PY 711 Fall 2010
Homework 8: Due Tuesday, October 26

1. In class we are learning about normal ordering and Wick's theorem in quantum field theory. In this homework problem we get some practice on a different application of normal ordering as it relates to quantum mechanics and coherent states. We consider a quantum harmonic oscillator with Hamiltonian

$$H = \frac{1}{2}\left(p^2 + q^2\right), \tag{1}$$

where p and q satisfy the usual commutation relation, $[q, p] = i$. We are being lazy about factors of m and ω, choosing instead to absorb them into the definitions of p and q. As usual we define annihilation and creation operators

$$a = \frac{1}{\sqrt{2}}(q + ip) \tag{2}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(q - ip). \tag{3}$$

Let $|\psi_0\rangle$ be the ground state of H with normalization $\langle\psi_0\,|\psi_0\rangle = 1$. For any complex number z we define the coherent state

$$|z\rangle = ce^{za^\dagger}|\psi_0\rangle, \tag{4}$$

where c is chosen so that $\langle z\,|z\rangle = 1$.

(a) (3 points) Find the normalization constant c in Eq. (4).

(b) (4 points) Coherent states for different z are not orthogonal. Compute $\langle z_1\,|z_2\rangle$ for complex numbers z_1 and z_2.

(c) (4 points) Show that $|z\rangle$ is an eigenstate of the annihilation operator a and find its eigenvalue.

(d) (4 points) We define normal ordering in the same manner as we did in class, annihilation operators go to the right-hand side and creation operators to the left-hand side. For general non-negative integers m, n determine the normal-ordered expectation value,

$$\langle z| : p^m q^n : |z\rangle. \tag{5}$$

KELLY PATION
HW #8

$\frac{15}{15}$

1. IN CLASS WE ARE LEARNING ABOUT NORMAL ORDERING AND WICK'S THEOREM IN QUANTUM FIELD THEORY. IN THIS HOMEWORK PROBLEM WE GET SOME PRACTICE ON A DIFFERENT APPLICATION OF NORMAL ORDERING AS IT RELATES TO QUANTUM MECHANICS AND COHERENT STATES. WE CONSIDER A QUANTUM HARMONIC OSCILLATOR WITH HAMILTONIAN

$$H = \tfrac{1}{2}(p^2 + q^2),$$

WHERE p AND q SATISFY THE USUAL COMMUTATION RELATION, $[q,p] = i$. WE ARE BEING LAZY ABOUT FACTORS OF m AND ω, CHOOSING TO ABSORB THEM INTO THE DEFINITIONS OF p AND q. AS USUAL WE DEFINE THE ANNIHILATION AND CREATION OPERATORS

$$a = \frac{1}{\sqrt{2}}(q + ip)$$

$$a^+ = \frac{1}{\sqrt{2}}(q - ip).$$

LET $|\psi_0\rangle$ BE THE GROUND STATE OF H WITH NORMALIZATION $\langle\psi_0|\psi_0\rangle = 1$. FOR ANY COMPLEX NUMBER z WE DEFINE THE COHERENT STATE

$$|z\rangle = c\, e^{za^+}|\psi_0\rangle,$$

WHERE c IS CHOSEN SO THAT $\langle z|z\rangle = 1$.

a. FIND THE NORMALIZATION CONSTANT c.

$$|z\rangle = c\exp[za^+]|\psi_0\rangle \qquad \langle z| = \langle\psi_0|\exp[z^* a]\, c^+$$

Recall $\quad (a^+)^n|\psi_0\rangle = \sqrt{n!}\,|\psi_0\rangle$

$$\langle z|z\rangle = 1 = \langle\psi_0|\exp[z^* a]\, c^+ c\, \exp[za^+]|\psi_0\rangle$$

$$1 = \sum_{n,m}\langle\psi_0|\frac{(z^* a)^n}{n!}\, c^+ c\, \frac{(za^+)^m}{m!}|\psi_0\rangle$$

$$1 = \sum_{n,m}\langle\psi_n|\frac{(z^*)^n}{\sqrt{n!}}\, c^+ c\, \frac{(z)^m}{\sqrt{m!}}|\psi_m\rangle$$

