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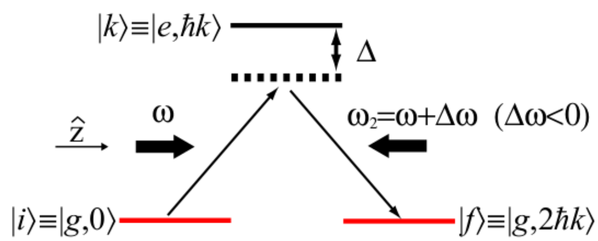
Course: **8.421 - AMO I**
Problem set: **#12**
Due: Monday, May 9, 2022.

I'd like to apologize to the grader in advance. This is possibly the most *sus* pset I've done so far in AMO 1. I'm unsure on some of my answers and some are definitely wrong, which bothers me quite a lot. This is possibly due to the fact that I have other things to take care of as this is the end of the semester. In any case, I have tried my best to give sensible responses.

1. Bragg Scattering. Consider he energy level diagram below where the states of the atom can be written as $|\text{internal, external}\rangle = |\text{internal}\rangle \otimes |\text{external}\rangle$. Here the internal state is either $|g\rangle$ or $|e\rangle$, and the momentum state is $|\boldsymbol{p}\rangle$. If two counter-propagating lasers are tuned as indicated, recoil momentum will be transferred to the atoms by redistributing photons between the beams. We want to look at this "Bragg scattering" in two ways:

- by describing it as a two-photon stimulated Raman process
- by considering the mechanical effect of the AC Stark shift potential seen by an atom



1. **Two-Photon Stimulated Raman Process.**
   (a) By conservation of energy, we have
$$\hbar\omega = \frac{(2\hbar k)^2}{2m} + \hbar(\omega + \Delta\omega),$$
   by comparing the energy of $|i\rangle$ and $|f\rangle$ to $|k\rangle$. From here, we find
$$\Delta\omega = -4\frac{\hbar k^2}{2m} = -4\omega_r$$
   where $\omega_r = \hbar k^2/2m$ is the recoil frequency.
   (b) Assume that the beams are counter-propagating along the z-axis and have the same polarization (which we may assume to be x), then
$$E_1 = E_0\cos(kz - \omega_1 t)$$
$$E_2 = E_0\cos(-kz - (\omega_1 + \Delta\omega)t).$$
   Since the electric fields E_j only couple the state $|j\rangle$ to the intermediate state $|k\rangle$, the interaction Hamiltonian is
$$\mathcal{H}_{\text{int}} = -e\vec{r}\cdot(\vec{E}_1 + \vec{E}_2)$$

where the relevant matrix elements are

$$\frac{\hbar\Omega_1}{2} = \langle i|\mathcal{H}_{\text{int}}|k\rangle = D_{ge}\langle 0|\cos(kz-\omega_1 t)|\hbar k\rangle \approx \frac{D_{ge}}{2}e^{i\hbar k^2 t/2m}e^{i\omega_1 t}$$
$$\frac{\hbar\Omega_2}{2} = \langle k|\mathcal{H}_{\text{int}}|f\rangle = D_{eg}\langle \hbar k|\cos(-kz-(\omega_1+\Delta\omega)t)|2\hbar k\rangle \approx \frac{D_{eg}}{2}e^{i3\hbar k^2 t/2m}e^{-(\omega_1+\Delta\omega)t}$$

where we have used the wavefunctions for a particle with definite momentum $p = \hbar k$ in a 1D box of length L in position space:

$$\psi(z) = \frac{1}{\sqrt{L}}e^{ipz/\hbar}e^{iEt/\hbar} = \frac{1}{\sqrt{L}}e^{ipz/\hbar}e^{i(p^2/2m)t/\hbar} = \frac{1}{\sqrt{L}}e^{ikz}e^{i\hbar k^2 t/2m}.$$

(c) See the last part of (a).

(d) The two-photon Rabi frequency is

$$\Omega_{R2} = \frac{\Omega_1\Omega_2}{2\Delta} = \frac{D_{eg}^2}{2\hbar^2\Delta}\exp\left(it\frac{4\hbar k^2}{2m}\right)e^{-i(\Delta\omega)t}.$$

Which definition of the two-photon Rabi frequency should I use? From what I've seen there are definitions where $\Omega_{2\text{R}}$ is real, or is of the form $\Omega_1^*\Omega_2$, etc. I don't know.

(e) If $\mathcal{H}'$ is the perturbation due to E_1, E_2, and if we treat the system as an effective two-level system, then we can simply identify $\hbar\langle i|\mathcal{H}'|f\rangle$ with the two-photon Rabi frequency $\Omega_{2\text{R}}$ found above. More precisely, if we identify

$$\frac{\hbar\Omega_{\text{R2}}}{2} = \langle i|\mathcal{H}'|f\rangle$$

then we have

$$\langle i|\mathcal{H}'|f\rangle = \frac{D_{eg}^2}{4\hbar^2\Delta}\exp\left(it\frac{4\hbar k^2}{2m}\right)\exp\left(-i(\Delta\omega)t\right).$$

2. **AC Stark Shift**

(a) Here we calculate the AC Stark shift $U(z,t)$ of an atom in the ground state $|g\rangle$ due to the total electric field $E_1 + E_2$. To keep things from going wrong (I hope) in Part (b), I will not be doing any time-averaging and save this to the very last step. The AC Stark shift $U(z,t)$ is:

$$U(z,t) \sim \frac{\hbar\omega_R^2}{2\Delta} = \frac{D_{eg}^2}{2\hbar^2\Delta}\left[E_1+E_2\right]^2 = \frac{D_{eg}^2}{2\hbar^2\Delta}\left[\cos(kz-\omega_1 t)+\cos(-kz-(\omega+\Delta\omega_1)t)\right]^2$$

where we have ignored the external part of the wavefunction.

