8.(3)09 Midterm Solutions 2018

100 Pts total

1. 15 total

(a) 3

$$L = L_0 + \dot{\vec{q}}^T \cdot \vec{a} + \frac{1}{2} \dot{\vec{q}}^T \cdot \hat{T} \cdot \dot{\vec{q}}$$

with $\hat{T} = m$, $\vec{a} = b$, $L_0 = -c x^4$ so

$$\boxed{H = \frac{1}{2m}(p-b)^2 + c x^4}$$

(b) 4

$H(p,q) = \alpha_1$ conserved since $\frac{\partial H}{\partial t} = 0$

$\therefore \quad p = p(q, \alpha_1)$

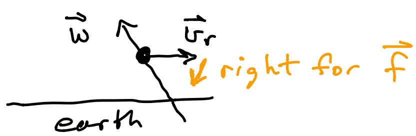
$$J = \oint p(q,\alpha_1)\, dq = J(\alpha_1) = \text{constant}$$

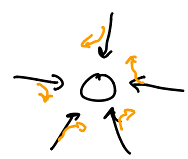
← definite integral

(c) 3

Northern hemisphere, Coriolis deflects right

$$\vec{F}_{\text{coriolis}} = -2m\, \vec{\omega} \times \vec{v}_r$$



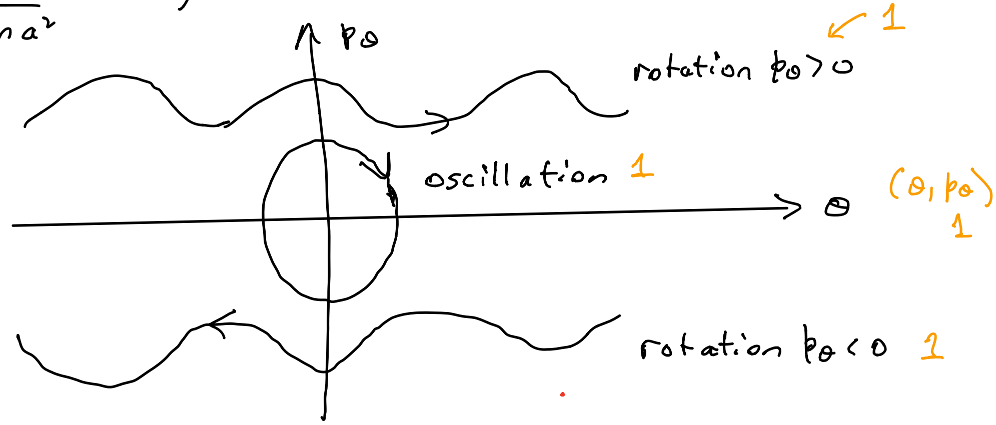


$\boxed{\text{counterclockwise}}$

(d) 5

$$E = \frac{p_\theta^2}{2ma^2} - mga\cos\theta$$

1 pt for proper arrows



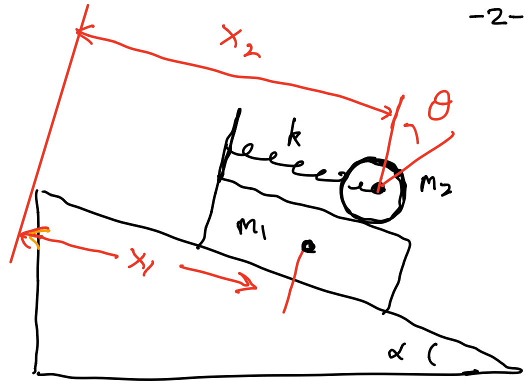
2. 25 total

Hoop m_2, radius a, $I = m_2 a^2$

Block m_1, frictionless to fixed wedge

Spring k, relaxed length @ $\frac{1}{2}$ block

gravity g



ⓐ Contact Hoop & Block Frictionless

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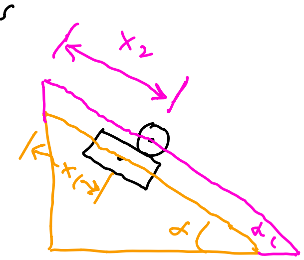

$$T = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{1}{2} m_2 a^2 \dot{\theta}^2 \qquad 2$$

$$V = \underbrace{\text{constant}}_{\text{drop}} - m_1 g x_1 \sin\alpha - m_2 g x_2 \sin\alpha + \frac{1}{2} k (x_1 - x_2)^2 \qquad 3$$

$$\boxed{L = T - V = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + \frac{1}{2} m_2 a^2 \dot{\theta}^2 + m_1 g \sin\alpha\, x_1 + m_2 g \sin\alpha\, x_2 - \frac{1}{2} k (x_1 - x_2)^2}$$

