Problem Set 8

Due: Friday 11:59pm, April 14th via Canvas upload
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Office hour: TBA

1 A model for Feshbach resonances

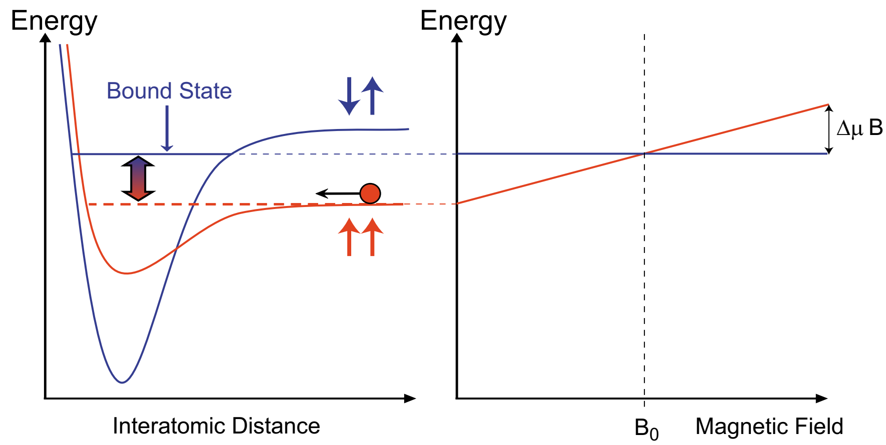

Figure 1: (Left) Sketch of interatomic singlet and triplet potentials, showing the incoming state of two colliding atoms in the triplet potential coupled to a bound state in the singlet potential. (Right) an applied magnetic field tunes the two states into resonance.

Collisions between ground-state alkali atoms occur within two electronic potentials, the singlet and the triplet potential. In atom-atom collisions, a Feshbach scattering resonance occurs when the energy of two colliding atoms entering (say) in the triplet channel is brought into resonance with the energy of a molecular state in (say) the singlet potential (see Fig. 1). Coupling between singlet and triplet potentials is provided by the hyperfine interaction that trades nuclear for electron spin. Magnetic fields only shift triplet levels, so one can tune these resonances by applying an external magnetic field.

In this problem we will treat a model for Feshbach resonances. The situation is, as in lecture and complement C_{I} of API, that of a discrete state - the molecular state $|m\rangle$ - coupled to a continuum. The continuum is here the states $\left|\vec{k}\right\rangle$ describing two free atoms (say of identical mass m) with relative momentum $\hbar\vec{k}$. The energy of state $\left|\vec{k}\right\rangle$ is $E_k = 2\epsilon_k = \frac{\hbar^2 k^2}{m}$, where $\epsilon_k = \frac{\hbar^2 k^2}{2m}$ is the energy of a single atom of momentum $\hbar\vec{k}$. (E_k is the kinetic energy of the effective particle of effective mass $\mu = m/2$ describing the two-particle scattering.) Note that in this problem, the continuum has a threshold at $E_k = 0$, it does not extend to negative energies.

The energy δ of the uncoupled (or "bare") molecular state, relative to the continuum threshold, is tuned by the external magnetic field. In the absence of coupling to the continuum, the molecular state would thus enter the continuum ("cross threshold") at $\delta = 0$. We should thus find a regime where the coupled (or dressed) molecule is still a discrete state, namely when its energy $E < 0$, and a regime where the molecule is dissolved in the continuum, for $E > 0$.

We thus have for the free Hamiltonian

$$H_0 \left|\vec{k}\right\rangle = 2\epsilon_k \left|\vec{k}\right\rangle \tag{1}$$

$$H_0 \left|m\right\rangle = \delta \left|m\right\rangle \tag{2}$$

We only consider a coupling V acting between $|m\rangle$ and the states $\left|\vec{k}\right\rangle$, and we consider that coupling to be momentum independent. This is an excellent assumption in the case of purely s-wave (head-on) scattering.

$$\langle m| V \left|\vec{k}\right\rangle = g \tag{3}$$

We take g to be real. The task is to solve the Schrödinger equation with the total Hamiltonian $H = H_0 + V$.

