1. We start with some properties of unitary matrices that will turn out to be very useful.

   (a) Consider the d-dimensional quantum space with basis is $|1\rangle$, $|2\rangle$, ..., $|d\rangle$. Suppose you have a different orthonormal basis $|v_1\rangle$, $|v_2\rangle$, ..., $|v_d\rangle$. for it. Show that there is a unitary transformation U such that $U|j\rangle = |v_j\rangle$. For this problem, you should assume that the definition of a unitary matrix is a matrix for which $UU^\dagger = I$, or equivalently, $U^\dagger U = I$.

   If you don't recall from linear algebra what an orthonormal basis is, it's one that satisfies $\langle v_i|v_j\rangle = \delta_{i,j}$, where δ is the Kronecker delta function.

   Solution Consider the following matrix.

   $$U = \sum_{i=1}^{d} |v_i\rangle \langle i| .$$

   Clearly, it satisfies $U|i\rangle = |v_i\rangle$. We now show that it is unitary. We have

   $$UU^\dagger = (\sum_{i=1}^{d} |v_i\rangle \langle i|)(\sum_{i=1}^{d} |i\rangle \langle v_i|) = \sum_{i=1}^{d} |v_i\rangle \langle v_i| .$$

   In the following part, we show that this matrix is identity.

   (b) Suppose $|v_1\rangle$, $|v_2\rangle$, ..., $|v_d\rangle$ are an orthonormal basis of a d-dimensional quantum state space. Show that

   $$\sum_{i=1}^{d} |v_i\rangle\langle v_i| = I.$$

   Solution Fix any vector $|v\rangle$ in the d-dimensional space. Since $|v_i\rangle$ form an orthonormal basis, we can write $|v\rangle$ as a linear combination of $|v_i\rangle$, such that $|v\rangle = \sum_{i=1}^{d} c_i |v_i\rangle$ for some $c_i \in \mathbb{C}$. Now we have

   $$(\sum_{i=1}^{d} |v_i\rangle\langle v_i|) |v\rangle = (\sum_{i=1}^{d} |v_i\rangle\langle v_i|)(\sum_{i=1}^{d} c_i |v_i\rangle) = \sum_{i=1}^{d} c_i |v_i\rangle = |v\rangle .$$

   This implies that $\sum_{i=1}^{d} |v_i\rangle\langle v_i| = I$.

2. In this problem, we will see the relation between the angle between two quantum states and the angle between the associated points of the Bloch sphere.

   (a) Recall from lecture that the point $p_i = (x_i, y_i, z_i)$ on the Bloch sphere associated with the quantum state of a qubit $|v_i\rangle$ satisfies

   $$x_i\sigma_x + y_i\sigma_y + z_i\sigma_z = |v_i\rangle\langle v_i| - |\bar{v}_i\rangle\langle \bar{v}_i| , \tag{1}$$

   where $|\bar{v}_i\rangle$ is a state orthogonal to $|v_i\rangle$. (There are many orthogonal states $|\bar{v}_i\rangle$, but they all give the same value of $|\bar{v}_i\rangle\langle \bar{v}_i|$.) Find an expression for $|v_i\rangle\langle v_i|$ in terms of x_i, y_i, z_i, the three Pauli matrices and the 2×2 identity matrix.

Solution Let's make use of the previous problem. Observe that $|v_i\rangle, |\bar{v}_i\rangle$ form an orthonormal basis of $\mathbb{C}^2$, which means that $|v_i\rangle\langle v_i| + |\bar{v}_i\rangle\langle \bar{v}_i| = I$. Therefore

$$|v_i\rangle\langle v_i| = x_i\sigma_x + y_i\sigma_y + z_i\sigma_z + |\bar{v}_i\rangle\langle\bar{v}_i| = x_i\sigma_x + y_i\sigma_y + z_i\sigma_z + I - |v_i\rangle\langle v_i|$$
$$|v_i\rangle\langle v_i| = \frac{x_i\sigma_x + y_i\sigma_y + z_i\sigma_z + I}{2} = \frac{1}{2}\begin{bmatrix} 1+z_1 & x_1 - y_1 i \\ x_1 + y_1 i & 1 - z_1 \end{bmatrix}.$$

(b) Use this formula and the half-angle formula from trigonometry to find a relation between $\arccos|\langle v_1|v_2\rangle|$ and $\arccos p_1 \cdot p_2$. You may need the fact that

$$|\langle v_1|v_2\rangle|^2 = \langle v_1|v_2\rangle\langle v_2|v_1\rangle = \mathrm{Tr}\left(|v_1\rangle\langle v_1|\ |v_2\rangle\langle v_2|\right),$$

which can be proved using the cyclic property of trace, $\mathrm{Tr}\, ABC = \mathrm{Tr}\, CBA$.

