Lecture 4 - Quantization of the electromagnetic field

2.5 Quantization of the e.m. field

Take cubic box of side length L, volume $V = L^3$, with periodic boundary conditions:

$$k_{\mathrm{x,\,y,\,z}} = \frac{2\pi}{L} n_{\mathrm{x,\,y,\,z}}$$

Therefore $\alpha_\epsilon(\vec{k}, t) \to \alpha_{\vec{k},\epsilon}(t)$ or simply α_i ($i = (\vec{k}_i, \vec{\epsilon}_i)$).

Correspondence

$$\int \mathrm{d}^3 k \sum_\epsilon f(\vec{k}, \vec{\epsilon}) \leftrightarrow \left(\frac{2\pi}{L}\right)^3 \Sigma_i f(\vec{k}_i, \vec{\epsilon}_i)$$

- Analogy with harmonic oscillator:

$$\begin{array}{c|c|c}
\text{One e.m. field mode} & \text{Correspondence} & \text{Harmonic Oscillator} \\
\dot{\mathcal{A}}_i = -\mathcal{E}_i & \mathcal{A}_i \hat{=} x & \dot{x} = \dfrac{p}{m} \\
\dot{\mathcal{E}}_i = \omega_i^2 \mathcal{A}_i & \mathcal{E}_i \hat{=} -\dfrac{p}{m} & \dfrac{\dot{p}}{m} = -\omega^2 x \\
H_i = \frac{\epsilon_0}{2} \frac{(2\pi)^3}{V} \left(|\mathcal{E}_i|^2 + \omega_i^2 |\mathcal{A}_i|^2\right) & \epsilon_0 \frac{(2\pi)^3}{V} \hat{=} m & H = \frac{m}{2}\left(\left(\frac{p}{m}\right)^2 + \omega^2 x^2\right) \\
\alpha_i = \mathcal{N}_i \left(\mathcal{A}_i - \dfrac{i}{\omega_i} \mathcal{E}_i\right) & & \alpha = \mathcal{N}\left(x + i\dfrac{p}{m\omega}\right) \\
\dfrac{d\alpha_i}{dt} = -i\omega_i \alpha_i & & \dfrac{d\alpha}{dt} = -i\omega\alpha
\end{array}$$

- Quantization and Commutation relations:

One e.m. field mode	Harmonic Oscillator
$\mathcal{A}_i \to \hat{\mathcal{A}}_i$	$x \to \hat{x}$
$\mathcal{E}_i \to \hat{\mathcal{E}}_i$	$p \to \hat{p}$
$\left[\hat{\mathcal{A}}_i, \hat{\mathcal{E}}_i\right] = -\frac{V}{(2\pi)^3\epsilon_0} i\hbar$	$[\hat{x}, \hat{p}] = i\hbar$
$\hat{a}_i$ annihilation operator associated to α_i	$\hat{a}$ annihilation operator associated to α
$\left[\hat{a}_i, \hat{a}_i^\dagger\right] = 1$ for	$\left[\hat{a}, \hat{a}^\dagger\right] = 1$ for
$\mathcal{N} = \sqrt{\frac{\epsilon_0\omega_i}{2\hbar}\frac{(2\pi)^3}{V}}$	$\mathcal{N} = \sqrt{\frac{m\omega}{2\hbar}}$

- Physical Operators:
- Hamiltonian:

$$H_i = \frac{\hbar\omega_i}{2}\left(\alpha_i^*\alpha_i + \alpha_i\alpha_i^*\right) \ \left(= \hbar\omega_i|\alpha_i|^2\right) \qquad H = \frac{\hbar\omega}{2}\left(\alpha^*\alpha + \alpha\alpha^*\right)$$

$$\hat{H}_i = \frac{\hbar\omega_i}{2}\left(\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger\right) \qquad \hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)$$

$$\hat{H} = \sum_i \hbar\omega_i\left(\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\right) \qquad \hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

