Classical Mechanics III (8.09)

Assignment 3: Solutions

September 27, 2021

1. Rotation Angle in the Euler Theorem [10 points]

(a) [4 points] Let us pick $\vec{\xi}_a$ and $\vec{\xi}_b$ to be real vectors, such that

$$\begin{aligned}\vec{\xi}_1 &= \frac{1}{\sqrt{2}}(\vec{\xi}_a + i\vec{\xi}_b)\\ \vec{\xi}_2 &= \frac{1}{\sqrt{2}}(\vec{\xi}_a - i\vec{\xi}_b).\end{aligned}$$

Aside from the normalizing factor of $1/\sqrt{2}$ that we will justify shortly, note that picking $\vec{\xi}_a$ and $\vec{\xi}_b$ this way is always possible since $\vec{\xi}_2 = \vec{\xi}_1^*$. Then we have

$$\begin{aligned}\vec{\xi}_a &= \frac{1}{\sqrt{2}}(\vec{\xi}_1 + \vec{\xi}_2)\\ \vec{\xi}_b &= \frac{1}{i\sqrt{2}}(\vec{\xi}_1 - \vec{\xi}_2)\end{aligned}$$

We can now check that $(\vec{\xi}_a, \vec{\xi}_b, \vec{\xi}_3)$ are an orthonormal basis: using the old relations

$$\vec{\xi}_1^* \cdot \vec{\xi}_1 = \vec{\xi}_2^* \cdot \vec{\xi}_2 = \vec{\xi}_3^2 = 1$$

$$\vec{\xi}_1^* \cdot \vec{\xi}_2 = \vec{\xi}_2^* \cdot \vec{\xi}_1 = \vec{\xi}_1^* \cdot \vec{\xi}_3 = \vec{\xi}_2^* \cdot \vec{\xi}_3 = \vec{\xi}_1 \cdot \vec{\xi}_3 = \vec{\xi}_2 \cdot \vec{\xi}_3 = 0,$$

we have (recalling $\vec{\xi}_a^* = \vec{\xi}_a$ and $\vec{\xi}_b^* = \vec{\xi}_b$ since the vectors are real)

$$\vec{\xi}_a^2 = \vec{\xi}_a^* \cdot \vec{\xi}_a = \frac{1}{2}(\vec{\xi}_1^* \cdot \vec{\xi}_1 + \vec{\xi}_1^* \cdot \vec{\xi}_2 + \vec{\xi}_2^* \cdot \vec{\xi}_1 + \vec{\xi}_2^* \cdot \vec{\xi}_2) = 1$$

$$\vec{\xi}_b^2 = \vec{\xi}_b^* \cdot \vec{\xi}_b = \frac{1}{2}(\vec{\xi}_1^* \cdot \vec{\xi}_1 - \vec{\xi}_1^* \cdot \vec{\xi}_2 - \vec{\xi}_2^* \cdot \vec{\xi}_1 + \vec{\xi}_2^* \cdot \vec{\xi}_2) = 1$$

$$\vec{\xi}_a \cdot \vec{\xi}_b = \vec{\xi}_a^* \cdot \vec{\xi}_b = \frac{1}{2i}(\vec{\xi}_1^* \cdot \vec{\xi}_1 - \vec{\xi}_1^* \cdot \vec{\xi}_2 + \vec{\xi}_2^* \cdot \vec{\xi}_1 - \vec{\xi}_2^* \cdot \vec{\xi}_2) = 0$$

and $\vec{\xi}_a \cdot \vec{\xi}_3 = \vec{\xi}_b \cdot \vec{\xi}_3 = 0$ obviously since $\vec{\xi}_a$ and $\vec{\xi}_b$ are linear combinations of $\vec{\xi}_1$ and $\vec{\xi}_2$.

(Note that $\vec{\xi}_1$ and $\vec{\xi}_2$ are only unique up to sign: we could have defined $\vec{\xi}_b = \frac{i}{\sqrt{2}}(\vec{\xi}_1 - \vec{\xi}_2)$, for example.)

(b) [2 points] Here $\vec{u}$ has components (u_a, u_b, u_3) in the $\{\vec{\xi}_a, \vec{\xi}_b, \vec{\xi}_3\}$ basis and $\vec{s}$ has components (s_1, s_2, s_3) in the $\{\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3\}$ basis. From part (a) we see that

$$\begin{aligned} s_1\vec{\xi}_1 + s_2\vec{\xi}_2 + s_3\vec{\xi}_3 &= \frac{1}{\sqrt{2}}(s_1 + s_2)\vec{\xi}_a + \frac{i}{\sqrt{2}}(s_a - s_b)\vec{\xi}_b + s_3\vec{\xi}_3 \\ &\equiv u_a\vec{\xi}_a + u_b\vec{\xi}_b + u_3\vec{\xi}_3 \end{aligned}$$

so we find

$$\begin{pmatrix} u_a \\ u_b \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \equiv W \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}.$$

