Physics 8.321, Fall 2021
Homework #3

Due **Wednesday, October 13** by 8:00 PM.

1. Consider the following two matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix},$$

(a) Show that A and B commute.

(b) Find the eigenvalues and eigenvectors of A and B.

(c) Find the unitarity transformation which simultaneously diagonalizes A and B.

Solution:

(a) It is straightforward to check that

$$AB = BA = \begin{pmatrix} 3 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 3 \end{pmatrix} \blacksquare$$

Note that the matrix A is singular ($\det A = 0$) and any matrix M which has the following form

$$M = \begin{pmatrix} a & b & c \\ d & e & -d \\ c & -b & a \end{pmatrix}$$

commutes with A:

$$AM = MA = (a+c)A$$

Our B is just a special case with $a = 2$, $b = 1$, $c = 1$, $d = 1$, and $e = 0$.

(b) Eigenvalues and eigenvectors of A, and B

$$\begin{array}{llll} 2, & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}; & -1, & \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \\ 0, & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; & 2, & \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \\ 0, & \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; & 3, & \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \end{array}$$

(c) Use the eigenvectors of B as the column vectors of the unitarity transformation matrix U:

$$U = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Then $U^\dagger AU$, and $U^\dagger BU$ are simultaneously diagonalized.

2. In this problem we consider some simple *spin chains*, which are a simple example of a type of system that appear in the context of condensed matter physics.

N spin-1/2 particles have a total Hilbert space

$$\mathcal{H} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)} \otimes \cdots \otimes \mathcal{H}_2^{(N)}$$

where $\mathcal{H}_2^{(i)}$ is the (two-dimensional) Hilbert space of the ith particle.

(a) What is the dimension of $\mathcal{H}$?

(b) Define

$$S_z = S_z^{(1)} + S_z^{(2)} + \cdots S_z^{(N)} .$$

What is the spectrum and degeneracy of S_z?

(c) Define an operator I coupling N spins to their nearest neighbors in a ring through

$$I = \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} + \mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)} + \mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)} + \cdots + \mathbf{S}^{(N-1)} \cdot \mathbf{S}^{(N)} + \mathbf{S}^{(N)} \cdot \mathbf{S}^{(1)}$$

Are S_z and I compatible observables? Prove your answer for any N.

(d) Find the spectrum and degeneracies of I for $N = 2, 3, 4$.

(e) Find the largest positive eigenvalue $\lambda_{\text{max}}^{(N)}$ of I for an arbitrary value of N, and identify an eigenvector with this eigenvalue.

(f) Find the smallest (most negative) eigenvalue $\lambda_{\text{min}}^{(N)}$ for small values of $N = 1, 2, \ldots$. What is the largest $N = n$ for which you can compute this quantity (analytically/numerically) in reasonable time? What can you say about $\lambda_{\text{min}}^{(N)}$ and its associated eigenvector(s) for general N?

(g) Consider N spin-1/2 particles in an external magnetic field, interacting with the external field and one another according to the Hamiltonian

$$H = b\,xS_z - a(1-x)I$$

where a, b are numerical constants with $b = a\hbar$, S_z and I are defined in parts (b, c), and $x \in [0,1]$ is a real number. Graph the spectrum of H for $N = 2, 3, 4$ for x in the range $0 \le x \le 1$. Check that your results agree with your answers to the previous parts.

Solution: Solution:

(a) $\dim \mathcal{H} = 2^N$. $\mathcal{H}$ is spanned by the orthogonal basis vectors $|\sigma_1 \ldots \sigma_n\rangle \equiv |\sigma_1\rangle \otimes \ldots \otimes |\sigma_n\rangle$, where $\sigma_n = \uparrow$ represents an up spin and $\sigma_n = \downarrow$ a down spin at site n.

(b) Max $S_z = N/2$, Min $S_z = -N/2$. $S_z = -N/2, -N/2+1, \ldots, N/2$. For any possible value of $S_z = N/2 - p$, $0 \le p \le N$ ($p =$ the number of down spins), the degeneracy is $C_p^N = \frac{n!}{p!(n-p)!}$.

Two different expressions of $\mathbf{S}^{(n)} \cdot \mathbf{S}^{(n+1)}$ can help us understand more about this part and part(c) below:(The site index n is moved lower so I have space for other notations.)

