Lecture 11 - Atoms and Cavities

This lecture introduces methods for studying one two-level atom, interacting with a single mode of light. We begin with a brief derivation of the interaction Hamiltonian needed, known as the Jaynes-Cummings Hamiltonian, starting from quantum electrodynamics (QED). We then review the physics of a classically controlled spin. Studying the same scenario, but with a full quantum treatment based on the Jaynes-Cummings Hamiltonian then allows us to appreciate some of the richness of atom-photon interactions, and the limitations of semiclassical approximations, particularly in the context of cavity QED.

The QED Hamiltonian

Consider a single electron charge interacting with a single mode of the electromagnetic field. From QED, we know this interaction is governed by the Hamiltonian

$$H = \frac{1}{2m}\left[\ \vec{p} - e\vec{A}(\vec{r},t)\right]^2 + e\phi(\vec{r},t) + V(r) + H_{\rm field}$$

where p is the electron's momentum, m its mass, e its charge; $\vec{A}(\vec{r},t)$ is the vector potential of the electromagnetic field at the position $\vec{r}$ of the electron; $\phi(\vec{r},t)$ is the scalar potential; $V(r)$ is the potential binding the electron to a certain position (eg as in an atom), and $H_{\rm field}$ is the free field Hamiltonian which we have previously modeled as being $\hbar\omega\, a^\dagger\, a$.

Recall that the electric and magnetic fields are related to the vector and scalar potentials through $\vec{E} = -\nabla\phi - \partial_t\vec{A}$ and $\vec{B} = \nabla\times\vec{A}$, and that we may choose a gauge such that $\phi = 0$ and $\nabla\cdot\vec{A} = 0$ (the Coulomb, or "radiation" gauge).

Suppose the field is a plane wave, interacting with the atom binding the charge. Because the atom is typically much smaller than the wavelength $1/k$ of the field, we may approximate $\vec{k}\cdot\vec{r} \ll 1$, so that $\vec{A}(\vec{r},t) \sim \vec{A}(\vec{r}_0,t)$, where $\vec{r}_0$ is the position of the atom. The Schrödinger equation for this system,

$$H|\psi\rangle = i\hbar\partial_t|\psi\rangle$$

is not immediately solvable, through direct exponentiation of H, because H is time varying (due to the field).

Solution of this equation of motion may be accomplished by transforming into a moving frame of reference, in a manner which is useful for later reference. Specifically, we may define the moving frame state

$$|\psi\rangle = \exp\left[\frac{ie}{\hbar}\left(\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right)\right]|\phi\rangle$$

motivated by the fact that $\exp\left[\frac{ie}{\hbar}\left(\vec{A}(\vec{r}_0,t)\cdot\vec{r}\right)\right]$ is a unitary operator which shifts the momentum by amount $e\vec{A}(\vec{r}_0,t)$; this is precisely what is needed to remove the time varying field from H. In particular, after substitution and simplification, we find that the equation of motion for $|\phi\rangle$ is

$$i\hbar\partial_t|\phi\rangle = \left[\left(\frac{p^2}{2m} + V(r) + H_{\text{field}}\right) - e\vec{r}\cdot\vec{E}\right]|\phi\rangle$$

where the first term in parentheses on the right is the free system Hamiltonian H_0, and $e\vec{r}\cdot\vec{E}$ is interpreted as the dipole interaction Hamiltonian.

Let us focus on H_I, in the case of a two-level atom. Note that $\vec{r}$ is an operator. For a two-level system, with energy levels $|e\rangle$ and $|g\rangle$, it is usually the case that $\langle r\rangle = 0$ for both of these eigenstates. $\langle r\rangle$ is nonzero for superpositions, such as $(|g\rangle + |e\rangle)/\sqrt{2}$. Without loss of generality, we may thus let

$$\vec{r} = d\hat{x}(|g\rangle\langle e| + |e\rangle\langle g|)$$

or in terms of the Pauli matrices σ_x, σ_y, and σ_z, we may write $\vec{r} = \hat{x}d\sigma_x$. Assuming the electric field is also along the $\hat{x}$ direction, such that

