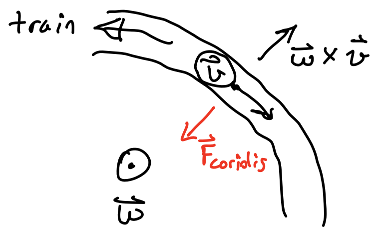
8.(3)09 Midterm Solutions 2021

100 Pts Total

1. 10 total

(a) 2 About the x-axis since $\overset{\hat{z}}{\tfrac{1}{3}I} < \overset{\hat{x}}{\tfrac{2}{3}I} < \overset{\hat{y}}{\tfrac{7}{3}I}$

(b) 3



$\vec{F}_{coriolis} = -2m\,\vec{\omega}\times\vec{v}$

inward towards center of circle

(c) 5

$$F = \underbrace{-Q_1P_1 + R(q_1, P_1)}_{\text{type } F_2} + \underbrace{G(q_2, Q_2)}_{\text{type } F_1}$$

$$p_1 = \frac{\partial R}{\partial q_1}, \quad Q_1 = \frac{\partial R}{\partial P_1}$$

$$p_2 = \frac{\partial G}{\partial q_2}, \quad P_2 = -\frac{\partial G}{\partial Q_2}$$

Not needed but can also write:

$$\dot{q}_i p_i - H = \dot{Q}_i P_i - K + \frac{dF}{dt}$$

$$\dot{q}_1 p_1 + \dot{q}_2 p_2 - H = \cancel{\dot{Q}_1 P_1} + \dot{Q}_2 P_2 - K - \cancel{\dot{Q}_1 P_1} - Q_1 \dot{P}_1 + \frac{\partial R}{\partial q_1}\dot{q}_1 + \frac{\partial R}{\partial P_1}\dot{P}_1 + \frac{\partial G}{\partial q_2}\dot{q}_2 + \frac{\partial G}{\partial Q_2}\dot{Q}_2$$

matching terms gives above

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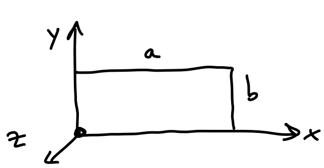
2. 27 total

(a) 7



$z=0$ for rectangle $\therefore$ $I_{xz} = I_{yz} = 0$

$$I_{xx} = \int_0^a dx \int_0^b dy\, \frac{M}{ab}\,(y^2+0) = \frac{M}{3}b^2 = A$$

$$I_{yy} = \text{''} \quad \text{''} \quad (x^2+0) = \frac{M}{3}a^2 = B$$

$$I_{xy} = \text{''} \quad \text{''} \quad (-xy) = -\frac{M}{4}ab = C$$

$$I_{zz} = \text{''} \quad \text{''} \quad (x^2+y^2) = \frac{M}{3}(a^2+b^2) = D$$

(b) [10] Two possible approaches i) rotate or ii) eigenvectors

Method i) $U \hat{I} U^T$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} A & C & 0\\ C & B & 0 \\ 0&0&D\end{pmatrix}\begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0&0&1\end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix} A\cos\theta + C\sin\theta & -A\sin\theta + C\cos\theta & 0\\ C\cos\theta + B\sin\theta & -C\sin\theta + B\cos\theta & 0 \\ 0&0&D\end{pmatrix}$$

only need to compute one off diagonal element

$$= \begin{pmatrix} A\cos^2\theta + B\sin^2\theta + 2C\sin\theta\cos\theta & \underline{C\cos^2\theta - C\sin^2\theta + (B-A)\sin\theta\cos\theta} & 0\\ C\cos^2\theta - C\sin^2\theta + (B-A)\sin\theta\cos\theta & A\sin^2\theta + B\cos^2\theta - 2C\sin\theta\cos\theta & 0 \\ 0&0&D\end{pmatrix}$$

off diagonal $= 0 = C\cos(2\theta) + \frac{(B-A)}{2}\sin 2\theta$ $\therefore$ $\boxed{\tan 2\theta = \frac{2C}{A-B}}$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2C}{A-B}\right)$$

