FINAL EXAM

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MA434: Algebraic Geometry
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Problem	Earned	Total
1		10
4		20
5		20
6		10
11		30
12		30
Total	/100	120

References: For this exam I referred extensively to the Moodle reading guides, videos, and problems, and Reid's *Undergraduate Algebraic Geometry*. I also used Gallian's *Contemporary Abstract Algebra, 8th edition* for theorems related to ring homomorphisms.

Problem 1 (10 pts)

Show that the affine variety in $\mathbb{A}^2$ defined by $xy = 1$ is not isomorphic to $\mathbb{A}^1$.

<u>*Solution*</u>*:* Let $V = V(xy - 1) \subset \mathbb{A}^2$ denote the affine variety defined by $xy = 1$. We know that V is isomorphic to $\mathbb{A}^2$ if and only if the coordinate rings $k[V]$ and $k[\mathbb{A}^2] = k[t]$ are also isomorphic. It suffices to show $k[V] = k[x, y]/\langle xy - 1\rangle$ is **not** isomorphic to $k[t]$. To this end, consider any ring homomorphism $\Phi : k[x, y] \to k[t]$ with $\ker \Phi = \langle xy - 1\rangle$. This is possible because every ideal I of a ring R is the kernel of some ring homomorphism of R.

Let $\Phi(x) = \alpha(t) \in k[t]$ and $\Phi(y) = \beta(t) \in k[t]$, then

$$0 = \Phi(xy - 1) = \Phi(x)\Phi(y) - 1 = \alpha(t)\beta(t) - 1.$$

This means

$$\alpha(t)\beta(t) = 1.$$

Since $\alpha(t)$ and $\beta(t)$ must be polynomials in t, the equation holds only if $\alpha(t)$ and $\beta(t)$ are elements of k, which are just constants. Now, since Φ is a ring homomorphism, for any $f \in k[V]$, $\Phi(f)$ must also be a constant (in k). So, Φ is not surjective, i.e., $\Phi(k[x, y])$ is not isomorphic to $k[V]$.

The first isomorphism theorem for rings says $k[x, y]/\langle xy-1\rangle$ is isomorphic to $\Phi(k[x, y])$. But we just showed $\Phi(k[x, y])$ is not isomorphic to $k[t]$, so $k[V]$ is not isomorphic to $k[t]$. This implies V is not isomorphic to $\mathbb{A}^1$. $\square$

Problem 4 **(20 pts)**

Suppose that f is a rational function on $\mathbb{P}^1$.

(a) Show that if f is regular at every point of $\mathbb{P}^1$ then it is constant. (Hint: consider two affine pieces $\mathbb{A}^1_{(0)}$ and $\mathbb{A}^1_{(1)}$.)

(b) Show that there are no non-constant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$.

<u>*Solution:*</u>

(a) Let $f \in k(\mathbb{P}^1)$ be given such that f is regular at every point in $\mathbb{P}^1$. From the last exam/the beginning of chapter 5, we know that $\mathbb{P}^1$ can be thought of as two copies of $\mathbb{A}^1$ glued together. Call x_0, y_1 the coordinates of the two $\mathbb{A}^1$, respectively. The "glueing" action is given by the isomorphism $\mathbb{A}^1_{(0)} - \{x_0 = 0\} \to \mathbb{A}^1_{(1)} - \{y_1 = 0\}$:

$$x_0 \mapsto y_1 = \frac{1}{x_0}$$

Explicitly, $\mathbb{P}^1 = \mathbb{A}^1_{(0)} \cup \mathbb{A}^1_{(1)}$ where

$$\mathbb{A}^1_{(0)} = \mathbb{A}^1 - (x_0 = 0), \quad \mathbb{A}^1_{(1)} = \mathbb{A}^1 - (y_1 = 0).$$

Applying theorem 4.8 (II) (which says $\operatorname{dom}(f) = V \iff f \in k[V]$) to the affine piece $\mathbb{A}^1_{(0)}$, we get $f = p(x_0) \in k[x_0]$. Applying theorem 4.8 (II) to the affine piece $\mathbb{A}^1_{(1)}$ and applying the "change of variables" $x_0 = 1/y_1$ we get $f = p(1/y_1) \in k[y_1]$. Now, the only way $p(1/y_1)$ can be a polynomial is that p is a constant. So, f is constant.

