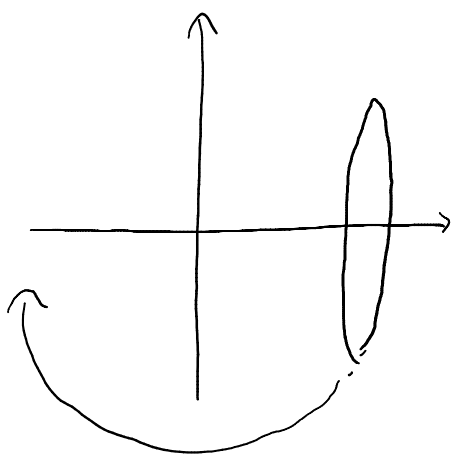
How to measure Squeezing? Homodyne Detection

The squeezed state rotates at an optical frequency, extraordinarily difficult to measure directly.

Trick: Need to "compare" it with a "local oscillator" oscillating at the same (or only slightly different) frequency. If the frequency is equal: homo-dyning
unequal: hetero-dyning.

What does "compare" mean? Mixing!



Classically:

$X(t) = c_0 \cos(\omega t) + s_0 \sin(\omega t)$

Reference oscillator

$B(t) = B_0 \cos(\omega t + \delta)$

Signal: $I = \frac{1}{T}\int X(t) \cdot B(t)\, dt$

$$I = B_0 c_0 \frac{1}{T}\int dt \cos(\omega t)\cos(\omega t + \delta) + B_0 s_0 \frac{1}{T}\int dt \sin(\omega t)\cos(\omega t + \delta)$$

$$= B_0 c_0 \frac{1}{T}\int dt \left(\cos(\delta) - \cancel{\cos(2\omega t + \delta)}\right)$$

$$+ B_0 s_0 \frac{1}{T}\int dt \left(\sin(\delta) + \cancel{\sin(2\omega t + \delta)}\right)$$

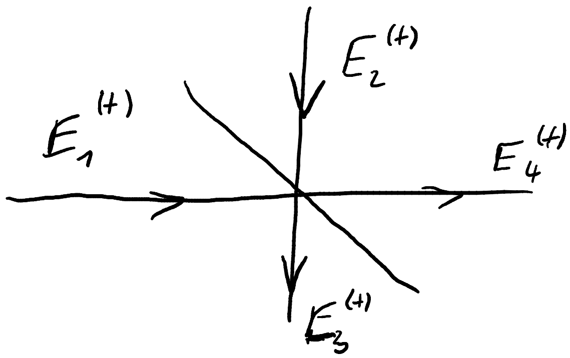
fast components average to 0

$$= B_0 \left(c_0 \cos(\delta) + s_0 \sin(\delta)\right)$$

signal prop. to local oscillator

phase controls whether I measure the c_0 or s_0 quadrature.

- How to mix? Use beamsplitter, i.e. semi-refl. mirror. In classical optics, continuity relations are imposed on the classical e.m. field at the interface of the layers deposited on mirror

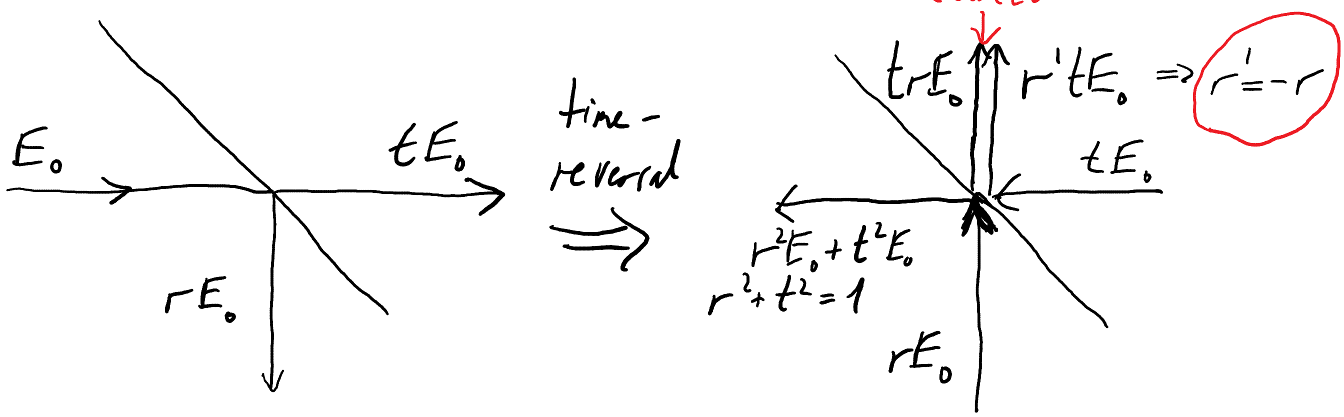


$$E_3^{(+)} = \frac{1}{\sqrt{2}}\left(E_1^{(+)} + E_2^{(+)}\right)$$
$$E_4^{(+)} = \frac{1}{\sqrt{2}}\left(E_1^{(+)} - E_2^{(+)}\right)$$

