Lecture 5 - Quantum States of Light

Review from last lecture

• Analogy with harmonic oscillator:

One e.m. field mode
$$\dot{\mathcal{A}}_i = -\mathcal{E}_i \\ \dot{\mathcal{E}}_i = \omega_i^2 \mathcal{A}_i \\ H_i = \frac{\epsilon_0}{2} \frac{1}{V} \left(|\mathcal{E}_i|^2 + \omega_i^2 |\mathcal{A}_i|^2 \right) \\ \alpha_i = \mathcal{N}_i \left(\mathcal{A}_i - \frac{i}{\omega_i} \mathcal{E}_i \right) \\ \frac{d\alpha_i}{dt} = -i\omega_i \alpha_i$$
 Correspondence
$$\lambda_i \hat{=} x \\ \mathcal{E}_i \hat{=} x \\ \mathcal{E}_i \hat{=} x \\ \mathcal{E}_i \hat{=} -\frac{p}{m} \\ \mathcal{E}_i \hat{=} -\frac{p}{m} \\ \mathcal{E}_i \hat{=} -\frac{p}{m} \\ H = \frac{m}{2} \left(\left(\frac{p}{m} \right)^2 + \omega^2 x^2 \right) \\ \alpha = \mathcal{N} \left(x + i \frac{p}{m\omega} \right) \\ \frac{d\alpha}{dt} = -i\omega \alpha$$

Quantization and Commutation relations:

One e.m. field mode
$$\mathcal{A}_i
ightarrow \hat{\mathcal{A}}_i \ \mathcal{E}_i
ightarrow \hat{\mathcal{E}}_i \ \hat{\mathcal{E}}_i = -rac{V}{\epsilon_0}i\hbar \ \hat{a}_i \ \mathrm{annihilation\ operator} \ \mathrm{associated\ to}\ \alpha_i \ \hat{a}_i, \hat{a}_i^{\dagger} = 1 \ \mathrm{for} \ \mathcal{N} = \sqrt{\frac{\epsilon_0 \omega_i}{2\hbar}} \ \mathcal{N} \ \mathcal{N} = \sqrt{\frac{m\omega}{2\hbar}} \ \mathcal{N} \ \mathcal{N} = \sqrt{\frac{m\omega}{2\hbar}} \ \mathcal{N} \ \mathcal{N} = \sqrt{\frac{m\omega}{2\hbar}} \ \mathcal{N} \ \mathcal{$$

• Fields:

$$egin{aligned} ec{E}_{\perp}(ec{r}) &= i \sum_{j} \mathcal{E}_{j} ec{\epsilon}_{j} \left(a_{j} e^{i ec{k}_{j} ec{r}} - a_{j}^{\dagger} e^{-i ec{k}_{j} ec{r}}
ight) \ ec{B}(ec{r}) &= i \sum_{j} \mathcal{E}_{j} rac{ec{k}_{j} imes ec{\epsilon}_{j}}{\omega_{j}} \left(a_{j} e^{i ec{k}_{j} \cdot ec{r}} - a_{j}^{\dagger} e^{-i ec{k}_{j} \cdot ec{r}}
ight) \ ec{A}_{\perp}(ec{r}) &= \sum_{i} rac{\mathcal{E}_{j}}{\omega_{j}} ec{\epsilon}_{j} \left(a_{j} e^{i ec{k}_{j} \cdot ec{r}} + a_{j}^{\dagger} e^{-i ec{k}_{j} \cdot ec{r}}
ight) \end{aligned}$$

where $\mathcal{E}_j=\sqrt{\frac{\hbar\omega_j}{2\epsilon_0 V}}$ (careful: This pure number should not to be confused with the complete Fourier transform of the classical field mode $\vec{\mathcal{E}}_\perp$, which is a function of \vec{k} .)

Uncertainty relation:

Just like $\Delta x \Delta p \geq \frac{\hbar}{2}$ we have, using our correspondence $\mathcal{A}_j \equiv x$ and $\mathcal{E}_j \equiv -p/m$ as well as $\epsilon_0/V \equiv m$:

$$\Delta \mathcal{A}_{\perp,j} \Delta \mathcal{E}_{\perp,j} \geq rac{\hbar}{2} rac{V}{\epsilon_0}$$

Quick reminder on units: $\vec{\mathcal{E}}_{\perp}(\vec{k}_j) = \int \mathrm{d}^3 r \, \vec{E}_{\perp}(\vec{r}) e^{-i\vec{k}_j \cdot \vec{r}}$ has units of [EV]. Also, $\dot{\vec{A}}$ has units of an electric field, so $[\mathcal{A}] = [EVt]$. So $[\epsilon_0 \mathcal{A}\mathcal{E}] = [\epsilon_0 E^2 V^2 t]$. $\epsilon_0 E^2$ is an energy density, so $[\epsilon_0 \mathcal{A}\mathcal{E}] = [\mathrm{Energy} \cdot \mathrm{time} \cdot \mathrm{Volume}] = [\hbar V]$.

