QUANTUM FIELD THEORY

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Preface. These notes come from Prof. Patton's QFT notes, Peskin-Schröder QFT, and "Quantum field theory & Condensed matter" - Shankar.

Enjoy!

Conventions

- $\hbar = c = 1$
- $[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1}$
- $g_{\mu\nu} = g^{\mu\nu} = \text{diag}(1\ -1\ -1\ -1)$
- $p^2 = p_\mu p^\mu = E^2 - |p|^2 = mc^2 = m$
- $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$
- $x^\mu = g^{\mu\nu} x_\nu = (x^0, \vec{x})$
- $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial x^0}, \nabla\right)$
- $\varepsilon^{0123} = +1$; $\varepsilon_{0123} = -1$

  $\varepsilon^{1230} = -1$
- $E = i\frac{\partial}{\partial x^0}$, $\vec{p} = -i\vec{\nabla}$
- $p^\mu = i\partial^\mu$

- $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- $\sigma^i \sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

- $\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

- Heaviside step fn:

$$\theta(x) = \begin{cases} 0 & x \cancel{<} 0 \\ 1 & x > 0 \end{cases}$$

- Dirac delta fn: $\delta(x) = \frac{d}{dx}\theta(x)$

- n-dimensional Dirac δ-fn:

$$\int d^n x \, \delta^{(n)}(x) = 1$$

- FT:

$$f(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x} \tilde{f}(x)$$

$$\tilde{f}(x) = \int d^4k \, e^{ik\cdot x} f(x)$$

- $\int d^4x \, e^{ik\cdot x} = (2\pi)^4 \delta^{(4)}(k)$

- <u>EM</u> $\Phi = \frac{Q}{4\pi r} \leftarrow$ Coulomb potential

- Fine structure constant:

$$\alpha = \frac{e^2}{4\pi} = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$$

- Maxwell's eqn:

$$\varepsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \;, \quad \partial_\mu F^{\mu\nu} = e j^\nu$$

where

$$A^\mu = (\Phi, \vec{A}) \;; \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Elements of Classical Field Theory

Lagrangian Field Theory:

Lag-density

$$S = \int \underline{L}\, dt = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

$$(\underline{L} = \mathcal{L}\, d^3x)$$

(Lagrangian)

Principle of least action:

$$0 = \delta S$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\delta\phi + \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right] \right\}$$

(E-L eqn)

$$\Rightarrow \boxed{0 = \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right] - \frac{\partial \mathcal{L}}{\partial \phi}}$$

FTC → term vanishes @ boundary

Euler - Lagrange Equations

$$\frac{\partial \mathcal{L}}{\partial \phi} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\}$$

Ex

$$\mathcal{L} = \phi^2 : \quad \frac{\partial \mathcal{L}}{\partial \phi} = 2\phi \, ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 0 \Rightarrow 2\phi = 0$$

$$\left.\begin{aligned} &\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi) \\ &\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = 0 \, ; \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = 2\partial^\mu \phi \end{aligned}\right\} \Rightarrow \partial^\mu \phi = 0.$$

Ex

Klein - Gordon field: (real)

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi .$$

$$\text{E-L} \Rightarrow \quad -m^2 \phi = \partial_\mu \partial^\mu \phi = \Box \phi$$

$$\rightarrow \boxed{(\Box + m^2)\phi = 0}$$

Klein - Gordon Eqn.

relativistic particle of mass m.

Ex $\phi = e^{-ip\cdot x} \Rightarrow (-p^2 + m^2) = 0 \Rightarrow p^2 = m^2$

Noether's Thm

Thm $\boxed{\text{For every continuous symmetry, there exists a conserved current } j^\mu \text{ which implies a local conservation law}}$

$$\partial_0 j^0 = -\vec{\nabla}\cdot\vec{j} \quad \Leftrightarrow \quad \partial_\mu j^\mu = 0$$

Ex Conservation of charge ...

(gauss' law)

$$Q = \int j^0 \, d^3x$$

$$\Rightarrow \frac{\partial Q}{\partial t} = \int \frac{dj^0}{dt} d^3x = -\int \vec{\nabla}\cdot\vec{j}\, d^3x$$

$$= -\oint \vec{j}\, d^2s$$

//

Idea Consider continuous transf. → infinitesimally...
(local)

$$(\star) \qquad \phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

small

$(\star)$ is a symmetry if EOM invariant under $(\star)$.

$\Leftrightarrow$ S is invariant.

$\Rightarrow$ $\mathcal{L}$ must be invariant, up to $\alpha \partial_\mu J^\mu(x)$.

for some J^μ.

Let us compare this expectation for $\Delta\mathcal{L}$ to the result obtained by varying the fields ...

$$\alpha \Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} (\alpha \Delta \phi) + \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \partial_\mu (\alpha \Delta \phi)$$

$$= \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi \right) + \alpha \underbrace{\left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right\} \right)}_{0} \Delta \phi$$

$$\Rightarrow \quad \Delta \mathcal{L} = \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi \right\}$$

So $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi$ is the desired J^μ.

So that $\partial_\mu j^\mu(x) = 0$ where

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - J^\mu .$$

Ex massless KG field

$$\mathcal{L} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi)$$

Consider transformation:

$$\phi(x) \longrightarrow \phi(x) + \alpha \quad , \Delta \phi = 1$$

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi = \partial^\mu \phi .$$

Check $\partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = \Box \phi = 0$ since

$$(m^2 + \Box) \phi = 0 \; ; \; m = 0$$

Ex **Complex KG field**

$$\mathcal{L} = (\partial_\mu \phi^\dagger)(\partial^\mu \phi) - m^2 \phi^\dagger \phi .$$

again, EOM is

$$(m^2 + \Box)\phi = 0.$$

Symmetry: $\phi \to e^{i\alpha}\phi$.

For infinitesimal transf we have:

$$\alpha \Delta \phi = i\alpha\phi \qquad \text{(Taylor expand)}$$
$$\alpha \Delta \phi^\dagger = -i\alpha\phi^\dagger$$

One can check that

$$j^\mu = i\left[(\partial^\mu \phi^\dagger)\phi - \phi^\dagger(\partial^\mu \phi)\right]$$

is the conserved current.

$\boxed{\text{Ex}}$

Space-time translation:

Consider infi. translation:

$$x^\mu \to x^\mu - a^\mu$$

in field transforms this becomes:

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

Lagrangian is a scalar ⇒ must transform the same way:

$$\mathcal{L}(x) \to \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Compare this to the eqn:

$$\mathcal{L} \to \mathcal{L} + a \partial_\mu J^\mu ,$$

we have
$$J^\mu = \delta^\mu_\nu \mathcal{L}$$

→ apply them, we find:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} (\partial_\mu \phi) - \delta^\nu_\mu \mathcal{L}$$

Make μ explicit...

$$\boxed{T^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)} \partial_\mu \phi - \delta^\nu_\mu \mathcal{L}}$$

STRESS-ENERGY TENSOR. (or Energy-Momentum tensor)

<u>Conserved charge</u> is the Hamiltonian...

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x \quad \text{(time-translation)}$$

momentum → $$p^i = \int T^{0i} d^3x = -\int \pi \partial_i \phi \, d^3x \quad \text{(spatial translation)}$$

Ex Klein-Gordon field again...

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 .$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} = \dots$$

$$= \frac{1}{2}(\partial_0 \phi)(\partial^0 \phi) - \frac{1}{2}(\partial_i \phi)(\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

So

$$H = \int T^{00} d^3x = \int \left\{ \frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{2} m^2 \phi^2 \right\} d^3x$$

↳ note: all terms are positive ... (sum of squares)

→ can't fall into arbitrary negative energy.

$$\boxed{\text{THE KLEIN-GORDON FIELD} \sim \text{HARMONIC OSCILLATOR}}$$

promote: ϕ, π to operators & impose suitable commutation relations...

Recall ...

$$[q_i, p_j] = i\delta_{ij}$$

$$[q_i, q_j] = [p_i, p_j] = 0 .$$

Harmonic oscillator: $H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 q^2$

Ladder operators:

- annihilation: $a = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} + ip\frac{1}{\sqrt{m\omega}}\right)$

- creation: $a^\dagger = \frac{1}{\sqrt{2}}\left(q\sqrt{m\omega} - ip\frac{1}{\sqrt{m\omega}}\right)$

- $\underset{\uparrow}{a^\dagger a} = \frac{1}{\omega}H - \frac{1}{2} \Leftrightarrow H = \omega\left(a^\dagger a + \frac{1}{2}\right)$

  # operator...

- $|0\rangle \;,\; a|0\rangle = 0.$

  $|n\rangle = \frac{1}{\sqrt{n!}}\underbrace{a^\dagger \ldots a^\dagger}_{n}|0\rangle$

- $[a^\dagger a, a] = -a$

- $[a^\dagger a, a^\dagger] = a^\dagger$

- $[H, a] = -\omega a \quad ; \quad [H, a^\dagger] = \omega a^\dagger$

  a lowers by ω

  $a^\dagger$ raises by ω

- $Ha^\dagger|n\rangle = (E_n + \omega)a^\dagger|n\rangle$

- $H|0\rangle = \frac{1}{2}\omega|0\rangle \rightarrow E_0 = \frac{1}{2}\omega$

- $E_n = \left(n + \frac{1}{2}\right)\omega$

For continuum systems ... commutation relations become ..

$$\boxed{\begin{gathered} [\phi(x), \pi(y)] = i\delta^{(3)}(x-y) \\ [\phi(x), \phi(y)] = [\pi(x), \pi(y)] = 0 . \end{gathered}}$$

Next, the Hamiltonian is now also an operator ...
To find spec(H), Fourier transf $\phi(x)$

$$\longrightarrow \quad \phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \, e^{ip\cdot x} \phi(\vec{p},t)$$

Recall KG eqn: $(m^2 + \Box)\phi = 0$

$$\Rightarrow \left\{ \frac{\partial^2}{\partial t^2} + (|p|^2 + m^2) \right\} \phi(\vec{p},t) = 0$$

$\Rightarrow$ This is the same eqn as SHO with freq

$$\omega_p = \sqrt{|\vec{p}|^2 + m^2} .$$

Now, $H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}w^2\phi^2 \qquad (m=1)$

$\rightarrow$ know spectrum! $\quad (n+\frac{1}{2})w .$

$$\phi = \frac{1}{\sqrt{2w}}(a^\dagger + a) \quad ; \quad p = -i\sqrt{\frac{w}{2}}(a - a^\dagger)$$

$$[a, a^\dagger] = 1 .$$

Since it's more convenient to work in position space

$$\boxed{\begin{aligned} \phi(x) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{ip\cdot x} + a_p^+ e^{-ip\cdot x} \right) \\ \pi(x) &= \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p e^{ip\cdot x} - a_p^+ e^{-ip x} \right) \end{aligned}}$$

note

$$\begin{cases} a_p \text{ goes with } e^{+ip\cdot x} \\ a_p^+ \text{ goes with } e^{-ip\cdot x}. \end{cases}$$

- Easy to show that

$$\begin{cases} [\phi(\vec{x}), \phi(\vec{y})] = 0 = [\pi(\vec{x}), \pi(\vec{y})] \\ [\phi(\vec{x}), \pi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}). \end{cases}$$

- Can re-arrange...

$$\begin{cases} \phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p + a_{-p}^+ \right) e^{ip\cdot x}. \\ \pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} \left(a_p - a_{-p}^+ \right) e^{+ip\cdot x}. \end{cases}$$

→ get commutation relation between a_p:

$$\boxed{[a_p ; a_{p'}^+] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{p}\,')}$$

Now, can check that

$$[\phi(x), \pi(x')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{w_{p'}}{w_p}} \times e^{i(p\cdot x + p'\cdot x')} \times$$

$$\left([a^\dagger_{-p}, a_{p'}] - [a_p, a^\dagger_{-p}] \right)$$

$$= i\delta^{(3)}(x - x') \quad \checkmark$$

- Now, can express Hamiltonian in terms of ladder ops

Recall that $\longrightarrow$ KG field, lost time

$$H = \int d^3x \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L} \right\}$$

$$= \int d^3x \left\{ \frac{1}{2} (\partial_0 \phi)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right\}$$

To quantize, need to define π ... Turns out that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi(x) \rightarrow \left(\text{like } p = \frac{\partial \mathcal{L}}{\partial \dot{q}} \right)$$

So ...

$$H = \int d^3x \left\{ \frac{1}{2} \pi^2 + \frac{1}{2} (\nabla\phi)\cdot(\nabla\phi) + \frac{1}{2} m^2 \phi^2 \right\}$$

with $\pi(x) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{w_p}{2}} \left(a_p - a^\dagger_{-p}\right) e^{ip\cdot x}$ we get

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2w_p}} \left(a_p + a^\dagger_{-p}\right) e^{ip\cdot x}$$

(14)

$\to$ results in $\delta(p-p')$ $\Rightarrow p = p'$

$$H = \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6}\, e^{i(p+p')\cdot x} \left\{ -\frac{\sqrt{\omega_p \omega_{p'}}}{4}(a_p - a^\dagger_{-p})(a_{p'} - a^\dagger_{-p'}) \right.$$

$$\left. + \frac{-p\cdot p' + m^2}{4\sqrt{\omega_p \omega_{p'}}}(a_p + a^\dagger_{-p})(a_{p'} + a^\dagger_{-p'}) \right\}$$

Some $\delta^{(3)}$ will appear...

$$= \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)$$

So

$$\boxed{H = \int \frac{d^3p}{(2\pi)^3}\, \omega_p \left(a_p^\dagger a_p + \frac{1}{2}[a_p, a_p^\dagger] \right)}$$

$\to$ can evaluate commutators...

$$\boxed{[H, a_p^\dagger] = \omega_p a_p^\dagger \; ; \; [H, a_p] = -\omega a_p}$$

With H, can find momentum operator...

KG field $\longrightarrow$ from $P^i = \int d^3x\, T^{0i} = -\int \pi \partial_i \phi\, d^3x$, we get

$$\boxed{\begin{aligned} \vec{P} &= -\int d^3x\, \pi(x) \nabla \phi(x) \\ &= \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a_p^\dagger a_p . \end{aligned}}$$

$E_p > 0$

$a_p^\dagger$ creates momentum $\vec{p}$ & energy $\omega_p = \sqrt{|p|^2 + m^2}$.

Excitation: $a_p^\dagger a_q^\dagger \ldots |0\rangle \equiv$ "particles".

$\hookrightarrow$ <u>each excitation $a_p^\dagger$ is a particle.</u>

$\Rightarrow$ get particle statistics...

Consider 2-particle state... $a_p^\dagger a_q^\dagger |0\rangle$.

Since $[a_p^\dagger, a_q^\dagger] = 0$, we have

$$\boxed{a_p^\dagger a_q^\dagger |0\rangle = a_q^\dagger a_p^\dagger |0\rangle}$$

$\rightarrow$ Klein Gordon particles follow Bose-Einstein stats.

- Normalization $\langle 0|0\rangle = 1$.

$$|p\rangle \propto a_p^\dagger |0\rangle .$$

This normalization $\longrightarrow$ $\langle q|p\rangle = (2\pi)^3 \delta^{(3)}(q-p)$ is NOT Lorentz inv

Pf Under a Lorentz boost $\rightarrow$ $p_3' = \gamma(p_3 + \beta E)$

$E' = \gamma(E + \beta p_3)$

using the identity $\rightarrow$ $\delta(f(x) - f(x_0)) = \dfrac{1}{|f'(x_0)|}\delta(x - x_0)$

we compute: $\delta^{(3)}(p-q) = \delta^{(3)}(p'-q') \cdot \left(\dfrac{dp_3'}{dp_3}\right)$

$\underbrace{\delta(p_1-q_1)\delta(p_2-q_2)}_{\text{same}}\underbrace{\delta(p_3-q_3)}_{\text{boosted}}$

$$= \delta^{(3)}(p'-q') \cdot \gamma\left(1 + \beta \, dE/dp_3\right)$$

$$= \delta^{(3)}(p'-q') \frac{\gamma}{E}(E + \beta p_3)$$

$$= \delta^{(3)}(p'-q')(E'/E)$$

$1 \Leftrightarrow E = E_p = E'$

So $\boxed{\delta^{(3)}(p-q) = \delta^{(3)}(p'-q')\left(\frac{E'}{E}\right)}$

For normalization to work → use E_p ($\parallel$ w_p), not E.

→ define: $\boxed{|p\rangle = \sqrt{2E_p}\, a_p^\dagger |0\rangle}$

so that $\boxed{\langle p|q\rangle = 2E_p\,(2\pi)^3\,\delta^{(3)}(p-q)}$

Completeness relation...

1 particle → $\boxed{1 = \int \frac{d^3p}{(2\pi)^3}\, |p\rangle \frac{1}{2E_p} \langle p|}$

Interpret $\phi(x)|0\rangle$ .... we know that $a_p^\dagger$ creates momentum p & energy $E_p = w_p$.

