

Physics 8.321, Fall 2020

Final Exam

You have **3 hours** to complete the exam and a grace period of an additional 20 minutes to get your solutions uploaded to canvas. You may use your books and notes including the notes on canvas from the course, you may freely use any results derived in homework assignments this semester, and you may use symbolic manipulation tools like mathematica and matlab, but you may not consult other online resources, and you may not communicate with other people in any way while doing the final. You also may not communicate any information about the exam to anyone after you have completed it until the exam period is over at the end of the day on 12/15/20.

Note: you are expected to upload your completed exam to the canvas website immediately after completing the exam and within 200 minutes or less of downloading it. You have a few extra minutes in case of technical complications. The system will log your download and upload times. If for some reason you have difficulty uploading your exam after completion, please email it immediately to one of the course staff.

1. Particle in a linear symmetric potential (40 points)

Consider a particle moving in one dimension in the potential $V = A|x|$, where A is a constant. You may fix $m = \hbar = 1$.

- (a) Estimate the ground state energy using the variational principle for the family of Gaussian wavefunctions $e^{-ax^2/2}$ parameterized by a , suitably normalized.
- (b) Estimate the ground state energy using the WKB approximation and compare with your result from part (a).

2. Rotating wave trick for a time-dependent Hamiltonian (30 points)

Consider the system with *time-dependent* Hamiltonian

$$H(t) = \frac{\hbar}{2} \begin{bmatrix} \omega & \Omega e^{-i\nu t} \\ \Omega e^{i\nu t} & -\omega \end{bmatrix} \quad \text{where } \omega, \Omega, \nu \in \mathbb{R}. \quad (1)$$

- (a) Consider the unitary operator

$$U_R(t) = \begin{bmatrix} e^{iat} & 0 \\ 0 & e^{-iat} \end{bmatrix}, \quad (2)$$

with $a \in \mathbb{R}$. Show that $U_R(t)H(t)U_R^\dagger(t)$ is time independent for suitable choice of a . Find a . Henceforth, we are going to use this a in $U_R(t)$.

- (b) Let $|\psi(t)\rangle$ be the solution of the time-dependent Schrodinger equation for the Hamiltonian $H(t)$ above. That is

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H(t) |\psi(t)\rangle. \quad (3)$$

Show that the state $U_R(t)|\psi(t)\rangle$ solves a Schrodinger equation with a *time-independent* Hamiltonian H_{eff} . Find H_{eff} .

- (c) Use the results above to write a formal expression for $|\psi(t)\rangle$ for given initial state $|\psi(0)\rangle$ and in terms of H_{eff} and $U_R(t)$. You don't have to evaluate any matrix exponential explicitly.
- (d) Now consider the situation $\omega = \nu$, and let the initial state be

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (4)$$

Find the time-evolved state $|\psi(t)\rangle$. Are there time(s) $t = t_c$ for which collapsing to the initial state after measurement of the operator

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (5)$$

is certain? If so, when does that happen (i.e. find t_c)?

3. Quantum Rigid Rotor (30 points)

Consider a rigid rotor immersed in a magnetic field, with Hamiltonian

$$H = \frac{\mathbf{L}^2}{2I} + \omega_0 L_z. \quad (6)$$

Suppose the system is in an initial state $|\psi_0\rangle$ such that

$$\langle\theta, \phi|\psi_0\rangle = \sqrt{3/4\pi} \sin\theta \sin\phi. \quad (7)$$

- (a) What values of L_z will be obtained if a measurement is carried out in the initial state, and with what probability do these values occur?
- (b) What is $\langle\theta, \phi|\psi(t)\rangle$?
- (c) What are $\langle L_x(t)\rangle, \langle L_y(t)\rangle, \langle L_y(t)^2\rangle, \langle\Delta L_y(t)\rangle$ in this state?

4. Interacting spins in a magnetic field (30 points)

Consider the Hamiltonian

$$H = H_0 + BH_1 = -J \mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)} - BS_z^{(1)}, \quad (8)$$

with $J > 0$.

- (a) Solve for the spectrum exactly when $B = 0$, including multiplicities, when $S^{(1)}, S^{(2)}$ are each systems of general spin j_1, j_2 .
- (b) Find the corrections to the energy eigenvalues using perturbation theory at leading order in small B in the case $j_1 = j_2 = 1/2$.
- (c) Solve exactly in the case $j_1 = j_2 = 1/2$ and check your result from part (b).

5. Three entangled quantum spins (30 points)

- (a) Consider a system of three spin-1/2 particles in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle - |--\rangle). \quad (9)$$

Compute the entanglement entropy for a subsystem containing one or two of the spins. (Recall that the entanglement entropy of a subsystem is the entropy of the density matrix of the subsystem after tracing out the degrees of freedom of the complement of the subsystem, and quantifies the randomness/uncertainty in the subsystem taken in isolation.)

- (b) Now consider a system of three spin-1/2 particles in the state

$$|\psi\rangle = \frac{1}{2}(|++\rangle + |+-\rangle - |-+-\rangle - |--\rangle). \quad (10)$$

Again, compute the entanglement entropy for a subsystem containing each subset of one or two of the spins. Explain your answer in terms of how the density matrix for the different subsystems is described by mixed or pure states. [Hint: Try factorizing the state!]

6. Simple Harmonic Oscillator & Magnetic Field in 2D (40 points)

Consider a spinless particle of mass m and charge q moving in two dimensions, in the xy -plane. The particle feels a uniform magnetic field $\mathbf{B} = B\hat{z}$ and is also subject to a harmonic potential that is independent of x

$$V(x, y) = \frac{1}{2}m\omega^2 y^2. \quad (11)$$

Find the energy eigenvalues and corresponding energy eigenfunctions. You can express your answer in terms of ψ_n , $n = 0, 1, 2, \dots$, which are the energy eigenfunctions corresponding to the one-dimensional simple harmonic oscillator. That is you don't have to derive the full explicit expression for all states, but other than that the answer should depend on the given parameters, fundamental constants, and/or quantum numbers needed to label the state. Check that your results for energy eigenvalues make sense in the limit of vanishing magnetic field $B \rightarrow 0$, and separately for the case without harmonic oscillator potential, i.e. $\omega \rightarrow 0$.