Summarizing:

$$S_{fi} = \delta_{fi} - 2\pi i\, \delta^{(T)}(E_f - E_i)\left[V_{fi} + \lim_{\eta\to 0^+} \sum_k \frac{V_{fk} V_{ki}}{E_i - E_k + i\eta}\right] + \mathcal{O}(V^3)$$

$$S_{fi} = \delta_{fi} - 2\pi i\, \delta(E_f - E_i)\ \mathcal{T}_{fi}$$

$$\mathcal{T}_{fi} = \langle \varphi_f | V | \varphi_i \rangle + \langle \varphi_f | V \frac{1}{E_i - H + i\eta} V | \varphi_i \rangle$$

check: $\dfrac{1}{A} = \dfrac{1}{B} + \dfrac{1}{B}(B-A)\dfrac{1}{A}$

$$A = E_i - H + i\eta$$
$$B = E_i - H_0 + i\eta$$

$$\Rightarrow \frac{1}{E_i - H + i\eta} = \frac{1}{E_i - H_0 + i\eta} + \frac{1}{E_i - H_0 + i\eta} V \frac{1}{E_i - H + i\eta}$$

Transition Probabilities:

a) $|\varphi_f\rangle$ different $|\varphi_i\rangle$

$$P_{fi}(T) = |S_{fi}|^2 = 4\pi^2 \left[\delta^{(T)}(E_i - E_f)\right]^2 \times$$
$$\times \left| V_{fi} + \lim_{\eta\to 0^+} \sum_k \frac{V_{fk} V_{ki}}{E_i - E_k + i\eta} + \dots \right|^2$$

$$\left|\delta^{(T)}(E_f - E_i)\right|^2 = \frac{1}{\pi^2} \frac{\sin^2((E_f - E_i)T/2\hbar)}{(E_f - E_i)^2}$$

$$\int_{-\infty}^{\infty} dE_f \left(\delta^{(T)}(E_f - E_i)\right)^2 = \frac{T}{2\pi\hbar}$$

b) Two discrete states:

$$P_{fi}(T) = \frac{4|V_{fi}|^2}{(E_f - E_i)^2} \sin^2\left(\frac{(E_f - E_i)T}{2\hbar}\right)$$

exact formula:

$$P_{fi}(T) = \frac{4|V_{fi}|^2}{(E_f - E_i)^2 + 4|V_{fi}|^2} \sin^2\left(\frac{T}{2\hbar}\sqrt{(E_f - E_i)^2 + 4|V_{fi}|^2}\right)$$

same! To lowest order in V_{fi}

$E_f = E_i$: $\quad P_{fi}(T) = |V_{fi}|^2 T^2/\hbar^2$

grows like $T^2 \to$ beginning of Rabi oscillation

Atom - light interaction: Assume only discrete states:

$$|\varphi_i\rangle = |a, n_j\rangle$$
$$|\varphi_f\rangle = |b, n_j - 1\rangle$$

Neglect spontaneous emission

$$\langle \varphi_f | \hat{H}_{I_1} | \varphi_i \rangle = \frac{\hbar \Omega_1}{2}$$

$$\Omega_1 = -\frac{q}{m} \frac{2}{\sqrt{2\varepsilon_0 \hbar \omega_j \cdot V}} \langle b | \hat{\vec{p}} \cdot \vec{\varepsilon}_j | a \rangle \sqrt{n_j}$$

$$P_{i \to f} = \frac{\Omega_1^2}{\Omega_1^2 + \delta^2} \sin^2\left(\sqrt{\Omega_1^2 + \delta^2}\, \frac{T}{2}\right)$$

detuning $\delta = \omega_j - \frac{E_b - E_a}{\hbar}$

Transition into a continuum:

$$\delta P(E_f, \beta, T) = \int_{\substack{E \in \delta E_f \\ \beta \in \delta\beta_f}} dE\, d\beta\, \rho(E,\beta) \, |\langle E, \beta | \tilde{U}(T) | \varphi_i \rangle|^2$$

$$= \int dE\, d\beta \quad 4\pi^2 \, |v(E, \beta; \varphi_i)|^2 \underbrace{\left[\delta^{(T)}(E - E_i)\right]^2}_{\approx \frac{T}{2\pi\hbar}\, \delta^{(T)}(E-E_i)}$$

