

Classical Mechanics III (8.09 & 8.309) Fall 2021

Assignment 2

Massachusetts Institute of Technology
Physics Department
Mon. September 20, 2021

*Due Mon. September 27, 2021
6:00pm*

Announcements

This week we will finish our discussion of problems with constraints, including the use of Lagrange multipliers, the topic of this problem set. Next up is the motion of Rigid Bodies.

On this problem set, **8.09 students** should do problems 1 to 5, and **8.309 students** should do problems 1 to 4 and 6. Note that problems 5 and 6 start out the same, but 8.309 students have two additional parts to do, 6.(f) and 6.(g). (8.09 students can consider these additional two parts as optional and not for credit.)

Reading Assignment for this week

- Systems with Constraints: D'Alembert's principle and systems with holonomic and non-holonomic constraints. Use of Lagrange multipliers to determine forces of constraint. The reading is **Goldstein**: sections 1.3-1.6, 2.4, 2.5, 8.5.
- Next up are Rigid Bodies which we'll start this week and continue next week (they are not on problem set #2). The main readings for this entire topic are **Goldstein** Ch.4, sections 4.1, 4.2, 4.4, 4.6, 4.9. Then read Ch.5, sections 5.1, 5.3–5.7. The Ch.4 reading is on rigid body kinematics where many physics topics will be familiar to you. Goldstein emphasizes vector notation and discusses rotations as matrices. Ch.5 is on rigid body dynamics. (Note that 4.3 is a linear algebra review. Other sections from Ch.4 and Ch.5 may be of interest, but the above are the most important ones.)

Problem Set 2

These five problems involve friction, forces of constraint, and Lagrange multipliers. Hint: If you get stuck on a problem, think about what the conservation of H implies in the coordinates you are using.

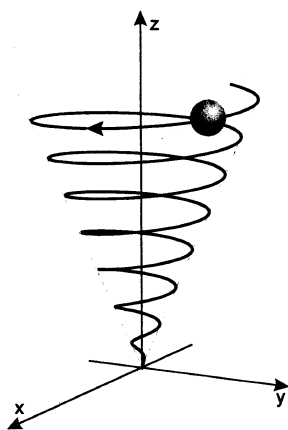
1. Spherical Pendulum with Friction [8 points, everyone]

Consider a spherical pendulum, consisting of a mass m attached to a massless rod of length ℓ . The spherical mass is immersed in a viscous fluid, and it moves with a small velocity obeying Stokes law. It is acted upon by gravity.

- [4 points] What is the dissipation function for this pendulum?
- [4 points] What are its equations of motion? Include the effect of the viscous friction force.

Note: A spherical pendulum means that the mass moves in three dimensions up to the constraint imposed by the rod. You do not need to use Lagrange multipliers for this problem.

2. Bead Spiraling on a Helix [12 points, everyone]



Consider the motion of a bead of mass m on the helix shown in the figure. In cylindrical coordinates the helix has a radius and angle described by $r = bz$ and $\theta = az$. At $t = 0$ let $\dot{z} = v_0 > 0$ and the height be $z = h$. There is no gravity in this problem.

- [8 points] By integrating the equation of motion find \dot{z} as a function of z , and then determine the constraint torque Z_θ as a function of a, b, m, z, v_0, h .

[Hint: $\int dt \ddot{z}/\dot{z} = \ln(\dot{z})$.]

- [4 points] Show that the particle's energy is constant, but that its angular momentum about the \hat{z} axis increases with increasing z .

3. Hoop Rolling on a Cylinder [12 points, everyone]

Do Goldstein Ch.2 #14 starting with 3 generalized coordinates (try one radial and two angular coordinates). Use Lagrange multipliers for the two constraints. Note that up until the point that the hoop leaves the cylinder that these constraints are holonomic or semi-holonomic.

4. **An Infinite Rope** [8 points, everyone] (Adapted from Goldstein Ch.2 #9)

Consider a rope of infinite length which passes freely over pulleys at heights y_1 and y_2 above the earth, where the pulleys are at horizontal distances x_1 and x_2 respectively. The rope has a uniform linear mass density and is affected by gravity. Demonstrate that the problem of finding the curve the rope assumes between the two pulleys is equivalent to a minimization problem. Using Euler-Lagrange equations or an equivalent method, solve to find the form of this curve. Write down the equations you would solve to fix the boundary conditions.

5. **A Falling Ladder for 8.09 students** [20 points, 8.09 ONLY]

Consider a uniform ladder of length L and mass M which has one end against a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest at an angle θ_0 to the floor and is acted on by gravity. We will be interested in studying the forces of constraints, so we will handle the constraints with Lagrange multipliers.

- (a) [4 points] Use the CM coordinates (x, y) for the ladder, and angle θ that the ladder makes to the floor. The kinetic energy is given by a term for motion of the CM, plus a term for rotation about the CM. What is the moment of inertia of the ladder for rotations about the CM? Find the full Lagrangian for the ladder and the equations for any constraints.
- (b) [3 points] Derive the equations of motion using the method of Lagrange multipliers.
- (c) [5 points] Solve for the angle $\theta(t)$ of the ladder to the floor by finding a formula $t = t(\theta)$ (which may involve an integral that you leave undone).
- (d) [4 points] At what angle θ does the ladder leave the wall? What is the height of the upper end of the ladder at this angle?
- (e) [4 points] What are the equations of motion after the ladder loses contact with the wall?

6. **A Falling Ladder for 8.309 students** [20 points, 8.309 ONLY]

Consider a uniform ladder of length L and mass M which has one end against a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest at an angle θ_0 to the floor and is acted on by gravity. We will be interested in studying the forces of constraints, so we will handle the constraints with Lagrange multipliers.

- (a) [3 points] Use the CM coordinates (x, y) for the ladder, and angle θ that the ladder makes to the floor. The kinetic energy is given by a term for motion of the CM, plus a term for rotation about the CM. What is the moment of inertia of the ladder for rotations about the CM? Find the full Lagrangian for the ladder and the equations for any constraints.

- (b) [2 points] Derive the equations of motion using the method of Lagrange multipliers.
- (c) [3 points] Solve for the angle $\theta(t)$ of the ladder to the floor by finding a formula $t = t(\theta)$ (which may involve an integral that you leave undone).
- (d) [2 points] At what angle θ does the ladder leave the wall? What is the height of the upper end of the ladder at this angle?
- (e) [2 points] What are the equations of motion after the ladder loses contact with the wall?
- (f) [4 points] Show that the ladder never loses contact with the floor. (You can assume that when $\theta = 0$ it does not bounce.) Your solution could be analytic or numeric, whichever you prefer.
- (g) [4 points] Solve the equations of motion in (a) and (e) numerically to find $\theta(t)$. Assume that the ladder is $L = 3$ meters long and that the starting angle is $\theta_0 = 60^\circ$. Make two plots of $\theta(t)$: i) for times t when the ladder is in contact with the wall, and ii) for times t between when it leaves the wall and hits the floor. (Use the functions `NDSolve` and `FindRoot` in `mathematica` or something equivalent in another program. You might find it interesting to make other plots but you will only be graded on the above.)