Ideal Quantum Gases

1. *Numerical estimates:* The following table provides typical values for the Fermi energy and Fermi temperature for (i) Electrons in a typical metal; (ii) Nucleons in a heavy nucleus; and (iii) He^3 atoms in liquid He^3 (atomic volume = $46.2\mathring{A}^3$ per atom).

	$n(1/\mathrm{m}^3)$	$m(\mathrm{Kg})$	$\varepsilon_F(\mathrm{eV})$	$T_F(\mathrm{K})$
electron	10^{29}	9×10^{-31}	4.4	5×10^4
nucleons	10^{44}	1.6×10^{-27}	1.0×10^8	1.1×10^{12}
liquid He^3	2.6×10^{28}	4.6×10^{-27}	10^{-3}	10^1

(a) Estimate the ratio of the electron and phonon heat capacities at room temperature for a typical metal.

• For an electron gas, $T_F \approx 5 \times 10^4$K,

$$T_F \gg T_{\mathrm{room}}, \quad \Longrightarrow \quad \frac{C_{\mathrm{electron}}}{Nk_B} \approx \frac{\pi^2}{2} \cdot \frac{T}{T_F} \approx 0.025.$$

For the phonon gas in iron, the Debye temperature is $T_D \approx 470$K, and hence

$$\frac{C_{\mathrm{phonon}}}{Nk_B} \approx 3\left[1 - \frac{1}{20}\left(\frac{T}{T_D}\right)^2 + \dots\right] \approx 3,$$

resulting in

$$\frac{C_{\mathrm{electron}}}{C_{\mathrm{phonon}}} \approx 8 \times 10^{-3}.$$

(b) Compare the thermal wavelength of a neutron at room temperature to the minimum wavelength of a phonon in a typical crystal.

• Thermal wavelengths are given by

$$\lambda \equiv \frac{h}{\sqrt{2\pi m k_B T}}.$$

For a neutron at room temperature, using the values

$$m = 1.67 \times 10^{-27}\mathrm{kg}, \quad T = 300\,\mathrm{K},$$
$$k_B = 1.38 \times 10^{-23}\mathrm{JK}^{-1}, \quad h = 6.67 \times 10^{-34}\mathrm{Js},$$

we obtain $\lambda = 1\mathring{A}$.

The typical wavelength of a phonon in a solid is $\lambda = 0.01\,m$, which is much longer than the neutron wavelength. The minimum wavelength is, however, of the order of atomic spacing $(3-5\,\AA)$, which is comparable to the neutron thermal wavelength.

(c) Estimate the degeneracy discriminant, $n\lambda^3$, for hydrogen, helium, and oxygen gases at room temperature and pressure. At what temperatures do quantum mechanical effects become important for these gases?

• Quantum mechanical effects become important if $n\lambda^3 \geq 1$. In the high temperature limit the ideal gas law is valid, and the degeneracy criterion can be reexpressed in terms of pressure $P = nk_BT$, as

$$n\lambda^3 = \frac{nh^3}{(2\pi m k_B T)^{3/2}} = \frac{P}{(k_BT)^{5/2}}\frac{h^3}{(2\pi m)^{3/2}} \ll 1.$$

It is convenient to express the answers starting with an imaginary gas of 'protons' at room temperature and pressure, for which

$$m_p = 1.7\times 10^{-34}\mathrm{Kg}, \qquad P = 1\,\mathrm{atm.} = 10^5\,\mathrm{Nm}^{-2},$$
$$\text{and} \quad (n\lambda^3)_{\mathrm{proton}} = \frac{10^{-5}}{(4.1\times 10^{-21})^{5/2}}\frac{(6.7\times 10^{-34})^3}{(2\pi\cdot 1.7\times 10^{-27})^{3/2}} = 2\times 10^{-5}.$$

The quantum effects appear below $T = T_Q$, at which $n\lambda^3$ becomes order of unity. Using

$$n\lambda^3 = (n\lambda^3)_{\mathrm{proton}}\left(\frac{m_p}{m}\right)^{3/2}, \quad \text{and} \quad T_Q = T_{\mathrm{room}}(n\lambda^3)^{3/2},$$

we obtain the following table:

Hydrogen H_2	$\frac{m}{m_p} = 2$	$n\lambda^3 = 0.7\times 10^{-5}$	$T_Q = 2.6K$
Helim He	$\frac{m}{m_p} = 4$	$n\lambda^3 = 3.0\times 10^{-6}$	$T_Q = 1.9K$
Oxygen O_2	$\frac{m}{m_p} = 32$	$n\lambda^3 = 0.1\times 10^{-6}$	$T_Q = 0.5K$

(d) **(Optional)** Experiments on He^4 indicate that at temperatures below 1K, the heat capacity is given by $C_V = 20.4T^3 JKg^{-1}\,\mathrm{K}^{-1}$. Find the low energy excitation spectrum, $\mathcal{E}(k)$, of He^4. (Hint: There is only one non-degenerate branch of such excitations.)

• A spectrum of low energy excitations scaling as

$$\mathcal{E}(k) \propto k^s,$$

in d-dimensional space, leads to a low temperature heat capacity that vanishes as

$$C \propto T^{d/s}.$$

Therefore, from $C_V = 20.4\, T^3 \mathrm{JKg}^{-1}\,\mathrm{K}^{-1}$ in $d = 3$, we can conclude $s = 1$, i.e. a spectrum of the form

$$\mathcal{E}(k) = \hbar c_s |\vec{k}|,$$

corresponding to sound waves of speed c_s. Inserting all the numerical factors, we have

$$C_V = \frac{12\pi^4 N k_B}{5} \left(\frac{T}{\Theta}\right)^3, \quad \text{where} \quad \Theta = \frac{\hbar c_s}{k_B} \left(\frac{6\pi^2 N}{V}\right)^{1/3}.$$

Hence, we obtain

$$\mathcal{E} = \hbar c_s k = k_B \left(\frac{2\pi^2 k_B V}{5} \frac{T^3}{C_V}\right)^{1/3} k = (2 \times 10^{-32}\mathrm{Jm})\ k,$$

corresponding to a sound speed of $c_s \approx 2 \times 10^2 \mathrm{ms}^{-1}$.

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2. *Solar interior:* According to astrophysical data, the plasma at the center of the sun has the following properties:

Temperature:	$T = 1.6 \times 10^7\,\mathrm{K}$
Hydrogen density:	$\rho_H = 6 \times 10^4$ kg m^{-3}
Helium density:	$\rho_{He} = 1 \times 10^5$ kg m^{-3}.

(a) Obtain the thermal wavelengths for electrons, protons, and α-particles (nuclei of He).

• The thermal wavelengths of electrons, protons, and α-particles in the sun are obtained from

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}},$$

and $T = 1.6 \times 10^7$K, as

$$\lambda_{\mathrm{electron}} \approx \frac{6.7 \times 10^{-34}\,\mathrm{J/s}}{\sqrt{2\pi \times (9.1 \times 10^{-31}\,\mathrm{Kg}) \cdot (1.4 \times 10^{-23}\,\mathrm{J/K}) \cdot (1.6 \times 10^7\,\mathrm{K})}} \approx 1.9 \times 10^{-11}\,\mathrm{m},$$

$$\lambda_{\mathrm{proton}} \approx \frac{6.7 \times 10^{-34}\,\mathrm{J/s}}{\sqrt{2\pi \times (1.7 \times 10^{-27}\,\mathrm{Kg}) \cdot (1.4 \times 10^{-23}\,\mathrm{J/K}) \cdot (1.6 \times 10^7\,\mathrm{K})}} \approx 4.3 \times 10^{-13}\,\mathrm{m},$$

and

$$\lambda_{\alpha-\mathrm{particle}} = \frac{1}{2}\lambda_{\mathrm{proton}} \approx 2.2 \times 10^{-13}\,\mathrm{m}.$$

(b) Assuming that the gas is ideal, determine whether the electron, proton, or α-particle gases are degenerate in the quantum mechanical sense.

