Magnetic Fields as a Consequence of Relativity

TA: Enrique Mendez (eqm@mit.edu)

8.311 Electromagnetic THeory

Side Note: Action at a Distance

Action at a Distance is a quite unintuitive phenomena. The only reason we have any familiarity at all with such a concept, is that our society has become so advanced that we have powerful magnets readily available for the consumer, to play with the mysterious property of nature that is action at a distance...

In disagreement with Professor Fu, I am going to stress that one cannot distinguish between action at a distance, and a field theory. Faraday invented the topic to make sense of action at a distance, but it is by no means required. It makes calculations easier, but that does not distinguish between two different physical theories. In short, as much effort as Professor Fu is using to get rid of charges in his lectures, we could take as much effort trying to remove the fields and write things only as a function of charge distributions and charge separations, which one could argue as being more physical, tangible, and readily measurable.

The Importance of Different Pictures

One could say, for this lecture, that I am going to take the point of view that electric force fields are caused via action at a distance. In which case, the Electric and Magnetic fields, will simply take the role of bookkeeping devices, for helping us keep track of what force a particle would experience if it was there.

I'm doing this to stress, that currently, there is no way to distinguish between an action at a distance theory, and a field theory. And while mathematically, this makes no difference, as Feynman pointed out in his Auckland (Cornell?) lectures, these different theories when extrapolated to new regimes can suggest new theories that the other theory would not. So this is a useful thing to keep in mind if you're trying to understand some new physical phenomena that seems opaque and mysterious.

Still, to prepare ourselves for the question, what is a magnetic field, I am going to analyze the case of the electric monopole.

Electric Forces

We're going to take electric forces from a point charge for granted, or more importantly as an axiom. Given above, we intuitively justify this by saying that we have experience with action at a distance, namely via playing with magnets. More justifiably, we can say we know that Newtonian gravity is an action at a distance theory, and the similarity between gravity and charge interactions, might help us postulate that it is the more fundamental behaviour in the universe.

Principle of Relativity

Another reasonable property of nature, is the principal of relativity.

At its' core, it says that all this physics we discovered was not special to humanity. An alien species on some other planet would discover the same physics.

The most illuminating, and consequential bit of physics we discovered, is that one can derive the speed of light from Maxwell's equations. Since, the alien's would have discovered this, and their planet would most definitely be moving at some speed relative to us, and since they would have calculated the same speed of light, we must surmise that the speed of light is independent of the velocity of an observer.

This symmetry principle has far reaching consequences, which I will assume you have seen before. But once again, I'd like to emphasize, at its heart, the principle of relativity, is nothing more than the statement that physics is the same for all inertial observers.

Another intuitive fact that we should take as an axiom. Given these two principles, both apparently fundamental, and both experimentally verified. We will show we can derive the consequence of a magnetic field. Hopefully, illustrating to you, that magnetism is an emergent phenomena far less fundamental than the electric field of a monopole.

Consequences

We can deduce, that if this is true, this will then help us understand why it is that Electricity and Magnetism are actually the same thing, and why it is that one needs a `tensor` to describe it rather than a simple four-vector like every other nice (i.e., simple) physical quantity in relativistic mechanics. Surely a nice fact, for those who've wondered why such complicated mathematics were needed to describe the fields. I believe the reasoning will also make you doubt the existence of monopoles. In stark, contradiction for those who've taken 8.321 from Senthil, and learned of Dirac's theorem. Lastly, we can get rid of the magnetic field in our heads as some fundamental property.

Why do we even want to explain the Magnetic Fields?

It's one thing to accept the Coulomb Law but another to accept the Biot-Savart Law. Let's examine them, behold the Coulomb Law:

$$\overrightarrow{F}_{E,1} = k\frac{q_1 q_2}{\sqrt{\left|\vec{r}_1 - \vec{r}_2\right|}}\hat{r}_{12}$$

where $\hat{r}_{12}$ is the unit vector pointing form charge 1 to 2. Note in depends on charge, a property of fundamental constants, and only on the separation and the force is inline with their separation.

