How do we find the Lie algebra of a MLG?

Some useful properties of the matrix exponential:

- $(e^X)^{-1} = e^{-X}$
- $(e^X)^T = e^{X^T}$
- $(e^X)^* = e^{X^*}$
- $\overline{(e^X)} = e^{\bar{X}}$
- $\det(e^X) = e^{tr(X)}$
- $e^{\varepsilon X} = e^{\varepsilon Y} \quad \forall \varepsilon \in \mathbb{R} \Leftrightarrow X = Y$

Proof: $\Leftarrow$ trivial
$\Rightarrow$ take derivative on both sides and evaluate at $\varepsilon = 0$

Example: $U(n) = \{ A \in GL(n, \mathbb{C}) \mid A^* A = \mathbb{1} \}$

$$\Rightarrow \mathfrak{u}(n) = \{ X \in M_n(\mathbb{C}) \mid e^{\varepsilon X} \in U(n) \quad \forall \varepsilon \in \mathbb{R} \}$$

$$e^{\varepsilon X} \in U(n) \;\forall \varepsilon \in \mathbb{R} \Leftrightarrow (e^{\varepsilon X})^* e^{\varepsilon X} = \mathbb{1} \quad \forall \varepsilon \in \mathbb{R} \Leftrightarrow e^{\varepsilon X^*} e^{\varepsilon X} = \mathbb{1} \quad \forall \varepsilon \in \mathbb{R}$$

$$\Leftrightarrow e^{\varepsilon X^*} = e^{-\varepsilon X} \quad \forall \varepsilon \in \mathbb{R} \Leftrightarrow X^* = -X$$

$$\Rightarrow \mathfrak{u}(n) = \{ X \in M_n(\mathbb{C}) \mid X^* = -X \} \quad \circledast$$

Dimension as a real vector space? Let $X = A + iB$ with A and B real.

$$X^* = -X \Leftrightarrow A^T - iB^T = -A - iB \Leftrightarrow \underbrace{A^T = -A}_{\frac{n(n-1)}{2} \text{ independent entries}}, \; B^T = B \rightarrow \frac{n(n+1)}{2} \text{ independent entries}$$

$$\Rightarrow \dim(\mathfrak{u}(n)) = \frac{n(n-1)}{2} + \frac{n(n+1)}{2} = n^2$$

Exercise: Find out the explicit form (the equivalent of $\circledast$) and dimension of the Lie algebras of the following groups:

$GL(n, \mathbb{R})$, $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $O(n)$, $SO(n)$, $SU(n)$

More exercises (you can do this on your own later)

Do the same for the following groups

$SO(n,\mathbb{C}) = \{ A \in SL(n,\mathbb{C}) \mid A^T A = \mathbb{1} \}$ complex special orthogonal group

$SO(p,q) = \left\{ A \in SL(p+q,\mathbb{R}) \mid A^T \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \right\}$ indefinite special orthogonal group

$Sp(2n,\mathbb{R}) = \left\{ A \in SL(2n,\mathbb{R}) \mid A^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right\}$

$Sp(2n,\mathbb{C}) = \left\{ A \in SL(2n,\mathbb{C}) \mid A^T \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & \mathbb{1}_n \\ -\mathbb{1}_n & 0 \end{pmatrix} \right\}$

real and complex symplectic group

$Sp(n) = Sp(2n,\mathbb{C}) \cap U(2n)$ compact symplectic group

<u>example</u>: $U(p,q) = \left\{ A \in GL(p+q,\mathbb{C}) \mid A^* \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \right\}$

$$\cdot\ e^{\varepsilon A^*} \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} e^{\varepsilon A} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \quad \forall \varepsilon \in \mathbb{R}$$

$C e^A C^{-1} = e^{CAC^{-1}}$

$$\Leftrightarrow\ e^{\varepsilon A^*} = \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} e^{-\varepsilon A} \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = \exp\left[-\varepsilon \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \right] \quad \forall \varepsilon \in \mathbb{R}$$

$$\Leftrightarrow\ A^* = -\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \quad \text{or} \quad A^* \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = -\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} A$$

if we write A in block form $A = \begin{pmatrix} \overset{p\times p}{X} & \overset{p\times q}{Y} \\ \underset{q\times p}{Z} & \underset{q\times q}{W} \end{pmatrix}$ we get

$$\begin{pmatrix} X^* & Z^* \\ Y^* & W^* \end{pmatrix} \begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} = -\begin{pmatrix} \mathbb{1}_p & 0 \\ 0 & -\mathbb{1}_q \end{pmatrix} \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} X^* & -Z^* \\ Y^* & -W^* \end{pmatrix} = \begin{pmatrix} -X & -Y \\ Z & W \end{pmatrix} \rightarrow$$

$X^* = -X \rightarrow p^2$ independent real entries
$W^* = -W \rightarrow q^2$ independent real entries
$Z^* = Y \rightarrow 2pq$ independent real entries

$$\rightarrow \mathfrak{u}(p,q) = \left\{ \begin{pmatrix} X & Y \\ Y^* & W \end{pmatrix} \middle|\ X \in \mathfrak{u}(p),\ W \in \mathfrak{u}(q),\ Y \in M_{p\times q}(\mathbb{C}) \right\}$$

$$\dim(\mathfrak{u}(p,q)) = (p+q)^2$$