$$\vec{E} = E\hat{x}(a + a^\dagger),$$

we have that

$$H_I = dE(|g\rangle\langle e| + |e\rangle\langle g|)(a + a^\dagger)$$

Of the four terms in this expression, the $a|g\rangle\langle e|$ and $a^\dagger|e\rangle\langle g|$ terms involve removing and adding two quanta of energy (one photon and one atomic transition). When those two energies are nearly equal, those two interactions are much more unlikely to occur than the $a|e\rangle\langle g|$ and $a^\dagger|g\rangle\langle e|$ terms, which move quanta of energy between the field and atom, conserving energy. It is thus a reasonable to drop the two-quanta terms (the "rotating wave approximation"), leaving us with the interaction Hamiltonian

$$H_I = g\ \left[\ a^\dagger\,\sigma^- + a\sigma^+\right]$$

where $\sigma^- = |g\rangle\langle e|$ and $\sigma^+ = |e\rangle\langle g|$. This is the Jaynes-Cummings interaction Hamiltonian, and will be the basis for all the following discussion, as well as much of the fields of quantum optics and atomic physics. It describes the interaction of one atom with a single mode of the electromagnetic field, with no decay mechanisms (in particular, no spontaneous emission), and no photon loss. Physically, you can think of the scenario governed as being an infinitely massive atom held fixed in the middle of a single mode optical cavity with perfect mirrors.

Classical control of a spin

We would now like to consider some of the physics of the Jaynes-Cummings interaction Hamiltonian, in the limit of a classical electromagnetic field. This will provide us with some intuition about how a two-level system behaves, in the absence of complication about the quantum nature of the field. It will also let us review some basic atomic physics, specifically the physics of resonance and two-state spins, using the language which will later be employed in our study of the optical Bloch equations.

When the electromagnetic field is a strong coherent state $|\alpha\rangle$ with $\alpha \gg 1$, we may approximate that $a|\alpha\rangle\ \sim \alpha|\alpha\rangle$ and $a^\dagger\ |\alpha\rangle\ \sim \alpha^*|\alpha\rangle$, so for $\alpha = \alpha_0 e^{-i\omega t}$. This gives us an atom-field Hamiltonian (letting $\hbar = 1$):

$$H = \frac{\omega_0}{2}\sigma_z + g_0\alpha_0\ \ \left[e^{i\omega t}\sigma^- + e^{-i\omega t}\sigma^+\right]$$

where the first term is the free Hamiltonian of the atom, with transition frequency ω_0, and the field has frequency ω. Letting $g = g_0\alpha_0$ (this turns out to be half the Rabi frequency), and rewriting the atomic raising and lowering operators with Pauli operators, we find that

$$H = \frac{\omega_0}{2}\sigma_z + g(\sigma_x \cos\omega t + \sigma_y \sin\omega t)$$

Define $|\phi(t)\rangle = e^{i\omega t\sigma_z/2}|\chi(t)\rangle$, such that the Schrödinger equation

$$i\partial_t|\chi(t)\rangle = H|\chi(t)\rangle$$

can be re-expressed as

$$i\partial_t|\phi(t)\rangle = \left[e^{i\omega\sigma_z t/2} H e^{-i\omega\sigma_z t/2} - \frac{\omega}{2}\sigma_z \right] |\phi(t)\rangle\,.$$

Since

$$e^{i\omega\sigma_z t/2}\sigma_x e^{-i\omega\sigma_z t/2} = (\sigma_x \cos\omega t - \sigma_y \sin\omega t)$$

this simplifies to become

$$i\partial_t|\phi(t)\rangle = \left[\frac{\omega_0 - \omega}{2}\sigma_z + g\sigma_x \right] |\phi(t)\rangle$$

where the terms on the right multiplying the state can be identified as the effective `rotating frame' Hamiltonian. The solution to this equation is

$$|\phi(t)\rangle = e^{i\left[\frac{\omega_0-\omega}{2}\sigma_z + g\sigma_x\right]t}|\phi(0)\rangle\,.$$

