1 Ramsey Fringes Overview:

Following a double pulse, the population of the excited state is:

$$\boxed{P_2 = 4\sin^2\theta\sin^2\frac{\Omega'\tau}{2}\left\{\cos\frac{\Omega'\tau}{2}\cos\frac{\Delta_0 T}{2} - \cos\theta\sin\frac{\Omega'\tau}{2}\sin\frac{\Delta_0 T}{2}\right\}}$$

under the assumption that initially,

$$\begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The final state vector is:

$$\begin{pmatrix} C_1(2\tau+T) \\ C_2(2\tau+T) \end{pmatrix} = \rho_2 D \rho_1 \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix}$$

where $|C_1|^2 + |C_2|^2 = 1$ for all value of time, and ρ_1 and ρ_2 are propagators associated with Pulse 1 and Pulse 2 (both with width τ), respectively. D is a propagator associated with the field-free evolution of duration T.

Specifically, in the interaction representation:

$$\rho_1 = e^{-i\overline{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} \\ ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}$$

$$\rho_2 = e^{-i\overline{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{-i\Delta_0(\tau+T)} \\ ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{i\Delta_0(\tau+T)} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that D, in the interaction representation, is the identity matrix. This is different from Ramsey's original approach in which the state vector does evolve and change during the delay time T. The angle θ is defined as:

$$\sin\theta = \frac{{\Omega_0}^*}{\Omega'}$$

and

$$\cos\theta = \frac{\Delta_0 + \Delta_d}{\Omega'}$$

where ${\Omega_0}^*$ is the complex conjugate of the Rabi rate, and Ω' can be defined as the "effective Rabi rate."

$$\Omega' = \sqrt{|{\Omega_0}^*| + (\Delta_0 + \Delta_d)^2}$$

2 Detailed Derivation for P_f

In the interaction representation,

$$\boxed{i\begin{pmatrix}\dot{a}_i(t)\\ \dot{a}_f(t)\end{pmatrix} = \begin{pmatrix}\Delta_i & -\frac{\Omega_0^*}{2}e^{-i\Delta_0 t}\\ -\frac{\Omega_0}{2}e^{i\Delta_0 t} & \Delta_f\end{pmatrix}\begin{pmatrix}a_i(t)\\ a_f(t)\end{pmatrix}} \tag{1}$$

where Δ_i is the ac Stark shift in the $|i\rangle$ state and Δ_f is the ac Stark shift in the $|f\rangle$ state. We first solve for a_i:

$$i\ddot{a}_i = \Delta_i\dot{a}_i - \frac{\Omega_0^*}{2}e^{-i\Delta_0 t}\dot{a}_f + \frac{i\Delta_0}{2}\Omega_0^* e^{-i\Delta_0 t}a_f, \tag{2}$$

where

$$a_f = \left(\frac{\Omega_0^*}{2}e^{-\Delta_0 t}\right)^{-1}(\Delta_i a_i - i\dot{a}_i) = \frac{2}{\Omega_0^*}e^{i\Delta_0 t}(\Delta_i a_i - i\dot{a}_i). \tag{3}$$

From Eq. (1), we also have

$$\begin{aligned}\dot{a}_f &= i^{-1}\left(-\frac{\Omega_0}{2}e^{i\Delta_0 t}a_i + \Delta_f a_f\right)\\ &= i^{-1}\left[-\frac{\Omega_0}{2}e^{i\Delta_0 t}a_i + \Delta_f(\Delta_i a_i - i\dot{a}_i)\left(\frac{2}{\Omega_0^*}e^{i\Delta_0 t}\right)\right]\\ &= \frac{2i}{\Omega_0^*}e^{i\Delta_0 t}\left(\frac{|\Omega_0|^2}{4}a_i - \Delta_f\Delta_i a_i + i\Delta_f\dot{a}_i\right).\end{aligned} \tag{4}$$