Now consider instead the Biot-Savart Law. The easiest form is to calculate the magnetic field first.

$$d\overrightarrow{B}(\vec{r}) = \frac{\mu_0}{4\pi}I \cdot \frac{d\vec{l} \times \vec{r}'}{\left|\vec{r}'\right|^3}$$

Notice it relies on the usage of the current of a wire, an experimental quantity that seems to obfuscate the fundamental nature of the source of the Magnetic Field. Its' description relies on the use of wires. What's worse still is it's impossible to cut up a section of wire without disrupting the current. So we can't even have an elementary source of magnetic field!!! We can't even verify if this differential form is fundamental! Maybe if you could cut up a wire and still have the current flow, it would deviate from this formula!!! If we ever want a magnetic field, we must have whole loops of wire... Perhaps this is just an effective equation that gives us an empirical relation alone :(

Add on to this the strange magnetic part of the Lorentz Force $q\overrightarrow{v} \times \overrightarrow{B}$, and you've got a recipe for something that seems to give more questions than answers.

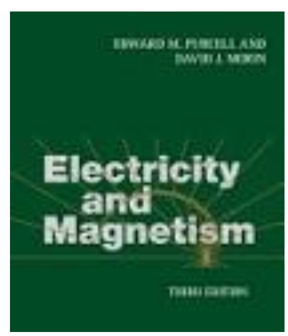
Furthermore,

It's so different in character, rather than being simple force fields pointing radially outward, which is necessary from a spherical symmetry principle, the Biot-Savart law requires one to take into consideration the velocities of the charges moving, and not only that but to rely on this strange thing: the cross product. Why should nature care about cross products?!

Hopefully, all these questions will instill in you the confusion I had as an undergraduate. That way we can enjoy the solution together.

A Map of the Derivation

Since this is going to be a not so straightforward derivation...



(I'll be following Electricty and Magnetism, Third Edition [Green Book], Chapter Five, of Purcell and Morin's book... Page numbers will be mentioned occasionally for sources)

We're going to explain magnetic force part of the experimental fact that the forces on a charge experience the Lorentz Force.

$$\underline{F} = q\,\underline{E} + q\,\underline{v} \times \underline{B}$$

where underlines or arrows above signify vectors.

... I'm going to draw a map of the facts, lemma's, and theorems we'll need to prove it.

- Assumption of the Coulomb Law
- Assumption of the Principle of Special Relativity
  - Gauss's Law must be Relativistically Invariant (p.242)

- Relativistic Invariance of Charge
  - An electron has the same value of charge in all frames.
- Transformation Law for Electric Field
  - $E'_{||} = E_{||}$
  - $E'_{\perp} = \gamma E_{\perp}$
  - Can be derived from Gauss' Law and Length Contraction
- Transformation Law for Forces
  - $\dfrac{dp_{||}}{dt} = \dfrac{dp'_{||}}{dt'}$
  - Note that these formulas are for converting between Newtonian Forces, i.e., where the velocity of the particle is ZERO to some other frame. It relies on the congruence of Newtonian momentum and Four Momentum at small velocities to discover the transformation law.
  - $\dfrac{dp_{\perp}}{dt} = \dfrac{1}{\gamma}\dfrac{dp'_{\perp}}{dt'}$

- Calculating the Force between Two Moving Charges in a frame with the Charges at Rest, and finally,
- Transforming Back to the Initial Frame to Deduce the existence of a magnetic field.

