Name: **Huan Q. Bui**
Course: **8.321 - Quantum Theory I**
Problem set: **#1**

1.

(a) After measuring S_z gives $\hbar/2$, we know that the electron spin state must be $|S_z,+\rangle$. Next, we measure along $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The operator associated with this measurement is

$$S_{\mathbf{n}} = \mathbf{S}\cdot\widehat{\mathbf{n}} = \frac{\hbar}{2}\sin\theta\cos\phi\sigma_x + \frac{\hbar}{2}\sin\theta\sin\phi\sigma_y + \frac{\hbar}{2}\cos\theta\sigma_z.$$

Working in the z-basis where $|+\rangle = (1 \quad 0)^\top$ and $|-\rangle = (0 \quad 1)^\top$ we find

$$S_{\mathbf{n}} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}.$$

We now diagonalize this, to make sure that it *actually* has an eigenvalue of $\hbar/2$, and to find the $\hbar/2$-eigenvector. Letting $\lambda \in \mathbb{C}$ and setting $S_{\mathbf{n}} - (\hbar/2)\lambda\mathbb{I} = 0$ gives

$$\det\left(S_{\mathbf{n}} - \frac{\hbar}{2}\lambda\mathbb{I}\right) = 0 \iff \frac{\hbar^2}{4}\left(\lambda^2 - \cos^2\theta\right) - \frac{\hbar^2}{4}\sin^2\theta = 0 \iff \lambda = \pm 1.$$

The $\hbar/2$-eigenvector $(a \quad b)$ solves the equation

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta - 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0,$$

which means that

$$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta}e^{i\phi} = \frac{1 + 2\sin^2(\theta/2) - 1}{2\sin(\theta/2)\cos(\theta/2)}e^{i\phi} = \frac{e^{i\phi}\sin(\theta/2)}{\cos(\theta/2)}.$$

After ensuring the normalization condition, we find that the electron spin state following measuring $\hbar/2$ in the $\widehat{\mathbf{n}}$ direction is

$$\boxed{|S_{\mathbf{n}},+\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix}}$$

The probability of measuring $|S_Z,+\rangle$ to be in $|S_{\mathbf{n}},+\rangle$ is

$$\boxed{|\langle S_{\mathbf{n}},+|S_z,+\rangle|^2 = \cos^2\theta/2}$$

(b) Measuring $\hbar/2$ in the $\mathbf{n}$ direction means that the electron spin state is now in $|S_{\mathbf{n}},+\rangle$. The probability that a subsequent measurement in S_z finds $\hbar/2$ is

$$\boxed{|\langle S_z,+|S_{\mathbf{n}},+\rangle|^2 = |\langle S_{\mathbf{n}},+|S_z,+\rangle|^2 = \cos^2\theta/2}$$

This makes sense since this is the equivalent to observing $\hbar/2$ after measuring $|S_z,+\rangle$ in $\widehat{\mathbf{n}}'$ where $\widehat{\mathbf{n}}' = \widehat{\mathbf{n}}(-\theta,\phi)$. In other words, we just call $\widehat{\mathbf{n}}$ the z-axis and measure in $\widehat{\mathbf{n}}'$. □

2. Suppose (to get a contradiction) that it is possible for the spin state of the electron to be such that

$$\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0.$$

Working in the z-basis, let the electron be in

$$\left|\psi\right\rangle = c_1 \left|S_z, +\right\rangle + c_2 \left|S_z, -\right\rangle \equiv c_1 \left|+\right\rangle + c_2 \left|-\right\rangle.$$

