Microscopic Theory of Superfluid Helium*

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This paper discusses the properties of an interacting condensed Bose system, emphasizing those aspects which do not depend on the weakness of the potential, and which therefore apply to superfluid helium. A physical and mathematical characterization of Bose condensation is presented in terms of an additional macroscopic quantity, the wave function of the condensed mode, which is defined in terms of microscopic quantities. It is shown that in equilibrium the assumption of Bose condensation leads to a two-fluid model. The hydrodynamic generalization applicable to slowly varying disturbances from equilibrium is then discussed and rigorous microscopic expressions are derived for the parameters of this theory (including dissipative coefficients). The elementary excitation spectrum in this collision dominated regime is exhibited. The Landau quasi-particle theory is examined, as well as the relation of condensation and the excitation spectrum, to the property of superfluidity. Under certain regularity assumptions the form of the long-wavelength excitation spectrum at vanishing temperature is deduced. The corresponding derivation at finite temperature is presented and criticized. Finally, techniques are discussed for evaluating properties of the Bose system starting from the interaction potential. Approximation schemes consistent with the conservation laws and with the absence of a gap in the elementary excitation spectrum are discussed. Previous approximations for the weakly interacting Bose gas are classified and summarized and additional approximations are examined.

I. INTRODUCTION

Generally speaking, the properties of liquid helium may be discussed on three levels. The first and most striking level begins with the unique *macroscopic* aspects of the superfluid state. This level, to which London (*1*), Tisza (*2*), and Landau

(*3*) have made important contributions, may be briefly summarized by the statement that a superfluid system is one whose dynamic and equilibrium description requires a *macroscopic wave function*. Among the consequences of this fact are the existence of an additional thermodynamic variable (the superfluid velocity v_s), additional parameters in the differential equation of state (the normal fluid density ρ_n) and two fluid hydrodynamic equations with additional dissipative coefficients (the viscosities ζ_1 and ζ_3). As extended by Onsager (*4*) and Feynman (*5*), this description also leads to the existence of quantized vorticity, and the various additional thermodynamic and hydrodynamic properties this vorticity demands.

The second level of understanding is a *semiphenomenological* one. It relates the values of the macroscopic parameters, and the stability requirement of the macroscopic wave function to certain microscopically measured quantities. This level is based on the theory of Landau (*3, 6*) which states that the properties of liquid helium may be computed by treating it as a gas of weakly interacting elementary excitations (quasiparticles): the phonons and rotons. In this theory the macroscopic parameters of helium are expressed in terms of the energies of these quasiparticles. Indeed, from the measured energies of the excitations the macroscopic parameters have been computed with impressive accuracy (*7, 28*). On the basis of a Boltzmann-like kinetic equation for the gas of excitations, Landau and Khalatnikov (*8*) have also determined temperature dependent transport coefficients of the system in terms of the interactions between quasiparticles; they have then deduced two fluid hydrodynamic behavior at frequencies lower than the collision frequencies associated with these transport coefficients. This kinetic facet of the Landau theory is precisely parallel to the kinetic treatment of a classical gas by the Boltzmann equation. Both kinetic equations permit an evaluation of the transport parameters at low densities and both imply (but are not necessary for) the existence of hydrodynamic equations.

The third level on which helium may be discussed is a fully *microscopic* one. Ideally, at this level all the macroscopic properties, including those unique to the superfluid state, and those properties which are necessary for the semiphenomenological theory, would be defined and then calculated from the van der Waals force between helium atoms. Ideally, this theory would give a rigorous basis for, and limitations of, the semiphenomenological Landau theory. Even in this ideal theory, not the wave function of the system (depending on 10^{23} particles), but only certain correlation functions would be relevant.

From both a mathematical and physical point of view, each of these levels has a close parallel in the discussion of electromagnetic theory in media. The first is the level of the macroscopic Maxwell equations. The second, the level where the parameters of the constitutive equations and the thermodynamic properties of a photon gas, are expressed in terms of *empirically* determined

quantities: a photon dispersion relation, the atomic levels, the interactions of photons with one another, and multipole moments characterizing their interaction with atoms. The third level is the one in which the properties of atoms and the thermodynamic properties of the system are determined from a microscopic Hamiltonian, consisting of charges interacting with the electromagnetic field according to the microscopic Maxwell equations and Newton's laws (or the quantum mechanical microscopic dynamical equation for the charges). This parallel between helium and charged media has been extended to great length in a lecture to be published elsewhere (9); we shall not pursue it here, except to say that the coherent radiation and the photons play the role of superfluid and quasiparticles respectively.

In this paper we are concerned with the third or microscopic level. We develop the mathematical framework for discussing the equilibrium and nonequilibrium properties of superfluids from a unified microscopic point of view. Our primary aim is the elucidation of the macroscopic and semiphenomenological Landau theories (3, 10, 6, 8) of Helium II. At the outset, however, let us stress that our microscopic theory is not the ideal one described above. It falls short in making certain mathematically well defined but unverified assertions about the microscopic properties of the system. The paper is divided into five sections, each of which contains a number of subsections.

In Section II we discuss Bose condensation and introduce the mathematical formalism appropriate for describing the equilibrium and nonequilibrium properties of a condensed system. In part A we discuss the condensate, in part B the "restricted" statistical ensemble which replaces the grand canonical ensemble in describing condensed systems. In part C we modify this ensemble for *mathematical convenience* by introducing a source η^{ext} coupled to the quantum field, and argue that the "η-ensemble" thus obtained yields the same answers for physical quantities as the "restricted" ensemble of part B. In the η-ensemble we are able to write down in a simple manner the general equations of quantum hydrodynamics, valid for time varying situations.

In Section III we limit ourselves to equilibrium systems and for the most part to systems with uniform condensate. We consider cases in which the condensate (the superfluid) moves relative to the remainder (the normal fluid). We give microscopic definitions of the parameters of the ensuing two-fluid model (3). We are then able to show in Appendix B that the expressions hypothesized by Landau (10) for the currents of the conserved quantities result from these microscopic definitions. In part D we make some tentative remarks on the property of superfluidity and the related question of the stability of the equilibrium described by the two-fluid model. In part E we discuss very briefly cases in which the condensate or superfluid is not spatially uniform, that is, cases in which there are vortices.

In Section IV we study small and slow deviations from the uniform equilibrium described in Section III. We consider the implications of the two-fluid hydrodynamic equations (3, 10) which hold in this domain, and deduce the form of the correlation functions of conserved quantities, in the long wavelength, low-frequency limit. We find, in particular, correlation function expressions for the superfluid transport coefficients (Kubo relations), and for the parameters of the static two-fluid model of Section III.

In Section V we study the semiphenomenological quasi-particle theory of Landau (6, 8). We first describe without proof an effectively rigorous Landau theory which we believe is valid whenever there exist well-defined long-lived excitations. In part B we discuss the additional assumptions made by Landau (6), in describing the normal fluid as an ideal gas, and show some of the limitations of this picture. In part C we indicate some advantages, not generally appreciated, of using the model of a weakly interacting quasi-particle gas to carry out microscopic calculations. In part D we make further remarks on the Landau criterion for superfluidity, and in part E we give our version of the "proof" that at zero temperature the phonon velocity is $c = [dp/d\rho]^{1/2}$. We examine critically the assumptions that enter the proof, and show, in Appendix C how this velocity might be altered at finite temperatures.

Finally, in Section VI we review the formal procedures for carrying out microscopic calculations in superfluid systems. We prove various general identities, one of which is the generalization to finite temperature of the Hugenholtz-Pines theorem (12) on the spectrum of one-particle excitations. We examine the compatibility of this theorem with the conservation laws in any given approximation, and discuss in a unified manner the various calculations of correlation functions and excitation energies which have been carried out on the dilute gas. We have nothing to say on the difficult problem of the phase transition.

II. BOSE CONDENSATION

A. The Condensate

The fundamental feature of the microscopic theory of superfluidity rests on the existence of a condensed mode. The mathematical description of such a mode, for interacting systems, was first discussed by Penrose and Onsager (13). Let us begin by recalling their argument. Consider first, for simplicity the ground state Ψ_0 of a Bose system. Its one-particle reduced density matrix is defined by[1]

$$G_1(r, r') = \frac{N}{V} \int dr_2 \cdots \int dr_N \Psi_0(r, r_2, \cdots r_N)\Psi_0^*(r', r_2, \cdots r_N) \quad (2.1)$$

[1] Boldface type is employed in this article to distinguish operators from their expectation values, not to differentiate vectors from scalars. We believe there is no difficulty in making the latter distinction from context alone. In (2.1), for example, all coordinates, r, are vectors.

(N is the number of particles in the system, V the volume). The value of $G_1(r, r)$ is the density $n(r)$. Typically we would expect the function $G_1(r, r')$ to fall off rapidly (as an inverse power of the number of particles) when r and r' are far apart.

However, in the *ideal Bose gas* at $T = 0$ the wave function is a simple product wave function

$$\Psi_0(r_1, \cdots r_N) = \alpha(r_1) \cdots \alpha(r_N) \tag{2.2}$$

where $\alpha(r)$ is the single particle state of lowest energy consistent with the boundary conditions. Therefore the one-particle reduced density matrix of the perfect Bose gas which is given by

$$G_1(r, r') = \frac{N}{V} \alpha(r)\alpha^*(r') \tag{2.3}$$

does not become small when r and r' are far apart. If we separate the wave function into an amplitude and phase, writing

$$\sqrt{\frac{N}{V}} \alpha(r) \equiv \sqrt{n(r)} e^{i\phi(r)} \tag{2.4}$$

then we may write

$$G_1(r, r') = \sqrt{n(r)} e^{i\phi(r)} \sqrt{n(r')} e^{-i\phi(r')}. \tag{2.5}$$

Onsager and Penrose proposed that an *interacting Bose system* would, under certain conditions, have a similar density matrix, namely,

$$G_1(r, r') = \sqrt{n_0(r)} e^{i\phi(r)} \sqrt{n_0(r')} e^{-i\phi(r')} + \tilde{G}_1(r, r'). \tag{2.6}$$

Here the first term is like the one in an ideal Bose gas and the remainder $\tilde{G}_1(r, r')$, which vanishes when r and r' are far apart, is analogous to the density matrix for a normal system. We call such an interacting system condensed, and $\sqrt{n_0(r)} e^{i\phi(r)}$, its condensed mode or condensate wave function. The density of the condensed mode is $n_0(r)$ and the remainder

$$n'(r) \equiv n(r) - n_0(r) = \tilde{G}_1(r, r) \tag{2.7}$$

is called the depletion.

No aspect of this characterization requires that we be concerned with the ground state or with a stationary state or ensemble of states. Indeed if a system is in a steady state characterized by a condensed mode, it is reasonable to expect that a nonstationary state, prepared from it by a small, slow perturbation will have a time dependent density matrix of the form

$$G_1(r, r', t) = \sqrt{n_0(r, t)} e^{i\phi(r, t)} \sqrt{n_0(r', t)} e^{-i\phi(r', t)} + \tilde{G}_1(r, r', t). \tag{2.8}$$

B. The Restricted Ensemble

As in any other large system, the precise state of the condensed system is neither interesting nor measurable. We shall thus be interested in *ensembles* of states having the same measurable properties. In normal systems these properties may include the local temperature, the density, or energy density. The point about condensed systems is that since the condensate wave function is macroscopic it is effectively measurable. To put it differently, states with different condensate wave functions will, in general, have different values for measurable properties and it shall behoove us to treat these physically different systems separately. We therefore shall *restrict* ourselves to ensembles of states having the *same* condensed mode, or more precisely to ensembles in which the condensed mode has small dispersion in phase and amplitude. The fact that the restricted ensembles of states exist and do not have large dispersion in their correlations, is borne out by rigorous mathematical arguments on simple models (*14, 15*) and by approximate calculations for more realistic systems. The difference in measurable properties of systems with different condensate phase is dramatically demonstrated by the physical phenomena of coherence in condensed systems (*15, 16*). These different systems will, nevertheless, have similar thermodynamic properties when isolated. Unfortunately, the density matrix[1a] for the restricted ensembles of real systems may not be constructed *explicitly* since we cannot *prove* that a system has a condensate wave function and deduce all *possible forms* without solving many-particle Schroedinger equations with general initial conditions. It seems clear, however, that the condensed mode will have many different forms, in different physical situations; it may be time dependent or independent, spatially uniform, or nonuniform, etc.

A similar discussion applies to electromagnetic phenomena. In the usual ensembles (canonical, grand canonical, etc.) the electromagnetic field vanishes. The possibility of coherent electromagnetic waves requires us to describe separately the measurably different physical states associated with well defined time dependent or time independent electromagnetic fields. A *particular subset* of these interesting ensembles are the stationary ones associated with equilibrium nonvanishing electromagnetic fields in the absence of external currents. These arise in the discussion of ferromagnetism. Even this subset contains many different ensembles since for example ferromagnets with different gross domain structure require different ensembles. The spontaneous transformation from one gross domain structure to another generally occurs extremely slowly because of a large "potential barrier." Correspondingly, externally induced "rapid" transformations display hysteresis. The need for restricted ensembles is twofold: In the first place

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[1a] By density matrix we here mean not the function G_1 of equation (2.1) but the operator ρ such that $\langle A \rangle = \text{Tr}\,\rho A$ is the ensemble average of A. Whenever a confusion may arise we shall refer to the one as the reduced one particle density matrix and to the other simply as the density matrix.

we need them physically because we perform measurements over times short compared to relaxation times and as a result of the macroscopic coherence these measurements do not average over a set of subsystems with sufficiently random domain structure. In the second place we need restricted ensembles mathematically in order to perform calculations. Perturbation calculations express the properties of interacting systems in terms of correlation functions in the ensemble employed. In ensembles like the canonical ensemble, which average over the direction of magnetization, these correlations or fluctuations are infinite when there is macroscopic coherence. As a result the radius of convergence of perturbation theory is vanishingly small in these ensembles and it is impossible to calculate.

This second requirement applies even when the measurements considered are correctly described by expectation values in a canonical ensemble; that is, when the measurements involve astronomically long times in closed systems, or when they are repeated on an ensemble of systems whose preparation results in equal probabilities for different domain structures.

The same notions apply to superfluids: different domains correspond to different configurations of the condensed mode, namely different patterns of superfluid flow. In the context of a torus carrying a supercurrent, the two points mentioned in the preceding paragraph become the statements:

1. To describe the time dependent properties of the superfluid we need ensembles more restrictive than those appropriate to normal fluids, since the angular momentum and the superfluid flow pattern will be constant or vary slowly over experimental times.

2. Even if we wait astronomically long times until absolute equilibrium is restored or make many measurements which average over different quasi-equilibrium flow patterns, a *calculation* of the distribution in equilibrium properties using the grand canonical ensemble and a perturbation series, will be impossible since the series will have zero radius of convergence.[2]

C. A Modification of the Restricted Ensemble—The η-Ensemble

Having restricted our ensemble for the mathematical and physical reasons just discussed it is useful to modify it for reasons which do not have a parallel

[2] This may be illustrated by considering the ideal Bose gas at $T = 0$ in the grand canonical ensemble. We then have rigorously

$$\langle N^2 \rangle = 2\langle N \rangle^2 = 2\langle N_0 \rangle^2.$$

This means that there are large fluctuations in the particle number. A calculation of the equilibrium properties of the interacting gas by a perturbation series in the interparticle potential, typically involves terms like $v[\langle N^2 \rangle - \langle N \rangle^2]$. In first order we use $\langle N^2 \rangle$ for the ideal gas and obtain an infinite correction, which means that no series solution is valid. In practice all calculations on the nonideal Bose gas have been performed using some kind of restricted ensemble, for instance an ensemble with given N_0.

in the ferromagnet. From this restricted ensemble we shall construct a new restricted ensemble composed of states Ψ' which have the same physical properties as the state Ψ of the original restricted ensemble, but which are technically easier to handle. Since the phase and amplitude of the condensed mode play a central role, the new states are constructed so that in them the condensate wave function is the diagonal matrix element of some operator.

In particular, by taking linear combinations of states $\Psi_{N+\alpha}$ with $N + \alpha$ particles which differ only by having different numbers of particles in the same condensed mode, and adding them with constant relative phase

$$\Psi' = \frac{1}{\sqrt{2m+1}} \sum_{\alpha=-m}^{m} \Psi_{N+\alpha}, \qquad \sqrt{N_0} \gg m \gg 1$$

we obtain a state Ψ' whose measurable properties are the same as those of the original state, when $N_0 \equiv n_0 V \gg 1$. To this accuracy, the states Ψ' transform in time in accordance with the same Hamiltonian as Ψ. (That is, $H = \sum_i T_i + \frac{1}{2} \sum_{ij} v_{ij}$.) Furthermore a state Ψ' has the property that the diagonal matrix element of the destruction operator, $\psi(r, t)$, and the creation operator, $\psi^+(r, t)$, satisfy

$$\langle \psi(r, t) \rangle_{\Psi'} = \sqrt{n_0(r, t)} e^{i\phi(r, t)} \tag{2.9a}$$

$$\langle \psi^+(r, t) \rangle_{\Psi'} = \sqrt{n_0(r, t)} e^{-i\phi(r, t)}. \tag{2.9b}$$

In fact the states Ψ' may also be constructed by introducing an external source which adds to the Hamiltonian the term

$$H'(t) = \int \eta^{\text{ext}*}(r, t) \psi(r, t) dr + \int \eta^{\text{ext}}(r, t) \psi^+(r, t) \, dr. \tag{2.10}$$

The latter prescription for generating the state Ψ' is given in Appendix A. In equilibrium, the magnitude of the source turns out to be negligibly small (of order $1/\sqrt{N_0}$). It is determined by the requirement that the one-particle reduced density matrix

$$\langle (\psi^+(r,t) - \langle \psi^+(r,t) \rangle_{\eta^{\text{ext}}}) (\psi(r,t) - \langle \psi(r, t) \rangle_{\eta^{\text{ext}}}) \rangle_{\eta^{\text{ext}}} \equiv \tilde{G}_1(r, r', t; \eta^{\text{ext}}) \tag{2.11}$$

should not contain a condensed mode. For that purpose it is sufficient to take $\eta^{\text{ext}} \sim 1/\sqrt{N_0}$, but with a prescribed phase variation. The expectation value of an operator in a state Ψ' will differ from its value in a state Ψ by a term of order $1/\sqrt{N_0}$, which is negligible.

We may summarize this discussion by saying that an ensemble of states $\{\Psi\}$, which are restricted to have the same condensed mode, may be modified to form an ensemble of states $\{\Psi'\}$. Such a modified restricted ensemble will be called an η-ensemble. The use of an η-ensemble is justified because it predicts in a more convenient form the same properties as a restricted ensemble, *not* because it predicts the properties of a canonical ensemble.

D. The Dynamical Equations for the Condensed Mode

If the Hamiltonian of our system is $(\hbar = 1)$

$$
H = -\int dr \psi^{+}(r, t) \frac{\nabla^2}{2m} \psi(r, t) + \int dr \ U^{\text{ext}}(r, t)\psi^{+}(r, t)\psi(r, t)
$$
$$
+ \tfrac{1}{2} \int dr \ dr' \ v(r - r')\psi^{+}(r, t)\psi^{+}(r', t)\psi(r, t)\psi(r', t)
$$
(2.12)

then the equation of motion for the expectation value of the field operator in a state Ψ' or an η-ensemble of such states is

$$
i \frac{\partial}{\partial t} \langle \psi(r, t) \rangle_{\eta^{\text{ext}}} + \frac{\nabla^2}{2m} \langle \psi(r, t) \rangle_{\eta^{\text{ext}}} - U^{\text{ext}}(r, t) \langle \psi(r, t) \rangle_{\eta^{\text{ext}}}
$$
$$
- \int d\bar{r} \ v(r - \bar{r}) \langle \psi^{+}(\bar{r}, t)\psi(\bar{r}, t)\psi(r, t) \rangle_{\eta^{\text{ext}}} = \eta^{\text{ext}}(r, t).
$$
(2.13)

In order to calculate in an η-ensemble we determine the $v\langle\psi^{+}\psi\psi\rangle$ term as a functional of $\langle\psi\rangle$ and then solve (2.13) with the condition that η^{ext} be negligibly small.[3] The fact that η^{ext} is negligibly small of course does not mean that $\langle\psi\rangle$ vanishes identically as it would in the grand canonical ensemble. Rather, $\langle\psi\rangle$ is determined by (2.13).

This *formal calculational procedure* has its analog in the ferromagnetic or anti-ferromagnetic problem, where it is convenient to introduce a spatially varying magnetic field, calculate physical quantities in terms of the external field, and then let the field go to zero in such a way that the fluctuations of the magnetization should be small. There, however, a restricted density matrix constructed from a subset of possible states of the system is used, and there is no need to modify the restricted ensemble (to introduce an analogous η-ensemble).

A very formal way of describing this calculational procedure is by a variational principle (17). It is possible to construct an action integral W which is a functional of η^{ext} and whose property of stationary action leads to the equation of motion (2.13) we have just considered. In addition, considered as a functional of $\langle\psi\rangle$, W is stationary. The reason we mention this very formal possibility is that its parallel is the basis of the thermodynamic discussion of the equilibrium system to be carried out in Section III. In that case the quantity corresponding to the action is the thermodynamic potential, its stationarity of action gives rise to the equation that determines the chemical potential, and its stationarity with respect to $\langle\psi\rangle$ corresponds to an extremal condition on the free energy. We believe that provided the lifetime of the coherent mode may be regarded as infinite, η may be set to zero and the ensemble applied in time dependent situations.

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[3] This is essentially the procedure used by Bogoliubov (14) to define "quasi averages." He pointed out that in one exactly soluble model where the limits $\eta^{\text{ext}} \to 0$ and $V \to \infty$ are not interchangeable, averages of physical quantities still are the same in both limits.

In the presence of an external vector potential coupled to the momentum density, which we denote by $v_n^{\text{ext}}(r, t)$, we have

$$i \frac{\partial}{\partial t} \langle \psi(r, t) \rangle - \frac{1}{2m} \left(\frac{1}{i} \nabla - m v_n^{\text{ext}}(r, t) \right)^2 \langle \psi(r, t) \rangle - U^{\text{ext}}(r, t) \langle \psi(r, t) \rangle$$
$$- \int d\bar{r}\, v(r - \bar{r}) \langle \psi^+(\bar{r}, t) \psi(\bar{r}, t) \psi(r, t) \rangle = \eta^{\text{ext}}(r, t) \tag{2.14}$$

or

$$\left(i \frac{\partial}{\partial t} \right) \sqrt{n_0(r, t)} - \frac{1}{2m} \left[\frac{1}{i} \nabla - (m v_n^{\text{ext}} - \nabla \phi) \right]^2 \sqrt{n_0(r, t)}$$
$$- \left(U^{\text{ext}} + \frac{\partial \phi}{\partial t} \right) \sqrt{n_0(r, t)} - \int d\bar{r}\, v(r - \bar{r}) \tag{2.15}$$
$$\cdot \langle \psi^+(\bar{r}, t) \psi(\bar{r}, t)\, \psi(r, t) \rangle e^{-i\phi(r, t)} = \eta^{\text{ext}}(r, t) e^{-i\phi(r, t)}.$$

The two real equations that result can be written in the form

$$\frac{\partial}{\partial t} \frac{\phi(r, t)}{m} + \frac{1}{2} \left(v_n^{\text{ext}} - \frac{\nabla \phi}{m} \right)^2 = - \mu^{\text{loc}}(r, t) + \frac{1}{2m^2 \sqrt{n_0(r, t)}} \nabla^2 \sqrt{n_0(r, t)} \tag{2.16}$$

with

$$\mu^{\text{loc}}(r, t) \equiv \frac{1}{m} U^{\text{ext}}(r, t) + \text{Re} \left[\frac{\eta^{\text{ext}}(r, t) e^{-i\phi(r, t)}}{m \sqrt{n_0(r, t)}} \right.$$
$$\left. + \int d\bar{r}\, v(r - \bar{r})\, \langle \psi^+(\bar{r}, t) \psi(\bar{r}, t) \psi(r, t) \rangle \frac{e^{-i\phi(r, t)}}{m \sqrt{n_0(r, t)}} \right] \tag{2.17}$$

and

$$\frac{\partial}{\partial t} n_0(r, t) + \nabla \cdot \left[\left(v_n^{\text{ext}} - u - \frac{\nabla \phi}{m} \right) n_0 \right] = 0 \tag{2.18}$$

with

$$\nabla \cdot (u n_0) \equiv 2\, \text{Im} \left[\eta^{\text{ext}} e^{-i\phi(r, t)} \sqrt{n_0} \right.$$
$$\left. + \int d\bar{r}\, v(r - \bar{r})\, \langle \psi^+(\bar{r}, t) \psi(\bar{r}, t) \psi(r, t) \rangle e^{-i\phi(r, t)} \sqrt{n_0} \right]. \tag{2.19}$$

These two equations are the quantum hydrodynamic analogs of Euler's equations, valid for the condensed Bose system in both equilibrium and slowly varying nonequilibrium situations. In the next section we shall discuss equilibrium solutions both for a spatially uniform condensed mode, from which

the two fluid hydrodynamics is obtained in Section IV, and for a nonuniform condensed mode which may give rise to vortices.

