$$u^{-s}(-\vec{p}) = -i\begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix} [u^s(\vec{p})]^*$$

$$= -\gamma^1\gamma^3 [u^s(\vec{p})]^*$$

Similarly $\quad v^{-s}(-\vec{p}) = -\gamma^1\gamma^3 [v^s(\vec{p})]^*$.

Now can define the time reversal operation on the creation and annihilation operators

$$T^\dagger a^s_{\vec{p}} T = a^{-s}_{-\vec{p}}$$

$$T^\dagger b^s_{\vec{p}} T = b^{-s}_{-\vec{p}}$$

where $\quad a^{-s}_{-\vec{p}} = (a^{\downarrow}_{-\vec{p}}, -a^{\uparrow}_{-\vec{p}})$

$\qquad\quad b^{-s}_{-\vec{p}} = (b^{\downarrow}_{-\vec{p}}, -b^{\uparrow}_{-\vec{p}})$

So $\quad T^\dagger \psi(x) T =$

$$\int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s T^\dagger (a^s_{\vec{p}} u^s(\vec{p}) e^{-ip\cdot x} + b^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{ip\cdot x}) T$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \Big(a^{-s}_{-\vec{p}} \underbrace{[u^s(\vec{p})]^*}_{\gamma^1\gamma^3 u^{-s}(-\vec{p})} e^{ip\cdot x} + b^{-s\dagger}_{-\vec{p}} \underbrace{[v^s(\vec{p})]^*}_{\gamma^1\gamma^3 v^{-s}(-\vec{p})} e^{-ip\cdot x} \Big)$$

[$\gamma^1\gamma^3$ is the inverse of $-\gamma^1\gamma^3$]

$$= \gamma^1\gamma^3 \psi(x_T) \quad \text{where} \quad x_T = (-t, \vec{x})$$

$$T^\dagger \bar\psi T = T^\dagger \psi^\dagger \gamma^0 T = T^\dagger \psi^\dagger T \underset{\uparrow \text{ real}}{\gamma^0}$$

$$= \left(\gamma^1\gamma^3\psi(x_T)\right)^\dagger \gamma^0 = \psi^\dagger(x_T)\gamma^3\gamma^1\gamma^0$$

$$(\gamma^{1\dagger} = -\gamma^1,\ \gamma^{3\dagger} = -\gamma^3)$$

$$= \underbrace{\psi^\dagger(x_T)\gamma^0}_{\bar\psi(x_T)}\gamma^3\gamma^1 = -\bar\psi(x_T)\gamma^1\gamma^3$$

Therefore

$$T^\dagger \bar\psi\psi(x) T = \bar\psi(x_T)(-\gamma^1\gamma^3)(\gamma^1\gamma^3)\psi(x_T)$$

$$= \bar\psi(x_T)\psi(x_T)$$

$$T^\dagger(i\bar\psi\gamma^5\psi)T = -i(T^\dagger\bar\psi T)\gamma^5(T^\dagger\psi T)$$

$$= -i\bar\psi(-\gamma^1\gamma^3)\gamma^5(\gamma^1\gamma^3)\psi$$

$$= -i\bar\psi\gamma^5\psi(x_T)$$

$$T^\dagger\bar\psi\gamma^0\psi T = (T^\dagger\bar\psi T)\gamma^0(T^\dagger\psi T) = \bar\psi(-\gamma^1\gamma^3)\gamma^0(\gamma^1\gamma^3)\psi$$

$$= \bar\psi\gamma^0\psi(x_T)$$

$$T^\dagger\bar\psi\gamma^i\psi T = \underset{\uparrow}{\pm}(T^\dagger\bar\psi T)\gamma^i(T^\dagger\psi T)$$

