

MATRIX ANALYSIS

A Quick Guide

Huan Bui

Colby College
Physics & Statistics
Class of 2021

March 13, 2019

Preface

Greetings,

Matrix Analysis: A Quick Guide to is compiled based on my MA353: Matrix Analysis notes with professor Leo Livshits. The sections are based on a number of resources: *Linear Algebra Done Right* by Axler, *A Second Course in Linear Algebra* by Horn and Garcia, *Matrices and Linear Transformations* by Cullen, *Matrices: Methods and Applications* by Barnett, *Problems and Theorems in Linear Algebra* by Prasolov, *Matrix Operations* by Richard Bronson, and professor Leo Livshits' own textbook (in the making). Prerequisites: some prior exposure to a first course in linear algebra.

The development of this text will come in layers. The first layer, one that I am working on during the course of S'19 MA353, will be an overview of the key topics listed in the table of contents. As the semester progresses, I will be constantly updating the existing notes, as well as adding prof. Livshits' problems and my solutions to the problems. The second layer will come after the course is over, when concepts will have hopefully "come together."

I will decide how much narrative I should put into the text as text is developed over the semester. I'm thinking that I will only add detailed explanations wherever I find fit or necessary for my own studies. I will most likely keep the text as condensed as I can.

Enjoy!

Contents

Preface	1
1 List of Special Matrices & Their Properties	5
2 List of Operations	7
3 List of Algorithms	8
4 Complex Numbers	9
4.1 A different point of view	9
4.2 Relevant properties and definitions	11
5 Vector Spaces & Linear Functions	13
5.1 Review of Linear Spaces and Subspaces	13
5.2 Review of Linear Maps	13
5.3 Review of Kernels and Images	14
5.4 Atrices	14
5.5 Linear Independence, Span, and Bases	16
5.5.1 Linear Independence	16
5.5.2 Span	17
5.5.3 Bases	17
5.6 Linear Bijections and Isomorphisms	18
5.7 Finite-Dimensional Linear Spaces	18
5.7.1 Dimension	18
5.7.2 Rank-Nullity Theorem	19
5.8 Infinite-Dimensional Spaces	19
6 Sums of Subspaces & Products of vector spaces	20
6.1 Direct Sums	20
6.2 Products of Vector Spaces	20
6.3 Products & Direct Sums	21
6.4 Rank-Nullity Theorem	21
6.5 Nullspaces & Ranges of Operator Powers	22
6.6 Generalized Eigenvectors and Eigenspaces	23
6.7 Nilpotent Operators	24
6.8 Weyr Characteristic	24
7 Idempotents & Resolutions of Identity	25
8 Block-representations of operators	26
8.1 Direct sums of operators	26
8.2 Coordinatization & matricial representation of linear functions .	26
8.3 Equality of rank and trace for idempotents; Resolution of Identity Revisited	26

9	Invariant subspaces	27
9.1	Reducing subspaces	27
10	Polynomials applied to operators	28
10.1	Minimal polynomials of block- Δ^r operators	28
10.2	Minimal polynomials at a vector	28
11	Eigentheory	29
11.1	Spectral Mapping Theorem	29
12	Triangularization	30
12.1	Compression to invariant subspaces	30
12.2	Simultaneously Δ -ity of commuting families	30
13	Diagonalization	31
13.1	Spectral resolutions	31
13.2	Compressions to reducing subspaces	31
13.3	Simultaneous diagonalizability for commuting families	31
14	Primary decomposition over \mathbb{C} and generalized eigenspaces	32
15	Cyclic decomposition and Jordan form	33
15.1	Square roots of operators	33
15.2	Similarity of a matrix and its transpose	33
15.3	Similarity of a matrix and its conjugate	33
15.4	Jordan forms of \mathcal{AB} and \mathcal{BA}	33
15.5	Power-convergent operators	33
15.6	Power-bounded operators	33
15.7	Row-stochastic matrices	33
16	Determinant & Trace	34
16.1	Classical adjoints	34
16.2	Cayley-Hamilton theorem	34
17	Inner products and norms	35
17.1	Riesz representation theorem	36
17.2	Adjoint	36
17.3	Grammians	36
17.4	Orthogonal complements and orthogonal decompositions	36
17.5	Ortho-projections	36
17.6	Closest point solutions	36
17.7	Gram-Schmidt and orthonormal bases	36
18	Isometries and unitary operators	37
19	Ortho-triangularization	38

20 Spectral resolutions	39
21 Ortho-diagonalization; self-adjoint and normal operators; spectral theorems	40
22 Positive (semi-)definite operators	41
22.1 Classification of inner products	41
22.2 Positive square roots	41
23 Polar decomposition	42
24 Single value decomposition	43
24.1 Spectral/operator norm	43
24.2 Singular values and approximation	43
24.3 Singular values and eigenvalues	43
25 Problems and Solutions	44
25.1 Problem set 1 (under corrections)	44
25.2 Problem set 2	49
25.3 Problem set 3	64
25.4 Problem set 4	89

1 List of Special Matrices & Their Properties

1. **Hermitian/Self-adjoint:** $H = H^\dagger$. A Hermitian matrix is matrix that is equal to its own conjugate transpose:

$$H \text{ is Hermitian} \iff H_{ij} = \bar{H}_{ji}$$

Properties 1.1.

- (a) H is Hermitian $\iff \langle w, Hv \rangle = \langle Hw, v \rangle$, where \langle, \rangle denotes the inner product.
- (b) H is Hermitian $\iff \langle v, Hv \rangle \in \mathbb{R}$.
- (c) H is Hermitian \iff it is *unitarily diagonalizable* with *real eigenvalues*.

Unitary: $U^*U = UU^* = I = U^\dagger U = UU^\dagger$. The real analogue of a unitary matrix is an orthogonal matrix. The following list contain the properties of U :

- (a) U preserves the inner product:

$$\langle Ux, Uy \rangle = \langle x, y \rangle.$$

- (b) U is normal: it commutes with $U^* = U^\dagger$.
- (c) U is diagonalizable:

$$U = VDV^*,$$

where D is diagonal and unitary, and V is unitary.

- (d) $|\det(U)| = 1$ (hence the real analogue to U is an orthogonal matrix)
- (e) Its eigenspaces are orthogonal.
- (f) U can be written as

$$U = e^{iH},$$

where H is a Hermitian matrix.

- (g) Any square matrix with unit Euclidean norm is the average of two unitary matrices.

2. **Idempotent:** M idempotent $\iff M^2 = M$.

- (a) Singularity: its number of independent rows (and columns) is less than its number of rows (and columns).
- (b) When an idempotent matrix is subtracted from the identity matrix, the result is also idempotent.

“Proof”.

$$[I - M][I - M] = I - M - M + M^2 = I - M - M + M = I - M.$$

□

- (c) M is idempotent $\iff \forall n \in \mathbb{N}, A^n = A$.
- (d) Eigenvalues: an idempotent matrix is always diagonalizable and its eigenvalues are either 0 or 1. (think “projection”)
- (e) Trace: the trace of an idempotent matrix equals the rank of the matrix and thus is always an integer. So

$$\text{tr}(A) = \dim(\text{Im } A).$$

3. **Nilpotent:** a nilpotent matrix is a square matrix N such that

$$N^k = 0$$

for some positive integer k . The smallest such k is sometimes called the **index** of N .

The following statements are equivalent:

- (a) N is nilpotent.
- (b) The minimal polynomial for N is x^k for some positive integer $k \leq n$.
- (c) The characteristic polynomial for N is x^n .
- (d) The only complex eigenvalue for N is 0.
- (e) $\text{tr } N^k = 0$ for all $k > 0$.

Properties 1.2.

- (a) The degree of an $n \times n$ nilpotent matrix is always less than or equal to n .
- (b) $\det N = \text{tr}(N) = 0$.
- (c) Nilpotent matrices are not invertible.
- (d) The only nilpotent diagonalizable matrix is the zero matrix.

2 List of Operations

1. **Conjugate transpose** is what its name suggests.
2. **Classical adjoint/Adjugate/adjunct** of a square matrix is the transpose of its cofactor matrix.

3 List of Algorithms

4 Complex Numbers

4.1 A different point of view

We often think of complex numbers as

$$a + ib$$

where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. While there is nothing “bad” about this way of thinking - in fact thinking of complex numbers as $a + ib$ allows us to very quickly and intuitively do arithmetics operations on them - a “matrix representation” of complex numbers can give us some insights on “what we actually do” when we perform complex arithmetics.

Let us think of

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

as a different representation of the same object - the same complex number “ $a + ib$.” Note that it does not make sense to say the matrix representation **equals** the complex number itself. But we shall see that a lot of the properties of complex numbers are carried into this matrix representation under interesting matricial properties.

$$\boxed{\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \sim a + ib}$$

First, let us break the matrix down:

$$a + ib = a \times 1 + i \times b \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = aI + b\mathcal{I}.$$

Right away, we can make some “mental connections” between the representations:

$$\begin{aligned} I &\sim 1 \\ \mathcal{I} &\sim i. \end{aligned}$$

Now, we know that complex number multiplications commute:

$$(a + ib)(c + id) = (c + id)(a + ib).$$

Matrix multiplications are not commutative. So, we might wonder whether commutativity holds under the this new representation of complex numbers. Well, the answer is yes. We can readily verify that

$$(aI + b\mathcal{I})(cI + b\mathcal{I}) = (cI + b\mathcal{I})(aI + b\mathcal{I}).$$

How about additions? Let's check:

$$(a + ib) + (c + id) = (a + c) + i(b + d) \sim \begin{pmatrix} a + c & -(b + d) \\ (b + d) & a + c \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix}.$$

Ah! Additions work. So, the new representation of complex numbers seems to be working flawlessly. However, we have yet to gain any interesting insights into the connections between the representations. To do that, we have to look into changing the form of the matrix. First, let's see what conjugation does:

$$(a + ib)^* = a - ib \sim \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^\top$$

Ah, so conjugation to a complex number in the traditional representation is the same as transposition in the matrix representations. What about the amplitude square? Let us call

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We have

$$(a + ib)(a - ib) \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = MM^\top = (a^2 + b^2)I = \det(M)I$$

Interesting. But observe that if $\det(M) \neq 0$

$$\frac{1}{\det(M)} MM^\top = I.$$

This tells us that

$$M^\top = M^{-1},$$

where M^{-1} is the inverse of M , and, not surprisingly, it corresponds to the reciprocal to the complex number $a + ib$. We can readily show that

$$M^{-1} \sim (a + ib)^{-1} = \frac{1}{a^2 + b^2}(a - ib).$$

Remember that we can also think of a complex number as a column vector:

$$c + id \sim \begin{pmatrix} c \\ d \end{pmatrix}.$$

Let us look back at complex number multiplication under matrix representation:

$$(a + ib)(c + id) = (ac - bd) + i(bc + ad) \sim \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac - bd \\ bc + ad \end{pmatrix}.$$

Multiplication actually works in this “mixed” way of representing complex numbers as well. Now, observe that what we just did was performing a linear transformation on a vector in \mathbb{R}^2 . It is always interesting to look at the geometrical interpretation of this transformation. To do this, let us call N the “normalized” version of M :

$$N = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

We immediately recognize that N is an orthogonal matrix. This means N is an orthogonal transformation (length preserving). Now, it is reasonable to define

$$\begin{aligned} \cos \theta &= \frac{a}{\sqrt{a^2 + b^2}} \\ \sin \theta &= \frac{b}{\sqrt{a^2 + b^2}}. \end{aligned}$$

We can write N as

$$N = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},$$

which is a physicists’ favorite matrix: the rotation by θ . So, let us write M in terms of N :

$$M = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \sqrt{a^2 + b^2} N = \sqrt{a^2 + b^2} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

We can interpret M as a rotation by θ , followed by a scaling by $\sqrt{a^2 + b^2}$. But what $\sqrt{a^2 + b^2}$ exactly is just the “length” or the “amplitude” of the complex number $a + ib$, if we think of it as an arrow in a plane.

4.2 Relevant properties and definitions

1. The *modulus* of $z = a + ib$ is the “amplitude” of z , denoted by $|z| = \sqrt{a^2 + b^2} = z\bar{z}$.
2. The modulus is *multiplicative*, i.e.

$$|wz| = |w||z|.$$

3. Triangle inequality:

$$|z + w| \leq |z| + |w|.$$

We can readily show this geometrically, or algebraically.

4. The *argument* of $z = a + ib$ is θ , where

$$\theta = \begin{cases} \tan^{-1} \left(\frac{b}{a} \right), & \text{if } a > 0 \\ \frac{\pi}{2} + k2\pi, k \in \mathbb{R} & \text{if } a = 0, b > 0 \\ -\frac{\pi}{2} + k2\pi, k \in \mathbb{R} & \text{if } a = 0, b < 0 \\ \text{Undefined} & \text{if } a = b = 0. \end{cases}$$

5. The *conjugate* of $a+ib$ is $a-ib$. Conjugation is *additive* and *multiplicative*, i.e.

$$\begin{aligned} z + w &= \bar{z} + \bar{w} \\ \bar{w}z &= \bar{w}\bar{z}. \end{aligned}$$

Note that we can also show the multiplicative property with the matrix representation as well:

$$\bar{w}z \sim (WZ)^\top = Z^\top W^\top \sim \bar{z}\bar{w} = \bar{w}\bar{z}.$$

6. Euler's identity, generalized to de Moivre's formula:

$$z^n = r^n e^{in\theta}.$$

5 Vector Spaces & Linear Functions

5.1 Review of Linear Spaces and Subspaces

Properties 5.1. of linear spaces:

1. Commutativity and associativity of addition
2. Existence of an additively neutral element (null element). Zero multiples of elements give the null element: $0 \cdot V = \mathbf{0}$
3. Every element has an (unique) additively antipodal element
4. Scalar multiplication distributes over addition
5. Multiplicative identity:
6. $ab \cdot V = a \cdot (bV)$
7. $(a + b)V = aV + bV$

W is a subspace of V if

1. $S \subseteq V$
2. S is non-empty
3. S is closed under addition and scalar multiplication

Properties 5.2. that are interesting/important/maybe-not-so-obvious:

1. If S is a subspace of V and $S \neq V$ then S is a proper subspace of V .
2. If X is a subspace of Y and Y is a subspace of Z , then X is a subspace of Z .
3. Non-trivial linear (non-singleton) spaces are infinite.

5.2 Review of Linear Maps

Consider linear spaces V and W and elements $v \in V$ and $w \in W$ and scalars $\alpha, \beta \in \mathbb{R}$, a function $F : V \rightarrow W$ is a linear map if

$$F[\alpha v + \beta w] = \alpha F[v] + \beta F[w].$$

Properties 5.3.

1. $F[\mathbf{0}_V] = \mathbf{0}_W$
2. $G[w] = (\alpha \cdot F)[w] = \alpha \cdot F[w]$
3. Given $F : V \rightarrow W$ and $G : V \rightarrow W$, $H[v] = F[v] + G[v] = (F + G)[v]$ is called the sum of the functions F and G .

4. Linear combinations of linear maps are linear.
5. Compositions of linear maps are linear.
6. Compositions distributes over linear combinations of linear maps.
7. Inverses of linear functions (if they exist) are linear.
8. Inverse of a bijective linear function is a bijective linear function

5.3 Review of Kernels and Images

Definition 5.1. Let $F : V \rightarrow W$ be given. The kernel of F is defined as

$$\ker(F) = \{v \in V \mid F[v] = \mathbf{0}_W\}.$$

Properties 5.4. Let $F : V \rightarrow W$ a linear map be given. Also, consider a linear map G such that $F \circ G$ is defined

1. F is null $\iff \ker(F) = V \iff \text{Im}(F) = \mathbf{0}_W$
2. $\ker(F)$ is a subspace of V
3. $\text{Im}(F)$ is a subspace of W
4. F is injective $\iff \ker(F) = \mathbf{0}_V$
5. $\ker(F) \subseteq \ker(F \circ G)$.
6. F injective $\implies \ker(F) = \ker(F \circ G)$

Definition 5.2. Let $F : V \rightarrow W$ be given. The image of F is defined as

$$\text{Im}(F) = \{w \in W \mid \exists v \in V, F[v] = w\}.$$

5.4 Atrices

Definition 5.3. Atrix functions: Let $V_1, \dots, V_m \in \mathbf{V}$ be given. Consider $f : \mathbb{R}^m \rightarrow \mathbf{V}$ be defined by

$$f \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} = \sum_{i=1}^m a_i V_i.$$

We denote f by

$$(V_1 \quad V_2 \quad \dots \quad V_m).$$

We refer to the V_i 's as the **columns** of f , even though there doesn't have to be any columns. Basically, f is simply a function that takes in an ordered list of coefficients and returns a linear combination of V_i with the respective coefficients. A matrix is a special atrix. Not every atrix is a matrix.

Properties 5.5. of atrices

1. The V_i 's - the columns of an atrix - are the images of the standard basis tuples.
2. $\mathbf{e}_j \in \ker(V_1 \dots V_m) \iff V_j = \mathbf{0}_V$, where \mathbf{e}_j denotes a standard basis tuple with a 1 at the j^{th} position. To put in words, a kernel of an atrix contains a standard basis if and only if one of its columns is a null element.
3. $\text{Im}(f) \equiv \text{Im}(V_1 \dots V_m) = \text{span}(V_1 \dots V_m)$
4. B is a null atrix $\iff \ker(B) = \mathbb{R}^m \iff \text{Im}(B) = \mathbf{0}_V \iff V_j = \mathbf{0}_V \forall j = 1, 2, \dots, m$.
5. f is a linear function $\mathbb{R}^m \rightarrow V \iff f$ is an atrix function $\mathbb{R}^m \rightarrow V$.
6. An atrix A is bijective/invertible, then its inverse A^{-1} is a linear function, but is an atrix only if A is a matrix.
7. Linear combinations of atrices are atrices:

$$\begin{aligned} & \alpha \cdot (V_1 \quad \dots \quad V_m) + \beta \cdot (W_1 \quad \dots \quad W_m) \\ &= (\alpha V_1 + \beta W_1 \quad \dots \quad \alpha V_m + \beta W_m) \end{aligned}$$

8. Compositions of two atrices are NOT defined unless the atrix going first is a matrix. Consider $F : \mathbb{R}^m \rightarrow W$ and $G : \mathbb{R}^n \rightarrow T$. $F \circ G$ is only defined if $T = \mathbb{R}^m$. This makes G an $n \times m$ matrix. It follows that the atrix $F \circ G$ has the form

$$(f(g_1) \quad f(g_2) \quad \dots \quad f(g_m)).$$

9. Consider $F : \mathbb{R}^m \rightarrow V$ and $G : \mathbb{R}^k \rightarrow V$. $\text{Im}(F) \subseteq \text{Im}(G) \iff F = G \circ C$, with $C \in \mathbb{M}_{k \times m}$, i.e. C is an $k \times m$ matrix.
10. Consider an atrix $A : \mathbb{R}^m \rightarrow V$. $A = (V_1 \dots V_m)$. A is NOT injective.
 $\iff \exists$ a non-trivial linear combination of the columns of A that gives $\mathbf{0}_V$
 $\iff A = [\mathbf{0}_v]$ or $\exists j | V_j$ is linear combination of other columns of A
 \iff The first column of A is $\mathbf{0}_V$ or $\exists j | V_j$ is a linear combination some of $V_i, i < j$.
11. If atrix $A : \mathbb{R} \rightarrow V$ has a single column then it is injective if the column is not $\mathbf{0}_V$.

Properties 5.6. of elementary column operations for atrices. Elementary operations on the columns of F can be expressed as a composition of F and an appropriate elementary matrix E , $F \circ E$.

1. Swapping i^{th} and j^{th} columns: $F \circ E^{[i] \leftrightarrow [j]}$.
2. Scaling the j^{th} column by α : $F \circ E^{\alpha \cdot [j]}$.

3. Adjust the j^{th} column by adding to it $\alpha \times i^{th}$ column: $F \circ E^{[i] \xleftarrow{\pm} \alpha \cdot [j]}$.
4. Elementary column operations do not change the injectivity nor image of F .
5. If a column of F is a linear combination of some of the other columns then elementary column operations can turn it into $\mathbf{0}_V$.
6. Removing/Inserting null columns or columns that are linear combinations of other columns does not change the image of F .
7. Given matrix A , it is possible to eliminate (or not) columns of A to end up with a matrix B with $\text{Im}(B) = \text{Im}(A)$.
8. If B is obtained from insertion of columns into matrix A , then $\text{Im}(B) = \text{Im}(A) \iff$ the insert columns $\in \text{Im}(A)$.
9. Inserting columns to a surjective A does not destroy surjectivity of A . (A is already having extra or just enough columns)
10. A surjective $\iff A$ is obtained by inserting columns (or not) into an invertible matrix \iff deleting some columns of A (or not) gives an invertible matrix.
11. If A is injective, then column deletion does not destroy injectivity. (A is already “lacking” or having just enough columns)
12. The new matrix obtained from inserting columns from $\text{Im}(A)$ into A is injective $\iff A$ is injective.

