Some Magnetic Traps for Neutral Atoms & The Majorana Spin-flip for $J = 1/2$.

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August 12, 2021

1 Notebook

The Mathematica notebook which contains all calculations and sketches can be downloaded via this link. Right-click and "Save" to save the notebook to your computer.

2 Quadrupole trap (anti-Helmholtz)

2.1 Calculation

In this configuration, there are two coils of radius a placed at distance $2b$ apart. Running through the coils are equal and opposite currents I and $-I$, respectively. Here, we set $I = NI_0$ where N is the number of coils and I_0 is the current going through each coil (or equivalently through all the coils).

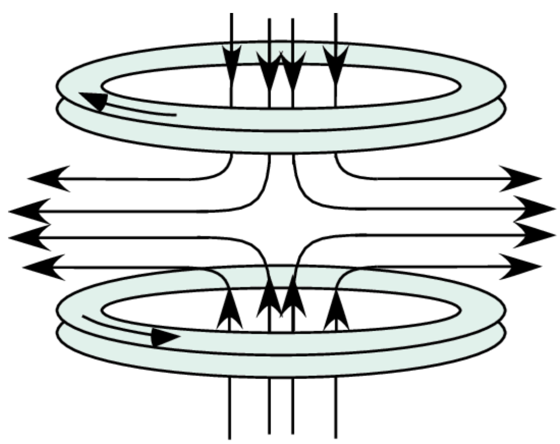

Figure 1: From [1]

To calculate the magnetic field for one coil, we can use Biot-Savart law because the current is constant. We will integrate along the closed loop C defined by the coil. The relative position between the point $\mathbf{r}$ and the point $\vec{l}$ on the wire is given by $\vec{r}' = \vec{r} - \vec{l}$

$$\vec{B}(\vec{r}) = \frac{\mu_0 I}{4\pi} \int_C \frac{d\vec{l} \times \vec{r}'}{|\vec{r}'|^3} = \frac{\mu_0 I}{4\pi} \int_C \frac{d\vec{l} \times (\vec{r} - \vec{l})}{|\vec{r} - \vec{l}|^3}.$$

We now make the approximation $|\vec{r}| \ll \vec{l}$ (i.e., we're interested in points far from the coil). With this we have the following expansion[1]

$$\frac{1}{|\vec{r}-\vec{l}|^3} \approx \frac{1}{|\vec{l}|^3} + \frac{3\vec{r}\cdot\vec{l}}{|\vec{l}|^5} + \dots$$

Plugging this back in for $\vec{B}(\vec{r})$ we find

$$\vec{B}(\vec{r}) \approx \frac{\mu_0 I}{4\pi}\int_C d\vec{l}\times\frac{\vec{r}-\vec{l}}{|\vec{l}|^3} + \frac{3\mu_0 I}{4\pi}\int_C d\vec{l}\times(\vec{r}-\vec{l})\frac{\vec{r}\cdot\vec{l}}{|\vec{l}|^5}.$$

Suppose that the center of the coil is at $+d$ from the XY plane. Going into cylindrical coordinates, we have

$$\vec{l} = a\cos\theta\hat{x} + a\sin\theta\hat{y} + d\hat{z}$$

from which we find

$$d\vec{l} = -a\sin\theta d\theta\hat{x} + a\cos\theta d\theta\hat{y}$$

So, we have

$$d\vec{l}\times(\vec{r}-\vec{l}) = (-a\sin\theta, a\cos\theta, 0)\times(x-a\cos\theta, y-a\sin\theta, z-d)$$

and

$$\vec{r}\cdot\vec{l} = xa\cos\theta + ya\sin\theta + za.$$

So, we have

$$\frac{\mu_0 I}{4\pi}\int_C d\vec{l}\times\frac{\vec{r}-\vec{l}}{|\vec{l}|^3} = \frac{\mu_0 I}{4\pi}\frac{2a^2\pi}{(d^2+a^2)^{3/2}}\hat{z} = \frac{\mu_0 I a^2}{2(d^2+a^2)^{3/2}}\hat{z}.$$

