# Matrices in Quantum Computing

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Matrix Analysis

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## Presentation layout

Background

2 Motivation

Some Matrix Theory

## Qubits & Quantum Gates

Qubit: A quantum system with measurable eigenstates  $|0\rangle$  and  $|1\rangle$ ,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \rightarrow \mathsf{like} \ \mathsf{a} \ \mathsf{Classical} \ \mathsf{Bit}.$$

But before measurement,

Wavefunction : 
$$|\psi\rangle = a|0\rangle + b|1\rangle \in \mathbb{C}^2$$
,  $|a|^2 + |b|^2 = 1$ .

Probabilistic:

$$P(|\psi\rangle \rightarrow |0\rangle) = |a|^2 \quad P(|\psi\rangle \rightarrow |1\rangle) = |b|^2.$$

Quantum gate: unitary transformation on  $|\psi\rangle$  of one of many qubits.

## Multiple Qubits

How to express the state of two qubits,  $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$ ?

$$|\psi_1\psi_2\rangle\stackrel{?}{\sim} |\psi_1\rangle\,, |\psi_2\rangle$$

More than two,  $|\psi_i\rangle \in \mathbf{V}_i$ ?

$$|\psi_1\psi_2\dots\psi_n\rangle \stackrel{?}{\sim} |\psi_1\rangle, |\psi_2\rangle,\dots, |\psi_n\rangle$$

#### Questions:

- What is the vector space containing  $|\psi_1\psi_2\dots\psi_n\rangle$ ?
- How does  $|\psi_1\psi_2\dots\psi_n\rangle$  change w.r.t  $\mathcal{A}_1|\psi_1\rangle$  where  $\mathcal{A}_1\in\mathfrak{L}(\mathbf{V})$ ?
- What about for  $A_1 | \psi_1 \rangle, \dots A_n | \psi_n \rangle$ , where  $A_i \in \mathfrak{L}(\mathbf{V})$ ?

### Tensor Product

**Postulate:** [Mike & Ike] The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

For  $|\psi_1\rangle\in \mathbf{V}_1,\ldots,|\psi_n\rangle\in \mathbf{V}_n$ ,

$$|\psi_1\dots\psi_n\rangle\in\mathbf{V}_1\otimes\cdots\otimes\mathbf{V}_n,$$

where the joint state  $|\psi_1 \dots \psi_n\rangle$  is given by

$$|\psi_1 \dots \psi_n\rangle = |\psi_1\rangle |\psi_2\rangle \dots |\psi_n\rangle = |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle.$$

 $|\psi_1 \dots \psi_n\rangle$  is an elementary tensor in  $\mathbf{V}_1 \otimes \dots \otimes \mathbf{V}_n$ .

Not all  $|\phi\rangle \in \mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n$  are elementary.



### Tensor Product: Construction

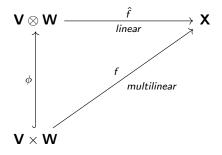
#### **Definition**

The *tensor product* of **V** and **W** is a vector space  $\mathbf{V} \otimes \mathbf{W}$  with the *bilinear map*  $\phi : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{V} \otimes \mathbf{W}$ , such that for every vector space **X** and every bilinear map  $f : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{X}$ , there exists a *unique linear map*  $\hat{f} : \mathbf{V} \otimes \mathbf{W} \longrightarrow \mathbf{X}$  such that  $f = \hat{f} \circ \phi$ .

### In other words...

Giving the  $\hat{f}: \mathbf{V} \otimes \mathbf{W} \stackrel{\text{linear}}{\longrightarrow} \mathbf{X}$  is the same as giving  $f: \mathbf{V} \times \mathbf{W} \stackrel{\text{bilinear}}{\longrightarrow} \mathbf{X}$ .

### Tensor Product: Construction



### Tensor Product: Some Properties

Let  $v_1, \ldots, v_n$  be a basis of **V** and  $w_1, \ldots, w_m$  be a basis of **W**,

• For  $i \in [1, n], j \in [1, m], \{v_i \otimes w_j\}$  is a basis of  $\mathbf{V} \otimes \mathbf{W}$ :

$$v \otimes w = \sum_{i}^{n} \alpha_{i} v_{i} \otimes \sum_{j}^{m} \beta_{j} w_{j} = \sum_{i,j}^{n,m} \alpha_{i} \beta_{j} (v_{i} \otimes w_{j})$$

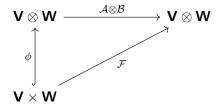
•  $\dim(\mathbf{V} \otimes \mathbf{W}) = \dim(\mathbf{V}) \dim(\mathbf{W}) = nm$ .

### Tensor Product: Some Properties

Let  $A \otimes B \in \mathfrak{L}(V \otimes W)$ , where  $A \in \mathfrak{L}(V)$ , and  $B \in \mathfrak{L}(W)$ .

$$(\mathcal{A} \otimes \mathcal{B})(v \otimes w) \stackrel{?}{\sim} \mathcal{F}(v, w) \stackrel{\Delta}{=} \mathcal{A}(v) \otimes \mathcal{B}(w).$$

One way to see this...



$$(\mathcal{A} \otimes \mathcal{B}) \circ \phi = \mathcal{F} \iff \boxed{(\mathcal{A} \otimes \mathcal{B})(v \otimes w) = \mathcal{A}(v) \otimes \mathcal{B}(w)}$$

### Tensor Product: Some Properties

Another way to see this...

## Why Tensor Product?