(1)

$$\mathbf{S}_n \cdot \mathbf{S}_{n+1} = \frac{1}{2}(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+) + S_n^z S_{n+1}^z$$

where $S_n^\pm \equiv S_n^x \pm S_n^y$ are spin flip operators. $\frac{1}{2}(S_n^+ S_{n+1}^- + S_n^- S_{n+1}^+)$ swaps the two spins at site n and $n+1$ but does not change the total number of down spins. $S_n^z S_{n+1}^z$ does not change the number of down spins either.

(2)

$$\mathbf{S}_n \cdot \mathbf{S}_{n+1} = \frac{1}{2}\mathcal{P}_{n,n+1} - \frac{1}{4}\mathbb{1} \otimes \mathbb{1}$$

where $\mathcal{P}_{n,n+1}$ is permutation matrix, which is defined by

$$\mathcal{P} = \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \\ 0\,1\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix}$$

For any

$$x \otimes y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_1 y_2 \\ x_2 y_1 \\ x_2 y_2 \end{pmatrix}$$

$$\mathcal{P}(x \otimes y) = \begin{pmatrix} x_1 y_1 \\ x_2 y_1 \\ x_1 y_2 \\ x_2 y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ y_2 \end{pmatrix} = y \otimes x$$

(c) A direct calculation specific to $z-$axis is:

$$\begin{aligned}
&[I, S_z] \\
&= \sum_i \sum_j [\mathbf{S}^{(i)} \cdot \mathbf{S}^{(i+1)}, S_z^{(j)}] \\
&= \sum_{i,j} [S_x^{(i)} S_x^{(i+1)} + S_y^{(i)} S_y^{(i+1)}, S_z^{(j)}] \\
&= \sum_{i,j} ([S_x^{(i)} S_x^{(i+1)}, S_z^{(j)}] + [S_y^{(i)} S_y^{(i+1)}, S_z^{(j)}]) \\
&= \sum_{i,j} (S_x^{(i)} [S_x^{(i+1)}, S_z^{(j)}] + [S_x^{(i)}, S_z^{(j)}] S_x^{(i+1)} + S_y^{(i)} [S_y^{(i+1)}, S_z^{(j)}] + [S_y^{(i)}, S_z^{(j)}] S_y^{(i+1)}) \\
&= \sum_{i,j} (S_x^{(i)}\, \delta_{i+1,j}\, (-i\hbar S_y^{(i+1)}) + (-i\hbar S_y^{(i)})\, \delta_{i,j}\, S_x^{(i+1)} + S_y^{(i)}\, \delta_{i+1,j}\, (i\hbar S_x^{(i+1)}) + (i\hbar S_x^{(i)})\, \delta_{i,j}\, S_y^{(i+1)}) \\
&= 0 \quad \text{(1st and 4th terms cancel each other; 2nd and 3rd terms cancel each other.)}
\end{aligned}$$

Notice each term in I is an inner product, so I is compatible with $S_{\mathbf{n}}$, for any direction $\mathbf{n}$.

(d) The explicit matrix expression for $\mathbf{S}^{(n)} \cdot \mathbf{S}^{(n+1)}$ is

$$\mathbf{S}^{(n)} \cdot \mathbf{S}^{(n+1)} = \begin{pmatrix} \frac{\hbar^2}{4} & 0 & 0 & 0 \\ 0 & -\frac{\hbar^2}{4} & \frac{\hbar^2}{2} & 0 \\ 0 & \frac{\hbar^2}{2} & -\frac{\hbar^2}{4} & 0 \\ 0 & 0 & 0 & \frac{\hbar^2}{4} \end{pmatrix}$$

For $N = 2$, $I = 2\,\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$ (the prefactor 2 is from the periodic boundary condition.) and has eigenvalues and eigenvectors

$$\begin{aligned} \text{Eigenvalue} &= -\frac{3\hbar^2}{4} \\ \text{Degeneracy} &= 1 \\ \text{Eigenvector(s)} &= \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \frac{1}{\sqrt{2}}(|+-\rangle - |-+\rangle) \end{aligned}$$

and

$$\begin{aligned} \text{Eigenvalue} &= \frac{\hbar^2}{4} \\ \text{Degeneracy} &= 3 \\ \text{Eigenvector(s)} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad |--\rangle, \frac{1}{\sqrt{2}}(|+-\rangle + |-+\rangle), |++\rangle \end{aligned}$$