EOM

x_1// $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} - \frac{\partial L}{\partial x_1} = 0 = m_1 \ddot{x}_1 - m_1 \sin\alpha\, g - k(x_2 - x_1)$ 1

x_2// $d/dt \frac{\partial L}{\partial \dot{x}_2} - \frac{\partial L}{\partial x_2} = 0 = m_2 \ddot{x}_2 - m_2 \sin\alpha\, g + k(x_2 - x_1)$ 1

ϕ// $d/dt \frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi} = 0 \Rightarrow d/dt\,(m_2 a^2 \dot{\phi}) = 0 \qquad \dot{\phi} = \text{constant}$ 1

ⓑ No slip constraint? $v = a\dot{\phi} = \dot{x}_2 - \dot{x}_1$

2

or $\boxed{g = \dot{x}_2 - \dot{x}_1 - a\dot{\phi} = 0}$ or $f = x_2 - x_1 - a\phi = 0$

(c) Add Lagrange Mult $\underline{\lambda \frac{\partial g}{\partial \dot{q}_i}}$ on RHS of (a)

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$$m_1 \ddot{x}_1 - m_1 \sin\alpha\, g - k(x_2 - x_1) = -\lambda \qquad (1) \qquad 1$$

$$m_2 \ddot{x}_2 - m_2 \sin\alpha\, g + k(x_2 - x_1) = +\lambda \qquad (2) \qquad 1$$

$$m_2 a^2 \ddot{\phi} = -\lambda a \qquad (3) \qquad 1$$

constraint gives $a\ddot{\phi} = \ddot{x}_2 - \ddot{x}_1$ (4)

So
$$\ddot{x}_1 - g\sin\alpha - \frac{k}{m_1}(x_2 - x_1) = -\frac{\lambda}{m_1}$$
$$\ddot{x}_2 - g\sin\alpha + \frac{k}{m_2}(x_2 - x_1) = \frac{\lambda}{m_2}$$
$$a\ddot{\phi} = -\frac{\lambda}{m_2}$$

plug into (4)

3

$$\underbrace{\ddot{x}_2 - \ddot{x}_1}_{a\ddot{\phi} = -\frac{\lambda}{m_2}} + k\left(\frac{1}{m_2} + \frac{1}{m_1}\right)(x_2 - x_1) = \lambda\left(\frac{1}{m_2} + \frac{1}{m_1}\right) \qquad (*)$$

$$\frac{k}{\mu}(x_2 - x_1) = \frac{\lambda}{\mu} + \frac{\lambda}{m_2} = \lambda\left(\frac{m_2 + \mu}{m_2 \mu}\right)$$

$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$\boxed{\lambda = \frac{m_2}{m_2 + \mu} k (x_2 - x_1)} \qquad 3$$

or
$$\lambda = \frac{\cancel{m_2}}{\cancel{m_2} + \frac{\cancel{m_2} m_1}{m_1 + m_2}} k (x_2 - x_1) \qquad \boxed{\lambda = \frac{m_1 + m_2}{m_2 + 2m_1} k (x_2 - x_1)}$$

(d) Freq. of oscillation $\Delta x = x_2 - x_1$

6

$$\ddot{x}_2 - \ddot{x}_1 + \frac{k}{\mu}(x_2 - x_1) = \frac{\lambda}{\mu} = \frac{m_2 k}{\mu(m_2 + \mu)}(x_2 - x_1) \qquad (*)$$

3

$$\ddot{x}_2 - \ddot{x}_1 + \frac{k}{\mu}\underbrace{\left(1 - \frac{m_2}{m_2 + \mu}\right)}_{\to\ \frac{\mu}{m_2 + \mu}}(x_2 - x_1) = 0$$

$$\boxed{\ddot{x}_2 - \ddot{x}_1 + \frac{k}{m_2 + \mu}(x_2 - x_1) = 0} \qquad 3$$

frequency $\widetilde{}$

$$\boxed{\Omega = \sqrt{\frac{k}{m_2+\mu}} \overset{\text{OR}}{=} \sqrt{\frac{k\,(m_1+m_2)}{m_2\,(2m_1+m_2)}}}$$