(b) From the previous item we should able to deduce that there are no non-constant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$: A morphism f on $\mathbb{P}^1$ must be such that $\mathbb{P}^1 \subset \operatorname{dom}(f)$. This means each component f_i (which is a rational function) of f must also be regular at every point in $\mathbb{P}^1$. The first item tells us that each f_i must be constant. So, we conclude f must also be constant, i.e., there are no non-constant morphisms $\mathbb{P}^1 \to \mathbb{A}^m$.

$\square$

Problem 5 **(20 pts)**

Below are three formulas that possibly define rational maps $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Decided whether the formulas do define rational maps. If they do, determine dom(f) and decide whether f is birational.

(i) $f([x : y : z]) = [1/x : 1/y : 1/z]$

(ii) $f[(x : y : z)] = [x : y : 1]$

(iii) $f([x : y : z]) = [(x^3 + y^3)/z^3 : y^2/z^2 : 1]$.

Rationals maps must be ratio(s) of homogeneous polynomials of the same degree. On first glance we see that (ii) does not define a rational map because there is no way to write its output into ratios of homogeneous polynomials of the same degree:

$$[x : y : 1] = [1 : y/x : 1/x] = [x/y : 1 : 1/y].$$

On the other hand, the outputs in (i) and (iii) can be written in the desired forms:

$$\left[\frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right] = \left[1 : \frac{x}{y} : \frac{x}{z}\right] = \left[\frac{y}{x} : 1 : \frac{y}{z}\right] = \left[\frac{z}{x} : \frac{z}{y} : 1\right] = \ldots$$

and

$$\left[\frac{x^3 + y^3}{z^3} : \frac{y^2}{z^2} : 1\right] = \left[\frac{x^3 + y^3}{zy^2} : 1 : \frac{z^2}{y^2}\right] = \left[1 : \frac{y^2 z}{x^3 + y^3} : \frac{z^3}{x^3 + y^3}\right] = \ldots$$

Now we want to find dom(f) for (i) and (iii). By definition,

$$\text{dom}(f) = \{P \in \mathbb{P}^2 | f \text{ is regular at } P\}.$$

Find the domain:

- For (i), clearly f is regular at all points with $x, y, z \neq 0$. Without loss of generality, suppose $x = 0$ and $y, z \neq 0$ then we write the output as $[1 : x/y : x/z] = [1 : 0 : 0]$. So f is also regular there. Similarly, we can see f is also regular at $[x : y : z]$ where only $z = 0$ and only $y = 0$. However, when two of x, y, z are zero, $f([x, y, z])$ is no longer defined. So, for (i),

$$\boxed{\text{dom}(f_{(i)}) = \mathbb{P}^2 - \{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}}$$

- For (iii), we are interested in cases where $z = 0$, $y = 0$, and $x^3 + y^3 = 0$. By writing the output of f in different forms above, we see that f is still regular at $[x : y : z]$ where only **one** of the possibilities $z = 0$, $y = 0$, or $x^3 + y^3 = 0$ occurs, or if only $z = y = 0$, $x = y = x^3 + y^3 = 0$ occurs. However, since we have the factor $[(x^3 + y^3)/z]^{\pm 1}$ in all of the three representations of the output of f, we see that f fails to be regular when $z = 0$ and $x^3 + y^3 = 0$. So, for (iii),

$$\boxed{\text{dom}(f_{(iii)}) = \mathbb{P}^2 - \{[-1 : 1 : 0]\}}$$

Birational? Next, $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is *birational* if there exists a rational (inverse) map $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}$ such that $f \circ g = \mathrm{id}_{\mathbb{P}^2}$ and $g \circ f = \mathrm{id}_{\mathbb{P}^2}$.

