8.09/8.309, Classical Mechanics III, Fall 2019
MIDTERM
Tuesday October 29, 7:30-9:30pm
You have 120 minutes.

Answer all problems in the white books provided. Write YOUR NAME on EACH book you use.

There are four problems, totalling 100 points. You should do all four. The problems are worth 18, 26, 28, 28 points.

None of the problems requires extensive algebra. If you find yourself lost in a calculational thicket, stop and think.

No books, notes, or calculators allowed.

Some potentially useful information

- Euler-Lagrange equations for generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j}\ , \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot q_j}$$

  constraints: holonomic $f_\alpha(q,t)=0$ or semiholonomic $g_\beta = \sum_j a_{\beta j}(q,t)\dot q_j + a_{\beta t}(q,t) = 0$

- Generalized forces: $d/dt(\partial L/\partial \dot q_j) - \partial L/\partial q_j = R_j$

  Friction forces: $\vec f_i = -h(v_i)\vec v_i/v_i$, $\vec v_i = \dot{\vec r}_i$ gives $R_j = -\partial \mathcal{F}/\partial \dot q_j\ ,\ \ \mathcal{F} = \sum_i \int_0^{v_i} dv_i'\, h(v_i')$

- Hamilton's equations for canonical variables (q_j, p_j): $\quad \dot q_j = \dfrac{\partial H}{\partial p_j}\ ,\quad \dot p_j = -\dfrac{\partial H}{\partial q_j}$

- Hamiltonian for a Lagrangian quadratic in velocities

$$L = L_0(q,t) + \dot{\vec q}^{\,T}\cdot \vec a + \frac{1}{2}\dot{\vec q}^{\,T}\cdot \hat T \cdot \dot{\vec q} \quad \Rightarrow \quad H = \frac{1}{2}(\vec p - \vec a)^T \cdot \hat T^{-1}\cdot(\vec p - \vec a) - L_0(q,t)$$

- The Moment of Inertia Tensor and its relations:

$$I_{ab} = \int dV\, \rho(\vec r)[\vec r^{\,2}\delta_{ab} - r_a r_b] \qquad \text{or} \qquad I^{ab} = \sum_i m_i[\delta^{ab}\vec r_i^{\,2} - r_i^a r_i^b]$$

$$I_{ab}^{(Q)} = M(\delta_{ab}\,\vec R^2 - R_a R_b) + I_{ab}^{(\mathrm{CM})}\ , \qquad \hat I' = \hat U\,\hat I\,\hat U^T$$

- Euler's Equations:

$$\begin{aligned} I_1\dot\omega_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1 \\ I_2\dot\omega_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2 \\ I_3\dot\omega_3 - (I_1 - I_2)\omega_1\omega_2 &= \tau_3 \end{aligned}$$

- Vibrations: $L = \frac{1}{2}\dot{\vec\eta}^{\,T}\cdot\hat T\cdot\dot{\vec\eta} - \frac{1}{2}\vec\eta^{\,T}\cdot\hat V\cdot\vec\eta$ has Normal modes $\vec\eta^{\,(k)} = \vec a^{\,(k)}\exp(-i\omega^{(k)}t)$

$$\det(\hat V - \omega^2\hat T) = 0\ , \qquad \left(\hat V - [\omega^{(k)}]^2\,\hat T\right)\cdot\vec a^{\,(k)} = 0\ , \qquad \vec\eta = \mathrm{Re}\sum_k C_k\,\vec\eta^{\,(k)}$$

- Generating functions for Canonical Transformations: $K = H + \partial F_i/\partial t$ and

$$F_1(q,Q,t):\quad p_i = \frac{\partial F_1}{\partial q_i}\ ,\ P_i = -\frac{\partial F_1}{\partial Q_i}\ , \qquad F_2(q,P,t):\quad p_i = \frac{\partial F_2}{\partial q_i}\ ,\ Q_i = \frac{\partial F_2}{\partial P_i}$$

- Poisson Brackets: $\quad [u,v]_{q,p} = \sum_j \left[\dfrac{\partial u}{\partial q_j}\dfrac{\partial v}{\partial p_j} - \dfrac{\partial u}{\partial p_j}\dfrac{\partial v}{\partial q_j}\right]\ , \qquad \dfrac{du}{dt} = [u,H] + \dfrac{\partial u}{\partial t}$

- Relations for Hamilton's Principle function, $S = S(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n,t)$

$$K = 0\ , \qquad P_i = \alpha_i\ , \qquad Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i}\ , \qquad p_i = \frac{\partial S}{\partial q_i}$$

- Relations for Hamilton's Characteristic function, $W = W(q_1,\ldots,q_n;\alpha_1,\ldots,\alpha_n)$

$$K = H = \alpha_1\ , \qquad P_i = \alpha_i\ , \qquad \beta_1 + t = \frac{\partial W}{\partial \alpha_1}\ , \qquad \beta_{i>1} = \frac{\partial W}{\partial \alpha_i}\ , \qquad p_i = \frac{\partial W}{\partial q_i}$$

- Action Angle Variables: $\quad J = \oint p\,dq\ , \qquad w = \dfrac{\partial W(q,J)}{\partial J}\ , \qquad \dot w = \dfrac{\partial H(J)}{\partial J} = \nu(J)$

1. **Semi-short answer problems** [18 points]

   These problems require less algebra, and a correct answer with no work shown will receive full credit. Your answers should be short.