$\Rightarrow$ transition probability per unit time:

$$\delta w(E_f, \beta_f) = \frac{1}{T}\, \delta P(E_f, \beta_f, T)$$

$$= \frac{2\pi}{\hbar} \int dE\, d\beta\, \rho(E,\beta)\, |v(E,\beta;\varphi_i)|^2\, \delta^{(T)}(E-E_i)$$

$$\boxed{\frac{\delta w}{\delta \beta} = \frac{2\pi}{\hbar}\, |v(E_f = E_i; \beta_f; \varphi_i)|^2\, \rho(E_f = E_i, \beta_f)}$$

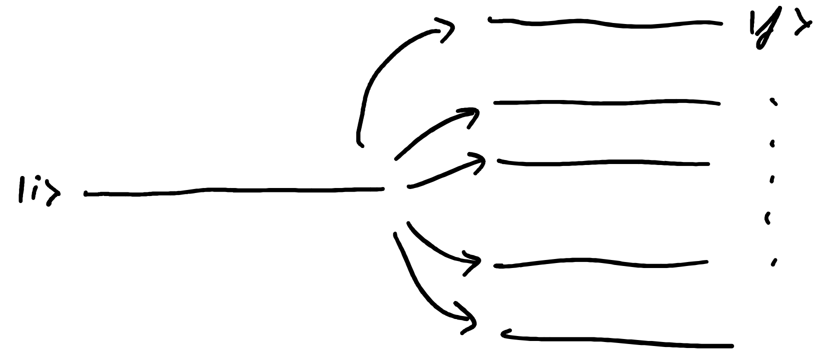
Fermi's Golden rule

Transition between continua:

divide $\frac{dw}{d\Omega}$ by incoming flux $\Phi_i = \frac{c}{L^3} n_e$

$$\frac{d\Gamma}{d\Omega} = c\, \frac{n_e}{L^3}\, \frac{d\sigma}{d\Omega} = \Phi_i\, \frac{d\sigma}{d\Omega}$$

$\frac{d\sigma}{d\Omega}$ – scattering cross section per unit angle



P_{fi} A $P_{fi} \sim T$?

B $P_{fi} \sim T^2$?

C $P_{fi} \sim$ more complicated ?

$$S_{fi} = \delta_{fi} - 2\pi i \, \delta^{(T)}(E_f - E_i) V_{fi} + \dots$$

$$\forall f \neq i: \; P_{fi} = |2\pi i \, \delta^{(T)}(E_f - E_i) \, V_{fi}|^2$$

$$= T \cdot \frac{2\pi}{\hbar} |V_{fi}|^2 \delta^{(T)}(E_f - E_i)$$

$$E_f \approx E_i: \quad P_{if} = T^2 \frac{|V_{fi}|^2}{\hbar^2}$$

Quadratic in T!
Beginning of Rabi oscillation

$P_{ii} = ?$ A $P_{ii} = 1 - \# T$

B $P_{ii} = 1 - \# T^2$

C   more complicated

$$P_{ii} = 1 - \sum_{f \neq i} P_{fi}$$

$$= 1 - T \underbrace{\frac{2\pi}{\hbar} \sum_{f \neq i} |V_{fi}|^2 \delta^{(T)}(E_f - E_i)}_{\Gamma}$$

$$= 1 - \Gamma T$$ linear in T!

$(= e^{-\Gamma T}$ for small $T)$ beginning of exponential decay

$$S_{fi} = \delta_{fi} - 2\pi i\, \delta^{(T)}(E_i - E_f)\left[V_{fi} + \sum_k \frac{V_{ik} V_{kf}}{E_i - E_k + i\eta} \dots \right]$$

$$P_{ii}(T) = \left| 1 - 2\pi i\, \delta^{(T)}(0)\left[\underset{\substack{\downarrow \\ 0 \text{ incorporate into } H_0}}{V_{ii}} + \sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} + \dots \right] \right|^2$$

$$= \left| 1 - 2\pi i \frac{T}{2\pi\hbar} \sum_k \frac{|V_{ki}|^2}{E_i - E_k + i\eta} + \dots \right|^2$$

$$= \underline{\left| 1 - i T \frac{1}{\hbar} \sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} + \dots \right|^2}$$

