Some Topics in Measure Theory
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July 3, 2020

1 Introduction

This is a collection of concepts in measure theory that serves as my crash course to the subject. Read at your own risk!!!

2 Some Theorems on Subsets of $\mathbb{R}^n$

Theorem 2.1. *Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a countable union of disjoint open intervals.*

Theorem 2.2. *Every open subset $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.*

3 Exterior measure

Definition 3.1. *If $E \subseteq \mathbb{R}^d$, then the exterior measure of E is*

$$m_*(E) = \inf \sum_{n=1}^{\infty} |Q_j| \in [0, \infty]$$

where the infimum is taken over all countable coverings $E \subseteq \bigcup_{j=1}^{\infty} Q_j$ by closed cubes.

Proposition 3.1. (Monotonicity) If $E_1 \subset E_2$ then $m_*(E_1) \leq m_*(E_2)$.

Proposition 3.2. (Countable Sub-additivity) If $E = \bigcup_{j=1}^{\infty} E_j$ then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

Proposition 3.3. If $E \subseteq \mathbb{R}^d$, $m_*(E) \leq \inf m_*(\mathcal{O})$ where the infimum is taken over all open $\mathcal{O} \supseteq E$.

Proposition 3.4. If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$ then $m_*(E) = m_*(E_1) + m_*(E_2)$.

Proposition 3.5. If E is a countable union of almost disjoint cubes $E = \cup_{i=1}^{\infty} Q_i$ then $m_*(E) = \sum_{i=1}^{\infty} |Q_i|$.

4 Measurable sets and the Lebesgue measure

Definition 4.1 (Lebesgue measurable)**.** *A subset E of $\mathbb{R}^d$ is Lebesgue measurable if for any $\epsilon > 0$ there exists an open set $\mathcal{O}$ containing E such that*

$$m_*(\mathcal{O} \setminus E) \leq \epsilon,$$

in which case, the Lebesgue measure of E is given by

$$m(E) = m_*(E).$$

Proposition 4.1. Every open set in $\mathbb{R}^d$ is measurable.

Proposition 4.2. If $m_*(E) = 0$ then E is measurable. In particular, if $F \subseteq E$ with $m_*(E) = 0$ then F is measurable.

Proposition 4.3. A countable union of measurable sets is measurable.

Proposition 4.4. Closed sets are measurable.

Proposition 4.5. The complement of a measurable set is measurable.

Proposition 4.6. A countable intersection of measurable sets is measurable.

Theorem 4.2. *If $E_1, \ldots,$ are disjoint measurable sets, and $E = \bigcup_{i=1}^{\infty} E_i$ then*

$$m(E) = \sum_{i=1}^{\infty} m(E_i)$$

Corollary 4.3. *Suppose $E_1, \ldots,$ are measurable subsets of $\mathbb{R}^d$,*

- *If $E_k \uparrow E$ then $m(E) = \lim_{n\to\infty} m(E_n)$.*
- *If $E_k \downarrow E$ and $m(E_k) < \infty$ for some k then $m(E) = \lim_{n\to\infty} m(E_n)$.*

Theorem 4.4. *If E is a measurable subset of $\mathbb{R}^d$ then for every $\epsilon > 0$,*

- *There exists an open set $\mathcal{O}$ with $E \subseteq \mathcal{O}$ and $m(\mathcal{O} \setminus E) \leq \epsilon$.*
- *There exists a closed set F with $F \subseteq E$ and $m(E \setminus F) \leq \epsilon$.*
- *If $m(E)$ is finite, there exists a compact set K with $K \subseteq E$ and $m(E \setminus K) \leq \epsilon$.*
- *If $m(E)$ is finite, then there exists a finite union $F = \bigcup_{i=1}^{N} Q_i$ of closed cubes such that $m(E\Delta F) \leq \epsilon$, where the notation $E\Delta F$ stands for the symmetric difference between the sets E and F:*

$$E\Delta F = (E \setminus F) \cup (F \setminus E).$$

Corollary 4.5. *A subset E of $\mathbb{R}^d$ is measurable*

- *if and only if E differs from a G_δ set by a set of measure zero. Here a G_δ set is a countable intersection of open sets,*
- *if and only if E differs from a F_σ by a set of measure zero. Here an F_σ set is a countable union of closed sets.*

5 σ-algebra

Definition 5.1. *Consider a set X an its power set $\mathcal{P}(X)$. A set $\mathcal{A} \subseteq \mathcal{P}(X)$ is a σ-algebra if*

- *$\varnothing, X \in \mathcal{A}$*
- *$A \in \mathcal{A} \implies A^c = X \setminus A \in \mathcal{A}$.*
- *$A_j \in \mathcal{A}, i \in \mathbb{N} \implies \bigcup_{i=1}^{\infty} A_j \in \mathcal{A}$.*

Any set $A \in \mathcal{A}$ is called an $\mathcal{A}$-measurable set.

