Some Magnetic Traps for Neutral Atoms & The Majorana Spin-flip for $J = 1/2$.

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1 Quadrupole trap (anti-Helmholtz)

1.1 Calculation

In this configuration, there are two coils of radius a placed at distance $2b$ apart. Running through the coils are equal and opposite currents I and $-I$, respectively. Here, we set $I = NI_0$ where N is the number of coils and I_0 is the current going through each coil (or equivalently through all the coils).

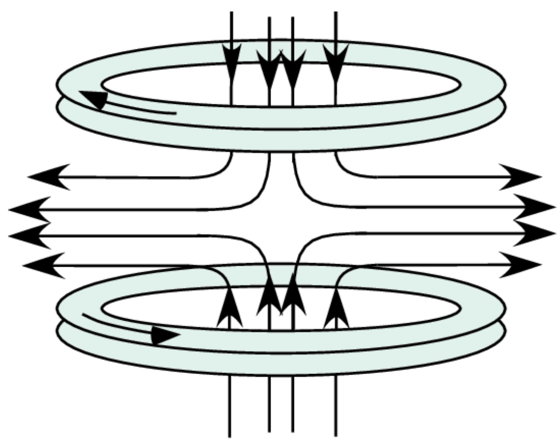

Figure 1: From [1]

To calculate the magnetic field for one coil, we can use Biot-Savart law because the current is constant. We will integrate along the closed loop C defined by the coil. The relative position between the point $\mathbf{r}$ and the point $\vec{l}$ on the wire is given by $\vec{r}' = \vec{r} - \vec{l}$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{l}\times\vec{r}'}{|\vec{r}'|^3} = \frac{\mu_0 I}{4\pi}\int_C \frac{d\vec{l}\times(\vec{r}-\vec{l})}{|\vec{r}-\vec{l}|^3}.$$

We now make the approximation $|\vec{r}| \ll \vec{l}$ (i.e., we're interested in points far from the coil). With this we have the following expansion

$$\frac{1}{|\vec{r}-\vec{l}|^3} \approx \frac{1}{|\vec{l}|^3} + \frac{3\vec{r}\cdot\vec{l}}{|\vec{l}|^5} + \dots$$

Plugging this back in for $\vec{B}(\vec{r})$ we find

$$\vec{B}(\vec{r}) \approx \frac{\mu_0 I}{4\pi}\int_C d\vec{l}\times\frac{\vec{r}-\vec{l}}{|\vec{l}|^3} + \frac{3\mu_0 I}{4\pi}\int_C d\vec{l}\times(\vec{r}-\vec{l})\frac{\vec{r}\cdot\vec{l}}{|\vec{l}|^5}.$$

Suppose that the center of the coil is at $+d$ from the XY plane. Going into cylindrical coordinates, we have

$$\vec{l} = a\cos\theta\hat{x} + a\sin\theta\hat{y} + d\hat{z}$$

from which we find

$$d\vec{l} = -a\sin\theta d\theta\hat{x} + a\cos\theta d\theta\hat{y}$$

So, we have

$$d\vec{l}\times(\vec{r}-\vec{l}) = (-a\sin\theta, a\cos\theta, 0)\times(x-a\cos\theta, y-a\sin\theta, z-d)$$

and

$$\vec{r}\cdot\vec{l} = xa\cos\theta + ya\sin\theta + za.$$

So, we have

$$\frac{\mu_0 I}{4\pi}\int_C d\vec{l}\times\frac{\vec{r}-\vec{l}}{|\vec{l}|^3} = \frac{\mu_0 I}{4\pi}\frac{2a^2\pi}{(d^2+a^2)^{3/2}}\hat{z} = \frac{\mu_0 I a^2}{2(d^2+a^2)^{3/2}}\hat{z}.$$

and

$$\begin{aligned}&\frac{3\mu_0 I}{4\pi}\int_C d\vec{l}\times(\vec{r}-\vec{l})\frac{\vec{r}\cdot\vec{l}}{|\vec{l}|^5}\\ =\ &\frac{3\mu_0 I a^2}{4(d^2+a^2)^{5/2}}\left(-x(d-z)\hat{x} - y(d-z)\hat{y} - (x^2+y^2-2dz)\hat{z}\right).\end{aligned}$$

Keeping only the linear terms and combining everything, we find the total field:

$$\begin{aligned}\vec{B}(\vec{r}) &= \frac{\mu_0 I a^2}{2(d^2+a^2)^{3/2}}\hat{z} + \frac{3\mu_0 I a^2}{4(d^2+a^2)^{5/2}}\left(-xd\hat{x} - yd\hat{y} - 2dz\hat{z}\right)\\ &= \frac{\mu_0 I a^2}{2(d^2+a^2)^{3/2}}\hat{z} + \frac{3\mu_0 I a^2 d}{2(d^2+a^2)^{5/2}}\left(-\frac{x}{2}\hat{x} - \frac{y}{2}\hat{y} - z\hat{z}\right).\end{aligned}$$