What about operator $\phi(x)$?

$$\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|p\rangle \quad , \quad a_p^\dagger|0\rangle = \frac{1}{\sqrt{2E_p}}|p\rangle$$

Pf. By defn...

annihilates.

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x}\right)$$

$$\to \phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot x}|0\rangle \qquad \square$$

→ $\phi(x)|0\rangle$ is a lin. superposition of single-particle states

~~Note HW1, HW2 easy~~

that have well-defn momentum.

When nonrelativistic $\rightarrow E_p \simeq$ Constant!

$\Rightarrow$ [$\phi(x)$ acting on the vacuum, "creates a particle at position x".]

$\hookrightarrow$ Confirm this by computing.

$$\langle 0|\, \phi(x)\, |p\rangle = \langle 0| \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} \left(a_{p'} e^{ip'\cdot x} + a_{p'}^\dagger e^{-ip'\cdot x}\right) \sqrt{2E_p}\, a_p^\dagger |0\rangle$$

$$\sim \boxed{\langle 0|\, \phi(x)\, |p\rangle = e^{ip\cdot x}}.$$

$\hookrightarrow$ Interpretation: position-space representation of the single-particle wfn of the state $|p\rangle$, just like ...

$\boxed{\langle x|p\rangle \propto e^{ip\cdot x}}$ in QM!

$\langle 0|\, \phi(x) \sim \langle x|$ ... (don't take this literally, ofc).

Note HW1, HW2 are easy, so we'll skip for now.

ep 14, 2020

THE KLEIN - GORDON FIELD IN SPACETIME

Last time → we quantized KG field in the Schrödinger picture.

→ Now, switch to Heisenberg picture.

Recall... Schrödinger picture:

$$\mathcal{U}(t) = e^{-iHt} \text{ is the time evolution.}$$

$$|\psi(t)\rangle = e^{-iHt}|\psi(0)\rangle. \longrightarrow \underline{\text{State}} \text{ evolve in time}$$

→ In the Heisenberg picture, ... $\underline{\text{Operators}}$ evolve in time

$$\mathcal{O}(t) = \mathcal{U}^\dagger(t)\,\mathcal{O}(0)\,\mathcal{U}(t).$$

So that

$$\langle\psi_1|\mathcal{O}(t)|\psi_1\rangle = \langle\psi_1(t)|\mathcal{O}|\psi_1(t)\rangle$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

Heisenberg $\qquad\qquad$ Schrödinger.

→ make the operators ϕ & π time-dependent like this

$$\boxed{\begin{aligned} \phi(x) &\to \phi(x,t) = e^{iHt}\phi(x)e^{-iHt} \\ \pi(x) &\to \pi(x,t) = e^{iHt}\pi(x)e^{-iHt} \end{aligned}}$$

Recall Heisenberg eqn of motion $\quad i\frac{\partial}{\partial t}\mathcal{O} = [\mathcal{O}, H]$

which gives, upon substituting in $\phi(x,t)$, $\pi(x,t)$

$$i\frac{\partial}{\partial t}\phi(x,t) = \left[\phi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

$(\phi \leftrightarrow \phi)$ → only nontrivial term is 1st.

$$= \int d^3x' \left(i\delta^{(3)}(x-x')\,\pi'(x,t) \right)$$

$$= i\pi(x,t) \Rightarrow \boxed{\frac{\partial}{\partial t}\phi(x,t) = \pi(x,t)}$$

and

$$i\frac{\partial}{\partial t}\pi(x,t) = \left[\pi(x,t), \int d^3x' \left\{ \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right\}\right]$$

(integrate by parts here)

$$= \int d^3x' \left(-i\delta^{(3)}(x-x')\left(-\nabla^2 + m^2\right)\phi(x',t) \right)$$

$$= -i\left(-\nabla^2 + m^2\right)\phi(x,t)$$

$$\Rightarrow \boxed{\frac{\partial}{\partial t}\pi(x,t) = \left(m^2 - \nabla^2\right)\phi(x,t)}$$

Combining these 2 results we get...

$$\boxed{\frac{\partial^2}{\partial t^2}\phi(x,t) = \left(\nabla^2 - m^2\right)\phi(x,t)}$$

rearranging this gives

$$\boxed{\left(\Box + m^2\right)\phi(x,t) = 0} \longrightarrow$$ just the KG eqn...

- Now, can better understand the time dependence of $\phi(x)$, $\pi(x)$ by writing them in terms of creation & annihilation ops.

<u>Recall</u>: $H a_p = a_p (H - E_p)$ → from comm. rule.

$$\Longrightarrow \left(\begin{array}{c}\text{proof by}\\ \text{induction}\end{array}\right) \quad \boxed{H^n a_p = a_p (H - E_p)^n}$$

Similarly,

$$\boxed{H^n a_p^\dagger = a_p^\dagger (H + E_p)^n}$$

→ So, we have

$$\boxed{e^{iHt} a_p e^{-iHt} = a_p e^{-iE_p t}} \rightarrow \text{from above...}$$

and

$$\boxed{e^{iHt} a_p^\dagger e^{-iHt} = a_p^\dagger e^{+iE_p t}}$$

→ Now... we want to write $\phi(x,t)$ in terms of these operators. (since $\phi(x)$, $\pi(x)$ is a combo of a & $a^\dagger$)

- We know that $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$.

and from before...

$$\phi(x) = \phi(x,0) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip\cdot x} + a_p^\dagger e^{-ip\cdot x} \right)$$

substitute this into $\phi(x,t) = e^{iHt} \phi(x) e^{-iHt}$ we find ...

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-iE_pt} e^{ip\cdot x} + a_p^\dagger e^{iE_pt} e^{-ip\cdot x} \right\}$$

now, note that $p^0 = E_p$

$$\Rightarrow \quad p\cdot x = E_p \cdot x^0 - \vec{p}\cdot\vec{x} \Big|_{p^0=E_p} = E_pt - \vec{p}\cdot\vec{x}\Big|_{p^0=E_p}$$

So...

$$\boxed{\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a_p e^{-i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right\} \Big|_{p^0=E_p}}$$

Similarly, we can find

$$\boxed{\pi(x,t) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{E_p}{2}} \left(a_p e^{-i\vec{p}\cdot\vec{x}} - a_p^\dagger e^{+i\vec{p}\cdot\vec{x}} \right) \Big|_{p^0=E_p}}$$

from defn or from $\pi(x,t) = \frac{\partial}{\partial t}\phi(x,t)$.

Note we can also do everything but starting from P and not H. But we won't worry about that.

Causality Note that causality is broken ~~when~~ without the presence of fields.

$\rightarrow$ QFT resolves causality problem.

In our present formalism, still in Heisenberg picture...

...the amplitude for a particle to propagate from y to x is given by

$$\langle 0|\phi(x)\phi(y)|0\rangle$$

"Two-point correlation function"

Let $\boxed{\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \langle 0| \text{ in terms of } a_p \,\&\, a_p^\dagger \ldots |0\rangle$$

$$= \langle 0| a_p a_q^\dagger |0\rangle =$$

$$= (2\pi)^3 \delta^{(3)}(p-q)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)}$$

More explicitly...

@ $p^0 = E_p$
$p^{0'} = E_{p'}$

$$\mathcal{D}(x-y) = \langle 0|\phi(x)\phi(y)|0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{\sqrt{2E_p}} \frac{1}{\sqrt{2E_{p'}}} \langle 0| e^{-ip\cdot x} e^{+ip'\cdot y} a_p a_{p'}^\dagger |0\rangle$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^{\cancel{6}}} \left(\frac{1}{\sqrt{2E_p}}\right) \frac{1}{\sqrt{2E_{p'}}} e^{-ip\cdot x + ip'\cdot y} \cancel{(2\pi)^3} \delta^{(3)}(p-p')$$

$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \cdot e^{-ip\cdot(x-y)} \quad \square .$$

Note that this integral is Lorentz-invariant.

→ Now, let us evaluate this integral for some particular value of $x-y$...

(1) Suppose that $x-y = (t, 0, 0, 0)$, then

$$\mathcal{D}(x-y) = \frac{4\pi}{(2\pi)^3} \int_0^\infty dp \, \frac{p^2}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\,t}$$

(timelike)

$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$

$$\underset{t\to\infty}{=} e^{-imt}.$$

→ dominated by region where $p \approx 0$.

(2) Suppose that $x-y = (0, \vec{x}-\vec{y}) = (0, \vec{r})$ then

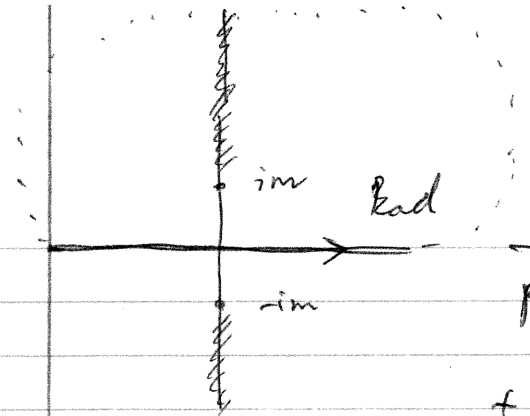
$$\mathcal{D}(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{i\vec{p}\cdot\vec{r}}$$

$$= \frac{2\pi}{(2\pi)^3} \int_0^\infty dp \, \frac{p^2}{2E_p} \frac{e^{ipr} - e^{-ipr}}{ipr}$$

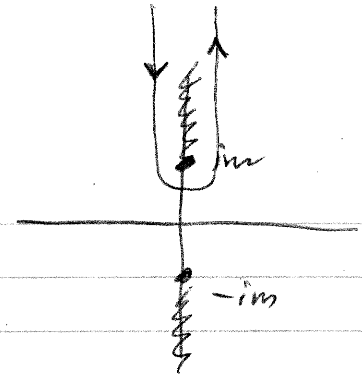
$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \, \frac{p e^{ipr}}{\sqrt{p^2+m^2}}$$

→ branch cut @ $\pm im$ (singularity)...

→ must change contour... 2 Wick rotate



push contour + Wick rotate ...



To get

$$\mathcal{D}(x-y) = \frac{1}{4\pi^2 r}\int_m^\infty d\rho \, \frac{\rho e^{-i\rho r}}{\sqrt{\rho^2 - m^2}}$$

$\rho = -ip$ (Wick rotate)

$$\Rightarrow \boxed{\mathcal{D}(x-y) \underset{r\to\infty}{=} \frac{e^{-mr}}{r}} \longrightarrow \underline{\text{non zero!}}$$

(more details about integrals like this can be found in Zee's QFT in a nutshell ...)

What does it mean for $\mathcal{D}(x-y)$ to be nonzero when $x-y$ is spacelike?

We saw that when $(x-y)^\mu (x-y)_\mu = -(\vec{x}-\vec{y})^2 < 0$ is spacelike, cannot have causality between x & y.

$\mathcal{D}(x-y) \neq 0 \Rightarrow$??? paradox?

$\rightarrow$ <u>NO!</u> To discuss causality, we should ask not whether particles can propagate over spacelike intervals ...

... but whether a "<u>measurement</u>" ~~at~~ at one point can affect a measurement at another point whose separation from the first is spacelike -

→ No violation of causality b/c no signal can be transmitted → no info is exchanged.

→ Suppose we have a local measurement @ x, call this $\phi(x)$, & a local measurement @ y, called $\phi'(y)$

So long as $[\phi(x), \phi'(y)] = 0$, the 2 measurements don't affect one another.

~ measure the field ϕ @ x & @ y,

If $[\phi(x), \phi(y)] = 0$ ~~when~~ $(x-y)^2 < 0$ then we're good

→ $$\cancel{[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x}[a_p, a^+_{p'}] e^{ip'\cdot y} + e^{ip\cdot x}[a^+_p, a_{p'}] e^{-ip'\cdot y} \right\}}$$

→ $$[\phi(x), \phi(y)] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \times \left\{ \left[(a_p e^{-ip\cdot x} + a_p^+ e^{ip\cdot x}), (a_{p'} e^{-ip'\cdot y} + a^+_{p'} e^{ip'\cdot y}) \right] \right.$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{1}{2\sqrt{E_p E_{p'}}} \left\{ e^{-ip\cdot x} \underbrace{[a_p, a^+_{p'}]}_{(2\pi)^3 \delta^3(p-p')} e^{ip'\cdot y} + e^{ip\cdot x} \underbrace{[a^+_p, a_{p'}]}_{-(2\pi)^3 \delta^3(p-p')} e^{-ip'\cdot y} \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2E_p}\right) \left\{ e^{-ip(x-y)} - e^{-ip(y-x)} \right\} = D(x-y) - D(y-x)$$

So $[\phi(x), \phi(y)] = D(x-y) - D(y-x)$.

Since $D(y-x)$ is Lorentz invariant, can take

$y-x \rightarrow x-y$. (possible b/c $(x-y)^2 < 0$)

note that when $(x-y)^2 > 0 \rightarrow$ there's no continuous transf that takes $y-x \rightarrow x-y$

$\rightarrow$ so this is only possible because $(x-y)^2 < 0$ (spacelike).

So $D(y-x) = D(x-y)$

So, $[\phi(x), \phi(y)] = 0$ ✓

The Klein-Gordon Propagator

~~Let's look at $[\phi(x), \phi(y)]$ in more details...~~

~~$[\phi(x), \phi(y)]$ is just a number~~

~~$\Rightarrow$ can write $[\phi(x), \phi(y)] \equiv \langle 0| [\phi(x), \phi(y)] |0\rangle$~~

~~$\rightarrow \langle 0| [\phi(x), \phi(y)] |0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \left\{ e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right\}$~~

~~(assuming $x^0 > y^0$)~~

Poles @ $p^2 = m^2$ ⇕ $p^0 = \pm E_p$

~~$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_p} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_p} - \frac{1}{2E_p} e^{-ip(x-y)} \Big|_{p^0 = -E_p}\right.$~~

The Klein-Gordon Propagator

→ Before looking at this ... need to look at Green's Functions & Contour Integrals ...

→ Consider function

$$\frac{1}{(p^0 - E_p)(p^0 + E_p)} \qquad E_p = \sqrt{|p|^2 + m^2}.$$

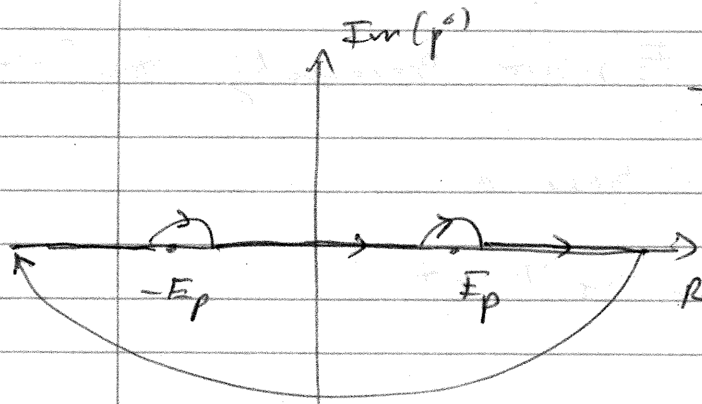
→ Poles at $p^0 = \pm E_p$.

Look at FT:

$$\boxed{f(t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}}$$

→ How to integrate this?

If $t > 0$ →



$$\rightarrow f(t) = (-2\pi i) \sum \text{Res}(f)$$

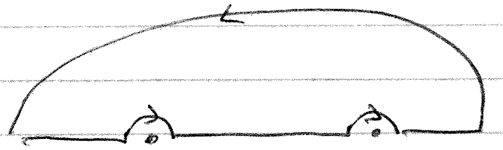
$$= (-2\pi i) \left\{ \frac{e^{iE_p t}}{2\pi(-2E_p)} + \frac{e^{-iE_p t}}{2\pi(2E_p)} \right\}$$

↑ close contour ...

but clockwise → reverse sign

$$\rightarrow \boxed{f(t) = \frac{-i}{2E_p}\left(e^{+iE_p t} - e^{-iE_p t}\right)} \qquad (t > 0)$$

If $t<0$ close contour above poles



$\rightarrow f(t) = 0.$

$\rightarrow$ So, altogether, we have...

$$f(t) = \int \frac{1}{2\pi} \frac{dp^0}{(p^0 - E_p)(p^0 + E_p)} e^{-ip^0 t}$$

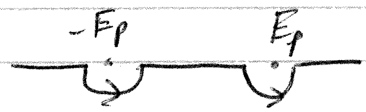
$$= \Theta(t) \left(\frac{-i}{2E_p}\right)\left(e^{-iE_p t} - e^{+iE_p t}\right)$$

where $\Theta(t)$ is the Heaviside Step fn...

$$\Theta(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0. \end{cases}$$

$\longrightarrow$ Retarded / Forward Propagating Green's fn

Suppose the contour is taken as



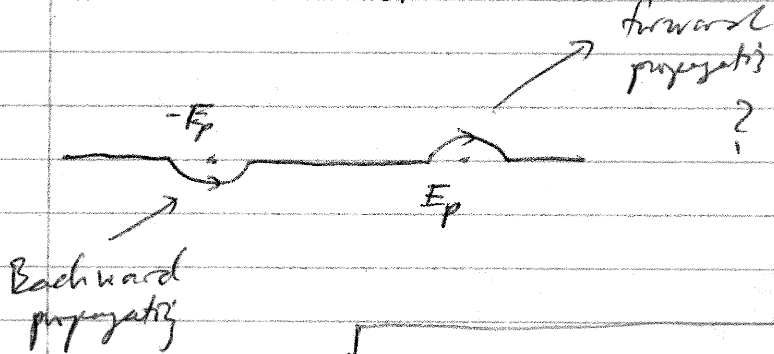
, then we'll get

$t > 0 \rightarrow f(t) = 0$

$t < 0 \rightarrow f(t) = \Theta(-t) \frac{i}{2E_p}\left(e^{-iE_p t} - e^{iE_p t}\right)$

$\rightarrow$ Advanced / Backward Propagating Green's fn.

