Problem Set 10

Due: Friday, April 28th, 11:59pm via Canvas upload
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1 Optical Bloch Equations: weak and short-time limits

The time-independent form of the optical Bloch equations (see e.g. lecture notes and API p. 359), including spontaneous emission, and the rotating wave approximation, is:

$$\dot{\rho}_{ee} = i\frac{\Omega}{2}(\rho_{eg} - \rho_{ge}) - \Gamma\rho_{ee} \tag{1}$$

$$\dot{\rho}_{ge} = i(\omega_0 - \omega_L)\rho_{ge} - i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg}) - \frac{\Gamma}{2}\rho_{ge}\,, \tag{2}$$

where the remaining two components of the density matrix are given by $\rho_{gg} = 1 - \rho_{ee}$, and $\rho_{eg} = \rho_{ge}^*$. It is insightful to study these equations in the limit of weak excitation, and for short evolution times.

a) Show that the solution of these equations to lowest order in $|\Omega|$, and in the limit $|\Omega| \ll \Gamma$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{\frac{1}{4}|\Omega|^2}{(\omega_0 - \omega_L)^2 + \left(\frac{\Gamma}{2}\right)^2}\left[1 + e^{-\Gamma t} - 2\cos[(\omega_0 - \omega_L)t]e^{-\Gamma t/2}\right]. \tag{3}$$

What does this solution reduce to in the limit of an infinitely narrow linewidth ($\Gamma \to 0$)?

b) Show that the solution of these equations to lowest order in $|\Omega|$ in the limit $|\Omega|t \ll 1$, with the initial conditions $\rho_{ee} = 0$ and $\rho_{ge} = 0$, gives

$$\rho_{ee} = \frac{1}{4}|\Omega|^2 t^2\,. \tag{4}$$

This result is independent of the detuning $\omega_0 - \omega_L$ and the decay rate Γ. Why?

2 One atom and one photon: spontaneous emission

A single atom coupled to a single mode of electromagnetic radiation undergoes spontaneous emission. What is the state of the atom during such spontaneous emission?
Let us model the interaction of one atom with a single optical mode using the Jaynes-Cummings interaction,

$$H = \hbar\omega a^\dagger a + \delta\sigma_z + g(a^\dagger\sigma_- + a\sigma_+)\,, \tag{5}$$

where δ is the detuning of the cavity from the atom, ω is the cavity frequency, and g is the coupling of the atom to the field. Restricted to the case where at most one quantum is exchanged with the optical mode, we may write this Hamiltonian as a matrix,

$$H = -\begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & g \\ 0 & g & -\delta \end{bmatrix}\,, \tag{6}$$

where the basis states are $|0g\rangle$, $|0e\rangle$, and $|1g\rangle$, where $|e\rangle$ and $|g\rangle$ are the ground and excited states of the atom, and the left 0 and 1 labels denote the number of photons in the optical mode. Note: The Hamiltonian is written in the rotating frame.

a) Compute the full unitary transform for evolution under this Hamiltonian, $U = e^{-iHt}$ and obtain

$$\begin{aligned} U &= e^{-i\delta t}|0g\rangle\langle 0g| + \left(\cos\Omega t + i\frac{\delta}{\Omega}\sin\Omega t\right)|0e\rangle\langle 0e| \\ &\quad + \left(\cos\Omega t - i\frac{\delta}{\Omega}\sin\Omega t\right)|1g\rangle\langle 1g| - i\frac{g}{\Omega}\sin\Omega t\left(|0e\rangle\langle 1g| + |1g\rangle\langle 0e|\right), \end{aligned} \tag{7}$$

where $\Omega = \sqrt{g^2+\delta^2}$ is the Rabi frequency.

b) Suppose the atom starts out in the excited state $|e\rangle$, and the cavity with no photon, $|0\rangle$. What is the state of the atom after time t, *if the cavity is measured and found to have no photon*? What if one photon is found to be in the cavity?

c) A reduced density matrix describes the state of part of a system, averaged over the possible states of the remainder. Give a reduced density matrix describing the state of the atom at time t.

d) Let $|e\rangle$ and $|g\rangle$ be depicted as the south and north poles of a Bloch sphere representation of the atom. Plot the points $(|e\rangle + |g\rangle)/\sqrt{2}$, $(|e\rangle - |g\rangle)/\sqrt{2}$, and $|e\rangle$. Suppose that the atom interacts with the cavity for a short time t (and the cavity starts in $|0\rangle$), after which the cavity is measured. Recall that a two-dimensional density matrix ρ can be represented by a point $\vec{r}$ *inside* the Bloch sphere, using

$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}. \tag{8}$$

Plot how these three initial states evolve under repeated short evolutions with the cavity. The cavity state is reset to $|0\rangle$ after each interaction. What is the fixed point of this process? This shows how spontaneous emission emerges as the limiting case of a continuous readout of the photon field.

3 Driven two-level atom: dressed states

A two-level atom driven by a classical laser field is often conveniently studied in a stationary basis, that is, the basis defined by the eigenstates of the Hamiltonian. These basis states are known as *dressed states*, and they are useful for interpreting many phenomena and solving problems in atomic physics.

Let the Hamiltonian for the classically driven atom be

$$H = \frac{\hbar\omega_0}{2}Z + \frac{\hbar\Omega_1}{2}\left[X\cos(\omega_L t) + Y\sin(\omega_L t)\right], \tag{9}$$

where X, Y, and Z are the usual Pauli matrices, $\hbar\omega_0$ is the energy difference between the atomic levels $|e\rangle$ and $|g\rangle$, ω_L is the laser frequency, and $\Omega_1 = eE_0\langle g|z|e\rangle/\hbar$ is the Rabi frequency.

a) Write the coupled time-dependent Schrödinger equations using a solution anzatz of the form $|\psi(t)\rangle = ae^{i\omega_1 t}|g\rangle + be^{i\omega_2 t}|e\rangle$. Choose the frequencies such that the equations are steady-state, containing no oscillating terms.

b) The Schrödinger equations you have just written are identical to those for a system with a Hamiltonian that is a function of the Rabi frequency and the detuning $\delta_L = \omega_L - \omega_0$ only. Give this Hamiltonian; denote it as H'.

c) Write H' in terms of trigonometric functions, where $\sin 2\theta = \Omega_1/\Omega$, where $\Omega = \sqrt{\Omega_1^2 + \delta_L^2}$ is the "effective" Rabi frequency.

d) Diagonalize H' to find the eigenvalues and associated eigenvectors.

e) Use these results to find the time-dependent solutions to the Schrödinger equations for H, in the original frame of reference.