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$$1 = \sum_{n,m}\frac{(z^*)^n (z)^m}{\sqrt{n!}\sqrt{m!}}\, c^+ c\, \underbrace{\langle\psi_n|\psi_m\rangle}_{\delta_{nm}}$$

$$1 = \sum_n \frac{|z|^2}{n!}\, c^+ c$$

$$1 = \exp[|z|^2]\, c^+ c$$

$$c^+ c = \exp[-|z|^2] \quad \Rightarrow \quad \boxed{c = \exp[-|z|^2/2]}$$

b. COHERENT STATES FOR DIFFERENT z ARE NOT ORTHOGONAL. COMPUTE $\langle z_1 | z_2 \rangle$ FOR COMPLEX NUMBERS z_1 AND z_2.

$$\langle z_1 | z_2 \rangle = \langle \psi_0 | \exp[z_1^* a] \exp[-|z_1|^2/2] \exp[-|z_2|^2/2] \exp[z_2 a^+] | \psi_0 \rangle$$

$$= \exp[-(|z_1|^2 + |z_2|^2)/2] \sum_{n,m} \langle \psi_0 | \frac{(z_1^* a)^n}{n!} \frac{(z_2 a^+)^m}{m!} | \psi_0 \rangle$$

$$= \exp[-(|z_1|^2 + |z_2|^2)/2] \sum_{n,m} \langle \psi_n | \frac{(z_1^*)^n}{\sqrt{n!}} \frac{(z_2)^m}{\sqrt{m!}} | \psi_m \rangle$$

$$= \exp[-(|z_1|^2 + |z_2|^2)/2] \sum_{n,m} \frac{(z_1^*)^n (z_2)^m}{\sqrt{n!}\sqrt{m!}} \underbrace{\langle \psi_n | \psi_m \rangle}_{\delta_{nm}}$$

$$= \exp[-(|z_1|^2 + |z_2|^2)/2] \sum_{n} \frac{(z_1^* z_2)^n}{n!}$$

$$\boxed{\langle z_1 | z_2 \rangle = \exp[-(|z_1|^2 + |z_2|^2)/2] \exp[z_1^* z_2]}$$

Check: If $z_1 = z_2$

$$\langle z | z \rangle = \exp[-2|z|^2/2] \exp[|z|^2] = 1 \checkmark$$

$\frac{4}{4}$

C. SHOW THAT $|z\rangle$ IS AN EIGENSTATE OF THE ANNIHILATION OPERATOR a AND FIND ITS EIGENVALUE.

$$a|z\rangle = a\left(\exp[-|z|^2/2]\exp[za^+]\,|\psi_0\rangle\right)$$
$$= \exp[-|z|^2/2]\sum_n a\,\frac{(za^+)^n}{n!}|\psi_0\rangle$$

At this point, it will be helpful to look at $[a,(a^+)^n]$

$$[a,a^+] = 1$$
$$[a,(a^+)^2] = a^+[a,a^+] + [a,a^+]a^+ = 2a^+$$
$$[a,(a^+)^3] = (a^+)^2[a,a^+] + [a,(a^+)^2]a^+ = 3(a^+)^2$$
$$\vdots$$
$$[a,(a^+)^n] = (a^+)^{n-1}[a,a^+] + [a,(a^+)^{n-1}]a^+ = n(a^+)^{n-1}$$

If we use this commutator, we see

$$a|z\rangle = \exp[-|z|^2/2]\left(\sum_n n\frac{z^n}{n!}(a^+)^{n-1}|\psi_0\rangle + \sum_n \frac{(za^+)^n}{n!}\cancelto{0}{a|\psi_0\rangle}\right)$$

$$= \exp[-|z|^2/2]\,z\sum_n \frac{(za^+)^{n-1}}{n!}\,n\,|\psi_0\rangle$$

$$\frac{n}{n!} = \frac{1}{(n-1)!}$$

$$= \exp[-|z|^2/2]\,z\sum_n \frac{(za^+)^{n-1}}{(n-1)!}|\psi_0\rangle$$

$$= z\,\underbrace{\exp[-|z|^2/2]\exp[za^+]|\psi_0\rangle}_{|z\rangle}$$

$$\boxed{a|z\rangle = z|z\rangle}$$

So $|z\rangle$ is an eigenstate of a with eigenvalue of z.

