Convolution powers of complex-valued functions on $\mathbb{Z}^d$

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The Classical Local Limit Theorem

Given iid random vectors $X_1, X_2, \ldots, X_n \in \mathbb{Z}^d$ from a probability distribution ϕ:

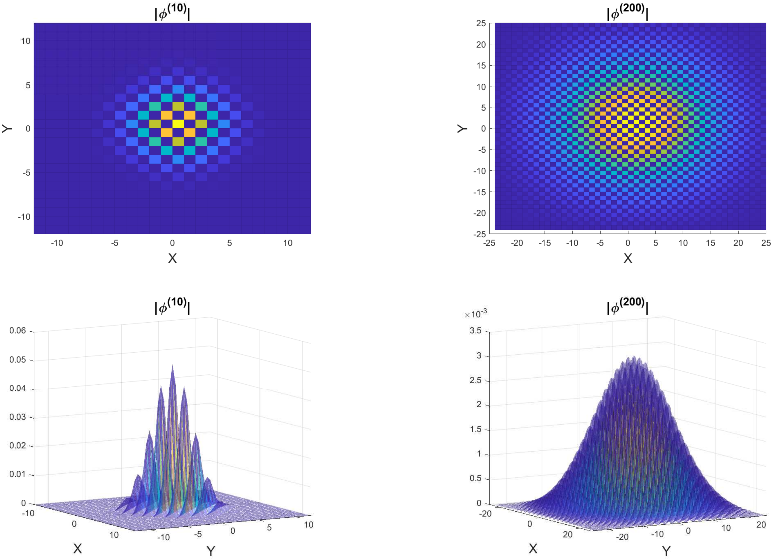
$$\phi(x) = \mathbb{P}(X_i = x).$$

The random walk $S_n = X_1 + X_2 + \ldots X_n$ has distribution

$$\phi^{(n)}(x) = \sum_{y \in \mathbb{Z}^d} \phi^{(n-1)}(x-y)\phi(y) = \phi^{(n-1)} * \phi^{(1)}.$$

How does $\phi^{(n)}$ behave when $n \to \infty$?

Example: Simple random walk in $\mathbb{Z}^2$



The Classical Local Limit Theorem

If ϕ is a "nice" probability distribution on $\mathbb{Z}^d$ with finite variance then

- Global decay: There are positive constants C_1, C_2 for which

$$C_1 n^{-d/2} \leq \|\phi^{(n)}\|_\infty \leq C_2 n^{-d/2}, \quad \forall n \in \mathbb{N}_+.$$

- Local description for large n:

$$\phi^{(n)}(x) = n^{-d/2}\Phi_\phi\left(xn^{-d/2}\right) + o\left(n^{-d/2}\right), \qquad \text{uniformly for } x \in \mathbb{Z}^d$$

  where Φ_ϕ is the generalized Gaussian associated with ϕ.

- Global estimate: There are positive constants C and M for which

$$\phi^{(n)}(x) \leq \frac{C}{n^{d/2}} \exp\left(-\frac{M|x|^2}{n}\right), \quad \forall x \in \mathbb{Z}^d, n \in \mathbb{N}_+$$

Beyond the Classical LLT

What if positivity is dropped?

Consider $\phi : \mathbb{Z}^d \to \mathbb{C}$ and define

$$\phi^{(n)}(x) = \sum_{y\in\mathbb{Z}^d} \phi^{(n-1)}(x-y)\phi(y) = \phi^{(n-1)} * \phi^{(1)}.$$

About the asymptotic behavior of $\phi^{(n)}$ as $n \to \infty$, can we still ask for

- A global decay?
- A local description?
- A global estimate?

