Physics 8.321, Fall 2020
Final Exam

You have **3 hours** to complete the exam and a grace period of an additional 20 minutes to get your solutions uploaded to canvas. You may use your books and notes including the notes on canvas from the course, you may freely use any results derived in homework assignments this semester, and you may use symbolic manipulation tools like mathematica and matlab, but you may not consult other online resources, and you may not communicate with other people in any way while doing the final. You also may not communicate any information about the exam to anyone after you have completed it until the exam period is over at the end of the day on 12/15/20.

Note: you are expected to upload your completed exam to the canvas website immediately after completing the exam and within 200 minutes or less of downloading it. You have a few extra minutes in case of technical complications. The system will log your download and upload times. If for some reason you have difficulty uploading your exam after completion, please email it immediately to one of the course staff.

1. Particle in a linear symmetric potential (40 points)

Consider a particle moving in one dimension in the potential $V = A|x|$, where A is a constant. You may fix $m = \hbar = 1$.

okay easy, just remember to normalize.

(a) Estimate the ground state energy using the variational principle for the family of Gaussian wavefunctions $e^{-ax^2/2}$ parameterized by a, suitably normalized.

(b) Estimate the ground state energy using the WKB approximation and compare with your result from part (a).

2. Rotating wave trick for a time-dependent Hamiltonian (30 points)

Consider the system with *time-dependent* Hamiltonian

$$H(t) = \frac{\hbar}{2}\begin{bmatrix} \omega & \Omega e^{-i\nu t} \\ \Omega e^{i\nu t} & -\omega \end{bmatrix} \quad \text{where} \quad \omega, \Omega, \nu \in \mathbb{R}. \tag{1}$$

(a) Consider the unitary operator

$$U_R(t) = \begin{bmatrix} e^{iat} & 0 \\ 0 & e^{-iat} \end{bmatrix}, \tag{2}$$

with $a \in \mathbb{R}$. Show that $U_R(t)H(t)U_R^\dagger(t)$ is time independent for suitable choice of a. Find a. Henceforth, we are going to use this a in $U_R(t)$.

(b) Let $|\psi(t)\rangle$ be the solution of the time-dependent Schrodinger equation for the Hamiltonian $H(t)$ above. That is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H(t)|\psi(t)\rangle. \tag{3}$$

Show that the state $U_R(t)|\psi(t)\rangle$ solves a Schrodinger equation with a *time-independent* Hamiltonian H_{eff}. Find H_{eff}.

(c) Use the results above to write a formal expression for $|\psi(t)\rangle$ for given initial state $|\psi(0)\rangle$ and in terms of H_{eff} and $U_R(t)$. You don't have to evaluate any matrix exponential explicitly.

(d) Now consider the situation $\omega = \nu$, and let the initial state be

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{4}$$

Find the time-evolved state $|\psi(t)\rangle$. Are there time(s) $t = t_c$ for which collapsing to the initial state after measurement of the operator

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5}$$

is certain? If so, when does that happen (i.e. find t_c)?

3. Quantum Rigid Rotor (30 points)

Consider a rigid rotor immersed in a magnetic field, with Hamiltonian

$$H = \frac{\mathbf{L}^2}{2I} + \omega_0 L_z. \tag{6}$$

Suppose the system is in an initial state $|\psi_0\rangle$ such that

$$\langle \theta, \phi | \psi_0 \rangle = \sqrt{3/4\pi} \sin\theta \sin\phi. \tag{7}$$

(a) What values of L_z will be obtained if a measurement is carried out in the initial state, and with what probability do these values occur?

(b) What is $\langle \theta, \phi | \psi(t) \rangle$?

(c) What are $\langle L_x(t) \rangle, \langle L_y(t) \rangle, \langle L_y(t)^2 \rangle, \langle \Delta L_y(t) \rangle$ in this state?

4. Interacting spins in a magnetic field (30 points)

Consider the Hamiltonian

$$H = H_0 + BH_1 = -J\, \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} - BS_z^{(1)}, \tag{8}$$

with $J > 0$.

(a) Solve for the spectrum exactly when $B = 0$, including multiplicities, when $S^{(1)}, S^{(2)}$ are each systems of general spin j_1, j_2.

(b) Find the corrections to the energy eigenvalues using perturbation theory at leading order in small B in the case $j_1 = j_2 = 1/2$.

(c) Solve exactly in the case $j_1 = j_2 = 1/2$ and check your result from part (b).