The matrix W transforms from coordinates in the $(\vec{\xi}_1, \vec{\xi}_2, \vec{\xi}_3)$ basis (which we call $\vec{s} = \begin{pmatrix} s_1 & s_2 & s_3 \end{pmatrix}^T$) to coordinates in the $(\vec{\xi}_a, \vec{\xi}_b, \vec{\xi}_3)$ basis ($\vec{u} = \begin{pmatrix} u_a & u_b & u_3 \end{pmatrix}^T$). We can check that it is unitary (as it must, to map from one orthonormal basis to another):

$$W^\dagger W = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(c) [4 points] If $\vec{r}' = U\vec{r}$, then (using $\vec{u} = WX^\dagger\vec{r}$ and $\vec{r} = XW^\dagger\vec{u}$)

$$\vec{u}' = WX^\dagger\vec{r}' = WX^\dagger U\vec{r} = WX^\dagger UXW^\dagger\vec{u}$$

or $\vec{u}' = \tilde{U}\vec{u}$ with $\tilde{U} = WX^\dagger UXW^\dagger$. We can evaluate $\tilde{U}$ directly:

$$
\begin{aligned}
\tilde{U} &= WX^\dagger UXW^\dagger = W\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} W^\dagger \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\Phi} & 0 & 0 \\ 0 & e^{-i\Phi} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2}(e^{i\Phi}+e^{-i\Phi}) & \frac{1}{2i}(e^{i\Phi}-e^{i\Phi}) & 0 \\ -\frac{1}{2i}(e^{i\Phi}-e^{i\Phi}) & \frac{1}{2}(e^{i\Phi}+e^{-i\Phi}) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos\Phi & \sin\Phi & 0 \\ -\sin\Phi & \cos\Phi & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

which is the matrix for a rotation about the $\vec{\xi}_3$ axis with angle Φ. (Either sign for Φ is fine, as it depends on the basis we chose in (a).)

2. Foucault Pendulum and the Coriolis Effect [13 points]

(a) [9 points] Let us express the unit vectors in our spherical coordinate system in terms of fixed unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$, where $+z$ points upwards (radially out from the Earth's center) and$+y$ points North perpendicular to the ground:

$$
\begin{aligned}
\hat{r} &= \sin\theta\sin\phi\hat{x} + \sin\theta\cos\phi\hat{y} - \cos\theta\hat{z} \\
\hat{\theta} &= \cos\theta\sin\phi\hat{x} + \cos\theta\cos\phi\hat{y} + \sin\theta\hat{z} \\
\hat{\phi} &= \cos\phi\hat{x} - \sin\phi\hat{y}.
\end{aligned}
$$

(Be careful: our choice of the coordinates θ and ϕ here, as seen in the diagram, is nonstandard.) Inverting the above, we have

$$
\begin{aligned}
\hat{x} &= \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi} \\
\hat{y} &= \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi} \\
\hat{z} &= -\cos\theta\hat{r} + \sin\theta\hat{\theta}.
\end{aligned}
$$

(The easiest way to see this is to remember that transformations between coordinate axes are orthogonal, and so if we write the first set of relations in matrix form then we need only take the transpose to find the inverse.) In the inertial frame, the angular velocity of the Earth is

$\vec{\omega} = \omega(\cos\lambda\hat{y} + \sin\lambda\hat{z})$, and hence in our terrestrial spherical coordinates

$$\vec{\omega} = \omega[(\cos\lambda\sin\theta\cos\phi - \sin\lambda\cos\theta)\hat{r} + (\cos\lambda\cos\theta\cos\phi + \sin\lambda\sin\theta)\hat{\theta} - \cos\lambda\sin\phi\hat{\phi}].$$

Recall that R_e is the distance from the top of the pendulum to the Earth's center. Then if the velocity of the mass in the terrestrial frame is $\vec{v} = \dot{\vec{r}}$, its velocity in the inertial frame is $\vec{v} + \vec{\omega} \times (R_e\hat{z} + \vec{r})$. Thus

$$\begin{aligned} L &= \frac{m}{2}[\vec{v} + \vec{\omega} \times (R_e\hat{z} + \vec{r})]^2 - V \\ &= \frac{m}{2}\vec{v}^2 + m\vec{v} \cdot (\vec{\omega} \times \vec{r}) + m\vec{v} \cdot (\vec{\omega} \times R_e\hat{z}) + [\vec{\omega} \times (R_e\hat{z} + \vec{r})]^2 - V. \end{aligned}$$