Therefore,

$$\begin{aligned}\ddot{a}_i &= -i\Delta_i a_i + \frac{i\Omega_0^*}{2}e^{-i\Delta_0 t}\left[\frac{2i}{\Omega_0^*}e^{i\Delta_0 t}\left(\frac{|\Omega_0|^2}{4}a_i - \Delta_f\Delta_i a_i + i\Delta_f\dot{a}_i\right)\right]\\ &+ \frac{\Delta_0}{2}\Omega^* e^{-i\Delta_0 t}\frac{2}{\Omega_0^*}e^{i\Delta_0 t}(\Delta_i a_i - i\dot{a}_i)\\ &= -i\Delta_i\dot{a}_i - \frac{|\Omega_0|^2}{4}a_i + \Delta_f\Delta_i a_i - i\Delta_f\dot{a}_i + \Delta_0\Delta_i a_i - i\Delta_0\dot{a}_i\\ &= -i(\Delta_0 + \Delta_i + \Delta_f)\dot{a}_i - \left[\frac{|\Omega_0|^2}{4} - \Delta_i(\Delta_f + \Delta_0)\right]a_i\\ &= -i(\Delta_0 + \Delta_i + \Delta_f)\dot{a}_i - \frac{1}{4}\left(|\Omega_0|^2 - 4\Delta_i\Delta_f - 4\Delta_i\Delta_0\right)a_i.\end{aligned} \tag{5}$$

We obtain the first second-order homogeneous differential equation:

$$\ddot{a}_i + i(\Delta_0 + \Delta_i + \Delta_f)\dot{a}_i + \left[\frac{|\Omega_0|^2}{4} - \Delta_i(\Delta_f + \Delta_0)\right]a_i = 0. \tag{6}$$

Let a guess solution be $a_i(t) = a_0 e^{i\omega t}$. The characteristic equation is:

$$\begin{aligned}-\omega^2 + i(\Delta_0 + \Delta_i + \Delta_f)(i\omega) + \frac{1}{4}(|\Omega_0|^2 - 4\Delta_0\Delta_i - 4\Delta_i\Delta_f) = 0\\ -\omega^2 - (\Delta_0 + \Delta_i + \Delta_f)\omega + \frac{1}{4}(|\Omega_0|^2 - 4\Delta_0\Delta_i - 4\Delta_i\Delta_f) = 0.\end{aligned} \tag{7}$$

Solving the quadratic equation (7) and obtain w:

$$\begin{aligned}\omega &= -\frac{\Delta_0+\Delta_f+\Delta_i}{2} \pm \frac{1}{2}\sqrt{(\Delta_0+\Delta_f+\Delta_i)^2+|\Omega_0|^2-4\Delta_0\Delta_i-4\Delta_i\Delta_f} \\ &= -\frac{\Delta_0+\Delta_f+\Delta_i}{2} \\ &\pm \frac{1}{2}\sqrt{(\Delta_0+\Delta_f+\Delta_i)^2+\Delta_0^2+|\Omega_0|^2+(\Delta_i-\Delta_f)^2-2\Delta_0(\Delta_i-\Delta_f)}\end{aligned} \tag{8}$$

Let $\bar{\Delta}=(\Delta_i+\Delta_f)/2$ and $\Delta_d=\Delta_f-\Delta_i$, this gives

$$\boxed{\omega = -\frac{\Delta_0}{2}-\bar{\Delta}\pm\frac{1}{2}\sqrt{|\Omega_0|^2+(\Delta_0+\Delta_d)^2}} \tag{9}$$

Next, let the "effective Rabi rate" be Ω', defined as

$$\Omega' = \sqrt{|\Omega_0|^2+(\Delta_0+\Delta_d)^2}. \tag{10}$$

The general solution to eq. (7) is:

$$\begin{aligned}a_i &= a_+e^{i\omega_+t}+a_-e^{i\omega_-t} \\ &= e^{-i\bar{\Delta}t}e^{-i\frac{\Delta_0}{2}t}\left(a_+e^{i\frac{\Omega'}{2}t}+a_-e^{-i\frac{\Omega'}{2}t}\right)\end{aligned} \tag{11}$$

So,

$$\boxed{a_i = e^{-i\bar{\Delta}t}e^{-i\frac{\Delta_0}{2}t}\left(A\cos\frac{\Omega't}{2}+B\sin\frac{\Omega't}{2}\right)} \tag{12}$$

