8.311 . EM theory

HW 2 : due Wed Feb 20, 2022

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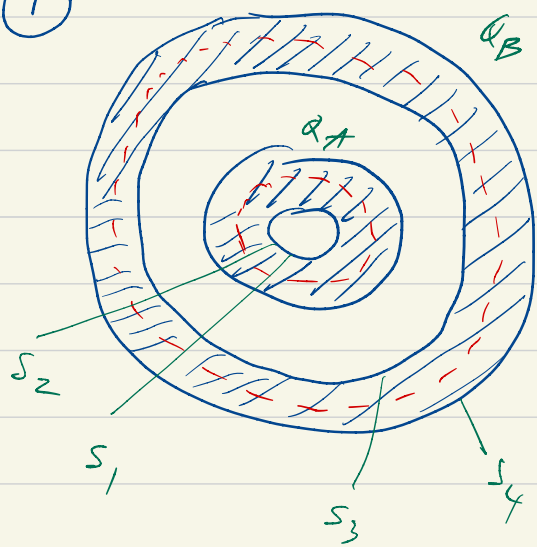
(1)



$\vec{E}$ field inside conductor must be zero...

So consider 2 Gaussian surfaces (red)

$\Rightarrow$ require that charge enclosed in them be zero...

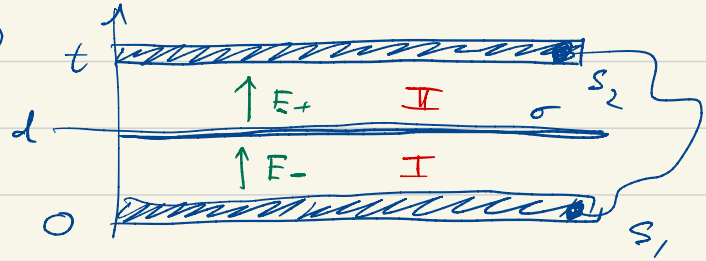
Recall further that charge on conductor must all be on the surface... So $\boxed{S_1 \text{ has } 0 \text{ charge}}$

By conservation of charge, $\boxed{S_2 \text{ must have } Q_A}$

For $\vec{E} = 0$ on the outer red surface, $\boxed{S_3 \text{ must have } -Q_A}$

This leaves S_4 with $\boxed{Q_B + Q_A}$

(2)



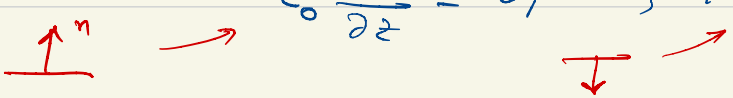
- Setup Laplace's eqns for regions I & II...

- Region I: ϕ_1 ; Region II: ϕ_2.

$$\partial_z^2 \phi_i = 0 \quad \text{for } i = 1, 2$$

$$\Rightarrow \phi_i = m_i z + b_i$$

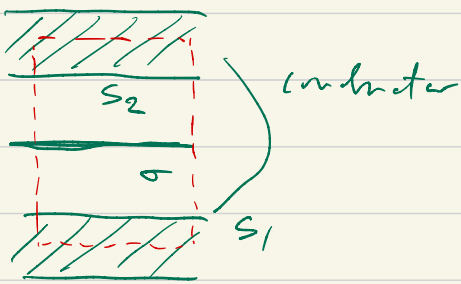
- Boundary conditions:
  1) Let's take $\phi_1(0) = 0$
  2) Plates are connected: $\phi_1(0) = \phi_2(t)$
  3) ϕ must be continuous...
  $$\phi_1(d) = \phi_2(d)$$
  4) Let S_1, S_2 be surface charge density on bottom & top plate respectively, then
  $$-\varepsilon_0 \frac{\partial \phi}{\partial z} = S_1 \quad ; \quad +\varepsilon_0 \frac{\partial \phi}{\partial z} = S_2$$



- From here we have

$$\begin{cases} \phi_1(z) = -\dfrac{S_1}{\varepsilon_0} z \\ \phi_2(z) = +\dfrac{S_2}{\varepsilon_0}(z-t) \end{cases}$$

Moreover, we must also satisfy Gauss' law...