Beyond the Classical LLT

Example: Look at $\phi^{(n)}$ for

$$\phi(x,y) = \frac{1}{768} \times \begin{cases} 602 - 112i & (x,y) = (0,0) \\ 56 + 32i & (x,y) = (-1,0) \\ 72 + 32i & (x,y) = (1,0) \\ -16 & (x,y) = (\pm 2, 0) \\ 56 + 32i & (x,y) = (0, \pm 1) \\ -28 - 8i & (x,y) = (0, \pm 2) \\ 56 & (x,y) = (0, \pm 3) \\ -1 & (x,y) = (0, \pm 4) \\ 4 & (x,y) = (-1, \pm 1) \\ -4 & (x,y) = (1, \pm 1) \\ 0 & \text{otherwise.} \end{cases}$$

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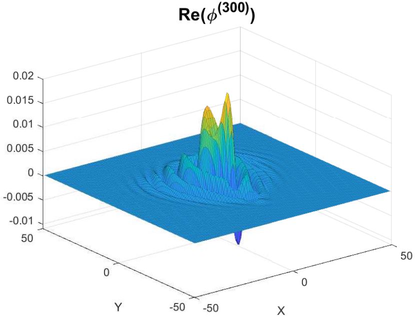
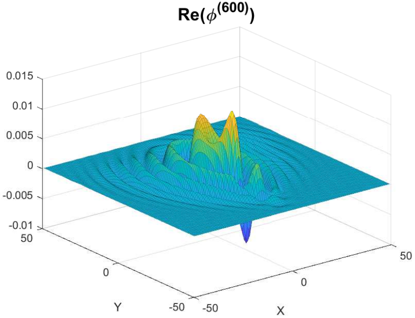
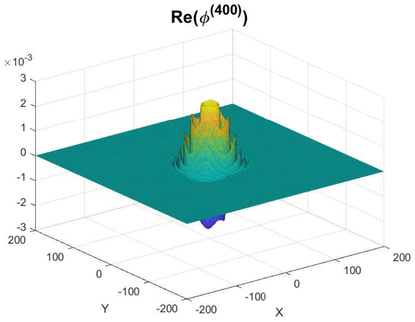


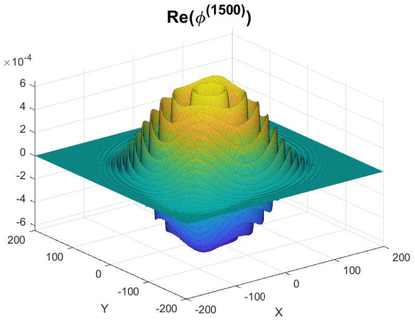


Example:

$$\phi(x,y) = \frac{1}{192} \times \begin{cases} 144 - 64i & (x,y) = (0,0) \\ 16 + 16i & (x,y) = (\pm 1, 0) \text{ or } (0, \pm 1) \\ -4 & (x,y) = (\pm 2, 0) \text{ or } (0, \pm 2) \\ i & (x,y) = \pm(1,1) \\ -i & (x,y) = \pm(1,-1) \\ 0 & \text{otherwise.} \end{cases}$$

Beyond the Classical LLT





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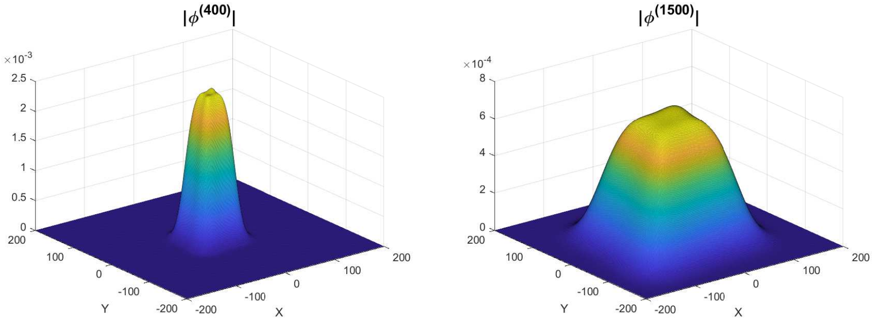

What if positivity is dropped?

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About the asymptotic behavior of $\phi^{(n)}$ as $n \to \infty$, can we still ask for

- **A global decay?** $\Longleftarrow$
- A local description?
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HOW?

