Atom-Photon Interaction

Reading Notes

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Oct 18, 2021

I. TRANSITION AMPLITUDES IN ELECTRODYNAMICS

(A) Probability amplitudes

Propagators $\mathcal{U}(t_2, t_1)$

State @ t_1 $\rightsquigarrow$ State @ t_2

$|\psi_i\rangle$ $|\psi_f\rangle$

Prob. finding $|\psi_f\rangle$ @ t_2 is

$$\boxed{P_r = \langle \psi_f | \mathcal{U}(t_2, t_1) | \psi_i \rangle}$$

generalize ... $t_1 \to t_2 \to t_3$ ...

$$\langle \psi_3 | U(t_3, t_2) \underbrace{|\psi_2\rangle\langle \psi_2|}\, \mathcal{U}(t_2, t_1) | \psi_1 \rangle$$

But can get interference

$\rightarrow$ sum over all intermediate states

$$\Rightarrow \Pr = \langle \psi_3 | \mathcal{U}_3(t_3, t_2) \underbrace{|\varphi_n\rangle\langle\varphi_n|}_{\substack{\text{summed} \\ \overline{} \\ \mathbb{1}}} \mathcal{U}_2(t_2, t_1) | \psi_1 \rangle$$

where $\{|\varphi_n\rangle\} \rightsquigarrow$ ONB complete

i.e.

$$\sum_n |\varphi_n\rangle\langle\varphi_n| = \mathbb{1}.$$

How to calculate $\langle \psi_2 | \mathcal{U}(t_2, t_1) | \psi_1 \rangle$?

$\Rightarrow$ need Hamiltonian.

In general, $\mathcal{H} = H_0 + V$ ← perturbation

H_0: unperturbed with ONB $\{|\varphi_n\rangle\} \leftrightarrow \{E_n\}$

Since any $|\Psi_i\rangle$ can be written in the $\mathcal{H}_0$-basis i.e.

$$|\Psi_i\rangle = \sum_j a_{ij} |\phi_j\rangle$$

$\Rightarrow$ $\langle \Psi_2 | U(t_2, t_1) | \Psi_1 \rangle$ will be some function of the terms

$$\langle \varphi_p | \mathcal{U}(t_2, t_1) | \varphi_n \rangle$$

<u>transition amplitudes</u> between unperturbed & perturbed states.

B. TIME DEPENDENCE OF TRANSITION AMPLITUDES

① Coupling between Discrete Isolated States

- Consider $\mathcal{H}_0$ where there is one or several discrete eigenstates that are well-isolated from all other eigenstates ...

- Consider just one state ...

$$|\varphi_1\rangle \underline{\qquad\qquad\qquad} E_1$$

Consider amplitude $\tilde{\mathcal{U}}_{11}(T) = \langle\varphi_1|\tilde{U}(T)|\varphi_1\rangle$

where

$$\tilde{\mathcal{U}}(T) = e^{i\mathcal{H}_0 T/2\hbar}\,\mathcal{U}(T)\,e^{-i\mathcal{H}_0 T/2\hbar}$$

evolves $t_i = -T/2 \longrightarrow t_f = T/2$

Heisenberg picture / interaction representation.

$\Rightarrow$ $\tilde{U}_{11}(T)$ represents Pr that system

@ $|\varphi_1\rangle$ @ t_1 <u>remains</u> @ $|\varphi_1\rangle$ after T

can show, by inserting $\sum_n |\phi_n\rangle\langle\phi_n| = \mathbb{1}$

on the right / left of $U(T)$ in $\tilde{U}(T)$

$$\boxed{\tilde{U}_{11}(T) \simeq |\langle\varphi_1|\psi_1\rangle|^2 e^{-i\delta E_1 T/\hbar}}$$

Here $|\psi_1\rangle$ is an eig-state of $\mathcal{H}$

which $\longrightarrow |\varphi_1\rangle$ as $V \to 0$.

and

$$\delta E_1 = \langle\varphi_1|V|\varphi_1\rangle + \sum_{n\neq 1} \frac{\langle\varphi_1|V|\varphi_n\rangle\langle\varphi_n|V|\varphi_1\rangle}{E_1 - E_n} + \cdots$$

shift of $|\varphi_1\rangle \to |\psi_1\rangle$ due to V (coupling)

pert. theory 1st order.

• What if now $\mathcal{H}_0$ has 2 levels ?

$|\varphi_1\rangle$ ———— E_1 & $|\varphi_2\rangle$ ———— E_2

degenerate levels ...

• Look at amplitudes

$$\tilde{U}_{21}(T) = \langle \varphi_2 | \tilde{\mathcal{U}}(T) | \varphi_1 \rangle$$

$$\Downarrow$$

$$\boxed{P_{21}(T) = |\tilde{U}_{21}(T)|^2}$$

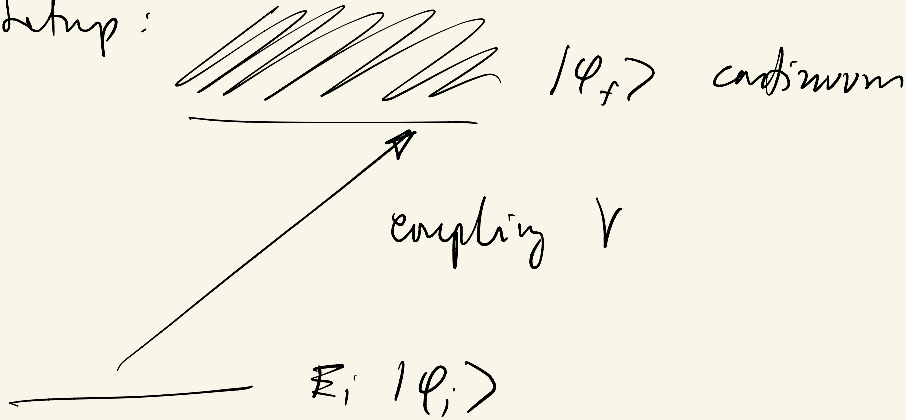
this is oscillatory, or "Rabi oscillation", between $|\varphi_1\rangle$ & $|\varphi_2\rangle$

→ see other places for details ...
Basically freqs (Rabi freqs) depends on the coupling / perturbation V.

