# 8.512 Recitation 6

- Pset 3 due 03/14 at 5pm, on Canvas

**Today:** 1) Gap eq. at finite T ← (from last time)
2) Critical temperature $T_c$ and field $H_{c2}$
3) SC density of states
4) SC heat capacity

**Ref:** Tinkham Sec. 3.6–3.7

3) Close to the Fermi surface, SC density of states can be approximated as

$$g_s(E) = \frac{1}{N}\sum_k \delta(E-E_k) = \frac{1}{N}\sum_k \delta\left(E-\sqrt{\varepsilon_k^2+\Delta_0^2}\right)$$

$$= \int_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon)\, \delta\left(E-\sqrt{\varepsilon^2+\Delta_0^2}\right) \approx g(\varepsilon_F)\int_{-\infty}^{\infty} d\varepsilon\, \delta\left(E-\sqrt{\varepsilon^2+\Delta_0^2}\right)$$

assuming $g(\varepsilon) \approx g(\varepsilon_F) = \text{const}$ for the normal state close to Fermi surface.

$$\int_{-\infty}^{\infty} dx\, \delta(g(x)) = \sum_i \frac{1}{|g'(x_i)|} \quad \text{where} \quad g(x_i)=0$$

$$\int_{-\infty}^{\infty} d\varepsilon\, \delta\left(E-\sqrt{\varepsilon^2+\Delta_0^2}\right) = \int_{-\infty}^{\infty} d\varepsilon \left[\frac{\delta\left(\varepsilon+\sqrt{E^2-\Delta_0^2}\right)}{\frac{|\varepsilon|}{\sqrt{\varepsilon^2+\Delta_0^2}}} + \frac{\delta\left(\varepsilon-\sqrt{E^2-\Delta_0^2}\right)}{\frac{\varepsilon}{\sqrt{\varepsilon^2+\Delta_0^2}}}\right]$$

$$\Rightarrow \boxed{g_s(E) = \begin{cases} g(\varepsilon_F)\cdot \dfrac{2E}{\sqrt{E^2-\Delta_0^2}} & \text{for } E>\Delta_0 \\ 0 & \text{for } E<\Delta_0 \end{cases}}$$

As $\Delta_0 \to 0$, $E \to |\varepsilon|$ contains two branches of particle-hole excitations, thus doubling the density of states!

4) Recall that $H_{BCS} = \sum_{k,\sigma} E_k \gamma^{+}_{k\sigma}\gamma_{k\sigma} + E_{BCS}$

internal energy $\Rightarrow$ $U = \langle H_{BCS}\rangle = 2\sum_k E_k n_k + E_{BCS}$

where $n_k \equiv n_F(E_k) = \dfrac{1}{1+e^{\beta E_k}}$

- $C = \dfrac{dU}{dT}$ but $E_{BCS}$ depends on $T$ through $\langle c_{-k\downarrow} c_{k\uparrow}\rangle$

$\Rightarrow$ Better to consider entropy $S$; no contribution from condensate

$$C = T\frac{dS}{dT} = -\beta\frac{dS}{d\beta}$$

- Partition function for Fermionic system

$$Z = \prod_{k,\sigma}\left(1+e^{-\beta E_k}\right) \qquad (E_k \text{ is relative to } \mu)$$

$$\Omega = -k_B T \ln Z$$

$$S = -\frac{\partial\Omega}{\partial T} = -k_B \sum_{k,\sigma}\left[n_k \ln n_k + (1-n_k)\ln(1-n_k)\right]$$

$$= -2k_B \sum_k \left[n_k \ln n_k + (1-n_k)\ln(1-n_k)\right]$$

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$$\Rightarrow C = -\beta \frac{dS}{d\beta} = 2\beta k_B \sum_k \left(\frac{dn_k}{d\beta}\right)\left[-\ln(1-n_k) - 1 + \ln n_k + 1\right]$$

$$= 2\beta k_B \sum_k \left(\frac{dn_k}{d\beta}\right) \ln\left(\frac{n_k}{1-n_k}\right) = -2\beta^2 k_B \sum_k E_k \left(\frac{dn_k}{d\beta}\right)$$

$$\frac{dn_k}{d\beta} = \frac{\partial n_k}{\partial \beta} + \frac{\partial n_k}{\partial E_k} \cdot \frac{\partial E_k}{\partial \beta} = \frac{\partial n_k}{\partial E_k}\left( \frac{E_k}{\beta} + \frac{1}{2E_k} \cdot \frac{d\Delta_k^2}{d\beta} \right)$$

$$\frac{\partial n_k}{\partial E_k} = -\frac{\beta e^{\beta E_k}}{(e^{\beta E_k}+1)^2} = -\beta n_k (1-n_k)$$

$$\Rightarrow \boxed{C = 2k_B \beta^2 \sum_k n_k(1-n_k)\left(E_k^2 + \frac{\beta}{2}\frac{d\Delta_k^2}{d\beta}\right)}$$

- For $T > T_c$, $\Delta_k = 0$ (normal metal)
  $E_k = \varepsilon_k$

  $2\sum_k \to \int_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon)$ and assume $g(\varepsilon) = g(\varepsilon_F) = \text{const}$

$$C = 2k_B\beta^2 \sum_k \frac{\varepsilon_k^2 e^{\beta\varepsilon_k}}{(e^{\beta\varepsilon_k}+1)^2} = k_B\beta^2 \int_{-\infty}^{\infty} d\varepsilon\, g(\varepsilon) \frac{\varepsilon^2 e^{\beta\varepsilon}}{(e^{\beta\varepsilon}+1)^2}$$

$$\overset{x=\beta\varepsilon}{\approx} \frac{k_B}{\beta} g(\varepsilon_F) \int_{-\infty}^{\infty} dx \frac{x^2 e^x}{(e^x+1)^2} = \boxed{\frac{\pi^2}{3} k_B^2 T g(\varepsilon_F)}$$

Same as Sommerfeld expansion!

