2022 Midterm Practice Problems 8.370/18.435/2.111

1. Suppose we have two qubits. We can apply a measurement on the first qubit using the basis

$$\left\{ \frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle , \quad \frac{-1}{\sqrt{5}} |0\rangle + \frac{2}{\sqrt{5}} |1\rangle \right\}$$

In the projection-matrix description of a von Neumann measurement on the two-qubit system, what two 4×4 projection matrices does this correspond to?

2. Suppose we have two qubits on system AB in the state

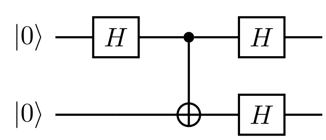
$$\frac{1}{3} |01\rangle_{AB} + \frac{2}{3} |10\rangle_{AB} + \frac{2}{3} |11\rangle_{AB}$$

We now apply a measurement on the first qubit, A, using the basis

$$\left\{ \frac{2}{\sqrt{5}} |0\rangle + \frac{1}{\sqrt{5}} |1\rangle , \quad \frac{-1}{\sqrt{5}} |0\rangle + \frac{2}{\sqrt{5}} |1\rangle \right\}$$

What is the probability of getting the first element of this basis, and if we do, what is the resulting state of qubit B?

3. Suppose we have the quantum circuit:



What is the state of the two qubits at the output?

4. (a) What is the density matrix ρ representing an equal mixture (probability $\frac{1}{2}$ each) of the quantum states:

$$\frac{1}{\sqrt{3}} (|00\rangle + |01\rangle + |10\rangle) \quad \text{and} \quad \frac{1}{\sqrt{3}} (|00\rangle - |01\rangle + |10\rangle)$$

   (b) What is $\mathrm{Tr}_A \rho$, where ρ is as in part (a)?

   (c) What is $\mathrm{Tr}_B \rho$?

5. Show that for any density matrix ρ

$$\tfrac{1}{4} \left(\rho + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z \right) = \tfrac{1}{2} I \,.$$

6. What is the value of: $\langle 0001111100 | \, H^{\otimes 10} \, | 1111110000 \rangle$?

7. Suppose we have two quantum circuits, C_1 and C_2, made up of unitary gates. If you input the state $|00000000\rangle$ into C_1, the output state is $\sum_{j=0}^{255} \alpha_j \,|j\rangle$. If you input the state $|00000001\rangle$ into C_2, the output state is $\sum_{k=0}^{255} \beta_k \,|k\rangle$. Now, suppose you make up a new quantum circuit C_3 by taking circuit C_1 and appending the conjugate transpose of the gates of circuit C_2 in reverse order. If you input $|00000000\rangle$ into C_3, and measure the output of C_3, what is the probability that you see $|00000001\rangle$?

8. Give a unitary transformation on one qubit that takes $|0\rangle$ to $\frac{1}{\sqrt{2}}\big(\,|0\rangle + i\,|1\rangle\,\big)$ and takes $\frac{1}{\sqrt{2}}\big(\,|0\rangle + i\,|1\rangle\,\big)$ to $|0\rangle$.

9. Suppose that we have four parties, Alice, Bob, Cathy, and David. Alice and Cathy share a pair of qubits in one of the four Bell states

$$\frac{1}{\sqrt{2}}\big(\,|01\rangle \pm |10\rangle\,\big), \quad \frac{1}{\sqrt{2}}\big(\,|00\rangle \pm |11\rangle\,\big),$$

but they don't know what state it's in. Bob and David share a pair of qubits in the same Bell state. Suppose further that Alice and Bob share a qubit in the state $\frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big)$. Show that there is a protocol that lets Cathy and David end up sharing a pair of qubits in the state $\frac{1}{\sqrt{2}}\big(\,|01\rangle - |10\rangle\,\big)$.

Justify your answer.