   (a) [5 points] Consider Kepler motion with energy $E < 0$. Describe physically what the 5 independent constants of motion represent. Pictures are encouraged.

   (b) [3 points] Let (q, p) be canonical variables, and consider the generating function $F(q, Q) = q\, e^{Q}$. What are the new canonical variables $Q = Q(q, p)$ and $P = P(q, p)$ that this F generates?

   (c) [4 points] Describe two different uses for the method of Lagrange Multipliers.

   (d) [6 points] Consider a vertical disk of radius a rolling on a horizontal plane. What are the 4 coordinates that are needed to describe the motion of the disk? What are the two no slip constraints expressed in terms of time derivatives of these coordinates?

2. **Moment of Inertia for 3 particles** [26 points]

   Consider three point particles in 3-dimensions with masses and (x, y, z) coordinates:

$$
\begin{aligned}
m_1 &: \quad (b, 0, b) \\
m_2 &: \quad (b, b, -b) \\
m_3 &: \quad (-b, b, 0)
\end{aligned}
$$

   The distance between the masses are fixed so that they form a rigid body.

   (a) [8 points] Calculate the components of the moment of inertia tensor $\hat{I}$ about the origin for the (x, y, z) axes.

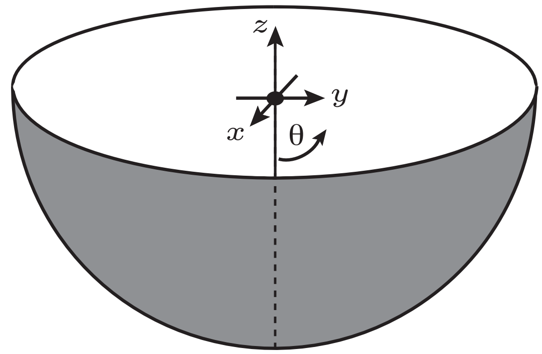
   (b) [12 points] Take $m_1 = m_2 = m_3 = m$. Determine the principal moments of inertia and principal axes for the system about the origin.

   (c) [6 points] Take $m_1 = m_2 = m_3 = m$ and solve the Euler Equations with zero torque to characterize the motion of the rigid body about the fixed origin in the body frame.

(continue)

3. **Two Interacting Particles in a Spherical Bowl** [28 points]

Consider two particles with equal mass m moving frictionlessly on the surface of a hemispherical bowl of radius R. The particles are attached together by a massless spring, giving rise to a potential that depends on the distance between them: $V_{\text{spring}} = \frac{1}{2}k\,|\vec{r}_1 - \vec{r}_2|^2$. Gravity also acts on the two masses.



For this problem use spherical coordinates $\vec{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, -r\cos\theta)$ which ensures the bottom of the bowl sits at $\theta = 0$ as shown. Note that $\dot{\vec{r}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$.

(a) [10 points] Assuming that the particles never leave the surface of the spherical bowl, what is the Lagrangian $L(\theta_i, \dot{\theta}_i, \phi_i, \dot{\phi}_i)$ for this system? [You may find the identity $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ useful to simplify your answer.]

(b) [3 points] What is the equation of motion for θ_1?

(c) [5 points] Identify two conserved quantities for the motion of the particles in this system. Either a mathematical or physical argument is acceptable.

(d) [10 points] Now assume that circular tracks are introduced such that the angles θ_i are held fixed at constant values with $0 < \theta_1 < \theta_2 < \pi/2$. Find the normal mode frequencies for small oscillations of the system in ϕ_1 and ϕ_2 about the point $\phi_1 = \phi_2 = 0$. If appropriate, also find the corresponding normal mode eigenvector. Draw a picture for the motion given by each of your frequencies.

(continue)

4. **Hamilton-Jacobi for a Relativistic Oscillator** [28 points]

Consider a relativistic particle of mass m, momentum p, in a potential $V(x) = k|x|$, with constant $k > 0$. Its Hamiltonian is given by

$$H(q,p) = \sqrt{p^2c^2 + m^2c^4} + k|x|\,,$$

where c is the constant speed of light. We will analyze this as a standard classical Hamiltonian. Assume that the system has a fixed energy $E > mc^2$. If you are comfortable with using natural units you can set $c = 1$ below.

(a) [2 points] What is the Hamilton-Jacobi equation for this system?

(b) [5 points] Solve the Hamilton-Jacobi equation to obtain an expression for Hamilton's characteristic function $W(x, \alpha_1)$. Leave your result in terms of an integral.

(c) [3 points] This system will undergo oscillations. What are the two x values which are turning points for this motion?

(d) [6 points] Find an expression for the variable β that is conjugate to α_1. Use this result to obtain the solution $x = x(t)$ for the part of the motion where $x > 0$ and $p > 0$.

(e) [10 points] Lets analyze the oscillations with action-angle variables. Derive an expression for the action variable J, which again you can leave as an integral. Next find an expression for the frequency of oscillations as a function of the energy, $\nu = \nu(E)$. For this expression any integrals should be done. [You may find it useful to recall that $\partial E/\partial J = (\partial J/\partial E)^{-1}$.]

(f) [2 points] How is the constant β from part (d) related to the period $\tau = 1/\nu$ for one full oscillation?

(the end)