Remark 1. *If $\mathcal{A}_i$ is a σ-algebra on X, then for $i \in I$ where I is any index set, then $\bigcap_{i\in I} \mathcal{A}_i$ is also a σ-algebra on X.*

This is important especially when we want to construct some σ-algebra that has all the properties of other σ-algebras.

6 Borel σ-algebra

Definition 6.1. *For $\mathcal{M} \subset \mathcal{P}(X)$, there is a smallest σ-algebra that contains $\mathcal{M}$:*

$$\sigma(M) := \bigcap_{\mathcal{A}\supseteq\mathcal{M}} \mathcal{A}, \quad \mathcal{A} \text{ is a } \sigma\text{-algebra}$$

called the σ-algebra generated by $\mathcal{M}$.

Definition 6.2. *Let X be a topological space (or a metric space, or a subset of $\mathbb{R}^n$), so that we have "open sets." The Borel σ-algebra $\mathcal{B}(X)$ is the smallest σ-algebra generated by the open sets.*

7 Measure(able) space

Definition 7.1. *Consider a set X and a σ-algebra $\mathcal{A}$ on X. $(X, \mathcal{A})$ is a measurable space. The map $\mu : \mathcal{A} \to [0, \infty] = [0, \infty) \cup \{\infty\}$ is called a measure if it satisfies:*

- $\mu(\varnothing) = 0$
- *σ-additivity*

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$$

  whenever $A_i \cap A_j = \varnothing$ for $i \neq j$ and $A_i \in \mathcal{A}$ for all i.

Once the measure μ is defined on the measurable space $(X, \mathcal{A})$, the triple $(X, \mathcal{A}, \mu)$ is called a measure space.

8 Measurable functions

Definition 8.1. *Given measurable spaces $(\Omega_1, \mathcal{A}_1)$ and $(\Omega_2, \mathcal{A}_2)$. Consider a map $f : \Omega_1 \to \Omega_2$. f is measurable (w.r.t $\mathcal{A}_1, \mathcal{A}_2$) if the pre-image $f^{-1}(A_2) \in \mathcal{A}_1$ for all $A_2 \in \mathcal{A}_2$.*

Example 8.1. Given $(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. It is easy to check that the characteristic function (or the indicator function) $\chi_A : \Omega \to \mathbb{R}$ is a measurable function.

Proposition 8.1. If f, g are measurable functions, then $f \circ g$ (if it is defined) is also a measureable function. This can be checked by looking at subsequent pre-images.

Proposition 8.2. Given $(\Omega, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and measureable functions $f, g : \Omega \to \mathbb{R}$. Then $f \pm g, f \circ g, |f|$ are measurable functions.

Proposition 8.3. The finite-value function f is measurable if and only if $f^{-1}(\mathcal{O})$ is measurable for every open set $\mathcal{O}$ and if and only if $f^{-1}(F)$ is measurable for every closed set F.

Proposition 8.4. If f is continuous on $\mathbb{R}^d$, then f is measurable. If f is measurable and finite-valued, an Φ is continuous, then $\Phi \circ f$ is measurable.

Proposition 8.5. Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of measurable functions then

$$\sup_n f_n(x), \quad \inf_n f_n(x), \quad \limsup_{n \to \infty} f_n(x), \quad \liminf_{n \to \infty} f_n(x)$$

are all measurable.

Proposition 8.6. Suppose $\{f_n\}_{n \in \mathbb{N}}$ is a sequence of measurable functions and

$$\lim_{n \to \infty} f_n(x) = f(x)$$

then f is measurable.

Proposition 8.7. If f, g are measurable, then

- The integer powers f^k, $k \geq 1$, are measurable.
- $f + g, fg$ are measurable if both f, g are finite-valued.

Proposition 8.8. If f is measurable and $f = g$ for almost every x then g is measurable.