- Momentum:

$$\vec{P} = \sum_i \frac{\hbar\vec{k}_i}{2}\left(\alpha_i^*\alpha_i + \alpha_i\alpha_i^*\right)$$

$$\vec{P} = \sum_i \frac{\hbar\vec{k}_i}{2}\left(\hat{a}_i^\dagger\hat{a}_i + \hat{a}_i\hat{a}_i^\dagger\right)$$

$$\vec{P} = \sum_i \hbar\vec{k}_i\left(\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\right)$$

but $\sum_i \vec{k}_i = 0$ so

$$\hat{\vec{P}} = \sum_i \hbar\vec{k}_i\hat{a}_i^\dagger\hat{a}_i$$

- Electric Field:

$$\hat{\vec{E}}(\vec{r}) = i\sum_i \mathcal{E}_i\left(\vec{\epsilon}_i\hat{a}_i e^{i\vec{k}_i\vec{r}} - \vec{\epsilon}_i\hat{a}_i^\dagger e^{-i\vec{k}_i\vec{r}}\right)$$

with $\mathcal{E}_i = \sqrt{\frac{\hbar\omega_i}{2\epsilon_0 V}}$

Note: We see that $\left[\hat{\vec{E}}(\vec{r}), \hat{\vec{E}}(\vec{r}')\right] \neq 0$. This implies that there are no eigenstates of $\hat{\vec{E}}(\vec{r})$ for all $\vec{r}$!

- Comment:

  Within the Lagrangian formalism one sees that the momentum conjugate to $\mathcal{A}_{\perp\epsilon}$ is $\Pi_\epsilon = \epsilon_0 \dot{\mathcal{A}}_{\perp\epsilon} = -\epsilon_0 \mathcal{E}_{\perp\epsilon}$. The canonical commutation relations are then

  $\left[\mathcal{A}_\epsilon(\vec{k}), \Pi^\dagger_{\epsilon'}(\vec{k}')\right] = i\hbar\delta_{\epsilon\epsilon'}\delta(\vec{k} - \vec{k}')$

  This agrees with $[\,\mathcal{A}_i, \mathcal{E}_j] = -\dfrac{V}{(2\pi)^3}\dfrac{i\hbar}{\epsilon_0}\delta_{ij}$ as

  $1 = \int \mathrm{d}^3k\, \delta(\vec{k} - \vec{k}') \leftrightarrow \frac{(2\pi)^3}{V}\sum_k \frac{V}{(2\pi)^3}\delta_{kk'}$ or

  $\delta(\vec{k} - \vec{k}') \leftrightarrow \frac{V}{(2\pi)^3}\delta_{kk'}$

2.6 Total Hamiltonian and Momentum:

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left(\vec{p}_\alpha - q_\alpha \vec{A}_\perp(\vec{r}_\alpha)\right)^2 + \sum_\alpha \left(-g_\alpha \frac{q_\alpha}{2m_\alpha}\right)\vec{S}_\alpha \cdot \vec{B}(\vec{r}_\alpha)$$

$$+V_{\text{Coulomb}} + H_{\text{R}}$$

$$V_{\text{Coulomb}} = \sum_\alpha \epsilon^\alpha_{\text{Coulomb}} + \frac{1}{8\pi\epsilon_0}\sum_{\alpha\neq\beta}\frac{q_\alpha q_\beta}{|\vec{r}_\alpha - \vec{r}_\beta|}$$

$$\epsilon^\alpha_{\text{Coulomb}} = \frac{q_\alpha^2}{2\epsilon_0(2\pi)^3}\int \mathrm{d}^3k\,\frac{1}{k^2} = \frac{q_\alpha^2}{4\pi^2\epsilon_0}k_c$$

$$H_{\text{R}} = \frac{\epsilon_0}{2}\int \mathrm{d}^3r\left(\vec{E}_\perp^2 + c^2\vec{B}^2\right) = \sum_i \hbar\omega_i\left(\hat{a}_i^\dagger\hat{a}_i + \frac{1}{2}\right)$$