• The corresponding number densities are given by

$$\begin{aligned}
&\rho_H \approx 6\times 10^4\,\mathrm{kg/m^3} \quad \Longrightarrow \quad n_H \approx 3.5\times 10^{31}\,\mathrm{m^{-3}},\\
&\rho_{He} \approx 1.0\times 10^5\,\mathrm{Kg/m^3} \quad \Longrightarrow \quad n_{He} = \frac{\rho_{He}}{4m_H} \approx 1.5\times 10^{31}\,\mathrm{m^{-3}},\\
&n_e = 2n_{He} + n_H \approx 8.5\times 10^{31}\,\mathrm{m^3}.
\end{aligned}$$

The criterion for degeneracy is $n\lambda^3 \geq 1$, and

$$\begin{aligned}
n_H \cdot \lambda_H^3 &\approx 2.8\times 10^{-6} \ll 1,\\
n_{He} \cdot \lambda_{He}^3 &\approx 1.6\times 10^{-7} \ll 1,\\
n_e \cdot \lambda_e^3 &\approx 0.58 \sim 1.
\end{aligned}$$

Thus the electrons are weakly degenerate, and the nuclei are not.

(c) Estimate the total gas pressure due to these gas particles near the center of the sun.

• Since the nuclei are non-degenerate, and even the electrons are only weakly degenerate, their contributions to the overall pressure can be approximately calculated using the ideal gas law, as

$$\begin{aligned}
P &\approx (n_H + n_{he} + n_e)\cdot k_B T \approx 13.5\times 10^{31}(\mathrm{m^{-3}})\cdot 1.38\times 10^{-23}(\mathrm{J/K})\cdot 1.6\times 10^7(\mathrm{K})\\
&\approx 3.0\times 10^{16}\,\mathrm{N/m^2}.
\end{aligned}$$

(d) Estimate the total radiation pressure close to the center of the sun. Is it matter, or radiation pressure, that prevents the gravitational collapse of the sun?

• The Radiation pressure at the center of the sun can be calculated using the black body formulas,

$$P = \frac{1}{3}\frac{U}{V}, \qquad \text{and} \qquad \frac{1}{4}\frac{U}{V}c = \frac{\pi^2 k^4}{60\hbar^3 c^3}T^4 = \sigma T^4,$$

as

$$P = \frac{4}{3c}\sigma T^4 = \frac{4\cdot 5.7\times 10^{-8}\,\mathrm{W/(m^2K^4)}\cdot(1.6\times 10^7\,\mathrm{K})^4}{3\cdot 3.0\times 10^8\,\mathrm{m/s}} \approx 1.7\times 10^{13}\,\mathrm{N/m^2}.$$

Thus at the pressure in the solar interior is dominated by the particles.

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3. *Density of states:* Consider a system of *non-interacting identical* degrees of freedom, with a set of single-particle energies $\{\varepsilon_n\}$, and ground state $\varepsilon_0 = 0$. In a grand canonical ensemble at temperature $T = (k_B\beta)^{-1}$, the number of particles N is related to the chemical potential μ by

$$N = \sum_n \frac{1}{e^{\beta(\varepsilon_n-\mu)} - \eta} = \int_0^\infty d\varepsilon \rho(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} - \eta} \quad ,$$

where $\rho(\varepsilon)$ is the density of single-particle states of energy ε, and $\eta = +1(-1)$ for bosons (fermions).

(a) Write a corresponding expression (in terms of $\rho(\varepsilon)$, β, and μ) for the total energy E of the system.

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$$E = \sum_n \frac{\varepsilon_n}{e^{\beta(\varepsilon_n-\mu)} - \eta} = \int_0^\infty d\varepsilon \rho(\varepsilon) \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)} - \eta} \quad .$$

(b) For bosons write an implicit (integral) equation whose solution gives the critical temperature for Bose condensation.

• At the onset of Bose condensation $\mu = \varepsilon_0 = 0$ (the lowest energy state), and since the occupation of this state is still not macroscopic, we have

$$N = \int_0^\infty d\varepsilon \rho(\varepsilon) \frac{1}{e^{\beta_c \varepsilon} - 1} \quad .$$

Solving this implicit integral equation leads to β_c and hence the condensation temperature T_c.

For any function $g(x)$, the Sommerfeld expansion indicates that as $\beta \to \infty$,

$$\int_0^\infty dx \frac{g(x)}{e^{\beta(x-\mu)} + 1} \simeq \int_0^\mu dx\ g(x) + \frac{\pi^2}{6\beta^2}\ g'(\mu) + \cdots \quad .$$

(c) Use the above expansion to express the low temperature behavior of $\mu - E_F$, where E_F is the Fermi energy, in terms of β, $\rho(E_F)$ and $\rho'(E_F)$.

• Performing the Sommerfeld expansion on the expression for N gives

$$N = \int_0^\mu dE\ \rho(E) + \frac{\pi^2}{6\beta^2}\ \rho'(\mu) + \cdots .$$

Since $\mu = E_F$ when $T = 0$ ($\beta \to \infty$), expanding μ around E_F to the lowest order yields

$$N = \int_0^{E_F} dE\ \rho(E) + (\mu - E_F)\rho(E_F) + \frac{\pi^2}{6\beta^2}\ \rho'(E_F) + \cdots .$$

The initial integral is simply N, since this is how E_F is determined at zero temperatures. Setting the rest of the series to zero gives

$$\mu - E_F = -\frac{\pi^2}{6\beta^2}\,\frac{\rho'(E_F)}{\rho(E_F)} + \cdots.$$

(d) As in the last part, find an expression for the increase in energy, $E(T) - E(T=0)$, at low temperatures.

- Applying the Sommerfeld expansion to the expression for E gives

$$E(T) = \int_0^{\mu} dE\ E\rho(E) + \frac{\pi^2}{6\beta^2}\,\left(\rho(\mu) + \mu\rho'(\mu)\right) + \cdots.$$

Expanding μ around E_F to the lowest order yields

$$E(T) = \int_0^{E_F} dE\ \rho(E)E + (\mu - E_F)\rho(E_F)E_F + \frac{\pi^2}{6\beta^2}\,\left(\rho(E_F) + E_f\rho'(E_F)\right) + \cdots.$$

The first integral defines $E(T=0)$, and using the expression for $\mu - E_F)$ from the previous part leads to

$$\begin{aligned} E(T) - E(T=0) &= -\frac{\pi^2}{6\beta^2}\,\frac{\rho'(E_F)}{\rho(E_F)} \times \rho(E_F)E_F + \frac{\pi^2}{6\beta^2}\,\left(\rho(E_F) + E_f\rho'(E_F)\right) + \cdots \\ &= \frac{\pi^2}{6\beta^2}\,\rho(E_F)\,. \end{aligned}$$

(e) Find the low temperature heat capacity of this system of fermions.

- The low temperature heat capacity is given by

$$C(T) = \frac{dE}{dT} = \frac{\pi^2}{3}k_B\rho(E_f)k_BT + \cdots\,,$$

i.e. it vanish linearly with a coefficient proportional to the density of states at the Fermi energy.

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4. *Quantum point particle condensation:* Consider a *quantum gas* of N spin-less point particles of mass m at temperature T, and volume V. An unspecified weak pairwise attraction between particles reduces the energy of any state by an amount $-uN^2/(2V)$ with $u > 0$, such that the partition function is

$$Z(T,N,V) = Z_0(T,N,V) \times \exp\left(\frac{\beta u N^2}{2V}\right),$$

where $Z_0(T,N,V)$ is the partition function of the ideal quantum gas, and $\beta = (k_BT)^{-1}$.

(a) Using the above relation between partition functions relate the pressure $P(n,T)$, as a function of the density $n = N/V$, to the corresponding pressure $P_0(n,T)$ of an ideal quantum gas.

