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Course: **8.370 - QC**
Problem set: **#1**
Due: Wednesday, Sep 21, 2022
Collaborators: None, but reviewed answers with Christina Yu and Shira Asa-El

1. Useful properties of unitary matrices

(a) Consider a d-dimension quantum space with orthonormal bases $\{|1\rangle, |2\rangle, \dots, |d\rangle\}$ and $\{|v_1\rangle, |v_2\rangle, \dots, |v_d\rangle\}$. We shall construct a unitary matrix U for which $U|j\rangle = |v_j\rangle$. To this end, we use the standard basis $\{|e_1\rangle, |e_2\rangle, \dots, |e_d\rangle\}$ as an intermediate basis. The matrix that transforms $|e_j\rangle$ to $|j\rangle$ is simply one whose jth-column has the components of $|j\rangle$ in the standard basis:

$$|j\rangle = U_A |e_j\rangle \;\; \forall j = 1, 2, \dots, d \quad \text{if} \quad U_A = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ |1\rangle & |2\rangle & \dots & |d\rangle \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

Similarly,

$$|v_j\rangle = U_B |e_j\rangle \;\; \forall j = 1, 2, \dots, d \quad \text{if} \quad U_B = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ |v_1\rangle & |v_2\rangle & \dots & |v_d\rangle \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix}.$$

Since the provided bases are orthonormal, it is clear by definition of U_A and U_B that $U_A^\dagger U_A = U_B^\dagger U_B = \mathbb{I}$, so both U_A and U_B are unitary. Our desired matrix U is then given by

$$U = U_B U_A^\dagger = \begin{pmatrix} \uparrow & \uparrow & \dots & \uparrow \\ |v_1\rangle & |v_2\rangle & \dots & |v_d\rangle \\ \downarrow & \downarrow & \dots & \downarrow \end{pmatrix} \begin{pmatrix} \leftarrow & \langle 1| & \rightarrow \\ \leftarrow & \langle 2| & \rightarrow \\ \vdots & \vdots & \vdots \\ \leftarrow & \langle d| & \rightarrow \end{pmatrix},$$

which is also unitary since $U^\dagger U = U_A U_B^\dagger U_B U_A^\dagger = U_A U_A^\dagger = \mathbb{I}$. It is clear that $U|j\rangle = |v_j\rangle$, but to see explicitly, suppose we apply U to $|1\rangle$. The application of $U_A^\dagger$ returns the column vector $|e_1\rangle = (1\;0\;0\;\dots)^\top$. The subsequent application of U_B therefore returns its first column, which is $|v_1\rangle$, as desired.

More succinctly, we can write

$$\boxed{U = \sum_{i=1}^{d} |v_i\rangle \langle i|}$$

It is clear that $U|j\rangle = \sum_i |v_i\rangle \delta_{i,j} = |v_j\rangle$, as desired.

(b) Let an orthonormal basis $\{|v_1\rangle, |v_2\rangle, \dots, |v_d\rangle\}$ be given. In the standard basis $\{|e_k\rangle\}$, we may write

$$|v_i\rangle = \sum_{k=1}^{d} (v_i)_k |e_k\rangle,$$

so that

$$\sum_{i=1}^{d} |v_i\rangle\langle v_i| = \sum_{i=1}^{d} \left[\sum_{k=1}^{d} (v_i)_k |e_k\rangle\right] \left[\sum_{l=1}^{d} (v_i)_l^* \langle e_l|\right].$$

Using the fact that $|e_m\rangle\langle e_n| = 0$ if $m \neq n$ and $|e_m\rangle\langle e_m| = \Pi_m$ we have

$$\sum_{i=1}^{d} |v_i\rangle\langle v_i| = \sum_{i=1}^{d}\sum_{k=1}^{d} |(v_i)_k|^2 \Pi_k = \sum_{k=1}^{d}\sum_{i=1}^{d} |(v_i)_k|^2 \Pi_k = \sum_{k=1}^{d} \Pi_k = \mathbb{I},$$

where we have used the fact that the given basis is orthonormal in the third equality and resolution of identity with standard projections Π_k in the last equality.