Signs are crucial!
Incident power $|E_1|^2 + |E_2|^2$
$=$ Outgoing power $|E_3|^2 + |E_4|^2$

Simple way to see this:

these two must cancel



Quantum beamsplitter: We could either evolve the states or evolve the operators (Heisenberg or Schrödinger picture) Heisenberg is easier.

Field operators transform like the classical description:

$$\hat{E}_3^{(+)} = \frac{1}{\sqrt{2}}\left(\hat{E}_1^{(+)} + \hat{E}_2^{(+)}\right)$$

$$\hat{E}_4^{(+)} = \frac{1}{\sqrt{2}}\left(\hat{E}_1^{(+)} - \hat{E}_2^{(+)}\right)$$

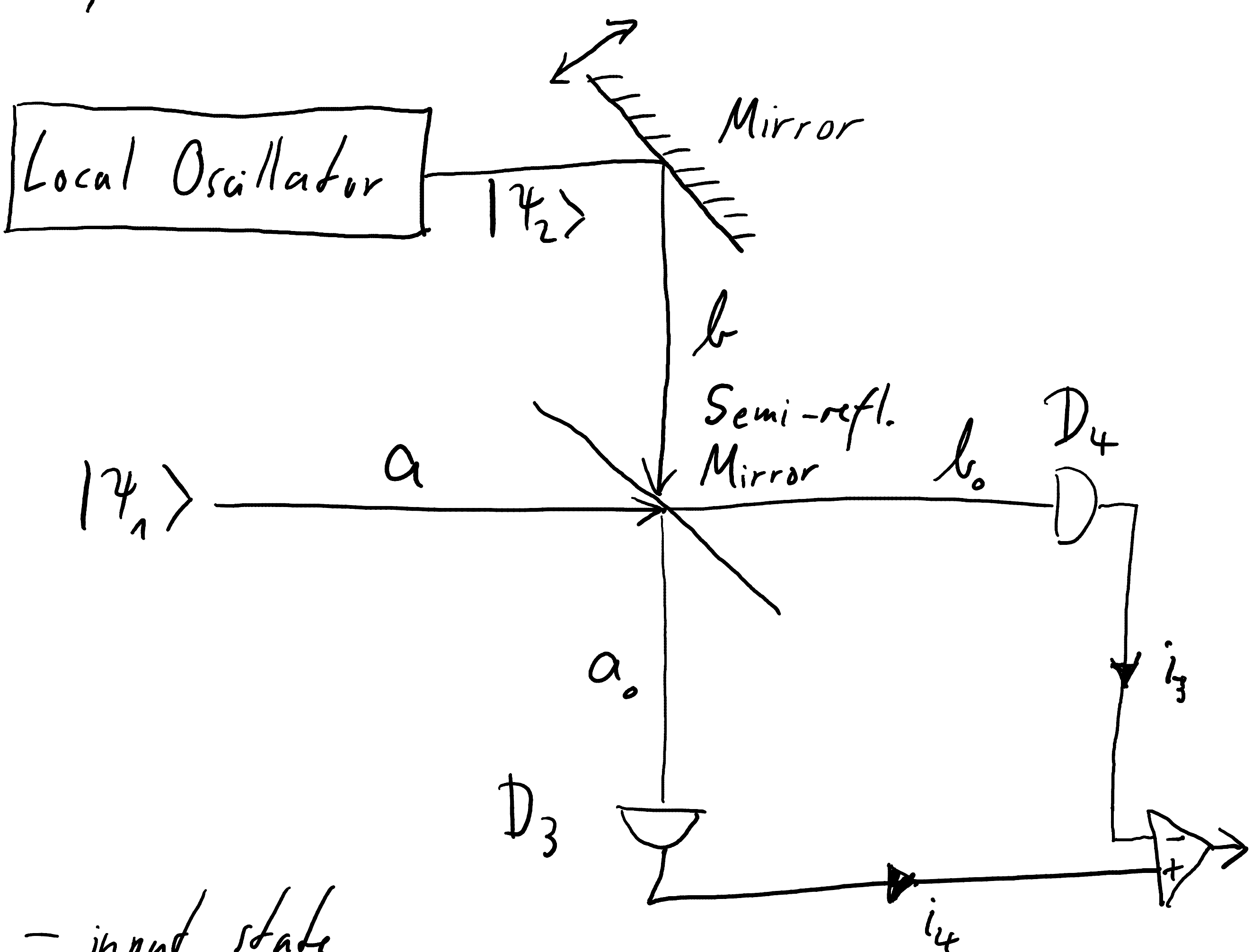
This ensures $[a_3, a_3^+] = 1$, $[a_4, a_4^+] = 1$

or $$a_3 = \frac{a_1 + a_2}{\sqrt{2}}$$

$$a_4 = \frac{a_1 - a_2}{\sqrt{2}}$$

- How to measure Squeezing:

Homodyne Detection



$|\psi_1\rangle$ – input state

$|\psi_2\rangle \equiv |\beta\rangle$ strong coherent state, eigenstate of b with eigenvalue β

All frequencies are equal $\Rightarrow$ same $\mathcal{E}_e$.

detector measures $\overline{i_3 - i_4}$, proportional to

$$\bar{d} = \langle\Psi|\hat{E}^{(-)}(\vec{r}_3)\,\hat{E}^{(+)}(\vec{r}_3)|\Psi\rangle - \langle\Psi|\hat{E}^{(-)}(\vec{r}_4)\hat{E}^{(+)}(\vec{r}_4)|\Psi\rangle$$

$$\hat{E}^{(+)}(\vec{r}_3) = i\,\mathcal{E}\,e^{i\vec{k}_3\cdot\vec{r}_3}\,a_0 \quad \text{with} \quad a_0 = \frac{a+b}{\sqrt{2}}$$

$$\hat{E}^{(+)}(\vec{r}_4) = i\,\mathcal{E}\,e^{i\vec{k}_4\cdot\vec{r}_4}\,b_0 \quad \text{with} \quad b_0 = \frac{a-b}{\sqrt{2}}$$

$$|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle = |\Psi_1\rangle \otimes |\beta\rangle$$

$$\bar{d} = -\mathcal{E}^2\langle\Psi(t)|a_0^+ a_0 - b_0^+ b_0|\Psi(t)\rangle$$

$$= -\mathcal{E}^2\langle\Psi(t)|a^+ b + b^+ a|\Psi(t)\rangle$$

Now $\langle\beta(t)|b|\beta(t)\rangle = \beta\, e^{-i\omega t} = |\beta| e^{i\varphi} e^{-i\omega t}$

If $|\Psi_1\rangle$ describes state in single mode of the field,

$\langle\Psi_1(t)|a|\Psi_1(t)\rangle = e^{-i\omega t}\langle\Psi_1(0)|a|\Psi_1(0)\rangle$

$\Rightarrow$ $\bar{d}$ is independent on time and can be written as a function of the state $|\Psi_1(0)\rangle$ of the input mode of radiation at time $t=0$:

$$\bar{d} = -\mathcal{E}^2|\beta|\langle\Psi_1(0)|(e^{-i\varphi}a + e^{i\varphi}a^+)|\Psi_1(0)\rangle$$

Introduce mode a quadrature operators

$$\hat{E}_Q = \mathcal{E}(a + a^+) = \mathcal{E}\sqrt{\tfrac{2}{\hbar}}\,\hat{x}$$

$$\hat{E}_p = -i\mathcal{E}(a - a^+) = \mathcal{E}\sqrt{\tfrac{2}{\hbar}}\,\hat{p}$$

$$\Rightarrow \bar{d} = -|\beta|\,\mathcal{E}\Big(\cos\varphi \langle\Psi_1(0)|\hat{E}_Q|\Psi_1(0)\rangle + \sin\varphi\langle\Psi_1(0)|\hat{E}_p|\Psi_1(0)\rangle\Big)$$

By choosing $\varphi = 0$ or $\frac{\pi}{2}$, we can measure the expectation value of either $\hat{E}_Q$ or $\hat{E}_p$, resp.

Analogously, one can show that

$$\overline{(i_3 - i_4)^2} = \overline{i_3^2} + \overline{i_4^2} - 2\,\overline{i_3 i_4}$$

gives

$$\overline{d^2} = \mathcal{E}^4\Big(|\beta|^2\langle\Psi_1(0)|(e^{-i\varphi}a + e^{i\varphi}a^+)^2|\Psi_1(0)\rangle + \langle\Psi_1(0)|a^+a|\Psi_1(0)\rangle\Big)$$

$\Rightarrow$ If the second term is negligible, i.e. for strong β, we can directly measure the fluctuations of the quadratures $\hat{E}_Q$ and $\hat{E}_p$.