5.5 Linear Independence, Span, and Bases

5.5.1 Linear Independence

- $X_1 \dots X_m$ are linearly independent
 $\iff F = [X_1 \dots X_m]$ injective
 $\iff \sum a_i X_i = 0 \iff a_i = 0 \forall i$
 $\iff \mathbf{0}_V \notin \{X_i\}$ and none are linear combinations of some of the others.
 $\iff X_1 \neq \mathbf{0}_V$ and X_j is not a linear combination of any of X_i 's for $i < j$.

Properties 5.7.

1. The singleton list is linearly independent if its entry is not the null element
2. Sublists of a linearly independent list are linearly independent
3. List operations cannot create/destroy linearly independence.
4. If a linear map $L : X \rightarrow W$ injective, then $X_1 \dots X_m$ linearly independent $\iff L(X_1) \dots L(X_m)$ linearly independent.

5.5.2 Span

Properties 5.8.

1. Spans are subspaces. $\text{span}(X_1 \dots X_m), X_j \in V$ is a subspace of V .
2. $\text{span}(X_1 \dots X_j) \subseteq \text{span}(X_1 \dots X_k)$ if $j \leq k$.
3. Adding elements to a list that spans V produces a list that spans V .
4. The following list operations do not change the span of V : removing the null/linearly dependent element, inserting a linearly dependent element, scaling element(s), adding (multiples) of an element to another element.
5. It is possible to reduce a list that spans to a list that spans AND have linearly independent elements.
6. Consider $A : V \rightarrow W$. If X_i span V then $A(X_i)$ span $\text{Im}(A)$. $i = 1, 2, \dots, m$.
7. If $A : V \rightarrow W$ invertible, then X_i span $V \iff A(X_i)$ span W , $i = 1, 2, \dots, m$.

5.5.3 Bases

Properties 5.9.

1. A list is a basis of V if the elements are linearly independent and they span V .
2. A singleton linear space has no basis.
3. Re-ordering the elements of a basis gives another basis.
4. $\{X_1 \dots X_m\}$ is a basis of V if $(X_1 \dots X_m)$ is injective AND $\text{Im}(X_1 \dots X_m) = V$, i.e. $(X_1 \dots X_m)$ invertible.
5. If $\{X_i\}$ is a basis of V and $\{Y_i\}$ is a list of elements in W , then there exists a unique linear function $L : V \rightarrow W$ satisfying

$$L[X_i] = Y_i$$

6. If $A : V \rightarrow W$ bijective, then $\{V_i\}$ forms a basis of V and $\{A(X_i)\}$ forms a basis of W .
7. Elementary operations on bases give bases.

5.6 Linear Bijections and Isomorphisms

Definition 5.4. Let linear spaces V, W be given. V is **isomorphic** to W if $\exists F : V \rightarrow W$ bijective. We say $V \sim W$.

Properties 5.10.

1. A non-zero scalar multiple of an isomorphism is an isomorphism.
2. A composition of isomorphism is an isomorphism.
3. “Isomorphism” behaves like an equivalence relation:
 - (a) Reflexivity: $V \sim V$.
 - (b) Symmetry: if $V \sim W$ then $W \sim V$.
 - (c) Transitivity: if $V \sim W$ and $W \sim Z$ then $V \sim Z$.
4. Consider $F : V \rightarrow W$ an isomorphism.
 - (a) Isomorphisms preserve linear independence. V_i ’s are linearly independent in $V \iff F(V_i)$ ’s are linearly independent in W .
 - (b) isomorphisms preserve spanning. V_i ’s span $V \iff F(V_i)$ ’s span W .
 - (c) Isomorphisms preserve bases. $\{V_i\}$ is a basis of $V \iff F\{V_i\}$ is a basis of W .
5. If $V \sim W$ then $\dim(V) = \dim(W)$ (finite or infinite).
6. If a linear map $A : V \rightarrow W$ is given and $A(X_i)$ ’s are linearly independent, then X_i ’s are linearly independent.
7. If a linear map $A : V \rightarrow W$ is injective and $\{X_i\}$ is a basis of V then $\{A(X_i)\}$ is a basis of $\text{Im}(A)$

5.7 Finite-Dimensional Linear Spaces

5.7.1 Dimension

Properties 5.11.

1. $\mathbb{R}^m \sim \mathbb{R}^n \iff m = n$.
2. Isomorphisms $F : \mathbb{R}^n \rightarrow V$ are bijective atrices.
3. Isomorphisms $G : W \rightarrow \mathbb{R}^m$ are inverses of bijective atrices.
4. Consider a non-singleton linear space V
 - (a) V has a basis with n elements.
 - (b) $V \sim \mathbb{R}^n$.
 - (c) $V \sim W$, where W is any linear space with a basis of n elements.

5. If V is a linear space with a basis with n elements, then any basis of V has n elements.
6. Linear space $V \sim W$ where W is n -dimensional if V is n -dimensional.
7. For a non-singleton linear space V , $V \sim \mathbb{R}^n \iff \dim(V) = n$.
8. $V \sim W \iff \dim(V) = \dim(W)$.
9. If W is a subspace of V , then $\dim(W) \leq \dim(V)$. Equality holds when $W = V$.

5.7.2 Rank-Nullity Theorem

Let finite-dimensional linear space V and linear map $F : V \rightarrow W$ be given. Then $\text{Im}(F)$ is finite-dimensional and

$$\dim(\text{Im}(F)) + \dim(\ker(F)) = \dim(V)$$

A stronger statement: If a linear map $F : V \rightarrow W$ has finite rank and finite nullity $\iff V$ is finite-dimensional, then

$$\text{Im}(F) + \ker(F) = \dim(V).$$

5.8 Infinite-Dimensional Spaces

Consider a non-singleton linear space V . The following statements are equivalent:

1. V is infinite-dimensional.
2. Every linearly independent list in V can be enlarged to a strictly longer linearly independent set in V .
3. Every linearly independent list in V can be enlarged to an arbitrarily long (finite) linearly independent set in V .
4. There are arbitrarily long (finite) linearly independent lists in V .
5. There are linearly independent lists in V of any (finite) length.
6. No list of finitely many elements of V spans V .

6 Sums of Subspaces & Products of vector spaces

6.1 Direct Sums

Definition 6.1. Let U_j , $j = 1, 2, \dots, m$ are subspaces of V . $\sum_1^m U_j$ is a *direct sum* if each $u \in \sum U_j$ can be written in only one way as $u = \sum_1^m u_j$. The direct sum $\sum_i^m U_j$ is denoted as $U_1 \oplus \dots \oplus U_m$.

Properties 6.1.

1. Condition for direct sum: If all U_j are subspaces of V , then $\sum_1^m U_j$ is a direct sum \iff the only way to write 0 as $\sum_1^m u_j$, where $u_j \in U_j$ is to take $u_j = 0$ for all j .
2. If U, W are subspaces of V and $U \cap W = \{0\}$ then $U + W$ is a direct sum.

6.2 Products of Vector Spaces

Definition 6.2. Product of vectors spaces

$$V_1 \times \dots \times V_m = \{(v_1, \dots, v_m) : v_j \in V_j, j = 1, 2, \dots, m\}.$$

Definition 6.3. Addition on $V_1 \times \dots \times V_m$:

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, v_m + u_m).$$

Definition 6.4. Scalar multiplication on $V_1 \times \dots \times V_m$:

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

Properties 6.2.

1. Product of vectors spaces is a vector space.
 V_j are vectors spaces over $\mathbb{F} \implies V_1 \times \dots \times V_m$ is a vector space over \mathbb{F} .
2. Dimension of a product is the sum of dimensions:

$$\dim(V_1 \times \dots \times V_m) = \sum_1^m \dim(V_j)$$

3. Vector space products are NOT commutative:

$$W \times V \neq V \times W.$$

However,

$$V \times W \sim W \times V.$$

4. Vector space products are NOT associative:

$$V \times (W \times Z) \neq (V \times W) \times Z$$

6.3 Products & Direct Sums

Properties 6.3.

1. Let U_1, \dots, U_m be subspaces of V . Define a linear map $\Gamma : U_1 \times \dots \times U_m \rightarrow U_1 + \dots + U_m$ by:

$$\Gamma(u_1, \dots, u_m) = \sum_1^m u_j.$$

$U_1 + \dots + U_m$ is a direct sum $\iff \Gamma$ is injective.

2. Let U_j be finite-dimensional and are subspaces of V .

$$U_1 \oplus \dots \oplus U_m \iff \dim(U_1 + \dots + U_m) = \sum_1^m \dim(U_j)$$

6.4 Rank-Nullity Theorem

Suppose Z_1 and Z_2 are subspaces of a finite-dimensional vector space W . Consider $z_1 \in Z_1$, $z_2 \in Z_2$, and a function $\phi : Z_1 \times Z_2 \rightarrow Z_1 + Z_2 \prec W$ defined by

$$\phi \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 + z_2.$$

First, ϕ is a linear function, as it satisfies the linearity condition:

$$\phi \left(\alpha \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \beta \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix} \right) = \alpha \phi \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \beta \phi \begin{pmatrix} z'_1 \\ z'_2 \end{pmatrix}.$$

By rank-nullity theorem,

$$\dim(Z_1 \times Z_2) = \dim(Z_1 + Z_2) + \dim(\ker(\phi)).$$

But this is equivalent to

$$\dim(Z_1) + \dim(Z_2) = \dim(Z_1 + Z_2) + \dim(\ker(\phi))$$

The kernel of ϕ is:

$$\ker(\phi) = \left\{ \begin{pmatrix} v \\ -v \end{pmatrix} \middle| v \in z \in Z_1, z \in Z_2 \right\} = \left\{ \begin{pmatrix} v \\ -v \end{pmatrix} \middle| v \in z \in Z_1 \cap Z_2 \right\}$$

We can readily verify that $Z_1 \cap Z_2$ is a subspace of W . With this, $\dim(\ker(\phi)) = \dim(Z_1 \cap Z_2)$. So we end up with

$$\dim(Z_1 + Z_2) = \dim(Z_1) + \dim(Z_2) - \dim(Z_1 \cap Z_2).$$

Properties 6.4.

1. When $Z_1 \cap Z_2$ is trivial, then $Z_1 + Z_2$ is direct.
2. When $\dim(\ker(\phi)) = 0$, ϕ is injective. But ϕ is also surjective by definition, this implies ϕ is a bijection, in which case

$$Z_1 \oplus Z_2 \sim Z_1 + Z_2.$$

6.5 Nullspaces & Ranges of Operator Powers

1. Sequence of increasing null spaces: Suppose $T \in \mathcal{L}(V)$, i.e., T is some linear function mapping $V \rightarrow V$, then

$$\{0\} = \ker(T^0) \subset \ker(T^1) \subset \ker(T^2) \subset \cdots \subset \ker(T^k) \subset \ker(T^{k+1}) \subset \cdots$$

Proof Outline. Let k be a nonnegative integer and $v \in \ker(T^k)$. Then $T^k v = 0$, so $T^{k+1} v = T(T^k v) = T(0) = 0$, so $v \in \ker T^{k+1}$. So $\ker(T^k) \subset \ker(T^{k+1})$. \square

2. Equality in the sequence of null spaces: Suppose m is a nonnegative integer such that $\ker(T^m) = \ker(T^{m+1})$, then

$$\ker(T^m) = \ker(T^{m+1}) = \ker(T^{m+2}) = \cdots$$

Proof Outline. We want to show

$$\ker(T^{m+k}) = \ker(T^{m+k+1}).$$

We know that $\ker T^{m+k} \subset \ker T^{m+k+1}$. Suppose $v \in \ker T^{m+k+1}$, then

$$T^{m+1}(T^k v) = T^{m+k+1} v = 0.$$

So

$$T^k v \in \ker T^{m+1} = \ker T^m.$$

So

$$0 = T^m(T^k v) = T^{m+k} v,$$

i.e., $v \in \ker T^{m+k}$. So $\ker T^{m+k+1} \subset \ker T^{m+k}$. This completes the proof. \square

3. Null spaces stop growing: If $n = \dim(V)$, then

$$\ker(T^n) = \ker(T^{n+1}) = \ker(T^{n+2}) = \cdots$$

Proof Outline. To show:

$$\ker T^n = \ker T^{n+1}.$$

Suppose this is not true. Then the dimension of the kernel has to increase by at least 1 every step until $n+1$. Thus $\dim \ker T^{n+1} \geq n+1 > n = \dim(V)$. This is a contradiction. \square

4. V is the direct sum of $\ker(T^{\dim(V)})$ and $\text{Im}(T^{\dim(V)})$: If $n = \dim(V)$, then

$$V = \ker(T^n) \oplus \text{Im}(T^n).$$

Proof Outline. To show:

$$\ker T^n \cap \operatorname{Im} T^n = \{0\}.$$

Suppose $v \in \ker T^n \cap \operatorname{Im} T^n$. Then $T^n v = 0$ and $\exists u \in V$ such that $v = T^n u$. So

$$T^n v = T^{2n} u = 0.$$

So

$$T^n u = 0.$$

But this means $v = 0$. □

5. IT IS NOT TRUE THAT $V = \ker(T) \oplus \operatorname{Im}(T)$ in general.

6.6 Generalized Eigenvectors and Eigenspaces

Definition 6.5. Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T . A vector $v \in V$ is called a **generalized eigenvector** of T corresponding to λ if $v \neq 0$ and

$$(T - \lambda I)^j v = 0$$

for some positive integer j .

Definition 6.6. Generalized Eigenspace: Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. The **generalized eigenspace** of T corresponding to λ , denoted $G(\lambda, T)$, is defined to be the set of all generalized eigenvectors of T corresponding to λ , along with the 0 vector.

Properties 6.5. 1. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Then

$$G(\lambda, T) = \ker(T - \lambda I)^{\dim(V)}.$$

Proof Outline. Suppose $v \in \ker(T - \lambda I)^{\dim(V)}$. Then $v \in G(\lambda, T)$. So, $\ker(T - \lambda I)^{\dim V} \subset G(\lambda, T)$. Next, suppose $v \in G(\lambda, T)$. Then there is a positive integer j such that

$$v \in \ker(T - \lambda I)^j.$$

But if this is true, then

$$v \in \ker(T - \lambda I)^{\dim V},$$

since $\ker(T - \lambda I)^{\dim V}$ is the largest possible kernel, in a sense. □

2. Linearly independent generalized eigenvectors: Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T and v_1, \dots, v_m are corresponding generalized eigenvectors. Then v_1, \dots, v_m is linearly independent.

6.7 Nilpotent Operators

Definition 6.7. An operator is called **nilpotent** if some power of it equals 0.

Properties 6.6.

Nilpotent operator raised to dimension of domain is 0: Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then

$$N^{\dim(V)} = 0.$$

Matrix of a nilpotent operator: Suppose N is a nilpotent operator on V . Then there is a basis of V with respect to which the matrix of N has the form

$$\begin{pmatrix} 0 & & & * \\ & \ddots & & \\ 0 & & & 0 \end{pmatrix};$$

here all entries on and below the diagonal are 0's.

6.8 Weyr Characteristic

7 Idempotents & Resolutions of Identity

8 Block-representations of operators

8.1 Direct sums of operators

8.2 Coordinatization & matricial representation of linear functions

Consider a finite-dimensional linear space V with basis $\{V_i\}$, $i = 1, 2, \dots, m$. An element \tilde{V} in V can be expressed in exactly one way:

$$\tilde{V} = \sum_{i=1}^m a_i V_i,$$

where $\{a_i\}$ is unique. We call $\{a_i\}$ the coordinate tuple of \tilde{V} and a_i 's the coordinates.

Properties 8.1.

1. Inverse of a bijective matrix outputs the coordinates. Suppose $A = [V_i]$. Then

$$\tilde{V} = \sum_{i=1}^m a_i V_i \iff A^{-1}(Z) = (a_1 \quad \dots \quad a_m)^\top$$

8.3 Equality of rank and trace for idempotents; Resolution of Identity Revisited

9 Invariant subspaces

9.1 Reducing subspaces

10 Polynomials applied to operators

10.1 Minimal polynomials of block- Δ^r operators

10.2 Minimal polynomials at a vector

11 Eigentheory

11.1 Spectral Mapping Theorem

12 Triangularization

12.1 Compression to invariant subspaces

12.2 Simultaneously Δ -ity of commuting families

13 Diagonalization

13.1 Spectral resolutions

13.2 Compressions to reducing subspaces

13.3 Simultaneous diagonalizability for commuting families

14 Primary decomposition over \mathbb{C} and generalized eigenspaces

15 Cyclic decomposition and Jordan form

15.1 Square roots of operators

15.2 Similarity of a matrix and its transpose

15.3 Similarity of a matrix and its conjugate

15.4 Jordan forms of AB and BA

15.5 Power-convergent operators

15.6 Power-bounded operators

15.7 Row-stochastic matrices

16 Determinant & Trace

16.1 Classical adjoints

16.2 Cayley-Hamilton theorem

17 Inner products and norms

- 17.1 Riesz representation theorem
- 17.2 Adjoints
- 17.3 Grammians
- 17.4 Orthogonal complements and orthogonal decompositions
- 17.5 Ortho-projections
- 17.6 Closest point solutions
- 17.7 Gram-Schmidt and orthonormal bases

18 Isometries and unitary operators

19 Ortho-triangularization

20 Spectral resolutions

21 Ortho-diagonalization; self-adjoint and normal operators; spectral theorems

22 Positive (semi-)definite operators

22.1 Classification of inner products

22.2 Positive square roots

23 Polar decomposition

24 Single value decomposition

24.1 Spectral/operator norm

24.2 Singular values and approximation

24.3 Singular values and eigenvalues

25 Problems and Solutions

25.1 Problem set 1 (under corrections)

Problem. 1. Equivalent formulations of directness of a subspace sum:

Suppose that $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{315}$ are subspaces of a vector space \mathbf{W} . Argue that the following claims are equivalent.

1. The subspace sum $\mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_{315}$ is direct.
2. If $x_i \in \mathbf{V}_i$ and $x_1 + x_2 + \dots + x_{315} = \mathbf{0}_{\mathbf{W}}$, then $x_i = \mathbf{0}_{\mathbf{W}}$, for every i .
3. If $x_i, y_i \in \mathbf{V}_i$ and $x_1 + x_2 + x_3 + \dots + x_{315} = y_1 + y_2 + y_3 + \dots + y_{315}$, then $x_i = y_i$, for every i .
4. For any i , no non-null element of \mathbf{V}_i can be express as a sum of the elements of the other \mathbf{V}_j 's.
5. For any i , no non-null element of \mathbf{V}_i can be expressed as a sum of the elements of the preceding \mathbf{V}_j 's.

Problem. 2. Sub-sums of direct sums are direct: Suppose that $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_{315}$ are subspaces of a vector space \mathbf{W} , and the subspace sum $\mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_{315}$ is direct.

1. Suppose that for each i , \mathbf{Z}_i is a subspace of \mathbf{V}_i . Argue that the subspace sum $\mathbf{Z}_1 + \mathbf{Z}_2 + \dots + \mathbf{Z}_{315}$ is direct.
2. Argue that the sum $\mathbf{V}_2 + \mathbf{V}_5 + \mathbf{V}_7 + \mathbf{V}_{12}$ is also direct.

Problem. 3. Associativity of directness of subspace sums: Suppose that

$$\mathbf{Y}, \mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_5, \mathbf{U}_1, \mathbf{U}_2, \dots, \mathbf{U}_{12}, \mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_4, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{53}$$

are subspaces of a vector space \mathbf{W} . Argue that the following claims are equivalent.

1. The subspace sum

$$\mathbf{Y} + \mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_5 + \mathbf{U}_1 + \mathbf{U}_2 + \dots + \mathbf{U}_{12} + \mathbf{Z}_1 + \mathbf{Z}_2 + \dots + \mathbf{Z}_4 + \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_{53}$$

is direct.

2. The subspace sums

$$[\mathbf{V} :=] \mathbf{V}_1 + \mathbf{V}_2 + \dots + \mathbf{V}_5$$

$$[\mathbf{U} :=] \mathbf{U}_1 + \mathbf{U}_2 + \dots + \mathbf{U}_{12}$$

$$[\mathbf{Z} :=] \mathbf{Z}_1 + \mathbf{Z}_2 + \dots + \mathbf{Z}_4$$

$$[\mathbf{X} :=] \mathbf{X}_1 + \mathbf{X}_2 + \dots + \mathbf{X}_{53}$$

$$\mathbf{Y} + \mathbf{V} + \mathbf{U} + \mathbf{X} + \mathbf{Z}$$

are all direct.

Problem. 4. Direct sums preserve linear independence: Suppose that $\mathbf{U}, \mathbf{V}, \mathbf{W}$ are subspaces of a vector space \mathbf{Z} , and the sum $\mathbf{U}, \mathbf{V}, \mathbf{W}$ is direct.

1. Suppose that U_1, U_2, \dots, U_{13} is a linearly independent list in \mathbf{U} , V_1, V_2, \dots, V_6 is a linearly independent list in \mathbf{V} , and W_1, W_2, \dots, W_{134} is a linearly independent list in \mathbf{W} . Argue that the concatenated list

$$U_1, U_2, \dots, U_{13}, V_1, V_2, \dots, V_6, W_1, W_2, \dots, W_{134}$$

is linearly independent.