- At $T = T_c$ there is a discontinuity ("jump") in $C$ characteristic of $2^{nd}$ order transitions

Close to $T_c$:

$$\Delta(T) = \begin{cases} 3.06\, k_B T_c \sqrt{1 - \frac{T}{T_c}}, & T < T_c \\ 0, & T > T_c \end{cases}$$

so $\frac{\beta}{2}\frac{d\Delta^2}{d\beta} = -\frac{T}{2}\frac{d\Delta^2}{dT}$ has a discontinuity of $4.68\,(k_B T_c)^2$ at $T = T_c$

$$\Rightarrow \Delta C(T_c) = k_B \beta_c^2 \int_{-\infty}^{\infty} d\varepsilon\; g(\varepsilon) \cdot \frac{e^{\beta_c \varepsilon}}{(e^{\beta_c \varepsilon}+1)^2} \cdot 4.68\,(k_B T_c)^2$$

$$= 4.68\, k_B^2 T_c\, g(\varepsilon_F) \int_{-\infty}^{\infty} dx\, \frac{e^x}{(e^x+1)^2} = \boxed{4.68\, k_B^2 T_c\, g(\varepsilon_F)}$$

$$\frac{\Delta C(T_c)}{C(T_c)} = \frac{4.68}{\pi^2/3} = 1.42 \quad \Leftrightarrow \quad 142\% \text{ jump}$$

- For $T \ll T_c$: $\quad C = 2 k_B \beta^2 \sum_k \frac{E_k^2 e^{\beta E_k}}{(e^{\beta E_k}+1)^2} \simeq 2 k_B \beta^2 \sum_k E_k^2 e^{-\beta E_k}$

  $\Delta \approx \Delta_0 \approx \text{const}$

Naively, sum is dominated by the term
$$\min E_k = \Delta_0$$

However, there are infinitely many such terms!
We showed that $g(\Delta_0) \to \infty$

- Need to perform the integral more carefully

$$C = k_B \beta^2 \int dE\, g_s(E)\, E^2 e^{-\beta E} = 2 k_B \beta^2 g(\varepsilon_F) \int_{\Delta_0}^{\infty} dE\, \frac{E^3 e^{-\beta E}}{\sqrt{E^2 - \Delta_0^2}}$$

Recall that $g_s(E)$ was computed close to $\Delta_0$

$$\Delta_0 \le E \le \sqrt{(\hbar\omega_D)^2 + \Delta_0^2}$$

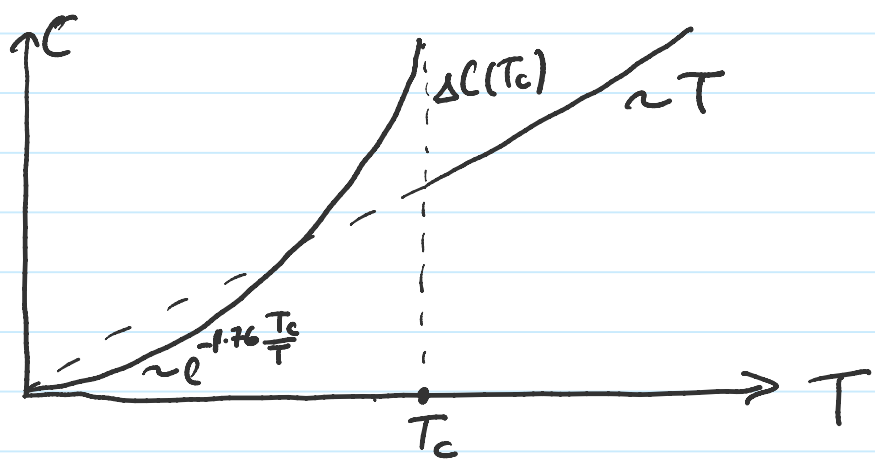
$$\Delta_0 = 2\hbar\omega_D\, e^{-\frac{1}{U_0 g(\varepsilon_F)}} \ll \hbar\omega_D$$

Approximate and upper bound the integral

$$C \simeq 2 k_B \beta^2 g(\varepsilon_F) \int_{\Delta_0}^{\sqrt{\Delta_0^2 + (\hbar\omega_D)^2}} dE \, \frac{E^3 e^{-\beta E}}{\sqrt{E^2 - \Delta_0^2}} \leq 2 k_B \beta^2 g(\varepsilon_F) e^{-\beta\Delta_0} \int_{\Delta_0}^{\sqrt{\Delta_0^2 + (\hbar\omega_D)^2}} dE \, \frac{E^3}{\sqrt{E^2 - \Delta_0^2}}$$

$$= 2 k_B \beta^2 g(\varepsilon_F) e^{-\beta\Delta_0} \cdot \frac{1}{3} \hbar\omega_D \left(3\Delta_0^2 + (\hbar\omega_D)^2\right)$$

$$\simeq \frac{2}{3} k_B \beta^2 (\hbar\omega_D)^3 g(\varepsilon_F) e^{-\beta\Delta_0} \sim \beta^2 e^{-\beta\Delta_0}$$

$\Rightarrow C$ is exp. small at low-temperatures

This is generic for systems w/ energy gap!