$$K(x_N,t_N;\ x_0,t_0) = \int \prod_{k=1}^{N-1} dx_k \langle x_N | \mathcal{U}(t_N,t_{N-1}) | x_{N-1}\rangle \langle x_{N-1} | \mathcal{U}(t_{N-1},t_{N-2}) | x_{N-2}\rangle \cdots \langle x_1 | \mathcal{U}(t_1,t_0) | x_0\rangle$$

$$= \int \prod_{k=1}^{N-1} dx_k \langle x_N | e^{-\frac{i\varepsilon}{\hbar}H} | x_{N-1}\rangle\langle x_{N-1} | e^{-\frac{i\varepsilon}{\hbar}H} | x_{N-2}\rangle \cdots \langle x_1 | e^{-\frac{i\varepsilon}{\hbar}H} | x_0\rangle$$

$[\varepsilon = \Delta t]$

Note: easy to include t-dependent H, time ordering works out automatically but will ignore for clarity.

write

$$\langle x_k | e^{-\frac{i\varepsilon}{\hbar}H(p,x)} | x_{k-1}\rangle = \int dp_k \langle x_k | p_k\rangle\langle p_k | e^{-\frac{i\varepsilon}{\hbar}H(p,x)} | x_{k-1}\rangle$$

Introduce notation: <u>Normal ordering.</u>

$\mathcal{O}(p,x)$ is a normal-ordered operator if p's on left, x's on right.
[often use NO notation for a^+, a's]

Ex. $H = \frac{p^2}{2m} + V(x)$ is normal ordered.

Write $:\mathcal{O}(p,x):$ for normal-ordered form of $\mathcal{O}$.

Ex. $xp = px + i\hbar = :xp: + i\hbar$

[normal ordering introduces commutators]

Ex: $H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A}(\vec{x}))^2$ [particle in EM field]

$$H = \frac{1}{2m}\left(\vec{p}^2 - \frac{e}{c}[\vec{p}\cdot\vec{A}(x) + \vec{A}(x)\cdot\vec{p}] + \frac{e^2}{c^2}\vec{A}^2(x)\right)$$

$$= :H: - \frac{ie\hbar}{2mc}\vec{\nabla}\cdot\vec{A}(x)$$

would like $e^{-\frac{i\varepsilon}{\hbar}H(p,x)}$ to be normal ordered.

For $H = \frac{p^2}{2m} + V(x)$,

$$e^{-\frac{i\varepsilon}{\hbar}H(p,x)} = 1 - \frac{i\varepsilon}{\hbar}\left[\frac{p^2}{2m} + V(x)\right] - \frac{\varepsilon^2}{2\hbar^2}\left[\left(\frac{p^2}{2m}\right)^2 + \frac{p^2}{2m}V(x) + V(x)\frac{p^2}{2m} + V(x)^2\right] + \dots$$

$$= :e^{-\frac{i\varepsilon}{\hbar}H(p,x)}: \underbrace{- \frac{\varepsilon^2}{2\hbar^2}\left[V(x), \frac{p^2}{2m}\right]}_{-\frac{\varepsilon^2}{4m}\left[2\frac{i}{\hbar}V'(x)p - V''(x)\right]} + \dots$$

Generally, if $H(p,x)$ is normal-ordered,

$$e^{-\frac{i\varepsilon}{\hbar}H(p,x)} = :e^{-\frac{i\varepsilon}{\hbar}H(p,x)}: + \mathcal{O}(\varepsilon^2).$$

as $\Delta t \to 0$, replace $e^{-\frac{i\varepsilon}{\hbar}H(p,x)} \longrightarrow :e^{-\frac{i\varepsilon}{\hbar}H(p,x)}:$

So
$$\int dp_k \langle x_k | p_k \rangle \langle p_k | e^{-\frac{i\varepsilon}{\hbar}H(p,x)} | x_{k-1} \rangle$$

becomes

$$\int dp_k \left(\frac{1}{2\pi\hbar}\right) e^{\frac{i}{\hbar}p_k(x_k - x_{k-1}) - \frac{i\varepsilon}{\hbar}H(p_k, x_{k-1})} + \mathcal{O}(\varepsilon^2)$$

so
$$K(x_N, t_N; x_0, t_0) \approx \int \left(\prod_{k=1}^{N-1} dx_k \prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\right) e^{\sum_{k=1}^{N}\left[\frac{i}{\hbar}p_k(x_k - x_{k-1}) - \frac{i\varepsilon}{\hbar}H(p_k, x_{k-1})\right]}$$