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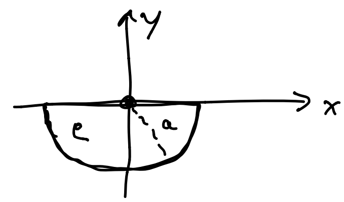
3. Half Disk $\boxed{30 \text{ total}}$ density ρ



ⓐ Mass = ? $\quad \frac{1}{2}\pi a^2 \rho = M$

$\boxed{2}$

ⓑ $\quad \hat{I}$ about x, y, z axes shown

$\boxed{8}$

$z = 0$ so $I_{xz} = I_{yz} = 0$

symmetric under $x \to -x$ so $I_{xy} = 0$ $\quad 2$

$$x^2+y^2 = r^2, \quad x^2+z^2 = r^2\cos^2\theta, \quad y^2+z^2 = r^2\sin^2\theta$$

$$I_{zz} = \rho \int_0^a r\,dr \int_\pi^{2\pi} d\theta\, r^2 = \rho \frac{a^4}{4}\pi = \frac{Ma^2}{2} \quad 2$$

$$I_{xx} = \rho \int_0^a r\,dr \int_\pi^{2\pi} d\theta\, r^2\cos^2\theta = \rho \frac{a^4}{4}\frac{\pi}{2} = \frac{Ma^2}{4} \quad 2$$

$$I_{yy} = \rho \int_0^a r\,dr \int_\pi^{2\pi} d\theta\, r^2\sin^2\theta = Ma^2/4 \quad \text{too} \quad 2$$

$$\boxed{I^{(Q)} = \frac{Ma^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

Q = origin of x, y, z axes

ⓒ

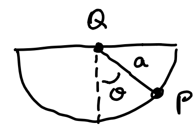
$\boxed{10}$ Need to use // Axes Theorem twice, relate through $I^{(cm)}$

First About cm point $\vec{R} = -d\,\hat{y}$

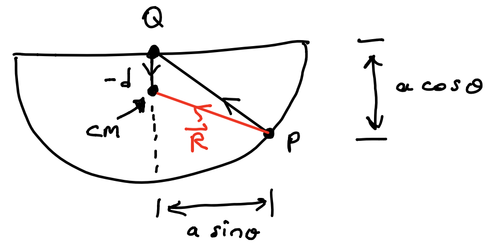
$$I^{(cm)} = I^{(Q)} - M(\vec{R}^2\delta^{ab} - R^a R^b) = I^{(Q)} - M\begin{pmatrix} d^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d^2 \end{pmatrix}$$

$$= M\begin{pmatrix} \frac{a^2}{4} - d^2 & 0 & 0 \\ 0 & \frac{a^2}{4} & 0 \\ 0 & 0 & \frac{a^2}{2} - d^2 \end{pmatrix}$$

Second About point on edge of disk



$$I^{(P)} = I^{(cm)} + M\left(\vec{R}^2 \delta^{ab} - R^a R^b\right)$$



$$\vec{R} = -d\hat{y} - a\sin\theta\,\hat{x} + a\cos\theta\,\hat{y}$$

$$\vec{R}^2 = a^2\sin^2\theta + (a\cos\theta - d)^2$$

$$M\left(\vec{R}^2\delta^{ab} - R^a R^b\right) = M\begin{pmatrix} (a\cos\theta - d)^2 & -(a\cos\theta - d)\,a\sin\theta & 0 \\ -(a\cos\theta - d)\,a\sin\theta & a^2\sin^2\theta & 0 \\ 0 & 0 & a^2\sin^2\theta + (a\cos\theta - d)^2 \end{pmatrix}$$

