So, we infer that

$$\boxed{\langle a|H|-a\rangle \simeq e^{-S_{cl}/\hbar}}$$

where

$$S_{cl} = \int (T+V)\,d\tau \quad \text{because} \quad E_0 = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V = T - V$$

conserved

$$= \int 2T\,d\tau \longleftarrow$$

→ want 0-energy solution → $T = V$

$$= \int m\dot{x}\dot{x}\,d\tau$$

$$= \int_{-a}^{a} p(x)\,dx = \int_{-a}^{a} \sqrt{2mV(x)}\,dx$$

→ Tunneling amplitude is $e^{-\frac{1}{\hbar}\int_{-a}^{a}\sqrt{2mV(x)}\,dx}$ which agrees with Schrödinger's approach.

Note that S_{cl} does not depend on τ_0 ⇒ every path has the same action.

Sum over everything gives...

$$\langle a|U(\tau)|-a\rangle \simeq \tau e^{-S_{cl}/\hbar}$$

Compare this to $\langle a|e^{-\frac{1}{\hbar}\tau H}|-a\rangle \simeq 0 - \frac{\tau}{\hbar}\langle a|H|-a\rangle + O(\tau^2)$

$$\Rightarrow \boxed{H_{+-} \simeq e^{-\frac{1}{\hbar}S_{cl}}}$$ From here we have...

$$|S\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle + |-a\rangle\right) \quad ; \quad |A\rangle = \frac{1}{\sqrt{2}}\left(|a\rangle - |-a\rangle\right)$$

$$E_S = -e^{-\frac{1}{\hbar}S_{cl}} \quad ; \quad E_A = +e^{-\frac{1}{\hbar}S_{cl}}$$

(f) Spontaneous Symmetry Breaking

Feb 21, 2021

Consider a Hamiltonian which has a symmetry, say under parity.

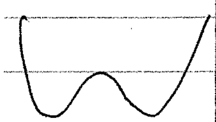
If the lowest-energy state of the problem is itself not invariant under the symmetry, we say symmetry is spontaneously broken

Ex

- Single-well oscillator. Hamiltonian is invariant under parity.

  Gnd state is particle sitting at the bottom of the well

  → This state respects the symmetry.



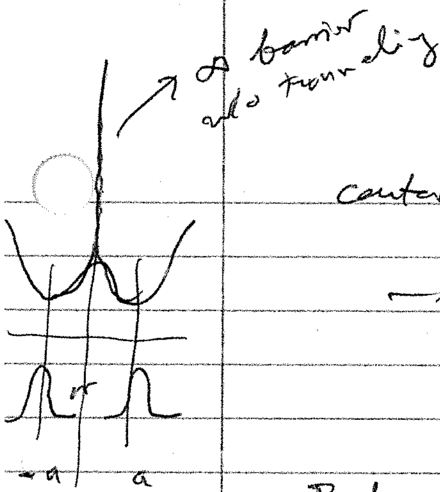
- Now consider the double well with minima at $x = \pm a$. → 2 lowest energy configurations available → does not obey parity symmetry.

  "Spontaneous" in that particle has to make a choice

  → ∃ more than one gnd state. States are not invariant under sym. Rather, one gnd state gets mapped to the other ...

- Consider the quantum case, but with an infinite barrier between the wells. (so that no tunneling occurs -- so the barrier is not δ-function).

→ particle has 2 choices: being Gaussian-like functions



centered at either one of the 2 troughs : $|\pm a\rangle$.

→ this has feature of symmetry breaking (degenerate 2 non-invariant under transf by parity i.e. $\langle x \rangle \neq 0$).

But since this is QM we can have a superposition...

$$|S/A\rangle = \frac{\left[|+a\rangle \pm |-a\rangle\right]}{\sqrt{2}}$$

and the parity transformation goes like

$$\Pi\,|S/A\rangle = \pm\,|S/A\rangle$$

So $|S/A\rangle$ are eigenstates of parity Π, which implies that:

$$\boxed{[\Pi, H] = 0}$$

which (circularly) implies that $|S/A\rangle$ can be formed

But should they be formed? $\boxed{\text{No}}$ b/c of the ∞ barrier

→ symmetry is spontaneously broken.

Ex Now, back to the finite barrier problem → $|S/A\rangle$ are possible and are not degenerate.

Now note that $|S\rangle$, in general, will be the unique gnd state. The instanton calculation (tunneling) tells us that the double well, despite having 2 classical states that break symmetry, has in QM a unique, symmetric gnd state.

⇒ The symmetry of the Hamiltonian is the symmetry of the gnd state

Sym. breaking does not take place in the double-well problem (finite)

→ We have that tunneling restores symmetry.

Ex Consider periodic potential $V(x) = 1 - \cos 2\pi x$
→ minima @ $x = n \in \mathbb{Z}$.

Symmetry is $x \to x+1$.

The approx states $|n\rangle$ which are Gaussian centered @ a classical minima n breaks the symmetry & are converted to each other by the translation operator T:

$$T|n\rangle = |n+1\rangle$$

Now, due to tunneling, we can form the symmetric ground state: (assuming that size = N)

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |n\rangle$$

$\boxed{\text{Ex}}$ Assume that

$$\mathcal{H} = \sum_{1}^{N} E_0 |n\rangle\langle n| - t\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right)$$

describes the low-energy Hamiltonian of a particle in a periodic potential w/ minima @ ~~$n \in \mathbb{Z}$~~. $n \in \{1, \dots, N\}$ on a ring.

→ Problem has sym under translation by 1 site

- First term of $\mathcal{H}$ represents the energy of the Gaussian state centered @ n.
- Second term represents the tunneling to adjacent minima with tunneling amplitude t.

→ Consider $\boxed{|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle}$.

Show that $|\theta\rangle$ is an eigstate of T & eigvalue. Use $T^N = 1$ to restrict the allowed values of θ & make sure we still have N states

Show that $|\theta\rangle$ is eigstate of $\mathcal{H}$ w/ eigv $\boxed{E(\theta) = E_0 - 2t\cos\theta}$

Let, $N = 2$ & regain the double-well result

~~Solution.~~ $\cancel{T|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\theta n} T|n\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\theta n} |n+1\rangle}$

$\cancel{= \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{i\theta} |n\rangle = |\theta\rangle}$

Now, $\frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n+1\rangle$

$$= \frac{1}{\sqrt{N}} e^{-i\theta} \sum_{n=1}^{N} e^{i(n+1)\theta} |n+1\rangle$$

$$= \frac{1}{\sqrt{N}} e^{-i\theta} \left\{ e^{i2\theta}|2\rangle + \dots + e^{i(N+1)\theta} |N+1\rangle \right\}$$

$e^{i\theta}|1\rangle$ since $T^N = 1$

$\rightarrow e_j v = e^{-i\theta}$. , $T^N = 1 \Rightarrow e^{-iN\theta} = 1 \rightarrow \boxed{\cos(N\theta) = 1}$

$\rightarrow \theta = 2\pi k,\ k \in \mathbb{Z}$ (?)

$$H(A) = \sum_{n=1}^{N} E_0 |n\rangle\langle n| - t\left(|n\rangle\langle n+1| + |n+1\rangle\langle n|\right)$$
$$\times \left\{ \sum_{m=1}^{N} e^{im\theta} |m\rangle \right\}$$

$$= \sum_{n=1}^{N} \sum_{m=1}^{N} E_0 |n\rangle\langle n|m\rangle e^{im\theta} - t\left(|n\rangle\langle n+1|m\rangle + |n+1\rangle\langle n|m\rangle\right) e^{im\theta}$$

$$= \sum_{n=1}^{N} E_0 |n\rangle e^{in\theta} - t\left(|n+1\rangle e^{in\theta} + |n\rangle e^{i(n+1)\theta}\right)$$

$$= \sum_{n=1}^{N} E_0 |n\rangle e^{in\theta} - t \sum_{n=1}^{N} |n\rangle e^{i(n+1)\theta} - t \sum_{n=1}^{N} e^{in\theta} |n+1\rangle$$

$= d + u + (-t) \sum$

→ this ~~problem~~ is phrased a bit weirdly, but we can expect the answer to be the symmetric wfn

$$|s\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |n\rangle .$$

Since θ can only be $2\pi k$, $k \in \mathbb{Z}$.

#

→ Will the gnd state always be invariant under the symmetry operation that commutes with $\mathcal{H}$?

↳ Yes, as long as the barrier is finite. (or as long as tunneling can occur)

#

Instantons in the d=1 Ising model

Recall that the Hamiltonian is (in the σ_z basis)

$$\mathcal{H} = -\begin{pmatrix} 0 & K^* \\ K^* & 0 \end{pmatrix}$$

Suppose that $K^* = e^{-2K} = T \to 0$. We can do a "semi-classical" approximation in which the sum over paths is dominated by the one of least action.

↳ The "instanton" that connects the $|\uparrow\rangle$ & $|\downarrow\rangle$ gnd state is a configuration in which the spin is up until some time when it flips to down & stay down. This tunneling restores symmetry @ $T \neq 0$.

(9) The Classical Limit of Quantum Stat Mech

Consider particle of mass m in potential V, then

$$Z(\beta) = \int dx \int_x^x [\mathcal{D}x] \exp\left\{ \frac{-1}{\hbar} \int_0^{\beta\hbar} \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V[x(\tau)] \right] d\tau \right\}$$

in imaginary time, and we're considering paths starting & ending on the same point x.

Consider $\beta\hbar \to 0$ (i.e. $\hbar \to 0$ (classical limit) or high temp $\beta \to 0 \Leftrightarrow \frac{1}{kT} \to \infty$)

Look at any x. Need to look @ particle @ x, go somewhere in time $\beta\hbar$, and return to x.

If the particle goes a distance Δx, the KE is like

$$\simeq m\left(\frac{\Delta x}{\beta\hbar}\right)^2$$

$\rightarrow$ Boltzmann factor for this is

$$\sim e^{-1/\hbar\, m(\Delta x/\beta\hbar)^2 \beta\hbar}$$

So we have

$$\Delta x \simeq \sqrt{\frac{\beta}{m}}\,\hbar$$

so that $e^{-1/\hbar\, m\, 1/\beta m \cdot \beta\hbar} = e \sim O(1)$.

If the potential does not vary much over such length scale, then we can pull it out ...

$$Z(\beta) \simeq \int dx\, e^{-\beta V(x)} \int_x^x [\mathcal{D}x] \exp\left\{ \frac{-1}{\hbar} \int_0^{\beta\hbar} \left[\frac{m}{2} \left(\frac{dx}{d\tau} \right)^2 \right] d\tau \right]$$

$$= \int dx\, e^{-\beta V(x)} \sqrt{\frac{m}{2\pi\hbar\beta\hbar}}$$

where the last step is obtained by viewing that integral as the amplitude for a free particle to go from x to x in time $\beta\hbar$ (view previous pages in the book for more details)

→ How does this compare with classical sol'n? Recall that if we replace the sum over states by an integral over phase space...

$$Z = A \int dx \int dp \exp\left\{ -\beta \left(\frac{p^2}{2m} + V(x) \right) \right\}$$

→ do the p-integral & compare the result to the classical limit of the path integral we see that QM fixes

$$A = \frac{1}{2\pi\hbar} \quad \text{~~in accordance~~}$$

What is A? ⟹ A reflects one's freedom to multiply Z by a constant w/o changing anything physical

→ A corresponds to the fact that the # of classical states in the region $(dx\,dp)$ of phase space is not uniquely defined.

⟹ if $A = \frac{1}{2\pi\hbar}$ then we see that it is in accordance with the uncertainty principle.

COHERENT STATE PATH INTEGRALS for SPINS, BOSONS, and FERMIONS

(a) Spin Coherent State Path Integral

Consider spin-S particle. Hilbert space is $(2S+1)$-dim. Let S_z eigenstates be basis.

