Assignment 1; MA353; S19

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Last modified at 10:43 on February 20, 2019

1 Preliminaries

Test Your Comprehension 1.1

Suppose that $\boldsymbol{V}$ is a finite-dimensional vector space. Consider the following subspace $\boldsymbol{W}$ of the vector space $\boldsymbol{V} \times \boldsymbol{V}$:

$$\boldsymbol{W} = \left\{ \left(\begin{smallmatrix} v \\ -v \end{smallmatrix}\right) \;\middle|\; v \in \boldsymbol{V} \right\} .$$

Argue that

$$\dim \boldsymbol{W} = \dim \boldsymbol{V} .$$

Test Your Comprehension 1.2

Argue that $\boldsymbol{V} + \{O_{\boldsymbol{W}}\} = \boldsymbol{V}$, for any subspace $\boldsymbol{V}$ of a vector space $\boldsymbol{W}$.

Definition 1.3

Suppose that $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_{315}$ are subspaces of a vector space $\boldsymbol{W}$. We say that the subspace sum $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}$ is **direct** if the linear* function

$$\varphi : \boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \ldots \times \boldsymbol{V}_{315} \longrightarrow \boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315},$$

defined by the formula

$$\varphi \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_{315} \end{pmatrix} := v_1 + v_2 + v_3 + \ldots + v_{315} ,$$

is an injection (and hence, being a surjection by definition, is a bijection).

When a subspace sum $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}$ is direct, we denote it by

$$\boldsymbol{V}_1 \uplus \boldsymbol{V}_2 \uplus \ldots \uplus \boldsymbol{V}_{315} .$$

<u>When $\boldsymbol{W}$ is finite-dimensional</u>, so are $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}$ and $\boldsymbol{V}_1 \times \boldsymbol{V}_2 \times \ldots \times \boldsymbol{V}_{315}$. As we know, finite-dimensional vector spaces (over $\mathbb{C}$) are isomorphic exactly when they have the same dimension.

Hence, <u>when $\boldsymbol{W}$ is finite-dimensional</u>, the subspace sum $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}$ is direct exactly when

$$\dim\left(\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}\right) = \dim\left(\boldsymbol{V}_1\right) + \dim\left(\boldsymbol{V}_2\right) + \ldots + \dim\left(\boldsymbol{V}_{315}\right) .$$

*The linearity of φ is easy to verify.

2 Problems

Problem 1 ☞ **Equivalent formulations of directness of a subspace sum**

Suppose that $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_{315}$ are subspaces of a vector space $\boldsymbol{W}$. Argue that the following claims are equivalent.

1. The subspace sum $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}$ is direct.
2. If $x_i \in \boldsymbol{V}_i$ and
$$x_1 + x_2 + x_3 + \ldots + x_{315} = 0_{\boldsymbol{W}} ,$$
then $x_i = 0_{\boldsymbol{W}}$, for every i.
3. If $x_i, y_i \in \boldsymbol{V}_i$ and
$$x_1 + x_2 + x_3 + \ldots + x_{315} = y_1 + y_2 + y_3 + \ldots + y_{315} ,$$
then $x_i = y_i$, for every i.
4. For any i, no non-null element of $\boldsymbol{V}_i$ can be expressed as a sum of the elements of the other $\boldsymbol{V}_j$'s.
5. For any i, no non-null element of $\boldsymbol{V}_i$ can be expressed as a sum of the elements of the preceding* $\boldsymbol{V}_j$'s.

*Here "preceding" refers to the order of the list $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_{315}$.

Problem 2 ☞ **Sub-sums of direct sums are direct**

Suppose that $\boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_{315}$ are subspaces of a vector space $\boldsymbol{W}$, and the subspace sum $\boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_{315}$ is direct.

1. Suppose that for each i, $\boldsymbol{Z}_i$ is a subspace of $\boldsymbol{V}_i$. Argue that the subspace sum $\boldsymbol{Z}_1 + \boldsymbol{Z}_2 + \ldots + \boldsymbol{Z}_{315}$ is direct.

