Problem Set 9

Due: Friday 5pm, April 15th, via Canvas upload or in envelope outside 26-255
TA: Botond Oreg, bzo20@mit.edu

1 Transition Lifetimes and Blackbody Radiation

a) Do you have to worry about blackbody radiation when you trap a Bose-Einstein condensate in a magnetic trap? Any transition to another state will spin-flip the atoms and/or give the atoms recoil energy, causing the atom to be ejected from the trap. Assume that after each absorption event, it will lose one atom in the trap, and you want to ensure a trapping time of at least one minute in a closed, cubic box. Consider sodium, which has a dominant electronic excitation at a wavelength of 590 nm and a lifetime of 16 ns.

i. What is the average number of photons per mode from blackbody radiation at this transition which leads to an absorption rate of 1 photon per minute? Hint: You might find Einstein's coefficients useful for this problem.

Solution

The spontaneous absorption rate is $R_{\text{BEC}} = 3A\bar{n} = 3\bar{n}/\tau_{\text{Na}}$ where the factor of 3 comes from the degeneracy of sodium as the excited state has same number of nuclear spin states and 3 times more electronic states. This gives an average mode occupancy of $\bar{n} = R_{\text{BEC}}\tau_{\text{Na}}/3 = 8.9 \cdot 10^{-11}$.

ii. What is the corresponding blackbody temperature? Do you need to shield the vacuum system from room temperature radiation or cool the vacuum system to cryogenic temperatures (*e.g.* 4 K)?

Solution

The average photon number is

$$\bar{n} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} \tag{1}$$

and the corresponding transition frequency is $\omega = 2\pi c/\lambda = 2\pi \cdot 508\text{THz}$. This gives a temperature of

$$T = 1100\text{K}. \tag{2}$$

This is way above the room temperature so blackbody temperature is not a problem for a typical atomic physics experiment. However some precise experiments with molecules which have many and broad transitions do require some cooling though not necessarily down to 4K.

b) Here we estimate the lifetime of the hydrogen in the $F = 1$ hyperfine level of the $1S$ state. The decay of $F = 1$ to $F = 0$ gives rise to the famous 21 cm line of radio astronomy.

i. What is the dominant coupling mechanism for this process, *e.g.* is it an electric dipole transition, or a magnetic dipole transition? or it is actually an electric octupole transition?

Solution

The dominant coupling mechanism is the magnetic dipole transition.

ii. What is the lifetime of the $F = 1$ state? Assume that the matrix element is ea_0 for electric dipole transition, μ_B for magnetic dipole transition, and ea_0^2 for electric quadrupole transition, where a_0 is the Bohr radius, and μ_B is the Bohr magneton.

Solution

The lifetime is given by the formula derived during class after modifying the appropriate matrix element. The matrix element squared for an electric dipole transition is $e\mathbf{d}_{ba} \cdot \mathbf{E}^2_{\text{rms}}$ while for a magnetic dipole transition it is $\mu_{ba} \cdot \mathbf{B}^2_{\text{rms}}$. Noting that for a mode of the electromagnetic field $\mathbf{B}^2_{\text{rms}} = \mathbf{E}^2_{\text{rms}}/c^2$ we get the appropriate formula for the decay rate by replacing $e^2\mathbf{d}^2_{ba}$ with μ_B^2/c^2 and we get (in SI units)

$$\Gamma = \frac{\omega^3}{3\pi\varepsilon_0\hbar c^3}\frac{\mu_B^2}{c^2} = 2.9 \cdot 10^{-15} 1/\text{s}. \tag{3}$$

This corresponds to a 10 million year lifetime.

c) A hydrogen Bose-Einstein condensate has been created in the $F = 1$ state at MIT in the group of Dan Kleppner and Tom Greytak (Fried *et al.*, *Phys. Rev. Lett.* **81**, 3811 (1998)).

i. What is the average number of photons per mode from blackbody radiation at the 21 cm line at 300 K and 4 K?

Solution

The average photon number is given by

$$\bar{n} = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}. \tag{4}$$

Substitution gives

$$\bar{n}_{300\text{K}} = 4400 \tag{5}$$
$$\bar{n}_{4\text{K}} = 58 \tag{6}$$

which are much larger than in the case of the sodium transitions in part a).

ii. For blackbody radiation at 300 K and 4 K, at what rate does it induce transitions to the $F = 0$ state respectively, which cannot be magnetically trapped?

Solution

The stimulated and spontaneous transition rate is $3(\bar{n}+1)\Gamma$ which are

$$R_{300\mathrm{K}} = 3.8 \cdot 10^{-11} 1/\mathrm{s} \tag{7}$$

$$R_{4\mathrm{K}} = 5.1 \cdot 10^{-13} 1/\mathrm{s}. \tag{8}$$

These correspond to very long times compared to any experimental timescales.

iii. Should you be concerned about blackbody radiation from the environment limiting your experiment with hydrogen in the $F = 1$ state, if you need a trapping time of 1 minute?