We shall ignore the fourth term (the centrifugal term) since it is second order in ω. Moreover, note that the third term $m\vec{v} \cdot (\vec{\omega} \times R_e\hat{z}) = \frac{d}{dt}[m\vec{r} \cdot (\vec{\omega} \times R_e\hat{z})]$ is a total derivative of the time, and (Assignment 1, Problem 6) can be neglected without changing the dynamics. Therefore we use

$$L = \frac{m}{2}\vec{v}^2 + m\vec{v} \cdot (\vec{\omega} \times \hat{r}) - V.$$

In spherical coordinates

$$\vec{v} = \dot{\vec{r}} = \ell\frac{d\hat{r}}{dt} = \ell\Big(\frac{\partial\hat{r}}{\partial\theta}\dot{\theta} + \frac{\partial\hat{r}}{\partial\phi}\dot{\phi}\Big) = \ell\big(\dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}\big)$$

and hence the first term in the Lagrangian is $\frac{m}{2}\vec{v}^2 = \frac{m}{2}\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2)$. For the second term, using the relations $\hat{r} \times \hat{\theta} = \hat{\phi}$, $\hat{\theta} \times \hat{\phi} = \hat{r}$, and $\hat{\phi} \times \hat{r} = \hat{\theta}$ we have

$$\begin{aligned} \vec{\omega} \times \vec{r} &= \omega[(\cos\lambda\sin\theta\cos\phi - \sin\lambda\cos\theta)\hat{r} + (\cos\lambda\cos\theta\cos\phi + \sin\lambda\sin\theta)\hat{\theta} - \cos\lambda\sin\phi\hat{\phi}] \times (\ell\hat{r}) \\ &= -\omega\ell[\cos\lambda\sin\phi\hat{\theta} + (\cos\lambda\cos\theta\cos\phi + \sin\lambda\sin\theta)\hat{\phi}] \end{aligned}$$

so that

$$\begin{aligned} m\vec{v} \cdot (\vec{\omega} \times \vec{r}) &= -m\omega\ell^2(\dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi}) \cdot [\cos\lambda\sin\phi\hat{\theta} + (\cos\lambda\cos\theta\cos\phi + \sin\lambda\sin\theta)\hat{\phi}] \\ &= -m\omega\ell^2[\dot{\theta}\cos\lambda\sin\phi + \dot{\phi}\sin\theta(\cos\lambda\cos\theta\cos\phi + \sin\lambda\sin\theta)]. \end{aligned}$$

Finally, taking the potential to be zero at the top of the pendulum we have $V = -mg\ell\cos\theta$. Putting everything together,

$$L = \frac{1}{2}m\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) - m\omega\ell^2\big[\dot{\theta}\cos\lambda\sin\phi + \dot{\phi}\sin\theta(\cos\lambda\cos\theta\cos\phi + \sin\lambda\sin\theta)\big] + mg\ell\cos\theta.$$

The Lagrange equation for θ is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} - \frac{\partial L}{\partial\theta} \\ &= m\ell^2[\frac{d}{dt}(\dot{\theta} - \omega\cos\lambda\sin\phi) - \sin\theta\cos\theta\dot{\phi}^2 + \omega\cos\lambda(\cos^2\theta - \sin^2\theta)\cos\phi\dot{\phi} + 2\omega\sin\lambda\sin\theta\cos\theta\dot{\phi} + \frac{g}{l}\sin\theta] \end{aligned}$$

or after simplification,

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta - \sin\theta\cos\theta\dot{\phi}^2 + 2\omega\dot{\phi}\sin\theta(\sin\lambda\cos\theta - \cos\lambda\sin\theta\cos\phi) = 0 \tag{1}$$

The Lagrange equation for ϕ is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} - \frac{\partial L}{\partial\phi} \\ &= m\ell^2\{\frac{d}{dt}[\sin^2\theta\dot{\phi} - \omega\sin\theta(\cos\lambda\cos\theta\cos\phi + \sin\lambda\sin\theta)] + \omega(\dot{\theta}\cos\lambda\cos\phi - \dot{\phi}\cos\lambda\sin\theta\cos\theta\sin\phi)]\} \end{aligned}$$

or after simplification (and division by $m\ell^2\sin\theta$),

$$\sin\theta\ddot{\phi} + \dot{\theta}(2\dot{\phi}\cos\theta + 2\omega\cos\lambda\sin\theta\cos\phi - 2\omega\sin\lambda\cos\theta) = 0. \tag{2}$$

(1) and (2) are our desired equations of motion.

