Classical Mechanics III 8.(3)09
Assignment 7: Solutions

November 7, 2021

1. Perturbation Theory for Two Springs [10 points]

(a) [3 points] Letting x be the displacement of the mass from equilibrium, we have $x = a\tan\theta$ and $\dot{x} = a\dot{\theta}\sec^2\theta$. Hence $T = \frac{m}{2}\dot{x}^2 = \frac{m}{2}a^2\dot{\theta}^2\sec^4\theta$. The stretched length of the spring is $s = a\sec\theta$, so $V = 2\frac{k}{2}(s-b)^2 = k(a\sec\theta - b)^2$. Hence

$$L = T - V = \frac{m}{2}a^2\dot{\theta}^2\sec^4\theta - k(a\sec\theta - b)^2.$$

This immediately gives

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta}\sec^4\theta$$

and

$$H = T + V = \frac{p_\theta^2}{2ma^2\sec^4\theta} + k(a\sec\theta - b)^2.$$

(b) [2 points] Recalling the Taylor expansion $\sec\theta = \frac{1}{\cos\theta} = 1 + \frac{\theta^2}{2} + \frac{5}{24}\theta^4 + O(\theta^4)$, we have that

$$\begin{aligned}
V = k(a\sec\theta - b)^2 &= k((a-b) + a(\frac{\theta^2}{2} + \frac{5}{24}\theta^4))^2 + O(\theta^6) \\
&= k(a-b)^2 + 2ka(a-b)(\frac{\theta^2}{2} + \frac{5}{24}\theta^4) + ka^2\theta^4(\frac{1}{2} + \frac{5}{24}\theta^2)^2 + O(\theta^6) \\
&= k(a-b)^2 + ka(a-b)\theta^2 + ka(\frac{2}{3}a - \frac{5}{12}b)\theta^4 + O(\theta^6).
\end{aligned}$$

The kinetic energy is much easier to deal with: since $\cos\theta = 1 - \frac{\theta^2}{2} + O(\theta^4)$,

$$\begin{aligned}
T = \frac{p_\theta^2}{2ma^2}\cos^4\theta &= \frac{p_\theta^2}{2ma^2}(1 - \frac{\theta^2}{2})^4 + O(p_\theta^2\theta^4) \\
&= \frac{p_\theta^2}{2ma^2} - \frac{p_\theta^2\theta^2}{ma^2}.
\end{aligned}$$

Therefore ignoring terms of order $O(\theta^6, p_\theta^2\theta^4)$ and above, we get

$$H = T + V = H_0 + \Delta H$$

where

$$H_0 = \frac{p_\theta^2}{2ma^2} + ka(a-b)\theta^2 + k(a-b)^2 = \frac{p_\theta^2}{2I} + \frac{1}{2}I\omega^2\theta^2 + k(a-b)^2$$

is the Hamiltonian for a harmonic oscillator with inertia $I = ma^2$ and angular frequency $\omega = \sqrt{\frac{2k(a-b)}{ma}}$ (the constant term $k(a-b)^2$ may be dropped), and

$$\Delta H = -\frac{p_\theta^2\theta^2}{ma^2} + ka(\frac{2}{3}a - \frac{5}{12}b)\theta^4$$

is the next order correction.

(c) [5 points] As was done in class, we can take as canonical coordinates the action variable J and the phase variable β of the unperturbed Hamiltonian H_0, defined by

$$\begin{aligned} \theta &= \sqrt{\frac{J}{\pi I\omega}}\sin(2\pi(\nu t + \beta)) \\ p_\theta &= \sqrt{\frac{IJ\omega}{\pi}}\cos(2\pi(\nu t + \beta)) \end{aligned}$$

where as before,

$$I = ma^2,\ \omega = \sqrt{\frac{2k(a-b)}{ma}},\ \nu = \frac{\omega}{2\pi}.$$

For the unperturbed Hamiltonian, J and β are constant and the transformed Hamiltonian $K_0(J,\beta)$ is identically zero. However with the addition of the perturbation term ΔH we have $K = K_0 + \Delta H = \Delta H$, and hence Hamilton's equations give

$$\dot\beta = \frac{\partial\Delta H(J,\beta,t)}{\partial J}, \qquad \dot J = -\frac{\partial\Delta H(J,\beta,t)}{\partial\beta}.$$

