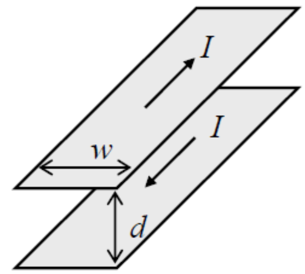
Problem 5.2. Two straight, plane, parallel, long, thin conducting strips of width w, separated by distance d, carry equal but oppositely directed currents I – see the figure on the right. Calculate the magnetic field in the plane located in the middle between the strips, assuming that the flowing currents are uniformly distributed across the strip widths.



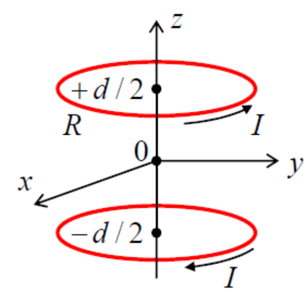
(30 pts)

Problem 5.3. For the system studied in the previous problem, but now only in the limit $d << w$, calculate:

(i) the distribution of the magnetic field in space,
(ii) the vector-potential of the field,
(iii) the magnetic force (per unit length) exerted on each strip, and
(iv) the magnetic energy and self-inductance of the loop formed by the strips (per unit length).

(30 pts)

Problem 5.4. Calculate the magnetic field distribution near the center of the system of two similar, plane, round, coaxial wire coils, carrying equal but oppositely directed currents – see the figure on the right.



(20 pts)

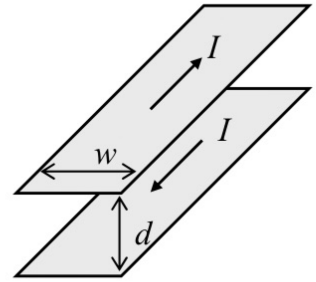
Problem 5.7. A thin round disk of radius R, carrying electric charge of a constant areal density σ, is being rotated around its axis with a constant angular velocity ω. Calculate:

(i) the induced magnetic field on the disk's axis,
(ii) the magnetic moment of the disk,

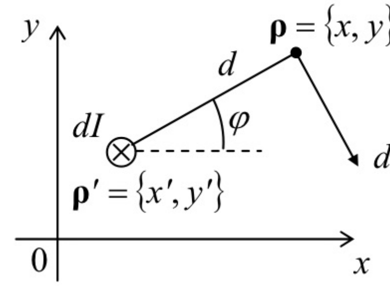
and relate these results.

(30 pts)

<u>Problem 5.2</u>. Two straight, plane, parallel, long, thin conducting strips of width w, separated by distance d, carry equal but oppositely directed currents I – see the figure on the right. Calculate the magnetic field in the plane located in the middle between the strips, assuming that the flowing currents are uniformly distributed across the strip widths.



Solution: Due to the linear superposition principle, we may calculate the total field **B** as a vector sum of elementary fields $d\mathbf{B}$ induced by elementary currents $dI = (I/w)dw$, flowing in each elementary segment dw of the strip widths, considering it as a thin wire. The magnetic field of such a wire was calculated in Sec. 5.1 of the lecture notes (see Eq. (5.20) and its derivation), and we need just to rewrite it in a form more convenient for our current purposes. Directing the z-axis along the current (see the figure on the right), we get

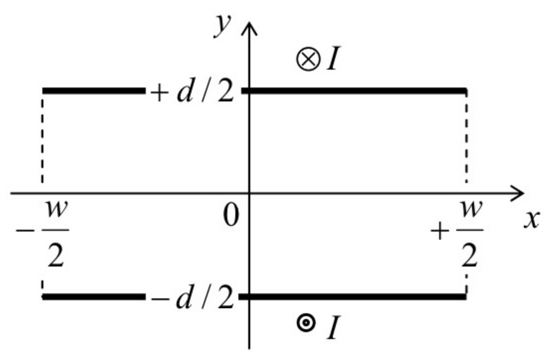


$$dB_x = dB\sin\varphi = \frac{\mu_0}{2\pi d} dI \frac{\Delta y}{d} = \frac{\mu_0}{2\pi} \frac{y - y'}{(x - x')^2 + (y - y')^2} dI,$$

$$dB_y = -dB\cos\varphi = -\frac{\mu_0}{2\pi d} dI \frac{\Delta x}{d} = -\frac{\mu_0}{2\pi} \frac{x - x'}{(x - x')^2 + (y - y')^2} dI,$$

where $\boldsymbol{\rho} = \{x, y\}$ is the 2D radius vector of the field observation point, while $\boldsymbol{\rho}' = \{x', y'\}$ is that of the source (wire segment).