The concept of resonance arises from the behavior of this time evolution, which can be understood as being a single qubit rotation about the axis

$$\hat{n} = \frac{\hat{z} + \frac{2g}{\omega_0-\omega}\hat{x}}{\sqrt{1+\left(\frac{2g}{\omega_0-\omega}\right)^2}}$$

by an angle

$$|\vec{n}| = t\sqrt{\left(\frac{\omega_0 - \omega}{2}\right)^2 + g^2}\,.$$

When ω is far from ω_0, the qubit is negligibly affected by the laser field; the axis of its rotation is nearly parallel with $\hat{z}$, and its time evolution is nearly exactly that of the free atom Hamiltonian. On the other hand, when $\omega_0 \approx \omega$, the free atom contribution becomes negligible, and a small laser field can cause large changes in the state, corresponding to rotations about the $\hat{x}$ axis. The enormous effect a small field can have on the atom, when tuned to the appropriate frequency, is responsible for the concept of atomic `resonance,' as well as nuclear magnetic resonance.

Let $\delta = \omega_0 - \omega$ be the detuning between atom and field. For $\delta = 0$, the on-resonance case, the coherent field causes a rotation of the atomic state by $\exp(igt\sigma_x)$, such that for $gt = \pi/4$ we have a 90° rotation of the spin about the $\hat{x}$ axis. In the limit of large δ, the far off-resonance case, the spin is rotated by $\exp(i\delta t\sigma_z)$. The leading correction to it, proportional to g^2/δ, is the AC Stark shift.

These spin dynamics are widely observed, but nevertheless, still just an approximation. When the control field $|\alpha\rangle$ is weak, then the original assumptions made, specifically that $a^\dagger\ |\alpha\rangle\ \sim a^*|\alpha\rangle$, are no longer good. For example, when the mean photon number in the control field, $|\alpha|^2$ is, say 16, the true dynamics of the system are far from the semiclassical NMR-like picture given here.

Jaynes-Cummings Hamiltonian

The full Jaynes-Cummings Hamiltonian, describing the quantum evolution of a single two-level atom with a single mode electromagnetic field, is given by

$$H = \frac{\omega_0}{2}\sigma_z + \omega\, a^\dagger\, a + g\ \left[a^\dagger\, \sigma^- + a\sigma^+\right]$$

where ω_0 is the transition frequency of the atom, and the field has frequency ω, and g is the coupling constant (we called it g_0 above). One of the most important facts about this Hamiltonian is that it is fully solvable. Here, we provide a solution in the interaction picture, obtained at zero detuning, $\delta = \omega_0 - \omega = 0$, in the frame of reference of bare Hamiltonians of the atom and field. The Hamiltonian in this frame is simply the Jaynes-Cummings interaction Hamiltonian,

$$H_I = g\ \left[a^\dagger\, \sigma^- + a\sigma^+\right]$$

which is easily exponentiated using the fact that for $\sigma^+ = |e\rangle\ \langle g|$ and $\sigma^- = |g\rangle\ \langle e|$,

$$\begin{array}{rcl} \sigma^+\sigma^- & = & |e\rangle\ \langle e| \\ \sigma^-\sigma^+ & = & |g\rangle\ \langle g|\,. \end{array}$$

From this, it follows that

$$\begin{array}{rcl} \left[a^\dagger\, \sigma^- + a\sigma^+\right]^{2k} & = & (a\, a^\dagger)^k |e\rangle\ \langle e| + (a^\dagger\, a)^k |g\rangle\ \langle g| \\ \left[a^\dagger\, \sigma^- + a\sigma^+\right]^{2k+1} & = & (a\, a^\dagger)^k a |e\rangle\ \langle g| +\ a^\dagger\ (a\, a^\dagger)^k |g\rangle\ \langle e|\,. \end{array}$$

Thus, letting $n = a^\dagger\ a$, we find for the time evolution operator U:

$$\begin{array}{rcl} U & = & e^{-iHt} \\ & = & \sum_k \frac{(-iHt)^k}{k!} \\ & = & \cos(gt\sqrt{n+1})|e\rangle\ \langle e| + \cos(gt\sqrt{n})|g\rangle\ \langle g| \\ & & -i\frac{\sin(gt\sqrt{n+1})}{\sqrt{n+1}} a|e\rangle\ \langle g| - i\ a^\dagger\ \frac{\sin(gt\sqrt{n+1})}{\sqrt{n+1}} |g\rangle\ \langle e|\,. \end{array}$$

An arbitrary state of the atom and field can be written as

$$|\psi\rangle\ = \sum_n \alpha_n |e, n\rangle\ + \beta_n |g, n\rangle\,.$$

so that the state at time t is given by $|\psi(t)\rangle\ = U(t)|\psi\rangle$.

