You may find the following information helpful:

Physical Constants

Electron mass	$m_e \approx 9.1 \times 10^{-31} kg$	Proton mass	$m_p \approx 1.7 \times 10^{-27} kg$
Electron Charge	$e \approx 1.6 \times 10^{-19} C$	Planck's const./2π	$\hbar \approx 1.1 \times 10^{-34} Js^{-1}$
Speed of light	$c \approx 3.0 \times 10^{8} ms^{-1}$	Stefan's const.	$\sigma \approx 5.7 \times 10^{-8} Wm^{-2}K^{-4}$
Boltzmann's const.	$k_B \approx 1.4 \times 10^{-23} JK^{-1}$	Avogadro's number	$N_0 \approx 6.0 \times 10^{23} mol^{-1}$

Conversion Factors

$1atm \equiv 1.0 \times 10^5 Nm^{-2}$ $\quad$ $1\mathring{A} \equiv 10^{-10} m$ $\quad$ $1eV \equiv 1.1 \times 10^4 K$

Thermodynamics

$dE = TdS + đW$ $\quad$ For a gas: $đW = -PdV$ $\quad$ For a wire: $đW = Jdx$

Mathematical Formulas

$\int_0^\infty dx\, x^n\, e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$ $\quad$ $\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}$

$\int_{-\infty}^{\infty} dx \exp\left[-ikx - \frac{x^2}{2\sigma^2}\right] = \sqrt{2\pi\sigma^2} \exp\left[-\frac{\sigma^2 k^2}{2}\right]$ $\quad$ $\lim_{N\to\infty} \ln N! = N\ln N - N$

$\left\langle e^{-ikx}\right\rangle = \sum_{n=0}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle$ $\quad$ $\ln \left\langle e^{-ikx}\right\rangle = \sum_{n=1}^{\infty} \frac{(-ik)^n}{n!} \left\langle x^n \right\rangle_c$

$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$ $\quad$ $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$

Surface area of a unit sphere in d dimensions $\quad$ $S_d = \frac{2\pi^{d/2}}{(d/2-1)!}$

1. *Gas:* The temperature of a gas is found to depend only on its pressure as $T(p,V) = c\,p^n$, while its internal energy is given by $E(p,V) = D\;pV$, where c, D and n are constants.

(a) Give the expression for the differential changes in entropy as $dS(p,V)$.

• **(1 points)** Starting from $dE = TdS - pdV$, we obtain

$$dS = \frac{dE}{T} + \frac{p}{T}dV = \frac{DpdV + DVdp}{T} + \frac{p}{T}dV = \frac{(D+1)}{c}p^{1-n}dV + \frac{D}{c}p^{-n}V\,dp\,.$$

(b) Noting that entropy is a function of state, find the relation between n and D.

• **(1 points)** From the expression for dS, we can construct the Maxwell relation:

$$\frac{\partial}{\partial V}\left[\frac{D}{c}p^{-n}V\right] = \frac{\partial}{\partial p}\left[\frac{(D+1)}{c}p^{1-n}\right],$$

leading to

$$\frac{D}{c} = \frac{(D+1)(1-n)}{c}, \quad \Rightarrow \quad n = \frac{1}{D+1}\,.$$

(c) Find the form of adiabatic curves as $p_S(V)$.

• **(1 points)** Adiabatic curves satisfy $dQ = 0$ and $dS = 0$, leading to

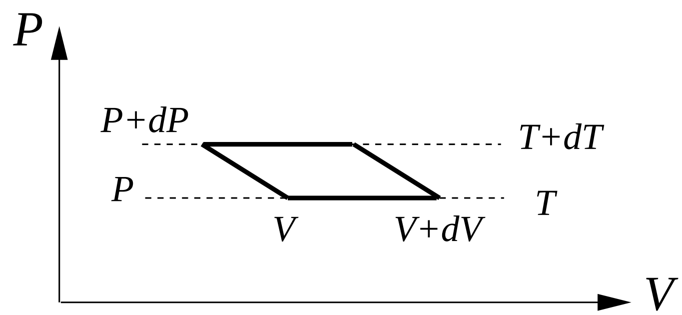
$$dS = 0, \quad \Rightarrow \quad (D+1)pdV + DV\,dp = 0, \quad \Rightarrow \quad \frac{dp}{p} + \frac{D+1}{D}\frac{dV}{V} = 0,$$

which can be integrated to

$$\ln\left[pV^{\frac{D+1}{D}}\right] = \text{constant}, \quad \Rightarrow \quad p_S(V) \propto V^{-\frac{D+1}{D}}\,.$$

(d) Draw an infinitesimal Carnot cycle in the (p,V) coordinates.

• **(1 points)**



(e) How much heat is extracted in the above Carnot cycle at the temperature $T(p)$?

• **(1 points)**

$$dQ_H = TdS = T\frac{(D+1)}{c}p^{1-n}dV = (D+1)pdV.$$

****

2. *Constant heat capacities:* Consider two bodies with temperature independent heat capacities C_1 and C_2, and initial temperatures $T_1 > T_2$.

