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Course: **8.370 - QC**
Problem set: **#2**
Due: Wednesday, Sep 28, 2022
Collaborators:

1. Tensor products

Starting with

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

we have

$$\begin{aligned}
|\psi_x\rangle &= \sigma_x \otimes \mathbb{I}\,|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle)\\
\left|\psi_y\right\rangle &= \sigma_y \otimes \mathbb{I}\,|\psi\rangle = \frac{1}{\sqrt{2}}(i\,|11\rangle + i\,|00\rangle) = \frac{i}{\sqrt{2}}(|11\rangle + |00\rangle)\\
|\psi_z\rangle &= \sigma_z \otimes \mathbb{I}\,|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)
\end{aligned}$$

Now we check that they are all orthogonal:

$$\begin{aligned}
\langle\psi_x|\psi_x\rangle &= \frac{1}{2}(\langle 11|11\rangle + \langle 00|00\rangle - \langle 00|11\rangle - \langle 11|00\rangle) = 1\\
\left\langle\psi_y\middle|\psi_y\right\rangle &= \frac{1}{2}(\langle 11|11\rangle + \langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle) = 1\\
\langle\psi_z|\psi_z\rangle &= \frac{1}{2}(\langle 01|01\rangle + \langle 10|10\rangle + \langle 01|10\rangle + \langle 10|01\rangle) = 1\\
\left\langle\psi_x\middle|\psi_y\right\rangle &= \frac{i}{2}(\langle 11|11\rangle - \langle 00|00\rangle + \langle 11|00\rangle - \langle 00|11\rangle) = 0\\
\left\langle\psi_y\middle|\psi_z\right\rangle &= \frac{-i}{2}(\langle 11|01\rangle + \langle 11|10\rangle + \langle 00|01\rangle + \langle 00|10\rangle) = 0\\
\langle\psi_z|\psi_x\rangle &= \frac{1}{2}(\langle 01|11\rangle + \langle 10|11\rangle - \langle 01|00\rangle - \langle 10|00\rangle) = 0
\end{aligned}$$

2. Observable with repeated eigenvalues

$$M = \begin{pmatrix} 2 & 2 & 2\\ 2 & -1 & 1\\ 2 & 1 & -1 \end{pmatrix}$$

(a) Computing the eigenvalues associated with the with given eigenvalues is easy:

$$\begin{aligned}
\vec{v}_1 = (2\ 1\ 1)^\top: &\qquad \lambda = 4\\
\vec{v}_2 = (1\ -1\ -1)^\top: &\qquad \lambda = -2\\
\vec{v}_3 = (0\ 1\ -1)^\top: &\qquad \lambda = -2
\end{aligned}$$

Before forming orthogonal projections, we need to make sure that the provided vectors form an orthogonal basis. By inspection, we need to first normalize the vectors to get

$$|\psi_1\rangle = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{6}}(2\ 1\ 1)^\top, \quad |\psi_2\rangle = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{3}}(1\ -1\ -1)^\top, \quad |\psi_3\rangle = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{2}}(0\ 1\ -1)^\top.$$

Now we check for orthonormality. We can do this by inspection so I won't write out the algebra.

$$\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = \langle\psi_3|\psi_3\rangle = 1$$
$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_3\rangle = \langle\psi_3|\psi_1\rangle = 0.$$

With these conditions satisfied, the orthogonal projections are:

$$\Pi_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{6}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$
$$\Pi_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{3}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$
$$\Pi_3 = |\psi_3\rangle\langle\psi_3| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Sanity check:

$$\Pi_1 + \Pi_2 + \Pi_3 = \mathbb{I} \checkmark$$
$$\Pi_1\Pi_2 = \Pi_2\Pi_3 = \Pi_3\Pi_1 = \mathcal{O} \checkmark$$

