A Generalized Polar-Coordinate Integration Formula with Applications to the Study of Convolution Powers of Complex-Valued Functions on $\mathbb{Z}^d$

Huan Q. Bui[1] · Evan Randles[2]

Abstract

In this article, we consider a class of functions on $\mathbb{R}^d$, called positive homogeneous functions, which interact well with certain continuous one-parameter groups of (generally anisotropic) dilations. Generalizing the Euclidean norm, positive homogeneous functions appear naturally in the study of convolution powers of complex-valued functions on $\mathbb{Z}^d$. As the spherical measure is a Radon measure on the unit sphere which is invariant under the symmetry group of the Euclidean norm, to each positive homogeneous function P, we construct a Radon measure σ_P on $S = \{\eta \in \mathbb{R}^d : P(\eta) = 1\}$ which is invariant under the symmetry group of P. With this measure, we prove a generalization of the classical polar-coordinate integration formula and deduce a number of corollaries in this setting. We then turn to the study of convolution powers of complex functions on $\mathbb{Z}^d$ and certain oscillatory integrals which arise naturally in that context. Armed with our integration formula and the Van der Corput lemma, we establish sup-norm-type estimates for convolution powers; this result is new and partially extends results of Randles and Saloff-Coste (J Fourier Anal Appl 21(4):754–798, 2015; Rev Mat Iberoam 33(3):1045–1121, 2017).

Keywords Polar-coordinate integration formula · Spherical measure · Oscillatory integrals · Convolution powers

Mathematics Subject Classification Primary 28A25 · 58C35 · Secondary 42B20 · 42A85

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1 Introduction

1.1 A Motivating Application

Denote by $\ell^1(\mathbb{Z}^d)$ the set of functions $\phi : \mathbb{Z}^d \to \mathbb{C}$ for which

$$\|\phi\|_1 := \sum_{x \in \mathbb{Z}^d} |\phi(x)| < \infty.$$

For $\phi \in \ell^1(\mathbb{Z}^d)$, we define its convolution powers $\phi^{(n)} \in \ell^1(\mathbb{Z}^d)$ iteratively by putting $\phi^{(1)} = \phi$ and, for $n \geq 2$,

$$\phi^{(n)}(x) = \sum_{y \in \mathbb{Z}^d} \phi^{(n-1)}(x - y)\phi(y)$$

for $x \in \mathbb{Z}^d$. For a given $\phi \in \ell^1(\mathbb{Z}^d)$, we are interested in the asymptotic behavior of $\phi^{(n)}(x)$ as $n \to \infty$.

In the special case that ϕ is a probability distribution on $\mathbb{Z}^d$, i.e., $\phi \geq 0$ and $\|\phi\|_1 = \sum \phi(x) = 1$, the convolution powers $\phi^{(n)}$ are well-studied objects in random walk theory [18, 27, 32]. From this perspective, the probability distribution ϕ defines a discrete-time Markov process on $\mathbb{Z}^d$ with transition kernel $K = K_{(\cdot)}(\cdot, \cdot)$ given by

$$K_n(x, y) = \phi^{(n)}(y - x) \tag{1}$$

for $x, y \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+ := \{1, 2, 3, \dots\}$. This Markov process is called the random walk on $\mathbb{Z}^d$ driven by ϕ and (1) represents the probability that the random walker goes from position x to position y in n steps. In particular, $\phi^{(n)}(0) = K_n(x, x)$ for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$ and thus the value of the nth convolution power $\phi^{(n)}$ at the origin describes to so-called return probability of the random walk. In the case that ϕ is truly d-dimensional,[1] and has finite second moments, there exists a positive constant $C > 0$ for which

$$\phi^{(n)}(0) \leq Cn^{-d/2} \tag{2}$$

for all $n \in \mathbb{N}_+$. In fact, under these hypotheses, one can find positive constants C_1 and C_2 for which

$$C_1 n^{-d/2} \leq \sup_{x \in \mathbb{Z}^d} \left| \phi^{(n)}(x) \right| \leq C_2 n^{-d/2} \tag{3}$$

for all $n \in \mathbb{N}_+$ (see Section 7.6 of [24]). In its essence, the above two-sided estimate was used by G. Pólya to settle the questions of recurrence/transience for simple random

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[1] In the language of Spitzer [27] a probability distribution ϕ on $\mathbb{Z}^d$ is said to be truly d-dimensional if ϕ is not supported on any affine hyperplane of $\mathbb{R}^d$. The more common assumption that a random walk on $\mathbb{Z}^d$ is irreducible implies that it is truly d-dimensional.

walk: the random walk is recurrent when $d = 1, 2$ and transient for $d \geq 3$ [21]. Going beyond the estimates (2) and (3), under the hypotheses that ϕ is truly d-dimensional and has finite second moments, the pointwise behavior of $\phi^{(n)}(x)$ is described by an appropriately scaled Gaussian density; this is the local (central) limit theorem (see Theorem 7.6 of [24] and Theorem 2.2.1 of [18]).

When $\phi \in \ell^1(\mathbb{Z}^d)$ is complex valued (or even real valued and signed), its convolution powers $\phi^{(n)}$ can exhibit striking and peculiar behavior not seen in the probabilistic setting [8, 22, 24]. In the context of one dimension, understanding the asymptotic behavior of convolution powers for $\phi \in \ell^1(\mathbb{Z})$ dates back to its initial investigation by E. L. de Forest in the nineteenth century driven by applications to data smoothing [8]. In fact, understanding the pointwise asymptotics of $\phi^{(n)}(x)$ for a finitely-supported and real-valued function ϕ on $\mathbb{Z}$ became known as *de Forest's problem* and was subsequently investigated by Schoenberg [26] and Greville [15]. In the 1960s, the study was reinvigorated by its applications to numerical solutions to partial differential equations (in particular, finite difference schemes) tied to advancements in scientific computing; for an account and references to the literature, we recommend the excellent survey [31] by Thomée (see also the recent article [7]). Following the previous works of Schoenberg [26], Greville [15], Thomée [30], Diaconis and Saloff-Coste [8], the second author and Saloff-Coste, established the following theorem concerning sup-norm estimates.

Theorem 1.1 (Theorem 1.1 of [22]) *Let $\phi : \mathbb{Z} \to \mathbb{C}$ have support which is finite and consists of more than one point. Assume also that ϕ is normalized so that $\sup_{\xi} |\widehat{\phi}(\xi)| = 1$ where $\widehat{\phi}(\xi) = \sum_{x \in \mathbb{Z}} \phi(x) e^{ix\xi}$ is ϕ's Fourier transform. Then, there is an integer $m \geq 2$ and positive constants C_1 and C_2 for which*

$$C_1 n^{-1/m} \leq \sup_{x \in \mathbb{Z}} \left| \phi^{(n)}(x) \right| \leq C_2 n^{-1/m} \tag{4}$$

for all $n \in \mathbb{N}_+$.

This result coincides with the estimate (3) in the case that $\phi \geq 0$ where, necessarily, $m = 2$. The article [22] also gives a complete description of local limit theorems, thus furnishing a full resolution to de Forest's problem. In the present article, we shall however focus only on estimates of the form (4) in the context of $\mathbb{Z}^d$; a forthcoming article will treat local limit theorems. To motivate the utility of the generalized polar-coordinate integration formula to be studied in this article, it is necessary to discuss, at least formally, the ideas of the proof of Theorem 1.1 given in [22]. Using the Fourier inversion formula, understanding the asymptotic behavior of convolution powers in one dimension reduces to the analysis of integrals of the form

$$I(n, x) = \int_{-\delta}^{\delta} e^{-n\Gamma(\xi)} e^{-ix\xi} \, d\xi$$

where $\delta > 0$ and Γ is a smooth function having

$$\Gamma(\xi) = i\alpha\xi - \beta\xi^m + o(\xi^m)$$

as $\xi \to 0$; here $\alpha \in \mathbb{R}$, $m \geq 2$ is an integer, and β is a non-zero complex number with $\mathrm{Re}(\beta) \geq 0$. In his seminal paper on the stability of finite difference schemes in the maximum norm [30], V. Thomée showed that, to determine the asymptotic behavior of convolution powers of a finitely-supported function, it was sufficient to analyze integrals of the above form and he separated their analysis into two cases. The case that $\mathrm{Re}(\beta) > 0$ (and necessarily m is even) corresponds to Thomée's case of "Type γ" and the case that $\mathrm{Re}(\beta) = 0$ corresponds to Thomée's case of "Type β". Using the homogeneity of the function $\xi \mapsto \xi^m$, the change of variables $u = n^{1/m}\xi$, yields

$$I(n, x) = n^{-1/m} \int_{-n^{1/m}\delta}^{n^{1/m}\delta} e^{-\beta u^m + o(u^m)} e^{-iy_n u} \, du \tag{5}$$

where $y_n = (x - n\alpha)/n^{1/m}$. In the case that $\mathrm{Re}(\beta) > 0$ and m is even, $u \mapsto e^{-\beta u^m}$ is absolutely integrable on $\mathbb{R}$ and from this it follows easily that, for some positive constant C, $|I(n, x)| \leq Cn^{-1/m}$ for $x \in \mathbb{Z}$ and $n \in \mathbb{N}_+$. When $\mathrm{Re}(\beta) = 0$, i.e., $\beta = iq$ for some $q \in \mathbb{R} \setminus \{0\}$, the integral in (5) is, by contrast, oscillatory in nature. In this case, establishing an estimate of the form $|I(n, x)| \leq Cn^{1/m}$ requires oscillatory-integral techniques and, in particular, its analysis in [22] makes use of the Van der Corput lemma (see Proposition 3.4). The combination of these estimates for $I(n, x)$, in both cases, is then used to establish the upper estimate in [22].

Moving beyond one dimension, much less is known about convolution powers of a complex-valued function $\phi \in \ell^1(\mathbb{Z}^d)$. The recent article [24] develops a theory for convolution powers of a subclass of functions $\phi \in \ell^1(\mathbb{Z}^d)$ whose behavior (or, more precisely, the behavior of its Fourier transform $\widehat{\phi}$) is driven by cases akin to those of "Type γ" of V. Thomée. Analogous to the one-dimensional setting, this analysis centers on integrals of the form

$$I(n, x) = \int_{\mathcal{O}} e^{-n\Gamma(\xi)} e^{-ix \cdot \xi} \, d\xi \tag{6}$$

where $\mathcal{O}$ is a neighborhood of 0 in $\mathbb{R}^d$ and Γ is a smooth function having

$$\Gamma(\xi) = i\alpha \cdot \xi - P(\xi) + o(P(\xi))$$

as $\xi \to 0$; here $\alpha \in \mathbb{R}^d$ and P is a multivariate polynomial which, primarily, enjoys two properties:

1. It is homogeneous in the sense that there exists a family of (generally anisotropic) dilations $\{T_r\}_{r>0} \subseteq \mathrm{Gl}(\mathbb{R}^d)$ for which

$$rP(\xi) = P(T_r\xi)$$

   for all $r > 0$ and $\xi \in \mathbb{R}^d$.
2. The function $\xi \mapsto e^{-P(\xi)}$ is absolutely integrable, i.e., $e^{-P(\xi)} \in L^1(\mathbb{R}^d)$.

Exploiting the homogeneity of P, a change of variables yields

$$I(n, x) = n^{-\mu} \int_{T_n(\mathcal{O})} e^{-P(\xi)+o(P(\xi))} e^{-iy_n \cdot \xi} \, d\xi$$

for $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$ where $y_n = (T_n^{-1})^*(x - n\alpha) \in \mathbb{R}^d$ and μ is a positive number called the homogeneous order of P and introduced in the next subsection. By virtue of the absolute integrability of $e^{-P(\xi)}$, we obtain a positive number C for which $|I(n, x)| \leq Cn^{-\mu}$ or all $n \in \mathbb{N}_+$ and $x \in \mathbb{Z}^d$. By this method, for any $\phi \in \ell^1(\mathbb{Z}^d)$ which satisfies the hypotheses of Theorem 1.4 of [24], there is a positive constant μ_ϕ, called the homogeneous order of ϕ, and $C > 0$ for which

$$\left| \phi^{(n)}(x) \right| \leq Cn^{-\mu_\phi} \tag{7}$$

for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}_+$; when $d = 1$, $\mu_\phi = 1/m$ for an even integer m and so this result partially extends the upper estimate of Theorem 1.1. In Sect. 3, we partially extend the above sup-norm type estimate to allow for cases akin to Thomée's cases of Type β. Specifically, we analyze integrals of the form (6) to include the possibility that

$$\Gamma(\xi) = i\alpha \cdot \xi - iQ(\xi) - o(Q(\xi))$$

as $\xi \to 0$ where $\alpha \in \mathbb{R}^d$ and $Q(\xi)$ is a real-valued function vanishing only at 0 and enjoying a homogeneous property analogous to that of P. That is, for some family of dilations $\{T_r\}_{r>0} \in \mathrm{Gl}(\mathbb{R}^d)$,

$$rQ(\xi) = Q(T_r\xi)$$

for all $r > 0$ and $\xi \in \mathbb{R}^d$. Given that $Q(\xi)$ is real valued, $\left| e^{-iQ(\xi)} \right| = 1$ and so $e^{-iQ(\xi)}$ is not absolutely integrable. Analogous to the one-dimensional case of Type β, the integral (6) is oscillatory in nature. To handle such oscillatory integrals, we make use of a generalized polar-coordinate integration formula (see Theorem 1.4), which is adapted to Q, to break the integration into a "radial" part and an "angular" part. By doing this, the we are able to handle the integrand's oscillations almost entirely through the radial integral for which we have sophisticated one-dimensional oscillatory integral techniques at our disposal. By this approach, we are able to establish Theorem 3.2, a sup-norm type estimate which partially extends the estimate (7) of [24].

1.2 A Generalized Polar-Coordinate Integration Formula

The spherical measure and the related polar-coordinate integration formula are important tools and objects of study in several areas of mathematical analysis [3, 16, 28]. To have them at our fingertips, let us first denote by $\mathbb{S}$ and $\mathbb{B}$ the unit sphere and open unit ball in $\mathbb{R}^d$, respectively, let m be the Lebesgue measure on $\mathbb{R}^d$, and write $dx = m(dx) = dm(x)$. The spherical measure is the canonical Radon measure on $\mathbb{S}$

for which $\Theta(\mathbb{S}) = d \cdot m(\mathbb{B})$ and $\Theta(OF) = \Theta(F)$ for every orthogonal transformation O and Borel set $F \subseteq \mathbb{S}$. With this measure, we state the classical polar-coordinate integration formula as follows: For every $f \in L^1(\mathbb{R}^d)$ (or non-negative measurable f),

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_{\mathbb{S}} \left(\int_0^\infty f(r\eta) r^{d-1}\, dr \right) \Theta(d\eta) = \int_0^\infty \left(\int_{\mathbb{S}} f(r\eta) \Theta(d\eta) \right) r^{d-1}\, dr.$$

(8)

Precise formulations of this classical result can be found in [29] and [12]. For two interesting applications which provide some useful context, we encourage the reader to see [3] and [11]. In this article, we generalize the polar-coordinate integration formula (8) and the spherical measure in a way that is well-adapted to the analysis of oscillatory integrals of the type discussed in the previous subsection. Making use of our integration formula, this analysis will be used to establish the advertised sup-norm-type estimate (Theorem 3.2) for convolution powers of complex-valued functions and, in a forthcoming article, a theory of local limit theorems.

To describe the generalized polar-coordinate integration formula treated in this article, we must first introduce a class of functions on $\mathbb{R}^d$ which share several desirable properties with the Euclidean norm. The first such property is that such functions "play well" with spatial dilations of the following form: Let $\{T_r\}_{r>0} \subseteq \mathrm{Gl}(\mathbb{R}^d)$ be a continuous one-parameter group, i.e., T is a continuous group homomorphism from the multiplicative group of positive real numbers into the general linear group, $\mathrm{Gl}(\mathbb{R}^d)$. It is well-known (c.f., [9, 10, 24]) that every continuous one-parameter group $\{T_r\}$ has the unique representation

$$T_r = r^E = \exp((\ln r)E) = \sum_{k=0}^\infty \frac{(\ln r)^k}{k!} E^k$$

for some $E \in \mathrm{End}(\mathbb{R}^d)$; here, $\mathrm{End}(\mathbb{R}^d)$ is the algebra of endomorphisms of $\mathbb{R}^d$ which we take equipped with the operator norm $\|\cdot\|$. E is called the (infinitesimal) generator of $\{T_r\}$ and $\{T_r\}$ is said to be generated by E. This, of course, gives a one-to-one correspondence between $\mathrm{End}(\mathbb{R}^d)$ and the collection of continuous one-parameter groups. A continuous one-parameter group $\{T_r\}$ is said to be **contracting** if

$$\lim_{r \to 0} \|T_r\| = 0.$$

We note that, if E is the generator of a contracting group $\{T_r\}$, then $\mathrm{tr}\, E = \mathrm{tr}(E) > 0$. This fact (Proposition A.7) and its proof can be found in the appendix along with some other basic results on one-parameter groups. Also, we encourage the reader to look at the excellent texts [9] and [10] on one-parameter (semi) groups.

Given a function $P : \mathbb{R}^d \to \mathbb{R}$, we shall call

$$S = \{\eta \in \mathbb{R}^d : P(\eta) = 1\}$$

the **unital level set of** P. We say that P is **positive definite** if P is non-negative and $P(x) = 0$ only when $x = 0$. Given a continuous one-parameter group $\{T_r\}$, we say that P is **homogeneous with respect to** $\{T_r\}$ if

$$r P(x) = P(T_r x) = P(r^E x) \tag{9}$$

for all $r > 0$ and $x \in \mathbb{R}^d$. By an abuse of language, we will also say that P is homogeneous with respect to E whenever (9) is satisfied. The set of all such $E \in \mathrm{End}(\mathbb{R}^d)$ for which (9) holds is denoted by $\mathrm{Exp}(P)$ and called the **exponent set of** P. We have the following characterization whose proof can be found in Sect. 2.

Proposition 1.2 *Let* $P : \mathbb{R}^d \to \mathbb{R}$ *be continuous, positive definite, and have* $\mathrm{Exp}(P) \neq \varnothing$. *The following are equivalent:*

(a) *S is compact.*
(b) *There is a positive number M for which*

$$P(x) > 1$$

  for all $|x| \geq M$.
(c) *For each $E \in \mathrm{Exp}(P)$, $T_r = r^E$ is contracting.*
(d) *There exists $E \in \mathrm{Exp}(P)$ for which $T_r = r^E$ is contracting.*
(e) *We have*

$$\lim_{x \to \infty} P(x) = \infty.$$

Definition 1.3 Let $P : \mathbb{R}^d \to \mathbb{R}$ be continuous, positive definite and have $\mathrm{Exp}(P) \neq \varnothing$. If any one (and hence all) of the equivalent conditions in Proposition 1.2 are fulfilled, we say that P is positive homogeneous.

Before we introduce several examples of positive homogeneous functions, it is helpful to fix some notation and introduce some basic topological objects connected to positive homogeneous functions. We shall denote by $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$ the set of integers, (non-negative) natural numbers, and positive natural numbers, respectively; the d-tuples formed by elements of these sets will be denoted by $\mathbb{Z}^d$, $\mathbb{N}^d$, and $\mathbb{N}_+^d$. Throughout this article, our setting is the d-dimensional Euclidean space $\mathbb{R}^d$ with coordinates $(x^1, x^2, \ldots, x^d)$ and equipped with the inner product $x \cdot y = \sum_{k=1}^{d} (x^k)(y^k)$ and associated Euclidean norm $|x| = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^d)^2}$. We take $\mathbb{R}^d$ to be equipped with its usual topology and oriented smooth structure. Given $x \in \mathbb{R}^d$ and $r > 0$, the open (Euclidean) ball with center x and radius r is denoted by $\mathbb{B}_r(x)$ and its corresponding sphere is denoted by $\mathbb{S}_r(x)$. In the case that $x = 0$, we write $\mathbb{B}_r = \mathbb{B}_r(0)$ and $\mathbb{S}_r = \mathbb{S}_r(0)$ for $r > 0$. For a subset A of a topological space, we denote by $\mathrm{Int}\,(A)$, $\overline{A}$, and ∂A its interior, closure, and boundary, respectively. Given a positive homogeneous function P, we define

$$B_r = \{\xi \in \mathbb{R}^d : P(\xi) < r\} \text{ and } A_s^r = \{\xi \in \mathbb{R}^d : s \leq P(\xi) < r\}$$

for $r > 0$ and $0 \le s < r$; these are P-adapted analogues of the Euclidean ball, $\mathbb{B}_r$, and the annulus of inner radius s and outer radius r, respectively. In view of Proposition 1.2 and the continuity of P, we see that, for each $r > 0$, B_r is open and $\overline{B_r}$ is compact. Further, by setting $B = B_1$, it is a straightforward exercise to see that $\overline{B} = B \cup S$ where $\partial B = S$ is the unital level set associated to P.

Example 1 For any $\alpha > 0$, the αth-power of the Euclidean norm $x \mapsto |x|^\alpha$ is positive homogeneous. In this case, the unital level set S is the standard unit sphere $\mathbb{S} = \mathbb{S}_1$, $B = \mathbb{B} = \mathbb{B}_1$, and

$$\mathrm{Exp}(|\cdot|^\alpha) = \frac{1}{\alpha} I + \mathfrak{o}(d)$$

where I is the identity and $\mathfrak{o}(d)$ is the Lie algebra of the orthogonal group $\mathrm{O}(\mathbb{R}^d)$ and is characterized by the set of skew-symmetric matrices.

Example 2 Consider the polynomial

$$P(x) = \sum_{|\alpha:\mathbf{n}|=1} a_\alpha x^\alpha, \tag{10}$$

where $\mathbf{n} = (n_1, n_2, \ldots, n_d)$ is a d-tuple of positive even natural numbers,[2] and, for each multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}^d$,

$$|\alpha : \mathbf{n}| := \sum_{k=1}^{d} \frac{\alpha_k}{n_k},$$

and

$$x^\alpha = \left(x^1\right)^{\alpha_1} \left(x^2\right)^{\alpha_2} \cdots \left(x^d\right)^{\alpha_d}$$

for $x = \left(x^1, x^2, \ldots, x^d\right) \in \mathbb{R}^d$. In the language of Hörmander [17], a (generally complex-valued) polynomial of the form (10) is said to be *semi-elliptic* if it vanishes only at the origin. If we consider $E \in \mathrm{End}(\mathbb{R}^d)$ whose standard matrix representation is $\mathrm{diag}(1/n_1, 1/n_2, \ldots, 1/n_d)$, we have

$$P\left(r^E x\right) = \sum_{|\alpha:\mathbf{n}|=1} a_\alpha \left(r^{1/n_1} x^1\right)^{\alpha_1} \left(r^{1/n_2} x^2\right)^{\alpha_2} \cdots \left(r^{1/n_d} x^d\right)^{\alpha_d}$$

$$= \sum_{|\alpha:\mathbf{n}|=1} a_\alpha r^{|\alpha:\mathbf{n}|} x^\alpha = r P(x)$$

for all $x \in \mathbb{R}^d$ and $r > 0$ and therefore $E \in \mathrm{Exp}(P)$. It is easy to see that $T_r = r^E$ is a contracting group and so we have the following statement by virtue of Proposition 1.2:

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[2] In Sect. 3.1, we will write this as $\mathbf{n} = 2\mathbf{m}$ for $\mathbf{m} \in \mathbb{N}_+^d$.
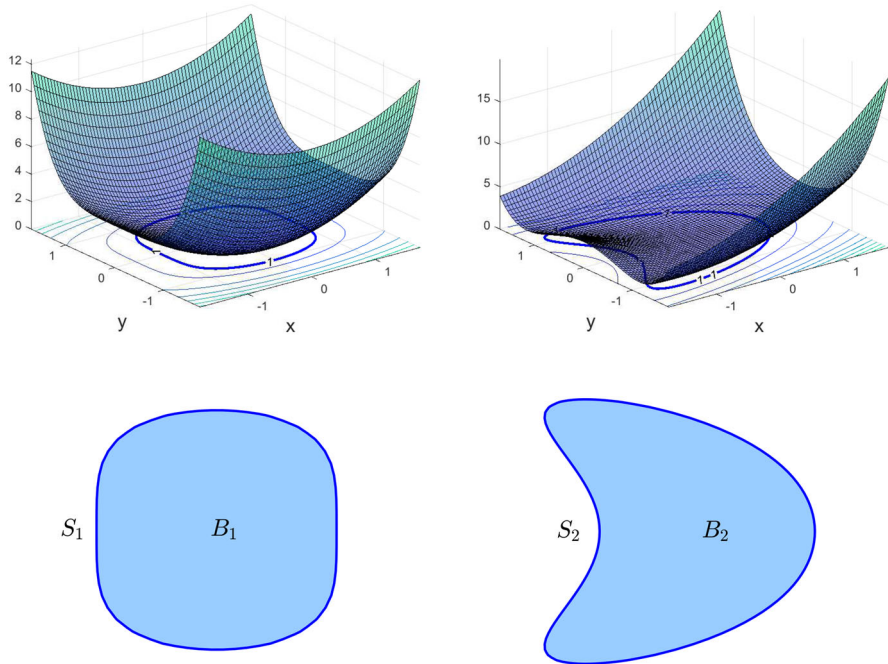
Fig. 1 The left column illustrates the graph of P_1 with its associated S_1 and convex B_1. The right column illustrates the graph of P_2 with its associated S_2 and non-convex B_2

If a semi-elliptic polynomial $P(x)$ of the form (10) *is positive definite, then it is positive homogeneous.*

For two concrete examples, consider the polynomials P_1 and P_2 on $\mathbb{R}^2$ defined by

$$P_1(x, y) = x^2 + y^4 \text{ and } P_2(x, y) = x^2 + \frac{3}{2}xy^2 + y^4$$

defined for $(x, y) \in \mathbb{R}^2$. It is straightforward to see that P_1 and P_2 are both positive definite and semi-elliptic of the form (10) with $\mathbf{n} = (2, 4)$. Figure 1 illustrates P_1 and P_2 along with their associated unital level sets S_1 and S_2 and corresponding sets $B_1 = \{(x, y) \in \mathbb{R}^2 : P_1(x, y) < 1\}$ and $B_2 = \{(x, y) \in \mathbb{R}^2 : P_2(x, y) < 1\}$ written with a slight abuse of notation.

Remark 1 In [24], a positive homogeneous polynomial P is, by definition, a complex-valued multivariate polynomial on $\mathbb{R}^d$ for which $\text{Exp}(P)$ contains an element of $\text{End}(\mathbb{R}^d)$ whose spectrum is purely real and for which $R = \text{Re } P$ is positive definite (see Proposition 2.3 below). By virtue of Proposition 2.2 of [24], for each such polynomial P and $E \in \text{Exp}(P)$ with real spectrum, there exists $A \in \text{Gl}(\mathbb{R}^d)$ (representing a change of basis of $\mathbb{R}^d$) and a d-tuple of even positive natural numbers $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}_+^d$ for which $A^{-1}EA$ has standard matrix representation $\text{diag}(1/n_1, 1/n_2, \ldots, 1/n_d)$ and $(P \circ A)(x)$ is semi-elliptic of the form (10) with, in

this case, complex coefficients. It follows that every real-valued positive homogeneous polynomial (in the sense of [24]) is a positive homogeneous function in the sense of the present article. Of course, the semi-elliptic polynomials discussed above are positive homogeneous polynomials in the sense of [24] where $A = I$. We refer the reader to Section 7.3 of [24] which presents a real-valued positive homogeneous polynomial which is not semi-elliptic (and so $A \neq I$).

Example 3 Let Q be a positive homogeneous function with exponent set $\mathrm{Exp}(Q)$ and compact unital level set $S_Q = \{\eta : Q(\eta) = 1\}$. Given any $f \in C^0(S_Q)$ for which $f(\eta) > 0$ for all $\eta \in S_Q$ and $E \in \mathrm{Exp}(Q)$, define $P = P_{f,E,Q} : \mathbb{R}^d \to \mathbb{R}$ by

$$P(x) = \begin{cases} Q(x) f\left((Q(x))^{-E} x\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

for $x \in \mathbb{R}^d$. We claim that P is positive homogeneous and $E \in \mathrm{Exp}(P)$.

Proof To see this, we first observe that, for any $x \in \mathbb{R}^d \setminus \{0\}$, $Q((Q(x))^{-E} x) = Q(x)/Q(x) = 1$ and hence $Q(x)^{-E} x \in S_Q$ and so the above formula makes sense and ensures that P is continuous on $\mathbb{R}^d \setminus \{0\}$. Furthermore, because f is continuous and positive on the compact set S_Q, we have $0 < \min f \leq \max f < \infty$. From this it follows that P is positive definite and, by virtue of the squeeze theorem, continuous at $x = 0$. For any $r > 0$ and $x \in \mathbb{R}^d$, we have

$$P(r^E x) = Q(r^E x) f(Q(r^E x)^{-E} r^E x) = r Q(x) f(Q(x)^{-E} x) = r P(x)$$

and therefore $E \in \mathrm{Exp}(P)$. Upon noting that $\{r^E\}$ is contracting by virtue of Proposition 1.2, we conclude that P is positive homogeneous. $\square$

The utility of this construction allows us to see that "most" positive homogeneous functions are not smooth. To see this, we fix a positive homogeneous function $Q \in C^\infty(\mathbb{R}^d)$ and remark that S_Q is necessarily a compact smooth embedded hypersurface of $\mathbb{R}^d$ (see Proposition 5.1). If $P = P_{f,Q}$ is $C^\infty(\mathbb{R}^d)$, $P|_{S_Q} = f$ is necessarily $C^\infty(S_Q)$. It follows that $P \notin C^\infty(\mathbb{R}^d)$ whenever f is chosen from $C^0(S_Q) \setminus C^\infty(S_Q)$. By precisely the same argument, we see that $P \in C^0(\mathbb{R}^d) \setminus C^k(\mathbb{R}^d)$ whenever $f \in C^0(S_Q) \setminus C^k(S_Q)$ for each $k \in \mathbb{N}_+$.
As a straightforward example, consider $Q(x, y) = |(x, y)| = \sqrt{x^2 + y^2}$ on $\mathbb{R}^2$ with $S_Q = \mathbb{S}$ and define

$$f(x, y) = w(\mathrm{Arg}(x, y)) + 3$$

where $w : \mathbb{R} \to \mathbb{R}$ is defined by

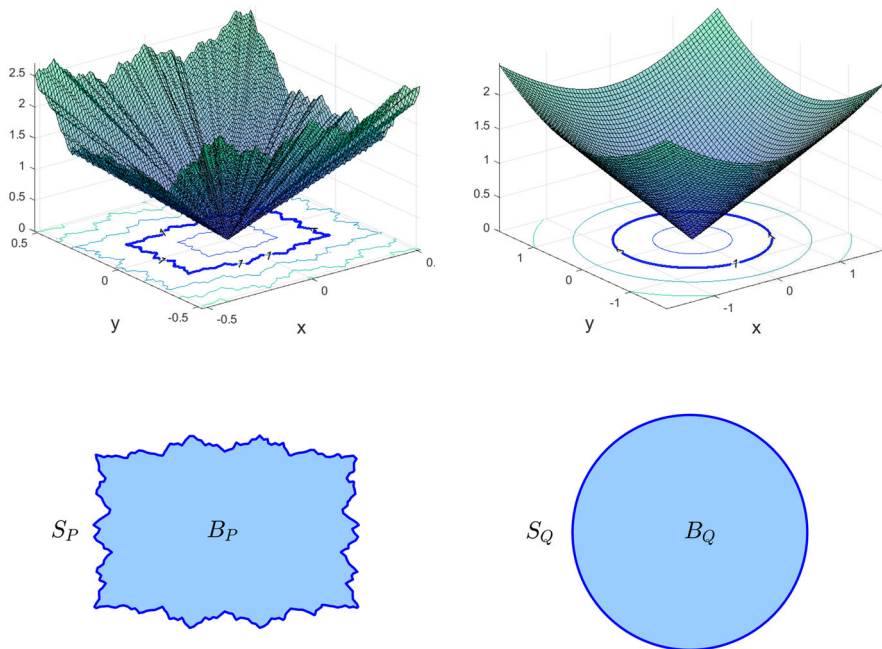$$w(t) = \sum_{n=0}^{\infty} 2^{-n} \cos\left(3^n t\right)$$

Fig. 2 The left column illustrates P's graph and associated level set S_P containing B_P. The right column illustrates Q's graph and associated level set S_Q containing B_Q

for $t \in \mathbb{R}$; w is a continuous 2π-periodic version of the Weierstrass function. The resulting positive homogeneous function P is continuous but nowhere differentiable. Figure 2 illustrates this function P alongside Q and together with their associated unital level sets. We note that $S_P \neq S_Q$ and this is generally the case unless $f \equiv 1$. Given a positive homogeneous function P, let $\mathrm{Sym}(P)$ be the set of $O \in \mathrm{End}(\mathbb{R}^d)$ for which

$$P(Ox) = P(x)$$

for all $x \in \mathbb{R}^d$ and observe that $OS \subseteq S$ whenever $O \in \mathrm{Sym}(P)$. By virtue of the positive definiteness of P, it is straightforward to verify that $\mathrm{Sym}(P)$ is a subgroup of $\mathrm{Gl}(\mathbb{R}^d)$. For this reason, $\mathrm{Sym}(P)$ is said to be the **symmetry group associated to** P. In fact, we will show that $\mathrm{Sym}(P)$ is a compact subgroup of $\mathrm{Gl}(\mathbb{R}^d)$; this is Proposition 2.1. As a consequence, we will prove that $\mathrm{tr} E = \mathrm{tr} E'$ for all $E, E' \in \mathrm{Exp}(P)$ (Corollary 2.2) and this allows us to define the **homogeneous order of** P to be the unique positive number μ_P for which

$$\mu_P = \mathrm{tr} E$$

for all $E \in \mathrm{Exp}(P)$. As we will see, the "radial" measure $r^{d-1} \, dr$ in (8) will be replaced by $r^{\mu_P - 1} \, dr$ in our generalized polar-coordinate integration formula; they coincide when P is the Euclidean norm.

