Calculus of Variations
&
Partial Differential Equations

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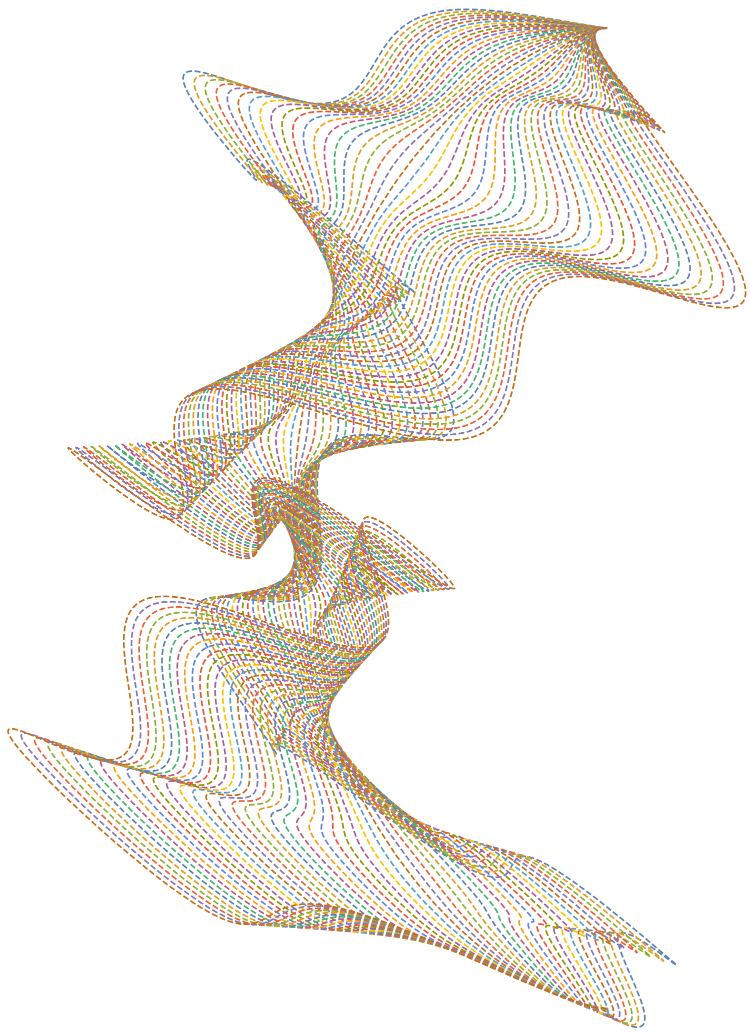

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1 Introduction

From an applied viewpoint, calculus of variations poses a fascinating alternative approach to solving optimization problems: by considering small deviations from a solution (variations) and finding solutions based on what makes non-solutions are sub-optimal. In a way, the idea of calculus of variations is much like the scientific process where the necessary conditions for the truth - a solution to a problem, or a theory of a natural phenomenon - is often obtained by trial and error. It is thus not surprising that calculus of variation gives a *natural* and appealing way for finding physical laws. Calculus of variations appears in almost all corners of physics, often under the name of the Principle of Least Action: from Lagrangian's formalism of classical mechanics to the foundations of classical and quantum field theory and even the Standard Model of physics and its extension (SME).

In physics, calculus of variations and differential equations go almost hand-in-hand. Physical laws are often written as differential equations. For example, Newton's law of gravitation relates the spatial gradient of a force field to the rate of change of the velocity of a particle in the field. On the other hand, while many theories such as the Schrödinger's equation or the Eintein's field equations, were originally postulated, it has been shown from time to time that these laws could be obtained from variational methods which are often purely mathematical. For example, the Einstein's field equations can be obtained through variational methods applied to the metric tensor $g_{\mu\nu}$.

This paper attempts to provide an overview of the connection between calculus of variations and (partial) differential equations. Starting with physically-motivated examples, this paper will derive the Euler-Lagrange equations and show how physical laws (differential equations) can be obtained in general from variational methods in Lagrangian mechanics. Through a general initial boundary value problem of an inhomogeneous Laplace's equation with Dirichlet boundary condition, the paper will also show how this problem in partial differential equation can be written as a minimization problem solvable by variational methods.