What about



$$\rightarrow \boxed{f(t) = \Theta(t)\left(\frac{-i}{2E_p}\right)e^{-iE_pt} + \Theta(-t)\left(\frac{-i}{2E_p}\right)e^{+iE_pt}}$$

↑ Time-ordered Green's fn.

With this, we can study the commutator $[\phi(x), \phi(y)]$ ...

Consider this quantity...

$$\langle 0|\,[\phi(x), \phi(y)]\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\underbrace{\frac{1}{2E_p}\left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\left\{ \frac{1}{2E_p} e^{-ip(x-y)} + \frac{1}{-2E_p} e^{-ip(x-y)} \right\}$$

pole @ $p_0 = E_p$ pole @ $p_0 = -E_p$

(4D) integral →

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3}\underbrace{\int \frac{dp^0}{2\pi i}\frac{-1}{p^2 - m^2}e^{-ip(x-y)}}$$

$f(t)$ before, where

$$(p^0 - E_p)(p^0 + E_p) = p^{0\,2} - |p|^2 - m^2 = p^2 - m^2 .$$

Now, call ...

$$\boxed{D_R(x-y) \equiv \theta(x^0-y^0)\langle 0|[\phi(x),\phi(y)]|0\rangle}$$

Then

$$(\Box+m^2)D_R(x-y) = \Box D_R(x-y) + m^2 D_R(x-y)$$

$$= (\Box\theta(x^0-y^0))\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ 2(\partial_\mu\theta(x^0-y^0))\partial^\mu\langle 0|[\phi(x),\phi(y)]|0\rangle$$

$$+ \theta(x^0-y^0)(\Box+m^2)\underbrace{\langle 0|[\phi(x),\phi(y)]|0\rangle}_{\cancel{\text{constant}}}$$

$$= -\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$+ 2\delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle + 0$$

needs verification (easy)

$$= \delta(x^0-y^0)\langle 0|[\pi(x),\phi(y)]|0\rangle$$

$$= -i\delta^{(4)}(x-y)$$

$$-i\delta^{(3)}(x-y)$$

So

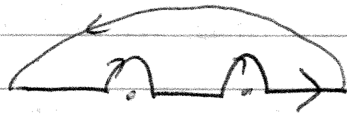
$$\boxed{(\Box+m^2)D_R(x-y) = -i\delta^{(4)}(x-y)}$$

$\longrightarrow$ $D_R(x-y)$ is a Green's fn of the Klein-Gordon operator.

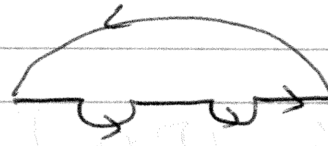
Since $D_R(x-y) = 0$ @ $x^0 < y^0$

$\Rightarrow$ $D_R(x-y) \equiv$ "Retarded" Green's fn

Now... As we have seen, there are many ways to take the contour...



or



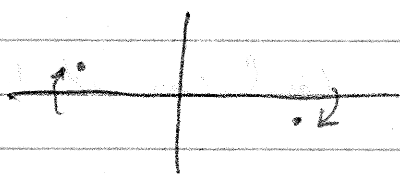
or

etc

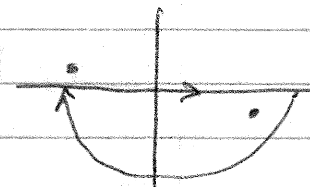
→ Use the Feynman prescription instead

$$\boxed{D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip\cdot(x-y)}}$$

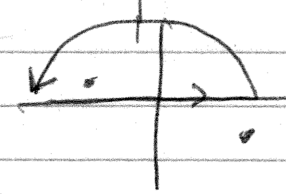
→ Convenient! B/c now poles are $p^0 = \pm(E_p - i\varepsilon)$



→ when $x^0 > y^0$ → take contour below



when $x^0 < y^0$ → take contour above.



→ get same expression but with $x \leftrightarrow y$.

So,

$$\boxed{\begin{aligned} D_F(x-y) &= \begin{cases} D(x-y) & x^0 > y^0 \\ D(y-x) & x^0 < y^0 \end{cases} \\ &= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle \\ &\equiv \langle 0|T\{\phi(x)\phi(y)\}|0\rangle \end{aligned}}$$

"time-ordering" symbol.

Time-ordering symbol $\Longrightarrow$ instructs us to place the operators that follows in order with the latest to the left.

$\rightarrow$ apply $(\Box + m^2)$ to last line, get D_F is Green's fn of Klein-Gordon operator.

$\hookrightarrow$ $D_F(x-y)$ is called the "Feynman Propagator" for a Klein-Gordon operator...

$\hookrightarrow$ propagation amplitude

$\rightarrow$ But we can't much calculation at this point just yet.

$\rightarrow$ B/c We're only looked at the free KG theory

$\rightarrow$ field eqn in this case is linear $\therefore$ there are no interactions...

$\rightarrow$ this theory is too simple to make any predictions...

$\rightarrow$ need perturbation ....

One kind of interaction that ~~ex~~ can als be solved ...

$\hookrightarrow$

Particle Creation by a classical Source

Consider the source $j(x)$

Recall... free field: $(\Box + m^2)\phi = 0$

→ Now... $(\Box + m^2)\phi = j(x)$ fixed fn in space time.

$j(x)$ is nonzero only for a finite time interval

The associated Lagrangian is

$$\mathcal{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

If $j(x)$ is turned on for only a finite time, it is easiest to solve...

Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}\right)$$

With a source...

$$\phi(x) = \phi_0(x) + i\int d^4y\, D_R(x-y)\, j(y)$$

⋮

We won't worry about this for now...

Some problems & Insights

(1) Classical EM (no sources) follows from the action

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

(a) Identify $\begin{cases} E^i = -F^{0i} \\ \varepsilon^{ijk} B^k = -F^{ij} \end{cases}$

→ Derive the E-L eqn (Maxwell's eqn)

→ easy to show that

$$\boxed{\partial_\mu F^{\mu\nu} = 0} ; \quad \vec{\nabla} \cdot \vec{E} = 0 \quad (\nu = 0)$$

$$-\partial_t \mathcal{E} + \vec{\nabla} \times \vec{E} = 0 \quad (\nu = i)$$

(2) Complex scalar field

$$S = \int d^4x \left(\partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi \right)$$

Derive E-L eqn:

$$\boxed{\begin{aligned} &i\partial_t \phi^+ - \frac{1}{2m} \nabla^2 \phi^+ = 0 \\ &-i\partial_t \phi - \frac{1}{2m} \nabla^2 \phi = 0 \end{aligned}}$$

now... write $\phi \to e^{-i\theta}\phi$, $\phi^\dagger \to e^{i\theta}\phi^\dagger$

$\sim \phi - i\theta\phi$ $\qquad \sim \phi^\dagger + i\theta\phi^\dagger$

$\to \Delta\phi \sim -i\theta$ $\qquad \Delta\phi^\dagger \sim i\phi^\dagger$

<u>So that</u>

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\Delta\phi^\dagger$$

conserved current.

↳ can find conjugate momenta:

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \to \ldots \text{compute} \ldots$$

→ can get Hamiltonian... there's a Ham. formula in book, but we won't worry abt this.

$$\mathcal{H} = \int d^3x \left(\pi^\dagger\pi + (\nabla\phi)^\dagger(\nabla\phi) + m^2\phi^\dagger\phi\right)$$

(3) If we take $(x-y)^2 = -r^2$ → can implicitly write

$$\langle 0|\phi(x)\phi(y)|0\rangle = D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

↳ when $(x-y)^2 < -r^2 \to D(x-y)$ can be written ~~as~~ in terms of Bessels Functions...

THE DIRAC FIELD

(1) Lorentz Invariance in Wave Eqn

→ Lorentz transformation... $\boxed{x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu}$

→ what happens to $\phi(x)$ under Λ?

we require that $\phi'(x') = \phi(x)$

$$\to \boxed{\phi(x) \to \phi'(x) = \phi(\Lambda^{-1} x)} \text{ so that}$$

$$\phi'(\Lambda x) = \phi(\Lambda^{-1}\Lambda x) = \phi(x) \checkmark$$

→ what about $\partial_\mu \phi(x)$?

Under transform --- $\partial_\mu \phi(x) \to \partial_\mu(\phi(\Lambda^{-1}x))$

$$\to \boxed{\partial_\mu(\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x)}$$

Recall that ...

$$(\Lambda^{-1})^\rho{}_\mu (\Lambda^{-1})^\sigma{}_\nu g^{\mu\nu} = g^{\rho\sigma}$$

$$\to (\partial_\mu \phi(x))^2 \to g^{\mu\nu}(\partial_\mu \phi'(x))(\partial_\nu \phi'(x))$$

$$= g^{\mu\nu}[\partial_\mu(\phi(\Lambda^{-1}x))][\partial_\nu(\phi(\Lambda^{-1}x))]$$

$$= g^{\mu\nu}\{(\Lambda^{-1})^\rho{}_\mu \partial_\rho \phi\}\{(\Lambda^{-1})^\sigma{}_\nu \partial_\sigma \phi\}(\Lambda^{-1}x)$$

$$= (\partial\phi)^2(\Lambda^{-1}x) \leftarrow = g^{\rho\sigma}\partial_\rho \phi\, \partial_\sigma \phi\, (\Lambda^{-1}x) \to \text{evaluated}$$

Recall

$$\boxed{\begin{aligned} \phi(x) &\to \phi(\Lambda^{-1}x) \\ \partial_\mu \phi(x) &\to (\Lambda^{-1})^\nu{}_\mu (\partial_\nu \phi)(\Lambda^{-1}x) \\ (\partial_\mu \phi(x))^2 &\to (\partial_\mu \phi)^2(\Lambda^{-1}x) \end{aligned}}$$

So it is clear that

$$\boxed{\mathcal{L} \longrightarrow \mathcal{L}(\Lambda^{-1}x)}$$

Lagrangian is Lorentz-invariant.

$\Rightarrow$ The action $S = \int d^4x\, \mathcal{L}$ is also Lorentz inv.

$\to$ also clear that EOM is also Lorentz inv.

$$(\Box + m^2)\,\phi'(x) = \left((\Lambda^{-1})^\mu{}_\nu \partial_\mu (\Lambda^{-1})^{\sigma\mu} \partial_\sigma + m^2\right)\phi(\Lambda^{-1}x)$$

$$= (\partial^\sigma \partial_\sigma + m^2)\,\phi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Sep 20, 2020

$\to$ How do we find Lorentz-invariant theories, in general?

$\to$ for simplicity, restrict attention to lin. transf

$\to$ $\Phi_a = v \in \mathbb{C}^n$,

matrix giving Lorentz transform Λ.

$\to$ $\boxed{\Phi_a(x) \Rightarrow \underbrace{M_{ab}(\Lambda)}_{n\times n}\, \Phi_b(\Lambda^{-1}x)}$

→ the most general nonlinear transf can be built out of linear ones ⇒ suffices to consider M only.

↳ for short, write $\Phi \longmapsto M(\Lambda)\Phi$.

→ What are the possible allowed $M(\Lambda)$?

Obs $\{\Lambda\}$ form a group $M(\Lambda'')M(\Lambda) = M(\Lambda')$ → $\Lambda''\Lambda = \Lambda'$

→ the correspondence between Λ & M must be preserved under multiplication.

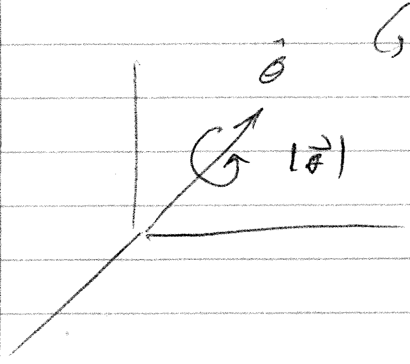
$\{\Lambda\}$ Lorentz group $\longmapsto$ $\{M(\Lambda)\}$ → n-dim representation of the Lorentz group

↳ $\boxed{?}$ What are the finite-dim matrix reps of the Lorentz group?

Ex in QM ... spin $\frac{1}{2}$ → $\{M\}$ are the 2×2 unitary matrices with determinant 1.

→ $\boxed{U = e^{-i\theta^i\sigma^i/2}}$ → $\sum_i \theta^i\sigma^i/2$

3 arbitrary parameters & Pauli matrices.



$U(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{\sigma}/2}$

→ In the case for arbitrary spin representations...

$$\mathcal{U}(\vec{\theta}) = e^{-i\vec{\theta}\cdot\vec{J}} \quad \text{where } \vec{J} = (J^1, J^2, J^3)$$

$$\text{and } [J^i, J^j] = i \sum_l \epsilon^{jkl} J^l$$

→ check that this works for spin 1/2:

$$\left[\frac{\sigma^j}{2}, \frac{\sigma^k}{2}\right] = i \sum_l \epsilon^{jkl} \frac{\sigma^l}{2} \quad ✓$$

→ for spinless particles... $\psi(\vec{x})$ can be decomposed into orbital angular momentum states $J = 0, 1, 2, \ldots$
(no intrinsic spin ⇒ $J = L$)

- $\vec{J} = \vec{L} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$
- $J^j = i \sum_l \epsilon^{jkl} x^k \nabla^l$
- $\nabla^l = -\nabla_l = -\dfrac{\partial}{\partial x^l}$

But the cross product is special to 3D case.

→ write operator as antisymmetric tensor...

$$\boxed{J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)}$$

→ reproduces the cross product.

so that $J^3 = J^{12}$, etc.

→ generalize to 4D: → 6 operators that generate {3 boosts, 3 rotations, of the Lorentz group.

$$\boxed{J^{\mu\nu} = +i(x^\mu \partial^\nu - x^\nu \partial^\mu)}$$

→ Spatial Rotations : $J^{jk} = i(x^j \partial^k - x^k \partial^j)$

→ Lorentz boosts along x^j axis : $J^{0j} = i(x^0 \partial^j - x^j \partial^0)$

→ Now, want to get commutation rules...
→ compute the commutators of differential ops

to get

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho} J^{\mu\sigma} - g^{\mu\sigma} J^{\nu\sigma} - g^{\nu\sigma} J^{\mu\rho} + g^{\mu\sigma} J^{\nu\rho}\right)}$$

Ex 3 rotations :

$J^{12} = -J^{21}$
$J^{23} = -J^{32}$
$J^{13} = -J^{31}$

3 boosts

$J^{01} = -J^{10}$
$J^{02} = -J^{20}$
$J^{03} = -J^{30}$

→ 6 total generating matrices...

Ex Consider the 4x4 matrices $(J^{\mu\nu})_{\alpha\beta}$ where μ, ν label which of the 6 matrices, while α, β label the component / matrix element...

$$\boxed{(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right)}$$

→ can verify that $(J^{\mu\nu})_{\alpha\beta}$ satisfies the comm. relation...

→ These are matrices that act on ordinary Lorentz 4-vectors...

to see this...

→ Look at elements of the Lorentz group...

$$\mathcal{U}(\omega_{\mu\nu}) = \operatorname{Exp}\left[\frac{-i}{2}\,\omega_{\mu\nu} J^{\mu\nu}\right]$$

infinitesimally →
$$\simeq \mathbb{I} + \frac{-i}{2}\omega_{\mu\nu} J^{\mu\nu}$$
$$\simeq \delta^\alpha_\beta + \frac{-i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta$$

So, infinitesimally...

$$V^\alpha \longrightarrow \delta^\alpha_\beta + \frac{-i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\alpha{}_\beta V^\beta$$

→ $\omega_{\mu\nu}$ is an anti-symmetric tensor that gives the infinitesimal angles...

V_α, V_β → 4-vectors...

