Due: **Monday, March 14 by 5pm**
Reading: Michael Cross' Lecture 14 on Ginzburg-Landau theory. Anthony J Leggett Lecture 10 (Ginzburg-Landau theory) Tinkham Chap 3; Superconductivity class notes, quasiparticles; phase transition; (note active links)

1. Type I and type II superconductivity in Ginzburg-Landau theory.
We begin with the Ginzburg-Landau free energy

$$F = \int d^3r \left[\alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 + \frac{\hbar^2}{4m}\left|\left(i\nabla + \frac{2e}{\hbar c}\mathbf{A}\right)\psi\right|^2\right] \tag{1}$$

and consider T slightly below T_c. The quadratic term changes sign at $T = T_c$, such that $\alpha(T) = a(T - T_c)$.

(a) [10 pts] Calculate the difference ΔF in free energies of the superconducting state and the normal state (in the absence of magnetic field) and show that the thermodynamic critical field H_c is given by

$$\frac{1}{8\pi}H_c^2(T) = \Delta F$$

Find the temperature dependence $H_c(T)$ near T_c.

(b) [10 pts] Use the expression of the London penetration depth in terms of the superfluid density $n_s = |\psi|^2$ and particle mass parameter m found in HW2 to show that the thermodynamic critical field can be written as

$$H_c = \eta\frac{\phi_0}{2\pi\xi(T)\lambda_L(T)}, \quad \xi(T) = \sqrt{\frac{\hbar^2}{4m|\alpha(T)|}},$$

where $\phi_0 = hc/2e$ is the superconducting flux quantum, and $\xi(T)$ is the correlation length defined by balancing the gradient term and the ψ^2 term of the GL functional. Determine the numerical prefactor η in the above expression.

(c) [20 pts] Now turn on a uniform magnetic field H such that we are in the normal state ($\psi = 0$) and gradually reduce H. We look for an instability towards $\psi \neq 0$. The field at which the instability happens is called the upper critical field and denoted $H_{c2}(T)$. We calculate $H_{c2}(T)$ by the following steps.

- Using the condition $\delta F = 0$ under a variation of ψ, show that ψ satisfies

$$-\xi^2\left(i\nabla + \frac{2e}{\hbar c}\mathbf{A}\right)^2\psi + \psi + \frac{\beta}{\alpha}|\psi|^2\psi = 0, \tag{2}$$

- Near the critical point, the last term in Eq.(2) can be ignored and we have a linearized equation. Instability ($\psi \neq 0$) occurs when the linearized equation has a negative eigenvalue. Notice that this equation has the same form as that of a single electron in a magnetic field, where the solution is known to be Landau levels. Use this fact to show that

$$H_{c2}(T) = \frac{\phi_0}{2\pi\xi^2(T)}$$

where $\phi_0 = hc/2e$ is the superconducting flux quantum. This formula is very useful in determining the coherence length ξ because it relates ξ to the quantity that can be easily measured in experiment: the upper critical field.

- Show that the condition $H_{c2} > H_c$ implies

$$\kappa = \frac{\lambda_L(T)}{\xi(T)} > \frac{1}{\sqrt{2}}$$

which is nothing but the Abrikosov criterion for type II superconductivity.

2. The BCS self-consistency equation

Superconducting pairing can be described by a Hamiltonian with a BCS-truncated interaction of a form that accounts only for pair scattering near the Fermi level,

$$H = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\sigma} c_{\mathbf{k}\sigma} - \frac{1}{V} \sum_{k,k'} U(\mathbf{k}', \mathbf{k}) c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}. \tag{3}$$

Here $\epsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m}$ is free-particle dispersion, V is system volume, and $U(\mathbf{k}', \mathbf{k})$ are scattering amplitudes for the processes $(\mathbf{k}\uparrow, -\mathbf{k}\downarrow) \to (\mathbf{k}'\uparrow, -\mathbf{k}'\downarrow)$. Summation over different plane-wave states $\sum_{\mathbf{k}} \ldots$ denotes $V \int \frac{d^3k}{(2\pi)^3} \ldots$, as usual.

The conventional s-wave pairing is described by isotropic pairing interaction

$$U(\mathbf{k}', \mathbf{k}) = \begin{cases} U_0 & |\xi_{\mathbf{k}}|, |\xi_{\mathbf{k}'}| < \omega_D \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ are energies of noninteracting electrons measured relative to the Fermi level.

