Classical Field Theory

Feb 7, 2011

$\boxed{\text{Action Principle}}$

Action: $S = \int_a^b L\,dt \qquad L = T - V$

(See E-L method in Farlow)

$\delta L = m\dot{x}\,\delta(\dot{x}) - \frac{dV}{dx}\delta x$

$= m\dot{x}\,\dot{(\delta x)} - \frac{dV}{dx}\delta x$

$= m\left[-\ddot{x}\,\delta x + \frac{d}{dt}(\dot{x}\,\delta x)\right] - \frac{dV}{dx}\delta x \qquad \delta x = 0 \text{ @ } t = a, b$

$= -m\ddot{x}\,\delta x - \frac{d}{dt}(\dot{x}\,\delta x)\,m - \frac{dV}{dx}\delta x$

$\delta S = \int_a^b \delta L\,\cancel{dt} = \cdots = -\int_a^b \left(m\ddot{x} + \frac{dV}{dx}\right)\delta x\,dt = 0 \;\forall\,\delta x$

So $m\ddot{x} + \frac{dV}{dx} = 0$

"t"

$m\ddot{x} = -\frac{dV}{dx} = -\vec{\nabla}V$

Claim $\boxed{\text{All fundamental physics obeys least action principle.}}$

To do this relativistically, use $L = \int d^3x\,\mathcal{L}(x)$ → Lagrangian density

$\boxed{S = \int L\,dt = \int \mathcal{L}(x)\,d^4x}$ ⟶ In Sean Carroll's

In flat spacetime $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}$

In Carroll's book, $\eta_{\mu\nu} = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix}$

Fields $\longrightarrow$ Scalar field

"Everything is fields in field theory"

spin 0

$\longrightarrow$ has 1 component, 1 degree of freedom

simplest case $\longrightarrow$ moving field in 1D

$$\boxed{\phi(x) \sim e^{-i\vec{k}\vec{x}}}$$

$k^\mu = (k^0, \vec{k})$

$x^\mu = (t, \vec{x})$

$$k \cdot x = k_\mu x^\mu = \eta_{\mu\nu} k^\nu x^\mu$$

$(\hbar = c = 1)$

$\rightarrow [E] = [\omega] = [k^0]$

So $\phi(x) \sim e^{-i\vec{k}\vec{x}} = e^{-ik_\mu x^\mu}$

$$= \exp\left[-ik^0 t + i\vec{k}\cdot\vec{x}\right]$$

$$= e^{-i\omega t} e^{i\vec{k}\cdot\vec{x}}$$

Massive Scalar fields $\qquad E^2 = m^2 + \vec{p}^2$

So $(k^0)^2 = m^2 + (\vec{k})^2$

$$\underbrace{(k^0)^2 - (\vec{k})^2}= m^2$$

$$\boxed{k^\mu k_\mu = m^2} \longrightarrow \text{massive particle}$$

$k \rightarrow$ wave number

$$\boxed{k^\mu k_\mu = 0} \longrightarrow \text{massless obeys this.}$$

harmonic osc.

How does this motivate Lagrangian density for a scalar field $\phi(x)$

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2} \rightarrow \text{in S.C. book}$$

Action $S = \int \mathcal{L}\, d^4x$ w.r.t $\phi \rightarrow \delta\phi$

$$\delta S = \int \underbrace{\delta\mathcal{L}}_{\rightarrow 0}\, d^4x$$

Do $\delta\mathcal{L}$ w.r.t ϕ

$$\delta\mathcal{L} = \frac{1}{2}(\partial_\mu \delta\phi)(\partial^\mu \phi) + \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \delta\phi) - m^2 \phi\, \delta\phi$$

same

$$= (\partial_\mu \delta\phi)(\partial^\mu \phi) - m^2 \phi\, \delta\phi$$

..int by parts $= -(\partial_\mu \partial^\mu \phi)\, \delta\phi - m^2 \phi\, \delta\phi$

(dropping total derivative)

Call $\partial_\mu \partial^\mu \equiv \square \rightarrow$ d'Alembertian

$$= \partial_0 \partial^0 + \partial_j \cdot \partial^j = \frac{d^2}{dt^2} - \vec{\nabla}^2$$

So $\delta\mathcal{L} = -(\square + m^2)\phi\, \delta\phi$

Solution $\Rightarrow \delta\mathcal{L} = 0 \ \forall\ \delta\phi$

So Klein-Gordon Eqn

$$\boxed{(\square + m^2)\phi = 0}$$

What are the solutions to $(\Box + m^2)\,\phi(x) = 0$ [$\phi(x)$]

- try $\phi(x) = e^{-ik_\mu x^\mu} = e^{-ik_\alpha x^\alpha}$

$$\partial_\mu \phi = -i\,\partial_\mu (k_\alpha x^\alpha)\, e^{-ik_\beta x^\beta}$$

$$= -i k_\alpha \partial_\mu x^\alpha e^{-ik\cdot x}$$

$$= -i k_\alpha \delta^\alpha_\mu \phi = -ik_\mu \phi$$

$\underbrace{\partial^\mu = \eta^{\mu\nu}\partial_\nu}$ only flat spacetime @ this pt.

$$\partial^\mu \partial_\mu \phi = (-i)^2 k_\mu k^\mu \phi$$

So $\Box\phi = -k_\mu k^\mu \phi = -m^2\phi$ (required)

So it's a solution as long as $\boxed{k_\mu k^\mu = m^2}$

massive particle

Vector fields (spin 1) (S.C Book as well)

Instead of $\phi \rightarrow A_\mu \rightarrow$ vector field (photon)

Lagrangian density $\rightarrow$ $\boxed{\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu}$

$j^\mu = (\rho, \vec{J})$, $A^\mu = (V, \vec{A}) \longrightarrow$ vector potential

$\boxed{\begin{aligned}\vec{E} &= -\vec{\nabla} V \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}}$ (static)

Full $\boxed{\begin{aligned}\vec{E} &= -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla}\times\vec{A}\end{aligned}}$ $\longrightarrow$ static + dynamic

$$\boxed{F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ +E^1 & 0 & -B^3 & -B^2 \\ +E^2 & B^3 & 0 & -B^1 \\ +E^3 & B^2 & B^1 & 0 \end{pmatrix}$$

↳ E-M stress tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

With def. $\longrightarrow$ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

<u>Hence</u> $\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$ holds

This identity yields $\left\{ \begin{array}{l} \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 \end{array} \right\}$

The remaining Maxwell eqns come from varying the action $\delta S = 0$ w.r.t $A_\mu(x)$

$$\delta(j^\mu A_\mu) = j^\mu \delta A_\mu$$

and

$$\delta\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) \text{ w.r.t } A_\mu$$

well

$$F^{\mu\nu}F_{\mu\nu} = (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$= 2\partial^\mu A^\nu \partial_\mu A_\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu$$

So

$$\delta\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\right) = -\frac{1}{2}\delta\left(\partial^\mu A^\nu \partial_\nu A_\mu - \partial_\mu A_\nu \partial^\nu A^\mu\right)$$

$$= \cdots -\frac{1}{2}\Big[(\partial_\mu \delta A_\nu)\partial^\mu A^\nu + \partial_\mu A_\nu(\partial^\mu(\delta A^\nu))$$

$$- \underbrace{\partial_\mu(\delta A_\nu)\partial^\nu A^\mu - \partial_\mu A_\nu \partial^\nu(\delta A^\nu)}_{\text{same after indexing} \ldots}\Big]$$

↑ ~~chain~~ product rule

$$= \partial_\mu A_\nu (\partial^\nu \delta A^\mu) - (\partial_\mu \delta A_\nu)\partial^\mu A^\nu$$

Want $\delta S = 0 = \int (\mathcal{L})\, \delta A \, d^4x = 0$

$$= (+\partial_\mu \partial^\mu A^\nu)\, \delta A_\nu + \text{total deriv} - (\partial^\nu \partial_\mu A_\nu)\, \delta A^\mu$$

cov - raise

$$\delta\left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) = +\partial_\nu \partial^\nu A^\mu \, \delta A_\mu - (\partial^\nu \partial^\mu A_\nu)\, \delta A_\mu$$

$$= (\Box A^\mu - \partial^\nu \partial^\mu A_\nu)\, \delta A_\mu = 0 \quad \forall\ \delta A_\mu$$

&

$$\boxed{\Box A^\mu - \partial^\nu \partial^\mu A_\nu = \Box A^\mu - \partial_\nu \partial^\mu A^\nu = 0}$$

Can write this as $\partial_\nu F^{\mu\nu} = 0$

with current $-j^\mu A_\mu$ get

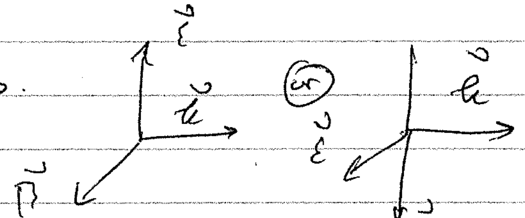
$\boxed{\partial_\nu F^{\mu\nu} = j^\mu}$ → give the remaining Maxwell's eqn

<u>Claim</u> For free photons (EM waves) then $\rho = 0$, $\vec{J} = 0$

& $j^\mu = 0$ eqn then is just $\boxed{\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0}$

How many independent EM waves? 2.

2 transverse polarizations.



2 massless modes for photons (2 polarizations)

→ massless $k_\mu k^\mu = 0$ for photons $E = cp$

But A_μ has 4 df (not 2!)

To be relativistic $\rightarrow$ must use scalars : ϕ

vectors : A_μ

tensors : $g_{\mu\nu}$

There must be 2 degrees of freedom in A_μ that don't matter. Have

$$A_\mu = (A_0, A_j) \text{ has too many degrees of freedom}$$

Turns out $\rightarrow$ physical waves have wave eqn

$$\Box A + \cdots = 0$$

$\Box = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \cdots$; $\cdots$ gives $e^{-ik\cdot x}$

Look at $\mu = 0 \rightarrow A_0 \Rightarrow \Box A_0 - \partial_0 \partial^\nu A_\nu = 0$

$$(\partial^0\partial_0 + \partial_j \partial^j) A_0 - \partial_0 \partial^\nu A_\nu = 0$$

time 2'nd derivs $= 0$ $\longrightarrow$ $\cancel{\partial^0\partial_0 A_0} + \partial^j \partial_j A_0 - \cancel{\partial_0 \partial^0 A_0} - \partial_0 \partial^j A_j = 0$

$\Rightarrow$ $\boxed{A_0 \text{ is not a propagating mode}}$

$\Rightarrow$ Auxiliary mode. (not propagating)

$\rightarrow$ NOT physical.

- This is good, because $(\Box A^\mu + \cdots)\,\delta A_\mu$

$$A^0 = A_0 \text{ but } A^j = -A_j$$

If all 4 were allowed to propagate, we will have a bad sign compared to the other ?

$\Rightarrow$ get a "ghost" mode. "Ghost" has wrong sign KE

. The desire to use $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ was made to eliminate the potential "ghost" mode. Recall scalar: $\mathcal{L} \sim \partial_\mu \phi \partial^\mu \phi$

for A, we might have guessed $\partial_\mu A_\nu \partial^\mu \partial^\nu$ only. But this has

$$\Box A_\mu + \dots = 0 \quad \forall \text{ four modes} \rightarrow \text{"ghost"}$$

$\rightarrow$ Use $-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ so that it don't allow $\frac{\partial^2}{\partial t^2} A_0$.

. The $\frac{\partial^2}{\partial t^2} A_0$ terms cancel.

Now Start w/ $A_\mu \longrightarrow 4$

Find A_0 is aux $\rightarrow 1$

?

} So that get 2 ?

$\boxed{\text{GAUGE SYMMETRY}}$ $\rightarrow$ gauge mode that can be eliminate

Finally $4-1-1 = 2 \rightarrow$ physical.

A_μ aux. gauge symmetry

Look at $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

2 transform $\boxed{A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda(x)}$ $\rightarrow$ gauge transformation

then

$$F_{\mu\nu} \rightarrow \boxed{\begin{aligned} F'_{\mu\nu} &= \partial_\mu (A_\nu + \partial_\nu \Lambda) - \partial_\nu (A_\mu + \partial_\mu \Lambda) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu} \end{aligned}}$$

Can choose $\Lambda(x)$ to eliminate A_μ mode leaving 2.