There are many other ways to solve the Jaynes-Cummings interaction, with $\delta = 0$ or even otherwise. The approach given here is sufficient for our goal, to explore some of the non-classical behavior of a single atom with a single mode field.

Cavity QED

Two of the most important features of a single atom interacting with a single mode electromagnetic field, in the absence of decay and loss, may be obtained from the above solution of the Jaynes-Cummings Hamiltonian. In particular, we find that an initial state with the atom being in $|e\rangle$, and the field being arbitrary evolves to become

$$\begin{array}{rcl} U(t)\sum_n \alpha_n^0 |e,n\rangle & = & \sum_n \alpha_n^0 \left[\; \cos(gt\sqrt{n+1})|e,n\rangle - i\sin(gt\sqrt{n+1})|g,n+1\rangle \right] \\ & = & \sum_n \left[\, \alpha_n(t)|e,n\rangle \; + \beta_{n+1}(t)|g,n+1\rangle \, \right] \end{array}$$

where

$$\begin{array}{rcl} \alpha_n(t) & = & \alpha_n^0 \cos(gt\sqrt{n+1}) \\ \beta_{n+1}(t) & = & -i\alpha_n^0 \sin(gt\sqrt{n+1})\,. \end{array}$$

Let $P(t) = \sum_n |\alpha_n(t)|^2 - |\beta_n(t)|^2$ be the polarization of the atom. Defining $\Omega_n^2 = \delta^2 + 4g^2(n+1)$, one can show that at finite detuning $\delta = \omega_0 - \omega$, this polarization is

$$P(t) = \sum_n \alpha_n^2 \; \left[\; \frac{\delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2}\cos(\Omega_n t) \right] \; .$$

Vacuum Rabi Oscillations

Suppose initially there are no photons, so only $\alpha_0 = 1$. Then

$$P(t) = \frac{\delta^2 + 4g^2\cos(\Omega_0 t)}{\Omega_0^2}$$

meaning that the atom in its ground or excited states is not in a stationary state. Specifically, the state of the system oscillates between $|e,0\rangle$, an excited atom with no photon in the cavity, and $|g,1\rangle$, a ground state atom with a single photon in the cavity. The frequency of this oscillation at $\delta = 0$ is $2g$, a quantity known as the vacuum Rabi splitting, and the oscillations are known as vacuum Rabi oscillations. Such oscillations have been observed in a wide variety of experimental systems, including solid state devices.

After a time $t = \pi/2g$ the system is in an eigenstate of the number operator $a^\dagger a \left|\psi\right\rangle = -ia^\dagger a \left|g\right\rangle \left|1\right\rangle = \left|\psi\right\rangle$.

The Fock state with exactly one photon is very different from coherent states created by a classical current; for example the classical coherence vanishes for the Fock state: $\langle \hat{a} \rangle = 0$.

The state evolution may appear similar to the spontaneous emission of a single photon from an excited atom, but there is an important difference. Whereas spontaneous emission produces a single photon wavepacket distributed over an infinite number of free space modes, the vacuum Rabi oscillation produces a single photon in a well defined mode. The emission process is thus fully reversible, and the photon can be reabsorbed by the atom. Such reabsorption is impossible in the Wigner Weisskopf theory, but it is predicted by this analysis because we have disregarded incoherent interactions. We have implicitly assumed that the vacuum Rabi frequency g is much greater than the rate of dissipation or decoherence. Achieving this so-called "strong coupling regime" is a significant challenge to experimentalists.