$+$ for $i = 1,3$
$-$ for $i = 2$

$$= \pm\, \bar{\psi}(-\gamma^1\gamma^3)\,\gamma^i\,(\gamma^1\gamma^3)\psi$$

$$= -\,\bar{\psi}\,\gamma^i\,\psi$$

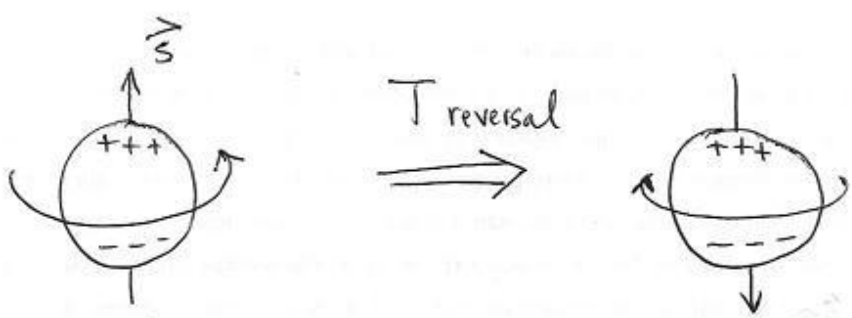
$\uparrow$ minus for all $i = 1,2,3$

This makes sense ... $\bar{\psi}\gamma^0\psi$ is charge density, should be the same under T

$\bar{\psi}\gamma^i\psi$ is current density, should reverse under T.

<u>Current work in our physics department ...</u>

An electric dipole for the neutron would violate T-invariance



neutron

Charge conjugation

Charge conjugation interchanges particles + antiparticles. Spin + momentum are left the same.

Let $C^\dagger a^s_{\vec{p}} C = b^s_{\vec{p}}$

$C^\dagger b^s_{\vec{p}} C = a^s_{\vec{p}}$

Should be clear that $C^\dagger \psi C$ cannot equal Matrix$\cdot\psi$ since we need something with b and $a^\dagger$.

So we can try $C^\dagger \psi C = \text{Matrix}\cdot\psi^*$.

Note: C is linear operator, even though it "looks" like it does something anti-linear.

Compare with T, which is anti-linear but "looks" like it does something linear.

$$C^\dagger \psi C = \text{Matrix}\cdot\psi^*, \quad C^\dagger i\psi C = i\,\text{Matrix}\,\psi^*$$

$$T^\dagger \psi T = \gamma^1\gamma^3\psi(x_T), \quad T^\dagger i\psi T = -i\gamma^1\gamma^3\psi(x_T)$$

If we want to find the matrix that connects ψ^* and $C^\dagger \psi C$ we need to connect $v^{s*}(p)$ with $u^s(p)$.

$$v^{s*}(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi^{-s} \\ -\sqrt{p\cdot\bar\sigma}\ \xi^{-s} \end{pmatrix}^* = \begin{pmatrix} \sqrt{p\cdot\sigma}\ (-i\sigma^2 \xi^{s*}) \\ -\sqrt{p\cdot\bar\sigma}\ (-i\sigma^2 \xi^{s*}) \end{pmatrix}^*$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma^*}\ (i\sigma^2 \xi^{s}) \\ -\sqrt{p\cdot\bar\sigma^*}\ (i\sigma^2 \xi^{s}) \end{pmatrix}$$

Note that $\vec{\sigma}^*\sigma^2 = -\sigma^2\vec{\sigma}$

and so $\sigma^{\mu *}\sigma^2 = +\sigma^2\bar{\sigma}^\mu$

Thus $$v^{s*}(p) = \begin{pmatrix} -i\sigma^2\sqrt{p\cdot\bar\sigma}\ \xi^{s} \\ +i\sigma^2\sqrt{p\cdot\sigma}\ \xi^{s} \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} u^s(p) \end{pmatrix}$$

$$= -i\gamma^2 u^s(p)$$

If we take the complex conjugate of this we get

$$v^s(p) = -i\gamma^2 u^{s*}(p)$$

since $(-i\gamma^2)(-i\gamma^2) = 1$, $\quad u^{s*}(p) = -i\gamma^2 v^s(p)$

So

$$C^\dagger \psi(x) C$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3\sqrt{2E_{\vec{p}}}} \sum_s \Big[b^s_{\vec{p}} \underbrace{u^s(p)}_{-i\gamma^2 v^{s*}(p)} e^{-ip\cdot x} + a^{s\dagger}_{\vec{p}} \underbrace{v^s(p)}_{-i\gamma^2 u^{s*}(p)} e^{ip\cdot x} \Big]$$

$$= -i\gamma^2 \psi^*(x)$$

It will be convenient to write this as

$$C^\dagger \psi(x) C = -i\gamma^2 (\psi^\dagger(x))^T = -i\gamma^2(\bar{\psi}\gamma^0)^T$$
$$= -i(\bar{\psi}\gamma^0\gamma^2)^T$$
(since $\gamma^{2T} = \gamma^2$)