Resolution of Id: $I = \sum_{-S}^{S} |S_z\rangle\langle S_z|$

→ Consider the spin-coherent state:

$$\boxed{|\Omega\rangle \equiv |\theta,\varphi\rangle = \mathcal{U}[R(\Omega)]\,|SS\rangle}$$

around x-axis, around z-axis, rotation, fully polarized state

Given that $\langle SS|\vec{S}|SS\rangle = \hat{k}S$, we have

$$\langle\Omega|\vec{S}|\Omega\rangle = \langle SS|\,\mathcal{U}^\dagger(R(\Omega))\,\vec{S}\,\mathcal{U}(R(\Omega))\,|SS\rangle$$

$$= S\left(\hat{\imath}\sin\theta\cos\varphi + \hat{\jmath}\sin\theta\sin\varphi + \hat{k}\cos\theta\right)$$

If $S=1$, then $\sigma(S_z) = \{0, \pm 1\}$.

The coherent state $\equiv$ one in which the spin operator has a nice exp value = equal classical spin of length S, but point in the direction of Ω.

(not an eigvec though)

Now, look at the following eqn...

$$\langle \Omega_2 | \Omega_1 \rangle = \left(\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + e^{i(\phi_1-\phi_2)}\sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \right)^{2S}$$

How is this true?
~~can check this pretty easily~~

If $S = \frac{1}{2}$, ~~then~~ the up spinor along (θ, ϕ) is

$$|\Omega\rangle = |\theta\phi\rangle = \cos\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle$$

~~Then~~ Suppose $2S$ spin-$\frac{1}{2}$ particles join to form a spin-s state $\rightarrow$ there's only one product state with $S_z = S$, which is where all spin-$\frac{1}{2}$ are $\uparrow$.

$$\rightarrow \quad |SS\rangle = \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \otimes \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle \otimes \cdots \otimes \left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle$$

If we rotate this, it becomes a tensor product of rotated states, and ~~we~~ when we form the inner product on the LHS of ______, we obtain the RHS of ______

The resolution of the identity in terms of these states is

$$I = \frac{2S+1}{4\pi}\int d\Omega\, |\Omega\rangle\langle\Omega| \quad \text{where } d\Omega = d\cos\theta\, d\phi$$

Try for $S = \frac{1}{2}$, then we have

$$|\Omega\rangle\langle\Omega| = \left(\cos\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle\right)$$

~~Ie~~

$$\left(\cos\frac{\theta}{2}\left\langle\tfrac{1}{2}\,\tfrac{1}{2}\right| + e^{-i\phi}\sin\frac{\theta}{2}\left\langle\tfrac{1}{2}\,\tfrac{-1}{2}\right|\right)$$

$$= \cos^2\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\left\langle\tfrac{1}{2}\,\tfrac{1}{2}\right| + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}\left|\tfrac{1}{2}\,\tfrac{1}{2}\right\rangle\left\langle\tfrac{1}{2}\,\tfrac{-1}{2}\right|$$

$$+ \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle\left\langle\tfrac{1}{2}\,\tfrac{1}{2}\right| + \sin^2\frac{\theta}{2}\left|\tfrac{1}{2}\,\tfrac{-1}{2}\right\rangle\left\langle\tfrac{1}{2}\,\tfrac{-1}{2}\right|$$

$$\int \cos^2\frac{\theta}{2}\, d\cos\theta = -\int \cos^2\frac{\theta}{2}\sin\theta\, d\theta = 1$$

$$\int \sin^2\frac{\theta}{2}\, d\cos\theta = -\int \sin^2\frac{\theta}{2}\sin\theta\, d\theta = 1.$$

other terms are zero.

So $$\int d\Omega\, |\Omega\rangle\langle\Omega| = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cdot 2\pi$$

$$\rightarrow \frac{2\cdot\frac{1}{2}+1}{4\pi}\int d\Omega\, |\Omega\rangle\langle\Omega| = \frac{2\cdot\frac{1}{2}+1}{4\pi}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\cdot 2\pi$$

$$= I \quad ✓.$$

(b) $\boxed{\text{Real-Time Path Integral for Spin}}$

Feb 26 2021

Approximately, the path integral looks like (as terms)

$$\ldots \langle \Omega(t+\varepsilon)|\, I - \frac{i\varepsilon}{\hbar}H(\vec{S})\,|\Omega(t)\rangle.$$

where $|\Omega\rangle$ is a coherent state. Now there are 2 parts to this...

$$\langle\Omega(t+\varepsilon)|\Omega(t)\rangle \quad \text{and} \quad \langle\Omega(\varepsilon+t)|\, -\frac{i\varepsilon}{\hbar}H\,|\Omega(t)\rangle.$$

- $\langle \Omega(t+\varepsilon) | -\frac{i\varepsilon}{\hbar} \mathcal{H}(\vec{S}) | \Omega(t) \rangle$ (up to order ε)

$$\simeq -\frac{i\varepsilon}{\hbar} \langle \Omega(t) | \mathcal{H}(\vec{S}) | \Omega(t) \rangle$$

$$\equiv -i\varepsilon \, \mathcal{H}(\Omega) \quad \leftarrow \text{definition}$$

- $\langle \Omega(t+\varepsilon) | \Omega(t) \rangle \simeq 1 - i\varepsilon S(1-\cos\theta)\dot{\phi}$

$$\simeq e^{iS(\cos\theta - 1)\dot{\phi}\varepsilon}$$

we got this by expanding the expression (which we derived before)

$$\langle \Omega_2 | \Omega_1 \rangle = \left(\cos\frac{\theta_2}{2}\cos\frac{\theta_1}{2} + e^{i(\phi_1-\phi_2)} \sin\frac{\theta_2}{2}\sin\frac{\theta_1}{2} \right)^{2S}$$

to first order in $\Delta\theta$ & $\Delta\phi$.

(we can check this quite easily, but I won't do it out here to save time...)

From there, we can calculate the "representation" of the propagator $\mathcal{U}(t)$ in the continuum limit...

$$\boxed{\langle \Omega_f | \mathcal{U}(t) | \Omega_i \rangle = \int [\mathcal{D}\Omega] \exp\left\{ i\int_{t_1}^{t_2} \left(S\cos\theta\,\dot{\phi} - \mathcal{H}(\Omega) \right) dt \right\}}$$

<u>How?</u>

We'll come back to the rest of the details on spins later... Since we're mostly interested in bosonic/fermionic systems, we will ~~at the~~ look at these topics first...

(c) Bosonic Coherent States

Recall harmonic oscillator...

$$\begin{cases} \mathcal{H} = \left(a^\dagger a + \frac{1}{2}\right)\hbar\omega \\ [a^\dagger, a] = \mathbb{1} \\ \mathcal{H}|n\rangle = E_n|n\rangle \qquad n = 0, 1, 2, \ldots \\ E_n = \left(n + \frac{1}{2}\right)\hbar\omega. \end{cases}$$

Rather than saying the HO is in nth state, we could ignore $n=0$ and say that there is one state of energy $\hbar\omega$ ~ n quanta in it.

→ This is how photons / phonons / etc are viewed

→ Any level $\hbar\omega$ can be occupied by any number of quanta n ⇒ BOSONS.

→ Bosonic coherent state is defined as

$$\boxed{|z\rangle = e^{a^\dagger z}|0\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle}$$

We can check that $\boxed{a|z\rangle = z|z\rangle}$

and $\boxed{\langle z|a^\dagger = \langle z|z^*}$

$$a|z\rangle = a\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n\rangle$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}|n-1\rangle\sqrt{n} \qquad n=0 \rightarrow 0$$

$$= z\sum_{n=1}^{\infty} \frac{z^{n-1}}{\sqrt{(n-1)!}}|n-1\rangle$$

$$= z|z\rangle \checkmark$$

Next, using the identity $\underline{e^A e^B = e^B e^A e^{[A,B]}}$

where $[A,B]$ is a c-number, we can show that

$$\langle z_2|z_1\rangle = e^{z_2^* z_1}$$

$$\boxed{\langle z_2|z_1\rangle} = \langle 0|e^{z_2^* a} e^{a^\dagger z_1}|0\rangle$$

$$= \langle 0|e^{a^\dagger z_1} e^{z_2^* a} e^{[z_2^* a,\, a^\dagger z_1]}|0\rangle$$

$$= \langle 0|e^{a^\dagger z_1} e^{z_2^* a} e^{z_2^* z_1 \underbrace{(aa^\dagger - a^\dagger a)}_{\mathbb{1}}}|0\rangle$$

$$= e^{z_2^* z_1}\langle 0|e^{a^\dagger z_1} \underbrace{e^{z_2^* a}|0\rangle}_{|0\rangle}$$

$$= e^{z_2^* z_1}\langle 0|e^{a^\dagger z_1}|0\rangle = \boxed{e^{z_2^* z_1}}$$

March 1 2021

Resolution of Identity:

$$I = \int \frac{dz\,dz^*}{2\pi i} e^{-z^* z} |z\rangle\langle z|$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{\pi} e^{-z^* z} |z\rangle\langle z|$$

$$= \int_0^{\infty}\int_0^{2\pi} \frac{r\,dr\,d\theta}{\pi} e^{-z^* z} |z\rangle\langle z|$$

verification is left as an exercise, but the idea is to ~~rere~~ write $|z\rangle$ as $|n\rangle$ and $\langle z|$ as $\langle m|$, then to see angular ~~part~~ & radial parts to show that I reduces to $\sum_n |n\rangle\langle n|$.

Need to recall the defn of the Γ function.

With this, we can write down the path integral.

$$\boxed{\langle z_f| \exp\left\{\frac{-i}{\hbar} : \mathcal{H}(a^\dagger, a) : t\right\} |z_i\rangle}$$

where $:\mathcal{H}(a^\dagger, a):$ → normal-order Hamiltonian

{ all creation ops: left
all annihilation ops: right

→ chop t into N pieces ... $\varepsilon = t/N$ & repeatedly insert resolution of ID we get we'll get a string with factors of the form...

$$\ldots |z_{n+1}\rangle\langle z_{n+1}| \left\{ 1 - \frac{i\varepsilon}{\hbar} : \mathcal{H}(a^\dagger, a) : |z_n\rangle\langle z_n| \frac{dz_n\, dz_n^*\, e^{-z_n^* z_n}}{2\pi i} \right\}$$

$$\times \left\{ 1 - \frac{i\varepsilon}{\hbar} \ldots \right\} \ldots$$

$$= \ldots |z_{n+1}\rangle \frac{dz_n\, dz_n^*}{2\pi i} \exp\left[(z_{n+1}^* - z_n^*) z_n - \frac{i\varepsilon}{\hbar} : \mathcal{H}(z_n^*, z_n) : \right] \langle z_n|$$

where we've used the facts that...

→ $a, a^\dagger$ in $:\mathcal{H}:$ are acting as their eigenvalues on the left & right, and with $\varepsilon \ll 1$, we may set $z_{n+1} = z_n$ inside it...

So now here we get the path integral in the continuum limit

$$\boxed{\begin{aligned} &\langle z_f| \exp\left\{ \frac{-i}{\hbar} : \mathcal{H}(a^\dagger, a) : t \right\} |z_i\rangle \\ &= \int_{z_i}^{z_f^*} [\mathcal{D}z^* \mathcal{D}z] \exp\left\{ \frac{+i}{\hbar} \int_0^t \left(i\hbar z^* \frac{dz}{dt} - : H(z^*, z) : \right) dt \right\} \end{aligned}}$$

(There's some sloppiness in the math, but we won't worry too much about it for now...)

(d) $\boxed{\text{THE FERMION PROBLEM}}$

Unlike bosons, fermions have "fermionic oscillator", which has only one level.

↳ which can contain of only one or zero quanta due to Pauli exclusion principle.

