8.422 Pset 4 Solution, 2023

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This solution benefits a great deal from the work of the previous TA (Thomas Hartke, Joshua Ramette and possibly in 2021, and possibly many more in earlier years). Their solutions are very rigorous and complete, but sometimes there is also a short-cut which many of you used. I try to mention both in this revised solution.

1 Problem 1

This problem solves the Hamiltonian for the *inverted* simple harmonic oscillator.

As stated in the problem, the quantum commutation relations and Hamiltonian in normalized units are $[\tilde{x}, \tilde{p}] = i$ and $H = \hbar\omega(\tilde{p}^2 - \tilde{x}^2)/2$. These operators are still related to the usual position and momentum up to a scaling factor.

Now we will rotate our basis by defining new operators $p \equiv (\tilde{p} - \tilde{x})/\sqrt{2}$ and $x \equiv (\tilde{p} + \tilde{x})/\sqrt{2}$. We can quickly check that $[x, p] = ([\tilde{x}, \tilde{p}] - [\tilde{p}, \tilde{x}])/2 = i$ as before, and $H = \hbar\omega(xp - i/2)$. Having a complex constant offset is odd, but one can see why it is necessary to ensure H is a hermitian operator

$$\frac{\tilde{H}^\dagger}{\hbar\omega} = p^\dagger x^\dagger + \frac{i}{2} = px + \frac{i}{2} = xp - [x, p] + \frac{i}{2} = xp - \frac{i}{2} = \frac{\tilde{H}}{\hbar\omega}.$$

Indeed, the complex constant offset produces a rescaling of the wavefunction with time, which is necessary to ensure that the wavefunction remains normalized in time. If dropping the constant offset bothers you, you can work with the symmetrized Hamiltonian $H = \hbar\omega(xp + px)/2$.

Sec. 4.1 has a brief discussion of the derivation of the form of the operator $\hat{p}$ in the $\hat{x}$ basis, and the transformation of the wavefunction from x to p space. These operator forms and basis transformations are directly implied by the commutation relation $[\hat{x}, \hat{p}] = i$ alone, regardless of the physical interpretation of $\hat{x}$ and $\hat{p}$.

1.1 (a)

Quick solution: If using our physical intuition, knowing this saddle-like potential of the Hamiltonian will cause squeezing, the wave function should be simply a re-scaled version of the initial wave-function. We assume the ansatz:

$$\psi(x, t) = \psi_0(x(t))$$

Then we can simply solve the Schrodinger's equation

$$i\hbar\frac{\partial}{\partial t}\psi = \hbar\omega x(-i\frac{\partial}{\partial x})\psi$$

$$\frac{\partial}{\partial t}\psi = -\omega x\frac{\partial}{\partial x}\psi$$

Now we substitute our ansatz $\psi(x, t) = \psi_0(x(t))$, we get

$$\frac{d\psi}{dx_0}\frac{dx}{dt} = -\omega x\frac{d\psi}{dx_0}$$

For this equation to hold, we need

$$\frac{dx}{dt} = -\omega x$$

Solving this give us

$$x(t) = x_0 e^{(} - \omega t)$$

Rigorous solution We would like to solve

$$i\hbar \frac{\partial |\psi\rangle}{\partial t} = H|\psi\rangle \tag{1}$$

given some initial state $|\psi\rangle = \int dx \psi(x)|x\rangle$. We know that $\hat{p} = -i\frac{\partial}{\partial x}$ when acting on the wavefunction in the x basis (see Sec. 4.1). Care will be taken to distinguish operators such as $\hat{x}$ from numbers such as x.

$$\begin{aligned}\hat{H}|\psi\rangle = \hat{H}\int dx\psi(x)|x\rangle &= \frac{\hbar\omega}{2}\left(\hat{x}\hat{p} + \hat{p}\hat{x}\right)\int dx\psi(x)|x\rangle \\ &= -i\frac{\hbar\omega}{2}\int dx\left(x\frac{\partial}{\partial x}(\psi(x)) + \frac{\partial}{\partial x}(x\psi(x))\right)|x\rangle = -i\frac{\hbar\omega}{2}\int dx\left(\psi(x) + 2x\left(\frac{\partial}{\partial x}\psi(x)\right)\right)|x\rangle. \end{aligned} \tag{2}$$

Projecting Schrodinger's equation for the state evolution onto a specific state $\langle y|$ gives

$$\frac{\partial}{\partial t}\psi(y,t) = -\frac{\omega}{2}\psi(y) - \omega y\frac{\partial}{\partial y}\psi(y). \tag{3}$$

We can guess a solution $\psi(x,t) = \alpha(t)\psi_0(x_0(x,t))$ with initial conditions/definitions $\alpha(t{=}0) = 1$ and $x_0(x,t{=}0) = x$ such that $\psi(x,t{=}0) = \psi_0(x)$. Note that x is simply an argument of the wavefunction, and is not dependent on time, while $x_0(x,t)$ is a function with two arguments. To solve this equation, we require

$$\begin{aligned}\frac{\partial}{\partial t}\left(\alpha(t)\psi_0(x_0(x,t))\right) &= -\frac{\omega}{2}\alpha(t)\psi_0(x_0(x,t)) - \omega x\frac{\partial}{\partial x}\left[\alpha(t)\psi_0(x_0(x,t))\right] \\ \alpha'(t)\psi_0(x_0(x,t)) + \alpha(t)\psi_0'(x_0(x,t))\left[\partial_t x_0(x,t)\right] &= -\frac{\omega}{2}\alpha(t)\psi_0(x_0(x,t)) - \omega x\alpha(t)\psi_0'(x_0(x,t))\left[\partial_x x_0(x,t)\right].\end{aligned}$$

This is equivalent to

$$\begin{aligned}\alpha'(t) &= -\frac{\omega}{2}\alpha(t) \\ \left[\partial_t x_0(x,t)\right] &= -\omega x\left[\partial_x x_0(x,t)\right]\end{aligned}$$

and is solved (with the correct initial conditions) by

$$\begin{aligned}\alpha(t) &= e^{-\frac{\omega}{2}t} \\ x_0(x,t) &= xe^{-\omega t} \\ \psi(x,t) &= e^{-\frac{\omega}{2}t}\psi_0(xe^{-\omega t}).\end{aligned}$$

Note that both the $\alpha(t)$ factor is necessary to preserve normalization of the state over time evolution.

$$\langle\psi(t)|\psi(t)\rangle = \int dxe^{-\omega t}\left|\psi_0(xe^{-\omega t})\right|^2 = \int d\left(xe^{-\omega t}\right)\left|\psi_0(xe^{-\omega t})\right|^2 = \int dy\left|\psi_0(y)\right|^2 = 1. \tag{4}$$

Because the scaling of x as an argument to $\psi(x,t)$ becomes smaller with time, the extent of the wavefunction in x-space gets larger in time, though the shape is unchanged.

1.2 (b)

Quick solution: Similar to part a, in momentum basis, x becomes $i\frac{\partial\tilde{\psi}}{\partial p}$, and the ansatz has the form

$$\tilde{\psi}(p,t) = \tilde{\psi}_0(p(t))$$

. Then we can simply solve the Schrodinger's equation

$$i\hbar\frac{\partial}{\partial t}\tilde{\psi} = i\hbar\omega\frac{\partial}{\partial p}(p\tilde{\psi})$$

$$\frac{\partial}{\partial t}\tilde{\psi} = \omega p\frac{\partial}{\partial p}\tilde{\psi} + \omega\tilde{\psi}$$

Note the last term is related to the fact that we threw away a constant term in the Hamiltonian that is important for preserving normalization. We can ignore this term. Now we substitute our ansatz, we get

$$\frac{d\tilde{\psi}}{dp_0}\frac{dp}{dt} = \omega p\frac{d\tilde{\psi}}{dp_0}$$

For this equation to hold, we need

$$\frac{dp}{dt} = \omega p$$

Solving this give us

$$p(t) = p_0 e^(\omega t)$$

Rigorous solution We are also free to solve the time evolution in the eigenbasis of the operator $\hat{p}$. Let's first reason through the form of the evolution equation in this basis, regardless of the initial state (and regardless of the transformation of a specific state between the $\hat{x}$ and $\hat{p}$ basis). A discussion of this transformation is in Sec. 4.1.

The simple important fact is, in the $\hat{p}$ eigenbasis, the operation of $\hat{x}$ on the wavefuntion $\tilde{\psi}(p)$ must have the opposite sign of the operation of $\hat{p}$ in the $\hat{x}$ eigenbasis, ie. $\hat{x} = i\frac{\partial}{\partial p}$ when acting on p-space wavefunction $\tilde{\psi}(p)$ (the derivation from the commutator in Sec. 4.1 is identical, but has the opposite sign from the antisymmetric property of the commutator). Then the operator $\hat{x}\hat{p}+\hat{p}\hat{x}$ becomes $i\frac{\partial}{\partial p}(p\cdot) + ip\frac{\partial}{\partial p}(\cdot)$ in the $\hat{p}$ basis instead of $-i\frac{\partial}{\partial x}(x\cdot) + ix\frac{\partial}{\partial x}(\cdot)$ in the $\hat{x}$ basis. Quite simply, $\hat{H}$ acting on the $\hat{p}$ eigenbasis looks identical to $\hat{H}$ acting on the $\hat{x}$ basis (as a mapping between wavefunctions) except that it *has the opposite sign*, which is equivalent to flipping the sign of ω. We can therefore take our derived solutions in x-space and simply flip the sign of ω to obtain the solution

$$\tilde{\psi}(p,t) = e^{\frac{\omega}{2}t}\tilde{\psi}_0(pe^{\omega t}).$$

Now, as argued in Sec. 4.1, if we have some initial state described by a wavefunction $\psi_0(x)$ in the $\hat{x}$ basis, this corresponds to an initial wavefunction

$$\tilde{\psi}_0(p) = \frac{1}{\sqrt{2\pi}}\int dx\psi_0(x)e^{-ipx} \tag{5}$$

in the $\hat{p}$ basis, which we can plug into the time dependent solution above. Alternatively to get the time dependence above, we could transform the solution for $\psi(x,t)$ to p-space.

1.3 (c), (d), and (e)

We will now consider specific initial conditions. Set the initial state as $\psi_0(x) = e^{-x^2/2}\pi^{-1/4}$, the simple harmonic oscillator ground state in normalized units. Note that this wavefunction looks identical in the x or p frame, according to the transformation above, ie. $\tilde{\psi}_0(p) = e^{-p^2/2}\pi^{-1/4}$.

