III.G Conservation Laws

• *Approach to equilibrium:* We now address the third question posed in the introduction, of *how* the gas reaches its final equilibrium. Consider a situation in which the gas is perturbed from the equilibrium form described by eq.(III.56), and follow its relaxation to equilibrium. There is a hierarchy of mechanisms that operate at different time scales.

(i) The fastest processes are the two body collisions of particles in immediate vicinity. Over a time scale of the order of τ_c, $f_2(\vec{q}_1, \vec{q}_2, t)$ relaxes to $f_1(\vec{q}_1, t) f_1(\vec{q}_2, t)$ for separations $|\vec{q}_1 - \vec{q}_2| \gg d$. Similar relaxations occur for the higher order densities f_s.

(ii) At the next stage, f_1 relaxes to a *local equilibrium* from, as in eq.(III.53), over the time scale of the mean free time $\tau_\times$. This is the intrinsic scale set by the collision term on the right hand side of the Boltzmann equation. After this time interval, quantities conserved in collisions achieve a state of local equilibrium. We can then define at each point a (time dependent) local density by integrating over all momenta as

$$n(\vec{q}, t) = \int d^3\vec{p} f_1(\vec{p}, \vec{q}, t), \tag{III.69}$$

as well as a local expectation value for any operator $\mathcal{O}(\vec{p}, \vec{q}, t)$

$$\langle \mathcal{O}(\vec{q}, t) \rangle = \frac{1}{n(\vec{q}, t)} \int d^3\vec{p} f_1(\vec{p}, \vec{q}, t) \mathcal{O}(\vec{p}, \vec{q}, t). \tag{III.70}$$

(iii) After the densities and expectation values have relaxed to their local equilibrium forms in the intrinsic time scales τ_c and $\tau_\times$, there is a subsequent slower relaxation to global equilibrium over extrinsic time and length scales. This final stage is governed by the smaller streaming terms on the left hand side of the Boltzmann equation. It is most conveniently expressed in terms of the time evolution of conserved quantities according to *hydrodynamic equations.*

Conserved quantities are left unchanged by the two body collisions, i.e. satisfy

$$\chi(\vec{p}_1, \vec{q}, t) + \chi(\vec{p}_2, \vec{q}, t) = \chi(\vec{p}_1{}', \vec{q}, t) + \chi(\vec{p}_2{}', \vec{q}, t), \tag{III.71}$$

where $(\vec{p}_1, \vec{p}_2)$ and $(\vec{p}_1{}', \vec{p}_2{}')$ refer to the momenta before and after a collision, respectively. For such quantities, we have

$$J_\chi(\vec{q}, t) = \int d^3\vec{p}\, \chi(\vec{p}, \vec{q}, t) \left.\frac{df_1}{dt}\right|_{\text{coll.}} (\vec{p}, \vec{q}, t) = 0. \tag{III.72}$$

- **Proof:** Using the form of the collision integral, we have

$$J_\chi = -\int d^3\vec{p}_1 d^3\vec{p}_2 d^2\vec{b}\,|\vec{v}_1 - \vec{v}_2|\,[f_1(\vec{p}_1)f_1(\vec{p}_2) - f_1(\vec{p}_1{}')f_1(\vec{p}_2{}')]\,\chi(\vec{p}_1). \tag{III.73}$$

(The implicit arguments $(\vec{q}, t)$ are left out for ease of notation.) We now perform the same set of changes of variables that were used in the proof of the H-theorem. The first step is averaging after exchange of the dummy variables $\vec{p}_1$ and $\vec{p}_2$, leading to

$$J_\chi = -\frac{1}{2}\int d^3\vec{p}_1 d^3\vec{p}_2 d^2\vec{b}\,|\vec{v}_1 - \vec{v}_2|\,[f_1(\vec{p}_1)f_1(\vec{p}_2) - f_1(\vec{p}_1{}')f_1(\vec{p}_2{}')]\,[\chi(\vec{p}_1) + \chi(\vec{p}_2)]\,. \tag{III.74}$$

