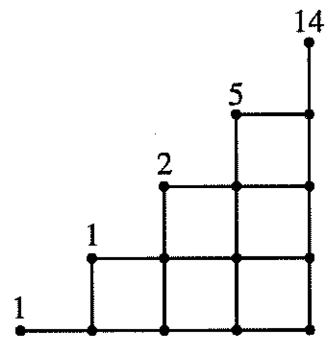
➔ · 51. Notice that a lattice path from $(0,0)$ to (n,n) stays inside (or on the edges of) the square whose sides are the x-axis, the y-axis, the line $x = n$ and the line $y = n$. In this problem we will compute the number of lattice paths from (0,0) to (n,n) that stay inside (or on the edges of) the triangle whose sides are the x-axis, the line $x = n$ and the line $y = x$. Such lattice paths are called *Catalan paths.* For example, in Figure 1.7 we show the grid of points with integer coordinates for the triangle whose sides are the x-axis, the line $x = 4$ and the line $y = x$.

Figure 1.7: The Catalan paths from $(0,0)$ to (i,i) for $i = 0,1,2,3,4$. The number of paths to the point (i,i) is shown just above that point.



(a) Explain why the number of lattice paths from $(0,0)$ to (n,n) that go outside the triangle described previously is the number of lattice paths from $(0,0)$ to (n,n) that either touch or cross the line $y = x + 1$.

Solution: If a lattice path between $(0,0)$ and (n,n) goes outside the triangle, it can only do so on an upstep. (A step from (i,j) to $(i,j+1)$.) And an upstep must originate at a point with integer coordinates. If $j < i$, an upstep from (i,j)) cannot leave the triangle. Thus to leave the triangle, the upstep must leave from a point of the form (i,i), and go to $(i,i+1)$, which is on the line $y = x + 1$. ∎

(b) Find a bijection between lattice paths from $(0,0)$ to (n,n) that touch (or cross) the line $y = x + 1$ and lattice paths from $(-1,1)$ to (n,n).

Solution: Suppose we have a lattice path from $(0, 0)$ to (n, n) which touches or crosses the line $y = x + 1$. Let $(k, k + 1)$ be the first point on the line $y = x + 1$ that the lattice path touches. From that point, work backwards, replacing every upstep with a step one unit to the left and every rightstep with a step one unit down. The segment of the path you just changed will have moved left $k+1$ times, so its leftmost x coordinate will be -1, and it will have moved down k times, so its lowest y coordinate will be 1. Thus we now have a lattice path from $(-1, 1)$ to (n, n). Further, given a lattice path from $(-1, 1)$ to (n, n), it must cross the line $y = x + 1$ at least once, because it starts above the line and ends below it. At the first point where such a path touches the line $y = x+1$, say $(k', k'+1)$, work backwards, replacing every upstep with a step to the left and every rightstep with a step downwards. The leftmost point on this path will have x coordinate 0, and the lowest point will have y coordinate 0, so the new path will be a lattice path from $(0, 0)$ to (n, n) that touches the line $y = x + 1$. Clearly these two processes reverse each other, and so they give us a bijection between paths from $(0, 0)$ to (n, n) that touch or cross the line $y = x + 1$ and lattice lattice paths from $(-1, 1)$ to (n, n). Notice that geometrically what we are doing to get the bijection is to take the portion of a lattice path that goes from the initial point till the first touch of the line $y = x + 1$ and reflecting it around that line. This idea of reflection was introduced by Feller, and is called Feller's reflection principle. ▮

(c) Find a formula for the number of lattice paths from $(0, 0)$ to (n, n) that do not go above the line $y = x$. The number of such paths is called a *Catalan Number* and is usually denoted by C_n.
Solution: $C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}$. ▮

➔52. Your formula for the Catalan Number can be expressed as a binomial coefficient divided by an integer. Whenever we have a formula that calls for division by an integer, an ideal combinatorial explanation of the formula is one that uses the quotient principle. The purpose of this problem is to find such an explanation using diagonal lattice paths. A diagonal lattice path that never goes below the y-coordinate of its first point is called a *Dyck Path*. We will call a Dyck Path from $(0, 0)$ to $(2n, 0)$ a (diagonal) *Catalan Path* of length $2n$. Thus the number of (diagonal) Catalan Paths of length $2n$ is the Catalan Number C_n. We normally can decide from context whether the phrase Catalan Path refers to a diagonal path, so we normally leave out the word diagonal.

