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 Course: **8.421 - AMO I**
 Problem set: **#8**
 Due: Friday, April 8, 2022.

1. Optical Traps and Scattering. What are the proper power and wavelength needed to trap an ultracold atomic gas? Consider an alkali atom with resonance frequency ω_0 on the principal $nS \rightarrow nP$ transition. A sample of atoms in the ground state nS are exposed to monochromatic radiation of intensity I and frequency $\omega_L < \omega_0$. Using the fact that essentially all of the oscillator strength out of the ground state comes from the $nS \rightarrow nP$ transition, we have

$$\alpha(\omega_L) \approx \frac{2e^2}{\hbar} |\langle nP | z | nS \rangle|^2 \frac{\omega_0}{\omega_0^2 - \omega_L^2} \implies |\langle nP | z | nS \rangle|^2 = \frac{\hbar \alpha(\omega_L)}{2e^2} \frac{\omega_0^2 - \omega_L^2}{\omega_0}.$$

(a) AC Stark shift:

(i) From lecture, the AC Stark shift U_i from time-dependent perturbation theory is given by

$$U_i = -\frac{1}{4} \alpha(\omega_L) \mathcal{E}^2 = -\frac{2I\alpha(\omega_L)}{4c\epsilon_0} = -\frac{I\alpha(\omega_L)}{2c\epsilon_0}.$$

(ii) Now, we want to use the rotating wave approximation to obtain the AC Stark shift. This can be done by first writing down the true (symmetrized) Hamiltonian:

$$\mathcal{H} = \frac{\hbar}{2} \begin{pmatrix} -\omega_0 & \omega_R e^{i\omega_L t} \\ \omega_R^* e^{-i\omega_L t} & \omega_0 \end{pmatrix}.$$

By going into the rotating frame, plus making the rotating wave approximation, we find that

$$\mathcal{H}_{\text{rot}}^{\text{RWA}} = \frac{\hbar}{2} \begin{pmatrix} -\delta & \omega_R \\ \omega_R & \delta \end{pmatrix}$$

where $\delta = \omega_0 - \omega_L$. The energy shifts can be obtained from the eigenvalues:

$$\Delta E = \pm \frac{\hbar}{2} \sqrt{\omega_R^2 + \delta^2} = \pm \frac{\hbar}{2} \sqrt{\omega_R^2 + (\omega_0 - \omega_L)^2} \approx \pm \frac{\hbar(\omega_0 - \omega_L)}{2} \pm \frac{\hbar\omega_R^2}{4(\omega_0 - \omega_L)}$$

where we have used the fact that $\omega_R \ll |\omega_0 - \omega_L|$. From here, we find that the shift is

$$U_{ii} = -\frac{\hbar\omega_R^2}{4(\omega_0 - \omega_L)}$$

In particular, the energy of the lower state gets shifted down while the energy of the higher state gets shifted up (since we're red-detuning).

(iii) From the previous two parts, we find that

$$\frac{U_i}{U_{ii}} = \frac{I\alpha(\omega_L)}{2c\epsilon_0} \frac{4(\omega_0 - \omega_L)}{\hbar\omega_R^2} = \frac{I}{c\epsilon_0\omega_R^2} \frac{4e^2}{\hbar^2} |\langle nP | z | nS \rangle|^2 \frac{\omega_0}{\omega_0 + \omega_L}.$$

To simplify this, we must write the Rabi frequency in terms of the intensity:

$$\omega_R = \frac{e\mathcal{E} |\langle nS | z | nP \rangle|}{\hbar} \implies \omega_R^2 = \frac{e^2 |\langle nS | z | nP \rangle|^2}{\hbar^2} \frac{2I}{c\epsilon_0}.$$

With this, we have

$$\boxed{\frac{U_i}{U_{ii}} = \frac{2\omega_0}{\omega_0 + \omega_L}}$$

When $\omega_L \approx 0$, we have

$$\frac{U_i}{U_{ii}} \approx 2.$$

When $\omega_L \approx \omega_0$, we may write $\omega_L + \omega_0 = 2\omega_0$, so that

$$\frac{U_i}{U_{ii}} \approx 1.$$

We see that if the intensity has spatial structure, with the appropriate detuning, the AC Stark shift can have energy minima where the atoms can be trapped.