Method ii) $\det(\hat{I} - \lambda \mathbb{1}) = 0 = \begin{vmatrix} A-\lambda & C & 0 \\ C & B-\lambda & 0 \\ 0 & 0 & D-\lambda \end{vmatrix} = (D-\lambda)\left[\underbrace{(A-\lambda)(B-\lambda) - C^2}_{\text{this part}}\right]$

$$\lambda^2 - \lambda(A+B) + AB - C^2 = 0$$

$$\lambda_\pm = \frac{(A+B)}{2} \pm \frac{1}{2}\sqrt{(A+B)^2 - 4AB + 4C^2} = \frac{A+B}{2} \pm \frac{1}{2}\sqrt{(A-B)^2 + 4C^2}$$

x', y' axes $\begin{pmatrix} A-\lambda_\pm & C \\ C & B-\lambda_\pm \end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix} = 0$ $\therefore$ $x = \frac{B-\lambda_\pm}{-C}$, $y = 1$ $\quad \tan\theta = \frac{-C}{B-\lambda_\pm} = \frac{A-\lambda_\pm}{-C}$

$$\tan\theta = \frac{-2C}{B-A \mp \sqrt{(B-A)^2 + 4C^2}}$$

use λ_- : smaller θ

$$\boxed{\theta = \tan^{-1}\left(\frac{-2C}{B-A+\sqrt{(B-A)^2+4C^2}}\right)}$$

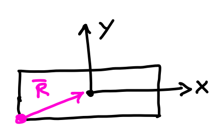
Check equivalent

$$\tan 2\theta = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \frac{2\tan\theta}{1-\tan^2\theta} = \left(\frac{-4C}{\cancel{B-A+\sqrt{\ }}}\right)\frac{(B-A+\sqrt{\ })^{\cancel{2}}}{(B-A+\sqrt{\ })^2 - 4C^2} = \frac{-4C(B-A+\sqrt{\ })}{2(B-A)^2 + 2(B-A)\sqrt{\ }}$$

$$= \frac{-2C}{B-A} \checkmark$$

(c) [10] Must use $\hat{I}^{(cm)}$. $\quad T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\vec{\omega}^T \cdot \hat{I}^{(cm)} \cdot \vec{\omega}$

Translate from (a) (or calculate again) $\vec{R} = \frac{a}{2}\hat{x} + \frac{b}{2}\hat{y}$



$$I^{(cm)}_{ab} = \hat{I}_{ab} - M(\vec{R}^2\delta_{ab} - R_a R_b) = \hat{I}_{ab} - M\begin{pmatrix} \frac{b^2}{4} & -\frac{ab}{4} & 0 \\ -\frac{ab}{4} & \frac{a^2}{4} & 0 \\ 0 & 0 & 0\end{pmatrix}$$

$$= M\begin{pmatrix} b^2/12 & 0 & 0 \\ 0 & a^2/12 & 0 \\ 0 & 0 & \frac{a^2+b^2}{12}\end{pmatrix}$$

$$\vec{\omega} = \dot{\phi}\,\frac{(a\hat{x} + b\hat{y})}{\sqrt{a^2+b^2}} \quad , \qquad \frac{1}{2} m \dot{\vec{R}}^2 = \frac{1}{2} m v^2$$

$$\vec{\omega}^T \cdot \hat{I}^{(cm)} \cdot \vec{\omega} = \frac{m\dot{\phi}^2}{12(a^2+b^2)} \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \frac{m\dot{\phi}^2}{12(a^2+b^2)} \begin{pmatrix} ab^2 & ba^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= \frac{m\dot{\phi}^2 \, 2a^2b^2}{12(a^2+b^2)} = \frac{m a^2 b^2}{6(a^2+b^2)} \dot{\phi}^2$$

$$\therefore \quad T = \frac{M}{2} v(t)^2 + \frac{m a^2 b^2}{12(a^2+b^2)} \left(\dot{\phi}(t)\right)^2$$

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3. [29 points total] $\qquad I = ma^2$

(a) [6] coord x, y, θ where θ = rotation angle of ring 2.

$$L = T - V \quad , \quad T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{ma^2}{2}\dot{\theta}^2 \qquad 2.$$

$$V = \frac{k}{2}\left[x^2 + (b+y)^2\right] \qquad 2.$$

(b) [3]

$$y = \frac{\alpha x^2}{2}$$

$$v = a\dot{\theta} = \sqrt{\dot{x}^2 + \dot{y}^2} \qquad 3.$$

Assuming $\dot{\theta} > 0$ is okay here. Really $a\dot{\theta} = \pm\sqrt{\dot{x}^2 + \dot{y}^2}$ since $\dot{\theta} > 0$ & $\dot{\theta} < 0$ during full motion.