5. Three entangled quantum spins (30 points)

(a) Consider a system of three spin-1/2 particles in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+++\rangle - |---\rangle)\,. \tag{9}$$

Compute the entanglement entropy for a subsystem containing one or two of the spins. (Recall that the entanglement entropy of a subsystem is the entropy of the density matrix of the subsystem after tracing out the degrees of freedom of the complement of the subsystem, and quantifies the randomness/uncertainty in the subsystem taken in isolation.)

(b) Now consider a system of three spin-1/2 particles in the state

$$|\psi\rangle = \frac{1}{2}(|+++\rangle + |+-+\rangle - |-+-\rangle - |---\rangle)\,. \tag{10}$$

Again, compute the entanglement entropy for a subsystem containing each subset of one or two of the spins. Explain your answer in terms of how the density matrix for the different subsystems is described by mixed or pure states. [Hint: Try factorizing the state!]

6. Simple Harmonic Oscillator & Magnetic Field in 2D (40 points)

Consider a spinless particle of mass m and charge q moving in two dimensions, in the $xy-$plane. The particle feels a uniform magnetic field $\mathbf{B} = B\hat{z}$ and is also subject to a harmonic potential that is independent of x

$$V(x,y) = \frac{1}{2}m\omega^2 y^2. \tag{11}$$

Find the energy eigenvalues and corresponding energy eigenfunctions. You can express your answer in terms of ψ_n, $n = 0, 1, 2, \ldots$, which are the energy eigenfunctions corresponding to the one-dimensional simple harmonic oscillator. That is you don't have to derive the full explicit expression for all states, but other than that the answer should depend on the given parameters, fundamental constants, and/or quantum numbers needed to label the state. Check that your results for energy eigenvalues make sense in the limit of vanishing magnetic field $B \to 0$, and separately for the case without harmonic oscillator potential, i.e. $\omega \to 0$.

$V = A|x|$

$\psi \sim Ce^{-ax^2/2} \longrightarrow$ normalized

$p = -i\hbar\,\partial_x$

$= -i\,\partial_x$

$p^2 = -\partial_x^2$

$\mathcal{H} = p^2/2m + A|x|$

$= p^2 + A|x|$

$= -\partial_x^2 + A|x|$

$\partial_x\, e^{-ax^2/2}$

$= e^{-ax^2/2}\left[\partial_x \; -\frac{ax^2}{2}\right]$

$= e^{-ax^2/2}\left[-\frac{2a}{2}x\right]$

$= e^{-ax^2/2}\left[-ax\right]$

$\partial_x^2\, e^{-ax^2/2} = \ldots$

Then $\langle \mathcal{H} \rangle = \int \psi^* [\mathcal{H}]\, \psi\, dx =$ something

then minimize . . .

WKB

$$\pm\int \sqrt{2m(E_n - V)}\,dx = \pi\hbar\left(\cancel{n} + \frac{1}{2}\right)$$

$n \to 0$

$A(x)$

$\to$ get $\mathcal{E}_n$ for $n = 0$.

$$\mathcal{H} = \frac{\hbar}{2}\begin{pmatrix} w & \Omega e^{-i\nu t} \\ \Omega e^{i\nu t} & -w \end{pmatrix}$$

$$\mathcal{U}_R(t) = \begin{pmatrix} e^{iat} & 0 \\ 0 & e^{-iat} \end{pmatrix}$$

$$\mathcal{U}_R \mathcal{H} \mathcal{U}_R^{\dagger} = \frac{\hbar}{2}\begin{pmatrix} e^{i(at - \bar{a}t)}\omega & \Omega e^{-i\nu t + 2i\,\mathrm{Re}(at)} \\ \Omega e^{i(\nu t - 2\mathrm{Re}(at))} & -w e^{-i(at - \bar{a}t)} \end{pmatrix}$$

Well ... $a \in \mathbb{R}$, and $-i\nu t + 2i\,\mathrm{Re}(at) = 0$

$\nu = 2a \Rightarrow \boxed{a = \frac{\nu}{2}}$ $\Rightarrow$ $\cancel{-i}\nu\cancel{t} = -2\cancel{i}\,a\cancel{t}$

$$\boxed{a = \nu/2}$$

$$i\hbar\, \partial_t |\psi(t)\rangle = \mathcal{H}(t)|\psi(t)\rangle$$

$$i\hbar\, \partial_t \left[\mathcal{U}_R |\psi(t)\rangle \right] = \mathcal{H}_{eff}\, \mathcal{U}_R |\psi(t)\rangle$$

what is $\mathcal{H}_{eff}$?