Now we should integrate this expression over both current-carrying strips. Selecting the reference frame in the natural way shown in the figure on the right, so that the mid-plane in question corresponds to $y = 0$, we see that due to the problem's symmetry, the integral of dB_y vanishes, while the "horizontal" component of the field is twice that of a single strip:

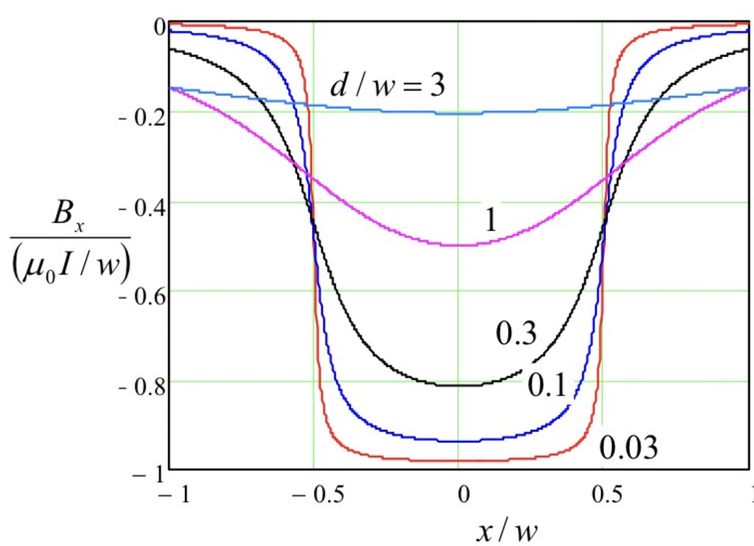


$$B_x = 2\int_{x'=-w/2}^{x'=w/2} dB_x = -\frac{\mu_0}{\pi}\int_{x'=-w/2}^{x'=w/2} \frac{d/2}{(x - x')^2 + (d/2)^2} dI$$

$$= -\frac{\mu_0}{\pi}\frac{I}{w}\frac{d}{2}\int_{-w/2}^{+w/2} \frac{dx'}{(x - x')^2 + (d/2)^2}$$

$$= -\frac{\mu_0 I}{\pi w}\left(\tan^{-1}\frac{x - w/2}{d/2} - \tan^{-1}\frac{x + w/2}{d/2}\right).$$

The figure on the right shows the plots of B_x as a function of x, for several values of the d/w ratio. If $d >> w$, the field is relatively low, distributed along axis x over a broad interval $\Delta x \sim d >> w$, and may be well approximated by the sum of fields of two thin wires.



On the other hand, as the distance between the strips is reduced, the field becomes localized within the gap between them. (At $d/w \to 0$, the so-called "fringe fields" at $|x| > w/2$ become negligible.) Moreover, the field becomes uniform and tends to the d-independent value

$$B_x = -\frac{\mu_0 I}{w}.$$

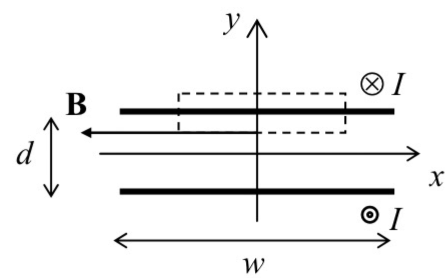
A simple explanation of this result, and the discussion of consequences of this very important result are the subjects of the next problem.

Problem 5.3. For the system studied in the previous problem, but now only in the limit $d << w$, calculate:

(i) the distribution of the magnetic field in space,
(ii) the vector-potential of the field,
(iii) the magnetic force (per unit length) exerted on each strip, and
(iv) the magnetic energy and self-inductance of the loop formed by the strips (per unit length).

