8.333: Statistical Mechanics I Problem Set # **6** Due: **12/3/21 @ mid-night†**

† According to MIT regulations, no problem set can have a due date after 12/3/21, and I have extended the due date to latest allowed. However, the final exam can cover material that is presented in December. The optional problems are part of such material; credit for them can be used to boost your overall homework grade.

Ideal Quantum Gases

1. *Numerical estimates:* The following table provides typical values for the Fermi energy and Fermi temperature for (i) Electrons in a typical metal; (ii) Nucleons in a heavy nucleus; and (iii) He^3 atoms in liquid He^3 (atomic volume $= 46.2\AA^3$ per atom).

	$n(1/\mathrm{m}^3)$	$m(\mathrm{Kg})$	$\varepsilon_F(\mathrm{eV})$	$T_F(\mathrm{K})$
electron	10^{29}	9×10^{-31}	4.4	5×10^4
nucleons	10^{44}	1.6×10^{-27}	1.0×10^8	1.1×10^{12}
liquid He^3	2.6×10^{28}	4.6×10^{-27}	10^{-3}	10^1

(a) Estimate the ratio of the electron and phonon heat capacities at room temperature for a typical metal.

(b) Compare the thermal wavelength of a neutron at room temperature to the minimum wavelength of a phonon in a typical crystal.

(c) Estimate the degeneracy discriminant, $n\lambda^3$, for hydrogen, helium, and oxygen gases at room temperature and pressure. At what temperatures do quantum mechanical effects become important for these gases?

(d) **(Optional)** Experiments on He^4 indicate that at temperatures below 1K, the heat capacity is given by $C_V = 20.4T^3 JKg^{-1}\,\mathrm{K}^{-1}$. Find the low energy excitation spectrum, $\mathcal{E}(k)$, of He^4. (Hint: There is only one non-degenerate branch of such excitations.)

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2. *Solar interior:* According to astrophysical data, the plasma at the center of the sun has the following properties:

Temperature: $T = 1.6\times 10^7\,\mathrm{K}$

Hydrogen density: $\rho_H = 6\times 10^4$ kg m^{-3}

Helium density: $\rho_{He} = 1\times 10^5$ kg m^{-3}.

(a) Obtain the thermal wavelengths for electrons, protons, and α-particles (nuclei of He).

(b) Assuming that the gas is ideal, determine whether the electron, proton, or α-particle gases are degenerate in the quantum mechanical sense.

(c) Estimate the total gas pressure due to these gas particles near the center of the sun.

(d) Estimate the total radiation pressure close to the center of the sun. Is it matter, or radiation pressure, that prevents the gravitational collapse of the sun?

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3. *Density of states:* Consider a system of *non-interacting identical* degrees of freedom, with a set of single-particle energies $\{\varepsilon_n\}$, and ground state $\varepsilon_0 = 0$. In a grand canonical ensemble at temperature $T = (k_B\beta)^{-1}$, the number of particles N is related to the chemical potential μ by

$$N = \sum_n \frac{1}{e^{\beta(\varepsilon_n - \mu)} - \eta} = \int_0^\infty d\varepsilon \rho(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} - \eta} \quad ,$$

where $\rho(\varepsilon)$ is the density of single-particle states of energy ε, and $\eta = +1(-1)$ for bosons (fermions).

(a) Write a corresponding expression (in terms of $\rho(\varepsilon)$, β, and μ) for the total energy E of the system.

(b) For bosons write an implicit (integral) equation whose solution gives the critical temperature for Bose condensation.

For any function $g(x)$, the Sommerfeld expansion indicates that as $\beta \to \infty$,

$$\int_0^\infty dx \frac{g(x)}{e^{\beta(x-\mu)} + 1} \simeq \int_0^\mu dx\ g(x) + \frac{\pi^2}{6\beta^2}\ g'(\mu) + \cdots \quad .$$

(c) Use the above expansion to express the low temperature behavior of $\mu - E_F$, where E_F is the Fermi energy, in terms of β, $\rho(E_F)$ and $\rho'(E_F)$.

(d) As in the last part, find an expression for the increase in energy, $E(T) - E(T = 0)$, at low temperatures.

(e) Find the low temperature heat capacity of this system of fermions.

