# 8.421 AMO I Lecture Notes

# Contents

## Preface

The original incarnation of these notes was developed to accompany the lectures in the MIT graduate courses in atomic physics. AMO I was created in the late 1960s as a one-term introductory course to prepare graduate students for research in atomic physics in the Physics department. Over the years Dan Kleppner and David Pritchard changed the contents of the course to reflect new directions of research, though the basic concepts remained as a constant thread. With the growth of interest in atom cooling and quantum gases, a second one-term course, AMO II, was designed by Wolfgang Ketterle in the late 1990s and presented with AMO I in alternating years. We still teach AMO I in the traditional way. These lecture notes combine the (g)olden notes of Dan and Dave. As part of the Joint Harvard/MIT Center for Ultracold Atoms summer school in Atomic Physics in 2002, John Doyle got involved and improved the notes. They have been circulated since and at some point were put into the form of an AMO Wiki. At this moment in time, I'd like to resurrect them in their traditional paper form, and only carefully add topics as I see fit.

Martin Zwierlein,

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# Chapter 1
# The Two-State System: Resonance

## 1.1 Introduction

The cornerstone of major areas of contemporary Atomic, Molecular and Optical Physics (AMO Physics) is the study of atomic and molecular systems through their resonant interaction with applied oscillating electromagnetic fields. The thrust of these studies has evolved continuously since Rabi performed the first resonance experiments in 1938. In the decade following World War II the edifice of quantum electrodynamics was constructed largely in response to resonance measurements of unprecedented accuracy on the properties of the electron and the fine and hyperfine structure of simple atoms. At the same time, nuclear magnetic resonance and electron paramagnetic resonance were developed and quickly became essential research tools for chemists and solid state physicists. Molecular beam magnetic and electric resonance studies yielded a wealth of information on the properties of nuclei and molecules, and provided invaluable data for the nuclear physicist and physical chemist. With the advent of lasers and laser spectroscopy these studies evolved into the creation of new species, such as Rydberg atoms, to studies of matter in ultra intense fields, to fundamental studies in the symmetries of physics, to new types of metrology. With the advent of laser cooling and trapping, these techniques led to the creation of novel atomic quantum fluids, from Bose-Einstein condensates to strongly interacting Fermi gases.

Resonance techniques may be used not only to learn about the structure of a system, but also to prepare it in a precisely controlled way. Because of these two facets, resonance studies have led physicists through a fundamental change in attitude - from the passive study of atoms to the active control of their internal quantum state and their interactions with the radiation field. The chief technical legacy of the early work on resonance spectroscopy is the family of lasers which have sprung up like the brooms of the sorcerer's apprentice. The scientific applications of these devices have been prodigious. They caused the resurrection of physical optics - what we now call quantum optics - and turned it into one of the liveliest fields in physics. They have had a similar impact on atomic and molecular spectroscopy. In addition, lasers have led to new families of physical studies such as single particle spectroscopy, multiphoton excitation, cavity quantum electrodynamics, and laser cooling and trapping. This chapter is about the interactions of a two-state system with a sinusoidally oscillating field whose frequency is close to the natural resonance frequency of the system. The term "two-level" system is sometimes used, but this is less accurate than the term two-state, because the levels could be degenerate, comprising several states. However, its misusage is so widespread that we adopt it anyway. The oscillating field will be treated classically, and the linewidth of both states will be taken as zero until near the end of the chapter where relaxation will be treated phenomenologically. The organization of the material is historical because this

happens to be also a logical order of presentation. The classical driven oscillator is discussed first, then the magnetic resonance of a classical spin, and then a quantized spin. The density matrix is introduced last and used to treat systems with damping - this is a useful prelude to the application of resonance ideas at optical frequencies and to the many real systems which have damping.

## 1.2 Resonance Studies and Q.E.D.

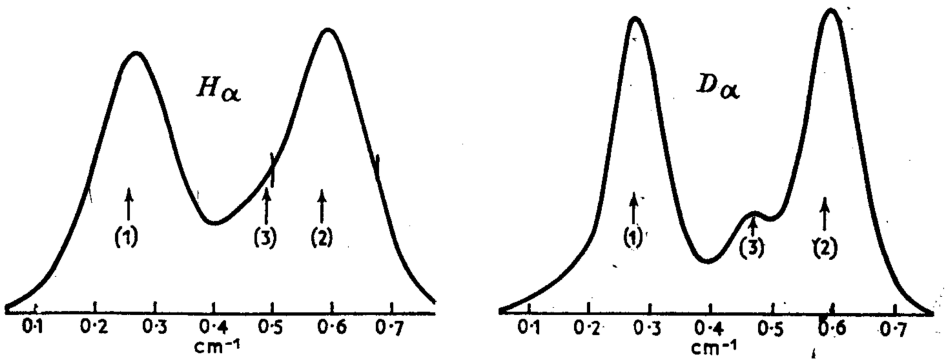


**Figure 1**. Spectral profile of the $H_\alpha$ line of atomic hydrogen by conventional absorption spectroscopy. Components 1) and 2) arise from the fine structure splitting. The possibility that a third line lies at position 3) was suggested to indicate that the Dirac theory might need to be revised. (From "The Spectrum of Atomic Hydrogen"-Advances. G.W. Series ed., World Scientific, 1988).