From the above time dependent solutions, we have

$$\psi(x,t) = e^{-\frac{\omega}{2}t}\psi_0(xe^{-\omega t}) = \frac{1}{\pi^{1/4}}e^{-\frac{\omega}{2}t}e^{-(xe^{-\omega t})^2/2} \tag{6}$$

$$\tilde{\psi}(p,t) = e^{\frac{\omega}{2}t}\tilde{\psi}_0(pe^{\omega t}) = \frac{1}{\pi^{1/4}}e^{\frac{\omega}{2}t}e^{-(pe^{\omega t})^2/2}. \tag{7}$$

The standard deviation of x (of p) exponentially grows (shrinks) by a factor of $e^{\omega t}$ with time, $\sqrt{\langle\hat{x}^2\rangle} = \sqrt{1/2}e^{\omega t}$ ($\sqrt{\langle\hat{p}^2\rangle} = \sqrt{1/2}e^{-\omega t}$). The product $\sqrt{\langle\hat{x}^2\rangle}\sqrt{\langle\hat{p}^2\rangle} = 1/2$ is clearly unchanged in time.

1.4 (f)

As discussed above, the full Hamiltonian is in fact $H = \hbar\omega(\hat{x}\hat{p}+\hat{p}\hat{x})/2$. If we define $a = \frac{\hat{x}+i\hat{p}}{\sqrt{2}}$ and $a^\dagger = \frac{\hat{x}-i\hat{p}}{\sqrt{2}}$ then we can quickly show that $[a,a^\dagger] = 1$ so that these operators commute as the usual bosonic annihilation and creation operators do. We can also rewrite $\hat{x} = \frac{a+a^\dagger}{\sqrt{2}}$ and $\hat{p} = -i\frac{a-a^\dagger}{\sqrt{2}}$ and

$$H = -i\frac{\hbar\omega}{2}\left(\frac{a+a^\dagger}{\sqrt{2}}\frac{a-a^\dagger}{\sqrt{2}} + \frac{a-a^\dagger}{\sqrt{2}}\frac{a+a^\dagger}{\sqrt{2}}\right) = -i\frac{\hbar\omega}{2}\left(a^2 - (a^\dagger)^2\right). \tag{8}$$

The time evolution operator is therefore $U = e^{-iHt/\hbar}$ or

$$U = e^{\frac{\omega t}{2}\left((a^\dagger)^2 - a^2\right)}. \tag{9}$$

We recognize as the same form as the squeezing operator in Problem 2, with $r = \omega t$.

1.5 Comment on what happens in real space

We solved for the time evolution of the wavefunction of x and p, where x and p are not the physical position and momentum. What does this mean for the original physical position and momentum of the inverted harmonic oscillator wavefunction in the original frame? Sec. 4.2 has some discussion and examples of the time-dependent solutions.

2 Problem 2

We will first demonstrate a few commutation relations, which will let us abstract away from specific structures and implementations of operators to focus on the underlying properties of the algebra of operators.

2.1 (a)

$$\begin{aligned}
[a^2,(a^\dagger)^2] &= a^2(a^\dagger)^2-(a^\dagger)^2a^2 \\
&= a(1+a^\dagger a)a^\dagger - a^\dagger(aa^\dagger-1)a \\
&= aa^\dagger + a^\dagger a + aa^\dagger aa^\dagger - a^\dagger aa^\dagger a \\
&= (1+a^\dagger a) + a^\dagger a + (1+a^\dagger a)(1+a^\dagger a) - a^\dagger aa^\dagger a \\
&= 4a^\dagger a + 2
\end{aligned}$$

$$\begin{aligned}
[a^2,a^\dagger a] &= a^2a^\dagger a - a^\dagger a^3 \\
&= a(1+a^\dagger a)a - a^\dagger a^3 \\
&= aa^\dagger a^2 - a^\dagger a^3 + a^2 \\
&= (1+a^\dagger a)a^2 - a^\dagger a^3 + a^2 \\
&= 2a^2
\end{aligned}$$

$$\begin{aligned}
[(a^\dagger)^2,a^\dagger a] &= (a^\dagger)^3a - a^\dagger a(a^\dagger)^2 \\
&= (a^\dagger)^3a - a^\dagger(1+a^\dagger a)a^\dagger \\
&= (a^\dagger)^3a - (a^\dagger)^2aa^\dagger - (a^\dagger)^2 \\
&= (a^\dagger)^3a - (a^\dagger)^2(1+a^\dagger a) - (a^\dagger)^2 \\
&= -2(a^\dagger)^2
\end{aligned}$$

You can also use the product rule of commutators. Showing only one example here:

$$\begin{aligned}
[(a^\dagger)^2,a^\dagger a] &= a^\dagger[(a^\dagger)^2,a] + [(a^\dagger)^2,a^\dagger]a \\
&= a^\dagger(-2a^\dagger) + 0 \\
&= -2(a^\dagger)^2
\end{aligned}$$

2.2 (b)

The definition of $\sigma_\pm$ might differ from definitions used elsewhere by a factor of 2. It is easy to perform matrix calculations to see we have $[\sigma_+,\sigma_-]=4\sigma_z$, $[\sigma_z,\sigma_\pm]=\pm2\sigma_\pm$ and $[\sigma_j,\sigma_k]=2i\epsilon_{jkl}\sigma_l$.

We recognize this as the same algebra as that of the Pauli matrices, with the identification

$$\begin{aligned}
\sigma_x &\sim \frac{-a^2+(a^\dagger)^2}{2} \\
\sigma_y &\sim i\frac{-a^2-(a^\dagger)^2}{2} \\
\sigma_z &\sim \left(a^\dagger a + \frac{1}{2}\right) \\
\sigma_- = \sigma_x - i\sigma_y &\sim -a^2 \\
\sigma_+ = \sigma_x + i\sigma_y &\sim (a^\dagger)^2
\end{aligned}$$

Proceeding with the problem, we know that we would like to solve

$$e^{\frac{r}{2}\left((a^\dagger)^2-a^2\right)} = e^{\frac{u}{2}(a^\dagger)^2} e^{t(a^\dagger a+\frac{1}{2})} e^{\frac{v}{2}a^2}. \tag{10}$$

If we expand out this expression and reorganize terms according to the commutation relations for the bosonic operators, we by definition obtain an equivalent expression (this is just rewriting something in a different way). This is what we want to do. The key insight is that the objects we want to commute can be given any label we would like (call them apples, oranges, and bananas), but if we expand out the expression in terms of these new labels, rearrange things using the correct commutation relations, and then replace the new labels with the old labels, we still have the same original expression (and therefore equality).

The usefulness of "labelling" each operator with a 2×2 matrix is that we don't have to remember every commutation relation separately for each pair of operators that we encounter: instead we only have to do a single type of operation, namely matrix multiplication. Then at the end we can replace the matrices via the correct inverse transformation.

In summary, the above expression is therefore entirely equivalent, in terms of operator algebras, to

$$e^{r\sigma_x} = e^{\frac{u}{2}\sigma_+} e^{t\sigma_z} e^{-\frac{v}{2}\sigma_-}. \tag{11}$$

We need only find the values $u(r)$, $t(r)$, and $v(r)$ to satisfy this equation, whether in terms of creation and annihilation operators or in terms of the Pauli matrices (which we can more easily explicitly compute), and the solution will hold for either Eqn. 10 or Eqn. 11.

2.3 (c) and (d)

A useful set of identities is $\sigma_i^2 = \mathbb{1}\ \forall i \in \{x, y, z\}$ and that $\sigma_+^2 = \sigma_-^2 = 0$, which can be derived directly from the expressions for the Pauli matrices.

Let's compute

$$\begin{aligned}
e^{r\sigma_x} &= \sum_{n=0}^{\infty} \frac{r^n}{n!}\sigma_x^n \\
&= \sum_{n=0}^{\infty} \frac{r^{2n}}{(2n)!}\sigma_x^{2n} + \sum_{n=0}^{\infty} \frac{r^{2n+1}}{(2n+1)!}\sigma_x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{r^{2n}}{(2n)!}\mathbb{1} + \sum_{n=0}^{\infty} \frac{r^{2n+1}}{(2n+1)!}\sigma_x \\
&= \cosh(r)\mathbb{1} + \sinh(r)\sigma_x
\end{aligned}$$

$$\begin{aligned}
e^{t\sigma_z} &= \cosh(t)\mathbb{1} + \sinh(t)\sigma_z \\
&= e^{t}\frac{\mathbb{1}+\sigma_z}{2} + e^{-t}\frac{\mathbb{1}-\sigma_z}{2} \\
&\equiv T,
\end{aligned}$$

and

$$\begin{aligned}
e^{\frac{u}{2}\sigma_+} &= \sum_{n=0}^{\infty} \frac{u^n}{n!}\sigma_+^n \\
&= \mathbb{1} + \frac{u}{2}\sigma_+ \equiv U, \\
e^{-\frac{v}{2}\sigma_-} &= \mathbb{1} - \frac{v}{2}\sigma_- \equiv V.
\end{aligned}$$

Writing the matrix explicitly out we have:

$$e^{r\sigma_x} = \begin{pmatrix} cosh(r) & sinh(r) \\ sinh(r) & cosh(r) \end{pmatrix}$$

$$T = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

$$V = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}$$

Fast solution via matrix multiplication One may choose to start by explicitly writing out the 2×2 matrices corresponding to the two sides of Eqn. 11, and the matrices U, T, and V, and multiply,

$$\begin{aligned} \begin{pmatrix} \cosh(r) & \sinh(r) \\ \sinh(r) & \cosh(r) \end{pmatrix} &= \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^t & ue^{-t} \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^t - uve^{-t} & ue^{-t} \\ -ve^{-t} & e^{-t} \end{pmatrix}, \end{aligned}$$

and then match terms in the matrices on either side of the equation. One obtains a set of equations to solve. The solutions are

$$\begin{aligned} t &= -\log(\cosh(r)) \\ u &= \tanh(r) \\ v &= -\tanh(r). \end{aligned}$$

Note: the same solution can also be obtained by slowly working out all the commutator relations instead of using matrix multiplication.

2.4 (e)

We now know u and t and v in terms of r. As a result we have proved

$$\boxed{e^{r\sigma_x} = e^{\tanh(r)\sigma_+} e^{-\log(\cosh(r))\sigma_z} e^{\tanh(r)\sigma_-}.} \tag{12}$$

This equation holds true for any set of operators with the same algebra, and thus, comparing with Eqn. 10, we have

$$\boxed{e^{\frac{r}{2}\left((a^\dagger)^2 - a^2\right)} = e^{\tanh(r)(a^\dagger)^2/2} e^{-\log(\cosh(r))\left(a^\dagger a + \frac{1}{2}\right)} e^{-\tanh(r)a^2/2}.} \tag{13}$$

Note carefully the sign of the last exponential term in each equation. Although $u = -v$, the coefficents of the σ_+ and σ_- terms are identical because of the $-v$ in the exponent of Eqn. 11. The sign is reversed again upon the replacement $\sigma_- \sim -a^2/2$.