(a) If a Dyck Path has n steps (each an upstep or downstep), why do the first k steps form a Dyck Path for each nonnegative $k \leq n$?
Solution: If no points on the path are lower than the first point, then no points among the first k steps are lower than the first point. ▮

(b) Thought of as a curve in the plane, a diagonal lattice path can have many local maxima and minima, and can have several absolute maxima and minima, that is, several highest points and several lowest points. What is the y-coordinate of an absolute minimum point of a Dyck Path starting at $(0, 0)$? Explain why a Dyck Path whose rightmost absolute minimum point is its last point is a Catalan Path.

Solution: Since the path starts at $(0,0)$ and can't go below it, the y coordinate of an absolute minimum must be zero. If the last point is an absolute minimum, then (because it ends with the same y coordinate with which it starts) the path has an even number $2k$ of steps and ends at $(2k,0)$, so it is a Catalan path. ∎

(c) Let D be the set of all diagonal lattice paths from $(0,0)$ to $(2n,0)$. (Thus these paths can go below the x-axis.) Suppose we partition D by letting B_i be the set of lattice paths in D that have i upsteps (perhaps mixed with some downsteps) following the last absolute minimum. How many blocks does this partition have? Give a succinct description of the block B_0.

Solution: The path must have n upsteps total, and so can have any number between 0 and n upsteps after the rightmost absolute minimum. Thus the partition has $n+1$ blocks. Block B_0 consists of the Catalan Paths. ∎

(d) How many upsteps are in a Catalan Path?

Solution: n. ∎

*(e) We are going to give a bijection between the set of Catalan Paths and the block B_i for each i between 1 and n. For now, suppose the value of i, while unknown, is fixed. We take a Catalan path and break it into three pieces. The piece F (for "front") consists of all steps before the ith upstep in the Catalan path. The piece U (for "up") consists of the ith upstep. The piece B (for "back") is the portion of the path that follows the ith upstep. Thus we can think of the path as FUB. Show that the function that takes FUB to BUF is a bijection from the set of Catalan Paths onto the block B_i of the partition. (Notice that BUF can go below the x axis.)

Solution: Since we are starting with a Catalan path, the point on the path at the beginning of the ith upstep must have y coordinate greater or equal to than zero. Thus wherever we start the sequence F of upsteps and downsteps, a path constructed by this sequence never goes lower than its starting point. Thus in BUF the last absolute minimum is either right before the U or earlier. But B is the final segment of a Catalan Path, so its final point is at least as low as its starting point. Thus the point at the beginning of the U in BUF is an absolute minimum, and there are i upsteps after that absolute minimum. If we take two different sequences and rearrange them in the same way, we get two different sequences, so the function we just described is a one-to-one function. If we take an arbitrary diagonal lattice path from $(0,0)$ to $(2n,0)$, let U' be the first upstep after the last absolute minimum, F' be the portion of the path that follows U', and B' be the portion that precedes U', then $F'U'B'$ is a Catalan Path, and U' is its ith upstep if and only if in $B'U'F'$ there are i upsteps after the last absolute minimum. Thus the mapping from FUB to BUF is a bijection. ∎

(f) Explain how you have just given another proof of the formula for the Catalan Numbers.

