MA434, Spring 2020 — Final Exam

This final exam is due on or before **May 15, 2020**. This is a hard deadline, because grades are due shortly after that date.

The test contains 12 questions, but you don't need to solve all of them. The number of points for each question is indicated. The problems are *not* arranged in order of difficulty. Several of them are either straight out of our textbook or closely related to material in the textbook.

You should solve enough problems to get at least 100 points, but the total value of the problems (full problems, not fractions of problems) you turn in can be no more than 120 points. The maximum score you can get is 100, so doing an extra 20 points' worth only buys you insurance. When I grade, I will go until I have graded problems worth up to the limit of 120 points, then stop.

Rules of the Game: While you work on this test, you may consult Miles Reid's *Undergraduate Algebraic Geometry*, your class notes and everything posted on the course Moodle site. You may use *Sage* or GP or other mathematical software to do computations. You may use an Abstract Algebra textbook for reference; if you do, please tell me what book you have used. No other reference materials are allowed. In particular, you should not search the internet for solutions.

You should work entirely by yourself. You may talk to me by Zoom or consult me by email (though I don't promise to answer every question you might have), but you may not contact anyone else. (Complaining and asking for sympathy are ok, but don't discuss the actual content of the test.)

Finally, you should really *write up* (and not just write down) your solutions: they should read as if they were an example in a well-written textbook. Make sure that you explain carefully your line of reasoning at each point; your text should be such that another student at about your level could follow the steps without having to ask you for help. To achieve this goal, be as verbose as necessary — it is better to write too much than too little. Solutions should be written as carefully and legibly as possible. (The professor has old eyes.) *Don't turn in first-draft material.*

Good luck!

1. [10 points] Show that the affine variety in $\mathbb{A}^2$ defined by $xy = 1$ is not isomorphic to $\mathbb{A}^1$.

2. [20 points] Given two affine varieties V and W and a polynomial map $f : V \longrightarrow W$, we get a ring homomorphism $f^* : k[W] \longrightarrow k[V]$. When is f^* injective?

3. [20 points] Consider the polynomial map $\varphi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$ defined by $\varphi(t) = (t^2, t^5)$ (one of the maps considered in problem 4.3). Let C_0 be the image of φ. We showed in that problem that there is a rational map $\pi : \mathbb{A}^2 \dashrightarrow \mathbb{A}^1$ so that $\pi|_{C_0}$ is the inverse of φ.

a. Show that C_0 is an irreducible algebraic subset of $\mathbb{A}^2$, i.e., an affine variety.

b. Find a rational map $\tilde{\varphi} : \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$ that extends φ.

c. Let C be the projective completion C_0, so that C is a projective variety. Show that C is the image of $\tilde{\varphi}$.

d. Find a rational map $\tilde{\pi} : C \dashrightarrow \mathbb{P}^1$ that extends $\pi|_{C_0}$.

e. Determine the domains of $\tilde{\pi}$ and $\tilde{\varphi}$.

4. [20 points] Suppose that f is a rational function on $\mathbb{P}^1$.

a. Show that if f is regular at every point of $\mathbb{P}^1$ then it is constant. (Hint: consider the two affine pieces $\mathbb{A}^1_{(0)}$ and $\mathbb{A}^1_{(1)}$.)

b. Show that there are no non-constant morphisms $\mathbb{P}^1 \longrightarrow \mathbb{A}^m$.

5. [20 points] Below are three formulas that possibly define rational maps $f : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$. Decide whether the formulas do define rational maps. If they do, determine $\mathrm{dom}(f)$ and decide whether f is birational.

a. $f([x : y : z]) = [1/x : 1/y : 1/z]$.

b. $f([x : y : z]) = [x : y : 1]$.

c. $f([x : y : z]) = [\frac{x^3+y^3}{z^3} : \frac{y^2}{z^2} : 1]$.

6. [10 points] Prove statements (i), (ii), (iii), (iv) from Example I from section 5.7 of *Undergraduate Algebraic Geometry*.

7. [20 points] Consider the rational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by

$$f([u : v : w]) = [uw : w^2 : vw].$$

Describe the domain and the image of f.

8. [25 points] Solve problem 5.2 in *Undergraduate Algebraic Geometry*.

9. [25 points] Solve problem 5.11 in *Undergraduate Algebraic Geometry*.

10. [25 points] Recall that a hyperfurface is a variety in $\mathbb{P}^n$ defined by a single polynomial (equivalently, by a principal ideal). Show that every variety is birational to a hypersurface. (Hint: Use the Noether Normalization Theorem.)

11. [30 points] Given an invertible matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with complex coefficients, define a function $f_A : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ by

$$f_A([u : v]) = [au + bv : cu + dv].$$

a. Show that f_A is a morphism.

b. How does f_{AB} relate to f_A and f_B?

c. Show that f_A has an inverse morphism, so that f_A defines an automorphism of $\mathbb{P}^1_C$.

d. If we identify $\mathbb{C}$ with the standard $\mathbb{A}^1 \subset \mathbb{P}^1$ defined by $v \neq 0$, show that the restriction of f_A to $\mathbb{C}$ is a rational function, and find its formula.

12. [30 points] Show that any automorphism of $\mathbb{P}^1_{\mathbb{C}}$ is of the form f_A as in the previous problem.