Continuity: Exercises 4.1 - 4.10, Baby Rudin

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4.1 *Proof.* Let f a real function on $\mathbb{R}$ which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)] = 0$$

for every $x \in \mathbb{R}$ be given. To prove: f is not continuous. Consider this counterexample:

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Clearly, f satisfies the conditions above, but f is not continuous at 0. □

4.2 *Proof.* Let f a continuous mapping of a metric space X into a metric space Y be given. To prove: $f(\bar{E}) = \overline{f(E)}$ for every set $E \subset X$. Let subset $E \subset X$ be given. If $f(\bar{E}) = \emptyset$ then there's nothing to prove. If $f(\bar{E}) \neq \emptyset$, then pick $y \in f(\bar{E})$ and so there is some $x \in \bar{E}$ such that $y = f(x)$. Now, $x \in \bar{E} = E \cup E'$, so $x \in E'$ or $x \in E$. If $x \in E$ then $y = f(x) \in f(E) \subset \overline{f(E)}$. If $x \in E'$, then x is a limit point of E. We now want to show $f(x)$ is a limit point of $f(E)$. Let $\epsilon > 0$ be given, then because f is continuous, $\exists \delta > 0$ such that $d(f(x_0), f(x)) < \epsilon$ whenever $d(x_0, x) < \delta$, for all $x_0 \in X$. x is a limit point of E, so for some $\delta > \delta' > 0$, there is $x_1 \in E$. This means $f(x_1) \in \mathcal{N}_\epsilon(x)$ for some $f(x_1) \in f(E)$. This means $f(p)$ is a limit point of $f(E)$. So, $f(p) \in \overline{f(E)}$. Therefore, $f(\bar{E}) \subset \overline{f(E)}$.

An example in which $f(\bar{E}) \subsetneq \overline{f(E)}$. Let $E = \mathbb{N} \subsetneq \mathbb{R}$. We know that $\mathbb{N} = \overline{\mathbb{N}}$. Now, define $f : \mathbb{R} \to \mathbb{R}$ by $f(n) = \frac{1}{n}$. Obviously, $f(\mathbb{N}) = f(\bar{\mathbb{N}}) = \{1/n : n \in \mathbb{N} \setminus \{0\}\}$ since the input sets are the same. Now, $\overline{f(\mathbb{N})} = \overline{\{1/n : n \in \mathbb{N}\}} \cup \{0\}$. So, $f(\bar{\mathbb{N}}) \subsetneq \overline{f(\mathbb{N})}$. □

4.3 *Proof.* Let f a continuous real function on a metric space X be given. Consider the zero set $Z(f)$ of f. We want to show $Z(f)$ is closed. We notice that $Z(f) \equiv f^{-1}(\{0\})$, where the set $\{0\}$ is closed. Theorem 4.8 says $f : X \to Y$ is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y. In this problem, take $C = \{0\} \subset X$. C is closed, so $f^{-1}(C) = f^{-1}(\{0\}) = Z(f)$ is closed. □

4.4 *Proof.* Let $f, g : X \xrightarrow{\text{cont.}} Y$ and $E \overset{\text{dense}}{\subset} X$ be given. To prove: $f(E)$ dense in $f(X)$. Since E dense in X, $\bar{E} = X$. Pick $y \in f(X)$. To show $f(E)$ dense in $f(X)$, we want to show that if $y \neq f(E)$, $y \in f(E)'$. Assume $y \in f(X) \setminus f(E)$, then there is an x such that $y = f(x)$. If $x \in E$ then $y = f(x) \in f(E)$. This cannot happen, so $x \in X \setminus E$. $x \notin E$, which is dense in X, so there is a sequence $\{x_n\} \subset E$ such that $x_n \to x \in X \setminus E$. Since f is continuous, $f(x_n) \to f(x)$. If $f(x_n) = f(x) \in f(E)$ for some n, then we get a contradiction. So $f(x_n) \neq f(x)$ for all n. This means $y = f(x)$ is a limit point of $f(E)$, i.e., $y \in f(E)'$. So $f(E)$ is dense in $f(X)$.

