Classical Mechanics III (8.(3)09)
Assignment 9: Solutions

November 17, 2021

1. Viscous Flow on an Inclined Plane [10 points]

(a) [2 points]
Let us take the $+x$-direction in the direction of fluid flow parallel to the plane (as shown on the picture), and the $+y$-direction normal to it (so tilted relative to gravity). Take $y = 0$ to be at the surface on the inclined plane. The Navier-Stokes equation with an external force is

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v} + \frac{\vec{f}}{\rho}$$

where $\vec{f}$ is the force per fluid volume. In our case we have $\frac{\partial \vec{v}}{\partial t} = 0$ (steady flow) and $\vec{f} = \rho \vec{g}$, so

$$(\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho} + \nu \nabla^2 \vec{v} + \vec{g}$$

where with our coordinate system $\vec{g} = g(\hat{x} \sin\theta - \hat{y}\cos\theta)$.
(b) [4 points]
By symmetry both $\vec{v}$ and p are functions of y only. Moreover $\vec{v}$ is parallel to the x-direction ($\vec{v} \cdot \hat{z} = 0$ by assumption, since $\hat{x}$ is the direction of fluid flow; and $\vec{v} \cdot \hat{y} = 0$ can be seen by invoking continuity on a pillbox from $y = 0$ to $y = y'$). Hence

$$p = p(y), \qquad \vec{v} = v(y)\hat{x}.$$

Now for the boundary conditions. On the surface of the inclined plane we require the no-slip condition $v(y = 0) = 0$. The boundary condition on the water-air surface is trickier, but we can make the approximation that since air is nearly inviscid there is no shear stress at that surface:

$$\sigma'_{xy}|_{y=h} = 0$$

which implies that

$$0 = \left.\frac{\partial(\vec{v} \cdot \hat{x})}{\partial y}\right|_{y=h} = \frac{\partial v(y = h)}{\partial y}.$$

Hence the boundary conditions for v are

$$v(y=0)=0, \qquad \frac{\partial v(y=h)}{\partial y}=0$$

(c) [4 points]
Note that $(\vec{v}\cdot\vec{\nabla})\vec{v}=(\vec{v}\cdot\hat{x})\frac{\partial}{\partial x}\vec{v}=0$. (No surprise here: $\frac{\partial\vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=\frac{d\vec{v}}{dt}=0$ since by symmetry the velocity on each streamline is constant – note the streamlines are all parallel to $\hat{x}$– and we already know $\frac{\partial\vec{v}}{\partial t}=0$.) Therefore the Navier-Stokes equation in the x- and y- directions give us respectively

$$\nu\frac{\partial^2 v(y)}{\partial y^2}+g\sin\theta=0$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial y}-g\cos\theta=0$$

Thus we must have

$$v(y)=-\frac{g\sin\theta}{2\nu}y^2+Ay+B \qquad \text{and} \qquad p=-\rho gy\cos\theta+C$$

for some constants A, B, and C. Matching the boundary conditions given in (b), and also $p(y=h)=p_{atm}$, we get finally:

$$\vec{v}=\frac{gh^2\sin\theta}{\nu}\left(\frac{y}{h}-\frac{y^2}{2h^2}\right)\hat{x}\,,$$

$$p=p_{atm}+(h-y)\rho g\cos\theta\,.$$

2. Chaos in a Nonlinear Circuit [13 points]

Solutions for this problem are given by the pages from mathematica attached at the end of this document.

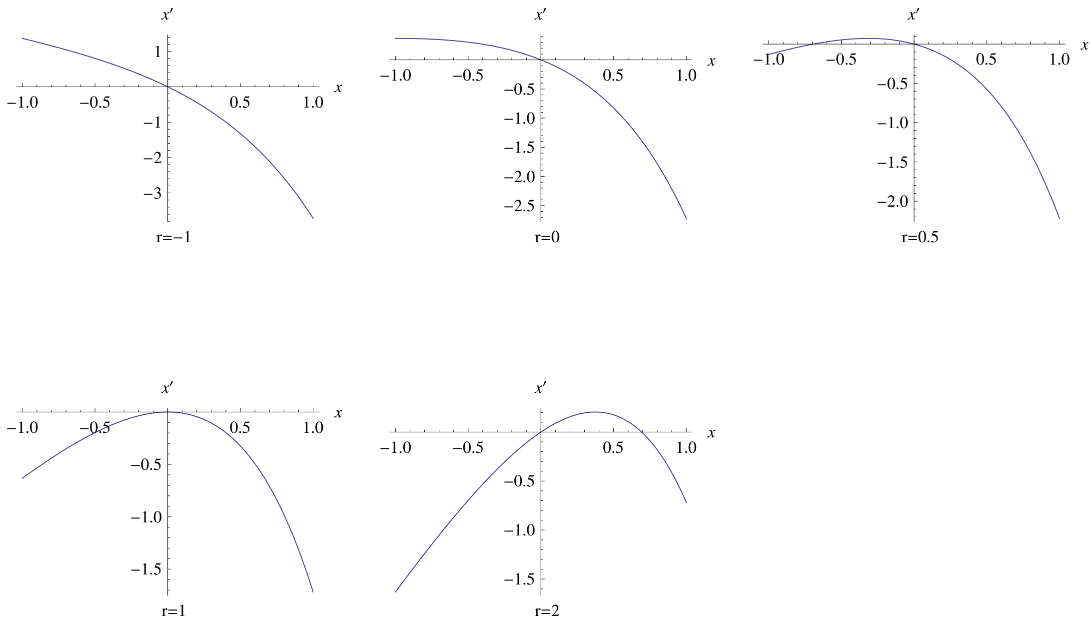
3. Bifurcations [12 points, 4 points each]