2.5 (f)

We will now use this expression, Eqn. 13, to show that

$$e^{\frac{r}{2}\left((a^\dagger)^2 - a^2\right)}|0\rangle = \frac{1}{\sqrt{\cosh(r)}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} \tanh(r)^n |2n\rangle. \tag{14}$$

Note that $e^{\hat{X}+a} = e^a e^{\hat{X}}$ for some operator $\hat{X}$ and constant a. Here, this relation is useful to rewrite

$$e^{-\log(\cosh(r))\left(a^\dagger a + \frac{1}{2}\right)} = e^{-\frac{1}{2}\log(\cosh(r))} e^{-\log(\cosh(r))a^\dagger a} = \frac{1}{\sqrt{\cosh(r)}} e^{-\log(\cosh(r))a^\dagger a}. \tag{15}$$

Note that the exponential of $a^\dagger a$ or an exponential of a will do nothing when applied to the ground state. We therefore have, as desired,

$$\begin{aligned}
e^{\frac{r}{2}\left((a^\dagger)^2-a^2\right)}|0\rangle &= e^{\tanh(r)(a^\dagger)^2/2}e^{-\log(\cosh(r))\left(a^\dagger a+\frac{1}{2}\right)}e^{-\tanh(r)a^2/2}|0\rangle \\
&= e^{\tanh(r)(a^\dagger)^2/2}e^{-\log(\cosh(r))\left(a^\dagger a+\frac{1}{2}\right)}|0\rangle \\
&= \frac{1}{\sqrt{\cosh(r)}}e^{\tanh(r)(a^\dagger)^2/2}e^{-\log(\cosh(r))a^\dagger a}|0\rangle \\
&= \frac{1}{\sqrt{\cosh(r)}}e^{\tanh(r)(a^\dagger)^2/2}|0\rangle \\
&= \frac{1}{\sqrt{\cosh(r)}}\sum_{n=0}^{\infty}\tanh(r)^n\frac{1}{2^n n!}(a^\dagger)^{2n}|0\rangle \\
&= \frac{1}{\sqrt{\cosh(r)}}\sum_{n=0}^{\infty}\tanh(r)^n\frac{\sqrt{(2n)!}}{2^n n!}\frac{(a^\dagger)^{2n}}{\sqrt{(2n)!}}|0\rangle \\
&= \frac{1}{\sqrt{\cosh(r)}}\sum_{n=0}^{\infty}\tanh(r)^n\frac{\sqrt{(2n)!}}{2^n n!}|2n\rangle
\end{aligned}$$

2.6 Summary of the relation between Problem 1 and Problem 2

Let's briefly review what we found. We have solved the time evolution of the wavefunction under the Hamiltonians $H = \hbar\omega(\hat{x}\hat{p}+\hat{p}\hat{x})/2$ and $H = -i\hbar\omega\left(a^2-(a^\dagger)^2\right)/2$ for a time $t = \omega/r$ such that the transformation of the state is

$$|\psi\rangle \to U|\psi\rangle = e^{\frac{r}{2}\left((a^\dagger)^2-a^2\right)}|\psi\rangle = e^{-i\frac{r}{2}(\hat{x}\hat{p}+\hat{p}\hat{x})}|\psi\rangle \tag{16}$$

We have operators $\hat{x}$ and $\hat{p}$ and we have the operators a and $a^\dagger$ with the usual transformation between them ($a = \frac{\hat{x}+i\hat{p}}{\sqrt{2}}$ and $a^\dagger = \frac{\hat{x}-i\hat{p}}{\sqrt{2}}$) such that a and $a^\dagger$ obey bosonic commutation relations. The simple idea is if you are told some state and some Hamiltonian in one basis, you are free to solve the system in either frame, after transforming the initial state and Hamiltonian appropriately. If you solve for the evolution in either the $\hat{x}$ and $\hat{p}$ basis (Problem 1) or the a and $a^\dagger$ basis (Problem 2), you can transform back to the other basis at the end to get a solution there with the equivalent initial conditions.

If the initial state is the harmonic oscillator ground state in Problem 1, this corresponds to the vacuum of the Fock basis in Problem 2. We can thus time evolve the state in either frame, and then transform everything back to the basis of $\hat{x}$ (replace Fock states with harmonic oscillator states with the correct normalization) to obtain

$$\frac{1}{\sqrt{e^r}}e^{-\frac{x^2}{2e^{2r}}} = \frac{1}{\sqrt{\cosh(r)}}e^{-\frac{x^2}{2}}\sum_{n=0}^{\infty}\frac{1}{2^{2n}n!}\tanh(r)^n H_{2n}(x), \tag{17}$$

where $H_n(x)$ are the Hermite polynomials, as stated in the problem.

3 Problem 3 Generation of Squeezed States by Two-Photon Interactions

Consider a mode $(\vec{k}, \vec{\varepsilon})$ with wavevector $\vec{k}$ and polarization vector $\vec{\varepsilon}$ of the electromagnetic field with frequency ω whose Hamiltonian H is given by

$$H = \hbar\omega a^\dagger a + i\hbar\Lambda\left[(a^\dagger)^2e^{-2i\omega t} - a^2e^{2i\omega t}\right] \tag{18}$$

where $a^\dagger$ and a are the creation and annihilation operators of the mode and Λ is real.

3.1 (a)

Write the equation of motion for $a(t)$ using the Heisenberg picture and find equations of motion for $b(t)$ and $b^\dagger(t)$.

In the Heisenberg picture, the equation of motion for $a(t)$ is

$$\dot{a} = \frac{1}{i\hbar}[a, H] \tag{19}$$

$$\dot{a} = -i\omega\left[a, a^\dagger a\right] + \Lambda e^{-2i\omega t}\left[a, \left(a^\dagger\right)^2\right] + \Lambda e^{2i\omega t}\left[a, a^2\right] \tag{20}$$

Using the fact that $\left[a, a^\dagger\right] = 1$, we compute

$$\left[a, a^\dagger a\right] = aa^\dagger a - a^\dagger aa = \left[a, a^\dagger\right]a = a \tag{21}$$

$$\left[a, \left(a^\dagger\right)^2\right] = aa^\dagger a^\dagger - a^\dagger a^\dagger a = aa^\dagger a^\dagger - a^\dagger aa^\dagger + a^\dagger aa^\dagger - a^\dagger a^\dagger a = umber$$

$$\left[a, a^\dagger\right]a^\dagger + a^\dagger\left[a, a^\dagger\right] = 2a^\dagger \tag{22}$$

$$\left[a, a^2\right] = aaa - aaa = 0 \tag{23}$$

and that

$$\boxed{\dot{a} = -i\omega a + 2\Lambda a^\dagger e^{-2i\omega t}} \tag{24}$$

Since $a(t) = b(t)e^{-i\omega t}$, $\dot{a} = \dot{b}e^{-i\omega t} - i\omega b e^{-i\omega t}$. Substituting these relations into Eq. 24 gives equations of motion for b (and $b^\dagger$, by taking the adjoint equation):

$$\boxed{\dot{b} = 2\Lambda b^\dagger \text{ and } \dot{b}^\dagger = 2\Lambda b} \tag{25}$$

Alternatively, you can substitute b into the Hamiltonian (i.e. going into the interaction picture) and get

$$H = \hbar\omega b^\dagger b + i\hbar\Lambda((b^\dagger)^2 - b^2)$$

But note now $b = ae^{i\omega t}$ has explicit time dependence. The time evolution can be calculated as:

$$\frac{d}{dt}b = -\frac{i}{\hbar}[b, H] + \frac{\partial}{\partial t}b$$

$$\frac{d}{dt}b = -i\omega b + 2\Lambda b^\dagger + i\omega b$$

$$\dot{b} = 2\Lambda b^\dagger$$

3.2 (b)

Show that b_P and b_Q are two quadrature components of the contribution of the mode $(\vec{k}, \vec{\epsilon})$. Find equations of motion for $b_P(t)$ and $b_Q(t)$, solve them in terms of $b_P(0)$ and $b_Q(0)$. Give solutions for $b(t)$ and $b^\dagger(t)$ in terms of $b(0)$ and $b^\dagger(0)$.

Start by re-expressing the contribution of the mode $(\vec{k}, \vec{\varepsilon})$ to the electric field in terms of $b(t)$ and $b^\dagger(t)$:

$$\frac{1}{i\mathcal{E}_\omega}\vec{E}(\vec{r}, t)\cdot\vec{\varepsilon} = a(t)e^{i\vec{k}\cdot\vec{r}} - a^\dagger(t)e^{-i\vec{k}\cdot\vec{r}} \tag{26}$$

$$= b(t)e^{i\vec{k}\cdot\vec{r} - i\omega t} - b^\dagger(t)e^{-i\vec{k}\cdot\vec{r} + i\omega t}umber$$

Using the definitions of $b_P(t)$ and $b_Q(t)$, we find

$$b(t) = b_P(t) + ib_Q(t) \text{ and } b^\dagger(t) = b_P(t) - ib_Q(t) \tag{27}$$

Using this and taking $x = \vec{k}\cdot\vec{r} - \omega t$,

$$\begin{aligned}
&= [b_P(t) + ib_Q(t)]\, e^{ix} - [b_P(t) - ib_Q(t)]\, e^{-ix} \\
&= b_P(t)\left[e^{ix} - e^{-ix}\right] + ib_Q(t)\left[e^{ix} + e^{-ix}\right] \\
&= 2ib_P(t)\sin x + 2ib_Q(t)\cos x \\
&= 2i\,[b_P(t)\sin x + b_Q(t)\cos x]
\end{aligned} \tag{28}$$

This shows that $b_P(t)$ and $b_Q(t)$ are two quadrature components of the field produced by the two-photon interaction.

