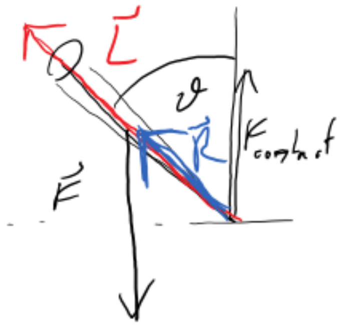
Consider spinning top: $\vec{R}$ - Center of mass coordinate.



$$\vec{\tau} = \vec{R} \times \vec{F}$$

total $F = 0$!

$$= -Mg\,\vec{R} \times \hat{z}$$

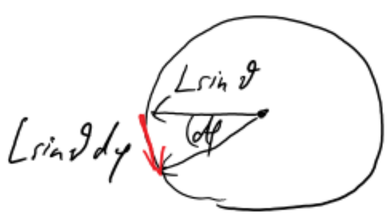
$$= -\frac{MgR}{L}\,\vec{L} \times \hat{z}$$

$$\vec{\tau} = \dot{\vec{L}} = \vec{\Omega} \times \vec{L}$$

with $\vec{\Omega} = \frac{MgR}{L}\,\hat{z}$.

Does precession depend on tipping angle?

No! Look from top:



$$dL = L\sin\vartheta\, d\varphi$$

$$\frac{dL}{dt} = L\sin\vartheta\,\dot{\varphi}$$

$$= \Omega L \cancel{\sin\vartheta}$$

$$\overset{!}{=} MgR\,\cancel{\sin\vartheta}$$

$$\Rightarrow \boxed{\Omega = \frac{MgR}{L}}$$ independent of ϑ.

Energy: $E = MgR\cos\vartheta$

$$= \frac{MgR}{L}\,\vec{L} \cdot \hat{z}$$

$$= \vec{\Omega}\cdot\vec{L}$$

So $\vec{\Omega}\cdot\vec{L}$ hamiltonian generates rotation of $\vec{L}$.

$$\dot{\vec{L}} = \vec{\Omega}\times\vec{L}$$

Magnetic moments:

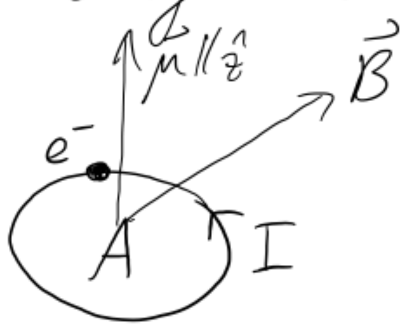
$$E = -\vec{\mu}\cdot\vec{B}$$

force $\vec{F} = -\vec{\nabla}E = 0$ for uniform fields

torque: $\vec{\tau} = \vec{\mu}\times\vec{B}$ non-zero!

For classical charge distribution:

μ



$$\vec{\mu} = I\cdot A\,\hat{z}$$

$$\left.\begin{aligned} I &= \frac{e}{T} = \frac{ev}{2\pi r} \\ A &= \pi r^2 \end{aligned}\right\} \quad \mu = \frac{ev}{2\pi r}\pi r^2 = \frac{1}{2}evr$$

$$= \frac{1}{2}\frac{e}{m} L$$

$$= \frac{1}{2}\frac{e\hbar}{m} \frac{L}{\hbar}$$

$$\mu_B = \frac{e\hbar}{2m} = 1.4 \frac{\text{MHz}}{\text{G}}$$

Lorentz force produces torque

$$\vec{\tau} = \dot{\vec{L}} = \vec{\mu} \times \vec{B}$$

We saw above that $\vec{\mu} \propto \vec{L}$ for a classical charge distribution. Let's assume that's true for quantum objects as well:

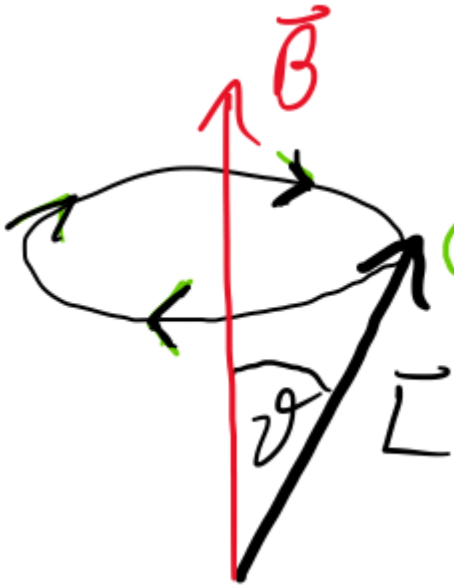
Write $\vec{\mu} = \gamma \vec{L}$

where γ - gyromagnetic ratio

So then $\boxed{\dot{\vec{L}} = \vec{L} \times \gamma \vec{B}}$

$\Rightarrow$ Precession of $\vec{L}$ (and thus $\vec{\mu}$) about Magnetic Field with angular frequency

$\Omega_L = -\gamma B$, the Larmor Frequency.



$$\odot \quad \dot{\vec L} = \vec L \times \gamma \vec B$$

Tipping angle ϑ is constant.