- The pressure is related to the partition function by

$$\beta P = \frac{\partial \ln Z}{\partial V} = \frac{\partial \ln Z_0}{\partial V} - \frac{\beta u N^2}{2V^2}, \quad \Longrightarrow \quad P(n,T) = P_0(n,T) - \frac{un^2}{2}.$$

(b) Use standard results for the non-relativistic gas to show that

$$\left.\frac{\partial P}{\partial n}\right|_T = -un + k_BT\frac{f^\eta_{3/2}(z)}{f^\eta_{1/2}(z)}, \quad \text{with} \quad f^\eta_{3/2}(z) = n\lambda^3 \quad \text{and} \quad \lambda = \frac{h}{\sqrt{2\pi m k_B T}}.$$

- For a non-relativistic quantum gas, the pressure and density are related to the fugacity z by

$$P_0 = \frac{1}{\lambda^3} f^\eta_{5/2}(z) \quad \text{and} \quad n = \frac{1}{\lambda^3} f^\eta_{3/2}(z).$$

Taking derivatives of both expressions with respect to n at constant T yields

$$\left.\frac{\partial P_0}{\partial n}\right|_T = \frac{1}{\lambda^3 z} f^\eta_{3/2}(z) \left.\frac{\partial z}{\partial n}\right|_T \quad \text{and} \quad 1 = \frac{1}{\lambda^3 z} f^\eta_{1/2}(z) \left.\frac{\partial z}{\partial n}\right|_T.$$

Eliminating $\partial z/\partial n|_T$ between the above two expressions yields

$$\left.\frac{\partial P}{\partial n}\right|_T = -un + \left.\frac{\partial P_0}{\partial n}\right|_T = -un + k_BT\frac{f^\eta_{3/2}(z)}{f^\eta_{1/2}(z)}.$$

(c) Find the critical value of the coupling $u_c(n,T)$ at which the gas becomes unstable, in the low density (non-degenerate) limit $n\lambda^3 \ll 1$, including the *first* correction that distinguishes between fermi and bose statistics.

• Stability of the gas requires $\partial P/\partial n|_T > 0$, and thus instability sets in at the critical coupling

$$u_c = \frac{k_B T}{n} \frac{f^\eta_{3/2}(z)}{f^\eta_{1/2}(z)}.$$

To express the result in terms of density, we must solve for $z(n)$. Using the standard expansions of the functions $f^\eta_m(z)$, we find

$$n\lambda^3 = f^\eta_{3/2}(z) = z + \eta\frac{z^2}{2^{3/2}} + \cdots, \quad \Longrightarrow \quad z = n\lambda^3 - \eta\frac{(n\lambda^3)^2}{2^{3/2}} + \cdots.$$

Expanding the result for u_c,

$$u_c = \frac{k_B T}{n} \frac{z + \eta z^2/2^{3/2} + \cdots}{z + \eta z^2/2^{1/2} + \cdots} = \frac{k_B T}{n}\left[1 - \eta\frac{z}{2^{3/2}} + \cdots\right],$$

we observe that the first order correction in z already distinguishes between fermions and bosons, and thus to this order

$$u_c = \frac{k_B T}{n}\left[1 - \eta\frac{n\lambda^3}{2^{3/2}} + \cdots\right].$$

The critical coupling is diminished for bosons and increased for fermions.

(d) For fermions, relate the limiting behavior of $u_c(n, T)$ in the low temperature (degenerate limit $n\lambda^3 \gg 1$) to the fermi energy ϵ_F. (This is somewhat similar to the Chandrashekar instability of neutron stars.)

• In the degenerate limit for fermions $z \gg 1$, and expansions of $f^-_m(z)$ in this limit yield

$$u_c = \frac{k_B T}{n} \frac{(\ln z)^{3/2}/(3/2)!}{(\ln z)^{1/2}/(1/2)!} = \frac{k_B T}{n}\frac{\ln z}{3/2} = \frac{2}{3}\frac{\epsilon_F}{n},$$

since in this limit $\ln z = \beta\epsilon_F$.

(e) What happens to u_c for bosons as temperature is decreased towards to quantum degenerate regime?

• On approaching the condensation point of the ideal bose gas, the fugacity $z \to 1$, $f^+3/2(z) \to \zeta_{3/2}$, $f^+1/2(z) \to \infty$, and thus $u_c \to 0$.

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5. *Harmonic confinement of Fermions:* A classical gas of fermions of mass m is confined in a d-dimensional anisotropic harmonic potential

$$U(\vec{r}) = \frac{m}{2}\sum_\alpha \omega_\alpha^2 x_\alpha^2\,,$$

with different restoring frequencies $\{\omega_\alpha\}$ along the different directions. We are interested in the limit of wide traps such that $\hbar\omega_\alpha \ll k_B T$, and the discreteness of the allowed energies can be ignored.

(a) Show that in this limit, the number of states $N(E)$ with energy less than or equal to E, and the density of states $\rho(E)$, are respectively given by

$$N(E) = \frac{1}{d!}\prod_{\alpha=1}^{d}\left(\frac{E}{\hbar\omega_\alpha}\right), \qquad \text{and} \qquad \rho(E) = \frac{1}{(d-1)!}\frac{E^{d-1}}{\prod_\alpha \hbar\omega_\alpha}.$$

- The energy of the quantized harmonic oscillator is given by

$$E = \sum_\alpha \hbar\omega_\alpha\left(n_\alpha + \frac{1}{2}\right).$$

Ignoring the effects of discreteness (and ground state energies), states with energy less than or equal to E are confined within a hyper-pyramid with sides of length $\mathcal{N}_\alpha = E/(\hbar\omega_\alpha)$. The number of states is simply the volume of this pyramid, which in d dimensions is given by

$$N(E) \approx \frac{1}{d!}\prod_{\alpha=1}^{d}\mathcal{N}_\alpha = \frac{1}{d!}\prod_{\alpha=1}^{d}\left(\frac{E}{\hbar\omega_\alpha}\right).$$

The density of states is the derivative of this expression, and hence

$$\rho(E) = \frac{dN(E)}{dE} = \frac{1}{(d-1)!}\frac{E^{d-1}}{\prod_\alpha \hbar\omega_\alpha}.$$

(b) Show that in a grand canonical ensemble, the number of particles in the trap is

$$\langle N\rangle = f_d^-(z)\prod_\alpha\left(\frac{k_BT}{\hbar\omega_\alpha}\right).$$

- Ignoring discreteness effects, the number of particles is given by

$$\langle N\rangle = \sum_{\{n_\alpha\}}\frac{1}{z^{-1}e^{\beta E}+1} \approx \int_0^\infty \frac{\rho(E)dE}{z^{-1}e^{\beta E}+1} = \frac{1}{\prod_\alpha \hbar\omega_\alpha}\frac{1}{(d-1)!}\int_0^\infty \frac{dE\ E^{d-1}}{z^{-1}e^{\beta E}+1}.$$

After the change of variables to $x = \beta E$, we are left with a standard integral, and

$$\langle N\rangle = \prod_\alpha\left(\frac{k_BT}{\hbar\omega_\alpha}\right) f_d^-(z).$$

(c) Compute the energy of E in the grand canonical ensemble. (Ignore the zero point energy of the oscillators.)

• Ignoring the ground state contribution, the amount of energy in the trap is given by

$$E = \langle \mathcal{H} \rangle = \sum_{\{n_\alpha\}} \frac{E}{z^{-1}e^{\beta E}+1} \approx \int_0^\infty \frac{\rho(E)EdE}{z^{-1}e^{\beta E}+1} = \frac{1}{\prod_\alpha \hbar\omega_\alpha} \frac{1}{(d-1)!} \int_0^\infty \frac{dE\ E^d}{z^{-1}e^{\beta E}+1}.$$

After the change of variables to $x = \beta E$, we are left with a standard integral, and

$$E = \prod_\alpha \left(\frac{k_BT}{\hbar\omega_\alpha}\right) dk_BTf_{d+1}^{-}(z).$$

(d) From the limiting forms of the expressions for energy and number, compute the leading term for energy per particle in the high temperature limit.

• In the high temperature limit, $f_d^-(z) \approx z$, and hence

$$N = \prod_\alpha \left(\frac{k_BT}{\hbar\omega_\alpha}\right) z, \quad \text{while} \quad E = \prod_\alpha \left(\frac{k_BT}{\hbar\omega_\alpha}\right) dk_BTz.$$

Dividing the two expressions, we find the usual results of $E/N = dk_BT$.

(e) Compute the limiting value of the chemical potential at zero temperature.

