Physics 8.321, Fall 2021
Homework #3

Due **Wednesday, October 13** by 8:00 PM.

1. Consider the following two matrices:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix},$$

(a) Show that A and B commute.

(b) Find the eigenvalues and eigenvectors of A and B.

(c) Find the unitarity transformation which simultaneously diagonalizes A and B.

2. In this problem we consider some simple *spin chains*, which are a simple example of a type of system that appear in the context of condensed matter physics.

N spin-1/2 particles have a total Hilbert space

$$\mathcal{H} = \mathcal{H}_2^{(1)} \otimes \mathcal{H}_2^{(2)} \otimes \cdots \otimes \mathcal{H}_2^{(N)}$$

where $\mathcal{H}_2^{(i)}$ is the (two-dimensional) Hilbert space of the ith particle.

(a) What is the dimension of $\mathcal{H}$?

(b) Define

$$S_z = S_z^{(1)} + S_z^{(2)} + \cdots S_z^{(N)}.$$

What is the spectrum and degeneracy of S_z?

(c) Define an operator I coupling N spins to their nearest neighbors in a ring through

$$I = \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} + \mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)} + \mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)} + \cdots + \mathbf{S}^{(N-1)} \cdot \mathbf{S}^{(N)} + \mathbf{S}^{(N)} \cdot \mathbf{S}^{(1)}$$

Are S_z and I compatible observables? Prove your answer for any N.

(d) Find the spectrum and degeneracies of I for $N = 2, 3, 4$.

(e) Find the largest positive eigenvalue $\lambda_{\max}^{(N)}$ of I for an arbitrary value of N, and identify an eigenvector with this eigenvalue.

(f) Find the smallest (most negative) eigenvalue $\lambda_{\min}^{(N)}$ for small values of $N = 1, 2, \ldots$. What is the largest $N = n$ for which you can compute this quantity (analytically/numerically) in reasonable time? What can you say about $\lambda_{\min}^{(N)}$ and its associated eigenvector(s) for general N?

(g) Consider N spin-1/2 particles in an external magnetic field, interacting with the external field and one another according to the Hamiltonian

$$H = b\,xS_z - a(1-x)I$$

where a, b are numerical constants with $b = a\hbar$, S_z and I are defined in parts (b, c), and $x \in [0, 1]$ is a real number. Graph the spectrum of H for $N = 2, 3, 4$ for x in the range $0 \leq x \leq 1$. Check that your results agree with your answers to the previous parts.

3. In this problem we consider a quantum system of multiple *qubits*, such as are used as the fundamental units in quantum computers and quantum communication systems.

Consider 4 spin-1/2 particles, each of which is in an eigenstate $S_x^{(i)} = \hbar/2$. In each part of this problem, a sequence of measurements is performed on these 4 particles. For each part of the problem, give all possible sequences of outcomes of the experiments, and calculate the probability for each sequence of outcomes. In each case, calculate the total probability that the final measurement of the quantity $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ gives each of the possible values. Each part of this problem should be done independently, starting with all spins in the eigenstate $S_x^{(i)} = \hbar/2$ as mentioned above.

(a) $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(b) $S_z^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(c) $S_z^{(2)}$ is measured, then $\mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(d) $S_z^{(1)}$ is measured, then $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$ is measured, then $\mathbf{S}^{(2)} \cdot \mathbf{S}^{(3)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

(e) $S_z^{(1)}$ is measured, then $\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}$ is measured, and then $\mathbf{S}^{(3)} \cdot \mathbf{S}^{(4)}$ is measured.

You might find it amusing to try to prove that the probability that the final measurement gives $-3\hbar^2/4$ is precisely $1/2^n$ where n is the total number of particles, measured in the same pattern as parts (b, c, d)!