Quasiparticles in a superconductor

Elementary excitations

Add/take an electron, momentum *k*, spin σ

Populate state **+** or **–** of a *k*-box

$$u_k \left| 0_{\vec{k}} \right\rangle + v_k \left| 2_{\vec{k}} \right\rangle \;\rightarrow\; \left| + \right\rangle, \left| - \right\rangle$$

States **+** or **–** do not take part in superconductivity (unpaired states), energies same as for *U*=0

excitation energy?

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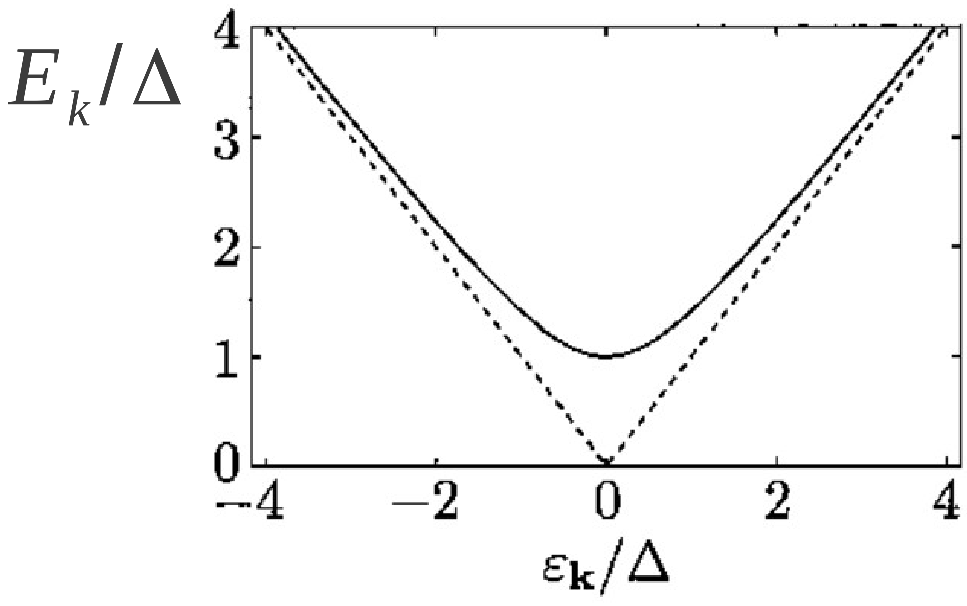
excitation energy

$$E_k = \epsilon_k - \tilde{\epsilon}_k = \epsilon_k - \left(\epsilon_k - \sqrt{\epsilon_k^2 + \Delta^2}\right)$$

$$\boxed{E_k = \sqrt{\epsilon_k^2 + \Delta^2} > 0}$$



Multiple excitations: noninteracting **quasiparticles**

Excitations discussion

Extra holes or electrons, but since $|g_s\rangle$ is not a number state, the $\hat{c}^{+}_{\vec{k},\sigma}, \ \hat{c}_{\vec{k},\sigma}$ are wrong operators

Quasiparticles $$|q_{\vec{k},\sigma}\rangle = \left(u^{*}\,\hat{c}^{+}_{\vec{k},\sigma} - v^{*}_{\vec{k}}\,\hat{c}_{-\vec{k},-\sigma}\right)|g_s\rangle$$

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QP's carry indefinite electron number and hence indefinite electric charge

$$Q_{qp}(\vec{k}) = e\left(|u_k|^2 - |v_k|^2\right) = e\,\frac{\epsilon_k}{E_k}$$

Quasiparticles from Bogoliubov transform

$$\hat{H}_{BCS}=\sum_{k,\sigma}\epsilon_k\hat{c}^{+}_{\vec{k}\sigma}\hat{c}_{\vec{k}\sigma}+\sum_{\vec{k},\vec{k}'}V_{k,k'}\hat{c}^{+}_{\vec{k}\uparrow}\hat{c}^{+}_{-\vec{k}\downarrow}\hat{c}_{-\vec{k}'\downarrow}\hat{c}_{\vec{k}'\uparrow}$$

