Name: **Huan Q. Bui**
Course: **8.309 - Classical Mechanics III**
Problem set: **#9**

1. Viscous Flow on an Inclined Plane

(a) We pick the z-axis to be perpendicular to the inclined and the x-axis along the incline. The NS equation for incompressible viscous laminar flow in full generality is

$$\cancel{\frac{\partial \mathbf{v}}{\partial t}} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{\boldsymbol{\nabla} P}{\rho} - \frac{\eta}{\rho} \nabla^2 \mathbf{v} = \frac{\mathbf{f}}{\rho}.$$

(b) By symmetry, the flow velocity has only the x-component and varies in z, so we may write

$$\mathbf{v} = v_x(z)\hat{x}.$$

This also means that the nonlinear term $\mathbf{v} \cdot \nabla \mathbf{v} = 0$. Moreover, since v does not depend on x, there is no gradient in the pressure in the x-direction, and the flow is driven entirely by gravity. So, With these, we have two equations:

$$\cancel{\frac{\partial P}{\partial x}} - \eta \frac{\partial^2 v_x(z)}{\partial z^2} = \rho g \sin\theta.$$

We thus have

$$-\eta \frac{\partial^2 v_x(z)}{\partial z^2} = \rho g \sin\theta \implies v_x(z) = -\frac{g \rho z^2 \sin(\theta)}{2\eta} + C_2 z + C_1$$

where C_2, C_1 are constants which we will find through the boundary conditions. On the surface of the inclined plane, we have

$$\hat{x} \cdot \mathbf{v}\big|_{z=0} = v_x(z=0) = 0.$$

Also, on the surface that is open to the air, we must have that the change in v_x with respect to z must vanish, so

$$\left.\frac{dv_x(z)}{dz}\right|_{z=h} = 0.$$

(c) With these conditions, we can solve for C_1, C_2. In Mathematica, we may just plug in the boundary conditions to find

$$\boxed{v_x(z) = \frac{g\rho \sin\theta}{2\eta} z(2h - z)}$$

Mathematica code:

```
In[23]:= DSolve[{-\[Eta]*v''[z] == \[Rho]*g*Sin[\[Theta]], v[0] == 0,
v'[h] == 0}, v[z], z] // FullSimplify

Out[23]= {{v[z] -> (g (2 h - z) z \[Rho] Sin[\[Theta]])/(2 \[Eta])}}
```

2. Chaos in a Nonlinear Circuit. The equation of motion is given by

$$\ddot{x} + \frac{1}{q_c}\dot{x} + x^3 = B\cos(\omega_D t).$$

The equation is already non-dimensionalized, so we have the following system

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2/q_c - x_1^3 + B\cos(x_3) \\ \omega_D \end{pmatrix}$$

where

$$x_1(t) = x(t), \quad x_2(t) = x_1'(t), \quad x_3(t) = \omega_D t$$

To get consistent result with the Mathematica notebook, we may match our coefficients q_c, B to the notebook's q, a using the following rules:

$$B = \alpha a, \quad q_c = \beta Q$$

where α, β are constant of proportionality. Since we're interested in $q_c \in [0, 20]$ while the notebook has $Q \in [0, 4]$, we may set $\beta = 5$. Since we're interested in $B \in [0, 12]$ while the notebook has $a \in [0, 2]$ we may set $\alpha = 6$. With these, the equations which we will give the notenook are

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_2/5Q - x_1^3 + 6a\cos(x_3) \\ \omega_D \end{pmatrix}$$

Modified Mathematica code:

```
eq1 = x1'[t] == x2[t];
If[isDamped,
eq2 = x2'[t] == -(1/(5 q)) x2[t] - x1[t]^3 + 6 a Cos[x3[t]],
eq2 = x2'[t] == -x1[t]^3 + 6 a Cos[x3[t]]
];

eq3 = x3'[t] == omega;
ic = {x1[0] == x10, x2[0] == x20, x3[0] == phase};
```

(a) We can leave $\omega = 2/3$ at its default value. To set $q_c = 10 = 5Q$ we put $Q = 2$. For $6 < B < 11$ we need $1 < a < 11/6$, so we set the interval for a to start at 1 and has length 1. Figure 1 is a bifurcation plot showing at least $6 < B < 11$.

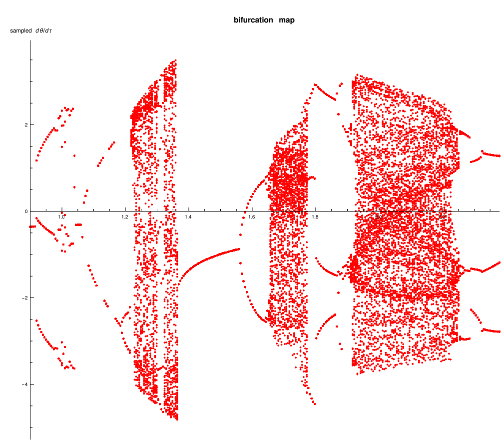


Figure 1: Problem 2a – Bifurcation plot showing at least $6 < B < 11$. Notice a bifurcation at $6a = 1.6$.

