A generalized polar coordinate integration formula

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In this section, we establish a generalized polar coordinate integration formula associated to a positive-homogeneous polynomial. In what follows, we fix a positive-homogeneous polynomial $P : \mathbb{R}^d \to [0, \infty)$ and set

$$S = \{\eta \in \mathbb{R}^d : P(\eta) = 1\} \qquad \text{and} \qquad B = \{\eta \in \mathbb{R}^d : P(\eta) < 1\}.$$

For $E \in \operatorname{Exp}(P)$, the results of [3] guarantee that, $T_t = t^E$ is a dilation[1] of $\mathbb{R}^d$, S is a compact hypersurface, i.e., a compact smooth manifold of dimension $d-1$, and B is a bounded open region. I'm not sure if this matters, but S is connected never contains 0 and is not necessarily convex, B does contains 0 and, I believe, is connected and simply connected. Also, we should think of P being the essential thing which defines both E and S here – and we can write $\mu_P = \operatorname{tr} E$ throughout.

This section needs to contain a description of the standard polar coordinate integration formula and, therein, cite Folland or St.-Sh. For reference, our description should make use of the formula in the following theorem:

Theorem 0.1. *Let σ_d be the canonical surface measure on the unit sphere $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$, i.e., σ_d is the unique rotation-invariant Radon measure on $\mathbb{S}^{d-1}$ for which $\sigma(\mathbb{S}^{d-1}) = d \cdot m(\mathbb{B}) = (d \cdot \pi^{d/2})/\Gamma(d/2+1)$; here $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^d$, m is the Lebesgue measure on $\mathbb{R}^d$ and we will write $dm(x) = dx$. Let $f \in L^1(\mathbb{R}^d)$. Then, for σ_d-almost every η, $t \mapsto f(t\eta)$ is (absolutely) integrable with respect to the measure $t^{d-1}dt$ on $(0,\infty)$, $\eta \mapsto \int_0^\infty f(t\eta)t^{d-1}\,dt$ is (absolutely) integrable with respect to σ_d on $\mathbb{S}^{d-1}$ and*

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{S}^{d-1}} \left(\int_0^\infty f(t\eta)t^{d-1}\,dt \right) d\sigma_d(\eta). \tag{1}$$

1 A surface measure on S and an S-adapted integration formula for $\mathbb{R}^d$

Our goal in this section is to define a measure σ on the surface S (which is adapted to P and E) and establish an analogue of the the formula (1) to integrate functions on $\mathbb{R}^d$. We believe this will be useful in proving local limit theorems. Our construction if purely measure-theoretic. It proceeds by replacing the standard isotropic group of dilations on $\mathbb{R}^d$ by a generally anisotropic group of dilations $t \mapsto T_t = t^E$ where $E \in \operatorname{Exp}(P)$. As the standard isotropic one-paramter group of dilation $(0,\infty) \ni t \mapsto tI \in \operatorname{Gl}_d(\mathbb{R})$ is well-fitted to the unit sphere $\mathbb{S}^{d-1}$ and allows every non-zero $x \in \mathbb{R}^d$ to be written uniquely as $x = t\eta$ for $t \in (0,\infty)$ and $\eta \in \mathbb{S}^{d-1}$, $(0,\infty) \ni t \mapsto T_t = t^E \in \operatorname{Gl}_d(\mathbb{R})$ is well-fitted to to S and, as we see below, has the property that every non-zero $x \in \mathbb{R}^d$ can be written uniquely as $x = t^E\eta$ where $t \in (0,\infty)$ and

[1] We should really think hard about how much our construction depends on E

$\eta \in S$. With this group of dilations as a tool, we define a surface measure σ on S by taking sufficiently nice sets $F \subseteq S$, stretching them into a quasi-conical region of the associated "ball" B, and computing the Lebesgue measure of the result. Once the measure σ is constructed, we turn our focus to an associated product measure and establish our analogue of (1); this is Theorem 1.5. As a consequence of the theorem, we are able to show that σ is a Radon measure on S.

We shall take S to be equipped with the relative topology inherited from $\mathbb{R}^d$ and, given $(0,\infty)$ with its usual topology, we take $S \times (0,\infty)$ to be equipped with the product topology. Consider the map $\psi : S \times (0,\infty) \to \mathbb{R}^d \setminus \{0\}$ defined by

$$\psi(\eta, t) = t^E \eta \tag{2}$$

for $\eta \in S$ and $t > 0$. As ψ is the restriction of the continuous function $\mathbb{R}^d \times (0,\infty) \ni (\xi, t) \mapsto t^E \xi \in \mathbb{R}^d$ to $S \times (0,\infty)$, it is necessarily continuous. As the following proposition shows, ψ is, in fact, a homeomorphism.

Proposition 1.1. *The map $\psi : S \times (0,\infty) \to \mathbb{R}^d \setminus \{0\}$, defined by (2) is a homeomorphism with continuous inverse $\psi^{-1} : \mathbb{R}^d \setminus \{0\} \to S \times (0,\infty)$ given by*

$$\psi^{-1}(\xi) = ((P(\xi)^{-E}\xi, P(\xi))$$

for $\xi \in \mathbb{R}^d \setminus \{0\}$.

