In the Weyl representation we found

$$S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$
$$S^{ij} = \frac{1}{2}\varepsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

This is a <u>reducible</u> representation (meaning that the transformations have a block diagonal structure)

Let $\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$ ← two-component "left" Weyl spinor / ← two-component "right" Weyl spinor

Using the forms above for S^{0i} and S^{ij},

under an infinitesmal rotation $\vec{\theta}$ and infinitesmal boost $\vec{\beta}$ $\left(\tanh|\vec{\beta}| = \frac{|\vec{v}|}{c}\right.$ ← relative velocity between frames$\left.\right)$

$$\psi_L \to \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L$$
$$\psi_R \to \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_R$$

The transformation of ψ_R is equivalent to the transformation of ψ_L^* ...

$$\psi_L^* \to \left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

noting that $\underset{\uparrow\ \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{\sigma^2}\ \vec{\sigma}^* = -\vec{\sigma}\,\sigma^2$

we see that

$$\sigma^2\psi_L^* \to \sigma^2\left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$
$$= \underbrace{\left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)}_{\text{just like } \psi_R \text{ transformation}}\sigma^2\psi_L^*$$

The Dirac equation has the form

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

or
$$\begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0$$

When $\underline{m=0}$ there is no coupling of ψ_L + ψ_R

$$i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0$$
$$i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0 \qquad \underline{\text{Weyl equations}}$$

For later convenience let us define

$$\sigma^\mu = (\underset{\substack{\uparrow \\ \text{2x2 identity}}}{1}, \vec{\sigma}) , \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

Then $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$

The Dirac equation is

$$\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = 0$$

and the Weyl equations are

$$i\bar{\sigma}\cdot\partial\, \psi_L = 0$$
$$i\sigma\cdot\partial\, \psi_R = 0$$

We found that solutions of Dirac equation are solutions of the Klein-Gordon equation.
Try the form...

$$\psi(x) = u(p)\, e^{-ip\cdot x} \qquad \text{where } p^0 = \sqrt{\vec{p}^{\,2} + m^2} = E_{\vec{p}}$$

These are positive frequency solutions $(e^{-i\omega t})$

$$(i\gamma^\mu \partial_\mu - m)\, u(p) e^{-ip\cdot x} = (\gamma^\mu p_\mu - m)\, u(p) e^{-ip\cdot x}$$

and so $\quad (\gamma^\mu p_\mu - m)\, u(p) = 0$

Let's assume $m \neq 0$ and go to the rest frame where $p^0 = m$, $\vec{p} = 0$.

Then we have

$$(m\gamma^0 - m)\, u(p) = 0$$

$$\Rightarrow \quad m \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} u(p) = 0$$

So $\quad u(p) = \begin{pmatrix} a \\ b \\ a \\ b \end{pmatrix}$ for any a, b

or $\quad u(p) = \sqrt{m} \begin{pmatrix} \xi \\ \xi \end{pmatrix}$ where $\xi = \begin{pmatrix} a \\ b \end{pmatrix}$
with $|a|^2 + |b|^2 = 1$

Our choice of normalization will be convenient later.

Our rotation generators are

$$S^{ij} = \frac{1}{2} \varepsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

In particular $S_z = S^{12} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

So $\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ gives spin-z $= +\frac{1}{2}$

$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives spin-z $= -\frac{1}{2}$

Since we are in the rest frame

$$p^\mu = \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{matrix} \leftarrow p^0 \\ \leftarrow p^1 \\ \leftarrow p^2 \\ \leftarrow p^3 \end{matrix}$$

Suppose we boost to the frame where the particle has velocity $\vec{v} = v \cdot \hat{z}$. Let $\tanh \eta = \frac{v}{c}$. η is called the rapidity. Then

$$p^\mu = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix} \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} m\cosh\eta \\ 0 \\ 0 \\ m\sinh\eta \end{pmatrix}$$

In this frame $E = m\cosh\eta$, $p^3 = m\sinh\eta$

Let us now see how our $u(p)$ looks when we boost it by rapidity η in the $+z$ direction.

$$S^{03} = -\frac{i}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}$$

The boost transformation is

$$M = \exp\left[-i\eta S^{03}\right] = \exp\left[-\frac{\eta}{2}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]$$

Since $\sigma^3\sigma^3 = 1$,

$$\exp\left[x\cdot\sigma^3\right] = 1 + x\sigma^3 + \frac{x^2}{2!}1 + \frac{x^3}{3!}\sigma^3 + \frac{x^4}{4!}1 + \cdots$$
$$= \cosh x \cdot 1 + \sinh x\, \sigma^3$$

So $M = \begin{pmatrix} \cosh\frac{\eta}{2} - \sinh\frac{\eta}{2}\sigma^3 & 0 \\ 0 & \cosh\frac{\eta}{2} + \sinh\frac{\eta}{2}\sigma^3 \end{pmatrix}$

