

# Matrices in Quantum Computing

Huan Q. Bui

Matrix Analysis

Professor Leo Livshits

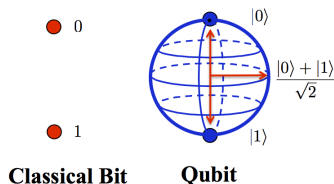
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# Presentation layout

- 1 Background
- 2 Motivation
- 3 Some Matrix Theory
- 4 Example: A 2-Qubit Entangler
- 5 Simulation on IBM-Q
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# Qubits

*Qubit*: A quantum system with measurable eigenstates  $|0\rangle$  and  $|1\rangle$ ,



$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Wavefunction :  $|\psi\rangle = a|0\rangle + b|1\rangle$ ,  $|a|^2 + |b|^2 = 1$ .

Probabilistic:  $Pr(|\psi\rangle \rightarrow |0\rangle) = |a|^2$

# Quantum Gates

*Quantum gate*: linear transformation on  $|\psi\rangle$  of one or many qubits.

A common single-qubit quantum gate: Hadamard gate.

$$H \triangleq \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

For example, applying  $H$  to  $|0\rangle$ :

$$H|0\rangle = H \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle.$$

# Multiple Qubits

Like classical circuits, quantum circuits require multiple qubits.

→ How to express the quantum state of two qubits  $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$ ?

$$|\psi_1\psi_2\rangle \stackrel{?}{\sim} |\psi_1\rangle, |\psi_2\rangle$$

What if there are more than two  $|\psi_i\rangle$ 's  $\in \mathbf{V}_i$ 's

$$|\psi_1\psi_2\ldots\psi_n\rangle \stackrel{?}{\sim} |\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle?$$

Mathematically,

- Is there a vector space that contains  $|\psi_1\psi_2\ldots\psi_n\rangle$ ?
- What is the vector space containing  $|\psi_1\psi_2\ldots\psi_n\rangle$ ?
- How does  $|\psi_1\psi_2\ldots\psi_n\rangle$  change w.r.t  $\mathcal{A}_1 |\psi_1\rangle$  where  $\mathcal{A}_1 \in \mathfrak{L}(\mathbf{V})$ ?
- What about for  $\mathcal{A}_1 |\psi_1\rangle, \ldots, \mathcal{A}_n |\psi_n\rangle$ , where  $\mathcal{A}_i \in \mathfrak{L}(\mathbf{V})$ ?

# Tensor Product

## Postulate (QM): [NC02]

The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

For  $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$ ,

$$|\psi_1\psi_2\rangle \in \mathbf{V}_1 \otimes \mathbf{V}_2,$$

where the joint state  $|\psi_1\psi_2\rangle$  is given by

$$|\psi_1\psi_2\rangle = |\psi_1\rangle |\psi_2\rangle = |\psi_1\rangle \otimes |\psi_2\rangle .$$

# Tensor Product: Definition

What is this “ $\otimes$ ” object?

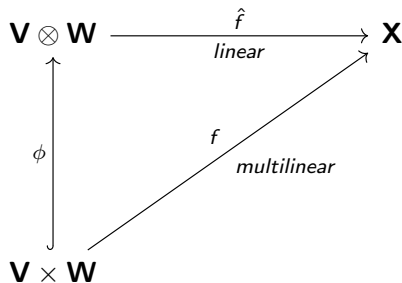
## Definition [Kam]

The *tensor product* of  $\mathbf{V}$  and  $\mathbf{W}$  is a vector space  $\mathbf{V} \otimes \mathbf{W}$  with the *bilinear map*  $\phi : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{V} \otimes \mathbf{W}$ , such that for every vector space  $\mathbf{X}$  and every bilinear map  $f : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{X}$ , there exists a *unique linear map*  $\hat{f} : \mathbf{V} \otimes \mathbf{W} \longrightarrow \mathbf{X}$  such that  $f = \hat{f} \circ \phi$ .

In other words...

Giving the  $\hat{f} : \mathbf{V} \otimes \mathbf{W} \xrightarrow{\text{linear}} \mathbf{X}$  is the same as giving  $f : \mathbf{V} \times \mathbf{W} \xrightarrow{\text{bilinear}} \mathbf{X}$ .