A simple example that shows magnetic fields are illusions

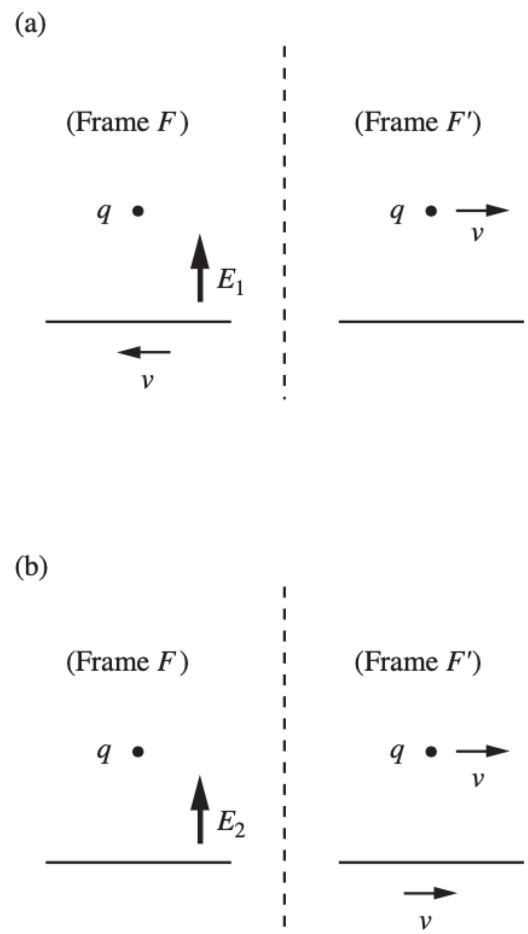


Figure 5.21.
(a) A point charge and a sheet moving relative to each other. (b) A point charge and a sheet at rest with respect to each other.

If we calculate the forces in F, and F' in Fig (a), we can calculate the force in both frames from knowing its value in the Newtonian Frame.

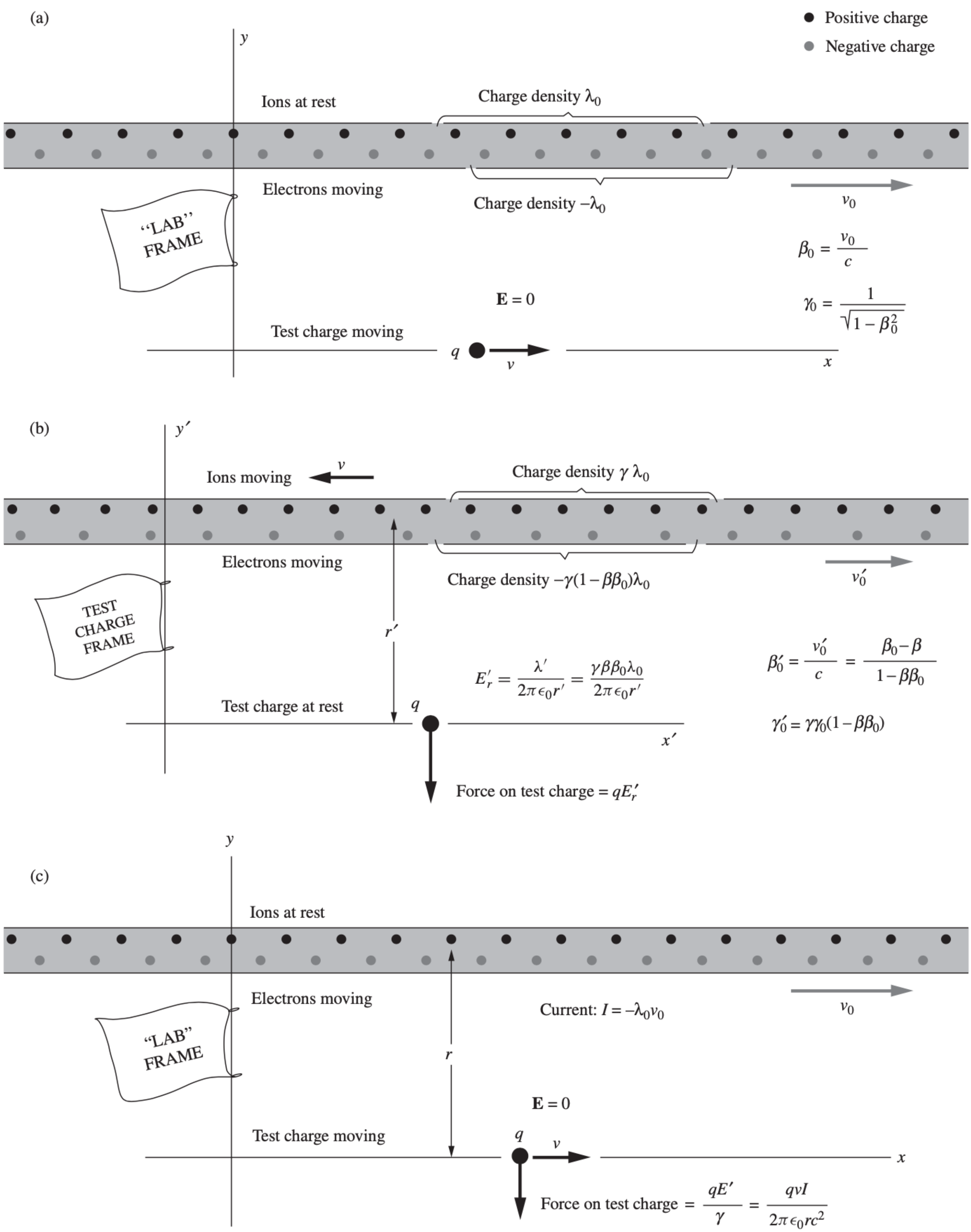
And calculate the electric field in both frames. Then we can verify that in both cases the forces can be explained as electric forces.

