Convolution powers of complex-valued functions on $\mathbb{Z}^d$

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CLAS, April 28, 2021

Random walk theory

Problem: A random walker takes a sequence of steps $X_1, X_2, \ldots$ on $\mathbb{Z}^d$. Each X_i is independent of previous steps and follows some distribution ϕ.

$$\phi(x) = \mathbb{P}(X = x).$$

$\boxed{?}$ What is the distribution which $S_n = X_1 + X_2 + \cdots + X_n$ follows?

$$\begin{aligned}
S_1 &\quad \text{follows} \quad \phi(x) \\
S_2 &\quad \text{follows} \quad \phi^{(2)}(x) = \sum_{y \in \mathbb{Z}^d} \phi(y)\phi(x-y) \\
&\vdots \\
S_n &\quad \text{follows} \quad \phi^{(n)}(x) = \sum_{y \in \mathbb{Z}^d} \phi^{(n-1)}(y)\phi(x-y)
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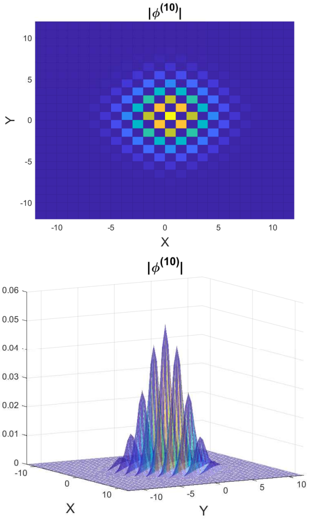
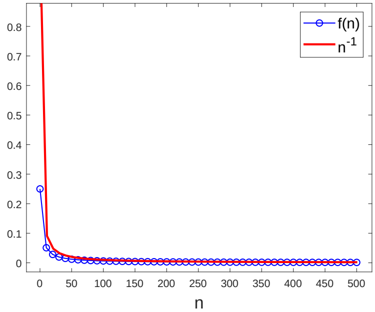
How does $\phi^{(n)}$ behave as $n \to \infty$?

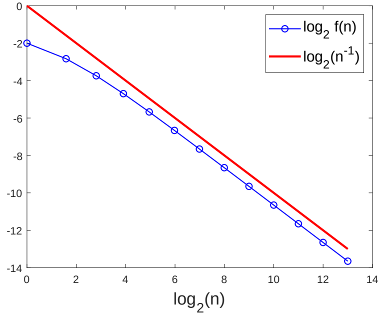
Example: Simple random walk in $\mathbb{Z}^2$



Example: Simple random walk in $\mathbb{Z}^2$.

$$f(n) = \max_{\mathbb{Z}^d} \phi^{(n)} \quad \text{decays like} \quad 1/n$$





Random walk theory

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Random walk theory

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- **Global decay:** There are positive constants C_1, C_2 for which

$$C_1 n^{-d/2} \leq \|\phi^{(n)}\|_\infty \leq C_2 n^{-d/2}, \quad \forall n \in \mathbb{N}_+.$$

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- **Local description** for large n (like CLT!):

$$\phi^{(n)}(x) = \frac{1}{n^{d/2}} \Phi_\phi\left(\frac{x}{\sqrt{n}}\right) + o\left(\frac{1}{n^{d/2}}\right), \quad \text{uniformly for } x \in \mathbb{Z}^d$$

where Φ_ϕ is the generalized Gaussian associated with ϕ.

Beyond probability theory

What if positivity is dropped?

Consider $\phi : \mathbb{Z}^d \to \mathbb{C}$ and define $\phi^{(n)}$ as before

$$\phi^{(n)}(x) = \sum_{y \in \mathbb{Z}^d} \phi^{(n-1)}(x-y)\phi(y).$$

About the asymptotic behavior of $\phi^{(n)}$ as $n \to \infty$, can we still ask for

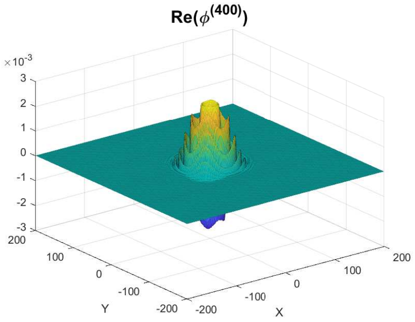
- A global decay?
- A local description?