- For (i), we consider the rational function $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $g([u : v : w]) = [1/u : 1/v : 1/w]$. So, g is just f. For $[u : v : w] \in \mathrm{dom}(g) = \mathrm{dom}(f)$, we have

$$f \circ g([u : v : w]) = f([1/u : 1/v : 1/w]) = [u : v : w].$$

  for all $[u : v : w]$ in $\mathrm{dom}(g) = \mathrm{dom}(f)$. Similarly, $g \circ f$ is also the identity function on $\mathrm{dom}(f) = \mathrm{dom}(g)$. Finally, since f and g are really the same function, it remains to show f is dominant. By definition, f is dominant if $f(\mathrm{dom}(f))$ is dense in $\mathbb{P}^2$. This is equivalent to saying $f(\mathrm{dom}(f)) \cap \mathcal{O} \neq \varnothing$ for any nonempty open set $\mathcal{O} \subset \mathbb{P}^2$. It is clear that the output of f is not only all tuples $[1/x : 1/y : 1/z]$ with $x, y, z \neq 0$ but also $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$:

$$f([1 : 1 : 0]) = [0 : 0 : 1], \quad f([1 : 0 : 1]) = [0 : 1 : 0], \quad f([0 : 1 : 1]) = [1 : 0 : 0].$$

  So, $f(\mathrm{dom}(f)) = \mathbb{P}^2$. It follows that f is dominant. With the dominant rational inverse g (which is just f itself), we conclude that f is birational.

  (†) Alternatively, we can see that the induced k-algebra homomorphism $f^* : k(\mathbb{P}^2) \to k(\mathbb{P}^2)$ given by $g \mapsto g \circ f$ is an isomorphism. This (I believe) is easy to see because for any $g \in k[\mathbb{P}^2]$, if $g([1/x : 1/y : 1/z]) = 0$ then $g = 0$ necessarily (because the factors $1/x, 1/y, 1/z$ are in some sense "independent"), which shows f^* is injective. Further, any element of $k(\mathbb{P}^2)$ (which has the form of a ratio of two homogeneous polynomials of the same degree) can be put into the form $g \circ f$ where $g \in (\mathbb{P}^2)$. So f^* is an isomorphism. This combined with the fact that f is dominant is equivalent to f being birational.

- For (iii), we claim that f is not birational. This is because the induced k-algebra homomorphism $f^* : k(\mathbb{P}) \to k(\mathbb{P}^2)$ is **not** onto (hence not an isomorphism). Consider the element $x/y \in k(\mathbb{P}^2)$. There is no $g \in k(\mathbb{P}^2)$ such that $g \circ f[x : y : z] = x/y$ because x always appears as x^3 in the output of f. We conclude f is not birational.

$\square$

Problem 6 **(10 pts)**

Prove statements (i), (ii), (iii), (iv) from Example I from section 5.7 of *Undergraduate Algebraic Geometry*

<u>Solution</u>: Define $f : \mathbb{P}^1 \to \mathbb{P}^m$ by

$$[U : V] \mapsto [U^m : U^{m-1}V : \cdots : V^m]$$

(i) f is a rational map: We notice that while U^iV^j are it rational functions (since they are not given by a ratio of homogeneous polynomials of the same degree), we can re-write the definition of f as

$$[U : V] \overset{f}{\mapsto} \left[\frac{U^m}{V^m} : \frac{U^{m-1}}{V^{m-1}} : \cdots : 1\right].$$

Now, each component f_i is a rational function, so we have a rational map.

