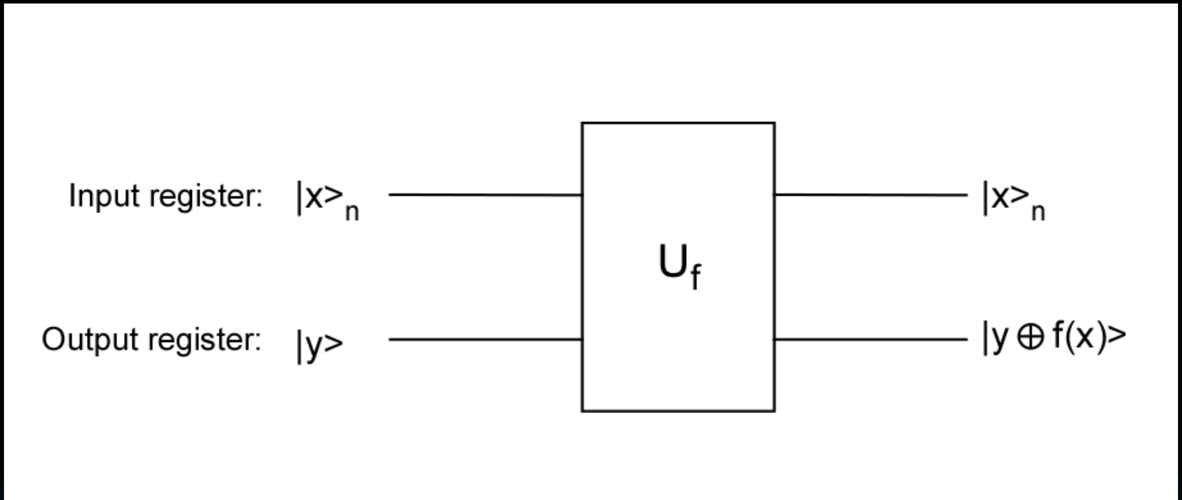
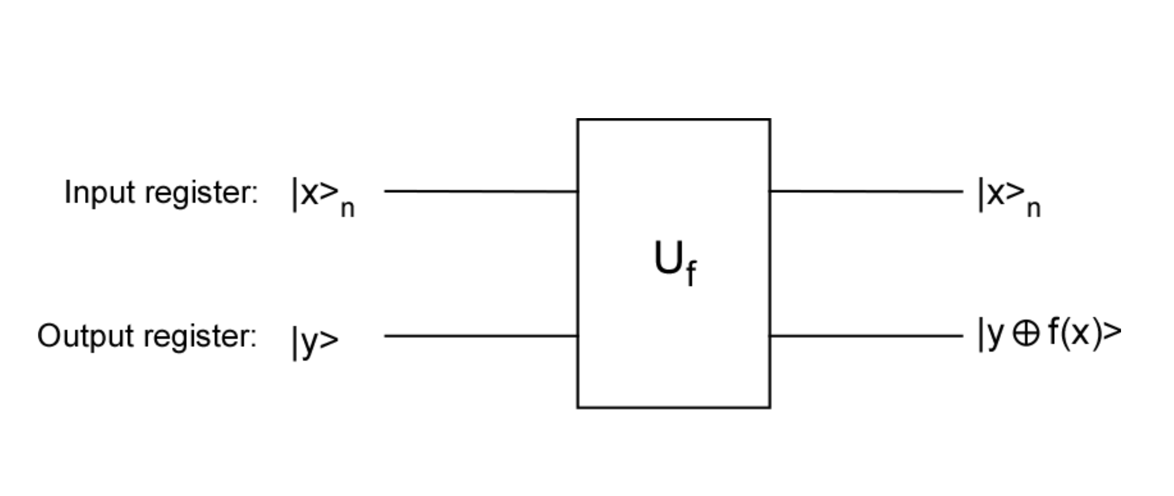
D. The Bernstein-Vazirani problem

Background: The Deutsch-Jozsa problem and the Bernstein-Vazirani try to uncover the behavior of an “oracle.”



Goal: To show how we can uncover the behavior of an oracle using quantum parallelism and phase kickback.