There are a few things we need to sort out before getting to the path integral for fermions...

1. Fermionic oscillator: spectrum & thermodynamics
2. Resolution of Id.
3. Non-interacting fermions only... we won't worry about interacting fermions...

(e) $\boxed{\text{FERMIONIC OSCILLATOR: SPECTRUM \& THERMODY.}}$

↳ Hamiltonian: $H_0 = \Omega_0 \Psi^\dagger \Psi$

↳ Fermions obey the anti-comm relations:

$$\boxed{\begin{aligned} \{\Psi^\dagger, \Psi\} &= \Psi^\dagger\Psi + \Psi\Psi^\dagger = \mathbb{1} \\ \{\Psi, \Psi\} &= \{\Psi^\dagger, \Psi^\dagger\} = 0 \end{aligned}}$$

↳ $\Psi\Psi = \Psi^2 = (\Psi^\dagger)^2 = 0.$

↳ we will use this all the time...

This is Pauli exclusion principle...

Also have the number operator... $\boxed{N = \psi^\dagger \psi}$

which is idempotent. $\boxed{N^2 = N}$

$$N^2 = \psi^\dagger\psi\psi^\dagger\psi = \psi^\dagger(1-\psi^\dagger\psi)\psi = \psi^\dagger\psi = N.$$

So eigs of N can only be $\underline{0 \text{ or } 1}$.

$$\boxed{N|0\rangle = 0|0\rangle \quad ; \quad N|1\rangle = 1|1\rangle}$$

$\underline{\text{So}}$

$$\psi^\dagger|0\rangle = \psi^\dagger(1-\psi^\dagger\psi)|0\rangle = \psi^\dagger\psi\psi^\dagger|0\rangle = N\psi^\dagger|0\rangle$$

$\Rightarrow \boxed{\psi^\dagger|0\rangle = |1\rangle} \rightarrow$ with unity norm.

Similarly, can show that $\boxed{\psi|1\rangle = |0\rangle}$

$\Rightarrow$ there are no other vectors in the Hilbert space.

since $\psi^2 = \psi^{\dagger 2} = 0$.

$\rightarrow$ States are either singly occupied or empty.

And so the Fermionic oscillator

$$\boxed{\mathcal{H}_0 = \Omega_0 \Psi^\dagger \Psi} \qquad \sigma(\mathcal{H}_0) = \{0, \Omega_0\}.$$

has eigvalues $\boxed{0 \,,\, \Omega_0}$

But we won't work with $\mathcal{H}_0$. Rather, we work with

$$\boxed{\mathcal{H} = \mathcal{H}_0 - \underbrace{\mu}_{\text{chemical potential}} N = (\Omega_0 - \mu)\psi^\dagger\psi}$$

Grand partition function

$$\boxed{Z = Tr\left(e^{-\beta(H_0 - \mu N)}\right)}$$

trace over any complete set of eigenstates.

in the N basis, this is easy...

$$Z = Tr\left(e^{-\beta(H_0-\mu N)}\right) = \underbrace{1}_{N=0} + \underbrace{e^{-\beta(\Omega_0-\mu)}}_{N=1}$$

From here, we can find the rest...

Mean occupation number...

$$\langle N \rangle = \frac{1}{\beta Z}\frac{\partial Z}{\partial \mu} = \frac{1}{\beta}\frac{\partial \ln Z}{\partial \beta} = \frac{1}{1+e^{\beta(\Omega_0-\mu)}} = n_F$$

this is Fermi-Dirac statistics -

At $T=0$ (zero temperature...)

$$\boxed{\langle N \rangle = \theta(\mu - \Omega_0) \quad @\ T=0}$$

Heaviside step fn, which says that the fermion is present if it energy $\Omega_0 < \mu$ & absent if $\Omega_0 > \mu$.

(f) COHERENT STATE FOR FERMIONS

Mar 8, 2021

Coherent states are eigenstates of the annihilation op.

for fermions, we have

$$\Psi|\psi\rangle = \psi|\psi\rangle$$

and more importantly $\psi^2 = 0$ since $\boxed{\Psi^2 = 0}$

$\Rightarrow$ But ψ wouldn't be interesting if itself is 0.

$\Rightarrow$ $\boxed{\psi \text{ is a Grassmann Variable}}$

↳ Defn variable that anti-commute with each other & w/ all fermionic annihilation-creation operators.

(and therefore commute with a string containing an even number of such operators).

→ What, then, is the coherent state?

Well... $\boxed{|\psi\rangle = |0\rangle - \psi|1\rangle}$ → Fermionic coherent state.

grassmann number.

Note that we can show $\Psi|\psi\rangle = \psi|\psi\rangle$ as follows

$$\Psi|\psi\rangle = \Psi(|0\rangle - \psi|1\rangle)$$

(−) → (+) by anti-commutativity

$$= 0 + \psi\Psi|1\rangle$$

$$= \psi|0\rangle = \psi(|0\rangle - \psi|1\rangle) = \psi|\psi\rangle \checkmark$$

We can similarly verify that $\langle\bar{\Psi}|\Psi^{\dagger} = \langle\bar{\Psi}|\bar{\Psi}$

where $\langle\bar{\Psi}| = \langle 0| - \langle 1|\bar{\Psi} = \langle 0| + \bar{\Psi}\langle 1|$

anti-commutativity.

- Note that fermionic coherent states are not in the complex vector space since its coefs have grassman numbers.
- $\bar{\Psi}$ is NOT the complex conjugate of Ψ, Ψ^*
- $\langle\bar{\Psi}|$ is NOT the adjoint of $|\Psi\rangle$, $\langle\Psi|$

↳ Inner product of 2 coherent states...

$$\langle\bar{\Psi}|\Psi\rangle = \left(\langle 0| - \langle 1|\bar{\Psi}\right)\left(|0\rangle - \Psi|1\rangle\right)$$

$$= \langle 0|0\rangle + \langle 1|\bar{\Psi}\Psi|1\rangle$$

$(-)$ $(-)$

$$= 1 + \bar{\Psi}\Psi$$

$$\Rightarrow \langle\bar{\Psi}|\Psi\rangle = e^{\bar{\Psi}\Psi}$$

→ This is formally similar to $\langle z_2|z_1\rangle = e^{z_2^* z_1}$ for bosons.

We remark that any function of a Grassmann variable can be expanded as follows.

$F(\Psi) = F_0 + F_1\Psi$, there being no higher power possible.

(g) Integration over Grassmann Numbers

→ we need to know how to do this before learning the path integral for fermions...

→ have to define integrals over Grassmann numbers

↳ these have no geometrical interpretations and we only define them formally.

But the nice thing is that we only need to define these integrals over 1 & ψ, since any function $F(\psi)$ can be written as

$$F(\psi) = F_0 + F_1\psi .$$

↳ There are only 2 integrals we need to know:

$$\boxed{\begin{aligned}\int \psi\, d\psi &= 1 \\ \int 1\, d\psi &= 0\end{aligned}}$$

and that's it.

The integral is postulated to be translationally invariant under a shift by another Grassmann number η:

$$\boxed{\int F(\psi+\eta)\, d\psi = \int F(\psi)\, d\psi}$$

Note that since $F(\psi) = F_0 + F_1\psi$, we must have that

$$\boxed{\int \eta\, d\psi = 0} \quad \forall\ \eta \in \text{Grassmann}$$

In general, for a collection of Grassmann #
$(\psi_1, \psi_2, \dots, \psi_N)$, we postulate that

$$\boxed{\int \psi_i \, d\psi_j = \delta_{ij}}$$

→ A few things to note here:

- ⊕ There are no limits on the integrals
- ⊕ $d\psi_j$ is also a Grassmann number so

$$\int d\psi_j \, \psi_j = -\delta_{ij}$$

- ⊕ Linear: the integral is linear...

• What about change of variables? i.e. Jacobians?

↳ Start with $\boxed{\chi = a\phi}$

Grassmann ordinary number

Since χ is Grassmann,

$$1 = \int \chi \, d\chi = \int a\phi \frac{d\chi}{d\phi} d\phi = a \frac{d\chi}{d\phi} \int \phi \, d\phi = a \frac{d\chi}{d\phi}$$

and so $\boxed{\dfrac{d\chi}{d\phi} = \dfrac{1}{a}}$, rather than $\dfrac{d\chi}{d\phi} = a$ as you would expect.

More generally, for a linear transformation...

$$\boxed{\phi_i = \sum_j M_{ij} \chi_j \implies d\phi_i = \sum_j d\chi_j M^{-1}_{ji}}$$

which ensures that new variables have the same integral value as the old, i.e.

$$\int \phi_i \, d\phi_j = \delta_{ij} = \int \chi_i \, d\chi_j$$

↳ We will often use the following result:

$$\int \bar{\psi} \psi \, d\bar{\psi} d\psi = 1$$

and

$$\int \bar{\psi} \psi \, d\psi d\bar{\psi} = -1$$

With these, we can do Gaussian integrals...

$$\boxed{\begin{aligned} \int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi &= a \\ \int e^{-\bar{\psi} M \psi} [d\bar{\psi} d\psi] &= \det M \end{aligned}}$$

{ To check this, simply expand the exponentials and do the integrals.

We also will need the "averages" over the Gaussian integrals, i.e. higher moments of the Gaussian integrals...

$$\boxed{\langle \bar{\psi} \psi \rangle = \frac{\int \bar{\psi} \psi e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi}{\int e^{-a\bar{\psi}\psi} d\bar{\psi} d\psi} = \frac{1}{a} = -\langle \psi \bar{\psi} \rangle}$$

The proof of this is actually quite straightfwd.

Now, let's consider a more general problem with 2 sets of Grassmann numbers.

$$\begin{cases} \Psi = [\Psi_1 \dots \Psi_N] \\ \bar{\Psi} = [\bar{\Psi}_1 \dots \bar{\Psi}_N] \end{cases}$$

and a <u>Gaussian action</u>. $S = -\bar{\Psi}M\Psi = -\sum_{ij} \bar{\Psi}_i M_{ij} \Psi_j$.

Assume that M is Hermitian, we will show that

$$\boxed{\int e^{-\bar{\Psi}M\Psi} [\mathcal{D}\bar{\Psi}\mathcal{D}\Psi] = \det M} \quad (*)$$

where $[\mathcal{D}\bar{\Psi}\mathcal{D}\Psi] = \prod_1^N d\bar{\Psi}_i \, d\Psi_i$

<u>Proof</u>. To do this, we ortho-diag M, so that we have

$$\begin{Bmatrix} MV_n = \lambda_n V_n \quad , \quad V_n^\dagger M = V_n^\dagger \lambda_n \\ V_n^\dagger V_m = \delta_{mn} \end{Bmatrix}.$$

and write $\Psi = \sum x_n V_n$; $\bar{\Psi} = \sum_n \bar{x}_n V_n^\dagger$

Grassmann numbers

<u>Then</u>,

$$Z = \exp\left\{ -\sum_n \lambda_n \bar{x}_n x_n \right\} [d\bar{X} dX] = \prod_n \lambda_n = \det(M).$$

where we use $\int e^{-a\bar{\psi}\psi}\, d\bar{\psi}\, d\psi = a$ for a single pair & the fact that under a unitary transform $[V_{ij}]$ of Grassmann numbers the Jacobian is <u>unity</u>.

(we won't go into details here since the proofs are actually not too hard and probably not entirely instructive).