(a)$\dot{x}=x(r-e^x)$

We first find the fixed points, i.e. those points $x=x^*$ for which $\dot{x}=x^*(r-e^{x^*})=0$. It is clear that $x^*=0$ is always a fixed point. The other fixed point x^*, if it exists, is such that $r-e^{x^*}=0$; this can only occur if $r>0$, in which case $x^*=\ln r$. Thus the fixed points of the system are

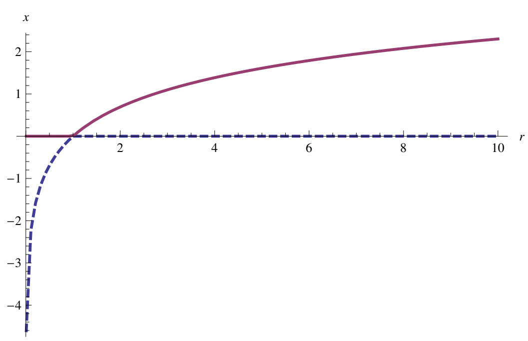
$$\begin{aligned} x^* &= 0 \quad , \qquad \text{if } r\le 0 \\ x^* &= 0,\ln r \quad , \qquad \text{if } r>0. \end{aligned}$$

Note that the fixed points $x^*=0$ and $x=\ln r$ merge at $r=1$, so we expect a bifurcation at $r=1$. Let's therefore plot $\dot{x}$ versus x for the four cases $r<0$, $r=0$, $0<r<1$, $r=1$, and $r>1$:



- For $r \leq 0$, there is one fixed point at $x^* = 0$.
- For $0 < r < 1$, the fixed point $x^* = \ln r$ is unstable while $x^* = 0$ is stable.
- At $r = 1$, the two fixed points merge and there is only a single fixed point at $x^* = 0$; it is half-stable (as can be seen from the above graph, or by noting that $\dot{x} = -\frac{1}{2}x^2 + O(x^3)$ for $r = 1$). As r increases the two fixed points swap stabilities:
- For $r > 1$, the fixed point $x^* = \ln r$ is stable while $x^* = 0$ is unstable.

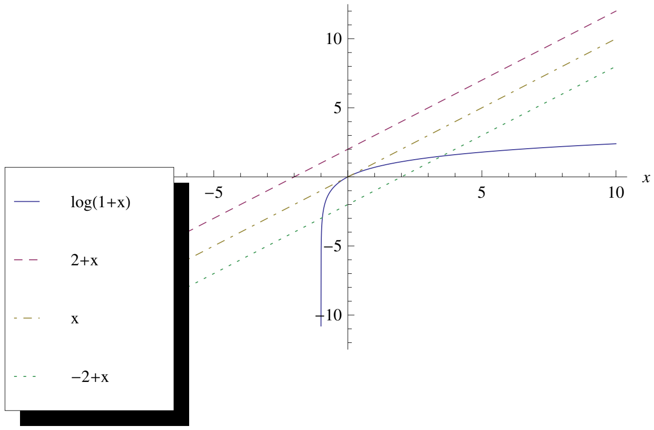
The bifurcation at $r = 1$ is a transcritical bifurcation. The bifurcation diagram is shown below.



Bifurcation diagram for x'=x(r-exp(x))

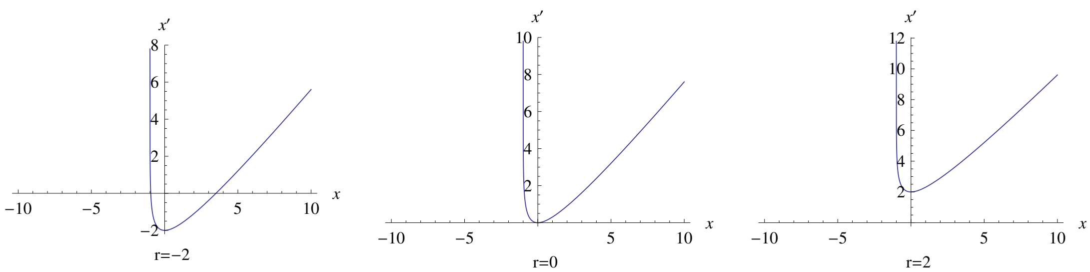
(b) $\dot{x} = r + x - \ln(1 + x)$

To determine the number and location of the fixed points, we plot $y = r + x$ and $y = \ln(1 + x)$. The x-coordinates of the intersections are the fixed points.



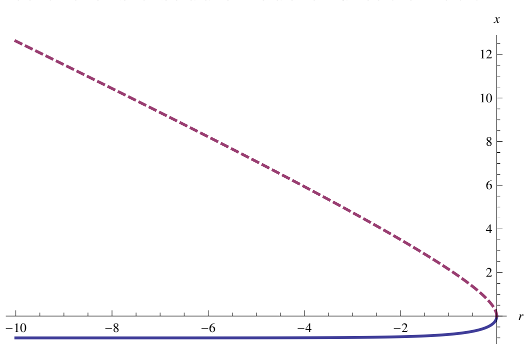
Note that $\frac{d\ln(1+x)}{dx} = \frac{1}{1+x}$, whose maximum matches the slope of $y = r + x$ (namely, 1) at $x = 0$. Hence we expect a bifurcation when $x = 0$ is a fixed point, and this happens when $r = 0$.