$$\left[i\hbar\, \partial_t \mathcal{U}_R \right] |\psi(t)\rangle + i\hbar\, \mathcal{U}_R \left[\partial_t |\psi(t)\rangle \right] = \mathcal{H}_{eff}\, \mathcal{U}_R |\psi(t)\rangle$$

$$\mathcal{U}_R\, \mathcal{H}(t) |\psi(t)\rangle$$

$$\frac{\hbar}{2}\begin{pmatrix} w & \Omega \\ \Omega & -w \end{pmatrix}$$

$$\frac{\hbar}{2}\begin{pmatrix} -\nu & 0 \\ 0 & \nu \end{pmatrix}$$

$$(i\hbar\, \partial_t \mathcal{U}_R)\mathcal{U}_R^\dagger + \mathcal{U}_R \mathcal{H}(t) \mathcal{U}_R^\dagger = \mathcal{H}_{eff}\, \mathcal{U}_R \mathcal{U}_R^\dagger$$

$$\rightarrow \quad \mathcal{H}_{eff} = \mathcal{U}_R \mathcal{H}(t) \mathcal{U}_R^\dagger + (i\hbar\, \partial_t \mathcal{U}_R)\mathcal{U}_R^\dagger$$

So

$$\boxed{\mathcal{H}_{eff} = \frac{\hbar}{2}\begin{bmatrix} \omega - \nu & \Omega \\ \Omega & \nu - \omega \end{bmatrix}}$$

$|\psi(t)\rangle$ in terms of $|\psi(0)\rangle$, $\mathcal{H}_{eff}$, $\mathcal{U}_R(t)$

well

$$i\hbar\,\partial_t\,|U_R(t)\psi(t)\rangle$$

$$= \mathcal{H}_{eff}\,|\mathcal{U}_R(t)|\psi(t)\rangle\rangle$$

$$\Rightarrow \quad \mathcal{U}_R(t)|\psi(t)\rangle = e^{-i\mathcal{H}_{eff}(t)\,t/\hbar}\,\underbrace{\mathcal{U}_R(0)}_{id}|\psi(0)\rangle$$

$$\boxed{|\psi(t)\rangle = \mathcal{U}_R^{\dagger}(t)\,e^{-i\mathcal{H}_{eff}(t)\,t/\hbar}\,|\psi(0)\rangle}$$

$\mathcal{U}_R$ = something known.

$\omega = \nu \ldots \delta_0$

$$\mathcal{H}_{eff} = \frac{\hbar}{2}\begin{pmatrix} 0 & \Omega \\ \Omega & 0 \end{pmatrix} = \frac{\hbar\Omega}{2}\sigma_x$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$U_R = \begin{pmatrix} e^{i\nu t/2} & 0 \\ 0 & e^{-i\nu t/2} \end{pmatrix}$$

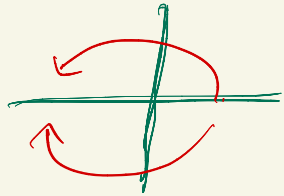
$$|\psi(t)\rangle = U_R^{\dagger} \exp\left[-i\frac{\mathcal{H}_{eff}\, t}{\hbar}\right]\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\nu t/2} & 0 \\ 0 & e^{i\nu t/2} \end{pmatrix} \exp\left[-\frac{i\Omega t}{2}\sigma_x\right]|\psi(0)\rangle$$

$$= \begin{pmatrix} e^{-i\nu t/2} & 0 \\ 0 & e^{i\nu t/2} \end{pmatrix} \exp\left(\frac{-i\Omega t}{2}\right)|\psi(0)\rangle$$

$$= \frac{1}{\sqrt{2}} e^{-i\Omega t/2}\begin{pmatrix} e^{-i\nu t/2} \\ e^{+i\nu t/2} \end{pmatrix}$$

Back to initial state ...



only if

$$e^{-i\nu t/2} = e^{+i\nu t/2}$$

$$e^{ix} = e^{-ix}$$

$$\pi n$$

$$\Leftrightarrow \quad \nu t/2 = \pi n$$

$$\nu t = 2\pi n$$

$$\boxed{t_c = \frac{2\pi n}{\nu}}$$

Quantum Rigid Rotor

$$\mathcal{H} = \frac{L^2}{2I} + \omega_0 L_z$$

$$\langle \theta, \phi | \psi_0 \rangle = \sqrt{3/4\pi}\, \sin\theta \sin\phi$$

what is L_z ?