Since $\langle S_z \rangle = 0$ and that the eigenvalues of S_z are $\pm\hbar/2$, we have $|c_1| = |c_2| \implies c_2 = e^{i\phi}c_1$ where $\phi \in \mathbb{R}$. Under normalization condition and the irrelevance of global phase, we may set $c_1 = 1/\sqrt{2}$. The condition that $\langle S_x \rangle = 0$ puts a constraint on ϕ. For ease of computation, we may rewrite S_x, S_y in terms of $\left|+\right\rangle, \left|-\right\rangle$, and their bras:

$$S_x = \frac{\hbar}{2}\left|-\right\rangle\left\langle+\right| + \frac{\hbar}{2}\left|+\right\rangle\left\langle-\right| \quad \text{and} \quad S_y = \frac{i\hbar}{2}\left|-\right\rangle\left\langle+\right| - \frac{i\hbar}{2}\left|+\right\rangle\left\langle-\right|.$$

These definitions give rise to the same matrix elements of S_x, S_y in the z-basis. With these, we find

$$\begin{aligned}
0 &= \langle S_x \rangle \\
&= \frac{\hbar}{4}\left(\left\langle+\right| + e^{-i\phi}\left\langle-\right|\right)\left(\left|-\right\rangle\left\langle+\right| + \left|+\right\rangle\left\langle-\right|\right)\left(\left|+\right\rangle + e^{i\phi}\left|-\right\rangle\right) \\
&= \frac{\hbar}{4}\left(e^{-i\phi} + e^{i\phi}\right) \implies e^{i\phi} = -e^{-i\phi} = \pm i
\end{aligned}$$

which implies

$$\begin{aligned}
\langle S_y \rangle &= \frac{i\hbar}{4}\left(\left\langle+\right| + e^{-i\phi}\left\langle-\right|\right)\left(\left|-\right\rangle\left\langle+\right| - \left|+\right\rangle\left\langle-\right|\right)\left(\left|+\right\rangle + e^{i\phi}\left|-\right\rangle\right) \\
&= \frac{i\hbar}{4}\left(\left\langle+\right| \mp i\left\langle-\right|\right)\left(\left|-\right\rangle\left\langle+\right| - \left|+\right\rangle\left\langle-\right|\right)\left(\left|+\right\rangle \pm i\left|-\right\rangle\right) \\
&= \frac{i\hbar}{4}\left(\mp i \mp i\right) \\
&= \pm\frac{\hbar}{2} \neq 0.
\end{aligned}$$

Thus we found a contradiction. The electron cannot be in a state such that $\langle S_x \rangle = \langle S_y \rangle = \langle S_z \rangle = 0$. □

3. Since all electrons are in the same state $|\alpha\rangle = s_+ |+\rangle + s_- |-\rangle$ such that $\langle S_z \rangle = 0$, we know (from the previous problem) that $s_- = e^{i\phi} s_+$ for some $\phi \in \mathbb{R}$. From the previous problem, we also found that $\langle S_x \rangle = (\hbar/4)\left(e^{-i\phi} + e^{i\phi}\right)$. Setting this equal to $\hbar/4$, we find that $e^{-i\phi} + e^{i\phi} = 1$, which gives $e^{i\phi} = 1/2 \pm i\sqrt{3}/2$.

(a) With $e^{i\phi} = 1/2 \pm i\sqrt{3}/2$, we can calculate S_y:

$$\begin{aligned}
\langle S_y \rangle &= \frac{i\hbar}{4}\left[\langle +| + (1/2 \mp i\sqrt{3}/2)\langle -|\right]\left(|-\rangle\langle +| - |+\rangle\langle -|\right)\left[|+\rangle + (1/2 \pm i\sqrt{3}/2)|-\rangle\right] \\
&= \frac{i\hbar}{4}\left[(1/2 \mp i\sqrt{3}/2) - (1/2 \pm i\sqrt{3}/2)\right] \\
&= \frac{-\hbar}{4}\left[\mp\frac{\sqrt{3}}{2} \mp \frac{\sqrt{3}}{2}\right] \\
&= \boxed{\pm\frac{\sqrt{3}}{4}\hbar}
\end{aligned}$$

(b) We recall that $\widehat{\mathbf{n}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and that the eigenstates of $S_{\mathbf{n}}$ are

$$|S_{\mathbf{n}}, +\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \quad \text{and} \quad |S_{\mathbf{n}}, -\rangle = \begin{pmatrix} -\sin\theta/2 \\ e^{i\phi}\cos\theta/2 \end{pmatrix}.$$

From Part (a), we want $\cos\theta/2 = \sin\theta/2 \implies \boxed{\theta = \pi/2 \mod 2\pi}$ and $\boxed{\phi = \pm\pi/3 \mod 2\pi}$ □

4. Sakurai and Napolitano Problem 1.19 (page 62). All expectation values are taken using the $|S_z, +\rangle \equiv |+\rangle$ state for this problem.