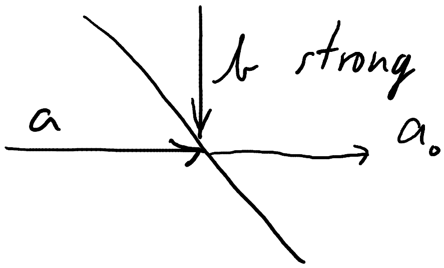
E.g. if $\varphi = 0$, we measure

$$\overline{d^2} = (\mathcal{E}|\beta|)^2\langle\Psi_1(0)|E_Q^2|\Psi_1(0)\rangle$$

$|\beta|$ allows to increase signal

Unbalanced homodyne:

$|t|^2 = 99\%$



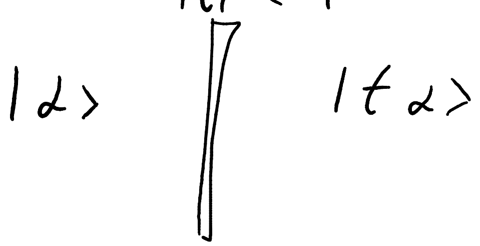
(replace by c-number β)

$$
\begin{aligned}
a_0 &= t\,a + \sqrt{1-t^2}\,b \\
&\approx \underbrace{\cancel{t}}_{\approx 1}\left(a + \sqrt{1-t^2}\,\beta\right) \\
&= D\left(\sqrt{1-t^2}\,\beta\right)\, a \, D\left(\sqrt{1-t^2}\,\beta\right)^{\dagger}
\end{aligned}
$$

$\Rightarrow$ implementation of the displacement operator.

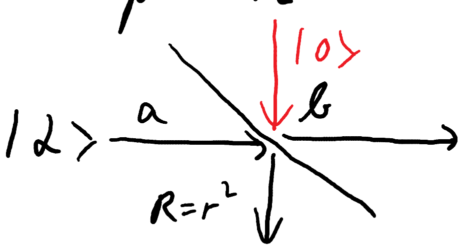
Attenuator (see homework)

$|t|^2 < 1$



Incorrect picture:

Correct picture: Beamsplitter mixes in vacuum:



$a_0 = t\,a + r\,b$

$$\langle \hat{n}_0 \rangle = \langle \alpha, 0 | (ta + rb)^+ (ta + rb) | \alpha, 0 \rangle$$
$$= |t|^2 |\alpha|^2$$

$$\langle \hat{n}_0^2 \rangle = \langle \alpha, 0 | (ta + rb)^+ (ta + rb)(ta + rb)^+ (ta + rb) | \alpha, 0 \rangle$$

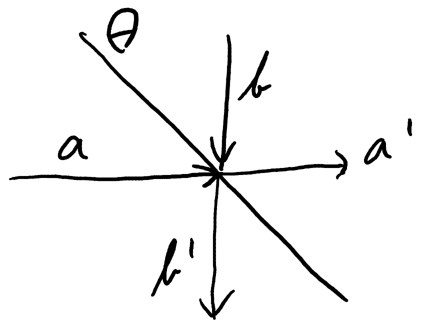
$$[ta + rb, t^* a^+ + r^* b^+] = |t|^2 + |r|^2 = 1$$

$$\Rightarrow \langle \hat{n}_0^2 \rangle = |t|^4 |\alpha|^4 + |t|^2 |\alpha|^2$$

$$\Delta n_0^2 = \langle \hat{n}_0^2 \rangle - \langle \hat{n} \rangle^2 = |t|^2 |\alpha|^2 = \langle \hat{n}_0 \rangle$$ as it should be.

Single Photons: $|0\rangle$ = no photon $|1\rangle$ = single photon st.

General Beamsplitter (see HW)



Mixing of modes:

$$H = i\theta (a b^+ - a^+ b)$$

$$B = e^{iH} = \exp(\theta (a^+ b - b^+ a))$$

H coherently transfers a photon from b to a and vice versa. It's like a spin-flip, but here we have ∞ many levels.

