

## The Origin of the Refractive Index

### 31-1 The index of refraction

We have said before that light goes slower in water than in air, and slower, slightly, in air than in vacuum. This effect is described by the index of refraction  $n$ . Now we would like to understand how such a slower velocity could come about. In particular, we should try to see what the relation is to some physical assumptions, or statements, we made earlier, which were the following:

- (a) That the total electric field in any physical circumstance can always be represented by the sum of the fields from all the charges in the universe.
- (b) That the field from a single charge is given by its acceleration evaluated with a retardation at the speed  $c$ , *always* (for the *radiation* field).

But, for a piece of glass, you might think: "Oh, no, you should modify all this. You should say it is retarded at the speed  $c/n$ ." That, however, is not right, and we have to understand why it is not.

It is approximately true that light or any electrical wave *does appear* to travel at the speed  $c/n$  through a material whose index of refraction is  $n$ , but the fields are still produced by the motions of *all* the charges—including the charges moving in the material—and with these basic contributions of the field travelling at the ultimate velocity  $c$ . Our problem is to understand how the *apparently* slower velocity comes about.

We shall try to understand the effect in a very simple case. A source which we shall call "the *external source*" is placed a large distance away from a thin plate of transparent material, say glass. We inquire about the field at a large distance on the opposite side of the plate. The situation is illustrated by the diagram of Fig. 31-1, where  $S$  and  $P$  are imagined to be very far away from the plate. According to the principles we have stated earlier, an electric field anywhere that is far from all moving charges is the (vector) sum of the fields produced by the external source (at  $S$ ) and the fields produced by *each* of the charges in the plate of glass, *every one with its proper retardation at the velocity  $c$* . Remember that the contribution of each charge is not changed by the presence of the other charges. These are our basic principles. The field at  $P$  can be written thus:

$$\mathbf{E} = \sum_{\text{all charges}} \mathbf{E}_{\text{each charge}} \quad (31.1)$$

or

$$\mathbf{E} = \mathbf{E}_s + \sum_{\text{all other charges}} \mathbf{E}_{\text{each charge}}, \quad (31.2)$$

where  $\mathbf{E}_s$  is the field due to the source alone and would be precisely the field at  $P$  if there were no material present. We expect the field at  $P$  to be different from  $\mathbf{E}_s$  if there are any other moving charges.

Why should there be charges moving in the glass? We know that all material consists of atoms which contain electrons. When the electric field of the source acts on these atoms it drives the electrons up and down, because it exerts a force on the electrons. And moving electrons generate a field—they constitute new radiators. These new radiators are related to the source  $S$ , because they are driven by the field of the source. The total field is not just the field of the source  $S$ , but it is modified by the additional contribution from the other moving charges. This means that the field is not the same as the one which was there before the glass was there, but is modified, and it turns out that it is modified in such a way that

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### 31-2 The field due to the material

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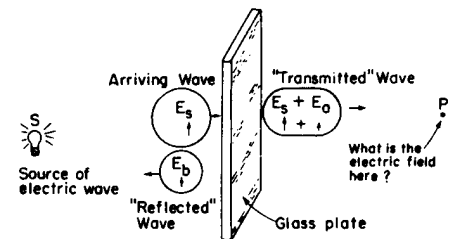


Fig. 31-1. Electric waves passing through a layer of transparent material.

the field inside the glass appears to be moving at a different speed. That is the idea which we would like to work out quantitatively.

Now this is, in the exact case, pretty complicated, because although we have said that all the other moving charges are driven by the source field, that is not quite true. If we think of a particular charge, it feels not only the source, but like anything else in the world, it feels *all* of the charges that are moving. It feels, in particular, the charges that are moving somewhere else in the glass. So the total field which is acting on a *particular charge* is a combination of the fields from the other charges, *whose motions depend on what this particular charge is doing!* You can see that it would take a complicated set of equations to get the complete and exact formula. It is so complicated that we postpone this problem until next year.

Instead we shall work out a very simple case in order to understand all the physical principles very clearly. We take a circumstance in which the effects from the other atoms is very small relative to the effects from the source. In other words, we take a material in which the total field is not modified very much by the motion of the other charges. That corresponds to a material in which the index of refraction is very close to 1, which will happen, for example, if the density of the atoms is very low. Our calculation will be valid for any case in which the index is for any reason very close to 1. In this way we shall avoid the complications of the most general, complete solution.

Incidentally, you should notice that there is another effect caused by the motion of the charges in the plate. These charges will also radiate waves back toward the source  $S$ . This backward-going field is the light we see reflected from the surfaces of transparent materials. It does not come from just the surface. The backward radiation comes from everywhere in the interior, but it turns out that the total effect is equivalent to a reflection from the surfaces. These reflection effects are beyond our approximation at the moment because we shall be limited to a calculation for a material with an index so close to 1 that very little light is reflected.

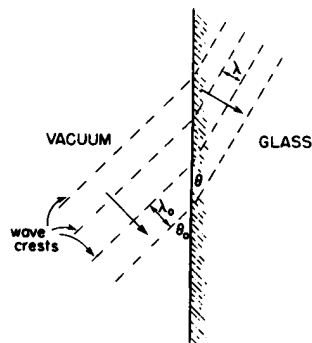


Fig. 31-2. Relation between refraction and velocity change.

Before we proceed with our study of how the index of refraction comes about, we should understand that all that is required to understand refraction is to understand why the apparent wave *velocity* is different in different materials. The *bending* of light rays comes about just *because* the effective speed of the waves is different in the materials. To remind you how that comes about we have drawn in Fig. 31-2 several successive crests of an electric wave which arrives from a vacuum onto the surface of a block of glass. The arrow perpendicular to the wave crests indicates the direction of travel of the wave. Now all oscillations in the wave must have the same *frequency*. (We have seen that driven oscillations have the same frequency as the driving source.) This means, also, that the wave crests for the waves on both sides of the surface must have the *same spacing along the surface* because they must travel together, so that a charge sitting at the boundary will feel only one frequency. The *shortest* distance between crests of the wave, however, is the wavelength which is the velocity divided by the frequency. On the vacuum side it is  $\lambda_0 = 2\pi c/\omega$ , and on the other side it is  $\lambda = 2\pi v/\omega$  or  $2\pi c/\omega n$ , if  $v = c/n$  is the velocity of the wave. From the figure we can see that the only way for the waves to “fit” properly at the boundary is for the waves in the material to be travelling at a different angle with respect to the surface. From the geometry of the figure you can see that for a “fit” we must have  $\lambda_0/\sin \theta_0 = \lambda/\sin \theta$ , or  $\sin \theta_0/\sin \theta = n$ , which is Snell’s law. We shall, for the rest of our discussion, consider only why light has an effective speed of  $c/n$  in material of index  $n$ , and no longer worry, in this chapter, about the bending of the light direction.

We go back now to the situation shown in Fig. 31-1. We see that what we have to do is to calculate the field produced at  $P$  by all the oscillating charges in the glass plate. We shall call this part of the field  $E_a$ , and it is just the sum written as the second term in Eq. (31.2). When we add it to the term  $E_s$ , due to the source, we will have the total field at  $P$ .

This is probably the most complicated thing that we are going to do this year, but it is complicated only in that there are many pieces that have to be put together; each piece, however, is very simple. Unlike other derivations where we say, "Forget the derivation, just look at the answer!" in this case we do not need the answer so much as the derivation. In other words, the thing to understand now is the physical machinery for the production of the index.

To see where we are going, let us first find out what the "correction field"  $E_a$  would have to be if the total field at  $P$  is going to look like radiation from the source that is slowed down while passing through the thin plate. If the plate had no effect on it, the field of a wave travelling to the right (along the  $z$ -axis) would be

$$E_s = E_0 \cos \omega(t - z/c) \quad (31.3)$$

or, using the exponential notation,

$$E_s = E_0 e^{i\omega(t-z/c)}. \quad (31.4)$$

Now what would happen if the wave travelled more slowly in going through the plate? Let us call the thickness of the plate  $\Delta z$ . If the plate were not there the wave would travel the distance  $\Delta z$  in the time  $\Delta z/c$ . But if it appears to travel at the speed  $c/n$  then it should take the longer time  $n \Delta z/c$  or the *additional* time  $\Delta t = (n - 1) \Delta z/c$ . After that it would continue to travel at the speed  $c$  again. We can take into account the extra delay in getting through the plate by replacing  $t$  in Eq. (31.4) by  $(t - \Delta t)$  or by  $[t - (n - 1) \Delta z/c]$ . So the wave after insertion of the plate should be written

$$E_{\text{after plate}} = E_0 e^{i\omega[t - (n-1)\Delta z/c - z/c]}. \quad (31.5)$$

We can also write this equation as

$$E_{\text{after plate}} = e^{-i\omega(n-1)\Delta z/c} E_0 e^{i\omega(t-z/c)}, \quad (31.6)$$

which says that the wave after the plate is obtained from the wave which could exist without the plate, i. e., from  $E_s$ , by multiplying by the factor  $e^{-i\omega(n-1)\Delta z/c}$ . Now we know that multiplying an oscillating function like  $e^{i\omega t}$  by a factor  $e^{i\theta}$  just says that we change the phase of the oscillation by the angle  $\theta$ , which is, of course, what the extra delay in passing through the thickness  $\Delta z$  has done. It has retarded the phase by the amount  $\omega(n - 1) \Delta z/c$  (retarded, because of the minus sign in the exponent).