E in conductor (plates) = 0
$\Rightarrow$ charge enclosed in the red gaussian surface $\to$ zero.

$$\Rightarrow \quad S_1 + S_2 + \sigma = 0$$

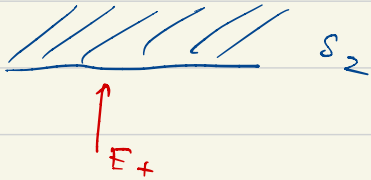
So we have $\phi_1(d) = \phi_2(d)$ implies

$$-\frac{S_1}{\varepsilon_0} d = +\frac{(-\sigma - S_1)}{\varepsilon_0}(d-t) \Rightarrow \boxed{S_1 = \frac{d-t}{t}\sigma}$$

$$\Rightarrow \boxed{S_2 = -\frac{d\sigma}{t}}$$

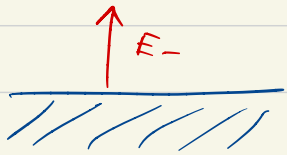
- With these can calculate force on the film

⊕ <u>Field due to S_2</u>:



$$E_+ = \frac{-S_2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(\frac{d\sigma}{t}\right)$$

⊕ <u>Field due to S_1</u>:



$$E_- = \frac{+S_1}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\frac{(d-t)}{t}\sigma$$

⇒ Field @ $z = d$ is

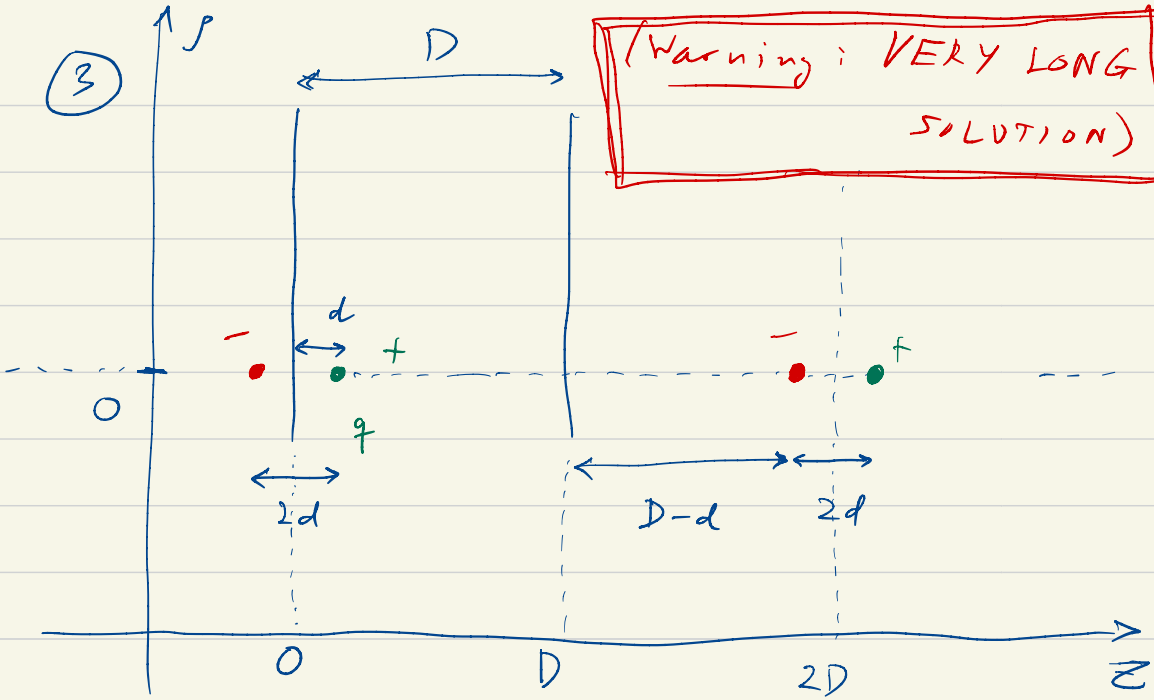
$$E(d) = \frac{\sigma}{2\varepsilon_0}\left[\frac{d}{t} + \frac{d-t}{t}\right] = \frac{\sigma}{2\varepsilon_0}\left(\frac{2d-t}{t}\right)$$

→ Force on film per unit area is

$$\boxed{F = \frac{\sigma^2}{2\varepsilon_0}\left(\frac{2d-t}{t}\right)\hat{z}}$$