Global decay estimate for $\left|\phi^{(n)}\right|$

$$\boxed{\mathrm{FT}\{\phi^{(n)}\} = (\mathrm{FT}\{\phi\})^n}$$

Define the Fourier transform for ϕ in $\mathcal{S}_d$:

$$\widehat{\phi}(\xi) = \sum_{x \in \mathbb{Z}^d} \phi(x) e^{ix\cdot\xi}$$

The asymptotic behavior of $\phi^{(n)}$ is characterized by how $\widehat{\phi}$ behaves near where $|\widehat{\phi}|$ is maximized:

$$\Omega(\phi) = \left\{\xi \in \mathbb{T}^d : \left|\widehat{\phi}(\xi)\right| = 1\right\}, \quad \mathbb{T}^d = (-\pi, \pi]^d$$

For each $\xi_0 \in \Omega(\phi)$, look at $\widehat{\phi}$ near ξ_0...

$$\widehat{\phi}(\xi + \xi_0) = \widehat{\phi}(\xi_0)e^{\Gamma_{\xi_0}(\xi)}$$

Need info about Q, R to find a global estimate. Why?

Recall $\widehat{\phi^{(n)}} = \widehat{\phi}^n$. So, $\phi^{(n)} = \mathrm{FT}^{-1}\left\{\widehat{\phi}^n\right\} \sim \mathrm{FT}^{-1}\left\{e^{n\Gamma_{\xi_0}(\xi)}\right\}$.

$\Longrightarrow$ The structure of Γ determines the asymptotic behavior of $\widehat{\phi}$

Taylor expand Γ_{ξ_0}...

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - iQ_{\xi_0}(\xi) - R_{\xi_0}(\xi) + \text{ h.o.t.}, \quad Q_{\xi_0}, R_{\xi_0} \text{ real polynomials}$$

Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

<u>In 1 dimension</u>:

ξ_0 is of **positive homogeneous type** if

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0}\xi - \beta\xi^m + \text{ h.o.t.}, \quad \mathrm{Re}\{\beta\} > 0$$

$\implies \phi^{(n)}$ is easy to estimate.

ξ_0 is of **imaginary homogeneous type** if

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0}\xi - i\xi^m p(\xi) - \gamma\xi^k + \text{ h.o.t.},$$

$\implies \widehat{\phi}^n$ is highly oscillatory. $\phi^{(n)}$ is more difficult to estimate.

<u>Remark</u>: In $d = 1$, these two types are collectively exhaustive for f.s. ϕ's.

Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

[RSC15] has completely solved the 1-dimensional problem.

Theorem (Global decay estimate, Theorem 1.1 of [RSC15])

Let $\phi : \mathbb{Z} \to \mathbb{C}$ be finitely supported and whose support contains more than one point. Then there is $\mathbb{N} \ni m \geq 2$, and A, C, $C' > 0$ such that

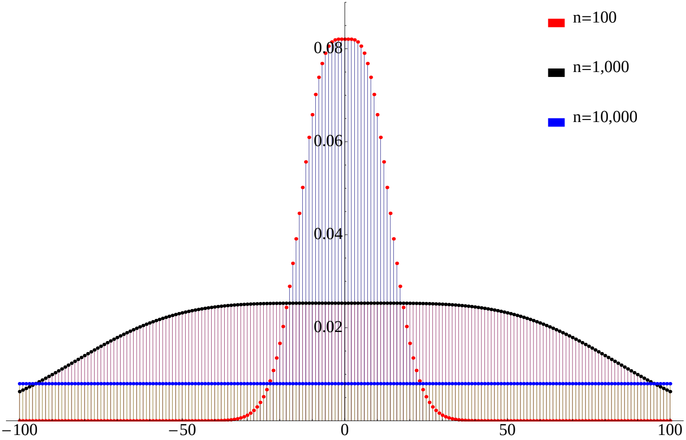
$$Cn^{-1/m} \leq A^{-n} \|\phi^{(n)}\|_\infty \leq C' n^{-1/m}, \quad \forall n \in \mathbb{N}$$

Here, $A = \sup |\widehat{\phi}(\xi)|$.

Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

Example: $\phi : \mathbb{Z} \to \mathbb{C}$ defined below. $\sup|\phi^{(n)}|$ decays like $n^{-1/2}$.

$$\phi(0) = \frac{5-2i}{8} \quad \phi(\pm 1) = \frac{2+i}{8} \quad \phi(\pm 2) = -\frac{1}{16} \quad \phi = 0 \text{ otherwise.}$$



Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

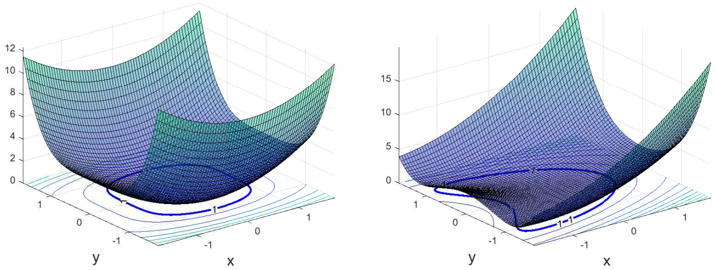
How to generalize to d dimensions?
$\implies$ Need **positive homogeneous functions**

Definition

Let $P : \mathbb{R}^d \to \mathbb{R}$ be continuous, positive definite, and $E \in \mathrm{Gl}(\mathbb{R}^d)$ s.t. $P(r^E\eta) = rP(\eta)$. If $S = \{\eta \in \mathbb{R}^d : P(\eta) = 1\}$ is compact then we say that P is **positive homogeneous***.

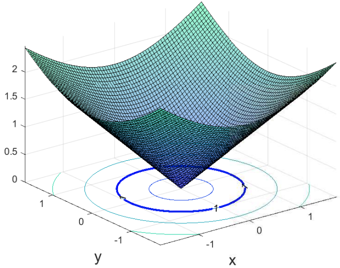
(*) see equivalent definitions in [BR21]

Examples:

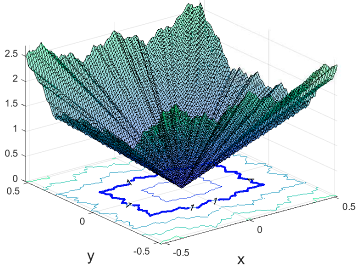


(a) $P_1(x, y) = x^2 + y^4$ (b) $P_2(x, y) = x^2 + 3xy^2/2 + y^4$

Examples: S doesn't have to be smooth



(a) $Q(x,y)=\sqrt{x^2+y^2}$



(b) $P(\xi)=Q(\xi)\times\text{Weierstrass}(\xi)$

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

In d dimensions:

ξ_0 is of **positive homogeneous type** if

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - P_{\xi_0}(\xi) + \text{ h.o.t.}$$

where $P_{\xi_0}(\xi)$ is a positive homogeneous *polynomial*

ξ_0 is of **imaginary homogeneous type** if

$$\Gamma_{\xi_0}(\xi) \sim i\alpha_{\xi_0} \cdot \xi - iP_{\xi_0}(\xi) + \text{ h.o.t.}$$

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

[RSC17] has partially solved the d-dimensional problem.

Theorem (Global decay estimate, Theorem 1.4 of [RSC17])

Let $\phi \in \mathcal{S}_d$ be such that $\sup|\widehat{\phi}(\xi)| = 1$ and suppose that each $\xi \in \Omega(\phi)$ is of **positive homogeneous type** *for $\widehat{\phi}$. There are μ_ϕ, C, $C' > 0$ for which*

$$C' n^{-\mu_\phi} \leq \|\phi^{(n)}\|_\infty \leq C n^{-\mu_\phi}, \quad \forall n \in \mathbb{N}$$

We now extend this to ξ of imaginary homogeneous type.