9 Lebesgue Integral

Definition 9.1 (Simple Functions)**.** *A function f is a simple function if we can find measurable sets $A_i, \ldots, A_n$ and numbers $c_1, \ldots, c_n \in \mathbb{R}$ such that we can write*

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x).$$

Remark 2. *Simple functions are measurable.*

Suppose a simple function f is given by the representation

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x).$$

then the "Lebesgue integral" of f is given by

$$I(f) \coloneqq \sum_{i=1}^{n} c_i \mu(A_i)$$

where μ is the Lebesgue measure. However, this becomes problematic when some measures are infinite and the c_i are not all negative or positive. One way to refine this to introduce the set of nonnegative simple functions:

$$\mathcal{S}^+ \coloneqq \{f : X \to \mathbb{R} | \, f \text{ simple}, f \geq 0\}.$$

Definition 9.2 (Lebesgue integral of a nonnegative simple function)**.** *Let $f \in \mathcal{S}^+$ be given and choose a representation:*

$$f(x) = \sum_{i=1}^{n} c_i \chi_{A_i}(x).$$

The Lebesgue integral of f with respect to the measure μ is

$$\int_X f \, d\mu \equiv \int_X f(x) \, d\mu(x) = I(f) = \sum_{i=1}^{n} c_i \mu(A_i) \in [0, \infty].$$

Theorem 9.3. *Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of simple functions $\{\varphi_k\}_{k\in\mathbb{N}}$ that satisfies*

$$|\varphi_k(x)| \leq |\varphi_{k+1}(x)|, \quad \lim_{k\to\infty} \varphi_k(x) = f(x), \forall x$$

In particular, we have $|\varphi_k(x)| \leq |f(x)|$ for all x, k.

Theorem 9.4. *Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k\in\mathbb{N}}$ that converges pointwise to $f(x)$ for almost every x.*

Definition 9.5 (L-integrals for nonnegative functions)**.** *Let $f : X \to [0, \infty)$ be a measurable function.*

$$\int_X f \, d\mu = \sup\{I(h) | \, h \in \mathcal{S}^+, h \leq f\} \in [0, \infty]$$

f is μ-integrable if $\int_X f \, d\mu < \infty$.

Proposition 9.1. Given measurable nonnegative functions $f, g : X \to [0, \infty)$, $f = g$ μ-almost everywhere (a.e.) $\implies \int_X f \, d\mu = \int_X g \, \mu$. By $f = g$ μ-a.e., we mean $\mu(\{x \in X | f(x) \neq g(x)\}) = 0$.

Proposition 9.2 (Linearity)**.** $I(\alpha f + \beta g) = \alpha I(f) + \beta I(g)$ for $\alpha, \beta \geq 0$.

Proposition 9.3 (Monotonicity)**.** $f \leq g \implies I(f) \leq I(g)$ for $f, g \in \mathcal{S}^+$.

Proposition 9.4. Given measurable nonnegative functions $f, g : X \to [0, \infty)$, $f \leq g$ μ-a.e. $\implies \int_X f \, d\mu \leq \int_X g \, d\mu$.

Proposition 9.5. Given measurable nonnegative functions $f, g : X \to [0, \infty)$, $f = 0$ μ-a.e. $\iff \int_X f \, d\mu = 0$.

Proposition 9.6 (Additivity)**.** If E and F are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{E \cup F} \varphi = \int_e \varphi + \int_F \varphi.$$

Proposition 9.7 (Triangle Inequality)**.** If φ is a simple function, then so is $|\varphi|$, and

$$\left| \int \varphi \right| \leq \int |\varphi|.$$

Proposition 9.8. The propositions above hold for functions f, g which are bounded and supported on sets of finite measure.

10 Bounded Convergence Theorem

Theorem 10.1 (BCT)**.** *Suppose that $\{f_n\}$ is a sequence of measurable functions that are all bounded by M, are supported on a set E of finite measure, and $f_n(x) \to f(x)$ for almost every x as $n \to \infty$. Then f is measurable, bounded, supposed on E for almost every x, and*

$$\int |f_n - f| \to 0, \quad n \to \infty.$$

Consequently,

$$\int f_n \to \int f, \quad n \to \infty.$$

11 Monotone Convergence Theorem

Theorem 11.1 (MCT)**.** *Let a measure space $(X, \mathcal{A}, \mu)$ and measurable functions $f_n, f : X \to [0, \infty)$ be given for all $n \in \mathbb{N}$ with*