For electron spin angular momentum, we have

$$\gamma_e = 2\pi \cdot 2.8 \ \frac{\text{MHz}}{G}$$

For protons, we have

$$\gamma_p = 2\pi \cdot 4.2 \ \frac{\text{kHz}}{G}.$$

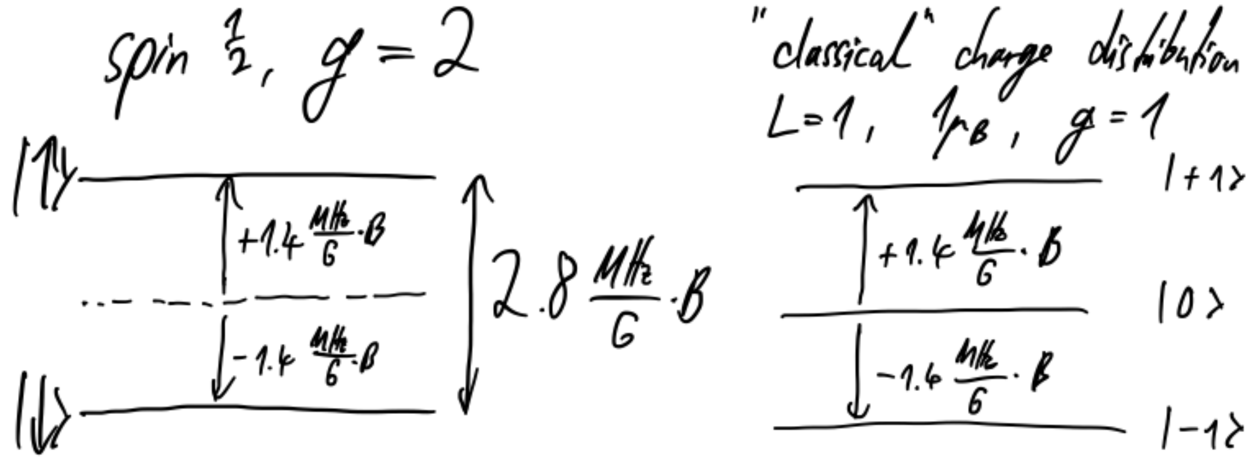
For a classical charge distribution with mass m_e, and orbital angular momentum with $L=1$, we have

$$\gamma = \frac{e\hbar}{2m} = 1.4 \ \frac{\text{MHz}}{G} \equiv \mu_B$$

Why the factor of 2 for e^- spin?

It turns out that yes, the magnetic moment of an electron is $+\mu_B$ in state $|\uparrow\rangle$ (spin up) and $-\mu_B$ in state $|\downarrow\rangle$. So then the energy difference between $|\uparrow\rangle$ and $|\downarrow\rangle$ in a magnetic field is $2\mu_B B = 2.8 \ \frac{\text{MHz}}{G} \cdot B$.

The precession frequency is the frequency at which the superposition oscillates, so:



For the classical charge distribution (i.e. standard vectors, i.e. $L=1$), there are three states with $m_L = \pm 1$ and 0. The energy of $|\pm\rangle$ is $1.4\,\frac{MHz}{G}\cdot B$, but state $|0\rangle$ stays at zero.

As one tries to tilt the magnetic moment away from say the $|-1\rangle$ state (pointing down), the superposition $\alpha|-1\rangle + \beta|0\rangle$ oscillates at $1.4\,\frac{\text{MHz}}{\text{G}} \cdot B$, so that's the precession frequency.