Total Momentum:

$$\vec{P} = \sum_\alpha \vec{p}_\alpha + \vec{P}_\mathrm{R}$$

$$\vec{P}_\mathrm{R} = \sum_i \hbar \vec{k}_i \, \hat{a}_i^\dagger \hat{a}_i$$

$$H = H_\mathrm{P} + H_\mathrm{R} + H_\mathrm{I}$$

Particle Hamiltonian:

$$H_\mathrm{P} = \sum_\alpha \frac{\vec{p}_\alpha^2}{2m} + V_\mathrm{Coulomb}$$

Interaction:

$$H_\mathrm{I} = H_\mathrm{I1} + H_\mathrm{I2} + H_{I1}^S$$

$$H_\mathrm{I1} = -\sum_\alpha \frac{q_\alpha}{m_\alpha} \vec{p}_\alpha \cdot \vec{A}_\perp(\vec{r}_{\,\alpha})$$

$$H_\mathrm{I1}^S = -\sum_\alpha g_\alpha \frac{q_\alpha}{2m_\alpha} \vec{S}_\alpha \cdot \vec{B}(\vec{r}_{\,\alpha})$$

$$H_\mathrm{I2} = \sum_\alpha \frac{q_\alpha^2}{2m_\alpha} \vec{A}_\perp^2(\vec{r}_{\,\alpha})$$

2.7 State Space

The Hilbert space is the tensor product of that of Particles and that of Radiation:

$$\mathcal{H} = \mathcal{H}_\mathrm{Particles} \otimes \mathcal{H}_\mathrm{Radiation}$$

where

$$\mathcal{H}_{\text{Particles}} = \cdots \otimes \mathcal{H}_\alpha \otimes \ldots$$

with $\mathcal{H}_\alpha$ the Hilbert space for particle α, and

$$\mathcal{H}_{\text{Radiation}} = \cdots \otimes \mathcal{H}_i \otimes \ldots$$

with $\mathcal{H}_i$ the Hilbert space for mode i of the electromagnetic field.

An orthonormal basis for $\mathcal{H}_i$ is $\{|n_i\rangle\}$ of energy eigenstates of the oscillator at i. Writing $|\{n_i\}\rangle$ for the many-photon state $|n_1\rangle \ldots |n_i\rangle \ldots$ containing n_1 photons in mode 1, n_i photons in state i, etc.

$H_{\text{R}} |\{n_i\}\rangle = \left[\sum_i (n_i + \frac{1}{2})\hbar\omega_i\right] |\{n_i\}\rangle$

$\vec{P}_{\text{R}} |\{n_i\}\rangle = \left(\sum_i n_i \hbar \vec{k}_i\right) |\{n_i\}\rangle$

where we used the usual actions of the creation/annihilation operators:

$a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle$

$a|0\rangle = 0$

The vacuum is the state $|0\rangle$, meaning $n_1 = 0, \ldots, n_j = 0, \ldots$ etc., with the property

$a_i |0\rangle = 0 \quad \forall i$

Is the vacuum empty? The vacuum is an energy eigenstate: $H|0\rangle = E_{\text{V}}|0\rangle$ with $E_{\text{V}} = \sum_i \frac{1}{2}\hbar\omega_i$ the vacuum energy.

This is non-zero, although one could argue that an arbitrary energy offset (even if infinite) can never be measured. All that we can measure in the lab is energy differences. However, there is a certain reality to the vacuum energy, in the sense that it depends on the volume I use for quantization, and that volume may well be something real, for example the spacing between two mirrors. While I cannot directly measure the vacuum energy in between the mirrors, I should be able to measure how this energy *changes* with the distance between the mirrors. Indeed, an energy *change* with distance is a force, and there is a measurable force due to the vacuum, which is the Casimir force which we will discuss in detail.

What about the electric field content of the vacuum? Since $\vec{E}(\vec{r}) = i\sum_i \mathcal{E}_i \left(\vec{\epsilon}_i a_i e^{i\vec{k}_i\vec{r}} - \vec{\epsilon}_i a_i^\dagger e^{-i\vec{k}_i\vec{r}}\right)$ we have (I leave out the hat $\hat{\vec{E}}$ over operators if there is no confusion)

$$\langle 0| \vec{E} |0\rangle = 0$$

since $a_i |0\rangle = 0$ and $\langle 0| a_i^\dagger = 0$. So the expectation value of the electric field in the vacuum is zero.

