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Course: **8.370 - QC**
Problem set: **#2**
Due: Wednesday, Sep 28, 2022
Collaborators:

1. Tensor products

Starting with

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

we have

$$\begin{aligned}
|\psi_x\rangle &= \sigma_x \otimes \mathbb{I}\,|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle - |00\rangle) \\
\left|\psi_y\right\rangle &= \sigma_y \otimes \mathbb{I}\,|\psi\rangle = \frac{1}{\sqrt{2}}(i\,|11\rangle + i\,|00\rangle) = \frac{i}{\sqrt{2}}(|11\rangle + |00\rangle) \\
|\psi_z\rangle &= \sigma_z \otimes \mathbb{I}\,|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)
\end{aligned}$$

Now we check that they are all orthogonal:

$$\begin{aligned}
\langle\psi_x|\psi_x\rangle &= \frac{1}{2}(\langle 11|11\rangle + \langle 00|00\rangle - \langle 00|11\rangle - \langle 11|00\rangle) = 1 \\
\left\langle\psi_y\middle|\psi_y\right\rangle &= \frac{1}{2}(\langle 11|11\rangle + \langle 00|00\rangle + \langle 00|11\rangle + \langle 11|00\rangle) = 1 \\
\langle\psi_z|\psi_z\rangle &= \frac{1}{2}(\langle 01|01\rangle + \langle 10|10\rangle + \langle 01|10\rangle + \langle 10|01\rangle) = 1 \\
\left\langle\psi_x\middle|\psi_y\right\rangle &= \frac{i}{2}(\langle 11|11\rangle - \langle 00|00\rangle + \langle 11|00\rangle - \langle 00|11\rangle) = 0 \\
\left\langle\psi_y\middle|\psi_z\right\rangle &= \frac{-i}{2}(\langle 11|01\rangle + \langle 11|10\rangle + \langle 00|01\rangle + \langle 00|10\rangle) = 0 \\
\langle\psi_z|\psi_x\rangle &= \frac{1}{2}(\langle 01|11\rangle + \langle 10|11\rangle - \langle 01|00\rangle - \langle 10|00\rangle) = 0
\end{aligned}$$

2. Observable with repeated eigenvalues

$$M = \begin{pmatrix} 2 & 2 & 2 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

(a) Computing the eigenvalues associated with the with given eigenvalues is easy:

$$\begin{aligned}
\vec{v}_1 = (2\ 1\ 1)^\top : &\qquad \lambda = 4 \\
\vec{v}_2 = (1\ \ -1\ \ -1)^\top : &\qquad \lambda = -2 \\
\vec{v}_3 = (0\ 1\ \ -1)^\top : &\qquad \lambda = -2
\end{aligned}$$

Before forming orthogonal projections, we need to make sure that the provided vectors form an orthogonal basis. By inspection, we need to first normalize the vectors to get

$$|\psi_1\rangle = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{6}}(2\ 1\ 1)^\top, \quad |\psi_2\rangle = \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{3}}(1\ \ -1\ \ -1)^\top, \quad |\psi_3\rangle = \frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{1}{\sqrt{2}}(0\ 1\ \ -1)^\top.$$

Now we check for orthonormality. We can do this by inspection so I won't write out the algebra.

$$\langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = \langle\psi_3|\psi_3\rangle = 1$$
$$\langle\psi_1|\psi_2\rangle = \langle\psi_2|\psi_3\rangle = \langle\psi_3|\psi_1\rangle = 0.$$

With these conditions satisfied, the orthogonal projections are:

$$\Pi_1 = |\psi_1\rangle\langle\psi_1| = \frac{1}{6}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$
$$\Pi_2 = |\psi_2\rangle\langle\psi_2| = \frac{1}{3}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$
$$\Pi_3 = |\psi_3\rangle\langle\psi_3| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Sanity check:

$$\Pi_1 + \Pi_2 + \Pi_3 = \mathbb{I} \checkmark$$
$$\Pi_1\Pi_2 = \Pi_2\Pi_3 = \Pi_3\Pi_1 = \mathcal{O} \checkmark$$

