MULTIPARAMETER FOURIER ANALYSIS

Robert Fefferman

Introduction

The article which follows is an attempt to give an exposition of some of the recent progress in that part of Fourier Analysis which deals with classes of operators commuting with multiparameter families of dilations. In some sense, this field is not that new, since already in the early 1930's the properties of the strong maximal function were being investigated by Saks, Zygmund, and others. However, for many of the problems in this area which seem quite classical, answers have either not been found at all, or only quite a short time ago, so that our knowledge of the area is still fragmentary at this time.

The article is divided into six sections. The first treats some basic issues in the classical one-parameter theory whose multiparameter theory is then discussed in the remaining sections. Since the reader is no doubt quite familiar with the main elements of the classical theory, we have omitted references to the materials in section one. The book "Singular Integrals and Differentiability Properties of Functions" by E. M. Stein is an excellent reference for virtually all of the material there.

Finally, it is a pleasure to thank Professors M. T. Cheng and E. M. Stein for all of their hard work in organizing the Summer Symposium in Analysis in China, as well as many others whose generous hospitality made the visit to China such a very enjoyable one.

1. *The maximal function, Calderón-Zygmund decomposition, and Littlewood-Paley-Stein theory*

We hope here to review briefly some aspects of the classical 1-parameter theory of these topics. The three are inseparable and we hope to stress this.

We begin with the fundamental

CALDERÓN-ZYGMUND LEMMA. *Let* $f(x) \geq 0$, $f \in L^1(R^n)$, and $\alpha > 0$. *Then there exist disjoint cubes* Q_k *such that*

$$(1) \quad \alpha < \frac{1}{|Q_k|} \int_{Q_k} f(x)\,dx \leq 2^n \alpha$$

$$(2) \quad f(x) \leq \alpha \ \text{ a.e. } for\ x \notin \cup Q_k$$

and

$$(3) \quad |\cup Q_k| \leq \frac{1}{\alpha} \int_{R^n} |f(x)|\,dx\ .$$

Proof. Let R^n be subdivided into a grid of congruent cubes so large that $\frac{1}{|Q|}\int_Q f \leq \alpha$ for all of them. Subdivide each cube in this collection into 2^n congruent subcubes. Select from these the cubes Q' such that $\frac{1}{|Q'|}\int_{Q'} f > \alpha$. For these Q' we stop the bisection process. For the rest, we continue until we first arrive at a cube Q' such that $\frac{1}{|Q'|}\int_{Q'} f > \alpha$, at which point we stop.

The cubes Q' at which we stop are then our Q_k. By construction $\frac{1}{|Q_k|}\int_{Q_k} f > \alpha$. Let $\tilde{Q}_k$ be the cube containing Q_k which was bisected to produce Q_k. Then $|\tilde{Q}_k| = 2^n|Q_k|$ and since we did not stop at $\tilde{Q}_k$, $\frac{1}{|\tilde{Q}_k|}\int_{\tilde{Q}_k} f \leq \alpha$. It follows that

$$\frac{1}{|Q_k|}\int_{Q_k} f \leq \frac{|\tilde{Q}_k|}{|Q_k|}\frac{1}{|\tilde{Q}_k|}\int_{\tilde{Q}_k} f \leq 2^n\alpha ,$$

proving (1). Notice that (3) follows from (1) because $|Q_k| \leq \frac{1}{\alpha}\int_{Q_k} f$ so summing on k, we have

$$|\cup Q_k| \leq \frac{1}{\alpha}\sum \int_{Q_k} f \leq \frac{1}{\alpha}\int_{R^n} f .$$

Finally, (2) follows, since for each $x \notin \cup Q_k$, x belongs to a sequence of cubes C_k whose diameters converge to zero and such that $\frac{1}{C_k}\int_{C_k} f \leq \alpha$. It follows from Lebesques theorem on differentiation of integrals that $f(x) \leq \alpha$ a.e. for such x.

We all know how important the maximal operator of Hardy-Littlewood is in the subject of Fourier analysis. This is the operator given by

$$Mf(x) = \sup_{r>0}\frac{1}{|B(x;r)|}\int_{B(x;r)} |f(t)|\,dt .$$

Going along with this we also define

$$M_\delta f(x) = \sup_{x\varepsilon Q \text{ dyadic cube}} \frac{1}{|Q|}\int_Q |f(t)|\,dt .$$

(Recall that a dyadic interval of R^1 is one of the form $[j2^k,(j+1)2^k]$ $j,k \varepsilon Z$ and a dyadic cube is a product of dyadic intervals of equal length; recall also the basic property of dyadic cubes—if Q_1, Q_2 are dyadic either $Q_1 \cap Q_2 = \emptyset$, $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$.) Then the following simple lemma sheds some light on the relationship of the Calderón-Zygmund lemma to the Maximal Operator.

LEMMA. *Let* $f \geq 0 \in L^1(R^n)$ *and* $\alpha > 0$. *Let* Q_k *be Calderón-Zygmund cubes as above, and let* $\tilde{Q}_k$ *denote the double of* Q_k. *Then there exist positive numbers* c *and* C *such that*

(1) $\cup Q_k \subseteq \{Mf > c\alpha\}$

(2) $\cup \tilde{Q}_k \supseteq \{Mf > C\alpha\}$

(3) *Furthermore, if* Q_k *are dyadic, then* $\cup Q_k \supseteq \{M_\delta f > C\alpha\}$.

Proof. (1) Let $x \in Q_k$. Then there exists a ball $B(x;r)$ such that $x \in Q_k \subseteq B(x;r)$ and $\frac{|B(x;r)|}{|Q_k|} \leq C_n$. Then

$$Mf(x) \geq \frac{1}{|B(x;r)|} \int_{B(x;r)} |f| \geq \left(\frac{|Q_k|}{|B(x;r)|}\right) \frac{1}{|Q_k|} \int_{Q_k} |f| \geq \frac{1}{C_n} \alpha .$$

(2) Let $x \notin \cup \tilde{Q}_k$. Let $r > 0$. Then we estimate

$$\int_{B(x;r)} f = \int_{B(x;r) \cap^c [\cup Q_k]} f + \sum_{Q_j \cap B \neq \emptyset} \int_{B(x;r) \cap Q_j} f$$

$$\leq \alpha |B(x;r)| + \sum_{Q_j \cap B \neq \emptyset} \int_{Q_j} f$$

$$\leq \alpha |B(x;r)| + 2^n \alpha \sum_{Q_j \cap B \neq \emptyset} |Q_j| .$$

Key point: if $Q_j \cap B(x;r) \neq \emptyset$ then $Q_j \subseteq B(x;10r)$ so that

$$\sum_{Q_j \cap B \neq \emptyset} |Q_j| \leq C|B(x;r)| \text{ and } \int_{B(x;r)} f \leq C_n \alpha |B(x;r)| .$$

That is, $Mf(x) \leq C_n \alpha$.

The use of the Calderón-Zygmund lemma is apparent in the Calderón-Zygmund Theorem on Singular Integrals: Let X and N be Banach spaces and let B(X,N) denote the bounded operators from X to N. Let $K : R^n \times R^n/\{x=y\} \to B(X,N)$ satisfy

$$(1) \qquad |K(x,y+h)-K(x,y)|_{B(X,N)} \leq \frac{|h|^{\delta}}{|x-y|^{n+\delta}} \text{ for } |h| < \frac{|x-y|}{2}$$

and for some $\delta > 0$.

(2) if $Tf(x) = \int_{R^n} K(x,y)f(y)dy$ for $f \in L^P(X)$, and suppose for some $p_0 > 1$ $\|Tf\|_{L^{p_0}(N)} \leq C\|f\|_{L^{p_0}(X)}$.

Then, for T we have

$$|\{x|\ |Tf(x)|_N > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{L^1(X)}$$

and

$$\|Tf\|_{L^p(N)} \leq C_p\|f\|_{L^p(X)} \quad \text{for } 1 < p < p_0 .$$

Proof. Let $\alpha > 0$ and $f \in L^1(X)$. Set

$$g(x) = \begin{cases} \dfrac{1}{|Q_k|} \displaystyle\int_{Q_k} f\,dt & \text{if } x \in Q_k , \\ f(x) & \text{if } x \notin Q_k . \end{cases}$$

and $b(x) = f(x)-g(x)$. Then

$$|\{|Tg(x)|_N > \alpha\}| \leq \frac{C}{p_0} \|g\|^{p_0}_{L^{p_0}(X)} \leq \frac{C'}{\alpha} \|g\|_{L^1(X)} \leq \frac{C'}{\alpha} \|f\|_{L^1(X)} .$$

As for $Tb(x)$, suppose $x \notin \cup\widetilde{Q}_k$.

Let $b_k(x) = \chi_{Q_k}(x)\,b(x)$; $\int_{Q_k} b_k(x)\,dx = 0$. Then

$$Tb_k(x) = \int_{Q_k} K(x,y)\,b_k(y)\,dy\ .$$

Let $\bar{y}_k$ be the center of Q_k; then

$$\int_{Q_k} K(x,\bar{y}_k)\,b_k(y)\,dy = K(x,\bar{y}_k)\int_{Q_k} b_k(y)\,dy = 0$$

so

$$Tb_k(x) = \int_{Q_k} \{K(x,y) - K(x,\bar{y}_k)\}\,b_k(y)\,dy$$

and

$$|Tb_k(x)|_N \le \frac{\operatorname{diam}(Q_k)^\delta}{|x-\bar{y}_k|^{n+\delta}}\,\|b_k\|_{L^1(X)}\ ;$$

summing over k we have

$$\int_{x\notin\cup\tilde{Q}_k} |T(b)(x)|_N \le \sum_k \int_{x\notin\tilde{Q}_k} \frac{\operatorname{diam}(Q_k)^\delta}{\operatorname{dist}(x,Q_k)^{n+\delta}}\,\|b_k\|_{L^1}\,dx \le C\,\|f\|_{L^1(X)}\ .$$

Thus

$$|\{|Tb(x)|_N > \alpha\}| \le |\cup\tilde{Q}_k| + \frac{1}{\alpha}\,\|f\|_1 \le \frac{C'}{\alpha}\,\|f\|_1\ .$$

From this weak (1,1) estimate, interpolate to get the L^p result.

We now quote some important examples:

1. *Classical Calderon-Zygmund Convolution Operators*. Here $Tf = f * K$ where $K(x)$ is a complex-valued function satisfying

(α) $|K(x)| \le C/|x|^n$;

(β) $$\int_{\rho_1 \le |x| \le \rho_2} K(x)\,dx = 0 \text{ for all } 0 < \rho_1 < \rho_2 ;$$

and

(γ) $$|K(x+h) - K(x)| \le C \frac{|h|^\delta}{|x|^{n+\delta}} \qquad |h| < \frac{1}{2}|x| .$$

The Riesz transforms $R_j f = f * x_j/|x|^{n+1}$ are especially important since they are related to H^p spaces and analytic functions.

For a Calderón-Zygmund singular integral T, it is easily seen to be bounded on $L^2(R^n)$, since $\hat{K}(\xi) \in L^\infty$. Also, since T^* is also a Calderón-Zygmund singular integral, T is bounded on the full range of $L^p(R^n)$, $1 < \rho < \infty$.

2. *Littlewood-Paley-Stein Functions.* The most basic, simplest of these are the g-function and S function defined as follows: Let $\psi \in C_c^\infty(R^n)$, $\int_{R^n} \psi = 0$. Let $\psi_t(x) = t^{-n}\psi\left(\frac{x}{t}\right)$ for $t > 0$. Then

$$g^2(f)(x) = \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t}$$

$$S^2(f)(x) = \iint_{\Gamma(x)} |f * \psi_t(y)|^2 \frac{dt\,dy}{t^{n+1}}$$

where $\Gamma(x) = \{(y,t) \mid |y-x| < t\}$. Then it is a basic fact that $\|S(f)\|_{L^p} \le C_p\|f\|_{L^p}$ and $\|g(f)\|_{L^p} \le C_p\|f\|_{L^p}$, when $1 < p < \infty$. If, say, ψ is suitably non-trivial, (radial, non-zero is good enough) then the reverse inequalities hold:

$$\|S(f)\|_{L^p} \ge c_p\|f\|_{L^p} \text{ and } \|g(f)\|_{L^p} \ge c_p\|f\|_{L^p} .$$

Now take $S(f)$. We want to point out here that S is a singular integral. In fact define $K:R^n \mapsto L^2(\Gamma(0); dydt)$ by $K(x)(y,t) = \psi_t(x-y)$. Then

$$S(f)(x) = |f * K(x)|_{L^2(\Gamma; dtdy/t^{n+1})}$$

and K satisfies

$$|K(x+h)-K(x)| \leq C \frac{|h|}{|x|^{n+1}} \qquad |h| < \frac{1}{2}|x| .$$

Also by a Fourier transform argument $\|Sf\|_{L^2(R^n)} \leq c\|f\|_{L^2(R^n)}$ so the Calderón-Zygmund theorem applies. In fact, the adjoint operator also maps $L^p(L^2(\Gamma)) \to L^p(R^n)$, $1 < p < 2$, because it is also C-Z, so again this explains why we get boundedness of S on the full range $1 < p < \infty$.

3. *The Hardy-Littlewood Maximal Operator as a Singular Integral.* Let $\phi(x) \in C^\infty(R^n)$ and suppose for $|x| < 1$, $\phi(x) = 1$ and for $|x| > 2$, $\phi(x) = 0$. Then define $K:R^n \mapsto L^\infty((0,\infty); dt)$ by $K(x)(t) = \phi_t(x) = t^{-n}\phi(x/t)$. Then

$$|\nabla_x K(x)(t)| = |t^{-(n+1)}\nabla\phi \ \frac{x}{t}| \leq C\|\nabla\phi\|_{L^\infty(R^n)} \frac{1}{|x|^{n+1}}$$

(since if $t > |x|/2$, $\nabla\phi(x/t) = 0$).

Again

$$|K(x+h)-K(x)|_{L^\infty} \leq C \frac{|h|}{|x|^{n+1}} \text{ if } |h| < \frac{1}{2}|x| ,$$

and we also have $\|f*K\|_{L^\infty(L^\infty)} \leq C\|f\|_{L^\infty}$ since $|f*\phi_t(x)| \leq \|\phi\|_1 \|f\|_\infty$. Then $Mf(x) \sim |f*K(x)|_{L^\infty}$ so M is bounded on $L^p(R^n)$, $p > 1$ and weak 1-1.

4. *The Estimates for Pointwise Convergence of Singular Integrals on* $L^1(R^n)$. Suppose that $K(x)$ is a classical Calderón-Zygmund kernel and let $K_\varepsilon(x) = K(x)\cdot\chi_{|x|>\varepsilon}(x)$, for $\varepsilon > 0$. We are interested in the existence a.e. of $\lim_{\varepsilon\to 0} f*K_\varepsilon(x)$ for $f \in L^1(R^n)$. In order to know this, it

is enough to show that $T^*f(x) = \sup_{\varepsilon>0} |f * K_\varepsilon(x)|$ satisfies the weak type estimate $|\{T^*f(x) > \alpha\}| \leq \frac{C}{\alpha} \int_{R^n} |f|$. It turns out that by using the Hardy Littlewood maximal operator it is not difficult to prove $T^*f(x) \leq C\{M(Tf)(x) + Mf(x)\}$ which immediately gives the boundedness of T^* on $L^p(R^n)$ for $p > 1$. However, it fails to give the weak type inequality for functions on $L^1(R^n)$. This inequality follows easily from the observation that T^* is a singular integral.

Let

$$\phi(x) \in C_c^\infty(R^n), \phi(x) = \begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 2 \end{cases}$$

and set $\widetilde{K}_\varepsilon(x) = K(x)\left[1 - \phi\left(\frac{x}{\varepsilon}\right)\right]$. Then $|\widetilde{K}_\varepsilon(x) - K(x)| \leq \frac{C}{|x|^n} \chi_{\varepsilon \leq |x| \leq 2\varepsilon}(x)$ so that $T^*f(x) \leq \sup_{\varepsilon>0} |f * \widetilde{K}_\varepsilon(x)| + Mf(x)$, and so we need only show that $\sup_{\varepsilon>0} |f * \widetilde{K}_\varepsilon|$ is weak type (1,1). In order to do this let $H: R^n \to L^\infty((0,\infty); d\varepsilon)$ be given by $H(x)(\varepsilon) = \widetilde{K}_\varepsilon(x)$. Then $|H(x) - H(x+h)|_{L^\infty} \leq \frac{C|h|}{|x|^{n+1}}$ and H is bounded from $L^2 \to L^2(L^\infty)$, so H is weak 1-1.

5. *The Maximal Function as a Littlewood-Paley-Stein Function.* Let $f \in L^2(R^n)$, $f(x) \geq 0$ for all x. Use the Calderón-Zygmund decomposition with $\alpha = C^j$, $j \in Z$ for some $C > 0$ sufficiently large, to get (dyadic) cubes Q_k^j where $\frac{1}{|Q_k^j|} \int_{Q_k^j} f \sim C^j$. Define f_j as in the Calderón-Zygmund decomposition,

$$\begin{cases} \dfrac{1}{|Q_k^j|} \displaystyle\int_{Q_k^j} f & \text{if } x \in Q_k^j \\ f(x) & \text{if } x \notin \bigcup_k Q_k^j \end{cases}$$

and $\Delta_j f = f_{j+1} - f_j$, then observe that:

(1) $\Delta_j f$ lives on $\bigcup_k Q_k^j$ and has mean value 0 on each Q_k^j.

(2) $\Delta_i f$ is constant on every Q_k^j for $i < j$.

(3) $f_j \to 0$ as $j \to -\infty$ and $f_j \to f$ as $j \to +\infty$ so $f = \sum_{j=-\infty}^{+\infty} \Delta_j f$.

From (1) and (2) it is clear that the $\Delta_j f$ are orthogonal so that

$$\|f\|_{L^2(R^n)} = \left(\sum_j \|\Delta_j f\|_{L^2}^2\right)^{1/2} = \left\| \left(\sum_j |\Delta_j f(x)|^2\right)^{1/2} \right\|_{L^2} .$$

Finally, observe that the square function $(\sum_j |\Delta_j f(x)|^2)^{1/2}$ is essentially just the dyadic maximal function. In fact, if $C^j << M_\delta(f)(x)$ then $x \in Q_k^j$ for some k and $\Delta_j(f)(x) \sim C^j$. It follows that

$$\left(\sum_j |\Delta_j f(x)|^2\right)^{1/2} \geq cM_\delta f(x) .$$

Before finishing this section, we shall need estimates near L^1 for the maximal function.

If Q denotes the unit cube in R^n then for k a positive integer

$$\int_Q Mf(\log^+ Mf)^{k-1} dx < \infty \text{ if and only if } \int_Q |f|(\log^+ |f|)^k dx < \infty .$$

The proof runs as follows: If $f \in L(\log^+ L)^k$ then

$$|\{x | Mf(x) > \alpha\}| \leq \frac{C}{\alpha} \int_{|f(x)| > \alpha/2} f(x)\, dx$$

and so

$$\|Mf\|_{L(\log L)^{k-1}} \le$$

$$\int_1^\infty (\log \alpha)^{k-1} \frac{1}{\alpha} \int_{|f(x)|>\alpha/2} |f(x)|dx\, d\alpha \le \int_{Q_0} |f(x)| \int_1^{2|f(x)|} \frac{1}{\alpha} (\log \alpha)^{k-1} d\alpha\, dx$$

$$\le \|f\|_{L(\log L)^k} .$$

Conversely, (Stein) Calderón-Zygmund decompose R^n at height $\alpha > 0$. We have

$$\int_{M_\delta f(x)>C\alpha} f(x)dx \le \int_{\cup Q_k} f(x)dx \le \sum_k \int_{Q_k} f \le C\alpha \sum_k |Q_k| \le C\alpha|\{Mf > c\alpha\}| .$$

This yields

$$\int_{Q_k} |f(x)|(\log^+ M_\delta f(x))^k dx \le \int_1^\infty \frac{1}{\alpha} \int_{M_\delta f(x)>C_n\alpha} |f(x)|dx(\log \alpha)^{k-1} d\alpha$$

$$\le \int_1^\infty \frac{1}{\alpha} \int_{M_\delta f(x)>C_n\alpha} |f(x)|dx \cdot (\log \alpha)^{k-1} d\alpha$$

$$\le \int_{Q_k} Mf(x)\,[\log^+ Mf(x)]^{k-1} dx .$$

2. *Multi-parameter differentiation theory*

During the first lecture we discussed some fundamentally important operators of classical (and sometimes, not so classical) harmonic analysis: the maximal operator, singular integrals, and Littlewood-Paley-

Stein operator. These operators all had one thing in common. They all commute in some sense with the one-parameter family of dilations on R^n, $x \to \delta x$, $\delta > 0$. The nature of the real variable theory involved does not seem to depend at all on the dimension n. In marked contrast, it turns out that a study of the analogous operators commuting with a multi-parameter family of dilations reveals that the number of parameters is enormously important, and changes in the number of parameters drastically change the results.

