PY 711 Fall 2010
Homework 10: Due Tuesday, November 9

1. (15 points) As in the previous homework assignment, we use the same Lagrange density involving three real scalar fields,

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}(\partial_\mu \phi_X)(\partial^\mu \phi_X) - \frac{1}{2}m_X^2\phi_X^2 + \frac{1}{2}(\partial_\mu \phi_Y)(\partial^\mu \phi_Y) - \frac{1}{2}m_Y^2\phi_Y^2 \\ &\quad + \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 - \lambda\Phi\phi_X\phi_Y. \end{aligned} \tag{1}$$

This time we consider only the case with three spatial dimensions. In the center-of-mass frame we consider elastic scattering of a ϕ_X particle and ϕ_Y particle to lowest non-vanishing order in λ. By elastic scattering we mean the process

$$\phi_X + \phi_Y \to \phi_X + \phi_Y. \tag{2}$$

Let $\vec{p}$ and $-\vec{p}$ be the incoming momenta for the ϕ_X and ϕ_Y particles respectively. Let $\vec{p}'$ and $-\vec{p}'$ be the outgoing momenta for the ϕ_X and ϕ_Y particles respectively. Determine the differential cross section $\frac{d\sigma}{d\Omega}$ as a function of the following parameters: the magnitude of the momentum $p = |\vec{p}|$; the angle θ between $\vec{p}$ and $\vec{p}'$; the coupling constant λ; and the particle masses m_X, m_Y, and M.

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HW#10

12.5/15

1. AS IN THE PREVIOUS HOMEWORK ASSIGNMENT, WE USE THE SAME LAGRANGIAN DENSITY INVOLVING THREE REAL SCALAR FIELDS,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_x)(\partial^\mu \phi_x) - \frac{1}{2}m_x^2\phi_x^2 + \frac{1}{2}(\partial_\mu \phi_y)(\partial^\mu \phi_y) - \frac{1}{2}m_y^2\phi_y^2$$
$$+ \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - \frac{1}{2}M^2\Phi^2 - \lambda \Phi \phi_x \phi_y .$$

THIS TIME WE CONSIDER ONLY THE CASE WITH THREE SPATIAL DIMENSIONS. IN THE CENTER-OF-MASS FRAME WE CONSIDER ELASTIC SCATTERING OF A ϕ_x PARTICLE AND ϕ_y PARTICLE TO LOWEST NON-VANISHING ORDER IN λ. BY ELASTIC SCATTERING WE MEAN THE PROCESS

$$\phi_x + \phi_y \longrightarrow \phi_x + \phi_y .$$

LET $\vec{p}$ AND $-\vec{p}$ BE THE INCOMING MOMENTA FOR THE ϕ_x AND ϕ_y PARTICLES RESPECTIVELY. LET $\vec{p}\,'$ AND $-\vec{p}\,'$ BE THE OUTGOING MOMENTA FOR THE ϕ_x AND ϕ_y PARTICLES RESPECTIVELY. DETERMINE THE DIFFERENTIAL CROSS SECTION $\frac{d\sigma}{d\Omega}$ AS A FUNCTION OF THE FOLLOWING PARAMETERS: THE MAGNITUDE OF THE MOMENTUM $p = |\vec{p}|$; THE ANGLE θ BETWEEN $\vec{p}$ AND $\vec{p}\,'$; THE COUPLING CONSTANT λ; AND THE PARTICLE MASSES m_x, m_y, AND M.

We know the cross section is given by

$$d\sigma = d\Pi_2 \frac{|\mathcal{M}|^2}{(2E_x)(2E_y)|\vec{v}_x - \vec{v}_y|}$$

Where

$$d\Pi_2 = \frac{d\Omega}{16\pi^2}\frac{p'}{E_{cm}} \qquad (p' = |\vec{p}\,'| \;,\; E_{cm} = E_x + E_y)$$

E_x, E_y are the initial energies of ϕ_x and ϕ_y

$|\vec{v}_x - \vec{v}_y|$ is the relative speed of the particles in the lab frame

and $\vec{p}\,'$ is chosen such that $\sqrt{p'^2 + m_x^2} + \sqrt{p'^2 + m_y^2} = E_{cM}$

(final energy = initial energy)