The essential restriction on the equations is that the time of phase coherence (lifetime of the condensate) be long compared to the microscopic times determined from these equations.

In Section VI we discuss the perturbation expansion of (2.16)–(2.19). The functionals μ^{loc} and u are expanded in powers of the interparticle potential, in terms of the condensate ϕ, $\sqrt{n_0}$ and the condensate fluctuations which are the Green's functions.

III. EQUILIBRIUM PROPERTIES

The question of what density matrix describes equilibrium for given external force, or for given value of certain constants of the motion, is of course something that cannot be decided from the Hamiltonian alone. As we argued in Section II it is necessary to distinguish between superfluid systems having different condensed modes, and we do this by the device of the η-ensemble. Our equilibrium density matrix is therefore chosen as

$$
e^{-W} \equiv \mathrm{tr}\ \rho = \mathrm{tr}\ \exp\Bigg\{ -\beta\Bigg[H - P \cdot v_n - mN\nu \\
- \int \eta^{\mathrm{ext}*}(r)\psi(r)\ dr - \int \eta^{\mathrm{ext}}(r)\psi^+(r)\ dr \Bigg]\Bigg\}
\tag{3.1}
$$

where β is the inverse temperature and v_n, ν, η^{ext} are undetermined Lagrange multipliers, and P is the momentum operator.

The function W satisfies

$$
dW = Ed\beta - mNd(\beta\nu) - P.\ d(\beta v_n) - \int dr\ \langle \psi^+(r)\rangle\ \delta\beta\eta^{\mathrm{ext}}(r) \\
- \int dr\ \langle \psi(r)\rangle\ \delta\beta\eta^{\mathrm{ext}*}(r);
\tag{3.2}
$$

(we denote by N and P both the operators and their expectation values). Using (2.9) this may be rewritten in the form

$$
d\Bigg[W + \beta \int \langle\psi(r)\rangle\ \eta^{\mathrm{ext}*}(r)\ dr + \beta \int \langle\psi^+(r)\rangle\ \eta^{\mathrm{ext}}(r)\ dr \Bigg] \\
= Ed\beta - P.\ d(\beta v_n) - mNd(\beta\nu) + \beta \int dr\alpha(r)\delta\sqrt{n_0(r)} \\
+ \beta \int dr\theta(r)\delta\phi(r)
\tag{3.3}
$$

where

$$\alpha(r) = 2 \operatorname{Re} [\eta^{\text{ext}}(r)e^{-i\phi(r)}] \qquad (3.4a)$$

$$\theta(r) = 2 \operatorname{Im} [\eta^{\text{ext}}(r)e^{-i\phi(r)} \sqrt{n_0(r)}]. \qquad (3.4b)$$

The last term on the right of (3.3) may be rewritten as

$$- \beta \int \theta(r) \left(\nabla^2 \frac{1}{4\pi |r - r'|} \right) \delta\phi(r') \, dr \, dr' \equiv - \beta \int \lambda(r').\delta v_s(r') \, dr' \qquad (3.5)$$

where

$$\lambda(r') \equiv \int m\nabla\theta(r) \frac{dr}{4\pi |r - r'|} = + \int m\nabla\theta(r) \frac{e^{ik\cdot(r-r')}}{k^2} \, dr \, \frac{d^3k}{(2\pi)^3}, \qquad (3.6)$$

$$\nabla \cdot \lambda = -m\theta,$$

$$v_s(r) \equiv \frac{1}{m} \nabla\phi(r). \qquad (3.7)$$

Thus (3.2) now reads

$$d \left[W + \beta \int \langle \psi(r) \rangle \, \eta^{\text{ext}*}(r) \, dr + \beta \int \langle \psi^+(r) \rangle \, \eta^{\text{ext}}(r) \, dr \right]$$

$$= Ed\beta - P \cdot d(\beta v_n) - mNd(\beta v) + \beta \int dr\alpha(r)\delta \sqrt{n_0(r)} \qquad (3.8)$$

$$- \beta \int \lambda(r) \cdot \delta v_s(r) \, dr.$$

A. Uniform Condensate

We remarked earlier that it is not possible to know the form of the "constant of the motion" $\langle \psi(r) \rangle$ before solving the problem completely, whereas the other constants which we specify here, E, N, P, may be known properties of the system. In order to derive the two-fluid model we make the specification of a uniform condensate

$$\sqrt{n_0(r)} = \text{constant (independent of } r),$$

$$v_s \equiv \frac{\nabla\phi(r)}{m} = \text{constant (independent of } r). \qquad (3.9)$$

Whether or not this constraint corresponds to physically realizable equilibrium situations should be demonstrated by an accurate solution of the microscopic equations. We have not attempted this demonstration and shall content ourselves with presenting at a later stage (Section III, E) some very heuristic argu-

ments (due to Gross (*18*)) which at least show a tendency of the condensate towards uniformity when there is no rotation.

In fact we not only assume that $\sqrt{n_0}$ is constant, but also that it need not be specified independently. We thus fix $\sqrt{n_0}$ and do not consider it as an independent thermodynamic variable. This is done by minimizing (3.8) with respect to $\sqrt{n_0}$, i.e., by setting $\alpha = 0$. All the definitions and calculations of this section are *consistent* with the unique value of $\sqrt{n_0}$. Once again, however, the *validity* of not considering $\sqrt{n_0}$ as another thermodynamic variable cannot be decided a priori. It requires demonstrating that other values of $\sqrt{n_0}$ are not dynamically maintained in equilibrium.

Equation (3.8) with $v_s = $ constant and $\alpha = 0$ yields a density matrix ρ appropriate for the description of the superfluid in equilibrium with uniform condensate. The use of this density matrix corresponds to the choice of a definite η-ensemble. As we shall show below, it leads to expressions for the currents of the conserved quantities which are similar to those of the Landau two-fluid theory (*10*). Before giving our derivation, however, we must make a number of definitions.

B. Definitions

From (3.8) it follows in the usual way that the density of states may be written

$$Td(\rho s) = -\lambda \cdot dv_s - \nu d\rho - v_n \cdot dg + d\epsilon \tag{3.10}$$

thus defining the entropy per unit mass $s = S/mN$. The quantity $\rho = mn = mN/V$ is the mass density, $\epsilon = E/V$ the energy per unit volume, $g = P/V$ the momentum density, and $\lambda = (1/V)\int \lambda(r)dr$. We may introduce μ by the relation[4]

$$\nu = \mu + \frac{1}{2}(v_n - v_s)^2 - \frac{1}{2}v_n^2 \tag{3.11}$$

and also define ρ_n and ρ_s by the relation

$$\rho = \rho_n + \rho_s \tag{3.12}$$

$$g = \rho_s v_s + \rho_n v_n . \tag{3.13}$$

To see that (3.11) is the correct definition for the chemical potential we observe that if we express the energy ϵ and momentum g in terms of their values

[4] The definitions used here are the standard ones in discussions of superfluid dynamics (*10*) but they differ by a factor of the mass m from the more usual thermodynamic definitions (*25*). In particular, our quantity μ which is that of (*10*) is related to the chemical potential of (*12*) and (*34*) (which we denote by $\bar{\mu}$) by the equation $\mu = \bar{\mu}/m$.

(ϵ_0 and g_0) in the frame in which $v_s = 0$

$$\epsilon = \epsilon_0 + g_0 \cdot v_s + \tfrac{1}{2}\rho v_s^{\,2} \tag{3.14}$$

$$g = g_0 + \rho v_s = \rho_n(v_n - v_s) + \rho v_s \tag{3.15}$$

we obtain

$$Td(\rho s) = [-\lambda + \rho_s(v_s - v_n)] \cdot dv_s - (v_n - v_s) \cdot dg_0 + d\epsilon_0 - \mu d\rho. \tag{3.16}$$

Now the condensate, which is a single coherent mode, cannot contribute to the entropy; this fact was proved formally in (*17*) Eq. (69). This implies that v_s is not an independent variable in (3.16) and therefore that

$$\lambda = \rho_s(v_s - v_n). \tag{3.17}$$

The resulting thermodynamic relation is the one by which the chemical potential μ is ordinarily defined.

Our definition of ρ_n given in (3.13) is particularly simple in the uniform case. In Sections IV and V we shall show how to relate ρ_n to the correlation functions in the uniform system at rest. At this time we merely wish to point out that the ensuing quantity $\rho_s \equiv \rho - \rho_n$ is *not* the condensate density mn_0. The latter depends on the strength of the interparticle interactions in the system at rest. In helium it is estimated (*13, 19*) that $n_0/n \sim 0.1$ at $T = 0$. The superfluid density, on the other hand, is defined in terms of the momentum of the system in uniform motion (*6*) when $v_n - v_s$ is nonzero. At zero temperature $\rho_s/\rho = 1$ in helium (although $n_0/n \ll 1$). In the ideal and nonideal Bose gases (for which $n_0/n \lesssim 1$), ρ_s/ρ is also unity.

The thermodynamic identity appropriate to the frame in which $v_s = 0$ is written in an alternative form in terms of the pressure, defined by,

$$p = -\epsilon_0 + \mu\rho + g_0 \cdot (v_n - v_s) + \rho Ts, \tag{3.18}$$

namely,

$$d\mu = -s dT + \frac{1}{\rho} dp - \frac{\rho_n}{\rho}(v_n - v_s) \cdot d(v_n - v_s) - v_s \cdot d(\lambda - \rho_s(v_s - v_n)). \tag{3.19`}$$

We may also write the pressure as

$$p = \frac{W}{\beta v} + \lambda \cdot v_s$$
$$= T\rho s - \epsilon + \mu\rho + \tfrac{1}{2}\rho v_s^{\,2} + \rho_n(v_n - v_s) \cdot v_n. \tag{3.20}$$

When the momentum, and therefore v_n are constant and nonvanishing we may discuss the system in the frame in which $v_n = 0$ by performing a Galilean trans-

formation. In the resulting frame it is useful to note the thermodynamic identity

$$d\left(\frac{\epsilon - T\rho s}{\rho}\right) = (\epsilon - \rho Ts)d\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\,d\epsilon - \frac{T}{\rho}\,d(\rho s) - s\,dT$$

$$= (\epsilon - \rho Ts)d\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\,[\lambda \cdot dv_s + (\mu + \tfrac{1}{2}v_s^2)\,d\rho] - s\,dT \quad (3.21)$$

$$= -p\,d\left(\frac{1}{\rho}\right) + \frac{\rho_s}{\rho}\,v_s \cdot dv_s - s\,dT.$$

That is to say in the frame in which $v_n = 0$ the thermodynamic identity in terms of v_s, T, ρ is particularly simple since it does not involve μ (54).

C. The Two-Fluid Thermodynamics

In order to derive the two-fluid model (10) valid for small v_s and v_n it is necessary to prove that with the definitions given previously, the currents j^ϵ, T_{ij}, and μ^{loc} of the conserved quantities ϵ, g, and v_s are given respectively by

$$j^\epsilon = (\mu + \tfrac{1}{2}v_s^2)g + T\rho s v_n + \rho_n v_n v_n \cdot (v_n - v_s) \quad (3.22)$$

$$T_{ij} = \rho_n v_{n_i} v_{n_j} + \rho_s v_{s_i} v_{s_j} + p\delta_{ij} \quad (3.23)$$

$$\mu^{\text{loc}} = \mu. \quad (3.24)$$

With the machinery we have developed the proofs of these facts are relatively simple. The first two facts are most conveniently proven by evaluating the currents in the frame in which v_n vanishes. In this frame the dependence on v_s is simple since it is a property of the condensate and the condensate is simple. We shall show that in the frame in which $v_n = 0$,

$$j^\epsilon = (\mu + \tfrac{1}{2}v_s^2)\rho_s v_s \quad (3.25)$$

$$T_{ij} = \rho_s v_{s_i} v_{s_j} + p\delta_{ij}. \quad (3.26)$$

To do this we first write (3.1) as

$$-W(\beta, \mu, v_s) = \log \text{tr} \exp\left\{-\beta\left[H - mN(\mu + \tfrac{1}{2}v_s^2)\right.\right.$$

$$- \int \alpha(r)\tfrac{1}{2}(\psi(r)e^{-i\phi} + \psi^+(r)e^{i\phi})\,dr \quad (3.27)$$

$$\left.\left. - \int \lambda(r) \cdot \nabla \frac{1}{2mi\sqrt{n_0(r)}}\,(\psi(r)e^{-i\phi} - \psi^+(r)e^{i\phi})\,dr\right]\right\}$$

$$\equiv \log \text{tr}\,\varrho_{v_s}$$

where we have used the definitions (3.4). We then expand the density matrix

in powers of v_s, remembering that $\lambda(r)$ is proportional to v_s, by (3.17). We keep terms involving the operator

$$\mathbf{v}_s \equiv \frac{1}{2im}\left[\frac{\langle\psi^+\rangle\,\nabla\psi - \langle\psi\rangle\,\nabla\psi^+}{\langle\psi^+\rangle\,\langle\psi\rangle}\right]$$

and omit those with $\sqrt{n_0} \equiv \tfrac{1}{2}(\psi e^{-i\phi} + \psi^+ e^{i\phi})$, whose effect is smaller. This yields (with $\mathbf{n} = \psi^+\psi$)

$$
\begin{aligned}
\varrho_{v_s} = \varrho_{v_s=0}\bigg\{ &1 + \tfrac{1}{2}\int dr'\int^\beta m\mathbf{n}(r',-i\beta')v_s^2\,d\beta' \\
&+ \int dr'\int_0^\beta \lambda(r')\cdot\mathbf{v}_s(r',-i\beta)\,d\beta' \\
&+ \tfrac{1}{2}\int_0^\beta dr'\,d\beta'\int_0^\beta dr''\,d\beta''\,\lambda_i(r')\lambda_j(r'')(\mathbf{v}_{s_i}(r',-i\beta')\mathbf{v}_{s_j}(r'',-i\beta''))_+\bigg\}.
\end{aligned}
\tag{3.28}
$$

(We use in this part $\mathbf{A}$ for an operator and $\langle\mathbf{A}\rangle_{v_s} \equiv A_{v_s}$ for its expectation value as a function of the parameter v_s.) The parameters ρ_s and $\sqrt{n_0}$ depend on v_s^2 so that their variation only contributes in third order. We note that although ρ_s and n_0 are different their variation is on the same scale so that the criterion for the convergence of the expansion (3.28) is $v_s^2(\partial\rho_s/\partial(v_s^2)) \ll \rho_s$.

Using (3.28) we may now derive the following expressions for the currents (Appendix B):

$$\langle\mathbf{g}\rangle_{v_s} = \langle\mathbf{g}\rangle_0 + \lambda = \langle\mathbf{g}\rangle_0 + \rho_s v_s \tag{3.29}$$

$$\langle\mathbf{j}^\epsilon\rangle_{v_s} = \langle\mathbf{j}^\epsilon\rangle_0 + (\tfrac{1}{2}v_s^2 + \mu^{\text{loc}})\rho_s v_s \tag{3.30}$$

$$\langle\mathbf{T}_{ij}\rangle_{v_s} = \langle\mathbf{T}_{ij}\rangle_0 + \tfrac{1}{2}\rho v_s^2\delta_{ij} + \rho_s v_{s_i}v_{s_j} - \tfrac{1}{2}\rho_s v_s^2\delta_{ij}. \tag{3.31}$$

The first relation checks that the choice (3.17) is consistent with (3.13). The second verifies (3.25). To deduce (3.26) from (3.31) requires a slight manipulation. We note from (3.19) that, at constant μ and T and with $\lambda = \rho_s v_s$ (for $v_n = 0$) we have $dp = \rho_n v_s\cdot dv_s$ so that $(\partial p/\partial\tfrac{1}{2}v_s^2)_{T,\mu} = \rho_n$ and

$$p(\mu,\beta,v_s,v_n=0) = p(\mu,\beta,0,0) + \tfrac{1}{2}\rho_n v_s^2 + o(v_s^4). \tag{3.32}$$

Since the pressure at rest with no relative flow is given by

$$p(\mu,\beta,0,0)\delta_{ij} = \langle\mathbf{T}_{ij}\rangle_0,$$

we obtain

$$\langle\mathbf{T}_{ij}\rangle_{v_s} = p(\mu,\beta,v_s,0)\delta_{ij} + \rho_s v_{s_i}v_{s_j}. \tag{3.33}$$

Equations (3.30) and (3.33) are the required two-fluid expressions. They were obtained on the basis of a microscopic calculation in Appendix B, *not* by any Galilean transformation. The dependence on the parameter v_s could be determined exactly, because v_s is simply related to the field operators, whose commutation properties are known.

In order to obtain the general expressions for the currents we must now make a Galilean transformation to a frame in which $v_n \neq 0$. As in any system, this is straightforward, and may be done by using the unitary operator that generates Galilean transformations

$$\mathbf{U}(v) = \exp\left[iv \cdot \left(\mathbf{P}t + m \int dr \, \boldsymbol{\psi}^+(r, t)\boldsymbol{\psi}(r, t)r \right) \right]. \tag{3.34}$$

One has for example, for the total momentum

$$\begin{aligned}
\text{Tr } \mathbf{P}\varrho_{v_{s'},v_{n'}} &= \text{Tr } \mathbf{UU}^{-1}\mathbf{PU } \mathbf{U}^{-1}\varrho_{v_{s'},v_{n'}} \\
&= \text{Tr } \mathbf{U}^{-1}\mathbf{PU}\varrho_{v_{s'}-v,v_{n'}-v} = \text{Tr}(\mathbf{P} + Mv)\varrho_{v_{s'}-v,v_{n'}-v} \tag{3.35} \\
&= \text{Tr } (\mathbf{P} + Mv)\varrho_{v_s v_n}
\end{aligned}$$

which we may abbreviate as

$$\langle\mathbf{P}\rangle' = \langle\mathbf{P}\rangle + Mv.$$

By this or any of a number of other methods one deduces

$$\langle\mathbf{g}_i\rangle' = \rho v_i + \langle\mathbf{g}_i\rangle \tag{3.36a}$$

$$\langle\boldsymbol{\varepsilon}\rangle' = \tfrac{1}{2}\rho v^2 + v \cdot \langle\mathbf{g}\rangle + \langle\boldsymbol{\varepsilon}\rangle \tag{3.36b}$$

$$\langle\mathbf{j}_i^{\epsilon}\rangle' = \langle\boldsymbol{\varepsilon}\rangle v_i + \tfrac{1}{2}v^2\langle\mathbf{g}_i\rangle + \langle\mathbf{T}_{ij}\rangle v_j + \langle\mathbf{j}^{\epsilon}\rangle \tag{3.36c}$$

$$\langle\mathbf{T}_{ij}\rangle' = \rho v_i v_j + v_i\langle\mathbf{g}_j\rangle + \langle\mathbf{g}_i\rangle v_j + \langle\mathbf{T}_{ij}\rangle. \tag{3.36d}$$

Applying these formulas to (3.30) and (3.31) with $v = v_{n'}$ and $v_{s'} = v_s + v_{n'}$ we deduce the desired expressions (3.22) and (3.23) for the currents in an arbitrary (primed) frame.

We turn now to the proof of (3.24). From (2.16), when n_0 is constant, $U^{\text{ext}} = 0$ and v_n^{ext} is constant, we see that $\mu^{\text{loc}}(r, t)$ is the current of v_s. We must show that this is just the chemical potential μ. To do this, we introduce a new "time" dependence of the operators by the relation

$$\boldsymbol{\psi}(z) = e^{-\beta H_{\text{eff}}z}\boldsymbol{\psi}(0)e^{\beta H_{\text{eff}}z} \tag{3.37}$$

where

$$H_{\text{eff}} = H - P \cdot v_n - Nm[\mu + \tfrac{1}{2}(v_n - v_s)^2 - \tfrac{1}{2}v_n^2] \tag{3.38}$$

$$\begin{aligned}
H_{\text{eff}} &= \int dr \, \frac{1}{2m} \boldsymbol{\psi}^+(r) \left(\frac{1}{i}\nabla - mv_n \right)^2 \boldsymbol{\psi}(r) \\
&\quad + \frac{1}{2} \int dr \, dr' \boldsymbol{\psi}^+(r)\boldsymbol{\psi}^+(r')v(r - r')\boldsymbol{\psi}(r)\boldsymbol{\psi}(r') \tag{3.39} \\
&\quad - [\mu + \tfrac{1}{2}(v_n - v_s)^2] \int dr \, \boldsymbol{\psi}^+(r)\boldsymbol{\psi}(r)
\end{aligned}$$

The "equation of motion" for $\langle \psi(z) \rangle$ is

$$-\frac{1}{\beta}\frac{\partial}{\partial z}\langle \psi(r,z)\rangle + \frac{1}{2m}(\nabla - imv_n)^2\langle \psi(r,z)\rangle$$

$$+ [\mu + (v_n - v_s)^2]\, m\,\langle \psi(r,z)\rangle - \int dr'\, v(r-r')\langle \psi^+(r',z)\psi(r',z)\psi(r,z)\rangle = 0. \quad (3.40)$$

From its definition,

$$\langle \psi(z)\rangle = e^{-W}\,\mathrm{Tr}\,e^{-\beta H_{\mathrm{eff}}}\psi(z)$$

is independent of z. Furthermore since we have taken n_0 and v_s to be spatially uniform, we have

$$\frac{-m^2}{2m}(v_n - v_s)^2\sqrt{n_0} + [\mu + \tfrac{1}{2}(v_n - v_s)^2]m\sqrt{n_0}$$

$$= \int dr' v(r-r')\,\langle \psi^+(r',z)\psi(r',z)\psi(r,z)\rangle\, e^{-imv_s\cdot r} \quad (3.41)$$

so that

$$\mu = \mathrm{Re}\,\frac{1}{m\sqrt{n_0}}\int dr'v(r-r')\,\langle \psi^+(r',z)\psi(r',z)\psi(r,z)\rangle e^{-imv_s\cdot r} = \mu^{\mathrm{loc}}(r)$$

$$0 = \mathrm{Im}\,\frac{1}{m\sqrt{n_0}}\int dr'v(r-r')\,\langle \psi^+(r',z)\psi(r',z)\psi(r,z)\rangle\, e^{-imv_s\cdot r}, \quad (3.42)$$

μ^{loc} being given by (2.17).

D. SUPERFLUIDITY AND THE STABILITY CRITERION

In this section we wish to tie together various statements concerning the stability of the equilibrium described here in terms of the η-ensemble. We shall not have much to add to the difficult problem of calculating critical velocities, but we do wish to examine and clarify the various statements which have been made about flow instabilities. We wish to warn the reader that this discussion is tentative and is not backed up by any quantitative results. For concreteness, consider a superfluid system which has periodic boundary conditions in one dimension and is bounded in the other two (a superfluid "torus"). Let us call L the circumference of the torus and D^2 its cross section. If we assume that the condensate is uniform, the permissible values of v_s are separated by h/mL.