Ex 1 when $\omega_{12} = -\omega_{21} = \theta$, $\omega_{\mu\nu} = 0$ else, we get

$$[V^\mu] \longrightarrow \begin{array}{c c} & \\ t & \\ x & \\ y & \\ z & \end{array}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\theta & 0 \\ 0 & \theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}[V^\mu]$$
$$\quad\quad t \quad x \quad y \quad z$$

→ Infinitesimal $\boxed{\text{ROTATION on } xy \text{ plane.}}$

Ex 2 when $\omega_{01} = -\omega_{10} = \beta$ ⇒ get
$\omega_{\mu\nu} = 0$ else

$$[V^\mu] \overset{\sim}{\longrightarrow} \begin{pmatrix} 1 & \beta & & \\ \beta & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}[V^\mu] \rightarrow \boxed{\text{BOOST along } x}$$

$\boxed{\text{THE DIRAC EQUATION}}$

→ Now that we have seen one f.d. representation of the Lorentz group

→ need to develop formalism for finding all other ~~formalisms~~ representations...

(Problem 3.1)

→ focus on spin $1/2$ systems...

→ In this case, use Dirac's trick ~~due to~~ -

Suppose we had a set of 4 $n \times n$ matrices γ^μ satisfying:

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} * I}$$

Then we could write down an n-dim representation of the Lorentz algebra...

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

These matrices satisfy the commutation relation...

→ $[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$

→ Verify that this trick works in 3D Euclidean space:

in which case, $\boxed{\gamma^j = i\sigma^j}$ → $\boxed{\{\gamma^i, \gamma^j\} = -2\delta^{ij}}$

→ The matrices representing the Lorentz algebra are then

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\sum_k \varepsilon^{ijk}\sigma^k = J^l$$

Which is what we saw before ⇒ angular momentum.

$$\left\{ \begin{array}{l} J^3 = S^{12} = \frac{1}{2}\sigma^3 \\ J^2 = S^{31} = \frac{1}{2}\sigma^2 \\ J^1 = S^{23} = \frac{1}{2}\sigma^1 \end{array} \right\}$$

→ now, want $S^{\mu\nu}$ for 4D Minkowski space...

→ matrices γ^μ must be at least 4x4.

→ suffices to write one explicit realization of the Dirac algebra since all reps are unitarily equiv

Ex

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & \mathbb{I}_{2\times2} \\ \mathbb{I}_{2\times2} & 0 \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}}$$

"Weyl" / "Chiral" representation.

→ In this case, the boost & rotation generators are...

Boosts → $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotations → $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \underbrace{\sum_k}_{\text{(sum)}} \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \sum_k \frac{1}{2}\varepsilon^{ijk}\Sigma^k$ (sum)

"

Digression: Group theory & Representation Theory

Why are we interested in this?

→ Recall that we want to look at ~~all~~ all transformations under which the Action S ~~Lagrangian~~ ~~is~~ is "invariant"

→ In particular, we want S to be Lorentz invariant

→ can consider this simple Lorentz transformation

$$\begin{cases} \phi(x) \longrightarrow \phi(\Lambda^{-1}x) \\ \text{i.e. } \phi(x^\mu) \rightarrow \phi(\Lambda^{-1\,\mu}{}_\nu\, x^\nu). \end{cases} \rightarrow \left(\text{check that } S \text{ is invariant}\right.$$

→ But this is very simple ... ⇒ There are many more transforms that leave S Lorentz invariant.

→ How do we find all of them?

⇒ For simplicity, we'll just restrict ourselves to linear ~~combinations~~ transformations

→ look at transformations of the form.

$$\phi_a(x) \longrightarrow \sum_b M_{ab}(\Lambda)\, \Phi_b(\Lambda^{-1}x)$$

→ more succinctly ...

$$\boxed{\phi \longrightarrow M(\Lambda)\,\phi}$$

These matrices M must be "nice" in the sense that M must obey...

$$\boxed{\phi \rightarrow M(\Lambda'')M(\Lambda)\phi = M(\Lambda''\Lambda)\phi}$$

This says that $\{M\}$ (the collection of M's) must be a <u>representation</u> of the Lorentz group.

<u>What??</u> So... recall that $\{\Lambda\}$ is a collection of Lorentz transforms, and they form a group

$$\rightarrow \quad \boxed{\{\Lambda\} \equiv \text{Lorentz group}}$$

<u>A representation</u> π of a group G is a function satisfying the property

$$\pi(g_1)\,\pi(g_2) = \pi(g_1 g_2)$$

$$\uparrow \qquad \uparrow \qquad \uparrow$$

$$\in G \qquad \in G \qquad \in G$$

With this, it is clear that

$$\boxed{\{\Lambda\}\ \text{Lorentz group} \Longrightarrow \{M\}\ \text{is a representation of}\ \{\Lambda\}}$$

So... what are these M's?

→ <u>Ex</u> Rotation group for spin 1/2 particles

For spin-$\frac{1}{2}$, the most important nontrivial representation is the 2D representation.

↳ these are unitary matrices with det = 1
(2x2)

⇒ In general: $\mathcal{U} = e^{-i\vec{\sigma}\cdot\vec{\theta}/2}$

$\vec{\sigma} \rightarrow$ Pauli matrices
$\vec{\theta} \rightarrow$ angle.

For infinitesimal rotations, we can write

$$\mathcal{U} = 1 - \frac{i}{\hbar}\vec{J}\cdot\vec{\theta} \equiv 1 - \vec{J}\cdot\vec{\theta}$$

$\{\mathcal{U}\}$ form a Lie-algebra of the L-group.

$\vec{J}$ here are the "generators" of the Lie algebra

when $\{\mathcal{U}\}$ is a representation of the rotational group, we identify

$$\vec{J} \leftrightarrow \frac{\vec{\sigma}}{2}$$

→ $\vec{J}$ as the quantum angular momentum operator

→ satisfies the commutation relation

$$[J^i, J^j] = i\varepsilon^{ijk} J^k$$

like the generators of SO(3), namely the Pauli matrices...

→ finite rotations are formed by matrix exp.

$$R = \exp\left[-i\theta^i\sigma^i\right]$$

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Back to present problem...

to get generator of the Lie algebra of the Lorentz group, first look at how the angular momentum operators are written in 4D:

(3D) $\vec{J} = \vec{x}\times\vec{p} = \vec{x}\times(-i\vec{\nabla})$

(4D) $\boxed{J^{\mu\nu} = i(x^\mu\partial^\nu - x^\nu\partial^\mu)}$

with commutation relation:

$$\boxed{[J^{\mu\nu}, J^{\rho\sigma}] = i\left(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}\right)}$$

→ any matrices that are ~~re~~ represent this algebra must obey the same comm. relation.

→ look at matrices of the form

$$(J^{\mu\nu})_{\alpha\beta} = i\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha\right)$$

→ by symmetry, μ,ν ~~table~~ label which of the six matrices we want;

. α,β label components.

The Dirac Eqn

What are the representations of the Lorentz group?

, especially for spin-$\frac{1}{2}$?

<u>Dirac's trick</u> if we have a set of 4 $n \times n$ matrices γ^μ which satisfies:

Dirac algebra $\rightarrow$ $$\boxed{\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} * I_{n\times n}}$$

Then the n-dim representation of the Lorentz algebra:

$$\boxed{S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]}$$

$\rightarrow$ In other words, $S^{\mu\nu}$ satisfies ...

$$[S^{\mu\nu}, S^{\rho\sigma}] = i(g^{\nu\rho}S^{\mu\sigma} + g^{\mu\sigma}S^{\nu\rho} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho})$$

(*) Note that this trick works ~~in~~ in any dim.

e.g. take $\gamma^j = i\sigma^j$ so that $\{\gamma^i, \gamma^j\} = -2\delta^{ij}$

$$\Rightarrow S^{ij} = \frac{1}{2}\varepsilon^{ijk}S^k$$ $\rightarrow$ just as before.

2D representation of the rotation group.

Spin $\frac{1}{2}$: $J^3 = J^{12} = \frac{1}{2}\sigma^3$; $J^2 = \frac{1}{2}\sigma^2 = S^{31}$, $J^1 = S^{23} = \frac{1}{2}\sigma^1$

One such representation for the Dirac algebra is

$$\boxed{\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}_{4\times4} \; ; \; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}_{4\times4}}$$

Weyl / chiral representation.

get

Boosts $\quad S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotations

$$S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\varepsilon^{ijk}\Sigma^k$$

Hermitian Defn not ~~[illegible]~~, but Ψ is also classical field, not a wfn

All 4-component field Ψ that transforms under boosts & rotations according to is called a Dirac spinor

S^{ij} are Hermitian
S^{0i} are anti-Hermitian

fine b/c Ψ is a classical field, not a ~~k~~ wfn.

//

Now, what is the field eqn for ψ?

→ try $(\Box + m^2)\psi = 0$ ← KG field eqn.

But this obviously works because the representation are block-diagonal...

→ need a stronger equation that implies the KG eqn but also contains additional info.

To do this, look ~~the~~ at transformation of γ matrices

In an expression we can think of...

$$[\cdots]\Lambda_{\frac{1}{2}}^{-1}\left[4\times 4\right]\Lambda_{\frac{1}{2}}\left[\vdots\right]$$

$\frac{1}{2}$: for spin $\frac{1}{2}$

where $\Lambda_{\frac{1}{2}} = \exp\left\{\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right\}$

$$\simeq 1 - \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}$$

$$\Rightarrow [\gamma^\mu] \to [\Lambda_{\frac{1}{2}}^{-1}][\gamma^\mu][\Lambda_{\frac{1}{2}}]$$

$$= \left(1 + \frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)\gamma^\mu\left(1 - \frac{i}{2}\omega_{\alpha\beta}S^{\alpha\beta}\right)$$

$$= \cdots \quad \text{(some terms of higher order cancelled ...)}$$

$$= \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}\underbrace{[\gamma^\mu, S^{\alpha\beta}]}_{?}$$

where a quick computation shows that

$$\boxed{[\gamma^\mu, S^{\alpha\beta}] = (\mathcal{J}^{\alpha\beta})^\mu{}_\nu \gamma^\nu}$$

where

$$\mathcal{J}^{\alpha\beta} = +i\left(g^{\alpha\mu}\delta^\beta_\nu - g^{\beta\mu}\delta^\alpha_\nu\right)$$

So...

$$\boxed{\gamma^\mu \longrightarrow \gamma^\mu - \frac{i}{2}\omega_{\alpha\beta}(\mathcal{J}^{\alpha\beta})^\mu{}_\nu \cancel{\gamma}^\nu} = \Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}$$

→ γ^μ transforms like 4-vectors...!

⇒ γ^μ are invariant under simultaneous rotations of their <u>vectors</u> & <u>spinor</u> indices...

→ can treat "μ" on γ^μ as a vector index!

→ can dot γ^μ into ∂_μ to form a Lorentz-inv. differential operator...

<u>Dirac eqn</u>

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}$$

Check that this is Lorentz-inv:

Let $\Psi(x) \longrightarrow \Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$ then

$$i\gamma^\mu\partial_\mu\Psi \to \left(i\gamma^\mu\Lambda^{+1}_{\frac{1}{2}}\right)\left(\partial_\mu\left(\Psi(\Lambda^{-1}x)\right)\right)$$

$$= i\Lambda_{\frac{1}{2}}\left(\Lambda^{-1}_{\frac{1}{2}}\gamma^\mu\Lambda_{\frac{1}{2}}\right)\cdot(\Lambda^{-1})^\alpha{}_\mu(\partial_\alpha\Psi)(\Lambda^{-1}x)$$

~~same~~ Lorentz transform

$$= i\Lambda_{\frac{1}{2}} (\Lambda)^{\mu}{}_{\nu} \gamma^{\nu} \cdot (\Lambda^{-1})^{\alpha}{}_{\mu} \; (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \underbrace{(\Lambda)^{\mu}{}_{\nu}(\Lambda)^{\alpha}{}_{\mu}}_{\delta^{\alpha}_{\nu}} (\partial_\alpha \Psi)(\Lambda^{-1}x)$$

$$= i\Lambda_{\frac{1}{2}} \gamma^{\mu} \partial_\mu \Psi(\Lambda^{-1}x)$$

$$\Rightarrow \quad i\gamma^{\mu}\partial_\mu \Psi(x) \to \Lambda_{\frac{1}{2}} i\gamma^{\mu}\Psi(\Lambda^{-1}x)$$

$\to$ transforms the same way as $\Psi(\Lambda^{-1}x)$.

Cleaner way:

Let $[i\gamma^{\mu}\partial_\mu - m]\Psi(x) \to [i\gamma^{\mu}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m]\Lambda_{\frac{1}{2}}\Psi(\Lambda^{-1}x)$

$$= \Lambda_{\frac{1}{2}} \underbrace{\Lambda^{-1}_{\frac{1}{2}} [i\gamma^{\mu}\Lambda^{-1\,\nu}{}_{\mu}\partial_\nu - m] \Lambda_{\frac{1}{2}}} \Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}} \{ i\Lambda^{\mu}{}_{\sigma}\gamma^{\sigma}(\Lambda^{-1})^{\nu}{}_{\mu}\partial_\nu - m \} \Psi(\Lambda^{-1}x)$$

$$= \Lambda_{\frac{1}{2}} \{ i\gamma^{\nu}\partial_\nu - m \} \Psi(\Lambda^{-1}x)$$

$$= 0 \quad \checkmark$$

Now, can show that Dirac eqn implies KG eqn:

$$0 = (i\gamma^{\mu}\partial_\mu - m)\Psi$$

$$\to \quad 0 = (-i\gamma^{\mu}\partial_\mu - m)(+i\gamma^{\nu}\partial_\nu - m)\Psi$$

$$= (\gamma^{\mu}\gamma^{\nu}\partial_\nu\partial_\mu + m^2)\Psi \qquad = \ldots$$

$$= (\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu + m^2)\psi$$

$$= \left[\frac{1}{2}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)\partial_\mu\partial_\nu + m^2\right]\psi$$

$$= \left[\frac{1}{2}\{\gamma^\mu,\gamma^\nu\}\partial_\mu\partial_\nu + m^2\right]\psi$$

KG eqn.

$$= (g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\psi = (\Box + m^2)\psi = 0$$

What is the Lagrangian for the Dirac theory?

→ need a way to multiply two Dirac spinors to get a Lorentz scalar.

$\psi^\dagger\psi$ doesn't work b/c under a boost,

$$\psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \Lambda_{\frac{1}{2}} \psi \neq \psi^\dagger\psi \quad \text{since } \Lambda_{\frac{1}{2}} = \exp\{-i\omega_{\mu\nu}S^{\mu\nu}\}$$

not unitary ... since not all $S^{\mu\nu}$ are Herm.

→ to fix this, define

$$\boxed{\bar{\psi} = \psi^\dagger\gamma^0}$$

Then under infinitesimal transform, get

$$\bar{\psi} \to \psi^\dagger\Lambda_{\frac{1}{2}}^\dagger\gamma^0 \cong \psi^\dagger\left(1 + \frac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger\right)\gamma^0$$

when ~~[illegible]~~ $\mu\neq 0$ & $\nu\neq 0$, $(S^{\mu\nu})^\dagger = (S^{\mu\nu})$

$$\& \ (S^{\mu\nu} \leftrightarrow \gamma^0)$$

When $\mu = 0$ or $\nu = 0$, $(S^{\mu\nu})^\dagger = -S^{\mu\nu}$

$\qquad$ 2 $S^{\mu\nu}$ anti-commutes w/ γ^0.

$$\rightarrow \quad \bar{\psi} \rightarrow \psi^\dagger \left(1 + \frac{i}{2} w_{\mu\nu} (S^{\mu\nu})^\dagger\right) \gamma^0$$

$$= \underbrace{\psi^\dagger \gamma^0}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right)$$

$$= \bar{\psi}\left(1 + \frac{i}{2} w_{\mu\nu} S^{\mu\nu}\right) = \bar{\psi} \Lambda_{\frac{1}{2}}^{-1} \text{ as desired.}$$

$$\rightarrow \quad \boxed{\bar{\psi} \rightarrow \bar{\psi} \Lambda_{\frac{1}{2}}^{-1}}$$

and so $\boxed{\bar{\psi}\psi = \psi^\dagger \gamma^0 \psi}$ is a Lorentz scalar.

Similarly, can show that

$\boxed{\bar{\psi}\gamma^\mu \psi}$ is a Lorentz vector.

$\rightarrow$ the correct Lorentz-inv Dirac Lagrangian is

$$\boxed{\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi}$$

$$\begin{cases} \text{E-L eqn for } \bar{\psi} \text{ gives } (i\gamma^\mu \partial_\mu - m)\psi = 0 \\ \text{E-L eqn for } \psi \text{ gives } -i\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0 \end{cases}$$

WEYL SPINOR

Recall that

$$\boxed{\begin{aligned} S^{0i} &= \frac{-i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} \\ S^{ij} &= \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \end{aligned}}$$

Since block-diagonal $\Rightarrow$ Dirac representation of the Lorentz group is reducible.