<u>Rotating coordinate system:</u>

If a vector rotates with $\vec{\Omega}$, then

$$\dot{\vec{A}} = \vec{\Omega} \times \vec{A}\,.$$

To relate an arbitrary rate of change of $\vec{A}$ in the inertial frame to the rate of change in a rotating frame rotating at $\vec{\Omega}$, we have

$$\left(\dot{\vec{A}}\right)_{in} = \left(\dot{\vec{A}}\right)_{rot} + \vec{\Omega} \times \vec{A}_{in}$$

(If $\vec{A}$ is constant in the rotating frame then $\dot{\vec{A}}_{in} = \vec{\Omega} \times \vec{A}_{in}$. If $\vec{\Omega} = 0$ then $\dot{\vec{A}}_{in} = \dot{\vec{A}}_{rot}$. General relationship must be linear.)

So we have the operator equation:

$$\left(\frac{d}{dt}\right)_{Rot} = \left(\frac{d}{dt}\right)_{In} - \vec{\Omega} \times$$

$$\Rightarrow \dot{\vec{L}}_{rot} = \dot{\vec{L}}_{in} - \vec{\Omega} \times \vec{L}$$

$$= \vec{L} \times \left(\gamma \vec{B} + \vec{\Omega}\right)$$

$$= \gamma \vec{L} \times \underbrace{\left(\vec{B} + \underbrace{\frac{\vec{\Omega}}{\gamma}}_{\vec{B}_{fictitious}}\right)}_{\vec{B}_{effective}}$$

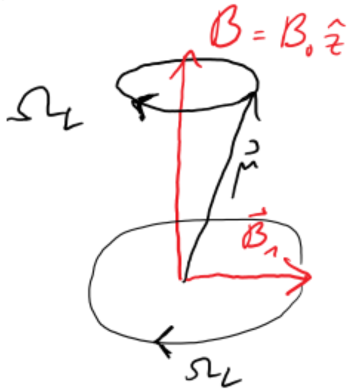
Choose $\vec{\Omega} = \vec{\Omega}_L = -\gamma \vec{B}$

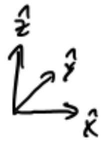
Then: $\Rightarrow \vec{L}$ is constant in the rotating frame.

Motion in a rotating magnetic field, resonance:

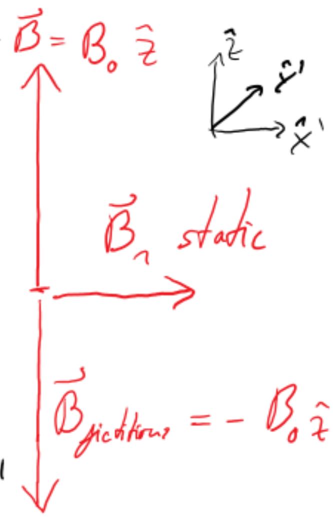
$\vec{B}_1 = B_1 \left(\cos(\Omega_L t)\, \hat{x} + \sin(\Omega_L t)\, \hat{y} \right)$

Larmor Frequency $\Omega_L = -\gamma B$





$\Longrightarrow$ rot. frame Ω_L



$\vec{\mu}$ will precess around $\vec{B}_1 = B_1 \hat{x}'$
at Rabi Frequency $\omega_R = \gamma B_1$

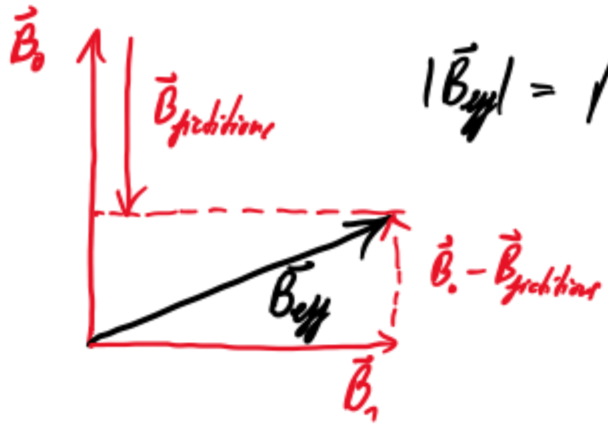
at $t = 0$: $\vec{\mu} = \mu \hat{z}$
at $t = \frac{\pi}{\omega_R}$: $\vec{\mu} = -\mu \hat{z}$ "π pulse" "spin flip"

Off-resonant rotating field

Now $\vec{B}_1 = B_1 \left(\hat{x} \cos(\omega t) - \hat{y} \sin(\omega t) \right)$ ← convenience

$$\vec{B}_{eff} = B_1 \hat{x}' + \underbrace{\left(B_0 - \frac{\omega}{\gamma} \right)}_{\neq 0} \hat{z}$$

↑ effective $\vec{B}$-field in the frame rotating at ω



$$|\vec{B}_{\text{eff}}| = \sqrt{B_1^2 + \left(B_0 - \frac{\omega}{\gamma}\right)^2}$$