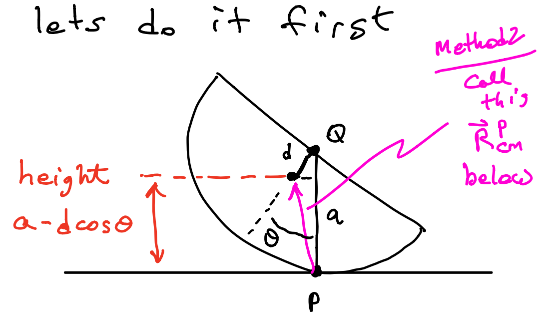
$$\boxed{I^{(P)} = M\begin{pmatrix} \frac{a^2}{4} - d^2 + (a\cos\theta - d)^2 & -(a\cos\theta - d)\,a\sin\theta & 0 \\ -(a\cos\theta - d)\,a\sin\theta & \frac{a^2}{4} + a^2\sin^2\theta & 0 \\ 0 & 0 & \frac{a^2}{2} - d^2 + a^2\sin^2\theta + (a\cos\theta - d)^2 \end{pmatrix}}$$

(d) 6

$L = T - V$, V is easy, so lets do it first

$\dot{\theta}$

$$V = mg\,(a - d\cos\theta)$$ 2.



Method 1 for T: contact point P is instantaneously at rest, rotates about z-axis going through P $\equiv \hat{z}'$

(easier)

Angular Velocity $\quad \omega = \dot{\theta}\,\hat{z}'$ 4.

KE about Point P:

$$T = \frac{m}{2}\dot{\theta}^2\left[\frac{a^2}{2} - d^2 + a^2\sin^2\theta + (a\cos\theta - d)^2\right] = \frac{m}{2}\dot{\theta}^2\left[\frac{3}{2}a^2 - 2da\cos\theta\right]$$

either fine, no need to simplify

OR

Method 2 for T: (harder) $T = \frac{1}{2}M\vec{v}_{cm}^{\,2} + \frac{1}{2}\vec{\omega}^T \cdot I^{(cm)}\vec{\omega}^T$, $\vec{\omega} = \dot{\theta}\hat{z}$

this is more complicated because we need $\vec{v}_{cm}$ relative to some inertial axes. Find $\vec{x}_{cm}$ relative to x_I-y_I axes fixed by axes at P for $\theta = 0$.

From above figure, the point P is displaced by $a\theta\hat{x}_I$, & hence $\vec{x}_{cm} = a\theta\hat{x}_I + \vec{R}^P_{cm}$ for the inertial frame

$$= (a\theta - d\sin\theta)\hat{x}_I + (a - d\cos\theta)\hat{y}_I$$

$$\vec{v}_{cm} = \dot{\vec{x}}_{cm} = (a - d\cos\theta)\dot{\theta}\hat{x}_I + d\sin\theta\,\dot{\theta}\hat{y}_I$$

$$\vec{v}_{cm}^{\,2} = \dot{\theta}^2\left[(a - d\cos\theta)^2 + d^2\sin^2\theta\right] = \dot{\theta}^2\left[a^2 + d^2 - 2ad\cos\theta\right]$$

Here

$$T = \frac{M}{2}\dot{\theta}^2\left[\left(a^2 + d^2 - 2ad\cos\theta\right) + \left(\frac{a^2}{2} - d^2\right)\right] = \frac{M}{2}\dot{\theta}^2\left[\frac{3a^2}{2} - 2ad\cos\theta\right]$$

same

(e) Yes/No suffices

4 pts

- Would your answers in (a) change if you half the density of the disk? Yes (doubles M) 1 pt
- How about (1)? No (written in terms of doubled M) results are the same 1 pt

  Yes answer accepted if explained rel to original M
- Would answers in (b) change if you attached another ½ disk of density $\rho/2$ in the upper half plane to create a full disk? No 2 pts

$\hat{I}$ linear in Mass & this is superposition of original & (b) answer masses simply add to new total mass, same matrix

4. 30 total

$$H = \frac{q^4 p^2}{2\mu} + \frac{\lambda}{q^2} \qquad \mu > 0, \ \lambda > 0$$

(a) H-Eqtns?

2

$$\boxed{\dot{q} = \frac{\partial H}{\partial p} = \frac{q^4 p}{\mu}}, \qquad \boxed{\dot{p} = -\frac{\partial H}{\partial q} = -\frac{2q^3 p^2}{\mu} + \frac{2\lambda}{q^3}}$$

1   1

(b) Consider $Q = \gamma q^a$, $P = q^b p$ a, b, γ are constants.

5

Canonical: $[Q, P] = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - 0 = a\gamma q^{a-1} q^b = 1$

$\therefore$ $\boxed{a\gamma = 1}$ $\boxed{a + b = 1}$

$$Q = \frac{1}{a} q^a$$
$$P = q^{1-a} p$$

(c) Use this transformation to transform H to a H. Osc.

