spin by a potential $V(\vec{x}) = -\langle\vec{\mu}\rangle\cdot\vec{B}(\vec{x})$
where $\langle\vec{\mu}\rangle$ is the magnetic moment.

So we conclude that

$$\langle\vec{\mu}\rangle = \frac{e}{m}\left[\underset{\overset{\shortparallel}{1}}{F_1(0)} + F_2(0)\right]\xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi$$

It is conventional to define

$$\langle\vec{\mu}\rangle = g\left(\frac{e}{2m}\right)\underset{\text{spin}}{\vec{S}}$$

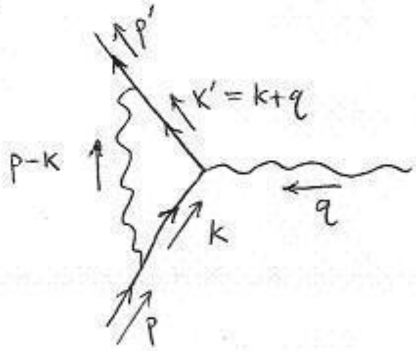
where g is called the Landé g-factor.

We find that $g = 2 + \mathcal{O}(\alpha)$. The 2 was what Dirac was able to explain from his Dirac equation, and made people notice what he was doing seemed correct... hard to get $g = 2$ from a classical picture of rotating charged object. The $\mathcal{O}(\alpha)$ is what Feynman, Schwinger, and Tomonaga was able to explain from QED, and made people notice that QED seemed correct despite the awful infinities.

We note that for an intrinsically composite "bound-state" particle like the proton, g is not close to 2 (not <u>very</u> close like for the electron). This is because the magnetic field "sees" the spins + masses of the constituent particles (quarks + gluons).

One-loop magnetic moment

We calculate the one-loop correction to the magnetic moment



$$\text{Let} \quad \Gamma^\mu = \underset{\text{(lowest order)}}{\gamma^\mu} + \delta\Gamma^\mu$$

Then $\bar{u}(p')\,\delta\Gamma^\mu(p',p)\,u(p)$

$$= \int \frac{d^4k}{(2\pi)^4} \frac{-i g_{\nu\beta}}{(k-p)^2+i\varepsilon}\, \bar{u}(p')(-ie\gamma^\nu)\frac{i(\not{k}'+m)}{k'^2-m^2+i\varepsilon}\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\varepsilon}(-ie\gamma^\beta)u(p)$$

Using $\gamma^\nu\gamma^\mu\gamma_\nu = -2\gamma^\mu$, $\gamma^\nu\gamma^\alpha\gamma^\beta\gamma_\nu = 4g^{\alpha\beta}$,
and $\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma_\nu = -2\gamma^\gamma\gamma^\beta\gamma^\alpha$,

we have

$$2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p)}{((k-p)^2+i\varepsilon)(k'^2-m^2+i\varepsilon)(k^2-m^2+i\varepsilon)}$$

This integral looks hard. We need some integration tricks.

Identity: (Feynman's trick for combining denominators)

$$\frac{1}{A \cdot B} = \int_0^1 dx \, \frac{1}{[xA + (1-x)B]^2}$$

check: For $A \neq B$, integral gives $-\frac{1}{(xA+(1-x)B)(A-B)}\Big|_0^1$

$$= -\frac{1}{A-B}\left(\frac{1}{A} - \frac{1}{B}\right) = \frac{1}{A \cdot B}$$

For $A = B$, integral gives $\int_0^1 dx \, \frac{1}{A^2} = \frac{1}{A^2} = \frac{1}{A \cdot B}$

<u>Generalizations:</u>

$$\frac{1}{A_1 A_2 \cdots A_n} = \int_0^1 dx_1 \cdots dx_n \, \frac{\delta(\sum_i x_i - 1) \cdot (n-1)!}{[x_1 A_1 + x_2 A_2 + \cdots x_n A_n]^n}$$

By differentiating this repeatedly we get

$$\frac{1}{A_1^{m_1} A_2^{m_2} \cdots A_n^{m_n}} = \int_0^1 dx_1 \cdots dx_n \, \frac{\delta(\sum_i x_i - 1) \cdot \prod_i x_i^{m_i - 1}}{[\sum_i x_i A_i]^{\sum_i m_i}} \frac{\Gamma(m_1 + \cdots + m_n)}{\Gamma(m_1)\Gamma(m_2) \cdots \Gamma(m_n)}$$