For this problem, using the defining relations of (J,β) we have

$$\Delta H = -\frac{J^2}{\pi^2 I}\sin^2(2\pi(\nu t+\beta))\cos^2(2\pi(\nu t+\beta)) + ka(\frac{2}{3}a - \frac{5}{12}b)\frac{J^2}{\pi^2 I^2\omega^2}\sin^4(2\pi(\nu t+\beta)).$$

Hence to first order (that is, substituting the original values $\beta^{(0)}$ and $J^{(0)} = E/\nu$ in, after taking derivatives), we have

$$\dot\beta^{(1)} = \left.\frac{\partial\Delta H}{\partial J}\right|_0 = -\frac{2J^{(0)}}{\pi^2 I}\sin^2(2\pi(\nu t+\beta))\cos^2(2\pi(\nu t+\beta^{(0)})) + ka(\frac{2}{3}a - \frac{5}{12}b)\frac{2J}{\pi^2 I^2\omega^2}\sin^4(2\pi(\nu t+\beta^{(0)}))$$

Now using the integrals $\frac{1}{2\pi}\int_0^{2\pi}\sin^4\theta d\theta = \frac{3}{8}$ and $\frac{1}{2\pi}\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta = \frac{1}{8}$, we can immediately take the time average to calculate the secular change:

$$\begin{aligned}\overline{\dot{\beta}^{(1)}} &= \frac{1}{\tau}\int_0^\tau \dot{\beta}^{(1)}(t)dt \\ &= -\frac{J^{(0)}}{4\pi^2 I} + ka(\frac{2}{3}a - \frac{5}{12}b)\frac{3J^{(0)}}{4\pi^2 I^2\omega^2} \\ &= \frac{J^{(0)}}{4\pi^2 I}\left[-1 + 3ka(\frac{2}{3}a - \frac{5}{12}b)\frac{1}{ma^2}\frac{ma}{2k(a-b)}\right] \\ &= \frac{J^{(0)}}{4\pi^2 I}\frac{3b}{8(a-b)},\end{aligned}$$

where $J^{(0)} = E/\nu = 2\pi E/\omega$, $I = ma^2$, and $\omega = \sqrt{\frac{2k(a-b)}{ma}}$. For the first order correction to J, we have

$$\dot{J}^{(1)} = -\left.\frac{\partial \Delta H}{\partial \beta}\right|_0 = -\left.\frac{1}{\nu}\frac{\partial \Delta H}{\partial t}\right|_0$$

since β only appears in ΔH in the combination $\nu t + \beta$. Therefore the secular change is

$$\begin{aligned}\overline{\dot{J}^{(1)}} &= -\frac{1}{\tau}\int_0^\tau \dot{J}^{(1)}(t)dt \\ &= -\frac{1}{\nu\tau}\int_0^\tau \frac{\partial \Delta H}{\partial t}dt \\ &= \Delta H(J^{(0)}, \beta^{(0)}, 0) - \Delta H(J^{(0)}, \beta^{(0)}, \tau) \\ &= 0,\end{aligned}$$

as desired.

Aside: Looking at these steps again, we see that

$$\frac{1}{\tau}\int_0^\tau \dot{J}(t)dt = \frac{1}{\nu\tau}\Big[\Delta H(J(0), \beta(0), \tau) - \Delta H(J(\tau), \beta(\tau), 0)\Big]$$

holds to all orders, and since the motion is periodic we have in fact $\overline{\dot{J}} = \frac{1}{\tau}\int_0^\tau \dot{J}(t)dt = 0$, exactly. Here τ is the exact period including ΔH, so $\nu\tau \neq 1$ for higher orders. We have essentially shown that if J is the action variable for periodic motion under a 1D time-independent Hamiltonian H, then under a time-independent perturbation ΔH, the secular rate of change is zero: $\overline{\dot{J}} = 0$ (assuming the motion is still periodic). Physically, this means J does not grow or decay with time, which should be clear from energy conservation since J is a measure of the amplitude of the oscillations.