Example 4 In Example 1, $\mathrm{Sym}(|\cdot|^\alpha)$ is precisely the orthogonal group $O(\mathbb{R}^d)$ and $\mu_{|\cdot|^\alpha} = d/\alpha$. In Example 2, the symmetric set of a semi-elliptic polynomial P of the form (10) depends on the specific nature of the polynomial in question. Concerning the polynomials P_1 and P_2 in that example, it is easily shown that $\mathrm{Sym}(P_1)$ is the four-element dihedral group D_2 and $\mathrm{Sym}(P_2)$ is the two-element group consisting of the identity and the transformation $(x, y) \mapsto (x, -y)$. For a semi-elliptic polynomial P of the form (10),

$$\mu_P = |\mathbf{1} : \mathbf{n}| = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_d}$$

and, in particular, $\mu_{P_1} = \mu_{P_2} = 1/2 + 1/4 = 3/4$.

Armed with the notion of positive homogeneous functions and their associated contracting groups, we are ready to introduce our generalization of spherical measure and the polar-coordinate integration formula. To this end, denote by $\mathcal{M}_d$ the Lebesgue σ-algebra on $\mathbb{R}^d \setminus \{0\}$ and by $\mathcal{L}$ the Lebesgue σ-algebra on $(0, \infty)$. Given a positive homogeneous function P with homogeneous order $\mu_P > 0$, we let λ_P denote the σ-finite measure on $((0, \infty), \mathcal{L})$ with $\lambda_P(dr) = r^{\mu_P-1}\, dr$. Our main theorem is as follows.[3]

Theorem 1.4 *Let P be a positive homogeneous function on $\mathbb{R}^d$ and let S, $\mathrm{Exp}(P)$, $\mathrm{Sym}(P)$, and μ_P be P's associated unital level set, exponent set, symmetric group, and homogeneous order, respectively. There exists a σ-algebra Σ_P on S containing the Borel σ-algebra on S, $\mathcal{B}(S)$, and a finite Radon measure σ_P on (S, Σ_P) which satisfies the following properties:*

1. *(S, Σ_P, σ_P) is the completion of $(S, \mathcal{B}(S), \sigma_P)$. In particular, (S, Σ_P, σ_P) is a complete measure space.*
2. *For any $F \in \Sigma_P$ and $O \in \mathrm{Sym}(P)$, $OF \in \Sigma_P$ and $\sigma_P(OF) = \sigma_P(F)$.*
3. *For $F \subseteq S$, $F \in \Sigma_P$ if and only if $\widetilde{F_E} := \{r^E \eta \in \mathbb{R}^d \setminus \{0\} : 0 < r < 1, \eta \in F\} \in \mathcal{M}_d$ for every $E \in \mathrm{Exp}(P)$. In this case*

$$\sigma_P(F) = \mu_P \cdot m(\widetilde{F_E})$$

for all $E \in \mathrm{Exp}(P)$.

Further, denote by $\big((0, \infty) \times S, (\mathcal{L} \times \Sigma_P)', \lambda_P \times \sigma_P\big)$ the completion of the product measure space $((0, \infty) \times S, \mathcal{L} \times \Sigma_P, \lambda_P \times \sigma_P)$. We have

1. *Given any $E \in \mathrm{Exp}(P)$, the map $\psi_E : (0, \infty) \times S \to \mathbb{R}^d \setminus \{0\}$, defined by $\psi_E(r, \eta) = r^E \eta$ for $r > 0$ and $\eta \in S$, is a point isomorphism of the measure spaces $\big((0, \infty) \times S, (\mathcal{L} \times \Sigma_P)', \lambda_P \times \sigma_P\big)$ and $(\mathbb{R}^d \setminus \{0\}, \mathcal{M}_d, m)$. That is*

$$\mathcal{M}_d = \left\{ A \subseteq \mathbb{R}^d \setminus \{0\} : \psi_E^{-1}(A) \in (\mathcal{L} \times \Sigma_P)' \right\}$$

---

[3] We refer the reader to Sections 3.6 and 9.2 of [4] which provide some basic context and vocabulary.

and, for each $A \in \mathcal{M}_d$,

$$m(A) = (\lambda_P \times \sigma_P)(\psi_E^{-1}(A)).$$

2. *Given any Lebesgue measurable function $f : \mathbb{R}^d \to \mathbb{C}$ and $E \in \mathrm{Exp}(P)$, $f \circ \psi_E$ is $(\mathcal{L} \times \Sigma_P)'$-measurable and the following statements hold:*

(a) *If $f \geq 0$, then*

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_0^\infty \left(\int_S f(r^E \eta)\,\sigma_P(d\eta) \right) r^{\mu_P - 1}\,dr$$
$$= \int_S \left(\int_0^\infty f(r^E \eta) r^{\mu_P - 1}\,dr \right) \sigma_P(d\eta). \qquad (11)$$

(b) *When f is complex-valued, we have*

$$f \in L^1(\mathbb{R}^d) \text{ if and only if } f \circ \psi_E \in L^1\left((0,\infty) \times S, (\mathcal{L} \times \Sigma_P)', \lambda_P \times \sigma_P \right)$$

and, in this case, (11) holds.

For comparison with the above theorem, we would like to highlight two related (but distinct) results which also generalize (8) and have found utility in their respective contexts. The first appears in the context of Riemannian manifolds and can be found as (III.3.4) of [6]. In that context, S is replaced by the geodesic unit sphere S_q centered at a point q in a Riemannian manifold M (with metric g and connection ∇), the paths $r \mapsto r^E \eta$ are replaced by geodesics $r \mapsto \gamma_\eta(r)$ for each $\eta \in S_q$, and the product measure $r^{\mu_P - 1}\,dr\,\sigma_P(d\eta)$ is replaced by a measure of the form $\det(\mathcal{A}(r : \eta))\,dr\,d\mathrm{Vol}_{S_q}(\eta)$ where $d\mathrm{Vol}_{S_q}$ is the Riemannian volume measure on S_q and $\mathcal{A}(r; \eta)$ is determined by the Riemannian curvature R along the geodesic paths $\gamma_\eta(r)$. The second related result appears in the context of homogeneous (Lie) groups and can be found as Proposition 1.15 of [13]. In that context, P is replaced by a homogeneous norm $| \cdot |_G$ on the homogeneous group G, S is replaced by the unit sphere S_G in the homogeneous norm, and μ_P is replaced by the homogeneous dimension of G. Perhaps obviously, the second result on homogeneous groups shares the most in common with Theorem 1.4 but differs in context and in that we ask much less of P and, consequently, S. In particular, P can be continuous yet nowhere differentiable (in contrast to $| \cdot |_G$) and S can be non-smooth and B non-convex (in contrast to S_G and B_G).

Before we discuss the ideas behind the construction of σ_P and the proof of Theorem 1.4, we first discuss two corollaries with well-known analogues in the classical setting.

Corollary 1.5 *Given $0 \leq a < b$, suppose that $f : \overline{A_a^b} \to \mathbb{C}$ is continuous and define $\mathcal{I} : [a, b] \to \mathbb{C}$ by*

$$\mathcal{I}(r) = \int_{A_a^r} f(x)\,dx$$

for $a \leq r \leq b$. Then $\mathcal{I}$ is continuously differentiable and

$$\mathcal{I}'(r) = r^{\mu_P - 1} \int_S f(r^E \eta) \sigma_P(d\eta)$$

on $[a, b]$ (or on $(0, b]$ provided that $a = 0$ and $\mu_P < 1$) where $E \in \operatorname{Exp}(P)$. In particular, if f is a complex-valued function which is continuous on some open neighborhood of $\overline{B}$, then

$$\int_S f(\eta) \sigma_P(\eta) = \frac{d}{dr} \left(\int_{B_r} f(x) \, dx \right) \Big|_{r=1}$$

where this derivative is two-sided.

Proof By virtue of Theorem 1.4, we have

$$\mathcal{I}(r) = \int_0^\infty \int_S \chi_{A_a^r}(t^E \eta) f(t^E \eta) \, \sigma_P(d\eta) t^{\mu_P - 1} \, dt = \int_a^r g(t) t^{\mu_P - 1} \, dt$$

where

$$g(t) = \int_S f(t^E \eta) \, \sigma_P(d\eta).$$

By virtue of the continuity of f and the compactness of S, it is easy to see that $g(t)$ is continuous on $[a, b]$. By an appeal to the fundamental theorem of calculus, it follows that $\mathcal{I}$ is differentiable on $[a, b]$ (or on $(0, b]$ provided that $a = 0$ and $\mu_P < 1$) and

$$\mathcal{I}'(r) = g(r) r^{\mu_P - 1} = r^{\mu_P - 1} \int_S f(r^E \eta) \, \sigma_P(d\eta).$$

To prove the final assertion, we first claim that there exists $\epsilon > 0$ for which $\overline{B} \subseteq B_{1+\epsilon} \subseteq \mathcal{O}$. To see this, we assume to reach a contradiction that, for each $n \in \mathbb{N}$, there exists $x_n \in B_{1+1/n} \setminus \mathcal{O}$. Because $\overline{B_2}$ is compact, the sequence $\{x_n\}$ is bounded and thus has a convergent subsequence by the Bolzano-Weierstrass theorem. By a (possible) reassignment of $\{x_n\}$, we may therefore assume that $\lim_{n \to \infty} x_n = x$. Because P is continuous and $1 \leq P(x_n) \leq 1 + 1/n$ for all $n \in \mathbb{N}$, $P(x) = \lim_{n \to \infty} P(x_n) = 1$ and therefore $x \in S \subseteq \overline{B} \subseteq \mathcal{O}$. This implies that $\mathcal{O}$ contains an accumulation point x of $\mathbb{R}^d \setminus \mathcal{O}$ which is impossible because $\mathcal{O}$ is an open set. As a consequence, the second assertion of the corollary follows immediately from the first where the derivative at $r = 1$ is two-sided. $\qquad \square$

As an application of the preceding corollary, information can be exchanged between the Fourier transforms $\{\widehat{\chi_{B_r}}\}_{r>0}$ of the characteristic functions $\{\chi_{B_r}\}_{r>0}$ and the Fourier transform of the surface measure σ_P defined by

$$\widehat{\sigma_P}(x) = \frac{1}{(2\pi)^d} \int_S e^{-i\eta \cdot x} \sigma_P(d\eta)$$

for $x \in \mathbb{R}^d$. Specifically, we have

$$\widehat{\sigma_P}(x) = \frac{d}{dr}\widehat{\chi_{B_r}}(x)\Big|_{r=1} \quad \text{and} \quad \widehat{\chi_B}(x) = \int_0^1 \widehat{\sigma_P}(r^{E^*}x)r^{\mu_P - 1}\, dr \qquad (12)$$

for $x \in \mathbb{R}^d$ where E^* is the adjoint of $E \in \text{Exp}(P)$. It is well known that decay estimates for the Fourier transform of surface-carried measures (of the form σ_P) have rich applications to the theory of maximal averages [28]. With such applications in mind, we refer the reader to the recent article [14] which adroitly utilizes relationships of the form (12).

Corollary 1.6 *Given $g : S \to \mathbb{C}$ and $E \in \text{Exp}(P)$, define $f : \mathbb{R}^d \to \mathbb{C}$ by*

$$f(x) = \begin{cases} \mu_P \cdot \chi_{(0,1)}(P(x))g(P(x)^{-E}x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

for $x \in \mathbb{R}^d$ where $\chi_{(0,1)}(\cdot)$ is the indicator function of the interval $(0, 1)$. Then $g \in L^1(S, \Sigma_P, \sigma_P)$ if and only if $f \in L^1(\mathbb{R}^d)$ and, in this case,

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_S g(\eta)\sigma_P(d\eta). \qquad (13)$$

Proof Observe that, for the $(\mathcal{L} \times \Sigma_P)'$-measurable function $k(r, \eta) = \chi_{(0,1)}(r)g(\eta)$,

$$k \circ \psi_E^{-1}(x) = k(P(x), P(x)^{-E}x) = f(x)$$

for $x \in \mathbb{R}^d \setminus \{0\}$; here we have used the fact that $\psi_E^{-1}(x) = (P(x), P(x)^{-E}x)$ which is guaranteed by Proposition 4.1. By virtue of Theorem 1.4, it follows that f is Lebesgue measurable and

$$\int_{\mathbb{R}^d} |f(x)|\, dx = \int_S \left(\int_{(0,\infty)} |k(r, \eta)|r^{\mu_P - 1}\, dr \right) \sigma_P(d\eta)$$

$$= \left(\int_S |g(\eta)|\sigma_P(d\eta) \right) \left(\int_0^1 \mu_P r^{\mu_P - 1}\, dr \right)$$

$$= \int_S |g(\eta)|\, \sigma_P(d\eta)$$

and therefore $f \in L^1(\mathbb{R}^d)$ if and only if $g \in L^1(S, \Sigma_P, \sigma_P)$ and $\|f\|_{L^1(\mathbb{R}^d)} = \|g\|_{L^1(S)}$. By an analogous computation (for f instead of $|f|$), we have

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_S g(\eta)\sigma_P(d\eta),$$

by virtue of Property 2 of Theorem 1.4. □

As it turns out, (13) is characterizing in the sense that it can be used to construct σ_P via the Riesz representation theorem. This is the approach taken, for example, in [13] and [3]; in fact, it is asserted in [3] that viewing (13) as a consequence of Theorem 1.4 (in the classical setting) is folkloric. We have decided to follow this so-called folkloric approach, which avoids the Riesz representation theorem and is that suggested by [25] and [11] (in the classical setting), mainly because it is constructive, illustrative, and allows us to precisely describe the σ-algebra Σ_P.

For the remainder of this introductory section, we outline this article and herein describe the heuristics of our construction of σ_P. Section 2 is a short section which contains a proof of Proposition 1.2, outlines the basics of positive homogeneous functions, and introduces the related notion of subhomogeneous functions (see Sect. 2.1). In Sect. 3, we focus on the study of convolution powers of complex-valued function on $\mathbb{Z}^d$. Therein, we make precise the discussion of the previous subsection in terms of positive homogeneous functions and present the main result of that section, Theorem 3.2. Partially extending results of [22] and [24], our result is new and its proof makes use of Theorem 1.4 and the Van der Corput lemma. Section 3.1 treats several concrete examples to which Theorem 3.2 applies. A forthcoming article will treat local limit theorems for complex-valued functions satisfying the hypotheses of Theorem 3.2. Section 4 focuses on the proof of Theorem 1.4 and is broken into several subsections. In Sect. 4.1, we take $E \in \mathrm{Exp}(P)$ and define a measure $\sigma_{P,E}$ on S by adapting the classical construction of the spherical measure, described in Remark 2 of [11], to our positive homogeneous setting. Specifically, for a given set $F \subseteq S$, we contract F into a wedge-like region of B by putting $\widetilde{F} = \widetilde{F}_E = \{r^E \eta : 0 < r < 1, \eta \in F\}$ and setting

$$\sigma_{P,E}(F) = \mu_P \cdot m(\widetilde{F})$$

provided that $\widetilde{F}$ is Lebesgue measurable. Figures 3 and 4 illustrate these wedge-like regions in $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.

In Sect. 4.2, we turn our focus to an associated product measure $\lambda_P \times \sigma_{P,E}$ on $(0, \infty) \times S$ with which we are able to formulate and prove a generalization of (8)

Fig. 3 Wedge-like region $\widetilde{F} = \widetilde{F}_E$ (in red) for $F \subseteq S$. Here, S is the unital level set of $P = P_2$ from Example 2 and $E \in \mathrm{Exp}(P_2)$ has the standard representation $\mathrm{diag}(1/2, 1/4)$
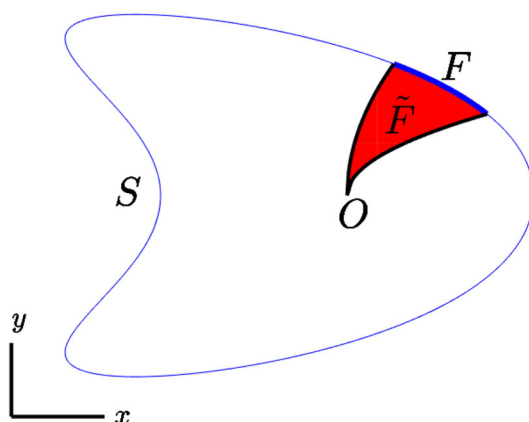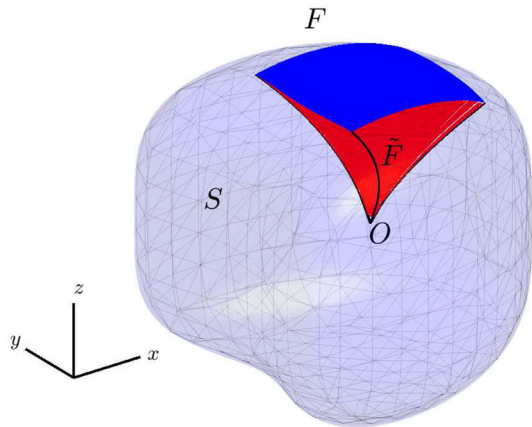
Fig. 4 Wedge-like region $\widetilde{F} = \widetilde{F}_E$ (in red) for $F \subseteq S$. Here, S is the unital level set of $P(x, y, z) = x^2 + xy^2 + y^4 + z^4$ and $E \in \mathrm{Exp}(P)$ has standard representation $\mathrm{diag}(1/2, 1/4, 1/4)$

using the measure $\sigma_{P,E}$; this is Theorem 4.4. We then derive a number of corollaries of Theorem 4.4, including the result that $\sigma_{P,E}$ is a Radon measure on S. As everything done in Sects. 4.1 and 4.2 is done using the contracting group $\{r^E\}$ for a chosen $E \in \mathrm{Exp}(P)$, it isn't clear, *a priori*, exactly how $\sigma_{P,E}$ is dependent on the choice of $E \in \mathrm{Exp}(P)$, if at all. In Sect. 4.3, we prove that $\sigma_{P,E}$ is, in fact, independent of the choice of $E \in \mathrm{Exp}(P)$ and, upon writing $\sigma_P = \sigma_{P,E}$, we immediately obtain Theorem 1.4. All throughout Sect. 4, our construction uses only tools from point-set topology and measure theory. In Sect. 5, we study the special case in which a positive homogeneous function P is additionally smooth. In that case, we find that S is a smooth compact embedded hypersurface of $\mathbb{R}^d$ and the measure σ_P is characterized by a differential form $d\sigma_P$ and is closely related to the Riemannian volume on S; this is Theorem 5.4. The section also contains a number of results helpful to the computation of integrals on S with respect to σ_P. Finally, the appendix outlines some basic results concerning one-parameter contracting groups.

2 Positive Homogeneous Functions

In this short section, we treat some basic results on positive homogeneous functions and introduce the useful and related concept of subhomogeneous functions. Several times throughout, we will appeal to results concerning contracting groups which can be found in the appendix.

Proof of Proposition 1.2 In the case that $d = 1$, it is easy to see that every function satisfying the hypotheses is of the form

$$P(x) = \begin{cases} P(1)x^\alpha & \text{for } x \geq 0 \\ P(-1)(-x)^\alpha & \text{for } x < 0 \end{cases}$$

for some $\alpha > 0$ where $P(1), P(-1) > 0$ and $\mathrm{Exp}(P)$ consists only of the linear function $x \mapsto x/\alpha$. In this setting, it is easy to see that Conditions (a)–(e) are satisfied

(always and) simultaneously. We shall therefore assume that $d > 1$ for the remainder of the proof.

(a)$\Rightarrow$(b) Given that S is compact, it is bounded and so we have a positive number M for which $P(x) \neq 1$ for all $|x| \geq M$. Observe that, if for two points $x_1, x_2 \in \mathbb{R}^d \setminus \mathbb{B}_M$, $P(x_1) < 1 < P(x_2)$ or $P(x_2) < 1 < P(x_1)$, then by virtue of the path connectedness of $\mathbb{R}^d \setminus \mathbb{B}_M$ and the intermediate value theorem, we would be able to find $x_0 \in \mathbb{R}^d \setminus \mathbb{B}_M$ for which $P(x_0) = 1$, an impossibility. Therefore, to show that Condition (b) holds, we must simply rule out the case in which $P(x) < 1$ for all $|x| \geq M$. Let us therefore assume, to reach a contradiction, that this alternate condition holds. In this case, we take $E \in \mathrm{Exp}(P)$ and $y \in \mathbb{R}^d \setminus \{0\}$ and observe that

$$\lim_{r \to \infty} P(r^E y) = \lim_{r \to \infty} r P(y) = \infty.$$

By virtue of our supposition, we find that $|r^E y| < M$ for all sufficiently large r. In particular, there exists a sequence $r_k \to \infty$ for which $|r_k^E y| \leq M$ for all k and

$$\lim_{k \to \infty} P(r_k^E y) = \infty.$$

Because $\overline{\mathbb{B}_M}$, the closure of $\mathbb{B}_M$, is compact, $\{r_k^E y\}$ has a convergent subsequence which we also denote by $\{r_k^E y\}$ by a slight abuse of notation. In view of the continuity of P at $\eta = \lim_{k \to \infty} r_k^E y$, we have

$$P(\eta) = \lim_{k \to \infty} P(r_k^E y) = \lim_{k \to \infty} r_k P(y) = \infty,$$

which is impossible. Thus Condition (b) holds.

(b)$\Rightarrow$(c) We shall prove the contrapositive statement. Suppose that, for $E \in \mathrm{Exp}(P)$, $\{r^E\}$ is not contracting. In this case, by virtue of Proposition A.2, there exists $x \in \mathbb{R}^d \setminus \{0\}$ and a sequence $r_k \to 0$ for which $r_k^E x$ does not converge to zero in $\mathbb{R}^d$. If our sequence $\{r_k^E x\}$ is bounded, then it must have a convergent subsequence $\{r_{k_m}^E x\}$ with non-zero subsequential limit $\eta = \lim_{m \to \infty} r_{k_m}^E x$. By the continuity of P, we have

$$P(\eta) = \lim_{m \to \infty} P(r_{k_m}^E x) = \lim_{k \to \infty} r_{k_m} P(x) = 0$$

which cannot be true for it would violate the positive definiteness of P. We must therefore consider the other possibility: The sequence $\{r_k^E x\}$ is unbounded. In particular, there must be some k_0 for which $r_{k_0} < 1/P(x)$ and $|r_{k_0}^E x| > M$. Upon putting $y = r_{k_0}^E x$, we have $|y| > M$ and $P(y) = P(r_{k_0}^E x) = r_{k_0} P(x) < 1$ which shows that Condition (b) cannot hold.

(c)$\Rightarrow$(d) This is immediate.

(d)$\Rightarrow$(e) Let $E \in \mathrm{Exp}(P)$ be such that the one-parameter group $\{r^E\}$ is contracting and let $\{x_k\} \subseteq \mathbb{R}^d$ be such that $\lim_{k \to \infty} |x_k| = \infty$. By virtue of Proposition A.5, there exist sequences $\{r_k\} \subseteq (0, \infty)$ and $\{\eta_k\} \in \mathbb{S}$ for which $r_k^E \eta_k = x_k$ for all k and

$\lim_{k\to\infty} r_k = \infty$. Given that P is continuous and strictly positive on the compact set $\mathbb{S}$, we have $\inf_{\eta\in\mathbb{S}} P(\eta) > 0$ and therefore

$$\liminf_k P(x_k) = \liminf_k r_k P(\eta_k) \geq \liminf_k r_k \left(\inf_{\eta\in\mathbb{S}} P(\eta)\right) = \infty$$

showing that $\lim_{k\to\infty} P(x_k) = \infty$, as desired.

(e)$\Rightarrow$(a) Because S is the preimage of the closed singleton $\{1\}$ under the continuous function P, it is closed. By virtue of Condition (e), S is also be bounded and thus compact in view of the Heine-Borel theorem. $\quad\square$

Proposition 2.1 *For each positive homogeneous function P, $\mathrm{Sym}(P)$ is a compact subgroup of $\mathrm{Gl}(\mathbb{R}^d)$.*

Proof As discussed in the introduction, it is clear that $\mathrm{Sym}(P)$ is a subgroup of $\mathrm{Gl}(\mathbb{R}^d)$. To see that $\mathrm{Sym}(P)$ is compact, it suffices to prove that $\mathrm{Sym}(P)$ is closed and bounded by virtue of the Heine-Borel theorem. To this end, let $\{O_n\} \subseteq \mathrm{Sym}(P)$ be a sequence converging to $O \in \mathrm{Gl}(\mathbb{R}^d)$. For each $x \in \mathbb{R}^d$, the continuity of P guarantees that

$$P(Ox) = P\left(\lim_{n\to\infty} O_n x\right) = \lim_{n\to\infty} P(O_n x) = \lim_{n\to\infty} P(x) = P(x).$$

Hence, $O \in \mathrm{Sym}(P)$ and so $\mathrm{Sym}(P)$ is closed.

We assume, to reach a contradiction, that $\mathrm{Sym}(P)$ is not bounded. In this case, there is a sequence $\{\eta_n\} \subseteq \mathbb{S}$ for which $\lim_{n\to\infty} |O_n\eta_n| = \infty$. Given that $\mathbb{S}$ is compact, by passing to a subsequence if needed, we may assume without loss in generality that $\lim_{n\to\infty} \eta_n = \eta \in \mathbb{S}$. By virtue of Proposition 1.2 and the continuity of P,

$$P(\eta) = \lim_{n\to\infty} P(\eta_n) = \lim_{n\to\infty} P(O_n\eta_n) = \infty$$

which is impossible. Hence $\mathrm{Sym}(P)$ is bounded. $\quad\square$

Corollary 2.2 *Let P be a positive homogeneous function, then*

$$\mathrm{tr}\,E = \mathrm{tr}\,E' > 0$$

for all $E, E' \in \mathrm{Exp}(P)$.

Proof By virtue of Propositions 1.2 and A.7, $\mathrm{tr}\,E > 0$ for all $E \in \mathrm{Exp}(P)$. It remains to show that the trace map is constant on $\mathrm{Exp}(P)$. To this end, let $E, E' \in \mathrm{Exp}(P)$. Then, for all $r > 0$ and $x \in \mathbb{R}^d$,

$$P(x) = r(1/r)P(x) = r P((1/r)^{E'} x) = P(r^E (1/r)^{E'} x) = P(r^E r^{-E'} x).$$

Thus $O_r = r^E r^{-E'} \in \mathrm{Sym}(P)$ for each $r > 0$. In view of Proposition 2.1, Proposition A.1 and the homomorphism property of the determinant,

$$1 = |\det(O_r)| = \left|\det(r^E)\det(r^{-E'})\right| = r^{\operatorname{tr}E} r^{-\operatorname{tr}E'} = r^{\operatorname{tr}E - \operatorname{tr}E'}$$

for all $r > 0$ and therefore $\operatorname{tr}E = \operatorname{tr}E'$. $\square$

For a continuous, positive definite function P for which $\operatorname{Exp}(P)$ is non-empty, the following proposition gives a sufficient condition for P to be positive homogeneous. As discussed in Example 3 above, it is this condition that was used to define "positive homogeneous polynomial" in [24].

Proposition 2.3 *If P is continuous, positive definite and $\operatorname{Exp}(P)$ contains an $E \in \operatorname{End}(\mathbb{R}^d)$ with real spectrum, then $\{r^E\}$ is contracting and hence all of the above conditions are (simultaneously) met.*

Proof Since $\operatorname{Spec}(E)$ is real, the characteristic polynomial of E factors completely over $\mathbb{R}$ and so we may apply the Jordan-Chevalley decomposition to write $E = D + N$ where D is diagonalizable, N is nilpotent, and $DN = ND$. Let $v_1, v_2, \dots, v_d \in \mathbb{R}^d$ be an eigenbasis of D whose corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_d$ satisfy $\lambda_k \le \lambda_{k+1}$ for all $k = 1, 2 \dots, d - 1$.