1 CONTINUED

First, I will find the value of $p' = |\vec{p}'|$

$$\sqrt{p'^2 + m_x^2} + \sqrt{p'^2 + m_y^2} = E_{CM}$$

$$2p'^2 + m_x^2 + m_y^2 + 2\sqrt{p'^2 + m_x^2}\sqrt{p'^2 + m_y^2} = E_{CM}^2$$

$$2\sqrt{p'^2 + m_x^2}\sqrt{p'^2 + m_y^2} = E_{CM}^2 - 2p'^2 - m_x^2 - m_y^2$$

$$\begin{aligned} 4(p'^2 + m_x^2)(p'^2 + m_y^2) = {} & E_{CM}^4 + 4p'^4 + m_x^4 + m_y^4 \\ & + 4p'^2(m_x^2 + m_y^2 - E_{CM}^2) \\ & - 2E_{CM}^2(m_x^2 + m_y^2) + 2m_x^2 m_y^2 \end{aligned}$$

$$\begin{aligned} 4(p'^4 + m_x^2 m_y^2 + p'^2(m_x^2 + m_y^2)) = {} & E_{CM}^4 + 4p'^4 + m_x^4 + m_y^4 \\ & + 4p'^2(m_x^2 + m_y^2 - E_{CM}^2) \\ & - 2E_{CM}^2(m_x^2 + m_y^2) + 2m_x^2 m_y^2 \end{aligned}$$

$$4p'^2 E_{CM}^2 = E_{CM}^4 + m_x^4 + m_y^4 - 2E_{CM}^2(m_x^2 + m_y^2) - 2m_x^2 m_y^2$$

$$p' = \frac{1}{2E_{CM}}\sqrt{E_{CM}^4 + m_x^4 + m_y^4 - 2E_{CM}^2(m_x^2 + m_y^2) - 2m_x^2 m_y^2}$$

Using Mathematica, substitute $E_{CM} = \sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2}$

$$p' = p\,\frac{\sqrt{\left(\sqrt{p^2 + m_x^2} + \sqrt{p^2 + m_y^2}\right)^2}}{\left(\sqrt{p^2 + m_y^2} + \sqrt{p^2 + m_y^2}\right)}$$

$$\Rightarrow \quad p' = p$$

1 CONTINUED.

Next, find $|\vec{v}_x - \vec{v}_y|$.

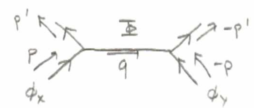
$$\vec{p}_x = \gamma_x m_x \vec{v}_x \qquad \vec{p}_y = \gamma_y m_y \vec{v}_y$$

$$|\vec{v}_x - \vec{v}_y| = \left| \frac{\vec{p}}{\gamma_x m_x} + \frac{+\vec{p}}{\gamma_y m_y} \right| = |\vec{p}| \left(\frac{1}{\gamma_x m_x} + \frac{1}{\gamma_y m_y} \right)$$

$$= |\vec{p}| \left(\frac{1}{E_x} + \frac{1}{E_y} \right)$$

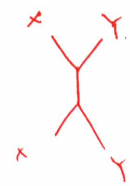
$$= |\vec{p}| \left(\frac{E_x + E_y}{E_x E_y} \right)$$

Now, determine $|\mathcal{M}|^2$



$q = p - p'$

+



$$i\mathcal{M} = (-i\lambda)^2 \frac{i}{q^2 - M^2 + i\epsilon}$$ (2 vertices, 1 propagator, scalar particles)

$$\mathcal{M} = \frac{-\lambda^2}{q^2 - M^2}$$

in future problems, don't assume non-relativistic limit

$$q^2 = (m_x - m_x)^2 - (\vec{p} - \vec{p}\,')^2$$

$$= -(p^2 + p'^2 - 2pp'\cos(\theta))$$

$$= -2p^2(1 - \cos(\theta)) \qquad (p = p')$$

$$\mathcal{M} = \frac{+\lambda^2}{+2p^2(1 - \cos(\theta)) + M^2}$$

$$|\mathcal{M}|^2 = \frac{\lambda^4}{(2p^2(1 - \cos(\theta)) + M^2)^2}$$

1 CONTINUED

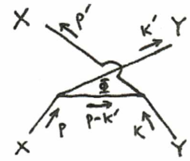
Finally, put everything together.

$$\frac{d\sigma}{d\Omega} = \frac{1}{16\pi^2} \frac{p'}{(E_x+E_y)} \frac{|\mathcal{M}|^2}{(2E_x)(2E_y)|\vec{v}_x - \vec{v}_y|}$$