# Tensor Product: Construction





# Tensor Product: Vectors [CER]

Let  $v_1, \dots, v_n$  be a basis for  $\mathbf{V}$  and  $w_1, \dots, w_m$  be a basis for  $\mathbf{W}$ ,

- $v_i \otimes w_j$ 's are elementary.
- $\{v_i \otimes w_j\}$  is a basis of  $\mathbf{V} \otimes \mathbf{W}$ :

$$v \otimes w = \sum_i^n \alpha_i v_i \otimes \sum_j^m \beta_j w_j = \sum_{i,j}^{n,m} \alpha_i \beta_j (v_i \otimes w_j).$$

- Not all  $x \in \mathbf{V} \otimes \mathbf{W}$  are elementary.
- $\dim(\mathbf{V} \otimes \mathbf{W}) = \dim(\mathbf{V}) \dim(\mathbf{W}) = nm$ .

# Tensor Product: Operators

Let  $\mathcal{L} \otimes \mathcal{M} \in \mathfrak{L}(\mathbf{V} \otimes \mathbf{W})$ , where  $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$ , and  $\mathcal{M} \in \mathfrak{L}(\mathbf{W})$ .

$$\begin{array}{ccc} \mathbf{V} \otimes \mathbf{W} & \xrightarrow{\mathcal{L} \otimes \mathcal{M}} & \mathbf{V} \otimes \mathbf{W} \\ \uparrow \phi & \nearrow \mathcal{F} & \\ \mathbf{V} \times \mathbf{W} & & \end{array}$$

$\mathcal{F}(v, w) = \mathcal{L}(v) \otimes \mathcal{M}(w)$ . By uniqueness,

$$\boxed{(\mathcal{L} \otimes \mathcal{M})(v \otimes w) = \mathcal{L}(v) \otimes \mathcal{M}(w)}$$

# Tensor Product to Kronecker Product

Let  $\Gamma$  be a basis for  $\mathbf{V} \otimes \mathbf{W}$ , and  $\{\cdot\}_\Gamma = \mathcal{A}_\Gamma^{-1}$  is the coordinatization from  $\mathbf{V} \otimes \mathbf{W}$  to  $\mathbb{C}^{nm}$ , where  $n = \dim(\mathbf{V})$ ,  $m = \dim(\mathbf{W})$ .

$$\begin{array}{ccc} \mathbf{V} \otimes \mathbf{W} & \xrightarrow[\text{linear}]{\mathcal{L} \otimes \mathcal{M}} & \mathbf{V} \otimes \mathbf{W} \\ \downarrow \{\cdot\}_\Gamma & & \uparrow \mathcal{A}_\Gamma \\ \mathbb{C}^{nm} & \xrightarrow[\text{linear}]{\{\mathcal{L} \otimes \mathcal{M}\}_\Gamma \leftarrow \Gamma} & \mathbb{C}^{nm} \end{array}$$

# Kronecker Product

$$[\mathcal{L} \otimes \mathcal{M}]_{\Gamma \leftarrow \Gamma} = [\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma}.$$

If

$$[\mathcal{L}]_{\Gamma \leftarrow \Gamma} = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \quad \text{and} \quad [\mathcal{M}]_{\Gamma \leftarrow \Gamma} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

then the *Kronecker product*  $[\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma}$  is defined as

$$\begin{aligned} [\mathcal{L}]_{\Gamma \leftarrow \Gamma} \otimes [\mathcal{M}]_{\Gamma \leftarrow \Gamma} &= \begin{bmatrix} l_{11}\mathcal{M} & l_{12}\mathcal{M} \\ l_{21}\mathcal{M} & l_{22}\mathcal{M} \end{bmatrix} \\ &= \begin{bmatrix} l_{11} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{12} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \\ l_{21} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} & l_{22} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \end{bmatrix}. \end{aligned}$$

# Kronecker Product

Doesn't care where scalar goes...

$$(\alpha \mathcal{A}) \otimes \mathcal{B} = \mathcal{A} \otimes (\alpha \mathcal{B}) = \alpha(\mathcal{A} \otimes \mathcal{B})$$

Associative:

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$$

Left-distributive:

$$\mathcal{A} \otimes (\mathcal{B} + \mathcal{C}) = \mathcal{A} \otimes \mathcal{B} + \mathcal{A} \otimes \mathcal{C}$$

Right-distributive:

$$(\mathcal{A} + \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes \mathcal{C} + \mathcal{B} \otimes \mathcal{C}$$

Not commutative.

# Multiple Qubits with Kronecker Product

Example: Representing 2-qubits with the Kronecker Product:

$$|01\rangle = |0\rangle \otimes |1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ 0 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = [0 \ 0 \ 1 \ 0]^T$$

$$|11\rangle = [0 \ 0 \ 0 \ 1]^T.$$

$\rightarrow |00\rangle, |01\rangle, |10\rangle, |11\rangle$  form a basis for a 2-qubit system.

