Classical Mechanics III (8.09 & 8.309) Fall 2021
Assignment 4

Massachusetts Institute of Technology
Physics Department
Mon. October 4, 2021

Due Tues. October 12, 2021
6:00pm

Announcements

This week we finished our discussion of Rigid Bodies and started on Oscillations which will be a short chapter, but also the focus of much of this problem set. At the end of the coming week we will likely begin our discussion of Canonical Transformations.

- Due to the Indigenous Peoples' day holiday on Monday I have made this assignment **due on Tuesday Oct. 12**. Note however that your next assignment (#5) will be posted as usual on Monday Oct. 11 and due on Monday Oct. 18.
- On this problem set, **8.09 students** should do problems 1 to 4, and **8.309 students** should do problems 1 to 3 and 5.

Reading Assignment for this week

- The reading for Oscillations is **Goldstein** Ch.6 sections 6.1-6.4.
- We will spend a few weeks on our next subject: Canonical Transformations, the Hamilton-Jacobi equations, and Action-Angle Variables. The complete reading for this material is **Goldstein** Ch.9 sections 9.1-9.7, and then Ch.10 sections 10.1-10.6, and 10.8.

Problem Set 4

In the first problem we look at a symmetric top, and in the final three problems we study oscillations. The use of Mathematica to help with the algebra is encouraged.

1. **A Heavy Symmetric Top** [everyone, 10 points]

   A heavy symmetric top ($I_1 = I_2$) with one point fixed is precessing at a steady angular velocity Ω about the vertical fixed inertial axis z_I. The Euler angle coordinates are defined as in lecture (and Goldstein), and here $\dot{\theta} = 0$. The top's mass is m and its center of mass is a distance R from the fixed point. Gravity acts on the top. Define $\omega' \equiv \dot{\psi}$. Consider components for axes in the body frame.

   (a) [3 points] Determine the components of the torque in terms of Euler angles.

   (b) [2 points] Write the angular velocities in terms of Euler angles. Explain why ω' is constant in time.

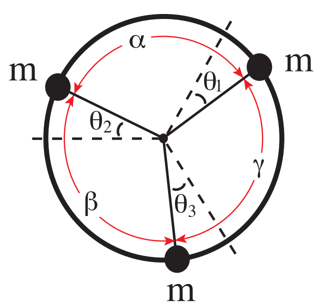
   (c) [5 points] Derive a minimum condition for ω'. Describe what type of tops will satisfy this condition for all possible ω's.

2. **Three Point Masses on a Circle** [everyone, 16 points]

   Three particles of equal mass m move on a circle with radius a under forces that can be derived from the potential

$$V(\alpha, \beta, \gamma) = V_0\left(e^{-2\alpha} + e^{-2\beta} + e^{-2\gamma}\right).$$

   Here α, β, and γ are the angular separations of the masses in radians as shown in the figure. An equilibrium position is indicated by the dashed lines and has $\alpha = \beta = \gamma = 2\pi/3$.



   (a) [6 points] Find the normal mode frequencies using the small amplitude approximation for oscillations about equilibrium. Determine the corresponding normalized normal modes.

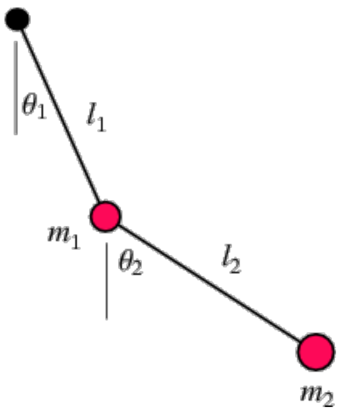
   (b) [3 points] What are the corresponding normal coordinates and equations of motion for the normal coordinates?

   (c) [3 points] Sketch the corresponding motion for each normal mode.

   (d) [4 points] Consider the following initial conditions at $t = 0$: $\theta_1 = \theta_2 = \theta_3 = 0$ and $\dot{\theta}_1 = -2\dot{\theta}_2 = -2\dot{\theta}_3 = 2\omega_0$. Use your results above to find $\theta_i(t)$ for $i = 1, 2, 3$.