(3)

(Warning: VERY LONG SOLUTION)



We notice that we have repeated reflection which leads to a system of charges $\pm q$ at points

$$\boxed{z_j^{\pm} = 2Dj \pm d}$$

e.g. $z_0^{+} = d$ (charge q)

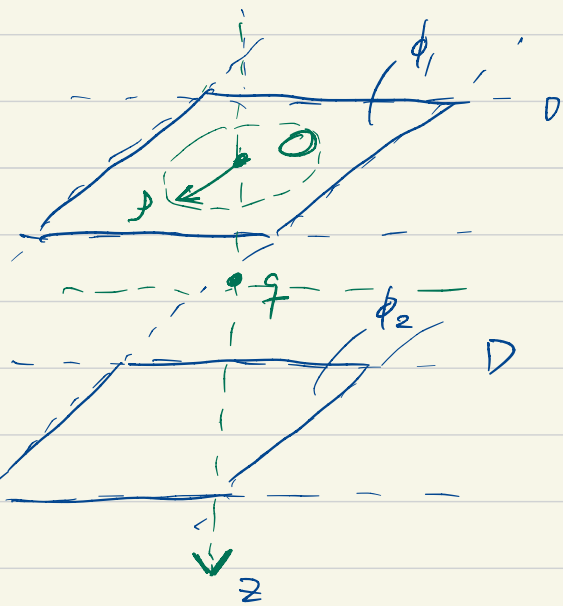
$z_0^{-} = -d$ (img of q)

$z_1^{+} = 2D + d$ (img^2 of q ...)

$\vdots$

- This results in an ∞-sum for the potential of the real charge q, created by the fields of its images

- For this problem we use cylindrical coords to get σ ... $\sigma = \sigma(\rho)$



pick the ρ of the charge q to be 0

- Since the plane capacitor is voltage unbiased, we require that

$$\phi\,(@\text{ plates}) = 0$$

$\boxed{Q}$ What is ϕ ?

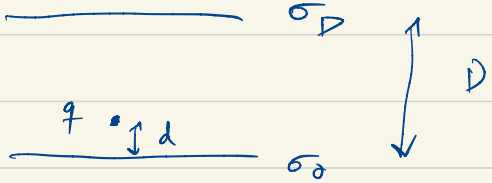
- Well... ϕ is obtained by adding contributions from the charge q and all image charges...

$$\boxed{\phi(\rho,z) = \frac{q}{4\pi\varepsilon_0}\sum_{j=-\infty}^{\infty}\left[\frac{1}{\sqrt{\rho^2+(z-(2Dj+d))^2}} - \frac{1}{\sqrt{\rho^2+(z-(2Dj-d))^2}}\right]}$$

a pair separated by $2d$

- Can easily see that $\phi(\rho,0)=0$ ✓
  $\phi(\rho,D)=0$ ✓

- Now from here, can evaluate surface charges densities σ_0 and σ_D



$$\sigma_0 = -\varepsilon_0 \left.\frac{\partial\phi}{\partial z}\right|_{z=0}$$

$$\sigma_D = +\varepsilon_0 \left.\frac{\partial\phi}{\partial z}\right|_{z=D}$$

From the densities, we can calculate total surface charges... $q = \int_{\mathbb{R}^2} \sigma \, dA$ ...

$$q_0 = -\varepsilon_0 \int_{\mathbb{R}^2} \left. \frac{\partial \phi}{\partial z} \right|_{z=0} dA = \text{oops...}$$

- How to manipulate the series to get convergence?

- Okay... upon inspection, the series indeed diverges, so we actually can't solve the problem legitimately this way...

- Of course, we may try to manipulate the series to get an answer we want...

→ But that's not mathematically sound

So...