Let us plot $\dot{x}$ versus x:



- For $r < 0$, there are two fixed points. The fixed point $x^* > 0$ is unstable, while the fixed point $x^* < 0$ is stable.
- For $r = 0$, there is one fixed point at $x^* = 0$. From the plot we see it is half-stable; this can also be seen from $\dot{x} = x - \ln(1+x) = \frac{x^2}{2} + O(x^3)$ around $x^* = 0$.
- For $r > 0$, there are no fixed points.

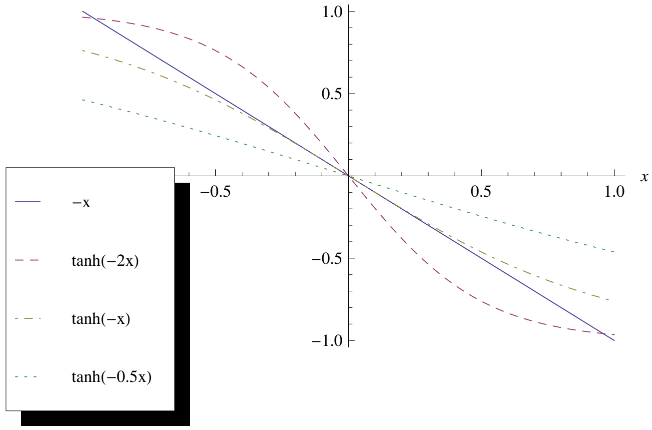
Hence there is a saddle-node bifurcation at $r = 0$. The bifurcation diagram is shown below.



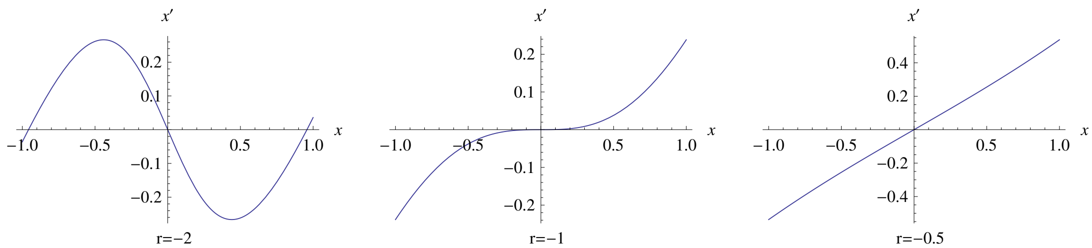
Bifurcation diagram for x'=r+x-ln(1+x)

(c) $\dot{x} = x + \tanh(rx)$

Note that for $r \geq 0$ we have that $\dot{x}$ and x always have the same sign, and the only fixed point is $x^* = 0$. For the case when $r < 0$, let us plot $y = -x$ and $y = \tanh(rx)$; their intersections are the fixed points.

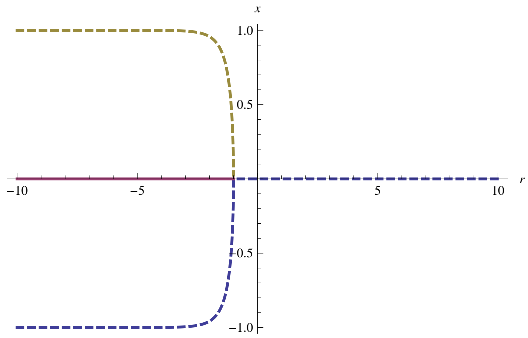


The derivative of $\tanh(rx)$ is $\frac{d\tanh(rx)}{dx} = r\mathrm{sech}^2(rx)$, whose extremum value occurs at $x = 0$ and is r. Thus the slopes of $y = \tanh(x)$ and $y = -x$ match at $x = 0$ when $r = -1$, and we expect a bifurcation there. Let us plot $\dot{x}$ versus x:



- For $r < -1$, there are three fixed points: the fixed point at $x^* = 0$ is stable, while the fixed points at $x^* = \pm\tilde{x}$, for some $\tilde{x} > 0$, are unstable.
- For $r \geq -1$, there is a single fixed point at $x^* = 0$, which is unstable. (The stability of the case $r = -1$ may be seen from the graph, or by calculating that $\dot{x} = x + \tanh(-x) = \frac{x^3}{3} + O(x^5)$, so $\dot{x}$ and $x - x^*$ have the same sign around $x^* = 0$.)

There is a subcritical pitchfork at $r = -1$. The bifurcation diagram is shown below.



Bifurcation diagram for x'=x+tanh(rx)

4. Damped Nonlinear Oscillator [15 points]

(a) [2 points] Let θ' be the deviation from a fixed point, i.e. $\theta = n\pi + \theta'$ for some fixed $n\pi$. Then around this fixed point, we have

$$\begin{aligned} \sin\theta &= \sin(n\pi + \theta') = \sin(n\pi)\cos(\theta') + \cos(n\pi)\sin(\theta') \\ &= (-1)^n \sin(\theta') \\ &\approx (-1)^n \theta' \end{aligned}$$

and so to first order in θ' and ω we have

$$\begin{aligned} \dot{\theta}' &= \omega \\ \dot{\omega} &= -\frac{\omega}{q} - (-1)^n \theta'. \end{aligned}$$

(b) [4 points] First let us consider the case that n is even. In this case we have the harmonic oscillator equations $\dot{\theta}' = \omega$, $\dot{\omega} = -\theta'$ with spring constant 1, so the solutions are

$$\theta' = A\cos(t+\delta), \qquad \omega = -A\sin(t+\delta)$$

for some constants A and δ. We therefore have elliptical oscillations around the fixed point.