In the eigenvectors, the former is the tensor product matrix representation of the latter. Please refer to the Mathematica notebook homework3.nb for complete solutions for the cases $N = 3, 4, 5$. The results are

N	2		3		4			
Eigenvalue($\hbar^2$)	$-\frac{3}{2}$	$\frac{1}{2}$	$-\frac{3}{4}$	$\frac{3}{4}$	0	-2	-1	1
Degeneracy	1	3	4	4	7	1	3	5

N	5					
Eigenvalue($\hbar^2$)	$-\frac{3}{4}$	$\frac{5}{4}$	$-\frac{\sqrt{5}}{4}$	$\frac{\sqrt{5}}{4}$	$-\frac{(3+2\sqrt{5})}{4}$	$\frac{(-3+2\sqrt{5})}{4}$
Degeneracy	2	6	8	8	4	4

(e) Note that the largest diagonal matrix element of $\mathbf{S}_i \cdot \mathbf{S}_j$ (in the space spanned by the two spins at site i, and j) is S^2, while the smallest diagonal matrix element is $-S(S+1)$. S is the spin at each site and is equal to $\frac{1}{2}$ in this problem set. This can be obtained via

$$\begin{aligned} \mathbf{S}_i \cdot \mathbf{S}_j &= \frac{1}{2}((\mathbf{S}_i + \mathbf{S}_j)^2 - \mathbf{S}_i^2 - \mathbf{S}_j^2) \\ &= (\bar{S}(\bar{S}+1) - S(S+1) - S(S+1))\hbar^2 \end{aligned}$$

Max $\bar{S} = 2S$, Min $\bar{S} = 0$, $\Rightarrow$ Max $\mathbf{S}_i \cdot \mathbf{S}_j = S^2\hbar^2$, with eigenvectors $|++\rangle$, $|--\rangle$, and $|+-\rangle + |-+\rangle$; Min $\mathbf{S}_i \cdot \mathbf{S}_j = -S(S+1)\hbar^2$, with eigenvector $|+-\rangle - |-+\rangle$.

It then follows that $\lambda^{(N)}_{\text{max}} = \frac{\hbar^2}{4}N$, with eigenvectors $|++\cdots+\rangle$, or $|--\cdots-\rangle$. This result is intuitively sensible as the Hamiltonian is maximized when the spins are parallel.

(f) $\lambda^{(6)}_{\text{min}} = -\frac{1}{2}(2+\sqrt{13})$. For $N = 7, 8$, and 9, we give numeric values here (which you can compute using the codes by converting the matrix elements to decimals first): $\lambda^{(7)}_{\text{min}} = -2.85518$, $\lambda^{(8)}_{\text{min}} = -3.65109$, $\lambda^{(9)}_{\text{min}} = -3.7973$.

There is no exact formula for $\lambda^{(N)}_{\text{min}}$ for general N, but only the asymptotic value for $\lim_{N\to\infty} \lambda^{(N)}_{\text{min}}/N$ is known.

Some words about calculating the eigenvalues and eigenvectors of the operator $I = \sum_n \mathbf{S_n} \cdot \mathbf{S_{n+1}}$. This problem is an important *exactly sovable* model and is usually referred to as "Heisenberg spin chain" or "Heisenberg antiferromagnetism". The task of the matrix diagonalization is one of the most time consuming problems in the current scientific calculations.

The computational time grows as $N^2 \log N$ or N^3, depending upon which part of the computation dominates (N^3 will always dominate for large N, of course). N is the size of the matrix, and is equal to 2^n in our case, n the number of spins. Therefore, we are facing a computation problem in which the computation time exponentially grows as the size of the chain. Of course, the $N^2 \log N$ or N^3 argument is from diagonalizing a *general* matrix; in this problem we have (two) symmetries to simplify the matrix. Furthermore, since only nearest neighbors are included, the Hamiltonian matrix actually is sparse, i.e., many matrix elements are zero. But this still couldn't change the exponential growth behavior. A supercomputer in this regards wouldn't be of much help than your laptop could. People have developed various methods to calculate the spectrum of this problem, one of them is "Bethe Ansatz" and has become influential in various fields.

We can give upper and lower bounds to $\lambda^{(N)}_{\text{min}}$:

$$-\frac{1}{2}NS(S+1)\hbar^2 \leq \lambda^{(N)}_{\text{min}} \leq -\frac{1}{2}NS^2\hbar^2$$

The left part (lower bound) can be deduced directly from part (e); the right part (upper bound) can be deduced using the trial wave function $|+-+-\cdots\rangle$ or $|-+-+\cdots\rangle$. This is true for even or odd N.