Let us begin by giving the most basic example, which dates back to Jessen, Marcinkiewicz, and Zygmund. We are referring to a maximal operator on R^n which commutes with the full n-parameter group of dilations $(x_1, x_2, \cdots, x_n) \to (\delta_1 x_1, \delta_2 x_2, \cdots, \delta_n x_n)$, where $\delta_i > 0$ is arbitrary. This is the "strong maximal operator," $M^{(n)}$, defined by

$$M^{(n)} f(x) = \sup_{x \epsilon R} \frac{1}{|R|} \int_R |f(t)|\, dt$$

where R is a rectangle in R^n whose sides are parallel to the axes. Unlike the case of the Hardy-Littlewood operator, $M^{(n)}$ does not satisfy

$$|\{x | M^{(n)}(f)(x) > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{L^1(R^n)} .$$

For instance when $n = 2$ and when $f_\delta = \delta^{-2} \chi_{(|x_1| < \delta/2) \times (|x_2| < \delta/2)}$, then for $|x_1|, |x_2| > 2\delta$,

$$M^{(2)}(f_\delta)(x_1, x_2) = M^{(1)}(\chi_{|x_1| < \delta/2})(x_1) M^{(1)}(\chi_{|x_2| < \delta/2})(x_2) \sim \frac{1}{|x_1|} \frac{1}{|x_2|}$$

and

$$|\{x \epsilon Q_0 | M^{(2)}(f_\delta) > \alpha\}| \geq |\{x|\ |x_1|\,|x_2| < \tfrac{1}{\alpha},\ \text{and}\ \delta < |x_i| < 1\}| \sim \frac{1}{\alpha} \log \frac{1}{\alpha} ,$$

if $\alpha = 1/\delta$.

If we have a weak type inequality, we must have

$$|\{M^{(2)}(f_\delta) > \alpha\}| \leq \frac{1}{\alpha} \|f_\delta\|$$

so that $\|f_\delta\| \geq c \log \frac{1}{\delta}$ and the smallest Orlicz norm for which this holds is the $L(\log L)$ norm. A similar computation in R^n reveals that for $M^{(n)}$ to map L_Φ boundedly to Weak L^1 we must have $L_\Phi \subseteq L(\log L)^{n-1}$. The next theorem shows that indeed $M^{(n)}$ does indeed map $L(\log L)^{n-1}$ boundedly into Weak L^1.

THEOREM OF JESSEN-MARCINKIEWICZ-ZYGMUND (1935) [1]. *For functions* $f(x)$ *in the unit cube of* R^n *we have*

$$|\{x \epsilon Q_0, M^{(n)}(f)(x) > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{L(\log L)^{n-1}(Q_0)} .$$

The proof is strikingly simple. Define M_{x_i} to be the 1-dimensional maximal function in the i^{th} coordinate direction. Consider the case $n = 2$, which is already entirely typical. Let R be a rectangle containing the point $(\overline{x}_1, \overline{x}_2)$, say $R = I \times J$. Then

$$\frac{1}{R} \iint_R |f(x_1,x_2)|\, dx_1 dx_2 = \frac{1}{|I|} \int_I \left(\frac{1}{|J|} \int_J f(x_1,x_2)\, dx_2\right) dx_1 \tag{2.1}$$

$$\frac{1}{|J|} \int_J |f(x_1,x_2)| dx_2 \leq M_{x_2} f(x_1,\overline{x}_2)$$

so (2.1) is

$$\leq \frac{1}{|I|} \int_I M_2 f(x_1,\overline{x}_2)\, dx_1 \leq M_{x_1}(M_{x_2} f)(\overline{x}_1,\overline{x}_2) .$$

Thus, for all $(x_1,x_2) \epsilon Q_0$,

$$M^{(2)}f(x_1,x_2) \leq M_{x_1} \circ M_{x_2}(f)(x_1,x_2) .$$

We have seen that M_{x_2} maps $L(\log L)(Q_0)$ boundedly into $L^1(Q_0)$ so that

$$\|M_{x_2} f\|_{L^1(Q_0)} \leq C\|f\|_{L(\log L)(Q_0)} ,$$

and finally

$$|\{x \epsilon Q_0 | M_{x_1} M_{x_2} f(x) > \alpha\}| \leq \frac{C'}{\alpha} \|M_{x_2} f\|_{L^1(Q_0)} \leq \frac{C''}{\alpha} \|f\|_{L(\log L)(Q_0)} .$$

Now, this method of iteration in the proof above gives sharp estimates for $M^{(n)}$, and it may be suspected that the whole story of the harmonic analysis of several parameters can be told by applying this iteration technique. That this is not the case should become clear as this lecture proceeds. We want to describe some of the multi-parameter theory and to do this, let us begin with maximal functions. For many years after the Jessen-Marcinkiewicz-Zygmund theorem, there was no machinery around to treat problems here, and then, only fairly recently, two such machines were created. The one we describe here proceeds by means of covering lemmas while the other, due to Nagel, Stein, and Wainger, which Wainger has described in detail, uses the Fourier transform [2]. Though the two methods seem totally different on the surface, they are really quite closely related and have in common the main theme of reducing higher parameter, complicated operators to lower parameter simpler ones, which are already well understood.

The model for our method is the following, where the operator $M^{(n)}$ is controlled by $M^{(n-1)}$.

COVERING LEMMA FOR RECTANGLES OF THE STONG MAXIMAL OPERATOR [3]. *Let* $\{R_k\}$ *be a given sequence of rectangles in* $R^n \subseteq B(0,1)$ *whose sides are parallel to the axes. Then there is a subsequence* $\{\widetilde{R}_k\}$ of $\{R_k\}$ *so that*

(1) $|\cup\tilde{R}_k| \geq c_n |\cup R_k|$

(2) $\left\|\exp\left(\sum \chi_{\tilde{R}_k}\right)^{\frac{1}{n-1}}\right\|_{L^1(B)} \leq C$.

Before we prove this theorem, let us show that it implies the Jessen-Marcinkiewicz-Zygmund result. Let $\alpha > 0$, and for each point $x \in \{M^{(n)}f(x) > \alpha\}$ there is a rectangle R_x containing x with

$$\frac{1}{|R_x|}\int_{R_x} |f| > \alpha . \tag{2.2}$$

Without loss of generality we assume $\cup R_x = \cup R_k$ where R_k are certain if the R_x's. Apply the covering lemma to get $\tilde{R}_k$ with properties (1) and (2) above. Then by virtue of (1) we need only show that

$$|\cup\tilde{R}_k| \leq \frac{C}{\alpha} \|f\|_{L(\log L)^{n-1}(Q_0)} .$$

By (2.2), $|\tilde{R}_k| \leq \frac{1}{\alpha}\int_{\tilde{R}_k} |f|$ and summing we have

$$|\cup\tilde{R}_k| \leq \frac{1}{\alpha}\int_{Q_0} |f| \sum \chi_{\tilde{R}_k} \leq \frac{1}{\alpha} \|f\|_{L(\log L)^{n-1}} \left\|\sum \chi_{\tilde{R}_k}\right\|_{\exp(L^{1/(n-1)})} .$$

Now, let us prove the covering theorem. We shall proceed by induction on n. Assume the case $n-1$. Let $R_1, R_2, \cdots, R_k, \cdots$ be ordered such that the x_n side length decreases. For a rectangle R, let R_d denote the rectangle whose center and x_i side lengths, $i < n$, are the same as those of R, but whose x_n side length is multiplied by 5. Then we describe the procedure for selecting the $\tilde{R}_k$ from the R_k: Let $\tilde{R}_1 = R_1$. Suppose $\tilde{R}_1, \tilde{R}_2, \cdots, \tilde{R}_k$ have already been chosen. We continue along the list, and each time we consider the rectangle R we ask whether or not

$$|R \cap [\cup (\tilde{R}_j)_d]| < \frac{1}{2} |R|$$

where the above union is taken over all the $\tilde{R}_j$, $j \le k$ for which $\tilde{R}_j \cap R \neq \emptyset$. If the answer is no, we move on to consider the next rectangle on the list. If the answer is yes, we make the rectangle $R = \tilde{R}_{k+1}$, and start the process over again.

Now, we prove (1) as follows: If R is an unselected rectangle, then

$$\left| R \cap \left[\bigcup_{\substack{\tilde{R}_j \cap R = \emptyset \\ \tilde{R}_j \text{ before } R}} (\tilde{R}_j)_d \right] \right| \ge \frac{1}{2} |R| .$$

Let us slice all rectangles with a hyperplane perpendicular to the x_n axis. Then if slices are indicated by using S's instead of R's,

$$|S \cap [\cup (\tilde{S}_j)_d]| \ge \frac{1}{2} |S|$$

so that $M^{(n-1)}(\chi_{\cup(\tilde{R}_j)_d}) > \frac{1}{2}$ on $\cup R_j$, where $M^{(n-1)}$ is acting in the $x_1, x_2, \cdots, x_{n-1}$ coordinates. By the boundedness of $M^{(n-1)}$ on, say, L^2 (by induction) we have

$$|\cup R_j| \le C |\cup(\tilde{R}_k)_d| \le C' \sum |\tilde{R}_k| \le C' |\cup \tilde{R}_k| .$$

To obtain (2), notice that the $\tilde{R}_j$'s satisfy

$$\left| \tilde{R}_k \cap \left[\bigcup_{\substack{\tilde{R}_j \cap \tilde{R}_k \neq \emptyset \\ j<k}} (\tilde{R}_j)_d \right] \right| < \frac{1}{2} |\tilde{R}_k| .$$

If we again slice with a hyperplane perpendicular to the x_n axis,

$$|\tilde{S}_k \cap \bigcup_{j<k} \tilde{S}_j| < \frac{1}{2} |\tilde{S}_k|$$

so that, if $\tilde{E}_k = \tilde{S}_k - \bigcup_{j<k} \tilde{S}_j$ we have $|\tilde{E}_k| > \frac{1}{2} |\tilde{S}_k|$ and if $\phi \in L(\log L)^{n-1}(S_0)$ we shall show that

$$\int_{S_0} \sum \chi_{\tilde{S}_k} \phi \leq C\|\phi\|_{L(\log L)^{n-1}} \tag{2.3}$$

which will give

$$\int_{S_0} \exp\left[c\left(\sum \chi_{\tilde{S}_k}\right)^{1/(n-1)}\right] dx_1 \cdots dx_{n-1} \leq C .$$

Integrating this estimate in x_n finishes things.

To obtain (2.3) we write

$$\int_{S_0} \sum \chi_{\tilde{S}_k} \phi dx_1 \cdots dx_{n-1} \leq \sum_k \int_{\tilde{S}_k} \phi \leq C \sum_k |\tilde{E}_k| \frac{1}{|\tilde{S}_k|} \int \phi \leq$$

$$\leq C \int_{S_0} M^{(n-1)}(\phi) dx_1 dx_2 \cdots dx_{n-1} \leq C' \|\phi\|_{L(\log L)^{n-1}} .$$

Notice that in our argument above the slicing was the most important idea. If you try the proof without it, you will not wind up with the estimate you want, on the $\exp(\ )^{1/(n-1)}$norm, but rather on the $\exp(\ )^{1/n}$ norm instead. Also the slicing is the mechanism by which we control $M^{(n)}$ by the lower parameter operator $M^{(n-1)}$ and here this enables us to proceed by induction. Of course, in the end the theory of the boundedness of $M^{(n)}$ had been known for some 40 years before the covering lemma. But the lowering of the number of parameters, and the induction procedure will be used in what follows as the key ingredient to prove new theorems.

To illustrate the method of the machine, we consider a maximal operator whose relation to multiplier theory will be studied below. Suppose in R^2 we consider the class $\mathfrak{A}$ of all rectangles of arbitrary side lengths which are oriented in one of the directions $\theta_k = 2^{-k}$, $k = 1,2,\cdots$, measured from some fixed direction, say the positive x-direction. Define the maximal operator $\mathfrak{m}$ by

$$\mathfrak{m}f(x) = \sup_{x \epsilon R \epsilon \mathfrak{A}} \frac{1}{|R|} \int_R |f(t)|\, dt .$$

This operator was considered following the ideas of the covering approach outlined above by Stromberg [4] and Cordoba-Fefferman [5]. Somewhat later Nagel, Stein, and Wainger [6] used Fourier transform methods to extend the result we shall discuss below.

What we prove is that

$$|\{\mathfrak{m}f(x) > \alpha\}| \le \frac{C}{\alpha^2} \|f\|^2_{L^2(R^2)} .$$

The proof consists of showing that, given a sequence of rectangles $\{R_k\}$ belonging to $\mathfrak{A}$, there exists a subfamily $\{\tilde{R}_k\}$ such that

(1) $|\cup \tilde{R}_k| \ge c |\cup R_k|$

and

(2) $\|\sum \chi_{\cup \tilde{R}_k}\|_{L^2(R^2)} \le |\cup R_k|^{1/2}$.

To prove this we give a rule for selecting $\tilde{R}_k$, given that we have already selected $\tilde{R}_j$ for $j < k$. Assume that the R_k all have their longest side in a direction in the 1^{st} quadrant, and are ordered so that their longer side lengths are decreasing. Then consider the rectangle R following $\tilde{R}_{k-1}$. Consider in particular

$$\frac{1}{|R|} \sum_{j<k} |R \cup \tilde{R}_j| = \frac{1}{|R|} \int_R \sum_{j<k} \chi_{\tilde{R}_j}\, dx .$$

If this is less than $1/2$, select R as $\tilde{R}_k$. If not go to the next rectangle on the list, and apply the same test to it. In this way we obtain the desired $\tilde{R}_k$. Notice that

$$\begin{aligned}\|\sum \chi_{\tilde{R}_k}\|_2^2 &= \int \sum_{j,k} \chi_{\tilde{R}_j}\chi_{\tilde{R}_k} = 2\int_j \sum_{j<k} \chi_{\tilde{R}_k} + \sum_k |\tilde{R}_k| \\ &= 2\sum_k \int_{\tilde{R}_k} \sum_{j<k} \chi_{\tilde{R}_j} + \sum_k |\tilde{R}_k| \le 2\sum |\tilde{R}_k| \le C|\cup\tilde{R}_k| .\end{aligned} \tag{2.4}$$

This is (2). To show (1), we let R be some rectangle which was unselected. This implies that

$$\frac{1}{|R|} \sum_{\tilde{R}_j \text{ before } R} |R \cap \tilde{R}_j| \ge \frac{1}{2} .$$

Now draw the following picture. Let S be the envelope rectangle to R whose sides are parallel to the coordinate axes.

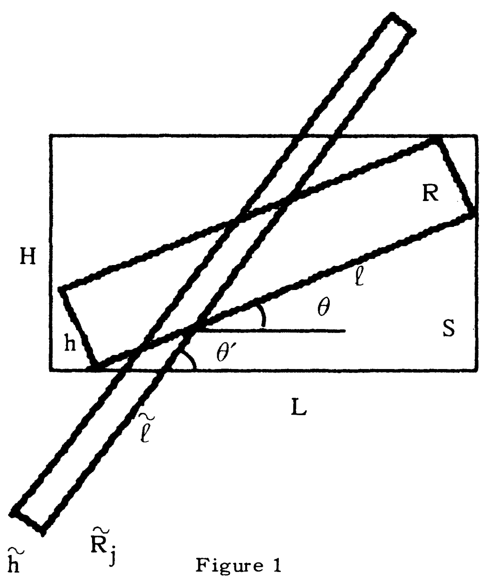

Figure 1

Then by dilating S if necessary, we can assume that $\tilde{R}_j$ is centered at the same center as S. Now the point is that

$$\frac{|\tilde{R}_j \cap S|}{|S|} \geq c\,\frac{|\tilde{R}_j \cap R|}{|R|}\,. \tag{2.5}$$

In fact,

$$\frac{|\tilde{R}_j \cap S|}{|S|} \sim \frac{\tilde{h}H/\theta'}{hL} = \frac{\tilde{h}}{L\theta'}$$

and

$$\frac{|\tilde{R}_j \cap R|}{|R|} \sim \frac{\tilde{h}\,h(\theta'-\theta)}{h\ell} = \frac{\tilde{h}}{\ell(\theta'-\theta)}$$

and our inequality (2.5), taking into account that $\ell \sim L$, is

$$\frac{1}{\theta'} \geq \frac{1}{(\theta'-\theta)} \quad \text{or} \quad \theta'-\theta \geq c\theta'$$

and this in our case of $\theta_k = 2^{-k}$ is valid with $c = 1/2$. (If $\theta_k = (1-\varepsilon)^k$, $c = \varepsilon$.) Then summing over j in (2.5) we have

$$\frac{1}{|S|}\int_S \sum_{\tilde{R}_j \text{ before } R} \chi_{\tilde{R}_j} \geq \frac{1}{|R|}\int_R \sum_{\tilde{R} \text{ before } R} \chi_{\tilde{R}_j} \geq \frac{1}{2}\,c$$

in other words

$$M^{(2)}\Big(\sum \chi_{\tilde{R}_j}\Big) > \frac{1}{2}\,c \quad \text{on} \quad \cup R_j$$

and by the boundedness of $M^{(2)}$ we see that

$$|\cup R_j| \leq C\|\sum \chi_{\tilde{R}_j}\|_2^2 \leq C'|\cup\tilde{R}_j|$$

by (2.5). We have controlled $\mathfrak{m}$ by $M^{(2)}$ here in just exactly the way $M^{(n)}$ was previously controlled by $M^{(n-1)}$. And while $\mathfrak{m}$ is a 3-parameter maximal operator, $M^{(2)}$ is a 2-parameter one.

Finally, we should note that, as before, this covering lemma implies the maximal theorem for $\mathfrak{m}$ as claimed. In fact, suppose we have shown that given $\{R_k\}$ there exists $\{\tilde{R}_k\}$ a subsequence so that

(1) $|\cup\tilde{R}_k| \geq c|\cup R_k|$

(2) $\|\sum \chi_{R_k}\|_{p'} \leq c|\cup R_k|^{1/p'}$, (here $\frac{1}{p}+\frac{1}{p'}=1$).

Then

$$|\{\mathfrak{m}f>\alpha\}| \leq \left(C\frac{\|f\|_p}{\alpha}\right)^p .$$

In fact we have, by definition $\{R_k\}$ so that $\{\mathfrak{m}f>\alpha\} \subseteq \cup R_k$ and $\frac{1}{|R_k|}\int_{R_k}|f| > \alpha$ for all k. By the covering lemma, select the class $\{\tilde{R}_k\}$. Then $|\tilde{R}_k| \leq \frac{1}{\alpha}\int_{\tilde{R}_k}|f|$ and so

$$|\cup\tilde{R}_k| \leq \frac{1}{\alpha}\int f \sum \chi_{\tilde{R}_k} \leq \frac{1}{\alpha}\|f\|_p\|\sum\chi_{\tilde{R}_k}\|_{p'}$$

$$\leq \frac{C}{\alpha}\|f\|_p|\cup R_k|^{1/p'} \leq C'\|f\|_p|\cup\tilde{R}_k|^{1/p'}$$

and the estimate on $|\{\mathfrak{m}f>\alpha\}|$ follows by a division of both sides by $|\cup\tilde{R}_k|^{1/p'}$.

Our last topic for this lecture will be the so-called Zygmund Conjecture. I believe it was Zygmund who was the first to realize the difference between the one-parameter and several parameter harmonic analysis. Particularly, he remarked that in differentiation theory a "big picture" was evolving. He considered n functions $\phi_1, \phi_2, \cdots, \phi_n$ of the positive real variable t, with each $\phi_i(t)$ increasing and the family of rectangles

$\{R_t\}_{t\geq 0}$ given by $R_t = \prod_{i=1}^{n}\left[-\frac{\phi_i(t)}{2}, \frac{\phi_i(t)}{2}\right]$. Form a maximal operator M defined by

$$M(f)(x) = \sup_{t>0} \frac{1}{|R_t|}\int_{R_t} |f(x+y)|\,dy \ .$$

Then M is of weak type 1-1, just as in the special case of the Hardy-Littlewood operator where $\phi_i(t) = t$. Zygmund noticed that the proof of this was virtually the same as the Hardy-Littlewood theorem. All one had to do was to prove a Vitali-type covering lemma for R_t's using the fact that if $\mathfrak{P}$ is the class of all translates of the R_t and if $R, S \in \mathfrak{P}$ and $R \cap S \neq \emptyset$ and if R corresponds to a bigger value of t than does S, then $S \subseteq \widetilde{R}$, the 5-fold dilation of R. Next, he considered the collection of rectangles $R_{s,t}$, $s,t > 0$ where

$$R_{s,t} = \left[-\frac{s}{2}, \frac{s}{2}\right] \times \left[-\frac{t}{2}, \frac{t}{2}\right] \times \left[-\frac{\phi(s,t)}{2}, \frac{\phi(s,t)}{2}\right]$$

where ϕ is a function increasing in each variable separately, fixing the other variable. In other words, Zygmund next conjectured that since $\mathfrak{R}$ is a 2-parameter family of rectangles in R^3, the corresponding maximal operator, which we shall call M_Z should behave like the model 2-parameter operator $M^{(2)}$ in R^2:

$$|\{M_Z f(x) > \alpha, |x| < 1\}| \leq \frac{C}{\alpha} \|f\|_{L(\log L)(|x|<1)} \ .$$

Not long ago, using the methods we have just discussed, Cordoba was able to prove this [7]. Let us give the proof. Suppose $\{R_k\}$ is a sequence of rectangles with side lengths s, t, and $\phi(s,t)$ in R^3. We must show there exists a subcollection $\{\widetilde{R}_k\}$ such that

(1) $|\cup \widetilde{R}_k| > c|\cup R_k|$

(2) $\|\sum \chi_{\widetilde{R}_k}\|_{\exp(L)} \leq C$.

