Exercises for Lecture 2
Foundations of Stat Mech

May 25, 2020

1 Swap operator

One of the problems of the entanglement entropies is that to know them one has to know all the eigenvalues of the reduced density operator, which amounts to knowing the wave-function. In other words, entanglement is not the expectation value of an observable. Define the swap operator as the operator on two copies of the original Hilbert space, namely $T^{(2)} \in \mathcal{H} \otimes \mathcal{H}$. This operator is defined from the action on the basis states as

$$T^{(2)} |\phi_i \phi_j\rangle = |\phi_j \phi_i\rangle \tag{1}$$

1.1 Purity as expectation value

Given a state $\sigma \in \mathcal{B}(\mathcal{H})$, show that $\mathrm{tr}\sigma^2 = \mathrm{tr}(T^{(2)}\sigma^{\otimes 2})$.

1.2 Trace of $T^{(2)}$

Compute the trace of $T^{(2)}$.

1.3 Frobenius norm

Show that, for any linear operator X, the Frobenius norm reads $\|X\|_2^2 = \mathrm{tr}(T^{(2)} X \otimes X^\dagger)$.

2 Unitary evolution

2.1 Norm invariance

Show that the Frobenius norm and the Operator norm are preserved by unitary operations.

2.2 Entropy invariance

Show that all Entropies are preserved by unitary evolution.

3 Entanglement and partial trace

Show that, if a state $\sigma(t)$ is evolving in time, the purity of the marginal state $\sigma_A(t)$ is given by

$$\mathrm{tr}\sigma_A^2(t) = \mathrm{tr}(T^{(2)} \otimes I_B^{\otimes 2} U^{\otimes 2} \sigma^{\otimes 2} U^{\dagger \otimes 2}) \tag{2}$$

where U is the unitary evolution operator.

4 Time evolution by Hamiltonian

Be $\mathcal{U}(X) = e^{iHt} X e^{-iHt} \equiv X_t$ the time evolution generated by a non degenerate Hamiltonian $H = \sum_n E_n P_n$

4.1 Time averaged operator

define the time average as $\overline{X} = \lim_{T\to\infty} \frac{1}{T} \int_0^T X_t dt$. Show that

$$\overline{X} = \sum_n P_n X P_n \tag{3}$$

5 Quantum measurement

Show that, after measuring on the state σ some observable $\Omega = \sum_i o_i |i\rangle\langle i| \equiv \sum_i o_i \omega_i$ the resulting state is

$$\sigma' = \sum_i \omega_i \sigma \omega_i \tag{4}$$