One characteristic of atomic resonance is that the results, if you can obtain them at all, are generally of very high accuracy, so high that the information is qualitatively different from other types. The hydrogen fine structure is a good example.

In the late 1930s there was extensive investigation of the Balmer series of hydrogen, ($n > 2 \rightarrow n = 2$). The Dirac Theory was thought to be in good shape, but some doubts were arising. Careful study of the Balmer-alpha line ($n = 3 \rightarrow n = 2$) showed that the line shape might not be given accurately by the Dirac Theory.

Pasternack, in 1939, suggested that the $2\,^2\mathrm{S}_{1/2}$ and $2\,^2\mathrm{P}_{1/2}$ states were not degenerate, but that the energy of the $2\,^2\mathrm{S}_{1/2}$ state was greater than the Dirac value by $\sim .04\,\mathrm{cm}^{-1}$ (or, in frequency, $\sim 1200\,\mathrm{MHz}$). However, there was no rush to throw out the Dirac theory on such flimsy evidence.

In 1947, Lamb found a splitting between the $^2\mathrm{S}_{1/2}$ and $^2\mathrm{P}_{1/2}$ levels using a resonance method. The experiment is one of the classics of physics. Although his very first observa- tion was relatively crude, it was nevertheless accurate to one percent. He found

$$S_H = \frac{1}{h}\left[E(^2\mathrm{S}_{1/2}) - E(^2\mathrm{P}_{1/2})\right] = 1050(10)\,\mathrm{MHz} \tag{1.1}$$

The inadequacy of the Dirac theory was inescapably demonstrated.

The magnetic moment of the electron offers another example. In 1925, Uhlenbeck and Goudsmit suggested that the electron has intrinsic spin angular momentum $s = 1/2$ (in units of $\hbar$) and magnetic moment

$$\mu_e = \frac{e\hbar}{2m} = \mu_B \tag{1.2}$$

where $\mu_B$ is the Bohr magneton. The evidence was based on studies of the multiplicity of atomic lines (in particular, the Zeeman structure). The proposal was revolutionary, but the accuracy of the prediction that $\mu_e = \mu_B$ was poor, essentially one significant figure. According to the Dirac theory, $\mu_e = \mu_B$, exactly. However, our present understanding is

$$\frac{\mu_e}{\mu_B} - 1 = 1.1596521884(43) \times 10^{-3} \qquad \text{(experiment, U. of Washington)} \tag{1.3}$$

This result is in good agreement with theory, the limiting factor in the comparison being possible doubts about the value of the fine structure constant.

The Lamb shift and the departure of $\mu_e$ from $\mu_B$ resulted in the award of the 1955 Nobel prize to Lamb and Kusch, and provided the experimental basis for the theory of quantum electrodynamics for which Feynman, Schwinger and Tomonaga received the Nobel Prize in 1965.

### 1.2.1 The language of resonance: a classical damped system

Because the terminology of classical resonance, as well as many of its features, are carried over into quantum mechanics, we start by reviewing an elementary resonant system. Consider a harmonic oscillator composed of a series RLC circuit. The charge obeys

$$\ddot{q} + \gamma\dot{q} + \omega_0^2 q = 0 \tag{1.4}$$

where $\gamma = R/L$, $\omega_0^2 = 1/LC$. Assuming that the system is underdamped (i.e. $\gamma^2 < 4\omega_0^2$), the solution for $q$ is a linear combination of

$$\exp\left(-\frac{\gamma}{2}\right)\exp\left(\pm i\omega' t\right) \tag{1.5}$$

where $\omega' = \omega_0\sqrt{1 - \gamma^2/4\omega_0^2}$. If $\omega_0 \gg \gamma$, which is often the case, we have $\omega' \equiv \omega_0$. The energy in the circuit is

$$W = \frac{1}{2C}q^2 + \frac{1}{2}L\dot{q}^2 = W_0 e^{-\gamma t} \tag{1.6}$$

where $W_0 = W(t = 0)$. The decay time of the stored energy is $\tau = \frac{1}{\gamma}$. If the circuit is driven by a voltage $E_0 e^{i\omega t}$, the steady state solution is $q_0 e^{i\omega t}$ where

$$q_0 = \frac{E_0}{2\omega_0 L}\frac{1}{(\omega_0 - \omega + i\gamma/2)}. \tag{1.7}$$