(ii) f is a morphism: f is a morphism if we can show $\mathbb{P}^1 \subset \operatorname{dom}(f)$, i.e., f is regular at every point of $\mathbb{P}^1$. If $V \neq 0$ then there's nothing to prove because of the formula we just wrote down. If $U \neq 0$ then we can just rewrite the definition of f as

$$[U : V] \overset{f}{\mapsto} \left[1 : \frac{V}{U} : \cdots : \frac{V^m}{U^m}\right]$$

which shows that f is also regular at these points. When $U, V \neq 0$, there's nothing to worry about. So, f is indeed regular at every point in $\mathbb{P}$, i.e., $f : \mathbb{P}^1 \to \mathbb{P}^m$ is a morphism.

(iii) The image of f is the set of points $[X_0 : \cdots : X_m] \in \mathbb{P}^m$ such that

$$[X_0 : X_1] = [X_1 : X_2] = \cdots = [X_{m-1} : X_m]$$

that is

$$X_0X_2 = X_1^2; \quad X_0X_3 = X_1X_2; \quad X_0X_4 = X_1X_3; \quad \text{etc.}$$

We notice that for every input $[U : V]$, the output looks like

$$[X_0 : X_1 : \cdots : X_m] = [U^m : U^{m_1}V : \ldots V^m]$$

So, we have that

$$[X_0 : X_1] = [U^m : U^{m-1}V] = [U : V]$$
$$[X_1 : X_2] = [U^{m-1}V : U^{m-2}V^2] = [U : V]$$

and so forth. So, we end up with

$$[X_0 : X_1] = [X_1 : X_2] = \cdots = [X_{m-1} : X_m]$$

From here it is not hard to generalize:

$$[X_0 : X_1] = [X_{n-1} : X_n]$$

and so we have a chain of equalities $X_0 X_n = X_1 X_{n-1}$ for different values of n. This means any 2×2 matrix of the form

$$\begin{bmatrix} X_0 & X_{n-1} \\ X_1 & X_n \end{bmatrix}$$

has vanishing determinant. This leads to the condition

$$\operatorname{rank}\begin{bmatrix} X_0 & X_1 & X_2 & \dots & X_{m-1} \\ X_1 & X_2 & X_3 & \dots & X_m \end{bmatrix} \leq 1.$$

This condition coincides exactly with the all-vanishing determinant condition above: If the matrix rank is zero, the matrix is the zero matrix, in which case there is nothing interesting (in fact this case won't happen because at least one X_i has to be nonzero – $[X_0 : \cdots : X_m] \in \mathbb{P}^m$). If the matrix has rank one, then one row is a constant multiple of the other. After writing, say, the first row as some multiple of the second row, we see that any 2×2 minor has the form $a(X_n X_m - X_m X_n)$, which vanishes identically. When the matrix has rank 2, the both rows are linearly independent, and we no longer have the vanishing 2×2 minor condition.

(iv) There is an inverse morphism $g : C \to \mathbb{P}^1$. The inverse morphism takes a point of C into the common ratio:

$$[X_0 : \cdots : X_m] \overset{g}{\mapsto} [X_0 : X_1]$$

where $[X_0 : X_1]$ is "common" in the sense of the previous item. We want to check that this is actually a morphism, i.e., it is a rational map that is regular at every point in C. Clearly, we can write

$$[X_0 : \cdots : X_m] \overset{g}{\mapsto} \left[1 : \frac{X_1}{X_0}\right] \text{ or } \left[\frac{X_{m-1}}{X_m} : 1\right]$$

depending on whether $X_1 = 0$ or $X_0 = 0$ (or both). In any case, we see that g is a rational function (as given by ratios of homogeneous polynomials of the same degree) that is regular at every point on C. □

Problem 11 (30 pts)

Given an invertible matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with complex coefficients, define a function $f_A : \mathbb{P}^1_{\mathbb{C}} \to \mathbb{P}^1_{\mathbb{C}}$ by

$$f_A([u:v]) = [au + bv : cu + dv].$$

(a) Show that f_A is a morphism.

(b) How does f_{AB} relate to f_A and f_B?

