8.512 Recitation 12

Today: 1) Signatures of Anderson localization
2) Scattering matrix for a δ-potential
3) Transfer matrix approach for disordered 1D system

Ref: arXiv 1005.0915 and 1710.01234

1) Consider a Hamiltonian w/ a disordered potential

$$H = K + V$$

- Classically: if $K \gg |V|$ particles move ballistically, ignoring potential landscape
  if $K \ll |V|$ particles localize in potential well

- In QM, wavefunctions can tunnel through potential barriers and can even be reflected by small potentials. Wavepacket splits into transmitted and reflected part after each scattering ⇒ random walk ⇒ diffusion

- However, phase coherence can lead to localization

$$|\psi(x)|^2 \sim e^{-|x|/\xi_{loc}}$$

ξ_{loc} → localization length

This also implies absence of transport $\sigma \sim e^{-L/\xi_{loc}}$

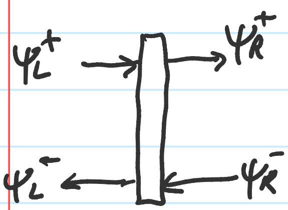
- Today we will see this scaling in a 1D non-interacting system with disorder.

- Surprisingly, localization seems to persist even in the presence of interactions (MBL) (somewhat debated?)

• Consider a simple scattering problem off a δ-potential in 1D

$$V(x) = V_0\,\delta(x)$$

Split wf. into left and right-moving components



$$\psi(x) = \begin{cases} \psi_L^+ e^{ikx} + \psi_L^- e^{-ikx} &, \quad x<0 \\ \psi_R^+ e^{ikx} + \psi_R^- e^{-ikx} &, \quad x>0 \end{cases}$$

$\psi_R^\pm$ can be related to $\psi_L^\pm$ using the boundary conditions at $x=0$: a) $\psi(x)$ is continuous

b) $\psi'(0^+) - \psi'(0^-) = \frac{2mV_0}{\hbar^2}\psi(0)$

$$\Rightarrow \underbrace{\begin{pmatrix} \psi_L^- \\ \psi_R^+ \end{pmatrix}}_{\text{outgoing part}} = \underbrace{\begin{pmatrix} r & t' \\ t & r' \end{pmatrix}}_{\text{scattering matrix } S} \underbrace{\begin{pmatrix} \psi_L^+ \\ \psi_R^- \end{pmatrix}}_{\text{incoming part}} \quad \Leftrightarrow \quad \boxed{\psi_{out} = S\,\psi_{in}}$$

Probability conservation requires that S be unitary!

• Introduce reflection and transmission coeficients:

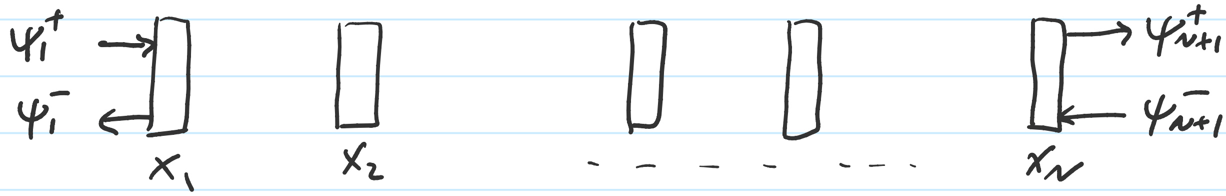
$$R = |r|^2 \qquad T = |t|^2 \qquad R + T = 1 \qquad \text{since} \quad S^\dagger S = I$$
$$R' = |r'|^2 \qquad T' = |t'|^2 \qquad R' + T' = 1$$

• For the δ-potential we find

$$S = \frac{1}{1+if}\begin{pmatrix} -if & 1 \\ 1 & if \end{pmatrix} \qquad \text{with} \quad f = \frac{mV_0}{\hbar^2 k}$$

and $\quad T = T' = \frac{1}{1+f^2} \quad , \quad R = R' = \frac{f^2}{1+f^2}$

3) Now consider a line of potentials at random x_i



and potentials are well-separated: $n = \frac{N}{L} \ll k$

- It's more convenient to define a transfer matrix

$$\begin{pmatrix} \psi_R^+ \\ \psi_R^- \end{pmatrix} = M \begin{pmatrix} \psi_L^+ \\ \psi_L^- \end{pmatrix} \qquad \text{with} \qquad M = \begin{pmatrix} \frac{1}{t^*} & -\frac{r^*}{t^*} \\ -\frac{r}{t} & \frac{1}{t} \end{pmatrix}$$

$\det M = 1$

$$\Rightarrow \begin{pmatrix} \psi_{N+1}^+ \\ \psi_{N+1}^- \end{pmatrix} = M_N \cdots M_2 M_1 \begin{pmatrix} \psi_1^+ \\ \psi_1^- \end{pmatrix}$$

- First consider only 2 obstacles

$$M_{12} = M_2 M_1 = \begin{pmatrix} \frac{1}{t_{12}^*} & -\frac{r_{12}^*}{t_{12}^*} \\ -\frac{r_{12}}{t_{12}} & \frac{1}{t_{12}} \end{pmatrix}$$

with $t_{12} = \dfrac{t_1 t_2}{1 - r_1' r_2} = t_2 t_1 + \underbrace{t_2 r_1' r_2 t_1 + t_2 (r_1' r_2)^2 t_1 + \ldots}_{\text{series of internal reflections}}$

Let $r_1' r_2 \equiv \sqrt{R_1 R_2}\, e^{i\theta}$ where θ is a phase accumulated during internal reflections.