(a) [10pts] Consider the BCS ground state and show that its energy $W = \langle \Psi | H - \mu N | \Psi \rangle$ is equal to

$$W = \sum_{\mathbf{k}} 2(\epsilon_{\mathbf{k}} - \mu)|v_{\mathbf{k}}|^2 - \frac{U_0}{V} \sum_{k,k'} u^*_{\mathbf{k}'} v_{\mathbf{k}'} u^*_{\mathbf{k}} v_{\mathbf{k}} \tag{5}$$

where $|\Psi\rangle$ is the BCS wavefunction.

(b) [10pts] The characteristic energy scale ω_D that determines the width of the band of states where pairing takes place, $-\omega_D < \xi_{\mathbf{k}} < \omega_D$, is usually of order of Debye's energy and is therefore much smaller than the Fermi energy. Assuming that at these energies the density of states $N(\epsilon)$ differs negligibly from a constant $\nu_0 = N(\epsilon = \mu)$ derive the BCS self-consistency equation

$$1 = \nu_0 U_0 \int_0^{\omega_D} d\xi \frac{1}{\sqrt{\xi^2 + \Delta^2}} \tag{6}$$

by minimizing the function W with respect to $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ subject to the constraint $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$. In doing so it is useful to parameterize $u_{\mathbf{k}} = \cos\theta_{\mathbf{k}}$ and $v_{\mathbf{k}} = \sin\theta_{\mathbf{k}}$, and show that $\theta_{\mathbf{k}}$ obeys the relation $\tan 2\theta_{\mathbf{k}} = \frac{\Delta}{\xi_{\mathbf{k}}}$.

(c) [10pts] Evaluate the integral over ξ, solve the BCS self-consistency equation and, assuming weak coupling $\nu_0 U_0 \ll 1$, derive an expression for the energy gap Δ. Take into account that in the weak-coupling limit we have $\Delta \ll \omega_D$. [Hint: Use the substitution $x = \sinh \xi/\Delta$.]

3. Gapless excitations for unconventional pairing

(a) [10 pts] Now consider fermionic pairing described by the Hamiltonian in Eq.3, with a more general angle-dependent pairing interaction $U(\mathbf{k}, \mathbf{k}')$. We will see below that the angle dependence of $U(\mathbf{k}, \mathbf{k}')$ can be chosen to describe Cooper pairs with an arbitrary nonzero angular momentum. Generalize the approach of Question 2 to show that now pairing is described by an angle-dependent gap function $\Delta_{\mathbf{k}}$ that (at temperature $T = 0$) satisfies

$$\Delta_{\mathbf{k}} = \frac{1}{2V} \sum_{\mathbf{k}'} U(\mathbf{k}, \mathbf{k}') \frac{\Delta_{\mathbf{k}'}}{\sqrt{\Delta_{\mathbf{k}'}^2 + \epsilon_{\mathbf{k}'}^2}}, \tag{7}$$

where $\epsilon_{\mathbf{k}} = \frac{\hbar^2 \mathbf{k}^2}{2m} - \epsilon_F$ are the energies of non-interacting electrons.

Show that if $U(\mathbf{k}, \mathbf{k}') = A_{\mathbf{k}} A_{\mathbf{k}'}$, where $A_{\mathbf{k}}$ is a specified function of $\mathbf{k}$, then the above equation is solved by a gap function of the form $\Delta_{\mathbf{k}} = A_{\mathbf{k}} \Delta$, where Δ is independent of $\mathbf{k}$. Derive a closed-form integral equation for Δ.

(b) [10 pts] Now consider a two-dimensional superconductor such that $A_{\mathbf{k}}$ is of the form $A_{\mathbf{k}} = A_0 \cos(2\phi)$ for $-\omega_0 < \epsilon_{\mathbf{k}} < \omega_0$ and zero otherwise. Here the angle ϕ describes vector $\mathbf{k}$ orientation, $\mathbf{k} = (k \cos \phi, k \sin \phi)$. Derive an integral equation which determines Δ in this case, however you need not solve this to obtain Δ explicitly.

(c) [10 pts] The Bogoliubov quaisparticles discussed in class are elementary excitations which have the dispersion relation $E_{\mathbf{k}} = \sqrt{\epsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$. Show that these excitations are gapless — that is, they have vanishing energies at certain specific values of $\mathbf{k}$ (NOT at all values of $\mathbf{k}$!). What are those values of $\mathbf{k}$?