1. Deutsch-Jozsa and Bernstein-Vazirani

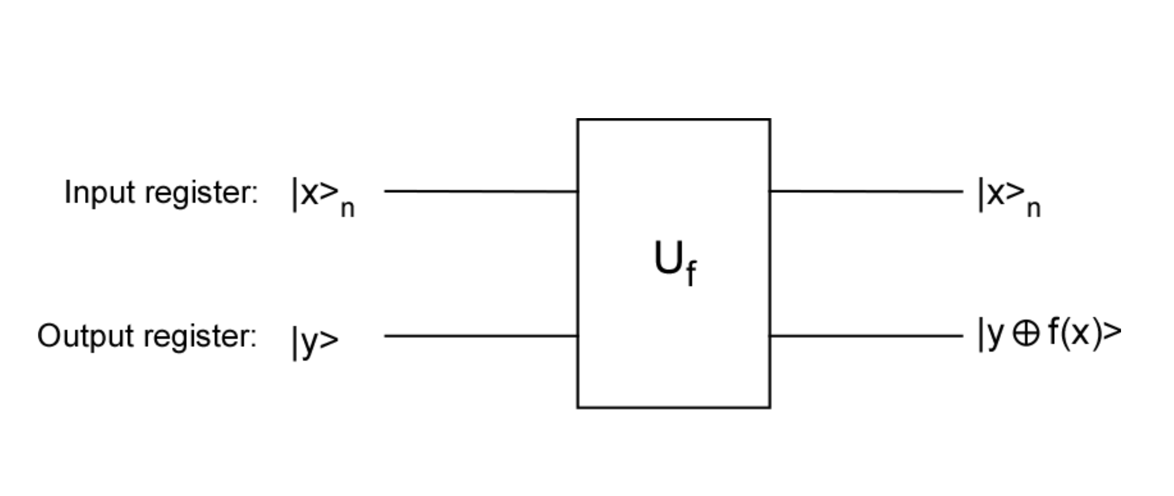


$$f : \{0,1\}^n \to \{0,1\}$$

Deutsch-Jozsa: $f(x)$ is either constant or balanced. Which is it?

Bernstein-Vazirani: $f(x) = a \cdot x$ for some a. What is a?

1. Deutsch-Jozsa and Bernstein-Vazirani



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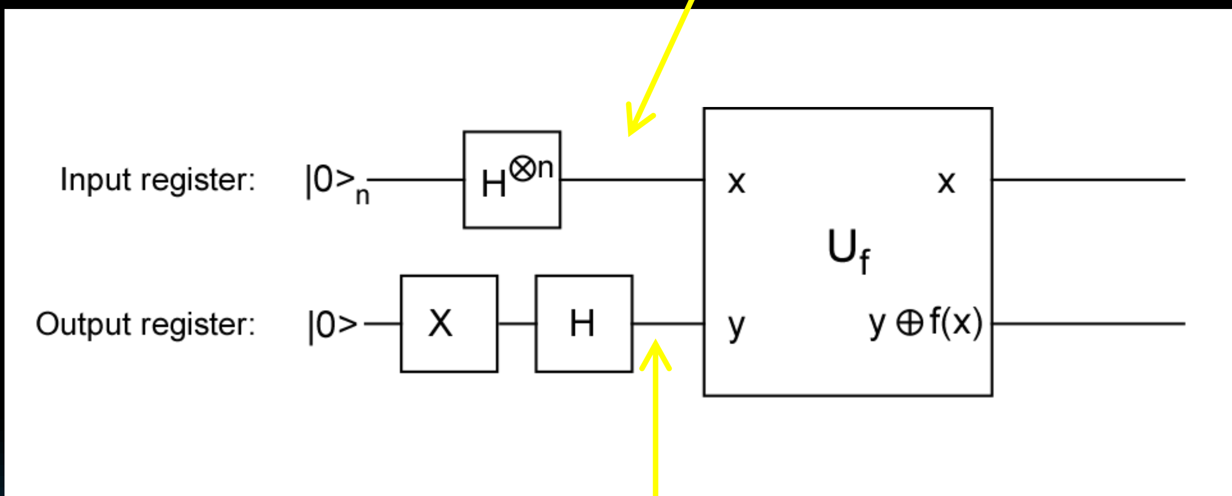
They are solved in ***exactly*** the same way!