(We have made the usual resonance approximation: $\omega_0^2 - \omega^2 \approx 2\omega_0(\omega_0 - \omega)$.) The average power delivered to the circuit is

$$P = \frac{1}{2}\frac{E_0^2}{R}\frac{1}{1 + \left(\frac{\omega - \omega_0}{\gamma/2}\right)^2} \tag{1.8}$$

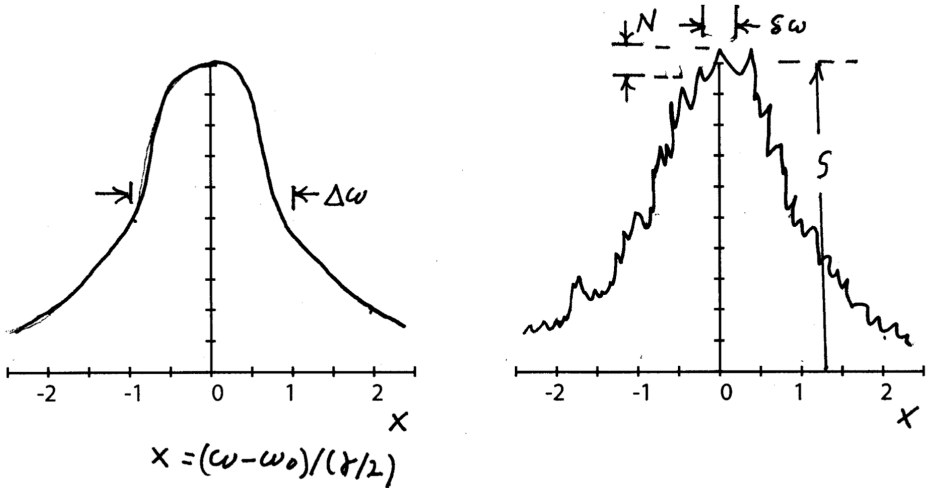


**Figure 2**. Sketch of a Lorentzian curve, the universal response curve for damped oscillators and for many atomic systems. The width of the curve (full width at half maximum) is $\Delta\omega = \gamma$, where $\gamma$ is the decay constant. The time constant for decay is $\tau = \gamma$. In the presence of noise (right), the frequency precision with which the center can be located, $\delta\omega$, depends on the signal-to-noise ratio, $S/N$: $\delta\omega = \Delta\omega/(S/N)$.

The plot of $P$ vs $\omega$ (Fig. 2) is universal resonance curve often called a "Lorentzian curve". The full width at half maximum ("FWHM") is $\Delta\omega = \gamma$. The quality factor of the oscillator is

$$Q = \frac{\omega_0}{\Delta\omega} \tag{1.9}$$

Note that the decay time of the free oscillator and the linewidth of the driven oscillator obey

$$\tau\Delta\omega = 1 \tag{1.10}$$

This can be regarded as an uncertainty relation. Assuming that energy and frequency are related by $E = \hbar\omega$ then the uncertainty in the energy is $\Delta E = \hbar\Delta\omega$ and

$$\tau\Delta E = \hbar \tag{1.11}$$

It is important to realize that the Uncertainty Principle merely characterizes the spread of individual measurements. Ultimate precision depends on the experimenter's skill: the Uncertainty Principle essentially sets the scale of difficulty for his or her efforts.

The precision of a resonance measurement is determined by how well one can "split" the resonance line. This depends on the signal to noise ratio (S/N)

(see Fig. 2). As a rule of thumb, the uncertainty $\delta\omega$ in the location of the center of the line is

$$\delta\omega = \frac{\Delta\omega}{S/N} \tag{1.12}$$

In principle, one can make $\delta\omega$ arbitrarily small by acquiring enough data to achieve the required statistical accuracy. In practice, systematic errors eventually limit the precision. Splitting a line by a factor of $10^4$ is a formidable task which has only been achieved a few times, most notably in the measurement of the Lamb shift. A factor of $10^3$, however, is not uncommon, and $10^2$ is child's play.

