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Course: **8.422 - AMO II**
Problem set: **#3**
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1. Classical Coherence of Light.

Consider a classical light field. The classical expressions for first-order and second-order coherence are:

$$g^{(1)}(\tau) = \frac{\langle E^*(t)E(t+\tau)\rangle}{\langle E^*(t)E(t)\rangle}$$
$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} = \frac{\langle E^*(t)E^*(t+\tau)E(t+\tau)E(t)\rangle}{\langle E^*(t)E(t)\rangle^2}$$

where $\langle\cdot\rangle$ denotes a statistical averaging over many measurements, which can be interpreted as a time average. Moreover, $\langle f^*(t)g(t)\rangle$ defines a scalar product.

a) Here we show that $|g^{(1)}(\tau)| \le 1$. In view of Cauchy's inequality, and the fact that $\langle E^*(t+\tau)E(t+\tau)\rangle = \langle E^*(t)E(t)\rangle$

$$|g^{(1)}(\tau)|^2 = \frac{|\langle E^*(t)E(t+\tau)\rangle|^2}{|\langle E^*(t)E(t)\rangle|^2} \le \frac{\langle E^*(t)E(t)\rangle\langle E^*(t+\tau)E(t+\tau)\rangle}{|\langle E^*(t)E(t)\rangle|^2} = \frac{\langle E^*(t)E(t)\rangle\langle E^*(t)E(t)\rangle}{|\langle E^*(t)E(t)\rangle|^2} = 1.$$

So, $|g^{(1)}(\tau)| \le 1$ as desired.

b) Here we show that for zero time-delay, the second-order coherence obeys $g^{(2)}(0) \ge 1$. To this end, we notice that

$$0 \le \langle [I(t) - \langle I(t)\rangle]^2\rangle = \langle I(t)^2 - 2I(t)\langle I(t)\rangle + \langle I(t)\rangle^2\rangle = \langle I(t)^2\rangle - 2\langle I(t)\rangle^2 + \langle I(t)\rangle^2 = \langle I(t)^2\rangle - \langle I(t)\rangle^2.$$

Thus, for $\tau = 0$, we have

$$g^{(2)}(0) = \frac{\langle I(t)^2\rangle}{\langle I(t)\rangle^2} \ge \frac{\langle I(t)\rangle^2}{\langle I(t)\rangle^2} = 1,$$

as desired. This implies that light in a number state, which has $g^{(2)}(0) < 1$, has no classical analog.

c) Here we show that $g^{(2)}(\tau) \le g^{(2)}(0)$. Viewing $\langle I(t)I(t+\tau)\rangle$ as the inner product $\langle I(t), I(t+\tau)\rangle$ (which is possible since intensities are real numbers), we see immediately that $\langle I(t)I(t+\tau)\rangle \le \langle I(t)I(t)\rangle$, which gives

$$g^{(2)}(\tau) = \frac{\langle I(t)I(t+\tau)\rangle}{\langle I(t)\rangle^2} \le \frac{\langle I(t)I(t)\rangle}{\langle I(t)\rangle^2} = g^{(2)}(0).$$

This implies that anti-bunched light, which has $g^{(2)}(\tau) > g^{(2)}(0)$, has no classical analog.

d) Consider chaotic classical light generated by an ensemble of ν atoms. The total electric field can be expressed as $E(t) = \sum_{i=1}^{\nu} E_i(t)$, where the phases of E_i are random. Here we show that when ν is large,

$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2.$$

Since statistical averages in which the random electric field phases do not cancel are zero, we have

$$\langle E^*(t)E(t+\tau)\rangle =$$

2. Quantum Coherence of Light.

Consider light in a single mode of the radiation field. The quantum mechanical expressions for first-order and second-order coherence are

$$g^{(1)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2) = \frac{\langle \hat{E}^-(\mathbf{r}_1, t_1)\hat{E}^+(\mathbf{r}_2, t_2)\rangle}{\langle \hat{E}^-(\mathbf{r}_1, t_1)\hat{E}^+(\mathbf{r}_1, t_1)\rangle^{1/2}\langle \hat{E}^-(\mathbf{r}_2, t_2)\hat{E}^+(\mathbf{r}_2, t_2)\rangle^{1/2}}$$
$$g^{(2)}(\mathbf{r}_1, t_1, \mathbf{r}_2, t_2; \mathbf{r}_2, t_2, \mathbf{r}_1, t_1) = \frac{\langle \hat{E}^-(\mathbf{r}_1, t_1)\hat{E}^-(\mathbf{r}_2, t_2)\hat{E}^+(\mathbf{r}_2, t_2)\hat{E}^+(\mathbf{r}_1, t_1)\rangle}{\langle \hat{E}^-(\mathbf{r}_1, t_1)\hat{E}^+(\mathbf{r}_1, t_1)\rangle\langle \hat{E}^-(\mathbf{r}_2, t_2)\hat{E}^+(\mathbf{r}_2, t_2)\rangle}$$

where

$$\hat{E}^+(\mathbf{r}, t) = i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\, a e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$
$$\hat{E}^-(\mathbf{r}, t) = -i\sqrt{\frac{\hbar\omega}{2\epsilon_0 V}}\, a^\dagger e^{+i(\omega t - \mathbf{k}\cdot\mathbf{r})}$$

a) Here we show that

$$g^{(2)}(0) = \frac{\langle a^\dagger a^\dagger a a\rangle}{\langle a^\dagger a\rangle^2} = \frac{\langle n^2\rangle - \langle n\rangle}{\langle n\rangle^2}$$

b) Next we show that for number states of light $|n\rangle$ where $n > 2$,

$$|g^{(1)}| = 1 \qquad g^{(2)} = 1 - \frac{1}{n}$$

independent of space-time separation.