Vacuum properties:

We have
$$\langle 0|\, \vec{E}_\perp(\vec{r})\, |0\rangle = \langle 0|\, \vec{B}(\vec{r})\, |0\rangle = \langle 0|\, \vec{A}(\vec{r})\, |0\rangle = 0$$
 but since $\langle 0|\, a_j a_k^\dagger\, |0\rangle = \delta_{jk}$ we find the

Vacuum fluctuations:

$$\left(\Delta ec{E}_{\perp}
ight)^2 = ra{0} (ec{E}_{\perp}(ec{r}))^2 \ket{0} = \sum_j \mathcal{E}_j^2 = \sum_j rac{\hbar \omega_j}{2\epsilon_0 V}$$

$$\left(\Delta ec{B}_{\perp}
ight)^2 = ra{0} \left(ec{B}(ec{r})
ight)^2 \ket{0} = \sum_j rac{1}{c^2} \mathcal{E}_j^2 = \sum_j rac{\hbar \omega_j}{2\epsilon_0 c^2 V}$$

$$\left(\Delta ec{A}_{\perp}
ight)^2 = ra{0} (ec{A}_{\perp}(ec{r}))^2 \ket{0} = \sum_j rac{1}{\omega_j^2} \mathcal{E}_j^2 = \sum_j rac{\hbar}{2\epsilon_0 V \omega_j}$$

Field values for number states:

$$|n_1=0,\dots,n_{j-1}=0,n_j,n_{j+1}=0,\dots
angle\equiv |n_j
angle$$
 $\langle n_j|\,ec E_\perp(ec r)\,|n_j
angle=\langle n_j|\,ec B(ec r)\,|n_j
angle=0$ $c^2\Delta B^2=\Delta E_\perp^2=\langle n_j|\,ec E_\perp(ec r)^2\,|n_j
angle=(2n_j+1)\mathcal{E}_j^2$ So this is $(2n_j+1)$ times the ΔE_\perp^2 in vacuum.

Coherent States (Quasi-classical states, Glauber states)

I want a state in which the E-field and B-field are as close to a classical state as possible.

$$ec{E}_{
m classical} = \sum_{j} \mathcal{E}_{j} ec{\epsilon} \left(lpha_{j} e^{i ec{k}_{j} \cdot ec{r}} + {
m c.c.}
ight)$$

$$ec{E}_{ ext{quantum}} = \sum_{j} \mathcal{E}_{j} ec{\epsilon} \left(a_{j} e^{i ec{k}_{j} \cdot ec{r}} + ext{c.c.}
ight)$$

This leads us to define the coherent state $|\alpha\rangle$ as an eigenstate of the annihilation operator with eigenvalue α_i :

$$oxed{a_j\ket{lpha_j}=lpha_j\ket{lpha_j}}$$

(for one mode, say mode j - let's drop the index j in the following to unclutter).

Expand over number states:

$$\ket{lpha} = \sum_{n=0}^{\infty} c_n \ket{n}$$

$$egin{aligned} \ket{lpha} &= \sum_{n=1}^{\infty} c_n \sqrt{n} \ket{n-1} = \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} \ket{n} \ &\stackrel{!}{=} lpha \sum_{n=0}^{\infty} c_n \ket{n} \end{aligned}$$

$$\Rightarrow c_{n+1}\sqrt{n+1} = lpha c_n \qquad ext{or} \qquad c_n = rac{lpha}{\sqrt{n}} c_{n-1}$$

$$\Rightarrow c_n = rac{lpha^n}{\sqrt{n!}} c_0$$

$$\Rightarrow \ket{lpha} = c_0 \sum_{n=0}^{\infty} rac{lpha^n}{\sqrt{n!}} \ket{n}$$

Normalization:

$$\left\langle lpha |lpha
ight
angle =1=\leftert c_{0}
ightert ^{2}\sum_{m=0}^{\infty}rac{lpha ^{n}lpha ^{st m}}{\sqrt{n!m!}}\left\langle m|n
ight
angle =\leftert c_{0}
ightert ^{2}\sum_{n=0}^{\infty}rac{\leftert lpha
ightert ^{2n}}{n!}=\leftert c_{0}
ightert ^{2}e^{\leftert lpha
ightert ^{2}}$$

$$\Rightarrow c_0 = e^{-rac{|lpha|^2}{2}}$$

up to a phase factor. So we finally have found

$$\ket{lpha} = e^{-rac{|lpha|^2}{2}} \sum_{n=0}^{\infty} rac{lpha^n}{\sqrt{n!}} \ket{n}$$