Let us assume, to reach a contradiction, that $\{r^E\}$ is not contracting. Repeating the same argument given in (b)$\Rightarrow$(c) in the proof of Proposition 1.2, leaves us with only one possibility: There is a non-zero $x = \sum_{i=1}^d \alpha_i v_i \in \mathbb{R}^d$, and a sequence $r_k \to 0$ for which $|r_k^E x| \to \infty$. Let $n + 1$ denote the index of N, then we have

$$r_k^E x = r_k^{N+D} x = r_k^N r_k^D x = \sum_{j=0}^n \sum_{i=1}^d \frac{r_k^{\lambda_i}(\log r_k)^j}{j!} \alpha_i N^j v_i$$

for all k. Since $\left|r_k^E x\right| \to \infty$ and $r_k \to 0$, at least one eigenvalue of D must be non-positive. To see this, suppose $\lambda_i > 0$ for all $i = 1, 2, \dots, d$, then in view of L'Hôpital's rule we have

$$\lim_{r_k \to 0} (\log r_k)^j r_k^{\lambda_i} = 0$$

for any $j = 0, 1, 2, \dots, n$ and $i = 1, 2, \dots, d$, which implies that $\left|r_k^E x\right| \not\to \infty$ as $r_k \to 0$, contradicting our assumption. Thus, $\lambda_1 = \min\{\operatorname{Spec}(D)\} \le 0$. Let k be such that $N^k v_1 \ne 0$ but $N^{k+1} v_1 = 0$, then

$$r^E N^k v_1 = r^D r^N N^k v_1 = r^D \sum_{j=0}^\infty \frac{(\log r)^j}{j!} N^j N^k v_1 = r^D N^k v_1 = N^k r^D v_1 = r^{\lambda_1} N^k v_1$$

where we have used the fact that $DN = ND$. If $\lambda_1 = 0$, then

$$\infty = \lim_{r \to \infty} r P(N^k v_1) = \lim_{r \to \infty} P(r^E N^k v_1) = \lim_{r \to \infty} P(r^0 N^k v_1)$$
$$= \lim_{r \to \infty} P(N^k v_1) = P(N^k v_1)$$

which is impossible since P is continuous at $N^k v_1$. On the other hand, if $\lambda_1 < 0$, then

$$\infty = \lim_{r \to \infty} r P(N^k v_1) = \lim_{r \to \infty} P(r^E N^k v_1) = \lim_{r \to \infty} P(r^{\lambda_1} N^k v_1) = P(0) = 0$$

which is also impossible. $\qquad\square$

Proposition 2.4 *For any $O \in \mathrm{Sym}\,(P)$*

$$\mathrm{Exp}(P) = O^{-1} \mathrm{Exp}(P) O.$$

In other words, the set $\mathrm{Exp}(P)$ is invariant under conjugation by $\mathrm{Sym}(P)$.

Proof Let $O \in \mathrm{Sym}(P)$. In view of Proposition A.1, it is easy to see that the map $\mathrm{Exp}(P) \ni E \mapsto O^{-1} E O \in \mathrm{Exp}(P)$ is a bijection and hence $\mathrm{Exp}(P) = O^{-1} \mathrm{Exp}(P) O$. $\qquad\square$

2.1 Subhomogeneous Functions

In this subsection, we introduce the notions of subhomogeneous functions and strongly subhomogeneous functions with respect to a given endomorphism $E \in \mathrm{End}(\mathbb{R}^d)$. As discussed in the following section, these notions appear naturally in the study of convolution powers of complex-valued functions on $\mathbb{Z}^d$.

Definition 2.5 Let Q be a continuous and complex-valued function defined on an open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^d$ and let $E \in \mathrm{End}(\mathbb{R}^d)$ be such that $\{r^E\}$ is a contracting group.

1. We say that Q is **subhomogeneous with respect to E** if, for each $\epsilon > 0$ and compact set $K \subseteq \mathbb{R}^d$, there is a $\delta > 0$ for which

$$\left| Q(r^E \xi) \right| \leq \epsilon r$$

   for all $0 < r < \delta$ and $\xi \in K$.
2. Given $l \in \mathbb{N}_+$, we say that Q is **strongly subhomogeneous with respect to E of order l** if $Q \in C^l(\mathcal{O})$ and, for each $\epsilon > 0$ and compact set $K \subseteq \mathbb{R}^d$, there is a $\delta > 0$ for which

$$\left| r^j \partial_r^j Q(r^E \xi) \right| \leq \epsilon r$$

   for all $j \in \{0, 1, 2, \ldots, l\}$, $0 < r < \delta$, and $\xi \in K$.

When the endomorphism E is understood (and fixed), we will say that Q is subhomogeneous if it is subhomogeneous with respect to E. Also, we will say that Q is l-strongly subhomogeneous if it is strongly subhomogeneous with respect to E of order l.

From the definition, it is clear that all strongly subhomogeneous functions are sub-homogeneous and that it is reasonable to interpret subhomogeneous as "strongly subhomogeneous of order 0". We remark that our supposition that $\{r^E\}$ is a contracting group ensures (in view of Proposition A.6) that, for some $r_0 > 0$, $r^E \xi \in \mathcal{O}$ for all $\xi \in K$ and $0 < r < r_0$ and so the inequalities above make sense. The following proposition connects the notion of subhomogeneity with that of a function being $o(P(\xi))$ as $\xi \to 0$.

Proposition 2.6 *Let P be positive homogeneous and $\widetilde{P}$ be complex valued and continuous on a neighborhood of 0 in $\mathbb{R}^d$. The following are equivalent:*

(a) $\widetilde{P}(\xi) = o(P(\xi))$ *as $\xi \to 0$.*
(b) *For every $E \in \mathrm{Exp}(P)$, $\widetilde{P}$ is subhomogeneous with respect to E.*
(c) *There exists $E \in \mathrm{Exp}(P)$ for which $\widetilde{P}$ is subhomogeneous with respect to E.*

Proof (a) $\Rightarrow$ (b) Let $\epsilon > 0$, K be a compact set and choose $E \in \mathrm{Exp}(P)$. Given our supposition that $\widetilde{P}(\xi) = o(P(\xi))$ as $\xi \to 0$, we can find an open neighborhood $\mathcal{O}$ of 0 for which

$$\left|\widetilde{P}(\xi)\right| \le \frac{\epsilon}{1 + \sup_{\eta \in K} P(\eta)} P(\xi)$$

for all $\xi \in \mathcal{O}$. Now, because r^E is contracting in view of Proposition 1.2, we can find a $\delta > 0$ for which $r^E \xi \in \mathcal{O}$ for all $0 < r < \delta$ and $\xi \in K$ by virtue of Proposition A.6. Consequently, for all $0 < r < \delta$ and $\xi \in K$,

$$\left|\widetilde{P}(r^E \xi)\right| \le \frac{\epsilon}{1 + \sup_{\eta \in K} P(\eta)} P(r^E \xi) = \epsilon r \frac{P(\xi)}{1 + \sup_{\eta \in K} P(\eta)} \le r\epsilon.$$

(b) $\Rightarrow$ (c) This implication is trivial.
(c) $\Rightarrow$ (a) Let $\epsilon > 0$. Choose $E \in \mathrm{Exp}(P)$ and let $S = \{\eta \in \mathbb{R}^d : P(\eta) = 1\}$. Using the supposition that $\widetilde{P}$ is subhomogeneous with respect to E, we may choose $\delta > 0$ for which

$$\left|\widetilde{P}(r^E \eta)\right| \le \epsilon r$$

for all $0 < r < \delta$ and $\eta \in S$. We remark that, in view of the continuity of $\widetilde{P}$ and the fact that r^E is contracting, this inequality ensures that $\widetilde{P}(0) = 0$. We fix $\mathcal{O}$ to be the open set $B_\delta = \{\xi \in \mathbb{R}^d : P(\xi) < \delta\}$. For each non-zero $\xi \in \mathcal{O}$, we observe that $\xi = r^E \eta$ where $0 < r = P(\xi) < \delta$ and $\eta = P(\xi)^{-E} \xi \in S$ and therefore

$$\left|\widetilde{P}(\xi)\right| = \left|\widetilde{P}(r^E \eta)\right| \le r\epsilon = \epsilon P(\xi)$$

If $\xi = 0$, obviously, $\left|\widetilde{P}(\xi)\right| = 0 = \epsilon P(0) = \epsilon P(\xi)$. Thus, for all $\xi \in \mathcal{O}$,

$$\left|\widetilde{P}(\xi)\right| \leq \epsilon P(\xi),$$

as desired. $\qquad\square$

Proposition 2.7 *Let $E \in \mathrm{End}(\mathbb{R}^d)$ be for which $\{r^E\}$ is contracting and suppose that Q is strongly subhomogeneous with respect to E of order 2. Given $\alpha > 0$, set $F = \alpha E$. Then, for any $\epsilon > 0$ and compact set K, there exists $\delta > 0$ for which*

$$\left|\theta \partial_\theta Q(\theta^F \eta)\right| \leq \epsilon \theta^\alpha$$

and

$$\left|\theta^2 \partial_\theta^2 Q(\theta^F \eta)\right| \leq \epsilon \theta^\alpha$$

for all $0 < \theta \leq \delta^{1/\alpha}$ and $\eta \in K$.

Proof Let $\epsilon > 0$ and $K \subseteq \mathbb{R}^d$ be a compact set. By virtue of the strong subhomogeneity of Q, let $\delta > 0$ be such that

$$\left|r \partial_r Q(r^E \eta)\right| \leq \epsilon' r \text{ and } \left|r^2 \partial_r^2 Q(r^E \eta)\right| \leq \epsilon' r$$

for $0 < r < \delta$ and $\eta \in K$ where $\epsilon' = \epsilon/(2\alpha^2 + \alpha)$. We set $r = \theta^\alpha$ so that $r^E = \theta^F$ and observe that

$$
\begin{aligned}
\left|\theta \partial_\theta Q(\theta^F \eta)\right| &= \left|\theta \partial_r Q(r^E \eta) \frac{\partial r}{\partial \theta}\right| = \left|\theta \partial_r Q(r^E \eta) \alpha \theta^{\alpha-1}\right| \\
&= \alpha r \left|\partial_r Q(r^E \eta)\right| \leq \alpha \epsilon' r < \epsilon \theta^\alpha
\end{aligned}
$$

for all $0 < \theta \leq \delta^{1/\alpha}$ and $\eta \in K$. Further, we have

$$
\begin{aligned}
\left|\theta^2 \partial_\theta^2 Q(\theta^F \eta)\right| &= \theta^2 \left|\partial_r^2 Q(r^E \eta)\left(\frac{\partial r}{\partial \theta}\right)^2 + \partial_r Q(r^E \eta) \frac{\partial^2 r}{\partial \theta^2}\right| \\
&= \theta^2 \left|\partial_r^2 Q(r^E \eta)\alpha^2 \theta^{2\alpha-2} + \partial_r Q(r^E \eta)\alpha(\alpha-1)\theta^{\alpha-2}\right| \\
&\leq \alpha^2 \left|r^2 Q(r^E \eta)\right| + |\alpha(\alpha-1)| \left|r \partial_r Q(r^E \eta)\right| \\
&< \alpha^2 \epsilon' r + \left|\alpha^2 - \alpha\right| \epsilon' r \\
&< \epsilon \theta^\alpha
\end{aligned}
$$

for all $0 < \theta < \delta^{1/\alpha}$ and $\eta \in K$. $\qquad\square$

3 An Application: Estimates for Convolution Powers

In this section, we focus on sup-norm-type estimates for convolution powers of complex-valued functions on $\mathbb{Z}^d$. As discussed in the introduction, our main result, Theorem 3.2, partially extends results of [22] and [24], and its proof makes use of Theorem 1.4 and the Van der Corput lemma. A forthcoming article will present a theory of local limit theorems for complex-valued functions on $\mathbb{Z}^d$ satisfying the hypotheses of Theorem 3.2. The Fourier transform is essential to our analysis and is defined as follows: Given $\phi \in \ell^1(\mathbb{Z}^d)$, the Fourier transform of ϕ is the function $\widehat{\phi} : \mathbb{R}^d \to \mathbb{C}$ defined by

$$\widehat{\phi}(\xi) = \sum_{x \in \mathbb{Z}^d} \phi(x) e^{ix \cdot \xi}$$

for $\xi \in \mathbb{R}^d$. As in [24], we shall focus on the subspace $\mathcal{S}_d$ of $\ell^1(\mathbb{Z}^d)$ consisting of those $\phi : \mathbb{Z}^d \to \mathbb{C}$ for which

$$\|x^\beta \phi(x)\|_1 = \sum_{x \in \mathbb{Z}^d} \left| x^\beta \phi(x) \right|$$
$$= \sum_{x \in \mathbb{Z}^d} \left| (x^1)^{\beta_1} (x^2)^{\beta_2} \cdots (x^d)^{\beta_d} \phi(x^1, x^2, \ldots, x^d) \right| < \infty$$

for each multi-index $\beta = (\beta_1, \beta_2, \ldots, \beta_d) \in \mathbb{N}^d$; we remark that $\mathcal{S}_d$ contains all finitely supported complex-valued functions on $\mathbb{Z}^d$. It is easy to see that $\widehat{\phi} \in C^\infty(\mathbb{R}^d)$ whenever $\phi \in \mathcal{S}_d$. As discussed in [8, 22, 24, 30], the asymptotic behavior of the iterative convolution powers $\phi^{(n)}$ of $\phi \in \mathcal{S}_d$ is characterized by the local behavior of $\widehat{\phi}$ near points at which $\widehat{\phi}$ is maximized in absolute value. For simplicity of our analysis, we shall focus on those $\phi \in \mathcal{S}_d$ which have been suitably normalized so that $\sup_\xi |\widehat{\phi}(\xi)| = 1$ and, in this case, we define

$$\Omega(\phi) = \left\{ \xi \in \mathbb{T}^d : \left| \widehat{\phi}(\xi) \right| = 1 \right\}$$

where $\mathbb{T}^d = (-\pi, \pi]^d$. For each $\xi_0 \in \Omega(\phi)$, consider $\Gamma_{\xi_0} : \mathcal{U} \to \mathbb{C}$ defined by

$$\Gamma_{\xi_0}(\xi) = \log \left(\frac{\widehat{\phi}(\xi + \xi_0)}{\widehat{\phi}(\xi_0)} \right)$$

for $\xi \in \mathcal{U}$ where $\mathcal{U} \subseteq \mathbb{R}^d$ is a convex open neighborhood of 0 which is small enough to ensure that $z \mapsto \log(z)$, the principal branch of the logarithm, is defined and continuous on $\{\widehat{\phi}(\xi + \xi_0)/\widehat{\phi}(\xi_0) : \xi \in \mathcal{U}\}$. Because $\widehat{\phi}$ is smooth, $\Gamma_{\xi_0} \in C^\infty(\mathcal{U})$ and so we can use Taylor's theorem to approximate Γ_{ξ_0} near 0. More precisely, we can write

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - i \left(Q_{\xi_0}(\xi) + \widetilde{Q}_{\xi_0}(\xi) \right) - \left(R_{\xi_0}(\xi) + \widetilde{R}_{\xi_0}(\xi) \right) \tag{14}$$

where $\alpha_{\xi_0} \in \mathbb{R}^d$, Q_{ξ_0} and R_{ξ_0} are real-valued polynomials which vanish at 0 and contain no linear terms, and $\widetilde{Q}_{\xi_0}$ and $\widetilde{R}_{\xi_0}$ are real-valued smooth functions on $\mathcal{U}$ which vanish at 0. The fact that this expansion contains no real linear part is seen necessary because ξ_0 is a local maximum for $|\widehat{\phi}|$. The vector $\alpha_{\xi_0} \in \mathbb{R}^d$ is said to be the **drift**[4] associated to ξ_0. Motivated by Thomée [30], we introduce the following definition.

Definition 3.1 Let $\phi \in \mathcal{S}_d$ with $\sup_\xi |\widehat{\phi}(\xi)| = 1$ and, given $\xi_0 \in \Omega(\phi)$, consider the expansion (14) above.

1. We say that ξ_0 is of **positive homogeneous type** for $\widehat{\phi}$ if R_{ξ_0} is positive homogeneous and, there exists $E \in \mathrm{Exp}(R_{\xi_0})$ for which Q_{ξ_0} is homogeneous with respect to E and both $\widetilde{R}_{\xi_0}$ and $\widetilde{Q}_{\xi_0}$ are subhomogeneous with respect to E. In this case, we will write $\mu_{\xi_0} = \mu_{R_{\xi_0}}$.
2. We say that ξ_0 is of **imaginary homogeneous type** for $\widehat{\phi}$ if $|Q_{\xi_0}|$ and R_{ξ_0} are both positive homogeneous and, there exists $E \in \mathrm{Exp}(|Q_{\xi_0}|)$ and $k > 1$ for which R_{ξ_0} is homogeneous with respect to E/k, $\widetilde{Q}_{\xi_0}$ is strongly subhomogeneous with respect to E of order 2, and $\widetilde{R}_{\xi_0}$ is strongly subhomogeneous with respect to E/k of order 1. In this case, we write $\mu_{\xi_0} = \mu_{|Q_{\xi_0}|}$.

In either case, μ_{ξ_0} is said to be the homogeneous order associated to ξ_0.

As we mentioned in the introduction, in his study of approximation schemes to solutions of parabolic partial differential equations, V. Thomée introduced the notions of points of Type γ and Type β, arising in local approximations of the Fourier transforms of (schemes) $\phi : \mathbb{Z} \to \mathbb{C}$, to dichotomize the stability of approximation schemes in the maximum norm [30]. Thomée's definition provided a key insight which led to the complete description of the asymptotic behavior of convolution powers of finitely supported functions on $\mathbb{Z}$ given in [22]. The points of positive homogeneous type and imaginary homogeneous type in the definition above parallel (and generalize) Thomée's points of Type γ and Type β (and points of Type 1 and Type 2 of [22]), respectively.

In Definition 1.3 of [24], a point $\xi_0 \in \Omega(\phi)$ is said to be of positive homogeneous type for $\widehat{\phi}$ provided that the expansion for Γ_{ξ_0} is of the form

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - P_{\xi_0}(\xi) - \widetilde{P}_{\xi_0}(\xi) \qquad (15)$$

for $\xi \in \mathcal{U}$ where P_{ξ_0} is a positive homogeneous polynomial in the sense of Example 2 and $\widetilde{P}_{\xi_0}(\xi) = o(R_{\xi_0}(\xi))$ as $\xi \to 0$ where $R_{\xi_0} = \mathrm{Re}\, P_{\xi_0}$. To put this into context with our definition above, let's write $P_{\xi_0}(\xi) = R_{\xi_0}(\xi) + iQ_{\xi_0}(\xi)$ and $\widetilde{P}_{\xi_0}(\xi) = \widetilde{R}_{\xi_0}(\xi) + i\widetilde{Q}_{\xi_0}(\xi)$ in which case (14) coincides with (15). If ξ_0 is of positive homogeneous type for $\widehat{\phi}$ in the sense of the definition above, it follows that P_{ξ_0} is a complex-valued polynomial which is homogeneous with respect to E (and so $\mathrm{Exp}(P_{\xi_0})$ contains $E \in \mathrm{End}(\mathbb{R}^d)$ for which $\{r^E\}$ is contracting) and $R_{\xi_0} = \mathrm{Re}\, P_{\xi_0}$ is positive definite. In view of Remark 1, this is consistent with (and perhaps generalizes) the assumption in which P_{ξ_0} is a positive homogeneous polynomial. Further, the assumption that $\widetilde{Q}_{\xi_0}$

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[4] In the case that ϕ defines a probability measure and a $\mathbb{Z}^d$-valued random vector X has this measure as its distribution, then α_{ξ_0} is X's mean. For a precise statement and details, see Proposition 7.4 of [24].

and $\widetilde{R}_{\xi_0}$ are subhomogeneous with respect to E guarantees that $\widetilde{P}_{\xi_0}(\xi) = o(R_{\xi_0}(\xi))$ as $\xi \to 0$ by virtue of Proposition 2.6. With these two observations, we see that our definition, which is stated in terms of subhomogeneity, is consistent with that of [24]. The essential difference between the cases in Definition 3.1 concerns the nature of the dominant (at low order) term in the expansion. When ξ_0 is of positive homogeneous type for $\widehat{\phi}$, the dominant term P_{ξ_0} contains the real-valued positive definite polynomial R_{ξ_0}. In this case, local limit theorems for $\phi^{(n)}(x)$ contains attractors/approximants of the form

$$H^n_{P_{\xi_0}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-nP_{\xi_0}(\xi)-ix\cdot\xi} \, d\xi$$

which can be seen, for example, in Theorem 1.5 of [24]. These are necessarily Schwartz functions and appear as fundamental solutions to the higher-order partial differential equations discussed in [23]. When ξ_0 is of imaginary homogeneous type for $\widehat{\phi}$, the dominant term in the expansion is the purely imaginary polynomial $iQ_{\xi_0}(\xi)$ and its existence (without a real counterpart) profoundly affects the asymptotic behavior of $\phi^{(n)}(x)$ (e.g., see [22]). In fact, as will be shown in a forthcoming article, local limit theorems for $\phi^{(n)}(x)$ will contain approximants/attractors which are (formally) given by the oscillatory integral

$$H^n_{iQ_{\xi_0}}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-inQ_{\xi_0}(\xi)-ix\cdot\xi} \, d\xi$$

whose convergence is a delicate matter.

Our theorem will be stated under the assumption that, for $\phi \in \mathcal{S}_d$ with $\sup_\xi |\widehat{\phi}(\xi)| = 1$, each $\xi_0 \in \Omega(\phi)$ is either of positive homogeneous type or of imaginary homogeneous type for $\widehat{\phi}$. In both cases, the positive definiteness of R_{ξ_0} guarantees that each $\xi_0 \in \Omega(\phi)$ is an isolated point of $\mathbb{T}^d$. Consequently, if each $\xi_0 \in \Omega(\phi)$ is of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$, the set $\Omega(\phi)$ is finite and we set

$$\mu_\phi = \min_{\xi \in \Omega(\phi)} \mu_\xi.$$

Theorem 3.2 *Let $\phi \in \mathcal{S}_d$ be such that $\sup |\widehat{\phi}| = 1$ and suppose that each $\xi_0 \in \Omega(\phi)$ is of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$. If $\alpha_{\xi_0} = 0$ and $\mu_{\xi_0} < 1$ for each $\xi_0 \in \Omega(\phi)$ which is of imaginary homogeneous type for $\widehat{\phi}$, then, for any compact set K, there is a constant $C_K > 0$ for which*

$$\left|\phi^{(n)}(x)\right| \leq \frac{C_K}{n^{\mu_\phi}}$$

for all $x \in K$ and $n \in \mathbb{N}_+$.

The above result partially extends the upper estimate of Theorem 1.1 (Theorem 1.1 of [22]) into the d-dimensional setting. Specifically, this upper estimate (appearing as Theorem 3.6 of [22]) guarantees that, for each $\phi : \mathbb{Z} \to \mathbb{C}$ for which $\sup_{\xi \in \mathbb{T}} |\widehat{\phi}(\xi)| = 1$

and whose support is finite and contains more than one point, there is a constant C and a positive integer $m \geq 2$ for which

$$\left| \phi^{(n)}(x) \right| \leq C n^{-1/m}$$

for all $x \in \mathbb{Z}$ and $n \in \mathbb{N}_+$. Under these hypotheses concerning ϕ's support, it follows from a basic result of complex analysis that every point ξ_0 of $\Omega(\phi)$ is necessarily[5] of positive homogeneous type or imaginary homogeneous type for $\widehat{\phi}$ with $\mu_{\xi_0} \leq 1/2 < 1$. In this way, we see that Theorem 3.2 partially extends Theorem 3.6 of [22] in the sense that it guarantees a spatially uniform estimate over compact sets and is stated under the additional hypotheses that the drift is zero for each point of imaginary homogeneous type for $\widehat{\phi}$. Though we expect that this is not the final result on the matter, the more limited scope of Theorem 3.2 is not surprising in light of the natural complexity of $\mathbb{R}^d$ (and $\mathbb{Z}^d$).

Concerning the existing theory in $\mathbb{Z}^d$, Theorem 3.2 is stated under weaker hypotheses than is the analogous result in [24]. Specifically, Theorem 1.4 of [24] is stated under the assumption that, given $\phi \in \mathcal{S}_d$ with $\sup_\xi |\widehat{\phi}(\xi)| = 1$, every point $\xi_0 \in \Omega(\phi)$ is of positive homogeneous type for $\widehat{\phi}$ and, in this case, the theorem gives positive constants C and C', for which

$$C' n^{-\mu_\phi} \leq \sup_{x \in \mathbb{Z}^d} \left| \phi^{(n)}(x) \right| \leq C n^{-\mu_\phi} \tag{16}$$

where $n \in \mathbb{N}_+$. Though we have not stated it this way, our proof of Theorem 3.2 (see Lemma 3.3), guarantees the upper estimate in (16) in the case that there are no points $\xi_0 \in \Omega(\phi)$ of imaginary homogeneous type for $\widehat{\phi}$. It is the presence of points $\xi_0 \in \Omega(\phi)$ of imaginary homogeneous type that makes the analysis significantly more difficult (even in one dimension) and leads to the slightly weaker conclusion. It is our belief that a uniform estimate of the type (16) is valid when all points are either of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$ (perhaps still with some restriction on homogeneous order) but a resolution of such a conjecture will require further analysis and a thorough study of local limits. In Sect. 3.1, we give several examples illustrating the conclusion of Theorem 3.2, none of which satisfy the hypotheses of Theorem 1.4 of [24].

Our proof of Theorem 3.2 will make use of the Fourier inversion formula

$$\phi^{(n)}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}_\phi^d} \widehat{\phi}(\xi)^n e^{-ix \cdot \xi} \, d\xi \tag{17}$$

which is valid for all $n \in \mathbb{N}_+$ and $x \in \mathbb{R}^d$; here $\mathbb{T}_\phi^d = \mathbb{T}^d + \xi_\phi \subseteq \mathbb{R}^d$ is a representation of the d-dimensional torus chosen so that $\Omega(\phi) \subseteq \text{Int}(\mathbb{T}_\phi^d)$; this can always be arranged, i.e., some $\xi_\phi \in \mathbb{R}^d$ can be selected, because $\Omega(\phi)$ is a finite set (see Remark 3 of [24]). As discussed in [24], the asymptotic behavior of $\phi^{(n)}$ is characterized by

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[5] This result is due to V. Thomée [30]. It is easy to see that the assertion fails when $d > 1$.

the contributions to the above integral produced by integration over neighborhoods of points $\xi_0 \in \Omega(\phi)$. Specifically, we shall study integrals of the form

$$\frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi} \, d\xi \tag{18}$$

where $\mathcal{O}_{\xi_0}$ is some (small and to be determined) neighborhood of $\xi_0 \in \Omega(\phi)$. When ξ_0 is of positive homogeneous type for $\widehat{\phi}$, such integrals are very well behaved (the integrand is dominated uniformly by $e^{-nR_{\xi_0}(\xi)/2}$, a member of the Schwartz class). When ξ_0 is of imaginary homogeneous type, such integrals are oscillatory in nature and therefore much more difficult to handle. Our first lemma below handles the "easy" case in which ξ_0 is of positive homogeneous type for $\widehat{\phi}$. This lemma appears, essentially, as Lemma 4.3 of [24]. For illustrative purposes, we have decided to present a distinct proof here which makes use of the polar-coordinate integration formula in Theorem 1.4.

Lemma 3.3 *Let $\xi_0 \in \Omega(\phi)$ be of positive homogeneous type for $\widehat{\phi}$ with homogeneous order μ_{ξ_0}. Then, there exists an open neighborhood $\mathcal{O}_{\xi_0} \subseteq \mathrm{Int}(\mathbb{T}_\phi^d)$ of ξ_0, which can be taken as small as desired, and a constant $C = C_{\xi_0}$ for which*

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi} \, d\xi \right| \leq C_{\xi_0} n^{-\mu_{\xi_0}}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{R}^d$.

Proof For simplicity, we write $R = R_{\xi_0}$, $\widetilde{R} = \widetilde{R}_{\xi_0}$ and $\mu = \mu_{\xi_0}$. Given that ξ_0 is of positive homogeneous type for $\widehat{\phi}$, there is an open neighborhood $\mathcal{U}$ of 0 for which

$$\left| \widehat{\phi}(\xi + \xi_0) \right| = \left| \widehat{\phi}(\xi_0) e^{\Gamma_{\xi_0}(\xi)} \right| = e^{-(R(\xi) + \widetilde{R}(\xi))}$$

for $\xi \in \mathcal{U}$. Using the fact that $\widetilde{R}(\xi) = o(R(\xi))$ as $\xi \to 0$ in view of Proposition 2.6, we can further restrict $\mathcal{U}$ so that

$$\left| \widehat{\phi}(\xi + \xi_0) \right| \leq e^{-R(\xi)/2}$$

for all $\xi \in \mathcal{U}$. Take $E \in \mathrm{Exp}(R)$ and let σ_R be the surface measure on $S = \{\eta \in \mathbb{R}^d : R(\eta) = 1\}$ guaranteed by Theorem 1.4. We fix an open neighborhood $\mathcal{O}_{\xi_0}$ of ξ_0 which is as small as desired and has the property that

$$\mathcal{O} := \mathcal{O}_{\xi_0} - \xi_0 \subseteq \mathcal{U}.$$

With this, we observe that

$$\int_{\mathcal{O}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi} \, d\xi = \int_{\mathcal{O}} \widehat{\phi}(\xi + \xi_0)^n e^{-ix\cdot(\xi + \xi_0)} \, d\xi \tag{19}$$

and therefore

$$
\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi} \, d\xi \right| \leq \frac{1}{(2\pi)^d} \int_{\mathcal{O}} \left| \widehat{\phi}(\xi + \xi_0)^n e^{-ix\cdot(\xi+\xi_0)} \right| \, d\xi
$$

$$
\leq \frac{1}{(2\pi)^d} \int_{\mathcal{U}} e^{-nR(\xi)/2} \, d\xi
$$

$$
\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-(n/2)R(\xi)} \, d\xi
$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$. By virtue of Theorem 1.4, we have

$$
\int_{\mathbb{R}^d} e^{-(n/2)R(\xi)} \, d\xi = \int_S \int_0^\infty e^{-(n/2)r} r^{\mu-1} \, dr \, \sigma_R(d\eta)
$$

$$
= \int_S \frac{2^\mu \Gamma(\mu)}{n^\mu} \sigma_R(d\eta) = 2^\mu \Gamma(\mu) \sigma_R(S) n^{-\mu}
$$

where Γ denotes the Euler's Gamma function. Consequently,

$$
\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi} \, d\xi \right| \leq C n^{-\mu}
$$

for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}_+$ where $C = 2^\mu \Gamma(\mu) \sigma_R(S)/(2\pi)^d$. □

We shall now focus on the case in which $\xi_0 \in \Omega(\phi)$ is of imaginary homogeneous type for $\widehat{\phi}$. As discussed above, (18) is oscillatory in nature; this is due to the fact that the "principal" behavior of $\Gamma_{\xi_0}(\xi)$, for small ξ, is characterized by the purely imaginary polynomial iQ_{ξ_0}. Our main estimate is presented in Lemma 3.7 and its proof makes use of (11) and the following version of the Van der Corput lemma.