and

$$\begin{aligned}&\frac{3\mu_0 I}{4\pi}\int_C d\vec{l}\times(\vec{r}-\vec{l})\frac{\vec{r}\cdot\vec{l}}{|\vec{l}|^5}\\ &= \frac{3\mu_0 I a^2}{4(d^2+a^2)^{5/2}}\left(-x(d-z)\hat{x} - y(d-z)\hat{y} - (x^2+y^2-2dz)\hat{z}\right).\end{aligned}$$

Keeping only the linear terms and combining everything, we find the total field:

$$\begin{aligned}\vec{B}(\vec{r}) &= \frac{\mu_0 I a^2}{2(d^2+a^2)^{3/2}}\hat{z} + \frac{3\mu_0 I a^2}{4(d^2+a^2)^{5/2}}\left(-xd\hat{x} - yd\hat{y} - 2dz\hat{z}\right)\\ &= \frac{\mu_0 I a^2}{2(d^2+a^2)^{3/2}}\hat{z} + \frac{3\mu_0 I a^2 d}{2(d^2+a^2)^{5/2}}\left(-\frac{x}{2}\hat{x} - \frac{y}{2}\hat{y} - z\hat{z}\right).\end{aligned}$$

[1] a more accurate expansion will be presented in the section on the Ioffe-Pritchard trap. For our purposes in this section, it suffices to not include the term $-3|\vec{r}|^2/2|\vec{l}|^2$

In the anti-Helmholtz configuration, we have two coils of radius a placed a distance $2b$ apart from each other. When summing the two fields to get the total field, the first term cancels. So we get

$$\vec{B}_{\text{tot}}(\vec{r}) = \vec{B}_{+b}(\vec{r}) + \vec{B}_{-b}(\vec{r}) = -\frac{3\mu_0 I a^2 b}{2(a^2+b^2)^{5/2}}\,(x, y, -2z) \equiv B_0(x, y, -2z).$$

The field strength is given by

$$|B(\vec{r})| = B_0\sqrt{x^2+y^2+4z^2}.$$

2.2 Trap parameters

2.3 Simulation

3 TOP trap

3.1 Calculation

As will be discussed later, the quadrupole or anti-Helmholtz trap suffers from the “Majorna spin-flip problem” which occurs due to the presence of a zero-magnetic field point in the trap. To overcome this issue, one can add a rotating magnetic field to the existing anti-Helmholtz field so that the time-averaged magnetic field no longer has a zero at the center. This trick gives us the TOP (time orbiting potential) trap. The total field is given by

$$\vec{B}(\vec{r},t) = \vec{B}_{\text{quad}}(\vec{r}) + \vec{B}_b(\vec{r}) = B_0(x,y,-2z) + B_b(\cos\Omega t, \sin\Omega t, 0),$$

where Ω is the angular frequency of the rotating field.

The field strength is given by

$$B(\vec{r},t) = \sqrt{(B_0 x + B_b\cos\Omega t)^2 + (B_0 y + B_b \sin\Omega t)^2 + 4B_0^2 z^2}$$

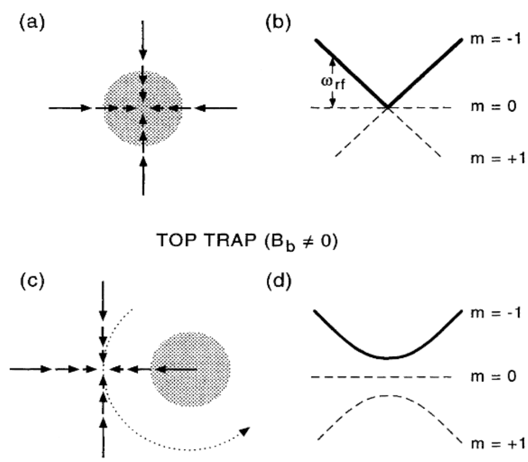

Figure 2: From [2]

For this trap to work Ω can’t be too small or too large. Ω must be larger than the oscillation frequency of the trapped particles (which is on the order of 100 Hz) so that the particles feel an effective time-averaged magnetic field. Ω should also be smaller than the frequency associated with the transition between two adjacent internal quantum states (which is on the order of 1 MHz) in order to prevent particle losses due to Majorana spin-flips.

