MA439: Functional Analysis
Tychonoff Spaces: Exercises 1-6 on p.36, Ben Mathes

Huan Q. Bui

Due: Wed, Oct 7, 2020

Exercise 1 (Ex 1, p.36)**.** *Let $\mathcal{X}$ be a topological space. Prove that if d is a continuous pseudometric, then the sets $\{y \in \mathcal{X} : d(x,y) > \delta\}$ are open, where $x \in \mathcal{X}$ and $\delta \in \mathbb{R}$.*

Proof. Let $O = \{y \in \mathcal{X} : d(x,y) > \delta\}$. We want to show that each $y \in O$ is an interior point of O. Let $y \in O$ be given, then $d(x,y) > \delta$. This means that $d(x,y) \geq \delta + \epsilon$ for some $\epsilon > 0$. d is a continuous pseudometric, so every d-ball is an open subset of $\mathcal{X}$. In particular, $B_d(y, \epsilon/2)$ is an open subset of $\mathcal{X}$. By the triangle inequality, for any $z \in B_d(y, \epsilon/2)$, $z \in O$. Thus, $B_d(y, \epsilon/2) \subseteq O$. So, O is open as desired. $\square$

Exercise 2 (Ex 2, p.36)**.** *Let $\mathcal{X}$ be a topological space. Prove that d is a continuous pseudometric on $\mathcal{X}$ if and only if the function $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$.*

Proof. ($\implies$) Suppose that d is a continuous pseudometric on $\mathcal{X}$. Let $\epsilon > 0$ and $x \in \mathcal{X}$. f_x^d is continuous at $y \in \mathcal{X}$ if and only if for every $\epsilon > 0 \exists f(y) \in G \subseteq \mathcal{X}$ open for which $\left|f_x^d(y) - f_x^d(y')\right| < \epsilon$ whenever $y' \in G$. Note that $\left|f_x^d(y) - f_x^d(y')\right| = |d(x,y) - d(x,y')| \leq d(y,y')$. So, we just take $G = B_d(y, \epsilon)$.

($\impliedby$) Let d be a pseudometric and suppose that $f_x^d = d(x, \cdot)$ is continuous for every $x \in \mathcal{X}$. We want to show that every d-ball is open in $\mathcal{X}$. To this end, let $x \in \mathcal{X}$ and $\delta > 0$ be given and consider $B_d(x, \delta) = \{y \in \mathcal{X} : d(x,y) < \delta\} = \{y \in \mathcal{X} : f_x^d(y) < \delta\} = \{y \in \mathcal{X} : f_x^d(y) \in (-\delta, \delta)\}$ which is open by continuity of f_x^d. So we're done. $\square$

Exercise 3 (Ex 3, p.36)**.** *Let $\mathcal{X}$ be a Tychonoff space whose topology is generated by the family of pseudometrics $\mathcal{G}$. Prove that the topology on $\mathcal{X}$ is the same as the weak topology induced by the family of functions f_x^d where $x \in \mathcal{X}$, $d \in \mathcal{G}$.*

Proof. One inclusion is trivial. It remains to show the other inclusion. A topological space is Tychonoff means that for every closed set $F \subseteq \mathcal{X}$ and every $x \in F$, there exists a continuous function $f : \mathcal{X} \to \mathbb{R}$ for which $f[F] = \{0\}$ and $f(x) = 1$. From $\mathcal{G}$, we use open balls as a subbase and build the topology from those balls. Alternatively, we can build the functions $\{f_x^d : x \in \mathcal{X}, d \in \mathcal{G}\}$ and build the (open-ball) topology by taking inverse images of open sets. From the previous exercise, we have that the weak topology $\implies$ f_x^d are all continuous, which implies that all balls are open relative to the weak topology, which implies that the new (open-ball) topology is contained in the weak topology. Since the weak topology is by definition *weak*, this open-ball topology must be the weak topology itself. $\square$

Exercise 4 (Ex 4, p.36)**.** *Assume $\mathcal{X}$ is a Tychonoff space with generating family $\mathcal{G}$. If E is a subset of $\mathcal{X}$, let $\mathcal{G}_E$ denote the set of restrictions of elements of $\mathcal{G}$ to E. Prove that the resulting Tychonoff Topology on E generated by the family $\mathcal{G}_E$ is the same as the topological* ***subspace topology*** *that E inherits from the topology on $\mathcal{X}$.*

Proof. (Ideas) Get base from finite intersection of balls. G open iff for every $x \in G$ there exist finitely many $d_1, \dots, d_k \in \mathcal{G}$ and $\epsilon_1, \dots, \epsilon_k > 0$ such that $\cap_{i=1}^k B_{d_i}(x, \epsilon_i) \subseteq G$. Try: Let τ denote the topology on $\mathcal{X}$. The subspace topology on E is given by $\tau_E = \{E \cap U : U \in \tau\}$. $\boxed{?}$ □