To prove this, order the R_k so that the z side lengths are decreasing. With no loss of generality, we may assume that $|R_k \cap [\bigcup_{j<k} R_j]| < \frac{1}{2}|R_k|$, that there are finitely many R_k and that the R_k are dyadic. (In fact, we may assume this because if $\frac{1}{|R|}\int_R |f| > \alpha$ for some $R \in \mathfrak{R}$ containing x, then there exists a dyadic R_1 whose $\tilde{R}_1$ (double) contains x such that $\frac{1}{|R_1|}\int_{R_1} |f| > \frac{\alpha}{C}$.) Now let $\tilde{R}_1 = R$ and, given $\tilde{R}_1, \cdots, \tilde{R}_k$, select $\tilde{R}_{k+1}$ as follows: Let $\tilde{R}_{k+1}$ be the first R on the list of R_k so that

$$\frac{1}{|R|}\int_R \exp\left(\sum_{j\leq k} \chi_{\tilde{R}_j}\right) dx \leq C .$$

We claim that the $\tilde{R}_k$ satisfy $\int_{|x|\leq 1} \exp(\sum \chi_{\tilde{R}_k}) \leq C'$. To see this let the $\tilde{R}_k$ be $\tilde{R}_1, \cdots, \tilde{R}_N$ and let $\tilde{E}_j = \tilde{R}_j - \bigcup_{k>j} \tilde{R}_k$. Then

$$\int_{\cup\tilde{R}_j} \exp\left(\sum_{k=1}^{N} \chi_{\tilde{R}_k}\right) = \int_{\tilde{E}_N} \exp\left(\sum_{k=1}^{N} \chi_{\tilde{R}_k}\right) dx + \int_{\tilde{E}_{N-1}} \exp\left(\sum_{k=1}^{N-1} \chi_{\tilde{R}_n}\right) + \cdots + \int_{\tilde{E}_1} \exp(\chi_{\tilde{R}_1})$$

and

$$\int_{\tilde{E}_j} \exp\left(\sum_{k=1}^{j} \chi_{\tilde{R}_k}\right) \leq C \int_{\tilde{R}_j} \exp\left(\sum_{k<j} \chi_{\tilde{R}_k}\right) \leq C|\tilde{R}_j| ,$$

so we have

$$\exp\left(\sum \chi_{\tilde{R}_k}\right) \leq C \sum |\tilde{R}_j| \leq C'|\cup\tilde{R}_j| .$$

Now let us show that $|\cup\tilde{R}_j| > c|\cup R_j|$. Let R be an unselected rectangle. Then

$$\frac{1}{|R|}\int_R \exp\left(\sum \chi_{\tilde{R}_k}\right)dx \geq C$$

where the sum extends only over those chosen $\tilde{R}_k$ which precede R. Let us slice R with a hyperplane in the x_1, x_2 direction. Call $S, \tilde{S}_j$ the slices of R and $\tilde{R}_j$. Then

$$\frac{1}{|S|}\int_S \exp\left(\sum \chi_{\tilde{S}_j}\right)dx_1 dx_2 \geq C\ .$$

(Again we sum only over those $\tilde{S}_j$ which appear before S.) Now, each $\tilde{R}_j$ appearing before R has the property that its x_3 (or z) side length exceeds that of R. It follows that each corresponding $\tilde{S}_j$ has either its x_1 or x_2 side length longer than that of S. Call those $\tilde{S}_j$ having longer x_1 side length than x_1 length of S *of type I*, and the other of type II. Put $S = I \times J$. Then

$$C < \frac{1}{|I\times J|}\iint_{I\times J} \exp\left(\sum \chi_{\tilde{S}_j}\right)dx_1 dx_2 = \frac{1}{|I|\,|J|}\int_I\int_J \exp\left(\sum_{I} \chi_{\tilde{S}_j} + \sum_{II} \chi_{\tilde{S}_j}\right)dx_1 dx_2$$

$$= \frac{1}{|I|}\int_I \exp\left(\sum_{II} \chi_{\tilde{R}_j}\right)dx_1\ \frac{1}{|J|}\int_J \exp\left(\sum_{I} \chi_{\tilde{R}_j}\right)d$$

so it follows that

$$M_{x_1}\left[\exp\left(\sum \chi_{\tilde{R}_j}\right)\right] M_{x_2}\left[\exp\left(\sum \chi_{\tilde{R}_j}\right)\right] > C$$

on $\cup R_j$; hence

$$\cup R_j \subseteq \left\{M_{x_1}\left[\exp\left(\sum \chi_{\tilde{R}_j}\right)\right] > \sqrt{C}\right\} \cup \left\{M_{x_2}\left[\exp\left(\sum \chi_{\tilde{R}_j}\right)\right] > \sqrt{C}\right\}$$

so

$$|UR_j| \leq C' \left\|\exp\left(\sum \chi_{\tilde{R}_j}\right)\right\|_{L^1} \leq C''|U\tilde{R}_j| .$$

So far what has been done suggests the following general conjecture of Zygmund which says: Let $\{\phi_i(t_1,t_2,\cdots,t_k)\} = \Phi$, $i = 1,2,\cdots,n$ be functions which are increasing in each of the variables $t_i > 0$ separately. Define a k-parameter family of rectangles $R_{t_1 t_2,\cdots,t_k}$ by

$$R_{t_1,t_2,\cdots,t_k} = \prod_{i=1}^{n} \left[\frac{-\phi(t_1,\cdots,t_k)}{2}, \frac{+\phi(t_1,\cdots,t_k)}{2}\right],$$

and a maximal operator on R^n by

$$M_\Phi(f)(x) = \sup_{t_1,t_2,\cdots,t_k>0} \frac{1}{|R_{t_1,\cdots,t_k}|} \int_{R_{t_1,\cdots,t_k}} |f(x+y)|\,dy .$$

Then

$$|\{x \,|\, |x| < 1, M_\Phi(f)(x) > \alpha\}| \leq \frac{C}{\alpha} \|f\|_{L(\log L)^{k-1}} .$$

Quite recently a beautifully simple counterexample to the general conjecture was given by Fernando Soria of the University of Chicago [8]. Soria's counterexample was: in R^3, consider all rectangles $\mathfrak{S}$ of the form $s \times t\phi_1(s) \times t\phi_2(s)$ where ϕ_i are increasing on $[0,1]$. In fact we claim that given any small rectangle R with side length x,y and z such that $x < y < z \leq 1$ in R^3, there exists a rectangle $S \in \mathfrak{S}$ such that $R \subset S$ and $|S| \leq C|R|$. Clearly, then $M_{\mathfrak{S}}$ cannot be any better than the 3-parameter operator $M^{(3)}$. To do this let ϕ_1 and ϕ_2 be increasing, $\gneq 0$, and continuous on $[0,1]$ so that on each interval $2^{-(k+1)} < s < 2^{-k}$, $\frac{\phi_2(s)}{\phi_1(s)}$ takes every value between 1 and $C \cdot 2^{(k+1)}$. Then given three side lengths $\rho_1, \rho_2, \rho_3 \leq 1$ of R, simply choose s according to the following: say $2^{-k-1} < \rho_1 \leq 2^{-k}$. Choose $s \in (2^{-k-1}, 2^{-k})$ satisfying $\frac{\phi_2(s)}{\phi_1(s)} = \frac{\rho_3}{\rho_2}$ (since $1 \leq \frac{\rho_3}{\rho_2} \leq \rho_1 < 2^{k+1}$ we can do this). Then choose t so that $t\phi_1(s) = \rho_2$. This guarantees $t\phi_2(s) = \rho_3$, and we are finished.

We can make $\frac{\phi_2(s)}{\phi_1(s)}$ assume every value between 1 and $C2^k$ on $[2^{-k-1}, 2^{-k}]$ bt letting $\phi_1(s) = e^{-1/s}$ and $\phi_2(s) = \phi_1(s)$ on $\left[\frac{3}{2} \cdot 2^{-k-1}, 2^{-k}\right]$ and

$$\phi_2(s) = \phi_1\left(\frac{3}{2} \cdot 2^{-k-1}\right) \text{ for all } s \in \left[2^{-k-1}, \frac{3}{2} \cdot 2^{-k}\right] .$$

3. *Multiparameter weight-norm inequalities and applications to multipliers*

In this lecture we want to describe further applications of the ideas centering around the covering lemma for rectangles previously described. We shall begin with more about maximal operators, and then move on to multiparameter multiplier operators, and the connection they have with our maximal functions.

The first topic we take up is that of classical weight norm inequalities, which have proven of enormous importance throughout Fourier analysis. Here, we want to know which locally integrable non-negative weight functions $w(x)$ on R^n have the property that some operator T is bounded on $L^p w(x)dx$. The most basic examples are the Hardy-Littlewood maximal operator, and Calderón-Zygmund singular integrals $Tf = f * K$. The theory was developed in R^1 by Muckenhoupt [9] and Hunt, Muckenhoupt and Wheeden [10], and in R^n by Coifman and C. Fefferman [11]. We will present only a small segment of that theory now and list some relevant facts for which the interested reader should see the Studia article of Coifman-C. Fefferman [11].

It is no coincidence that the class of weights w for which the Hardy-Littlewood maximal operator is bounded on $L^p(w)$ is exactly the same as the class of w for which all Calderon-Zygmund operators are bounded on $L^p(w)$. This is the so-called A^p class of Muckenhoupt.

A nonnegative locally integrable function $w(x)$ on R^n is said to belong to A^p if and only if for each cube $Q \subseteq R^n$

$$\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^{p-1} \leq C .$$

The smallest such C is called the A^p norm. We say that $w \in A^\infty$ if and only if, whenever Q is a cube and $E \subset Q$, if $|E|/|Q| > 1/2$ then $w(E)/w(Q) > \eta$ for some $\eta > 0$.

Let us list some properties of A^p classes:

(α) If $\rho > 0$ and $w \in A^p$ then $\rho w \in A^p$ with the same norm as w.

(β) If $w \in A^p$ and $\delta > 0$ then $w(\delta x) \in A^p$ with the same norm as w.

(γ) If $w \in A^p$ then $w^{-1/(p-1)} \in A^{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

(δ) If $w \in A^p$ then $w \in A^\infty$. In fact, if $w \in A^p$ by (α) and (β) it is enough to show that if $|Q| = 1$, and $\int_Q w = 1$ then $|E| > 1/2$ implies $w(E) > \eta$.

(For, in general if Q is arbitrary of side δ and $|E| > 1/2|Q|$, consider $w(\delta x)$ on Q/δ and multiply $w(\delta x)$ by the right constant ρ to have $\int_{1/\delta} \rho w(\delta x) = 1$. Then $\rho w(\delta x)$ on E/δ would have

$$\frac{[\rho w(\delta x)](E/\delta)}{[\rho w(\delta x)](Q/\delta)} > \eta \iff \frac{w(E)}{w(Q)} > \eta \cdot \Big)$$

But by the A^p condition,

$$c\left(\int_E w\right)^{-1} \leq \left(\int_E w^{-1/(p-1)}\right)^{p-1} \leq \left(\int_Q w^{-1/(p-1)}\right)^{p-1} \leq C$$

and so

$$\int_E w \geq \frac{c}{C} \cdot$$

So far all the properties of A^p weights listed are obvious and follow straight from the definitions. There are some deeper properties which though not difficult to prove are not immediate.:

(ε) If $w \in A^p$ then w satisfies a Reverse Hölder Inequality:

$$\left(\frac{1}{|Q|}\int_Q w^{1+\delta}\right)^{1/(1+\delta)} \leq C_\delta\left(\frac{1}{|Q|}\int_Q w\right)$$

for all cubes Q with $\delta > 0$. The constant C_δ may be taken arbitrarily close to 1 as $\delta > 0$.

(ζ) From (ε) it is immediate (see also (γ)) that $w \in A^p$ implies $w \in A^q$ for some $q < p$.

(η) If f is a locally integrable function in some L^p space and $0 < \alpha < 1$ then $(Mf)^\alpha \in A^1$, i.e., $M((Mf)^\alpha)(x) \leq C(Mf)^\alpha(x)$ (for $w \in A^1$ implies $w \in A^p$ for all $p > 1$).

To prove this let $f \in L^p(R^n)$ be given and $\alpha \in (0,1)$. Let Q be a cube centered at $\overline{x}$, and $\tilde{Q}$ its double. Then write $f = \chi_{\tilde{Q}} f + \chi_{c\tilde{Q}} f = f_I + f_0$. We must show that

$$\frac{1}{|Q|}\int_Q M(f_I)^\alpha dx \leq CM(f)^\alpha(\overline{x})$$

and

$$\frac{1}{|Q|}\int_Q M(f_0)^\alpha dx \leq CM(f)^\alpha(\overline{x}) .$$

As for the first inequality,

$$\left(\frac{1}{|Q|}\int_Q M(f_I)^\alpha dx\right)^{1/\alpha} \leq \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f_I| dx\right) = \frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f| .$$

(This is an immediate consequence of the weak type estimate for M on L^1.) This shows that

$$\left(\frac{1}{|Q|}\int_Q M(f_I)^{\alpha}dx\right) \leq \left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f|\right)^{\alpha} \leq M(f)^{\alpha}(\bar{x}) .$$

As for the second inequality, choose a cube C centered at $\bar{x}$ such that

$$\frac{1}{|C|}\int_C |f_0|\,dx \geq \frac{1}{2} M(f_0)(\bar{x}) .$$

Now, if $x \in Q$ then any cube C' centered at x which intersects ${}^c\tilde{Q}$ is contained inside a cube of comparable volume centered at $\bar{x}$ so that

$$\frac{1}{|C'|}\int_{C'} |f_0| \leq A\,\frac{1}{|C|}\int_C |f_0|\,dx .$$

We have proven that for all $x \in Q$, $M(f_0)(x) \leq A\,\frac{1}{|C|}\int_C |f_0|\,dx$ so that

$$\frac{1}{|Q|}\int_Q M(f_0)^{\alpha}dx \leq A^{\alpha}\left(\frac{1}{|C|}\int_C |f_0|\,dx\right)^{\alpha} \leq A^{\alpha}M(f)^{\alpha}(\bar{x}) .$$

Let us begin to discuss the weight theory by showing that the Hardy-Littlewood maximal operator is bounded on $L^p(w)$ if and only if $w \in A^p$, $1 < p < \infty$ [12]. In the first place if $f = w^{-1/(p-1)}\chi_Q$ and if M is bounded on $L^p(w)$ one sees right away that

$$w(Q)\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}dx\right)^p \leq C\int_Q w^{-p/(p-1)}w\,dx$$

or

$$\frac{w(Q)}{|Q|}\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)}\right)^{p-1} \le C .$$

Conversely, assume $w \in A^p$. Calderón-Zygmund decompose $f \in L^p(w)$ at heights C^k where $k \in Z$, and C is large (to be described later) and get Calderon-Zygmund cubes $\{Q_j^k\}_{j=1}^{\infty}$ so that

$$\frac{1}{|Q_j^k|}\int_{Q_j^k} f \sim C^k \quad \text{and} \quad \{Mf > \gamma C^k\} \subseteq \cup \tilde{Q}_j^k$$

(γ is a large constant dependent only on n). Then

$$\int_{R^n} (Mf)^p w\,dx \le C' \sum_{k,j} w(\tilde{Q}_j^k) C^k \le C' \sum_{k,j} w(\tilde{Q}_j^k)\left(\frac{1}{|Q_j^k|}\int_{Q_j^k} f\right)^p .$$

Now $w \in A^p \Longrightarrow w \in A^\infty$ therefore $w(\tilde{Q}_j^k) \le C'' w(Q_j^k)$ and so the above expression is

$$\le C''' \sum_{k,j} w(Q_j^k)\left(\frac{\sigma(Q_j^k)}{|Q_j^k|}\right)\left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} f\sigma^{-1}\sigma dx\right)^p \le \text{by the } A^p \text{ condition on}$$

$$(*) \quad \le C'''' \sum_{k,j} \sigma(Q_j^k)\left(\frac{1}{\sigma(Q_j^k)}\int_{Q_j^k} (f\sigma^{-1})\,\sigma dx\right)^p \quad \text{where } \sigma = w^{-1/(p-1)} .$$

So far we have used only arithmetic. Now we come to the main point. If $E_j^k = Q_j^k - \bigcup_{\ell > k} Q_j^\ell$ then choosing C large enough insures that $|E_j^k| > \frac{1}{2}|Q_j^k|$, and since $\sigma \in A^{p'} \Longrightarrow \sigma \in A^\infty$ we have $\sigma(E_j^k) > \eta\sigma(Q_j^k)$ and so

$$(*) \qquad \leq C \sum \sigma(E_j^k)\left(\frac{1}{\sigma(Q_j^k)} \int_{Q_j^k} (f\sigma^{-1})\, d\sigma\right) \leq C \int_{R^n} M_\sigma^p(f\sigma)\, d\sigma$$

where $M_\sigma(f)(x) = \sup_{x \epsilon Q} \frac{1}{\sigma(Q)} \int_Q |f| d\sigma$. Now the same proof that works to show that the Hardy-Littlewood maximal operator is bounded on $L^p(R^n)$ proves that if σ satisfies a doubling condition, then for $p > 1$, M_σ is bounded on $L^p(d\sigma)$.

It follows that the second inequality is

$$\leq C \int f^p \sigma^{1-p} dx = \int f^p w\, dx .$$

Note that the operator $M_\mu f(x) = \sup_{x \epsilon Q} \frac{1}{\mu(Q)} \int_Q |f| d\mu$ and its boundedness on $L^p(d\mu)$ enter in a crucial way the proof of the weight norm inequalities for the Hardy-Littlewood operator. This operator M_μ is interesting in its own right, since it is natural to ask what happens if we replace the God-given Lebesgue measure by another measure $d\mu$.

In fact, if μ is any measure finite on compact sets and if, in the definition of M_μ we insist that the balls be *centered* at x then M_μ is bounded on $L^p(\mu)$, $p > 1$. The proof of this remarkable theorem relies on a refinement of the usual Vitali covering lemma due to Besocovitch. We should also remark that the original proof of the weight norm inequalities for M on R^n made use of M_μ as well. In fact, if $w \epsilon A^p$ it is not hard to see that

$$M(f)(x) \leq CM_w(f^p)^{1/p}(x) .$$

(Indeed,

$$\frac{1}{|Q|}\int_Q f\,dx = \frac{1}{|Q|}\int_Q f w^{1/p} w^{-1/p} dx \le \frac{1}{|Q|}\left(\int_Q f^p w\,dx\right)^{1/p}\left(\int_Q w^{-p'/p} dx\right)^{1/p'}$$

$$= \frac{w(Q)}{|Q|}^{1/p}\left(\frac{1}{w(Q)}\int_Q f^p w\right)^{1/p}\left(\frac{1}{|Q|}\int_Q w^{-1/(p-1)} dx\right)^{(p-1)/p} \le CM_w(f^p)^{1/p}(x)\ .)$$

The proof would be complete if $M_w(f^p)^{1/p}$ were bounded on $L^p(w)$. Unfortunately, $M_w(f^p)^{1/p}$ is bounded on $L^q(w)$ only for $q > p$. But if we notice that $w \in A_p \Longrightarrow w \in A_{p-\varepsilon}$ then the proof is complete.

Anyway, it is clear that the operator M_μ arises naturally. Next we shall study its n-parameter variant $M_\mu^{(n)}$ just the strong maximal operator, only taken relative to the measure μ :

$$M_\mu^{(n)}(f)(x) = \sup_{x \in R} \frac{1}{\mu(R)} \int_R |f|\, d\mu\ .$$

R a rectangle with sides parallel to the axes. Unlike the case where $n = 1$, restrictions must be placed on μ whether or not R is centered at x.

The following result gives conditions on μ which are rather unrestrictive, and which guarantee the boundedness on $L^p(\mu)$ on $M_\mu^{(n)}$ [13]:

THEOREM. *Suppose* μ *is absolutely continuous and non-negative on* R^n, *and that the Radon-Nikodym derivative of* μ, $w(x)$ *satisfies an* A^∞ *condition in each variable separately, uniformly in the other variables. Then* $M_\mu^{(n)}$ *is bounded on* $L^p(d\mu)$ *for all* $p > 1$.

We require the following lemma.

LEMMA. *If* $w(x_1,\cdots,x_n)$ *is uniformly in* A^∞ *in each of the variables separately, then* w *satisfies the following: If* R *is any rectangle with its sides parallel to the axes and* $E \subseteq R$ *is such that* $\frac{|E|}{|R|} > \frac{1}{2}$, *then* $\frac{\int_E w}{\int_R w} > \eta$, *for some* $\eta > 0$.

Proof. The proof is by induction on n. Assume this for $n-1$. Consider a rectangle R as above, $R = I \times J$ where I is n-1 dimensional and J one-dimensional, and a subset E of R such that

$$\frac{|E|}{|R|} > \frac{1}{2}\,.$$

For each $x \in I$ let $J_x = \{(x,y) | y \in J\}$ be a vertical segment. Since $\frac{|E|}{|R|} > \frac{1}{2}$ it is easy to see that for x in a set I' of measure $\geq \frac{1}{100}|I|$ we have $|E \cap J_x| > \frac{1}{100}|J_x|$. Now since w is uniformly A^∞ in the x_n variable,

$$\int_{J_{x'} \cap E} w \, dx_n \geq \eta \int_{J_{x'}} w \, dx_n\,, \quad x \in I'. \tag{3.1}$$

But also if we fix any $x_n \in J$ then

$$\int_{I'} w \, dx' \geq \eta' \int_I w \, dx' \tag{3.2}$$

by induction. It follows by integrating (3.1) in $x' \in I'$ that

$$\int_E w \, dx \geq \eta \int_{I' \times J} w \, dx$$

and integrating (3.2) in $x_n \epsilon J$ gives,

$$\int_{I' \times J} w\, dx \geq \eta' \int_{I \times J} w\, dx \ ,$$

which shows that $\int_E w > \eta\eta' \int_R w$ and proves the lemma.

