Practice Final: 8.310, 18.435, 2022

1. **The Bloch Sphere**

   (a) Each point on the Bloch sphere is associated with a quantum state. Explain how you would find the quantum state associated with the point $\frac{1}{\sqrt{3}}(1, 1, 1)$. You don't have to do the computations, especially if you don't have a calculator. Just explain the steps. (Although points will not be deducted if you give the state.)

   (b) What is the unitary that you apply to a state $|\psi\rangle$ on the Bloch sphere that rotates it 180° around the $x$-axis? That rotates it 90° counterclockwise around the $z$-axis?

2. Suppose a state $|\psi\rangle$ is encoded in the afternoon 7-qubit code. Then the unitary matrix

$$\begin{pmatrix} .8e^{i\theta} & .6i \\ .6i & .8e^{-i\theta} \end{pmatrix}$$

   is applied to both qubit 1 and qubit 2 of the code, after which the error correction procedure is applied. What is the probability that there is an error in the corrected encoded qubit? Assuming that there is an error, what are the probabilities for the resulting state of the encoded qubit, after the error is corrected.

3. **Grover's Algorithm**
   Recall that Grover's algorithm works with an oracle $O$ that takes $|x\rangle$ to $-|x\rangle$ if $x$ is a "marked state" and does not change the sign of $|x\rangle$ if $x$ is not marked.

   (a) Grover's algorithm works by repeatedly applying a *Grover iteration*. What is the Grover iteration?

   (b) Suppose that there are $M$ marked items out of $N$ items total, where $M$ is much smaller than $N$. How many times should we apply the Grover iteration to obtain a large probability of measuring a marked item?

   (c) Suppose we have $N$ items total, but we don't know how many marked items there are. How should we proceed to find a marked item with large probability?

4. Consider the operator

$$T = \sigma_x \otimes \sigma_y + \sigma_y \otimes \sigma_x + \sigma_z \otimes \sigma_z \, .$$

   (a) Show that $T^2 = 2T + 3I$, where $I$ is the $4 \times 4$ identity matrix.

   (b) Explicitly compute $T\,|00\rangle$ and $T^2\,|00\rangle$ and check that they satisfy the equation in part (a)

5. Both classical and quantum error correcting codes can be used to detect errors, as well as correct them. For a CSS code, if the minimum weight of a codeword in $C_1$ is $d_1$, then the code can detect $d_1 - 1$ bit errors, and if the minimum weight of a codeword in $C_2^\perp$ is $d_2$, then the code can detect $d_2 - 1$ phase errors. Explain in more detail how this works, and sketch the procedure you would use to detect the errors.

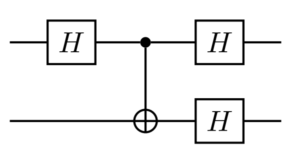
6. Let the gate $P_\theta$ be
$$P_\theta = \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}.$$
   (a) Suppose we initialize a qubit in the state $|0\rangle$ and apply the unitary operations $H$, followed by $P_\theta$, followed by $H$, to it, where $H$ is the Hadamard gate. We then measure it in the $\{|0\rangle, |1\rangle\}$ basis. What are the probabilities of each outcome?

   (b) Suppose we initialize a qubit in the state $|+\rangle$ and apply the same unitary operations as above: $H$, followed by $P_\theta$, followed by $H$, to it. We then measure it in the $\{|+\rangle, |-\rangle\}$ basis. What are the probabilities of each outcome?

7. (a) Put the state
$$|\psi\rangle = \frac{1}{\sqrt{6}}\left(2\,|00\rangle + |01\rangle + |10\rangle\right)$$
   in the $|\pm\rangle$ basis.

   (b) Suppose this state is measured with the observable
$$J_x = \frac{1}{2}(\sigma_x \otimes \mathrm{id} + \mathrm{id} \otimes \sigma_x)$$
   What is the expectation of the result?

8. The quantum circuit




   produces $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ when the state $|\psi\rangle$ is input. What is $|\psi\rangle$?

9. Consider the quantum state
$$|\psi\rangle = \frac{1}{2}(|001\rangle + |011\rangle + |100\rangle - |111\rangle)$$
   where the qubits are labeled $A$, $B$, $C$, in that order. What is $\mathrm{Tr}_B\, |\psi\rangle\langle\psi|$?