We have said earlier that the plate should *add* a field  $E_a$  to the original field  $E_s = E_0 e^{i\omega(t-z/c)}$ , but we have found instead that the effect of the plate is to *multiply* the field by a factor which shifts its phase. However, that is really all right because we can get the same result by adding a suitable complex number. It is particularly easy to find the right number to add in the case that  $\Delta z$  is small, for you will remember that if  $x$  is a small number then  $e^x$  is nearly equal to  $(1 + x)$ . We can write, therefore,

$$e^{-i\omega(n-1)\Delta z/c} = 1 - i\omega(n - 1) \Delta z/c. \quad (31.7)$$

Using this equality in Eq. (31.6), we have

$$E_{\text{after plate}} = \underbrace{E_0 e^{i\omega(t-z/c)}}_{E_s} - \underbrace{\frac{i\omega(n-1)\Delta z}{c} E_0 e^{i\omega(t-z/c)}}_{E_a}. \quad (31.8)$$

The first term is just the field from the source, and the second term must just be equal to  $E_a$ , the field produced to the right of the plate by the oscillating charges of the plate—expressed here in terms of the index of refraction  $n$ , and depending, of course, on the strength of the wave from the source.

What we have been doing is easily visualized if we look at the complex number diagram in Fig. 31-3. We first draw the number  $E_s$  (we chose some values for  $z$  and  $t$  so that  $E_s$  comes out horizontal, but this is not necessary). The delay due to

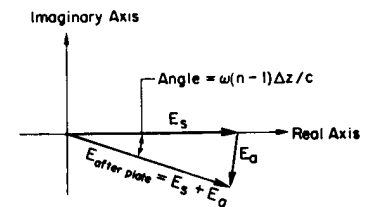


Fig. 31-3. Diagram for the transmitted wave at a particular  $t$  and  $z$ .

slowing down in the plate would delay the phase of this number, that is, it would rotate  $E_s$  through a negative angle. But this is equivalent to adding the small vector  $E_a$  at roughly right angles to  $E_s$ . But that is just what the factor  $-i$  means in the second term of Eq. (31.8). It says that if  $E_s$  is real, then  $E_a$  is negative imaginary or that, in general,  $E_s$  and  $E_a$  make a right angle.

### 31-2 The field due to the material

We now have to ask: Is the field  $E_a$  obtained in the second term of Eq. (31.8) the kind we would expect from oscillating charges in the plate? If we can show that it is, we will then have calculated what the index  $n$  should be! [Since  $n$  is the only nonfundamental number in Eq. (31.8).] We turn now to calculating what field  $E_a$  the charges in the material will produce. (To help you keep track of the many symbols we have used up to now, and will be using in the rest of our calculation, we have put them all together in Table 31-1.)

**Table 31-1**

#### Symbols used in the calculations

$E_s$	= field from the source
$E_a$	= field produced by charges in the plate
$\Delta z$	= thickness of the plate
$z$	= perpendicular distance from the plate
$n$	= index of refraction
$\omega$	= frequency (angular) of the radiation
$N$	= number of charges per unit volume in the plate
$\eta$	= number of charges per unit area of the plate
$q_e$	= charge on an electron
$m$	= mass of an electron
$\omega_0$	= resonant frequency of an electron bound in an atom

If the source  $S$  (of Fig. 31-1) is far off to the left, then the field  $E_s$  will have the same phase everywhere on the plate, so we can write that in the neighborhood of the plate

$$E_s = E_0 e^{i(\omega t - z/c)} \quad (31.9)$$

Right at the plate, where  $z = 0$ , we will have

$$E_s = E_0 e^{i\omega t} \text{ (at the plate)} \quad (31.10)$$

Each of the electrons in the atoms of the plate will feel this electric field and will be driven up and down (we assume the direction of  $E_0$  is vertical) by the electric force  $qE$ . To find what motion we expect for the electrons, we will assume that the atoms are little oscillators, that is, that the electrons are fastened elastically to the atoms, which means that if a force is applied to an electron its displacement from its normal position will be proportional to the force.

You may think that this is a funny model of an atom if you have heard about electrons whirling around in orbits. But that is just an oversimplified picture. The correct picture of an atom, which is given by the theory of wave mechanics, says that, *so far as problems involving light are concerned*, the electrons behave as though they were held by springs. So we shall suppose that the electrons have a linear restoring force which, together with their mass  $m$ , makes them behave like little oscillators, with a resonant frequency  $\omega_0$ . We have already studied such oscillators, and we know that the equation of their motion is written this way:

$$m \left( \frac{d^2 x}{dt^2} + \omega_0^2 x \right) = F, \quad (31.11)$$

where  $F$  is the driving force.

For our problem, the driving force comes from the electric field of the wave from the source, so we should use

$$F = q_e E_s = q_e E_0 e^{i\omega t}, \quad (31.12)$$

where  $q_e$  is the electric charge on the electron and for  $E_s$  we use the expression  $E_s = E_0 e^{i\omega t}$  from (31.10). Our equation of motion for the electron is then

$$m \left( \frac{d^2 x}{dt^2} + \omega_0^2 x \right) = q_e E_0 e^{i\omega t}. \quad (31.13)$$

We have solved this equation before, and we know that the solution is

$$x = x_0 e^{i\omega t}, \quad (31.14)$$

where, by substituting in (31.13), we find that

$$x_0 = \frac{q_e E_0}{m(\omega_0^2 - \omega^2)}, \quad (31.15)$$

so that

$$x = \frac{q_e E_0}{m(\omega_0^2 - \omega^2)} e^{i\omega t}. \quad (31.16)$$

We have what we needed to know—the motion of the electrons in the plate. And it is the same for every electron, except that the mean position (the “zero” of the motion) is, of course, different for each electron.

Now we are ready to find the field  $E_a$  that these atoms produce at the point  $P$ , because we have already worked out (at the end of Chapter 30) what field is produced by a sheet of charges that all move together. Referring back to Eq. (30.19), we see that the field  $E_a$  at  $P$  is just a negative constant times the velocity of the charges retarded in time the amount  $z/c$ . Differentiating  $x$  in Eq. (31.16) to get the velocity, and sticking in the retardation [or just putting  $x_0$  from (31.15) into (30.18)] yields

$$E_a = - \frac{\eta q_e}{2\epsilon_0 c} \left[ i\omega \frac{q_e E_0}{m(\omega_0^2 - \omega^2)} e^{i\omega(t-z/c)} \right]. \quad (31.17)$$

Just as we expected, the driven motion of the electrons produced an extra wave which travels to the right (that is what the factor  $e^{i\omega(t-z/c)}$  says), and the amplitude of this wave is proportional to the number of atoms per unit area in the plate (the factor  $\eta$ ) and also proportional to the strength of the source field (the factor  $E_0$ ). Then there are some factors which depend on the atomic properties ( $q_e$ ,  $m$ , and  $\omega_0$ ), as we should expect.

The most important thing, however, is that this formula (31.17) for  $E_a$  looks very much like the expression for  $E_a$  that we got in Eq. (31.8) by saying that the original wave was delayed in passing through a material with an index of refraction  $n$ . The two expressions will, in fact, be identical if

$$(n - 1) \Delta z = \frac{\eta q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)}. \quad (31.18)$$

Notice that both sides are proportional to  $\Delta z$ , since  $\eta$ , which is the number of atoms *per unit area*, is equal to  $N \Delta z$ , where  $N$  is the number of atoms *per unit volume* of the plate. Substituting  $N \Delta z$  for  $\eta$  and cancelling the  $\Delta z$ , we get our main result, a formula for the index of refraction in terms of the properties of the atoms of the material—and of the frequency of the light:

$$n = 1 + \frac{N q_e^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)}. \quad (31.19)$$

This equation gives the “explanation” of the index of refraction that we wished to obtain.

### 31-3 Dispersion

Notice that in the above process we have obtained something very interesting. For we have not only a number for the index of refraction which can be computed from the basic atomic quantities, but we have also learned how the index of refraction should vary with the frequency  $\omega$  of the light. This is something we would never understand from the simple statement that "light travels slower in a transparent material." We still have the problem, of course, of knowing how many atoms per unit volume there are, and what is their natural frequency  $\omega_0$ . We do not know this just yet, because it is different for every different material, and we cannot get a general theory of that now. Formulation of a general theory of the properties of different substances—their natural frequencies, and so on—is possible only with quantum atomic mechanics. Also, different materials have different properties and different indexes, so we cannot expect, anyway, to get a general formula for the index which will apply to all substances.

