PY 711 Fall 2010
Homework 11: Due Tuesday, November 16

1. (15 points) Consider a Lagrange density involving a real scalar field ϕ_A, another real scalar field ϕ_B, a spin-1/2 fermion field ψ, and a Yukawa coupling with interaction coefficient g between ψ and each scalar field. We assume that the scattering process is at sufficiently high energies that all particles can be considered massless. Written out explicitly, the Lagrange density is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_A)(\partial^\mu \phi_A) + \frac{1}{2}(\partial_\mu \phi_B)(\partial^\mu \phi_B) + i\bar{\psi}\gamma^\mu \partial_\mu \psi - g\bar{\psi}\psi\phi_A - g\bar{\psi}\psi\phi_B. \tag{1}$$

In the center-of-mass frame we consider unpolarized scattering of a ψ particle and $\bar{\psi}$ antiparticle which produces a ϕ_A particle and ϕ_B particle,

$$\psi + \bar{\psi} \to \phi_A + \phi_B. \tag{2}$$

Recall that unpolarized scattering means we average over all possible initial spin configurations. To lowest non-vanishing order in g, determine the differential cross section $\frac{d\sigma}{d\Omega}$ in the center-of-mass frame. Let $\vec{p}$ and $-\vec{p}$ be the incoming momenta for the ψ particle and $\bar{\psi}$ antiparticle respectively. Let $\vec{p}\,'$ and $-\vec{p}\,'$ be the outgoing momenta for the ϕ_A and ϕ_B particles respectively. Determine the differential cross section $\frac{d\sigma}{d\Omega}$ as a function of g, $|\vec{p}|$, and θ, the angle between $\vec{p}$ and $\vec{p}\,'$. You should simplify your final answer as much as possible.

KELLY PATTON
HW #11

15/15

1. CONSIDER A LAGRANGE DENSITY INVOLVING A REAL SCALAR FIELD ϕ_A, ANOTHER REAL SCALAR FIELD ϕ_B, A SPIN-1/2 FERMION FIELD ψ, AND A YUKAWA COUPLING WITH INTERACTION COEFFICIENT g BETWEEN ψ AND EACH SCALAR FIELD. WE ASSUME THAT THE SCATTERING PROCESS IS AT SUFFICIENTLY HIGH ENERGIES THAT ALL PARTICLES CAN BE CONSIDERED MASSLESS. WRITTEN OUT EXPLICITLY, THE LAGRANGE DENSITY IS

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_A)(\partial^\mu \phi_A) + \frac{1}{2}(\partial_\mu \phi_B)(\partial^\mu \phi_B) + i\bar{\psi}\gamma^\mu \partial_\mu \psi$$

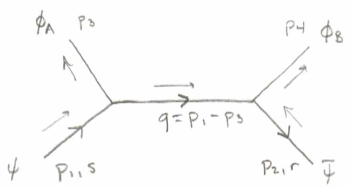
$$- g\bar{\psi}\psi\phi_A - g\bar{\psi}\psi\phi_B .$$

IN THE CENTER-OF-MASS FRAME WE CONSIDER UNPOLARIZED SCATTERING OF A ψ PARTICLE AND $\bar{\psi}$ ANTIPARTICLE WHICH PRODUCES A ϕ_A PARTICLE AND ϕ_B PARTICLE,

$$\psi + \bar{\psi} \rightarrow \phi_A + \phi_B .$$

RECALL THAT UNPOLARIZED SCATTERING MEANS WE AVERAGE OVER ALL POSSIBLE INITIAL SPIN CONFIGURATIONS. TO LOWEST NON-VANISHING ORDER IN g, DETERMINE THE DIFFERENTIAL CROSS SECTION $\frac{d\sigma}{d\Omega}$ IN THE CENTER-OF-MASS FRAME. LET $\vec{p}$ AND $-\vec{p}$ BE THE INCOMING MOMENTA FOR THE ψ PARTICLE AND $\bar{\psi}$ ANTIPARTICLE RESPECTIVELY. LET $\vec{p}\,'$ AND $-\vec{p}\,'$ BE THE OUTGOING MOMENTA FOR THE ϕ_A AND ϕ_B PARTICLES, RESPECTIVELY. DETERMINE THE DIFFERENTIAL CROSS SECTION $\frac{d\sigma}{d\Omega}$ AS A FUNCTION OF g, $|\vec{p}|$, AND θ, THE ANGLE BETWEEN $\vec{p}$ AND $\vec{p}\,'$. YOU SHOULD SIMPLIFY YOUR ANSWER AS MUCH AS POSSIBLE.

