**PY 711  Fall 2010**
**Homework 6: Due Tuesday, October 5**

1. (8 points) The free Dirac bispinor field $\psi$ can be written as

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left[ a^r_{\vec{p}} u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}} v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}. \tag{1}$$

Using the anticommutation relations

$$\left\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{p}'} \right\} = \left\{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{p}'} \right\} = (2\pi)^3 \, \delta^{rs} \delta^{(3)}(\vec{p} - \vec{p}'), \tag{2}$$

and all other anticommutators equal to zero, derive the anticommutation relation for $\psi_a$ and $\psi^\dagger_b$,

$$\left\{ \psi_a(\vec{x}), \psi^\dagger_b(\vec{y}) \right\} = \delta_{ab} \delta^{(3)}(\vec{x} - \vec{y}). \tag{3}$$

2. (7 points) The momentum operator $\vec{P}$ is the Noether charge associated with spatial translations. In terms of $\psi$ and $\psi^\dagger$ it has the form

$$\vec{P} = -i \int d^3\vec{x} \; \psi^\dagger(\vec{x}) \vec{\nabla} \psi(\vec{x}). \tag{4}$$

Show that $\vec{P}$ can be written as

$$\vec{P} = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p} \left( a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} + b^{r\dagger}_{\vec{p}} b^r_{\vec{p}} \right). \tag{5}$$

HW #6

$\frac{15}{15}$

1. THE FREE DIRAC BISPINOR FIELD $\psi$ CAN BE WRITTEN AS

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left( a^r_{\vec{p}} u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}} v^r(-\vec{p}) \right) e^{i\vec{p}\cdot\vec{x}}$$

USING THE ANTICOMMUTATION RELATIONS

$$\{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{p}'} \} = \{ b^r_{\vec{p}}, b^{s\dagger}_{\vec{p}'} \} = (2\pi)^3 \delta^{rs} \delta^3(\vec{p}-\vec{p}')$$

AND ALL OTHER ANTICOMMUTATORS EQUAL TO ZERO, DERIVE THE ANTICOMMUTATION RELATION FOR $\psi_a$ AND $\psi_b^\dagger$,

$$\{ \psi_a(\vec{x}), \psi_b^\dagger(\vec{y}) \} = \delta_{ab} \delta^3(\vec{x}-\vec{y}).$$

$$\{ \psi_a(\vec{x}), \psi_b^\dagger(\vec{y}) \} = \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}} E_{\vec{q}}}} e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} \times$$

$$\sum_{r,s} \left( \left\{ \left(a^r_{\vec{p}} u^r_a(\vec{p}) + b^{r\dagger}_{-\vec{p}} v^r_a(-\vec{p})\right), \left(a^{s\dagger}_{\vec{q}} u^{s\dagger}_b(\vec{q}) + b^s_{-\vec{q}} v^{s\dagger}_b(-\vec{q})\right) \right\} \right)$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}} E_{\vec{q}}}} e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} \times$$

$$\sum_{r,s} \Big( \{ a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}} \} u^r_a(\vec{p}) u^{s\dagger}_b(\vec{q}) + \cancel{\{ a^r_{\vec{p}}, b^s_{-\vec{q}} \}}^{\to 0} u^r_a(\vec{p}) v^{s\dagger}_b(-\vec{q})$$

$$+ \cancel{\{ b^{r\dagger}_{-\vec{p}}, a^{s\dagger}_{\vec{q}} \}}^{\to 0} v^r_a(-\vec{p}) u^{s\dagger}_b(\vec{q}) + \{ b^{r\dagger}_{-\vec{p}}, b^s_{-\vec{q}} \} v^r_a(-\vec{p}) v^{s\dagger}_b(-\vec{q}) \Big)$$

✓

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}} E_{\vec{q}}}} e^{i(\vec{p}\cdot\vec{x} - \vec{q}\cdot\vec{y})} \times$$

$$\sum_{r,s} \Big( (2\pi)^3 \delta^{rs} \delta^3(\vec{p}-\vec{q}) u^r_a(\vec{p}) u^{s\dagger}_b(\vec{q})$$

$$+ (2\pi)^3 \delta^{rs} \delta^3(\vec{p}-\vec{q}) v^r_a(-\vec{p}) v^{s\dagger}_b(-\vec{q}) \Big)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \sum_r \Big( \big(u^r_a(\vec{p}) u^{r\dagger}_b(\vec{p})\big) + \big(v^r_a(-\vec{p}) v^{r\dagger}_b(-\vec{p})\big) \Big)$$