(b) From time-dependent perturbation theory, the amplitude of the excited state is

$$c_2(t) = \frac{e\mathcal{E}}{2\hbar} \langle nS | z | nP \rangle \left[\frac{e^{i(\omega_0 + \omega_L)t} - 1}{\omega_0 - \omega_L} + \frac{e^{i(\omega_0 - \omega_L)t} - 1}{\omega_0 + \omega_L} \right].$$

Ignoring the -1 terms which are associated with transients, we have

$$P_e(t) = \frac{e^2 \mathcal{E}^2}{4\hbar^2} |\langle nS | z | nP \rangle|^2 \frac{2[\omega_0^2 + \omega_L^2 + (\omega_0^2 - \omega_L^2) \cos(2\omega_L t)]}{(\omega_0^2 - \omega_L^2)^2}.$$

After time-averaging, this quantity is

$$P_{e,i} = \boxed{\frac{\omega_R^2}{2} \frac{\omega_0^2 + \omega_L^2}{(\omega_0^2 - \omega_L^2)^2}}$$

In the RWA picture, we know that

$$P_{e,ii}(t) = \frac{\omega_R^2}{\omega_R^2 + \delta^2} \sin^2 \left(\frac{\sqrt{\omega_R^2 + (\omega_0 - \omega_L)^2} t}{2} \right).$$

After time-averaging this is

$$P_{e,ii} = \frac{\omega_R^2}{2(\omega_R^2 + \delta^2)} \approx \boxed{\frac{\omega_R^2}{2(\omega_0 - \omega_L)^2}}$$

where we have used the approximation that the Rabi frequency is much less than the detuning.

(c) Calculate the photon scattering rate:

(i) Starting with

$$P = \frac{ck^4 |d|^2}{3} = \frac{\omega_L^4}{3c^3} |d|^2,$$

if we say $d = \alpha(\omega_L)\mathcal{E}$ then we have

$$R_{sc} = \frac{P}{\hbar\omega_L} = \frac{\omega_L^3}{3\hbar c^3} |\alpha(\omega_L)|^2 \mathcal{E}^2 = \frac{\omega_L^3}{3\hbar c^3} |\alpha(\omega_L)|^2 \frac{8\pi I}{c}$$

where we have converted the intensity into CGS units. Now we recall from perturbation theory that

$$\alpha(\omega_L) = \frac{2e^2}{\hbar} |\langle nS | z | nP \rangle|^2 \frac{\omega_0}{\omega_0^2 - \omega_L^2} = \frac{e^2}{\hbar} |\langle nS | z | nP \rangle|^2 \left(\frac{1}{\omega_0 - \omega_L} + \frac{1}{\omega_0 + \omega_L} \right).$$

From here we find that

$$|\alpha(\omega_L)|^2 = \frac{e^4}{\hbar^2} |\langle nS | z | nP \rangle|^4 \left(\frac{1}{\omega_0 - \omega_L} + \frac{1}{\omega_0 + \omega_L} \right)^2.$$

Putting everything together, we find

$$\begin{aligned} R_{\text{sc},i} &= \frac{\omega_L^3}{3\hbar c^3} \frac{8\pi I}{c} \frac{e^4}{\hbar^2} |\langle nS | z | nP \rangle|^4 \left(\frac{1}{\omega_0 - \omega_L} + \frac{1}{\omega_0 + \omega_L} \right)^2 \\ &= \frac{8\pi I \omega_L^3 e^4}{3c^4 \hbar^3} |\langle nS | z | nP \rangle|^4 \left(\frac{1}{\omega_0 - \omega_L} + \frac{1}{\omega_0 + \omega_L} \right)^2. \end{aligned}$$

Under RWA, we simply drop the counter-rotating term to find

$$R_{\text{sc},ii} = \frac{8\pi I \omega_L^3 e^4}{3c^4 \hbar^3} |\langle nS | z | nP \rangle|^4 \left(\frac{1}{\omega_0 - \omega_L} \right)^2.$$

(ii) Let us write $R_{\text{cs},i}$ and $R_{\text{sc},ii}$ in terms of the spontaneous emission rate $\Gamma = 4e^2 \omega_0^3 |\langle nS | z | nP \rangle|^2 / \hbar c^3$:

$$\begin{aligned} R_{\text{sc},i} &= \frac{8\pi I \omega_L^3 e^4}{3c^4 \hbar^3} |\langle nS | z | nP \rangle|^4 \left(\frac{1}{\omega_0 - \omega_L} + \frac{1}{\omega_0 + \omega_L} \right)^2 \\ &= \frac{8\pi I \omega_L^3}{3c^4 \hbar^3} \frac{9\Gamma^2 \hbar^2 c^6}{16\omega_0^6} \left(\frac{1}{\omega_0 - \omega_L} + \frac{1}{\omega_0 + \omega_L} \right)^2 \\ &= \frac{3\pi c^2}{2\hbar \omega_0^3} \left(\frac{\omega_L}{\omega_0} \right)^3 \left(\frac{\Gamma}{\omega_0 - \omega_L} + \frac{\Gamma}{\omega_0 + \omega_L} \right)^2 I, \end{aligned}$$

which is a well-known result given in many textbooks. Making the RWA, we find

$$R_{\text{sc},ii} = \frac{3\pi c^2}{2\hbar \omega_0^3} \left(\frac{\omega_L}{\omega_0} \right)^3 \left(\frac{\Gamma}{\omega_0 - \omega_L} \right)^2 I.$$

