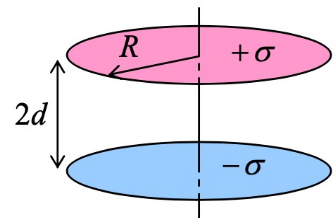
Electromagnetic Theory
Problem Set #1 Solutions (8.311) [Due Wed. Feb 9, 2022]

Problem 1.1. Calculate the electric field created by a thin, long, straight filament, electrically charged with a constant linear density λ, using two approaches:

(20 pts)

(i) directly from the Coulomb law, and
(ii) using the Gauss law.

Problem 1.11. Two similar thin, circular, coaxial disks of radius R, separated by distance $2d$, are uniformly charged with equal and opposite areal densities $\pm\sigma$ – see the figure on the right. Calculate and sketch the distribution of the electrostatic potential and the electric field of the disks along their common axis.



(10 pts)

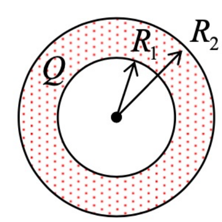
Problem 1.17. Prove the following *reciprocity theorem of electrostatics*:[14] if two spatially-confined charge distributions $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ induce, respectively, distributions $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ of the electrostatic potential, then

(30 pts)

$$\int \rho_1(\mathbf{r})\phi_2(\mathbf{r})d^3r = \int \rho_2(\mathbf{r})\phi_1(\mathbf{r})d^3r .$$

Hint: Consider integral $\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3r$.

Problem 1.19. Calculate the electrostatic energy U of a (generally, thick) spherical shell, with charge Q uniformly distributed through its volume – see the figure on the right. Analyze and interpret the dependence of U on the inner cavity's radius R_1, at fixed Q and R_2.



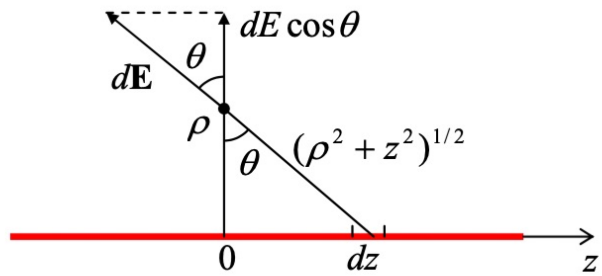
(30 pts)

Problem 1.1. Calculate the electric field created by a thin, long, straight filament, electrically charged with a constant linear density λ, using two approaches:

(i) directly from the Coulomb law, and
(ii) using the Gauss law.

Solutions:

(i) From the translational and axial symmetries of the problem, it is clear that $\mathbf{E}(\mathbf{r}) = \mathbf{n}_\rho E(\rho)$, where ρ is the shortest distance from the observation point to the filament.[1] Let us select the plane of drawing so that it contains both the filament and the observation point, and take the line of filament for the z-axis – see the figure on the right. Then, according to the linear superposition principle, the field's magnitude may be calculated as



$$E(\rho) = \int_{z=-\infty}^{z=+\infty} dE_\rho = \int_{z=-\infty}^{z=+\infty} dE\cos\theta = \int_{z=-\infty}^{z=+\infty} dE \frac{\rho}{\left(\rho^2+z^2\right)^{1/2}},$$

where dE is the magnitude of the elementary contribution to the field, created by a small segment dz of the filament, with the electric charge λdz. According to Eq. (1.7) of the lecture notes,

$$dE = \lambda dz \frac{1}{4\pi\varepsilon_0}\frac{1}{\rho^2+z^2},$$

so that the total field

$$E(\rho) = \frac{\lambda\rho}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{dz}{\left(\rho^2+z^2\right)^{3/2}} \equiv \frac{\lambda}{2\pi\varepsilon_0\rho}\int_0^{+\infty}\frac{d\xi}{\left(1+\xi^2\right)^{3/2}} = \frac{\lambda}{2\pi\varepsilon_0\rho}. \qquad (*)$$

For the last step, I have used the well-known value (1) of this table integral – see, e.g., MA Eq. (6.5b).[2]

(ii) Taking a round cylinder of radius ρ and length l, with its axis on the filament, for the Gaussian volume, we ensure that on its round walls the electric field E is constant, and normal to the volume boundary, while the field flux through the cylinder's "lids" is zero. As a result, Eq. (1.16) yields

$$2\pi\rho l E = \frac{\lambda l}{\varepsilon_0},$$

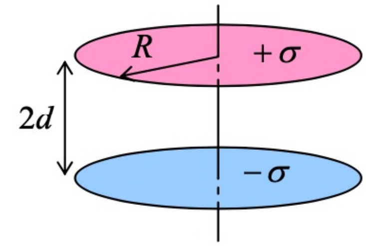
[1] I regret using the same letter (ρ) as for the charge density per unit volume (used in lecture notes, but not in this problem), but both notations are traditional. The difference between these notions will be always very clear from the context.