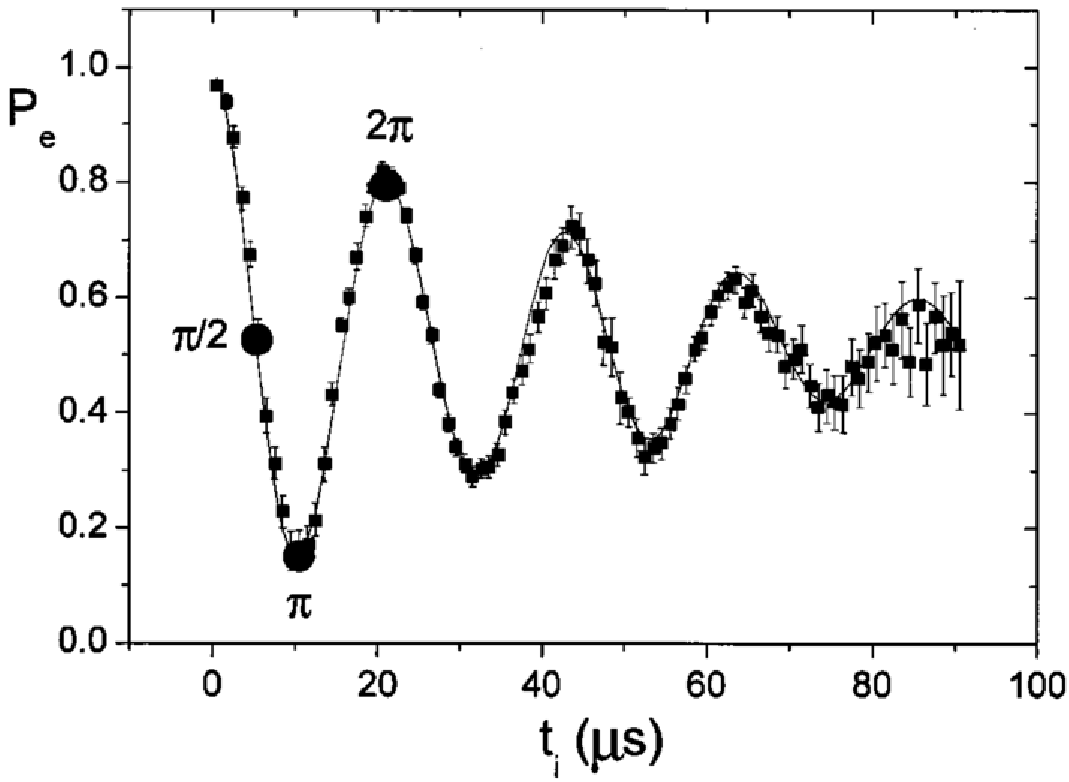


FIG. 3. Vacuum Rabi oscillations. The atom in state e enters an empty resonant cavity. P_e denotes the probability for detecting the atom in e as a function of the effective interaction time t_i. Three important interaction times (corresponding to the $\pi/2$, π, and 2π Rabi rotations) are indicated.

Reference: M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 76, 1800

Review: J. M. Raimond, M. Brune, and S. Haroche, Manipulating quantum entanglement with atoms and photons in a cavity, Rev. Mod. Phys. **73**, 565 (2001)

Collapse and Revival

Finally, let us return to the approximation made in studying the classical control of the two-level atom. Our solution of the Jaynes-Cummings Hamiltonian allows us to now compute what happens when the control field is a coherent state, but instead of being a strong, it has few photons. At zero detuning,

$$P(t) = \sum_n \alpha_n^2 \, \cos(\Omega_n t) \, .$$

where $\Omega_n = 2g\sqrt{n+1}$ may be interpreted as being the Rabi frequency induced by n photons. For a strong coherent state, the photon number distribution α_n^2 is strongly peaked about $|\alpha|^2$, with a width of $|\alpha|$, so that the width is much smaller than the mean for large α. For small α, however, the fields oscillating at different frequencies can interfere with each other, causing the net atomic polarization to decay, in sharp contrast to the continuous rotations expected in the semiclassical picture. Moreover, because of the discreteness of the number of oscillating frequencies, there can be Poincare recurrences in the polarization.