✓

1 CONTINUED

$$\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \sum_r \Big( u_a^r(\vec{p}) u_b^{r\dagger}(\vec{p}) \gamma^0 + v_a^r(-\vec{p}) v_b^{r\dagger}(-\vec{p}) \gamma^0 \Big) \gamma^0$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \times \sum_r \Big( u_a^r(\vec{p}) \bar{u}_b^r(\vec{p}) + v_a^r(-\vec{p}) \bar{v}_b^r(-\vec{p}) \Big) \gamma^0$$

Recall

$$\sum_r u^r(p) \bar{u}^r(p) = \gamma \cdot p + m$$

$$\sum_r v^r(p) \bar{v}^r(p) = \gamma \cdot p - m$$

So

$$\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \times \Big( \big( (\gamma^0 E_{\vec{p}} - \vec{\gamma}\cdot\vec{p} + m) + (\gamma^0 E_{\vec{p}} + \vec{\gamma}\cdot\vec{p} - m) \big) \gamma^0 \Big)_{ab}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\cancel{2E_{\vec{p}}}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \Big( \cancel{2E_{\vec{p}}} \underbrace{\gamma^0 \gamma^0}_{=1} \Big)_{ab}$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} \delta_{ab}$$

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$$\boxed{\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta_{ab} \delta^3(\vec{x}-\vec{y})}$$

2. THE MOMENTUM OPERATOR $\vec{P}$ IS THE NOETHER CHARGE ASSOCIATED WITH SPATIAL TRANSLATIONS. IN TERMS OF $\psi$ AND $\psi^\dagger$ IT HAS THE FORM

$$\vec{P} = -i \int d^3x\, \psi^\dagger(\vec{x})\, \nabla \psi(\vec{x}).$$

SHOW THAT $\vec{P}$ CAN BE WRITTEN AS

$$\vec{P} = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, \left(a_{\vec{p}}^{r\dagger} a_{\vec{p}}^r + b_{\vec{p}}^{r\dagger} b_{\vec{p}}^r\right).$$

$$\nabla\psi(\vec{x}) = \sum_r \int \frac{d^3p}{(2\pi)^3} \frac{(i\vec{p})}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}^r u^r(\vec{p}) - b_{-\vec{p}}^{r\dagger} v^r(-\vec{p})\right) e^{i\vec{p}\cdot\vec{x}}$$

$$\vec{P} = -i \int d^3x \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{(i\vec{p})}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\, e^{i\vec{x}\cdot(\vec{p}-\vec{q})} \times$$

$$\sum_{r,s} \left( \left(a_{\vec{q}}^{s\dagger} u^{s\dagger}(\vec{q}) + b_{-\vec{q}}^{s} v^{s\dagger}(-\vec{q})\right)\left(a_{\vec{p}}^r u^r(\vec{p}) - b_{-\vec{p}}^{r\dagger} v^r(-\vec{p})\right)\right)$$

$$= \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{\vec{p}}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \times$$

$$\sum_{r,s} \left( \left(a_{\vec{q}}^{s\dagger} u^{s\dagger}(\vec{q}) + b_{-\vec{q}}^{s} v^{s\dagger}(-\vec{q})\right)\left(a_{\vec{p}}^r u^r(\vec{p}) - b_{-\vec{p}}^{r\dagger} v^r(-\vec{p})\right)\right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{2E_{\vec{p}}} \sum_{r,s} \left( \left(a_{\vec{p}}^{s\dagger} u^{s\dagger}(\vec{p}) + b_{-\vec{p}}^{s} v^{s\dagger}(-\vec{p})\right)\left(a_{\vec{p}}^r u^r(\vec{p}) - b_{-\vec{p}}^{r\dagger} v^r(-\vec{p})\right)\right)$$

