A Quick Guide to
Geometric Phase: Theory and Experimental Observation

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1 Adiabatic theorem

Let a quantum system be described by a Hamiltonian $\mathcal{H} = \mathcal{H}(\boldsymbol{R}(t))$ where $\boldsymbol{R}$ is a family of slowly-varying parameters and t denotes time. Let the system be in the n^{th} eigenstate of $\mathcal{H}(\boldsymbol{R}(0))$. Then at time t the system will be in the n^{th} instantaneous eigenstate of $\mathcal{H}(\boldsymbol{R}(t))$. cite Shankar here.

2 Berry phase

Consider the Hamiltonian described in the adiabatic theorem. Let $|\psi(0)\rangle = |n(\boldsymbol{R}(0))\rangle$ where $|n(\boldsymbol{R}(0))\rangle$ is the n^{th} eigenstate of $\mathcal{H}(\boldsymbol{R}(0))$. In view of the adiabatic theorem, $|\psi(t)\rangle$ must be $|n(\boldsymbol{R}(t))\rangle$, the n^{th} instantaneous eigenstate of $\mathcal{H}(t)$, up to a phase factor:

$$|\psi(t)\rangle = \exp\left(-\frac{i}{\hbar}\int_0^t E_n(\boldsymbol{R}(t'))\,dt'\right)\exp(i\gamma_n(t))\,|n(\boldsymbol{R}(t))\rangle\,,$$

where $\gamma_n(t)$ is a function of time and is called the **Berry phase** or **geometric phase**. Since $|\psi(t)\rangle$ solves the Schrödinger equation $\mathcal{H}(\boldsymbol{R}(t))\,|\psi(t)\rangle = i\hbar(d/dt)\,|\psi(t)\rangle$, we have by the chain rule in calculus:

$$\dot{\gamma}_n(t) = i\,\langle n(\boldsymbol{R}(t))|\,\nabla_{\boldsymbol{R}}\,|n(\boldsymbol{R}(t))\rangle\cdot\dot{\boldsymbol{R}}(t).$$

In particular, we find that at some final time t_f, the Berry phase is given by

$$\gamma_n(t_f) = \int_0^{t_f}\dot{\gamma}_n(t')\,dt' = \int_{\boldsymbol{R}_i}^{\boldsymbol{R}_f} i\,\langle n(\boldsymbol{R})|\,\nabla_{\boldsymbol{R}}\,|n(\boldsymbol{R})\rangle\cdot d\boldsymbol{R},$$

which depends only on the path in parameter space over which the evolution takes place. Define the **Berry connection**,

$$\boldsymbol{A}_n(\boldsymbol{R}) = i\,\langle n(\boldsymbol{R})|\,\nabla_{\boldsymbol{R}}\,|n(\boldsymbol{R})\rangle$$

and consider gauge transformation in parameter phase of instantaneous eigenstates. The Berry connection is also known as the Berry potential since it transforms like the electromagnetic vector potential:

$$|n(\boldsymbol{R})\rangle \to |\widetilde{n}(\boldsymbol{R})\rangle = e^{-i\beta(\boldsymbol{R})}\,|n(\boldsymbol{R})\rangle \implies \boldsymbol{A}_n(\boldsymbol{R}) \to \widetilde{\boldsymbol{A}_n}(\boldsymbol{R}) = \boldsymbol{A}_n(\boldsymbol{R}) + \nabla_{\boldsymbol{R}}\beta(\boldsymbol{R}).$$

Meanwhile the Berry phase transforms as

$$\widetilde{\gamma_n}(\boldsymbol{R}) = \int_{\boldsymbol{R}_i}^{\boldsymbol{R}_f}\widetilde{A_n}(\boldsymbol{R})\cdot d\boldsymbol{R} = \gamma_n(\boldsymbol{R}_f) + \beta(\boldsymbol{R}_f) - \beta(\boldsymbol{R}_i)$$

which is gauge-invariant exactly when the Hamiltonian evolution is cyclical in parameter space, i.e., $\boldsymbol{R}(t_f) = \boldsymbol{R}(0)$. A remarkable consequence of cyclic evolutions is that the Berry phase is well-defined and is a measurable quantity. To see this, consider the overlap integral between the initial and final state of the system $\langle\psi(0)|\psi(t_f)\rangle \sim e^{i\gamma_n(t_f)}$. We see that this relative phase can be observed via interferometry.

3 Example: Two-level system

Consider a spinor in a magnetic field. The (parametric) Hamiltonian is given by

4 Aharonov-Bohm Effect

Consider a charged particle in a magnetic field... The (parametric) Hamiltonian is given by

what is the parameter of the Hamiltonian? What is the closed loop here? Answer these questions.

5 Aharonov-Casher Effect

A dual to the Aharonov-Bohm effect. This is more of a sideline. I will mention only in passing.

6 Observation of a Gravitational Aharonov-Bohm effect

6.1 Experimental Techniques

6.1.1 Atom interferometry: The Mach-Zehnder Interferometer for Atoms

6.1.2 Bragg transition