Introduce states $|k\rangle$ as a continuum:

$$|V_{ik}|^2 \longrightarrow |V(E)|^2$$

$$\sum_k \longrightarrow \int dE \int d\beta\, \rho(E,\beta)$$

$$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} \longrightarrow \int dE\, d\beta\, \rho(E,\beta) \frac{|V(E)|^2}{E_i - E + i\eta}$$

Math excursion: $\frac{1}{x + i\eta} = \frac{1}{x + i\eta} \frac{x - i\eta}{x - i\eta} = \frac{x}{x^2 + \eta^2} - i \frac{\eta}{x^2 + \eta^2}$

$$\lim_{\eta \to 0} \int dx\, f(x) \frac{x}{x^2 + \eta^2} \approx \lim_{\eta \to 0} \left(\int_{-\infty}^{-\eta} dx + \int_{\eta}^{\infty} dx \right) \frac{f(x)}{x} + \int_{-\eta}^{\eta} dx\, f(x) \frac{x}{x^2 + \eta^2}$$

$$= P \int_{-\infty}^{\infty} dx \frac{f(x)}{x} + f(0) \underbrace{\int_{-\eta}^{\eta} dx \frac{x}{x^2 + \eta^2}}_{0}$$

So $\lim_{\eta \to 0} \int dx \frac{x f(x)}{x^2 + \eta^2} = P \int dx \frac{f(x)}{x}$

$$\lim_{\eta \to 0} \int dx\, f(x) \underbrace{\frac{\eta}{x^2 + \eta^2}}_{\substack{\text{peaked at } x = 0 \\ \text{height} : \frac{1}{\eta};\ \text{width} \sim \eta}} \approx f(0) \int_{-\infty}^{\infty} dx \frac{\eta}{x^2 + \eta^2} = \pi f(0)$$

$$\Rightarrow \lim_{\eta \to 0} - \frac{i\eta}{x^2 + \eta^2} = - i\pi\, \delta(x)$$

$$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} = \mathcal{P} \int dE\, d\beta\, \rho(E,\beta) \frac{|V(E,\beta)|^2}{E_i - E}$$

$$- i\pi \int dE\, d\beta\, \rho(E,\beta)\, |V(E,\beta)|^2\, \delta(E_i - E)$$

$$= \Delta - i\pi \int d\beta\, \rho(E_i,\beta)\, |V(E_i,\beta)|^2$$

$$= \Delta - i\,\frac{\hbar\Gamma}{2}$$

$$P_{ii} = \left| 1 - iT \frac{1}{\hbar} \left(\Delta - i\hbar \frac{\Gamma}{2} \right) \right|^2$$

$$= \left| 1 - T\frac{\Gamma}{2} - iT\frac{\Delta}{\hbar} \right|^2$$

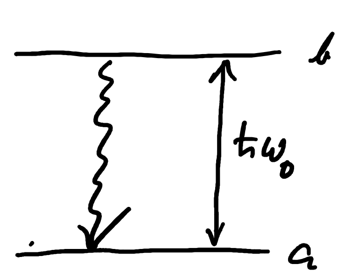
$$= 1 - \Gamma T + \text{higher order terms}$$

Non-perturbative

$$S_{ii} = \underset{\substack{\uparrow \\ \text{decay}}}{e^{-\Gamma T/2}}\ \underset{\substack{\uparrow \\ \text{Level shift}}}{e^{-i\Delta T/\hbar}}$$

$P_{ii} = |S_{ii}|^2 = e^{-\Gamma T}$ exponential decay.

Spontaneous Emission



Continuum of final states

$$\vec{k}_j = \frac{2\pi}{L}\left(n_x \vec{e}_x + n_y \vec{e}_y + n_z \vec{e}_z\right)$$

$$\omega_j = c k_j$$

Fermi's Golden Rule:

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar}\left|\langle a, \vec{k}\vec{\varepsilon}\,|H_I|\, b, 0\rangle\right|^2 \rho(\vartheta, \varphi, \hbar\omega - \hbar\omega_0)$$

$$\langle a, \vec{k}\vec{\varepsilon}\,|-\hat{\vec{d}}\cdot\hat{\vec{E}}\,|\, b, 0\rangle = \langle a|\hat{\vec{d}}|b\rangle \cdot \langle \vec{k}\vec{\varepsilon}\,|-i\sum_\ell \mathcal{E}_\ell \vec{\varepsilon}_\ell (a_\ell - a_\ell^\dagger)|0\rangle$$

$$(\hbar c k = \hbar\omega_0) \qquad = i\, d\, \vec{e}_z \cdot \mathcal{E}_k \vec{\varepsilon}$$

$$= i\sqrt{\frac{\hbar\omega_0}{2\varepsilon_0 V}}\; d\, (\vec{\varepsilon}\cdot\vec{e}_z)$$

density of states: $dN = \rho\, dE\, d\Omega \qquad E = \hbar c k$

$$= \left(\frac{L}{2\pi}\right)^3 d^3k = \left(\frac{L}{2\pi}\right)^3 k^2 dk\, d\Omega$$

$$= \frac{V}{(2\pi)^3}\frac{E^2}{(\hbar c)^3}\, dE\, d\Omega$$

$$\Rightarrow \quad \rho(\vartheta, \varphi, E) = \frac{V}{(2\pi)^3}\frac{E^2}{(\hbar c)^3}$$

$$\frac{d\Gamma}{d\Omega} = \frac{2\pi}{\hbar}\,\frac{\hbar\omega_0}{2\varepsilon_0 \cancel{V}}\, d^2(\vec{\varepsilon}\cdot\vec{e}_z)^2 \cdot \frac{\cancel{V}}{(2\pi)^3}\frac{(\hbar\omega_0)^2}{(\hbar c)^3}$$

$$= \frac{1}{8\pi^2\varepsilon_0}\,\frac{\omega_0^3}{\hbar c^3}\, d^2(\vec{\varepsilon}\cdot\vec{e}_z)^2$$

Sum up over all directions ϑ, φ and all polarizations.

$\vec{\varepsilon}, \vec{\varepsilon}'$ and $\vec{k}/|\vec{k}|$ are ortho normal:

$$1 = \underbrace{(\vec{\varepsilon}\cdot\vec{e}_z)^2 + (\vec{\varepsilon}'\cdot\vec{e}_z)^2}_{1-\cos^2\vartheta = \sin^2\vartheta} + \underbrace{\left(\frac{\vec{k}}{|\vec{k}|}\cdot\vec{e}_z\right)^2}_{\cos^2\vartheta}$$

$$\int d\Omega \, \sin^2\vartheta = 2\pi \int_{-1}^{1} d(\cos\vartheta)\, \sin^2\vartheta = 2\pi \int_{-1}^{1} du\, (1-u^2)$$

$$= 2\pi \left(2 - \frac{2}{3}\right) = \frac{8\pi}{3}$$

$$\boxed{\Gamma = \frac{d^2 \omega_0^3}{3\pi \varepsilon_0 \hbar c^3}}$$

$d = q\, z_{ab}$ and $\frac{q^2}{4\pi\varepsilon_0 \hbar c} = \alpha \approx \frac{1}{137}$

$$\Gamma = \frac{4}{3}\, \alpha \, \frac{\omega_0^3 \, z_{ab}^2}{c^2}$$

What's the Q of the atomic oscillator?

$$\frac{1}{Q} = \frac{\Gamma}{\omega_0} = \frac{4}{3}\, \alpha \underbrace{\frac{\omega_0^2\, z_{ab}^2}{c^2}}_{\alpha^2}$$

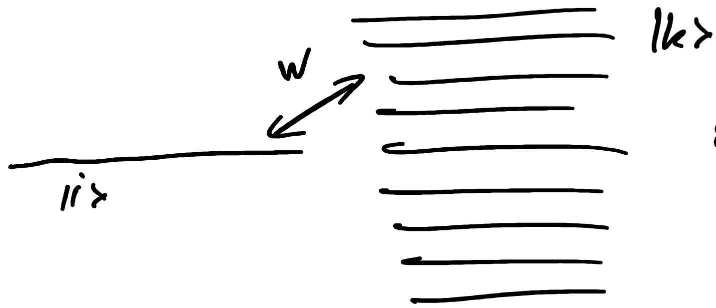
$$\hbar\omega_0 = \alpha^2 m c^2$$

$$z_{ab} = a_0 = \frac{\hbar^2}{me^2} = \frac{1}{\alpha}\frac{\hbar}{mc}$$

$$\frac{1}{Q} = \frac{\Gamma}{\omega_0} \approx \alpha^3$$

$$Q = \frac{1}{\alpha^3} \simeq \text{few } 10^6$$

Wigner – Weisskopf treatment of spont. emission



$$\varepsilon_k = k\,\varepsilon \quad \text{(say)}$$

Fermi's Golden Rule: $\Gamma = \frac{2\pi}{\hbar} |W|^2 \cdot \frac{1}{\varepsilon}$