✓

Recall, from Peskin and Schroeder,

$$u^{s\dagger}(\vec{p})\, v^r(-\vec{p}) = 0 \qquad v^{s\dagger}(-\vec{p})\, u^r(\vec{p}) = 0$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{2E_{\vec{p}}} \sum_{r,s} \left( a_{\vec{p}}^{s\dagger} a_{\vec{p}}^r u^{s\dagger}(\vec{p}) u^r(\vec{p}) - b_{-\vec{p}}^{s} b_{-\vec{p}}^{r\dagger} v^{s\dagger}(-\vec{p}) v^r(-\vec{p})\right)$$

2 CONTINUED

Now, $u^{r\dagger}(\vec{p})\, u^{s}(\vec{p}) = 2E_{\vec{p}}\, \delta^{rs}$

$v^{r\dagger}(\vec{p})\, v^{s}(\vec{p}) = 2E_{\vec{p}}\, \delta^{rs}$

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \frac{\vec{p}}{2E_{\vec{p}}} \sum_r \left( (2E_{\vec{p}}) \left( a^{r\dagger}_{\vec{p}} a^{r}_{\vec{p}} - b^{r}_{-\vec{p}} b^{r\dagger}_{-\vec{p}} \right)\right)$$

Since $\{ b^{r\dagger}_{\vec{p}}, b^{r}_{\vec{p}} \} = (2\pi)^3 \delta^{rr} \delta^3(\vec{p}-\vec{p}) \Rightarrow -b^{r}_{-\vec{p}} b^{r\dagger}_{-\vec{p}} = b^{r\dagger}_{-\vec{p}} b^{r}_{-\vec{p}} - (2\pi)^3 \delta^{rr} \delta(\vec{p}-\vec{p})$

The term with the $\delta^3(0)$ is the same kind of infinity as in the Klein-Gordon Hamiltonian, so we'll ignore it. The term $b^{r\dagger}_{-\vec{p}} b^{r}_{-\vec{p}} \rightarrow b^{r\dagger}_{\vec{p}} b^{r}_{\vec{p}}$ since we're integrating over all $\vec{p}$ (Parity)

$$\boxed{\vec{P} = \int \frac{d^3p}{(2\pi)^3}\, \vec{p} \sum_r \left( a^{r\dagger}_{\vec{p}} a^{r}_{\vec{p}} + b^{r\dagger}_{\vec{p}} b^{r}_{\vec{p}} \right)}$$

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# Solutions #6

1. Let $\tilde{\psi}(\vec{p}) = \sum_{r=1,2} \frac{1}{\sqrt{2E_{\vec{p}}}} \left( a^r_{\vec{p}} u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}} v^r(-\vec{p}) \right)$

so that $\psi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \tilde{\psi}(\vec{p}) e^{i\vec{p}\cdot\vec{x}}$.

$$\{\tilde{\psi}_a(\vec{p}), \tilde{\psi}^\dagger_b(\vec{p}')\} = \frac{1}{\sqrt{2E_{\vec{p}}\cdot 2E_{\vec{p}'}}} \left\{ \sum_{r=1,2} a^r_{\vec{p}} u^r_a(\vec{p}), \sum_{r'=1,2} a^{r'\dagger}_{\vec{p}'} u^{r'\dagger}_b(\vec{p}') \right\}$$
$$+ \frac{1}{\sqrt{2E_{\vec{p}}\cdot 2E_{\vec{p}'}}} \left\{ \sum_{r=1,2} b^{r\dagger}_{-\vec{p}} v^r_a(-\vec{p}), \sum_{r'=1,2} b^{r'}_{-\vec{p}'} v^{r'\dagger}_b(-\vec{p}') \right\}$$

$$= (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \frac{1}{2E_{\vec{p}}} \Big[ \underbrace{\sum_{r=1,2} u^r_a(\vec{p}) u^{r\dagger}_b(\vec{p})}_{} + \underbrace{\sum_{r=1,2} v^r_a(-\vec{p}) v^{r\dagger}_b(-\vec{p})}_{} \Big]$$

using $\sum_{r=1,2} u^r_a(\vec{p}) \bar{u}^r_b(\vec{p}) = \not{p} + m$ $\quad = \left[ (E_{\vec{p}}\gamma^0 - \vec{p}\cdot\vec{\gamma} + m)\gamma^0 \right]_{ab}$

using $\sum_{r=1,2} v^r_a(\vec{p}) \bar{v}^r_b(\vec{p}) = \not{p} - m$ (with $-\vec{p}$) $\quad = \left[ (E_{\vec{p}}\gamma^0 + \vec{p}\cdot\vec{\gamma} - m)\gamma^0 \right]_{ab}$