2. Suppose that U_1, U_2, \dots, U_{13} is a basis of \mathbf{U} , V_1, V_2, \dots, V_6 is a basis of \mathbf{V} , and W_1, W_2, \dots, W_{134} is a basis of \mathbf{W} . Argue that the concatenated list

$$U_1, U_2, \dots, U_{13}, V_1, V_2, \dots, V_6, W_1, W_2, \dots, W_{134}$$

is a basis of $\mathbf{U} \oplus \mathbf{V} \oplus \mathbf{W}$.

Problem. 5. Suppose that x_1, x_2, \dots, x_{14} are non-null elements of a linear space \mathbf{W} .

1. Argue that the following claims are equivalent.

- (a) x_1, x_2, \dots, x_{14} are linearly independent.
- (b) The subspace sum

$$\text{span}(x_1) + \text{span}(x_2) + \cdots + \text{span}(x_{14})$$

is direct.

2. Argue that the following claims are equivalent.

- (a) x_1, x_2, \dots, x_{14} is a basis of \mathbf{W} .
- (b) $\mathbf{W} = \text{span}(x_1) \oplus \text{span}(x_2) \oplus \cdots \oplus \text{span}(x_{14})$

25.2 Problem set 2

Problem. 1. Finite unions of subspaces are rarely a subspace

Suppose that $\mathbf{W}_1, \mathbf{W}_2, \dots, \mathbf{W}_n$ are subspaces of a vector space \mathbf{V} . Prove that the following are equivalent:

1. $\mathbf{W}_1 \cup \mathbf{W}_2 \cup \dots \cup \mathbf{W}_n$ is a subspace of \mathbf{V} .
2. One of the \mathbf{W}_i 's contains all the others.

Solution. 1.

- [1. \implies 2.] : Suppose 2. is false and define

$$\mathbf{S} = \bigcup_{i=1}^{n_0} \mathbf{W}_i$$

a subspace of \mathbf{V} , where none of the \mathbf{W}_i 's $\prec \mathbf{V}$ contains all the others, and n_0 is minimal. If $n_0 = 1$, then the implication 1. \implies 2. is true because \mathbf{W}_1 contains itself. Therefore, in order for this implication to fail, $n_0 \geq 2$.

Because n_0 is minimal, \mathbf{S} cannot be obtained from a union of less than n_0 of the \mathbf{W}_i 's. Therefore, each \mathbf{W}_i contains an element w_i that does not belong to any other \mathbf{W}_j 's.

Take $w_2 \in \mathbf{W}_2 \setminus \mathbf{W}_1$ and $w_1 \in \mathbf{W}_1$ such that $w_1 \notin \bigcup_{j \neq 1}^{n_0} \mathbf{W}_j$. It follows that $w_1, w_2 \in \mathbf{S}$ because $\mathbf{W}_1, \mathbf{W}_2 \subseteq \mathbf{S}$. And since \mathbf{S} is a subspace, $nw_1 + w_2 \in \mathbf{S}$ for any $n \in \mathbb{C}$ by closure under addition.

Consider a collection of n_0 linear combinations of w_1 and w_2 , defined as

$$T = \{w_1 + w_2, 2w_1 + w_2, \dots, n_0 w_1 + w_2\} \subseteq \mathbf{S}.$$

Consider a typical element of T : $nw_1 + w_2 \in \mathbf{S}$, $n \in \{1, 2, \dots, n_0\}$. Suppose $nw_1 + w_2 \in \mathbf{W}_1$. By closure under addition,

$$(nw_1 + w_2) - nw_1 = w_2 \in \mathbf{W}_1.$$

But this contradicts the choice of w_2 . Therefore, $nw_1 + w_2 \notin \mathbf{W}_1$, for any $n \in \{1, 2, \dots, n_0\}$. It follows that $T \subseteq \bigcup_{i=2}^{n_0} \mathbf{W}_i$.

By construction, T has n_0 elements, while $\bigcup_{i=2}^{n_0} \mathbf{W}_i$ has $(n_0 - 1)$ \mathbf{W}_i 's. By the pigeonhole principle, a \mathbf{W}_k of $\mathbf{W}_2, \dots, \mathbf{W}_{n_0}$ must contain $nw_1 + w_2$ for two distinct values of n . Let these values be n_a and n_b . By closure under addition, we have

$$(n_a w_1 + w_2) - (n_b w_1 + w_2) = (n_a - n_b)w_1 \in \mathbf{W}_k.$$

But since $n_a \neq n_b$, $w_1 \in \mathbf{W}_k$. This contradicts our initial choice of w_1 and thus rules out the possibility that the implication 1. \implies 2. fails to hold. Therefore, the implication 1. \implies 2. must be true.

- [2. \implies 1.] : Without loss of generality, assume that \mathbf{W}_1 contains all the other \mathbf{W}_j 's. It follows that

$$\bigcup_{i=1}^{n_0} \mathbf{W}_i = \mathbf{W}_1.$$

Since \mathbf{W}_1 is a subspace of \mathbf{V} , $\bigcup_{i=1}^{n_0} \mathbf{W}_i$ is also a subspace of \mathbf{V} . Therefore, 2. \implies 1. must hold.

Problem. 2. Images of pre-images and pre-images of images: Is there a 3×3 matrix \mathcal{A} such that

$$\mathbf{W} := \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \mid x, y \in \mathbb{C} \right\}$$

is an invariant subspace for \mathcal{A} , and

$$\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] \subsetneq \mathbf{W} \subsetneq \mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]?$$

Justify your answer.

Solution. 2.

First, we observe that $\dim(\mathbf{W}) = 2$, because a basis set of \mathbf{W} ,

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\},$$

has two elements. Next, consider the vector space \mathbb{C}^3 over the complex numbers. Since \mathbf{W} is a subspace and any element of \mathbf{W} is contained in \mathbb{C}^3 by construction, $\mathbf{W} \prec \mathbb{C}^3$. In fact, $\mathbf{W} \subsetneq \mathbb{C}^3$ because $\mathbb{C}^3 \ni (0 \ 0 \ 1)^\top \notin \mathbf{W}$.

By the condition $\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] \subsetneq \mathbf{W} \subsetneq \mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]$, it must be true that

$$\dim(\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]]) < \dim(\mathbf{W}) < \dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) .$$

Now, since $\dim(\mathbf{W}) = 2$ and \mathcal{A} is a 3×3 matrix (i.e., neither $\dim(\ker(\mathcal{A}))$ nor $\dim(\text{Im}(\mathcal{A}))$ can exceed $3^{(\dagger)}$), it is required that $\dim(\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]]) \leq 1$ and $\dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) = 3$.

Because \mathbf{W} is invariant under \mathcal{A} , $\mathcal{A}[\mathbf{W}] \subseteq \mathbf{W}$. It follows that $\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]] \subseteq \mathcal{A}^{-1}[\mathbf{W}]$. Therefore,

$$3 = \dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) \leq \dim(\mathcal{A}^{-1}[\mathbf{W}]) .$$

It follows, by fact (\dagger) , that

$$3 \leq \dim(\mathcal{A}^{-1}[\mathbf{W}]) \leq 3,$$

which implies $\dim(\mathcal{A}^{-1}[\mathbf{W}]) = 3 = \dim(\mathbb{C}^3)$. Consider $v \in \mathcal{A}^{-1}[\mathbf{W}]$. v can have the form $(x \ y \ z)^\top$, $x, y, z \in \mathbb{C}$, so $v \in \mathbb{C}^3$. This implies $\mathcal{A}^{-1}[\mathbf{W}] \subseteq \mathbb{C}^3$. But since their dimensions are both 3, $\mathcal{A}^{-1}[\mathbf{W}] = \mathbb{C}^3$. It follows that

$$\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] = \mathcal{A}[\mathbb{C}^3] \supseteq \mathcal{A}[\mathbf{W}],$$

where the second relation comes from the fact that \mathbf{W} is a subspace of \mathbb{C}^3 . Hence,

$$1 \geq \dim(\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]]) \geq \dim(\mathcal{A}[\mathbf{W}]),$$

which implies $\dim(\mathcal{A}[\mathbf{W}]) = 0$ or 1 .

So, such a 3×3 matrix \mathcal{A} that satisfies the given condition is

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

First, \mathbf{W} is invariant under \mathcal{A} because take any $(x \ y \ 0)^\top \in \mathbf{W}$, $\mathcal{A}((x \ y \ 0)^\top) = (x \ 0 \ 0)^\top \in \mathbf{W}$. This implies $\mathcal{A}[\mathbf{W}] \subseteq [\mathbf{W}]$ (a).

Second, it is easy to see that the pre-image of \mathbf{W} under \mathcal{A} is \mathbb{C}^3 , since take any $(x \ y \ z)^\top \in \mathbb{C}^3$, we have $\mathcal{A}((x \ y \ z)^\top) = (x \ 0 \ 0)^\top \in \mathbf{W}$. But observe that $\text{Im}(\mathcal{A})$ is a proper subset of \mathbf{W} , because any $(x \ y \ 0)^\top \in \mathbf{W}$ with $y \neq 0$ is not contained in $\text{Im}(\mathcal{A})$. This implies $\mathcal{A}[\mathcal{A}^{-1}[\mathbf{W}]] \subsetneq \mathbf{W}$ (b).

Lastly, as we have pointed out, because $\dim(\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]]) = 3$ and $\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]] \prec \mathbb{C}^3$, $\mathcal{A}^{-1}[\mathcal{A}[\mathbf{W}]] = \mathbb{C}^3 \supsetneq \mathbf{W}$ (c).

From (a), (b), (c), we see that this 3×3 matrix \mathcal{A} works as desired.

Problem. 3. Invariant subspaces form a “lattice”:

1. Argue that $\mathfrak{Lat}(\mathcal{L})$ is closed under intersection; i.e., that the intersection of any two invariant subspaces for \mathcal{L} is also an invariant subspace for \mathcal{L} .
2. Argue that $\mathfrak{Lat}(\mathcal{L})$ is closed under subspace sums; i.e., that a subspace sum of two invariant subspaces for \mathcal{L} is again an invariant subspace for \mathcal{L} .
3. Show that an image under \mathcal{L} of an invariant subspace for \mathcal{L} is again an invariant subspace for \mathcal{L} .
4. Show that a pre-image under \mathcal{L} of an invariant subspace for \mathcal{L} is again an invariant subspace for \mathcal{L} .

Solution. 3.

1. Let subspaces $\mathbf{V}, \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ be given. $\mathfrak{Lat}(\mathcal{L})$ is closed under intersection if $\mathbf{V} \cap \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$. Consider $u \in \mathbf{V} \cap \mathbf{W}$, then $u \in \mathbf{V}$ and $u \in \mathbf{W}$. It follows that $\mathcal{L}(u) \in \mathcal{L}[\mathbf{V}]$ and $\mathcal{L}(u) \in \mathcal{L}[\mathbf{W}]$.

Now, since $\mathcal{L}[\mathbf{V}] \subseteq \mathbf{V}$ and $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$ because $\mathbf{V}, \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$, we have $\mathcal{L}(u) \in \mathbf{V}$ and $\mathcal{L}(u) \in \mathbf{W}$, which implies $\mathcal{L}(u) \in \mathbf{V} \cap \mathbf{W}$. Since this implication holds for any $u \in \mathbf{V} \cap \mathbf{W}$, $\mathcal{L}[\mathbf{V} \cap \mathbf{W}] \subseteq \mathbf{V} \cap \mathbf{W}$. Therefore, $\mathbf{V} \cap \mathbf{W}$ is invariant under \mathcal{L} ; i.e., $\mathbf{V} \cap \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$. This completes the argument.

2. Let subspaces $\mathbf{V}, \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ be given. $\mathfrak{Lat}(\mathcal{L})$ is closed under subspace sums if $\mathbf{V} + \mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$. Consider $v \in \mathbf{V}, w \in \mathbf{W}$. Then $v + w \in \mathbf{V} + \mathbf{W}$. Next, consider $l \in \mathcal{L}[\mathbf{V} + \mathbf{W}]$ given by

$$l = \mathcal{L}(v + w) = \mathcal{L}(v) + \mathcal{L}(w) = v' + w',$$

where $v' \in \mathbf{V}$ and $w' \in \mathbf{W}$ because \mathbf{V}, \mathbf{W} are invariant under \mathcal{L} . So $l \in \mathbf{V} + \mathbf{W}$. This implies $\mathcal{L}[\mathbf{V} + \mathbf{W}] \subseteq \mathbf{V} + \mathbf{W}$; i.e., $\mathfrak{Lat}(\mathcal{L})$ is closed under subspace sums.

3. Let $\mathbf{V} \in \mathfrak{Lat}(\mathcal{L})$ be given. By definition, $\mathcal{L}[\mathbf{V}] \subseteq \mathbf{V}$. It follows that $\mathcal{L}[\mathcal{L}[\mathbf{V}]] \subseteq \mathcal{L}[\mathbf{V}]$. So, $\mathcal{L}[\mathbf{V}]$ is invariant under \mathcal{L} .

4. Let $\mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ be given. Claim: $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \subseteq \mathcal{L}^{-1}[\mathbf{W}]$.

It follows from the definition of the pre-image of \mathbf{W} under $\mathcal{L} : \mathbf{V} \rightarrow \mathbf{W}$ that $\mathcal{L}^{-1}[\mathbf{W}] = \{v \in \mathbf{V} | \mathcal{L}(v) \in \mathbf{W}\}$. This implies $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \subseteq \mathbf{W}$ (i).

Next, consider $w \in \mathbf{W} \subseteq \mathbf{V}$. By definition, $\mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]] = \{v \in \mathbf{V} | \mathcal{L}(v) \in \mathcal{L}[\mathbf{W}]\}$, which implies $w \in \mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]]$, because $\mathcal{L}(w) \in \mathcal{L}[\mathbf{W}]$. Therefore,

$$\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]] \text{ (ii).}$$

Moreover, because $\mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$, $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$ (iii).

From (i), (ii), and (iii) we have

$$\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \stackrel{(i)}{\subseteq} \mathbf{W} \stackrel{(ii)}{\subseteq} \mathcal{L}^{-1}[\mathcal{L}[\mathbf{W}]] \stackrel{(iii)}{\subseteq} \mathcal{L}^{-1}[\mathbf{W}].$$

Therefore, $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{W}]] \subseteq \mathcal{L}^{-1}[\mathbf{W}]$, verifying the claim.

Problem. 4. Cyclic invariant subspaces For a given $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$ and a fixed $v_0 \in \mathbf{V}$, define

$$\mathbf{P}(\mathcal{L}, v_0) := \left\{ (a_0\mathcal{L}^0 + a_1\mathcal{L}^1 + a_2\mathcal{L}^2 + \cdots + a_k\mathcal{L}^k)(v_0) \mid k \geq 0, a_i \in \mathbb{C} \right\}.$$

1. Argue that $\mathbf{P}(\mathcal{L}, v_0)$ is a subspace of \mathbf{V} .
2. Argue that $\mathbf{P}(\mathcal{L}, v_0)$ is invariant under \mathcal{L} .
3. Argue that $\mathbf{P}(\mathcal{L}, v_0)$ is the smallest invariant subspace for \mathcal{L} that contains v_0 . We refer to $\mathbf{P}(\mathcal{L}, v_0)$ as **the cyclic invariant subspace for \mathcal{L} generated by v_0** .
4. Argue that $\mathbf{P}(\mathcal{L}, v_0)$ is 1-dimensional exactly when v_0 is an eigenvector of \mathcal{L} .
5. Argue that a subspace \mathbf{W} of \mathbf{V} is invariant under \mathcal{L} exactly when it is a union of cyclic invariant subspaces for \mathcal{L} .

Solution. 4.

1. Consider a typical element $P(\mathcal{L}, v_0) \in \mathbf{P}(\mathcal{L}, v_0)$. Since $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$, $\mathcal{L}^i(v_0) \in \mathbf{V}$ for all i and $v_0 \in \mathbf{V}$. By closure under scalar multiplication and addition, for any $a_i \in \mathbb{C}$, $v_0 \in \mathbf{V}$, and $k \geq 0$,

$$P(\mathcal{L}, v_0) = \sum_{i=0}^k a_i \mathcal{L}^i(v_0) \in \mathbf{V}.$$

Therefore $\mathbf{P}(\mathcal{L}, v_0) \subseteq \mathbf{V}$ (i).

With $a_i = 0$ for all i , $\mathbf{P}(\mathcal{L}, v_0) \ni P(\mathcal{L}, v_0) = \sum_{i=1}^k 0 \cdot \mathcal{L}^i(v_0) = \mathbf{0}_{\mathbf{P}}$. So $\mathbf{P}(\mathcal{L}, v_0)$ contains the null element (ii).

Consider $P_1(\mathcal{L}, v_0) = \sum_{i=0}^k b_i \mathcal{L}^i(v_0)$ and $P_2(\mathcal{L}, v_0) = \sum_{j=0}^l c_j \mathcal{L}^j(v_0)$, with $b_i, c_j \in \mathbb{C}$ for all i, j . Without loss of generality, assume that $k \geq l \geq 0$. It is clear that $\mathbf{P}(\mathcal{L}, v_0)$ is closed under addition because,

$$\begin{aligned} P_1(\mathcal{L}, v_0) + P_2(\mathcal{L}, v_0) &= \sum_{i=0}^k b_i \mathcal{L}^i(v_0) + \sum_{j=0}^l c_j \mathcal{L}^j(v_0) \\ &= \sum_{i=0}^k b_i \mathcal{L}^i(v_0) + \sum_{i=0}^k c_i \mathcal{L}^i(v_0) \quad \text{with } c_i = 0 \text{ for all } i > l \\ &= \sum_{i=0}^k (b_i + c_i) \mathcal{L}^i(v_0) \\ &= \sum_{i=0}^k d_i \mathcal{L}^i(v_0) \in \mathbf{P}(\mathcal{L}, v_0), \end{aligned}$$

where $d_i = b_i + c_i \in \mathbb{C}$. (iii).

It is also clear that $\mathbf{P}(\mathcal{L}, v_0)$ is closed under scalar multiplication because given $\mu \in \mathbb{C}$,

$$\mu P_1(\mathcal{L}, v_0) = \mu \sum_{i=0}^k b_i \mathcal{L}^i(v_0) = \sum_{i=0}^k \mu b_i \mathcal{L}^i(v_0) = \sum_{i=0}^k e_i \mathcal{L}^i(v_0) \in \mathbf{P}(\mathcal{L}, v_0),$$

where $e_i = \mu b_i \in \mathbb{C}$ (iv).

By (i), (ii), (iii), and (iv), $\mathbf{P}(\mathcal{L}, v_0)$ is a subspace of \mathbf{V} .

2. Let $P(\mathcal{L}, v_0)$ be given. It suffices to show that $\mathcal{L}(P(\mathcal{L}, v_0)) \in \mathbf{P}(\mathcal{L}, v_0)$, for all $P(\mathcal{L}, v_0) \in \mathbf{P}(\mathcal{L}, v_0)$.

$$\mathcal{L}(P(\mathcal{L}, v_0)) = \mathcal{L} \left(\sum_{i=0}^k a_i \mathcal{L}^i(v_0) \right) = \sum_{i=0}^k a_i \mathcal{L}^{i+1}(v_0),$$

where $a_i \in \mathbb{C}$ and $k \geq 0$. Define new coefficients $b_j = a_i$ where $j = i + 1$ and $b_0 = 0$, then

$$\mathcal{L}(P(\mathcal{L}, v_0)) = \sum_{j=1}^{k+1} b_j \mathcal{L}^j(v_0) = \sum_{j=0}^{k+1} b_j \mathcal{L}^j(v_0) \in \mathbf{P}(\mathcal{L}, v_0).$$

Therefore, $\mathcal{L}[\mathbf{P}(\mathcal{L}, v_0)] \subseteq \mathbf{P}(\mathcal{L}, v_0)$; i.e., $\mathbf{P}(\mathcal{L}, v_0)$ is invariant under \mathcal{L} .

3. Let $\mathbf{P}(\mathcal{L}, v_0)$ be given. $\mathbf{P}(\mathcal{L}, v_0)$ is an invariant subspace under \mathcal{L} that contains v_0 by definition. Suppose $\mathbf{Q}(\mathcal{L}, v_0)$ is some invariant subspace under \mathcal{L} that also contains v_0 . It suffices to show $\mathbf{P}(\mathcal{L}, v_0) \subseteq \mathbf{Q}(\mathcal{L}, v_0)$ for any such $\mathbf{Q}(\mathcal{L}, v_0)$.

Consider $\mathcal{L}^j(v_0)$ for some $j \geq 0$. $\mathcal{L}^j(v_0) \in \mathbf{P}(\mathcal{L}, v_0)$ by definition. We claim that $\mathcal{L}^j(v_0) = q \in \mathbf{Q}(\mathcal{L}, v_0)$ for any non-negative j . Assume that this is true. The base case where $j = 0$ is true since $v_0 \in \mathbf{Q}(\mathcal{L}, v_0)$. The inductive case is also true because

$$\mathcal{L}^{(j+1)}(v_0) = \mathcal{L}(\mathcal{L}^j(v_0)) = \mathcal{L}(q) \in \mathbf{Q}(\mathcal{L}, v_0)$$

where the last relation comes from the fact that $\mathbf{Q}(\mathcal{L}, v_0)$ is invariant under \mathcal{L} . Therefore, by the principle of induction, $\mathcal{L}^j(v_0) \in \mathbf{Q}(\mathcal{L}, v_0)$ for any non-negative j . This implies $\mathbf{P}(\mathcal{L}, v_0) \subseteq \mathbf{Q}(\mathcal{L}, v_0)$ for any such $\mathbf{Q}(\mathcal{L}, v_0)$. Thus $\mathbf{P}(\mathcal{L}, v_0)$ must be the smallest invariant subspace under \mathcal{L} that contains v_0 .