(c) [12] impose $y = \frac{\alpha x^2}{2}$, $\dot{y} = \alpha x \dot{x}$

3. $g = a\dot{\theta} - \dot{x}\sqrt{1+\alpha^2 x^2} = 0$ $\qquad$ [← true for both signs $\dot{\theta}, \dot{x}$ since sign $\dot{\theta}$ = sign $\dot{x}$]

now semi-holonomic

use Lagrange Multiplier λ

3. $$L = \frac{m}{2}\dot{x}^2(1+\alpha^2x^2) + \frac{ma^2}{2}\dot{\theta}^2 - \frac{k}{2}\left[x^2 + \left(b + \frac{\alpha x^2}{2}\right)^2\right]$$

$$= \frac{m}{2}\dot{x}^2(1+\alpha^2x^2) + \frac{ma^2\dot{\theta}^2}{2} - \frac{k}{2}b^2 - x^2 \underbrace{\frac{k}{2}(1+\alpha b)}_{A} - \underbrace{\frac{k\alpha^2}{8}}_{B} x^4$$

$\boxed{x}$ $$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda \frac{\partial g}{\partial \dot{x}}$$

$$\frac{d}{dt}\left(m\dot{x}(1+\alpha^2x^2)\right) - m\dot{x}^2\alpha^2 x + 2Ax + 4Bx^3 = -\lambda\sqrt{1+\alpha^2x^2}$$

$$m\ddot{x}(1+\alpha^2x^2) + m\dot{x}^2\alpha^2x + k(1+\alpha b)x + \frac{k\alpha^2}{2}x^3 = -\lambda\sqrt{1+\alpha^2x^2} \qquad (1)$$

3.

$\boxed{\theta}$ $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = \lambda\frac{\partial g}{\partial \dot{\theta}} \; : \quad ma^2\ddot{\theta} - 0 = \lambda a \equiv f \qquad (2)$$

3.

(d) $\boxed{8}$ (3) $a\dot{\theta} = \dot{x}\sqrt{1+\alpha^2x^2}$ Solve (1),(2),(3) for λa

(3): $a\ddot{\theta} = \ddot{x}\sqrt{1+\alpha^2x^2} + \dfrac{\alpha^2\dot{x}^2x}{\sqrt{1+\alpha^2x^2}} = \dfrac{\ddot{x}(1+\alpha^2x^2)+\dot{x}^2\alpha^2x}{\sqrt{1+\alpha^2x^2}}$

(2):
$$\lambda = ma\ddot{\theta} = \frac{m\ddot{x}(1+\alpha^2x^2)+m\dot{x}^2\alpha^2x}{\sqrt{1+\alpha^2x^2}} \underset{(1)}{=} \frac{-\lambda\sqrt{1+\alpha^2x^2}-2Ax-4Bx^3}{\sqrt{1+\alpha^2x^2}}$$

$$\therefore f = a\lambda = \left(\frac{-Ax-2Bx^3}{\sqrt{1+\alpha^2x^2}}\right)a = \left[\frac{-\frac{k}{2}(1+\alpha b)x - \frac{k\alpha^2}{4}x^3}{\sqrt{1+\alpha^2x^2}}\right]a$$

= torque from no slip constraint

Note: when $x<0$, $f>0$ as expected. When $x>0$, $f<0$ & $\ddot{\theta}<0$.

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4. $\boxed{34 \text{ points total}}$ $H = \dfrac{1}{2ma^2}\left(p_\theta^2 + \dfrac{p_\phi^2}{\sin^2\theta}\right) + k\sin^2\theta$

(a) $\boxed{2}$ $\dfrac{\partial H}{\partial t} = 0$, L quad. in velocities, H quad in p's $\therefore$ $E = H$ conserved 1.