Now consider instead the case where n is odd. Then instead we have $\dot{\theta}' = \omega$, $\dot{\omega} = \theta'$. Thus $\ddot{\theta}' = \theta'$ and $\ddot{\omega} = \omega$, and the solutions to these equations are

$$\theta' = Ae^t + Be^{-t}, \qquad \omega = Ae^t - Be^{-t}$$

for some constants A and B. Note that

$$\theta' + \omega = 2Ae^t$$

$$\theta' - \omega = 2Be^{-t}$$

so the $\theta' + \omega$ direction corresponds to the growing solution, and the $\theta' - \omega$ direction corresponds to the decaying solution. (We could also have gotten this more systematically by putting this in matrix form, see part (c).)

(c) [4 points] For finite q the fixed points where n is even become attractors, as we'll see in a moment. (This isn't surprising, since we already know that for a damped harmonic oscillator the fixed point is an attractor.) In this case we have

$$\frac{d}{dt}\begin{pmatrix} \theta' \\ \omega \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -1/q \end{pmatrix}\begin{pmatrix} \theta' \\ \omega \end{pmatrix}$$

Let us write $\vec{x} = \begin{pmatrix} \theta' \\ \omega \end{pmatrix}$ and $M = \begin{pmatrix} 0 & 1 \\ -1 & -1/q \end{pmatrix}$, so

$$\frac{d\vec{x}}{dt} = M\vec{x}, \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & -1/q \end{pmatrix}$$

We can try a solution of the form $\vec{x} = e^{\lambda t}\vec{a}$, where $\vec{a}$ is a constant vector (i.e. independent of time). Then this gives

$$M\vec{a} = \lambda\vec{a}$$

i.e. λ is an eigenvalue of M, with corresponding eigenvector $\vec{a}$. We can directly solve the characteristic equation

$$\det(M - \lambda\mathbb{I}) = \begin{vmatrix} -\lambda & 1 \\ -1 & -1/q - \lambda \end{vmatrix} = \lambda(\lambda + \frac{1}{q}) + 1 = 0$$

which has the solutions

$$\lambda_\pm = -\frac{1}{2q} \pm \sqrt{\left(\frac{1}{2q}\right)^2 - 1}.$$

We now break into cases [Note to grader: if less detail is given, relying on results from 8.03, then that is fine.]:

1. $q > 1/2$. In this case the factor in the square root is negative, so we have two complex eigenvalues:

$$\lambda_\pm = -\frac{1}{2q} \pm i\sqrt{1 - \left(\frac{1}{2q}\right)^2} \equiv -\frac{1}{2q} + i\Omega_0$$

where $\Omega_0 = \sqrt{1 - \left(\frac{1}{2q}\right)^2}$. The general solution is then a linear combination of the specific solutions $e^{\lambda_\pm t}\vec{a}_\pm$, i.e.

$$\begin{aligned} \vec{x} &= A_+\vec{a}_+e^{\lambda_+ t} + A_-\vec{a}_-e^{\lambda_- t} \\ &= e^{-t/(2q)}[A_+\vec{a}_+(\cos(\Omega_0 t) + i\sin(\Omega_0 t)) + A_-\vec{a}_-(\cos(\Omega_0 t) - i\sin(\Omega_0 t)) \\ &= e^{-t/(2q)}[(A_+\vec{a}_+ + A_-\vec{a}_-)\cos(\Omega_0 t) + i(A_+\vec{a}_+ - A_-\vec{a}_-)\sin(\Omega_0 t)] \\ &= e^{-t/(2q)}[\vec{a}_1\cos(\Omega_0 t) + \vec{a}_2\sin(\Omega_0 t)] \end{aligned}$$

for some vectors $\vec{a}_1$ and $\vec{a}_2$. (Both vectors must be real to be physically meaningful.) Therefore we can write, for some constants A_1 and A_2,

$$\begin{aligned}\theta' &= e^{-t/(2q)}[A_1\cos(\Omega_0 t) + A_2\sin(\Omega_0 t)]\\ \omega &= e^{-t/(2q)}[(A_2\Omega_0 - \frac{A_1}{2q})\cos(\Omega_0 t) + (-A_1\Omega_0 - \frac{A_2}{2q})\sin(\Omega_0 t)]\end{aligned}$$

(we got the solution for ω by differentiating that of θ' by t). This is an oscillating solution that exponentially decays to the fixed point, i.e. it is the underdamped case. (We could also have written $\theta' = Ae^{-t/(2q)}\cos(\Omega_0 t + \delta)$ and differentiated that for the solution for ω.) Figure is below.

2. $q < 1/2$. Then the factor in the square root is positive, and we have two distinct real eigenvalues:

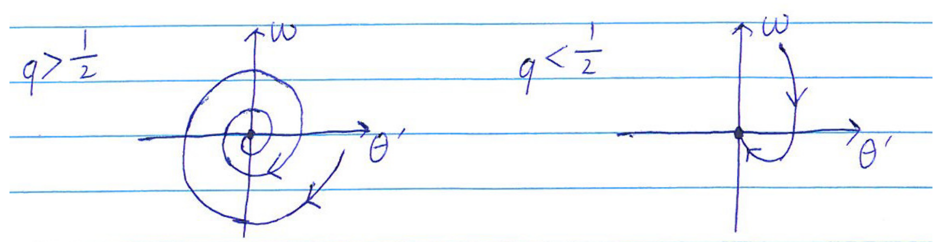
$$\lambda_\pm = -\frac{1}{2q} \pm \sqrt{\left(\frac{1}{2q}\right)^2 - 1}$$

and the general solution is simply $\vec{x} = A_+ e^{\lambda_+ t}\vec{a}_+ + A_- e^{\lambda_- t}\vec{a}_-$ for some constants A_+ and A_-, or

$$\begin{aligned}\theta' &= A_1 e^{\lambda_+ t} + A_2 e^{\lambda_- t}\\ \omega &= \lambda_+ A_1 e^{\lambda_+ t} + \lambda_- A_2 e^{\lambda_- t}\end{aligned}$$

for some constants A_1 and A_2. Note that both eigenvalues are smaller than zero: $\lambda_+, \lambda_- < 0$, so this is the overdamped case. Figure is below.