Feb 14 2019

Alex' Last week → Real Scalar Fields
→ Vector fields

Note $\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0 \longleftarrow \delta S = 0$

- A_μ has 4 components $\quad 4$
- $A_0 \to$ auxiliary $\quad -1$
- gauge field $\quad -1$

$\quad 2$ degrees of freedom

Gauge

Can fix the gauge

$$A_\mu \Rightarrow A_\mu + \partial_\mu \Lambda(x)$$

Pick $\Lambda(x)$ to remove a degree of freedom

Suppose $\underbrace{\partial^\nu A_\nu}_{\mathrm{div}(A)} \neq 0$ → Can pick a gauge that sets $\partial^\nu A_\nu \to 0$

Let $A_\nu \to A'_\nu = A_\nu + \partial_\nu \Lambda$

then $\partial^\nu A'_\nu \to \partial^\nu (A_\nu + \partial_\nu \Lambda)$

$$= \partial^\nu A_\nu + \Box \Lambda$$

If we pick Λ s.t. $\boxed{\Box \Lambda = -\partial^\nu A_\nu}$, then in this gauge

$\partial^\nu A'_\nu = 0$, then drop the prime.

In the fixed gauge, EOM: $\begin{cases} \Box A_\mu = 0 \\ \partial^\nu A_\nu = 0 \rightarrow \text{Lorentz gauge} \end{cases}$

To solve, assume $A_\mu = \varepsilon_\mu e^{-i\tilde{k}\cdot\tilde{x}}$

polarization vector.

- $\Box A_\mu = 0 \Rightarrow$ find that $\underline{k_\mu k^\mu = 0 \text{ must hold}}$

$\rightarrow$ massless vector field (good)

- With $\partial^\nu A_\nu = 0 \rightarrow$ removes 1 degree of freedom that we chose

$$\Box \Lambda = -\partial^\nu A_\nu$$

But this doesn't completely fix Λ → Have a residual gauge freedom → Can use to set $A_0 = 0$

Residual gauge transform $\rightarrow A_\mu \rightarrow A_\mu + \partial_\mu \Lambda$

but with $\Box \Lambda = 0 \longrightarrow \partial^\nu A_\nu = 0$ alone

Look at $A_0 = \varepsilon_0 e^{-i\tilde{k}\cdot\tilde{x}}$ with $k_\mu k^\mu = 0$

- If we pick $\Lambda = \lambda e^{-i\tilde{k}\cdot\tilde{x}}$ then $\Box \Lambda \propto k_\mu k^\mu = 0$

Then $A_0 \longrightarrow A_0 + \partial_0 \Lambda \rightarrow 0$

$$\underbrace{A_0}_{\varepsilon_0 e^{-ikx}} + \underbrace{\partial_0 \Lambda}_{-i\lambda k_0 e^{-ikx}}$$

$$\varepsilon_0 - i\lambda k_0 = 0$$

pick $\lambda = \dfrac{\varepsilon_0}{ik_0}$ then $A_0 = 0 \Rightarrow \partial^\nu A_\nu = \underbrace{\partial^0 A_0}_{0} + \underbrace{\tilde{\nabla}\cdot\tilde{A}}_{0} = 0$

Complete gauge fixing $\begin{cases} A_0 = 0 \\ \vec{\nabla}\cdot\vec{A} = 0 \end{cases} \rightarrow$ Coulomb

Now have $A_\mu \rightarrow 4$ df

$$\vec{\nabla}\cdot\vec{A} \rightarrow -1 \text{ df}$$
$$A_0 \rightarrow -\cancel{0}\ \text{df}$$

$$2 \text{ df} \Rightarrow \text{physical}$$

Now $A_\mu = \varepsilon_\mu e^{-i\vec{k}\cdot\vec{x}}$

$$\varepsilon_\mu = (\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3)$$

$A_0 = 0 \Rightarrow$ lets $\varepsilon_0 = 0$

$\vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{k}\cdot\vec{A} = 0$

Consider γ's moving in the z direction

then $k^\mu = (k, 0, 0, k)$ since $k_\mu k^\mu = 0$

so $\vec{k}\cdot\vec{A} \sim \vec{k}\cdot\vec{\varepsilon} = k\varepsilon^3 = 0$

So, no longitudinal component

So $\varepsilon_\mu = (0, \varepsilon_1, \varepsilon_2, 0)$

So $A_\mu = \varepsilon_\mu e^{-i\vec{k}\cdot\vec{x}} = (0, \varepsilon_1, \varepsilon_2, 0)\, e^{-ik\cdot x}$ ← 4-vector.

So the independent modes can be chosen as $\begin{cases} \varepsilon_\mu^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \\ \varepsilon_\mu^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \end{cases}$

→ 2 physical modes $\left\{ \begin{array}{l} A_\mu^{(1)} = \varepsilon_\mu^{(1)} e^{-ik\cdot x} \\ A_\mu^{(2)} = \varepsilon_\mu^{(2)} e^{-ik\cdot x} \end{array} \right\}$

→ 2 massless ($a_\mu k^\mu = 0$) <u>transverse modes</u>

☒ Why are photons massless? → $\boxed{\text{Because of gauge symmetry}}$

Suppose

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu$$

Under gauge transform → $A_\mu \to A_\mu + \partial_\mu \Lambda$

Have $\left\{ \begin{array}{l} F_{\mu\nu} \to F_{\mu\nu} \\ A_\mu A^\mu \to (A_\mu + \partial_\mu \Lambda)(A^\mu + \partial^\mu \Lambda) \\ \qquad \neq A_\mu A^\mu \end{array} \right.$

→ no longer gauge invariant

→ massive vectors do not have <u>gauge symmetry</u>

Can't add $m^2 A_\mu A^\mu$ if want gauge invariance

$\boxed{\text{massless} \Leftrightarrow \text{gauge invariance}}$

Complex Scalars

→ have 2 degrees of freedom

$$\phi = \phi_1 + i\phi_2$$

$$\phi^* = \phi_1 - i\phi_2$$

or just use ϕ & ϕ^* as 2 df

Could write down

$$\mathcal{L} = (\partial_\mu \phi)(\partial_\mu \phi)^* - m^2 \phi\phi^*$$

can vary w.r.t. to ϕ & ϕ^*

Does this have symmetry? → Yes! (phase symmetry)

$$\text{if } \phi \Rightarrow \phi e^{i\alpha} \longrightarrow \text{constant}$$

so $|\phi|^2 = \phi^*\phi \longrightarrow \phi^*\phi$

likewise $(\partial_\mu \phi)(\partial_\mu \phi)^* \longrightarrow (\partial_\mu \phi)(\partial_\mu \phi)^*$

since $|e^{i\alpha}|^2 = 1$

→ this is a symmetry of $\mathcal{L}$. global $U(1)$ sym.

Group Theory

(1) Consider N-dim complex vector,

$z = (z_1 \dots z_N)^T$. Norm $= \sum_{i=1}^{N} z_i^* z_i = z^\dagger z$

"unitary"

A gauge transformation U that takes $|z|^2 \to |z^2|$

is called a $U(N)$ gauge transform

if $z \to Uz$, then $|z|^2 \to (Uz)^\dagger (Uz)$

$(Uz)^\dagger (Uz) \to z^\dagger U^\dagger U z = |z|^2$

true if $U^\dagger U = 1$

mean that $U^\dagger = U^{-1}$ → Unitary matrix

<u>Note</u> $e^{i\alpha}$ is a 1-by-1 Unitary matrix

<u>So</u> Complex scalars have an $U(1)$ gauge invariance

Special case is when also $\det U = 1$

Special unitary group $SU(N)$

Consider instead N-dim real vector,

$X = (x_1 \ldots x_N) \to$ real

$|x| = \vec{x} \cdot \vec{x} = \sum_{i=1}^{N} x_i x_i$

A transformation $O: x \to Ox$ which leave

$\vec{x}^2$ alone → <u>orthogonal transformation</u>

group → $O(N)$

$|x|^2 = x^T x \to (Ox)^T (Ox) = x^T O^T O x = x^T x$

true if $O^T O = 1$ → <u>orthogonal matrix</u>

Special group → $SO(N)$ $\det O = 1$

rotations → $O(3)$

Lorentz group → $x^\mu = (x^0 \dots x^3) = x^2 = x_\mu x^\mu$

$= (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$

a LT keeps $|x^2|$ unchanged

$x \to \Lambda x$ $\Lambda^T \Lambda = 1$

Lorentz group $SO(3,1)$

- The theory $\mathcal{L} = |\partial_\mu \phi|^2 - m^2 |\phi|^2$ has a global $U(1)$ sym → α = constant

$\phi \to U\phi = e^{i\alpha}\phi$

local $U(1)$ transform → $\alpha = \alpha(x)$

$\phi \to e^{i\alpha(x)}\phi$ different locally

Is this a symmetry? → No.

But if $\phi \to e^{i\alpha(x)}\phi$ then $\partial_\mu \phi \to \partial_\mu (e^{i\alpha(x)}\phi)$

$= i(\partial_\mu \alpha)\phi + e^{i\alpha}\partial_\mu \phi$

$(\partial_\mu \phi)(\partial_\mu \phi)^* \not\to (\partial_\mu \phi)(\partial^\mu \phi)$ no local $U(1)$

But can "fix" "derivative" → Introducing →

Gauge-covariant derivative

$\to$ charge-coupling

$$\boxed{D_\mu = \partial_\mu + iqA_\mu}$$

gauge field

gauge field has symmetry $A_\mu = A_\mu + \partial_\mu \Lambda$

pick $\Lambda = \frac{-\alpha(x)}{q} \to A_\mu \to A_\mu - \frac{1}{q}\partial_\mu \alpha$

with both $U(1)$
$$\begin{cases} \phi \to \phi e^{-i\alpha(x)} \\ A_\mu \to A_\mu - \frac{1}{q}\partial_\mu \alpha \end{cases}$$

then
$$D_\mu \phi = (\partial_\mu + iqA_\mu)\phi$$
$$\Rightarrow \left(\partial_\mu + iq\left(A_\mu - \frac{1}{q}\partial_\mu \alpha\right)\right) e^{i\alpha(x)}\phi$$
$$\to e^{i\alpha}\partial_\mu \phi + \cancel{i(\partial_\mu \alpha)e^{i\alpha}\phi}$$
$$+ iqA_\mu e^{i\alpha}\phi - \cancel{i(\partial_\mu \alpha)e^{i\alpha}\phi}$$
$$= e^{i\alpha}(\partial_\mu \phi + iqA_\mu \phi)$$

$$\boxed{D_\mu \phi \Rightarrow D_\mu \phi\, e^{i\alpha}}$$

$\mathcal{L} \quad (D_\mu \phi)(D^\mu \phi)^* \to (D_\mu \phi)(D_\mu \phi)^*$

(good)

Since we add A_μ, can make it dynamical by also adding $\frac{-1}{4}F_{\mu\nu}F^{\mu\nu}$

<u>Full theory</u> $\boxed{\mathcal{L} = |D_\mu \phi|^2 - m^2|\phi|^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$

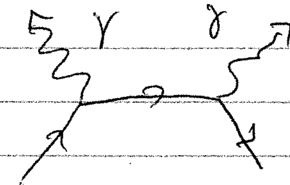
charged scalar field in EM

<u>QED</u> propagating massive scalar $\longrightarrow$

propagating photon $\rightsquigarrow$

$|D_\mu \phi|^2$ contains $A \cdot A \phi^2$

or $A \partial_\mu \phi \phi$



<u>Note</u> $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$

N=1 $U(1), O(1)$ $e^{i\alpha} e^{i\beta} = e^{i\beta} e^{i\alpha}$ $\rightarrow$ <u>commute</u> $\rightarrow$ <u>abelian</u>

N=2 $U(N), SU(N), O(N), N > 1$ $\rightarrow$ <u>non-abelian</u> groups

<u>e.g.</u> strong & weak forces $\rightarrow$ <u>non-abelian</u>

$SU(3)$ <u>$SU(2)$</u>

complicated

<u>Yang-Mills gauge theory</u>

Lie-group, Lie-algebra ...

SPONTANEOUS SYMMETRY BREAKING

↳ mechanism where symmetry still holds dynamically, but the solutions break the symmetry.

$\mathcal{L} \Rightarrow$ has a symmetry. But the solutions break it.

