Matrices in Quantum Computing

Huan Q. Bui

Matrix Analysis

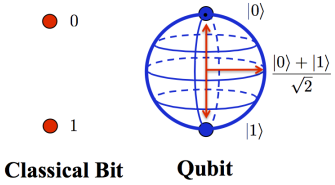
Professor Leo Livshits

CLAS, May 2, 2019

Presentation layout

Qubits

Qubit: A quantum system with measurable eigenstates $|0\rangle$ and $|1\rangle$,



$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\text{Wavefunction : } |\psi\rangle = a\,|0\rangle + b\,|1\rangle\,, \qquad |a|^2 + |b|^2 = 1.$$

$$\text{Probabilistic: } Pr(|\psi\rangle \to |0\rangle) = |a|^2$$

Quantum Gates

Quantum gate: linear transformation on $|\psi\rangle$ of one or many qubits.

A common single-qubit quantum gate: Hadamaar gate.

$$H \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

For example, applying H to $|0\rangle$:

$$H|0\rangle = H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle .$$

Multiple Qubits

Like classical circuits, quantum circuits require multiple qubits.
$\rightarrow$ How to express the quantum state of two qubits $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$?

$$|\psi_1\psi_2\rangle \overset{?}{\sim} |\psi_1\rangle, |\psi_2\rangle$$

What if there are more than two $|\psi_i\rangle$'s $\in \mathbf{V}_i$'s

$$|\psi_1\psi_2\dots\psi_n\rangle \overset{?}{\sim} |\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_n\rangle?$$

Mathematically,

- Is there a vector space that contains $|\psi_1\psi_2\dots\psi_n\rangle$?
- What is the vector space containing $|\psi_1\psi_2\dots\psi_n\rangle$?
- How does $|\psi_1\psi_2\dots\psi_n\rangle$ change w.r.t $\mathcal{A}_1|\psi_1\rangle$ where $\mathcal{A}_1 \in \mathfrak{L}(\mathbf{V})$?
- What about for $\mathcal{A}_1|\psi_1\rangle, \dots \mathcal{A}_n|\psi_n\rangle$, where $\mathcal{A}_i \in \mathfrak{L}(\mathbf{V})$?

Tensor Product

Postulate (QM): [NC02]

The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

For $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$,

$$|\psi_1\psi_2\rangle \in \mathbf{V}_1 \otimes \mathbf{V}_2,$$

where the joint state $|\psi_1\psi_2\rangle$ is given by

$$|\psi_1\psi_2\rangle = |\psi_1\rangle\,|\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle\,.$$

Tensor Product: Definition

What is this "$\otimes$" object?

Definition [Kam]

The *tensor product* of $\mathbf{V}$ and $\mathbf{W}$ is a vector space $\mathbf{V} \otimes \mathbf{W}$ with the *bilinear map* $\phi : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{V} \otimes \mathbf{W}$, such that for every vector space $\mathbf{X}$ and every bilinear map $f : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{X}$, there exists a *unique linear map* $\hat{f} : \mathbf{V} \otimes \mathbf{W} \longrightarrow \mathbf{X}$ such that $f = \hat{f} \circ \phi$.

In other words...

Giving the $\hat{f} : \mathbf{V} \otimes \mathbf{W} \xrightarrow{\text{linear}} \mathbf{X}$ is the same as giving $f : \mathbf{V} \times \mathbf{W} \xrightarrow{\text{bilinear}} \mathbf{X}$.

Tensor Product: Construction

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\hat{f}", "\textit{linear}"'] & & \mathbf{X} \\
& & \\
\mathbf{V} \times \mathbf{W} \arrow[uu, hook, "\phi"] \arrow[uurr, "f", "\textit{multilinear}"'] & &
\end{tikzcd}
$$

Tensor Product: Vectors [CER]

Let $v_1, \ldots, v_n$ be a basis for $\mathbf{V}$ and $w_1, \ldots, w_m$ be a basis for $\mathbf{W}$,

- $v_i \otimes w_j$'s are elementary.
- $\{v_i \otimes w_j\}$ is a basis of $\mathbf{V} \otimes \mathbf{W}$:

$$v \otimes w = \sum_{i}^{n} \alpha_i v_i \otimes \sum_{j}^{m} \beta_j w_j = \sum_{i,j}^{n,m} \alpha_i \beta_j (v_i \otimes w_j).$$

- Not all $x \in \mathbf{V} \otimes \mathbf{W}$ are elementary.
- $\dim(\mathbf{V} \otimes \mathbf{W}) = \dim(\mathbf{V}) \dim(\mathbf{W}) = nm$.