Proposition 3.4 *Let $g \in C^1([a, b])$ be complex valued and $f \in C^2([a, b])$ be real valued and such that $f''(x) \neq 0$ for all $x \in [a, b]$. Then*

$$
\left| \int_a^b e^{if(x)} g(x) \, dx \right| \leq \min \left\{ \frac{4}{\lambda_1}, \frac{8}{\sqrt{\lambda_2}} \right\} \left(\|g\|_{L^\infty[a,b]} + \|g'\|_{L^1[a,b]} \right).
$$

where $\lambda_1 = \inf_{x \in [a,b]} \left| f'(x) \right|$ and $\lambda_2 = \inf_{x \in [a,b]} \left| f''(x) \right|$.

For a proof of the above proposition, we refer the reader to Chapter 8 of [28] or Section 3 of [22] (see Lemma 3.4 therein). To effectively make use of the proposition above to estimate (18) in the case that ξ_0 is of imaginary homogeneous type, we first treat two preliminary lemmas.

Lemma 3.5 *Let $Q : \mathbb{R}^d \rightarrow \mathbb{R}$ be a continuous function for which $|Q|$ is positive homogeneous with $\mu := \mu_{|Q|} < 1$. Given a compact subset S of $\mathbb{R}^d$ for which $0 \notin S$, set*

$$
\rho = \inf_{\eta \in S} |Q(\eta)|/3 > 0.
$$

For an open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^d$, suppose that $\widetilde{Q} : \mathcal{O} \to \mathbb{R}$ is a twice continuously differentiable function which is strongly subhomogeneous with respect to E of order 2, set $F = E/\mu$, and define

$$f_{n,\eta,x}(\theta) = -nQ(\theta^F \eta) - n\widetilde{Q}(\theta^F \eta) - x \cdot \theta^F \eta$$
$$= -n\theta^{1/\mu} Q(\eta) - n\widetilde{Q}(\theta^F \eta) - x \cdot \theta^F \eta$$

for $n \in \mathbb{N}_+$, $\eta \in S$, $x \in \mathbb{R}^d$ and $\theta > 0$ sufficiently small so that $\theta^F \eta \in \mathcal{O}$. Then, given any compact set K, there is a $\delta > 0$ for which $\partial_\theta^2 f_{n,x,\eta}(\theta) \neq 0$ and

$$|\partial_\theta f_{n,x,\eta}(\theta)| \geq \frac{\rho}{\mu} n^\mu$$

for all $n \in \mathbb{N}_+$, $\eta \in S$, $x \in K$, and $\theta > 0$ for which $n^{-\mu} \leq \theta \leq \delta^\mu$.

Proof Let E and S be as in the statement of the lemma and write $f = f_{n,\eta,x}$. Because θ^F is contracting, let $\delta_1 > 0$ be such that

$$\left| x \cdot \theta^F F\eta \right| \leq \frac{\rho}{\mu} \text{ and } \left| x \cdot \theta^F (F - I) F\eta \right| \leq \frac{\rho}{\mu} \left(\frac{1}{\mu} - 1 \right) \qquad (20)$$

for all $0 < \theta < \delta_1^\mu$, $x \in K$ and $\eta \in S$. By virtue of Proposition 2.7, there exists $\delta_2 > 0$ such that

$$\left| \theta \partial_\theta \widetilde{Q}(\theta^F \eta) \right| \leq \frac{\rho}{\mu} \theta^{1/\mu} \text{ and } \left| \theta^2 \partial_\theta^2 \widetilde{Q}(\theta^F \eta) \right| \leq \frac{\rho}{\mu} \left(\frac{1}{\mu} - 1 \right) \theta^{1/\mu} \qquad (21)$$

for all $0 < \theta < \delta_2^\mu$ and $\eta \in S$. Set $\delta = \min\{\delta_1, \delta_2\}$. By virtue (20) and (21), we have

$$\left| \theta^2 \partial_\theta^2 f(\theta) \right| = \theta^2 \left| n\partial_\theta^2 \left(\theta^{1/\mu} Q(\eta) + \widetilde{Q}(\theta^F \eta) \right) + \partial_\theta^2 \left(x \cdot \theta^F \eta \right) \right|$$
$$= \left| \frac{n}{\mu} \left(\frac{1}{\mu} - 1 \right) \theta^{1/\mu} Q(\eta) + n\theta^2 \partial_\theta^2 \widetilde{Q}(\theta^F \eta) + \left(x \cdot \theta^F (F - I) F\eta \right) \right|$$
$$\geq \frac{n}{\mu} \left(\frac{1}{\mu} - 1 \right) \theta^{1/\mu} |Q(\eta)| - n \left| \theta^2 \partial_\theta^2 \widetilde{Q}(\theta^F \eta) \right| - \left| x \cdot \theta^F (F - I) F\eta \right|$$
$$\geq \frac{3\rho n}{\mu} \left(\frac{1}{\mu} - 1 \right) \theta^{1/\mu} - \frac{\rho n}{\mu} \left(\frac{1}{\mu} - 1 \right) \theta^{1/\mu} - \frac{\rho}{\mu} \left(\frac{1}{\mu} - 1 \right)$$
$$\geq \frac{\rho}{\mu} \left(\frac{1}{\mu} - 1 \right) \left(2n\theta^{1/\mu} - 1 \right)$$

for all $n \in \mathbb{N}_+$, $\eta \in S$, $x \in K$ and $0 < \theta < \delta^\mu$. Given that $\theta \mapsto \theta^{1/\mu}$ is increasing, it follows that

$$\left| \theta^2 \partial_\theta^2 f(\theta) \right| \geq \frac{\rho}{\mu} \left(\frac{1}{\mu} - 1 \right) (2n\theta^{1/\mu} - 1)$$

$$\geq \frac{\rho}{\mu} \left(\frac{1}{\mu} - 1 \right) (2n(n^{-\mu})^{1/\mu} - 1) = \frac{\rho}{\mu} \left(\frac{1}{\mu} - 1 \right) > 0$$

and, in particular, $\partial_\theta^2 f(\theta) \neq 0$ for all $n \in \mathbb{N}_+$, $\eta \in S$, $x \in K$ and $\theta > 0$ for which $n^{-\mu} \leq \theta \leq \delta^\mu$. By another appeal to (20) and (21), we find

$$|\partial_\theta f(\theta)| = \left| n \partial_\theta \left(\theta^{1/\mu} Q(\eta) + \widetilde{Q}(\theta^F \eta) \right) + \partial_\theta \left(x \cdot \theta^F \eta \right) \right|$$

$$= \left| \frac{n}{\mu} \theta^{1/\mu - 1} Q(\eta) + n \partial_\theta \widetilde{Q}(\theta^F \eta) + \theta^{-1} \left(x \cdot \theta^F F \eta \right) \right|$$

$$\geq \frac{3\rho n}{\mu} \theta^{1/\mu - 1} - n\theta^{-1} \left| \theta \partial_\theta \widetilde{Q}(\theta^F \eta) \right| - \theta^{-1} \left| x \cdot \theta^F F \eta \right|$$

$$\geq \frac{3\rho n}{\mu} \theta^{1/\mu - 1} - \frac{\rho n}{\mu} \theta^{1/\mu - 1} - \frac{\rho}{\mu} \theta^{-1}$$

$$\geq \frac{\rho}{\mu} \left(2n\theta^{1/\mu - 1} - \theta^{-1} \right)$$

for all $n \in \mathbb{N}_+$, $\eta \in S$, $x \in K$ and $0 < \theta < \delta^\mu$. Given our supposition that $\mu < 1$, $\theta \mapsto \left(2n\theta^{1/\mu - 1} - \theta^{-1} \right)$ is increasing for $\theta > 0$ and therefore

$$|\partial_\theta f(\theta)| \geq \frac{\rho}{\mu} \left(2n\theta^{1/\mu - 1} - \theta^{-1} \right) \geq \frac{\rho}{\mu} \left(2n(n^{-\mu})^{1/\mu - 1} - (n^{-\mu})^{-1} \right) = \frac{\rho}{\mu} n^\mu$$

for all for all $n \in \mathbb{N}_+$, $\eta \in S$, $x \in K$ and $\theta > 0$ for which $n^{-\mu} \leq \theta \leq \delta^\mu$, as was asserted. $\square$

Lemma 3.6 *Let R be a positive homogeneous function with $G \in \mathrm{Exp}(R)$, $\widetilde{R} : \mathcal{O} \to \mathbb{R}$ be once continuously differentiable on a neighborhood $\mathcal{O}$ of 0 which is strongly subhomogeneous with respect to G of order 1, and let k and μ be positive real numbers. Set $F = (k/\mu)G$ and, for each $\eta \in S = \{\eta : R(\eta) = 1\}$ and $n \in \mathbb{N}_+$, set*

$$g_{n,\eta}(\theta) = e^{-n\left(R(\theta^F \eta) + \widetilde{R}(\theta^F \eta) \right)}$$

for $\theta > 0$ which is sufficiently small so that $\theta^F \eta \in \mathcal{O}$. Then, for each $\beta > 1$, there is $\delta > 0$ for which

$$\|g_{n,\eta}\|_{L^\infty[\theta_1, \theta_2]} \leq 1$$

and

$$\|\partial_\theta g_{n,\eta}\|_{L^1[\theta_1, \theta_2]} \leq \beta$$

uniformly for $\eta \in S$, $n \in \mathbb{N}$ and $0 < \theta_1 \leq \theta_2 \leq \delta^\mu$.

Proof By virtue of the strong subhomogeneity of $\widetilde{R}$ and the fact that r^G is a contracting group, we may choose $\delta > 0$ for which

$$R(r^G \eta) + \widetilde{R}\left(r^G \eta\right) = r + \widetilde{R}\left(r^G \eta\right) \geq (1 - \epsilon)r > 0 \tag{22}$$

and

$$\left| \partial_r \left(R(r^G \eta) + \widetilde{R}(r^G \eta) \right) \right| \leq 1 + \epsilon \tag{23}$$

for all $0 < r < \delta^k$ and $\eta \in S$ where

$$\epsilon = \frac{\beta - 1}{\beta + 1} \in (0, 1).$$

In view of (22), for any $0 < \theta_1 \leq \theta_2 < \delta^\mu$, $\eta \in S$ and $n \in \mathbb{N}_+$,

$$\|g_{n,\eta}\|_{L^\infty[\theta_1,\theta_2]} \leq \sup_{0 < \theta \leq \delta^\mu} \left| g_{n,\eta}(\theta) \right|$$

$$= \sup_{0 < r \leq \delta^k} \left| g_{n,\eta}(r^{\mu/k}) \right| \leq \sup_{0 < r \leq \delta^k} e^{-nr(1-\epsilon)} = 1$$

where we have used the fact that $(r^{\mu/k})^F = r^{(\mu/k)F} = r^G$ for $r > 0$. By virtue of (22) and (23), we find that

$$\|\partial_\theta g_{n,\eta}\|_{L^1[\theta_1,\theta_2]} = \int_{\theta_1}^{\theta_2} \left| \partial_\theta g_{n,\eta}(\theta) \right| d\theta$$

$$= \int_{\theta_1^{k/\mu}}^{\theta_2^{k/\mu}} \left| \partial_r \left(g_{n,\eta}(r^{\mu/k}) \right) \right| dr$$

$$= \int_{\theta_1^{k/\mu}}^{\theta_2^{k/\mu}} \left| \partial_r \left(e^{-n\left(R(r^G \eta) + \widetilde{R}(r^G \eta)\right)} \right) \right| dr$$

$$= \int_{\theta_1^{k/\mu}}^{\theta_2^{k/\mu}} n \left| \partial_r \left(R(r^G \eta) + \widetilde{R}(r^G \eta) \right) \right| \left| e^{-n(R(r^G \eta) + \widetilde{R}(r^G \eta))} \right| dr$$

$$\leq \int_{\theta_1^{k/\mu}}^{\theta_2^{k/\mu}} n(1 + \epsilon) e^{-n(1-\epsilon)r} dr$$

$$\leq \frac{1 + \epsilon}{1 - \epsilon} \int_0^\infty e^{-r} dr = \beta$$

for all $n \in \mathbb{N}_+$, $\eta \in S$ and $0 < \theta_1 \leq \theta_2 \leq \delta^\mu$, as desired. □

Lemma 3.7 *Suppose that ξ_0 is of imaginary homogeneous type for $\widehat{\phi}$ with associated drift α_{ξ_0} and homogeneous order μ_{ξ_0}. If $\alpha_{\xi_0} = 0$ and $\mu_{\xi_0} < 1$, then, for each compact*

set K, there is an open neighborhood $\mathcal{O}_{\xi_0} \subseteq \text{Int}(\mathbb{T}^d_\phi)$ of ξ_0, which can be taken as small as desired, and a constant C_{ξ_0} for which

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_{\xi_0}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi} \, d\xi \right| \leq C_{\xi_0} n^{-\mu_{\xi_0}}$$

for all $x \in K$ and $n \in \mathbb{N}_+$.

Proof For simplicity of notation, we will write $Q = Q_{\xi_0}$, $\widetilde{Q} = \widetilde{Q}_{\xi_0}$, $R = R_{\xi_0}$, $\widetilde{R} = \widetilde{R}_{\xi_0}$ and $\mu = \mu_{\xi_0}$. We fix a compact set $K \subseteq \mathbb{R}^d$ and let E and k as given in Definition 3.1. In studying the proof of Lemma 3.3 and (19), in particular, it is evident that we may assume $\xi_0 = 0$ and $\widehat{\phi}(0) = 1$ without loss of generality. Given that $G := E/k \in \text{Exp}(R)$, set

$$F = (k/\mu)G = E/\mu.$$

Using the positive-homogeneous structure of R, let σ_R be the measure on $S = \{\eta \in \mathbb{R}^d : R(\eta) = 1\}$ as guaranteed by Theorem 1.4. By setting

$$\rho = \inf_{\eta \in S} |Q(\eta)|/3,$$

an appeal to Lemma 3.5 guarantees a $\delta_1 > 0$ for which

$$\left| \partial_\theta f_{n,\eta,x}(\theta) \right| \geq \frac{\rho}{\mu} n^\mu \text{ and } \partial_\theta^2 f_{n,\eta,x}(\theta) \neq 0 \tag{24}$$

for all $n \in \mathbb{N}_+$, $\eta \in S$, $x \in K$ and $\theta > 0$ for which $n^{-\mu} \leq \theta \leq \delta_1^\mu$. An appeal to Lemma 3.6 guarantees $\delta_2 > 0$ for which

$$\|g_{n,\eta}\|_{L^\infty[\theta_1,\theta_2]} + \|\partial_\theta g_{n,\eta}\|_{L^1[\theta_1,\theta_2]} \leq 3 \tag{25}$$

for all $n \in \mathbb{N}_+$, $\eta \in S$ and $0 < \theta_1 \leq \theta_2 \leq \delta_2^\mu$. We set $\mathcal{O} = \{\eta \in \mathbb{R}^d : R(\eta) < \delta^k\}$ where $0 < \delta \leq \min\{\delta_1, \delta_2\}$ is as small as desired; this is necessarily an open neighborhood of 0. We have

$$\int_{\mathcal{O}} \widehat{\phi}(\xi)^n e^{-ix\cdot\xi} \, d\xi = \int_S \int_0^{\delta^k} \widehat{\phi}(r^G \eta)^n e^{-ix\cdot r^G \eta} r^{\mu/k-1} \, dr \, \sigma_R(d\eta)$$

$$= \frac{k}{\mu} \int_S \int_0^{\delta^\mu} \widehat{\phi}(\theta^F \eta)^n e^{-ix\cdot\theta^F \eta} \, d\theta \, \sigma_R(d\eta)$$

$$= \frac{k}{\mu} \int_S I_{n,x}(\eta) \, \sigma_R(d\eta)$$

where we have made the change of variables $\theta = r^{\mu/k}$ and set

$$I_{n,x}(\eta) = \int_0^{\delta^\mu} \widehat{\phi}(\theta^F \eta)^n e^{-ix\cdot\theta^F \eta} \, d\theta.$$

For each $n \in \mathbb{N}_+$, $\eta \in S$, and $x \in \mathbb{R}^d$, we have

$$
\begin{aligned}
\left| I_{n,x}(\eta) \right| &\leq \left| \int_{n^{-\mu}}^{\delta^\mu} \widehat{\phi}(\theta^F \eta)^n e^{-ix \cdot \theta^F \eta} \, d\theta \right| + \int_0^{n^{-\mu}} \left| \widehat{\phi}(\theta^F \eta)^n \right| \, d\theta \\
&\leq \left| \int_{n^{-\mu}}^{\delta^\mu} e^{-i\left(nQ(\theta^F \eta) + n\widetilde{Q}(\theta^F \eta) + x \cdot \theta^F \eta\right)} e^{-n\left(R(\theta^F \eta) + \widetilde{R}(\theta^F \eta)\right)} \, d\theta \right| + n^{-\mu} \\
&= \left| \int_{n^{-\mu}}^{\delta^\mu} e^{if_{n,\eta,x}(\theta)} g_{n,\eta}(\theta) \, d\theta \right| + n^{-\mu}.
\end{aligned}
$$

In view of (24) and (25), an appeal to Proposition 3.4 guarantees that, for any $n \in \mathbb{N}_+$, $\eta \in S$ and $x \in K$,

$$
\begin{aligned}
\left| \int_{n^{-\mu}}^{\delta^\mu} e^{if_{n,\eta,x}(\theta)} g_{n,\eta}(\theta) \, d\theta \right| &\leq 4 \frac{\|g_{n,\eta}\|_{L^\infty[n^{-\mu},\delta^\mu]} + \|\partial_\theta g_{n,\eta}\|_{L^1[n^{-\mu},\delta^\mu]}}{\inf_{n^{-\mu} \leq \theta \leq \delta^\delta} |\partial_\theta f_{n,x,\eta}(\theta)|} \\
&\leq 4 \frac{3}{(\rho/\mu)n^\mu} = \frac{12\mu}{\rho} n^{-\mu}
\end{aligned}
$$

and so

$$
\left| I_{n,x}(\eta) \right| \leq \left(\frac{12\mu}{\rho} \right) n^{-\mu} + n^{-\mu} \leq \left(\frac{12\mu}{\rho} + 1 \right) n^{-\mu}.
$$

Thus, for all $n \in \mathbb{N}_+$ and $x \in K$,

$$
\begin{aligned}
\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}} \widehat{\phi}(\xi)^n e^{-i\xi \cdot x} \, d\xi \right| &= \frac{1}{(2\pi)^d} \frac{k}{\mu} \left| \int_S I_{n,x}(\eta) \, \sigma_R(d\eta) \right| \\
&\leq \frac{1}{(2\pi)^d} \frac{k}{\mu} \int_S \left| I_{n,x}(\eta) \right| \sigma_R(d\eta) \\
&\leq C n^{-\mu}
\end{aligned}
$$

where

$$
C = \frac{1}{(2\pi)^d} \frac{k}{\mu} \left(\frac{12\mu}{\rho} + 1 \right) \sigma_R(S).
$$

$\square$

Proof of Theorem 3.2 Let $K \subseteq \mathbb{R}^d$ be a compact set. As we discussed in the paragraph preceding the theorem, the set $\Omega(\phi)$ is finite and so we may write

$$
\Omega(\phi) = \{\xi_1, \xi_2, \ldots, \xi_N, \xi_{N+1}, \xi_{N+2}, \ldots, \xi_M\}
$$

where our labeling assumes that the points $\xi_1, \xi_2, \ldots, \xi_N$ are of imaginary homogeneous type for $\widehat{\phi}$ and the points $\xi_{N+1}, \xi_{N+2}, \ldots, \xi_M$ are of positive homogeneous type

for $\widehat{\phi}$. In view of the theorem's hypotheses, for each $j = 1, 2, \ldots, N$, the point ξ_j, which is of imaginary homogeneous type for $\widehat{\phi}$, has drift $\alpha_{\xi_j} = 0$ and homogeneous order $\mu_j := \mu_{\xi_j} < 1$. Thus, for each $j = 1, 2, \ldots, N$, an appeal to Lemma 3.7 guarantees an open neighborhood $\mathcal{O}_j = \mathcal{O}_{\xi_j} \subseteq \mathrm{Int}(\mathbb{T}_\phi^d)$ of ξ_j and a constant $C_j = C_{\xi_j}$ for which

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_j} \widehat{\phi}(\xi)^n e^{-ix \cdot \xi} \, d\xi \right| \leq C_j n^{-\mu_j} \tag{26}$$

for all $n \in \mathbb{N}_+$ and $x \in K$. For each $j = N+1, N+2, \ldots M$, an appeal to Lemma 3.3 guarantees an open neighborhood $\mathcal{O}_j = \mathcal{O}_{\xi_j} \subseteq \mathrm{Int}(\mathbb{T}_\phi^d)$ of ξ_j and a constant $C_j = C_{\xi_j}$ for which

$$\left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_j} \widehat{\phi}(\xi)^n e^{-ix \cdot \xi} \, d\xi \right| \leq C_j n^{-\mu_j} \tag{27}$$

for all $n \in \mathbb{N}_+$ and $x \in \mathbb{R}^d$ where $\mu_j := \mu_{\xi_j}$ is the homogeneous order associated to ξ_j. As guaranteed by the lemmas, let us take this collection of open sets $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_M \subseteq \mathbb{T}_\phi^d$ to be mutually disjoint and define

$$\mathcal{G} = \mathbb{T}_\phi^d \setminus \left(\bigcup_{j=1}^{M} \mathcal{O}_j \right). \tag{28}$$

Given that $\mathcal{G}$ is a closed set which contains no elements of $\Omega(\phi)$,

$$s := \sup_{\xi \in \mathcal{G}} \left| \widehat{\phi}(\xi) \right| < 1.$$

By virtue of (17), (26), (27), and the disjointness of the collection $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_M$, we have

$$\left| \phi^{(n)}(x) \right| = \left| \left[\sum_{j=1}^{M} \frac{1}{(2\pi)^d} \int_{\mathcal{O}_j} \widehat{\phi}(\xi)^n e^{-ix \cdot \xi} \, d\xi \right] + \frac{1}{(2\pi)^d} \int_{\mathcal{G}} \widehat{\phi}(\xi)^n e^{-ix \cdot \xi} \, d\xi \right|$$

$$\leq \sum_{j=1}^{M} \left| \frac{1}{(2\pi)^d} \int_{\mathcal{O}_j} \widehat{\phi}(\xi)^n e^{-ix \cdot \xi} \, d\xi \right| + \left| \frac{1}{(2\pi)^d} \int_{\mathcal{G}} \widehat{\phi}(\xi)^n e^{-ix \cdot \xi} \, d\xi \right|$$

$$\leq \sum_{j=1}^{M} C_j n^{-\mu_j} + s^n \tag{29}$$

for all $n \in \mathbb{N}_+$ and $x \in K$. Upon noting that $\mu_\phi = \min\{\mu_1, \mu_2, \ldots, \mu_M\}$, we have

$$n^{-\mu_j} = O(n^{-\mu_\phi})$$

as $n \to \infty$ for each $j = 1, 2, \ldots M$. Also, because $s < 1$, $s^n = o(n^{-\mu_\phi})$ as $n \to \infty$. With these two observations, the theorem follows immediately from (29). $\qquad\square$

3.1 Examples

In this subsection, we give a number of examples illustrating the results of Theorem 3.2, all of which are beyond the scope of validity of the results of [24]. First, we treat a useful proposition which gives sufficient conditions for a point $\xi_0 \in \Omega(\phi)$ to be of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$ in terms of the Taylor expansion for Γ_{ξ_0}.

Proposition 3.8 *Let $\phi \in \mathcal{S}_d$ with $\sup_\xi |\widehat{\phi}(\xi)| = 1$ and let $\xi_0 \in \Omega(\phi)$. Suppose that there exists $\mathbf{m} \in \mathbb{N}_+^d$ and some $k \geq 1$ such that the Taylor expansion of $\Gamma_{\xi_0} : \mathcal{U} \to \mathbb{C}$ centered at 0 is a series of the form*

$$
\begin{aligned}
\Gamma_{\xi_0}(\xi) &= i\alpha_{\xi_0} \cdot \xi - i \left(\sum_{|\beta:2\mathbf{m}|\geq 1} A_\beta \xi^\beta \right) - \sum_{|\beta:2\mathbf{m}|\geq k} B_\beta \xi^\beta \\
&= i\alpha_{\xi_0} \cdot \xi - i \left(\sum_{|\beta:2\mathbf{m}|=1} A_\beta \xi^\beta + \sum_{|\beta:2\mathbf{m}|>1} A_\beta \xi^\beta \right) \\
&\quad - \left(\sum_{|\beta:2\mathbf{m}|=k} B_\beta \xi^\beta + \sum_{|\beta:2\mathbf{m}|>k} B_\beta \xi^\beta \right) \\
&= i\alpha_{\xi_0} \cdot \xi - i \left(Q_{\xi_0}(\xi) + \widetilde{Q}_{\xi_0}(\xi) \right) - \left(R_{\xi_0}(\xi) + \widetilde{R}_{\xi_0}(\xi) \right),
\end{aligned} \tag{30}
$$

where $\alpha_{\xi_0} \in \mathbb{R}^d$; Q_{ξ_0} and R_{ξ_0} are real-valued polynomials for which R_{ξ_0} is positive definite; and $\widetilde{Q}_{\xi_0}$, and $\widetilde{R}_{\xi_0}$ are real multivariate power series which are absolutely and uniformly convergent on $\mathcal{U}$. If $k = 1$, then ξ_0 is of positive homogeneous type for $\widehat{\phi}$. If $k > 1$ and $|Q_{\xi_0}|$ is positive definite, then ξ_0 is of imaginary homogeneous type for $\widehat{\phi}$. In either case, ξ_0 has drift α_{ξ_0} and homogeneous order

$$
\mu_{\xi_0} = |\mathbf{1} : 2\mathbf{m}| = \sum_{j=1}^{d} \frac{1}{2m_j}.
$$

Before proving the proposition, we shall first take care of the following useful lemma.

Lemma 3.9 *Given an open neighborhood $\mathcal{U}$ of 0 in $\mathbb{R}^d$, suppose that $Q : \mathcal{U} \to \mathbb{C}$ is real-analytic on $\mathcal{U}$ with absolutely and uniformly convergent series expansion*

$$
Q(\xi) = \sum_{|\beta:\mathbf{n}|>1} A_\beta \xi^\beta
$$

for some $\mathbf{n} \in \mathbb{N}_+^d$. *Consider* $E \in \mathrm{End}(\mathbb{R}^d)$ *with standard representation* $\mathrm{diag}(1/n_1, 1/n_2, \dots, 1/n_d)$. *Then, for each* $l \in \mathbb{N}_+$, Q *is strongly subhomogeneous with respect to* E *of order* l.

Proof It suffices to show that, for each, $j \in \mathbb{N}$, $\epsilon > 0$ and compact set $K \subseteq \mathbb{R}^d$, there is a $\delta > 0$ for which

$$\left| r^j \partial_r^j Q(r^E \eta) \right| \leq \epsilon r$$

for all $0 < r < \delta$ and $\eta \in K$. To this end, we fix j, ϵ, and K as above and write $Q = Q_1 + Q_2$ where

$$Q_1(\xi) = \sum_{1+\rho \leq |\beta:\mathbf{n}| \leq 2j+4} A_\beta \xi^\beta \quad \text{and} \quad Q_2(\xi) = \sum_{|\beta:\mathbf{n}| > 2j+4} A_\beta \xi^\beta$$

where $\rho := \min\{|\beta : \mathbf{n}| : A_\beta \neq 0\} - 1 > 0$. For each $q \geq 1$ and $l \in \mathbb{N}$, define

$$\mathcal{P}(q, l) = q(q-1)(q-2)\cdots(q-(l-1)).$$

where we assume the convention that $\mathcal{P}(q, 0) = 1$. In this notation, we observe that

$$\partial_r^j (r^E \xi)^\beta = \partial_r^j \left(r^{|\beta:\mathbf{n}|} \xi^\beta \right) = \mathcal{P}(|\beta : \mathbf{n}|, j) r^{|\beta:\mathbf{n}| - j} \xi^\beta$$

for $\xi \in \mathbb{R}^d$, $r > 0$ and $\beta \in \mathbb{N}^d$. Because Q_1 is a polynomial and K is compact, we have

$$M_1 := \sup_{\eta \in K} \left(\sum_{1+\rho \leq |\beta:\mathbf{n}| \leq 2j+4} \left| A_\beta \mathcal{P}(|\beta : \mathbf{n}|, j) \eta^\beta \right| \right) < \infty.$$

Given that Q is absolutely and uniformly convergent on $\mathcal{U}$, let $\mathcal{O} \subseteq \overline{\mathcal{O}} \subseteq \mathcal{U}$ be an open neighborhood of 0 and set

$$M_2 := \sup_{\xi \in \overline{\mathcal{O}}} \left(\sum_{|\beta:\mathbf{n}| > 2j+4} \left| A_\beta \xi^\beta \right| \right) < \infty.$$

We now specify δ. First, given that $\{r^E\}$ and $\{r^{E/4}\}$ are contracting and the set K is compact, we may find a $0 < \delta_1 \leq 1$ for which $r^E \eta$ and $r^{E/4} \eta$ belong to $\mathcal{O}$ whenever $0 < r < \delta_1$ and $\eta \in K$. Also, there exists $\delta_2 > 0$ for which

$$|\mathcal{P}(q, j)| r^{q/4} \leq 1 \tag{31}$$

for all $q > j$ and $0 < r \leq \delta_2$; it is sufficient to take $\delta_2 = e^{-4j}$. Finally, given that $\rho > 0$, let $\delta_3 > 0$ be such that

$$M_1 r^\rho + M_2 r < \epsilon$$

for all $0 < r < \delta_3$. Set $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and observe that for all $\eta \in K$ and $0 < r < \delta$, we have

$$
\begin{aligned}
\left| r^j \partial_r^j Q_1(r^E \eta) \right| &= r^j \left| \sum_{1+\rho \leq |\beta:\mathbf{n}| \leq 2j+4} A_\beta \partial_r^j \left(r^E \eta \right)^\beta \right| \\
&\leq r^j \sum_{1+\rho \leq |\beta:\mathbf{n}| \leq 2j+4} \left| A_\beta \mathcal{P}(|\beta:\mathbf{n}|, j) r^{|\beta:\mathbf{n}|-j} \eta^\beta \right| \\
&\leq r^{1+\rho} \sum_{1+\rho \leq |\beta:\mathbf{n}| \leq 2j+4} \left| A_\beta \mathcal{P}(|\beta:\mathbf{n}|, j) \eta^\beta \right| \\
&\leq r M_1 r^\rho.
\end{aligned}
$$

