MA439: Functional Analysis
Tychonoff Spaces: Exercises 9, 10 on p.8
& 5,6,7,9,10,13,14,15 on p.19, Ben Mathes

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Exercise 1 (Ex 9 p.8)**.** *Assume $\mathcal{F}$ is a filter in $\mathcal{X}$ and $E \subset \mathcal{X}$. Prove that $\mathcal{F} \cup \{E\}$ is contained in a filter if and only if $E \cap F \neq \varnothing$ for every $F \in \mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a filter in $\mathcal{X}$ and $E \subset \mathcal{X}$. Suppose that $\mathcal{F} \cup \{E\}$ is contained in a filter $\mathcal{G}$. Then by definition, $G \cap E \neq \varnothing$ for all $G \in \mathcal{G}$, and in particular $F \cap E \neq \varnothing$ for all $F \in \mathcal{F}$. Conversely, if $F \cap E \neq \varnothing$ for all $F \in \mathcal{F}$, then $\cap_{\mathcal{F}} F \subseteq E$. Now, since $\cap_{\mathcal{F}} F \in \mathcal{F}$, we have $E \in \mathcal{F}$. Thus, $\mathcal{F} \cup \{E\} = \mathcal{F}$, which is a filter. □

Exercise 2 (Ex 10 p.8)**.** *Assume that $\mathcal{F}$ is a filter in a set $\mathcal{X}$. Prove that $\mathcal{F}$ is an ultrafilter if and only if for every subset $E \subset \mathcal{X}$ either $E \in \mathcal{F}$ or $\mathcal{X} \setminus E \in \mathcal{F}$. (We denote the **complement** of E in $\mathcal{X}$ by $\mathcal{X} \setminus E$ and it is defined as the set $\mathcal{X} \setminus E \equiv \{x \in \mathcal{X} : x \notin E\}$.)*

Proof. Suppose that $\mathcal{F}$ is an ultrafilter and $E \neq \varnothing$ is a subset of $\mathcal{X}$. Suppose $E \notin \mathcal{F}$. Since $\mathcal{F}$ is maximal, the collection $\mathcal{F} \cup \{E\}$ can neither be a filter nor be contained in a filter (in view of the previous exercise). It follows that there is an $F \in \mathcal{F}$ for which $F \cap E = \varnothing$. This gives $F \subseteq (\mathcal{X} \setminus E)$, which implies that $\mathcal{X} \setminus E \in \mathcal{F}$. Repeating this argument starting with $\mathcal{X} \setminus E$, we obtain $E \in \mathcal{F}$.

Conversely, if $\mathcal{F}$ is not an ultrafilter, then there is a filter $\mathcal{G}$ properly containing $\mathcal{F}$. Let $\{E\} = \mathcal{G} \setminus \mathcal{F}$. In view of the previous exercise, we have that $E \cap F \neq \varnothing$ for all $F \in \mathcal{F}$. In particular, $\mathcal{X} \setminus E \notin \mathcal{F}$ because $E \cap (\mathcal{X} \setminus E) = \varnothing$. □

Exercise 3 (Ex 5 p.19)**.** *Prove that every set E gives rise to two sets $\overline{E}$ and E° such that $E^\circ \subseteq E \subseteq \overline{E}$ where $\overline{E}$ is the smallest closed set containing E, and E° is the largest open set contained in E. Give examples to show that there might not exist a smallest open set containing E, and there might not exist a largest closed set contained in E. (The set $\overline{E}$ is called the **closure** of E and E° is called the **interior** of E.)*

Proof. Let $E^\circ = \{x \in E : B_d(x, \epsilon) \subseteq E, \epsilon > 0\}$ and $\overline{E} = \{x \in \mathcal{X} : x = \lim_{n\to\infty} x_n,$ where $(x_n) \subseteq E\}$. It is clear that $E^\circ \subseteq E \subseteq \overline{E}$.

Here we show that E° is the largest open set contained in E. Let $x \in E^\circ$ be given. By construction, there is an $\epsilon > 0$ for which $B_d(x, \epsilon) \subseteq E$. Because $B_d(x, \epsilon)$ is open, for any $y \subseteq B_d(x, \epsilon)$, there is a $\delta > 0$ for which $B_d(y, \delta) \subseteq B_d(x, \epsilon) \subseteq E$. Thus, $B_d(x, \epsilon) \subseteq E^\circ$, and so E° is open. Now, consider any open set $O \subseteq E$. For any point $x \in O$, there is an $r > 0$ for which $B_d(x, r) \subseteq O$. Thus, $x \in E^\circ$. It follows that $O \subseteq E^\circ$. So, E° is the largest open set contained in E.

Next, we show that $\overline{E}$ is the smallest closed set containing E. First, S is closed because any sequence (x_n) in S converges within S. Now, let C be a closed set in $\mathcal{X}$ such that $E \subseteq C$. Choose an $x \in \overline{E}$. By definition, there is a sequence (x_n) in E that converges to x. Since C is closed and contains E, $x \in C$. Thus, $\overline{E} \subseteq C$. Thus, $\overline{E}$ is the smallest closed set containing E.