$\rightarrow$ Can form 2-D representations by considering each block separately...

$\rightarrow$ write $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ $\begin{matrix} \rightarrow \text{Left-handed} \\ \rightarrow \text{RH} \end{matrix}$ Weyl spinors

Under infinitesimal boost $\vec{\beta}$ & rotation $\vec{\theta}$, these transform as

$$\boxed{\begin{aligned} \psi_L &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 - \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_L \\ \psi_R &\rightarrow \left(1 - i\vec{\theta}\cdot\vec{\sigma}/2 + \vec{\beta}\cdot\vec{\sigma}/2\right)\psi_R . \end{aligned}}$$

Recall that $\left(\tanh(\vec{\beta}) = \frac{|\vec{v}|}{c}\right)$.

$\rightarrow$ Transf of ψ_R is equiv to transf of ψ_L^*

By noting that

$$\psi_L^* \longrightarrow \left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

noting that $\underset{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{\sigma^2}\vec{\sigma}^* = -\vec{\sigma}\underset{\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{\sigma^2}$ $\quad(\sigma^2 = \sigma^y)$

we find.

$$\sigma^2\psi_L^* \longrightarrow \sigma^2\left[1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right]\psi_L^*$$

$$= \underbrace{\left[1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right]}_{\text{like } \psi_R \text{ transform.}}\sigma^2\psi_L^*$$

So $\sigma^2\psi_L^*$ transform like ψ_R ...

With $\psi = \begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$, the Dirac eqn has form.

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0 \iff \begin{pmatrix}-m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = 0}$$

When $m = 0$, the eqns for ψ_L & ψ_R decouple to give us

$$\left\{\begin{aligned} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0\\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0\end{aligned}\right\} \longrightarrow \text{Weyl eqns.}$$

$\rightarrow$ important for neutrinos & weak force studies...

For convenience let us define...

$$\boxed{\sigma^\mu \equiv (1, \vec{\sigma}) \quad ; \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})}$$

so that

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$\sigma^\mu = (1, \sigma^1, \sigma^2, \sigma^3)$

With this, can simplify notation... Dirac eqn becomes...

$$\boxed{\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0}$$

$i\sigma\cdot\partial = i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})$

$i\bar{\sigma}\cdot\partial = i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})$

& the Weyl eqns become:

$$\boxed{\begin{aligned} (i\bar{\sigma}\cdot\partial)\psi_L &= 0 \\ (i\sigma\cdot\partial)\psi_R &= 0 \end{aligned}}$$

Free-particle solution of Dirac Eqn

Since Dirac field ψ satisfies KG eqn, ψ can be written as a lin. comb. of plane waves:

$$\psi(x) = u(p)\, e^{-ip\cdot x} \quad , \quad p^2 = m^2$$

so that

$p^0 = \sqrt{\vec{p}^2 + m^2} = E_{\vec{p}}$.

Look only solutions with positive frequency ... that is $E_p = p^0 > 0$ ....

ψ solves Dirac eq $\Rightarrow$ $(i\gamma^\mu \partial_\mu - m)\psi = 0$

$$\Leftrightarrow \quad (i\gamma^\mu \partial_\mu - m)\, u(p)\, e^{-ip\cdot x} = 0$$

$$\Rightarrow \quad \boxed{(i\gamma^\mu p_\mu - m)\, u(p) = 0}$$

go to rest frame $\Rightarrow$ $p = p_0 = (m, \vec{0})$. The soln for generic p can be obtained by boosting with $\Lambda_{\frac{1}{2}}$.

In rest frame, we have

$$(i\gamma^\mu p_\mu - m)\, u(p) \rightarrow (m\gamma^0 - m)\, u(p_0) = m(\gamma^0 - 1)\, u(p_0) = 0$$

$$\Rightarrow \quad m \begin{pmatrix} -I & I \\ I & -I \end{pmatrix} u(p_0) = 0$$

$$\Rightarrow \quad \boxed{u(p_0) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \quad \text{two-component spinor}}$$

just a factor, with norm. constraint.

$$\xi^\dagger \xi = 1.$$

<u>What are these ξ?</u>

Look at <u>rotation</u> generators...

$$\boxed{S^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}}$$

In particular, $S^z = S^{12} = \frac{1}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}$

So if $\left\{ \begin{array}{l} \xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{+1}{2} \\ \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rightarrow \text{gives spin-}z = \frac{-1}{2} \end{array} \right\}$

Now, we're in rest frame, so $p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Now, boost to frame where particle has velocity...

$\vec{v} = v \cdot \vec{z}$. Let $\tanh(\eta) = \frac{v}{c}$.

"rapidity"

Then

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = p^\mu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh(\eta) \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m\cosh\eta \\ 0 \\ 0 \\ m\sinh\eta \end{pmatrix}$$

(infinitesimal $\Lambda_{\frac{1}{2}}$)

$\Lambda_{\frac{1}{2}} \rightarrow$ just the Lorentz transform.

$\rightarrow$ In this frame, $\left\{ \begin{array}{l} E = m\cosh\eta \\ p^3 = m\sinh\eta \end{array} \right.$

Now, apply the same boost to $u(p)$ ...

$$u(p) = \Lambda_{\frac{1}{2}} \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix} \qquad \left(\Lambda_{\frac{1}{2}}\right) = \exp\left(\frac{-i}{2}\omega_{\mu\nu}S^{\mu\nu}\right)$$

$$= \exp\left(\frac{-1}{2}\eta \begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right) \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

S^{03} $\quad -i \,(\sigma^3 \quad \sigma^3)$

So, infitesimally...

$$\exp\left\{\frac{-1}{2}\eta\begin{pmatrix}\sigma^3 & 0\\ 0 & -\sigma^3\end{pmatrix}\right\} \simeq \begin{pmatrix}\cosh \eta/2 - \sinh \eta/2\, \sigma^3 & 0 \\ 0 & \cosh \eta/2 + \sinh \frac{\eta}{2}\sigma^3\end{pmatrix}$$

So that

$$u(p) \simeq \sqrt{m}\begin{pmatrix}\left(\cosh \eta/2 - \sinh \eta/2\, \sigma^3\right)\xi \\ \left(\cosh \eta/2 + \sinh \eta/2\, \sigma^3\right)\xi\end{pmatrix} \ldots$$

Simplify... note that

$$\left(\cosh \eta/2 - \sinh \eta/2\, \sigma^3\right)^2 = \ldots$$

$$= \cosh \eta - \sinh \eta\, \sigma^3$$

$$= E/m - p^3/m\, \sigma^3 = \frac{p\cdot\sigma}{m}$$

$$= \frac{p_\mu \sigma^\mu}{m} \quad \text{where } \sigma^\mu = (1, \vec{\sigma})$$

$p_\mu \sigma^\mu / m$

So...

$$\left(\cosh \eta/2 - \sinh \eta/2\, \sigma^3\right) = \sqrt{\frac{p\cdot\sigma}{m}}$$

&

$$\left(\cosh \eta/2 + \sinh \eta/2\, \sigma^3\right) = \sqrt{\frac{p\cdot\bar{\sigma}}{m}}$$

So...

$$\boxed{u(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi\end{pmatrix}} \longrightarrow \text{compact \& valid for any arbitrary direction of } \vec{p}.$$

Fact $(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0)^2 - \vec{p}^{\,2} = p^2 = m^2.$

Now, back to example...

$$p = (E, 0, 0, p^3)$$

$$\Rightarrow \quad p \cdot \sigma = \cdots = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$$

so

$$\sqrt{p \cdot \sigma} = \begin{pmatrix} \sqrt{E - p^3} & 0 \\ 0 & \sqrt{E + p^3} \end{pmatrix}$$

and

$$\sqrt{p \cdot \bar{\sigma}} = \begin{pmatrix} \sqrt{E + p^3} & 0 \\ 0 & \sqrt{E - p^3} \end{pmatrix}$$

pick $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ then (spin $\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E - p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

Pick $\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ then (spin $-\frac{1}{2}$)

$$u(p) = \begin{pmatrix} \sqrt{E + p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E - p^3} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

In the massless limit, $E \to p^3$ $\left(E^2 = \sqrt{m^2c^4 + (p^3)^2}\right)$

$$\rightarrow \boxed{\begin{array}{ll} u(p) = \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} & \text{spin } \frac{1}{2} \\ u(p) = \begin{pmatrix} \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} & \text{spin } -\frac{1}{2} \end{array}}$$

These states: $u(p) = \sqrt{2E}\begin{pmatrix}0\\0\\1\\0\end{pmatrix}$, $u(p) = \sqrt{2E}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ are eigenstates of the <u>helicity</u> operator

$$\boxed{h \equiv \hat{p}\cdot\vec{S} = \sum_i \frac{1}{2}\hat{p}^i\begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix}} = \frac{1}{2}\hat{p}\cdot\begin{pmatrix}\vec{\sigma} & 0\\ 0 & \vec{\sigma}\end{pmatrix}$$

When $\begin{cases} h = \frac{1}{2} \Rightarrow \text{call Right-handed} \\ h = \frac{-1}{2} \Rightarrow \text{call Left-handed}\end{cases}$

<u>Note</u>: Dirac helicity is frame-dependent... (for massive particle). — since can boost so that momentum is in the opposite direction.

(This can't happen for massless particles).

Back to Weyl's eqn:

$$\begin{cases} i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = i(\bar{\sigma}\cdot\partial)\psi_L = 0 \\ i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = i(\sigma\cdot\partial)\psi_R = 0 \end{cases}$$

Plug $\psi = u(p)e^{-ip\cdot x}$ in, $\partial_0 \longrightarrow -iE$

$\vec{\nabla} \longrightarrow i\vec{p}$.

So, with $m=0$, $\vec{p} = E\hat{p}$.

$$\Rightarrow \text{get} \begin{cases} (E + E\underbrace{\hat{p}\cdot\sigma}_{})\psi_L = 0 \Rightarrow (E)(1+2h)\psi_L = 0 \Rightarrow h = \frac{-1}{2} \\ (E - E\hat{p}\cdot\sigma)\psi_R = 0 \Rightarrow (E)(1-2h)\psi_R = 0 \longrightarrow h = \frac{1}{2}\end{cases}$$

$\Rightarrow \begin{cases} \psi_L \text{ is left-handed} \\ \psi_R \text{ is right-handed} \end{cases}$, as expected

$\#$

Recap... $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 > 0)$ $\rightarrow$ positive frequency

$\rightarrow$ $u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$ $\longrightarrow$ spinor.

when $\psi(x) = u(p)\, e^{-ip\cdot x}$ $(p^0 < 0)$ $\longrightarrow$ negative frequency

$\Rightarrow$ $(u(p)) = \ldots = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ -\sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$

$\#$

Now, note that ($p^0 > 0$ again)

$$u^\dagger u = \left(\xi^\dagger\sqrt{p\cdot\sigma},\ \xi^\dagger\sqrt{p\cdot\bar\sigma}\right)\cdot\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar\sigma}\,\xi \end{pmatrix}$$

$$= \xi^\dagger\left[(p\cdot\sigma) + (p\cdot\bar\sigma)\right]\xi$$

$\rightarrow$ depends on p!

$\Rightarrow$ $u^\dagger u = 2E_p\, \xi^\dagger\xi$ ~~ddd~~

$\rightarrow$ ~~dd~~ $u^\dagger u$ is <u>not</u> a Lorentz-inv scalar. just like $\psi^\dagger\psi$.

$\Rightarrow$ to make one such Lorentz-inv scalar, define

$\boxed{\bar u(p) = u^\dagger(p)\,\gamma^0}$ $\Rightarrow$ $\boxed{\bar u u = 2m\,\xi^\dagger\xi}$ Lorentz-inv (indep of $\vec p$)

So, with ~~a~~ $\bar{u}u = u^\dagger\gamma^0 u = 2m\underbrace{\xi^\dagger\xi}_{1} = 2m$

→ convenient to choose ONB spinors. ξ^1 & ξ^2.

This gives 2 linearly indep solution for $u(p)$:

$$\boxed{u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s \end{pmatrix} \qquad s = 1,2}$$

Normalize:

$$\bar{u}^r(p)\,u^s(p) = 2m\,\delta^{rs} \Leftrightarrow u^{r\dagger}(p)\,u^s(p) = 2E_p\,\delta^{rs}$$

For the negative-freq solns, we get

$$\bar{v}^r(p)\,v^s(p) = -2m\,\delta^{rs} \Leftrightarrow v^{r\dagger}(p)\,v^s(p) = +2E_p\,\delta^{rs}$$

and

v, u are orthogonal to each other...

$$\bar{u}^r(p)\,v^s(p) = \bar{v}^r(p)\,u^s(p) = 0$$

Finally, talk about $\boxed{\text{spin sums}}$

→ useful when evaluating Feynman diagrams.

→ when we need to sum all spin-$\frac{1}{2}$ polarizatin

Since $\{\xi^s\}$ form an ONB, ...

$$\sum_{s=1,2} \xi^s \xi^{s\dagger} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

Using this, we find that

$$\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) = \sum_s \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}\cdot\left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\sigma}\right)$$

$$= \sum_s \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s\\ \sqrt{p\cdot\bar\sigma}\,\xi^s\end{pmatrix}\cdot\left(\xi^{s\dagger}\sqrt{p\cdot\sigma},\ \xi^{s\dagger}\sqrt{p\cdot\bar\sigma}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

"completeness" $\longrightarrow$

$$= \begin{pmatrix}p\cdot\sigma & m\\ m & p\cdot\bar\sigma\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar\sigma & m\end{pmatrix}$$

$$\left(\begin{aligned}\sqrt{p\cdot\bar\sigma}\,\sqrt{p\cdot\sigma} &= \cancel{\tfrac{1}{2}\left((p\cdot\bar\sigma)+(p\cdot\sigma)\right)^2 - p\cdot\sigma - p\cdot\bar\sigma} \\ &= \sqrt{(p\cdot\bar\sigma)(p\cdot\sigma)} = \sqrt{\left((p^0,\vec{p})\cdot(1,-\vec\sigma)\right)\left((p^0,\vec{p})\cdot(1,\vec\sigma)\right)} \\ &= \sqrt{(p^0)^2 - p^2} = m.\end{aligned}\right)$$

So,

$$\boxed{\begin{aligned}\sum_{s=1,2} u^s(p)\,\bar{u}^s(p) &= \begin{pmatrix}m & p\cdot\sigma\\ p\cdot\bar\sigma & m\end{pmatrix} = \overbrace{p\cdot\gamma}^{\not p} + m\,I \\ \sum_{s=1,2} v^s(p)\,\bar{v}^s(p) &= \begin{pmatrix}-m & p\cdot\sigma\\ p\cdot\bar\sigma & -m\end{pmatrix} = \underbrace{p\cdot\gamma}_{\not p} - m\,I\end{aligned}}$$

Feynman's slash notation

→ The combo $\gamma \cdot p$ occur so often that Feynman introduced the notation:

$$\boxed{\not{p} \equiv \gamma^\mu p_\mu = p_\mu \gamma^\mu}$$

#

<u>Exercise</u> Recall that $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

Let ψ_L^* be the complex conjugate of ψ_L.
The Majorana eqn is given by

$$i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$$

where

$$\sigma^2 = \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

and

$$\bar{\sigma} = (1, -\vec{\sigma})$$

$$m = \text{Majorana mass.}$$

(a) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal rotation.

(b) Show that $i\bar{\sigma} \cdot \partial \psi_L(x) - im\sigma^2 \psi_L^*(x) = 0$ is inv under infinitesimal boosts.