A more general pairing interaction: $V_{k,k'}$ instead of U

Define: $b_k=\langle\hat{c}_{-\vec{k}\downarrow}\hat{c}_{\vec{k}\uparrow}\rangle,\quad \Delta_k=\sum_{\vec{k}'}V_{k,k'}b_k$

Quasiparticles from Bogoliubov transform

$$\hat{H}_{BCS} = \sum_{k,\sigma} \epsilon_k \hat{c}^+_{\vec{k}\sigma} \hat{c}_{\vec{k}\sigma} + \sum_{\vec{k},\vec{k}'} V_{k,k'} \hat{c}^+_{\vec{k}\uparrow} \hat{c}^+_{-\vec{k}\downarrow} \hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow}$$

A more general pairing interaction: $V_{k,k'}$ instead of U

Define: $b_k = \left\langle \hat{c}_{-\vec{k}\downarrow} \hat{c}_{\vec{k}\uparrow} \right\rangle, \quad \Delta_k = \sum_{\vec{k}'} V_{k,k'} b_k$

A mean field approximation:

$$\hat{H}^{(2)} = \sum_{\vec{k},\vec{k}'} V_{kk'} \left(\hat{c}^+_{\vec{k}\uparrow} \hat{c}^+_{-\vec{k}\downarrow} - b_k^* + b_k^* \right) \left(\hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow} - b_{k'} + b_{k'} \right)$$

$$= \sum_{\vec{k},\vec{k}'} V_{kk'} b_k^* b_{k'} + V_{kk'} \left(\hat{c}^+_{\vec{k}\uparrow} \hat{c}^+_{-\vec{k}\downarrow} - b_k^* \right) b_{k'}$$

$$+ V_{kk'} \left(\hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow} - b_{k'} \right) b_k^* + V_{kk'} \left(\hat{c}^+_{\vec{k}\uparrow} \hat{c}^+_{-\vec{k}\downarrow} - b_k^* \right) \left(\hat{c}_{-\vec{k}'\downarrow} \hat{c}_{\vec{k}'\uparrow} - b_k \right)$$

Quasiparticles from Bogoliubov transform

$$\hat{H}_{BCS}=\sum_{k,\sigma}\epsilon_k \hat{c}^{+}_{\vec{k}\sigma}\hat{c}_{\vec{k}\sigma}+\sum_{\vec{k},\vec{k}'}V_{k,k'}\hat{c}^{+}_{\vec{k}\uparrow}\hat{c}^{+}_{-\vec{k}\downarrow}\hat{c}_{-\vec{k}'\downarrow}\hat{c}_{\vec{k}'\uparrow}$$

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Define: $b_k=\langle \hat{c}_{-\vec{k}\downarrow}\hat{c}_{\vec{k}\uparrow}\rangle,\quad \Delta_k=\sum_{\vec{k}'}V_{k,k'}b_k$

A mean field approximation:

$$\hat{H}^{(2)}=\sum_{\vec{k},\vec{k}'}V_{kk'}\left(\hat{c}^{+}_{\vec{k}\uparrow}\hat{c}^{+}_{-\vec{k}\downarrow}-b^*_k+b^*_k\right)\left(\hat{c}_{-\vec{k}'\downarrow}\hat{c}_{\vec{k}'\uparrow}-b_{k'}+b_{k'}\right)$$

$$=\sum_{\vec{k},\vec{k}'}V_{kk'}b^*_k b_{k'}+V_{kk'}\left(\hat{c}^{+}_{\vec{k}\uparrow}\hat{c}^{+}_{-\vec{k}\downarrow}-b^*_k\right)b_{k'}$$

$$+V_{kk'}\left(\hat{c}_{-\vec{k}'\downarrow}\hat{c}_{\vec{k}'\uparrow}-b_{k'}\right)b^*_k+V_{kk'}\left(\hat{c}^{+}_{\vec{k}\uparrow}\hat{c}^{+}_{-\vec{k}\downarrow}-b^*_k\right)\left(\hat{c}_{-\vec{k}'\downarrow}\hat{c}_{\vec{k}'\uparrow}-b_k\right)$$

$$\boxed{\hat{H}^{(2)}=-\sum_{\vec{k}}\Delta_k b^*_k+\sum_k \Delta_k\hat{c}^{+}_{\vec{k}\uparrow}\hat{c}^{+}_{-\vec{k}\downarrow}+\Delta^*_k\hat{c}_{-\vec{k}\downarrow}\hat{c}_{\vec{k}\uparrow}}$$

