MATRIX ANALYSIS
MA353

Feb 7, 2019

1 Complex Numbers

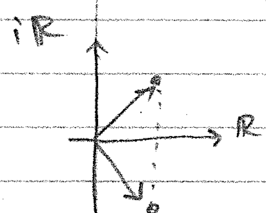
"Traditional way to think"

- $\mathbb{C}$ writing "$a+ib$" is equiv to writing a pair $(a,b) \in \mathbb{R}^2$, $a, b \in \mathbb{R}$
- Addition $(a,b) + (c,d) = (a+c,\ b+d)$
- Multiplication $(a,b)\cdot(c,d) = (ac-bd,\ ad+bc)$

↳ Same as $(a+ib)(c+id) = (ac-bd) + i(ad+bc)$

Think of i as $\sqrt{-1}$, a multiplication helper.

→ "better" multiplication than element wise multiplication, because there are properties we're familiar with... in real multiplication.



"i" ~ $(0,1)$, "1" ~ $(1,0)$, "0" ~ $(0,0)$

- Conjugate of $a+ib$ is $a-ib$ → reflecting about the real axis.

"New way to think abt complex no."

↳ $$a+ib \sim \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I} + b\underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{"}i\text{"}}$$

real-entried matrices

$$= aI + b\mathring{I}$$

So

$$(aI + b\mathring{I})(cI + d\mathring{I}) \overset{?}{=} (cI + d\mathring{I})(aI + b\mathring{I}) \ ✓$$

$$\rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix},\ \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$ commute! same as complex no. multiplication

Addition works exactly the same way as before. → adding matrices.

Taking the adjoint (conjugating) ≡ Transpose them.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = MM^T = \begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix} = \underline{\det(M)} \cdot I$$

$$= (a^2+b^2)\, I$$

Similarly $\rightarrow (a+ib)(a-ib) = a^2+b^2$

Note $\boxed{A \cdot [\ \ ] = (\det A)\, I}$

Note . $\det A = 0 \iff a = b = 0$

. $A \cdot [\ \ ]/\det A = I \Rightarrow [\ \ ] = A^{-1} \equiv$ "reciprocal" of complex no.

Note $A^T = A^{-1}$ here if $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

$$\underbrace{\frac{1}{3+5i}}_{\text{number}} = \frac{3-5i}{3-5i}\,\frac{1}{3+5i} = \underbrace{\frac{1}{3^2+5^2}}_{1/\text{length}}\underbrace{(3-5i)}_{\text{new number...}}$$

We will show that there always exists a classical adjoint B s.t $AB = (\det A) \cdot I$

Geometric interpretation

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac-bd \\ bc+ad \end{pmatrix} \rightarrow \text{multiplication is some linear transformation}$$

"$a+ib$" "$c+id$" = "$(a+ib)(c+id)$"

Think normalize

$A = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \rightarrow$ is a $\boxed{\text{Unitary matrix}}$

$\rightarrow$ preserves length, orthogonality, angles...

Can write $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ → rotation by θ

So complex number multiplication ≡ rotate + scale by length.

↳ Euler notation $re^{i\theta}$

↳ $\boxed{re^{i\varphi}\rho e^{i\theta} = r\rho e^{i(\theta+\varphi)}}$ → captures multiplication.

↳ Formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

why complex numbers? → Closed - completes fundamental theorem of algebra

↳ poly. w/ real coeffs. $p(x) = \sum a_i x_i^{th}$

let $z \in \mathbb{C}$ be a root $\sum^{n} a_i z_0^i = 0$

So $\left(\sum^{n} a_i z_0^i\right)^* = 0 = \sum^{n} (a_i z_0^i)^*$ $(AB)^T = B^T A^T$

$= \sum^{n} (a_i)^*(z_0^i)^* = 0$

$= \sum^{n} (a_i)^*(z_0^*)^i = 0$

$= \sum^{n} (a_i)(z_0^*)^i = 0$

So if z_0 is a root, then z_0^* is also a root.

So $P(x) = (x-r_1)(x-r_2)\cdots\underbrace{(x-z_1)(x-z_1^*)}_{\text{quadratic}}\cdots(x-z_n)(x-z_n^*)$

So "real" fundamental theorem of algebra ⇒ any poly ~~ex~~ factors as ... linear terms + quadratic terms

Feb 12, 2019

$\boxed{\text{Review of Linear spaces \& Linear Functions}}$

- Linear Space $\equiv$ Vector Space. $\rightarrow$ Ex $\mathbb{C}^n$, $\mathbb{P}_n$ (Poly. deg $\leq n$ complex coeffs)

  $\mathbb{P} \rightarrow$ polynomials no restrictions on degrees

  closed under addition \& multiplication scaler

- Coordinate system $\equiv$ Basis

More examples • $C(X, \mathbb{C})$ $\rightarrow$ continuous functions $[0,1] \mapsto \mathbb{C}$

$\mathbb{P}_n$ subspace of $\mathbb{P}$ subspace of $\mathbb{C}$

- Space of $n \times k$ matrixes $\mathbb{M}_{n\times k}(\mathbb{C})$

- $\mathcal{G}^2 \rightarrow$ geometric vectors in a plane

$\boxed{\text{Linear Function}}$ $\quad L: V \mapsto W$

$$\Leftrightarrow \begin{cases} L(\alpha \cdot v) = \alpha \hat{\cdot} L(v) \\ L(v_1 + v_2) = L(v_1) \hat{+} L(v_2) \end{cases}$$

operations in W

Ex • $\mathbb{M}_{2\times 3} : \mathbb{C}^3 \mapsto \mathbb{C}^2$

$$A = [V_1 \quad V_2 \quad V_3]$$

$$A\begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix} = \alpha V_1 + \beta V_2 + \gamma V_3$$

"Atrix"

- $\varphi : \mathbb{C}^3 \xrightarrow{\text{linear}} \mathbb{P}_2$

We write $\boxed{\Phi = [\, x^2 + ix \quad 3x - i \quad ix^2 + 2x - 7 \,]}$

$$\hookrightarrow \varphi\begin{pmatrix}\alpha \\ \beta \\ \gamma\end{pmatrix} = \alpha(x^2 + ix) + \beta(3x - i) + \gamma(ix^2 + 2x - 7)$$

"Atrix" $\varphi : \mathbb{C}^n \longmapsto V$ → always linear.

Define $\varphi = [\,V_1\ V_2 \ldots V_n\,]$

$$\varphi(a^i_q) = \sum_{i=1}^{n} a_i V_i = a^i V_i$$

$\boxed{\text{Theorem}}$

↳ Every linear function $\Psi : \mathbb{C}^n \xrightarrow{\text{lin}} V$ is an atrix

key $\Psi(\alpha^i) = \alpha^i \Psi(\vec{e}_i)$

So $\Psi = [\,\Psi(\vec{e}_i)\,]$ — So atrices are linear functions.

Why are atrixs useful? → $\left\{ \begin{array}{l} \text{linear independence} \\ \text{span} \\ \text{bases.} \end{array} \right.$

- $\mathrm{Im}(\Psi) = \mathrm{span}(\{V_i\})$ If $\Psi = [\,V_1 \ldots V_m\,]$

Linear independence . $\boxed{a^i V_i = 0_V \Leftrightarrow a^i = 0 \ \forall i}$

- $\Psi = [\,V_1 \ldots V_m\,]$ injective when V_i's linearly independent

$\Updownarrow$

$\ker(\Psi) = 0_V$

- "Basis" = "linear independence" + "span"

- Ψ bijective $\Leftrightarrow$ $\{V_i\}$ is a basis of V

$\boxed{\text{Elementary Column Operations}}$ $\left\{ \begin{array}{l} \text{swap} \\ \text{scale} \\ \text{add one} \cdot \alpha \text{ to another.} \end{array} \right.$

- $[\,V_1 \ldots V_n\,] : \mathbb{C}^n \mapsto V$

Note Let $\varphi = [V_1 \cdots V_n] \quad \mathbb{C}^n \longmapsto V$

$$\text{Consider} \quad \underbrace{\mathbb{C}^m \xrightarrow[\text{lin}]{\underset{n\times m}{M}} \mathbb{C}^n \xrightarrow[\text{lin}]{\varphi} V}\xrightarrow[\text{lin}]{} W$$

"Every atrix can be combined with a <u>matrix</u>"

Ex

$$\left.\begin{array}{l} A = [V_1 \ V_2 \ V_3]_{V\times 3} \\ M = [\vec{v}_1 \ \vec{v}_2 \ \cancel{\vec{v}_3}]_{3\times \cancel{3}} \end{array}\right\} \quad A \circ M = \left[A\vec{v}_1 \ ? \ A\vec{v}_2\right]_{V\times 2}$$

↳ Use the standard tuples

$$(A\circ M)(\vec{e}_1) = A(M(\vec{e}_1)) = A\vec{v}_1$$

So $\boxed{A \circ \underset{\downarrow\,[v_k]}{M} = [A(\vec{v}_1) \ldots A(\vec{v}_n)]}$

Elementary Glue Operation

Ex

$$Ⓐ = [V_1 \ V_2 \ V_3] \circ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = [V_3 \ V_2 \ V_1] = Ⓑ$$

"swap" is equiv to composing with bijection

→ "jectivity" is preserved.

→ $\text{Im } A = \text{Im } B$

→

$$[V_1 \ V_2 \ V_3] \circ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3i & 0 \\ 0 & 0 & 1 \end{pmatrix} = [V_1 \ \ 3i V_2 \ \ V_3]$$

$$[V_1 \ V_2 \ V_3] \circ \begin{pmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [V_1 - 4V_2 \ \ V_2 \ \ V_3]$$

So, performing elementary column operations <u>don't alter</u>

$\left\{ \begin{array}{c} \text{'jectivity} \\ \text{Image} \end{array} \right\}$ (composing w bijection)

$\operatorname{Im}[V_1 \ldots V_n] = \operatorname{Im}[V_1 \ldots V_n \, 0_V]$ same thing

$\operatorname{Im}[V_1 \ldots V_n] = \operatorname{Im}[V_1 \ldots V_{n-1}] \Leftrightarrow V_n \in \operatorname{Ran}[V_1 \ldots V_{n-1}]$

- $(\Leftarrow)$ If $V_n \in \operatorname{Ran}(V_1 \ldots V_n) = \operatorname{Im}(V_1 \ldots V_n)$

  then $\operatorname{Im}[V_1 \ldots V_n]$

  $= \operatorname{Im}\left[V_1 \ldots V_{n-1} \ \sum^{n-1} V_i a_i\right]$ E.C.O

  $= \operatorname{Im}[V_1 \ldots V_{n-1} \ 0_V]$

  $= \operatorname{Im}[V_1 \ldots V_{n-1}]$

- $(\Rightarrow)$ $V_n \in \operatorname{Ran}[V_1 \ldots V_n] = \operatorname{Ran}[V_1 \ldots V_{n-1}]$

  "Ran" $\equiv$ "Im"

If not all $\{V_i\}$'s are null, by removing some of them, we can arrive at a linearly independent column with the same <u>span</u>.

Consequently, If $\{V_i\}$ <u>spans</u> V then $\{V_i\}$ contains a <u>basis</u>.

<u>Note</u> NOT THIS $\{0_V\} \to$ not a basis! Error!

$\boxed{\text{Every basis of } V \text{ has the same dimension}}$

(8)

$$\begin{array}{ccc} A & & B \end{array}$$

<u>Suppose</u> $[V_1 \ldots V_{13}]$ and $[W_1 \ldots W_{17}]$ ~~span~~ basis V

$$[V_1 \ldots V_{13}] : \mathbb{C}^{13} \xrightarrow[\text{linear}]{\text{bijection}} V \xleftarrow[\text{linear}]{\text{bijection}} [W_1 \ldots W_{17}] \; \mathbb{C}^{17}$$

<u>So</u> $[V_1 \ldots V_{13}] : \mathbb{C}^{13} \xrightarrow[\text{lin}]{\text{bij}} V \xrightarrow[\text{lin}]{\text{bij}} [W_1 \ldots W_{17}] \; \mathbb{C}^{17}$

<u>So</u> $A \; \mathbb{C}^{13} \xrightarrow{A} V \xrightarrow{B^{-1}} \mathbb{C}^{17}$

<u>Note</u> $AB^{-1} \; \mathbb{C}^{13} \to \mathbb{C}^{17}$ has to be a linear bijection

~~But~~ <u>So</u> AB^{-1} is a 17×13 matrix

But this can't happen. $\Rightarrow$ Contra(+) true.

<u>So</u>

{ Span sets have at least $\dim(V)$ elements
{ linearly independent set have at <u>most</u> $\dim(V)$ elements.

Claim basis $V_1 \ldots V_{13}$
lin. ind $W_1 \ldots W_{17}$

$$\begin{array}{c} \mathbb{C}^{17} \\ \Big\downarrow \text{inj} \;\; [W_1 \ldots W_{17}] \\ V \\ \text{bij} \Big\downarrow\Big\uparrow \;\; [V_1 \ldots V_{13}]^{-1} \\ \mathbb{C}^{13} \end{array}$$

$[\quad]_{13 \times 17}$

wide matrix cannot be inj!

$\boxed{\text{So}}$

- Every lin. ind in finite dim can be enlarged to a basis

Start w $[V_1 \ldots V_n]$ lin. ind.

If $\to$ Span $\to$ stop

not Span $\to \exists V_{n+1}$ s.t.

$V_{n+1} \notin \mathrm{Im}[V_1 \ldots V_n]$

$\to$ Bigger atrix $A = [V_1 \ldots V_{n+1}]$

Look at .tex

Claim $[V_1 \dots V_n\ V_{n+1}]$ lin. ind.

since $V_{n+1} \notin \operatorname{Im}[V_1, \dots V_n]$

$\rightarrow [V_1 \dots V_n\ V_{n+1}]$ lin. ind $(\Leftarrow)$

keep repeating until

$\#[\ ] = \dim(V)$

$V_1 \dots V_n$ lin. ind.

$\Leftrightarrow \sum a_i V_i = 0 \Leftrightarrow a_i = 0 \quad \forall i$

$\Leftrightarrow \begin{cases} V_1 \neq 0_V \\ \text{no } V_i \text{ is a lin ind of the previous } V_i\text{'s} \end{cases}$

Next time Isomorphisms, linear functions & matrices ...

$\rightarrow$ change of basis

then Rank-Nullity theorem

Then $\rightarrow$ Decomposition of vector space w/ a basis

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$\boxed{\text{Products of Vector Spaces}}$

recall $\mathbb{C}^2 \begin{pmatrix} a \\ b \end{pmatrix}$, $\mathbb{C}^5 \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \varepsilon \end{pmatrix}$ $\longrightarrow \mathbb{C}^7$

So $\mathbb{C}^2 \times \mathbb{C}^5 \leadsto \mathbb{C}^7$

Note $\mathbb{C}^5 \times \mathbb{C}^2 \neq \mathbb{C}^7$

So ... $\mathbb{C}^2 \times \mathbb{C}^5 \longrightarrow \mathbb{C}^7$

& ...

$$\alpha \begin{pmatrix} \bigcirc \\ \square \end{pmatrix} \underset{\sim}{+} \beta \begin{pmatrix} \hat{\bigcirc} \\ \hat{\square} \end{pmatrix} = \begin{pmatrix} \alpha \bigcirc + \beta \hat{\bigcirc} \\ \alpha \square + \beta \hat{\square} \end{pmatrix}$$

So, what if we have 2 vector spaces V & W

$$V \times W = \left\{ \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix} \middle| \ \vec{v} \in V, \ \vec{w} \in W \right\}$$

Define addition

$$\left\{ \begin{array}{l} \begin{pmatrix} V_1 \\ W_1 \end{pmatrix} \pm \begin{pmatrix} V_2 \\ W_2 \end{pmatrix} \triangleq \begin{pmatrix} V_1 \overset{V}{+} V_2 \\ W_1 \overset{W}{+} W_2 \end{pmatrix} \\ \alpha \begin{pmatrix} V \\ W \end{pmatrix} \triangleq \begin{pmatrix} \alpha V \\ \alpha W \end{pmatrix} \end{array} \right\} \text{This makes } V \times W \text{ a vector space.}$$

Neutral element $\rightarrow \begin{pmatrix} \phi_V \\ \phi_W \end{pmatrix}$

Suppose V is 3 dimensional, W is 5 dimensional $\}$ finite-dim

claim $\dim(V \times W) = \dim(V) + \dim(W)$

Proof let ~~suppose~~ $\{v_1, v_2, v_3\}$ be a basis of V

$\{w_1 \dots w_5\}$ be a basis of W

Consider $\begin{pmatrix} V_1 \\ \phi_W \end{pmatrix}, \begin{pmatrix} V_2 \\ \phi_W \end{pmatrix}, \begin{pmatrix} V_3 \\ \phi_W \end{pmatrix}, \begin{pmatrix} \phi_V \\ w_1 \end{pmatrix} \dots \begin{pmatrix} \phi_V \\ w_5 \end{pmatrix}$

Claim these span $V \times W$

Given $\begin{pmatrix} V \\ W \end{pmatrix}$ $\quad V = \sum_1^3 a_i v_i, \quad W = \sum_1^5 b_i w_i$

$$\begin{pmatrix} V \\ W \end{pmatrix} = a_1 \begin{pmatrix} V_1 \\ \phi_W \end{pmatrix} + a_2 \begin{pmatrix} V_2 \\ \phi_W \end{pmatrix} + \dots + b_5 \begin{pmatrix} \phi_V \\ W_5 \end{pmatrix} \Rightarrow \text{span}$$

<u>Lin-indep</u> $a_1\begin{pmatrix} v_1 \\ 0_W \end{pmatrix} + \cdots + a_8\begin{pmatrix} 0_V \\ w_5 \end{pmatrix}$

$$= \begin{pmatrix} \sum^{3} a_i v^i \\ \sum^{5} a_i w^i \end{pmatrix} = \begin{pmatrix} 0_W \\ 0_W \end{pmatrix} \iff \begin{cases} a=b=c=0 \\ d=e=f=g=i=0 \end{cases}$$

So all coeffs are <u>zero</u>

$\Rightarrow$ So $\dim(V+W) = \dim(V) + \dim(W) = 8$

<u>Caution</u> Vector products are $\boxed{\text{not}}$ commutative

- $\boxed{W\times V \neq V\times W}$ $\quad W\times V \sim V\times W$ But they're not the same
- Not associative

$$\boxed{V\times(W\times Z) \neq (V\times W)\times Z}$$

$\Big($ natural $\begin{pmatrix} (1) \\ \binom{1}{1} \end{pmatrix} \neq \begin{pmatrix} \binom{1}{1} \\ (1) \end{pmatrix}\Big)$

<u>Def</u>

$$\boxed{V_1\times\cdots\times V_n = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \,\middle|\, v_i\in V_i \right\}}$$

<u>Note</u>

$$\begin{pmatrix} v \\ w \end{pmatrix} \rightsquigarrow \begin{pmatrix} v \\ 0_W \end{pmatrix} + \begin{pmatrix} 0_V \\ w \end{pmatrix}$$

"subspace"

<u>Note</u>

$$\left\{ \begin{pmatrix} v \\ 0_W \end{pmatrix} \,\middle|\, v\in V \right\} < V\times W$$

$$\left\{ \begin{pmatrix} 0_V \\ w \end{pmatrix} \,\middle|\, w\in W \right\} < V\times W$$

Every object in $V\times W$ can be written in a <u>unique</u> way with $\begin{pmatrix} v \\ 0_W \end{pmatrix}, \begin{pmatrix} 0_V \\ w \end{pmatrix}$

→ this is something along the line of basis...