3. $q = 1/2$. (Note to grader: do not grade this case.) In this case we have only one real eigenvalue, and in fact there are no longer two distinct eigenvectors. Instead we have $\lambda = -1$ with a single eigenvector $\vec{a} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$. The two independent solutions in this case are actually $e^{\lambda t}\vec{a}$ and $te^{\lambda t}\vec{a}$, so $\theta' = e^{-t}(A_1 + A_2 t)$ for some A_1 and A_2. (This is the critically damped case.)



Aside: For first-order homogenous linear differential equations with constant coefficients, i.e. equations of the form $\frac{d\vec{x}}{dt} = M\vec{x}$ for some constant matrix M, we can usually express the general solution as a linear combination of solutions of the form $\vec{x} = e^{\lambda t}\vec{a}$, where $\vec{a}$ is constant. λ and $\vec{a}$ are the eigenvalue-eigenvector pairs of M. The only situation where this is insufficient to generate the required $\dim(M)$ independent solutions is when M does not have $\dim(M)$ independent eigenvectors, i.e. when M is not diagonalizable.

(d) [5 points] We now consider the case where n is odd. Then again, with $\vec{x} = \begin{pmatrix} \theta' \\ \omega \end{pmatrix}$,

$$\frac{d\vec{x}}{dt} = M\vec{x}, \qquad M = \begin{pmatrix} 0 & 1 \\ 1 & -1/q \end{pmatrix}$$

The eigenvalues of M in this case are given by

$$\det(M - \lambda\mathbb{I}) = \begin{vmatrix} -\lambda & 1 \\ 1 & -1/q - \lambda \end{vmatrix} = \lambda(\lambda + \frac{1}{q}) - 1 = 0$$

so

$$\lambda_{\pm} = -\frac{1}{2q} \pm \sqrt{\left(\frac{1}{2q}\right)^2 + 1}$$

Note that the eigenvalues are always real. Moreover $0 < \lambda_+ < 1$ and $-\infty < \lambda_- < -1$; λ_+ gives the growing mode and λ_- gives the decaying mode, so this fixed point is indeed a saddle point. The corresponding eigenvectors are

$$\vec{a}_{\pm} = \frac{1}{\sqrt{1+\lambda_{\pm}^2}} \begin{pmatrix} 1 \\ \lambda_{\pm} \end{pmatrix}$$

and the general solution is $\vec{x} = A_+ e^{\lambda_+ t}\vec{a}_+ + A_- e^{\lambda_- t}\vec{a}_-$, or

$$\begin{aligned} \theta' &= A_1 e^{\lambda_+ t} + A_2 e^{\lambda_- t} \\ \omega &= \lambda_+ A_1 e^{\lambda_+ t} + \lambda_- A_2 e^{\lambda_- t}. \end{aligned}$$

where A_1, A_2 are arbitrary constants. The growth rate of the growing mode is of course $\kappa = \lambda_+$, while the direction of the purely growing solution is given by the direction of $\vec{a}_+$, which makes an angle of

$$\tan^{-1}(\lambda_+) = \tan^{-1}\left(-\frac{1}{2q} + \sqrt{\left(\frac{1}{2q}\right)^2 + 1}\right)$$

with the θ-axis.

5. Lorenz Equations [10 points]

(a) [2 points] Recall that the rate of change of a phase space volume V is given by (we use ∂V to indicate the boundary of V)

$$\frac{dV}{dt} = \int_{\partial V} \vec{f} \cdot d\vec{A} = \int_V \vec{\nabla} \cdot \vec{f} dV$$

by the divergence theorem, where $\vec{f}(\vec{x}) = \dot{\vec{x}}$is the total time derivative of the coordinates of a phase space element. In our case, we have

$$\begin{aligned}\vec{\nabla} \cdot \vec{f} &= \frac{\partial}{\partial x}(\sigma y - \sigma x) + \frac{\partial}{\partial y}(rx - y - xz) + \frac{\partial}{\partial z}(-bz + xy) \\ &= -(\sigma + b + 1) < 0\end{aligned}$$

so $\frac{dV}{dt} < 0$, and the volume contracts. (In fact $\frac{dV}{dt} = -(\sigma + b + 1)V$, $V = V_0 \exp[-(\sigma + b + 1)t]$, and the volume contracts exponentially fast.)

(b) [3 points] We seek solutions $\vec{x}^* = \begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}$ such that $\vec{f}(\vec{x}^*) = 0$, i.e.

$$\begin{aligned}\dot{x}(x^*, y^*, z^*) &= \sigma y^* - \sigma x^* = 0 \\ \dot{y}(x^*, y^*, z^*) &= rx^* - y^* - x^* z^* = 0 \\ \dot{z}(x^*, y^*, z^*) &= -bz^* + x^* y^* = 0\end{aligned}$$

The equations $\dot{x} = 0$ and $\dot{z} = 0$ immediately give

$$x^* = y^*, \qquad z^* = \frac{x^{*2}}{b}.$$

Plugging this into $\dot{y} = 0$ gives

$$(r-1)x^* - \frac{1}{b}x^{*3} = 0.$$

and we obtain the three fixed point solutions

$$\begin{aligned}&(x^*, y^*, z^*) = (0, 0, 0) \\ &(x^*, y^*, z^*) = \Big(\sqrt{b(r-1)}, \sqrt{b(r-1)}, r-1\Big) \\ &(x^*, y^*, z^*) = \Big(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, r-1\Big).\end{aligned}$$

Note that the latter two solutions only exist if $r > 1$ (and if $r = 1$ they coincide with the first one).

