Assignment 3; MA353; S19

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The goal of this assignment is to have you work through the details of the ideas I had presented this week in class.

Please do NOT assume that a vector space is finite-dimensional, unless this is stated explicitly.

Definition 0.1

Suppose that $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ are non-trivial vector spaces, and for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \boldsymbol{V}_j \xrightarrow{\text{linear}} \boldsymbol{V}_i \,.$$

A **block-matrix** function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3 \longrightarrow \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3$$

is defined by

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \begin{pmatrix} x \\ y \\ z \end{pmatrix} := \begin{pmatrix} \mathcal{L}_{11}(x)+\mathcal{L}_{12}(y)+\mathcal{L}_{13}(z) \\ \mathcal{L}_{21}(x)+\mathcal{L}_{22}(y)+\mathcal{L}_{23}(z) \\ \mathcal{L}_{31}(x)+\mathcal{L}_{32}(y)+\mathcal{L}_{33}(z) \end{pmatrix}.$$

Definition 0.2

In a similar fashion, for example, one may define a function

$$\begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \mathcal{K}_{13} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{23} \end{bmatrix}_\times : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3 \longrightarrow \boldsymbol{Z}_1 \times \boldsymbol{Z}_2 \,,$$

constructed from functions $\mathcal{K}_{ij} : \boldsymbol{V}_j \xrightarrow{\text{linear}} \boldsymbol{Z}_i$.

In the context of Definition 0.1, functions such as

$$\left[\mathcal{I}_{\boldsymbol{V}_1}\ \mathcal{O}\ \mathcal{O}\right]_\times : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3 \longrightarrow \boldsymbol{V}_1$$

and

$$\begin{bmatrix} \mathcal{I}_{\boldsymbol{V}_1} \\ \mathcal{O} \\ \mathcal{O} \end{bmatrix}_\times : \boldsymbol{V}_1 \longrightarrow \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3 \ ,$$

play an important role. The primer is said to be **a coordinate projection of $\boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3$ onto $\boldsymbol{V}_1$** and is often denoted by Π_1, with coordinate projections Π_2 and Π_3 defined similarly.

The other function is often called **a coordinate injection of $\boldsymbol{V}_1$ into $\boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3$**, and we shall denote it by Υ_1, with coordinate injections Υ_2 and Υ_3 defined similarly.

It is a trivial exercise to check that coordinate projections and injections are linear functions. The reader is encouraged to verify these properties and is welcome to take them for granted thereafter.

Test Your Comprehension 0.3

Argue that coordinate injections and coordinate projections are linear functions, and that $\Pi_i \circ \Upsilon_i = \mathcal{I}_{\boldsymbol{V}_i}$.

Problem 1

Suppose that $\boldsymbol{V}_1, \boldsymbol{V}_2, \boldsymbol{V}_3$ are non-trivial vector spaces, and for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \boldsymbol{V}_j \xrightarrow{\text{linear}} \boldsymbol{V}_i \ .$$

1. Argue that $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times$ is a linear function.

2. Argue that every linear function $T : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3 \longrightarrow \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3$ can be expressed as $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times$, with $\mathcal{L}_{ij} : \boldsymbol{V}_j \xrightarrow{\text{linear}} \boldsymbol{V}_i$, in exactly one way.

3. If the function

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3 \longrightarrow \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \boldsymbol{V}_3$$

Hint: 2. Use coordinate projections and coordinate injections for a quick proof.

is defined similarly, find (with proof) the block-matrix form of the functions

$$2\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} +3\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times} \quad \text{and} \quad \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times} .$$

Definition 0.4

Suppose that $\boldsymbol{W}_1 \oplus \boldsymbol{W}_2 \oplus \boldsymbol{W}_3 = \boldsymbol{V}$ and the $\boldsymbol{W}_i$'s are non-trivial subspaces of $\boldsymbol{V}$. Suppose that for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \boldsymbol{W}_j \xrightarrow{\text{linear}} \boldsymbol{W}_i \ .$$

Let $\maltese : \boldsymbol{W}_1 \times \boldsymbol{W}_2 \times \boldsymbol{W}_3 \longrightarrow \boldsymbol{W}_1 + \boldsymbol{W}_2 + \boldsymbol{W}_3 = \boldsymbol{V}$ be our usual isomorphism. Let us write

$$\maltese\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\maltese} \quad \text{and} \quad x_1 + x_2 + x_3$$

interchangeably, when $x_i \in \boldsymbol{W}_i$.

A **block-matrix** function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$$

is defined by

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} := \maltese \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \maltese^{-1} . \tag{1}$$

Being a composition of linear functions, the function is itself linear.