Quadratic Hamiltonian

$$\hat{H}_{tot} = \sum_{k,\sigma} \epsilon_k \hat{c}^{+}_{\vec{k}\sigma} \hat{c}_{\vec{k}\sigma} + \sum_k \Delta_k \hat{c}^{+}_{\vec{k}\uparrow} \hat{c}^{+}_{-\vec{k}\downarrow} + \Delta^{*}_k \hat{c}_{-\vec{k}\downarrow} \hat{c}_{\vec{k}\uparrow}$$

rewrite (up to a constant) as

$$\hat{H}_{tot} = \sum_k \epsilon_k \hat{c}^{+}_{\vec{k}\uparrow} \hat{c}_{\vec{k}\uparrow} - \epsilon_k \hat{c}_{-\vec{k}\downarrow} \hat{c}^{+}_{-\vec{k}\downarrow} + \Delta_k \hat{c}^{+}_{\vec{k}\uparrow} \hat{c}^{+}_{-\vec{k}\downarrow} + \Delta^{*}_k \hat{c}_{-\vec{k}\downarrow} \hat{c}_{\vec{k}\uparrow}$$

Diagonalize?

Quadratic Hamiltonian

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$$\hat{H}_{tot}=\sum_k \epsilon_k \hat{c}^+_{\vec{k}\uparrow}\hat{c}_{\vec{k}\uparrow}-\epsilon_k \hat{c}_{-\vec{k}\downarrow}\hat{c}^+_{-\vec{k}\downarrow}+\Delta_k \hat{c}^+_{\vec{k}\uparrow}\hat{c}^+_{-\vec{k}\downarrow}+\Delta^*_k \hat{c}_{-\vec{k}\downarrow}\hat{c}_{\vec{k}\uparrow}$$

Diagonalize?

$$\boxed{\begin{aligned}\hat{c}_{\vec{k}\uparrow}&=u^*_k a_{k,0}+v_k a^+_{k,1}\\ \hat{c}^+_{-\vec{k}\downarrow}&=-v^*_k a_{k,0}+u_k a^+_{k,1}\end{aligned}}$$

$$|u_k{}^2|+|v_k{}^2|=1$$

Quadratic Hamiltonian

$$\hat{H}_{tot}=\sum_{k,\sigma}\epsilon_k \hat{c}^{+}_{\vec{k}\sigma}\hat{c}_{\vec{k}\sigma}+\sum_{k}\Delta_k \hat{c}^{+}_{\vec{k}\uparrow}\hat{c}^{+}_{-\vec{k}\downarrow}+\Delta^{*}_k \hat{c}_{-\vec{k}\downarrow}\hat{c}_{\vec{k}\uparrow}$$

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$$|u_k{}^2|+|v_k{}^2|=1$$

Canonical transformation: $\left[\hat{a}^{+}_i,\hat{a}_j\right]_{+}=\delta_{ij},\ \left[\hat{a}_i,\hat{a}_j\right]_{+}=0$

1) $\hat{c}_i=\sum_j U_{ij}\hat{a}_j,\ \hat{c}^{+}_i=\sum_j U^{*}_{ij}\hat{a}^{+}_j,\ \sum_j U^{*}_{ij}U_{ji'}=\delta_{ii'}$ unitary transformation

2) $\hat{c}_i=\hat{a}^{+}_i,\ \hat{c}^{+}_i=\hat{a}_i$ particle-hole transformation

Quasiparticle Hamiltonian

Terms $\hat{a}_i^+, \hat{a}_j^+, \ \hat{a}_i, \hat{a}_j$ vanish when

$$\frac{u_k}{v_k} = \frac{E_k - \epsilon_k}{\Delta}, \quad E_k = \sqrt{\epsilon_k^2 + |\Delta^2|}$$

Diagonal H! $\hat{H} = \sum_{k,i=0,1} E_k \hat{a}_{\vec{k}i}^+ \hat{a}_{\vec{k}i}$