4. (a) (\implies) : Let a $\mathbf{P}(\mathcal{L}, v_0)$ be given such that $\dim(\mathbf{P}(\mathcal{L}, v_0)) = 1$. Consider the subspace $\text{span}(v_0)$. We know that $\dim(\text{span}(v_0)) = 1$. Consider $v \in \text{span}(v_0)$. Then v can be expressed as $v = av_0$ where $a \in \mathbb{C}$. It follows immediately that $v \in \mathbf{P}(\mathcal{L}, v_0)$. Therefore, $\text{span}(v_0) \subseteq \mathbf{P}(\mathcal{L}, v_0)$. But because $\dim(\mathbf{P}(\mathcal{L}, v_0)) = 1 = \dim(\text{span}(v_0))$, $\mathbf{P}(\mathcal{L}, v_0) = \text{span}(v_0)$; i.e., all elements of $\mathbf{P}(\mathcal{L}, v_0)$ are some scalar multiples of v_0 .

Now, it suffices to show $\mathcal{L}(v_0) = \lambda v_0$, where $\lambda \in \mathbb{C}$. Consider $P(\mathcal{L}, v_0) = \mathcal{L}^0(v_0) + \mathcal{L}(v_0) \in \mathbf{P}(\mathcal{L}, v_0) = \text{span}(v_0)$, $v_0 \neq \mathbf{0}_V$. By closure under addition,

$$(\mathcal{L}^0(v_0) + \mathcal{L}(v_0)) - v_0 = (\mathcal{L}^0(v_0) + \mathcal{L}(v_0)) - \mathcal{L}^0(v_0) = \mathcal{L}(v_0) \in \text{span}(v_0).$$

This implies $\mathcal{L}(v_0) = \lambda v_0$ for some $\lambda \in \mathbb{C}$; i.e., v_0 is an eigenvector of \mathcal{L} .

- (b) (\impliedby) : If v_0 is an eigenvector of \mathcal{L} , then $\mathcal{L}^i(v_0) = \lambda^i v_0$, where $\lambda^i \in \mathbb{C}$ is the \mathcal{L} 's v_0 -eigenvalue raised to the i^{th} power. It follows that a typical element of $\mathbf{P}(\mathcal{L}, v_0)$ is

$$P(\lambda, v_0) = \sum_{i=0}^k a_i \mathcal{L}^i(v_0) = \left(\sum_{i=0}^k a_i \lambda^i \right) v_0 = \mu v_0,$$

where $\mu = \sum_{i=0}^k a_i \lambda^i \in \mathbb{C}$. Therefore, $\mathbf{P}(\mathcal{L}, v_0) = \text{span}(v_0)$; i.e., $\dim(\mathbf{P}(\mathcal{L}, v_0)) = 1$. This completes the argument.

5. (a) (\implies) : Let $\mathbf{W} \in \mathfrak{Lat}(\mathcal{L})$ be given. Then $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$. Let $w_1 \in \mathbf{W}$ be given. We first show that $\mathcal{L}^k(w_1) \in \mathbf{W}$ for all non-negative k by induction. Assume that $\mathcal{L}^k(w_1) = v_1 \in \mathbf{W}$ for all non-negative k holds. The base case where $k = 0$ is true since $\mathcal{L}^0(w_1) = w_1 \in \mathbf{W}$. The inductive case is also true because

$$\mathcal{L}^{(k+1)}(w_1) = \mathcal{L}(\mathcal{L}^k(w_1)) = \mathcal{L}(v_1) \in \mathbf{W}.$$

By the principle of induction, $\mathcal{L}^k(w_1) \in \mathbf{W}$ for all $w_1 \in \mathbf{W}$ and $k \geq 0$ (\dagger). Now, consider $P \in \mathbf{P}(\mathcal{L}, w_1)$. By the definition of $\mathbf{P}(\mathcal{L}, w_1)$, P is a linear combination of $\mathcal{L}^0(w_1), \dots, \mathcal{L}^k(w_1)$ for some $k \geq 0$. So, by (\dagger), $P \in \mathbf{W}$. This implies $\mathbf{P}(\mathcal{L}, w_1) \subseteq \mathbf{W}$.

We can repeat the argument above, starting with $w_2, w_3, \dots \in \mathbf{W}$, and conclude that $\mathbf{P}(\mathcal{L}, w_2) \subseteq \mathbf{W}$, $\mathbf{P}(\mathcal{L}, w_3) \subseteq \mathbf{W}$, and so on. This means \mathbf{W} contains a union of $\mathbf{P}(\mathcal{L}, w_1), \mathbf{P}(\mathcal{L}, w_2), \mathbf{P}(\mathcal{L}, w_3), \dots$. Moreover, observe the fact that any element $w_n \in \mathbf{W}$ can be used to generate a $\mathbf{P}(\mathcal{L}, w_n)$ that is contained in the union of itself and the other $\mathbf{P}(\mathcal{L}, w_m)$'s, which is ultimately contained in \mathbf{W} . Therefore, \mathbf{W} is itself a union of cyclic invariant subspaces for \mathcal{L} .

(b) (\Leftarrow) : Let $\mathbf{W} = \mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n$ be a subspace of \mathbf{V} , where \mathbf{P}_i denotes $\mathbf{P}(\mathcal{L}, v_i)$ and v_i is an arbitrary element of \mathbf{V} . By problem 1, because $\mathbf{P}_1 \cup \mathbf{P}_2 \cup \cdots \cup \mathbf{P}_n$ is a subspace of \mathbf{V} , a \mathbf{P}_i , $1 \leq i \leq n$, must contain all the other \mathbf{P}_j 's. Without loss of generality, assume that such a \mathbf{P}_i is \mathbf{P}_1 . It follows that $\mathbf{W} = \mathbf{P}_1$. By part 2. of this problem, we know that \mathbf{P}_1 is invariant under \mathcal{L} . Therefore, \mathbf{W} is invariant under \mathcal{L} .

Problem. 5. Invariant subspaces of commuting operators: Suppose that linear functions $\mathcal{L}, \mathcal{M} \in \mathfrak{L}(\mathbf{V})$ commute; i.e.,

$$\mathcal{L} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{L}.$$

1. Argue that for any non-negative integer k , $\text{Im}(\mathcal{M}^k)$ and $\ker(\mathcal{M}^k)$ are invariant subspaces for \mathcal{L} .
2. Argue that every eigenspace of \mathcal{M} is an invariant subspace for \mathcal{L} .
3. By giving a general example (with justification, of course!) show that for each $n > 1$ there are commuting matrices \mathcal{A} and \mathcal{B} in \mathbb{M}_n such that

$$\mathfrak{Lat}(\mathcal{A}) \neq \mathfrak{Lat}(\mathcal{B}).$$

Solution. 5.

1. Because $\mathcal{L} \circ \mathcal{M} = \mathcal{M} \circ \mathcal{L}$, $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$ for any non-negative k . We can verify this by a short proof by induction. Assume that $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$ holds for any non-negative k . The base case where $k = 0$ is true, for $\mathcal{L} \circ \mathcal{M}^0 = \mathcal{L} \circ \mathcal{I} = \mathcal{I} \circ \mathcal{L} = \mathcal{M}^0 \circ \mathcal{L}$, where \mathcal{I} is the identity operator. The inductive case is also true because

$$\mathcal{L} \circ \mathcal{M}^{k+1} = \mathcal{L} \circ \mathcal{M}^k \circ \mathcal{M} = \mathcal{M}^k \circ \mathcal{L} \circ \mathcal{M} = \mathcal{M}^k \circ \mathcal{M} \circ \mathcal{L} = \mathcal{M}^{k+1} \circ \mathcal{L}.$$

Therefore, by the principle of induction, $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$ is indeed true for all $k \geq 0$.

(a) To show: $\mathcal{L}[\text{Im}(\mathcal{M}^k)] \subseteq \text{Im}(\mathcal{M}^k)$.

Since $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$, $\mathcal{L} \circ \mathcal{M}^k[\mathbf{V}] = \mathcal{M}^k \circ \mathcal{L}[\mathbf{V}]$; i.e., $\mathcal{L}[\text{Im}(\mathcal{M}^k)] = \mathcal{M}^k[\text{Im}(\mathcal{L})]$. But since $\text{Im}(\mathcal{L}) \subseteq \mathbf{V}$ (because $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$),

$$\mathcal{M}^k[\text{Im}(\mathcal{L})] \subseteq \mathcal{M}^k[\mathbf{V}] = \text{Im}(\mathcal{M}^k).$$

It follows that $\mathcal{L}[\text{Im}(\mathcal{M}^k)] \subseteq \text{Im}(\mathcal{M}^k)$. So, $\text{Im}(\mathcal{M}^k)$ is an invariant subspace for \mathcal{L} .

(b) To show: $\mathcal{L}[\ker(\mathcal{M}^k)] \subseteq \ker(\mathcal{M}^k)$.

Since $\mathcal{L} \circ \mathcal{M}^k = \mathcal{M}^k \circ \mathcal{L}$,

$$\mathcal{L} \circ \mathcal{M}^k[\ker(\mathcal{M}^k)] = \{\mathbf{0}_\mathbf{V}\} = \mathcal{M}^k \circ \mathcal{L}[\ker(\mathcal{M}^k)].$$

It follows that $\mathcal{L}[\ker(\mathcal{M}^k)] \subseteq \ker(\mathcal{M}^k)$. This completes the argument.

2. Let \mathbf{E}_λ , the eigenspace of \mathcal{M} associated with eigenvalue λ , be given. Consider $e \in \mathbf{E}_\lambda$, we have

$$\mathcal{M} \circ \mathcal{L}(e) = \mathcal{L} \circ \mathcal{M}(e) = \mathcal{L}(\lambda e) = \lambda \mathcal{L}(e).$$

This implies $\mathcal{L}(e) \in \mathbf{E}_\lambda$. Since this relation holds for any $e \in \mathbf{E}_\lambda$, $\mathcal{L}[\mathbf{E}_\lambda] \subseteq \mathbf{E}_\lambda$. Therefore, \mathbf{E}_λ is an invariant subspace for \mathcal{L} for any eigenvalue λ . This proves the claim that *every* eigenspace of \mathcal{M} is an invariant subspace for \mathcal{L} .

3. It suffices to find commuting matrices \mathcal{A} and \mathcal{B} such that there exists an invariant subspace under \mathcal{A} , $\mathbf{J} \in \mathfrak{Lat}(\mathcal{A})$, but $\mathbf{J} \notin \mathfrak{Lat}(\mathcal{B})$.

For $n = 2$, consider

$$\mathcal{A} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

\mathcal{A} and \mathcal{B} commute:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Consider $\mathbf{J} = \text{span}\{(1 \ 1)^\top\}$. $\mathbf{J} \in \mathfrak{Lat}(\mathcal{A})$ since $\mathcal{A}[\mathbf{J}] = \{\mathbf{0}\} \subseteq \mathbf{J}$. However, consider $j = (1 \ 1)^\top \in \mathbf{J}$, we have

$$\mathcal{B}(j) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \notin \text{span}\{(1 \ 1)^\top\},$$

which implies $\mathbf{J} \notin \mathfrak{Lat}(\mathcal{B})$. Therefore, $\mathfrak{Lat}(\mathcal{A}) \neq \mathfrak{Lat}(\mathcal{B})$.

Observe that we can give a more general example for any $n > 1$ based on the 2×2 example above. Let \mathcal{A} be an $n \times n$ zero matrix, and \mathcal{B} be an $n \times n$ matrix of the form

$$\mathcal{B} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}.$$

It is clear that \mathcal{A} and \mathcal{B} commute, since \mathcal{A} is the zero matrix:

$$\mathcal{A}\mathcal{B} = [0]_{n \times n} = \mathcal{B}\mathcal{A}.$$

Consider $j = (1 \ 1 \ \dots \ 1)^\top \in \mathbf{J} = \text{span}\{(1 \ 1 \ \dots \ 1)^\top\} \subseteq \mathbb{C}^n$. It is clear that $\mathcal{B}(j) = (1 \ 0 \ \dots \ 0)^\top \notin \mathbf{J}$ but $\mathcal{A}(j') = \mathbf{0} \in \mathbf{J}$ for any $j' \in \mathbf{J}$. So, $\mathbf{J} \in \mathfrak{Lat}(\mathcal{A})$ but $\mathbf{J} \notin \mathfrak{Lat}(\mathcal{B})$; i.e.,

$$\mathfrak{Lat}(\mathcal{A}) \neq \mathfrak{Lat}(\mathcal{B}).$$

Problem. 6. Invariant subspace chains: Suppose that $\mathcal{L} \in \mathfrak{L}(\mathbf{V})$ and \mathbf{W} is an invariant subspace of \mathcal{L} .

1. Argue that the following inclusions hold:

$$\cdots \subseteq \mathcal{L}^3[\mathbf{W}] \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \mathcal{L}^{-3}[\mathbf{W}] \subseteq \cdots$$

Note that by Problem 3, all of the subspaces listed in the chain are invariant under \mathcal{L} .

2. Argue that the following implications hold:

$$\mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}] \implies \mathcal{L}^m[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}] \text{ for any } m \geq k,$$

and

$$\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+1)}[\mathbf{W}] \implies \mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-m}[\mathbf{W}] \text{ for any } m \geq k,$$

and then explain why the chain in part 1 stabilizes in both directions, and has at most $\dim(\mathbf{V})$ proper inclusions.

3. Argue that any subspace \mathbf{M} that falls between two consecutive subspaces in the chain shown in part 1. is also invariant under \mathcal{L} .

Solution. 6.

1. (a) To show: $\cdots \subseteq \mathcal{L}^3[\mathbf{W}] \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$.

Assume that $\mathcal{L}^{k+1}[\mathbf{W}] \subseteq \mathcal{L}^k[\mathbf{W}]$ for all $k \geq 0$. The base case $k = 0$ is true because \mathbf{W} is an invariant subspace under \mathcal{L} . The inductive case is also true because

$$\mathcal{L}^{k+2}[\mathbf{W}] = \mathcal{L}[\mathcal{L}^{k+1}[\mathbf{W}]] \subseteq \mathcal{L}[\mathcal{L}^k[\mathbf{W}]] = \mathcal{L}^{k+1}[\mathbf{W}],$$

where the second relation follows from assumption. By principle of induction, $\cdots \subseteq \mathcal{L}^3[\mathbf{W}] \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$.

- (b) To show: $\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \mathcal{L}^{-3}[\mathbf{W}] \subseteq \cdots$

Assume that $\mathcal{L}^{-k}[\mathbf{W}] \subseteq \mathcal{L}^{-k-1}[\mathbf{W}]$ for all $k \geq 0$. The base case where $k = 0$ is true: Given $w \in \mathbf{W} \subseteq \mathbf{V}$, $\mathcal{L}(w) \in \mathbf{W}$ because \mathbf{W} is invariant under \mathcal{L} . But since $\mathcal{L}^{-1}[\mathbf{W}] = \{v \in \mathbf{V} \mid \mathcal{L}(v) \in \mathbf{W}\}$ and $\mathcal{L}[\mathbf{W}] \subseteq \mathbf{W}$, $w \in \mathcal{L}^{-1}[\mathbf{W}]$. Therefore, $\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}]$ (i).

The inductive case is also true:

$$\mathcal{L}^{-k-1}[\mathbf{W}] = \mathcal{L}^{-1}[\mathcal{L}^{-k}[\mathbf{W}]] \subseteq \mathcal{L}^{-1}[\mathcal{L}^{-k-1}[\mathbf{W}]] \subseteq \mathcal{L}^{-k-2}[\mathbf{W}].$$

So by the principle of induction, $\mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \cdots$ (ii).

From (i) and (ii),

$$\cdots \subseteq \mathcal{L}^2[\mathbf{W}] \subseteq \mathcal{L}[\mathbf{W}] \subseteq \mathbf{W} \subseteq \mathcal{L}^{-1}[\mathbf{W}] \subseteq \mathcal{L}^{-2}[\mathbf{W}] \subseteq \cdots$$

2. (a) Assume the hypothesis that $\mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$ for some $k \geq 0$. We can prove by induction that $\mathcal{L}^{k+j}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$ for all non-negative j . The base case where $j = 0$ is true. The inductive case is also true since:

$$\mathcal{L}^{k+j+1}[\mathbf{W}] = \mathcal{L}[\mathcal{L}^{k+j}[\mathbf{W}]] = \mathcal{L}[\mathcal{L}^k[\mathbf{W}]] = \mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}].$$

By the principle of induction, $\mathcal{L}^m[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$ must hold for any $m \geq k \geq 0$ if $\mathcal{L}^{k+1}[\mathbf{W}] = \mathcal{L}^k[\mathbf{W}]$.

- (b) Assume that hypothesis that $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+1)}[\mathbf{W}]$ for all $k \geq 0$. Also assume that $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+j)}[\mathbf{W}]$ for all non-negative j . The base case where $j = 0$ is true. In inductive case is also true since:

$$\begin{aligned} \mathcal{L}^{-(k+j+1)}[\mathbf{W}] &= \mathcal{L}^{-1}[\mathcal{L}^{-(k+j)}[\mathbf{W}]] \\ &= \mathcal{L}^{-1}[\mathcal{L}^{-k}[\mathbf{W}]] \\ &= \mathcal{L}^{-(k+1)}[\mathbf{W}] \\ &= \mathcal{L}^{-k}[\mathbf{W}]. \end{aligned}$$

By the principle of induction, $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-m}[\mathbf{W}]$ must hold for $m \geq k$ if $\mathcal{L}^{-k}[\mathbf{W}] = \mathcal{L}^{-(k+1)}[\mathbf{W}]$.

- (c) It follows from part 1. that

$$\dots \leq \dim(\mathcal{L}[\mathbf{W}]) \leq \dim(\mathbf{W}) \leq \dim(\mathcal{L}^{-1}[\mathbf{W}]) \leq \dots$$

Because $\dim(\mathcal{L}^{-k}[\mathbf{W}]) \leq \dim(\mathbf{V})$ for any $k \geq 0$, there exists an $\mathcal{L}^{-h}[\mathbf{W}]$, $h \geq 0$, that is an invariant subspace in the chain with the highest dimension where h is minimal. It follows that $\dim(\mathcal{L}^{-(h+1)}[\mathbf{W}]) = \dim(\mathcal{L}^{-h}[\mathbf{W}])$ is maximal. But because $\mathcal{L}^{-h}[\mathbf{W}] \subseteq \mathcal{L}^{-(h+1)}[\mathbf{W}]$, $\mathcal{L}^{-h}[\mathbf{W}] = \mathcal{L}^{-(h+1)}[\mathbf{W}]$. As a result, $\mathcal{L}^{-h}[\mathbf{W}] = \mathcal{L}^{-j}[\mathbf{W}]$ for any $j \geq h$, which follows from (b). Therefore, the chain in part 1. stabilizes in the “inclusion” direction.

A similar argument can be given to show that the chain in part 1. also stabilizes in the “inclusion by” direction. Since $0 \leq \dim(\mathcal{L}^l[\mathbf{W}])$ for any $l \geq 0$, there exists an $\mathcal{L}^l[\mathbf{W}]$ in the chain with the lowest dimension but l is minimal. It follows that $\dim(\mathcal{L}^{(l+1)}[\mathbf{W}]) = \dim(\mathcal{L}^l[\mathbf{W}])$ is minimal. But because $\mathcal{L}^{(l+1)}[\mathbf{W}] \subseteq \mathcal{L}^l[\mathbf{W}]$, $\mathcal{L}^{(l+1)}[\mathbf{W}] = \mathcal{L}^l[\mathbf{W}]$. As a result, $\mathcal{L}^k[\mathbf{W}] = \mathcal{L}^l[\mathbf{W}]$ for any $k \geq l$, which follows from (a). Therefore, the chain in part 1. stabilizes in the “inclusion by” direction as well.