All the algebra is from above is verified in Matheatica. Mathematica calculations:

```
(*eigv check*)
In[53]:= M = {{2, 2, 2}, {2, -1, 1}, {2, 1, -1}};
In[5]:= Eigenvalues[M]
Out[5]= {4, -2, -2}

(*vectors in the ONB*)
p1 = {2, 1, 1}/Norm[{2, 1, 1}];
p2 = {1, -1, -1}/Norm[{1, -1, -1}];
p3 = {0, 1, -1}/Norm[{0, 1, -1}];

(*check ONB*)
In[14]:= Dot[p1, p2]
Out[14]= 0
In[15]:= Dot[p2, p3]
Out[15]= 0
In[16]:= Dot[p3, p1]
Out[16]= 0
In[21]:= Dot[p1, p1]
Out[21]= 1
In[22]:= Dot[p2, p2]
Out[22]= 1
In[23]:= Dot[p3, p3]
Out[23]= 1

(*Compute projectors*)
In[46]:= M1 = KroneckerProduct[p1, p1]
Out[46]= {{2/3, 1/3, 1/3}, {1/3, 1/6, 1/6}, {1/3, 1/6, 1/6}}
In[47]:= M2 = KroneckerProduct[p2, p2]
Out[47]= {{1/3, -(1/3), -(1/3)}, {-(1/3), 1/3, 1/3}, {-(1/3), 1/3, 1/3}}
In[48]:= M3 = KroneckerProduct[p3, p3]
Out[48]= {{0, 0, 0}, {0, 1/2, -(1/2)}, {0, -(1/2), 1/2}}

(*Check resolution of identity:*)
In[52]:= M1 + M2 + M3
Out[52]= {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
In[205]:= M1 . M2
Out[205]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
In[206]:= M2 . M3
Out[206]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
In[207]:= M3 . M1
Out[207]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

(b) We are given

$$|\psi\rangle = \frac{2}{3}|0\rangle + \frac{2}{3}|1\rangle - \frac{1}{3}|2\rangle .$$

When this qutrit is measured, the possible outcomes are 4 and −2, with probabilities:

$$\Pr(4) = \langle\psi|\,\Pi_1\,|\psi\rangle = \frac{4}{9} \quad \text{and} \quad \Pr(-2) = \langle\psi|\,\Pi_2\,|\psi\rangle + \langle\psi|\,\Pi_3\,|\psi\rangle = \frac{5}{9}.$$

There are two ways to get the answer. By inspection, we can immediately see that the probability of measuring 4 is 4/9, since the coefficient for $|0\rangle$ is 2/3. From there, we can conclude that the probability of measuring −2 is simply $1 - 4/9 = 5/9$. The other way to find these values is by directly doing the algebra. The Mathematica code below has the explicit calculations.

```
In[58]:= \[Psi] = (2/3)*p1 + (2/3)*p2 - (1/3)*p3;

(*Pr(4)*)
In[71]:= Transpose[\[Psi]] . M1 . \[Psi] // FullSimplify
Out[71]= 4/9

(*Pr(-2)*)
In[72]:= Transpose[\[Psi]] . M2 . \[Psi] +  Transpose[\[Psi]] . M3 . \[Psi] // FullSimplify
Out[72]= 5/9
```

3. Spin-1 particle

We are given a spin-1 particle with three quantum states $|1\rangle, |0\rangle, |-1\rangle$. The observables corresponding to the spin along the three spatial directions are J_x, J_y, J_z:

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(a) We will show that J_x, J_z cannot be measured simultaneously by showing that they do not commute:

$$[J_x, J_z] = J_x J_z - J_z J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -iJ_y \neq O.$$

Mathematica code:

```
In[83]:= Jz = {{1, 0, 0}, {0, 0, 0}, {0, 0, -1}};
In[84]:= Jx = (1/Sqrt[2])*{{0, 1, 0}, {1, 0, 1}, {0, 1, 0}};
In[85]:= Jy = (1/Sqrt[2])*{{0, -I, 0}, {I, 0, -I}, {0, I, 0}};
In[87]:= Jx . Jz - Jz . Jx
Out[87]= {{0, -(1/Sqrt[2]), 0}, {1/Sqrt[2], 0, -(1/Sqrt[2])}, {0, 1/Sqrt[2], 0}}
```

(b) However, the observables J_x^2, J_y^2, J_z^2 all commute. We can do this by hand or use Mathematica again:

```
(*[Jx^2,Jy^2]*)
In[91]:= (Jx . Jx) . (Jy . Jy) - (Jy . Jy) . (Jx . Jx)
Out[91]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

(*[Jy^2,Jz^2]*)
In[92]:= (Jy . Jy) . (Jz . Jz) - (Jz . Jz) . (Jy . Jy)
Out[92]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

(*[Jz^2,Jx^2]*)
In[93]:= (Jz . Jz) . (Jx . Jx) - (Jx . Jx) . (Jz . Jz)
Out[93]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