(c) [5 points] Around the fixed point $(0, 0, 0)$ we neglect all terms of quadratic order, giving

$$\dot{z} = -bz$$

and

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \equiv M \begin{pmatrix} x \\ y \end{pmatrix}.$$

The variable z decouples and has the decaying solution $z = A\exp(-bt)$. For the other two variables, the eigenvalues of M are given by

$$0 = \det(M - \lambda \mathbb{I}) = (\lambda + \sigma)(\lambda + 1) - \sigma r$$

which has the solutions

$$\lambda_{\pm} = -\frac{1+\sigma}{2} \pm \frac{1}{2}\sqrt{(1+\sigma)^2 - 4\sigma(1-r)}.$$

Notice that the term inside the square root is equal to $(1-\sigma)^2 + 4\sigma r$, and so is always positive; there are always two distinct real roots. The corresponding eigenvectors are easily determined: they are (we omit normalization)

$$\vec{a}_{\pm} = \begin{pmatrix} 1 \\ 1 + \frac{\lambda_{\pm}}{\sigma} \end{pmatrix}$$

and the general solution is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = A_+ \vec{a}_+ e^{\lambda_+ t} + A_- \vec{a}_- e^{\lambda_- t}.$$

Now we want to determine the nature of this fixed point.

1. If $r < 1$ then $\lambda_+, \lambda_- < 0$, and all solutions (including the z-direction) decay (i.e. approaches the fixed point): the fixed point is an attractor.

2. If $r > 1$, then $\lambda_+ > 0 > \lambda_-$, and solutions in the $\vec{a}_+$ direction grow (are repelled from the origin), while solutions in the $\vec{a}_-$ and z-directions decay: the fixed point is a saddle point.

3. If $r = 1$, then $\lambda_+ = 0 > \lambda_-$. This is a critical case, and keeping only terms up to linear order is insufficient to determine the nature of the fixed point. [Grader: give full credit even if this case was ignored]

Problem Set #9, Problem 2, Nonlinear Circuit

First Order equations :

$$\dot{x} = w,$$

$$\dot{\phi} = w_D,$$

$$\dot{w} = \frac{-1}{q_C} w - x^3 - B\cos[\phi]$$

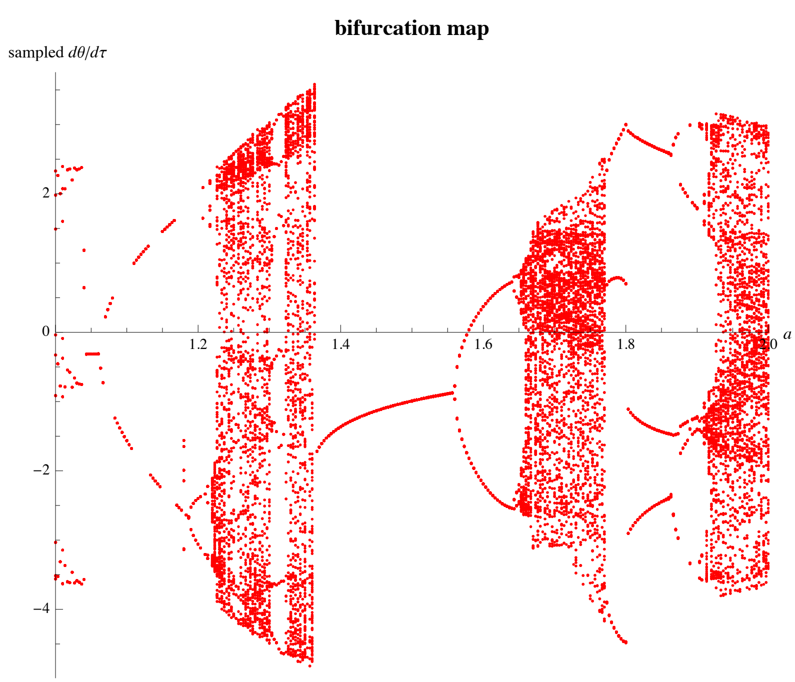
$$= \frac{-1}{5\,q} w - x^3 - 6\,a\cos[\phi]$$

where my Map is : $B = 6\,a$, $q_C = 5\,q$ (other possibilities fine too).

