Spring, 2021

Physics 312: Physics of Fluids
Assignment #6 (Solutions)

Background Reading

Friday, Mar. 19: Tritton 5.8,
Kundu & Cohen 4.18

Monday, Mar. 22: Tritton 7.1 - 7.4,

Wednesday, Mar. 24: Tritton 8.1 - 8.3

Informal Written Reflection

Due: Thursday, March 25 (8 am)

Same overall approach, format, and goals as before!

Formal Written Assignment

Due: Friday, March 26 (in class)

1. Early in the course, when we were first learning about pipe flow and flow around an obstacle, we got our first look at the Reynolds number. This week we met the Reynolds number again, this time as part of a broader introduction to *dimensional analysis* and *dynamical similiarity.* The following review articles will help you deepen your understanding and appreciation of these lessons. After reading these, please briefly summarize what you learned:

- D. Bolster, R. E. Hershberger, and R. J. Donnelly, "Dynamic similarity, the dimensionless science", *Physics Today*, September issue, page 42 (2011).
- T. Hecksher, "Insights through dimensions", *Nature Physics*, volume 13, page 1026 (2017).

2. For any basic flow problem involving characteristic velocity and length scales, U and L, we know that dimensional analysis of the momentum equation,
$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla P + \nu \nabla^2 \mathbf{u},$$
leads to a nondimensional equation of the form
$$\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' = -\nabla' P' + \frac{1}{\mathrm{Re}} \nabla'^2 \mathbf{u}',$$
where the Reynolds number
$$\mathrm{Re} = \frac{UL}{\nu}$$
describes the relative importance of inertial and viscous effects. In this problem, you will explore how other dimensionless numbers arise in similar problems involving more complicated governing equations. In particular, the magnetic field $\mathbf{B}$ plays an important role in the dynamics of electrically conducting or charged fluids. Ignoring thermal effects, steady flow of these fluids is described by a combination of conservation laws and expressions derived from Maxwell's equations:
$$\begin{aligned}
\nabla \cdot \mathbf{u} &= 0 \\
\nabla \cdot \mathbf{B} &= 0 \\
\mathbf{u} \cdot \nabla \mathbf{u} &= -\frac{1}{\rho} \nabla P + \frac{1}{\rho \mu} (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{u} \\
\mathbf{u} \cdot \nabla \mathbf{B} &= \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}
\end{aligned}$$
Here, $\mathbf{B}$ is the magnetic field, μ is the magnetic permeability, η is the magnetic diffusivity, and all other variables and parameters have their usual meanings.

(i) Using characteristic velocity and length scales, U and L, as well as a characteristic magnetic field strength B_0, nondimensionalize these *magnetohydrodynamic* equations.

(Hint: Even though these equations are less familiar, the process of switching to dimensionless variables is the same...)

(ii) Group all the dimensional parameters together and interpret the *three* nondimensional numbers that emerge from this procedure.

(Hint: One of these nondimensional groups will be the standard Reynolds number Re, which you already know how to interpret. Another will be a magnetic Reynolds number Re_m, for which you should be able to guess an interpretation. The last will involve the magnetic energy density B_0^2/μ and, again, you should be able to guess an interpretation...)

Solution:

In this problem, we use a characteristic length and velocity scales and a characteristic magnetic field strength to to nondimensionalize the relevant variables:

$$\mathbf{u} = U\mathbf{u}', \quad \mathbf{B} = B_0, \mathbf{B}',$$
$$\frac{D}{Dt} = \frac{U}{L}\frac{D}{Dt'}, \quad \text{and} \quad \nabla = \frac{1}{L}\nabla'.$$

Plugging these into our magnetohydrodynamic equations, we find

$$\frac{U}{L}\left(\nabla' \cdot \mathbf{u}'\right) = 0$$
$$\frac{B_0}{L}\left(\nabla' \cdot \mathbf{B}'\right) = 0$$
$$\frac{U^2}{L}\mathbf{u}' \cdot \nabla'\mathbf{u}' = -\frac{1}{\rho L}\nabla' P + \frac{B_0^2}{\rho\mu L}(\nabla' \times \mathbf{B}') \times \mathbf{B}' + \frac{\nu U}{L^2}\nabla'^2\mathbf{u}'$$
$$\frac{UB_0}{L}\mathbf{u}' \cdot \nabla'\mathbf{B}' = \frac{UB_0}{L}\mathbf{B}' \cdot \nabla'\mathbf{u}' + \frac{\eta B_0}{L^2}\nabla'^2\mathbf{B}'.$$

If we choose ρU^2 as our characteristic pressure scale then, once again, the dimensional parameters appearing in the pressure term are the same as those appearing in the inertia term. Finally, dividing each equation by the dimensional parameters appearing in the first term on the left, we arrive at the following nondimensional equations:

$$\begin{aligned}
\nabla' \cdot \mathbf{u}' &= 0 \\
\nabla' \cdot \mathbf{B}' &= 0 \\
\mathbf{u}' \cdot \nabla' \mathbf{u}' &= -\nabla' P' + \frac{B_0^2}{\rho \mu U^2} (\nabla' \times \mathbf{B}') \times \mathbf{B}' + \frac{\nu}{LU} \nabla'^2 \mathbf{u}' \\
\mathbf{u}' \cdot \nabla' \mathbf{B}' &= \mathbf{B}' \cdot \nabla' \mathbf{u}' + \frac{\eta}{LU} \nabla'^2 \mathbf{B}'.
\end{aligned}$$

Three nondimensional numbers emerge from this process:

- In the viscous term of the momentum equation we find, right where we expect it, the usual *Reynolds number*,

  $$\frac{1}{\mathrm{Re}} = \frac{\nu}{LU},$$

  which compares the relative importance of inertial and viscous effects.

- The momentum equation also has an extra body force term, which gives rise to a second nondimensional number,

  $$\mathrm{N} = \frac{B_0^2}{\rho \mu U^2},$$

  which may be interpreted as comparing the energy stored in the magnetic field, B_0^2/μ, to the kinetic energy density, ρU^2, (which is also our characteristic pressure scale).

- The fourth equation, derived from Maxwell's equations, introduces a *magnetic Reynolds number*,

  $$\frac{1}{\mathrm{R}_m} = \frac{\eta}{LU},$$

  which is analogous to the usual Reynolds; the magnetic Reynolds number compares the relative importance of inertia and the diffusion of magnetic properties.