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4. *Quantum point particle condensation:* Consider a *quantum gas* of N spin-less point particles of mass m at temperature T, and volume V. An unspecified weak pairwise attraction between particles reduces the energy of any state by an amount $-uN^2/(2V)$ with $u > 0$, such that the partition function is

$$Z(T, N, V) = Z_0(T, N, V) \times \exp\left(\frac{\beta u N^2}{2V}\right),$$

where $Z_0(T, N, V)$ is the partition function of the ideal quantum gas, and $\beta = (k_B T)^{-1}$.

(a) Using the above relation between partition functions relate the pressure $P(n, T)$, as a function of the density $n = N/V$, to the corresponding pressure $P_0(n, T)$ of an ideal quantum gas.

(b) Use standard results for the non-relativistic gas to show that

$$\left.\frac{\partial P}{\partial n}\right|_T = -un + k_B T \frac{f^\eta_{3/2}(z)}{f^\eta_{1/2}(z)}, \quad \text{with} \quad f^\eta_{3/2}(z) = n\lambda^3 \quad \text{and} \quad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}.$$

(c) Find the critical value of the coupling $u_c(n, T)$ at which the gas becomes unstable, in the low density (non-degenerate) limit $n\lambda^3 \ll 1$, including the *first* correction that distinguishes between fermi and bose statistics.

(d) For fermions, relate the limiting behavior of $u_c(n, T)$ in the low temperature (degenerate limit $n\lambda^3 \gg 1$) to the fermi energy ϵ_F. (This is somewhat similar to the Chandrashekar instability of neutron stars.)

(e) What happens to u_c for bosons as temperature is decreased towards to quantum degenerate regime?

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5. *Harmonic confinement of Fermions:* A classical gas of fermions of mass m is confined in a d-dimensional anisotropic harmonic potential

$$U(\vec{r}) = \frac{m}{2}\sum_\alpha \omega_\alpha^2 x_\alpha^2,$$

with different restoring frequencies $\{\omega_\alpha\}$ along the different directions. We are interested in the limit of wide traps such that $\hbar\omega_\alpha \ll k_B T$, and the discreteness of the allowed energies can be ignored.

(a) Show that in this limit, the number of states $N(E)$ with energy less than or equal to E, and the density of states $\rho(E)$, are respectively given by

$$N(E) = \frac{1}{d!}\prod_{\alpha=1}^{d}\left(\frac{E}{\hbar\omega_\alpha}\right), \qquad \text{and} \qquad \rho(E) = \frac{1}{(d-1)!}\frac{E^{d-1}}{\prod_\alpha \hbar\omega_\alpha}.$$

(b) Show that in a grand canonical ensemble, the number of particles in the trap is

$$\langle N\rangle = f_d^-(z)\prod_\alpha\left(\frac{k_BT}{\hbar\omega_\alpha}\right).$$

(c) Compute the energy of E in the grand canonical ensemble. (Ignore the zero point energy of the oscillators.)

(d) From the limiting forms of the expressions for energy and number, compute the leading term for energy per particle in the high temperature limit.

(e) Compute the limiting value of the chemical potential at zero temperature.

(f) Give the expression for the heat capacity of the gas at low temperatures, correct up to numerical factors that you need not compute.

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6. *Anharmonic trap:* A collection of *non-interacting identical* particles is placed in an *anharmonic* trap, with one-particle Hamiltonian

$$H_1 = \frac{p^2}{2m} + Kr^n\,,$$

where r is the radial distance (in 3 dimensions) from the center of the trap.

(a) Show that the one particle density of state can be written as $\rho(\varepsilon) = \frac{C}{(p-1)!}\varepsilon^{p-1}$, where $p = 3/2 + 3/n$ and C is an amplitude *that you do not need to evaluate.*

(b) At high densities, quantization of one particle energy levels, $\{\varepsilon_j\}$, can be ignored, such that in a grand canonical ensemble at temperature $T = (k_B\beta)^{-1}$ and chemical potential μ, the number of particles is given by

$$N = \sum_j \frac{1}{e^{\beta(\varepsilon_j-\mu)}-\eta} = \int_0^\infty d\varepsilon\rho(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}-\eta}\quad,$$

with $\eta = +1(-1)$ for bosons (fermions). Give the expression for N in terms of the functions $f_m^\eta(z)$ with $z = e^{\beta\mu}$.

(c) Obtain the corresponding expression for the total energy E of the system.

(d) For a gas of N fermions, find the Fermi energy $E_F = \lim_{T\to 0}\mu$.

