Adding interactions

To preserve causality, we consider only local interactions. For example,

$$H_{int} = \int d^3\vec{x}\ \underbrace{\mathcal{H}_{int}(\phi(x))}_{\substack{\text{function of}\\ \text{fields at the}\\ \text{same point}}} = -\int d^3\vec{x}\ \mathcal{L}_{int}(\phi(x))$$

Common example in particle + condensed matter physics,

$$\mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4$$

So $\quad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$

Note that $\Pi(x)$ is still $\partial_0\phi(x)$ since there are no interactions involving $\partial_0\phi$.

In general adding interactions will produce divergences.

These are ultraviolet (high momentum) divergences which signal that this is not the fundamental theory at arbitrarily short distances. (could be string theory or something else).

However no matter what the true physics looks like at high momenta or short distances, the low momentum or long distance physics is well approximated by an "effective" field theory with only "renormalizable" interactions.
These are interactions where the coupling constant has dimensions $[\text{Mass}]^d$ where $d \geq 0$.

Example: In 3 space + 1 time dimensions $(3+1)$ it turns out that $\phi(x)$ has dimensions $[\text{Mass}]^1$.
If I compare $-\frac{1}{2}m^2\phi^2$ and $-\frac{\lambda}{4!}\phi^4$ ... both must have the same mass dimension. So

$$\lambda \sim [\text{Mass}]^0,$$

and this is renormalizable.

On the other hand, something like $-\frac{\lambda_6}{6!}\phi^6$ would give

$$\lambda_6 \sim [\text{Mass}]^{-2},$$

which is not renormalizable.

Perturbation Expansion

Let $$\underbrace{H}_{} = H_0 + H_{int}$$ ← for example, $H_{int} = \int d^3\vec{x}\, \frac{\lambda}{4!}\phi^4(x)$

H_0: free Klein-Gordon, which we have be studying

We will generate a power series in λ.

At any fixed time t_0, we can write

$$\phi(t_0, \vec{x}) = \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}}\right)$$

we can absorb the $e^{-iE_{\vec{p}}t_0}$ in the definition of $a_{\vec{p}}$

The Heisenberg field $\phi(t, \vec{x})$ is then given by

$$\phi(t, \vec{x}) = e^{iH\cdot(t-t_0)}\, \phi(t_0, \vec{x})\, e^{-iH\cdot(t-t_0)}$$

If we shut off the interaction we have

free Hamiltonian

$$e^{iH_0(t-t_0)}\,\phi(t_0,\vec{x})\,e^{-iH_0(t-t_0)}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}e^{-ip\cdot x} + a^\dagger_{\vec{p}}e^{ip\cdot x}\right)\Big|_{\substack{x^0=t-t^0\\ p^0=E_{\vec{p}}}}$$

... we define this to be $\phi_I(t,\vec{x})$, the interaction picture field.

This interaction picture field coincides with the Heisenberg field when $\lambda = 0$.

The Heisenberg field for $\lambda \neq 0$ is

$$\phi(t,\vec{x}) = e^{iH\cdot(t-t_0)}\phi(t_0,\vec{x})e^{-iH(t-t_0)}$$

$$= e^{iH\cdot(t-t_0)}e^{-iH_0(t-t_0)}\,\phi_I(t,\vec{x})\,e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$$

$$= U^\dagger(t,t_0)\,\phi_I(t,\vec{x})\,U(t,t_0)$$

where $U(t,t_0) = e^{iH_0(t-t_0)}e^{-iH(t-t_0)}$

(we evolve the operators as $\phi_I(t,\vec{x})$
we evolve the states
by $U(t,t_0)$ ...)

$U(t,t_0)|v\rangle$

Note:

$$i\frac{\partial}{\partial t}\,U(t,t_0) = e^{iH_0(t-t_0)}\left(-H_0+H\right)e^{-iH(t-t_0)}$$

$$= e^{iH_0(t-t_0)}\,H_{int}\,e^{-iH(t-t_0)}$$

$$= \underbrace{e^{iH_0(t-t_0)}\,H_{int}\,e^{-iH_0(t-t_0)}}_{H_{int}(\phi_I(t,\vec{x}))}\;\underbrace{e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}}_{U(t,t_0)}$$

$$= \underset{\substack{\parallel \\ H_I(t)}}{H_{int}(\phi_I(t,\vec{x}))}\,U(t,t_0)$$

So $$U(t,t_0) = T\exp\left\{-i\int_{t_0}^{t}dt'\,H_I(t')\right\}$$ Dyson's formula

time ordering symbol

Why time ordering? Because $H_I(t_1)$ + $H_I(t_2)$ don't commute for different times t_1 + t_2.

The time ordering puts the <u>latest</u> operators on the <u>left</u>. That's why the $H_I(t)$ is on the left.

$$i\frac{\partial}{\partial t}\,U(t,t_0) = H_I(t)\,U(t,t_0)$$

As a power series in λ:

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t}dt'\,H_I(t') + \frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t}dt_1\,dt_2\,T\left\{H_I(t_1)\,H_I(t_2)\right\} + \cdots$$

Let us now generalize U...