We claim that for every *proper* inclusion, there is a dimension loss. Revisiting $\mathcal{L}^{-h}[\mathbf{W}]$, we know that $\dim(\mathcal{L}^{-h}[\mathbf{W}]) \leq \dim(\mathbf{V})$. Consider the first proper inclusion:

$$\dots \mathcal{L}^{-(h-j)}[\mathbf{W}] \subsetneq \mathcal{L}^{-(h-j+1)}[\mathbf{W}] \subseteq \dots \subseteq \mathcal{L}^{-h}[\mathbf{W}] \subseteq \dots$$

This implies $\dim(\mathcal{L}^{-(h-j)}[\mathbf{W}]) < \dim(\mathcal{L}^{-h}[\mathbf{W}]) \leq \dim(\mathbf{V})$. Suppose $\dim(\mathcal{L}^{-h}[\mathbf{W}]) = \dim(\mathbf{V})$, i.e., maximal, then

$$\dim(\mathcal{L}^{-(h-j)}[\mathbf{W}]) \leq \dim(\mathbf{V}) - 1.$$

By assuming equality is satisfied and consider the next proper inclusion in the chain, a similar argument shows that the invariant subspace of interest has at most $\dim(\mathbf{V}) - 2$ dimensions. It follows that, at the $\dim(\mathbf{V})^{\text{th}}$ proper inclusion, the considered invariant subspace has at most $\dim(\mathbf{V}) - \dim(\mathbf{V}) = 0$ dimensions; i.e. the subspace is $\{\mathbf{0}_{\mathbf{V}}\}$. At this point, no further proper inclusion is possible. Therefore, the chain in part 1. has at most $\dim(\mathbf{V})$ proper inclusion.

3. Let $\mathcal{L}^{(k+1)}[\mathbf{W}] \subseteq \mathbf{M} \subseteq \mathcal{L}^k[\mathbf{W}]$ be given for some nonnegative integer k . \mathbf{M} is a subspace. We have that

$$\mathcal{L}[\mathbf{M}] \subseteq \mathcal{L}[\mathcal{L}^k[\mathbf{W}]] = \mathcal{L}^{(k+1)}[\mathbf{W}] \subseteq \mathbf{M}^{(i)}.$$

Let $\mathcal{L}^{-h}[\mathbf{W}] \subseteq \mathbf{M} \subseteq \mathcal{L}^{-(h+1)}[\mathbf{W}]$ be given for some nonnegative integer h . We have that

$$\mathcal{L}[\mathbf{M}] \subseteq \mathcal{L}[\mathcal{L}^{-(h+1)}[\mathbf{W}]] = \mathcal{L}[\mathcal{L}^{-1}[\mathcal{L}^{-h}[\mathbf{W}]]] \subseteq \mathcal{L}[\mathcal{L}^{-1}[\mathbf{M}]].$$

But because $\mathcal{L}^{-1}[\mathbf{M}] = \{v \in \mathbf{V} | \mathcal{L}(v) \in \mathbf{M}\}$, $\mathcal{L}[\mathcal{L}^{-1}[\mathbf{M}]] \subseteq \mathbf{M}$. Therefore, we have $\mathcal{L}[\mathbf{M}] \subseteq \mathbf{M}^{(ii)}$.

From (i) and (ii), \mathbf{M} is invariant under \mathcal{L} if \mathbf{M} is a subspace that falls between two consecutive subspaces in the chain shown in part 1..

25.3 Problem set 3

Problem. 1. Suppose that $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ are non-trivial vector spaces, and for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i.$$

1. Argue that

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$$

is a linear function.

2. Argue that every linear function $\mathcal{T} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \rightarrow \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$ can be expressed as

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times},$$

with $\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i$, in exactly one way.

3. If the function

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \rightarrow \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$$

is defined similarly, find (with proof) the block-matrix form of the functions

$$2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times}$$

and

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times}.$$

Solution. 1.

1. (a) It suffices to show that $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$ satisfies the linearity conditions.

Consider

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$$

where $a_j, b_j \in \mathbf{V}_j$ for $j \in \{1, 2, 3\}$. By definition,

$$\begin{aligned} & \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_{11}(a_1) + \mathcal{L}_{12}(a_2) + \mathcal{L}_{13}(a_3) \\ \mathcal{L}_{21}(a_1) + \mathcal{L}_{22}(a_2) + \mathcal{L}_{23}(a_3) \\ \mathcal{L}_{31}(a_1) + \mathcal{L}_{32}(a_2) + \mathcal{L}_{33}(a_3) \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{11}(b_1) + \mathcal{L}_{12}(b_2) + \mathcal{L}_{13}(b_3) \\ \mathcal{L}_{21}(b_1) + \mathcal{L}_{22}(b_2) + \mathcal{L}_{23}(b_3) \\ \mathcal{L}_{31}(b_1) + \mathcal{L}_{32}(b_2) + \mathcal{L}_{33}(b_3) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{L}_{11}(a_1) + \mathcal{L}_{12}(a_2) + \mathcal{L}_{13}(a_3) + \mathcal{L}_{11}(b_1) + \mathcal{L}_{12}(b_2) + \mathcal{L}_{13}(b_3) \\ \mathcal{L}_{21}(a_1) + \mathcal{L}_{22}(a_2) + \mathcal{L}_{23}(a_3) + \mathcal{L}_{21}(b_1) + \mathcal{L}_{22}(b_2) + \mathcal{L}_{23}(b_3) \\ \mathcal{L}_{31}(a_1) + \mathcal{L}_{32}(a_2) + \mathcal{L}_{33}(a_3) + \mathcal{L}_{31}(b_1) + \mathcal{L}_{32}(b_2) + \mathcal{L}_{33}(b_3) \end{pmatrix} \\ &= \begin{pmatrix} [\mathcal{L}_{11}(a_1) + \mathcal{L}_{11}(b_1)] + [\mathcal{L}_{12}(a_2) + \mathcal{L}_{12}(b_2)] + [\mathcal{L}_{13}(a_3) + \mathcal{L}_{13}(b_3)] \\ [\mathcal{L}_{21}(a_1) + \mathcal{L}_{21}(b_1)] + [\mathcal{L}_{22}(a_2) + \mathcal{L}_{22}(b_2)] + [\mathcal{L}_{23}(a_3) + \mathcal{L}_{23}(b_3)] \\ [\mathcal{L}_{31}(a_1) + \mathcal{L}_{31}(b_1)] + [\mathcal{L}_{32}(a_2) + \mathcal{L}_{32}(b_2)] + [\mathcal{L}_{33}(a_3) + \mathcal{L}_{33}(b_3)] \end{pmatrix}, \text{ by the linearity of } \mathcal{L}_{ij} \\ &= \begin{pmatrix} \mathcal{L}_{11}(a_1 + b_1) + \mathcal{L}_{12}(a_2 + b_2) + \mathcal{L}_{13}(a_2 + b_2) \\ \mathcal{L}_{21}(a_1 + b_1) + \mathcal{L}_{22}(a_2 + b_2) + \mathcal{L}_{23}(a_2 + b_2) \\ \mathcal{L}_{31}(a_1 + b_1) + \mathcal{L}_{32}(a_2 + b_2) + \mathcal{L}_{33}(a_2 + b_2) \end{pmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix} \\ &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \left(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \right). \end{aligned}$$

Therefore, the additivity condition is satisfied.

- (b) Next, we consider the scaling condition. Let $c \in \mathbb{C}$ be given,

$$\begin{aligned}
& \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \left(c \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \right) \\
&= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} ca_1 \\ ca_2 \\ ca_3 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}_{11}(ca_1) + \mathcal{L}_{12}(ca_2) + \mathcal{L}_{13}(ca_3) \\ \mathcal{L}_{21}(ca_1) + \mathcal{L}_{22}(ca_2) + \mathcal{L}_{23}(ca_3) \\ \mathcal{L}_{31}(ca_1) + \mathcal{L}_{32}(ca_2) + \mathcal{L}_{33}(ca_3) \end{pmatrix} \\
&= \begin{pmatrix} c\mathcal{L}_{11}(a_1) + c\mathcal{L}_{12}(a_2) + c\mathcal{L}_{13}(a_3) \\ c\mathcal{L}_{21}(a_1) + c\mathcal{L}_{22}(a_2) + c\mathcal{L}_{23}(a_3) \\ c\mathcal{L}_{31}(a_1) + c\mathcal{L}_{32}(a_2) + c\mathcal{L}_{33}(a_3) \end{pmatrix}, \text{ by the linearity of } \mathcal{L}_{ij} \\
&= c \begin{pmatrix} \mathcal{L}_{11}(a_1) + \mathcal{L}_{12}(a_2) + \mathcal{L}_{13}(a_3) \\ \mathcal{L}_{21}(a_1) + \mathcal{L}_{22}(a_2) + \mathcal{L}_{23}(a_3) \\ \mathcal{L}_{31}(a_1) + \mathcal{L}_{32}(a_2) + \mathcal{L}_{33}(a_3) \end{pmatrix}.
\end{aligned}$$

Therefore, the scalar multiplication condition is also satisfied.

From (a) and (b), $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$ is a linear function.

2. (a) Existence: Let Π_i be a coordinate projection on $\mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$ onto \mathbf{V}_i defined in Definition 0.2. Let γ_i be a coordinate injection of \mathbf{V}_i into $\mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$ also defined in Definition 0.2.

Consider the block-matrix \mathcal{L}_\times ,

$$\mathcal{L}_\times = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times.$$

Let the linear function $\mathcal{T} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$ be given. Let each $\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i$ be defined as

$$\mathcal{L}_{ij} = \Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{V}_j}.$$

Then we have

$$\begin{aligned} \mathcal{L}_\times &= \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \\ &= \begin{bmatrix} \Pi_1 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_1 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_1 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \\ \Pi_2 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_2 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_2 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \\ \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_3 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \end{bmatrix}_\times. \end{aligned}$$

Consider $(w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \in \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$ with $w_1 \in \mathbf{V}_1$.

$$\begin{aligned} &\begin{bmatrix} \Pi_1 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_1 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_3} \\ \Pi_2 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_2 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_3} \\ \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} & \Pi_3 \circ \mathcal{T} \circ \gamma_2 \Big|_{\mathbf{V}_2} & \Pi_3 \circ \mathcal{T} \circ \gamma_3 \Big|_{\mathbf{V}_3} \end{bmatrix}_\times \begin{pmatrix} w_1 \\ \mathbf{0}_\mathbf{V} \\ \mathbf{0}_\mathbf{V} \end{pmatrix} \\ &= \begin{bmatrix} \Pi_1 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} (w_1) \\ \Pi_2 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} (w_1) \\ \Pi_3 \circ \mathcal{T} \circ \gamma_1 \Big|_{\mathbf{V}_1} (w_1) \end{bmatrix} \\ &= \begin{bmatrix} \Pi_1 \circ \mathcal{T} (w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \\ \Pi_2 \circ \mathcal{T} (w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \\ \Pi_3 \circ \mathcal{T} (w_1 \quad \mathbf{0}_\mathbf{V} \quad \mathbf{0}_\mathbf{V})^\top \end{bmatrix}. \end{aligned}$$

Let $\mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)$ be $\begin{pmatrix} a & b & c \end{pmatrix}^\top \in \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3$. By definition,

$$\begin{aligned} & \begin{bmatrix} \Pi_1 \circ \mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)^\top \\ \Pi_2 \circ \mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)^\top \\ \Pi_3 \circ \mathcal{T}\left(\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top\right)^\top \end{bmatrix} \\ &= \begin{bmatrix} \Pi_1\left(\begin{pmatrix} a & b & c \end{pmatrix}^\top\right) \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ \Pi_2\left(\begin{pmatrix} a & b & c \end{pmatrix}^\top\right) \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \\ \Pi_3\left(\begin{pmatrix} a & b & c \end{pmatrix}^\top\right) \end{bmatrix} \\ &= \begin{bmatrix} a \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ b \\ \mathbf{0}_{\mathbf{V}} \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathcal{T}\left(\begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}\right) \end{aligned}$$

Therefore,

$$\mathcal{L}_\times \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix} = \mathcal{T} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}.$$

In general, the equality above also holds for any $v = w_1 + w_2 + w_3 \in \mathbf{V}$ with $w_i \in \mathbf{V}_i$, since v can be expressed as

$$v = \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{\mathbf{V}} \\ w_2 \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix} + \begin{pmatrix} \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \\ w_3 \end{pmatrix},$$

and the linearity of \mathcal{L}_\times and \mathcal{T} ensures that $\mathcal{L}_\times(v) = \mathcal{T}(v)$. Hence, \mathcal{T} can be expressed as the matrix \mathcal{L}_\times whose elements \mathcal{L}_{ij} 's are defined as

$$\mathcal{L}_{ij} = \Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{V}_j}.$$

- (b) Uniqueness: We observe that all of the steps in the “Existence part” are reversible. This means if we were given the linear map \mathcal{T} , we could consider applying \mathcal{T} to $\begin{pmatrix} w_1 & \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top$, $\begin{pmatrix} \mathbf{0}_{\mathbf{V}} & w_2 & \mathbf{0}_{\mathbf{V}} \end{pmatrix}^\top$, and $\begin{pmatrix} \mathbf{0}_{\mathbf{V}} & \mathbf{0}_{\mathbf{V}} & w_3 \end{pmatrix}^\top \in \mathbf{V}$; and by reversing the steps in part (a), we know that \mathcal{T} can be expressed as a 3×3 block-matrix whose elements must have the form $\Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{V}_j} = \mathcal{L}_{ij}$.

From parts (a) and (b), the linear function \mathcal{T} can be expressed as the matrix \mathcal{L}_\times in exactly one way.

3. Let

$$\mathcal{L} := \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$$

and

$$\mathcal{M} := \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$$

be given.

- (a) Let \mathcal{L} be the matrix of the linear function \mathcal{T} above and \mathcal{M} be the matrix of a linear function $\mathcal{K} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$. Consider the expression $2\mathcal{L}_{ij} + 3\mathcal{M}_{ij}$. Let \mathcal{L}_{ij} be constructed as in Part 1 for the linear function \mathcal{T} . Let \mathcal{M} be constructed similarly for \mathcal{K} . Then, we have

$$\begin{aligned} 2\mathcal{L}_{ij} + 3\mathcal{M}_{ij} &= 2\Pi_i \circ \mathcal{T} \circ \gamma_j \Big|_{\mathbf{w}_j} + 3\Pi_i \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{w}_j} \\ &= \Pi_i \circ 2\mathcal{T} \circ \gamma_j \Big|_{\mathbf{w}_j} + \Pi_i \circ 3\mathcal{K} \circ \gamma_j \Big|_{\mathbf{w}_j} \\ &= \Pi_i \circ \left(2\mathcal{T} \circ \gamma_j \Big|_{\mathbf{w}_j} + 3\mathcal{K} \circ \gamma_j \Big|_{\mathbf{w}_j} \right) \\ &= \Pi_i \circ (2\mathcal{T} + 3\mathcal{K}) \circ \gamma_j \Big|_{\mathbf{w}_j} \\ &= (2\mathcal{T} + 3\mathcal{K})_{ij}, \text{ by definition.} \end{aligned}$$

This shows that each element of the matrix of the linear function $(2\mathcal{L} + 3\mathcal{K})$ is given by

$$\boxed{(2\mathcal{T} + 3\mathcal{K})_{ij} = 2\mathcal{L}_{ij} + 3\mathcal{M}_{ij}}$$

i.e.,

$$\begin{aligned} &2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\times} \\ &= \begin{bmatrix} 2\mathcal{L}_{11} + 3\mathcal{M}_{11} & 2\mathcal{L}_{12} + 3\mathcal{M}_{12} & 2\mathcal{L}_{13} + 3\mathcal{M}_{13} \\ 2\mathcal{L}_{21} + 3\mathcal{M}_{21} & 2\mathcal{L}_{22} + 3\mathcal{M}_{22} & 2\mathcal{L}_{23} + 3\mathcal{M}_{23} \\ 2\mathcal{L}_{31} + 3\mathcal{M}_{31} & 2\mathcal{L}_{32} + 3\mathcal{M}_{32} & 2\mathcal{L}_{33} + 3\mathcal{M}_{33} \end{bmatrix}_{\times}. \end{aligned}$$

- (b) Let \mathcal{T} , \mathcal{M}_{ij} , \mathcal{K} , and \mathcal{L}_{ij} be defined similarly: \mathcal{L} is the matrix of the linear function \mathcal{T} and \mathcal{M} is the matrix of the linear function \mathcal{K} . Consider the expression $\sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj}$, we have

$$\begin{aligned} \sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj} &= \sum_{k=1}^3 \Pi_i \circ \mathcal{T} \circ \gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= \sum_{k=1}^3 \Pi_i \circ \mathcal{T} \circ \left(\gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \right) \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= \Pi_i \circ \mathcal{T} \circ \left(\sum_{k=1}^3 \gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \right) \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j}. \end{aligned}$$

Consider $v = (w_1 \ w_2 \ w_3)^\top \in \mathbf{V}$ such that $w_i \in \mathbf{W}_i$ for $i \in \{1, 2, 3\}$. We can show that the summation in the expression above is simply the identity function:

$$\begin{aligned} \left(\sum_{k=1}^3 \gamma_k \Big|_{\mathbf{W}_k} \circ \Pi_k \right) (v) &= \gamma_1 \Big|_{\mathbf{W}_1} \circ \Pi_1(v) + \gamma_2 \Big|_{\mathbf{W}_2} \circ \Pi_2(v) + \gamma_3 \Big|_{\mathbf{W}_3} \circ \Pi_3(v) \\ &= \gamma_1(w_1) + \gamma_2(w_2) + \gamma_3(w_3) \\ &= \begin{pmatrix} w_1 \\ \mathbf{0}_V \\ \mathbf{0}_V \end{pmatrix} + \begin{pmatrix} \mathbf{0}_V \\ w_2 \\ \mathbf{0}_V \end{pmatrix} + \begin{pmatrix} \mathbf{0}_V \\ \mathbf{0}_V \\ w_3 \end{pmatrix} \\ &= \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \\ &= v. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj} &= \Pi_i \circ \mathcal{T} \circ \mathcal{K} \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= \Pi_i \circ (\mathcal{T} \circ \mathcal{K}) \circ \gamma_j \Big|_{\mathbf{W}_j} \\ &= (\mathcal{T} \circ \mathcal{K})_{ij}. \end{aligned}$$

This shows that each element of the matrix of the linear function $(\mathcal{T} \circ \mathcal{K})$ is given by

$$(\mathcal{T} \circ \mathcal{K})_{ij} = \sum_{k=1}^3 \mathcal{L}_{ik} \circ \mathcal{M}_{kj}$$

which gives the full block-matrix form:

$$\begin{aligned}
& \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \\
&= \begin{bmatrix} \sum_{k=1}^3 \mathcal{L}_{1k} \circ \mathcal{M}_{k1} & \sum_{k=1}^3 \mathcal{L}_{1k} \circ \mathcal{M}_{k2} & \sum_{k=1}^3 \mathcal{L}_{1k} \circ \mathcal{M}_{k3} \\ \sum_{k=1}^3 \mathcal{L}_{2k} \circ \mathcal{M}_{k1} & \sum_{k=1}^3 \mathcal{L}_{2k} \circ \mathcal{M}_{k2} & \sum_{k=1}^3 \mathcal{L}_{2k} \circ \mathcal{M}_{k3} \\ \sum_{k=1}^3 \mathcal{L}_{3k} \circ \mathcal{M}_{k1} & \sum_{k=1}^3 \mathcal{L}_{3k} \circ \mathcal{M}_{k2} & \sum_{k=1}^3 \mathcal{L}_{3k} \circ \mathcal{M}_{k3} \end{bmatrix}_{\times}
\end{aligned}$$

Problem. 2. Suppose that $\mathbf{V} = \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3$ and the \mathbf{W}_i 's are non-trivial subspaces of \mathbf{V} . Suppose that for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i.$$

1. Verify for formula

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\boxtimes} = \left(\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right)_{\boxtimes},$$

where we have used as few brackets as possible without losing clarity.

2. Argue that every $T : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$ can be expressed as $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}$,

with $\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$, in exactly one way.

3. If the function

$$\begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus} : \mathbf{V} \longrightarrow \mathbf{V}$$

is defined similarly, find (with proof) the corresponding black-matrix form of the functions

$$2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}$$

and

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}.$$

4. Suppose that $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$ is a resolution of the identity, where $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$. If

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus},$$

express (with proof, of course) \mathcal{L}_{ij} in terms of $\mathcal{L}, \mathcal{E}_k$'s and \mathbf{W}_k 's.

5. Continuing with the set-up of part 4, find $\mathcal{M}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$ such that

$$\mathcal{E}_1 = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}.$$

Then do the same for \mathcal{E}_2 and \mathcal{E}_3 .

Solution. 2.

1. By definition,

$$\begin{aligned}
\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\boxtimes} &= \begin{bmatrix} \boxtimes \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \boxtimes^{-1} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{\boxtimes} \\
&= \begin{bmatrix} \boxtimes \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \circ \boxtimes^{-1} \circ \boxtimes \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \boxtimes \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \boxtimes \left(\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right) \\
&= \left(\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right)_{\boxtimes}.
\end{aligned}$$

2. (a) Existence: Let the linear functions $\mathcal{E}_i := \Pi_i \circ \boxtimes^{-1} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{W}_i$ and $\mathcal{K}_i := \boxtimes \circ \gamma_i : \mathbf{W}_i \xrightarrow{\text{linear}} \mathbf{V}$ be given, where the Π_i 's and γ_i 's are defined the same way as in Problem 1.