Next, we solve for a_f. From eq. (3):

$$\begin{aligned}a_f &= \frac{2}{\Omega_0^*}e^{i\Delta_0t}\left(\Delta_ia_i-i\dot{a}_i\right) \\ &= \frac{2}{\Omega_0^*}e^{i\Delta_0t}\left[\Delta_ie^{-i\bar{\Delta}t}e^{-i\frac{\Delta_0}{2}t}\left(A\cos\frac{\Omega't}{2}+B\sin\frac{\Omega't}{2}\right)-i\dot{a}_i\right] \\ &= \frac{2}{\Omega_0^*}e^{i\frac{\Delta_0}{2}t}\left[\Delta_ie^{-i\bar{\Delta}t}\left(A\cos\frac{\Omega't}{2}+B\sin\frac{\Omega't}{2}\right)-i\dot{a}_i\right]\end{aligned} \tag{13}$$

where

$$\begin{aligned}-i\dot{a}_i &= (-i)^2\left(\bar{\Delta}+\frac{\Delta_0}{2}\right)e^{-i\bar{\Delta}t}e^{-i\frac{\Delta_0}{2}t}\left[\left(A\cos\frac{\Omega't}{2}+B\sin\frac{\Omega't}{2}\right)\right. \\ &\left.+-i\frac{\Omega'}{2}\left(-A\sin\frac{\Omega't}{2}+B\cos\frac{\Omega't}{2}\right)\right] \\ &= e^{-i\bar{\Delta}t}e^{-i\frac{\Delta_0}{2}t}\left[-\left(\bar{\Delta}+\frac{\Delta_0}{2}\right)\left(A\cos\frac{\Omega't}{2}+B\sin\frac{\Omega't}{2}\right)\right. \\ &\left.+i\frac{\Omega'}{2}\left(A\sin\frac{\Omega't}{2}-B\cos\frac{\Omega't}{2}\right)\right].\end{aligned} \tag{14}$$

Assume that at $t = 0$, $A = a_i(0)$ and

$$B = i\frac{\Omega_0^*}{\Omega'}a_f(0) + i\frac{\Delta_d + \Delta_0}{\Omega'}a_i(0). \tag{15}$$

So, from Eq. (12):

$$\boxed{a_i(t) = e^{-i(\bar{\Delta}+\frac{\Delta_0}{2})t}\left\{a_i(0)\left[\cos\frac{\Omega' t}{2} + i\frac{\Delta_0+\Delta_d}{\Omega'}\sin\frac{\Omega' t}{2}\right] + a_f(0)\frac{i\Omega_0^*}{\Omega'}\sin\frac{\Omega' t}{2}\right\}} \tag{16}$$

From Eq. (13) and (14), we obtain an expression for $a_f(t)$:

$$\begin{aligned}
a_f(t) = &\frac{2}{\Omega_0^*}e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{\Delta_i\left(a_i(0)\cos\frac{\Omega' t}{2} + \left(\frac{i\Omega_0^*}{\Omega'}a_f(0) + i\frac{\Delta_d+\Delta_0}{\Omega'}a_i(0)\right)\sin\frac{\Omega' t}{2}\right)\right.\\
&-\left(\bar{\Delta} + \frac{\Delta_0}{2}\left(a_i\cos\frac{\Omega' t}{2} + \left(\frac{i\Omega_0^*}{\Omega'}a_f(0) + \frac{i}{2}\frac{\Delta_d+\Delta_0}{\Omega'}a_i(0)\right)\sin\frac{\Omega' t}{2}\right)\right)\\
&\left. i\frac{\Omega'}{2}\left(a_i\sin\frac{\Omega' t}{2} - \left(\frac{i\Omega_0^*}{\Omega'}a_f(0) + \frac{i}{2}\frac{\Delta_d+\Delta_0}{\Omega'}a_i(0)\right)\cos\frac{\Omega' t}{2}\right)\right\}\\
&= \frac{2}{\Omega_0^*}e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{\left(\Delta_i - \bar{\Delta} - \frac{\Delta_0}{2}\right)\left(A\cos\frac{\Omega' t}{2} + B\sin\frac{\Omega' t}{2}\right)\right.\\
&\left.\frac{i\Omega'}{2}\left(A\sin\frac{\Omega' t}{2} - B\cos\frac{\Omega' t}{2}\right)\right\}.
\end{aligned} \tag{17}$$