[2] Actually, this integral may be easily worked out by substitution $\xi \equiv \tan\varphi$, giving $d\xi = d\varphi/\cos^2\varphi = d\varphi\,(1 + \tan^2\varphi) = d\varphi\,(1+\xi^2)$, so that $d\xi/(1+\xi^2)^{3/2} = d\varphi/(1+\xi^2)^{1/2} = \cos\varphi\, d\varphi = d(\sin\varphi)$.

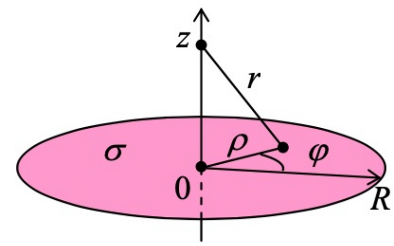
immediately giving the same result (*).

We see that for this highly symmetric problem both approaches are straightforward, but the Gauss law makes calculations easier.

Problem 1.11. Two similar thin, circular, coaxial disks of radius R, separated by distance $2d$, are uniformly charged with equal and opposite areal densities $\pm\sigma$ – see the figure on the right. Calculate and sketch the distribution of the electrostatic potential and the electric field of the disks along their common axis.



Solution: Let us start by calculating the electrostatic potential of one disk, with a constant areal charge density σ, at the axis point separated by distance z from the disk's plane – see the figure on the right. In the polar coordinates $\{\rho, \varphi\}$ within the plane of the disk (with angle φ referred to an arbitrary horizontal axis}, the elementary disk area is $\rho d\varphi d\rho$, and its electric charge is $\sigma\rho d\rho d\varphi$, so that Eq. (1.38) of the lecture notes takes the following form:



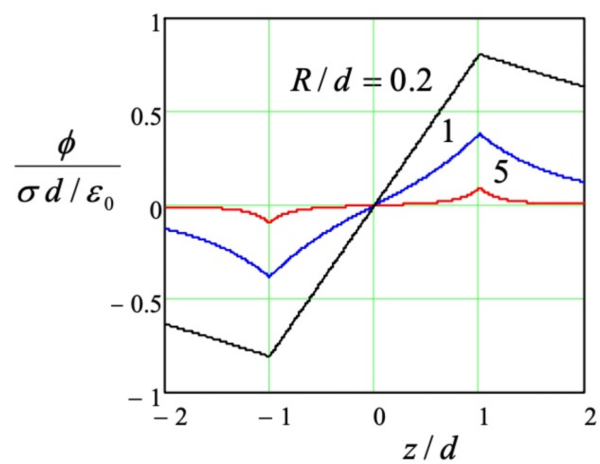
$$\phi(z) = \frac{\sigma}{4\pi\varepsilon_0}\int_0^{2\pi} d\varphi \int_0^R \frac{\rho d\rho}{r},$$

where r is the distance between the elementary charge's location and the observation point on the disks' common axis. As the figure on the right shows, $r = (\rho^2 + z^2)^{1/2}$, so that the function under the integral is independent of φ, and it may be easily worked out:

$$\phi(z) = \frac{\sigma}{4\pi\varepsilon_0} 2\pi \int_0^R \frac{\rho\, d\rho}{\left(\rho^2 + z^2\right)^{1/2}} = \frac{\sigma}{4\varepsilon_0}\int_{\rho=0}^{\rho=R} \frac{d\left(\rho^2 + z^2\right)}{\left(\rho^2 + z^2\right)^{1/2}} = \frac{\sigma}{2\varepsilon_0}\left[\left(R^2 + z^2\right)^{1/2} - |z|\right].$$

Now using this formula, and the linear superposition principle, we may readily write down an expression describing the potential created by both disks, at distance z from the center of the system (in this new reference frame, the disk center positions are $\pm d$):

$$\phi = \frac{\sigma}{2\varepsilon_0} \times \left\{ \begin{aligned} &\left[R^2 + (z-d)^2\right]^{1/2} - |z-d| \\ &-\left[R^2 + (z+d)^2\right]^{1/2} + |z+d| \end{aligned} \right\}.$$