There are possibly multiple ways (including clever math tricks) to find the simultaneous eigenvectors for J_x^2, J_y^2, J_z^2. However, it turns out that we could also do this by inspection:

$$J_x^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad J_y^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad J_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the form of the matrices, we can guess that the three normalized simultaneous eigenvectors are

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle), \quad |0\rangle = |0\rangle\,.$$

The corresponding eigenvalues can be found from the results below:

$$\begin{aligned}
J_x^2\,|+\rangle &= |+\rangle \\
J_x^2\,|-\rangle &= 0 \\
J_x^2\,|0\rangle &= |0\rangle \\
J_y^2\,|+\rangle &= 0 \\
J_y^2\,|-\rangle &= |-\rangle \\
J_y^2\,|0\rangle &= |0\rangle \\
J_z^2\,|+\rangle &= |+\rangle \\
J_z^2\,|-\rangle &= |-\rangle \\
J_z^2\,|0\rangle &= 0
\end{aligned}$$

So, J_i^2 has spectrum $\{0, 1\}$ for all $i = x, y, z$. Finally, we have

$$J^2 = J_x^2 + J_y^2 + J_z^2 = 2\mathbb{I}.$$

While a lot of the calculations in this problem could be done by hand, it is faster and more accurate to do them in Mathematica:

```
(*squaring*)
Jx2 = Jx . Jx;
Jy2 = Jy . Jy;
Jz2 = Jz . Jz;

In[117]:= Jx2
Out[117]= {{1/2, 0, 1/2}, {0, 1, 0}, {1/2, 0, 1/2}}
In[118]:= Jy2
Out[118]= {{1/2, 0, -(1/2)}, {0, 1, 0}, {-(1/2), 0, 1/2}}
In[119]:= Jz^2
Out[119]= {{1, 0, 0}, {0, 0, 0}, {0, 0, 1}}


(*eigenvalues calcs*)
In[140]:= plus = (1/Sqrt[2]) {1, 0, 1};
In[152]:= minus = (1/Sqrt[2])*{1, 0, -1};
In[153]:= zero = {0, 1, 0};
In[145]:= Jx2 . plus
Out[145]= {1/Sqrt[2], 0, 1/Sqrt[2]}
In[154]:= Jx2 . minus
Out[154]= {0, 0, 0}
In[149]:= Jx2 . zero
Out[149]= {0, 1, 0}
In[150]:= Jy2 . plus
Out[150]= {0, 0, 0}
In[155]:= Jy2 . minus
Out[155]= {1/Sqrt[2], 0, -(1/Sqrt[2])}
In[156]:= Jy2 . zero
Out[156]= {0, 1, 0}
In[157]:= Jz2 . plus
Out[157]= {1/Sqrt[2], 0, 1/Sqrt[2]}
In[159]:= Jz2 . minus
Out[159]= {1/Sqrt[2], 0, -(1/Sqrt[2])}
In[160]:= Jz2 . zero
Out[160]= {0, 0, 0}

(*finally, total spin*)
In[94]:= Jx . Jx + Jy . Jy + Jz . Jz
Out[94]= {{2, 0, 0}, {0, 2, 0}, {0, 0, 2}}
```

4. Deriving Spin-1 Observables

In this problem we derive the matrix J_x in the previous problem. Suppose we have two qubits A and B. The observable giving the spin in the x-direction is

$$S_x = \frac{1}{2}\left(\sigma_x^A \otimes \mathbb{I}^B + \mathbb{I}^A \otimes \sigma_x^B\right).$$

The 3-dimensional subspace of the 4-dimensional state space of two qubits which corresponds to the state space of a spin-1 particle is the subspace orthogonal to the state $(|01\rangle - |10\rangle)/\sqrt{2}$. To avoid confusion, let us replace 0 with $\uparrow$ and 1 with $\downarrow$

Since we have two qubits, we can treat them as two spin-1/2 particles, each denoted by $|s, m_s\rangle$. In this notation, we have

$$\begin{aligned}
|\uparrow\uparrow\rangle &= |1/2, +1/2\rangle \otimes |1/2, +1/2\rangle \\
|\uparrow\downarrow\rangle &= |1/2, +1/2\rangle \otimes |1/2, -1/2\rangle \\
|\downarrow\uparrow\rangle &= |1/2, -1/2\rangle \otimes |1/2, +1/2\rangle \\
|\downarrow\downarrow\rangle &= |1/2, -1/2\rangle \otimes |1/2, -1/2\rangle
\end{aligned}$$

