8.311 Electromagnetic Theory

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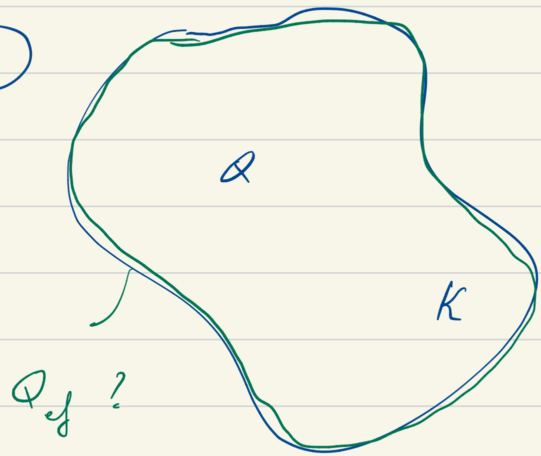


What is Q_{ef}?

Gauss law

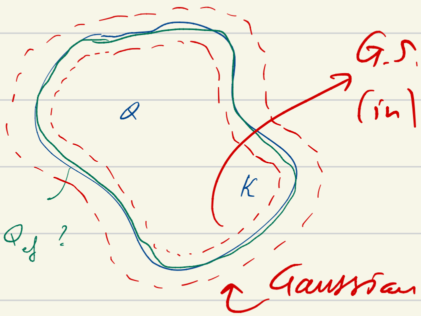
$$\vec{\nabla} \times \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{D} = 0$$

- Q_{ef} causes a discontinuity in E_n ...
- So we have $\dfrac{Q_{ef}}{\varepsilon_0} = \oint_{out} E_n \, dA - \oint_{in} E_n \, dA$

- Now note that $\oint_{out} E_n \, dA = \dfrac{Q}{\varepsilon_0}$

usual Gauss law stuff



- For inside ... $\vec{D} = \varepsilon \vec{E} = \varepsilon_0 K \vec{E}$

So
$$\oint_{in} E_n \, dA = \oint_{in} \frac{D_n}{\varepsilon_0 K} dA = \frac{1}{\varepsilon_0 K} Q$$

- So
$$\frac{Q_{ef}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} - \frac{Q}{\varepsilon_0 K} = \frac{Q}{\varepsilon_0}\left[1 - \frac{1}{K}\right]$$

or
$$\boxed{Q_{ef} = Q\left[\frac{K-1}{K}\right]}$$

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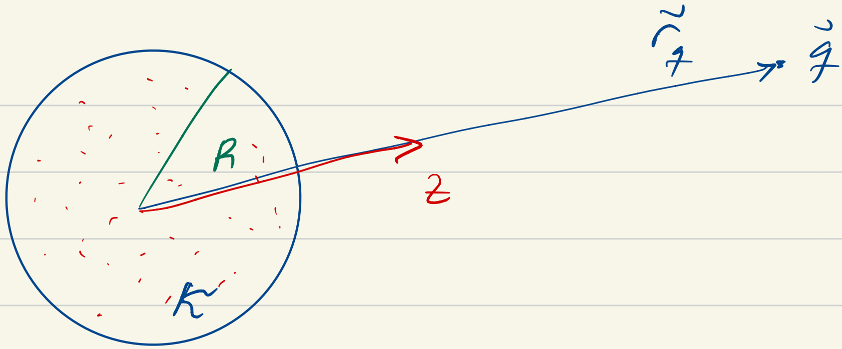

In the limit that $r \gg R$, we may treat the external field felt by the sphere to be uniform ...

Then we may solve the Laplace's eqn for $r \le R$ and $r \ge R$ to find

$\phi \sim -E_0 x$ at $r \to \infty$

$$\phi_{r \ge R} = -E_0 r \cos\theta + \sum_{\ell=1}^{\infty} \frac{b_\ell}{r^{\ell+1}} P_\ell(\cos\theta)$$

$$\phi_{r \le R} = \sum_{\ell=1}^{\infty} a_\ell r^\ell P_\ell(\cos\theta)$$

With boundary conditions at $r = R$
$$\begin{cases} (\varepsilon E_n)_1 = (\varepsilon E_n)_2 \\ \vec{\nabla} \times \vec{E} = \vec{0} \end{cases}$$