(a) If the two bodies are brought into contact such that the only heat exchange is between them, what is the final temperature T_F, and what is the change in entropy.

• **(2 points)** If the only heat exchange is between the two bodies,

$$0 = đQ_1 + đQ_2 = C_1 dT_1 + C_2 dT_2.$$

Integrating the above equation from the initial to final state gives

$$C_1T_1 + C_2T_2 = (C_1 + C_2)T_F, \quad \Longrightarrow \quad T_F = \frac{C_1T_1 + C_2T_2}{C_1 + C_2}.$$

The overall change in entropy is

$$\Delta S = \int_{T_1}^{T_F} \frac{đQ_1}{T_1} + \int_{T_2}^{T_F} \frac{đQ_2}{T_2} = C_1 \ln\frac{T_F}{T_1} + C_2 \ln\frac{T_F}{T_2} = \ln\frac{(C_1T_1 + C_2T_2)^{C_1+C_2}}{(C_1 + C_2)^{C_1+C_2}T_1^{C_1}T_2^{C_2}}.$$

(b) What is the final temperature if a Carnot engine is used to transfer heat between the two bodies? What is the amount of work done by the engine in this case?

• **(2 points)** With a Carnot engine, the heat transfers are related by

$$0 = dS = \frac{đQ_1}{T_1} + \frac{đQ_2}{dT_2} = C_1\frac{dT_1}{T_1} + C_2\frac{dT_2}{dT_2}.$$

Integrating the above equation from the initial to final state gives

$$0 = C_1 \ln\frac{T_F}{T_1} + C_2 \ln\frac{T_F}{T_2}, \quad \Longrightarrow \quad T_F = T_1^{\frac{C_1}{C_1+C_2}} T_2^{\frac{C_2}{C_1+C_2}}.$$

The work done by the Carnot engine is equal to

$$\begin{aligned} W &= \int_{T_F}^{T_1} đQ_1 + \int_{T_F}^{T_2} đQ_2 = C_1(T_1 - T_F) + C_2(T_2 - T_F) \\ &= C_1T_1 + C_2T_2 - (C_1 + C_2)T_1^{\frac{C_1}{C_1+C_2}} T_2^{\frac{C_2}{C_1+C_2}}. \end{aligned}$$

****

3. *Probability:* Calculate the characteristic function, the mean, and the variance of the following probability density functions:

(a) $$p(x) = \frac{1}{2b}\exp\left(-\frac{|x-a|}{b}\right) \quad \text{for} \quad -\infty < x < \infty.$$

• **(3 points)** This is the *Laplace* PDF shifted so that the mean has moved to $x = a$,

$$\begin{aligned}
\tilde{p}(k) &= \frac{1}{2b}\int_{-\infty}^{\infty} dx \exp\left(-ikx - \frac{|x-a|}{b}\right) \\
&= \frac{e^{-ika}}{2b}\int_{0}^{\infty} dx \exp(-ikx - x/b) + \frac{e^{-ika}}{2b}\int_{-\infty}^{0} dx \exp(-ikx + x/b) \\
&= \frac{e^{-ika}1}{2b}\left[\frac{1}{-ik+1/b} - \frac{1}{-ik-1/b}\right] = \frac{e^{-ika}}{1+(bk)^2}.
\end{aligned}$$

Cumulants are generated by

$$\begin{aligned}
\ln \tilde{p}(k) &= -ika - \ln(1+(bk)^2) \\
&= -ika - (bk)^2 + \cdots.
\end{aligned}$$

Therefore,

$$\langle x\rangle_c = a, \qquad \text{and} \qquad \langle x^2\rangle_c = b^2.$$

$$m_1 = \langle x\rangle = a, \qquad \text{and} \qquad m_2 = \langle x^2\rangle = a^2 + b^2.$$

(b) $$p(x) = \frac{|x|}{2a^2}\exp\left(-\frac{|x|}{a}\right) \quad \text{for} \quad -\infty < x < \infty.$$

• **(3 points)**

$$\begin{aligned}
\tilde{p}(k) &= \frac{1}{2a^2}\int_{-\infty}^{\infty} dx|x| \exp\left(-ikx - \frac{|x|}{a}\right) \\
&= \frac{1}{2a^2}\int_{0}^{\infty} dx x \exp(-ikx - x/a) - \frac{1}{2a^2}\int_{-\infty}^{0} dx x \exp(-ikx + x/a) \\
&= \frac{1}{2a^2}\left[\frac{1}{(-ik+1/a)^2} + \frac{1}{(-ik-1/a)^2}\right] = \frac{1-(ak)^2}{\left(1+(ak)^2\right)^2} \\
&= 1 - 3(ak)^2 + 5(ak)^4 - \cdots.
\end{aligned}$$

Therefore,

$$m_1 = \langle x\rangle = 0, \qquad \text{and} \qquad m_2 = \langle x^2\rangle = 6a^2.$$

****