However, we have

$$\langle 0| \vec{E}^2 |0\rangle \neq 0$$

since $\langle 0| a_i a_i^\dagger |0\rangle = 1$.

We find

$$\langle 0| \vec{E}^2 |0\rangle = \sum_i |\mathcal{E}_i|^2 = \sum_i \frac{\hbar\omega_i}{2\epsilon_0 V}$$

and therefore also the uncertainty of the value of the electric field, $\Delta E \neq 0$, in other words we find fluctuations of the vacuum!

This is in origin precisely the some as the zero point fluctuations of a harmonic oscillator, deriving from the uncertainty relation between momentum and position (here, between the transverse electric field and the transverse vector potential).

The vacuum is therefore not empty!

$$|0\rangle \neq 0$$

A direct application of this is that there must be a "Vacuum Stark effect". Indeed, this is the origin of the Lamb shift splitting the $2s_{1/2}$ and $2p_{1/2}$ states in hydrogen.

2.8 The Dipole Interaction

Here we will sketch how one obtains the typical form of the dipole interaction ($-\vec{d} \cdot \vec{E}$) from the total Hamiltonian we found above. We will assume wavelengths λ much larger than the size of an atom, a_0, the Bohr radius. This is the long wavelength (or dipole) approximation. The dipole of the system of charges is

$$\vec{d} = \sum_\alpha q_\alpha \vec{r}_\alpha$$

In our approximation, we have

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left(\vec{p}_\alpha - q_\alpha \vec{A}_\perp(0)\right)^2 + V_{\text{Coulomb}} + \sum_j \hbar\omega_j \left(a_j^\dagger a_j + \frac{1}{2}\right)$$

We will apply a unitary transformation to the Hamiltonian, which will generate a simultaneous translation of momenta $\vec{p}_\alpha$ and of the transverse electric field (and thus the creation / annihilation operators):

$$T = \exp\left(-\frac{i}{\hbar}\vec{d}\cdot\vec{A}_\perp(\vec{0})\right)$$

$$= \exp\left(\sum_j \left(\lambda_j^* a_j - \lambda_j a_j^\dagger\right)\right)$$

with $\lambda_j = \frac{i}{\sqrt{2\epsilon_0\hbar\omega_j V}}\vec{\epsilon}_j \cdot \vec{d}$.

We have

$$\begin{aligned} T\vec{r}_\alpha T^\dagger &= \vec{r}_\alpha \\ T\vec{p}_\alpha T^\dagger &= \vec{p}_\alpha + q_\alpha \vec{A}(0) \\ Ta_j T^\dagger &= a_j + \lambda_j \\ Ta_j^\dagger T^\dagger &= a_j^\dagger + \lambda_j^* \end{aligned}$$

The momentum translation turns the canonical momentum back into the mechanical momentum. Indeed,

$$\vec{v}_\alpha' = T\vec{v}_\alpha T^\dagger = \frac{\vec{p}_\alpha}{m_\alpha}.$$

The transformed Hamiltonian is

$$\begin{aligned} H' = THT^\dagger = \sum_\alpha \frac{\vec{p}_\alpha^2}{2m_\alpha} + V_{\text{Coulomb}} + \epsilon_{\text{dipole}} + \sum_j \hbar\omega_j \left(a_j^\dagger a_j + \frac{1}{2}\right) - \\ - \vec{d}\cdot\sum_j \mathcal{E}_j \left(i a_j \vec{\epsilon}_j - i a_j^\dagger \vec{\epsilon}_j\right) \end{aligned}$$

Here, $\epsilon_{\text{dipole}} = \sum_j \frac{1}{2\epsilon_0 V}\left(\vec{\epsilon}_j \cdot \vec{d}\right)^2$ is the dipolar self-energy.