The ground state configuration is *NOT* $|+-+-\cdots+-\rangle$, neither $|-+-+\cdots-+\rangle$ or their linear combinations. Please refer to 2(d) for values of $\lambda^{(N)}_{\text{min}}$.

(g) Refer to homework3.nb.

3. In this problem we consider a quantum system of multiple *qubits*, such as are used as the fundamental units in quantum computers and quantum communication systems.

Consider 4 spin-1/2 particles, each of which is in an eigenstate $S_x^{(i)} = \hbar/2$. In each part of this problem, a sequence of measurements is performed on these 4 particles. For each part of the problem, give all possible sequences of outcomes of the experiments, and calculate the probability for each sequence of outcomes. In each case, calculate the total probability that the final measurement of the quantity $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ gives each of the possible values. Each part of this problem should be done independently, starting with all spins in the eigenstate $S_x^{(i)} = \hbar/2$ as mentioned above.

(a) $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(b) $S_z^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(c) $S_z^{(2)}$ is measured, then $\mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(d) $S_z^{(1)}$ is measured, then $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$ is measured, then $\mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(e) $S_z^{(1)}$ is measured, then $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

Solution: Solution:

The projection operators corresponding to an outcome λ for an observable O are given by $M_\lambda = \sum_i |\lambda_i\rangle \langle\lambda_i|$, where $|\lambda_i\rangle$ are the eigenstates of O with eigenvalue λ. If λ is a non-degenerate eigenvalue of O, this sum has only one term, but we will need to consider the more general case for this problem. If we have a composite system $L = A \otimes B$, and we measure an observable $\mathbf{1}_A \otimes O_B$ supported only on B, the projection operator corresponding to measurement outcome λ is given by $\mathbf{1}_A \otimes M_{\lambda B}$.

If the initial state is $|\psi\rangle$, the state created by the measurement when the outcome is λ is

$$|\psi'\rangle = M_\lambda |\psi\rangle / \sqrt{\langle\psi| M_\lambda |\psi\rangle}, \tag{1}$$

and the probability of this measurement outcome is

$$p(\lambda) = \langle\psi| M_\lambda |\psi\rangle \tag{2}$$

Now in the current problem, the full state lies in the Hilbert space of four qubits, and is initially given by

$$|\psi_0\rangle = |+_x\rangle |+_x\rangle |+_x\rangle |+_x\rangle \tag{3}$$

a) Recall that the eigenvalues and eigenvectors of $\mathbf{S_3} \cdot \mathbf{S_4}$ are given by:

$$\begin{aligned} |{+}{+}\rangle_{34}\,, |{-}{-}\rangle_{34}\,, \frac{|{+}{-}\rangle_{34} + |{-}{+}\rangle_{34}}{\sqrt{2}} \quad &\text{with eigenvalue } \hbar^2/4 \\ \frac{|{+}{-}\rangle_{34} - |{-}{+}\rangle_{34}}{\sqrt{2}} \quad &\text{with eigenvalue } -3\hbar^2/4 \end{aligned} \tag{4}$$

In a case like this with degenerate eigenvectors, we can define the projection operator for measuring $\hbar^2/4$ on qubits 3 and 4 as

$$M^{3,4}_{\hbar^2/4} = [|{+}{+}\rangle \langle{+}{+}| + |{-}{-}\rangle \langle{-}{-}| + \frac{(|{+}{-}\rangle + |{-}{+}\rangle)}{\sqrt{2}} \frac{(\langle{+}{-}| + \langle{-}{+}|)}{\sqrt{2}}]_{34} \tag{5}$$

The projection operator for measuring $-3\hbar^2/4$ on qubits 3 and 4 as

$$M^{3,4}_{-3\hbar^2/4} = [\frac{(|+-\rangle - |-+\rangle)}{\sqrt{2}} \frac{(\langle +-| - \langle -+|)}{\sqrt{2}}]_{34} \tag{6}$$