By virtue of (31), for each $q = |\beta : \mathbf{n}| > 2j + 4$, we have

$$
\begin{aligned}
\left| \partial_r^j \left(A_\beta (r^E \eta)^\beta \right) \right| &= |A_\beta| \, |\mathcal{P}(|\beta:\mathbf{n}|, j)| \, r^{|\beta:\mathbf{n}|-j} \left| \eta^\beta \right| \\
&= r^{|\beta:\mathbf{n}|/2 - j} |A_\beta| \left| \mathcal{P}(|\beta:\mathbf{n}|, j) r^{|\beta:\mathbf{n}|/4} \right| \left| (r^{E/4} \eta)^\beta \right| \\
&\leq r^2 \left| A_\beta (r^{E/4} \eta)^\beta \right|
\end{aligned}
$$

for all $0 < r < \delta \leq \delta_2$ and $\eta \in K$. It follows that

$$
\begin{aligned}
\left| \partial_r^j Q_2(r^E \eta) \right| &= \left| \sum_{|\beta:\mathbf{n}|>2j+4} \partial_r^j \left(A_\beta (r^E \eta)^\beta \right) \right| \\
&\leq \sum_{|\beta:\mathbf{n}|>2j+4} \left| \partial_r^j \left(A_\beta (r^E \eta)^\beta \right) \right| \\
&\leq \sum_{|\beta:\mathbf{n}|>2j+4} r^2 \left| A_\beta (r^{E/4} \eta)^\beta \right| \\
&\leq r^2 M_2
\end{aligned}
$$

for all $0 < r < \delta$ and $\eta \in K$. Therefore, for each $0 < r < \delta$ and $\eta \in K$, we have

$$
\begin{aligned}
\left| r^j \partial_r^j Q(r^E \eta) \right| &\leq \left| r^j \partial_r^j Q_1(r^E \eta) \right| + \left| r^j \partial_r^j Q_2(r^E \eta) \right| \\
&\leq r r^\rho M_1 + r^{j+2} M_2 \\
&\leq r(M_1 r^\rho + M_2 r) \\
&< r\epsilon.
\end{aligned}
$$

$\square$

Proof of Proposition 3.8. It is easy to see that $E \in \text{Exp}(Q_{\xi_0}) \cap \text{Exp}(|Q_{\xi_0}|)$ and $E/k \in \text{Exp}(R_{\xi_0})$ for $E \in \text{End}(\mathbb{R}^d)$ with standard matrix representation

$$\text{diag}((2m_1)^{-1}, (2m_2)^{-1}, \ldots, (2m_d)^{-1}).$$

If $k = 1$, R_{ξ_0} is positive homogeneous with $E \in \text{Exp}(R_{\xi_0}) \cap \text{Exp}(Q_{\xi_0})$. By virtue of the preceding lemma (with $\mathbf{n} = 2\mathbf{m}$), $\widetilde{Q}_{\xi_0}$ and $\widetilde{R}_{\xi_0}$ are strongly subhomogeneous with respect to E of order 1. In this case, we may conclude that ξ_0 is of positive homogeneous type for $\widehat{\phi}$ with drift α_{ξ_0} and homogeneous order

$$\mu_{\xi_0} = \text{tr}\,E = |\mathbf{1} : 2\mathbf{m}| = \sum_{j=1}^{d} \frac{1}{2m_j}.$$

If $k > 1$, our supposition guarantees that $|Q_{\xi_0}|$ is positive homogeneous with respect to E and R_{ξ_0} is positive homogeneous with respect to E/k. By virtue of the preceding lemma, $\widetilde{Q}_{\xi_0}$ is strongly subhomogeneous with respect to E of order 2 and $\widetilde{R}_{\xi_0}$ is strongly subhomogeneous with respect to E/k of order 1. Consequently, ξ_0 is of imaginary homogeneous type for $\widehat{\phi}$ with drift α_{ξ_0} and homogeneous order $\mu_{\xi_0} = \text{tr}\,E = |\mathbf{1} : 2\mathbf{m}|$ as in the previous case. $\qquad\square$

Example 5 Consider the function $\phi : \mathbb{Z}^2 \to \mathbb{C}$ defined by

$$\phi(x, y) = \frac{1}{512} \times \begin{cases} 372 - 96i & (x, y) = (0, 0) \\ 56 + 32i & (x, y) = (\pm 1, 0) \text{ or } (0, \pm 1) \\ -28 - 8i & (x, y) = (\pm 2, 0) \text{ or } (0, \pm 2) \\ 8 & (x, y) = (\pm 3, 0) \text{ or } (0, \pm 3) \\ -1 & (x, y) = (\pm 4, 0) \text{ or } (0, \pm 4) \\ 0 & \text{otherwise.} \end{cases}$$

It is easily verified that $\sup_\xi |\widehat{\phi}| = 1$ and $\Omega(\phi) = \{\xi_0\}$ where $\xi_0 = (0, 0)$. Since ϕ is finitely supported, $\Gamma_0 = \Gamma_{\xi_0}$ is holomorphic and so its Taylor series converges absolutely and uniformly on an open neighborhood $\mathcal{U} \subseteq \mathbb{R}^2$ of 0. By a straightforward computation, we find

$$\begin{aligned} \Gamma_0(\xi) &= -i\left(\frac{\tau^4}{64} + \frac{\zeta^4}{64}\right) - i \sum_{|\beta : (4,4)| \geq 2} A_\beta \xi^\beta \\ &\quad - \left(\frac{15\tau^8}{8192} - \frac{\tau^4 \zeta^4}{4096} + \frac{15\zeta^8}{8192}\right) - \sum_{|\beta : (4,4)| \geq 6} B_\beta \xi^\beta \\ &= -i\left(Q_0(\xi) + \widetilde{Q}_0(\xi)\right) - \left(R_0(\xi) + \widetilde{R}_0(\xi)\right) \end{aligned}$$

where

$$
Q_0(\xi) = \sum_{|\beta:(4,4)|=1} A_\beta \xi^\beta = \frac{\tau^4}{64} + \frac{\zeta^4}{64},
$$

$$
R_0(\xi) = \sum_{|\beta:(4,4)|=2} B_\beta \xi^\beta = \frac{15\tau^8}{8192} - \frac{\tau^4 \zeta^4}{4096} + \frac{15\zeta^8}{8192},
$$

$$
\widetilde{Q}_0(\xi) = \sum_{|\beta:(4,4)|\geq 3/2} A_\beta \xi^\beta = -\frac{\tau^6 + \zeta^6}{384} + \frac{\tau^8 + \zeta^8}{5120} + \frac{7(\tau^4 \zeta^8 + \tau^8 \zeta^4)}{262144} + \cdots,
$$

and

$$
\widetilde{R}_0(\xi) = \sum_{|\beta:(4,4)|\geq 5/2} B_\beta \xi^\beta = \frac{(\tau^4 \zeta^6 + \tau^6 \zeta^4)}{24576} - \frac{\tau^6 \zeta^6}{147456} - \frac{(\tau^4 \zeta^8 + \tau^8 \zeta^4)}{327680} \cdots,
$$

for $\xi = (\tau, \zeta) \in \mathcal{U}$. Observe that this expansion is of the form (30) with $\alpha_0 = (0, 0)$, $\mathbf{m} = (2, 2)$, and $k = 2$. It is readily verified that $|Q_0| = Q_0$ and R_0 are positive definite and, by virtue of Proposition 3.8, we conclude that $\xi_0 = 0$ is of imaginary homogeneous type for $\widehat{\phi}$ with drift $\alpha_0 = 0$ and homogeneous order

$$
\mu_\phi = \mu_0 = |\mathbf{1} : 2\mathbf{m}| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.
$$

By an appeal to Theorem 3.2 we obtain, to each compact set $K \subseteq \mathbb{R}^2$, a positive constant C for which

$$
|\phi^{(n)}(x, y)| \leq \frac{C}{n^{\mu_\phi}} = \frac{C}{n^{1/2}} \tag{32}
$$

for all $n \in \mathbb{N}_+$ and $(x, y) \in K$. To illustrate this result, we consider the compact set $K = [-700, 700] \times [-700, 700]$ and define $f(n) = f_{\phi,K}(n) = \max_{(x,y)\in K} |\phi^{(n)}(x, y)|$. Figure 5 illustrates this result by capturing the decay of $f(n) = f_{\phi,K}(n) = \max_{(x,y)\in K} |\phi^{(n)}(x, y)|$ relative to that of $n^{-\mu_\phi}$. Also, Fig. 6 illustrates the graph of $\mathrm{Re}\, \phi^{(n)}(x, y)$ for $(x, y) \in K$ and $n = 200$ and $n = 1000$.
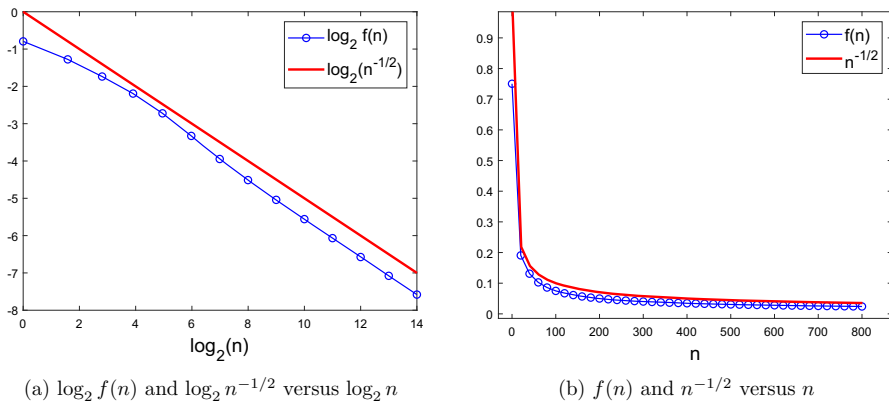
(a) $\log_2 f(n)$ and $\log_2 n^{-1/2}$ versus $\log_2 n$

(b) $f(n)$ and $n^{-1/2}$ versus n

Fig. 5 Behavior of $f(n) = f_{\phi,K}(n) = \max_{(x,y)\in K} \left| \phi^{(n)}(x,y) \right|$

(a) $n = 100$.

(b) $n = 1000$.

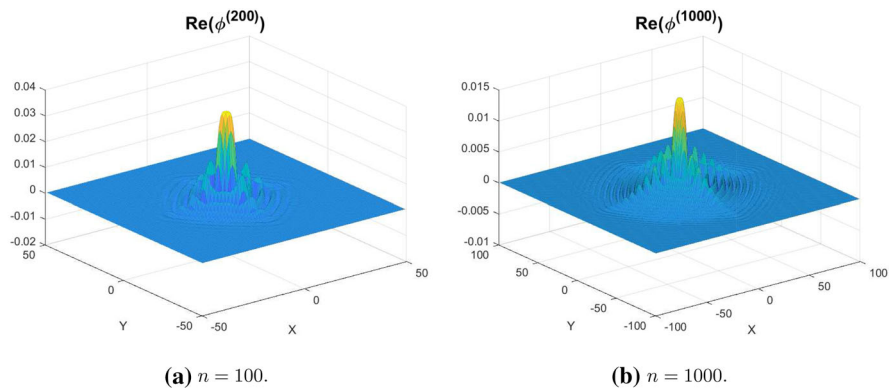**Fig. 6** $\operatorname{Re} \phi^{(n)}(x,y)$ for $n = 200$ and 1000

Example 6 Consider the function $\phi : \mathbb{Z}^2 \to \mathbb{C}$ defined by

$$
\phi(x,y) = \frac{1}{768} \times
\begin{cases}
602 - 112i & (x,y) = (0,0) \\
56 + 32i & (x,y) = (0,\pm 1) \text{ or } (-1,0) \\
72 + 32i & (x,y) = (1,0) \\
-28 - 8i & (x,y) = (0,\pm 2) \\
-16 & (x,y) = (\pm 2, 0) \\
56 & (x,y) = (0,\pm 3) \\
-1 & (x,y) = (0,\pm 4) \\
4 & (x,y) = (-1,\pm 1) \\
-4 & (x,y) = (1,\pm 1) \\
0 & \text{otherwise.}
\end{cases}
$$

As with the preceding examples, it is easy to see that $\sup_\xi |\widehat{\phi}(\xi)| = 1$, $\Omega(\phi) = \{\xi_0\} = \{(0,0)\}$ and $\Gamma_0 = \Gamma_{\xi_0}$ has the absolutely and uniformly convergent Taylor expansion

$$\Gamma_0(\xi) = -i\left(Q_0(\xi) + \widetilde{Q}_0(\xi)\right) - \left(R_0(\xi) + \widetilde{R}_0(\xi)\right)$$

where

$$Q_0(\xi) = \sum_{|\beta:(2,4)|=1} A_\beta \xi^\beta = \frac{\tau^2}{24} - \frac{\tau\zeta^2}{96} + \frac{\zeta^4}{96},$$

$$R_0(\xi) = \sum_{|\beta:(2,4)|=2} B_\beta \xi^\beta = \frac{23\tau^4}{1152} + \frac{\tau^3\zeta^2}{2304} - \frac{\tau^2\zeta^4}{2048} + \frac{\tau\zeta^6}{9216} + \frac{23\zeta^8}{18432},$$

$$\widetilde{Q}_0(\xi) = \sum_{|\beta:(2,4)|\geq 3/2} A_\beta \xi^\beta = -\frac{\tau^4}{288} + \frac{\tau\zeta^4}{1152} + \frac{\tau^3\zeta^2}{576} - \frac{\tau^3\zeta^4}{6912} + \cdots,$$

and

$$\widetilde{R}_0(\xi) = \sum_{|\beta:(2,4)|\geq 5/2} B_\beta \xi^\beta = -\frac{\tau^3\zeta^4}{27648} + \frac{\tau^4\zeta^4}{18432} + \frac{\tau^4\zeta^4}{18432} + \cdots$$

for $\xi = (\tau, \zeta) \in \mathcal{U}$ where $\mathcal{U} \subseteq \mathbb{R}^2$ is a neighborhood of 0. In this case, the above expansion is of the form (30) with $\alpha_0 = (0,0)$, $\mathbf{m} = (1,2)$ and $k = 2$. Here, as with the previous examples, it is readily verified that $|Q_0| = Q_0$ and R_0 are positive definite and so an appeal to Proposition 3.8 guarantees that $\xi_0 = (0,0)$ is of imaginary homogeneous type for $\widehat{\phi}$ with drift $\alpha_0 = (0,0)$ and homogeneous order

$$\mu_\phi = \mu_0 = |\mathbf{1}:2\mathbf{m}| = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

By an appeal to Theorem 3.2, we obtain, to each compact set K, a positive constant C for which

$$\left|\phi^{(n)}(x,y)\right| \leq \frac{C}{n^{\mu_\phi}} = \frac{C}{n^{3/4}} \tag{33}$$

for all $n \in \mathbb{N}$ and $(x,y) \in K$. Figure 7 illustrates this result by capturing the decay of $f(n) = f_{\phi,K}(n) = \max_{(x,y)\in K}\left|\phi^{(n)}(x,y)\right|$ where $K = [-300, 300] \times [-300, 300]$ relative to that of $n^{-\mu_\phi}$. Also, Fig. 8 illustrates the graph of $\mathrm{Re}\,\phi^{(n)}(x,y)$ for $(x,y) \in K$ and $n = 300$ and $n = 600$.

Example 7 This example illustrates a complex-valued function ϕ on $\mathbb{Z}^2$ whose Fourier transform is maximized in absolute value at two distinct points in $\mathbb{T}^2$, one of which is a point of imaginary homogeneous type for $\widehat{\phi}$ with homogeneous order 2/3 and the
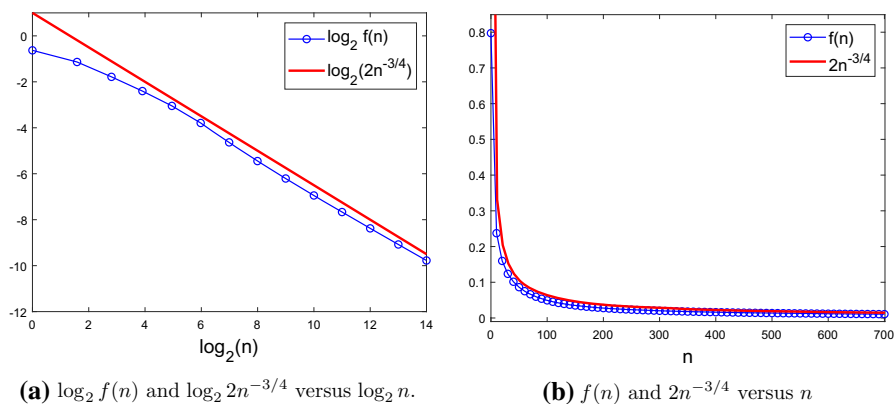
(a) $\log_2 f(n)$ and $\log_2 2n^{-3/4}$ versus $\log_2 n$.   **(b)** $f(n)$ and $2n^{-3/4}$ versus n

Fig. 7 Behavior of $f(n) = f_{\phi,K}(n) = \max_{(x,y) \in K} \left| \phi^{(n)}(x,y) \right|$
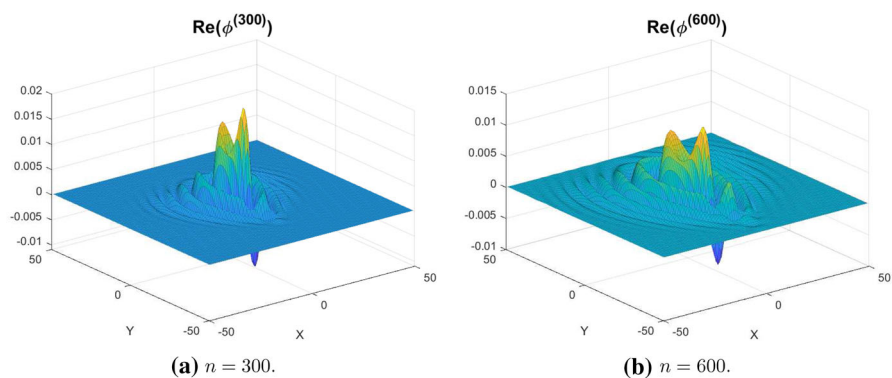

(a) $n = 300$.   **(b)** $n = 600$.

Fig. 8 Re $\phi^{(n)}$ for $n = 300$ and $n = 600$

other is a point of positive homogeneous type $\widehat{\phi}$ of homogeneous order 1. We define $\phi : \mathbb{Z}^2 \to \mathbb{C}$ by $\phi = 2^{-7}\phi_1 - i2^{-11}\phi_2 + 2^{-21}\phi_3$ where

$$\phi_1(x,y) = \begin{cases} 15 + 15i & (x,y) = (\pm 1, 0) \\ 16 + 16i & (x,y) = (0, \pm 1) \\ 1 + 1i & (x,y) = (\pm 3, 0) \\ 0 & \text{otherwise} \end{cases}, \quad \phi_2(x,y) = \begin{cases} 682 & (x,y) = (0,0) \\ 152 & (x,y) = (\pm 2, 0) \\ -28 & (x,y) = (\pm 4, 0) \\ 8 & (x,y) = (\pm 6, 0) \\ -1 & (x,y) = (\pm 8, 0) \\ 60 & (x,y) = (0, \pm 2) \\ -24 & (x,y) = (0, \pm 4) \\ 4 & (x,y) = (0, \pm 6) \\ 0 & \text{otherwise} \end{cases},$$

and

$$
\phi_3(x, y) = \begin{cases}
1387004 & (x, y) = (0, 0) \\
-106722 & (x, y) = (\pm 2, 0) \\
3960 & (x, y) = (\pm 4, 0) \\
-1045 & (x, y) = (\pm 6, 0) \\
138 & (x, y) = (\pm 8, 0) \\
-9 & (x, y) = (\pm 10, 0) \\
-131072 & (x, y) = (0, \pm 2) \\
0 & \text{otherwise}
\end{cases}
$$

for $(x, y) \in \mathbb{Z}^2$. Though this example is slightly more complicated than the previous ones considered, it is straightforward to verify that $\sup_\xi |\widehat{\phi}(\xi)| = 1$ and, in this case, the supremum is attained at two points in $\mathbb{T}^2$. Specifically, $\Omega(\phi) = \{\xi_1, \xi_2\}$ where $\xi_1 = (0, 0)$ and $\xi_2 = (\pi, \pi)$. For ξ_1, $\Gamma_1 = \Gamma_{\xi_1}$ has an absolutely and uniformly convergent Taylor series of the form

$$
\Gamma_1(\xi) = -i \left(Q_1(\xi) - \widetilde{Q_1}(\xi) \right) - \left(R_1(\xi) + \widetilde{R_1}(\xi) \right)
$$

for $\xi = (\tau, \zeta) \in \mathcal{U}_1$ where $\mathcal{U}_1 \subseteq \mathbb{R}^2$ is an open neighborhood of $(0, 0)$ and

$$
Q_1(\xi) = \sum_{|\beta:(6,2)|=1} A_\beta \xi^\beta = \frac{\tau^6}{128} + \frac{\zeta^2}{8},
$$

$$
R_1(\xi) = \sum_{|\beta:(6,2)|=2} B_\beta \xi^\beta = \frac{111\tau^{12}}{32768} - \frac{\tau^6 \zeta^2}{1024} + \frac{3\zeta^4}{128},
$$

$$
\widetilde{Q_1}(\xi) = \sum_{|\beta:(6,2)|\geq 4/3} A_\beta \xi^\beta = -\frac{65\tau^8}{512} - \frac{\zeta^4}{96} + \frac{\tau^6 \zeta^4}{8192} + \cdots,
$$

and

$$
\widetilde{R_1}(\xi) = \sum_{|\beta:(6,2)|\geq 7/3} B_\beta \xi^\beta = \frac{65\tau^8 \zeta^2}{4096} + \frac{\tau^6 \zeta^4}{12288} + \cdots
$$

for $\xi = (\tau, \zeta) \in \mathcal{U}_1$. It is straightforward to verify that $Q_1 = |Q_1|$ and R_1 are positive definite and so Proposition 3.8 guarantees that $\xi_1 = (0, 0)$ is of imaginary homogeneous type for $\widehat{\phi}$ with $\mathbf{m}_1 = (3, 1)$, $k_1 = 2$, drift $\alpha_{\xi_1} = (0, 0)$ and homogeneous order

$$
\mu_{\xi_1} = |\mathbf{1} : 2\mathbf{m}_1| = \frac{1}{6} + \frac{1}{2} = \frac{2}{3}.
$$

For $\xi_2 = (\pi, \pi)$, $\Gamma_2 = \Gamma_{\xi_2}$ has an absolutely and uniformly convergent Taylor series of the form

$$\Gamma_2(\xi) = -i\left(Q_2(\xi) + \widetilde{Q}_2(\xi)\right) - \left(R_2(\xi) + \widetilde{R}_2(\xi)\right)$$

for $\xi = (\tau, \zeta) \in \mathcal{U}_2$ where $\mathcal{U}_2 \subseteq \mathbb{R}^2$ is an open neighborhood of $(0, 0)$ and

$$Q_2(\xi) = \sum_{|\beta:(2,2)|=1} A_\beta \xi^\beta = -\left(\frac{3\tau^2}{8} + \frac{\zeta^2}{4}\right),$$

$$R_2(\xi) = \sum_{|\beta:(2,2)|=1} B_\beta \xi^\beta = \frac{\tau^2}{8} + \frac{3\zeta^2}{8},$$

$$\widetilde{Q}_2(\xi) = \sum_{|\beta:(2,2)|\geq 2} A_\beta \xi^\beta = \frac{\tau^4}{64} - \frac{9\tau^2\zeta^2}{64} + \frac{\zeta^4}{48} + \cdots,$$

and

$$\widetilde{R}_2(\xi) = \sum_{|\beta:(2,2)|\geq 2} B_\beta \xi^\beta = -\frac{\tau^4}{8} - \frac{3\tau^2\zeta^2}{64} - \frac{13\zeta^4}{384} + \cdots,$$

for $\xi = (\tau, \zeta) \in \mathcal{U}_2$. Thus, the expansion is of the form (30) with $\mathbf{m}_2 = (1, 1)$ and $k_2 = 1$. Since R_2 is clearly positive definite, Proposition 3.8 guarantees that $\xi_2 = (\pi, \pi)$ is of positive homogeneous type for $\widehat{\phi}$ with drift $\alpha_{\xi_2} = (0, 0)$ and homogeneous order

$$\mu_{\xi_2} = |\mathbf{1} : 2\mathbf{m}_2| = \frac{1}{2} + \frac{1}{2} = 1.$$

Upon noting that $\mu_\phi = \min\{\mu_{\xi_1}, \mu_{\xi_2}\} = 2/3$, an appeal to Theorem 3.2 guarantees, to each compact set K, a constant C for which

$$\left|\phi^{(n)}(x, y)\right| \leq \frac{C}{n^{\mu_\phi}} = \frac{C}{n^{2/3}} \tag{34}$$

for all $n \in \mathbb{N}$ and $(x, y) \in K$. Figure 9 illustrates this result by capturing the decay of $f(n) = f_{\phi,K}(n) = \max_{(x,y)\in K}\left|\phi^{(n)}(x, y)\right|$ where $K = [-500, 500] \times [-500, 500]$ relative to that of $n^{-\mu_\phi}$. Also, Fig. 10 illustrates the graph of $\mathrm{Re}\,\phi^{(n)}(x, y)$ for $(x, y) \in K$ and $n = 200$ and $n = 700$. $\qquad\square$
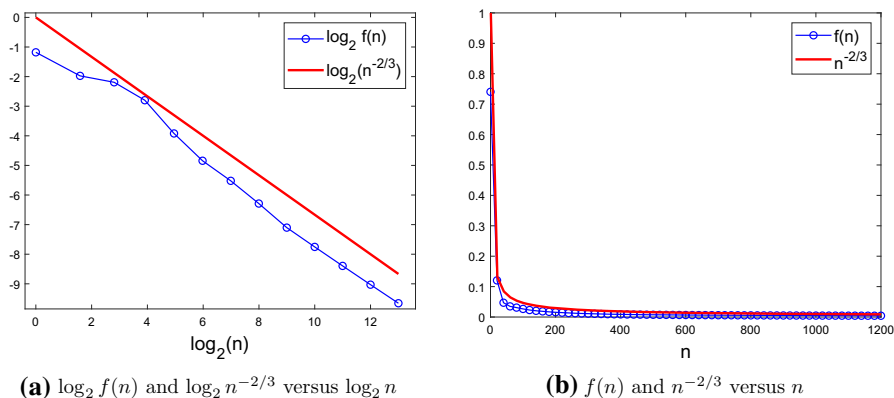
(a) $\log_2 f(n)$ and $\log_2 n^{-2/3}$ versus $\log_2 n$ **(b)** $f(n)$ and $n^{-2/3}$ versus n

Fig. 9 Behavior of $f(n) = f_{\phi, K}(n) = \max_{(x,y) \in K} \left| \phi^{(n)}(x, y) \right|$
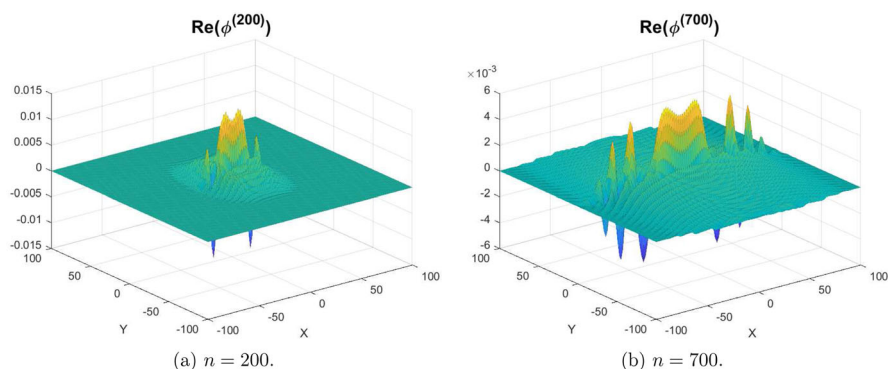

(a) $n = 200$.   (b) $n = 700$.

Fig. 10 Re $\phi^{(n)}$ for $n = 200$ and $n = 700$

4 Proof of Theorem 1.4

4.1 Construction of $\sigma_{P,E}$

Throughout this section, we fix $E \in \mathrm{Exp}(P)$. Define $\psi_E : (0, \infty) \times S \to \mathbb{R}^d \setminus \{0\}$ by

$$\psi_E(r, \eta) = r^E \eta \tag{35}$$

for $r > 0$ and $\eta \in S$. As ψ_E is the restriction of the continuous function $(0, \infty) \times \mathbb{R}^d \ni (r, x) \mapsto r^E x \in \mathbb{R}^d$ to $(0, \infty) \times S$, it is necessarily continuous. As the following proposition shows, ψ_E is, in fact, a homeomorphism.

Proposition 4.1 *The map* $\psi_E : (0, \infty) \times S \to \mathbb{R}^d \setminus \{0\}$ *is a homeomorphism with continuous inverse* $\psi_E^{-1} : \mathbb{R}^d \setminus \{0\} \to (0, \infty) \times S$ *given by*

$$\psi_E^{-1}(x) = (P(x), (P(x))^{-E}x)$$

for $x \in \mathbb{R}^d \setminus \{0\}$.

Proof Given that P is continuous and positive definite, $P(x) > 0$ for each $x \in \mathbb{R}^d \setminus \{0\}$ and the map $\mathbb{R}^d \setminus \{0\} \ni x \mapsto (P(x))^{-E}x \in \mathbb{R}^d$ is continuous. Further, in view of the homogeneity of P,

$$P\left((P(x))^{-E}x\right) = P(x)^{-1}P(x) = 1$$

for all $x \in \mathbb{R}^d \setminus \{0\}$. It follows from these two observations that

$$\rho(x) = (P(x), (P(x))^{-E}x),$$

defined for $x \in \mathbb{R}^d \setminus \{0\}$, is a continuous function taking $\mathbb{R}^d \setminus \{0\}$ into $(0, \infty) \times S$. We have

$$(\psi_E \circ \rho)(x) = \psi_E(P(x), (P(x))^{-E}x) = (P(x))^E (P(x))^{-E}x = x$$

for every $x \in \mathbb{R}^d \setminus \{0\}$ and

$$(\rho \circ \psi_E)(r, \eta) = \rho(r^E \eta) = (P(r^E \eta), (P(r^E \eta))^{-E}(r^E \eta))$$
$$= (r P(\eta), (r P(\eta))^{-E}(r^E \eta)) = (r, \eta)$$

for every $(r, \eta) \in (0, \infty) \times S$. Thus ρ is a (continuous) inverse for ψ_E and so it follows that ψ_E is a homeomorphism and $\rho = \psi_E^{-1}$. □

We shall now construct the σ-algebra $\Sigma_{P,E}$ on S; later, we will show that it is independent of our choice of E. As in the statement of Theorem 1.4, for each $F \subseteq S$, define

$$\widetilde{F_E} = \bigcup_{0 < r < 1} \left(r^E F\right) = \{r^E \eta : 0 < r < 1, \eta \in F\}.$$

We shall denote by $\Sigma_{P,E}$ the collection of subsets F of S for which $\widetilde{F_E} \in \mathcal{M}_d$, i.e.,

$$\Sigma_{P,E} = \{F \subseteq S : \widetilde{F_E} \in \mathcal{M}_d\}.$$

Proposition 4.2 *$\Sigma_{P,E}$ is a σ-algebra on S containing the Borel σ-algebra on S, $\mathcal{B}(S)$.*

Proof Throughout the proof, we write $\Sigma = \Sigma_{P,E}$ and $\widetilde{F} = \widetilde{F_E}$ for each $F \subseteq S$. We first show that Σ is a σ-algebra. Since $\widetilde{S} = B \setminus \{0\}$, it is open in $\mathbb{R}^d \setminus \{0\}$ and therefore Lebesgue measurable. Hence $S \in \Sigma$. Let $G, F \in \Sigma$ be such that $G \subseteq F$. Then,

$$\widetilde{F \setminus G} = \bigcup_{0<r<1} r^E (F \setminus G) = \bigcup_{0<r<1} \left(r^E F \setminus r^E G \right)$$

$$= \left(\bigcup_{0<r<1} r^E F \right) \setminus \left(\bigcup_{0<r<1} r^E G \right) = \widetilde{F} \setminus \widetilde{G}$$

where we have used the fact that the collection $\{r^E F\}_{0<r\leq 1}$ is mutually disjoint to pass the union through the set difference. Consequently $\widetilde{F} \setminus \widetilde{G}$ is Lebesgue measurable and therefore $F \setminus G \in \Sigma$. Now, given a countable collection $\{F_n\} \subseteq \Sigma$, observe that

$$\widetilde{\bigcup_{n=1}^{\infty} F_n} = \bigcup_{0<r<1} r^E \left(\bigcup_{n=1}^{\infty} F_n \right) = \bigcup_{0<r<1} \bigcup_{n=1}^{\infty} r^E F_n = \bigcup_{n=1}^{\infty} \bigcup_{0<r<1} r^E F_n = \bigcup_{n=1}^{\infty} \widetilde{F_n} \in \mathcal{M}_d$$

whence $\cup_n F_n \in \Sigma$. Thus Σ is a σ-algebra.