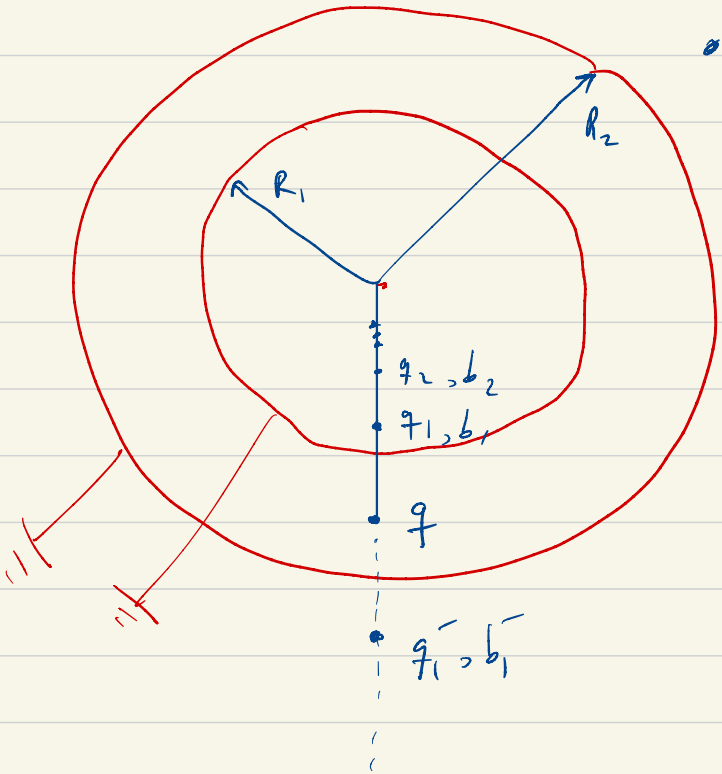
SOLUTION

- I found this pdf from MIT OCW (link here) which provides an alternative approach ... https://ocw.'mit.edu/resource's/res-6-002-electromagnetic-field-theory-a-problem-solving-approach-spring-2008/textbook-contents/MITRES_6_002S08_chp02_pset.pdf u

- Textbook: Electromagnetic field theory - a problem-solving approach (Markus Zahn)

- Following pg 129... we consider a point charge q @ radial distance R_0 from the center of 2 concentric grided conducting spheres of radii R_1, R_2

By method of images, we have



- To do the parallel plates problem, let us start with this problem

- By method of images, we see that we also require infinitely many image charges.

- Let q_i denote charge in the small sphere, @ distance b_i from the centre

- Similarly, q_i^-, b_i^- for charges outside the large sphere ...

- Then we know that (from the sphere problem)

$$\boxed{\begin{aligned} q_{n+1} &= \frac{-R_1}{b_n^-} q_n^- \quad ; \quad q_{n+1}^- = -\frac{R_2}{b_n} q_n \\ b_{n+1} &= \frac{R_1^2}{b_n^-} \quad ; \quad b_{n+1}^- = \frac{R_2^2}{b_n} \end{aligned}}$$

- From here, we see that

$$q_{n+1} = \frac{-R_1}{R_2^2 / b_{n-1}} \left(\frac{-R_2}{b_{n-1}} \right) q_{n-1}$$

$$\Rightarrow \boxed{q_{n+1} - q_{n-1} \left(\frac{R_1}{R_2} \right) = 0}$$

$$b_{n+1} = \frac{R_1^2}{R_2^2 / b_{n-1}} \Rightarrow \boxed{b_{n+1} - b_{n-1} \left(\frac{R_1}{R_2} \right)^2 = 0}$$

- What is q_n? Let us try a power law solution...

$$\boxed{q_n = A\lambda^n \quad , \quad b_n = B\alpha^n}$$

- Then $A\lambda^{n+1} - A\lambda^{n-1}\left(\frac{R_1}{R_2}\right) = 0$

$$\Rightarrow \quad \lambda^2 = \frac{R_1}{R_2} \Rightarrow \boxed{\lambda = \sqrt{\frac{R_1}{R_2}}}$$

and

$$B\alpha^{n+1} - B\alpha^{n-1}\left(\frac{R_1}{R_2}\right)^2 = 0$$

$$\Rightarrow \quad \alpha^2 = \left(\frac{R_1}{R_2}\right)^2 \Rightarrow \boxed{\alpha = \frac{R_1}{R_2}}$$

- Now try to find A, B ... Well... since q_n changes sign depending on mod $(2, n)$, we must say that

$$q_n = \lambda^n\left[A_1 + (-1)^n A_2\right]$$
$$b_n = \alpha^n\left[B_1 + (-1)^n B_2\right]$$