Hence, the probabilities for the two measurements are given by

$$\begin{aligned} P(\hbar^2/4) &= \langle \psi_0 | (\mathbf{1}_{12} \otimes M^{3,4}_{\hbar^2/4}) . |\psi_0\rangle = 1 \\ P(-3\hbar^2/4) &= \langle \psi_0 | (\mathbf{1}_{12} \otimes M^{3,4}_{-3\hbar^2/4}) . |\psi_0\rangle = 0 \end{aligned} \tag{7}$$

In this part, it may be reasonable to calculate the above probabilities by hand, but for later parts, it will be very useful to do the calculations in Mathematica. See an outline of how to do this in the attached Mathematica file.

b) In this case, for the first measurement where $S^{(3)}_z$ is measured, the projection operators for the full system are given by

$$\begin{aligned} M_{\hbar/2} &= \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes (|+\rangle \langle +|)_3 \otimes \mathbf{1}_4 \\ M_{-\hbar/2} &= \mathbf{1}_1 \otimes \mathbf{1}_2 \otimes (|-\rangle \langle -|)_3 \otimes \mathbf{1}_4 \end{aligned} \tag{8}$$

Using (1) and (2), the final states and probabilities after these measurements are

$$\begin{aligned} |\psi^{\text{up}}\rangle &= |+_x\rangle |+_x\rangle |+\rangle |+_x\rangle \qquad p = 1/2 \\ |\psi^{\text{down}}\rangle &= |+_x\rangle |+_x\rangle |-\rangle |+_x\rangle \qquad p = 1/2 \end{aligned} \tag{9}$$

Now we can use the two possible final states above as initial states for the next measurement of $\mathbf{S_3} \cdot \mathbf{S_4}$. For this, we can again use the projection operators (5) and (6). So for instance, the probability of measuring $+\hbar/2$ in the first step and subsequently measuring $+\hbar^2/4$ is given by

$$P(\hbar/2 \to \hbar^2/4) = \frac{1}{2} \langle \psi^{\text{up}} | (\mathbf{1}_{12} \otimes M^{3,4}_{\hbar^2/4}) |\psi^{\text{up}}\rangle = 3/8 \tag{10}$$

Similarly,

$$\begin{aligned} P(\hbar/2 \to -3\hbar^2/4) &= \frac{1}{2} \langle \psi^{\text{up}} | (\mathbf{1}_{12} \otimes M^{3,4}_{-3\hbar^2/4}) |\psi^{\text{up}}\rangle = 1/8 \\ P(-\hbar/2 \to \hbar^2/4) &= \frac{1}{2} \langle \psi^{\text{down}} | (\mathbf{1}_{12} \otimes M^{3,4}_{\hbar^2/4}) |\psi^{\text{up}}\rangle = 3/8 \\ P(-\hbar/2 \to -3\hbar^2/4) &= \frac{1}{2} \langle \psi^{\text{down}} | (\mathbf{1}_{12} \otimes M^{3,4}_{-3\hbar^2/4}) |\psi^{\text{down}}\rangle = 1/8 \end{aligned} \tag{11}$$

The total probabilities of the two outcomes in the final measurement are

$$\begin{aligned} P(-3\hbar^2/4) &= P(\hbar/2 \to -3\hbar^2/4) + P(-\hbar/2 \to -3\hbar^2/4) = 1/4 \\ P(\hbar^2/4) &= P(\hbar/2 \to \hbar^2/4) + P(-\hbar/2 \to \hbar^2/4) = 3/4 \end{aligned} \tag{12}$$

c) The two possible final states after the measurement of $S^{(2)}_z$ are given by

$$\begin{aligned} \hbar/2: &\quad |\psi^{\text{up}}\rangle = |+_x\rangle |+\rangle |+_x\rangle |+_x\rangle \qquad p = 1/2 \\ -\hbar/2: &\quad |\psi^{\text{down}}\rangle = |+_x\rangle |-\rangle |+_x\rangle |+_x\rangle \qquad p = 1/2 \end{aligned} \tag{13}$$