(b) [4 points] With the small angle approximation we will only keep terms first order in θ, so $\sin\theta \approx \theta$ and $\cos\theta \approx 1$. Then (2) gives

$$\theta\ddot{\phi} + 2\dot{\theta}\dot{\phi} - 2\omega\sin\lambda\dot{\theta} + 2\omega\cos\lambda\theta\dot{\theta} = 0.$$

We can drop the last term here, since it is second order in θ (all the other terms are first order in θ). After neglecting this term and multiplying by θ, we get

$$\theta^2\ddot{\phi} + 2\theta\dot{\theta}\dot{\phi} - 2\omega\sin\lambda\theta\dot{\theta} = 0$$

which can be integrated to give

$$\theta^2(\dot{\phi} - \omega\sin\lambda) = C$$

for some constant C. We want to consider the perturbation caused by ω to planar motion of the pendulum (this must be enforced by suitable initial conditions), so we must eliminate the potential blowup of $\dot{\phi} \to \infty$ as $\theta \to 0$ by setting $C = 0$. This therefore gives

$$\dot{\phi} = \omega\sin\lambda$$

i.e. the pendulum undergoes precession at an angular velocity of $\omega\sin\lambda$. Now looking at (1) and using the small angle approximation gives

$$\ddot{\theta} + \frac{g}{\ell}\theta - \theta\dot{\phi}^2 + 2\omega\theta\dot{\phi}(\sin\lambda - \theta\cos\lambda\cos\phi) = 0$$

but since $\dot{\phi} \sim \omega$ all terms except the first two are at least second order in ω; neglecting those terms:

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0$$

which says simply that the pendulum undergoes simple harmonic motion in the swing plane.

3. Angular Velocity with Euler Angles [9 points]

(a) [2 points] Let us recall first how to perform the transofrmation from space axes to coordinate axes under the Euler rotations. Starting from the space coordinates (x, y, z), we first rotate about the z-axis (which is also the $\tilde{z}$ axis) by an angle of ϕ:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = D \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad D = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then from the coordinates $(\tilde{x}, \tilde{y}, \tilde{z})$, we rotate about the $\tilde{x}$-axis (which is also the x''-axis) by an angle of θ:

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = C \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

and finally from the coordinates (x'', y'', z''), we rotate about the z''-axis (which is also the z'-axis) by an angle of ψ:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = B \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}, \qquad B = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The complete transformation is given by $\vec{x}' = B\vec{x}'' = BC\vec{\tilde{x}} = BCD\vec{x}$, or $\vec{x}' = A\vec{x}$ with $A = BCD$.

Now for a given angular velocity $\vec{\omega}$ of the body, we can decompose this vector (or the infinitesimal rotation associated with it) into three successive rotations corresponding to the ones above, with angular velocities $\vec{\omega}_\phi = \dot{\phi}\hat{z} = \dot{\phi}\hat{\tilde{z}}$, $\vec{\omega}_\theta = \dot{\theta}\hat{\tilde{x}} = \dot{\theta}\hat{x}''$, $\vec{\omega}_\psi = \dot{\psi}\hat{z}'' = \dot{\psi}\hat{z}'$. We then have $\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$. We now need to express $\vec{\omega}_\phi$, $\vec{\omega}_\theta$, and $\vec{\omega}_\phi$ in terms of body coordinates (x', y', z'), using our transformation matrices above. We already have $\vec{\omega}_\psi = \dot{\psi}\hat{z}'$. For $\vec{\omega}_\theta$:

$$B \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos\psi \\ -\dot{\theta}\sin\psi \\ 0 \end{pmatrix}$$

and hence $\vec{\omega}_\theta = \dot{\theta}\cos\psi\hat{x}' - \dot{\theta}\sin\psi\hat{y}'$. Finally, for $\vec{\omega}_\phi$ we need to apply C then B:

$$BC \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = B \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ \dot{\phi}\sin\theta \\ \dot{\phi}\cos\theta \end{pmatrix} = \begin{pmatrix} \dot{\phi}\sin\theta\sin\psi \\ \dot{\phi}\sin\theta\cos\psi \\ \dot{\phi}\cos\theta \end{pmatrix}$$

and hence $\vec{\omega}_\phi = \dot{\phi}\sin\theta\hat{y}'' + \dot{\phi}\cos\theta\hat{z}'' = \dot{\phi}\sin\theta\sin\psi\hat{x}' + \dot{\phi}\sin\theta\cos\psi\hat{y}' + \dot{\phi}\cos\theta\hat{z}'$. Adding these up, we get finally

$$\begin{aligned}\vec{\omega} &= \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi \\ &= \hat{x}'(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi) + \hat{y}'(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi) + \hat{z}'(\dot{\phi}\cos\theta + \dot{\psi})\end{aligned}$$

as desired.

(b) [4 points] We can also convert $\vec{\omega}_\phi$, $\vec{\omega}_\theta$, and $\vec{\omega}_\phi$ into inertial coordinates (x, y, z) by using the inverse transformations $B^{-1} = B^T$, $C^{-1} = C^T$, and $D^{-1} = D^T$. (Rotation matrices are orthogonal matrices, so their inverses are just their transposes.) Here we don't need to do anything for $\vec{\omega}_\phi = \dot{\phi}\hat{z}$. For $\vec{\omega}_\theta$,

$$D^{-1}\begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\dot{\theta}\\0\\0\end{pmatrix} = \begin{pmatrix}\dot{\theta}\cos\phi\\ \dot{\theta}\sin\phi\\ 0\end{pmatrix}$$