Consider the block matrix \mathcal{L}_{\oplus}

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

Let the linear function $\mathcal{T} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$ be given. Let each $\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$ be defined as:

$$\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{W}_j}.$$

By definition,

$$\begin{aligned}
\mathcal{L}_{ij} &= \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{W}_j} \\
&= \Pi_i \circ \boxtimes^{-1} \circ \mathcal{T} \circ \boxtimes \circ \gamma_j \Big|_{\mathbf{W}_j}.
\end{aligned}$$

Applying \mathcal{L}_\oplus to $(w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \in \mathbf{V}$, we get

$$\begin{aligned}
\mathcal{L} \begin{pmatrix} w_1 \\ \mathbf{0}_V \\ \mathbf{0}_V \end{pmatrix}_{\mathfrak{X}} &= \left[\begin{array}{c} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} \circ \gamma_1 \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} \circ \gamma_1 \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} \circ \gamma_1 \end{array} \middle| \begin{array}{c} (w_1) \\ \mathbf{w}_1 \\ \mathbf{w}_1 \end{array} \right]_{\mathfrak{X}} \\
&= \left[\begin{array}{c} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} (w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} (w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T} \circ \mathfrak{X} (w_1 \quad \mathbf{0}_V \quad \mathbf{0}_V)^\top \end{array} \right]_{\mathfrak{X}} \\
&= \left[\begin{array}{c} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \end{array} \right]_{\mathfrak{X}} \\
&= \mathfrak{X} \left(\begin{bmatrix} \Pi_1 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_2 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \\ \Pi_3 \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1) \end{bmatrix} \right) \\
&= \mathfrak{X} \circ \mathfrak{X}^{-1} \circ \mathcal{T}(w_1), \text{ by Part 2(a) of Problem 1} \\
&= \mathcal{T}(w_1)
\end{aligned}$$

By the definition of \mathfrak{X} , we can write this as

$$\mathcal{L}(w_1 + \mathbf{0}_V + \mathbf{0}_V) = \mathcal{L}(w_1) = \mathcal{T}(w_1).$$

In general, the equality above also holds if we start with any $v = w_1 + w_2 + w_3 \in \mathbf{V}$ with $w_i \in \mathbf{W}_i$ for $i = \{1, 2, 3\}$, because as we have shown in Part 2(a) of Problem 1, v can be written as

$$v = \begin{pmatrix} w_1 \\ \mathbf{0}_V \\ \mathbf{0}_V \end{pmatrix}_{\mathfrak{X}} + \begin{pmatrix} \mathbf{0}_V \\ w_2 \\ \mathbf{0}_V \end{pmatrix}_{\mathfrak{X}} + \begin{pmatrix} \mathbf{0}_V \\ \mathbf{0}_V \\ w_3 \end{pmatrix}_{\mathfrak{X}},$$

and the linearity of \mathcal{L} and \mathcal{T} ensures $\mathcal{L}(v) = \mathcal{T}(v)$. Hence, \mathcal{T} can be expressed as the matrix \mathcal{L} whose elements are defined as

$$\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{W}_j}.$$

- (b) Uniqueness: Once again, we observe that all of the steps in the “Existence” part are reversible. We can consider applying \mathcal{T} to v , expressed in Part 3(a), and by reversing the steps in Part (a) show that the 3×3 block-matrix of \mathcal{T} must consist elements defined by

$$\mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_j \Big|_{\mathbf{W}_j} = \mathcal{L}_{\oplus ij} \text{ for } i, j \in \{1, 2, 3\}.$$

From parts (a) and (b), the linear function \mathcal{T} can be expressed as the matrix \mathcal{L}_\oplus defined in the problem statement in exactly one way.

3. (a) Let $\mathcal{L}_\oplus = [\mathcal{L}_{\oplus ij}]$ denote the matrix $\mathcal{L}_\oplus : \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3 \xrightarrow{\text{linear}} \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3$. Let $\mathcal{M}_\oplus = [\mathcal{M}_{\oplus ij}]$ be defined similarly for \mathcal{M} . By part 1, we can re-express the matrix

$$[(2\mathcal{L}_\oplus + 3\mathcal{M}_\oplus)_{ij}] = 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\oplus + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\oplus$$

as

$$\begin{aligned} & 2 \mathfrak{X} \circ \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} + 3 \mathfrak{X} \circ \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} + \mathfrak{X} \circ 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ \left(2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \circ \mathfrak{X}^{-1} \right) \\ &= \mathfrak{X} \circ \left(2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\times + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\times \right) \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ \left(2[\mathcal{L}_{\times ij}] + 3[\mathcal{M}_{\times ij}] \right) \circ \mathfrak{X}^{-1}, \end{aligned}$$

where the notation $[A_{ij}]$ denotes the block-matrix of elements A_{ij} where A is a linear function, and the last relation:

$$[(2\mathcal{L}_\times + 3\mathcal{M}_\times)_{ij}] = 2[\mathcal{L}_{\times ij}] + 3[\mathcal{M}_{\times ij}].$$

follows from Problem 1. Then, we have

$$\begin{aligned} [(2\mathcal{L}_\oplus + 3\mathcal{M}_\oplus)_{ij}] &= \mathfrak{X} \circ \left(2[\mathcal{L}_{\times ij}] + 3[\mathcal{M}_{\times ij}] \right) \circ \mathfrak{X}^{-1} \\ &= \mathfrak{X} \circ [2\mathcal{L}_{\times ij}] \circ \mathfrak{X}^{-1} + \mathfrak{X} \circ [3\mathcal{M}_{\times ij}] \circ \mathfrak{X}^{-1} \\ &= 2[\mathcal{L}_{\oplus ij}] + 3[\mathcal{M}_{\oplus ij}]. \end{aligned}$$

So, in the block-matrix form:

$$\begin{aligned} & 2 \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_\oplus + 3 \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_\oplus \\ &= \begin{bmatrix} 2\mathcal{L}_{11} + 3\mathcal{M}_{11} & 2\mathcal{L}_{12} + 3\mathcal{M}_{12} & 2\mathcal{L}_{13} + 3\mathcal{M}_{13} \\ 2\mathcal{L}_{21} + 3\mathcal{M}_{21} & 2\mathcal{L}_{22} + 3\mathcal{M}_{22} & 2\mathcal{L}_{23} + 3\mathcal{M}_{23} \\ 2\mathcal{L}_{31} + 3\mathcal{M}_{31} & 2\mathcal{L}_{32} + 3\mathcal{M}_{32} & 2\mathcal{L}_{33} + 3\mathcal{M}_{33} \end{bmatrix}_\oplus. \end{aligned}$$

- (b) Let \mathcal{L}_\oplus and \mathcal{M}_\oplus denote the same matrices in part (a), and let \mathcal{L}_\oplus represent the linear function \mathcal{T} , \mathcal{M}_\oplus represent the linear function \mathcal{N} . Let the matrix $[(\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij}]$ express the composition $\mathcal{T} \circ \mathcal{N}$. Its the elements $[\mathcal{L}_\oplus \circ \mathcal{M}_\oplus]$ can be constructed as

$$(\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij} = \mathcal{E}_i \circ (\mathcal{L}_\oplus \circ \mathcal{M}_\oplus) \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} = \mathcal{E}_i \circ (\mathcal{T} \circ \mathcal{N}) \circ \mathcal{K}_j \Big|_{\mathbf{w}_j}, \quad (\dagger)$$

as we have done in Part 1. Consider the expression $\sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj}$. By part 1, we have

$$\mathcal{L}_{\oplus ik} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{K}_k \Big|_{\mathbf{w}_k} \quad \text{and} \quad \mathcal{M}_{\oplus kj} = \mathcal{E}_k \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j}.$$

This gives

$$\begin{aligned} \sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj} &= \sum_{k=1}^3 \mathcal{E}_i \circ \mathcal{T} \circ \left(\mathcal{K}_k \Big|_{\mathbf{w}_k} \circ \mathcal{E}_k \right) \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} \\ &= \mathcal{E}_i \circ \mathcal{T} \circ \left(\sum_{k=1}^3 \mathcal{K}_k \Big|_{\mathbf{w}_k} \circ \mathcal{E}_k \right) \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} \end{aligned}$$

By definition, we have

$$\mathcal{K}_k \Big|_{\mathbf{w}_k} = \mathfrak{K} \circ \gamma_k \Big|_{\mathbf{w}_k} \quad \text{and} \quad \mathcal{E}_k = \Pi_k \circ \mathfrak{K}^{-1},$$

which imply

$$\begin{aligned} \sum_{k=1}^3 \mathcal{K}_k \Big|_{\mathbf{w}_k} \circ \mathcal{E}_k &= \sum_{k=1}^3 \mathfrak{K} \circ \gamma_k \Big|_{\mathbf{w}_k} \circ \Pi_k \circ \mathfrak{K}^{-1} \\ &= \mathfrak{K} \circ \left(\sum_{k=1}^3 \gamma_k \Big|_{\mathbf{w}_k} \circ \Pi_k \right) \circ \mathfrak{K}^{-1} \\ &= \mathfrak{K} \circ \mathfrak{K}^{-1}, \quad \text{by Part 3(b) of Problem 1} \\ &= \text{Identity.} \end{aligned}$$

It follows that,

$$\sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj} = \mathcal{E}_i \circ \mathcal{T} \circ \mathcal{N} \circ \mathcal{K}_j \Big|_{\mathbf{w}_j} = (\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij}, \quad \text{by } (\dagger).$$

Therefore, the matrix elements of $[\mathcal{L}_\oplus \circ \mathcal{M}_\oplus]$ are given by

$$\boxed{(\mathcal{L}_\oplus \circ \mathcal{M}_\oplus)_{ij} = \sum_{k=1}^3 \mathcal{L}_{\oplus ik} \circ \mathcal{M}_{\oplus kj}}$$

Since the full block-matrix expression for $[\mathcal{L}_\oplus \circ \mathcal{M}_\oplus]$ is very similar to what we have found with $\mathcal{L}_\times \circ \mathcal{M}_\times$, we will not produce it here.

4. Let $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$ be a resolution of identity, with $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$, be given. Furthermore, let the linear map $\mathcal{L} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$ be given where

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

Claim:

$$\boxed{\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{L} \Big|_{\mathbf{W}_j} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i}$$

Consider applying \mathcal{L} to $(w_1 \quad \mathbf{0}_{\mathbf{V}} \quad \mathbf{0}_{\mathbf{V}})_{\mathfrak{X}}^{\top} \in \mathbf{V}$, we have

$$\mathcal{L} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \begin{pmatrix} \mathcal{L}_{11}(w_1) \\ \mathcal{L}_{21}(w_1) \\ \mathcal{L}_{31}(w_1) \end{pmatrix}_{\mathfrak{X}}.$$

To extract an element \mathcal{L}_{i1} , we can apply \mathcal{E}_i to this result:

$$\mathcal{E}_i \circ \mathcal{L} \begin{pmatrix} w_1 \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \mathcal{E}_i \begin{pmatrix} \mathcal{L}_{11}(w_1) \\ \mathcal{L}_{21}(w_1) \\ \mathcal{L}_{31}(w_1) \end{pmatrix}_{\mathfrak{X}} = \mathcal{L}_{i1}(w_1).$$

This shows that by considering the restricted map $\mathcal{L} \Big|_{\mathbf{W}_1}$, we are able to

obtain the \mathcal{L}_{i1} element of \mathcal{L} by composing the idempotent \mathcal{E}_i with $\mathcal{L} \Big|_{\mathbf{W}_1}$.

So in general, if we consider the restricted map $\mathcal{L} \Big|_{\mathbf{W}_j}$, the same procedure will give us the element \mathcal{L}_{ij} . Hence, we have verified the claim that

$$\mathcal{L}_{ij} = \mathcal{E}_i \circ \mathcal{L} \Big|_{\mathbf{W}_j}.$$

5. Let

$$\mathcal{E}_1 = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ \mathcal{M}_{21} & \mathcal{M}_{22} & \mathcal{M}_{23} \\ \mathcal{M}_{31} & \mathcal{M}_{32} & \mathcal{M}_{33} \end{bmatrix}_{\oplus}.$$

be given, where $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$ and $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$ is a resolution of identity. Continuing with the set-up of part 4, we have

$$\mathcal{M}_{ij} = \mathcal{E}_i \circ \mathcal{E}_1 \Big|_{\mathbf{W}_j}.$$

Since $\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 = \mathcal{I}_{\mathbf{V}}$ is a resolution of identity, and $\text{Im}(\mathcal{E}_i) = \mathbf{W}_i$ where $\mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3 = \mathbf{V}$, $\ker(\mathcal{E}_1) = \mathbf{W}_2 \oplus \mathbf{W}_3$. This implies the restricted map $\mathcal{E}_1 \Big|_{\mathbf{W}_j}$ gives $\mathbf{0}_{\mathbf{V}}$ for $j \neq 1$ and acts as an identity on \mathbf{W}_j if $j = 1$. Therefore, for $j \neq 1$, M_{ij} is a zero function.

If $j = 1$, then

$$\mathcal{M}_{ij} = \mathcal{E}_i \circ \mathcal{E}_1 \Big|_{\mathbf{W}_1} = \mathcal{E}_i \Big|_{\mathbf{W}_1},$$

since \mathcal{E}_1 acts as identity on \mathbf{W}_1 . The same argument for the restricted map $\mathcal{E}_i \Big|_{\mathbf{W}_1}$ shows that \mathcal{M}_{ij} is the zero function if $i \neq 1$ and the identity on \mathbf{W}_1 if $i = 1$. Therefore,

$$\text{For } \mathcal{E}_1 : \quad \mathcal{M}_{ij} = \begin{cases} Id_{\mathbf{W}_1}, & i, j = 1 \\ \mathcal{O}, & i \text{ or } j \neq 1 \end{cases}$$

The exact same procedure can be used to find the elements M_{ij} of \mathcal{E}_2 and \mathcal{E}_3 , simply replacing all 1's with 2's and 3's. So,

$$\begin{aligned} \text{For } \mathcal{E}_2 : \quad \mathcal{M}_{ij} &= \begin{cases} Id_{\mathbf{W}_2}, & i, j = 2 \\ \mathcal{O}, & i \text{ or } j \neq 2 \end{cases} \\ \text{For } \mathcal{E}_3 : \quad \mathcal{M}_{ij} &= \begin{cases} Id_{\mathbf{W}_3}, & i, j = 3 \\ \mathcal{O}, & i \text{ or } j \neq 3 \end{cases} \end{aligned}$$

Problem. 3.

1. Using the set-up of Definition 0.7, verify the identity

$$[\mathcal{T}(\mathcal{X})]_{\Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma} [\mathcal{X}]_{\Gamma}.$$

2. Use part 1 to prove that

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma} = \begin{bmatrix} [\mathcal{T}(v_1)]_{\Omega} & [\mathcal{T}(v_2)]_{\Omega} & \dots & [\mathcal{T}(v_n)]_{\Omega} \end{bmatrix}.$$

3. If \mathbf{U} is a finite-dimensional vector space with a coordinate system Δ , and $\mathcal{S} : \mathbf{Z} \xrightarrow{\text{linear}} \mathbf{U}$, argue that

$$[\mathcal{S} \circ \mathcal{T}]_{\Delta \leftarrow \Gamma} = [\mathcal{S}]_{\Delta \leftarrow \Omega} [\mathcal{T}]_{\Omega \leftarrow \Gamma}.$$

4. Verify that in the case \mathcal{T} is invertible (in which case $m = n$), so is $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$, and

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1}.$$

Solution. 3.

1. By the set-up in Definition 0.3, we have $[\mathcal{T}]_{\Omega \leftarrow \Gamma} = [\]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma}$, and $[\mathcal{X}]_{\Gamma} = [\]_{\Gamma}(\mathcal{X})$. Hence,

$$\begin{aligned} [\mathcal{T}]_{\Omega \leftarrow \Gamma} [\mathcal{X}]_{\Gamma} &= [\]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \circ [\mathcal{X}]_{\Gamma} \\ &= [\]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \circ [\]_{\Gamma}(\mathcal{X}) \\ &= [\]_{\Omega} \circ \mathcal{T} \circ (\mathcal{A}_{\Gamma} \circ [\]_{\Gamma})(\mathcal{X}) \\ &= [\]_{\Omega} \circ \mathcal{T}(\mathcal{X}) \\ &= [\]_{\Omega}(\mathcal{T}(\mathcal{X})) \\ &= [\mathcal{T}(\mathcal{X})]_{\Omega}. \end{aligned}$$

2. Claim:

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma} = \begin{bmatrix} [\mathcal{T}(v_1)]_{\Omega} & [\mathcal{T}(v_2)]_{\Omega} & \dots & [\mathcal{T}(v_n)]_{\Omega} \end{bmatrix}.$$

Let \mathcal{X} be $v_i \in \{v_1, v_2, \dots, v_n\}$ a coordinate system of Γ . Then we can create a bijective matrix

$$A_{\Gamma} = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

whose inverse is $[\]_{\Gamma}$. To find the columns of the matrix $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$, we want to apply $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ to the standard basis vectors. To generate these standard basis vectors, we simply coordinatize \mathcal{X} :

$$[\mathcal{X}]_{\Gamma} = [v_i]_{\Gamma} = [\]_{\Gamma}(v_i) = A_{\Gamma}^{-1}(v_i) = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}^{-1}(v_i) = \vec{e}_i \in \mathbb{C}^n.$$

Thus, the i^{th} column of $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ is given by

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma}(\vec{e}_i) = [\mathcal{T}]_{\Omega \leftarrow \Gamma}[v_i]_{\Gamma} = [\mathcal{T}(v_i)]_{\Gamma}, \text{ by Part 1.}$$

This verifies the claim.

3. Let the finite-dimensional vector space \mathbf{U} with coordinate system Δ be given. Let $\mathcal{S} : \mathbf{Z} \xrightarrow{\text{linear}} \mathbf{U}$ be given. Claim:

$$[\mathcal{S} \circ \mathcal{T}]_{\Delta \leftarrow \Gamma} = [\mathcal{S}]_{\Delta \leftarrow \Omega} [\mathcal{T}]_{\Omega \leftarrow \Gamma}.$$

Let $\Delta = (w_1, w_2, \dots, w_l)$ be a coordinate system for \mathbf{U} . We also define the bijective matrix

$$A_{\Delta} := \begin{bmatrix} w_1 & w_2 & \dots & w_l \end{bmatrix} : \mathbb{C}^l \rightarrow \mathbf{U}.$$

Since A_{Δ} is invertible, we can write $[\]_{\Delta}$ for A_{Δ}^{-1} .

By the set-up in definition 0.7, we have the following

$$\begin{aligned} [\mathcal{T}]_{\Omega \leftarrow \Gamma} &= [\]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ [\mathcal{S}]_{\Delta \leftarrow \Omega} &= [\]_{\Delta} \circ \mathcal{S} \circ \mathcal{A}_{\Omega}. \end{aligned}$$

It follows that

$$\begin{aligned} [\mathcal{S}]_{\Delta \leftarrow \Omega} [\mathcal{T}]_{\Omega \leftarrow \Gamma} &= [\]_{\Delta} \circ \mathcal{S} \circ \mathcal{A}_{\Omega} \circ [\]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ &= [\]_{\Delta} \circ \mathcal{S} \circ (\mathcal{A}_{\Omega} \circ [\]_{\Omega}) \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ &= [\]_{\Delta} \circ \mathcal{S} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma} \\ &= [\]_{\Delta} \circ (\mathcal{S} \circ \mathcal{T}) \circ \mathcal{A}_{\Gamma} \\ &= [\mathcal{S} \circ \mathcal{T}]_{\Delta \leftarrow \Gamma}, \end{aligned}$$

where we have used the fact that $A_{\Omega}[\]_{\Omega} = A_{\Omega}A_{\Omega}^{-1}$ is the identity function. This completes the argument.

4. (a) To show: $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ is invertible when \mathcal{T} is invertible.

The matrix $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ can be expressed as

$$[\mathcal{T}]_{\Omega \leftarrow \Gamma} = [\]_{\Omega} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma}.$$

Since T , $[\]_{\Omega}$, and \mathcal{A}_{Γ} are invertible, the above composition is also invertible, i.e., $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ is invertible.

(b) Let \mathcal{T} be invertible (in which case $m = n$). Claim:

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1}.$$

Since \mathcal{T} , \mathcal{A}_Ω , \mathcal{A}_Γ are all invertible, their inverses are defined. By the set-up in definition 0.7, the matrix $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$ can be written as the composition $[\]_\Omega \circ \mathcal{T} \circ \mathcal{A}_\Gamma$ for the linear function $T : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{Z}$. Since $\mathcal{T}^{-1} : \mathbf{Z} \xrightarrow{\text{linear}} \mathbf{V}$ is defined, we can express its matrix $[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega}$ as

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\]_\Gamma \circ \mathcal{T}^{-1} \circ \mathcal{A}_\Omega. \quad (\dagger)$$

Consider the inverse of the matrix $[\mathcal{T}]_{\Omega \leftarrow \Gamma}$:

$$\begin{aligned} [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1} &= ([\]_\Omega \circ \mathcal{T} \circ \mathcal{A}_\Gamma)^{-1} \\ &= \mathcal{A}_\Gamma^{-1} \circ ([\]_\Gamma \circ \mathcal{T})^{-1} \\ &= \mathcal{A}_\Gamma^{-1} \circ \mathcal{T}^{-1} \circ [\]_\Omega^{-1} \\ &= [\]_\Gamma \circ \mathcal{T}^{-1} \circ \mathcal{A}_\Omega. \quad (\dagger\dagger) \end{aligned}$$

From (\dagger) and $(\dagger\dagger)$, we have

$$[\mathcal{T}^{-1}]_{\Gamma \leftarrow \Omega} = [\mathcal{T}]_{\Omega \leftarrow \Gamma}^{-1}$$

as claimed.