Now, notice that

$$\Delta_i - \bar{\Delta} = \Delta_i - \frac{\Delta_i + \Delta_f}{2} = -\frac{\Delta_d}{2}. \tag{18}$$

So,

$$\begin{aligned}
a_f(t) &= e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{A\left(-\frac{\Delta_d+\Delta_0}{\Omega_0^*}\cos\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\sin\frac{\Omega' t}{2}\right)\right.\\
&\quad\left.-B\left(\frac{\Delta_d+\Delta_0}{\Omega_0^*}\sin\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\cos\frac{\Omega' t}{2}\right)\right\}\\
&= e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{a_i(0)\left(-\frac{\Delta_d+\Delta_0}{\Omega_0^*}\cos\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\sin\frac{\Omega' t}{2}\right)\right.\\
&\quad\left.-\left(i\frac{\Omega_0^*}{\Omega'}a_f(0) + i\frac{\Delta_d+\Delta_0}{\Omega'}a_i(0)\right)\left(\frac{\Delta_d+\Delta_0}{\Omega_0^*}\sin\frac{\Omega' t}{2} + \frac{i\Omega'}{\Omega_0^*}\cos\frac{\Omega' t}{2}\right)\right\}\\
&= e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{a_i\sin\frac{\Omega' t}{2}\left(\frac{i\Omega'}{\Omega_0^*\Omega'} - \frac{i(\Delta_d+\Delta_0)^2}{\Omega_0^*\Omega'}\right)\right.\\
&\quad\left. a_f(0)\left(\cos\frac{\Omega' t}{2} - i\frac{\Delta_d+\Delta_0}{\Omega'}\sin\frac{\Omega' t}{2}\right)\right\}.
\end{aligned} \tag{19}$$

Next, note that

$$\Omega'^2 = (\Delta_0 + \Delta_d)^2 + |\Omega_0|^2 = (\Delta_0 + \Delta_d)^2 + \Omega_0\Omega_0^*. \tag{20}$$

So

$$\boxed{a_f(t) = e^{-i\bar{\Delta}t}e^{i\frac{\Delta_0}{2}t}\left\{a_i(0)\frac{i\Omega_0}{\Omega'}\sin\frac{\Omega' t}{2} + a_f(0)\left(\cos\frac{\Omega' t}{2} - i\frac{\Delta_d+\Delta_0}{\Omega'}\sin\frac{\Omega' t}{2}\right)\right\}}. \tag{21}$$

Finally, let us put everything together in matrix form:

$$\begin{pmatrix} a_i(t) \\ a_f(t) \end{pmatrix} = \mathcal{M}\begin{pmatrix} a_i(0) \\ a_f(0) \end{pmatrix}, \tag{22}$$

where $\mathcal{M}(t)$ is the matrix

$$\boxed{e^{-i\bar{\Delta}t}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}t}\left(\cos\frac{\Omega' t}{2} + i\frac{\Delta_d+\Delta_0}{\Omega'}\sin\frac{\Omega' t}{2}\right) & e^{-i\frac{\Delta_0}{2}t}\frac{i\Omega_0^*}{\Omega'}\sin\frac{\Omega' t}{2} \\ e^{i\frac{\Delta_0}{2}t}\frac{i\Omega_0}{\Omega'}\sin\frac{\Omega' t}{2} & e^{i\frac{\Delta_0}{2}t}\left(\cos\frac{\Omega' t}{2} - i\frac{\Delta_d+\Delta_0}{\Omega'}\sin\frac{\Omega' t}{2}\right) \end{pmatrix}} \tag{23}$$

Let's define more terms:

$$\begin{aligned} \Omega_0 &= |\Omega_0|e^{i\phi_0} \\ \Omega_0^* &= |\Omega_0|e^{-i\phi_0}. \end{aligned} \tag{24}$$

Therefore,

$$\begin{aligned} \cos\theta &= \frac{\Delta_0+\Delta_d}{\Omega'} \\ \sin\theta &= \frac{|\Omega_0|}{\Omega'} = \frac{\Omega_0}{\Omega'}e^{i\phi_0}. \end{aligned} \tag{25}$$