If Z & U are subspaces of a vector space W
then
$$Z \boxplus U \triangleq \{ z \underset{W}{+} u \mid z \in Z, u \in U \}$$
is a subspace of W

↳ idea → decomposing a vector space...

Question is $Z \boxplus U$ a subspace? → not empty $\emptyset_Z + \emptyset_U$

Yes. $\alpha(z+u) = \alpha z + \alpha u \in W$

$\quad \in Z \quad \in U \qquad \in Z \quad \in U$

$$(z_1+u_1)+(z_2+u_2) = \overbrace{(z_1+z_2)}^{\in Z} + \overbrace{(u_1+u_2)}^{\in U}$$

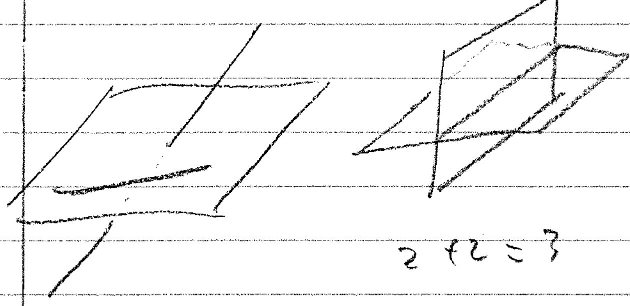
So $Z \boxplus U$ subspace

Question is $Z \boxplus U$ commutative? Yes
$= U \boxplus Z$

↳ $\boxed{Z \boxplus U = U \boxplus Z}$

Question ~~Associativity~~? **Yes.** $\boxed{(Z \boxplus U) \boxplus W = Z \boxplus (U \boxplus W)}$

Dimensions? → they don't add! { span, linear ind



$2+1=?$ $\quad 2+2=?$

☑ Suppose Z_1 & Z_2 are subspaces of W

Consider $Z_1 \overset{\otimes}{\times} Z_2 \xrightarrow{\varphi} Z_1 \overset{\oplus}{+} Z_2 < W$

defined by $\varphi\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{\Delta}{=} z_1 + z_2$

(1) → φ is a linearity function. ⟶ φ is linear

$$\varphi\left(\alpha\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{\otimes}{+} \beta\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix}\right) = \varphi\begin{pmatrix} \alpha z_1 \overset{\oplus}{+} \beta \hat{z}_1 \\ \alpha z_2 \overset{\oplus}{+} \beta \hat{z}_2 \end{pmatrix} = \alpha z_1 \overset{\oplus}{+} \beta \hat{z}_1 \overset{W}{+} \alpha z_2 \overset{W}{+} \beta \hat{z}_2$$

$$= \alpha \varphi\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \overset{\oplus}{+} \beta \varphi\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = \ldots$$

(2) If W finite dim, then Z_1, Z_2 finite dim

So $Z_1 \times Z_2$ and $Z_1 + Z_2 < W$ finite dim

By Rank-Nullity Theorem (fin → fin, linear

$$\underbrace{\dim(Z_1 \times Z_2)} = \dim(Z_1 + Z_2) + \dim(\ker(\varphi))$$

$$\dim(Z_1) + \dim(Z_2) = \dim(Z_1 + Z_2) + \dim(\ker \varphi)$$

What is $\ker(\varphi)$?

$$\ker(\varphi) = \left\{ \begin{pmatrix} z \\ -z \end{pmatrix} \middle| \begin{array}{l} z \in Z_1 \\ +z \in Z_2 \end{array} \right\} = \left\{ \begin{pmatrix} z \\ -z \end{pmatrix} \middle| z \in \underbrace{\overset{?}{Z_1 \cap Z_2}} \right\}$$

↓ also a subspace of W

From Assignment 1, $\boxed{\dim\left\{\begin{pmatrix} u \\ -u \end{pmatrix} \middle| u \in U\right\} = \dim(U)}$

start with

So $\dim(\ker(\varphi)) = \dim(Z_1 \cap Z_2)$

↳ A sieve formula

$$\boxed{\dim(Z_1 + Z_2) = \dim(Z_1) + \dim(Z_2) - \dim(Z_1 \cap Z_2)}$$

when $Z_1 \cap Z_2$ is trivial, then $Z_1 + Z_2$ is direct.

→ $\dim(\ker\varphi) = 0$ → φ is injective.

But φ is also surjective by def

⇒ φ is a bijection. → $Z_1 \otimes Z_2 \underset{\substack{\uparrow \\ \text{isomorphic}}}{\sim} Z_1 \oplus Z_2$

Feb 19 2019

Warm up $W_1 + W_2 = V$ direct when $W_1 \cap W_2 = \{0_V\}$

φ bijective ⇔ $u = 0_V$, $v = 0_V$

Contraposition: If $\{0_V\} \subsetneqq W_1 \cap W_2$, then $\exists\, u \neq 0_V$ s.t. $u \in W_1 \cap W_2$

then $u + (-u) = 0_V$ (contradiction)

$\boxed{\text{Why direct Sum?}}$ → generalize the idea of Basis.

↳ decomposes a big vector space into understandable ones.

Now let V be a f.dim vector space

$L \in \mathcal{L}(V)$ $\quad (L : V \xrightarrow{\text{lin}} V)$

$$\begin{array}{ccccc} Im(L^2) & \subseteq & Im(L) & \subseteq & V \\ \| & & \| & & \| \\ L^2[V] & \subseteq & L[V] & \subseteq & I[V] \\ & & & & \| \\ & & & & L^0[V] \end{array}$$

what about $L^2[V]$? $\rightarrow$ $L[V] \supset L[L[V]]$

Claim $\boxed{\text{if } Im(L^p) = Im(L^{p+1}), \text{ then } Im(L^{p+1}) = Im(L^{p+2})}$

(i.e. if $Im(L^p) = Im(L^{p+1})$ then

automatic

$(Im\, L^{p+2} \subset Im\, L^{p+1})$

then $Im(L^{p+1}) \subset Im(L^{p+2})$

To Show $L^{p+1}(x) \in Im\, L^{p+2}$ $\forall x \in V$, supp $Im(L^p) = Im(L^{p+1})$

$L^{p+1}(x) = L(L^p(x)) = L(L^{p+1}(v))$ for some $v \in V$

$= L^{p+2}(v) \in Im(L^{p+2})$

~~Stop~~ So

Claim true

"If you get equality, then you'll get equality forever..."

Theorem If $\dim(V) = n$, and $L \in \mathcal{L}(V)$, then

$Im\, L^n = Im\, L^{n+1} = Im\, L^{n+2} = Im^{n+3} = \cdots$

and $\ker L^n = \ker L^{n+1} = \ker L^{n+2} = \cdots$

"losing dimension"

(kernel grows ... $\{0_V\} \subseteq \ker(L) \subseteq \ker(L^2) \subseteq \cdots = \cdots =$

Image shrinks ... $= \cdots = \cdots \cdots \subseteq Im(L^2) \subseteq Im(L) \subseteq V$

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<u>What about</u> $\boxed{\begin{array}{c}\mathrm{Im}(L^n) \text{ and } \ker(L^n) \text{ and } V? \\ \text{for any } n = \dim(V)\end{array}}$

<u>Claim</u>

$\boxed{\text{Theorem}}$ If V is finite dimensional and $L \in \mathcal{L}(V)$, $\dim(V) = n$

Then $V = \mathrm{Im}(L^n) \oplus \ker(L^n)$

can't we use sieve formula ..

- By rank-nullity

$$\dim(V) = \dim(\mathrm{Im}(L^n)) + \dim(\ker(L^n))$$

- If we can show that $\mathrm{Ran}\, L^n \cap \ker L^n = \{0_V\}$

then the sum is direct, and

$$\dim(\underbrace{\mathrm{Ran}\, L^n \oplus \ker L^n}_{\text{subspace of } V}) = \dim(\mathrm{Ran}\, L^n) + \dim(\ker L^n)$$

$$= n = \dim(V)$$

<u>So</u> $\mathrm{Ran}(L^n) \oplus \ker(L^n) = V$

- <u>Now</u> if $x \in \mathrm{Im}\, L^n \cap \ker L^n$, then

$$x = L^n(v),\ v \in V,\ L^n(x) = 0_V$$

<u>So</u> $L^n(L^n(v)) = 0_V = L^{2n}(v)$

<u>So</u> $v \in \ker(L^{2n}) \Rightarrow v \in \ker(L^n) \Rightarrow x = 0_V$

$\boxed{\text{Note}}$ only true if $n = \dim(V)$

If $\dim V = ?$, then $\ker(L^2) \overset{?}{\oplus} \operatorname{Ran}(L^2) \to$ may not give V.

#

"The loss in dimension" $V \to \operatorname{Im}(L) \to \operatorname{Im}(L^2) \to \cdots$

↳ $\boxed{\text{Weyr number}}$ → can change each step.

What about gain in dimension kernel?

↳ By nullity, Weyr for loss of dim of range (image) is the same of # gain in dimension of kernel ...

#

$$\begin{array}{ccc} a & & b \\ W_1 & \times & W_2 \end{array}$$

$$\varphi \downarrow \quad \uparrow \varphi^{-1}$$

$$W_1 \oplus W_2 = V \xrightarrow{\ ?\ }$$

<u>Projection</u> $(a,b) \xrightarrow{\pi_2} b \in W_2$, $(a,b) \xrightarrow{\pi_1} a \in W_1$

What are projections on V? $W_1 \oplus W_2 \longrightarrow V$

"change of basis" $\xleftarrow{\ ?\ }$ $a+b$

$$\boxed{\begin{array}{ll} W_1 \oplus W_2 \xrightarrow{\varphi^{-1}} W_1 \times W_2 \xrightarrow{\pi_1} W_1 & a \\ \quad\searrow^{\varphi^{-1}}\ W_1 \times W_2 \xrightarrow{\pi_2} W_2 & b \end{array}}$$

$(a+b)$

Let $E : W_1 \oplus W_2 \longmapsto W_1$ "$E(a+b) = a$"

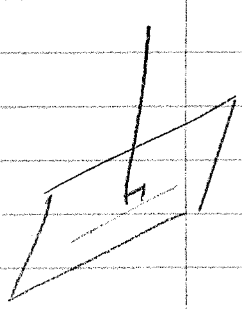
defined as $E(v) = a \in W_1$ s.t $\exists\, b \in W_2$ s.t $a + b = v$

So $E(\underbrace{E(v)}_{a}) = E(v) = a$ $\quad \underset{W_1}{a} = \underset{W_1}{a} + \underset{W_2}{0}$ unique since $\oplus$

true for any $v \in V$

So $\boxed{E^2 = E}$ → "idempotent" (squares to themselves)

↳ Note $\left\{ \begin{array}{l} \mathrm{Ran}(E_1) = \mathrm{Im}(E_1) = W_1 \\ \ker(E_1) = W_2 \end{array} \right\}$



So

Decomposition of $W_1 \oplus W_2$ gives rise to idempotent whose $\mathrm{Im} = W_1$, $\ker = W_2$

→ Are there other idempotents like this? No

↳ For idempotent (F) → $\ker(F) \oplus \mathrm{Im}(F) = V$

In fact Idempotent ⟷ direct sum

↳ coding space by functions $E_1 + \dots + E_k = I$

↳ "resolution of Identity"

Ex $\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 1 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}$

span$\{\vec{e}_1\}$ span$(\vec{e}_2)$ span$(\vec{e}_3)$

Why functions code for space? → So we can do algebra on space!

Feb 21 2019

$\boxed{\text{Idempotents}} \to$ linear operator.

Operator: linear fn from vector space onto itself.

$$E : V \longmapsto V$$

Def

{ Idempotents are operators with the property $E^2 = E$, i.e. $E \circ E = E$ }

Last time

$$\boxed{\begin{array}{l}\text{If } V = W \oplus Z \text{, then there exists an idempotent } E \in \mathcal{L}(V) \text{ such that} \\ \quad W = \operatorname{Im}(E) \text{ ("projection")} \\ \quad Z = \ker(E)\end{array}}$$

In fact, $V = W \oplus Z$ then there are at least 2 idempotents E, $W = \operatorname{Im}(E)$, $Z = \ker(E)$ and F, $W = \ker(F)$, $Z = \operatorname{Im}(F)$

Now (?) $\to$ Is every idempotent $E \in \mathcal{L}(V)$ generated this way?

The answer is Yes $\to$ Two parts to answer { existence (I), uniqueness (II) }

I// Theorem:

$$\boxed{\begin{array}{l}\text{If } E \circ E = E^2 = E, \text{ then } V = \operatorname{Ran}(E) \oplus \ker(E) \\ \qquad\qquad\qquad\qquad\quad V = \operatorname{Im}(E) \oplus \ker(E)\end{array}}$$

Proof 1) Show $\operatorname{Im}(E) + \ker(E) = V$

2) Show $\operatorname{Im}(E) \oplus \ker(E) = V$

(1) First, $v = \underbrace{E(v)}_{\operatorname{Im}(E)} + \underbrace{(v - E(v))}_{\ker(E)}$. Now $E(v - E(v)) = Ev - E^2v = Ev - Ev = 0$

Now (2) directness. Show $Im(E) \oplus ker(E) = V$

i.e. $Im(E) \cap ker(E) = \{0_V\}$

Proof Let $x \in Im(E) \to E(\underbrace{E(x)}) = 0_V$

generic element in ~~range~~ image

so $E(x) = 0_V$

so $E(x) \in ker(E)$ //

$\boxed{\text{Key Product}}$

$$\boxed{\begin{aligned} E(E(x)) &= E(x) \quad \text{for all } x \in V \\ \Leftrightarrow \quad E &= E^2 \end{aligned}}$$

i.e. E is an idempotent exactly when it acts as an identity fn on its own image.

Theorem

II//

$$\boxed{\begin{aligned} &\text{if } E^2 = E \text{ and } G^2 = G \text{ and } \left\{ \begin{aligned} Im(G) &= Im(E) \\ ker(E) &= ker(G) \end{aligned} \right. \\ &\quad \text{in } \mathcal{L}(V) \\ &\quad \text{then} \quad E = G \end{aligned}}$$

$\in Im(E)$

$E(w) = w$

$E(E(v)) = E v$

proof Let $W = Im(E) = Im(G)$ Note $W \oplus Z = V$
$Z = ker(E) = ker(G)$

$v \in V$

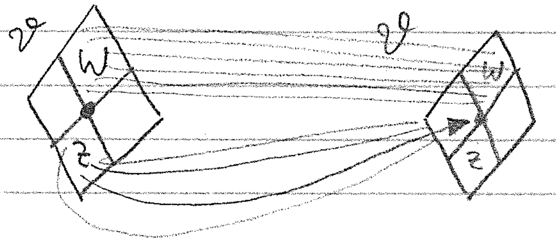
Then $E(v) = E(w+z) = E(w) + \underbrace{E(z)}_{0_V} = E(w) = w$

where $v = w + z$ is unique

Can do the same thing with $G \to G(v) = w = E(v) \; \forall v$

$\to G = E$

$\boxed{\text{Picture of idempotent}}$ $\longrightarrow$ think "<u>Projections</u>"



$W = \mathrm{Im}(E)$
$Z = \ker(E)$

<u>So</u>

$\boxed{\text{For each decomposition } V = W \oplus Z, \ \exists \text{ a unique idempotent whose Im is } W \text{ and kernel is } Z}$

<u>Now</u>, notice:

$$w + z \xleftarrow{I} v = w + z \quad \xrightarrow{E} w, \quad \xrightarrow{F} z$$

<u>So</u> $E + F = I$

<u>Now</u>, E, F idempotents s.t. $\mathrm{Im}(E) = \ker(F)$
$\ker(E) = \mathrm{Im}(F)$, then

$E + F = I \rightsquigarrow$ identity

<u>So</u>

(1) If $E^2 = E \in \mathcal{L}(V)$, then $(I - E)^2 = (I - E)$

(2) $\mathrm{Im}(E) = \ker(I - E)$
$\ker(E) = \mathrm{Im}(I - E)$

<u>So</u> idempotents <u>come in pairs</u>

<u>Summary</u> If $V = W \oplus Z$, then there is a pair of idempotents ...
more concretely ...

unique

⇒ If $V = W \oplus Z$, then there exist unique idempotents E & F such that $E + F = I$ and

$$\begin{cases} W = \operatorname{Im}(E) = \ker(F) \\ Z = \ker(E) = \operatorname{Im}(F) \end{cases}$$

⇐ If $\exists$ exist idempotents ... E & F ... such th.. then $V = W \oplus Z$

All great, but what about $W \oplus Z \oplus U = V$?

Well $W \oplus (Z \oplus U) = V \Rightarrow$

By $\exists$ unique $E_W^2 = E_W$ such that $\operatorname{Im}(E_W) = W$, $\ker(E_W) = Z \oplus U$

(★) Similarly $E_U^2 = E_U$ s.t. $\operatorname{Im}(E_U) = U$, $\ker(E_U) = Z \oplus W$

$E_Z^2 = E_Z$ s.t. $\operatorname{Im}(E_Z) = Z$, $\ker(E_Z) = U \oplus W$

So there exists

Observation ① $E_W + E_U + E_u = ?$

$$(E_Z + E_U + E_W)(v) = (E_W + E_U + E_Z)(u + w + z)$$

$$= E_W(w + u + z) + E_U(w + u + z) + E_Z(u + w + z)$$

$$= w + u + z$$

$$= v$$

So $\boxed{E_W + E_U + E_Z = I}$

Obs ② $E_W(E_Z(v)) = E_W(z) = \theta_V$

So $E_W \circ E_Z$ is null.

In fact compositions of different E_i's get θ_V

compositions of the same E_i's get E_i

Now other direction

If we know $E_W^2 = E_W$, $E_U^2 = E_U$, $E_Z^2 = E_Z$ such that ... (page 22), then

$$V = W \oplus U \oplus Z$$

Result ~~(Suppose E_W, E_U, E_Z satisfy (*) for some $W, U, Z \prec V$~~

(**)

Suppose E_1 E_2 E_3 are idempotents such that

$E_1 + E_2 + E_3 = I$ and $E_iE_j = \delta_j^i E_i$, then

$Im(E_1) \oplus Im(E_2) \oplus Im(E_3) = V$ and

$\ker(E_1) = Im(E_2) \oplus Im(E_3)$, etc

Once we established (*) and (**), we have the following assertions are equivalent for ss $W, Z, U \prec V$

① $W \oplus Z \oplus U = V$

② There are unique idempotents $E_1, E_2, E_3 \in \mathcal{L}(V)$ such that

$Im(E_1) = W$, $Im(E_2) = Z$, $Im(E_3) = U$

and $E_1 + E_2 + E_3 = I$, $E_jE_i = \delta_j^i E_i$

① → ② is already shown.

② → ① We will show now.

(1) We have $\mathrm{Im}(E_1) + \mathrm{Im}(E_2) + \mathrm{Im}(E_3) = V$

$$v = \underset{\mathrm{Im}(I) = V}{\underset{\cap}{id(v)}} = (E_1 + E_2 + E_3)(v) = \underset{\mathrm{Im}(E_1)}{\underset{\cap}{E_1(v)}} + \underset{\mathrm{Im}(E_2)}{\underset{\cap}{E_2(v)}} + \underset{\mathrm{Im}(E_3)}{\underset{\cap}{E_3(v)}}$$

So $V = \mathrm{Ran}(E_1) + \mathrm{Ran}(E_2) + \mathrm{Ran}(E_3)$

(2) Show directness. Suppose $x_1 + x_2 + x_3 = 0_V$, $x_i \in \mathrm{Im}(E_i)$

To show: $x_i = 0_V \ \forall i$

Then $E_1(x_1) + E_2(x_2) + E_3(x_3) = 0_V$

So $E_1\big(E_1(x_1) + E_2(x_2) + E_3(x_3)\big) = E_1(0_V) = 0_V$

So $E_1(E_1(x_1)) = 0_V$

So $E(x_1) = 0_V$

$x_1 = 0_V$

Same thing, get $x_2 = x_3 = x_1 = 0_V$

So get directness. $\mathrm{Im}(E_1) \oplus \mathrm{Im}(E_2) \oplus \mathrm{Im}(E_3) = V$

(there's more ...)