All the algebra is from above is verified in Matheatica. Mathematica calculations:

```
(*eigv check*)
In[53]:= M = {{2, 2, 2}, {2, -1, 1}, {2, 1, -1}};
In[5]:= Eigenvalues[M]
Out[5]= {4, -2, -2}

(*vectors in the ONB*)
p1 = {2, 1, 1}/Norm[{2, 1, 1}];
p2 = {1, -1, -1}/Norm[{1, -1, -1}];
p3 = {0, 1, -1}/Norm[{0, 1, -1}];

(*check ONB*)
In[14]:= Dot[p1, p2]
Out[14]= 0
In[15]:= Dot[p2, p3]
Out[15]= 0
In[16]:= Dot[p3, p1]
Out[16]= 0
In[21]:= Dot[p1, p1]
Out[21]= 1
In[22]:= Dot[p2, p2]
Out[22]= 1
In[23]:= Dot[p3, p3]
Out[23]= 1

(*Compute projectors*)
In[46]:= M1 = KroneckerProduct[p1, p1]
Out[46]= {{2/3, 1/3, 1/3}, {1/3, 1/6, 1/6}, {1/3, 1/6, 1/6}}
In[47]:= M2 = KroneckerProduct[p2, p2]
Out[47]= {{1/3, -(1/3), -(1/3)}, {-(1/3), 1/3, 1/3}, {-(1/3), 1/3, 1/3}}
In[48]:= M3 = KroneckerProduct[p3, p3]
Out[48]= {{0, 0, 0}, {0, 1/2, -(1/2)}, {0, -(1/2), 1/2}}

(*Check resolution of identity:*)
In[52]:= M1 + M2 + M3
Out[52]= {{1, 0, 0}, {0, 1, 0}, {0, 0, 1}}
In[205]:= M1 . M2
Out[205]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
In[206]:= M2 . M3
Out[206]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
In[207]:= M3 . M1
Out[207]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

(b) We are given

$$|\psi\rangle = \frac{2}{3}|0\rangle + \frac{2}{3}|1\rangle - \frac{1}{3}|2\rangle\,.$$

When this qutrit is measured, the possible outcomes are 4 and −2, with probabilities:

$$\Pr(4) = \langle\psi|\,\Pi_1\,|\psi\rangle = \frac{4}{9} \quad \text{and} \quad \Pr(-2) = \langle\psi|\,\Pi_2\,|\psi\rangle + \langle\psi|\,\Pi_3\,|\psi\rangle = \frac{5}{9}.$$

There are two ways to get the answer. By inspection, we can immediately see that the probability of measuring 4 is 4/9, since the coefficient for $|0\rangle$ is 2/3. From there, we can conclude that the probability of measuring −2 is simply $1 - 4/9 = 5/9$. The other way to find these values is by directly doing the algebra. The Mathematica code below has the explicit calculations.

```
In[58]:= \[Psi] = (2/3)*p1 + (2/3)*p2 - (1/3)*p3;

(*Pr(4)*)
In[71]:= Transpose[\[Psi]] . M1 . \[Psi] // FullSimplify
Out[71]= 4/9

(*Pr(-2)*)
In[72]:= Transpose[\[Psi]] . M2 . \[Psi] +  Transpose[\[Psi]] . M3 . \[Psi] // FullSimplify
Out[72]= 5/9
```

3. Spin-1 particle

We are given a spin-1 particle with three quantum states $|1\rangle\,,|0\rangle\,,|-1\rangle$. The observables corresponding to the spin along the three spatial directions are J_x, J_y, J_z:

$$J_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad J_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

(a) We will show that J_x, J_z cannot be measured simultaneously by showing that they do not commute:

$$[J_x, J_z] = J_xJ_z - J_zJ_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} = -iJ_y \neq O.$$

Mathematica code:

```
In[83]:= Jz = {{1, 0, 0}, {0, 0, 0}, {0, 0, -1}};
In[84]:= Jx = (1/Sqrt[2])*{{0, 1, 0}, {1, 0, 1}, {0, 1, 0}};
In[85]:= Jy = (1/Sqrt[2])*{{0, -I, 0}, {I, 0, -I}, {0, I, 0}};
In[87]:= Jx . Jz - Jz . Jx
Out[87]= {{0, -(1/Sqrt[2]), 0}, {1/Sqrt[2], 0, -(1/Sqrt[2])}, {0, 1/Sqrt[2], 0}}
```

(b) However, the observables J_x^2, J_y^2, J_z^2 all commute. We can do this by hand or use Mathematica again:

```
(*[Jx^2,Jy^2]*)
In[91]:= (Jx . Jx) . (Jy . Jy) - (Jy . Jy) . (Jx . Jx)
Out[91]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

(*[Jy^2,Jz^2]*)
In[92]:= (Jy . Jy) . (Jz . Jz) - (Jz . Jz) . (Jy . Jy)
Out[92]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

(*[Jz^2,Jx^2]*)
In[93]:= (Jz . Jz) . (Jx . Jx) - (Jx . Jx) . (Jz . Jz)
Out[93]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}
```

There are possibly multiple ways (including clever math tricks) to find the simultaneous eigenvectors for J_x^2, J_y^2, J_z^2. However, it turns out that we could also do this by inspection:

$$J_x^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad J_y^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \quad J_z^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By the form of the matrices, we can guess that the three normalized simultaneous eigenvectors are

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |-1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |-1\rangle), \quad |0\rangle = |0\rangle\,.$$

The corresponding eigenvalues can be found from the results below:

$$\begin{aligned}
J_x^2\,|+\rangle &= |+\rangle \\
J_x^2\,|-\rangle &= 0 \\
J_x^2\,|0\rangle &= |0\rangle \\
J_y^2\,|+\rangle &= 0 \\
J_y^2\,|-\rangle &= |-\rangle \\
J_y^2\,|0\rangle &= |0\rangle \\
J_z^2\,|+\rangle &= |+\rangle \\
J_z^2\,|-\rangle &= |-\rangle \\
J_z^2\,|0\rangle &= 0
\end{aligned}$$

So, J_i^2 has spectrum $\{0, 1\}$ for all $i = x, y, z$. Finally, we have

$$J^2 = J_x^2 + J_y^2 + J_z^2 = 2\mathbb{I}.$$

While a lot of the calculations in this problem could be done by hand, it is faster and more accurate to do them in Mathematica:

```
(*squaring*)
Jx2 = Jx . Jx;
Jy2 = Jy . Jy;
Jz2 = Jz . Jz;

In[117]:= Jx2
Out[117]= {{1/2, 0, 1/2}, {0, 1, 0}, {1/2, 0, 1/2}}
In[118]:= Jy2
Out[118]= {{1/2, 0, -(1/2)}, {0, 1, 0}, {-(1/2), 0, 1/2}}
In[119]:= Jz^2
Out[119]= {{1, 0, 0}, {0, 0, 0}, {0, 0, 1}}


(*eigenvalues calcs*)
In[140]:= plus = (1/Sqrt[2]) {1, 0, 1};
In[152]:= minus = (1/Sqrt[2])*{1, 0, -1};
In[153]:= zero = {0, 1, 0};
In[145]:= Jx2 . plus
Out[145]= {1/Sqrt[2], 0, 1/Sqrt[2]}
In[154]:= Jx2 . minus
Out[154]= {0, 0, 0}
In[149]:= Jx2 . zero
Out[149]= {0, 1, 0}
In[150]:= Jy2 . plus
Out[150]= {0, 0, 0}
In[155]:= Jy2 . minus
Out[155]= {1/Sqrt[2], 0, -(1/Sqrt[2])}
In[156]:= Jy2 . zero
Out[156]= {0, 1, 0}
In[157]:= Jz2 . plus
Out[157]= {1/Sqrt[2], 0, 1/Sqrt[2]}
In[159]:= Jz2 . minus
Out[159]= {1/Sqrt[2], 0, -(1/Sqrt[2])}
In[160]:= Jz2 . zero
Out[160]= {0, 0, 0}