Problem. 4. Let us write Γ_0 for the standard coordinate system $(1, x, x^2, x^3)$ of the vector space \mathbb{P}_3 of all polynomials of degree at most 3.

1. Use *Mathematica* and the fact that $[\]_{\Gamma_0}$ is an isomorphism (and isomorphisms map bases to bases) to verify that

$$\Delta := (1 + x + x^2 + x^3, 1 + 2x + 4x^2 + 8x^3, \\ 1 + 3x + 9x^2 + 27x^3, 1 + 4x + 16x^2 + 64x^3)$$

and

$$\Omega := (x + x^2, x - 6x^3, 1 + 4x^2, 1 + 8x^3)$$

are also coordinate systems of \mathbb{P}_3 .

2. Verify that general identity

$$[\mathcal{T}]_{\Delta \leftarrow \Omega} = [\mathcal{I}_{\mathbb{P}_3}]_{\Delta \leftarrow \Gamma_0} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\ = \left([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta} \right)^{-1} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega},$$

for $\mathcal{T} : \mathbb{P}_3 \xrightarrow{\text{linear}} \mathbb{P}_3$.

3. Consider $\mathcal{T} : \mathbb{P}_3 \rightarrow \mathbb{P}_3$ defined by

$$\mathcal{T}(p(x)) := xp'(2x) + p(x).$$

Prove that \mathcal{T} is a linear function and use part 2 to find $[\mathcal{T}]_{\Delta \leftarrow \Omega}$.

Solution. 4.

1. (a) Let

$$\Delta := (1 + x + x^2 + x^3, 1 + 2x + 4x^2 + 8x^3, \\ 1 + 3x + 9x^2 + 27x^3, 1 + 4x + 16x^2 + 64x^3)$$

be given. To show that Δ is a coordinate system of \mathbf{V} , it suffices to show there exists an isomorphism $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta} : \mathbb{C}_{\Delta_0}^4 \rightarrow \mathbb{C}_{\Gamma_0}^4$.

We can construct $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta}$ column-by-column by letting $[\]_{\Gamma_0}$ act on the elements of Δ , by Part 2 of Problem 3:

$$[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta} = \begin{bmatrix} [(1 + x + x^2 + x^3)]_{\Gamma_0} \\ [(1 + 2x + 4x^2 + 8x^3)]_{\Gamma_0} \\ [(1 + 3x + 9x^2 + 27x^3)]_{\Gamma_0} \\ [(1 + 4x + 16x^2 + 64x^3)]_{\Gamma_0} \end{bmatrix}^{\top} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}.$$

To show $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta}$ is an isomorphism, it suffices to check if its determinant is nonzero, which can be done in *Mathematica*:

$$\det([\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Delta}) = 12 \neq 0.$$

Therefore, Δ is a coordinate system of \mathbb{P}_3 .

Mathematica code:

```
In[4]:= Det[{{1, 1, 1, 1}, {1, 2, 3, 4}, {1, 4, 9, 16},
             {1, 8, 27, 64}}]
Out[4]= 12
```

- (b) For $\Omega := (x + x^2, x - 6x^3, 1 + 4x^2, 1 + 8x^3)$, we repeat the procedure laid out above to determine whether Ω is a coordinate system for \mathbb{P}_3 .

$$[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Omega} = \begin{bmatrix} [(x + x^2)]_{\Gamma_0} \\ [(x - 6x^3)]_{\Gamma_0} \\ [(1 + 4x^2)]_{\Gamma_0} \\ [(1 + 8x^3)]_{\Gamma_0} \end{bmatrix}^\top = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & -6 & 0 & 8 \end{pmatrix}.$$

We can calculate the determinant of $[\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Omega}$ in *Mathematica*:

$$\det([\mathcal{I}\mathcal{I}]_{\Gamma_0 \leftarrow \Omega}) = -32 \neq 0.$$

Therefore, Ω is a coordinate system of \mathbb{P}_3 .

Mathematica code:

```
In[6]:= Det[{{0, 0, 1, 1}, {1, 1, 0, 0},
             {1, 0, 4, 0}, {0, -6, 0, 8}}]
Out[6]= -32
```

2. Let $\mathcal{T} : \mathbb{P}_3 \xrightarrow{\text{linear}} \mathbb{P}_3$ be given.

- (a) First identity:

$$\begin{aligned} & [\mathcal{I}_{\mathbb{P}_3}]_{\Delta \leftarrow \Gamma_0} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\ &= ([\]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Gamma_0}) \circ ([\]_{\Gamma_0} \circ \mathcal{T} \circ \mathcal{A}_{\Gamma_0}) \circ ([\]_{\Gamma_0} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Omega}) \\ &= [\]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{T} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Omega} \\ &= [\]_{\Delta} \circ \mathcal{T} \circ \mathcal{A}_{\Omega} \\ &= [\mathcal{T}]_{\Delta \leftarrow \Omega}. \end{aligned}$$

- (b) The second identity follows from the fact that

$$\begin{aligned} ([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta})^{-1} &= ([\]_{\Gamma_0} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Delta})^{-1} \\ &= [\]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3}^{-1} \circ \mathcal{A}_{\Gamma_0} \\ &= [\]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Gamma_0}. \end{aligned}$$

So we have

$$\begin{aligned}
[\mathcal{T}]_{\Delta \leftarrow \Omega} &= [\mathcal{I}_{\mathbb{P}_3}]_{\Delta \leftarrow \Gamma_0} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\
&= ([\]_{\Delta} \circ \mathcal{I}_{\mathbb{P}_3} \circ \mathcal{A}_{\Gamma_0}) \circ [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} \\
&= ([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta})^{-1} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega}
\end{aligned}$$

3. Let $\mathcal{T} : \mathbb{P}_3 \rightarrow \mathbb{P}_3$ be given. T is defined as

$$\mathcal{T}(p(x)) := xp'(2x) + p(x).$$

(a) Claim: \mathcal{T} is a linear function.

\mathcal{T} is a linear function if it satisfies the linearity conditions. Consider $p(x), q(x) \in \mathbb{P}_3$. Then we have

$$\begin{aligned}
\mathcal{T}(p(x) + q(x)) &= x(p(2x) + q(2x))' + (p(x) + q(x)) \\
&= xp'(2x) + xq'(2x) + p(x) + q(x) \\
&= (xp'(2x) + p(x)) + (xq'(2x) + q(x)) \\
&= \mathcal{T}(p(x)) + \mathcal{T}(q(x)). \quad (\dagger)
\end{aligned}$$

Consider $c \in \mathbb{C}$ and $p(x) \in \mathbb{P}_3$,

$$\begin{aligned}
\mathcal{T}(cp(x)) &= x(cp(2x))' + cp(x) \\
&= c xp'(2x) + cp(x) \\
&= c(xp'(2x) + p(x)) \\
&= c\mathcal{T}(p(x)). \quad (\dagger\dagger)
\end{aligned}$$

From (\dagger) and $(\dagger\dagger)$, \mathcal{T} is a linear function.

(b) Find $[\mathcal{T}]_{\Delta \leftarrow \Omega}$.

By part 2 of Problem 3

$$[\mathcal{T}]_{\Delta \leftarrow \Omega} = \left([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta}\right)^{-1} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} [\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega}.$$

In part 1, we have found the isomorphisms:

$$\begin{aligned}
\left([\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Delta}\right)^{-1} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \\ 1 & 8 & 27 & 64 \end{pmatrix}^{-1} = \begin{pmatrix} 4 & -13/3 & 3/2 & -1/6 \\ -6 & 19/2 & -4 & 1/2 \\ 4 & -7 & 7/2 & -1/2 \\ -1 & 11/6 & -1 & 1/6 \end{pmatrix} \\
[\mathcal{I}_{\mathbb{P}_3}]_{\Gamma_0 \leftarrow \Omega} &= \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & -6 & 0 & 8 \end{pmatrix}.
\end{aligned}$$

To find $[\mathcal{T}]_{\Delta \leftarrow \Omega}$ we only need to find $[\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0}$. To do this, we use part 2 of Problem 3, which says,

$$[\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} = \begin{bmatrix} [\mathcal{T}(1)]_{\Gamma_0} & [\mathcal{T}(x)]_{\Gamma_0} & [\mathcal{T}(x^2)]_{\Gamma_0} & [\mathcal{T}(x^3)]_{\Gamma_0} \end{bmatrix},$$

where

$$\mathcal{T}(1) = x \frac{d}{dx} 2 + 1 = 1$$

$$\mathcal{T}(x) = x \frac{d}{dx} (2x) + x = 3x$$

$$\mathcal{T}(x^2) = x \frac{d}{dx} (2x)^2 + x^2 = 9x^2$$

$$\mathcal{T}(x^3) = x \frac{d}{dx} (2x)^3 + x^3 = 25x^3.$$

Therefore,

$$\begin{aligned} [\mathcal{T}]_{\Gamma_0 \leftarrow \Gamma_0} &= \begin{bmatrix} [T(1)]_{\Gamma_0} & [T(x)]_{\Gamma_0} & [T(x^2)]_{\Gamma_0} & [T(x^3)]_{\Gamma_0} \end{bmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix}. \end{aligned}$$

Finally, we can compute $[\mathcal{T}]_{\Delta \leftarrow \Omega}$:

$$\begin{aligned} [\mathcal{T}]_{\Delta \leftarrow \Omega} &= \begin{pmatrix} 4 & -13/3 & 3/2 & -1/6 \\ -6 & 19/2 & -4 & 1/2 \\ 4 & -7 & 7/2 & -1/2 \\ -1 & 11/6 & -1 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 25 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 \\ 0 & -6 & 0 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1/2 & 12 & 58 & -88/3 \\ -15/2 & -93/2 & -150 & 94 \\ 21/2 & 54 & 130 & -96 \\ -7/2 & -39/2 & -37 & 97/3 \end{pmatrix}. \end{aligned}$$

Mathematica code:

```
A =
Inverse[{{1, 1, 1, 1}, {1, 2, 3, 4}, {1, 4, 9, 16}, {1, 8, 27, 64}}]

{{4, -(13/3), 3/2, -(1/6)}, {-6, 19/2, -4, 1/2}, {4, -7, 7/
2, -(1/2)}, {-1, 11/6, -1, 1/6}}

B = {{1, 0, 0, 0}, {0, 3, 0, 0}, {0, 0, 9, 0}, {0, 0, 0, 25}}

CC = {{0, 0, 1, 1}, {1, 1, 0, 0}, {1, 0, 4, 0}, {0, -6, 0, 8}}

In[7]:= A.B.CC

Out[7]= {{1/2, 12, 58, -(88/3)}, {-(15/2), -(93/2), -150, 94},
{21/2, 54, 130, -96}, {-(7/2), -(39/2), -37, 97/3}}
```

Problem. 5. Suppose that $\mathbf{V} = \mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3$ and Γ_i is a coordinate system of \mathbf{W}_i . Suppose that for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i.$$

Let

$$\mathcal{L} := \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

Let Δ be the concatenation $\Gamma_1 || \Gamma_2 || \Gamma_3$ of the coordinate systems Γ_i . As we know, Δ is a coordinate system of \mathbf{V} .

Prove that $[\mathcal{L}]_{\Delta \leftarrow \Delta}$ equals the partitioned matrix

$$\begin{bmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} & [\mathcal{L}_{12}]_{\Gamma_1 \leftarrow \Gamma_2} & [\mathcal{L}_{13}]_{\Gamma_1 \leftarrow \Gamma_3} \\ [\mathcal{L}_{21}]_{\Gamma_2 \leftarrow \Gamma_1} & [\mathcal{L}_{22}]_{\Gamma_2 \leftarrow \Gamma_2} & [\mathcal{L}_{23}]_{\Gamma_2 \leftarrow \Gamma_3} \\ [\mathcal{L}_{31}]_{\Gamma_3 \leftarrow \Gamma_1} & [\mathcal{L}_{32}]_{\Gamma_3 \leftarrow \Gamma_2} & [\mathcal{L}_{33}]_{\Gamma_3 \leftarrow \Gamma_3} \end{bmatrix}.$$

Solution. 5. Consider a 3×3 block-matrix, which consists of just the linear function $\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i$ at the i^{th} row and j^{th} column and the zero function everywhere else. Let $[\mathcal{L}_{ij}]_{\oplus} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}$ denote this matrix. By this construction, we have

$$\mathcal{L} = \sum_{i=1}^3 \sum_{j=1}^3 [\mathcal{L}_{ij}]_{\oplus} = \begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}.$$

It follows that the matrix $[\mathcal{L}]_{\Delta \leftarrow \Delta}$ of \mathcal{L} can be written as

$$\begin{aligned} [\mathcal{L}]_{\Delta \leftarrow \Delta} &= \left[\sum_{i=1}^3 \sum_{j=1}^3 [\mathcal{L}_{ij}]_{\oplus} \right]_{\Delta \leftarrow \Delta} \\ &= \sum_{i=1}^3 \sum_{j=1}^3 [[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta} \end{aligned}$$

Consider the coordinate system Γ_k where $k \in \{1, 2, 3\}$. Let Γ_k be formed by a (finite) collection of basis elements, called g_{lk} 's. Consider the mapping $[\mathcal{L}_{ij}]_{\oplus}(g_{lk})_{\oplus}$ and the matrix $[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta}$.

We observe that $[\mathcal{L}_{ij}]_{\oplus}(g_{lk})_{\oplus}$ is $\mathbf{0}_{\mathbf{V}}$ if $k \neq j$ and $\mathbf{X}([\mathcal{L}_{ij}]_{\oplus}(g_{lj}) + \mathbf{0}_{\mathbf{V}} + \mathbf{0}_{\mathbf{V}})$ if $j = k$, where \mathbf{X} is defined in Definition 0.4. What this says is that $[\mathcal{L}_{ij}]_{\oplus}(g_{lk})_{\oplus}$ gives a column vector with $\mathcal{L}_{ij}(g_{lj})$ on the i^{th} row and $\mathbf{0}_{\mathbf{V}}$ everywhere else. This means that if we consider the coordinatization $[[\mathcal{L}_{ij}]_{\oplus}(g_{lj})_{\oplus}]_{\Delta}$, where $\Delta = \Gamma_1 || \Gamma_2 || \Gamma_3$, we will end up with a column vector with the zero function everywhere except $[[\mathcal{L}_{ij}]_{\oplus}(g_{lj})]_{\Gamma_i}$ on the i^{th} row. (\dagger)

Now, we can construct $[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta}$ column-by-column as

$$[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta} = \begin{bmatrix} [[[\mathcal{L}_{ij}]_{\oplus}(g_{11})]_{\oplus}]_{\Delta}^{\top} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{12})]_{\oplus}]_{\Delta}^{\top} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{13})]_{\oplus}]_{\Delta}^{\top} \\ \vdots & \vdots & \vdots \\ [[[\mathcal{L}_{ij}]_{\oplus}(g_{a1})]_{\oplus}]_{\Delta} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{b2})]_{\oplus}]_{\Delta} & [[[\mathcal{L}_{ij}]_{\oplus}(g_{c3})]_{\oplus}]_{\Delta} \end{bmatrix}^{\top}$$

where a, b, c are the number of elements in $\Gamma_1, \Gamma_2, \Gamma_3$, respectively. From (\dagger) , we know that two of three matrix-blocks in the expression above will be the zero-block because j has a fixed value of either 1, 2, or 3. Consider the only non-zero block (located at the j^{th} column)

$$\begin{bmatrix} [[[\mathcal{L}_{ij}]_{\oplus}(g_{1j})]_{\oplus}]_{\Delta}^{\top} \\ \vdots \\ [[[\mathcal{L}_{ij}]_{\oplus}(g_{aj})]_{\oplus}]_{\Delta}^{\top} \end{bmatrix} = [[[\mathcal{L}_{ij}]_{\oplus}(g_{1j})]_{\oplus}]_{\Delta} \quad \dots \quad [[[\mathcal{L}_{ij}]_{\oplus}(g_{aj})]_{\oplus}]_{\Delta}. \quad (\dagger\dagger)$$

Recalling that $[[[\mathcal{L}_{ij}]_{\oplus}(g_{lj})]_{\oplus}]_{\Delta}$ is a column vector with $\mathcal{L}_{ij}(g_{lj})$ on the i^{th} row and $\mathbf{0}_{\mathbf{V}}$ everywhere else, it follows that we can represent this matrix block as one with two rows of zeros and an i^{th} row of

$$[\mathcal{L}_{ij}(g_{1j}) \quad \mathcal{L}_{ij}(g_{1j}) \quad \dots \quad \mathcal{L}_{ij}(g_{aj})],$$

which precisely expresses the matrix $[\mathcal{L}_{ij}]_{\Gamma_i \leftarrow \Gamma_j}$. Therefore, we can treat the expression $(\dagger\dagger)$ as a 3-element column of the matrix $[[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta}$ whose element on the i^{th} row and j^{th} column is $[\mathcal{L}_{ij}]_{\Gamma_i \leftarrow \Gamma_j}$, and the zero function everywhere else. Hence,

$$\begin{aligned} [\mathcal{L}]_{\Delta \leftarrow \Delta} &= \sum_{i=1}^3 \sum_{j=1}^3 [[[\mathcal{L}_{ij}]_{\oplus}]_{\Delta \leftarrow \Delta} \\ &= \begin{bmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} & [\mathcal{L}_{21}]_{\Gamma_2 \leftarrow \Gamma_1} & [\mathcal{L}_{31}]_{\Gamma_3 \leftarrow \Gamma_1} \\ [\mathcal{L}_{12}]_{\Gamma_1 \leftarrow \Gamma_2} & [\mathcal{L}_{22}]_{\Gamma_2 \leftarrow \Gamma_2} & [\mathcal{L}_{32}]_{\Gamma_3 \leftarrow \Gamma_2} \\ [\mathcal{L}_{13}]_{\Gamma_1 \leftarrow \Gamma_3} & [\mathcal{L}_{23}]_{\Gamma_2 \leftarrow \Gamma_3} & [\mathcal{L}_{33}]_{\Gamma_3 \leftarrow \Gamma_3} \end{bmatrix}. \end{aligned}$$

This completes the argument.

An example can help justify this argument. Consider $[\mathcal{L}_{11}]_{\oplus}$,

$$[\mathcal{L}_{11}] = \begin{bmatrix} \mathcal{L}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\oplus},$$

where we are using the 0's to denote the zero functions $\mathbf{W}_i \rightarrow \mathbf{W}_j$. Then consider g_{11} the first vector in the coordinate system Γ_1 . We have

$$\begin{bmatrix} \mathcal{L}_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\oplus} \begin{pmatrix} g_{11} \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}} = \begin{pmatrix} \mathcal{L}_{11}(g_{11}) \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{X}}.$$

Now consider $[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta}$. This matrix can be constructed column-by-column with each column being $[\mathcal{L}_{11}(g_{ij})]_{\Delta \leftarrow \Delta}$, where j indicates from which Γ_j g_{ij} is taken. We observe that if j is not 1, then $\mathcal{L}_{11}(g_{ij}) = \mathbf{0}_{\mathbf{V}}$. This means the all columns of $[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta}$ are the zero columns unless $j = 1$. Consider the $j = 1$ columns. We know that the coordinatization

$$\left[\begin{pmatrix} \mathcal{L}_{11}(g_{11}) \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix}_{\mathfrak{K}} \right]_{\Delta \leftarrow \Delta} = \begin{pmatrix} [\mathcal{L}_{11}(g_{11})]_{\Gamma_1} \\ \mathbf{0}_{\mathbf{V}} \\ \mathbf{0}_{\mathbf{V}} \end{pmatrix},$$

because the $\mathcal{L}_{11}(g_{11}) \in \mathbf{W}_1$. So it follows that the concatenation of the columns

$$\begin{pmatrix} [\mathcal{L}_{11}(g_{11})]_{\Gamma_1} & \cdots & \mathcal{L}_{11}(g_{a1})_{\Gamma_1} \\ \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \cdots & \mathcal{O} \end{pmatrix}$$

in the matrix $[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta}$ is nothing but

$$\begin{pmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} \\ \mathcal{O} \\ \mathcal{O} \end{pmatrix}.$$

So we have

$$[[\mathcal{L}_{11}]_{\oplus}]_{\Delta \leftarrow \Delta} = \begin{bmatrix} [\mathcal{L}_{11}]_{\Gamma_1 \leftarrow \Gamma_1} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O} \end{bmatrix},$$

where \mathcal{O} denotes the zero function.

We can equivalently generate other examples for $[\mathcal{L}_{ij}]_{\oplus}$, $i, j \in \{1, 2, 3\}$, and in the end add up the results (with different combinations of i, j 's of course) to get $[\mathcal{L}]_{\Delta \leftarrow \Delta}$, expressed in the form given in the problem statement as desired.

25.4 Problem set 4

Problem. 1.