(a) Since $|+\rangle = (1/\sqrt{2})(|+\rangle + 0\,|-\rangle)$, we have, according to past calculations,

$$\langle +|\, S_x\, |+\rangle = \frac{\hbar}{4}(0+0) = 0 \implies \langle S_x \rangle^2 = 0.$$

Using the fact that $S_x^2 = (\hbar^2/4)\mathbb{I}$ (since S_x flips $|+\rangle \to |-\rangle$ and $|-\rangle \to |+\rangle$), we have $\langle S_x^2 \rangle = (\hbar/2)^2 \langle +|+\rangle = \hbar^2/4$. Thus,

$$\langle (\Delta S_x)^2 \rangle \equiv \langle S_x^2 \rangle - \langle S_x \rangle^2 = \boxed{\frac{\hbar^2}{4}}$$

We now do the same for S_y. Once again, since $S_y^2 = (\hbar^2/4)\mathbb{I}$, we have $\langle S_y^2 \rangle = \hbar^2/4$. We're left with $\langle S_y \rangle^2$, but we've also done this calculation in the previous problem. Observe that since the amplitude of $|-\rangle$ is zero, we have $\langle S_y \rangle = 0$ which implies $\langle S_y \rangle^2 = 0$. Thus,

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = \frac{\hbar^4}{16}.$$

Finally, since $[S_x, S_y] = i\hbar S_z$ and that $|+\rangle$ is the $\hbar/2$-eigenstate of S_z, we can see by inspection that $(1/4)\left|\langle [S_x, S_y] \rangle\right|^2 = (1/4)|i\hbar^2/2|^2 = \hbar^4/16$, and therefore, the uncertainty relation

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle \geq \frac{1}{4}\left|\langle [S_x, S_y] \rangle\right|^2$$

is satisfied.

(b) $|S_x, +\rangle = 1/\sqrt{2}\,|+\rangle + 1/\sqrt{2}\,|-\rangle$. Since $S_x^2 = (\hbar^2/4)\mathbb{I}$ we have $\langle S_x^2 \rangle_x = \hbar^2/4$. We can also quickly find $\langle S_x \rangle^2 = \hbar^2/4$ since $|S_x, +\rangle$ is an eigenstate of S_x. Thus, $\langle (\Delta S_x)^2 \rangle = 0$ and therefore,

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = 0.$$

This leaves the LHS of the uncertainty to calculate. Since $[S_x, S_y] = i\hbar S_z$, we have

$$\langle +|_x\, [S_x, S_y]\, |+\rangle_x = i\hbar \,\langle +|_x\, S_z\, |+\rangle_x = \frac{i\hbar^2}{4}\begin{pmatrix} 1 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$

Thus,

$$\langle (\Delta S_x)^2 \rangle \langle (\Delta S_y)^2 \rangle = 0 = \frac{1}{4}\left|\langle [S_x, S_y] \rangle\right|^2,$$

and the uncertainty relation still holds. $\square$

5. Sakurai and Napolitano Problem 1.20 (page 62).

Without loss of generality, assume that the desired linear combination has the form

$$\left|\psi\right\rangle = \left|\psi(\theta,\phi)\right\rangle = \cos\frac{\theta}{2}\left|+\right\rangle + \sin\frac{\theta}{2}e^{i\phi}\left|-\right\rangle,$$

where we once again work in the z-basis. We note that $\left|\psi(\theta,\psi)\right\rangle$ spans the full spin-1/2 state space, so this assumption is valid.