Consider real scalar field ϕ

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - V(\phi)$$

Suppose that $V(-\phi) = V(\phi)$

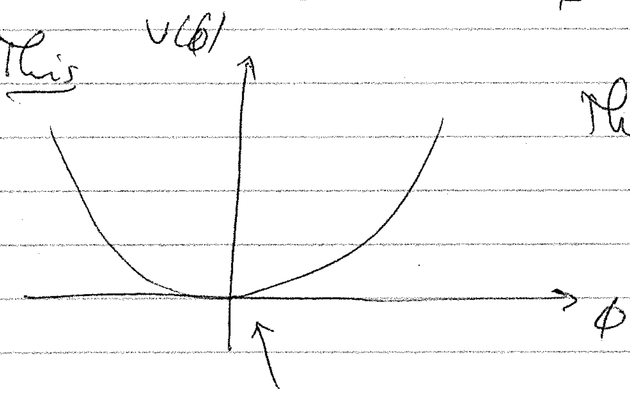
Then $\mathcal{L}$ is invariant under parity transformation.

$$\phi \to -\phi$$

So $\mathcal{L} \to \mathcal{L} \to$ a symmetry.

Ex $V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \begin{cases} \lambda > 0 \\ m^2 > 0 \end{cases}$

This



There is a unique minimum

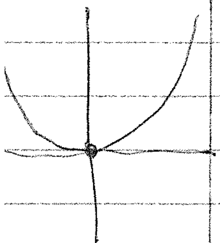
→ ground state of field theory is called the $\boxed{\text{vacuum expectation value}}$

↳ (state of lowest energy)

$$\boxed{\langle \phi \rangle = 0} \quad (\text{VeV})$$

At $\phi = 0$, we still have a symmetry.

~~But now suppose~~ Now, look at small excitation around the vacuum



$$\rightarrow \phi = \langle\phi\rangle + \varepsilon = 0 + \varepsilon = \varepsilon$$

In which case, the Lagrangian becomes

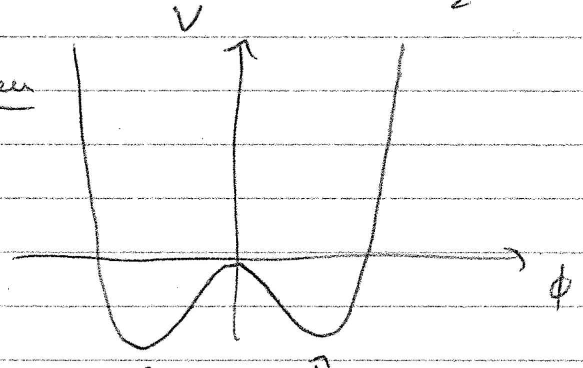
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varepsilon)(\partial^\mu \varepsilon) + \frac{1}{2} m^2 \varepsilon^2 + \cancel{\varepsilon^4}^{\,0}$$

massive particle → real scalar field

But now suppose

Consider $V(\phi) = -\frac{1}{2} m^2 \phi^2 + \frac{1}{4} \phi^4$

then



2 possible vacuum solutions - which one?

→ Nature spontaneously pick one.

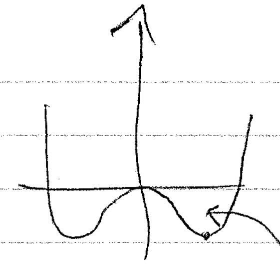
$\mathcal{L}$ still has $\phi \to -\phi$ symmetry

Look at

$$\frac{dV}{d\phi} = m^2\phi + \lambda\phi^3 = 0 \quad (m^2 < 0)$$

$$\rightarrow \langle\phi\rangle = \pm\sqrt{\frac{-m^2}{\lambda}}$$

Suppose it picks the one on the right



$$\langle\phi\rangle = \sqrt{\frac{-m^2}{\lambda}} \equiv v$$

We can shift to a field defined w.r.t. to vacuum.

$$\phi' = \phi - \langle\phi\rangle = \phi - v$$

Then $\langle\phi'\rangle = 0$

In terms of ϕ', the $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi')(\partial^\mu\phi') - (-m^2)\left[\frac{\phi'^4}{4v^2} + \frac{\phi'^3}{v} + \phi'^2 - \frac{v^2}{4}\right]$$

has no symmetry in terms of ϕ' (parity transform)

Symmetry is hidden.

Look at small excitations about $\langle\phi\rangle$,

$$\phi' = \langle\phi'\rangle + \varepsilon = 0 + \varepsilon = \varepsilon$$

Plug in... $\mathcal{L} = \frac{1}{2}(\partial_\mu\varepsilon)(\partial^\mu\varepsilon) - \frac{1}{2}(-2m^2)\varepsilon^2$

again, assuming $m^2 < 0$

acts as a massive particle scalar with mass $(-2m^2) > 0$

→ By breaking symmetry ... → get massive particle.

Verify

$V = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4$

Let $\phi = \langle \phi \rangle + \varepsilon$

$\langle \phi \rangle = \sqrt{\frac{-m^2}{\lambda}} \equiv v$ $= v + \varepsilon$

$\circ$ $\partial_\mu \phi = \partial_\mu \varepsilon$

$\circ$ $V = \frac{1}{2} m^2 (v+\varepsilon)^2 + \frac{1}{4} \lambda (v+\varepsilon)^4$

$= \frac{1}{2} m^2 (v^2 + 2v\varepsilon + \varepsilon^2) + \frac{1}{4} \lambda (v^4 + 4v^3 \varepsilon + 6 v^2 \varepsilon^2 + 4 v \varepsilon^3 + \varepsilon^4)$

keep linear terms $\cong \varepsilon (m^2 v + \lambda v^3) + \varepsilon^2 (\frac{1}{2} m^2 + \frac{3}{2} \lambda v^2) + \varepsilon^3 (\cancel{\ })^{0} + \varepsilon^4 (\cancel{\ })^{0}$

$= \varepsilon v (\underbrace{m^2 + \lambda v^2}) + \varepsilon^2 (-m^2)$

$m^2 - \lambda \frac{m^2}{\lambda} = 0$

So $\boxed{V(\varepsilon) \cong -\varepsilon^2 m^2 = +\frac{1}{2} (-2m^2) \varepsilon^2}$

"Discrete Symmetry" $\longrightarrow$ this symmetry is a discrete symmetry

$\hookrightarrow$ not continuous, unlike rotation

$\boxed{\text{Theorem}}$

Goldstone, MIT: In a theory with a continuous sym that is spontaneously broken, then Goldstone's theorem says there will be a massless particle

(Nambu - Goldstone mode)

(NG)

Consider 2 scalar particles

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \cdot \partial^\mu \phi - V(\phi \cdot \phi)$$

- This theory has a global $O(2)$ symmetry (continuous)

~~Rlt~~ $\phi \longrightarrow \phi' = R\phi$

$$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- θ is continuous, constant (global symmetry)

- Under R $\underbrace{\phi \cdot \phi}_{\phi^T \phi} = \phi'^T \phi' = (R\phi)^T (R\phi)$

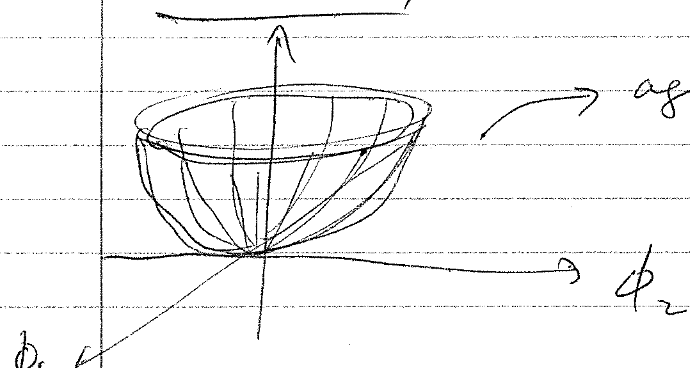
$$= \phi^T R^T R \phi$$

$$= \phi^T \phi$$

$\therefore$ $\mathcal{L} \rightarrow \mathcal{L}$ under $O(2)$

- Suppose $V(\phi \cdot \phi) = \frac{1}{2} m^2 \phi \cdot \phi + \frac{1}{4} \lambda (\phi \cdot \phi)^2$

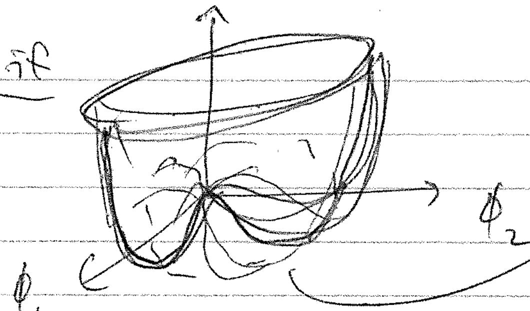
- If $m^2 > 0$, then

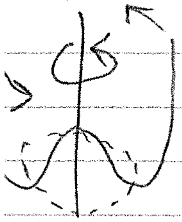


$\rightarrow$ again, has unique vacuum solution.

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

But if





→ have circle of possible ground state.

~ Nature spontaneously picks a vacuum

Let's pick $\langle\phi\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}$ $\quad\quad \langle\phi'\rangle = 0$

Can look at excitations about ϕ

↳ $\phi' = \phi - \langle\phi\rangle = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} - \begin{pmatrix} v \\ 0\end{pmatrix} = \begin{pmatrix}\phi_1 - v \\ \phi_2\end{pmatrix}$

For small excitations

↳ $\phi' = \langle\phi'\rangle + \epsilon = \langle\phi'\rangle + \begin{pmatrix}\eta \\ \xi\end{pmatrix} = \begin{pmatrix}\eta \\ \xi\end{pmatrix}$

Can express $\mathcal{L}$ in terms of these.

Find that → one is massless → NG mode
the other is massive → Higgs particle

Feb 28 2019

$\boxed{\text{Continuous global symmetry}}$

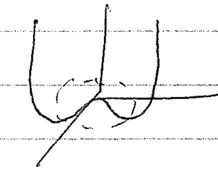
2 scalar fields $\phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$ Consider $\phi' \to R\phi$

where R is a rotation matrix, θ has no x dependent

Look at the $\mathcal{L}$ $\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi\cdot\phi)}$

$m^2 < 0 \Rightarrow \quad V(\phi\cdot\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4$

has min at $\quad \langle\phi\rangle^2 = -\frac{m^2}{\lambda} = v^2$



let's pick $\quad \langle\phi\rangle = \begin{pmatrix} v \\ 0 \end{pmatrix}$

Can shift $\quad \phi' = \phi - \langle\phi\rangle$

so that $\quad \langle\phi'\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Excitation round the vacuum $\quad \phi' = \overset{0}{\langle\phi'\rangle} + \begin{pmatrix} \eta \\ \xi \end{pmatrix} = \begin{pmatrix} \eta \\ \xi \end{pmatrix}$

and $\quad \phi = \langle\phi\rangle + \phi' = \begin{pmatrix} v+\eta \\ \xi \end{pmatrix}$

So $\quad \phi\cdot\phi = (v+\eta)^2 + \xi^2$

and $\quad \partial_\mu\phi = \partial_\mu\phi' = \begin{pmatrix} \partial_\mu\eta \\ \partial_\mu\xi \end{pmatrix}$

Can write $\mathcal{L}$ in terms of the excitations, dropping cubic & higher powers.

$$V(\phi\cdot\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\lambda\phi^4$$

$$= \frac{1}{2}m^2\left[(v+\eta)^2 + \xi^2\right] + \frac{1}{4}\lambda\left[(v+\eta)^2 + \xi^2\right]^2$$

$$= \frac{1}{2}m^2\left[v^2 + 2v\eta + \eta^2 + \xi^2\right] + \frac{1}{4}\lambda\left[v^2 + 2v\eta + \eta^2 + \xi^2\right]^2$$

$$= \frac{1}{2}m^2\left[v^2 + 2v\eta + \eta^2 + \xi^2\right] + \frac{1}{4}\lambda\left[4v^3\eta + 4v^2\eta^2 + 2v^2\xi^2 + 2v^2\eta^2 + \dots\right]$$

$$= \eta\left[\frac{1}{2}m^2\cdot 2v + \frac{1}{4}\lambda 4v^3\right] + \eta^2\left[\frac{1}{2}m^2 + \frac{3}{2}\lambda v^2\right]$$