Proof of the Theorem. We prove a covering lemma, namely, if $\{R_k\}$ are rectangles with sides parallel to the axes, there exists $\{\widetilde{R}_k\}$ so that $w(\cup R_k L \leq Cw(\cup \widetilde{R}_k)$ and

$$\|\sum \chi_{\widetilde{R}_k}\|_{L^{p'}(w)} \leq Cw(\cup \widetilde{R}_k)^{1/p'} .$$

To prove this order R_k by decreasing x_n side length, and then select a rectangle R when

$$|R \cap [\cup(\widetilde{R}_k)_d]| < \frac{1}{2}\ |R|$$

where the union is taken over all those k such that $\widetilde{R}_k$ precedes R and $\widetilde{R}_k \cap R = \emptyset$.

Now if we slice R, an unselected rectangle, with a hyperplane perpendicular to the x_n direction we have

$$|S \cap [\cup(\widetilde{S}_k)_d]| > \frac{1}{2}\ |S| \Longrightarrow w(S \cap [\cup(\widetilde{S}_k)_d]) > \eta w(S)$$

so that $M_w^{(n-1)}(\chi_{\cup(\widetilde{R}_k)_d}) > \eta$ on $\cup R_k$. By induction, $w(\cup R_k) \leq Cw(\cup(\widetilde{R}_k)_d)$. Now the $\widetilde{R}_k$ have disjoint parts property with respect to dx and so they have this property with respect to $w\, dx$ also. It follows that

$$w(\cup(\widetilde{R}_k)_d) \leq \sum w(\widetilde{R}_k)_d \leq C \sum w(\widetilde{R}_k) \leq C' w(\cup \widetilde{R}_k) .$$

Therefore

$$w(\cup R_k) \leq Cw(\cup \widetilde{R}_k) .$$

Now to estimate $\|\Sigma\, \chi_{\widetilde{R}_k}\|_{L^{p'}(w)}$, let us slice the $\widetilde{R}_k$ with a hyperplane perpendicular to x_n, calling the slices $\widetilde{S}_k$. Then $|\widetilde{S}_k - \bigcup_{j<k} \widetilde{S}_j| > \frac{1}{2}\, |\widetilde{S}_k|$ so since $w \in A^\infty$ in $x_1,\cdots,x_{n-1}$ we have $w(\widetilde{S}_k - \bigcup_{j<k} \widetilde{S}_j) > \eta w(\widetilde{S}_k)$ (if $w(E) = \int_E w(x_1,\cdots,x_{n-1},x_n)dx_1,\cdots,dx_{n-1})$ and we test

$$\int_{R^{n-1}} \sum \chi_{\widetilde{S}_k} \phi w = \sum \int_{\widetilde{S}_k} \phi w dx \le C \sum w(\widetilde{E}_k)\, \frac{1}{w(\widetilde{S}_k)} \int_{\widetilde{S}_k} \phi w\, dx \le$$

$$\le C \int_{\widetilde{S}_k} M_w^{(n-1)}(\phi) w\, dx\ .$$

$(\widetilde{E}_k = \widetilde{S}_k - \bigcup_{j<k} \widetilde{S}_j)$. By induction $M_w^{(n-1)}$ is bounded on $L^p(w)$ so this last integral is

$$\le \|\phi\|_{L^p(w)}\, \|\chi_{\cup \widetilde{S}_k}\|_{L^{p'}(w)} \le \|\phi\|_{L^p(w)}\, w(\cup \widetilde{S}_k)^{1/p}\ .$$

This shows that $\|\Sigma\, \chi_{\widetilde{S}_k}\|_{L^{p'}(w)} \le C w(\cup \widetilde{S}_k)^{1/p'}$. Raising this to the p^{th} power and integrating in x_n we have

$$\|\sum \chi_{\widetilde{R}_k}\|_{L^p(w)} \le C w(\cup \widetilde{R}_k)\ .$$

REMARK. Given this covering lemma, cover the set $\{M_w^{(n)}(f) > \alpha\}$ by R_k such that $\frac{1}{w(R_k)} \int_{R_k} fw > \alpha$. Then we need only estimate $w(\cup \widetilde{R}_k)$ of the covering lemma. But

$$w(\cup \widetilde{R}_k) \le \sum w(\widetilde{R}_k) \le \frac{1}{\alpha} \int f \sum \chi_{\widetilde{R}_k} w \le \frac{1}{\alpha}\, \|f\|_{L^p(w)}\, \|\sum \chi_{\widetilde{R}_k}\|_{L^{p'}(w)}$$

$$\le \frac{1}{2}\, \|f\|_{L^p(w)}\, w(\cup \widetilde{R}_k)^{1/p'}\ .$$

The maximal operator is weak type (p,p), $p > 1$ and we are finished by interpolation.

One application of this theorem is that with it, one can obtain weighted norm inequalities for multi-parameter maximal operators which cannot be handled directly through iteration. We give the following example.

Suppose $\mathfrak{R}$ denotes the family of rectangles with side lengths of the form s, t, and $s \cdot t$ in R^3, where s and $t > 0$ are arbitrary. (Suppose the sides are also parallel to the axes.) Define the corresponding maximal operator M by

$$Mf(x) = \sup_{x \in R \in \mathfrak{R}} \frac{1}{|R|} \int_R |f| \, dt .$$

Then it is natural to ask for which weights w do we have

$$\int Mf^p w \le C \int f^p w .$$

The answer is $A^p(\mathfrak{R})$ where this class is defined in the obvious way [14]:

$$w \in A^p(\mathfrak{R}) \text{ if and only if } \left(\frac{1}{|R|} \int_R w\right) \left(\frac{1}{|R|} \int_R w^{-1/(p-1)}\right)^{p-1} \le C$$

for all $R \in \mathfrak{R}$. To prove this result we need the following.

LEMMA. *If* $w \in A^p(\mathfrak{R})$ *then* w *satisfies a reverse Hölder inequality.*

Proof. Since w is uniformly A^p in the x_1 variable, w will satisfy a reverse Hölder inequality uniformly in that variable:

$$\left(\frac{1}{|I|} \int_I w(x_1,p)^{1+\delta} dx_1\right)^{1/(1+\delta)} \le C\left(\frac{1}{|I|} \int_I w(x_1,p)\, dx_1\right) .$$

(C independent of p). Fix I, an interval of the x_1-axis, and define a measure in the x_2, x_3 plane whose Radon-Nikodym derivative W(p) is defined by

$$W(p) = \left(\frac{1}{|I|}\int_I w^{1+\delta}(x_1,p)\,dx_1\right)^{1/(1+\delta)} .$$

We claim that W satisfies an A^∞ condition uniformly (in I) relative to the class of rectangles whose side lengths are t, $|I|t$ in the x_2,x_3 plane.

Then let S be such a rectangle in the x_2,x_3 plane and $E \subseteq S$ such that $|E|/|S| \leq 1/2$. Then

$$\int_E W(p)\,dp \leq C\int_E\left(\frac{1}{|I|}\int_I w(x_1,p)\,dx_1\right)dp = C\,\frac{1}{|I|}\iint_{E\times I} w(x_1,p)\,dx_1 dp .$$

Since $w \in A^\infty(\mathfrak{R})$, $\iint_{E\times I} w \leq (1-\eta)\iint_R w$ and so $\int_E W(p)\,dp \leq C(1-\eta)\int_S W(p)\,dp$, and by taking δ small enough, C will be so close to 1 that $C(1-\eta) < 1$ and W is uniformly A^∞ on the collection of all such S. Therefore since S is just 1-parameter (just a linear change of variable in one of the x_2 or x_3 variable away from squares) we have that W satisfies a reverse Hölder inequality: (For δ' some value $< \delta$)

$$\left(\frac{1}{|S|}\int_W W^{1+\delta}\,dp\right)^{1/(1+\delta')} \leq C\,\frac{1}{|S|}\int_S W .$$

This shows that

$$\left(\frac{1}{|S|}\int_S\left(\frac{1}{|I|}\int_I w^{1+\delta}dx_1\right)^{(1+\delta)^{-1}(1+\delta')} dp\right)^{(1+\delta')^{-1}} \leq C'\,\frac{1}{|S|}\int_S\left(\frac{1}{|I|}\int_I w\,dx_1\right)dp$$

and so

$$\left(\frac{1}{|R|}\int_R w^{1+\delta'}\right)^{1/(1+\delta')} \leq C'\frac{1}{|R|}\int_R w\,, \quad R \in \mathfrak{R}\,.$$

Now on to the theorem. Because $w \in A^p(\mathfrak{R})$ and $w^{-1/(p-1)} \in A^{p'}(\mathfrak{R})$ satisfies a reverse Holder inequality, $w \in A^{p-\varepsilon}(\mathfrak{R})$ for some $\varepsilon > 0$. It follows that

$$Mf(x) \leq CM_w(f^{p-\varepsilon})^{1/(p-\varepsilon)}\,,$$

and it just remains to show that M_w is bounded on $L^p(w)$.

A quick review of the proof that $M_w^{(n)}$, $n = 3$, is bounded on $L^p(w)$ reveals that all we really used was that w satisfy an A^∞ condition in the x_1 and x_2 variables as well as a doubling condition in the x_3 variable: $w((R)_d) \leq Cw(R)$. All of these are satisfied by our w here, and this concludes the proof since $M_w(f) \leq M_w^{(3)}(f)$.

Now we wish to relate some of our results on multi-parameter maximal functions to the theory of multiplier operators.

We shall work in R^2, and consider the following basic question: For which sets $S \subseteq R^2$ is $\chi_S(\xi)$ a multiplier on $L^p(R^2)$ for some $p \neq 2$? For χ_S to be a multiplier of course means that, if for $f \in C_c^\infty(R^2)$ we set $Tf\hat{}(\xi) = \chi_S(\xi)\hat{f}(\xi)$ then we have the a priori estimate

$$\|Tf\|_{L^p(R^2)} \leq C\|f\|_{L^p(R^2)}\,.$$

In his celebrated theorem, Charles Fefferman showed that if S is a nice open set in R^2 whose boundary has some curvature then $\chi_S(\xi)$ will only be an L^2 multiplier [15]. The other nice regions left are those whose boundaries are comprised of polygonal segments. If S is a convex polygon then χ_S will obviously be an L^p multiplier for all p, $1 < p < \infty$, just because of the boundedness of the Hilbert transform on the L^p spaces in R^1. The case that remains is the one we consider here. Let $\theta_1 > \theta_2 > \theta_3 \cdots > \theta_n > \theta_{n-1} \to 0$ be a given sequence of angles θ and let

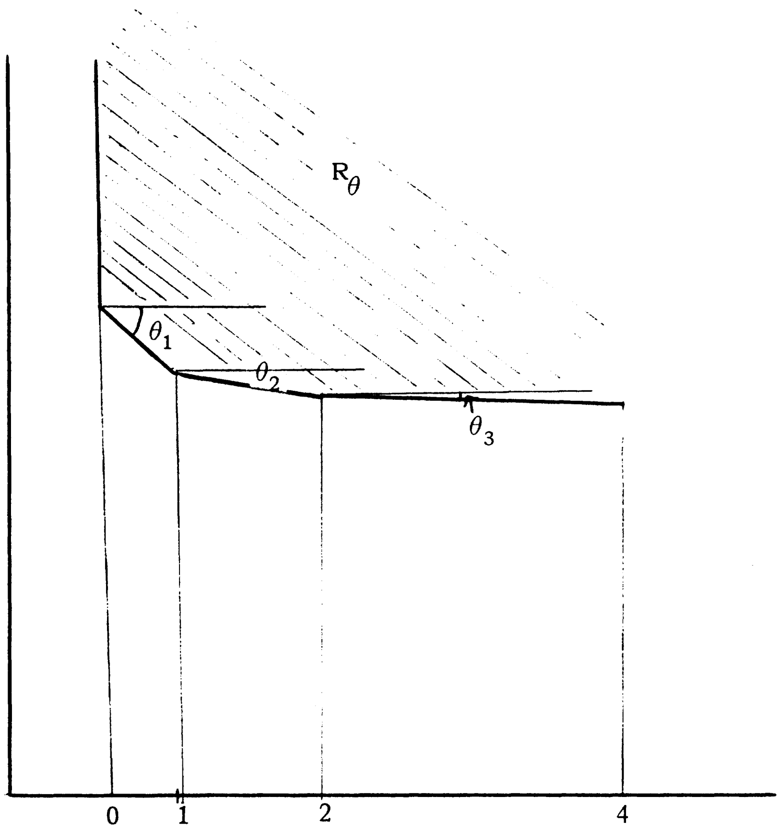

Figure 2

R_θ be the polygonal region pictured above. Then we shall define T_θ by

$$T_\theta f\,\hat{}\,(\xi) = \chi_{R_\theta}(\xi)\hat{f}(\xi) .$$

Consider as well the maximal operator M_θ defined by

$$M_\theta f(x) = \sup_{x \epsilon R \epsilon B_\theta} \frac{1}{|R|} \int_R |f| dt$$

where B_θ is the family of all rectangles in R^2 which are oriented in one of the directions θ_k, but whose side lengths are arbitrary. We claim that the behavior of T_θ on $L^p(R^2)$ for $p > 2$ is linked with the behavior of M_θ on $L^{(p/2)'}((p/2)'$ is the exponent dual to $p/2$). More precisely, suppose that T_θ is bounded on $L^p(R^2)$ and we assume the weakest possible estimate on M_θ, namely $|\{M_\theta \chi_E > \frac{1}{2}\}| \leq C|E|$. Then M_θ is of weak type on $L^{(p/2)'}$. Conversely, if M_θ is bounded on $L^{(p/2)'}(R^2)$ then T_θ is bounded on $L^q(R^2)$, for $p' < q < p$ [16].

To prove this assume first that T_θ is bounded on $L^p(R^2)$. Then the first step is to notice that this implies that

$$\left\| \left(\sum |T_k f_k|^2 \right)^{1/2} \right\|_{L^p(R^2)} \leq C \left\| \left(\sum |f_k|^2 \right)^{1/2} \right\|_{L^p(R^2)}.$$

This is proven by observing that if we dilate the region R_θ by a huge factor ρ to get R_θ^ρ and translate R_θ^ρ properly (by τ_k) then R_θ^{ρ,τ_k} looks like a half plane with boundary line making an angle of θ_k with the positive x axis. Then if $r_k(t)$ are the Rademacher functions, and $\widehat{T_\theta^{\rho,\tau_k} f} = \chi_{R_\theta^{\rho,\tau_k}} \hat{f}$ then

$$T_\theta^{\rho,\tau_k} f = e^{-i\tau_k \cdot x} T_\theta^\rho (e^{i\tau_k \cdot x} f)$$

and

$$\left\| \left(\sum |T_\theta^{\rho,\tau_k} f_k|^2 \right)^{1/2} \right\|_{L^p}^p = \left\| \left(\sum |T_\theta^\rho (e^{i\tau_k \cdot x} f_k)|^2 \right)^{1/2} \right\|_{L^p}^p$$

$$= \int_{R^2} \int_0^1 \left| \sum r_k(t) T_\theta^\rho (e^{i\tau_k \cdot x} f_k)(x) \right|^p dt\, dx = \int_0^1 \int_{R^2} \left| T_\theta^\rho \left(\sum r_k(t) e^{i\tau_k \cdot x} f_k \right) \right|^p dx\, dt$$

$$\leq C \int_{R^2} \int_0^1 \left| \sum r_k(t) e^{i\tau_k \cdot x} f_k(x) \right|^p dt\, dx \leq C' \left\| \left(\sum |f_k(x)|^2 \right)^{1/2} \right\|_{L^p}.$$

Taking the limit as $\rho \to \infty$, we have

$$\left\| \left(\sum |T_k f_k|^2 \right)^{1/2} \right\|_{L^p(R^2)} \le C \left\| \left(\sum |f_k|^2 \right)^{1/2} \right\|_{L^p(R^2)},$$

where T_k is the Hilbert transform in the direction θ_k.

The next step is to use the above inequality to prove a covering lemma for rectangles in $\mathfrak{B}_\theta$. Let $\{R_k\}$ be a sequence of such rectangles. Select R given that $\widetilde{R}_1, \widetilde{R}_2, \cdots, \widetilde{R}_{k-1}$ have been chosen provided $|R \cap [\bigcup_{j<k} \widetilde{R}_j]| < \frac{1}{2}|R|$. Then if R is unselected $M_\theta(\chi_{\cup \widetilde{R}_j}) \ge \frac{1}{2}$ on R so that

$$|\cup R_k| \le \left|\left\{M_\theta(\chi_{\cup \widetilde{R}_k}) > \frac{1}{2}\right\}\right| \le C|\cup \widetilde{R}_k| .$$

Let $E_k = \widetilde{R}_k - \bigcup_{j<k} \widetilde{R}_j$ and let $f_k = \chi_{\widetilde{E}_k}$. Then looking at the picture below, since $|\widetilde{E}_k| \ge \frac{1}{2}|\widetilde{R}_k|$ on at least a set of measure $> \frac{1}{100}|\widetilde{I}_k|$ the

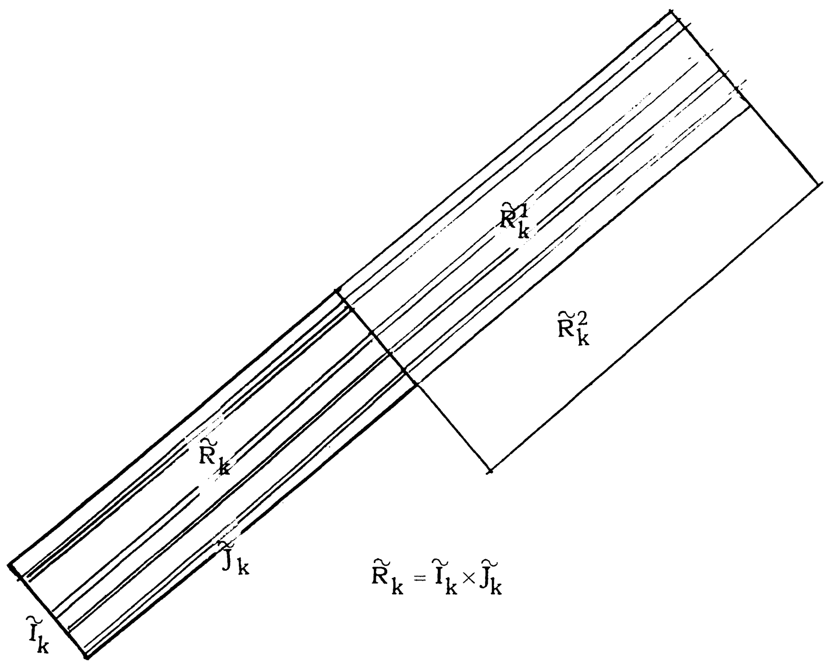

Figure 3

segments pictured contain at least $1/100$ of their measure in $\tilde{E}_k$. If we duplicate the rectangle $\tilde{R}_k$ as shown, on these segments $T_k f_k > 1/100$. Applying $T_k' =$ Hilbert transform in the direction perpendicular to θ_k to $T_k f_k$ we see that $T_k'(T_k f_k) > 1/100$ on all of $\tilde{R}_k^2$. Repeating twice more we get

$$\left\|\left(\sum \chi_{\tilde{R}_k}\right)^{1/2}\right\|_{L^p} \leq C\left\|\left(\sum |\chi_{\tilde{E}_k}|^2\right)^{1/2}\right\|_{L^p} \leq C\|\chi_{\cup \tilde{R}_k}\|_{L^p}.$$

This shows that M_θ is of weak type $(p/2)'$.

Conversely, assume that M_θ is bounded on $L^{(p/2)'}$. Define $S_k = \{\xi = (\xi_1, \xi_2) | 2^k \leq \xi_1 < 2^{k+1}\}$. Then if S_k also stands for the multiplier operator corresponding to S_k, we see that

$$\|T_\theta f\|_q \sim \left\|\left(\sum |S_k T_\theta f|^2\right)^{1/2}\right\|_q = \left\|\left(\sum |S_k T_k f|^2\right)^{1/2}\right\|_q .$$

To estimate $\|(\Sigma |S_k T_k f|^2)^{1/2}\|_q^2$, let $\|\phi\|_{(q/2)'} = 1$ and let us estimate

$$\int \sum |T_k S_k f|^2 \phi \leq \sum \int |T_k (S_k f)|^2 \phi .$$

But in R^1 we have the classical weight norm inequality for the Hilbert transform:

$$\int |Hf|^2 \phi \leq C \int |f|^2 M(\phi^{1+\varepsilon})^{1/(1+\varepsilon)} \text{ for all } \varepsilon > 0 .$$

It follows that (3.3) is

$$\leq \sum \int |S_k(f)|^2 M_\theta (\phi^{1+\varepsilon})^{1/(1+\varepsilon)} . \tag{3.4}$$

M_θ is bounded on $L^{(q/2)'/(1+\varepsilon)}$ if ε is sufficiently small. It follows that (3.4) is

$$\leq C\left\| \left(\sum |S_k f|^2\right)\right\|_{L^{q/2}} \leq C'\|f\|^2_{L^q}$$

proving that T_θ is bounded on L^q.