(e) *Without explicit calculation* state the behavior of heat capacity for this Fermi gas at temperatures $T \ll E_F/k_B$.

(f) For bosons find the expression for the heat capacity C close to zero temperature.

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7. (Optional) *Fermi gas in two dimensions:* Consider a non-relativistic gas of non-interacting spin 1/2 fermions of mass m in two dimensions.

(a) Find an explicit relation between the fugacity z and the areal density $n = N/A$. (If needed, note that $f_1^-(z) = \ln(1+z)$.)

(b) Give an explicit expression for the chemical potential $\mu(n,T)$, and provide its limiting forms at zero and high temperatures.

(c) Find the temperature at which $\mu = 0$.

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8. (Optional) *Partitions of Integers:* In mathematics, the partition $P(E)$ of an integer E refers to the number of ways the integer can be written as sums of smaller integers. For example, $P(5) = 7$ since 5= [5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1]. A celebrated result due to Hardy and Ramanujan is that asymptotically at large E,

$$P(E) \sim \frac{1}{4E\sqrt{3}} \exp\left(\pi\sqrt{\frac{2E}{3}}\right) .$$

The leading asymptotic dependence can in fact be obtained by considering the statistical mechanics of a gas of 'photons' in one dimension. Working in a system of units such that $k_B = \hbar = c = 1$, the single particle energies of such a gas are integers, i.e. $\epsilon_k = k$ for $k = 1, 2, 3, \cdots$, and

$$E = \sum_{k=1}^{\infty} k n_k \quad .$$

(a) Compute a 'partition function' $Z(\beta)$ in the limit $\beta \to 0$, after replacing sums over k with integrals. (Note that the number of 'photons' is arbitrary.)

(b) Compute the average energy, and use the result to find $T \equiv 1/\beta$ as a function of E, to leading order for $E \gg 1$.

(c) Compute the entropy $S(E)$ for $E \gg 1$. Is your result consistent with the Hardy–Ramanujan formula for partitions of integers?

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9. (Optional) *Fermions pairing into Bosons:* As a primitive model of superconductivity, consider a gas of non-interacting non-relativistic electrons (spin–1/2 and mass m). Assume that electrons of opposite spin can bind into a composite boson (spin 0, mass $2m$) of rest energy $-\epsilon$ with $\epsilon > 0$.

(a) In a grand canonical ensemble with chemical potential μ *for each electron*, write down the expressions for the densities n_e and n_b of free electrons, and bound electron pairs. (You do not need to derive the relevant expressions. Express your answer in terms of the fugacity $z = e^{\beta\mu}$, $y = e^{\beta\epsilon}$, and the electron thermal wavelength $\lambda = h/\sqrt{2\pi m k_B T}$.)

(b) What is the value of $z = z_c$ at the onset of Bose condensation? Write the expression for the total (bound plus unbound) electron density n_c at the onset of Bose condensation.

(c) Give the expression for pressure P in the condensed phase ($n > n_c$).

(d) Find the dimensionless ratio $(\beta P_c/n_c)$ at the *onset of condensation*. Evaluate its limiting values for $\beta\epsilon \to \infty$ and $\beta\epsilon \to 0$.

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10. (Optional) *Ring diagrams mimicking bosons:* Motivated by the statistical attraction between bosons, consider a *classical* system of identical particles, interacting with a pairwise potential $V(|\vec{q} - \vec{q}\,'|)$, such that

$$f(\vec{r}) = e^{-\beta V(r)} - 1 = \exp\left(-\frac{\pi r^2}{\lambda^2}\right) \quad , \text{ and } \quad \tilde{f}(\vec{\omega}) = \lambda^3 \exp\left(-\frac{\lambda^2 \omega^2}{4\pi}\right) ,$$

where $\tilde{f}(\vec{\omega})$ is the Fourier transform of $f(\vec{r})$.