• Using the Sommerfeld expansion, the number of particles can be written as

$$N = \frac{1}{(\beta\epsilon)^d} \frac{(\ln z)^d}{d!} \left[1 + \frac{\pi^2}{6} \frac{d(d-1)}{(\ln z)^2} + \cdots\right],$$

where we have introduced the energy scale $\epsilon = (\prod_\alpha \hbar\omega_\alpha)^{1/d}$. Inverting this expression, we find

$$\beta\epsilon_F \equiv \ln z = \beta\epsilon \left(d!N\right)^{1/d} \left[1 - \frac{\pi^2}{6} \frac{(d-1)}{(\ln z)^2} + \cdots\right] = \beta\epsilon_F \left[1 - \frac{\pi^2}{6} \frac{(d-1)}{(\beta\epsilon_F)^2} + \cdots\right],$$

where we have defined a Fermi energy

$$\epsilon_F = \lim_{T\to 0} k_BT \ln z = \left(d!N \prod_\alpha \hbar\omega_\alpha\right)^{1/d}.$$

(f) Give the expression for the heat capacity of the gas at low temperatures, correct up to numerical factors that you need not compute.

• Since the confined gas has a finite Fermi energy, its specific heat must scale as usual, as

$$\frac{C}{Nk_B} = a\frac{k_BT}{\epsilon_F},$$

where ϵ_F computed above. The numerical coefficient a is obtained from an analysis that incorporates the subleading correction in the Sommerfeld expansion, as

$$\beta E = \frac{d}{(\beta\epsilon)^d}\frac{(\ln z)^{d+1}}{(d+1)!}\left[1 + \frac{\pi^2}{6}\frac{(d+1)d}{(\ln z)^2} + \cdots\right].$$

Dividing by the similar expansion for N, we obtain

$$\frac{\beta E}{N} = \frac{d\ln z}{(d+1)}\left[1 + \frac{\pi^2}{6}\frac{2d}{(\ln z)^2} + \cdots\right].$$

Substituting the value of $\ln z$ from the earlier part gives

$$\frac{\beta E}{N} = \frac{d\beta\epsilon_F}{(d+1)}\left[1 + \frac{\pi^2}{6}\frac{d+1}{(\beta\epsilon_F)^2} + \cdots\right].$$

The heat capacity is then obtained as

$$\frac{C}{N} = \frac{1}{N}\frac{dE}{dT} = \frac{d\epsilon_F}{(d+1)}\frac{\pi^2}{6}(d+1)\frac{2k_B^2T}{\epsilon_F^2} + \cdots = k_B\left[d\frac{\pi^2}{3}\frac{k_BT}{\epsilon_F} + \cdots\right].$$

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6. *Anharmonic trap:* A collection of *non-interacting identical* particles is placed in an *anharmonic* trap, with one-particle Hamiltonian

$$H_1 = \frac{p^2}{2m} + Kr^n,$$

where r is the radial distance (in 3 dimensions) from the center of the trap.

(a) Show that the one particle density of state can be written as $\rho(\varepsilon) = \frac{C}{(p-1)!}\varepsilon^{p-1}$, where $p = 3/2 + 3/n$ and C is an amplitude *that you do not need to evaluate.*

• The surface of constant one particle energy ε is a covex shape in the six-dimensional phase space that extends to $p_{max} = (2m\varepsilon)^{1/2}$ in the momentum direction, and $r_{max} = (\varepsilon/K)^{1/n}$ in the radial coordinate. The volume of this shape in 6 dimensions is $\Omega(\varepsilon) = g(p_{max}r_{max})^3$, where g is a dimensionless constant. Thus $\Omega(\varepsilon) \propto \varepsilon^p$ with $p = 3(1/2 + 1/n)$. The density of space is then obtained from $\rho(\varepsilon) = d\Omega/d\varepsilon$, and can be written as $\rho(\varepsilon) = \frac{C}{(p-1)!}\varepsilon^{p-1}$ with $C = gp!(2m)^{3/2}K^{-3/n}$.

(b) At high densities, quantization of one particle energy levels, $\{\varepsilon_j\}$, can be ignored, such that in a grand canonical ensemble at temperature $T = (k_B\beta)^{-1}$ and chemical potential μ, the number of particles is given by

$$N = \sum_j \frac{1}{e^{\beta(\varepsilon_j-\mu)} - \eta} = \int_0^\infty d\varepsilon \rho(\varepsilon) \frac{1}{e^{\beta(\varepsilon-\mu)} - \eta} \quad ,$$

with $\eta = +1(-1)$ for bosons (fermions). Give the expression for N in terms of the functions $f_m^\eta(z)$ with $z = e^{\beta\mu}$.

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$$N = \int_0^\infty d\varepsilon \frac{C\varepsilon^{p-1}}{(p-1)!} \frac{1}{e^{\beta(\varepsilon-\mu)} - \eta} \quad .$$

Changing variables to $x = \beta\varepsilon$ (i.e. setting $\varepsilon = k_B T x$) gives

$$N = C(k_BT)^p \int_0^\infty \frac{dx}{(p-1)!} \frac{x^{p-1}}{z^{-1}e^x - \eta} \quad = C(k_BT)^p f_p^\eta(z)\,.$$

(c) Obtain the corresponding expression for the total energy E of the system.

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$$E = \sum_n \frac{\varepsilon_n}{e^{\beta(\varepsilon_n-\mu)} - \eta} = \int_0^\infty d\varepsilon \rho(\varepsilon) \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)} - \eta} = \int_0^\infty d\varepsilon \frac{C\varepsilon^{p-1}}{(p-1)!} \frac{\varepsilon}{e^{\beta(\varepsilon-\mu)} - \eta} \,.$$

Setting $\varepsilon = k_B T x$ now results in

$$E = C(k_BT)^{p+1} \int_0^\infty \frac{dx}{(p-1)!} \frac{x^p}{z^{-1}e^x - \eta} \quad pC(k_BT)^{p+1} f_{p+1}^\eta(z)\,.$$

(d) For a gas of N fermions, find the Fermi energy $E_F = \lim_{T\to 0} \mu$.

• Using the Summerfeld expansion, we find

$$N = C(k_BT)^p \frac{(\ln z)^p}{p!} \left[1 + \frac{\pi^2}{6} \frac{p(p+1)}{\ln^2 z} + \cdots\right] .$$

The leading asymptotic term leads to

$$E_F = \lim_{T\to 0} (k_BT \ln z) = \left(\frac{p!N}{C}\right)^{1/p} \quad .$$

(e) *Without explicit calculation* state the behavior of heat capacity for this Fermi gas at temperatures $T \ll E_F/k_B$.

• For $T \ll E_F/k_B$ only particles within a distance k_BT of the Fermi energy are excited. As the excited fraction is k_BT/E_F, (up to a proportionality constant) the heat capacity behaves as

$$C \propto k_B N \frac{k_B T}{E_F} \,.$$

(f) For bosons find the expression for the heat capacity C close to zero temperature.

• Since $p > 1$, the Bose gas will undergo Bose-Einstein condensation at some temperature T_c, such that $z = 1$ close to zero temperature. The expression for energy thus simplifies to $E = pC(k_BT)^{p+1}\zeta_{p+1}$, where ζ_{p+1} is the zeta-function at $(p+1)$. The heat capacity is then obtained as

$$C = \frac{dE}{dT} = k_B p(p+1)C(k_BT)^p \zeta_{p+1} \,.$$

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7. (Optional) *Fermi gas in two dimensions:* Consider a non-relativistic gas of non-interacting spin 1/2 fermions of mass m in two dimensions.

(a) Find an explicit relation between the fugacity z and the areal density $n = N/A$. (If needed, note that $f_1^-(z) = \ln(1+z)$.)

• Using the Fermi occupation number, the total number of particles in the grand-canonical ensemble ia

$$N = 2\sum_{\vec{k}} < n(\vec{k}) >_- = 2A \int \frac{d^2p}{h^2} \frac{1}{z^{-1}e^{\frac{\beta p^2}{2m}} + 1} = 2A \int_0^\infty \frac{2\pi}{h^2} \frac{pdp(\beta/m)}{z^{-1}e^{\frac{\beta p^2}{2m}} + 1} mk_BT \,.$$

After a change of variable to $x = \beta p^2/2m$, the integral is easily evaluated to yield

$$n = \frac{N}{A} = \frac{4\pi m k_B T}{h^2} \int_0^\infty dx \frac{ze^{-x}}{1 + ze^{-x}} = \frac{2}{\lambda^2} \ln(1+z) \,.$$

(b) Give an explicit expression for the chemical potential $\mu(n,T)$, and provide its limiting forms at zero and high temperatures.