- Now look at $n=1$, $n=2$

$\boxed{n=1}$ $$q_1 = -q\frac{R_1}{R_0} = (A_1 - A_2)\sqrt{\frac{R_1}{R_2}}$$

$\boxed{n=2}$ $$q_2 = q\frac{R_1}{R_2} = (A_1 + A_2)\frac{R_1}{R_2}$$

$$\Rightarrow \begin{cases} A_1 - A_2 = -q\sqrt{R_1R_2}/R_0 \\ A_1 + A_2 = q \end{cases}$$

$\Rightarrow$ Solve for A_1, A_2 (by hand or Mathematica)

$$\Rightarrow \boxed{q_n = q\left[\frac{R_0 - \sqrt{R_1R_2}}{2R_0} + (-1)^n\frac{R_0 + \sqrt{R_1R_2}}{2R_0}\right]\left(\frac{R_1}{R_2}\right)^{n/2}}$$

- Same thing for B ... (b_n alternating so...)

$$b_1 = \frac{R_1^2}{R_0} = (B_1 - B_2)\frac{R_1}{R_2}$$

$$b_2 = \frac{R_1^2 R_0}{R_2^2} = (B_1 + B_2)\left(\frac{R_1}{R_2}\right)^2$$

$$\Rightarrow \begin{cases} B_1 + B_2 = R_0 & (n \text{ even}) \\ B_1 - B_2 = \dfrac{R_2 R_1}{R_0} & (n \text{ odd}) \end{cases}$$

$\Rightarrow$ Solve for B_1, B_2 (by hand or Mathematica)

$$\Rightarrow \boxed{b_n = \left[\frac{R_0^2 + R_1 R_2}{2R_0} + (-1)^n \frac{R_0^2 - R_1 R_2}{2R_0}\right]\left(\frac{R_1}{R_2}\right)^n}$$

- Now we calculate the total charge induced on the inner sphere ...

Well...

- Total charge on inner sphere is ...

$$Q_{inner} = \sum_{n=1}^{\infty} \left[q_n \right]$$

$$= \sum_{n=1}^{\infty} q \left[\frac{R_0 - \sqrt{R_1 R_2}}{2R_0} + (-1)^n \frac{R_0 + \sqrt{R_1 R_2}}{2R_0} \right] \left(\frac{R_1}{R_2} \right)^{n/2}$$

$$= -q \frac{R_1 (R_2 - R_0)}{R_0 (R_2 - R_1)}$$ (using Mathematica or geometric series ...)

- Now let's use Gauss' law to find Q_{outer} (it turns out that Q_{outer} cannot be found using the same method, as we get a divergent series ...)

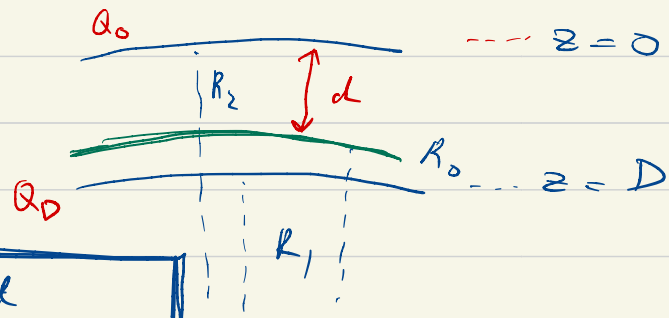
- By Gauss's law, the total induced charge must be $-q$ ... so

$$Q_{outer} = -q \frac{R_2 (R_0 - R_1)}{R_0 (R_2 - R_1)}$$

- We now return to the original problem, by taking

$$\left[\begin{array}{l} R_2,\ R_1 \to \infty \quad \text{such that} \\ D = R_2 - R_1 \quad \text{and} \quad d = R_2 - R_0 < \infty \end{array}\right.$$