From the previous problem(s), we know that $S_x^2 = S_y^2 = (\hbar^2/4)\mathbb{I}$, so we have

$$\langle S_x^2\rangle = \langle S_y^2\rangle = \frac{\hbar^2}{4}.$$

Next, we compute

$$\begin{aligned}\left\langle\psi\right| S_x \left|\psi\right\rangle &= \left[\cos\frac{\theta}{2}\left\langle+\right| + \sin\frac{\theta}{2}e^{-i\phi}\left\langle-\right|\right]\left(\frac{\hbar}{2}\left|-\right\rangle\left\langle+\right| + \frac{\hbar}{2}\left|+\right\rangle\left\langle-\right|\right)\left[\cos\frac{\theta}{2}\left|+\right\rangle + \sin\frac{\theta}{2}e^{i\phi}\left|-\right\rangle\right]\\ &= \frac{\hbar}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{-i\phi}+e^{i\phi}\right)\\ &= \frac{\hbar}{2}\sin\theta\cos\phi \implies \langle S_x\rangle^2 = \frac{\hbar^2}{4}\sin^2\theta\cos^2\phi.\end{aligned}$$

Similarly,

$$\begin{aligned}\left\langle\psi\right| S_y \left|\psi\right\rangle &= \left[\cos\frac{\theta}{2}\left\langle+\right| + \sin\frac{\theta}{2}e^{-i\phi}\left\langle-\right|\right]\left(\frac{i\hbar}{2}\left|-\right\rangle\left\langle+\right| - \frac{i\hbar}{2}\left|+\right\rangle\left\langle-\right|\right)\left[\cos\frac{\theta}{2}\left|+\right\rangle + \sin\frac{\theta}{2}e^{i\phi}\left|-\right\rangle\right]\\ &= \frac{i\hbar}{2}\cos\frac{\theta}{2}\sin\frac{\theta}{2}\left(e^{-i\phi}-e^{i\phi}\right)\\ &= -i\frac{i\hbar}{2}\sin\theta\sin\phi\\ &= \frac{\hbar}{2}\sin\theta\sin\phi \implies \langle S_y\rangle^2 = \frac{\hbar^2}{4}\sin^2\theta\sin^2\phi.\end{aligned}$$

So,

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^2}{4}\left(1-\sin^2\theta\cos^2\phi\right)\left(1-\sin^2\theta\sin^2\phi\right).$$

An obvious choice which maximizes this quantity for (θ,ϕ) is $(0 \mod \pi, \phi\in\mathbb{R})$. To find other solutions, assume that $\theta \neq 0 \mod \pi$ is fixed. Then we let $a = 1-\sin^2\theta\cos^2\phi$ and $b = 1-\sin^2\theta\sin^2\phi$. Since $ab \leq a^2/2+b^2/2$ with equality occurring if and only if $a=b$, we want $\sin^2\theta\cos^2\phi = \sin^2\theta\sin^2\phi$, or $\cos^2\phi = \sin^2\phi = 1/2 \implies \phi = \pi/4 \mod \pi/2$. We're then left to maximize $\left(1-(1/2)\sin^2\theta\right)\left(1-(1/2)\sin^2\theta\right)$, and it is clear that $\theta = 0 \mod \pi$ is the solution. In summary, we want $\theta = 0 \mod \pi$ and $\phi = \pi/4 \mod \pi/2$.