2. Second Order Perturbation Theory [18 points]

(a) [2 points] Using $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + O(\theta^8)$, we have (neglecting terms of order $O(\theta^8)$)

$$\begin{aligned}H &= -I\omega^2 + \frac{p_\theta^2}{2I} + \frac{I\omega^2\theta^2}{2} - \frac{I\omega^2\theta^4}{24} + \frac{I\omega^2\theta^6}{720} \\ &= -I\omega^2 + H_0 + \Delta H_1 + \Delta H_2\end{aligned}$$

where

$$H_0 = \frac{p_\theta^2}{2I} + \frac{I\omega^2\theta^2}{2}$$

$$\Delta H_1 = -\frac{I\omega^2\theta^4}{24}$$

$$\Delta H_2 = \frac{I\omega^2\theta^6}{720}.$$

(b) [4 points] Using the same change of variables into (J, β) as in the previous problem ($\nu = \omega/2\pi$),

$$\begin{aligned} \theta &= \sqrt{\frac{J}{\pi I\omega}}\sin(2\pi(\nu t + \beta)) \\ p_\theta &= \sqrt{\frac{IJ\omega}{\pi}}\cos(2\pi(\nu t + \beta)), \end{aligned}$$

we get

$$\Delta H_1 = -\frac{J^2}{24\pi^2 I}\sin^4(2\pi(\nu t + \beta)).$$

Therefore using Hamilton's equations,

$$\dot{\beta}^{(1)} = \left.\frac{\partial \Delta H_1}{\partial J}\right|_0 = -\frac{J^{(0)}}{12\pi^2 I}\sin^4(2\pi(\nu t + \beta^{(0)}))$$

and integrating with respect to time,

$$\begin{aligned} \beta^{(1)}(t) &= \int \dot{\beta}^{(1)}(t')dt' \\ &= \beta^{(0)} - \frac{J^{(0)}}{12\pi^2 I}\left[\frac{3}{8}t - \frac{\sin(4\pi(\nu t + \beta^{(0)}))}{4\omega} + \frac{\sin(8\pi(\nu t + \beta^{(0)}))}{32\omega}\right] \\ &= \beta^{(0)} + \nu_1 t + \beta_1(t) \end{aligned}$$

where

$$\nu_1 = \overline{\dot{\beta}^{(1)}} = -\frac{J^{(0)}}{32\pi^2 I}$$

is the first-order change to the velocity, and

$$\beta_1 = \frac{J^{(0)}}{384\pi^2 I\omega}\left[8\sin(4\pi(\nu t + \beta^{(0)})) - \sin(8\pi(\nu t + \beta^{(0)}))\right]$$

is the periodic term. (We've used $\omega = 2\pi\nu$, and set the constant of integration to $\beta^{(0)}$, the zeroth order constant value.)

Similarly,

$$\dot{J}^{(1)} = -\left.\frac{\partial \Delta H_1}{\partial \beta}\right|_0 = \frac{(J^{(0)})^2}{3\pi I}\sin^3(2\pi(\nu t + \beta^{(0)}))\cos(2\pi(\nu t + \beta^{(0)}))$$

so after integrating,

$$J^{(1)}(t) = J^{(0)} + J_1(t)$$

where

$$J_1(t) = \frac{(J^{(0)})^2}{12\pi I\omega}\sin^4(2\pi(\nu t + \beta^{(0)})).$$

(Again, the result for $J^{(1)}$ can be found more efficiently if one notices $\dot{J}^{(1)} = -\frac{1}{\nu}\frac{\partial \Delta H_1}{\partial t}\Big|_0$ and hence $J^{(1)}(t) = J^{(0)} - \frac{1}{\nu}\Delta H_1(J^{(0)}, \beta^{(0)}, t)$.)