• From the previous result we find

$$z = e^{\beta\mu} = e^{n\lambda^2/2} - 1 \qquad \Longrightarrow \qquad \mu = k_B T \ln[e^{n\lambda^2/2} - 1]\,.$$

In the high temperature limit, $n\lambda^2 \to 0$, and $\mu \approx k_B T \ln(n\lambda^2/2)$. In the low temperature limit $n\lambda^2 \to \infty$, and $\mu \approx k_B T n\lambda^2/2 = \frac{nh^2}{4\pi m}$.

(c) Find the temperature at which $\mu = 0$.

• Setting $\mu = 0$ and $z = 1$ yields

$$n = \frac{4\pi m k_B T_0}{h^2} \ln 2 \qquad \Longrightarrow \qquad k_B T_0 = \frac{nh^2}{4\pi \ln 2m}\,.$$

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8. (Optional) *Partitions of Integers:* In mathematics, the partition $P(E)$ of an integer E refers to the number of ways the integer can be written as sums of smaller integers. For example, $P(5) = 7$ since 5= [5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1]. A celebrated result due to Hardy and Ramanujan is that asymptotically at large E,

$$P(E) \sim \frac{1}{4E\sqrt{3}} \exp\left(\pi\sqrt{\frac{2E}{3}}\right)\,.$$

The leading asymptotic dependence can in fact be obtained by considering the statistical mechanics of a gas of 'photons' in one dimension. Working in a system of units such that $k_B = \hbar = c = 1$, the single particle energies of such a gas are integers, i.e. $\epsilon_k = k$ for $k = 1, 2, 3, \cdots$, and

$$E = \sum_{k=1}^{\infty} k n_k \quad .$$

(a) Compute a 'partition function' $Z(\beta)$ in the limit $\beta \to 0$, after replacing sums over k with integrals. (Note that the number of 'photons' is arbitrary.)

• Since there is no constraint on the number of photons, the partition function is

$$Z(\beta) = \sum_{\{n_k\}} e^{-\beta \sum_k k n_k} = \prod_{k=1}^{\infty} \sum_{n_k=0}^{\infty} e^{-\beta k n_k} = \prod_{k=1}^{\infty} [1 - e^{-\beta k}]^{-1}.$$

Using the approximation,

$$\sum_{k=1}^{\infty} f(\beta k) = \sum_{k=0}^{\infty} f(\beta(k+1)) \simeq \int_0^{\infty} dk f(\beta + \beta k) + \frac{1}{2} f(\beta) + \cdots,$$

$\ln Z$ can be expanded as

$$\ln Z(\beta) = -\sum_{k=1}^{\infty} \ln\left(1 - e^{-\beta k}\right) = -\int_0^{\infty} dk \ln\left[1 - e^{-\beta(k+1)}\right] + \frac{1}{2}\ln\left[1 - e^{-\beta}\right] + \cdots.$$

The first integral can be evaluated through integration by parts, and

$$\begin{aligned}
\ln Z(\beta) &= +\int_0^{\infty} dk k \frac{\beta e^{-\beta(k+1)}}{1 - e^{-\beta(k+1)}} + \frac{1}{2}\ln\left[1 - e^{-\beta}\right] + \cdots \\
&= \int_0^{\infty} dk k \frac{\beta}{e^{\beta} e^{\beta k} - 1} + \frac{1}{2}\ln\left[1 - e^{-\beta}\right] + \cdots \\
&= \frac{1}{\beta}\int_0^{\infty} \frac{x dx}{e^{\beta} e^{x} - 1} + \frac{1}{2}\ln\left[1 - e^{-\beta}\right] + \cdots \\
&= \frac{1}{\beta} f_2^{+}(e^{-\beta}) + \frac{1}{2}\ln\left[1 - e^{-\beta}\right] + \cdots \\
&= \frac{1}{\beta}\zeta_2 + \frac{1}{2}\ln\beta + \cdots,
\end{aligned}$$

with the last line obtain by taking the limit of $\beta \to 0$ in the arguments of the two functions.

(b) Compute the average energy, and use the result to find $T \equiv 1/\beta$ as a function of E, to leading order for $E \gg 1$.

- The average energy is obtained as

$$E = -\frac{d\ln Z}{d\beta} = \frac{\zeta_2}{\beta^2} - \frac{1}{2\beta} + \cdots.$$

Note that for $E \gg 1$, $\beta \ll 1$, and to leading order

$$T = \frac{1}{\beta} = \sqrt{\frac{E}{\zeta_2}}.$$

(c) Compute the entropy $S(E)$ for $E \gg 1$. Is your result consistent with the Hardy–Ramanujan formula for partitions of integers?

• Since in the thermodynamic limit $\ln Z = -\beta(E - TS)$, we have

$$S(E) = \ln Z + \beta E = \frac{\zeta_2}{\beta} + \frac{1}{2}\ln\beta + \frac{\zeta_2}{\beta} + \cdots = \frac{2\zeta_2}{\beta} + \frac{1}{2}\ln\beta + \cdots = 2\sqrt{\zeta_2 E} + \frac{1}{2}\ln\sqrt{\frac{\zeta_2}{E}} + \cdots .$$

Since $\zeta_2 = \pi^2/6$, we find

$$S(E) = \pi\sqrt{\frac{2E}{3}} - \frac{1}{4}\ln E + \mathcal{O}(E^0).$$

While the leading order result for $S(E)$ agrees with the Hardy–Ramanujan formula, the subleading correction computed here, $-\ln E/4$ is distinct from the correct result of $-\ln E$. To obtain the latter exactly, one has to evaluate $\Omega(E)$ from the inverse Laplace transform of $Z(\beta)$ including the range of integration in β in addition to the saddle-point value of the integrand.

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9. (Optional) *Fermions pairing into Bosons:* As a primitive model of superconductivity, consider a gas of non-interacting non-relativistic electrons (spin–1/2 and mass m). Assume that electrons of opposite spin can bind into a composite boson (spin 0, mass $2m$) of rest energy $-\epsilon$ with $\epsilon > 0$.

(a) In a grand canonical ensemble with chemical potential μ *for each electron*, write down the expressions for the densities n_e and n_b of free electrons, and bound electron pairs. (You do not need to derive the relevant expressions. Express your answer in terms of the fugacity $z = e^{\beta\mu}$, $y = e^{\beta\epsilon}$, and the electron thermal wavelength $\lambda = h/\sqrt{2\pi m k_B T}$.)

• Using standard formulae for the grand canonical ensemble, and the definitions provided in the problem, the electron density is given by

$$n_e = \frac{2}{\lambda^3} f^{-}_{3/2}(z)\,.$$

To create a bound pair, two electrons are needed (hence 2μ), resulting in a composite object with single particle energies $\varepsilon(\vec{k}) = -\epsilon + (\hbar k)^2/(4m)$ (since mass is $2m$). Due to doubling of mass, the thermal wavelength for the composite boson is $\lambda_b = \lambda/\sqrt{2}$. Including the binding energy, the effective fugacity of the bosons is $z_b = z^2 y$, resulting in

$$n_b = \frac{2^{3/2}}{\lambda^3} f^{+}_{3/2}(z^2 y)\,.$$

(b) What is the value of $z = z_c$ at the onset of Bose condensation? Write the expression for the total (bound plus unbound) electron density n_c at the onset of Bose condensation.

• Bose condensation occurs when the argument of $f^+_{3/2}$ is unity, which occurs for

$$z_c = y^{-1/2} = e^{-\beta\epsilon/2} .$$

Summing the expressions for n_e and n_b from the previous part (including the factor of two for bound pairs), gives

$$n_c = n_e(z_c) + 2n_b(z_c) = \frac{2}{\lambda^3}\left[f^-_{3/2}(y^{-1/2}) + 2^{3/2}\zeta_{3/2}\right] .$$

(c) Give the expression for pressure P in the condensed phase ($n > n_c$).

• Throughout the condensate phase, $z = z_c$. As the condensed fraction exerts no pressure, the net pressure is the sum of contributions from free electrons $\beta P_e = 2f^-_{5/2}(z_c)/\lambda^3$, and bound (non-condensate) electrons $\beta P_b = 2^{3/2} f^+_{5/2}(1)/\lambda^3$, leading to

$$P = \frac{1}{\beta\lambda^3}\left[2f^-_{5/2}(y^{-1/2}) + 2^{3/2}\zeta_{5/2}\right] .$$

(d) Find the dimensionless ratio $(\beta P_c/n_c)$ at the *onset of condensation*. Evaluate its limiting values for $\beta\epsilon \to \infty$ and $\beta\epsilon \to 0$.