(b) From this we can calculate the coupling $\langle i|U(z,t)|f\rangle$ due to the mechanical potential presented by the AC Stark shift. To do this, we include the external part of the wavefunction:

$$\langle 0|U(z,t)|2\hbar k\rangle = \text{...with time averaging...} = \frac{D_{eg}^2}{4\hbar^2\Delta}\exp\left(it\frac{4\hbar k^2}{2m}\right)\exp\left(-i(\Delta\omega)t\right),$$

which agrees with what we found in Part 1. This result illustrates that forces due to the AC Stark effect correspond in the photon picture to a stimulated Raman process which redistributes photons between laser beams.

2. Spontaneous Two-Photon Emission

1. **The Göppert-Mayer formula.** From second order perturbation theory, the amplitude for state $|b\rangle$ after excitation for a time t is:

$$a_b^{[2]} = \frac{1}{4\hbar^2}\sum_f \left\{ \frac{H_{bf,2}H_{fa,1}}{\omega_1-\omega_{fa}} \frac{e^{i(\omega_{ba}-\omega_1-\omega_2)}-1}{\omega_{ba}-\omega_1-\omega_2} + \frac{H_{bf,1}H_{fa,2}}{\omega_2-\omega_{fa}} \frac{e^{i(\omega_{ba}-\omega_1-\omega_2)}-1}{\omega_{ba}-\omega_1-\omega_2} \right\}.$$

(i) The excitation rate $\Gamma_{ab}(\omega_1)$ when one beam as frequency ω_1 can be calculated using the Fermi's golden rule

$$\begin{aligned}
\Gamma_{ab}(\omega_1) &= \frac{2\pi}{\hbar}\left|a_b^{[2]}\right|^2 \delta(\omega_{ba}-(\omega_1+\omega_2)) \\
&= \frac{\pi E_1^2 E_2^2 e^4}{8\hbar^4}\delta(\omega_{ba}-(\omega_1+\omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \left|f\right\rangle \left\langle f\right| \boldsymbol{r}\cdot\boldsymbol{e}_2\, |b\rangle}{\omega_1-\omega_{fa}} + \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \left|f\right\rangle \left\langle f\right| \boldsymbol{r}\cdot\boldsymbol{e}_1\, |b\rangle}{\omega_2-\omega_{fa}} \right\} \right|^2.
\end{aligned}$$

(ii) Since we have the constraint:

$$\omega_{ba} = \omega_b - \omega_a = \omega_1 + \omega_2,$$

we must have that

$$\omega_1 - \omega_{fa} = \omega_1 - \omega_f + \omega_a = \omega_b - \omega_2 - \omega_f = -(\omega_2 + \omega_{fb})$$

and similarly

$$\omega_2 - \omega_{fa} = -(\omega_1 + \omega_{fb}).$$

With these,

$$\begin{aligned}
\Gamma_{ab}(\omega_1) &= \frac{\pi E_1^2 E_2^2 e^4}{8\hbar^4}\delta(\omega_{ba}-(\omega_1+\omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_1 \left|f\right\rangle \left\langle f\right| \boldsymbol{r}\cdot\boldsymbol{e}_2\, |b\rangle}{\omega_2+\omega_{fb}} + \frac{\langle a|\, \boldsymbol{r}\cdot\boldsymbol{e}_2 \left|f\right\rangle \left\langle f\right| \boldsymbol{r}\cdot\boldsymbol{e}_1\, |b\rangle}{\omega_1+\omega_{fb}} \right\} \right|^2,
\end{aligned}$$

as desired.

(iii) The terms in the sum are simply the amplitudes of different possible decay paths via an arbitrary intermediate state $\left|f\right\rangle$. Each summand has two terms due to the fact that the two beams can switch roles.