## 1.3 Magnetic Resonance: Classical Spin in Time-Varying B-Field

### 1.3.1 The classical motion of spins in a static magnetic field

Note: angular momentum will always be expressed in a form such as $\hbar\boldsymbol{J}$, where the vector $\boldsymbol{J}$ is dimensionless. The interaction energy and equation of motion of a classical spin in a static magnetic field are given by

$$W = -\boldsymbol{\mu}\cdot\boldsymbol{B}, \tag{1.13}$$

$$\boldsymbol{F} = -\boldsymbol{\nabla}W = \boldsymbol{\nabla}(\boldsymbol{\mu}\cdot\boldsymbol{B}), \tag{1.14}$$

$$\text{torque} = \boldsymbol{\mu}\times\boldsymbol{B} \tag{1.15}$$

In a uniform field, $\boldsymbol{F} = 0$. The torque equation ($\mathrm{d}\hbar\boldsymbol{J}/\mathrm{d}t = \text{torque}$) gives

$$\frac{\mathrm{d}\hbar\boldsymbol{J}}{\mathrm{d}t} = \boldsymbol{\mu}\times\boldsymbol{B}. \tag{1.16}$$

Since $\boldsymbol{\mu} = \gamma\hbar\boldsymbol{J}$ (where $\gamma$ is called the gyromagnetic ratio - not to be confused with the different meaning of $\gamma$ in the previous section), we have

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \gamma\boldsymbol{J}\times\boldsymbol{B} = -\gamma\boldsymbol{B}\times\boldsymbol{J}. \tag{1.17}$$

To see that the motion of $\boldsymbol{J}$ is a pure precession about $\boldsymbol{B}$, imagine that $\boldsymbol{B}$ is along $\hat{\boldsymbol{z}}$ and that the spin, $\boldsymbol{J}$, is tipped at an angle $\theta$ from this axis, and then rotated at an angle $\phi(t)$ from the $x$-axis (i.e., $\theta$ and $\phi$ are the conventionally chosen angles in spherical coordinates). The torque, $-\gamma\boldsymbol{B}\times\boldsymbol{J}$, has no component along $\boldsymbol{J}$ (that is, along $\hat{\boldsymbol{r}}$), nor along $\hat{\boldsymbol{\theta}}$ (because the $\boldsymbol{J}-\boldsymbol{B}$ plane contains $\hat{\boldsymbol{\theta}}$), hence $-\gamma\boldsymbol{B}\times\boldsymbol{J} = -\gamma B|\boldsymbol{J}|\sin(\theta)\hat{\boldsymbol{\phi}}$. This implies that $\boldsymbol{J}$ maintains constant magnitude and constant tipping angle $\theta$. Generally, for an infinitesimal change $\mathrm{d}\phi$ the component of $\mathrm{d}\boldsymbol{J}$ in the direction of $\hat{\boldsymbol{\phi}}$ is $|\boldsymbol{J}|\sin(\theta)\mathrm{d}\phi$, and so we can see that $\phi(t) = -\gamma Bt$. This solution shows that the moment precesses with angular velocity

$$\Omega_L = -\gamma B \tag{1.18}$$

where $\Omega_L$ is called the *Larmor frequency.*

For electrons, $\gamma_e/2\pi = 2.8\,\mathrm{MHz/G}$, for protons, $\gamma_p/2\pi = 4.2\,\mathrm{kHz/G}$. Note that Planck's constant does not appear in the equation of motion: the motion is classical.
*Note: G stands for gauss - it is part of the gaussian (cgs) system of units, and ubiquitous in atomic physics labs, as it is a much more typical laboratory field than the SI unit for magnetic field, 1 tesla.* $10^4\,\mathrm{G} = 1\,\mathrm{T}$.

### 1.3.2 Rotating coordinate transformation

A second way to find the motion is to look at the problem in a rotating coordinate system. If some vector $\boldsymbol{A}$ rotates with angular velocity $\boldsymbol{\Omega}$, then

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{A}. \tag{1.19}$$

If the rate of change of the vector in a system rotating at $\Omega$ is $(\mathrm{d}\boldsymbol{A}/\mathrm{d}t)_{\mathrm{rot}}$, then the rate of change in an inertial system is the motion *in* plus the motion *of* the rotating coordinate system.

$$\left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_{\mathrm{inert}} = \left(\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right)_{\mathrm{rot}} + \boldsymbol{\Omega} \times \boldsymbol{A}. \tag{1.20}$$

The operator prescription for transforming from an inertial to a rotating system is thus

$$\left(\frac{\mathrm{d}\cdot}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left(\frac{\mathrm{d}\cdot}{\mathrm{d}t}\right)_{\mathrm{inert}} - \boldsymbol{\Omega} \times \cdot \tag{1.21}$$

Applying this to Eq. 1.17 gives

$$\left(\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \gamma \boldsymbol{J} \times \boldsymbol{B} - \boldsymbol{\Omega} \times \boldsymbol{J} = \gamma \boldsymbol{J} \times (\boldsymbol{B} + \boldsymbol{\Omega}/\gamma)\,. \tag{1.22}$$

If we let

$$\boldsymbol{B}_{\mathrm{eff}} = \boldsymbol{B} + \boldsymbol{\Omega}/\gamma, \tag{1.23}$$

Eq. 1.22 becomes

$$\left(\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \gamma \boldsymbol{J} \times \boldsymbol{B}_{\mathrm{eff}}. \tag{1.24}$$

If $\boldsymbol{B}_{\mathrm{eff}} = 0$, $\boldsymbol{J}$ is constant in the rotating system. The condition for this is

$$\boldsymbol{\Omega} = -\gamma \boldsymbol{B} \tag{1.25}$$

as we have previously found in Eq. 1.18.