Solutions:

(i) As was shown in the previous problem, in the limit $d/w \to 0$, the magnetic field is localized in the gap between the strips, and is uniform. Applying the Ampère law, given by Eq. (5.37) of the lecture notes, to the contour shown with the dashed line in the figure on the right (which shows the cross-section of the system), we get the same result as was obtained in the previous problem,



$$B = -B_x = \mu_0 \frac{I}{w}, \qquad (*)$$

in a much simpler way, and for arbitrary y between $-d/2$ and $+d/2$.

(ii) According to Eq. (5.28) of the lecture notes, the vector-potential has to be directed along the z-axis, i.e. along the current, and be independent of z. From the structure of that equation, it is also clear that at $d << w$, well inside the gap, $\mathbf{A}$ should not depend on x either (because the strip edges are not "visible" from those points). Hence we may look for the vector-potential in the form

$$\mathbf{A} = A(y)\mathbf{n}_z.$$

Calculating the curl of such a vector,

$$\nabla \times \mathbf{A} = \mathbf{n}_x \frac{\partial A}{\partial y},$$

and requiring it to be equal to the vector $\mathbf{B}$ described by Eq. (*), we get

$$\mathbf{A} = -\mu_0 \frac{I}{w} y\mathbf{n}_z + \text{const}.$$

This result could be also obtained by the direct integration of Eq. (5.28) along x' and z'. (Alternatively, we can integrate Eq. (5.51) written for each current component $dI = Idx'/w$, along the x'-axis.)

Note, however, that a uniform field, like our $\mathbf{B} = -B\mathbf{n}_x$, cannot "tell" one transverse coordinate (say, y) from another one (z), and hence may be the same for different distributions of the vector potential, for example[91]

$$\mathbf{A} = \mu_0 \frac{I}{w} z\mathbf{n}_y + \text{const},$$

or a linear combination of these two functions. This is one more manifestation of the gauge invariance of the magnetic field with respect to any transformation $\mathbf{A} \to \mathbf{A} + \nabla\chi$ – see Eq. (5.46).

(iii) As it follows from Eq. (5.1), the total magnetic force acting on each strip is directed along the y-axis, and corresponds to strips' repulsion. To calculate its magnitude, it would be wrong to plug Eq. (*) into Eq. (5.15), because an elementary current (like a point electric charge) does exert force on itself.[92] As we can readily check using the Ampère law, the field created by a single strip is twice lower:

$$B_1 = \mu_0 \frac{I}{2w},$$

so that, according to Eq. (5.15), the force magnitude (per unit length)

$$\frac{F}{l} = \frac{1}{l} \int_{\text{strip}} jB_1 d^3r = IB_1 = \mu_0 \frac{I^2}{2w}.$$

This expression may be also represented as the integral, over the strip area, of the positive pressure of the magnetic field:

$$\mathcal{P} \equiv \frac{F}{lw} = \frac{\mu_0 I^2}{2w^2} = \frac{B^2}{2\mu_0} = u,$$

where $u = U/lwd = U/V_{\text{gap}}$ is the magnetic energy density – see Eq. (5.57b).

(iv) Since the full magnetic field is uniform inside the gap (with the cross-section area dw) and vanishes outside of it, the magnetic energy per unit length is just[93]

$$\frac{U}{l} = \frac{1}{l} \int_{\text{gap}} \frac{B^2}{2\mu_0} d^3r = \frac{B^2}{2\mu_0} dw = \mu_0 \frac{d}{w} \frac{I^2}{2} = u \frac{V_{\text{gap}}}{l}.$$

From this result and Eq. (5.72) we immediately get the following expression for the self-inductance (also per unit length):

$$\frac{L}{l} = \mu_0 \frac{d}{w}.$$

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[91] Note that this distribution, of course, does *not* satisfy Eq. (5.28).

[92] Of course, each component jdx of the current distributed in a strip does exert a nonvanishing force on each other component jdx' of current in the same strip, but this force is directed along axis x, rather than y.

