Matrices in Quantum Computing

Huan Q. Bui

Matrix Analysis

Professor Leo Livshits

CLAS, May 2, 2019

Presentation layout

Qubits & Quantum Gates

Qubit: A quantum system with measurable eigenstates $|0\rangle$ and $|1\rangle$,

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \rightarrow \text{like a Classical Bit.}$$

But before measurement,

$$\text{Wavefunction} : |\psi\rangle = a\,|0\rangle + b\,|1\rangle \in \mathbb{C}^2, \quad |a|^2 + |b|^2 = 1.$$

Probabilistic:

$$P(|\psi\rangle \rightarrow |0\rangle) = |a|^2 \quad P(|\psi\rangle \rightarrow |1\rangle) = |b|^2.$$

Quantum gate: unitary transformation on $|\psi\rangle$ of one of many qubits.

Multiple Qubits

How to express the state of two qubits, $|\psi_1\rangle \in \mathbf{V}_1, |\psi_2\rangle \in \mathbf{V}_2$?

$$|\psi_1\psi_2\rangle \overset{?}{\sim} |\psi_1\rangle, |\psi_2\rangle$$

More than two, $|\psi_i\rangle \in \mathbf{V}_i$?

$$|\psi_1\psi_2\ldots\psi_n\rangle \overset{?}{\sim} |\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle$$

Questions:

- What is the vector space containing $|\psi_1\psi_2\ldots\psi_n\rangle$?
- How does $|\psi_1\psi_2\ldots\psi_n\rangle$ change w.r.t $\mathcal{A}_1|\psi_1\rangle$ where $\mathcal{A}_1 \in \mathfrak{L}(\mathbf{V})$?
- What about for $\mathcal{A}_1|\psi_1\rangle, \ldots \mathcal{A}_n|\psi_n\rangle$, where $\mathcal{A}_i \in \mathfrak{L}(\mathbf{V})$?

Tensor Product

Postulate: [Mike & Ike] The state space of a composite physical system is the *tensor product* of the state spaces of the component physical systems.

For $|\psi_1\rangle \in \mathbf{V}_1, \dots, |\psi_n\rangle \in \mathbf{V}_n$,

$$|\psi_1 \dots \psi_n\rangle \in \mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n,$$

where the joint state $|\psi_1 \dots \psi_n\rangle$ is given by

$$|\psi_1 \dots \psi_n\rangle = |\psi_1\rangle |\psi_2\rangle \dots |\psi_n\rangle = |\psi_1\rangle \otimes \cdots \otimes |\psi_n\rangle .$$

$|\psi_1 \dots \psi_n\rangle$ is an *elementary tensor* in $\mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n$.

Not all $|\phi\rangle \in \mathbf{V}_1 \otimes \cdots \otimes \mathbf{V}_n$ are elementary.

Tensor Product: Construction

Definition

The *tensor product* of $\mathbf{V}$ and $\mathbf{W}$ is a vector space $\mathbf{V} \otimes \mathbf{W}$ with the *bilinear map* $\phi : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{V} \otimes \mathbf{W}$, such that for every vector space $\mathbf{X}$ and every bilinear map $f : \mathbf{V} \times \mathbf{W} \longrightarrow \mathbf{X}$, there exists a *unique linear map* $\hat{f} : \mathbf{V} \otimes \mathbf{W} \longrightarrow \mathbf{X}$ such that $f = \hat{f} \circ \phi$.

In other words...

Giving the $\hat{f} : \mathbf{V} \otimes \mathbf{W} \overset{\text{linear}}{\longrightarrow} \mathbf{X}$ is the same as giving $f : \mathbf{V} \times \mathbf{W} \overset{\text{bilinear}}{\longrightarrow} \mathbf{X}$.

Tensor Product: Construction

$$
\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\hat{f}", "\textit{linear}"'] & & \mathbf{X} \\
& & \\
\mathbf{V} \times \mathbf{W} \arrow[uu, hook, "\phi"] \arrow[uurr, "f", "\textit{multilinear}"'] & &
\end{tikzcd}
$$

Tensor Product: Some Properties

Let $v_1, \ldots, v_n$ be a basis of $\mathbf{V}$ and $w_1, \ldots, w_m$ be a basis of $\mathbf{W}$,

- For $i \in [1, n]$, $j \in [1, m]$, $\{v_i \otimes w_j\}$ is a basis of $\mathbf{V} \otimes \mathbf{W}$:

$$v \otimes w = \sum_{i}^{n} \alpha_i v_i \otimes \sum_{j}^{m} \beta_j w_j = \sum_{i,j}^{n,m} \alpha_i \beta_j (v_i \otimes w_j)$$

- $\dim(\mathbf{V} \otimes \mathbf{W}) = \dim(\mathbf{V}) \dim(\mathbf{W}) = nm$.

Tensor Product: Some Properties

Let $\mathcal{A} \otimes \mathcal{B} \in \mathfrak{L}(\mathbf{V} \otimes \mathbf{W})$, where $\mathcal{A} \in \mathfrak{L}(\mathbf{V})$, and $\mathcal{B} \in \mathfrak{L}(\mathbf{W})$.

$$(\mathcal{A} \otimes \mathcal{B})(v \otimes w) \overset{?}{\sim} \mathcal{F}(v, w) \triangleq \mathcal{A}(v) \otimes \mathcal{B}(w).$$

One way to see this...

$$\begin{tikzcd}
\mathbf{V} \otimes \mathbf{W} \arrow[rr, "\mathcal{A}\otimes\mathcal{B}"] & & \mathbf{V} \otimes \mathbf{W} \\
\mathbf{V} \times \mathbf{W} \arrow[u, hook, "\phi"] \arrow[urr, "\mathcal{F}"'] & &
\end{tikzcd}$$

$$(\mathcal{A} \otimes \mathcal{B}) \circ \phi = \mathcal{F} \iff \boxed{(\mathcal{A} \otimes \mathcal{B})(v \otimes w) = \mathcal{A}(v) \otimes \mathcal{B}(w)}$$

Tensor Product: Some Properties

Another way to see this...

Why Tensor Product?