4. H^p *spaces – one and several parameters*

In this lecture we wish to discuss another chapter of harmonic analysis relating to differentiation theory and singular integrals, namely Hardy Space theory. In this lecture, we shall discuss the one-parameter theory, and, in the next, the theory in several parameters. In the beginning when H^p spaces were first considered, they were spaces of complex analytic functions in $R^2_+ = \{z = x + iy \,|\, x \in R^1, y \in R^1, y > 0\}$ which satisfies the size restriction

$$\left(\int_{-\infty}^{+\infty} |F(x+iy)|^p dx\right)^{1/p} \leq C \quad \text{for all } y \in R^1_+ .$$

One of the main reasons for introducing these spaces was the connection with the Hilbert transform. If $F(z) = u(z) + iv(z)$ is analytic with u and v real, and if F is sufficiently nice then F will have boundary value $u(x) + iv(x)$ where $v(x)$ is the Hilbert transform of $u(x)$. It turns out, since

$$\int_{-\infty}^{+\infty} |F(x+it)|\, dx$$

increases as $t \to 0$, we have

$$\|F\|_{H^1} \overset{\text{def}}{=} \sup_{t>0} \int_{-\infty}^{+\infty} |F(x+it)| dx \sim \|u\|_{L^1(R^1)} + \|v\|_{L^1(R^1)} .$$

So we may view the space H^1 through its boundary values as the space of all real valued functions $f \in L^1(R^1)$ whose Hilbert transforms are L^1 as well.

If we want a theory of $H^p(R^n)$ then, following Stein and Weiss we may consider the functions $F(x,t)$ in $R_+^{n+1} = \{(x,t) | x \in R^n, t > 0\}$ whose values lie in $R^{n+1} : F(x,t) = (u_0(x,t), \cdots, u_n(x,t))$ where the $u_i(x,t)$ satisfy the "Generalized Cauchy-Riemann equations,"

$$\sum_{i=0}^{n} \frac{\partial u_i}{\partial x_i}(x,t) \equiv 0 \quad (t = x_0)$$

and

$$\frac{\partial u_i}{\partial x_j} \equiv \frac{\partial u_j}{\partial x_i} \quad \text{for all } i,j .$$

These Stein-Weiss analytic functions are then said to be $H^p(R_+^{n+1})$ if and only if

$$\sup_{t>0} \left(\int_{R^n} |F(x,t)|^p dx \right)^{1/p} = \|F\|_{H^p(R^n)} \qquad [17] .$$

Again, these functions have an interpretation in terms of singular integrals, since if a Stein-Weiss analytic function $F(x,t)$ is sufficiently "nice" on $\bar{R}_+^{n+1}$, then the boundary values $u_i(x)$ satisfy $u_i(x) = R_i[u_0](x)$ where R_i is the i^{th} Riesz transform given by $R_i(f)(x) = f * \frac{c_n x_i}{|x|^{n+1}}$. In particular we may consider an $H^1(R_+^{n+1})$ function (by identifying functions in R_+^{n+1} with their boundary values) as a function f with real values in $L^1(R^n)$ each of whose Riesz transforms $R_i f$ also belong to $L^1(R^n)$. An interesting feature of H^p spaces is that they are intimately connected to differentiation theory as well as singular integrals. To discuss this, let us make some well-known observations. For a harmonic function $u(x,t)$ which is continuous on $\overline{R_+^{n+1}}$ and bounded there, u is given as an average of its boundary values according to the Poisson integral:

$$u(x,t) = f * P_t(x);\ f(x) = u(x,0) \text{ and } P_t(x) = \frac{c_n t}{(|x|^2+t^2)^{(n+1)/2}} .$$

Let $\Gamma(x) = \{(y,t) \mid |x-y| < t\}$. Then since convolving with P_t at a point x can be dominated by an appropriate linear combination of averages of f over balls centered at x of different radii, it follows that

$$\text{if } u^*(x) = \sup_{(y,t)\epsilon\Gamma(x)} |u(y,t)| , \text{ then } u^*(x) \le cMf(x) .$$

Unfortunately, if $u(x,t)$ is harmonic, for $p \le 1$, $u = P[f]$, and $\int_{R^n} |u(x,t)|^p dx \le C$ then the domination $u^* \le CMf$ is not useful, since M is not bounded on L^p, and it is not true in general that $u^*(x) < \infty$ for a.e. $x \epsilon R^n$. On the other hand, suppose F is Stein-Weiss analytic, say $F \epsilon H^1(R_+^{n+1})$.

Then a beautiful computation shows that if $1 > \alpha > 0$ is close enough to 1 $\left(\alpha \ge \frac{n-1}{n}\right)$ then $\Delta(|F|^\alpha) \ge 0$ so that $|F|^\alpha$ is subharmonic. If $s(x,t)$ is subharmonic and has boundary values $h(x)$ then s is dominated by the averages of h, i.e.,

$$s(x,t) \le P[h](x,t) .$$

Applying this to $G = |F|^\alpha$ (which has $\int G^{1/\alpha}(x,t)dx \le C$ for all $t > 0$) we see that $G^* \le M(h)$ for some $h \epsilon L^{1/\alpha}$. Now M *is* bounded on $L^{1/\alpha}$ so that $M(h) \epsilon L^{1/\alpha}$ and so $G^* \epsilon L^{1/\alpha}$. It follows that $F^* \epsilon L^1$. Just as for a random $f \epsilon L^1(R^n)$ we do not necessarily have $R_i f \epsilon L^1(R^n)$ (singular integrals do not preserve L^1) it is also not true that for an arbitrary L^1 function f that for $u = P[f]$, $u^* \epsilon L^1$. But if $f \epsilon H^1(R_+^{n+1})$ then $u^* \epsilon L^1(R^n)$. Thus the nontangential maximal function

$$F^*(x) = \sup_{(y,t)\epsilon\Gamma(x)} |F(y,t)| \ \epsilon\ L^1(R^n)$$

if and only if the analytic function $f \epsilon H^1(R_1^{n+1})$.

We know so far that we can characterize H^p functions in terms of singular integrals and maximal functions. There is another characterization which is of great importance. To discuss it, let us return to H^p functions in R^2_+ as complex analytic functions, $F = u + iv$. It is an interesting question as to whether the maximal function characterization of H^p can be reformulated entirely in terms of u. That is, is it true that $F^* \epsilon L^p$ if and only if $u^* \epsilon L^p$? In fact, this is true, and the best way to see this is by introducing a special singular integral, the Lusin-Littlewood-Paley-Stein area integral,

$$S^2(u)(x) = \iint_{\Gamma(x)} |\nabla u|^2(y,t)\,dt\,dy$$

which we already considered in the first lecture. As we shall see later, for a harmonic function $u(x,t)$, $\|S(u)\|_{L^p} \approx \|u^*\|_{L^p}$ for all $p > 0$ [18]. The importance of S here is that the area integral is invariant under the Hilbert transform, i.e.,

$$S(u) \equiv S(v)\,, \quad \text{since} \quad |\nabla v| = |\nabla u|\ .$$

When we combine the last two results, we immediately see that

$$\|u^*\|_{L^p} < \infty \Longleftrightarrow \|F^*\|_{L^p} < \infty \Longleftrightarrow F \,\epsilon\, H^p(R^2_+)\ .$$

It is interesting to note that the first proof of $\|S(u)\|_{L^p} \sim \|u^*\|_{L^p}$, $1 \geq p > 0$ was obtained by Burkholder, Gundy, and Silverstein [19] by using probabilistic arguments involving Brownian motion. Nowadays direct real variable proofs of this exist as we shall see later on.

To summarize, we can view functions f in H^p spaces by looking at their harmonic extensions u to R^{n+1}_+ and requiring that u^* or $S(u)$ belong to $L^p(R^n)$.

It turns out that there is another important idea which is very useful concerning H^p spaces and their real variable theory. So far, we have spoken of H^p functions only in connection with certain differential equations. Thus, if we wanted to know whether or not $f \in H^p$ we could take $u = P[f]$ which of course satisfies $\Delta u = 0$.

This is not necessary. If f is a function and $\phi \in C_c^\infty(R^n)$ with $\int_{R} \phi_n = 1$, then we may form $f^*(x) = \sup_{(t,y)\in\Gamma(x)} |f*\phi_t(y)|$, $\phi_t(x) = t^{-n}\phi(x/t)$ and if $\psi \in C_c^\infty(R^n)$ is suitably non-trivial (say radial, non-zero) and $\int\psi = 0$ we may form

$$S_\psi^2(f)(x) = \iint_{\Gamma(x)} |f*\psi_t(y)|^2 \frac{dy\,dt}{t^{n+1}} \quad [18] .$$

Then C. Fefferman and E. M. Stein have shown that

$$\|f\|_{H^p(R^n)} \sim \|f^*\|_{L^p(R^n)} \sim \|S_\psi(f)\|_{L^p(R^n)} \quad \text{for } 0<p<\infty .$$

Thus, it is possible to think of H^p spaces without any reference to particular approximate identities like $P_t(x)$ which relate to differential equations.

In addition to understanding the various characterizations of H^p spaces, another important aspect is that of duality of H^1 with BMO, which we shall now discuss.

A function $\phi(x)$, locally integrable on R^n is said to belong to the class BMO of functions of bounded mean oscillation provided

$$\frac{1}{|Q|}\int_Q |\phi(x)-\phi_Q|\,dx \le M \quad \text{for all cubes } Q \text{ in } R^n ,$$

where $\phi_Q = \frac{1}{|Q|}\int_Q \phi$. The BMO functions are really functions defined modulo constants and $\| \ \|_{BMO}$ is defined to be $\sup \frac{1}{|Q|}\int_Q |\phi-\phi_Q|$.

According to a celebrated theorem of C. Fefferman and Stein, BMO is the dual of H^1 [18]. This result's original proof involves knowing that singular integrals map L^∞ to BMO and also a characterization of BMO functions in terms of their Poisson integrals which we now describe.

Suppose $\mu \geq 0$ is a measure in R^{n+1}_+ and $Q \subseteq R^n$ is a cube. Let $S(Q) = \{y,t) | y \in Q, 0 < t < \text{side length } (Q)\}$. Then we say that μ is a Carleson measure on R^{n+1}_+ iff $\mu(S(Q)) \leq C|Q|$. The basic property that characterizes Carleson measures is

$$\iint_{R^{n+1}_+} |u(x,t)|^p d\mu(x,t) \leq C_p \int_{R^n} u^*(x)^p dx \qquad p > 0$$

for all functions u on R^{n+1}_+. In connection with this type of measure there is the characterization of functions in $BMO(R^n)$ in terms of their Poisson integrals. A function $\phi(x)$ on R^n with Poisson integral $u(x,t)$ is in BMO if and only if the associated measure

$$d\mu(x,t) = |\nabla u|^2(x,t)\, t\, dx\, dt$$

is a Carleson measure. C. Fefferman proved this and used it to prove that every function in BMO acts continuously on H^1 :

$$\left| \int_{R^n} f(x)\phi(x)\, dx \right| \leq C \|f\|_{H^1(R^n)} \|\phi\|_{BMO(R^n)} .$$

These are the basic facts of H^p spaces that will concern us here and which we shall later generalize to product spaces.

Let us now prove that for a harmonic function $u(x,t)$ in R^{n+1}_+,

$$\|S(u)\|_{L^p} \sim \|u^*\|_{L^p} \quad \text{for } p > 0 .$$

We begin with the estimate $\|u^*\|_p \le C_p \|S(u)\|_p$, and to do this we shall show that

$$|\{u^* > C\alpha\}| \le C\left[\frac{1}{\alpha^2}\int_{S(u)\le\alpha} S^2(u)(x)\,dx + |\{S(u)>\alpha\}|\right].$$

From this our claim follows. This is because for $\lambda_g(\alpha) = |\{|g| > \alpha\}|$ we have

$$\|u^*\|^p_{L^p} \sim \int_0^\infty \alpha^{p-1}\lambda_{u^*}(\alpha)d\alpha \lesssim \int_0^\infty \alpha^{p-1}\left\{\frac{1}{\alpha^2}\int_0^\alpha \beta\lambda_{S(u)}(\beta)d\beta + \lambda_{S(u)}(\alpha)\right\}d\alpha$$

$$\le \int_0^\infty \beta\lambda_{S(u)}(\beta)\int_\infty^\infty \alpha^{p-3}d\alpha d\beta + \int_0^\infty \alpha^{p-1}\lambda_{S(u)}(\alpha)d\alpha .$$

Assuming $p < 2$ as we clearly may, this is

$$\lesssim \int_0^\infty \beta^{p-1}\lambda_{S(u)}(\beta)d\beta \sim \|S(u)\|^p_p .$$

To prove the estimate on $|\{u^* > C\alpha\}|$, we set the notation that $\tilde{E} = \left\{M(\chi_E) > \frac{1}{2}\right\}$, and then claim

I. $\displaystyle\int_{S(u)\le\alpha} S^2(u)(x)\,dx \ge c\iint_R |\nabla u(y,t)|^2\, t\; dt\, dy$ where $\displaystyle\mathfrak{R} = \bigcup_{x\notin\{S(u)>\alpha\}}\Gamma(x)$.

In fact, if $(y,t)\in\mathfrak{R}$ then

$$|B(y,t)\cap\{S(u)>\alpha\}| < \tfrac{1}{2}\,|B(y,t)| .$$

Then

$$\int_{S(u)\le\alpha} S^2(u)(x)\,dx = \int_{x\notin\{S(u)\le\alpha\}}\left(\int_{T(x)} |\nabla u|^2(y,t)t^{1-n}dy\,dt\right)dx$$

(4.1)

$$= \iint_{R^{n+1}_+} |\nabla u|^2(y,t)t^{1-n}|\{x\,|(y,t)\in\Gamma(\cancel{x}),\, x\notin\{S(u)>\alpha\}\}|\,dy\,dt\ .$$

But for $(y,t)\in R$,

$$|\{x\,|(y,t)\in\Gamma(x), x\notin\{S(u)>\alpha\}\}| = |B(y,t)\cap{}^c\{S(u)>\alpha\}| \ge \frac{1}{2}\,|B(y,t)|$$

and (4.1) is

$$(4.1)\ge c\iint |\nabla u|^2(y,t)\,t\,dy\,dt$$

as claimed.

II. The next step is to write $|\nabla u|^2 = \Delta(u^2)$, and apply Green's theorem to $\mathfrak{R}$:

$$\iint_R \Delta(u^2)(y,t)\,t\,dy\,dt = \int_{\partial R} \frac{\partial(u^2)}{\partial n}\,t - u^2\,\frac{\partial(t)}{\partial n}\,d\sigma\,.$$

Now $\frac{-\partial t}{\partial n}\ge c>0$ for some c so the above gives

$$\int_{\partial R} u^2 d\sigma \le C\left\{\int_{S(u)\le\alpha} S^2(u)(x)\,dx + \int_{\partial R} u(\nabla u)\,t\,d\sigma\right\}.$$

Since, for purposes of all estimates we may assume that u is rather nice, we may assume $u(\nabla u)t$ vanishes at $t=0$, so

$$\int_{\partial R} u(\nabla u)t\, d\sigma = \int_{\partial R} u(\nabla u)t\, d\sigma$$

where ∂R is the part of ∂R above $\{S(u) > \alpha\}$. It is not hard to see that $|\nabla u|t \leq \alpha$ on ∂R so that

$$\int_{\partial R} |u|\, |\nabla u|t\, d\sigma \leq \alpha|\{S(u) > \alpha\}|^{1/2} \left(\int_{\partial R} u^2 d\sigma\right)^{1/2} .$$

Putting all of our estimates together, we see that

$$\int_{\partial R} u^2 d\sigma \leq C \left\{ \int_{S(u) \leq \alpha} S^2(u)(x)dx + \alpha^2\, |S(u) > \alpha\}| \right\} .$$

III. Next we wish to define a function $\tilde{f}$ by $\tilde{f}(x) = u(x, \tau(x))$ where $(x, \tau(x)) \in \partial R$ defines the function τ. We claim that in the region R

$$|u| \leq P[\tilde{f}] + C\alpha . \tag{4.2}$$

This is done by harmonic majorization. It is enough to show this on ∂R, and this in turn is just saying that for any point $p \in \partial R$, $|U(p)|$ is dominated by the average over a relative ball on ∂R of $u + C\alpha$. This follows from the estimate $|\nabla u|t \leq C\alpha$ on ∂R. Anyway, from (4.2) we have, for $x \notin \{S(u) > \alpha\}$, $u^*(x) \leq CP[f]^*(x) + C\alpha$, so that finally

$$|\{u^* > C'\alpha\}| \leq \{M\tilde{f}(x) > \alpha\}| \leq \frac{C}{\alpha^2} \|\tilde{f}\|_2^2$$

$$\leq \frac{C}{\alpha^2} \int_{S(u) \leq \alpha} S^2(u)(x)\, dx + C|\{S(u) > \alpha\}| .$$

This completes the proof.

The proof that $\|u^*\|_p \le C_p\|S(u)\|_p$ which we just gave has been lifted from Charles Fefferman and E. M. Stein's Acta paper [18]. To prove the reverse inequality we want to go via a different route, and we shall follow Merryfield here [20]. We prove the following lemma. In the next lecture we show how this lemma proves $\|u^*\|_p \ge C_p\|S(u)\|_p$.

LEMMA. *Let* $f(x)$ *and* $g(x) \in L^2(R^n)$, *and suppose* $\phi \in C_c^\infty(R^n)$ *radial and* $u = P[f]$. *Then*

$$\iint_{R_+^{n+1}} |\nabla u|^2(x,t)|g * \phi_t(x)|^2 t\, dt\, dx$$

$$\le \int_{R^n} f^2(x)\phi^2(x)\, dx + \iint_{R_+^{n+1}} u^2(x,t)|g * \psi_t(x)|^2 \frac{dt}{t}\, dx$$

for some $\psi \in C_c^\infty(R^n)$ *with* $\int\psi = 0$ (ψ *real-valued*).

Proof.

$$\iint_{R_+^{n+1}} \Delta(u^2)(x,t)|g * \phi_t(x)|^2 t\, dt\, dx = \int_{R_+^{n+1}} \nabla \cdot \nabla (u^2)|g*\phi_t(x,t)|^2 t\, dt\, dx$$

$$= -\iint_{R_+^{n+1}} \nabla(u^2)(x,t) \cdot \nabla[(g * \phi_t(x,t)^2)t]dt\, dx = -2\iint_{R_+^{n+1}} u(x,t)\nabla u(x,t) \cdot \nabla[(g * \phi_t)^2](x,t)\, t\, dt\, dx$$

$$-2\iint_{R_+^{n+1}} u(x,t)\frac{\partial u}{\partial t} u(x,t)(g * \phi_t)^2(x,t)\, dt\, dx$$

$$= -2\iint_{R_+^{n+1}} u\nabla u \cdot 2(g*\phi_t)(x)\nabla[g*\phi_t(x)]t\, dt\, dx - 2\iint_{R_2^{n+1}} u\frac{\partial u}{\partial t}(g*\phi_t)^2 dt\, dx = I + II ,$$

where

$$I = \iint_{R_+^{n+1}} u(t\nabla(g*\phi_t))\, t^{-1/2} \cdot \nabla u(g*\phi_t)\, t^{1/2} dt\, dx$$

$$\leq \left(\iint_{R_+^{n+1}} u^2 |g*\psi_t|^2 \frac{dt}{t}\, dx \right)^{1/2} \left(\iint_{R_+^{n+1}} |\nabla u|^2 |g*\phi_t(x)|^2\, t\; dt\, dx \right)^{1/2} ,$$

and

$$II = \iint_{R_+^{n+1}} u \frac{\partial u}{\partial t} (g*\phi_t)^2\, dt\, dx = -\iint_{R_+^{n+1}} u \frac{\partial}{\partial t} (u \cdot (g*\phi_t)^2)\, dt\, dx + \int_{R^n} u^2(x,0) g^2(x)\, dx$$

$$= -\iint_{R_+^{n+1}} u \frac{\partial u}{\partial t} (g*\phi_t)^2\, dt\, dx - \iint u^2 2(g*\phi_t) \frac{\partial}{\partial t} (g*\phi_t)\, dt\, dx + \int_{R^n} f^2 g^2 dx\ .$$

We see that

$$2 \iint_{R_+^{n+1}} u \frac{\partial u}{\partial t} (g*\phi_t)^2 dt\, dx \leq \iint_{R_+^{n+1}} u^2 (g*\phi_t) \frac{\partial}{\partial t} (g*\phi_t) dt\, dx + \int_{R^n} f^2 g^2\, dx$$

but

$$\iint_{R_+^{n+1}} u^2 (g*\phi_t) \frac{\partial}{\partial t} (g*\phi_t) dt\, dx = \iint_{R_+^{n+1}} u^2 (g*\phi_t)\, t \frac{\partial}{\partial t} (g*\phi_t) \frac{dt}{t}\, dx\ .$$

But

$$t\frac{\partial}{\partial t}(g*\phi_t) = \left[\sum_i \frac{\partial}{\partial x_i}(x_i\phi)\right]_t * g = \sum_i t\frac{\partial}{\partial x_i}[(x_i\phi)_t * g]$$

so

$$S = \sum \iint -2u\frac{\partial u}{\partial x_i}(g*\phi_t)[(x_i\phi)_t * g]\,dt\,dx + \int -u^2 t\frac{\partial}{\partial x_i}(g*\phi_t)(g*(x_i\phi))\frac{dt\,dx}{t}$$

$$\leq \sum\left\{\left(\iint u^2(g*(x_i\phi)_t)^2\frac{dx\,dt}{t}\right)^{1/2}\left(\iint |\nabla u|^2(g*\phi_t)^2\,t\,dt\,dx\right)^{1/2}\right.$$

$$\left. + \left(\iint u^2\left(g*\left(\frac{\partial\phi}{\partial x_i}\right)_t\right)^2\frac{dt\,dx}{t}\right)^{1/2}\left(\iint u^2(g*(x_i\phi))^2\frac{dt\,dx}{t}\right)^{1/2}\right\}.$$

Putting this together gives

$$\iint |\nabla u|^2(g*\phi_t)^2\,t\,dx\,dt$$

$$\leq C\left[\sum\left\{\iint u^2\left(g*\left(\frac{\partial\phi}{\partial x_i}\right)_t\right)^2\frac{dt\,dx}{t} + \iint u^2(g*(x_i\phi)_t)^2\frac{dt\,dx}{t}\right\} + \int_{R^n} f^2g^2\right]$$

$$+ \iint u^2\left(g*\sum\frac{\partial}{\partial x_i}(\alpha_i\phi)\right)^2\frac{dt\,dx}{t}.$$

5. *More on* H^p *spaces*

At this point we wish to discuss the theory of multi-parameter H^p spaces and BMO. We saw, in the last lecture, that $H^p(R^n)$ could be defined either by maximal functions or by Littlewood-Paley-Stein theory. All of these spaces, H^p and BMO were invariant under the usual dilations on R^n, $x \to \delta x$, and this is hardly a surprise, since they can be defined by the maximal functions and singular integrals which are invariant under these dilations. Here we shall define H^p and BMO spaces which are invariant under the dilations (in R^2) $(x_1,x_2) \to (\delta_1 x_1, \delta_2 x_2)$, $\delta_1, \delta_2 > 0$. For convenience we shall work in R^2 but all of this could just as well be carried out in $R^n \times R^m$, $n, m > 1$. We shall call our H^p and BMO spaces "product H^p and BMO" and denote them by $H^p(R^2_+ \times R^2_+)$ and $BMO(R^2_+ \times R^2_+)$ so that we reserve $H^p(R^2)$ for the one-parameter space of functions on R^2.