(b) $\boxed{\text{Resolution of Identity \& Trace}}$

Recall that the resolution of identity in the bosonic land is

$$I = \int |z\rangle\langle z|\, e^{-z^* z}\, \frac{dz^* dz}{2\pi i}$$

Here, we claim that $\boxed{I = \int |\psi\rangle\langle\bar{\psi}|\, e^{-\bar{\psi}\psi}\, d\bar{\psi}\, d\psi}$

<u>Proof.</u>

$$\int |\psi\rangle\langle\bar{\psi}|\, e^{-\bar{\psi}\psi}\, d\bar{\psi}\, d\psi$$

$$= \int |\psi\rangle\langle\bar{\psi}|\, (1 - \bar{\psi}\psi)\, d\bar{\psi}\, d\psi$$

$$= \int (|0\rangle - \psi|1\rangle)(\langle 0| - \langle 1|\bar{\psi})\, (1 - \bar{\psi}\psi)\, d\bar{\psi}\, d\psi$$

$$= \int (|0\rangle\langle 0| + \psi|1\rangle\langle 1|\bar{\psi})\, (1 - \bar{\psi}\psi)\, d\bar{\psi}\, d\psi$$

$$= |0\rangle\langle 0| \int (-\bar{\psi}\psi)\, d\bar{\psi}\, d\psi \;+\; |1\rangle\langle 1| \int \psi\bar{\psi}\, d\bar{\psi}\, d\psi$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}.$$

The trace of any bosonic operator Ω (an operator made up of an even # of Fermi operators)...

$$\boxed{\mathrm{Tr}\,\Omega = \int \langle -\bar\psi|\Omega|\psi\rangle e^{-\bar\psi\psi}\, d\bar\psi\, d\psi}$$

The proof this similar to that for the resolution of id

Lastly, we want the δ-function for Grassmann #:

↳ Suppose ψ, χ, η are Grassmann #, then

$$\boxed{\int e^{(\eta-\chi)\psi} F(\eta)\, d\psi\, d\eta = F(\chi)}$$

so that

$$\boxed{\int e^{(\eta-\chi)\psi}\, d\psi = \delta(\eta-\chi)}$$

To see that, we write $F(\eta) = F_0 + F_1\eta$, and expand everything, keeping only terms that survive integration over η & ψ.

e.g.
$$\begin{cases} e^{(\eta-\chi)\psi} = 1 + (\eta-\chi)\psi. \\ F(\eta) = F_0 + F_1\eta. \end{cases}$$

(This is straight fwd, so we'll skip over the details).

(i) Thermodynamics of a Fermi Oscillator

→ consider partition fn of a single oscillator ...

$$Z = \mathrm{Tr}\, \exp\{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi\}$$

(by defn) $$= \int \langle -\bar\Psi | \exp\{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi\}|\Psi\rangle e^{-\bar\Psi\Psi} d\bar\Psi d\Psi$$

- But notice that we can't expand this... b/c there are infinitely many terms + Grassmann variables getting mixed up...

⇒ have to rewrite the operator to support using Grassmann numbers...

$$\boxed{\exp\{-\beta(\Omega_0-\mu)\Psi^\dagger\Psi\} = 1 + (e^{-\beta(\Omega_0-\mu)}-1)\Psi^\dagger\Psi.}$$

note that RHS = LHS if $\Psi^\dagger\Psi = 0$ or $\mathbb{1}$, which is good...

With this, we can expand the integral...

$$Z = \int \langle -\bar\Psi | 1 + \{e^{-\beta(\Omega_0-\mu)}-1\}\Psi^\dagger\Psi|\Psi\rangle e^{-\bar\Psi\Psi} d\bar\Psi d\Psi$$

$$= \int \underbrace{\langle -\bar\Psi|\Psi\rangle}\left[1 + (e^{-\beta(\Omega_0-\mu)}-1)(-\bar\Psi\Psi)\right] e^{-\bar\Psi\Psi} d\bar\Psi d\Psi$$

$$= \int \left\{1 - \underbrace{(e^{-\beta(\Omega_0-\mu)}-1)}_{A}(\bar\Psi\Psi)\right\} e^{-2\bar\Psi\Psi} d\bar\Psi d\Psi$$

$$= 2 - A\int \bar\Psi\Psi e^{-2\bar\Psi\Psi} d\bar\Psi d\Psi$$

as expected ✓

$$= 2 - A\left(\frac{-1}{2}\right)\cdot 2 = 2 + A = \boxed{1 + e^{-\beta(\Omega_0-\mu)}}$$

✓

(j) Fermionic Path Integral

We're now ready to map the quantum problem of fermions to a path integral.

→ Begin with

$$\boxed{Z = e^{-\beta H}}$$

where H is the normal-ordered operator $H(\Psi^\dagger, \Psi)$

→ Next, write the exponent in discrete time:

$$e^{-\beta H} = \lim_{N\to\infty} \left(e^{\{-\beta H/N\}}\right)^N$$

$$= \underbrace{(1-\varepsilon H)\cdots(1-\varepsilon H)}_{N \text{ times} \to \infty}, \quad \varepsilon = \beta/N$$

we have, by the defn of the trace & inserting res of Id. $N-1$ times

$$Z = \mathrm{Tr}\left(e^{-\beta H}\right) = \mathrm{Tr}\left[(1-\varepsilon H)\cdots(1-\varepsilon H)\right]$$

$$= \int \langle -\bar{\psi}_0|(1-\varepsilon H)|\psi_{N-1}\rangle e^{-\bar{\psi}_{N-1}\psi_{N-1}} \langle\bar{\psi}_{N-1}|(1-\varepsilon H)$$

$$|\psi_{N-2}\rangle e^{-\bar{\psi}_{N-2}\psi_{N-2}} \langle\psi_{N-2}|\cdots|\psi_1\rangle e^{-\bar{\psi}_1\psi_1}$$

$$\langle\bar{\psi}_1|(1-\varepsilon H)|\psi_0\rangle e^{-\bar{\psi}_0\psi_0} \prod_{i=0}^{N-1} d\bar{\psi}_i\, d\psi_i$$

to see this, expand everything out...

Now,

$$\langle\bar{\psi}_{i+1}|(1-\varepsilon H)|\psi_i\rangle = \langle\bar{\psi}_{i+1}|1-\varepsilon H(\Psi^\dagger,\Psi)|\psi_i\rangle$$

$$= \langle\bar{\psi}_{i+1}|1-\varepsilon H(\bar{\psi}_{i+1},\psi_i)|\psi_i\rangle \sim e^{\bar{\psi}_{i+1}\psi_i} e^{-\varepsilon H(\bar{\psi}_{i+1},\psi)}$$

where in the last step we've anticipating the limit $\varepsilon \to 0$

Now, let us *define* an additional pair of vars...

$$\boxed{\bar{\psi}_N = -\bar{\psi}_0 \quad \& \quad \psi_N = -\psi_0}$$

With these, we can replace $\langle -\bar{\psi}_0| \to \langle \bar{\psi}_N|$, putting everything together, we find...

$$\mathcal{Z} = \int \prod_{i=0}^{N-1} e^{\bar{\psi}_{i+1}\psi_i} e^{-\varepsilon H(\bar{\psi}_{i+1}, \psi_i)} e^{-\bar{\psi}_i \psi_i} d\bar{\psi}_i d\psi_i$$

$$= \int \prod_{i=0}^{N-1} \exp\left\{ \varepsilon \left[\frac{\bar{\psi}_{i+1} - \bar{\psi}_i}{\varepsilon} \psi_i - H(\bar{\psi}_{i+1}, \psi_i) \right] \right\} d\bar{\psi}_i \psi_i$$

$$\Rightarrow \boxed{\mathcal{Z} \simeq \int e^{S(\bar{\psi}, \psi)} [\mathcal{D}\bar{\psi}\mathcal{D}\psi]}$$, where

$$S = \int_0^\beta \Big[\underbrace{\bar{\psi}(\tau) \left(\frac{-\partial}{\partial \tau}\right) \psi(\tau) - H(\bar{\psi}(\tau), \psi(\tau))}_{\underbrace{p\dot{q} - H}_{\mathcal{L}}} \Big] d\tau.$$

From here, we find that, for $\tau_1 > \tau_2$

$$\boxed{\begin{aligned} \langle \psi(\tau_1) \bar{\psi}(\tau_2) \rangle &= \frac{\mathrm{Tr}\left[e^{-H(\beta - \tau_1)} \Psi e^{-H(\tau_1 - \tau_2)} \Psi^\dagger e^{-H\tau_2} \right]}{\mathrm{Tr}\left[e^{-\beta H} \right]} \\ &= \frac{\mathrm{Tr}\left[e^{-\beta H} \Psi(\tau_1) \Psi^\dagger(\tau_2) \right]}{\mathrm{Tr}\, e^{-\beta H}} \end{aligned}}$$

upon invoking $\Psi(\tau_i) = e^{H\tau_i} \Psi e^{-H\tau_i}$, likewise for $\Psi^\dagger$

If $\tau_1 < \tau_2$, have to reorder the Grassmann numbers...

$$\psi(\tau_1)\bar{\Psi}(\tau_2) = -\bar{\psi}(\tau_2)\psi(\tau_1)$$

And in general we write:

$$\langle \psi(\tau_1)\bar{\psi}(\tau_2)\rangle = \frac{\mathrm{Tr}\left(e^{-\beta H}\mathcal{T}[\Psi(\tau_1)\Psi^\dagger(\tau_2)]\right)}{\mathrm{Tr}\, e^{-\beta H}}$$

with:

$$\boxed{\begin{aligned}\mathcal{T}(\psi(\tau_1)\psi^\dagger(\tau_2)) = {} & \theta(\tau_1-\tau_2)\,\Psi(\tau_1)\Psi^\dagger(\tau_2) \\ & -\theta(\tau_2-\tau_1)\,\Psi^\dagger(\tau_2)\Psi(\tau_1)\end{aligned}}$$

« remember this »

⟨j1⟩ Finite - Temperature Green's function:

The Green's function is directly related to $\langle\Psi(\tau_1)\Psi^\dagger(\tau_2)\rangle$

$$G(\tau = \tau_1-\tau_2) = -\frac{\mathrm{Tr}\left[e^{-\beta H}\mathcal{T}\{\Psi(\tau_1)\Psi^\dagger(\tau_2)\}\right]}{\mathrm{Tr}\, e^{-\beta H}}$$

$$= -\langle\psi(\tau_1)\bar{\psi}(\tau_2)\rangle = +\langle\bar{\psi}(\tau_2)\psi(\tau_1)\rangle$$

the (−) sign is just a convention.

Due to anti-commutativity, we have that G is anti-periodic, i.e.

$$\boxed{G(\tau-\beta) = -G(\tau) \quad , \quad 0 \leq \tau \leq \beta}$$

Proof. Choose $\tau_1 = \beta$, $\tau_2 > 0$ with $\beta > \tau_2$.

$$Z\,G(\beta,\tau_2) = -\mathrm{Tr}\left[e^{-\beta H}\left(e^{-\beta H}\Psi e^{-\beta H}\right)\left(e^{H\tau_2}\Psi^{+}e^{-H\tau_2}\right)\right]$$

$$= -\mathrm{Tr}\left[e^{-\beta H}\Psi^{+}(\tau_2)\Psi(0)\right]$$

$$= -\mathrm{Tr}\left[e^{-\beta H}\,T\left\{\Psi^{+}(\tau_2)\Psi(0)\right\}\right]$$

$$= +\mathrm{Tr}\left[e^{-\beta H}\,T\left\{\Psi(0)\Psi^{+}(\tau_2)\right\}\right]$$

$$= -Z\,G(0,\tau_2)$$

Since a sign change by β flips sign of G, it is periodic on $[-\beta,\beta] \Rightarrow$ take FT:

$$\begin{cases} G(\tau) = \displaystyle\sum_{m=-\infty}^{\infty} e^{-i\omega_m\tau}\,G(\omega_m) \quad \text{where } \omega_m = \frac{2\pi m}{2\beta} = \frac{m\pi}{\beta} \\ \text{and} \quad G(\omega_m) = \displaystyle\frac{1}{2\beta}\int_{-\beta}^{\beta} G(\tau)\,e^{i\omega_m\tau}\,d\tau \end{cases}$$

But we can actually split $G(\omega_m)$ in half b/c of the anti-periodicity of G...