(c) Show that f_A has an inverse morphism, so that f_A defines an automorphism of $\mathbb{P}^1_{\mathbb{C}}$.

(d) If we identify $\mathbb{C}$ with the standard $\mathbb{A}^1 \subset \mathbb{P}^1$ defined by $v \neq 0$, show that the restriction of f_A to $\mathbb{C}$ is a rational function, and find its formula.

Solution:

(a) f_A is a morphism if $\mathbb{P}^1_{\mathbb{C}} \subset \operatorname{dom}(f)$, i.e., f is regular at every point in $\mathbb{P}^1_{\mathbb{C}}$, i.e., $au + bv$ and $cu + dv$ are never simultaneously zero for any u, v. Now, we don't have the possibility $u = v = 0$ because $[u : v] \in \mathbb{P}^1_{\mathbb{C}}$. So, $au + bv = 0 = cu + dv$ for some pair u, v if and only if $\det(A) = 0$. But this never happens because A is invertible. So, f is regular at every point $[u : v] \in \mathbb{P}^1_{\mathbb{C}}$, i.e., f is a morphism.

(b) We claim that $f_{AB} = f_A \circ f_B$. Let an invertible matrix B be given,

$$B = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \implies AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{bmatrix}.$$

Then

$$\begin{aligned} f_{AB}([u:v]) &= [(aa' + bc')u + (ab' + bd')v : (ca' + dc')u + (cb' + dd')v] \\ &= [a(a'u + b'v) + b(c'u + d'v) : c(a'u + b'v) + d(c'u + d'v)] \\ &= f_A[a'u + b'v : c'u + d'v] \\ &= f_A \circ f_B([u:v]). \end{aligned}$$

(c) To show that f_A has an inverse morphism, it suffices to construct one. Consider $f_{A^{-1}}$ defined by A^{-1}, the matrix inverse of A:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We know that $\det(A) \neq 0$, so A^{-1} exists. Also, since the scaling factor $1/\det(A) \neq 0$ appears at every entry of A^{-1}, we can ignore it in the definition of $f_{A^{-1}}$:

$$\begin{aligned} f_{A^{-1}}([u:v]) &= \left[\frac{d}{\det(A)}u + \frac{-b}{\det(A)}v : \frac{-c}{\det(A)}u + \frac{a}{\det(A)}v\right] \\ &= [du - bv : -cu + av]. \end{aligned}$$

Next we check that f_A and fA^{-1} are inverses. By the previous item, we know that $f_{A^{-1}A} = f_{AA^{-1}} = f_I$ where I is the 2×2 identity matrix. Since

$$f_I([u:v]) = [u + 0v : 0u + v] = [u:v],$$

f_A and $f_{A^{-1}}$ are inverses. So, f_A is an isomorphism from (the entire) $\mathbb{P}^1_{\mathbb{C}}$ to itself. This makes f_A an automorphism.

(d) We want to look at $f_A : \mathbb{C} \to \mathbb{C}$ where $\mathbb{C}$ is identified with the standard $\mathbb{A}^1 \subset \mathbb{P}^1$ defined by $v \neq 0$ (here the restriction is at both ends). We want to show that f_A in this case is a rational function and find its formula. Now, when $v \neq 0$, we can write the input $[u:v]$ as $[u/v:1] = [t:1]$ where $t \in \mathbb{C}$. With this,

$$f_A([t:1]) = [at + b : ct + d] = \left[\frac{at+b}{ct+d} : 1\right].$$

Restricting both ends to $\mathbb{C}$, we can identify a rational function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(t) = \frac{at+b}{ct+d}.$$

This is a function from $\mathbb{C}$ to $\mathbb{C}$ (or equivalently from $\mathbb{A}^1 \to \mathbb{A}^1$). Also, because it is a ratio of polynomials, it is a rational function.

$\square$

Problem 12 (30 pts)

Show that any automorphism of $\mathbb{P}^1_{\mathbb{C}}$ is of the form f_A as in the previous problem.

