8.(3)09 Midterm Solutions 2019

100 Pts total

1. 18 total

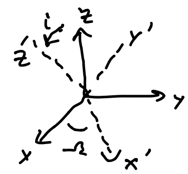
(a) [5] Various options but must pick 5 <u>independent</u> constants

- energy E, $\vec{L}$ & Laplace-Runge-Lenz vector $\vec{A}$ (1 indep)

or

- 2 angles specify plane x'-y' of orbit: i, Ω

  & 3 specify shape of orbit (a, e, ω)



or ••• (combos)

Relations (for grading): $a = -\frac{k}{2E}$, $e = \left[1 + \frac{2E\ell^2}{\mu k^2}\right]^{1/2}$, two $\vec{L}$'s $\Leftrightarrow$ plane of orbit

(b) [3] $F_1(q, Q) = q\, e^{Q}$, $p = \frac{\partial F_1}{\partial q} = e^{Q}$, $P = -\frac{\partial F_1}{\partial Q} = -q\, e^{Q}$

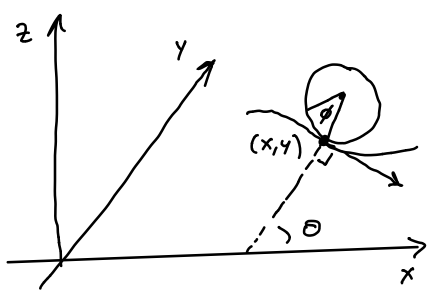
so $Q = \ln p$ & $P = -q\, p$

(c) [4] Lagrange multipliers can be used to

2. ① determine or study forces of constraint

2. ② solve problems with semi-holonomic constraints

(d) [6]



4. (x, y, θ, ϕ)

$\underbrace{x, y}_{\text{touches floor}}$ θ: orientation, ϕ: how much its rotated

(picture <u>or</u> words fine)

$$v = (\dot{x}^2 + \dot{y}^2)^{1/2} = a\dot{\phi}$$] -1 point if <u>only</u> this

1. $\dot{x} = a \sin\theta\, \dot{\phi}$

1. $\dot{y} = -a \cos\theta\, \dot{\phi}$

] full points

2. 26 total

$m_1 \quad (b, 0, b)$

$m_2 \quad (b, b, -b)$

$m_3 \quad (-b, b, 0)$

(a) 8

$$I^{ab} = \sum_i m_i \left(\delta^{ab} \vec{r}_i^{\,2} - r_i^a r_i^b \right)$$

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2) = m_1 b^2 + m_2 2b^2 + m_3 b^2$$

$$I_{yy} = \sum_i m_i (x_i^2 + z_i^2) = m_1 2b^2 + m_2 2b^2 + m_3 b^2$$

$$I_{zz} = \sum_i m_i (x_i^2 + y_i^2) = m_1 b^2 + m_2 2b^2 + m_3 2b^2$$

$$I_{xy} = \sum_i m_i (-x_i y_i) = m_1 (0) + m_2 (-b^2) + m_3 (b^2)$$

$$I_{xz} = \sum_i m_i (-x_i z_i) = m_1 (-b^2) + m_2 (b^2) + m_3 (0)$$

$$I_{yz} = \sum_i m_i (-y_i z_i) = m_1 (0) + m_2 b^2 + m_3 (0)$$

(b) 12

$$\hat{I} = m b^2 \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 5 \end{pmatrix} \qquad m_1 = m_2 = m_3 = m$$

principal moments of inertia $\det(\hat{I} - \lambda \mathbb{1}) = 0$

$$0 = \begin{vmatrix} 4mb^2 - \lambda & 0 & 0 \\ 0 & 5mb^2 - \lambda & mb^2 \\ 0 & mb^2 & 5mb^2 - \lambda \end{vmatrix} = (4mb^2 - \lambda)\left[(5mb^2 - \lambda)^2 - (mb^2)^2\right]$$

$$= (mb^2)^3 (4 - \hat{\lambda}) \left[(5 - \hat{\lambda})^2 - 1 \right]$$

$$\hat{\lambda} = \frac{\lambda}{mb^2}$$

$$0 = (4 - \hat{\lambda})(24 - 10\hat{\lambda} + \hat{\lambda}^2)$$

$$= (4 - \hat{\lambda})(4 - \hat{\lambda})(6 - \hat{\lambda})$$

$$\boxed{\begin{aligned} I_1 = I_2 &= 4mb^2 \\ I_3 &= 6mb^2 \end{aligned}}$$

6.