When the spins are added, we can express the total spin and its projection as $|s, m\rangle \in \mathcal{H}^{\otimes 2}$ where

$$|s, m\rangle = \sum_{m_{s,1}=-s_1}^{s_1} \sum_{m_{s,2}=-s_2}^{s_2} C^{s,m}_{s_1, m_{s,1}, s_2, m_{s,2}} |s_1, m_{s,1}\rangle |s_2, m_{s,2}\rangle$$

where $C^{\dots}_{\dots}$'s are the Clebsch-Gordan coefficients. For this problem, the solution is rather simple. In the two-qubit Hilbert space, there is one state (the singlet) for which the total spin is zero ($s = 0$), and this state is one given in the problem: $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, and there are three states (triplet) which correspond to $s = 1$ (total spin equal to 1). It turns out that these are

$$\begin{aligned}
|s = 1, m = +1\rangle &= |\uparrow\uparrow\rangle \\
|s = 1, m = \ \ 0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
|s = 1, m = -1\rangle &= |\downarrow\downarrow\rangle
\end{aligned}$$

With this information, we can now construct a unitary matrix which transforms the standard basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ into the new basis where the first elements has spin 0 and the subsequent three has spin 1: $\{|0,0\rangle, |1,1\rangle, |1,0\rangle, |1,-1\rangle\}$. By inspection, this matrix is

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We expect that a similarity transformation on S_x by U will take the form of a (2×2)-block diagonal matrix of the form $\mathrm{diag}(0, Jx)$. And indeed, using Mathematica, we find that

$$U^\dagger S_x U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \\ & J_x \end{pmatrix}.$$

With this, we have

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

as desired.

Mathematica calculations:

```
In[220]:=
U = {{0, 1, 0, 0}, 1/Sqrt[2]*{1, 0, 1, 0},
    1/Sqrt[2]*{-1, 0, 1, 0}, {0, 0, 0, 1}};

In[222]:= ConjugateTranspose[U] . SX . U
Out[222]= {{0, 0, 0, 0}, {0, 0, 1/Sqrt[2], 0}, {0, 1/Sqrt[2], 0, 1/
        Sqrt[2]}, {0, 0, 1/Sqrt[2], 0}}

(*displaying the result in matrix form gives Jx*)
```

5. Generalized Measurements

Here we derive an example of a non-von Neumann measurement. We're given one of the three states

$$|\psi_1\rangle = |0\rangle\,, \quad |\psi_2\rangle = -\frac{1}{2}\,|0\rangle + \frac{\sqrt{3}}{2}\,|1\rangle\,, \quad |\psi_3\rangle = -\frac{1}{2}\,|0\rangle - \frac{\sqrt{3}}{2}\,|1\rangle$$

with equal probabilities.

(a) Suppose that we make a measurement of the state in the following arbitrary basis:

$$\{|A\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,, |B\rangle = -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle\}$$

Let us focus on when we find state $|A\rangle$ after the measurement. Suppose we guess that the input state is $|\psi_1\rangle$ whenever we see state $|A\rangle$, then the probability of success is given by

$$\Pr_{|A\rangle \implies |\psi_1\rangle} = \frac{|\langle\psi_1|A\rangle|^2}{|\langle\psi_1|A\rangle|^2 + |\langle\psi_2|A\rangle|^2 + |\langle\psi_3|A\rangle|^2}$$

which comes from the fact that there are three ways we could find state $|A\rangle$ after the measurement, each contributing some probability. Plugging in the numbers, we find that

$$\Pr_{|A\rangle \implies |\psi_1\rangle} = \frac{2}{3}\cos^2\theta \le \frac{2}{3}, \quad \forall\theta \in [0, 2\pi]$$

Similarly, if we guess it is state $|\psi_2\rangle$ or $|\psi_3\rangle$ then the probability of success in each case is

$$\Pr_{|A\rangle \implies |\psi_2\rangle} = \frac{1}{6}\left(\cos\theta - \sqrt{3}\sin\theta\right)^2 \le \frac{2}{3}, \quad \forall\theta \in [0, 2\pi]$$
$$\Pr_{|A\rangle \implies |\psi_3\rangle} = \frac{1}{6}\left(\cos\theta + \sqrt{3}\sin\theta\right)^2 \le \frac{2}{3}, \quad \forall\theta \in [0, 2\pi]$$

From the last three inequalities, we see that the best success probability is 2/3.

