Differentiation: 5.1, 2, 3, 6, 9, 12, 22, Baby Rudin

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5.1 *Proof:* Let f be defined for all reals such that $\left|f(x)-f(y)\right| \leq (x-y)^2$ for all $x, y \in \mathbb{R}$. Let $\delta > 0$ be given. Pick $0 < x-y < \delta$, which gives $0 \leq \left|f(x)-f(y)\right|/(x-y) \leq x-y < \delta$. This holds for any $\delta > 0$, so $f'(x) = 0$ for all $x \in \mathbb{R}$. This means f is constant. □

5.2 *Proof:* Suppose $f'(x) > 0$ on (a,b). Pick $x, y \in (a,b)$ such that $y > x$. Then by the MVT $(f(y)-f(x))/(y-x) = f'(c) > 0$ for some $c \in (x,y)$. Since $y > x$, this holds if $f(y) > f(x)$, so f is a strictly increasing function.

Let g be its inverse function, so $g(f(x)) = x$.

$$\frac{g(y)-g(x)}{y-x} = \frac{g(f(u))-g(f(v))}{f(u)-f(v)} = \frac{u-v}{f(u)-f(v)}$$

where $y = f(u), x = f(v), y \neq x$, and $f(u) \neq f(v)$ (because f is strictly increasing hence is one-to-one). And so

$$\lim_{y\to x}\frac{g(y)-g(x)}{y-x} = \lim_{u\to v}\frac{u-v}{f(u)-f(v)} = \frac{1}{f'(v)}. \tag{1}$$

The limit exists because $f'(v) > 0$, and so $g'(x = f(v)) = 1/f'(v)$. □

5.3 *Proof:* Suppose g is a real function on $\mathbb{R}$, with $\left|g'\right| \leq M$. Fix $\epsilon > 0$ and define $f(x) = x + \epsilon g(x)$. We want to show f is one-to-one if ϵ is small enough. Let $a < b$ be given. Then $f(b)-f(a) = (b-a) + \epsilon(g(b)-g(a))$. By MVT, $\exists c \in (a,b)$ such that $(b-a)g'(c) = g(b)-g(a)$. With this, $f(b)-f(a) = (b-a)(1-\epsilon g'(c))$. Pick $\epsilon < 1/M$, then $\left|\epsilon g'(c)\right| < 1$, which means $f(b)-f(a) \neq 0$. So, f is one-to-one. □

5.6 *Proof:* Suppose f is continuous for $x \geq 0$, f' exists for $x > 0$, $f(0) = 0$, and f' is monotonically increasing. We want to show $g(x) = f(x)/x, (x > 0)$ is increasing. Well we know that $g(x)$ is differentiable for $x > 0$. So it suffices to show $g'(x) > 0$ for all $x > 0$. Well, $g'(x) = -f(x)/x^2 + f'(x)/x$. Now it comes down to showing the function $h(x) = xf'(x) - f(x)$ is positive for all $x > 0$. Now, $h(0) = 0f'(x) - 0 = 0$. And so $h(x)$ is positive for all $x > 0$ if $h'(x) > 0$ for all $x > 0$. Well, $h'(x) = f'(x) + xf''(x) - f'(x) = xf''(x) > 0$ for all $x > 0$ because f' is monotonically increasing. So we're done. □

5.9 *Proof:* f is a continuous real function on $\mathbb{R}$. f' exists for all $x \neq 0$ and $f'(x) \to 3$ as $x \to 0$. It DOES follow that $f'(0) = 3$. Consider the function $h(x) = f(x) - f(0)$ and $g(x) = x - 0$. Then both approach zero as $x \to 0$. This means by l'Hopital's rule:

$$\frac{h(x)}{g(x)} = \frac{f(x)-f(0)}{x} \to \frac{f'(x)}{1} = 3 \text{ as } x \to 0^{\pm}.$$

Thus,

$$\lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = 3 = f'(0).$$

$\square$

5.12 *Proof:* Suppose

$$f(x) = |x|^3 = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0. \end{cases}$$

It is not difficult to see that

$$f'(x) = \begin{cases} 3x^2, & x > 0 \\ -3x^2, & x < 0 \end{cases} \quad . \tag{2}$$

When $x = 0$, we look at the limits:

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x|^3}{x} = +0^2 = 0$$
$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x|^3}{x} = -0^2 = 0.$$

So, $f'(0) = 0$. Next, it is also not difficult to see that

$$f''(x) = \begin{cases} 6x, & x > 0 \\ -6x, & x < 0 \end{cases} \quad .$$

When $x = 0$, we look at the limits:

$$\lim_{x \to 0+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0+} \frac{3x^2}{x} = 0$$
$$\lim_{x \to 0-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0+} \frac{-3x^2}{x} = 0.$$

So, $f''(0) = 0$. However, $f'''(0)$ does not exist because

$$\lim_{x \to 0+} \frac{f''(x) - f''(0)}{x - 0} = \lim_{x \to 0+} \frac{6x}{x} = 6$$
$$\lim_{x \to 0-} \frac{f''(x) - f''(0)}{x - 0} = \lim_{x \to 0+} \frac{-6x}{x} = -6 \neq \lim_{x \to 0+} \frac{f''(x) - f''(0)}{x - 0},$$

i.e., the limit as $x \to 0$ of the difference quotient does not exist. $\square$

5.22 *Proof:* Suppose f is a real function $(-\infty, \infty)$. x is a fixed point if $f(x) = x$.