Finally, we show that

$$\mathcal{B}(S) \subseteq \Sigma.$$

As the Borel σ-algebra is the smallest σ-algebra containing the open subsets of S, it suffices to show that $\mathcal{O} \in \Sigma$ whenever $\mathcal{O}$ is open in S. Armed with Proposition 4.1, this is an easy task: Given an open set $\mathcal{O} \subseteq S$, observe that

$$\widetilde{\mathcal{O}} = \{r^E \eta : 0 < r < 1, \eta \in \mathcal{O}\} = \psi_E((0, 1) \times \mathcal{O}).$$

Upon noting that $(0, 1) \times \mathcal{O}$ is an open subset of $(0, \infty) \times S$, Proposition 4.1 guarantees that $\widetilde{\mathcal{O}} = \psi_E((0, 1) \times \mathcal{O}) \subseteq \mathbb{R}^d \setminus \{0\}$ is open and therefore $\widetilde{\mathcal{O}} \in \mathcal{M}_d$. Thus, $\mathcal{O} \in \Sigma$. $\square$

We are now ready to specify a measure on the measurable space $(S, \Sigma_{P,E})$. For each $F \in \Sigma_{P,E}$, we define

$$\sigma_{P,E}(F) = \mu_P \cdot m(\widetilde{F_E})$$

where m is the Lebesgue measure on $\mathbb{R}^d$ and $\mu_P = \mathrm{tr} E > 0$ is the homogeneous order associated to P.

Proposition 4.3 $\sigma_{P,E}$ *is a finite measure on* $(S, \Sigma_{P,E})$.

Proof Throughout the proof, we will write $\sigma = \sigma_{P,E}$, $\Sigma = \Sigma_{P,E}$, and, $\widetilde{F} = \widetilde{F_E}$ for each $F \subseteq S$. It is clear that σ is non-negative and $\sigma(\varnothing) = 0$. Let $\{F_n\}_{n=1}^{\infty} \subseteq \Sigma$ be a mutually disjoint collection. We claim that $\{\widetilde{F_n}\}_{n=1}^{\infty} \subseteq \mathcal{M}_d$ is also a mutually disjoint collection. To see this, suppose that $x = r_n^E \eta_n = r_m^E \eta_m \in \widetilde{F_n} \cap \widetilde{F_m}$, where $r_n, r_m \in (0, 1)$, $\eta_n \in F_n$, and $\eta_m \in F_m$. Then

$$r_n = P(r_n^E \eta_n) = P(x) = P(r_m^E \eta_m) = r_m,$$

implying that $\eta_n = \eta_m \in F_n \cap F_m$. Because $\{F_n\}_{n=1}^{\infty}$ is mutually disjoint, we must have $n = m$ which verifies our claim. By virtue of the countable additivity of Lebesgue measure, we therefore have

$$\sigma\left(\bigcup_{n=1}^{\infty} F_n\right) = \mu_P \cdot m\left(\widetilde{\bigcup_{n=1}^{\infty} F_n}\right) = \mu_P \cdot m\left(\bigcup_{n=1}^{\infty} \widetilde{F_n}\right) = \mu_P \sum_{n=1}^{\infty} m(\widetilde{F_n}) = \sum_{n=1}^{\infty} \sigma(F_n).$$

Therefore σ is a measure on (S, Σ). In view of Condition (b) of Proposition 1.2, $\widetilde{S} = B \setminus \{0\}$ is a bounded subset of $\mathbb{R}^d \setminus \{0\}$ and hence $\sigma(S) = \mu_P \cdot m(B \setminus \{0\}) < \infty$ showing that σ is finite. $\square$

By virtue of the two preceding propositions, $\sigma_{P,E}$ is a finite Borel measure on S. In fact, as a consequence of next subsection's main result, Theorem 4.4, we will see that $\sigma_{P,E}$ is independent of our choice of $E \in \mathrm{Exp}(P)$ and is a Radon measure; see Sect. 4.3.

4.2 Product Measure and Point Isomorphism

Throughout this subsection, $E \in \mathrm{Exp}(P)$ will remain fixed and $(S, \Sigma_{P,E}, \sigma_{P,E})$ will denote the finite measure space of Proposition 4.3. We recall that $\mathcal{L}$ denotes the σ-algebra of Lebesgue measurable subsets of $(0, \infty)$ and λ_P denotes the measure on $(0, \infty)$ with $\lambda_P(dr) = r^{\mu_P - 1} \, dr$, i.e., for each $L \in \mathcal{L}$,

$$\lambda_P(L) = \int_0^{\infty} \chi_L(r) r^{\mu_P - 1} \, dr.$$

It is easy to see that λ_P is σ-finite and so, in view of the finiteness of the measure $\sigma_{P,E}$, there exists a unique product measure $\lambda_P \times \sigma_{P,E}$ on $(0, \infty) \times S$ equipped with the product σ-algebra $\mathcal{L} \times \Sigma_{P,E}$ which satisfies

$$(\lambda_P \times \sigma_{P,E})(L \times F) = \lambda_P(L)\sigma_{P,E}(F)$$

for all $L \in \mathcal{L}$ and $F \in \Sigma_{P,E}$. We shall denote by $((0, \infty) \times S, (\mathcal{L} \times \Sigma_{P,E})', \lambda_P \times \sigma_{P,E})$ the completion of the measure space $((0, \infty) \times S, \mathcal{L} \times \Sigma_{P,E}, \lambda_P \times \sigma_{P,E})$. Our primary goal in this subsection is to prove the theorem below. We note that Properties 1 and 2 in Theorem 4.4 differ only from Properties 1 and 2 in Theorem 1.4 in that, a priori, the σ-algebra $\Sigma_{P,E}$ and the measure $\sigma_{P,E}$ in Theorem 4.4 depends on our choice of $E \in \mathrm{Exp}(P)$. As a consequence of Theorem 4.4, we shall see in Sect. 4.3 that $\Sigma_{P,E_1} = \Sigma_{P,E_2}$ and $\sigma_{P,E_1} = \sigma_{P,E_2}$ for all $E_1, E_2 \in \mathrm{Exp}(P)$ and so this apparent dependence is superficial; this is Proposition 4.11. As a consequence of the proposition, we shall obtain Properties 1 and 2 of Theorem 1.4 immediately from Properties of 1 and 2 in Theorem 4.4.

Theorem 4.4 *Let* $((0, \infty) \times S, (\mathcal{L} \times \Sigma_{P,E})', \lambda_P \times \sigma_{P,E})$ *be as above.*

1. *The map* $\psi_E : (0, \infty) \times S \to \mathbb{R}^d \setminus \{0\}$, *defined by (35), is a point isomorphism of the measure spaces* $((0, \infty) \times S, (\mathcal{L} \times \Sigma_{P,E})', \lambda_P \times \sigma_{P,E})$ *and* $(\mathbb{R}^d \setminus \{0\}, \mathcal{M}_d, m)$.

That is

$$\mathcal{M}_d = \{A \subseteq \mathbb{R}^d \setminus \{0\} : \psi_E^{-1}(A) \in (\mathcal{L} \times \Sigma_{P,E})'\}$$

and, for each $A \in \mathcal{M}_d$,

$$m(A) = (\lambda_P \times \sigma_{P,E})(\psi_E^{-1}(A)).$$

2. *If $f : \mathbb{R}^d \to \mathbb{C}$ is Lebesgue measurable, then $f \circ \psi_E$ is $(\mathcal{L} \times \Sigma_{P,E})'$-measurable and the following statements hold:*

   (a) *If $f \geq 0$, then*

   $$\begin{aligned}\int_{\mathbb{R}^d} f(x)\, dx &= \int_0^\infty \left(\int_S f(r^E \eta)\, \sigma_{P,E}(d\eta) \right) r^{\mu_P - 1}\, dr \\ &= \int_S \left(\int_0^\infty f(r^E \eta) r^{\mu_P - 1}\, dr \right) \sigma_{P,E}(d\eta). \qquad (36)\end{aligned}$$

   (b) *When f is complex-valued, we have*

   $$f \in L^1(\mathbb{R}^d) \text{ if and only if } f \circ \psi_E \in L^1((0, \infty) \times S, (\mathcal{L} \times \Sigma_{P,E})', \lambda_P \times \sigma_{P,E})$$

   and, in this case, (36) holds.

To prove Theorem 4.4, we shall first treat several lemmas. These lemmas isolate and generalize several important ideas used in standard proofs of (8) (see, e.g., [12] and [29]).

Lemma 4.5 *Let $A \subseteq \mathbb{R}^d$ and $r > 0$. A is Lebesgue measurable if and only if $r^E A = \{x = r^E a : a \in A\}$ is Lebesgue measurable and, in this case,*

$$m(r^E A) = r^{\mu_P} m(A).$$

Proof Because $x \mapsto r^E x$ is a linear isomorphism, $r^E A$ is Lebesgue measurable if and only if A is Lebesgue measurable. Observe that $x \in r^E A$ if and only if $r^{-E} x \in A$ and therefore

$$m(r^E A) = \int_{\mathbb{R}^d} \chi_{r^E A}(x)\, dx = \int_{\mathbb{R}^d} \chi_A(r^{-E} x)\, dx$$

where $\chi_{r^E A}$ and χ_A respectively denote the indicator functions of the sets $r^E A$ and A. Now, by making the linear change of variables $x \mapsto r^E x$, we have

$$m(r^E A) = \int_{\mathbb{R}^d} \chi_A(x) |\det(r^E)|\, dx = r^{\mu_P} m(A),$$

because $\det(r^E) = r^{\operatorname{tr} E} = r^{\mu_P} > 0$ by virtue of Proposition A.1 and Corollary 2.2. $\qquad \square$

Lemma 4.6 *Let $F \in \Sigma_{P,E}$. If $I \subseteq (0,\infty)$ is open, closed, G_δ, or F_σ, then $\psi_E(I \times F) \in \mathcal{M}_d$ and*

$$m(\psi_E(I \times F)) = (\lambda_P \times \sigma_{P,E})(I \times F) = \lambda_P(I)\sigma_{P,E}(F). \tag{37}$$

Proof To simplify notation, we shall write $\lambda = \lambda_P$ and $\sigma = \sigma_{P,E}$ throughout the proof. We fix $F \in \Sigma_{P,E}$ and consider several cases for I. If $I = (0,b)$ for $0 < b < \infty$, we have

$$\psi_E(I \times F) = \{r^E\eta : 0 < r < b, \eta \in F\} = b^E\{r^E\eta : 0 < r < 1, \eta \in F\} = b^E\widetilde{F_E}.$$

By virtue of Lemma 4.5, it follows that $\psi_E(I \times F) \in \mathcal{M}_d$ and

$$\begin{aligned}
(\lambda \times \sigma)(I \times F) &= \lambda(I)\sigma(F) \\
&= \left(\int_0^b r^{\mu_P-1}\,dr\right)\left(\mu_P \cdot m(\widetilde{F_E})\right) \\
&= b^{\mu_P}m(\widetilde{F_E}) \\
&= m(b^E\widetilde{F_E}) \\
&= m(\psi_E(I \times F)).
\end{aligned}$$

Using this result and the continuity of the measures $\lambda \times \sigma$ and m, standard arguments guarantee that $\psi_E(I \times F) \in \mathcal{M}_d$ and

$$(\lambda \times \sigma)(I \times F) = \left(b^{\mu_P} - a^{\mu_P}\right)\sigma(F) = m(\psi_E(I \times F))$$

whenever I is an interval of the form (a,b), $(a,b]$, $[a,b)$, and $[a,b]$ for $0 \le a \le b \le \infty$. Further, using the fact that every open subset of $(0,\infty)$ is countable union of disjoint open intervals, another standard argument using the continuity of the measures $\lambda \times \sigma$ and m guarantees that $\psi_E(I \times F) \in \mathcal{M}_d$ and

$$(\lambda \times \sigma)(I \times F) = m(\psi_E(I \times F))$$

whenever I is an open set. We then extend this to the case in which I is a G_δ by virtue of the continuity of measure. Finally, by taking complements and using the continuity of measure, we find that the assertion holds whenever I is an F_σ set. $\qquad\square$

Lemma 4.7 *For any $L \in \mathcal{L}$ and $F \in \Sigma_{P,E}$, $\psi_E(L \times F) \in \mathcal{M}_d$ and*

$$m(\psi_E(L \times F)) = (\lambda_P \times \sigma_{P,E})(L \times F).$$

Proof Fix $L \in \mathcal{L}$ and $F \in \Sigma_{P,E}$. It is easy to see that λ_P and the Lebesgue measure dr on $(0,\infty)$ are mutually absolutely continuous. It follows that $((0,\infty), \mathcal{L}, \lambda_P)$ is a complete measure space and, further, that there exists an F_σ set $L_\sigma \subseteq (0,\infty)$ and

a G_δ set $L_\delta \subseteq (0, \infty)$ for which $L_\sigma \subseteq L \subseteq L_\delta$ and $\lambda_P(L_\delta \setminus L_\sigma) = 0$. Note that, necessarily, $\lambda_P(L) = \lambda_P(L_\sigma) = \lambda_P(L_\delta)$. We have

$$\psi_E(L \times F) = \psi_E(L_\sigma \times F) \cup \psi_E((L \setminus L_\sigma) \times F) \tag{38}$$

where, by virtue of the preceding lemma, $\psi_E(L_\sigma \times F) \in \mathcal{M}_d$ and

$$m(\psi_E(L_\sigma \times F)) = (\lambda_P \times \sigma_{P,E})(L_\sigma \times F) = \lambda_P(L_\sigma)\sigma_{P,E}(F) = \lambda_P(L)\sigma_{P,E}(F)$$
$$= (\lambda_P \times \sigma_{P,E})(L \times F). \tag{39}$$

Observe that

$$\psi_E((L \setminus L_\sigma) \times F) \subseteq \psi_E((L_\delta \setminus L_\sigma) \times F)$$

where, because $L_\delta \setminus L_\sigma$ is an G_δ set, the latter set is a member of $\mathcal{M}_d$ and

$$m(\psi_E((L_\delta \setminus L_\sigma) \times F)) = (\lambda_P \times \sigma_{P,E})((L_\delta \setminus L_\sigma) \times F)$$
$$= \lambda_P(L_\delta \setminus L_\sigma)\sigma_{P,E}(F) = 0$$

by virtue of the preceding lemma. Using the fact that $(\mathbb{R}^d \setminus \{0\}, \mathcal{M}_d, m)$ is complete, we conclude that $\psi_E((L \setminus L_\sigma) \times F) \in \mathcal{M}_d$ and $m(\psi_E((L \setminus L_\sigma) \times F)) = 0$. It now follows from (38) and (39) that $\psi_E(L \times F) \in \mathcal{M}_d$ and

$$m(\psi_E(L \times F)) = m(\psi_E(L_\sigma \times F)) + m(\psi_E((L \setminus L_\sigma) \times F)) = (\lambda_P \times \sigma_{P,E})(L \times F),$$

as desired. □

Lemma 4.8 *Every open subset $U \subseteq \mathbb{R}^d \setminus \{0\}$ can be written as a countable union of open sets of the form $\psi_E(\mathcal{U})$ where $\mathcal{U} = I \times \mathcal{O}$ is an open rectangle in $(0, \infty) \times S$.*

Proof Let $\{r_k\}_{k=1}^\infty$ and $\{\eta_j\}_{j=1}^\infty$ be countably dense subsets of $(0, \infty)$ and S, respectively. For each triple of natural numbers $j, l, n \in \mathbb{N}_+$, consider the open set

$$\mathcal{U}_{j,l,n} = \{|r - r_j| < 1/n\} \times \mathcal{O}_{l,n} \subseteq (0, \infty) \times S$$

where

$$\mathcal{O}_{l,n} = \{\eta \in S : |\eta - \eta_l| < 1/n\}.$$

Let $U \subseteq \mathbb{R}^d \setminus \{0\}$ be open. We will show that

$$U = \bigcup_{\substack{j,l,n \\ \psi_E(\mathcal{U}_{j,l,n}) \subseteq U}} \psi_E(\mathcal{U}_{j,l,n}), \tag{40}$$

where, in view of Proposition 4.1, each $\psi_E(\mathcal{U}_{j,l,n})$ is open. It is clear that any element of the union on the right hand side of (40) belongs to some $\psi_E(\mathcal{U}_{j,l,n}) \subseteq U$ and so

the union is a subset of U. To prove (40), it therefore suffices to prove that, for each $x \in U$, there exists a triple j, l, n with

$$x \in \psi_E(\mathcal{U}_{j,l,n}) \subseteq U.$$

To this end, fix $x \in U$ and let $\delta > 0$ be such that $\mathbb{B}_\delta(x) \subseteq U$. Consider $(r_x, \eta_x) = \psi_E^{-1}(x) \in (0, \infty) \times S$ and set $M = \|r_x^E\| > 0$ and $C = \|E\| > 0$. Observe that

$$\|I - \alpha^E\| = \left\| \sum_{k=1}^{\infty} \frac{(\ln \alpha)^k}{k!} E^k \right\| \leq \sum_{k=1}^{\infty} \frac{|\ln \alpha|^k}{k!} \|E\|^k = e^{(C|\ln \alpha|)} - 1$$

for all $\alpha > 0$. Since $\alpha \mapsto e^{(C|\ln \alpha|)} - 1$ is continuous and 0 at $\alpha = 1$, we can choose $\delta' > 0$ for which

$$\|I - \alpha^E\| < \frac{\delta}{2M(|\eta_x| + 2)}$$

whenever $|\alpha - 1| < \delta'$. Fix an integer $n > \max\left\{ 1/\delta' r_x, 4M/\delta \right\}$ and, using the density of the collections $\{r_j\}$ and $\{\eta_l\}$, let r_j and η_l be such that $|r_j - r_x| < 1/n$ and $|\eta_l - \eta_x| < 1/n$. It follows that the corresponding open set $\mathcal{U}_{j,l,n}$ contains $\psi_E^{-1}(x)$, or, equivalently, $x \in \psi_E(\mathcal{U}_{j,l,n})$. Thus, it remains to show that $\psi_E(\mathcal{U}_{j,l,n}) \subseteq \mathbb{B}_\delta(x)$. To this end, let $y = \psi_E(r_y, \eta_y) \in \psi_E(\mathcal{U}_{j,l,n})$ and observe that

$$\begin{aligned} |x - y| &\leq |\psi_E(r_x, \eta_x,) - \psi_E(r_x, \eta_y)| + |\psi_E(r_x, \eta_y) - \psi_E(r_y, \eta_y)| \\ &= |r_x^E(\eta_x - \eta_y)| + |(r_x^E - r_y^E)\eta_y| \\ &\leq M|\eta_x - \eta_y| + \|r_x^E - r_y^E\| |\eta_y|. \end{aligned}$$

Since both $(\eta_x, r_x), (\eta_y, r_y) \in \mathcal{U}_{j,l,n}$, we have

$$|\eta_x - \eta_y| \leq |\eta_x - \eta_j| + |\eta_j - \eta_y| < \frac{2}{n} \text{ and } |\eta_y| \leq |\eta_y - \eta_x| + |\eta_x| < |\eta_x| + \frac{2}{n}.$$

Also, since $|r_x - r_y| < 1/n$, it follows that $r_y = \alpha r_x$ where $|1 - \alpha| < 1/nr_x < \delta'$ by our choice of n. Consequently,

$$\begin{aligned} |x - y| &< \frac{2}{n}M + \left(|\eta_x| + \frac{2}{n} \right) \|r_x^E - r_x^E \alpha^E\| \\ &< \frac{2}{n}M + (|\eta_x| + 2)\, M \|I - \alpha^E\| \\ &< \frac{2}{n}M + \frac{\delta M (|\eta_x| + 2)}{2M(|\eta_x| + 2)} \\ &< \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

and so we have established (40). Finally, upon noting that $\{\mathcal{U}_{j,l,n}\}_{(j,l,n)\in\mathbb{N}_+^3}$ is a countable collection of open rectangles, the union in (40) is necessarily countable and we are done with the proof.   □

In our final lemma preceding the proof of Theorem 4.4, we treat a general measure-theoretic statement which gives sufficient conditions concerning two measure spaces to ensure that their completions are isomorphic. Though we suspect that this result is well-known, we present its proof for completeness.

Lemma 4.9 *Let (X_1, Σ_1, ν_1) and (X_2, Σ_2, ν_2) be measure spaces, let $\varphi : X_1 \to X_2$ be a bijection and denote by (X_i, Σ_i', ν_i') the completion of the measure space (X_i, Σ_i, ν_i) for $i = 1, 2$. Assume that the following two properties are satisfied:*

1. *For each $A_1 \in \Sigma_1$, $\varphi(A_1) \in \Sigma_2'$ and $\nu_2'(\varphi(A_1)) = \nu_1(A_1)$.*
2. *For each $A_2 \in \Sigma_2$, $\varphi^{-1}(A_2) \in \Sigma_1'$ and $\nu_1'(\varphi^{-1}(A_2)) = \nu_2(A_2)$.*

Then the measure spaces (X_1, Σ_1', ν_1') and (X_2, Σ_2', ν_2') are isomorphic with point isomorphism φ. Specifically,

$$\Sigma_2' = \{A_2 \subseteq X_2 : \varphi^{-1}(A_2) \in \Sigma_1'\} \tag{41}$$

and

$$\nu_2'(A_2) = \nu_1'(\varphi^{-1}(A_2)) \tag{42}$$

for all $A_2 \in \Sigma_2'$.

Proof Let us first assume that $A_2 \in \Sigma_2'$. By definition, $A_2 = G_2 \cup H_2$ where $G_2 \in \Sigma_2$ and $H_2 \subseteq G_{2,0} \in \Sigma_2$ with $\nu_2'(A_2) = \nu_2(G_2)$ and $\nu_2'(H_2) = \nu_2(G_{2,0}) = 0$. Consequently, $\varphi^{-1}(A_2) = \varphi^{-1}(G_2) \cup \varphi^{-1}(H_2)$ and $\varphi^{-1}(H_2) \subseteq \varphi^{-1}(G_{2,0})$. In view of Property 2, $\varphi^{-1}(G_2), \varphi^{-1}(G_{2,0}) \in \Sigma_1'$ and we have

$$\nu_1'(\varphi^{-1}(G_2)) = \nu_2(G_2) = \nu_2'(A_2) \text{ and } \nu_1'(\varphi^{-1}(G_{2,0})) = \nu_2(G_{2,0}) = 0.$$

In view of the fact that $(X_1', \Sigma_1', \nu_1')$ is complete, $\varphi^{-1}(H_2) \in \Sigma_1'$ and $\nu_1'(\varphi^{-1}(H_2)) = 0$. Consequently, we obtain $\varphi^{-1}(A_2) = \varphi^{-1}(G_2) \cup \varphi^{-1}(H_2) \in \Sigma_1'$ and

$$\begin{aligned}
\nu_2'(A_2) = \nu_1'(\varphi^{-1}(G_2)) &\leq \nu_1'(\varphi^{-1}(A_2)) \\
&\leq \nu_1'(\varphi^{-1}(G_2)) + \nu_1'(\varphi^{-1}(H_2)) = \nu_2(G_2) + 0 = \nu_2'(A_2).
\end{aligned}$$

From this we obtain that $\Sigma_2' \subseteq \{A_2 \subseteq X_2 : \varphi^{-1}(A_2) \in \Sigma_1'\}$ and, for each $A_2 \in \Sigma_2'$, $\nu_2'(A_2) = \nu_1'(\varphi^{-1}(A_2))$. It remains to prove that

$$\{A_2 \subseteq X_2 : \varphi^{-1}(A_2) \in \Sigma_1'\} \subseteq \Sigma_2'.$$

To this end, let A_2 be a subset of X_2 for which $\varphi^{-1}(A_2) \in \Sigma_1'$. By the definition of Σ_1', we have $\varphi^{-1}(A_2) = G_1 \cup H_1$ where $G_1 \in \Sigma_1$, $H_1 \subseteq G_{1,0} \in \Sigma_1$ and

$v_1'(H_1) = v_1(G_{1,0}) = 0$. In view of Property 1, $\varphi(G_1) \in \Sigma_2'$, $\varphi(H_1) \subseteq \varphi(G_{1,0}) \in \Sigma_2'$ and $v_2'(\varphi(G_{1,0})) = v_1(G_{1,0}) = 0$. Because (X_2', Σ_2', v_2') is complete, we have $\varphi(H_1) \in \Sigma_2'$ and so

$$A_1 = \varphi(\varphi^{-1}(A_2)) = \varphi(G_1) \cup \varphi(H_1) \in \Sigma_2',$$

as desired. $\square$

We are finally in a position to prove Theorem 4.4.

Proof of Theorem 4.4 Denote by $\mathcal{C}$ the collection of sets $G \subseteq (0, \infty) \times S$ for which $\psi_E(G) \in \mathcal{M}_d$ and $m(\psi_E(G)) = (\lambda_P \times \sigma_{P,E})(G)$. By virtue of Lemma 4.7, it follows that $\mathcal{C}$ contains all elementary sets, i.e., finite unions of disjoint measurable rectangles. Using the continuity of measure (applied to the measures m and $\lambda_P \times \sigma_{P,E}$) and the fact that ψ_E is a bijection, it is straightforward to verify that $\mathcal{C}$ is a monotone class. By the monotone class lemma (Theorem 8.3 of [25]), it immediately follows that $\mathcal{L} \times \Sigma_{P,E} \subseteq \mathcal{C}$. In other words, for each $G \in \mathcal{L} \times \Sigma_{P,E}$,

$$\psi_E(G) \in \mathcal{M}_d \text{ and } m(\psi_E(G)) = (\lambda_P \times \sigma_{P,E})(G). \tag{43}$$

We claim that, for each Borel subset A of $\mathbb{R}^d \setminus \{0\}$, $\psi_E^{-1}(A) \subseteq \mathcal{L} \times \Sigma_{P,E}$. To this end, we write

$$\psi_E(\mathcal{L} \times \Sigma_{P,E}) = \{\psi_E(G) : G \in \mathcal{L} \times \Sigma_{P,E}\}$$

for the σ-algebra on $\mathbb{R}^d \setminus \{0\}$ induced by ψ_E. In view of Lemma 4.8, $\psi_E(\mathcal{L} \times \Sigma_{P,E})$ contains every open subset of $\mathbb{R}^d \setminus \{0\}$ and therefore

$$\mathcal{B}(\mathbb{R}^d \setminus \{0\}) \subseteq \psi_E(\mathcal{L} \times \Sigma_{P,E}).$$

where $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^d \setminus \{0\}$ thus proving our claim.

Together, the results of the two preceding paragraphs show that, for each $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $\psi_E^{-1}(A) \subseteq \mathcal{L} \times \Sigma_{P,E}$ and $m(A) = (\lambda_P \times \sigma_{P,E})(\psi_E^{-1}(A))$. Upon noting that $\mathcal{L} \times \Sigma_{P,E} \subseteq (\mathcal{L} \times \Sigma_{P,E})'$, we immediately obtain the following statement: For each $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$,

$$\psi_E^{-1}(A) \in (\mathcal{L} \times \Sigma_{P,E})' \text{ and } m(A) = (\lambda_P \times \sigma_{P,E})(\psi_E^{-1}(A)). \tag{44}$$

In comparing (43) and (44) with Properties 1 and 2 of Lemma 4.9 and, upon noting that $((0, \infty) \times S, (\mathcal{L} \times \Sigma_{P,E})', \lambda_P \times \sigma_{P,E})$ is the completion of $((0, \infty) \times S, \mathcal{L} \times \Sigma_{P,E}, \lambda_P \times \sigma_{P,E})$ and $(\mathbb{R}^d \setminus \{0\}, \mathcal{M}_d, m)$ is the completion of $(\mathbb{R}^d \setminus \{0\}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}), m)$, Property 1 of Theorem 4.4 follows immediately from Lemma 4.9.

It remains to prove Property 2. To this end, let $f : \mathbb{R}^d \to \mathbb{C}$ be Lebesgue measurable. Because $\mathcal{M}_d = \{A \subseteq \mathbb{R}^d \setminus \{0\} : \psi_E^{-1}(A) \in (\mathcal{L} \times \Sigma_{P,E})'\}$, it follows that $f \circ \psi_E$ is $(\mathcal{L} \times \Sigma_{P,E})'$-measurable. In the case that $f \geq 0$, we may approximate f monotonically by

simple functions and, by invoking Property 1 and the monotone convergence theorem, we find that

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d \setminus \{0\}} f(x)\,dx = \int_{(0,\infty)\times S} f \circ \psi_E \, d(\lambda_P \times \sigma_{P,E}). \qquad (45)$$

From this, (36) follows from Fubini's theorem (see, e.g., Part (a) of Theorem 8.8 and Theorem 8.12 of [25]). Finally, by applying the above result to $|f| \geq 0$, we obtain $f \in L^1(\mathbb{R}^d)$ if and only if $f \circ \psi_E \in L^1((0,\infty) \times S, (\mathcal{L} \times \Sigma_{P,E})', \lambda_P \times \sigma_{P,E})$. In this case, by applying (45) to $\mathrm{Re}(f)_+, \mathrm{Re}(f)_-, \mathrm{Im}(f)_+$ and $\mathrm{Im}(f)_-$, we find that (45) holds for our integrable f and, by virtue Fubini's theorem (see, e.g., Part (b) of Theorem 8.8 and Theorem 8.12 of [25]), the desired result follows. □

Our next result, Proposition 4.10, guarantees that, in particular, $\sigma_{P,E}$ is a Radon measure.

Proposition 4.10 *We have:*

1. *$(S, \Sigma_{P,E}, \sigma_{P,E})$ is the completion of the measure space $(S, \mathcal{B}(S), \sigma_{P,E})$. In particular, the measure space $(S, \Sigma_{P,E}, \sigma_{P,E})$ is complete and every $F \in \Sigma_{P,E}$ is of the form $F = G \cup H$ where G is a Borel set and H is a subset of a Borel set Z with $\sigma_{P,E}(Z) = 0$.*
2. *For each $F \in \Sigma_{P,E}$,*

$$\sigma_{P,E}(F) = \inf\{\sigma_{P,E}(\mathcal{O}) : F \subseteq \mathcal{O} \subseteq S \text{ and } \mathcal{O} \text{ is open}\} \qquad (46)$$

and

$$\sigma_{P,E}(F) = \sup\{\sigma_{P,E}(K) : K \subseteq F \subseteq S \text{ and } K \text{ is compact}\}. \qquad (47)$$

Remark 2 This proposition can be seen as an application of Proposition 4.2 and Theorem 2.18 of [25]. The proof we give here is distinct and nicely illustrates the utility of (36) of Theorem 4.4.