Wave function: $|\psi(t)\rangle = \gamma_i(t)\,|i\rangle + \sum_{k=-\infty}^{\infty} \gamma_k(t)\, e^{-i\varepsilon k t/\hbar} |k\rangle$

$$\begin{cases} i\hbar \frac{d}{dt}\gamma_i(t) = W \sum_{k=-\infty}^{k} \gamma_k(t)\, e^{-ik\varepsilon t/\hbar} \\ i\hbar \frac{d}{dt}\gamma_k(t) = W\, e^{ik\varepsilon t/\hbar}\, \gamma_i(t) \end{cases}$$

Initially $\gamma_k(0) = 0$ so

$$\gamma_k(t) = \frac{W}{i\hbar} \int_0^t dt'\, \gamma_i(t')\, e^{ik\varepsilon t'/\hbar}$$

$$\Rightarrow \quad \frac{d}{dt}\gamma_i(t) = -\frac{\Gamma}{2\pi\hbar} \int_0^t dt'\, \gamma_i(t') \underbrace{\left[\sum_{k=-\infty}^{\infty} \varepsilon\, e^{ik\varepsilon(t'-t)/\hbar}\right]}_{\simeq \int_{-\infty}^{\infty} dE\, e^{iE(t'-t)/\hbar} = 2\pi\hbar\,\delta(t'-t)}$$

$\tau = t' - t$

$$\frac{d}{dt}\gamma_i(t) = -\Gamma \int_{-t}^{0} d\tau\, \delta(\tau)\, \gamma_i(t+\tau)$$

$$= -\frac{\Gamma}{2}\gamma_i(t)$$

$$\Rightarrow \quad \gamma_i(t) = e^{-\frac{\Gamma}{2}t}$$

$$P_{ii}(t) = e^{-\Gamma t}$$

Notes on spontaneous emission:

- $P_b = e^{-\Gamma t}$

  (not like $1 - \Gamma T + \frac{\Gamma^2 t^2}{2} + \dots$, as perturbation theory)

- Q: What about shifts of the excited level b?

  A - There is a shift

  B - There is no shift

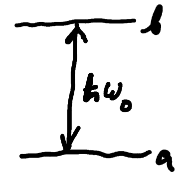
  $$\sum_k \frac{|V_{ik}|^2}{E_i - E_k + i\eta} = \Delta - i\frac{\hbar\Gamma}{2}$$

- Γ can be modified! Dan Kleppner PRL 47, 233

  e.g. use cavity $\Rightarrow$ discretizes spectrum

  If one of the modes is resonant

  $|\varphi_i\rangle = |b, 0\rangle$; $|\varphi_f\rangle = |a, 1\rangle$

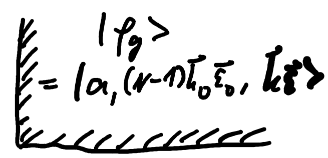


  $\Rightarrow$ Rabi oscillations

  If all modes are different: atom cannot emit!

- Consider absorption of a monochromatic wave

  $$|\varphi_i\rangle = |a; N\vec{k}_0\vec{\varepsilon}_0\rangle \underset{V_{if} = \frac{\hbar\Omega}{2}}{\longleftrightarrow} |\varphi_f\rangle = |b, (N-1)\vec{k}_0\vec{\varepsilon}_0\rangle \underset{V_{fg}}{\longleftrightarrow}$$



$V_{fg} = 0$: Rabi oscillations $\Omega \propto dE \propto d\sqrt{N}$

$V_{if} = 0$: $|\varphi_f\rangle$ decays spontaneously $e^{-\Gamma t}$

both on:

$\Omega \gg \Gamma$: atom oscillates between a and b

oscillations slowly damped with $\tau \sim \frac{1}{\Gamma}$

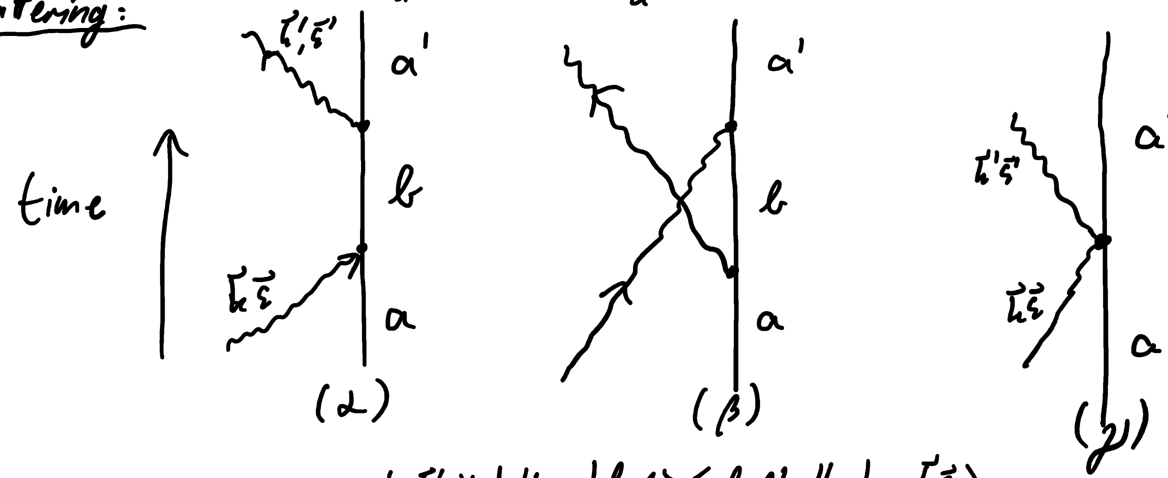
$\Omega \ll \Gamma$: $|\varphi_f\rangle$ "dissolved" in the continuum $|\varphi_g\rangle$

density of states of $|\varphi_f\rangle$ in $|\varphi_g\rangle$ is $\frac{1}{\hbar\Gamma}$

$|\varphi_i\rangle$ decays via $\Gamma_i = \frac{2\pi}{\hbar}|V_{if}|^2 \cdot \frac{1}{\hbar\Gamma} \approx \frac{\Omega^2}{\Gamma}$

Scattering:

$E_{a'} + \hbar\omega' = E_a + \hbar\omega$ $\qquad$ $E_a + \hbar\omega = E_{a'} + \hbar\omega'$



$$\tau_{fi}^{\alpha} = \sum_b \lim_{\eta \to 0^+} \frac{\langle a', \vec{k}'\vec{\varepsilon}' | H_{I_1} | b, 0 \rangle \langle b, 0 | H_{I_1} | a, \vec{k}\vec{\varepsilon} \rangle}{E_a + \hbar\omega - E_b + i\eta}$$

$$\tau_{fi}^{\beta} = \sum_b \lim_{\eta \to 0^+} \frac{\langle a', \vec{k}'\vec{\varepsilon}' | H_{I_1} | b, \vec{k}\vec{\varepsilon}\, \vec{k}'\vec{\varepsilon}' \rangle \langle b, \vec{k}\vec{\varepsilon}\, \vec{k}'\vec{\varepsilon}' | H_{I_1} | a, \vec{k}\vec{\varepsilon} \rangle}{E_a - \hbar\omega' - E_b + i\eta}$$

$$\tau_{fi}^{\gamma} = \langle a', \vec{k}'\vec{\varepsilon}' | H_{I_2} | a; \vec{k}\vec{\varepsilon} \rangle$$

(using $-\vec{d} \cdot \vec{E}_\perp$ only α and β exists)

intermediate state in β is never discrete due to 2 photons

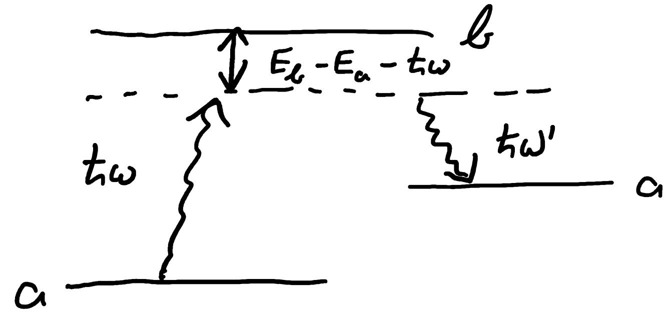
" " in α could be discrete if b is discrete.