Replacing
$$\frac{x_k - x_{k-1}}{\varepsilon} \longrightarrow \dot{x}$$
$$\sum_k \varepsilon f_k \longrightarrow \int dt\, f(t)$$
$$\left(\prod_{k=1}^{N-1} dx_k\right)\left(\prod_{k=1}^{N} \frac{dp_k}{2\pi\hbar}\right) \longrightarrow \mathcal{D}[x(t)]\,\mathcal{D}[p(t)]$$

[Functional measure defined by limit]

gives phase space form of path integral:

$$\boxed{K(x_N, t_N; x_0, t_0) = \int \mathcal{D}[x(t)]\mathcal{D}[p(t)]\, e^{\frac{i}{\hbar}\int dt\,[p(t)\dot{x}(t) - H(p(t), x(t))]}}$$

<u>Lagrangian form of PI</u>

say $H = \frac{p^2}{2m} + V(x)$

$$\frac{1}{2\pi\hbar}\int dp_k\, e^{\frac{i}{\hbar}p_k(x_k - x_{k-1}) - \frac{i\varepsilon}{\hbar}H(p_k, x_{k-1})}$$
$$= \frac{1}{2\pi\hbar}\int dp_k\, e^{\frac{-i\varepsilon}{2m\hbar}\left[\left(p_k - \frac{m}{\varepsilon}(x_k - x_{k-1})\right)^2 - \frac{m^2}{\varepsilon^2}(x_k - x_{k-1})^2\right] - \frac{i\varepsilon}{\hbar}V}$$

$\int e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$

$$= \sqrt{\frac{m}{2\pi i\hbar\varepsilon}}\; e^{+\frac{im}{2\hbar\varepsilon}(x_k - x_{k-1})^2 - \frac{i\varepsilon}{\hbar}V(x)_{k-1}}$$

so

$$K(x,t;x_0,t_0) \approx \underbrace{\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{N/2}\int\prod_{k=1}^{N-1} dx_k}_{\mathcal{D}[x(t)]}\; e^{\sum_{k=1}^{N} -\frac{i\varepsilon}{\hbar}V(x)_{k-1} + \frac{im}{2\hbar\varepsilon}(x_k - x_{k-1})^2}$$

$$K(x,t;x_0,t_0) = \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}\int dt\left[\frac{1}{2}m\dot{x}(t)^2 - V(x)\right]}$$

$$= \int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}\int dt\, \mathcal{L}(x(t),\dot{x}(t))}$$

$$= \boxed{\int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar}S[x(t)]}}\,.$$

Check formalism: Calculate free particle prop explicitly

Choose $N = 2^A$ for simplicity

Calc.

$$K_N = \left(\frac{mN}{2\pi i\hbar t}\right)^{N/2} \int \prod_{k=1}^{N-1} dx_k\, e^{\sum_k \frac{imN}{2\hbar t}(x_k - x_{k-1})^2}$$

Exponent is $\frac{imN}{2\hbar t}\Big[x_0^2 + 2x_1^2 + 2x_2^2 + \dots + 2x_{N-1}^2 + x_N^2$
$-2x_0x_1 - 2x_1x_2 \cdots - 2x_{N-1}x_N\Big]$

Do odd integrals first

$$\int \prod_{n=1}^{\frac{N}{2}} dx_{2n}$$

$$K_N = \prod_{\substack{k\ \mathrm{odd}\\ k=1}}^{N-1}\left[\left(\frac{m2^A}{2\pi i\hbar t}\right)\int_{(k\ \mathrm{odd})} dx_k\, e^{\frac{im2^A}{2\hbar t}\underbrace{(2x_k^2 + x_{k-1}^2 + x_{k+1}^2 - 2x_k(x_{k-1}+x_{k+1}))}_{\left[2\{x_k - \frac{1}{2}(x_{k-1}+x_{k+1})\}^2 + \frac{1}{2}x_{k-1}^2 + \frac{1}{2}x_{k+1}^2 - x_{k-1}x_{k+1}\right]}}\right.$$