$+ \xi^2\left(\frac{m^2}{2} + \frac{\lambda}{4}v^2 + \dots\right.$

Recall that minimum at $v^2 = -m^2/\lambda$

So $$V(\phi^2) = v\eta\underbrace{(m^2+\lambda v^2)}_{0} + \eta^2\underbrace{\left[\tfrac{1}{2}m^2 - \tfrac{3}{2}m^2\right]}_{-m^2}$$

$$+\frac{1}{2}\varphi^2\underbrace{\left(m^2+(-m^2)\right)}_{0}$$

So to 2nd order, $V(\phi^2) = -m^2\eta^2 = \frac{1}{2}(-2m^2\eta^2)$

Back to $\mathcal{L}$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi\cdot\phi)$$

$$= \frac{1}{2}\begin{pmatrix}\partial_\mu\eta & \partial_\mu\varphi\end{pmatrix}\begin{pmatrix}\partial^\mu\eta \\ \partial^\mu\varphi\end{pmatrix} - \left[\frac{1}{2}(-2m^2\eta^2)\right] + \ldots$$

$$= \frac{1}{2}\left[\partial_\mu\eta\partial^\mu\eta + \partial_\mu\varphi\partial^\mu\varphi\right] - \frac{1}{2}(-2m^2)\eta^2 + \ldots$$

We started with 2 scalars... $\phi = (\phi_1\ \phi_2)^T$ and wrong sign $m^2<0$
But after spontaneous symmetry breaking around a physical vacuum, we have

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}(-2m^2\eta^2)}_{} + \underbrace{\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi}_{}$$

1 massive scalar η with mass $-2m^2 > 0$ — This is called $\boxed{\text{Higgs Boson}}$

1 massless scalar φ — NG mode

$\boxed{\text{Goldstone}}$ → { For every continuous global symmetry that is spontaneously broken you get a massless mode }

Ex EM → U(1) local gauge symmetry → massless photon

But Weak interactions → we'd like to describe these as a gauge theory as well. SU(2) → 3 gauge fields

But the WI is too weak + short ranged
→ Maybe the weak force is carried by 3 massive vector fields.

But can't have both gauge sym and massive terms for the gauge fields.

Note Any interacting massless particle is detectable because it's got a long-range int

↳ NG mode → detectable, but not seen. ...

$\boxed{\text{Comes Higgs + others}}$ Worked Spon. Sym. Breaking of a local gauge theory
→ Found a mechanism where the massless NG modes get "eaten" and the gauge fields acquire mass

Higgs Mechanism

Take local O(2)

Consider $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ ∼ real. And rotation is local

$$\phi \Rightarrow \phi' = R(x)\phi$$

Take $R(x) = (2\times 2)$ matrix

$$R(x) = \begin{pmatrix} \cos\alpha(x) & -\sin(\alpha(x)) \\ \sin(\alpha(x)) & \cos(\alpha(x)) \end{pmatrix}$$ $\alpha \to \alpha(x)$ local

Note $\det(R(x)) = 1 \to SO(2)$ (abelian)

Can write these as $R = e^{i\alpha(x)T} \rightsquigarrow$ 2×2 matrix (generator)

For $O(2) \to T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ observed that $T^2 = I$
$T^T = -T$
$\to (T$ is Hermitian$)$ $T^\dagger = T$

For small α $e^{i\alpha(x)T} = R \simeq I + i\alpha(x)T + \dots$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\alpha(x)\begin{pmatrix} 0 & +i \\ -i & 0 \end{pmatrix}$$

So $R \simeq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -\alpha(x) \\ \alpha(x) & 0 \end{pmatrix} + \dots$

$$= \begin{pmatrix} \cos\alpha(x) & -\sin\alpha(x) \\ \sin(\alpha(x)) & \cos\alpha(x) \end{pmatrix}$$ when all powers included

If $\alpha \ll 1$, then $R \simeq \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$

Now, R has to be orthogonal

$R \simeq I + i\alpha T$

$R^T = I + i\alpha T^T = I - i\alpha T$

$\to \left\{ \begin{matrix} R \text{ is still} \\ \text{orthogonal} \end{matrix} \right\}$

So $R^T R = (I - i\alpha T)(I + i\alpha T) = I + \cancelto{0}{\alpha^2} \dots$

Consider $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \cdot \partial^\mu \phi - V(\phi \cdot \phi)$

Let $\phi \to \phi' = R(x)\phi$

so $\partial_\mu \phi \to \partial_\mu \phi' = R\partial_\mu \phi + (\partial_\mu R)\phi$ $\neq 0$ for local transform

There's no local symmetry $\Rightarrow$ fix by changing the deriv..

Need to define a gauge-covariant derivative $\boxed{D_\mu = \partial_\mu + ig A_\mu}$

$g \to$ coupling parameter
$A_\mu \to$ function
} like EM

Now $D_\mu = \partial_\mu + igA_\mu$ has to be 2×2 to act on $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$

$\boxed{D_\mu = I\partial_\mu + TigA_\mu}$ $\quad T = \begin{pmatrix} 0 & i \\ -i & 0\end{pmatrix}$

We want $D_\mu \phi \longmapsto D'_\mu \phi' = R D_\mu \phi$ so that

$$D_\mu \phi \cdot D^\mu \phi \Rightarrow (RD_\mu\phi)^T (RD^\mu \phi)$$

$$= (D_\mu \phi)^T R^T R (D^\mu \phi)$$

$$= (D_\mu \phi)^T (D^\mu \phi)$$

$$= (D_\mu \phi)\cdot(D^\mu \phi)$$

How must A_μ transform to get this?

Look at $D'_\mu \phi' = (\partial_\mu + igA'_\mu)R\phi$

$$= R\partial_\mu \phi + (\partial_\mu R)\phi + igA'_\mu R\phi$$

set

Let $D'_\mu \phi' = R\partial_\mu \phi + (\partial_\mu R)\phi + igA'_\mu R\phi = RD_\mu \phi$

$= R(\partial_\mu + igA_\mu)\phi$

$= R\partial_\mu \phi + R\,ig A_\mu \phi$

So these equal if $igA'_\mu R\phi = igRA_\mu \phi - \cancel{igA'_\mu R\phi}$ $(\partial_\mu R)\phi$

So $\frac{ig}{ig} A'_\mu \cancel{R} = RA_\mu - \frac{1}{ig}\partial_\mu R$

So $A'_\mu R = RA_\mu - \frac{1}{ig}\partial_\mu R$

So $A'_\mu RR^{-1} = RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R)R^{-1}$

So

$$\boxed{A'_\mu = RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R)R^{-1}}$$

under local $O(2)$ gauge transf $A_\mu \to A'_\mu$

Ex with $R = e^{i\alpha(x)T}$

$R^{-1} = R^T = e^{i\alpha(x)T^T} = e^{-i\alpha(x)T}$ $\quad (T^T = -T)$

So $RR^{-1} = I$

So $A_\mu \to A'_\mu = RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R)R^{-1}$

$= e^{i\alpha T}A_\mu e^{-i\alpha T} + \frac{i}{g}\left[i(\partial_\mu \alpha)T R\right]R^{-1}$

really, $A'_\mu = A'_\mu T$
fn $\to A_\mu = A_\mu T$
matrix

Notice $e^{i\alpha T}T = Te^{i\alpha T}$

So $A'_\mu T = A_\mu T e^{i\alpha T} e^{-i\alpha T} - \frac{1}{g}(\partial_\mu \alpha) T$

Take away the T's, the functions obey

$$A'_\mu = A_\mu - \frac{1}{g}(\partial_\mu \alpha)$$

Call $\Lambda(x) = -\frac{\alpha(x)}{g}$

get $\boxed{A'_\mu = A_\mu + \partial_\mu \Lambda(x)}$ → same as $U(1)$ case.

So this is still Maxwell's theory. Note, also get

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \to F'_{\mu\nu} = F_{\mu\nu} \quad (\text{g. invariant})$$

So Need A_μ in D_μ to have local gauge sym.

— Make A_μ dynamical by adding $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

So then a local $O(2)$ gauge theory is

$$\boxed{\mathcal{L} = \frac{1}{2}(D_\mu \phi)\cdot(D^\mu \phi) - V(\phi\cdot\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}}$$

This is invariant under local $O(2)$

$$\begin{cases} \phi \to R(x)\phi \\ A_\mu \to RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R)R^{-1} \end{cases}$$

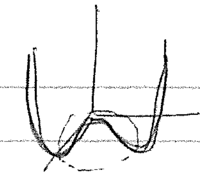
where $D_\mu = \partial_\mu + igA_\mu$

Now

Let $V(\phi\cdot\phi) = \frac{1}{2}m^2 \phi\cdot\phi + \frac{1}{4}\lambda(\phi\cdot\phi)^2$

$\boxed{\text{If } m^2 > 0}$ → no SSB.

Have massless A_μ → 2 modes
And 2 massive scalars ϕ_1, ϕ_2 $\Bigg\}$ 4 → ?

$\boxed{\text{If } m^2 < 0}$ $\Rightarrow$  get SSB.

Goldstone theorem says, for a global symmetry that you get one massive Higgs scalar + massless NG mode (2) (not 4)

$\rightarrow$ Not true for local symmetry.

$\hookrightarrow$ with SSB of local $O(2)$ can have Higgs mechanism.

Result NG mode gets eaten and $A_\mu \rightarrow A_\mu'$, which is massive

Left with massive A_μ' ——— 3 (no NG mode)
+
massive Higgs scalar $\rightarrow$ 1
———
4

———

$\boxed{\text{Higgs Mechanism}}$

if we have $\mathcal{L} = \frac{1}{2} D_\mu \phi \cdot D^\mu \phi - V(\phi \cdot \phi) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$

Ex: $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ and $D_\mu \phi = (\partial_\mu - igA_\mu)\phi$

$\hookrightarrow (\mathbb{I}\partial_\mu - igTA_\mu)\phi$

plug A_μ' into $F_{\mu\nu}F^{\mu\nu}$

This has local $O(2)$ invariance

$$\begin{cases} \phi \rightarrow \phi' = R\phi \\ A_\mu \rightarrow A_\mu' = RA_\mu R^{-1} + \frac{i}{g}(\partial_\mu R)R^{-1} \end{cases}$$

with SSB $\rightarrow$ get Higgs mechanism

let $V(\phi \cdot \phi) = \frac{1}{2} m^2 \phi^2 + \frac{1}{4} \lambda \phi^4 \quad (\phi^2 = \phi \cdot \phi)$

if $m^2 < 0$, we have minimum at $\langle\phi\rangle^2 = -\frac{m^2}{\lambda} = v^2$

Pick $\langle\phi\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix} \Rightarrow$ spontaneously breaks $O(2)$

Recall Generator for $O(2)$ is $T = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

An infinitesimal gauge transformation is

$$e^{i\alpha T} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

For small α $\longrightarrow \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}$

Now, we can write an arbitrary ϕ in a special way

$$\phi = R^{-1}\phi' = R^{-1}\begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix}$$ and let $\alpha = \frac{\varphi}{v}$

reparameterization

$$R^{-1} = e^{-i\alpha T} \simeq \begin{pmatrix} 1 & \alpha \\ -\alpha & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{\varphi}{v} \\ -\frac{\varphi}{v} & 1 \end{pmatrix}$$

This is what we have earlier

So $$\phi = R^{-1}\phi' = \begin{pmatrix} 1 & +\varphi/v \\ -\frac{\varphi}{v} & 1 \end{pmatrix}\begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix} \simeq \begin{pmatrix} \varphi \\ v+\varepsilon \end{pmatrix}$$

excitation, small

Recall $\langle\phi\rangle + \begin{pmatrix} \varphi \\ \varepsilon \end{pmatrix}$

$$\downarrow$$

$$\begin{pmatrix} 0 \\ v \end{pmatrix}$$

Note

This parameterization is called the "unitary gauge"

Can put $\phi = R^{-1}\phi'$ into $\mathcal{L}$.

$$\mathcal{L} = \frac{1}{2} D_\mu (R^{-1}\phi') \cdot D^\mu (R^{-1}\phi') - V\left((R^{-1}\phi')\cdot(R^{-1}\phi')\right) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu}$$

Since this is gauge invariant, we can perform a gauge transformation.

$$\phi \to R\phi = (R(R^{-1}\phi')) = \phi' \leftarrow \begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix}$$

At the same time $A_\mu \longrightarrow A'_\mu$ (new gauge field in the new gauge)

This gives

$$\mathcal{L} = \frac{1}{2} D'_\mu \phi' \cdot D'^\mu \phi' - V(\phi'\cdot\phi') - \frac{1}{4} F'_{\mu\nu}F'^{\mu\nu}$$

Let's look at $V(\phi'\cdot\phi') =$ ~~xxxx~~ ?