3. **Small Oscillations of the Double Pendulum** [everyone, 14 points]

Consider the double pendulum in a plane that you analyzed on problem set #1. Use results from that problem as a starting point for this one. Take $m_2 = m$ and $m_1 = m$.



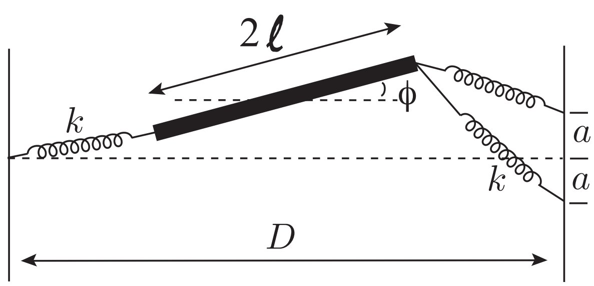
a) [4 points] Make a small angle approximation for θ_1 and θ_2, and determine results for the kinetic and potential energies which are quadratic in $\dot{\theta}_i$ and θ_i.

b) [4 points] What are the normal mode frequencies of this system? Confirm that the eigenvalues are positive and frequencies are real.

c) [6 points] Compute the corresponding eigenvectors and hence determine the normal modes. Sketch the corresponding motion of the pendulum for each one.

4. **A Rigid Oscillating Bar** [8.09 ONLY, 20 points]

Consider a thin uniform rigid bar of length L=2ℓ and mass m suspended by three equal springs with force constant k and zero relaxed length. They are attached to fixed walls a distance D apart. The two springs on the right have ends fixed at heights $\pm a$ relative to the fixed end of the spring on the left. In this problem we will consider the small oscillation modes of the bar in the plane *without gravity*. When the bar is at rest at equilibrium it is horizontal and we have $\phi = 0$. At a given instant the bar has rotated about its center from a horizontal position by the angle denoted by ϕ.



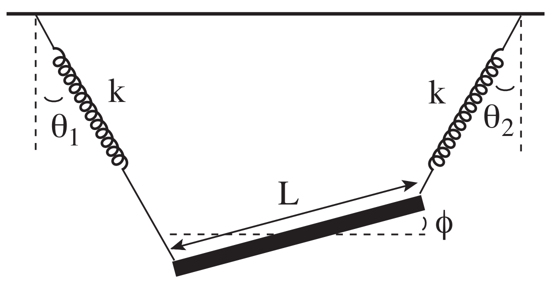
(a) [6 points] What are a suitable set of coordinates for describing the motion of the bar in the plane? Take ϕ as one of your coordinates. What are the lengths of the springs in terms of your variables and the given parameters? Using these coordinates determine the Lagrangian $L = T - V$ (without making a small amplitude approximation).

(b) [6 points] Determine a suitable form for T and V to study small amplitude oscillations. Take $D = 3\ell$ for this part and part (c). Write your answer in terms of matrices that depend only on k, m, a, and ℓ.

(c) [8 points] What are the normal modes of small oscillation? Make a sketch of each of these oscillations being sure to indicate any coupled motion.

5. **A Rigid Oscillating Bar** [8.309 ONLY, 20 points] (Adapted from Goldstein Ch.6 #11)

   Consider a thin uniform rigid bar of length L=2ℓ and mass m suspended by two equal springs with force constant k. In this problem we will consider the small oscillation modes of the bar in the plane with gravity. When the bar is at rest at equilibrium we have $\theta_1 = \theta_2 = \theta_0$ and $\phi = 0$, and the length of the springs is a. At a given instant the bar has rotated about its center from a horizontal position by the angle denoted by ϕ.



   (a) [7 points] What is the equilibrium length of the springs without the bar attached in terms of the given parameters? What are a suitable set of coordinates for describing the motion of the bar in the plane? Using these coordinates determine the Lagrangian $L = T - V$ (without making a small amplitude approximation).

   (b) [5 points] Determine a suitable form for T and V to study small amplitude oscillations. Write your answer in terms of matrices that depend only on k, m, g, a, ℓ, and θ_0. For simplicity, to answer this problem and the problem below, assume θ_0 is small and only work to linear order in θ_0.

   (c) [8 points] What are the normal modes of small oscillation? Make a sketch of each of these oscillations. What would differ if $\theta_0 = 0$?