Aside: What is the physical significance of this choice of $J^{(0)}$? Well, our choice corresponds to $J(t) = J^{(0)}$ when $\nu t + \beta = 0$, or $\theta = 0$. Hence $(p_\theta)_{max} = \sqrt{\frac{IJ_0\omega}{\pi}}$. At this point all the energy (that is, the energy ignoring the constant $-I\omega^2$ term) is in the momentum, so $E = \frac{(p_\theta)^2_{max}}{2I} = \nu J_0$. Thus in terms of initial conditions, $J^{(0)}$ is the ratio of the total energy to the zeroth order frequency: $J^{(0)} = E/\nu$.

Note: I've kept the $\beta^{(0)}$ term above explicitly to keep track of the meaning of $J^{(0)}$; we see above that it's not the value of J at $t = 0$ in general, but rather the non-periodic portion of $J^{(1)}(t)$. In the following I will set $\beta^{(0)} = 0$.

(c) [4 points] Substituting θ for J and β, we get

$$\Delta H_2 = \frac{J^3}{720\pi^3 I^2\omega}\sin^6(2\pi(\nu t + \beta)).$$

Thus we can immediately use Hamilton's equations:

$$\dot{\beta}_b^{(2)} = \frac{\partial \Delta H_2}{\partial J}\bigg|_0 = \frac{3(J^{(0)})^2}{720\pi^3 I^2\omega}\sin^6(2\pi\nu t)$$

or after taking the time average (and using $\frac{1}{2\pi}\int_0^{2\pi}\sin^6\theta d\theta = \frac{5}{16}$),

$$\overline{\dot{\beta}_b^{(2)}} = \frac{(J^{(0)})^2}{768\pi^3 I^2\omega}.$$

Similarly,

$$\dot{J}_b^{(2)} = -\frac{\partial \Delta H_2}{\partial \beta}\bigg|_0 = -\frac{(J^{(0)})^2}{60\pi^2 I^2\omega}\sin^5(2\pi\nu t)\cos(2\pi\nu t)$$

which averages to zero,

$$\overline{\dot{J}_b^{(2)}} = 0,$$

as expected (see discussion at end of problem 1).

(d) [8 points] Just as in (b), we take derivatives of ΔH_1 and substitute in our lower-order result:

$$\begin{aligned}
\dot{\beta}_a^{(2)} = \frac{\partial \Delta H_1}{\partial J}\bigg|_1 &= -\frac{J^{(1)}}{12\pi^2 I}\sin^4(2\pi(\nu t + \beta^{(1)})) \\
\dot{J}_a^{(2)} = -\frac{\partial \Delta H_1}{\partial \beta}\bigg|_1 &= \frac{(J^{(1)})^2}{3\pi I}\sin^3(2\pi(\nu t + \beta^{(1)}))\cos(2\pi(\nu t + \beta^{(1)})).
\end{aligned}$$

Now, however, $J^{(1)}$ and $\beta^{(1)}$ are functions of time. Moreover (and more subtle), the correct period to average over is no longer the original period $\tau = 1/\nu$, but $\tau^{(1)} = 1/\nu^{(1)}$, where

$$\nu^{(1)} = \nu + \nu_1 = \nu - \frac{J^{(0)}}{32\pi^2 I}$$

is the frequency correct to first order (we've incorporated the secular change to β). This can be seen by reexpressing the phase:

$$\nu t + \beta^{(1)} = \nu t + (\beta^{(0)} + \nu_1 t + \beta_1(t)) = \nu^{(1)} t + \beta^{(0)} + \beta_1(t)$$

and since $\beta_1(t)$ is periodic, the period must be $\tau^{(1)} = 1/\nu^{(1)}$, as claimed. Now (setting $\beta^{(0)} = 0$),

$$\dot{\beta}_a^{(2)} = -\frac{J^{(0)} + J_1(t)}{12\pi^2 I} \sin^4(2\pi(\nu^{(1)} t + \beta_1(t))).$$

