Final

1. Differentiation:

   (a) Assume that
   $$f(x) = \begin{cases} \frac{g(x)}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$
   and assume that $g(0) = g'(0) = 0$ and $g''(0) = 17$. With no further assumptions, find $f'(0)$, justifying everything.

   (b) Assuming only that $f'(0) > 0$ and f' is continuous at 0, prove that there exists an interval containing 0 on which f is increasing. (This f is in no way related to the previous f in part (a).)

   (c) Show that there exists a continuous function f with $f'(0) > 0$, but f is not increasing on any interval containing 0.

   (d) Assume that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

2. Series:

   (a) Prove that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then
   $$|\sum_{n=1}^{\infty} a_n| \leq \sum_{n=1}^{\infty} |a_n|.$$

   (b) Show that if $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and b_n is a subsequence of a_n, then $\sum_{n=1}^{\infty} b_n$ is absolutely convergent. Give and example that shows this statement is false if $\sum_{n=1}^{\infty} a_n$ is assumed to be only conditionally convergent.

   (c) Assume a_n is a decreasing sequence of positive numbers, and that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $\lim_{n\to\infty} n a_n = 0$.

   (d) Prove that every positive rational number can be written as a finite sum of *distinct* numbers of the form $\frac{1}{k}$, with $k \in \mathbb{N}$.

3. Hilbert Space:

   (a) Let V denote the set of continuous functions that map $[0, 1]$ into the complex numbers $\mathbb{C}$. With $f \in V$, each complex number $f(x)$ can be written in terms of it's real and imaginary parts
   $$f(x) = \text{Re}f(x) + i\ \text{Im}f(x).$$
   The real valued functions Ref and Imf are called the real part of f and the imaginary part of f (respectively). We define the integral of a complex valued function by
   $$\int_0^1 f(x)dx \equiv \int_0^1 \text{Re}f(x)dx + i\int_0^1 \text{Im}f(x)dx.$$

Show that the assignment

$$(f, g) \equiv \int_0^1 f(x)\overline{g(x)}dx$$

satisfies the axioms of a complex inner product (find the axioms in a book or on the internet).

(b) Assume V is a complex inner product space with inner product (x, y) and its associated metric

$$d(x, y) = \sqrt{(x - y, x - y)},$$

and let $\mathcal{H}$ denote the metric completion of V. Thus we may think of V as a dense subset of the metric space $\mathcal{H}$. The purpose of the following exercises is to show how one may extend the vector space structure of V to $\mathcal{H}$, and how to extend the inner product to $\mathcal{H}$, which shows that the metric completion of an inner product space is a Hilbert space.

i. Given $x, y \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$, we define $\alpha x + \beta y$ to be the limit of the sequence $\alpha x_i + \beta y_i$, where x_i is any sequence in V converging to x, and y_i is any sequence in V converging to y. Show that this definition is well defined.

ii. Imitate the procedure above to show how to extend the inner product so that (x, y) is defined for all $x, y \in \mathcal{H}$. (Hint: extend one variable at a time.)

4. Isometries:

(a) Assume that $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}$. Prove that

$$f(x) = mx + b$$

with $m = 1$ or $m = -1$.

(b) Prove that there does not exist a function $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfies $|f(x) - f(y)| = ||x - y||$ for all $x, y \in \mathbb{R}^2$.

(c) Prove that if $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $||f(x) - f(y)|| = ||x - y||$ for all $x, y \in \mathbb{R}^n$, then f is onto.

(d) Let $\mathcal{H}$ denote an infinite dimensional (real or complex) Hilbert space. Give an example of a function $f : \mathcal{H} \to \mathcal{H}$ that satisfies

$$||f(x) - f(y)|| = ||x - y||$$

for all $x, y \in \mathcal{H}$, but f is *not* onto.