Note $\phi'\cdot\phi' = (v+\varepsilon)^2$

$$\hookrightarrow V(\phi\cdot\phi) = \frac{1}{2}m^2(v+\varepsilon)^2 + \frac{1}{4}\lambda(v+\varepsilon)^4$$

$$= \frac{1}{2}m^2\left(v^2+2v\varepsilon+\varepsilon^2\right) + \frac{1}{4}\lambda(v^2+2v\varepsilon+\varepsilon^2)^2$$

Take quadratic in ε

$$= \frac{1}{2}m^2\left(v^2+2v\varepsilon+\varepsilon^2\right) + \frac{1}{4}\lambda\left(v^4+4v^3\varepsilon+6v^2\varepsilon^2 + 4v\varepsilon^3+\varepsilon^4\right)$$

$$= \varepsilon\left(m^2v+\lambda v^3\right) + \varepsilon^2\left(\frac{1}{2}m^2+\frac{3}{2}\lambda v^2+\dots\right)$$

Recall $v^2 = -\frac{m^2}{\lambda}$

$$= \underbrace{\varepsilon\left(-\lambda v^3+\lambda v^3\right)}_{0} + \varepsilon^2\left(\frac{1}{2}m^2-\frac{3}{2}m^2+\dots\right)$$

$$\simeq \left(-\varepsilon^2 m^2\right)$$

And so the new potential $V(\phi'\cdot\phi') \simeq -m^2\varepsilon^2 \simeq \frac{1}{2}(-2m^2)\varepsilon^2$

We also need to look at D'_μ ...

$$D'_\mu = (\partial_\mu + igA'_\mu) \quad \text{where } A'_\mu = TA_\mu$$

$$= (\mathbb{1}\partial_\mu + igTA'_\mu)$$

$$= \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} + ig\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} A'_\mu$$

$$= \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} + \begin{pmatrix} 0 & -igA'_\mu \\ igA'_\mu & 0 \end{pmatrix}$$

So
$$D'_\mu \phi' = \left[\begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix} + \begin{pmatrix} 0 & -igA'_\mu \\ igA_\mu & 0 \end{pmatrix}\right]\begin{pmatrix} 0 \\ v+\varepsilon \end{pmatrix}$$

$$= \begin{pmatrix} -gA'_\mu(v+\varepsilon) \\ \partial_\mu \varepsilon \end{pmatrix}$$

So
$$D'_\mu \phi' \cdot D'^\mu \phi' = \begin{pmatrix} -gA'_\mu(v+\varepsilon) \\ \partial_\mu \varepsilon \end{pmatrix}^T \begin{pmatrix} -gA'^\mu(v+\varepsilon) \\ \partial^\mu \varepsilon \end{pmatrix}$$

$$= g^2 A'_\mu A'^\mu (v+\varepsilon)^2 + \partial_\mu \varepsilon \cdot \partial^\mu \varepsilon$$

<u>Then the $\mathcal{L}$ becomes</u>

$$\mathcal{L} = \frac{1}{2}g^2 A'_\mu A'^\mu (v+\varepsilon)^2 + \frac{1}{2}\partial_\mu \varepsilon \cdot \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\varepsilon^2 - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$$

<u>Expand this out</u>

$$\boxed{\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\partial_\mu \varepsilon \cdot \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\varepsilon^2 - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} \\ & + \frac{g^2 v^2}{2} A'_\mu A'^\mu + \frac{g^2}{2}(2v\varepsilon + \varepsilon^2) A'_\mu A'^\mu + \cdots\end{aligned}}$$

$\boxed{\text{This theory describes}}$

(1) $\boxed{\frac{1}{2}\partial_\mu \varepsilon\, \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\,\varepsilon^2}$ → has $-2m^2 > 0$

↳ massive scalar particle → Higgs boson (scalar, spin 0)

(2) $\boxed{-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} + \frac{g^2 v^2}{2}A'_\mu A'^\mu}$ → massive vector gauge field

(3) $\boxed{\frac{g^2}{2}(2v\varepsilon + \varepsilon^2)\,A'_\mu A'^\mu}$ Interaction between ε and A'_μ

Note no massless NG mode (ξ is gone)

↳ got eaten resulting in A_μ gains mass → A'_μ

We can count degrees of freedom

↳ Before SSB: $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\phi \cdot D^\mu\phi - V(\phi\cdot\phi)$

$$\left.\begin{array}{l} \text{massless} \to A_\mu \to 2 \\ \phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \to 2 \end{array}\right\} \text{4 total}$$

After SSB → $\mathcal{L} = \frac{1}{2}\partial_\mu \varepsilon\, \partial^\mu \varepsilon - \frac{1}{2}(-2m^2)\,\varepsilon^2 - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}$
$+ \frac{g^2 v^2}{2}A'_\mu A'^\mu + \ldots$

$$\left.\begin{array}{l} \text{massive scalar } \varepsilon \to 1 \\ \text{massive gauge field } A'_\mu \to 3 \end{array}\right\} \text{4 total}$$

4

Next look at $U(1)$ → similar to $SO(2)$

NOETHER'S THEOREM

Apr 14 2014

Let the action be.

$$S = \int_\Omega d^4x \, \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)$$

Consider infinitesimal spacetime + internal transformation

$$x^\mu \longmapsto x'^\mu = x^\mu + \delta x^\mu$$

$$\phi^A(x) \to \phi'^A(x') = \phi^A(x) + \delta\phi^A(x)$$

So

$$\delta S = \int_{\Omega'} d^4x' \, \mathcal{L}\left(\phi'^A(x'), \partial'_\mu \phi'^A(x')\right) - \int_\Omega d^4x \, \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)$$

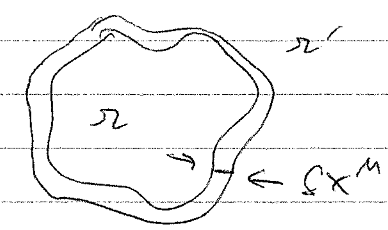
relabel $x' \to x$

But Ω' is the new volume

$$\delta S = \int_{\Omega'} d^4x \, \mathcal{L}\left(\phi'^A(x), \partial_\mu \phi'^A(x)\right) - \int_\Omega d^4x \, \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)$$

$$= \int_\Omega d^4x \left[\mathcal{L}\left(\phi'^A(x), \partial_\mu \phi'^A(x)\right) - \mathcal{L}\left(\phi^A(x), \partial_\mu \phi^A(x)\right)\right]$$

$$+ \int_{\Omega' - \Omega} d^4x \, \mathcal{L}\left(\phi'^A(x), \partial_\mu \phi'^A(x)\right)$$



Note $\displaystyle\int_{\Omega' - \Omega} d^4x = \int_{\partial\Omega} dS_\lambda \, \delta x^\lambda$

So

$$\int_{\Omega' - \Omega} d^4x \, \mathcal{L}\left(\phi'^A, \partial_\mu \phi'^A\right) = \int_{\delta\Omega} dS_\lambda \, \delta x^\lambda \, \mathcal{L}\left(\phi'^A, \partial_\mu \phi'^A\right)$$

where to leading order $\delta x^\lambda \mathcal{L}(\phi'^A, \partial_\mu \phi'^A)$ $\quad$ old Lag

$$\simeq \delta x^\lambda \mathcal{L}(\phi^A, \partial_\mu \phi^A)$$

Gauss' Law:

$$\int_{\partial\Omega} d\Sigma_\lambda \, \delta x^\lambda \mathcal{L}(\phi^A, \partial_\mu \phi^A) = \int_\Omega d^4x \, \underbrace{\partial_\lambda}_{\text{divergence}} \left(\delta x^\lambda \mathcal{L}(\phi^A, \partial_\mu \phi^A)\right)$$

Define

$$\bar\delta f(x) = f'(x) - f(x)$$

$$= \left[f'(x') - f(x)\right] - \left[f'(x') - f'(x)\right]$$

$$= \delta f(x) - \partial_\mu f(x) \, \delta x^\mu$$

where $\delta f(x) = f'(x') - f(x)$

prime is NOT the derivative $f' \neq \partial_\mu f$ $\quad$ and

$$f'(x') = f'(x + \delta x) = f'(x) + \delta x^\mu \partial_\mu (f'(x))$$

$$\simeq f'(x) + \delta x^\mu \partial_\mu (f(x))$$

Then

$$\mathcal{L}(\phi'^A(x), \partial_\mu \phi'^A(x)) - \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))$$

$$\cancel{= \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x)) + \bar\delta}$$

$$= \mathcal{L}(\phi^A(x) + \bar\delta \phi^A(x), \partial_\mu \phi^A + \bar\delta \partial_\mu \phi^A) - \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))$$

commute

$$= \mathcal{L}(\phi^A(x) + \bar\delta \phi^A(x), \partial_\mu \phi^A + \partial_\mu \bar\delta \phi^A) - \mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))$$

$$\simeq \cancel{\mathcal{L}(\phi^A(x), \partial_\mu \phi^A)} + \frac{\delta \mathcal{L}}{\delta \phi^A} \bar\delta \phi^A + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \partial_\mu \bar\delta \phi^A - \cancel{\mathcal{L}(\phi^A(x), \partial_\mu \phi^A(x))}$$

$$= \underbrace{\left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A}\right)}_{\text{Euler-Lagrange}} \bar{\delta} \phi^A + \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \cdot \bar{\delta} \phi^A + \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \cdot \partial_\mu \phi^A$$

$$= \underbrace{\left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A}\right)}_{\text{Euler Lagrange Eqn, 0 on shell}} \bar{\delta} \phi^A + \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A\right)$$

Putting things together

$$\delta S = \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A}\right) \bar{\delta} \phi^A + \int_\Omega d^4x\, \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A\right)$$

$$+ \int_{\Omega' - \Omega} d^4x\, \partial_\mu (\delta x^\mu \mathcal{L})$$

$$= \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A}\right) \bar{\delta} \phi^A + \int_\Omega d^4x\, \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A + \mathcal{L} \delta x^\mu\right]$$

Now, use $\bar{\delta} \phi^A = \delta \phi^A - \partial_\mu \phi^A \delta x^\mu$ then

$$\partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \bar{\delta} \phi^A + \mathcal{L} \delta x^\mu\right) = \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta \phi^A - \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \partial_\nu \phi^A \delta x^\nu + \mathcal{L} \delta x^\mu\right]$$

$$= \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta \phi^A\right) - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L}\right) \delta x_\nu$$

Call $\boxed{T^{\mu\nu} = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \partial^\nu \phi^A - \eta^{\mu\nu} \mathcal{L}}$ energy momentum

So

$$\delta S = \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \right) \delta\phi^A + \int_\Omega \overset{\partial_\mu}{d^4x} \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta\phi^A - T^{\mu\nu} \delta x_\nu \right)$$

Define

$$\boxed{J^\mu = \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} \delta\phi^A - T^{\mu\nu} \delta x_\nu} \leftarrow \text{current}$$

get

$$\delta S = \int_\Omega d^4x \left(\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi} \right) \delta\phi^A + \int_\Omega d^4x \, \partial_\mu J^\mu \longrightarrow \text{div}(J)$$

Result: if $\phi^A \longmapsto \phi^A + \delta\phi^A$
and/or
$x^\mu \longmapsto x^\mu + \delta x^\mu$

} are/is a symmetry transform

$\rightarrow$ $\boxed{\delta S = 0}$ Then if also ϕ^A is on shell, obeys eqs of motion, then

$$\boxed{\frac{\delta \mathcal{L}}{\delta \phi^A} - \partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^A} = 0}$$

div $(J^\mu) = 0$

These together give $\boxed{\partial_\mu J^\mu = 0}$

This is the Noether's Theorem result

$\rightarrow$ | when a theory has a symmetry & the equations of motion hold you get a conserved quantity |

Consider $\partial_\mu J^\mu = 0$

Integrate over space $\rightarrow$ use $J^\mu = (\rho, \vec{J})$ then,

$$\int d^3x\ \partial_\mu J^\mu = 0$$

$$\int d^3x\ (\partial_0 J^0 + \partial_j J^j) = 0 \longrightarrow \vec{\nabla}\cdot\vec{J}$$

$$\frac{d}{dt}\int d^3x\ J^0 + \int d^3x\ \underbrace{\partial_j J^j} = 0$$

So

$$\frac{d}{dt}\int d^3x\ \underbrace{J^0}_{\rho} + \int d^3x\ \vec{\nabla}\cdot\vec{J} = 0$$

$$\underbrace{\frac{d}{dt}\int d^3x\ \rho}_{Q} + \underbrace{\int_S d\vec{A}\cdot\vec{J}}_{0 \text{ if } \vec{J}\to\vec{0} \text{ on the boundary}} \longrightarrow \text{Gauss's Law.}$$

So $\boxed{\dfrac{dQ}{dt} = 0}$ → conservation of charge.