Using Eqs. 25 and the definitions of $b_P(t)$ and $b_Q(t)$, we find equations of motion for $b_P(t)$ and $b_Q(t)$:

$$\dot{b_P}(t) = \frac{\dot{b} + \dot{b}^\dagger}{2} = \frac{2\Lambda b^\dagger + 2\Lambda b}{2} = \boxed{2\Lambda b_P(t)} \tag{29}$$

$$\dot{b_Q}(t) = \frac{\dot{b} - \dot{b}^\dagger}{2i} = \frac{2\Lambda b^\dagger - 2\Lambda b}{2i} = \boxed{-2\Lambda b_Q(t)} \tag{30}$$

whose solutions are

$$\boxed{b_P(t) = b_P(0)e^{2\Lambda t} \text{ and } b_Q(t) = b_Q(0)e^{-2\Lambda t}} \tag{31}$$

To find $b(t)$, we plug in the solutions for $b_P(t)$ and $b_Q(t)$ from Eqs. 31 into Eqs. 27.

$$\begin{aligned}
b(t) &= b_P(t) + ib_Q(t) = b_P(0)e^{2\Lambda t} + ib_Q(0)e^{-2\Lambda t} \\
&= \frac{b(0) + b^\dagger(0)}{2}e^{2\Lambda t} + \frac{b(0) - b^\dagger(0)}{2}e^{-2\Lambda t} \\
&= b(0)\frac{e^{2\Lambda t} + e^{-2\Lambda t}}{2} + b^\dagger(0)\frac{e^{2\Lambda t} - e^{-2\Lambda t}}{2} \\
&= \boxed{b(0)\cosh(2\Lambda t) + b^\dagger(0)\sinh(2\Lambda t)}
\end{aligned} \tag{32}$$

Similarly, for $b^\dagger(t)$:

$$\begin{aligned}
b^\dagger(t) &= b_P(t) - ib_Q(t) = b_P(0)e^{2\Lambda t} - ib_Q(0)e^{-2\Lambda t} \\
&= \frac{b(0) + b^\dagger(0)}{2}e^{2\Lambda t} - \frac{b(0) - b^\dagger(0)}{2}e^{-2\Lambda t} \\
&= b(0)\frac{e^{2\Lambda t} - e^{-2\Lambda t}}{2} + b^\dagger(0)\frac{e^{2\Lambda t} + e^{-2\Lambda t}}{2} \\
&= \boxed{b(0)\sinh(2\Lambda t) + b^\dagger(0)\cosh(2\Lambda t)}
\end{aligned} \tag{33}$$

Since $b(0) = a(0)$, we also have equations of motion for $a(t)$ and $a\dagger(t)$:

$$a(t) = a(0)\cosh(2\Lambda t) + a^\dagger(0)\sinh(2\Lambda t) \tag{34}$$

$$a^\dagger(t) = a(0)\sinh(2\Lambda t) + a^\dagger(0)\cosh(2\Lambda t) \tag{35}$$

3.3 (c)

Assuming that at $t = 0$ the field is in the vacuum state, calculate $\langle N\rangle$, $\langle\Delta b_P(t)\rangle$ and $\langle\Delta b_Q(t)\rangle$ and explain your results.

The field is initialized in the vacuum state so that $|\psi\rangle_{t=0} = |0\rangle$. In the Heisenberg picture, states do not evolve, so $|\psi\rangle_t = |\psi\rangle_{t=0} = |0\rangle$.

The mean number of photons $\langle N(t)\rangle = \langle a^\dagger a\rangle = \langle b^\dagger b\rangle$. Using Eqs. 32 and 33, this is:

$$\begin{aligned}
\langle N(t)\rangle = \langle &\big(b(0)\cosh(2\Lambda t) + b^\dagger(0)\sinh(2\Lambda t)\big) \times \\
&\big(b(0)\sinh(2\Lambda t) + b^\dagger(0)\cosh(2\Lambda t)\big)\rangle
\end{aligned} \tag{36}$$

Noting that $\langle b(0)b(0)\rangle = \langle b^\dagger(0)b(0)\rangle = \langle b^\dagger(0)b^\dagger(0)\rangle = 0$, this simplifies to

$$\begin{aligned}
\langle N(t)\rangle &= \langle b(0)b^\dagger(0)\rangle \sinh^2(2\Lambda t) = \langle \left(b^\dagger(0)b(0)+1\right)\rangle \sinh^2(2\Lambda t) && (37)\\
&= \boxed{\sinh^2(2\Lambda t)} && (38)
\end{aligned}$$

The quadrature variances $\langle \Delta b_P^2(t)\rangle$ and $\langle \Delta b_Q(t)\rangle^2$ are:

$$\begin{aligned}
\langle \Delta b_P^2(t)\rangle &= \langle b_P^2(t)\rangle - \langle b_P(t)\rangle^2 = e^{4\Lambda t}\left(\langle b_P^2(0)\rangle - \langle b_P(0)\rangle^2\right)\\
&= \frac{e^{4\Lambda t}}{4}\left(\langle \left(b(0)+b^\dagger(0)\right)^2\rangle - \langle b(0)+b^\dagger(0)\rangle^2\right) = \frac{e^{4\Lambda t}}{4} && (39)\\
\langle \Delta b_Q^2(t)\rangle &= \langle b_Q^2(t)\rangle - \langle b_Q(t)\rangle^2 = e^{-4\Lambda t}\left(\langle b_Q^2(0)\rangle - \langle b_Q(0)\rangle^2\right)\\
&= \frac{-e^{-4\Lambda t}}{4}\left(\langle \left(b(0)-b^\dagger(0)\right)^2\rangle - \langle b(0)-b^\dagger(0)\rangle^2\right) = \frac{e^{-4\Lambda t}}{4} && (40)
\end{aligned}$$

where I have used Eqs 31 and the fact that $\langle b(0)b(0)\rangle = \langle b^\dagger(0)b(0)\rangle = \langle b^\dagger(0)b^\dagger(0)\rangle = 0$ and $\langle b(0)b^\dagger(0)\rangle = 1$. The dispersion relationships then are

$$\boxed{\Delta b_P(t) = \frac{e^{2\Lambda t}}{2} \text{ and } \Delta b_Q(t) = \frac{e^{-2\Lambda t}}{2}} \qquad (41)$$

$\boxed{\text{Interpretation:}}$ The parametric interaction keeps the system in a minimum uncertainty state because While the uncertainty product $\Delta b_P \Delta b_Q = \frac{1}{4}$ does not change in time. However, it does reduce the variance in one quadrature - here b_Q - at the expense of increased variance in the other quadrature, squeezing the state.

Making this squeezing takes energy, which we observe as an increase in the mean photon number $\langle N\rangle = \sinh^2(2\Lambda t)$, which becomes exponential in the limit of large t.

3.4 (d)

Show that the squeezing Hamiltonian (Eq. 18) makes the state $\exp\left[\frac{1}{2}\epsilon^* a^2 - \frac{1}{2}\epsilon a^{\dagger 2}\right]|0\rangle$ *when applied to the vacuum for a time* $t = t_0$*. Find* ϵ*.*

This problem is easiest to solve in the interaction picture, where we take $H = H_0 + V(t)$ and convert Schrödinger operators and states via the unitary transformation $U = exp\left[-iH_0 t/\hbar\right]$ so that $A_I = U^{-1}A_S U$ and $U^{-1}|\psi\rangle_S$. States are solutions to $i\hbar\frac{\partial}{\partial t}|\psi\rangle_I = V_I|\psi\rangle_I$.

Choose $H_0 = \hbar\omega a^\dagger a$ and $V(t) = i\hbar\Lambda\left[(a^\dagger)^2 e^{-2i\omega t} - a^2 e^{2i\omega t}\right]$. V_I is given by

$$\begin{aligned}
V_I &= U^{-1}V_S U && (42)\\
&= i\hbar\Lambda\left[\left(U^{-1}a^\dagger U\right)^2 e^{-2i\omega t} - \left(U^{-1}aU\right)^2 e^{2i\omega t}\right] && (43)
\end{aligned}$$

The interaction picture creation and annihilation operators are

$$\begin{aligned}
U^{-1}a^\dagger U &= e^{iH_0 t/\hbar}a^\dagger e^{-iH_0 t/\hbar} = e^{i\omega t a^\dagger a}a^\dagger e^{-i\omega t a^\dagger a} && (44)\\
&= a^\dagger + i\omega t[a^\dagger a, a^\dagger] + \frac{(i\omega t)^2}{2!}[a^\dagger a,[a^\dagger a, a^\dagger]] + ... && (45)\\
&= a^\dagger \sum_n \frac{(i\omega t)^n}{n!} = a^\dagger e^{i\omega t} && (46)\\
U^{-1}aU &= ae^{-i\omega t} && (47)
\end{aligned}$$

so $V_I = i\hbar\Lambda\left[\left(a_I^\dagger\right)^2 - a_I^2\right]$. Since $|\psi(t=0)\rangle = |0\rangle$, the time evolved state is

$$|\psi(t)\rangle_I = e^{-iV_I t/\hbar}|\psi(t=0)\rangle_I = e^{\Lambda t\left[\left(a_I^\dagger\right)^2 - a_I^2\right]}|0\rangle \qquad (48)$$

so that $\boxed{\epsilon = -2\Lambda t_0}$ if we apply the Hamiltonian for a time $t = t_0$.

3.5 (e)

Plot the Q function of (i) squeezed vacuum, (ii) a displaced squeezed state and (iii) a squeezed coherent state for $\epsilon = 0.2, 1.2$ and 4. Compare the plots, especially: are the displaced squeezed state and the squeezed coherent state the same? If not how do they differ? Does this Hamiltonian produce amplitude or phase squeezing?

First, let's find closed expressions for the plots.