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

Theorem (Theorem 3.2 of [BR21])

Let $\phi \in \mathcal{S}_d$ be such that $\sup|\widehat{\phi}| = 1$ and suppose that each $\xi_0 \in \Omega(\phi)$ is of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$. Then, for any compact set K, there are $C_K, \mu_\phi > 0$ for which***

$$\left|\phi^{(n)}(x)\right| \leq \frac{C_K}{n^{\mu_\phi}}$$

for all $x \in K$ and $n \in \mathbb{N}_+$.

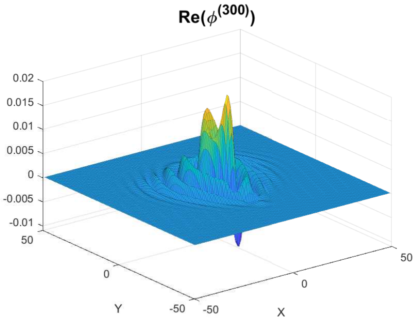
(*) and some additional conditions
(**) see [BR21] for how to calculate μ_ϕ

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

Example: From earlier...

$$\phi(x,y) = \frac{1}{768} \times \begin{cases} 602 - 112i & (x,y) = (0,0) \\ 56 + 32i & (x,y) = (0,\pm 1) \text{ or } (-1,0) \\ 72 + 32i & (x,y) = (1,0) \\ -28 - 8i & (x,y) = (0,\pm 2) \\ -16 & (x,y) = (\pm 2, 0) \\ 56 & (x,y) = (0,\pm 3) \\ -1 & (x,y) = (0,\pm 4) \\ 4 & (x,y) = (-1,\pm 1) \\ -4 & (x,y) = (1,\pm 1) \\ 0 & \text{otherwise.} \end{cases}$$

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions



Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

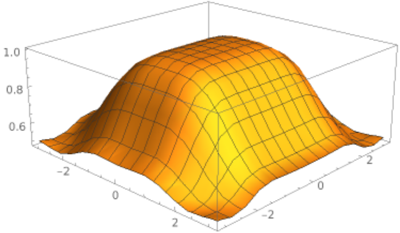


Figure: $|\widehat{\phi}|$ on $(-\pi, \pi] \times (-\pi, \pi]$

- $\sup|\widehat{\phi}| = 1$ and $\Omega(\phi) = \{\xi_0\} = \{(0,0)\}$

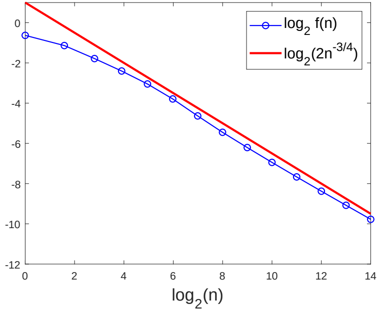
$$\Gamma_0(\xi) = -i\left(\frac{\tau^2}{24} - \frac{\tau\zeta^2}{96} + \frac{\zeta^4}{96}\right) + \text{ h.o.t.}$$

- $\mu_\phi = 1/2 + 1/4 = 3/4$

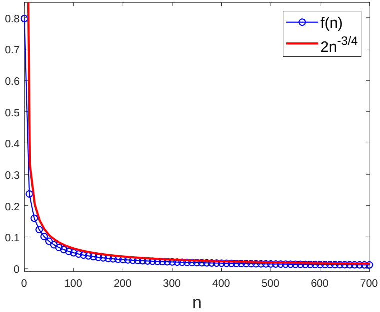
Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

Let $K = [-300, 300] \times [-300, 300]$ and pick $C = 2$.

$$f(n) := \max_{K} \left|\phi^{(n)}\right| \leq 2n^{-\mu_\phi} = 2n^{-3/4}$$



(a) $\log_2 f(n)$, $\log_2 2n^{-3/4}$ vs $\log_2 n$.