In the anti-Helmholtz configuration, we have two coils of radius a placed a distance $2b$ apart from each other. When summing the two fields to get the total field, the first term cancels. So we get

$$\vec{B}_{\text{tot}}(\vec{r}) = \vec{B}_{+b}(\vec{r}) + \vec{B}_{-b}(\vec{r}) = -\frac{3\mu_0 I a^2 b}{2(a^2+b^2)^{5/2}}(x, y, -2z) \equiv B_0(x, y, -2z).$$

The field strength is given by

$$|B(\vec{r})| = B_0\sqrt{x^2+y^2+4z^2}.$$

1.2 Trap parameters

1.3 Simulation

2 TOP trap

2.1 Calculation

As will be discussed later, the quadrupole or anti-Helmholtz trap suffers from the "Majorna spin-flip problem" which occurs due to the presence of a zero-magnetic field point in the trap. To overcome this issue, one can add a rotating magnetic field to the existing anti-Helmholtz field so that the time-averaged magnetic field no longer has a zero at the center. This trick gives us the TOP (time orbiting potential) trap. The total field is given by

$$\vec{B}(\vec{r},t) = \vec{B}_{\text{quad}}(\vec{r}) + \vec{B}_b(\vec{r}) = B_0(x, y, -2z) + B_b(\cos\Omega t, \sin\Omega t, 0),$$

where Ω is the angular frequency of the rotating field.

The field strength is given by

$$B(\vec{r},t) = \sqrt{(B_0 x + B_b\cos\Omega t)^2 + (B_0 y + B_b \sin\Omega t)^2 + 4B_0^2 z^2}$$

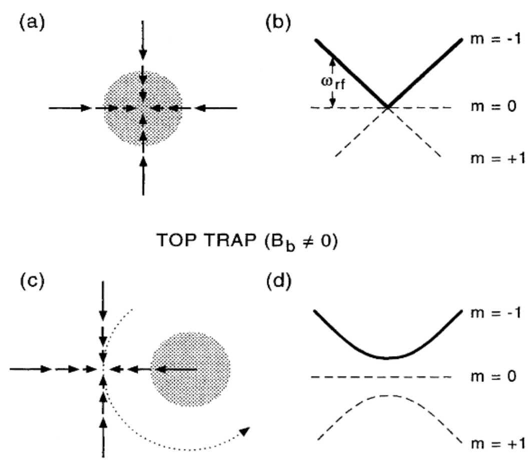

Figure 2: From [2]

For this trap to work Ω can't be too small or too large. Ω must be larger than the oscillation frequency of the trapped particles (which is on the order of 100 Hz) so that the particles feel an effective time-averaged magnetic field. Ω should also be smaller than the frequency associated with the transition between two adjacent internal quantum states (which is on the order of 1 MHz) in order to prevent particle losses due to Majorana spin-flips.

We are interested in dynamics near the center of the trap, so we can make the approximation $r = \sqrt{x^2+y^2+z^2} \ll a$, under which

$$B(\vec{r},t) \approx B_b + \frac{B_0^2}{2B_b^2}(x^2+y^2+4z^2) + \frac{B_0}{B_b}(x\cos\Omega t + y\sin\Omega t) - \frac{B_0^2}{2B_b^2}(x\cos\Omega t + y\sin\Omega t)^2.$$

The time-averaged magnetic field strength is thus, by inspection,

$$\begin{aligned}\langle B \rangle &= B_b + \frac{B_0^2}{2B_b^2}(x^2+y^2+4z^2) - \frac{B_0^2}{2B_b^2}\left(\frac{x^2}{2}+\frac{y^2}{2}\right) \\ &= B_b + \frac{B_0^2}{4B_b}(x^2+y^2+8z^2).\end{aligned}$$

2.2 Trap parameters

2.3 Simulation

3 Ioffe-Pritchard traps

3.1 Calculation

There are many variants of the IP trap, the simplest being one with two coils in the anti-Helmholtz configuration and four wires in the z-direction. The four wires are suited at the corners of a square, with the currents flowing along adjacent wires being of opposite sign. A generalization of the IP trap is often called the "all coils Ioffe-Pritchard trap." The all-coil trap consists of the following set of coils:

- Big Ioffe Coils (BI), anti-Helmholtz, along z
- Small Ioffe Coils (SI), anti-Helmholtz, along x
- Pinch coils (PI), Helmholtz, along y
- Compensation coils (CO), Helmholtz, along y, opposite current to SI.