We may immediately note that $|\alpha\rangle$ is *not* an eigenstate of the creation operator:

$$a^\dagger\ket{lpha}=e^{-rac{|lpha|^2}{2}}\sum_{n=0}^{\infty}rac{lpha^n}{\sqrt{n!}}\sqrt{n+1}\ket{n+1}=e^{-rac{|lpha|^2}{2}}\sum_{n=1}^{\infty}rac{nlpha^{n-1}}{\sqrt{n!}}\ket{n}$$

which is clearly not proportional to |lpha
angle. In fact, one sometimes works with an alternative normalization where $||lpha
angle=e^{\frac{|lpha|^2}{2}}|lpha
angle$, which allows one to right simply

$$a^{\dagger}\ket{\ket{lpha}}=rac{\partial}{\partiallpha}\ket{\ket{lpha}}$$

(simply from $rac{\partial}{\partial lpha} lpha^n = n lpha^{n-1}$)

• Time evolution:

Let's start at t=0 with $|\alpha\rangle$. At time t we find:

$$egin{aligned} \ket{\psi(t)} &= e^{-iHt/\hbar}\ket{lpha} \ &= e^{-rac{|lpha|^2}{2}} \sum_{n=0}^{\infty} rac{lpha^n}{\sqrt{n!}} e^{-(n+rac{1}{2})\omega t}\ket{n} \ &= e^{-irac{\omega t}{2}} e^{-rac{|lpha|^2}{2}} \sum_{n=0}^{\infty} rac{(lpha e^{-i\omega t})^n}{\sqrt{n!}}\ket{n} \ &= e^{-irac{\omega t}{2}}\ket{lpha e^{-i\omega t}} \end{aligned}$$

Probability of finding the value $(n+\frac{1}{2})\hbar\omega$ for the energy (or the value n for the photon number) is time independent:

$$P(n)=\leftert c_{n}
ightert ^{2}=e^{-\leftert lpha
ightert ^{2}}rac{\leftert lpha
ightert ^{2n}}{n!}$$

Mean photon number: $ar{n}=\left<\hat{N}\right>=\left<lpha\right|a^\dagger a\ket{lpha}=lpha^*lpha=\ket{lpha}^2$

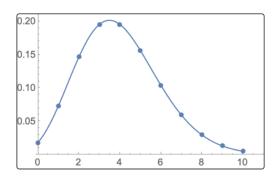
Equivalently we can calculate:

$$\left\langle \hat{N}
ight
angle = \sum_n n P(n) = \sum_n n e^{-|lpha|^2} rac{\left|lpha
ight|^{2n}}{n!} = \left|lpha
ight|^2 \sum_n e^{-|lpha|^2} rac{\left|lpha
ight|^{2n}}{n!} = \left|lpha
ight|^2$$

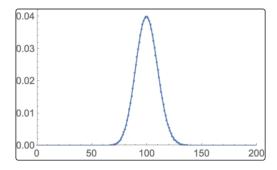
So
$$ar{n}=\leftert lpha
ightert ^{2}$$
 and

$$P(n) = e^{-ar{n}} rac{ar{n}^n}{n!}$$
 Poisson Distribution

$$P(n)$$
 for $\bar{n}=4$:



$$P(n)$$
 for $\bar{n}=100$



Standard deviation:

$$egin{aligned} \Delta n^2 &= \left< \Delta N
ight)^2 = \left< N^2 \right> - \left< N
ight>^2 = \left< lpha
ight| a a^\dagger a a^\dagger a \left| lpha \right> - \left| lpha
ight|^4 \ &= \left< lpha
ight|^4 + \left< N \right> - \left| lpha
ight|^4 \ &= \left< N \right> = ar{n} \end{aligned}$$

So $\Delta n^2 = ar{n}$ as it should be for the Poisson distribution.

ullet Electric field: Use $|\psi(t)
angle=e^{-irac{\omega t}{2}}\,|lpha e^{-i\omega t}
angle$ and find (again, we omit the index j):

$$ra{\psi(t)}ec{E}_{\perp}(ec{r})\ket{\psi(t)}=i\mathcal{E}ec{\epsilon}\left\{lpha e^{i(ec{k}\cdotec{r}-\omega t)}-lpha^*e^{-i(ec{k}\cdotec{r}-\omega t)}
ight\}$$

(which is just the classical field)

$$egin{aligned} ra{\psi(t)}ec{E}_{ot}(ec{r})^2\ket{\psi(t)} &= ra{\psi(t)}(\ldots a \cdots - a^\dagger \ldots)(\ldots a \cdots - \ldots a^\dagger \ldots)\ket{\psi(t)} \ &= \left(ra{\psi(t)}ec{E}_{ot}(ec{r})\ket{\psi(t)}
ight)^2 + \mathcal{E}^2 \end{aligned}$$

Therefore

$$\Rightarrow oldsymbol{ec{\Delta}ec{E}_{oldsymbol{\perp}}^2 = \mathcal{E}^2}$$

which is the same fluctuation as in the vacuum state!