<u>for $4mb^2$</u> (dble root)

$$mb^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 0$$

any ξ_1

$\xi_2 + \xi_3 = 0$

solutions: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\boxed{\hat{x}} \qquad \boxed{\frac{(\hat{y} - \hat{z})}{\sqrt{2}}}$

2. 2.

for $6mb^2$

$$mb^2 \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & +1 \\ 0 & +1 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0 \qquad q_1 = 0, \quad q_2 = q_3$$

solution $\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $\boxed{\frac{\hat{y}+\hat{z}}{\sqrt{2}}}$ 2.

(c) Euler Equations $I_1 = I_2 = 4mb^2$, $I_3 = 6mb^2$

$\boxed{6}$

set $I_2 = I_1$ $\qquad \tau_i = 0$

$I_3 \dot{\omega}_3 = 0 \qquad \therefore \boxed{\omega_3 = \text{constant}}$ 2.

$I_1 \dot{\omega}_1 - (I_1 - I_3)\omega_2\omega_3 = 0 \qquad \dot{\omega}_1 = -\left[\frac{(I_3 - I_1)\omega_3}{I_1}\right]\omega_2$

$I_1 \dot{\omega}_2 - (I_3 - I_1)\omega_1\omega_3 = 0 \qquad \dot{\omega}_2 = +\underbrace{\left[\frac{(I_3 - I_1)\omega_3}{I_1}\right]}_{\equiv \Omega > 0}\omega_1$

$\dot{\omega}_1 = -\Omega\omega_2$

$\dot{\omega}_2 = +\Omega\omega_1$ so $\ddot{\omega}_2 = -\Omega^2\omega_2$, $\underline{\omega_2 = A\sin(\Omega t + \beta)}$ 2.

$\underline{\omega_1} = \frac{\dot{\omega}_2}{\Omega} = \underline{A\cos(\Omega t + \beta)}$ 2. $\qquad \Omega = \frac{2mb^2}{4mb^2}\omega_3 = \frac{\omega_3}{2}$

$\vec{\omega}$ precesses about body 3-axis $\frac{\hat{y}+\hat{z}}{\sqrt{2}}$

General motion is spinning & precessing

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3. $\boxed{28 \text{ total}}$

(a) $r_1 = r_2 = R$ fixed, $\dot{r}_1 = \dot{r}_2 = 0$

$\boxed{10}$

$$T = \frac{m}{2}\dot{\vec{r}}_1^{\,2} + \frac{m}{2}\dot{\vec{r}}_2^{\,2} = \frac{m}{2}R^2\left[\dot{\theta}_1^2 + \dot{\theta}_2^2 + \sin^2\theta_1\,\dot{\phi}_1^2 + \sin^2\theta_2\,\dot{\phi}_2^2\right]$$ 3.

gravity $V_{grav} = mgz_1 + mgz_2 = -mgR(\cos\theta_1 + \cos\theta_2)$ 2.

spring $V_{spring} = \frac{k}{2}(\vec{r}_1 - \vec{r}_2)^2$,

$\vec{r}_1 - \vec{r}_2 = R(\sin\theta_1\cos\phi_1 - \sin\theta_2\cos\phi_2,\ \sin\theta_1\sin\phi_1 - \sin\theta_2\sin\phi_2,\ -\cos\theta_1 + \cos\theta_2)$

$$V_{spring} = \frac{k}{2}R^2\Big[\sin^2\theta_1\cos^2\phi_1 + \sin^2\theta_2\cos^2\phi_2 - 2\sin\theta_1\sin\theta_2\cos\phi_1\cos\phi_2$$
$$+ \sin^2\theta_1\sin^2\phi_1 + \sin^2\theta_2\sin^2\phi_2 - 2\sin\theta_1\sin\theta_2\sin\phi_1\sin\phi_2$$
$$+ \cos^2\theta_1 + \cos^2\theta_2 - 2\cos\theta_1\cos\theta_2\Big]$$