However, we shall discuss the formula we have obtained, in various possible circumstances. First of all, for most ordinary gases (for instance, for air, most colorless gases, hydrogen, helium, and so on) the natural frequencies of the electron oscillators correspond to ultraviolet light. These frequencies are higher than the frequencies of visible light, that is,  $\omega_0$  is much larger than  $\omega$  of visible light, and to a first approximation, we can disregard  $\omega^2$  in comparison with  $\omega_0^2$ . Then we find that the index is nearly constant. So for a gas, the index is nearly constant. This is also true for most other transparent substances, like glass. If we look at our expression a little more closely, however, we notice that as  $\omega$  rises, taking a little bit more away from the denominator, the index also rises. So  $n$  rises slowly with frequency. The index is higher for blue light than for red light. That is the reason why a prism bends the light more in the blue than in the red.

The phenomenon that the index depends upon the frequency is called the phenomenon of *dispersion*, because it is the basis of the fact that light is "dispersed" by a prism into a spectrum. The equation for the index of refraction as a function of frequency is called a *dispersion equation*. So we have obtained a dispersion equation. (In the past few years "dispersion equations" have been finding a new use in the theory of elementary particles.)

Our dispersion equation suggests other interesting effects. If we have a natural frequency  $\omega_0$  which lies in the visible region, or if we measure the index of refraction of a material like glass in the ultraviolet, where  $\omega$  gets near  $\omega_0$ , we see that at frequencies very close to the natural frequency the index can get enormously large, because the denominator can go to zero. Next, suppose that  $\omega$  is greater than  $\omega_0$ . This would occur, for example, if we take a material like glass, say, and shine x-ray radiation on it. In fact, since many materials which are opaque to visible light, like graphite for instance, are transparent to x-rays, we can also talk about the index of refraction of carbon for x-rays. All the natural frequencies of the carbon atoms would be much lower than the frequency we are using in the x-rays, since x-ray radiation has a very high frequency. The index of refraction is that given by our dispersion equation if we set  $\omega_0$  equal to zero (we neglect  $\omega_0^2$  in comparison with  $\omega^2$ ).

A similar situation would occur if we beam radiowaves (or light) on a gas of free electrons. In the upper atmosphere electrons are liberated from their atoms by ultraviolet light from the sun and they sit up there as free electrons. For free electrons  $\omega_0 = 0$  (there is no elastic restoring force). Setting  $\omega_0 = 0$  in our dispersion equation yields the correct formula for the index of refraction for radiowaves in the stratosphere, where  $N$  is now to represent the density of free electrons (number per unit volume) in the stratosphere. But let us look again at the equation, if we beam x-rays on matter, or radiowaves (or any electric waves) on free electrons the term  $(\omega_0^2 - \omega^2)$  becomes *negative*, and we obtain the result that  $n$  is *less than one*. That means that the effective speed of the waves in the substance is *faster* than  $c$ ! Can that be correct?

It is correct. In spite of the fact that it is said that you cannot send signals any faster than the speed of light, it is nevertheless true that the index of refraction of materials at a particular frequency can be either greater or less than 1. This

just means that the *phase shift* which is produced by the scattered light can be either positive or negative. It can be shown, however, that the speed at which you can send a *signal* is not determined by the index at one frequency, but depends on what the index is at *many* frequencies. What the index tells us is the speed at which the *nodes* (or crests) of the wave travel. The *node* of a wave is not a signal by itself. In a perfect wave, which has no modulations of any kind, i.e., which is a steady oscillation, you cannot really say when it “starts,” so you cannot use it for a timing signal. In order to send a *signal* you have to change the wave somehow, make a notch in it, make it a little bit fatter or thinner. That means that you have to have more than one frequency in the wave, and it can be shown that the speed at which *signals* travel is not dependent upon the index alone, but upon the way that the index changes with the frequency. This subject we must also delay (until Chapter 48). Then we will calculate for you the actual speed of *signals* through such a piece of glass, and you will see that it will not be faster than the speed of light, although the nodes, which are mathematical points, do travel faster than the speed of light.

Just to give a slight hint as to how that happens, you will note that the real difficulty has to do with the fact that the responses of the charges are opposite to the field, i.e., the sign has gotten reversed. Thus in our expression for  $x$  (Eq. 31.16) the displacement of the charge is in the direction opposite to the driving field, because  $(\omega_0^2 - \omega^2)$  is negative for small  $\omega_0$ . The formula says that when the electric field is pulling in one direction, the charge is moving in the opposite direction.

How does the charge happen to be going in the opposite direction? It certainly does not start off in the opposite direction when the field is first turned on. When the motion first starts there is a transient, which settles down after awhile, and only *then* is the phase of the oscillation of the charge opposite to the driving field. And it is then that the *phase* of the transmitted field can appear to be *advanced* with respect to the source wave. It is this *advance in phase* which is meant when we say that the “phase velocity” or velocity of the nodes is greater than  $c$ . In Fig. 31-4 we give a schematic idea of how the waves might look for a case where the wave is suddenly turned on (to make a signal). You will see from the diagram that the *signal* (i.e., the *start* of the wave) is *not* earlier for the wave which ends up with an advance in phase.

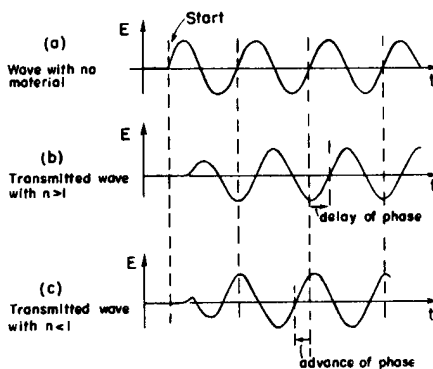


Fig. 31-4. Wave “signals.”

Let us now look again at our dispersion equation. We should remark that our analysis of the refractive index gives a result that is somewhat simpler than you would actually find in nature. To be completely accurate we must add some refinements. First, we should expect that our model of the atomic oscillator should have some damping force (otherwise once started it would oscillate forever, and we do not expect that to happen). We have worked out before (Eq. 23.8) the motion of a damped oscillator and the result is that the denominator in Eq. (31.16), and therefore in (31.19), is changed from  $(\omega_0^2 - \omega^2)$  to  $(\omega_0^2 - \omega^2 + i\gamma\omega)$ , where  $\gamma$  is the damping coefficient.

We need a second modification to take into account the fact that there are several resonant frequencies for a particular kind of atom. It is easy to fix up our

dispersion equation by imagining that there are several different kinds of oscillators, but that each oscillator acts separately, and so we simply add the contributions of all the oscillators. Let us say that there are  $N_k$  electrons per unit of volume, whose natural frequency is  $\omega_k$  and whose damping factor is  $\gamma_k$ . We would then have for our dispersion equation

$$n = 1 + \frac{q_e^2}{2\epsilon_0 m} \sum_k \frac{N_k}{\omega_k^2 - \omega^2 + i\gamma_k \omega}. \quad (31.20)$$

We have, finally, a complete expression which describes the index of refraction that is observed for many substances.\* The index described by this formula varies with frequency roughly like the curve shown in Fig. 31-5.

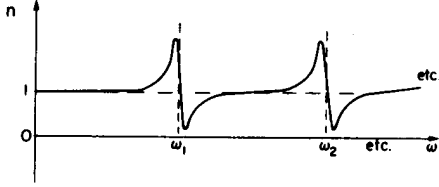


Fig. 31-5. The index of refraction as a function of frequency.

You will note that so long as  $\omega$  is not too close to one of the resonant frequencies, the slope of the curve is positive. Such a positive slope is called “normal” dispersion (because it is clearly the most common occurrence). Very near the resonant frequencies, however, there is a small range of  $\omega$ 's for which the slope is negative. Such a negative slope is often referred to as “anomalous” (meaning abnormal) dispersion, because it seemed unusual when it was first observed, long before anyone even knew there were such things as electrons. From our point of view both slopes are quite “normal”!

### 31-4 Absorption

Perhaps you have noticed something a little strange about the last form (Eq. 31.20) we obtained for our dispersion equation. Because of the term  $i\gamma$  we put in to take account of damping, the index of refraction is now a *complex number*! What does *that* mean? By working out what the real and imaginary parts of  $n$  are we could write

$$n = n' - in'', \quad (31.21)$$

where  $n'$  and  $n''$  are real numbers. (We use the minus sign in front of the  $in''$  because then  $n''$  will turn out to be a positive number, as you can show for yourself.)