Exercise 5 (Ex 5, p.36)**.** *Give an example of a continuous pseudometric on* $(0, 1)$ *that is not the restriction of a continuous pseudometric on* $\mathbb{R}$ *to* $(0, 1)$*.*

Proof. Consider the continuous function $f(x) = 1/x$ defined on $(0, 1)$. This function induces a continuous pseudometric $d(x, y) = |f(x) - f(y)| = |1/x - 1/y|$ on $(0, 1)$ since d-balls are open. Now, this cannot be a restriction of a continuous pseudometric on $\mathbb{R}$ to $(0, 1)$ because $d(x, y)$ is undefined when x or $y = 0$. □

Exercise 6 (Ex 6, p.36)**.** *Prove that a bounded continuous pseudometric on* $(0, 1)$ *is the restriction of a continuous pseudometric on* $\mathbb{R}$ *to* $(0, 1)$*. (?CHECK?)*

Proof. Ben said he found a counter-example to this? □

Exercise 7 (Ex 7, p.36)**.** *If* d_1 *and* d_2 *are continuous relative to a topology on* $\mathcal{X}$*, prove that* $d_1 + d_2$ *is continuous also.*

Proof. We want to show that any $(d_1 + d_2)$-ball is open. To this end, let $x \in \mathcal{X}, \epsilon > 0$ and consider $B_{d_1+d_2}(x, \epsilon) = \{y \in \mathcal{X} : d_1(x, y) + d_2(x, y) < \epsilon\} = \{y \in \mathcal{X} : d_1(x, y) \leq \delta \wedge d_2(x, y) \leq \epsilon - \delta : \forall \delta \in [0, \epsilon)\}$. We can write this set as

$$B_{d_1+d_2}(x, \epsilon) = \bigcup_{\delta \in [0, \epsilon)} \left[B_{d_1}(x, \delta) \cap B_{d_2}(x, \epsilon - \delta)\right].$$

Since d_1, d_2 are both continuous, any intersection between a d_1 ball and a d_2 ball is open. It follows that any arbitrary union of these balls is also open. So $d_1 + d_2$ is continuous. □

Exercise 8 (Ex 8, p.36)**.** *Assume that the topology on* $\mathcal{X}$ *is generated by the family of pseudometrics* $\mathcal{G}$*, and let* $\mathcal{G}'$ *be the set of all finite sums of elements of* $\mathcal{G}$*. Show that the set of* d*-balls with* $d \in \mathcal{G}'$ *forms a base for the topology.*

Proof. Let $d_1, d_2 \in \mathcal{G}'$ be given. Assume to avoid triviality that $B_{d_1}(x, \epsilon_1) \cap B_{d_2}(y, \epsilon_2) \neq \varnothing$. Let $z \in B_{d_1}(x, \epsilon_1) \cap B_{d_2}(y, \epsilon_2)$. We want to show that there is some $d \in \mathcal{G}'$ and $\epsilon > 0$ such that $B_d(z, \epsilon) \subseteq B_d \subseteq B_{d_1}(x, \epsilon_1) \cap B_{d_2}(y, \epsilon_2)$ (the ball $B_d(z, \epsilon)$ obviously contains z, so these two conditions make the collection of d-ball a base for $\mathcal{X}$). Now, let $\epsilon = \min\{\epsilon_1, \epsilon_2\} - \max\{d_1(x, z), d_2(z, y)\}$ and $d = d_1 + d_2$, which is in $\mathcal{G}'$. For any $u \in B_d(z, \epsilon)$, we have

$$d(u, z) = d_1(u, z) + d_2(u, z) < \epsilon = \min\{\epsilon_1, \epsilon_2\} - \max\{d_1(x, z), d_2(z, y)\}$$

which implies that

$$\begin{cases} d_1(u, x) < d_1(u, z) + d_1(z, x) + d_2(u, z) < \min\{\epsilon_1, \epsilon_2\} \\ d_2(u, y) < d_2(u, z) + d_2(z, y) + d_1(u, z) < \min\{\epsilon_1, \epsilon_2\} \end{cases}$$

so $u \in B_{d_1}(x, \epsilon_1) \cap B_{d_2}(y, \epsilon_2)$. Thus, $B_d(z, \epsilon) \subseteq B_{d_1}(x, \epsilon_1) \cap B_{d_2}(y, \epsilon_2)$ as desired. So the collection of d-balls where $d \in \mathcal{G}'$ forms a base the given topology. □

Exercise 9 (Ex 9, p.36)**.** *Two pseudometrics are* ***topologically equivalent*** *if they give rise to the same open sets. Prove that two pseudometrics are topologically equivalent if and only if each is continuous relative to the topology generated by the other.*