• Dividing expressions obtained in the previous parts gives

$$\frac{\beta P_c}{n_c} = \frac{2f^-_{5/2}(y^{-1/2}) + 2^{3/2}\zeta_{5/2}}{2f^-_{3/2}(y^{-1/2}) + 2^{5/2}\zeta_{3/2}} = \frac{f^-_{5/2}(y^{-1/2}) + 2^{1/2}\zeta_{5/2}}{f^-_{3/2}(y^{-1/2}) + 2^{3/2}\zeta_{3/2}} .$$

In the limit of $\beta\epsilon \to \infty$, $y^{-1/2} \to 0$, and since $f^-_m(0) = 0$, we find

$$\frac{\beta P_c}{n_c} = \frac{\zeta_{5/2}}{\zeta_{3/2}} .$$

In the limit of $\beta\epsilon \to 0$, $y^{-1/2} \to 1$, and since $f^-_m(1) = \left(1 - 2^{1-m}\right)\zeta_m$, we find

$$\frac{\beta P_c}{n_c} = \frac{\zeta_{5/2}}{\zeta_{3/2}} \cdot \frac{1 - 2^{-3/2} + 2^{1/2}}{1 - 2^{-1/2} + 2^{3/2}} = \frac{\zeta_{5/2}}{\zeta_{3/2}} \cdot \frac{3 + 2\sqrt{2}}{6 + 2\sqrt{2}} .$$

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10. (Optional) *Ring diagrams mimicking bosons:* Motivated by the statistical attraction between bosons, consider a *classical* system of identical particles, interacting with a pairwise potential $V(|\vec{q}-\vec{q}\,'|)$, such that

$$f(\vec{r}) = e^{-\beta V(r)} - 1 = \exp\left(-\frac{\pi r^2}{\lambda^2}\right) \quad , \text{ and } \quad \tilde{f}(\vec{\omega}) = \lambda^3 \exp\left(-\frac{\lambda^2 \omega^2}{4\pi}\right) ,$$

where $\tilde{f}(\vec{\omega})$ is the Fourier transform of $f(\vec{r})$.

(a) In a perturbative cluster expansion of the partition function, we shall retain only the diagrams forming a ring, which (after a summation over all powers of V between any pair of points) are proportional to

$$R_\ell = \int \frac{d^3\vec{q}_1}{V} \cdots \frac{d^3\vec{q}_\ell}{V} f(\vec{q}_1 - \vec{q}_2) f(\vec{q}_2 - \vec{q}_3) \cdots f(\vec{q}_\ell - \vec{q}_1).$$

Use properties of Fourier transforms to show that

$$R_\ell = \frac{1}{V^{\ell-1}} \int \frac{d^3\vec{\omega}}{(2\pi)^3} \tilde{f}(\vec{\omega})^\ell = \frac{\lambda^{3\ell}}{\ell^{3/2} \lambda^3 V^{\ell-1}} .$$

- The contribution of the ring diagrams to the partition function is

$$\begin{aligned} R_\ell &= \int \frac{d^3\vec{q}_1}{V} \frac{d^3\vec{q}_2}{V} \cdots \frac{d^3\vec{q}_\ell}{V} f(\vec{q}_1 - \vec{q}_2) f(\vec{q}_2 - \vec{q}_3) \cdots f(\vec{q}_\ell - \vec{q}_1) \\ &= \frac{1}{V^\ell} \int \cdots \int d^3\vec{x}_1 d^3\vec{x}_2 \cdots d^3\vec{x}_{\ell-1} d^3\vec{q}_\ell f(\vec{x}_1) f(\vec{x}_2) \cdots f(\vec{x}_{\ell-1}) f\left(-\sum_{i=1}^{\ell-1} \vec{x}_i\right), \end{aligned}$$

where we introduced the new set of variables $\{\vec{x}_i \equiv \vec{q}_i - \vec{q}_{i+1}\}$, for $i = 1, 2, \cdots, \ell - 1$. Note that since the integrand is independent of $\vec{q}_\ell$,

$$R_\ell = \frac{1}{V^{\ell-1}} \int \cdots \int d^3\vec{x}_1 d^3\vec{x}_2 \cdots d^3\vec{x}_{\ell-1} f(\vec{x}_1) f(\vec{x}_2) \cdots f\left(-\sum_{i=1}^{\ell-1} \vec{x}_i\right).$$

Using the inverse Fourier transform

$$f(\vec{q}) = \frac{1}{(2\pi)^3} \int d^3\vec{\omega}\ \tilde{f}(\vec{\omega}) e^{-i\vec{q}\cdot\vec{\omega}},$$

the integral becomes

$$\begin{aligned} R_\ell = \frac{1}{(2\pi)^{3\ell} V^{\ell-1}} \int \cdots \int d^3\vec{x}_1 \cdots d^3\vec{x}_{\ell-1} \tilde{f}(\vec{\omega}_1) e^{-i\vec{\omega}_1\cdot\vec{x}_1} \tilde{f}(\vec{\omega}_2) e^{-i\vec{\omega}_2\cdot\vec{x}_2} \\ \cdots \tilde{f}(\vec{\omega}_\ell) \exp\left(-i \sum_{k=1}^{\ell-1} \vec{\omega}_\ell \cdot \vec{x}_k\right) d^3\vec{\omega}_1 \cdots d^3\vec{\omega}_\ell. \end{aligned}$$

Since

$$\int \frac{d^3\vec{q}}{(2\pi)^3} e^{-i\vec{\omega}\cdot\vec{q}} = \delta^3(\vec{\omega}\,),$$

we have

$$R_\ell = \frac{1}{(2\pi)^3 V^{\ell-1}} \int \cdots \int \left(\prod_{k=1}^{\ell-1} \delta(\vec{\omega}_k - \vec{\omega}_\ell) \tilde{f}(\vec{\omega}_k) d^3\vec{\omega}_k \right) d^3\vec{\omega}_\ell,$$

resulting finally in

$$R_\ell = \frac{1}{V^{\ell-1}} \int \frac{d^3\vec{\omega}}{(2\pi)^3} \tilde{f}(\vec{\omega})^\ell = \frac{1}{V^{\ell-1}} \int \frac{d^3\vec{\omega}}{(2\pi)^3} \lambda^{3\ell} \exp\left(-\frac{\ell\lambda^2\omega^2}{4\pi}\right) = \frac{\lambda^{3\ell}}{\ell^{3/2}\lambda^3 V^{\ell-1}}\,.$$

Note that there is no ring diagram for $\ell = 2$. Including the multiplicity of ring diagrams, and the special case of $\ell = 2$, the expansion for the partition function can be written as

$$\ln Z_{\text{rings}} = \ln Z_0 + \frac{N^2\lambda^3}{2V} + \sum_{\ell=3}^{\infty} \frac{N^\ell}{2\ell} R_\ell\,.$$

(You do not have to prove this expression.)

(b) Show that in the ring approximation, the partition function is given by

$$\ln Z_{\text{rings}} = \ln Z_0 + \frac{V}{2\lambda^3} f^+_{5/2}(n\lambda^3) - \frac{N}{2} + \frac{N^2\lambda^3}{2V}(1 - 2^{-5/2})\,,$$

where Z_0 is the partition function of the non-interacting gas, and $n = N/V$ is the number density.

- Inserting the result for R_ℓ in the expansion for the partition function yields

$$\begin{aligned}
\ln Z_{\text{rings}} &= \ln Z_0 + \frac{N^2\lambda^3}{2V} + \sum_{\ell=3}^{\infty} \frac{N^\ell}{2\ell} \frac{\lambda^{3\ell}}{\ell^{3/2}\lambda^3 V^{\ell-1}} \\
&= \ln Z_0 + \frac{N^2\lambda^3}{2V} + \frac{V}{2\lambda^3} \sum_{\ell=3}^{\infty} \frac{(n\lambda^3)^\ell}{\ell^{5/2}} \\
&= \ln Z_0 + \frac{N^2\lambda^3}{2V} + \frac{V}{2\lambda^3} \left[f^+_{5/2}(n\lambda^3) - n\lambda^3 - \frac{(n\lambda^3)^2}{2^{5/2}} \right] \\
&= \ln Z_0 + \frac{V}{2\lambda^3} f^+_{5/2}(n\lambda^3) - \frac{N}{2} + \frac{N^2\lambda^3}{2V}(1 - 2^{-5/2})\,.
\end{aligned}$$

The missing terms for $\ell = 1$ and $\ell = 2$ are added to make the series correspond to that of the function $f^+_{5/2}$.