We can see what such a complex index means by going back to Eq. (31.6), which is the equation of the wave after it goes through a plate of material with an index  $n$ . If we put our complex  $n$  into this equation, and do some rearranging, we get

$$E_{\text{after plate}} = \underbrace{e^{-\omega n'' \Delta z / c}}_A \underbrace{e^{-i\omega(n' - 1)\Delta z / c} E_0 e^{i\omega(t - z/c)}}_B \quad (31.22)$$

The last factors, marked B in Eq. (31.22), are just the form we had before, and again describe a wave whose phase has been delayed by the angle  $\omega(n' - 1)\Delta z/c$  in traversing the material. The first term (A) is new and is an exponential factor with a *real* exponent, because there were two  $i$ 's that cancelled. Also, the exponent is negative, so the factor is a real number less than one. It describes a *decrease* in the magnitude of the field and, as we should expect, by an amount which is more the larger  $\Delta z$  is. As the wave goes through the material, it is weakened. The material is “absorbing” part of the wave. The wave comes out the other side with less energy. We should not be surprised at this, because the damping we put in for the oscillators is indeed a friction force and must be expected to cause a loss of energy. We see that the imaginary part  $n''$  of a complex index of refraction represents an absorption (or “attenuation”) of the wave. In fact,  $n''$  is sometimes referred to as the “absorption index.”

We may also point out that an imaginary part to the index  $n$  corresponds to bending the arrow  $E_a$  in Fig. 31-3 toward the origin. It is clear why the transmitted field is then decreased.

\* Actually, although in quantum mechanics Eq. (31.20) is still valid, its interpretation is somewhat different. In quantum mechanics even an atom with one electron, like hydrogen, has several resonant frequencies. Therefore  $N_k$  is not really the number of electrons having the frequency  $\omega_k$ , but is replaced instead by  $Nf_k$ , where  $N$  is the number of atoms per unit volume and  $f_k$  (called the oscillator strength) is a factor that tells how strongly the atom exhibits each of its resonant frequencies  $\omega_k$ .



Normally, for instance as in glass, the absorption of light is very small. This is to be expected from our Eq. (31.20), because the imaginary part of the denominator,  $i\gamma_k\omega$ , is much smaller than the term  $(\omega_k^2 - \omega^2)$ . But if the light frequency  $\omega$  is very close to  $\omega_k$  then the resonance term  $(\omega_k^2 - \omega^2)$  can become small compared with  $i\gamma_k\omega$  and the index becomes almost completely imaginary. The absorption of the light becomes the dominant effect. It is just this effect that gives the dark lines in the spectrum of light which we receive from the sun. The light from the solar surface has passed through the sun's atmosphere (as well as the earth's), and the light has been strongly absorbed at the resonant frequencies of the atoms in the solar atmosphere.

The observation of such spectral lines in the sunlight allows us to tell the resonant frequencies of the atoms and hence the chemical composition of the sun's atmosphere. The same kind of observations tell us about the materials in the stars. From such measurements we know that the chemical elements in the sun and in the stars are the same as those we find on the earth.

### 31-5 The energy carried by an electric wave

We have seen that the imaginary part of the index means absorption. We shall now use this knowledge to find out how much energy is carried by a light wave. We have given earlier an argument that the energy carried by light is proportional to  $\overline{E^2}$ , the time average of the square of the electric field in the wave. The decrease in  $E$  due to absorption must mean a loss of energy, which would go into some friction of the electrons and, we might guess, would end up as heat in the material.

If we consider the light arriving on a unit area, say one square centimeter, of our plate in Fig. 31-1, then we can write the following energy equation (if we assume that energy is conserved, as we *do!*):

$$\text{Energy in per sec} = \text{energy out per sec} + \text{work done per sec.} \quad (31.23)$$

For the first term we can write  $\alpha\overline{E_s^2}$ , where  $\alpha$  is the as yet unknown constant of proportionality which relates the average value of  $E^2$  to the energy being carried. For the second term we must include the part from the radiating atoms of the material, so we should use  $\alpha(\overline{E_s + E_a})^2$ , or (evaluating the square)  $\alpha(\overline{E_s^2} + 2\overline{E_s E_a} + \overline{E_a^2})$ .

All of our calculations have been made for a thin layer of material whose index is not too far from 1, so that  $E_a$  would always be much less than  $E_s$  (just to make the calculations easier). In keeping with our approximations, we should, therefore, leave out the term  $\overline{E_a^2}$ , because it is much smaller than  $\overline{E_s E_a}$ . You may say: "Then you should leave out  $\overline{E_s E_a}$  also, because it is much smaller than  $\overline{E_s^2}$ ." It is true that  $\overline{E_s E_a}$  is much smaller than  $\overline{E_s^2}$ , but we must keep  $\overline{E_s E_a}$  or our approximation will be the one that would apply if we neglected the presence of the material completely! One way of checking that our calculations are consistent is to see that we always keep terms which are proportional to  $N\Delta z$ , the area density of atoms in the material, but we leave out terms which are proportional to  $(N\Delta z)^2$  or any higher power of  $N\Delta z$ . Ours is what should be called a "low-density approximation."

In the same spirit, we might remark that our energy equation has neglected the energy in the reflected wave. But that is OK because this term, too, is proportional to  $(N\Delta z)^2$ , since the amplitude of the reflected wave is proportional to  $N\Delta z$ .

For the last term in Eq. (31.23) we wish to compute the rate at which the incoming wave is doing work on the electrons. We know that work is force times distance, so the *rate* of doing work (also called power) is the force times the velocity. It is really  $\mathbf{F} \cdot \mathbf{V}$ , but we do not need to worry about the dot product when the velocity and force are along the same direction as they are here (except for a possible minus sign). So for each atom we take  $q_e\overline{E_s v}$  for the average rate of

doing work. Since there are  $N \Delta z$  atoms in a unit area, the last term in Eq. (31.23) should be  $N \Delta z q_e \overline{E_s v}$ . Our energy equation now looks like

$$\alpha \overline{E_s^2} = \alpha \overline{E_s^2} + 2\alpha \overline{E_s E_a} + N \Delta z q_e \overline{E_s v}. \quad (31.24)$$

The  $\overline{E_s^2}$  terms cancel, and we have

$$2\alpha \overline{E_s E_a} = N \Delta z q_e \overline{E_s v}. \quad (31.25)$$

We now go back to Eq. (30.19), which tells us that for large  $z$

$$E_a = \frac{N \Delta z q_e}{2\epsilon_0 c} v(\text{ret by } z/c) \quad (31.26)$$

(recalling that  $\eta = N \Delta z$ ). Putting Eq. (31.26) into the left-hand side of (31.25), we get

$$2\alpha \frac{N \Delta z q_e}{2\epsilon_0 c} \overline{E_s(\text{at } z) \cdot v(\text{ret by } z/c)}.$$

However,  $E_s(\text{at } z)$  is  $E_s$  (at atoms) retarded by  $z/c$ . Since the average is independent of time, it is the same now as retarded by  $z/c$ , or is  $\overline{E_s}(\text{at atom}) \cdot v$ , the same average that appears on the right-hand side of (31.25). The two sides are therefore equal if

$$\frac{\alpha}{\epsilon_0 c} = 1, \quad \text{or} \quad \alpha = \epsilon_0 c. \quad (31.27)$$

We have discovered that if energy is to be conserved, the energy carried in an electric wave per unit area and per unit time (or what we have called the *intensity*) must be given by  $\epsilon_0 c \overline{E^2}$ . If we call the intensity  $\overline{S}$ , we have

$$\overline{S} = \left\{ \begin{array}{c} \text{intensity} \\ \text{or} \\ \text{energy/area/time} \end{array} \right\} = \epsilon_0 c \overline{E^2}, \quad (31.28)$$

where the *bar* means the *time average*. We have a nice bonus result from our theory of the refractive index!

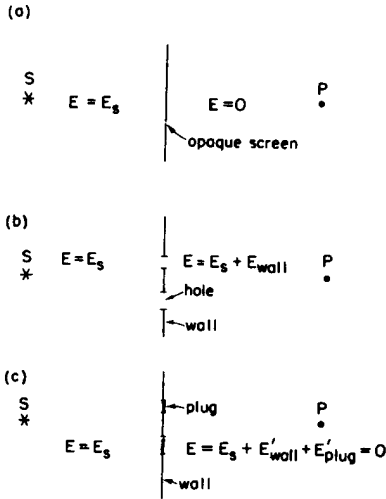


Fig. 31-6. Diffraction by a screen.

### 31-6 Diffraction of light by a screen

It is now a good time to take up a somewhat different matter which we can handle with the machinery of this chapter. In the last chapter we said that when you have an opaque screen and the light can come through some holes, the distribution of intensity—the diffraction pattern—could be obtained by imagining instead that the holes are replaced by sources (oscillators) uniformly distributed over the hole. In other words, the diffracted wave is the same as though the hole were a new source. We have to explain the reason for that, because the hole is, of course, just where there are *no* sources, where there are *no* accelerating charges.