There are two diagrams to consider



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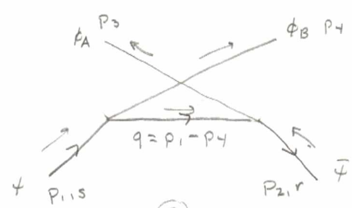


②

I'm calling the momenta p_1, p_2, p_3, and p_4 for now. I will define them later in the calculation.

The amplitude $\mathcal{M}$ for this scattering process will be

$$\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$$

1 CONTINUED

Using the Feynman rules for Yukawa theory, we see

$$i\mathcal{M}_1 = (-ig)^2\, \bar{v}^r(p_2) \left(\frac{i(\not{p}_1 - \not{p}_3)}{(p_1-p_3)^2} \right) u^s(p_1)$$

We can simplify this using the Dirac equation for a massless particle

$$\not{p}\, u^s(p) = 0$$

$$i\mathcal{M}_1 = \frac{+ig^2}{(p_1-p_3)^2}\, \bar{v}^r(p_2)\, \not{p}_3\, u^s(p_1)$$ ✓

Similarly for $\mathcal{M}_2$,

$$i\mathcal{M}_2 = (-ig)^2\, \bar{v}^r(p_2) \left(\frac{i(\not{p}_1 - \not{p}_4)}{(p_1-p_4)^2} \right) u^s(p_1)$$

$$= \frac{+ig^2}{(p_1-p_4)^2}\, \bar{v}^r(p_2)\, \not{p}_4\, u^s(p_1)$$ ✓

So $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$ is

$$\mathcal{M} = g^2\, \bar{v}^r(p_2) \left(\frac{\not{p}_3}{(p_1-p_3)^2} + \frac{\not{p}_4}{(p_1-p_4)^2} \right) u^s(p_1)$$ ✓

There are four possible initial spin states.

$$\frac{1}{4}\sum_{r,s} |\mathcal{M}|^2 = \frac{g^4}{4}\sum_{r,s} \mathrm{Tr}\left(\bar{v}^r(p_2) \left(\frac{\not{p}_3}{(p_1-p_3)^2} + \frac{\not{p}_4}{(p_1-p_4)^2} \right) u^s(p_1)\, \bar{u}^s(p_1) \left(\frac{\not{p}_3}{(p_1-p_3)^2} + \frac{\not{p}_4}{(p_1-p_4)^2} \right) v^r(p_2) \right)$$

$$= \frac{g^4}{4}\mathrm{Tr}\left(\left(\frac{\not{p}_3}{(p_1-p_3)^2} + \frac{\not{p}_4}{(p_1-p_4)^2} \right) \not{p}_1 \left(\frac{\not{p}_3}{(p_1-p_3)^2} + \frac{\not{p}_4}{(p_1-p_4)^2} \right) \not{p}_2 \right)$$

In the last step, I used the cyclic property of the trace and the fact that

$$\sum_s u^s(p)\,\bar{u}^s(p) = \not{p} \qquad \sum_s v^s(p)\,\bar{v}^s(p) = \not{p}$$