The desired linear combinations are therefore

$$\boxed{\left|\psi\right\rangle = \pm\left|+\right\rangle \equiv \left|+\right\rangle \quad\text{and}\quad \left|\psi\right\rangle = i^k e^{i\pi/4}\left|-\right\rangle \equiv \left|-\right\rangle}$$

Plugging these values for θ,ϕ into $\left\langle\psi\right| S_x\left|\psi\right\rangle$ and $\left\langle\psi\right| S_y\left|\psi\right\rangle$ we find $\langle S_x\rangle^2 = \langle S_y\rangle^2 = 0$ and therefore

$$\langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle = \frac{\hbar^4}{16}.$$

The uncertainty relation is **not violated** because

$$\frac{1}{4}\left|\langle[S_x,S_y]\rangle\right|^2 = \frac{1}{4}\hbar^2\left|\left\langle\pm\right| S_z\left|\pm\right\rangle\right|^2 = \frac{\hbar^4}{16} \leq \langle(\Delta S_x)^2\rangle\langle(\Delta S_y)^2\rangle.$$

□

6. Let $A, B \in \mathbb{M}_{n\times n}$, the (finite-dimensional) space of $n \times n$ matrices (they have to be square matrices so that the commutator is defined). The trace function is defined for A, B, AB, BA and satisfies the following properties

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \quad \text{and} \quad \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA),$$

whose proofs can be obtained by explicitly writing down the matrix elements of AB, BA and comparing the sum of the diagonal entries of AB to that of BA. It is clear that for $n \geq 1$,

$$\mathrm{tr}([A, B]) = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = \mathrm{tr}(AB) - \mathrm{tr}(AB) = 0 \neq n = \mathrm{tr}(\mathbb{I}).$$

Therefore $AB - BA = \mathbb{I}$ cannot be satisfied. □

7.

(a) Treating

$$e^A = \sum_{a=0}^{\infty} \frac{1}{a!} A^a,$$

we have

$$e^A B e^{-A} = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} (-1)^b \frac{1}{a!b!} A^a B A^b.$$

Writing this sum order-by-order, we we have

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \frac{1}{k!(n-k)!} A^k B A^{n-k}.$$

We thus want to show that

$$A^n\{B\} = \sum_{k=0}^{n} (-1)^{n-k} \frac{n!}{k!(n-k)!} A^k B A^{n-k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A^k B A^{n-k}$$

using induction. The base case $n = 0$ is trivial $A^0\{B\} = B = (A^0/0!)B(A^0/0!)$. Assume that this holds for n. We now show that it is true for $n+1$. By definition, we have

$$\begin{aligned} A^{n+1}\{B\} &= [A, A^n\{B\}] \\ &= A\left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A^k B A^{n-k}\right) - \left(\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A^k B A^{n-k}\right) A. \end{aligned}$$

Taking the last term of the first summand and the first term of the second summand out of their sums, we get

$$A^{n+1}\{B\} = A^{n+1}B + \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} A^{k+1} B A^{n-k} - \sum_{k=1}^{n} (-1)^{n-k} \binom{n}{k} A^k B A^{n-k+1} - (-1)^n B A^{n+1}.$$

Shift the indices of the first sum by $k \longrightarrow k-1$ and multiply the last two terms by $(+1) \times (-1)$ to get

$$A^{n+1}\{B\} = A^{n+1}B + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k-1} A^k B A^{n-k+1} + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{n}{k} A^k B A^{n-k+1} + (-1)^{n+1} B A^{n+1}.$$

From combinatorics/Pascal triangle, we know that

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$

Thus,

$$A^{n+1}\{B\} = A^{n+1}B + \sum_{k=1}^{n} (-1)^{n-k+1} \binom{n+1}{k} A^k B A^{n-k+1} + (-1)^{n+1} B A^{n+1} = \sum_{k=0}^{n} (-1)^{(n+1)-k} \binom{n+1}{k} A^k B A^{(n+1)-k},$$

as desired. As a result, we have

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \frac{1}{k!(n-k)!} A^k B A^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} A^k B A^{n-k} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\}.$$

To prepare for the next problem, we will introduce the following notation:

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\{B\} = e^{[A,\cdot]}(B)$$

(b) **Note to the grader:** Some of the ideas I used for solving this problem came from Theorem 5.4 (Derivative of Exponential) in Brian Hall's *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, 2nd Edition. I wonder if this problem could be solved using a similar approach used in Part (a), i.e., treating everything as power series.

We can do this problem by expanding both sides into power series and look at how the low-order terms match. While it is a legitimate way to check that the power series representation is "good enough," we're not really *showing* that the equality is true.