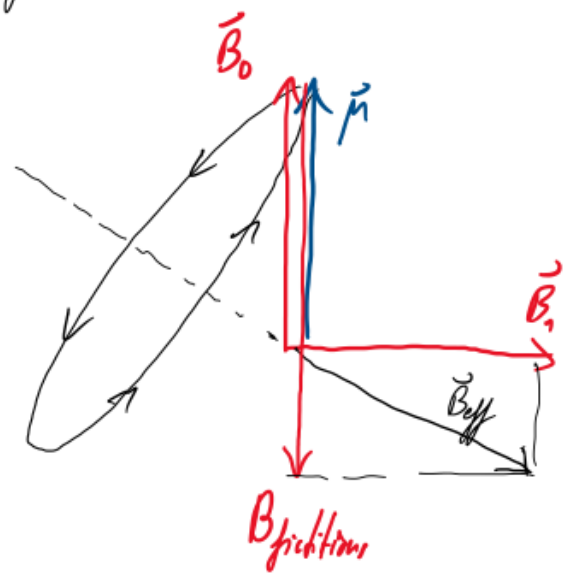
Precession Frequency

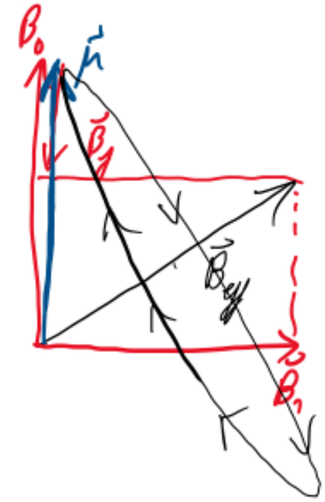
$$\Omega_R = \gamma B_{\text{eff}} = \sqrt{\omega_R^2 + \underbrace{(\Omega_L - \omega)^2}_{\delta^2}}$$

$$\Omega_R = \sqrt{\omega_R^2 + \delta^2}$$

generalized Rabi frequency

Evolution of the magnetic moment
for $\omega > \Omega_L$: $\omega < \Omega_L$:

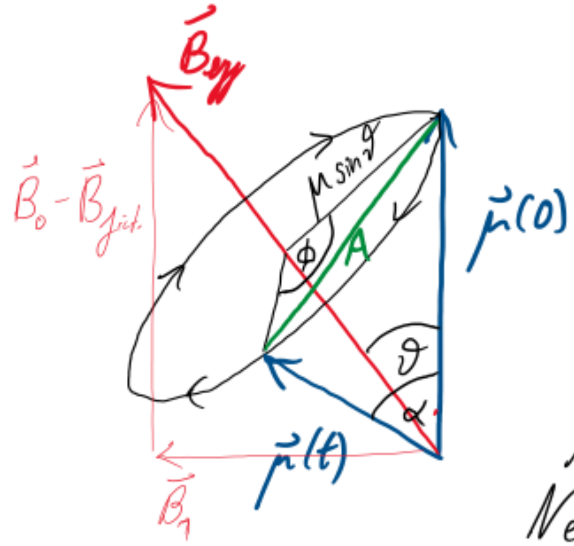




- In both cases, precession frequency $\Omega_R > \omega_R$
- In both cases, μ does never fully invert.

Time evolution:

At $t=0$, $\vec{\mu} = \mu\hat{z}$



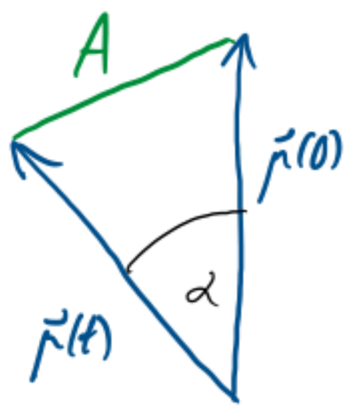
$$\phi = \Omega_R \cdot t$$

$$\mu \sin\vartheta = \mu \cdot \frac{B_1}{B_{eff}} = \mu \frac{\omega_R}{\Omega_R}$$

$$\mu_z(t) = \mu \cos\alpha$$

Need to find α.

