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Course: **8.370 - QC**
Problem set: **#5**
Due: Wednesday, Oct 26, 2022
Collaborators/References:

1. Teleportation

Suppose the qubit Alice wants to teleport has state $|\psi\rangle_A = \alpha|0\rangle + \beta|1\rangle$. If Alice and Bob start with $(1/\sqrt{2})(|01\rangle_{AB} - |10\rangle_{AB})$ then Bob's states for each of Alice's measurement outcome and Bob's required unitary transformations are

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) &\to |\psi\rangle_B = +\alpha|1\rangle - \beta|0\rangle \implies \sigma_y \\
\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) &\to |\psi\rangle_B = -\alpha|0\rangle + \beta|1\rangle \implies \sigma_z \\
\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) &\to |\psi\rangle_B = +\alpha|1\rangle + \beta|0\rangle \implies \sigma_x \\
\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) &\to |\psi\rangle_B = +\alpha|0\rangle + \beta|1\rangle \implies \text{id}
\end{aligned}$$

Of course, another way to do this problem is to notice that the new EPR pair which Alice and Bob share is the original EPR pair transformed by σ_y. Therefore, Bob needs to apply a σ_y to his original unitary transformations to obtain the new unitary transformations.

2. Deutsch-Jozsa

In the Deutsch-Jozsa algorithm, the probability for a state $|k\rangle$ to be measured is

$$\left|\frac{1}{2^n}\sum_{j=0}^{2^n-1}(-1)^{f(j)}(-1)^{j\cdot k}\right|^2$$

In the algorithm, we are interested in the probability of measuring $k = 0$, which is

$$\Pr = \left|\frac{1}{2^n}\sum_{j=0}^{2^n-1}(-1)^{f(j)}\right|^2.$$

For $f(j)$ constant, this quantity is 1. For $f(j)$ balanced, this quantity is 0.

In the case that $f(j) = 0$ for r times and $f(j) = 1$ for s times, the probability that the algorithm tells us that f is constant is the probability of measuring $|0\rangle$, which is:

$$\Pr = \left|\frac{1}{2^n}(r-s)\right|^2 = \boxed{\frac{(r-s)^2}{2^{2n}}}$$

3. Two qubits

(a) It suffices to check that the states are mutually orthogonal:

$$\begin{aligned}
\langle 00|01\rangle &= \langle 0|0\rangle \langle 0|1\rangle = 0\\
\langle 00|1+\rangle &= \langle 0|1\rangle \langle 0|+\rangle = 0\\
\langle 00|1-\rangle &= \langle 0|1\rangle \langle 0|-\rangle = 0\\
\langle 01|1+\rangle &= \langle 0|1\rangle \langle 1|+\rangle = 0\\
\langle 01|1-\rangle &= \langle 1|1\rangle \langle 1|-\rangle = 0\\
\langle 1+|1-\rangle &= \langle 1|1\rangle \langle +|-\rangle = 0.
\end{aligned}$$

Since the states also span the 4-dimensional Hilbert space for states of a 2-qubit system, the states provided form an orthonormal basis.

(b) Since each of the given states are product states, Alice and Bob can make sequential measurements to find what state their qubits are in originally. Consider the following procedure: Alice makes a measurement in the $\{|0\rangle, |1\rangle\}$ basis. If she sees $|0\rangle$, she calls Bob and tells him to measure his qubit in the $\{|0\rangle, |1\rangle\}$ basis. Else if Alice sees $|1\rangle$, then she calls and tells Bob to measure his qubit in the $\{|+\rangle, |-\rangle\}$ basis. They will be able to unambiguously identify the state they had in the beginning.

(c) Alice can always tell with certainty the state of her qubit by measuring in the $\{|0\rangle, |1\rangle\}$ basis. However, Bob cannot, simply because $|\pm\rangle$ is a superposition of $|0\rangle$ and $|1\rangle$ and vice versa. There is no measurement basis with which Bob (alone) can use to unambiguously identity the state of his qubit.

(d) No? Essentially we want to make a non-local controlled-Hadamard gate. The simultaneous measurement condition disallows us from doing this. Gut feeling says this is possible but I can't seem to find a way to get an answer.

4. GHZ

Alice, Bob, and Charlie hold the GHZ state:

$$\frac{1}{\sqrt{2}}\left(|000\rangle_{ABC} + |111\rangle_{ABC}\right)$$

(a) In order to arrange so that Alice and Charlie share the Bell pair $(1/\sqrt{2})(|00\rangle + |11\rangle)$, Alice, Bob, and Charlie must somehow "factor" Bob's qubit out of the GHZ state. This requires a controlled-unitary gate or a classical controlled-operation whose control bit is Alice's or Charlie's qubit and target bit is Bob's. This is not possible if the trio are not allowed to communicate.

(b) For Alice and Charlie to have the Bell pair state in the problem, Bob first applies a Hadamard on his qubit, so that the system becomes

$$\frac{1}{\sqrt{2}}\left(|0+0\rangle + |1-1\rangle\right) = \frac{1}{2}\left(|000\rangle + |010\rangle + |101\rangle - |111\rangle\right).$$

Now Bob measures his qubit in the $\{|0\rangle, |1\rangle\}$ basis. The system after this measurement is in one of the two states:

$$\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{AC} |0\rangle_B \qquad \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)_{AC} |1\rangle_B$$

with equal probability. If Bob measures $|0\rangle$, he does nothing, Alice and Charlie already share the correct EPR state. If Bob measures $|1\rangle$, then he lets either Charlie or Alice knows so that one of them applies a σ_z on their qubit, so that $(|00\rangle - |11\rangle)_{AC} \to (|00\rangle + |11\rangle)_{AC}$.