(c) Compute the pressure P of the gas within the ring approximation.

• Since $\beta P = \partial \ln Z/\partial V$, and using the standard result for the ideal gas, we obtain

$$\beta P = \frac{\partial \ln Z}{\partial V} = n + \frac{1}{2\lambda^3} f^+_{5/2}(n\lambda^3) + \frac{V}{2\lambda^3}\left(\frac{-N\lambda^3}{V^2}\right)\frac{f^+_{3/2}(n\lambda^3)}{n\lambda^3} - \frac{N^2\lambda^3}{2V^2}(1-2^{-5/2}),$$

where we have taken advantage of $df^+_m(x)/dx = f^+_{m-1}(x)/x$, and thus

$$\beta P = n\left[1 + \frac{1}{2n\lambda^3}\left(f^+_{5/2}(n\lambda^3) - f^+_{3/2}(n\lambda^3)\right) - (1-2^{-5/2})\frac{n\lambda^3}{2}\right].$$

(d) By examining the compressibility, or equivalently $\partial P/\partial n|_T$, show that this classical system of interacting particles must undergo a condensation transition.

• Taking a derivative of the expression for P at constant T (and hence constant β and λ), we find

$$\beta \left.\frac{\partial P}{\partial n}\right|_T = 1 + \frac{1}{2n\lambda^3}\left[f^+_{3/2}(n\lambda^3) - f^+_{1/2}(n\lambda^3)\right] - (1-2^{-5/2})n\lambda^3.$$

As density increases towards the point $n\lambda^3 = 1$, the term proportional to $f^+_{1/2}(n\lambda^3)$ becomes highly negative (and divergent). The compressibility $\partial V/\partial P = -\frac{N}{n^2}\partial n/\partial V$ thus becomes positive indicating a mechanical instability. The gas must thus undergo a phase transition at a point $n\lambda^3 < 1$ to avoid this instability.

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11. (Optional) *Relativistic Bose gas in d dimensions:* Consider a gas of non-interacting (spinless) bosons with energy $\epsilon = c\,|\vec{p}|$, contained in a box of "volume" $V = L^d$ in d dimensions.

(a) Calculate the grand potential $\mathcal{G} = -k_B T \ln \mathcal{Q}$, and the density $n = N/V$, at a chemical potential μ. Express your answers in terms of d and $f^+_m(z)$, where $z = e^{\beta\mu}$, and

$$f^+_m(z) = \frac{1}{(m-1)!}\int_0^\infty \frac{x^{m-1}}{z^{-1}e^x - 1}dx.$$

(Hint: Use integration by parts on the expression for $\ln \mathcal{Q}$.)

• We have

$$\begin{aligned}\mathcal{Q} &= \sum_{N=0}^{\infty} e^{N\beta\mu} \sum_{\{n_i\}}^{\sum_i n_i = N} \exp\left(-\beta\sum_i n_i\epsilon_i\right)\\ &= \prod_i \sum_{\{n_i\}} e^{\beta(\mu-\epsilon_i)n_i} = \prod_i \frac{1}{1-e^{\beta(\mu-\epsilon_i)}}\end{aligned},$$

whence $\ln \mathcal{Q} = -\sum_i \ln\left(1 - e^{\beta(\mu-\epsilon_i)}\right)$. Replacing the summation $\sum_i$ with a d dimensional integration $\int_0^\infty V d^d k/(2\pi)^d = \left[V S_d/(2\pi)^d\right] \int_0^\infty k^{d-1} dk$, where $S_d = 2\pi^{d/2}/(d/2-1)!$, leads to

$$\ln \mathcal{Q} = -\frac{V S_d}{(2\pi)^d} \int_0^\infty k^{d-1} dk \ln\left(1 - z e^{-\beta\hbar c k}\right).$$

The change of variable $x = \beta\hbar c k$ results in

$$\ln \mathcal{Q} = -\frac{V S_d}{(2\pi)^d} \left(\frac{k_B T}{\hbar c}\right)^d \int_0^\infty x^{d-1} dx \ln\left(1 - z e^{-x}\right).$$

Finally, integration by parts yields

$$\ln \mathcal{Q} = \frac{V S_d}{(2\pi)^d} \frac{1}{d} \left(\frac{k_B T}{\hbar c}\right)^d \int_0^\infty x^d dx \frac{z e^{-x}}{1 - z e^{-x}} = V \frac{S_d}{d} \left(\frac{k_B T}{hc}\right)^d \int_0^\infty dx \frac{x^d}{z^{-1} e^x - 1},$$

leading to

$$\mathcal{G} = -k_B T \ln \mathcal{Q} = -V \frac{S_d}{d} \left(\frac{k_B T}{hc}\right)^d k_B T d! f_{d+1}^+(z),$$

which can be somewhat simplified to

$$\mathcal{G} = -k_B T \frac{V}{\lambda_c^d} \frac{\pi^{d/2} d!}{(d/2)!} f_{d+1}^+(z),$$

where $\lambda_c \equiv hc/(k_B T)$. The average number of particles is calculated as

$$N = -\frac{\partial \mathcal{G}}{\partial \mu} = -\beta z \frac{\partial \mathcal{G}}{\partial z} = \frac{V}{\lambda_c^d} \frac{\pi^{d/2} d!}{(d/2)!} f_d^+(z),$$

where we have used $z \partial f_{d+1}(z)/\partial z = f_d(z)$. Dividing by volume, the density is obtained as

$$n = \frac{1}{\lambda_c^d} \frac{\pi^{d/2} d!}{(d/2)!} f_d^+(z).$$

(b) Calculate the gas pressure P, its energy E, and compare the ratio $E/(PV)$ to the classical value.

• We have $PV = -\mathcal{G}$, while

$$E = -\left.\frac{\partial \ln \mathcal{Q}}{\partial \beta}\right|_z = +d \frac{\ln \mathcal{Q}}{\beta} = -d\mathcal{G}.$$

Thus $E/(PV) = d$, identical to the classical value for a relativistic gas.

(c) Find the critical temperature, $T_c(n)$, for Bose-Einstein condensation, indicating the dimensions where there is a transition.

• The critical temperature $T_c(n)$ is given by

$$n = \frac{1}{\lambda_c^d}\frac{\pi^{d/2}d!}{(d/2)!}f_d^+(z=1) = \frac{1}{\lambda_c^d}\frac{\pi^{d/2}d!}{(d/2)!}\zeta_d.$$

This leads to

$$T_c = \frac{hc}{k_B}\left(\frac{n(d/2)!}{\pi^{d/2}d!\zeta_d}\right)^{1/d}.$$

However, ζ_d is finite only for $d > 1$, and thus a transition exists for all $d > 1$.

(d) What is the temperature dependence of the heat capacity $C(T)$ for $T < T_c(n)$?

• At $T < T_c$, $z = 1$ and $E = -d\mathcal{G} \propto T^{d+1}$, resulting in

$$C(T) = \left.\frac{\partial E}{\partial T}\right|_{z=1} = (d+1)\frac{E}{T} = -d(d+1)\frac{\mathcal{G}}{T} = d(d+1)\frac{V}{\lambda_c^d}k_B\frac{\pi^{d/2}d!}{(d/2)!}\zeta_{d+1} \propto T^d.$$

(e) Evaluate the dimensionless heat capacity $C(T)/(Nk_B)$ at the critical temperature $T = T_c$, and compare its value to the classical (high temperature) limit.

• We can divide the above formula of $C(T \leq Tc)$, and the one obtained earlier for $N(T \geq Tc)$, and evaluate the result at $T = T_c$ $(z = 1)$ to obtain

$$\frac{C(T_c)}{Nk_B} = \frac{d(d+1)\zeta_{d+1}}{\zeta_d}.$$

In the absence of quantum effects, the heat capacity of a relativistic gas is $C/(Nk_B) = d$; this is the limiting value for the quantum gas at infinite temperature.