→ most common when atom in monochromatic EM field (laser)

② Resonant Coupling between a Discrete Level & a Continuum

- Setup:



- In this case, transition amplitude is

$\underline{\tilde{\mathcal{U}}_{fi}(T) = \langle \varphi_f | \tilde{\mathcal{U}}(T) | \varphi_i \rangle}$ is well known

To lowest orders ...

δ-fn of width $\hbar/T$

$$\boxed{\begin{aligned} \tilde{\mathcal{U}}_{fi}(T) &= \langle \varphi_f | \tilde{\mathcal{U}}(T) | \varphi_i \rangle \\ &\simeq \delta_{fi} - 2\pi i \, \delta^{(T)}(E_f - E_i) V_{fi} + \cdots \end{aligned}}$$

$V_{fi} = \langle \varphi_f | V | \varphi_i \rangle$

- $\Pr(|\varphi_i\rangle \to |\varphi_f\rangle)$ is then $\propto$ duration of interaction $\Rightarrow$ can define <u>transition rate</u> as

$$w_{fi} = \frac{1}{T}\left|\tilde{U}_{fi}(T)\right|^2 \simeq \frac{2\pi}{\hbar}|V_{fi}|^2 \delta^{(T)}(E_f - E_i)$$

- However, $|\varphi_f\rangle$ are not normalizable ...

  $\to$ only makes sense to talk about transition to a group of final states

$\Rightarrow$ "Physical" transition rate

$$\Gamma = \sum_f w_{fi} = \frac{2\pi}{\hbar}\sum_f |V_{fi}|^2 \delta^{(T)}(E_f - E_i)$$

$$\Rightarrow \boxed{\Gamma = \frac{2\pi}{\hbar}|V_{fi}|^2 \rho(E_f = E_i)}$$

assumed that $|V_{fi}|^2$ only depends on E_f

density of final states evaluated at $E_f = E_i$

(Fermi's Golden Rule)

- How does "lifetime" come out of this?

↳ need to calculate the prob. that the system remains in $|\psi_i\rangle$ ...

$$|\tilde{U}_{ii}(T)|^2 = 1 - \sum_f |\tilde{U}_{fi}(T)|^2 \simeq 1 - \Gamma T$$

↳ We will show much later on that in fact (non perturbatively)

$$\boxed{\tilde{U}_{ii}(T) = e^{-\Gamma T/2} e^{-i\delta E_i T/\hbar}}$$

decays exponentially as time with "lifetime"

$$\tau = 1/\Gamma$$

- Here the shift δE also occurs due to coupling to continuum

$$\delta E_i = \mathcal{P} \sum_f \frac{|V_{fi}|^2}{E_i - E_f}$$

$\mathcal{P}$: principal part ...

③ Coupling inside a Continuum or between Continua

↳ Both $|\varphi_i\rangle \rightsquigarrow |\varphi_f\rangle$ are continuous eigenstates of $\mathcal{H}_0$

$\Rightarrow$ frequently seen in the study of <u>scattering theory</u> of photons by atoms

Here $|\varphi_i\rangle, |\varphi_f\rangle \rightarrow$ represents atom in a given E-level

photons come in & scatter, but they can carry a continuum of energies...

$\rightarrow$ The theory here can be found in Sakurai Ch. 7 (Scattering Theory)

But to quickly summarize ---

$$\tilde{U}_{fi}(T) = \delta_{fi} - 2\pi i \underbrace{\delta^{(T)}(E_f - E_i)}_{\delta\text{-fn with width } \hbar/T} \mathcal{T}_{fi}$$

transition matrix from the Born expansion ---

$$\mathcal{T}_{fi} = (\varphi_f|V|\varphi_i)$$
$$+ (\varphi_f|V \frac{1}{E_i - \mathcal{H}_0 + i\eta} V|\varphi_i)$$
$$+ (\varphi_f|V \frac{1}{E_i - \mathcal{H}_0 + i\eta} V \frac{1}{E_i - \mathcal{H}_0 + i\eta} V|\varphi_i)$$
$$+ \cdots$$

where η is infinitesimal.

(cf. <u>Lippmann-Schwinger eqn</u>).

- In any case, can still define transition rate ...

$$\begin{cases} w_{fi} = \frac{1}{T} |\tilde{u}_{fi}(T)|^2 \\ \quad = \frac{2\pi}{\hbar} |\mathcal{T}_{fi}|^2 \delta^{(T)}(E_f - E_i) \end{cases}$$

- Once again we'll have to sum over f because $|\varphi_f\rangle$ not normalizable.
  → causes $\rho(E_f = E_i)$ to appear.

- $|\varphi_i\rangle$ also not normalizable ...
  But it turns out that the "scattering cross section" takes care of this worry ...

⇒ It's possible to derive cross-section from transition amplitudes.
↳ we'll see some examples soon ...

Ⓒ APPLICATIONS TO ELECTRODYNAMICS

① Coulomb Gauge Hamiltonian