After the second measurement, the possible final states are:

$$\begin{aligned} \hbar/2 \to -3\hbar^2/4: &\quad |\psi_f^1\rangle = (\mathbf{1}_1 \otimes M^{2,3}_{-3\hbar^2/4} \otimes \mathbf{1}_4)\,|\psi^{\text{up}}\rangle \,/\sqrt{(1/4)}, \qquad p = 1/8 \\ \hbar/2 \to \hbar^2/4: &\quad |\psi_f^2\rangle = (\mathbf{1}_1 \otimes M^{2,3}_{\hbar^2/4} \otimes \mathbf{1}_4)\,|\psi^{\text{up}}\rangle \,/\sqrt{(3/4)}, \qquad p = 3/8 \\ -\hbar/2 \to -3\hbar^2/4: &\quad |\psi_f^3\rangle = (\mathbf{1}_1 \otimes M^{2,3}_{-3\hbar^2/4} \otimes \mathbf{1}_4)\,|\psi^{\text{down}}\rangle \,/\sqrt{(1/4)}, \qquad p = 1/8 \\ -\hbar/2 \to \hbar^2/4: &\quad |\psi_f^4\rangle = (\mathbf{1}_1 \otimes M^{2,3}_{-3\hbar^2/4} \otimes \mathbf{1}_4)\,|\psi^{\text{up}}\rangle \,/\sqrt{(3/4)}, \qquad p = 3/8 \end{aligned} \tag{14}$$

Now using each of the four final states, which can be evaluated in Mathematica, we find the probabilities of all possible sequences of outcomes after the last measurement

$$\begin{aligned} \hbar/2 \to -3\hbar^2/4 \to -3\hbar^2/4: &\quad p = 1/32 \;\; \langle\psi_f^1|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{-3\hbar^2/4})\,|\psi_f^1\rangle \\ \hbar/2 \to -3\hbar^2/4 \to \hbar^2/4: &\quad p = 3/32 \;\; \langle\psi_f^1|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{\hbar^2/4})\,|\psi_f^1\rangle \\ \hbar/2 \to \hbar^2/4 \to -3\hbar^2/4: &\quad p = 1/32 \;\; \langle\psi_f^2|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{-3\hbar^2/4})\,|\psi_f^2\rangle \\ \hbar/2 \to \hbar^2/4 \to \hbar^2/4: &\quad p = 11/32 \;\; \langle\psi_f^2|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{\hbar^2/4})\,|\psi_f^2\rangle \\ -\hbar/2 \to -3\hbar^2/4 \to -3\hbar^2/4: &\quad p = 1/32 \;\; \langle\psi_f^3|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{-3\hbar^2/4})\,|\psi_f^3\rangle \\ -\hbar/2 \to -3\hbar^2/4 \to \hbar^2/4: &\quad p = 3/32 \;\; \langle\psi_f^3|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{\hbar^2/4})\,|\psi_f^3\rangle \\ -\hbar/2 \to \hbar^2/4 \to -3\hbar^2/4: &\quad p = 1/32 \;\; \langle\psi_f^4|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{-3\hbar^2/4})\,|\psi_f^4\rangle \\ -\hbar/2 \to \hbar^2/4 \to \hbar^2/4: &\quad p = 11/32 \;\; \langle\psi_f^4|\,(\mathbf{1}_{1,2} \otimes M^{3,4}_{\hbar^2/4})\,|\psi_f^4\rangle \end{aligned} \tag{15}$$

Evaluating each of the above probabilities using mathematica, we find that the total probabilities of the two outcomes after the final measurement are

$$\begin{aligned} P(\frac{-3\hbar^2}{4}) = 1/8 \\ P(\frac{\hbar^2}{4}) = 7/8 \end{aligned} \tag{16}$$

d) The calculation here is very similar to the previous part with an added step. The final answers are

$$\begin{aligned} P(\frac{-3\hbar^2}{4}) = 1/16 \\ P(\frac{\hbar^2}{4}) = 15/16 \end{aligned} \tag{17}$$

e) In this case, the spins 3 and 4 are undisturbed by the first two measurements and the initial state is a product state, so the probabilities should be the same as in part (a). This can also be checked explicitly

$$\begin{aligned} P(\frac{-3\hbar^2}{4}) = 0 \\ P(\frac{\hbar^2}{4}) = 1 \end{aligned} \tag{18}$$

3. Alternative solution:

(a) As discussed previously

$$\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This has the following eigenvalues and corresponding eigenvectors:

$$\begin{aligned} \frac{\hbar^2}{4} : \qquad & |++\rangle,\ |+-\rangle + |-+\rangle,\ |--\rangle \\ -\frac{3\hbar^2}{4} : \qquad & |+-\rangle - |-+\rangle \end{aligned}$$

Note that the eigenvectors with eigenvalue $\hbar^2/4$ are all symmetric under particle exchange, while the eigenvector with eigenvalue $-3\hbar^2/4$ is antisymmetric.