(d) [10 pts] Let $\mathbf{k}_0$ be a nodal point for the excitation dispersion $E_{\mathbf{k}}$, and let $\delta\mathbf{k} = \mathbf{k} - \mathbf{k}_0$. Show that, upon a suitable choice of the coordinate axes, the energies of the elementary excitations near the point $\mathbf{k}_0$ have the approximate form $E(\delta\mathbf{k}) = \sqrt{C_1 \delta k_1^2 + C_2 \delta k_2^2}$ (massless Dirac particles). What are the values C_1, C_2?

(e) [10 pts] Argue that the nodes in the pairing function $\Delta_{\mathbf{k}}$ of the form found above are robust with respect to small changes of the pairing interaction $U(\mathbf{k}, \mathbf{k}')$. This type of a gap function is believed to describe some of the cuprate-based high-temperature superconductors.

4. Specific heat of a superconductor: conventional vs. exotic pairing

The electronic specific heat of quasiparticles in a superconductor can be evaluated as

$$C_{\mathrm{el}} = T \frac{dS}{dT} \tag{8}$$

where S is the entropy given by the standard result for a system of noninteracting fermions,

$$S = -2k_{\mathrm{B}} \sum_{\mathbf{k}} \left[(1 - f_{\mathbf{k}}) \ln(1 - f_{\mathbf{k}}) + f_{\mathbf{k}} \ln f_{\mathbf{k}} \right], \tag{9}$$

where the factor of 2 accounts for spin degeneracy. Here $f_{\mathbf{k}} = \frac{1}{e^{\beta E_{\mathbf{k}}+1}}$ and $E_{\mathbf{k}} = \sqrt{\Delta_{\mathbf{k}}^2 + \epsilon_{\mathbf{k}}^2}$.

(a) [10 pts] Show that the specific heat can be written as

$$C_{\mathrm{el}} = 2\beta k_{\mathrm{B}} \sum_{\mathbf{k}} \left(-\frac{\partial f_{\mathbf{k}}}{\partial E_{\mathbf{k}}} \right) \left(E_{\mathbf{k}}^2 + \frac{\beta}{2} \frac{d\Delta_{\mathbf{k}}^2}{d\beta} \right) \tag{10}$$

where $\beta = 1/k_{\mathrm{B}} T$.

Apply these results to an s-wave superconductor with a constant $\mathbf{k}$-independent Δ. Show explicitly that, if Δ is temperature-independent, the specific heat is exponentially small at low temperatures, i.e. it varies as $\exp(-\Delta/k_{\mathrm{B}} T)$ multiplied by a function which varies more slowly with temperature. (This activation T dependence remains approximately true even when the gap is a function of temperature, provided that $\Delta(T)$ remains finite at $T = 0$).

(b) [10 pts] Next, consider a superconductor with exotic pairing such as that discussed in Question 3. What is the density of states of the gapless elementary excitations found in Question 3 at very low energy?

(c) [10 pts] If we neglect the temperature dependence of the gap variable Δ at very low temperatures, $T \ll T_c$, what is the functional form of the temperature dependence of the specific heat at such temperatures for a superconductor with this type of energy gap?

5. A universal relation between $\Delta_{T=0}$ and the critical temperature T_c

In class we derived the BCS self-consistency equation for the order parameter $\Delta(T)$ at an arbitrary temperature

$$\Delta = \frac{U_0}{2}\sum_{\mathbf{k}'}(1-2f_{\mathbf{k}'})\frac{\Delta}{E_{\mathbf{k}'}}, \quad f_{\mathbf{k}} = \frac{1}{e^{\beta E_{\mathbf{k}}}+1}, \quad E_{\mathbf{k}} = \sqrt{\Delta^2+\epsilon_{\mathbf{k}}^2} \tag{11}$$

Here $f_{\mathbf{k}}$ is the Fermi distribution function, and $\beta = 1/k_{\mathrm{B}}T$.

(a) [10pts] Assuming weak coupling, $\nu_0 U_0 \ll 1$, show that the value of the energy gap at zero temperature, $2\Delta_0$, is related to the superconducting transition temperature, T_c , by

$$\frac{2\Delta_0}{T_c} = 8e^I, \quad I = \int_0^\infty \frac{\ln x}{\cosh^2 x}dx \tag{12}$$

(b) [10pts] Evaluate the integral I numerically or otherwise. Look up the significance of, and the numerical value of, the Euler's Gamma constant, usually denoted by γ.

Many superconductors were found to have a value for this ratio of around 3.5. The agreement with this data was one of the first major successes of the BCS theory. There are some materials with larger values of this ratio. These materials are called "strong-coupling" superconductors, and require a more sophisticated theory than BCS.