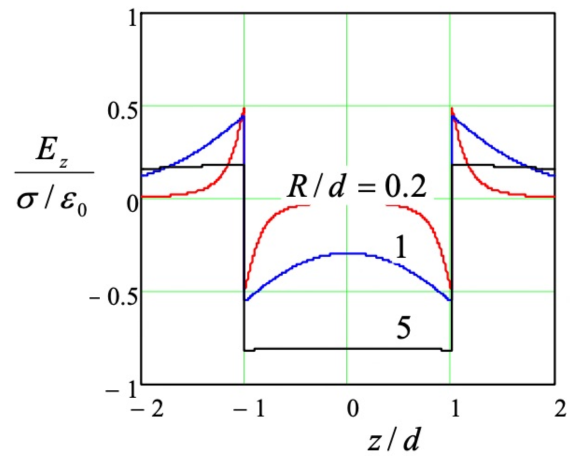


This function is plotted in the figure on the right for three values of the R/d ratio. For small values of this ratio, the potential is clearly separated into two peaks, of opposite polarity, created by each disk. On the other hand, at $R \gg d$, the result tends to the one for two infinite planes, with ϕ between the disks being a linear function of z, with the slope corresponding to the electric field – see the model solution of Problem 14 below.

Now the electric field at the axis (which has only one, vertical component due to the axial symmetry of the problem) may be calculated by differentiation of the electrostatic potential – see Eq. (1.33) of the lecture notes:

$$E_z = -\frac{\partial\phi}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left\{ -\frac{z-d}{\left[R^2 + (z-d)^2\right]^{1/2}} + \operatorname{sgn}(z-d) + \frac{z+d}{\left[R^2 + (z+d)^2\right]^{1/2}} - \operatorname{sgn}(z+d) \right\}.$$

In the figure on the right, this function is plotted for the same three values of the R/d ratio as the potential in the figure above. The plots show that at $R/d = 5$, the field between the disks is already pretty uniform, though its magnitude is still noticeably smaller than that (σ/ε_0) between two infinite planes. Note also that the ratio does not affect the electric field's jump by $\pm\sigma/\varepsilon_0$ as the observation point crosses a disk – just as it should be, according to Eq. (1.24) of the lecture notes.



On the technical side, this solution illustrates again the advantage of calculating the electrostatic potential (a scalar function) first, and only then the electric field from it.

Problem 1.17. Prove the following *reciprocity theorem of electrostatics*:[14] if two spatially-confined charge distributions $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ induce, respectively, distributions $\phi_1(\mathbf{r})$ and $\phi_2(\mathbf{r})$ of the electrostatic potential, then

$$\int \rho_1(\mathbf{r})\phi_2(\mathbf{r})d^3r = \int \rho_2(\mathbf{r})\phi_1(\mathbf{r})d^3r .$$

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[13] This rule has a simple geometric meaning – see, e.g., CM Sec. 2.1, in particular the figure 2.2.
[14] This is only the simplest one of the whole family of reciprocity theorems in electromagnetism – see, e.g., Sec. 6.8 of the lecture notes.

Hint: Consider integral $\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3r$.

Solution: Applying Eq. (1.33) of the lecture notes to $\mathbf{E}_1(\mathrm{r})$, let us transform the integral mentioned in the *Hint* as

$$\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3r = -\int \nabla \phi_1 \cdot \mathbf{E}_2 d^3r .$$

Now we may use the rule of spatial differentiation of a vector-by-scalar function product[15] to continue as follows:

$$-\int \nabla \phi_1 \cdot \mathbf{E}_2 d^3r = \int \phi_1 (\nabla \cdot \mathbf{E}_2) d^3r - \int \nabla (\phi_1 \mathbf{E}_2) d^3r.$$

Next, we may use the inhomogeneous Maxwell equation (1.27) in the first integral on the right-hand side, and the well-known divergence theorem[16] to transform the second integral to that of $(\phi_1 \mathbf{E}_2)_n$ over some distant closed surface S that limits the volume of our spatial integration. As a result, our expression becomes

$$\frac{1}{\varepsilon_0}\int \phi_1 \rho_2 d^3r - \oint_S (\phi_1 \mathbf{E}_2)_n d^2r .$$

Since the charge (and hence the field) distributions are space-confined, we may always select the surface S so distant that the surface integral is negligible,[17] and our chain of transformations may be summarized as

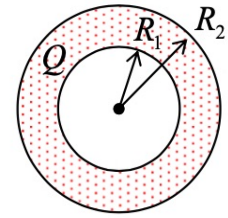
$$\int \mathbf{E}_1 \cdot \mathbf{E}_2 d^3r = \frac{1}{\varepsilon_0}\int \phi_1 \rho_2 d^3r .$$

Now repeating the same calculation with swapped indices, we arrive at the reciprocity theorem.

