\phi(t)$?

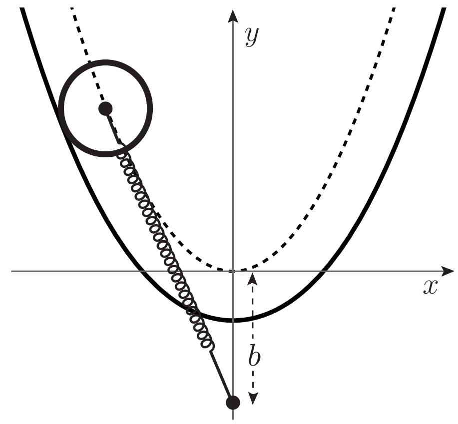


Note: You might find $\cos(2\alpha) = \cos^2(\alpha) - \sin^2(\alpha)$ and $\sin(2\alpha) = 2\sin(\alpha)\cos(\alpha)$ useful.

(continue)

3. **A Rolling Ring with a Spring** [29 points]

Consider a ring of mass m, radius a, moment of inertia $I = ma^2$, and center of mass coordinates (x, y). The ring rolls without slipping on a curve. The curve yields the constraint $y = \alpha x^2/2$ with constant $\alpha > 0$ (shown by the dashed line in the figure). A spring is attached between the ring's center of mass and the point $-b\hat{y}$, as shown. The spring has zero relaxed length and a spring constant k. [There is no gravity in this problem.]



(a) [6 points] Using a suitable set of generalized coordinates, what is the Lagrangian for this system prior to imposing any constraints?

(b) [3 points] One constraint is $y = \alpha x^2/2$. What is the other constraint for this system in terms of your generalized coordinates?

(c) [12 points] Suppose we wish to find the generalized force f for the no slipping constraint. Setup a suitable set of Euler-Lagrange equations for this problem. Identify what term in your equations is equal to the desired force f.

(d) [8 points] Solve to find f as a function of x, m, k, α, a, and b. Your result should not involve time derivatives.

(continue)

4. **Hamilton-Jacobi for Motion on a Sphere** [34 points]

Consider a particle of mass m that is constrained to move on a sphere of radius a. It has Hamiltonian in spherical coordinates (r, θ, ϕ) that is given by

$$H = \frac{1}{2ma^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right) + k\sin^2\theta\,, \tag{1}$$

where $k > 0$ is a constant.

(a) [2 points] Identify two conserved quantities for this system.

(b) [12 points] Construct a suitable Hamilton-Jacobi equation, and solve for Hamilton's characteristic function. Identify new constant momenta P_1 and P_2. What are the transformation equations for the new coordinates Q_1 and Q_2? You should leave results in terms of integrals. Assuming the integrals could be done, explain in words how you would use your results to solve for the motion in $\theta(t)$ and $\phi(t)$.

The remaining parts of this problem do not depend on solving (b). Also part (d) can be solved without solving part (c).

Consider a situation where at time $t = 0$ we have $0 < \theta < \pi/2$ and $p_\phi \neq 0$. To simplify the algebra for the remaining parts define $A = p_\phi^2/(2ma^2)$ and feel free to leave answers in terms of A.

(c) [12 points] The variables for this system will undergo oscillations or rotations, and hence can be treated with action-angle variables. Determine explicit formulas for the action variables J_θ and J_ϕ. You may leave one of your results in terms of an integral, but must specify explicitly the upper and lower integration limits. You should give results for these limits for two special cases, one where the particle never leaves the region $0 < \theta < \pi/2$, and one where it does. What are the allowed ranges for the energy for each of these two cases?

(d) [8 points] For the case where $0 < \theta(t) < \pi/2$ there is a minimum value for the energy. Find explicit results for $\theta(t)$ and $\phi(t)$ for this minimum energy case.

Note: If it helps to have a physical picture for this problem, the Hamiltonian considered here is obtained by considering a mass on rigid pendulum of length a. A spring is then attached between the mass and a frictionless ring that moves freely on the z-axis.

(the end)