Here is a plot of the case when α_n is the photon distribution for a $|\alpha|^2 = 16$ coherent state:

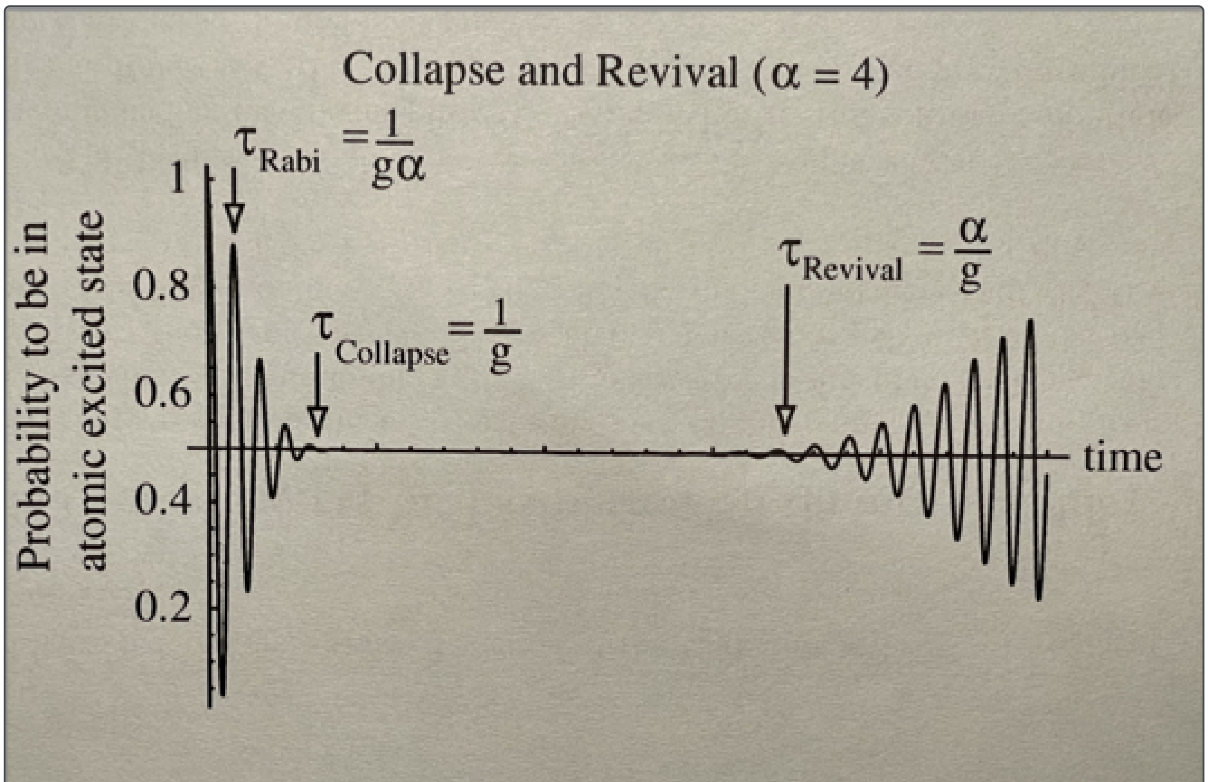


Figure 9.3: Collapse and revival of Rabi oscillations for a cavity field initially in a coherent state with $\alpha = 4$.

Let's look at the phase of the sine and cosine functions in detail:

$$\phi_n = g\sqrt{n+1}t \approx g\sqrt{\bar{n}+\delta n t}$$

$$\approx gt\sqrt{\bar{n}}\left(1+\frac{\delta n}{2\bar{n}}\right) = gt\sqrt{\bar{n}}\left(\frac{\bar{n}+n}{2\bar{n}}\right)$$

To first order in $\delta n/n$, the phase depends linearly on n. This observation allows us to capture the main effect of photon number fluctuations, because it shows that at long times even small δn will have a non-negligible contribution to the phase. Qualitatively, these corrections introduce phases into the sum which depend on the precise value of δn, thereby dephasing the wavepacket.

The photon number fluctuations first become significant when the phase corrections $gt\delta n/\sqrt{\bar{n}}$ approach unity:

$$\tau_{\text{Collapse}} \sim \frac{\sqrt{\bar{n}}}{\delta n\, g} \sim \frac{1}{g}$$

The different Fock states get out of phase with each other, so the Rabi oscillations damp out. If we evolve for even longer times, however, the Rabi oscillations revive. The revival time,

$$\tau_{\text{Revival}} \sim 2\pi \frac{\sqrt{\bar{n}}}{g} \sim 2\frac{\pi|\alpha|}{g}$$

corresponds to the time required for the phase corrections $gt\sqrt{\bar{n}}\frac{\delta n}{2\bar{n}}$ to equal multiples of π for the dominant Fock states (it can be seen that multiples of π rather than 2π are sufficient since the sin and cos terms will cause at most an overall phase shift).