Note that if some parts of these two charge distributions reside on some surface(s) S, and may be well described by surface charge densities $\sigma_1(\mathbf{r})$ and $\sigma_2(\mathbf{r})$ (as is very instrumental, for example, in systems with good conductors, to be discussed in Chapter 2 of the lecture notes), the reciprocity theorem may be rewritten as

$$\int_V \rho_1(\mathbf{r})\phi_2(\mathbf{r}) d^3r + \int_S \sigma_1(\mathbf{r})\phi_2(\mathbf{r}) d^2r = \int_V \rho_2(\mathbf{r})\phi_1(\mathbf{r}) d^3r + \int_S \sigma_2(\mathbf{r})\phi_1(\mathbf{r}) d^2r ,$$

where $\rho_1(\mathbf{r})$ and $\rho_2(\mathbf{r})$ are the remaining, "genuinely-volume" parts of the distributions. (In this form, it is sometimes called the "Green's reciprocity theorem".)



Problem 1.19. Calculate the electrostatic energy U of a (generally, thick) spherical shell, with charge Q uniformly distributed through its volume – see the figure on the right. Analyze and interpret the dependence of U on the inner cavity's radius R_1, at fixed Q and R_2.

Solution: Calculating the only (radial) component E of the electric field $\mathbf{E} = \mathbf{n}_r E(r)$ (say, using the Gauss law), we readily get

$$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2} \times \begin{cases} 0, & \text{for } 0 \leq r < R_1, \\ (r^3 - R_1^3)/(R_2^3 - R_1^3), & \text{for } R_1 < r < R_2, \\ 1, & \text{for } R_2 < r, \end{cases}$$

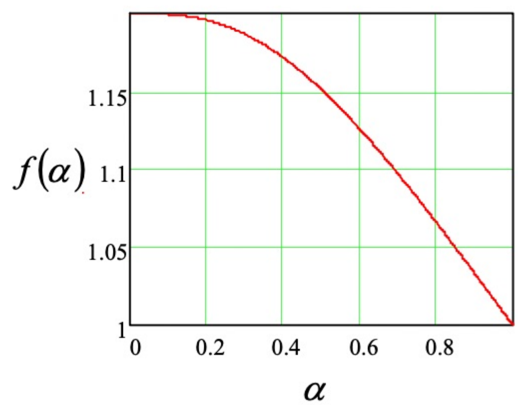
so that Eq. (1.65) for the electrostatic energy yields

$$U = \frac{\varepsilon_0}{2}\int E^2(r)\,dr^3 = \frac{\varepsilon_0}{2}4\pi\int_0^\infty E^2(r)r^2 dr = \frac{\varepsilon_0}{2}4\pi\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\left[\int_{R_1}^{R_2}\left(\frac{r^3 - R_1^3}{R_2^3 - R_1^3}\right)^2\frac{dr}{r^2} + \int_{R_2}^{\infty}\frac{dr}{r^2}\right]$$

$$= \frac{Q^2}{8\pi\varepsilon_0 R_2}\left[\int_\alpha^1\left(\frac{\xi^3 - \alpha^3}{1-\alpha^3}\right)^2\frac{d\xi}{\xi^2} + \int_1^\infty\frac{d\xi}{\xi^2}\right] = \frac{Q^2}{8\pi\varepsilon_0 R_2}f(\alpha), \quad \text{with } f(\alpha) \equiv \frac{1/5 - \alpha^3 + (9/5)\alpha^5 - \alpha^6}{(1-\alpha^3)^2} + 1,$$

where $\alpha \equiv R_1/R_2 \leq 1$.

A plot of the function $f(\alpha)$, i.e. of the normalized electrostatic energy as a function of R_1 at fixed Q and R_2, is shown in the figure on the right. The function reaches its maximum $f(0) = 6/5$ (so that U is given by Eq. (1.66) of the lecture notes, with $R = R_2$) at $\alpha = 0$, i.e. for a solid sphere, and tends to $f(1) = 1$, giving



$$U_{\min} = \frac{Q^2}{8\pi\varepsilon_0 R_2}$$

at $\alpha \to 1$, i.e. for the ultimately thin spherical shell. This is very natural because the elementary charges of the sphere repulse each other and try to go apart as far as possible, in particular increasing its inner radius.