- $f_1 \leq f_2 \leq f_3 \leq \dots$ *μ-a.e.*
- $\lim_{n \to \infty} f_n(x) = f(x)$ *μ-a.e.*

then

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int \lim_{n \to \infty} f_n \, d\mu = \int_X f \, d\mu.$$

Corollary 11.2. *Suppose $(g_n)_{n \in \mathbb{N}}$ with $g_n : X \to [0, \infty]$ measurable for all n be given. Then $\sum_{i=1}^{\infty} g_n : X \to [0, \infty]$ is measurable. By the MCT*

$$\int_X \sum_{i=1}^{\infty} g_n \, d\mu = \sum_{i=1}^{\infty} \int_X g_n \, d\mu.$$

12 Fatou's Lemma

Lemma 12.1 (Fatou's)**.** *Let a measure space $(X, \mathcal{A}, \mu)$ be given with $f_n : X \to [0, \infty]$ be measurable for all $n \in \mathbb{N}$. Then*

$$\int_X \liminf_{n \to \infty} f_n \, d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu$$

where $\liminf_{n\to\infty} f_n : X \to [0,\infty]$ *is defined by*

$$g(x) := \left(\liminf_{n\to\infty} f_n\right)(x) = \lim_{n\to\infty} \underbrace{\left(\inf_{k\geq n} f_k(x)\right)}_{g_k(x)} \in [0,\infty].$$

It turns out that $g_1 \leq g_2 \leq \dots$ are measurable and thus $g(x)$ is also measurable.

13 Lebesgue's Dominated Convergence Theorem

Theorem 13.1 (LDCT)**.** *Let a measure space* $(X, \mathcal{A}, \mu)$ *be given. Consider the set of all Lebesgue-integrable functions:*

$$\mathcal{L}^1(\mu) := \{f : X \to \mathbb{R} \text{ measureable} | \int_X |f|^1 \, d\mu < \infty\}.$$

For $f \in \mathcal{L}^1(\mu)$*, write* $f = f^+ - f^-$ *with* $f^+, f^- \geq 0$ *and define*

$$\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.$$

Consider $f_n : X \to \mathbb{R}$ *a sequence of measurable functions and* $f : X \to \mathbb{R}$ *with* $\lim_{n\to\infty} f_n(x) = f(x)$ *for* $x \in X$*,* μ*-a.e. where* $|f_n| \leq g$ *with* $g \in \mathcal{L}^1(\mu)$ *for all* n*. Then* $f_1, \dots, \in \mathcal{L}^1(\mu)$*,* $f \in \mathcal{L}^1(\mu)$ *and*

$$\lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

14 σ-finite measure

Definition 14.1. *Let* $(X, \mathcal{A}, \mu)$ *be a measure space. The measure* μ *is called a* σ*-finite measure if it satisfies one of the following equivalent criteria:*

- *The set* X *can be covered with at most countably many measurable sets with finite measure, i.e., there are sets* $A_1, \dots, \in \mathcal{A}$ *with* $\mu(A_n) < \infty$ *for all* n *such that* $\bigcup_{n\in\mathbb{N}} A_n = X$*.*
- *The set* X *can be covered with at most countable many measureable disjoint sets with finite measure, i.e., there are sets* $B_1, \dots, \in \mathcal{A}$ *with* $\mu(B_n) < \infty$ *for all* n *and* $B_i \cap B_j = \varnothing$ *for* $i \neq j$ *that satisfy* $\bigcup_{n\in\mathbb{N}} B_n = X$*.*
- *The set* X *can be covered with a monotone sequence of measurable sets with finite measure, i.e., there are sets* $C_1, \dots, \in \mathcal{A}$ *with* $C_1 \subseteq C_2 \subseteq \dots$ *and* $\mu(C_n) < \infty$ *for all* n *that satisfy* $\bigcup_{n\in\mathbb{N}} C_n = X$*.*
- *There exists a strictly positive measurable function* f *whose integral is finite, i.e.,* $f(x) > 0$ *for all* $x \in X$ *and* $\int_X f \, d\mu < \infty$*.*

If μ is a σ-finite measure, the measure space $(X, \mathcal{A}, \mu)$ is called a σ-finite measure space.