(a) In general, infinitesimal Lorentz transforms on ψ_L has the form

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

$\rightarrow$ Rotation has the form:

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(x)$$

$$\Rightarrow \sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L(x) \quad \text{(notes)}$$

Lorentz transform:

$$\psi_L(x) \rightarrow \Lambda_{\frac{1}{2}}\psi_L(\Lambda^{-1}x)$$

$$\partial_\mu\psi_L(x) \longrightarrow (\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

$\rightarrow$ put these together...

$$\psi_L(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L(\Lambda^{-1}x)$$

$$\sigma^2\psi_L^*(x) \longrightarrow \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

$$\rightarrow \quad -im\sigma^2\psi_L^*(x) \longrightarrow -im\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*(\Lambda^{-1}x)$$

Next, $i\bar{\sigma}\cdot\partial\psi_L(x) = i\bar{\sigma}^\mu\partial_\mu\psi_L(x)$

$$\Rightarrow i\bar{\sigma}^\mu(\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= i\bar{\sigma}^\mu\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)(\Lambda^{-1})^\alpha{}_\mu\,\partial_\alpha\psi_L(\Lambda^{-1}x)$$

use trick: multiply:

$$1 \equiv \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right) \quad \text{(rot x inv\_rot)}$$

$$\Rightarrow \to i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\overbrace{\left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)}$$

$$\times\ (\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\boxed{?}\,(\Lambda^{-1})^{\alpha}_{\ \mu}\,\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

what is $\boxed{?}$

$$\to \boxed{?} = \left(1+i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\mu}\left(1-i\vec{\theta}\cdot\vec{\sigma}/2\right)$$

$$\approx \bar{\sigma}^{\mu} + i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\bar{\sigma}^{\mu} - i\vec{\theta}\cdot\bar{\sigma}^{\mu}\frac{\vec{\sigma}}{2}$$

$$= \bar{\sigma}^{\mu} - \frac{i}{2}\vec{\theta}\left[\bar{\sigma}^{\mu},\vec{\sigma}\right]$$

can show that

$$= \bar{\sigma}^{\mu} - i\theta\left[J_4^{ij}\right]^{\mu}_{\ \nu}\bar{\sigma}^{\nu}$$

$$\downarrow$$

$$i\left(g^{i\mu}\delta^{j}_{\nu} - g^{j\mu}\delta^{i}_{\nu}\right)$$

$$\Rightarrow \boxed{?} = (\Lambda_4)^{\mu}_{\ \nu}\,\bar{\sigma}^{\nu} \to \bar{\sigma}^{\mu} \text{ transforms like 4-vector}$$

$$\Rightarrow i\bar{\sigma}\cdot\partial\Psi_L(x) \Rightarrow i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Lambda^{\mu}_{\ \nu}\bar{\sigma}^{\nu}(\Lambda^{-1})^{\alpha}_{\ \mu}\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\delta^{\alpha}_{\nu}\bar{\sigma}^{\nu}\partial_{\alpha}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^{\nu}\partial_{\nu}\Psi_L(\Lambda^{-1}x)$$

$$= i\left(1-i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}\cdot\partial\Psi_L(\Lambda^{-1}x)$$

$\rightarrow \quad i\bar{\sigma}\cdot\partial\psi_L(x) - im\,\sigma^2\psi_L^*(x) = 0$

$\rightarrow$ leads to infinitesimal rotation...

$$\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\Big\{\underbrace{i\bar{\sigma}\cdot\partial\psi_L(\Lambda^{-1}x) - im\sigma^2\psi_L^*(\Lambda^{-1}x)}_{=0}\Big\} = 0$$

$\Rightarrow$ done! So Majorana eqn is invariant under infinitesimal rotations...

(b) <u>Boosts</u> (proceed in a similar way...)

<u>Key</u>

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= \bar{\sigma}^\mu - \frac{1}{2}\vec{\beta}\{\bar{\sigma}^\mu, \vec{\sigma}\}$$

$$\rightarrow \quad \bar{\sigma}^\mu \rightarrow \bar{\sigma}^\mu - i\beta\,[J_4^{0i}]^\mu{}_\nu\,\bar{\sigma}^\nu$$

$$\rightarrow \quad \left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = (\Lambda)^\mu{}_\nu\,\bar{\sigma}^\nu \quad \text{as before}$$

so plug this back into the original eqn.

$\rightarrow \quad i\bar{\sigma}\cdot\partial\psi_L(x) - im\sigma^2\psi_L^*(x) = 0$

$\rightarrow$ leads to infinitesimal rotation...

$$\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}\right)\left\{\underbrace{i\bar{\sigma}\cdot\partial\psi_L(\Lambda^{-1}x) - im\sigma^2\psi_L^*(\Lambda^{-1}x)}_{=0}\right\} = 0$$

$\rightarrow$ done! So Majorana eqn is invariant under infinitesimal rotations...

(b) Boosts (proceed in a similar way...)

Key

$$\left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)$$

$$= \bar{\sigma}^\mu - \frac{1}{2}\vec{\beta}\left\{\bar{\sigma}^\mu, \vec{\sigma}\right\}$$

$$\rightarrow \quad \bar{\sigma}^\mu \rightarrow \bar{\sigma}^\mu - i\vec{\beta}\left[J_4^{0i}\right]^\mu{}_\nu\,\bar{\sigma}^\nu$$

$$\rightarrow \quad \left(1 - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\bar{\sigma}^\mu\left(1 + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right) = (\Lambda)^\mu{}_\nu\,\bar{\sigma}^\nu \quad \text{as before}$$

so plug this back into the original eqn.

Sep 28, 2020 $\boxed{\text{Dirac Matrices \& Dirac Field Bilinears}}$

Oct 2, 2020 Recall that $\bar{\Psi}\Psi$ is Lorentz scalar...

Recall that $\bar{\Psi}\gamma^\mu\Psi$ is also a 4-vector.

$\Rightarrow$ $\boxed{?}$ Consider $\bar{\Psi}\Gamma\Psi$, where Γ is any 4×4

→ can we decompose into terms that have definite transformation properties under the Lorentz group?

↳ Γ can be written as combo of 16-element basis defined by

$$
\begin{array}{lll}
1: & \mathbb{1} & \rightarrow 1 \\
4: & \gamma^\mu & \rightarrow 4C2 \\
6: & \gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu,\gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} & \rightarrow 4C3 \\
4: & \gamma^{\mu\nu\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^{\sigma]} & \rightarrow 4C2 \\
1: & \gamma^{\mu\nu\sigma\rho} = \gamma^{[\mu}\gamma^\nu\gamma^\sigma\gamma^{\rho]} & \rightarrow 4C2
\end{array}
$$

16 total.

→ all are anti-symmetric products.

→ Each set of matrices transform as an antisym tensor of successively higher rank.

→ Introduce

$$\boxed{\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3}$$

$$= \frac{-i}{4!}\varepsilon^{\mu\nu\sigma\rho}\gamma_\mu\gamma_\nu\gamma_\sigma\gamma_\rho$$

0123 → 1

1023 → -1

↳ totally anti-symmetric

Note that $\rightarrow$ $\boxed{(\gamma^5)^2 = \mathbb{1}}$

$$\rightarrow \boxed{\begin{aligned}(\gamma^5)^\dagger &= -i(\gamma^3)^\dagger \cdots (\gamma^0)^\dagger \\ &= +i\gamma^3\gamma^2\gamma^1\gamma^0 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \gamma^5\end{aligned}}$$

also

$$\boxed{\begin{aligned}\{\gamma^5,\gamma^\mu\} &= i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu + i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 \quad (-1) \\ &= 0\end{aligned}}$$

and thus

$$\boxed{[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}\{\gamma^\mu,\gamma^\nu\}] = 0}$$

$\Rightarrow$ Eigenstates of γ^5 with different eigv don't mix under Lorentz transform.

$\rightarrow$ In basis, can write...

$$\gamma^5 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \begin{array}{l} \rightarrow \text{ for } \psi_L \text{ (left-hd)} \\ \rightarrow \text{ for } \psi_R \text{ (right-hd)} \end{array}$$

$\rightarrow$ a Dirac spinor with only L/R component is an eigenstate of γ^5 with eigv $(-1)/(1)$.

With γ^5, can rewrite the table of 4×4 matrices as

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu} = \frac{i}{2}\{\gamma^\mu,\gamma^\nu\}$	tensor	6
$\gamma^\mu\gamma^5$	pseudo vector	4
γ^5	pseudo scalar	1
		16

pseudo-vector/scalar is due to the fact that they transform like vector/scalar, BUT with an additional sign change under parity-transf.
under Lorentz transf

Ex Parity transf: $\vec{x} \to -\vec{x}$

$$(x^0, x^i) \to (x^0, -x^i)$$

If instead $(x^0; \vec{x}) \to -(x^0; -\vec{x}) = (-x^0; \vec{x})$ under parity, we call this a <u>pseudo-vector</u>

→ pseudo vect/scalar flips sign under parity transf.

→ From vector & pseudo-vector we can form 2 currents out of Dirac field bilinears...

$$\boxed{\begin{aligned} j^\mu(x) &= \bar{\Psi}(x)\gamma^\mu\Psi(x) \to \text{vector current} \\ j^{\mu 5}(x) &= \bar{\Psi}(x)\gamma^\mu\gamma^5\Psi(x) \to \text{pseudo vector current} \end{aligned}}$$

Assume that Ψ satisfies Dirac eqn... $\bar{\Psi} = \Psi^\dagger\gamma^0$

$\to i\partial\!\!\!/\Psi = m\Psi$ & $-i\partial\!\!\!/\bar{\Psi} = m\bar{\Psi}$ (since $\bar{\gamma}^\mu = \gamma^\mu$)

→ compute div of these currents...

$$\partial_\mu j^\mu = (\partial_\mu\bar{\Psi})\gamma^\mu\Psi + \bar{\Psi}\gamma^\mu(\partial_\mu\Psi)$$

$$= im\bar{\Psi}\Psi + \bar{\Psi}(-im\Psi) = 0$$

$$\to \boxed{\partial_\mu j^\mu = 0}$$

→ j^μ is always <u>conserved</u> if $\psi(x)$ satisfies Dirac eqn

→ It is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha}\psi(x)$$

<u>Similarly</u>

$$\partial_\mu j^{\mu 5} = (\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi$$

$$= (\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi + (-1)\bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi$$

$$= im\bar{\psi}\gamma^5\psi + (-1)(-i)m\bar{\psi}\gamma^5\psi$$

→ $\boxed{\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\gamma}$ ⇝ "axial vector current

→ if $m=0$ then $\partial_\mu j^{\mu 5}$ is conserved.

→ When $m=0$, $j^{\mu 5}$ is the Noether current for the symmetry

$$\psi(x) \to e^{i\alpha\gamma^5}\psi(x).$$

(we worry about ~~that~~ the rest of this section in ~~Weyl~~ Peskin's ...)

QUANTIZATION OF THE DIRAC FIELD

→ now, ready to construct quantum theory of the Dirac field.

Recall Lagrangian:

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi .$$

→ Canonical momentum conjugate to Ψ is

$$\frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)} = i\bar{\Psi}\gamma^0 = i\Psi^\dagger .$$

→ Hamiltonian is given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial(\partial_0\Psi)}\partial_0\Psi - \mathcal{L} = \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi} - \cancel{i\bar{\Psi}\gamma^0\partial_0\Psi}$$

$$- i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$

$$= -i\bar{\Psi}\vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi$$

Thus, $\boxed{H = \int \mathcal{H} d^3x = \int d^3x\, \bar{\Psi}(-i\vec{\gamma}\cdot\vec{\nabla} + m)\Psi}$

→ now let's figure out the commutators to make this a quantum field theory...

→ **HOW NOT TO QUANTIZE THE DIRAC FIELD**

This won't work!

<u>Guess</u> $$[\psi_a(\vec{x}),\, i\psi_b^\dagger(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

spinor components $(a, b = 1, 2, 3, 4)$

i.e.
$$[\psi_a(\vec{x}),\, \psi_b^\dagger(\vec{y})] = \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab}$$

In matrix notation...

$$[\psi_a(\vec{x}),\, \psi_a^\dagger(\vec{y})] = \mathbb{1}_{4\times4}\,\delta^{(3)}(\vec{x}-\vec{y})$$

$$\begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix} \qquad [\,-\,-\,-\,-\,]$$

Also guess $$[\psi_a(\vec{x}),\, \psi_b(\vec{y})] = 0$$

$$= [\psi_a^\dagger(\vec{x}),\, \psi_b^\dagger(\vec{y})] = 0$$

Note that

$$[\psi(\vec{x}),\, \bar{\psi}(\vec{y})] = [\psi(x),\, \psi^\dagger(y)\gamma^0]$$

$$= [\psi(x),\, \psi^\dagger(y)]\gamma^0 = \gamma^0\delta^{(3)}(\vec{x}-\vec{y})$$

With these... we recall that for bosons we wrote...

(real field) $$\phi(\vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\vec{a}_p + \vec{a}^{\,\dagger}_{-p}\right\} e^{i\vec{p}\cdot\vec{x}} \quad \text{(FT)}$$

↳ For <u>complex</u> field, we get

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left\{\vec{a}_p + \vec{b}^{\,\dagger}_{-p}\right\} e^{+i\vec{p}\cdot\vec{x}}$$

In the case of Dirac field, need spin degree of freedom...

Try ...

$$\psi(\vec{x}) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a^r_{\vec{p}}\, u^r(\vec{p}) + b^r_{-\vec{p}}\, v^r(-\vec{p}) \right\} e^{+i\vec{p}\cdot\vec{x}}$$

Spin degrees of freedom

Fourier components : $\psi(x) = u(p)e^{ip}$

2

$$\psi^\dagger(\vec{x}) = \sum_{r=1,2} \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ a^{r\dagger}_{\vec{p}}\, u^{r\dagger}(\vec{p}) + b^{r\dagger}_{-\vec{p}}\, v^{r\dagger}(-\vec{p}) \right\} e^{-i\vec{p}\cdot\vec{x}}$$

Recall that u, v also solves Dirac eqn in the sense that (in momentum space...)

$$\boxed{p^\mu \gamma_\mu u^r(\vec{p}) = m u^r(p) \quad \& \quad p^\mu \gamma_\mu v^r(\vec{p}) = -m v^r(p)}$$

We can try the commutators ...

$$[a^r_{\vec{p}}, a^{s\dagger}_{\vec{p}'}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[b^r_{\vec{p}}, b^s_{\vec{p}'}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{p}')$$

$$[a^r_{\vec{p}}, a^s_{\vec{p}'}] = 0$$

⋮

the rest are all zero...

We find that $\longrightarrow$ as desired...

$$[\psi_a(\vec{x}), \psi_b(\vec{y})] = 0 = [\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})]$$

We also find that

$$[\Psi_a(\vec{x}), \Psi_b^\dagger(\vec{y})] = \delta_{ab}\, \delta^{(3)}(\vec{x}-\vec{y}).$$

as desired.

With these ... we can try computing the Hamiltonian...

$$\mathcal{H} = \int d^3x \left[-i \bar{\Psi} \vec{\gamma}\cdot\vec{\nabla}\Psi + m\bar{\Psi}\Psi \right]$$

$$= \int d^3x \left\{ \Psi^\dagger \gamma^0 \left[-i\vec{\gamma}\cdot\vec{\nabla} + m \right] \Psi \right\}$$

just coefs

Now, with $p^\mu \gamma_\mu u^r(p) = m u^r(p)$

$$\longrightarrow \quad (\vec{p}\cdot\vec{\gamma} + m)\, u^r(p) = p^0 \gamma^0 u^r(p) = E_p \gamma^0 u^r(p)$$

Similarly, b/c $p^\mu \gamma_\mu v^{(r)}(p) = -m v^{(r)}(p)$

$$\longrightarrow \quad (\vec{p}\cdot\vec{\gamma} + m)\, v^r(p) = -E_p \gamma^0 v^{(r)}(p).$$

So ....

$$\rightarrow \quad [-i\vec{\gamma}\cdot\vec{\nabla} + m]\Psi = [-i\vec{\gamma}\cdot\vec{\nabla} + m] \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_p^r u_p^r + b_{-p}^{r\dagger} v^r \right] e^{i\vec{p}\cdot\vec{x}}$$

$$= \gamma^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ E_p a_p^r u^r(\vec{p}) - E_{+p} b_{-p}^{r\dagger} v^r(\vec{p}) \right\} e^{i\vec{p}\cdot\vec{x}}$$

So....

$$\mathcal{H} = \int d^3x \left\{ \Psi^\dagger \gamma^0 \gamma^0 \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left\{ \cdots \right\} e^{i\vec{p}\cdot\vec{x}} \right\}$$

plug in ...

$$\rightarrow \mathcal{H} = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} E_p \left\{ a_p^{r\dagger} a_p^r - \underbrace{b_{+p}^r b_{+p}^{r\dagger}}_{\substack{\parallel \\ b_{+p}^{r\dagger} b_{+p}^r + \cancel{\text{const}}}} \right\}$$

$\boxed{!}$

→ By creating more and more particles with $b^\dagger$, we can lower the energy indefinitely

→ This is <u>bad</u>...

→ $\boxed{\text{So we should use Fermi-Dirac statistics instead → anti-commutators instead of commutators...}}$

→ Require that

$$\boxed{\{a_{\vec{p}}^r, a_{\vec{q}}^{s\dagger}\} = \{b_{\vec{p}}^r, b_{\vec{q}}^{s\dagger}\} = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q})}$$

no longer bosonic!

+ all other anti-commutators are zero...