Note
- for finite dim vector space $\sum E_i = I \not\Rightarrow E_i E_j = \delta_j^i E_i$
- Resolution of identity, code direct sum as algebra ...

<u>Next, kernel</u>

<u>Note</u> $E_1(E_2(v)) = 0_V$

<u>So</u> $\mathrm{Im}(E_2) \subset \ker(E_1)$

Similarly, $\mathrm{Im}(E_3) \subset \ker(E_1)$

<u>So</u> $\mathrm{Im}(E_3) + \mathrm{Im}(E_2) \subset \ker E_1$

But this is also a direct sum, so

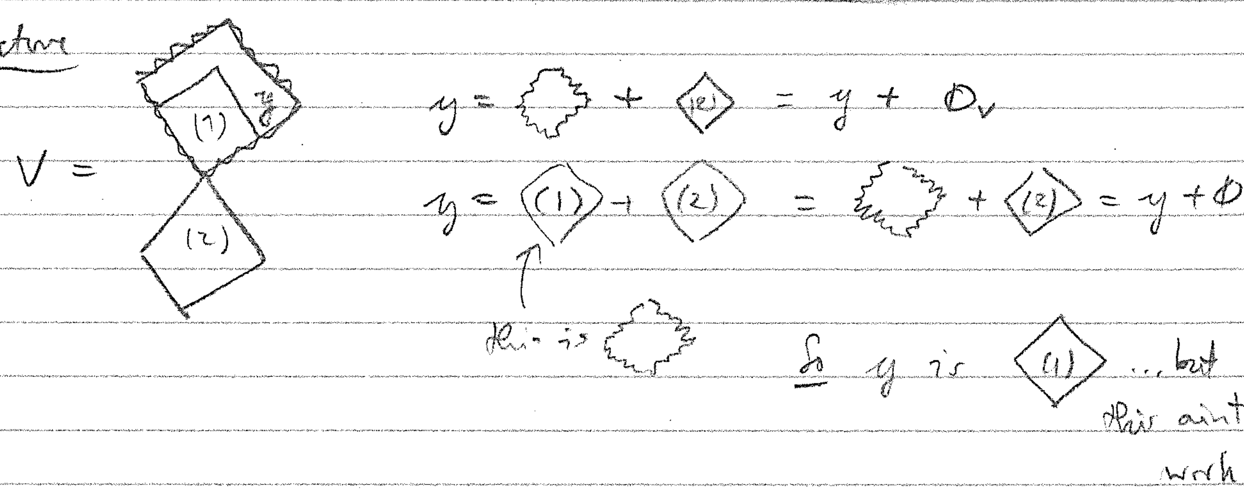
$\mathrm{Im}(E_3) \oplus \mathrm{Im}(E_2) \subset \ker E_1$ → cannot be proper → must be equality

<u>Now</u> $\underline{\bigoplus \mathrm{Im}(E_k) \quad = \quad \bigoplus \mathrm{Im}(E_1)}$

$V \quad = \quad V$

<u>So</u> $\mathrm{Im}(E_2) \oplus \mathrm{Im}(E_3) = \ker(E_1)$ $\square$

<u>Picture</u>



Feb 26, 2019

Review of idempotents.

TFAE for $A \in \mathcal{L}(V)$

1. $A^2 = A$
2. $Ax = x$ for $x \in \operatorname{Im}(A)$
3. $(I-A)^2 = (I-A)$
4. $\operatorname{Im}(A) = \ker(I-A)$
5. $\operatorname{Im}(I-A) = \ker(A)$

Today $\rightsquigarrow$ $\boxed{\text{Representing linear functions as matrices}}$

Consider $\underset{f.d.}{V} \longmapsto V$ and coordinate system, i.e. order basis.

$\beta = (v_1, \dots, v_m)$ ordered basis of V (coord. sys of V)

Then if $z \in V$, $z = \sum_{i=1}^{m} a_i v_i$ unique. But long notation $\rightarrow$ new notation

$$[z]_\beta = (a_1 \dots a_m)^T$$

So consider atrix $[v_1 \dots v_m]$, then $[v_1 \dots v_m][z]_\beta = z$

i.e. $\underbrace{[v_1 \dots v_m]}\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = a_1 v_1 + a \dots + a_m v_m = z$

$\boxed{A_\beta [z]_\beta = z}$ $\rightsquigarrow$ ~~gives the address of z~~ gives z from $[z]_\beta$

Image

$$\begin{array}{ccc} z & & V \overset{f.d.}{\longmapsto} V \\ {\scriptstyle [\ ]_\beta}\downarrow & & {\scriptstyle A_\beta}\uparrow\downarrow{\scriptstyle A_\beta^{-1} = [\ ]_\beta} \\ [z]_\beta & & \mathbb{C}^m \end{array}$$

Note $[v_1 \dots v_m] = A_\beta$ is a bijection, obviously

More fully, Consider $\mathcal{L} : V \xmapsto{\text{lin}} V$

Get

$$\begin{tikzcd} V \arrow[r, mapsto, "\mathcal{L}", "\text{linear}"'] \arrow[d, shift left, "{[\ ]_\beta}"] & V \arrow[d, shift left, "{[\ ]_\beta}"] \\ \mathbb{C}^m \arrow[u, shift left, "A_\beta"] \arrow[r, "M"'] & \mathbb{C}^m \arrow[u, shift left, "A_\beta"] \end{tikzcd} \quad \Big\}\ \text{isomorphisms}$$

- M is a composition of linear fns mapping $\mathbb{C}^m \longmapsto \mathbb{C}^m$, so M is a matrix. Note $A_\beta^{-1} = [\ ]_\beta$ are isomorphisms

- We note $M \equiv [\mathcal{L}]_{\beta\leftarrow\beta}$

- Def $\boxed{[\mathcal{L}]_{\beta\leftarrow\beta} := [\ ]_\beta \circ \mathcal{L} \circ A_\beta = A_\beta^{-1} \circ \mathcal{L} \circ A_\beta}$

Now

$$\begin{tikzcd} z \arrow[r] \arrow[d] & \mathcal{L}(z) \arrow[d] \\ {[z]_\beta} \arrow[r] & {[\mathcal{L}(z)]_\beta} \end{tikzcd}$$

- Claim $[\mathcal{L}(z)]_\beta = [\mathcal{L}]_{\beta\leftarrow\beta}[z]_\beta$

$\hookrightarrow$ $[\ ]_\beta \circ \mathcal{L}(z)$ vs $([\ ]_\beta \circ \mathcal{L} \circ \underbrace{A_\beta) \circ [\ ]_\beta}_{I} z$

So $([\ ]_\beta \circ \mathcal{L})(z) = ([\ ]_\beta \circ \mathcal{L})(z) \longrightarrow$ true.

So, observe

$$\begin{tikzcd} (z) \arrow[r, "\mathcal{L}"] & \mathcal{L}(z) \\ {[z]_\beta} \arrow[r, "M"] & {[\mathcal{L}(z)]_\beta} \end{tikzcd}$$

$\boxed{\text{Consider 2 bases}}$

$$\begin{tikzcd}
\mathbb{C}^m \arrow[rr, "{[\mathcal{L}]_{\gamma\leftarrow\gamma}}"] \arrow[d, shift left, "A_\gamma"] \arrow[dd, bend right=60, "\text{isomorphism}"'] & & \mathbb{C}^m \arrow[d, shift left, "A_\gamma"] \\
V \arrow[u, shift left, "{[\ ]_\gamma}"] \arrow[rr, "\mathcal{L}"] \arrow[d, shift left, "{[\ ]_\beta}"] & & V \arrow[u, shift left, "{[\ ]_\gamma}"] \arrow[d, shift right, "{[\ ]_\beta}"'] \\
\mathbb{C}^m \arrow[u, shift left, "A_\beta"] \arrow[rr, "{[\mathcal{L}]_{\beta\leftarrow\beta}}"'] & & \mathbb{C}^m \arrow[u, shift right, "A_\beta"'] \arrow[uu, bend right=60, "\text{isomorphism}"']
\end{tikzcd}$$

☒ Are $[\mathcal{L}]_{\gamma\leftarrow\gamma}$ the same as $[\mathcal{L}]_{\beta\leftarrow\beta}$

$$\begin{aligned}
[\mathcal{L}]_{\gamma\leftarrow\gamma} &= [\ ]_\gamma \circ \mathcal{L} \circ A_\gamma \\
&= [\ ]_\gamma \circ \left(A_\beta \circ [\mathcal{L}]_{\beta\leftarrow\beta} \circ [\ ]_\beta \right) \circ A_\gamma \\
&= \left([\ ]_\gamma \circ A_\beta \right) \circ [\mathcal{L}]_{\beta\leftarrow\beta} \circ \left([\ ]_\beta \circ A_\gamma \right)
\end{aligned}$$

☒ Note
$$\left.\begin{aligned}
[\ ]_\gamma \circ A_\beta &= \left([\ ]_\beta \circ A_\gamma \right)^{-1} \\
&= A_\gamma^{-1} \circ [\ ]_\beta^{-1} \\
&= [\ ]_\gamma \circ A_\beta
\end{aligned}\right\} \quad \underline{\text{nice}}$$

But note $[\ ]_\gamma \circ A_\beta$ and $[\ ]_\beta \circ A_\gamma$ are invertible matrices $\mathbb{C}^m \longrightarrow \mathbb{C}^m$.

☒ So if we choose different bases. (invertible matrix)

$$\boxed{[\mathcal{L}]_{\gamma\leftarrow\gamma} = M^{-1} \circ [\mathcal{L}]_{\beta\leftarrow\beta} \circ M}$$

☒ So $[\mathcal{L}]_{\gamma\leftarrow\gamma}$ is similar to $[\mathcal{L}]_{\beta\leftarrow\beta}$?

☒ Question If $B \sim [\mathcal{L}]_{\beta\leftarrow\beta}$, does there exist basis γ such that $B = [\mathcal{L}]_{\gamma\leftarrow\gamma}$

$\boxed{\text{The answer is YES}}$

If $B = K^{-1}[\mathcal{L}]_{\beta\leftarrow\beta} K$, we will show that there is a basis γ of V such that

$K = [\ ]_{\beta} \circ A_\gamma$, so that $B = [\mathcal{L}]_{\gamma\leftarrow\gamma}$.

i.e. we will show that there is a basis γ of V s.t.

$$A_\beta \circ K = A_\gamma$$

Next $\underbrace{\underbrace{A_\beta}_{\substack{V \leftarrow C^m \\ \text{isom}}} \circ \underbrace{K}_{\substack{C^m \leftarrow C^m \\ \text{isom}}}}_{\substack{V \leftarrow C^m \\ \text{isom}}}$. So $A_\beta \circ K$ must be a bijective matrix & so is

$$A_\gamma = [\underbrace{\ \cdots\ }_{\text{some basis } \gamma \text{ of } V}]$$

So similar matrices represent the same linear function. But which one? What is this linear function?

Sore the answer is unsatisfying... Let $A \in M_{153}$

↳ itself $A = [A]_{\varepsilon\leftarrow\varepsilon}$

$$\begin{array}{ccc} \mathbb{C}^{153} & \xrightarrow{A} & \mathbb{C}^{153} \\ I = A_\varepsilon \Big\uparrow\Big\downarrow [\ ]_\varepsilon = I & & [\ ]_\varepsilon \Big\downarrow\Big\uparrow A_\varepsilon = I \\ \mathbb{C}^{153} & \xrightarrow{A} & \mathbb{C}^{153} \end{array}$$

$\varepsilon \to$ std basis

So if $A \sim B$ then B represents A with respect to some different basis (not the standard basis)

$\boxed{\text{Big Idea}}$ Similarity is all about change of basis.
Similar matrices ~~give~~ represent the same linear fn.
The goal is to find __better__ representation...

$\boxed{Ex}$ $\mathbb{P}_3 \xrightarrow{\ L\ } \mathbb{P}_3$

defined as $L(p) = p' + 2p$

Find $[\mathcal{L}]_{\beta\leftarrow\beta}$ where $\beta = \{1, x+1, x^2, x^3\}$

$A_\beta = [\,1 \quad x+1 \quad x^2 \quad x^3\,]$

First column of $[\mathcal{L}]_{\beta\leftarrow\beta} = [\ ]_\beta \circ \mathcal{L} \circ A_\beta\, e_1$

$= [\mathcal{L}(e_1)]_{\beta_1}$

So i^{th} column of $[\mathcal{L}]_{\beta\leftarrow\beta} = [\mathcal{L}(e_i)]_\beta$

So

1^{st} col of $[\mathcal{L}]_{\beta\leftarrow\beta} = (2\ 0\ 0\ 0)^T$
2^{nd} col of $[\mathcal{L}]_{\beta\leftarrow\beta} = (1\ 2\ 0\ 0)^T$
3^{rd} col of $[\mathcal{L}]_{\beta\leftarrow\beta} = (-2\ 2\ 2\ 0)^T$
4^{th} col of $[\mathcal{L}]_{\beta\leftarrow\beta} = (0\ 0\ 3\ 2)^T$

So $[\mathcal{L}]_{\beta\leftarrow\beta} = \begin{pmatrix} 2 & 1 & -2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Side note

$\alpha[L]_{\beta\leftarrow\beta} + [M]_{\beta\leftarrow\beta} = [\alpha L + M]_{\beta\leftarrow\beta}$

$[L]_{\beta\leftarrow\beta} \circ [M]_{\beta\leftarrow\beta} = [L \circ M]_{\beta\leftarrow\beta}$

Notice

~~$\mathcal{L} \longmapsto [\mathcal{L}]_{\beta\leftarrow\beta}$ is injective~~

Note $\mathcal{L} \in \mathcal{L}(V) \longmapsto M_n(\mathbb{C}) \ni [L]_{\beta\leftarrow\beta}$

This map is isomorphic. But there's also multiplicative, &
$\Rightarrow$ called an algebra ... $\mathcal{L}(V)$ & M_n are algebra

BLOCK MATRICES

Feb 28, 2019

Start with Cartesian product of 2 vector spaces

$$V \times W \xrightarrow[\text{lin}]{L} V \times W$$

$$\begin{pmatrix} v \\ w \end{pmatrix} \downarrow \begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ w \end{pmatrix}$$

So formal...

$$\boxed{\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{pmatrix} v \\ w \end{pmatrix} \triangleq \begin{pmatrix} F_{11} v + F_{12} w \\ F_{21} v + F_{22} w \end{pmatrix}}$$

So

$$\left.\begin{aligned} F_{11} &: V \longmapsto V \\ F_{12} &: W \longmapsto V \end{aligned}\right\} \text{ since } F_{11}(v) + F_{12}(w) \in V$$

and

$$\left.\begin{aligned} F_{21} &: V \longmapsto W \\ F_{22} &: W \longmapsto W \end{aligned}\right\} \text{ since } F_{21}(v) + F_{22}(w) \in W$$

In diagram

	V	W
V	↲	↲
W	↲	↲

If F_{ij} are all linear, then $\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$ is a linear fn as well

$$\begin{bmatrix} F_{12} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} : V \times W \longrightarrow V \times W$$

$\boxed{Q}$ → can every linear fn be represented this way?

Note $\begin{array}{|c|c|}\hline F_{11} & F_{12} \\ \hline F_{21} & F_{22} \\ \hline \end{array} \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} F_{11}(v) \\ F_{21}(v) \end{pmatrix}$

In particular $\pi_1\left(\underbrace{\boxplus}_{L} \begin{pmatrix} v \\ 0 \end{pmatrix} \right) = F_{11}(v)$ (π : projection)

So

Given $L : V \times W \xrightarrow{\text{lin.}} V \times W$

Let $F_{11}(v) := \pi_1 \circ L\begin{pmatrix} v \\ 0 \end{pmatrix}$

$F_{12}(w) := \pi_1 \circ L\begin{pmatrix} 0 \\ w \end{pmatrix}$

$F_{21}(v) := \pi_2 \circ L\begin{pmatrix} v \\ 0 \end{pmatrix}$

$F_{22}(w) := \pi_2 \circ L\begin{pmatrix} 0 \\ w \end{pmatrix}$

Do we have $L = \begin{array}{|c|c|}\hline F_{11} & F_{12} \\ \hline F_{21} & F_{22} \\ \hline \end{array}$?

by linearity

Well

$$\begin{cases} F_{11}(v) + F_{12}(w) = \pi_1 \circ L\begin{pmatrix} v \\ 0 \end{pmatrix} + \pi_1 \circ L\begin{pmatrix} 0 \\ w \end{pmatrix} = \pi_1 \circ L\begin{pmatrix} v \\ w \end{pmatrix} \\ F_{21}(v) + F_{22}(w) = \pi_2 \circ L\begin{pmatrix} v \\ 0 \end{pmatrix} + \pi_2 \circ L\begin{pmatrix} 0 \\ w \end{pmatrix} = \pi_2 \circ L\begin{pmatrix} v \\ w \end{pmatrix} \end{cases}$$

So $\begin{pmatrix} F_{11}(v) + F_{12}(w) \\ F_{21}(v) + F_{22}(w) \end{pmatrix} = \begin{pmatrix} \pi_1 \circ L\begin{pmatrix} v \\ w \end{pmatrix} \\ \pi_2 \circ L\begin{pmatrix} v \\ w \end{pmatrix} \end{pmatrix} = L\begin{pmatrix} v \\ w \end{pmatrix}$ obviously...

So If we start with any L, we can represent L as a matrix of linear functions...

$\binom{w}{z}_+$ is actually $\varphi\binom{w}{z} = w+z$, $W \times Z \longrightarrow W \oplus Z$

▨ But we don't often work with Cartesian products. So we want to use this idea to break a vector space into direct sums

$$\boxed{\text{What about direct sums?}}$$

▨ Bad notation Given $W \oplus Z = V$. Let us write $\binom{w}{z}_\oplus$ instead of $w+z$

So $\binom{w}{z}_+ \sim w+z$ so that we can mimick our previous idea

▨ Recall

$$\begin{array}{c|c|c|} & W & Z \\ \hline W & \circlearrowleft & \circlearrowleft \\ \hline Z & \circlearrowleft & \circlearrowleft \\ \hline \end{array}$$

Given
$F_{11} : W \longmapsto W$
$F_{12} : Z \longmapsto W$
$F_{21} : W \longmapsto Z$
$F_{22} : Z \longmapsto Z$

we can define linear fns

same as before...

$$\begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} : V \longmapsto V \quad \text{by} \quad \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{pmatrix} w \\ z \end{pmatrix}_+ = \begin{pmatrix} \cdots \\ \cdots \end{pmatrix}_+$$

▨ Now, we want to find something that is similar to π_i (projection)

→ the answer is idempotent (analogous to projections)

☆ Think

$$\begin{array}{ccccc} W & \oplus & Z & = & V \\ \| & & \| & & \\ \operatorname{Ran}(E_1) & & \operatorname{Ran}(E_2) & & \end{array}$$

where E_i are idempotents.

where $E_2 = I - E$

▨ If $L = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix}$, then

$$L\begin{pmatrix} w \\ 0 \end{pmatrix}_+ = \begin{pmatrix} F_{11}(w) \\ F_{21}(w) \end{pmatrix}_+ \begin{matrix} \in W \\ \in Z \end{matrix}$$

So

$E \circ L(w) = F_{11}(w) \in W$

$(I-E) \circ L(w) = F_{21}(w) \in Z$

Same thing $L\begin{pmatrix} 0 \\ z \end{pmatrix}_+ = \begin{pmatrix} F_{12}(z) \\ F_{22}(z) \end{pmatrix}_+$

So $E(L(z)) = F_{12}(z) \in W$

$(I-E)L(z) = F_{22}(z) \in Z$

So can again check that

$$L = \begin{array}{c|c|c} & W & Z \\ \hline W & E\circ L\big|_W & E\circ L\big|_Z \\ \hline Z & (I-E)\circ L\big|_W & (I-E)\circ L\big|_Z \end{array}$$

restricted maps...

