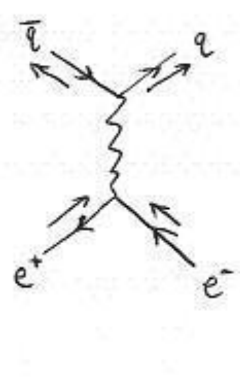
cannot be seen individually, the total cross section $e^+e^- \to$ any hadrons (strongly interacting particles) can be approximated at high energies by $e^+e^- \to q\bar{q}$ (quark antiquark).

The reason is that quantum chromodynamics or the strong interactions becomes weak at high energies. This is called asymptotic freedom, which was discovered by Gross, Politzer, and Wilczek (Nobel, 2004).

The idea is that the high energy produces a $q\bar{q}$ pair and then "afterwards" the strong interactions determines which hadrons form (hadronization). To a good approximation the total cross-section is given by

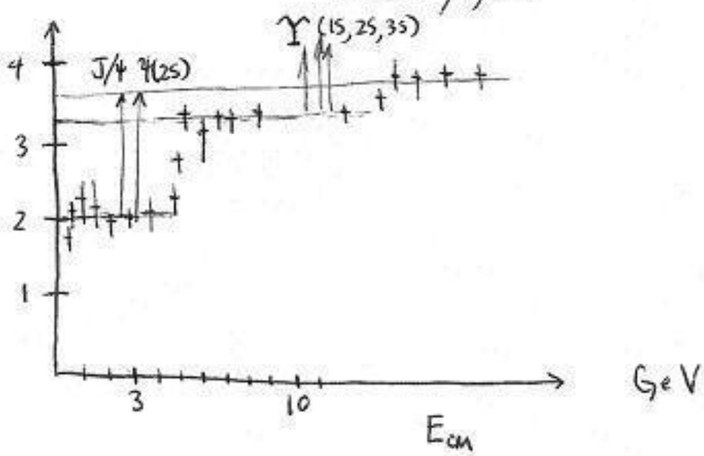


So we expect $\sigma(e^+e^- \to \text{hadrons}) \approx \sum_{i=\{\text{all quarks}\}} Q_i^2 \, \sigma(e^+e^- \to \mu^+\mu^-)$

at high energies where Q_i is the electric charge of quarks i.

One observes for

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}$$



Below ~4 GeV we have 3 light quark flavors

$$u \; \ldots . \quad Q = +\tfrac{2}{3}$$
$$d \; \ldots . \quad Q = -\tfrac{1}{3}$$
$$s \; \ldots . \quad Q = -\tfrac{1}{3}$$

But there are three colors for each flavor and so

$$R = 3 \times \left(\left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 \right) = 2$$

Between ~4 GeV and ~10 GeV we have a fourth flavor

$$c \ldots \quad Q = +\tfrac{2}{3}$$

So $$R = 3 \times \left(\left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 \right) = 3\tfrac{1}{3}$$

Above ~10 GeV we have a fifth flavor

$$b \ldots \quad Q = -\tfrac{1}{3}$$

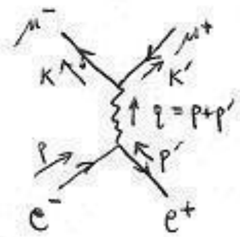
So $$R = 3 \times \left(\left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 + \left(\tfrac{2}{3}\right)^2 + \left(-\tfrac{1}{3}\right)^2 \right) = 3\tfrac{2}{3}$$

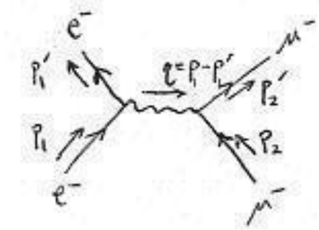
The heaviest quark, t, has mass ~180 GeV and so requires $E_{cm} > 360$ GeV.

