Spring, 2021

Physics 312: Physics of Fluids

Assignment #7

Background Reading

Friday, Mar. 26: Tritton 9.1, 9.2,
Kundu & Cohen 9.1 - 9.5

Monday, Mar. 29: Tritton 9.3,
Kundu & Cohen 9.6

Wednesday, Mar. 31: Tritton 9.4, 9.5,
Kundu & Cohen 9.12

Informal Written Reflection

Due: Thursday, April 1 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, April 2 (in class)

1. In this problem, we'll take a look at viscous smoothing of a discontinuity in the velocity field. This is our first opportunity to consider *unsteady* laminar flow…

(a) We learned in class that, for steady shear flows with parallel streamlines, the Navier-Stokes equation reduces to a very simple form,

$$\nu \frac{\partial^2 u}{\partial y^2} = 0,$$

when there are no imposed pressure gradients. The unsteady version of this problem is instructive because, in this case, the Navier-Stokes equation becomes the diffusion equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}.$$

Consider the following flow field, which has a discontinuity in velocity at $y = 0$:

$$u(y, 0) = \begin{cases} +U & \text{if } y > 0 \\ -U & \text{if } y < 0. \end{cases}$$

We intuitively expect that viscous effects will smooth over this discontinuity... In sections 9.7 and 9.8, Kundu and Cohen derive the following solution to this problem:

$$u(\eta) = U\mathrm{erf}(\eta) = U\frac{2}{\sqrt{\pi}} \int_0^\eta \exp(-\alpha^2)\, \mathrm{d}\alpha,$$

where $\mathrm{erf}(\eta)$ is known as the *error function* and $\eta = y/\sqrt{4\nu t}$ is a dimensionless combination of the vertical coordinate and time. Does this solution behave physically the way we expect it to?

(Hint: $\mathrm{erf}(\eta)$ is known to Wolfram Alpha and other similar programs. Plotting this function for different choices of t and comparing the results will help you answer this question...)

(b) In a previous assignment, we learned that a general solution to the one-dimensional diffusion equation can be written as a convolution:

$$u(y, t) = \int_{-\infty}^{\infty} G(y - y', t)\, u(y', 0)\, \mathrm{d}y',$$

$$\text{where} \quad G(y-y') = \frac{1}{\sqrt{4\pi\nu t}} \exp(-\frac{(y-y')^2}{4\nu t}).$$

Show, for the discontinuous initial condition $u(y,0)$ given above, that this general solution reduces to $u(\eta) = U\mathrm{erf}(\eta)$, as expected. Thus, viscous smoothing really is a diffusion problem!

(Hint: This is a tricky change of variables calculation! Start with $\alpha = (y-y')/\sqrt{4\nu t}$ and be careful with the limits of integration. Note also that

$$\int_0^\infty \exp(-\alpha^2)\ \mathrm{d}\alpha = \frac{\sqrt{\pi}}{2}$$

and that this identity may come in handy more than once!)

(c) This phenomenon can also be described as *vorticity* diffusion. Show that the diffusion equation can be rewritten

$$\frac{\partial \omega}{\partial t} = \nu \frac{\partial^2 \omega}{\partial y^2},$$

where ω is the only nonzero component of the vorticity. Using your expression for ω and the above result for $u(\eta)$, show that the vorticity field is described by

$$\omega(y,t) = -\frac{U}{\sqrt{\pi\nu t}} \exp(-\frac{y^2}{4\nu t}).$$

How does this function change shape over time? How does the *total* vorticity of the flow change over time?

(d) As with $u(y,t)$ we can rederive our result for the vorticity field directly from a convolution integral. First, using the relationship between circulation and vorticity, show that our flow discontinuity at $y=0$ (our initial condition on velocity) can be represented as a "sheet" vortex: $\omega(y) = -2U\delta(y)$. Combining this result with the appropriate convolution integral, you should be able to reproduce your answers in (b).

(Hint: Start by looking through Kundu and Cohen, section 3.8.)

2. In this problem, we will work through Kundu and Cohen's (rather challenging!) derivation of drag due to low Re flow over a sphere...

   (a) The streamfunction formulation of the Navier-Stokes equation for creeping flow in spherical coordinates (Kundu and Cohen, equation (9.64)) is written

   $$\left[\frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right]^2 \psi = 0.$$

   (This looks messy but it's still a massive simplification.) Where does this equation come from? Don't worry about every step of the derivation. Just highlight the key moves: What happened to the pressure term? Where are the velocities? What's this function ψ? And so on.

   (b) Guess a separable solution of the form $\psi(r,\theta) = f(r)\sin^2\theta$ and solve the boundary value problem for creeping flow over a rigid sphere. You should find (Kundu and Cohen, equation (9.68)):

   $$\psi(r,\theta) = Ur^2\sin^2\theta\left[\frac{1}{2} - \frac{3a}{4r} + \frac{a^3}{4r^3}\right],$$

   Verify that this gives you:

   $$u_r = U\cos\theta\left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3},\right]$$
   $$u_\theta = -U\sin\theta\left[1 - \frac{3a}{4r} - \frac{a^3}{4r^3},\right].$$

   (c) Using the solution you found in (b), solve the creeping flow equation $\nabla p = \nabla^2\mathbf{u}$ for the pressure field. You should be able to reproduce (Kundu and Cohen, equation (9.70)):

   $$p - p_\infty = -\mu U\cos\theta\frac{3a}{2r^2}.$$

   (Hint: Be careful with differentiation in spherical coordinates! The appendices will help you identify all of the necessary terms.)

(d) Calculate the stress components σ_{rr} and $\sigma_{r\theta}$ at the surface of the sphere and integrate over the surface to rederive Stokes' famous drag law for creeping flow over a sphere:

$$D = 6\pi\mu a U.$$

(Hint: Once again, you will need the appendices. Look for the stress tensor components in spherical coordinates.)