#

Mar 21, 2019

GRAVITATIONAL ACTIONS

- Flat spacetime $S = \int \mathcal{L}\, d^4x$
- → metric $= \eta_{\mu\nu}$

But in curved space, $g_{\mu\nu} \neq \eta_{\mu\nu}$.

- if $X^\mu \longmapsto X^{\mu'}$ then $dX^{\mu'} = \Sigma^{\mu'}_{\nu} X^\nu$

and $[\Sigma_\nu{}^{\mu'}] = \dfrac{\partial X^{\mu'}}{\partial X^\nu}$ the Jacobian

and $g_{\mu'\nu'} = \Sigma^{\alpha}_{\mu'}\Sigma^{\beta}_{\nu'}\, g_{\alpha\beta}$

volume elements → $d^4x \longrightarrow d^4x' = \left|\dfrac{\partial X^{\mu'}}{\partial X^{\nu}}\right| d^4x$ ↗ $\det|\Sigma^{\mu'}_{\nu}|$

⇒ Need an invariant volume element ⇒ compensate with a factor of

$$g = \det(g_{\mu\nu}) = |g_{\mu\nu}|$$

Since

$$g_{\mu\nu} = \begin{pmatrix} + & & & \\ & - & & \\ & & - & \\ & & & - \end{pmatrix}, \quad \det(g) < 0$$

So $-g > 0$

For the determinants ⇒ $g' = \left|\frac{\partial x}{\partial x'}\right|^{2} g$

So $g' = \left|\frac{\partial x'}{\partial x}\right|^{-2} g$

So $\sqrt{-g'} = \left|\frac{\partial x'}{\partial x}\right|^{-1} \sqrt{-g}$

Then

$$d^4x \sqrt{-g} = d^4x' \left|\frac{\partial x'}{\partial x}\right|^{-1} \left|\frac{\partial x'}{\partial x}\right| \sqrt{-g'} = d^4x' \sqrt{-g'}$$

So $\boxed{d^4x \sqrt{-g} \text{ is an invariant volume element}}$

If $g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$ then $g = -1$ & $\sqrt{-g} = 1$

So, for curved spaces

coord-invariant

$$\boxed{S = \int d^4x \sqrt{-g}\, \mathcal{L}}$$

The action for pure gravity (no matter) in GR is $(c=1)$

$$\boxed{S = \int d^4x \sqrt{-g} \frac{1}{16\pi G} R}$$ ← Einstein-Hilbert action

Here $R^{\mu}{}_{\mu}$ where $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = g^{\lambda\beta} R_{\beta\mu\lambda\nu}$

$$\hookrightarrow R = R^{\mu\nu}{}_{\mu\nu} = g^{\mu\nu} R_{\mu\nu}$$

Recall

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu} \Gamma^{\rho}_{\nu\sigma} - \partial_{\nu} \Gamma^{\rho}_{\mu\sigma} - \Gamma^{\rho}_{\mu\lambda} \Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda} \Gamma^{\lambda}_{\mu\sigma}$$

where

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\nu\sigma} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu} \right)$$

We need to vary $\mathcal{L} = \int d^4x \sqrt{-g}\, R$ with respect to $g_{\mu\nu}$ or $g^{\mu\nu}$

Need to pick either $g_{\mu\nu}$ or $g^{\mu\nu}$ as the fundamental field.

- These obey

$$g^{\mu\nu} g_{\nu\sigma} = \delta^{\nu}_{\sigma} \leftarrow \underline{\text{constant}}.$$

$$\hookrightarrow \text{So } (\delta g^{\mu\nu}) g_{\nu\sigma} + (g^{\mu\nu})(\delta g_{\nu\sigma}) = 0$$

So $$\boxed{(\delta g^{\mu\nu})\, g_{\nu\sigma} = -(g^{\mu\nu})(\delta g_{\nu\sigma})}$$

Now, multiply with $g^{\sigma\rho}$

$$\hookrightarrow (\delta g^{\mu\nu})\, g_{\nu\sigma} g^{\sigma\rho} = -g^{\mu\nu} g^{\sigma\rho} (\delta g_{\nu\sigma})$$

$$(\delta g^{\mu\rho}) = -g^{\mu\nu} g^{\sigma\rho} \delta g_{\nu\sigma}$$

re Write $$\boxed{\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}}$$

Likewise,

$$\boxed{\delta g_{\mu\nu} = -g_{\mu\alpha} g_{\nu\beta} \delta g^{\alpha\beta}}$$

Carroll uses $g^{\mu\nu}$ as fundamental

<u>Look at</u> $\sqrt{-g}\mathcal{L} = \sqrt{-g}R = \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}$

<u>So</u> $\delta(\sqrt{-g}\mathcal{L}) = \delta\left[\sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}\right]$

$$= (\delta\sqrt{-g})\, g^{\mu\nu} R_{\mu\nu} + \sqrt{-g}\,(\delta g^{\mu\nu})\, R_{\mu\nu} + \sqrt{-g}\, g^{\mu\nu}(\delta R_{\mu\nu})$$

we know this

<u>Want</u> $\delta S = \int d^4x\, (\ )_{\mu\nu}\, \delta g^{\mu\nu} = 0$

<u>so that</u> $(\ )_{\mu\nu} = 0$ should give Einstein's eqn

$\boxed{\text{What is } \delta\sqrt{-g} \text{ in terms of } \delta g^{\mu\nu}?}$

Here $g = \det(g_{\mu\nu})$

There's an identity for matrices $\rightarrow$ $\boxed{\ln(\det M) = \mathrm{Tr}(\ln M)}$

verify with silly example $\sim$ $M = \begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix}$

then $\det M = abc \rightarrow \ln(\det M) = \ln(abc) = \ln(a) + \ln(b) + \ln(c)$

$$= \mathrm{Tr}\begin{pmatrix} \ln a & & \\ & \ln b & \\ & & \ln c \end{pmatrix} = \mathrm{Tr}(\ln M)$$

We can use $M = [g_{\mu\nu}] \rightarrow M^{-1} = [g^{\mu\nu}]$

$g = \det(g_{\mu\nu}) = \det(M)$

<u>So</u>, vary the identity, get

$$\frac{1}{\det M}\,\delta \det M = \mathrm{Tr}(M^{-1}\delta M)$$

<u>So</u> $\frac{1}{g}(\delta g) = \mathrm{Tr}(g^{\mu\nu}\delta g_{\mu\nu}) = g^{\mu\nu}\delta g_{\mu\nu}$ a number

<u>So</u> $\boxed{\delta g = g\, g^{\mu\nu}\delta g_{\mu\nu}} \boxed{= -g\, g_{\mu\nu}\delta g^{\mu\nu}}$

Then $\delta\sqrt{-g} = \delta(-g)^{1/2} = \frac{1}{2}(-g)^{-\frac{1}{2}}\,\delta(-g) = \frac{-1}{2}\frac{1}{\sqrt{-g}}\,\delta g$

$$= \frac{-1}{2}\frac{1}{\sqrt{-g}}\cdot(-g)\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

$$\boxed{\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu} = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\,\delta g_{\mu\nu}}$$

Now, what about δR ? , i.e. what is $(\delta R_{\mu\nu})$?

Recall $\delta(\sqrt{-g}R) = (\delta\sqrt{-g})\underbrace{g^{\mu\nu}R_{\mu\nu}}_{R} + \sqrt{-g}\,\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\delta R_{\mu\nu}$

$$= -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,\delta g^{\mu\nu}$$

So $\delta(\sqrt{-g}R) = \sqrt{-g}\underbrace{\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)}\delta g^{\mu\nu} + \sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu}$

Einstein tensor $\to G_{\mu\nu}$. Recall Einstein eq: $G_{\mu\nu} = 8\pi G\,T_{\mu\nu}$

For no matter $\to$ $T_{\mu\nu} = 0$ so $G_{\mu\nu} = 0$

Turns out that $\boxed{\int d^4x\,\sqrt{-g}\,g^{\mu\nu}\,\delta R_{\mu\nu} = 0}$ (page 162, Carroll)

(derive this eq) (int. by parts)

Notice it is NOT that $\delta R_{\mu\nu} = 0$

Remember $\to$ keep everything in an integral ...

With

$$S = \int d^4x\,\sqrt{-g}\,\frac{1}{16\pi G}\,R$$

then

$$(16\pi G)\,\delta S = \int d^4x\,\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} = 0$$

So $\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0}$ $\to$ Einstein equations with no matter... (pure gravity) $(\Lambda = 0)$

<u>With $\Lambda \neq 0$ and no matter</u>

$$S = \frac{1}{16\pi G}\int d^4x \sqrt{-g}\,(R - 2\Lambda)$$

get $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0$ → verify this (easy)

$\boxed{\text{Now, how do we add matter?}}$

matter terms

$$\hookrightarrow \quad S = \int d^4x \sqrt{-g}\left(\frac{1}{16\pi G}(R-2\Lambda) + \mathcal{L}_{matter}\right)$$

most simple scalar field $\mathcal{L} \stackrel{?}{=} \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$

<u>But really, we have</u>

$$\mathcal{L} \stackrel{?}{=} \frac{1}{2}(\partial_\mu\phi)g^{\mu\nu}(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2$$

↳ notice that there's some interaction ...

<u>Recall</u>

For scalars $D_\mu\phi = \partial_\mu\phi = \phi_{;\mu} = \phi_{,\mu} = \nabla_\mu\phi$

If we vary this action, we also need to include

$$\delta\left(\sqrt{-g}\left[\stackrel{?}{=}\frac{1}{2}(\partial_\mu\phi)g^{\mu\nu}(\partial_\nu\phi) - \frac{1}{2}m^2\phi^2\right]\right)$$

☒ How does this give $T_{\mu\nu}$? By definition!

⇒ Any matter fields add extra stuff to vary w.r.t. $g^{\mu\nu}$
↳ in $\sqrt{-g}\mathcal{L}_M$

☒ Simply define

$$\delta S = \delta S_g + \delta S_M = \int d^4x \sqrt{-g}\frac{1}{16\pi G}(R-2\Lambda) + \int d^4x\sqrt{-g}\,\mathcal{L}_M$$

$$\delta S_M = \int d^4x \; \delta(\sqrt{-g}\,\mathcal{L}_M)$$

Define $$\boxed{\delta\left(+\sqrt{-g}\,\mathcal{L}_M\right) \equiv -\frac{1}{2}\sqrt{-g}\,T_{\mu\nu}\,\delta g^{\mu\nu}}$$

or $$\boxed{T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\left(+\sqrt{-g}\,\mathcal{L}_M\right)}$$

With that, $(\Lambda = 0)$

$$S = \int d^4x \sqrt{-g}\left[\frac{1}{16\pi G}R + \mathcal{L}_M\right]$$

$$\Rightarrow \quad \delta S = \int d^4x \left\{\frac{\sqrt{-g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\delta g^{\mu\nu} + \delta(\sqrt{-g}\,\mathcal{L}_M)\right\}$$

$$= \int d^4x \left[\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) - \frac{1}{2}T_{\mu\nu}\right]\sqrt{-g}\;\delta g^{\mu\nu} = 0$$

Therefore

$$\frac{1}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = \frac{1}{2}T_{\mu\nu}$$

so

$$\boxed{R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\,T_{\mu\nu}}$$

If $\mathcal{L}_M = \frac{-1}{4}F^{\mu\nu}F_{\mu\nu} \rightarrow \sqrt{-g}\,\mathcal{L}_M = \frac{-1}{4}\sqrt{-g}\,F_{\mu\nu}\,g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}$

and we can show

$$F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Vary wrt $g^{\mu\nu}$ to get $T^{\mu\nu}$ from this (verify!)

$$\boxed{T_{\mu\nu} = F_{\mu\lambda} F_{\nu}{}^{\lambda} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}}$$

How do we know this is actually energy-momentum?