The final parts of shows a glimpse of the deeper mathematical connection between partial differential equation and calculus of variations. In particular, the paper will briefly discuss how some of the questions can lead to the study of functional analysis.

1.1 Euler-Lagrange Equation

In this section, we will look at how calculus of variations is applied to a simple problem, and how the Euler-Lagrange equation arises as a necessary condition for a solution to an optimization problem.

Suppose we want to find the shortest arc joining two points on a plane. We are certain that the arc is a straight line. But how is a straight line joining two points is the shortest arc? The idea of calculus of variations is to consider a solution $\bar{y}(x)$ that minimizes the distance S between the two points, say $A(x_1, y_1)$ and $B(x_2, y_2)$, and some deviation $\eta(x)$ from this correct curve $\bar{y}(x)$. Thus any deviation from the correct path associated with $\eta(x)$ can be written as

$$y(x) = \bar{y}(x) + \epsilon\eta(x) \tag{1}$$

where η is a constant parameter controlling for the magnitude of the deviation $\eta(x)$. Since we want the end points of any general path to be the same as the correct path, it is required that $\eta(x_1) = \eta(x_2) = 0$.

The distance $S(\epsilon)$ between A and B for any given $\eta(x)$ can be found by a little bit of calculus:

$$S(\epsilon) = \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{2}$$

Before solving this problem, let us think about a more general case. Let the integrand be $f = f(y', y, x) = f(\bar{y}' + \epsilon\eta', y + \epsilon\eta, x)$, where x is the "independent" variable and f is admissible. Since we require that S is minimized,

$dS(\epsilon)/d\epsilon = 0$. Thus it is necessary that

$$\begin{aligned}
0 &= \frac{dS}{d\epsilon} \\
&= \frac{d}{d\epsilon}\int_{x_1}^{x_2} f(\bar{y}' + \epsilon\eta', y + \epsilon\eta, x)\,dx \\
&= \int_{x_1}^{x_2} \frac{d}{d\epsilon} f(\bar{y}' + \epsilon\eta', y + \epsilon\eta, x)\,dx \\
&= \int_{x_1}^{x_2} \frac{d}{d\epsilon} f(\bar{y}' + \epsilon\eta', y + \epsilon\eta, x)\,dx \\
&= \int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'} + \eta \frac{\partial f}{\partial y}\,dx && (3)
\end{aligned}$$

Consider the first term in the integrand. Integrating by parts gives.

$$\begin{aligned}
\int_{x_1}^{x_2} \eta' \frac{\partial f}{\partial y'}\,dx &= \eta \frac{\partial f}{\partial y'}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial f}{\partial y}\,dx \\
&= -\int_{x_1}^{x_2} \frac{d}{dx}\frac{\partial f}{\partial y}\,dx, && (4)
\end{aligned}$$

where the boundary term vanishes due to the constraint $\eta(x_1) = \eta(x_2) = 0$. From (3) and (4), we have

$$0 = \int_{x_1}^{x_2} \eta \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) dx. \tag{5}$$

Now, because we require that $\bar{y}(x)$ minimizes S and that this holds for any deviation $\eta(x)$ from $\bar{y}(x)$, we obtain the **Euler-Lagrange equation**.

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0} \tag{6}$$

Back to our original problem with finding the shortest arc joining two points on a plane. We have that

$$f(y', y, x) = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{7}$$

Thus,

$$\begin{aligned}
\frac{\partial f}{\partial y} &= 0 && (8) \\
\frac{\partial f}{\partial y'} &= \frac{1}{2}\frac{2y'}{\sqrt{1 + y'^2}} = \frac{y'}{\sqrt{1 + y'^2}}. && (9)
\end{aligned}$$

By the Euler-Lagrange equation,

$$0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1 + y'^2}} \right) = 0, \tag{10}$$

i.e., $(y')\sqrt{1 + y'^2}$ is some constant C, i.e.,

$$\begin{aligned}
y' &= C\sqrt{1 + y'^2} \\
y'^2 &= C^2(1 + y'^2) \\
(1 - C^2)y'^2 &= C^2. && (11)
\end{aligned}$$

This says $y' = dy/dx$ is a constant, which means

$$y(x) = ax + b \tag{12}$$

for some constants a, b. This is nothing but an equation for a line on a plane as expected.