15 Lebesgue's Decomposition Theorem & Radon-Nikodym Theorem

Consider the special measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where λ is the Lebesgue reference measure: $\lambda([a,b)) = b - a$. Also, consider another measure $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$.

Definition 15.1.

- μ *is called* ***absolutely continuous*** *with respect to the reference measure* λ *if* $\lambda(A) = 0 \implies \mu(A) = 0$ *for all* $A \in \mathcal{B}(\mathbb{R})$*. To denote this, one writes* $\mu \ll \lambda$*.*

- *μ is called **singular** with respect to the reference measure λ if there is $N \in \mathcal{B}([R])$ with $\lambda(N) = 0$ and $\mu(\mathbb{R} \setminus N) = 0$. To denote this, one writes $\mu \perp \lambda$.*

Theorem 15.2. *Consider some measure $\mu : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ that is σ-finite.*

- *(Lebesgue's Decomposition Theorem). There are two measures (uniquely determined) $\mu_{ac}, \mu_s : \mathcal{B}(\mathbb{R}) \to [0, \infty]$ with*
$$\mu = \mu_{ac} + \mu_s$$
where $\mu_{ac} \ll \lambda$ and $\mu_s \perp \lambda$.
- *(Radon-Nikodym Theorem). There is a measurable map $h : \mathbb{R} \to [0, \infty)$ with*
$$\mu_{ac}(A) = \int_A h \, d\lambda$$
*for all $A \in \mathcal{B}(\mathbb{R})$. We call such an h a **density function**.*

16 Product Measure

Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces. We have the product σ-algebra $\mathcal{M} \otimes \mathcal{N}$ on $X \times Y$., so now we construct the a measure on $\mathcal{M} \otimes \mathcal{N}$ that is the product of μ and ν.

A measurable **rectangle** is a set of the form $A \times B$ where $A \in \mathcal{M}$ and $B \in \mathcal{N}$. The collection $\mathcal{A}$ of finite disjoint unions of rectangles is an algebra. The σ-algebra it generates is $\mathcal{M} \otimes \mathcal{N}$.

Suppose $A \times B$ is a rectangle that is a finite/countable **disjoint** union of rectangles $A_j \times B_j$. Then for $x \in X, y \in Y$,
$$\chi_A(x)\chi_B(y) = \chi_{A\times B}(x, y) = \sum \chi_{A_j \times B_j}(x, y) = \chi\chi_{A_j}(x)\chi_{B_j}(y).$$
Integrating wrt x to get
$$\mu(A)\chi_B(y) = \cdots = \sum \mu(A_j)\chi_{B_j}(y).$$
By symmetry, we get
$$\mu(A)\nu(B) = \sum \mu(A_j)\nu(B_j)$$
Thus, if $E \in \mathcal{A}$ is a disjoint union of rectangles $A_1 \times B_1, \ldots, A_n \times B_n$ and we set
$$\pi(E) = \sum_1^n \mu(A_j)\nu(B_j)$$
then π is well-defined on $\mathcal{A}$ and π is a premeasure (a function that satisfies $\mu(\varnothing) = 0$ and σ-additivity but isn't necessarily defined on a σ algebra) on $\mathcal{A}$. Theorem 1.4 of Folland says that π generates an exterior measure on $X \times Y$ whose restriction to $\mathcal{M} \otimes \mathcal{N}$ is a measure that extends π. This measure is the product of μ and ν and we denote it by $\mu \times \nu$.

If μ, ν are σ-finite, then $\mu \times \nu$ is also σ-finite. In this case, Theorem 1.4 of Folland also tells us that $\mu \times \nu$ is a **unique** measure on $\mathcal{M} \otimes \mathcal{N}$ such that
$$\mu \times \nu(A \times B) = \mu(A)\nu(B)$$
for all rectangles $A \times B$.

17 “Measurable Slice” Theorems

Definition 17.1. *Let* $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ *be measure spaces. If* $E \subset X \times Y$*, for* $x \in X, y \in Y$ *we define the* x*-section* E_x *and* y*-section* E^y *of* E *by*

$$E_x = \{y \in Y : (x, y) \in E\} \subseteq Y, \quad E^y = \{x \in X : (x, y) \in E\} \subseteq X$$

Also, if f *is a function on* $X \times Y$ *we define the* x*-section* f_x *and* y*-section* f^y *of* f *by*

$$f_x(y) = f^y(x) = f(x, y).$$

Example 17.1. One can check that

$$(\chi_E)_x = \chi_{E_x}, \quad (\chi_E)^y = \chi_{E^y}$$

Proposition 17.1.