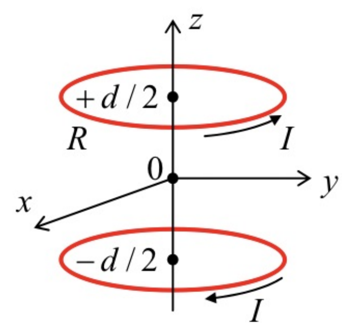
[93] This result poses an additional question for the reader: how can the strip *repulsion* (i.e. their "desire" to *increase* d, and hence volume V_{gap} and product $U = uV_{\text{gap}}$) be compatible with the apparent general trend for any system to *decrease* its potential energy? As a reminder, for the *electrostatic* force exerted on a conductor, this paradox does not exist because the force direction corresponds to *negative* pressure – see, e.g., the solution of Problem 2.1. Thinking about this issue may be a good primer for the general discussion, in Sec. 6.2 of the lecture notes, of the relation between the two magnetic energies given by Eqs. (5.53) and (5.54).

The same result may be also calculated as the ratio $(\Phi/l)/I$, where Φ is the magnetic flux in the gap between the strips:

$$\frac{\Phi}{l}=\frac{1}{l}\int_{\text{gap}} B_n d^2r = Bt = \mu_0 \frac{I}{w}t .$$

This simple formula, which shows a clear way for the reduction of current loop inductances (bring the counterpart conductors as close to each other as possible!), is very important for applications in electronics, because self-inductance frequently plays a negative role in high-speed integrated circuits, reducing the interconnect bandwidth.

<u>Problem 5.4</u>. Calculate the magnetic field distribution near the center of the system of two similar, plane, round, coaxial wire coils, carrying equal but oppositely directed currents – see the figure on the right.



Solution: In order to find the field $\mathbf{B}(x, y, z)$ exactly on the system's axis, we may combine two versions of Eq. (5.23) of the lecture notes, applied to each of the loops:

$$\mathbf{B}(0,0,z)=\frac{\mu_0 I}{2}\left\{-\frac{R^2}{\left[R^2+(d/2+z)^2\right]^{3/2}}+\frac{R^2}{\left[R^2+(d/2-z)^2\right]^{3/2}}\right\}\mathbf{n}_z ,$$

where z is the axis directed along the system axis, and its center is taken for the reference frame origin (see the figure above). At the center of the system ($z = 0$), the field evidently vanishes, while for small ($|z| << R, d$) but nonvanishing deviations it may be found from the linear term of the Taylor expansion of the right-hand side:

$$\mathbf{B}(0,0,z)\approx\frac{\mu_0 I}{2}\frac{\partial}{\partial z}\left\{\frac{R^2}{\left[R^2+(d/2+z)^2\right]^{3/2}}-\frac{R^2}{\left[R^2+(d/2-z)^2\right]^{3/2}}\right\}_{z=0} z\mathbf{n}_z = \frac{3\mu_0 IdR^2}{2\left[R^2+(d/2)^2\right]^{5/2}} z\mathbf{n}_z .$$

Evidently, this expression may be rewritten as

$$\left.\frac{\partial B_z}{\partial z}\right|_{\mathbf{r}=0}=\frac{3\mu_0 IdR^2}{2\left[R^2+(d/2)^2\right]^{5/2}} .$$

Now, in the Cartesian coordinates (see the figure above), the general Eq. (5.29) reads

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0 .$$

Due to the axial symmetry of the system, we may argue that at $z = 0$, the first two partial derivatives have to be equal to each other. Hence, the last two relations yield

$$\left.\frac{\partial B_x}{\partial x}\right|_{\mathbf{r}=0}=\left.\frac{\partial B_y}{\partial y}\right|_{\mathbf{r}=0}=-\frac{1}{2}\left.\frac{\partial B_z}{\partial z}\right|_{\mathbf{r}=0}=-\frac{3\mu_0 IdR^2}{4\left[R^2+(d/2)^2\right]^{5/2}} ,$$

so that in the linear approximation in small x, y, and z we may write

$$\mathbf{B}(x,y,z) \approx \frac{3\mu_0 I d R^2}{4\left[R^2 + (d/2)^2\right]^{5/2}}\left(-x\mathbf{n}_x - y\mathbf{n}_y + 2z\mathbf{n}_z\right), \qquad \text{for } r << d, R.$$

Such *quadrupole magnetic field* pattern may provide trapping of an atom with zero net electric charge but a non-zero total spin (and hence nonvanishing magnetic moment), within a certain range of its initial velocity. A brief discussion of other magnetic traps, and references to additional literature on the subject, may be found in Sec. 9.7 of the lecture notes.