At time τ, the matrix $\mathcal{M}(\tau)$ is:

$$e^{-i\bar{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\frac{\Delta_0}{2}\tau}e^{-i\phi_0}\sin\theta\sin\frac{\Omega'\tau}{2} \\ ie^{i\frac{\Delta_0}{2}\tau}e^{i\phi_0}\sin\theta\sin\frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix} \tag{26}$$

Further simplification gives the matrix $\mathcal{M}(\tau)$:

$$\boxed{e^{-i\bar{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} \\ ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}} \tag{27}$$

What is an interpretation of $\mathcal{M}(\tau)$? The matrix $\mathcal{M}(\tau)$ represents what a laser pulse of width τ and intensity Ω_0 does to an initial state vector $(a_i(0)\, a_f(0))^\top$.

Now, our goal is to derive the final state vector following a (i) a pulse of width τ, (ii) another a pulse of width τ after some wait time T. Let us call the propagator associated with the second pulse $\mathcal{N}$. The next step is to derive $\mathcal{N}$.

$$\begin{pmatrix} a_i(2\tau+T) \\ a_f(2\tau+T) \end{pmatrix} = \mathcal{N} \begin{pmatrix} a_i(\tau+T) \\ a_f(\tau+T) \end{pmatrix}. \tag{28}$$

We assume that, at $t = 0$, the probability amplitude of finding the atom in the ground state is 1 and in the excited state is 0:

$$\boxed{\begin{pmatrix} a_i(0) \\ a_f(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \tag{29}$$

This gives the state amplitudes after time τ:

$$\begin{aligned} a_i(\tau) &= e^{-i\frac{\Delta_0}{2}\tau}\left\{ e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right)\right\} \\ a_f(\tau) &= e^{-i\bar{\Delta}\tau}\left\{ ie^{i\left(\frac{\Delta_0\tau}{2}+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}\right\} \end{aligned} \tag{30}$$

The derivation of $\mathcal{N}$ should be quite similar to that of $\mathcal{M}$. However, we should also take into account the wait time T. It turns out that we only need to add the extra terms $e^{i\Delta_0 t}$ and $e^{-i\Delta_0 t}$ to the off-diagonals. These terms represent how the state vector evolves over the "rest-time" T. The matrix $\mathcal{N}(T)$ has the following form:

$$\boxed{e^{-i\bar{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{-i\Delta_0 T} \\ ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{i\Delta_0 T} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}} \tag{31}$$

So, the final state vector, as represented by the initial state vector and $\mathcal{M}(\tau)$ and $\mathcal{N}(\tau+T)$ is:

$$\begin{pmatrix} a_i(2\tau+T) \\ a_f(2\tau+T) \end{pmatrix} = \mathcal{N}(\tau+T)\mathcal{M}(\tau)\begin{pmatrix} a_i(0) \\ a_f(0) \end{pmatrix} = \mathcal{N}(\tau+T)\mathcal{M}(\tau)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{32}$$

Since we're only interested in the final state probability amplitude, we can ignore the initial state amplitude:

$$
\begin{aligned}
a_f(2\tau+T) = e^{-i\bar{\Delta}\tau} \Bigg\{ & ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)} \sin\theta \sin\frac{\Omega'\tau}{2} e^{i\Delta_0(\tau+T)} a_i(\tau) \\
& e^{i\frac{\Delta_0}{2}\tau} \left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) a_f(\tau) \Bigg\} \\
= e^{-i\bar{\Delta}\tau} \Bigg\{ & ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)} \sin\theta\sin\frac{\Omega'\tau}{2} e^{i\Delta_0(\tau+T)} \\
\times & e^{-i\frac{\Delta_0\tau}{2}} \left[e^{-i\frac{\Delta_0}{2}\tau} \left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \right] \\
& e^{i\frac{\Delta_0}{2}\tau} \left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) e^{-i\bar{\Delta}\tau} \left[ie^{i\left(\frac{\Delta_0\tau}{2}+\phi_0\right)} \sin\theta\sin\frac{\Omega'\tau}{2} \right] \Bigg\}.
\end{aligned} \tag{33}
$$