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12. (Optional) *Surface adsorption of an ideal Bose gas:* Consider adsorption of particles of an ideal (spin-less) Bose gas onto a two dimensional surface.

(a) Treating the ambient gas as a *non-degenerate* ideal gas of temperature T and pressure P, find its chemical potential $\mu(T, P)$.

• In the grand canonical ensemble for a gas

$$\beta P = \frac{\ln \mathcal{Q}}{V} = \frac{1}{\lambda^3}f_{5/2}^{\eta}(z) \approx \frac{z}{\lambda^3}, \qquad \Longrightarrow \qquad \mu = k_BT\ln z = k_BT\ln\left(\frac{P\lambda^3}{k_BT}\right).$$

(b) The gas is in contact with a attractive surface, such that a particle gains an energy u upon adsorption to the surface. Treating the particles on the surface as a two dimensional ideal gas (in equilibrium with the ambient gas), find the areal density n_2 as a function of P, u, and temperature (T, β, and/or λ).

• The average number of particles on the surface is

$$N_2 = \sum_{\vec{k}} \frac{1}{z^{-1}e^{\beta\epsilon(\vec{k}\,)} - 1}, \quad \text{where} \quad \epsilon(\vec{k}\,) = -u + \frac{\hbar^2 k^2}{2m}.$$

Converting the sum to an integral, the surface density is obtained as

$$n_2 = \frac{N_2}{A} = \int \frac{d^2\vec{k}}{(2\pi)^2} \frac{1}{z^{-1}e^{-\beta u + \beta\frac{\hbar^2 k^2}{2m}} - 1}.$$

Changing variables to $x = \beta\hbar^2 k^2/2m$, and noting $kdk = dx(2\pi/\lambda^2)$, we find

$$n_2 = \frac{1}{\lambda^2}\int_0^\infty dx \frac{1}{z^{-1}e^{-\beta u + x} - 1} = \frac{1}{\lambda^2} f_1^+\left(ze^{\beta u}\right) = \frac{1}{\lambda^2} f_1^+\left(\beta P\lambda^3 e^{\beta u}\right).$$

(c) Find the maximum pressure P^* before complete condensation to the surface.

• The function f_1^+ diverges when its argument approaches unity, indicating adsorption of all particles to the surface. Thus the maximum possible pressure is

$$P^* = \frac{k_B T}{\lambda^3} e^{\frac{u}{k_B T}}.$$

(d) Find the singular behavior of n_2 for $\delta P = P^* - P \to 0$.

• For P close to P^*, the argument of f_1^+ is

$$\frac{\beta P}{\lambda^3} e^{\beta u} = \frac{\beta P^*}{\lambda^3} e^{\beta u} - \frac{\delta P \beta}{\lambda^3} e^{\beta u} = 1 - \frac{\delta P}{P^*}.$$

Since $f_1^+(z) = -\ln(1 - z)$, we conclude that the density diverges as $P \to P^*$ as

$$n_2 = \frac{1}{\lambda^2} \ln\left(\frac{P^*}{\delta P}\right).$$

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13. (Optional) *Inertia of superfluid helium:* Changes in frequency of a torsional oscillator immersed in liquid helium can be used to track the "normal fraction" of the liquid as a function of temperature. This problem aims at computing the contribution of phonons (dominant at low temperatures) to the fraction of superfluid that moves with the oscillator plates. Consider a superfluid confined between two parallel plates moving with velocity $\vec{v}$.

(a) The isolated stationary superfluid has a branch of low energy excitations characterized by energy $\epsilon(p)$, where $p = |\vec{p}|$ is the magnitude of the momentum $\vec{p}$. Show that for excitations produced by walls (of large mass M) moving with velocity $\vec{v}$, this spectrum is modified (due to consideration of momentum and energy of the walls) to $\epsilon_{\vec{v}}(\vec{p}) = \epsilon(p) - \vec{p} \cdot \vec{v}$.

• Due to momentum conservation, upon creation of an excitation of momentum $\vec{p}$ within the superfluid, the velocity of the walls is reduced to $\vec{v}' = \vec{v} - \vec{p}/M$.This corresponds to a reduction in the kinetic energy of the walls, such that the net energy required to create the excitation is

$$\epsilon_{\vec{v}}(\vec{p}) = \epsilon(p) + \frac{M}{2}v'^2 - \frac{M}{2}v^2 = \epsilon(p) + \frac{M}{2}\left[\left(\vec{v} - \frac{\vec{p}}{M}\right)^2 - v^2\right] = \epsilon(p) - \vec{p} \cdot \vec{v} + \mathcal{O}\left(\frac{1}{M}\right).$$

(b) Using the standard Bose occupation number for particles of energy $\epsilon_{\vec{v}}(\vec{p})$, obtain an integral expression for the net momentum $\vec{P}$ carried by the excitations in the superfluid. (**Hint:** $\sum_{\vec{p}} = V \int d^3\vec{p}/h^3$, where V is the volume.)

• Since the occupation number of a Boson is $\langle n(\vec{p},)\rangle \left[e^{\beta\epsilon_{\vec{v}}(\vec{p})} - 1\right]^{-1}$, the net momentum carried by these excitations is

$$\vec{P} = \sum_{\vec{p}} \frac{\vec{p}}{e^{\beta\epsilon_{\vec{v}}(\vec{p})} - 1} = \frac{V}{h^3}\int d^3\vec{p}\, \frac{\vec{p}}{e^{\beta\epsilon_{\vec{v}}(\vec{p})} - 1}.$$

(c) Expanding the result for small velocities, show that $P_\alpha = V\rho_n v_\alpha$, and give an integral expression for ρ_n. (**Hint:** The angular average of $p_\alpha p_\gamma$ is $p^2\delta_{\alpha\gamma}/3$.)

• Using $\epsilon_{\vec{v}}(\vec{p}) = \epsilon(p) - \vec{p} \cdot \vec{v}$, and expanding the denominator for small $\vec{v}$, we obtain

$$P_\alpha = \frac{V}{h^3}\int d^3\vec{p}\left[\frac{p_\alpha}{e^{\beta\epsilon(p)} - 1} + \frac{p_\alpha \beta e^{\beta\epsilon(p)}\vec{p} \cdot \vec{v}}{\left(e^{\beta\epsilon(p)} - 1\right)^2} + \cdots\right].$$

Since $\epsilon(p)$ is spherically symmetric, the first integer zero. To evaluate the second integral, note that the angular average of $p_\alpha p_\beta = p^2\delta_{\alpha\beta}/3$ to get

$$P_\alpha = \frac{V}{h^3}\int_0^\infty 4\pi p^2 dp\, \frac{\beta e^{\beta\epsilon(p)} v_\alpha p^2/3}{\left(e^{\beta\epsilon(p)} - 1\right)^2} + \cdots = v_\alpha V\rho_n + \cdots,$$

where

$$\rho_n = \frac{4\pi}{3}\frac{\beta}{h^3}\int_0^\infty \frac{p^4 e^{\beta\epsilon(p)}dp}{\left(e^{\beta\epsilon(p)}-1\right)^2}.$$

(d) Compute the contribution of phonons, with $\epsilon(p) = cp$, to ρ_n. (An answer that is correct up to a numerical coefficient is sufficient.)

- Using $\epsilon(p) = cp$, and changing variables to $x = \beta cp$, leads to

$$\rho_n = \frac{4\pi}{3}\frac{\beta}{h^3}\frac{1}{(\beta c)^5}\int_0^\infty \frac{x^4 e^x dx}{(e^x-1)^2}.$$

The final integral can be evaluated through integration by parts, yielding

$$\rho_n = \frac{4\pi}{3}\frac{(k_BT)^4}{h^3c^5}\int_0^\infty \frac{4x^3dx}{e^x-1} = \frac{16\pi}{3}\frac{(k_BT)^4}{h^3c^5}3!\zeta_4 = 32\pi\frac{(k_BT)^4}{h^3c^5}\frac{\pi^4}{90} = \frac{2\pi^2}{45}\frac{(k_BT)^4}{\hbar^3c^5}.$$

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