Now, to prove: if $g(p) = f(p)$ for all $p \in E$, then $g(p) = f(p)$ for all $p \in X$. Well, if $p \in E$ then obviously, $g(p) = f(p)$. Consider $p \in E^c$. Since E dense in X, there is a sequence $\{p_n\}$ in E such that $p_n \to p \in E^c$. Now, $f(p_n) = g(p_n)$ for all n by hypothesis, so $f(p) = f(\lim_{n\to\infty} p_n) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} g(p_n) = g(\lim_{n\to\infty} p_n) = g(p)$. This means $f(p) = g(p)$ for all $p \in X$. □

4.5 *Proof.* Let f be a real continuous function defined on a closed set $E \subset \mathbb{R}$. We want to construct a real function g on $\mathbb{R}$ such that $f(x) = g(x)$ for all $x \in E$. Before we do this, we use a fact from Exercise 29, Chap 2 which says that because $E \subset X$ is closed,

$$E^c = \bigcup_{i=1} (a_i, b_i)$$

where the union is at most countable and $a_i < b_i < a_{i+1} < b_{i+1}$ for any $i \in \mathbb{N}$. With this, $g(x)$ can be given by:

$$g(x) = \begin{cases} f(x), & x \in E \\ f(a_i) + (x - a_i)\frac{f(b_i) - f(a_i)}{b_i - a_i}, & x \in E^c \end{cases}$$

Obviously $g(x)$ is a continuous function because $f(x)$ is continuous on E, and g is just a linear function (hence continuous) on E^c open.

When the word "closed" is omitted, we run into trouble. Consider $f(x) = 1/x$ on the open set $E = \mathbb{R} \setminus \{0\}$. Then there is no way for us to assign a real value to $g(0)$ and require that g be continuous.

For vector-valued functions, the result is the following: for $f(x) = (f_1(x), \ldots, f_d(x))$, where each $f_i(x)$ is a real continuous function on a closed set $E \subset \mathbb{R}$, we can extend each $f_i(x)$ by g_i given by a similar definition above, to get an extension g for f given by $g(x) = (g_1(x), \ldots, g_d(x))$. g is continuous on $\mathbb{R}^d$ because each g_i is continuous on $\mathbb{R}$. □

4.6 *Proof.* Let f defined on E be given. Assume $E \subset \mathbb{R}$ is compact. We want to show f is continuous on E iff its graph, $\mathcal{G} = \{(x, f(x)) : x \in E\}$ is compact.

Before doing anything, we have to define the metric for the space $E \times f(E)$ in which the graph lives. For $x_1, x_2 \in E$ and $f(x_1), f(x_2) \in f(E)$, define

$$d((x_1, f(x_1)), (x_2, f(x_2)) = \sqrt{d^2(x_1, x_2) + d^2(f(x_1), f(x_2))}.$$

Okay with this we can start with the proof.

($\longrightarrow$) Suppose E is compact and f is continuous. To show $\mathcal{G}$ is compact, we define a map $\mathcal{F} : E \longrightarrow \mathcal{G}$ given by $\mathcal{F}(x) = (x, f(x))$. Since E is compact, Theorem 4.14 tells us that if $\mathcal{F}$ is continuous on E then $\mathcal{F}(E) = \mathcal{G}$ is compact. Well, let $\epsilon > 0$ be given. Pick a point $x_0 \in E$. Since f is continuous, there is a $\delta > 0$ such that $d(f(x), f(x_0)) < \epsilon/\sqrt{2}$ whenever $d(x, x_0) < \delta$. Choose $\delta < \epsilon/\sqrt{2}$, then

$$d(\mathcal{F}(x), \mathcal{F}(x_0)) = \sqrt{d^2(x, x_0) + d^2(f(x), f(x_0))} < \sqrt{2\epsilon^2/2} = \epsilon. \tag{1}$$

So, $\mathcal{F}$ is continuous on E, and we're done.