- If $E \in \mathcal{M} \otimes \mathcal{N}$, then $E_x \in \mathcal{N}$ for all $x \in X$ and $E^y \in \mathcal{M}$ for all $y \in Y$.
- If f is $\mathcal{M} \otimes \mathcal{N}$-measurable, then f_x is $\mathcal{N}$-measurable for all $x \in X$ and f^y is $\mathcal{M}$-measurable for all $y \in Y$

18 Monotone Class Lemma

Definition 18.1 (Algebra in a set)**.** *Let* X *be a set. An algebra in* X *is a non-empty collection of subsets of* X *that is closed under complements, finite unions, and finite intersections.*

Definition 18.2 (Premeasure)**.** *Let* $\mathcal{A}$ *be an algebra in* X*. A premeasure on an algebra* $\mathcal{A}$ *is a function* $\mu_0 : \mathcal{A} \to [0, \infty]$ *that satisfies*

- $\mu_0(\varnothing) = 0$.
- *If* $E_1, \ldots$ *is a countable collection of disjoint sets in* $\mathcal{A}$ *with* $\bigcup_{n=1}^{\infty} E_n \in \mathcal{A}$ *then*

$$\mu_0\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu_0(E_k)$$

In particular, μ_0 *is finitely additive on* $\mathcal{A}$*.*

Definition 18.3 (Monotone class)**.** *A monotone class on a space* X *is a subset* $\mathcal{C} \subset \mathcal{P}(X)$ *that is closed under increasing unions and countable decreasing intersections. That is, if* $E_j \in \mathcal{C}$ *and* $E_1 \subset E_2 \subset \ldots$ *then* $\bigcup E_j \in \mathcal{C}$*, and likewise for intersections.*

Remark 3. *Every* σ*-algebra is a monotone class.*

Remark 4. *The intersection of any family of monotone classes is a monotone class.*

Definition 18.4 (Monotone class Generated by a subset of $\mathcal{P}(X)$)**.** *For any* $\mathcal{E} \subset \mathcal{P}(X)$*, there is a unique smallest monotone class containing* $\mathcal{E}$*, called the monotone class* ***generated by*** $\mathcal{E}$*.*

Definition 18.5 (σ-algebra generated by a family of subsets)**.** *Let* F *be an arbitrary family of subsets of* X*. Then there exists a unique smallest* σ*-algebra which contains every set in* F*. It is, in fact, the intersection of all* σ*-algebras containing* F*. This* σ*-algebra is denoted* $\sigma(F)$ *and is called the* σ*-algebra generated by* F*.*

Lemma 18.6 (Monotone Class Lemma)**.** *If* $\mathcal{A}$ *is an algebra of subsets of* X*, then the monotone class* $\mathcal{C}$ *generated by* $\mathcal{A}$ *coincides with the* σ*-algebra* $\mathcal{M}$ *generated by* $\mathcal{A}$*.*

Proof. $\mathcal{M}$ is a σ-algebra, so it is a monotone class. As a result, $\mathcal{C} \subset \mathcal{M}$. To show the reverse containment, we show that $\mathcal{C}$ is a σ-algebra. To do this, let $E \in \mathcal{C}$ be given and define

$$\mathcal{C}(E) = \{F \in \mathcal{C} : E \setminus F, F \setminus E, E \cap F \in \mathcal{C}\}.$$

Check that $\varnothing, E \in \mathcal{C}(E)$, and $E \in \mathcal{C}(F) \iff F \in \mathcal{C}(E)$. Check that $\mathcal{C}(E)$ is a monotone class. Because $\mathcal{A}$ is an algebra, $E \in \mathcal{A} \implies F \in \mathcal{C}(E) \forall F \in \mathcal{A}$. This means $\mathcal{A} \subset \mathcal{C}(E) \implies \mathcal{C} \subset \mathcal{C}(E)$. This means if $F \in \mathcal{C}$ then $F \in \mathcal{C}(E) \iff E \in \mathcal{C}(F) \forall E \in \mathcal{A}$, and by a similar argument we get $\mathcal{C} \in \mathcal{C}(F)$. So, if $E, F\mathcal{C}, E \setminus F, E \cap F \in \mathcal{C}$. Now, $X \in \mathcal{A} \subset \mathcal{C}$, so $\mathcal{C}$ is an algebra. Finally, since $\mathcal{C}$ is closed under countable increasing unions, $\mathcal{C}$ is a σ-algebra. ☐