Let us revisit the calculation we've done for Helmholtz and anti-Helmholtz coils, but now we will expand to higher orders (still assuming that $\vec{r}$ is near the origin). In particular, we will use

$$\frac{1}{|\vec{r}-\vec{l}|^3} \approx \frac{1}{|\vec{l}|^3} + \frac{3\vec{r}\cdot\vec{l}}{|\vec{l}|^5} + \frac{15}{2}\frac{(\vec{r}\cdot\vec{l})^2}{|\vec{l}|^7} + \frac{35}{2}\frac{(\vec{r}\cdot\vec{l})^3}{|\vec{l}|^9}.$$

The expansion can be obtained by following Eq. 3.88 of [3]. For a single coil with current I placed a vertical distance $+d$ from the origin, we find the factor $|\vec{l}| = \sqrt{a^2+d^2}$ to be constant. The field, order-by-order to third order, is thus

$$B^{(0)}(\vec{r}) = \frac{\mu_0 I}{4\pi|\vec{l}|^3}\int_C d\vec{l} \times (\vec{r}-\vec{l}) = \frac{1}{2}\frac{\mu_0 I a^2}{(a^2+d^2)^{3/2}}\hat{z} \equiv \frac{1}{2}\mathbb{F}\hat{z}$$

$$
\begin{aligned}
B^{(1)}(\vec{r}) &= \frac{3\mu_0 I}{4\pi|\vec{l}|^5}\int_C d\vec{l}\times(\vec{r}-\vec{l})\vec{r}\cdot\vec{l} \\
&= \frac{3\mu_0 I a^2}{4(d^2+a^2)^{5/2}}\left(-x(d-z)\hat{x}-y(d-z)\hat{y}-(x^2+y^2-2dz)\hat{z}\right) \\
&\approx \frac{3\mu_0 I a^2 d}{2(d^2+a^2)^{5/2}}\left(-\frac{x}{2}\hat{x}-\frac{y}{2}\hat{y}+z\hat{z}\right) \\
&\equiv \mathbb{G}\left(-\frac{x}{2}\hat{x}-\frac{y}{2}\hat{y}+z\hat{z}\right)
\end{aligned}
$$

$$
\begin{aligned}
B^{(2)}(\vec{r}) &= \frac{15\mu_0 I}{8\pi|\vec{l}|^7}\int_C d\vec{l}\times(\vec{r}-\vec{l})(\vec{r}\cdot\vec{l})^2 \\
&=
\end{aligned}
$$

$$
B^{(3)}(\vec{r}) = \frac{35\mu_0 I}{8\pi|\vec{l}|^9}\int_C d\vec{l}\times(\vec{r}-\vec{l})(\vec{r}\cdot\vec{l})^3 =
$$

3.2 Trap parameters

3.3 Simulation

References

[1] Reina Maruyama. *Optical trapping of ytterbium atoms*. University of Washington, 2003.

[2] Wolfgang Petrich, Michael H. Anderson, Jason R. Ensher, and Eric A. Cornell. Stable, tightly confining magnetic trap for evaporative cooling of neutral atoms. *Phys. Rev. Lett.*, 74:3352–3355, Apr 1995.

[3] David J Griffiths. Introduction to electrodynamics, 2005.

[4] Danyel Cavazos. All coils ioffe-pritchard magnetic trap. Summer 2015.

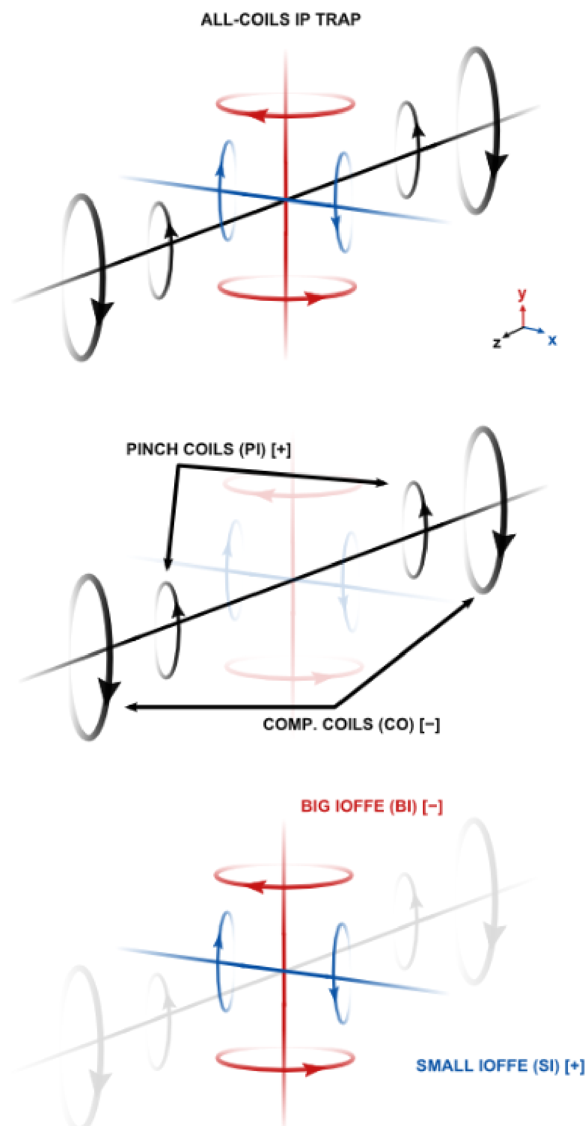

Figure 3: From [4]