Solution

Even at 300 K the lifetime of an atom in the $F = 1$ state is thousands of years so blackbody radiation is not a limitation.

d) You probably noticed from parts *a*) and *c*) that the average number of photons per mode at their respective relevant frequencies and temperatures are rather different. In particular, there are many more photons per mode for one of them than the other. Yet, the lifetime is much longer for the one with more photons per mode, for these two examples. Why is that?

Solution

One reason is the ratio of the photon number and the density of states. For the sodium transition there are much more photon modes the excited state can emit to. Another reason is that the hydrogen transition is a forbidden transition so it decays via a magnetic rather than an electric dipole transition. This gives an extra factor of α^2 in the decay rate.

2 Saturation Intensity

We define the saturation intensity of a laser for an optical transition as the intensity (power/area) at which a monochromatic beam excites the transition at a rate equal to one half of its natural line width. In this problem, we compute the saturation intensity for the principal transition in sodium, 590 nm.

a) Express the Einstein A coefficient by the oscillator strength f, the fine structure constant α and the transition frequency ω. Estimate the lifetime of sodium by assuming an oscillator strength of unity.

Solution
The dipole matrix element expressed with the oscillator strength is

$$|\langle b|z|a\rangle|^2 = \frac{\hbar}{2m\omega} f. \tag{9}$$

Accounting for the factor of 3 in the degeneracies of the ground and excited states and also for the three coordinate axes ($|\langle b|\mathbf{r}|a\rangle|^2 = |\langle b|x|a\rangle|^2 + |\langle b|y|a\rangle|^2 + |\langle b|z|a\rangle|^2$) we get a decay rate of

$$\Gamma = \frac{e^2\omega^3}{3\pi\varepsilon_0\hbar c^3}\frac{\hbar}{2m\omega} f = \frac{2\alpha}{3}\frac{\hbar\omega}{mc^2} f\omega \tag{10}$$

where α is the fine structure constant. Assuming an oscillator strength of unity we get a lifetime of $1/\Gamma = 16$ns which is very close to the actual value.

b) Find the saturation intensity for the principal transition in sodium. Treat the atom as a two-level system, neglecting fine and hyperfine structure.

Solution
The stimulated absorption rate is Ω^2/Γ where Ω is the Rabi frequency of the transition given by in terms of the driving field amplitude as $\hbar\Omega = er_{ab}E_0$. The dipole moment can be written using the oscillator strength of unity (assuming that light is polarized for a given transition) and the driving electric field amplitude can be written using the laser intensity to give

$$\Omega^2 = \frac{e^2}{\hbar^2}\frac{\hbar}{2m\omega}\frac{2I}{\varepsilon_0 c} = \frac{4\pi\alpha}{m\omega} I \tag{11}$$

At saturation the stimulated absorption rate is half of the natural decay rate ($\Omega^2/\Gamma = \Gamma/2$) which gives

$$\Omega^2 = \Gamma^2/2 \tag{12}$$

$$\frac{4\pi\alpha}{m\omega} I_{\mathrm{SAT}} = \frac{1}{2}\left[\frac{2\alpha}{3}\frac{\hbar\omega}{mc^2}\omega\right]^2 \tag{13}$$

$$I_{\mathrm{SAT}} = \frac{\alpha}{18\pi}\left(\frac{\hbar\omega}{mc^2}\right)^2 m\omega^3. \tag{14}$$

Substitution gives $I_{\mathrm{SAT}} = 65\ \mathrm{W/m^2} = 6.5\ \mathrm{mW/cm^2}$.

3 Saturation of Atomic Transitions

We discussed excitation of atoms via weak radiation. In this limit the atom scatters incident radiation at a rate proportional to the light intensity, corresponding to a fixed cross-section.

We also discussed the excitation of atoms via strong radiation and showed that in this limit the atom performs Bloch oscillations between the ground and excited states. Since during these oscillations the mean excited state population population is at most $1/2$ and the excited state decays with rate Γ, the atom can scatter at most $\Gamma/2$ photons per unit time. To obtain a fixed scattering rate, as the radiation intensity increases, the photon-scattering cross-section decreases, becoming very low at high light intensities.

This problem will motivate this saturation of atomic transitions by considering broadband excitation. The obtained results can be exactly extended to narrowband transitions.

a) In the case of broadband excitation, the atom dynamics is correctly described by the Einstein rate equations. Consider a two-state atom with $R_{ge} = R_{eg}$ the stimulated absorption/emission rate and $A = \Gamma$ the spontaneous emission rate. Define the saturation parameter s as $s = 2R_{ge}/\Gamma$. Show that in equilibrium the ratio of the excited state to the ground state populations is $N_b/N_a = s/(s+2)$.

