The Riemann-Stieltjes Integral: 6.1, 2, 3, 4, 5, 8, Baby Rudin

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6.1 *Proof:* f is clearly bounded on $[a, b]$ and is discontinuous at exactly the point x_0, where α is continuous. Theorem 6.10 says these conditions imply $f \in \mathscr{R}(\alpha)$. So, for any partition P of $[a, b]$, we have $\int_a^b f\,d\alpha = \sup L(P, f, \alpha) = \sup \sum_{i=1}^n \Delta\alpha_i \inf_{x \in [x_{i-1}, x_i]} f$. Look at each interval, $[x_{i-1}, x_i]$. If the interval has nonzero length then $\inf f$ on it is zero. If the interval is just the point x_0 then $\Delta\alpha_i$ is zero. So in any case, $\sup L(P, f, \alpha) = 0$, which means $\int f\,d\alpha = 0$. $\square$

6.2 *Proof:* We have $f \geq 0$ continuous on $[a, b]$ and $\int_a^b f\,dx = 0$. We first note that for $c, d \in [a, b]$ such that $c \leq d$, $\int_c^d f\,dx \geq 0$ because $f \geq 0$ for all $x \in [a, b]$. Now, suppose $f(x_0) > 0$ for some $x_0 \in [a, b]$. Let $\epsilon = f(x_0)/2 > 0$ be given. By continuity, there exists a small enough $\delta > 0$ such that $\left|f(x) - f(x_0)\right| < \epsilon = f(x_0)/2 \implies f(x) > f(x_0)/2$ for some x in $(x_0 - \delta, x_0 + \delta)$. With this, we write

$$\int_a^b f\,dx = \int_a^{x_0-\delta} f\,dx + \int_{x_0-\delta}^{x_0+\delta} f\,dx + \int_{x_0+\delta}^b f\,dx \geq 0 + \delta f(x_0) + 0 > 0,$$

which is a contradiction. So $f = 0$ on $[a, b]$. $\square$

6.3 *Proof:* Define three functions $\beta_1, \beta_2, \beta_3$ as: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = 1/2$. f is a bounded function on $[-1, 1]$.

1. We want to show $f \in \mathscr{R}(\beta_1) \iff \lim_{x\to 0+} f(x) \equiv f(0+) = f(0)$ and that then $\int f\,d\beta_1 = f(0)$.

   (a) ($\rightarrow$) Suppose $f \in \mathscr{R}(\beta_1)$. To prove the implication we want to look at what happens to f as $x \to 0+$. Since $f \in \mathscr{R}(\beta_1)$ on $[-1, 1]$, $f \in \mathscr{R}(\beta_1)$ on $[0, 1]$ as well. Let $\epsilon > 0$ be given. Theorem 6.6. says that $f \in \mathscr{R}(\beta_1)$ on $[0, 1] \iff \forall\epsilon > 0 \exists$ a partition P such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. For any $x \in [0, \delta]$ where $0 < \delta < 1$, we have that

   $$L(P, f, \beta_1) \leq f(x) \leq U(P, f, \beta_1).$$

   Further, since $0 \in [0, 1]$

   $$L(P, f, \beta_1) \leq f(0) \leq U(P, f, \beta_1).$$

   So, $\left|f(x) - f(0)\right| \leq U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. Since ϵ and δ can be made arbitrarily small, we have that $\lim_{x\to 0+} f(x) = f(0+) = f(0)$.

   (b) ($\leftarrow$) Let $\epsilon > 0$ be given. Suppose $\lim_{x\to 0+} f(x) = f(0)$, then there exists $\delta > 0$ such that whenever $0 \leq x < \delta$, $\left|f(x) - f(0)\right| < \epsilon$. Okay, fix any $y \in (0, \delta)$, set $M = \sup_{y\in(0,\delta)} f(y), m = \inf_{y\in(0,\delta)} f(y)$. Then clearly, for any $y \in (0, \delta)$, $M \geq f(y)$ and $m \leq f(y)$. This combines with $f(0+) = f(0)$ mean we can remove the absolute value sign and write $M - f(y) < \epsilon$ and $f(y) - m < \epsilon$. This imply

   $$M - m < 2\epsilon.$$

Let a partition P of $[-1, 1]$ be given. Then we immediately have $U(P, f, \beta_1) = M$ and $L(P, f, \beta_1) = m$ (because $\beta_1(x) = 0$ for all $x < 0$, which means there's no contribution from $d\beta_1$ from $x < 0$). So, because the following holds for any arbitrary P of $[-1, 1]$

$$U(P, f, \beta_1) - L(P, f, \beta_1) = M - n < 2\epsilon,$$

$f \in \mathscr{R}(\beta_1)$ on $[-1, 1]$. So we're done.

(c) Showing $\int f\, d\beta_1 = f(0)$ is easy. Since we have shown that for any partition P of $[-1, 1]$ and $\epsilon > 0$, $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. And because $L(P, f, \beta_1) \le f(0) \cdot (\beta_1(x_j > 0) - \beta_1(0)) = f(0) \le U(P, f, \beta_1)$, we must have that $f(0) = U(P, f, \beta_1) = L(P, f, \beta_1) = \int f\, d\beta_1$.

2. For β_2, the statement becomes $f \in \mathscr{R}(\beta_2) \iff f(0-) = f(0)$ and that then $\int f\, d\beta_2 = f(0)$. The proof is very similar to that in the previous item, except that we look at what happens when $x \to 0-$. The difference comes from the fact that $\beta_1(0) = 0$ while $\beta_2(0) = 1$, that is the "jump" occurs at a different location.