Beyond probability theory

Example: Consider $\phi : \mathbb{Z}^2 \to \mathbb{C}$ given by

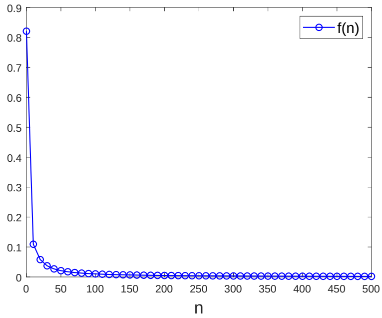
$$\phi(x,y) = \frac{1}{192} \times \begin{cases} 144 - 64i & (x,y) = (0,0) \\ 16 + 16i & (x,y) = (\pm 1, 0) \text{ or } (0, \pm 1) \\ -4 & (x,y) = (\pm 2, 0) \text{ or } (0, \pm 2) \\ i & (x,y) = \pm(1,1) \\ -i & (x,y) = \pm(1,-1) \\ 0 & \text{otherwise.} \end{cases}$$

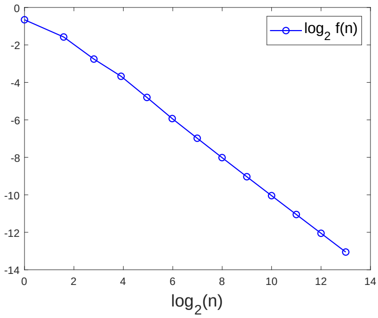
Beyond probability theory



Beyond probability theory

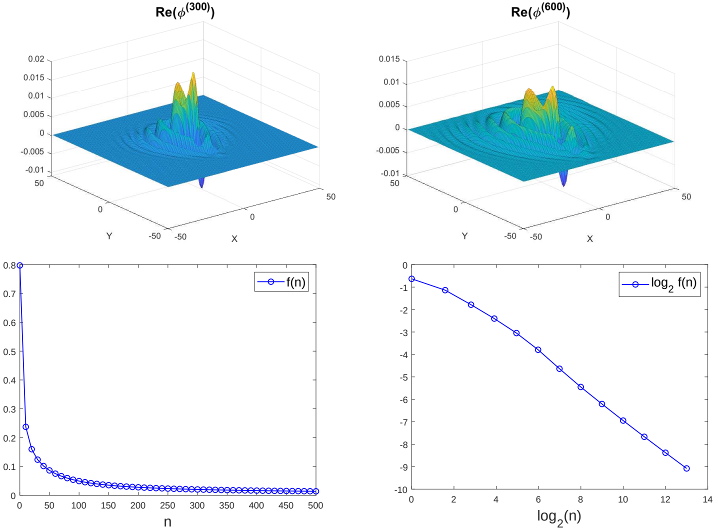
$$f(n) = \max|\phi^{(n)}|$$





Beyond probability theory

A different ϕ gives a completely different behavior.



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About the asymptotic behavior of $\phi^{(n)}$ as $n \to \infty$, can we still ask for

- **A global decay?** $\Longleftarrow$ Focus of Bui and Randles (2021)
- A local description?

Global decay estimate for $\left|\phi^{(n)}\right|$

Define the Fourier transform for[1] $\phi : \mathbb{Z}^d \to \mathbb{C}$

$$\widehat{\phi}(\xi) = \sum_{x \in \mathbb{Z}^d} \phi(x) e^{ix\cdot\xi}$$

$$\boxed{\mathrm{FT}\{\phi^{(n)}\} = (\mathrm{FT}\{\phi\})^n}$$

The asymptotic behavior of $\phi^{(n)}$ is characterized by how $\widehat{\phi}$ behaves near where $|\widehat{\phi}|$ is maximized.