Define $U(t,t') = T\exp\left\{-i\int_{t'}^{t} dt'' H_I(t'')\right\}$

for any $t \geq t'$

Then $i\frac{\partial}{\partial t} U(t,t') = H_I(t)\, U(t,t')$

$i\frac{\partial}{\partial t'} U(t,t') = -U(t,t')\, H_I(t')$

Easy to see that

$$U(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)}$$

will work...

$$i\frac{\partial}{\partial t} U(t,t') = e^{iH_0(t-t_0)}(-H_0 + H)\, e^{-iH(t-t')}\, e^{-iH_0(t'-t_0)}$$

$$= H_I(t)\, U(t,t')$$

$$i\frac{\partial}{\partial t'} U(t,t') = e^{iH_0(t-t_0)}\, e^{-iH(t-t')}(-H + H_0)\, e^{-iH_0(t'-t_0)}$$

$$= -\,U(t,t')\, H_I(t')$$

Note that $U(t,t')$ is unitary,

$$U^{\dagger}(t,t') = U^{-1}(t,t')$$

Also for $t_1 \geq t_2 \geq t_3$,

$$U(t_1,t_2)\,U(t_2,t_3) = U(t_1,t_3)$$

Let $|0\rangle$ be the ground state of H_0

Let $|\Omega\rangle$ be the ground state of H.

Let $|n\rangle$ label all the energy states of H
($n=0$ corresponds with $|\Omega\rangle$)

Let the corresponding energies be E_n.

Then $e^{-iHT}|0\rangle =$

$$e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle + \sum_{n\neq 0} e^{-iE_nT}|n\rangle\langle n|0\rangle$$

We will zero out H_0 so that $H_0|0\rangle = 0$.

Now let us consider the limit as $T\to\infty$.

Subtle point... instead of considering $T \to +\infty$, we will consider $T \to (1 - i\varepsilon)\cdot\infty$ (small positive)

(what this means is that we use analytic continuation to get the physical result)

Assuming there is a gap between E_0 + other E_n's, then e^{-iE_nT} dies slowest for $n=0$.

As $T \to (1-i\varepsilon)\cdot\infty$,

$$e^{-iHT}|0\rangle \longrightarrow e^{-iE_0T}|\Omega\rangle\langle\Omega|0\rangle.$$

So $$|\Omega\rangle = \lim_{T\to(1-i\varepsilon)\infty} \left(e^{-iE_0T}\langle\Omega|0\rangle\right)^{-1} e^{-iHT}|0\rangle.$$

Also

$$|\Omega\rangle = \lim_{T\to(1-i\varepsilon)\infty} \left(e^{-iE_0(T+t_0)}\langle\Omega|0\rangle\right)^{-1} e^{-iH(T+t_0)}|0\rangle$$

$$= \lim_{T\to(1-i\varepsilon)\infty} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} e^{-iH(t_0-(-T))} \underbrace{e^{-iH_0(-T-t_0)}}_{=1 \text{ since } H_0|0\rangle = 0}|0\rangle$$

$$= \lim_{T\to(1-i\varepsilon)\infty} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} U(t_0,-T)|0\rangle$$

Also we have

$$\langle\Omega| = \lim_{T\to(1-i\epsilon)\infty} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \langle 0| e^{-iH(T-t_0)}$$

$$= \lim_{T\to(1-i\epsilon)\infty} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \langle 0| e^{-iH_0(T-t_0)} e^{-iH(T-t_0)}$$

$$= \lim_{T\to(1-i\epsilon)\infty} \left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \langle 0| U(T,t_0)$$

For $x_0 > y^0 > t_0$, we then have

$$\langle\Omega|\phi(x)\,\phi(y)|\Omega\rangle$$

$$= \lim_{T\to(1-i\epsilon)\infty} \left[\left(e^{-iE_0(T-t_0)}\langle 0|\Omega\rangle\right)^{-1} \left(e^{-iE_0(t_0-(-T))}\langle\Omega|0\rangle\right)^{-1} \langle 0| U(T,t_0)\, \underbrace{\phi(x)\,\phi(y)}_{U^\dagger(x^0,t_0)\,\phi_I(x)\,U(x^0,t_0)\,U^\dagger(y^0,t_0)\,\phi_I(y)\,U(y^0,t_0)}\, U(t_0,-T)|0\rangle \right]$$

$$= \lim_{T\to(1-i\epsilon)\infty} \left(|\langle 0|\Omega\rangle|^2 e^{-iE_0 2T}\right)^{-1} \times \langle 0| U(T,x^0)\,\phi_I(x)\, U(x^0,y^0)\,\phi_I(y)\, U(y^0,-T)|0\rangle$$

We note that

$$\langle\Omega|\Omega\rangle = \lim_{T\to(1-i\epsilon)\infty} \left[\left(|\langle 0|\Omega\rangle|^2 e^{-iE_0 2T}\right)^{-1} \langle 0| U(T,-T)|0\rangle \right]$$

So if $|\Omega\rangle$ normalized,

$$|\langle 0|\Omega\rangle|^2 e^{-iE_0 2T} \rightarrow \langle 0|U(T,-T)|0\rangle$$

Therefore

$$\langle\Omega|\phi(x)\phi(y)|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|U(T,x^0)\,\phi_I(x)\,U(x^0,y^0)\,\phi_I(y)\,U(y^0,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}$$

$$(x^0 > y^0)$$

For $x^0 < y^0$ then roles of x + y on the right hand side are reversed. So

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle$$

$$= \lim_{T\to\infty(1-i\varepsilon)} \frac{\langle 0|T\{\phi_I(x)\,\phi_I(y)\exp[-i\int_{-T}^{T}dt\; H_I(t)]\}|0\rangle}{\langle 0|T\{\exp[-i\int_{-T}^{T}dt\; H_I(t)]\}|0\rangle}$$