For such a system it is true (20) but not very relevant to note that the free energy

$$-\frac{1}{\beta V}\log\sum_p e^{-\beta(E(p)-p\cdot v_n)} = \epsilon - T\rho s - g\cdot v_n + \frac{1}{2}\rho v_n^2 \quad (3.43)$$

is a periodic function of v_n with period h/mL. This is a property of the unrestricted ensemble and reflects the fact that if we wait astronomic times, the system will go into some state in which $v_n = v_s$, thus minimizing the free energy with value $\epsilon_0(v_n = 0) - T\rho s$. On a macroscopic scale this periodic function is essentially constant. On a finer scale it looks roughly as in Fig. 1. It is clear that in equilibrium the value of v_n will correspond to one of the minima on the curve. The free energy at intermediate points is extremely large. Since we are considering a situation in which $v_s = v_n$ the maxima may be estimated by saying that the whole condensate must move with a velocity $\delta v_s \sim h/mL$. This leads to an energy

$$E \sim mN_0 \left(\frac{h}{mL}\right)^2 \sim \frac{n_0}{n} \frac{mN}{V} \frac{D^2}{L} \left(\frac{h}{m}\right)^2 \sim \frac{n_0}{n} 10^9 \frac{D^2}{L} \deg \qquad (3.44)$$

(if D and L are in centimeters; we have used the density mN/V of helium).

A more relevant picture is given by considering the possibility of metastable states (21), that is, by plotting the free energy for states with prescribed v_s. This curve is depicted in Fig. 2, for small v_n and for $v_s = 0$. The curve in Fig. 1 is the sum of curves such as in Fig. 2, for different values of v_s. A state with $g = \rho_s v_s + \rho_n v_n = \rho_n(v_n - v_s) + \rho v_s$ and $v_n \neq v_s$ must "tunnel" through a barrier of height $\sim (n_0/n)10^9(D^2/L)$ degrees in order to reach the absolute minimum of free energy $(v_n = v_s)$. This impenetrable barrier ceases to be operative only when the quantity n_0 is negligible.

Actually n_0 and ρ_s are functions of $(v_s - v_n)^2$. Indeed as $(v_s - v_n)$ is changed from zero velocity, n_0 and ρ_s decrease. In other words, particles are transferred
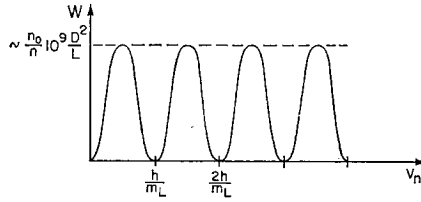

FIG. 1. The free energy as a function of v_n in absolute equilibrium.
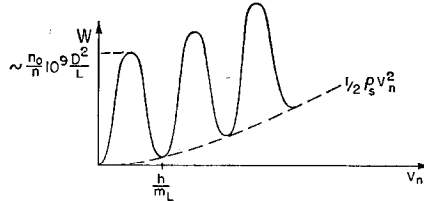

FIG. 2. The free energy as a function of v_n for fixed v_s $(v_s = 0)$.

from the condensate to the excitations. As we see from the pictures, this effect will not eliminate superfluidity until $n_0/n \sim 10^{-9}$. This change in the condensate with relative motion may be likened to the increase in the number of spin waves and decrease in magnetization as a function of temperature in a ferromagnet. This variation in the order parameter does not affect the fundamental property, ferromagnetism.

For sufficiently large velocities of uniform relative flow $(v_s - v_n)$, the superfluid undergoes a transition. How this comes about is not easily determined. Roughly speaking, two features may be discussed: (1) How the uniform condensate in relative flow "tunnels" into the normal state or into another condensed state which is nonuniform. (2) What would happen if the tunnelling could be neglected until "the mountain was removed," i.e. until n_0 became zero.

If tunnelling occurs it may be to a vortex state (which is known to exist experimentally and is a possible solution of our equations (see Part E). This is a macroscopic process, not unlike crystallization of a liquid or liquefaction of a gas. The way in which it occurs is difficult to discuss. Indeed, like condensation in a supercooled gas or liquid, vortex formation in a superfluid may occur slowly. Until it does the system will be metastable. It is most likely that the system will tunnel before the barrier is completely removed (a "first order" transition) never reaching the point of "microscopic instability" at which the barrier disappears (and the transition is "second order").

In a uniform supercooled gas this point of "microscopic instability" is the one at which the gas would be unstable to microscopically frequent density fluctuations, i.e., the point (only attained on the isotherm through the critical point) at which the compressibility of the uniform system vanishes.

In a ferromagnet it is the point at which the susceptibility vanishes. In a superfluid it is the point at which the second derivative of the free energy (or condensate susceptibility) vanishes. It is given by

$$- \frac{\partial^2}{\partial \lambda^2} \frac{W}{\beta V} = \frac{\partial}{\partial \lambda} v_s \to 0. \tag{3.45}$$

in a superfluid, which corresponds to

$$- \frac{\partial^2}{\partial \mu^2} \frac{W}{\beta V} = \frac{\partial n}{\partial \mu}\bigg)_T = n \frac{\partial n}{\partial p}\bigg)_T \to 0$$

in a gas, and to

$$- \frac{\partial^2}{\partial H^2} \frac{W}{\beta V} = \frac{\partial M}{\partial H} \to 0$$

in a ferromagnet. The condition that (3.45) be positive is a *necessary* one for superfluid flow. A more stringent condition, but still only a necessary one, is the

requirement that

$$-\frac{\delta^2}{\delta\lambda_k\ \delta\lambda_k}\frac{W}{\beta V} > 0 \qquad (3.46)$$

for all k. This is to be compared with

$$-\frac{\delta^2}{\delta H_k\ \delta H_k}\frac{W}{\beta V} > 0$$

which is violated in an antiferromagnetic substance in a state with no magnetization below the Néel temperature. It is also violated in an antiferromagnet at low temperature and strong uniform fields, when the uniform field strength which produces ferromagnetic spin alignment is reduced below some critical value. This is perhaps the closest parallel to the instability envisaged by Landau $(3, 6)$ in real helium, since his instability is also associated with a finite wave number instability in an ordered state.

Indeed, as this example demonstrates, the appearance of a microscopic instability in one ordered phase does not necessarily augur the disappearance of ordering. A different ordered phase may set in. Thus in helium, the instability of the uniform condensate to roton excitations (6) may be symptomatic of a transition to another superfluid phase with vortex structure. In this case the critical velocity for the instability of the uniform condensate phase can occur at a velocity even lower than the true critical velocity, the velocity at which superfluidity ceases. In this connection it is important to note the remarks we shall make in Section V, D on the *computation* of the critical velocity for the roton instability. At that time we shall comment further on the Landau criterion. It may be remarked that many of the possibilities we have mentioned seem relevant to superconductors of the second kind, in the presence of magnetic fields.

E. NONUNIFORM CONDENSATE

Thus far we have discussed equilibrium properties of uniformly condensed systems. In the presense of external forces, or rotating walls, even the equilibrium condensate will not be uniform. In consequence the solution of (2.13) which minimizes the free energy will have variable $n_0(r)$ and $v_s(r)$. These solutions have been investigated by Gross (18), Pitaevskii (22), and Fetter (23) in a model in which the interparticle potential is supposed to be a weak point repulsion. In that case, Eq. (2.13) reduces to

$$i\frac{\partial}{\partial t}\langle\psi(r,t)\rangle + \frac{\nabla^2}{2m}\langle\psi(r,t)\rangle - v\mid\langle\psi(r,0)\rangle\mid^2\langle\psi(r,t)\rangle = 0. \qquad (3.47)$$

On the basis of this equation Gross and Pitaevski found a solution with $v_s \sim 1/r$ and an energy per unit length like that of a classical vortex. For an assembly

of these vortices, Fetter verified Feynman's prediction that a rotating system with frequency ω would have its minimum free energy when the number of vortices per unit area was $2m\omega/h$ and they were uniformly distributed. In helium, (2.13) may not be approximated by (3.47) but there is no reason to doubt the existence of vortex solutions.

An interesting qualitative feature of the nonlinearity, pointed out by Gross, is its tendency to force the condensate to be uniform. This nonlinearity results in changing the ideal gas result for a system with walls, and the condensed mode, as in Fig. 3(a), into one in which the condensed mode has the form depicted in Fig. 3(b). It also leads to the uniformity of the condensate where there is no centrifugal barrier in a rotating system, a fact which explains qualitatively the vortex solution.
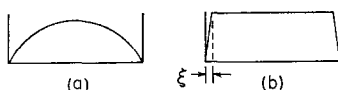

Fig. 3. The form of the condensed mode: (a) for the ideal gas; (b) for the interacting gas; ξ is a coherence length.

Fig. 4. The different approximations in the low density expansion are depicted diagrammatically. The notation is as follows: —— is the matrix $\tilde{G}_1$; ⁓ is $G_{1/2}$; and ------ the potential v.

When the equilibrium condensed mode has vortex structure the thermodynamic definitions given above do not all apply. One may still define v_n and v_s, but in the frame in which $v_s = 0$, the momentum g_0 is not parallel to $v_n - v_s$. We may define ρ_n in terms of its component $(g_0)_\parallel$ along $v_n - v_s$ as

$$g_0 = (g_0)_\parallel + (g_0)_\perp = \rho_n(v_n - v_s) + (g_0)_\perp. \tag{3.48}$$

Thermodynamic expressions in terms of the variables v_n, v_s, $\nabla \times v_s$, as well as generalizations to slowly varying situations, have been obtained by Bekarevitch and Khalatnikov (24). We shall not pursue them here. We may, however, remark on the influence of vortices on stability criteria. If the condensate is nonuniform the quantity v_s is not the only parameter describing its state. The microscopic instability condition (3.45) is then replaced by

$$\frac{-\delta}{\delta\eta^{\text{ext}}} \frac{\delta}{\delta\eta^{\text{ext}*}} \left[\frac{W}{V}\right] < 0. \tag{3.49}$$

This condition can be related to one on the excitations of the system. It turns out that even an overly simple estimate of the critical velocity for creation of classical vortex rings has the same order of magnitude as the experimentally observed critical velocity and a qualitatively correct dependence on the radius of the pipe (5). Note particularly that the vortices and their effects on the energy spectrum depend on the size of the vessel. Stability properties are therefore size dependent and indeed in an infinite system the critical velocity vanishes.

IV. THE HYDRODYNAMIC REGION[5]

In order to determine the quasi-equilibrium properties of a system it is necessary that the dynamic processes naturally divide into distinct groups. For example, if all dynamical variables other than the conserved quantities relax over times shorter than some time τ, the response for frequencies $\omega\tau \ll 1$ will be determined by the conserved properties alone. Characteristic of the superfluid is the fact that in addition to the usual conserved variables, the properties of the condensate also persist for very long times. If we considered the behavior of the system for frequencies less than the characteristic relaxation frequency of the superfluid, ordinary hydrodynamics would apply. When the system rotates rapidly, this characteristic relaxation time, related to the microscopic vorticity, is almost experimentally attainable so that the usual hydrodynamic description is almost applicable. Typically, however, we are concerned with phenomena that take place rapidly compared to relaxation of the condensate but slowly compared

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[5] Preliminary accounts of the work in this section were given in (52) and (47). After the work was completed two preprints (54, 55), were brought to our attention, in which many of the problems treated here were also discussed. However, we do not agree with the results of (55); in particular, they seem to contradict the work of Khalatnikov (8).

to the relaxation of other nonconserved properties. When the condensate is uniform, and we neglect the very slow macroscopic relaxation processes of the metastable state with given v_s, we have two rigorously conserved densities. These are the ordinary density ρ and the condensate density mn_0 which are conserved up to terms proportional to the square of the wave number, k, since their currents also obey conservation laws (see (4.2)). To order k^2, there are variations in the condensate phase, the momentum, and the energy. These are represented by the conservation laws with dissipative terms involving the currents (4.3). It is because n_0 is related exactly to a conserved current at long wavelengths that we did not consider it as a separate thermodynamic variable in Section III, and fixed it by taking $\alpha = 0$. We did, however, keep the velocity v_s as an independent variable in calculating the density of states (or entropy) as a function of λ.

We turn then to the discussion of the form of the correlation functions appropriate to this hydrodynamic domain ($\omega\tau \ll 1$, $kl \ll 1$). The discussion closely parallels that given elsewhere (25) (referred to in what follows as KM) for normal systems. In particular, we derive no new information, not contained in the (linearized) two-fluid hydrodynamic equations. The point is simply that there exist two ways of describing the response of a system to small and slow variations about the equilibrium state. The first method employs the hydrodynamic equations, which may be solved in linear approximation (10); the second description involves the linear response functions, which are expectation values in the equilibrium ensemble of products of those operators which satisfy the hydrodynamic equations. By requiring that these two descriptions coincide we may infer the form of the correlation functions in the long-wavelength low-frequency domain. The correlation functions may in principle be calculated from the microscopic theory. However it is very difficult to calculate them accurately for a real liquid; and we make no attempt in this direction. What follows is merely a statement of what the correlation functions must be, because they must agree with the phenomenological description in terms of the two-fluid hydrodynamic equations. Like the equilibrium description, this discussion is restricted to vortex free systems in which there is only one additional hydrodynamic variable v_s.

Our starting point is the fundamental thermodynamic relation which for normal fluids (25) is

$$\rho T ds = d\epsilon - (\mu + Ts)d\rho - v \cdot dg$$

and for superfluids, to linear approximation in v_n, v_s, according to (3.10), is

$$\rho T ds = d\epsilon - (\mu + Ts)d\rho - \rho_s(v_s - v_n) \cdot dv_s - v_n \cdot dg. \qquad (4.1)$$

The two-fluid hydrodynamic equations are

$$\frac{\partial v_s}{\partial t} + \nabla\left(\mu^{\text{loc}} + \frac{1}{2} v_s^2\right) = 0 \qquad (4.2a)$$

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot j^{\epsilon} = 0 \qquad (4.2\text{b})$$

$$\frac{\partial g}{\partial t} + \nabla \cdot T = 0 \qquad (4.2\text{c})$$

and, in addition,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot g = 0 \qquad (4.2\text{d})$$

$$\frac{\partial n_0}{\partial t} + \nabla \cdot n_0 (v_s - v_n) = 0. \qquad (4.2\text{e})$$

The linearized constitutive equations are, according to (3.22)–(3.24),

$$\mu^{\text{loc}}(r, t) = \mu - \zeta_2 \nabla \cdot \rho_s (v_s - v_n) - \zeta_4 \nabla \cdot v_n , \qquad (4.3\text{a})$$

$$j^{\epsilon}(r, t) = \mu \rho_s (v_s - v_n) + (\mu + Ts)\rho v_n - \kappa \nabla T, \qquad (4.3\text{b})$$

$$\begin{aligned} T_{ij}(r, t) = p\delta_{ij} &- \delta_{ij}(\zeta_1 \nabla \cdot \rho_s (v_s - v_n) + \zeta_2 \nabla \cdot v_n) \\ &- \eta(\nabla_i v_{nj} + \nabla_j v_{ni} - \tfrac{2}{3}\delta_{ij}\nabla \cdot v_n). \end{aligned} \qquad (4.3\text{c})$$

We have inserted the dissipative terms containing the viscosities ζ_1, ζ_2, ζ_3, ζ_4, η and the thermal conduction κ, in accordance with the definitions of Khalatnikov (8, 26).

The first three equations may be solved in linear approximation, for the quantities $g(k, z), q(k, z), v_s(k, z)$ in terms of the appropriate conjugate variables [cf. KM (56, 57)]. We use g instead of $\rho = mn$ since the longitudinal part of g (related to ρ by the continuity equation) is the quantity which varies in the long wavelength limit to order k^2. We shall not write down the intermediate equations corresponding to KM (56, 57), since they are rather cumbersome, but go directly to a discussion of the correlation functions.

As was done in KM, we shall construct a disturbance $\delta\mathcal{H}$ under which the system remains in local thermodynamic equilibrium for negative times.

Since we are interested in situations which differ little from equilibrium, we may suppose the disturbances produce small deviations in the uniform system, and also take the velocities to be small quantities. We may thus write, to first order

$$\mathbf{q} = \boldsymbol{\epsilon} - (\mu + Ts)\varrho, \qquad (4.4)$$

$$\mathbf{v}_s = \frac{1}{2im} \frac{\langle\psi^+\rangle \nabla\psi - \langle\psi\rangle \nabla\psi^+}{\langle\psi^+\rangle \langle\psi\rangle} . \qquad (4.5)$$

Just as for normal systems (cf. KM (69)) we now take an external disturbance

of the form

$$\delta\mathfrak{K}(t) = - \int dr[b_q(r)\mathbf{q}(r, t) + b_g(r)\cdot\mathbf{g}(r, t) + b_{v_s}(r) - \mathbf{v}_s(r, t)]e^{\epsilon t}\eta(-t) \quad (4.6)$$

where $\eta(x)$ is the step function (that is, $\eta(x) = 1$ for $x > 0$ and $\eta(x) = 0$ for $x < 0$) and where the quantities b_A are unspecified. We do not include terms proportional to $\varrho(rt) = m\mathbf{n}(rt)$ and $\sqrt{\mathbf{n}_0(rt)}$ since these operators are related to $\mathbf{g}$ and $\mathbf{v}_s$ by the conservation laws. In order to identify the b's in (4.6) we may *prove* a relation analogous to KM (64), namely that the first order change in a quantity $\langle\mathbf{A}\rangle$ induced by (4.6) is

$$\delta\langle\mathbf{A}(r, t = 0)\rangle = \frac{\partial A}{\partial T}\bigg)_{v_n, \rho_s(v_s - v_n)} Tb_q(r) + \frac{\partial A}{\partial v_n}\bigg)_{T, \rho_s(v_s - v_n)} \cdot b_g(r)$$
$$+ \frac{\partial A}{\partial \rho_s(v_s - v_n)}\bigg)_{T, v_n} \cdot b_{v_s}(r). \quad (4.7)$$

The proof of KM (64) makes use of the fluctuation dissipation theorem which is derived very easily in the grand canonical ensemble (see KM Appendix B), and similarly in the η-ensemble used here, by using the invariance properties of the trace. In the original restricted ensemble the density matrix is not known so that the algebraic proof cannot be given. However the theorem still holds; a dynamical proof may be given, which expresses the detailed balancing property in that equilibrium ensemble.

Now we may *define* the local temperature, $\delta T(r)$, and the local velocities $\delta v_n(r)$, $\delta v_s(r)$ by the relations for **A** equal to **q**, **g**, and $\mathbf{v}_s$,

$$\delta\langle\mathbf{A}(r, t = 0)\rangle = \frac{\partial A}{\partial T}\bigg)_{v_n, \rho_s(v_s - v_n)} \delta T(r) + \frac{\partial A}{\partial v_n}\bigg)_{T, \rho_s(v_s - v_n)} \cdot \delta v_n(r)$$
$$+ \frac{\partial A}{\partial \rho_s(v_s - v_n)}\bigg)_{T, v_n} \cdot \rho_s[\delta v_s(r) - \delta v_n(r)]. \quad (4.7a)$$

This is certainly a reasonable definition for $\delta T(r)$, for instance, since when $\delta T(r)$ is independent of r it reduces to the change in the temperature of the system. A comparison of (4.7a) with (4.7) permits us to identify the b's and to rewrite (4.6) in the form

$$\delta\mathfrak{K}(t) = - \int dr\left[\frac{\delta T(r)}{T}\mathbf{q}(r, t) + \delta v_n(r)\cdot\mathbf{g}(r, t)\right.$$
$$\left. + \rho_s[\delta v_s(r) - \delta v_n(r)]\cdot\mathbf{v}_s(r, t)\right]e^{\epsilon t}\eta(-t). \quad (4.6a)$$

The *purely mechanical* disturbance (4.6a) is the one which leads to hydrody-

namic behavior, and to the correlation functions in the hydrodynamic limit. It is analogous to the form KM (63) for normal systems, except that in the superfluid the additional mode with velocity v_s may be excited. We have repeated in some detail the various steps which lead to (4.6a) since the similar discussion for normal systems seems to have been largely misunderstood.

Let us now find the linear response of the system to $\delta\mathfrak{K}(t)$. This can be expressed as in KM (71) by relating the expectation values of the operators $\mathbf{g}$, $\mathbf{q}$, and $\mathbf{v}_s$ to the variations in the conjugate variables $\delta v_n(r)$, $\delta T(r)/T$, and $\rho_s[\delta v_s(\mathbf{r}) - \delta v_n(r)]$, respectively. The coefficients will then involve the correlation functions of the operators, evaluated in the equilibrium η-ensemble with no flow (for which $\delta\mathfrak{K}(t) = 0$). The correlation functions are defined by[6]

$$i\eta(t - t') \langle [A(r, t), B(r',t')]\rangle \equiv \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \chi_{AB}(k, \omega) e^{ik\cdot(r-r')-i\omega(t-t')} \quad (4.8\text{a})$$

$$\chi_{AB}(k, z) = \int \frac{d\omega}{\pi} \frac{\chi_{AB}''(k, \omega)}{\omega - z}, \quad (4.8\text{b})$$

$$\frac{1}{2} \langle [A(r, t) B(r't')]\rangle = \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \chi_{AB}''(k, \omega) e^{ik\cdot(r-r')-i\omega(t-t')}; \quad (4.9)$$

(A and B denote the operators, $\mathbf{q}$, $\mathbf{g}$, $\mathbf{v}_s$.) We refer to χ as the response or correlation function and to χ'' as the absorptive response function.

By comparing the solutions of the linearized hydrodynamic equations (cf. KM (56)) with the linear response to the disturbance $\delta\mathfrak{K}$ (cf. KM (71)) we can obtain expressions for the correlation functions, in the limit of wavelengths long compared to mean free paths and frequencies low compared to inverse collision times. In order to make this identification one must go through a formidable amount of algebraic manipulation which we shall of course not reproduce here. It is simply the solution of a homogeneous system of linear equations.

The function $\chi_{g_n g_m}$ may be broken up into longitudinal and transverse parts[7]

$$\chi_{g_n g_m}(k, \omega) \equiv \chi_{gg}^l(k, \omega) \frac{k_n k_m}{k^2} + \chi_{gg}^t(k, \omega) \left[\delta_{nm} - \frac{k_n k_m}{k^2} \right] \quad (4.10)$$

and χ_{gg}^t decouples completely from the other functions. These correlation functions, which are solutions of the system of homogeneous equations mentioned above, may all be written in the form (for ω in the upper half-plane),

$$\chi_{A,B}(k, \omega) = \frac{[-\lambda_{A,B}\omega^2 + \alpha_{A,B}c_1^2 c_2^2 k^2 + iL_{A,B}\omega^3 - i\gamma_{A,B}\omega k^2]k^2}{(\omega^2 - c_1^2 k^2 + iD_1 k^2\omega)(\omega^2 - c_2^2 k^2 + iD_2 k^2\omega)} \quad (4.11\text{a})$$

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[6] Unless otherwise specified $\int d\omega$ means $\int_{-\infty}^{\infty} d\omega$.

[7] We shall also use the notation g^l and g^t for the longitudinal and transverse momenta.

which for small D_1, D_2 is equivalent to

$$\chi_{A,B}(k,\omega) = \frac{(-\lambda_{A,B}\omega^2 + \alpha_{A,B}c_1^2c_2^2k^2 + iL_{A,B}\omega^3 - i\gamma_{A,B}\omega k^2)}{c_1^2 - c_2^2}$$

$$\cdot \left[\frac{1}{\omega^2 - c_1^2k^2 + iD_1k^2\omega} - \frac{1}{\omega^2 - c_2^2k^2 + iD_2k^2\omega}\right] \qquad (4.11\mathrm{b})$$

$$\times \left[1 + \frac{i\omega(D_1 - D_2)}{c_1^2 - c_2^2}\right].$$

This yields, to lowest order in the D's

$$\chi''_{A,B}(k,\omega) \equiv \operatorname{Im} \chi(k,\omega + i\epsilon) = \frac{(\lambda_{A,B}\omega^2 - \alpha_{A,B}c_1^2c_2^2k^2)}{c_1^2 - c_2^2}$$

$$\cdot \left[\frac{D_1k^2\omega}{(\omega^2 - c_1^2k^2)^2 + (D_1k^2\omega)^2} - \frac{D_2k^2\omega}{(\omega^2 - c_2^2k^2)^2 + (D_2k^2\omega)^2}\right]$$

$$+ \left[\frac{\omega(D_1 - D_2)(-\lambda_{A,B}\omega^2 + \alpha_{A,B}c_1^2c_2^2k^2)}{(c_1^2 - c_2^2)^2} + \frac{L_{A,B}\omega^3 - \gamma_{A,B}\omega k^2}{c_1^2 - c_2^2}\right] \qquad (4.12)$$

$$\cdot \left[\frac{\omega^2 - c_1^2k^2}{(\omega^2 - c_1^2k^2)^2 + (D_1k^2\omega)^2} + \frac{\omega^2 - c_2^2k^2}{(\omega^2 - c_2^2k^2)^2 + (D_2k^2\omega)^2}\right].$$

Here A and B refer to pairs of the quantities v_s, g^l, and q with the same time reversal properties.