(b) **Period Doubling:** From Figure 1 we notice a period doubling at $6a = 1.6$. So we will fix the other parameters as in Part (a) and generate Poincaré section and phase portrait for this (see Figures 2 and 3)

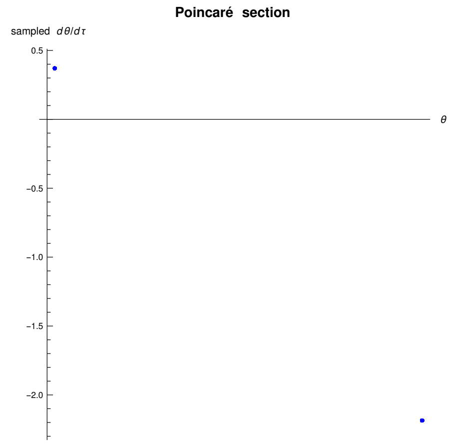


Figure 2: Poincaré section for period doubling at $6a = 1.6$.

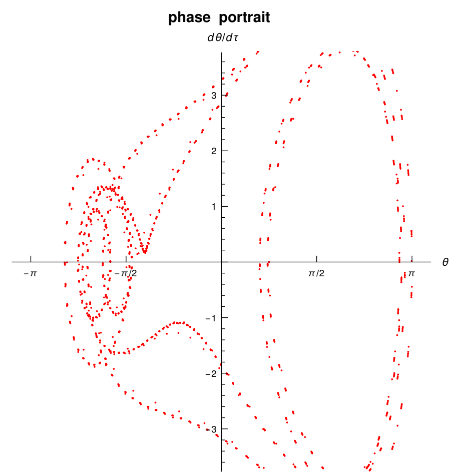


Figure 3: Phase portrait for period doubling at $6a = 1.6$.

Period Quadrupling: A trick to find period quadrupling is to look a bit further after a period doubling. Upon zooming into the region where $1.64 < 6a < 1.65$ we found a period quadrupling, shown in Figure 4.

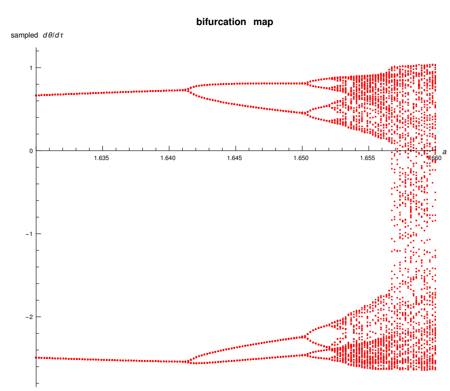


Figure 4: Bifurcation plot showing a period quadrupling at $1.64 < 6a < 1.65$.

With this we can repeat and generate Poincaré sections and phase portrait for this period quadrupling.

See Figures 5 and 6.

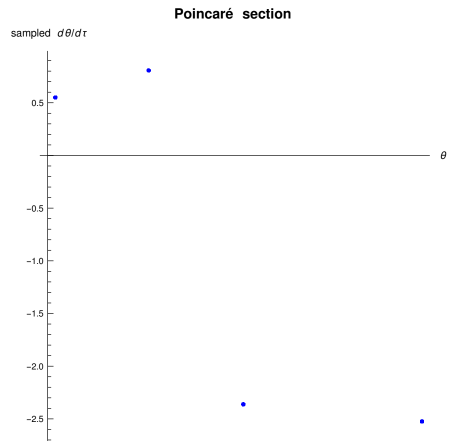


Figure 5: Poincaré section for period quadrupling at $1.64 < 6a < 1.65$.

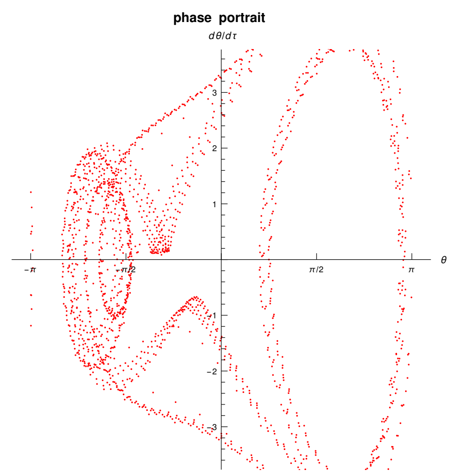
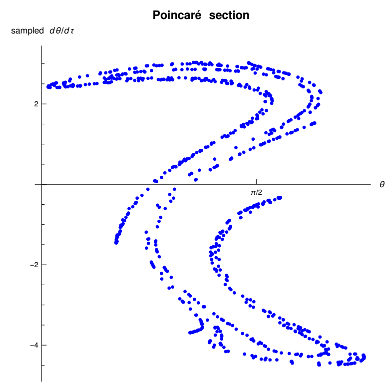
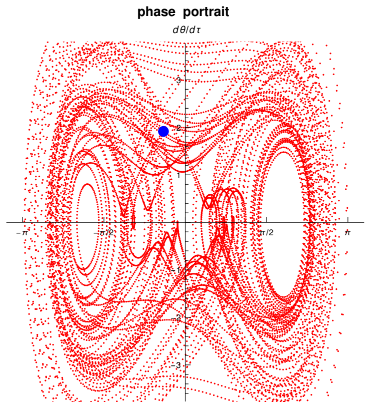


Figure 6: Phase portrait for period quadrupling at $1.64 < 6a < 1.65$.

