Note: looks different from $J_i = i\hbar K_i$ above, since in this basis J_z is diagonal. Otherwise, just related by orthogonal change of basis.

- For General j, if $\overset{=l}{j} \in \mathbb{Z}$, representation of $SO(3)$ since $e^{-i/\hbar\, 2\pi J_i} = \mathbb{1}$.
  if $j + 1/2 \in \mathbb{Z}$, $e^{i/\hbar\, 2\pi J_i} = -\mathbb{1}$, not a rep. of $SO(3)$.

3.3 Spherical harmonics

Consider functions on $S^2 = \{x, y, z : x^2 + y^2 + z^2 = 1\}$, (parameterized by θ, ϕ.)

An $SO(3)$ rotation gives a linear transformation on the set of functions on S^2. The set of homogeneous polynomials of degree N (in x, y, z) is invariant under $SO(3)$ & must form a representation of $so(3)$, hence of $SU(2)$. (with $j \in \mathbb{Z}$)

Counting functions:

		total
constant :	1	1
linear :	x, y, z	3
quadratic :	$x^2, y^2, z^2, xy, yz, zx$	$6 = 5 + 1$ from $x^2+y^2+z^2$

Total # of independent polynomials of degree N:

$$\binom{N+2}{2} + \binom{N+1}{2} = \sum_{\substack{k\,\text{odd} \\ k=1}}^{2N+1} k = (N+1)^2$$

$\underset{x}{\cdots}|\underset{y}{\cdots}|\underset{z}{\cdots}$ $\left[\text{or } + \frac{(N+2)(N+1)}{2}\right]$

Theorem:
At degree N, acquire $2N+1$ polynomials living in a spin $j=N$ (irreducible) representation.

Associated eigenfunctions of J^2, J_z: $Y_{lm}(\theta, \phi)$

Can explicitly construct $Y_{\ell m}(\theta,\phi)$ from general representation theory.

Defining $\vec{L} = \vec{r} \times \vec{p}$, generators of $so(3)$

$$L^i = -i\hbar\, \varepsilon_{ijk}\, x^j \frac{\partial}{\partial x^k},$$

have

$$L_z = \hbar/i\ \partial/\partial\phi$$
$$L_\pm = \hbar e^{\pm i\phi}\left(i \cot\theta \frac{\partial}{\partial\phi} \pm \frac{\partial}{\partial\theta}\right).$$
$$L^2 = -\hbar^2\left[\csc^2\theta \frac{\partial^2}{\partial\phi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial}{\partial\theta}\right)\right].$$

Looking for functions solving

$$L^2 Y_{\ell m}(\theta,\phi) = \hbar^2 \ell(\ell+1)\, Y_{\ell m}(\theta,\phi)$$
$$L_z Y_{\ell m}(\theta,\phi) = \hbar m\, Y_{\ell m}(\theta,\phi).$$

(prop. to assoc. Legendre polys)

From L_z clearly $Y_{\ell m}(\theta,\phi) = e^{im\phi} P_{\ell m}(\theta)$.

$$L_+ Y_{\ell\ell}(\theta,\phi) = \hbar e^{i(\ell+1)\phi}\left[-\ell\cot\theta + \frac{\partial}{\partial\theta}\right] P_{\ell m}(\theta) = 0$$

$$\Rightarrow \quad P_{\ell\ell} = \text{const.}\ (\sin\theta)^\ell$$

so $$Y_{\ell\ell} = c_\ell\, e^{i\ell\phi} (\sin\theta)^\ell = c_\ell (x+iy)^\ell$$

Normalization $$\underbrace{\int d\Omega}_{\int_0^{2\pi} d\phi \int_0^\pi d\theta} |Y_{\ell m}(\theta,\phi)|^2 = 1$$

$$\Rightarrow \quad c_\ell = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)\ (2\ell)!}{4\pi}}$$ [sign by convention]

Generate $Y_{\ell m}$ by acting with L_-

$$Y_{\ell m}(\theta,\phi) = \frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}\, e^{im\phi}\frac{1}{\sin^m\theta}\frac{d^{\ell-m}}{d(\cos\theta)^{\ell-m}}(\sin\theta)^{2\ell}$$

Exs: $Y_{00} = \frac{1}{\sqrt{4\pi}}$ ((constant))

$$Y_{11} = -\sqrt{\frac{3}{8\pi}}\, e^{i\phi} \sin\theta = -\sqrt{\frac{3}{8\pi}}\,[x+iy]$$

$$Y_{10} = \frac{1}{\sqrt{2}} e^{-i\phi} \left(i(\cos\theta)\, i - \cos\theta\right)\left(-\sqrt{\frac{3}{8\pi}} e^{i\phi}\right)$$

$$= \sqrt{\frac{3}{4\pi}} \cos\theta = \sqrt{\frac{3}{4\pi}}\,[z]$$

$$Y_{1-1} = \frac{1}{\sqrt{2}} e^{-i\phi} \cdot \sqrt{\frac{3}{4\pi}} \sin\theta = \sqrt{\frac{3}{8\pi}}\, e^{-i\phi} \sin\theta = \sqrt{\frac{3}{8\pi}}\,[x-iy]$$

$\ell = 2$: Homework.

