Exercises for Lecture 1
Fundamentals of Quantum Theory

May 25, 2020

1 Density operators

Show that given the distribution of states $(q_j, |\psi_j\rangle)$ the probability of getting an outcome a_i when measuring $A = \sum_i a_i |a_i\rangle\langle a_i|, \equiv \sum_i a_i P_i$ is given by

$$\Pr(a_i) = \mathrm{Tr}(\psi P_i) \tag{1}$$

where we have defined the density matrix of the system as

$$\psi = \sum_j q_j \psi_j \tag{2}$$

2 Partial Trace

Mixed states appear naturally also when considering subsystems of a composite quantum system. Consider

$$A = \tilde{A}_S \otimes I_{\overline{S}} \tag{3}$$

in a Hilbert space with tensor product structure $\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_{\overline{S}}$. It is a natural question to ask what is the state $\rho_S \in \mathcal{B}(\mathcal{H}_S)$ onto the subsystem $\mathcal{H}_S$ that can return all the right expectation values when performing any measurement of operators with that support, namely the $\tilde{A}_S$. We ask that

$$tr(A\rho) = tr_S(\tilde{A}_S \rho_S) \tag{4}$$

Show that the solution of this equation is

$$\rho_S = tr_{\overline{S}}\rho \tag{5}$$

where the operation of partial trace is defined by

$$tr_{\overline{S}}X = tr_{\overline{S}} \sum_{i_S i_{\overline{S}} j_S j_{\overline{S}}} X_{i_S i_{\overline{S}} j_S j_{\overline{S}}} \left|i_S i_{\overline{S}}\right\rangle \left\langle j_S j_{\overline{S}}\right| = \sum_{i_S i_{\overline{S}} j_S} X_{i_S i_{\overline{S}} j_S j_S} \left|i_S\right\rangle \left\langle j_S\right| \tag{6}$$

3 Entropies

The Rényi entropy for Rényi index q is defined as

$$S_q(\rho) = \frac{1}{1-q} \log \mathrm{Tr}\rho^q = \frac{1}{1-q} \log \left(\sum_i \lambda_i^q \right), \tag{7}$$

where the logarithm is again taken base two and λ_i are the eigenvalues of ρ.

3.1 Properties of Rényi entanglement entropies

a) Show that for the completely mixed density operator $\rho = I/d$ in d-dimensional Hilbert space, the von Neumann entropy is given by

$$S(I/d) = \log d. \tag{8}$$

b) Show that

$$S_0(\rho) = \lim_{q \to 0^+} S_q(\rho) = \log(\text{rank}\, \rho), \tag{9}$$

where $\text{rank}\, \rho$ is the rank of the density matrix ρ.

c) Show that

$$\lim_{q \to 1} S_q(\rho) = S_1(\rho), \tag{10}$$

where $S_1(\rho)$ is the von Neumann entropy.

d) Consider the Rényi entropy for the case where the Rényi index is a integer n with $n \geq 2$. If ρ corresponds to a pure state, calculate $S_n(\rho)$. Show that if ρ_{pure} corresponds to a pure state, then

$$S_n(\rho_{\text{pure}}) = 0. \tag{11}$$

3.2 Entropy of the Gibbs state

Compute the von Neumann entropy of the Gibbs state.