Expanding this out, and only keeping terms to second order in $J^{(0)}$ (so only linear order in $J_1(t)$ and $\beta_1(t)$), we have

$$\begin{aligned}
\dot{\beta}_a^{(2)} &= -\frac{1}{12\pi^2 I}\left[J^{(0)} \sin^4(2\pi\nu^{(1)}t) + J_1(t)\sin^4(2\pi\nu^{(1)}t) + 8\pi J^{(0)}\beta_1(t)\cos(2\pi\nu^{(1)}t)\sin^3(2\pi\nu^{(1)}t)\right] \\
&= -\frac{J^{(0)}\sin^4(2\pi\nu^{(1)}t)}{12\pi^2 I} - \frac{1}{12\pi^2 I}\frac{(J^{(0)})^2\sin^8(2\pi\nu^{(1)}t)}{12\pi I\omega} - \\
&\quad -\frac{8\pi J^{(0)}}{12\pi^2 I}\frac{J^{(0)}}{384\pi^2 I}\left[8\sin(4\pi\nu t) - \sin(8\pi\nu t)\right]\cos(2\pi\nu^{(1)}t)\sin^3(2\pi\nu^{(1)}t)
\end{aligned}$$

Now setting the phase to be $\phi = 2\pi\nu^{(1)}t$, we can take the average by $\frac{1}{\tau^{(1)}}\int_0^{\tau^{(1)}} dt = \int_0^{\tau^{(1)}} \nu^{(1)} dt = \frac{1}{2\pi}\int_0^{2\pi} d\phi$, i.e. we can average over ϕ instead. Note also that the difference between ν and $\nu^{(1)}$ is $\nu_1 = O(J^{(0)})$, so we can set $2\pi\nu t = \phi$ as well with an error of only $O((J^{(0)})^3)$ in $\dot{\beta}_a^{(2)}$. We now use the averages

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\sin^4\phi = \frac{3}{8}; \qquad \int_0^{2\pi}\frac{d\phi}{2\pi}\sin^8\phi = \frac{35}{128}$$

$$\int_0^{2\pi}\frac{d\phi}{2\pi}\sin(2\phi)\cos\phi\sin^3\phi = \frac{1}{8}; \qquad \int_0^{2\pi}\frac{d\phi}{2\pi}\sin(4\phi)\cos\phi\sin^3\phi = -\frac{1}{16}$$

to obtain

$$\begin{aligned}
\overline{\dot{\beta}_a^{(2)}} &= -\frac{J^{(0)}}{32\pi^2 I} - \frac{(J^{(0)})^2}{\pi^3 I^2\omega}\frac{35}{18432} - \frac{(J^{(0)})^2}{\pi^3 I^2\omega}\frac{1}{576} - \frac{(J^{(0)})^2}{\pi^3 I^2\omega}\frac{1}{9216} \\
&= -\frac{J^{(0)}}{32\pi^2 I} - \frac{23(J^{(0)})^2}{6144\pi^3 I^2\omega}
\end{aligned}$$

Summing this up with our result from (c), we get, for the total secular rate of change of $\beta^{(2)}$,

$$\overline{\dot{\beta}^{(2)}} = \overline{\dot{\beta}_a^{(2)}} + \overline{\dot{\beta}_b^{(2)}} = -\frac{J^{(0)}}{32\pi^2 I} - \frac{5(J^{(0)})^2}{2048\pi^3 I^2\omega}.$$

Now to work with J. Recall that we have

$$\dot{J}_a^{(2)} = \frac{(J^{(0)} + J_1(t))^2}{3\pi I}\sin^3(2\pi(\nu^{(1)}t + \beta_1(t)))\cos(2\pi(\nu^{(1)}t + \beta_1(t))).$$

Once again expanding to linear order in $J_1(t)$ and $\beta_1(t)$, we have

$$\begin{aligned}
\dot{J}_a^{(2)} &= \frac{(J^{(0)})^2}{3\pi I}\cos(2\pi\nu^{(1)}t)\sin^3(2\pi\nu^{(1)}t) + \frac{2J^{(0)}J_1(t)}{3\pi I}\cos(2\pi\nu^{(1)}t)\sin^3(2\pi\nu^{(1)}t) \\
&\quad + \frac{2(J^{(0)})^2}{I}\beta_1(t)\cos^2(2\pi\nu^{(1)}t)\sin^2(2\pi\nu^{(1)}t) - \frac{2(J^{(0)})^2}{3I}\beta_1(t)\sin^4(2\pi\nu^{(1)}t) \\
&= \frac{J^{(0)}}{3\pi I}\left[J^{(0)} + 2J_1(t)\right]\cos\phi\sin^3\phi + \frac{2(J^{(0)})^2}{I}\beta_1(t)\left[\cos^2\phi\sin^2\phi - \frac{1}{3}\sin^4\phi\right]
\end{aligned}$$