So if "$L : V \longmapsto V$ and wrt to the composition $V = W \oplus Z$ L has the block form $\begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array}$"

then it means

$$A : E\circ L\big|_W : W \longmapsto W \quad \text{etc}$$

where E is an idempotent with $Im(E) = W$

$ker(E) = Z$

Caveat direct sum is commutative
BUT here order matters!

[Order of the direct sum matters in our exploration here.]

Example ex G : idem $V \longmapsto V$, $V = Ran\,G \oplus ker\,G$

So

$$G = \begin{array}{c|c|c} & Im\,G & ker\,G \\ \hline Im\,G & I\big|_{Im\,G} & 0 \\ \hline ker\,G & 0 & 0 \end{array}$$

Properties of Block-representation...

Algebra of Block-matrices

↳ All our block-matrices go from $V \mapsto V$ and $V = W \oplus Z$

(1)

$$\underset{(L)}{\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array}} + \underset{(S)}{\begin{array}{|c|c|}\hline P & Q \\ \hline R & T \\ \hline\end{array}} = \underset{(L+S)}{\begin{array}{c|c|c|}& w & z \\ \hline w & ?_{11} & ?_{12} \\ \hline z & ?_{21} & ?_{22} \\ \hline\end{array}}$$

$W = Im(E)$, $E^2 = E$
$Z = ker(E)$

$$?_{11}(w) = E\Big[(L+S)(w)\Big] = EL(w) + ES(w) = A(w) + P(w)$$

So... the algebra is very simple :-)

$$\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} + \begin{array}{|c|c|}\hline P & Q \\ \hline R & T \\ \hline\end{array} = \begin{array}{|c|c|}\hline A+P & B+Q \\ \hline C+R & D+T \\ \hline\end{array}$$

(2) Scaling also works...

$$\alpha \begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} = \begin{array}{|c|c|}\hline \alpha A & \alpha B \\ \hline \alpha C & \alpha D \\ \hline\end{array}$$

(3) Composition

$$\underset{L}{\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array}} \circ \underset{S}{\begin{array}{|c|c|}\hline P & Q \\ \hline R & T \\ \hline\end{array}} = \underset{L \circ S}{\begin{array}{c|c|c|}& w & z \\ \hline w & ?_{11} & \\ \hline z & & \\ \hline\end{array}}$$

$$?_{11}(w) = E\Big[L \circ S(w)\Big] = ELS(w)$$

$$= EL\Big[E + (id - E)\Big]S(w)$$

$$= ELES(w) + EL(id-E)S(w)$$

$$\cancel{= ELE(ES(w)) + EL(id-E)(id-E)S(w)}$$

$$= EL\,\underset{\Uparrow\, W}{P(w)} + EL\,\underset{\Uparrow\, Z}{R(w)}$$

$$= A(P(w)) + B(R(w))$$

So $\boxed{?_{11}(w) = [A \circ P + B \circ R](w)}$ multiply like matrices!

So
$$\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} \circ \begin{array}{|c|c|}\hline P & Q \\ \hline R & T \\ \hline\end{array} = \begin{array}{|c|c|}\hline A \circ P + B \circ R & A \circ Q + B \circ T \\ \hline C \circ P + D \circ R & C \circ Q + D \circ T \\ \hline\end{array}$$

//

(4) Suppose I have
$$L = \begin{array}{c|c|c|} & W & Z \\ \hline W & A & B \\ \hline Z & C & D \\ \hline\end{array}$$
. If I pick a basis β for W and γ for Z

then

$$[L]_{\beta\|\gamma \leftarrow \beta\|\gamma} = \begin{bmatrix} c_1 & \cdots \end{bmatrix}$$

concatenate

$$C_1 = [L(b_1)]_{\beta\|\gamma}$$ $b_1 \to$ first vector of β

$$= \left[\begin{array}{|c|c|}\hline A & B \\ \hline C & D \\ \hline\end{array} \begin{pmatrix} b_1 \\ 0 \end{pmatrix}_+\right]_{\beta\|\gamma} = \left[\begin{pmatrix} A(b_1) \\ C(b_1) \end{pmatrix}_+\right]_{\beta\|\gamma}$$

$\in W$, $\in Z$

$A(b_1) =$ lin comb of b_i
$C(b_i) =$ lin co of g_i

same thing

$$= \begin{bmatrix} [A(b_1)]_\beta \\ [C(b_1)]_\gamma \end{bmatrix}$$

If we keep doing this ... $\rightarrow$ Note $W \sim \beta$, $Z \sim \gamma$

$$[L]_{\beta\|\gamma \leftarrow \beta\|\gamma} = \begin{array}{c|c|c|} & \beta\}W & \gamma\}Z \\ \hline W\}\beta & [A]_{\beta \leftarrow \beta} & [B]_{\beta \leftarrow \gamma} \\ \hline Z\}\gamma & [C]_{\gamma \leftarrow \beta} & [D]_{\gamma \leftarrow \gamma} \\ \hline\end{array}$$

Note $[C]_{\gamma \leftarrow \beta}$ is generated

(1) Take sth in beta
(2) $C(b)$
(3) coord in γ

$$\rightarrow [C]_{\gamma \leftarrow \beta} = \begin{bmatrix} [C(\beta_1)]_\gamma & [C(\beta_2)]_\gamma \cdots & [C(\beta_n)]_\gamma \end{bmatrix}$$

Mar 5, 2019

Representing linear functions as matrix-like stuff...

$$\mathcal{L}: \overset{F_1}{V_1} \oplus \overset{F_2}{V_2} \oplus \overset{F_3}{V_3} \longmapsto \overset{E_1}{W_1} \oplus \overset{E_2}{W_2}$$

	V_1	V_2	V_3
W_1	$\mathcal{L}_{11}$	$\mathcal{L}_{12}$	$\mathcal{L}_{13}$
W_2	$\mathcal{L}_{21}$	$\mathcal{L}_{22}$	$\mathcal{L}_{23}$

$$\mathcal{L}_{13} = E_1 \circ \mathcal{L}\Big|_{V_2} = E_1 \circ \mathcal{L} \circ F_2\Big|_{V_2}$$

Back to idempotents

Consider idempotent matrices. $E = E^2 \in M_n$

Ex $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ is such a matrix.

↳ we know that $\text{Im}(E) \oplus \ker(E) = \mathbb{C}^n$

So $E: \text{Im}(E) \oplus \ker(E) \longmapsto \text{Im}(E) \oplus \ker(E)$

So we can represent E as

		E	$I-E$
		$\text{Im} E$	$\ker E$
(E)	$\text{Im} E$	E_{11}	E_{12}
$(I-E)$	$\ker E$	E_{21}	E_{22}

$$E_{12} = E \circ E\Big|_{\ker E} = 0$$

$$E_{22} = \underbrace{(I-E) \circ E}_{0}\Big|_{\ker E} = 0$$

$$E_{11} = \underbrace{E \circ E}_{E}\Big|_{\text{Im} E} = Id$$

$$E_{21} = \underbrace{(I-E) \circ E}_{0}\Big|_{\text{Im} E} = 0$$

So "E" = $\begin{array}{|c|c|} \hline Id & 0 \\ \hline 0 & 0 \\ \hline \end{array}$

Idea

Can pick a basis for $\text{Im}(E)$

& $\ker(E)$

→ then concatenate & get basis for whole space

$$[E]_{\Gamma_1 \| \Gamma_2 \leftarrow \Gamma_1 \| \Gamma_2} = \begin{bmatrix} [Id]_{\Gamma_1 \leftarrow \Gamma_1} & [0]_{\Gamma_1 \leftarrow \Gamma_2} \\ [0]_{\Gamma_2 \leftarrow \Gamma_1} & [0]_{\Gamma_2 \leftarrow \Gamma_2} \end{bmatrix}$$

Now $[Id]_{\Gamma_i \leftarrow \Gamma_i} = I$ so $E = \begin{array}{|c|c|} \hline I & 0 \\ \hline 0 & 0 \\ \hline \end{array}$

$[0]_{\Gamma_i \leftarrow \Gamma_j} = 0$

(1) So I can write $[E]_{\Gamma_0 \leftarrow \Gamma_0} = E = E^2 \in M_n$

↑ standard basis

(2) But we also know:

$$[E]_{\Gamma_1 \| \Gamma_2 \leftarrow \Gamma_1 \| \Gamma_2} = \begin{array}{|c|c|} \hline I & 0 \\ \hline 0 & 0 \\ \hline \end{array}$$

From (1) & (2), we have a theorem: Every idempotent $E \in M_n$ is similar to a matrix of the form

$$\begin{pmatrix} \underbrace{1 \; 1 \; 1}_{k} & & & & 0 \\ & \ddots & & & \\ & & 0 & & \\ 0 & & & 0 & \\ & & & & 0 \end{pmatrix}$$

where $k = \text{Rank}(E)$
$= \text{Tr}(E)$

So $\text{Rank}(E) = \text{Trace}(E)$

Cons-quence? { So since $\text{Rank}(E) = \text{Trace}(E)$, if $\text{Trace}(M)$ is not an integer or negative ... then M is NOT idempotent

(PRE-) $\boxed{\text{Corollary}}$ Suppose $\underbrace{E_1 + E_2 + \dots + E_m}_{m \text{ idempotents}} = I_{\mathbb{C}^n}$

Then $\text{Trace}(E_1 + E_2 + \dots + E_m) = \text{Trace}(I_{\mathbb{C}^n}) = n$

↳ $\text{Tr}(E_1) + \text{Tr}(E_2) + \dots + \text{Tr}(E_m) = n$

So

$$\boxed{\text{Rank}(E_1) + \text{Rank}(E_2) + \dots + \text{Rank}(E_m) = n}$$

Now, consider $\mathrm{Im}(E_1 + \dots + E_m)$

and

$\mathrm{Im}\, E_1 + \mathrm{Im}\, E_2 + \dots + \mathrm{Im}\, E_m$

Consider $(E_1 + \dots E_m)(x) = E_1 x + E_2 x + \dots + E_m x \in \sum \mathrm{Im}\, E_i$

So $\mathbb{C}^n = \mathrm{Im}(I_{\mathbb{C}^n}) = \mathrm{Im}\left(\sum_1^m E_i\right) \subseteq \sum_1^m \mathrm{Im}\, E_i \subseteq \mathbb{C}^n$

So $\sum_1^m \mathrm{Im}\, E_i = \mathbb{C}^n$ (i)

But we also know $\sum_1^m \dim(\mathrm{Im}(E_i))$

$= \sum_1^m \mathrm{rank}(E_i) = n$ (ii)

From (i) & (ii), then

$$\boxed{\bigoplus_{i=1}^m \mathrm{Im}(E_i) = \mathbb{C}^n} \longleftarrow \text{direct sum}$$

In particular,

$$\dim\left(\bigoplus_{i=2}^m \mathrm{Im}(E_i)\right) = n - \dim(\mathrm{Im}\, E_1)$$

$$= \mathrm{nullity}(E_1) = \dim(\ker E_1)$$

Now,

$$\mathrm{Im}\left(\sum_2^m E_i\right) = \mathrm{Im}(I - E_1)$$

$$\cap \qquad\qquad \|$$

Now $\bigoplus_{i=2}^m \mathrm{Im}(E_i) = \ker(E_1)$

So since $\ker(E_1) \subseteq \bigoplus_{i=2}^m \mathrm{Im}(E_i)$

But

$\dim \underset{\ker}{\wedge} (E_1) = \dim\left(\bigoplus_{i=2}^m \mathrm{Im}(E_i)\right)$

$$\boxed{\ker(E_1) = \bigoplus_2^m \mathrm{Im}\, E_i}$$

This implies $\mathrm{Im}\, E_2 \subseteq \ker E_1$

So $E_1 [E_2(\square)] = \mathcal{O}_n \quad \forall \square$

Hence $E_1 \circ E_2 = \mathcal{O}$

So, in general $\boxed{E_i \circ E_j = \mathcal{O}_{n\times n} \quad \text{if } i \neq j}$

So Corollary: If $E_1 \dots E_m$ are idempotents and $\sum_1^m E_i = I$, then $\left\{ \begin{array}{l} E_i E_j = 0 \text{ for } i \neq j \\ \& \\ \bigoplus_{i=1}^m \mathrm{Im}(E_i) = \mathbb{C}^n \end{array} \right\}$

Note

$\bigoplus_{i=2}^m \mathrm{Im}\, E_i = \ker(E_1)$

Note $\mathrm{Trace}(AB) = \mathrm{Trace}(BA)$

So

$\left.\begin{array}{l} \mathrm{Trace}(S^{-1}AS) = \mathrm{Trace}(ASS^{-1}) \\ \mathrm{Trace}(B) = \mathrm{Trace}(A) \end{array}\right\}$ $\mathrm{Trace}(AB^T) = \mathrm{Trace}\, b_A \cdot b_B = \mathrm{Trace}(B^T A)$
$\hookrightarrow$ use indices.

So similar matrices have the same trace & rank.

Now, let's revisit E → what we've done is change of basis to get a better matrix (simpler...)

Now, suppose $\mathcal{L} : V \xrightarrow{\text{lin}} V$ → finite dim

Suppose $\{0\} \neq \underset{\neq V}{W} \in \mathrm{Lat}(\mathcal{L})$

$V = W \oplus \boxed{?}$

Let $w_1 \dots w_m \in W$ basis of W, then $\mathrm{span}\{w_{m+1} \dots w_n\}$ → basis of $\square$

Now think $V \quad v_1 \ldots v_{37} \mid v_{38} \ldots v_{135}$

Then $\text{span}(v_1 \ldots v_{37}) + \text{span}(v_{38} \ldots v_{135}) = V$

because $$v = \underbrace{\sum^{37} a_i v_i}_{\substack{w_1 \\ \in \\ W_1}} + \underbrace{\sum_{38}^{135} a_i v_i}_{\substack{w_2 \\ \in \\ W_2}} \qquad (\checkmark)$$

So $\text{span}(v_1 \ldots v_{37}) \oplus \text{span}(v_{38} \ldots v_{135}) = V$

since $\dim W_1 + \dim W_2 = V$

$$\begin{array}{ccc} & E & I-E \end{array}$$

Now $V = W \oplus Z \qquad (W \in \text{Lat}(\mathcal{L}))$

Then
$$\begin{array}{c|c|c|} & W & Z \\ \hline W & & \\ \hline (I-E)Z & \Theta & \\ \hline \end{array}$$

$? = (I-E) \circ \mathcal{L}\big|_W$

$= 0$

Note $\mathcal{L}(w) \in W = \text{Im}\, E = \ker(I-E)$

$\Rightarrow (I-E) \circ \mathcal{L}\big|_W = \Theta$

In fact, converse also true

$(I-E) \circ \mathcal{L}\big|_W = \Theta \Rightarrow \text{Im}(\mathcal{L}\big|_W) \subseteq \ker(I-E) = \text{Ran}(E)$
$= W$

So $W \in \text{Lat}(\mathcal{L})$

So

Now, consider

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So $\begin{array}{c|cc} & W & Z \\ \hline W & & \\ Z & 0 & \end{array} \Leftrightarrow W \in Lat(\mathcal{L})$

Observe $\begin{array}{c|cc} & W & Z \\ \hline W & & 0 \\ Z & & \end{array} \Leftrightarrow Z \in Lat(\mathcal{L})$

So if $W, Z \in Lat(\mathcal{L})$, $W \oplus Z = V$

then $\begin{array}{c|cc} & W & Z \\ \hline W & A & 0 \\ Z & 0 & B \end{array} \to$ in essence, there's no mixing W & Z

$\boxed{\text{Q: how do we know there's an invariant subspace to start?}}$

$\mathcal{L}: V \longmapsto V \to$ fin-dim (m)

How do I find a good invariant subspace.

We can construct one... for $v_0 \neq 0$, we want $v_0 \in W \in Lat(\mathcal{L})$ small

Want
$v_0 \in W$
$\mathcal{L}(v_0) \in W$ — If $m = \dim(V)$
$\mathcal{L}^2(v_0) \in W$ — then there are at most m
$\mathcal{L}^3(v_0) \in W$ — things...
$\vdots$
$\mathcal{L}^n(v_0) \in W$

Say $\mathcal{L}^k(v_0)$ is the 1st one that is a lin. combo. of previous ones, i.e.

$$\mathcal{L}^n(v_0) = \sum_0^{n-1} a_i \mathcal{L}^i(v_0)$$

then $$\mathcal{L}^{n+1}(v_0) = \mathcal{L}\left(\sum_0^{n} \mathcal{L}^n(v_0)\right)$$

$$= \sum \mathcal{L}^{i+1}(v_0)\, a_i \;\in \text{Span}\left(\mathcal{L}^i(v_0)\right)_{i=1\cdots n-1}$$

So other way $\cancel{\mathcal{L}}$

<u>Consider</u> $(\mathcal{L}^3 - a_0\mathcal{L}^0 - a_1\mathcal{L}^1 - a_2\mathcal{L}^2)\, v_0 = 0$ → complex r_i

<u>So</u> $(\mathcal{L} - r_1 I)(\mathcal{L} - r_2 I)(\mathcal{L} - r_3 I)(v_0) = 0$

but since $v_0 \neq 0$

At least one of $\mathcal{L} - r_1 I$, $\mathcal{L} - r_2 I$, $\mathcal{L} - r_3 I$ is NOT injective.

If $\mathcal{L} - r_i I$ not injective

$\Rightarrow \mathcal{L}(\hat{v}) = r_i \hat{v}$ for some $\hat{v} \neq 0$

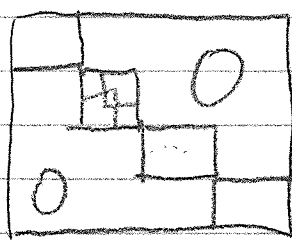
eigenvalue

eigenvector

We have just proved, every linear fn on fin dim vector space on $\mathbb{C}$ has an eigenvector

↳ $\exists$ 1 dim invariant subspace.

So we can always have $\{0\} \neq W \subsetneq V$



<u>Mar 7, 2019</u>

<u>Idea</u>

$$\mathbb{P} \longmapsto \text{"}\mathbb{P}(L)\text{"}$$

$$c_0 z^0 + c_1 z^1 + \dots + c_n z^n \qquad c_0 I + c_1 \mathcal{L} + \dots + c_n \mathcal{L}^n$$

$(\mathcal{L} : V \xrightarrow{lin} V)$

$$\mathbb{P} \xmapsto{\rho_{\mathcal{L}}} \mathcal{L}(V)$$

what sort of properties does this map have?

