Chapter 6

Light Forces

6.1 Light forces from steady-state solutions

6.1.1 Optical Bloch Equations – Review

Let σ be the reduced density matrix for the atom. The optical bloch equation is:

$$\dot{\sigma} = \frac{1}{i\hbar}\Big[H_0 + H_I\sigma\Big] - \text{“}\Gamma\text{”}(\sigma - \sigma_e q)\,, \tag{6.1.1}$$

where “Γ” is a matrix representing the dissipation or loss. Recall that we previously used

$$\rho = \frac{I + \boldsymbol{r}\cdot\sigma}{2}\,. \tag{6.1.2}$$

Note that API uses σ for the reduced density matrix, whereas we've used it for the Pauli matrices; be careful and don't be confused by this. The specific elements of the Bloch vector are

$$r_x = \rho_{ge} + \rho_{eg} \tag{6.1.3}$$
$$r_y = (\rho_{ge} - \rho_{eg})/i \tag{6.1.4}$$
$$r_z = \rho_{gg} - \rho_{ee}\,. \tag{6.1.5}$$

Often in the literature, and in API, these components are written as u, v, and w, where $u = r_x/2 = \langle S_x\rangle$, $v = r_y/2 = \langle S_y\rangle$, and $w = -r_z/2 = \langle S_z\rangle$. Note the extra factor of 1/2. Explicitly, using API's notation of σ for the reduced density matrix,

$$u = \frac{1}{2}\sigma_{ge} + \sigma_{eg} \tag{6.1.6}$$
$$v = \frac{1}{2}(\sigma_{ge} - \sigma_{eg})/i \tag{6.1.7}$$
$$w = \frac{1}{2}\sigma_{ee} - \sigma_{gg}\,. \tag{6.1.8}$$

What can we get from the optical bloch equations? One set of solutions is the steady state ones; there are steady state solutions when nothing changes, those when something is only slowly changing. For example, an atom under light forces can cause the atom to move, but at every point in time the atom is nearly in equilibrium with the field.

A second aspect that can be discussed is the spectrum of the scattered light, as we have seen with the Mollow triplet, in which the frequencies and linewidths of the observed spectra are obtained from the optical bloch equations.

Today, we'll discuss light forces; this is from API p.370-378. We'll also need the amount of light absorbed from an atom; this is from API p.369.

6.1.2 Absorbed light

The energy per unit time absorbed by an atom from the light field is

$$\frac{dW}{dt} = q\epsilon_0 \cos\omega_L t \, \frac{dr}{dt}\,, \tag{6.1.9}$$

representing a dipole $q\,dr$ being driven by an electric field.

Recall u and v are off-diagonal elements of the atomic density matrix, known as coherences. Component u is in phase with the driving electric field, and v is in quadrature (90 degrees out of phase) with the driving field. Only the v component results in absorption. Averaging, we get

$$\left\langle \frac{dW}{dt} \right\rangle = \hbar\Omega\omega_L v \tag{6.1.10}$$

giving

$$\left\langle \frac{dN}{dt} \right\rangle = \Omega v\,. \tag{6.1.11}$$

Now back to the OBE. From this equation,

$$\dot{\sigma}_{ee} = \Omega v - \Gamma\sigma_{ee} \tag{6.1.12}$$

we note that

$$\frac{dN}{dt} = \Gamma\sigma_{ee}\,, \tag{6.1.13}$$

which shows that the excited state decays with rate Γ. That is nice to see, since it means the optical Bloch equations are consistent with our expected classical understanding of energy loss and absorption.

6.1.3 Light forces: radiation

An electric field can extert a force on an atom because it has an electric dipole moment. This moment can be in-phase with the field, or out-of-phase with the field. As we know classically from a simple harmonic oscillator, a driving force in phase with the oscillation does not contribute energy to the oscillator, whereas the drive out of phase with the oscillator does transfer energy. We will now identify how this happens with light driving an atom, in terms of the u and v components of the Bloch vector.

The force is

$$\begin{aligned} \boldsymbol{F} &= \dot{\boldsymbol{p}} && (6.1.14) \\ &= -\frac{\partial H}{\partial R} && (6.1.15) \\ &= \sum_{j=x,y,z} d_j \nabla_R \left(E_{ej}(\boldsymbol{R}, t) + E_{\perp j}(\boldsymbol{R}) \right) , && (6.1.16) \end{aligned}$$

where the subscript e on E emphasizes that the electric field is applied externally, and the subscript $\perp$ distinguishes the vacuum field. For more about this, read Exercise #17 in the back of API – this shows a canonical transform in which the electric field can be transformed, such that a coherent state field can be represented by a classical c-number field, plus fluctuations from the vacuum.

We now make some approximations. First, the vacuum field is even in $\boldsymbol{R}$, and thus does not contribute any force. Thus,

$$\boldsymbol{F} \approx \sum_{j=x,y,z} d_j \nabla_R \left(E_{ej}(\boldsymbol{R}, t) \right) . \tag{6.1.17}$$

Next, we assume the atom is sufficiently localized such that we may replace the operator $\boldsymbol{R}$ with $\langle \boldsymbol{R} \rangle = \boldsymbol{r}_G$, a c-number. This gives us

$$\boldsymbol{F} \approx \sum_{j=x,y,z} d_j \nabla_R E_{ej}(\boldsymbol{r}_G, t) . \tag{6.1.18}$$

This is a simple expression for light forces which we shall now expand on, by inserting steady state solutions for the optical Bloch equation.