10

$a = -1$ $Q = -1/q$, $P = q^2 p$

$$K(Q, P) = \frac{P^2}{2\mu} + \lambda Q^2$$ 4.

$$\dot{P} = -\frac{\partial H}{\partial Q} = -2\lambda Q, \quad \dot{Q} = +\frac{\partial H}{\partial P} = \frac{P}{\mu}, \quad \ddot{Q} = -\frac{2\lambda}{\mu} Q$$ 3.

$$Q(t) = A \sin\left(\sqrt{\frac{2\lambda}{\mu}}\, t + \phi\right), \quad P(t) = \mu \sqrt{\frac{2\lambda}{\mu}}\, A \cos\left(\sqrt{\frac{2\lambda}{\mu}}\, t + \phi\right)$$

$$q(t) = -\frac{1}{A} \csc\left(\sqrt{\frac{2\lambda}{\mu}}\, t + \phi\right)$$
$$p(t) = \mu \sqrt{\frac{2\lambda}{\mu}}\, A^3 \cos\left(\sqrt{\frac{2\lambda}{\mu}}\, t + \phi\right) \sin^2\left(\sqrt{\frac{2\lambda}{\mu}}\, t + \phi\right)$$ 3.

(d) H-J eqtn for original H. $p = \frac{\partial W}{\partial q}$

$\boxed{5}$

2 $$\boxed{\frac{q^4}{2\mu}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{\lambda}{q^2} = \alpha} > 0 \qquad \left(\frac{\partial W}{\partial q}\right)^2 = \frac{2\mu}{q^4}\left[\alpha - \frac{\lambda}{q^2}\right]$$

3. $$\boxed{W = \pm \int dq \sqrt{\frac{2\mu}{q^4}\left(\alpha - \frac{\lambda}{q^2}\right)}}$$

(e) $\boxed{8}$

$$\beta + t = \frac{\partial W}{\partial \alpha} = \pm \int dq \, \frac{\mu}{q^4} \frac{1}{\sqrt{\frac{2\mu}{q^4}(\alpha - \lambda/q^2)}} \qquad \$ \quad P = \alpha$$

$$= \pm \sqrt{\frac{\mu}{2}} \int dq \frac{1}{(\alpha q^4 - \lambda q^2)^{1/2}} = \pm \sqrt{\frac{\mu}{2\alpha}} \int \frac{dq}{q\sqrt{q^2 - \frac{\lambda}{\alpha}}}$$

form given
$z = q$
$z_0^2 = \frac{\lambda}{\alpha}$
$z_0 = \sqrt{\frac{\lambda}{\alpha}}$

$$= \pm \sqrt{\frac{\mu}{2\cancel{\alpha}}}\left(-\sqrt{\frac{\cancel{\alpha}}{\lambda}}\right) \sin^{-1}\left(\frac{\sqrt{\lambda/\alpha}}{q}\right)$$

$$= \mp \sqrt{\frac{\mu}{2\lambda}} \sin^{-1}\left(\frac{\sqrt{\lambda/\alpha}}{q}\right)$$ drop $\mp$ by shifting β

$$q = \sqrt{\frac{\lambda}{\alpha}} \csc\left(\sqrt{\frac{2\lambda}{\mu}}(t+\beta)\right) = \sqrt{\frac{\lambda}{\alpha}}\left[\sin\left(\sqrt{\frac{2\lambda}{\mu}}(t+\beta)\right)\right]^{-1}$$ ✓

$\boxed{\text{NOT FOR CREDIT}}$ (but for completeness)

$$\csc^2(\theta) - 1 = \frac{1 - \sin^2}{\sin^2} = \frac{\cos^2}{\sin^2}$$

$$p = \frac{\partial W}{\partial q} = \pm \sqrt{\frac{\mu}{2\alpha}} \frac{1}{q} \frac{1}{\sqrt{q^2 - \lambda/\alpha}}$$

$$= \pm \sqrt{\frac{\mu}{2\alpha}} \sqrt{\frac{\alpha}{\lambda}} \sin^2\left(\sqrt{\frac{2\lambda}{\mu}}(t+\beta)\right) \cos\left(\sqrt{\frac{2\lambda}{\mu}}(t+\beta)\right)$$ ✓