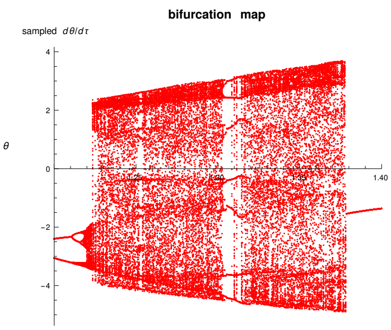
Two examples with chaos: For this we can pick two regions $1.2 < 6a < 1.4$ and $1.9 < 6a < 2.2$. See Figure 7.



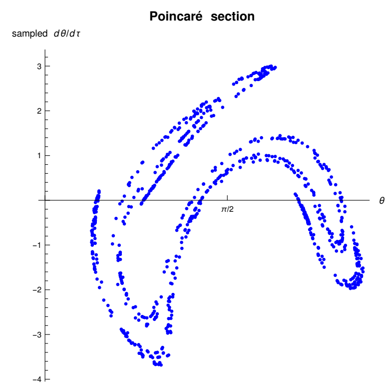
(a) Poincaré section for chaos at $6a = 1.3$



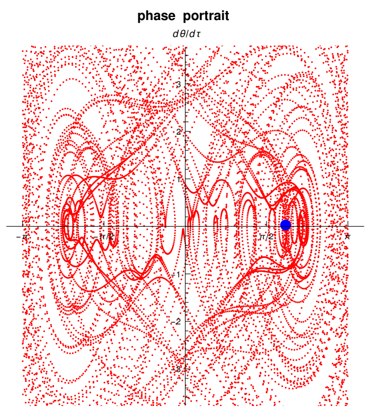
(b) Phase portrait for chaos at $6a = 1.3$.



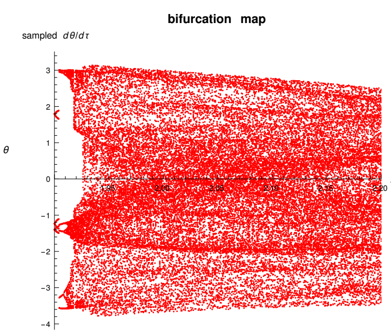
(c) Bifurcation plot with chaos at $1.2 < 6a < 1.4$.



(d) Poincaré section for chaos at $6a = 2.0$.



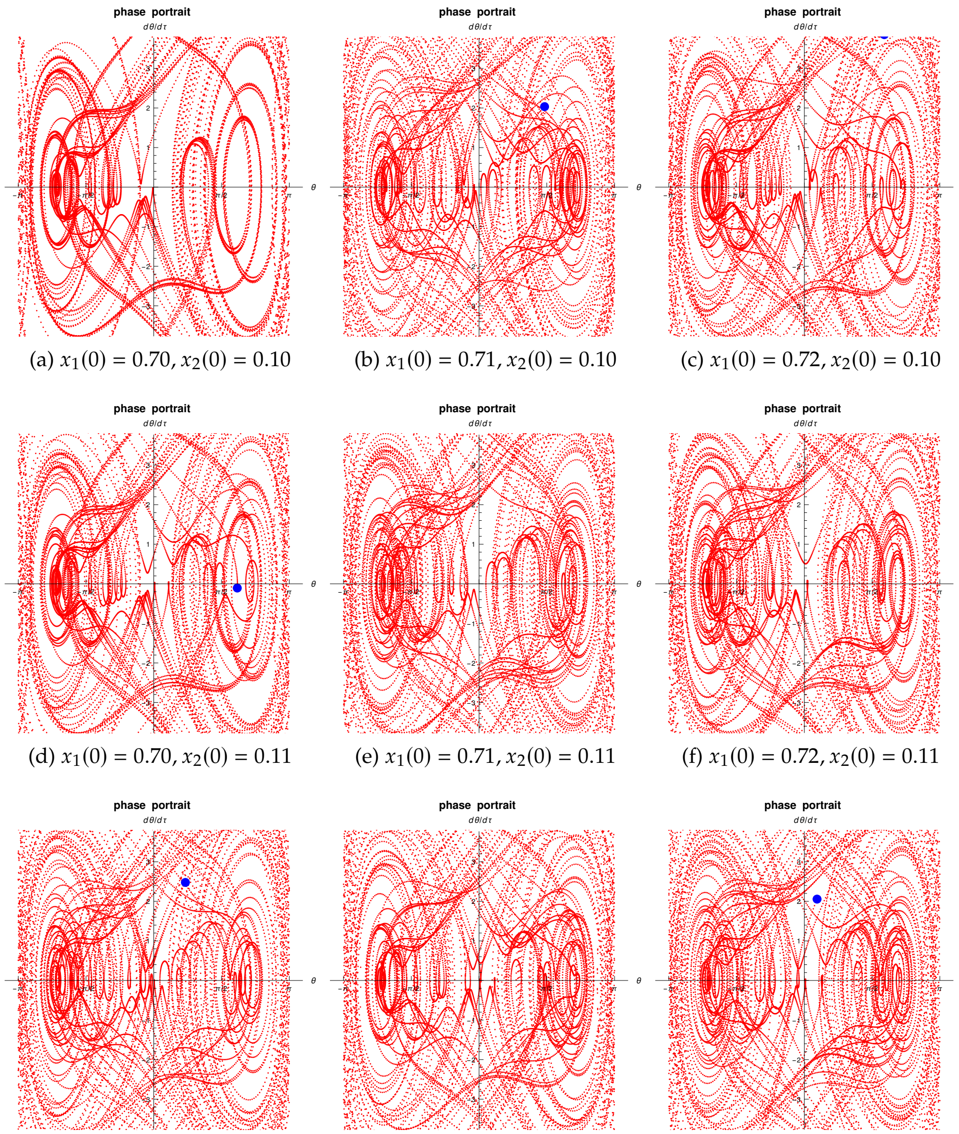
(e) Phase portrait for chaos at $6a = 2.0$.