- In any case, following the book/lecture notes we find that

$$\begin{cases} \phi_{r \geq R} = E_0 \left(-r + \dfrac{k-1}{k+2} \dfrac{R^3}{r^2} \right) \cos\theta \\ \\ \phi_{r \leq R} = -E_0 \dfrac{3}{k+2} r \cos\theta \end{cases}$$

$$\vec{p} = 4\pi R^3 \frac{k-1}{k+2} \varepsilon_0 \vec{E_0}$$ $\rightsquigarrow$ dipole moment

- From here, we have to calculate $\vec{E}$ field produced by the dipole ... this is given by

$$\vec{E}_d = \frac{p}{4\pi\varepsilon_0 r^3} \left(2\hat{n}_r \cos\theta + \hat{n}_\theta \sin\theta \right)$$

- By choice, the charge q is at $\theta = 0$, so the force on q is given by

$$\vec{F} = q\vec{E}_d \Rightarrow F_z = q \cdot \frac{2p}{4\pi\varepsilon_0 r_q^3}$$

$$= \frac{2q}{\cancel{4\pi\varepsilon_0} r_q^3} \cdot \cancel{4\pi} R^3 \frac{K-1}{K+2} \cancel{\varepsilon_0} E_0$$

$$= \frac{2R^3}{r_q^3} \frac{K-1}{K+2} \frac{q^2}{4\pi\varepsilon_0 r_q^2}$$

$$\boxed{F = \frac{2q^2}{4\pi\varepsilon_0} \frac{R^3}{r_q^5} \frac{K-1}{K+2}}$$

The force is <u>attractive</u>, since we can think of the induced charge distribution as something like

Energy of interaction

$$\mathcal{U} = -\frac{1}{2}\vec{p}\cdot\vec{E}_{ext} = -\frac{1}{2} p_z E_z$$

$$= -\frac{1}{2} 4\pi R^3 \frac{k-1}{k+2} \varepsilon_0 \left(\frac{q}{4\pi\varepsilon_0 r_q^2}\right)^2$$

$$\boxed{\mathcal{U} = -\frac{1}{2}\frac{1}{4\pi\varepsilon_0}\frac{k-1}{k+2} q^2 R^3 \frac{1}{r_q^4}}$$

Can also do this ...

$$\mathcal{U} = \int_\infty^r F\,dr$$

$$= \int_\infty^{r_q} \frac{2q^2}{4\pi\varepsilon_0}\frac{k-1}{k+2}\frac{R^3}{r^5}\,dr$$

$$\boxed{\mathcal{U} = -\frac{q^2}{8\pi\varepsilon_0}\frac{k-1}{k-2} R^3 \frac{1}{r_q^4}}$$

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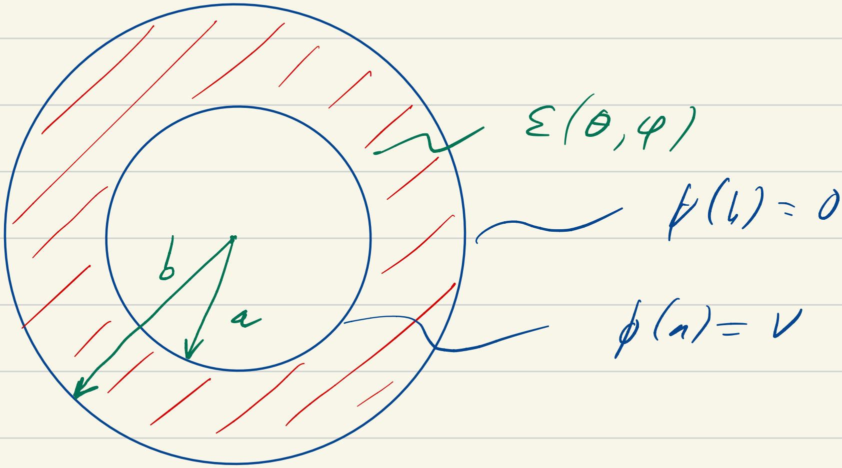

- To calculate C ... let the inner plate have potential $\phi(a)=V$, $\phi(b)=0$

- What is the $\vec{E}$ field inside capacitor?