2. Angle between quantum states and angle between associated points on the Bloch sphere

(a) The point $p_i = (x_i, y_i, z_i)$ on the Bloch sphere is associated with the quantum state of a qubit $|v_i\rangle$ where

$$|v_i\rangle\langle v_i| - |\bar{v}_i\rangle\langle \bar{v}_i| = x_i\sigma_x + y_i\sigma_y + z_i\sigma_z$$

where $|\bar{v}_i\rangle$ is orthogonal to $|v_i\rangle$. Because the quantum system is 2-dimensional and $|v_i\rangle \perp |\bar{v}_i\rangle$ are normalized quantum states, we have that $\{|v_i\rangle, |\bar{v}_i\rangle\}$ is an orthonormal basis. This implies

$$|v_i\rangle\langle v_i| + |\bar{v}_i\rangle\langle \bar{v}_i| = \mathbb{I}.$$

Combine this with the equation above, we find

$$|v_i\rangle\langle v_i| = \frac{\mathbb{I} + x_i\sigma_x + y_i\sigma_y + z_i\sigma_z}{2} = \boxed{\frac{\mathbb{I} + \vec{p}_i \cdot \vec{\sigma}}{2}}$$

(b) Using the fact that

$$|\langle v_1|v_2\rangle|^2 = \langle v_1|v_2\rangle\langle v_2|v_1\rangle = \mathrm{Tr}\left(|v_1\rangle\langle v_1|v_2\rangle\langle v_2|\right),$$

which can be proved using the cyclic property of the trace, we find that

$$|\langle v_1|v_2\rangle|^2 = \mathrm{Tr}\left(\frac{\mathbb{I} + \vec{p}_1 \cdot \vec{\sigma}}{2}\frac{\mathbb{I} + \vec{p}_2 \cdot \vec{\sigma}}{2}\right) = \frac{1 + \vec{p}_1 \cdot \vec{p}_2}{2}. \qquad \text{(using Mathematica)}$$

Let θ denote the angle between $|v_1\rangle$ and $|v_2\rangle$ and θ' denote the angle between $\vec{p}_1$ and $\vec{p}_2$, then

$$\theta = \arccos|\langle v_1|v_2\rangle| = \arccos\left(\sqrt{\frac{1 + |\vec{p}_1||\vec{p}_2|\cos\theta'}{2}}\right) = \arccos\left(\sqrt{\frac{1+\cos\theta'}{2}}\right) = \arccos\left(\left|\cos\frac{\theta'}{2}\right|\right) \to \frac{\theta'}{2}$$

If we ignore a possible minus sign due to relative orientation, the angle θ between quantum states is **half** the angle between associated points on the Bloch sphere. This makes sense, as *orthogonal* quantum states occupy *opposite poles* on the Bloch sphere.

Mathematica code:

```
In[7]:= Id = {{1, 0}, {0, 1}};

In[8]:= \[Sigma]x = PauliMatrix[1];

In[9]:= \[Sigma]y = PauliMatrix[2];

In[10]:= \[Sigma]z = PauliMatrix[3];

In[11]:= \[Sigma] = {\[Sigma]x, \[Sigma]y, \[Sigma]z};

In[15]:= p1 = {x1, y1, z1};

In[16]:= p2 = {x2, y2, z2};

In[32]:= M = (Id + Dot[p1, \[Sigma]]) . (Id + Dot[p2, \[Sigma]])/4;

In[29]:= Tr[M] // Simplify

Out[29]= 1/2 (1 + x1 x2 + y1 y2 + z1 z2)
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3. von Neumann measurement

We have

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1+i}{\sqrt{3}}|1\rangle .$$

Suppose we make a von Neumann measurement in the basis

$$\left\{\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right\}$$

then

$$\Pr\left(\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)\right) = \left|\left(\frac{1}{\sqrt{2}}(\langle 0| - i\langle 1|)\right)\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{1+i}{\sqrt{3}}|1\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{6}} - i\frac{i+1}{\sqrt{3}}\right|^2 = \left|\frac{2-i}{\sqrt{6}}\right|^2 = \boxed{\frac{5}{6}}$$
$$\Pr\left(\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)\right) = \left|\left(\frac{1}{\sqrt{2}}(\langle 0| + i\langle 1|)\right)\left(\frac{1}{\sqrt{3}}|0\rangle + \frac{1+i}{\sqrt{3}}|1\rangle\right)\right|^2 = \left|\frac{1}{\sqrt{6}} + i\frac{1+i}{\sqrt{3}}\right|^2 = \left|\frac{i}{\sqrt{6}}\right|^2 = \boxed{\frac{1}{6}}$$

4. Qutrit

(a) The quickest way to do this problem is writing the down matrix U that transforms $\{|0\rangle, |1\rangle, |2\rangle\}$ to $\{|a\rangle, |b\rangle, |c\rangle\}$ and check that it is unitary. Since unitary matrices are invertible and preserve orthonormality, we can conclude that $\{|a\rangle, |b\rangle, |c\rangle\}$ is an orthonormal basis. From the problem statement, we immediately have that

$$U = \begin{pmatrix} 1/2 & 1/2 & 1/\sqrt{2} \\ 1/\sqrt{2} & -1\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \end{pmatrix}$$

Since $U^\dagger U = \mathbb{I}$ (checked in Mathematica), U is indeed unitary and we're done.

Mathematica code:

```
In[26]:= U = {{1/2, 1/2, 1/Sqrt[2]}, {1/Sqrt[2], -1/Sqrt[2], 0}, {1/2,
         1/2, -1/Sqrt[2]}};

In[25]:= ConjugateTranspose[U] . U

Out[25]= {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
```

Alternatively, we could also verify that $\langle a|b\rangle = \langle b|c\rangle = \langle c|a\rangle = 0$ and $\langle a|a\rangle = \langle b|b\rangle = \langle c|c\rangle = 1$:

$$\langle a|b\rangle = \frac{1}{2}\frac{1}{2} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{2} = 0 \checkmark$$
$$\langle b|c\rangle = \frac{1}{2}\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}0 - \frac{1}{2}\frac{1}{\sqrt{2}} = 0 \checkmark$$
$$\langle c|a\rangle = \frac{1}{\sqrt{2}}\frac{1}{2} + 0\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\frac{1}{2} = 0 \checkmark$$
$$\langle a|a\rangle = \frac{1}{2}\frac{1}{2} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{2} = 1 \checkmark$$
$$\langle b|b\rangle = \frac{1}{2}\frac{1}{2} + \frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{2}} + \frac{1}{2}\frac{1}{2} = 1 \checkmark$$
$$\langle c|c\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{2}} = 0 \checkmark$$

(b) Given

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle - |2\rangle),$$

we find

$$\begin{aligned}
\Pr(|a\rangle) &= |\langle a|\psi\rangle|^2 = \left|\frac{1}{2}\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} + \frac{1}{2}\frac{-1}{\sqrt{3}}\right|^2 = \boxed{\frac{1}{6}} \\
\Pr(|b\rangle) &= |\langle b|\psi\rangle|^2 = \left|\frac{1}{2}\frac{1}{\sqrt{3}} + \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{3}} + \frac{1}{2}\frac{-1}{\sqrt{3}}\right|^2 = \boxed{\frac{1}{6}} \\
\Pr(|c\rangle) &= |\langle c|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\frac{1}{\sqrt{3}} + \frac{-1}{\sqrt{2}}\frac{-1}{\sqrt{3}}\right|^2 = \boxed{\frac{2}{3}}
\end{aligned}$$

5. Perfect polarizing filter

(a) Suppose the incoming photons have polarization state $|\psi\rangle = \cos\theta\,|H\rangle + e^{i\phi}\sin\theta\,|V\rangle$ (where H stands for horizontal and V stands for vertical). The first polarizing filter is horizontal, so $\cos^2\theta$ of the photons make it through and become $|H\rangle$. The second filter is at 45° relative to the first polarizing filter (and thus $|H\rangle$). So, half the photons make it through and their state becomes $|D\rangle$ where D stands for diagonal. A similar scenario applies for the last polarizing filter. So, **25%** of photons coming out of the first polarizing filter make it out of the final one.

(b) Intuition tells us that we get the most light through if the angle of the polarizing filter relative to the polarization state is as small as possible, so we want to put the 30°-rotated polarizing filter after the horizontal polarizing filter, followed by the 60°-rotated one. Let's check.

For the case described above, $\cos^2 30° = 3/4$ of the photons coming through the horizontal filter make it through the 30°-rotated filter. Next 60°-rotated filter is at $60° - 30° = 30°$ relative to the polarization state of photons leaving the 30°-rotated filter, so we once again lose 1/4 of the incoming photons. Finally, the vertical polarizing filter is at 30° relative to the incoming photons, so we lose yet another 1/4. So,

$$\frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$$

of the photons coming out of the first polarizing filter make it out of the final one. Assuming the incoming photons are randomly polarized, we end up with **27/128** of the initial light.

On the other hand, if we put the 60°-rotated filter first, then only $\cos^2 60° = 1/4$ of the photons leaving the horizontally rotated filter make it through. The subsequent 30°-rotated filter is $-30°$ relative to the polarization state leaving the 60°-rotated filter, so 3/4 of the incoming photon make it through. Finally, the vertical filter is at 60° relative to the polarization state leaving the 30° filter, so 1/4 of the incoming photons make it through. We see that only