$T_{00} \sim (\vec{E}^2 + \vec{B}^2) \sim$ energy density

$T_{0j} \sim$ Poynting vector...

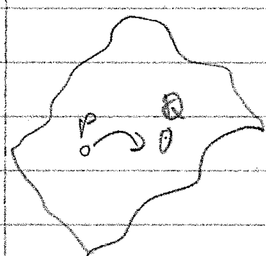
April 11, 2019

DIFFEOMORPHISMS

A diffeomorphism is a mapping of one manifold to another.

In GR $\Rightarrow$ mapping of spacetime to itself.

Carroll's book describes all the math...



P is at x^μ then $x^\mu \to x^\mu + \xi^\mu$
Q is at $x^\mu + \xi^\mu$ under a diffeomorphism

Want to know $\rightarrow$ how scalars, vectors tensors change under diffs. + how it's a symm of GR

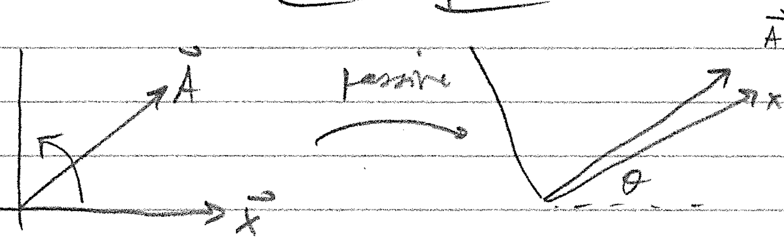
changes in tensors are given by the Lie derivative

We'll look at the "passive" version of coord. transformation.

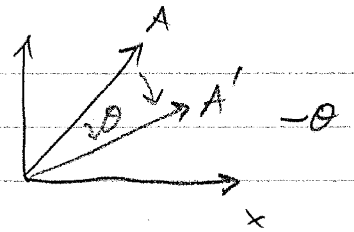
$\hookrightarrow$ Coord. transf can be active or passive.

Rotations

"passive"



"Active"



See that the new component of $\vec{A}$ in X' frame under a passive transformation are the same as the components $A^{i'}$ of $\vec{A}'$ under an active transformation

Symmetries always involve active transformations. With unbroken symmetries these are inverses of passive transformations

Even though passive transformations are just observer changes, we can still use them to find the form of "-transformation"

In GR, diffeos is an active transform: $A^\mu \to A'^\mu$ under moving or translating $x^\mu \to x^\mu + \xi^\mu$

The passive version is a general transformation

$$x^\mu \to x^{\mu'}(x) = x^\mu - \xi^\mu \qquad (\text{— because inverse})$$

We can use the inverse general coord. transf to find the form of Lie derivative

$\mathcal{L}_\xi$ → denotes a Lie deriv using ξ^μ (4 transformations)

$$\begin{cases} A_\mu \to A_\mu + \mathcal{L}_\xi A_\mu & \text{under diffs} \\ A^\mu \to A^\mu + \mathcal{L}_\xi A^\mu \end{cases}$$

If $\delta S = 0$ under diffs, then the theory is diffs. invariant → GR is diffeomorphism invariant

Consider an infinitesimal general coord. transform:

$$X^{\mu'} = X^{\mu'}(x) = x^\mu - \xi^\mu$$

↗ Jacob. matrix

Now, a vector under GCT obeys $A^{\mu'}(x') = \Xi^{\mu'}_{\ \nu} A^\nu(x)$

$$\Xi^{\mu'}_{\ \nu} = \frac{\partial X^{\mu'}}{\partial X^\nu} = \frac{\partial}{\partial x^\nu}\left(x^\mu - \xi^\mu\right) = \delta^\mu_\nu - \partial_\nu \xi^\mu$$

<u>Then</u>

$$A^{\mu'}(x') = \left(\delta^\mu_\nu - \partial_\nu \xi^\mu\right) A^\nu(x) =$$

↑ $x' = x - \xi$ + Taylor expand

$$A^{\mu'}(x - \xi) \cong A^{\mu'}(x) - \xi^\nu \partial_\nu A^{\mu'}(x) + \dots$$

Then use $\xi^\nu \partial_\nu A^{\mu'}(x) \simeq \xi^\nu \partial_\nu A^\mu(x)$ + 2nd order...

So $A^{\mu'}(x') = A^{\mu'}(x) - \xi^\nu \partial_\nu A^\mu(x) = A^\mu(x) - \left(\partial_\nu \xi^\mu\right) A^\nu(x)$

So $\boxed{A^{\mu'}(x) = A^\mu(x) - \left(\partial_\nu \xi^\mu\right) A^\nu(x) + \xi^\nu\left(\partial_\nu A^\mu(x)\right)}$

This gives the same result as for the active diff.

$$A^\mu(x) \to A^{*\mu'}(x) = A^\mu(x) + \mathcal{L}_\xi A^\mu(x)$$

So we get $\boxed{\mathcal{L}_\xi A^\mu(x) = -\left(\partial_\nu \xi^\mu\right) A^\nu(x) + \xi^\nu\left(\partial_\nu A^\mu(x)\right)}$

↖ Lie derivative of a contravariant vector.

(this is without parallel transport ...)

For a scalar $\phi(x)$, under $X^{\mu'} = x^\mu - \xi^\mu$

we know that $\phi(x) = \phi'(x') = \phi'(x - \xi) \simeq \phi'(x) - \xi^\nu \partial_\nu \phi'(x)$

$$= \phi'(x) - \xi^\nu \partial_\nu \phi(x)$$

So $\boxed{\phi'(x) = \phi(x) + \xi^\nu \partial_\nu \phi(x)}$

Under a diff, (active) $\phi(x) \to \phi'(x') = \phi(x) + \mathcal{L}_\xi \phi(x)$
$(x^\mu \to x^\mu + \xi^\mu)$

So $\boxed{\mathcal{L}_\xi \phi(x) = \xi^\nu \partial_\nu \phi(x)}$ $\rightsquigarrow$ Lie deriv of a scalar...

- For a covariant vector, $A_\mu(x)$

$$A_{\mu'}(x') = \Xi^{\nu}_{\ \mu'} A_\nu(x) \quad , \quad \Xi^{\nu}_{\ \mu'} \text{ is the inverse of } \Xi^{\mu'}_{\ \nu} = \delta^\mu_\nu - \partial_\nu \xi^\nu$$

We can verify that $\boxed{\Xi^{\nu}_{\ \mu'} = \delta^\nu_\mu + \partial_\mu \xi^\nu}$

by multiplying the two...

$$\Xi^{\nu}_{\ \mu'} \Xi^{\mu'}_{\ \gamma} \stackrel{?}{=} \delta^\nu_\gamma \ ?$$

$$(\delta^\nu_\mu + \partial_\mu \xi^\nu)(\delta^\mu_\gamma - \partial_\gamma \xi^\mu) = \delta^\nu_\gamma - \cancel{\partial_\gamma \xi^\nu} + \cancel{\partial_\gamma \xi^\nu} - 0$$

$$= \delta^\nu_\gamma \quad \text{works!}$$

So then

$$A_{\mu'}(x') = \Xi^{\nu}_{\ \mu'} A_\nu(x)$$

$$= (\delta^\nu_\mu + \partial_\mu \xi^\nu) A_\nu(x) = A_\mu(x) + (\partial_\mu \xi^\nu) A_\nu(x)$$

For $x' = x - \xi$, $A_{\mu'}(x - \xi) \simeq A_{\mu'}(x) - \xi^\nu \partial_\nu A_{\mu'}(x)$

$$\simeq A_{\mu'}(x) - \xi^\nu \partial_\nu A_\mu(x) + \cdots$$

Then,

$$\boxed{A_{\mu'}(x) = A_\mu(x) + (\partial_\mu \xi^\nu) A_\nu(x) + \xi^\nu \partial_\nu A_\mu(x)}$$

Claim, this is the same as $A_\mu(x) \to A_{\mu'}(x) = A_\mu(x) + \mathcal{L}_\xi A_\mu(x)$
when $x^\mu \to x^\mu + \xi^\mu$

So $\boxed{\mathcal{L}_\xi A_\mu(x) = (\partial_\mu \xi^\nu) A_\nu(x) + \xi^\nu \partial_\nu A_\mu(x)}$ $\rightsquigarrow$ Lie deriv of cov. vector

Given these, can guess the form for a tensor, say $T^{\mu\nu}{}_{\sigma}$

$$\boxed{\mathcal{L}_\xi T^{\mu\nu}{}_{\sigma} = -(\partial_\alpha \xi^\mu) T^{\alpha\nu}{}_{\sigma} - (\partial_\alpha \xi^\nu) T^{\mu\alpha}{}_{\sigma} + (\partial_\sigma \xi^\alpha) T^{\mu\nu}{}_{\alpha} + \xi^\alpha \partial_\alpha T^{\mu\nu}{}_{\sigma}}$$

See Carroll's book on page p.434 to get the full mathematical rigor...
But he uses V^μ for ξ^μ.

$\boxed{\text{Exercise}}$ Show that the same formulas hold for covariant derivatives everywhere in place of ∂_μ

① Show $\mathcal{L}_\xi \phi = \xi^\alpha \partial_\alpha \phi = \xi^\alpha D_\alpha \phi$

where $D_\alpha \phi = \phi_{;\alpha} = \phi_{,\alpha} = \partial_\alpha \phi$ (done)

② $\mathcal{L}_\xi A^\mu = -(\partial_\alpha \xi^\mu) A^\alpha + \xi^\alpha \partial_\alpha A^\mu$

$= -(D_\alpha \xi^\mu) A^\alpha + \xi^\alpha (D_\alpha A^\mu)$ (connections...)

③ $\mathcal{L}_\xi A_\mu = (\partial_\mu \xi^\alpha) A_\alpha + \xi^\alpha (\partial_\alpha A_\mu)$

$= (D_\mu \xi^\alpha) A_\alpha + \xi^\alpha (D_\alpha A_\mu)$

So Lie derivs of tensors are tensors.

Now, let's look at $g_{\mu\nu}$. Under a diff. $g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}$

where $\mathcal{L}_\xi g_{\mu\nu} = (D_\mu \xi^\alpha) g_{\alpha\nu} + (D_\nu \xi^\alpha) g_{\mu\alpha} + \xi^\alpha D_\alpha g_{\mu\nu}$

Recall, $D_\lambda g_{\mu\nu} = 0$ (metric tensor is cov. constant)

So $\mathcal{L}_\xi g_{\mu\nu} = D_\mu (g_{\alpha\nu} \xi^\alpha) + D_\nu (g_{\mu\alpha} \xi^\alpha) + 0$

$$= \boxed{D_\mu \xi_\nu + D_\nu \xi_\mu}$$

So, under a diffs, $\boxed{g_{\mu\nu} \to g_{\mu\nu} + D_\mu \xi_\nu + D_\nu \xi_\mu}$

However, using partial derivatives, ...

$$\mathcal{L}_\xi g_{\mu\nu} = (\partial_\mu \xi^\alpha) g_{\alpha\nu} + (\partial_\nu \xi^\alpha) g_{\mu\alpha} + \xi^\alpha \partial_\alpha g_{\mu\nu}$$ does not simplify ...

$\to \partial_\lambda g_{\mu\nu} \neq 0 \to$ doesn't simplify nicely.

Next $\to$ See how diffs are a symmetry of GR. What does it mean to break diffeomorphism?

April 18, 2019

$\boxed{\text{Spacetime Symmetry}}$

~~Wha~~ Want to consider:

1. Global LTs in Minkowski space (no gravity)
2. Diffeomorphisms in curved spacetime (with gravity)
3. Local LTs in curved spacetime. (w/ gravity)

I. Global LTs in Minkowski Space

You know LT's are coords transform $\to x^\mu \to x^{\mu'}$

$$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu \quad \text{where } \Lambda^{\mu'}{}_\nu \text{ are constants...}$$

Vectors: $V_{\mu'} = \Lambda_{\mu'}{}^\nu V_\nu \; ; \; V^{\mu'} = \Lambda^{\mu'}{}_\nu V^\nu$

- We typically write $\Lambda^{\mu'}_{\nu}$ with indices on top of another ... Have inverses ...

$$\Lambda^{\mu'}_{\nu} \Lambda^{\nu}_{\alpha'} = \delta^{\mu'}_{\alpha'} = \delta^{\mu}_{\alpha} = \Lambda^{\mu}_{\nu'} \Lambda^{\nu'}_{\alpha}.$$

- Also $\eta_{\mu'\nu'} = \Lambda^{\alpha}_{\mu'} \Lambda^{\beta}_{\nu'} \eta_{\alpha\beta}$

But . All this is the passive point of view.