Of course the identity (1) can be also expressed as

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\maltese} = \left(\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right)_{\maltese} = \begin{pmatrix} \mathcal{L}_{11}(x)+\mathcal{L}_{12}(y)+\mathcal{L}_{13}(z) \\ \mathcal{L}_{21}(x)+\mathcal{L}_{22}(y)+\mathcal{L}_{23}(z) \\ \mathcal{L}_{31}(x)+\mathcal{L}_{32}(y)+\mathcal{L}_{33}(z) \end{pmatrix}_{\maltese} .$$

Test Your Comprehension 0.5

In the context of Definitions 0.2 and 0.4, examine the linear functions

$$\mathcal{E}_i := \Pi_i \circ \maltese^{-1} : \boldsymbol{V} \longrightarrow \boldsymbol{W}_i \quad \text{and} \quad \maltese \circ \Upsilon_i : \boldsymbol{W}_i \longrightarrow \boldsymbol{V} \ ,$$

and argue that

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\boldsymbol{V}}$$

is the resolution of the identity on $\boldsymbol{V}$ corresponding to the direct sum decomposition

$$\boldsymbol{W}_1 \oplus \boldsymbol{W}_2 \oplus \boldsymbol{W}_3 = \boldsymbol{V} \ .$$

Problem 2

Suppose that $\boldsymbol{W}_1 \oplus \boldsymbol{W}_2 \oplus \boldsymbol{W}_3 = \boldsymbol{V}$ and the $\boldsymbol{W}_i$'s are non-trivial subspaces of $\boldsymbol{V}$. Suppose that for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \boldsymbol{W}_j \overset{\text{linear}}{\longrightarrow} \boldsymbol{W}_i \ .$$

1. Verify the formula

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\maltese} = \left(\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right)_{\maltese} ,$$

   where we have used as few brackets as possible without losing clarity.

2. Argue that every $T : \boldsymbol{V} \overset{\text{linear}}{\longrightarrow} \boldsymbol{V}$ can be expressed as $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}$, with $\mathcal{L}_{ij} : \boldsymbol{W}_j \overset{\text{linear}}{\longrightarrow} \boldsymbol{W}_i$, in exactly one way.*

3. If the function

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus} : \boldsymbol{V} \longrightarrow \boldsymbol{V}$$

   is defined similarly, find (with proof) the corresponding block-matrix form of the functions

$$2\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} + 3\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus} \quad \text{and} \quad \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus} .$$

4. Suppose that $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\boldsymbol{V}}$ is a resolution of the identity, where $Range(\mathcal{E}_i) = \boldsymbol{W}_i$. If

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\uplus},$$

express (with proof, of course) $\mathcal{L}_{ij}$ in terms of $\mathcal{L}$, $\mathcal{E}_k$'s and $\boldsymbol{W}_k$'s.

5. Continuing with the set-up of part 4, find $\mathcal{M}_{ij} : \boldsymbol{W}_j \xrightarrow{\text{linear}} \boldsymbol{W}_i$ such that

$$\mathcal{E}_1 = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\uplus}.$$

Then do the same for $\mathcal{E}_2$ and $\mathcal{E}_3$.

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*In this case we say that this is a **block-matrix representation of $\mathcal{T}$ with respect to the direct sum decomposition $\boldsymbol{W}_1 \uplus \boldsymbol{W}_2 \uplus \boldsymbol{W}_3 = \boldsymbol{V}$**.

Comment 0.6

In a similar fashion, for example, one may define a function

$$\begin{bmatrix} \mathcal{K}_{11} & \mathcal{K}_{12} & \mathcal{K}_{13} \\ \mathcal{K}_{21} & \mathcal{K}_{22} & \mathcal{K}_{23} \end{bmatrix}_{\uplus} : \boldsymbol{W}_1 \uplus \boldsymbol{W}_2 \uplus \boldsymbol{W}_3 \longrightarrow \boldsymbol{Z}_1 \uplus \boldsymbol{Z}_2,$$

constructed from functions $\mathcal{K}_{ij} : \boldsymbol{W}_j \xrightarrow{\text{linear}} \boldsymbol{Z}_i$.

Definition 0.7

If $\boldsymbol{V}$ and $\boldsymbol{Z}$ are finite-dimensional subspaces, with coordinate systems $\Gamma = (v_1, v_2 \ldots, v_n)$ and $\Omega = (z_1, z_2, \ldots, z_m)$ respectively, we define bijective atrices

$$\mathcal{A}_\Gamma := [v_1 \ v_2 \ \cdots \ v_n] \quad \text{and} \quad \mathcal{A}_\Omega := [z_1 \ z_2 \ \cdots \ z_m],$$

and write $[\ \ ]_\Gamma$ and $[\ \ ]_\Omega$ for $\mathcal{A}_\Gamma^{-1}$ and $\mathcal{A}_\Omega^{-1}$. We usually write $[X]_\Gamma$ instead of $[\ \ ]_\Gamma(X)$.

For any $\mathcal{T} : \boldsymbol{V} \xrightarrow{\text{linear}} \boldsymbol{Z}$, the composition $[\ \ ]_\Omega \circ \mathcal{T} \circ \mathcal{A}_\Gamma$ is a linear function from $\mathbb{C}^n$ to $\mathbb{C}^m$, and is therefore a matrix. We denote this matrix by $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$.