When this is true, we find that

$$\boxed{\begin{aligned} \{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} &= \delta^{(3)}(\vec{x}-\vec{y})\,\delta_{ab} \\ \{\psi_a(\vec{x}), \psi_b(\vec{y})\} &= \{\psi_a^\dagger(x), \psi_b^\dagger(y)\} = 0 \end{aligned}}$$

where we're using

$$\psi(x) = \sum_{r=1,2} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left[a_{\vec{p}}^r u^r(\vec{p}) + b_{-\vec{p}}^{r\dagger} v^r(-\vec{p}) \right] e^{i\vec{p}\cdot\vec{x}}$$

Compute the Hamiltonian again, we find that

$$\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_{\vec p} \left(a_p^{r\dagger} a_p^r - \underbrace{b^r_{-\vec p} b^{r\dagger}_{-\vec p}}_{} \right)$$

$$- b^{r\dagger}_{-p} b^r_{-p} + \cancel{\text{const}}$$

$$\Rightarrow \boxed{\mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} E_p \left\{ a_{\vec p}^{r\dagger} a_{\vec p}^r + b^{r\dagger}_{-\vec p} b^r_{-\vec p} \right\}}$$

now good, b/c E is bdd below...

→ also can compute...

$$\boxed{\vec P = \int \frac{d^3p}{(2\pi)^3} \sum_{r=1,2} \vec p \left(a_{\vec p}^{r\dagger} a_{\vec p}^r + b^{r\dagger}_{+\vec p} b^r_{+\vec p} \right)}$$

To avoid sign confusion, we will usually write...

$$\boxed{\Psi(\vec x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a^r_{\vec p} u^r(\vec p) e^{+i\vec p\cdot\vec x} + b^{r\dagger}_{\vec p} v^r(\vec p) e^{-i\vec p\cdot\vec x} \right)}$$

As a Heisenberg field,

$$\boxed{\Psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{r=1,2} \left(a_p^r u^r(\vec p) e^{-i\vec p\cdot\vec x} + b_p^{r\dagger} v^r(\vec p) e^{i\vec p\cdot\vec x} \right)}$$

where:

$$\begin{cases} a_p^r & : \text{annihilates particles} \\ a_p^{r\dagger} & : \text{creates particles} \\ b_p^r & : \text{annihilates anti-particles} \\ b_p^{r\dagger} & : \text{creates anti-particles.} \end{cases}$$

Vacuum state as $|0\rangle$ where

$$\begin{cases} a_p^s |0\rangle = 0 \\ b_p^s |0\rangle = 0 \end{cases}$$

Define one-particle excitation state w/ covariant norm:

$$\boxed{|\vec{p}, s\rangle = \sqrt{2E_p}\, a_p^{s\dagger} |0\rangle}$$

so that

$$\boxed{\langle \vec{p}, s | \vec{q}, r \rangle = \sqrt{2E_p}\sqrt{2E_{\vec{q}}}\; \delta^{(3)}(\vec{p} - \vec{q})\, \delta^{rs}}$$

Now, look at Lorentz transform ...

$$\psi(x) \xrightarrow{\text{Lortz}} \psi'(x) = \Lambda_{\frac{1}{2}} \psi(\Lambda_4^{-1} x)$$

recall that with $\quad \rightsquigarrow \exp\{-i\omega_{\mu\nu} S^{\mu\nu}/2\} \simeq \mathbb{1} - \frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}$

$$\begin{cases} \omega_{12} = -\omega_{21} = \theta \\ S^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \end{cases}$$

$= \mathbb{1} - \frac{i}{2}\vec{\theta}\cdot(\vec{\sigma}$

$$\rightarrow \text{ and } \quad \psi(\Lambda_4^{-1} x) \simeq [1 - \vec{\theta}\cdot\vec{J}]\, \psi(x)$$

$$\vec{J} = \vec{x} \times (-i\vec{\nabla})$$

So ~~Wtlk~~ $\psi \to \psi + \delta\psi$ where

$$\delta\psi = \psi' - \psi = \left(-\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}\right)\psi(x) - \vec{\theta}\cdot(\vec{x}\times\vec{\nabla})\,\psi(x)$$

By Noether's thm,

$$\vec{J}_{total}\ (\text{total spin}) = \int \frac{d^3x}{2}\left[\psi^\dagger(-i\vec{x}\times\vec{\nabla})\psi + \frac{1}{2}\psi^\dagger\vec{\Sigma}\psi\right].$$

We won't worry abt the rest of this section about propagators

→ we'll come back to them later when looking at Feynman diagrams.

$$\boxed{\text{DISCRETE SYMMETRIES OF THE DIRAC THEORY}}$$

Basically, we have

Parity —— Time Reversal

Charge conjugation

Recall that before, we looked at implementation of continuous Lorentz transform -

→ found that $\forall \Lambda \in$ Lorentz group

$\exists\ \mathcal{U}(\Lambda)$ unitary for which

$$\mathcal{U}(\Lambda)\psi(x)\mathcal{U}^{-1}(\Lambda) = \Lambda_{\frac{1}{2}}^{-1}\psi(\Lambda x).$$

→ Now, we'll talk about <u>Discrete</u> symmetries on the Dirac field.

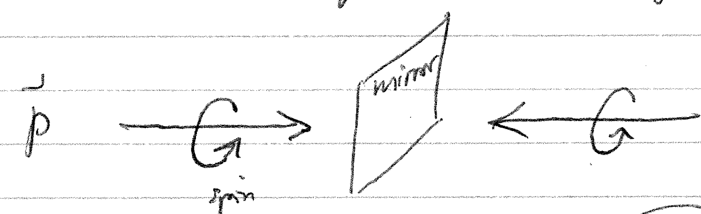
Apart from continuous Lorentz transforms, there are other spacetime-transformations for which the Lagrangian might remain invariant:

→ e.g. $\left\{ \begin{array}{c} \text{time-reversal} \\ \text{parity} \end{array} \right\}$.

Parity (P) : flips direction of spatial vectors

$$\boxed{P : (t, \vec{x}) \longrightarrow (t, -\vec{x})}.$$

⇝ mirror sym → change the handedness.



⇝ Note: momentum flips sign, but spin is unchanged.

Time-reversal

$$\boxed{T : (t, \vec{x}) \to (-t, \vec{x})}$$

P, T don't belong to the "proper" Lorentz group $L_+^\uparrow$

→ the full Lorentz group breaks into 4 disconnected subsets ...

↳

$$\begin{tikzcd} L_+^{\uparrow} \arrow[r, leftrightarrow, "P"] \arrow[d, leftrightarrow, "T"] & L_-^{\uparrow} = PL_+^{\uparrow} \arrow[d, leftrightarrow, "T"] \\ L_+^{\downarrow} = TP_+^{\downarrow} \arrow[r, leftrightarrow, "P"] & L_-^{\downarrow} = PTL_+^{\uparrow} \end{tikzcd}$$

"orthochronous"

"non-orthochronous"

"proper" "improper"

$\boxed{\text{charge conjugation}}$ $\longrightarrow$ interchange particles & anti-particles.

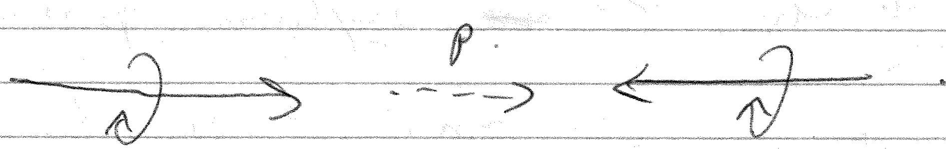
↳ non-space-time.

Let's look at $\boxed{\text{Parity}}$.

Note that because $P: (t, \vec{x}) \to (t, -\vec{x})$

$\to$ momentum flips sign

but not spin! $\Rightarrow$ What is P? As an operator?



As an operator on creation/annihilation ops, we want

$$\boxed{P^{\dagger} a_{\vec{p}}^{s} P = a_{-\vec{p}}^{s} \quad \& \quad P^{\dagger} b_{\vec{p}}^{s} P = b_{-\vec{p}}^{s}}$$

where, as discussed, P must be unitary.

$$PP^\dagger = P^\dagger P = \mathbb{1}.$$

Taking adjoint, get

$$\boxed{P^\dagger a^{s\dagger}_{\vec{p}} P = a^{s\dagger}_{-\vec{p}} \quad \& \quad P^\dagger b^{s\dagger}_{+\vec{p}} P = b^{s\dagger}_{-\vec{p}}}$$

But these might be too restrictive ... we can get better constraints by requiring that.

$$P^\dagger a^s_{\vec{p}} P = \eta_a\, a^s_{-\vec{p}} \quad \& \quad P^\dagger b^s_{\vec{p}} P = \eta_b\, b^s_{-\vec{p}}$$

as long as $|\eta_a|^2 = |\eta_b|^2 = \mathbb{1}$ are "phases"!

Why? b/c ultimately, all observables will have fermion operators in pairs and the phases η_a, η_b will cancel:

$$\left\{\begin{aligned} P^\dagger a^{s'\dagger}_{\vec{p}} a^s_{\vec{p}} P &= a^{s'\dagger}_{-\vec{p}} a^s_{-\vec{p}} \\ P^\dagger b^{s'\dagger}_{\vec{p}} b^{s'}_{\vec{p}} P &= b^{s\dagger}_{-\vec{p}} b^s_{-\vec{p}}. \end{aligned}\right\}$$

With this, let's implement parity condition on $\psi(x)$

$$P^\dagger \psi P = ?$$

(to find out what these η_a, η_b must be....)

$$P^\dagger \psi(x) P = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(\eta_a\, a^s_{-\vec{p}}\, u^s(p)\, e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^*\, b^{s\dagger}_{-\vec{p}}\, v^s(\vec{p})\, e^{i\tilde{p}\cdot\tilde{x}} \Big)$$

Define
$$\begin{cases} \tilde{p} = (E_p, -\vec{p}) \\ \tilde{x} = (t, -\vec{x}). \end{cases}$$

Note that

$$u^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ \sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} \qquad \text{where } \sigma = (\mathbb{1}, \vec{\sigma}), \quad \bar{\sigma} = (\mathbb{1}, -\vec{\sigma})$$

$$= \begin{pmatrix} \sqrt{\tilde{p}\cdot\bar{\sigma}}\, \xi^s \\ \sqrt{\tilde{p}\cdot\sigma}\, \xi^s \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\, \xi^s \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\, \xi^s \end{pmatrix}}_{u^s(-\vec{p})} = \underbrace{\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}}_{\gamma^0} u^s(-\vec{p})$$

$$\Rightarrow \boxed{u^s(p) = \gamma^0 u^s(-\vec{p})}$$

and

$$v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\, \xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\, \xi^s \end{pmatrix} = \cdots = \begin{pmatrix} 0 & -\mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} v^s(-\vec{p})$$

$$\Rightarrow \boxed{v^s(p) = -\gamma^0 v^s(p)}$$

With these, we find that ...

$\tilde{p}\cdot\vec{x} = p\cdot\tilde{x}$ (86)

$$P^\dagger \psi(x) P = \gamma^0 \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1,2} \Big(\eta_a a^s_{-p} u^s(-p) e^{-i\tilde{p}\cdot\tilde{x}} + \eta_b^* b^{s\dagger}_{-p} v^s(-p) e^{i\tilde{p}\cdot\tilde{x}} \Big)$$

now, notice that if $\eta_a = \eta_b^*$ then it's "nice":

$$\boxed{(\eta_a = \eta_b^*) \Rightarrow P\psi^\dagger(x)P = \eta_a \gamma^0 \psi(\tilde{x})}$$ → P-transf in final form

→ suffices to choose $\eta_a = 1 = -\eta_b$.

relative sign between fermions & antifermions ...

Now, useful to know how various Dirac field bilinears transform under parity ...

<u>Recall ... 5 of them</u>:

$$\bar{\psi}\psi, \quad \bar{\psi}\gamma^\mu\psi, \quad i\bar{\psi}[\gamma^\mu,\gamma^\nu]\psi$$

$$\bar{\psi}\gamma^\mu\gamma^5\psi, \quad i\bar{\psi}\gamma^5\psi.$$

to find these, first compute: $P\bar{\psi}(x)P$ ...

$$P^\dagger\bar{\psi}(x)P = P^\dagger\psi^\dagger(x)\gamma^0 P = (P^\dagger\psi P)^\dagger\gamma^0 \qquad (\gamma^{0\dagger} = \gamma^0)$$

$$= \eta_a^*(\gamma^0\psi(\tilde{x}))^\dagger\gamma^0 = \eta_a^*\psi^\dagger(\tilde{x})\gamma^0\gamma^0$$

$$\rightarrow \boxed{P^\dagger\bar{\psi}P = \eta_a^*\bar{\psi}(\tilde{x})\gamma^0}$$

With this---

$$P^\dagger \underset{(x)}{\bar\Psi} \underset{(x)}{\Psi} P = \underbrace{P^\dagger \bar\Psi(x) P}\overset{\mathbb{1}}{\widetilde{P^\dagger}} \Psi(x) P .$$

$$= \eta_a^* \bar\Psi(\tilde x)\gamma^0 \eta_a \gamma^0 \Psi(\tilde x)$$

$$= |\eta_a|^2 \bar\Psi(\tilde x)\Psi(\tilde x)$$

Scalar $\boxed{P^\dagger \bar\Psi \Psi P(x) = \bar\Psi\Psi(\tilde x)}$ (scalar) bilinear.

can also show...

$$\boxed{\begin{aligned} & P^\dagger \underbrace{\bar\Psi(x)\gamma^\mu}\Psi P = \bar\Psi\gamma^0\gamma^\mu\gamma^0\Psi(\tilde x) \\ & \text{(vector) } \cancel{\text{bilinear}} \quad = \begin{cases} +\bar\Psi\gamma^\mu\Psi(\tilde x) & \mu = 0 \\ -\bar\Psi\gamma^\mu\Psi(\tilde x) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

$$\boxed{P^\dagger (i\bar\Psi\gamma^5\Psi) P = i\bar\Psi(\tilde x)\gamma^0\gamma^5\gamma^0\Psi(\tilde x) = -i\bar\Psi\gamma^5\Psi(\tilde x)}$$

pseudo scalar ↳ (−)

$$\cancel{= \begin{cases} -\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 0 \\ \bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 1,2,3 \end{cases}}$$

$$\boxed{\begin{aligned} & P^\dagger \bar\Psi\gamma^\mu\gamma^5\Psi(x) P = \bar\Psi(\tilde x)\gamma^0\gamma^\mu\gamma^5\gamma^0\Psi(\tilde x) \\ & = \begin{cases} -\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 0 \\ +\bar\Psi\gamma^\mu\gamma^5\Psi(\tilde x) & \mu = 1,2,3 \end{cases} \end{aligned}}$$

pseudo vector.

(−)

Note the relative sign: $\boxed{-\eta_a = \eta_b^*}$ is important.

for the relationship between fermions - anti - fermi

Consider ~~and~~ fermion - antifermion state...

$$a_p^{s\dagger} b_q^{s\dagger} |0\rangle \xrightarrow{P} P\left(a_p^{s\dagger} b_q^{s\dagger} |0\rangle\right)$$

$$= P^\dagger \left(a_p^{s\dagger} b_q^{s\dagger}\right) P |0\rangle$$

$$= \underbrace{P^\dagger a_p^{s\dagger} P} P^\dagger b_q^{s\dagger} P |0\rangle$$

$$= \eta_a \, a_{-p}^{s\dagger} \, \eta_b \, b_{-q}^{s\dagger} |0\rangle$$

$$= -\left(\eta_b \eta_b^*\right) a_{-p}^{s\dagger} b_{-q}^{s\dagger} |0\rangle$$

$$= - a_{-p}^{s\dagger} b_{-q}^{s\dagger} |0\rangle$$

→ a state containing a fermion - antifermion pair gets an extra (−1) under parity transform.

TIME REVERSAL

if T is unitary ~ $[T, H] = 0$

→ $T^\dagger e^{iHt} T = e^{iHt} T^\dagger T = e^{iHt}$

→ no good...

what if $T^\dagger H T = -H$? or $\{T, H\} = 0$?

But this is no good either since implies that
H is unbounded ...

→ Assume this ...

$\boxed{\text{"Time-reversal is conjugate-linear / anti-linear"}}$

Assume:

T is unitary

$$T^\dagger c T = c^* \qquad (c \in \mathbb{C}).$$

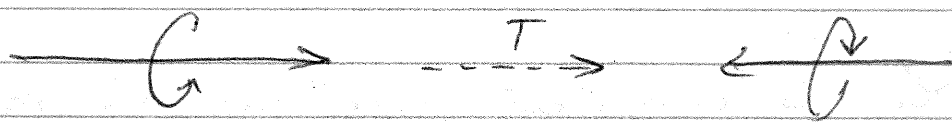
$$[T, H] = 0$$

With these ...

$$T^\dagger e^{-iHt} T = e^{iHt} \quad \checkmark$$

→ time reversal: momentum & spin are reversed

→ like watching a movie played backwards



Flipping momentum is easy.

what abt flipping spinor s? We know that

In some basis ...

$$\xi(\uparrow) = \begin{pmatrix} \cos\theta/2 \\ e^{i\phi}\sin\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\xi(\downarrow) = \begin{pmatrix} -e^{-i\phi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix} \xrightarrow{\text{basis}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Let $\xi^s = (\xi(\uparrow), \xi(\downarrow))$ for $s = 1, 2$ & define

reversed spin

$$\boxed{\xi^{-s} = -i\sigma^2 (\xi^s)^*.}$$

$\sigma^2 \to \sigma^y$

↳ this is the flipped spinor ...