1. f is differentiable and $f'(t) \neq 1$ for every real t. We want to show f has at most one fixed point. Suppose f has at least two fixed points a and b, then

   $$\frac{f(b) - f(a)}{b - a} = 1.$$

   By MVT, there exists $c \in (a, b)$ such that $f'(c) = 1$, which is a contradiction. So, f at most one fixed point.

2. We want to show $f(t) = t + (1+e^t)^{-1}$ has no fixed point, although $0 < f'(t) < 1$ for all real t, i.e., we want to show $t \neq t + (1+e^t)^{-1} \forall t \in \mathbb{R}$. Obviously, $1/(1+e^t) \neq 0$ for all $t \in \mathbb{R}$. Now,

$$f'(t) = 1 - \frac{e^t}{(1+e^t)^2} = 1 - \frac{e^t}{1+2e^t+e^{2t}}.$$

The quantity $\frac{e^t}{(1+e^t)^2} > 0$, so $f'(t) < 1$. Also, $\frac{e^t}{(1+e^t)^2} < 1$ because $0 < e^{t/2} << e^t < 1 + e^t$ (as e^t is an increasing function). So, $0 < f'(t) < 1$. So, even though $0 < f'(t) < 1$, f does not have any fixed point.

3. Suppose $\left|f'(t)\right| \leq A$ for all real t for some constant $A < 1$. We want to show f has a fixed point x and that $x = \lim x_n$ where x_1 is an arbitrary real number and $x_{n+1} = f(x_n)$ for $n = 1, 2, 3, \ldots$. Well, if $x_n = x_{n+1}$ then $\{x_n\}$ is identically the sequence $\{x\}$ and x_n is a fixed point of f for any n. Otherwise, MVT says

$$f(x_{n+1}) - f(x_n) = f'(t)(x_{n+1} - x_n)$$

for some t between x_n and x_{n+1}. Since $\left|f'(t)\right| \leq A < 1$ and $f(x_{n+1}) = x_{n+2}$, we have that

$$|x_{n+2} - x_{n+1}| \leq A|x_{n+1} - x_n| \leq \cdots \leq A^{n-1}|x_2 - x_1|.$$

With this, for any positive $n > m$,

$$\begin{aligned}
|x_n - x_m| &\leq |x_n - x_{n-1}| + \cdots + |x_{m+1} - x_m| \\
&\leq |x_2 - x_1| \left(A^{n-2} + A^{n-3} + \cdots + A^m + A^{m-1}\right) \\
&= |x_2 - x_1| A^{m-1} \left(A^{n-m-1} + \cdots + 1\right) \\
&= |x_2 - x_1| A^{m-1} \frac{1 - A^{n-m}}{1 - A} \\
&\leq |x_2 - x_1| \frac{A^{m-1}}{1 - A}.
\end{aligned}$$

For $\epsilon > 0$, there exists $N = 2 + \log_A \epsilon(1-A)/|x_2 - x_1|$ such that whenever $n > m > N$, we have $|x_n - x_m| < \epsilon$. So, $\{x_n\} \to x$, and so because f is continuous,

$$x = \lim_{n\to\infty} x_{n+1} = f\left(\lim_{n\to\infty} x_n\right) = f(x).$$

This means $x = f(x)$, or x is a fixed point of f. Why does this choice of N work? Well, if $n > m > N$ then

$$\begin{aligned}
|x_n - x_m| &\leq |x_2 - x_1| \frac{A^{m-1}}{1 - A} \\
&\leq |x_2 - x_1| \frac{A^{N-1}}{1 - A} \\
&= |x_2 - x_1| \frac{A\epsilon(1 - A)}{A(1 - A)|x_2 - x_1|} \\
&= \epsilon.
\end{aligned}$$

4. We want to show that the process described in item 3. can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \dots$$

We start with the x-coordinate x_1, then $f(x_1) = x_2$, so we get the first point (x_1, x_2). If x_1 is a fixed point of f then $x_1 = x_2$, so we move to (x_2, x_2). If $(x_2, x_2) \neq (x_1, x_2)$, then we keep going by repeating: look at point $(x_2, f(x_2) = x_3)$, and so on.

$\square$