Also $C^\dagger \bar{\psi} C = C^\dagger \psi^\dagger C \gamma^0$

$$= (C^\dagger \psi C)^\dagger \gamma^0 = -i\psi^T \gamma^2 \gamma^0$$
$(\gamma^{2\dagger} = -\gamma^2)$

$$= -i(\gamma^0\gamma^2\psi)^T \quad (\gamma^{0T} = \gamma^0,\ \gamma^{2T} = \gamma^2)$$

How about bilinears?

$$C^\dagger \bar{\psi}\psi C = (-i(\gamma^0\gamma^2\psi)^T)(-i(\bar{\psi}\gamma^0\gamma^2)^T)$$
$$= \underset{\uparrow}{-} ((-i\bar{\psi}\gamma^0\gamma^2)(-i\gamma^0\gamma^2\psi))^T$$

since $\bar{\psi}$ + ψ anticommute (there is some subtlety here... just regard $\bar{\psi}$ + ψ as anticommuting variables for now)

$$= +\bar{\Psi}\gamma^0\gamma^2\gamma^0\gamma^2\psi$$ (transpose of 1×1 matrix is same as the matrix)

$$= -\bar{\Psi}\underbrace{\gamma^0\gamma^0}_{1}\underbrace{\gamma^2\gamma^2}_{-1}\psi$$

$$= \bar{\Psi}\psi$$

$$C\, i\bar{\Psi}\gamma^5\psi\, C = i\left(-i(\gamma^0\gamma^2\psi)^T\right)\gamma^5\left(-i(\bar{\Psi}\gamma^0\gamma^2)^T\right)$$

$$= -\,i(-i)^2\,\bar{\Psi}\gamma^0\gamma^2\gamma^5\gamma^0\gamma^2\psi \qquad -1\ -1$$

($\gamma^{5T}=\gamma^5$) anticommuting

$$= i\bar{\Psi}\gamma^5\psi$$

$$C^\dagger\,\bar{\Psi}\gamma^{0,2}\psi\, C = (-i)^2(\gamma^0\gamma^2\psi)^T\gamma^{0,2}(\bar{\Psi}\gamma^0\gamma^2)^T$$

$$= +\bar{\Psi}\gamma^0\gamma^2\gamma^{0,2}\gamma^0\gamma^2\psi$$ ($\gamma^{0,2}$ anticommutes with $\gamma^0\gamma^2$)

($\gamma^{0T}=\gamma^0$, $\gamma^{2T}=\gamma^2$)

$$= -\bar{\Psi}\gamma^{0,2}\psi$$

$$C^\dagger\bar{\Psi}\gamma^{1,3}\psi\, C = (-i)^2(\gamma^0\gamma^2\psi)^T\gamma^{1,3}(\bar{\Psi}\gamma^0\gamma^2)^T$$

$$= -\bar{\Psi}\gamma^0\gamma^2\gamma^{1,3}\gamma^0\gamma^2\psi$$ ($\gamma^{1,3}$ commutes with $\gamma^0\gamma^2$)

($\gamma^{1T}=-\gamma^1$, $\gamma^{3T}=-\gamma^3$)

$$= -\bar{\Psi}\gamma^{1,3}\psi$$

So $C^\dagger\bar{\Psi}\gamma^\mu\psi\, C = -\bar{\Psi}\gamma^\mu\psi$

Similarly

$$C^\dagger\bar{\Psi}\gamma^{0,2}\gamma^5\psi\, C = (-i)^2(\gamma^0\gamma^2\psi)^T\gamma^{0,2}\gamma^5(\bar{\Psi}\gamma^0\gamma^2)^T$$

$$= \bar{\psi}\gamma^0\gamma^2 \underbrace{(\gamma^{0,2}\gamma^5)^T}_{= -\gamma^{0,2}\gamma^5} \gamma^0\gamma^2\psi \qquad \begin{pmatrix}\gamma^{0,2}\gamma^5 \text{ anticommutes} \\ \text{with } \gamma^0\gamma^2\end{pmatrix}$$