Problem 5.7. A thin round disk of radius R, carrying electric charge of a constant areal density σ, is being rotated around its axis with a constant angular velocity ω. Calculate:

(i) the induced magnetic field on the disk's axis,
(ii) the magnetic moment of the disk,

and relate these results.

Solutions:

(i) The rotating disk may be fairly represented as a set of narrow elementary rings of radius ρ and width $d\rho << \rho$, each carrying a ring current $dI = Jd\rho = \sigma v d\rho = \sigma\omega\rho d\rho$ and creating the elementary field dB described by Eq. (5.23) of the lecture notes (with the notation replacements $B \to dB$, $I \to dI$, $R \to \rho$):

$$dB = \frac{\mu_0 dI}{2}\frac{\rho^2}{\left(\rho^2 + z^2\right)^{3/2}} = \frac{\mu_0 \sigma\omega}{2}\frac{\rho^3 d\rho}{\left(\rho^2 + z^2\right)^{3/2}},$$

where z is the distance of the field observation point from the ring's plane. Since the fields of all rings have the same direction (along the disk's axis), we may sum up their contributions to the total field as scalars:

$$B = \int_{\rho=0}^{\rho=R} dB = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{\rho^3 d\rho}{\left(\rho^2 + z^2\right)^{3/2}} = \frac{\mu_0\sigma\omega}{2}\frac{|z|}{2}\int_{\rho=0}^{\rho=R}\frac{\xi d\xi}{(\xi+1)^{3/2}},$$

where $\xi \equiv \rho^2/z^2$. The last integral may be readily worked out as

$$\int \frac{\xi d\xi}{(\xi+1)^{3/2}} = \int \frac{(\xi+1)d\xi}{(\xi+1)^{3/2}} - \int \frac{d\xi}{(\xi+1)^{3/2}} = \int \frac{d\xi}{(\xi+1)^{1/2}} - \int \frac{d\xi}{(\xi+1)^{3/2}} = 2(\xi+1)^{1/2} + \frac{2}{(\xi+1)^{1/2}} \equiv \frac{2(\xi+2)}{(\xi+1)^{1/2}},$$

and we finally get

$$B = \frac{\mu_0\sigma\omega}{2}\frac{|z|}{2}\left[\frac{2\left(\rho^2/z^2 + 2\right)}{\left(\rho^2/z^2 + 1\right)^{1/2}}\right]_{\rho=0}^{\rho=R} = \mu_0\sigma\omega\left[\frac{R^2 + 2z^2}{2\left(R^2 + z^2\right)^{1/2}} - |z|\right]. \qquad (*)$$

(ii) Very similarly, the magnitude of the magnetic moment $\mathbf{m}$ of the system (evidently, also directed along the symmetry axis z) may be calculated by summing up the elementary ring contributions given by Eq. (5.97):

$$dm = AdI = \pi\rho^2 dI = \pi\sigma\omega\rho^3 d\rho,$$

so that

$$m = \int_{\rho=0}^{\rho=R} dm = \pi\sigma\omega\int_0^R \rho^3 d\rho = \frac{\pi\sigma\omega R^4}{4}. \qquad (**)$$

In order to relate the above results, let us find the limit of Eq. (*) at large distances, $|z| >> R$:

$$B = \mu_0\sigma\omega|z|\left[\frac{1+R^2/2z^2}{\left(1+R^2/z^2\right)^{1/2}}-1\right] \to \mu_0\sigma\omega|z|\left[\left(1+\frac{R^2}{2z^2}\right)\left(1-\frac{R^2}{2z^2}+\frac{3R^4}{8z^4}\right)-1\right] \to \mu_0\sigma\omega\frac{R^4}{8|z|^3}. \quad (***)$$

On the other hand, the general expression for the magnetic dipole field, following from Eq. (5.99) of the lecture notes for the observation point on the magnetic dipole's axis ($\mathbf{r} = |z|\mathbf{m}/m$), is

$$B = \frac{\mu_0}{4\pi}\frac{2m}{|z|^3}.$$

With the m given by Eq. (**), this expression coincides with that Eq. (***), i.e. our two results do match.