Solution Using the above formulas, we have

$$|\langle v_1|v_2\rangle|^2 = \mathrm{Tr}\left(|v_1\rangle\langle v_1|\ |v_2\rangle\langle v_2|\right) = \frac{1}{4}\mathrm{Tr}\left(\begin{bmatrix} 1+z_1 & x_1 - y_1 i \\ x_1 + y_1 i & 1 - z_1 \end{bmatrix}\begin{bmatrix} 1+z_2 & x_2 - y_2 i \\ x_2 + y_2 i & 1 - z_2 \end{bmatrix}\right).$$
$$= \frac{1}{4}(2 + 2p_1 \cdot p_2) = (1 + p_1 p_2)/2.$$

Using the half angle formula for $\cos$, we see that $\arccos|\langle v_1|v_2\rangle| = \frac{1}{2}\arccos p_1 \cdot p_2$.

3. Suppose a qubit is in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1+i}{\sqrt{3}}|1\rangle$$

If a von Neumann measurement is applied using the basis

$$\left\{\tfrac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \tfrac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right\},$$

what are the probabilities of the various outcomes? Remember that when you go from the ket to the bra, you have to take the complex conjugate.

Solution The probability of observing $|\mathrm{in}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ is

$$|\langle \mathrm{in}|\psi\rangle|^2.$$

Recall that we need to conjugate the coefficients when going from $|\mathrm{in}\rangle$ to $\langle\mathrm{in}|$, so

$$\langle\mathrm{in}|\psi\rangle = \frac{1}{\sqrt{6}}\left(\langle 0| - i\langle 1|\right)\left(|0\rangle + (1+i)|1\rangle\right)$$
$$= \frac{2-i}{\sqrt{6}},$$

because $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

Therefore

$$|\langle\mathrm{in}|\psi\rangle|^2 = \frac{(2-i)(2+i)}{6} = 5/6.$$

While you can do a similar calculation to obtain the probability of observing $|\mathrm{out}\rangle$, you can also compute that it is $\frac{1}{6}$ because these two probabilities have to add to 1.

4. A qutrit is a three-state quantum system.

   (a) Show that

$$|a\rangle = \frac{1}{2}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle,$$
$$|b\rangle = \frac{1}{2}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{2}|2\rangle,$$
$$|c\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|2\rangle,$$

   is an orthonormal basis for a qutrit.

   Solution One way to do this is to check that $|a\rangle$, $|b\rangle$, $|c\rangle$ have unit length, and that $\langle a|b\rangle = \langle a|c\rangle = \langle b|c\rangle = 0$.

   (b) Suppose a qutrit is in the state $\frac{1}{\sqrt{3}}(|0\rangle+|1\rangle-|2\rangle)$ and is measured using the von Neumann measurement associated with the basis in part (a). What are the probabilities of the various outcomes?

   Solution

$$\left\langle a \middle| \left(\frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle - \frac{1}{\sqrt{3}}|2\rangle \right) \right\rangle = \left(\frac{1}{2}\langle 0| + \frac{1}{\sqrt{2}}\langle 1| + \frac{1}{2}\langle 2| \right) \left(\frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle - \frac{1}{\sqrt{3}}|2\rangle \right)$$
$$= \frac{1}{2\sqrt{3}} + \frac{1}{\sqrt{6}} - \frac{1}{2\sqrt{3}} = \frac{1}{\sqrt{6}}$$

   so the probability of observing $|a\rangle$ is $1/6$. Similar calculations show that the probability of observing $|b\rangle$ is $1/6$ and the probability of observing $|c\rangle$ is $2/3$.

5. Imagine perfect polarizing filters that let all horizontally polarized photons through and filter out all vertically polarized photons. If we put two such polarizing filters on top of each other, and rotate the top one by $90°$, no light will get through.

   (a) Now, suppose we put a polarizing filter between them at an angle of $45°$. What fraction of the photons make it through, on average?

   (b) Now, put two filters between them, one rotated by $30°$ and the other by $60°$. What order should you put them in to make the most light come through, and what fraction of the photons will make it through?

   Solution A photon in a random position has a probability $\frac{1}{2}$ of passing through the first polarizing filter. Now, assume that the first polarizing filter is horizontal. The second polarizing filter will be rotated 45°, so the second polarizing filter measures the polarization in the basis $\{|\swarrow\!\!\!\nearrow\rangle, |\nwarrow\!\!\!\searrow\rangle\}$, and lets (say) the $|\swarrow\!\!\!\nearrow\rangle$ photons through. The probability of measuring a horizontal photon as $|\swarrow\!\!\!\nearrow\rangle$ is

$$|\langle\swarrow\!\!\!\nearrow|\leftrightarrow\rangle|^2 = \left| \left(\frac{1}{\sqrt{2}}\langle\leftrightarrow| + \frac{1}{\sqrt{2}}\langle\updownarrow| \right) |\leftrightarrow\rangle \right|^2$$
$$= \frac{1}{2}.$$

   When we apply the vertical polarizing filter after the diagonal one, the same calculation shows that half the remaining photons get through. Thus, inserting a single polarizing filter at 45° lets $\frac{1}{8}$ of the photons get through instead of none.