Take $E = \{0\} \subseteq \mathbb{R}$. Suppose an open set O contains E. Then there is an open interval $I = (a, b) \subseteq O$ that contains 0. It follows that there is open interval $I' = (a/2, b/2) \subseteq I$ that contains 0. So, there is no smallest open set containing E.

Take $E = [0, 1] \subseteq \mathbb{R}$. Suppose that S is a closed set contained in E. It is clear that $E \setminus S$ is a non-empty open set. Take an interval $I \subseteq E \setminus S$. Then $S \cup I$ is closed and is larger than S. $\square$

Exercise 4 (Ex 6 p.19)**.** *A point x is called an **interior point** of a subset $E \subseteq X$ if there exists $\epsilon > 0$ so that $B(x, \epsilon) \subseteq E$. Prove that E° is exactly the set of interior points of E.*

Proof. Let E° be the largest open set contained in E and let $\mathrm{Int}(E) = \{x \in E : B(x, \epsilon) \subseteq E, \epsilon > 0\}$. By virtue of the previous problem, $\mathrm{Int}(E) = E^\circ$. $\square$

Exercise 5 (Ex 7 p.19)**.** *Given any subset E of a metric space, let $Bd(E)$ denote the set of x with the property that every ball containing x intersects both E and $\mathcal{X} \setminus E$. (This set is called the **boundary** of E.) Prove that $E^\circ, Bd(E)$, and $(\mathcal{X} \setminus E)^\circ$ form a partition of $\mathcal{X}$. (The latter set is often called the **exterior** of E.)*

Proof. Let $E \subseteq \mathcal{X}$ be given. Since $E^\circ \subseteq E$, $E \cap (\mathcal{X} \setminus E) = \varnothing$, and $(\mathcal{X} \setminus E)^\circ \subseteq \mathcal{X} \setminus E$, we have that $E^\circ \cap (\mathcal{X} \setminus E)^\circ = \varnothing$. It is also clear that $\mathrm{Bd}(E) \cap E^\circ = \varnothing = \mathrm{Bd}(E) \cap (\mathcal{X} \setminus E)$. It remains to show that $\mathcal{X} = E^\circ \cup \mathrm{Bd}(E) \cup (\mathcal{X} \setminus E)^\circ$. Let $x \in \mathcal{X}$ be given. If $x \in E$, then $x \in E^\circ$ or $x \in E \setminus E^\circ$. In the latter case, for every $\epsilon > 0$, $B(x, \epsilon)$ can intersect both E and $(\mathcal{X} \setminus E)$. This means that $x \in \mathrm{Bd}(E)$. If $x \in (\mathcal{X} \setminus E)$, then $x \in (\mathcal{X} \setminus E)^\circ$ or $x \in (\mathcal{X} \setminus E) \setminus (\mathcal{X} \setminus E)^\circ$. In the latter case, we get $x \in \mathrm{Bd}(E)$ by a similar argument. Thus, $\{E^\circ, \mathrm{Bd}(E), (\mathcal{X} \setminus E)^\circ\}$ form a partition of $\mathcal{X}$. $\square$

Exercise 6 (Ex 9 p.19)**.** *If E is a subset of a metric space $(\mathcal{X}, d)$ and $x \in \mathcal{X}$, we will say that x is a **limit point** of E when, for every $\epsilon > 0$, the ball $B(x, \epsilon)$ contains an element of E other than x. If x is a limit point of E, prove that the set $B(x, \epsilon) \cap E$ is infinite.*

Proof. Suppose that there is an $\epsilon > 0$ for which $B(x, \epsilon) \cap E$ is finite. Let $\epsilon' = \inf_{x' \in B(x,\epsilon) \cap E} d(x, x')$. Then, $B(x, \epsilon') \cap E = \varnothing$, which implies that x is not a limit point of E. The claim follows as desired. $\square$

Exercise 7 (Ex 10 p.19)**.** *Let E' denote the set of limit points of E. Prove that the closure of E equals $E \cup E'$.*

Proof. Let E be given. Then we have $E^\circ \subseteq E \subseteq \overline{E}$. Let $x \in \overline{E}$. If $x \in E$ or $x \in \overline{E} \setminus E$. When $x \notin E$, $x \notin E^\circ$ as well. Thus, there is no $\epsilon > 0$ for which $B(x, \epsilon) \subseteq E$. Thus, for all $\epsilon > 0$, $B(x, \epsilon) \cap E \neq \varnothing$, i.e., $x \in E'$. So $\overline{E} \subseteq E \cup E'$.