1.2 The Brachistochrone Problem

Perhaps one of the most well-known examples of the superiority of Lagrangian mechanics over the conventional methods in Newtonian mechanics is the problem of find the frictionless path for an object to slide down with the shortest (*brachistos*) amount of time (*chronos*). This problem was originally posed by John Bernoulli in 1696 and attracted the attention of many eminent mathematicians and physicists (or more accurately, *natural philosophers*) at the end time including Newton, Leibniz, L'Hopital, and Johann Bernoulli, John Bernoulli's brother.

Here we are minimizing time, so we must first find an express for time. Assuming that the object travels from initial height a to final height b. With a bit of calculus and basic mechanics, we have

$$T = \int_0^L \frac{ds}{v} = \int_a^b \frac{\sqrt{1+x'^2}}{v}\, dy, \tag{13}$$

where v is the speed. To express v in terms of height y, we use conservation of energy. Assuming the total energy is zero, we get

$$[\text{Potential Energy}] - [\text{Kinetic energy}] = mgy - \frac{1}{2}mv^2 = 0. \tag{14}$$

Thus,

$$v = \sqrt{2gy}. \tag{15}$$

Putting this into (13), we get

$$T = \frac{1}{\sqrt{2g}} \int_a^b \sqrt{\frac{1+x'^2}{y}}\, dy. \tag{16}$$

Let the integrand be $f[x', x, y]$, by the Euler-Lagrange equation we have

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0 - \frac{d}{dy}\frac{1}{\sqrt{y}} \cdot \frac{x'}{\sqrt{1+x'^2}} = 0. \tag{17}$$

Thus,

$$\frac{x'}{\sqrt{y}\sqrt{1+x'^2}} = \frac{1}{2a} \tag{18}$$

where a is some non-zero constant. It follows that

$$\begin{aligned} 0 &= (2ax')^2 - y(1+x'^2) \\ &= x'^2(2a-y) - y. \end{aligned} \tag{19}$$

Rearranging and integrating both sides, we get

$$x = \int \sqrt{\frac{y}{2a-y}}\, dy. \tag{20}$$

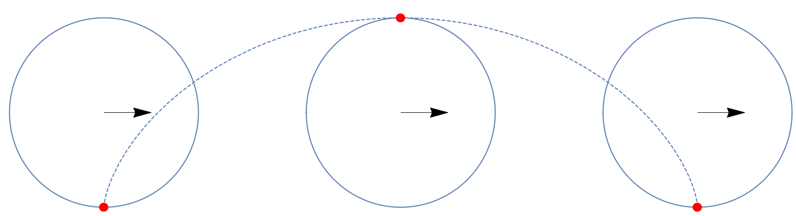
Now, we notice that $0 \leqslant y \leqslant 2a$, so we can make the substitution $y = a(1-\cos\theta)$. Then, $dy = a\sin\theta\, d\theta$. And so,

$$\begin{aligned} x &= \int \sqrt{\frac{1-\cos\theta}{1+\cos\theta}} a\sin\theta\, d\theta \\ &= a\int \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \sin\theta\, d\theta \\ &= a\int \frac{1-\cos\theta}{\sin\theta}\sin\theta\, d\theta \\ &= a\int 1-\cos\theta\, d\theta \\ &= a(\theta - \sin\theta) + C. \end{aligned} \tag{21}$$

We can define locations such that initially, the object is at $(x, y) = (0, 0)$. This gives $C = 0$. And so,

$$\begin{cases} x = a(\theta - \sin\theta) \\ y = a(1-\cos\theta). \end{cases} \tag{22}$$

This is the parameterization for a **cycloid**, the curved traced by a point on a circle as is rolls without slipping in a straight line.