(i) Starting from the expression of the squeezed vacuum state in the hint (Note here α is taken to be real):

$$\begin{aligned}
S(\epsilon)|0\rangle &= \frac{1}{\pi}\frac{e^{\epsilon/2}}{\sqrt{e^{2\epsilon}-1}}\int_{-\infty}^{\infty} d\alpha\, e^{-[\alpha^2/(e^{2\epsilon}-1)]}|\alpha\rangle \\
|\langle\beta|S(\epsilon)|0\rangle|^2 &= \frac{1}{\pi^2}\frac{e^{\epsilon}}{e^{2\epsilon}-1}\left|\int_{-\infty}^{\infty} d\alpha\, e^{[-\alpha^2/(e^{2\epsilon}-1)]}\langle\beta|\alpha\rangle\right|^2 \\
&= Ce^{-|\beta|^2}\left|\int_{-\infty}^{\infty} d\alpha\, \exp\left[-\frac{\alpha^2}{(e^{2\epsilon}-1)} - \frac{\alpha^2}{2} + \alpha\beta^*\right]\right|^2
\end{aligned}$$

where $C = \frac{1}{\pi^2}\frac{e^{\epsilon}}{e^{2\epsilon}-1}$. Here we used the overlap of two coherent state

$$\langle\beta|\alpha\rangle = e^{-1/2(|\beta|^2+|\alpha|^2-\beta^{*2}\alpha)}$$

Evaluating the integral and taking $\beta = x + iy$ gives

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} d\alpha\, \exp\left[\frac{-\alpha^2}{2\tanh\epsilon} + \alpha\beta^*\right]\right|^2 &= \left|\exp\left[\frac{\beta^{*2}}{2}\tanh\epsilon\right]\sqrt{2\pi\tanh\epsilon}\right|^2 \\
&= \exp\left[\left(\beta^2+\beta^{*2}\right)\frac{\tanh\epsilon}{2}\right]2\pi\tanh\epsilon \\
\implies Q_1(\beta) &= \frac{1}{\pi\cosh\epsilon}\exp\left[\Re(\beta^2)\tanh\epsilon - |\beta|^2\right] \\
\implies Q_1(x,y) &= \frac{1}{\pi\cosh\epsilon}\exp\left[-x^2\left(1-\tanh\epsilon\right) - y^2\left(1+\tanh\epsilon\right)\right]
\end{aligned}$$

(ii) To compute $Q_2(\alpha)$, first note that $\langle\beta|D(\gamma)$ may be written as

$$\langle 0|D(-\beta)D(\gamma) = \langle\beta-\gamma|\exp\left[-i\Im[-\gamma\beta^*]\right]$$

using the property $D(\beta+\gamma) = D(\beta)D(\gamma)\exp\left[-i\Im[\gamma\beta^*]\right]$. Therefore, from (i), we can write down $Q_2(\alpha)$ by replacing β by $\beta-\gamma$ in the solution for $Q_1(\alpha)$. Note that the additional phase factor $\exp\left[-i\Im[-\gamma\beta^*]\right]$ disappears after we take the absolute value.

$$\begin{aligned}
|\langle\beta|D(\gamma)S(\epsilon)|0\rangle|^2 &= \frac{1}{\pi\cosh\epsilon}\exp\left[\Re((\beta-\gamma)^2)\tanh\epsilon - |\beta-\gamma|^2\right] \qquad (49)\\
\implies Q_2(x,y) &= \frac{1}{\pi\cosh\epsilon}\exp\left[-\left(x-\Re[\gamma]\right)^2\left(1-\tanh\epsilon\right) - \left(y-\Im[\gamma]\right)^2\left(1+\tanh\epsilon\right)\right]
\end{aligned}$$

(iii) Using the operator identity $S(\epsilon)D(\gamma) = D(\gamma_-)S(\epsilon)$ with $\gamma_-(\epsilon) = \cosh\epsilon\gamma - \sinh\epsilon\gamma^*$, we find

$$\begin{aligned}
|\langle\beta|S(\epsilon)D(\gamma)|0\rangle|^2 &= |\langle\beta|D(\gamma_-)S(\epsilon)|0\rangle|^2 \qquad (50)\\
&= \frac{1}{\pi\cosh\epsilon}\exp\left[\Re((\beta-\gamma_-)^2)\tanh\epsilon - |\beta-\gamma_-|^2\right] \qquad (51)\\
\implies Q_3(x,y) &= \frac{1}{\pi\cosh\epsilon}\exp\left[-\left(x-\Re[\gamma_-]\right)^2\left(1-\tanh\epsilon\right) - \left(y-\Im[\gamma_-]\right)^2\left(1+\tanh\epsilon\right)\right]
\end{aligned}$$

Since $D(\gamma)$ and $S(z)$ do not commute, $Q_2(\alpha)$ and $Q_3(\alpha)$ not generally the same. If the displacement operator $D(\gamma)$is first applied and then the state is squeezed, there is an additional translation to γ_-.

In general, this squeezing operator produces neither phase nor amplitude squeezed light; it always squeezes the same quadrature independent of the amplitude of the coherent state that it is acting on.

You can also calculate these functions numerically. The distribution shown below are calculated with the QuTiP package, but you are free to use other language / code you like. Since squeezing populates states with much higher photon numbers, it is important to have a large enough computation basis to avoid truncation artifacts.

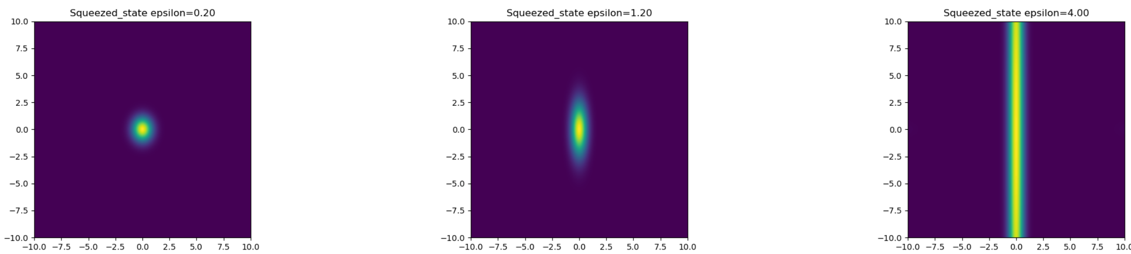

Figure 1: Squeezed Vaccum

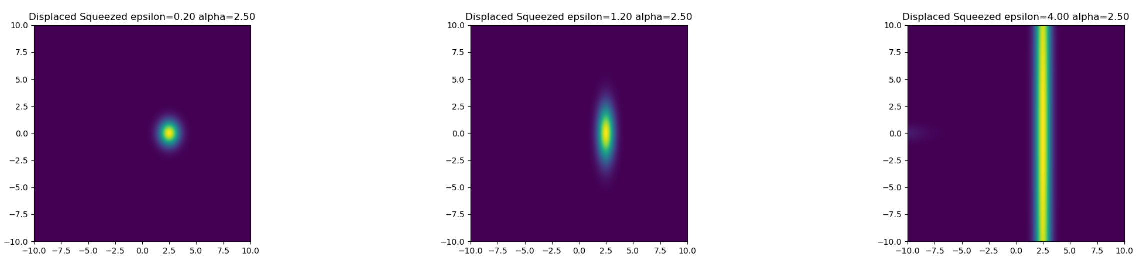

Figure 2: Displaced Squeezed State

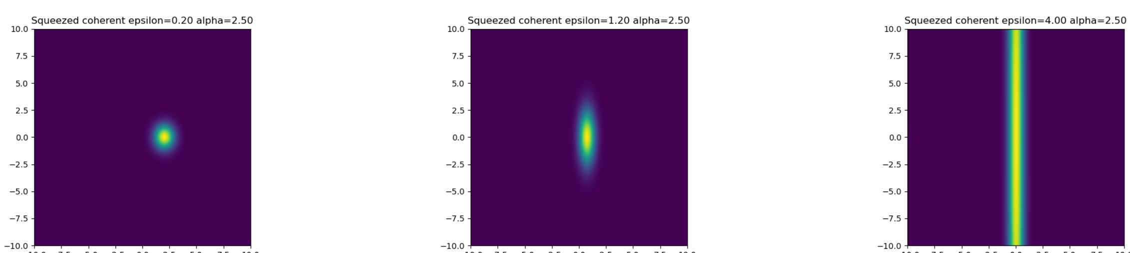

Figure 3: Squeezed Displaced (Coherent) State

Qualitatively, it is important to understand that the displacement and squeezing operator do not commute. For the squeezed coherent state, the displacement is also squeezed.

4 Appendix

A useful resource is https://doi.org/10.1016/0003-4916(86)90142-9 (G. Barton, Quantum mechanics of the inverted oscillator potential, Annals of Physics, Volume 166, Issue 2, 1986) for additional details. In particular one can check the Heisenberg equations of motion for the inverted oscillator in equations 2.4-2.8 and the analytic solution for the time evolution starting from an initial harmonic oscillator ground state in equations 4.6-4.10. Supplementary info more likely to have algebra mistakes.

4.1 Addendum to Problem 1(b): Aside on the form of the operator p in the x basis, and the basis transformation

We can write any initial state as an expansion over the eigenstates of the operator $\hat{x}$ (this is a complete basis). The $\hat{x}$ notation is used to emphasize the difference between a coordinate x (argument of a wavefunction) and an operator $\hat{x} = \int dz z|z\rangle\langle z|$ (acting on states in the Hilbert space). For any state $|\psi\rangle$, there will be some wavefunction $\psi(x)$ such that $|\psi\rangle = \int dx \psi(x)|x\rangle$. The state can alternatively be expressed as $|\psi\rangle = \psi(\hat{x}) \int dz|z\rangle = \int dz \psi(\hat{x})|z\rangle$.

4.1.1 Operator form

The form of the operator $\hat{p}$ acting on the wavefunction in the $\hat{x}$ eigenbasis is implied by the commutation relation $[\hat{x}, \hat{p}] = i$. We will need two relations. First, note that $[\hat{x}^n, \hat{p}] = n[\hat{x}, \hat{p}]\hat{x}^{n-1} = in\hat{x}^{n-1}$. Therefore given any function $f(\hat{x})$, to find $[f(\hat{x}), \hat{p}]$ we can Taylor expand $f(\cdot)$, commute each term $\hat{x}^n$ with $\hat{p}$, and re-sum the series to obtain $[f(\hat{x}), \hat{p}] = if'(\hat{x})$ (meaning plug in the operator $\hat{x}$ to the function which is the derivative of $f(\cdot)$). Second, let's examine $\langle y|[\hat{x}, \hat{p}]|z\rangle$. The commutation relation implies $(y-z)\langle y|\hat{p}|z\rangle = i\langle y|z\rangle = i\delta(y-z)$. Aside from the difficulty of working with distributions, this tells us $\langle y|\hat{p}|z\rangle$ is an odd function of $y - z$.