Let us first ask: "What is an opaque screen?" Suppose we have a completely opaque screen between a source  $S$  and an observer at  $P$ , as in Fig. 31-6(a). If the screen is "opaque" there is no field at  $P$ . Why is there no field there? According to the basic principles we should obtain the field at  $P$  as the field  $E_s$  of the source delayed, plus the field from all the other charges around. But, as we have seen above, the charges in the screen will be set in motion by the field  $E_s$ , and these motions generate a new field which, if the screen is opaque, must *exactly cancel* the field  $E_s$  on the back side of the screen. You say: "What a miracle that it balances *exactly*! Suppose it was not exactly right!" If it were not exactly right (remember that this opaque screen has some thickness), the field toward the rear part of the screen would not be exactly zero. So, not being zero, it would set into motion some other charges in the material of the screen, and thus make a little more field, trying to get the total balanced out. So if we make the screen thick enough, there is no residual field, because there is enough opportunity to finally get the thing quieted down. In terms of our formulas above we would say that the

screen has a large and imaginary index, so the wave is absorbed exponentially as it goes through. You know, of course, that a thin enough sheet of the most opaque material, even gold, is transparent.

Now let us see what happens with an opaque screen which has holes in it, as in Fig. 31-6(b). What do we expect for the field at  $P$ ? The field at  $P$  can be represented as a sum of two parts—the field due to the source  $S$  plus the field due to the wall, i.e., due to the motions of the charges in the walls. We might expect the motions of the charges in the walls to be complicated, but we can find out *what fields they produce* in a rather simple way.

Suppose that we were to take the same screen, but plug up the holes, as indicated in part (c) of the figure. We imagine that the plugs are of exactly the same material as the wall. Mind you, the plugs go where the holes were in case (b). Now let us calculate the field at  $P$ . The field at  $P$  is certainly zero in case (c), but it is *also* equal to the field from the source plus the field due to all the motions of the atoms in the walls and in the plugs. We can write the following equations:

$$\text{Case (b): } E_{\text{at } P} = E_s + E_{\text{wall}},$$

$$\text{Case (c): } E'_{\text{at } P} = 0 = E_s + E'_{\text{wall}} + E'_{\text{plug}},$$

where the primes refer to the case where the plugs are in place, but  $E_s$  is, of course, the same in both cases. Now if we subtract the two equations, we get

$$E_{\text{at } P} = (E_{\text{wall}} - E'_{\text{wall}}) - E'_{\text{plug}}.$$

Now if the holes are not too small (say many wavelengths across), we would not expect the presence of the plugs to change the fields which arrive at the walls except possibly for a little bit around the edges of the holes. Neglecting this small effect, we can set  $E_{\text{wall}} = E'_{\text{wall}}$  and obtain that

$$E_{\text{at } P} = -E'_{\text{plug}}.$$

We have the result that the field at  $P$  *when there are holes* in a screen (case  $b$ ) is the same (except for sign) as the field that is produced by *that part* of a complete opaque wall which is *located where the holes are!* (The sign is not too interesting, since we are usually interested in intensity which is proportional to the square of the field.) It seems like an amazing backwards-forwards argument. It is, however, not only true (approximately for not too small holes), but useful, and is the justification for the usual theory of diffraction.

The field  $E'_{\text{plug}}$  is computed in any particular case by remembering that the motion of the charges *everywhere* in the screen is just that which will cancel out the field  $E_s$  on the back of the screen. Once we know these motions, we add the radiation fields at  $P$  due just to the charges in the plugs.

We remark again that this theory of diffraction is only approximate, and will be good only if the holes are not too small. For holes which are too small the  $E'_{\text{plug}}$  term will be small and then the difference between  $E'_{\text{wall}}$  and  $E_{\text{wall}}$  (which difference we have taken to be zero) may be comparable to or larger than the small  $E'_{\text{plug}}$  term, and our approximation will no longer be valid.

## ***Radiation Damping. Light Scattering***

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### **32-1 Radiation resistance**

In the last chapter we learned that when a system is oscillating, energy is carried away, and we deduced a formula for the energy which is radiated by an oscillating system. If we know the electric field, then the average of the square of the field times  $\epsilon_0 c$  is the amount of energy that passes per square meter per second through a surface normal to the direction in which the radiation is going:

$$S = \epsilon_0 c \langle E^2 \rangle. \quad (32.1)$$

Any oscillating charge radiates energy; for instance, a driven antenna radiates energy. If the system radiates energy, then in order to account for the conservation of energy we must find that power is being delivered along the wires which lead into the antenna. That is, to the driving circuit the antenna acts like a *resistance*, or a place where energy can be “lost” (the energy is not really lost, it is really radiated out, but so far as the circuit is concerned, the energy is lost). In an ordinary resistance, the energy which is “lost” passes into heat; in this case the energy which is “lost” goes out into space. But from the standpoint of circuit theory, without considering *where* the energy goes, the net effect on the circuit is the same—energy is “lost” from that circuit. Therefore the antenna appears to the generator as having a resistance, even though it may be made with perfectly good copper. In fact, if it is well built it will appear as almost a pure resistance, with very little inductance or capacitance, because we would like to radiate as much energy as possible out of the antenna. This resistance that an antenna shows is called the *radiation resistance*.

If a current  $I$  is going to the antenna, then the average rate at which power is delivered to the antenna is the average of the square of the current times the resistance. The rate at which power is *radiated* by the antenna is proportional to the square of the current in the antenna, of course, because all the fields are proportional to the currents, and the energy liberated is proportional to the square of the field. The coefficient of proportionality between radiated power and  $\langle I^2 \rangle$  is the radiation resistance.

An interesting question is, what is this radiation resistance due to? Let us take a simple example: let us say that currents are driven up and down in an antenna. We find that we have to put work in, if the antenna is to radiate energy. If we take a charged body and accelerate it up and down it radiates energy; if it were not charged it would not radiate energy. It is one thing to calculate from the conservation of energy that energy is lost, but another thing to answer the question, *against what force* are we doing the work? That is an interesting and very difficult question which has never been completely and satisfactorily answered for electrons, although it has been for antennas. What happens is this: in an antenna, the fields produced by the moving charges in one part of the antenna react on the moving charges in another part of the antenna. We can calculate these forces and find out how much work they do, and so find the right rule for the radiation resistance. When we say “We can calculate—” that is not quite right—we cannot, because we have not yet studied the laws of electricity at short distances; only at large distances do we know what the electric field is. We saw the formula (28.3), but at present it is too complicated for us to calculate the fields inside the wave zone. Of course, since conservation of energy is valid, we can calculate the result all right without knowing the fields at short distances. (As a matter of fact, by using this argument backwards it turns out that one can find the formula for the forces at short

### **32-1 Radiation resistance**

### **32-2 The rate of radiation of energy**

### **32-3 Radiation damping**

### **32-4 Independent sources**

### **32-5 Scattering of light**

distances only by knowing the field at very large distances, by using the laws of conservation of energy, but we shall not go into that here.)

The problem in the case of a single electron is this: if there is only one charge, what can the force act on? It has been proposed, in the old classical theory, that the charge was a little ball, and that one part of the charge acted on the other part. Because of the delay in the action across the tiny electron, the force is not exactly in phase with the motion. That is, if we have the electron standing still, we know that "action equals reaction." So the various internal forces are equal, and there is no net force. But if the electron is accelerating, then because of the time delay across it, the force which is acting on the front from the back is not exactly the same as the force on the back from the front, because of the delay in the effect. This delay in the timing makes for a lack of balance, so, as a net effect, the thing holds itself back by its bootstraps! This model of the origin of the resistance to acceleration, the radiation resistance of a moving charge, has run into many difficulties, because our present view of the electron is that it is *not* a "little ball"; this problem has never been solved. Nevertheless we can calculate exactly, of course, what the net radiation resistance force must be, i.e., how much loss there must be when we accelerate a charge, in spite of not knowing directly the mechanism of how that force works.

### 32-2 The rate of radiation of energy

Now we shall calculate the total energy radiated by an accelerating charge. To keep the discussion general, we shall take the case of a charge accelerating any which way, but nonrelativistically. At a moment when the acceleration is, say, vertical, we know that the electric field that is generated is the charge multiplied by the projection of the retarded acceleration, divided by the distance. So we know the electric field at any point, and we therefore know the square of the electric field and thus the energy  $\epsilon_0 c E^2$  leaving through a unit area per second.

The quantity  $\epsilon_0 c$  appears quite often in expressions involving radiowave propagation. Its reciprocal is called the *impedance of a vacuum*, and it is an easy number to remember: it has the value  $1/\epsilon_0 c = 377$  ohms. So the power in watts per square meter is equal to the average of the field squared, divided by 377.