Quasiparticle Hamiltonian

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Diagonal H!

$$\boxed{\hat{H} = \sum_{k,i=0,1} E_k \hat{a}_{\vec{k}i}^+ \hat{a}_{\vec{k}i}}$$

Coherence amplitudes:

$$u_k^2, v_k^2 = \frac{1}{2}\left(1 \pm \frac{\epsilon_k}{E_k}\right)$$

(agrees w/ our prev result)

Excitations discussion

$$|g_s\rangle = \prod_k \left(u_k + v_k c^+_{k\uparrow} c^+_{-k\downarrow}\right)|0\rangle$$

Extra holes or electrons, but since $|g_s\rangle$ is not a number state, the $\hat{c}^+_{\vec{k},\sigma}, \hat{c}_{\vec{k},\sigma}$ are wrong operators

Quasiparticles $\quad a^+_{\vec{k},\sigma}|g_s\rangle = \left(u^*_k \hat{c}^+_{\vec{k},\sigma} - v^*_k \hat{c}_{-\vec{k},-\sigma}\right)|g_s\rangle$

QP energy $\quad E_k = \sqrt{\epsilon^2_k + \Delta^2} \; > \; 0$

QP's carry indefinite electron number and hence indefinite electric charge

$$Q_{qp}(\vec{k}) = e\left(|u_k|^2 - |v_k|^2\right) = e\frac{\epsilon_k}{E_k}, \quad -|e| \le Q_{qp} \le |e|$$

QP density of states

$$\nu(\epsilon)\,d\epsilon = N(E)\,dE\,,\quad E=\sqrt{\epsilon^2+\Delta^2}$$

$$N(E)=\nu_0\frac{E}{\sqrt{E^2-\Delta^2}},\quad |E|\geq\Delta \qquad N(E)=0,\quad |E|<\Delta$$

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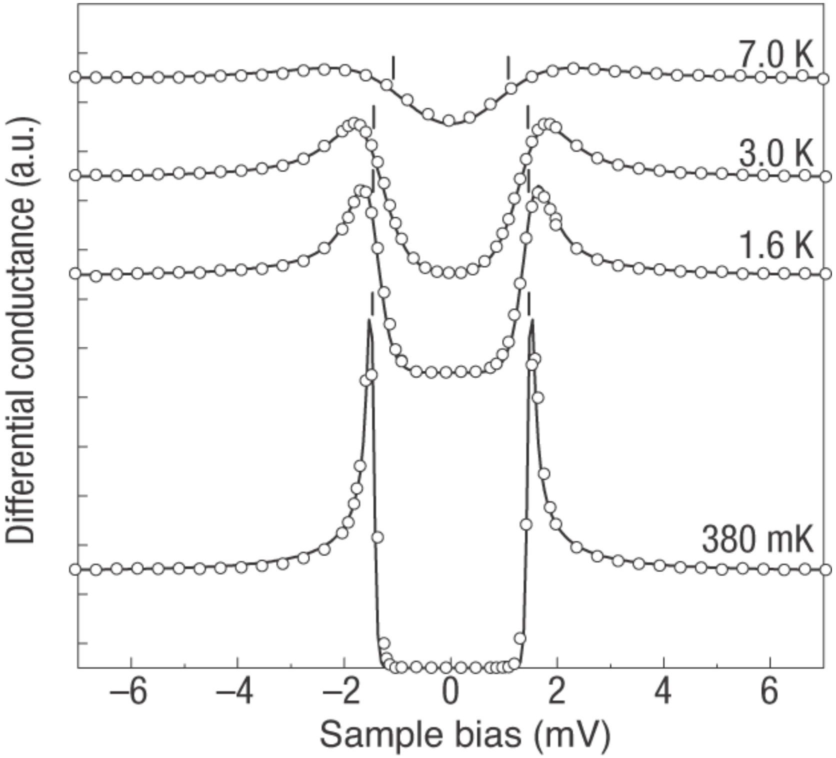
Scanning tunneling spectroscopy (STS) in Niobium (fits by BCS theory)



High-Tc SC (non-BCS mechanism)