Inspired by Theorem 5.4 of Brian Hall's *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, 2nd Edition, we first start by treating some lemmas. The first lemma allows us to convert the derivative of the exponential map into something that is nicer to handle.

Lemma 1.

$$\frac{d}{dt}e^{X(t)} = \frac{d}{d\tau}e^{X(t)+\tau\frac{dX(t)}{dt}}\bigg|_{\tau=0}.$$

Proof. We show this using power series:

$$\begin{aligned}
\frac{d}{d\tau}e^{X(t)+\tau\frac{dX}{dt}}\bigg|_{\tau=0} &= \frac{d}{d\tau}\sum_{n=0}^{\infty}\frac{1}{n!}\left[X(t)+\tau\frac{dX(t)}{dt}\right]^n\bigg|_{\tau=0} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}\left[X(t)+0\right]^k\left[X(t)+0\right]\left[X(t)+0\right] \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n-1}X^k\frac{dX(t)}{dt}X(t)^{n-1-k} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\frac{d}{dt}X(t)^k \\
&= \frac{d}{dt}e^{X(t)}.
\end{aligned}$$

Here we note that that product rule must be written out explicitly since $X(t)$ and $dX(t)/dt$ don't necessarily commute. □

The second lemma gives us the right hand side of the equality in the problem, as we will see later. This lemma lets us convert the right hand side into a geometric series of more convenient terms.

Lemma 2.

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1}\left(e^{-X/m}\right)^k = \frac{1-e^{-X}}{X} = \sum_{k=0}^{\infty}(-1)^k\frac{X^k}{(k+1)!}$$

Proof. We note here is that while the first equality makes sense when X is a number, it does not when X is a matrix (or, What do we mean by diving by a matrix?). However, we won't worry about that. All we care about the equality between the first and third expressions, which makes when we replace e^X with $e^{[A,\cdot]}(B)$ – something we introduced in Part (a). This will be useful later in the derivation/proof. The proof of this lemma uses geometric series and power series.

$$\frac{1}{m}\sum_{k=0}^{m-1}\left(e^{-X/m}\right)^k = \frac{1}{m}\frac{1-e^{-X}}{1-e^{-X/m}}.$$

Taking $m \longrightarrow \infty$, the denominator of the right hand side becomes unity:

$$\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \left(e^{-X/m}\right)^k = \frac{1-e^{-X}}{X}.$$

The rest is power series:

$$\frac{1-e^{-X}}{X} = -\frac{1}{X} \sum_{n=1}^{\infty} \frac{1}{n!}(-1)^n X^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} X^n.$$

$\square$

The forthcoming lemma allows us to deal with a technical detail in the main proof. The motivation for this lemma will be clear later (I will point out where each lemma helps various steps of the main proof).

Lemma 3.

$$\frac{d}{d\tau} \exp\left(\frac{X(t)}{m} + \frac{\tau}{m}\frac{dX(t)}{dt}\right)\bigg|_{\tau=0} = \frac{1}{m}\frac{d}{d\tau} \exp\left(\frac{X(t)}{m} + \tau\frac{dX(t)}{dt}\right)\bigg|_{\tau=0}$$

Proof. The point of this manipulation is so that the term dX/dt does not vanish when we take $m \longrightarrow \infty$. To show this, we once again use power series:

$$\begin{aligned}
\frac{d}{d\tau} \exp\left(\frac{X(t)}{m} + \frac{\tau}{m}\frac{dX(t)}{dt}\right)\bigg|_{\tau=0} &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-1} \left[\frac{X(t)}{m} + 0\right]^k \left\{\frac{1}{m}\frac{dX(t)}{dt} + 0\right\} \left[\frac{X(t)}{m} + 0\right]^{n-k-1} \\
&= \frac{1}{m} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n-1} \left(\frac{X(t)}{m}\right)^k \frac{dX(t)}{dt} \left(\frac{X(t)}{m}\right)^{n-k-1} \\
&= \frac{1}{m}\frac{d}{d\tau} \exp\left(\frac{X(t)}{m} + \tau\frac{dX(t)}{dt}\right)\bigg|_{\tau=0},
\end{aligned}$$

where the last equality follows from Lemma 1. $\square$

We are now ready for the main proof.