$$= \left(\frac{m2^{A-1}}{2\pi i\hbar t}\right)^{2^{A-2}} \int \prod_{n=1}^{2^{A-1}-1} dx_{2n}\, e^{\sum_n^{2^{A-1}} \frac{im2^{A-1}}{2\hbar t}(x_{2n} - x_{2n-2})^2}$$

$$= K_{N/2}\,.$$

So by induction,

$$K_N = K_1 = \sqrt{\frac{m}{2\pi i \hbar t}}\, e^{\frac{im}{2\hbar t}(X_N - X_0)^2}$$

$$= K(X_N, t; X_0, 0) \quad \text{as promised [Exactly].}$$

Feynman path integral approach:

- Alternative formulation of quantum theories.
- Requires "classical" action $S[x(t)]$ as starting point.
- Requires definition of measure $\mathcal{D}[x(t)]$
- Not practical for most QM calculations
- Highly useful in formulating quantum field theory ("Feynman diagrams")
- Avoids conceptual problems of Hamiltonian formalism of QM.

→ Not a "realist" approach: no $|\psi(t)\rangle$, can replace by correlation function $\int e^{iS/\hbar}\, \theta(t_2)\theta(t$

→ Thus, no collapse of wavefunction.

<u>Stationary phase</u>

Given a function $g(x)$, so $\frac{dg}{dx}(x_c) = 0$ at a unique $x = x_c$.

consider $\int dx\, e^{\frac{i}{\varepsilon} g(x)}$, ε small.

$$g(x) = g(x_c) + \frac{1}{2} g''(x_c)(x - x_c)^2 + \frac{1}{6} g'''(x_c)(x - x_c)^3 + \dots$$

$$\int dx\, e^{\frac{i}{\varepsilon} g(x)} = e^{\frac{i}{\varepsilon} g(x_c)} \sqrt{\frac{2\pi i \varepsilon}{g''(x_c)}} \left[1 + O(\varepsilon^2) \right]$$

Integral dominated by part near x_c.

Similarly, $\int \mathcal{D}[x(t)]\, e^{\frac{i}{\hbar} S[x(t)]}$

dominated by x_{class} where $\frac{\delta S}{\delta x}[x_{class}] = 0$.

$$\text{so} \quad K(x,t;x_0,t_0) \cong e^{iS[x_{class}(t)]/\hbar}$$

$$\cong e^{iS(x,t;x_0,t_0)/\hbar}$$

For free particle, $S(x,t;x_0,t_0) = \frac{m(x-x_0)^2}{2(t-t_0)}$,

so this is exactly right.

2.4 Quantum particles in potentials and EM fields

Potentials

In Classical & Quantum mech. shifting potential by overall constant

$$V \to V + V_0$$

has no effect on measurable quantities

Classical: EOM all involve derivatives of V

Newtonian: $F = m - \nabla V$
Hamiltonian: $\{p, V(x)\}$ involves $\partial V / \partial x$
Lagrangian: S shifts by $\int V(t)\,dt$, no effect on δS.

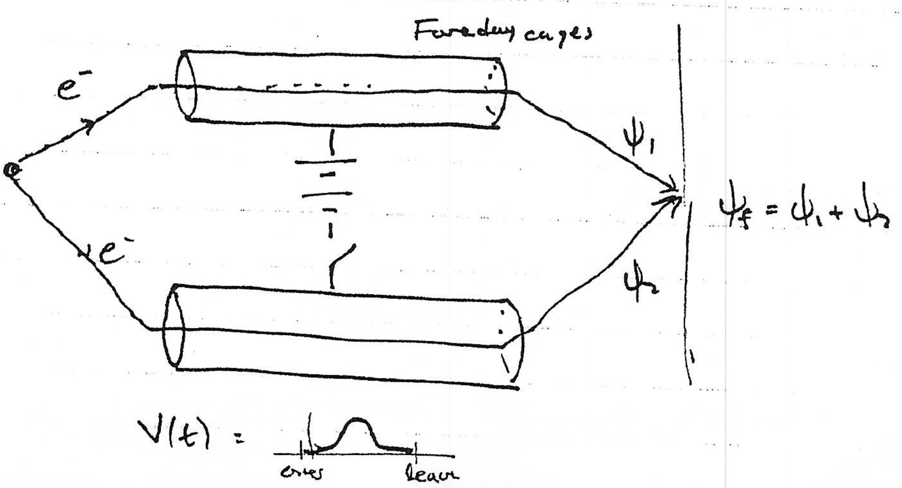
Quantum: $H \to H + V_0 \Rightarrow$

$$|\psi(t)\rangle \Rightarrow e^{-\frac{i}{\hbar}\int V_0 dt} |\psi(t)\rangle$$

Overall phase $e^{i\theta}$ not observable.

since in expansion $|\psi\rangle = \sum C_a |a\rangle$,
just changes $C_a \to e^{i\theta} C_a$.

