Discrete state coupled to a continuum

Eigenvalue equation: $\quad (|\varphi\rangle \equiv |i\rangle)$

E_μ and $|\psi_\mu\rangle$ eigenvalue and eigenvector of H

$$H|\psi_\mu\rangle = E_\mu |\psi_\mu\rangle \qquad H = H_0 + V$$

Project: $E_k \langle k|\psi_\mu\rangle + v\langle\varphi|\psi_\mu\rangle = E_\mu \langle k|\psi_\mu\rangle$ ⓐ

$$\sum_k v\langle k|\psi_\mu\rangle = E_\mu \langle\varphi|\psi_\mu\rangle \quad ⓑ$$

ⓐ $\Rightarrow \quad \langle k|\psi_\mu\rangle = v\dfrac{\langle\varphi|\psi_\mu\rangle}{E_\mu - E_k}$

into ⓑ $\Rightarrow \quad \sum_k \dfrac{v^2}{E_\mu - E_k} = E_\mu \quad$ Eigenvalue eq.

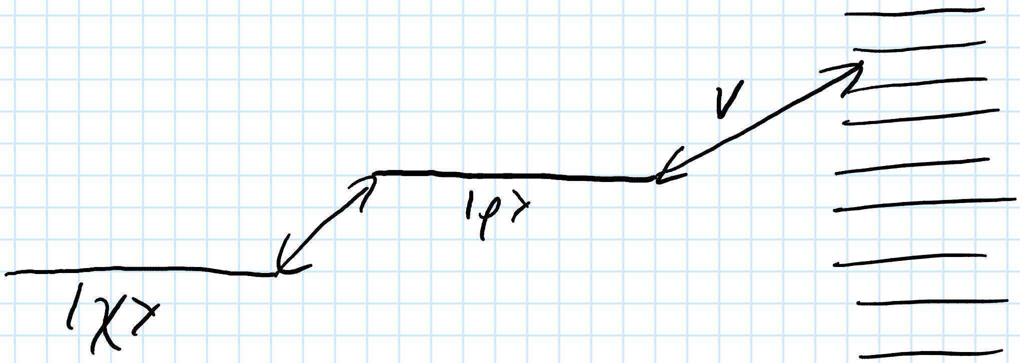
Normalization: $\sum_k |\langle k|\psi_\mu\rangle|^2 + |\langle\varphi|\psi_\mu\rangle|^2 = 1$

$$\Rightarrow \langle\varphi|\psi_\mu\rangle = \left(1 + \sum_{k'}\left(\frac{v}{E_\mu - E_{k'}}\right)^2\right)^{-1/2}$$

$$\langle k|\psi_\mu\rangle = \frac{\frac{v}{E_\mu - E_k}}{\left(1 + \sum_{k'}\left(\frac{v}{E_\mu - E_{k'}}\right)^2\right)^{1/2}}$$

See discussion in complement C_I of API

Recall discrete state coupled to continuum:



$$H|\psi_\mu\rangle = E_\mu|\psi_\mu\rangle$$

$$\langle\varphi|\psi_\mu\rangle = \frac{V}{\left[V^2 + \left(\frac{\hbar\Gamma}{2}\right)^2 + E_\mu^2\right]^{1/2}}$$

Now: Couple a new state $|\chi\rangle$ to $|\varphi\rangle$.

$$\langle\varphi|W|\chi\rangle = w$$

$$\langle k|W|\chi\rangle = 0$$

$|\chi\rangle$ will acquire a finite lifetime due to the fact that $|\varphi\rangle$ is coupled to a continuum.

