➔ 76. An inductive definition of a^n for nonnegative n is given by $a^0 = 1$ and $a^n = aa^{n-1}$. (Notice the similarity to the inductive definition of $n!$.) We remarked above that inductive definitions often give us easy proofs of useful facts. Here we apply this inductive definition to prove two useful facts about exponents that you have been using almost since you learned the meaning of exponents.

(a) Use this definition to prove the rule of exponents $a^{m+n} = a^m a^n$ for nonnegative m and n.

Solution: We use induction on n to prove this. When $n = 0$, the formula gives us $a^{m+0} = a^m a^0 = a^m \cdot 1 = a^m$, so the rule of exponents holds when $n = 0$. Now assume it holds when $n = k-1$ so that $a^{m+k-1} = a^m a^{k-1}$. Then, starting and ending with our inductive definition, we may write

$$a^{m+n} = aa^{m+n-1} = aa^m a^{k-1} = a^m \cdot a \cdot a^{k-1} = a^m a^k.$$

Thus the truth of our law for $n = k - 1$ implies its truth for $n = k$. Therefore, by the principle of mathematical induction, $a^{m+n} = a^m a^n$ for all nonnegative integers n. ∎

(b) Use this definition to prove the rule of exponents $a^{mn} = (a^m)^n$.

Solution: We will use induction on n and part (a) of this problem to prove that $a^{mn} = (a^m)^n$. First, when $n = 0$ the left and right hand sides of the equation are both 1, so $a^{mn} = (a^m)^n$ holds when $n = 0$. Now assume that $a^{m(k-1)} = (a^m)^{k-1}$. This may be rewritten as $a^{mk-m} = (a^m)^{k-1}$. Multiply both sides by a^m and apply part (a) of the problem and then the inductive definition (with a^m replacing a) to get

$$\begin{aligned} a^{mk-m}a^m &= (a^m)^{k-1}a^m \\ a^{mk} &= (a^m)^{k-1}a^m \\ a^{mk} &= (a^m)^k. \end{aligned}$$

Thus the truth of our formula when $n = k - 1$ implies its truth when $n = k$. Therefore by the principle of mathematical induction, the formula is true for all nonnegative integers n. ▮

→78. Give an inductive definition of the summation notation $\sum_{i=1}^{n} a_i$. Use it and the distributive law $b(a + c) = ba + bc$ to prove the distributive law

$$b\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ba_i.$$

Solution: We define $\sum_{i=1}^{1} a_i = a_1$ and for $n > 1$, $\sum_{i=1}^{n} a_i = \left(\sum_{i=1}^{n-1} a_i\right) + a_n$. When $n = 1$, $b\sum_{i=1}^{1} a_i = ba_1$ by the base step of our inductive definition. Assume that $k > 1$ and $b\sum_{i=1}^{k-1} a_i = \sum_{i=1}^{k-1} ba_i$. Now we can write

$$b\sum_{i=1}^{k} a_i = b\left[\left(\sum_{i=1}^{k-1} a_i\right) + a_k\right] = \left(b\sum_{i=1}^{k-1} a_i\right) + ba_k = \left(\sum_{i=1}^{k-1} ba_i\right) + ba_k = \sum_{i=1}^{k} ba_i,$$

where the last step is justified by the inductive step of our inductive definition with a_i replaced by ba_i. Thus the truth of our statement for $k-1$ implies its truth for $i = k$, and therefore by the principle of mathematical induction, for all positive integers n, $b\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} ba_i$. ▮

+80. Prove the general form of the product principle from the partition form of the product principle.

Solution: We prove by induction that if S is a set of functions defined on $[m]$ such that

- there are k_1 choices for $f(1)$ and
- when $2 \le i \le m$, for each choice of $f(1)$, $f(2)$, $\ldots f(i-1)$, there are k_i choices for $f(i)$,

then there are $\prod_{i=1}^{m} k_i$ functions in S. When $m = 1$, the product is k_1 and there are k_1 functions in S. Now assume inductively that when S' is a set of functions defined on $[m-1]$ such that

- there are k_1 choices for $f(1)$ and
- when $2 \le i \le m-1$, for each choice of $f(1)$, $f(2)$, $\ldots f(i-1)$, there are k_i choices for $f(i)$,

then there are $\prod_{i=1}^{m-1} k_i$ functions in S'. Now partition S into k_1 sets S_j, where S_j is the set of functions f in S with $f(n) = y_j$ for each of the k_n values y_j that are possible for $f(1)$. Thus S is a union of k_n sets S_j each of size $\prod_{i=1}^{m-1} k_i$ (by the inductive hypothesis), and so by the product principle for unions of sets, S has size $\prod_{i=1}^{m} k_i$. Therefore, by the principle of mathematical induction, we have proved the general product principle. ∎