This is true even for non-integer m_i's

In our case we have

$$\frac{1}{((k-p)^2+i\varepsilon)(k'^2-m^2+i\varepsilon)(k^2-m^2+i\varepsilon)} = \int_0^1 \frac{dx\,dy\,dz\ \delta(x+y+z-1)\cdot \overset{2!=2}{\overset{\shortparallel}{\Gamma(3)}}}{D^3}$$

where $D = x(k^2-m^2) + y((\underbrace{k+q}_{k'})^2 - m^2) + z(k-p)^2 + i\varepsilon$

$$= k^2 + 2k\cdot(yq - zp) + yq^2 + zp^2 - (x+y)m^2 + i\varepsilon$$

We now complete the square...

$$D = \ell^2 - \Delta + i\varepsilon$$

where $\ell = k + yq - zp$ and $\Delta = +y^2q^2 - yq^2 + \underbrace{z^2p^2}_{z^2m^2} - \underbrace{zp^2}_{-zm^2}$
$-2yz\, q\cdot p + (x+y)m^2$

Since $p'^2 = (q+p)^2 = q^2+p^2+2q\cdot p$, $m^2 = q^2+m^2+2q\cdot p$

and so $2q\cdot p = -q^2$.

We can write $\Delta = (y^2-y+yz)q^2 + (\overbrace{z^2-z+x+y}^{z^2-2z+1})m^2$

$$= -xy\,q^2 + (1-z)^2 m^2$$

Since $q^2 = (p'-p)^2 = m^2+m^2-2p'\cdot p$

$$= 2m^2 - 2E'E + 2\vec{p}\,'\cdot\vec{p}$$

$$\le 2m^2 - 2\sqrt{\vec{p}\,'^2+m^2}\sqrt{\vec{p}^{\,2}+m^2} + 2|\vec{p}\,'||\vec{p}|$$

$$\le 0$$

proof: let $v_1 = \begin{pmatrix} m \\ |\vec{p}'| \end{pmatrix}$ $v_2 = \begin{pmatrix} m \\ |\vec{p}| \end{pmatrix}$

then $v_1 \cdot v_2 = m^2 + |\vec{p}'||\vec{p}|$, and $|v_1||v_2| = \sqrt{m^2+\vec{p}'^2}\sqrt{m^2+\vec{p}^2}$

Therefore $q^2 \leq 0$

Since $q^2 \leq 0$, $\Delta = -xyq^2 + (1-z)^2 m^2 \geq 0$

Let us work on the numerator of the integral.

Since D is even in ℓ, and only a function of ℓ^2,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu}{D^3} = 0$$

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = C \int \frac{d^4\ell}{(2\pi)^4} \frac{g^{\mu\nu}\ell^2}{D^3}$$

Contracting μ and ν, we find a factor of $\frac{1}{4}$,

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{\ell^\mu \ell^\nu}{D^3} = \frac{1}{4} \int \frac{d^4\ell}{(2\pi)^4} \frac{g^{\mu\nu}\ell^2}{D^3}$$

Our numerator was

$$\bar{u}(p')\left[\not{k}\gamma^\mu \not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu\right]u(p)$$

$\left[\begin{matrix}\text{in terms of } \ell^\mu, \\ \ell = k + yq - zp \\ k' = k+q\end{matrix}\right]$

$$= \bar{u}(p')\Big[\underbrace{(\not{\ell} - y\not{q} + z\not{p})\gamma^\mu(\not{\ell} + (1-y)\not{q} + z\not{p})} + m^2\gamma^\mu \underbrace{- 2m(k+k')^\mu}\Big]u(p)$$

$$= \underbrace{2\not{\ell}\ell^\mu}_{\substack{\text{upon integration} \\ \text{gives } \frac{1}{2}\ell^2\gamma^\mu}} - \ell^2\gamma^\mu + (-y\not{q} + z\not{p})\gamma^\mu((1-y)\not{q} + z\not{p})$$

$$= -2m(\underbrace{2\ell^\mu}_{\substack{\text{vanishes} \\ \text{upon integration}}} + (1-2y)q^\mu + 2zp^\mu)$$

We try to get the form

$$\gamma^\mu A + (p'^\mu + p^\mu) B + q^\mu C$$

Take the term...