Using our expression (29.1) for the electric field, we find that

$$S = \frac{q^2 a'^2 \sin^2 \theta}{16\pi^2 \epsilon_0 r^2 c^3} \quad (32.2)$$

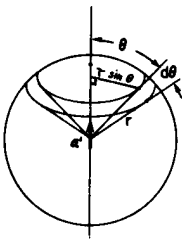


Fig. 32-1. The area of a spherical segment is  $2\pi r \sin \theta \cdot r d\theta$ .

is the power per square meter radiated in the direction  $\theta$ . We notice that it goes inversely as the square of the distance, as we said before. Now suppose we wanted the total energy radiated in all directions: then we must integrate (32.2) over all directions. First we multiply by the area, to find the amount that flows within a little angle  $d\theta$  (Fig. 32-1). We need the area of a spherical section. The way to think of it is this; if  $r$  is the radius, then the width of the annular segment is  $r d\theta$ , and the circumference is  $2\pi r \sin \theta$ , because  $r \sin \theta$  is the radius of the circle. So the area of the little piece of the sphere is  $2\pi r \sin \theta$  times  $r d\theta$ :

$$dA = 2\pi r^2 \sin \theta d\theta. \quad (32.3)$$

By multiplying the flux [(32.2), the power per square meter] by the area in square meters included in the small angle  $d\theta$ , we find the amount of energy that is liberated in this direction between  $\theta$  and  $\theta + d\theta$ ; then we integrate that over all the angles  $\theta$  from 0 to  $180^\circ$ :

$$P = \int S dA = \frac{q^2 a'^2}{8\pi \epsilon_0 c^3} \int_0^\pi \sin^3 \theta d\theta. \quad (32.4)$$

By writing  $\sin^3 \theta = (1 - \cos^2 \theta) \sin \theta$  it is not hard to show that  $\int_0^\pi \sin^3 \theta d\theta = 4/3$ . Using that fact, we finally get

$$P = \frac{q^2 a'^2}{6\pi \epsilon_0 c^3}. \quad (32.5)$$

This expression deserves some remarks. First of all, since the vector  $\mathbf{a}'$  had a certain direction, the  $a'^2$  in (32.5) would be the square of the vector  $\mathbf{a}'$ , that is,  $\mathbf{a}' \cdot \mathbf{a}'$ , the length of the vector, squared. Secondly, the flux (32.2) was calculated using the retarded acceleration; that is, the acceleration at the time at which the energy now passing through the sphere was radiated. We might like to say that this energy was in fact liberated at this earlier time. This is not exactly true; it is only an approximate idea. The exact time when the energy is liberated cannot be defined precisely. All we can really calculate precisely is what happens in a complete motion, like an oscillation or something, where the acceleration finally ceases. Then what we find is that the total energy flux per cycle is the average of acceleration squared, for a complete cycle. This is what should really appear in (32.5). Or, if it is a motion with an acceleration that is initially and finally zero, then the total energy that has flown out is the time integral of (32.5).

To illustrate the consequences of formula (32.5) when we have an oscillating system, let us see what happens if the displacement  $x$  of the charge is oscillating so that the acceleration  $a$  is  $-\omega^2 x_0 e^{i\omega t}$ . The average of the acceleration squared over a cycle (remember that we have to be very careful when we square things that are written in complex notation—it really is the cosine, and the average of  $\cos^2 \omega t$  is one-half) thus is

$$\langle a'^2 \rangle = \frac{1}{2} \omega^4 x_0^2.$$

Therefore

$$P = \frac{q^2 \omega^4 x_0^2}{12 \pi \epsilon_0 c^3}. \quad (32.6)$$

The formulas we are now discussing are relatively advanced and more or less modern; they date from the beginning of the twentieth century, and they are very famous. Because of their historical value, it is important for us to be able to read about them in older books. In fact, the older books also used a system of units different from our present mks system. However, all these complications can be straightened out in the final formulas dealing with electrons by the following rule: The quantity  $q_e^2/4\pi\epsilon_0$ , where  $q_e$  is the electronic charge (in coulombs), has, historically, been written as  $e^2$ . It is very easy to calculate that  $e$  in the mks system is numerically equal to  $1.5188 \times 10^{-14}$ , because we know that, numerically,  $q_e = 1.60206 \times 10^{-19}$  and  $1/4\pi\epsilon_0 = 8.98748 \times 10^9$ . Therefore we shall often use the convenient abbreviation

$$e^2 = \frac{q_e^2}{4\pi\epsilon_0}. \quad (32.7)$$

If we use the above numerical value of  $e$  in the older formulas and treat them as though they were written in mks units, we will get the right numerical results. For example, the older form of (32.5) is  $P = \frac{2}{3} e^2 a^2 / c^3$ . Again, the potential energy of a proton and an electron at distance  $r$  is  $q_e^2/4\pi\epsilon_0 r$  or  $e^2/r$ , with  $e = 1.5188 \times 10^{-14}$  mks.

### 32-3 Radiation damping

Now the fact that an oscillator loses a certain energy would mean that if we had a charge on the end of a spring (or an electron in an atom) which has a natural frequency  $\omega_0$ , and we start it oscillating and let it go, it will not oscillate forever, even if it is in empty space millions of miles from anything. There is no oil, no resistance, in an ordinary sense; no “viscosity.” But nevertheless it will not oscillate, as we might once have said, “forever,” because if it is charged it is radiating energy, and therefore the oscillation will slowly die out. How slowly? What is the  $Q$  of such an oscillator, caused by the electromagnetic effects, the so-called radiation resistance or radiation damping of the oscillator? The  $Q$  of any oscillating system is the total energy content of the oscillator at any time divided by the energy loss per radian:

$$Q = \frac{W}{dW/d\phi}.$$

Or (another way to write it), since  $dW/d\phi = (dW/dt)/(d\phi/dt) = (dW/dt)/\omega$ ,

$$Q = \frac{\omega W}{dW/dt}. \quad (32.8)$$

If for a given  $Q$  this tells us how the energy of the oscillation dies out,  $dW/dt = -(\omega/Q)W$ , which has the solution  $W = W_0 e^{-\omega t/Q}$  if  $W_0$  is the initial energy (at  $t = 0$ ).

To find the  $Q$  for a radiator, we go back to (32.8) and use (32.6) for  $dW/dt$ .

Now what do we use for the energy  $W$  of the oscillator? The kinetic energy of the oscillator is  $\frac{1}{2}mv^2$ , and the *mean* kinetic energy is  $m\omega^2 x_0^2/4$ . But we remember that for the total energy of an oscillator, on the average half is kinetic and half is potential energy, and so we double our result, and find for the total energy of the oscillator

$$W = \frac{1}{2}m\omega^2 x_0^2. \quad (32.9)$$

What do we use for the frequency in our formulas? We use the natural frequency  $\omega_0$  because, for all practical purposes, that is the frequency at which our atom is radiating, and for  $m$  we use the electron mass  $m_e$ . Then, making the necessary divisions and cancellations, the formula comes down to

$$\frac{1}{Q} = \frac{4\pi e^2}{3\lambda m_e c^2}. \quad (32.10)$$

(In order to see it better and in a more historical form we write it using our abbreviation  $q_e^2/4\pi\epsilon_0 = e^2$ , and the factor  $\omega_0/c$  which was left over has been written as  $2\pi/\lambda$ .) Since  $Q$  is dimensionless, the combination  $e^2/m_e c^2$  must be a property only of the electron charge and mass, an intrinsic property of the electron, and it must be a *length*. It has been given a name, the *classical electron radius*, because the early atomic models, which were invented to explain the radiation resistance on the basis of the force of one part of the electron acting on the other parts, all needed to have an electron whose dimensions were of this general order of magnitude. However, this quantity no longer has the significance that we believe that the electron really *has* such a radius. Numerically, the magnitude of the radius is

$$r_0 = \frac{e^2}{m_e c^2} = 2.82 \times 10^{-15} \text{ m}. \quad (32.11)$$

Now let us actually calculate the  $Q$  of an atom that is emitting light—let us say a sodium atom. For a sodium atom, the wavelength is roughly 6000 angstroms, in the yellow part of the visible spectrum, and this is a typical wavelength. Thus

$$Q = \frac{3\lambda}{4\pi r_0} \approx 5 \times 10^7, \quad (32.12)$$

so the  $Q$  of an atom is of the order  $10^8$ . This means that an atomic oscillator will oscillate for  $10^8$  radians or about  $10^7$  oscillations, before its energy falls by a factor  $1/e$ . The frequency of oscillation of light corresponding to 6000 angstroms,  $\nu = c/\lambda$ , is on the order of  $10^{15}$  cycles/sec, and therefore the lifetime, the time it takes for the energy of a radiating atom to die out by a factor  $1/e$ , is on the order of  $10^{-8}$  sec. In ordinary circumstances, freely emitting atoms usually take about this long to radiate. This is valid only for atoms which are in empty space, not being disturbed in any way. If the electron is in a solid and it has to hit other atoms or other electrons, then there are additional resistances and different damping.