Further simplification gives:

$$
\begin{aligned}
a_f(2\tau+T) = & ie^{-2i\bar{\Delta}\tau} e^{i(\Delta_0\tau+\phi_0)} \sin\theta\sin\frac{\Omega'\tau}{2} \Bigg\{ e^{iT(\Delta_0-\Delta_i)} \times \\
& \left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) + e^{-i\Delta_f\tau} \left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \Bigg\}
\end{aligned} \tag{34}
$$

In order to calculate the transition probability $P_2 = |a_f|^2 = a_f^* a_f$, we have to find the complex conjugate of a_f. Consider this term:

$$
E = e^{iT(\Delta_0-\Delta_i)} \left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) + e^{-i\Delta_f\tau} \left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right).
$$

Let

$$
a = \cos\frac{\Omega'\tau}{2} \tag{35}
$$

$$
b = \cos\theta\sin\frac{\Omega'\tau}{2} \tag{36}
$$

It follows that

$$
\begin{aligned}
E &= e^{iT(\Delta_0-\Delta_i)}(a+ib) + e^{-i\Delta_f\tau}(a-ib) \\
&= \left[\cos\left((\Delta_0-\Delta_i)\tau\right) + i\sin\left((\Delta_0-\Delta_i)\tau\right)\right](a+ib) \\
&+ \left[\cos\Delta_f\tau - i\sin\Delta_f\tau\right](a-ib) \\
&= R + iI.
\end{aligned} \tag{37}
$$

The Real part R is:

$$\begin{aligned}R &= a\cos\left((\Delta_0-\Delta_i)\tau\right) - b\sin\left((\Delta_0-\Delta_i)\tau\right) + a\cos\Delta_f\tau - b\sin\Delta_f\tau \\ &= 2a\cos\left(\frac{\tau(\Delta_0-\Delta_i+\Delta_f)}{2}\right)\cos\left(\frac{\tau(\Delta_0-\Delta_i-\Delta_f)}{2}\right) \\ &- 2b\sin\left(\frac{\tau(\Delta_0-\Delta_i+\Delta_f)}{2}\right)\sin\left(\frac{\tau(\Delta_0-\Delta_i-\Delta_f)}{2}\right) \\ &= 2\cos\left[T\left(\frac{\Delta_0}{2}-\bar{\Delta}\right)\right]\left[\cos\frac{\Omega'\tau}{2}\cos\frac{T(\Delta_0+\Delta_d)}{2} - \cos\theta\sin\frac{\Omega'\tau}{2}\sin\frac{T(\Delta_0+\Delta_d)}{2}\right].\end{aligned} \tag{38}$$

And deriving in a similar fashion, the imaginary part I is:

$$I = 2\sin\left[T\left(\frac{\Delta_0}{2}-\bar{\Delta}\right)\right]\left[\cos\frac{\Omega'\tau}{2}\cos\frac{T(\Delta_0+\Delta_d)}{2} - \cos\theta\sin\frac{\Omega'\tau}{2}\sin\frac{T(\Delta_0+\Delta_d)}{2}\right]. \tag{39}$$

Therefore,

$$\begin{aligned}P_2 &= a_f^* a_f \\ &= R^2 + I^2 \\ &= 4\sin^2\theta\sin^2\frac{\Omega'\tau}{2}\left[\cos\frac{\Omega'\tau}{2}\cos\frac{T(\Delta_0+\Delta_d)}{2} - \cos\theta\sin\frac{\Omega'\tau}{2}\sin\frac{T(\Delta_0+\Delta_d)}{2}\right]^2\end{aligned} \tag{40}$$

To complete our derivation and match our version with Ramsey's, we make one approximation:

$$\Delta_d = \Delta_f - \Delta_i \ll \Delta_0, \tag{41}$$

this basically says that the difference in the ac Stark shift between the states is much smaller than the detuning. This leaves us with:

$$\boxed{P_2 = 4\sin^2\theta\sin^2\frac{\Omega'\tau}{2}\left(\cos\frac{\Omega'\tau}{2}\cos\frac{T\Delta_0}{2} - \cos\theta\sin\frac{\Omega'\tau}{2}\sin\frac{T\Delta_0}{2}\right)^2} \tag{42}$$