If you put a polarizing filter at $30°$, then the basis you are measuring in is now

$$\{\cos 30° \,|{\leftrightarrow}\rangle + \sin 30° \,|{\updownarrow}\rangle\,, -\sin 30° \,|{\leftrightarrow}\rangle + \cos 30° \,|{\updownarrow}\rangle\},$$

and you find that $\cos^2 30° = \frac{3}{4}$ of the photons pass through. The next filter is at $60°$, which is another difference of $30°$ compared to the previous filter. So the fraction of photons that pass through the third filter is once again $\frac{3}{4}$ (compared to all photons arriving at the filter,) and the fraction of photons that pass through all four polarizing filters is $\frac{1}{2}(\cos^2 30°) = \frac{1}{2}\left(\frac{3}{4}\right)^3 = \frac{27}{128} > \frac{1}{8}$.
If the first polarizing filter is at an angle of $0°$, the second at $60°$, the third at an angle of $30°$, and the last at $90°$, the probability of a photon going through is

$$\frac{1}{2} \cdot \cos^2 60° \cdot \cos^2 30° \cdot \cos^2 60° = \frac{3}{128} < \frac{1}{8}.$$

6. Suppose somebody gives me a qubit which is equally likely to be $|0\rangle$, $|1\rangle$, $|+\rangle$, and $|-\rangle$, and challenges me to make a copy of it. There is a no-cloning theorem in quantum mechanics that says I cannot succeed 100% of the time.

   (a) One thing I could do is measure it in the $\{|0\rangle\,, |1\rangle\}$ basis, and make two copies of the resulting state. I do this, and hand the challenger both qubits. They measure them in either the $\{|0\rangle\,, |1\rangle\}$ basis, or the $\{|+\rangle\,, |-\rangle\}$ basis, depending on which basis their original qubit was in. I succeed only if both qubits are measured to be equal to the challenger's original qubit. What is the probability I pass the test?

   Solution: If the photon he gives is in the state $|0\rangle$ or $|1\rangle$, then we always succeed. If it's in the $|+\rangle$ or $|-\rangle$ state, then we give him two photons which are either both $|0\rangle$ or both $|1\rangle$. The probability that he measures them one of them in the same state that he gave us is $\frac{1}{2}$, so the probability that he measures them both in that state is $1/4$. The total probability that we succeed is thus $\frac{5}{8}$.

   (b) I could try to do better by measuring in an intermediate basis, say

   $$\{\cos(\theta)\,|0\rangle + \sin(\theta)\,|1\rangle\,, -\sin(\theta)\,|0\rangle + \cos(\theta)\,|1\rangle\}$$

   What is the probability that I succeed if I use this basis (as a function of θ)?

   Solution: Suppose he gives us a photon in the state $|0\rangle$. We will measure it in the state $\cos(\theta)\,|0\rangle + \sin\theta\,|1\rangle$ with probability $cos^2\theta$. When we give a photons in this state back to him, the chance that he measures the it in the state $|0\rangle$ is $\cos^2\theta$. Thus, if we give two photons in this state back to him, the probability that he measures them both in the state $|0\rangle$ is $\cos^4\theta$. Similarly, we will measure the photon in the state $= \sin(\theta)\,|0\rangle + \cos\theta\,|1\rangle$ with probability $\sin^2\theta$, and in this case, he measures both photons in $|0\rangle$ with probability $\sin^4\theta$. Putting this all together, the probability that we succeed if he gives us a photon in the state $|0\rangle$ or $|1\rangle$ is

   $$\cos^6\theta + \sin^6\theta.$$

   An analogous argument shows that if he gives us a photon in the state $|+\rangle$ or the state $|-\rangle$, the probability of success is

   $$\cos^6(\pi/4-\theta) + \sin^6(\pi/4-\theta),$$

   so the answer is

   $$\frac{1}{2}\left(\cos^6\theta + \sin^6\theta + \cos^6(\pi/4-\theta) + \sin^6(\pi/4-\theta)\right).$$

(c) How should I pick θ in part b to maximize my probability of success? Does it matter?

Solution: It doesn't matter. You can plot the expression above, and find that it is constant at $5/8$. Or you can use trigonometric identities (or have Mathematica do this) to show that it is always $5/8$. So the choice of θ doesn't change the probability of our succeeding.