and hence $\vec{\omega}_\theta = \dot{\theta}\cos\phi\hat{x} + \dot{\theta}\sin\phi\hat{y}$. Finally, for $\vec{\omega}_\psi$ we need to apply C^{-1} then D^{-1}:

$$D^{-1}C^{-1}\begin{pmatrix}0\\0\\\dot{\psi}\end{pmatrix} = D^{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}0\\0\\\dot{\psi}\end{pmatrix} = \begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0\\ -\dot{\psi}\sin\theta\\ \dot{\psi}\cos\theta\end{pmatrix}$$
$$= \begin{pmatrix}\dot{\psi}\sin\theta\sin\phi\\ -\dot{\psi}\sin\theta\cos\phi\\ \dot{\psi}\cos\theta\end{pmatrix}$$

and hence $\vec{\omega}_\psi = -\dot{\psi}\sin\theta\hat{\tilde{y}} + \dot{\psi}\cos\theta\hat{\tilde{z}} = \dot{\psi}\sin\theta\sin\phi\hat{x} - \dot{\psi}\sin\theta\cos\phi\hat{y} + \dot{\psi}\cos\theta\hat{z}$. Again adding these up,

$$\vec{\omega} = \hat{x}(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi) + \hat{y}(-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi) + \hat{z}(\dot{\psi}\cos\theta + \dot{\phi}).$$

(c) [3 points] We assume the body axes (x', y', z') are principle axes for the moment of inertia and use the result from (a). Then

$$\begin{aligned}T &= \frac{I_{x'}}{2}\omega_{x'}^2 + \frac{I_{y'}}{2}\omega_{y'}^2 + \frac{I_{z'}}{2}\omega_{z'}^2 \\ &= \frac{I_{x'}}{2}(\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)^2 + \frac{I_{y'}}{2}(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)^2 + \frac{I_{z'}}{2}(\dot{\phi}\cos\theta + \dot{\psi})^2.\end{aligned}$$

Hence

$$\frac{\partial T}{\partial\dot{\psi}} = I_{z'}(\dot{\phi}\cos\theta + \dot{\psi}) = I_{z'}\omega_{z'}$$

and

$$
\begin{aligned}
\frac{\partial T}{\partial \psi} &= I_{x'}(\dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi)(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi) + I_{y'}(\dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi)(-\dot\phi\sin\theta\sin\psi - \dot\theta\cos\psi) \\
&= I_{x'}\omega_{x'}\omega_{y'} - I_{y'}\omega_{y'}\omega_{x'}
\end{aligned}
$$

and therefore the Lagrange equation for ψ gives

$$
\frac{d}{dt}\frac{\partial T}{\partial \dot q} - \frac{\partial T}{\partial q} = I_{z'}\dot\omega_{z'} - \omega_{x'}\omega_{y'}(I_{x'} - I_{y'}) = Q_\psi = \tau_{z'}
$$

which is an Euler equation of motion.

4. Point Mass on a Disk [12 points]

(a) [4 points] About the center of the disk, we can obviously take the x-, y- and z-axes as principle axes for the moment of inertia. Assuming the disk is uniform, the area density of the disk is $\sigma = \frac{M}{\pi R^2}$, and the moment of inertia about the z-axis is

$$I_{zz} = \int \sigma(x^2 + y^2)dA = \int_0^R \sigma r^2 (2\pi r dr) = \frac{\pi\sigma R^4}{2} = \frac{MR^2}{2}$$

About the x-axis,

$$I_{xx} = \int \sigma y^2 dA = \frac{1}{2}\int \sigma(x^2 + y^2)dA = \frac{I_{zz}}{2} = \frac{MR^2}{4}$$

and by symmetry $I_{yy} = I_{xx} = \frac{MR^2}{4}$. Therefore

$$I^{disk}_{center} = \frac{MR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

To find the moment of inertia tensor of the disk about the point A, we use the parallel axis theorem

$$\begin{aligned} I^{disk}_A &= I^{disk}_{center} + M(\delta_{ab}r^2 - r_a r_b), \qquad \vec{r} = R\hat{y} \\ &= \frac{MR^2}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + MR^2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{MR^2}{4}\begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \end{aligned}$$

The point mass is at a displacement $x_1\hat{x} + y_1\hat{y} + z_1\hat{z} = R\hat{x} + R\hat{y}$ from the point A, and hence the moment of intertia tensor for the point mass is

$$I^m_A = m\begin{pmatrix} y_1^2 + z_1^2 & -x_1y_1 & -x_1z_1 \\ -x_1y_1 & x_1^2 + z_1^2 & -y_1z_1 \\ -x_1z_1 & -y_1z_1 & x_1^2 + y_1^2 \end{pmatrix} = \frac{3}{8}MR^2\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