1.3 Necessary and almost-sufficient conditions

It is very important to keep in mind that the Euler-Lagrange equation gives a *necessary condition* for some function to be a solution; i.e., if some function is a solution to the optimization problem in consideration then the function must satisfy the Euler-Lagrange equation. The Euler-Lagrange equation is not a sufficient condition [2]. By considering the Euler-Lagrange equation, we are only concerned with the first-order change (or first variation) in the action. To obtain sufficient conditions, we need to look for second-order change (or second-variation) in the action.

It turns out that in most cases, especially in physical systems, a first variation in the Lagrangian is sufficient for finding a minimizing solution. [3]

2 Euler-Lagrange equations as PDE's

Euler-Lagrange equation, in its form, is a partial differential equation. In practice, we often don't solve the Euler-Lagrange equation but instead we solve the differential equation *obtained from* the Euler-Lagrange equation for a given Lagrangian. In this section, We will look at two examples: one in classical mechanics and one in classical field theory. The point of these examples is to demonstrate how the Euler-Lagrange equation is used in physics, and how powerful a method it is for finding physical laws/equation of motion when the solution is not obvious.

2.1 Solutions as Equations of Motion

> Mr. Bader told me the following: Suppose you have a particle (in a gravitational field, for instance) which starts somewhere and moves to some other point by free motionyou throw it, and it goes up and comes down. It goes from the original place to the final place in a certain amount of time. Now, you try a different motion. Suppose that to get from here to there, but got there in just the same amount of time. Then he said this: If you calculate the kinetic energy at every moment on the path, take away the potential energy, and integrate it over the time during the whole path, youll find that the number youll get is *bigger* than that for the actual motion. In other words, the laws of Newton could be stated not in the form $\vec{F} = m\vec{a}$ but in the form: the average kinetic energy less the average potential energy is as little as possible for the path of an object going from one point to another.
> - Feynman Lectures on Physics, *The Principle of Least Action* [1]

One aspect of the Brachistochrone problem that makes it a "classic" apart from its unexpected solution and interesting history is the prescription it provides for finding the "equation of motion" for certain physical systems. The prescription is as follows:

1. Determine the Lagrangian, which in most cases is the "kinetic" pieces minus the "potential" pieces. The same principle applies to physics beyond Lagrangian (classical) mechanics.
2. (a) Either apply the Euler-Lagrange equation to the Lagrangian to get a system of relationships among the physical quantities
   (b) Or if it is unclear how to proceed with the Euler-Lagrange equations, vary the action functional with respect to some independent variable and obtain the Euler-Lagrange equation.
3. Simplify and obtain the physical law.

For example, suppose we would like to "derive" Hooke's law from only the energy terms. We first set up the Lagrangian:

$$\mathcal{L} = [\text{Potential energy}] - [\text{Kinetic energy}] = \frac{1}{2}kx^2 - \frac{1}{2}m\dot{x}^2. \tag{23}$$

where x is the position and $\dot{x}$ is the velocity of the particle. In independent variable here is time, t. Applying the Euler-Lagrangian equation:

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \implies kx = -\frac{d}{dt}(m\dot{x}) = -m\ddot{x}. \tag{24}$$

This is of course a silly example just to demonstrate how this method works. We can only really see how this method becomes useful when the Lagrangian is known but it is not clear what the "equation of motion" is. Let us look at a slightly more complicated example in the following subsection.

2.2 Example: Klein-Gordon equation for photons (wave equation)

In classical field theory, the independent variable in the Euler-Lagrange equation becomes the field ϕ, which essentially is a function that associates every point in space and time with some number. Here, we will use Einstein's notation for summation, and so the gradient operator ∇ is replaced by ∂_μ, where $\mu = 0, 1, 2, 3$ is an index, and ∂_μ is defined as

$$\partial_\mu = \frac{\partial}{\partial t} - \nabla. \tag{25}$$

This sign convention is defined by the Minkowski metric

$$[\eta_{\mu\nu}] = \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}. \tag{26}$$

With this notation and the change in the independence variable, we have that for a given Lagrangian $\mathcal{L}$,

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{27}$$

For massless photons, it can be shown from variational methods that

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi), \tag{28}$$

where this "quadratic" term is the analogue of kinetic energy. Because photons are massless, we don't have a potential term here. Plugging this Lagrangian into the Euler-Lagrange equation, we get