$$= \bar{\psi}\gamma^{0,2}\gamma^5\psi$$

$$C^\dagger \bar{\psi}\gamma^{1,3}\gamma^5\psi C = (-i)^2 (\gamma^0\gamma^2\psi)^T \gamma^{1,3}\gamma^5 (\bar{\psi}\gamma^0\gamma^2)^T$$

$$= \bar{\psi}\gamma^0\gamma^2 \underbrace{(\gamma^{1,3}\gamma^5)^T}_{= \gamma^{1,3}\gamma^5} \gamma^0\gamma^2\psi$$

$$\begin{pmatrix}\gamma^{1,3}\gamma^5 \text{ commutes with} \\ \gamma^0\gamma^2\end{pmatrix}$$

$$= \bar{\psi}\gamma^{1,3}\gamma^5\psi$$

So $\quad C^\dagger \bar{\psi}\gamma^\mu\gamma^5\psi C = \bar{\psi}\gamma^\mu\gamma^5\psi$

Summary of C, P, T: $\quad (-1)^\mu = \begin{cases} 1 & \text{for } \mu = 0 \\ -1 & \text{for } \mu = 1,2,3 \end{cases}$

	$\bar{\psi}\psi$	$i\bar{\psi}\gamma^5\psi$	$\bar{\psi}\gamma^\mu\psi$	$\bar{\psi}\gamma^\mu\gamma^5\psi$	$\bar{\psi}\sigma^{\mu\nu}\psi$	∂^μ
P	$+1$	-1	$(-1)^\mu$	$-(-1)^\mu$	$(-1)^\mu(-1)^\nu$	$(-1)^\mu$
T	$+1$	-1	$(-1)^\mu$	$(-1)^\mu$	$-(-1)^\mu(-1)^\nu$	$-(-1)^\mu$
C	$+1$	$+1$	-1	$+1$	-1	$+1$
CPT	$+1$	$+1$	-1	-1	$+1$	-1

Note that under CPT, the operator get $(-1)^{\#}$ where # is the number of Lorentz indices.

Invariance under CPT is required for any Lorentz invariant local Hermitian operator!

Correlation functions for Dirac fields

$$\langle 0 | \psi_a(x) \bar{\psi}_b(y) | 0 \rangle$$

a term only contributes $a^\dagger$ term only contributes

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \underbrace{\sum_s u_a^s(\vec{p}) \bar{u}_b^s(\vec{p})}_{(\not{p}+m)_{ab}} e^{-ip\cdot(x-y)} \qquad \not{p} \equiv p_\mu \gamma^\mu$$

$$= (i\not{\partial}_x + m)_{ab} \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} e^{-ip\cdot(x-y)}$$

$(\not{\partial} \equiv \partial_\mu \gamma^\mu)$

$$= (i\not{\partial}_x + m)_{ab} D(x-y)$$

Similarly   b term $b^\dagger$ term

$$\langle 0 | \bar{\psi}_b(y) \psi_a(x) | 0 \rangle$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} \underbrace{\sum_s v_a^s(\vec{p}) \bar{v}_b^s(\vec{p})}_{(\not{p}-m)_{ab}} e^{-ip\cdot(y-x)}$$

$$= \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} (\not{p}-m)_{ab}\, e^{-ip\cdot(y-x)} = -(i\not{\partial}_x+m)_{ab}\, D(y-x)$$

Feynman propagator:

$$S_F^{ab}(x-y) = \begin{cases} \langle 0|\, \psi_a(x)\, \bar{\psi}_b(y)\, |0\rangle & \text{for } x^0 > y^0 \\ -\langle 0|\, \bar{\psi}_b(y)\, \psi_a(x)\, |0\rangle & \text{for } x^0 < y^0 \end{cases}$$

$$= \langle 0|\, T\{\psi_a(x)\, \bar{\psi}_b(y)\}\, |0\rangle$$

where $T\{\psi_a(x)\,\bar{\psi}_b(y)\}$

$$= \theta(x^0-y^0)\, \psi_a(x)\, \bar{\psi}_b(y)$$
$$- \theta(y^0-x^0)\, \bar{\psi}_b(y)\, \psi_a(x)$$

minus sign for fermions