5.
$$= \frac{k}{2}R^2\left[2 - 2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) - 2\cos\theta_1\cos\theta_2\right]$$

$$L = T - V = \frac{MR^2}{2}\left[\dot\theta_1^2 + \dot\theta_2^2 + \sin^2\theta_1\,\dot\phi_1^2 + \sin^2\theta_2\,\dot\phi_2^2\right]$$
$$+ mgR\,(\cos\theta_1 + \cos\theta_2)$$
$$+ kR^2\left[-1 + \sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) + \cos\theta_1\cos\theta_2\right]$$

(b)
$\boxed{3}$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta_1} - \frac{\partial L}{\partial\theta_1} = 0$$

$$mR^2\ddot\theta_1 - \frac{MR^2}{\cancel{2}}\cancel{2}\sin\theta_1\cos\theta_1\,\dot\phi_1^2 + mgR\sin\theta_1$$
$$- kR^2\left[\cos\theta_1\sin\theta_2\cos(\phi_1-\phi_2) - \sin\theta_1\cos\theta_2\right] = 0$$

(c)
$\boxed{4}$

H time indep ($\therefore$ conserved) & quadratic in velocities ($\therefore H=E$)

2. Energy conserved

3.
Force of constraint in $-\hat r$ direction, gravity in $-\hat z$
so no torque in $\hat\phi$ $\therefore$ L_z conserved

$\boxed{\text{or}}$ Note that $\phi_1-\phi_2$ appears in L, but $\phi_1+\phi_2$ cyclic
rotational symmetry $\phi_i \to \phi_i + \alpha$ $\Rightarrow$ L_z conserved

(d)
$\boxed{10}$

$0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ fixed $\qquad \frac{1}{2}(\phi_1-\phi_2)^2 = \frac{1}{2}(\phi_1^2+\phi_2^2-2\phi_1\phi_2)$

$$L = \frac{1}{2}(\dot\phi_1\ \dot\phi_2)\underbrace{\begin{pmatrix} MR^2\sin^2\theta_1 & 0 \\ 0 & MR^2\sin^2\theta_2\end{pmatrix}}_{\equiv \hat T}\begin{pmatrix}\dot\phi_1\\ \dot\phi_2\end{pmatrix} - \frac{1}{2}(\phi_1\ \phi_2)\underbrace{kR^2\sin\theta_1\sin\theta_2\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}}_{\equiv \hat V}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$$

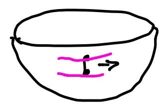
Write $\hat{T} = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$, $\hat{V} = \begin{pmatrix} a & -a \\ -a & a \end{pmatrix}$

$$\hat{V} - \lambda \hat{T} = \begin{pmatrix} a - \lambda b & -a \\ -a & a - \lambda c \end{pmatrix} \rightarrow \det(\hat{V} - \lambda \hat{T}) = 0$$

$$= (a - \lambda b)(a - \lambda c) - a^2$$

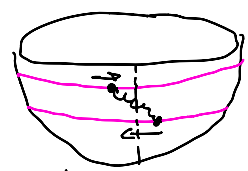
$$= bc\lambda^2 - \lambda a (b+c)$$

$$0 = \lambda \left(\lambda - \frac{(b+c)a}{bc} \right)$$

- $\lambda_1 = 0 = \omega_1^2$ travelling solution 

- $\lambda_2 = \frac{a}{b} + \frac{a}{c} = \frac{K R^2 \sin\theta_1 \sin\theta_2}{m R^2 \sin^2\theta_1} + \frac{K R^2 \sin\theta_1 \sin\theta_2}{m R^2 \sin^2\theta_2}$

$$\omega_2^2 = \frac{K}{m} \left(\frac{\sin\theta_2}{\sin\theta_1} + \frac{\sin\theta_1}{\sin\theta_2} \right)$$ osc. solution



$$\hat{V} - \lambda_2 \hat{T} = \begin{pmatrix} a - a - \frac{ab}{c} & -a \\ -a & a - a - \frac{ac}{b} \end{pmatrix} \Rightarrow a \begin{pmatrix} -\frac{b}{c} & -1 \\ -1 & -\frac{c}{b} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0$$

$$\therefore \frac{b}{c} q_1 + q_2 = 0 \qquad \vec{q} \propto \begin{pmatrix} 1 \\ -\frac{b}{c} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{\sin^2\theta_1}{\sin^2\theta_2} \end{pmatrix}$$

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4. $H(q,p) = \sqrt{p^2c^2 + m^2c^4} + K|x| = E = \alpha_1 > mc^2$