(*finally, total spin*)
In[94]:= Jx . Jx + Jy . Jy + Jz . Jz
Out[94]= {{2, 0, 0}, {0, 2, 0}, {0, 0, 2}}
```

4. Deriving Spin-1 Observables

In this problem we derive the matrix J_x in the previous problem. Suppose we have two qubits A and B. The observable giving the spin in the x-direction is

$$S_x = \frac{1}{2}\left(\sigma_x^A \otimes \mathbb{I}^B + \mathbb{I}^A \otimes \sigma_x^B\right).$$

The 3-dimensional subspace of the 4-dimensional state space of two qubits which corresponds to the state space of a spin-1 particle is the subspace orthogonal to the state $(|01\rangle - |10\rangle)/\sqrt{2}$. To avoid confusion, let us replace 0 with $\uparrow$ and 1 with $\downarrow$

Since we have two qubits, we can treat them as two spin-1/2 particles, each denoted by $|s, m_s\rangle$. In this notation, we have

$$\begin{aligned}
|\uparrow\uparrow\rangle &= |1/2, +1/2\rangle \otimes |1/2, +1/2\rangle \\
|\uparrow\downarrow\rangle &= |1/2, +1/2\rangle \otimes |1/2, -1/2\rangle \\
|\downarrow\uparrow\rangle &= |1/2, -1/2\rangle \otimes |1/2, +1/2\rangle \\
|\downarrow\downarrow\rangle &= |1/2, -1/2\rangle \otimes |1/2, -1/2\rangle
\end{aligned}$$

When the spins are added, we can express the total spin and its projection as $|s, m\rangle \in \mathcal{H}^{\otimes 2}$ where

$$|s, m\rangle = \sum_{m_{s,1}=-s_1}^{s_1} \sum_{m_{s,2}=-s_2}^{s_2} C^{s,m}_{s_1,m_{s,1},s_2,m_{s,2}} |s_1, m_{s,1}\rangle |s_2, m_{s,2}\rangle$$

where $C^{\cdots}_{\cdots}$'s are the Clebsch-Gordan coefficients. For this problem, the solution is rather simple. In the two-qubit Hilbert space, there is one state (the singlet) for which the total spin is zero ($s = 0$), and this state is one given in the problem: $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$, and there are three states (triplet) which correspond to $s = 1$ (total spin equal to 1). It turns out that these are

$$\begin{aligned}
|s = 1, m = +1\rangle &= |\uparrow\uparrow\rangle \\
|s = 1, m = 0\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) \\
|s = 1, m = -1\rangle &= |\downarrow\downarrow\rangle
\end{aligned}$$

With this information, we can now construct a unitary matrix which transforms the standard basis $\{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ into the new basis where the first elements has spin 0 and the subsequent three has spin 1: $\{|0,0\rangle, |1,1\rangle, |1,0\rangle, |1,-1\rangle\}$. By inspection, this matrix is

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We expect that a similarity transformation on S_x by U will take the form of a (2×2)-block diagonal matrix of the form $\mathrm{diag}(0, Jx)$. And indeed, using Mathematica, we find that

$$U^\dagger S_x U = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \\ & J_x \end{pmatrix}.$$

With this, we have

$$J_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

as desired.

Mathematica calculations:

```
In[220]:=
U = {{0, 1, 0, 0}, 1/Sqrt[2]*{1, 0, 1, 0},
    1/Sqrt[2]*{-1, 0, 1, 0}, {0, 0, 0, 1}};

In[222]:= ConjugateTranspose[U] . SX . U
Out[222]= {{0, 0, 0, 0}, {0, 0, 1/Sqrt[2], 0}, {0, 1/Sqrt[2], 0, 1/
        Sqrt[2]}, {0, 0, 1/Sqrt[2], 0}}

