➔ 65. Show that in a set of six people, there is a set of at least three people who all know each other, or a set of at least three people none of whom know each other. (We assume that if person 1 knows person 2, then person 2 knows person 1.)

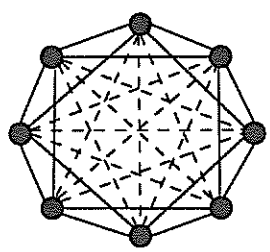
Solution: By the generalized pigeonhole principle, person 1 either knows at least three people or doesn't know at least three people. Suppose person 1 knows three people. Then either two of these people know each other, giving us, with person 1, three mutual acquaintances, or no two of these people know each other, giving us three mutual strangers. On the other hand if there are three people person 1 does not know, then either two of these people don't know each other, giving us, with person 1, three mutual strangers, or all three of these people know each other, giving us three mutual acquaintances. ∎

·69. Show that among an odd number of people there is at least one person who is an acquaintance of an even number of people and therefore also a stranger to an even number of people.

Solution: Suppose we add, for each person, the number of people with whom he or she is acquainted. Then we get twice the number of acquaintance edges in the graph of acquaintance and non-acquaintance relationships. Thus the sum must be even. But if each person among an odd number of people were acquainted with an odd number of people, then the sum would be odd. Since this is a contradiction, among an odd number of people, there must be at least one who is acquainted with an even number of people. Since the number of people different from this person is even, the number of people with whom this person is not acquainted is also even. ▮

·70. Find a way to color the edges of a K_8 with red and green so that there is no red K_4 and no green K_3.

Solution: In the graph



there is no K_3 whose edges are dashed, and no K_4 whose edges are solid. By symmetry, to verify this you need only look at vertex 1 and vertices connected to it by either dashed lines or by solid lines. ▮

➔·71. Find $R(4,3)$.

Solution: $R(4,3) = 9$. In Problem 70 we showed that $R(4,3)$ is more than 8. So we must show that if we have nine people, we either have 4 mutual acquaintances or three mutual strangers. By Problem 69 there is at least one person (say person A) who is acquainted with an even number of people. If person A is acquainted with six or more people, then among these six people, there are either three mutual acquaintances or three mutual strangers. If there are three mutual strangers, we are done; if there are three mutual acquaintances, they, together with Person A are four mutual acquaintances. Thus we may assume Person A is acquainted with at most four people. Thus person A is a stranger to at least four people. If two of these people are strangers, then they, together with person A form three mutual strangers and we are done. Otherwise all of these people know each other and we have at least four mutual acquaintances, and so in every possible situation, we have either four mutual acquaintances or three mutual strangers. ▮

10. A town has n streetlights running along the north side of Main Street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green?

Solution: We can think of first choosing the set of even size of poles to be painted green, and the painting the remaining poles red, white, and blue. We may do this in $\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} 3^{n-2k}$ ways. ∎