It is clear that

$$\xi^{-s} = -i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} (\xi(\uparrow)^*, \xi(\downarrow)^*)^T$$

$$= (\xi(\downarrow), -\xi(\uparrow))^T$$

where
$$\left\{ \begin{aligned} \xi^2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \xi^{1*} \\ \xi^1 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \xi^{2*} \end{aligned} \right.$$

→ This is convenient since our time reversal op. involves complex conjugation ...

→ Can show:
$$\boxed{u^{-s}(-\vec{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\xi^{s*} \end{pmatrix}}$$

So if we use the identity ...

$$\boxed{\sqrt{\tilde{p}\cdot\sigma}\,\sigma^2 = \sigma^2\sqrt{p\cdot\sigma^*}}$$

(prove using $\sigma^2\vec{\sigma} = -\vec{\sigma}^*\sigma^2$

then we get

$$u^{-s}(\tilde{p}) = \begin{pmatrix} \sqrt{\tilde{p}\cdot\sigma}\,(-i\sigma^2)\,\xi^{s*} \\ \sqrt{\tilde{p}\cdot\bar{\sigma}}\,(-i\sigma^2)\,\xi^{s*} \end{pmatrix} = \begin{pmatrix} (-i\sigma^2)\sqrt{\tilde{p}\cdot\sigma^*}\,\xi^{s*} \\ (-i\sigma^2)\sqrt{p\cdot\bar{\sigma}^*}\,\xi^{s*} \end{pmatrix}$$

$$= (-i)\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}[u^s(p)]^* = -\gamma^1\gamma^3[u^s(p)]^*$$

$\sigma^2 = \sigma^y$

$$\Rightarrow \boxed{u^{-s}(\tilde{p}) = -\gamma^1\gamma^3[u^s(p)]^*}$$

element-wise complex conjugation

Similarly, ...

$$\boxed{v^{-s}(\tilde{p}) = -\gamma^1\gamma^3[v^s(p)]^*}$$

in this relation, v^{-s} contains $\boxed{\xi^{-(-s)} = -\xi^s}$

a 360° flip introduces a $(-)$ sign.

Now we can define time reversal operation on the creation - annihilation operators...

spins don't matter here... $\longrightarrow$ $\boxed{T^\dagger a^s_{\vec p} T = a^{-s}_{-\vec p} \quad \text{e} \quad T^\dagger b^s_{\vec p} T = b^{-s}_{-\vec p}}$ → flip spin → flip momentum

where we now define

$$a^{-s}_{-\vec p} = (a^{\downarrow}_{-\vec p}, -a^{\uparrow}_{-\vec p})$$
$$b^{-s}_{-\vec p} = (b^{\downarrow}_{-\vec p}, -b^{\uparrow}_{-\vec p})$$

just like what we did with $\xi^{s} = (\xi(\downarrow), -\xi(\downarrow))$

if $a^s_{\vec p} = (a^{\uparrow}_{\vec p}, a^{\downarrow}_{\vec p})$
$b^s_{\vec p} = (b^{\uparrow}_{\vec p}, b^{\downarrow}_{\vec p})$

analogous to what we did before...

With this, let's evaluate $T^\dagger \psi(x) T$:

$$\rightarrow T^\dagger \psi(x) T = \int \frac{d^3 p}{(2\pi)^3} \underbrace{\frac{1}{\sqrt{2E_p}}} \sum_{s=1,2} T^\dagger \left(a^s_p u^s(p) e^{-ip\cdot x} + b^{s\dagger}_p v^s(p) e^{+ip\cdot x} \right) T$$

$$= \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_{s=1}^{2} \left\{ a^{-s}_{-\vec p} [u^s(p)]^* e^{ip\cdot x} + b^{-s\,\dagger}_{-p} [v^s(p)]^* e^{-ip\cdot x} \right]$$

where under T, $\qquad = \gamma^1\gamma^3 \psi(x_T)$, $x_T = (-t, \vec x)$

- $a^s_p \xrightarrow{T} a^{-p}_{-p}$
- ~~$u^{-s}(\tilde p) = -\gamma_1\gamma_3 [u(p)]^*$~~
- $\rightarrow$ ~~$\gamma_1\gamma_3 u^{-s}(\tilde p) = [u^s(p)]^*$~~
- $T^\dagger e^{-ip\cdot x} T = e^{+ip\cdot x}$; $T^\dagger u^s_p T = [u^s_p]^*$

note sign change here

Because $\{u^s(p)\}^* = \gamma_1\gamma_3 u^{-s}(\tilde{p})$, we have

$$T^\dagger \psi(x) T = \gamma^1\gamma^3 \int \frac{d^3\tilde{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\tilde{p}}}} \sum_{s=1}^{2} \left\{ a_{\tilde{p}}^{-s} u^{-s}(\tilde{p}) e^{i\tilde{p}(t,-\vec{x})} + b_{\tilde{p}}^{-s} v^{-s}(\tilde{p}) e^{-i\tilde{p}(t,-\vec{x})} \right\}$$

$$= \gamma^1\gamma^3 \psi(-t, x) \qquad = -\tilde{p}(-t,\vec{x})$$

$$\Rightarrow \boxed{T^\dagger \psi(x,t) T = \gamma^1\gamma^3 \psi(x,-t)}$$

Next, can check the action of T on bilinears...

$$T^\dagger \bar{\psi} T = T^\dagger \psi^\dagger \gamma^0 T = T^\dagger \psi^\dagger T \gamma^0 \quad \text{real}$$

$$= (\gamma^1\gamma^3\psi(x_T))^\dagger \gamma^0 = \psi^\dagger(x_T) \underbrace{\gamma^{3\dagger}}_{-\gamma^3} \underbrace{\gamma^{1\dagger}}_{-\gamma^1} \gamma^0$$

$$= \psi^\dagger(x_T)\gamma^3\gamma^1\gamma^0$$

$$= +\psi^\dagger(x_T)\gamma^0 \underbrace{\gamma^3\gamma^1}$$

$$\Rightarrow \boxed{T^\dagger \bar{\psi} T = -\bar{\psi}(x_T)\gamma^1\gamma^3}$$

with this, can compute the rest....

Scalar $\boxed{T\bar{\psi}\psi(x) T = \bar{\psi}\underbrace{(-\gamma^1\gamma^3)(\gamma^1\gamma^3)}_{1}\psi(x_T) = \bar{\psi}(x_T)\psi(x_T)}$

Pseudoscalar $\to$ get $(-)$

"pseudo"

$$\boxed{T^\dagger i\bar{\psi}\gamma^5\psi T = -i\bar{\psi}(-\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3)\psi(x_T) = -i\bar{\psi}(x_T)\gamma^5\psi(x_T)}$$

Vector

$$\boxed{\begin{aligned} T^\dagger \bar{\Psi}\gamma^\mu\Psi T &= \bar{\Psi}(-\gamma^1\gamma^3)(\gamma^\mu)^*(\gamma^1\gamma^3)\Psi \\ &\quad (x) \\ &= \begin{cases} +\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu=0 \\ -\bar{\Psi}\gamma^\mu\Psi(x_T) & \mu=1,2,3. \end{cases} \end{aligned}}$$

This makes sense... Recall that $\bar{\Psi}\gamma^0\Psi$ is the charge density

↳ $\bar{\Psi}\gamma^0\Psi$ should be the same under T.

as we saw: $T^\dagger\bar{\Psi}\gamma^0\Psi T = \bar{\Psi}\gamma^0\Psi$.

but current density (time-dep) must reverse so

$\rightarrow \quad T^\dagger\bar{\Psi}\gamma^i\Psi T = -\bar{\Psi}\gamma^i\Psi$ ✓.

$\boxed{\text{Charge Conjugation – Matter-anti-matter flip}}$

↳ $\left\{\begin{array}{l}\text{anti particles + particles are swapped.} \\ \text{spin + momentum are the same.}\end{array}\right.$

Let $\left\{\begin{aligned} C^\dagger a_p^s C &= b_p^s \\ C^\dagger b_p^s C &= a_p^s . \end{aligned}\right\} \rightarrow$ ignore phases...

How should C act on $\Psi(x)$?

First, look at relation...

$$(v^s(p))^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\ (-i\sigma^2)\xi^{s*} \\ \sqrt{p\cdot\bar\sigma}\ (-i\sigma^2)\xi^{s*} \end{pmatrix}^* = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma^*}\,\xi^{s*} \\ i\sigma^2\sqrt{p\cdot\sigma^*}\,\xi^* \end{pmatrix}^*$$

$$= \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} =$$

$\rightarrow$ set

$$\boxed{\begin{aligned} u^s(p) &= -i\gamma^2(v^s(p))^* \\ v^s(p) &= -i\gamma^2(u^s(p))^* \end{aligned}}$$

$$\rightarrow\quad C^\dagger\psi(x)C = \int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}\sum_{s=1}^{2}\Big\{-i\gamma^2 b_p^s(v^s(p))^* e^{-ip\cdot x} - i\gamma^2 a_p^{s\dagger}(u^s(p))^* e^{ip\cdot x}\Big)$$

$$= -i\gamma^2\psi^*(x) = -i\gamma^2(\psi^\dagger)^T = -i(\bar\psi\gamma^0\gamma^2)^T$$

$\Rightarrow$ $\boxed{C^\dagger\psi(x)C = -i(\bar\psi\gamma^0\gamma^2)^T}$ $\rightarrow$ C is a unitary op.

On bilinears... first, find $\quad \bar\psi^\dagger = (\psi^\dagger)^\dagger\gamma^0 = \psi\gamma^0$

$$\boxed{\begin{aligned} C^\dagger\bar\psi(x)C &= C^\dagger\psi^\dagger\gamma^0 C = \underbrace{C^\dagger\psi^\dagger C}\gamma^0 = -i\psi^T\gamma^2\gamma^0 \\ &= (-i\gamma^2\psi)^T\gamma^0 = (-i\gamma^0\gamma^2\psi)^T \end{aligned}}$$

Next ....

$$C^\dagger\bar\psi\psi C = (-i(\gamma^0\gamma^2\psi)^T)(-i(\bar\psi\gamma^0\gamma^2)^T) = \cdots =$$

$$= -\left[(-i\bar\psi\gamma^0\gamma^2)(-i\gamma^0\gamma^2\psi)\right]^T = +\bar\psi\gamma^0\gamma^2\gamma^0\gamma^2\psi$$

$$= -+\bar\psi\gamma^0\gamma^2\gamma^2\gamma^0\psi = +\bar\psi\psi$$

So... $\boxed{C^\dagger \bar{\psi}\psi C = \bar{\psi}\psi}$ $\longrightarrow$ scalar

vector $\boxed{C^\dagger i \bar{\psi}\gamma^5\psi C = i(-i\gamma^0\gamma^2\psi)^T\gamma^5(-i\bar{\psi}\gamma^0\gamma^2)^T = i\bar{\psi}\gamma^5\psi}$

$$\boxed{\begin{aligned} &\text{pseudo-scalar} \quad C^\dagger\bar{\psi}\gamma^\mu\psi C = -\bar{\psi}\gamma^\mu\psi \\ &\text{pseudo vector.} \quad C^\dagger\bar{\psi}\gamma^\mu\gamma^5\psi C = +\bar{\psi}\gamma^\mu\gamma^5\psi \end{aligned}}$$

(I'll skip the derivations... to save time)

<u>Summary</u>

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂_μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

notice that
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \quad \text{is invariant}$$
under C, P, T separately

$\rightarrow$ in general, can't build a Lorentz inv QFT with a Hermitian Hamiltonian that violates CPT!

$\downarrow$ Invariance under CPT is required for any Lorentz invariant local Hermitian op.

$\boxed{\text{Problem 5}}$

$\hookrightarrow$ (to be continued...)

$\boxed{\text{Correlation functions for Dirac fields}}$

$\langle 0|\psi_a(x)\bar{\psi}_b(y)|0\rangle \rightsquigarrow$ Dirac propagation amplitudes...

only "a" terms contributes — only "$a^\dagger$" term contributes

Recall...

$$\to \psi_A(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{a^s_{Ap}\, u^s_A(p)\, e^{-ip\cdot x} + \cancel{b^{s\dagger}_{Ap}\, v^s_A(p)\, e^{-ip\cdot x}}\right\} \quad \langle 0|$$

$$\to \bar{\psi}_B(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_{s=1,2}\left\{\cancel{b^s_{Bp}\, \bar{v}^s_B(p)\, e^{-ip\cdot x}} + a^{s\dagger}_{Ap}\, \bar{u}^s_B(p)\, e^{ip\cdot x}\right\} \quad \to |0\rangle$$

where $\{a^r_p, a^{s\dagger}_q\} = \{b^r_p, b^{s\dagger}_q\} = (2\pi)^3\delta^{(3)}(p-q)\delta^{rs}$

$$\to \langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\bar{u}^s_B(p)}_{(\not{p}+m)_{AB}}e^{-ip(x-y)}$$

$$= (i\not{\partial}_x + m)_{AB}\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}e^{-ip(x-y)}$$

$$\boxed{\langle 0|\psi_A(x)\bar{\psi}_B(x)|0\rangle = (i\not{\partial}_x + m)_{AB}\,\mathcal{D}(x-y)}$$

- - -

$$\langle 0|\,\bar{\Psi}_B(y)\,\Psi_A(x)\,|0\rangle = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\,\underbrace{\sum_s v_A^s(p)\,\bar{v}_B^s(p)}_{(\not{p}-m)_{AB}}\,e^{-ip(x-y)}$$

b terms contribute

$b^\dagger$ terms contribute

$$= \int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2E_p}\right)(\not{p}-m)_{AB}\;e^{-ip(x-y)}$$

$$\boxed{\langle 0|\,\bar{\Psi}_B(y)\Psi_A(x)\,|0\rangle = -\left(i\not{\partial}_x+m\right)_{AB}\,\mathcal{D}(y-x)}$$

Feynman Propagator

$$S_F^{AB}(x-y) = \begin{cases} \langle 0|\,\Psi_A(x)\,\bar{\Psi}_B(y)\,|0\rangle & \text{for } x^0>y^0 \\ -\langle 0|\,\bar{\Psi}_B(y)\,\Psi_A(x)\,|0\rangle & \text{for } x^0<y^0 \end{cases}$$

$$= \langle 0|\,T\{\Psi_A(x),\,\bar{\Psi}_B(y)\}\,|0\rangle$$

--- time-ordering ---

where $T\{\Psi_A(x),\,\bar{\Psi}_B(y)\}$

$$= \theta(x^0-y^0)\,\Psi_A(x)\,\bar{\Psi}_B(y)$$

$$-\,\theta(y^0-x^0)\,\bar{\Psi}_B(y)\,\Psi_A(x)$$

$\boxed{\text{minus sign for Fermions}}$

Let's check the calculations...

$$\langle 0|\psi_A(x)\bar{\psi}_B(y)|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\frac{1}{\sqrt{2E_{p'}}}\langle 0|\left\{\sum_s a^s_{A\vec{p}}\, u^s_A(p)e^{-ip\cdot x} + \cancel{b^{s\dagger}_{A\vec{p}}\, v^s_A(p)e^{ip\cdot x}}\right\}$$

$$\times\left\{\cancel{\sum_s b^s_{B\vec{p}'}\,\bar{v}^s_B(p')e^{-ip'\cdot y}} + a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{ip'\cdot y}\right\}|0\rangle$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\langle 0|\sum_s a^s_{A\vec{p}}\,u^s_A(p)\sum_s a^{s\dagger}_{B\vec{p}'}\,\bar{u}^s_B(p')e^{-i(p\cdot x-p'\cdot y)}|$$

$$= \int \frac{d^3p\,d^3p'}{(2\pi)^6}\frac{1}{2}\frac{1}{\sqrt{E_pE_{p'}}}\sum_s\langle 0|u^s_A(p)\bar{u}^s_B(p')|0\rangle\,(2\pi)^3\delta^{(3)}(p-p')$$

$$\times\, e^{-i(p\cdot x-p'\cdot y)}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\underbrace{\sum_s u^s_A(p)\bar{u}^s_B(p)}\,e^{-ip(x-y)}$$

$$(\not{p}+m)_{AB} = (\gamma^\mu p_\mu + m)_{AB}$$ (spin sum relations)

$$= (i\not{\partial}_x+m)_{AB}\int\frac{d^3p}{(2\pi)^3\,2E_p}\,e^{-ip(x-y)}. \checkmark$$

Similarly, we can get the other relation too...