↳ We see that



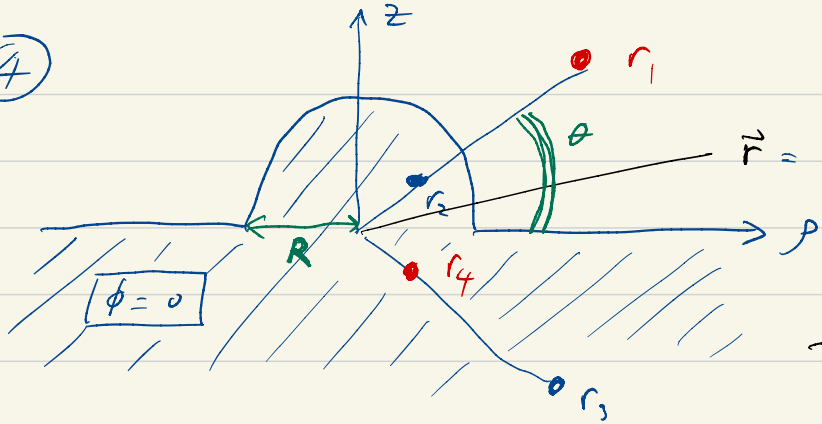
$$\boxed{\begin{aligned} Q_D &= \lim_{R_1 \to R_0} = -q\,\frac{d}{D} \\ Q_0 &= \lim_{R_2 \to R_0} = -q\,\frac{(D-d)}{D} \end{aligned}} \Longleftarrow \underline{\text{Answer}}$$

The correspondence is $Q_D \sim Q_{inner}$

$Q_0 \sim Q_{outer}$

<u>Note</u>: This problem could be solved efficiently using Green's reciprocation theorem

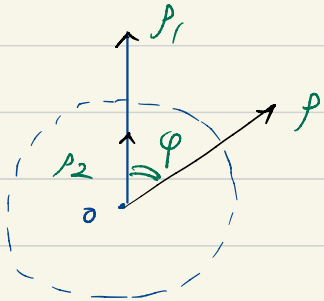
④



$\Downarrow$

use cylindrical coords...

From top



Most generally, we have

$$G(\vec{r}) = \frac{1}{|\vec{r} - \vec{r}_1|} - \frac{1}{|\vec{r} - \vec{r}_3|} - \frac{R/r_1}{|\vec{r} - \vec{r}_2|} + \frac{R/r_1}{|\vec{r} - \vec{r}_4|}$$

where $|\vec{r} - \vec{r}_i| = \left[(z - z_i)^2 + (\rho^2 + \rho_i^2 - 2\rho\rho_i \cos\varphi) \right]^{1/2}$

$\longrightarrow$

where we will write everything in terms of $\vec{r}_1$ which is defined by 2 numbers ...

(ρ_1, z_1) ... assuming $\varphi_1 = 0$

$\Rightarrow$ need to know z_i ; ρ_i ...

Well...

$$\boxed{\begin{aligned} z_3 &= -z_1 \quad ; \quad z_2 = z_1 \frac{r_2}{r_1} = z_1 \cdot \frac{R^2}{r_1^2} = -z_4 \\ \rho_3 &= \rho_1 \quad ; \quad \rho_2 = \rho_1 \frac{r_2}{r_1} = \rho_1 \frac{R^2}{r_1^2} = +\rho_4 \end{aligned}}$$

- Getting the full answer is only a matter of substituting the quantities here into the 2nd eqn and then into the 1st ...

  So I won't do that here ...

- As a sanity check... let's plug in some special locations ...

Suppose we're on the sphere, then we have that $z^2 + \rho^2 = R^2$

$$G(\text{sphere}) = \left[R^2 + r_1^2 - 2zz_1 - 2\rho\rho_1\cos\varphi\right]^{-1/2}$$

$$- \left[R^2 + r_1^2 + 2zz_1 - 2\rho\rho_1\cos\varphi\right]^{-1/2}$$

$$- \frac{R}{r_1}\left[R^2 - 2zz_1\frac{R^2}{r_1^2} + \left(\frac{R}{r_1}\right)^4 r_1^2 - 2\rho\rho_1\frac{R^2}{r_1^2}\cos\varphi)\right]^{-1/2}$$

$$+ \frac{R}{r_1}\left[R^2 + 2zz_1\frac{R^2}{r_1^2} + \left(\frac{R}{r_1}\right)^4 r_1^2 - 2\rho\rho_1\frac{R^2}{r_1^2}\cos\varphi)\right]^{-1/2}$$

- By inspection (bringing R/r_1 into the $\frac{1}{\sqrt{\ldots}}$)

We see easily that $G(\text{sphere}) = 0$, or $\phi(\text{sphere}) = 0$, as expected.