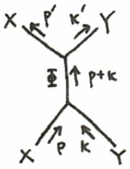
$$= \frac{1}{16\pi^2} \frac{\cancel{p}}{(E_x+E_y)} \left(\frac{\lambda^4}{(2p^2(1-\cos(\theta)) + M^2)^2} \right) \frac{1}{4\cancel{E_x E_y}} \frac{\cancel{E_x E_y}}{\cancel{p}(E_x + E_y)}$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \frac{1}{\left(\sqrt{p^2+m_x^2} + \sqrt{p^2+m_y^2}\right)^2} \frac{\lambda^4}{(2p^2(1-\cos(\theta)) + M^2)^2}}$$

Solutions #10 PY711

1. 

$$= i\mathcal{M}_1 = (-i\lambda)^2 \frac{i}{(p-k')^2 - M^2 + i\varepsilon}$$



$$= i\mathcal{M}_2 = (-i\lambda)^2 \frac{i}{(p+k)^2 - M^2 + i\varepsilon}$$

$$i\mathcal{M} = i\mathcal{M}_1 + i\mathcal{M}_2 = -i\lambda^2 \left[\frac{1}{(p-k')^2 - M^2 + i\varepsilon} + \frac{1}{(p+k)^2 - M^2 + i\varepsilon}\right]$$

In the center of mass frame

$$p = (E_X, \vec{p}) \qquad k = (E_Y, -\vec{p})$$
$$p' = (E_X, \vec{p}\,') \qquad k' = (E_Y, -\vec{p}\,') \qquad |\vec{p}| = |\vec{p}\,'|$$

where $E_X = \sqrt{\vec{p}^2 + m_X^2}$, $E_Y = \sqrt{\vec{p}^2 + m_Y^2}$, $E_{cm} = E_X + E_Y$

From lecture,

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\vec{p}|\; |\mathcal{M}|^2}{2E_X\, 2E_Y\, |\vec{v}_X - \vec{v}_Y|\, 16\pi^2 E_{cm}}$$

Note that $\vec{v}_X = \frac{\vec{p}}{E_X}$, $\vec{v}_Y = \frac{-\vec{p}}{E_Y}$ and so $|\vec{v}_X - \vec{v}_Y| = |\vec{p}|\left(\frac{1}{E_X} + \frac{1}{E_Y}\right)$

$$\Rightarrow 2E_X 2E_Y |\vec{v}_X - \vec{v}_Y| = 4|\vec{p}|(E_X + E_Y)$$

Therefore $\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{|\mathcal{M}|^2}{64\pi^2 (E_X + E_Y)^2}$

Using $(p-k')^2 = (E_x - E_Y)^2 - (\vec{p}+\vec{p}')^2 = (E_x - E_Y)^2 - (\vec{p}^2 + \vec{p}'^2 + 2\vec{p}\cdot\vec{p}')$

$$= (E_x - E_Y)^2 - (2\vec{p}^2 + 2\vec{p}^2 \cos\theta)$$

$$= (E_x - E_Y)^2 - 2\vec{p}^2(1+\cos\theta)$$

and $(p+k)^2 = (E_x + E_Y)^2$, we get

$$|\mathcal{M}|^2 = \lambda^4 \left[\frac{1}{(E_x - E_Y)^2 - 2\vec{p}^2(1+\cos\theta) - M^2} + \frac{1}{(E_x + E_Y)^2 - M^2} \right]^2$$

So $$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^4}{64\pi^2 (E_x + E_Y)^2} \left[\frac{1}{(E_x - E_Y)^2 - 2\vec{p}^2(1+\cos\theta) - M^2} + \frac{1}{(E_x + E_Y)^2 - M^2} \right]^2$$

In terms of $|\vec{p}|$, θ, m_x, m_Y, and M,

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{\lambda^4}{64\pi^2 \left[\sqrt{\vec{p}^2 + m_x^2} + \sqrt{\vec{p}^2 + m_Y^2}\right]^2}$$

$$\times \left[\frac{1}{\left[\sqrt{\vec{p}^2 + m_x^2} - \sqrt{\vec{p}^2 + m_Y^2}\right]^2 - 2\vec{p}^2(1+\cos\theta) - M^2} + \frac{1}{\left[\sqrt{\vec{p}^2 + m_x^2} + \sqrt{\vec{p}^2 + m_Y^2}\right]^2 - M^2} \right]^2$$