Are coherent states orthonormal? Let's find out:

$$egin{aligned} \left| \left\langle eta | lpha
ight
angle
ight|^2 &= \left| e^{-rac{|lpha|^2}{2}} e^{-rac{|eta|^2}{2}} \sum_{m,n} rac{lpha^n eta^{*m}}{\sqrt{n!} \sqrt{m!}} \left\langle m | n
ight
angle
ight|^2 \ &= e^{-|lpha|^2} e^{-|eta|^2} \left| e^{lphaeta^*}
ight| \ &= e^{-|lpha-eta|^2} \end{aligned}$$

They are not orthogonal!

But they do obey the closure relation:

$$\frac{1}{\pi} \int d(\operatorname{Re} \alpha) d(\operatorname{Im} \alpha) |\alpha\rangle \langle \alpha| = 1$$

Equivalently we can write

$$\frac{1}{\pi} \int \mathrm{d}^2 \alpha \ket{lpha} ra{lpha} = 1$$

Proof:

$$rac{1}{\pi}\int \mathrm{d}^2lpha \ket{lpha}ra{lpha}=rac{1}{\pi}\sum_{m,n}rac{\ket{m}ra{n}}{\sqrt{m!n!}}\int \mathrm{d}^2lpha\,e^{-|lpha|^2}lpha^m(lpha^*)^n$$

$$\int \mathrm{d}^2 lpha \, e^{-|lpha|^2} lpha^m (lpha^*)^n = \int_0^\infty \mathrm{d} r \, e^{-r^2} r^{m+n+1} \int_0^{2\pi} e^{i(m-n) heta} \mathrm{d} heta \ = 2\pi \int_0^\infty \mathrm{d} r \, e^{-r^2} r^{m+n+1} \delta_{mn} \ = 2\pi \delta_{mn} \int_0^\infty \mathrm{d} r \, e^{-r^2} r^{2n+1} \ = 2\pi \delta_{mn} rac{1}{2} \int_0^\infty \mathrm{d} u \, e^{-u} u^n \ = \pi \, n! \, \delta_{mn}$$

$$rac{1}{\pi}\int\mathrm{d}^2lpha\ket{lpha}ra{lpha}=rac{1}{\pi}\sum_{m\,n}rac{\ket{m}ra{n!}}{\sqrt{m!n!}}\pi\,n!\,\delta_{mn}=\sum_n\ket{n}ra{n!}\,ra{n!}=1$$

Application to number state:

$$|n
angle = rac{1}{\pi} \int \mathrm{d}^2 lpha \ket{lpha} ra{lpha} = rac{1}{\pi} \int \mathrm{d}^2 lpha \, e^{-rac{|lpha|^2}{2}} rac{(lpha^*)^n}{\sqrt{n!}} \ket{lpha}$$

The weight of the state $|\alpha\rangle$ can be seen to be peaked at $|\alpha|=\sqrt{n}$, which is independent of the argument of α .

Application to a coherent state:

$$\ket{eta} = rac{1}{\pi} \int \mathrm{d}^2 lpha \ket{lpha} ra{lpha} = rac{1}{\pi} \int \mathrm{d}^2 lpha \, e^{-rac{1}{2} \left(|lpha|^2 + |eta|^2
ight) + lpha^* eta} \ket{lpha}$$

So a coherent state can be written as a superposition of coherent states! The basis is said to be overcomplete. Still, we have

$$\langle eta | eta
angle = rac{1}{\pi} \int \mathrm{d}^2 lpha \, e^{-|lpha - eta|^2} = 1$$

as it should be.

Decomposition: $|\psi\rangle=\int \mathrm{d}^2 \alpha\, c_{lpha}\, |lpha
angle$ with $c_{lpha}=rac{1}{\pi}\, \langlelpha|\psi
angle$

This decomposition is not unique as

$$\langle lpha |lpha'
angle
eq 0$$
, but $\left|\langle lpha |lpha'
angle
ight|^2=e^{-\left|lpha-lpha'
ight|^2} o 0$

if α and α' are widely separated.

• We say that the basis of coherent states $|\alpha\rangle$ is overcomplete.

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