$$\bar u(p') \left[(-y \underset{\substack{\uparrow \\ \not p' - \not p}}{\not q} + z\not p)\, \gamma^\mu \,((1-y) \underset{\substack{\uparrow \\ \not p' - \not p}}{\not q} + z\not p) \right] u(p)$$

$$= \bar u(p') \left[(-y \underset{\substack{\uparrow \\ \text{replace} \\ \text{with } m}}{\not p'} + (y+z)\not p)\, \gamma^\mu \,((1-y)\not p' + (\overbrace{y+z-1}^{-x}) \underset{\substack{\uparrow \\ \text{replace with } m}}{\not p}) \right] u(p)$$

anticommute

$$= \bar u(p') \left[\begin{array}{ll} ym\,\gamma^\mu\, xm & + \; (-y) m\, \gamma^\mu (1-y)\not p' \\ -(y+z)\not p\, \gamma^\mu xm & + \; (y+z)\not p\, \gamma^\mu (1-y)\not p' \end{array} \right] u(p)$$

$$= \bar u(p') \left[\begin{array}{l} + yx m^2 \gamma^\mu \; + \; (+y) m \underset{= m}{\not p'} \gamma^\mu (1-y) \; + \; (-y)(1-y) m \cdot 2p'^\mu \\ (y+z)\gamma^\mu \not p\, xm \; - \; 2(y+z) x m p^\mu \; + \; 2(y+z)\, 2p^\mu (1-y) \underset{= m}{\not p'} \\ \qquad - \; (y+z)\gamma^\mu \underbrace{\not p (1-y) \not p'}_{\text{anticommute}} \end{array} \right] u(p)$$

a little more work gives

$$\text{Numerator} = \bar u(p') \Big[\gamma^\mu \cdot \left(-\tfrac{1}{2} l^2 + (1-x)(1-y) q^2 + (1-2z-z^2) m^2 \right) + (p'^\mu + p^\mu) \cdot m z (z-1) + q^\mu \cdot m (z-2)(x-y) \Big] u(p)$$

Notice that the $q^\mu m(z-2)(x-y)$ term vanishes since the integral is symmetric under $x \leftrightarrow y$ exchange

We can use the Gordon identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p)$$

to put the $p'^\mu + p^\mu$ term in terms of γ^μ and $i\sigma^{\mu\nu}q_\nu$

So we now have

$$\bar{u}(p')\,\delta\Gamma^\mu(p',p)\,u(p)$$
$$= 2ie^2 \int \frac{d^4l}{(2\pi)^4} \int_0^1 \frac{dx\,dy\,dz\ \delta(x+y+z-1)}{D^3} \times 2$$
$$\cdot\ \bar{u}(p')\Big[\gamma^\mu\big(-\tfrac{1}{2}l^2 + (1-x)(1-y)q^2\big) + (1-4z+z^2)m^2$$
$$+ \frac{i\sigma^{\mu\nu}q_\nu}{2m}\big(2m^2 z(1-z)\big)\Big]u(p)$$

$$\bar{u}(p')\,\delta\Gamma^{\mu}(p',p)\,u(p)$$

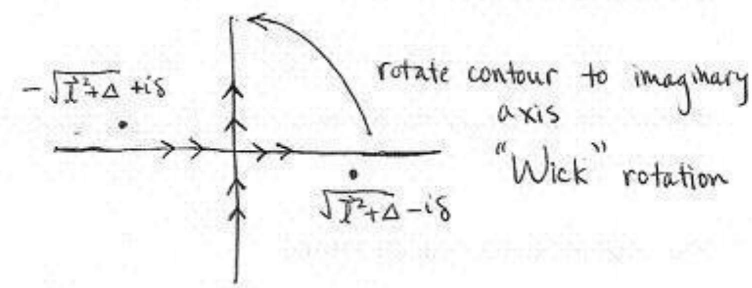
$$= 2ie^2 \int \frac{d^4 l}{(2\pi)^4} \int_0^1 dx\,dy\,dz \;\frac{\delta(x+y+z-1)\cdot 2\cdot N}{D^3}$$

numerator

$D = l^2 - \Delta + i\varepsilon$

where we showed $\Delta > 0$

The l^0 integral has poles at $\pm\left(\sqrt{\vec{l}^{\,2}+\Delta} - i\delta\right)$



Let $l^0 = i\,l_E^0\,,\quad l^{1,2,3} = l_E^{1,2,3}$ E stands for Euclidean

We integrate l_E^0 from $-\infty$ to ∞. Then

$$\int \frac{d^4 l}{(2\pi)^4}\,\frac{1}{\left[l^2-\Delta+i\varepsilon\right]^m} = \int \frac{i\,dl_E^0\, d^3\vec{l}_E}{(2\pi)^4}\,\frac{1}{\left[-l_E^2-\Delta\right]^m}$$