Functions on S^2 spanned by $|\ell, m\rangle$,

$$Y_{\ell m}(\theta,\phi) = \langle \theta, \phi | \ell, m \rangle$$

Completeness:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi)\, Y^*_{\ell m}(\theta',\phi') = \frac{\delta(\theta-\theta')\,\delta(\phi-\phi')}{\sin\theta}$$

$$\int d\Omega\, Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'}$$

<u>Application of spherical harmonics:</u> separation of variables.

If $V(r)$ is spherically symmetric, $H\psi = E\psi$

for $H = \frac{p^2}{2m} + V(r)$

solutions are of form $\psi_{E,\ell,m} = \frac{u_{E\ell}(r)}{r} Y_{\ell m}(\theta,\phi)$

where $$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \left[\frac{\hbar^2 \ell(\ell+1)}{2mr^2} + V(r)\right]\right] u_{E\ell}(r) = E\, u_{E\ell}(r)$$

— reduces to 1D problem with new potential (HW)

3.4 Addition of angular momenta

Reducible representations

A representation $\mathcal{D}$ of an algebra $\mathfrak{g}$, $\mathcal{D}(K): \mathcal{H} \to \mathcal{H} \;\; \forall K \in \mathfrak{g}$ is reducible if

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 ,$$

where

$$\mathcal{D}(K) = \mathcal{D}_1(K) \oplus \mathcal{D}_2(K),$$
$$\mathcal{D}_1(K): \mathcal{H}_1 \to \mathcal{H}_1 \qquad \forall K \in \mathfrak{g}$$
$$\mathcal{D}_2(K): \mathcal{H}_2 \to \mathcal{H}_2$$

i.e., $\mathcal{D}(K)$ is block-diagonal $\forall K$.

$$\left(\begin{array}{c:c} \mathcal{D}_1 & 0 \\ \hdashline 0 & \mathcal{D}_2 \end{array}\right)$$

The representation is <u>irreducible</u> if this is not possible.

Spin j reps are all irreducible representations of $SU(2)$. Any other representation is a direct sum of irreps.

$$\mathcal{H} = \mathcal{H}_{j_1} \oplus \mathcal{H}_{j_2} \oplus \mathcal{H}_{j_3} \oplus \cdots$$

<u>Question:</u> Given two systems, one ($\mathcal{H}_1$) with spin j_1, the other ($\mathcal{H}_2$) with spin j_2, how can we classify angular momentum of the combined system [recall tensor product spaces]

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

One basis: $\quad |m_1, m_2\rangle = |j_1, m_1\rangle \otimes |j_2, m_2\rangle .$

Total angular momentum is given by

$$J_i = J_i^{(1)} + J_i^{(2)} \qquad \left[= J_i^{(1)} \otimes \mathbb{1} + \mathbb{1} \otimes J_i^{(2)} \right]$$

$$J^2 = J_{(1)}^2 + J_{(2)}^2 + 2\vec{J}_{(1)} \cdot \vec{J}_{(2)} .$$

Now, $[J^2, J_z^{(i)}] \neq 0$,

so J^2 not a good quantum number in basis $|j_1, m_1; j_2, m_2\rangle$

For total system, want to diagonalize J^2, J_z.
use total j, m as quantum numbers.

What are possible values of j, m given j_1, j_2?

<u>Example</u>: two spin-1/2 particles ($j_1 = j_2 = 1/2$)

States & J_z eigenvalues $\quad (m = m_1 + m_2)$

states	m
$\|++\rangle$	1
$\|+-\rangle \quad \|-+\rangle$	0
$\|--\rangle$	-1

Clearly, quantum numbers are those of

one spin-1 multiplet $(m = -1, 0, 1)$
one spin ϕ multiplet $(m = 0)$

So another basis is $|j, m\rangle = \begin{matrix} |1,1\rangle \\ |1,0\rangle \\ |1,-1\rangle \end{matrix}, \quad |0,0\rangle.$

What are coefficients for a change of basis

$\langle j, m \mid m_1, m_2 \rangle$ (given j_1, j_2: often written as $\langle j_1 j_1, j_2, m \mid j_1, m_1; j_2, m_2 \rangle$ or $\langle j_1, j_2; j, m \mid j_1, j_2; m_1, m_2 \rangle$ (book))

Clebsch-Gordan coefficients

Can calculate by recursion, using J_-.