Let $\bar{\tau} = \tau + \beta$, we may write ...

$$\frac{1}{2\beta}\int_{-\beta}^{0} G(\tau)\, e^{i\omega_m \tau}\, d\tau = \frac{1}{2\beta}\int_{0}^{\beta} G(\bar{\tau}-\beta)\, e^{-i\omega_m \beta}\, e^{i\omega_m \bar{\tau}}\, d\bar{\tau}$$

$$= \underbrace{(-1)}_{\{\text{anti-periodicity}\}} \frac{1}{2\beta}\, e^{-i\omega_m \beta} \int_{0}^{\beta} G(\bar{\tau})\, e^{i\omega_m \bar{\tau}}\, d\bar{\tau}$$

Note further that $\omega_m = \frac{m\pi}{\beta} \Rightarrow e^{i\omega_m \beta} = (-1)^m$.

$\Rightarrow$ If m even, then the integral $\equiv 0$ when in $[0,\beta]$
If $m = (2n+1)$, the integrals are equal in both ranges

$$\boxed{\begin{aligned} &\Rightarrow G(\omega_n) = \frac{1}{\beta}\int_0^\beta G(\tau)\, e^{i\omega_n \tau}\, d\tau \\ &\text{where } \omega_n = \frac{(2n+1)\pi}{\beta} \\ &G(\tau) = \sum_n e^{-i\omega_n \tau}\, G(\omega_n) \\ &\int_0^\beta e^{i\omega_n \tau}\, e^{-i\omega_m \tau} = \beta\, \delta_{mn} \end{aligned}}$$

$\rightarrow$ we will use these extensively later.

$\rightarrow$ {Matsubara frequency}

(Fermionic frequency)

#

(j2) $\underline{\underline{G(\tau) \text{ for a free-fermion}}}$

Consider a free-fermion for which

$$\boxed{\mathcal{H} = (\Omega_0 - \mu)\, \Psi^\dagger \Psi}.$$

chemical potential

In Euclidean time, the EOM for the Heisenberg op $\Psi(\tau)$ is

$$\boxed{\frac{d}{d\tau}\Psi(\tau) = [\mathcal{H}, \Psi(\tau)] = -(\Omega_0 - \mu)\Psi(\tau)}$$

with solution

$$\begin{cases} \Psi(\tau) = e^{-(\Omega_0-\mu)\tau}\,\Psi. \\ \Psi^\dagger(\tau) = e^{(\Omega_0-\mu)\tau}\,\Psi^\dagger \end{cases}$$

→ Choose $\tau_1 = \tau$, $\tau_2 = 0$, WLOG we find

$$\boxed{\begin{aligned} G(\tau) &= -\theta(\tau)\frac{\mathrm{Tr}\{e^{-\beta H}\Psi(\tau)\Psi^\dagger(0)\}}{Z} + \theta(-\tau)\frac{\mathrm{Tr}\{e^{-\beta H}\Psi^\dagger(0)\Psi(\tau)\}}{Z} \\ &= -\theta(\tau)\, e^{-(\Omega_0-\mu)\tau}\left(1 - n_F(\Omega_0-\mu)\right) \\ &\quad + \theta(-\tau)\, e^{-(\Omega_0-\mu)\tau}\, n_F(\Omega_0-\mu) \end{aligned}}$$

where $n_F(\Omega_0-\mu) = \dfrac{\mathrm{Tr}\{e^{-\beta H}\Psi^\dagger\Psi\}}{Z} = \dfrac{1}{e^{\beta(\Omega_0-\mu)}+1}$

thermally averaged occupation number

From here, we calculate $G(\omega_n)$ by defn:

$$G(w_n) = \frac{1}{\beta}\int_0^\beta e^{i w_n \tau} e^{-(\Omega_0-\mu)\tau}\left(1-n_F(\Omega_0-\mu)\right) d\tau$$

$$\boxed{G(w_n) = \frac{1}{\beta}\frac{1}{i w_n - (\Omega_0-\mu)}}$$

As $T \to 0$, we have

$$G(\tau) = -\theta(\tau)\, e^{-(\Omega_0-\mu)\tau} \quad (\mu < \Omega_0)$$

$$= +\theta(\tau)\, e^{-(\Omega_0-\mu)\tau} \quad (\mu > \Omega_0)$$

(j3) Fermion Path Integral in frequency space

Recall the path integral for fermions:

$$Z = \int e^{S(\bar\Psi, \Psi)} [\mathcal{D}\bar\Psi\, \mathcal{D}\Psi] \text{ where}$$

$$S = \int_0^\beta \left[\bar\Psi(\tau)\left(\frac{-\partial}{\partial\tau}\right)\Psi(\tau) - H(\bar\Psi(\tau), \Psi(\tau))\right] d\tau$$

with $\mathcal{H} = (\Omega_0-\mu)\, \Psi^\dagger \Psi$, we have ...

$$S = \int_0^\beta \bar\Psi(\tau)\left(\frac{-\partial}{\partial\tau} - w_0 + \mu\right)\Psi(\tau)\, d\tau.$$

Here, we want to express S in frequency space.

To do this, need to write $\Psi(\tau)$, $\bar\Psi(\tau)$ in terms of Matsubara freqs ...

$$w_n = \frac{(2n+1)\pi}{\beta} ..$$

$$\left\{ \begin{aligned} \bar{\psi}(\tau) &= \sum_n e^{i\omega_n \tau} \bar{\psi}(\omega_n) \\ \psi(\tau) &= \sum_n e^{-i\omega_n \tau} \psi(\omega_n) \end{aligned} \right.$$

these are NOT conjugated!

We then get inversions...

$$\left\{ \begin{aligned} \psi(\omega_n) &= \frac{1}{\beta}\int_0^\beta \psi(\tau) e^{i\omega_n \tau} d\tau \\ \bar{\psi}(\omega_n) &= \frac{1}{\beta}\int_0^\beta \bar{\psi}(\tau) e^{-i\omega_n \tau} d\tau \end{aligned} \right\}$$

where we use the orthogonality property:

$$\frac{1}{\beta}\int_0^\beta e^{i(\omega_n - \omega_m)\tau} d\tau = \delta_{nm}$$

if $\beta \to \infty$ then become $\omega_n = \frac{(2n+1)\pi}{\beta}$ implying that when $n' = n+1$, ω_n changes by $2\pi/\beta$, so that

$$\frac{1}{\beta}\sum_n \xrightarrow{\text{continuum}} \int \frac{d\omega}{2\pi}$$

With this, we can see how the action of the Fermi oscillator transforms under FT:

$$\boxed{\begin{aligned} S &= \int_0^\beta \bar{\psi}(\tau)\left(\frac{-\partial}{\partial \tau} - (\Omega_0 - \mu)\right)\psi(\tau)\, d\tau \\ &= \beta \sum_n \bar{\psi}(\omega_n)\left[i\omega_n - (\Omega_0 - \mu)\right]\psi(\omega_n) \end{aligned}}$$

In the $\beta \to \infty$ limit, $\omega_n \to$ continuous ω.

If we introduce rescaled Grassmann vars

$$\overline{\Psi}(\omega) = \beta\,\overline{\Psi}(\omega_n)\quad,\quad \Psi(\omega) = \beta\,\Psi(\omega_n)$$

and we $\frac{1}{\beta}\sum_n \to \int \frac{d\omega}{2\pi}$, we find that the action when $\beta \to \infty$ is

$$S = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\overline{\Psi}(\omega)\left[i\omega - \Omega_0 + \mu\right]\Psi(\omega)$$

Because the Jacobian for $(\overline{\Psi}(\tau), \Psi(\tau)) \to (\overline{\Psi}(\omega), \Psi(\omega))$ is $\underline{1}$, the path integral is

$$\boxed{Z = \int \exp\left\{\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\,\overline{\Psi}(\omega)\left[i\omega - \Omega_0 + \mu\right]\Psi(\omega)\right\}\left[\mathcal{D}\overline{\Psi}(\omega)\,\mathcal{D}\Psi(\omega)\right]}$$

What about correlation function?

$$\langle \overline{\Psi}(\omega_1)\Psi(\omega_2)\rangle = \frac{1}{Z}\int \overline{\Psi}(\omega_1)\Psi(\omega_2)\exp\left[\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\overline{\Psi}(\omega)\left(i\omega - \Omega_0 + \mu\right)\Psi(\omega)\right]\left[\mathcal{D}\overline{\Psi}\mathcal{D}\Psi\right]$$

$$\Rightarrow \boxed{\langle \overline{\Psi}(\omega_1)\Psi(\omega_2)\rangle = \frac{2\pi\,\delta(\omega_1 - \omega_2)}{i\omega_1 - \Omega_0 + \mu}}$$

because $G(\omega) = \beta^2 G(\omega_n)$ due to the rescaling

And, in particular ...

$$\boxed{G(\omega) = \langle \overline{\Psi}(\omega)\Psi(\omega)\rangle = \frac{2\pi\,\delta(0)}{i\omega - \Omega_0 + \mu} = \frac{\beta}{i\omega - \Omega_0 + \mu}}$$

With this, we can calculate the mean occupation #:

$$\langle N \rangle = \frac{1}{\beta \mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \mu}$$

single-particle system, so outer integral vanishes.

$$= \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \langle \bar{\Psi}(\omega) \Psi(\omega) \rangle$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{i\omega 0^+}}{i\omega - \Omega_0 + \mu}$$

allows us to close contour in upper half plane. (to make integral converge).

$$= \theta(\mu - \Omega_0)$$

which is what we found in the operator approach.

(k) $\boxed{\text{Generating Functions } Z(J) \text{ \& } W(J)}$

→ generating functions allow us to generate "moments" i.e. <u>all</u> correlation functions.

We will discuss generating fns for
- Ising spins
- real bosonic vars
- complex bosonic vars
- Grassmann vars.

(k1) <u>Ising Correlators</u>

Recall spins in a magnetic field... $h = (h_1, \ldots, h_N)$

$$\boxed{Z(h_1, \ldots, h_N) \equiv Z(h) = \sum_{s_i} \exp\left\{ -E(K, s_i) + \sum_i h_i s_i \right\}}$$

where $E(K, s_i) \rightsquigarrow$ energy as fn of spins, other params, ...

From here, we have...

$$\begin{cases} \langle s_i \rangle = \dfrac{1}{Z}\dfrac{\partial Z}{\partial h_i} \\ \langle s_i s_j \rangle = \dfrac{1}{Z}\dfrac{\partial^2 Z}{\partial h_i \partial h_j} \\ \vdots \end{cases}$$

Just like in probability where we also have the MGF, we have Z.

We can also define a fn $W(h)$ by

$$\boxed{Z(h) = e^{-W(h)} = e^{-\beta F}}$$

We can see that derivatives of $W(h)$ give the <u>connected</u> correlation fns...

$$-\frac{\partial^2 W}{\partial h_i \partial h_j} = \langle s_i s_j \rangle_c = \langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle$$

$$\frac{-\partial^4 W}{\partial h_i \partial h_j \partial h_k \partial h_\ell} = \langle s_i s_j s_k s_\ell \rangle_c$$

$$= \langle s_i s_j s_k s_\ell \rangle_c - \Big[\langle s_i s_j \rangle \langle s_k s_\ell \rangle + \langle s_i s_k \rangle \langle s_j s_\ell \rangle + \langle s_i s_\ell \rangle \langle s_j s_k \rangle \Big]$$

$$\vdots$$

(k2) Real Scalar Vars

This is introductory QFT, so we won't go into much detail here...