There are two different timescales involved here. The external motion of an atom occurs on a timescale of $\hbar/E_{recoil}$, whereas the internal states evolve with time Γ^{-1}. These two are very well separated for atomic sodium, and most alkali atoms. But for certain atoms, like metastable helium, they are not, so one must be careful (such systems have really not been studied much in the community yet).

We will be considering forces when velocity $v \approx 0$ and $r = 0$. We assume that the electric field is given by a traveling or standing wave,

$$\boldsymbol{E}_e(\boldsymbol{r}, t) = \hat{e}\mathcal{E}_0(r) \cos \left[\omega_L t + \phi(r) \right] , \tag{6.1.19}$$

where $\hat{e}$ is the polarization. Assume $\phi(0) = 0$. The gradient of this is

$$\nabla E_{ej} = e_j \left[\cos \omega_l t \, \nabla \mathcal{E}_0 - \sin \omega_L t \mathcal{E}_0 \nabla \phi \right] . \tag{6.1.20}$$

This is one part of the expression. The other comes from the atomic dipole moment, given by the optical Bloch equations. We have

$$\langle d_j \rangle = 2 d_{ab} \left[u_{st} \cos \omega_L t - v_{st} \sin \omega_L t \right] . \tag{6.1.21}$$

The subscript "st" emphasizes these are steady state values. Integrating over one cycle of the radiation field (such that cross terms with $\cos \times \sin$ go away) gives us

$$\boldsymbol{F} = \hat{e} \cdot \boldsymbol{d}_{ab} \left[u_{st} \nabla \mathcal{E}_0 + v_{st} \mathcal{E}_0 \nabla \phi \right] . . \tag{6.1.22}$$

The two terms of this can be identified as the reactive force, and the dissipative force. Specifically, the reactive term $\sim u_{st} \nabla \mathcal{E}_0$, and the dissipative term $\sim v_{st} \mathcal{E}_0 \nabla \phi$.

We may usefully re-express this in terms of the Rabi frequency

$$\Omega = -\boldsymbol{d}_{ge} \cdot \hat{e} \mathcal{E}_0 / \hbar , \tag{6.1.23}$$

in terms of which the reactive force can be expressed using

$$\boldsymbol{\alpha} = \frac{\nabla \Omega}{\Omega} , \tag{6.1.24}$$

giving

$$\boldsymbol{F}_{react} = -\hbar \Omega u_{st} \boldsymbol{\alpha} . \tag{6.1.25}$$

Similarly, the dissipative force can be expressed using

$$\boldsymbol{\beta} = \nabla \phi , , \tag{6.1.26}$$

giving

$$\boldsymbol{F}_{diss} = -\hbar \Omega v_{st} \boldsymbol{\beta} . \tag{6.1.27}$$

It is useful to consider situations where we have just the dissipative, or just the reactive forces.

6.1.4 Radiation-pressure force

Consider an atom in a traveling wave,

$$\boldsymbol{E}_e = \hat{e} \mathcal{E}_0 \cos(\omega_L - \boldsymbol{k}_L \cdot r) . \tag{6.1.28}$$

For this, $\boldsymbol{\alpha} = 0$, whereas $\boldsymbol{\beta} = -\boldsymbol{k}_L$. This gives

$$\boldsymbol{F}_{diss} = \Omega v_{st} \hbar k_L . \tag{6.1.29}$$

This expression was seen earlier, the discussion above about energy transfer from the field to the atom,

$$\begin{aligned} \boldsymbol{F}_{diss} &= \left\langle \frac{dN}{dt} \right\rangle \hbar \boldsymbol{k}_L & (6.1.30) \\ &= \Gamma \sigma_{ee}^{st} \hbar k_L . & (6.1.31) \end{aligned}$$

σ_{ee}^{st} is the excited state fraction, so we get

$$\boldsymbol{F}_{diss} = \hbar \boldsymbol{k}_L \frac{\Gamma}{2} \frac{\Omega^2/2}{(\omega_L - \omega_0)^2 + \Gamma^2/4 + \Omega^2/2} . \tag{6.1.32}$$

This is an expression we saw right at the start of the class, argued intuitively, and now derived rigorously. The maximum value is

$$\boldsymbol{F}_{diss,max} = \hbar \boldsymbol{k}_L \frac{\Gamma}{2} \,. \tag{6.1.33}$$