(iv) Now we obtain the expression of $A(\omega_1)$. To do this, let us first write E_1, E_2 in terms of the photon number raising and lowering operators:

$$E_j = \sqrt{\frac{\hbar\omega_j}{2V\epsilon_0}}(a-a^\dagger).$$

With this, we find

$$\langle n|\, E_j^2\, |n\rangle = \frac{\hbar\omega_j}{V\epsilon_0}\left(n_{\omega_j}+\frac{1}{2}\right).$$

We can take this as E_j^2. Now we replace n_j with the densities of modes at frequency ω_j:

$$n_{\omega_j} \longrightarrow \frac{2\omega_j^2}{\pi c^3}$$

We may ignore the zero-point energy to get

$$\begin{aligned}
\Gamma_{ab}(\omega_1) &= \frac{\pi E_1^2 E_2^2 e^4}{8\hbar^4}\delta(\omega_{ba} - (\omega_1 + \omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a|\boldsymbol{r}\cdot\boldsymbol{e}_1\left|f\right\rangle\left\langle f\right|\boldsymbol{r}\cdot\boldsymbol{e}_2\left|b\right\rangle}{\omega_2 + \omega_{fb}} + \frac{\langle a|\boldsymbol{r}\cdot\boldsymbol{e}_2\left|f\right\rangle\left\langle f\right|\boldsymbol{r}\cdot\boldsymbol{e}_1\left|b\right\rangle}{\omega_1 + \omega_{fb}} \right\} \right|^2 \\
&= \frac{e^4\omega_1^3\omega_2^3}{2\pi\hbar^2 c^6}\delta(\omega_{ba} - (\omega_1 + \omega_2)) \\
&\quad \times \left| \sum_f \left\{ \frac{\langle a|\boldsymbol{r}\cdot\boldsymbol{e}_1\left|f\right\rangle\left\langle f\right|\boldsymbol{r}\cdot\boldsymbol{e}_2\left|b\right\rangle}{\omega_2 + \omega_{fb}} + \frac{\langle a|\boldsymbol{r}\cdot\boldsymbol{e}_2\left|f\right\rangle\left\langle f\right|\boldsymbol{r}\cdot\boldsymbol{e}_1\left|b\right\rangle}{\omega_1 + \omega_{fb}} \right\} \right|^2
\end{aligned}$$

From here, we find that

$$A(\omega_1)\,d\omega_1 = \frac{e^4\omega_1^3\omega_2^3}{2\pi\hbar^2 c^6}\left\langle \left| \sum_f \left\{ \frac{\langle a|\boldsymbol{r}\cdot\boldsymbol{e}_1\left|f\right\rangle\left\langle f\right|\boldsymbol{r}\cdot\boldsymbol{e}_2\left|b\right\rangle}{\omega_2 + \omega_{fb}} + \frac{\langle a|\boldsymbol{r}\cdot\boldsymbol{e}_2\left|f\right\rangle\left\langle f\right|\boldsymbol{r}\cdot\boldsymbol{e}_1\left|b\right\rangle}{\omega_1 + \omega_{fb}} \right\} \right|^2 \right\rangle_{\text{avg}} d\omega_1$$

which is off by some numerical factor from the answer given in the problem.

Remark: I feel like I'm doing something weird/missing some step here.

2. $2S$ **natural lifetime.**

   (i) Here we use $2P$ as the only intermediate state and assume $\omega_{2S-2P} = \omega_{fb} = 0$. Moreover, we use the Bohr radius a_0 for both relevant matrix elements of z. With these, we find

$$\begin{aligned}
\frac{1}{\tau} &= A_\tau \\
&= \frac{1}{2}\int_0^{\omega_{ba}} A(\omega_1)\,d\omega_1 \\
&= \frac{1}{2}\frac{8e^4}{3\pi\hbar^2 c^6}\int_0^{\omega_{ba}} \omega_1^3(\omega_{ba} - \omega_1)^3 \left| a_0^2\left(\frac{1}{\omega_1} + \frac{1}{\omega_{ba} - \omega_1}\right)\right|^2 d\omega_1 \\
&= \frac{1}{2}\frac{8e^4}{3\pi\hbar^2 c^6}\frac{a_0^4\omega_{ba}^5}{6},
\end{aligned}$$

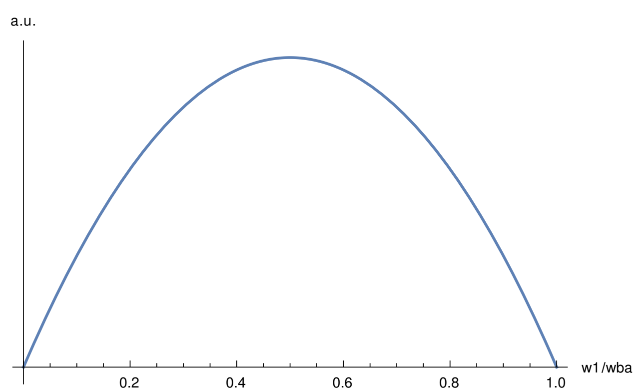
   where we have used the formula provided by Breit and Teller provided at the end of Part 1.

   (ii) To actually get a sensible lifetime out of this, we have to convert the formula above back to SI units by including a factor of $1/(4\pi\epsilon_0)^2$:

$$\tau = \left(\frac{1}{(4\pi\epsilon_0)^2}\frac{1}{2}\frac{8e^4}{3\pi\hbar^2 c^6}\frac{a_0^4\omega_{ba}^5}{6}\right)^{-1} \approx 0.095 \text{ s}.$$

   Given that the actual natural lifetime of $2S$ is 0.122 s, this result is quite remarkable.

   (iii) Here we plot $A(\omega_1)$ over its range:


a.u.
0.2
0.4
0.6
0.8
1.0
w1/wba