Key is to express A in two ways:

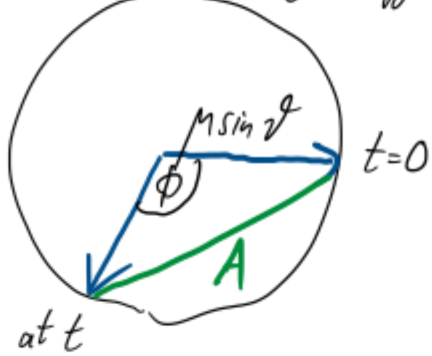


$$a^2 = b^2 + c^2 - 2bc\cos(\sphericalangle \vec{b}, \vec{c})$$

$$A^2 = 2\mu^2 (1 - \cos\alpha)$$

$$=$$

Look down along $\vec{B}_{eff}$:



$$A^2 = 2\mu^2 \sin^2\vartheta \,(1 - \cos\phi)$$

$$= 4\mu^2 \sin^2\vartheta \sin^2\frac{\phi}{2}$$

$$\Rightarrow \cos\alpha = 1 - 2\sin^2\vartheta \sin^2\frac{\Phi}{2}$$

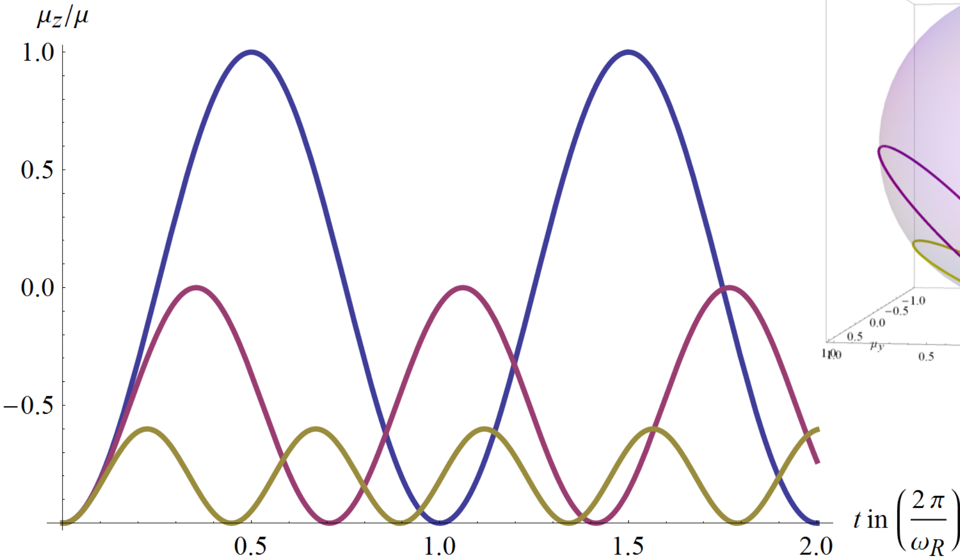
$$\Rightarrow \boxed{\mu_z(t) = \mu\left(1 - 2\frac{\omega_R^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R t}{2}\right)\right)}$$

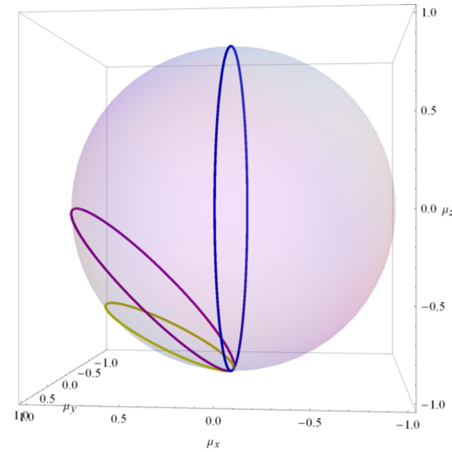
- exact result
- same as for expectation values in QM.

Full inversion only for $\omega = \Omega_L$ (then $\omega_R = \Omega_R$!)

Off-resonant case: precession is faster
inversion incomplete

Plot of Rabi oscillations:





In all these examples, we start with the magnetic moment initially pointing down, in the -z direction (then the curves also look identical to the later probability to excite an atom from the ground-state in a two-level system, with the y-axis now going from 0 to 1).

The blue curve is for a resonant drive, with $\delta = 0$. We see that after half a Rabi period, $\frac{T}{2} = \frac{\pi}{\omega_R}$, the magnetic moment got inverted (so-called π-pulse) and points up. After one Rabi period, $T = \frac{2\pi}{\omega_R}$, the magnetic moment is back to pointing in the original direction, down.