2. Argue that the sum $\boldsymbol{V}_2 + \boldsymbol{V}_5 + \boldsymbol{V}_7 + \boldsymbol{V}_{12}$ is also direct.

Problem 3 ☞ **Associativity of directness of subspace sums**

Suppose that

$$\boldsymbol{Y}, \boldsymbol{V}_1, \boldsymbol{V}_2, \ldots, \boldsymbol{V}_5, \boldsymbol{U}_1, \boldsymbol{U}_2, \ldots, \boldsymbol{U}_{12}, \boldsymbol{Z}_1, \boldsymbol{Z}_2, \ldots, \boldsymbol{Z}_4, \boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_{53}$$

are subspaces of a vector space $\boldsymbol{W}$.

Argue that the following claims are equivalent.

1. The subspace sum

   $$\boldsymbol{Y}+\boldsymbol{V}_1+\boldsymbol{V}_2+\ldots+\boldsymbol{V}_5+\boldsymbol{U}_1+\boldsymbol{U}_2+\ldots+\boldsymbol{U}_{12}+\boldsymbol{Z}_1+\boldsymbol{Z}_2+\ldots+\boldsymbol{Z}_4+\boldsymbol{X}_1+\boldsymbol{X}_2+\ldots+\boldsymbol{X}_{53}$$

   is direct.

2. The subspace sums

   $$\left[\boldsymbol{V} :=\right] \ \boldsymbol{V}_1 + \boldsymbol{V}_2 + \ldots + \boldsymbol{V}_5$$

   $$\left[\boldsymbol{U} :=\right] \ \boldsymbol{U}_1 + \boldsymbol{U}_2 + \ldots + \boldsymbol{U}_{12}$$

   $$\left[\boldsymbol{Z} :=\right] \ \boldsymbol{Z}_1 + \boldsymbol{Z}_2 + \ldots + \boldsymbol{Z}_4$$

   $$\left[\boldsymbol{X} :=\right] \ \boldsymbol{X}_1 + \boldsymbol{X}_2 + \ldots + \boldsymbol{X}_{53}$$

   $$\boldsymbol{Y} + \boldsymbol{V} + \boldsymbol{U} + \boldsymbol{X} + \boldsymbol{Z}$$

   are all direct.

Problem 4 ☞ **Direct sums preserve linear independence**

Suppose that $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are subspaces of a vector space $\boldsymbol{Z}$, and the sum $\boldsymbol{U} + \boldsymbol{V} + \boldsymbol{W}$ is direct.

1. Suppose that $U_1, U_2, \ldots, U_{13}$ is a linearly independent list in $\boldsymbol{U}$, $V_1, V_2, \ldots, V_6$ is a linearly independent list in $\boldsymbol{V}$, and $W_1, W_2, \ldots, W_{134}$ is a linearly independent list in $\boldsymbol{W}$.

   Argue that the concatenated list

   $$U_1, U_2, \ldots, U_{13}, V_1, V_2, \ldots, V_6, W_1, W_2, \ldots, W_{134}$$

   is linearly independent.

2. Suppose that $U_1, U_2, \ldots, U_{13}$ is a basis of $\boldsymbol{U}$, $V_1, V_2, \ldots, V_6$ is a basis of $\boldsymbol{V}$, and $W_1, W_2, \ldots, W_{134}$ is a basis of $\boldsymbol{W}$.

   Argue that the concatenated list

   $$U_1, U_2, \ldots, U_{13}, V_1, V_2, \ldots, V_6, W_1, W_2, \ldots, W_{134}$$

   is a basis of $\boldsymbol{U} \oplus \boldsymbol{V} \oplus \boldsymbol{W}$.

Problem 5

Suppose that $x_1, x_2, \ldots, x_{14}$ are <u>non-null</u> elements of a linear space $\boldsymbol{W}$.

1. Argue that the following claims are equivalent.

   (a) $x_1, x_2, \ldots, x_{14}$ are linearly independent.

   (b) The subspace sum

   $$Span\,(x_1) + Span\,(x_2) + \cdots + Span\,(x_{14})$$

   is direct.

2. Argue that the following claims are equivalent.

   (a) $x_1, x_2, \ldots, x_{14}$ is a basis of $\boldsymbol{W}$.

   (b) $\boldsymbol{W} = Span\,(x_1) \oplus Span\,(x_2) \oplus \cdots \oplus Span\,(x_{14})$.