### 1.3.3 Larmor's theorem

Treating the effects of a magnetic field on a magnetic moment by transforming to a rotating coordinate system is closely related to Larmor's theorem, which asserts that the effect of a magnetic field on a free charge can be eliminated by a suitable rotating coordinate transformation.

Consider the motion of a particle of mass $m$, charge $q$, under the influence

of an applied force $\boldsymbol{F}_0$ and the Lorentz force due to a static field $\boldsymbol{B}$:

$$\boldsymbol{F} = \boldsymbol{F}_0 + q\boldsymbol{v} \times \boldsymbol{B}. \tag{1.26}$$

Now consider the motion in a rotating coordinate system. By applying Eq. 1.21 twice to $\boldsymbol{r}$, we have

$$\ddot{\boldsymbol{r}}_{\rm rot} = \ddot{\boldsymbol{r}}_{\rm inert} - 2\boldsymbol{\Omega} \times \boldsymbol{v}_{\rm rot} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}). \tag{1.27}$$

$$\boldsymbol{F}_{\rm rot} = \boldsymbol{F}_{\rm inert} - 2m(\boldsymbol{\Omega} \times \boldsymbol{v}_{\rm rot}) - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}), \tag{1.28}$$

where $\boldsymbol{F}_{\rm rot}$ is the apparent force in the rotating system, and $\boldsymbol{F}_{\rm inert}$ is the true or inertial force. Substituting Eq. 1.26 gives

$$\boldsymbol{F}_{\rm rot} = \boldsymbol{F}_{0,\rm inert} + q\boldsymbol{v} \times \boldsymbol{B} + 2m\boldsymbol{v} \times \boldsymbol{\Omega} - m\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}). \tag{1.29}$$

If we choose $\boldsymbol{\Omega} = -(q/2m)\boldsymbol{B}$, and take $\boldsymbol{B} = \hat{\boldsymbol{z}}B$, we have

$$\boldsymbol{F}_{\rm rot} = \boldsymbol{F}_{0,\rm inert} - m\left(\frac{qB}{2m}\right)^2 \hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{r}). \tag{1.30}$$

The last term is usually small. If we drop it we have

$$\boldsymbol{F}_{\rm rot} = \boldsymbol{F}_{0,\rm inert} \tag{1.31}$$

The effect of the magnetic field is removed by going into a system rotating at the Larmor frequency $qB/2m$.

Although Larmor's theorem is suggestive of the rotating co-ordinate transformation, Eq. 1.22, it is important to realize that the two transformations, though identical in form, apply to fundamentally different systems. A magnetic moment is not necessarily charged- for example a neutral atom can have a net magnetic moment, and the neutron possesses a magnetic moment in spite of being neutral - and it experiences no net force in a uniform magnetic field. Furthermore, the rotating co-ordinate transformation is exact for a mag- netic moment, whereas Larmor's theorem for the motion of a charged particle is only valid when the $\propto B^2$ term is neglected.

## 1.4 Motion in a Rotating Magnetic Field

### 1.4.1 Exact resonance

Consider a moment $\boldsymbol{\mu}$ precessing about a static field $\boldsymbol{B}_0$, which we take to lie along the $z$ axis. Its motion might be described by

$$\mu_z = \mu\ \cos\theta, \quad \mu_x = \mu\ \sin\theta\ \cos\omega_0 t, \quad \mu_y = -\mu\ \sin\theta\ \sin\omega_0 t \tag{1.32}$$

where $\omega_0$ is the Larmor frequency, and $\theta$ is the angle the moment makes with $\boldsymbol{B}_o$.