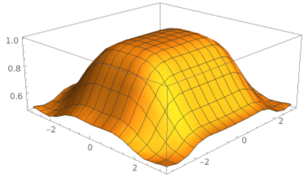
$$\Omega(\phi) = \left\{\xi \in \mathbb{T}^d : \left|\widehat{\phi}(\xi)\right| \text{ is maximized}\right\}, \quad \mathbb{T}^d = (-\pi, \pi]^d$$

[1] Actually, we take $\phi \in \mathcal{S}_d$ – a discrete analogue of the Schwartz class.

Global decay estimate for $\left|\phi^{(n)}\right|$

For each $\xi_0 \in \Omega(\phi)$, look at $\widehat{\phi}$ near ξ_0...

$$\widehat{\phi}(\xi + \xi_0) = \widehat{\phi}(\xi_0) e^{\Gamma_{\xi_0}(\xi)}$$



A $|\widehat{\phi}|$ on $\mathbb{T}^2$

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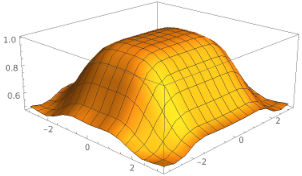

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Recall $\widehat{\phi^{(n)}} = \widehat{\phi}^n$. So, $\phi^{(n)} \sim \mathrm{FT}^{-1}\left\{ e^{n\Gamma_{\xi_0}(\xi)} \right\}$.

$\Longrightarrow$ **The structure of Γ determines the asymptotic behavior of $\phi^{(n)}$.**

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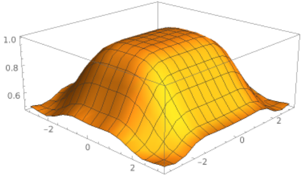

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So, Taylor expand Γ_{ξ_0}...

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - P_{\xi_0}(\xi) + \text{ h.o.t.}, \quad P_{\xi_0} \text{ a polynomial.}$$

The nature of this expansion is characterizing.

Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

In 1 dimension: 2 types

ξ_0 is of **positive homogeneous type** if

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0}\xi - \beta\xi^m + \text{ h.o.t.}, \quad \mathrm{Re}\{\beta\} > 0$$

$\implies \phi^{(n)}$ is easy to estimate.

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ξ_0 is of **imaginary homogeneous type** if

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0}\xi - i\beta\xi^m + \text{ h.o.t.}, \quad \beta \in \mathbb{R} \setminus \{0\}$$

$\implies \widehat{\phi}^n$ is highly oscillatory. $\phi^{(n)}$ is more difficult to estimate.

- These types are collectively exhaustive for f.s. ϕ's (Thomée – 1965)

Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

Randles & Saloff-Coste sorted out the 1-dimensional problem.

Theorem (Global decay estimate, Randles & Saloff-Coste – 2015)

Let $\phi : \mathbb{Z} \to \mathbb{C}$ be finitely supported and whose support contains more than one point. Then there is $\mathbb{N} \ni m \geq 2$, and A, C, $C' > 0$ such that

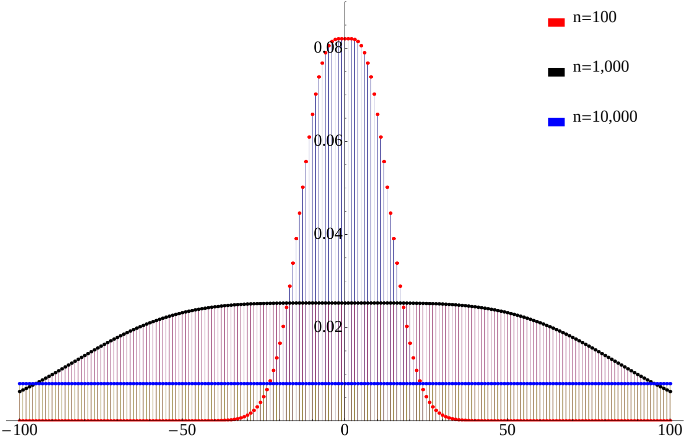
$$Cn^{-1/m} \leq A^{-n} \|\phi^{(n)}\|_\infty \leq C' n^{-1/m}, \quad \forall n \in \mathbb{N}.$$