$\boxed{\text{Properties of } \quad \mathbb{P} \xrightarrow{S_{\mathcal{L}}} \mathcal{L}(V)}$

1. Linear

2. Multiplicative (product of ~~linear~~ polynomial $\to$ products of lin. fn)

<u>Consider</u> something from previous proof

$$\to \left(-a_0 I - a_1 \mathcal{L} - \dots + \mathcal{L}^n\right)(v_0) = \Theta_V$$

<u>So</u> $\quad \left(P(L)\right)(v_0) = \Theta_V$

$\boxtimes$ $\quad$ By fundamental theorem of algebra,

unique ... $\to$ $P(z) = \underbrace{(z-\lambda_1)^{\mu_1} \dots (z-\lambda_k)^{\mu_k}}_{\text{distinct}}$, $\quad \lambda_k$ are the roots

μ_k multiplicity of root λ_k

<u>So</u> since $S_{\mathcal{L}}$ multiplicative

<u>Hence</u> $\quad (\mathcal{L}-\lambda_1 I)^{\mu_1} \dots (\mathcal{L}-\lambda_k I)^{\mu_k}$

So $\quad (\mathcal{L}-\lambda_1 I)^{\mu_1} \dots (\mathcal{L}-\lambda_k I)^{\mu_k}(v_0) = 0$

i.e. $\quad \underbrace{(\mathcal{L}-\lambda_1 I_1) \dots (\mathcal{L}-\lambda_1 I)}_{\mu_1} \dots \underbrace{(\mathcal{L}-\lambda_k I) \dots (\mathcal{L}-\lambda_k I)}_{\mu_k}(v_0) = \Theta_V$

<u>So</u> $\quad (\mathcal{L}-\lambda_1 I_1) \dots (\mathcal{L}-\lambda_1 I) \dots (\mathcal{L}-\lambda_k I) \dots (\mathcal{L}-\lambda_n I)$

<u>not injective</u> for at least one i,

i.e. there exists some $V_i \neq 0$ such that $(\mathcal{L}-\lambda_i I)(v_i) = \Theta_V$

<u>or</u> $\quad \mathcal{L}(v_i) = \lambda_i v_i$

<u>So</u> one of λ_i's is an eigenvalue of $\mathcal{L}$.

(*) $\boxed{\text{Every } \mathcal{L}: V \xrightarrow{\text{lin}} V \text{ has an eigenvalue}}$ (finite dim)

$\boxed{\text{Every } \mathcal{L}: V \xrightarrow{\text{lin}} V \text{ has a 1-dimensional invariant subspace}}$ (fin. dim)

<u>Consider</u> $V = \underbrace{W_1}_{(1\text{-dim}) \,\in\, Lat(\mathcal{L})} \oplus Z$ $\quad V$ 2-dimensional

Then

$$\mathcal{L} = \begin{array}{c|cc} & W_1 & Z \\ \hline W_1 & A & B \\ Z & 0 & D \end{array} = \begin{pmatrix} \text{"}\lambda\text{"} & B \\ 0 & D \end{pmatrix}$$

1 - dim

<u>What about D?</u> $\quad D: Z \to Z$ or

If W_2 not one-dimensional then $D: Z \to Z$, then there is an $Z = \underbrace{W_2}_{1\text{-dim invariant, } \in Lat(D)} \oplus \hat{W}_4$

<u>Then</u>

$$\mathcal{L} = \begin{array}{c|cc} & W_1 & Z \\ \hline W_1 & \lambda & B \\ Z & 0 & D \end{array} = \begin{array}{c|ccc} & W_1 & W_2 & \\ \hline W_1 & \text{"}\lambda_1\text{"} & B_1 & B_2 \\ W_2 & 0 & \text{"}\lambda_2\text{"} & \\ & 0 & 0 & \hat{D} \end{array} = \ldots$$

$\boxed{\text{UPPER TRIANGULAR MATRIX}}$

$\boxed{\text{Schur's Theorem}}$ $\quad \boxed{\text{Every } n \times n \text{ matrix is } \underline{\text{similar}} \text{ to an upper triangular matrix}}$

<u>Proof by induction</u> Base case $n = 1 \to$ trivial

* Suppose n_0 is smallest size for which there is a counter example A ($n_0 \geq 2$)

• Now, A is in ℂ $\to$ A has an eigenvector by (*), called w_0 and span(w_0) is invariant under A. Then

$\mathbb{C}^{n_0} = \mathrm{Span}(w) \oplus Z$

w.r.t. to this decomposition, A has the form
$$\begin{array}{c|c|c|} & W & Z \\ \hline W & \lambda & B \\ \hline Z & 0 & D \\ \hline \end{array}$$

Pick a basis of Z : $\overbrace{w_1 \ldots w_{n_0-1}}^{\Gamma_2}$, $\overbrace{w_0}^{\Gamma_1} = \Gamma$

$w_0 \ldots\ldots w_{n_0-1}$

A $\quad w_0 \; \vdots \; w_{n_0-1}$

Recall $[A]_\Gamma = \begin{bmatrix} [A(w_0)]_\Gamma & \cdots & [A(w_{n_0-1})]_\Gamma \end{bmatrix}$

$= \left[\begin{array}{c|c} \lambda & \\ \hline 0 & \\ 0 & B \\ 0 & \\ 0 & \end{array}\right]$

So
$$\begin{array}{cc} & \begin{array}{cc} (1) & (n_0-1) \\ \Gamma_1 & \Gamma_2 \end{array} \\ \begin{array}{c} (1)\ \Gamma_1 \\ (n_0-1)\ \Gamma_2 \end{array} & [A]_\Gamma = \left[\begin{array}{c|c} \lambda & \cdots\cdots \\ \hline 0 & [T]_{\Gamma_2 \to \Gamma_2} \end{array}\right] \end{array}$$

A is the smallest counterexp

So $A \sim [A]_\Gamma$ $\qquad$ Since B smaller than A

$B = S \circ T \circ S^{-1}$ for some

But A is a counterexample $\to$ $\qquad$ upper Δ^r T

So
$$A \sim \left[\begin{array}{c|c} \lambda & M \\ \hline 0 & S^{-1}TS \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S^{-1} \end{array}\right] \left[\begin{array}{c|c} \lambda & MS^{-1} \\ \hline & T \\ & 0 \end{array}\right] \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & S \end{array}\right]$$

inverses

So
$$A \sim \left[\begin{array}{c|c} \lambda & MS^{-1} \\ \hline & T \end{array}\right] \rightsquigarrow \text{upper triangular} \to \text{contradiction}$$

■ Now, back to polynomials ...

① For each $v_0 \in V$ there is some $P \neq O_p \in \mathbb{P}$ such that
$\neq O_v$

$$(P(\mathcal{L}))(v_0) = O_v$$

Consider $\qquad p \xmapsto{P_{\mathcal{L}}} P(L)$

$$\{ P \in \mathbb{P} \mid (P(\mathcal{L}))(v_0) = O_v \} \subseteq \mathbb{P}$$

Closed under scaling: closed under scaling

closed under multiplication, absorption property w.r.t multiplication: if $(P(\mathcal{L}))(v_0) = 0$ then $(B \circ P(\mathcal{L}))(v_0) = 0$

in particular

$$(q(\mathcal{L}) \circ P(\mathcal{L}))(v_0) = O_v \text{ for any } q \in \mathbb{P}$$

i.e

$$(q \cdot P)(\mathcal{L})(v_0) = O_v \text{ for any } q \in \mathbb{P}$$

closed under addition: closed under addition

↳ $\boxed{\text{Ideals}}$ ← subspace that is closed under multiplication

■ There's the smallest unique monic polynomial

■ ∃ unique $P(\mathcal{L})$ smallest that kills (v_0)

↳ Now, each v_0, can associate a polynomial to it.

■ Is there a polynomial that kills them all?

Consider

$$\begin{array}{ccc} V & \xrightarrow{\mathcal{L}} & V \\ \downarrow & & \\ \mathbb{C}^n & \xrightarrow{[\mathcal{L}]_{\Gamma\in\Gamma}} & \mathbb{C}^n \end{array}$$

linear + multiplicative

$$[P(\mathcal{L})]_{\Gamma\in\Gamma} = P\left([\mathcal{L}]_{\Gamma\in\Gamma}\right)$$

Recall $[\mathcal{L}^2]_{\Gamma\in\Gamma} = [\mathcal{L}\circ\mathcal{L}]_{\Gamma\in\Gamma}$

$$= [\mathcal{L}]_{\Gamma\in\Gamma} \circ [\mathcal{L}]_{\Gamma\in\Gamma} = [\mathcal{L}]^2_{\Gamma\in\Gamma}$$

Can ~~also~~ add $[\alpha^2+\alpha]_{\Gamma\in\Gamma} = [\alpha]^2_{\Gamma\in\Gamma} + [\alpha]_{\Gamma\in\Gamma}$

Reach to this

$$[P(\alpha)]_{\Gamma\in\Gamma} = P\left([\alpha]_{\Gamma\in\Gamma}\right)$$

To kill linear fn → kill matrix $\in \mathbb{M}_{n\times n}$

But a matrix is just a $v_0 \in V$

So sooner or later $A^n - a_0 A^0 - \cdots = \theta_{n\times n}$

Look at $\{P \in \mathbb{P} \mid (P(A)) = \theta_{n\times n}\}$

↳ an ideal again ... (

↳ ∃ single minimal monic element that kills A ...

But if $P(A) = \theta_{n\times n}$ then $(P(A))(v_0) = \theta_V$

So $P(A) \in \{P \in \mathbb{P} \mid (P(A))(v_0) = \theta_V\}$

So all $P(A)(v_0)$ are divisors ~~of~~ $P(A)$ when $P(A) = \theta_{n\times n}$

Nov 12, 2019

Review Let $M \in M_n$ = n^2 - dimensional vector space (k)

Look at $I, M, M^2, M^3, \ldots$ eventually come to the first power of M such that ~~[illegible]~~ → smallest $k_0 \in \mathbb{N}$ such that M^{k_0} is a lin. combination of $I, M, M^2, \ldots M^{k_0-1}$

Note $k_0 \le n^2$ | So $M^{k_0} = \sum_{i}^{k_0-1} a_i M^i$ So

So $M^{k_0} + (-a_{k_0-1})M^{k_0-1} + (-a_{k_0-2})M^{k_0-2} + \dots + (-a_0)I = \mathcal{O} = P(M)$

↳ This polynomial annihilates matrix M. In fact, it is the smallest degree polynomial (monic) to do so.

Now

for any matrix we can do this → now look at the set of all polys that annihilate M.

$$W = \{ p \in \mathbb{P} \mid P(M) = \mathcal{O} \}$$

This is a vector space. Recall: $\mathbb{P} \xrightarrow[\text{linear multiplicative}]{f_M} \mathbb{M}_n$

(non-trivial) subspace absorption property $\quad p \longmapsto p(M)$

So W is also an ideal. Why? → singly generated.

↳ We say that it is generated by a single core monic polynomial.

Called: $\boxed{\mu_M \longrightarrow \text{the minimal polynomial of } M.}$

global

Recall that

$$Z = \{ p \in \mathbb{P} \mid (P(M))(v_0) = \mathcal{O} \} \text{ is also an ideal of } \mathbb{P}$$

$\left[\text{Here } M \in M_n,\ \mathcal{O} \neq v_0 \in \mathbb{C}^n \right]$

μ_M is a multiple of μ_{M,v_0} ...

Local → and we have the minimal poly μ_{M,v_0} as the generator of this ideal.

Observe that if $(\mu_M(M))(v_0) = \mathcal{O}(v_0) = \mathcal{O}$

So the global μ_M is in Z, generated by μ_{M,v_0}

So, μ_{M,v_0} divides μ_M

So $Z \subseteq W$

Also note that a polynomial has at most finitely many monic divisors

Suppose $\mu_M(z) = (z - r_1)^{\gamma_1} \cdots (z - r_k)^{\gamma_k}$

Consider $\mu_{M,v_1}(z), \ldots \mu_{M,v_{135}}(z)$

Consider $\left\{ v \,\middle|\, \left(\mu_{M,v_7}(M)\right)(v) = \Theta \right\} = \ker\left(\mu_{M,v_7}(M)\right)$

$= \text{subspace}$

The union of these 135 subspaces is
$\longrightarrow$ the whole space. So, by PSET 2, one of these subspaces equals the whole union, i.e. equals the whole space.

So $\mu_{M,v_i}(M) = \Theta$ for at least one i

i.e. μ_{M,v_i} annihilates M

But μ_{M,v_i} divides $\mu_M = q \cdot \mu_{M,v_i}$

So $\boxed{\mu_{M,v_i} = \mu_M \text{ for at least one } i}$

Why do we want to look at these polynomials?

(1) Suppose $\mu_A(z) = (z - \alpha_1)^{\gamma_1} \cdots (z - \alpha_k)^{\gamma_k}$

Then $(A - \alpha_1 I)^{\gamma_1} \circ \cdots \circ (A - \alpha_k I)^{\gamma_k} = \Theta$

But none of these factors can be invertible (proof by contradiction)

$\hookrightarrow$ So ~~$\forall i$ $(A - \alpha_i I)$~~ So $(A - \lambda I)v = \Theta \Rightarrow A(v) = \lambda v \rightarrow v$ eigenvector
$v \neq 0$

So $\boxed{\text{All roots of } \mu_A \text{ are eigenvalues of } A}$ Theorem/2

? Are there other eigenvalues that are NOT roots of μ_A?

Consider $(z-\lambda)$ ⟶ eigen- of A and $\lambda \neq \alpha_i$

NO

So $\mu_A(z)$ and $(z-\lambda)$ relatively prime. $\rightarrow \exists\, p, q$ such that

$$p(z)(z-\lambda) + q(z)\mu_A(z) = \mathbb{1}$$

So $$p(A)(A-\lambda I) + q(A)\underbrace{(\mu_A(A))}_{0} = I$$

So $p(A)(A-\lambda I) = I$ ⟶ ker $(A-\lambda I) = \{0\}$

So $(A-\lambda I)$ invertible $\rightarrow$ λ not eigenvalue of A

$(p^{-1}(A) = (A-\lambda I))$

So $\boxed{\text{All roots of } \mu_A \text{ are } \underline{\text{exactly}} \text{ eigenvalues of } A}$

☒ much easier than finding eigenvalues using characteristic polynomials since the degrees are often much smaller.

Recap

$\mu_A(z)$ $\qquad\qquad$ $\mu_{A,v}(z)$ ⟶ characteristic polynomial

$I, A, A^2, \ldots$ $\qquad\qquad$ $v, Av, A^2v, \ldots$

at most n^2 $\qquad\qquad$ at most n

But $\mu_{A,v}(z)$ divides $\mu_A(z)$

So Thm. $\boxed{\text{For } A \in M_n,\ \deg(\mu_A) \leq n}$

Block-diagonal matrices

Observe $T = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ $\quad T^2 = \begin{pmatrix} A^2 & 0 \\ 0 & D^2 \end{pmatrix} \cdots \quad T^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$

So $\quad P(T) = \begin{pmatrix} P(A) & 0 \\ 0 & P(D) \end{pmatrix}$

What is μ_T?

P annihilates $T \iff P$ annihilates A and D
$\iff \mu_A | P$ and $\mu_D | P$
$\iff P$ is a common multiple of μ_A, μ_D

So $\boxed{\mu_T = LCM(\mu_A, \mu_D)}$

For a diagonal matrix, $\quad T = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_k \end{pmatrix}$

So $\quad \mu_T = LCM\left(z-\alpha_1, z-\alpha_2, \ldots, z-\alpha_k\right)$

some α_i can be repeated

So $\boxed{\mu_T = LCM(\,) = (z-\lambda_1)\ldots(z-\lambda_k)}$ $\to$ power 1

where $\lambda_i \neq \lambda_j \ \forall\, i,j$

VERY IMPORTANT

$\boxed{?}$ Diagonalizability

Recall (we shown)

$$\begin{tikzcd} {}_\Gamma V \arrow[r, "\mathcal{L}"] \arrow[d, shift left] & V_\Gamma \arrow[d, shift left] \\ \mathbb{C}^n \arrow[r, "{[\mathcal{L}]}"'] \arrow[u, shift left] & \mathbb{C}^n \arrow[u, shift left] \end{tikzcd}$$

$\Rightarrow \quad [P(\mathcal{L})]_{\Gamma \leftarrow \Gamma} = P\left([\mathcal{L}]_{\Gamma \leftarrow \Gamma}\right)$

i.e. $\boxed{\text{Similar matrices have the same minimal poly.}}$

"Polynomials belong to linear fn, not the basis/matrix"

So, for $P\left([\mathcal{L}]_{\Gamma \leftarrow \Gamma}\right) = 0, \quad P(\mathcal{L}) = 0$

If $A \in M_n$ and $a_0 I + a_1 A + \dots + a_k A^k = O$

Apply transposition $\to$ $\sum (a_i A^i)^T$

$$\hookrightarrow \sum_k^i a_k (A^k)^T = \sum_k^i a_k (A^T)^k = O$$

So if $P(A) = O$, then $P(A^T) = 0$

But if $P(A^T) = 0$, then $P(A) = 0$

So $\boxed{\mu_{A^T} = \mu_A}$

---

what abt $T = \begin{pmatrix} A & B \\ O & D \end{pmatrix} \rightsquigarrow T^2 = \begin{pmatrix} A^2 & \\ O & D^2 \end{pmatrix} \dots$

$$T^0 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

So $P(T) = \begin{pmatrix} P(A) & \\ 0 & P(D) \end{pmatrix}$

not least

only 1 direction $\to$ So $\left\{ \begin{array}{l} P(T) \text{ annihilates } T \Rightarrow P(A) = P(D) = 0 \\ \Rightarrow P = \underline{\text{common multiple}} \text{ of } \mu_A, \mu_D \end{array} \right.$

So $(\mu_A \cdot \mu_D)(T) = \mu_A(T) \cdot \mu_D(T)$

$$= \begin{pmatrix} O & \\ O & \mu_A(D) \end{pmatrix} \cdot \begin{pmatrix} \mu_D(A) & \\ O & O \end{pmatrix} = \begin{pmatrix} O & O \\ O & O \end{pmatrix}$$

So $\mu_A \cdot \mu_D$ annihilates T So $\begin{array}{l} \mu_A | \\ \mu_D | \end{array} T | \mu_A \mu_D$

Therefore, $\boxed{LCM(\mu_A, \mu_D) | T | \mu_A \mu_D}$

if $\mu(A), \mu_D$ relatively prime, then $T'' = \mu_A \mu_D$

Mar 14, 2019 Review

1. Minimal Polys
2. Eigenvalues are exactly the roots of the minimal polynomial
3. Local v Global min poly
   (a) Local divides Global
   (b) One of locals is global
4. Similar matrices have the same minimal poly.
   $(\mu_L = \mu_{[L]_{\beta\leftarrow\Gamma}})$

   assume T & assume A, B

5. $\boxed{\mu_A = \mu_{A^T} \Rightarrow \sigma_{\mathbb{C}}(T) = \sigma_{\mathbb{C}}(T^T)}$
6. if $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ then $\mu_T = LCM(\mu_A, \mu_B)$ gap due to multiplicity
7. if $T = \begin{bmatrix} A & D \\ 0 & B \end{bmatrix}$ then $\boxed{LCM(\mu_A, \mu_B) \mid \mu_T \mid \mu_A \mu_B}$

   ④ if μ_A, μ_B relatively prime $\Rightarrow$ $\boxed{LCM(\mu_A, \mu_B) = \mu_A \mu_B = \mu_T}$

b) $\boxed{\sigma_{\mathbb{C}}(T) = \sigma_{\mathbb{C}}(A) \cup \sigma_{\mathbb{C}}(B)}$

lin factors of T are exactly those that appear in μ_A or μ_B (or both)

(6a) μ_T of $T = \begin{pmatrix} \alpha_1 & & 0 \\ & \ddots & \\ 0 & & \alpha_n \end{pmatrix}$ is a product of $(x-\alpha_1)\cdots(x-\alpha_n)$ distinct

Ex: $T = \begin{pmatrix} 3 & & & \\ & 4 & & \\ & & 1 & \\ & & & 3 \end{pmatrix} \rightarrow \mu_T(z) = (z-3)(z-4)(z-1)$

$\hookrightarrow \sigma_{\mathbb{C}}(T) = \{1, 3, 4\}$

What if

$$\begin{pmatrix} A & D \\ 0 & B \end{pmatrix} = T = \left(\begin{array}{cc|cc} 3 & 1 & 2 & 8 \\ & 4 & 5 & 1 \\ \hline & & 2 & 7 \\ & & & 3 \end{array}\right) \Rightarrow \sigma_{\mathbb{C}}(T) = \sigma_{\mathbb{C}}\left(\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}\right) \cup \sigma_{\mathbb{C}}\left(\begin{pmatrix} 2 & 7 \\ 0 & 3 \end{pmatrix}\right)$$

$$= \sigma_{\mathbb{C}}([3]) \cup \sigma_{\mathbb{C}}([4]) \cup \sigma_{\mathbb{C}}([2]) \cup \sigma_{\mathbb{C}}([3])$$

$$= \{3, 4, 2\} \rightarrow \text{just the diagonal!}$$

on

So what is $\mu_T(z)$?

$$\mu_T(z) = \overset{1}{(z-2)}\overset{1\text{ or }2}{(z-3)}\overset{1}{(z-4)}$$

we $LCM(\mu_A, \mu_B) \mid \mu_T \mid \mu_A \mu_B$ {

Observe If $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $p(z) = (z-a)(z-d) - bc$ then p annihilates T

Proof $p(T) = (T - aI)(T - dI) - bcI$ ⟶ We can show this is the minimal poly if....

$$= \begin{pmatrix} 0 & b \\ c & d-a \end{pmatrix}\begin{pmatrix} a-d & b \\ c & 0 \end{pmatrix} - bcI = \begin{pmatrix} bc & 0 \\ 0 & bc \end{pmatrix} - \begin{pmatrix} bc & 0 \\ 0 & bc \end{pmatrix} = [0]$$

Thm $\boxed{\text{if } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is not a multiple of } I_2 \text{, then } \mu_T = (z-a)(z-d) - bc}$

Back to example

$$\mu_A = \mu_{\begin{bmatrix} 3 & 1 \\ 0 & 4 \end{bmatrix}}(z) = (z-3)(z-4)$$

$$LCM = (z-2)(z-3)(z-4)$$

$$\mu_B = \mu_{\begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}}(z) = (z-2)(z-3)$$

So $(z-2)(z-3)(z-4) \mid \mu_T \mid (z-2)(z-3)^2(z-4)$

So $\mu_T(z) = (z-2)(z-3)^{1 \text{ or } 2}(z-4)$

Question

$$\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \overset{?}{\rightsquigarrow} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \text{ or } \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$$

Well $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \sim \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$ So $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix} \overset{?}{\sim} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ Yes.