Crossing symmetry

$e^+e^- \to \mu^+\mu^-$ is related to $e^-\mu^- \to e^-\mu^-$

$(m_e = 0)$





$$i\mathcal{M} = \frac{ie^2}{q^2}\left(\bar{v}(p')\gamma^\mu u(p)\right)\left(\bar{u}(k)\gamma_\mu v(k')\right)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4(q^2)^2}\times \mathrm{tr}\left[\not{p}'\gamma^\mu \not{p}\gamma^\nu\right] \times \mathrm{tr}\left[(\not{k}+m_\mu)\gamma_\mu(\not{k}'-m_\mu)\gamma_\nu\right]$$

$$= \frac{8e^4}{(q^2)^2}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k) + m_\mu^2(p\cdot p')\right]$$

$$i\mathcal{M} = \frac{ie^2}{q^2}\left(\bar{u}(p_1')\gamma^\mu u(p_1)\right)\left(\bar{u}(p_2')\gamma_\mu u(p_2)\right)$$

$$\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{e^4}{4(q^2)^2}\times \mathrm{tr}\left[\not{p}_1'\gamma^\mu \not{p}_1\gamma^\nu\right] \times \mathrm{tr}\left[(\not{p}_2'+m_\mu)\gamma_\mu(\not{p}_2+m_\mu)\gamma_\nu\right]$$

$$= \frac{8e^4}{(q^2)^2}\left[(p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2') - m_\mu^2(p_1\cdot p_1')\right]$$

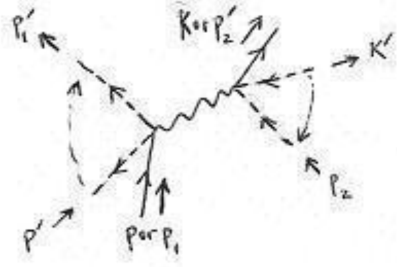
comparing

$p \longleftrightarrow p_1$

$k \longleftrightarrow p_2'$

$p' \longleftrightarrow -p_1'$

$k' \longleftrightarrow -p_2$



incoming $\longleftrightarrow$ outgoing

particle $\longleftrightarrow$ antiparticle

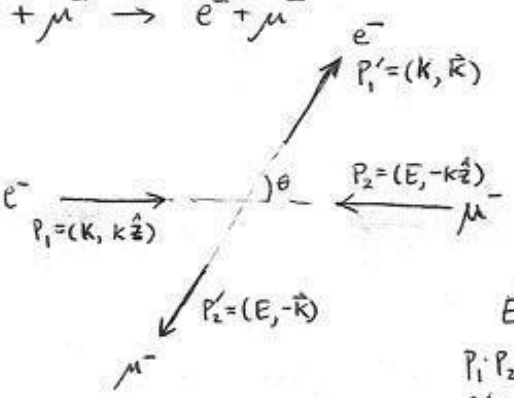
momentum $\longleftrightarrow$ $-$momentum

Once you have the unpolarized $\frac{1}{4}\sum_{spin}|\mathcal{M}|^2$ for one process you can get the $\frac{1}{4}\sum_{spin}|\mathcal{M}|^2$ for all possible "crossing" diagrams

$$e^+ + e^- \rightarrow \mu^+ + \mu^-$$
$$e^- + \mu^- \rightarrow e^- + \mu^-$$
$$e^+ + \mu^+ \rightarrow e^+ + \mu^+$$
$$e^+ + \mu^- \rightarrow e^+ + \mu^-$$
$$\vdots$$

But the Kinematic factors in the cross section formula are completely different.

For $e^- + \mu^- \rightarrow e^- + \mu^-$



$(m_e = 0)$

$K = |\vec{K}|$

$E^2 = K^2 + m_\mu^2$

$P_1 \cdot P_2 = P_1' \cdot P_2' = KE + K^2$

$P_1' \cdot P_2 = P_1 \cdot P_2' = KE + K^2\cos\theta$

$P_1 \cdot P_1' = K^2 - K^2\cos\theta$

$q = P_1 - P_1'$, $\quad q^2 = (P_1 - P_1')^2 = P_1^2 + P_1'^2 - 2P_1 \cdot P_1'$
$= -2P_1 \cdot P_1' = -2(K^2 - K^2\cos\theta)$

So
$$\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 = \frac{8e^4}{4K^4 \cdot (1-\cos\theta)^2}\left[K^2(E+K\cos\theta)^2 + K^2(E+K)^2 - m_\mu^2 K^2(1-\cos\theta)\right]$$

$$= \frac{2e^4}{K^2(1-\cos\theta)^2}\left[(E+K)^2 + (E+K\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right]$$

For two body final state...