Clearly $\boxed{|j=1, m=1\rangle = |++\rangle}$ (up to sign)

$$J_- |1,1\rangle = \hbar\sqrt{2}\,|1,0\rangle$$
$$= (J_-^{(1)} + J_-^{(2)})|++\rangle$$
$$= \hbar(|+-\rangle + |-+\rangle)$$

so $\boxed{|1,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle)}$

$$J_- |1,0\rangle = \hbar\sqrt{2}\,|1,-1\rangle$$
$$= (J_-^{(1)} + J_-^{(2)})\tfrac{1}{\sqrt{2}}(|+,-\rangle + |-,+\rangle)$$
$$= \hbar\sqrt{2}\,|--\rangle$$

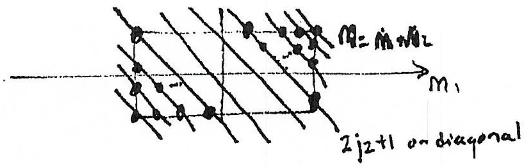
so $\boxed{|1,-1\rangle = |--\rangle.}$

By orthogonality,

$$|0,0\rangle = \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$ [up to conventional sign / phase]

Check: $J_z|0,0\rangle = 0$

$$J^2|0,0\rangle = (J_{(1)}^2 + J_{(2)}^2 + 2\vec{J}_{(1)}\cdot\vec{J}_{(2)})\tfrac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle)$$
$$= \left(3\left[\tfrac{1}{4}\hbar^2 + \tfrac{1}{4}\hbar^2 - \tfrac{1}{2}\hbar^2\right]\right)\tfrac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) = 0.$$



Generally, add spin j_1, spin j_2 — assume $j_1 \geq j_2$ wlog.
Diagonalize in $m = m_1 + m_2$

m	# states	States
j_1+j_2	1	$\|m_1 = j_1, m_2 = j_2\rangle$
j_1+j_2-1	2	$\|j_1, j_2-1\rangle$, $\|j_1-1, j_2\rangle$
j_1+j_2-2	3	$\|j_1, j_2-2\rangle$, $\|j_1-1, j_2-1\rangle$, $\|j_1-2, j_2\rangle$
$\vdots$		
j_1-j_2	$2j_2+1$	$\|j_1, -j_2\rangle$, $\ldots$ $\|j_1-2j_2, j_2\rangle$
$\vdots$	$(2j_2+1)$	
j_2-j_1	$2j_2+1$	$\|2j_2-j_1, -j_2\rangle$ $\cdots$ $\|-j_1, j_2\rangle$
j_2-j_1-1	$2j_2$	$\|2j_2-j_1-1, -j_2\rangle$ $\cdots$ $\|-j_1, j_2-1\rangle$
$\vdots$		$\vdots$
$-j_1-j_2+1$	2	$\|-j_1+1, -j_2\rangle$ $\|-j_1, -j_2+1\rangle$
$-j_1-j_2$	1	$\|-j_1, -j_2\rangle$

Gives all states associated with
one spin-j multiplet for each j: $|j_1-j_2| \leq j \leq j_1+j_2$

Counting # of states $(j_1 \geq j_2)$

$$\sum_{j=j_1-j_2}^{j_1+j_2} 2j+1 = (j_1+j_2+1)^2 - (j_1-j_2)^2$$
$$= (2j_1+1)(2j_2+1) \quad \checkmark.$$

Can calculate all Clebsch's $\langle j, m | j_1, m_1; j_2, m_2\rangle$
using J_-'s recursively as before.

First set $|j = j_1+j_2, m = j_1+j_2\rangle = |j_1, m_1 = j_1; j_2, m_2 = j_2\rangle$
Construct $|j_1+j_2, m\rangle$ using J_-.
$|j_1+j_2-1, j_1+j_2-1\rangle$ using orthog.
$|j_1+j_2-1, m\rangle$ using J_-, etc...

Generally,

$$\langle j, m | j_1, m_1; j_2, m_2 \rangle = 0$$

unless $m = m_1 + m_2$, $|j_1 - j_2| \le j \le j_1 + j_2$.