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{29}$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi. \tag{30}$$

So the Euler-Lagrange equation gives:

$$\partial_\mu(\partial^\mu \phi) = 0. \tag{31}$$

Now, the operator

$$\partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \eta_{\mu\nu} \partial^\mu \partial^\nu = \Box \tag{32}$$

is called the **d'Alembertian**. If we expand out the terms, with the 0 index associated with the time variable t, and $1, 2, 3$ associated with the spatial components x, y, z, then we get from (31)

$$\frac{\partial^2}{\partial t^2}\phi = \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi + \frac{\partial^2}{\partial z^2}\phi = \nabla^2 \phi. \tag{33}$$

Of course, this is the wave equation. This result makes good sense, because photons can be described as propagating electromagnetic waves. While this result might not be too surprising, we can see that if the Lagrangian is somehow modified (which is often the case if, say, we want to describe a different particle with mass and spin, etc), then we should expect an equation of motion from the Euler-Lagrange equation. It is truly remarkable that by specifying certain characteristics of a particle, we can find an equation describing exactly what the particle does in space and time.

3 PDE's as Minimization Problems

In this section we focus on how initial boundary value problems can be expressed as minimization problems and solved using variational methods. First, a Dirichlet-type initial boundary value problem is discussed. Then, an example in electrostatics is given to demonstrate how a PDE can be expressed as a minimization problem and vice versa.

3.1 A Dirichlet-type Initial Boundary Value Problem

Consider the following initial boundary value problem.

$$(*) \begin{cases} \nabla^2 u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \tag{34}$$

One with some familiarity with partial differential equation may recognize that this is a Laplace's equation with inhomogeneous boundary condition. One can solve this problem by transforming it to get homogeneous boundary condition, but in context of calculus of variations, it turns out that for some admissible function u,

$$\boxed{u \text{ solves } (*) \iff u \text{ minimizes } S[w] = \frac{1}{2}|\nabla w|^2 \, dx} \tag{35}$$

Sketch of Proof.

1. ($\Longrightarrow$) We want to show that for any solution u of $(*)$, $S[u] \leqslant S[w+u]$ for any admissible w. In context of calculus of variations, the function w satisfying the boundary conditions is the analogue of the deviation η in the derivation of the Euler-Lagrange equation. Thus, it is natural to consider the perturbed solution $u' = u + w$ and what the action functional associated with it. Let a solution u to $(*)$ be given. Then $\nabla^2 u = 0$. Let an admissible w satisfying the boundary condition $w = f$ in $\partial\Omega$ be given. Then it is true that

$$\int_\Omega w \nabla^2 u \, dx = 0. \tag{36}$$

Integration by parts gives

$$\begin{aligned}\int_\Omega w\nabla^2 u \, dx &= w\nabla u\Big|_{\partial\Omega} - \int_\Omega \nabla u \cdot \nabla w \, dx \\ &= -\int_\Omega \nabla u \cdot \nabla w \, dx.\end{aligned} \tag{37}$$

Now, we look at the action associated with the solution u perturbed by some amount w:

$$\begin{aligned}S[u'] = S[u+w] &= \frac{1}{2}\int_\Omega |\nabla(u+w)|^2 \, dx \\ &= \frac{1}{2}\int_\Omega \nabla(u+w)\cdot\nabla(u+w)\, dx \\ &= \frac{1}{2}\int_\Omega \left(\nabla u \cdot \nabla u + 2\nabla u \cdot \nabla w + \nabla w \cdot \nabla w\right) dx, \qquad \text{by linearity of } \nabla \\ &= \frac{1}{2}\int_\Omega \nabla u \cdot \nabla u + \nabla w \cdot \nabla w \, dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\int_\Omega |\nabla w|^2 \, dx \\ &= S[u] + S[w] \\ &\geqslant S[u].\end{aligned} \tag{38}$$

Thus, $S[u] \leqslant S[u+w]$ for any admissible perturbation w. This means u minimizes the action $S[\,]$.