1 CONTINUED

$$\frac{1}{4}\sum_{s,r}|\mathcal{M}|^2 = \frac{g^4}{4}\,\mathrm{Tr}\left(\frac{1}{((p_1-p_3)^2)^2}\left(\not{p}_3\not{p}_1\not{p}_3\not{p}_2\right) + \frac{1}{(p_1-p_3)^2(p_1-p_4)^2}\left(\not{p}_3\not{p}_1\not{p}_4\not{p}_2 + \not{p}_4\not{p}_1\not{p}_3\not{p}_2\right)\right.$$

$$\left. + \frac{1}{((p_1-p_4)^2)^2}\left(\not{p}_4\not{p}_1\not{p}_4\not{p}_2\right)\right)$$

We know that

$$\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\right) = 4\left(g^{\mu\nu}g^{\sigma\rho} - g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma}\right)$$

So

$$\mathrm{Tr}\left(\not{a}\not{b}\not{c}\not{d}\right) = \mathrm{Tr}\left(a_\alpha\gamma^\alpha b_\beta\gamma^\beta c_\delta\gamma^\delta d_\rho\gamma^\rho\right)$$

$$= a_\alpha b_\beta c_\delta d_\rho\left(4\left(g^{\alpha\beta}g^{\delta\rho} - g^{\alpha\delta}g^{\beta\rho} + g^{\alpha\rho}g^{\beta\delta}\right)\right)$$

$$= 4\left(a\cdot b\; c\cdot d - a\cdot c\; b\cdot d + a\cdot d\; b\cdot c\right)$$

This means that

$$\frac{1}{4}\sum_{s,r}|\mathcal{M}|^2 = g^4\left(\frac{1}{((p_1-p_3)^2)^2}\left(p_3\cdot p_1\; p_3\cdot p_2 - p_3\cdot p_3\; p_1\cdot p_2 + p_3\cdot p_2\; p_3\cdot p_1\right)\right.$$

$$+ \frac{1}{(p_1-p_3)^2(p_1-p_4)^2}\left(p_3\cdot p_1\; p_4\cdot p_2 - p_3\cdot p_4\; p_1\cdot p_2 + p_3\cdot p_2\; p_1\cdot p_4\right.$$

$$\left. + p_4\cdot p_1\; p_3\cdot p_2 - p_4\cdot p_3\; p_1\cdot p_2 + p_4\cdot p_2\; p_1\cdot p_3\right)$$

$$\left. + \frac{1}{((p_1-p_4)^2)^2}\left(p_4\cdot p_1\; p_4\cdot p_2 - p_4\cdot p_4\; p_1\cdot p_2 + p_4\cdot p_2\; p_4\cdot p_1\right)\right)$$

At this point, we should define p_1, p_2, p_3 and p_4.

good work

1 CONTINUED

$p_1 = (E_\psi, \vec{p})$ $\qquad$ $p_3 = (E_A, \vec{p}\,')$

$p_2 = (E_{\bar\psi}, -\vec{p})$ $\qquad$ $p_4 = (E_B, -\vec{p}\,')$

Since these are massless particles, $E_\psi = |\vec{p}| = E_{\bar\psi}$, $E_A = |\vec{p}\,'| = E_B$

Because we are in center of mass frame, and the particles are massless,

$$|\vec{p}| = |\vec{p}\,'|$$

$p_1 = (|\vec{p}|, \vec{p})$ $\qquad$ $p_3 = (|\vec{p}|, \vec{p}\,')$

$p_2 = (|\vec{p}|, -\vec{p})$ $\qquad$ $p_4 = (|\vec{p}|, -\vec{p}\,')$

$(p_1 - p_3) = (0, \vec{p} - \vec{p}\,')$

$$(p_1 - p_3)^2 = -\left(|\vec{p}|^2 + |\vec{p}\,'|^2 - 2|\vec{p}||\vec{p}\,'|\cos(\theta)\right) = -2|\vec{p}|^2(1 - \cos(\theta))$$

$(p_1 - p_4) = (0, \vec{p} + \vec{p}\,')$

$$(p_1 - p_4)^2 = -\left(|\vec{p}|^2 + |\vec{p}\,'|^2 + 2|\vec{p}||\vec{p}\,'|\cos(\theta)\right) = -2|\vec{p}|^2(1 + \cos(\theta))$$