where $J_1 = \frac{(J^{(0)})^2}{12\pi I\omega}\sin^4\phi$ and $\beta_1 = \frac{J^{(0)}}{384\pi^2 I}\left[8\sin(2\phi) - \sin(4\phi)\right]$ (making again the approximation $2\pi\nu t \approx \phi = 2\pi\nu^{(1)}t$). Now notice $\cos\phi\sin^3\phi$ is an odd function in ϕ while $J^{(0)} + 2J_1$ is even, and so the first term vanishes after averaging; similarly β_1 is odd while $\cos^2\phi\sin^2\phi - \frac{1}{3}\sin^4\phi$ is even, so the second term vanishes as well. Therefore

$$\overline{\dot{J}_b^{(2)}} = 0$$

and

$$\overline{\dot{J}^{(2)}} = \overline{\dot{J}_a^{(2)}} + \overline{\dot{J}_b^{(2)}} = 0$$

as expected.

Aside: We can use our discussion at the end of (b), and our result for $\overline{\dot{\beta}^{(2)}}$ above, to derive the frequency as a function of the amplitude. Specifically, let θ_0 be the maximum angle the pendulum makes with the horizontal. Then by the discussion at the end of (b), we know that

$$J^{(0)} = \frac{E}{\nu} = 2\pi I\omega(1 - \cos\theta_0) = 2\pi I\omega\left(\frac{\theta_0^2}{2} - \frac{\theta_0^4}{24} + O(\theta_0^6)\right)$$

Inserting this into our result for $\overline{\dot{\beta}^{(2)}}$, the frequency of the pendulum is, with an error of $O(\theta_0^6)$, (remember that $\nu = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$)

$$\nu^{(2)} = \nu + \overline{\dot{\beta}^{(2)}} = \nu\left[1 - \frac{\theta_0^2}{16} + \frac{\theta_0^4}{3072}\right]$$

Alternatively, if we take the inverse we immediately get that the period is (again up to fourth-order in θ_0)

$$\tau^{(2)} = \frac{1}{\nu^{(2)}} = \tau\left[1 + \frac{\theta_0^2}{16} + \frac{11\theta_0^4}{3072}\right]$$

a result which matches the literature.

3. Fluid Siphon Producing a Jet [18 points]

(a) [3 points] We apply Bernoulli's equation to a streamline connecting the top of reservoir to the siphon exit. At the top of the reservoir the pressure is the atmospheric pressure p_{atm} and the speed is zero (since the reservoir is infinitely large). At the exit the pressure is also the atmospheric pressure p_{atm} while the speed of the fluid is v_e. The height difference is H_1, and hence by Bernoulli's equation

$$g\rho H_1 + p_a = \frac{\rho v_e^2}{2} + p_a$$

or

$$v_e = \sqrt{2gH_1}.$$

(b) [3 points] Since the siphon has constant area, by continuity $A_T v_{top} = A_T v_e$, so the speed at the top of the siphon is $v_{top} = v_e = \sqrt{2gH_1}$. The pressure can be obtained through Bernoulli's equation (we apply this to the top of the siphon and to the siphon exit):

$$\frac{\rho v_e^2}{2} + g\rho(H_1 + H_2) + p_{top} = \frac{\rho v_e^2}{2} + p_a$$

so the pressure at the top is

$$p_{top} = p_a - \rho g(H_1 + H_2).$$

(c) [5 points] By energy conservation (or Bernoulli's equation), the velocity of the fluid as a function of the height, $v(y)$, is determined from

$$\frac{v(y)^2}{2} + gy = \frac{v_e^2}{2}$$

or

$$v(y) = \sqrt{v_e^2 - 2gy} = \sqrt{2g}\,\sqrt{H_1 - y}.$$

Continuity gives $v(y)A(y) = v_e A_t$, or

$$A(y) = \frac{v_e A_t}{v(y)} = A_t \sqrt{\frac{H_1}{H_1 - y}}.$$

(d) [2 points] The horizontal velocity $v_x(y)$ remains constant (since there's no horizontal force): $v_x(y) = v_e \cos\theta$. Therefore the vertical velocity is