Proceeding with finding the action of $\hat{p}$ in the $\hat{x}$ eigenbasis, we only need to know what the action of $\hat{p}$ is on the coordinates in the basis, ie. the wavefunction (any operator can be defined either by its operation on states, or on coordinates). By definition, the new state in the $\hat{x}$ eigenbasis after applying $\hat{p}$ is $|\phi\rangle = \int dy|y\rangle\langle y|\hat{p}|\psi\rangle = \int dy \phi(y)|y\rangle$, where we have defined the resulting wavefunction $\phi(y) = \langle y|\hat{p}|\psi\rangle$. Alternatively, using the commutator and the operator definition of the state ($|\psi\rangle = \int dz \psi(\hat{x})|z\rangle$), this new wavefunction can be expressed as

$$\begin{aligned}\phi(y) = \langle y|\hat{p}|\psi\rangle = \int dz\langle y|\hat{p}\psi(\hat{x})|z\rangle &= \int dz\langle y| - [\psi(\hat{x}), \hat{p}] + \psi(\hat{x})\hat{p}|z\rangle = \int dz\langle y| - i\psi'(\hat{x}) + \psi(\hat{x})\hat{p}|z\rangle \\ = \int dz\langle y| - i\psi'(y) + \psi(y)\hat{p}|z\rangle &= -i\psi'(y)\int dz\langle y|z\rangle + \psi(y)\int dz\langle y|\hat{p}|z\rangle = -i\psi'(y) + \psi(y)\int dz\langle y|\hat{p}|z\rangle,\end{aligned} \tag{52}$$

but we know $\langle y|\hat{p}|z\rangle$ is an odd function of $y - z$, so that $\int dz\langle y|\hat{p}|z\rangle = 0$. The final term vanishes, leaving $\phi(y) = -i\psi'(y)$ and therefore

$$\hat{p}|\psi(x)\rangle = | - i\psi'(x)\rangle, \tag{53}$$

meaning that we can use the formula $\hat{p} = -i\frac{\partial}{\partial x}$ when acting on the wavefunction of a state in the $\hat{x}$ basis.

4.1.2 Basis transformation

As a final note, we can ask what the transformation is between the $\hat{x}$ and $\hat{p}$ basis. By definition, the basis vectors in either basis are the eigenstates of their respective operator, $\hat{x}|x'\rangle = x'|x'\rangle$ and $\hat{p}|p'\rangle = p'|p'\rangle$. The wavefunctions are the coefficients of a specific state for each basis vector, and the wavefunctions in the two bases must be related in order to describe the same state. With an overabundance of notation,

$$|\psi(\cdot)\rangle_{(x)} = \int dy \psi(y)|y\rangle_x = \int dq \tilde{\psi}(q)|q\rangle_p = |\tilde{\psi}(\cdot)\rangle_{(p)}. \tag{54}$$

The notation $|\psi(\cdot)\rangle_{(x)}$ is really a shorthand for the full expansion $\int dy \psi(y)|y\rangle_x$. The objects $|y\rangle_x$ and $|q\rangle_p$ refer to the basis states of the Hilbert space, where $\langle z|_x|y\rangle_x = \delta(y - z)$.

The simplest way to see the transformation is to try to find a state in one basis (the $\hat{x}$ basis) which is the eigenstate of the other operator (the $\hat{p}$ operator). In other words, if we find $|q\rangle_p$ in terms of the basis $|y\rangle_x$, then we know how the two bases are related. This transformation will be unique ($\hat{p}$ is a hermitian operator, so eigenstates with different eigenvalues are distinct, and we know the basis is complete as well). Let's (of course) look a state $|\phi_p(\cdot)\rangle_{(x)} = |e^{ir(\cdot)}/\sqrt{2\pi}\rangle_{(x)}$ with wavefunction $\langle z|_x|e^{ir(\cdot)}/\sqrt{2\pi}\rangle_{(x)} = e^{irz}/\sqrt{2\pi}$. We explicitly know what $\hat{p}$ does to the wavefunction in the $\hat{x}$ basis from our knowledge that $\hat{p} = -i\frac{\partial}{\partial x}$ when acting on the x wavefunction: $\langle z|_x\hat{p}|e^{ir(\cdot)}/\sqrt{2\pi}\rangle_{(x)} = re^{irz}/\sqrt{2\pi}$. We see that $|e^{ir(\cdot)}/\sqrt{2\pi}\rangle_{(x)}$ is therefore an eigenstate of the $\hat{p}$ operator with eigenvalue r. It is also normalized, as one can quickly check. Therefore we have found the appropriate basis transformation:

$$|q\rangle_p = \left|\frac{e^{iq(\cdot)}}{\sqrt{2\pi}}\right\rangle_{(x)} = \int dy \frac{e^{iqy}}{\sqrt{2\pi}}|y\rangle_x \tag{55}$$

What does this tell us for the transformation between arbitrary wavefunctions?

$$|\psi(\cdot)\rangle_{(x)} = \int dy\psi(y)|y\rangle_x = \int dy\psi(y)\left(\int dq|q\rangle_p\langle q|_p\right)|y\rangle_x = \int dq\left(\int dy\psi(y)\langle q|_p|y\rangle_x\right)|q\rangle_p \tag{56}$$

$$\boxed{\tilde{\psi}(q) = \int dy\psi(y)\langle q|_p|y\rangle_x = \frac{1}{\sqrt{2\pi}}\int dy\psi(y)e^{-iqy}} \tag{57}$$

We recognize this as a Fourier transform (with a specific normalization and choice of exponent scaling).

In the end, we can relax the burden of the notation, and know that whatever basis we choose to work in, we are referring to the same state in the Hilbert space as long as we transform our coordinates correctly,

$$|\psi(\cdot)\rangle_{(x)} = \int dy\psi(y)|y\rangle_x = \int dq\tilde{\psi}(q)|q\rangle_p = |\tilde{\psi}(\cdot)\rangle_{(p)}. \tag{58}$$

The purpose of this section was to describe how the commutation relation sets the form of the operators in either basis, and therefore also the transformation between the two bases, regardless of what the physical operators mean.

4.2 Addendum to Problem 1: Writing the solution in real space

We found solutions for the wavefunction of x in the inverted harmonic oscillator. How can we transform back to the wavefunction of the real position $\tilde{x}$? In the notation from above, we need to find normalized states that are eigenstates of $\hat{\tilde{x}} = (\hat{x} - \hat{p})/\sqrt{2} \sim (x + i\frac{\partial}{\partial x})/\sqrt{2}$ when acting on the x wavefunction.

Let's guess the state

$$|\tilde{y}\rangle_{\tilde{x}} = \frac{2^{1/4}}{\sqrt{2\pi}}\int dye^{-i\sqrt{2}y\tilde{y}}e^{i(y^2+\tilde{y}^2)/2}|y\rangle_x \tag{59}$$

which immediately conversely implies that

$$|y\rangle_x = \frac{2^{1/4}}{\sqrt{2\pi}}\int dye^{i\sqrt{2}y\tilde{y}}e^{-i(y^2+\tilde{y}^2)/2}|\tilde{y}\rangle_{\tilde{x}}. \tag{60}$$

We will check that $|\tilde{y}\rangle_{\tilde{x}}$ an eigenstate of $\hat{\tilde{x}}$, as well as that this state is normalized:

$$\begin{aligned}\hat{\tilde{x}}|\tilde{y}\rangle_{\tilde{x}} &= \frac{\hat{x}-\hat{p}}{\sqrt{2}}\frac{2^{1/4}}{\sqrt{2\pi}}\int dye^{-i\sqrt{2}y\tilde{y}}e^{i(y^2+\tilde{y}^2)/2}|y\rangle_x = \frac{\hat{x}+i\frac{\partial}{\partial\hat{x}}}{\sqrt{2}}\frac{2^{1/4}}{\sqrt{2\pi}}\int dye^{-i\sqrt{2}y\tilde{y}}e^{i(y^2+\tilde{y}^2)/2}|y\rangle_x \\ &= \frac{2^{1/4}}{\sqrt{2\pi}}\int dy\frac{1}{\sqrt{2}}\left(y + ii(y-\sqrt{2}\tilde{y})\right)e^{-i\sqrt{2}y\tilde{y}}e^{i(y^2+\tilde{y}^2)/2}|y\rangle_x = \tilde{y}\frac{2^{1/4}}{\sqrt{2\pi}}\int dye^{-i\sqrt{2}y\tilde{y}}e^{i(y^2+\tilde{y}^2)/2}|y\rangle_x = \tilde{y}|\tilde{y}\rangle_{\tilde{x}}\end{aligned}$$

$$\begin{aligned}
\langle \tilde{z}|_{\tilde{x}}|\tilde{y}\rangle_{\tilde{x}} &= \frac{\sqrt{2}}{2\pi}\int dw_1 dw_2 e^{i\sqrt{2}w_1\tilde{z}}e^{-iw_1^2/2}e^{-i\sqrt{2}w_2\tilde{y}}e^{iw_2^2/2}\langle w_1|_x|w_2\rangle_x e^{i(\tilde{y}^2-\tilde{z}^2)/2} \\
&= e^{i(\tilde{y}^2-\tilde{z}^2)/2}\frac{1}{2\pi}\int d(w\sqrt{2})e^{iw\sqrt{2}(\tilde{z}-\tilde{y})} \\
&= e^{i(\tilde{y}^2-\tilde{z}^2)/2}\int \frac{d\beta}{2\pi}e^{i\beta(\tilde{z}-\tilde{y})} \\
&= \delta(\tilde{z}-\tilde{y}).
\end{aligned}$$

Therefore our states $|\tilde{y}\rangle_{\tilde{x}}$ are orthonormal. Conversely, we can check easily that $\hat{x}|y\rangle_x = y|y\rangle_x$ when using the form of the operator and wavefunction in the $\hat{\tilde{x}}$ basis (the operator $\hat{x}$ in the $\hat{\tilde{x}}$ basis is $\hat{x} = (\hat{\tilde{x}}+\hat{\tilde{p}})/\sqrt{2} \sim (\tilde{x} - i\frac{\partial}{\partial \tilde{x}})/\sqrt{2}$.[1]

If we have a wavefunction $\psi(y)$ in the x basis, then we can transform back to a wavefunction in $\tilde{x}$-space by inserting the identity $\int dy \psi(y)|y\rangle_x\langle\tilde{y}|$ as

$$\boxed{\psi_{\tilde{x}}(\tilde{y}) = \int dy\psi(y)\langle\tilde{y}|_{\tilde{x}}|y\rangle_x = \int dy\psi(y)\frac{2^{1/4}}{\sqrt{2\pi}}e^{i\sqrt{2}y\tilde{y}}e^{-i(y^2+\tilde{y}^2)/2}.} \tag{61}$$

However, before continuing, there is one more ambiguity which requires careful consideration. Do the states defined above provide the correct basis transformation corresponding to physical reality? This final question is tied up in the ambiguity we have in defining any basis: we could introduce a phase to each of our eigenstates, instead choosing a new basis $e^{if(\tilde{y})}|\tilde{y}\rangle_{\tilde{x}}$ for some real function $f(\tilde{y})$, and by the same reasoning above, our new states would still be eigenstates of $\hat{\tilde{x}}$ and would moreover still be orthonormalized (if you follow through the same derivation). They then enable an invertible basis change. So why is the choice we made for $f(\tilde{y})$ the *correct* choice (up to a possible global phase) for the *physical* basis change back to real position coordinates?