Theorem 18.7. *Suppose $(X, \mathcal{M}, \mu), (Y, \mathcal{N}, \nu)$ are σ-finite measure spaces. If $E \in \mathcal{M} \otimes \mathcal{N}$ then the functions $x \mapsto \nu(E_x)$ and $y \mapsto \mu(E^y)$ are measurable on X, Y respectively and*

$$\mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y)\, d\nu(y)$$

19 Fubini-Tonelli's Theorem

Theorem 19.1. *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces*

- *(Tonelli's) If $f \imath \mathcal{L}^+(X \times Y)$ then the functions $g(x) = \int f_x\, d\nu$ and $h(y) = \int f^y\, d\mu$ are in $\mathcal{L}^+(X), \mathcal{L}^+(Y)$, respectively, and*

$$\int f\, d(\mu \times \nu) = \int \left[\int f(x,y)\, d\nu(y)\right] d\mu(x) = \int \left[\int f(x,y)\, d\mu(x)\right] d\nu(y).$$

- *(Fubini's) If $f \in \mathcal{L}^1(\mu \times \nu)$ then $f_x \in \mathcal{L}^1(\nu)$ for a.e. $x \in X$, $f^y \in \mathcal{L}^1(\mu)$ for a.e. $y \in Y$, the a.e.-defined functions $g(x) = \int f_x\, d\nu$ and $h(y) = \int f^y\, d\mu$ are in $\mathcal{L}^1(\mu), \mathcal{L}^1(\nu)$ respectively, and*

$$\int f\, d(\mu \times \nu) = \int \left[\int f(x,y)\, d\nu(y)\right] d\mu(x) = \int \left[\int f(x,y)\, d\mu(x)\right] d\nu(y).$$

20 Fubini's Theorem for Complete Measures

Theorem 20.1. *Suppose that $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite and complete measure spaces, and let $(X \times Y, \mathcal{L}, \lambda)$ be the completion of $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$. If f is $\mathcal{L}$-measurable and either (a) $f \geq 0$ or (b) $f \in \mathcal{L}^1(\lambda)$, then f_x, f^y are also integrable for a.e. x, y. Moreover, $x \mapsto \int f_x\, d\nu, y \mapsto \int f^y\, d\mu$ are measurable, and in case (b) also integrable, and*

$$\int f\, d\lambda = \iint f(x,y)\, d\mu(x) d\nu(y) = \iint f(x,y)\, d\nu(y) d\mu(x).$$

21 Integration in Polar Coordinates

Denote the unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ by S^{n-1}. If $x \imath n \mathbb{R}^n \setminus \{0\}$ then the polar coordinates of x are

$$r = |x| \in (0, \infty), \quad x' = \frac{x}{|x|} \in S^{n-1}.$$

The map $\Phi(x) = (r, x')$ is a continuous bijection from $\mathbb{R}^n \setminus \{0\}$ to $(0, \infty) \times S^{n-1}$ whose continuous inverse is $\Phi^{-1}(r, x') = rx'$. Denote by m_* the Borel measure on $(0, \infty) \times S^{n-1}$ induce by Φ from the Lebesgue measure on $\mathbb{R}^n$, that is

$$m_*(E) = m(\Phi^{-1}(E)).$$

Moreover, define the measure $\rho = \rho_n$ on $(0, \infty)$ by

$$\rho(E) = \int_E r^{n-1}\, dr.$$

Theorem 21.1. *There is a unique Borel measure $\sigma = \sigma_{n-1}$ on S^{n-1} such that $m_* = \rho \times \sigma$. If f is Borel measurable on $\mathbb{R}^n$ and $f \geq 0$ or $f \in \mathcal{L}^1(m)$, then*

$$\int_{\mathbb{R}^n} f(x)\,dx = \int_0^\infty \int_{S^{n-1}} f(rx')r^{n-1}\,d\sigma(x')\,dr.$$

References

[1] Stein & Shakarchi's *Real Analysis.*
[2] G. Folland's *Real Analysis: Modern Techniques and Their Applications.*
[3] Rudin's *Principles of Mathematical Analysis.*