(a) In a perturbative cluster expansion of the partition function, we shall retain only the diagrams forming a ring, which (after a summation over all powers of V between any pair of points) are proportional to

$$R_\ell = \int \frac{d^3\vec{q}_1}{V} \cdots \frac{d^3\vec{q}_\ell}{V} \, f(\vec{q}_1 - \vec{q}_2) f(\vec{q}_2 - \vec{q}_3) \cdots f(\vec{q}_\ell - \vec{q}_1).$$

Use properties of Fourier transforms to show that

$$R_\ell = \frac{1}{V^{\ell-1}} \int \frac{d^3\vec{\omega}}{(2\pi)^3}\, \tilde{f}(\vec{\omega})^\ell = \frac{\lambda^{3\ell}}{\ell^{3/2}\lambda^3 V^{\ell-1}} \,.$$

(b) Show that in the ring approximation, the partition function is given by

$$\ln Z_{\text{rings}} = \ln Z_0 + \frac{V}{2\lambda^3} f^+_{5/2}(n\lambda^3) - \frac{N}{2} + \frac{N^2\lambda^3}{2V}(1 - 2^{-5/2})\,,$$

where Z_0 is the partition function of the non-interacting gas, and $n = N/V$ is the number density.

(c) Compute the pressure P of the gas within the ring approximation.

(d) By examining the compressibility, or equivalently $\partial P/\partial n|_T$, show that this classical system of interacting particles must undergo a condensation transition.

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11. (Optional) *Relativistic Bose gas in d dimensions:* Consider a gas of non-interacting (spinless) bosons with energy $\epsilon = c\,|\vec{p}|$, contained in a box of "volume" $V = L^d$ in d dimensions.

(a) Calculate the grand potential $\mathcal{G} = -k_\text{B}T \ln \mathcal{Q}$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of d and $f^+_m(z)$, where $z = e^{\beta\mu}$, and

$$f^+_m(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1}dx.$$

(Hint: Use integration by parts on the expression for $\ln \mathcal{Q}$.)

(b) Calculate the gas pressure P, its energy E, and compare the ratio $E/(PV)$ to the classical value.

(c) Find the critical temperature, $T_\text{c}(n)$, for Bose-Einstein condensation, indicating the dimensions where there is a transition.

(d) What is the temperature dependence of the heat capacity $C(T)$ for $T < T_\text{c}(n)$?

(e) Evaluate the dimensionless heat capacity $C(T)/(Nk_B)$ at the critical temperature $T = T_c$, and compare its value to the classical (high temperature) limit.

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12. (Optional) *Surface adsorption of an ideal Bose gas:* Consider adsorption of particles of an ideal (spin-less) Bose gas onto a two dimensional surface.

(a) Treating the ambient gas as a *non-degenerate* ideal gas of temperature T and pressure P, find its chemical potential $\mu(T, P)$.

(b) The gas is in contact with a attractive surface, such that a particle gains an energy u upon adsorption to the surface. Treating the particles on the surface as a two dimensional ideal gas (in equilibrium with the ambient gas), find the areal density n_2 as a function of P, u, and temperature (T, β, and/or λ).

(c) Find the maximum pressure P^* before complete condensation to the surface.

(d) Find the singular behavior of n_2 for $\delta P = P^* - P \to 0$.

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13. (Optional) *Inertia of superfluid helium:* Changes in frequency of a torsional oscillator immersed in liquid helium can be used to track the "normal fraction" of the liquid as a function of temperature. This problem aims at computing the contribution of phonons (dominant at low temperatures) to the fraction of superfluid that moves with the oscillator plates. Consider a superfluid confined between two parallel plates moving with velocity $\vec{v}$.

(a) The isolated stationary superfluid has a branch of low energy excitations characterized by energy $\epsilon(p)$, where $p = |\vec{p}|$ is the magnitude of the momentum $\vec{p}$. Show that for excitations produced by walls (of large mass M) moving with velocity $\vec{v}$, this spectrum is modified (due to consideration of momentum and energy of the walls) to $\epsilon_{\vec{v}}(\vec{p}) = \epsilon(p) - \vec{p} \cdot \vec{v}$.

(b) Using the standard Bose occupation number for particles of energy $\epsilon_{\vec{v}}(\vec{p})$, obtain an integral expression for the net momentum $\vec{P}$ carried by the excitations in the superfluid. (**Hint:** $\sum_{\vec{p}} = V \int d^3\vec{p}/h^3$, where V is the volume.)

(c) Expanding the result for small velocities, show that $P_\alpha = V \rho_n v_\alpha$, and give an integral expression for ρ_n. **(Hint:** The angular average of $p_\alpha p_\gamma$ is $p^2 \delta_{\alpha\gamma}/3$.)

(d) Compute the contribution of phonons, with $\epsilon(p) = cp$, to ρ_n. (An answer that is correct up to a numerical coefficient is sufficient.)

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