(f) Bifurcation plot with chaos at $1.9 < 6a < 2.2$.

Figure 7: Problem 2b.

(c) We will consider the chaotic case where $6a = 2$. To observe sensitivity to initial conditions we can generate an array of phase portraits corresponding to multiple nearby initial conditions. We will generate 9 images corresponding 3 different initial x_1's $\{0.7, 0.71, 0.72\}$ and 3 initial x_2's $\{0.1, 0.11, 0.12\}$. See Figure 8.



(a) $x_1(0) = 0.70, x_2(0) = 0.10$ (b) $x_1(0) = 0.71, x_2(0) = 0.10$ (c) $x_1(0) = 0.72, x_2(0) = 0.10$

(d) $x_1(0) = 0.70, x_2(0) = 0.11$ (e) $x_1(0) = 0.71, x_2(0) = 0.11$ (f) $x_1(0) = 0.72, x_2(0) = 0.11$

(g) $x_1(0) = 0.70, x_2(0) = 0.12$ (h) $x_1(0) = 0.71, x_2(0) = 0.12$ (i) $x_1(0) = 0.72, x_2(0) = 0.12$

Figure 8: Problem 2c.

3. Bifurcations.

(a) We have

$$\dot{x} = x(r - e^x).$$

It is clear that the fixed points are $x^* = 0$ and $x^* = \ln r$. Moreover, the critical value r_c satisfies

$$\frac{d}{dx}\left[x(r - e^x)\right]\bigg|_{x=x^*, r=r_c} = 0 \implies r_c - e^{x^*(r_c)}(1 + x^*(r_c)) = 0 \implies r_c = e^{x^*(r_c)}(1 + x^*(r_c)).$$

So we have two possibilities:

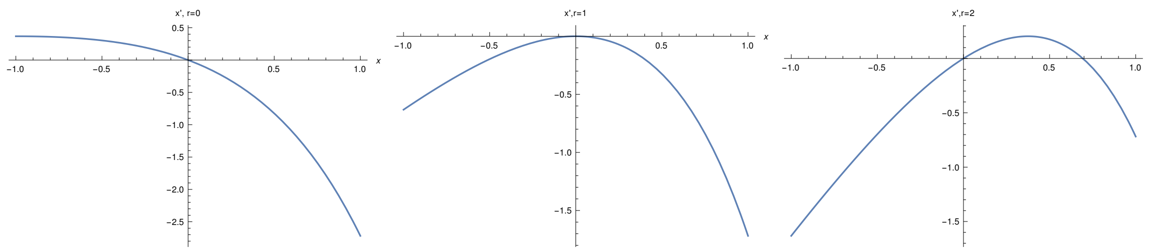
$$r_c = 1$$

and

$$r_c = r_c(1 + \ln r_c) \implies r_c = 1$$

We conclude that there is a unique critical value $\boxed{r_c = 1}$.

We may now sketch $\dot{x}$ versus x for $r = 0, 1, 2$. See Figure 9. When $r < 1$, there are two fixed points at $x^* = 0$ and at $x^* < 0$. From the $\dot{x}$ versus x plots we can see that $x^* = 0$ is stable while $x^* < 0$ is unstable. When $1 < r$, we have $x^* = 0$ is unstable and $x^* > 0$ is stable. As a result, we generate the bifurcation diagram as Figure 10, following the lecture notes' convention. Since a fixed point exists for all values of r but changes its stability as r is varied, we say that the bifurcation in this case is **transcritical.**



(a) $r = 0$, stable fixed point.

(b) $r = 1$, semi-stable fixed point.

(c) $r = 2$, unstable fixed point on the left, stable on the right.

Figure 9: $\dot{x}$ versus x with $r = 0, 1, 2$.

