Spring, 2021

Physics 312: Physics of Fluids
Assignment #3 (Solutions)

Background Reading

Friday, Feb. 26: Kundu & Cohen 2.10 - 2.12, 3.7 - 3.9

Monday, Mar. 1: Tritton 6.4,
Kundu & Cohen 3.8, 3.10 - 3.12

Wednesday, Mar. 3: Kundu & Cohen 2.6, 4.5, 4.6

Informal Written Reflection

Due: Thursday, March 4 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, March 5 (in class)

1. *Mass* is a fundamental quantity and *density* is not. And yet, when talking about fluids, we will often find the concept of density to be more useful. Why do you think that is?

   (Hint: Think about dividing a volume of fluid in half. What changes? What does not change?. . .)

   Solution:

   Many fluids can be thought of as having uniform density. In such cases, *mass* depends on how much fluid you're dealing with, while *density* does not. Imagine a cup of milk. If you drink half of the milk, what remains takes up half as much space and weighs half as much. Dividing mass by volume, these factors cancel out.

2. *Force* is a fundamental quantity and *stress* is not. And yet, when talking about fluids, we will often find the concept of stress to be more useful. Why do you think that is?

   (Hint: Consider the example of shear stresses, as covered in lecture. Now think about changing the size of the material being sheared. . .)

   Solution:

   Consider the difference between shearing one cube of fluid and shearing two cubes side by side. With two cubes there is more resistance to deformation so, unless the force is doubled, the deformation rate will be cut in half. Dividing force by the surface area on which the force acts also explains why a sharp knife is more useful than a dull one. You could also construct similar argument involving isotropic compression and pressure, by the way!

3. The line integral of the fluid velocity around a closed curve,

$$\Gamma = \oint_C \mathbf{u} \cdot \mathrm{d}\mathbf{s},$$

   is known as the *circulation*. As Kundu and Cohen describe in section 3.8 and Tritton describes in section 6.4, circulation is closed related to *vorticity*. Refer to these sections as needed and work through Kundu and Cohen, Chapter 3, Problem 3.

   (Hint: Use polar coordinates. If you write $\mathrm{d}\mathbf{s} = (-\sin\theta, \cos\theta)\,\mathrm{d}\theta$, the line integral around a unit circle can be written as an integral over θ from 0 to 2π. Show that $\Gamma = -\pi a$. Then show that you get the same answer using Kundu and Cohen equation (3.18).)

Solution:

As suggested in the hint, plane polar coordinates make the circulation around a circular path much easier to compute:

$$\Gamma = \oint_C \mathbf{u} \cdot \mathrm{d}\mathbf{s} = \int_0^{2\pi} (ay, 0) \cdot (-\sin\theta, \cos\theta)\, \mathrm{d}\theta$$
$$= \int_0^{2\pi} (a\sin\theta, 0) \cdot (-\sin\theta, \cos\theta)\, \mathrm{d}\theta$$
$$= -a \int_0^{2\pi} \sin^2\theta\, \mathrm{d}\theta = -a\pi.$$

We can double check this result, the problem suggests, using Stokes' theorem:

$$\Gamma = \int_A \boldsymbol{\omega} \cdot \mathrm{d}\mathbf{A} = \int_A \omega_3\, \mathrm{d}A = \int_A (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})\, \mathrm{d}A$$
$$= -a \int_A \mathrm{d}A = -a\pi.$$

4. The relative motion of nearby fluid elements can be expressed in terms of a symmetric tensor e_{ij} and an antisymmetric tensor r_{ij}. In this problem, you will take a closer look at these tensors and discuss how we interpret these pieces geometrically...

   (Hint: Since we highlighted these interpretations in class, you should refer to your class notes and section 3.9 in Kundu and Cohen for guidance on the steps.)

   (a) Consider a small spherical element of fluid and explain how we know that $e_{ij}\, \mathrm{d}x_j$ deforms this sphere into an ellipsoid.

   (b) In class, we decomposed $e_{ij}\, \mathrm{d}x_j$ into two pieces,

   $$e_{ij}\, \mathrm{d}x_j = \frac{1}{3} e_{kk}\delta_{ij}\, \mathrm{d}x_j + (e_{ij} - \frac{1}{3} e_{kk}\delta_{ij})\, \mathrm{d}x_j$$

   and interpreted these two pieces. Explain, using the tensor properties of each terms, how we arrived at these interpretations.

(c) Using the definition of vorticity ($\nabla \times \mathbf{u}$) and the epsilon-delta relation from our first problem set, confirm that

$$r_{ij} = -\epsilon_{ijk}\omega_k.$$

How do we interpret this result geometrically?

Solution:

(a) This is described pretty thoroughly in Kundu and Cohen, section 3.9. The basic idea is to consider the action of e_{ij} in a coordinate system for which e_{ij} is diagonal. In this coordinate system, the entries of e_{ij} are naturally associated with expansions or contractions along the (principal) coordinate axes. If these three diagonal entries are not equal, a sphere deforms into an ellipsoid instead of simply a sphere of a different size.

(b) The first term is proportional to δ_{ij} and therefore isotropic. Then, since the constant of proportionality contains $\nabla \cdot \mathbf{u}$, this term represents an isotropic expansion or compression. The second term can be interpreted along lines of your answer to (a) but, since the sum of the diagonal terms is zero, this term is volume-preserving.

(c) Plugging the definition of vorticity into the given expression for r_{ij}, we recover our original definition of tensor r_{ij}:

$$\begin{aligned} r_{ij} = -\epsilon_{ijk}\omega_k &= -\epsilon_{ijk}\epsilon_{klm}\frac{\partial u_m}{\partial x_l} \\ &= -(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial u_m}{\partial x_l} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}. \end{aligned}$$

Following the steps discussed in class and in the text, you should be able to convince yourself that $\frac{1}{2}r_{ij}\,\mathrm{d}x_j$ creates a rigid rotation. These steps explicitly connects vorticity to the rotation of infinitesimal fluid elements.

5. Consider a velocity field $\mathbf{u}(\mathbf{r}, t)$ for which both $\nabla \cdot \mathbf{u}$ and $\nabla \times \mathbf{u}$ equal zero everywhere...

   (a) Thinking back to our discussion of relative motion near a point, what can you say about the local description of this flow (in terms of translation, rotation, etc.).

   (b) Show that $\mathbf{u}$ can be written as the gradient of a potential function, $\mathbf{u} = \nabla\phi$, where ϕ solves Laplace's equation,

$$\nabla^2\phi = 0.$$

   Note the analogy with Maxwell's equations (which also lead to Laplace's equation in the case of electrostatics)!

Solution:

   (a) $\nabla \times \mathbf{u} = 0$ means zero vorticity and, therefore, the velocity gradient tensor must be symmetric. Since $\nabla \cdot \mathbf{u} = 0$, the isotropic part of e_{ij} vanishes. Thus, a small spherical fluid element may change position and deform into an ellipsoid but will not change its volume or orientation.

   (b) $\mathbf{u} = \nabla\phi$ automatically satisfies $\nabla \times \mathbf{u} = 0$, since the curl of a gradient vanishes. Then, plugging $\mathbf{u} = \nabla\phi$ into $\nabla \cdot \mathbf{u} = 0$ gives the answer, since $\nabla^2 = \nabla \cdot \nabla$. In electrostatics, we write the electric field in terms of the gradient of a potential function and, in the same way, Gauss' law leads us to Laplace's equation. Electrodynamics courses typically spend a good deal of time on the various mathematical tools and tricks used to solve Laplace's equation; many of these techniques have similar applications in fluid dynamics (!)