More precisely, can introduce $S_+ = a^+ b$

$$S_- = a b^+$$

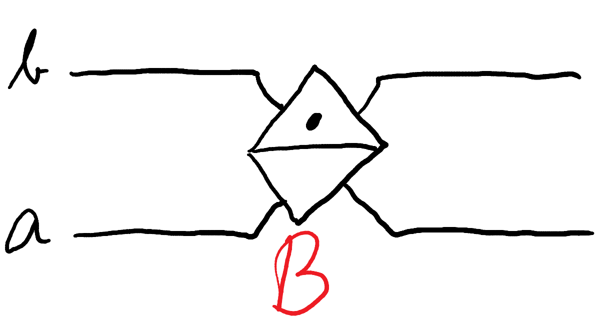
$$S_z = a^+ a - b^+ b$$

$$S_x = a b^+ + a^+ b$$

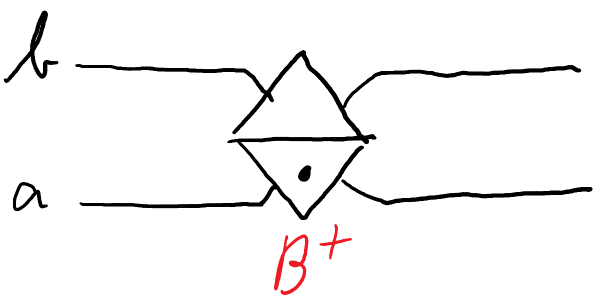
$$S_y = i(a b^+ - a^+ b)$$

$$\Rightarrow H = \theta S_y$$
$$B = \exp(\theta S_y)$$
= rotation about y-axis

$$B a B^{+} = a\cos\theta + b\sin\theta = a'$$
$$B b B^{+} = -a\sin\theta + b\cos\theta = b'$$
$$\theta = \frac{\pi}{4}$$



$$b' = \frac{b-a}{\sqrt{2}}$$
$$a' = \frac{b+a}{\sqrt{2}}$$



$$\frac{a+b}{\sqrt{2}}$$
$$\frac{a-b}{\sqrt{2}}$$

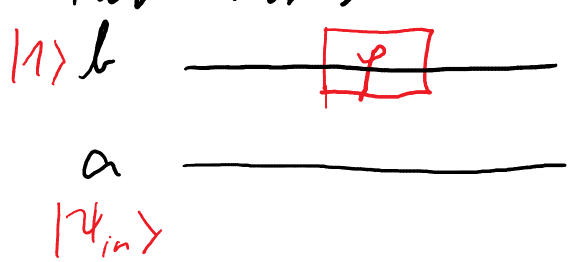
Matrix representation $\begin{bmatrix} a \\ b \end{bmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{"}B\text{"}} \begin{bmatrix} a \\ b \end{bmatrix}$

Phase Shifter

$$|1\rangle \xrightarrow[\text{Medium}]{\text{Time}} \begin{array}{l} e^{i\omega_0 t}|1\rangle \\ e^{i\omega_0 t + i\varphi}|1\rangle \end{array} \Rightarrow \text{phase shift}$$

Phase has to be compared to a reference

$\Rightarrow$ Two modes



$e^{i\varphi}|1\rangle$

$|\psi_{out}\rangle$

Single photons $|\overset{b}{1}\,\overset{a}{0}\rangle$

$|0\,1\rangle$

$$B\,\underbrace{|\overset{b}{1}\,\overset{a}{0}\rangle}_{|1\rangle\otimes|0\rangle} = B\,b^{+}\,B^{+}B\,\underbrace{|00\rangle}_{|0,0\rangle}$$

$$= -\sin\theta\,|01\rangle + \cos\theta\,|10\rangle$$

$$B\,|0\,1\rangle = \cos\theta\,|01\rangle + \sin\theta\,|10\rangle$$

B conserves the photon number

$$B\,|11\rangle = -\sqrt{2}\,\sin\theta\cos\theta\,|0\,2\rangle$$
$$+\sqrt{2}\,\sin\theta\cos\theta\,|2\,0\rangle$$
$$+(\cos^2\theta - \sin^2\theta)\,|11\rangle$$

Restrict ourselves to $\{\cancel{|00\rangle}\ |01\rangle\ |10\rangle\}$

always <u>one</u> photon

"Dual-rail photon state space"
spanned by $|01\rangle$ and $|10\rangle$
Two-level system

Arbitrary state $|\psi\rangle = \alpha|01\rangle + \beta|10\rangle$

Theorem: Any $|\psi\rangle$ can be created from $|01\rangle$ by beamsplitters and phase shifters.

Proof: $|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$B_\theta |\psi\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

Rotation around $\hat{y}$

Phase shifter $\underbrace{e^{-i\varphi/2}}_{\text{irrelevant global phase}} \underbrace{\begin{bmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{bmatrix}}_{\text{Rotation around } \hat{z}} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$B(\theta) = R_y(-2\theta)$ $\qquad P(\varphi) = R_z(-\varphi)$

$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta)$

Euler's angles

Bloch's theorem $\Rightarrow$ see Wiki for proof.

Modern language: qubit $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

arbitrary single qubit operations ("gates") can be performed by phase-shifters and beam splitters.

example: Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$