For $n = 0$, we calculate $g^{(1)}$ and $g^{(2)}$:

$$g^{(1)}_{n=0} =$$
$$g^{(2)}_{n=0} =$$

For $n = 1$, we calculate $g^{(1)}$ and $g^{(2)}$

$$g^{(1)}_{n=1} =$$
$$g^{(2)}_{n=1} =$$

c) Consider a coherent state $|\alpha\rangle$, we show that $|g^{(1)}| = |g^{(2)}| = 1$

$$|g^{(1)}_{|\alpha\rangle}| =$$

$$|g^{(2)}_{|\alpha\rangle}| =$$

d) Finally, consider chaotic light with density matrix

$$\hat{\rho} = \left(1 - e^{-\hbar\omega/k_B T}\right)\sum_n e^{-n\hbar\omega/k_B T}\,|n\rangle\langle n|\,.$$

Here we show that $|g^{(1)}| = 1$ and $|g^{(2)}| = 2$.

Compare these results to what we found in Problem 1... What about multi-mode chaotic light?

e) Consider the states

$$|\psi_\pm\rangle = \frac{|\alpha\rangle \pm |-\alpha\rangle}{\sqrt{2}\sqrt{1 \pm e^{-2|\alpha|^2}}}.$$

Let us compute $g^{(2)}(\tau)$ for these states as a function of α.

Do either of these two states show non-classical second-order coherence? Why (or why not)? (Make sure you agree with the normalization given)

3. The Quantum Beamsplitter.

Let the beamsplitter operator B, acting with angle $\angle$ on modes a and b, by defined by

$$B = \exp\left[\theta\left(a^\dagger b - ab^\dagger\right)\right].$$

a) Here we show that B conserves photon number and leaves coherent states as coherent states.

b) Let $|\alpha\rangle$ be a coherent state. Here we compute $B|0\rangle_b |\alpha\rangle_a$.

From here, we see that the output is a tensor product of coherent states for all θ. This makes sense: the beamsplitter has well-defined transmission and reflection coefficients. What are these in terms of θ?

c) There is close connection between the Lie group $SU(2)$ and the algebra of two coupled harmonic oscillators, which is useful for understanding B. Define

$$s_z = a^\dagger a - b^\dagger b \qquad s_+ = a^\dagger a \qquad s_- = b^\dagger b,$$

and let $s_\pm = (s_x \pm i s s_y)/\sqrt{2}$.

What is $B(\theta)$ in spin space?

What is $a^\dagger a + b^\dagger b$ in spin space?

We now show that s_x, s_y, s_z have the same commutation relations as Pauli matrices.

How does this explain why $a^\dagger a + b^\dagger b$ is invariant under B. It is the Casimir operator of the algebra.

d) Here we look at how a beamsplitter transforms an input of a photon-number eigenstate. Let

$$B(\theta) = \exp\left[\theta\left(-a^\dagger b + ab^\dagger\right)\right],$$

and $B = B(\pi/4)$ be a $50/50$ beamsplitter, such that

$$BaB^\dagger = \frac{a+b}{\sqrt{2}} \qquad BbB^\dagger = \frac{-a+b}{\sqrt{2}}.$$

Let us compute $B|0\rangle_b |n\rangle_a$.

Note that the result is NOT $|n/2\rangle|n/2\rangle$ since $|n\rangle$ is a number state and not a coherent state.

What photon number states have the largest amplitude?

How sharp is the distribution for $n = 10$, and $n = 100$, or as a function of n, if a general solution exists? To do this, we look at the binomial expansion for $(a^\dagger + b^\dagger)^n$.

4. The Hanbury-Brown and Twiss experiment and $g^{(2)}(\tau)$.

Here we look at how the HBT experiment measures $g^{(2)}(\tau)$. To this end, let $a, a^\dagger, b, b^\dagger$ be the raising and lowering operators for the two modes of light input to the beamsplitter, and let the unitary transformation performed by the beamsplitter be defined by

$$a_1 = UaU^\dagger = \frac{a+b}{\sqrt{2}} \qquad b_1 = UbU^\dagger = \frac{a-b}{\sqrt{2}}.$$

For light input in state $|\psi_a\rangle$, suppose that the output of the coincidence circuit is a voltage

$$V_{\psi_a,0_b} = V_0 \langle\psi_a, 0_b| \, a_1^\dagger a_1 b_1^\dagger b_1 \, |\psi_a, 0_b\rangle \, ,$$

which is nothing but the average of the product of the two detected photon signals. We will show that $V_{\psi_a,0_b}$ gives measure of $g^{(2)}(\tau)$ up to an additive offset and normalization.