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d. WE DEFINE NORMAL ORDERING IN THE SAME MANNER AS WE DID IN CLASS, ANNIHILATION OPERATORS GO TO THE RIGHT-HAND SIDE AND CREATION OPERATORS TO THE LEFT-HAND SIDE. FOR GENERAL NON-NEGATIVE INTEGERS m, n DETERMINE THE NORMAL-ORDERED EXPECTATION VALUE,

$$\langle z| : p^m q^n : |z\rangle .$$

Using the definitions of a and a^+, we can solve for p and q:

$$q = \frac{1}{\sqrt{2}} (a + a^+)$$

$$p = \frac{-i}{\sqrt{2}} (a - a^+).$$

Now, using the binomial theorem

$$: p^n q^m : = \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \binom{n}{k}\binom{m}{\ell} (-a^+)^k (a^+)^\ell a^{n-k} a^{m-\ell}$$

$$= \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \binom{n}{k}\binom{m}{\ell} (a^+)^{k+\ell} a^{n+m-k-\ell} (-1)^k$$

From part c., we know

$$a|z\rangle = z|z\rangle \qquad \text{so} \qquad \langle z| a^+ = \langle z| z^*$$

$$\langle z| : p^n q^m : |z\rangle = \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \binom{n}{k}\binom{m}{\ell} \langle z| (a^+)^{k+\ell} a^{n+m-k-\ell} |z\rangle (-1)^k$$

$$= \frac{(-i)^n}{2^{(n+m)/2}} \sum_{k=0}^{n} \sum_{\ell=0}^{m} \binom{n}{k}\binom{m}{\ell} (-1)^k (z^*)^{k+\ell} z^{n+m-k-\ell} \cancel{\langle z|z\rangle}^{1}$$

Notice this is the same as $: p^n q^m :$ except with z^* and z instead of a^+ and a. So, we can write it as

$$\boxed{\langle z| : p^n q^m : |z\rangle = \frac{(-i)^n}{2^{(n+m)/2}} (z - z^*)^n (z + z^*)^m}$$

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PY711
Solutions # 8

1. $$|z\rangle = c\, e^{z a^\dagger}|0\rangle = c\sum_{n=0}^{\infty}\frac{z^n (a^\dagger)^n}{n!}|0\rangle = c\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle$$

(a) $$\langle z|z\rangle = |c|^2\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\frac{(z^*)^n z^{n'}}{\sqrt{n!}\sqrt{n'!}}\langle n|n'\rangle = |c|^2\sum_{n=0}^{\infty}\frac{(|z|^2)^n}{n!}$$
$$= |c|^2 e^{|z|^2}$$

We can choose $c = e^{-\frac{|z|^2}{2}}$ (times arbitrary phase), then $\langle z|z\rangle = 1$.

(b) $$\langle z_1|z_2\rangle = e^{-\frac{|z_1|^2}{2}}e^{-\frac{|z_2|^2}{2}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\frac{(z_1^*)^{n_1}}{\sqrt{n_1!}}\frac{z_2^{n_2}}{\sqrt{n_2!}}\langle n_1|n_2\rangle$$
$$= e^{-\frac{|z_1|^2}{2}}e^{-\frac{|z_2|^2}{2}}\sum_{n=0}^{\infty}\frac{(z_1^* z_2)^n}{n!} = e^{-\frac{|z_1|^2}{2}}e^{-\frac{|z_2|^2}{2}}e^{z_1^* z_2}$$

(c) $$a|z\rangle = e^{-\frac{|z|^2}{2}}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{n!}}\sqrt{n}\,|n-1\rangle = e^{-\frac{|z|^2}{2}}\sum_{n=1}^{\infty}\frac{z^n}{\sqrt{(n-1)!}}|n-1\rangle$$
$$= z\, e^{-\frac{|z|^2}{2}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|n\rangle = z|z\rangle$$

eigenvalue is z

(d) Note that $a|z\rangle = z|z\rangle$ and $\langle z|a^\dagger = \langle z|z^*$.
In the normal-ordered product expectation value replace $a\to z$, $a^\dagger \to z^*$.
$$\langle z|:p^m q^n:|z\rangle = \langle z|:\left[\frac{a-a^\dagger}{i\sqrt{2}}\right]^m\left[\frac{a+a^\dagger}{\sqrt{2}}\right]^n:|z\rangle$$
$$= \langle z|:\left[\frac{z-z^*}{i\sqrt{2}}\right]^m\left[\frac{z+z^*}{\sqrt{2}}\right]^n:|z\rangle$$
$$= \left[\sqrt{2}\,\mathrm{Im}(z)\right]^m\cdot\left[\sqrt{2}\,\mathrm{Re}(z)\right]^n$$