$$v_y(y) = \pm\sqrt{v(y)^2 - v_x(y)^2} = \pm\sqrt{v_e^2 \sin^2\theta - 2gy}.$$

At the top of the jet the vertical velocity is zero: $v_y(H_3) = 0$, or

$$H_3 = \frac{v_e^2 \sin^2\theta}{2g} = H_1 \sin^2\theta.$$

(e) [5 points] This works just like finding the trajectory of a projectile under uniform gravity. On the pathline $\frac{dx}{dt} = v_x(y)$, $\frac{dy}{dt} = v_y(y)$, so

$$\frac{dy}{dx} = \frac{v_y(y)}{v_x(y)} = \frac{\pm\sqrt{v_e^2 \sin^2\theta - 2gy}}{v_e \cos\theta}$$

or

$$\pm\frac{dy}{\sqrt{v_e^2 \sin^2\theta - 2gy}} = \frac{dx}{v_e \cos\theta}$$

which integrates to (assuming the siphon exit is the origin)

$$\frac{v_e \sin\theta \mp \sqrt{v_e^2 \sin^2\theta - 2gy}}{g} = \frac{x}{v_e \cos\theta}$$

or after rearranging,

$$y = x\tan\theta - \frac{gx^2}{2v_e^2\cos^2\theta} = x\tan\theta - \frac{x^2}{4H_1\cos^2\theta}.$$

4. Fluid Angular Momentum and a Vortex without Vorticity [14 points]

(a) [5 points] Recall from lecture that

$$\frac{\partial(\rho\vec{v})}{\partial t} + \vec{\nabla}\cdot\hat{T} = \vec{f}, \qquad T_{ij} = \delta_{ij}p + v_i v_j \rho$$

where usually $\vec{f} = -\rho g\hat{z}$. Taking the cross product with $\vec{r}$ on the left (and using $\vec{r}\times(\rho\vec{v}) = \vec{\ell}$,

$$\frac{\partial\vec{\ell}}{\partial t} + \vec{r}\times(\vec{\nabla}\cdot\hat{T}) = \vec{\tau}$$

where $\vec{\tau} = \vec{r}\times\vec{f}$ is the torque per fluid volume. Let us expand the term $\vec{r}\times(\vec{\nabla}\cdot\hat{T})$ in components:

$$\begin{aligned}
(\vec{r}\times(\vec{\nabla}\cdot\hat{T}))_i &= \varepsilon_{ijk}x_j\frac{\partial}{\partial x_m}T_{mk} \\
&= \varepsilon_{ijk}x_j\frac{\partial}{\partial x_m}(\delta_{km}p + v_k v_m\rho) \\
&= \varepsilon_{ijk}\frac{\partial}{\partial x_m}[x_j(\delta_{km}p + v_k v_m\rho)] - \varepsilon_{ijk}\delta_{jm}(\delta_{km}p + v_k v_m\rho)
\end{aligned}$$

The second term is zero however, since $\varepsilon_{ijk}\delta_{jm}\delta_{km} = \varepsilon_{ijj} = 0$ and $\varepsilon_{ijk}\delta_{jm}v_k v_m = \epsilon_{ijk}v_j v_k = 0$. Hence

$$\begin{aligned}
(\vec{r}\times(\vec{\nabla}\cdot\hat{T}))_i &= \varepsilon_{ijk}\frac{\partial}{\partial x_m}[x_j(\delta_{km}p + v_k v_m\rho)] \\
&= \frac{\partial}{\partial x_m}[\varepsilon_{ijm}x_j p + \varepsilon_{ijk}x_j v_k v_m\rho] \\
&= \frac{\partial}{\partial x_m}[\varepsilon_{ijm}x_j p + v_m\ell_i] \\
&= \frac{\partial}{\partial x_m}J_{mi}
\end{aligned}$$

where the source term is given by

$$J_{mi} = \varepsilon_{mik} x_k p + v_m \ell_i.$$

(Note J_{ij} is not symmetric.) With this, we can write

$$\frac{\partial \ell_i}{\partial t} + \frac{\partial}{\partial x_m} J_{mi} = \tau_i.$$