From Part (b), we have that

$$P_{e,ii} = \frac{\omega_R^2}{2(\omega_0 - \omega_L)^2},$$

which gives

$$\begin{aligned} R_{\text{sc},ii} &= \frac{3\pi c^2}{2\hbar \omega_0^3} \left(\frac{\omega_L}{\omega_0} \right)^3 \left(\frac{\Gamma}{\omega_0 - \omega_L} \right)^2 I \\ &= \frac{3\pi c^2}{2\hbar \omega_0^3} \left(\frac{\omega_L}{\omega_0} \right)^3 \frac{\Gamma^2}{\omega_R^2} 2P_{e,ii} \\ &= \frac{3\pi c^2}{\hbar \omega_0^3} \left(\frac{\omega_L}{\omega_0} \right)^3 \Gamma^2 P_{e,ii} \frac{\hbar^2}{e^2} \frac{c}{8\pi} \frac{4e^2 \omega_0^3}{3\hbar c^3 \Gamma} \\ &= \boxed{\frac{1}{2} \left(\frac{\omega_L}{\omega_0} \right)^3 \Gamma P_{e,ii}} \end{aligned}$$

- (iii) (Optional) We describe scattering as spontaneous emission from a virtual energy level. The energy diagram for this process is:

The lifetime of this virtual state is:

blah

- (d) (i) The $D_{1,2}$ lines of Na have $\omega_0 \approx 2\pi \cdot 508$ THz. Now the infrared laser at 985 nm has $\omega_L = 2\pi \cdot 304$ THz, which corresponds to a detuning of $2\pi \cdot 204$ THz, which is much larger than the detuning of the yellow laser (a few GHz). From the RWA expression, we see that the trap depth goes like $I(0)/\delta$. Assuming that the waist radius is the same for both lasers, we see that to get the same trap depth, a far detuned laser requires more power and vice versa. However, the scattering rate goes like $I(0)/\delta^2$. The most ideal optical trap requires sufficient trap depth plus low scattering rate. From the two factors (U, R_{sc}) and their dependence on I and δ we conclude that infrared laser is more suitable than the yellow laser.
- (ii) Here we want to calculate the required power and scattering rate for each of the two types of lasers. Using results from time-dependent perturbation theory, we have two equations:

$$U_i = \frac{I(0)\alpha(\omega_L)}{2c\epsilon_0} = \frac{3\pi c^2}{2\omega_0^3} \left(\frac{\Gamma}{\omega_0 - \omega_L} + \frac{\Gamma}{\omega_0 + \omega_L} \right) I(0) = k_B \cdot 10 \mu\text{K}$$

$$R_{sc,i} = \frac{3\pi c^2}{2\hbar\omega_0^3} \left(\frac{\omega_L}{\omega_0} \right)^3 \left(\frac{\Gamma}{\omega_0 - \omega_L} + \frac{\Gamma}{\omega_0 + \omega_L} \right)^2 I(0).$$

where we have converted to CGS units in the first line. To solve for $P, R_{sc,ii}$, we will use $I(0) = 2P/\pi w^2$ and Mathematica. For each laser, we will have:

$$\text{Yellow laser: } P = 0.102 \text{ mW}; \quad R_{sc,ii} = 7637 \text{ s}^{-1}$$

$$\text{Infrared laser: } P = 9.871 \text{ mW}; \quad R_{sc,ii} = 17.1102 \text{ s}^{-1}$$

2. Magic Wavelength Optical Trap. Here we have a system with lower state $|S\rangle$, upper state $|P\rangle$ with bare energy separation $\hbar\omega_{PS}$. The dipole moment is d_{PS} .

- (a) (i) The AC Stark shift for the state S is given by

$$\Delta E_S = -\frac{1}{4}\alpha(\omega_L)\mathcal{E}^2 = -\frac{d_{PS}^2\mathcal{E}^2}{4\hbar} \left(\frac{1}{\omega_{PS} - \omega_L} + \frac{1}{\omega_{PS} + \omega_L} \right)$$

(ii)

(iii)

- (b) (i)

(ii)

3. Species-Dependent and Spin-Dependent AC Stark shift

- (a) (i)

(ii)

- (b) (i)

(ii)