The effective resistance term  $\gamma$  in the resistance law for the oscillator can be found from the relation  $1/Q = \gamma/\omega_0$ , and we remember that the size of  $\gamma$  determines how wide the resonance curve is (Fig. 23-2). Thus we have just computed the *widths of spectral lines* for freely radiating atoms! Since  $\lambda = 2\pi c/\omega$ , we find that

$$\begin{aligned} \Delta\lambda &= 2\pi c \Delta\omega/\omega^2 = 2\pi c \gamma/\omega_0^2 = 2\pi c/Q\omega_0 \\ &= \lambda/Q = 4\pi r_0/3 = 1.18 \times 10^{-14} \text{ m}. \end{aligned} \quad (32.13)$$

### 32-4 Independent sources

In preparation for our second topic, the scattering of light, we must now discuss a certain feature of the phenomenon of interference that we neglected to discuss previously. This is the question of when interference does *not* occur. If we have two sources  $S_1$  and  $S_2$ , with amplitudes  $A_1$  and  $A_2$ , and we make an observation in a certain direction in which the phases of arrival of the two signals are  $\phi_1$  and  $\phi_2$  (a combination of the actual time of oscillation and the delayed time, depending on the position of observation), then the energy that we receive can be found by compounding the two complex number vectors  $A_1$  and  $A_2$ , one at angle  $\phi_1$  and the other at angle  $\phi_2$  (as we did in Chapter 30) and we find that the resultant energy is proportional to

$$A_R^2 = A_1^2 + A_2^2 + 2A_1A_2 \cos(\phi_1 - \phi_2). \quad (32.14)$$

Now if the cross term  $2A_1A_2 \cos(\phi_1 - \phi_2)$  were not there, then the total energy that would be received in a given direction would simply be the sum of the energies,  $A_1^2 + A_2^2$ , that would be liberated by each source separately, which is what we usually expect. That is, the combined intensity of light shining on something from two sources is the sum of the intensities of the two lights. On the other hand, if we have things set just right and we have a cross term, it is not such a sum, because there is also some interference. If there are circumstances in which this term is of no importance, then we would say the interference is apparently lost. Of course, in nature it is always there, but we may not be able to detect it.

Let us consider some examples. Suppose, first, that the two sources are 7,000,000,000 wavelengths apart, not an impossible arrangement. Then in a given direction it is true that there is a very definite value of these phase differences. But, on the other hand, if we move just a hair in one direction, a few wavelengths, which is no distance at all (our eye already has a hole in it that is so large that we are averaging the effects over a range very wide compared with one wavelength) then we change the relative phase, and the cosine changes very rapidly. If we take the *average* of the intensity over a little region, then the cosine, which goes plus, minus, plus, minus, as we move around, averages to zero.

So if we average over regions where the phase varies very rapidly with position, we get no interference.

Another example. Suppose that the two sources are two independent radio oscillators—not a single oscillator being fed by two wires, which guarantees that the phases are kept together, but two independent sources—and that they are not *precisely* tuned at the same frequency (it is very hard to make them at exactly the same frequency without actually wiring them together). In this case we have what we call two *independent* sources. Of course, since the frequencies are not exactly equal, although they started in phase, one of them begins to get a little ahead of the other, and pretty soon they are out of phase, and then it gets still further ahead, and pretty soon they are in phase again. So the phase difference between the two is gradually drifting with time, but if our observation is so crude that we cannot see that little time, if we average over a much longer time, then although the intensity swells and falls like what we call “beats” in sound, if these swellings and fallings are too rapid for our equipment to follow, then again this term averages out.

In other words, in any circumstance in which the phase shift averages out, we get no interference!

One finds many books which say that two distinct light sources never interfere. This is not a statement of physics, but is merely a statement of the degree of sensitivity of the technique of the experiments at the time the book was written. What happens in a light source is that first one atom radiates, then another atom radiates, and so forth, and we have just seen that atoms radiate a train of waves only for about  $10^{-8}$  sec; after  $10^{-8}$  sec, some atom has probably taken over, then another atom takes over, and so on. So the phases can really only stay the same for about  $10^{-8}$  sec. Therefore, if we average for very much more than  $10^{-8}$  sec, we do not see an interference from two different sources, because they cannot hold their phases steady for longer than  $10^{-8}$  sec. With photocells, very high-speed detection



is possible, and one can show that there is an interference which varies with time, up and down, in about  $10^{-8}$  sec. But most detection equipment, of course, does not look at such fine time intervals, and thus sees no interference. Certainly with the eye, which has a tenth-of-a-second averaging time, there is no chance whatever of seeing an interference between two different ordinary sources.

Recently it has become possible to make light sources which get around this effect by making all the atoms emit *together* in time. The device which does this is a very complicated thing, and has to be understood in a quantum-mechanical way. It is called a *laser*, and it is possible to produce from a laser a source in which the interference frequency, the time at which the phase is kept constant, is very much longer than  $10^{-8}$  sec. It can be of the order of a hundredth, a tenth, or even one second, and so, with ordinary photocells, one can pick up the frequency between two different lasers. One can easily detect the pulsing of the beats between two laser sources. Soon, no doubt, someone will be able to demonstrate two sources shining on a wall, in which the beats are so slow that one can *see* the wall get bright and dark!

Another case in which the interference averages out is that in which, instead of having only *two* sources, we have *many*. In this case, we would write the expression for  $A_R^2$  as the sum of a whole lot of amplitudes, complex numbers, squared, and we would get the square of each one, all added together, plus cross terms between every pair, and if the circumstances are such that the latter average out, then there will be no effects of interference. It may be that the various sources are located in such random positions that, although the phase difference between  $A_2$  and  $A_3$  is also definite, it is very different from that between  $A_1$  and  $A_2$ , etc. So we would get a whole lot of cosines, many plus, many minus, all averaging out.

So it is that in many circumstances we do not see the effects of interference, but see only a collective, total intensity equal to the sum of all the intensities.

### 32-5 Scattering of light

The above leads us to an effect which occurs in air as a consequence of the irregular positions of the atoms. When we were discussing the index of refraction, we saw that an incoming beam of light will make the atoms radiate again. The electric field of the incoming beam drives the electrons up and down, and they radiate because of their acceleration. This scattered radiation combines to give a beam in the same direction as the incoming beam, but of somewhat different phase, and this is the origin of the index of refraction.

But what can we say about the amount of re-radiated light in some other direction? Ordinarily, if the atoms are very beautifully located in a nice pattern, it is easy to show that we get nothing in other directions, because we are adding a lot of vectors with their phases always changing, and the result comes to zero. But if the objects are *randomly* located, then the total intensity in any direction is the *sum* of the intensities that are scattered by each atom, as we have just discussed. Furthermore, the atoms in a gas are in actual motion, so that although the relative phase of two atoms is a definite amount now, later the phase would be quite different, and therefore *each* cosine term will average out. Therefore, to find out how much light is scattered in a given direction by a gas, we merely study the effects of *one atom* and multiply the intensity it radiates by the number of atoms.

Earlier, we remarked that the phenomenon of scattering of light of this nature is the origin of the blue of the sky. The sunlight goes through the air, and when we look to one side of the sun—say at  $90^\circ$  to the beam—we see blue light; what we now have to calculate is *how much* light we see and *why it is blue*.

If the incident beam has the electric field  $\mathbf{E} = \mathbf{E}_0 e^{i\omega t}$  at the point where the atom is located, we know that an electron in the atom will vibrate up and down in response to this  $\mathbf{E}$  (Fig. 32-2). From Eq. (23.8), the amplitude will be

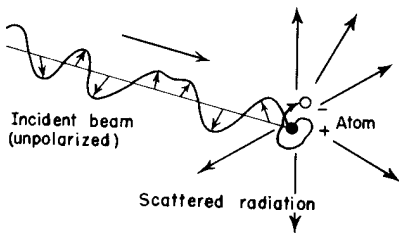


Fig. 32-2. A beam of radiation falls on an atom and causes the charges (electrons) in the atom to move. The moving electrons in turn radiate in various directions.

$$\hat{x} = \frac{q_e \mathbf{E}_0}{m(\omega_0^2 - \omega^2 + i\omega\gamma)}. \quad (32.15)$$

We could include the damping and the possibility that the atom acts like several oscillators of different frequency and sum over the various frequencies, but for simplicity let us just take one oscillator and neglect the damping. Then the response to the external electric field, which we have already used in the calculation of the index of refraction, is simply

$$\hat{\mathbf{x}} = \frac{q_e \hat{\mathbf{E}}_0}{m(\omega_0^2 - \omega^2)}. \quad (32.16)$$

We could now easily calculate the intensity of light that is emitted in various directions, using formula (32.2) and the acceleration corresponding to the above  $\hat{\mathbf{x}}$ .