These complicated expressions contain all the information inherent in the two-fluid hydrodynamics. This information is incorporated in the correlation functions in the following way:

The poles of χ represent the frequencies and damping constants, D, of the normal modes.

The residues represent the effectiveness of external disturbances in setting up these modes. The first two lines of (4.12) dominate when damping is small, and contain the resonant terms; the probability of exciting these resonances is seen to be proportional to $\lambda_{A,B}$ and $\alpha_{A,B}$. The antiresonant term involves $L_{A,B}$ and $\gamma_{A,B}$ and disappears when there is no damping (e.g., at $T = 0$). In discussing the residues one may distinguish two limits:

(a) The k-limit, in which the frequency goes to zero first leaving a function of k, after which the wave number k goes to zero. This yields the static susceptibilities which are thermodynamic derivatives of the conserved quantities with respect to their conjugate variables. These are the α's of (4.11).

(b) The ω-limit, in which the wave number goes to zero first and then the frequency. This determines the constitutive equation relating the current of a conserved quantity (e.g., j^g, the current of the momentum, or pressure) to the

gradient of the conjugate variable, (∇v_n, the velocity gradient). This may be written

$$\delta\langle j^A \rangle = \lim_{\omega \to 0} \lim_{k \to 0} \left\{ i \, \frac{\omega \chi_{AB}}{k^2} \, (k, \omega) \right\} \nabla b \qquad (4.13a)$$

$$= \lim_{\omega \to 0} \left\{ \frac{\lambda_{AB}}{i\omega} - L_{AB} \right\} \nabla b \qquad (4.13b)$$

where b is the variable conjugate to B.

From the conservation laws it follows that (4.13a) may be rewritten in terms of χ_{j_A, j_B}, the correlation function for the currents, j_A and j_B, of the conserved variables A and B, so that we have

$$\lim_{\omega \to 0} \lim_{k \to 0} \frac{\omega \chi_{AB}(k, \omega)}{k^2} = \lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{\omega} \left[\chi_{j_A j_B}(k, \omega) - \chi_{j_A j_B}(k, 0) \right]$$
$$= \lim_{\omega \to 0} \left\{ -\frac{\lambda_{AB}}{\omega} + iL_{AB} \right\}. \qquad (4.13c)$$

The real part of the curly bracket in (4.13b), the coefficient L_{AB}, is usually called a transport coefficient. The fact that the transport coefficients, which are naturally associated with the *poles* of the correlation functions (the D's in (4.11)), may also be obtained from the *residues* is the basis of expressions known as Kubo relations, namely,

$$\lim_{\omega \to 0} \lim_{k \to 0} \mathrm{Im} \left[\frac{\omega}{k^2} \chi_{AB}(k, \omega) \right] = L_{AB}. \qquad (4.13d)$$

Sum Rules

From the general dispersion relation (4.8) defining the absorptive response χ'', it is clear that we may relate the residues of the correlation functions χ, to certain integrals of χ''. We may also use an equation analogous to KM (70) to deduce other integral expressions for χ''. All these relations are known as sum rules.

Let us now discuss the results obtained for the specific correlation functions.

The Poles

They are the same for all the functions we are now discussing (i.e., those not involving g'). They are determined by the equations

$$\omega^2 = c_i^2 k^2 - iD_i k^2 \omega \qquad i = 1, 2 \quad (4.14)$$

$$c_1^2 + c_2^2 = \frac{T\rho_s s^2}{\rho_n c_v} + \frac{\partial p}{\partial \rho}\bigg)_s \qquad (4.15)$$

$$c_1^2 c_2^2 = \frac{T \rho_s s^2}{\rho_n c_v} \frac{\partial p}{\partial \rho}\bigg)_T \tag{4.16}$$

$$D_1 + D_2 = \frac{4}{3} \frac{\eta}{\rho_n} + \frac{\zeta_2}{\rho_n} - \frac{\rho_s}{\rho_n}(\zeta_1 + \zeta_4) + \frac{\zeta_3 \rho_s}{\rho_n} \rho + \frac{\kappa}{\rho c_v} \tag{4.17}$$

$$c_1^2 D_2 + c_2^2 D_1 = \left(\frac{\partial p}{\partial \rho}\right)_T \frac{\kappa}{\rho c_v} + \frac{\zeta_2 + \tfrac{4}{3}\eta}{\rho} \frac{\rho_s}{\rho_n} \left(\frac{Ts^2}{c_v} - \frac{2Ts}{\rho c_v}\left(\frac{\partial p}{\partial T}\right)_\rho + \left(\frac{\partial p}{\partial \rho}\right)_s\right)$$
$$+ \zeta_3 \frac{\rho_s}{\rho_n}\left(\frac{\partial p}{\partial \rho}\right)_s \rho - (\zeta_1 + \zeta_4)\frac{\rho_s}{\rho_n}\left[-\frac{Ts}{c_v\rho}\left(\frac{\partial p}{\partial T}\right)_\rho + \left(\frac{\partial p}{\partial \rho}\right)_s\right]. \tag{4.18}$$

These formulas may be simplified in helium since $c_p \approx c_v$, i.e., the thermal and mechanical derivatives are approximately uncoupled. With this replacement

$$c_1^2 \cong \frac{\partial p}{\partial \rho} \tag{4.19}$$

$$c_2^2 \cong -\frac{\rho_s}{\rho_n}\frac{\partial T}{\partial(1/s)} \tag{4.20}$$

$$D_1 \cong \frac{\zeta_2 + \tfrac{4}{3}\eta}{\rho} \tag{4.21}$$

$$D_2 \cong \frac{1}{\rho}\left\{\frac{\kappa}{T(\partial s/\partial T)} + \frac{\rho_s}{\rho_n}\left[\zeta_3 \rho^2 - \rho(\zeta_1 + \zeta_4) + \zeta_2 + \frac{4}{3}\eta\right]\right\}. \tag{4.22}$$

We shall work with the completely general formulas except when we state otherwise. Note incidentally that in normal fluids the formulas reduce to (cf. KM)

$$c_1^2 = \frac{dp}{d\rho}\bigg)_s; \qquad\qquad c_2 = 0;$$
$$D_1 = \frac{\tfrac{4}{3}\eta + \zeta_2}{\rho} + \frac{\kappa}{\rho}\left(\frac{1}{c_v} - \frac{1}{c_p}\right); \qquad D_2 = \frac{\kappa}{\rho c_p}. \tag{4.23}$$

The Correlation Functions Involving v_s and g^l

Their poles are given in (4.14)–(4.18), and the residues are as follows.
Static susceptibilities (k-limits)

$$\alpha_{v_s,v_s} = \frac{\partial v_s}{\partial \rho_s(v_s - v_n)}\bigg)_{v_n,T} = \frac{1}{\rho_s} = \lim_{k\to 0}\int \frac{d\omega}{\pi}\frac{\chi''_{v_s,v_s}}{\omega}(k,\omega)$$
$$= \lim_{k\to 0}\lim_{\omega\to 0}\chi_{v_s,v_s}(k,\omega); \tag{4.24}$$

$$\alpha_{v_s,g^l} = \frac{\partial g^l}{\partial \rho_s(v_s - v_n)}\bigg)_{v_n,T} = \frac{\partial[\rho_s(v_s - v_n) + \rho v_n]}{\partial \rho_s(v_s - v_n)}\bigg)_{v_n,T}$$

$$= 1 = \lim_{k \to 0} \int \frac{d\omega}{\pi} \frac{\chi''_{g^l, v_s}(k, \omega)}{\omega} ; \tag{4.25}$$

$$= \lim_{k \to 0} \lim_{\omega \to 0} \chi_{v_s, g^l}(k, \omega)$$

$$\alpha_{g^l, g^l} = \frac{\partial g^l}{\partial v_n}\Bigg)_{\rho_s(v_s - v_n), T} = \rho = \lim_{k \to 0} \int \frac{d\omega}{\pi} \frac{\chi''_{g^l, g^l}}{\omega}(k, \omega)$$

$$= \lim_{k \to 0} \lim_{\omega \to 0} \chi_{g^l, g^l}(k, \omega). \tag{4.26}$$

The above statements contain exact sum rules, valid when k is in the hydrodynamic region. Throughout this paper, when vector indices are not explicitly indicated $\chi_{v_s, v_s}(k, \omega)$ is understood to be the longitudinal part of the purely longitudinal $\chi_{v_{si}, v_{sj}}(k, \omega)$, i.e., $\chi_{v_{si}, v_{sj}}(k, \omega) = \chi_{v_s, v_s}(k, \omega) k_i k_j / k^2$.

Reactive Response and Kubo Relations (ω-limits). The strengths of interaction λ_{AB} may be obtained, as in KM, in the absence of dissipation. We find using (4.13)

$$\lambda_{v_s, v_s} = \frac{1}{\rho}\left[\left(\frac{\partial p}{\partial \rho}\right)_s - \frac{2Ts}{\rho c_v}\left(\frac{\partial p}{\partial T}\right)_\rho + \frac{Ts^2}{c_v}\right] = -\lim_{\omega \to 0}\lim_{k \to 0} \frac{\omega^2}{k^2}\chi_{v_s, v_s}(k, \omega);$$

$$\lambda_{v_s, g^l} = \frac{\partial p}{\partial \rho}\Bigg)_s - \frac{Ts}{\rho c_v}\left(\frac{\partial p}{\partial T}\right)_\rho = -\lim_{\omega \to 0}\lim_{k \to 0} \frac{\omega^2}{k^2}\chi_{v_s, g^l}(k, \omega); \tag{4.27}$$

$$\lambda_{g^l, g^l} = \rho\frac{\partial p}{\partial \rho}\Bigg)_s = -\lim_{\omega \to 0}\lim_{k \to 0} \frac{\omega^2}{k^2}\chi_{g^l, g^l}(k, \omega).$$

The Kubo relations for the transport coefficients are

$$L_{v_s, v_s} = \zeta_3 = \lim_{\omega \to 0}\lim_{k \to 0} \frac{\omega}{k^2}\chi''_{v_s, v_s}(k, \omega);$$

$$L_{v_s, g^l} = L_{g^l, v_s} = \zeta_1 = \zeta_4 = \lim_{\omega \to 0}\lim_{k \to 0} \frac{\omega}{k^2}\chi''_{g^l, v_s}(k, \omega) \tag{4.28}$$

$$L_{g^l, g^l} = \zeta_2 + \frac{4}{3}\eta = \lim_{\omega \to 0}\lim_{k \to 0} \frac{\omega}{k^2}\chi''_{g^l, g^l}(k, \omega)$$

where we have used (4.13d).

By the definition of v_s, we have an exact relation between χ_{v_s, v_s} and the one-particle Green's function $\mathcal{G}^{(0)}(k, \omega)$ to be introduced in (6.26)

$$\chi_{v_{si}, v_{sj}}(k, \omega) = -\frac{k_i k_j}{n_0 m^2} \mathcal{G}^{(0)}(k, \omega) \tag{4.29}$$

which allows us to give the full asymptotic form for $\mathcal{G}^{(0)}$ in the hydrodynamic region, from the knowledge of the coefficients α, λ, L, γ for χ_{v_s, v_s}.

$$\frac{\mathcal{G}^{(0)}(k,\,\omega)}{n_0\,m^2}$$

$$= \frac{\dfrac{\omega^2}{\rho}\left[\left(\dfrac{\partial p}{\partial \rho}\right)_s - \dfrac{2Ts}{\rho c_v}\left(\dfrac{\partial p}{\partial T}\right)_\rho + \dfrac{Ts^2}{c_v}\right] - \dfrac{Ts^2}{c_v\,\rho_n}\left(\dfrac{\partial p}{\partial \rho}\right)_T k^2 - i\zeta_3\,\omega^3 + i\gamma_{v_s,v_s}\omega k^2}{(\omega^2 - c_1^2 k^2 + iD_1 k^2\omega)(\omega^2 - c_2^2 k^2 + iD_2 k^2\omega)}\,. \tag{4.30}$$

The sum rules (4.27) and the relation in (4.28) on χ''_{v_s,v_s} can be immediately transcribed in terms of the weight function $\mathcal{Q}^{(0)}(k,\,\omega)$ of $\mathcal{G}^{(0)}$.

The Density-Density Correlation Function $\chi_{\rho,\rho}$. As we stated earlier, we may use the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot g = 0$$

and the commutation relation

$$[\rho(r),\, g(r')] = -i\nabla_r\delta(r - r')$$

to obtain

$$\chi_{\rho,\rho} = \frac{-\rho k^2}{\omega^2} + \frac{k^2}{\omega^2}\chi^l_{g,g}\,. \tag{4.31}$$

This enables us to write down the formula for $\chi_{\rho,\rho}$ from the expressions obtained above for the coefficients λ, α, L, γ of $\chi^l_{g,g}$

$$\frac{\chi_{\rho,\rho}(k,\,\omega)}{\rho k^2}$$

$$= \frac{-\omega^2 + k^2\dfrac{\rho_s}{\rho_n}\dfrac{Ts^2}{c_v} - ik^2\omega\left[\dfrac{\rho_s}{\rho_n}\left(\dfrac{\tfrac{4}{3}\eta + \zeta_2}{\rho} - \zeta_1 - \zeta_4 + \rho\zeta_3\right) + \dfrac{\kappa}{\rho c_v}\right]}{(\omega^2 - c_1^2 k^2 + iD_1 k^2\omega)(\omega^2 - c_2^2 k^2 + iD_2 k^2\omega)}\,. \tag{4.32}$$

We note that this formula is again of the type (4.10), since ρ is a conserved variable. In particular, the static susceptibility yields the "compressibility sum-rule" (*27*)

$$\alpha_{\rho,\rho} = \rho\,\frac{\partial \rho}{\partial p}\bigg)_T = \left(\frac{\partial \rho}{\partial \mu}\right)_T = \frac{\rho}{c_T^2} = \lim_{k\to 0}\lim_{\omega\to 0}\chi_{\rho,\rho}(k,\,\omega)$$

$$= \lim_{k\to 0}\int \frac{d\omega}{\pi}\,\frac{\chi''_{\rho,\rho}(k,\,\omega)}{\omega}\,. \tag{4.33}$$

The constitutive equation of type (4.13a) relating the current (momentum density g^l) to the force $\nabla\mu = (1/\rho)\nabla p$, is given by

$$\delta g^l \ = \ \left(\frac{\lambda_{\rho,\rho}}{i\omega} - L_{\rho,\rho} \right) \nabla \mu \ = \ \frac{\rho}{i\omega} \, \nabla \mu. \tag{4.34}$$

This is just the equation $\partial g^l / \partial t \ = - \nabla p$, and is in line with the fact that there is no momentum dissipation in a one component system ($L_{\rho,\rho} = 0$).

The function $\chi''_{\rho,\rho}$ [multiplied by $(2N)(1 - e^{-\beta\omega})^{-1}$] is just the liquid structure function $S(k, \omega)$ measured in neutron scattering experiments. It must be noted however, that such experiments involve wavenumbers k which are outside the range of validity of (4.32) by several orders of magnitude, so that no comparison is possible. The form predicted by our hydrodynamic analysis is rigorously correct at low frequencies and long wavelengths. It is instructive to see how forms more complicated than

$$\chi_{\rho,\rho}(k, \omega) \ = \ \frac{-\rho k^2}{\omega^2 - c_T^2 k^2} \tag{4.35}$$

can arise in the hydrodynamic region, without violating the sum rules (4.33) and

$$\lambda_{\rho,\rho} \ = \ \rho \ = \ \lim_{k \to 0} \int \frac{d\omega}{\pi} \, \frac{\omega}{k^2} \, \chi''_{\rho,\rho}(k, \omega) \ = \ \lim_{k \to 0} \int \frac{d\omega}{\pi} \, \frac{\chi''^{\,l}_{g,g}(k, \omega)}{\omega}. \tag{4.36}$$

The expression we have derived in (4.32) is in fact consistent, as it must be, with the rigorous low k and ω dispersion relation

$$\chi''_{\rho,\rho}(k, \omega) \ = \ \frac{\rho \Gamma(k, \omega) k^4 \omega}{\left\{ \omega^2 - k^2 \left(\dfrac{\partial p}{\partial \rho} \right)_T - \omega^2 k^2 P \displaystyle\int \frac{d\omega'}{\pi} \frac{\Gamma(k, \omega')}{\omega^2 - \omega'^2} \right\}^2 + \{ \omega k^2 \Gamma(k, \omega) \}^2} \tag{4.37}$$

given in KM (86). However the spectral function is

$$\Gamma(k, \omega) \cong \left[\pi \, | \, \omega \, | \, \delta \left(\omega^2 - k^2 \, \frac{\rho_s}{\rho_n} \, \frac{Ts^2}{c_v} \right) \right] \left(\frac{c_p}{c_v} - 1 \right) \left(\frac{\partial p}{\partial \rho} \right)_T \tag{4.38a}$$

or, more accurately,

$$\Gamma(k, \omega) \ = \ \frac{(\kappa/\rho c_v) k^2 \omega [(c_p/c_v) - 1](\partial p/\partial \rho)_T}{[\omega^2 - k^2(\rho_s/\rho_n)(Ts^2/c_v)]^2 + [(\kappa/\rho c_v)k^2\omega]^2} + \frac{\frac{4}{3}\eta + \zeta_2}{[1 + (\omega\tau)^2]\rho}. \tag{4.38b}$$

(τ can be expressed in terms of the other parameters.) The term,

$$\omega^2 k^2 P \int \frac{d\omega'}{\pi} \frac{\Gamma(k, \omega')}{\omega^2 - \omega'^2} \ = \ \frac{\omega^2 k^2}{\omega^2 - k^2 \rho_s \, Ts^2/\rho_n \, c_v} \left(\frac{\partial p}{\partial \rho} \right)_T \left(\frac{c_p}{c_v} - 1 \right), \tag{4.39}$$

although it vanishes as $\omega \to 0$ and as $k \to 0$, alters the function from one with a single pole at $\omega^2 = c_T^2 k^2$ to one with two poles at $\omega^2 = c_1^2 k^2$ and $\omega^2 = c_2^2 k^2$. The possibility of such irregularity is ignored in assuming that the mass operator may be expanded (see Section V, E).

Correlation Functions Involving the Heat Density q

Using the standard expression (4.11) for $\chi_{q,q}$ we find

$$\alpha_{q,q} = \lim_{k \to 0} \int \frac{d\omega}{\pi} \frac{\chi_{q,q}''(k, \omega)}{\omega} = \rho T c_p, \tag{4.40a}$$

$$\begin{aligned} \lambda_{q,q} &= \frac{\rho_s}{\rho_n} T^2 s^2 \rho = \alpha_{q,q} c_1^2 c_2^2 \left(\frac{\partial \rho}{\partial p}\right)_s \\ &= -\lim_{\omega \to 0} \lim_{k \to 0} [\chi_{jq,jq}(k, \omega) - \chi_{jq,jq}(k, 0)], \end{aligned} \tag{4.40b}$$

$$\begin{aligned} L_{q,q} &= \kappa T = \lim_{\omega \to 0} \lim_{k \to 0} \frac{\omega}{k^2} \chi_{q,q}''(k, \omega) \\ &= \lim_{\omega \to 0} \lim_{k \to 0} \mathrm{Im} \frac{1}{\omega} [\chi_{jq,jq}(k, \omega) - \chi_{jq,jq}(k, 0)]. \end{aligned} \tag{4.40c}$$

Because both $\lambda_{q,q}$ and $\alpha_{q,q} c_1^2 c_2^2$ vanish as ρ_s approaches zero, it is necessary to retain the term $\gamma_{q,q}$ in this case. Specifically, $\gamma \to L c_1^2$ in this limit, leading to the expression, valid for $T \geqq T_c$,

$$\chi_{q,q}(k, \omega) = \frac{i\kappa T k^2}{\omega + (i\kappa/\rho c_p)k^2} \tag{4.41}$$

to lowest order in k and ω. This of course agrees with KM.

The remaining elements of the correlation function matrix, which link the heat energy with the momentum density and superfluid velocity, are not of the form (4.11) because they link quantities with opposite time reversal properties. In virtue of this fact, they are simpler, and in particular reflect the Onsager-Casimir property that there are no dissipative coefficients, i.e.,

$$\lim_{\omega \to 0} \lim_{k \to 0} \frac{\omega}{k^2} \chi_{g,q}'' = 0; \qquad \lim_{\omega \to 0} \lim_{k \to 0} \frac{\omega}{k^2} \chi_{v_s,q}'' = 0 \tag{4.42}$$

$$\chi_{g,q}(k, \omega) = \frac{(\alpha_{g,q} c_1^2 c_2^2 k^2 - i\gamma_{g,q} \omega k^2)k\omega}{(\omega^2 - c_1^2 k^2 + iD_1 k^2 \omega)(\omega^2 - c_2^2 k^2 + iD_2 k^2 \omega)} \tag{4.43}$$

where the coefficient $\alpha_{g,q}$ satisfies

$$\begin{aligned} \alpha_{g,q} &= \lim_{k \to 0} \int \frac{\chi_{\rho,q}''(k, \omega)}{\omega} \frac{d\omega}{\pi} = \lim_{k \to 0} \int \frac{k\chi_{g,q}''(k, \omega)}{\omega^2} \frac{d\omega}{\pi} \\ &= T \left(\frac{\partial \rho}{\partial T}\right)_p = \rho^2 T \left(\frac{\partial s}{\partial p}\right)_T. \end{aligned} \tag{4.44}$$

Likewise

$$\chi_{q,v_s}(k, \omega) = \frac{(\alpha_{q,v_s} c_1^2 c_2^2 k^2 - \lambda_{q,v_s} \omega^2 - i\gamma_{q,v_s} \omega k^2)k\omega}{(\omega^2 - c_1^2 k^2 + iD_1 k^2 \omega)(\omega^2 - c_2^2 k^2 + iD_2 k^2 \omega)} \tag{4.45}$$

where

$$\lambda_{q,v_s} = -Ts \tag{4.46}$$

$$\alpha_{q,v_s} = \frac{T}{\rho}\frac{\partial \rho}{\partial T}\Big)_p - \frac{\rho_n c_p}{\rho_s s}. \tag{4.47}$$

In all these expressions, the functions $\gamma_{A,B}$ are linear combinations of transport coefficients, with thermodynamic derivatives for coefficients.

The Correlation Function for the Transverse Momentum g^t

As we said earlier, $\chi_{g^t g^t} \equiv \chi_{g,g}^t$ decouples completely from the other functions, so that its general expression is, at low frequency and wave number,

$$\chi_{g,g}^t(k,\omega) = \frac{(-\lambda_{g,g}^t + iL_{g,g}^t\omega)k^2}{\omega^2 - c_0^2 k^2 + iD_t k^2\omega}. \tag{4.48}$$

Since we are dealing with a liquid, there are no transverse momentum waves or shears, so that $c_0 = \lambda_{g,g}^t = 0$ and the expression may be simplified to yield

$$\chi_{g,g}^t(k,\omega) = \frac{i\eta k^2\omega}{\omega^2 + (i\eta k^2/\rho_n)\omega}. \tag{4.49}$$

The static susceptibility, or k-limit, yields the thermodynamic derivative $\partial g/\partial v_n)^t_{\rho_s(v_s-v_n)}$ and is equal to ρ_n. In Section V, E, we shall return to the fact that the transverse momentum correlation function is proportional to ρ_n in the static limit, when the two-fluid description is valid.

