MA439: Functional Analysis
Tychonoff Spaces: Exercises 1-6 on p.36, Ben Mathes

Huan Q. Bui

Due: Wed, Sep 30, 2020

Exercise 1 (Ex 1, p.36)**.** *Let $\mathcal{X}$ be a topological space. Prove that if d is a continuous pseudometric, then the sets $\{y \in \mathcal{X} : d(x,y) > \delta\}$ are open, where $x \in \mathcal{X}$ and $\delta \in \mathbb{R}$.*

Proof. Let $O = \{y \in \mathcal{X} : d(x,y) > \delta\}$. We want to show that each $y \in O$ is an interior point of O. Let $y \in O$ be given, then $d(x,y) > \delta$. This means that $d(x,y) \geq \delta + \epsilon$ for some $\epsilon > 0$. d is a continuous pseudometric, so every d-ball is an open subset of $\mathcal{X}$. In particular, $B_d(y, \epsilon/2)$ is an open subset of $\mathcal{X}$. By the triangle inequality, for any $z \in B_d(y, \epsilon/2)$, $z \in O$. Thus, $B_d(y, \epsilon/2) \subseteq O$. So, O is open as desired. □

Exercise 2 (Ex 2, p.36)**.** *Let $\mathcal{X}$ be a topological space. Prove that d is a continuous pseudometric on $\mathcal{X}$ if and only if the function $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$.*

Proof. ($\implies$) Suppose that d is a continuous pseudometric on $\mathcal{X}$. Let $\epsilon > 0$, then for $\delta = \epsilon$, $\left|f_x^d(y) - f_x^d(z)\right| = |d(x,y) - d(x,z)| \leq d(y,z) \leq \epsilon$ whenever $d(y,z) \leq \delta$. So f_x^d is continuous for all $x \in \mathcal{X}$.

($\impliedby$) Let d be a pseudometric and suppose that $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$. We want to show that every d-ball is open in $\mathcal{X}$. To this end, let $x \in \mathcal{X}$ and $\delta > 0$ be given and consider $B_d(x,\delta) = \{y \in \mathcal{X} : d(x,y) < \delta\} = \{y \in \mathcal{X} : f_x^d(y) < \delta\}$ then what??? □

Exercise 3 (Ex 3, p.36)**.** *Let $\mathcal{X}$ be a Tychonoff space whose topology is generated by the family of pseudometrics $\mathcal{G}$. Prove that the topology on $\mathcal{X}$ is the same as the weak topology induced by the family of functions f_x^d where $x \in \mathcal{X}$, $d \in \mathcal{G}$.*

Proof. This follows from the proof from □

Exercise 4 (Ex 4, p.36)**.** *Assume $\mathcal{X}$ is a Tychonoff space with generating family $\mathcal{G}$. If E is a subset of $\mathcal{X}$, let $\mathcal{G}_E$ denote the set of restrictions of elements of $\mathcal{G}$ to E. Prove that the resulting Tychonoff Topology on E generated by the family $\mathcal{G}_E$ is the same as the topological* ***subspace topology*** *that E inherits from the topology on $\mathcal{X}$.*

Proof. blah □

Exercise 5 (Ex 5, p.36)**.** *Give an example of a continuous pseudometric on $(0,1)$ that is not the restriction of a continuous pseudometric on $\mathbb{R}$ to $(0,1)$.*

Proof. blah □

Exercise 6 (Ex 6, p.36)**.** *Prove that a bounded continuous pseudometric on $(0,1)$ is the restriction of a continuous pseudometric on $\mathbb{R}$ to $(0,1)$. (?CHECK?)*

Proof. blah □