1. One simple argument is that we expect the basis transformation to be in some way "symmetric" in its treatment of y and $\tilde{y}$ (note that the above formula for the basis transformation is indeed symmetric).

   Why? We know the explicit form of each operator $\hat{\tilde{x}}$ and $\hat{x}$ in the other operator's basis, and after a basis transformation of an eigenstate in one basis to the other, the new operators and new states in the new basis must still have the correct eigenvalue. This is what we checked above, and it is easy to check that choosing a new phase factor $f(\tilde{y})$ would cause the states $|y\rangle_x$ to have the wrong eigenvalue under the operator $\hat{x}$ when everything is expressed in the $\hat{\tilde{x}}$ basis. The symmetry of the operator expressions ($\hat{\tilde{x}} \sim (x + i\frac{\partial}{\partial x})/\sqrt{2}$ and $\hat{x} \sim (\tilde{x} - i\frac{\partial}{\partial \tilde{x}})/\sqrt{2}$) then suggests we should expect a symmetric form of the overlap matrix element $\langle y|_x|\tilde{y}\rangle_{\tilde{x}}$, as we indeed observe.

2. Another argument that we have the correct basis transformation is that the annihilation operators $a = \frac{\hat{x}+i\hat{p}}{\sqrt{2}}$ in x space and $\tilde{a} = \frac{\hat{\tilde{x}}+i\hat{\tilde{p}}}{\sqrt{2}}$ in $\tilde{x}$ space are identical operators, up to an overall complex phase (as you can quickly check by substitution). Thus the ground state which is annihilated by a in x space should still be annihilated by the operator $\tilde{a}$ when everything is expressed in $\tilde{x}$ space, and thus must be the ground state in $\tilde{x}$ space as well. There is one unique choice of $f(\tilde{y})$ that performs this mapping from ground state to ground state (up to a phase rotation of the whole state $e^{i\phi}$, corresponding to a constant offset of $f(\tilde{y})$, or $f(\tilde{y}) \to f(\tilde{y}) + \phi$. Any non-constant change in $f(\tilde{y})$ directly introduces a spatially-dependent phase to the wavefunction in $\tilde{x}$ space after performing the basis transformation described below, and it is no longer the same state. This tells us the correct choice of $f(\tilde{y})$ is unique. As is shown below, the choice given does transform the harmonic ground state to the harmonic ground state, apart from an overall phase.

3. For another perspective, consider an analogous transformation: from the usual position coordinates to momentum coordinates. If we arbitrarily decide to add a linear phase to each of our position eigenstates (ie. multiply each state $|x\rangle$ by e^{iax}) as we perform the transformation, we know that physically we

[1] The change of sign in the exponent in the formula for $|y\rangle_x$ is accompanied by the change of sign of the derivative term in the operator $\hat{x}$ compared to the operator $\hat{\tilde{x}}$, so that the state still has the correct eigenvalue when everything is expressed in the new basis).

now have altered the resulting state, for example giving it additional momentum. Thus, in that case, the phase ambiguity of the basis corresponds to boosting to a frame at finite velocity. This example shows that adding a non-uniform phase (e.g. spatially-dependent or momentum-dependent, depending on the circumstance) to the basis transformation is not harmless, and requires thought.

4.3 Time-dependent solution starting from the harmonic ground state in x-space

4.3.1 Schrödinger picture solution

Let's specifically examine the solution for the time-dependent wavefunction we found in Problem 1. This was a wavefunction that corresponds to the initial state being the ground harmonic oscillator state in x-space.

$$\psi(x,t) = \frac{1}{\pi^{1/4}} e^{-\frac{\omega}{2}t} e^{-\frac{(xe^{-\omega t})^2}{2}}, \tag{62}$$

which therefore has a corresponding real space wavefunction of time

$$\begin{aligned}
\psi_{\tilde{x}}(\tilde{y},t) &= \frac{1}{\pi^{1/4}} \frac{2^{1/4}}{\sqrt{2\pi}} e^{-\frac{\omega}{2}t} \int dy e^{i\sqrt{2}y\tilde{y}} e^{-i(y^2+\tilde{y}^2)/2} e^{-\frac{(ye^{-\omega t})^2}{2}} \\
&= \frac{1}{(2\pi)^{1/4}} \frac{1}{\sqrt{\pi}} e^{-\frac{\omega}{2}t} e^{-i\tilde{y}^2/2} \int dy e^{-[(i+e^{-2\omega t})/2]y^2} e^{(i\sqrt{2}\tilde{y})y}.
\end{aligned}$$

We know how to solve Gaussian integrals of this form (even with factors of i): $\int dx e^{-Ax^2+bx} = e^{b^2/(4A)}\sqrt{\pi/A}$.

$$\begin{aligned}
\psi_{\tilde{x}}(\tilde{y},t) &= \frac{1}{(2\pi)^{1/4}} \frac{1}{\sqrt{\pi}} e^{-\frac{\omega}{2}t} e^{-i\tilde{y}^2/2} \frac{\sqrt{\pi}}{\sqrt{(i+e^{-2\omega t})/2}} \exp\left(\frac{(i\sqrt{2}\tilde{y})^2}{2(i+e^{-2\omega t})}\right) \\
&= \frac{1}{\pi^{1/4}} \frac{e^{-\frac{\omega}{2}t}}{\sqrt{i}\sqrt{\frac{1-ie^{-2\omega t}}{\sqrt{2}}}} \exp\left(-\frac{\tilde{y}^2}{2}\left[\frac{2}{i+e^{-2\omega t}}+i\right]\right) \\
&= \frac{1}{\pi^{1/4}} \frac{e^{-\frac{\omega}{2}t}}{\sqrt{i}\sqrt{\frac{1-ie^{-2\omega t}}{\sqrt{2}}}} \exp\left(-\frac{(\tilde{y}e^{-\omega t})^2}{2}\left[\frac{1-ie^{+2\omega t}}{1-ie^{-2\omega t}}\right]\right).
\end{aligned}$$

One can check this matches the analytic solution quoted in this paper. Let's examine the wavefunction and probability density at $t=0$,

$$\begin{aligned}
\psi_{\tilde{x}}(\tilde{y},0) &= \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{i}\sqrt{\frac{1-i}{\sqrt{2}}}} \exp\left(-\frac{\tilde{y}^2}{2}\right) \\
|\psi_{\tilde{x}}(\tilde{y},0)|^2 &= \frac{1}{\sqrt{\pi}} e^{-\tilde{y}^2},
\end{aligned}$$

so indeed we see that this wavefunction is normalized and has the same probability amplitude and probability as the usual harmonic oscillator ground state at $t=0$, although it has an overall phase.

Let's examine the probability density of the time-dependent solution:

$$
\begin{aligned}
|\psi_{\tilde{x}}(\tilde{y},t)|^2 &= \left| \frac{1}{\pi^{1/4}} \frac{e^{-\frac{\omega}{2}t}}{\sqrt{i}\sqrt{\frac{1-ie^{-2\omega t}}{\sqrt{2}}}} \exp\left(-\frac{(\tilde{y}e^{-\omega t})^2}{2} \left[\frac{1-ie^{+2\omega t}}{1-ie^{-2\omega t}} \right] \right) \right|^2 \\
&= \frac{1}{\sqrt{\pi}} \frac{e^{-\omega t}}{\sqrt{\frac{1+e^{-4\omega t}}{2}}} \exp\left[-\frac{(\tilde{y}e^{-\omega t})^2}{2} \left[\frac{1-ie^{+2\omega t}}{1-ie^{-2\omega t}} + \frac{1+ie^{+2\omega t}}{1+ie^{-2\omega t}} \right] \right] \\
&= \frac{1}{\sqrt{\pi}} \frac{e^{-\omega t}}{\sqrt{\frac{1+e^{-4\omega t}}{2}}} \exp\left[-\tilde{y}^2 \frac{e^{-2\omega t}}{2} \left(\frac{4}{1+e^{-4\omega t}} \right) \right] \\
&= \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\cosh(2\omega t)}} \exp\left[-\left(\frac{\tilde{y}}{\sqrt{\cosh(2\omega t)}} \right)^2 \right].
\end{aligned}
$$

We see that in $\tilde{x}$ space, the probability density spreads out exponentially in time at large times, similar to what is observed in x-space, although at short times the rate of expansion is slower and indeed begins quadratically with time. Specifically the solution is rescaled by a factor of $\sqrt{\cosh(2\omega t)}$ with time, which begins as $1+(\omega t)^2$ and asymptotes to $e^{\omega t}/\sqrt{2}$. Note that the position uncertainty in time grows as

$$
\langle \Psi(t)|\hat{\tilde{x}}^2|\Psi(t)\rangle = \frac{1}{2}\cosh 2\omega t. \tag{63}
$$

4.3.2 Heisenberg picture solution

We can instead consider the time evolution of $\tilde{x}$ and $\tilde{p}$ in the Heisenberg picture. With $H = \frac{\hbar\omega}{2}(\tilde{p}^2 - \tilde{x}^2)$, the Heisenberg equations of motion are simply

$$
\dot{\tilde{x}} = \frac{i}{\hbar}[H,\tilde{x}] = \omega\tilde{p} \qquad \dot{\tilde{p}} = \frac{i}{\hbar}[H,\tilde{p}] = \omega\tilde{x}, \tag{64}
$$

having used $[\tilde{p}^2,\tilde{x}] = -2i\tilde{p}$ and $[\tilde{x}^2,\tilde{p}] = 2i\tilde{x}$. We see that the Heisenberg operators obey the same equations of motion as the classical variables. The solutions are therefore

$$
\begin{aligned}
\tilde{x}(t) &= \tilde{x}_0 \cosh\omega t + \tilde{p}_0 \sinh\omega t \\
\tilde{p}(t) &= \tilde{p}_0 \cosh\omega t + \tilde{x}_0 \sinh\omega t.
\end{aligned}
$$

We have used the initial condition $\tilde{x}(t{=}0) = \tilde{x} \equiv \tilde{x}_0$ and $\tilde{p}(t{=}0) = \tilde{p} \equiv \tilde{p}_0$ (ie. we mean that $\tilde{x}_0$ and $\tilde{p}_0$ are the usual Schrödinger operators at $t{=}0$). Given this solution, one can ask how the position uncertainty changes in time in the Heisenberg picture, with the initial state $|\Psi\rangle$ being the harmonic ground state with property $\langle\Psi|\tilde{x}_0^2|\Psi\rangle = \langle\Psi|\tilde{p}_0^2|\Psi\rangle = 1/2$,

$$
\begin{aligned}
\langle\Psi|(\tilde{x}(t))^2|\Psi\rangle &= \langle\Psi|(\tilde{x}_0\cosh\omega t + \tilde{p}_0\sinh\omega t)^2|\Psi\rangle \\
&= \langle\Psi|\tilde{x}_0^2\cosh^2\omega t + \tilde{p}_0^2\sinh^2\omega t + \sinh\omega t\cosh\omega t(\tilde{x}_0\tilde{p}_0 + \tilde{p}_0\tilde{x}_0)|\Psi\rangle \\
&= \frac{\cosh^2\omega t + \sinh^2\omega t}{2} + \sinh\omega t\cosh\omega t\langle\Psi|(\tilde{x}_0\tilde{p}_0 + \tilde{p}_0\tilde{x}_0)|\Psi\rangle \\
&= \frac{1}{2}\cosh 2\omega t,
\end{aligned}
$$

where one can easily check that $\langle\Psi|(\tilde{x}_0\tilde{p}_0 + \tilde{p}_0\tilde{x}_0)|\Psi\rangle = 0$ for the harmonic ground state. This result matches the spread in $\langle\tilde{x}^2\rangle$ with time found above for the time-dependent solution for the wavefunction in the Schrödinger picture.