(*displaying the result in matrix form gives Jx*)
```

5. Generalized Measurements

Here we derive an example of a non-von Neumann measurement. We're given one of the three states

$$|\psi_1\rangle = |0\rangle\,, \quad |\psi_2\rangle = -\frac{1}{2}\,|0\rangle + \frac{\sqrt{3}}{2}\,|1\rangle\,, \quad |\psi_3\rangle = -\frac{1}{2}\,|0\rangle - \frac{\sqrt{3}}{2}\,|1\rangle$$

with equal probabilities.

(a) If we choose our measurement basis at arbitrary angles then whenever we measure $|0\rangle$ or $|1\rangle$, we have no way of telling which state is the input with certainty. So we may begin by making a random guess, and the probability of getting it right is just 1/3, since we have no extra information.

Can we improve? Now suppose we try to be clever and pick a special basis, say without loss of generality we pick $\{|0\rangle\,, |1\rangle\}$. If we measure $|0\rangle$, then again we have no way of knowing for sure which state was the input. However, if we guess that it is state $|0\rangle$ then the probability of getting that right is:

$$\Pr\big(\text{correctly guess } |0\rangle \mid \text{see } |0\rangle\big) = \frac{1}{1 + 1/4 + 1/4} = \frac{2}{3},$$

where 1/4 is the probability of measuring $|0\rangle$ if the other two states were the input, and 1 is the probability of measuring $|0\rangle$ when the input state is $|0\rangle$.

On the other hand, if we measure $|1\rangle$, then we know for sure the input is not $|0\rangle$. However, since the other two states are equally like to give $|1\rangle$ as the measurement result, we have no way of distinguishing which state is the input. The probability of correctly guessing it is one of the other two states, provided that our measurement results in $|1\rangle$ is 1/2:

$$\Pr\big(\text{correctly guess } |\psi_2\rangle \mid \text{see } |1\rangle\big) = \Pr\big(\text{correctly guess } |\psi_3\rangle \mid \text{see } |1\rangle\big) = \frac{1}{2}.$$

So, the probability that we make a correct guess cannot exceed

$$\frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{2} = \frac{7}{12} < \frac{2}{3},$$

as desired. Here, 1/2 is the probability of measuring $|0\rangle$ or $|1\rangle$ *overall*.

But is this the best we can do? We could be suspicious and say that perhaps we did not choose the most clever basis. So let us consider an arbitrary basis

$$\{|A\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle\,, |B\rangle = -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle\}$$

Without loss of generality, suppose we guess the input is $|0\rangle$ if our measurement results in $|A\rangle$. Then,

$$\Pr\big(\text{correctly guess } |0\rangle \mid \text{see } |A\rangle\big) = \frac{\cos^2\theta}{\cos^2\theta + \frac{1}{4}(\cos\theta - \sqrt{3}\sin\theta)^2 + \frac{1}{4}(\cos\theta + \sqrt{3}\sin\theta)^2} = \frac{2\cos\theta^2}{3},$$

where the denominator is the sum of the probabilities of measuring $|A\rangle$.

We can further assume that θ is small by symmetry, so that $\cos^2\theta > \sin^2\theta$, so that this is the more optimal way to guess. If we find $|1\rangle$ after our measurement, then we will want to guess one of the two states $|\psi_2\rangle$ or $|\psi_3\rangle$ that would maximize our chance of being correct. This probability is at most

$$\Pr\big(\text{correctly guess } |\psi_i\rangle, i = 2,3 \mid \text{see } |B\rangle\big) = \frac{\max\left[\frac{1}{4}(\sqrt{3}\cos\theta + \sin\theta)^2, \frac{1}{4}(-\sqrt{3}\cos\theta + \sin\theta)^2\right]}{\sin^2\theta + \frac{1}{4}(\sqrt{3}\cos\theta + \sin\theta)^2 + \frac{1}{4}(-\sqrt{3}\cos\theta + \sin\theta)^2},$$

where the denominator is the sum of the probabilities of measuring $|B\rangle$. Since the probabilities of measuring $|A\rangle$ or $|B\rangle$ are equally $1/2$, the probability that we succeed cannot exceed:

$$\max_{\theta\in[0,\pi/2]}\left[\frac{1}{2}\Pr\big(\text{correctly guess } |0\rangle \mid \text{see } |A\rangle\big) + \frac{1}{2}\Pr\big(\text{correctly guess } |\psi_i\rangle, i = 2,3 \mid \text{see } |B\rangle\big)\right] = 0.622 < \frac{2}{3}.$$

Here, the calculation was done using Mathematica. From this result, we conclude that the probability that we make a correct guess is strictly less than 2/3, as desired.

It turns out that any guessing strategy yields the same result. This is due to the symmetry of the original three states (the polarizations are spaced by 120°). This makes the whole argument general.

Mathematica calculation:

```
In[28]:= FindMaxValue[
Cos[\[Theta]]^2/(2*(Cos[\[Theta]]^2 +
1/4 (Cos[\[Theta]] - Sqrt[3] Sin[\[Theta]])^2 +
1/4 (Cos[\[Theta]] + Sqrt[3] Sin[\[Theta]])^2)) + (1/
2)*(Max[1/4 (Sqrt[3] Cos[\[Theta]] + Sin[\[Theta]])^2,
1/4 (-Sqrt[3] Cos[\[Theta]] +
Sin[\[Theta]])^2])/(Sin[\[Theta]]^2 +
1/4 (Sqrt[3] Cos[\[Theta]] + Sin[\[Theta]])^2 +
1/4 (-Sqrt[3] Cos[\[Theta]] + Sin[\[Theta]])^2), \[Theta]]