We are interested in dynamics near the center of the trap, so we can make the approximation $r = \sqrt{x^2 + y^2 + z^2} \ll a$, under which

$$B(\vec{r}, t) \approx B_b + \frac{B_0^2}{2B_b^2}(x^2+y^2+4z^2) + \frac{B_0}{B_b}(x\cos\Omega t + y\sin\Omega t) - \frac{B_0^2}{2B_b^2}(x\cos\Omega t + y\sin\Omega t)^2.$$

The time-averaged magnetic field strength is thus, by inspection,

$$\begin{aligned}\langle B \rangle &= B_b + \frac{B_0^2}{2B_b^2}(x^2 + y^2 + 4z^2) - \frac{B_0^2}{2B_b^2}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) \\ &= B_b + \frac{B_0^2}{4B_b}(x^2 + y^2 + 8z^2).\end{aligned}$$

3.2 Trap parameters

3.3 Simulation

4 Ioffe-Pritchard traps

This section will essentially follow [4].

4.1 Calculation

There are many variants of the IP trap, the simplest being one with two coils in the anti-Helmholtz configuration and four wires in the z-direction. The four wires are suited at the corners of a square, with the currents flowing along adjacent wires being of opposite sign. A generalization of the IP trap is often called the "all coils Ioffe-Pritchard trap." The all-coil trap consists of the following set of coils:

- Big Ioffe Coils (BI), anti-Helmholtz, along z
- Small Ioffe Coils (SI), anti-Helmholtz, along x
- Pinch coils (PI), Helmholtz, along y
- Compensation coils (CO), Helmholtz, along y, opposite current to SI.

Let us revisit the calculation we've done for Helmholtz and anti-Helmholtz coils, but now we will expand to higher orders (still assuming that $\vec{r}$ is near the origin). In particular, we will use

$$\frac{1}{|\vec{r} - \vec{l}|^3} \approx \frac{1}{|\vec{l}|^3}\left[1 - \frac{3}{2}\epsilon + \frac{15}{8}\epsilon^2 - \frac{35}{16}\epsilon^3\right]$$

where

$$\epsilon = \frac{|\vec{r}|^2}{|\vec{l}|^2} - \frac{2\vec{r}\cdot\vec{l}}{|\vec{l}|^2}.$$

The expansion can be obtained by following Eq. 3.88 of [3]. For a single coil with current I placed a vertical distance $+d$ from the origin, we find the factor $|\vec{l}| = \sqrt{a^2 + d^2}$ to be constant. The field, order-by-order to third order, is thus

$$B^{(0)}(\vec{r}) = \frac{\mu_0 I}{4\pi|\vec{l}|^3} \int_C d\vec{l} \times (\vec{r} - \vec{l}) = \frac{1}{2} \frac{\mu_0 I a^2}{(a^2 + d^2)^{3/2}} \hat{z}$$

$$B^{(1)}(\vec{r}) = \frac{3\mu_0 I}{4\pi|\vec{l}|^5} \int_C d\vec{l} \times (\vec{r} - \vec{l}) \left(\frac{|\vec{r}|^2}{|\vec{l}|^2} - \frac{2\vec{r} \cdot \vec{l}}{|\vec{l}|^2} \right)$$

$$B^{(2)}(\vec{r}) = \frac{15\mu_0 I}{8\pi|\vec{l}|^7} \int_C d\vec{l} \times (\vec{r} - \vec{l}) \left(\frac{|\vec{r}|^2}{|\vec{l}|^2} - \frac{2\vec{r} \cdot \vec{l}}{|\vec{l}|^2} \right)^2$$

$$B^{(3)}(\vec{r}) = \frac{35\mu_0 I}{8\pi|\vec{l}|^9} \int_C d\vec{l} \times (\vec{r} - \vec{l}) \left(\frac{|\vec{r}|^2}{|\vec{l}|^2} - \frac{2\vec{r} \cdot \vec{l}}{|\vec{l}|^2} \right)^3$$