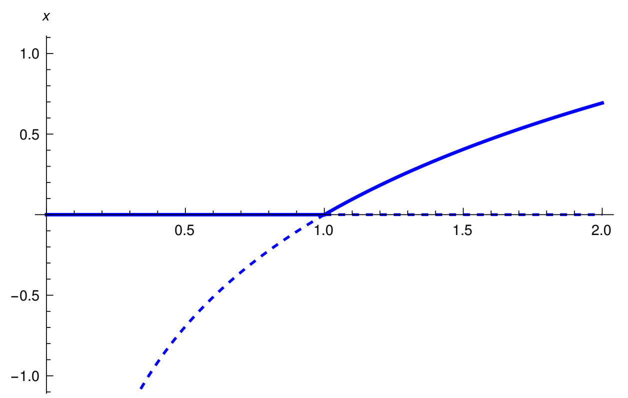


Figure 10: Problem 3a, bifurcation diagram.

(b) We have

$$\dot{x} = r + x - \ln(1 + x).$$

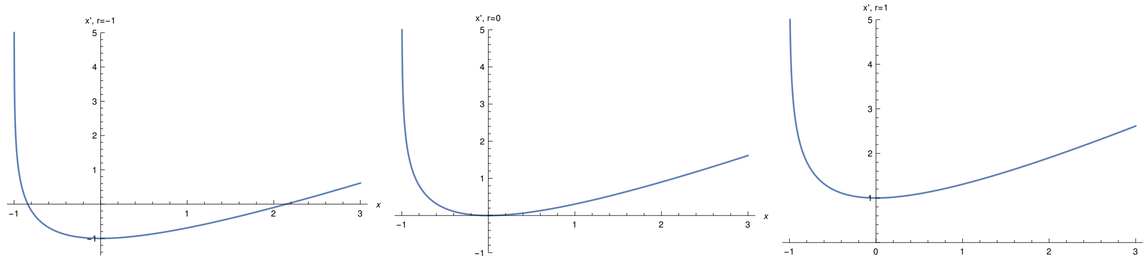
The fixed point solves the equation:

$$r = -x^* + \ln(1 + x^*)$$

The critical value r_c can be found via solving

$$0 = \frac{d}{dx}[r + x - \ln(1 + x)]\bigg|_{r_c, x^*} \implies \frac{x^*(r_c)}{1 + x^*(r_c)} = 0 \implies x^*(r_c) = 0 \implies \boxed{r_c = 0}$$

We may look at what happens when we set $r = -1, 0, 1$ in Figure 11. When $r < -1$, there is a stable fixed point on the left and an unstable fixed point on the right. There exists a unique stable fixed point



(a) $r = -1$, stable fixed point on the left, unstable on the right.

(b) $r = 0$, stable fixed point.

(c) $r = 1$, no fixed point.

Figure 11: $\dot{x}$ versus x with $r = -1, 0, 1$.

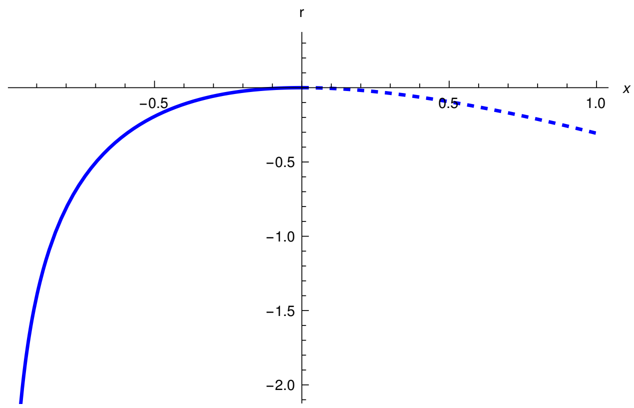


Figure 12: Problem 3b, bifurcation diagram. Here we're plotting r versus x to avoid inverting the equation $r = -x + \ln(1 + x)$

at $r = 0$ beyond which there is no fixed point. Since as we vary r two fixed points can either appear or disappear, with one stable and one unstable, we conclude that the bifurcation is of **saddle-node type.** The bifurcation diagram is shown in Figure 12, following the same convention as the lecture notes.

(c) We have

$$\dot{x} = x + \tanh rx.$$

The fixed point solves the equation

$$x^* + \tanh rx^* = 0 \implies -x^* = \tanh rx^*$$

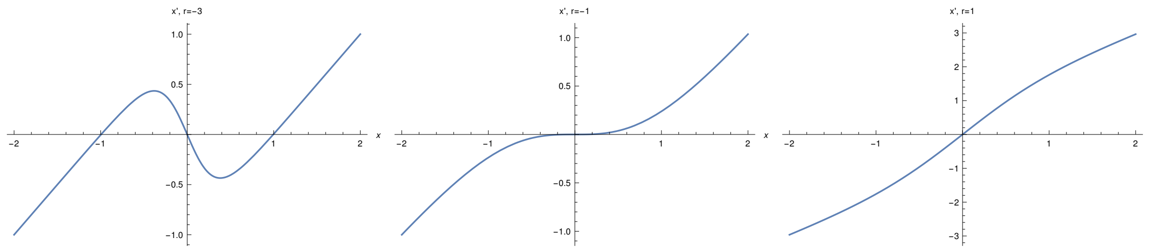
Te critical value r_c can be found via solving

$$\frac{d}{dx}[x + \tanh rx]\bigg|_{x^*, r_c} = 1 + r_c \operatorname{sech}^2(r_c x^*) = 0.$$