Now, we want to look at the active LT's where x^μ doesn't change.
(no prime indices...)

↳ Now we distinguish $\Lambda_\mu{}^\nu$ vs. $\Lambda^\mu{}_\nu$, these are inverses.
These obey
$$\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = \delta_\beta^\alpha$$
and
$$\Lambda^\alpha{}_\mu \Lambda_\beta{}^\mu = \delta_\beta^\alpha \ldots$$

. Also must have $\eta_{\mu\nu} \to \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta} = \eta_{\mu\nu}$

↳ Minkowski metric is unchanged.

. To consider infinitesimal LT's
$$\begin{cases} \Lambda_\mu{}^\nu = \delta_\mu^\nu + \varepsilon_\mu{}^\nu \\ \Lambda^\mu{}_\nu = \delta^\mu_\nu + \varepsilon^\mu{}_\nu \end{cases}$$

where $\varepsilon_\mu{}^\nu$ ⇝ small + constant and $(\varepsilon_\mu{}^\nu)^2 = 0$

<u>Check</u>: $\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta = (\delta_\mu^\alpha + \varepsilon_\mu{}^\alpha)(\delta^\mu_\nu + \varepsilon^\mu{}_\nu)$

$$= \delta^\alpha_\nu + \varepsilon^\alpha{}_\nu + \varepsilon_\nu{}^\alpha + \cancel{0}^{0}$$

$$= \delta^\alpha_\beta, \text{ provided that } \varepsilon^\alpha{}_\nu = -\varepsilon_\nu{}^\alpha$$

↳ provided that $\varepsilon^\alpha{}_\beta = -\varepsilon_\beta{}^\alpha$

<u>Check</u> $\Lambda^\mu{}_\alpha \Lambda_\nu{}^\alpha = (\delta^\mu_\alpha + \varepsilon^\mu{}_\alpha)(\delta_\nu^\alpha + \varepsilon_\nu{}^\alpha)$

$$= \delta^\mu_\nu + \varepsilon^\mu{}_\nu + \varepsilon_\nu{}^\mu + \cancel{0}^{0}$$

$$= \delta^\mu_\nu \text{ since } \varepsilon^\mu{}_\nu = -\varepsilon_\nu{}^\mu \qquad \text{O}$$

<u>Check</u>

$$\eta_{\mu\nu} \stackrel{?}{=} \Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta} = \eta_{\mu\nu} + \varepsilon_\mu{}^\alpha \eta_{\alpha\nu} + \varepsilon_\nu{}^\beta \eta_{\mu\beta} = \eta_{\mu\nu} + \varepsilon_{\mu\nu} + \varepsilon_{\nu\mu}$$
$$= \eta \ldots$$

So Minkowski metric is unchanged if $\boxed{\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}}$

The parameters $\varepsilon_{\mu\nu}$ are anti symmetric and 4-dimensional...

$$[\varepsilon_{\mu\nu}] = \begin{pmatrix} 0 & \varepsilon_{01} & & \\ - & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

only 6 independent component.

$\varepsilon_{0j} = -\varepsilon_{j0} \rightsquigarrow 3$ infinitesimal boosts

$\varepsilon_{jk} = -\varepsilon_{kj} \rightsquigarrow 3$ infinitesimal rotations...

To summarize how things transform under infitesimal LTs...

- Scalars $\phi \to \phi$
- Coordinate x^μ don't change. d^4x doesn't change.
- Minkowski metric $\eta_{\mu\nu} \to \eta_{\mu\nu}$ unchanged.

But all dynamical vectors and tensors change.

$$\begin{cases} A_\mu \to A_\mu + \varepsilon_\mu{}^\nu A_\nu \\ A^\mu \to A^\mu + \varepsilon^\mu{}_\nu A^\nu \end{cases}$$

For a tensor $\tau^{\mu\nu}{}_\lambda \to \tau^{\mu\nu}{}_\lambda + \varepsilon^\mu{}_\alpha \tau^{\alpha\nu}{}_\lambda + \varepsilon^\nu{}_\alpha \tau^{\mu\alpha}{}_\lambda + \varepsilon_\lambda{}^\alpha \tau^{\mu\nu}{}_\alpha$

Now, when will $S = \int d^4x\, \mathcal{L}$ be invariant ($\delta S = 0$) under global LT's?

If $\mathcal{L}$ is a scalar function then under a global LT, $\mathcal{L} \to \mathcal{L}$ so then $S \to S$ or $\delta S = 0$, which says it's a symmetry.

Ex Is $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2$ a scalar under LTs?

Under LTs.. $\phi \to \phi$, $\phi^2 \to \phi^2$, and so $\frac{1}{2}m^2\phi^2 \to \frac{1}{2}m^2\phi^2$

Now
$$\begin{cases} \partial_\mu\phi \to \partial_\mu\phi + \varepsilon_\mu{}^\alpha\partial_\alpha\phi & \text{(covariant rule)} \\ \partial^\mu\phi \to \partial^\mu\phi + \varepsilon^\mu{}_\alpha\partial^\alpha\phi \end{cases}$$

So
$$(\partial_\mu\phi)(\partial^\mu\phi) = \left(\varepsilon_\mu{}^\alpha\partial_\alpha\phi + \partial_\mu\phi\right)\left(\varepsilon^\mu{}_\alpha\partial^\alpha\phi + \partial^\mu\phi\right)$$
$$= \mathcal{O}^{\prime} + (\partial_\mu\phi)(\partial^\mu\phi) + (\partial_\mu\phi)(\varepsilon^\mu{}_\alpha\partial^\alpha\phi) + (\partial^\mu\phi)(\varepsilon_\mu{}^\alpha\partial_\alpha\phi)$$

$$= (\partial_\mu\phi)(\partial^\mu\phi) + \varepsilon^{\mu\alpha}(\partial_\alpha\phi)(\partial_\mu\phi) + \varepsilon^{\mu\alpha}(\partial_\alpha\phi)(\partial_\mu\phi)$$

$$= \cancel{(\partial_\mu\phi)(\partial^\mu\phi) + \varepsilon^{\mu\alpha}(\partial_\alpha\phi)(\partial_\mu\phi) - \varepsilon^{\alpha\mu}(\partial_\mu\phi)(\partial_\alpha\phi)}$$

Now,
$$\varepsilon^{\mu\alpha}(\partial_\alpha\phi)(\partial_\mu\phi) = -\varepsilon^{\alpha\mu}(\partial_\mu\phi)(\partial_\alpha\phi)$$
$$= -\varepsilon^{\alpha\mu}(\partial_\alpha\phi)(\partial_\mu\phi)$$

So $\varepsilon^{\mu\alpha}(\partial_\alpha\phi)(\partial_\mu\phi) = 0$

So $(\partial_\mu\phi)(\partial^\mu\phi) \to (\partial_\mu\phi)(\partial^\mu\phi)$

Exercise $\boxed{\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2A_\mu A^\mu. \text{ Show this is a scalar under global LTs..}}$

Use that $F_{\mu\nu}$ is a tensor....

$$F_{\mu\nu} \to F_{\mu\nu} + \varepsilon_\mu{}^\alpha F_{\alpha\nu} + \varepsilon_\nu{}^\alpha F_{\mu\alpha}$$

Likewise for $A_\mu A^\mu$.. Show $\mathcal{L} \to \mathcal{L}$

② $\boxed{\text{Diffeomorphism in Curved Spacetime}}$

$$S = \int d^4x \sqrt{-g}\, \mathcal{L}$$

Under diffeomorphism with ξ^μ, scalars & tensors transf with changes given by the Lie deriv...

<u>Scalars</u>: $\phi \to \phi + \xi^\alpha \partial_\alpha \phi$

or $\phi \to \phi + \xi^\alpha D_\alpha \phi$

(cov) <u>Vectors</u> $A_\mu \to A_\mu + (\partial_\mu \xi^\alpha) A_\alpha + \xi^\alpha \partial_\alpha A_\mu$

or $A_\mu \to A_\mu + (D_\mu \xi^\alpha) A_\alpha + \xi^\alpha D_\alpha A_\mu$

(contr) <u>Vectors</u> $A^\mu \to A^\mu - (\partial_\alpha \xi^\mu) A^\alpha + \xi^\alpha \partial_\alpha A^\mu$

or $A^\mu \to A^\mu - (D_\alpha \xi^\mu) A^\alpha + \xi^\alpha D_\alpha A^\mu$

<u>Tensors</u>

$$\tau^\mu{}_\nu \to \tau^\mu{}_\nu - (D_\alpha \xi^\mu) \tau^\alpha{}_\nu + (D_\nu \xi^\alpha) \tau^\mu{}_\alpha + \xi^\alpha D_\alpha \tau^\mu{}_\nu$$

<u>Now</u>

$$g_{\mu\nu} \to g_{\mu\nu} + D_\mu \xi_\nu + D_\nu \xi_\mu$$

So this means $\sqrt{-g}$ also transforms... We can find identities...

$$\boxed{\Gamma^\mu_{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g})}$$ ⟶ Show this...

With this, we can show that

$$\boxed{D_\mu V^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, V^\mu)}$$ ⟶ Show this

divergence

With these, the Lie derivs of $\sqrt{-g}$ can be found...

$$\boxed{\sqrt{-g} \longrightarrow \sqrt{-g} + \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right)}$$

This says

$$\boxed{\mathcal{L}_\xi \sqrt{-g} = \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right) = \sqrt{-g}\,D_\mu \xi^\mu}$$

If $\mathcal{L}$ is a scalar under diffs, then

$$\mathcal{L} \longrightarrow \mathcal{L} + \xi^\alpha \partial_\alpha \mathcal{L}$$

But what about S? where $S = \int d^4x\, \mathcal{L}\sqrt{-g}$?

Under diffs? $S \rightarrow S + \xi^\alpha \partial_\alpha S$

$$S \rightarrow \underbrace{\int d^4x}_{\text{don't chge}} \left[\sqrt{-g} + \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right)\right]\left(\mathcal{L} + \xi^\alpha \partial_\alpha \mathcal{L}\right)$$

$$= \int d^4x\,\sqrt{-g}\,\mathcal{L} + \int d^4x \left[\sqrt{-g}\,\xi^\alpha \partial_\alpha \mathcal{L} + \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha\right)\mathcal{L}\right]$$

$$+ \cancelto{0}{\int d^4x\, \xi^2}$$

$$= S + \int d^4x\, \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha \mathcal{L}\right)$$

So $$S \rightarrow S + \underbrace{\int d^4x\, \partial_\alpha\left(\sqrt{-g}\,\xi^\alpha \mathcal{L}\right)} \rightarrow \text{use 4D } \text{~~Stokes'~~ Gauss'} \ldots$$

$$S \rightarrow S + \left[\int_{\text{3D surface}} d^3x\, \hat{n}_\alpha \left(\sqrt{-g}\,\xi^\alpha \mathcal{L}\right)\right]$$

Push 3D surface to ∞ where $\xi^\alpha = 0$ So $S \rightarrow S$ And $\boxed{\delta S = 0}$

And action is unchanged under diffs $\rightarrow$ sym of GR...

Exercise

1 *Show* $\mathcal{L} = \frac{1}{2}(D_\mu \phi)(D^\mu \phi) - \frac{1}{2}m^2\phi^2$ is a

scalar under diffs. $\mathcal{L} \to \mathcal{L} + \xi^\alpha D_\alpha \mathcal{L}$

know ϕ^2 is a scalar... $D_\mu \phi = \cancel{\partial_\mu \phi} \to D_\mu \phi + (D_\mu \xi^\alpha) D_\alpha \phi$

$+ \xi^\alpha D_\alpha D_\mu \phi$

$D^\mu \phi \to \ldots$

Verify that $(D_\mu \phi)(D^\mu \phi) \to (D_\mu \phi)(D^\mu \phi) + \xi^\alpha D_\alpha$ (itself)