The formula for χ_{gg}^t reflects additionally the transport coefficient η. The fact that for an uncoupled mode we have

$$D_t = \frac{\eta}{\rho_n} = \frac{L_{gg}^t}{\partial g/\partial v_n)^t_{\rho_s(v_s-v_n)}} \tag{4.50}$$

is sometimes called an Einstein relation. This has its counterpart at zero temperature for the other modes which are also uncoupled, in the statements

$$D_1 = \frac{\frac{4}{3}\eta + \zeta_2}{\rho} = \frac{L_{g,g}^l}{\partial g/\partial v_n)^l_{\rho_s(v_s-v_n)}} \tag{4.51}$$

and in a normal fluid

$$D_2 = \frac{\kappa T}{\rho T^2 \partial s/\partial T} = \frac{L_{q,q}}{T\partial q/\partial T}. \tag{4.52}$$

The niceties of satisfying the high-frequency sum rule

$$[\delta_{ij}k^2 - k_i k_j]\rho_n c_\infty^2 \equiv \langle[(k \cdot T)^t, g^t]\rangle(k, t = 0)$$

$$= \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right)\int \frac{d\omega}{\pi}\chi_{g,g}''^t(k,\omega)\omega \tag{4.53}$$

(which involves the unknown equal time commutator of the stress tensor and momentum) can be achieved by using instead of (4.48) the interpolation formula

$$\chi_{g,g}^{t}(k, \omega) = \left(\frac{i\eta k^{2}}{1 - i\omega\eta/\rho_{n} c_{\infty}^{2}}\right) \Big/ \left(\omega + \frac{ik^{2}\eta/\rho_{n}}{1 - i\omega\eta/\rho_{n} c_{\infty}^{2}}\right) \qquad (4.54)$$

or a rigorous dispersion relation that incorporates the fact that the high-frequency response is reactive and not dissipative.

V. THE SEMIPHENOMENOLOGICAL QUASI-PARTICLE THEORY OF LANDAU

The Landau semiphenomenological theory of the Bose liquid (*3, 6, 8*) pictures helium as a gas of weakly interacting excitations, indefinite in number. The properties of this gas are supposed to be the thermodynamic and kinetic properties of liquid helium. The picture is analagous to the description of a crystal as a phonon gas, or a black body as a photon gas.

Rigorously, in each of these systems there may be a coherent background (the condensate, the equilibrium crystal coordinates, or a coherent electromagnetic field), which does not contribute to the equilibrium entropy. The latter and all differential thermodynamic properties are determined by the fluctuations of the field—in helium the particle field, in a crystal the displacement field, and in a black body the photon field. These frequency dependent field fluctuations are the single particle Green's functions.

The generalized Landau theories result when the field fluctuations can be treated as monochromatic, that is to say, when each excitation of wave number p can be treated as having a single energy $\epsilon(p)$.

It will be clear from our presentation of the theory, and it follows from experimental work with liquid helium (*7, 28*), that it is important to generalize the notions employed by Landau and to use elementary excitations, which, unlike the ones he discussed depend on temperature and relative velocity. We remark that in superconductors these dependences, and the variation of the excitation spectrum with magnetic field, are particularly significant. It is not correct therefore to think of these excitations simply as excited states of the helium (which would be temperature independent) any more than it is correct to think of the renormalized phonons, observed in neutron scattering from crystals, as defining energies of excited states of the crystal.

A. The Effectively Rigorous Landau Quasi-Particle Theory

This theory is obtained from the microscopic theory if one makes the following assumptions:

1. The frequency spectrum of the correlations of the quantum field has small, or at least symmetrical, dispersion at those frequencies which contribute to thermodynamic properties, or in other words the width of the spectral density

of the single particle Green's function can be neglected at these frequencies. Typically these are frequencies such that $\omega \lesssim kT$, but in helium, because the rotons have a large density of states, frequencies of order $\omega \sim 10kT$ will also contribute.

2. Despite the fact that there are nonresonant contributions to the field fluctuation spectrum, the thermodynamic and kinetic properties may be calculated in terms of the resonant parts alone. Moreover these properties depend only on the positions (frequencies) of the resonances; the heights of the resonances, as reflected in normalization constants, do not enter.

These assertions are the content of the Landau theory of Fermi liquids (29, 30), and they also enter here. For the condensed Bose liquid there is an important additional fact which must be borne in mind, namely that

3. The quasi-particle spectrum is the spectrum of the density-correlation function $\chi''_{\rho,\rho}(k, \omega)$. This spectrum, or more precisely that of

$$S(k, \omega) = 2N\chi''_{\rho,\rho}(k, \omega; \beta, \mu, v_n - v_s)(1 - e^{-\beta\omega})^{-1},$$

is the one measured in neutron diffraction experiments (7) (when $v_n - v_s = 0$). For each value of k this function is peaked at $\omega = \epsilon(k; \beta, \mu, v_n - v_s)$ and it is this energy which enters into the Landau theory. In the Fermi liquid the function $\chi''_{\rho,\rho}$ is not at all strongly peaked and the quasi-particle energies are only the poles of the field-correlation function (G_1). In the condensed Bose system the field-correlation function and the density-correlation function have resonances at the same frequency. The origin of this feature is that a density fluctuation may be created in the system by a field fluctuation accompanied by a change in the number of condensed particles. It is illustrated later in this section in Eq. (5.16) (in which $\chi_{g,g}$ is simply related to $\chi_{\rho,\rho}$, and χ_{v_s,v_s} is simply related to G_1) and in Section VI in Eq. (6.38). The fact that $\chi_{\rho,\rho}$ and χ_{v_s,v_s} have the same poles was already true in the hydrodynamic region and it persists in the quasi-particle region. In both regions the part of $\chi_{\rho,\rho}$ which has the same pole as G_1 is proportional to n_0 and therefore disappears when there is no condensation. As a result, when the Bose system is not condensed, the elementary excitations from which the thermodynamic properties are determined cannot be measured with neutrons.

Having made these preliminary remarks we now summarize the assertions of the Landau theory as the following. In equilibrium, the thermodynamic properties of the system are given, for small $(v_n - v_s)$, by

$$\rho s \cong \int \frac{d^3p}{(2\pi)^3}[(1 + \tilde{f}(p)) \log (1 + \tilde{f}(p)) - \tilde{f}(p) \log \tilde{f}(p)] \qquad (5.1)$$

where

$$\tilde{f}(p) = \frac{1}{\exp [\beta\epsilon(p; v_n - v_s, \mu, \beta)] - 1} \qquad (5.2)$$

and $\epsilon(p; v_n - v_s, \mu, \beta)$ is the quasi-particle energy (the peak of the propagator for the quantum field) in the frame in which $v_s = 0$, at temperature β^{-1}, chemical potential μ, and relative velocity $(v_n - v_s)$, and with uniform condensate. Under these conditions we also have for small $v_n - v_s$

$$j^\epsilon = T\rho s(v_n - v_s) + \rho_n(v_n - v_s)\mu \tag{5.3}$$

$$g = \int \frac{d^3p}{(2\pi)^3} \, p\tilde{f}(p) \equiv \rho_n(v_n - v_s). \tag{5.4}$$

It is also possible to treat disturbances which vary in space and time, by the kinetic equation

$$\frac{\partial}{\partial t} \tilde{f}(r, p, t) + \nabla_r\epsilon(r, p, t)\cdot\nabla_p\tilde{f}(r, p, t) - \nabla_p\epsilon(r, p, t)\cdot\nabla_r\tilde{f}(r, p, t) \tag{5.5}$$
$$= \text{collision terms,}$$

$$\frac{\partial\phi(r, t)}{m\partial t} + \left[\mu(r, t) + \frac{1}{2}\left(\nabla\frac{\phi}{m}(r, t)\right)^2\right] = 0, \tag{5.6}$$

where $\mu(r, t)$ and $\epsilon(r, p, t)$ are functionals of $\tilde{f}(r, p, t)$, $\nabla\phi(r, t)$, and $\phi(r, t)$. In terms of such functionals the form of $\tilde{f}$ given in (5.2) may be looked upon as the function minimizing the entropy subject to constraints on the energy, the number, and the momentum, all as functionals of $\tilde{f}$, ϕ, and $\nabla\phi$.

We have taken some liberties in this summary with the original Landau theory (3, 6) because we wish to distinguish between what we believe to be an effectively rigorous theory and the approximate theory that results when more specific properties of helium at low temperatures are introduced (see part B). To derive the equilibrium theory it is necessary to show the exact expressions

$$g_i = \int \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \, p_i \, A(p, \omega; v_n - v_s)f(\omega) \tag{5.7}$$

$$j_i^\epsilon = \int \frac{d\omega}{2\pi} \int \frac{d^3p}{(2\pi)^3} \, p_i[\omega + p\cdot(v_n - v_s) + \mu]A(p, \omega; v_n - v_s)f(\omega) \tag{5.8}$$

where

$$f(\omega) = \frac{1}{e^{\beta\omega} - 1} \tag{5.9}$$

lead to (5.1)–(5.4) when the weight function $A(p, \omega)$ of (6.25) has negligible or symmetric width. Although we do not at present have complete proofs which exhibit the range of validity (i.e., the expansion parameter) of this theory, we believe that these proofs entail arguments like those applied to the Landau theory of Fermi liquids (29, 30). The equilibrium theory in slightly different

form has been proven by Balian and DeDominicis (*31*). The nonequilibrium theory (5.5), (5.6) is just the Green's function description of Section VI in the case of slowly varying external disturbances, with appropriately defined quasi-particle interactions. It should be recalled that the matrix elements for interaction of quasi particles with external fields involve the so-called coherence factors, familiar in the theory of superconductivity.

B. THE IDEAL QUASI-PARTICLE GAS

Our understanding of Landau's original theory (*3, 6*) is that he made the further assumption that the quasi-particle gas behaves in every way like an ideal gas, which moves without friction through the superfluid. This implies that one can neglect the β and μ dependence of $\epsilon(p)$ and that its dependence on $v_n - v_s$ is the same as the dependence of $\epsilon(p)$ on the average velocity v in a gas of noninteracting particles. The latter dependence may be obtained by Galilean invariance. The dependence of ϵ on p therefore is *assumed* to be (for small $v_n - v_s$, neglecting β and μ dependence)

$$\epsilon(p; v_n - v_s) = \epsilon(p; 0) - p \cdot (v_n - v_s) \tag{5.10}$$

in this approximation. We stress here that from the point of view of the helium as a whole no Galilean invariance argument can determine the $v_n - v_s$ dependence. To obtain (5.10) Landau argued that the velocity dependence of the energy of each quasi-particle excitation was determined by Galilean invariance. Such an assertion is at best approximately true at small velocities and quasi-particle densities. On the basis of (5.10), Landau derived from (5.4) the expression

$$\rho_n = \int \frac{d^3p}{(2\pi)^3} \frac{1}{3} p^2 \tilde{f}(1 + \tilde{f}) \tag{5.11}$$

and also the other expressions of his original theory. For the dilute nonideal Bose gas at low temperatures these expressions which involve noninteracting quasi particles have been deduced microscopically (*32, 33*). The results obtained for the thermodynamic parameters of helium with this theory are remarkably good at low temperatures, when the ideal quasi-particle gas approximation is valid. At higher temperatures the temperature dependence of ϵ does come into play (*28*). Near T_c the approximation of small or symmetric width breaks down so that the theory does not hold, but it is not clear, either experimentally or theoretically, how close to T_c the theory is valid.

By arguing that the excitation gas carries all the entropy and by once more using "Galilean invariance" arguments to determine the stress tensor of this gas, Landau was able to determine the form of the currents j^ϵ and $j^{v_s} = \mu$. He thus derived the two fluid thermodynamics and hydrodynamics which we obtained in

Sections III and IV. It is interesting to note that our derivation of the two-fluid expressions (3.22), (3.23) differs in an important respect from the one given by Landau on the basis of the excitation picture. Landau's discussion is cast as far as possible in terms of the excitation gas, the superfluid being eliminated by working in the frame in which v_s vanishes. This was necessary because he did not acknowledge at the time he presented his theory that superfluidity was due to Bose condensation. In the London picture, which we employed in Section III we were able to derive the v_s dependence of the currents (for small v_s) by working in the frame in which $v_n = 0$ (see Eqs. (3.25) and (3.26)), thus eliminating the excitations. Having found this dependence, we were able to use rigorous Galilean invariance arguments *for the whole liquid*, to infer the dependence on v_n, just as Landau was able to find the v_s dependence rigorously once he knew the v_n dependence. In this way, starting from our assumptions about the microscopic nature of Bose condensation we were able to *deduce* the dependence of the parameters on v_s, for small v_s. This was possible because v_s involves essentially one degree of freedom whereas the dependence on v_n involves all the degrees of freedom of a gas moving through the superfluid. We do not understand how purely macroscopic considerations suffice to fix the dependence of the hydrodynamic currents on relative flow. We believe that microscopic arguments must be used to make the deductions Khalatnikov[8] claims follow from purely macroscopic considerations. These arguments may either involve the excitations, as in (*3*), or the condensate as in Section III.

C. The Landau Quantum Hydrodynamics

Having shown how we may use the microscopic theory to make some minor improvements on the rigor of the Landau theory, we would like to turn to an advantage of this theory over the microscopic approach, which does not seem to be generally appreciated. The semiphenomenological quasi-particle theory has a long wavelength limit which is in essential agreement with yet another Landau theory, the so-called Landau theory of quantum hydrodynamics (*3*, *8*). This is essentially a theory of a gas of interacting phonons whose anharmonic coupling is determined in terms of the differential properties of the phonon velocity as a function of density. Beginning with such a Hamiltonian[9] one may compute rather simply the following properties of the phonons: (see Fig. 5).

(a) their k^5 damping at zero temperature,

(b) their damping at finite temperature,

(c) the $T^4 \log T$ shift in the renormalized velocity with temperature as a result of anharmonicity, and

[8] Ref. *8*, chap. 8.

[9] The utility of Landau's quantum hydrodynamics for obtaining microscopic results was first pointed out by M. Nelkin (*11*). We warn the reader that Nelkin uses the convention of (*53*) that $2m = 1$ in his Eq. (9), but not in his other equations.

Approximation	$\tilde{G}_1$ Spectrum $k \to 0$	$\chi_{\rho\rho}$ Spectrum $k \to 0$	Ground state energy E_0/N	Compressional velocity $c_0 = \left[\frac{1}{m}\frac{d}{dn}n^2\frac{d}{dn}\left(\frac{E_0}{N}\right)\right]^{1/2}$	Depletion $n' \equiv n - n_0$
Ideal gas	$\omega = \dfrac{k^2}{2m}$	$\omega = \dfrac{k^2}{2m}$	0	0	0
Hartree	$\omega = \dfrac{k^2}{2m}$	$\omega^2 = \left(\dfrac{k^2}{2m}\right)^2 + \dfrac{nv}{m}k^2$	$\dfrac{E_0}{N} = \dfrac{1}{2}\,nv$	$\left[\dfrac{nv}{m}\right]^{1/2}$	0
Girardeau, Arnowitt (49)	Gap $\omega^2 = \left(\dfrac{k^2}{2m}\right)^2 + \dfrac{n_0 v}{m}k^2$	No gap $\omega = c_0 k + i\alpha k^5 + i\gamma k$ See Beliaev spectrum	$\dfrac{E_0}{N} = \dfrac{1}{2}\,nv\left[1 + \dfrac{16}{15\pi^2}\sqrt{n_0(mv)^3}\right]$	$c_0 = \sqrt{\dfrac{n_0 v}{m}}\left[1 + \dfrac{7}{6\pi^2}\sqrt{n_0(mv)^3}\right]$	$n' = \dfrac{n_0\sqrt{n_0(mv)^3}}{3\pi^2}$
Bogoliubov (51)	—	—	$\dfrac{E_0}{N} = \dfrac{1}{2}\,nv\left[1 + \dfrac{16}{15\pi^2}\sqrt{n_0(mv)^3}\right]$	$c_0 = \sqrt{\dfrac{n_0 v}{m}}\left[1 + \dfrac{7}{6\pi^2}\sqrt{n_0(mv)^3}\right]$	$n' = \dfrac{n_0\sqrt{n_0(mv)^3}}{3\pi^2}$
Beliaev (34, 46)	$\omega = c_0 k + i\alpha k^5 + i\gamma k$ $c_0 = \sqrt{\dfrac{n_0 v}{m}}\left[1 + \dfrac{7}{6\pi^2}\sqrt{n_0(mv^3)}\right]$ $\gamma = \dfrac{3\pi^3 m T^4}{40 n_0^3 v^2}$ $\alpha = \dfrac{3}{(320\pi)2mn_0}$	—	$\dfrac{E_0}{N} = \dfrac{1}{2}\,nv\left[1 + \dfrac{16}{15\pi^2}\sqrt{n_0(mv)^3} + o\{n_0(mv)^3 \ln n_0(mv)^3\}\right]$	$c_0 = \sqrt{\dfrac{n_0 v}{m}}\left[1 + \dfrac{7}{6\pi^2}\sqrt{n_0(mv)^3} + o\{n_0(mv)^3 \ln n_0(mv)^3\}\right]$	$n' = \dfrac{n_0\sqrt{n_0(mv)^3}}{3\pi^2}$ $\times [1 + o(\sqrt{n_0(mv)^3})]$

FIG. 5. The results of calculations in the various approximations, for a point repulsive potential $v(r) = v\delta(r)$

(d) using a cutoff, the $\lambda^4 \log \lambda$ shift in their zero point energy as a result of the anharmonicity.

These calculations reproduce, essentially identically, the results for weakly interacting Bose gases obtained much more tediously by field theoretic techniques involving the particle Hamiltonian. Specifically, when one inserts the velocity density relation $dc^2/d\rho = c^2/\rho$, (a) is a result of Belyaev (*34*), (b) one of Morita and Mohling (*35*), (c) one of Khalatnikov and Andreyev (*36*), and (d) a result of T. Wu (*37*).

In fact, *all low temperature measurable quantities* (as contrasted with n_0, or $n(k)$ the number of particles with a given momentum, or the ground state energy E_0), which have been calculated for models, using the quantized fields, may be calculated far more easily by using the Landau equations. This suggests that the appropriate procedure for carrying out manageable calculations on real helium should be based on equations which make the Landau hydrodynamic limit clearest. Specifically it suggests using for technical reasons the rigorous equations involving fluctuations of n_0 and ϕ instead of the equations involving ψ and ψ^+. It is the latter formalism, however, which has been most extensively studied, and which is summarized in Section VI.

D. The Landau Stability Criterion

Using the notion of an ideal quasi-particle gas Landau also formulated a stability criterion for superfluidity. However this criterion is considerably less well founded than the thermodynamic expressions (5.1)–(5.4), even neglecting vortex formation.

The foundation of the Landau criterion is the statement, already made in Section III, that condensate fluctuations must not be microscopically unstable. We argued there that the condition of microscopic stability was a necessary but not sufficient condition for the existence of superfluid flow with the given condensate structure and that it did not preclude superfluidity with a different condensate structure.

In Section III we described this condition in terms of the wavenumber dependent static susceptibility which (by the Kramers-Kronig relation) is a frequency integral of the wave-number dependent absorptive susceptibility (cf. Section IV). In fact, the absorptive susceptibility must be positive for all positive frequencies and all wave numbers (*56*). This requires in particular that

$$\tfrac{1}{2} \operatorname{tr} \omega A (r, r', \omega) \geqq 0 \tag{5.12}$$

which, when the excitations have long lifetimes, requires

$$\epsilon(p, \beta, \mu, v_n - v_s) \geqq 0. \tag{5.13}$$

With Landau's assumption (5.10) this condition could be written

$$\epsilon(p) - p \cdot (v_n - v_s) \geqq 0 \tag{5.14}$$

whence the critical velocity $(v_n - v_s)_{crit}$ was determined by [10]

$$(v_n - v_s)_{crit} = \min \frac{\epsilon(p)}{p} . \tag{5.15}$$

This assumption is far too drastic. Just as we know from models (corroborated by experiment in real helium (7)) that $\epsilon(p, \beta, \mu)$ varies substantially with $\rho_s(\beta)$ and is radically altered when $\beta \to \beta_c$, we can be sure that $\epsilon(p, \beta, \mu, v_n - v_s)$ varies radically as a function of $v_n - v_s$, as $v_n - v_s$ approaches the point of microscopic instability $(v_n - v_s)_{crit}$. Indeed, in the same way as the "density fluctuation energy" $\sqrt{(dp/(d\rho)_T}$ in a simple approximation is imaginary at the point of microscopic instability in a supercooled gas, representing exponential growth of fluctuations, the energy $\epsilon(p, \beta, \mu, v_n - v_s)$ must be imaginary when $(v_n - v_s) > (v_n - v_s)_{crit}$. In fact in the supercooled gas and in helium near T_c and $(v_s - v_n)_{crit}$ the fluctuations of a given wave number have a broad frequency spectrum— there are no quasi particles, so that the general relation (5.12) must be used.

For nonuniform condensates, this condition of microscopic instability (5.12) is also applicable, and thus, the microscopic instability of the most stable flow pattern, which corresponds to the occurrence of growing fluctuations in this state, provides an upper bound on $(v_n - v_s)_{crit}$.

E. The Phonon Spectrum

In Section IV we derived asymptotic expressions for the correlation functions for k and ω in the hydrodynamic region, namely, for systems which are in local thermodynamic equilibrium. As we saw, this region is characterized by the relations

$$\omega\tau \ll 1 \qquad kl \ll 1$$

where τ is the collision time and l the mean free path of the quasi-particles of the system. Experimental evidence on the attenuation of sound (26) indicates that the collision time is of the order of 10^{-7}–10^{-8} sec at $1°$ so that the hydrodynamic region is limited to very small frequencies ($<10^7$ cps). The quasi-particle excitations in helium that contribute most to the thermodynamic properties and transport coefficients are the phonons and rotons which can be measured by neutron diffraction. These have frequencies of the order of 10^{11}–10^{12} cps (energies of 1 to $10°K$). Therefore the information obtained in Section IV on the basis of thermodynamic and hydrodynamic arguments is not immediately relevant to

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[10] One case in which the criterion may be used is the ideal Bose gas in which the particle and quasi-particle excitations are the same. The system does condense, so that it is a super-fluid in the sense of Section II, but by (5.15) the critical velocity is zero, or at least related to the inverse dimensions of the container. The noninteracting gas condenses but the uniformly condensed ideal gas is unstable against any external perturbation which can couple to it. Whether an external disturbance produces a nonuniform superfluid flow is not answered by the examination of the Landau criterion (5.15).

the overwhelming majority of elementary excitations, that contribute to the thermodynamic parameters (ρ_n, $dp/d\rho$, c_v) and transport coefficients (ζ_1, ζ_2, ζ_3, η, κ) in helium. The *form* of the hydrodynamic equations and the values of the thermodynamic parameters and kinetic coefficients are, to be sure, apparent in the correlation functions of the hydrodynamic ("low-energy") region. The *calculation* of these parameters or correlation functions depends primarily on the quasi-particle spectrum in the collisionless ("high-energy") region. Note that our remark about the hydrodynamic modes applies also to vortex modes since they too occupy a negligible portion of phase space and are irrelevant in a calculation of thermodynamic parameters.

In this section we shall be concerned with the quasi-particle spectrum. In order to determine its form from first principles one must give microscopic arguments based on the particular properties of condensed helium. In the phonon region and at zero temperature such arguments have been presented by Gavoret and Nozières (*38*). However their calculations are plagued by spurious divergences. At finite temperature the assumptions made by them seem to be less valid, and the nature of the spectrum is less well understood. In what follows we wish to give a shorter derivation of the result of Gavoret and Nozières. In Appendix C we shall make some remarks on a possible extension to finite temperatures.