Both of these equations are transcendental, so a graphical approach suffices. We will deal with the first equation first, plotting $-x^*$ and $\tanh rx^*$ separately and looking for intersections of the curves to provide the position of the fied points. However, since we have Mathematica, we can also use the function `FindRoot` to find the simultaneous solution to the $\dot{x} = 0$ and $f'(x)|_c = 0$ equation. The answer is

$$\boxed{x^*(r_c) = 0 \qquad \text{and} \qquad r_c = -1}$$

We may investigate how the system behaves for $r = -3, -1, 1$ in Figure 13. When $r < -1$, there is an unstable fixed point $x^* < 0$, a stable $x^* = 0$, and an unstable $x^* > 0$. When $r = -1$, there is a semi-stable fixed point at $x^* = 0$. When $r > -1$, there is an unstable fixed point at $x^* = 0$. We see that as r is varied (is decreased to be exact), one fixed point ($x^* = 0$) is always present and changes from unstable to stable, while two unstable fixed points appear. We conclude that the bifurcation is of **subcritical pitchfork type.** The bifurcation diagram is shown in Figure 14.



(a) $r = -3$, unstable fixed point on the left, stable in the middle, unstable on the right.

(b) $r = -1$, semistable fixed point.

(c) $r = 1$, unstable fixed point.

Figure 13: $\dot{x}$ versus x with $r = -3, -1, 1$.

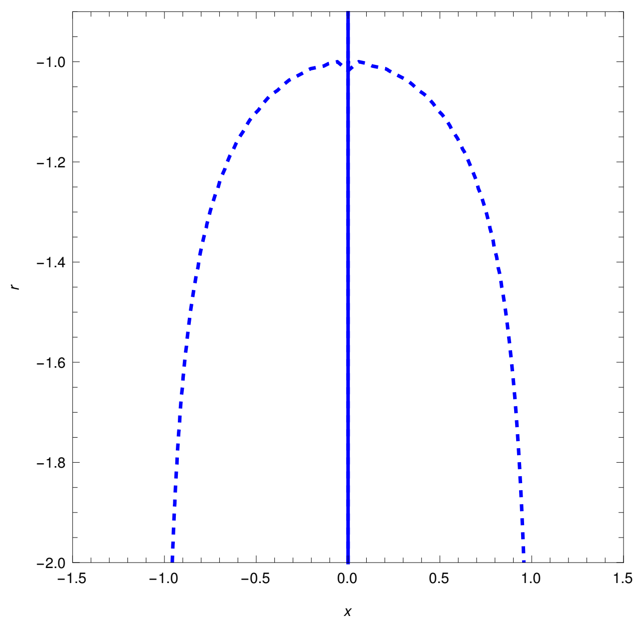


Figure 14: Problem 3c, bifurcation diagram.

4. Damped Nonlinear Oscillator.

(a) Expanding $\sin\theta$ about $n\pi$ gives

$$\sin\theta \approx (-1)^n(\theta - n\pi) + O(\theta^3)$$

If n even then we have

$$\dot{\theta} = \omega, \qquad \dot{\omega} = -\frac{1}{q}\omega - (\theta - n\pi)$$

If n odd then we have

$$\dot{\theta} = \omega, \qquad \dot{\omega} = -\frac{1}{q}\omega + (\theta - n\pi)$$

(b) Setting $q = \infty$. Then for n even, the differential equation and its solution are

$$\ddot{\theta} = -(\theta - n\pi) \implies \theta(t) = n\pi + C_1 \cos t + C_2 \sin t \implies \text{ elliptical oscillation about fixed point.}$$

For n odd, the differential equation and its solution are

$$\ddot{\theta} = (\theta - n\pi) \implies \theta(t) = n\pi + C_1 e^t + C_2 e^{-t} \implies \text{ fixed point is a saddle point.}$$

For the saddle point solution, we have

$$\omega(t) = \dot{\theta}(t) = C_1 e^t - C_2 e^{-t} \qquad \text{and} \qquad \theta(t) = n\pi + C_1 e^t + C_2 e^{-t}$$

Therefore the direction in phase space for which (ω, θ) are purely exponentially growing/decaying solution is along the $\omega = \pm(\theta - n\pi)$ line.

Mathematica code:

```
In[69]:= (*n even*)

In[43]:= DSolve[\[Theta]''[t] == -(\[Theta][t] - n*Pi), \[Theta][t], t]

Out[43]= {{\[Theta][t] -> n \[Pi] + C[1] Cos[t] + C[2] Sin[t]}}

In[70]:= (*n odd*)

In[44]:= DSolve[\[Theta]''[t] == (\[Theta][t] - n*Pi), \[Theta][t], t]

Out[44]= {{\[Theta][t] -> n \[Pi] + E^t C[1] + E^-t C[2]}}

In[45]:= D[n \[Pi] + E^t C[1] + E^-t C[2], t]

Out[45]= E^t C[1] - E^-t C[2]
```