Interestingly, the same result holds for $\langle\Psi|(\tilde{p}(t))^2|\Psi\rangle = \frac{1}{2}\cosh 2\omega t$, so that the wavefunction no longer saturates the Heisenberg uncertainty principle as the system evolves.

4.3.3 Classical solution

This section is by M.Z., any errors to be blamed on him.

For quadratic Hamiltonians there is a direct correspondence between the classical and the quantum evolution, as we have already seen above e.g. when looking at the evolution of operators in the Heisenberg picture.

To see the correspondence explicitly, let us ask the question: Given a *classical* phase space distribution $f_0(x,p)$ at time $t=0$, what is the phase space distribution at later times t?

Let's first use the Hamiltonian written in terms of the (x,p) coordinates (rotated 45 degrees in phase space with respect to the original $\tilde{x}$ and $\tilde{p}$ operators), i.e. $H=\omega xp$ (we don't know about $\hbar$ in classical mechanics, but we also don't need it here - the x and p simply have units of $\sqrt{\text{action}}$).

The equations of motion of some particle of coordinates (x,p) are

$$\begin{aligned}\dot{x} &= \{x,H\}=\omega x\\ \dot{p} &= \{p,H\}=-\omega p\end{aligned}$$

with $\{f,g\}=\frac{\partial f}{\partial x}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$ the Poisson bracket. The Poisson bracket acts algebraically in the same way as the commutator (or in other words, the quantum mechanical commutator (divided by $i\hbar$) is just also a Poisson bracket).

We find immediately the solution $x(t)=x_0e^{\omega t}$ and $p(t)=p_0e^{-\omega t}$.

Particles located at (x_0,p_0) in phase space at time $t=0$ will be found at $(x,p)=(x_0e^{\omega t},p_0e^{-\omega t})$ after time t, or, inverting this sentence, particles found at (x,p) at time t must have come from $(x_0,p_0)=(xe^{-\omega t},pe^{\omega t})$ at time $t=0$. So we have for the phase space distribution at time t:

$$f(x,p,t)=f_0(xe^{-\omega t},pe^{\omega t}) \tag{65}$$

We can also see this from Liouville's equation for the phase space distribution $\frac{\mathrm{d}f}{\mathrm{d}t}=0$ or

$$\frac{\mathrm{d}f}{\mathrm{d}t}=\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\dot{x}+\frac{\partial f}{\partial p}\dot{p}=\frac{\partial f}{\partial t}+\{f,H\}=\frac{\partial f}{\partial t}+\omega x\frac{\partial f}{\partial x}-\omega p\frac{\partial f}{\partial p}=0 \tag{66}$$

and thus

$$\left(\frac{\partial}{\partial\omega t}+\frac{\partial}{\partial\ln x}-\frac{\partial}{\partial\ln p}\right)f=0 \tag{67}$$

which is a way of saying that the function $f(x,p,t)$ can only depend on $xe^{-\omega t}$ and $pe^{\omega t}$.

Let us apply this to an initial gaussian distribution that mimics the quantum distribution (it corresponds to the Q distribution of the vacuum state).

$$f_0(x,p)=\frac{1}{\pi}e^{-x^2-p^2}$$

where we secretly chose units of x and p so that they are dimensionless (i.e. we chose some unit of action, and set that to unity - and for perfect correspondence that unit of action is $\hbar$). We immediately find

$$f(x,p,t)=f_0(xe^{-\omega t},pe^{\omega t})=\frac{1}{\pi}e^{-x^2e^{-2\omega t}-p^2e^{2\omega t}}$$

Now we can go back to the "physical" coordinates $(\tilde{x},\tilde{p})=(\frac{1}{\sqrt{2}}(x-p),\frac{1}{\sqrt{2}}(x+p))$, rotated by -45 degrees w.r.t. $(x,p)=(\frac{1}{\sqrt{2}}(\tilde{x}+\tilde{p}),\frac{1}{\sqrt{2}}(\tilde{p}-\tilde{x}))$ and find the distribution $\tilde{f}(\tilde{x},\tilde{p},t)=f(x,p,t)$:

$$\begin{aligned}\tilde{f}(\tilde{x},\tilde{p},t) &= \frac{1}{\pi}e^{-\frac{1}{2}(\tilde{x}+\tilde{p})^2e^{-2\omega t}-\frac{1}{2}(\tilde{p}-\tilde{x})^2e^{2\omega t}}\\ &= \frac{1}{\pi}e^{-(\tilde{x}^2+\tilde{p}^2)\cosh(2\omega t)+2\tilde{x}\tilde{p}\sinh(2\omega t)}\end{aligned}$$

Of course, the same result we could have obtained from the solution of the equation of motion of the original Hamiltonian $H=\frac{\omega}{2}\left(\tilde{p}^2-\tilde{x}^2\right)$, which are

$$\begin{aligned}\tilde{x}(t) &= \tilde{x}_0\cosh(\omega t)+\tilde{p}_0\sinh(\omega t)\\ \tilde{p}(t) &= \tilde{x}_0\sinh(\omega t)+\tilde{p}_0\cosh(\omega t),\end{aligned}$$

and again noting that any particle found at $(\tilde{x}, \tilde{p})$ in phase space at time t must have originated from a point $(\tilde{x}_0, \tilde{p}_0)$ at time $t = 0$ such that

$$\begin{aligned}
\tilde{x}_0 &= \tilde{x}\cosh(\omega t) - \tilde{p}\sinh(\omega t) \\
\tilde{p}_0 &= -\tilde{x}\sinh(\omega t) + \tilde{p}\cosh(\omega t)
\end{aligned} \tag{68}$$

which is - again - just the inverse time-evolution of the point (x, p) propagated backwards in time. With that, one immediately writes down

$$\begin{aligned}
\tilde{f}(\tilde{x}, \tilde{p}, t) &= f_0(\tilde{x}\cosh(\omega t) - \tilde{p}\sinh(\omega t), -\tilde{x}\sinh(\omega t) + \tilde{p}\cosh(\omega t)) \\
&= \frac{1}{\pi} e^{-(\tilde{x}\cosh(\omega t) - \tilde{p}\sinh(\omega t))^2 - (-\tilde{x}\sinh(\omega t) + \tilde{p}\cosh(\omega t)))^2} \\
&= \frac{1}{\pi} e^{-(\tilde{x}^2 + \tilde{p}^2)(\cosh^2(\omega t) + \sinh^2(\omega t)) + 4\tilde{x}\tilde{p}\sinh(\omega t)\cosh(\omega t)} \\
&= \frac{1}{\pi} e^{-(\tilde{x}^2 + \tilde{p}^2)\cosh(2\omega t) + 2\tilde{x}\tilde{p}\sinh(2\omega t)}
\end{aligned} \tag{69}$$

(thanks to “hyperbolic trig.” identities).
Now we are ready to find the probability to find a particle at position $\tilde{x}$ after time t, simply by integrating over all possible momenta $\tilde{p}$:

$$\begin{aligned}
P(x, t) &= \int \mathrm{d}\tilde{p}\, \tilde{f}(\tilde{x}, \tilde{p}, t) \\
&= \frac{1}{\pi} \int \mathrm{d}\tilde{p}\, e^{-(\tilde{x}^2 + \tilde{p}^2)\cosh(2\omega t) + 2\tilde{x}\tilde{p}\sinh(2\omega t)} \\
&= \frac{1}{\pi} \int \mathrm{d}\tilde{p}\, e^{-\left\{\left(\tilde{p} - \tilde{x}\frac{\sinh(2\omega t)}{\cosh(2\omega t)}\right)^2 \cosh(2\omega t) - \tilde{x}^2 \frac{\sinh^2(2\omega t)}{\cosh(2\omega t)} + \tilde{x}^2\cosh(2\omega t)\right\}} \\
&= \frac{1}{\sqrt{\pi\cosh(2\omega t)}} e^{-\tilde{x}^2\cosh(2\omega t)\left(1 - \tanh^2(2\omega t)\right)} \\
&= \frac{1}{\sqrt{\pi\cosh(2\omega t)}} \exp\left[-\frac{\tilde{x}^2}{\cosh(2\omega t)}\right]
\end{aligned} \tag{70}$$

which is the same solution we have found quantum mechanically.

4.4 Summary

We now know how to solve the inverted harmonic oscillator in various representations. If you ever see any of these Hamiltonians

$$H = \frac{1}{2}(\hat{p}^2 - \hat{x}^2) \qquad H = \hat{x}\hat{p} - \frac{i}{2} \qquad H = -\frac{i}{2}(a^2 - a^{\dagger 2}), \tag{71}$$

you can choose to think about the problem in various ways. You have the explicit basis transformations between the various representations, and you derived the (very general and simple) time-dependent solution in x-space.

1. Real space: $\tilde{x}$ and $\tilde{p}$. (Solved in Appendix)

   Benefits: these are the real physical variables for the inverted oscillator, sometimes of interest.

   Challenges: not simple time evolution

2. Rotated space: x and p. (Solved in Problem 1)

   Benefits: simple time evolution of the wavefunction of x via a rescaling exponentially in time.

   Challenges: not physical variables

3. Fock space of rotated coordinates: a and $a^\dagger$. (Solved in Problem 2)

   Benefits: allows interpretation of photon number squeezing.

Given any initial state, you can express it in x-space to find the time-dependent solution, and then return to your desired representation.