8.(3)09 Section 7

October 22, 2021

1 H-J Example

Consider a one-dimensional system governed by the Hamiltonian $H = a\,e^{-q}p + b\,e^{-2q}$ where $a,\,b > 0$

(a)

Write down the Hamilton-Jacobi equation for this system and find a general solution (involving constants that are determined by the initial conditions).

(b)

Suppose that at time $t = 0$ the system is in state $(q, p) = (0, 0)$. Determine $q(t)$, $p(t)$ completely.

2 Action-angle with a half-pipe potential

Consider a particle of mass m with the Hamiltonian

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{m\omega_0^2 x^2}{2} + V(y), \quad V(y) = \begin{cases} 0 & b < y \\ -V_0 & -b < y < b \\ 0 & v < -b \end{cases} \tag{1}$$

and $\omega_0 > 0$, $b > 0$, $V_0 > 0$ are constants. If the mass hits the potential wall at $y = \pm b$ then it bounces off elastically, with the same velocity in the x direction and opposite velocity in the y-direction. Take the initial conditions to be $(x, y) = (0, 0)$ and $(\dot{x}, \dot{y}) = (v_x, v_y)$ at time $t = 0$, with $v_x > 0, v_y > 0$. The following integral may be helpful:

$$\int_0^{z_0} dz \sqrt{z_0^2 - z^2} = \frac{\pi z_0^2}{4}$$

(a)

Determine the conserved energy $E = H$ in terms of the given constants.

(b)

Define action variables J_x and J_y for the periodic motion, and express the Hamiltonian as a function of J_x and J_y.

(c)

Find the frequencies ν_x and ν_y as functions of ω_0, v_x, v_y, and b.

(d)

Write down a condition that will guarantee that the four-dimensional phase space orbit for (x, p_x, y, p_y) forms a closed path.