Proof Throughout the proof, we shall write $\sigma = \sigma_{P,E}$, $\Sigma = \Sigma_{P,E}$ and, for each $F \subseteq S$, $\widetilde{F} = \widetilde{F}_E$. We remark that, by standard arguments using G_δ and F_σ sets, Item 1 follows immediately from Item 2. Also, given that S is compact and σ is finite, it suffices to prove (46), i.e., it suffices to prove the statement: For each $F \in \Sigma$ and $\epsilon > 0$, there is an open subset $\mathcal{O}$ of S containing F for which

$$\sigma(\mathcal{O} \setminus F) < \epsilon.$$

To this end, let $F \in \Sigma$ and $\epsilon > 0$. Given that $\widetilde{F} \in \mathcal{M}_d$ and m is outer regular, there exists an open set $U \subseteq \mathbb{R}^d \setminus \{0\}$ for which $\widetilde{F} \subseteq U$ and $m(U \setminus \widetilde{F}) < \epsilon/(2\mu_P)$. Since $\widetilde{F}$ is a subset of the open set $B \setminus \{0\}$, we may assume without loss of generality that $U \subseteq B \setminus \{0\}$ and so $m(\widetilde{F}) \leq m(U) < \infty$ and

$$m(U \setminus \widetilde{F}) = m(U) - m(\widetilde{F}) < \epsilon/(2\mu_P). \qquad (48)$$

For each $0 < r < 1$, consider the open set

$$\mathcal{O}_r = S \cap \left(r^{-E} U \right)$$

in S. Observe that, for each $x \in F$, $r^E x \in \widetilde{F} \subseteq U$ and therefore $x \in \mathcal{O}_r$. Hence, for each $0 < r < 1$, $\mathcal{O}_r$ is an open subset of S containing F.

We claim that there is at least one $r_0 \in (0, 1)$ for which

$$m(\widetilde{\mathcal{O}_{r_0}}) < m(U) + \epsilon/(2\mu_P). \tag{49}$$

To prove the claim, we shall assume, to reach a contradiction, that

$$m(\widetilde{\mathcal{O}_r}) \geq m(U) + \epsilon/(2\mu_P)$$

for all $0 < r < 1$. By virtue of (36) of Theorem 4.4,

$$m(U) = \int_0^\infty \left(\int_S \chi_U(r^E \eta) \, \sigma(d\eta) \right) r^{\mu_P - 1} \, dr.$$

Upon noting that $U \subseteq B \setminus \{0\}$, it is easy to see that

$$U = \bigcup_{0 < s < 1} s^E \mathcal{O}_s \text{ and } r^E \eta \in \bigcup_{0 < s < 1} s^E \mathcal{O}_s$$

if and only if $0 < r < 1$ and $\eta \in \mathcal{O}_r$. Consequently,

$$m(U) = \int_0^1 \left(\int_S \chi_{\mathcal{O}_r}(\eta) \, \sigma(d\eta) \right) r^{\mu_P - 1} \, dr$$
$$= \int_0^1 \sigma(\mathcal{O}_r) r^{\mu_P - 1} \, dr = \int_0^1 \mu_P \cdot m(\widetilde{\mathcal{O}_r}) \, r^{\mu_P - 1} \, dr.$$

Upon making use of our supposition, we have

$$\int_0^1 \mu_P \cdot m(\widetilde{\mathcal{O}_r}) r^{\mu_P - 1} \, dr \geq \int_0^1 \mu_P \cdot (m(U) + \epsilon/(2\mu_P)) r^{\mu_P - 1} \, dr$$
$$= m(U) + \epsilon/(2\mu_P)$$

and so

$$m(U) \geq m(U) + \epsilon/(2\mu_P),$$

which is impossible. Thus, the stated claim is true.

Given any such r_0 for which (49) holds, set $\mathcal{O} = \mathcal{O}_{r_0}$. As previously noted, $\mathcal{O}$ is an open subset of S which contains F. In view of (48) and (49), we have

$$m(\widetilde{\mathcal{O}}) - m(\widetilde{F}) < m(U) - m(\widetilde{F}) + \epsilon/(2\mu_P) < \epsilon/(2\mu_P) + \epsilon/(2\mu_P) = \epsilon/\mu_P$$

and therefore

$$\sigma(\mathcal{O} \setminus F) = \sigma(\mathcal{O}) - \sigma(F) = \mu_P(m(\widetilde{\mathcal{O}}) - m(\widetilde{F})) < \epsilon,$$

as desired.   □

4.3 The Construction is Independent of $E \in \mathrm{Exp}(P)$

In this subsection, we show that the Radon measure $\sigma_{P,E}$ is independent of the choice of $E \in \mathrm{Exp}(P)$ and complete the proof of Theorem 1.4. To set the stage for our first result, let $E_1, E_2 \in \mathrm{Exp}(P)$ and consider the associated (respective) measure spaces $(S, \Sigma_{P,E_1}, \sigma_{P,E_1})$ and $(S, \Sigma_{P,E_2}, \sigma_{P,E_2})$ produced via the construction in Sect. 4.1.

Proposition 4.11 *These measure spaces are the same, i.e.,* $\Sigma_{P,E_1} = \Sigma_{P,E_2}$ *and* $\sigma_{P,E_1} = \sigma_{P,E_2}$.

Proof Throughout the proof, we will write $\Sigma_i = \Sigma_{P,E_i}$ and $\sigma_i = \sigma_{P,E_i}$ for $i = 1, 2$. In view of the Proposition 4.10, it suffices to show that

$$\sigma_1(F) = \sigma_2(F)$$

for all $F \in \mathcal{B}(S) \subseteq \Sigma_1 \cap \Sigma_2$. To this end, we let $F \in \mathcal{B}(S)$ be arbitrary but fixed.

Given $n \in \mathbb{N}_+$, using the regularity of the measures σ_1 and σ_2, select open sets $\mathcal{O}_{n,1}, \mathcal{O}_{n,2}$ and compact sets $K_{n,1}, K_{n,2}$ for which

$$K_{n,j} \subseteq F \subseteq \mathcal{O}_{n,j} \text{ and } \sigma_j(\mathcal{O}_{n,j} \setminus K_{n,j}) < 1/n$$

for $j = 1, 2$. Observe that $K_n = K_{n,1} \cup K_{n,2}$ is a compact set, $\mathcal{O}_n = \mathcal{O}_{n,1} \cap \mathcal{O}_{n,2}$ is an open set and $K_n \subseteq F \subseteq \mathcal{O}_n$. Furthermore,

$$\sigma_j(\mathcal{O}_n \setminus K_n) \leq \sigma_j(\mathcal{O}_{n,j} \setminus K_{n,j}) < 1/n$$

for $j = 1, 2$. Given that $\mathcal{O}_n$ is open in S, $\mathcal{O}_n = S \cup U_n$ where U_n is an open subset of $\mathbb{R}^d$ and, because that S is compact, $K_n = K_n \cap S$ is a compact subset of $\mathbb{R}^d$. By virtue of Urysohn's lemma, let $\phi_n : \mathbb{R}^d \to [0, 1]$ be a continuous function which is compactly supported in U_n and for which $\phi_n(x) = 1$ for all $x \in K_n$. Using this sequence of functions $\{\phi_n\}$, we establish the following useful lemma.

Lemma 4.12 *For $j = 1, 2$ and $n \in \mathbb{N}$, define $g_{n,j} : (0, \infty) \to \mathbb{R}$ by*

$$g_{n,j}(r) = \int_S \phi_n(r^{E_j} \eta) \, \sigma_j(d\eta).$$

for $r > 0$. Then $g_{n,j}$ is continuous for each $n \in \mathbb{N}$ and $j = 1, 2$ and

$$\sigma_j(F) = \lim_{n \to \infty} g_{n,j}(1)$$

for $j = 1, 2$.

Proof First, we note that, for each $r \in (0, \infty)$, the above integral makes sense because $\eta \mapsto \phi_n(r^{E_j}\eta)$ is Borel measurable (because it's continuous on S) and non-negative. Let $\epsilon > 0$ and $r_0 \in (0, \infty)$ be arbitrary but fixed. It is clear that the function $(0, \infty) \times S \ni (r, \eta) \mapsto \phi_n(r^{E_j}\eta)$ is continuous on its domain and therefore, in view of the compactness of S, we can find a $\delta > 0$ for which

$$|\phi_n(r^{E_j}\eta) - \phi_n(r_0^{E_j}\eta)| \le \frac{\epsilon}{2\sigma_j(S)} \quad \text{whenever} \quad |r - r_0| < \delta$$

for all $\eta \in S$. The triangle inequality guarantees that

$$|g_{n,j}(r) - g_{n,j}(r_0)| \le \int_S |\phi_n(r^{E_j}\eta) - \phi_n(r_0^{E_j}\eta)|\,\sigma_j(d\eta) \le \epsilon/2 < \epsilon$$

whenever $|r - r_0| < \delta$. Thus, $g_{n,j}$ is continuous.

We observe that

$$g_{n,j}(1) = \int_S \phi_n(\eta)\,\sigma_j(d\eta)$$

because $1^{E_j} = I$. By construction, we have $\chi_{K_n}(\eta) \le \phi_n(\eta) \le \chi_{\mathcal{O}_n}(\eta)$ for all $\eta \in S$ and $n \in \mathbb{N}_+$ and therefore

$$\sigma_j(K_n) \le g_{n,j}(1) \le \sigma_j(\mathcal{O}_n)$$

by the monotonicity of the integral. Since

$$\sigma_j(F) = \lim_{n \to \infty} \sigma_j(K_n) = \lim_{n \to \infty} \sigma_j(\mathcal{O}_n)$$

in view of our choice of $\mathcal{O}_n$ and K_n, the remaining result follows immediately from the preceding inequality (and the squeeze theorem).

Let us now complete the proof of Proposition 4.11. Given any $0 < s < 1 < t$ and $n \in \mathbb{N}$, consider the function $f = f_{n,s,t} : \mathbb{R}^d \to [0, 1]$ given by

$$f(x) = \phi_n(x)\chi_{[s,t]}(P(x))$$

for $x \in \mathbb{R}^d$. It is clear that f is Lebesgue measurable on $\mathbb{R}^d$ and non-negative. By virtue of Theorem 4.4 (applied to the two measures σ_1 and σ_2), we have

$$\int_0^\infty \int_S f(r^{E_1}\eta)\,\sigma_1(d\eta)r^{\mu_P - 1}\,dr = \int_{\mathbb{R}^d} f(x)\,dx = \int_0^\infty \int_S f(r^{E_2}\eta)\,\sigma_2(d\eta)r^{\mu_P - 1}\,dr \tag{50}$$

Upon noting that

$$f(r^{E_j}\eta) = \phi_n(r^{E_j}\eta)\chi_{[s,t]}\left((P(r^{E_j}\eta)\right) = \phi_n(r^{E_j}\eta)\chi_{[s,t]}(rP(\eta)) = \chi_{[s,t]}(r)\phi_n(r^{E_j}\eta)$$

for $r \in (0, \infty)$, $\eta \in S$, and $j = 1, 2$, we have

$$
\int_0^\infty \int_S f(r^{E_j}\eta)\, \sigma_j(d\eta) r^{\mu_P - 1}\, dr
$$
$$
= \int_{[s,t]} \int_S \phi_n(r^{E_j}\eta)\, \sigma_j(d\eta) r^{\mu_P - 1}\, dr = \int_{[s,t]} g_{n,j}(r) r^{\mu_P - 1}\, dr
$$

for $j = 1, 2$. By virtue of the Lemma 4.12, $r \mapsto g_{n,j}(r) r^{\mu_P - 1}$ is continuous and necessarily bounded on $[s, t]$ and so the final integral above can be interpreted as a Riemann integral. In this interpretation, we have

$$
\int_s^t g_{n,j}(r) r^{\mu_P - 1}\, dr = \int_0^\infty \int_S f(r^{E_j}\eta)\, \sigma_j(d\eta) r^{\mu_P - 1}\, dr \tag{51}
$$

for $j = 1, 2$ and $0 < s < 1 < t$. In view of (50) and (51), we conclude that

$$
\int_s^t g_{n,1}(r) r^{\mu_P - 1}\, dr = \int_s^t g_{n,2}(r) r^{\mu_P - 1}\, dr
$$

for all $0 < s < 1 < t$. In view of continuity of the integrands, an application of the fundamental theorem of calculus now guarantees that $g_{n,1}(1) = g_{n,2}(1)$ for each $n \in \mathbb{N}$. Therefore

$$
\sigma_1(F) = \lim_{n \to \infty} g_{n,1}(1) = \lim_{n \to \infty} g_{n,2}(1) = \sigma_2(F)
$$

by virtue of Lemma 4.12.   □

In view of Proposition 4.11, we will denote by Σ_P and σ_P the unique σ-algebra and measure on S which, respectively, satisfy

$$
\Sigma_P = \Sigma_{P,E} \quad \text{and} \quad \sigma_P = \sigma_{P,E}
$$

for all $E \in \mathrm{Exp}(P)$. We will henceforth assume this notation.

Proposition 4.13 *For any $O \in \mathrm{Sym}(P)$ and $F \in \Sigma_P$, $OF \in \Sigma_P$ and*

$$
\sigma_P(OF) = \sigma_P(F).
$$

That is, the measure σ_P is invariant under the action by $\mathrm{Sym}(P)$.

Proof Let $O \in \mathrm{Sym}(P)$, $F \in \Sigma_P$ and, for $E \in \mathrm{Exp}(P)$, define $E' = O^{-1}EO$. In view of Proposition 2.4, we note that $E' \in \mathrm{Exp}(P)$. Observe that

$$
\widetilde{(OF)}_E = \bigcup_{0 < r < 1} r^E(OF) = \bigcup_{0 < r < 1} O\left(O^{-1}r^E OF\right) = O\left(\bigcup_{0 < r < 1} r^{E'}F\right) = O\widetilde{F_{E'}} \tag{52}
$$

thanks to Proposition A.1. In view of Proposition 4.11, we have $O\widetilde{F_{E'}} \in \mathcal{M}_d$ because $F \in \Sigma_P = \Sigma_{P,E'}$ and O is linear. Using (52), we find that $\widetilde{(OF)_E} \in \mathcal{M}_d$ and therefore $OF \in \Sigma_{P,E} = \Sigma_P$ by virtue of Proposition 4.11. Since $\text{Sym}(P)$ is compact, $\det(O) = \pm 1$ and hence O preserves the Lebesgue measure m. Thus,

$$\sigma_{P,E}(OF) = \mu_P \cdot m\left(\widetilde{(OF)_E}\right) = \mu_P \cdot m\left(O\widetilde{F_{E'}}\right) = \mu_P \cdot m\left(\widetilde{F_{E'}}\right) = \sigma_{P,E'}(F)$$

and therefore, a final appeal to Proposition 4.11 guarantees that

$$\sigma_P(OF) = \sigma_{P,E}(OF) = \sigma_{P,E'}(F) = \sigma_P(F),$$

as desired. $\square$

Proof of Theorem 1.4 Together, the results of Propositions 4.10, 4.11 and 4.13 , guarantee that σ_P is a Radon measure satisfying Properties 1 and 2. Property 3 follows directly from Proposition 4.11 and the definition of σ_P in terms of $\sigma_{P,E}$ for any $E \in \text{Exp}(P)$. Similarly, Properties 1 and 2 follow from Theorem 4.4 by virtue of Proposition 4.11.
$\square$

5 Using a Smooth Structure on S to Compute σ

In this section, we shall study the special case in which a positive homogeneous function P on $\mathbb{R}^d$ is smooth.[6,7] Under this additional assumption, we shall find that ∇P is everywhere non-vanishing on S and so S is a smooth compact embedded hypersurface in $\mathbb{R}^d$.

We first set up some notation: For a smooth manifold M, we shall denote by $\mathcal{A}(M)$ its unique maximal atlas. Also on M, the collection of smooth vector fields is denoted by $\mathfrak{X}(M)$ and, for each $k = 1, 2, \ldots$, the set of (smooth) differential k-forms on M will be denoted by $\Omega^k(M)$. In this section, we integrate non-smooth differential forms and, for the generality needed here, we shall refer the reader to [20] for background (Another perspective is given in [1]). To this end, let us denote the Lebesgue σ-algebra of measurable sets on M by $\mathcal{L}(M)$. We note that $F \in \mathcal{L}(M)$ if and only if,

$$\varphi(F \cap \mathcal{U}) \in \mathcal{M}_d$$

for every chart $(\varphi, \mathcal{U}) \in \mathcal{A}(M)$. An n-form ω on M, with $n = \dim(M)$, is said to be (Lebesgue) measurable, if in each coordinate system $(\varphi, \mathcal{U})$, the local representation

$$\omega = h_\varphi(x)dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$$

---

[6] Many of the results in this section remain valid (with appropriate modification) under the weaker assumption that $P \in C^k(\mathbb{R}^d)$ for $k = 1, 2, \ldots$. In this setting, S is easily seen to be a C^k manifold. Because working in the smooth category is sufficient for our purposes, we shall not pursue the greater level of generality but invite the reader to do so.

[7] To avoid trivialities, we assume that $d > 1$ throughout.

in the coordinates $\varphi = (x^1, x^2, \ldots, x^n)$ has $h_\varphi(x)$ a Lebesgue measurable function on $U = \varphi(\mathcal{U}) \subseteq \mathbb{R}^n$. The collection of measurable n-forms on M is denoted by $\mathcal{L}(\Lambda^d(M))$ and, naturally, $\Omega^n(M) \subseteq \mathcal{L}(\Lambda^n(M))$. As standard, we shall use Einstein's summation convention without explicit mention.

We view $\mathbb{R}^d$ as smooth oriented Riemannian manifold with its standard Euclidean metric $\overline{g}$, oriented smooth atlas $\mathcal{A}_+(\mathbb{R}^d)$, and Riemannian volume form $d\operatorname{Vol}_{\mathbb{R}^d}$. Given any $E \in \operatorname{Exp}(P)$, consider $\mathcal{E}_E \in \mathfrak{X}(\mathbb{R}^d)$ defined, at each $x \in \mathbb{R}^d$, by

$$(\mathcal{E}_E)_x(f) = \frac{d}{dt} f(x + t(Ex))\big|_{t=0}$$

for $f \in C^\infty(\mathbb{R}^d)$. In the standard (global) chart with coordinates $x = (x^1, x^2, \ldots, x^d)$, $(\mathcal{E}_E)_x \in T_x(\mathbb{R}^d)$ is given by

$$(\mathcal{E}_E)_x = (Ex)^\alpha \partial_{x^\alpha} = E^\alpha_\beta x^\beta \partial_{x^\alpha}$$

where (E^β_α) is the standard matrix representation for E and $\partial_{x^\alpha} = \partial/\partial x^\alpha$. By an abuse of notation, we shall write ∇P to denote both the function

$$\mathbb{R}^d \ni x \mapsto \nabla P(x) = \left(\frac{\partial P}{\partial x^1}, \frac{\partial P}{\partial x^2}, \ldots, \frac{\partial P}{\partial x^d} \right) \in \mathbb{R}^d,$$

where $\frac{\partial P}{\partial x^\alpha} = \frac{\partial P}{\partial x^\alpha}|_x$ for $\alpha = 1, 2 \ldots, d$, and its canonical identification $\nabla P \in \mathfrak{X}(\mathbb{R}^d)$ given by

$$\nabla P_x = \overline{g}^{\alpha\beta} \frac{\partial P}{\partial x^\alpha} \partial_{x^\beta} = \delta^{\alpha\beta} \frac{\partial P}{\partial x^\alpha} \partial_{x^\beta} = \sum_{\alpha=1}^d \frac{\partial P}{\partial x^\alpha} \partial_{x^\alpha}$$

in standard Euclidean coordinates $x = (x^\alpha)$. Of course, for each $x \in \mathbb{R}^d$, the Riemannian norm $|\nabla P_x|_{\overline{g}}$ of $\nabla P_x \in T_x(\mathbb{R}^d)$ coincides with the Euclidean norm $|\nabla P(x)|$ of $\nabla P(x) \in \mathbb{R}^d$. These equivalent quantities (functions) will be henceforth denoted by $|\nabla P|$.

Proposition 5.1 *For each $\eta \in S$,*

$$\overline{g}(\nabla P, \mathcal{E})_\eta = \nabla P(\eta) \cdot (E\eta) = 1.$$

In particular, ∇P (and dP) never vanishes on S and so S is a smooth compact embedded hypersurface in $\mathbb{R}^d$.

Proof Given that $E \in \operatorname{Exp}(P)$ and $P \in C^\infty(\mathbb{R}^d)$, we differentiate the identity $r P(x) = P(r^E x)$ to find that

$$P(x) = \frac{d}{dr} P(r^E x) = \nabla P(r^E x) \cdot \left(\left(r^{E-I} E \right) x \right)$$

for $r > 0$ and $x \in \mathbb{R}^d$. In particular, when $r = 1$ and $x = \eta \in S$, we have

$$
1 = \nabla P(\eta) \cdot (E\eta) = \frac{\partial P}{\partial x^\alpha}(E\eta)^\alpha = \overline{g}_{\alpha\beta}\left(\nabla P_\eta\right)^\alpha (\mathcal{E}_\eta)^\beta = \overline{g}(\nabla P, \mathcal{E})_\eta.
$$

Thus, our (necessarily) compact level set S is a smooth embedded hypersurface of $\mathbb{R}^d$ in view of the regular level set theorem. $\qquad\square$

We shall denote by $\iota : S \hookrightarrow \mathbb{R}^d$ the canonical inclusion map and set $d' = d - 1$. As an embedded submanifold of $\mathbb{R}^d$, S is a Riemannian submanifold of $\mathbb{R}^d$ with metric g^S given by

$$
g^S(X, Y) = \overline{g}(\iota_*(X), \iota_*(Y))
$$

for $X, Y \in \mathfrak{X}(S)$; here, for each $\eta \in S$, $\iota_* : T_\eta(S) \to T_\eta(\mathbb{R}^d)$ is the pushforward of ι. In view of the preceding proposition, $N := \nabla P/|\nabla P| \in \mathfrak{X}(\mathbb{R}^d)$ is a smooth unit normal vector field along S and it determines an orientation on the Riemannian manifold S. Equipped with this orientation, (S, g^S) is an oriented Riemannian manifold and we shall denote by $d\,\mathrm{Vol}_S$ the Riemannian volume form and by $\mathcal{A}_+(S)$ its corresponding (maximal) oriented atlas. By virtue of Proposition 15.21 of [19], $d\,\mathrm{Vol}_S = (N \lrcorner\, d\,\mathrm{Vol}_{\mathbb{R}^d})|_S$, i.e.,

$$
\begin{aligned}
d\,\mathrm{Vol}_S(X_1, X_2, \ldots, X_{d'}) &= d\,\mathrm{Vol}_{\mathbb{R}^d}(N, \iota_*(X_1), \iota_*(X_2), \ldots, \iota_*(X_{d'})) \\
&= \frac{1}{|\nabla P|} d\,\mathrm{Vol}_{\mathbb{R}^d}(\nabla P, \iota_*(X_1), \iota_*(X_2), \ldots, \iota_*(X_{d'}))
\end{aligned}
$$

for any collection $\{X_1, X_2, \ldots, X_{d'}\} \in \mathfrak{X}(S)$. Beyond $d\,\mathrm{Vol}_S \in \Omega^{d'}(S)$, we consider the following smooth d'-form(s): Given $E \in \mathrm{Exp}(P)$, define $d\sigma_{P,E} \in \Omega^{d'}(S)$ by

$$
d\sigma_{P,E}(X_1, X_2, \ldots, X_{d'}) = d\,\mathrm{Vol}_{\mathbb{R}^d}(\mathcal{E}_E, \iota_*(X_1), \iota_*(X_2), \ldots, \iota_*(X_{d'}))
$$

for $X_1, X_2, \ldots, X_{d'} \in \mathfrak{X}(S)$. We have

Proposition 5.2 *For any $E \in \mathrm{Exp}(P)$,*

$$
d\sigma_{P,E} = \frac{1}{|\nabla P|} d\,\mathrm{Vol}_S .
$$

In particular, $d\sigma_{P,E}$ is positively oriented and is independent of $E \in \mathrm{Exp}(P)$.

Before proving the proposition, we first treat a lemma of a purely linear algebraic nature.

Lemma 5.3 *Let $v_1, v_2, \ldots, v_{d'}$ be linearly independent vectors in $\mathbb{R}^d$ and suppose that $w \in \mathbb{R}^d \setminus \{0\}$ is such that $w \perp v_i$ for all i. Then, for any $z \in \mathbb{R}^d$ for which $z \cdot w = 1$,*

$$\det(z, v_1, v_2, \ldots, v_{d'}) = \frac{1}{|w|} \det(n, v_1, v_2, \ldots, v_{d'}) = \frac{1}{|w|^2} \det(w, v_1, v_2, \ldots, v_{d'}).$$

where $n := w/|w|$.

Proof Given $z \in \mathbb{R}^d$ such that $z \cdot w = 1$, it follows that

$$z = \frac{1}{|w|} n + a_1 v_1 + a_2 v_2 + \cdots a_{d'} v_{d'}.$$

By the multilinearity of the determinant map, we have

$$
\begin{aligned}
&\det(z, v_1, v_2, \ldots, v_{d'}) \\
&= \det\left(\frac{1}{|w|} n + a_1 v_2 + a_2 v_2 + \cdots a_{d'} v_{d'}, v_1, v_2, \ldots, v_{d'}\right) \\
&= \frac{1}{|w|} \det(n, v_1, v_2, \ldots, v_{d'}) + \det(a_1 v_1 + \cdots + a_{d'} v_{d'}, v_1, v_2, \ldots, v_{d'}) \\
&= \frac{1}{|w|} \det(n, v_1, v_2, \ldots, v_{d'}) + 0
\end{aligned}
$$

where we have used the fact that the columns of the matrix $(a_1 v_1 + \cdots + a_{d'} v_{d'}, v_1, v_2, \ldots, v_{d'})$ are linearly dependent to conclude that the final determinant is zero. $\qquad\square$

Proof of Proposition 5.2 We fix $E \in \mathrm{Exp}(P)$ and note that the assertion at hand is a local one. Thus, it suffices to verify that, for any $\eta \in S$ and $X_1, X_2, \ldots, X_{d'} \in T_\eta(S)$,

$$d\sigma_{P,E}(X_1, X_2, \ldots, X_{d'}) = \frac{1}{|\nabla P_\eta|} d\,\mathrm{Vol}_{\mathbb{R}^d}(N_\eta, \iota_*(X_1), \iota_*(X_2), \ldots, \iota_*(X_{d'})).$$

Fix $\eta \in S$ and let $(\mathcal{O}, \varphi)$ be a coordinate chart centered at η with local coordinates $u = (u^\alpha)$. As usual, denote by $x = (x^\alpha)$ the Euclidean coordinates on $\mathbb{R}^d$. For each $i = 1, 2, \ldots, d'$,

$$X_i = X_i^\alpha \partial_{u^\alpha} \quad \text{and} \quad \iota_*(X_i) = v_i^\beta \partial_{x^\beta}$$

where

$$v_i^\beta = X_i^\alpha \frac{\partial x^\beta}{\partial u^\alpha}.$$

For each $i = 1, 2, \ldots, d'$, we set $v_i = (v_i^1, v_i^2, \ldots, v_i^{d'}) \in \mathbb{R}^d$. Also, let $w = \nabla P(\eta) \in \mathbb{R}^d$ with $|w| = |\nabla P(\eta)| = |\nabla P_\eta|$, set $n = w/|w| \in \mathbb{R}^d$ and note that

$$N_\eta = \frac{1}{|\nabla P_\eta|} \sum_{k=1}^{d} \frac{\partial P}{\partial x^k} \partial_{x^k} = n^\mu \partial_{x^\mu}.$$

Given that ∇P is normal to S, we have

$$v_i \cdot w = \overline{g}_{\mu,\nu} v_i^\mu w^\nu = \overline{g}(\iota_*(X_i), \nabla P)_\eta = 0$$

and therefore $w \perp v_i$ for each $i = 1, 2, \ldots, d'$. Upon recalling that $(\mathcal{E}_E)_\eta = (E\eta)^\alpha \partial_{x^\alpha}$, set $z = ((E\eta)^1, (E\eta)^2, \ldots, (E\eta)^d) \in \mathbb{R}^d$ and observe that $z \cdot w = 1$ by virtue of Proposition 5.1. An appeal to the lemma guarantees that

$$
\begin{aligned}
d\sigma_{P,E}(X_1, X_2, \ldots, X_{d'}) &= d\,\mathrm{Vol}_{\mathbb{R}^d}(\mathcal{E}_E, \iota_*(X_1), \iota_*(X_2), \ldots, \iota_*(X_{d'})) \\
&= \det(z, v_1, v_2, \ldots, v_{d'}) \\
&= \frac{1}{|w|} \det(n, v_1, v_2, \ldots, v_{d'}) \\
&= \frac{1}{|\nabla P_\eta|} d\,\mathrm{Vol}_{\mathbb{R}^d}(N_\eta, \iota_*(X_1), \iota_*(X_2), \ldots, \iota_*(X_{d'})) \\
&= \frac{1}{|\nabla P|} d\,\mathrm{Vol}_S(X_1, X_2, \ldots, X_{d'}).
\end{aligned}
$$

$\square$

By virtue of the preceding proposition, we shall denote by $d\sigma_P$ the unique smooth d'-form on S which satisfies

$$d\sigma_P = d\sigma_{P,E} = \frac{1}{|\nabla P|} d\,\mathrm{Vol}_S \tag{53}$$

for all $E \in \mathrm{Exp}(P)$. In this notation, we have this section's central result.

Theorem 5.4 *Let P be a smooth positive homogeneous function and let $S = \{\eta \in \mathbb{R}^d : P(\eta) = 1\}$. Then S is a compact smooth embedded hypersurface of $\mathbb{R}^d$. Viewing $\mathbb{R}^d$ as an oriented Riemannian manifold with its usual orientation and metric $\overline{g}$, $N = \nabla P/|\nabla P|$ is a smooth unit normal vector field along S. As a submanifold of $\mathbb{R}^d$, S is a oriented Riemannian manifold of dimension $d' = d - 1$ with its induced Riemannian metric g^S, volume form $d\,\mathrm{Vol}_S \in \Omega^{d'}(S)$ and orientation determined by N. The σ-algebras Σ_P and $\mathcal{L}(S)$ on S coincide and the smooth d'-form $d\sigma_P \in \Omega^{d'}(S)$, defined by (53), coincides with the measure σ_P in the sense that*

$$\int_S g(\eta)\, \sigma_P(d\eta) = \int_S g\, d\sigma_P \tag{54}$$

for all $g \in L^1(S, \Sigma_P, \sigma_P)$; here, the left hand side represents the Lebesgue integral of g with respect to σ_P and the right hand side is the integral of the measurable d'-form $g\, d\sigma_P$. Furthermore, the measure σ_P and the canonical Riemannian volume measure Vol_S on S are mutually absolutely continuous.