(b) [4 points] By Stoke's theorem, if we take a curve in the xy-plane then

$$\oint_C \vec{v} \cdot d\vec{\ell} = \int_S \hat{z} \cdot (\vec{\nabla} \times \vec{v}) ds = 0$$

since we've assumed the flow is irrotational. Thus the line integral of the velocity around a closed curve (we call this the velocity circulation) is zero if the curve encloses a surface over which the fluid is irrotational.

By symmetry, the velocity is only dependent on the distance to the origin: $v_\theta = v_\theta(r)$ and $v_r = v_r(r)$. Let us take a curve formed by two arcs spanning angle ψ at $r = r_1$ and $r = r_2$, connected by straight-line segments. In polar coordinates, this curve starts at some point (r_1, θ), goes along an arc to $(r_1, \theta + \psi)$ (segment 1), goes along a radial segment to $(r_2, \theta + \psi)$ (segment 2), travels back an arc to (r_2, θ) (segment 3), and finally returns to (r_1, θ) along a radial segment (segment 4). If we integrate the velocity along this curve, then the segments 2 and 4 with integrals of $v_r(r)$ along the radial directions vanish since they have equal value but opposite sign. Thus we are left with

$$\int_1 v_\theta(r_1) r_1 d\theta - \int_3 v_\theta(r_2) r_2 d\theta = 0$$

Therefore we must have, for all r_1 and r_2,

$$r_1 v_\theta(r_1) = r_2 v_\theta(r_2)$$

or in otherwords $r v_\theta$ is constant (independent of r). Now since $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$, we have $v_\theta = r\dot{\theta}$, so

$$r^2 \dot{\theta} = \text{const.}$$

i.e. $\dot{\theta} \propto \frac{1}{r^2}$.

(c) [5 points] Expanding our result from (a) out,

$$\frac{\partial \ell_i}{\partial t} + \frac{\partial}{\partial x_m} (\varepsilon_{mik} x_k p + v_m \ell_i) = \tau_i$$

or

$$\frac{\partial \ell_i}{\partial t} + \varepsilon_{ikm} x_k \frac{\partial p}{\partial x_m} + \ell_i \frac{\partial v_m}{\partial x_m} + v_m \frac{\partial \ell_i}{\partial x_m} = \tau_i.$$

Written in vector form, this is

$$\frac{\partial \vec{\ell}}{\partial t} + \vec{r} \times (\vec{\nabla} p) + \vec{\ell}(\vec{\nabla} \cdot \vec{v}) + (\vec{v} \cdot \vec{\nabla})\vec{\ell} = \vec{\tau}.$$

We know that $\vec{\nabla} \cdot \vec{v} = 0$ for an incompressible fluid. Therefore this equation reduces to

$$\frac{\partial \vec{\ell}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{\ell} + \vec{r} \times (\vec{\nabla} p) = \frac{d\vec{\ell}}{dt} + \vec{r} \times (\vec{\nabla} p) = \vec{\tau}.$$

This holds for any incompressible fluid. Now specializing to our situation, we know that $\tau_z = 0$. Moreover $\hat{\theta} \cdot \vec{\nabla} p = 0$ (since by symmetry p is independent of θ, i.e. it is a function of only r and z), so

$$\hat{z} \cdot (\vec{r} \times (\vec{\nabla} p)) = \vec{r} \cdot ((\vec{\nabla} p) \times \hat{z}) = \vec{r}(\vec{\nabla} p)_\theta = 0.$$

Hence

$$\frac{d\ell_z}{dt} = 0$$

as desired: the vertical component of the angular momentum is constant along a pathline. Letting ℓ_z be this constant value, we have $\ell_z = r\hat{r} \times \rho r\dot{\theta}\hat{\theta} = \rho r^2\dot{\theta}$, so

$$\dot{\theta} = \frac{\ell_z}{\rho r^2}.$$