3. We want to prove $f \in \mathscr{R}(\beta_3) \iff f$ is continuous at 0, i.e., $f(0-) = f(0) = f(0+)$.

   (a) ($\rightarrow$) Suppose $f \in \mathscr{R}(\beta_3)$, then Theorem 6.6. says there is a partition P such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$. Consider the numbers $\gamma < 0 < \rho$ in the partition P. For $u \in (\gamma, 0]$ and $v \in [0, \rho)$, we have that

$$L(P, f, \beta_3) \le f(u)\underbrace{(\beta_3(0) - \beta_3(\gamma))}_{1/2} + f(0)\underbrace{(\beta_3(\rho) - \beta_3(0))}_{1/2} \le U(P, f, \beta_3)$$
$$L(P, f, \beta_3) \le f(v)\underbrace{(\beta_3(\rho) - \beta_3(0))}_{1/2} + f(0)\underbrace{(\beta_3(0) - \beta_3(\gamma))}_{1/2} \le U(P, f, \beta_3).$$

   In a similar fashion we also have

$$L(P, f, \beta_3) \le \frac{1}{2} f(0) + \frac{1}{2} f(0) = f(0) \le U(P, f, \beta_3).$$

   Combining these we have

$$\left| f(u) - f(0) \right| \le 2|U(\dots) - L(\dots)| < \epsilon$$
$$\left| f(v) - f(0) \right| \le 2|U(\dots) - L(\dots)| < \epsilon.$$

   So, $f(0-) = f(0) = f(0+)$.

   (b) ($\leftarrow$) Suppose $f(0-) = f(0) = f(0+)$. Then we just have $f(0) = f(0-)$ and $f(0) = f(0+)$ (duh). But this allows us to repeat the proof in part (a) and (b) to get $f \in (R)(\beta_3)$.

4. If f is continuous at 0 then (c) holds. Parts (a) and (b) hold automatically. So we're done.

$\square$

6.4 *Proof:* Let $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x. We want to show $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$. Well, let a partition P be given. Both the rationals and irrationals are dense in $[a, b]$. So, for every little interval $[x_i, x_{i+1}]$, $\sup f = 1$. So, $U(P, f) = \sum_{i=1}^{n} \sup_{[x_i, x_{i+1}]} f(x)\Delta x_i = b - a$. Also, for every little interval $[x_i, x_{i+1}]$, $\inf f = 0$, so $L(P, f) = \sum_{i=1}^{n} \inf_{[x_i, x_{i+1}]} f(x)\Delta x_i = 0$. Obviously, $\underline{\int} f = \sup_P L = 0 < \inf_P U = b - a = \overline{\int} f$, so $f \notin \mathscr{R}$ on $[a, b]$. □

6.5 *Proof:* Suppose f is a bounded real function on $[a, b]$ and $f^2 \in \mathscr{R}$ on $[a, b]$.

1. $f \notin \mathscr{R}$, because we can't "invert" f^2 to get f back. Consider the counter example:

$$f(x) = \begin{cases} 1, x \in [a, b] \cap \mathbb{Q} \\ -1, x \in [a, b] \cap \mathbb{Q}^c \end{cases}$$

   Then $f^2 = 1 \in \mathscr{R}$. However, similar to last problem, we can show $L(P, f) = -1$ and $U(P, f) = 1$ for any partition P of $[a, b]$. So, $f \notin \mathscr{R}$.

2. $f \in \mathscr{R}$ if $f^3 \in \mathscr{R}$. In this case we can "invert" f^3. Consider the continuous function ϕ on $[a, b]$ defined by $\phi(x) = x^{1/3}$. Since f is bounded, Theorem 6.11., the function $h(x) = \phi(f^3(x)) = f(x) \in \mathscr{R}$ on $[a, b]$.

□

6.8 *Proof:* Suppose $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. We want to show $\int_1^\infty f\,dx$ converges $\iff$ $\sum_{n=1}^{\infty} f(n)$ converges.

1. ($\rightarrow$) Suppose $\int_1^\infty f\,dx$ converges, that is, $\lim_{b\to\infty} \int_1^b f\,dx$ exists. We want to show $\sum_{n=1}^{\infty} f(n)$ converges, i.e., $\sum_{n=1}^{k} f(n)$ is bounded (f is monotonic & Theorem 3.14). Well,

$$\sum_{n=1}^{k} f(n) = f(1) + \sum_{n=2}^{k} f(n) \leq f(1) + \int_1^k f(x)\,dx.$$

   We note that $\lim_{k\to\infty} \int_1^k f\,dx$ exists, so $\sum_{n=1}^{k} f(n)$ is bounded for all k. And so, $\sum_{n=1}^{\infty} f(n)$ converges.

2. ($\leftarrow$) We also have that

$$\sum_{n=1}^{k} f(n) = f(1) + \sum_{n=2}^{k} f(n) \leq f(1) + \int_1^k f(x)\,dx \leq \sum_{n=1}^{k-1} f(n)$$

   which means if $\int_1^\infty f\,dx$ diverges, $\sum_{n=1}^{\infty} f(n)$ diverges as well. So, by contraposition, if $\sum_{n=1}^{\infty} f(n)$ converges, the integral $\int_1^\infty f\,dx$ also converges.

□