The idea is that with $|x\rangle = \{x_1 \ldots x_N\}$, $x_i \in \mathbb{R}$, we have

$$Z(J) = \prod_{i=1}^{N} \int_{-\infty}^{\infty} dx_i \, e^{-S_0(x) + \langle J|x\rangle} = e^{-W(J)}$$

$S_0(x)$: don't involve J

$\langle J|x\rangle$: source (generic)

$$\langle J|x\rangle = \sum J_i x_i$$

Correspondence w/ Ising correlators: $\begin{cases} s_i \to x_i \\ h_i \to J_i \end{cases}$

$$\left. \frac{-\partial^2 W}{\partial J_i \partial J_j} \right|_{J=0} = \langle x_i x_j \rangle_c = \langle x_i x_j \rangle$$

and so on...

$\boxed{\text{If}}$ $S_0(x)$ is quadratic then everything is nice.

i.e.

$$\boxed{\text{all connected correlators} = 0 \text{ except } \langle x_i x_j \rangle}$$

(Wick's Thm in QFT)

Recall that $$\int_{-\infty}^{\infty} e^{-\frac{1}{2}mx^2 + Jx} = \sqrt{\frac{2\pi}{m}} \exp\left[\frac{J^2}{2m}\right]$$

Consider $S(J) = -\frac{1}{2} \langle x|M|x\rangle + \langle J|x\rangle$

where M symmetric real. Then we can show (look at old notes...) that

$$\boxed{Z(J) = e^{\frac{1}{2}\langle J|M^{-1}|J\rangle} \cdot \frac{(2\pi)^{N/2}}{\sqrt{\det M}}}$$

and

$$W(J) = -\frac{1}{2}\langle J|M^{-1}|J\rangle + \frac{1}{2}\ln(\det M)) + \text{etc.}$$

$$\hookrightarrow \quad \langle x_i x_j\rangle = \left.\frac{-\partial^2 W}{\partial J_i \partial J_j}\right|_{J=0} = (M^{-1})_{ij} \equiv G_{ij}$$

(there are results about $\langle x_i x_j x_k x_l\rangle$ which we won't worry about here...)

If S is not quadratic, then we may expand the exponential in pwr series ~~then~~ and ~~do~~ calc. things perturbatively...

↳ There are also results about replacing $|x\rangle$ with the field $\phi(x)$, but that's QFT.

↳ We will pull results for QFT to here at some point, but perhaps not now.

(we won't worry about it now...

#

(h?) Complex Scalar Variables

↳ look at partition function

$$\mathcal{Z} = \int_{-\infty}^{\infty} \frac{dz\, dz^{*}}{2\pi i} e^{-m z^{*} z}.$$

Real Jerry

use $\frac{dz\,dz^{*}}{2\pi i} \longrightarrow \frac{dx\,dy}{\pi} = \frac{r\,dr\,d\theta}{\pi}$

and get $\mathcal{Z} = \int_{-\infty}^{\infty} \frac{dx\,dy}{\pi} e^{-m(x^2+y^2)} = \frac{1}{m}$

↳ & that

$$\langle z^{*} z\rangle = \frac{1}{\mathcal{Z}} \int_{-\infty}^{\infty} z^{*} z\, e^{-a z^{*} z}\, dz\, dz^{*}/2\pi i = \frac{1}{a}.$$

and $\langle z z\rangle = \langle z^{*} z^{*}\rangle = 0.$

- In general, if $z \in \mathbb{C}^d$, then we have (for Hermitian M)

$$\boxed{\mathcal{Z} = \int [\mathcal{D}z\, \mathcal{D}^{\dagger}z]\, e^{-z^{\dagger} M z}}$$

where $[\mathcal{D}z\, \mathcal{D}^{\dagger}z] = \prod_{i=1}^{N} \int dz_i\, dz_i^{*}/2\pi i$

eigvs of M.

$z = (z_1 \ldots z_N)$, $z^{\dagger} = (z_1^{*} \ldots z_N^{*})^T$

and

$$\boxed{\mathcal{Z} = \prod_i \frac{1}{m_i} = [\det(M)]^{-1}}$$

We can check that $\boxed{\langle z_i^\dagger z_j\rangle = \dfrac{\delta_{ij}}{m_i} \equiv \langle \bar{i}j\rangle}$

Wick's Thm: $\langle z_i^\dagger z_j z_k^\dagger z_l\rangle = \langle \bar{i}j\rangle\langle \bar{k}l\rangle + \langle \bar{i}l\rangle\langle \bar{k}j\rangle$

- Now, add the source $\vec{J}$: 2 sources: $\vec{J}$, $\vec{J}^\dagger$

$$\mathcal{Z} = \mathcal{Z}(J, J^\dagger) = e^{-W(J,J^\dagger)}$$
$$= \int [\mathcal{D}z\,\mathcal{D}^\dagger z]\, e^{-z^\dagger M z + J^\dagger z + z^\dagger J}$$

We can of course move to complex fields ϕ in 4-d. But we won't worry about that now...

(since there (standard) results can be found in any QFT text book...)

(h4) Grassmann Variables

All the preceding machinery can be applied to Grassmann integrals...

→ We just have to recall the appropriate defns first...

$$\boxed{\begin{aligned} &Z(J,\bar{J}) = \int \exp\left[S(\bar{\psi},\psi) + \bar{J}\psi + \bar{\psi}J\right][\mathcal{D}\bar{\psi}\mathcal{D}\psi] \\ &\bar{J}\psi = \sum_i \bar{J}_i \psi_i \qquad \text{where } (\bar{J},J) \text{ are also grassmann variables.} \\ &\bar{\psi}J = \sum_i \bar{\psi}_i J_i \\ &[\mathcal{D}\bar{\psi}\mathcal{D}\psi] = \prod_i^N d\bar{\psi}_i\, d\psi_i \end{aligned}}$$

We can readily check that

$$\langle \bar{\psi}_\beta \psi_\alpha \rangle = \frac{-1}{Z}\frac{\partial^2 Z}{\partial J_\beta \partial \bar{J}_\alpha}$$

by following these steps

- expand source term to linear ord in exp
- take derivatives, w/ $\partial/\partial J$, $\partial/\partial \bar{J}$ commute w/ even terms like $\bar{\psi}J$, $\bar{J}\psi$
- remember that $\langle \bar{\psi}_\beta \psi_a \rangle = -\langle \psi_a \bar{\psi}_\beta \rangle$

→ Next, can define $W(\bar{J},J)$ by

$$\boxed{Z(J,\bar{J}) = e^{-W(\bar{J},J)}}$$

from which we get the 2-pt connected correlation:

$$\left.\frac{\partial^2 W}{\partial J_\beta \partial \bar{J}_\alpha}\right|_{J,\bar{J}=0} = \langle \bar{\psi}_\beta \psi_\alpha \rangle_c = \langle \bar{\psi}_\beta \psi_\alpha \rangle \equiv \langle \bar{\beta}\alpha \rangle$$

where we have used $\langle \bar{\psi}_\beta \rangle = \langle \psi_a \rangle = 0$ when $(J,\bar{J}) = 0$

The connected 4-pt correlation fn is:

$$\boxed{\left.\frac{-\partial^4 W}{\partial J_\alpha \partial J_\beta \partial \bar{J}_\gamma \partial \bar{J}_\delta}\right|_{(\bar{J},J)=0} = \langle \bar{\alpha}\bar{\beta}\gamma\delta \rangle_c = \langle \bar{\alpha}\bar{\beta}\gamma\delta \rangle - \left[\langle \bar{\alpha}\delta \rangle \langle \bar{\beta}\gamma \rangle - \langle \bar{\alpha}\gamma \rangle \langle \bar{\beta}\delta \rangle \right]}$$

The central result, when Ψ becomes a tuple, is

$$\boxed{\int e^{-\bar{\Psi}M\Psi + \bar{J}\Psi + \bar{\Psi}J} [d\bar{\Psi} d\Psi] = \det(M)\, e^{\bar{J}M^{-1}J}}$$

<u>Proof.</u> making the following translation:

$$\Psi \to \Psi + M^{-1}J$$
$$\bar{\Psi} \to \bar{\Psi} + \bar{J}M^{-1}$$

And so, for a Gaussian action,

$$\boxed{W(J,\bar{J}) = -\ln \det(M) - \bar{J}M^{-1}J}$$

from which we find

$$\boxed{\langle \bar{\Psi}_\beta \Psi_\alpha \rangle_c = \frac{\partial^2 W}{\partial J_\beta \partial \bar{J}_\alpha} = -M^{-1}_{\alpha\beta}}$$

and

$$\boxed{\langle \bar{\alpha}\bar{\beta}\gamma\delta \rangle_c = 0}$$

since we can differentiate W only twice ...

→ ~~the~~ Only the connected 2-pt corr fn is non zero in a fermionic Gaussian theory, as in bosonic theories

7: THE 2D ISING MODEL

Mar 19 2021

→ This is the easiest example of a solvable system.

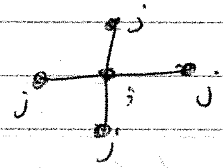
(a) Intro

→ Now, we'll focus on the magnetic transition in the Ising model.

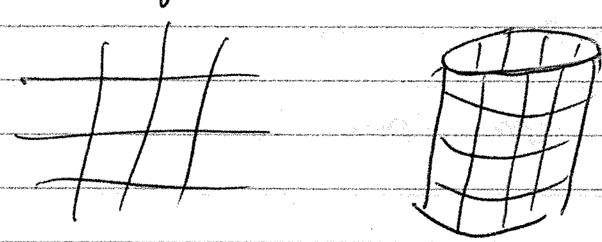
<u>Model</u>: Consider model with N cols & M rows

$$\mathcal{Z} = \sum_{s_i} \exp\left\{ K \sum_{\langle ij \rangle} s_i s_j \right\}$$

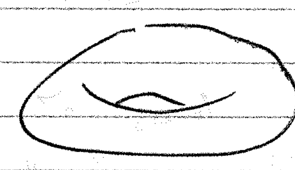
where $K = J/kT$, $\langle i,j \rangle$ means nearest neighbor



3 types of boundary conditions:



or



open

cylinder (periodic in 1 direction)

torus (periodic in 2 directions)

Assume total = $\mathcal{N} = M \cdot N \gg 1$

Assume that we're not near the "end" of the model

→ Total # bonds = $2\mathcal{N}$.

Extremal limits:

$K = J/kT \to \infty$ i.e. $T \to 0$ $\Rightarrow$ all spin up or down

→ system is magnetized
→ $\boxed{\langle M \rangle = \pm 1}$ → can pick $\langle M \rangle = +1$
↓
average spin per site

$K = J/kT \to 0$ i.e. $T \to \infty$ $\boxed{\langle M \rangle = 0}$ since Boltzmann weights will be 1 for all configurations

→ How does $\langle M \rangle$ look like as a fn of T?

$$\begin{cases} \langle M \rangle = 1 \ @ \ T = 0 \\ \langle M \rangle = 0 \ @ \ T = \infty \end{cases}$$

Now, if $\langle M \rangle = 0$ after some T_c then we have a phase transition.

If this is true then there must be a singularity @ T_c (since nice analytic functions don't do this)

→ To see this singularity, we must go to the thermodynamic limit.

{ → Particularly want to look @ "free energy per site"

In 1D → no phase transition @ finite T

In 2D however, there is finite-T phase transition.

(b) High-temp expansion (MIT ocw)

Basic idea: $K=0 \Leftrightarrow T=\infty \rightsquigarrow$ all Boltzmann weights are 1.