Let $(x_1,x_2) \in R^2$ and denote by $\Gamma(x)$ the set

$$\{(y_1,t_1,y_2,t_2) \mid |y_i - x_i| < t_i,\ t_i > 0\} \subseteq R^2_+ \times R^2_+ .$$

Let $u(x,t)$ be a function in $R^2_+ \times R^2_+$, $x \in R^2$, $t \in R_+ \times R_+$, which is biharmonic, i.e., harmonic in each half plane separately. Then the non-tangential maximal function and area integral of u are defined by

$$u^*(x_1,x_2) = \sup_{(y,t) \in \Gamma(x_1,x_2)} |u(y_1,t_1,y_2,t_2)|$$

and

$$S^2(u)(x) = \iint_{\Gamma(x)} |\nabla_1 \nabla_2 u(y,t)|^2 dy_1 dy_2 dt_1 dt_2 .$$

More generally, if $\phi \in C_c^\infty(R^2)$ and if

$$\int \phi = 1,\ \phi_{t_1,t_2}(x_1,x_2) = t_1^{-1} t_2^{-1} \phi\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}\right)$$

then for a function f on R^2

$$f^*(x) = \sup_{(y,t) \in \Gamma(x)} |f * \phi_{t_1,t_2}(x)|$$

and if $\psi \in C_c^\infty(R^2)$ and

$$\begin{cases} \displaystyle\int \psi(x_1,x_2)\,dx_1 = 0 & \text{for all } x_2 \in R^1 \\[2em] \displaystyle\int \psi(x_1,x_2)\,dx_2 = 0 & \text{for all } x_1 \in R^1\,, \end{cases}$$

then

$$S_\psi^2(f)(x) = \iint\limits_{\Gamma(x)} |f*\psi_{t_1,t_2}(y_1,y_2)|^2\,dy_1\,dy_2\,\frac{dt_1}{t_1{}^2}\,\frac{dt_2}{t_2{}^2}\,.$$

Given $f(x_1,x_2)$, we define its bi-Poisson integral by $u(x_1,t_1,x_2,t_2) = P[f](x,t) = f*P_{t_1,t_2}(x_1,x_2)$, where the bi-Poisson (or just Poisson for short) kernel is defined by $P_{t_1,t_2}(x_1,x_2) = t_1^{-1}t_2^{-1}\,P\left(\frac{x_1}{t_1}\right)\,P\left(\frac{x_2}{t_2}\right)$ and where P is the 1-dimensional Poisson kernel. Then, of course, $P[f]$ is bi-harmonic in $R_+^2 \times R_+^2$.

In analogy with the 1-parameter case, we define $f \in H^p(R_+^2 \times R_+^2)$ if and only if $u^* \in L^p(R^2)$ where $u = P[f]$. It is not hard to see, just as in the single parameter case that for any $\phi \in C_c^\infty(R^2)$, $\int \phi = 1$,

$$\|f^*\|_{L^p} \sim \|u^*\|_{L^p} \quad \text{for } p > 0\,,$$

so that we may use any approximate identity which is sufficiently nice to define product H^p spaces. In terms of area integrals, we also have, for $\psi(x_1,x_2)$ suitably nontrivial, say ψ even in x_1, x_2 and not $\equiv 0$,

$$\|S_\psi f\|_{L^p} \sim \|S(u)\|_{L^p} \qquad p > 0\,,\ u = P[f]\,.$$

To complete the chain of equivalences, we would like to know that $\|S(u)\|_{L^p} \sim \|u^*\|_{L^p}$.

In fact, this is true, but is not obvious, and so we intend to present the proof here. The proofs are by iteration, but they are not of the same totally straightforward nature as the iteration in the Jessen-Marcinkiewicz-Zygmund theorem. Often, this is the case in the analysis of product domains, namely, the proof is by iteration, but this requires a different way of looking at the one-parameter case than one is used to.

Proof that of $S(u) \in L^p$, *then,* $u^* \in L^p$ (Gundy-Stein) [21]. We can assume that $u = P[f]$. Then since the 2-parameter area integral is invariant under taking Hilbert transforms in each variable separately, we see that we may write

$$f = f_{++} + f_{+-} + f_{-+} + f_{--}$$

where $\hat{f}_{++}$ is supported in $\xi_1, \xi_2 > 0$ and $\hat{f}_{+-}$ is supported in $\xi > 0$, $\xi_2 < 0$, etc., and we have $S(f_{\pm\pm}) \in L^p$. By reflection we may assume f is analytic (i.e., $P[f]$ is bi-analytic) and then show that $f^* \in L^p$. But for $u = [f]$ a bi-analytic function, we know that for $\alpha > 0$, $|u(x_1,t_1,x_2,t_2)|^\alpha$ is subharmonic in each half plane $(x_i,t_i) \in R^2_+$ separately; this implies that

$$|u(x,t)|^\alpha \leq P[|f(x)|^\alpha] .$$

If $\alpha < p$,

$$u^*(x)^\alpha \leq M(|f|^\alpha) \in L^{p/\alpha}(R^2), \text{ if } f \in L^p .$$

So what we must show is that

$$\|f\|_{L^p} \leq C_p \|S(f)\|_{L^p}, \quad p > 0 .$$

To show this let us define some notation. In R^1, if $f(x)$ has Poisson integral u, we let Q_t be the operator (or kernel) which takes f to $t\nabla u(x,t)$, so that

$$S^2(f)(x) = \iint_{\Gamma(x)} |f * Q_t(y)|^2 \frac{dy\,dt}{t^2} .$$

Then going back to our present situation where f is given on R^2, we define

$$Q_t^1 f(x_1,x_2) = \int_{-\infty}^{\infty} f(x_1 - y, x_2) Q_t(y)\,dy$$

and Q_t^2 similarly. Then let us define a Hilbert space valued function

$$F(x_1,x_2) \in L^2\left(\Gamma(0); \frac{dy\,dt}{t^2}\right)$$

by

$$F(x_1,x_2)(y_2,t_2) = Q_{t_2}^2 f(x_1,x_2+y_2) .$$

Now we know that in the one-parameter case $\|S^1 f\|_p \geq c_p \|f\|_p$ and a glance at the proof of this fact reveals that it remains valid for Hilbert space valued functions. Fix x_2. Then

$$\int_{-\infty}^{\infty} S^1(F)(x_1,x_2)^p\,dx_1 \geq c_p \int_{-\infty}^{\infty} |F(x_1,x_2)|^p_{L^2(\Gamma)}\,dx_1$$

and integrating this in x_2,

$$\iint_{R^2} S^1(F)(x_1,x_2)^p\,dx_1 dx_2 \geq c_p \iint_{R^2} |F(x_1,x_2)|^p_{L^2(\Gamma)}\,dx_1 dx_2 . \tag{5.1}$$

But fixing x_1, since $|F(x_1,x_2)|_{L^2(\Gamma)}$ is the value of the one-parameter integral of $f(x_1,\cdot)$ at x_2, we have

$$\int_{R^1} |F(x_1,x_2)|^p_{L^2(\Gamma)} dx_2 \geq c_p \int_{R^1} |f(x_1,x_2)|^p dx_2$$

and so (5.1) is greater than or equal to

$$c'_p \iint_{R^2} |f|^p dx_1 dx_2 .$$

On the other hand when the S^1 operator acts on the first variable we have

$$S^1(F)(x_1,x_2) = S(f)(x_1,x_2) ,$$

the two-parameter area integral of f.

Next, let us prove that $\|S(u)\|_{L^p} \leq c_p \|u^*\|_{L^p}$ [21]. The proof is a simple iteration of the one-parameter case given previously. To begin with, we recall Merryfield's lemma: Let $\phi \in C_c^\infty(R^1)$ be supported in $[-1,+1]$ and have $\int \phi = 1$. Then there exists $\psi \in C_c^\infty(R^1)$ whose support is also contained in $[-1,+1]$ with $\int \psi = 0$ and such that if $u = P[f]$,

$$\iint_{R^2_+} |\nabla u|^2(x,t)\,(g * \phi_t(x))^2\, t\, dt\, dx \leq C \left\{ \int_{R^1} f^2(x)\, g^2(x)\, dx + \right.$$
$$\left. + \iint_{R^2_+} u^2(x,t)\,(g * \psi_t(x))^2 \frac{dt}{t}\, dx \right\} .$$

Introduce the notation $t\nabla u(x,t) = Q_t f(x)$, $u(x,t) = P_t f(x)$, $g * \phi_t(x) = \widetilde{P}_t(g)(x)$ and $g * \psi_t(x) = \widetilde{Q}_t(g)(x)$, Q_t^i, $i = 1,2$, will denote the operator acting in the i^{th} variable. Then we estimate

$$(5.2)\qquad \iint\limits_{R^2_+\times R^2_+} [Q^1_{t_1}Q^2_{t_2}f(x_1,x_2)]^2[\tilde{P}^1_{t_1}\tilde{P}^2_{t_2}g(x_1,x_2)]^2\,\frac{dt_1dt_2}{t_1t_2}\,dx_1dx_2\,.$$

Fix x_2, t_2. Then (5.2) is

$$\leq \iint\limits_{x_2,t_2}\iint\limits_{x_1,t_2}(P^1_{t_1}Q^2_{t_2}f(x))^2(\tilde{Q}^1_{t_1}\tilde{P}^2_{t_2}g(x))^2\,\frac{dt\,dx}{t}$$

$$+\iint\limits_{x_2t_2}\int\limits_{x_1}[Q^2_{t_2}f(x)]^2[\tilde{P}^2_{t_2}g(x)]^2\,dx_1\,\frac{dt_2dx_1}{t_2} = \mathrm{I}+\mathrm{II}\,.$$

Now

$$\mathrm{I} = \iint\limits_{x_1t_1}\iint\limits_{x_2t_2}P^2_{t_2}P^1_{t_1}f(x))^2(\tilde{Q}^2_{t_2}\tilde{Q}^1_{t_1}g(x))^2\,\frac{dt\,dx}{t}$$

$$+\iint\limits_{x_1,t_1}\int\limits_{x_2}(P^1_{t_1}f(x))^2(\tilde{Q}_{t_1}g(x))^2\,dx_2\,\frac{dt_1}{t_1}\,dx_1\,.$$

Then

$$\mathrm{II} = \int\limits_{x_1}\iint(P^2_{t_2}f(x))^2(\tilde{Q}_{t_2}g(x))^2\,dx_2\,dt_2/t_2\,dx_1 + \iint\limits_{x_1x_2}f^2(x)\,g^2(x)\,dx_1dx_2\,.$$

So the inequality we seek in the product case is

$$(5.3)\quad \int_{(R^2_+)^2} (Q_t f)^2(x)(\tilde{P}_t g)^2(x)\,dxdt/t \le c\Bigg\{ \int_{(R^2_+)^2} (P_t f(x))^2(\tilde{Q}_t g(x))^2 dxdt/t$$

$$+ \int_{x_1\epsilon R^1}\left(\int_{(x_2,t_2)\epsilon R^2_+} (P^2_{t_2} f(x))^2 (\tilde{Q}^2_{t_2})^2(x)\,dx_2 dt_2/t_2\right)$$

$$+ \int_{x_2\epsilon R^1}\left(\int_{(x_1,t_1)\epsilon R^2_+} P^{(1)}_{t_1} f(x)^2 \cdot \tilde{Q}_{t_1} g(x)^2\,dx_1 dt_1/t_1\right) dx_2 + \int_{R^2} f^2(x)\,g^2(x)\,dx \Bigg\}.$$

It is now easy to see that $\|S(u)\|_{L^p} \le C_p \|u^*\|_{L^p}$, $p > 0$.

In order to simplify things a little, we shall take a modified definition of u^* in what follows, namely

$$u^*(x) = \sup_{(y,t)\epsilon \Gamma_{10^{10}}(x)} |u(y,t)|$$

where

$$\Gamma_{10^{10}}(x) = \{(y,t) \mid |y_i - x_i| < 10^{10} t_i,\ i=1,2\}.$$

This is an irrelevant change, since a trivial computation shows that $\|u^*\|_p$ is, for a larger aperture, no more than a constant times $\|u^*\|_p$ for a smaller aperture, the constant depending only on the apertures involved.

In (5.3), take $\phi(x) = 1$ for all $|x| < 1/3$, and $g(x) = \chi_{u^*(x)\le\alpha}(x)$. Let us estimate

$$\int_{M(\chi_{u^*>\alpha})<1/200} S^2(u)(x)\,dx \quad \text{when} \quad u = P[f]$$

$$\leq \iint |\nabla_1 \nabla_2 u|^2(y,t)|R(y,t) \cap \{M(\chi_{u^*>\alpha}) < 1/200\}|\,dy\,dt$$

$$\leq \iint_{R^*} |\nabla_1 \nabla_2 u|^2(y,t)\, t_1 t_2\, dy\,dt$$

where $R^* = \{(y,t)|\ |R(y,t) \cap \{u^* > \alpha\}| < \frac{1}{200}\ |R(y,t)|\}$ and where $R(y;t)$ is the rectangle in R^2 with sides parallel to the axes and with side lengths $2t_1, 2t_2$ centered at y. Notice that if $|R(y;t) \cap \{u^* > \alpha\}| < \frac{1}{100}\ |R(y;t)|$ then $g * \phi_t(y) = \tilde{P}_t g(y) > c$ for some $c > 0$. It follows that

$$\int_{M(\chi_{\{u^*>\alpha\}})<1/200} S^2(u)(x)\,dx \leq \iint_{(R^2_+)^2} |\nabla_1 \nabla_2 u|^2(y,t)\tilde{P}_t(g)^2(y)\,dy\, t_1 t_2\, dt$$

$$\leq C \left\{ \iint_{(R^2_+)^2} u^2(y,t)\tilde{Q}_t(g)^2(y)dy\, \frac{dt}{t_1 t_2} + \int_{y_1} \iint_{(y_2,t_2)\epsilon R^2_+} P^2_{t_2} f(y)^2 \cdot \tilde{Q}^2_{t_2}(g)^2(y)dy\, \frac{dt_2}{t_2} \right\}$$

$$+ \int_{y_2} \iint_{(y_1,t_1)\epsilon R^2_+} P^{(1)}_{t_1} f(y)^2 \cdot \tilde{Q}_{t_1}(g)^2(y)\,dy\, \frac{dt_1}{t_1} + \iint_{R^2} f^2 g^2\, dy$$

$= i + ii + iii + iv$.

Consider i: If $\tilde{Q}_t(g)(y) \neq 0$ then $u^*(x) \leq \alpha$ for some $x \epsilon R(y;t)$. But then $|u(y,t)| \leq \alpha$ so i is less than or equal to

$$\alpha^2 \iint_{(R_+^2)^2} \tilde{Q}_t(g)^2(y)dy \frac{dt}{t_1 t_2} = \alpha^2 \iint_{(R_+^2)^2} \tilde{Q}_t(1-g)^2(y)dy \frac{dt}{t} \le \alpha^2 \|1-g\|_{L^2}^2 \le \alpha^2 |\{u^* > \alpha\}| .$$

Consider ii: If $\tilde{Q}_{t_2}^{(2)}(g)(y_1,y_2) \neq 0$, then there exists x_2 such that $|x_2 - y_2| < t_2$ and $u^*(y_1,x_2) \le \alpha$. This implies that $|P_{t_2}^{(2)} f(y_1,y_2)| \le \alpha$ so ii is less than or equal to

$$\alpha^2 \int_{y_1} \left(\iint_{(y_2,t_2)} \tilde{Q}_{t_2}^{(2)}(g)^2(y) \frac{dt_2}{t_2} dy_2 \right) dy_1 .$$

Again

$$\int_{y_1} \left(\iint_{y_2,t_2} \tilde{Q}_{t_2}^{(2)}(g)^2(y) \frac{dt_2}{t_2} dy_2 \right) dy_1 = \int_{y_1} \left(\iint_{y_2,t_2} \tilde{Q}_{t_2}^{(2)}(1-g)^2(y) dy_2 \frac{dt_2}{t_2} \right) dy_1$$

$$\le \int_{y_1} \left(\int_{y_2} (1-g)^2(y_1,y_2) dy_2 \right) dy_1 \le |\{u^* > \alpha\}| .$$

(iii) is similar to (ii).

Finally, (iv) is less than or equal to

$$\int_{u^*(x) \le \alpha} f^2(x) dx \le \int_{u^*(x) \le \alpha} (u^*)^2(x) dx .$$

So we have

$$\int_{\{u^* \le \alpha\}} S^2(u)(x) dx \le C \quad \alpha^2 |\{u^* > \alpha\}| + \int_{u^*(x) \le \alpha} u^{*2}(x) dx$$

and we have seen before that this implies that

$$\|S(u)\|_{L^p} \leq C_p \|u^*\|_{L^p}, \quad 2 > p > 0 .$$

The next topic that we shall consider is that of duality of H^1 and BMO in the product setting. In the classical case there were four results which expressed this duality.

1) The characterization of Carleson measures μ for which the Poisson transform $f \to P[f]$ is bounded from $L^p(dx)$ to $L^p(d\mu)$, $p > 1$.

2) The characterization of functions in $BMO(R^1)$ by a condition on their Poisson integrals in terms of Carleson measures.

3) The characterization of functions in the dual of H^1 by the BMO condition.

4) The atomic decomposition of H^1.

Let us try to guess what the analogous theory should look like in product spaces. For simplicity we consider the dual of $H^1(R^2_+ \times R^2_+)$. Then what should an element of $BMO(R^2_+ \times R^2_+)$ look like? We might look at tensor products of functions in $BMO(R^1)$ to get a feel for the answer. So, for example if ϕ_1 and ϕ_2 are in $BMO(R^1)$ then $\phi_1(x_1)\phi_2(x_2)$ might be our model. Of course, this function $\phi(x_1,x_2)$ satisfies

$$(5.4) \qquad \frac{1}{|R|}\int_R |\phi(x_1,x_2)-c_1(x_1)-c_2(x_2)|^2\, dx_2 dx_2 \leq C$$

for the appropriate choice of functions c_1 and c_2 of the x_1,x_2 variable.