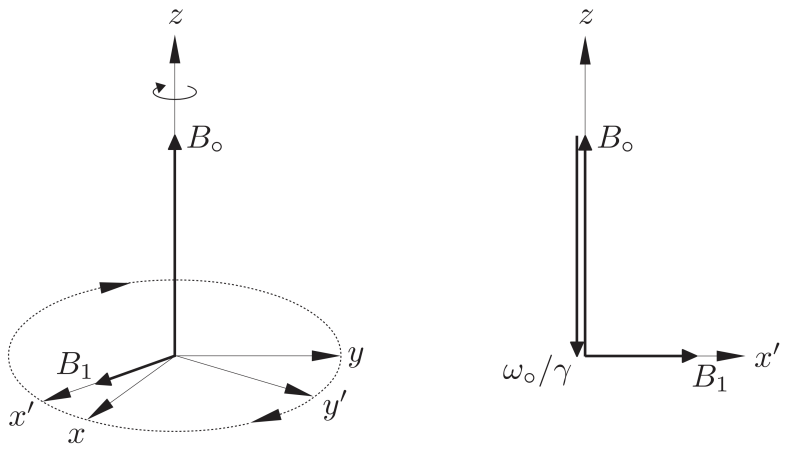


**Figure 3**. Rotating coordinate transformation to the primed system that is co-rotating with $B_1$ at $\omega$, with $x'$ chosen to lie along $B_1$. For the exact resonance case of $\omega = \omega_0$ considered here, the effective field around which the moment precesses is equal to $B_1$.

Now suppose we introduce a magnetic field $\boldsymbol{B}_1$ which rotates in the x-y plane at the Larmor frequency $\omega_0 = -\gamma B_0$. The magnetic field is

$$\boldsymbol{B}(t) = B_1(\hat{\boldsymbol{x}}\ \cos\omega_0 t - \hat{\boldsymbol{y}}\ \sin\omega_0 t)\ +\ B_0\ \hat{\boldsymbol{z}}. \tag{1.33}$$

The problem is to find the motion of $\boldsymbol{\mu}$. The solution is simple in a rotating coordinate system (see Fig. 3). Let system $(\hat{\boldsymbol{x}}', \hat{\boldsymbol{y}}', \hat{\boldsymbol{z}}' = \hat{\boldsymbol{z}})$ precess around the z-axis at rate $-\omega_0$. In this system the field $\boldsymbol{B}_1$ is stationary (and $\hat{\boldsymbol{x}}'$ is chosen to lie along $\boldsymbol{B}_1$), and we have

$$\boldsymbol{B}_{\text{eff}}(t) = \boldsymbol{B}(t) - (\omega_0/\gamma)\ \hat{\boldsymbol{z}} = B_1\hat{\boldsymbol{x}}' + (B_0 - \omega_0/\gamma)\hat{\boldsymbol{z}} = B_1\hat{\boldsymbol{x}}'. \tag{1.34}$$

The effective field is static and has the value of $B_1$. The moment precesses about the field at rate

$$\omega_R = \gamma B_1, \tag{1.35}$$

often called the *Rabi* frequency, in honor of Rabi's invention of the resonance technique.

If the moment initially lies along the z axis, then its tip traces a circle in the $\hat{\boldsymbol{y}}' - \hat{\boldsymbol{z}}$ plane. At time $t$ it has precessed through an angle $\phi = \omega_R t$. The moment's z-component is given by

$$\mu_z(t) = \mu\ \cos\omega_R t. \tag{1.36}$$

At time $T = \pi/\omega_R$, the moment points along the negative z-axis: it has "turned over".

### 1.4.2 Off-resonance behavior

Now suppose that the field $B_1$ rotates at frequency $\omega \neq \omega_0$. In a coordinate frame rotating with $B_1$ the effective field is

$$\boldsymbol{B}_{\text{eff}} = B_1\hat{\boldsymbol{x}}' + (B_0 - \omega/\gamma)\hat{\boldsymbol{z}}. \tag{1.37}$$

The effective field lies at angle $\theta$ with the z-axis, as shown in Fig. 4 The field is static, and the moment precesses about it at rate (called the *effective Rabi frequency* )

$$\Omega_R = \gamma B_{\text{eff}} = \gamma\sqrt{(B_0 - \omega/\gamma)^2 + B_1^2} = \sqrt{(\omega_0 - \omega)^2 + \omega_R^2} \tag{1.38}$$

where $\omega_0 = \gamma B_0, \omega_R = \gamma B_1$, as before.

Assume that $\boldsymbol{\mu}$ points initially along the +z-axis. Finding $\mu_z(t)$ is a straightforward problem in geometry. The moment precesses about $B_{\text{eff}}$ at rate $\Omega_R$, sweeping a circle as shown. The radius of the circle is $\mu\sin\theta$, where $\sin\theta = B_1/\sqrt{(B_0 - \omega/\gamma)^2 + B_1^2} = \omega_R/\sqrt{(\omega - \omega_0)^2 + \omega_R^2}$. In time $t$ the tip sweeps through angle $\phi = \Omega_R t$. The z-component of the moment is $\mu_z(t) = \mu\cos\alpha$ where $\alpha$ is the angle between the moment and the z-axis after it has precessed through angle $\phi$. As the drawing shows, $\cos\alpha$ is found from $A^2 = 2\mu^2(1 - \cos\alpha)$. Since $A = 2\mu\ \sin\theta\sin(\Omega_R t/2)$, we have $4\mu^2\ \sin^2\theta\sin^2(\Omega_R t/2) = 2\mu^2(1 - \cos\alpha)$ and