The vector potential does not transform under T, but the electric field gets shifted:

$$\vec{A}'_\perp = T\vec{A}_\perp T^\dagger = \vec{A}_\perp = \sum_j \mathcal{A}_j \left(a_j \vec{\epsilon}_j e^{i\vec{k}_j\cdot\vec{r}} + a_j^\dagger \vec{\epsilon}_j e^{-i\vec{k}_j\cdot\vec{r}}\right)$$

$$\begin{aligned}\vec{E}'_\perp = T\vec{E}_\perp T^\dagger &= \sum_j \mathcal{E}_j \left(i(a_j+\lambda_j)\vec{\epsilon}_j e^{i\vec{k}_j\cdot\vec{r}} + \text{c.c.}\right)\\ &= \vec{E}_\perp - \frac{1}{\epsilon_0}\vec{P}_\perp\end{aligned}$$

with $\vec{P}_\perp = \sum_j \frac{\vec{\epsilon}_j(\vec{\epsilon}_j\cdot\vec{d})}{V} e^{i\vec{k}_j\cdot\vec{r}}$

It turns out that $\vec{P}$ is the polarization density of the system of charges. From $\rho(\vec{r}) = \sum_\alpha q_\alpha \delta(\vec{r}-\vec{r}_\alpha)$ we have a Fourier transform $\rho(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\sum_\alpha q_\alpha e^{-i\vec{k}\cdot\vec{r}_\alpha}$ which is, in the long-wavelength approximation and for a neutral system of charges ($\sum_\alpha q_\alpha = 0$) equal to $\rho(\vec{k}) = -\frac{1}{(2\pi)^{3/2}} i\vec{k}\cdot\vec{d}$. Fourier-transforming back this gives $\rho(\vec{r}) = -\vec{\nabla}\cdot(\vec{d}\,\delta(\vec{r}))$ inviting the definition of the polarization density $\vec{P}(\vec{r}) = \vec{d}\,\delta(\vec{r})$ corresponding to the dipole localized at the origin. It's spatial Fourier transform is $\vec{\mathcal{P}}(\vec{k}) = \frac{1}{(2\pi)^{3/2}}\vec{d}$. We therefore have

$\rho(\vec{r}) = -\vec{\nabla}\cdot\vec{P}(\vec{r})$

This motivates the introduction of the displacement field $\vec{D} = \epsilon_0\vec{E} + \vec{P}$, for which Maxwell's equations directly give

$$\vec{\nabla}\cdot\vec{D} = 0,$$

showing that $\vec{D}$ is a transverse field. So $\vec{D} = \vec{D}_\perp = \epsilon_0\vec{E}_\perp + \vec{P}_\perp$.

We realize that the expression of $\vec{P}_\perp = \sum_j \frac{\vec{\epsilon}_j(\vec{\epsilon}_j\cdot\vec{d})}{V} e^{i\vec{k}_j\cdot\vec{r}}$ found above is indeed just the Fourier transform of the transverse part of $\vec{P}$.

Finally, we can calculate how the displacement field $\vec{D}$ transforms under the unitary transformation T:

$\vec{D}'_\perp = \epsilon_0\vec{E}'_\perp + \vec{P}'_\perp = \vec{E}_\perp = i\sum_j \mathcal{E}_j \left(a_j\vec{\epsilon}_j e^{i\vec{k}_j\cdot\vec{r}} - a_j^\dagger\vec{\epsilon}_j e^{-i\vec{k}_j\cdot\vec{r}}\right)$

The dipole Hamiltonian, describing the interaction between the electric dipole and the displacement field, is thus

$H'_\mathrm{I} = -\vec{d}\cdot\frac{\vec{D}'(0)}{\epsilon_0} = -\vec{d}\cdot\vec{E}_\perp(0)$

We note that the same mathematical operator (the $a_j \cdots - a_j^\dagger \ldots$ above) describes two different physical variables, depending on the represenation used: The electric field operator in the original representation, and the displacement field in the representation transformed with operator T.

The Hamiltonian in the transformed representation features an interaction between charges and radiation field that is only linear in the field. We no longer have a quadratic component such as H_{I2}. This is an important simplification.