($\longleftarrow$) Suppose $\mathcal{G}$ and E are compact. We want to show f is continuous. Consider the function $\mathcal{F}$ given by $\mathcal{F}(x) = (x, f(x))$ like that defined above. To show f is continuous, we can show $\mathcal{F}(x)$ is continuous, assuming that $\mathcal{G}, E$ are compact (since if $\mathcal{F}$ is continuous then its second component f must also be continuous). The function $\bar{g}(x, f(x)) = x$ is 1-1 and continuous. It's inverse mapping is just $\mathcal{F}(x)$. By theorem 4.17, $\mathcal{F}$ is a continuous mapping from E to $(E, f(E))$. It follows that f is also continuous. $\square$

4.7 *Proof.* f, g on $\mathbb{R}^2$ are given by $f(0,0) = g(0,0) = 0$, and if $(x, y) \neq 0$, $f(x, y) = xy^2/(x^2 + y^4)$, and $g(x, y) = xy^2/(x^2 + y^6)$. We want to show that f is bounded on $\mathbb{R}^2$. By completing the square, we know that $x^2 + y^4 \geq 2xy^2$, so $f(x, y) \leq 2$ for all $(x, y) \in \mathbb{R}^2$. So f is bounded.

Next, to show g is unbounded in every neighborhood of $(0,0)$, we look at sequences that converge to $(0,0)$. One such sequence is $\{(x_n, y_n) =\} = \{(1/n^3, 1/n)\}$. Clearly, $g(x_n, y_n) = n^6/2n^5 = n/2 \to \infty$ as $n \to \infty$. So g is unbounded in every neighborhood of $(0,0)$.

To show f is not continuous at $(0,0)$ we look at where $\{f(x_n, y_n)\}$ converges to when $(x_n, y_n) \to (0,0)$. Take the sequence $\{(x_n, y_n) = (1/n^2, 1/n)\}$. Then $f(x_n, y_n) = 1/2$ for all n. Obviously, $f(x_n, y_n) \to 1/2 \neq 0$ so f is not continuous at $(0,0)$.

Now we want to show the restrictions of f, g to any straight line in $\mathbb{R}^2$ are continuous. There are two cases: $x = c$ (the "vertical" line) and $y = ax + b$. If $x = c$ constant, then if $c \neq 0$, then $f(x, y) = cy^2/(x^2 + c^4)$ and $g(x, y) = cy^2/(c^2 + y^6)$ are both continuous in y and hence are continuous. If $c = 0$ then $f = g = 0$, also continuous.

Consider straight lines: $y = ax + b$. If $b = 0$, then if for nonzero (x, y), $f(x, y) = a^2x/(1 + a^4x^2)$ and $g = a^2x/(1 + a^6x^4)$. As $x \to 0$, it is clear that $f \to 0$ and $g \to 0$, so f, g are also continuous. If $b \neq 0$ then we don't have to worry because these lines don't pass the origin (which is where things can be bad). $\square$

4.8 *Proof.* Let f a real uniformly continuous function on the bounded set $E \subset \mathbb{R}$. We want to show f is bounded on E. Suppose E is bounded by $M > 0$. Let $\epsilon > 0$ be given, then there is a $\delta > 0$ such that $\left|f(p) - f(q)\right| < \epsilon$ for all $p, q \in E$ for which $\left|p - q\right| < \delta$. Since E is bounded, we can find a finite cover for E:

$$E \subset \bigcup_{1 \leq i \leq n} (x_i - \delta, x_i + \delta).$$

where $x_i \in E$. Now we look at all the $f(x_i)$. For every $x \in E$, $x \in (x_i - \delta, x_i + \delta)$ for some i. By uniform continuity, $\left|f(x) - f(x_i)\right| < \epsilon$. In other words, $\left|f(x)\right| < \epsilon + \left|f(x_i)\right|$. This holds for for all $x \in E$, so f is bounded above by $\sup_i\{f(x_i)\} + \epsilon$ and below by $\inf_i\{f(x_i)\} - \epsilon$. Since f is also continuous, it also makes sense to use max/min instead of sup/inf.