Since there's nothing special about whether we pick $|A\rangle$ or $|B\rangle$ as the "indicator," we expect the same result to hold if we use $|B\rangle$ instead of $|A\rangle$ and can stop here. However, it doesn't hurt to be explicit. So, following the same notation as in the argument above, we have

$$\Pr_{|B\rangle \implies |\psi_1\rangle} = \frac{2}{3}\sin^2\theta \le \frac{2}{3}, \quad \forall\theta \in [0, 2\pi]$$
$$\Pr_{|B\rangle \implies |\psi_2\rangle} = \frac{1}{6}\left(\sqrt{3}\cos\theta + \sin\theta\right)^2 \le \frac{2}{3}, \quad \forall\theta \in [0, 2\pi]$$
$$\Pr_{|B\rangle \implies |\psi_3\rangle} = \frac{1}{6}\left(-\sqrt{3}\cos\theta + \sin\theta\right)^2 \le \frac{2}{3}, \quad \forall\theta \in [0, 2\pi]$$

which is not surprisingly the same as before.

Mathematica code:

```
In[159]:= (*Problem 5a*)
In[133]:= (*define input states*)
In[128]:= v0 = {1, 0};
In[129]:= v1 = {-1/2, Sqrt[3]/2};
In[130]:= v2 = {-1/2, -Sqrt[3]/2};

In[134]:= (*define measurement basis*)
In[131]:= A = {Cos[\[Theta]], Sin[\[Theta]]};
B = {-Sin[\[Theta]], Cos[\[Theta]]};
In[142]:= (*calculate inner products*)
In[143]:= A0 = Dot[A, v0]^2 // FullSimplify
Out[143]= Cos[\[Theta]]^2
In[144]:= B0 = Dot[B, v0]^2 // FullSimplify
Out[144]= Sin[\[Theta]]^2
In[145]:= A1 = Dot[A, v1]^2 // FullSimplify
Out[145]= 1/4 (Cos[\[Theta]] - Sqrt[3] Sin[\[Theta]])^2
In[146]:= B1 = Dot[B, v1]^2 // FullSimplify
Out[146]= 1/4 (Sqrt[3] Cos[\[Theta]] + Sin[\[Theta]])^2
In[147]:= A2 = Dot[A, v2]^2 // FullSimplify
Out[147]= 1/4 (Cos[\[Theta]] + Sqrt[3] Sin[\[Theta]])^2
In[148]:= B2 = Dot[B, v2]^2 // FullSimplify
Out[148]= 1/4 (-Sqrt[3] Cos[\[Theta]] + Sin[\[Theta]])^2

In[155]:= (*if pick second basis vector as indicator*)
(*calculate success probabilities*)
In[151]:= PrBv0 = B0/(B0 + B1 + B2) // FullSimplify
Out[151]= (2 Sin[\[Theta]]^2)/3
In[152]:= PrBv1 = B1/(B0 + B1 + B2) // FullSimplify
Out[152]= 1/6 (Sqrt[3] Cos[\[Theta]] + Sin[\[Theta]])^2
In[153]:= PrBv2 = B2/(B0 + B1 + B2) // FullSimplify
Out[153]= 1/6 (-Sqrt[3] Cos[\[Theta]] + Sin[\[Theta]])^2

In[154]:= (*if pick first basis vector as indicator*)
(*calculate success probabilities*)
In[156]:= PrAv0 = A0/(A0 + A1 + A2) // FullSimplify
Out[156]= (2 Cos[\[Theta]]^2)/3
In[157]:= PrAv1 = A1/(A0 + A1 + A2) // FullSimplify
Out[157]= 1/6 (Cos[\[Theta]] - Sqrt[3] Sin[\[Theta]])^2
In[158]:= PrAv2 = A2/(A0 + A1 + A2) // FullSimplify
Out[158]= 1/6 (Cos[\[Theta]] + Sqrt[3] Sin[\[Theta]])^2

(*find Max*)
In[161]:= MaxValue[PrAv0, \[Theta]]
Out[161]= 2/3
In[162]:= MaxValue[PrAv1, \[Theta]]
Out[162]= 2/3
In[163]:= MaxValue[PrAv2, \[Theta]]
Out[163]= 2/3
In[164]:= MaxValue[PrBv0, \[Theta]]
Out[164]= 2/3
In[165]:= MaxValue[PrBv1, \[Theta]]
Out[165]= 2/3
In[166]:= MaxValue[PrBv2, \[Theta]]
Out[166]= 2/3
```