ϕ cyclic $\therefore$ p_ϕ = constant, conserved 1.

(b) $\boxed{12}$ $H = \alpha_1 = E_1$

2. $W = \underbrace{\alpha_\phi \phi}_{\text{cyclic}} + W_\theta[\theta,\alpha]$, $p_\phi = \alpha_\phi = E_2$

2. $\alpha_1 = \dfrac{1}{2ma^2}\left(\left(\dfrac{\partial W_\theta}{\partial\theta}\right)^2 + \dfrac{\alpha_\phi^2}{\sin^2\theta}\right) + k\sin^2\theta = E$ H-J eqtn

$$\frac{\partial W_\theta}{\partial \theta} = \pm\sqrt{2ma^2\left(\alpha_1 - k\sin^2\theta - \frac{\alpha_\phi^2}{2ma^2\sin^2\theta}\right)}$$

2. $W = \alpha_\phi\phi \pm \int d\theta \sqrt{2ma^2}\sqrt{\alpha_1 - k\sin^2\theta - \dfrac{\alpha_\phi^2}{2ma^2\sin^2\theta}}$ sol'n

2. $Q_1 = \beta_1 + t = \dfrac{\partial W}{\partial\alpha_1} = \pm\sqrt{2ma^2}\int d\theta\, \frac{1}{2}\left(\alpha_1 - k\sin^2\theta - \dfrac{\alpha_\phi^2}{2ma^2\sin^2\theta}\right)^{-1/2}$

2. $Q_2 = \beta_2 = \dfrac{\partial W}{\partial\alpha_\phi} = \phi \pm \sqrt{2ma^2}\int d\theta\, \frac{1}{2}\left(-\dfrac{\alpha_\phi}{ma^2\sin^2\theta}\right)$
$* \left(\alpha_1 - k\sin^2\theta - \alpha_\phi/_{2ma^2\sin^2\theta}\right)^{-1/2}$

2. Get $\beta_1 + t = f_1(\theta, \alpha_1, \alpha_\phi)$ invert $\theta = \theta(t, \alpha_1, \alpha_\phi, \beta_1)$

$\beta_2 = f_2(\phi, \theta, \alpha_1, \alpha_\phi)$ invert. $\phi = \phi(\theta, \alpha_1, \alpha_\phi, \beta_2)$

subst. $= \phi(t, \alpha_1, \alpha_\phi, \beta_1, \beta_2)$

(c) $\boxed{12}$ ϕ cyclic, $p_\phi = \text{const.}$ $J_\phi = \oint d\phi\, p_\phi = 2\pi\, \alpha_\phi$ (choice)

(ϕ, p_ϕ) exhibit rotations

(θ, p_θ) has oscillations. $p_\theta = \dfrac{\partial W_\theta}{\partial \theta} = \pm \sqrt{2ma^2} \sqrt{E - k\sin^2\theta - \dfrac{A}{\sin^2\theta}}$

Turning points $p_\theta = 0$: $E = k\sin^2\theta + \dfrac{A}{\sin^2\theta}$, $A \equiv \dfrac{\alpha_\phi^2}{2ma^2}$

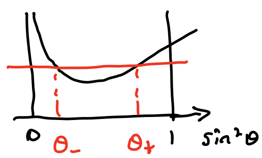
4. $y = \sin^2\theta$ then $E = ky + \dfrac{A}{y}$, $ky^2 - Ey + A = 0$

$y_\pm = \sin^2\theta_\pm = \dfrac{E \pm \sqrt{E^2 - 4Ak}}{2k}$ need $E \geq 2\sqrt{Ak}$ always

Case i both solutions $0 \leq \sin^2\theta_\pm \leq 1$, $y_+ = 1$ is $E = k + A$

so $0 \leq \theta \leq \frac{\pi}{2}$ and $\boxed{2\sqrt{Ak} \leq E \leq A + k}$.

4.



