Physics 8.321, Fall 2021
Homework #2

Due **Friday, October 1** by 6:00 PM.

1. A skew-Hermitian operator A is an operator satisfying $A^\dagger = -A$.

(a) Prove that A can have at most one real eigenvalue (which may be degenerate).

(b) Prove that the commutator of two Hermitian operators is skew-Hermitian.

2. Show that if H and K are both Hermitian operators with non-negative eigenvalues, then

$$\mathrm{Tr}\; HK \geq 0,$$

and that equality implies that $HK = 0$.

3. Consider a Hermitian operator H whose eigenvectors form a complete orthonormal set, and whose eigenvalues are all positive.

(a) Prove that for any two vectors $|\alpha\rangle, |\beta\rangle$

$$|\langle\alpha|H|\beta\rangle|^2 \leq \langle\alpha|H|\alpha\rangle\langle\beta|H|\beta\rangle$$

(b) Prove that $\mathrm{Tr}\;(H) > 0$.

4. Let U be a unitary operator. Consider the eigenvalue equation

$$U|\lambda\rangle = \lambda|\lambda\rangle\,.$$

(a) Prove that λ is of the form $e^{i\theta}$ with θ real.

(b) Show that if $\lambda \neq \mu$ then $\langle\mu|\lambda\rangle = 0$.

5. (a) Show that the set of N x N complex matrices form a vector space of dimension N^2.

(b) Show that $\mathrm{Tr}\;(A^\dagger B)$ defines an inner product on this vector space.

(c) Show that the set of 2 x 2 matrices is spanned by the basis

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

How can these matrices be used to form an orthonormal basis?

(d) Find the spectrum and eigenvectors for each of the matrices in (c)

(e) Prove that if $\mathbf{A}, \mathbf{B}$ are two vector operators that commute with σ, it follows that

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})\mathbb{1} + i\sigma \cdot \mathbf{A}\mathbf{x}\mathbf{B}$$

(f) Prove that

$$\exp(i\theta\,\sigma\cdot\mathbf{n}) = \cos\theta + i\,\sigma\cdot\mathbf{n}\,\sin\theta$$

where $\mathbf{n}$ is a unit 3-vector.

6. [Sakurai and Napolitano Problem 24, Chapter 1 (page 63)]

(a) Prove that $(1/\sqrt{2})(1+i\sigma_x)$ acting on a two-component spinor can be regarded as the matrix representation of the rotation operator about the x-axis by angle $-\pi/2$. (The minus sign signifies that the rotation is clockwise.)

(b) Construct the matrix representation of S_z when the eigenkets of S_y are used as base vectors.

7. [Sakurai and Napolitano Problem 26, Chapter 1 (page 63)]

Construct the transformation matrix that connects the S_z diagonal basis to the S_x diagonal basis. Show that your result is consistent with the general relation

$$U = \sum_r |b^{(r)}\rangle\langle a^{(r)}|\,.$$