Problem 3

1. Using the set-up of Definition 0.7, verify the identity
$$[\mathcal{T}(X)]_\Omega = [\mathcal{T}]_{\Omega\leftarrow\Gamma}\,[X]_\Gamma\ .$$
2. Use part 1 to prove that
$$[\mathcal{T}]_{\Omega\leftarrow\Gamma} = \Big[[\mathcal{T}(v_1)]_\Omega \quad [\mathcal{T}(v_2)]_\Omega \quad \cdots [\mathcal{T}(v_n)]_\Omega\Big]\,.$$
3. If $\boldsymbol{U}$ is a finite-dimensional vector space with a coordinate system Δ, and $\mathcal{S} : \boldsymbol{Z} \xrightarrow{\text{linear}} \boldsymbol{U}$, argue that
$$[\mathcal{S}\circ\mathcal{T}]_{\Delta\leftarrow\Gamma} = [\mathcal{S}]_{\Delta\leftarrow\Omega}\,[\mathcal{T}]_{\Omega\leftarrow\Gamma}\ .$$
4. Verify that in the case $\mathcal{T}$ is invertible (in which case $m = n$), so is $[\mathcal{T}]_{\Omega\leftarrow\Gamma}$, and
$$\left[\mathcal{T}^{-1}\right]_{\Gamma\leftarrow\Omega} = [\mathcal{T}]_{\Omega\leftarrow\Gamma}^{-1}\ .$$

Problem 4

Let us write Γ_o for the standard coordinate system $(1, x, x^2, x^3)$ of the vector space $\mathbb{P}_3$ of all polynomials of degree at most 3.

1. Use *Mathematica* and the fact that $[\ \ ]_{\Gamma_o}$ is an isomorphism (and isomorphisms map bases to bases) to verify that
$$\Delta := \big(1 + x + x^2 + x^3,\quad 1 + 2x + 4x^2 + 8x^3,\\ 1 + 3x + 9x^2 + 27x^3,\quad 1 + 4x + 16x^2 + 64x^3\big)$$
and
$$\Omega := \big(x + x^2,\quad x - 6x^3,\quad 1 + 4x^2,\quad 1 + 8x^3\big)$$
are also coordinate systems of $\mathbb{P}_3$.
2. Verify the general identity
$$[\mathcal{T}]_{\Delta\leftarrow\Omega} = \left[\mathcal{I}_{\mathbb{P}_3}\right]_{\Delta\leftarrow\Gamma_o}[\mathcal{T}]_{\Gamma_o\leftarrow\Gamma_o}\left[\mathcal{I}_{\mathbb{P}_3}\right]_{\Gamma_o\leftarrow\Omega}$$
$$= \left(\left[\mathcal{I}_{\mathbb{P}_3}\right]_{\Gamma_o\leftarrow\Delta}\right)^{-1}[\mathcal{T}]_{\Gamma_o\leftarrow\Gamma_o}\left[\mathcal{I}_{\mathbb{P}_3}\right]_{\Gamma_o\leftarrow\Omega}\ ,$$

for $\mathcal{T} : \mathbb{P}_3 \xrightarrow{\text{linear}} \mathbb{P}_3$.

3. Consider $\mathcal{T} : \mathbb{P}_3 \longrightarrow \mathbb{P}_3$ defined by

$$T(p(x)) := xp'(2x) + p(x) \ .$$

Prove that $\mathcal{T}$ is a linear function and use part 2 to find $[\mathcal{T}]_{\Delta\leftarrow\Omega}$.

Problem 5

Suppose that $\boldsymbol{W}_1 \uplus \boldsymbol{W}_2 \uplus \boldsymbol{W}_3 = \boldsymbol{V}$ and Γ_i is a coordinate system of $\boldsymbol{W}_i$. Suppose that for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \boldsymbol{W}_j \xrightarrow{\text{linear}} \boldsymbol{W}_i \ .$$

Let

$$L := \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\uplus} \ .$$

Let Δ be the concatenation $\Gamma_1 || \Gamma_2 || \Gamma_3$ of the coordinate systems Γ_i. As we know, Δ is a coordinate system of $\boldsymbol{V}$.

Prove that $[\mathcal{L}]_{\Delta\leftarrow\Delta}$ equals the partitioned matrix

$$\begin{bmatrix} [\mathcal{L}_{11}]_{\Gamma_1\leftarrow\Gamma_1} & [\mathcal{L}_{12}]_{\Gamma_1\leftarrow\Gamma_2} & [\mathcal{L}_{13}]_{\Gamma_1\leftarrow\Gamma_3} \\ [\mathcal{L}_{21}]_{\Gamma_2\leftarrow\Gamma_1} & [\mathcal{L}_{22}]_{\Gamma_2\leftarrow\Gamma_2} & [\mathcal{L}_{23}]_{\Gamma_2\leftarrow\Gamma_3} \\ [\mathcal{L}_{31}]_{\Gamma_3\leftarrow\Gamma_1} & [\mathcal{L}_{32}]_{\Gamma_3\leftarrow\Gamma_2} & [\mathcal{L}_{33}]_{\Gamma_3\leftarrow\Gamma_3} \end{bmatrix} .$$