$\longrightarrow$ resonant scattering $\rightarrow$ later

$a' = a$: $E_a = E_{a'}$; $\omega' = \omega \Rightarrow$ elastic scattering

$a' \neq a$: inelastic ; $\hbar\omega' - \hbar\omega = E_a - E_{a'}$

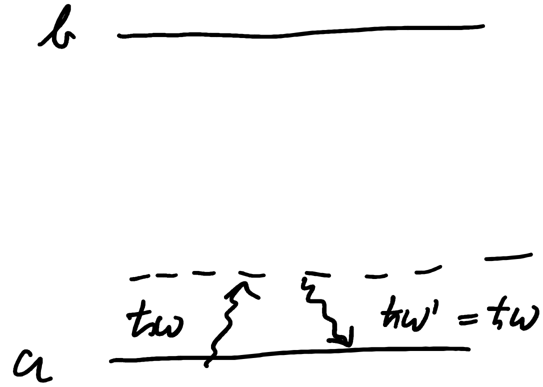
Other diagrammatic rep:



Rayleigh scattering

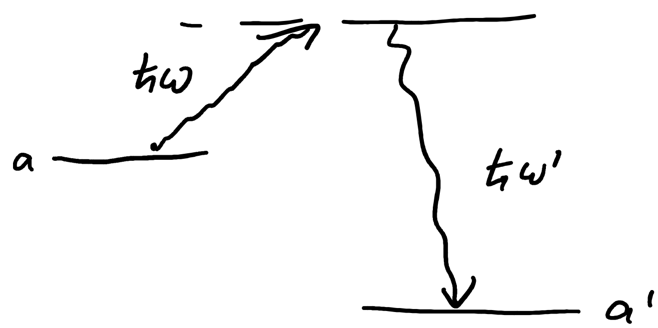
$\hbar\omega \ll |E_b - E_a|$

$\sigma \propto \omega^4$



Raman scattering

Stokes

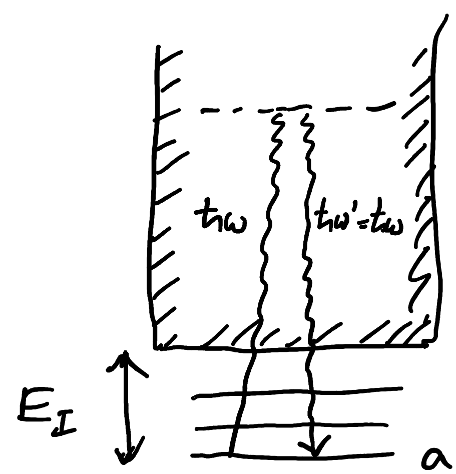


Anti-Stokes

Thompson scattering

$\hbar\omega \gg E_I$ (ionization energy)

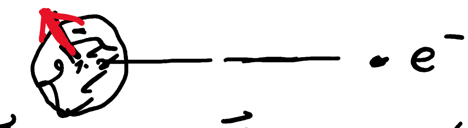
process γ dominates
(no $\frac{1}{\hbar\omega}$ denominator)



Van-der-Waals and Casimir forces

$e^- - e^-$ $\qquad \frac{e^2}{r}$

e^- - atom



$\vec{E}$ induces dipole moment $\vec{d} = \alpha \vec{E}$ in atom

Interaction: $\quad -\vec{d} \cdot \vec{E} = -\alpha |\vec{E}|^2$

$$|\vec{E}| = \frac{e}{r^2}$$

$$\Rightarrow V_{a-e^-} = -\alpha \frac{e^2}{r^4}$$

atom - atom ; mean value $\langle \vec{d} \rangle = 0$

In second order we find

$$V_{aa} = \frac{\left(\frac{\vec{d} \cdot \vec{d}'}{r^3} \right)^2}{\Delta E}$$

$$\Delta E \sim \frac{e^2}{a_0} \qquad |\vec{d}| \sim e a_0$$

$$V_{aa} \sim \frac{\left(\frac{(e a_0)^2}{r^3} \right)}{\frac{e^2}{a_0}} \sim -\frac{e^2}{a_0} \cdot \frac{a_0^6}{r^6}$$