Tensor Product: Operators

Let $\mathcal{L} \otimes \mathcal{M} \in \mathfrak{L}(\mathbf{V} \otimes \mathbf{W})$, where $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$, and $\mathcal{M} \in \mathfrak{L}(\mathbf{W})$.

$$\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\mathcal{L}\otimes\mathcal{M}"] & \mathbf{V} \otimes \mathbf{W} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[ur, "\mathcal{F}"'] &
\end{tikzcd}$$

$\mathcal{F}(v, w) = \mathcal{L}(v) \otimes \mathcal{M}(w)$. By uniqueness,

$$\boxed{(\mathcal{L} \otimes \mathcal{M})(v \otimes w) = \mathcal{L}(v) \otimes \mathcal{M}(w)}$$

Tensor Product to Kronecker Product

Let Γ be a basis for $\mathbf{V} \otimes \mathbf{W}$, and $\{\,\}_\Gamma = \mathcal{A}_\Gamma^{-1}$ is the coordinatization from $\mathbf{V} \otimes \mathbf{W}$ to $\mathbb{C}^{nm}$, where $n = \dim(\mathbf{V})$, $m = \dim(\mathbf{W})$.

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[r, "\mathcal{L}\otimes\mathcal{M}", "linear"'] \arrow[d, "\{\,\}_\Gamma"] & \mathbf{V} \otimes \mathbf{W} \\
\mathbb{C}^{nm} \arrow[r, "\{\mathcal{L}\otimes\mathcal{M}\}_{\Gamma\leftarrow\Gamma}", "linear"'] & \mathbb{C}^{nm} \arrow[u, "\mathcal{A}_\Gamma"']
\end{tikzcd}
$$

Kronecker Product

$$[\mathcal{L} \otimes \mathcal{M}]_{\Gamma\leftarrow\Gamma} = [\mathcal{L}]_{\Gamma\leftarrow\Gamma} \otimes [\mathcal{M}]_{\Gamma\leftarrow\Gamma}.$$

If

$$[\mathcal{L}]_{\Gamma\leftarrow\Gamma} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \quad \text{and} \quad [\mathcal{M}]_{\Gamma\leftarrow\Gamma} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

then the *Kronecker product* $[\mathcal{L}]_{\Gamma\leftarrow\Gamma} \otimes [\mathcal{M}]_{\Gamma\leftarrow\Gamma}$ is defined as

$$\begin{aligned} [\mathcal{L}]_{\Gamma\leftarrow\Gamma} \otimes [\mathcal{M}]_{\Gamma\leftarrow\Gamma} &= \begin{bmatrix} l_{11}\mathcal{M} & l_{12}\mathcal{M} \\ l_{21}\mathcal{M} & l_{22}\mathcal{M} \end{bmatrix} \\ &= \begin{bmatrix} l_{11}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{12}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \\ l_{21}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{22}\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \end{bmatrix}. \end{aligned}$$

Kronecker Product

Doesn't care where scalar goes...

$$(\alpha\mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\alpha\mathcal{B}) = \alpha(\mathcal{A} \otimes \mathcal{B})$$

Associative:

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$$

Left-distributive:

$$\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = \mathcal{A} \otimes \mathcal{B} + \mathcal{A} \otimes \mathcal{C}$$

Right-distributive:

$$(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes \mathcal{C} + \mathcal{B} \otimes \mathcal{C}$$

Not commutative.

Multiple Qubits with Kronecker Product

Example: Representing 2-qubits with the Kronecker Product:

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}^\top$$

$$|11\rangle = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top.$$

$\rightarrow |00\rangle, |01\rangle, |10\rangle, |11\rangle$ form a basis for a 2-qubit system.