Changing potential in one region is observable



without V, by superposition

$$\psi_f = \psi_1 + \psi_2$$

with U, $$\psi_f = \left(e^{-\frac{i}{\hbar}\int V(t)\,dt}\,\psi_1 + \psi_2\right)$$

gives phase difference, changes interference pattern.

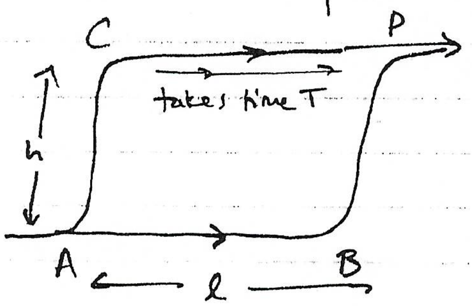
No effect in classical limit $\hbar \to 0$.

Note: no fields introduced in region with particles (!)
[this is a variation of Aharanov - Bohm]

<u>Example</u>: Gravity induced quantum interference

- No quantum theory of gravity
- Hard to see quantum effects where gravity is relevant
(Gravity $\sim 10^{-39}$x as strong as EM forces)
($e^- N$ vs. $e^- p^+$)

Possible to see quantum effect through phase difference

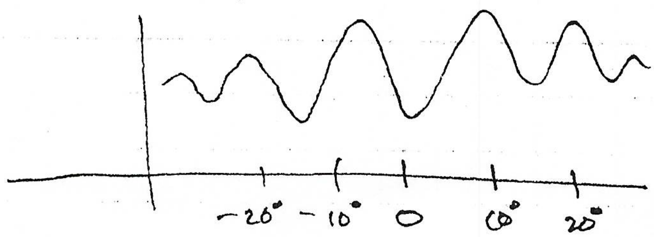


paths ABD vs. ACD:

$\delta V = mgh$

Phase difference: $e^{\frac{i}{\hbar} mghT}$

Interference seen using neutrons following loops rotated @ angle δ from horizontal



Colella, Overhauser, Werner
1975

Particles in EM fields

Recall Electromagnetism:

Fields $\quad F_{\mu\nu} = \partial_{[\mu} A_{\nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu .$

A_μ is 4-vector potential, $A_\mu = (-\phi, \vec{A})$

$(x^\mu = (ct, \vec{x}))$

$$F_{0i} = -F_{i0} = -E_i \qquad (i = 1,2,3)$$
$$F_{ij} = \varepsilon_{ijk} B^k$$

(Einstein summation convention)

or
$$E_i = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \frac{\partial \phi}{\partial x^i}$$
$$B^i = \varepsilon^{ijk} \partial_j A_k$$

Gauge invariance: $A_\mu \to A_\mu + \partial_\mu \Lambda$
leaves $F_{\mu\nu}$, thence E & B unchanged.

Lagrangian for charged particle in EM field is

Relativistic: $$S = -m \int d\tau \sqrt{\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}} + \underbrace{\frac{e}{c}\int d\tau\, A_\mu\, dx^\mu/d\tau}$$

note: δ is total derivative under $\delta A_\mu = \partial_\mu \Lambda$.

Nonrelativistic: $$\mathcal{L} = \frac{m}{2}\dot{x}^2 + \frac{e}{c} A_i \dot{x}^i - e\phi$$

Going to Hamiltonian formalism:

canonical momentum $$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = m\dot{x} + \frac{e}{c} A^i$$

$$\mathcal{H} = p_i \dot{x}^i - \mathcal{L}$$
$$= \frac{m}{2}\dot{x}^2 + e\phi$$

$$\boxed{\mathcal{H} = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2 + e\phi}$$