2. ($\Longleftarrow$) Here we want to show that if u minimizes the action functional $S[\,]$ then u solves $(*)$. Here the idea of calculus of variations comes in handy. Suppose that u satisfies the boundary condition $u = f$ in $\partial\Omega$ and minimizes the action $S[\,]$. Now, consider some perturbation in u, i.e., we let

$$u \to u + \epsilon w \tag{39}$$

for some constant ϵ and function w that satisfies the boundary condition of $(*)$. Once again, we consider the action associated with this perturbed u:

$$\begin{aligned}S[u+\epsilon w] &= \frac{1}{2}\int_\Omega |\nabla(u+\epsilon w)|^2 \, dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 + 2\epsilon \nabla u \nabla w + \epsilon^2 |\nabla w|^2 \, dx \qquad \text{by linearity of } \nabla.\end{aligned} \tag{40}$$

Now, because u minimizes $S[\,]$, $\partial S/\partial\epsilon = 0$ at $\epsilon = 0$, so we have

$$\begin{aligned}0 &= \frac{\partial}{\partial\epsilon} S[u+\epsilon w]\bigg|_{\epsilon=0}\\ &= \int_\Omega \nabla u\cdot\nabla w\,dx + \int_\Omega \epsilon|\nabla w|^2\,dx\bigg|_{\epsilon=0}\\ &= \int_\Omega \nabla u\cdot\nabla w\,dx. \end{aligned}\tag{41}$$

But recall we have argued that

$$\int_\Omega w\nabla^2 u\,dx = -\int_\Omega \nabla u\cdot\nabla w\,dx.\tag{42}$$

Therefore,

$$\int_\Omega \nabla u\cdot\nabla w\,dx = 0,\tag{43}$$

which must hold for any perturbation w. This means $\nabla^2 u = 0$. But since u also satisfies the boundary condition of $(*)$, u solves $(*)$.

□

3.2 Poisson's Equation in Electrostatics

Suppose we are given some charger distribution ρ and want to know what the potential (scalar) field associated with this charge distribution is. One way to solve this problem is to consider solving the Poisson's equation:

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}.\tag{44}$$

Alternatively, it turns out that this PDE can be expressed as the following problem: Find ϕ such that the action:

$$J = \frac{\epsilon_0}{2}\int |\nabla\phi|^2\,dV - \int \rho\phi\,dV\tag{45}$$

is minimum. We shall verify that the solution to this problem is indeed Poisson's equation for electrostatics, via variational methods.

Proof. Let us write a general potential ϕ as $\rho = \bar\phi + \epsilon F$ where $\bar\phi$ is the correct potential and F is the variation from this correct potential. It follows that

$$|\nabla\phi|^2 = |\nabla\bar\phi|^2 + 2\epsilon\nabla\bar\phi\cdot\nabla F + \epsilon^2|\nabla F|^2.\tag{46}$$

Like how we have argued before, because $\bar\phi$ minimizes J, $\partial J/\partial\epsilon = 0$ at $\epsilon = 0$, so we have

$$\begin{aligned}0 &= \frac{\partial}{\partial\epsilon}\left(\frac{\epsilon_0}{2}\int|\nabla\phi|^2\,dV - \int\rho\phi\,dV\right)\bigg|_{\epsilon=0}\\ &= \frac{\partial}{\partial\epsilon}\left(\frac{\epsilon_0}{2}\int|\nabla\bar\phi|^2 + 2\epsilon\nabla\bar\phi\cdot\nabla F + \epsilon^2|\nabla F|^2\,dV - \int\rho(\bar\phi+\epsilon F)\,dV\right)\bigg|_{\epsilon=0}\\ &= \dots\\ &= \int\left(\epsilon_0\nabla\bar\phi\cdot\nabla F - \rho F\right)\,dV.\end{aligned}\tag{47}$$

Using (42), we simplify this to

$$0 = \int -\epsilon_0 F \nabla^2 \bar{\phi} - \rho F \, dV = \int (-\epsilon_0 \nabla^2 \phi - \rho) F \, dV, \tag{48}$$

which has to be true for all F. Therefore,

$$\nabla^2 \bar{\phi} = -\frac{\rho}{\epsilon_0}, \tag{49}$$

which is the Poisson's equation for electrostatics as desired. [1] □

4 Above and Beyond

[4]

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