# Entangling 2 qubits

- Entanglement.
- Recipe.
- Running on IBM-Q.

# Entanglement

Not every  $|\psi\rangle \in \mathbf{V} \otimes \mathbf{W}$  is an elementary tensor.

Example: There are no states  $|c\rangle, |d\rangle \in \mathbb{C}^2$  such that

$$\begin{aligned} |c\rangle \otimes |d\rangle &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T \\ &= \frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \rightarrow \textbf{Entangled} \end{aligned}$$

Examples: Bell states, also entangled [CMTH]

$$\begin{aligned} &\frac{1}{\sqrt{2}} |00\rangle - \frac{1}{\sqrt{2}} |11\rangle \\ &\frac{1}{\sqrt{2}} |01\rangle + \frac{1}{\sqrt{2}} |10\rangle \\ &\frac{1}{\sqrt{2}} |01\rangle - \frac{1}{\sqrt{2}} |10\rangle \end{aligned}$$



# “Entangled” operators

For operators:  $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W}), \mathcal{A} \otimes \mathcal{B} \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$  is defined by

$$(\mathcal{A} \otimes \mathcal{B})(|v\rangle \otimes |w\rangle) = (\mathcal{A}|v\rangle) \otimes (\mathcal{B}|w\rangle).$$

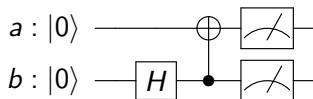
Not all  $C \in \mathcal{L}(\mathbf{V} \otimes \mathbf{W})$  can be written as  $\mathcal{A} \otimes \mathcal{B}$ ,  $\mathcal{A} \in \mathcal{L}(\mathbf{V}), \mathcal{B} \in \mathcal{L}(\mathbf{W})$ .

Example:

$$CNOT_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{cases} |00\rangle \rightarrow |00\rangle \\ |10\rangle \rightarrow |10\rangle \\ |01\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |01\rangle \end{cases}$$

## 2-Qubit Entanglement Circuit

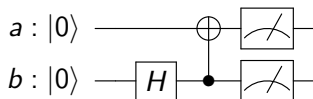
[EF04]



$$H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b = \frac{1}{\sqrt{2}} |0\rangle_b + \frac{1}{\sqrt{2}} |1\rangle_b$$

$$CNOT_b = C_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{cases} |00\rangle \rightarrow |00\rangle \\ |10\rangle \rightarrow |10\rangle \\ |01\rangle \rightarrow |11\rangle \\ |11\rangle \rightarrow |01\rangle \end{cases}$$

# Entanglement (cont.)



One way to see how this works...

$$\begin{aligned} CNOT_b \left( I \begin{bmatrix} 1 \\ 0 \end{bmatrix}_a \otimes H \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b \right) &= CNOT_b \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_a \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_b \right) \\ &= CNOT_b \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \\ &= \frac{1}{\sqrt{2}} |0\rangle \otimes |0\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes |1\rangle \\ &\rightarrow \textbf{Entangled} \end{aligned}$$

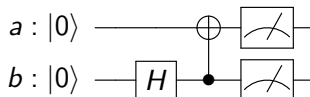
# Entanglement (cont.)

Another way to see how this works...

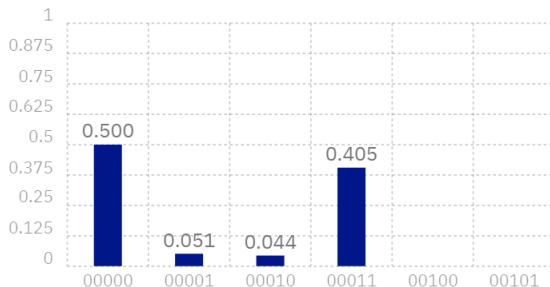
$$\begin{aligned} CNOT_b [(I|0\rangle) \otimes (H_b|0\rangle)] &= CNOT_b (I \otimes H_b)(|0\rangle \otimes |0\rangle) \\ &= CNOT_b \begin{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \mathcal{O} \\ \mathcal{O} & \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &= CNOT_b \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \end{bmatrix}^T \\ &= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}^T \end{aligned}$$

# Simulation on IBM-Q

## Entanglement circuit, revisited








## Quantum State: Computation Basis



# Recap

- Representing multi-qubit systems with tensor products.
- Representing multi-qubit operators with tensor products.
- Compute with the Kronecker product.
- Entanglement, mathematically.
- 2-qubit entangler, mathematically.
- Entanglement on IBM-Q.

# References

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