All these integrals can be done in Mathematica. The total field is found by summing the integrals. Collecting the terms order-by-order to third order (in z and ρ) and defining $\rho^2 = x^2 + y^2$ we find that

$$\begin{aligned} \frac{B_z(z,\rho)}{\mu_0 I} &\approx \frac{1}{2} \frac{a^2}{(a^2 + d^2)^{3/2}} + \frac{3da^2}{2(a^2 + d^2)^{5/2}} z \\ &\quad + \frac{3a^2(4d^2 - a^2)}{4(a^2 + d^2)^{7/2}} \left(z^2 - \frac{\rho^2}{2} \right) + \frac{5a^2 d(4d^2 - 3a^2)}{4(a^2 + d^2)^{9/2}} \left(z^3 - \frac{3z\rho^2}{2} \right) \\ &\equiv \frac{1}{2}\mathbb{F} + \mathbb{G}z + \frac{1}{4}\mathbb{H} \left(z^2 - \frac{\rho^2}{2} \right) + \frac{1}{2}\mathbb{I} \left(z^3 - \frac{3z\rho^2}{2} \right) \end{aligned}$$

where

$$\mathbb{F} = \frac{a^2}{(a^2 + d^2)^{3/2}}, \quad \mathbb{G} = \frac{3da^2}{2(a^2 + d^2)^{5/2}}, \quad \mathbb{H} = \frac{3a^2(4d^2 - a^2)}{(a^2 + d^2)^{7/2}}, \quad \mathbb{I} = \frac{5a^2 d(4d^2 - 3a^2)}{2(a^2 + d^2)^{9/2}}$$

are purely geometric factors. Following similar steps, we can find the field in the radial direction:

$$\begin{aligned} \frac{B_\rho}{\mu_0 I} &\approx \frac{\sqrt{B_x^2 + B_y^2}}{\mu_0 I} \\ &= \mathbb{G} \left(-\frac{\rho}{2} \right) + \frac{1}{4}\mathbb{H}(-\rho z) + \frac{1}{2}\mathbb{I} \left(\frac{3\rho^3}{8} - \frac{3\rho z^2}{2} \right). \end{aligned}$$

In these equations, we can interpret $\mathbb{F}$ as the component of the bias field, $\mathbb{G}$ of the field's gradient, and $\mathbb{H}$ of field curvature. By symmetry, there is no field in the ϕ direction.

4.1.1 Field by Helmholtz pair

To transform from one Helmholtz coil to the other in the Helmholtz configuration, we do the following:

$$I \to I, \quad d \to -d.$$

With this, we can easily compute the total field.

$$\begin{aligned} B_z(z,\rho) &\approx \mu_0 I \left[\mathbb{F} + \frac{1}{2}\mathbb{H}\left(z^2 - \frac{\rho^2}{2}\right)\right] \\ B_\rho(z,\rho) &\approx \mu_0 I \left[\frac{1}{2}\mathbb{H}(-\rho z)\right]. \end{aligned}$$

The other terms vanish since they are odd functions in d. We notice that this file only has a bias and curvature components. By letting $a = 2d$ we can make $\mathbb{H}$ vanish, producing a nearly constant field in $\hat{z}$ and almost no field in ρ. On the other hand, setting $a = d\sqrt{4/3}$ produces maximal $\mathbb{H}$ (maximal curvature).