Proof. The forward direction is automatic by definition. It remains to show the converse. Let pseudometrics d_1, d_2 be given such that d_1 is continuous relative to the topology τ_2 generated by d_2 and d_2 is continuous relative to the topology τ_1 generated by d_1. By continuity, for any $x \in \mathcal{X}$ and $\epsilon > 0$, $B_{d_1}(x, \epsilon)$ is d_2-open and $B_{d_2}(x, \epsilon)$ is d_1-open. Let O_1 be an open set generated by d_1. Then O_1 is some union of d_1-balls. But since each d_1-open ball is open in d_2, each of these balls is generated by d_2-balls. By symmetry, we see that, d_1, d_2 must generate the same open sets. $\square$

Exercise 10 (Ex 10, p.36)**.** *Assume d is a pseudometric on a set $\mathcal{X}$ and $d(x, y) = 0$ for some $x, y \in \mathcal{X}$. Prove that $d(x, z) = d(y, z)$ for all $z \in \mathcal{X}$.*

Proof. By the triangle inequality: $|d(x, z) - d(y, z)| \leq d(x, y) = 0 \quad \forall z \in \mathcal{X}$. So, $|d(x, z) - d(y, z)| = 0$ for all $z \in \mathcal{X}$. Thus, $d(x, z) = d(y, z)$ for all $z \in \mathcal{X}$ as desired. $\square$

Exercise 11 (Ex 11, p.36)**.** *Assume d is a pseudometric on $\mathcal{X}$, and define a relation by $x \sim y$ if and only if $d(x, y) = 0$. Verify that this defines an equivalence relation on $\mathcal{X}$, and show that the quotient topology on the quotient space is metrizable.*

Proof. We first check that $\sim$ is an equivalence relation on $\mathcal{X}$:

- Symmetry follows automatically since d is a pseudometric.
- Reflexivity follows because $d(x, x) = 0$ for all $x \in \mathcal{X}$
- Transitivity: follows from the previous exercise.

Thus, $\sim$ is an equivalence relation on $\mathcal{X}$. To prove that $\mathcal{X}/\sim$ is metrizable, we want to show that the open sets in $\mathcal{X}/\sim$ are generated by a single metric. Consider the following function $\mathfrak{d} : \mathcal{X}/\sim \times \mathcal{X}/\sim \to [0, \infty)$ defined by

$$\mathfrak{d}([x], [y]) = d(x, y).$$

for $x, y \in \mathcal{X}$ (and of course $[x], [y] \in \mathcal{X}/\sim$). It is clear that this is a metric because not only it inherits properties of the pseudometric d but also it satisfies the property that $\mathfrak{d}([x], [y]) = d(x, y) = 0 \iff x \sim y \iff [x] = [y]$. We also know that open sets of $\mathcal{X}/\sim$ are the subsets of $\mathcal{X}/\sim$ that have an open pre-image under the surjective map $q : x \to [x]$. As a result, because d-balls in $\mathcal{X}$ are open, we have that $\mathfrak{d}$-balls in $\mathcal{X}/\sim$ are also open. Putting the results together, we find that $\mathcal{X}/\sim$ is metrizable, as desired. $\square$

Exercise 12 (Ex 12, p.36)**.** *A topological space $\mathcal{X}$ is called **Hausdorff** if every pair of distinct points in $\mathcal{X}$ are contained in disjoint open subsets of $\mathcal{X}$. Prove that every Tychonoff space is Hausdorff.*

Proof. Let a Tychonoff space $\mathcal{X}$ be given. By definition, the topology of $\mathcal{X}$ is the weak topology generated by the d-balls of a separating family of pseudometrics. From here, it is clear that for any two distinct points x, y in $\mathcal{X}$, there is always some pseudometric d in the family for which $d(x, y) = \delta > 0$. Consider the open balls $B_d(x, \delta/4)$ and $B_d(y, \delta/4)$. Assume that some point $u \in \mathcal{X}$ is in the intersection, then $\delta d(x, y) \leq d(x, u) + d(u, y) < \delta/2$, which is a contradiction. So, these open balls cannot intersect. Therefore, $\mathcal{X}$ is Hausdorff. $\square$

Exercise 13 (Ex 14, p.36)**.** *A topological space is **completely regular** if every pair consisting of a closed set and a point not in that set can be separated with a continuous function. Prove that every Tychonoff space is completely regular.*

Proof. Let a $A \subseteq \mathcal{X}$ be closed and $x \in \mathcal{X} \setminus A$ be given. It suffices to define some function f that separates A and x. Choose d a pseudometric generating $\mathcal{X}$. Define $\delta_{A,d}(x) : \mathcal{X} \to [0, \infty)$ by $\delta_{A,d}(z) = \inf_{a \in A} d(z, a)$. By the triangle inequality property inherited from the pseudometric d, we can check that $\delta_{A,d}$ is continuous. Further, we see that $\delta_{A,d}(A) = 0$ and $\delta_{A,d}(x) \neq 0$ (since A is closed). Thus, $\mathcal{X}$ must be completely regular. $\square$