Physics 8.321, Fall 2021
Homework #7

Due **Friday, November 19** by 8:00 PM.

1. Use the WKB approximation to calculate the spectrum of energies for a particle in the following 1D potentials (in units $\hbar = m = 1$):

(a) Harmonic oscillator potential $V(x) = \frac{1}{2}x^2$.

(b) Box potential $V(x) = 0$ for $0 \le x \le L$, $V(x) = \infty$ otherwise.

(c) $V(x) = |kx^\alpha|$, with $\alpha, k > 0$.

(d) Compare with the exact values for the ground state and first excited state in each case (set $k = 1/4, \alpha = 4$ in case (c), and compare with the results $E_0 = 0.4208, E_1 = 1.5079$ from problem set 5). In which case does the WKB approximation do the worst? Why?

2. [Sakurai and Napolitano Problem 32, Chapter 2 (page 154)]

Define the partition function as

$$Z = \int d^3x' K(\mathbf{x}', t; \mathbf{x}', 0)|_{\beta = it/\hbar} \,,$$

as in (2.6.20)-(2.6.22). Show that the ground-state energy is obtained by taking

$$-\frac{1}{Z}\frac{\partial Z}{\partial \beta}, \; (\beta \to \infty) \,.$$

Illustrate this for a particle in a one-dimensional box, using the spectrum of energies for that system.

3. [Sakurai and Napolitano Problem 34, Chapter 2 (page 155)]

(a) Write down an expression for the action for a classical solution of the simple harmonic oscillator from $x(0) = x$ to $x(t') = x'$.

(b) Construct $\langle x_n, t_n | x_{n-1}, t_{n-1} \rangle$ for a simple harmonic oscillator using Feynman's prescription for $t_n - t_{n-1} = \Delta t$ small. Show that the leading terms in this propagator can can be expressed in the form

$$K(x', x, \Delta t) = K_{\text{free}}(x', x, \Delta t) \left[c_0(x, x') + c_1(x, x')\Delta t + \mathcal{O}((\Delta t)^2) \right] \,,$$

and find $c_0(x, x')$ and $c_1(x, x')$. Further, show that this matches with the leading terms in a term-by-term expansion of the propagator written in terms of the time development operator in the Hamiltonian picture.

4. (Optional, extra credit)

In this problem we use the semiclassical WKB-type approximation to gain some insight into the rate of nuclear fusion reactions between a pair of protons in the sun. At very close distances, nuclear forces between a pair of protons will overcome the Coulomb repulsion between their common positive charges and the nuclei can fuse into a deuteron, giving an initial stage towards the production of helium that releases energy and provides the sun's power. The Coulomb barrier for two protons to reach this distance classically requires energies of $V_C \sim 0.2$ MeV (1 MeV $\cong 1.6 \times 10^{-13}$ J). At the temperature of the sun, no protons have this much energy. So quantum tunneling is needed to drive the sun's nuclear reactions.

(a) Use the semiclassical approximation to show that the *tunneling probability* (per unit time) for a particle with energy E to pass through a classically forbidden barrier described by a potential $V(x)$ is given by

$$P_T(E) \sim \exp\left(-\frac{2}{\hbar}\int_{x_1(E)}^{x_2(E)} dx\sqrt{2m(V(x)-E)}\right)$$

where x_1, x_2 are the classical turning points.

(b) Using $m = M_p/2$ (the reduced mass in the two-proton) system and the Coulomb potential describing the repulsion between two protons, estimate the tunneling rate of a proton of energy E to penetrate the Coulomb barrier by reaching a distance $r \sim 0$ from another proton.

(c) Use the Boltzmann distribution $P_B(E) = e^{-E/k_BT}/Z$ giving the probability that a quantum system at temperature T has energy E to derive the Maxwell-Boltzmann distribution for the probability per unit energy that a particle in a hot 3D gas has a given energy (assuming energy can be taken as a continuous variable)

$$\frac{dP_B}{dE} = \left(\frac{2\pi}{(\pi k_BT)^{3/2}}\right)\sqrt{E}e^{-E/k_BT}\,.$$

(d) The temperature in the sun is roughly 1.5×10^7 K. Combine the Maxwell-Boltzmann distribution with the tunneling probability to derive the Gamow distribution

$$dP_{\text{Gamow}} = P_T \times dP_B$$

describing the window where nuclear fusion is possible. Use this to estimate the fraction of protons that can get close enough for a fusion reaction.

(e) The mass of the sun is 1.989×10^{30} kg, Earth's distance from the sun is $R_{\text{orbit}} \cong 1.5 \times 10^8$ km, and the insolation at the top of Earth's atmosphere is $I \cong 1366$ W/m^2. One nuclear fusion of four protons to a single helium nucleus, which involves two *pp* fusion events, releases roughly 26.7 MeV. Estimate the rate at which *pp* fusion events occur in the sun. The large discrepancy between this rate and that you computed in (c) results from the fact that this process involves the weak nuclear interactions, and are suppressed by a very small factor since these interactions are so weak (coupling constant $\sim 10^{-10}$).