Spring, 2021

Physics 312: Physics of Fluids
Assignment #2 (Solutions)

Background Reading

Friday, Feb. 19: Tritton 5.1 - 5.3, 5.5, 6.1, 6.2,
Kundu & Cohen 3.1 - 3.5

Monday, Feb. 22: Kundu & Cohen 2.1 - 2.4, 3.6, 3.7

Wednesday, Feb. 24: Kundu & Cohen 2.5, 2.7 - 2.9

Informal Written Reflection

Due: Thursday, February 25 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, February 26 (in class)

1. Explain why particle paths, streamlines, and streaklines are identical for *steady flow*. Do this in three steps...

   (a) First, using the definition of steady flow, explain why fluid particles always move along streamlines.

   (b) If a particle is moving along one streamline, can it ever move onto a different streamline? Why or why not?

   (c) Next, using a similar argument, explain why the fluid particles that make up a streaklines are all following the same path.

Solution:

(a) A fluid element's trajectory must be tangent to the velocity field at each point the element passes through. At any instant in time, there is a particular streamline that is tangent to the trajectory at the current position. These two curves part company if, in the time it takes the fluid element to move to another position, the velocity field is changing. For steady flow this isn't happening and, therefore, each particle's path is also a streamline.

(b) Since streamlines never cross, it is not possible for a particle path to hop from one streamline to another in steady flow.

(c) If hopping from one streamline to another isn't possible, then particles starting at different times on the same (steady) streamline will follow the same trajectory and, therefore, streaklines coincide with streamlines and particle paths in steady flow. This is helpful, since it means we can use dye injection to help us map out streamlines and/or particle paths under the right conditions.

2. Both of your textbooks note that a flow that is steady in one reference frame may not be steady in another. Read through Kundu and Cohen sections 3.4 and 3.5 and Tritton sections 6.1 and 6.2. Then give a rough interpretation of the flow fields plotted in Kundu and Cohen's figure 3.8. Why, for example, does the velocity point upstream in one frame and downstream in another?

   Solution:

   In Kundu and Cohen's example, the boat has to push fluid out of the way as it moves. This is why, in the reference frame of an observer on the river bank, the velocity field points forward ahead of the ship and circles around behind it as it passes. In the reference frame of the ship, however, fluid is constantly passing around the ship. The relationship between the two pictures get more complicated, of course, when you look carefully at the formation of eddies and their consequences for particle paths. Tritton discusses this in detail.

3. Developing a sense of comfort with index notation is an important part of your basic training in fluid dynamics. Indeed, without this notation, we would have had a much harder time deriving and understanding the fluid dynamical expressions of fundamental conservation laws. In this problem, you will get some more practice working with indices...

   Consider each of the following tensor expressions. Which are allowed under the standard rules of index notation? If any of these expressions are not allowed, explain what the problem is.

   (a) $a = b_i c_{ij} d_j$
   (b) $a = b_i c_i + d_j$
   (c) $a_i = \delta_{ij} b_i + c_i$
   (d) $a_k = b_k c + d_i e_{ik}$
   (e) $a_i = b_i + c_{ij} d_{ji} e_i$

   Solution:

   There are three basic rules. First, every term on both sides of any equation must have matching free indices, which are indices that appear exactly once. Second, double indices imply summation over a dummy index. Finally, no index can appear more than twice in the same term.

   (a) This formula is fine.
   (ii) The last term on the right has j as a free index, while none of the other terms have free indices (not allowed).
   (iii) It's unclear whether i is a free index or a dummy index here. The first term on the right has j as a free index and a sum over i, while every other term is indexed by a free i (not allowed).

(iv) This formula is fine.

(v) This formula has more than two i's in the last term (not allowed).

4. If A is a second-order tensor, show that the quantity A_{ii} is invariant under rotation of the coordinate axes. This quantity has a special name in linear algebra... Do you recognize it?

(Hint: Use the second-order transformation rule,

$$A'_{mn} = R_{mi} R_{nj} A_{ij},$$

and the fact that R is an orthogonal matrix: the inverse of R is equal to its transpose. Show that $A'_{ii} = A_{ii}$.)

Solution:

First, replace n with m everywhere in the transformation rule:

$$A'_{mm} = R_{mi} R_{mj} A_{ij}.$$

Next, make use of the fact that R is orthogonal:

$$\begin{aligned} A'_{mm} &= R^T_{im} R_{mj} A_{ij} \\ &= R^{-1}_{im} R_{mj} A_{ij}. \end{aligned}$$

The product of R with its inverse introduces a Kronecker delta, which demands $i = j$:

$$A'_{mm} = R^T_{im} R_{mj} A_{ij} = \delta_{ij} A_{ij} = A_{ii}.$$

In linear algebra, A_{ii} is known as the *trace* of the matrix A and, as you've seen here, trace is invariant under coordinate transformations.

5. Show that δ_{ij} is an isotropic tensor. That is, show that rotation of the coordinate axes does not change any of the components of δ_{ij}.

   (Hint: See the hint for previous problem...)

   Solution:

   Again, use the transformation rule:

$$\delta'_{mn} = R_{mi} R_{nj} \delta_{ij}.$$

   The Kronecker delta demands $i = j$ and, again, we can make use of the matrix transpose and inverse:

$$\delta'_{mn} = R_{mi} R_{ni} = R_{mi} R^T_{in} = R_{mi} R^{-1}_{in} = \delta_{mn}.$$