need to write $|\psi_0\rangle$ in terms of spherical harmonics $\longrightarrow$ to find l, m

$$\langle \theta, \phi | \psi_0 \rangle = \sqrt{3/4\pi}\, \sin\theta \sin\phi$$

$$= \sqrt{\frac{3}{4\pi}}\, \sin\theta\, \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

$$= \frac{1}{\sqrt{2}} \sqrt{\frac{3}{2\pi}}\, \sin\theta\, \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

$$= -\frac{1}{\sqrt{2}\, i} \left\{ -\frac{1}{2} \sqrt{\frac{3}{2\pi}}\, \sin\theta\, e^{i\varphi} + \frac{1}{2} \sqrt{\frac{3}{2\pi}}\, \sin\theta\, e^{-i\varphi} \right\}$$

$$= \frac{-1}{\sqrt{2}\,i}\left\{ Y_1^1 + Y_1^{-1} \right\}$$

So
$$\langle \theta, \phi | \psi_0 \rangle = \frac{-1}{\sqrt{2}} \left\{ |\overset{l}{1}, \overset{m}{1}\rangle + |\overset{l}{1}, \overset{m}{-1}\rangle \right\}$$

So $L_z |l, m\rangle = m\hbar\, |l, m\rangle$

gives
$$\boxed{\begin{array}{ll} \hbar \to \text{probability } 50\% \\ -\hbar \to \text{probability } 50\% \end{array}}$$

---

what is $\langle \theta, \phi | \psi(t) \rangle$?

Well

$$|\psi(t)\rangle = e^{-i\mathcal{H}t/\hbar}\, |\psi(0)\rangle$$

Can think of initial state as

$$\frac{1}{\sqrt{2}\, i}\left\{ |1,1\rangle + |1,-1\rangle \right\}$$

now evolve..

$$e^{-i\mathcal{H}t/\hbar}\, |1,1\rangle$$

$$\frac{L^2}{2I} + \omega_0 L_z$$

$$\ell(\ell+1)\hbar^2$$

$$m\hbar$$

since $L^2 \leftrightarrow L_z$, can do this

$$\underbrace{e^{-i\frac{L^2}{2I}t/\hbar}}\ \underbrace{e^{-i\omega_0 L_z t/\hbar}}\ |\ell, m\rangle$$

$$e^{-i\, c^2/2I\, t/\hbar}\ e^{-i\omega_0 t/\hbar\,(m\hbar)}\ |\ell, m\rangle$$

$$e^{-i/2I\, t/\hbar\, \ell(\ell+1)\hbar^2}\ e^{-i\omega_0 t/\cancel{\hbar}\,(m\cancel{\hbar})}\ |\ell, m\rangle$$

So ... get

$$e^{-iHt/\hbar}|\ell, m\rangle = e^{\frac{-it}{2I}\ell(\ell+1)\hbar}\, e^{-i\omega_0 t m}|\ell, m\rangle$$

get

$\ell=1, m=1$ $\ell=1, m=-1$

$$\langle\theta,\phi|\psi(t)\rangle = \frac{-1}{\sqrt{2}\,i}\left\{e^{\frac{-it}{2I}2\hbar}\right\}\left\{e^{-i\omega_0 t}|1,1\rangle + e^{+i\omega_0 t}|1,-1\rangle\right\}$$

$$= \frac{-1}{\sqrt{2}\,i}e^{\frac{-it\hbar}{I}}\left\{e^{-i\omega_0 t}|1,1\rangle + e^{+i\omega_0 t}|1,-1\rangle\right\}$$

convert this back to wfn...

$$= \frac{-1}{\sqrt{2}\,i}e^{\frac{-it\hbar}{I}}\left\{e^{-i\omega_0 t}Y_1^1 + e^{+i\omega_0 t}Y_1^{-1}\right\}$$

Done

(c) Call $\langle \theta, \phi | \psi(t) \rangle = \psi(\theta, \phi, t)$

$= \psi_{\theta\phi}(t)$

guess by symmetry

$$\boxed{\langle L_x(t) \rangle = 0 = \langle L_y(t) \rangle}$$

$L_x^2 + L_y^2 + L_z^2 = L^2$

- $\langle L_y^2(t) \rangle$

$= \langle L^2 - L_z^2 \rangle$

Complicated

$= \frac{\langle L^2 \rangle - \langle L_z^2 \rangle}{2}$

$= \frac{1}{2}\left[2\hbar \right.$

$\psi(t)$ in but 's

orthogonal

$|\psi(t)\rangle = \frac{-1}{\sqrt{2}} e^{\frac{it\hbar}{I}} \left\{ e^{-i\omega_0 t} |1,1\rangle + e^{+i\omega_0 t} |1,-1\rangle \right\}$

$\langle L^2 \rangle = \frac{1}{2}\left\{ \langle 1, \quad ^2 | 1,1 \rangle + \langle 1,-1 | L^2 | 1,-1 \rangle \right\}$

$= \frac{1}{\cancel{2}} \cdot \cancel{2}\, \ell \underbrace{}_{2}$ $\quad -^2($

Best method is to brute force

→ make a Mathematica notebook to handle this.