Here, $A = \max|\widehat{\phi}(\xi)|$.

Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

Example: $\phi : \mathbb{Z} \to \mathbb{C}$ defined below. $\|\phi^{(n)}\|_\infty$ decays like $n^{-1/2}$.

$$\phi(0) = \frac{5-2i}{8} \quad \phi(\pm 1) = \frac{2+i}{8} \quad \phi(\pm 2) = -\frac{1}{16} \quad \phi = 0 \text{ otherwise.}$$



Global decay estimate for $\left|\phi^{(n)}\right|$: In 1 dimension

How to generalize to d dimensions?
$\implies$ Need **positive homogeneous functions**

Definition

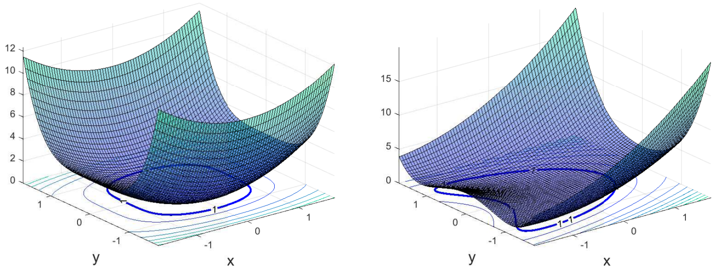
Let $P : \mathbb{R}^d \to \mathbb{R}$ be continuous, positive definite, and $d \times d$ matrix E s.t.

$$P(r^E\eta) = rP(\eta), \quad r > 0,\ \eta \in \mathbb{R}^d.$$

If $S := \{\eta \in \mathbb{R}^d : P(\eta) = 1\}$ is compact, then we say that P is **positive homogeneous***.

(*) see equivalent definitions in [BR21]

Examples:



(a) $P_1(x,y) = x^2 + y^4$ (b) $P_2(x,y) = x^2 + 3xy^2/2 + y^4$

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

In d dimensions:

ξ_0 is of **positive homogeneous type** if

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - P_{\xi_0}(\xi) + \text{ h.o.t.}$$

ξ_0 is of **imaginary homogeneous type** if

$$\Gamma_{\xi_0}(\xi) = i\alpha_{\xi_0} \cdot \xi - iP_{\xi_0}(\xi) + \text{ h.o.t.}$$

where $P_{\xi_0}(\xi)$ is a positive homogeneous *polynomial*.

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

A partial answer in d dimensions is due to Randles & Saloff-Coste.

Theorem (Global decay estimate, Randles & Saloff-Coste – 2017)

Let $\phi \in \mathcal{S}_d$ be such that $\sup |\widehat{\phi}| = 1$ *and suppose that each $\xi_0 \in \Omega(\phi)$ is of* **positive homogeneous type**. *There are μ_ϕ, C, $C' > 0$ for which*

$$C' n^{-\mu_\phi} \leq \|\phi^{(n)}\|_\infty \leq C n^{-\mu_\phi}, \quad \forall n \in \mathbb{N}_+.$$

We extend this to include points of imaginary homogeneous type.