(c) Now consider $q \neq 0$. We may solve the differential equations in Part (a) again to find that for n even,

$$\theta(t) = \pi n + c_1 e^{\frac{1}{2}\left(-\sqrt{\frac{1}{q^2}-4}-\frac{1}{q}\right)t} + c_2 e^{\frac{1}{2}\left(\sqrt{\frac{1}{q^2}-4}-\frac{1}{q}\right)t}$$

and for n odd,

$$\theta(t) = \pi n + c_1 e^{\frac{1}{2}\left(-\sqrt{\frac{1}{q^2}+4}-\frac{1}{q}\right)t} + c_2 e^{\frac{1}{2}\left(\sqrt{\frac{1}{q^2}+4}-\frac{1}{q}\right)t}$$

Mathematica code:

```
In[76]:= (*n even, with q*)

In[79]:= DSolve[\[Theta]''[
t] == -(1/q)*\[Theta]'[t] - (\[Theta][t] - n*Pi), \[Theta][t], t]

Out[79]= {{\[Theta][t] ->
n \[Pi] + E^(1/2 (-Sqrt[-4 + 1/q^2] - 1/q) t) C[1] +
E^(1/2 (Sqrt[-4 + 1/q^2] - 1/q) t) C[2]}}

In[81]:= (*n odd, with q*)

In[82]:= DSolve[\[Theta]''[
t] == -(1/q)*\[Theta]'[t] + (\[Theta][t] - n*Pi), \[Theta][t], t]

Out[82]= {{\[Theta][t] ->
n \[Pi] + E^(1/2 (-Sqrt[4 + 1/q^2] - 1/q) t) C[1] +
E^(1/2 (Sqrt[4 + 1/q^2] - 1/q) t) C[2]}}
```

We see that if $q > 1/2$ then $1/q < 2$ which means

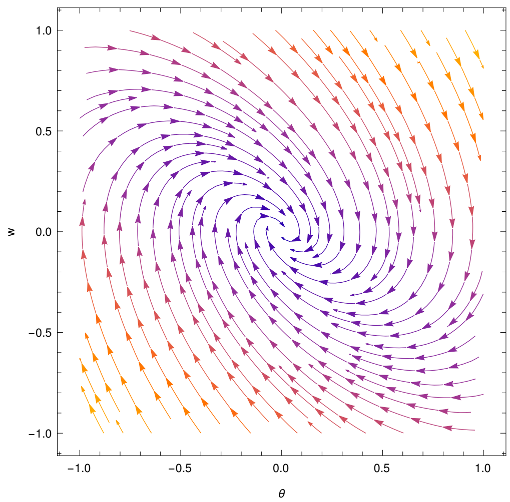
$$\sqrt{\frac{1}{q^2} - 4}$$

is imaginary (and we will call this quantity $i\Omega$), and the solution is oscillatory. We therefore see that for n odd, all fixed points are saddle points, but for n even, fixed points are attractors for both cases $q > 1/2$ (underdamped) and $q < 1/2$ (overdamped).

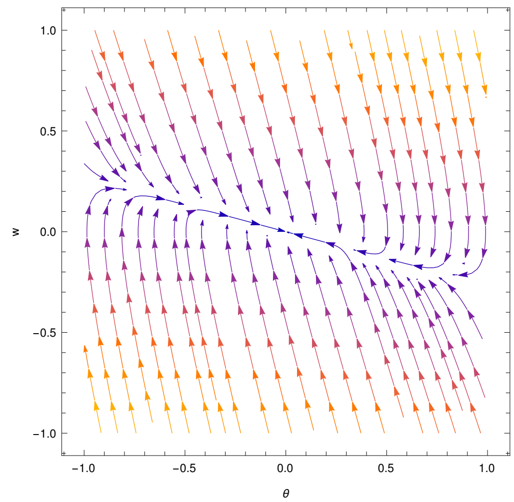
We thus consider only the n even case. If $q > 1/2$ (underdamped) then we have

$$\theta(t) = n\pi + c_1 e^{-t/2q} e^{-i\Omega t/2} + c_2 e^{-t/2q} e^{i\Omega t/2}.$$

The solution is obtained by taking the real part. If $q < 1/2$ (overdamped), then we just leave the solution as it is presented in Part (c).



(a) Underdamped



(b) Overdamped

Mathematica code:

```
(*underdamped*)

StreamPlot[{y, -1/(1/1)*y - (x - 0 Pi)}, {x, -1, 1}, {y, -1, 1},
FrameLabel -> {"\[Theta]", "w"}]

(*Overdamped*)

StreamPlot[{y, -1/(1/4)*y - (x - 0 Pi)}, {x, -1, 1}, {y, -1, 1},
FrameLabel -> {"\[Theta]", "w"}]
```

(d) For the saddle fixed points we consider n odd. The solution is once again

$$\theta(t) = \pi n + c_1 \underbrace{e^{\frac{1}{2}\left(-\sqrt{\frac{1}{q^2}+4}-\frac{1}{q}\right)t}}_{\text{exp. decay}} + c_2 \underbrace{e^{\frac{1}{2}\left(\sqrt{\frac{1}{q^2}+4}-\frac{1}{q}\right)t}}_{\text{exp. growth}}$$