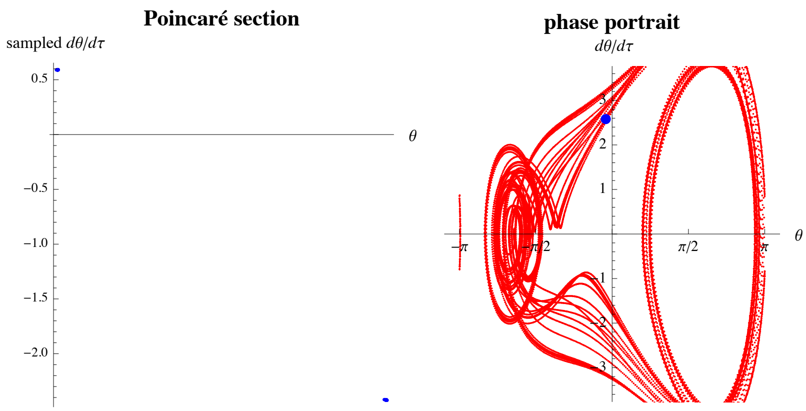
Except where otherwise indicated I use the default initial conditions :

$$\Theta(0) = 0.6184;\quad \frac{d\Theta(0)}{d\tau} = 0;\quad \phi(0) = 0$$

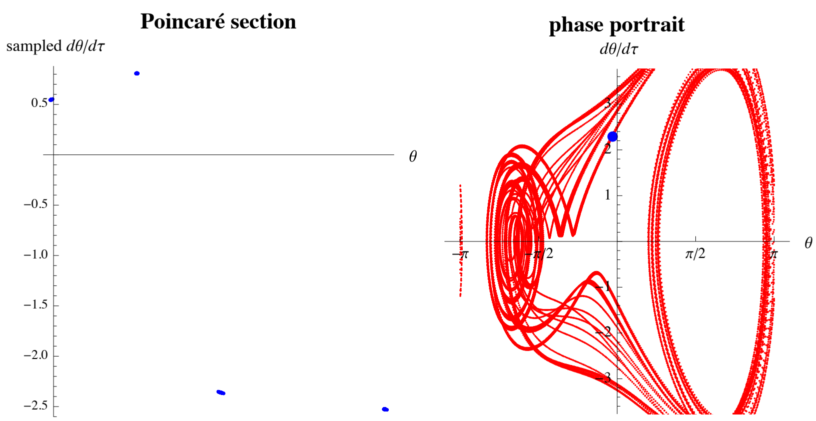
(a) First plot the Bifurcation map, $w_D = 2/3$, $qc = 10$
Showing the range $1 < a < 2$, so $6 < B < 12$:



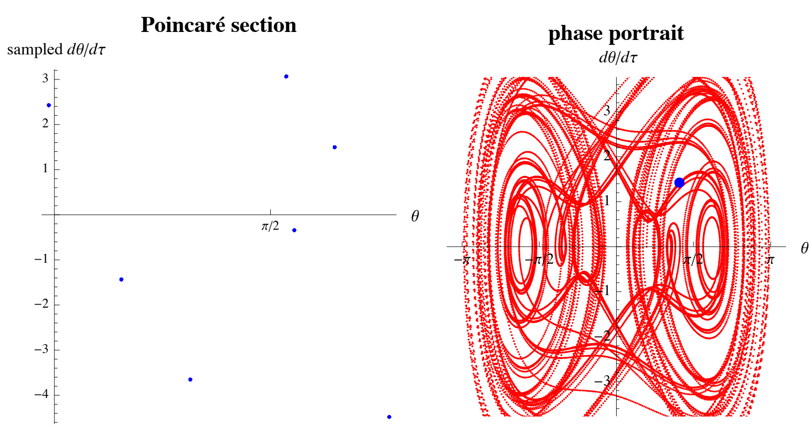
(b) Example with 2 periods (a=1.62, so B=9.72)



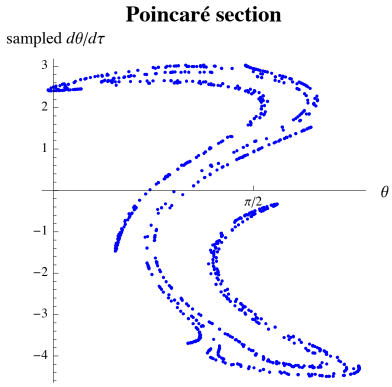
Example with 4 periods (a=1.645, so B=9.87)

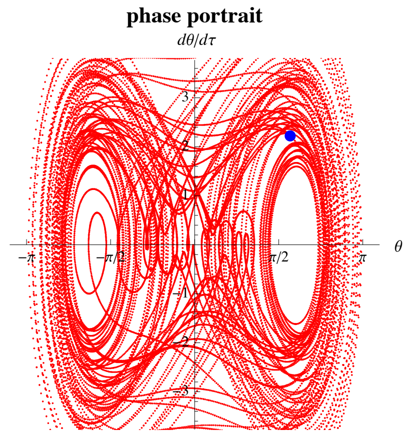


Here’s an example with 7 periods that you were not asked to provide
(a=1.31, so B=7.86)

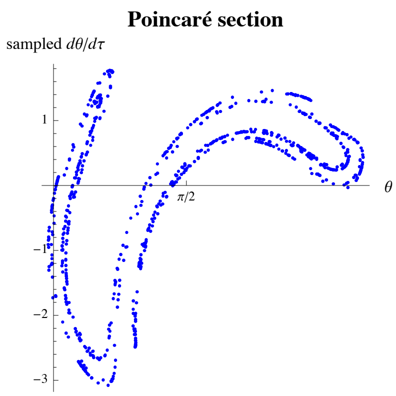


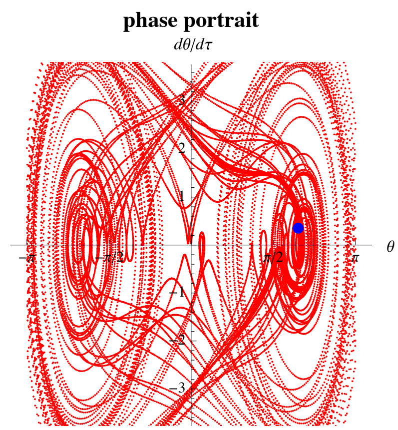
Example 1 with Chaos (a=1.3, so B=7.8)