$l_E^2 = (l_E^0)^2+(l_E^1)^2+(l_E^2)^2+(l_E^3)^2$

$$= \frac{i(-1)^m}{(2\pi)^4}\underbrace{\int d\Omega_4}_{\text{4-D spherical integration}}\int_0^\infty dl_E\,\frac{l_E^3}{\left[l_E^2+\Delta\right]^m}$$

What is $\int d\Omega_4 = ?$

We know $\int d\theta = 2\pi = \int d\Omega_2$

$$\int d\phi \int_{-1}^{1} d\cos\theta = 4\pi = \int d\Omega_3$$

Consider the Gaussian integral

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dl_E^1 \cdots dl_E^4 \, e^{-(l_E)^2} = (\sqrt{\pi})^4 = \pi^2$$

$$(l_E)^2 = (l_E^1)^2 + (l_E^2)^2 + (l_E^3)^2 + (l_E^4)^2$$

$$= \int d\Omega_4 \int_0^\infty dl_E \, l_E^3 \, e^{-l_E^2} = \left[-\tfrac{1}{2} l_E^2 e^{-l_E^2} - \tfrac{1}{2} e^{-l_E^2}\right]_0^\infty \cdot \int d\Omega_4$$

$$= \tfrac{1}{2} \int d\Omega_4 .$$

So $\int d\Omega_4 = 2\pi^2$.

Now we think about the dl_E integration. In the denominator we have $(l_E^2 + \Delta)^3$. In the numerator we have $\int l_E^3 dl_E \cdot (\text{something} \cdot l_E^2 + \cdots)$.

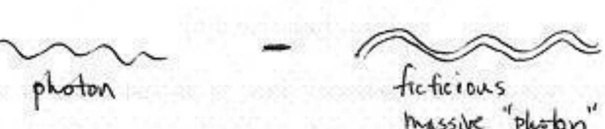
Notice that this integral diverges as $l_E \to \infty$.

It is a logarithmic divergence since if we cutoff the integral at M, we get $\sim \log M$ as $M \to \infty$.

We will hope that this divergence can be absorbed into an unknown constant (which is divergent), but then everything comes out convergent thereafter.

First we need to regulate the ultraviolet divergence by some means. The following is one type of regularization method called Pauli-Villars.

Idea: Modify the photon propagator ...



$$\frac{-i g_{\mu\nu}}{(k-p)^2 + i\varepsilon} - \frac{-i g_{\mu\nu}}{(k-p)^2 - \Lambda^2 + i\varepsilon}$$ we then take the limit $\Lambda \to \infty$

So our Δ in the denominator becomes,

for the massive photon, $\Delta_\Lambda = -xyq^2 + (1-z)^2 m^2 + z\Lambda^2$

Now we can do the integral.

$$\int \frac{d^4 \ell}{(2\pi)^4} \left[\frac{\ell^2}{[\ell^2 - \Delta]^3} - \frac{\ell^2}{(\ell^2 - \Delta_\Lambda)^3} \right) \sim \log\left(\frac{\Lambda^2}{\Delta}\right) \quad \text{as } \Lambda \to \infty$$

In any case the $F_2(q^2)$ part is finite.

$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\left[\ell^2-\Delta+i\varepsilon\right]^3} = \frac{i \times (-1)}{(4\pi)^2} \cdot \frac{1}{2\Delta}$$

$$\bar{u}(p')\,\delta\Gamma^\mu u(p) = \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\ \delta(x+y+z-1) \times \bar{u}(p')\Big[\underbrace{\cdots}_{F_1 \text{ part}} + \frac{i\sigma^{\mu\nu}q_\nu}{2m\Delta}\left[2m^2 z(1-z)\right]u(p)\Big]$$

This gives $F_2(0) = \frac{\alpha}{2\pi}\int_0^1 dx\,dy\,dz\ \delta(x+y+z-1)\,\frac{z}{1-z}$

$$= \frac{\alpha}{\pi}\int_0^1 dz \int_0^{1-z} dy\, \frac{z}{1-z} = \frac{\alpha}{2\pi}$$

So $\frac{g-2}{2} = F_2(0) \approx \frac{\alpha}{2\pi} \simeq 0.0011614$

first computed by Schwinger in 1948

Experimentally $F_2(0) = 0.0011597$