Suppose that $T \in \mathcal{L}(\mathbf{V})$ and

$$\operatorname{Im}(T) = \mathbf{W} + \mathbf{Z},$$

where \mathbf{W} and \mathbf{Z} are subspaces of \mathbf{V} . Argue that

$$\mathbf{V} = T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}].$$

Solution.

Since $\operatorname{Im}(T) = \mathbf{W} + \mathbf{Z}$, $T^{-1}[\mathbf{W} + \mathbf{Z}] = T^{-1}[\operatorname{Im}(T)] = \mathbf{V}$. We also have

$$\begin{aligned} T^{-1}[\mathbf{W}] &= \{w \in \mathbf{V} \mid T(w) \in \mathbf{W}\} \subseteq \mathbf{V} \\ T^{-1}[\mathbf{Z}] &= \{z \in \mathbf{V} \mid T(z) \in \mathbf{Z}\} \subseteq \mathbf{V}, \end{aligned}$$

which means

$$T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}] = \{w + z \in \mathbf{V} \mid T(w) \in \mathbf{W}, T(z) \in \mathbf{Z}\}.$$

It follows directly from this definition that

$$T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}] \subseteq \mathbf{V} = T^{-1}[\mathbf{W} + \mathbf{Z}] \quad (\dagger).$$

Consider $v \in T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}]$. v can be written as $w + z$ where $T(w) \in \mathbf{W}$ and $T(z) \in \mathbf{Z}$, which implies

$$T(v) = T(w + z) = T(w) + T(z) \in \mathbf{W} + \mathbf{Z},$$

where the second equality follows from the linearity of T . Hence,

$$v \in T^{-1}[\mathbf{W} + \mathbf{Z}]. \quad (\dagger\dagger)$$

From (\dagger) and $(\dagger\dagger)$, $\mathbf{V} = T^{-1}[\mathbf{W}] + T^{-1}[\mathbf{Z}]$ as desired.

Problem. 2.

1. Suppose that $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3$ are non-trivial vector spaces, and for each $i, j \in \{1, 2, 3\}$,

$$\mathcal{L}_{ij} : \mathbf{V}_j \xrightarrow{\text{linear}} \mathbf{V}_i.$$

Let \mathcal{L} be the block-matrix function:

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \xrightarrow{\text{linear}} \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3.$$

Suppose that it turns out that $\mathbf{V}_2 = \mathbf{V}_{2.1} \times \mathbf{V}_{2.2}$. Then \mathcal{L} maybe considered as a linear function on

$$\mathbf{V}_1 \times \mathbf{V}_{2.1} \times \mathbf{V}_{2.2} \times \mathbf{V}_3.$$

What is the corresponding block-matrix form of \mathcal{L} and how does it relate

to $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times}$? Justify your claims.

2. Suppose that $\mathbf{W}_1 \oplus \mathbf{W}_2 \oplus \mathbf{W}_3 \oplus \mathbf{V} = \mathbf{V}$ and the \mathbf{W}_i 's are non-trivial subspaces of \mathbf{V} . Suppose that for each $i, j \in \{1, 2, 3\}$:

$$\mathcal{L}_{ij} : \mathbf{W}_j \xrightarrow{\text{linear}} \mathbf{W}_i.$$

Let \mathcal{L} be the block-matrix function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus} : \mathbf{V} \xrightarrow{\text{linear}} \mathbf{V}.$$

Suppose it turns out that $\mathbf{W}_1 = \mathbf{W}_{1.1} \oplus \mathbf{W}_{1.2}$. Then

$$\mathbf{V} = \mathbf{W}_{1.1} \oplus \mathbf{W}_{1.2} \oplus \mathbf{W}_2 \oplus \mathbf{W}_3.$$

What is the block-matrix form of \mathcal{L} with respect to this direct sum de-

composition and how does it relate to $\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\oplus}$? Justify your claims.

Solution.

1. Recall from the previous problem set that the elements of the block matrix function

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} & \mathcal{L}_{13} \\ \mathcal{L}_{21} & \mathcal{L}_{22} & \mathcal{L}_{23} \\ \mathcal{L}_{31} & \mathcal{L}_{32} & \mathcal{L}_{33} \end{bmatrix}_{\times} : \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3 \xrightarrow{\text{linear}} \mathbf{V}_1 \times \mathbf{V}_2 \times \mathbf{V}_3.$$

can be expressed as

$$\mathcal{L}_{ij} = \Pi_i \circ \mathcal{L} \circ \gamma_j,$$

where the Π_i 's and γ_j 's are coordinate projections onto and injections into \mathbf{V} . Let $\mathbf{V}_2 = \mathbf{V}_{2.1} \times \mathbf{V}_{2.2}$. We claim that the matrix-block form of \mathcal{L} in this case is

$$\begin{bmatrix} \mathcal{L}_{11} & \mathcal{L}_{1|2.1} & \mathcal{L}_{1|2.2} & \mathcal{L}_{13} \\ \mathcal{L}_{2.1|1} & \mathcal{L}_{2.1|2.1} & \mathcal{L}_{2.1|2.2} & \mathcal{L}_{2.1|3} \\ \mathcal{L}_{2.2|1} & \mathcal{L}_{2.2|2.1} & \mathcal{L}_{2.2|2.2} & \mathcal{L}_{2.2|3} \\ \mathcal{L}_{31} & \mathcal{L}_{3|2.1} & \mathcal{L}_{3|2.2} & \mathcal{L}_{33} \end{bmatrix}_{\times}.$$

In particular, we claim the following relations between the matrix elements \mathcal{L}_{ij} in the former expression and the block-elements in the latter expression:

$$\mathcal{L}_{12} = [\mathcal{L}_{1|2.1} \quad \mathcal{L}_{1|2.2}], \mathcal{L}_{32} = [\mathcal{L}_{3|2.1} \quad \mathcal{L}_{3|2.2}],$$

$$\mathcal{L}_{21} = \begin{bmatrix} \mathcal{L}_{2.1|1} \\ \mathcal{L}_{2.2|1} \end{bmatrix}, \mathcal{L}_{23} = \begin{bmatrix} \mathcal{L}_{2.1|3} \\ \mathcal{L}_{2.2|3} \end{bmatrix}, \mathcal{L}_{22} = \begin{bmatrix} \mathcal{L}_{2.1|2.1} & \mathcal{L}_{2.1|2.2} \\ \mathcal{L}_{2.2|2.1} & \mathcal{L}_{2.2|2.2} \end{bmatrix}.$$

- 2.

Problem. 3.

Suppose that relatively prime polynomials p_1 and p_2 have degrees 6 and 11 respectively. Consider the function:

$$\psi : \mathbb{P}_{10} \times \mathbb{P}_5 \rightarrow \mathbb{P}_{16}$$

defined by

$$\psi \begin{pmatrix} f \\ g \end{pmatrix} := f \cdot p_1 - g \cdot p_2.$$

Verify each of the following claims.

1. ψ is a linear function.
2. ψ is injective.
3. ψ is surjective.
4. There exist polynomials q_1 and q_2 such that

$$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1},$$

where $\mathbb{1}$ is the constantly 1 polynomial.

Solution.

1. (a) Consider

$$\begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} h \\ k \end{pmatrix} \in \mathbb{P}_{10} \times \mathbb{P}_5.$$

Then

$$\begin{aligned} \psi \begin{pmatrix} f \\ g \end{pmatrix} + \psi \begin{pmatrix} h \\ k \end{pmatrix} &= (f \cdot p_1 - g \cdot p_2) + (h \cdot p_1 - k \cdot p_1) \\ &= (f + h) \cdot p_1 + (g + h) \cdot p_2 \\ &= \psi \begin{pmatrix} f + h \\ g + k \end{pmatrix} \\ &= \psi \left(\begin{pmatrix} f \\ g \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} \right) \quad (\dagger) \end{aligned}$$

- (b) Consider $c \in \mathbb{C}$,

$$\psi \begin{pmatrix} cf \\ cg \end{pmatrix} = cf \cdot p_1 - cg \cdot p_2 = c(f \cdot p_1 - g \cdot p_2) = c\psi \begin{pmatrix} f \\ g \end{pmatrix}. \quad (\dagger\dagger)$$

From (\dagger) and $(\dagger\dagger)$, ψ is a linear function.

2. Assume (to get a contradiction) that $\begin{pmatrix} f \\ g \end{pmatrix}$ is a non-zero element in $\ker(\psi)$.

Then

$$f \cdot p_1 - g \cdot p_2 = 0.$$

Since p_1, p_2 are both non-zero (they are relatively prime), if either one of f or g were zero, it would force the other polynomial to be zero so that $f \cdot p_1 - g \cdot p_2 = 0$, contradicting the requirement that $\begin{pmatrix} f \\ g \end{pmatrix}$ is non-zero element. Therefore, neither f nor g can be zero.

Since p_1 and p_2 are relatively prime (i.e., they have no common roots), the roots of p_2 must also be roots of f . Furthermore, the roots of p_2 must also be roots of f with equal or greater multiplicities, because otherwise there will at least one factor $(x - \lambda_k)$ in the factorization of p_2 that has a higher power (or multiplicity) than that in f (and not in p_1), resulting in a non-zero $f \cdot p_1 - g \cdot p_2$.

This implies that the degree of f has to be at least $\deg(p_2) = 11$. But since $\deg(f) \leq 10$, we have a contradiction. By symmetry, we will also have a contradiction if we repeat this argument for g and p_1 .

Hence, there is no such non-zero $\begin{pmatrix} f \\ g \end{pmatrix}$ in the kernel of ψ , i.e., ψ is injective.

3. Since both $\mathbb{P}_{10} \times \mathbb{P}_5$ and \mathbb{P}_{16} are finite-dimensional vector spaces, we can use the Rank-Nullity Theorem:

$$\dim(\ker(\psi)) + \dim(\text{Im}(\psi)) = \dim(\mathbb{P}_{10} \times \mathbb{P}_5).$$

From part 1., we know $\dim(\ker(\psi)) = 0$, so

$$\dim(\text{Im}(\psi)) = \dim(\mathbb{P}_{10} \times \mathbb{P}_5) = \dim(\mathbb{P}_{10}) + \dim(\mathbb{P}_5) = 11 + 6 = 17 = \dim(\mathbb{P}_{16}).$$

Furthermore, $\text{Im}(\psi) \prec \mathbb{P}_{16}$. Hence, $\text{Im}(\psi) = \mathbb{P}_{16}$. So, ψ is surjective.

4. Since ψ is surjective, there exist polynomials $f \in \mathbb{P}_{10}$ and $g \in \mathbb{P}_5$ such that

$$\psi \begin{pmatrix} f \\ g \end{pmatrix} = 1 \in \mathbb{P}_{16} = \text{Im}(\psi),$$

i.e., the given f and g satisfy

$$f \cdot p_1 - g \cdot p_2 = 1.$$

Let $q_1 = f \in \mathbb{P}_{10}$ and $q_2 = -g \in \mathbb{P}_5$, then

$$q_1 \cdot p_1 + q_2 \cdot p_2 = 1,$$

verifying the existence of q_1 and q_2 .

Problem. 4.

1. What would you do to prove the last claim of problem 2 in a general case of non-zero relatively prime polynomials p_1 and p_2 ? You do not need to carry out the proof, but you DO need to set it all up along the lines of Problem 2. Make sure you cover all o the cases!
2. Prove that the following claims are equivalent.
 - (a) Non-zero polynomials p_1 and p_2 are relatively prime.
 - (b) There exist polynomials q_1 and q_2 such that

$$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}.$$

3. Argue that for any non-zero polynomials f and g there exist polynomials q_1 and q_2 such that

$$q_1 \cdot f + q_2 \cdot g = \gcd(f, g).$$

4. Argue that the following claims are equivalent for any non-zero polynomials f , g , and h .
 - (a) There exist polynomials q_1 and q_2 such that

$$q_1 \cdot f + q_2 \cdot g = h.$$

- (b) h is a polynomial multiple of $\gcd(f, g)$.

Solution.

1. (a) Case 1: Both p_1 and p_2 are zero-degree polynomials.

In this case, p_1 and p_2 are two constant, non-zero polynomials. Let $p_2(z) = C \in \mathbb{C}$ for any $z \in \mathbb{C}$, $C \neq 0$. If we let $q_1(z)$ be the constant 0 and $q_2(z)$ be the constant $1/C$ for any $z \in \mathbb{C}$ then

$$q_1(z)p_1(z) + q_2(z)p_2(z) = 0 + \frac{1}{C}C = 0 + 1 = 1,$$

for any $z \in \mathbb{C}$, i.e., $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$. This shows there are polynomials q_1 and q_2 such that $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$ is satisfied.

- (b) Case 2: Exactly one of p_1 and p_2 is a zero-degree polynomial.

Assume that p_1 is the non-zero zero-degree polynomial of the two, i.e., it is a non-zero constant polynomial. Let $p_1(z) = C \in \mathbb{C}$, $C \neq 0$,

for any $z \in \mathbb{C}$. If we let $q_1(z)$ be the constant $1/C$ and $q_2(z)$ be the constant 0 for any $z \in \mathbb{C}$ then

$$q_1(z)p_1(z) + q_2(z)p_2(z) = \frac{1}{C}C + 0 = 1 + 0 = 1,$$

for any $z \in \mathbb{C}$, i.e., $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$. This shows there are polynomials q_1 and q_2 such that $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$ is satisfied.

(c) Case 3: Both p_1 and p_2 have degree greater than 0.

To show the existence of q_1 and q_2 such that $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$ we can construct a similar proof to that in Problem 3. Let the polynomials p_1 and p_2 be given with degrees $m, n > 0$ respectively. Consider the function:

$$\psi : \mathbb{P}_{n-1} \times \mathbb{P}_{m-1} \rightarrow \mathbb{P}_{m+n-1}, \quad (1)$$

defined by

$$\psi \begin{pmatrix} f \\ g \end{pmatrix} := f \cdot p_1 + g \cdot p_2. \quad (2)$$

We can then show that ψ is a linear function. Next, we show that it is injective, which implies surjectivity by the Rank-Nullity Theorem. By the surjectivity of ψ , we know that given $\mathbb{1} \in \text{Im}(\psi)$, there must exist a pair $f, -g \in \mathbb{P}_{n-1} \times \mathbb{P}_{m-1}$ such that $\psi \begin{pmatrix} f \\ -g \end{pmatrix} = \mathbb{1}$.

From parts (a), (b), and (c), we can conclude that there always exist polynomials q_1, q_2 such that for non-zero relatively prime polynomials p_1, p_2 , $q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1}$.

2.
 - $[a. \implies b.]$ This implication is true by Part 1. of this problem.
 - $[b. \implies a.]$ Let polynomials q_1 and q_2 be given such that

$$q_1 \cdot p_1 + q_2 \cdot p_2 = \mathbb{1},$$

where p_1 and p_2 are also polynomials. Claim: Non-zero p_1 and p_2 are relatively prime.

Assume that p_1 and p_2 are non-zero. From TYC 2.4, we can write p_1 and p_2 as

$$\begin{aligned} p_1 &= \hat{p}_1 \cdot \gcd(p_1, p_2) \\ p_2 &= \hat{p}_2 \cdot \gcd(p_1, p_2), \end{aligned}$$

where \hat{p}_1 and \hat{p}_2 are non-zero relatively prime polynomials. It follows that

$$q_1 \cdot \hat{p}_1 \cdot \gcd(p_1, p_2) + q_2 \cdot \hat{p}_2 \cdot \gcd(p_1, p_2) = 1,$$

i.e.,

$$(q_1 \cdot \hat{p}_1 + q_2 \cdot \hat{p}_2) \cdot \gcd(p_1, p_2) = 1,$$

which implies

$$\deg(q_1 \cdot \hat{p}_1 + q_2 \cdot \hat{p}_2) = \deg(\gcd(p_1, p_2)) = 0.$$

Therefore, $\gcd(p_1, p_2)$ has to be a constant polynomial. Moreover, since $\gcd(p_1, p_2)$ is monic by definition, it has to be the constantly 1 polynomial. Hence, p_1 and p_2 are relatively prime by definition.

3. Let non-zero polynomials f, g be given. Then we have

$$\begin{aligned} f &= \hat{f} \cdot \gcd(f, g) \\ g &= \hat{g} \cdot \gcd(f, g), \end{aligned}$$

where \hat{f}, \hat{g} are relatively prime. It follows that

$$\begin{aligned} q_1 \cdot f + q_2 \cdot g &= q_1 \cdot \hat{f} \cdot \gcd(f, g) + q_2 \cdot \hat{g} \cdot \gcd(f, g) \\ &= (q_1 \cdot \hat{f} + q_2 \cdot \hat{g}) \cdot \gcd(f, g). \end{aligned}$$

From Part 2., we know that for relatively prime polynomials \hat{f} and \hat{g} , there exist polynomials q_1, q_2 such that

$$q_1 \cdot \hat{f} + q_2 \cdot \hat{g} = 1.$$

This means such q_1 and q_2 will give

$$(q_1 \cdot \hat{f} + q_2 \cdot \hat{g}) \cdot \gcd(f, g) = 1 \cdot \gcd(f, g) = \gcd(f, g).$$

This completes the argument.

4. • [$a. \implies b.$] Let non-zero polynomials f, g, h, q_1, q_2 be given such that

$$q_1 \cdot f + q_2 \cdot g = h.$$

From Part 3., we know that

$$\begin{aligned} h &= q_1 \cdot f + q_2 \cdot g = q_1 \cdot \hat{f} \cdot \gcd(f, g) + q_2 \cdot \hat{g} \cdot \gcd(f, g) \\ &= (q_1 \cdot \hat{f} + q_2 \cdot \hat{g}) \cdot \gcd(f, g), \end{aligned}$$

for relatively prime \hat{f} and \hat{g} . This implies h is a polynomial multiple of $\gcd(p_1, p_2)$.

- [b. \implies a.] Let h be written as $k \cdot \gcd(f, g)$ where k is a non-zero polynomial. From Part 3., there exist polynomials s_1, s_2 such that

$$s_1 \cdot f + s_2 \cdot p_2 = \gcd(f, g).$$

Multiplying both sides by the polynomial k , we have

$$k \cdot s_1 \cdot f + k \cdot s_2 \cdot g = k \cdot \gcd(f, g) = h.$$

Hence, there exist polynomials q_1 and q_2 such that

$$q_1 \cdot f + q_2 \cdot g = h,$$

namely,

$$\begin{aligned} q_1 &= k \cdot s_1 \\ q_2 &= k \cdot s_2. \end{aligned}$$

Problem. 5.

Suppose that \mathbf{W} is a non- $\{0\}$ ideal in \mathbb{P} . Then \mathbf{W} contains some monic polynomials (why?) and among these there must be some of the smallest degree, say n_0 . Let p_0 be one such. (It is entirely possible that $p_0 = 1$.)

1. Suppose that p is a non-zero polynomial in \mathbf{W} , and using the Division Algorithm for Polynomials (see Axler p.121) we write

$$p = q \cdot p_0 + r,$$

where $q, r \in \mathbb{P}$ and $\deg(r) < \deg(p_0)$. Argue that $r \in \mathbf{W}$.

2. Use the result of part 1 to argue that every polynomial p in \mathbf{W} is a polynomial multiple of p_0 .
3. Argue that p_0 is the only monic polynomial of the smallest degree n_0 in \mathbf{W} , and that

$$\mathbf{W} = \{q \cdot p_0 \mid q \in \mathbb{P}\}.$$

This polynomial p_0 is said to be **the generator** of the ideal \mathbf{W} .

Solution.

1. Let $p \in \mathbf{W}$ a non-zero polynomial be given. Let $p_0 \in \mathbf{W}$ be the monic polynomial with the smallest degree. \mathbf{W} is an ideal in \mathbb{P} , so for $q \in \mathbb{P}$, $q \cdot p_0 \in \mathbf{W}$. \mathbf{W} is also subspace, so

$$r = p - q \cdot p_0 \in \mathbf{W}.$$

2. Let the above q, p_0, p, r be given again. We know that

$$p = q \cdot p_0 + r,$$

where $\deg(r) < \deg(p_0)$. Since $\deg(p_0)$ minimal and p is non-zero, r has to be the constantly zero polynomial. Therefore,

$$p = q \cdot p_0 \in \mathbf{W},$$

i.e., every $p \in \mathbf{W}$ is a polynomial multiple of p_0 .

3. Let some p'_0 be a monic polynomial of the smallest degree n_0 in \mathbf{W} be given. Since $p'_0 \in \mathbf{W}$, p'_0 is a polynomial multiple of p_0 , by the previous part. Let us write $p'_0 = q \cdot p_0$ for some $q \in \mathbb{P}$. Since $\deg(p'_0) = \deg(p_0) = n_0$, $\deg(q) = 1$, i.e., q is a constant polynomial. Furthermore, since both p_0 and p'_0 are monic, q has to be 1. Hence, $p'_0 = p_0$, i.e., p_0 is unique.

It follows from the previous part that \mathbf{W} is a collection of elements that are polynomial multiples of a unique monic polynomial of the smallest degree, called p_0 , i.e.,

$$\mathbf{W} = \{q \cdot p_0 \mid q \in \mathbb{P}\}.$$