Fig b, is where it gets interesting. We can once again calculate the forces on q_1 in both frames by transforming the Newtonian Force, and once again calculate the electric fields in both frames. What we'll find however, is that in Frame F', they won't match :) Despite looking at the same physical phenomena, a charged particle repulsed from a charged plate, the electric field isn't sufficient to accurately calculate the force in the moving frame. What was simple and just explained by the electric field, requires something new to explain the deviation, as the electric field is no longer sufficient.

IN OTHER WORDS, WE DEDUCE WE MUST HAVE A NEW FORCE FIELD TO SATISFY THE PRINCIPLE OF RELATIVITY.

Now while we can verify that it is precisely equal to $qv \times B$ that does not give us intuition as to how loops of wire induce 'magnetic' forces. Unfortunately, we ran out of time in lecture, so I'll give you just the intuition, as the book contains the derivation (p. 259), and as usual the details aren't enlightening as long as we can correctly interpret the physics once we calculate the answer.

The figure below carries a summary of the different frames to analyze.



(a)
Positive charge
Negative charge
y
Ions at rest
Charge density λ_0
Electrons moving
Charge density $-\lambda_0$
"LAB" FRAME
v_0
$\beta_0 = \frac{v_0}{c}$
$\gamma_0 = \frac{1}{\sqrt{1-\beta_0^2}}$
$\mathbf{E} = 0$
Test charge moving
q
v
x
(b)
y′
Ions moving
v
Charge density $\gamma\lambda_0$
Electrons moving
Charge density $-\gamma(1-\beta\beta_0)\lambda_0$
TEST CHARGE FRAME
v_0'
r′
$E_r' = \frac{\lambda'}{2\pi\epsilon_0 r'} = \frac{\gamma\beta\beta_0\lambda_0}{2\pi\epsilon_0 r'}$
$\beta_0' = \frac{v_0'}{c} = \frac{\beta_0-\beta}{1-\beta\beta_0}$
Test charge at rest
q
x′
$\gamma_0' = \gamma\gamma_0(1-\beta\beta_0)$
Force on test charge = qE_r'
(c)
y
Ions at rest
Electrons moving
Current: $I = -\lambda_0 v_0$
v_0
"LAB" FRAME
r
$\mathbf{E} = 0$
Test charge moving
q
v
x
Force on test charge $= \frac{qE'}{\gamma} = \frac{qvI}{2\pi\epsilon_0 rc^2}$

Why loops of wires exert Magnetic Forces

Ultimately, current carrying wires are two moving charge distributions of opposite sign and differing velocity. This difference in velocity causes DIFFERENT amounts of length contraction in the Newtonian Frame (rest frame of the charged particle). This means the wires pick up appear charged for the moving particle. Isn't that magic?

If we were riding along with the particle, imagining it initially at rest. We would see this neutral wire, and as we accelerated to some final velocity v. We would see it get more and more and more charged. I wonder how that happens...