(b) Now we take the first qubit and tensor it with a second qubit in $|0\rangle$. Consider the following states:

$$\{|a\rangle, |b\rangle, |c\rangle, |d\rangle\} = \left\{|11\rangle, -\frac{\alpha}{2}|00\rangle + \frac{\sqrt{3}\alpha}{2}|10\rangle + \beta|01\rangle, \alpha|00\rangle + \beta|01\rangle, -\frac{\alpha}{2}|00\rangle - \frac{\sqrt{3}\alpha}{2}|10\rangle + \beta|01\rangle\right\}.$$

In order for these to form an orthonormal basis, α and β must satisfy the following conditions:

$$|\alpha|^2 + |\beta|^2 = 1$$
$$-\frac{|\alpha|^2}{2} + |\beta|^2 = 0$$

From the first two equations, we find that $|\alpha|^2 = 2/3$ and $|\beta|^2 = 1/3$. Assuming $\alpha, \beta \in \mathbb{R}$, we can let $\alpha = \sqrt{2/3}$ and $\beta = \sqrt{1/3}$.

(c) With probability $1/3$ we are given $|\psi_1\rangle$, which we transform to $|\psi_1\rangle|0\rangle$. Measuring this state in the basis above, we find

$$\Pr(|a\rangle) = 0$$
$$\Pr(|b\rangle) = 1/6$$
$$\Pr(|c\rangle) = 2/3$$
$$\Pr(|d\rangle) = 1/6$$

With probability 1/3 we are given $|\psi_2\rangle$, which we transform to $|\psi_2\rangle\,|0\rangle$. Measuring this state in the basis above, we find

$$\Pr(|a\rangle) = 0$$
$$\Pr(|b\rangle) = 2/3$$
$$\Pr(|c\rangle) = 1/6$$
$$\Pr(|d\rangle) = 1/6$$

With probability 1/3 we are given $|\psi_3\rangle$, which we transform to $|\psi_3\rangle\,|0\rangle$. Measuring this state in the basis above, we find

$$\Pr(|a\rangle) = 0$$
$$\Pr(|b\rangle) = 1/6$$
$$\Pr(|c\rangle) = 1/6$$
$$\Pr(|d\rangle) = 2/3$$

Now we make the following rules for guessing:

- If we measure and find $|b\rangle$ then guess $|\psi_2\rangle$
- If we measure and find $|c\rangle$ then guess $|\psi_1\rangle$
- If we measure and find $|d\rangle$ then guess $|\psi_3\rangle$

Since the cases are symmetric, the success probability is simply given by

$$\Pr(\text{success}) = \frac{2/3}{2/3 + 1/6 + 1/6} = \frac{2}{3}$$

Mathematica calculations:

```
In[17]:= (*5b*)
In[60]:= a1 = {0, 0, 0, 1};
In[59]:= a3 = {\[Alpha], \[Beta], 0, 0};
In[56]:= a2 = {-\[Alpha]/2, \[Beta], \[Alpha]*Sqrt[3]/2, 0};
In[57]:= a4 = {-\[Alpha]/2, \[Beta], -\[Alpha]*Sqrt[3]/2, 0};
In[64]:= Psi1 = {1, 0, 0, 0};
In[62]:= Psi2 = {-1/2, 0, Sqrt[3]/2, 0};
In[63]:= Psi3 = {-1/2, 0, -Sqrt[3]/2, 0};
In[65]:= Dot[Psi2, a1]^2
Out[65]= 0
In[66]:= Dot[Psi2, a2]^2
Out[66]= \[Alpha]^2
In[67]:= Dot[Psi2, a3]^2
Out[67]= \[Alpha]^2/4
In[68]:= Dot[Psi2, a4]^2
Out[68]= \[Alpha]^2/4
In[52]:= Dot[Psi3, a1]^2
Out[52]= 0
In[53]:= Dot[Psi3, a2]^2
Out[53]= \[Alpha]^2/4
In[54]:= Dot[Psi3, a3]^2
Out[54]= \[Alpha]^2/4
In[55]:= Dot[Psi3, a4]^2
Out[55]= \[Alpha]^2
```