Solution: We have taken the set of all $\binom{2n}{n}$ diagonal lattice paths of length $2n$ from $(0,0)$ to $(2n,0)$ and partitioned it into $n+1$ blocks all of size C_n. Thus by the quotient principle, $C_n = \frac{1}{n+1}\binom{2n}{n}$. ∎

55. What is $\sum_{i=1}^{10} \binom{10}{i} 3^i$?

Solution: $\sum_{i=1}^{10} \binom{10}{i} 3^i = \sum_{i=0}^{10} \binom{10}{i} 3^i - \binom{10}{0} 3^0 = (1+3)^{10} - 1 = 4^{10} - 1$ ▮

56. What is $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n}$ if n is an integer bigger than zero?

Solution: The sum is 0 because it is $(-1+1)^n$. ▮

➔58. From the symmetry of the binomial coefficients, it is not too hard to see that when n is an odd number, the number of subsets of $\{1, 2, \ldots, n\}$ of odd size equals the number of subsets of $\{1, 2, \ldots, n\}$ of even size. Is it true that when n is even the number of subsets of $\{1, 2, \ldots, n\}$ of even size equals the number of subsets of odd size? Why or why not?

Solution: It is true, because if $n > 0$, when you expand $(1-1)^n$ by the binomial theorem, you get an alternating sum of binomial coefficients equal to 0, and so the sum of the binomial coefficients $\binom{n}{i}$ with i even must equal the sum of the binomial coefficients $\binom{n}{i}$ with i odd. ▮

➔59. What is $\sum_{i=0}^{n} i\binom{n}{i}$? (Hint: think about how you might use calculus.)

Solution: $\sum_{i=0}^{n} \binom{n}{i} x^i = (1+x)^n$. Taking derivatives of both sides gives us $\sum_{i=0}^{n} i\binom{n}{i} x^{i-1} = n(1+x)^{n-1}$. Now substitute 1 for x and you get $\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$. ▮

61. Show that if we have a function from a set of size n to a set of size less than n, then f is not one-to-one.

Solution: Let T be the set of size less than n, and S be the set of size n. Let $B_j = \{i | f(i) = j\}$ for each j in T. Then the nonempty sets among the B_js form a partition of S and the number of blocks is less than the size of S. Therefore by the pigeonhole principle, there is at least one block with at least two elements, so there are two elements i_1 and i_2 such that $f(i_1) = f(i_2)$. ▮

•62. Show that if S and T are finite sets of the same size, then a function f from S to T is one-to-one if and only if it is onto.

Solution: First suppose that f is a one-to-one function from S to T, sets which have the same size. Let $B_j = \{i | f(i) = j\}$ for each j in T. If f is not onto, then the number of nonempty sets B_j is smaller than the number of elements of T and thus is smaller than the size of S. The nonempty sets B_j are a partition of S. But then by the pigeonhole principle, some nonempty B_j has two or more elements, contradicting the assumption that f is one-to-one. Therefore if f is one-to-one, then it is onto. Now suppose that f is an onto function from S to T, sets of the same size. Again let $B_j = \{i | f(i) = j\}$ for each j in T. The size of the union of the sets B_j is, by the sum principle, the sum of their sizes. Since f is onto, each B_j has at least one element. Since the number of sets B_j is the number of elements of S, if one of those sets has more than one element, the the size of their union is more than the size of S, which is a contradiction since they are subsets of S. Therefore each set B_j has exactly one element and therefore f is one-to-one. ▮

·63. There is a *generalized pigeonhole principle* which says that if we partition a set with more than kn elements into n blocks, then at least one block has at least $k+1$ elements. Prove the generalized pigeonhole principle.

Solution: Suppose we partition a set S of more than kn elements into n blocks. If each block has at most k elements, the by the sum principle the size of S is at most kn. But this is a contradiction, so some block has at least $k+1$ elements. ▌

➔ 5. A list of parentheses is said to be balanced if there are the same number of left parentheses as right, and as we count from left to right we always find at least as many left parentheses as right parentheses. For example, (((()())) ()) is balanced and ((()) and (( ) ()))(() are not. How many balanced lists of n left and n right parentheses are there?

Solution: The number is the Catalan Number: we get a bijection between balanced lists of parentheses and Catalan paths by sending each left parenthesis to an upstep and each right parenthesis to a down-step. The condition that there are always as many left parentheses as right ensures we never go below the x axis. ▌

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