The moment tensor is linear in mass so we can simply add the two together,

$$I_A = I^{disk}_A + I^m_A = MR^2\begin{pmatrix} \frac{13}{8} & -\frac{3}{8} & 0 \\ -\frac{3}{8} & \frac{5}{8} & 0 \\ 0 & 0 & \frac{9}{4} \end{pmatrix}.$$

(b) [4 points] We need the eigenvalues and eigenvectors of I_A. The characteristic equation is

$$
\begin{aligned}
0 &= \det(I_A - \lambda\mathbb{I}) = \begin{vmatrix} \frac{13}{8}MR^2 - \lambda & -\frac{3}{8}MR^2 & 0 \\ -\frac{3}{8}MR^2 & \frac{5}{8}MR^2 - \lambda & 0 \\ 0 & 0 & \frac{9}{4}MR^2 - \lambda \end{vmatrix} \\
&= (\frac{9}{4}MR^2 - \lambda)[(\frac{13}{8}MR^2 - \lambda)(\frac{5}{8}MR^2 - \lambda) - \frac{9}{64}(MR^2)^2] \\
&= (\frac{9}{4}MR^2 - \lambda)(\lambda^2 - \frac{9}{4}MR^2\lambda + \frac{7}{8}(MR^2)^2) \\
&= (\frac{9}{4}MR^2 - \lambda)(\frac{1}{2}MR^2 - \lambda)(\frac{7}{4}MR^2 - \lambda)
\end{aligned}
$$

and hence the eigenvalues are $I_1 = \frac{9}{4}MR^2$, $I_2 = \frac{1}{2}MR^2$, $I_3 = \frac{7}{4}MR^2$.

For $\lambda = I_1 = \frac{9}{4}MR^2$, the corresponding eigenvector is obviously $\vec{\xi}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (there are no off-diagonal terms in the last row and column, and hence the z-axis remains a principal axis). As for $I_2 = \frac{1}{2}MR^2$ and its eigenvector $\vec{\xi}_2$,

$$
(I_A - I_2\mathbb{I})\vec{\xi}_2 = 0 = MR^2 \begin{pmatrix} \frac{9}{8} & -\frac{3}{8} & 0 \\ -\frac{3}{8} & \frac{1}{8} & 0 \\ 0 & 0 & \frac{7}{4} \end{pmatrix} \begin{pmatrix} \xi_{2x} \\ \xi_{2y} \\ \xi_{2z} \end{pmatrix}
$$

which yields $\xi_{2z} = 0$ and $3\xi_{2x} - \xi_{2y} = 0$, so we can take $\vec{\xi}_2 = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}$. (The normalization of the axes is not required for full marks.) Finally, for $I_3 = \frac{7}{4}MR^2$ and $\vec{\xi}_3$

$$
(I_A - I_3\mathbb{I})\vec{\xi}_3 = 0 = MR^2 \begin{pmatrix} -\frac{1}{8} & -\frac{3}{8} & 0 \\ -\frac{3}{8} & -\frac{9}{8} & 0 \\ 0 & 0 & \frac{9}{4} \end{pmatrix} \begin{pmatrix} \xi_{3x} \\ \xi_{3y} \\ \xi_{3z} \end{pmatrix}
$$

which yields $\xi_{3z} = 0$ and $\xi_{3x} + 3\xi_{3y} = 0$, so we can take $\vec{\xi}_3 = \frac{1}{\sqrt{10}} \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$. Altogether then, the principle moments of inertia and the principle axes about A are

$$
I_1 = \frac{9}{4}MR^2,\ \vec{\xi}_1 = \hat{z}; \quad I_2 = \frac{1}{2}MR^2,\ \vec{\xi}_2 = \frac{1}{\sqrt{10}}(\hat{x} + 3\hat{y}); \quad I_3 = \frac{7}{4}MR^2,\ \vec{\xi}_3 = \frac{1}{\sqrt{10}}(-3\hat{x} + \hat{y}).
$$

(The normalization of the axes is not required for full marks.) It can be checked that the principle axes $\vec{\xi}_1$, $\vec{\xi}_2$, and $\vec{\xi}_3$ are orthogonal, as they should be.

(c) [4 points] Now the angular velocity is $\vec{\omega} = \omega\hat{y}$, and hence the angular momentum is (in body coordinates (x', y', z') where $y' = y$)

$$\vec{L} = I_A \cdot \vec{\omega} = MR^2 \begin{pmatrix} \frac{13}{8} & -\frac{3}{8} & 0 \\ -\frac{3}{8} & \frac{5}{8} & 0 \\ 0 & 0 & \frac{9}{4} \end{pmatrix} \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} = MR^2\omega \begin{pmatrix} -\frac{3}{8} \\ \frac{5}{8} \\ 0 \end{pmatrix}$$