Why? Same minimal polynomial. $(z-3)(z-4)$

Suppose p_1, p_2, relatively prime.

Then $\ker\left(p_1(L)\right) \cap \ker\left(p_2(L)\right) = \{\theta\}$

"Proof" $\rightsquigarrow$ $\underline{\exists\ q_1, q_2 \text{ s.t.}}$ $q_1(z)p_1(z) + q_2(z)p_2(z) = \mathbb{1}$

So $q_1(L)p_1(L) + q_2(L)p_2(L) = I$

So $\left(q_1(L)p_1(L) + q_2(L)p_2(L)\right)(v) = v$

So $q_1(L)\left[(p_1(L))(v)\right] + q_2(L)\left[(p_2(L))(v)\right] = v$

if $v \in \ker(p_1(L)) \cap \ker(p_2(L))$ then

$$q_1(L)\vec{\theta} + q_2(L)\vec{\theta} = v \Rightarrow v = \vec{\theta}$$

So $\boxed{\{\theta\} = \ker(p_1(L)) \cap \ker(p_2(L))}$ if p_1, p_2 relatively prime

Observe

If $\mu_A(z) = \underbrace{(z-\lambda_1)^{k_1}}_{p_1} \cdots \underbrace{(z-\lambda_m)^{k_m}}_{p_2}$

Since p_1, p_2 relatively prime $\Rightarrow$ $\ker(p_1(A)) \cap \ker(p_2(A)) = \{\theta\}$

Also $p_1(A)p_2(A) = \mu_A(A) = \theta = p_2(A)p_1(A)$

so $\begin{cases} \operatorname{Im}(p_1(A)) \subseteq \ker(p_2(A)) \\ \operatorname{Im}(p_2(A)) \subseteq \ker(p_1(A)) \end{cases}$

Yet also $q_1(z)p_1(z) + q_2(z)p_2(z) = 1$ for some q_1, q_2

So $p_1(z)\cdot q_1(z) + p_2(z)\cdot q_2(z) = 1$

So $p_1(A)q_1(A) + p_2(A)\cdot q_2(A) = I$

hence, $\underbrace{P_1(A)\Big((q_1(A))(y)\Big)}_{\in \mathrm{Im}(P_1(A)) \subset \ker(P_2(A))} + \underbrace{P_2(A)\Big((q_2(A))(y)\Big)}_{\mathrm{Im}(P_2(A)) \subset \ker(P_1(A))} = y \quad \forall y \in V$

So

$$\ker(P_1(A)) + \ker(P_2(A)) = V$$

But since $\ker(P_2(A)) \cap \ker(P_1(A)) = \{0\}$

So $\boxed{V = \ker(P_1(A)) \oplus \ker(P_2(A))}$ $(*)$

Observe that $P_1(A) \leftrightarrow A$, ($P_1(A)$ commutes with A)

So $\ker(P_1(A)) \in Lat(A)$

Similarly, $\ker(P_2(A)) \in Lat(A)$

So, with respect to the decomposition, A can be expressed as

$$A = \begin{bmatrix} C & 0 \\ 0 & D \end{bmatrix} = \begin{array}{c|cc} & \ker(P_1(A)) & \ker(P_2(A)) \\ \hline \ker(P_1(A)) & C & 0 \\ \ker(P_2(A)) & 0 & D \end{array}$$

Observe what are μ_C & μ_D? $\left.\begin{array}{l}\mu_C = P_1 \\ \mu_D = P_2\end{array}\right\} \to$ as we keep going

[Primary decomposition]

Nov 19, 2019

Lemma

Summarize. If q_1, q_2 relatively prime and $q_1 q_2$ annihilates $A: V \mapsto V$ then $V = \ker(q_1(A)) \oplus \ker(q_2(A))$ and w.r.t this decomp A has the form

$$A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} \quad \left\{\begin{array}{l} q_1 \text{ annihilates } C \\ q_2 \text{ annihilates } D \end{array}\right.$$

"Proof Sketch" $\to$ (1) $q_1(A)\,q_2(A) = 0 = q_2(A)\,q_1(A) \Rightarrow \left\{\begin{array}{l}\mathrm{Im}(q_2(A)) \subset \ker q_1 \\ \mathrm{Im}(q_1(A)) \subset \ker q_2\end{array}\right.$

(2) $\exists\, f, g : f q_1 + g q_2 = \mathbb{1} \Rightarrow q_1(A) f(A) + q_2(A) g(A) = I$

So $\underbrace{q_1(A) f(A)(x)}_{} + q_2(A) g(A)(x) = x$

$$q_1(A)f(A)(x) \in \operatorname{Im} q_1(A) \subseteq \ker(q_2(A)), \qquad q_2(A)g(A)(x) \in \operatorname{Im} q_2(A) \subseteq \ker(q_1(A))$$

So $\ker(q_1(A)) + \ker(q_2(A)) = V$

(3) for any $y \neq 0$, $\underbrace{f(A) q_1(A)(y)}_{} + \underbrace{g(A) q_2(A)(y)}_{} = y$

cannot both be zero unless $y = 0$

$\Rightarrow \ker(q_1(A)) \cap \ker(q_2(A)) = \{0\}$

So $\ker(q_1(A)) \oplus \ker(q_2(A)) = V$

(4) $\left.\begin{array}{l} \ker(q_1(A)) \in \operatorname{Lat}(A) \\ \ker(q_2(A)) \in \operatorname{Lat}(A) \end{array}\right\}$

Hence wrt $V = \ker(q_1(A)) \oplus \ker(q_2(A))$,

$$A = \begin{pmatrix} B & \theta \\ \theta & C \end{pmatrix} \Rightarrow q_1(A) = \begin{pmatrix} q_1(B) & \theta \\ \theta & q_1(C) \end{pmatrix}$$

So,

$$A\begin{pmatrix} x \\ y \end{pmatrix}_+ = \begin{pmatrix} Bx \\ Cy \end{pmatrix}_+ = \underset{\ker(q_1(A))}{Bx} + \underset{\ker(q_2(A))}{Cy}$$

$\bullet\ q_1(B) : \ker(p_1(A)) \mapsto \ker(p_1(A$

Now

$$q_1(A)\begin{pmatrix} x \\ y \end{pmatrix}_+ = q_1(B)(x) + q_1(C)(y)$$

But $x \in \ker(q_1(A)) \Rightarrow q_1(B)(x) + q_1(C)(y) = q_1(A)(y)$

when $y = 0 \Rightarrow \boxed{q_1(B)(x) = \theta}\ \forall x \in \ker(q_1(A))$

So $q_1(B) = 0$: $\ker(q_1(A)) \longmapsto \ker(q_1(A))$

☒ So q_1 annihilates B

Same argument $\Rightarrow$ q_2 annihilates C

Corollary

Under the hypothesis of the previous lemma
If q_1 & q_2 are monic and $\mu_A = q_1 \cdot q_2$ then

$q_1 = \mu_B$, $q_2 = \mu_C$

"Proof" $\mu_A = \mu_{\begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix}} = LCM(\mu_B, \mu_C)$

We know that μ_B divides q_1 since q_1 annihilates B
μ_C divides q_2 since q_2 annihilates C

But since q_1, q_2 relatively prime $\Rightarrow$ μ_B, μ_C relatively prime

$\Rightarrow LCM(\mu_B, \mu_C) = \mu_B \cdot \mu_C$ So $\mu_A = \mu_B \cdot \mu_C$

But we also know $\mu_A = q_1 \cdot q_2 = \mu_B \cdot \mu_C$ and $\mu_B | q_1$, $\mu_C | q_2$ and that q_1, q_2 are monic

So $q_1 = \mu_B$ and $q_2 = \mu_C$

//

$\Rightarrow$ Primary Decomposition Theorem

(distinct)

Suppose $A : V \longmapsto V$ and $\mu_A(z) = (z-\lambda_1)^{p_1} \cdots (z-\lambda_k)^{p_k}$, then

$$V = \ker(A-\lambda_1 I)^{p_1} \oplus \cdots \oplus \ker(A-\lambda_k I)^{p_k}$$

and wrt this decomposition, A has the form — $\begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots \\ & & & A_k \end{bmatrix}$

where

$\mu_{A_i} = (z-\lambda_i)^{p_i}$

"**Proof**" Induct on k.

Base case "$k=1$" holds true trivially

Induction hypothesis ↳ Suppose "$k=1,2,\ldots,m$" hold

Show Case "$k=m+1$" holds.

Then $$\mu_A(z) = \underbrace{(z-\lambda_1)^{p_1}\cdots(z-\lambda_m)^{p_m}}_{q_1}\underbrace{(z-\lambda_{m+1})^{p_{m+1}}}_{q_2}$$

By the lemma or we have $A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ w.r.t $V = \overbrace{\ker(q_1(A))}^{W_1} \oplus \underbrace{\ker(q_2(A))}_{W_2}$

and $\mu_B = q_1$, $\mu_C = q_2$

By inductive hyp, $\ker(q_1(A)) = \ker(B-\lambda_1 I)^{p_1} \oplus \cdots \oplus \ker(B-\lambda_m I)^{p_m}$

"new V" new $V = \ker(q_1(A)) = W_1$

and $$B = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_m \end{pmatrix}$$ w.r.t this decomposition, with $\mu_{B_i}(z) = (z-\lambda_i)^{p_i}$

By assignment (?), A can be represented by

$$A = \left(\begin{array}{ccc|c} B_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & B_m & \\ \hline & 0 & & C \end{array}\right)$$ w.r.t the decomposition

$$V = \overbrace{\ker(B-\lambda_1 I)^{p_1} \oplus \cdots \oplus \ker(B-\lambda_m I)^{p_m}}^{\ker(q_1(A))} \oplus \ker(A-\lambda_{m+1} I)$$

So, the next step is to show $\ker(B-\lambda_i I)^{p_i} = \ker(A-\lambda_i I)^{p_i}$

Now, $\ker(A-\lambda_i I)^{p_i} \subset \ker\left(\underbrace{(A-\lambda_1 I)^{p_1}\cdots(A-\lambda_m I)^{p_m}}_{\text{commute}\ldots}\right)$ since more

Such that I can write $(A-\lambda_1 I)^{p_1}\cdots(A-\lambda_m I)^{p_m}(A-\lambda_i I)^{p_i}$

(1) So $\ker(A-\lambda_i I)^{p_i} \subset \ker(q_1(A)) = W_1$

(2) Similarly, $\ker(B-\lambda_j I)^{p_j} \subset \ker(q_1(A)) = W_1$

Recall

$$A = \begin{array}{c|cc} & W_1 & W_2 \\ \hline W_1 & B & 0 \\ W_2 & 0 & C \end{array}$$

So

$$(A-\lambda_i I)^{p_i} = \begin{array}{c|cc} & W_1 & W_2 \\ \hline W_1 & (B-\lambda_i I)^{p_i} & 0 \\ W_2 & 0 & \text{-Stuff-} \end{array}$$

Now $x \in W_1$ is in $\ker(A-\lambda_i I)^{p_i} \iff (A-\lambda_i I)^{p_i}(x) = 0 \iff \boxed{}\begin{pmatrix} x \\ 0 \end{pmatrix} = 0$

$\iff (B-\lambda_i I)^{p_i}(x) = 0 \iff x \in \ker\left((B-\lambda_i I)^{p_i}\right)$

So $x \in \ker(A-\lambda_i I)^{p_i} \iff x \in \ker((B-\lambda_i I)^{p_i})$

So $\boxed{\ker\left((A-\lambda_i I)^{p_i}\right) = \ker\left((B-\lambda_i I)^{p_i}\right)}$ for $i = 1 \ldots m$ //

$\boxed{\text{So, what does this theorem say?}}$

Think if $(A-\lambda I)^p = 0$ and $N = A - \lambda I$, N is nilpotent

$\iff A = \lambda I + N$, N nilpotent

Corollary

→ Every $L: V \xrightarrow{\text{lin}} V$, fin dim can be expressed as

$L = L_1 \oplus L_2 \oplus L_3 \oplus \cdots \oplus L_m$, where $L_i = \lambda_i I + N_i$

distinct nilpotent

NILPOTENTS

global

Structures of Nilpotents:

such that $N^8 = \theta \to \mu_N = z^8$

Given a nilpotent $N : V \longmapsto V$ and $v_0 \in V$, we write

$$\langle v_0 \rangle \equiv \text{span}\{v_0, N(v_0), N^2(v_0), \dots, N^7(v_0)\}$$
$$= \{ P(N)(v_0) \mid \deg(P) \le 7 \} \in Lat(N)$$

Theorem $\boxed{\text{For any } N \text{ as above, there exist some } v_1 \dots v_k \in V \text{ such that } V = \langle v_0 \rangle \oplus \langle v_1 \rangle \oplus \dots \oplus \langle v_k \rangle}$

we will look at proof later, but wrt this decomposition, N has the form

$$N = \begin{array}{c|cccc} & \langle v_0 \rangle & \langle v_1 \rangle & \dots & \langle v_k \rangle \\ \hline \langle v_0 \rangle & * & \theta & \dots & \theta \\ \langle v_1 \rangle & \theta & * & & \vdots \\ \vdots & \vdots & & \ddots & \theta \\ \langle v_k \rangle & \theta & \dots & \theta & * \end{array}$$

$\to$ N is block - diagonal

Suppose $\underbrace{v_0, N(v_0), \dots, N^5(v_0)}_{\text{lin. ind}}, N^6(v_0)$

So $N^6(v_0) = \sum_0^5 a_i N^i(v_0)$

So $\underbrace{(N^6 - a^5 N^5 \dots - a_0 I)}(v_0) = \theta$

So $q(N)(v_0) = \theta$ So $q_{(v_0)}(N) = N^k$ but deg at most 6

But local is some z^k, $k < 8$ but deg at least 6

$(v_0, N(v_0) \dots$ so $)$

So $N^6 = \theta$

So $(v_0, N(v_0) \ldots N^5(v_0)) =$ basis of $\langle v_0 \rangle$

So

$$\begin{array}{c|c}
 & \begin{matrix} v_0 & N(v_0) & \cdots & N^5(v_0) \end{matrix} \\
\begin{matrix} v_0 \\ N(v_0) \\ \vdots \\ \\ N^5(v_0) \end{matrix} & N_{\langle v_0 \rangle} = \begin{pmatrix} 0 & 0 & & 0 \\ 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & & 0 \end{pmatrix}
\end{array}$$

Mar 21, 2019

$\boxed{\text{Recall}}$ Primary Decomposition Theorem

If $\mathcal{L}: V \mapsto V$ has minimal polynomial $\mu_{\mathcal{L}} = (z-\lambda_1)^{p_1} \cdots (z-\lambda_k)^{p_k}$ then $V = \ker(\mathcal{L}-\lambda_1 I)^{p_1} \oplus \cdots \oplus \ker(\mathcal{L}-\lambda_k I)^{p_k}$ and wrt this decomposition $\mathcal{L}$ has the form $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_k$ with $\mu_{\mathcal{L}_i}(z) = (z-\lambda_i)^{p_i}$

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_1 & & 0 \\ & \ddots & \\ 0 & & \mathcal{L}_k \end{pmatrix}$$

(1) Next, observed that

$\mu_A(z) = (z-\alpha)^p$ then $A = \alpha I - N$ where $\mu_N(z) = z^p$ (N nilpotent)

we say that

"N is a nilpotent of order p."

(2) if N is nilpotent of order p. $N: W \mapsto W$, and $0 \neq w_0 \in W$ then $\mu_{N,w_0}(z) = z^m$, for some $m \leq p$.

Consider $\underbrace{\overset{\neq 0}{w_0}, N(w_0), N^2(w_0), \ldots, N^{m-1}(w_0)}_{\text{lin. ind (since otherwise we get another } \mu_N)}, 0, 0, 0, \ldots$

Recall $\langle w_0 \rangle_N = \{ q(N)(w_0) \mid q \in P \} \Leftarrow$ cyclic invariant subspace $\in Lat(N)$

So $= \{ q(N)(w_0) \mid q \in P_{m-1} \}$ & this invariant ss of N has basis $w_0 \ldots N^{m-1}(w_0)$

Let $\Gamma =$ basis set $w_0, N(w_0), \dots N^{m-1}(w_0)$.

<u>Note</u>: $N\big|_{\langle w_0\rangle} : \langle w_0\rangle \mapsto \langle w_0\rangle$ & $\left[N\big|_{\langle w_0\rangle}\right]_{\Gamma\leftarrow\Gamma} = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ 0 & & \ddots & \ddots & \\ \vdots & & & 1 & 0 \end{bmatrix}$

Let η: $N^{m-1}(w_0) \dots w_0$ → write basis in reverse

<u>then</u> $\left[N\big|_{\langle w_0\rangle}\right]_{\eta\leftarrow\eta} = \begin{array}{c} N^{m-1} \\ \vdots \\ \\ \\ w_0 \end{array} \begin{bmatrix} 0 & 1 & 0 & & 0 \\ \vdots & 0 & 1 & & \\ & \vdots & 0 & \ddots & \\ & & \vdots & \ddots & 1 \\ 0 & 0 & & & 0 \end{bmatrix}$

<u>Theorem</u> $\boxed{\text{Cyclic Decomposition for Nilpotents}}$

If $\underset{\text{f.d.}}{N : V} \mapsto \underset{\text{f.d.}}{V}$ nilpotent then $V = \bigoplus_{i=1}^{k} \langle v_i \rangle_N$ for some non-zero v_i.