$\rightarrow$ must have $\vec{\nabla}\times\vec{E}=0$ everywhere ...
$\vec{\nabla}\cdot\vec{D}=0$

So, since $E_{\theta,\varphi}(\text{for } r>b,\ r<a)=0$

$\Rightarrow$ $E_{\theta,\varphi}=0$ inside as well.
($E_{\parallel}$ is continuous)

- So $\vec{E}$ is only in the radial direction

$$\vec{E} = E(r)\,\hat{r}.$$

- Now $\vec{D} = \varepsilon(\theta,\varphi)\,\vec{E}(r)$ by definition

$$\rightarrow \boxed{D(r) = \varepsilon(\theta,\varphi)\,E(r)}$$

- Since $C = \frac{Q}{V}$, we want to calculate the total charge Q on surface with $r=a$

Well... $D(a) = \sigma_a = \varepsilon(\theta,\varphi)\,E(a)$

$\Rightarrow$ It remains to write $E(a)$ in terms of V...

- We know that $-\nabla\phi = \vec{E}$, so have to solve Laplace's eqn inside the capacitor...

We have $\nabla^2 \phi = 0$ (with boundary condition...)

Since $-\nabla\phi = \vec{E}$ is spherically symmetric,

ϕ is spherically symmetric too,

and so $\nabla^2\phi = \frac{1}{r^2}\partial_r\left(r^2\partial_r \phi\right) = 0$

$$\Rightarrow \partial_r\left(r^2 E(r)\right) = 0$$

$$\Rightarrow E(r) = \frac{A}{r^2}$$

Now... what is A? well... we must have that

$$-\nabla\phi = \vec{E} \Rightarrow -\partial_r\phi = E(r)$$

$$\Rightarrow \underbrace{\phi(b) - \phi(a)}_{-V} = -\int_a^b E(r)\,dr = A\left(\frac{1}{b} - \frac{1}{a}\right)$$

- $\mathcal{S}_v$ $\quad A = V\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$

- with this,

$$E(a) = \frac{V}{a^2}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$$

- We can now calculate σ_a ...

$$\sigma_a = D(a) = \varepsilon(\theta,\varphi)\,\frac{V}{a^2}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1}$$

- $\mathcal{S}_\sigma$

$$C = \frac{Q}{V} = \frac{1}{V}\oint_{r=a} \sigma_a \, dA$$

$$= \frac{1}{a^2}\left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \int_0^{\pi}\int_0^{2\pi} d\varphi\, d\theta\, \varepsilon(\theta,\varphi) \sin\theta \, a^2$$

$$\boxed{C = \left(\frac{1}{a} - \frac{1}{b}\right)^{-1} \int_0^{2\pi}\int_0^{\pi} \varepsilon(\theta,\varphi) \sin\theta \, d\theta \, d\varphi}$$

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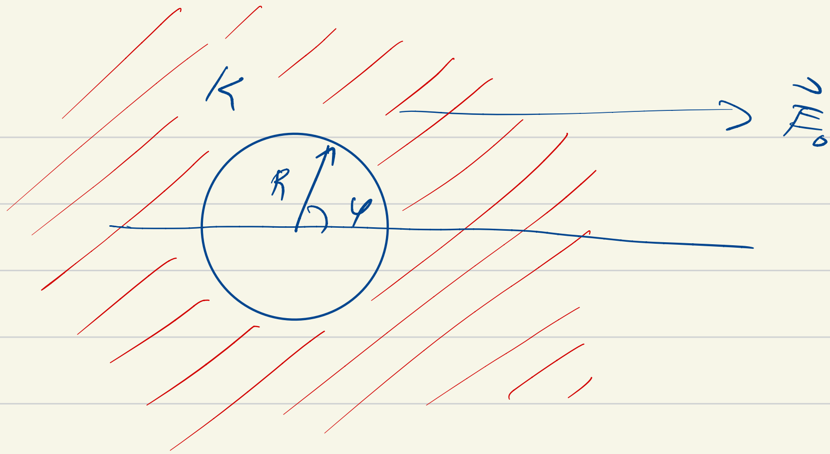

To find $\delta\vec{E}$, we must first find $\vec{E}$ induced on the hole by $\vec{E}_0$...

$\Rightarrow$ need to solve the Laplace's eqn...