Since $q > 0$, we must have that the c_1 term corresponds to the exponential decay while the c_2 term corresponds to the exponential growth term. The exponential **growth** rate is

$$\boxed{\kappa = \frac{1}{2}\left(\sqrt{\frac{1}{q^2}+4}-\frac{1}{q}\right)}$$

The angle ϕ between the direction of purely growing solution and the θ axis can be found by in the $t \longrightarrow \infty$ regime where

$$\tan\phi = \lim_{t\to\infty} \frac{\dot{\omega}(t)}{\dot{\theta}(t)} = \lim_{t\to\infty} \frac{\ddot{\theta}(t)}{\dot{\theta}(t)} = \kappa \implies \boxed{\phi = \arctan(\kappa)}$$

Mathematica code to check calculation:

```
In[104]:= dw =
D[n \[Pi] + E^(1/2 (-Sqrt[4 + 1/q^2] - 1/q) t) C[1] +
E^(1/2 (Sqrt[4 + 1/q^2] - 1/q) t) C[2], {t, 2}]

Out[104]=
1/4 E^(1/2 (-Sqrt[4 + 1/q^2] - 1/q) t) (-Sqrt[4 + 1/q^2] - 1/q)^2 C[
1] + 1/4 E^(
1/2 (Sqrt[4 + 1/q^2] - 1/q) t) (Sqrt[4 + 1/q^2] - 1/q)^2 C[2]

In[103]:= d\[Theta] =
D[n \[Pi] + E^(1/2 (-Sqrt[4 + 1/q^2] - 1/q) t) C[1] +
E^(1/2 (Sqrt[4 + 1/q^2] - 1/q) t) C[2], t]

Out[103]=
1/2 E^(1/2 (-Sqrt[4 + 1/q^2] - 1/q) t) (-Sqrt[4 + 1/q^2] - 1/q) C[
1] + 1/2 E^(
1/2 (Sqrt[4 + 1/q^2] - 1/q) t) (Sqrt[4 + 1/q^2] - 1/q) C[2]

In[107]:= Limit[dw/d\[Theta], t -> Infinity] // FullSimplify

Out[107]= ConditionalExpression[
1/2 (Sqrt[4 + 1/q^2] - 1/
q), (ConditionalExpression[1, \[Placeholder]] |
ConditionalExpression[2, \[Placeholder]]) \[Element] Reals &&
q + Sqrt[4 + 1/q^2] q^2 > 0]
```

5. Lorenz Equations.

(a) The Lorenz equations are given by

$$\begin{cases} \dot{x} = \sigma y - \sigma x \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}$$

Let $\vec{u} = (\dot{x}, \dot{y}, \dot{z})$ be the flow field. Then we have that

$$\nabla \cdot \vec{u} = \partial_x \dot{x} + \partial_y \dot{y} + \partial_z \dot{z} = -\sigma - 1 - b = -(\sigma + b + 1) < 0.$$

Therefore, the change in phase space volume over some Δt is negative:

$$\frac{\Delta V}{\Delta t} \sim \frac{dV}{dt} = \int_V \nabla \cdot \vec{u}\, dV < 0$$

We thus conclude that the phase space volime shrinks over time.

(b) The fixed points (x^*, y^*, z^*) solve the following system of equations

$$\begin{cases} 0 = \sigma y - \sigma x \\ 0 = rx - y - xz \\ 0 = -bz + xy \end{cases}$$

It is clear that $\boxed{(x^*, y^*, z^*) = 0}$ is a fixed point for all σ, b, r. For other solutions, we may solve by hand or consult Mathematica using the following command:

```
In[1]:= Solve[{0 == s*(y - x), 0 == r*x - y - x*z,
0 == -b*z + x*y}, {x, y, z}]

Out[1]= {{x -> 0, y -> 0, z -> 0}, {x -> -Sqrt[b] Sqrt[-1 + r],
y -> -Sqrt[b] Sqrt[-1 + r],
z -> -1 + r}, {x -> Sqrt[b] Sqrt[-1 + r], y -> Sqrt[b] Sqrt[-1 + r],
z -> -1 + r}}
```

From which we get

$$\boxed{(x^*, y^*, z^*) = \left(\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1\right)}$$

whenever $r > 1$.

(c) Linearing the Lorenz equations near $(x, y, z) = 0$ means that we ignore the quadratic terms xz and xy. With this we get the system

$$\begin{cases} \dot{x} = \sigma y - \sigma x \\ \dot{y} = rx - y \\ \dot{z} = -bz \end{cases} \implies \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The z-equation gives an exponential decay:

$$\boxed{z(t) = z_0 e^{-bt}}$$

so we will now only care about the xy-equations:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma \\ r & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The trace of this matrix is $\tau = -\sigma - 1 < 0$ and the determinant is $\Delta = \sigma(1 - r)$. The fixed point $(x^*, y^*, z^*) = 0$ is a saddle point if $r > 1$. Now we look at

$$\tau^2 - 4\Delta = (\sigma + 1)^2 - 4\sigma(1 - r) = 1 + \sigma(-2 + 4r + \sigma) > 0$$

if $r < 1$. And so at $r < 1$, $(x^*, y^*, z^*) = 0$ is a stable fixed point. At $r = 1$, As r is increased, the stable fixed point $(x^*, y^*, z^*) = 0$ becomes unstable while new stable fixed points emerge. Thus, at $r = 1$, we have a supercritical bifurcation.