MA355 Midterm 1

Colby College — Spring 2021

due Friday, March 19, by 8:30 EST

Please upload your solutions (preferably as a single PDF file) to Moodle.

Explain/justify all answers!

Problems

1. (25 points)
   (a) An instructor has 13 students in her combinatorics class and wants to divide them into four groups with at least three students in each group. How many ways are there to do that? What if the instructor also picks a spokesperson for each group?
   (b) Five weeks after the beginning of the semester and right after the first midterm, the instructor divides the students into a new set of four groups, again with at least three students in each group. The instructor would like the new groups to be such that no two students who previously were in the same group are again in the same group. How many ways are there to do that?
   (c) Answer all the above questions again, but for 12 students instead of 13.

2. (25 points)
   (a) Let (a_i, b_i, c_i) for $1 \leq i \leq 9$ be nine vectors in $\mathbb{R}^3$ with integer coordinates. Show that two of these vectors have a sum whose coordinates are all even integers.
   (b) Show that this result is best possible, i.e. the conclusion may fail if we have only eight vectors.

3. (25 points) Chess is a game played on an 8×8 board with alternating light and dark squares. The rows of the board are called "ranks" and are numbered 1 through 8. There are two players, white and black, and each player has a set of 16 pieces: 8 pawns, 2 rooks, 2 bishops, 2 knights, 1 queen, and 1 king. Count the number of possible starting positions for this game given the following constraints:
   (a) All the white pieces are placed in the first and second ranks, and all the black pieces are placed in the seventh and eighth ranks.
   (b) The black starting position is a mirror image of the white starting position.
   (c) All the white pawns are placed in the second rank.

(d) The white king is placed on any square between the two white rooks (not necessarily adjacent to either of them).

(e) The white bishops are placed on opposite-color squares.

4. (25 points)

(a) A circle of 17 friends has the property that no matter how we choose two from these 17 friends, those two people correspond with each other on one of three given subjects. Prove that there are three friends among the circle of these 17 friends such that any two of the three of them correspond with each other on the same subject.

(b) Let

$$n_k = k!\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!}\right) + 1$$

We color all edges of K_{n_k} with one of k colors. Prove that there will be a triangle with monochromatic edges (i.e., edges of a single color).