$$\begin{aligned}
\mu_z(t) &= \mu\cos\alpha = \mu(1 - 2\sin^2\theta\sin^2\Omega_R t/2) \\
&= \mu\left[1 - 2\frac{\omega_R^2}{(\omega - \omega_0)^2 + \omega_R^2}\sin^2\frac{1}{2}\sqrt{(\omega - \omega_0)^2 + \Omega_R}t\right] \qquad (1.39) \\
&= \mu\Big[1 - 2(\omega_R/\Omega_R)^2\sin^2(\Omega_R t/2)\Big] \qquad (1.40)
\end{aligned}$$

The z-component of $\boldsymbol{\mu}$ oscillates in time, but unless $\omega = \omega_0$, the moment

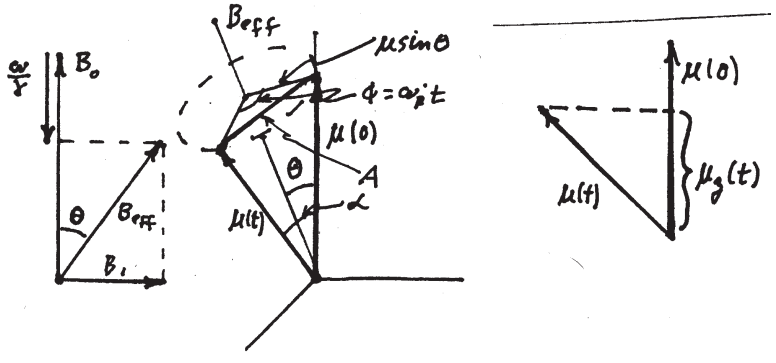


**Figure 4**. Constructions for viewing spin motion in a coordinate system rotating below the resonance frequency.

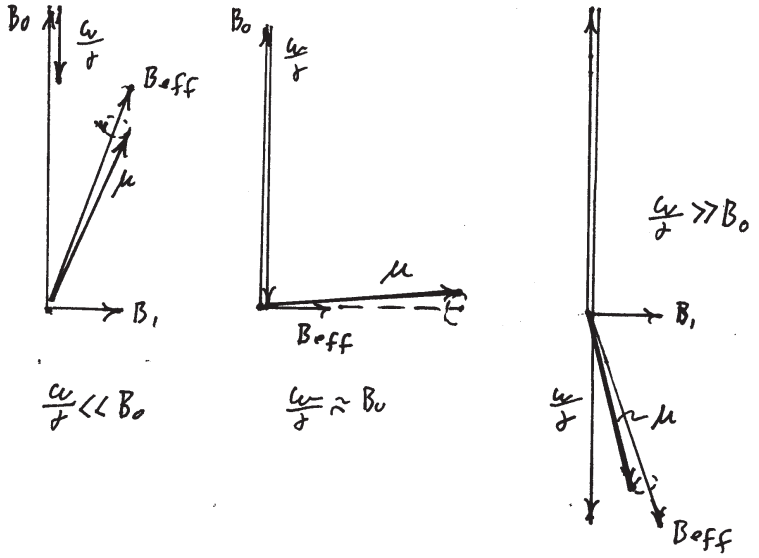


**Figure 5**. Motion of precessing moment in a rotating coordinate system whose frequency is swept from below resonance to above resonance.

never completely inverts. The rate of oscillation depends on the magnitude of the rotating field; the amplitude of oscillation depends on the frequency difference, $\omega - \omega_0$, relative to $\omega_R$. The quantum mechanical result will turn out to be identical.

## 1.5 Adiabatic Rapid Passage: Landau-Zener Crossing

Adiabatic rapid passage is a technique for inverting a spin population by sweeping the system through resonance. Either the frequency of the oscillating field or the transition frequency (e.g., by changing the applied magnetic field) is slowly varied. The principle is qualitatively simple in the rotating coordinate system. The problem can also be solved analytically. In this section we give the qualitative argument and then present the analytic quantum result. The solution is of quite general interest because this physical situation arises frequently, for example in inelastic scattering, where it is called a curve crossing.