A Carleson measure in $R^2_+ \times R^2_+$ would be a non-negative measure μ for which

$$(5.5) \qquad \iint_{(R^2_+)^2} P[f]^p\, d\mu \leq C_p \iint_{R^2} |f(x)|^p\, dx\,, \quad p > 1\,,$$

where P is the bi-Poisson integral. The obvious guess is that μ

satisfies (5.5) if and only if $\mu(S(R)) \leq C|R|$ for all rectangles $R \subseteq R^2$ with sides parallel to the axes, where the Carleson region $S(R)$ is defined by $S(I \times J) = S(I) \times S(J)$ for $R = I \times J$. In terms of these Carleson measures, it is not hard to show that ϕ satisfies (5.4) if and only if its bi-Poisson integral u satisfies

$$d\mu = |\nabla_1 \nabla_2 u|^2(y,t)\, t_1 t_2 dt \quad \text{is a Carleson measure.}$$

And finally, all of this in some sense is equivalent to asserting that every $f \epsilon H^1(R^2_+ \times R^2_+)$ can be written as $\Sigma \lambda_k a_k$ where $\Sigma |\lambda_k| \leq C\|f\|_{H^1}$ and $a_k(x_1,x_2)$ are "atoms," i.e., a_k is supported in a rectangle $R_k = I_k \times J_k$ such that

$$\begin{cases} \displaystyle\int_{I_k} a_k(x_1,x_2)\,dx_1 = 0 & \text{for all } x_2 \\[2ex] \displaystyle\int_{J_k} a_k(x_1,x_2)\,dx_2 = 0 & \text{for all } x_1 \end{cases}$$

and

$$\|a_k\|_2 \leq \frac{1}{|R_k|^{1/2}} \,.$$

In 1974 [22], L. Carleson showed that $\mu(S(R)) \leq C|R|$ was not sufficient to guarantee the inequality

$$\iint_{(R^2_+)^2} |P[f]|^p\, d\mu \leq C_p \int_{R^2} |f|^p\, dx \,.$$

From here it is not difficult to produce examples of functions $\phi(x_1,x_2)$ which satisfy

$$\frac{1}{|R|}\int_R |\phi(x_1,x_2)-C_1(x_1)-C_2(x_2)|^2 dx_1dx_2 \leq C$$

where C_1, C_2 depend on R, yet $\phi \notin L^p(R^2)$ for any $p > 2$. Therefore, this condition is not strong enough to force ϕ to belong to the dual of H^1.

In other words the simple picture of the structure of $H^1(R^2_+ \times R^2_+)$ and $BMO(R^2_+ \times R^2_+)$ suggested above as the obvious guess is completely wrong.

Rather one considers the role of rectangles to be played instead by arbitrary open sets. Although this may seem a bit frightening at first glance, it turns out, and this is of course the final test of the theory, that nearly all the classical theory of H^p and BMO can easily be carried out using the approach suggested here.

By way of introduction, we shall prove that for any function $\phi \in H^1(R^2_+ \times R^2_+)^*$, if $u = P[\phi]$ we have a Carleson condition with respect to open sets satisfied by the appropriate measure. To describe this result, we make the following definition ([23], [24], and [25]).

Let $\Omega \subseteq R^2$ be an arbitrary open set, and let $R(y;t)$ be the rectangle in R^2 centered at $(y_1,y_2) = y$ and with side lengths $2t_1$ and $2t_2$. Then $S(\Omega)$ the Carleson region above Ω is defined as

$$S(\Omega) = \bigcup_{R \subseteq \Omega} S(R) = \{(y,t) \in (R^2_+)^2 \,|\, R(y;t) \subseteq \Omega\}.$$

Then we say that $\mu \geq 0$ in $(R^2_+)^2$ is a Carleson measure if and only if $\mu(S(\Omega)) \leq C|\Omega|$ for every open set $\Omega \subseteq R^2 \cdot f \in H^1(R^2_+ \times R^2_+)^*$ if and only if for $u = P[f]$,

$$d\mu = |\nabla_1 \nabla_2 u|^2 t_1 t_2 dt_1 dt_2 dy_1 dy_2 \text{ is a Carleson measure.}$$

In fact, this follows immediately from the inequality (5.3). To see this, notice that $|\nabla_1 \nabla_2 u|$ is invariant under the Hilbert transform H_{x_i} $(i = 1,2)$ so that if we prove this when $f \in L^\infty(R^2)$, we will have proven it also when f is of the form

$$g_1 + H_{x_1}g_2 + H_{x_2}g_3 + H_{x_1}H_{x_2}g_4 \quad g_i \in L^\infty .$$

A function $a(x)$ on R^2 will be in $H^1(R^2_+ \times R^2_+)$ if and only if $a \in L^1(R^2)$, $H_{x_1}a, H_{x_2}a$, and $H_{x_1}H_{x_2}a \in L^1$.

In fact, if $a \in H^1(R^2_+ \times R^2_+)$ then $S(a) \in L^1(R^2)$ hence so are $S(H_{x_1}a)$, $S(H_{x_2}a)$ and $S(H_{x_1}H_{x_2}a)$; therefore $H_{x_i}a, H_{x_1}H_{x_2}a \in L^1$. Conversely, if $a, H_{x_i}a$, and $H_{x_1}H_{x_2}a \in L^1(R^2)$ then we can form F_{++}, F_{+-}, F_{-+}, and $F_{--} \in L^1(R^2)$ such that $a = \Sigma F_{\pm\pm}$ and reflections of the $F_{\pm\pm}$ are boundary values of bi-analytic functions. A bianalytic function F with (distinguished) boundary values in $L^1(R^2)$ has $F^* \in L^1$ by a subharmonicity argument applied to $|F|^\alpha$, $\alpha < 1$. So $a^* \in L^1$ and $a \in H^1$. Let $\Phi \in H^1(R^2_+ \times R^2_+)^*$. Define a map from $H^1(R^2_+ \times R^2_+) \xrightarrow{\vartheta} \bigoplus_{i=1}^{4} L^1(R^2)_i$ by

$$\vartheta(f) = (f, H_{x_1}f, H_{x_2}f, H_{x_1}H_{x_2}f) .$$

Then $\|\vartheta f\|_{\oplus L^1} \sim \|f\|_{H^1}$. ϑ is obviously one to one, so $\vartheta^{-1} = \mathfrak{I}$ exists and is bounded on $\mathrm{Im}(\vartheta)$. The map $\Phi \circ \mathfrak{I}$ extends, by Hahn-Banach to an element of the dual $\oplus L^1 = \oplus L^\infty$. Then

$$\Phi(f) = \Phi \circ \mathfrak{I}(f, H_{x_1}f, H_{x_2}f, H_{x_1}H_{x_2}f)$$

$$= \int fg_1 + H_{x_1}f \cdot g_2 + H_{x_2}f \cdot g_3 + H_{x_1}H_{x_2}f \cdot g_4$$

$$= \int f \cdot (g_1 H_{x_1}g_2 + H_{x_2}g_3 + H_{x_1}H_{x_2}g_4) \quad f \in H^1 .$$

Thus every element of $(H^1)^*$ is of the form

$$g_1 + H_{x_1}g_2 + H_{x_2}g_3 + H_{x_1}H_{x_2}g_4 \quad g_i \in L^\infty .$$

So it suffices to show that if $f \in L^\infty(R^2)$ with $u = P[f]$ then

$$d\mu = |\nabla_1 \nabla_2|^2(y,t)\, t_1 t_2 \, dy\, dt \quad \text{is a Carleson measure.}$$

But in (5.3), take $g = \chi_\Omega(x_1,x_2)$, and notice that if $\int \phi = 1$, $\operatorname{supp} \phi(x) \subseteq [-1,1]$ then $\widetilde{P}_t g(x) = 1$ if $(x,t) \in S(\Omega)$. This is because for such (x,t), $R(x;t) \subseteq \Omega$ and $g * \phi_t(x) = \int_{R^2} \phi_t(x-u)\,du = 1$. It follows from (5.3) that

$$\iint\limits_{S(\Omega)} |\nabla_1 \nabla_2 u|^2 t_1 t_2 \, dx\, dt \le C\|f\|_\infty^2 \left(\iint |\widetilde{Q}_t g|^2 | \frac{dt}{t}\, dx \right.$$

$$\left. + \int_{x_1} \left(\iint\limits_{(x_2 t_2)} |\widetilde{Q}_{t_2} g|^2 \frac{dt_2}{t_2} dx_2 \right) dx_1 + \int_{x_2} \left(\iint\limits_{(x_1, t_1)} |\widetilde{Q}_{t_1} g|^2 \frac{dt_1}{t_1}\, dx_1 \right) dx_2 + \int g^2 dx \right)$$

$$\le C\|f\|_\infty^2 \|g\|_2^2 \le C\|f\|_\infty^2 |\Omega| \, .$$

6. *Duality of* H^1 *and* BMO *and the atomic decomposition*

In this lecture we shall consider in greater detail the spaces $H^p(R_+^2 \times R_+^2)$ and $BMO(R_+^2 \times R_+^2)$, which we discussed briefly in section 5. There we saw that in product spaces, the most obvious guesses at characterizations of H^p atoms of BMO failed. In order to circumvent these difficulties we must take a slightly different approach than we are used to in the classical 1-parameter case.

In what follows we shall be working with functions in $H^p(R_+^2 \times R_+^2)$ or $BMO(R_+^2 \times R_+^2)$ only. The theory for $R_+^2 \times R_+^2 \times \cdots \times R_+^2$ or for $R_+^{n+1} \times R_+^{m+1}$ is quite similar and only requires minor changes. Now let $\mathfrak{R}$ be the family of all rectangles with sides parallel to the axes and $\mathfrak{R}_d$ be the subfamily of $\mathfrak{R}$ whose sides are dyadic intervals.

If $f(x_1,x_2)$ is a sufficiently nice function on R^2, and $\psi \in C^\infty(R^1)$, ψ is even, $\psi \not\equiv 0$ real valued and $\operatorname{supp}(\psi) \subseteq [-1,1]$, and ψ has a large number of moments vanishing, then for

$$\psi_{t_1t_2}(x_1,x_2) = \psi\left(\frac{x_1}{t_1}\right)\psi\left(\frac{x_2}{t_2}\right) t_1^{-1}t_2^{-1}, \quad \int_0^\infty |\hat{\psi}(\xi)|^2 d\xi/\xi = 1$$

we have

$$f(x_1,x_2) = \iint_{R_+^2\times R_+^2} f*\psi_{t_1,t_2}(y_1,y_2)\psi_{t_1,t_2}(x_1-y_1,x_2-y_2)\,dy_1dy_2\,\frac{dt_1dt_2}{t_1t_2}$$

In fact, taking Fourier transforms of both sides, for the right-hand side we have

$$\iint_{R_+^2\times R_+^2} \hat{f}(\xi)|\hat{\psi}(t_1\xi_1,t_2\xi_2)|^2\,\frac{dt_1dt_2}{t_1t_2} = \hat{f}(\xi)\int_0^\infty\int_0^\infty |\hat{\psi}(t_1\xi_1,t_2\xi_2)|^2\,\frac{dt_1dt_2}{t_1t_2} = \hat{f}(\xi)$$

We can use this representation to decompose the function f as follows: $R \in \mathfrak{R}_d$. Set $\mathfrak{A}(R) = \{(y,t) \in R_+^2\times R_+^2 | y \in R,\ \ell_1 < t_1 \leq 2\ell_i$ where ℓ_i, $i = 1,2$ is the side length of R in the x_i direction$\}$. Since $R_+^2\times R_+^2 = \bigcup_{R\in\mathfrak{R}_d} \mathfrak{A}(R)$, if we define

$$f_R(x_1,x_2) = \iint_{\mathfrak{A}(R)} f(y,t)\,\psi_{t_1t_2}(x_1-y_1,x_2-y_2)\,dy\,\frac{dt}{t_1t_2}$$

where $f(y,t) = f*\psi_t(y)$, then $f = \sum_{R\in\mathfrak{R}_d} f_R$, and each f_R is supported in $\tilde{R}$ the double of R and has the property that

$$\begin{cases} \displaystyle\int_{\tilde{I}} f_R(x_2,x_2)\,dx_1 = 0 & \text{for all } x_2 \\ \displaystyle\int_{\tilde{J}} f_R(x_1,x_2)\,dx_2 = 0 & \text{for all } x_1 \end{cases}$$

where $\tilde{R} = \tilde{I} \times \tilde{J}$.

It will be convenient to define a norm $|\ |_R$ on functions supported on a rectangle R, as follows.

$$|f|_R = \sum_{|\alpha|=0}^{N} \left\| \frac{\partial^{\alpha} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \right\|_{\infty} |I|^{\alpha_1} |J|^{\alpha_2}$$

where N is a large integer. With these preliminaries we can pass to a theorem characterizing $(H^1)^*$ in a number of useful ways.

THEOREM [25]. *For a function on* R^2 *the following are equivalent:*

(1) $\phi \in H^1(R^2_+ \times R^2_+)^*$.

(2) $\phi = g_1 + H_{x_1}(g_2) + H_{x_2}(g_3) + H_{x_1}H_{x_2}(g_4)$ *for some* g_1, g_2, g_3 *and* g_4 *in* $L^\infty(R^2)$.

(3) *If* $u = P[\phi]$ *in* $R^2_+ \times R^2_+$, *then*

$$\iint_{S(\Omega)} |\nabla_1 \nabla_2 u|^2(y,t)\, t_1 t_2\, dt \le C|\Omega|, \quad \textit{for all open sets } \Omega \subseteq R^2 .$$

(4) *If* $\phi(y,t) = \phi * \psi_t(y)$, *then*

$$\iint_{S(\Omega)} |\phi(y,t)|^2 dy\, \frac{dt}{t} \le C|\Omega|, \quad \textit{for all open sets } \Omega \subseteq R^2 .$$

(5) *ϕ can be written in the form* $\sum_{R \epsilon R_d} c_R b_R$ *where* $b_R(x_1,x_2)$ *are supported in* $\tilde{R}$, $|b_R|_R \leq 1$ *and* $\sum_{R \subseteq \Omega} c_R^2 \leq C|\Omega|$ *for all* Ω *open in* R^2.

Proof. To begin with, we proved in section 5 that (1) $\Longrightarrow$ (2). It is also trivial that (2) $\Longrightarrow$ (1), since if $f \epsilon H^1(R_+^2 \times R_+^2)$,

$$\int f(x) H_{x_1} H_{x_2}(g)(x)\,dx = \int H_{x_1} H_{x_2}(f)(x)\, g(x)\,dx$$

and since $f \epsilon H^1$, $H_{x_1} H_{x_2}(f) \epsilon L^1$.

Next, we recall that (2) $\Longrightarrow$ (3) was also proven in the preceding section.

Now we claim that (3) or (4) implies (1). We show that (4) implies (1), the other proof being similar. We do this via the atomic decomposition of H^1 which we shall describe here only enough to derive our implication. We shall present the decomposition of H^1 in greater detail later. Let $f \epsilon H^1(R_+^2 \times R_+^2)$. Then $S_\psi(f) \epsilon L^1(R^2)$ (this follows by vector iteration, just as in the argument that $S(f) \epsilon L^1$ implies $f \epsilon L^1$).

Consider the sets $\Omega_k = \{S_\psi(f) > 2^k\}$, $k \epsilon Z$. Set

$$a_k(x_1,x_2) = \sum_{\substack{R \epsilon \mathfrak{R}_d \\ |R \cap \Omega_k| > 1/2|R| \\ |R \cap \Omega_{k+1}| < 1/2|R|}} f_R(x) .$$

Then, as we shall elaborate later on $\tilde{a}_k(x_1,x_2) = \dfrac{a_k(x_1,x_2)}{2^k|\tilde{\Omega}_k|}$ is an $H^1(R_+^2 \times R_+^2)$ atom where

$$\tilde{\Omega}_k = \{(x_1,x_2) | M^{(2)}(\chi_{\Omega_k})(x_1,x_2) > 1/10\} .$$

Then $f = \Sigma \lambda_k a_k$ where $\lambda_k = 2^k|\tilde{\Omega}_k|$, and by the strong maximal theorem $|\tilde{\Omega}_k| \le C|\Omega_k|$ so that

$$\sum \lambda_k \le C\|S_\psi(f)\|_{L^1} \le C'\|f\|_{H^1} .$$

Now consider $\phi(x_1,x_2)$ satisfying (4). Then it will be enough to show that

$$\left| \int_{R^2} \tilde{a}_k(x_1,x_2)\,\phi(x_1,x_2)\,dx \right| \le C$$

and then simply sum over k. But

$$\int_{R^2} a_k(x_1,x_2)\,\phi(x_1,x_2)\,dx_1 dx_2 = \frac{1}{2^k|\tilde{\Omega}_k|}\int \sum f_R(x)\,\phi(x)\,dx \tag{6.1}$$

where the sum is taken over

$$\mathfrak{R}_k = \left\{ R \in \mathfrak{R}_d \mid |R \cap \Omega_k| > \tfrac{1}{2}\,|R| \ \text{ but } \ |R \cap \Omega_{k+1}| \le \tfrac{1}{2}\,|R| \right\} .$$

Then (6.1) becomes

$$\sum_{R \in \mathfrak{R}_k} \phi(x) \iint_{\mathfrak{U}(R)} f(y,t)\,\psi_t(x-y)\,\frac{dy\,dt}{t_1 t_2}\,dx \cdot \frac{1}{2^k|\tilde{\Omega}_k|}$$

$$= \frac{1}{2^k|\tilde{\Omega}_k|} \bigcup_{R \in \mathfrak{R}_k} \iint_{\mathfrak{U}(R)} f(y,t)\,\phi(y,t)\,dy\,\frac{dt}{t_1 t_2}$$

$$\le \frac{1}{2^k|\tilde{\Omega}_k|} \left(\bigcup_{R \in \mathfrak{R}_k} \iint_{\mathfrak{U}(R)} |f(y,t)|^2 dy\,\frac{dt}{t_1 t_2} \right)^{1/2} \left(\bigcup_{R \in \mathfrak{R}_k} \iint_{\mathfrak{U}(R)} |\phi(y,t)|^2 dy\,\frac{dt}{t_1 t_2} \right)^{1/2} .$$

$$(6.2)\qquad \int_{\tilde{\Omega}_k/\Omega_{k+1}} |S_\psi(f)^2(x)\,dx \geq \bigcup_{R\in\mathfrak{R}_k} \iint_{\mathfrak{A}(R)} |f(y,t)|^2 dy\,\frac{dt}{t_1 t_2}\,.$$

To show this

$$\int_{\tilde{\Omega}_k/\Omega_{k+1}} S_\psi^2(f)(x)\,dx = \int_{x\in\tilde{\Omega}_k/\Omega_{k+1}} \left(\iint_{\Gamma(x)} |f(y,t)|^2 dy\,\frac{dt}{t_1^2 t_2^2}\right) dx$$

$$\geq \iint |f(y,t)|^2 \,|\{x\in\tilde{\Omega}_k/\Omega_{k+1}\,|(y,t)\in\Gamma(x)\}|\,dy\,\frac{dt}{t_1^2 t_2^2}\,.$$

Suppose that $(y,t)\in\mathfrak{A}(R)$, $R\in\mathfrak{R}_k$. Then if the aperture of Γ is large enough, then $(y,t)\in\Gamma(x)$ for all $x\in R$. Since

$$|\{x\in(\tilde{\Omega}_k/\Omega_{k+1})\cap R\}| \geq \frac{1}{2}\,|R|\,,\quad R\in\mathfrak{R}_k$$

we see that

$$|\{x\in\tilde{\Omega}_k/\Omega_{k+1}, (y,t)\in\Gamma(x)\}| \geq \frac{1}{2}\, t_1 t_2 \quad (\text{observe that } t_1 t_2 \sim |R|\,)$$

for $(y,t)\in \bigcup_{R\in R_k} \mathfrak{A}(R)$, and this proves (6.1).

But then

$$\int_{\tilde{\Omega}_k/\Omega_{k+1}} S_\psi(f)^2(x)\,dx \leq (2^{k+1})^2|\tilde{\Omega}_k|$$

and combining this with (6.1) yields

$$\bigcup_{R\in\mathfrak{R}_k} \iint_{\mathfrak{A}(R)} |f(y,t)|^2 dy\,\frac{dt}{t_1^2 t_2^2} \leq C2^k\,|\tilde{\Omega}_k|\,.$$

As for

$$\bigcup_{R \epsilon \mathfrak{R}_k} \iint_{\mathfrak{A}(R)} |\phi(y,t)|^2 dy \frac{dt}{t_1 t_2},$$

if we observe that for any $R \epsilon \mathfrak{R}_k$, $\mathfrak{A}(R)$ is contained in $S(\tilde{\Omega}_k)$, we get

$$\bigcup_{R \epsilon \mathfrak{R}_k} \iint_{\mathfrak{A}(R)} |\phi(y,t)|^2 dy \frac{dt}{t_1 t_2} \leq \iint_{S(\tilde{\Omega}_k)} |\phi(y,t)|^2 dy \frac{dt}{t_1 t_2} \leq C|\tilde{\Omega}_k|$$

by (4). This shows that $|\int \tilde{a}_k \cdot \phi \, dx| \leq C$ and completes the proof that (4) implies (1).

We shall show next that (2) implies (4). Let $g \epsilon L^\infty(R^2)$. We claim that if $\Omega \subset R^2$, then

$$\iint_{S(\Omega)} |g(y,t)|^2 dy \frac{dt}{t_1 t_2} C \|g\|_\infty^2 |\Omega| ,$$

where $g(y,t) = g * \psi_{t_1 t_2}(y)$. To show this, observe that since $\text{supp}(\psi) \subseteq [-1,1]$, $\text{supp}(\psi_{t_1,t_2}(\cdot - y)) \subseteq R(y,t)$. Hence, if $(y,t) \epsilon S(\Omega)$, $g * \psi_t(y) = (g\chi_\Omega) * \psi_t(y)$ and so

$$\iint_{S(\Omega)} |g(y,t)|^2 dy \frac{dt}{t_1 t_2} \leq \iint_{(R_+^2)^2} |(g\chi_\Omega)(y,t)|^2 dy \frac{dt}{t_1 t_2} .$$

An easy application of Plancherel's formula says that this is, in turn,

$$\|g\chi_\Omega\|_2^2 \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} .$$

Since, $\int_{-1}^{+1} \psi = 0$,

$$\begin{cases} \hat{\psi}(0) = 0 \text{ and } \hat{\psi} \in C^{\infty} \\ \hat{\psi}(\xi) = 0(|\xi|^{-N}) \text{ as } |\xi| \to \infty, \text{ for each } N > 0 . \end{cases}$$

so

$$\int_0^{\infty} |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < \infty .$$

It follows that

$$\iint_{S(\Omega)} |g(y,t)|^2 dy \frac{dt}{t_1 t_2} \le C \|g\|_{\infty}^2 |\Omega|$$

as claimed.