The collapse and revival phenomenon represents a generic feature of many quantum non-linear systems which are isolated from their environments. In this case, the revival is imperfect, but there exists other nonlinear systems which exhibit complete revivals of oscillations.

The origin of the collapse and revival phenomenon lies in the field evolution. If we let a single-mode coherent field interact with a nonlinear medium (such as a two-level atom), at long times it will not remain in a coherent state. The situation is perhaps made most apparent by using our approximate value for ϕ_n to find the state at later times $t > 0$. Using

$$\phi_n \approx gt\sqrt{\bar{n}}\left(\frac{\bar{n}+n}{2\bar{n}}\right)$$

over the range of relevant n we can write

$$i\sin(g\sqrt{n+1}t) = \frac{1}{2}\left(e^{i\phi_n} - e^{-i\phi_n}\right)$$
$$\approx e^{ig\sqrt{\bar{n}}t/2}e^{ignt/2\sqrt{\bar{n}}} - e^{-ig\sqrt{\bar{n}}t/2}e^{-ignt/2\sqrt{\bar{n}}}$$

The ground state contribution to the wavefunction can then be written at arbitrary times as

$$\sum_n e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}} i\sin g\sqrt{n+1}t \left|g\right\rangle \left|n+1\right\rangle \approx$$

$$\frac{1}{2}\left|g\right\rangle \otimes \sum_n e^{-|\alpha^2|/2}\left(e^{ig\sqrt{\bar{n}}t/2}\frac{(\alpha e^{igt/2\sqrt{\bar{n}}})^n}{\sqrt{n!}} - e^{-ig\sqrt{\bar{n}}t/2}\frac{(\alpha e^{-igt/2\sqrt{\bar{n}}})^n}{\sqrt{n!}}\right) =$$

$$\frac{1}{2}\left|g\right\rangle \otimes \left(e^{ig\sqrt{\bar{n}}t/2}\left|\alpha e^{igt/2\sqrt{\bar{n}}}\right\rangle - e^{-ig\sqrt{\bar{n}}t/2}\left|\alpha e^{-igt/2\sqrt{\bar{n}}}\right\rangle\right)$$

A similar calculation can be done with the excited state term and yields the total state

$$\left|\psi(t)\right\rangle = \frac{1}{\sqrt{2}}\left(\left|e\right\rangle\left|\alpha_+\right\rangle + \left|g\right\rangle\left|\alpha_-\right\rangle\right)$$