We begin by making some additional comments on the momentum correlation function discussed in Section IV. When the condensate is uniform the momentum can be shown to be a functional of v_n and v_s (for given β and μ). Under these circumstances $\chi_{v_{s_i}, v_{s_j}}$ is unambiguous and longitudinal because the superfluid flow is irrotational. Under these conditions we shall prove below that

$$\chi_{g_i, g_j}(k, \omega) = \rho_n^{ij}(k, \omega) + \rho_s^{il}(k, \omega)\chi_{v_{s_l}, v_{s_m}}(k, \omega)\rho_s^{mj}(k, \omega) \qquad (5.16)$$

where the functions $\rho_n(k, \omega)$ and $\rho_s(k, \omega)$ have the property

$$\lim_{k \to 0} \lim_{\omega \to 0} \rho_n^{ij}(k, \omega) = \rho_n \delta_{ij} ; \qquad \lim_{k \to 0} \lim_{\omega \to 0} \rho_s^{ij}(k, \omega) = \rho_s \delta_{ij} . \qquad (5.17)$$

These expressions serve to explain and interrelate the various definitions proposed to characterize superfluidity. In particular, since the longitudinal part of the momentum-correlation function satisfies $\chi_{g_i, g_j}^l(k, 0) = \rho$ (the *f*-sum-rule) and $\chi_{v_{s_l}, v_{s_m}}(k, 0)$ approaches $k_l k_m / k^2 \rho_s$, we have

$$\lim_{k \to 0} \chi_{g_i, g_j}(k, 0) = \rho_n \delta_{ij} + \rho_s \frac{k_i k_j}{k^2} \qquad (5.18)$$

or

$$= \rho_n \left(\delta_{ij} - \frac{k_i k_j}{k^2} \right) + \rho \frac{k_i k_j}{k^2} . \qquad (5.19)$$

The first of these equations expresses the response to an external field coupled to g as the sum of a normal (isotropic) response proportional to ρ_n and a superfluid (longitudinal) response proportional to ρ_s. The second equation, (5.19), expresses it as a transverse response proportional to ρ_n plus a longitudinal response proportional to ρ. The fact that the longitudinal response is ρ independently of temperature follows from gauge invariance, or Galilean invariance, or the f-sum rule. The fact that the transverse response is less than the longitudinal expresses the fact that at frequencies too low to permit vortex creation the moment of inertia of a superfluid is reduced by the factor ρ_n/ρ. Furthermore the occurrence of a term $k_i k_j / k^2$ in (5.18) implies that the spatial integral of the momentum correlation function is infinite. This is not too shocking in a superfluid, and was stressed by Shafroth (39).

The method used by Landau to calculate ρ_n, (6), corresponds to the first equation, (5.18). It is the response of the system at fixed condensate, or fixed superfluid. This method and not the one involving transverse correlations may be used to generalize the definition of ρ_n to cases in which there is vorticity. (See (6.38).)

We turn now to the proof of Eq. (5.16) from which we shall deduce the properties of the phonon spectrum.

This equation could be obtained by the methods of the next section, as a relation between the two particle Green's function and the single particle Green's function. However we believe the following formal proof, based on the thermodynamic definitions of Section IV, gives more insight.

We may consider the momentum $g_i(r, t)$ to be a functional of the velocities v_n and v_s. We then have[11]

$$\delta g(1) = \frac{\delta g(1)}{\delta v_n(2)}\bigg)_{v_s} \delta v_n(2) + \frac{\delta g(1)}{\delta v_s(2)}\bigg)_{v_s} \delta v_s(2). \tag{5.20}$$

The variational derivatives in (5.20) are actually correlation functions which in equilibrium depend on the coordinate differences $r_1 - r_2$, $t_1 - t_2$, but this is not important for the manipulations. From (4.1) we see that

$$g \cdot \delta v_n + \delta[\rho_s(v_s - v_n)] \cdot v_s \tag{5.21}$$

is a total differential, so that

$$\frac{\delta v_s}{\delta v_n}\bigg)_{\rho_s(v_s - v_n)} = \frac{\delta g}{\delta \rho_s(v_s - v_n)}\bigg)_{v_n}. \tag{5.22}$$

[11] The index 1 includes all the space time variables which the quantity depends on, namely r, t, and in the case of a vector, the vector index. We shall also sum or integrate over repeated indices.

But by the chain rule we have

$$\frac{\delta g}{\delta \rho_s(v_s - v_n)}\bigg)_{v_n} = \frac{\delta g}{\delta v_s}\bigg)_{v_n} \frac{\delta v_s}{\delta \rho_s(v_s - v_n)}\bigg)_{v_n} \qquad (5.23)$$

and from (5.20)

$$\frac{\delta g}{\delta v_n}\bigg)_{\rho_s(v_s - v_n)} = \frac{\delta g}{\delta v_n}\bigg)_{v_s} + \frac{\delta g}{\delta v_s}\bigg)_{v_n} \frac{\delta v_s}{\delta v_n}\bigg)_{\rho_s(v_s-v_n)} \qquad (5.24)$$

which by (5.22) and (5.23) may be written

$$\frac{\delta g(1)}{\delta v_n(2)}\bigg)_{\rho_s(v_s-v_n)} = \frac{\delta g(1)}{\delta v_n(2)}\bigg)_{v_s} + \frac{\delta g(1)}{\delta v_s(3)}\bigg)_{v_n} \frac{\delta v_s(3)}{\delta \rho_s(v_s - v_n)(4)}\bigg)_{v_n} \frac{\delta g(4)}{\delta v_s(2)}\bigg)_{v_n} . \quad (5.25)$$

Since v_n is the variable conjugate to g and $\rho_s(v_s - v_n)$ the variable conjugate to v_s in the Hamiltonian (4.6a), we may identify the variational derivatives in (5.25) as

$$\frac{\delta g(1)}{\partial v_n(2)}\bigg)_{\rho_s(v_s-v_n)} = \chi_{g,g}(12)$$

$$\frac{\delta v_s(3)}{\delta \rho_s(v_s - v_n)(4)}\bigg)_{v_n} = \chi_{v_s,v_s}(34). \qquad (5.26)$$

The other derivatives are combinations of correlation functions which we shall denote by

$$\frac{\delta g(1)}{\delta v_n(2)}\bigg)_{v_s} = \rho_n(12)$$

$$\frac{\delta g(1)}{\delta v_s(3)}\bigg)_{v_n} = \rho_s(13). \qquad (5.27)$$

As the notation is intended to suggest these functions are related to the constants ρ_s and ρ_n. In particular the k-limit of the Fourier transform of (5.27) yields the thermodynamic derivatives, namely (see (5.17))

$$\lim_{k\to 0}\lim_{\omega\to 0} \rho_s^{ij}(k,\omega) = \frac{\partial g_i}{\partial v_{s_j}}\bigg)_{v_n} = \delta_{ij}\rho_s ,$$

$$\lim_{k\to 0}\lim_{\omega\to 0} \rho_n^{ij}(k,\omega) = \frac{\partial g_i}{\partial v_{n_j}}\bigg)_{v_s} = \delta_{ij}\rho_n . \qquad (5.28)$$

We thus have, in Fourier transform, the desired result

$$\chi_{g_i,g_j}(k,\omega) = \rho_n^{ij}(k,\omega) + \rho_s^{il}(k,\omega)\chi_{v_{s_l},v_{s_m}}(k,\omega)\rho_s^{mj}(k,\omega). \qquad (5.16)$$

Equation (5.16) gives a precise meaning to the qualitative division of the mo-

mentum response into a many quasi-particle contribution and a single quasi-particle contribution.

We shall now use this equation to deduce properties of the correlation functions χ_{v_s,v_s} and $\chi_{\rho,\rho}$ at zero temperature. We assume (as did Gavoret and Nozières (*38*)) that for small k and ω the numerator and denominator of χ_{v_s,v_s} may be expanded in a power series. (In terms of the Green's function $\tilde{G}$, to be defined in Section VI, this is an assumption on the behavior of the matrix mass operator $\tilde{G}_1^{-1}$ for small k and ω.) Taking account of the symmetry of the system we can write χ_{v_s,v_s}, for small k and ω, in the form

$$\chi_{v_{s_i},v_{s_j}}(k,\omega) \equiv \frac{\chi_{v_s,v_s}(k,\omega)k_i\,k_j}{k^2} = \frac{-\,ak_i\,k_j}{\omega^2 - v^2k^2} \tag{5.29}$$

where the constants a and v are to be determined. Since we are at zero temperature we have $\rho_s = \rho$ and $\rho_n = 0$. This fact must of course be checked by a microscopic calculation. Such a calculation, based on the perturbation expansion, is implicit in the work of Gavoret and Nozières (*38*). It also follows from (5.11) whose proof involves all the conservation laws.

Moreover we assume that for finite but small k and ω the functions $\rho_s(k,\omega)$ and $\rho_n(k,\omega)$ can be expanded, and therefore differ from their limiting values (5.28) by higher order terms. Again, only detailed microscopic considerations can justify this fully.[12] With these assumptions we can now use rigorous "sum-rule" arguments to determine a and v. Taking the longitudinal part of (5.16) and applying (4.31) we obtain at small k and ω, when $T = 0$ so that $\rho_n = 0$ and $\rho_s = \rho$,

$$\chi_{g,g}^l(k,\omega) = \frac{\omega^2}{k^2}\chi_{\rho,\rho} + \rho = \frac{-\rho^2ak^2}{\omega^2 - v^2k^2}. \tag{5.30}$$

From the k-limit of χ_{v_s,v_s} given in (4.24) we have

$$\lim_{k\to0}\lim_{\omega\to0}\chi_{v_s,v_s} = \alpha_{v_s,v_s} = \frac{1}{\rho_s} = \frac{1}{\rho} = \frac{a}{v^2} \tag{5.31}$$

and from (5.30) and (5.31)

$$\chi_{\rho,\rho} = \frac{-\rho k^2}{\omega^2 - v^2k^2}. \tag{5.32}$$

We then apply the "compressibility" sum rule (4.33)

$$\alpha_{\rho,\rho} = \rho\,\frac{d\rho}{dp}\bigg) = \frac{\rho}{c_0^2} = \frac{\rho}{v^2}. \tag{5.33}$$

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[12] Physically, this expansion corresponds to the statement that for small k and ω the only term that contributes to the density fluctuation spectrum is the excitation of a single phonon out of the condensate, the second term in (5.16). The many-phonon contributions which come into $\rho_n(k,\omega)$ give rise to higher order terms in k and ω.

This means that $v = c_0 = [dp/d\rho]^{1/2}$ is the (zero temperature) compressional sound velocity, and we may rewrite the correlation functions at $T = 0$ as

$$\chi_{v_s, v_s} = \frac{-(c_0^2/\rho)k^2}{\omega^2 - c_0^2 k^2} \qquad (5.34)$$

$$\chi_{\rho, \rho} = \frac{-\rho k^2}{\omega^2 - c_0^2 k^2}. \qquad (5.35)$$

This is the result obtained by Gavoret and Nozières on the basis of assumptions similar to ours.

These expressions agree, as they must, with the functions obtained by formally taking the limit $T \to 0$ in the hydrodynamic expressions (4.29), (4.30), (4.32). This is because we have here only used thermodynamic sum rules and an expansion assumption which is satisfied by the hydrodynamic expressions at $T = 0$. However the range of validity of hydrodynamics would, a priori, seem to shrink to zero as $T \to 0$ since τ and l become infinite. Our assumption that we could make *small* k and ω expansions at $T = 0$ implies that the relevant parameters are not $\omega\tau$ and kl, but rather ωt_m and ka, where $t_m \sim 10^{-11}$ sec and $a \sim 10^{-8}$ cm are *microscopic* quantities. The consistency of this assumption was verified in perturbation theory by us and more thoroughly by Gavoret and Nozières (*38*).[13] In Appendix C we indicate how this discussion might be modified at finite temperature.

VI. MICROSCOPIC THEORY—GREEN'S FUNCTION FORMULATION

As we mentioned in Section V the Green's function formulation on the basis of correlation functions of the quantized fields ψ and ψ^+, has not been very successful in calculating the properties of helium. Despite a considerable amount of work by many authors, thus far the rewards have been few. Even for the dilute gas model, few, if any, measureable results have been obtained which may not be derived more simply and understandably by the Landau quantum hydrodynamic description in terms of a phonon gas (see Section V, C). There are, however, a number of general statements and concepts that are conveniently discussed by using the field correlation functions in the η-ensemble, such as conservation

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[13] In a recent paper Huang and Klein (*48*) have repeated part of the Gavoret Nozières proof using the same functional techniques as we introduce in Section VI, rather than diagrammatic techniques. These authors make more numerous regularity assumptions than we have employed here; some of the additional ones are violated in the perturbation expansion, as noted by Gavoret and Nozières. Moreover, due to an error in their derivation the term $\rho_n(k, \omega)$ is left out of an equation (*167*) similar to (5.16). Since for pure He⁴ at $T = 0$, $\rho_n = 0$ they arrive at the desired result (5.34). One may understand however why their derivation is wrong by observing that it involves particle, but not momentum, conservation of the He⁴. The introduction of He³ impurities destroys momentum conservation and results in a nonzero ρ_n. In such a case their proof is essentially unchanged and its conclusion false.

laws (40), Ward identities (38), and renormalization (41). Moreover the use of the field-correlation functions will facilitate our discussion of the work of previous authors on the low density expansion. Much of the formalism we shall discuss here has already been described elsewhere (17) as part of a more general formalism, but no specific approximations were investigated in ref. 17.

Just as in normal systems, we shall be primarily concerned with the correlation functions evaluated in the equilibrium ensemble at rest. The techniques apply equally well to nonequilibrium situations with the modifications discussed by Schwinger (42) and Kadanoff (30). We shall use the η-ensemble (3.8) with $\lambda = 0$, $\alpha = 0$, $v_n = 0$. As we explained in Section II we generate the equations of motion in the η-ensemble by using the auxiliary source (2.10). This leads naturally to the appearance of "anomalous Green's functions" such as $\langle(\psi\psi)_+\rangle$, $\langle(\psi^+\psi\psi)_+\rangle$, which do not vanish when η^{ext} becomes negligibly small (like $\langle\psi\rangle$ itself). Since we are interested in equilibrium we use the Hamiltonian $H - \bar{\mu}N$[14] and then use periodic boundary conditions in imaginary time for the Green's functions. (For nonequilibrium, the integration path is modified and μ is not introduced (42)). In the presence of η^{ext} these equilibrium boundary conditions are proven by the same simple algebraic manipulation employed for normal systems (43). As we saw in Section III, C, the chemical potential may be determined by the condition that $\langle\psi\rangle$ be independent of "time" when its "time" variation is the one associated with the ensemble operator H_{eff}. In our case after we have set $\eta^{\mathrm{ext}} \to 0$ we have $v_s = 0$, $v_n = 0$, $\alpha = 0$ so that $H_{\mathrm{eff}} = H - \bar{\mu}N$ which is the time dependence we are using. Thus μ will be given by (3.42).

A. The Matrix Green's Function Equations

It is convenient to use a matrix representation (44), in which the single spinor

$$\Psi_{\alpha_1}(r_1, t_1) \equiv \Psi(\alpha_1, r_1, t_1) \equiv \Psi(1) = \begin{pmatrix} \psi(r_1, t_1) \\ \psi^+(r_1, t_1) \end{pmatrix} \qquad (6.1)$$

contains both creation and annihilation operators. We also introduce external sources coupled to the quantum field and the "density," thus adding to the Hamiltonian a term[15]

$$H_{\mathrm{ext}}(t_1) = \tfrac{1}{2} \int \Psi^+(1) U^{\mathrm{ext}}(12)\Psi(2) \, dr_1 \, dr_2 \, dt_2 + \int dr_1 \Psi^+(1)\eta^{\mathrm{ext}}(1) \quad (6.2)$$

[14] Note that we here use $\bar{\mu} = m\mu$ to conform with the definition which is usual in statistical mechanics.

[15] In matrix equations the single variable 1 will include the space time coordinates r, t, and the matrix index α_1. We shall often omit the summation signs over repeated indices. We shall also make the convention that when two operators are multiplied together at equal times the ψ^+'s are always to the left of the ψ's. Thus $\int d1 \Psi^+(1)\Psi(1)$ means $\int dr \, dt$ $[\psi^+(r, t)\psi(r, t) + \psi^+(r, t)\psi(r, t)]$.

It must also be noted that we have used the same symbols $(G_1, \tilde{G}_1)$ to denote the Green's

where $U^{\text{ext}}(12)$ is a 2×2 matrix proportional to $\delta(t_1 - t_2)$ and

$$\eta^{\text{ext}}(1) \equiv \begin{pmatrix} \eta^{\text{ext}}(r_1, t_1) \\ \eta^{\text{ext}*}(r_1, t_1) \end{pmatrix}. \tag{6.3}$$

Integration over r_i is implicitly understood to include summation over α_i.

By using the interaction representation with (6.2) as the "interaction term," we can generate the correlation functions[16]

$$G_{1/2}(1) \equiv \sqrt{-i} \, \langle \Psi(1) \rangle \equiv \frac{1}{\sqrt{-i}} \left. \frac{\delta W}{\delta \eta^{\text{ext}*}(1)} \right)_{U^{\text{ext}}} \tag{6.4}$$

$$G_1(11') = -i \langle (\Psi(1)\Psi^+(1'))_+ \rangle$$

$$\tilde{G}_1(11') \equiv G_1(11') - G_{1/2}(1)G_{1/2}^+(1') = \frac{1}{\sqrt{-i}} \left. \frac{\delta G_{1/2}(1)}{\delta \eta^{\text{ext}}(1')} \right)_{U^{\text{ext}}} \tag{6.5}$$

where

$$e^W = \text{Tr} \, \{ e^{-\beta H}(S)_+ \} \tag{6.6}$$

$$S = \exp\left\{ -i \int_0^{-i\beta} dt' H_{\text{ext}}(t') \right\}. \tag{6.7}$$

By successive differentiations with respect to the source η^{ext} we may generate higher order Green's functions. The equation of motion for the quantum field, together with its adjoint may be written

$$\int dr_2 \int dt_2 \left[\tau^{(3)}_{\alpha_1 \alpha_2} i \frac{\partial}{\partial t_1} + \left(\frac{\nabla_1^2}{2m} + \bar{\mu} \right) \delta_{\alpha_1 \alpha_2} \right] \delta(r_1 - r_2) \delta(t_1 - t_2) G_{1/2}(2)$$

$$= \frac{1}{2} \sqrt{-i} \int dr_2 \int dt_2 \, \langle \Psi(1)\Psi^+(2)\Psi(2) \rangle v(12) \tag{6.8}$$

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function matrices here as we used in Section II for the reduced density matrix (2.6). The reduced density matrix $G_1(r, r', t)$ is one element of the equal time Green's function matrix $G_1(r, t, r', t)$. Since we do not use the reduced density matrix at all in this section, no confusion will arise.

[16] The notation used here differs from that of ref. *17*, in order to make comparison with other work more convenient. In particular we shall not use the dimensionless time variable or the completely symmetrized potentials of (*17*) which are extremely convenient for discussing formal developments. In contrast with (*17*) our $\tilde{G}_1$ matrix has the anomalous functions ($\langle \psi\psi \rangle$ and $\langle \psi^+\psi^+ \rangle$) in its off-diagonal elements and our interparticle potential v is not a matrix. *Apart from normalization factors* the correspondence is as follows:

Our notation	Ref. *17*
$G_{1/2}$	$G_{1/2}$
$\tilde{G}_1$	$\tilde{G}_1 \tau^{(1)}$
Σ	$K_1 \tau^{(1)}$
η	$K_{1/2}$
η^{ext}	$u_{1/2}$

$$+ \int dr_2 \int dt_2 \; \bar{U}(12)G_{1/2}(2) + \sqrt{-i}\,\eta^{\text{ext}}(1).$$

The $\tau^{(i)}$ are Pauli matrices. Introducing a matrix notation for the indices $\alpha r t$, and a summation convention over repeated indices we may rewrite (6.8) as

$$G_0^{-1}(12)G_{1/2}(2) = \sqrt{-i}\eta(1) + \sqrt{-i}\eta^{\text{ext}}(1) + \bar{U}(12)G_{1/2}(2) \qquad (6.9)$$

defining $\eta(1)$, the source function for the condensate, and the operator G_0^{-1}. The matrix $\bar{U}$ is given in terms of U^{ext} and its transpose $U^{\text{ext T}}$ by the expression $\bar{U} = \frac{1}{2}[U^{\text{ext}} + \tau^{(1)}\,U^{\text{ext T}}\,\tau^{(1)}]$; the matrix $v(12)$ has elements $v(r_1 - r_2)$ $\delta(t_1 - t_2)$. The equation of motion for G_1 is obtained by differentiating (6.9) at constant U^{ext}, with respect to η^{ext}

$$G_0^{-1}(12)\widetilde{G}_1(21') = \delta(11') + \bar{U}(12)\widetilde{G}_1(21') + \frac{\delta}{\delta\eta^{\text{ext}}(1')}\,\eta(1). \qquad (6.10)$$

Thus far we have considered the Green's functions to be functionals of U^{ext} and η^{ext}. We may just as well change to $G_{1/2}$ as our independent functional variable; this is equivalent to making a Legendre transformation on W (17). The derivative in (6.10) becomes

$$\frac{\delta\eta(1)}{\delta\eta^{\text{ext}}(1')}\bigg)_{U^{\text{ext}}} = \frac{\delta\eta(1)}{\delta G_{1/2}(2)}\bigg)_{U^{\text{ext}}} \frac{\delta G_{1/2}(2)}{\delta\eta^{\text{ext}}(1')}\bigg)_{U^{\text{ext}}} \equiv \Sigma(12)\widetilde{G}(21'), \qquad (6.11)$$

defining the mass operator as $\Sigma = \sqrt{-i}\delta\eta/\delta G_{1/2}\,)_{U^{\text{ext}}}$. This permits us to rewrite (6.10) as

$$G_0^{-1}(12)\widetilde{G}_1(21') = \delta(11') + \bar{U}(12)\widetilde{G}_1(21') + \Sigma(12)\widetilde{G}_1(21'). \qquad (6.12)$$

To get the equilibrium Green's functions we set

$$U^{\text{ext}}(12) = U^{\text{ext}}(r)\delta(12) \quad \text{and} \quad \eta^{\text{ext}}(1) \to 0.$$

We then solve (6.9), (6.12) in some approximation. As we noted in Section II the fact that $\eta^{\text{ext}} \to 0$ does not mean $\langle\psi\rangle \equiv 0$.

It is possible to derive a perturbation expansion for Σ and η by using the properties of the external field U^{ext}, just as for normal systems (30). Moreover, there exist nonperturbative approximation schemes, in terms of the elementary vertex functions (17). As we shall see later, the equations we have derived provide a convenient language in which to discuss the approximations made by other authors.

B. Spectral Properties and Symmetry Properties

In the presence of condensation the spectral properties of $\widetilde{G}_1$ are somewhat more subtle than in the grand canonical ensemble. In order to discuss them we introduce the commutator A, which represents the rate at which work would

be done on the system by the external source η^{ext}

$$A(\alpha, r, t; \alpha', r', t') = \langle [\Psi_\alpha(r, t), \Psi_{\alpha'}^+(r', t')] \rangle \qquad (6.13)$$

where the expectation value is taken in the equilibrium η-ensemble.

The matrix A is hermitian, that is,

$$A(\alpha, r, t; \alpha', r', t') = A^*(\alpha', r', t'; \alpha, r, t). \qquad (6.14)$$

Its time Fourier transform

$$A(\alpha, r, \alpha', r'; \omega) \equiv \int d(t - t') e^{i\omega(t-t')} A(\alpha, r, t; \alpha', r', t') \qquad (6.15)$$

satisfies the relations

$$A^*(\alpha, r, \alpha', r'; \omega) = A(\alpha', r', \alpha, r; \omega) \qquad (6.16)$$

$$A(\alpha, r, \alpha', r'; \omega) = -[\tau^{(1)} A^T \tau^{(1)}](\alpha, r, \alpha', r'; -\omega) \qquad (6.17)$$

and the sum-rule

$$\int \frac{d\omega}{2\pi} A(\alpha, r, \alpha', r'; \omega) = \tau_{\alpha\alpha'}^{(3)} \delta(r - r') \qquad (6.18)$$

implied by the equal-time commutation relations. The density is also obtained from A by integrating over frequencies. More specifically, we have from the fluctuation dissipation theorem (see the remarks made before Eq. (4.7a)).

$$\int d(t - t') e^{i\omega(t-t')} \langle \{\Psi_\alpha(r, t), \Psi_{\alpha'}^+(r', t')\} \rangle$$
$$= \coth \frac{\beta\omega}{2} \int d(t - t') e^{i\omega(t-t')} \langle [\Psi_\alpha(r, t), \Psi_{\alpha'}^+(r', t')] \rangle. \qquad (6.19)$$

Therefore we know that

$$\int d(t - t') e^{i\omega(t-t')} \langle \Psi_{\alpha'}^+(r', t') \Psi_\alpha(r, t) \rangle = \frac{A(\alpha, r, \alpha', r'; \omega)}{e^{\beta\omega} - 1}. \qquad (6.20)$$

In order to discuss the symmetry properties in more detail it is convenient to introduce another representation

$$A(\alpha, r, \alpha', r'; \omega) = \sum_{i=0}^{3} \mathcal{C}^{(i)}(r, r', \omega) \tau_{\alpha\alpha'}^{(i)}. \qquad (6.21)$$

Then the symmetry properties of the $\tau^{(i)}$, together with (6.17), imply that

$$\mathcal{C}^{(0),(1),(2)}(r, r', \omega) = -\mathcal{C}^{(0),(1),(2)}(r', r, -\omega)$$
$$\mathcal{C}^{(3)}(r, r', \omega) = \mathcal{C}^{(3)}(r', r, -\omega). \qquad (6.22)$$

If the system is spatially invariant, the Fourier coefficients $\mathcal{C}^{(i)}(k, \omega)$ are pure numbers (not matrices), and since the $\tau^{(i)}$ are hermitian matrices we obtain the additional information that all the $\mathcal{C}^{(i)}(k, \omega)$ are real and that $\mathcal{C}^{(3)}(k, \omega)$ is even in ω, and $\mathcal{C}^{(0),(1),(2)}(k, \omega)$ are all odd (they are, of course, functions of the scalar k^2).

