3. Symmetries in QM: Angular momentum + discrete symmetries

3.1 SO(3) vs. SU(2)

We are interested in studying rotational symmetry group & its representations.

[group G: closed $gh \in G$, unit $1 \cdot g = g$, inverses $g^{-1}g = gg^{-1} = 1$, associative $f(gh) = (fg)h$]

What is rotational symmetry group?

Natural candidate: SO(3), rotation group of $\mathbb{R}^3$

SO(3): 3×3 (special) orthogonal matrices

$$R^T R = \mathbb{1} \qquad \text{(preserve inner product } \vec{a} \cdot \vec{b}\text{)}$$
$$\det R = 1 \qquad \text{(preserve orientation)}$$

Examples:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Can get any rotation in SO(3) by multiplying these.

Note: $R_x(\alpha) R_z(\beta) \neq R_z(\beta) R_x(\alpha)$

<u>nonabelian group</u>

Any rotation can be characterized by:

$\hat{n}$ axis of rotation

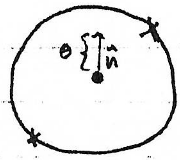
θ angle

$SO(3)$ is a 3-dimensional manifold (looks like $\mathbb{R}^3$ locally)

(circle bundle over $\mathbb{RP}^2$)

A group which is a manifold is called a <u>Lie group</u>

Picture:

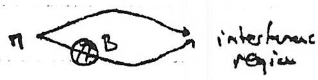


Ball in $\mathbb{R}^3$ of radius π, identify $(\hat{n}, \pi) \sim (-\hat{n}, \pi)$.

Seems like this is rotational symmetry group.

BUT...

Consider neutron interferometer



In B field $\vec{B} = B\hat{z}$,

neutron with magnetic moment $\frac{ge\hbar}{2mc}$ has coupling

$(g \sim -1.91)$

$$H = \frac{ge}{2mc} \vec{S} \cdot \vec{B}$$

$$= \omega S_z, \qquad \omega = \frac{geB}{2mc}$$

So if at $t = 0$, state is $\chi(0) = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}$ $[c_+|+\rangle + c_-|-\rangle]$

At time t, state is

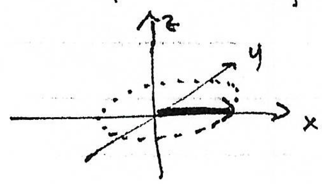
$$\chi(t) = e^{-\frac{iH}{\hbar}t} \chi(0) = \begin{pmatrix} e^{-\frac{i\omega t}{2}} c_+ \\ e^{\frac{i\omega t}{2}} c_- \end{pmatrix}.$$

Describes <u>precession</u> of spin, with angular frequency ω.

Ex. start in state with $S_x = +\hbar/2$

$$\chi(0) = |S_x, +\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

At time t, $\chi(t) = |S_n, +\rangle$, $\hat{n} = \hat{x}\cos\omega t + \hat{y}\sin\omega t$
up to a phase.



After time $T_- = 2\pi/\omega$,

$$\chi(T_-) = \begin{pmatrix} -c_+ \\ -c_- \end{pmatrix} = -\chi(0) = -|S_x, +\rangle.$$

State has rotated once, again has $S_x = +\hbar/2$.

<u>But</u> appearance of phase (-1) changes interference pattern!

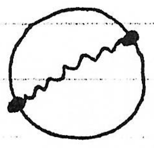
To get successive maxima at same point,

need $T_+ = \frac{4\pi}{\omega}$ $\left[\Delta B = \frac{4\pi\hbar c}{|e|g\, x\, l}\right]$

This demonstrates that rotation by 360° is not always a trivial transformation.

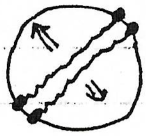
$\overline{\overline{\mathsf{F}}}$

In SO(3), rotation by 2π cannot be deformed into a trivial transformation.



[Demo]

<u>But</u> rotation by 4π <u>can be</u>.



[Demo]

[Technically, $\pi_1(SO(3)) = \mathbb{Z}_2$]

This leads us to consider a "larger" group: SU(2).

SU(2): 2×2 (special) unitary matrices

$$U^\dagger U = \mathbb{1} \qquad \text{(preserve inner product } x^\dagger x\text{)}$$
$$\det U = 1.$$

General SU(2) matrix:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \qquad a, b \in \mathbb{C} \text{ with } |a|^2 + |b|^2 = 1.$$

SU(2) is group describing rotations of an electron (spin ½) state.

Topologically, $SU(2) \cong S^3$, since $|a|^2 + |b|^2 = 1$ describes a sphere in $\mathbb{R}^4 = \mathbb{C}^2$.