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

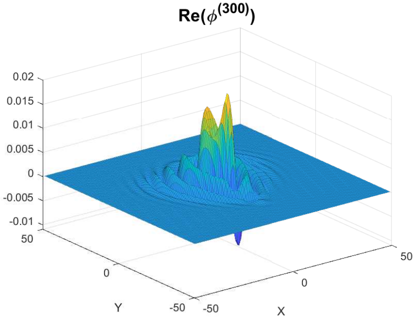
Theorem (B, Randles – 2021)

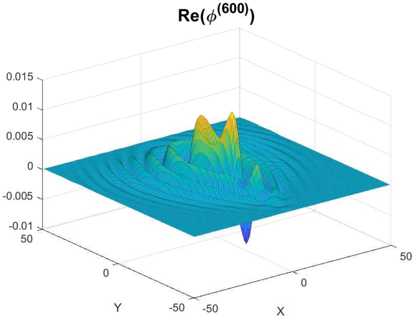
Let $\phi \in \mathcal{S}_d$ be such that $\sup|\widehat{\phi}| = 1$ and suppose that each $\xi_0 \in \Omega(\phi)$ is of positive homogeneous or imaginary homogeneous type for $\widehat{\phi}$. Then, for any compact set K, there are $C_K, \mu_\phi > 0$ for which***

$$\left|\phi^{(n)}(x)\right| \leq C_K n^{-\mu_\phi}, \quad \forall x \in K,\ n \in \mathbb{N}_+.$$

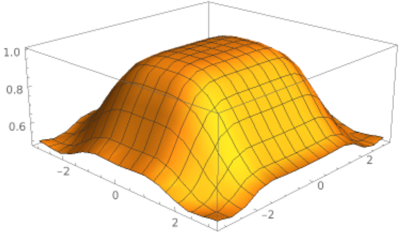
(*) and some additional conditions
(**) see [BR21] for how to calculate μ_ϕ

Example: From earlier...





Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions



$|\widehat{\phi}|$ on $(-\pi,\pi]\times(-\pi,\pi]$

- $\sup|\widehat{\phi}| = 1$ and $\Omega(\phi) = \{\xi_0\} = \{(0,0)\}$

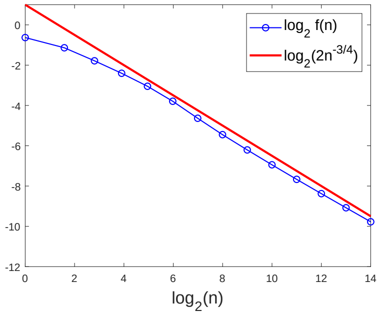
$$\Gamma_0(\xi) = -i\left(\frac{\tau^2}{24} - \frac{\tau\zeta^2}{96} + \frac{\zeta^4}{96}\right) + \text{ h.o.t.}$$

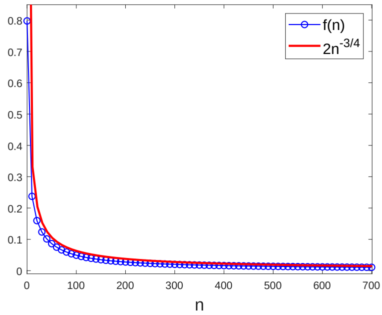
- $\mu_\phi = 3/4$

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

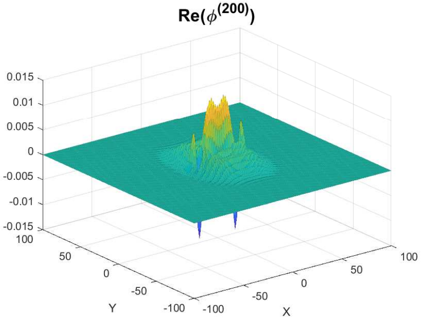
Let $K = [-300, 300] \times [-300, 300]$ and with $C = 2$,