$\theta_\pm = \sin^{-1}\left(\left[\dfrac{E \pm \sqrt{E^2 - 4Ak}}{2k}\right]^{1/2}\right)$

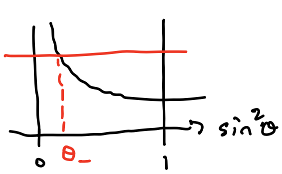
oscillates $\theta_- \xrightarrow[p_\theta > 0]{} \theta_+ \xrightarrow[p_\theta < 0]{} \theta_-$ factor of 2.

$$J_\theta = 2\int_{\theta_-}^{\theta_+} d\theta \sqrt{2ma^2} \sqrt{E - k\sin^2\theta - \frac{A}{\sin^2\theta}}$$

Case ii $\sin^2\theta_+ > 1$ so $\boxed{E > k + A}$

4. Check Always have $\sin^2\theta_- = \dfrac{E - \sqrt{E^2 - 4Ak}}{2k} < 1$, $E - 2k < \sqrt{E^2 - 4Ak}$

$(\cancel{E^2} - 4Ek + 4k^2) < \cancel{E^2} - 4Ak \iff A + k < E$ ✓



since $\sin^2(\pi - \theta_-) = \sin^2\theta_-$

oscillates $\theta_- \xrightarrow[p_\theta > 0]{} \pi - \theta_- \xrightarrow[p_\theta < 0]{} \theta_-$

$$J_\theta = 2\int_{\theta_-}^{\pi - \theta_-} d\theta \sqrt{2ma^2} \sqrt{E - k\sin^2\theta - \frac{A}{\sin^2\theta}}$$

$$= 4 \int_{\theta_-}^{\pi/2} d\theta \sqrt{2ma^2} \sqrt{E - k\sin^2\theta - \frac{A}{\sin^2\theta}}$$

(d) $\boxed{8}$

Using results from (c):

$$E_{min} = 2\sqrt{Ak} . \quad \sin^2\theta_\pm = \frac{E}{2k} = \sqrt{A/k} \equiv \sin^2\theta_0$$

Here $\theta = \theta_+ = \theta_- = \sin^{-1}\left(\frac{A}{K}\right)^{1/4} = \text{constant}$, at minimum $\theta = \theta_0$

$$p_\theta = 0$$

$$\dot\phi = +\frac{2H}{2p_\phi} = \frac{p_\phi}{ma^2\sin^2\theta_0} = \frac{\alpha_\phi}{ma^2}\sqrt{\frac{k}{A}}$$

$$\phi(t) = \frac{\alpha_\phi}{ma^2}\sqrt{\frac{k}{A}}\, t + \phi_0 = \frac{\alpha_\phi}{|\alpha_\phi|}\sqrt{\frac{2k}{ma^2}}\, t + \phi_0$$

or

Standalone solution:

$$E = \frac{p_\theta^2}{2ma^2} + V_{eff}(\theta), \quad V_{eff}(\theta) = k\sin^2\theta + \frac{A}{\sin^2\theta}$$

Min energy at min of potential

$$\frac{\partial V_{eff}}{\partial\theta} = 2k\sin\theta\cos\theta - \frac{2A\cos\theta}{\sin^3\theta} = \frac{2\cos\theta}{\sin\theta}\left(k\sin^4\theta - A\right)$$

$$= 0 \quad \text{at} \quad \theta = \theta_0$$

$$\sin\theta_0 = (A/k)^{1/4} \qquad V_{eff}(\theta_0) = k\sqrt{A/k} + A\sqrt{\frac{k}{A}} = 2\sqrt{Ak}$$

$$E_{min} = V_{eff}(\theta_0) = 2\sqrt{Ak}$$

$$\& \quad p_\theta = 0, \quad \theta(t) = \theta_0 \ \text{fixed}$$

then

$$\left.\begin{aligned} \dot\phi &= +\frac{2H}{2p_\phi} = \frac{p_\phi}{ma^2\sin^2\theta_0} = \frac{\alpha_\phi}{ma^2}\sqrt{\frac{k}{A}} \\ \phi(t) &= \frac{\alpha_\phi}{ma^2}\sqrt{\frac{k}{A}}\, t + \phi_0 = \frac{\alpha_\phi}{|\alpha_\phi|}\sqrt{\frac{2k}{ma^2}}\, t + \phi_0 \end{aligned}\right\} \text{same as above}$$