In fact, we can choose any v_1 to start with.

(and wrt to this decomposition, N has the form $N_1 \oplus N_2 \oplus \dots \oplus N_k$)

$\boxed{\text{Corollary}}$ If $\underset{\text{f.d.}}{N : V} \mapsto \underset{\text{f.d.}}{V}$ nilpotent then there is a basis of Ψ of V such that $[N]_{\Psi\leftarrow\Psi} = \begin{bmatrix} \boxed{\begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{smallmatrix}} & & & 0 \\ & \boxed{\begin{smallmatrix} 0 & 1 & \\ & \ddots & 1 \\ & & 0 \end{smallmatrix}} & & \\ 0 & & \ddots & \\ & & & \boxed{0} \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}$

{ Ψ is made by concatenation }

may or may not be there

Now → "improve", look at

$[\alpha I + N]_{\Psi\leftarrow\Psi} = \begin{bmatrix} \boxed{\begin{smallmatrix} \alpha & 1 & \\ & \ddots & 1 \\ & & \alpha \end{smallmatrix}} & & & 0 \\ & \boxed{\begin{smallmatrix} \alpha & 1 & \\ & \ddots & 1 \\ & & \alpha \end{smallmatrix}} & & \\ 0 & & \boxed{\begin{smallmatrix} \alpha & 1 & \\ & \ddots & 1 \\ & & \alpha \end{smallmatrix}} & \\ & & & \boxed{\alpha} \end{bmatrix}$

$\| $

$[\alpha I]_{\Psi\leftarrow\Psi} + [N]_{\Psi\leftarrow\Psi}$

where each $\begin{bmatrix} \alpha & 1 & \\ & \ddots & 1 \\ & & \alpha \end{bmatrix}$ is called a "Jordan block"

"$J_{\alpha,\,\text{size}}$" ← denoted

$[\alpha] \to J_{\alpha,1}$

Jordan canonical form theorem

unique up to changing order of appearance

Every matrix is similar to a direct sum of Jordan blocks

Consider $N = \begin{pmatrix} J_{0,9} & & \\ & \ddots & \\ & & J_{0,1} \end{pmatrix}$

$$\{0\} \subseteq \underbrace{\ker(N)}_{8} \subseteq \underbrace{\ker(N^2)}_{7} \subseteq \underbrace{\ker(N^3)}_{7} \subseteq \dots \subseteq \underbrace{\ker(N^8)}_{6} \subseteq \ker(N^9)$$

# of blocks

Given $\{0\} \subseteq \ker(N) \subseteq \dots \subseteq V$

if $\underbrace{\quad}_{10} \; \underbrace{\quad}_{10} \; \underbrace{\quad}_{8} \; \underbrace{\quad}_{6} \; \underbrace{\quad}_{\dots}$

10 blocks, no $J_{0,1}$ blocks, 2 $J_{0,2}$ blocks, 2 $J_{0,3}$ blocks ...

So we can reconstruct $N = \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}$ by Weyr characteristic...

↳ So, no matter how decomposition is different, ~~the~~ this Jordan construction is determined by just N.

So Similar N, M nilpotents then they have the same Weyr characteristics
→ they have the same Jordan form

But we also shown that if N, M have the same Jordan-block decomposition they are similar

Step 1

show true for nilpotents

$\boxed{\text{So}}$ Weyr characteristic of N completely determines the Jordan block sizes and their multiplicities, i.e. completely determines the J-form of N (up to order of appearance)

⇒ Similar matrices have the same Weyr characteristics; so similar nilpotent have the same J-form

Conversely ...

Conversely, if 2 nilpotents have the same J-form, then they are similar to it, and so to each other

So { the J-form is a "complete invariant" for similarity of nilpotent }

Now, what abt $A = \alpha I + N$?

{ If $A = \alpha I + N$ then the Weyr characteristic of $A - \alpha I$ determines the Jordan form of A. }

What if we don't know α? What if all we know is
$A =$ something $\cdot I$ + nilpotent

→ Use Trace! → $\boxed{\dfrac{Tr(A)}{\text{size of } A}}$ → the trace doesn't change in different representations...

("Trace" is similarity-invariant)

Step 2 Similar ~~(nilpotents)~~ matrices have the same Jordan forms.

(not just restricted to nilpotents)

April 2, 2019

Diagonalizability

① Recall Every $n \times n$ matrix is triangularizable, i.e. is similar to an upper or lower triangular matrix.

Δ-ty { Equivalently / Consequence $\mathcal{L} \in \mathcal{L}(V)$ can be represented by Δ^r matrix for some basis

② Which matrices are diagonalizable ?

Reverse engineer: If $A \sim D$, D diagonal, then, $\mu_A = \mu_D$.

and $\mu_D = \underbrace{(z - d_{i_1}) \cdots (z - d_{i_k})}_{\text{distinct } d_i}$, μ_D has no repeated roots
i.e. all roots have multiplicity 1

$\boxed{\text{Observe}}$

Ex $J_{\lambda,3} = \begin{bmatrix} \lambda & 1 & 0 \\ & \lambda & 1 \\ & & \lambda \end{bmatrix}$ $(z-\lambda) \mid \mu_J(z) \mid (z-\lambda)^3$

Know

$(J - \lambda I)^3 = 0$

$J_{\lambda,3} \Rightarrow \mu_J = (z) = (z-\lambda)^3$

So $\boxed{J_{\lambda,p} \Rightarrow \mu_J(z) = (z-\lambda)^p}$

Now, if $A \sim J_1 \oplus J_2 \oplus \cdots \oplus J_k$ then $\mu_A = LCM(\mu_{J_1}, \ldots, \mu_{J_k})$
So in particular, $\mu_{J_i} \mid \mu_A$ for each i

So if μ_A has no repeated roots, then no μ_{J_i} has repeated roots

i.e. all J_i is 1×1

So $A \sim J_1 \oplus \cdots \oplus J_k$ is similar to a diagonal matrix

$\boxed{\text{Theorem}}$ TFAE
1. A is diagonalizable
2. μ_A has no repeated roots
3. There is a basis of $\mathbb{C}^n$ made up entirely of eigenvectors of A (an A-eigenbasis of $\mathbb{C}^n$)

How to check if μ_A has no repeated roots? $\to$ look at μ_A, μ_A'
if μ_A has repeated roots, $\mu_A(\text{root}) = \mu_A'(\text{root}) = 0$

How to check if μ_A and μ_A' have no common roots? $\Rightarrow$ <u>division algorith</u>
$\rightarrow$ find gcd (...) (or Euclidean algorithm

If gcd $= \mathbb{1}$ then no common roots...

matrix $A \longrightarrow$ $\boxed{\sigma_{\mathbb{C}}(p(A)) = p[\sigma_{\mathbb{C}}(A)]}$

<u>Spectral Mapping Theorem</u>

<u>Exp</u> $\left\{ \begin{array}{l} \sigma_{\mathbb{C}}(\alpha(A)) = \alpha[\sigma_{\mathbb{C}}(A)] \\ \sigma_{\mathbb{C}}(A + \beta I) = \sigma_{\mathbb{C}}(A) + \beta \end{array} \right\}$

<u>Suppose</u> A invertible. what is $\sigma_{\mathbb{C}}(A^{-1})$?

$A^{-1}(v) = \gamma v \quad , v \neq 0$

$\Updownarrow$

$1/\gamma \, v = Av \quad$ ($\gamma \neq 0$ since...

<u>Theorem</u> $\boxed{\gamma \in \sigma_{\mathbb{C}}(A^{-1}) \Leftrightarrow 1/\gamma \in \sigma_{\mathbb{C}}(A)}$

<u>So</u> $\sigma_{\mathbb{C}}(A^{-1}) = \dfrac{1}{\sigma_{\mathbb{C}}(A)} = \dfrac{1}{z}\left[\sigma_{\mathbb{C}}(A)\right]$

$\downarrow$

$\frac{1}{z}[A]$

SMT holds for rational fn...

What about simultaneous...? Commute...?

$\boxed{\text{Simultaneous } \Delta\text{-ability, Diag-ability}}$

April 4, 2019

① Sim. Diag-bility

$V \xrightarrow{L, M} V$ Suppose $A \xrightarrow{S}$ diagonal

$B \xrightarrow{S}$ diagonal

$F^n \longrightarrow \mathbb{C}^n$ i.e.

$S^{-1}AS$ diag

$S^{-1}BS$ diag

Then

$$S^{-1}ASS^{-1}BS = S^{-1}ABS \iff \boxed{AB = BA}$$

$$\quad \| \qquad\qquad\qquad \|$$

$$S^{-1}BS\,S^{-1}AS = S^{-1}BAS$$

Necessary conditions for Simultaneous diagonalizability

(1) Individual diagonalizability

(2) Commutativity

In fact, these are also sufficient.

(†) **Theorem** TFAE for a collection of $\mathcal{F} \in M_n$

① $\mathcal{F}$ is a commutative collection and all elements $\in \mathcal{F}$ are individually diagonalizable

② There is invertible S such that $S^{-1}AS$ diagonal for every $A \in \mathcal{F}$

$AB = BA$, $BC = CB$

Pf We prove {Lemma} $\boxed{A \text{ diagonalizable} \iff A = \alpha_1 E_1 + \dots + \alpha_k E_k \text{ for some idempotents s.t. } E_1 + \dots + E_k = I \text{ (non zero)}}$

$(1 \Rightarrow 2)$ Pf A diag-able $\iff S^{-1}AS$ diag $= \beta_1 F_1 + \dots + \beta_m F_m$, $F_i \to$ idemp $\neq 0$

, $F_1 + \dots + F_m = I$

So $\quad A = S\left[\beta_1 F_1 + \dots + \beta_m F_m\right] S^{-1}$

$$= \beta_1 \underbrace{S F_1 S^{-1}}_{\text{idemp}} + \dots + \beta_m S F_m S^{-1}$$

$$= \beta_1 \mathcal{E}_1 + \dots + \beta_m \mathcal{E}_m$$

Now, $\quad \mathcal{E}_1 + \dots + \mathcal{E}_m = S^{-1}(F_1 + \dots + F_m) S = I$

(2) ⇒ (1) Suppose $A = \alpha_1 \mathcal{E}_1 + \dots + \alpha_k \mathcal{E}_k$ where $\mathcal{E}_1 + \dots + \mathcal{E}_k = I$, $\mathcal{E}_i$ idemp.
then

$$\bigoplus \operatorname{Im} \mathcal{E}_i = V = \mathbb{C}^p \quad (*)$$

Let Γ_i = a basis of $\operatorname{Im} \mathcal{E}_i$, $\quad \Gamma = \Gamma_1 \| \Gamma_2 \| \dots \| \Gamma_k$ a basis of $\mathbb{C}^p$

then $[A]_{\Gamma \leftarrow \Gamma}$? Express A as a block matrix w.r.t (*)

$$A_{11} = \mathcal{E}_1 \circ A \circ \mathcal{E}_1 \Big|_{\operatorname{Im}(\mathcal{E}_1)} = \alpha_1 \mathcal{E}_1 \Big|_{\operatorname{Im}(\mathcal{E}_1)} = \alpha I_{\operatorname{Ran} \mathcal{E}_1}$$

So...

$$A = \begin{pmatrix} \alpha_1 I & & \\ & \ddots & \\ & & \alpha_k I \end{pmatrix} \text{ w.r.t } \Gamma \quad \rightarrow \quad [A]_{\Gamma \leftarrow \Gamma} = \begin{pmatrix} \alpha_1 & & \\ & \ddots & \\ & & \alpha_k \end{pmatrix}$$

So A diagonalizable. $\square$

Corollary (to the proof) The α's are exactly the eigenvalues of A

<u>and</u> $\operatorname{Im} \mathcal{E}_i = \mathcal{E}_A(\alpha_i)$ ⇝ eigenspace of A corresponding to α_i

<u>and</u> the rep^n of A as
$A = \alpha_1 \mathcal{E}_1 + \dots + \alpha_k \mathcal{E}_k$ is <u>unique</u>
distinct

$\square$

If $A = \begin{pmatrix} \alpha_1 I & & \\ & \ddots & \\ & & \alpha_k I \end{pmatrix}$ wrt $\oplus \operatorname{Im} \mathcal{E}_i = \mathbb{C}^p$

then $p(A) = \boxed{\begin{matrix} p(\alpha_1) I & & \\ & \ddots & \\ & & p(\alpha_k) I \end{matrix}}$ α_i are distinct

Consider $\mathcal{E}_1 = \boxed{\begin{matrix} I & 0 & \\ 0 & \ddots & \\ & & 0 \end{matrix}}$ Now, $p(z) = \dfrac{(z - \alpha_2) \cdots (z - \alpha_k)}{(\alpha_1 - \alpha_2) \cdots (\alpha_1 - \alpha_k)}$ sends A to $\mathcal{E}_1$

Thm: Each spectral idempotent of a diagonalizable A is a polynomial in A, i.e. is of the form $p(A)$ for some poly. p.

Pf of Thm † → by induction ... Induct on n

(1)→(2) • Base case: $n = 1$ trivially true
• Suppose holds for all $k \leq n < n_0$. Show holds for n_0.

If $\mathcal{F}$ contains only scalar multiples of identity ⇒ done ...
So, let us assume $\mathcal{F}$ contains $A \neq \alpha I$

then $A = \sum^{k} \alpha_i \mathcal{E}_i$ for some ... ... distinct α_i

Since distinct α_i ..., $k \geq 2$

Recall $\mathcal{E}_i = p_i(A)$, for some poly p_i
and every $B \in \mathcal{F}$ commutes with A

So B commutes with $p_i(A)$, i.e with every $\mathcal{E}_i$.

So $B = \begin{bmatrix} \\ \\ \end{bmatrix}$ $B_{11} = \mathcal{E}_1 B \mathcal{E}_1 \Big|_{\operatorname{Im} \mathcal{E}_1} = \ldots$

$$B_{ij} = \mathcal{E}_i \circ B \circ \mathcal{E}_j \Big|_{\mathrm{Im}(\mathcal{E}_j)} = \underbrace{\mathcal{E}_i \circ \mathcal{E}_j}_{\delta^i_j} \circ B \Big|_{\cancel{\mathrm{Im}}}$$

So B block diagonal : $B = \begin{bmatrix} B_{11} & & \\ & \ddots & \\ & & B_{kk} \end{bmatrix}$

A B commute, so A_{ij} commutes w B_{ij}

$\Rightarrow$ ~~B$\mathcal{E}$~~ B_{ij} individually diagonalizable ... (by hypothesis)

$\hookrightarrow$ concatenate basis $\Rightarrow$ B diagonalizable $\Rightarrow$ //

$\square$

April 9, 2019

$\boxed{\text{INNER PRODUCTS}}$

Lots and lots of way to measure distance ... { Pythagorean theorem, sum of ...

~~But~~

Pythagorean distance is associated with the dot product ...

$$\mathrm{dist}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} a \\ b \end{pmatrix}\right)^2 = \begin{pmatrix} x-a \\ y-b \end{pmatrix} \cdot \begin{pmatrix} x-a \\ y-b \end{pmatrix}$$

Also, $\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\| \cdot \left\|\begin{pmatrix} a \\ b \end{pmatrix}\right\| \cdot \cos\theta = \begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}$ $\rightsquigarrow$ there's a angles captured and distance,

$\Rightarrow$ makes Pythagorean distance stand out ...

Defn An inner product $\varphi : V \times \mathbf{W} \mapsto \mathbb{C}$ on a vector space V is a function satisfying the following conditions ...

① φ is ~~bilinear~~ partially linear in the first slot & partially conjugate linear in the second

i.e.

$$\begin{cases} \varphi(\alpha v_1 + \beta v_2, w) = \alpha \varphi(v_1, w) + \beta \varphi(v_2, w) \quad \text{and} \\ \varphi(v, \alpha w_1 + \beta w_2) = \bar{\alpha}(\varphi)(v, w_1) + \bar{\beta} \varphi(v, w_2) \end{cases}$$

Standard inner product on $\mathbb{C}^n$:

$$\left\langle \begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix}\right\rangle = a\bar{\alpha} + b\bar{\beta} + c\bar{\gamma} = \dots$$

so that $\left\langle \begin{pmatrix}a\\b\\c\end{pmatrix}, \begin{pmatrix}a\\b\\c\end{pmatrix}\right\rangle = |a|^2 + |b|^2 + |c|^2 = \left\|\begin{pmatrix}a\\b\\c\end{pmatrix}\right\|^2$

---

(2) $\overline{\varphi(v,w)} = \varphi(w,v)$

(3) $\varphi(v,v) \geq 0$, equality holds iff $v = 0$ (positive definite)

#

$\boxed{\text{An inner product space is a vector space with an inner product}}$

Observe

If (V, φ) is an inner product space, then

$\|v\| := \sqrt{\varphi(v,v)}$ defines a "norm" (i.e. a sizing function)

Such that

(i) $\| \ \| : V \to [0, \infty)$

(ii) $\|v\| = 0 \iff v = 0$

(iii) $\|\alpha v\| = |\alpha| \cdot \|v\|$

(iv) $\|u + v\| \leq \|u\| + \|v\|$ (triangle inequality)

(v) $\|\langle v, u\rangle\| \leq \|v\| \cdot \|u\|$

Pf of (iv) $\|u + v\| = \dots = \sqrt{\|u\|^2 + 2\,\mathrm{Re}\,\varphi(u,v) + \|v\|^2}$

$$\cancel{\left(\leq \sqrt{\|u\|^2 + 2|\varphi(u,v)| + \|v\|^2} \right)}$$

$$\leq \sqrt{(\|u\| + \|v\|)^2} = \|u\| + \|v\|$$

☒ Can it be that we always have $|\varphi(u,v)| \leq \|u\| \cdot \|v\|$?