Next, change variables from the originators $(\vec{p}_1, \vec{p}_2, \vec{b}\,)$, to the products $(\vec{p}_1{}', \vec{p}_2{}', \vec{b}\,')$ of the collision. After relabeling the integration variables, the above equation is transformed to

$$J_\chi = -\frac{1}{2}\int d^3\vec{p}_1 d^3\vec{p}_2 d^2\vec{b}\,|\vec{v}_1 - \vec{v}_2|\,[f_1(\vec{p}_1{}')f_1(\vec{p}_2{}') - f_1(\vec{p}_1)f_1(\vec{p}_2)]\,[\chi(\vec{p}_1{}') + \chi(\vec{p}_2{}')]\,. \tag{III.75}$$

Averaging the last two equations leads to

$$\begin{aligned} J_\chi = -\frac{1}{4}\int d^3\vec{p}_1 d^3\vec{p}_2 d^2\vec{b}\,|\vec{v}_1 - \vec{v}_2|\,&[f_1(\vec{p}_1)f_1(\vec{p}_2) - f_1(\vec{p}_1{}')f_1(\vec{p}_2{}')] \\ &[\chi(\vec{p}_1) + \chi(\vec{p}_2) - \chi(\vec{p}_1{}') - \chi(\vec{p}_2{}')]\,, \end{aligned} \tag{III.76}$$

which is zero from eq.(III.71).

Let us explore the consequences of this result for the evolution of expectation values involving χ. Substituting for the collision term in eq.(III.72) the streaming terms on the left hand side of the Boltzmann equation, leads to

$$J_\chi(\vec{q}, t) = \int d^3\vec{p}\,\chi(\vec{p}, \vec{q}, t)\left[\partial_t + \frac{p_\alpha}{m}\partial_\alpha + F_\alpha\frac{\partial}{\partial p_\alpha}\right] f_1(\vec{p}, \vec{q}, t) = 0, \tag{III.77}$$

where we have introduced the notations $\partial_t \equiv \partial/\partial t$, $\partial_\alpha \equiv \partial/\partial q_\alpha$, and $F_\alpha = -\partial U/\partial q_\alpha$. We can manipulate the above equation into the form

$$\int d^3\vec{p}\left\{\left[\partial_t + \frac{p_\alpha}{m}\partial_\alpha + F_\alpha\frac{\partial}{\partial p_\alpha}\right](\chi f_1) - f_1\left[\partial_t + \frac{p_\alpha}{m}\partial_\alpha + F_\alpha\frac{\partial}{\partial p_\alpha}\right]\chi\right\} = 0. \tag{III.78}$$

The third term is zero, as it is a complete derivative. Using the definition of expectation values in eq.(III.70), the remaining terms can be rearranged into

$$\partial_t\left(n\left\langle\chi\right\rangle\right) + \partial_\alpha\left(n\left\langle\frac{p_\alpha}{m}\chi\right\rangle\right) - n\left\langle\partial_t\chi\right\rangle - n\left\langle\frac{p_\alpha}{m}\partial_\alpha\chi\right\rangle - nF_\alpha\left\langle\frac{\partial\chi}{\partial p_\alpha}\right\rangle = 0. \tag{III.79}$$

As discussed earlier, for elastic collisions, there are 5 conserved quantities: particle number, the three components of momentum, and kinetic energy. Each leads to a corresponding hydrodynamic equation, as constructed below:

(a) *Particle number:* Setting $\chi = 1$ in eq.(III.79) leads to

$$\partial_t n + \partial_\alpha \left(n u_\alpha \right) = 0, \tag{III.80}$$

where we have introduced the local velocity

$$\vec{u} \equiv \left\langle \frac{\vec{p}}{m} \right\rangle . \tag{III.81}$$

This equation simply states that the time variation of the local particle density is due to a particle current $\vec{J}_n = n\vec{u}$.

