Understanding Convolution Powers of Complex Functions on $\mathbb{Z}^d$

November 8, 2019

Huan Q. Bui

1 Some definitions

$L^1([a,b])$ is...

$\mathcal{C}^1[(a,b)]$ is...

2 van der Corput's & related lemmas

We are interested in integrals of the form

$$\int_{\mathbb{R}^d} e^{iP(\xi)-ix\cdot\xi}\,d\xi. \tag{1}$$

To understand the behavior of this integral in higher dimensions, it is helpful to understand the 1-dimensional version of these integrals, called oscillatory integrals:

$$\int_a^b g(\xi)e^{if(\xi)}\,d\xi. \tag{2}$$

The following five lemmas gives us a good understanding of the bounds of these integrals, under certain hypotheses.

Lemma 2.1. Let $h \in L^1([a,b])$ and $g \in \mathcal{C}^1([a,b])$ be complex valued. For any M such that

$$\left|\int_a^x h(u)\,du\right| \leq M \tag{3}$$

for all $x \in [a,b]$ we have

$$\left|\int_a^b g(u)h(u)\,du\right| \leq M(\|g\|_\infty + \|g'\|_1). \tag{4}$$

Lemma 2.2. Let $f \in \mathcal{C}^1([a,b])$ be real valued and suppose that f' is a monotonic function such that $f'(x) \neq 0$ for all $x \in [a,b]$.

$$\left|\int_a^b e^{if(u)}\,du\right| \leq \frac{4}{\lambda} \tag{5}$$

where

$$\lambda = \inf_{x\in[a,b]} |f'(x)|. \tag{6}$$

Lemma 2.3. Let $f \in \mathcal{C}^2([a,b])$ be real valued and suppose that $f''(x) \neq 0$ for all $x \in [a,b]$.

$$\left| \int_a^b e^{if(u)}\, du \right| \leq \frac{8}{\sqrt{\rho}} \tag{7}$$

where

$$\rho = \inf_{x \in [a,b]} |f''(x)|. \tag{8}$$

Lemma 2.4. Let $g \in \mathcal{C}^1([a,b])$ be complex valued and let $f \in \mathcal{C}^2([a,b])$ be real valued and such that $f''(x) \neq 0$ for all $x\imath[a,b]$.

$$\left| \int_a^b g(u) e^{if(u)}\, du \right| \leq \min\left\{ \frac{4}{\lambda}, \frac{8}{\sqrt{\rho}} \right\} (\|g\|_\infty + \|g'\|_1), \tag{9}$$

where

$$\lambda = \inf_{x \in [a,b]} |f'(x)|; \quad \rho = \inf_{x \in [a,b]} |f''(x)|. \tag{10}$$

Lemma 2.5. Let $\nu : \mathbb{R} \to \mathbb{C}$ be analytic on a neighborhood of ξ_0 where $|\nu(\xi_0)| = 1$. If ξ_0 is a point of order $m \geq 2$ for ν, then there is $\delta > 0$ such that

$$\frac{1}{2\pi} \int_{|\xi - \xi_0| \leq \delta} \nu(\xi) e^{-ix\xi}\, d\xi = O(n^{-1/m}) \tag{11}$$

where the limit is uniform in $x \in \mathbb{R}$.

3 Oscillatory Integrals