so $\vec{L} = MR^2\omega(-\frac{3}{8}\hat{x}' + \frac{5}{8}\hat{y}')$. Recall that the body axes are time dependent. The body axes are related to fixed axes by a rotation about the $\hat{y} = \hat{y}'$-axis:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega t & 0 & \sin\omega t \\ 0 & 1 & 0 \\ -\sin\omega t & 0 & \cos\omega t \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

which means in fixed inertial (x, y, z) axes,

$$\vec{L} = MR^2\omega \begin{pmatrix} \cos\omega t & 0 & \sin\omega t \\ 0 & 1 & 0 \\ -\sin\omega t & 0 & \cos\omega t \end{pmatrix} \begin{pmatrix} -\frac{3}{8} \\ \frac{5}{8} \\ 0 \end{pmatrix} = MR^2\omega \begin{pmatrix} -\frac{3}{8}\cos\omega t \\ \frac{5}{8} \\ \frac{3}{8}\sin\omega t \end{pmatrix}$$

or $\vec{L} = MR^2\omega(-\frac{3}{8}\hat{x}\cos\omega t + \frac{5}{8}\hat{y} + \frac{3}{8}\hat{z}\sin\omega t)$. The problem does not say which way the disk rotates about $\hat{y}$ so the solution with $\omega \to -\omega$ is also fine.

5. A Rolling Cone [16 points]

(a) [5 points] Let us choose body coordinates (x', y', z') such that the origin is on the tip of the cone and the y'-axis is the axis of symmetry. The radius of the base is $r = h\tan\alpha$, and the volume of the cone is $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3 \tan^2\alpha$. The mass of the cone is therefore $M = \rho V = \frac{1}{3}\pi\rho h^3 \tan^2\alpha$.

Let us first calculate the position of the center of mass. Obviously $x'_{CM} = z'_{CM} = 0$ by symmetry. To calculate y'_{CM}, we use cylindrical coordinates (y', r, θ) about the axis of symmetry:

$$\begin{aligned} y'_{CM} &= \frac{\rho}{M}\int_0^h dy' \int_0^{y'\tan\alpha} rdr \int_0^{2\pi} d\theta \cdot y = \frac{2\pi\rho}{M}\int_0^h \frac{1}{2}y'^3 \tan^2\alpha dy' \\ &= \frac{\pi\rho h^4 \tan^2\alpha}{4M} = \frac{3h}{4}. \end{aligned}$$

The center of mass is located on the symmetry axis at a distance $\frac{3h}{4}$ from the tip.

We now calculate the moment of inertia tensor about the tip. We can choose coordinates

$x' = r\cos\theta$ and $z' = r\sin\theta$. We evaluate the terms in the moment of inertia tensor one-by-one:

$$
\begin{aligned}
I_{x'x'} &= \rho\int(y'^2+z'^2)dV = \rho\int_0^h dy'\int_0^{y\tan\alpha} rdr\int_0^{2\pi} d\theta(y'^2+r^2\sin^2\theta) \\
&= \rho\int_0^h dy'\int_0^{y'\tan\alpha} rdr(2\pi y'^2+\pi r^2) \\
&= 2\pi\rho\int_0^h dy'\Big(\frac{1}{2}y'^4\tan^2\alpha+\frac{1}{8}y'^4\tan^4\alpha\Big) \\
&= \frac{\pi\rho h^5}{20}\tan^2\alpha(4+\tan^2\alpha) \\
I_{y'y'} &= \rho\int(x'^2+z'^2)dV = \rho\int_0^h dy'\int_0^{y\tan\alpha} rdr\int_0^{2\pi} d\theta\cdot r^2 \\
&= 2\pi\rho\int_0^h dy\frac{1}{4}y'^4\tan^4\alpha \\
&= \frac{\pi\rho h^5\tan^4\alpha}{10}
\end{aligned}
$$

and by symmetry $I_{z'z'} = I_{x'x'}$. The off-diagonal terms are all zero, as can be seen from symmetry: for example in the integral for $I_{x'y'} = -\rho\int x'y'dV$, each contribution by a point at (x', y') will be canceled by a point at $(-x', y')$. (In general, if a rigid body is symmetric about a plane, then any axis perpendicular to that plane can be taken to be a principle axis. This is the case here: since the body is symmetric about the yz- and xy-planes, we can take the x- and z- axes to be principle axes.) Therefore the moment of inertia tensor about the tip is

$$
I_{tip} = \frac{\pi\rho h^5}{20}\tan^2\alpha\begin{pmatrix} 4+\tan^2\alpha & 0 & 0 \\ 0 & 2\tan^2\alpha & 0 \\ 0 & 0 & 4+\tan^2\alpha \end{pmatrix}.
$$