4.1.2 Field by anti-Helmholtz pair

To transform from one Helmholtz coil to the other in the anti-Helmholtz configuration, we do the following:

$$I \to -I, \quad d \to -d.$$

With this, we can easily compute the total field:

$$\begin{aligned} B_z(z,\rho) &\approx \mu_0 I \left[2\mathbb{G}z + \mathbb{I}\left(z^3 - \frac{3z\rho^2}{2}\right)\right] \\ B_\rho(z,\rho) &\approx \mu_0 I \left[-\mathbb{G}\rho + \mathbb{I}\left(\frac{3\rho^3}{8} - \frac{3\rho z^2}{2}\right)\right] \end{aligned}$$

where now the terms that are even in d vanish. We notice that $\mathbb{I} = 0$ when $a = d\sqrt{4/3}$. Thus, configuring the field this way cancels out higher order terms. On the other hand, $\mathbb{G}$ is maximal at $a = 2d$.

4.1.3 Ioffe-Pritchard trap

With the fields calculated, we now just plug everything in and find the total field in the Ioffe-Pritchard configuration.

We start with the big Ioffe coils (BI) in the anti-Helmholtz configuration. We will impose the condition $a_{BI} = d_{BI}\sqrt{4/3}$ so that $\mathbb{I}_{BI} = 0$. In Cartesian coordinates, this field is

$$\vec{B}_{BI}(x,y,z) = \mu_0 I_{BI}\mathbb{G}_{BI}(-x,-y,2z).$$

In practice, BI coils have y as their symmetry axis, with current running in the opposite direction. The correct expression for the field is thus

$$\vec{B}_{BI}(x,y,z) = \mu_0 I_{BI}\mathbb{G}_{BI}(x,-2y,z).$$

For the small Ioffe coils, x is the symmetry axis. We also impose the condition $a_{SI} = d_{SI}\sqrt{4/3}$ so that $\mathbb{I}_{SI} = 0$. Since they're also in the anti-Helmholtz configuration we have

$$\vec{B}_{SI}(x, y, z) = \mu_0 I_{SI} \mathbb{G}_{SI}(2x, -y, -z).$$

For the pinch coils (PI) which are in the Helmholtz configuration, we find

$$\vec{B}_{PI}(x, y, z) = \mu_0 I_{PI} \left(-\frac{1}{2}\mathbb{H}_{PI} xz, -\frac{1}{2}\mathbb{H}_{PI} yz, \mathbb{F}_{PI} + \frac{1}{2}\mathbb{H}_{PI}\left(z^2 - \rho^2/2\right) \right).$$

Finally, we have the compensation coils (CO), which are in the Helmholtz configuration, but the current is in the opposite direction to PI's. The field is

$$\vec{B}_{CO}(x, y, z) = \mu_0 I_{CO} \left(\frac{1}{2}\mathbb{H}_{CO} xz, \frac{1}{2}\mathbb{H}_{CO} yz, -\mathbb{F}_{CO} - \frac{1}{2}\mathbb{H}_{CO}\left(z^2 - \rho^2/2\right) \right).$$

Adding everything up we find the full field:

$$\begin{aligned}\vec{B}(x, y, z) = &\begin{pmatrix} 0 \\ 0 \\ \mu_0 I_{PI}\mathbb{F}_{PI} - \mu_0 I_{CO}\mathbb{F}_{CO} \end{pmatrix} + \begin{pmatrix} (\mu_0 I_{BI}\mathbb{G}_{BI} + 2\mu_0 I_{SI}\mathbb{G}_{SI})x \\ -(2\mu_0 I_{BI}\mathbb{G}_{BI} + \mu_0 I_{SI}\mathbb{G}_{SI})y \\ (\mu_0 I_{BI}\mathbb{G}_{BI} - \mu_0 I_{SI}\mathbb{G}_{SI})z \end{pmatrix} \\ &+ \frac{1}{2}\left(\mu_0 I_{PI}\mathbb{H}_{PI} - \mu_0 I_{CO}\mathbb{H}_{CO}\right) \begin{pmatrix} -xz \\ -yz \\ z^2 - \rho^2/2 \end{pmatrix}.\end{aligned}$$