Thus $M u(p) = \sqrt{m}\begin{pmatrix} (\cosh\frac{\eta}{2} - \sinh\frac{\eta}{2}\sigma^3)\xi \\ (\cosh\frac{\eta}{2} + \sinh\frac{\eta}{2}\sigma^3)\xi \end{pmatrix}$

This looks complicated. Notice that

$$\left(\cosh\frac{\eta}{2} - \sinh\frac{\eta}{2}\sigma^3\right)^2 = \cosh^2\frac{\eta}{2} + \sinh^2\frac{\eta}{2} - 2\cosh\frac{\eta}{2}\sinh\frac{\eta}{2}\,\sigma^3$$
$$= \cosh\eta - \sinh\eta\,\sigma^3$$
$$= \frac{E}{m} - \frac{p^3}{m}\sigma^3 = \frac{p\cdot\sigma}{m} \quad (\sigma^\mu = (1, \vec{\sigma}))$$

So $\quad \sqrt{m}\,(\cosh\frac{\eta}{2} - \sinh\frac{\eta}{2}\sigma^3) = \sqrt{p\cdot\sigma}$

Similarly $\quad \sqrt{m}\,(\cosh\frac{\eta}{2} + \sinh\frac{\eta}{2}\sigma^3) = \sqrt{p\cdot\bar{\sigma}}$

(recall $\bar{\sigma}^\mu = (1, -\vec{\sigma})$)

So $\quad u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix}$ this is the general form for arbitrary p

Useful fact: $(\vec{p}\cdot\vec{\sigma})^2 = \vec{p}^{\,2}$ and so

$$(p\cdot\sigma)(p\cdot\bar{\sigma}) = (p^0)^2 - \vec{p}^{\,2} = p^2 = m^2$$

Back to our specific example with momentum

$$p = (E, 0, 0, p^3)$$

$$p\cdot\sigma = \begin{pmatrix} E - p^3 & 0 \\ 0 & E + p^3 \end{pmatrix}$$

And so $\quad \sqrt{p\cdot\sigma} = \begin{pmatrix} \sqrt{E-p^3} & 0 \\ 0 & \sqrt{E+p^3} \end{pmatrix}$

Similarly $\quad \sqrt{p\cdot\bar{\sigma}} = \begin{pmatrix} \sqrt{E+p^3} & 0 \\ 0 & \sqrt{E-p^3} \end{pmatrix}$

For spin-z $= +\frac{1}{2}$

$$u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix}$$

For spin-z $= -\frac{1}{2}$

$$u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix} \end{pmatrix}$$

Notice that in the massless limit $E \to p^3$ and so

$$u(p) = \begin{pmatrix} \begin{pmatrix}0\\0\end{pmatrix} \\ \sqrt{2E}\begin{pmatrix}1\\0\end{pmatrix} \end{pmatrix} \quad \text{for spin-z} = +\tfrac{1}{2}$$

$$u(p) = \begin{pmatrix} \sqrt{2E}\begin{pmatrix}0\\1\end{pmatrix} \\ \begin{pmatrix}0\\0\end{pmatrix} \end{pmatrix} \quad \text{for spin-z} = -\tfrac{1}{2}$$

The helicity operator

$$h \equiv \hat{p}\cdot\vec{S} \qquad \left(\text{where } \begin{aligned} S^1 &\equiv S^{23} \\ S^2 &\equiv S^{31} \\ S^3 &\equiv S^{12} \end{aligned}\right)$$

is the component of spin

in the direction of motion

$$h = \frac{1}{2}\hat{p}\cdot\begin{pmatrix}\vec{\sigma} & 0 \\ 0 & \vec{\sigma}\end{pmatrix}$$

When $h = +\frac{1}{2}$ we call it right-handed

$h = -\frac{1}{2}$ we call it left-handed

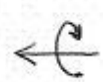
Helicity is frame dependent

Suppose we have a particle in our reference frame

left-handed

$h = -\frac{1}{2}$

If we now boost to velocity $\vec{v}' > \vec{v}$ along the direction of $\vec{v}$ then the particle now looks right-handed



$h = +\frac{1}{2}$

For massless particles h is not changeable...

you can't catch up to a massless particle since it goes at the speed of light.

Let's return to the Weyl equations.

$$i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0$$
$$i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0$$

The ∂_0 on $e^{-ip\cdot x}$ gives $-iE$
$\vec{\nabla}$ on $e^{-ip\cdot x}$ gives $i\vec{p}$

For a massless particle $\vec{p} = |\vec{p}|\,\hat{p} = E\,\hat{p}$

So the Weyl equations read

$$(E + E\,\hat{p}\cdot\vec{\sigma})\psi_L = 0 \Rightarrow E\,(1+2h)\psi_L = 0$$
$$(E - E\,\hat{p}\cdot\vec{\sigma})\psi_R = 0 \Rightarrow E\,(1-2h)\psi_R = 0$$

So ψ_L has $h = -\frac{1}{2}$
ψ_R has $h = +\frac{1}{2}$