Since potentials are placed at random distances $k\Delta x \gg 2\pi$

the phase θ can be taken as random uniform in $[0, 2\pi]$

$\Rightarrow$ expectation values can be computed as

$$\langle P(\theta) \rangle = \int^{2\pi} d\theta\, P(\theta)$$

$$\langle f(\theta)\rangle = \int_0^{2\pi} \frac{d\theta}{2\pi} f(\theta)$$

but what quantity $f(\theta)$ should be averaged?

a) Incoherent transmission — Ohm's law:

$$T_{12} = \frac{T_1 T_2}{|1-\sqrt{R_1R_2}\,e^{i\theta}|^2} \qquad \langle T_{12}\rangle = \frac{T_1T_2}{1-R_1R_2}$$

- This can be captured by an S-matrix directly relating prob. instead of amps.

$$\begin{pmatrix} |\psi_L^-|^2 \\ |\psi_R^+|^2 \end{pmatrix} = \tilde{S} \begin{pmatrix} |\psi_L^+|^2 \\ |\psi_R^-|^2 \end{pmatrix} \quad \text{with} \quad \tilde{S} = \begin{pmatrix} R & T \\ T & R \end{pmatrix}$$

Completely classical: phase gets scrambled between two scatterings.

- Define a element "resistance" of each barrier $\frac{R}{T} = \frac{1-T}{T}$

$$\frac{1-\langle T_{12}\rangle}{\langle T_{12}\rangle} = \frac{1-T_1}{T_1} + \frac{1-T_2}{T_2} \quad \Rightarrow \text{additive resistance!}$$

For N barriers, the resistance is

$$N\frac{R_1}{T_1} \equiv \frac{L}{\ell_1} \qquad \text{with} \quad \boxed{\ell_1 \equiv \frac{T_1}{nR_1}}$$

ℓ_1 — backscattering strength of each potential

- Equivalently, the problem can be viewed as a 1D random walk w/ prob T_1 to move right and $R_1 = 1-T_1$ to move left

- Diffussion constant is $D = v \cdot \frac{1}{n} \cdot \frac{T_1}{2R_1} = \frac{v \ell_1}{2}$

velocity
Between collisions

In 1D, ℓ_1 is twice the mean free path $\ell_1 = 2\ell$

B) Coherent transmission — localization:

Generalizing T_{12} to T_N is very complicated; need additive quantity!

- Introduce the extinction coefficient

$$\mathcal{K} = -\ln T = |\ln T|$$
$$\mathcal{K}_{12} = \mathcal{K}_1 + \mathcal{K}_2 + \ln|1 - \sqrt{R_1 R_2}\, e^{i\theta}|^2$$

But $\langle \ln|1-\sqrt{R_1R_2}\, e^{i\theta}|^2 \rangle = 0$ so $\mathcal{K}$ is additive!

$\Rightarrow \quad \langle |\ln T_N| \rangle = N \cdot |\ln T_1| \qquad N = nL$

$\Rightarrow \quad e^{\langle \ln T_N \rangle} = e^{-L/\xi_{loc}} \qquad |\ln T_N| = -\ln T_N$

with $\xi_{loc} = \frac{1}{n|\ln T_1|} = \frac{1}{n \ln(\frac{1}{T_1})} \approx \frac{1}{n(\frac{1}{T_1} - 1)} = \frac{T_1}{nR_1} = \ell_1$

$$\Leftrightarrow \quad \boxed{\xi_{loc} = \ell_1 = 2\ell}$$

- Note that T is not self averaging
$\langle T \rangle \neq T_{typ}$ (most probable value)

but instead $\langle \ln T \rangle = \ln T_{typ}$
This is because T obeys a log-normal distribution.

- The most probable value of conductivity is

- The most probable value of conductivity is

$$\frac{T_{typ}}{R_{typ}} = \frac{T_{typ}}{1-T_{typ}} \approx T_{typ} = e^{-L/\xi_{loc}}$$

- Depending on system size L, there are 3 transport regimes

$$L < \ell \quad \text{— Ballistic}$$
$$\ell < L < \xi_{loc} \quad \text{— diffusive} \qquad (\ell = \text{mean free path})$$
$$L > \xi_{loc} \quad \text{— localized}$$

In $d=1$, $\xi_{loc} = 2\ell$ so the system localizes with no room for diffusion.

In $d=2$, $\underset{(RG)}{\xi_{loc}} \gg 2\ell$ so the regimes are distinct.

In $d \geq 3$, can have a delocalization transition, depending on disorder strength.