Solution
In equilibrium the excitation and emission rates equal so

$$RN_a = (R+\Gamma)N_b. \tag{15}$$

Expressing R with s and Γ and rearranging gives $N_b/N_a = s/(s+2)$.

b) Express the equilibrium spontaneous emission rate per atom AN_b in terms of Γ and s. Show that the cross-section for photon absorption bleaches out as $\sigma(s) = \sigma(s=0)/(1+s)$.

Solution
The spontaneous scattering rate is $\Gamma N_b = \Gamma(N_a + N_b)s/(2(s+1)) \propto s/(s+1)$. The incoming photon flux is proportional to s. Therefore the cross section (the scattering rate divided by the incoming flux) is proportional to $1/(1+s)$ giving the relation $\sigma(s) = \sigma(s=0)/(1+s)$.

c) Find the energy density $\langle w \rangle_{SAT}$ per unit frequency corresponding to $s = 1$. Explain why $\langle w \rangle$ is independent of the atomic dipole matrix element $\langle g | er | e \rangle$.

Solution
At saturation $1 = 2\langle w \rangle_{SAT} B/\Gamma$. Since the ratio of the Einstein coefficients is independent of the microscopic system parameters (dipole matrix element), the energy density is also independent of that.

d) Use the relationship between Einstein's A and B coefficients to obtain an expression for $\langle w \rangle_{SAT}$ independent of the atomic dipole. For $s = 1$, what is the mean occupation number n per photon mode?

Solution

Using the relation $\Gamma/B = (\hbar\omega^3)/(\pi^2 c^3)$ we get for unpolarized light

$$\langle w \rangle_{SAT,unpol} = \frac{\hbar\omega^3}{2\pi^2 c^3}. \tag{16}$$

For appropriately polarized light $\langle w \rangle_{SAT}$ is 3 times smaller. Since $\langle w \rangle_{SAT} B/\Gamma = \bar{n}$ by definition the average mode occupation number at saturation is $\bar{n} = 1/2$.

e) Suppose that the light is provided by a laser beam of intensity I_0 and Lorentzian lineshape centered at the atomic transition frequency ω_0 and of FWHM $\Gamma' \gg \Gamma$. What is the energy density of this beam per frequency interval at ω_0? What beam intensity I_s corresponds to $s = 1$?

Solution

A Lorentzian lineshaped beam of total intensity I_0 has the spectrum of

$$I(\omega) = \frac{\Gamma'/2}{\Delta\omega^2 + (\Gamma'/2)^2} \frac{I_0}{\pi}. \tag{17}$$

The energy density is given by I/c and on resonance

$$\langle w \rangle = \frac{2}{\pi c \Gamma'} I_0. \tag{18}$$

At saturation

$$\langle w \rangle_{SAT} = \frac{\hbar\omega^3}{2\pi^2 c^3} = \frac{2}{\pi c \Gamma'} I_s \tag{19}$$

$$I_s = \frac{\hbar\omega^3}{4\pi c^2} \Gamma'. \tag{20}$$

f) Let ω_R be the Rabi frequency corresponding to a monochromatic beam with the same intensity I_0 as the broadband beam. Show that the stimulated broadband absorption rate can be written as $R = \omega_R^2/\Gamma'$. What is ω_R^2 corresponding to $s = 1$?

Solution

The Rabi frequency can be written as $\omega_R^2 = e^2(r_{ab}^2/3)E_0^2/\hbar^2$ where E_0 is the electric field amplitude of the laser beam and the factor of 3 comes from the fact that for unpolarized light only a third of the photons have the appropriate polarization for that transition. We can express the dipole moment using Γ and the electric field amplitude using the intensity to get

$$\omega_R^2 = \frac{8\pi\alpha}{3\hbar} I \frac{3c^2}{4\alpha\omega^3} \Gamma = 2\pi \frac{c^2}{\hbar\omega^3} \Gamma I_0 \tag{21}$$

The stimulated broadband absorption rate is

$$R = \frac{\Gamma}{2}\frac{\langle w\rangle}{\langle w\rangle_{SAT}} = 2\pi\frac{c^2}{\hbar\omega^3}\frac{\Gamma}{\Gamma'}I_0 \tag{22}$$

which is exactly ω_R^2/Γ'. At saturation $I_0 = I_s$ which gives

$$\omega_{R,SAT}^2 = \frac{\Gamma\Gamma'}{2} \tag{23}$$

g) If you set $\Gamma' = \Gamma$, you get exactly the saturation intensity of a monochromatic laser beam and the Rabi frequency at saturation. Argue why.

Solution
It is because the atom occupies the excited state only for a time $1/\Gamma$ so it cannot resolve frequencies to better precision than Γ. So from an atom's perspective any linewidth below Γ is not resolved and seen the same as a linewidth of Γ.