Define $L_x = \ldots$

$L_y = \ldots$

have $\psi_{\phi,\theta}(t) = \ldots$

Then just brute force ...

Do this in Mathematica

↳ code already made!

$$\mathcal{H} = \mathcal{H}_0 + B\mathcal{H}$$

$$= -J\,\vec{S}^{(1)}\cdot\vec{S}^{(2)} - B\,S_z^{(1)}$$

formal spin $\vec{j}$

$B = 0$. Addition of angular momentum.

$$S_j = S_j^{(1)} + S_j^{(2)}$$

$$\vec{S}^2 = \vec{S}^{(1)2} + \vec{S}^{(2)2} + 2\,\vec{S}^{(1)}\cdot\vec{S}^{(2)}$$

$$\Rightarrow \mathcal{H} = -\frac{J}{2}\left[\vec{S}^2 - \vec{S}^{(1)2} - \vec{S}^{(2)2}\right]$$

$$|j_1, m_1, j_2, m_2\rangle \rightsquigarrow |j, m_j\rangle$$

$$= -\frac{J}{2}\left[\vec{S}^2\right] + \frac{J}{2}\vec{S}^{(1)2} + \frac{J}{2}\vec{S}^{(2)2}$$

$$= \frac{J}{2}\hbar^2\left[j(j+1) - j_1(j_1+1) - j_2(j_2+1)\right]$$

<u>multiplicity</u>: let j_1, j_2 be fixed, then

look at $|j, m\rangle$

$\boxed{|j_1 - j_2| < j < j_1 + j_2}$ has $\boxed{2j+1}$ choices for fixed j

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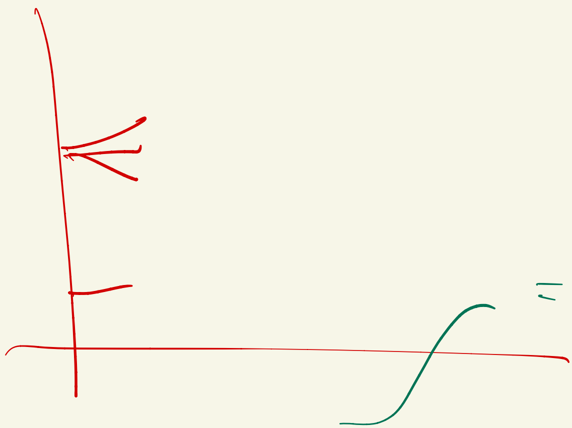
<u>Suppose $j_1 = j_2 = 1/2$ then back</u> to <u>familiar territory</u>

$\mathcal{H} =$ small B ...

Small B ... $|j, m\rangle$ good basis (is?) ...

$|1,0\rangle$, $|1,1\rangle$, $|1,-1\rangle$

$|0,0\rangle$



$= \frac{1}{\sqrt{2}}\left[|-+\rangle + |+-\rangle\right]$

$$|0,0\rangle = \left(|\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, \tfrac{1}{2}\rangle - |\tfrac{1}{2}, \tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle\right)\frac{1}{\sqrt{2}}$$

$$\Delta E_{00} = B\langle 0,0|\, S_z^{(1)}\, |0,0\rangle$$

$$= B\frac{\hbar}{2}\langle 0,0|\sigma_z^{(1)}|0,0\rangle$$

$\uparrow$

can we undergrad

pert th

=

$$= -\frac{B\hbar}{2}\,\frac{1}{\sqrt{2}}\left(\cancel{\langle +-|} \cancel{+} \cancel{\langle -+|}\right)\sigma_z^{(1)}\frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right)$$

$$= -\frac{B\hbar}{2}\cdot\frac{1}{2}\left(\langle +-| - \langle -+|\right)\left(|+-\rangle + |-+\rangle\right)$$

$$= -\frac{B\hbar}{2}\,\frac{1}{2}\left[1 + 0 - 0 - 1\right]$$

$$= 0 \quad \checkmark$$

For the other ones... have to do degenerate pert. theory...

