8.311 : Electromagnetic Theory

PSet 1, due Feb 9, 2022

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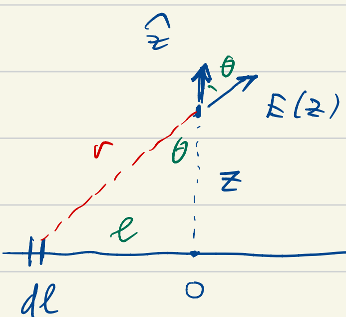
① $\vec{E}$ due to line of charge with charge density λ ...

(i) Coulomb's law

By symmetry, $\vec{E}$ points radially outward... and depends only on the distance to the wire (z)

$$dE = \frac{1}{4\pi\varepsilon_0}\lambda\, d\ell \frac{1}{r^2}\cos\theta$$

vert component only

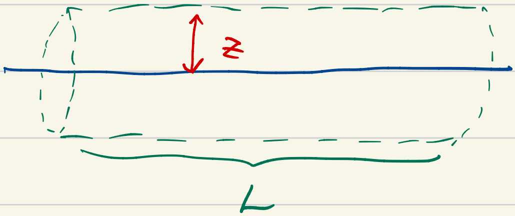


$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{1}{z^2+\ell^2}\, d\ell \frac{z}{\sqrt{z^2+\ell^2}}$$

$$\Rightarrow E(z) = \int dE = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{\infty}\frac{z}{(z^2+\ell^2)^{3/2}}\, d\ell = \frac{\lambda}{2\pi\varepsilon_0 z}$$

$$\to \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 z}\hat{z}}$$

(ii) Gauss law



→ Gaussian surface ...

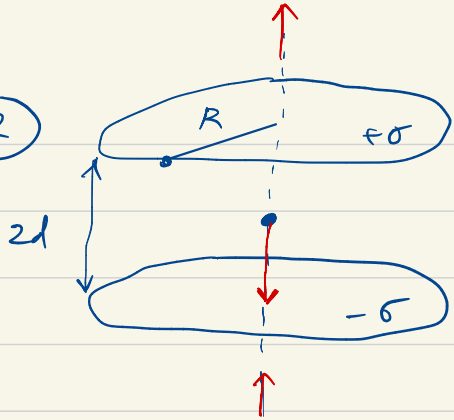
No flux through the endcaps, by symmetry.

Gauss law: $\oint \vec{E}\cdot d\vec{A} = \dfrac{Q_{encl}}{\varepsilon_0}$ (since $\vec{E} \parallel \hat{n}$)

$$\Rightarrow \quad E\,(2\pi z \cdot L) = \frac{\lambda L}{\varepsilon_0}$$

$$\Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 z} \quad \Rightarrow \quad \boxed{\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 z}\,\hat{z}}$$

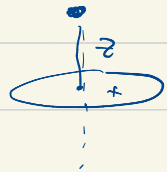
2



Find ϕ & E along the common axis...

→ Find $\phi(z)$, $\vec{E}(z)$

Find ϕ ... Let's do this for one plate ...

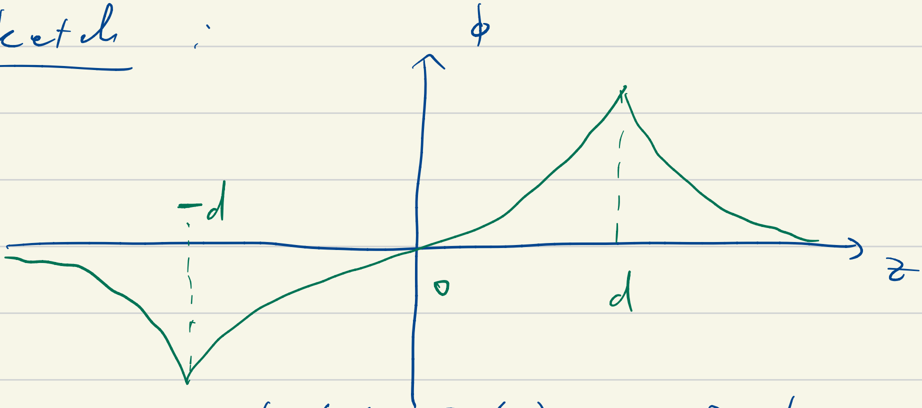


$$\phi(z) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\, dA}{r} = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r\,dr\,d\phi}{\sqrt{z^2+r^2}} = \frac{\sigma}{2\varepsilon_0}\left[-|z| + \sqrt{R^2+z^2}\right]$$

So for our problem...