If we wish to prove that the same Carleson condition holds for g replaced by $H_{x_1} H_{x_2}(g)$, then we proceed as follows. Observe that

$$\left(\iint_{S(\Omega)} |[H_{x_1} H_{x_2}(g)] * \psi_t(y)|^2 dy \frac{dt}{t_1 t_2} \right)^{1/2} = \left(\iint_{S(\Omega)} |g * \psi_t(y)|^2 dy \frac{dt}{t_1 t_2} \right)^{1/2}$$

where $\Psi = H_{x_1} H_{x_2}(\psi)$. The function Ψ splits into a product of $\Psi^{(1)}(x_1) \cdot \Psi^{(2)}(x_2)$ where $\Psi^{(i)}$ is odd, C^{∞} and decreasing at ∞ like $|x_i|^{-N}$ (depending on how many moments of ψ vanish). Now suppose we choose $\eta(x)$ on R^1 so that $\operatorname{supp}(\eta) \subseteq (1/4,4)$, $\eta \in C^{\infty}(R^1)$, η even and $\sum_{k=-\infty}^{\infty} \eta\left(\frac{x}{2^k}\right) \equiv 1$. Let $\eta_0(x) = \sum_{k \le 0} \eta\left(\frac{x}{2^k}\right)$ and for $k > 0$, let $\eta_k(x) = \eta\left(\frac{x}{2^k}\right)$. Set $\psi_{k,j}(x_1,x_2) = \Psi^{(1)}(x_1)\, \eta_k(x_1) \cdot \Psi^{(2)}(x_2)\, \eta_k(x_2)$. Then

(a) $\operatorname{supp}(\Psi_{k,j}) \subseteq 4R(0; 2^k, 2^j)$

(b) $\Psi_{k,j}$ is odd in each variable separately

(c) Ψ_{kj} is $C_c^\infty(R^2)$

and

(d) $\left\| \left(\frac{\partial}{\partial x}\right)^\alpha \Psi_{k,j} \right\|_\infty = 0(2^{-kN-jN})$ as $k,j \to \infty$ if $|\alpha| \le 2$

By Minkowski's inequality, we have

$$(6.3)\quad \left(\iint\limits_{S(\Omega)} |g * \Psi_t|^2 dy \frac{dt}{t_1 t_2} \right)^{1/2} \le \sum_{k,j} \left(\iint\limits_{S(\Omega)} |g * (\Psi_{k,j})_t|^2 dy \frac{dt}{t_1 t_2} \right)^{1/2} .$$

Now, to estimate

$$\sum_{k,j} |g * (\Psi_{k,j})_t|^2 dy \frac{dt}{t_1 t_2}$$

we use the same argument as that given above, except that now $\operatorname{supp}(\Psi_{k,j}) \le 4R(0; 2^k, 2^j)$ and not the unit square. If $(y,t) \in S(\Omega)$, then $R(y;t) \subseteq \Omega$ and the support of $(\Psi_{k,j})_t(\cdot - y)$ will be contained in

$$R(y, 2^k t_1, 2^j t_2) \subseteq \{M^{(2)}(\chi_\Omega) > 2^{-(k+j)}\} = \tilde{\Omega}_{k,j} .$$

Thus

$$\iint\limits_{S(\Omega)} |g * (\Psi_{k,j})_t(y)|^2 dy \frac{dt}{t_1 t_2} = \iint\limits_{S(\Omega)} |(g\chi_{\tilde{\Omega}_{kj}}) * \Psi_{kj}|^2 dy \frac{dt}{t_1 t_2}$$

$$\le \|g\chi_{\tilde{\Omega}_{kj}}\|_2^2 \int_0^\infty \int_0^\infty |\Psi_{kj}(\xi_1, \xi_2)|^2 \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} \le \|g\|_\infty^2 |\tilde{\Omega}_{kj}| \left(\int_0^\infty \int_0^\infty |\Psi_{kj}(\xi_1, \xi_2)|^2 \frac{d\xi_1}{\xi_1} \frac{d\xi_2}{\xi_2} \right) .$$

By the strong maximal theorem $|\tilde{\Omega}_{kj}| \leq C(k+j)2^{k+j}$, and it is easy to see that $\int_0^\infty \int_0^\infty |\hat{\Psi}_{kj}|^2 \, d\xi$ decreases like a large power of $2^{-(k+j)}$ as $k,j \to \infty$. So our desired estimate follows from (6.2).

So far we have proven the equivalence of (1), (2), (3) and (4). We shall not go into the details of the equivalence of (5) except to say the proof is given in the Annals paper of Chang-Fefferman [25]. Rather, let us point out a beautiful application of the equivalence of (5) with the other definitions of BMO which occurs already in the one-parameter setting. This is the theorem of A. Uchiyama [26], which tells us which families of multipliers homogeneous on R^n of degree 0 determine $H^1(R^n)$. He showed that for multiplier operators $I, K_1, K_2, \cdots, K_m$ with multipliers $1, \theta_i(\xi)$ that $f, K_i f \in L^1(R^n)$ implies $f \in H^1(R^n)$ if and only if the θ_i separate antipodal points of S^{n-1}, i.e., if and only if for every $\xi \in S^{n-1}$, there exists i such that $\theta_i(\xi) \neq \theta_i(-\xi)$.

The way Uchiyama proves this is to show that the dual statement is true, namely, every $\phi \in BMO(R^n)$ can be written as

$$g_0 + \sum K_i g_i \quad \text{for some } g_0, g_1, \cdots, g_m \in L^\infty . \tag{6.4}$$

This depends on a simple lemma.

LEMMA. *If θ_i are as above, then given $f \in L^2$, and a vector $\nu \in C^{m+1}$, there exist functions $g_0, \cdots, g_m \in L^2$ so that*

$$g_0 + \sum K_i g_i = f \quad \textit{and} \quad g(x) = (g_0(x), g_1(x), \cdots, g_m(x)) \quad \nu \quad \textit{for all } x \in R^n .$$

To prove the formula (6.4), Uchiyama decomposes $\phi = \Sigma\, C_I \phi_I$ as in our (5), and applies the lemma to get functions $g_I(x)$ such that $Kg_I(x) = C_I \phi_I(x)$ for which $g(x)$ is perpendicular to the correct ν, and the result, when modified only slightly to $\tilde{g}_I$, has the property that $\Sigma\, \tilde{g}_I \in L^\infty$. For the details see Uchiyama's recent paper in Acta [26].

Now, finally we wish to discuss the atomic decomposition of $H^1(R^2_+ \times R^2_+)$ in greater detail. There are interesting applications of this decomposition besides duality with $\mathrm{BMO}(R^2_+ \times R^2_+)$ which was presented above. We shall be content with one more application here which sheds a good deal of light on the nature of these atoms. Namely, we intend to give a second proof, directly by real variables, that on $R^2_+ \times R^2_+$ if $S_\psi(f) \in L^1(R^2)$ then $f^* \in L^1(R^2)$ [27]. Suppose $S_\psi(f) \in L^1(R^2)$. Then in our discussion of duality we defined atoms

$$\tilde{a}_k(x) = \frac{1}{2^k|\tilde{\Omega}_k|} \sum_{R \in R_k} f_R(x_1, x_2) .$$

To simplify this notation we define $\omega = \tilde{\Omega}_k$ and $A(x) = 2^k|\tilde{\Omega}_k|\tilde{a}_k(x)$. Let ϕ^1 and $\phi^2 \in C_c^\infty(R^1)$ with $\phi^i(x) \geq 0$, $\int \phi^i = 1$, and $\mathrm{supp}(\phi^i) \subset [-1, +1]$. Set

$$\phi_{t_1,t_2}(x_1,x_2) = t_1^{-1}t_2^{-1}\phi^1\left(\frac{x_1}{t_1}\right)\phi^2\left(\frac{x_2}{t_2}\right) .$$

Define $\tilde{\omega} = M^{(2)}(\chi_\omega) > \frac{1}{10^{10}}$. We need to estimate $A * \phi_{t_1,t_2}(x)$ for $x \notin \tilde{\omega}$. To do this let us make the following definitions. If R is a rectangle then R_1, R_2 will be its sides, so that $R = R_1 \times R_2$. Let

$$A_j^1(x) = \sum_{\substack{R \in \mathfrak{R}_k \\ |R_1| = 2^j}} f_R(x), \quad A_j^2(x) = \sum_{\substack{R \in \mathfrak{R}_k \\ |R_2| = 2^j}} f_R(x) .$$

Then to estimate $A * \phi_{t_1,t_2}(x)$, since $\mathrm{supp}(\phi_t(\cdot - x)) \subseteq R(x;t) = S$, in the definition of A we need only consider those f_R for which $\tilde{R} \cap S \neq 0$. For any such rectangle $R \in \mathfrak{R}_k$, since $R \subset \omega$,

$$\text{minimum}\left(\frac{|R_1|}{|S_1|}, \frac{|R_2|}{|S_2|}\right) < \left(\frac{|S \cap \tilde{\omega}|}{|S|}\right)^{1/2} = \rho$$

where $\tilde{\omega}$ denotes again $M^{(2)}(\chi_\omega) > \frac{1}{10^{10}}$. Then

$$A * \phi_{t_1 t_2}(x) = \sum_{2^j/|S_I|<\rho} A_j^1 * \phi_{t_1 t_2}(x) + \sum_{2^j/|S_2|<\rho} A_j^2 * \phi_{t_1 t_2}(x)$$

$$- \sum_{R \in \mathfrak{B}} f_R * \phi_{t_1 t_2}(x)$$

where $\mathfrak{B} \subset \mathfrak{R}_k$ consists of rectangles R so that $\frac{|R_1|}{|S_1|} < \rho$, $\frac{|R_2|}{|S_2|} < \rho$ and $\tilde{R} \cap S \neq \emptyset$. Thus $R \subseteq \tilde{S}$ for all $R \in \mathfrak{B}$, and the reason this subtracted term occurs is that we have double counted these f_R whose R sides are both very small.

In order to estimate $A_j^1 * \phi_{t_1 t_2}(x)$ we use the following trivial lemma.

LEMMA. *On* R^1 *suppose that* $\phi(x) \in C^\infty$ *and is supported in an interval* ϑ. *Suppose* $\alpha(x)$ *is supported on disjoint subintervals of* ϑ, I_k *whose lengths are all* $\leq \gamma|\vartheta|$. *Assume that* $\alpha(x)$ *has* N *vanishing moments over each* I_k. *Then*

$$\left| \int_\vartheta \alpha(x)\,\phi(x)\,dx \right| \leq C \|\phi^{(N+1)}\|_\infty (\gamma|\vartheta|)^{N+1} \int |\alpha(x)|\,dx\ .$$

We estimate $A_j^1 * \phi_{t_1 t_2}(x)$ using the fact that for each fixed x_2, $A_j^1(\cdot\,, x_2)$ has N vanishing moments over disjoint x_1 intervals over length $2 \cdot 2^j$. (Actually, we sould have to break up A_j^1 into 3 pieces to insure this, but we spare the reader this trivial complication.) It follows from the lemma that

$$|A_j^1 *_1 \phi_{t_1}^1(x)| \leq_{t_1} \left(\frac{2^j}{|S_1|}\right)^{N+1} (|A_j^1| *_1 \chi_{[-t_1 t_1]}(x))\ .$$

Convolving in the x_2 variable, we have

$$|A_j^1 * \phi_{t_1, t_2}(x)| \leq C\ \frac{2^j}{|S_1|}^{\ N+1} \frac{1}{|\tilde{S}|} \int_{\tilde{S}} |A_j^1|\,dx'\ .$$

For this we get

$$\left|\sum_{2^j/|S_1|<\rho} A_j^1 * \phi_{t_1,t_2}(x)\right| \le C\rho^{N/2}\frac{1}{|\tilde{S}|}\int_{\tilde{S}}\left(\sum |A_j^1|^2\right)^{1/2}\left(\sum_{2^j/|S_1|<\rho}\left(\frac{2^j}{|S_1|}\right)^N\right)^{1/2}.$$

$$\le C\rho^{N/2}\frac{1}{|\tilde{S}|}\int_{\tilde{S}}\left(\sum |A_j^1|^2\right)^{1/2}.$$

By symmetry

$$\left|\sum_{2^j/|S_2|<\rho} A_j^2 * \phi_{t_1 t_2}(x)\right| \le C\rho^{N/2}\frac{1}{|\tilde{S}|}\int_{\tilde{S}}\left(\sum |A_j^2|^2\right)^{1/2}.$$

Now let $R \subset \tilde{S}$, with $\frac{|R_1|}{|S_1|} \le \frac{|R_2|}{|S_2|}$. Then

$$|f_R * \phi_{t_1}(x)| \le C\left(\frac{|R|}{|S|}\right)^{N/2}\frac{1}{t_1}\chi_{[-t_1,t_2]} * |f_R|(x)$$

and also

$$|f_R * \phi_{t_1,t_2}(x)| \le C\left(\frac{|R|}{|S|}\right)^{N/2}\frac{1}{|\tilde{S}|}\int_{\tilde{S}} |f_R|.$$

Thus

$$\sum_{R\in\mathfrak{R}_k} |f_R * \phi_{t_1 t_2}(x)| \le C\frac{1}{|\tilde{S}|}\int_{\tilde{S}}\left(\sum_{R\in\mathfrak{R}_k} f_R^2(x)\right)^{1/2}\left(\sum_{R\in\mathfrak{R}_k}\left(\frac{|R|}{|S|}\right)^{N/2}\right)^{1/2}\cdot R^{N/4}$$

$$\le C\rho^{N/4}\frac{1}{|\tilde{S}|}\int_{\tilde{S}} \sum_{R\in\mathfrak{R}_k} |f_R|^2 {}^{1/2}.$$

To sum up our findings, we have seen that if $x \notin \tilde{\omega}$ then

$$(6.5) \qquad |A * \phi_{t_1t_2}(x)| \le C\left(\frac{|S \cap \omega|}{|\widetilde{S}|}\right)^{N/4}\left[\frac{1}{|\widetilde{S}|}\int_{\widetilde{S}} J(y)\,dy\right]$$

where

$$J(x) = \left(\sum (A_j^1)^2\right)^{1/2} + \left(\sum (A_j^2)^2\right)^{1/2} + \left(\sum_{R \in \mathfrak{R}_k} f_R\right)^{1/2}.$$

To finish the proof, we need another lemma:

LEMMA. *Let* $g(x) = \sum_{R \in \mathfrak{B}} f_R(x)$ *where* B *is a collection of dyadic rectangles. Then*

$$\|g\|_2^2 \le \bigcup_{R \in \mathfrak{B}} \int_{\mathfrak{A}(R)} |f(y,t)|^2\,dy\,\frac{dt}{t_1t_2}.$$

Proof. Let $\|h\|_{L^2(R^2)} = 1$. Then

$$\int g(x)\,h(x)\,dx = \int \sum_{R \in \mathfrak{B}} \iint_{\mathfrak{A}(R)} f(y,t)\,\psi_t(x-y)\,dy\,\frac{dt}{t_1t_2} \cdot h(x)\,dx$$

$$= \sum_{R \in \mathfrak{B}} \iint_{\mathfrak{A}(R)} f(y,t)\,h(y,t)\,dy\,\frac{dt}{t_1t_2}$$

$$\le \left(\bigcup_{R \in \mathfrak{B}} \iint_{\mathfrak{A}(R)} f(y,t)|^2 dy\,\frac{dt}{t_1t_2}\right)^{1/2}\left(\iint_{(R_+^2)^2} |h(y,t)|^2 dy\,\frac{dt}{t_1t_2}\right)^{1/2}$$

$$\le C\left(\iint |f(y,t)|^2 dy\,\frac{dt}{t_1t_2}\right)^{1/2} \|S_\psi(h)\|_2 \le C'\left(\iint |f(y,t)|^2 dy\,\frac{dt}{t_1t_2}\right)^{1/2}.$$

Now, notice that, by the lemma,

$$\|J\|_2^2 \le \iint_{R\epsilon_k^{(R)}} |f(y,t)|^2 dy \frac{dt}{t_1 t_2} \le \int_{\tilde{\Omega}_k/\Omega_{k+1}} S_\psi^2(f)(x)dx \le C \cdot 2^{2k} \cdot |\omega| .$$

The same estimate holds for $\|A\|_2^2$. Then

$$\int_{\tilde{\omega}} A^* \le |\omega|^{1/2}\left(\int_{R^2} A^{*2}\right)^{1/2} \le C|\omega|^{1/2} \|A\|_2 \le C2^k |\omega| .$$

Also away from $\tilde{\omega}$,

$$A^*(x) \le M^{(2)}(\chi_{\tilde{\omega}})^{10}(x) \cdot M^{(2)}(J)(x) ,$$

so

$$\int_{c\tilde{\omega}} A^* dx \le \left(\int_{R^2} M^{(2)}(\chi_{\tilde{\omega}})^{20}(x)\, dx\right)^{1/2} \left(\int_{R^2} M^{(2)}(J)^2(x)\, dx\right)^{1/2}$$

$$\le C|\omega|^{1/2}|\omega|^{1/2}2^k = C2^k|\omega| .$$

It follows that $\|A^*\|_1 \le C2^k|\omega|$ and also $\|\tilde{a}_k\|_1 \le C$.

ROBERT FEFFERMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CHICAGO
CHICAGO, ILLINOIS 60637

BIBLIOGRAPHY

[1] B. Jessen, J. Marcinkiewicz and A. Zygmund, Notes on the Differentiability of Multiple Integrals, Fund. Math. 24, 1935.

[2] E. M. Stein and S. Wainger, Problems in Harmonic Analysis Related to Curvature, Bull. AMS. 84, 1978.

[3] A. Cordoba and R. Fefferman, A Geometric Proof of the Strong Maximal Theorem, Annals of Math., 102, 1975.

[4] J. O. Stromberg, Weak Estimates on Maximal Functions with Rectangles in Certain Directions, Arkiv fur Math., 15, 1977.

[5] A. Cordoba and R. Fefferman, On Differentiation of Integrals, Proc. Nat. Acad. of Sci., 74, 1977.

[6] A. Nagel, E. M. Stein, and S. Wainger, Differentiation in Lacunary Directions, Proc. Nat. Acad. Sci., 75, 1978.

[7] A. Cordoba, Maximal Functions, Covering Lemmas, and Fourier Multipliers, Proc. Symp. in Pure Math., 35, Part I, 1979.

[8] F. Soria, Examples and Counterexamples to a Conjecture in the Theory of Differentiation of Integrals, to appear in Annals of Math.

[9] B. Muckenhoupt, Weighted Norm Inequalities for the Hardy Maximal Function, Trans. of the AMS, 165, 1972.

[10] R. Hunt, B. Muckenhoupt, and R. Wheeden, Weighted Norm Inequalities for the Conjugate Function and Hilbert Transform, Trans. AMS, 176, 1973.

[11] R. R. Coifman and C. Fefferman, Weighted Norm Inequalities for Maximal Functions and Singular Integrals, Studia Math., 51, 1974.

[12] M. Christ and R. Fefferman, A Note on Weighted Norm Inequalities for the Hardy-Littlewood Maximal Operator, Proceedings of the AMS, 84, 1983.

[13] R. Fefferman, Strong Differentiation with Respect to Measures, Amer. Jour. of Math., 103, 1981.

[14] ———, Some Weighted Norm Inequalities for Cordoba's Maximal Function, to appear in Amer. Jour. of Math.

[15] C. Fefferman, The Multiplier Problem for the Ball, Annals of Math., 94, 1971.

[16] A. Cordoba and R. Fefferman, On the Equivalence between the Boundedness of Certain Classes of Maximal and Multiplier Operators in Fourier Analysis, Proc. Nat. Acad. Sci., 74, No. 2, 1977.

[17] E. M. Stein and G. Weiss, On the Theory of H^p Spaces, Acta. Math., 103, 1960.

[18] C. Fefferman and E. M. Stein, H^p Spaces of Several Variables, Acta Math., 129, 1972.

[19] D. Burkholder, R. Gundy, and M. Silverstein, A Maximal Function Characterization of the Class H^p, Trans. AMS, 157, 1971.

[20] K. Merryfield, Ph.D. Thesis: H^p Spaces in Poly-Half Spaces, University of Chicago, 1980.

[21] R. Gundy and E. M. Stein, H^p Theory for the Polydisk, Proc. Nat. Acad. Sci., 76, 1979.

[22] L. Carleson, A Counterexample for Measures Bounded on H^p for the Bi-Disc, Mittag-Leffler Report No. 7, 1974.

[23] S.Y. Chang, Carleson Measure on the Bi-Disc, Annals of Math., 109, 1979.

[24] R. Fefferman, Functions on Bounded Mean Oscillation on the Bi-Disc, Annals of Math., 10, 1979.

[25] S.Y. Chang and R. Fefferman, A Continuous Version of the Duality of H^1 and BMO on the Bi-Disc, Annals of Math., 1980.

[26] A. Uchiyama, A Constructive Proof of the Fefferman-Stein Decomposition of $BMO(R^n)$, Acta. Math., 148, 1982.