Out[28]= 0.622008
```

(b) Now we take the first qubit and tensor it with a second qubit in $|0\rangle$. Consider the following states:

$$\{|a\rangle, |b\rangle, |c\rangle, |d\rangle\} = \left\{|11\rangle, -\frac{\alpha}{2}|00\rangle + \frac{\sqrt{3}\alpha}{2}|10\rangle + \beta|01\rangle, \alpha|00\rangle + \beta|01\rangle, -\frac{\alpha}{2}|00\rangle - \frac{\sqrt{3}\alpha}{2}|10\rangle + \beta|01\rangle\right\}.$$

In order for these to form an orthonormal basis, α and β must satisfy the following conditions:

$$|\alpha|^2 + |\beta|^2 = 1$$
$$-\frac{|\alpha|^2}{2} + |\beta|^2 = 0$$

From the first two equations, we find that $|\alpha|^2 = 2/3$ and $|\beta|^2 = 1/3$. Assuming $\alpha, \beta \in \mathbb{R}$, we can let $\alpha = \sqrt{2/3}$ and $\beta = \sqrt{1/3}$.

(c) With probability 1/3 we are given $|\psi_1\rangle$, which we transform to $|\psi_1\rangle|0\rangle$. Measuring this state in the basis above, we find

$$\Pr(|a\rangle) = 0$$
$$\Pr(|b\rangle) = 1/6$$
$$\Pr(|c\rangle) = 2/3$$
$$\Pr(|d\rangle) = 1/6$$

With probability 1/3 we are given $|\psi_2\rangle$, which we transform to $|\psi_2\rangle|0\rangle$. Measuring this state in the basis above, we find

$$\begin{aligned}
\Pr(|a\rangle) &= 0 \\
\Pr(|b\rangle) &= 2/3 \\
\Pr(|c\rangle) &= 1/6 \\
\Pr(|d\rangle) &= 1/6
\end{aligned}$$

With probability 1/3 we are given $|\psi_3\rangle$, which we transform to $|\psi_3\rangle|0\rangle$. Measuring this state in the basis above, we find

$$\begin{aligned}
\Pr(|a\rangle) &= 0 \\
\Pr(|b\rangle) &= 1/6 \\
\Pr(|c\rangle) &= 1/6 \\
\Pr(|d\rangle) &= 2/3
\end{aligned}$$

Now we make the following rules for guessing:

- If we measure and find $|b\rangle$ then guess $|\psi_2\rangle$
- If we measure and find $|c\rangle$ then guess $|\psi_1\rangle$
- If we measure and find $|d\rangle$ then guess $|\psi_3\rangle$

Since the cases are symmetric, the success probability is simply given by

$$\Pr(\text{success}) = \frac{2/3}{2/3 + 1/6 + 1/6} = \frac{2}{3}$$

Mathematica calculations:

```
In[17]:= (*5b*)
In[60]:= a1 = {0, 0, 0, 1};
In[59]:= a3 = {\[Alpha], \[Beta], 0, 0};
In[56]:= a2 = {-\[Alpha]/2, \[Beta], \[Alpha]*Sqrt[3]/2, 0};
In[57]:= a4 = {-\[Alpha]/2, \[Beta], -\[Alpha]*Sqrt[3]/2, 0};
In[64]:= Psi1 = {1, 0, 0, 0};
In[62]:= Psi2 = {-1/2, 0, Sqrt[3]/2, 0};
In[63]:= Psi3 = {-1/2, 0, -Sqrt[3]/2, 0};
In[65]:= Dot[Psi2, a1]^2
Out[65]= 0
In[66]:= Dot[Psi2, a2]^2
Out[66]= \[Alpha]^2
In[67]:= Dot[Psi2, a3]^2
Out[67]= \[Alpha]^2/4
In[68]:= Dot[Psi2, a4]^2
Out[68]= \[Alpha]^2/4
In[52]:= Dot[Psi3, a1]^2
Out[52]= 0
In[53]:= Dot[Psi3, a2]^2
Out[53]= \[Alpha]^2/4
In[54]:= Dot[Psi3, a3]^2
Out[54]= \[Alpha]^2/4
In[55]:= Dot[Psi3, a4]^2
Out[55]= \[Alpha]^2
```