Because of the periodic boundary conditions in time, the Green's function matrix G_1 is related to A by the equation

$$G_1(\alpha, r, t; \alpha', r', t')$$

$$= \eta(t - t') \int \frac{d\omega}{2\pi i} e^{-i\omega(t-t')} A(\alpha, r, \alpha', r'; \omega) \left[\frac{1}{2} + \frac{1}{2} \coth \frac{\beta\omega}{2}\right] \qquad (6.23)$$

$$+ \eta(t' - t) \int \frac{d\omega}{2\pi i} e^{-i\omega(t-t')} A(\alpha, r, \alpha', r'; \omega) \left[-\frac{1}{2} + \frac{1}{2} \coth \frac{\beta\omega}{2}\right].$$

Its Fourier coefficient, defined by

$$G_1(\alpha, r, t; \alpha', r', t') = \frac{i}{\beta} \sum_\nu e^{-\omega_\nu(t-t')} G_1(\alpha, r, \alpha', r'; \omega_\nu)$$

$$\omega_\nu = \frac{2\pi\nu}{-i\beta}, \qquad (6.24)$$

satisfies the relation

$$G_1(\alpha, r, \alpha', r'; \omega_\nu) = \int \frac{d\omega}{2\pi} \frac{A(\alpha, r, \alpha', r'; \omega)}{\omega_\nu - \omega}. \qquad (6.25)$$

If we expand in the basis of the τ-matrices we have

$$G_1(\alpha, r, \alpha', r'; \omega_\nu) = \sum_{i=0}^{3} \mathcal{G}^{(i)}(r, r'; \omega_\nu) \tau^{(i)}_{\alpha\alpha'}, \qquad (6.26)$$

$$\mathcal{G}^{(i)*}(r, r'; \omega_\nu) = \mathcal{G}^{(i)}(r', r; -\omega_\nu) \qquad (6.27a)$$

$$\mathcal{G}^{(3)}(r, r'; \omega_\nu) = -\mathcal{G}^{(3)}(r', r; -\omega_\nu) \qquad (6.27b)$$

$$\mathcal{G}^{(j)}(r, r'; \omega_\nu) = \mathcal{G}^{(j)}(r', r; -\omega_\nu) \quad j = 0, 1, 2. \quad (6.27c)$$

A similar discussion applies to $\tilde{G}_1$, $\tilde{A}_1$, $\tilde{\mathcal{C}}^{(i)}$, $\tilde{\mathcal{G}}^{(i)}$.

If the system is spatially invariant and the condensate at rest, we may impose a further symmetry. Since in the η-ensemble the parameter $\langle\psi(r,t)\rangle$ is specified we may state the following:

Under time reversal the operator $\psi(r, t)$ is transformed into $e^{i\alpha}\psi(r, t)$ with α arbitrary, and the c-number $\langle\psi(r,t)\rangle$ is transformed into its complex conjugate. Thus, if we employ an ensemble in which $\langle\psi(r,t)\rangle$ is always real, as is possible for

a spatially invariant system at rest, then in this ensemble the condition

$$\langle \psi(r, t) \rangle \rightarrow \langle \psi(r, t) \rangle$$

implies the additional relation

$$\mathcal{G}^{(1)}(r, r'; \omega_\nu) = \mathcal{G}^{(1)}(r', r; -\omega_\nu) \qquad (6.27\text{d})$$

$$\mathcal{G}^{(2)}(r, r'; \omega_\nu) = -\mathcal{G}^{(2)}(r', r; -\omega_\nu). \qquad (6.27\text{e})$$

Thus *in this ensemble* we deduce that $\mathcal{G}^{(2)}(r, r'; \omega_\nu)$ vanishes. We could equally have worked in an ensemble in which $\langle \psi(r) \rangle$ was purely imaginary, in which case $\mathcal{G}^{(1)}$ would vanish, and $\mathcal{G}^{(2)}$ would be equal to the $\mathcal{G}^{(1)}$ of the first ensemble. The possibility of finding an ensemble in which the off-diagonal part of G_1 is purely real (or imaginary) rests on the ability to choose the phase of a constant $\langle \psi(r) \rangle$ in a time-reversal invariant ensemble. (It is of course always possible to transform away the phase of a variable $\langle \psi(r, t) \rangle$ by a variable gauge transformation but this will introduce a vector potential which destroys time reversal invariance.) In particular in the presence of vortices the phase of $\langle \psi \rangle$ depends on r and no reality requirement on the off-diagonal part of G_1 can be imposed.

The statements (6.27) imply relations for the spatial Fourier transforms of $\mathcal{G}^{(i)}(r, r'; \omega_\nu)$ in a spatially invariant system. These are

$$\tilde{\mathcal{G}}^{(0)}(k, \omega_\nu) = \tilde{\mathcal{G}}^{(0)}(k, -\omega_\nu) = \tilde{\mathcal{G}}^{(0)*}(k, \omega_\nu) \qquad (6.28\text{a})$$

$$\tilde{\mathcal{G}}^{(3)}(k, \omega_\nu) = -\tilde{\mathcal{G}}^{(3)}(k, -\omega_\nu) = -\tilde{\mathcal{G}}^{(3)*}(k, \omega_\nu) \qquad (6.28\text{b})$$

$$\tilde{\mathcal{G}}^{(1)}(k, \omega_\nu) = \tilde{\mathcal{G}}^{(1)}(k, -\omega_\nu) = \tilde{\mathcal{G}}^{(1)*}(k, -\omega_\nu) \qquad (6.28\text{c})$$

$$\tilde{\mathcal{G}}^{(2)}(k, \omega_\nu) = 0. \qquad (6.28\text{d})$$

These properties of the Green's function imply the following statement about its inverse, or about Σ: In a spatially invariant system the mass operator Σ can be chosen in the following form

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix};$$

$$\begin{aligned} \Sigma_{12} &= \Sigma_{12}{}^* = \Sigma_{21} \\ \Sigma_{11} &= \sigma_1(\omega_\nu) + i\sigma_2(\omega_\nu) \\ \Sigma_{22} &= \sigma_1(\omega_\nu) - i\sigma_2(\omega_\nu) \\ \sigma_1(\omega_\nu) &= \sigma_1(-\omega_\nu) = \sigma_1{}^*(\omega_\nu) \\ \sigma_2(\omega_\nu) &= -\sigma_2(-\omega_\nu) = -\sigma_2{}^*(\omega_\nu). \end{aligned} \qquad (6.29)$$

When there is no spatial invariance Σ may require a complex off-diagonal part, i.e., the function analogous to the gap function of a superconductor may be complex.

C. Conservation Laws

Just as for normal systems $(45, 40)$, conservation laws may be discussed by constructing a functional Φ of the Green's functions (here of $\widetilde{G}_1$ and $G_{1/2}$), from which the mass operators may be obtained. The proof of the existence of this functional Φ is given in (17), Eq. (63), and again in (41), Eqs. (23), (25), (30), and we shall not repeat it here. Suffice it to say that there exists a functional of $\widetilde{G}_1$ and $G_{1/2}$ which has the properties

$$\frac{\delta}{\delta \widetilde{G}_1(11')} \, \Phi\{\widetilde{G}_1 \, , \, G_{1/2}\} \bigg)_{G_{1/2}} = \Sigma(11') \tag{6.30a}$$

$$\frac{1}{2\sqrt{-i}} \, \frac{\delta}{\delta G_{1/2}^+(1)} \, \Phi\{\widetilde{G}_1 \, , \, G_{1/2}\} \bigg)_{\widetilde{G}_1} = \eta(1). \tag{6.30b}$$

Just as for normal systems (45), it can be shown that a "Φ-derivable" approximation, namely, one which satisfies (6.30) for some Φ, obeys the conservation laws.

D. The Uniform System—Pines Hugenholtz Theorem—Ward Identities

In the equilibrium η-ensemble we have seen that $G_{1/2}$ is time independent and Eq. (6.9) with $\eta^{\text{ext}} = 0$, and U^{ext} independent of time, defines the chemical potential in terms of η and $G_{1/2}$. In the infinite homogeneous system, $U^{\text{ext}} = 0$, and we may pick

$$G_{1/2}(1) = \sqrt{-in_0} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \text{constant} \tag{6.31}$$

so that

$$\bar{\mu} = \frac{1}{2\sqrt{n_0}} \, (\eta_1 + \eta_2). \tag{6.31a}$$

In the spatially uniform system the simplified equations with a constant $G_{1/2}$ just reproduce the formalism of Beliaev (46), Eqs. (3.20), (5.6). The perturbation expansion for Σ and η are just his expansions of Σ and μ, (34), Eqs. (4.1–3).

Ward Identities

By noticing the invariance properties of the equations of motion (6.9) (6.12) of the system with external fields, it is possible to prove certain relationships which hold in the equilibrium uniform system.

If in addition to the field U^{ext} we introduce in (6.2) a vector potential $A(1)$ coupled to the momentum density $\mathbf{g}(rt)$, the equations of motion (6.9), (6.12) remain invariant under the following gauge transformation:

$$G_{1/2}(1) \rightarrow e^{i\Lambda(1)}G_{1/2}(1)$$

$$\tilde{G}_1(11') \rightarrow e^{i\Lambda(1)}\tilde{G}_1(11')e^{-i\Lambda(1')}$$

$$\Sigma(11') \rightarrow e^{-i\Lambda(1)}\Sigma(11')e^{i\Lambda(1')}$$

$$\eta(1) \rightarrow e^{i\Lambda(1)}\eta(1) \qquad (6.32)$$

$$\tfrac{1}{2}\text{tr } U^{\text{ext}}(1) \rightarrow \tfrac{1}{2}\text{tr } U^{\text{ext}}(1) - \frac{\partial}{\partial t_1}\lambda(r_1, t_1)$$

$$A(1) \rightarrow A(1) + \nabla_1\lambda(r_1, t_1)$$

$$\eta^{\text{ext}}(1) \rightarrow e^{i\Lambda(1)}\eta^{\text{ext}}(1)$$

(where $\Lambda(1) = \tau^{(3)}\lambda(r_1, t_1)$, and λ is a real function). We now consider all functions to be functionals of $G_{1/2}$, U^{ext}, and A.[17] Then we have for an infinitesimal $\lambda = \delta\lambda$

$$\delta\eta(1) = \frac{\delta\eta(1)}{\delta G_{1/2}(2)}\bigg)_{U^{\text{ext}},A} \delta G_{1/2}(2)$$

$$+ \frac{\delta\eta(1)}{\delta U^{\text{ext}}(2)}\bigg)_{G_{1/2},A} \delta U^{\text{ext}}(2) + \frac{\delta\eta(1)}{\delta A(2)}\bigg)_{G_{1/2},U^{\text{ext}}} \delta A(2). \qquad (6.33)$$

Equation (6.33) is the basic starting point for the proof of all the Ward identities.

First we take $\delta\lambda$ to be a constant, which implies $\delta U^{\text{ext}} = \delta A = 0$. From (6.32) we have, using the definition of Σ, (6.11),

$$-i\tau^{(3)}\delta\lambda\eta(1) = \sqrt{-i}\Sigma(12)\tau^{(3)}\delta\lambda G_{1/2}(2). \qquad (6.34)$$

As $\delta\lambda$ approaches zero, the functions η and $G_{1/2}$ remain constant. Thus (6.34) picks out the $k = 0$ and $\omega = 0$ Fourier component of Σ. Remembering (6.31) we have

$$\bar{\mu} = \Sigma_{11}(00) - \Sigma_{12}(00) \qquad (6.35)$$

which is the theorem of Pines and Hugenholtz (*12*) generalized to finite temperatures. It implies that G_1 has a pole at $k = \omega = 0$. Thus if the mass operator is not too badly behaved as k and ω go to zero there will be no gap in the excitation spectrum.

---

[17] Notice that in the presence of A the symmetry properties of Σ are more complicated than those mentioned in Section IV, B.

Physically this statement reflects the fact that small oscillations of the condensate wave function (whose spectrum appears in $\tilde{G}_1$) cost little energy. However nothing more specific can be concluded. The theorem does not indicate the nature of the spectrum, or even if well-defined single-particle modes exist. It is however related to another of the assumptions we made very early in this paper. We stated in Section III, A that in equilibrium with no sources $\alpha = 0$. Clearly that value minimizes the free energy. The same, however, is true of the parameter, $v_n - v_s$; since we saw in Section III, D that the state in which the free energy is minimum has $v_s = v_n$. There is, however, the following difference. If $\alpha \neq 0$, those poles of G_1 , which begin at $\omega = 0$ when $\alpha = 0$, would be complex. It is thus not possible to have local equilibrium states in which n_0 does not minimize the free energy. We have only checked this statement in lowest order but believe it to be generally true.

Next we may consider a gauge transformation depending either only on t or only on r. In Fourier transform the δU^{ext} and δA terms are proportional to ω and k respectively. For small k and ω this yields an expansion of Σ, in terms of the quantities $\delta\eta/\delta U^{\text{ext}}$ and $\delta\eta/\delta A$ which can be related to the two-particle function. These are the Ward identities proved by Gavoret and Nozières (38) and subsequently by ourselves (47) and Huang and Klein (48).

As we mentioned in Section V, E, the assumption that each element of the matrix Σ may be expanded for small k and ω is probably invalid even at $T = 0$. Certain physical combinations of these elements seem to be regular at $T = 0$, and irregular for finite temperature. It appears that the limit in (6.35) exists at all T.

From our proof of (6.34) it is clear that an approximation will surely satisfy the Pines Hugenholtz theorem if the mass operators are related by

$$\sqrt{-i} \left. \frac{\delta\eta(1)}{\delta G_{1/2}(1')} \right)_{U^{\text{ext}},A} = \Sigma(11'). \qquad (6.36)$$

We note that in (6.36) the derivative is not at constant $\tilde{G}_1$ but at constant U^{ext}. When (6.36) is violated a gap may (and generally does) appear.

It is perhaps instructive to write down in the notation of this section the relation used in Section V, E to investigate the quasi-particle spectrum. If we consider the momentum density $g(rt)$ (which is related to the Green's function $G_1 = \tilde{G}_1 + G_{1/2}G_{1/2}^+$) to be a functional of A and $G_{1/2}$, we have, holding U^{ext} constant,

$$\chi_{g,g}(12) = \frac{\delta g(1)}{\delta A(2)} = \left. \frac{\delta g(1)}{\delta A(2)} \right)_{G_{1/2}} + \left. \frac{\delta g(1)}{\delta G_{1/2}(3)} \right)_{A} \frac{\delta G_{1/2}(3)}{\delta A(2)} . \qquad (6.37)$$

The same partial derivative manipulations employed in Section V, E show that

we can rewrite this as

$$\chi_{g,g}(12) = \frac{\delta g(1)}{\delta A(2)}\bigg)_{G_{1/2}} + \frac{\delta g(1)}{\delta \tilde{G}_{1/2}(3)}\bigg)_A \tilde{G}_1(34) \frac{\delta g(2)}{\delta \tilde{G}_{1/2}(4)}\bigg)_A . \qquad (6.38)$$

For small k and zero frequency the first term on the right gives the Landau definition of ρ_n (in uniform systems with no vorticity or impurities it vanishes at $T = 0$). Thus in uniform systems if the vertices $\delta g/\delta G_{1/2}$ are well behaved it is clear that $\chi_{g,g}$ (and therefore also $\chi_{\rho,\rho}$) have the same poles as $\tilde{G}_1$. The coupling occurs via the vertex $\delta g/\delta G_{1/2}$ which may be shown to equal $\delta \eta/\delta A$. This vertex of course vanishes if there is no condensation. The Ward identities resulting from (6.33), along with (6.38), provide the Green's function derivation of the results of Section V, E. This is essentially the method used by Gavoret and Nozières (*38*) to relate Σ to the spectrum of $\chi_{g,g}$ or $\chi_{\rho,\rho}$.

E. The Perturbation Expansion-Discussion of Approximations

As we mentioned earlier, our formal expressions for Σ and η may be expanded in powers of the interparticle potential v (*30, 34*). A convenient way to describe a given approximation is to display the approximate functional Φ. From Φ, using (6.30b) we obtain the mass operator η. Then there are essentially two different methods for obtaining Σ. The first yields the so-called "Φ-derivable" (*45*) approximations which obey the conservation laws. In these Σ is obtained directly from Φ by (6.30a). In the second method, Σ is derived from (6.36), and the approximation satisfies the Pines Hugenholtz theorem. The difference between these methods lies in the fact that in (6.36) η is considered to be an explicit functional of $G_{1/2}$, and also to depend implicitly on $G_{1/2}$ and U^{ext} through $\tilde{G}_1$, while in (6.30a) the two independent variables are $\tilde{G}_1$ and $G_{1/2}$. These two methods, which we call "Φ-derivable" and "gapless" respectively, are different and lead to different single-particle spectra. From a given "Φ-derivable" approximation we can obtain the density-correlation function (two-particle spectrum) by differentiation with respect to U^{ext}, just as for normal systems (*40*) since

$$\frac{\delta G_1}{\delta U^{\text{ext}}} = \frac{1}{m^2} [\langle(\varrho\varrho)_+\rangle - \langle\varrho\rangle\langle\varrho\rangle].$$

This two-particle function will have a gapless spectrum, since that property follows for any density-correlation function which obeys the conservation laws. In other words both (6.36) and (6.38) will be violated in such an approximation, so that even if the spectrum of $\tilde{G}_1$ has a gap, the quasi-particle spectrum as calculated from $\chi_{\rho,\rho}$ or $\chi_{g,g}$ will not.

We may illustrate these general remarks by an examination of the first few orders of perturbation theory valid in the uniform dilute repulsive gas.[18] The situation is summarized in Figs. 4 and 5.

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[18] Our expansion in v is only valid for weak potentials. The formalism may however

1. Φ-Derivable Approximations

a. The Ideal Gas. If we take $v = 0$ then Φ is independent of $\tilde{G}_1$ and $G_{1/2}$ and we have $\eta = \Sigma = 0$. Our equations yield the usual $\omega = k^2/2m$ spectrum in $\tilde{G}_1$. In $\chi_{\rho,\rho}$ we have the $\omega = k^2/2m$ branch and also a two-particle branch which disappears at zero temperature for low k.

b. The Hartree Approximation. Here we have[19]

$$\Phi = \frac{i}{4} \left[\tilde{G}_1(11)v(12)\tilde{G}_1(22) + G_{1/2}(1)G_{1/2}^+(1)v(12)G_{1/2}(2)G_{1/2}^+(2) \right.$$
$$\left. + 2G_{1/2}(1)G_{1/2}^+(1)v(12)\tilde{G}_1(22) \right]. \tag{6.39}$$

This yields with the aid of (6.30)[20]

$$\Sigma(11') = \frac{i}{2} v(12)[\tilde{G}_1(22) + G_{1/2}(2)G_{1/2}^+(2)]\delta(11') \tag{6.40a}$$

$$\sqrt{-i}\eta(1) = \frac{i}{2} v(12)[G_{1/2}(1)G_{1/2}(2)G_{1/2}^+(2) + G_{1/2}(1)\tilde{G}_1(22)]. \tag{6.40b}$$

Since Σ and η are both independent of k and ω the $\tilde{G}_1$ spectrum is the same here as for the ideal gas. However, if we take the derivative with respect to U^{ext} to calculate $\chi_{\rho,\rho}$ we obtain the Bogoliubov spectrum

$$\omega = \left[\left(\frac{k^2}{2m} \right)^2 + \frac{nv(k)}{m} k^2 \right]^{1/2} \tag{6.41}$$

in this function. The phonon velocity $(nv(k)/m)^{1/2}$ is the compressional velocity

$$c_0 = \left[\frac{1}{m} \frac{d}{dn} n^2 \frac{d}{dn} \left(\frac{E_0}{N} \right) \right]^{1/2}$$

in this approximation.

c. Hartree with Exchange—The Girardeau Arnowitt Approximation. Here we take

$$\Phi = \frac{i}{4} G_{1/2}(1)G_{1/2}^+(1)v(12)G_{1/2}(2)G_{1/2}^+(2) + \frac{i}{2} G_{1/2}(1)G_{1/2}^+(1)v(12)\tilde{G}_1(22)$$

$$+ iG_{1/2}(1)G_{1/2}^+(2)v(12)\tilde{G}_1(12) \tag{6.42a}$$

$$+ \frac{i}{4} \tilde{G}_1(11)v(12)\tilde{G}_1(22) + \frac{i}{2} \tilde{G}_1(12)v(12)\tilde{G}_1(21)$$

---

easily be generalized to strong potentials, in the *dilute* gas, by a simultaneous expansion of v in powers of the two-body scattering matrix $(53, 46)$.

[19] We remind the reader that $v(12)$ has no matrix index. $v(12) = v(r_1 - r_2)\delta(t_1 - t_2)$. The equations written here are of course still matrix equations.

[20] It must be remembered that $G_{1/2}$ and $G_{1/2}^+$ are not independent.

which yields

$$\Sigma(11') = \frac{i}{2} v(12)[G_{1/2}(2)G_{1/2}^+(2) + \tilde{G}_1(22)]\delta(11')$$
$$+ iv(11')[G_{1/2}(1)G_{1/2}^+(1') + \tilde{G}_1(11')] \tag{6.42b}$$

$$\sqrt{-i}\eta(1) = \frac{i}{2} v(12)[G_{1/2}(1)G_{1/2}(2)G_{1/2}^+(2) + G_{1/2}(1)\tilde{G}_1(22)]$$
$$+ iv(12)G_{1/2}(2)\tilde{G}_1(12). \tag{6.42c}$$

The single-particle spectrum obtained from (6.42) is the one found by Girardeau and Arnowitt (49) namely[21]

$$\omega = \left\{\left[\frac{k^2}{2m} + f_1(k)\right]^2 - [f_2(k)]^2\right\}^{1/2} \tag{6.43}$$

where f_1 and f_2 are known functions of $v(k)$ which are unequal when $k = 0$. Thus (6.43) exhibits a gap $\omega_g = [f_1^2(0) - f_2^2(0)]^{1/2}$ at $k = 0$, which is not surprising since (6.42) does not satisfy (6.36). If we now differentiate (6.42) with respect to U^{ext} we can find coupled equations for $\delta\tilde{G}_1/\delta U^{\text{ext}}$ and $\delta G_{1/2}/\delta U^{\text{ext}}$ whose kernels involve $\tilde{G}_1$. If we use (6.42) for $\tilde{G}_1$, the resulting system of five homogeneous linear equations may be solved for a point potential (50) $v(r) = v\delta(r)$[22] and the resulting spectrum in $\chi_{\rho,\rho}$ is, for small k

$$\omega = ck + i \frac{k^2}{2m} \frac{k^3}{n_0} \frac{3}{320\pi} + i \frac{3\pi^3 m}{40n_0^3\beta^4 v^2} k \tag{6.44}$$

where

$$c = \sqrt{\frac{n_0 v}{m}} \left[1 + \frac{7}{12\sqrt{2\pi^2}} \sqrt{n_0(2mv)^3}\right]. \tag{6.45}$$

At $T = 0$ this is just the spectrum found by Beliaev (34) in a calculation of $\tilde{G}_1$ (see Subsection 2 below). At $T \neq 0$ the result was first found (albeit implicitly) by Mohling and Morita (35). Once again the velocity c is the compressional velocity (we have only calculated it at $T = 0$), to the approximation in which we are working. It is not surprising that a conserving ("Φ-derivable") approximation should lead to compressional waves in the density correlation function $\chi_{\rho,\rho}$.