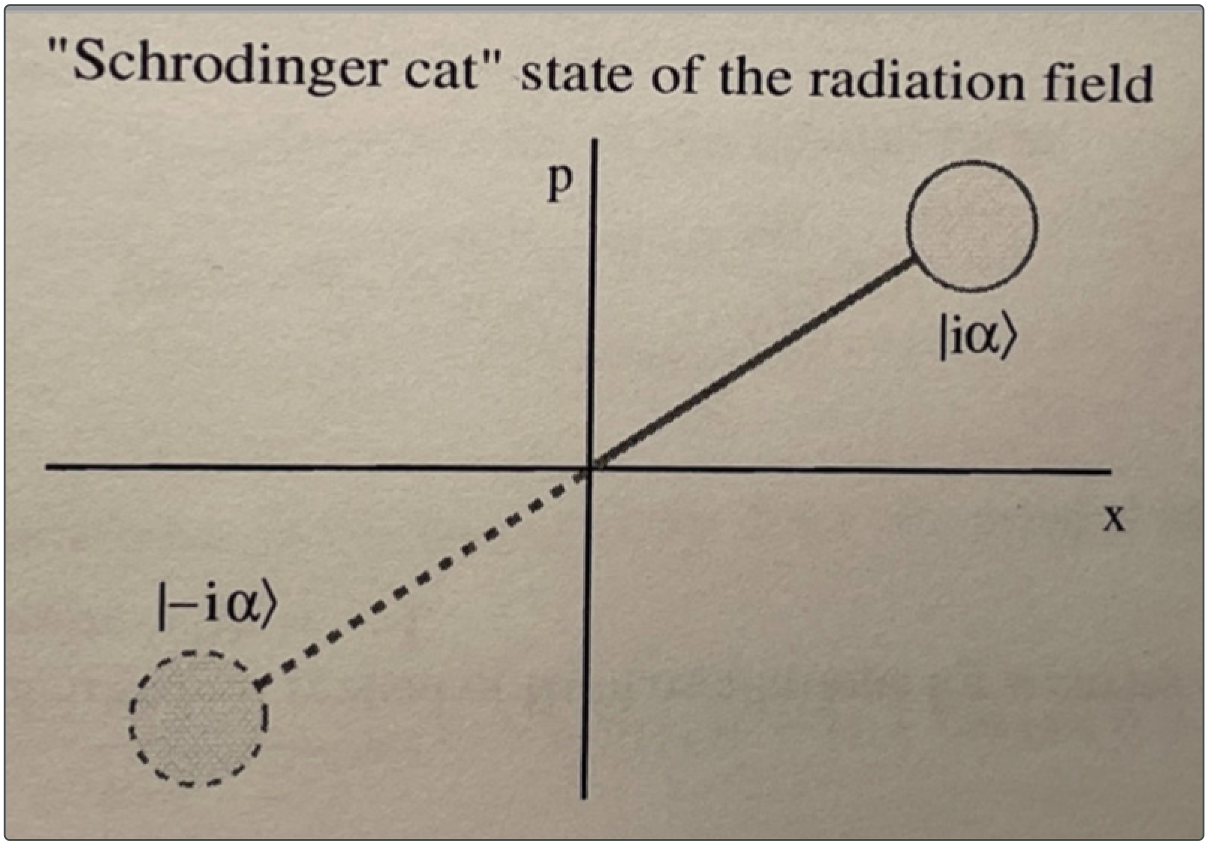
where

$$\left|\alpha_+\right\rangle = \frac{1}{\sqrt{2}}\left(e^{ig\sqrt{\bar{n}}t/2}\left|\alpha e^{igt/2\sqrt{\bar{n}}}\right\rangle + e^{-ig\sqrt{\bar{n}}t/2}\left|\alpha e^{-igt/2\sqrt{\bar{n}}}\right\rangle\right)$$

$$\left|\alpha_-\right\rangle = \frac{1}{\sqrt{2}}\left(e^{ig\sqrt{\bar{n}}t/2}\left|\alpha e^{igt/2\sqrt{\bar{n}}}\right\rangle - e^{-ig\sqrt{\bar{n}}t/2}\left|\alpha e^{-igt/2\sqrt{\bar{n}}}\right\rangle\right)$$

The atom and the field are in general no longer factorizable, and the field state is very different from a coherent state.

At time $gt = \pi\sqrt{\bar{n}} \sim \tau_{\text{Revival}}/2$, suppose we measure the internal state of the atom, which in turn projects the field component into either the state $\left|\alpha_+\right\rangle$ or $\left|\alpha_-\right\rangle$, where $\left|\alpha_\pm\right\rangle \sim \left|\alpha e^{i\pi/2}\right\rangle \mp \left|\alpha e^{-i\pi/2}\right\rangle = \left|i\alpha\right\rangle \mp \left|-i\alpha\right\rangle$.



A phase space depiction of the field state $\sim |i\alpha\rangle - |-i\alpha\rangle$ near $\tau_{\text{Revival}}/2$.

Note that the classical coherence of this state $\langle \hat{a} \rangle \sim e^{-|\alpha|^2}$ is exponentially small for large α, but it possesses a higher order quantum coherence. This kind of superposition state is often called a "Schrödinger cat": much like the infamous feline, the two superposed states $|\pm i\alpha\rangle$ are macroscopically distinguishable. Consequently this kind of state is very difficult to produce experimentally because it is easily destroyed by decoherence and dissipation. Nonetheless, such techniques have been demonstrated (see, e.g., Phys. Rev. Lett. **94**, 010401 (2005)).

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