$\phi(R) = 0$, $\nabla^2\phi = 0$ in cylindrical coords

- From chapter 2, Eqn 2.112 ... we have a general solution...

$$\phi(\rho,\varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty}\left(a_n\rho^n + \frac{b_n}{\rho^n}\right) \times \left(c_n \cos n\varphi + s_n \sin n\varphi\right)$$

- We want $\phi_{out}(\infty) = -E_0 \rho \cos\varphi$, so

$a_0 = b_0 = 0$, $a_n = 0 \ \forall n \geq 1$

$$\Rightarrow \phi_{out}(\rho, \varphi) = -E_0 \rho \cos\varphi + \sum_{n=1}^{\infty} \frac{b_n}{\rho^n} \cos(n\varphi)$$

By symmetry, we may drop

So $$\boxed{\phi_{out}(\rho, \varphi) = \left(-E_0 \rho + \frac{b_1}{\rho}\right) \cos\varphi}$$

- For $\phi_{in}(\rho, \varphi)$, want $\phi(0)$ finite, so

$$\boxed{\phi_{in}(\rho, \varphi) = a_1 \rho \cos\varphi}$$

<u>Boundary conditions</u>

$$\begin{cases} (D_n)_{out} = (D_n)_{in} \\ \phi \text{ continuous} \end{cases}$$

So

$$\begin{cases} \phi_{in}(R) = \phi_{out}(R) \\ \varepsilon_0 \left.\dfrac{\partial \phi_{in}}{\partial r}\right|_R = \varepsilon \left.\dfrac{\partial \phi_{out}}{\partial r}\right|_R \end{cases}$$

$$\Rightarrow \begin{cases} a_1 R = \left(-E_0 R + \dfrac{b_1}{R}\right) \\ \varepsilon_0 a_1 = \varepsilon\left(-E_0 - \dfrac{b_1}{R^2}\right) \end{cases}$$

$$\Rightarrow \begin{cases} a_1 = \dfrac{-2\varepsilon E_0}{\varepsilon + \varepsilon_0} = \dfrac{-2k}{k+1} E_0 \\ b_1 = \dfrac{-\varepsilon + \varepsilon_0}{\varepsilon + \varepsilon_0} E_0 R^2 = -\dfrac{k-1}{k+1} E_0 R^2 \end{cases}$$

So,

$$\boxed{\begin{aligned} \phi_{in}(\rho, \varphi) &= -\frac{2k}{k+1} E_0 \rho \cos\varphi \\ \phi_{out}(\rho, \varphi) &= -E_0\left(\rho + \frac{k-1}{k+1}\frac{R^2}{\rho}\right)\cos\varphi \end{aligned}}$$

- From here, can calculate field produced by the hole ...

- $$\phi_{in}(x,y) = -\frac{2k}{k+1} E_0 x$$

$$\boxed{\vec{E}_{in}(x,y) = \frac{2k}{k+1} E_0 \hat{x}}$$

- $$\phi_{out}(x,y) = -E_0 \left(\sqrt{x^2+y^2} + \frac{k-1}{k+1} \frac{R^2}{\sqrt{x^2+y^2}} \right) \frac{x}{\sqrt{x^2+y^2}}$$

$$= -E_0 x - E_0 \frac{k-1}{k+1} R^2 \frac{x}{x^2+y^2}$$

So
$$\boxed{\begin{aligned} \vec{E}_{out}(x,y) = {} & E_0 \left\{ 1 + \frac{k-1}{k+1} R^2 \frac{y^2-x^2}{(x^2+y^2)^2} \right\} \hat{x} \\ & -2E_0 \frac{k-1}{k+1} R^2 \frac{xy}{(x^2+y^2)^2} \hat{y} \end{aligned}}$$

With this ... can calculate change in field ...

$$\Delta \vec{E}_{in}(\rho, \varphi) = \vec{E}_{in, new} - \vec{E}_{in, old}$$

$$= \left(\frac{2\kappa}{\kappa+1} - 1 \right) E_0 \hat{x}$$

$$\boxed{\Delta \vec{E}_{in}(\rho, \varphi) = \frac{\kappa-1}{\kappa+1} E_0 \hat{x}}$$

$$\Delta \vec{E}_{out}(\rho, \varphi) = \vec{E}_{out, new} - \vec{E}_{in, new}$$

$$= E_0 \frac{\kappa-1}{\kappa+1} R^2 \frac{y^2 - x^2}{(x^2+y^2)^2} \hat{x}$$

$$-2E_0 \frac{\kappa-1}{\kappa+1} R^2 \frac{xy}{(x^2+y^2)^2} \hat{y}$$

$$\rightarrow \boxed{\Delta \vec{E}_{out}(\rho, \varphi) = \frac{\kappa-1}{\kappa+1} E_0 R^2 \frac{(y^2-x^2)\hat{x} - 2xy\hat{y}}{(x^2+y^2)^2}}$$