$$H \left|\Psi_\mu\right\rangle = E_\mu \left|\Psi_\mu\right\rangle \tag{4}$$

with

$$\left|\Psi_\mu\right\rangle = \alpha_\mu \left|m\right\rangle + \sum_{\vec{k}} c_{\vec{k},\mu} \left|\vec{k}\right\rangle \tag{5}$$

We will for all but the last part (f) be concerned only with $E < 0$. There is only one solution, the dressed molecular state, so here we can drop the index μ.

a) By projecting the Schrödinger equation onto the states $|m\rangle$ and $\left|\vec{k}\right\rangle$, find a system of two coupled equations for α and c_k. Solve for c_k in terms of α and find the eigenvalue equation for the energy E.

b) Using a large quantization volume $\mathcal{V} = L^3$, apply the limit

$$\sum_{\vec{k}} \to \mathcal{V} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} = \frac{\mathcal{V}}{2\pi^2} \int_0^\Lambda \mathrm{d}k\, k^2 \tag{6}$$

with a momentum-cutoff Λ, corresponding to a cutoff in energy $E_\Lambda = \frac{\hbar^2 \Lambda^2}{2m}$. You will need the integral $\int_0^y \frac{x^2}{1+x^2} = y - \arctan(y)$. Performing the integration, rewrite the eigenvalue equation for $|E|$ solely in terms of δ, δ_0 and E_0, where

$$\delta_0 = g^2 \rho(E_\Lambda) \qquad \text{a shift} \tag{7}$$

$$E_0 = \frac{1}{32\pi^2} \frac{g^4}{\Delta^3} \qquad \text{coupling energy scale} \tag{8}$$

$$\rho(\epsilon) = \frac{1}{\sqrt{2}\pi^2} \frac{\sqrt{\epsilon}}{\Delta^{3/2}} \qquad \text{energy density of states} \tag{9}$$

$$\Delta = \frac{\hbar^2}{mL^2} \qquad \text{scale of quantization of energy levels} \tag{10}$$

Hint: The arctan from the integration can be replaced by $\pi/2$ in the limit $E_\Lambda \gg |E|$.

c) The equation you found yields a quadratic equation for the molecular bound state energy $E < 0$. Solve it, thereby obtaining a result for the dressed energy of the molecular state, due to coupling to the continuum. Make a plot of the energy versus δ, and also include the bare molecular energy itself.

d) At what bare molecular energy δ does the Feshbach resonance occur, i.e. where is $E = 0$? Why is this no longer at $\delta = 0$?

e) Find a useful approximation of $E < 0$ for the near-resonant case where the detuning is very close to the resonance location on the scale of E_0. How does E scale with the detuning from resonance?

f) For δ larger than the resonance position the energy is no longer purely real but acquires an imaginary contribution $-i\hbar\Gamma/2$. The molecular state is unstable when embedded in the continuum, and it acquires a width. Obtain Γ from Fermi's Golden rule, and in particular bring out the dependence of Γ on the bare molecular energy δ (careful about the resonance shift). The behavior you find is known as the Wigner threshold law.

Notes and References: Feshbach resonances are an essential tool for quantum gas experiments. They enabled the observation of superfluidity in Fermi gases, the creation of dipolar molecules, coherent control of interactions in optical lattices etc. Their first observation was done here at MIT by Wolfgang Ketterle and his team in a Bose-Einstein condensate of sodium [S. Inouye, Nature 392, 151–154 (1998)], on the "BEC1" apparatus that still uses Feshbach resonances every day today (these days between fermionic atoms). The dissociation and decay of ultracold Feshbach molecules, and the Wigner threshold law was observed in T. Mukaiyama, J. R. Abo-Shaeer, K. Xu, J. K. Chin, and W. Ketterle, Phys. Rev. Lett. 92, 180402 (2004).
Review on resonant Fermi gases: W. Ketterle, M. Zwierlein, Ultracold Fermi Gases, Proceedings of the International School of Physics "Enrico Fermi", Course CLXIV, Varenna, 20 - 30 June 2006, edited by M. Inguscio, W. Ketterle, and C. Salomon (IOS Press, Amsterdam) 2008, arXiv:0801.2500
Review on Feshbach resonances: C. Chin, R. Grimm, P. Julienne, E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).