Conversely, if $x \in E \cup E'$, then $x \in \overline{E}$ if $x \in E$. If $x \in E'$, then for every $\epsilon > 0$, we have $B(x, \epsilon) \cap E \neq \varnothing$. Since $E \subseteq \overline{E}$, we have that $B(x, \epsilon) \cap \overline{E} \neq \varnothing$. So $x \in \overline{E}$ as well. Thus, $E \cup E' \subseteq \overline{E}$.

Thus $\overline{E} = E \cup E'$ as desired. $\square$

Exercise 8 (Ex 13 p.19)**.** *Assume $x \in \mathcal{X}$ and d is a metric on $\mathcal{X}$. We define a family of sets $\mathcal{F}_x$ by $H \in \mathcal{F}_x$ if and only if there exists ϵ such that $B_d(x, \epsilon) \subseteq H$. Prove that $\mathcal{F}_x$ is a filter. (It is called the **neighborhood filter** of x.)*

Proof. Let $H_1, H_2 \in \mathcal{F}_\S$ be given. Then there are ϵ_1, ϵ_2 for which $B(x, \epsilon_2) \subseteq H_2, B(x, \epsilon_1) \subseteq H_1$. Let $\epsilon' = \min\{\epsilon_1, \epsilon_2\}/2$. Then $B(x, \epsilon') \in H_1 \cap H_2$, whence $H_1 \cap H_2 \in \mathcal{F}_x$. Next, let $H \in \mathcal{F}_\S$ and $G \supseteq H$ be given. It is clear that there exists an $\epsilon > 0$ for which $B(x, \epsilon) \subseteq H \subseteq G$. So, $G \in \mathcal{F}_x$. Thus, $\mathcal{F}_x$ is a filter. $\square$

Exercise 9 (Ex 14 p.19)**.** *We will write $\mathcal{F} \to x$, and say that the* ***filter*** *$\mathcal{F}$* ***converges*** *to x, when $\mathcal{F}_x \subseteq \mathcal{F}$. Given a filter $\mathcal{F}$ in the domain of a function f, we denote by $f(\mathcal{F})$ the family of sets defined as follows: $H \in f(\mathcal{F})$ if and only if H contains a subset of the form $f(F)$ with $F \in \mathcal{F}$ Prove that $f(\mathcal{F})$ is a filter, and a mapping f between metric spaces is continuous if and only if filter convergence $\mathcal{F} \to x$ in the domain of f implies $f(\mathcal{F}) \to f(x)$.*

Proof. We first show that $f(\mathcal{F})$ is a filter. Let $H_1, H_2 \in f(\mathcal{F})$. Then there are $F_1, F_2 \subseteq F$ for which $f(F_1) \subseteq H_1$ and $f(F_2) \subseteq H_2$. It is clear that $f(F_1 \cap F_2) \subseteq f(F_1) \cap f(F_2) \subseteq H_1 \cap H_2$. So, $H_1 \cap H_2 \in f(\mathcal{F})$. Next, let $H \in f(\mathcal{F})$ be given and $G \supseteq H$. It follows that there is an $F \in \mathcal{F}$ for which $f(F) \subseteq H \subseteq G$, so $G \in f(\mathcal{F})$. Thus, $f(\mathcal{F})$ is a filter.

[1] Suppose that f is continuous at $x \in \mathcal{X}$. Let V be any neighborhood of $f(x)$ in $\mathcal{X}'$, i.e., $V \in f(\mathcal{F}_x)$. Then for some neighborhood U of x in $\mathcal{X}$, $f(U) \subset V$ (by continuity of f). Since $U \in \mathcal{F}_x \subseteq \mathcal{F}$, we have that $V \in f(\mathcal{F})$.

Conversely, suppose that $\mathcal{F} \to x$ implies $f(\mathcal{F}) \to f(x)$. Let $\mathcal{F}$ be the filter of all neighborhood of x in $\mathcal{X}$. It follows that each neighborhood V of $f(x)$ belongs to $f(\mathcal{F})$. Thus, for some neighborhood U of x_0, $f(U) \subseteq V$. So, f is continuous at x.

□

Exercise 10 (Ex 15 p.19)**.** *Prove that $f(\mathcal{F})$ is an ultrafilter when F is.*

Proof. Suppose that $\mathcal{F}$ is an ultrafilter and let $f : \mathcal{X} \to \mathcal{Y}$ be given. Let $E \subseteq \mathcal{Y}$. In view of Theorem 12.11 in Willard's *General Topology*, we have that for $f^{-1}(E) \in \mathcal{X}$, either $f^{-1}(E) \in \mathcal{F}$ or $\mathcal{X} \setminus f^{-1}(E) \in \mathcal{F}$. This implies that $E = f(f^{-1}(E)) \in f(\mathcal{F})$ or $\mathcal{Y} \setminus E = f(X \setminus f^{-1}(E)) \in f(\mathcal{F})$. In view of Theorem 12.11 in Willard's again we find that $f(\mathcal{F})$ is also an ultrafilter. □

[1] Referenced Willard's *General Topology*, Theorem 12.8.