The initial state of the system is a sum over all possible spin combinations with equal weight, since

$$|S_x;+\rangle = |+\rangle + |-\rangle\,.$$

In particular, for the last two spins the state is

$$(|+\rangle + |-\rangle\ ) \otimes (|+\rangle + |-\rangle\ ) = |++\rangle + |+-\rangle + |-+\rangle + |--\rangle$$

The full state is just this state tensored with the same thing for the first two spins. Since this state is symmetric, it's a linear combination of states with eigenvalue $\hbar^2/4$, so

$$P(\hbar^2/4) = 1, \qquad P(-3\hbar^2/4) = 0.$$

(b) If we first measure a + spin, then the state is

$$\frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|+-\rangle$$

This state gives

$$P(\hbar^2/4) = 3/4, \qquad P(-3\hbar^2/4) = 1/4.$$

The same is true if we first measure a - spin by symmetry, so these are the total probabilities.

(c) It is useful here to notice that if the previous measurement (of $\mathbf{S}^{(i-1)} \cdot \mathbf{S}^{(i)}$) gave $-3\hbar^2/4$, then the state of particles $i-1, i, i+1$ is proportional to

$$(|+-\rangle - |-+\rangle) \otimes (|+\rangle + |-\rangle) = |+-+\rangle + |+--\rangle + |-++\rangle - |-+-\rangle$$

in which case just as above we have a 3/4 chance of getting $\hbar^2/2$ when we measure $\mathbf{S}^{(i)} \cdot \mathbf{S}^{(i+1)}$. This is almost enough information to finish the problem. For this part we also need the probability of getting two $\hbar^2/4$ measurements in a row. After measuring a single $\hbar^2/4$ (either from the initial state or after a sequence of measurements with the last being $-3\hbar^2/4$), the particles $i-1, i$ are in a state of the form

$$\frac{\sqrt{2}}{\sqrt{3}}|++\rangle + \frac{1}{\sqrt{6}}(|+-\rangle + |-+\rangle)$$

Tensoring with the state of the $i+1$ particle and extracting the part which is antisymmetric in particles $i, i+1$, we get

$$\frac{1}{4\sqrt{3}}(|++-\rangle - |+-+\rangle + |-+-\rangle - |--+\rangle)$$

Thus after a single $\hbar^2/4$ measurement we have a probability of 1/12 of measuring $-3\hbar^2/4$. Combining these results we can tabulate the probability of each sequence of measurements (the first measurement is always independently 50% for each spin; the signs in this table represent $+ = \hbar^2/4, - = -3\hbar^2/4$):

$--:$	$1/16\ \ (= (1/4)(1/4))$
$-+:$	$3/16\ \ (= (1/4)(3/4))$
$+-:$	$3/16\ \ (= (3/4)(1/12))$
$++:$	$11/16\ \ (= (3/4)(11/12))$

The total probability that the last measurement is $-3\hbar^2/4$ is therefore

$$P(\frac{-3\hbar^2}{4}) = 1/8$$

(d) The only new piece of information we need is the probability of getting a third $\hbar^2/4$ after two previous measurements of $\hbar^2/4$. Using a similar argument to above, we find that after two $\hbar^2/4$ measurements the state of particles $i-2, i-1, i$ is

$$\frac{1}{4\sqrt{3}}(4|+++\rangle + 3(|++-\rangle + |+-+\rangle) \\ +2(|+--\rangle + |-++\rangle) + (|-+-\rangle + |--+\rangle))$$

tensoring with the state of particle $i+1$ and extracting the part antisymmetric in particles $i, i+1$ we find a total probability of measuring a $-3\hbar^2/4$ after two $\hbar^2/4$'s is 1/44. This gives us a final table:

$---:$	1/64
$--+:$	3/64
$-+-:$	1/64
$-++:$	11/64
$+--:$	1/64
$+-+:$	3/64
$++-:$	1/64
$+++:$	43/64

The total probability that the last measurement is $-3\hbar^2/4$ is therefore

$$P(\frac{-3\hbar^2}{4}) = 1/16.$$

(e) This is exactly the same as part (a) since neither spin 3 or 4 is measured before the final measurement.

You might find it amusing to try to prove that the probability that the final measurement gives $-3\hbar^2/4$ is precisely $1/2^n$ where n is the total number of particles, measured in the same pattern as parts (b, c, d)!