$p_1 \cdot p_2 = |\vec{p}|^2 + |\vec{p}|^2 = 2|\vec{p}|^2 = p_3 \cdot p_4$

$p_1 \cdot p_3 = |\vec{p}|^2 - \vec{p}\cdot\vec{p}\,' = |\vec{p}|^2(1 - \cos(\theta)) = p_2 \cdot p_4$

$p_1 \cdot p_4 = |\vec{p}|^2 + \vec{p}\cdot\vec{p}\,' = |\vec{p}|^2(1 + \cos(\theta)) = p_2 \cdot p_3$

$p_3 \cdot p_3 = |\vec{p}|^2 - |\vec{p}\,'|^2 = 0$ $\qquad$ $p_4 \cdot p_4 = |\vec{p}|^2 - |\vec{p}\,'|^2 = 0$ ✓

Now, use these definitions in our equation for $\frac{1}{4}\sum_{s,r} |M|^2$.

1 CONTINUED

$$\frac{1}{4}\sum_{s,r}|\mathcal{M}|^2 = g^4\Big(\frac{1}{4|\vec{p}|^4(1-\cos(\theta))^2}\left(2|\vec{p}|^4(1-\cos(\theta))(1+\cos(\theta)) - 0\right)$$

$$+\frac{2}{4|\vec{p}|^4(1-\cos(\theta))(1+\cos(\theta))}\left(|\vec{p}|^4(1-\cos(\theta))^2 + |\vec{p}|^4(1+\cos(\theta))^2 - 4|\vec{p}|^4\right)$$

$$+\frac{1}{4|\vec{p}|^4(1+\cos(\theta))^2}\left(2|\vec{p}|^4(1-\cos(\theta))(1+\cos(\theta)) - 0\right)\Big)$$

$$= \frac{g^4}{4}\left(2\left(\frac{1+\cos(\theta)}{1-\cos(\theta)}\right) + 2\left(\frac{1-\cos(\theta)}{1+\cos(\theta)}\right) + \frac{2}{\sin^2(\theta)}\left(-2\sin^2(\theta)\right)\right)$$

$$= \frac{g^4}{4}\left(8\cot^2(\theta)\right) \qquad \text{(simplified in Mathematica)}$$

$$\frac{1}{4}\sum_{s,r}|\mathcal{M}|^2 = 2g^4\cot^2(\theta)$$

We know from class that

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{p_{final}}{(2E_\psi)(2E_{\bar\psi})|\vec{v}_\psi - \vec{v}_{\bar\psi}|}\frac{1}{16\pi^2 E_{CM}}|\mathcal{M}|^2$$

$$p_{final} = |\vec{p}\,'| = |\vec{p}|$$

$$E_\psi = E_{\bar\psi} = |\vec{p}|$$

$$|\vec{v}_\psi - \vec{v}_{\bar\psi}| = 2|\vec{v}_\psi| = \frac{2|\vec{p}|}{E_\psi} = 2$$

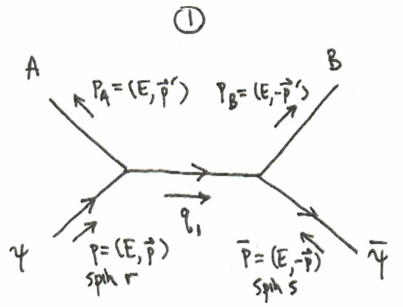
$$E_{CM} = E_\psi + E_{\bar\psi} = 2|\vec{p}|$$

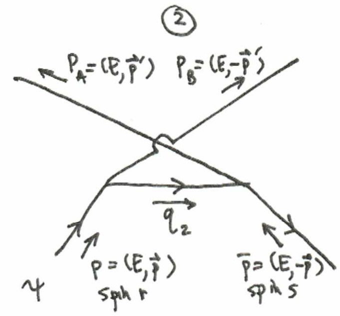
$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2}\frac{|\vec{p}|}{(2|\vec{p}|)(|\vec{p}|^2)(2)}\left(\cancel{2}g^4\cot^2(\theta)\right)$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{\underset{128}{\cancel{132}}\pi^2}\frac{g^4\cot^2(\theta)}{|\vec{p}|^2}}$$