Although $|\chi\rangle$ is not directly coupled to the bare continuum states $|k\rangle$, it is coupled to the new continuum $|\psi_\mu\rangle$:

$$\langle\psi_\mu|W|\chi\rangle = \langle\psi_\mu|\varphi\rangle\langle\varphi|W|\chi\rangle = \langle\psi_\mu|\varphi\rangle w$$

For small coupling w, Fermi's Golden rule gives the rate that the system leaves $|\chi\rangle$:

$$\Gamma_\chi = \frac{2\pi}{\hbar} |\langle \psi_\mu | W | \chi \rangle|^2 \frac{1}{\varepsilon}$$

(density of states of $|\psi_\mu\rangle$ for $E_\mu = E_\chi$ is $\frac{1}{\varepsilon}$).

$$\Rightarrow \Gamma_\chi = w^2 \frac{\Gamma}{\left(\frac{\hbar \Gamma}{2}\right)^2 + E_\chi^2}$$

If $|\chi\rangle$ and $|\varphi\rangle$ have same energy ($E_\chi = E_\varphi = 0$):

$$\Gamma_\chi = \frac{4 w^2}{\hbar^2 \Gamma}$$

This is valid for $w \ll \hbar \Gamma$, so $\Gamma_\chi \ll \Gamma$

otherwise: Rabi oscillations

Resonant scattering through a discrete level:

Take $|\chi_i\rangle$ and $|\chi_j\rangle$ of same energy E_χ.

Example: $|\chi_i\rangle = |a, \vec{k}\vec{\varepsilon}\rangle$

$|\chi_j\rangle = |a, \vec{k}'\vec{\varepsilon}'\rangle$

Let's couple $|\chi_i\rangle$ and $|\chi_j\rangle$ to $|\varphi\rangle$

$\langle\varphi|W|\chi_i\rangle = w_i \qquad \langle\chi_i|W|\chi_j\rangle = 0$

$\langle\varphi|W|\chi_j\rangle = w_j \qquad \langle k|W|\chi_{i,j}\rangle = 0$

We will show that even if $E_\chi = E_\varphi$, scattering amplitude doesn't diverge.
This is due to the fact that $|\varphi\rangle$ is now spread over the various states $|\varphi_\mu\rangle$.

lowest order:

$$\tau_{ji} = \lim_{\eta\to 0^+} \langle\chi_j|W \frac{1}{E_\chi - H_0 + i\eta} W|\chi_i\rangle$$

$$= \lim_{\eta\to 0^+} \frac{\langle\chi_j|W|\varphi\rangle\langle\varphi|W|\chi_i\rangle}{E_\chi - E_\varphi + i\eta}$$

diverges if $E_\chi = E_\varphi$

to all orders in V:

$$\tau_{ji} = \lim_{\eta\to 0^+} \langle\chi_j|W|\varphi\rangle\langle\varphi| \frac{1}{E_\chi - H + i\eta} |\varphi\rangle\langle\varphi|W|\chi_i\rangle$$

Insert closure relation $\langle \varphi | \frac{1}{E_\chi - H + i\eta} | \varphi \rangle = \sum_\mu \frac{|\langle \varphi | \psi_\mu \rangle|^2}{E_\chi - E_\mu + i\eta}$

No more divergence!

It's $\sum_\mu$ probability that $|\varphi\rangle$ is in $|\psi_\mu\rangle \cdot \frac{1}{E_\chi - E_\mu + i\eta}$.

$$\Rightarrow \tau_{ji} = \lim_{\eta \to 0^+} w_i w_j^* \, \varepsilon \sum_\mu \frac{\frac{\hbar\Gamma}{2\pi}}{(E_\chi - E_\mu + i\eta)\left(E_\mu^2 + \left(\frac{\hbar\Gamma}{2}\right)^2 + v^2\right)}$$

For $\varepsilon \to 0$: $\varepsilon \sum_\mu \longrightarrow \int dE$

$$\tau_{ji} = \lim_{\eta \to 0^+} w_i w_j^* \int_{-\infty}^{\infty} dE \, \frac{\frac{\hbar\Gamma}{2\pi}}{(E_\chi - E + i\eta)\left(E^2 + \left(\frac{\hbar\Gamma}{2}\right)^2\right)}$$

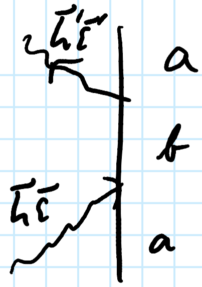
$$= w_j^* w_i \frac{1}{E_\chi + i\hbar\frac{\Gamma}{2}} \qquad \text{no divergence!}$$