Another useful example: $j_1 = l$, $j_2 = 1/2$

(spin-1/2 particle with orbital angular momentum)

Expect

$$|j = l + 1/2, m = 1/2 + m_1\rangle = \alpha\, |m_1, 1/2\rangle + \beta\, |m_1 + 1, -1/2\rangle$$

act with $J^2/\hbar^2$

$$(l + 1/2)(l + 3/2)\left[\alpha |m_1, 1/2\rangle + \beta |m_1 + 1, -1/2\rangle\right]$$

$$= \left(L^2 + S^2 + 2L_z S_z + L_+ S_- + L_- S_+\right)\left[\alpha |m_1, 1/2\rangle + \beta |m_1 + 1, -1/2\rangle\right]$$

$$= \left[\alpha\left[l(l+1) + \tfrac{3}{4} + m_1\right] + \beta\sqrt{l(l+1) - m_1(m_1+1)}\right] |m_1, 1/2\rangle$$
$$+ [\quad \cdots \quad]\, |m_1 + 1, -1/2\rangle$$

$$\Rightarrow \quad \alpha(l - m_1) = \beta\sqrt{(l + m_1 + 1)(l - m_1)} \qquad \left[\text{from } |m_1, 1/2\rangle \text{ or } |m_1 + 1, -1/2\rangle\right]$$

so $$\frac{\alpha}{\beta} = \sqrt{\frac{l + m_1 + 1}{l - m_1}}$$

Normalization: $\alpha^2 + \beta^2 = 1 \Rightarrow \alpha = \sqrt{\dfrac{l + m_1 + 1}{2l + 1}}$ $\quad \beta = \sqrt{\dfrac{l - m_1}{2l + 1}}$

so $$|j = l + 1/2, m = m_1 + 1/2\rangle = \sqrt{\frac{l + m_1 + 1}{2l + 1}}\, |m_1, 1/2\rangle + \sqrt{\frac{l - m}{2l + 1}}\, |m + 1, -1/2\rangle$$
$$= \sqrt{\frac{l + m_1 + 1}{2l + 1}}\, Y_{lm}\, |+\rangle + \sqrt{\frac{l - m_1}{2l + 1}}\, Y_{l m+1}\, |-\rangle$$

$$|j = l - 1/2, m = m_1 + 1/2\rangle = -\sqrt{\frac{l + m_1 + 1}{2l + 1}}\, |m_1 + 1, -1/2\rangle + \sqrt{\frac{l - m_1}{2l + 1}}\, |m_1, +1/2\rangle$$

by orthogonality.

Last time: discussed Clebsch-Gordan coefficients
$\langle jm \mid j_1 m_1; j_2, m_2 \rangle$

Given $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2}$. CG coefts give transformation between bases.

$|j, m\rangle$ eigenvectors of J^2, J_z
$|j_1 m_1; j_2 m_2\rangle$ eigenvectors of $J_z^{(1)}, J_z^{(2)}$.

$$\mathcal{D}^{(j_1)} \otimes \mathcal{D}^{(j_2)} = \mathcal{D}^{(j_1+j_2)} \oplus \mathcal{D}^{(j_1+j_2-1)} \oplus \cdots \oplus \mathcal{D}^{(|j_1-j_2|)}$$

$$\mathcal{D}^{(j_1)}_{m_1 m_1'}(R)\,\mathcal{D}^{(j_2)}_{m_2 m_2'}(R) = \langle j_1 m_1; j_2 m_2 | \mathcal{D}(R) | j_1, m_1'; j_2, m_2' \rangle$$
$$= \sum_{j, m, m'} \langle j_1 m_1; j_2 m_2 | j, m \rangle \mathcal{D}_{mm'} \langle j, m' | j_1 m_1'; j_2, m_2' \rangle \quad (*)$$

Showed how to compute $\langle j, m | j_1 m_1; j_2 m_2 \rangle$ recursively.

Closed form expression (Racah, etc.)

$$\langle j_1 m_1; j_2 m_2 | j, m \rangle = \delta_{m_1+m_2, m} \sqrt{2j+1} \left[\frac{(j_1+j_2-j)!\,(j_1-j_2+j)!\,(-j_1+j_2+j)!}{(j_1+j_2+j+1)!} \right]^{1/2} \times$$

$$\left[(j_1+m_1)!\,(j_1-m_1)!\,(j_2+m_2)!\,(j_2-m_2)!\,(j+m)!\,(j-m)! \right]^{1/2} \times$$

$$\sum_n \frac{1}{n!\,(j_1+j_2-j-n)!\,(j_1-m_1-n)!\,(j_2+m_2-n)!\,(j-j_2+m_1+n)!\,(j-j_1-m_2+n)!}$$

(Sum over all integers (+) so all !'s are nonnegative.

Note: all CG's are real

Note: symm under perm of $(j, m), (j_1, m_1), (j_2, m_2)$ up to sign $\longrightarrow$

$$= \sqrt{2j+1}\,(-1)^{j_1+j_2-m} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{pmatrix}$$ "3j" symbol