The purple curve shows the situation when the detuning is $\delta = 1\ \omega_R$. We see that the Rabi oscillations are now **incomplete** and are **faster**. We find in this special case the maximum value of the z-component of the magnetic moment to be

$$\frac{\mu_z}{\mu} = -1 + 2\frac{\omega_R^2}{\omega_R^2 + \delta^2} = -1 + 2\frac{1}{1+1} = 0.$$

The generalized Rabi frequency is here $\Omega_R = \sqrt{\omega_R^2 + \delta^2} = \sqrt{2}\,\omega_R$, so the magnetic moment will be back pointing perfectly down after only $T' = \frac{2\pi}{\Omega_R} = \frac{1}{\sqrt{2}}\frac{2\pi}{\omega_R} \approx 0.71\,T$

The yellow curve is for an even larger detuning of $\delta = 2\ \omega_R$. The oscillations are even faster, and the contrast even smaller. The maximum μ_z is now

$$\frac{\mu_z}{\mu} = -1 + 2\frac{\omega_R^2}{\omega_R^2 + \delta^2} = -1 + 2\frac{1}{1+4} = -\frac{3}{5} = -0.6.$$

The generalized Rabi frequency is here $\Omega_R = \sqrt{\omega_R^2 + \delta^2} = \sqrt{5}\,\omega_R$, so the magnetic moment will be back pointing down after only $T' = \frac{2\pi}{\Omega_R} = \frac{1}{\sqrt{5}}\frac{2\pi}{\omega_R} \approx 0.45\,T$

Note: The **initial** behavior of all the curves is identical! μ_z grows initially quadratically with time, with a curvature that's independent of detuning! Indeed, we have

$$\frac{\mu_z}{\mu} = -1 + 2\frac{\omega_R^2}{\Omega_R^2}\sin^2\left(\frac{\Omega_R t}{2}\right) \approx -1 + \frac{1}{2}\omega_R^2 t^2$$

This is valid for times such that $\Omega_R t \ll 1$, which for large detuning means $t \ll \frac{1}{\delta}$. This we can also see directly by considering the Fourier uncertainty relation between frequency and time: At times $t \ll \frac{1}{\delta}$, the system did not even "have enough time to realize" that it was detuned from resonance!
3D View on Bloch sphere:

Rapid adiabatic passage

Technique for inverting spin by (slowly) sweeping across the resonance

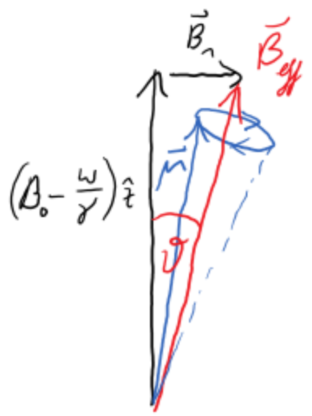
"slow" compared to the Larmor frequency γB_{eff}

"rapid" compared to relaxation processes

Physical picture:

A magnetic moment $\vec\mu$ is initially aligned with a static field $\vec B_0 = B_0 \hat z$, and there is a weak magnetic field $\vec B_1$ that rotates at frequency ω in the $x-y$ plane.