Solutions #11 PY 711

1.





$$q_1 = p - p_A = p_B - \bar p \qquad\qquad q_2 = p - p_B = p_A - \bar p$$

$$i\mathcal{M}_1 = -ig^2\, \bar v^s(\bar p)\, \not q_1\, u^r(p)\, \frac{1}{q_1^2} \qquad\qquad i\mathcal{M}_2 = -ig^2\, \bar v^s(\bar p)\, \not q_2\, u^r(p)\, \frac{1}{q_2^2}$$

$$i\mathcal{M} = i\mathcal{M}_1 + i\mathcal{M}_2 = -ig^2\, \bar v^s(\bar p)\left[\frac{\not q_1}{q_1^2} + \frac{\not q_2}{q_2^2}\right] u^r(p)$$

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = \frac{g^4}{4}\sum_{r,s} \mathrm{Tr}\left\{ v^s(\bar p)\bar v^s(\bar p)\left[\frac{\not q_1}{q_1^2} + \frac{\not q_2}{q_2^2}\right] u^r(p)\bar u^r(p)\left[\frac{\not q_1}{q_1^2} + \frac{\not q_2}{q_2^2}\right]\right\}$$

$$= \frac{g^4}{4}\,\mathrm{Tr}\left\{ \not{\bar p}\left[\frac{\not q_1}{q_1^2} + \frac{\not q_2}{q_2^2}\right] \not p \left[\frac{\not q_1}{q_1^2} + \frac{\not q_2}{q_2^2}\right]\right\}$$

$$= g^4\left\{ 2\left[\bar p\cdot\left(\frac{q_1}{q_1^2} + \frac{q_2}{q_2^2}\right)\right]\left[p\cdot\left(\frac{q_1}{q_1^2} + \frac{q_2}{q_2^2}\right)\right] - (\bar p\cdot p)\left[\left(\frac{q_1}{q_1^2} + \frac{q_2}{q_2^2}\right)\cdot\left(\frac{q_1}{q_1^2} + \frac{q_2}{q_2^2}\right)\right]\right\}$$

We use the following relations: $E = |\vec p| = |\vec p\,'|$, $\vec p\cdot\vec p\,' = |\vec p|^2\cos\theta = E^2\cos\theta$

$$q_1^2 = -2E^2(1-\cos\theta), \quad q_2^2 = -2E^2(1+\cos\theta)$$
$$\bar p\cdot q_1 = E^2(1-\cos\theta), \quad \bar p\cdot q_2 = E^2(1+\cos\theta)$$
$$p\cdot q_1 = -E^2(1-\cos\theta), \quad p\cdot q_2 = -E^2(1+\cos\theta)$$
$$q_1\cdot q_2 = 0, \quad \bar p\cdot p = 2E^2$$

we find
$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 = g^4\left\{ 2\left[-\tfrac{1}{2} - \tfrac{1}{2}\right]\cdot\left[\tfrac{1}{2} + \tfrac{1}{2}\right] - 2E^2\left(\frac{1}{q_1^2} + \frac{1}{q_2^2}\right)\right\}$$
$$= g^4\left(-2 + \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\right) = 2g^4\,\frac{\cos^2\theta}{\sin^2\theta}$$

The unpolarized differential cross section is

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{|\vec{p}\,'|}{2E_\psi 2E_{\bar\psi} |\vec{v}_\psi - \vec{v}_{\bar\psi}| 16\pi^2 E_{CM}} \cdot \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2$$

Since $E_{CM} = 2E$ and $|\vec{v}_\psi - \vec{v}_{\bar\psi}| = 2$, we have

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{E}{(2E)(2E)\,2\,(16\pi^2)(2E)} \cdot 2g^4 \frac{\cos^2\theta}{\sin^2\theta}$$

$$= \frac{g^4}{128\pi^2 E^2} \frac{\cos^2\theta}{\sin^2\theta}$$