(b) *Momentum:* Any linear function of the momentum $\vec{p}$ is conserved in the collision, and we shall explore the consequences of the conservation of

$$\vec{c} \equiv \frac{\vec{p}}{m} - \vec{u}. \tag{III.82}$$

Substituting c_α into eq.(III.79) leads to

$$\partial_\beta \left(n \left\langle \left(u_\beta + c_\beta \right) c_\alpha \right\rangle \right) + n \partial_t u_\alpha + n \partial_\beta u_\alpha \left\langle u_\beta + c_\beta \right\rangle - n \frac{F_\alpha}{m} = 0. \tag{III.83}$$

Taking advantage of $\langle c_\alpha \rangle = 0$, from eqs.(III.81) and (III.82), leads to

$$\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha = \frac{F_\alpha}{m} - \frac{1}{mn} \partial_\beta P_{\alpha\beta} \,, \tag{III.84}$$

where we have introduced the *pressure tensor*,

$$P_{\alpha\beta} \equiv mn \left\langle c_\alpha c_\beta \right\rangle . \tag{III.85}$$

The left hand side of the equation is the acceleration of an element of the fluid $d\vec{u}/dt$, which should equal $\vec{F}_{\rm net}/m$ according to Newton's equation. The net force includes an additional component due to the variations in the pressure tensor across the fluid.

(c) *Kinetic energy:* We first introduce an average local kinetic energy

$$\varepsilon \equiv \left\langle \frac{mc^2}{2} \right\rangle = \left\langle \frac{p^2}{2m} - \vec{p} \cdot \vec{u} + \frac{mu^2}{2} \right\rangle , \tag{III.86}$$

and then examine the conservation law obtained by setting χ equal to $mc^2/2$ in eq.(III.79). Since for space and time derivatives $\partial\varepsilon = mc_\beta \partial c_\beta = -mc_\beta \partial u_\beta$, we obtain

$$\partial_t(n\varepsilon) + \partial_\alpha \left(n \left\langle (u_\alpha + c_\alpha) \frac{mc^2}{2} \right\rangle \right) + nm\partial_t u_\beta \langle c_\beta \rangle + nm\partial_\alpha u_\beta \langle (u_\alpha + c_\alpha) c_\beta \rangle - nF_\alpha \langle c_\alpha \rangle = 0. \tag{III.87}$$

Taking advantage of $\langle c_\alpha \rangle = 0$, the above equation is simplified to

$$\partial_t(n\varepsilon) + \partial_\alpha (nu_\alpha \varepsilon) + \partial_\alpha \left(n \left\langle c_\alpha \frac{mc^2}{2} \right\rangle \right) + P_{\alpha\beta}\partial_\alpha u_\beta = 0. \tag{III.88}$$

We next take out the dependence on n in the first two terms of the above equation, finding

$$\varepsilon\partial_t n + n\partial_t \varepsilon + \varepsilon\partial_\alpha (nu_\alpha) + nu_\alpha \partial_\alpha \varepsilon + \partial_\alpha h_\alpha + P_{\alpha\beta} u_{\alpha\beta} = 0, \tag{III.89}$$

where we have also introduced the local *heat flux*

$$\vec{h} \equiv \frac{nm}{2} \left\langle c_\alpha c^2 \right\rangle, \tag{III.90}$$

and the *rate of strain tensor*

$$u_{\alpha\beta} = \frac{1}{2} \left(\partial_\alpha u_\beta + \partial_\beta u_\alpha \right). \tag{III.91}$$

Eliminating the first and third terms in eq.(III.89) with the aid of eq.(III.80) leads to

$$\partial_t \varepsilon + u_\alpha \partial_\alpha \varepsilon = -\frac{1}{n}\partial_\alpha h_\alpha - \frac{1}{n} P_{\alpha\beta} u_{\alpha\beta}. \tag{III.92}$$

Clearly to solve the hydrodynamic equations for n, $\vec{u}$, and ε, we need expressions for $P_{\alpha\beta}$ and $\vec{h}$, which are either given phenomenologically, or calculated from the density f_1, as in the next sections.