(b) $f(n)$, $2n^{-3/4}$ vs. n

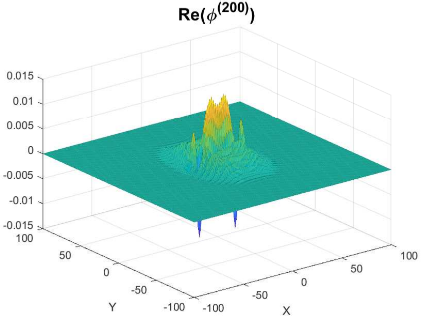
Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

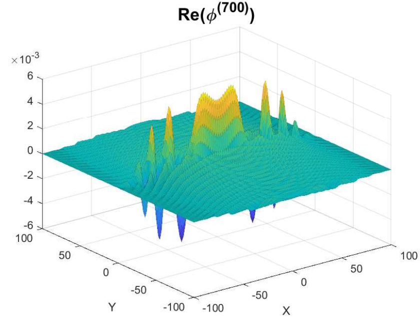
Example: $\phi : \mathbb{Z}^2 \to \mathbb{C}$ defined by $\phi = 2^{-7}\phi_1 - i2^{-11}\phi_2 + 2^{-21}\phi_3$ where

$$\phi_1(x,y) = \begin{cases} 15+15i & (x,y)=(\pm 1,0) \\ 16+16i & (x,y)=(0,\pm 1) \\ 1+1i & (x,y)=(\pm 3,0) \\ 0 & \text{otherwise} \end{cases}, \quad \phi_2(x,y) = \begin{cases} 682 & (x,y)=(0,0) \\ 152 & (x,y)=(\pm 2,0) \\ -28 & (x,y)=(\pm 4,0) \\ 8 & (x,y)=(\pm 6,0) \\ -1 & (x,y)=(\pm 8,0) \\ 60 & (x,y)=(0,\pm 2) \\ -24 & (x,y)=(0,\pm 4) \\ 4 & (x,y)=(0,\pm 6) \\ 0 & \text{otherwise} \end{cases},$$

$$\phi_3(x,y) = \begin{cases} 1387004 & (x,y)=(0,0) \\ -106722 & (x,y)=(\pm 2,0) \\ 3960 & (x,y)=(\pm 4,0) \\ -1045 & (x,y)=(\pm 6,0) \\ 138 & (x,y)=(\pm 8,0) \\ -9 & (x,y)=(\pm 10,0) \\ -131072 & (x,y)=(0,\pm 2) \\ 0 & \text{otherwise} \end{cases}$$

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions





Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

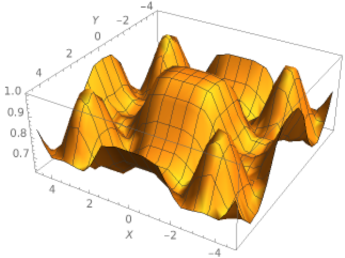


Figure: $|\widehat{\phi}|$ on $(-\pi, \pi] \times (-\pi, \pi]$

- $\sup |\widehat{\phi}| = 1$ and $\Omega(\phi) = \{\xi_0, \xi_1\} = \{(0,0), (\pi, \pi)\}$

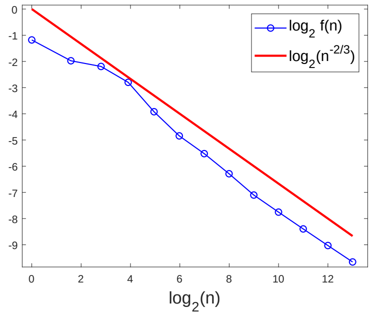
$$\Gamma_0(\xi) = -i\left(\frac{\tau^6}{128} + \frac{\zeta^2}{8}\right) + \dots \qquad \Gamma_1(\xi) = +i\left(\frac{3\tau^2}{8} + \frac{\zeta^2}{4}\right) + \dots$$