We now add another constraint:

$$\mu_0 I_{BI}\mathbb{G}_{BI} = \mu_0 I_{SI}\mathbb{G}_{SI}$$

so that the full field simplifies:

$$\vec{B}(x, y, z) = \delta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} x \\ -y \\ 0 \end{pmatrix} + \frac{1}{2}\beta \begin{pmatrix} -xz \\ -yz \\ z^2 - \rho^2/2 \end{pmatrix}$$

where

$$\begin{aligned}\alpha &= 3\mu_0 I_{BI}\mathbb{G}_{BI} = 3\mu_0 I_{SI}\mathbb{G}_{SI} \\ \beta &= \mu_0 I_{PI}\mathbb{H}_{PI} - \mu_0 I_{CO}\mathbb{H}_{CO} \\ \delta &= \mu_0 I_{PI}\mathbb{F}_{PI} - \mu_0 I_{CO}\mathbb{F}_{CO}.\end{aligned}$$

4.2 Trap parameters

4.3 Simulation

5 The Majorana spin-flip problem

References

[1] Reina Maruyama. *Optical trapping of ytterbium atoms.* University of Washington, 2003.

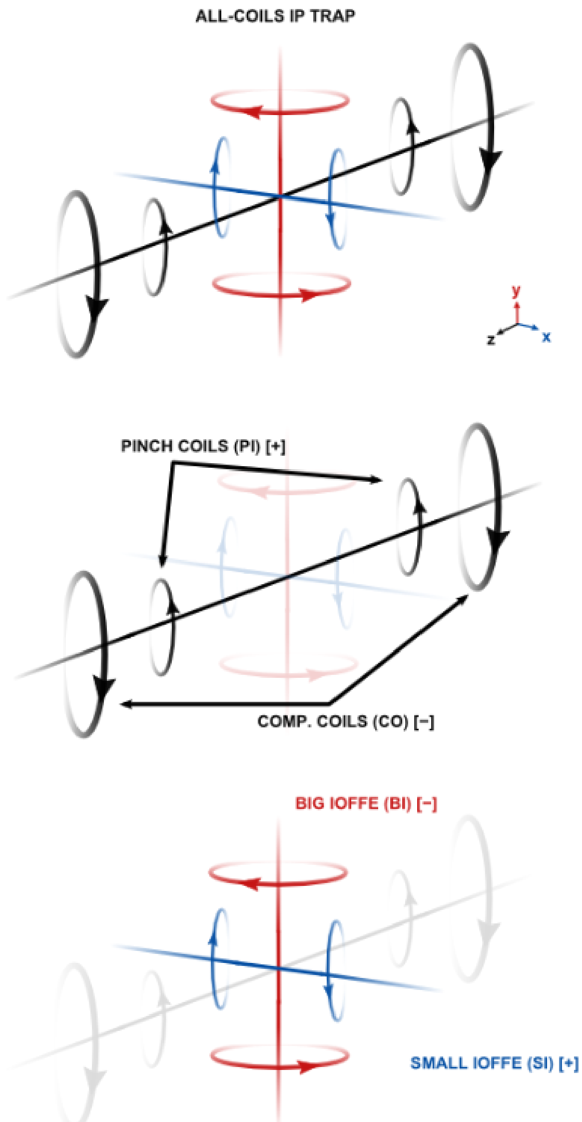

Figure 3: From [4]

[2] Wolfgang Petrich, Michael H. Anderson, Jason R. Ensher, and Eric A. Cornell. Stable, tightly confining magnetic trap for evaporative cooling of neutral atoms. *Phys. Rev. Lett.*, 74:3352–3355, Apr 1995.

[3] David J Griffiths. Introduction to electrodynamics, 2005.

[4] Danyel Cavazos. All coils ioffe-pritchard magnetic trap. Summer 2015.