To show that the conclusion is false if boundedness of E is omitted, we look at a counterexample. Look at the function $f(x) = x$ with $x \in \mathbb{R}$. f is as uniformly continuous as one would like, but f is not bounded. $\square$

4.9 *Proof.* We want to show that the definition of uniform continuity can be rephrased as: for every $\epsilon > 0$ there is a $\delta > 0$ such that $\operatorname{diam} f(E) < \epsilon$ for all $E \subset X$ with $\operatorname{diam} E < \delta$. To do this, we recall the definition of uniform continuity: $f : X \to Y$ is said to be *uniformly continuous* if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ for all $p, q \in X$ for which $d_X(q, p) < \delta$.

($\rightarrow$) Let $\epsilon > 0$ be given. Let $q, p \in X$ for which $d_X(q,p) < \delta$ for some δ. Take $E = \{p, q\}$, then $\text{diam}E = d_X(q,p) < \delta$. By the new definition $\text{diam} f(E) < \epsilon$. Further, by the definition of the diameter of a set, $\text{diam}E \geq d_X(q,p)$, so $\text{diam} f(E) \geq d_Y(f(p), f(q))$, because we're taking the sup over more terms. This implies $d_Y(f(p), f(q)) < \epsilon$ whenever $d_X(q,p) < \delta$. New definition implies old definition.

($\leftarrow$) Let $E \subset X$ be given with $\text{diam}E < \delta$. By definition, for any $p, q \in E$, $d_X(q,p) \leq \text{diam}E < \delta$. The old definition says that $d_Y(f(p), f(q)) < \epsilon/2$, for any $p, q \in E$, and so $\text{diam} f(E) = \sup_{p,q \in E} d_Y(f(q), f(p)) \leq \epsilon/2 < \epsilon$. This means for every $\epsilon > 0$ there is a $\delta > 0$ such that $\text{diam} f(E) < \epsilon$ for all $E \subset X$ for which $\text{diam}E < \delta$. So, the old definition implies the new definition. $\square$

4.10 *Proof.* Here we want to prove Theorem 4.19 in a different fashion. Theorem 4.19 says: If f is a continuous mapping of a compact metric space X into a metric space Y then f is uniformly continuous. The alternative proof goes by contradiction. Assume (to get a contradiction) that f is ~~uniformly~~ continuous. Since f is not uniformly continuous, for some $\epsilon > 0$ there are sequences $\{q_n\}$ and $\{p_n\}$ in X such that $d_X(p_n, q_n) \rightarrow 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$.

Consider the sequences $\{q_n\}$ and $\{p_n\}$ above. Theorem 2.37 says that if E is an infinite subset of a compact set K then E has a limit point in K. This means that the sequences $\{p_n\}$ and $\{q_n\}$ converge to points p and q in E, respectively. Now, since $d_X(p_n, q_n) \rightarrow 0$, $p = q$ which means that $f(p_n) \rightarrow f(p) = f(q) \leftarrow f(q_n)$ (because f is continuous). Also,

$$\begin{aligned} d_Y(f(p_n), f(q_n)) &\leq d_Y(f(p_n), f(p)) + d_Y(f(q_n), f(p)) \\ &= d_Y(f(p_n), f(p)) + d_Y(f(q_n), f(q)) \rightarrow 0 + 0 = 0. \end{aligned}$$

However, this contradicts $d_Y(f(p_n), f(q_n)) > \epsilon$. So, f must be uniformly continuous. $\square$