6.1.5 Moving atoms & friction force

Laser cooling requires us to include moving atoms in the picture. The simplest case is as follows. Let us go into a moving frame,

$$\omega_L \to \omega_L - \boldsymbol{k}_L \cdot \boldsymbol{v}_0 \,, \tag{6.1.34}$$

now including the Doppler shift. Substituting into the above expression, and taylor expanding, we get

$$F(v_0) = F(v_0 = 0) - \alpha v_0 \cdots \,, \tag{6.1.35}$$

where the friction coefficient α is

$$\alpha = -\hbar k_L^2 \frac{s}{(1+s)^2} \frac{\delta\Gamma}{\delta^2 + (\Gamma^2/4)} \,. \tag{6.1.36}$$

6.1.6 Reaction forces

Let us now consider the case of atoms in a standing wave. Earlier, we argued that this case could be viewed as the sum of two traveling wave interactions; we then took the force as the sum of the two dissipative forces. Now, we shall take a different viewpoint. Let the standing wave be

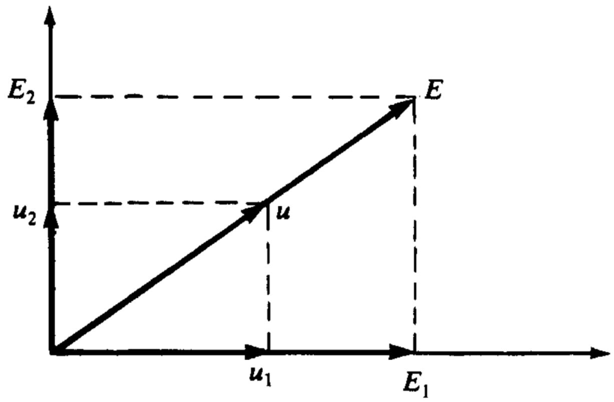
$$\boldsymbol{E}_e(r,t) = \hat{e}_x \mathcal{E}_0 \cos k_L z \, \cos \omega_L t \,. \tag{6.1.37}$$

The dissipative force for this standing wave is actually *zero*, because there is no phase, and β is the derivative of the phase. There is actually only a *reaction* force from a standing wave. What is happening physically is there is interference happening, with processes such as an atom absorbing a photon from one beam, and emitting into the other beam. Let us see what the resulting force is.

The reactive force comes from u_{st}. We get

$$\boldsymbol{F}_{react} = \frac{-\hbar(\omega_L - \omega_0)}{4} \frac{\nabla(\Omega^2)}{(\omega_L - \omega_0)^2 + \Gamma^2/4 + \Omega^2/2} \,. \tag{6.1.38}$$

What is the connection between the two traveling, counter-propaging fields E_1 and E_2, and the standing waves? Consider this phasor diagram:



The reactive force comes from u, the component in phase with the electric field. From the classical harmonic oscillator, we know that if the motion is phase shifted with respect to the drive between 0 and 180 degrees we absorb energy from the field, and if it is shifted between 180 and 360 degrees, we deliver energy to the field. The above phasor diagram shows that u is 45 degrees from E_1, but -45 degrees from E_2. This indicates that the atom is absorbing energy from one field, and delivering it to the other beam, and this is at the heart of stimulated forces.

6.1.7 Reactive force magnitudes

Let us go back to discussing the physics of the reactive force expression

$$\boldsymbol{F}_{react} = \frac{-\hbar(\omega_L - \omega_0)}{4} \frac{\nabla(\Omega^2)}{(\omega_L - \omega_0)^2 + \Gamma^2/4 + \Omega^2/2} \,. \tag{6.1.39}$$

We can write this as a conservative force, such that $\boldsymbol{F} = -\nabla U$, where

$$U = \frac{\hbar(\omega_L - \omega_0)}{2} \ln \left[1 + \frac{\Omega^2/2}{(\omega_L - \omega)^2 + \Gamma^2/4} \right] . \tag{6.1.40}$$

This is a very useful observation for laser cooling.

The absolute magnitiude of this force is useful to know. Given Ω^2, the magnitude of the force $|F_{react}|$ is maximum when $|\omega_L - \omega| \sim |\Omega|$. The maximum, for this optimal detuning, is

$$\max|F_{react}| \sim \hbar \frac{\nabla(\Omega)^2}{\Omega} \sim \hbar \nabla \Omega \sim \hbar k_L \Omega \,. \tag{6.1.41}$$

This makes intuitive sense; the strongest force you can get from an optical field is the photon momentum times the Rabi frequency. In each cycle, the atom absorbs a photon from one beam, and emits it into the other beam, giving it $2\hbar k_L$ per cycle.

6.1.8 Reactive forces at finite velocity

If the atom is traveling with velocity v_0 through a standing wave, what is α for $F_{react} = -\alpha v_0$? To derive the answer, we cannot take u to be the steady state value u_{st}, because the atom is moving rapidly across changes in the electric field. The atom can only respond on a timescale Γ^{-1}, so it has some memory, resulting in $u \sim u_{st}(\boldsymbol{r} - \boldsymbol{v}_0 \Delta t)$. Thus allows one to obtain the friction coefficient α for an atom in a standing wave. In the weak intensity limit, $\alpha = 2\alpha_{TW}$, where α_{TW} is the traveling wave result. However, in the strong intensity limit, things change considerably. The upshot is that *blue* detuned light may be needed to cool atoms, in a standing wave. See the experiment by A. Aspect, PRL 1986.