<u>*Solution:*</u> Let an automorphism f on $\mathbb{P}^1_{\mathbb{C}}$ be given. It is an automorphism so it is an isomorphism - a morphism with an inverse morphism. This means

$$f([u:v]) = [f_1(u,v) : f_2(u,v)]$$

where f_1, f_2 are necessarily ratios of homogeneous polynomials of the same degree. We look at two cases: either f maps the point at infinity to the point at infinity, i.e., $f([1:0]) = [1:0]$, or to some point not at infinity - without loss of generality assume this point is $f([1:0]) = [\epsilon : 1]$ where $\epsilon \in \mathbb{C}$.

- If f maps the point at infinity to the point at infinity, i.e., $f([1:0]) = [1:0]$, then because f is an isomorphism, it must map any "regular" point to a "regular point." This means we can make the restriction (at both ends, with $v \neq 0$, $f_2(u,v) \neq 0$) so that $f|_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$, with $t \mapsto f|_{\mathbb{C}}(t)$. With this we have

  $$f([t:1]) = [f|_{\mathbb{C}}(t) : 1],$$

  where $f|_{\mathbb{C}}$ must be defined for all $t \in \mathbb{C}$, is bijective in $\mathbb{C}$, and must be a ratio of homogeneous polynomials of the same degree. For such $f|_{\mathbb{C}}$ to be defined for all $t \in \mathbb{C}$, f is necessarily a polynomial (a non-constant denominator always has roots - not good). If this polynomial has degree 0 or greater than 1 then it fails to be bijective. So, $f|_{\mathbb{C}}$ is a polynomial of degree 1. With this, we write, for $a, b \in \mathbb{C}$, $a \neq 0$:

  $$f([t:1]) = [at + b : 1].$$

  We see that when we write the input as $[u:v] \in \mathbb{P}^1_{\mathbb{C}}$ where $[u:v] = [1:0]$ or $[u:v] = [u/v : 1] = [t:1]$, we can write the output of this f as

  $$f([u:v]) = [au + bv : cv], \quad c \neq 0$$

  which captures $[1:0] \mapsto [1:0]$ as well. We notice that the output can never have the form $[0:0]$. This corresponds exactly to

  $$\det \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \neq 0.$$

- If f maps the point at infinity to a "regular" point $[\epsilon : 1]$, then we can send this point back to the point at infinity using a known automorphism g. The composition $g \circ f$ is now an automorphism that sends $[1:0]$ to $[1:0]$. By the previous item, we know the form $g \circ f$ takes. To find the form of f, we want to find the form of g. To do this, we look at

  $$g \circ f([1:0]) = g([\epsilon : 1]) = [1:0].$$

Take

$$g([u:v]) = [v:u-\epsilon v].$$

The matrix associated with g is

$$G = \begin{bmatrix} 0 & 1 \\ 1 & -\epsilon \end{bmatrix}.$$

We see that $\det(G) = -1 \neq 0$, so by the previous problem we know g is indeed an automorphism on $\mathbb{P}^1_{\mathbb{C}}$. Now, the form of $g \circ f$, by the previous item, is

$$g \circ f([u:v]) = [cu+dv : ev] = [f_2(u/v) : f_1(u/v) - \epsilon f_2(u/v)].$$

where we are taking $v \neq 0$. Call $u/v = t$, then because $g \circ f$ only maps the point at infinity to the point at infinity, we know that $f_2(u/v)$ must be a polynomial of degree one in u/v (by our previous argument). This means $f_1(u/v)$ is of degree one as well. After homogenizing, we have

$$f([u:v]) = [au+bv : cu+dv].$$

Finally, we want conditions on a, b, c, d such that f is actually an automorphism. f fails to be an automorphism exactly when the matrix $F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible. So f is an automorphism exactly when $\det(F) \neq 0$.

In either case, we have shown that any automorphism of $\mathbb{P}^1_{\mathbb{C}}$ is of the form f_A as in the previous problem. □