$$\phi(z) = \frac{\sigma}{2\varepsilon_0}\Big[\overbrace{-|z-d| + \sqrt{R^2+(z-d)^2}}^{\text{top plate}} \; \underbrace{+\,|z+d| - \sqrt{R^2+(z+d)^2}}_{\text{bottom plate}}\Big]$$

Sketch:

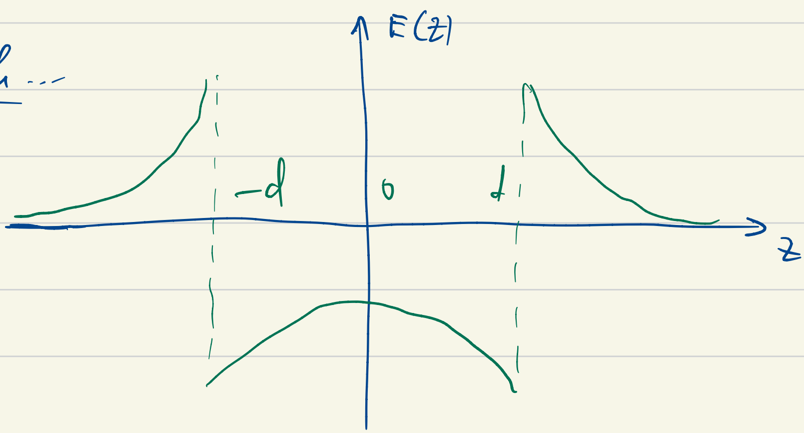


Can now calculate $E(z) = -\partial_z \phi$ ...

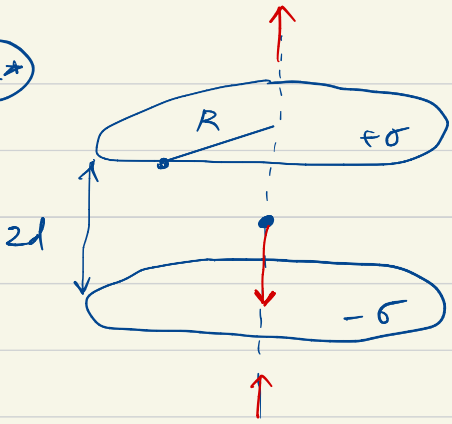
- $E(z) =$ (mathematical)

$$= \boxed{\frac{\sigma}{2\varepsilon_0}\left[-\frac{z-d}{\sqrt{R^2+(z-d)^2}} + \frac{z+d}{\sqrt{R^2+(z+d)^2}} + \frac{z-d}{|z-d|} - \frac{z+d}{|z+d|}\right]}$$

Sketch...

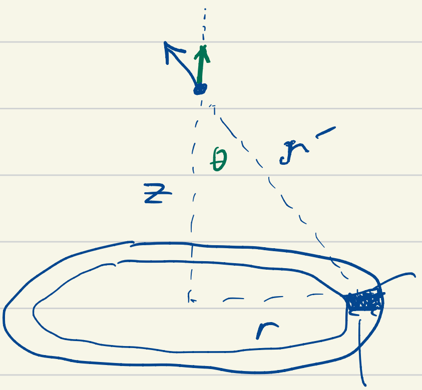


(2*)



Can also calculate $\vec{E}$ explicitly ...

$$r'^2 = r^2 + z^2$$



$$dE = \cos\theta \cdot \sigma r\,dr\,d\varphi \cdot \frac{1}{z^2 + r^2} \cdot \frac{1}{4\pi\varepsilon_0}$$

$$= \sigma r\,dr\,d\varphi \,\frac{z}{(z^2 + r^2)^{3/2}} \cdot \frac{1}{4\pi\varepsilon_0}$$

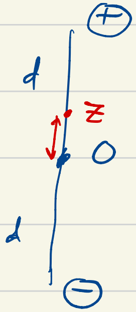
$dA = r\,dr\,d\varphi$

$$\Rightarrow \quad E = \int dE = \int_0^{2\pi} d\varphi \int_0^R \sigma r\,dr \,\frac{z}{(z^2 + r^2)^{3/2}} \,\frac{1}{4\pi\varepsilon_0}$$

$$= \frac{2\pi}{4\pi\varepsilon_0}\,\sigma \left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{R^2 + z^2}}\right)$$

With this ...