2. The Bernstein-Vazirani problem

a. Quantum Parallelism and phase kickback

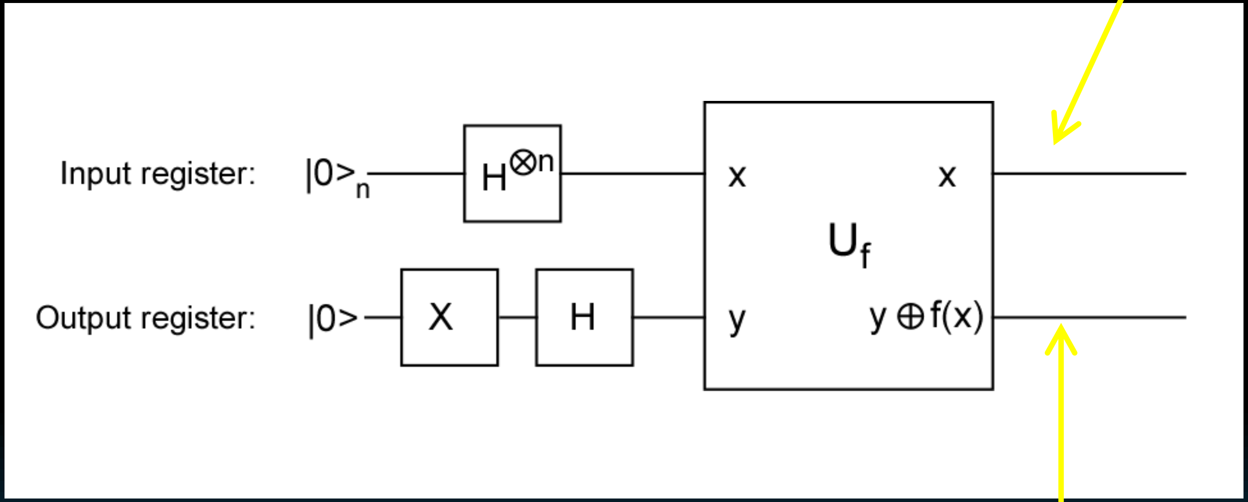
$$|\psi_i\rangle = \mathbf{H}^{\otimes n}|0\rangle_n = \frac{1}{\sqrt{2^n}} \sum_{0\leq x<2^n} |x\rangle_n$$



$$|\psi_o\rangle = \mathbf{HX}|0\rangle = \mathbf{H}|1\rangle = |-\rangle$$

b. After the oracle

$$|\psi_i\rangle = \frac{1}{\sqrt{2^n}} \sum_{0 \le x < 2^n} (-1)^{f(x)} |x\rangle_n$$
$$= \frac{1}{\sqrt{2^n}} \sum_{0 \le x < 2^n} (-1)^{a \cdot x} |x\rangle_n$$



$$|\psi_o\rangle = |-\rangle$$

c. What we learned earlier:

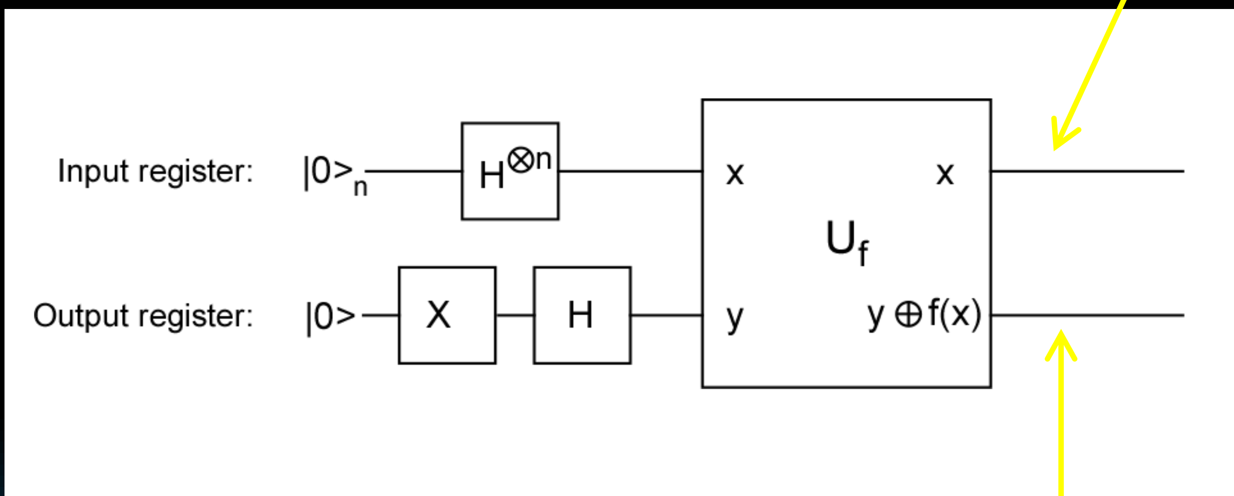
$$\begin{aligned}\mathbf{H}^{\otimes n}|x\rangle_n &= \mathbf{H}|x_{n-1}\rangle \ldots \mathbf{H}|x_1\rangle \mathbf{H}|x_0\rangle \\ &= \left\{\frac{|0\rangle + (-1)^{x_{n-1}}|1\rangle}{\sqrt{2}}\right\} \cdots \left\{\frac{|0\rangle + (-1)^{x_1}|1\rangle}{\sqrt{2}}\right\} \left\{\frac{|0\rangle + (-1)^{x_0}|1\rangle}{\sqrt{2}}\right\} \\ &= \frac{1}{\sqrt{2^n}} \sum_{0 \le y < 2^n} (-1)^{x \cdot y} |y\rangle_n\end{aligned}$$