→ For each bond we have (since $(\sigma_i\sigma_j)^2=1$)

$$\boxed{e^{K\sigma_i\sigma_j} = \frac{e^K+e^{-K}}{2} + \frac{e^K-e^{-K}}{2}\sigma_i\sigma_j}$$

(check this (easy))

so we can write

$$e^{K\sigma_i\sigma_j} = \cosh K\,(1+\tanh K\,\sigma_i\sigma_j)$$

Applying this to find Z:

$$\boxed{Z = \sum_{\{s_i\}} e^{K\sum_{\langle i,j\rangle} s_i s_j} = (\cosh K)^{\#\text{bonds}} \sum_{s_i}\prod_{\langle i,j\rangle}(1+\tanh K\, s_i s_j)}$$

For N_b bonds or N bonds in the lattice, the product generates 2^N terms, which can be represented diagrammatically by drawing a line connecting sites i,j for each factor of $\tanh K(s_i s_j)$

→ There can be at most 1 line for each bond
→ either empty or occupied.

→ each site obtains a factor of $s_i^{p_i}$ where p_i is # occupied bonds emanating from i.

Since $\sigma_i = \pm 1$, summing over gives a factor of

$$\begin{cases} 2 & \text{if } p_i \text{ even} \\ 0 & \text{if } p_i \text{ odd} \end{cases}$$

→ The only graphs that survive the sum have an <u>even</u> # of lines passing through each site

⇒ The resulting graphs are collections of closed paths and

(needs some explaining)

$$Z = 2^N \times (\cosh K)^{N} \sum_{\substack{\text{all} \\ \text{closed} \\ \text{graphs}}} (\tanh K)^{\#\text{ bonds in graph}}$$

For d-dimensional hypercubic lattice, the smallest closed graph is a square of 4 bonds which has $d(d-1)/2$ possible orientations, and so on...

$$\Rightarrow \; Z = 2^N (\cosh K)^{dN} \left\{ 1 + \frac{d(d-1)N}{2} (\tanh K)^4 + d(d-1)(2d-3)(\tanh K)^6 + \dots \right\}$$

→ when $d = 2$, we have

$$\boxed{Z = 2^N (\cosh K)^{2N} \left\{ 1 + N(\tanh K)^4 + 2N(\tanh K)^6 + (\tanh K)^8 \left(6N + \tfrac{1}{2} N(N-5)\right) + O(t^{10}) \right\}}$$

How do we get this expression, really?

$\rightarrow$ Look at $Z = \sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} \underbrace{(\cosh K)}_{A} (1 + s_i s_j \underbrace{\tanh K}_{B})$

$$= \sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} A(1 + s_i s_j B)$$

- There are 2^N terms in the product over bonds (each site has $s_i = \pm 1$).
- Each bond contributes a (1) or a ($\tanh K$).

$$= (\cosh K)^{2N} \sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} (1 + s_i s_j B)$$

Sum over all configs

product over links in the lattice. (2^N terms)

$\left\{ \begin{array}{l} (1) \text{ is represented by an empty edge} \\ \text{and} \\ (\tanh K) s_i s_j \text{ by an occupied edge} \end{array} \right\}$

$\rightarrow \sum_{s_i} \prod_{\substack{\text{bond}\\ ij}} (1 + s_i s_j B) = \sum_{s_i} \boxed{?}$

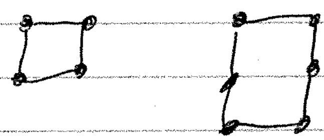
n.n only

This $\boxed{?}$ is $\displaystyle\sum_{G \subseteq \text{lattice}} (\tanh K)^{\#\text{edges of } G} \prod_{(ij)\in G} s_i s_j$

graphs on the lattice.

To get non zero contributions, every factor s_j should occur with even power, and sum over s_j gives a factor of 2.

↳ selects G which are collections of closed polygons



etc ...

Summing over spins gives...

$$Z = (\cosh K)^{2N}\, 2^{N} \sum_{G \in C} (\tanh K)^{\#\text{edges of } G}$$

set of closed ~~poo~~ graphs

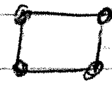
Now, what is

$$\sum_{G \in C} (\tanh K)^{\#\text{edges of } G} \ ?$$

Well... there is a factor of $2^N \cosh K^{2N}$ for every bond...

First nonzero contribution is $(\tanh K)^4$: picking 4

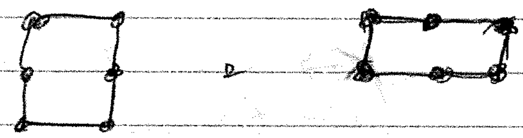
4 bonds to form a square. The spin @ each corner appears twice



$\hookrightarrow$ get $2^N (\cosh K)^{2N} \cdot N \tanh^4 K$

# of squares we can have (by labeling)

With 6 bonds, there are 2 possible orientations:



and so on for

So we have

$$\boxed{\frac{Z(K)}{2^N (\cosh K)^{2N}} = 1 + N \tanh^4 K + 2N \tanh^6 K + \cdots}$$

okay... so we have, in general...

$$\boxed{\frac{Z(K)}{2^N (\cosh K)^{2N}} = \sum_{\substack{\text{closed} \\ \text{loops}}} C(L) \tanh^L(K)}$$

loops of length L without covering any bond more than once

With this, can calculate free ~~energy~~ per site

$$\boxed{\begin{aligned} -\frac{f}{kT} = \frac{1}{N} \ln Z &= \ln\left[2\cosh^2 K\right] + \frac{1}{N} \ln\left(1 + N \tanh^4 K + \cdots\right) \\ &\cong \ln\left[2\cosh^2 K\right] + \tanh^4 K + \cdots \end{aligned}}$$

This is not too important now...

(c) Low Temp Expansion

- So before, we did the expansion when $T \gg 1$ in which case all Boltzmann weights are 1.

- When $T = 0$, $K \to \infty$. The spins tend to be aligned in one direction...

  ↳ Assume this direction is up.

Boltzmann weight now is e^K on each of the $2N$ bonds. In this case, the bonds are unbroken.

If, however, one spin is flipped down, 4 bonds linking to it will be reduced by e^{-8K}

(energy excitation)

↳ This can occur at any site, we have

$$Z = e^{2NK}(1 + Ne^{-8K} + \dots)$$

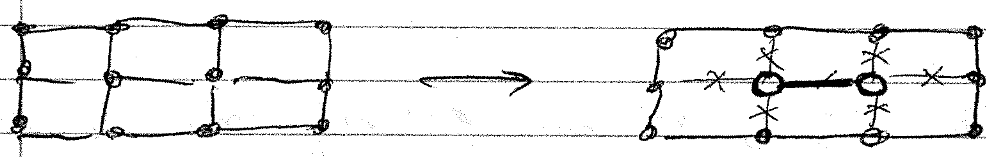
no flip — one flip

What if 2 spins flip? → we only care if the 2 are nearest neighbor... In this case, there are:

- N ways to choose the first spin.
- 2 ways to chose the 2^{nd} (not 4, since we'll double-count)

So we have: $\boxed{Z = e^{2NK}\left(1 + Ne^{-8K} + 2Ne^{-12K} + \cdots\right)}$

Why e^{-12K}: since the bond between the flipped spin isn't broken, but between the flipped spins & other (6) are.



→ things are reduced by e^{-12K}.

We can keep going to get

$$\boxed{Z = e^{2NK}\left[1 + Ne^{-8K} + 2Ne^{-12K} + \frac{N(N-3)}{2}e^{-16K} + \cdots\right]}$$

Can find the free energy per site by

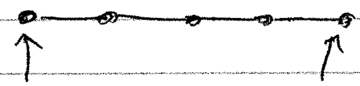
$$\left(-\frac{f}{kT} = \frac{1}{N}\ln Z\right).$$

See that if we create an island of spins pointing opposite to the majority, it costs an energy proportional to the perimeter of the island –

↳ reflected in the Boltzmann factor e^{-2KL}

(perimeter of island.)

- In the 1D model ... the cost of an island is always $\boxed{e^{-2KL}}$ regardless of size



perim $= 1 + 1 = 2$.

$\Downarrow$

the 1D model loses magnetization above $T = 0$.

- But in 2D, we can actually find T_c by asking when large islands will go unsuppressed.

↳ Imagine having a loop of length L ... at each stage we can go in 3 directions ... so

Energy cost is $\boxed{3^L e^{-2KL}}$ (roughly speaking)

↳ loops of arbitrary size no longer suppressed when we reach ...

$$e^{\underbrace{(-2K_c + \ln 3)}_{0} L} = 1 \Leftrightarrow \boxed{K_c \sim 0.5493}$$

(The exact result is $K_c = 0.4407$)

Can do similar analysis for high temp series to estimate K_c:

$$(\tanh K_c)^L \cdot 3^L \sim 1 \Rightarrow \boxed{K_c = 0.3466}$$

→ The correct answer:

$$0.3466 < 0.4407 < 0.5493$$

(d) Kramers - Wannier Duality

Kramers & Wannier discovered a hidden symmetry that relates properties of Ising model on the square lattice at low & high temps.

↳ Compare low & high temp expansion...

Low: $$Z = e^{2NK}\left[1 + Ne^{-4\times 2K} + 2Ne^{-6\times 2K} + \cdots\right]$$

$$= e^{2NK} \sum_{\substack{\text{Island} \\ \text{of } (-) \\ \text{spins}}} e^{-2K \times L}$$ ← perimeter of island

High: $$Z = 2^N (\cosh K)^{2N}\left[1 + N\tanh K^4 + 2N\tanh K^6 + \cdots\right]$$

$$= 2^N(\cosh K)^{2N} \sum_{\substack{\text{Graphs} \\ 2/4 \text{ lines} \\ \text{per site}}} (\tanh K)^{\text{length of graph}}$$

As the boundary of any island of spins serves as an acceptable graph

↳ there is a 1-1 correspondence between 2 series...

↳ Define function g to indicate the log of the above series...

↳ Free energy $= \dfrac{\ln Z}{N} = 2K + g(e^{-2K})$

$= \ln(2) + 2\ln\cosh K + g(\tanh K)$

The argument of g in the eqn above are related by the duality condition

$$\boxed{e^{-2\tilde{K}} \longleftrightarrow \tanh K \Rightarrow \tilde{K} = D(K) \equiv -\frac{1}{2} \ln \tanh K}$$

g must have a special symmetry that relates its value for dual arguments.—

(1) Low temp mapped to high temp & vice versa

(2) The map connects pairs of points since

$D(D(K)) = K$:

$$\sinh 2K = 2\sinh K \cosh K = 2\tanh K \cosh^2 K$$

$$= \frac{2\tanh K}{1-\tanh^2 K} = \frac{2e^{-2\tilde{K}}}{1-e^{-4\tilde{K}}}$$

$$= \frac{2}{e^{2\tilde{K}} - e^{-2\tilde{K}}} = \frac{1}{\sinh 2\tilde{K}}.$$

$$\boxed{(\sinh 2K)(\sinh 2\tilde{K}) = 1}$$

how the dual interactions are symmetrically related by this relation.

If $g(K)$ is singular at a point K_c, it must also be singular at $\tilde{K}_c$.

$\Rightarrow$ Since the free energy is expected to be analytic everywhere except at the transition

$\hookrightarrow$ Critical model must be SELF-DUAL.

$\Rightarrow$ At self-dual point,

$$e^{-2K_c} = \tanh K_c = \frac{1-e^{-2K_c}}{1+e^{-2K_c}}.$$

$\hookrightarrow$ $e^{-4K_c} + 2e^{-K_c} - 1 = 0 \Rightarrow e^{-2K_c} = -1 \pm \sqrt{2}$

$\Rightarrow$ Only positive solution is acceptable..

$$\boxed{K_c = -\frac{1}{2}\ln(\sqrt{2}-1) = \frac{1}{2}(\sqrt{2}+1) = 0.4407.}$$

If $K_c \sim 0.4407$, then we can work out what T_c is by the relation $K = J/kT$.