→ find matrix

$$V_{mm'} = \boxed{-\frac{\hbar B}{2}\langle 1, m|\,\sigma_z^{(1)}\,|1, m'\rangle}$$

Well

$$|1,1\rangle = |++\rangle \qquad (\text{annotation: } m)$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)$$

$$|1,-1\rangle = |--\rangle \qquad \begin{pmatrix} V_{11}\;0 & \cancel{V_{10}}\;0 & \cancel{V_{1-1}}\;0 \\ \cancel{V_{01}}\;0 & V_{00}\;0 & V_{0-1} \\ \cancel{V_{-11}}\;0 & V_{-10} & V_{-1-1} \end{pmatrix}$$

$$V_{11} = -\frac{\hbar B}{2}\langle ++|\,\sigma_z^{(1)}\,|++\rangle = -\hbar B/2$$

$$V_{10} = -\frac{\hbar B}{2}\langle ++|\,\sigma_z^{(1)}\left(|+-\rangle + |-+\rangle\right)\frac{1}{\sqrt{2}}$$

$$= -\frac{\hbar B}{2}\langle ++|\left[\,|+-\rangle - |-+\rangle\,\right]\frac{1}{\sqrt{2}}$$

$$= 0$$

$$V_{1-1} = -\frac{\hbar B}{2}\langle ++|\,\sigma_z^{(1)}\,|--\rangle = 0$$

$$V_{01} = V_{10} = 0 \;;$$

$$V_{00} = -\frac{\hbar B}{2}\,\frac{1}{2}\left(\langle +-| + \langle -+|\right)\left(|+-\rangle - |-+\rangle\right) = 0$$

$$V_{0,-1} = -\frac{\hbar B}{2}\frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle\right)\sigma_z^{(1)}|--\rangle$$

$$= 0$$

$$V_{-1,-1} = -\frac{\hbar B}{2}\langle --|\sigma_z^{(1)}|--\rangle$$

$$= \frac{\hbar B}{2}$$

<u>So</u> $V = \begin{pmatrix} -\hbar B/2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \hbar B/2 \end{pmatrix}$

No need to rediagonalize ---

$\Rightarrow$ get $\Delta E_{11} = -\hbar B/2$

$\Delta E_{-1,-1} = \hbar B/2$

So eigenvalue ---

$$-\frac{J}{2}\hbar^2\Big[\overset{1,\,0}{j(j+1)} - \overset{1/2}{j_1(j_1+1)} - j_2(j_2+1)\Big]$$

(1) $$= -\frac{J}{2}\hbar^2\Big[\underbrace{2 - \frac{1}{2}\,\frac{3}{2} - \frac{1}{2}\,\frac{3}{2}}_{2 - 3/2 \,=\, \frac{1}{2}}\Big]$$

$$= -\frac{J}{2}\hbar^2 \cdot \frac{1}{2}$$

$$= -\frac{J}{4}\hbar^2$$

(0) $$= -\frac{J}{2}\hbar^2\Big[0 - \frac{3}{2}\Big] = \frac{3}{4}J\hbar^2$$

$|11\rangle$:

$$E_{11} \simeq -\frac{J}{4}\hbar^2 - \hbar B/2$$

$$E_{10} \simeq -\frac{J}{4}\hbar^2$$

$$E_{1-1} \simeq -\frac{J}{4}\hbar^2 + \hbar B/2$$

$$E_{00} \simeq \frac{3}{4}J\hbar^2$$

To solve exactly ...

many ways to do this --

most straight fwd is to write the full Hamiltonian in the

$|j_1, m_1, j_2, m_2\rangle$ basis

$|\frac{1}{2}, m_1, \frac{1}{2}, m_2\rangle$ basis ...

So write

$$\vec{S}^{(1)} \cdot \vec{S}^{(2)} = \sigma_x^1 \sigma_x^2 + \sigma_y^1 \sigma_y^2 + \sigma_z^1 \sigma_z^2$$

get this matrix

then write $S_z^{(1)} = \sigma_z^{(1)} \otimes \mathbb{1}$.

get this matrix

add, then diagonalize to get eigenvalues ...