In our current notation – and turning it around:

$$\begin{aligned}&\frac{1}{\sqrt{2^n}} \sum_{0 \le x < 2^n} (-1)^{a \cdot x} |x\rangle_n \\ &= \left\{\frac{|0\rangle + (-1)^{a_{n-1}}|1\rangle}{\sqrt{2}}\right\} \cdots \left\{\frac{|0\rangle + (-1)^{a_1}|1\rangle}{\sqrt{2}}\right\} \left\{\frac{|0\rangle + (-1)^{a_0}|1\rangle}{\sqrt{2}}\right\} \\ &= \mathbf{H}|a_{n-1}\rangle \ldots \mathbf{H}|a_1\rangle \mathbf{H}|a_0\rangle \\ &= \mathbf{H}^{\otimes n}|a\rangle_n\end{aligned}$$

d. Back to the oracle

$$|\psi_i\rangle = \mathbf{H}^{\otimes n}|a\rangle_n$$

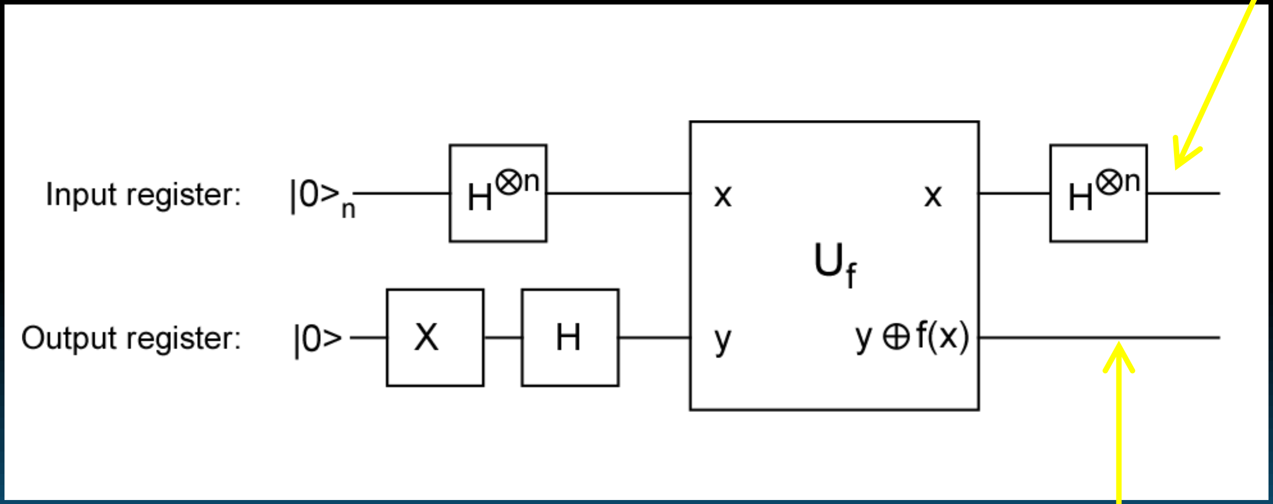


$$|\psi_o\rangle = |-\rangle$$

e. Manipulating the output to get an answer!

Hadamards are their own inverse!

$$\begin{aligned}|\psi_i\rangle &= \mathbf{H}^{\otimes n}\left\{\mathbf{H}^{\otimes n}|a\rangle_n\right\}\\ &= |a\rangle_n\end{aligned}$$



$$|\psi_o\rangle = |-\rangle$$