Let the origin O be equidistant from the disks... Look at E between plates...



then $E(z) = \dfrac{-\sigma_+}{2\varepsilon_0}\left[1 - \dfrac{d-z}{\sqrt{R^2+(d-z)^2}}\right]$

(−) since we're below the disk

$$+ \frac{\sigma_-}{2\varepsilon_0}\left[1 - \frac{d+z}{\sqrt{R^2+(d+z)^2}}\right] \qquad (\star)$$

$$= \frac{\sigma}{2\varepsilon_0}\left[-1 - \frac{z-d}{\sqrt{R^2+(z-d)^2}} - 1 + \frac{z+d}{\sqrt{R^2+(d+z)^2}}\right]$$

$$= \frac{\sigma}{2\varepsilon_0}\left[-2 - \frac{z-d}{\sqrt{R^2+(z-d)^2}} + \frac{z+d}{\sqrt{R^2+(z+d)^2}}\right]$$

$$(-d \leq z \leq d)$$

Next, look at E above and below the plates...

So, we have

$$E_{above}(z) = \frac{\sigma}{2\varepsilon_0}\left[\cancel{+1} - \frac{z-d}{\sqrt{R^2+(z-d)^2}} - \cancel{1} + \frac{z+d}{\sqrt{R^2+(d+z)^2}}\right]$$

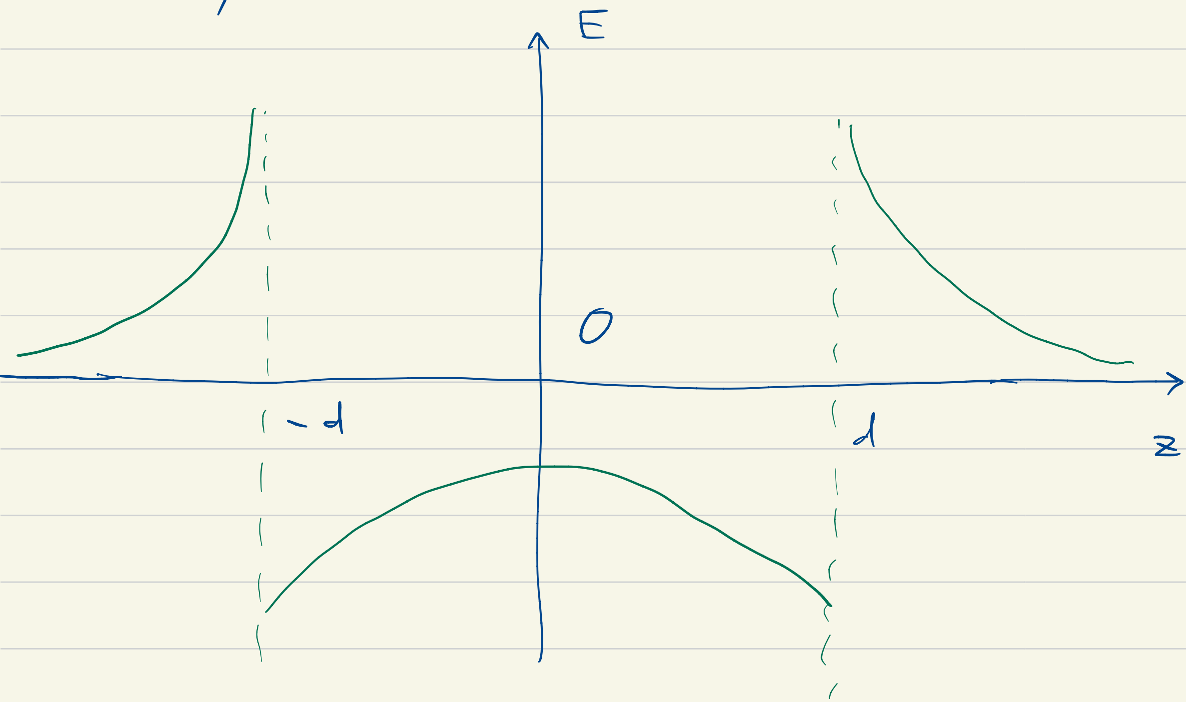
$(z \geq d)$

$$E_{below}(z) = \frac{\sigma}{2\varepsilon_0}\left[-1 + \frac{-z+d}{\sqrt{R^2+(-z+d)^2}} + 1 - \frac{-z-d}{\sqrt{R^2+(-z-d)^2}}\right]$$

$(z \leq -d)$

$(z \to -z)$

Summary ...



The full expression for $E(z)$ is

$$E(z) = \frac{\sigma}{2\varepsilon_0}\left[-\frac{z-d}{\sqrt{R^2+(z-d)^2}} + \frac{z+d}{\sqrt{R^2+(z+d)^2}} + \frac{z-d}{|z-d|} - \frac{z+d}{|z+d|} \right]$$

To find $\phi(z)$ ... Use definition.

