MA355: Combinatorics Final (Prof. Friedmann)

Huan Q. Bui

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1. 9 and [9]

(a) There are $\boxed{2}$ partitions of 9 with all their parts of size 2 or 3:

$$\begin{aligned} 9 &= 3+2+2+2 \\ &= 3+3+3. \end{aligned}$$

(b)

2. $\mathbf{F_i}$bonacci

(a) Claim: The number of subsets S of $[n]$ such that S contains no two consecutive integers is F_{n+2}.

Proof. $\square$

(b) Claim: The number of compositions of n into parts of size greater than 1 is F_{n-2}.

Proof. $\square$

3. $\mathbf{S(t,i)}$rling.

(a) Claim:

$$\boxed{S(k,k-2) = \sum_{i=3}^{k}(i-2)\binom{i-1}{2}}$$

Proof. From Problem 134, we know that

$$S(k,n) = S(k-1,n-1) + nS(k-1,n).$$

With $n = k-2$, we have

$$\begin{aligned} S(k,k-2) &= S(k-1,k-3) + (k-2)S(k-1,k-2) \\ &= S(k-1,k-3) + (k-2)S(k-1,(k-1)-1) \\ &= S(k-1,k-3) + (k-2)\binom{k-1}{2}. \end{aligned}$$

Let S_k denote $S(k,k-2)$, then we have a recurrence relation

$$S_k = S_{k-1} + (k-2)\binom{k-1}{2}, \quad k \geq 2.$$

From here, we find a formula for S_k:

$$S(k,k-2) = S_k = \underbrace{S_2}_{=S(2,0)=0} + \sum_{i=3}^{k}(i-2)\binom{i-1}{2} = \sum_{i=3}^{k}(i-2)\binom{i-1}{2}.$$

as desired. □

(b) Claim:

$$\boxed{S(k,2) = 2^{k-1}-1,\ k \geq 2 \qquad \text{and} \qquad S(k,3) =,\ k \geq 3}$$

Proof. From Problem 134, we know that

$$S(k,n) = S(k-1,n-1) + nS(k-1,n).$$

With $n = 2$, we have

$$\begin{aligned} S(k,2) &= S(k-1,1) + nS(k-1,2) \\ &= 1 + 2S(k-1,2). \end{aligned}$$

Let S_k denote $S(k,2)$ then we have a first-order linear recurrence:

$$S_k = 1 + 2S_{k-1}.$$

With $S_2 = S(2,2) = 1$, the formula for $S(k,2)$, due to Problem 98, is

$$S(k,2) = S_k = 2^{k-2}S_2 + 1 \times \left(\frac{2^{k-2}-1}{2-1}\right) = 2^{k-2} + 2^{k-2} - 1 = 2^{k-1} - 1, \quad k \geq 2,$$

as claimed[1]. Now, we will use this result and Problem 134 to find $S(k,3)$:

$$\begin{aligned} S(k,3) &= S(k-1,2) + 3S(k-1,3) \\ &= (2^{k-1} - 1) + 3S(k-1,3). \end{aligned}$$

Let T_k denote $S(k,3)$, then we have the recurrence relation

$$T_k = (2^{k-1} - 1) + 3T_{k-1}.$$

□

(c) <u>Claim</u>:

$$\boxed{S(k,n) = \sum_{i=1}^{k} S(k-i, n-1)\binom{k-1}{i-1}}$$

Proof. □

[1]Here, recurrence begins at S_2, so the exponent in the formula only goes up to $k-2$.

4. LattiC$_e$ paths. We break the journey from $(0, 0) \to (20, 30)$ into $(0, 0) \to (8, 15)$ followed by $(8, 15) \to (20, 30)$. We can do this because the lattice walker can't move backwards (i.e., to the left or down). The number of paths P_1 from $(0, 0)$ to $(8, 15)$ is given by

$$P_1 = \binom{8+15}{8} = \binom{23}{8}.$$

Now we want to go from $(8, 15)$ to $(20, 30)$ but avoid $(14, 23)$. Since a path from $(8, 15)$ to $(20, 30)$ either goes through $(14, 23)$ or not, the number of paths from $(8, 15)$ to $(20, 30)$ is combination of paths through $(14, 23)$ and not through $(14, 23)$. There are:

$$\binom{(20-8)+(30-15)}{(30-15)} = \binom{27}{15}$$

paths from $(8, 15)$ to $(20, 30)$, while there are

$$\binom{(14-8)+(23-15)}{(23-15)}\binom{(20-14)+(30-23)}{(30-23)} = \binom{14}{8}\binom{13}{7}$$

paths from $(8, 15)$ to $(20, 30)$ that go through $(14, 23)$. So, the number of paths from $(8, 15)$ to $(20, 30)$ that don't go through $(14, 23)$ is

$$P_2 = \binom{27}{15} - \binom{14}{8}\binom{13}{7}.$$

With this, we combine the two parts of the journey to find that there are

$$P = P_1 P_2 = \binom{8+15}{8} = \boxed{\binom{23}{8} \times \left\{\binom{27}{15} - \binom{14}{8}\binom{13}{7}\right\}}$$

paths from $(0, 0)$ to $(20, 30)$ that go through $(8, 15)$ but not $(14, 23)$.