Lecture 4 - Quantization of the electromagnetic field

2.5 Quantization of the e.m. field

Take cubic box of side length L, volume $V = L^3$, with periodic boundary conditions:

$$k_{\text{x, y, z}} = \frac{2\pi}{L} n_{\text{x, y, z}}$$

Therefore $\alpha_\epsilon(\vec{k}, t) \rightarrow \alpha_{\vec{k},\epsilon}(t)$ or simply α_i ($i = (\vec{k}_i, \vec{\epsilon}_i)$).

- Correspondence

$$\int \mathrm{d}^3k \sum_\epsilon f(\vec{k}, \vec{\epsilon}) \leftrightarrow \left(\frac{2\pi}{L}\right)^3 \Sigma_i f(\vec{k}_i, \vec{\epsilon}_i)$$

- Analogy with harmonic oscillator:

$$\begin{array}{c|c|c}
\text{One e.m. field mode} & \text{Correspondence} & \text{Harmonic Oscillator} \\
\dot{\mathcal{A}}_i = -\mathcal{E}_i & \mathcal{A}_i \hat{=} x & \dot{x} = \dfrac{p}{m} \\
\dot{\mathcal{E}}_i = \omega_i^2 \mathcal{A}_i & \mathcal{E}_i \hat{=} -\dfrac{p}{m} & \dfrac{\dot{p}}{m} = -\omega^2 x \\
H_i = \frac{\epsilon_0}{2}\frac{(2\pi)^3}{V}\left(|\mathcal{E}_i|^2 + \omega_i^2 |\mathcal{A}_i|^2\right) & \epsilon_0 \frac{(2\pi)^3}{V} \hat{=} m & H = \frac{m}{2}\left(\left(\frac{p}{m}\right)^2 + \omega^2 x^2\right) \\
\alpha_i = \mathcal{N}_i \left(\mathcal{A}_i + \dfrac{i}{\omega_i}\mathcal{E}_i\right) & & \alpha = \mathcal{N}\left(x + i\dfrac{p}{m\omega}\right) \\
\dfrac{d\alpha_i}{dt} = -i\omega_i \alpha_i & & \dfrac{d\alpha}{dt} = -i\omega\alpha
\end{array}$$

- Quantization and Commutation relations:

One e.m. field mode	Harmonic Oscillator
$\mathcal{A}_i \rightarrow \hat{\mathcal{A}}_i$	$x \rightarrow \hat{x}$
$\mathcal{E}_i \rightarrow \hat{\mathcal{E}}_i$	$p \rightarrow \hat{p}$
$\left[\hat{\mathcal{A}}_i, \hat{\mathcal{E}}_i\right] = -\frac{V}{(2\pi)^3\epsilon_0} i\hbar$	$[\hat{x}, \hat{p}] = i\hbar$
$\hat{a}_i$ annihilation operator associated to α_i	$\hat{a}$ annihilation operator associated to α
$\left[\hat{a}_i, \hat{a}_i^\dagger\right] = 1$ for	$\left[\hat{a}, \hat{a}^\dagger\right] = 1$ for
$\mathcal{N} = \sqrt{\frac{\epsilon_0 \omega_i}{2\hbar} \frac{(2\pi)^3}{V}}$	$\mathcal{N} = \sqrt{\frac{m\omega}{2\hbar}}$

- Physical Operators:
- Hamiltonian:

$$H_i = \frac{\hbar\omega_i}{2}\left(\alpha_i^* \alpha_i + \alpha_i \alpha_i^*\right) \quad \left(= \hbar\omega_i |\alpha_i|^2\right) \qquad H = \frac{\hbar\omega}{2}\left(\alpha^* \alpha + \alpha\alpha^*\right)$$

$$\hat{H}_i = \frac{\hbar\omega_i}{2}\left(\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger\right) \qquad \hat{H} = \frac{\hbar\omega}{2}\left(\hat{a}^\dagger \hat{a} + \hat{a}\hat{a}^\dagger\right)$$

$$\hat{H} = \sum_i \hbar\omega_i \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}\right) \qquad \hat{H} = \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)$$