(b) [2 points] To get the moment of inertia tensor about the CM, we simply apply the parallel axis theorem $I_{ab}^{CM} = I_{ab}^{tip} - M(\delta_{ab}R^2 - R_aR_b)$, where here $\vec{R} = \frac{3h}{4}\hat{y}'$. Hence

$$
\begin{aligned}
I_{CM} &= \frac{\pi\rho h^5}{20}\tan^2\alpha\begin{pmatrix} 4+\tan^2\alpha & 0 & 0 \\ 0 & 2\tan^2\alpha & 0 \\ 0 & 0 & 4+\tan^2\alpha \end{pmatrix} - \frac{1}{3}\pi\rho h^3\tan^2\alpha\begin{pmatrix} \frac{9}{16}h^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{9}{16}h^2 \end{pmatrix} \\
&= \frac{\pi\rho h^5}{80}\tan^2\alpha\begin{pmatrix} 1+4\tan^2\alpha & 0 & 0 \\ 0 & 8\tan^2\alpha & 0 \\ 0 & 0 & 1+4\tan^2\alpha \end{pmatrix}.
\end{aligned}
$$

(c) [4 points] We will assume that at this moment the x'- and x- axis coincide, and that the instantaneous axis of contact coincides with the y-axis. Then the transformation from body coordinates

(x', y', z') to inertial coordinates (x, y, z) is simply given by a rotation by α about the x-axis:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \equiv U \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

Therefore in inertial coordinates, the moment of inertia tensor is

$$\begin{aligned} I_{rolling} &= U I_{tip} U^T \\ &= \frac{\pi\rho h^5}{20}\tan^2\alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 4+\tan^2\alpha & 0 & 0 \\ 0 & 2\tan^2\alpha & 0 \\ 0 & 0 & 4+\tan^2\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix} \\ &= \frac{\pi\rho h^5}{20}\tan^2\alpha \begin{pmatrix} 4+\tan^2\alpha & 0 & 0 \\ 0 & \sin^2\alpha(6+\tan^2\alpha) & \cos\alpha\sin\alpha(\tan^2\alpha-4) \\ 0 & \cos\alpha\sin\alpha(\tan^2\alpha-4) & \sin^2\alpha(2\tan^2\alpha-3)+4 \end{pmatrix}. \end{aligned}$$

(d) [5 points] The motion of the cone is just a rotation around the instantaneous line of contact (the y-axis in (c)). Let us first compute the angular velocity of the cone; it must lie along the y-axis. In inertial coordinates, the center of mass is located at

$$(x_{CM}, y_{CM}, z_{CM}) = (0, \frac{3h}{4}\cos\alpha, \frac{3h}{4}\sin\alpha).$$

The center of mass of the cone is a distance $\frac{3}{4}h\cos\alpha$ from the z-axis, and hence its path traces out a circle of circumference $\frac{3\pi}{2}h\cos\alpha$. This is done in time τ, so the speed of the center of mass must be

$$v_{CM} = \frac{3(2\pi)}{4\tau}h\cos\alpha.$$

Finally, to compute the angular velocity we notice that instantaneously the CM rotates about the y-axis; it is a distance $z_{CM} = \frac{3h}{4}\sin\alpha$ away from the y-axis, and therefore the angular velocity is

$$\vec{\omega} = \frac{v_{CM}}{z_{CM}}\hat{y} = \frac{2\pi}{\tau}\cot\alpha\hat{y}.$$

The motion of the cone is just a rotation around the instantaneous line of contact (the y-axis), and hence using the result from (c) the kinetic energy is

$$\begin{aligned} T &= \frac{1}{2}\vec{\omega}^T \cdot I_{rolling} \cdot \vec{\omega}, \qquad \vec{\omega} = \frac{2\pi}{\tau}\cot\alpha\hat{y} \\ &= \frac{I_{yy}\omega^2}{2} = \frac{1}{2}(\frac{2\pi}{\tau}\cot\alpha)^2 \cdot \frac{\pi\rho h^5}{20}\tan^2\alpha\sin^2\alpha(6+\tan^2\alpha) \\ &= \frac{\pi^3\rho h^5}{10\tau^2}\sin^2\alpha(6+\tan^2\alpha) = \frac{\pi^3\rho h^5}{10\tau^2}\tan^2\alpha(1+5\cos^2\alpha). \end{aligned}$$

This is the desired result.

An equally good way of answering this problem is to consider T as broken into a translation and rotation of the CM. In this case we must decompose $\vec{\omega}$ for components along axes centered at the CM as in part (b): $\vec{\omega} = (0, \omega\cos\alpha, -\omega\sin\alpha)$. Here we find

$$\begin{aligned} T_{trans} &= \frac{m}{2} v_{CM}^2 = \frac{3\pi^3 \rho h^5}{8\tau^2} \tan^2\alpha \cos^2\alpha \\ T_{rot} &= \frac{1}{2} \vec{\omega}^T \cdot I_{cm} \cdot \vec{\omega} = \frac{\pi^3 \rho h^5}{10\tau^2} \tan^2\alpha \left[\sin^2\alpha + \frac{9}{4}\cos^2\alpha \right] \end{aligned}$$

and the sum gives the same total T as above.