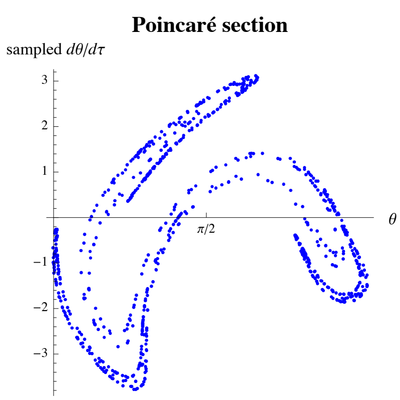


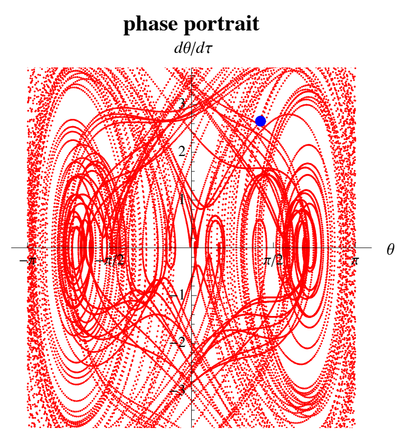
Example 2 with Chaos (a=1.7, so B=10.2)





Example 3 with Chaos (only two examples required) (a=1.95, so B=11.7)

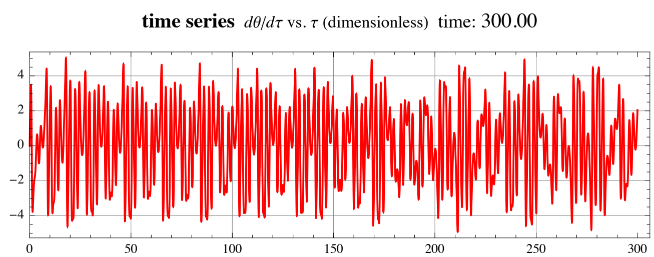


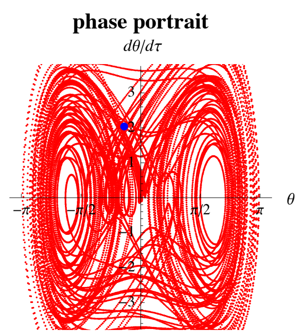


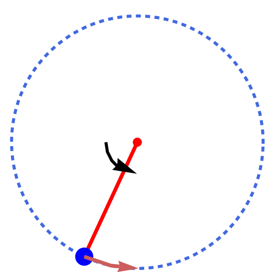
(c) Demonstration of Sensitivity to Initial Conditions

we consider the case (a=1.3, so B=7.8) and change the initial condition for $\theta(0)$ by ~ 10^{-4}. [Varying a different initial condition also fine.] There are clear differences in the the generated time series for time >120. One also observes that the phase space location of the oscillator is quite different for these cases.

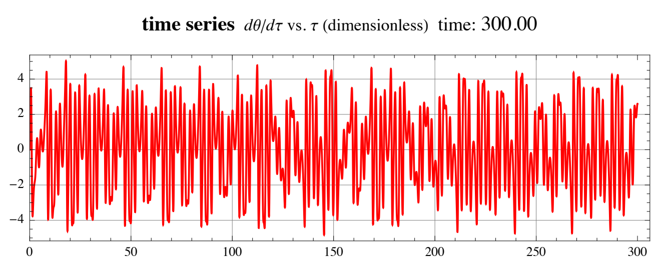
```
(* θ(0)=0.6146 *)
```

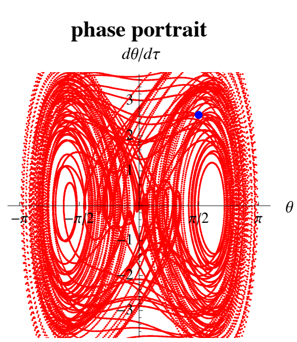


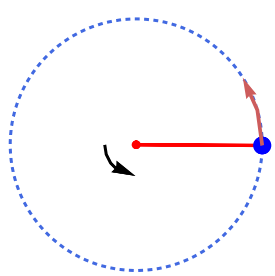




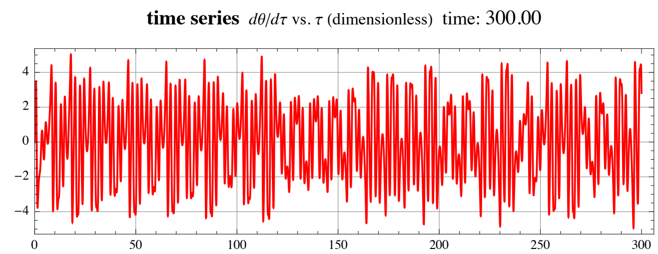
```
(* θ(0)=0.6147 *)
```

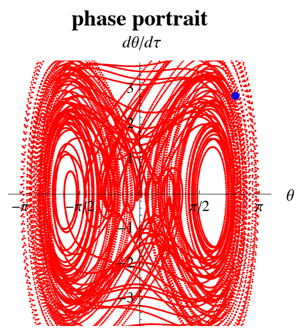


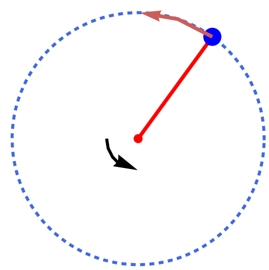




```
(* θ(0)=0.6148 *)
```







```
(* θ(0)=0.6149 *)
```