This takes a few seconds ...

get eigenvalues ...

$$-\frac{1}{4}\hbar(2B+\hbar J) \longrightarrow -\frac{\hbar^2 J}{4} - \frac{\hbar B}{2}$$

$$-\frac{1}{4}\hbar(-2B+\hbar J) \longrightarrow -\frac{\hbar^2 J}{4} + \frac{\hbar B}{2}$$

$$\left.\begin{array}{l} \frac{1}{4}\hbar\left(\hbar J - 2\sqrt{B^2+\hbar^2J^2}\right) \longrightarrow -\frac{\hbar^2 J}{4} \\ \frac{1}{4}\hbar\left(\hbar J + 2\sqrt{B^2+\hbar^2J^2}\right) \end{array}\right.$$

$$\downarrow$$

$$3\frac{\hbar^2 J}{4}$$

$\downarrow$

small B

$\downarrow$

ignore ✓

SHO in B field

magnetic field $\vec{B} = B\hat{z}$

$$V(x,y) = \frac{1}{2} m\omega^2 y^2$$

Find spectrum & eigenfns.

Well...

$$\mathcal{H} = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 + \frac{1}{2} m\omega^2 y^2$$

tactic... ignore the $\frac{1}{2} m\omega^2 y^2$ term

to get $\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2$

Pick gauge...

$\vec{A} = (0, xB, 0)$ so that $\vec{\nabla}\times\vec{A} = \vec{B}$.

$$\mathcal{H} = \frac{p_x^2}{2m} + \frac{mq^2B^2}{2m^2c^2}\left(\frac{c}{qB}\hat{p}_y - \hat{x}\right)^2 + \frac{1}{2}m\omega^2 y^2$$

or <u>different gauge</u> $\vec{A} = (-yB, 0, 0)$

$$= \frac{p_y^2}{2m} + \frac{mq^2B^2}{2m^2c^2}\left(\frac{c}{qB}\hat{p}_x \cancel{+} - \hat{y}\right)^2 + \frac{1}{2}m\omega^2 y^2$$

notice that $\hat{x}$ cyclic $\rightarrow$ $\hat{p}_x$ conserved

$\rightarrow$ $\hat{p}_x$ commutes with $\mathcal{H}$

$\rightarrow$ replace by $\hbar k_x$

$$\mathcal{H} = \frac{p_y^2}{2m} + \overbrace{\frac{mq^2B^2}{2m^2c^2}}^{b}\left(\overbrace{\frac{c\hbar k_x}{qB}}^{a_x} - \hat{y}\right)^2 + \frac{1}{2}m\omega^2 y^2$$

now complete the squares...

$$\mathcal{H} = \frac{p_y^2}{2m} + b(a+y)^2 + \frac{1}{2}m\omega^2 y^2$$

$$= \frac{p_y^2}{2m} + b(a^2 + 2ay + y^2) + \frac{1}{2}m\omega^2 y^2$$

$$= \frac{p_y^2}{2m} + y^2\left(\frac{1}{2}m\omega^2 + b\right)$$

$$+ 2aby + ba^2$$

$$= \frac{p_y^2}{2m} + \underbrace{\left(\frac{1}{2}m\omega^2 + b\right)}_{c}\left\{y^2 + 2y\frac{ab}{c} + \frac{ba^2}{c}\right\}$$

$$= \frac{p_y^2}{2m} + c\left\{\left(y^2 + 2y\frac{ab}{c} + \frac{a^2b^2}{c^2}\right) - \frac{a^2b^2}{c^2} + \frac{ba^2}{c}\right\}$$

$$= \frac{p_y^2}{2m} + c\left\{y + \frac{ab}{c}\right\}^2 + c\left\{-\frac{a^2b^2}{c^2} + \frac{ba^2}{c}\right\}$$

$$= \frac{p_y^2}{2m} + \left(\frac{1}{2} m w^2 + b\right)\left(y - \frac{ab}{\frac{1}{2} m w^2 + b}\right)^2$$

$$+ \underbrace{\left(- \frac{a^2 b^2}{c} + a^2 b\right)}_{\text{constants}}$$

$$a = \frac{c \hbar k_x}{q B} \qquad {}^{qB}/_{mc} = w_c$$

$$b = \frac{m q^2 B^2}{2 m^2 c^2} = \frac{1}{2} m w_c^2$$

$$\Rightarrow \quad \frac{1}{2} m w^2 + b = \frac{1}{2} m \left(w^2 + w_c^2\right) = C$$

$$ab = \frac{c \hbar k_x \, m}{m q B} \cdot \frac{1}{2} m w_c^2 = \frac{\hbar k_x}{m} \cdot \frac{1}{w_c} \cdot \frac{1}{2} m w_c^2$$

ok retry ...