All loops in S^3 are contractible, unlike in SO(3).

Can map $\begin{array}{ccc} SU(2) & \longrightarrow & SO(3) \\ \pm 1 & \longrightarrow & 1. \end{array}$ by group homomorphism

write $SO(3) = SU(2)/\mathbb{Z}_2$

For example,
$$\begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \to \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$SU(2)$ is simply connected, "universal covering group" of $SO(3)$.

3.2. Lie algebra & representations of $SU(2)$.

We want to understand how symmetry group $SU(2)$ works in QM.

Symmetry group acts through representations on $\mathcal{H}$.

[Representation $\mathcal{D}$:
$$\mathcal{D}(g) : \mathcal{H} \to \mathcal{H} \quad \text{linear map} \quad \forall g$$
$$\mathcal{D}(1) = \mathbb{1}.$$
$$\mathcal{D}(gh) = \mathcal{D}(g)\,\mathcal{D}(h)$$
]

To understand representations of a Lie Group, consider Lie Algebra

Associated with a Lie group G is an Lie algebra $\mathfrak{g}$, of infinitesimal elements of G.

For example, for $SO(3)$

$$\mathbb{1} + \varepsilon A \quad \text{is orthogonal if} \quad \text{(working to order } \varepsilon)$$

$$(\mathbb{1} + \varepsilon A)(\mathbb{1} + \varepsilon A^T) = \mathbb{1} + \varepsilon(A + A^T) = \mathbb{1},$$

so $A = -A^T$.

Basis of Lie algebra for $SO(3)$ given by

$$K_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$K_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$K_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For QM, want Hermitian operators, so write

$$J_\ell = i\hbar K_\ell.$$

[Note: will change basis later so J_z is diagonal.]

Lie algebra on space spanned by J_i defined by $[A, B]$; J_i are generators of algebra.

Properties of a general Lie algebra:

i) Closed $[J_i, J_j] = i f_{ijk} \hbar J_k$ (f_{ijk}: structure constants)

ii) linear in A, B

iii) $[A, B] = -[B, A]$

iv) $[[A,B],C] + [[B,C],A] + [[C,A],B] = 0$ (Jacobi)

Same algebra for $SO(3), SU(2)$: $f_{ijk} = \varepsilon_{ijk}$

$$[J_i, J_j] = i\varepsilon_{ijk}\hbar J_k$$

$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k. \qquad \left(S_i = \frac{\hbar}{2}\sigma_i\right)$$

Any element of $\begin{matrix} SU(2) \\ SO(3) \end{matrix}$ can be written as

$$g = e^{-\frac{i}{\hbar}\left(\begin{smallmatrix} \vec{S} \\ \vec{J} \end{smallmatrix} \cdot \hat{n}\right)\phi}$$

for g = rotation by ϕ about $\hat{n}$.

when $\phi = 2\pi$, $\quad g = \mathbb{1}$ in $SO(3)$,
$\quad g = -\mathbb{1}$ in $SU(2)$.

Representations of a Lie algebra:

$$\mathcal{D}(k) : \mathcal{H} \to \mathcal{H} \qquad \forall k \in \mathfrak{g}, \mathcal{D} \text{ linear in } k.$$
$$\mathcal{D}(0) = \mathbb{0}$$
$$\mathcal{D}([k, \ell]) = [\mathcal{D}(k), \mathcal{D}(\ell)]$$

To each representation of the group, there is a corresponding representation of the algebra (but not necessarily vice-versa if gp not simply connected.)

Classify representations of group by representations of the algebra.

[Notation: write $J_i = \mathcal{D}[\sigma_i]$ on general rep. space $\mathcal{H}$]

Representations of SU(2) algebra: $\quad [J_i, J_j] = i\hbar\,\varepsilon_{ijk} J_k$.

Define
$$J^2 = J_x^2 + J_y^2 + J_z^2$$
$$J_\pm = J_x \pm i J_y.$$

Can show:
$$[J^2, J_i] = 0$$
$$[J_z, J_\pm] = \pm\hbar J_\pm$$
$$[J_+, J_-] = 2\hbar J_z$$

and $\quad J^2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = J_z^2 + J_-J_+ + \hbar J_z$

with $J_\pm^{\dagger} = J_\mp$

Can simultaneously diagonalize J^2, J_z.

write
$$J^2|a,b\rangle = a|a,b\rangle$$
$$J_z|a,b\rangle = b|a,b\rangle$$

What values of a, b are allowed?

