Physics 8.321, Fall 2021
Homework #1

Due **Wednesday, September 22** by 8:00 PM.

The operator measuring the spin of a spin-1/2 particle along the axis parallel to a general unit vector $\widehat{\mathbf{n}}$ is given by

$$S_{\mathbf{n}} = \mathbf{S} \cdot \widehat{\mathbf{n}}$$

where $S_i = \sigma_i\, \hbar/2$ for $i = 1, 2, 3$, and

$$\sigma_1 = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

These operators are used in problems 1-5.

You may find it helpful to use the result mentioned in class that when an operator O is measured and the (normalized) initial state is the ket/column vector $|i\rangle$, the probability that the final state is $|f\rangle$ is just $|\langle f|i\rangle|^2$, where $\langle f|$ is the bra/row vector (*dual vector*) formed by the adjoint/transpose conjugate of $|f\rangle$, when $|f\rangle$ is a (normalized) eigenstate of O, and there are no eigenvalue degeneracies. (This is just a convenient way of picking out the coefficient α of $|f\rangle$ when writing $|i\rangle$ in a basis of eigenstates of O.)

1. (a) Measurement of an electron's spin along the z-axis (S_z) using a Stern-Gerlach apparatus gives the eigenvalue $\hbar/2$. What is the probability that a subsequent measurement of the spin in the direction $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ yields $\hbar/2$?

(b) Measurement of an electron's spin along the axis $\widehat{\mathbf{n}}$ gives the eigenvalue $\hbar/2$. What is the probability that a subsequent measurement of the spin along the z-axis yields $\hbar/2$?

2. The *expectation value* of an operator O in a state $|s\rangle$ is $\langle O\rangle = \langle s|O|s\rangle$. If $|\lambda_i\rangle$ is a basis of (normalized) eigenvectors of O with eigenvalues λ_i, then if $|s\rangle = \sum_i c_i|\lambda_i\rangle$ then $\langle O\rangle = \sum_i |c_i|^2\lambda_i$, i.e. the probabilistically weighted average of the measured values.

Show that it is impossible for an electron to be in a state such that

$$\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = 0\,.$$

3. A beam produced by a Stern-Gerlach filter contains electrons that are all in the same spin state, which can be written as

$$|\alpha\rangle = s_+|+\rangle + s_-|-\rangle$$

where $|+\rangle, |-\rangle$ are eigenstates of S_z with eigenvalues $\pm\hbar/2$.

Part of the beam is passed through an analyzer oriented in the z direction, giving

$$\langle S_z\rangle = 0\,.$$

The other part of the beam is passed through an analyzer oriented in the x direction, giving

$$\langle S_x\rangle = \hbar/4\,.$$

(a) Calculate $\langle S_y\rangle$.

(b) What are the possible directions along which the original Stern-Gerlach filter may have been oriented?

4. [Sakurai and Napolitano Problem 1.19 (page 62); typo there corrected]

(a) Compute

$$\langle(\Delta S_x)^2\rangle \cong \langle S_x^2\rangle - \langle S_x\rangle^2\,,$$

where the expectation value is taken for the S_z+ state. Using this result, check the generalized uncertainty relation

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geq \frac{1}{4}|\langle[A,B]\rangle|^2\,,$$

with $A \to S_x, B \to S_y$, and where $[A,B] = AB - BA$.

(b) Check the uncertainty relation with $A \to S_x, B \to S_y$ for the S_x+ state.

5. [Sakurai and Napolitano Problem 1.20 (page 62)]

Find the (normalized) linear combination of $|+\rangle$ and $|-\rangle$ kets that maximizes the uncertainty product

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle\,.$$

Verify explicitly that for the linear combination you found, the uncertainty relation for S_x and S_y is not violated.

6. Prove that the equation $AB - BA = \mathbb{1}$ cannot be satisfied by any finite-dimensional matrices A, B.

7. (a) Consider two operators A, B that do not necessarily commute. Show that

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\}$$

where

$$A^0\{B\} = B, \quad A^1\{B\} = [A,B], \quad A^2\{B\} = [A,[A,B]], \text{ etc.}$$

Hint: treat $e^A = 1 + A + A^2/2 + \cdots$ as a formal power series.

(b) Let $A(x)$ be an operator that depends on a continuous parameter x. Derive the following identity

$$e^{-iA(x)}\frac{d}{dx}e^{iA(x)} = i\sum_{n=0}^{\infty}\frac{(-i)^n}{(n+1)!}A^n\{\frac{dA}{dx}\}\,.$$