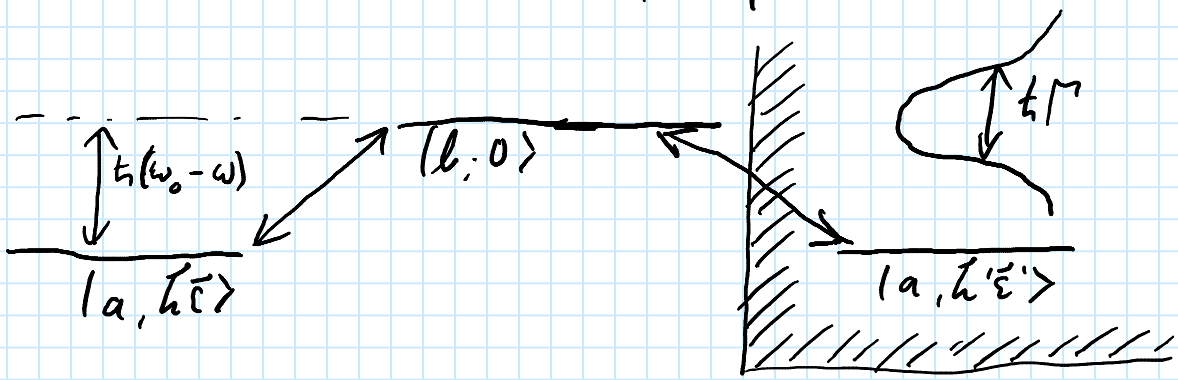
As if we had replaced E_φ by $E_\varphi - i\frac{\hbar\Gamma}{2}$.

$\Rightarrow \tau_{ji}$ has a resonance at $E_\chi = E_\varphi$ with width $\hbar\Gamma$.

Resonant Scattering

Case α dominates





$$T_{a\hbar\vec{k}'\vec{\varepsilon}',\,a\hbar\vec{k}\vec{\varepsilon}} = \langle a,\hbar\vec{k}'\vec{\varepsilon}'|H_{I_n}\frac{1}{E_a+\hbar\omega-H+i\eta}H_{I_n}|a;\hbar\vec{k}\vec{\varepsilon}\rangle$$

Intermediate state $|b,0\rangle$ "dissolves" in the continuum of eigenstates of H.

$\Rightarrow$ Sum over intermediate states = Sum over continuum

$\rightarrow$ no divergence as $\sum_\mu \frac{1}{x-E_\mu+i\eta} \to \int dE\, \mathcal{P}\left(\frac{1}{x-E} + i\pi\delta(x-E)\right)$

Involves density of $|b,0\rangle$ in the new continuum of eigenstates of H, a density that varies over a width Γ about E_b. $\Rightarrow$ Scattering has resonance about E_b when frequency ω of incident photon is swept about $\omega_0 = (E_b - E_a)/\hbar$.

To see this: $E_b \longrightarrow E_b - i\hbar\frac{\Gamma}{2}$

$$\exp(-iE_b t/\hbar) \longrightarrow \exp(-iE_b t/\hbar)\exp\left(-\frac{\Gamma}{2}t\right)$$

exponential decay!

$$T_{fi}^{res} = \frac{\langle a, \hbar \vec{k}' \vec{\varepsilon}' | H_{I_n} | b; 0 \rangle \langle b; 0 | H_{I_n} | a; \hbar \vec{\varepsilon} \rangle}{\underbrace{\hbar\omega + E_a - E_b}_{\hbar(\omega - \omega_0)} + i\hbar \frac{\Gamma}{2}}$$

Non-perturbative character clear from:

$$\frac{1}{\omega - \omega_0 + i\frac{\Gamma}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(i\frac{\Gamma}{2}\right)^n}{(\omega - \omega_0)^{n+1}}$$

Γ is 2^{nd} order in H_I, $\frac{1}{\ldots + i\frac{\Gamma}{2}} = \sum$ infinite orders in H_I.

Resonance fluorescence:

Incident monochromatic light at ω $\longrightarrow$ Scattered monochromatic light at ω intensity varies with detuning from resonance as Lorentzian with width Γ

Incident light with flat spectrum $\longrightarrow$ scattered light will have Lorentzian spectrum of width Γ centered at $\omega = \omega_0$.

Later: Include effects of high intensity