Yes! → $\boxed{\text{Cauchy-Schwartz inequality}}$

In any product space (V, φ) $|\varphi(v,w)| \leq \sqrt{\varphi(v,v)} \cdot \sqrt{\varphi(w,w)}$

$0 \leq \varphi(tu+w, tu+w) = \varphi(tu,tu) + 2\mathrm{Re}\,\varphi(tu,w) + \varphi(w,w)$

for $t \in \mathbb{R}$ $= t^2 \varphi(u,u) + 2t\,\mathrm{Re}\,\varphi(u,w) + \varphi(w,w)$

Hence no real roots except 0 ⇒ $\Delta < 0$

So $(2\mathrm{Re}\,\varphi(u,w))^2 - 4\varphi(u,u)\varphi(w,w) \leq 0$

So $\mathrm{Re}\,\varphi(u,w) \leq \sqrt{\varphi(u,u)}\sqrt{\varphi(w,w)}$ ☐ But we're not done...

know, for $|\alpha| = 1$,

$\mathrm{Re}\left(\varphi(\alpha u, w)\right) \leq |\alpha| \sqrt{\varphi(u,u)}\sqrt{\varphi(w,w)}$

$\mathrm{Re}\left(\alpha \underbrace{\varphi(u,w)}_{re^{i\theta}}\right) \leq \sqrt{\varphi(u,u)}\sqrt{\varphi(w,w)}$

for $\alpha = e^{-i\theta}$, I get

$\mathrm{Re}(r) \leq \sqrt{\varphi(u,u)}\sqrt{\varphi(w,w)}$, but $r = |\varphi(u,w)|$ real

So $\mathrm{Re}\,|\varphi(u,w)| \leq \sqrt{\varphi(u,u)}\sqrt{\varphi(w,w)}$

So $|\varphi(u,w)| \leq \sqrt{\varphi(u,u)}\sqrt{\varphi(w,w)}$ ■

▶ for $V = C[0,1]$
we can def inner product of 2 fns $\varphi(f,g) = \int_0^1 f \cdot \bar{g}$

April 11, 2019

ORTHOGONALITY

Recall : Inner product spaces $(V, \langle , \rangle)$

"Def" ORTHOGONAL $\Leftrightarrow$ $\langle , \rangle = 0$

☒ For any $S \subset V$, Let $S^\perp = \{ v \in V \mid \langle v, s \rangle = 0 \text{ for every } s \in S \}$

1. Observe that • $\theta_V \in S^\perp$
2. • $V^\perp = \{ v \in V \mid \langle v, w \rangle \overset{0}{=} \text{ for all } w \in V \} \ni \theta_V$

Suppose $v_0 \in V^\perp$, then $\langle v_0, v_0 \rangle = 0$, thus $v_0 = \theta$

3. • $(V \setminus \{\theta_V\})^\perp = \dots = \{\theta_V\}$
4. • $(V \setminus \{v_0\})^\perp = \dots = \{\theta_V\}$
5. • $(V \setminus \{v_0, v_1 \dots v_n\})^\perp = \dots = \{\theta_V\}$

countably many

6. For any $S \subset V$: $S^\perp \triangleleft V$ Pf $\begin{cases} \theta_V \in S^\perp \\ \text{partial lin. in } 1^{st} \text{ slot of } \langle v, w \rangle \end{cases}$

7. Thm $\boxed{\text{If } W \triangleleft V \text{ then } W \oplus W^\perp = V} \to$ f.d only...

☒ If we know $W + W^\perp = V$ then the sum is direct, since $W \cap W^\perp = \{\theta\}$

☒ finite-dim Let $w_1, w_2, \dots, w_k$ be a basis for W and let $v \in V$

To show : $\exists\, \alpha_1 \dots \alpha_k \in \mathbb{C}$ s.t. $\alpha_1 w_1 + \dots + \alpha_k w_k + y = v$ for some $y \in W^\perp$

i.e. $\exists\, \alpha_1 \dots \alpha_k \in \mathbb{C}$ such that $v - \alpha_1 w_1 - \dots - \alpha_k w_k \in W^\perp$

i.e. $\exists\, \alpha_1 \dots \alpha_k \in \mathbb{C}$ s.t.

$$\langle v - \alpha_1 w_1 - \dots - \alpha_k w_k, w_1 \rangle = 0$$
$$\langle v - \alpha_1 w_1 - \dots - \alpha_k w_k, w_2 \rangle = 0$$
$$\vdots$$

So $\langle v - \alpha_1 w_1 - \dots - \alpha_k w_k, w_j \rangle = 0$

So $\langle v, w_j \rangle = \alpha_1 \langle w_1, w_j \rangle + \alpha_2 \langle w_2, w_j \rangle + \dots + \alpha_k \langle w_k, w_j \rangle$

i.e.

GRAMIAN MATRIX $\rightsquigarrow$ $$\begin{bmatrix} \langle w_1, w_1 \rangle & \langle w_2, w_1 \rangle & \cdots & \langle w_k, w_1 \rangle \\ \vdots & & & \\ \langle w_1, w_k \rangle & & & \langle w_k, w_k \rangle \end{bmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{bmatrix} \langle v, w_1 \rangle \\ \vdots \\ \langle v, w_k \rangle \end{bmatrix}$$

We shall show it is invertible. But since it's square, we can just show it is injective.

Suppose $G(\gamma) = 0$ then

$$\sum_i \gamma_i \langle w_i, w_1 \rangle = \sum_i \gamma_i \langle w_i, w_2 \rangle = \dots = \sum_i \gamma_i \langle w_i, w_k \rangle = 0$$

So $\sum_i \gamma_i w_i$ is $\perp$ $w_1, \dots, w_k$

In particular, $\sum_i \gamma_i w_i \perp \sum_i \gamma_i w_i$, i.e. $\sum \gamma_i w_i = 0$

So $\gamma_i = 0 \ \forall i$ (since w_i form a basis)

So, the Gramian is injective, hence it is invertible. ✓ ☐

→ Note that in infinite dim → this theorem does not hold

Ex ↳ Let $V = \mathbb{C}^{\mathbb{N}} : \{ f : \mathbb{N} \to \mathbb{C} \mid f = \text{fnct} \}$ ← sequences of complex numbers

$$= \boxed{\begin{matrix} 1 & 2 & 3 & \cdots \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots \end{matrix}} \quad \alpha_i \in \mathbb{C}$$

We claim that V is a vector space (this is easy to check)

ℓ^2

Let $V_2 < V$ be defined by

$(\alpha_1, \ldots \alpha \ldots) \in V_2 \Leftrightarrow \sum_1^\infty |\alpha_i|^2 < \infty$

Define $\langle (\alpha_1 \ldots\ldots), (\beta_1 \ldots) \rangle = \alpha_1 \overline{\beta_1} + \alpha_2 \overline{\beta_2} + \ldots$

Now, we show that this sum doesn't diverges

Consider $|\alpha_1 \overline{\beta_1}| + |\alpha_2 \overline{\beta_2}| + \ldots$ i.e. $|\alpha_1| \cdot |\overline{\beta_1}| + |\alpha_2||\overline{\beta_2}| + \ldots$

Consider partial sums... $|\alpha_1| \cdot |\beta_1| + \ldots + |\alpha_{34}||\beta_{34}|$

$$= \left\langle \begin{pmatrix} |\alpha_1| \\ \vdots \\ |\alpha_{34}| \end{pmatrix}, \begin{pmatrix} |\beta_1| \\ \vdots \\ |\beta_{34}| \end{pmatrix} \right\rangle \leq \sqrt{\underbrace{\left\| \begin{pmatrix} |\alpha_1|^2 \\ \vdots \\ |\alpha_{34}|^2 \end{pmatrix} \right\|}_{A} \cdot \underbrace{\left\| \begin{pmatrix} |\beta_1|^2 \\ \vdots \\ |\beta_{34}|^2 \end{pmatrix} \right\|}_{B}}$$

$A \leq \sum^\infty |\alpha_i|^2$, $B \leq \sum^\infty |\beta_i|^2$

But $\sum^\infty |\alpha_i|^2 < \infty$, so both A, B converges.

→ so any inner product converges...

Therefore the inner product is well defined...

Back to proof. Let $W \subseteq \ell^2$ be defined by

W the set of all terminating sequences in ℓ^2

So (we can check) $W < \ell^2$, and $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \in W$, and

$\begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \in W$, $\begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \in W$ ...

$W + W^\perp = W$

So $\begin{pmatrix} \alpha_1 \\ \vdots \end{pmatrix} \in W^\perp \Rightarrow \alpha_j = 0 \ \forall j \Rightarrow W^\perp = \{0\}$

Yet $\begin{pmatrix} 1 \\ 1/2 \\ 1/4 \\ \vdots \end{pmatrix} \in \ell^2, \notin W$ So $\boxed{W + W^\perp \neq \ell^2}$

(read Axler)

April 16, 2019 **Recall**, $W < V$, V finite dimensional inner product space...
$W \oplus W^{\perp} = V$

Observation $\boxed{\text{Orthonormal list in inner product space is linearly independent.}}$

Suppose $\alpha_1 V_1 + \dots + \alpha_k V_k = 0$
where $V_1 \dots V_k$ mutually orthogonal unit vectors

Then $0 = \langle 0, V_i \rangle$

$= \alpha_i \langle V_i, V_i \rangle + 0 = \alpha_i$

So all V_i's are linearly independent.

Thm Every finite dimensional inner product space (nontrivial) has an orthonormal basis, and in fact any ~~list~~ orthonormal list can be enlarged to one.

Riesz Representation Theorem

Suppose V is fdips and $\underbrace{\rho : V \xrightarrow{\text{lin}} \mathbb{C}}_{\text{a linear functional on } V}$.
Then there exists ~~only~~ ~~one~~ exactly one $w_0 \in V$ s.t.

$\rho(v) = \langle v, w_0 \rangle$, all v, i.e. $\rho() = \langle (), w_0 \rangle$

Proof $\underbrace{\text{Rank}(\rho)}_{0 \text{ or } 1} + \text{Nullity}(\rho) = \dim(V)$

If rank $(\rho) = 0$, then $\rho \equiv 0$, so $w_0 = 0$ (unique) (easy)
If rank $(\rho) = 1$
then $\ker(\rho) \oplus \ker(\rho)^{\perp} = V$

then $\ker(\rho)^{\perp}$ has dimension one. Thus $\ker(\rho)^{\perp} = \text{span}(z_0)$, z_0 unit vector...

then $\rho(z_0) \neq 0 \Rightarrow \frac{1}{\rho(z_0)} z_0 \in \text{span}(z_0)$

and $\rho\Big(\underbrace{\frac{1}{\rho(z_0)} z_0}_{z_1}\Big) = \frac{1}{\rho(z_0)} \cdot \rho(z_0) = 1$

And so, $\text{span}(z_0) = \text{span}(z_1)$

Thus, any $v \in V$, $v = \underset{\ker(\rho)}{\underset{\cap}{w}} + \underset{\ker(\rho)^{\perp}}{\underset{\cap}{\alpha z_1}}$ (uniquely)

So $\rho(v) = \rho(w + \alpha z_1) = 0 + \alpha \rho(z_1) = \alpha \overset{?}{=} \langle v, ?? \rangle$

So now, $\langle w + \alpha z_1, \boxed{??} \rangle$ how to get α?

$= \langle w + \alpha z_1, ?? \rangle$

$= \langle w, ? z_1 \rangle + \alpha \langle z_1, ? z_1 \rangle$

0 So, $? = \frac{1}{\|z_1\|^2}$

And then, $\boxed{??} = \frac{1}{\|z_1\|^2} z_1$

So, Let $w_0 = \frac{z_1}{\|z_1\|^2}$

$w + \alpha z_1$

then $\rho(v) = \rho(w + \alpha z_1) = \langle v, \frac{z_1}{\|z_1\|^2} \rangle = (\alpha) = \langle v, w_0 \rangle$

for all v. //

Recap, if $v = w + \alpha z_1$, then $w_0 = \frac{z_1}{\|z_1\|^2}$ $\forall v$

Uniqueness Assume $\rho() = \langle (), w_0 \rangle = \langle (), w_1 \rangle$ then $0 = \langle (), w_0 - w_1 \rangle \Leftrightarrow w_0 = w_1$. //

$\boxed{\text{Consequences?}}$

① Suppose $\mathcal{L} : V \xrightarrow[\text{lin}]{\text{f dim's}} V$. Consider

$f() := \langle \mathcal{L}(), y_0 \rangle$, $f : V \xmapsto{\text{lin}} \mathbb{C}$. So, f must be like

$f() = \langle (), w_0 \rangle$ for some $\boxed{\text{UNIQUE}}$ $w_0 \in V$, w_0 depends on y_0 and $\mathcal{L}$.

So let us write $w_0 = w_{\mathcal{L}}(y_0)$

Consider $w_{\mathcal{L}} : V \longmapsto V$

Now $f() = \langle (), w_0 \rangle = \langle \mathcal{L}(), y_0 \rangle$

$\langle x, \alpha w_{\mathcal{L}}(y_0) \rangle = \bar{\alpha} \langle x, w_{\mathcal{L}}(y_0) \rangle$

So $\langle x, w_{\mathcal{L}}(\alpha y_0) \rangle = \langle \mathcal{L}x, \alpha y_0 \rangle = \bar{\alpha} \langle \mathcal{L}x, y_0 \rangle = $

So $\boxed{w_{\mathcal{L}}(\alpha y_0) = \bar{\alpha} w_{\mathcal{L}}(y_0)}$ (~~conjugate~~ linear)

- what about additivity?

$\langle x, w_{\mathcal{L}}(y_1 + y_2) \rangle = \langle \mathcal{L}(x), y_1 + y_2 \rangle = \langle x, w_{\mathcal{L}}(y_1) \rangle + \langle x, w_{\mathcal{L}}(y_2) \rangle$

$\langle x, y \rangle - \langle x, z \rangle = 0$

$\langle x, y - z \rangle = 0 \ \forall x$

$\Rightarrow y - z = 0$

Thus, we can show $w_{\mathcal{L}}(y_1 + y_2) = w_{\mathcal{L}}(y_1) + w_{\mathcal{L}}(y_2)$ by subtracting ...

And so, additivity checks ...

So $W_{\mathcal{L}} : V \xmapsto{\text{lin}} V$ has the property

$$\langle \mathcal{L}(), y \rangle = \langle (), W_{\mathcal{L}}(y) \rangle$$

$W_{\mathcal{L}}$ is the adjoint of $\mathcal{L}$, denoted $\mathcal{L}^*$ □

Note $\mathcal{L}^*$ depends on the inner product, but not on the basis.

☒ Now, $\underset{\text{f dim's}}{V} \xrightarrow{L} \underset{\text{f dim's}}{U}$

Let Γ be orthonorm basis of V $\quad v_1 \dots v_n$

Π be orthnorm basis of U $\quad u_1 \dots u_m$

then

$$\begin{tikzcd}[column sep=huge, row sep=huge] V \arrow[r, shift left, "L"] \arrow[d, shift left, "{[\ ]_\Gamma}"] & U \arrow[l, shift left, "L^*"] \arrow[d, shift right, "{[\ ]_\Pi}"'] \\ \mathbb{C}^n \arrow[u, shift left, "A_\Gamma"] \arrow[r, shift left, "{[L]_{\Pi\leftarrow\Gamma}}"] & \mathbb{C}^m \arrow[u, shift right, "A_\Pi"'] \arrow[l, shift left, "{[L^*]_{\Gamma\leftarrow\Pi}}"] \end{tikzcd}$$

What is the relationship between $[L]$ & $[L^*]$? Claim:

$$\boxed{\underbrace{[L^*]_{\Gamma\leftarrow\Pi}}_{n\times m} = \left(\overline{\underbrace{[L]_{\Pi\leftarrow\Gamma}}_{m\times n}}\right)^T}$$

Let write this $A = (\overline{B})^T$ $\quad$ do this entry-wise ...

$$A_{ij} = A[\![i,j]\!] = A(e_j)\cdot e_i = A(e_j)\,\bar{\cdot}\, e_i$$

(★) Observation $\alpha_1 V_1 + \dots + \alpha_n V_n$, V_i orthonormal

then $\langle \alpha_1 V_1 + \dots + \alpha_n V_n, \beta_1 V_1 + \dots + \beta_n V_n \rangle$

$$= \alpha_1\bar{\beta_1} + \dots + \alpha_n\bar{\beta_n} = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} \cdot \begin{pmatrix}\bar{\beta_1}\\ \vdots \\ \bar{\beta_n}\end{pmatrix} = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix} \,\bar{\cdot}\, \begin{pmatrix}\beta_1\\ \vdots \\ \beta_n\end{pmatrix}$$

standard inner product

↳ like dot product!

April 18, 2019

Def^n $L : V \xrightarrow{lin} V$ f.dips is normal if $LL^* = L^*L$; $L^* \leftrightarrow L$

is self-adjoint if $L = L^*$; $L^* = L$

is unitary if $LL^* = L^*L = I$ i.e. $L^* = L^{-1}$

NORMAL OPERATOR | SELF-ADJOINT & UNITARY

Peak into Future

(1) L is normal $\Leftrightarrow$ there is an ortho-basis Γ of V such that $[L]_{\Gamma\leftarrow\Gamma}$ is diagonal

① SPECTRAL THEOREM $\Leftrightarrow$ there is an ortho-eigenbasis of L for V (eigenvect

② $L = L^* \Leftrightarrow L$ is normal & $\sigma_{\mathbb{C}}(L) \subset \mathbb{R}$

③ $L^* = L^{-1} \Leftrightarrow L$ is normal & $\sigma_{\mathbb{C}}(L) \subset \mathbb{T} = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$

(all eigenvalues have modulus 1)

☆ **Schur's Thm** For each $L : V \xrightarrow{lin} V$ fdips there is an ortho-basis Γ of V s.t $[L]_{\Gamma\leftarrow\Gamma}$ = upper-triangular

pf 1-dim ✓

2-dim : There is a unit ~~eigenvector~~ eigenpair (λ, v_0) for L. (unit vector)

Let $W = \mathrm{Span}(v_0) \rightsquigarrow$ 1 dimensional

dim = 1

Then $V = W \oplus W^\perp$. Let v_1 be a unit vector in $W^\perp$

Then $\Gamma = (v_0, v_1)$ is an orth-basis of V

then

$$[L]_{\Gamma\leftarrow\Gamma} = \begin{array}{c c} & v_0 \\ v_0 & \begin{pmatrix} \lambda & m \\ 0 & m \end{pmatrix} \end{array} \rightsquigarrow \text{upper triangular...}$$

3-dim

$$[L]_{\Gamma\leftarrow\Gamma} = \left[\begin{array}{c|c} \lambda & \\ \hline 0 & \\ 0 & M \end{array}\right] \to M = \mathcal{E}_{W^\perp} \circ L \Big|_{W^\perp}$$

By 2-dim, there is an basis v_1, v_2 of $W^\perp$ such that (φ, ortho)

$$[M]_{\varphi\leftarrow\varphi} = \left[\begin{array}{c|c} ? & ? \\ \hline 0 & ? \end{array}\right]$$

Thus, let $\Gamma = (v_0, v_1, v_2)$. then Γ = ortho. basis of L

Thus $[L]_{\Gamma\leftarrow\Gamma} = \begin{bmatrix} [\lambda I] & [u] \\ 0 & [M] \end{bmatrix} = \begin{bmatrix} \lambda & \cdots \\ 0 & 0 & \ddots \end{bmatrix} \rightsquigarrow$ upper triangular.

Towards the Spectral Theorem

① Suppose $M \in M_{m\times m}$, and $M = \begin{array}{c|cc|c} & k & & \\ k & A & B & m-k \\ m-k & C & D & k \end{array}$ then $M^* = (\overline{M})^T = \begin{pmatrix} \overline{A} & \overline{B} \\ \overline{C} & \overline{D} \end{pmatrix}^T = \begin{pmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & \overline{D}^T \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}$

② Suppose M is normal, then $M \leftrightarrow M^*$ then

$$MM^* = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} AA^* + BB^* & \cdots \\ \cdots & \cdots \end{pmatrix}$$

$$\|$$

$$M^*M = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A^*A + C^*C & \cdots \\ \cdots & \cdots \end{pmatrix}$$

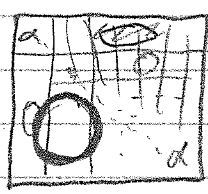
So $AA^* + BB^* = A^*A + C^*C$

So $\mathrm{Trace}(AA^* + BB^*) = \mathrm{Trace}(A^*A + C^*C)$

So $\cancel{\mathrm{Trace}(AA^*)} + \mathrm{Trace}(BB^*) = \cancel{\mathrm{Trace}(A^*A)} + \mathrm{Trace}(C^*C)$

Note $\boxed{\mathrm{Trace}(AA^*) = 0 \iff A = 0}$ Note $\mathrm{Tr}(BB^*) = \sum |b_{ij}|^2$

So

③ Suppose  is normal, then  is diagonal ...

 → every normal has a representation that is diagonal ...

④ Converse?