Before we are able to prove the above theorem, we first treat two lemmas which we will find useful in computation.

Lemma 5.5 *Let $(\mathcal{U}, \varphi) \in \mathcal{A}(S)$ and $E \in \mathrm{Exp}(P)$. Set $U = \varphi(\mathcal{U}) \subseteq \mathbb{R}^{d'}$, $V = (0, 1) \times U$ and define $\rho_{E,\varphi} : V \to \mathbb{R}^d$ and $h_{E,\varphi} : U \to \mathbb{R}$, respectively, by*

$$\rho_{E,\varphi}(y) = \psi_E(r, \varphi^{-1}) = r^E \varphi^{-1}(u)$$

for $y = (r, u) \in V$ and

$$h_{E,\varphi}(u) = \det \left(E\varphi^{-1}(u) \,\middle|\, D_u \varphi^{-1}(u) \right)$$

for $u \in U$; here, the vertical bar separates the first column of the (necessarily) $d \times d$ matrix from the rightmost $d \times d'$ submatrix and D_u denotes the Jacobian in the coordinates $u = (u^1, u^2, \dots, u^{d'}) \in U$. Then, $\rho_{E,\varphi}$ is a diffeomorphism onto its image $\rho_{E,\varphi}(V) = \widetilde{\mathcal{U}}_E$ and its Jacobian matrix $D\rho_{E,\varphi}$ has

$$\det(D\rho_{E,\varphi}(y)) = r^{\mu_P - 1} h_{E,\varphi}(u) \tag{55}$$

for all $y = (r, u) \in V$. Furthermore, $h_{E,\varphi}$ is everywhere non-zero, smooth and

$$d\sigma_P = h_{E,\varphi}(u)\, du^1 \wedge du^2 \wedge \cdots \wedge du^{d'} \tag{56}$$

in the coordinates $u = (u^1, u^2, \dots, u^{d'}) \in U$.

Proof The map $\rho_{E,\varphi}$ is smooth because φ is smooth. By virtually the same argument made in the proof of Proposition 4.1, which here uses the fact that $P \in C^\infty(\mathbb{R}^d)$, we conclude that $\rho_{E,\varphi}$ is a diffeomorphism onto

$$\rho_{E,\varphi}(V) = \bigcup_{0 < r < 1} r^E \mathcal{U} = \widetilde{\mathcal{U}}_E$$

with inverse $\rho_{E,\varphi}^{-1}(x) = (P(x), \varphi((P(x))^{-E} x))$. For $y = (r, u) \in V$, observe that

$$
\begin{aligned}
D\rho_{E,\varphi}(y) &= \left(\frac{d}{dr}(r^E \varphi^{-1}(u)) \,\middle|\, D_u \left[r^E \varphi^{-1}(u) \right] \right) \\
&= \left(r^{E-I} E\varphi^{-1}(u) \,\middle|\, r^E D_u \varphi^{-1}(u) \right) \\
&= r^E \left(\frac{1}{r} E\varphi^{-1}(u) \,\middle|\, D_u \varphi^{-1}(u) \right).
\end{aligned}
$$

Using properties of the determinant, we have

$$
\begin{aligned}
\det(D\rho_E(y)) &= \det(r^E) \det \left(\frac{1}{r} E\varphi^{-1}(u) \,\middle|\, D_u \varphi^{-1}(u) \right) \\
&= r^{\mathrm{tr}\, E} r^{-1} \det \left(E\varphi^{-1}(u) \,\middle|\, D_u \varphi^{-1}(u) \right) \\
&= r^{\mu_P - 1} h_{E,\varphi}(u)
\end{aligned}
$$

for all $y = (r, u) \in V$ thus proving (55). It is clear that $h_{E,\varphi}$ is smooth and, by virtue of the fact that $\rho_{E,\varphi} : (0, 1) \times U \to \widetilde{\mathcal{U}_E}$ is a diffeomorphism, (55) guarantees that $h_{E,\varphi}$ is everywhere non-vanishing. Finally, in view of Proposition 5.2 (and (53)), we find that

$$
\begin{aligned}
h_{E,\varphi}(u) &= \det \left(E\varphi^{-1}(u) \,\middle|\, D_u\varphi^{-1}(u) \right) \\
&= (d \operatorname{Vol}_{\mathbb{R}^d})_{\varphi^{-1}(u)} (\mathcal{E}_E, \iota_*(\partial_{u^1}), \iota_*(\partial_{u^2}), \ldots, \iota_*(\partial_{u^{d'}})) \\
&= (d\sigma_P)_{\varphi^{-1}(u)} (\partial_{u^1}, \partial_{u^2}, \ldots, \partial_{u^{d'}})
\end{aligned}
$$

for $u \in U$ and so (56) is satisfied. $\qquad\square$

Lemma 5.6 *Let $g \in L^1(S, \Sigma_P, \sigma_P)$ be supported on the domain of some chart on S. Then, for any $(\mathcal{U}, \varphi) \in \mathcal{A}^+(S)$ such that $\mathrm{Supp}(g) \subseteq \mathcal{U}$, the pushforward $(\varphi^{-1})^*(g) = g \circ \varphi^{-1}$ is Lebesgue measurable on $U = \varphi(\mathcal{U}) \subseteq \mathbb{R}^{d'}$ and*

$$
\int_S g(\eta)\sigma_P(d\eta) = \int_U (\varphi^{-1})^*(g \, d\sigma_P).
$$

Proof Let $(\mathcal{U}, \varphi)$ be a chart on S for which $\mathrm{Supp}(g) \subseteq \mathcal{U}$ and assume the notation of Lemma 5.5. Given $E \in \mathrm{Exp}(P)$, observe that

$$
f(x) = \mu_P \, \chi_{(0,1)}(P(x))g(P(x)^{-E}x),
$$

defined for $x \in \mathbb{R}^d \setminus \{0\}$, is supported on $\widetilde{\mathcal{U}_E} = \rho_{E,\varphi}(V)$. An appeal to Corollary 1.6 guarantees that f is absolutely integrable on $\rho_{E,\varphi}(V)$ and

$$
\int_S g(\eta) \, \sigma_P(d\eta) = \int_{\rho_{E,\varphi}(V)} f(x) \, dx. \tag{57}
$$

Given that $\rho_{E,\varphi}$ is a diffeomorphism, an appeal to Theorem 15.11 of [2] guarantees that

$$
\int_{\rho_{E,\varphi}(V)} f(x) \, dx = \int_V f(\rho_{E,\varphi}(y)) |\det(D\rho_{E,\varphi}(y))| \, dy; \tag{58}
$$

in particular, $V \ni y \mapsto f(\rho_{E,\varphi}(y)) |\det(D\rho_{E,\varphi}(y))$ is Lebesgue measurable on $\mathbb{R}^d$. Let us now view the Lebesgue measure dy on $\mathbb{R}^d$ as the completion of the product measure $dr \times du$ on the product space $\mathbb{R} \times \mathbb{R}^{d'}$. By virtue of Lemma 5.5, we have

$$
\begin{aligned}
f(\rho_{E,\varphi}&(y)) |\det(D\rho_{E,\varphi}(y))| \\
&= f(r^E\varphi^{-1}(u)) \, r^{\mu_P - 1} |h_{E,\varphi}(u)| \\
&= \mu_P \, \chi_{(0,1)}(r)g(\varphi^{-1}(u)) \, r^{\mu_P - 1} |h_{E,\varphi}(u)| \\
&= \mu_P \, r^{\mu_P - 1} g(\varphi^{-1}(u)) |h_{E,\varphi}(u)| \tag{59}
\end{aligned}
$$

for $y = (r, u) \in V$. By virtue of Fubini's theorem, dr-almost every $r \in (0, 1)$, the r-section

$$U \ni u \mapsto \mu_P r^{\mu_P - 1} g(\varphi^{-1}(u))|h_{E,\varphi}(u)|$$

is Lebesgue measurable and, upon recalling that $h_{E,\varphi}$ is smooth and everywhere nonzero, we conclude that $(\varphi^{-1})^*(g) = g \circ \varphi^{-1}$ is Lebesgue measurable on U. In view of (59), Fubini's theorem also guarantees that

$$
\begin{aligned}
\int_V & f(\rho_{E,\varphi}(y))|\det(D\rho_{E,\varphi}(y))|\, dy \\
&= \int_{(0,1)} \int_U \mu_P\, r^{\mu_P - 1} g(\varphi^{-1}(u))|h_{E,\varphi}(u)|\, du\, dr \\
&= \int_0^1 \mu_P\, r^{\mu_P - 1} \left(\int_U g(\varphi^{-1}(u))|h_{E,\varphi}(u)|\, du \right) dr \\
&= \int_U g(\varphi^{-1}(u))|h_{E,\varphi}(u)|\, du
\end{aligned}
\tag{60}
$$

By combining (57), (58) and (60), we have shown that $(\varphi^{-1})^*g = g \circ \varphi^{-1}$ is Lebesgue measurable on $U = \varphi(\mathcal{U})$ and

$$\int_S g(\eta)\sigma_P(d\eta) = \int_U g(\varphi^{-1}(u))|h_{E,\varphi}(u)|\, du.$$

Finally, if $(\mathcal{U}, \varphi) \in \mathcal{A}_+(S)$, an appeal to Proposition 5.2 and (56) of Lemma 5.5 guarantees that

$$|h_{E,\varphi}(u)| = h_{E,\varphi}(u) = (d\sigma_P)_{\varphi^{-1}(u)}(\partial_{u^1}, \partial_{u^2}, \ldots, \partial_{u^{d'}}) > 0$$

for all $u = (u^1, u^2, \ldots, u^{d'}) \in U$ and thus

$$
\begin{aligned}
\int_S g(\eta)\sigma_P(d\eta) &= \int_U g(\varphi^{-1}(u))h_{E,\varphi}(u)\, du \\
&= \int_U g(\varphi^{-1}(u))\, h_{E,\varphi}(u)\, du^1 \wedge du^2 \wedge \cdots \wedge du^{d'} \\
&= \int_U (\varphi^{-1})^*(g \cdot d\sigma_P),
\end{aligned}
$$

as desired.   □

Remark 3 In studying the proof of Lemma 5.6, we deduce the (slightly) more general statement: Given any chart $(\mathcal{U}, \varphi) \in \mathcal{A}(S)$ and $g \in L^1(S, \Sigma_P, \sigma_P)$ for which

$\text{Supp}(g) \in \mathcal{U}$, $(\varphi^{-1})^* g$ is Lebesgue measurable on $U = \varphi(\mathcal{U})$ and

$$
\int_S g(\eta)\sigma_P(d\eta) = \begin{cases} \displaystyle\int_U (\varphi^{-1})^*(g\,d\sigma_P) & \text{if } (\mathcal{U}, \varphi) \in \mathcal{A}_+(S) \\[4mm] -\displaystyle\int_U (\varphi^{-1})^*(g\,d\sigma_P) & \text{if } (\mathcal{U}, \varphi) \in \mathcal{A}_-(S) := \mathcal{A}(S) \setminus \mathcal{A}_+(S). \end{cases}
$$

Proof of Theorem 5.4 In view of Proposition 5.1 and the discussion following its proof, it remains to prove the assertions in the last two sentences in the statement of the theorem. Given any $F \in \Sigma_P$ and chart $(\mathcal{U}, \varphi) \subseteq \mathcal{A}(S)$, we have $\chi_{\varphi(F \cap \mathcal{U})} = (\varphi^{-1})^* (\chi_{F \cap \mathcal{U}})$ is Lebesgue measurable on $U = \varphi(\mathcal{U})$ and so it follows (Exercise 4.6.2 of [20]) that $F \in \mathcal{L}(S)$. Consequently, $\mathcal{B}(S) \subseteq \Sigma_P \subseteq \mathcal{L}(S)$.

Let $g \in L^1(S, \Sigma_P, \sigma_P)$, which is necessarily Lebesgue measurable on S in view of the results of the previous paragraph. Now, let $\{(\mathcal{U}_j, \varphi_j)\} \subseteq \mathcal{A}_+(S)$ be a countable atlas on S and let $\{\kappa_j\}$ be a smooth partition of unity subordinate to the cover $\{\mathcal{U}_j\}$. For each $j \in \mathbb{N}$, observe that $\kappa_j g \in L^1(S, \Sigma_P, \sigma_P)$ and has $\text{Supp}(\kappa_j g) \subseteq \mathcal{U}_j$. By virtue of Lemma 5.6, we have $(\varphi_j^{-1})^*(\kappa_j g\,d\sigma_P)$ is integrable on $U_j = \varphi(\mathcal{U}_j)$ and

$$
\int_S \kappa_j(\eta) g(\eta)\sigma_P(d\eta) = \int_{U_j} (\varphi_j^{-1})^*(\kappa_j g\,d\sigma_P) \tag{61}
$$

for each $j \in \mathbb{N}$. With the help of Proposition 5.2, it is easy to see that $g\,d\sigma_P$ and $\kappa_j g\,d\sigma_P$ (for $j \in \mathbb{N}$) are Lebesgue measurable d'-forms on S. In view of (4.4.6) of [20], (61) ensures that, for each $j \in \mathbb{N}$, $\kappa_j g\,d\sigma_P$ is integrable (in the sense of forms) on S and

$$
\int_S \kappa_j(\eta) g(\eta)\sigma_P(d\eta) = \int_S \kappa_j g\,d\sigma_P.
$$

By the monotone convergence theorem, it follows that

$$
\sum_{j=1}^\infty \left| \int_S \kappa_j g\,d\sigma_P \right| = \sum_{j=1}^\infty \left| \int_S \kappa_j(\eta) g(\eta)\sigma(d\eta) \right| \leq \sum_{j=1}^\infty \int_S \kappa_j(\eta) |g(\eta)|\sigma_P(d\eta)
$$
$$
= \|g\|_{L^1(S, \Sigma_P, \sigma_P)} < \infty.
$$

Therefore, in view of the construction on p. 242 of [20], we conclude that the d'-form $g\,d\sigma_P$ is integrable and

$$
\int_S g(\eta)\sigma_P(d\eta) = \sum_{j=1}^\infty \int_S \kappa_j(\eta) g(\eta)\sigma_P(d\eta) = \sum_{j=1}^\infty \int_S \kappa_j g\,d\sigma_P = \int_S g\,d\sigma_P
$$

by virtue of the dominated convergence theorem; this is (54).

Finally, given that $|\nabla P|$ is continuous and non-vanishing on the compact set S,

$$C_1 := \inf_S \frac{1}{|\nabla P|} \quad \text{and} \quad C_2 := \sup_S \frac{1}{|\nabla P|}$$

are both positive real numbers. For each $F \in \Sigma_P \subseteq \mathcal{L}(S)$, (54) guarantees that

$$\sigma_P(F) = \int_S \chi_F(\eta)\sigma_P(d\eta) = \int_S \chi_F \, d\sigma_P$$

By virtue of Proposition 5.2, it follows that

$$C_1 \operatorname{Vol}_S(F) = C_1 \int_S \chi_F \, d\operatorname{Vol}_S \leq \int_S \frac{\chi_F}{|\nabla P|} d\operatorname{Vol}_S = \int_S \chi_F \, d\sigma_P = \sigma_P(F)$$

and

$$\sigma_P(F) = \int_S \chi_F d\sigma_P = \int_S \frac{\chi_F}{|\nabla P|} d\operatorname{Vol}_S \leq C_2 \int_S \chi_F \, d\operatorname{Vol}_S = C_2 \operatorname{Vol}_S(F)$$

where we have used the definition of the Riemannian volume measure on S, c.f., [1]. In short, there are positive constants C_1 and C_2 for which

$$C_1 \operatorname{Vol}_S(F) \leq \sigma_P(F) \leq C_2 \operatorname{Vol}_S(F) \tag{62}$$

for all $F \in \Sigma_P$. In particular, (54) holds for all $F \in \mathcal{B}(S)$ and so it follows that the completions of the σ-algebra $\mathcal{B}(S)$ with respect to σ_P and Vol_S coincide. We know, however, that Vol_S, which is defined on $\mathcal{L}(S)$, is a Radon measure (Proposition 1.5 [1,Chapter XII]) and, by virtue of Proposition 4.10, it follows that $\Sigma_P = \mathcal{L}(S)$ and (62) holds for all F in this common σ-algebra. Thus σ_P and Vol_S are mutually absolutely continuous and the theorem is proved. □

We immediately obtain the following corollary which allows us to compute the Lebesgue integral with respect to σ_P in coordinates.

Corollary 5.7 *Let $g \in L^1(S, \mathcal{L}(S), \sigma_P)$. Then, given any countable (or finite) atlas $\{(\mathcal{U}_j, \varphi_j)\} \subseteq \mathcal{A}_+(S)$, smooth partition of unity $\{\kappa_j\}$ subordinate to $\{\mathcal{U}_j\}$, and $E \in \operatorname{Exp}(P)$,*

$$\int_S g(\eta)\sigma_P(d\eta) = \sum_j \int_S \kappa_j g \, d\sigma_P = \sum_j \int_{U_j} \kappa_j(\varphi^{-1}(u)) g(\varphi^{-1}(u)) h_{E,\varphi_j}(u) \, du$$

where, for each j, $U_j = \varphi_j(\mathcal{U}_j) \subseteq \mathbb{R}^{d'}$ and

$$h_{E,\varphi_j}(u) = \det(E\varphi^{-1}(u)|D_u\varphi^{-1})$$

for $u = (u^1, u^2, \ldots, u^{d'}) \in U_j$.

A Appendix

This appendix amasses some facts about continuous one-parameter subgroups of $\mathrm{Gl}(\mathbb{R}^d)$ and, in particular, those which are contracting.

Proposition A.1 (see Section 8 of [24]) *Let $E, G \in \mathrm{End}(\mathbb{R}^d)$ and $A \in \mathrm{Gl}(\mathbb{R}^d)$. Also, let E^* denote the adjoint of E. Then, for all $t, s > 0$, the following statements hold:*

- $1^E = I$
- $t^{E^*} = (t^E)^*$
- $(t^E)^{-1} = t^{-E}$
- If $EG = GE$, then $t^E t^G = t^{E+G}$
- $(st)^E = s^E t^E$
- $At^E A^{-1} = t^{AEA^{-1}}$
- $\det\left(t^E\right) = t^{\mathrm{tr} E}$

We recall from the introduction that a continuous one-parameter group $\{T_t\}$ is said to be contracting if $\lim_{t \to 0} \|T_t\| = 0$. The notion of a contracting group (or semigroup) is closely related to notion of asymptotic stability of solutions to linear systems [5]. By virtue of the Banach-Steinhaus theorem, we have the following useful characterization of contracting groups.

Proposition A.2 *Let $\{T_r\}$ be a continuous one-parameter group. Then $\{T_r\}$ is contracting if and only if*

$$\lim_{t \to 0} |T_r x| = 0 \tag{63}$$

for all $x \in \mathbb{R}^d$.

Proof It is clear that (63) is a necessary condition for $\{T_r\}$ to be contracting. We must therefore prove (63) is also sufficient. To this end, we assume that the continuous one-parameter group $\{T_r\}$ satisfies (63). By virtue of the continuity of $\{T_r\}$ and (63), we have

$$\sup_{0 < r \le 1} |T_r x| < \infty$$

for each $x \in \mathbb{R}^d$. From the Banach-Steinhaus theorem, it follows that $\|T_r\| \le C$ for all $0 < r \le 1$. Now, suppose that $\{T_r\}$ is not contracting. In this case, one can find a sequence $\{\eta_n\} \subseteq \mathbb{S}$ and a sequence $r_n \to 0$ for which $\lim_n |T_{r_n} \eta_n| > 0$. But because the unit sphere is compact, $\{\eta_n\}$ has a convergence subsequence $\eta_{n_k} \to \eta$ with $|\eta| = 1$. Observe that, for all n,

$$|T_{r_n}(\eta - \eta_n)| \le C |\eta - \eta_n|$$

and so it follows that

$$\lim_{k\to\infty} |T_{t_{r_k}}\eta| = \lim_{k\to\infty} |T_{r_{n_k}}\eta_{n_k}| > 0,$$

a contradiction.   □

Lemma A.3 *Let $\{T_r\} \subseteq Gl_d(\mathbb{R})$ be a continuous one-parameter group and let $E \in$ $\mathrm{End}(\mathbb{R}^d)$ be its generator, i.e., $T_r = r^E$ for all $r > 0$. If $\{T_r\}$ is contracting, then $E \in \mathrm{Gl}(\mathbb{R}^d)$ and there is a positive constant C for which*

$$\|T_r\| \leq C + r^{\|E\|}$$

for all $r > 0$.

Proof If for some non-zero vector η, $E\eta = 0$, then $r^E\eta = \eta$ for all $r > 0$ and this would contradict our assumption that $\{T_r\}$ is contracting. Hence $E \in \mathrm{Gl}(\mathbb{R}^d)$ and, in particular, $\|E\| > 0$. From the representation $T_r = r^E$, it follows immediately that $\|T_r\| \leq r^{\|E\|}$ for all $r \geq 1$ and so it remains to estimate $\|T_r\|$ for $r < 1$. Given that $\{T_r\}$ is continuous and contracting, the map $r \mapsto \|T_r\|$ is continuous and approaches 0 as $r \to 0$ and so it is necessarily bounded for $0 < r \leq 1$.   □

Proposition A.4 *Let $\{T_r\}_{r>0} \subseteq \mathrm{Gl}(\mathbb{R}^d)$ be a continuous one-parameter contracting group. Then, for all non-zero $x \in \mathbb{R}^d$,*

$$\lim_{r\to 0} |T_r x| = 0 \ \text{ and } \ \lim_{r\to\infty} |T_r x| = \infty.$$

Proof The validity of the first limit is clear. Upon noting that $|x| = |T_{1/r} T_r x| \leq \|T_{1/r}\||T_r x|$ for all $r > 0$, the second limit follows at once.   □

Proposition A.5 *Let $\{T_r\}_{r>0}$ be a continuous one-parameter contracting group. There holds the following:*

(a) *For each non-zero $x \in \mathbb{R}^d$, there exists $r > 0$ and $\eta \in \mathbb{S}$ for which $T_r\eta = x$. Equivalently,*

$$\mathbb{R}^d \setminus \{0\} = \{T_r\eta : r > 0 \text{ and } \eta \in \mathbb{S}\}.$$

(b) *For each sequence $\{x_n\} \subseteq \mathbb{R}^d$ such that $\lim_n |x_n| = \infty$, $x_n = T_{r_n}\eta_n$ for each n, where $\{\eta_n\} \subseteq \mathbb{S}$ and $r_n \to \infty$ as $n \to \infty$.*

(c) *For each sequence $\{x_n\} \subseteq \mathbb{R}^d$ such that $\lim_n |x_n| = 0$, $x_n = T_{r_n}\eta_n$ for each n, where $\{\eta_n\} \subseteq \mathbb{S}$ and $r_n \to 0$ as $n \to \infty$.*

Proof In view of Proposition A.4, the assertion (a) is a straightforward application of the intermediate value theorem. For (b), suppose that $\{x_n\} \subseteq \mathbb{R}^d$ is such that $|x_n| \to \infty$ as $n \to \infty$. In view of (a), take $\{\eta_n\} \subseteq \mathbb{S}$ and $\{r_n\} \subseteq (0, \infty)$ for which $x_n = T_{r_n}\eta_n$ for each n. In view of Lemma A.3,

$$\infty = \liminf_n |x_n| \leq \liminf_n \left(C + r_n^M\right)|\eta_n| \leq C + \liminf_n r_n^M,$$

where C, $M > 0$ and therefore $r_n \to \infty$. If instead $\lim_n x_n = 0$,

$$\infty = \lim_{n \to \infty} \frac{|\eta_n|}{|x_n|} = \lim_{n \to \infty} \frac{|T_{1/r_n} x_n|}{|x_n|} \leq \limsup_n \|T_{1/r_n}\| \leq \limsup_n (C + (1/r_n)^M)$$

from which we see that $r_n \to 0$, thus proving (c).   □

Proposition A.6 *Let $\{T_r\}$ be a continuous contracting one-parameter group. Then for any open neighborhood $\mathcal{O} \subseteq \mathbb{R}^d$ of the origin and any compact set $K \subseteq \mathbb{R}^d$, $K \subseteq T_r(\mathcal{O})$ for sufficiently large r.*

Proof Assume, to reach a contradiction, that there are sequences $\{x_n\} \subseteq K$ and $r_n \to \infty$ for which $x_n \notin T_{r_n}(\mathcal{O})$ for all n. Because K is compact, $\{x_n\}$ has a subsequential limit and so by relabeling, let us take sequences $\{\zeta_k\} \subseteq K$ and $\{t_k\} \subseteq (0, \infty)$ for which $\zeta_k \to \zeta$, $t_k \to \infty$ and $\zeta_k \notin T_{t_k}(\mathcal{O})$ for all k. Setting $s_k = 1/t_k$ and using the fact that $\{T_r\}$ is a one-parameter group, we have $T_{s_k} \zeta_k \notin \mathcal{O}$ for all k and so $\liminf_k |T_{s_k} \zeta_k| > 0$, where $s_k \to 0$. This is however impossible because $\{T_r\}$ is contracting and so

$$\lim_{k \to \infty} |T_{s_k} \zeta_k| \leq \lim_{k \to \infty} |T_{s_k}(\zeta_k - \zeta)| + \lim_{k \to \infty} |T_{s_k} \zeta| \leq C \lim_{k \to \infty} |\zeta_k - \zeta| + 0 = 0$$

in view of Lemma A.3.   □

Proposition A.7 *Let $\{T_r\} \subseteq \mathrm{Gl}(\mathbb{R}^d)$ be a continuous one-parameter group with generator E. If $\{T_r\}$ is contracting, then $\mathrm{tr}\, E > 0$.*

Proof The supposition that $\{T_r\}$ is a contracting group implies that $r^E \to \mathbf{0}$ as $r \to 0$ in the operator-norm topology on $\mathrm{End}(\mathbb{R}^d)$; here $\mathbf{0}$ is zero transformation. Because the determinant is a continuous function from $\mathrm{End}(\mathbb{R}^d)$, equipped with operator-norm topology, into $\mathbb{R}$, we have

$$0 = \det(\mathbf{0}) = \det\left(\lim_{r \to 0} r^E\right) = \lim_{r \to 0} \det\left(r^E\right) = \lim_{r \to 0} r^{\mathrm{tr}\, E}$$

in view of the preceding proposition. Therefore, $\mathrm{tr}\, E > 0$.   □

References

1. Amann, H., Escher, J.: Analysis III. Birkhäuser, Basel (2009)
2. Apostol, T.M.: Mathematical Analysis, 2nd edn. Addison-Wesley Publishing Company, Boston (1974)
3. Baker, J.A.: Integration over spheres and the divergence theorem for balls. Am. Math. Monthly **104**(1), 36–47 (1997)
4. Bogachev, V.I.: Measure Theory, vol. 1. Springer, New York (2007)
5. Braun, M.: Differential Equations and Their Applications Texts in Applied Mathematics, vol. 11, 4th edn. Springer, New York (1993)
6. Chavel, I.: Riemannian geometry: a modern introduction, volume 98 of Cambridge studies. In Advanced Mathematics. 2nd ed. Cambridge University Press, Cambridge (2006)
7. Coulombel, J.-F., Faye, G.: Generalized gaussian bounds for discrete convolution powers. arXiv:2012.09437 [math.NA] (2020)

8. Diaconis, P., Saloff-Coste, L.: Convolution powers of complex functions on $\mathbb{Z}$. Math. Nachrichten **287**(10), 1106–1130 (2014)
9. Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics, vol. 194. Springer, New York (2000)
10. Engel, K.-J., Nagel, R.: A Short Course on Operator Semigroups. Universitext. Springer, New York (2006)
11. Folland, G.B.: How to integrate a polynomial over a sphere. Am. Math. Monthly **108**(5), 446–448 (2001)
12. Folland, G.B.: Real Analysis: Modern Techniques and Their Applications. Pure and Applied Mathematics, 2nd edn. Wiley, New York (2013)
13. Folland, G.B., Stein, E.M.: Hardy Spaces on Homogeneous Groups. Princeton University Press, Princeton (1982)
14. Greenblatt, M.: Fourier transforms of indicator functions, lattice point discrepancy, and the stability of integrals. Mathematische Annalen (2021)
15. Greville, T.N.E.: On stability of linear smoothing formulas. SIAM J. Numer. Anal. **3**(1), 157–170 (1966)
16. Helms, L.L.: Potential Theory. Universitext. Springer, London (2009)
17. Hörmander, L.: The Analysis of Linear Partial Differential Operators II. Springer, Berlin Heidelberg, Berlin (1983)
18. Lawler, G.F., Limic, V.: Random walk: A Modern Introduction, vol. 123. Cambridge University Press, Cambridge (2010)
19. Lee, J.M.: Introduction to Smooth Manifolds. Graduate Texts in Mathematics, vol. 218. Springer, New York (2003)
20. Naber, G.L.: Topology, Geometry and Gauge fields. Texts in Applied Mathematics, vol. 25, 2nd edn. Springer, New York (2011)
21. Pólya, G.: Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz. Math. Ann. **84**, 149–160 (1921)
22. Randles, E., Saloff-Coste, L.: On the convolution powers of complex functions on $\mathbb{Z}$. J. Fourier Anal. Appl. **21**(4), 754–798 (2015)
23. Randles, E., Saloff-Coste, L.: Positive-homogeneous operators, heat kernel estimates and the Legendre–Fenchel transform. In Baudoin F., Peterson, J. (eds.). Stochastic Analysis and Related Topics: A Festschrift in Honor of Rodrigo Bañuelos, volume 72 of Progress in Probability, pp. 1–47. (2017)
24. Randles, E., Saloff-Coste, L.: Convolution powers of complex functions on $\mathbb{Z}^d$. Rev. Mat. Iberoam. **33**(3), 1045–1121 (2017)
25. Rudin, W.: Real and Complex Analysis. McGraw-Hill Series in Higher Mathematics, 3rd edn. McGraw-Hill Inc., New York (1987)
26. Schoenberg, I.J.: On smoothing operations and their generating functions. Bull. Am. Math. Soc. **59**(3), 199–230 (1953)
27. Spitzer, F.: Principles of Random Walk. Graduate Texts in Mathematics, vol. 34. Springer, New York (1964)
28. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals., volume 43 of Princeton Mathematical Series. Princeton University Press, Princeton (1993)
29. Stein, E.M., Shakarchi, R.: Real Analysis: Measure Theory, Integration, and Hilbert Spaces, volume 3 of Princeton Lectures in Analysis. Princeton University Press, Princeton (2009)
30. Thomée, V.: Stability of difference schemes in the maximum-norm. J. Differ. Equ. **1**(3), 273–292 (1965)
31. Thomée, V.: Stability theory for partial difference operators. Siam Rev. **11**(2), 152–195 (1969)
32. Woess, W.: Random Walks on Infinite Graphs and Groups, vol. 138. Cambridge University Press, Cambridge (2000)