- Momentum:

$$\vec{P} = \sum_i \frac{\hbar\vec{k}_i}{2}\left(\alpha_i^* \alpha_i + \alpha_i \alpha_i^*\right)$$

$$\vec{P} = \sum_i \frac{\hbar\vec{k}_i}{2}\left(\hat{a}_i^\dagger \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger\right)$$

$$\vec{P} = \sum_i \hbar\vec{k}_i \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}\right)$$

but $\sum_i \vec{k}_i = 0$ so

$$\hat{\vec{P}} = \sum_i \hbar\vec{k}_i \hat{a}_i^\dagger \hat{a}_i$$

- Electric Field:

$$\hat{\vec{E}}(\vec{r}) = i\sum_i \mathcal{E}_i \left(\vec{\epsilon}_i \hat{a}_i e^{i\vec{k}_i \vec{r}} - \vec{\epsilon}_i \hat{a}_i^\dagger e^{-i\vec{k}_i \vec{r}}\right)$$

with $\mathcal{E}_i = \sqrt{\frac{\hbar\omega_i}{2\epsilon_0 V}}$

- Comment:

  Within the Lagrangian formalism one sees that the momentum conjugate to $\mathcal{A}_{\perp\epsilon}$ is $\Pi_\epsilon = \epsilon_0 \dot{\mathcal{A}}_{\perp\epsilon} = -\epsilon_0 \mathcal{E}_{\perp\epsilon}$. The canonical commutation relations are then

  $$\left[\mathcal{A}_\epsilon(\vec{k}), \Pi^\dagger_{\epsilon'}(\vec{k}')\right] = i\hbar\delta_{\epsilon\epsilon'}\delta(\vec{k}-\vec{k}')$$

  This agrees with $[\,\mathcal{A}_i, \mathcal{E}_j] = -\dfrac{V}{(2\pi)^3}\dfrac{i\hbar}{\epsilon_0}\delta_{ij}$ as

  $1 = \int \mathrm{d}^3k\,\delta(\vec{k}-\vec{k}') \leftrightarrow \frac{(2\pi)^3}{V}\sum_k \frac{V}{(2\pi)^3}\delta_{kk'}$ or

  $\delta(\vec{k}-\vec{k}') \leftrightarrow \frac{V}{(2\pi)^3}\delta_{kk'}$

2.6 Total Hamiltonian and Momentum:

$$H = \sum_\alpha \frac{1}{2m_\alpha}\left(\vec{p}_\alpha - q_\alpha \vec{A}_\perp(\vec{r}_\alpha)\right)^2 + \sum_\alpha \left(-g_\alpha \frac{q_\alpha}{2m_\alpha}\right)\vec{S}_\alpha \cdot \vec{B}(\vec{r}_\alpha)$$

$$+V_{\rm Coulomb} + H_{\rm R}$$

$$V_{\rm Coulomb}=\sum_\alpha\epsilon^\alpha_{\rm Coulomb}+\frac{1}{8\pi\epsilon_0}\sum_{\alpha\ne\beta}\frac{q_\alpha q_\beta}{|\vr_\alpha-\vr_\beta|}$$

$$\epsilon_{\rm Coulomb}^\alpha = \frac{q_\alpha^2}{2\epsilon_0(2\pi)^3}\intk \frac{1}{k^2} = \frac{q_\alpha^2}{4\pi^2\epsilon_0} k_c$$

$$H_{\rm R} = \frac{\epsilon_0}{2} \intr \left(\vE_\perp^2 + c^2 \vB^2\right) = \sum_i \hbar\omega_i \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}\right)$$

Total Momentum:

$$\vec{P} = \sum_\alpha \vec{p}_\alpha + \vec{P}_{\rm R}$$

$$\vec{P}_{\rm R} = \sum_i \hbar \vk_i\, \hat{a}_i^\dagger \hat{a}_i$$

$$H = H_{\rm P} + H_{\rm R} + H_{\rm I}$$

Particle Hamiltonian: $$H_{\rm P} = \sum_\alpha \frac{\vp_\alpha^2}{2m} + V_{\rm Coulomb}$$

Interaction:

$$H_{\rm I} = H_{\rm I1} + H_{\rm I2} + H_{I1}^S$$

$$H_{\rm I1} = - \sum_\alpha \frac{q_\alpha}{m_\alpha} \vp_\alpha \cdot \vA_\perp(\vr_\alpha)$$

$$H_{\rm I1}^S = - \sum_\alpha g_\alpha \frac{q_\alpha}{2 m_\alpha} \vec{S}_\alpha \cdot \vec{B}(\vr_\alpha)$$

$$H_{\rm I2} = \sum_\alpha \frac{q_\alpha^2}{2m_\alpha} \vA_\perp^2(\vr_\alpha)$$