$$\frac{1}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{3}{64}$$

of the photons coming out of the first polarizing filter make it out of the final one, which is less than what we have in the first case.

6. Copying a qubit?

(a) After measuring the challenger's qubit in the $\{|0\rangle, |1\rangle\}$ basis and and making two copies of the resulting state, we are guaranteed to hand the challenger either two qubits in state $|0\rangle$ or two qubits in state $|1\rangle$.

With probability 1/4 the challenger gives us a qubit in state $|0\rangle$. After our measurement, we return two qubits in state $|0\rangle$ to the challenger with probability 1. The challenger measures both qubits in the $\{|0\rangle, |1\rangle\}$ basis and finds both to be in $|0\rangle$ with probability 1. We succeed. In this scenario we always succeed.

A similar argument applies if the challenger gives us a qubit in $|1\rangle$. We also always succeed

With probability 1/4 the challenger gives us a qubit in $|+\rangle$. We measure this qubit and obtain $|0\rangle$ with probability 1/2 and $|1\rangle$ with probability 1/2. No matter which two qubits we give the challenger ($|0\rangle|0\rangle$ or $|1\rangle|1\rangle$), he measures and finds them both in $|+\rangle$ with probability 1/4. In this scenario, we succeed 1/4 of the time.

A similar argument applies if the challenger gives us a qubit in $|-\rangle$. We also succeed with probability 1/4 in this scenario.

In summary, we pass the test with probability

$$\Pr(\text{pass}) = \frac{1}{4} \times 1 + \frac{1}{4} \times 1 + \frac{1}{4} \times \frac{1}{4} + \frac{1}{4} \times \frac{1}{4} = \boxed{\frac{5}{8}}$$

(b) Suppose that we now measure the provided qubits in the basis

$$\{|\theta_+\rangle, |\theta_-\rangle\} \equiv \{\cos\theta\,|0\rangle + \sin\theta\,|1\rangle, -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle\}.$$

With probability 1/4 the challenger gives us a qubit in $|0\rangle$. With probability $\cos^2\theta$ we find it in $|\theta_+\rangle$ and $\sin^2\theta$ in $|\theta_-\rangle$. If we give two $|\theta_+\rangle$ to the challenger, he finds them both in $|0\rangle$ with probability $\cos^4\theta$. If we give two $|\theta_-\rangle$, he finds them both in $|1\rangle$ with probability $\sin^4\theta$. So, in this scenario we succeed with probability $\cos^6\theta + \sin^6\theta$.

By the same analysis, we will also succeed the probability $\cos^6\theta + \sin^6\theta$ if the challenger give us $|1\rangle$.

With probability 1/4 the challenger gives us a qubit in $|+\rangle$. With probability $(\cos\theta + \sin\theta)^2/2$ we find it in $|\theta_+\rangle$ and $(\cos\theta - \sin\theta)^2/2$ in $|\theta_-\rangle$. If we give two $|\theta_+\rangle$ to the challenger, he finds them both in $|0\rangle$ with probability $(\cos\theta + \sin\theta)^4/4$. If we give two $|\theta_-\rangle$, he finds them both in $|1\rangle$ with probability $(\cos\theta - \sin\theta)^4/4$. So, in this scenario we succeed with probability $(\cos\theta + \sin\theta)^6/8 + (\cos\theta - \sin\theta)^6/8$.

By the same analysis, we will also succeed the probability $(\cos\theta + \sin\theta)^6/8 + (\cos\theta - \sin\theta)^6/8$ if the challenger give us $|-\rangle$.

In summary, we pass the test with probability:

$$\begin{aligned}\Pr(\text{pass}) =& \frac{1}{4} \times (\cos^6\theta + \sin^6\theta) + \frac{1}{4} \times (\cos^6\theta + \sin^6\theta) \\ &+ \frac{1}{4} \times \left[\frac{(\cos\theta + \sin\theta)^6}{8} + \frac{(\cos\theta - \sin\theta)^6}{8}\right] + \frac{1}{4} \times \left[\frac{(\cos\theta + \sin\theta)^6}{8} + \frac{(\cos\theta - \sin\theta)^6}{8}\right] \\ =& \boxed{\frac{5}{8}} \qquad \text{(simplified in Mathematica)}\end{aligned}$$

Mathematica code:

```
In[50]:= P = (1/2)*(Cos[\[Theta]]^6 + Sin[\[Theta]]^6) +
2*((1/4)*(1/8)*(Cos[\[Theta]] + Sin[\[Theta]])^6 + (1/4)*(1/
8)*(Cos[\[Theta]] - Sin[\[Theta]])^6) // FullSimplify

Out[50]= 5/8
```

(c) From this analysis in Part (b), the choice of θ does not matter.