If we take into account second order diagrams in Φ we obtain more complicated equations for G_1 and the ensuing equations for $\chi_{\rho,\rho}$ can certainly not be solved exactly for any potential.

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[21] Note that all four elements of the $\tilde{G}_1$ matrix have the same denominator, i.e., the same spectrum.

[22] The ensuing divergences are removed as in (53) and (46).

2. Gapless Approximations

a. The "Bogoliubov" Approximation (51). By this we mean the approximation which leads to the Bogoliubov spectrum (6.41) in $\tilde{G}_1$. It is obtained by computing Σ using (6.36) instead of (6.30a), starting with the lowest order η

$$\sqrt{-i}\,\eta(1) = \frac{i}{2}\, v(12)G_{1/2}(1)G_{1/2}(2)G_{1/2}^+(2) \tag{6.46a}$$

$$\Sigma(11') = \frac{i}{2}\, v(12)G_{1/2}(2)G_{1/2}^+(2)\delta(11') + iv(11')G_{1/2}(1)G_{1/2}^+(1'). \tag{6.46b}$$

The approximation (6.46) yields the Bogoliubov spectrum (6.41) in $\tilde{G}_1$. However the phonon velocity $c_{\mathrm{ph}} = (n_0 v(k)/m)^{1/2}$ is not the compressional velocity. The latter may be obtained from the ground state energy in this approximation and turns out to be (6.45).

b. The Beliaev Approximation. Instead of Hartree, we use a better approximation for η, namely (6.42c), and again compute Σ from (6.36). We use the relation

$$\frac{\delta\tilde{G}_1}{\delta G_{1/2}} = -\tilde{G}_1 \frac{\delta\tilde{G}_1^{-1}}{\delta G_{1/2}} \tilde{G}_1 = +\tilde{G}_1 \frac{\delta\Sigma}{\delta G_{1/2}} \tilde{G}_1 \tag{6.47}$$

and insert the first order expression for $\delta\Sigma/\delta G_{1/2}$. This leads to the second order expression

$$
\begin{aligned}
\Sigma(11') = \;& \tfrac{1}{2}iv(12)[G_{1/2}(2)G_{1/2}^+(2) + \tilde{G}_1(22)]\delta(11') \\
& + iv(11')[G_{1/2}(1)G_{1/2}^+(1') + \tilde{G}_1(11')] \\
& - \tfrac{1}{2}iv(13)v(21')\tilde{G}_1(11')[\tilde{G}_1(23)G_{1/2}(3)G_{1/2}^+(2) + G_{1/2}(2)G_{1/2}^+(3)\tilde{G}_1(32)] \\
& - v(13)v(41')\tilde{G}_1(14)[\tilde{G}_1(43)G_{1/2}(3)G_{1/2}^+(1') + G_{1/2}(4)G_{1/2}^+(3)\tilde{G}_1(31')] \\
& - \tfrac{1}{2}v(12)v(31')G_{1/2}(1)G_{1/2}^+(1')\tilde{G}_1(23)\tilde{G}_1(32) \\
& - v(12)v(31')G_{1/2}(1)G_{1/2}^+(3)\tilde{G}_1(32)\tilde{G}_1(21').
\end{aligned}
\tag{6.48}
$$

We have omitted some second order terms containing no condensate functions, which are of higher order at $T = 0$ (terms $\sim v\tilde{G}_1 v\tilde{G}_1\tilde{G}_1$). The approximation (6.42c), (6.48) is just the one originally used at $T = 0$ by Beliaev (*34*). The spectrum of $\tilde{G}_1$ is identical to the one given in (6.44). There is no gap since (6.36) is satisfied. The fact that the compressional velocity was obtained could not be guaranteed here a priori, as it could in the calculation of $\chi_{\rho,\rho}$ from (6.42) (where the conservation laws insure that the phonons have the correct velocity). The reason we have the same answer here for c is because Eq. (6.38) relating $\tilde{G}_1$ and $\chi_{g,g}$ (and therefore also $\chi_{\rho,\rho}$) is satisfied to the order in which we are calculating. However, if we again calculated the ground state energy from $\tilde{G}_1$ (or $\bar{\mu}$)

we would obtain the logarithmic correction of Wu (*37*) (see also (*12*)). The ensuing compressional velocity has corrections to the phonon velocity (6.45). It is not surprising that these "gapless," nonconserving approximations do not give the same value for the velocity of the elementary excitation and the thermodynamic velocity, since these approximations are not Φ-derivable (see ref. *45*).

The whole preceding discussion is summarized in Figs. 4 and 5.

APPENDIX A

Given a state of the system $\Psi(t)$ we construct a new state $\Psi'(t)$ by the operation

$$\Psi'_{t_0}(t) = U_{t_0}(t)\Psi_{t_0}(t) \tag{A.1}$$

where

$$U_{t_0}(t) = \exp\left\{-i\int_{t_0}^t dt' \int dr[\tilde{\psi}_I(r, t')\eta^{*\text{ext}}(r, t') + \tilde{\psi}_I^+(r, t)\eta^{\text{ext}}(r, t')]\right\}_+ \tag{A.2}$$

$$\tilde{\psi}_I(r, t) = \psi_{\eta=0}(r, t) - \sqrt{n_0(r, t)}e^{i\phi(r, t)}; \tag{A.3}$$

($\psi_{\eta=0}$ is the Heisenberg operator for the source-free system).

The function η^{ext} is then chosen in such a manner that $\langle(\psi - \sqrt{n_0}e^{i\phi})(\psi^+ - \sqrt{n_0}e^{-i\phi})\rangle_{\eta^{\text{ext}}}$ should have no long-range order (condensation), when the expectation value is taken in a state Ψ'. We stress once again that the prescription given here is quite implicit since the condensate $\sqrt{n_0}e^{i\phi}$ is not known *a priori*.

APPENDIX B

We derive here the expansions (3.29), (3.30), and (3.31) for the currents as a function of v_s.

Using (3.28) we have, for the momentum (denoting the operator as $\mathbf{g}$),

$$\delta\langle\mathbf{g}_j(r)\rangle_{v_s} = \int_0^\beta d\beta' \int dr'\lambda_i(r')[\langle\mathbf{v}_{s_i}(r', -i\beta')\mathbf{g}_j(r, 0)\rangle_0 \tag{B.1}$$
$$- \langle\mathbf{v}_{s_i}(r', -i\beta')\rangle_0\langle\mathbf{g}_j(r, 0)\rangle_0].$$

By the fluctuation-dissipation theorem in the ensemble in which $v_s = 0$, the product of operators in the square brackets of (B.1) is related to their commutator (*25*). We may thus rewrite (B.1) as

$$\delta g_j(r) = \int dr' \int \frac{d\omega}{\pi} \frac{\chi''_{g_j, v_{s_i}}(r, r'; \omega)}{\omega} \lambda_i(r'). \tag{B.2}$$

In a spatially invariant system we have

$$\chi''_{g_j, v_{s_i}}(r, r'; \omega) = \nabla_i \nabla_j f_1(r - r'; \omega). \tag{B.3}$$

In virtue of particle conservation

$$\omega\chi''_{\rho,v_{s_i}}(r-r';\omega) = -i\nabla_j\chi''_{\theta j,v_{s_i}}(r-r';\omega) \tag{B.4}$$

and the commutation relation at equal times,

$$\int \chi''_{\rho,v_{s_i}}(r,r';\omega)\,\frac{d\omega}{\pi} = -i\nabla_i\,\delta(r-r'), \tag{B.5}$$

we deduce

$$\nabla^2 \int f_1(r,r';\omega)\,\frac{d\omega}{\pi} = \delta(r-r'). \tag{B.6}$$

Since v_s is a gradient,

$$\delta g_j(r) = \int dr' \left[\nabla_i\nabla_j \frac{-1}{4\pi|r-r'|} \right] \lambda_i(r') = \lambda_j(r) = \rho_s v_{s_j}(r) \tag{B.7}$$

which proves (3.29). Similarly, for the energy current, we have the conservation law

$$\omega\chi''_{\epsilon,v_{s_i}}(r-r';\omega) = -i\nabla_l\chi''_{j_l\epsilon,v_{s_i}}(r-r';\omega) \equiv -\nabla_i\nabla^2 f_2(r-r';\omega) \tag{B.8}$$

and the commutation relation

$$\begin{aligned}
\int \chi''_{\epsilon,v_{s_i}}(r-r';\omega)\,\frac{d\omega}{\pi} &= \frac{-1}{2m}\left[\frac{1}{\langle\psi\rangle}\frac{\partial\langle\psi\rangle}{\partial t} - \frac{1}{\langle\psi^+\rangle}\frac{\partial\langle\psi^+\rangle}{\partial t} \right]\nabla_i\,\delta(r-r') \\
&= \left(\mu^{\text{loc}} + \frac{v_s^2}{2} \right)\nabla_i\,\delta(r-r')
\end{aligned} \tag{B.9}$$

which imply

$$\int \frac{d\omega}{\pi}\frac{\chi''_{j_j\epsilon,v_{s_i}}}{\omega}(r,r';\omega) = \left(\mu^{\text{loc}} + \frac{v_s^2}{2} \right)\nabla_i\nabla_j\frac{1}{4\pi|r-r'|}. \tag{B.10}$$

Therefore

$$\delta\langle\mathbf{j}_j^\epsilon(r)\rangle_{v_s} = \left(\mu^{\text{loc}} + \frac{v_s^2}{2} \right)\lambda_j(r) = \left(\mu^{\text{loc}} + \frac{v_s^2}{2} \right)\rho_s v_{s_j}(r) \tag{B.11}$$

which is (3.30).

The calculation of the stress tensor is slightly more complicated. Let

$$\delta\langle\mathbf{T}_{ij}\rangle = \delta\langle\mathbf{T}_{ij}\rangle^{(1)} + \delta\langle\mathbf{T}_{ij}\rangle^{(2)} \tag{B.12}$$

$$\delta\langle\mathbf{T}_{ij}\rangle^{(1)} = \int_0^\beta d\beta' \int dr'\,mv_s^2[\langle\mathbf{n}(r',-i\beta')\mathbf{T}_{ij}(r)\rangle - \langle\mathbf{n}(r',-i\beta')\rangle\langle\mathbf{T}_{ij}(r)\rangle] \tag{B.13}$$

$$= \int dr' \int \frac{d\omega}{\pi}\frac{\chi''_{T_{ij}\rho,}(r,r';\omega)}{\omega}v_s^2(r'). \tag{B.14}$$

In equilibrium we have

$$\chi''_{T_{ij},\rho}(k,\,\omega) \,=\, \delta_{ij} f_3(k,\,\omega), \tag{B.15}$$

and by the conservation law,

$$k_i \chi''_{T_{ij},\rho}(k,\,\omega)\omega \,=\, \omega \chi''_{g_j,g_l}(k,\,\omega) k_l \,. \tag{B.16}$$

In view of the commutation relation

$$\int \frac{d\omega}{\pi} \, \chi''_{\rho,g_i}(r,\,r';\,\omega) \,=\, -i\nabla_i \,\delta(r \,-\, r')\rho, \tag{B.17}$$

the first term in the stress tensor is

$$\delta\langle \mathbf{T}_{ij}\rangle^{(1)} \,=\, \delta_{ij} v_s{}^2 \int \frac{d\omega}{\pi} \frac{k_{l'}\,k_l}{k^2} \frac{\chi''_{g_{l'},g_l}(k,\,\omega)}{\omega} \,=\, \rho v_s{}^2 \delta_{ij}\,. \tag{B.18}$$

For the second term we must evaluate

$$
\begin{aligned}
\delta\langle\mathbf{T}_{ij}\rangle^{(2)} \,=\, \frac{1}{2} \int dr' \int_0^\beta d\beta' \int dr'' \int_0^\beta d\beta'' \lambda_k(r')\lambda_l(r'') \\
\cdot[\langle(\mathbf{v}_{s_k}(r',\,-i\beta')\mathbf{v}_{s_l}(r'',\,-i\beta'')\mathbf{T}_{ij}(r,\,0))_+\rangle \\
-\, \langle(\mathbf{v}_{s_k}(r',\,-i\beta')\mathbf{v}_{s_l}(r'',\,-i\beta''))_+\rangle\langle\mathbf{T}_{ij}(r,\,0)\rangle].
\end{aligned}
\tag{B.19}
$$

Ordinarily we would expect that in the long wavelength limit, these expressions vanish because they are equal to a second spatial derivative of correlation functions like

$$\langle(\psi(r',\,-i\beta')\psi(r'',\,-i\beta'')\mathbf{T}_{ij}(r,\,0))_+\rangle.$$

If this term is singular in wavenumber, however, there will be a contribution. We allow for this possibility by permitting the Fourier transform of

$$
\begin{aligned}
\langle(\{(\psi \,-\, \psi^+)(r',\,-i\beta')\}\{(\psi \,-\, \psi^+)(r'',\,-i\beta'')\}\mathbf{T}_{ij}(r,\,0))_+\rangle \\
-\, \langle(\{(\psi \,-\, \psi^+)(r',\,-i\beta')\}\{(\psi \,-\, \psi^+)(r'',\,-i\beta'')\})_+\rangle\langle\mathbf{T}_{ij}(r,\,0)\rangle \\
=\, \int \frac{d^3k'}{(2\pi)^3}\frac{d^3k''}{(2\pi)^3}\frac{d\omega'}{2\pi}\frac{d\omega''}{2\pi} f_{ij}(k',\,k'',\,\omega',\,\omega'')\,e^{ik'\cdot(r'-r)+ik''\cdot(r''-r)-\omega'\beta'-\omega''\beta''}
\end{aligned}
\tag{B.20}
$$

to contain such singular terms as

$$
\begin{aligned}
f_{ij}(k',\,k'',\,\omega',\,\omega'') \,=\, \frac{\delta_{ij}\,k'\cdot k''}{k'^2 k''^2} f_4(k',\,k'',\,\omega',\,\omega'') \\
+\, \frac{k_i'k_j'' + k_j'k_i''}{k'^2 k''^2} f_5(k',\,k'',\,\omega',\,\omega'').
\end{aligned}
\tag{B.21}
$$

This is the only form with the appropriate symmetry (in k' and k'' and in i and

j). Taking the divergence of this equation, we have

$$\langle(\{\{(\psi - \psi^+)(r', -i\beta')\}\{(\psi - \psi^+)(r'', -i\beta'')\}\{(\partial g_j/\partial t)(r, 0)\})_+\rangle$$

$$= \int \frac{d^3k'}{(2\pi)^3} \frac{d^3k''}{(2\pi)^3} \frac{d\omega'}{2\pi} \frac{d\omega''}{2\pi} (k' + k'')_i f_{ij} e^{ik'\cdot(r'-r)+ik''\cdot(r''-r)-\omega'\beta'-\omega''\beta''}$$

$$= \frac{1}{i}\left[\frac{\partial}{\partial\beta'} + \frac{\partial}{\partial\beta''}\right]\langle(\{\{(\psi - \psi^+)(r', -i\beta')\}\{(\psi - \psi^+)(r'', -i\beta'')\}g_j(r, 0))_+\rangle \quad \text{(B.22)}$$

$$+ \delta(-i\beta')\langle[\{\psi - \psi^+)(r', 0)\}, g_j(r, 0)]\{(\psi - \psi^+)(r'', -i\beta'')\}\rangle$$

$$+ \delta(-i\beta'')\langle\{(\psi - \psi^+)(r', -i\beta')\}[\{(\psi - \psi^+)(r'', 0)\}, g_j(r, 0)]\rangle.$$

In the limit in which ω'', $\omega' \to 0$ the Fourier transform of this expression is

$$\frac{(k' + k'')_j k'\cdot k''}{k'^2k''^2} f_4(k', k'', 0, 0)$$

$$+ \frac{(k'^2 + k'\cdot k'')k_j'' + (k'\cdot k'' + k''^2)k_j'}{k'^2k''^2} f_5(k', k'', 0, 0)$$

$$= k_j' \int dr' \int_0^\beta d\beta' \langle\{(\psi - \psi^+)(r', -i\beta')\}\{(\psi - \psi^+)(r, 0)\}\rangle e^{-ik'\cdot(r'-r)}$$

$$+ \text{same term with } k_j' \to k_j'' \quad \text{(B.23)}$$

$$= k_j' \chi_{\psi-\psi^+, \psi-\psi^+}(k', 0) + k_j''\chi_{\psi-\psi^+, \psi-\psi^+}(k'', 0);$$

$$\frac{(k'^2 + k''\cdot k')k'\cdot k''}{k'^2k''^2} f_4 + \frac{(k'^2 + k'\cdot k'')k'\cdot k'' + (k'\cdot k'' + k''^2)k'^2}{k'^2k''^2} f_5 \quad \text{(B.24)}$$

$$= k'^2\chi_{\psi-\psi^+, \psi-\psi^+}(k', 0) + k'\cdot k''\chi_{\psi-\psi^+, \psi-\psi^+}(k'', 0).$$

For small and perpendicular k' and k'' we find

$$f_5 = k'^2\chi_{\psi-\psi^+, \psi-\psi^+}(k', 0); \quad \text{(B.25)}$$

for small, equal, and parallel k' and k'',

$$f_4 + 2f_5 = k'^2\chi_{\psi-\psi^+, \psi-\psi^+}(k', 0). \quad \text{(B.26)}$$

It is shown in Section IV, $\chi_{v_s, v_s}(k, 0) \to 1/\rho_s$ for small k. This identification only requires the linear terms in v_s. It is therefore not circular to anticipate that result in this computation of the quadratic term and deduce

$$-f_4 = +f_5 = 4\chi_{v_{s\,l}, v_{s\,m}}(k, 0)n_0 m^2 \frac{k_l k_m}{k^2} \to \frac{4n_0 m^2}{\rho_s}. \quad \text{(B.27)}$$

We therefore have

$$
\int dr' \int dr'' \int_0^\beta d\beta' \int_0^\beta d\beta'' e^{-i[k'\cdot(r'-r)+k''\cdot(r''-r)]}
$$

$$
\cdot \tfrac{1}{2}[\langle((\mathbf{v}_{sk}(r',-i\beta')\,\mathbf{v}_{s_l}(r'',-i\beta'')\mathbf{T}_{ij}(r,0))_+\rangle
$$

$$
- \langle(\mathbf{v}_{sk}(r',-i\beta')\mathbf{v}_{s_l}(r'',-i\beta''))_+\rangle\langle\mathbf{T}_{ij}(r,0)\rangle]
$$

$$
= -\frac{\delta_{ij}}{2\rho_s}\frac{k'_k k''_l k'\cdot k''}{k'^2 k''^2} + \frac{1}{2\rho_s}\frac{(k_i' k_j'' + k_j' k_i'')k_k' k_l''}{k'^2 k''^2}. \tag{B.28}
$$

Substituting, we obtain

$$
\delta\langle\mathbf{T}_{ij}\rangle^{(2)} = -\frac{1}{2\rho_s}\lambda^2\delta_{ij} + \frac{1}{\rho_s}\lambda_i\lambda_j = -\tfrac{1}{2}\rho_s v_s^2\delta_{ij} + \rho_s v_{s_i} v_{s_j} \tag{B.29}
$$

thus proving (3.31).

<div align="center">APPENDIX C</div>

<div align="center">THE PHONON SPECTRUM AT FINITE TEMPERATURE</div>

At finite temperature the hydrodynamic region is present and the discussion of Section V, E is much more questionable. It is confined to frequencies large compared to hydrodynamic ones (10^7 cps) but small compared to microscopic ones (10^{12} cps). The assumption of a single pole in $\chi_{v_s,v_s}(\omega)$ made in (5.29) is certainly not valid at finite temperatures. As we have seen in (4.38) the denominator of $\chi_{\rho,\rho}$ in the hydrodynamic region is more complicated than (5.32) and no "single-pole" approximation is rigorously valid. This is obviously also true for χ_{v_s,v_s}. However we will assume that this complication, which appears in the hydrodynamic region, may be neglected at higher frequencies so that in the collisionless region we may write

$$
\chi_{v_s,v_s} = \frac{-ak^2}{\omega^2 - v^2 k^2} + A(k,\omega) \tag{C.1}
$$

where A contributes primarily at large k and $\omega(ka \gtrsim 1,\ \omega t_m \gtrsim 1)$. If we assume that for small ω, A is of order k^4 then by (5.31) we again have

$$
\frac{a}{v^2} = \frac{1}{\rho_s}. \tag{C.2}
$$

We may similarly separate $\rho_n(k,\omega)$ and $\rho_s(k,\omega)$ as

$$
\rho_n(k,\omega) = \rho_n\delta_{ij} + \delta\rho_n(k,\omega)
$$
$$
\rho_s(k,\omega) = \rho_s\delta_{ij} + \delta\rho_s(k,\omega)
\tag{C.3}
$$

and rewrite the longitudinal part of (5.16) as

$$\chi^l_{g,g}(k, \omega) = \rho_n - \frac{\rho_s v^2 k^2}{\omega^2 - v^2 k^2} + D^l(k, \omega) \tag{C.4}$$

where

$$D^l(k, \omega) = \delta\rho_n(k, \omega) + \rho_s^2(k, \omega)A(k, \omega) - [2\rho_s \delta\rho_s(k, \omega)$$
$$+ \{\delta\rho_s(k, \omega)\}^2]\frac{ak^2}{\omega^2 - v^2 k^2} \tag{C.5}$$

contains the part of $\chi^l_{g,g}$ which is not given by a simple pole. Applying the sum rule (4.26) to Eq. (C.4) we see that our assumption (C.2) implies $D^l(k, 0) \sim k^2$. We may then use (5.30) to obtain

$$\chi_{\rho,\rho} = \frac{-\rho_s k^2}{\omega^2 - v^2 k^2} + \frac{k^2}{\omega^2} D^l(k, \omega). \tag{C.6}$$

Now from the "f-sum rule" (4.36), which may also be written

$$\lim_{k\to 0} \lim_{\omega\to 0} \frac{\chi_{\rho,\rho}(k, \omega)}{k^2} = -\rho,$$

we see, by (C.6), that $D^l(0, \infty) \sim -\rho_n$. We now assume that

$$\lim_{k\to 0,} \lim_{\omega\to 0} \frac{k^2}{\omega^2} D^l(k, \omega) = \frac{\zeta\rho_n}{v^2}. \tag{C.7}$$

It may be shown that the dimensionless quantity ζ is positive. Applying the "compressibility sum-rule" (4.33) to (C.7) we have

$$\alpha_{\rho,\rho} = \lim_{k\to 0,} \lim_{\omega\to 0} \chi_{\rho,\rho}(k, \omega) = \frac{\rho}{c_T^2} = \frac{\rho_s}{v^2} + \frac{\zeta\rho_n}{v^2} \tag{C.8}$$

so that

$$v^2 = c_T^2 \left[\frac{\rho_s + \zeta\rho_n}{\rho}\right]. \tag{C.9}$$

(c_T is the isothermal sound velocity.) A number of crude calculations based on perturbation theory suggest that $\zeta \ll 1$ (which is not inconsistent with $D^l(k, 0) \sim k^2$). In this case the quasi-particle velocity in the phonon region would be smaller than the isothermal sound velocity by an amount which increases with ρ_n. Such a variation could be detected by neutron diffraction measurements near 1.8°K, or possibly by ultrasonic absorption measurements.[23] It should be noted that our conclusion, weak as it is, depends on a number of

[23] J. Wilks, private communication.

regularity assumptions on the functions $A(k, \omega)$, (C.1), and $D^l(k, \omega)$, (C.7), which are at best only approximate. In particular we have neglected the broadening of the quasi-particle pole, which will certainly occur at finite temperatures and which could mask the effect given in (C.9).[24]

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