③ Prove the reciprocity thm of electrostatics

Setup: We have $\rho_1(\vec{r})$ induces $\phi_1(\vec{r})$

$\rho_2(\vec{r})$ induces $\phi_2(\vec{r})$

To prove: $\displaystyle\int_{\mathbb{R}^3} \rho_1(\vec{r})\phi_2(\vec{r})\,d^3r = \int_{\mathbb{R}^3} \rho_2(\vec{r})\phi_1(\vec{r})\,d^3r$

Pf// Consider the integral $\displaystyle\int_{\mathbb{R}^3} \vec{E}_1(\vec{r})\cdot\vec{E}_2(\vec{r})\,d^3r$

We have

(int. by parts, take $S\to\infty$)

$$\int_{\mathbb{R}^3} \vec{E}_1(\vec{r})\cdot\vec{E}_2(\vec{r})\,d^3r = \int_{\mathbb{R}^3} \vec{\nabla}\phi_1(\vec{r})\cdot\vec{\nabla}\phi_2(\vec{r})\,d^3r$$

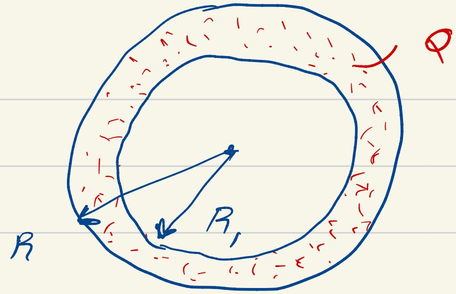
$$= \int_{\mathbb{R}^3} \left(-\nabla^2\phi_1(\vec{r})\right)\phi_2(\vec{r})\,d^3r = \int \phi_1(\vec{r})\left(-\nabla^2\phi_2(\vec{r})\right)d^3\vec{r}$$

$$= \frac{1}{\varepsilon_0}\int \rho_1(\vec{r})\phi_2(\vec{r})\,d^3r = \frac{1}{\varepsilon_0}\int \phi_1(\vec{r})\rho_2(\vec{r})\,d^3r \quad \checkmark$$

as desired.

$\square$

④ Calculate U of



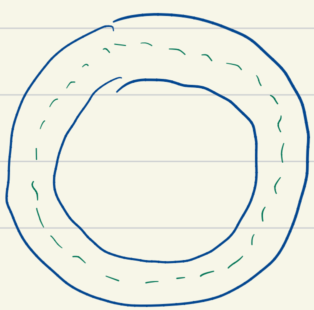
To do this, we first find $\vec{E}(\vec{r})$ ...

- By Gauss law, there's no charge enclosed at $r < R_1$, so $E_{inside} = 0$.

- For $r > R_2$, $E_{outside} = \dfrac{Q}{4\pi\varepsilon_0 r^2}$, again by Gauss law

  (which says that $E_{outside}$ looks like E due to a point charge, b/c of spherical symmetry).

- For $R_1 < r < R_2$ ... look at the Gaussian surface like below



It is easy to see that, by gauss law...

- $E_{between} = \dfrac{q}{4\pi\varepsilon_0 r^2}$, where q is the total charge enclosed

$$q = \underbrace{\frac{Q}{\frac{4}{3}\pi(R_2^3 - R_1^3)}}_{\text{charge density}} \cdot \underbrace{\frac{4}{3}\pi(r^3 - R_1^3)}_{\text{new volume}}$$

$$= Q\,\frac{(r^3 - R_1^3)}{R_2^3 - R_1^3}.$$

Summary

$r < R_1$ ... $E(r) = 0$

$R_1 < r < R_2$ ... $E(r) = \dfrac{Q}{4\pi\varepsilon_0 r^2}\,\dfrac{r^3 - R_1^3}{R_2^3 - R_1^3}$

$r > R_2$ ... $E(r) = \dfrac{Q}{4\pi\varepsilon_0 r^2}$

With this ... can calculate $\mathcal{U} = \frac{\varepsilon_0}{2}\int |E|^2 d^3r$

$$\mathcal{U} = \frac{\varepsilon_0}{2}(4\pi)\int r^2 \cdot |E|^2 dr$$

spherical symmetry

$$= (2\pi\varepsilon_0)\int_{R_1}^{R_2}\left(\frac{Q}{4\pi\varepsilon_0}\,\frac{r^3-R_1^3}{R_2^3-R_1^3}\right)^2\frac{1}{r^2}\,dr$$

$$+ (2\pi\varepsilon_0)\int_{R_2}^{\infty}\left(\frac{Q}{4\pi\varepsilon_0}\right)^2\frac{1}{r^2}\,dr$$

$$= \ldots \text{(mathematica)}$$

$$\boxed{\mathcal{U} = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{(R_2-R_1)(5R_1^3+6R_1^2R_2+3R_1R_2^2+R_2^3)}{5R_2(R_1^2+R_1R_2+R_2^2)^2} + \frac{1}{R_2}\right]}$$

$\boxed{?}$ Dependence of $\mathcal{U}$ on R_1, with $R_2 = \alpha$ fixed ...

