8.321 Recitation 1-2

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1 Hamiltonian Mechanics

- (Assume time independence)
- Positions q_i, momenta p_i, Hamiltonian $H\left[q_i, p_i\right]$.
- Equation of motion: $\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}$.
- Poisson bracket: $\{\omega, \lambda\} = \frac{\partial \omega}{\partial q_i}\frac{\partial \lambda}{\partial p_i} - \frac{\partial \omega}{\partial p_i}\frac{\partial \lambda}{\partial q_i}$.
- Time evolution: $\frac{d\omega}{dt} = \{\omega, H\}$.
- Canonical transformation (preserving EOM): $\bar{q}_i\left(q_j, p_j\right), \bar{p}_i\left(q_j, p_j\right)$ satisfying $\{\bar{q}_i, \bar{q}_j\} = \{\bar{p}_i, \bar{p}_j\} = 0, \{\bar{q}_i, \bar{p}_j\} = \delta_{ij}$.
- Exercise: Poisson brackets are invariant under canonical transformations.
- Noether's Theorem: H is invariant under infinitesimal $q_i \to q_i + \epsilon\frac{\partial g}{\partial p_i}, p_i \to p_i - \epsilon\frac{\partial g}{\partial q_i}$ iff g is conserved.
- Generator of symmetry g: $\delta\omega = \epsilon\{\omega, g\}$.

1.1 Correspondence to Quantum Mechanics

- Quantization rule: $\omega \to$ operators $\hat{\omega}$, $\{\omega, \lambda\} \to \left[\hat{\omega}, \hat{\lambda}\right]/i\hbar$.
- Heisenberg picture: $\frac{d\hat{A}}{dt} = \left[\hat{A}, \hat{H}\right]/i\hbar$.
- Canonical commutators: $\left[\hat{q}_i, \hat{q}_j\right] = \left[\hat{p}_i, \hat{p}_j\right] = 0, \left[\hat{q}_i, \hat{p}_j\right] = i\hbar\delta_{ij}$.
- Generator of symmetry: $\delta\hat{A} = \epsilon\left[\hat{A}, \hat{G}\right]/i\hbar$, $\hat{G}$ is the generator of a symmetry group.

2 Pauli Matrices

- (Anti)commutators: $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \{\sigma_i, \sigma_j\} = 2\delta_{ij}$.
- Product of Pauli matrices: $(\mathbf{A} \cdot \boldsymbol{\sigma})(\mathbf{B} \cdot \boldsymbol{\sigma}) = \mathbf{A} \cdot \mathbf{B} + i(\mathbf{A} \times \mathbf{B}) \cdot \boldsymbol{\sigma}$ where $\mathbf{A}, \mathbf{B}$ are vectors.
- Exponential: $\exp(i\theta\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}) = \cos\theta + i(\hat{\mathbf{n}} \cdot \boldsymbol{\sigma})\sin\theta$.
- Rotation of spin by angle θ about $\hat{\mathbf{n}}$ axis: $\hat{R}|s\rangle = \exp(-i\theta\hat{\mathbf{n}} \cdot \boldsymbol{\sigma}/2)|s\rangle$.

3 Exponentials

- BCH formula: $e^A e^B = \exp\left(A + B + \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [B, [B, A]]) + ...\right)$.
- $e^A e^B = e^{A+B}$ if $[A, B] = 0$.
- $e^A e^B = \exp\left(A + B + \frac{1}{2}[A, B]\right)$ if $[A, B]$ is central i.e. commutes with A and B.
- Exercise: $e^A e^B = \exp\left(A + \frac{s}{1-e^{-s}}B\right)$ if $[A, B] = sB$ where s is a constant.