Proof.

$$
\begin{aligned}
e^{-iA(t)}\frac{d}{dt}e^{iA(t)} &= e^{-iA(t)}\frac{d}{d\tau}e^{iA(t)+\tau\frac{idA(t)}{dt}}\bigg|_{\tau=0} \qquad \text{(by Lemma 1)}\\
&= e^{-iA(t)}\frac{d}{d\tau}\left[\exp\left(\frac{iA(t)}{m}+\frac{\tau}{m}\frac{idA(t)}{dt}\right)\right]^m\bigg|_{\tau=0}\\
\text{(product rule)}\quad &= e^{-iA(t)}\sum_{k=0}^{m-1}\left(e^{iA(t)/m}\right)^{m-k-1}\left[\frac{d}{d\tau}\exp\left(\frac{iA(t)}{m}+\frac{\tau}{m}\frac{idA(t)}{dt}\right)\right]\bigg|_{\tau=0}\left(e^{iA(t)/m}\right)^k\\
&= e^{-iA(t)}e^{(m-1)A(t)/m}\sum_{k=0}^{m-1}\left(e^{A(t)/m}\right)^{-k}\left[\frac{d}{d\tau}\exp\left(\frac{iA(t)}{m}+\frac{\tau}{m}\frac{idA(t)}{dt}\right)\right]\bigg|_{\tau=0}\left(e^{iA(t)/m}\right)^k\\
\text{(Lemma 3)}\quad &= e^{-iA(t)}e^{(m-1)A(t)/m}\frac{1}{m}\sum_{k=0}^{m-1}\underbrace{\left(e^{A(t)/m}\right)^{-k}\left[\frac{d}{d\tau}\exp\left(\frac{iA(t)}{m}+\tau\frac{idA(t)}{dt}\right)\right]\bigg|_{\tau=0}\left(e^{iA(t)/m}\right)^k}\\
\text{(Part (a))}\quad &= e^{-iA(t)}e^{(m-1)A(t)/m}\sum_{k=0}^{m-1}\frac{1}{m}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{-ikA(t)}{m}\right)^n\left\{\frac{d}{d\tau}\exp\left(\frac{iA(t)}{m}+\tau\frac{idA(t)}{dt}\right)\bigg|_{\tau=0}\right\}\\
\text{(Part (a))}\quad &= e^{-iA(t)}e^{(m-1)A(t)/m}\frac{1}{m}\sum_{k=0}^{m-1}\left(e^{[-iA(t)/m,\cdot]}\right)^k\left\{\frac{d}{d\tau}\exp\left(\frac{iA(t)}{m}+\tau\frac{idA(t)}{dt}\right)\bigg|_{\tau=0}\right\}\\
\text{(Lemma 2, } m\to\infty)\quad &= e^{-iA(t)}e^{iA(t)}\sum_{n=0}^{\infty}\frac{(-i)^n}{(n+1)!}A(t)^n\left\{\frac{idA(t)}{dt}\right\}\\
&= \boxed{i\sum_{n=0}^{\infty}\frac{(-i)^n}{(n+1)!}A(t)^n\left\{\frac{dA(t)}{dt}\right\}}
\end{aligned}
$$

as desired. In the equality preceding the penultimate equality, we have used

$$
\lim_{m\to\infty}\frac{d}{d\tau}\exp\left(\frac{iA(t)}{m}+\tau\frac{idA(t)}{dt}\right)\bigg|_{\tau=0} = i\frac{dA(t)}{dt}
$$

which can again be proven straightforwardly by first writing the exponential as a power series and take the $d/d\tau$ derivative, then set $\tau=0$ and ultimately send $m\to\infty$. □