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{k\,|\mathcal{M}|^2}{2E_A 2E_B\,|v_A - v_B|\,(2\pi)^2\,4E_{CM}}$$

Using $E_{CM} = E + K$,

(electron) $E_A = k$,

(muon) $E_B = E$,

$v_A = 1 \quad (m_e = 0)$,

$v_B = -\frac{k}{E}$,

we have $\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{k\,|\mathcal{M}|^2}{2k\cdot 2E\,(1+\frac{k}{E})\cdot 4\pi^2\cdot 4(E+K)} = \frac{|\mathcal{M}|^2}{64\pi^2(E+K)^2}.$

So $$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{\alpha^2}{2K^2(E+K)^2(1-\cos\theta)^2}\left[(E+K)^2 + (E+K\cos\theta)^2 - m_\mu^2(1-\cos\theta)\right]$$

Notice that as $\theta \to 0$, the cross-section diverges as $\theta \to 0$ as $\sim \frac{1}{\theta^4}$. This is because of the photon propagator being nearly on mass shell, $q^2 \approx 0$. The same result can be seen in non-relativistic Rutherford scattering. The divergent cross section is due to the fact that the Coulomb force has infinite range.

General Crossing Symmetry

For a scalar particle,

$$\mathcal{M}(\phi(p) + X \to Y) = \mathcal{M}(X \to Y + \bar{\phi}(-p))$$

just flip sign of p and you get
the new amplitude.

For fermion spinors, there is an additional minus sign for the unpolarized spin sums since

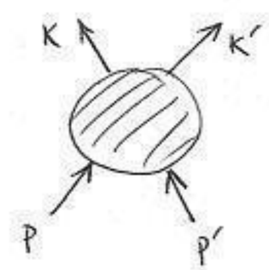
$$\sum_{\text{spins}} u(p)\bar{u}(p) = \not{p} + m \quad \text{while}$$

$$\sum_{\text{spins}} v(-p)\bar{v}(-p) = (-\not{p}) - m = -(\not{p}+m)$$

So $\times(-1)$ for each flipped fermion.

Mandelstam variables (convenient for crossing symmetries)

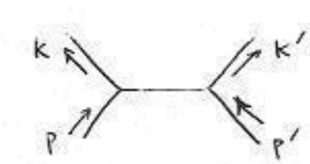
Two-body to two-body scattering



there is some ambiguity in defining K, K′ (which one is which)
If one of the outgoing particles is the same type as an incoming particle, call the momenta K + p respectively.

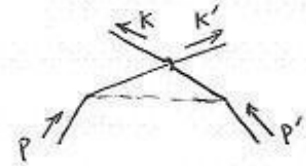
We define
$$s = (p+p')^2 = (k+k')^2$$
$$t = (p-k)^2 = (p'-k')^2$$
$$u = (p-k')^2 = (p'-k)^2$$

t - channel:



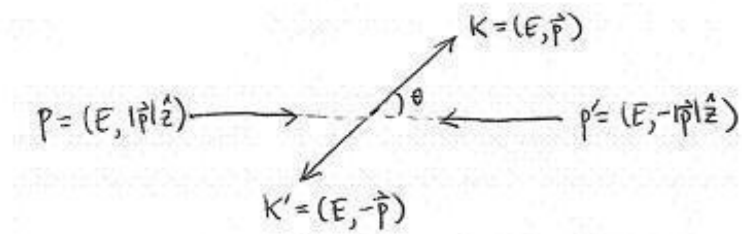
$\mathcal{M} \propto \frac{1}{t - m^2}$ t-channel pole

u- channel:



$\mathcal{M} \propto \frac{1}{u - m^2}$ u-channel pole

Let's take a look at s, t, u in the center of mass frame. For simplicity we assume all particles have mass m.



$$s = (p+p')^2 = (2E)^2 = E_{cm}^2$$

$$t = (p-k)^2 = -(|\vec{p}|\hat{z} - \vec{p}) \cdot (|\vec{p}|\hat{z} - \vec{p}) = -2|\vec{p}|^2(1-\cos\theta)$$

$$u = (p-k')^2 = -(|\vec{p}|\hat{z} + \vec{p}) \cdot (|\vec{p}|\hat{z} + \vec{p}) = -2|\vec{p}|^2(1+\cos\theta)$$

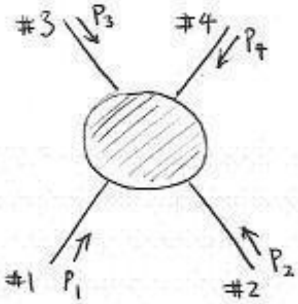
Note that $s + t + u = E_{cm}^2 - 4|\vec{p}|^2 = 4(E^2 - |\vec{p}|^2) = 4m^2$

This is a special case of the general statement

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

square of mass of incoming/outgoing particle

Proof:



If #3 & #4 are outgoing particles then the physical momenta are $-p_3$ and $-p_4$.