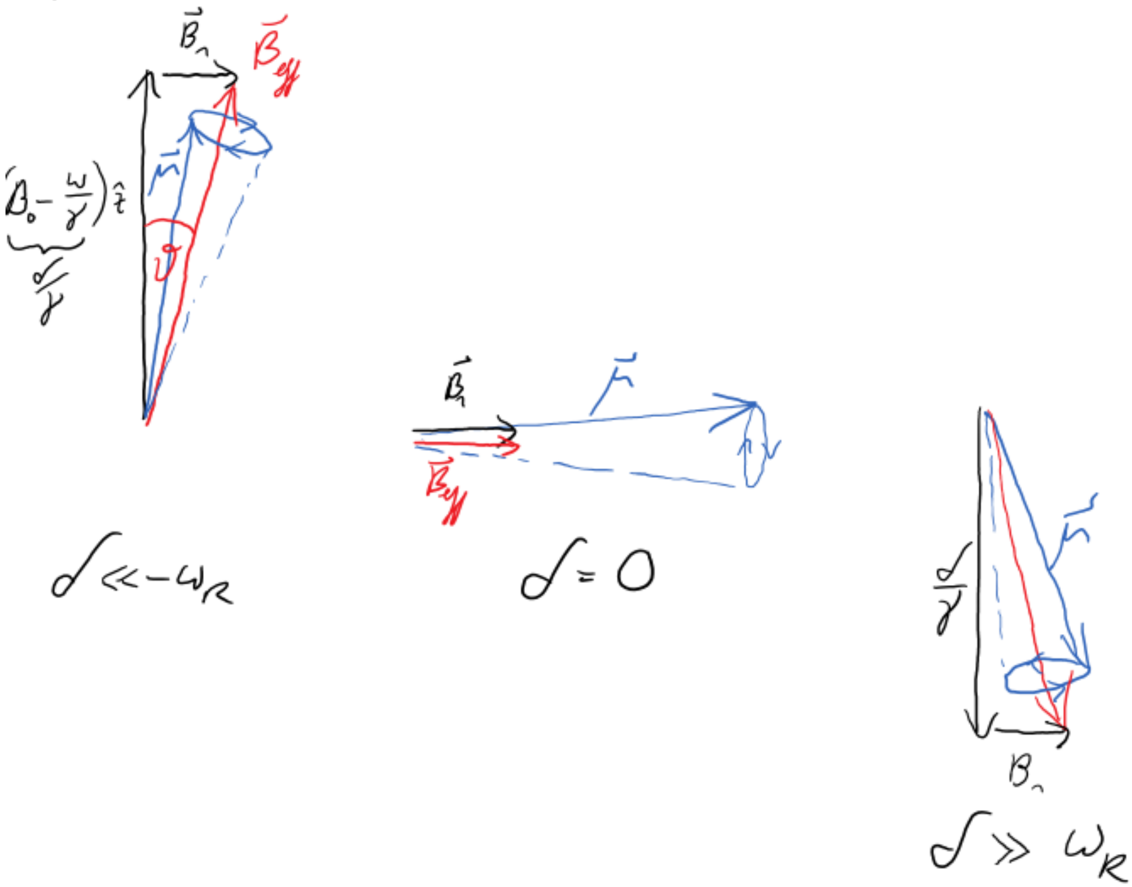
We start at a detuning $\delta \ll -\omega_R = \gamma B_1$



In the rotating frame (rotating at ω about the z-axis), the effective magnetic field points almost in the z-direction (as $|B_1| \ll |B_0 - \frac{\omega}{\gamma}| = \left|\frac{\delta}{\gamma}\right|$).

The magnetic moment $\vec\mu$ slightly precesses about the effective magnetic field, so is almost aligned in $\hat z$

Now let us slowly (we will see what that means) sweep the frequency of the rotating field $\vec{B}_1$ from $\delta \ll -\omega_R$ through $\delta = 0$ and up to $\delta \gg \omega_R$.
The situation in the rotating frame (rotating at the momentary frequency $\omega(t)$ of $\vec{B}_1$) looks like this:



$\Rightarrow$ The magnetic moment always precesses tightly around the – slowly changing – effective field.
At the end of the sweep, the magnetic moment finds itself <u>inverted</u>!

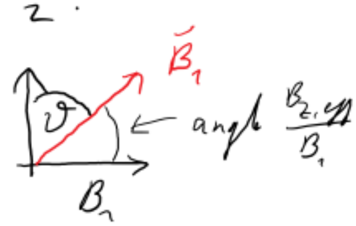
Adiabaticity requires that the magnetic moment always precesses tightly around B_{eff}, that means: Within one precession period $\frac{2\pi}{\Omega_{Larmor}}$, the angle ϑ of B_{eff} must not have advanced more than a few degrees, i.e. $\Delta\vartheta \ll 2\pi$.

$$\Delta\vartheta = \dot{\vartheta}\cdot\Delta t = \dot{\vartheta}\,\frac{2\pi}{\Omega_{Larmor}} \ll 2\pi$$

or just $\Omega_{Larmor} = \gamma B_{eff} \gg \dot{\vartheta}$.

Now the smallest Larmor frequency, and (for a linear sweep $\delta(t) = \alpha t$, say) the largest rate of change of ϑ occur around resonance, at $\delta = 0$, or $\vartheta = \frac{\pi}{2}$.

Here, $\vartheta = \frac{\pi}{2} - \frac{B_{z,eff}}{B_1}$ $\qquad \frac{\delta}{\gamma} = B_0 - \frac{\omega(t)}{\gamma}$



$$\dot{\vartheta} = \frac{|\dot{\omega}|}{\gamma B_1} = \frac{|\dot{\omega}|}{\omega_R}$$

$$\Omega_L(\delta=0) = \gamma B_1 = \omega_R$$

$$\Rightarrow \dot{\vartheta} = \frac{|\dot{\omega}|}{\omega_R} \ll \omega_R \quad \text{or} \quad \boxed{|\dot{\omega}| \ll \omega_R^2}$$