$$\mathcal{H} = \frac{p_y^2}{2m} + \frac{m}{2}\omega_c^2\left(\frac{\boxed{\frac{mc}{qB}}^{\nearrow 1/\omega_c}\hbar k_x}{m} + y\right)^2 + \frac{1}{2}m\omega^2 y^2$$

$$= \frac{p_y^2}{2m} + \frac{1}{2}m\omega_c^2\left(y + \frac{\hbar k_x}{m\omega_c}\right)^2 + \frac{1}{2}m\omega^2 y^2$$

$$= \frac{p_y^2}{2m} + \frac{1}{2}m\left\{\omega_c^2 y^2 + 2y\frac{\hbar k_x}{m\omega_c}\omega_c^2 + \cancel{\omega_c}^2\frac{\hbar^2 k_x^2}{m^2\cancel{\omega_c}^2} + \omega^2 y^2\right\}$$

$$= \frac{p_y^2}{2m} + \frac{1}{2}m\left\{\overbrace{(\omega_c^2+\omega^2)}^{\Omega^2}y^2 + 2y\left(\frac{\hbar k_x\omega_c}{m\Omega^2}\right)\Omega^2 + \frac{\hbar^2 k_x^2\omega_c^2}{m^2\Omega^2} - \frac{\hbar^2 k_x^2\omega_c^2}{m^2\Omega^2} + \Omega^2\hbar^2 k_x^2/m^2\Omega^2\right\}$$

$$= \frac{p_y^2}{2m} + \frac{1}{2}m\Omega^2\left\{y^2 + 2y\frac{\hbar k_x\omega_c}{m\Omega^2} + \left(\frac{\hbar k_x\omega}{m\Omega^2}\right)^2 - \left(\frac{\hbar k_x\omega_c}{m\Omega^2}\right)^2 + \frac{\hbar^2 k_x^2}{m^2\Omega^2}\right\}$$

$$= \frac{p_y^2}{2m} + \frac{1}{2} m \Omega^2 x$$

$$\left\{ \left(y + \frac{\hbar k_x \omega_c}{m \Omega^2} \right)^2 - \frac{\hbar^2 k_x^2 \omega_c^2}{m^2 \Omega^4} + \frac{\hbar^2 k_x^2}{m^2 \Omega^2} \right\}$$

$$= \frac{p_y^2}{2m} + \frac{1}{2} m \Omega^2 \left\{ \left(y + \frac{k_x \hbar \omega_c}{m \Omega^2} \right)^2 \right\}$$

$$- \frac{1}{2} m \Omega^2 \cdot \frac{\hbar^2 k_x^2 \omega_c^2}{m^2 \Omega^4} = \frac{-\hbar^2 k^2}{2m} \frac{\omega_c^2}{\Omega^2}$$

$$+ \frac{1}{2} m \Omega^2 \frac{\hbar^2 k_x^2}{m^2 \Omega^2} = \frac{\hbar^2 k^2}{2m} \cdot 1$$

$$= \frac{p_y^2}{2m} + \frac{m}{2} (\omega^2 + \omega_c^2) \left\{ \overbrace{y + \frac{\hbar k_x \omega_c}{m \Omega^2}}^{y'} \right\}^2 + \frac{\hbar^2 k_x^2}{2m} \frac{\omega^2}{\Omega^2}$$

$$= \underbrace{\frac{p_y^2}{2m} + \frac{m}{2}\Omega^2 y'^2}_{\text{SHO}} + \underbrace{\pi_x}_{\text{constant}}$$

$$E_n = \hbar\Omega\left(n+\frac{1}{2}\right) + \frac{\hbar^2 k_x^2}{2m}\frac{\omega^2}{\omega^2+\omega_c^2}$$

$$\Omega = \sqrt{\omega^2+\omega_c^2}$$

<u>Eigenfns</u> $e^{ik_x x}\phi(y') \sim \psi(x,y)$

$$\boxed{\psi(x,y) \sim e^{ik_x x}\phi\left(y + \frac{\hbar k_x \omega_c}{(\omega_c^2+\omega^2)m}\right)}$$

harmonic oscillator eigenfunctions.

<u>check</u> $B=0 \Rightarrow \omega_c = 0 \Rightarrow \Omega = \omega$

$$\Rightarrow \psi(x,y) \sim e^{ik_x x}\phi(y)$$

$$E_n = \hbar\omega(n+1/2) + \hbar^2 k^2/2m$$

Check $w = 0 \Rightarrow$ get Landau levels like before