- $\mu_\phi = \min\{1/6 + 1/2, 1/2 + 1/2\} = \min\{2/3, 1\} = 2/3$

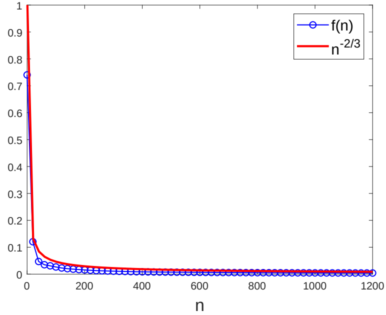
Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

Let $K = [-500, 500] \times [-500, 500]$ and pick $C = 1$.

$$f(n) := \max_K \left|\phi^{(n)}\right| \leq n^{-\mu_\phi} = n^{-2/3}$$



(a) $\log_2 f(n)$, $\log_2 n^{-2/3}$ vs $\log_2 n$.



(b) $f(n)$, $n^{-2/3}$ vs. n

Applications?

1. Numerical solutions to PDEs
    - Approximate solutions by taking convolution powers

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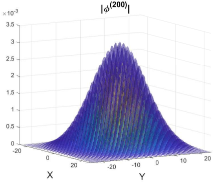
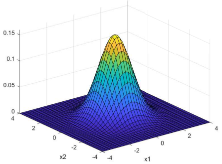
2. Quantum (field) theory
   - Oscillatory integrals are ubiquitous
   - Solutions to PDEs in QFT are often difficult to obtain/approximate
3. ...
4. For its own sake
   - Inspiration from examples/numerical evidence

What's next?

Classical result (for probability distributions):



$\phi^{(n)} \to$ Gaussian



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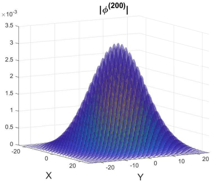

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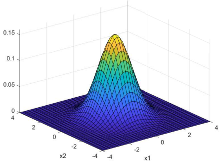
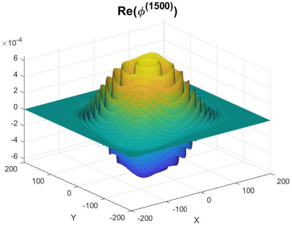
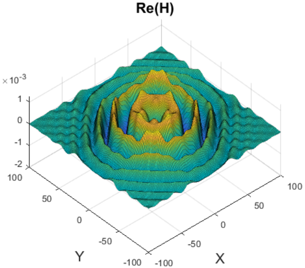


New conjecture: No positivity? No problem.



$\phi^{(n)} \to H_t^{iP}$



Global decay estimate for $\left|\phi^{(n)}\right|$: Extra

Proof ingredients:

1/ A generalized polar-coordinate integration formula (see [BR21])

2/ Van der Corput lemma

Lemma (Van der Corput lemma)

Let $g \in C^1([a,b])$ be complex-valued and let $\Phi \in C^2([a,b])$ be real-valued such that $\Phi''(x) \neq 0$ for all $x \in [a,b]$. Then

$$\left|\int_a^b g(u)e^{i\Phi(u)}\,du\right| \leq \min\left\{\frac{4}{\delta}, \frac{8}{\sqrt{\rho}}\right\}\left(\|g'\|_1 + \|g\|_\infty\right),$$

where $\delta = \inf_{x\in[a,b]}|\Phi'(x)|$ and $\rho = \inf_{x\in[a,b]}|\Phi''(x)|$.

- Integration by parts to bring the **amplitude** g out
- Integral dominated by the slowly-varying part of the **phase** Φ

References

Huan Q Bui and Evan Randles, *A generalized polar-coordinate integration formula with applications to the study of convolution powers of complex-valued functions on $\mathbb{Z}^d$*, arXiv preprint arXiv:2103.04161 (2021).

Evan Randles and Laurent Saloff-Coste, *On the Convolution Powers of Complex Functions on $\mathbb{Z}$*, J. Fourier Anal. Appl. **21** (2015), no. 4, 754–798.

______, *Convolution powers of complex functions on $\mathbb{Z}^d$*, Rev. Matemática Iberoam. **33** (2017), no. 3, 1045–1121.