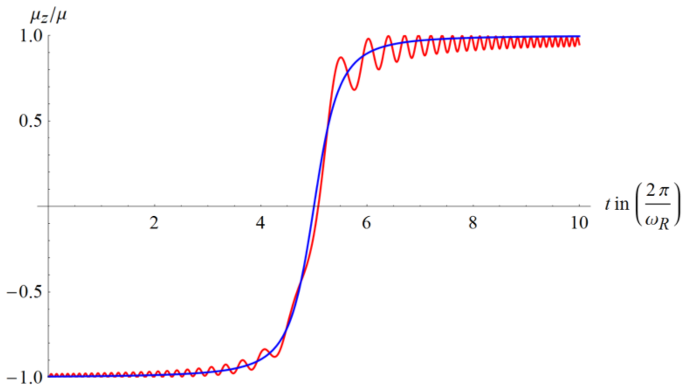
i.e. the change of ω in one Rabi period should be much smaller than the Rabi frequency.

This implies that this inversion will be <u>slower</u> than an on-resonance π-pulse!

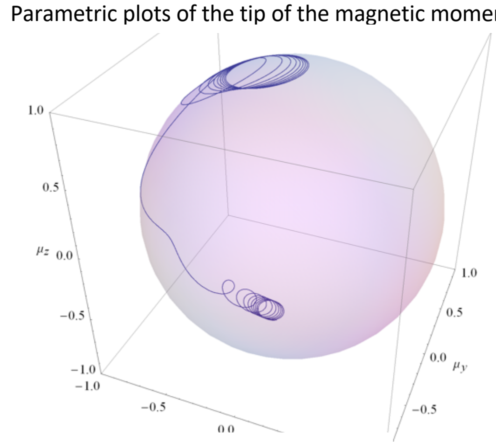
Rapid adiabatic passage with various parameters:

Blue: $\cos\theta = \frac{\delta(t)}{\sqrt{\delta(t)^2+\omega_R^2}}$ Cosine of B_{eff} with z; Red: z-component of the magnetic moment

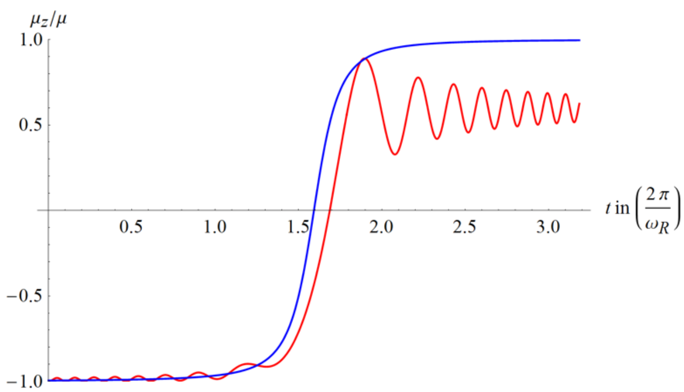
1. Start of sweep: $\delta = -10\ \omega_R$, End of sweep: $\delta = +10\ \omega_R$, Sweep rate: $\dot{\omega} = \frac{20\ \omega_R}{10\ \frac{2\pi}{\omega_R}} = \frac{1}{\pi}\omega_R^2$

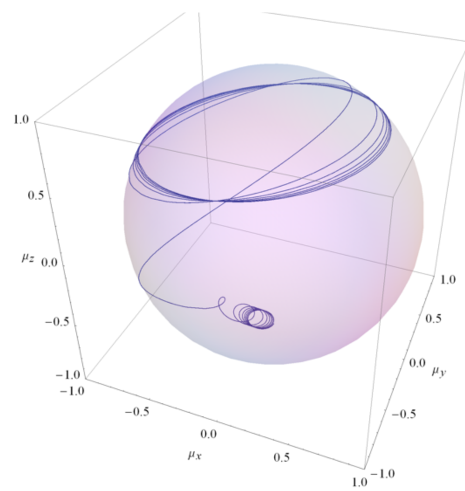


Parametric plots of the tip of the magnetic moment:



2. Start of sweep: $\delta = -10\ \omega_R$, End of sweep: $\delta = +10\ \omega_R$, Sweep rate: $\dot{\omega} = \pi\frac{20\ \omega_R}{10\ \frac{2\pi}{\omega_R}} = \omega_R^2$





3. Start of sweep: $\delta = -10\ \omega_R$, End of sweep: $\delta = +10\ \omega_R$, Sweep rate: $\dot{\omega} = \pi^2\frac{20\ \omega_R}{10\ \frac{2\pi}{\omega_R}} = \pi\omega_R^2$