### 1.5.1 Rotating frame argument

Consider a moment $\boldsymbol{\mu}$ in the presence of a static magnetic field $\boldsymbol{B}_0$ and a perpendicular rotating field $\boldsymbol{B}_1$ at some frequency $\omega$, originally far from resonance; $\omega \ll \gamma B_0$ In the frame rotating with $\boldsymbol{B}_1$ the magnetic moment "sees" an effective field $\boldsymbol{B}_{\text{eff}}$ whose direction is nearly parallel to $\boldsymbol{B}_0$. A magnetic moment $\boldsymbol{\mu}$ initially parallel to $\boldsymbol{B}_0$ precess around $\boldsymbol{B}_{\text{eff}}$, making only a small angle with $\boldsymbol{B}_{\text{eff}}$, as shown in Fig. 5.

If $\omega$ is *slowly* swept through resonance, $\boldsymbol{\mu}$ will continue to process tightly around $\boldsymbol{B}_{\text{eff}}$, as shown in Figs. 1.11b,c. and will follow its direction adiabati-

cally. In Fig L-Z-rot the effective field now points in the $-\hat{\boldsymbol{z}}$ direction, because $\omega \gg \gamma B_0$. Since the spin still precesses tightly around $\boldsymbol{B}_{\rm eff}$, its direction in the laboratory system has "flipped" from $+\hat{\boldsymbol{z}}$ to $-\hat{\boldsymbol{z}}$. The laboratory field $\boldsymbol{B}_0$ remains unchanged, so this represents a transition from spin up to spin down.

The requirement for $\boldsymbol{\mu}$ to follow the effective field $\boldsymbol{B}_{\rm eff}(t)$ is that the Larmor frequency $\Omega_L = \gamma B_{\rm eff}$ be large compared to $\dot{\theta}$, the rate at which $B_{\rm eff}(t)$ is changing direction. This requirement is most severe near exact resonance where $\theta = \pi/2$. Using $B_{\rm eff}(t) = B_0 - \omega(t)/\gamma$ we have in this case (from geometry)

$$\mid \dot{\theta}_{max} \mid = \frac{1}{B_1}\frac{dB_{\rm eff}(t)}{dt} = \frac{1}{B_1}\frac{1}{\gamma}\frac{d\omega}{dt} \ll \gamma B_1, \tag{1.41}$$

or using $\omega_R = \gamma B_1$ ,

$$\frac{d\omega}{dt} \ll \omega_R^2, \tag{1.42}$$

In this example we have shown that a slow change from $\omega \ll \gamma B_0$ to $\omega \gg \gamma B_0$ will flip the spin; the same argument shows that the reverse direction of slow change will also flip the spin.

For a two-state system the problem can be solved rigorously. Consider a spin 1/2 system in a magnetic field $\boldsymbol{B}_{\rm eff}$ with energies

$$W_\pm = \pm\frac{1}{2}\hbar\gamma B_{\rm eff}. \tag{1.43}$$

For a uniform field $B_0$ (with $B_1 = 0$), the effective field in the rotating frame is $B_0 - \omega/\gamma$, and

$$W_\pm = \pm\frac{1}{2}\hbar(\omega_0 - \omega), \tag{1.44}$$

where $\omega_0 = \gamma B_0$. As $\omega$ is swept through resonance, the two states move along their changing eigenenergies. The energies change, but the states do not. There is no coupling between the states, so a spin initially in one or the other will remain so indefinitely no matter how $\omega$ changes relative to $\omega_0$.

In the presence of a rotating field, $B_1$, however, the energy levels look quite different: instead of intersecting lines they form non-intersecting hyperbolas separated by energy $\pm\hbar\omega_R$. If the system moves along these hyperbolas, then $\uparrow\rightarrow\downarrow$ and $\downarrow\rightarrow\uparrow$.

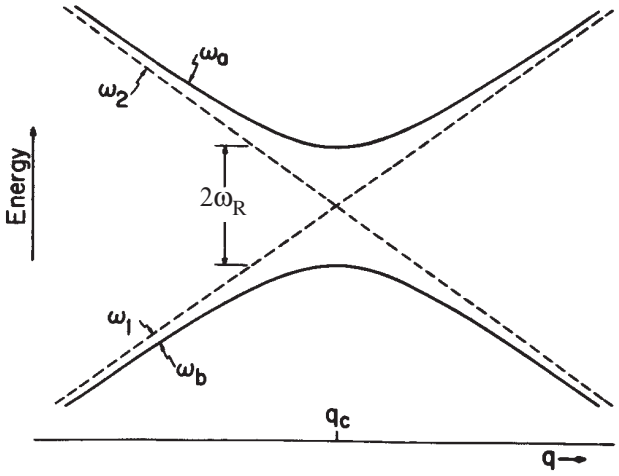


**Figure 6**. An avoided crossing. A system on one level can jump to the other if the parameter $q$ that governs the energy levels is swept sufficiently rapidly.