Neutrino Oscillations

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1 Neutrino Oscillations

Neutrinos are extremely light leptons with zero electric charge and thus, in the standard model, they only interact via the weak and gravitational forces. So far we have observed there to be three “flavour eigenstates” of neutrinos, one to interact which each of the electron, muon, and tau leptons via the weak force. For the purposes of this discussion we will ignore the tau neutrino, so that we can work with a two state system.[1]

One basis for our system may be constructed from the eigenstates of the weak Hamiltonian, namely, the electron neutrino $|\nu_e\rangle$ and the muon neutrino $|\nu_\mu\rangle$. In this basis the weak force Hamiltonian is diagonal. However, when neutrinos travel through space and are not interacting via the weak force, the evolution of their state is governed by the free Hamiltonian, H_0. In general, the eigenstates of H_0 might not coincide with the flavour eigenstates. Denoting the eigenstates of H_0 by $|\nu_1\rangle\,,\ |\nu_2\rangle$ with eigenvalues $E_1,\ E_2$, respectively. We may write, using the decomposition of (2):

$$\begin{aligned} |\nu_1\rangle &= \cos\theta\,|\nu_e\rangle + e^{i\phi}\sin\theta\,|\nu_\mu\rangle \\ |\nu_2\rangle &= -\sin\theta\,|\nu_e\rangle + e^{i\phi}\cos\theta\,|\nu_\mu\rangle \end{aligned} \tag{1}$$

for some $\theta,\ \phi \in [0, 2\pi)$, since $\{|\nu_1\rangle\,,\ |\nu_2\rangle\}$ forms an orthonormal basis as H_0 is Hermitian. Note that the phase of $|\nu_\mu\rangle$ is unphysical, so we may remove the phase $e^{i\phi}$ in (3) by simply re-defining what we call $|\nu_\mu\rangle$ to be $e^{-i\phi}\,|\nu_\mu\rangle$. This removes the phase[2] in (3) allowing us to

[1] Note that we will assume that we are working in the subspace of states of neutrinos with fixed momentum and spin so that the only degree of freedom is the flavour and hence, we have a two state system.

[2] The freedom to remove such phases depends on the general form of a state in the projective Hilbert space (2). An important consequence of this is with the proper treatment of this problem including all three flavour eigenstates. In the three state system version of this problem, not all phases can be removed so there end up being three mixing angles and one phase left over in the decomposition analogous to (4). This leftover phase, if non-zero, violates CP-symmetry and could help explain the abundance of matter over anti-matter in the universe! If you’re interested you might want to look up the “PMNS matrix”.

write the most general form of the H_0 eigenstates as

$$\begin{aligned}
|\nu_1\rangle &= \cos\theta\,|\nu_e\rangle + \sin\theta\,|\nu_\mu\rangle \\
|\nu_2\rangle &= -\sin\theta\,|\nu_e\rangle + \cos\theta\,|\nu_\mu\rangle \\
&\text{with inverse transformation} \\
|\nu_e\rangle &= \cos\theta\,|\nu_1\rangle - \sin\theta\,|\nu_2\rangle \\
|\nu_\mu\rangle &= \sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle
\end{aligned} \tag{2}$$

where the parameter θ is called a 'mixing angle'.

Suppose an electron neutrino is created by a nuclear reaction (at $t = 0$) (for example $p^+p^+ \to p^+ne^+\nu_e$ the first nuclear fusion reaction in stars). Such a neutrino will then propagate under the free particle Hamiltonian until it interacts again via the weak force. Under time evolution the $|\nu_1\rangle$ and $|\nu_2\rangle$ pieces of the state will pick up different phases:

$$|\nu_e(t)\rangle = \cos\theta e^{-iE_1t/\hbar}\,|\nu_1\rangle - \sin\theta e^{-iE_2t/\hbar}\,|\nu_2\rangle\,. \tag{3}$$

Thus the probability amplitude at time t for the neutrino to interact via the weak force as an electron neutrino is:

$$\langle\nu_e|\nu_e(t)\rangle = \cos^2\theta\; e^{-iE_1t/\hbar} + \sin^2\theta\; e^{-iE_2t/\hbar}. \tag{4}$$

Thus the probability that the neutrino is observed as an electron neutrino at time t is:

$$\begin{aligned}
P(\nu_e \to \nu_e) &= |\,\langle\nu_e|\nu_e(t)\rangle\,|^2 && (5)\\
&= \sin^4\theta + \cos^4\theta + \sin^2\theta\cos^2\theta(e^{-i(E_2-E_1)t/\hbar} + e^{i(E_2-E_1)t/\hbar}) && (6)\\
&= 1 - 2\sin^2\theta\cos^2\theta + 2\sin^2\theta\cos^2\theta\cos\left(\frac{\Delta Et}{\hbar}\right) && (7)\\
&= 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta Et}{2\hbar}\right). && (8)
\end{aligned}$$

Letting m_i be the mass of the i^{th} "mass eigenstate" of H_0, the eigenvalues of H_0 will be given by the relativistic formula:

$$E_i = \sqrt{p^2c^2 + m_i^2c^4} \approx pc\left(1 + \frac{m_i^2c^2}{2p^2}\right), \tag{9}$$

where p is the absolute value of the momentum. Thus at lowest order in m/p the energy difference is:

$$\Delta E = E_2 - E_1 = \frac{(m_2^2 - m_1^2)c^3}{2p} = \frac{\Delta m^2c^3}{p}, \tag{10}$$

where $\Delta m^2 = (m_2^2 - m_1^2)$. Thus, in terms of the difference of square masses the probability to observe an electron neutrino is:

$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2c^3t}{4p\hbar}\right). \tag{11}$$

Note that to lowest order in m/p the neutrino travels at c and has energy $E = pc$. Thus to travel a distance L, the time it takes is $t = L/c$, giving a formula:

$$P(\nu_e \to \nu_e) = 1 - \sin^2{(2\theta)} \sin^2\left(\frac{\Delta m^2 c^3 L}{4E\hbar}\right). \tag{12}$$

By measuring the proportion of neutrinos at a specific energy that arrive a distance L away from their production point as function of E one can determine the values of Δm^2 and θ.

Plugging in the experimentally measured values of $\Delta m^2 = 7.53 \pm 0.18 * 10^{-5} eV^2$ and $\sin^2{(2\theta)} = 0.846 \pm 0.021$ relevant for this problem[3] and a typical solar neutrino energy $E = 400keV$ we compute:

$$P(\nu_e \to \nu_e) = 1 - 0.846 \sin^2\left(9.54 * 10^{-4} L[m]\right). \tag{13}$$

Since the oscillation period is of order $10^4 m$ it doesn't make too much sense to plug in the distance to the sun $10^{11} m$ for L as we would have to know L to a precision of at least $10^3 m$ to make any kind of meaningful prediction; however, it is clear that $P(\nu_e \to \nu_e)$ will in general be different from one. Thus, even if we produce many electron neutrinos at one location, if we observe them after they have propagated for a while we will see less than we expect without oscillations! A lack of neutrinos from the sun was first observed in the 1960's by the 'Homestake experiment' and was called the 'solar neutrino problem'. This was resolved in the early 2000's as SNO and Kamiokande observed neutrino oscillations through other means.

Note that the dependence of (15) on Δm^2 is independent of its sign. This is related to the so called (neutrino mass) 'hierarchy problem' in which present day experiments have measured the differences in mass squared of two pairs of mass eigenstates; however the relationship between the other pair is unknown!

[3] Note that here we are ignoring the mixing with third neutrino state (which is okay for our purposes as the mixing angle θ_{13} is small)