Rather than do this, however, we shall simply calculate the *total amount* of light scattered in *all* directions, just to save time. The total amount of light energy per second, scattered in all directions by the single atom, is of course given by Eq. (32.7). So, putting together the various pieces and regrouping them, we get

$$\begin{aligned} P &= [(q_e^2 \omega^4 / 12 \pi \epsilon_0 c^3) q_e^2 E_0^2 / m^2 (\omega^2 - \omega_0^2)^2] \\ &= (\frac{1}{2} \epsilon_0 c E_0^2) (8 \pi / 3) (q_e^4 / 16 \pi^2 \epsilon_0^2 m^2 c^4) [\omega^4 / (\omega^2 - \omega_0^2)^2] \\ &= (\frac{1}{2} \epsilon_0 c E_0^2) (8 \pi r_0^2 / 3) [\omega^4 / (\omega^2 - \omega_0^2)^2] \end{aligned} \quad (32.17)$$

for the total scattered power, radiated in all directions.

We have written the result in the above form because it is then easy to remember: First, the total energy that is scattered is proportional to the square of the incident field. What does that mean? Obviously, the square of the incident field is proportional to the energy which is coming in per second. In fact, the energy incident per square meter per second is  $\epsilon_0 c$  times the average  $\langle E^2 \rangle$  of the square of the electric field, and if  $E_0$  is the maximum value of  $E$ , then  $\langle E^2 \rangle = \frac{1}{2} E_0^2$ . In other words, the total energy scattered is proportional to the energy per square meter that comes in; the brighter the sunlight that is shining in the sky, the brighter the sky is going to look.

Next, what *fraction* of the incoming light is scattered? Let us imagine a “target” with a certain area, let us say  $\sigma$ , in the beam (not a real, material target, because this would diffract light, and so on; we mean an imaginary area drawn in space). The total amount of energy that would pass through this surface  $\sigma$  in a given circumstance is proportional both to the incoming intensity and to  $\sigma$ , and would be

$$P = (\frac{1}{2} \epsilon_0 c E_0^2) \sigma. \quad (32.18)$$

Now we invent an idea: we say that the atom scatters a total amount of intensity which is the amount which would fall on a certain geometrical area, and we give the answer by giving that area. That answer, then, is independent of the incident intensity; it gives the ratio of the energy scattered to the energy incident per square meter. In other words, the ratio

$$\frac{\text{total energy scattered per second}}{\text{energy incident per square meter per second}} \text{ is an } \textit{area}.$$

The significance of this area is that, if all the energy that impinged on that area were to be spewed in all directions, then that is the amount of energy that would be scattered by the atom.

This area is called a *cross section for scattering*; the idea of cross section is used constantly, whenever some phenomenon occurs in proportion to the intensity of a beam. In such cases one always describes the amount of the phenomenon by saying what the effective area would have to be to pick up that much of the beam. It does not mean in any way that this oscillator actually *has* such an area. If there were nothing present but a free electron shaking up and down there would be no area directly associated with it, physically. It is merely a way of expressing the answer to a certain kind of problem; it tells us what area the incident beam would

have to hit in order to account for that much energy coming off. Thus, for our case,

$$\sigma_s = \frac{8\pi r_0^2}{3} \cdot \frac{\omega^4}{(\omega^2 - \omega_0^2)^2} \quad (32.19)$$

(the subscript  $s$  is for “scattering”).

Let us look at some examples. First, if we go to a very low natural frequency  $\omega_0$ , or to completely unbound electrons, for which  $\omega_0 = 0$ , then the frequency  $\omega$  cancels out and the cross section is a constant. This low-frequency limit, or the free electron cross section, is known as the *Thompson scattering cross section*. It is an area whose dimensions are approximately  $10^{-15}$  meter, more or less, on a side, i.e.,  $10^{-30}$  square meter, which is rather small!

On the other hand, if we take the case of light in the air, we remember that for air the natural frequencies of the oscillators are higher than the frequency of the light that we use. This means that, to a first approximation, we can disregard  $\omega^2$  in the denominator, and we find that the scattering is proportional to the *fourth power* of the frequency. That is to say, light which is of higher frequency by, say, a factor of two, is *sixteen times* more intensely scattered, which is a quite sizable difference. This means that blue light, which has about twice the frequency of the reddish end of the spectrum, is scattered to a far greater extent than red light. Thus when we look at the sky it looks that glorious blue that we see all the time!

There are several points to be made about the above results. One interesting question is, why do we ever see the *clouds*? Where do the clouds come from? Everybody knows it is the condensation of water vapor. But, of course, the water vapor is already in the atmosphere *before* it condenses, so why don't we see it then? After it condenses it is perfectly obvious. It wasn't there, now it *is* there. So the mystery of where the clouds come from is not really such a childish mystery as “Where does the water come from, Daddy?,” but has to be explained.

We have just explained that every atom scatters light, and of course the water vapor will scatter light, too. The mystery is why, when the water is condensed into clouds, does it scatter such a *tremendously greater amount* of light?

Consider what would happen if, instead of a single atom, we had an agglomerate of atoms, say two, very close together compared with the wavelength of the light. Remember, atoms are only an angstrom or so across, while the wavelength of light is some 5000 angstroms, so when they form a clump, a few atoms together, they can *be* very close together compared with the wavelength of light. Then when the electric field acts, *both of the atoms will move together*. The electric field that is scattered will then be the sum of the two electric fields in phase, i.e., double the amplitude that there was with a single atom, and the *energy* which is scattered is therefore *four times* what it is with a single atom, not twice! So lumps of atoms radiate or scatter more energy than they do as single atoms. Our argument that the phases are independent is based on the assumption that there is a real and large difference in phase between any two atoms, which is true only if they are several wavelengths apart and randomly spaced, or moving. But if they are right next to each other, they necessarily scatter in phase, and they have a coherent interference which produces an increase in the scattering.

If we have  $N$  atoms in a lump, which is a tiny droplet of water, then each one will be driven by the electric field in about the same way as before (the effect of one atom on the other is not important; it is just to get the idea anyway) and the amplitude of scattering from each one is the same, so the total field which is scattered is  $N$ -fold increased. The *intensity* of the light which is scattered is then the square, or  $N^2$ -fold, increased. We would have expected, if the atoms were spread out in space, only  $N$  times as much as 1, whereas we get  $N^2$  times as much as 1! That is to say, the scattering of water in lumps of  $N$  molecules each is  $N$  times more intense than the scattering of the single atoms. So as the water agglomerates the scattering increases. Does it increase *ad infinitum*? No! When does this analysis begin to fail? How many atoms can we put together before we cannot drive this argument any further? *Answer*: If the water drop gets so big that from one end to the other is a wavelength or so, then the atoms are no longer all in

phase because they are too far apart. So as we keep increasing the size of the drop-lets we get more and more scattering, until such a time that a drop gets about the size of a wavelength, and then the scattering does not increase anywhere nearly as rapidly as the drop gets bigger. Furthermore, the blue disappears, because for long wavelengths the drops can be bigger, before this limit is reached, than they can be for short wavelengths. Although the short waves scatter more per atom than the long waves, there is a bigger enhancement for the red end of the spectrum than for the blue end when all the drops are bigger than the wavelength, so the color is shifted from the blue toward the red.

Now we can make an experiment that demonstrates this. We can make particles that are very small at first, and then gradually grow in size. We use a solution of sodium thiosulfate (hypo) with sulphuric acid, which precipitates very fine grains of sulphur. As the sulphur precipitates, the grains first start very small, and the scattering is a little bluish. As it precipitates more it gets more intense, and then it will get whitish as the particles get bigger. In addition, the light which goes straight through will have the blue taken out. That is why the sunset is red, of course, because the light that comes through a lot of air to the eye has had a lot of blue light scattered out, so it is yellow-red.

Finally, there is one other important feature which really belongs in the next chapter, on polarization, but it is so interesting that we point it out now. This is that the electric field of the scattered light tends to vibrate in a particular direction. The electric field in the incoming light is oscillating in some way, and the driven oscillator goes in this same direction, and if we are situated about at right angles to the beam, we will see *polarized* light, that is to say, light in which the electric field is going only one way. In general, the atoms can vibrate in any direction at right angles to the beam, but if they are driven directly toward or away from us, we do not see it. So if the incoming light has an electric field which changes and oscillates in any direction, which we call unpolarized light, then the light which is coming out at  $90^\circ$  to the beam vibrates in only one direction! (See Fig. 32-3.)

There is a substance called polaroid which has the property that when light goes through it, only the piece of the electric field which is along one particular axis can get through. We can use this to test for polarization, and indeed we find the light scattered by the hypo solution to be strongly polarized.

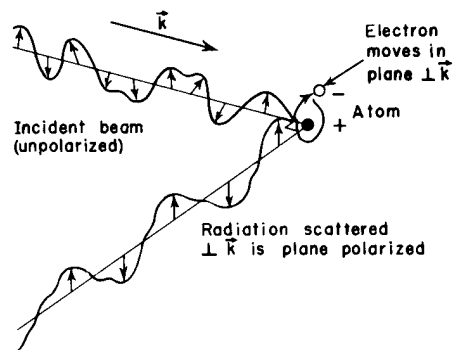


Fig. 32-3. Illustration of the origin of the polarization of radiation scattered at right angles to the incident beam.