(a) $\boxed{2}$

$$\left[\left(\frac{\partial W}{\partial x} \right)^2 c^2 + m^2 c^4 \right]^{1/2} + K|x| = \alpha_1$$

ok to set $c = 1$

(b) $\boxed{5}$

$$c^2 \left(\frac{\partial W}{\partial x} \right)^2 = (\alpha_1 - K|x|)^2 - m^2c^4$$

$$W = \pm \int dx \sqrt{\frac{(\alpha_1 - K|x|)^2}{c^2} - m^2c^2}$$

$\Rightarrow p = \pm \int$

(c) turning points have $p = 0$

3

$$\therefore \quad \alpha_1 - k|x| = mc^2 , \qquad k|x| = \alpha_1 - mc^2 > 0$$

$$x = \pm \frac{\alpha_1 - mc^2}{k} \equiv \pm x_0$$

(d)

6

$$\beta + t = \frac{\partial W}{\partial \alpha_1} = \pm \int dx \, \frac{\partial}{\partial \alpha_1} \sqrt{\frac{(\alpha_1 - k|x|)^2}{c^2} - m^2c^2}$$

$$= \pm \int dx \, \frac{(\alpha_1 - k|x|)}{c^2} \left[\frac{(\alpha_1 - k|x|)^2}{c^2} - m^2c^2 \right]^{-1/2}$$

$x > 0$, $p > 0$ is + sign & $|x| = x$ $\qquad \frac{\partial}{\partial \alpha_1} = -\frac{1}{k} \frac{\partial}{\partial x}$

$$\beta + t = + \left(-\frac{1}{k}\right) \int dx \, \frac{\partial}{\partial x} \sqrt{\cdots}$$

$$= -\frac{1}{k} \left[\frac{(\alpha_1 - kx)^2}{c^2} - m^2c^2 \right]^{1/2}$$

$$(\alpha_1 - kx)^2 = k^2c^2(\beta+t)^2 + m^2c^4$$

$$x(t) = \frac{\alpha_1}{k} - \frac{\sqrt{k^2c^2(\beta+t)^2 + m^2c^4}}{k}$$

minus sign is correct solution

(e)

10

$$J = \oint p \, dq = 4 \int_0^{x_0} dx \left[\frac{(\alpha_1 - kx)^2}{c^2} - m^2c^2 \right]^{+1/2}$$

$$\nu = \frac{\partial E}{\partial J} = \left(\frac{\partial J}{\partial E}\right)^{-1} , \qquad x_0 = \frac{E - mc^2}{k}$$

same trick as in (d)

$$\frac{\partial J}{\partial E} = 4 \left[\frac{\partial x_0}{\partial E} \frac{\partial}{\partial x_0} \int_0^{x_0} dx \sqrt{\;} + \int_0^{x_0} dx \frac{\partial}{\partial E} \left[\frac{(E - kx)^2}{c^2} - m^2c^2 \right]^{1/2} \right.$$

$$= 4 \left[\frac{1}{k} \underbrace{\sqrt{\;} \Big|_{x = x_0}}_{0} - \frac{1}{k} \left[\frac{(E - kx)^2}{c^2} - m^2c^2 \right]^{1/2} \Big|_{x=0}^{x_0} \right]$$

$$= \frac{4}{k} \left[E^2/c^2 - m^2c^2 \right]^{1/2} = \frac{4}{kc} \left[E^2 - m^2c^4 \right]^{1/2}$$

$$\therefore \quad \nu = \left(\frac{\partial J}{\partial E}\right)^{-1} = \frac{kc}{4[E^2 - m^2c^4]^{1/2}} = \nu(E)$$

(f) $\boxed{2}$

From (d) & (e) we see that $x(t)$ reaches turning point x_0

at $t_1 = -\beta$, $x(t) = \frac{\alpha_1}{u} - \frac{mc^2}{K} = x_0$

& $x = 0$ when $E^2 = K^2c^2\beta^2 + m^2c^4 \Rightarrow \beta = -\frac{\sqrt{E^2 - m^2c^4}}{Kc}$
$t = 0$

So $\boxed{\tau = 4t_1 = -4\beta} = \frac{1}{\nu} = \frac{1}{Kc}\left[E^2 - m^2c^4\right]^{1/2}$