$$\langle a,b|\underbrace{J^2}_{a}|a,b\rangle = \langle a,b|\underbrace{J_z^2}_{b^2} + \tfrac{1}{2}(\underbrace{J_+J_-}_{\geq 0} + \underbrace{J_-J_+}_{\geq 0})|a,b\rangle$$

$$\Rightarrow \quad a \geq b^2$$

compare
$$[J_z, J_\pm] = \pm \hbar J_\pm,$$
$$[N, {a^+ \atop a}] = \pm \hbar {a^+ \atop a}.$$

so $J_\pm$ are raising / lowering operators for J_z.

$$J_z(J_\pm|a,b\rangle) = (b \pm \hbar)(J_+|a,b\rangle).$$

but $[J^2, J_\pm] = 0$, so

$$J_\pm|a,b\rangle = C_\pm^{(a,b)}|a, b\pm\hbar\rangle$$

Since $a \geq b^2$, there must be a maximum b which can be reached for a fixed a. Call this $b_{max} = \hbar j$.

Then
$$\langle a,b|J_-J_+|a,b\rangle = \langle a,b|J^2 - J_z^2 - \hbar J_z|a,b\rangle$$
$$|C_+(a,b)|^2 = a - b^2 - \hbar b.$$

This must vanish for $b_{max} = \hbar j$, so $a = \hbar^2 j(j+1)$.

Similarly, must be a b_{min} which can be reached by acting with J_-.

$$|c_-(a,b)|^2 = a - b^2 + \hbar b$$

so $b_{min} = -b_{max}$.

It follows that $2b_{max} = n\hbar$, so $j = n/2$ is half-integral

For each $n = 2j \in \mathbb{Z}$, we have constructed an irreducible n-dimensional representation $\mathcal{H}_j$ of $su(2)$ algebra. $[J_i, J_j] = i\hbar \epsilon_{ijk} J_k$

$\mathcal{H}_j$ spanned by $\{|j,m\rangle,\ m = -j, -j+1, \ldots, j-1, j\}$

$J^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle \qquad J_z|j,m\rangle = m\hbar|j,m\rangle$

$$J_+|j,j\rangle = J_-|j,-j\rangle = 0$$

$$J_\pm|j,m\rangle = \sqrt{(j\mp m)(j\pm m+1)}\,\hbar\,|j,m\pm 1\rangle.$$

(Irreducible representation: no linear subspace is closed under the action of all J_i's.)

Can use representations of algebra to get group representation through

$$\mathcal{D}^{(j)}_{m'm}(g) = \langle j,m'|\, e^{-\frac{i}{\hbar}(\vec{J}\cdot\hat{n})\phi}\, |j,m\rangle, \qquad g = e^{-\frac{i}{\hbar}(\vec{J}\cdot\hat{n})\phi}$$

$\mathcal{D}^{(j)}_{m'm}(g)$ are Wigner functions on group G.

Theorem: dim. n irrep is unique up to unitary isomorphism.

SU(2):

Specific representations, j = "spin" of representation

$j=0$: only state is $|j,m\rangle = |0,0\rangle$

$$J^2|0,0\rangle = J_\pm|0,0\rangle = J_i|0,0\rangle = 0.$$

action of any group element is trivial $\mathcal{D}^{(0)}(g)\,|0,0\rangle = |0,0\rangle$.

$j=1/2$ (spin-1/2 system)

States $|j,m\rangle = |1/2, \pm 1/2\rangle$. (previously, $|\pm\rangle$ or $|S_z;\pm\rangle$)

$$J_i = S_i = \frac{\hbar}{2}\sigma_i, \qquad J_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad J_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathcal{D}^{(1/2)}(\hat{n},\phi) = \exp[-i(\vec{n}\cdot\vec{\sigma})\phi/2] = (\cos\tfrac{\phi}{2})\mathbb{1} + i\sin\tfrac{\phi}{2})(\vec{n}\cdot\vec{\sigma}).$$

as discussed in earlier lectures.

[This gives background for examples previously described].

$j=1$: (spin 1)

States $|1,1\rangle$, $|1,0\rangle$, $|1,-1\rangle$.

$$J_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad J_+ = \hbar\begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix} \quad J_- = \hbar\begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$J_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Note: looks different from $J_i = i\hbar K_i$ above, since in this basis J_z is diagonal. Otherwise, just related by orthogonal change of basis.

- For General j, if $\underset{}{\overset{=\ell}{j}} \in \mathbb{Z}$, representation of $SO(3)$ since $e^{-i/\hbar\, 2\pi J_i} = \mathbb{1}$.
  if $j + 1/2 \in \mathbb{Z}$, $e^{i/\hbar\, 2\pi J_i} = -\mathbb{1}$, not a rep. of $SO(3)$.