(e) Correlation Function in the Tanh Expansion

At high temp:

$$\langle s_i s_f \rangle = \frac{\sum_s s_i s_f \prod_{\text{bonds}} (1 + s_m s_n \tanh K)}{\sum_s \prod_{\text{bonds}} (1 + s_m s_n \tanh K)}$$

$$= (\tanh K)^{|i-f|} (1 + \cdots)$$

We won't worry about this for now

Mar 20, 2021

8: EXACT SOLUTION OF THE 2D ISING MODEL

(a) Transfer Matrix in Terms of Pauli Matrices

Consider model:

$$\boxed{Z = \sum_{S} \exp\left[\sum_{i} K_x S_i S_{i+x} + K_\tau (S_i S_{i+\tau} - 1)\right]}$$

where $i+x$ & $i+\tau$ are neighbors of site i in the x & τ directions, where we have subtracted the 1 from $S_i S_{i+\tau}$ so we can borrow 1D results.

↳ Get transfer matrix for a lattice with N columns is:

$$T = \underbrace{\frac{\exp\left[\sum_{n=1}^{N} K_\tau^{*} \sigma_1(n)\right]}{\left[\cosh K_\tau^{*}\right]^N}}_{V_1} \cdot \underbrace{\exp\left[\sum_{n=1}^{N} K_x \sigma_3(n)\sigma_3(n+1)\right]}_{V_3}$$

$$\boxed{T \equiv V_1 V_3}$$

where $\sigma_1(n)$, $\sigma_3(n)$ are Pauli matrices @ site n.

<u>Check</u>

eigenvalues of $\sigma_3(n)$, $n = 1, 2, 3, \dots N$

$$\langle s_1' \, s_2' \dots s_N' | T | s_1 \, s_2 \dots s_n \rangle$$

$$\| $$

$$\langle s_1' s_2' \dots s_N' | V_1 V_3 | s_1 s_2 \dots s_n \rangle$$

↓

$\sim \sigma_3(n)\sigma_3(n+1) \Rightarrow$ turn into s_n and so on

→ give Boltzmann weights associated w/ <u>horizontal bonds</u> of the row containing s_n's

For V_1 --- we can factorize, and look at contribution

$$\langle s_n'| \frac{e^{K_\tau^* \sigma_1(n)}}{\cosh K_\tau^*} |s_n\rangle = \langle s_n'| (1 + \sigma_1(n) \tanh K_\tau^*) |s_n\rangle$$

$$= (\delta_{s_n s_n'} + \tanh K_\tau^* \, \delta_{s_n, -s_n'})$$

$$= (\delta_{s_n s_n'} + e^{-2K_\tau} \delta_{s_n, -s_n'})$$

which is the Boltzmann weight due to vertical bond at site ~~s_n~~ n

So we roughly see that V_3 transfer horizontally, V_1 transfer vertically.

→ note that T not hermitian, so we can use the following version

$$\boxed{T = V_3^{1/2} V_1 V_3^{1/2}}$$

But we might ignore the hermiticity since it won't matter in the continuum limit.

(b) The Jordan-Wigner transformation & Majorana Fermions

It's hard to diagonalize the transfer matrix T.

↳ why? B/c Pauli matrices are neither Bosonic nor Fermionic, since they anti-commute at one site but commute at ~~the~~ different sites.

Ex

So, we need to trade Pauli matrices for Majorana Fermions, which have canonical anti-comm relation

↳ even before & after FT.

- How to make Majorana Fermions?

↳ start w/ Fermions (Dirac) Ψ which obey the usual anti-comm relation

$$\begin{cases} \{\Psi, \Psi^{+}\} = 1, \quad \{\Psi, \Psi\} = \{\Psi^{+}, \Psi^{+}\} = 0. \\ n_F = \Psi^{+}\Psi = \{0, 1\}. \end{cases}$$

→ Make Majorana Fermions by ~~be~~ combining these

$$\boxed{\begin{aligned} \psi_1 &= \frac{1}{\sqrt{2}}(\Psi + \Psi^{+}) \\ \psi_2 &= \frac{1}{\sqrt{2}i}(\Psi - \Psi^{+}) \end{aligned}}$$

So that $\boxed{\{\psi_i, \psi_j\} = \delta_{ij}}$

The algebra Majorana fermions obey is called the Clifford Algebra.

<u>Inverse relation:</u>

$$\begin{cases} \Psi = \dfrac{\psi_1 + i\psi_2}{\sqrt{2}} \\ \Psi^{+} = \dfrac{\psi_1 - i\psi_2}{\sqrt{2}}. \end{cases}$$

At site n, suppose we have a pair of Majorana Fermions

$\psi_1(n)$ & $\psi_2(n)$ $\Rightarrow$ live in 2D space.

→ If we have a lattice the fermions will need a Hilbert space of dimension 2^N.

↳ This is of course also the dimensionality of Pauli matrices, However, the relation between Pauli matrices & Majorana fermions is non-local

$$\boxed{\begin{aligned} \psi_1(n) &= \begin{cases} \frac{1}{\sqrt{2}}\left[\prod_{l=1}^{n-1}\sigma_1(l)\right]\sigma_2(n) & n>1 \\ \frac{1}{\sqrt{2}}\,\sigma_2(1) & n=1 \end{cases} \\ \psi_2(n) &= \begin{cases} \frac{1}{\sqrt{2}}\left[\prod_{l=1}^{n-1}\sigma_1(l)\right]\sigma_3(n) & n>1 \\ \frac{1}{\sqrt{2}}\,\sigma_3(1) & n=1 \end{cases} \end{aligned}}$$

This is the Jordan-Wigner Transformation

We can check that

$$\boxed{\{\psi_i(n),\ \psi_j(n')\} = \delta_{ij}\,\delta_{nn'}}$$

The role of the string (the product $\prod \sigma_{\dots}$) is to ensure that ψ's @ different sites n & n' anti-commute. Example : $n' > n$ ⇒ then in the product of 2 fermions, the $\sigma_1(n)$ in $\psi_{1,2}(n')$ will anti-commute with both the $\sigma_2(n)$ & $\sigma_3(n)$ in $\psi_{1,2}(n)$.

Nice properties

$$\boxed{\begin{aligned} \sigma_1(n) &= -2i\,\psi_1(n)\,\psi_2(n) \\ \sigma_3(n)\,\sigma_3(n+1) &= 2i\,\psi_1(n)\,\psi_2(n+1) \\ \sigma_3(n) &= \sqrt{2}\left[\prod_{\ell=1}^{n-1}\left(-2i\,\psi_1(\ell)\,\psi_2(\ell)\right)\right]\psi_2(n) \end{aligned}}$$

This is why no one has been able to solve the 2D Ising in a magnetic field using fermions.

—//—

With these, we can return to the transfer matrix T (dropping the $(\cosh K_\tau^*)^{-N}$ factor ...

After JW transform →

$$T = \exp\left[\sum_{n=1}^{N} -2i K_\tau^* \,\psi_1(n)\psi_2(n)\right]\exp\left[\sum_{n=1}^{N} 2i K_x\, \psi_1(n)\psi_2(n+1)\right]$$

$$\equiv V_1 V_3$$

→ See that V_1, V_3 both contain expressions quadratic in fermions

↳ which means we can diagonalize them in momentum space.

(!) But also see that we have 2 exponentials ...

→ These issues will be resolved after we do a Fourier Transform (go to k-space).

(c) Boundary Conditions.

We will impose periodic boundary condition in order to analyze finite-chain Ising models.

↳ We might expect that ↳ $1+N=1$

$$\sigma_3(N)\sigma_3(1) = 2i\,\psi_1(N)\,\psi_2(1)$$

But this is WRONG, since $\psi_2(1)$ has no string of σ_1's to cancel the string that $\psi_1(N)$ has.

(can check this)

The correct result is (we can check this, by defn)

$$\boxed{\begin{aligned}\sigma_3(N)\sigma_3(1) &= -\left[\prod_{l}^{N}\sigma_1(l)\right] 2i\,\psi_1(N)\,\psi_2(1)\\ &\equiv (-1)\cdot\underbrace{\mathcal{P}}_{\left(\prod_{l}^{N}\sigma_1(l)\right)\ \text{(non-local)}}\cdot\, 2i\,\psi_1(N)\,\psi_2(1)\end{aligned}}$$

$\mathcal{P}$ commutes with the transfer matrix T

($\mathcal{P}$ is a symmetry of T, since it flips all spins)

$$\mathcal{P}^2 = 1 \Rightarrow \sigma(\mathcal{P}) = \{1,-1\}$$

$\Rightarrow$ Simultaneous eigenstates of $\mathcal{P}$ & T can be divided into those where $\mathcal{P}=1$ (even sector) & $\mathcal{P}=-1$ (odd sector)

In the odd sector,

$$\sigma_3(N)\,\sigma_3(1) = 2i\,\psi_1(N)\,\psi_2(1)$$

$$= 2i\,\psi_1(N)\,\psi_2(1+N)$$

since $\psi_2(1+N) \equiv \psi_2(1)$

$\Rightarrow$ The periodic spin-spin interaction becomes

(odd) $$\boxed{\sum_{n=1}^{N}{}' \sigma_3(n)\,\sigma_3(n+1) = 2i \sum_{n=1}^{N} \psi_1(n)\,\psi_2(n+1)}$$

In the even sector, we want $\psi_2(N+1) = -\psi_2(1)$

in order to obtain $\sum_{n=1}^{N} \sigma_3(n)\sigma_3(n+1) = 2i\sum_{n=1}^{N} \psi_1(n)\,\psi_2(n+1)$

$\Rightarrow$ This means that fermions return to minus itself when we go around the loop & come back to the same point.

$\hookrightarrow$ just like spinors do after a rotation by 2π.

$\longrightarrow$ Need (ANTI) PERIODIC Boundary conditions for the fermion in the (even) odd sector with $P = (-1)\cdot 1$

Now need to characterize the sectors in the fermionic language. (not Majorana)

At site n, we first form a Dirac fermion:

$$\Psi(n) = \frac{\psi_1(n) + i\psi_2(n)}{\sqrt{2}}, \quad \Psi^\dagger(n) = \frac{\psi_1(n) - i\psi_2(n)}{\sqrt{2}}$$

$\hookrightarrow$ so $N_\Psi(n) = \Psi^\dagger(n)\Psi(n)$

$$= \frac{1 + 2i\psi_1(n)\psi_2(n)}{2} = \frac{1 - \sigma_1(n)}{2}.$$

If we now use $\sigma_1 = -i e^{i\frac{\pi}{2}\sigma_1} = e^{i\frac{\pi}{2}(\sigma_1 - 1)}$

we find that

$$\mathcal{P} = \prod_{\ell}^{N} \sigma_1(\ell) = e^{i\frac{\pi}{2}\sum_n(\sigma_1(n) - 1)}$$

$$= e^{-i\frac{\pi}{2}\sum_n\left(2\Psi^\dagger(n)\Psi(n)\right)}$$

$$= e^{-i\pi N_\Psi} = (-1)^{N_\Psi}$$

where $N_\Psi = \sum_{n=1}^{N} \Psi^\dagger(n)\,\Psi(n)$ = total fermion number associated w/ Dirac field Ψ

$\Rightarrow$ $\mathcal{P}$ is the fermion parity of the state.

$\mathcal{P} = -1$ @ odd N_Ψ

$\mathcal{P} = 1$ @ even N_Ψ.

⇒ ~~But note that~~ periodic solution in the even sector & anti-periodic solutions in the odd sector are physically irrelevant & should be discarded

↳ This is key to proving why, in the thermodynamic limit, the model has $\boxed{2}$ degenerate gnd states at low temp (T, x) and only $\boxed{1}$ in high-T state (disordered).

(d) <u>Solution by Fourier Transform</u>

The rest of the details here can be found in notes on "Quantum Ising Chain for beginners"

Reading notes

April 4, 2021

$\boxed{\text{Chapter 10: Gauge Theories}}$