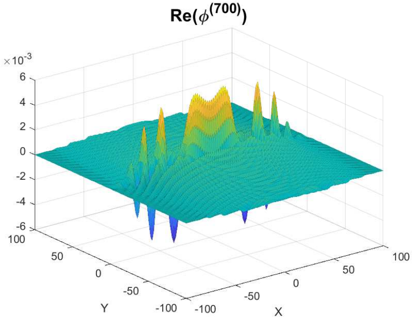
$$f(n) := \max_K \left|\phi^{(n)}\right| \leq 2n^{-\mu_\phi} = 2n^{-3/4}$$



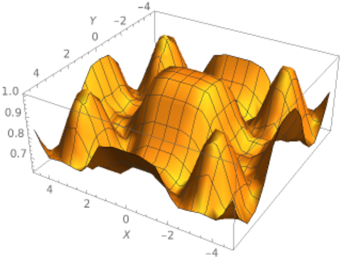


Example: Ω has more than one point...





Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions



$|\widehat{\phi}|$ on $(-\pi, \pi] \times (-\pi, \pi]$

- $\sup |\widehat{\phi}| = 1$ and $\Omega(\phi) = \{\xi_0, \xi_1\} = \{(0,0), (\pi, \pi)\}$

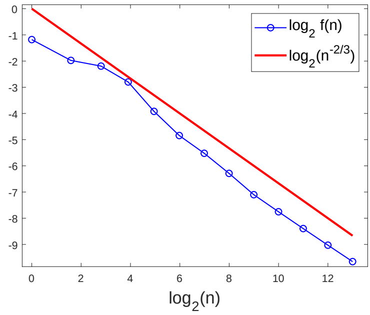
$$\Gamma_0(\xi) = -i\left(\frac{\tau^6}{128} + \frac{\zeta^2}{8}\right) + \dots \qquad \Gamma_1(\xi) = +i\left(\frac{3\tau^2}{8} + \frac{\zeta^2}{4}\right) + \dots$$

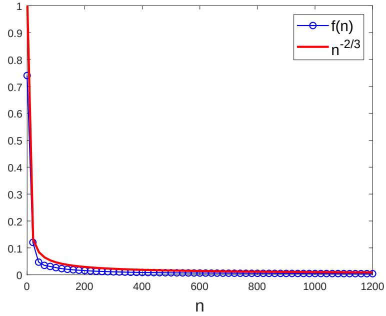
- $\mu_\phi = 2/3$

Global decay estimate for $\left|\phi^{(n)}\right|$: In d dimensions

Let $K = [-500, 500] \times [-500, 500]$ and with $C = 1$,

$$f(n) := \max_K \left|\phi^{(n)}\right| \leq n^{-\mu_\phi} = n^{-2/3}$$





Applications?

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   - Approximate solutions via convolution power schemes
2. . . .

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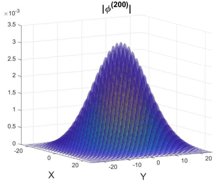
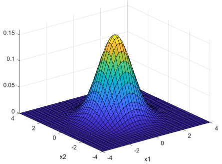
2. . . .
3. IT'S NICE.

What's next?

Classical result (for probability distributions):



$\phi^{(n)} \to$ Gaussian



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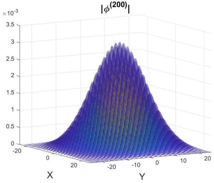

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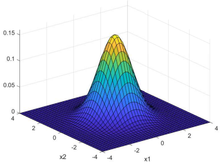
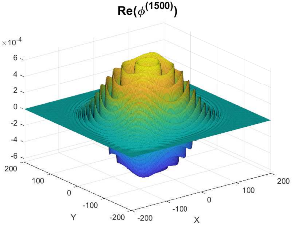
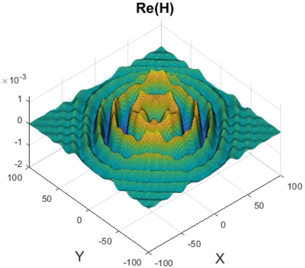


New conjecture: No positivity? No problem.



$\phi^{(n)} \to H_t^{iP}$



References

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