Entangling 2 qubits

- Entanglement.
- Recipe.
- Running on IBM-Q.

Entanglement

Not every $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$ is an elementary tensor.

Example: There are no states $|c\rangle, |d\rangle \in \mathbb{C}^2$ such that

$$\begin{aligned}|c\rangle \otimes |d\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top \\ &= \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \to \textbf{Entangled}\end{aligned}$$

Examples: Bell states, also entangled [CMTH]

$$\frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle$$
$$\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle$$
$$\frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle$$

“Entangled” operators

For operators: $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$, $\mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}\,|v\rangle) \otimes (\mathcal{B}\,|w\rangle).$$

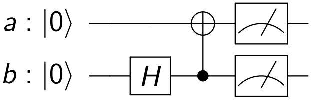
Not all $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$ can be written as $\mathcal{A} \otimes \mathcal{B}$, $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$.

Example:

$$CNOT_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{cases} |00\rangle \to |00\rangle \\ |10\rangle \to |10\rangle \\ |01\rangle \to |11\rangle \\ |11\rangle \to |01\rangle \end{cases}$$

2-Qubit Entanglement Circuit

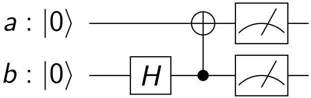
[EF04]



$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \qquad \begin{cases} |00\rangle \to |00\rangle \\ |10\rangle \to |10\rangle \\ |01\rangle \to |11\rangle \\ |11\rangle \to |01\rangle \end{cases}$$

Entanglement (cont.)



One way to see how this works...

$$
\begin{aligned}
CNOT_b\left(I\begin{bmatrix}1\\0\end{bmatrix}_a \otimes H\begin{bmatrix}1\\0\end{bmatrix}_b\right) &= CNOT_b\left(\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_a \otimes \frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}_b\right)\\
&= CNOT_b\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}\\
&= \frac{1}{\sqrt{2}}|0\rangle\otimes|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\otimes|1\rangle\\
&\rightarrow \textbf{Entangled}
\end{aligned}
$$

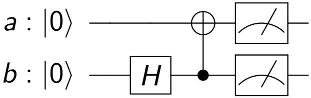
Entanglement (cont.)

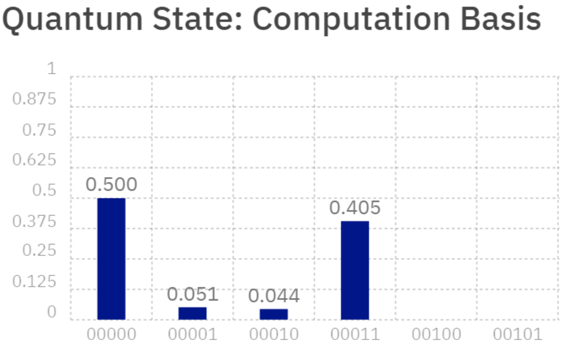
Another way to see how this works...

$$
\begin{aligned}
CNOT_b\left[(I\,|0\rangle) \otimes (H_b\,|0\rangle)\right] &= CNOT_b(I \otimes H_b)(|0\rangle \otimes |0\rangle) \\
&= CNOT_b \begin{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \mathcal{O} \\ \mathcal{O} & \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= CNOT_b \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}^\top \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^\top
\end{aligned}
$$

Simulation on IBM-Q

Entanglement circuit, revisited





Recap

- Representing multi-qubit systems with tensor products.
- Representing multi-qubit operators with tensor products.
- Compute with the Kronecker product.
- Entanglement, mathematically.
- 2-qubit entangler, mathematically.
- Entanglement on IBM-Q.

References

CERN, *Appendix a: Linear algebra for quantum computation*.

Chih-Sheng Chen Chao-Ming Tseng and Chua-Huang Huang, *Quantum gates revisited: A tensor product based interpretation model*.

Bryan Eastin and Steven T Flammia, *Q-circuit tutorial*, arXiv preprint quant-ph/0406003 (2004).

Joel Kamnitzer, *Tensor products*.

Michael A Nielsen and Isaac Chuang, *Quantum computation and quantum information*, 2002.