Then
$$2(s+t+u) = \underbrace{(p_1+p_2)^2 + (p_3+p_4)^2}_{2s} + \underbrace{(p_1+p_3)^2 + (p_2+p_4)^2}_{2t}$$
$$+ \underbrace{(p_1+p_4)^2 + (p_2+p_3)^2}_{2u}$$
$$= 3\sum_{i=1}^{4} p_i^2 + 2\sum_{i>j} p_i \cdot p_j$$

Since $\sum_{i=1}^{4} p_i = 0$, this means that $\left(\sum_{i=1}^{4} p_i\right)^2 = 0$
$$= \sum_{i=1}^{4} p_i^2 + 2\sum_{i>j} p_i \cdot p_j$$

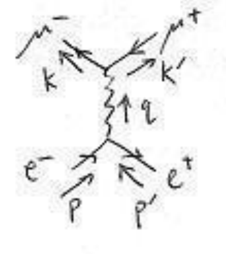
So $2\sum_{i>j} p_i p_j = -\sum_{i=1}^{4} p_i^2$. Therefore

$2(s+t+u) = 2\sum_{i=1}^{4} p_i^2$, and so $s+t+u = \sum_{i=1}^{4} p_i^2$
$$= \sum_{i=1}^{4} m_i^2. \quad //$$

Compton Scattering

In 1923 Compton studied angular dependence of scattering of

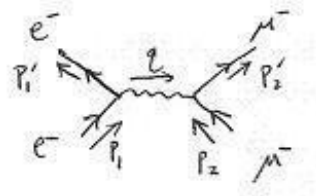
It is always possible to write Lorentz invariant quantities in terms of s, t, u



$$s = (p+p')^2 = q^2$$
$$t = (p-k)^2 = p^2+k^2-2\,p\cdot k \;=\; m_e^2+m_\mu^2-2p\cdot k$$
$$\underset{(m_e=m_\mu=0)}{\text{when}} \quad = -2p\cdot k = -2p'\cdot k'$$
$$u = (p-k')^2 = m_e^2+m_\mu^2-2p\cdot k' = -2p\cdot k' = -2p'\cdot k$$

We find $\displaystyle \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{(q^2)^2}\left[(p\cdot k)(p'\cdot k') + (p\cdot k')(p'\cdot k)\right]$

$$= \frac{8e^4}{s^2}\left[\frac{1}{4}t^2+\frac{1}{4}u^2\right] = \frac{2e^4}{s^2}(t^2+u^2)$$



(when $m_e=m_\mu=0$)
$$s = (p_1+p_2)^2 = 2p_1\cdot p_2 = 2p_1'\cdot p_2'$$
$$t = (p_1-p_1')^2 = -2p_1\cdot p_1' = -2p_2\cdot p_2' = q^2$$
$$u = (p_1-p_2')^2 = -2p_1\cdot p_2' = -2p_2\cdot p_1'$$

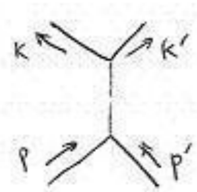
We find $\displaystyle \frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2 = \frac{8e^4}{(q^2)^2}\left[(p_1\cdot p_2')(p_1'\cdot p_2) + (p_1\cdot p_2)(p_1'\cdot p_2')\right]$

$$= \frac{8e^4}{t^2}\left[\frac{1}{4}u^2+\frac{1}{4}s^2\right] = \frac{2e^4}{t^2}(u^2+s^2)$$

You can now easily see how s and t interchange roles in the two scattering processes.

If a Feynman diagram has only one virtual particle (internal line) then we can say it is either an s-channel, t-channel, or u-channel diagram.

s-channel:



$\mathcal{M} \propto \frac{1}{s-m^2}$ s-channel pole