- If $R_1 = 0$ then we just have a spherical blob of charge with total charge Q ...

$$\boxed{\mathcal{U}(R_1 \to 0) = \frac{Q^2}{8\pi\varepsilon_0}\left[\frac{R_2^3}{5R_2^4} + \frac{1}{R_2}\right] = \frac{3Q^2}{20\pi\varepsilon_0 R_2}}$$

- If $R_1 = R_2$, then get thin shell

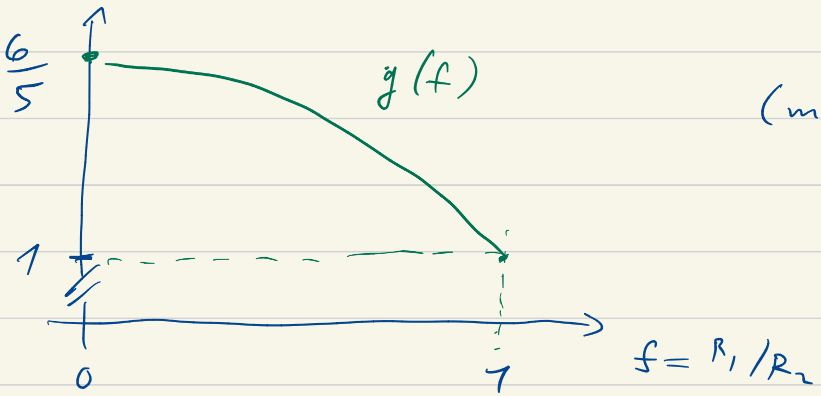
$$\boxed{\mathcal{U}(R_1 = R_2) = \frac{Q^2}{8\pi\varepsilon_0 R_2}}$$

We can also plot $g(R_1/R_2) = g(f)$

$$g(f) = \frac{\cancel{R_2}(R_2 - R_1)(5R_1^3 + 6R_1^2R_2 + 3R_1R_2^2 + R_2^3)}{5\cancel{R_2}(R_1^2 + R_1R_2 + R_2^2)^2} + 1$$

$$= 1 + \frac{(1-f)(5f^3 + 6f^2 + 3f + 1)}{5(f^2 + f + 1)^2}$$

where $f = R_1/R_2$ to see other behavior, if any ...



(mathematica)

So $\mathcal{U}$ max when $R_1 = 0$, $\mathcal{U} = \dfrac{3Q^2}{20\pi\varepsilon_0 R_2}$

$\mathcal{U}$ min when $R_1 = R_2$, $\mathcal{U} = \dfrac{Q^2}{8\pi\varepsilon_0 R_2}$

(Bonus...) Can also do the problem this way ... continuously adding thin shells of charge to form the thick shell...

$$\mathcal{U} = \int d\mathcal{U} \quad ; \quad d\mathcal{U} = \frac{Q(r)\,dQ}{4\pi\varepsilon_0 r}$$

like point charge

Now, $$Q(r) = \frac{Q(r^3 - R_1^3)}{R_2^3 - R_1^3}$$

$$dQ = \frac{Q}{\frac{4}{3}\pi(R_2^3 - R_1^3)} \cdot 4\pi r^2$$

$$\mathcal{U} = \int_{R_1}^{R_2} \frac{1}{4\pi\varepsilon_0 r} \frac{(r^3 - R_1^3)}{R_2^3 - R_1^3} \cdot \frac{Q}{\frac{4}{3}\pi(R_2^3 - R_1^3)} \cdot 4\pi r^2\,dr$$

$$= \frac{3Q^2}{40\pi\varepsilon_0} \cdot \frac{(3R_1^3 + 6R_1^2 R_2 + 4R_1 R_2^2 + 2R_2^3)}{(R_1^2 + R_1 R_2 + R_2^2)^2}$$

$$\lim_{R_1 \to 0} \mathcal{U} = \frac{3Q^2}{20\pi\varepsilon_0 R_2} \quad ; \quad \lim_{R_1 \to R_2} \mathcal{U} = \frac{Q^2}{8\pi\varepsilon_0 R_2}$$