So $\{\tilde{\psi}_a(\vec{p}), \tilde{\psi}^\dagger_b(\vec{p}')\} = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}') \frac{1}{\cancel{2E_{\vec{p}}}} (\cancel{2E_{\vec{p}}} \underbrace{\gamma^0\gamma^0}_{=1})_{ab} = (2\pi)^3 \delta_{ab} \delta^{(3)}(\vec{p}-\vec{p}')$

Therefore $\{\psi_a(\vec{x}), \psi^\dagger_b(\vec{y})\} = \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{d^3\vec{p}'}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} e^{-i\vec{p}'\cdot\vec{y}} (2\pi)^3 \delta_{ab} \delta^{(3)}(\vec{p}-\vec{p}')$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3} \delta_{ab} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} = \delta_{ab} \delta^{(3)}(\vec{x}-\vec{y}).$$

2. We start with some preliminaries...

$$u^{r\dagger}(\vec{p}) v^s(-\vec{p}) = \left( \xi^{r\dagger}\sqrt{p\cdot\sigma} \quad \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \right) \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\,\xi^s \\ -\sqrt{p\cdot\sigma}\,\xi^s \end{pmatrix} = 0$$

$$v^{r\dagger}(-\vec{p}) u^s(\vec{p}) = \left( \xi^{r\dagger}\sqrt{p\cdot\bar{\sigma}} \quad -\xi^{r\dagger}\sqrt{p\cdot\sigma} \right) \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} = 0$$

As before we let $\tilde{\psi}(\vec{p}) = \sum_{r=1,2} \frac{1}{\sqrt{2E_{\vec{p}}}} (a^r_{\vec{p}} u^r(\vec{p}) + b^{r\dagger}_{-\vec{p}} v^r(-\vec{p}))$.

Then
$$\tilde{\psi}^\dagger(\vec{p})\,\psi(\vec{p}) = \frac{1}{2E_{\vec{p}}} \sum_{r=1,2} \sum_{s=1,2} \Big[ u^{r\dagger}(\vec{p}) u^s(\vec{p})\, a^{r\dagger}_{\vec{p}} a^s_{\vec{p}} + u^{r\dagger}(\vec{p}) v^s(-\vec{p})\, a^{r\dagger}_{\vec{p}} b^{s\dagger}_{-\vec{p}} + v^{r\dagger}(-\vec{p}) u^s(\vec{p})\, b^r_{-\vec{p}} a^s_{\vec{p}} + v^{r\dagger}(-\vec{p}) v^s(-\vec{p})\, b^r_{-\vec{p}} b^{s\dagger}_{-\vec{p}} \Big]$$
$$= \frac{1}{2E_{\vec{p}}} \Big( 2E_{\vec{p}} \sum_{r=1,2} a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} - 2E_{\vec{p}} \sum_{r=1,2} b^{r\dagger}_{-\vec{p}} b^r_{-\vec{p}} + 2E_{\vec{p}} \sum_{r=1,2} \delta^{(3)}(\vec{0})(2\pi)^3 \Big)$$
$$= \sum_{r=1,2} a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} - \sum_{r=1,2} b^\dagger_{-\vec{p}} b_{-\vec{p}} + \sum_{r=1,2} (2\pi)^3 \delta^{(3)}(\vec{0})$$

Therefore
$$\vec{P} = -i \int d^3\vec{x}\, \psi^\dagger(\vec{x}) \vec{\nabla} \psi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3} \tilde{\psi}^\dagger(\vec{p})\, \vec{p}\, \psi(\vec{p})$$
$$= \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p}\, (a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} - b^{r\dagger}_{-\vec{p}} b^r_{-\vec{p}})$$
$\delta^{(3)}(\vec{0})$ term is odd in $\vec{p}$
$$= \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \vec{p}\, (a^{r\dagger}_{\vec{p}} a^r_{\vec{p}} + b^{r\dagger}_{\vec{p}} b^r_{\vec{p}})$$