Proof. Given that P is continuous and positive-definite, $P(\xi) > 0$ for each $\xi \in \mathbb{R}^d \setminus \{0\}$ and the map $\mathbb{R}^d \setminus \{0\} \ni \xi \mapsto P(\xi)^{-E}\xi \in \mathbb{R}^d$ is continuous. Further, in view of the homogeneity of P,

$$P\left(P(\xi)^{-E}\xi\right) = P(\xi)^{-1}P(\xi) = 1$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$. It follows from these two observations that

$$\rho(\xi) = (P(\xi)^{-E}\xi, P(\xi),$$

defined for $\xi \in \mathbb{R}^d \setminus \{0\}$, is a continuous function taking $\mathbb{R}^d \setminus \{0\}$ into $S \times (0,\infty)$. We have

$$(\psi \circ \rho)(\xi) = \psi((P(\xi)^{-E}\xi, P(\xi)) = P(\xi)^E(P(\xi)^{-E}\xi) = \xi$$

for every $\xi \in \mathbb{R}^d \setminus \{0\}$ and

$$(\rho \circ \psi)(\eta, t) = \rho(t^E\eta) = P(t^E\eta)^{-E}t^E\eta, P(t^E\eta)) = ((tP(\eta))^{-E}t^E\eta, tP(\eta) = (\eta, t)$$

for every $(\eta, t) \in S \times (0,\infty)$. Thus ρ is a (continuous) inverse for ψ and so it follows that ψ is a homeomorphism and $\rho = \psi^{-1}$. $\square$

Remark 1. *We shall later (In the next section – should have a label) discuss manifold structures on S and $S \times (0,\infty)$ at which point we'll see that, in fact, ψ is a diffeomorphism.*

As our immediate goal is to construct a measure on the surface S, we shall first construct an appropriate σ-algebra on S. To this end, a set $F \subseteq S$ is said to be *measurable* if

$$\widetilde{F} := \bigcup_{0<t<1} t^E F = \{t^E\eta \in \mathbb{R}^d : \eta \in F, 0 < t < 1\}$$

is Lebesgue measurable and the collection of all such subsets F of S shall be denoted by Σ_S. In other words, if $\mathcal{M}_d$ denotes the σ-algebra of Lebesgue measurable sets in $\mathbb{R}^d \setminus \{0\}$, then

$$\Sigma_S = \{F \subseteq S : \widetilde{F} \in \mathcal{M}_d\}.$$

Proposition 1.2. *Σ_S is a σ-algebra on S and contains the Borel σ-algebra on S, $\mathcal{B}(S)$.*

Proof. We first show that Σ_S is a σ-algebra. Since $\widetilde{S} = B \setminus \{0\}$, it is open and therefore Lebesgue measurable. Hence $S \in \Sigma_S$. Let $G, F \in \Sigma_S$ be such that $G \subseteq F$. Then,

$$\widetilde{F \setminus G} = \bigcup_{0<t<1} t^E (F \setminus G) = \bigcup_{0<t<1} \left(t^E F \setminus t^E G\right)$$
$$= \left(\bigcup_{0<t<1} t^E F\right) \setminus \left(\bigcup_{0<t<1} t^E G\right) = \widetilde{F} \setminus \widetilde{G}$$

where we have used the fact that the collection $\{t^E F\}_{0<t<1}$ is mutually disjoint to pass the union through the set difference. Consequently $\widetilde{F} \setminus \tilde{G}$ is Lebesgue measurable and therefore $F \setminus G \in \Sigma_S$. Now, let $\{F_n\}_{n\in\mathbb{N}}$ be a countable collection of measurable sets on S, i.e., $\{F_n\} \subseteq \Sigma_S$. Then

$$\widetilde{\bigcup_{n=1}^{\infty} F_n} = \bigcup_{0<t<1} t^E \left(\bigcup_{n=1}^{\infty} F_n\right) = \bigcup_{0<t<1} \bigcup_{n=1}^{\infty} t^E F_n = \bigcup_{n=1}^{\infty} \bigcup_{0<t<1} t^E F_n = \bigcup_{n=1}^{\infty} \widetilde{F_n} \in \mathcal{M}_d$$

and so $\bigcup_n F_n \in \Sigma_S$. Thus Σ_S is a σ-algebra.

Finally, we show that

$$\mathcal{B}(S) \subseteq \Sigma_S.$$

As the Borel σ-algebra is the smallest σ-algebra containing the open subsets of S, it suffices to show that $\mathcal{O} \in \Sigma_S$ whenever $\mathcal{O}$ is open in S. Armed with Proposition 1.1, this is an easy task: Given an open set $\mathcal{O} \subseteq S$, observe that

$$\widetilde{O} = \{t^E \eta : 0 < t < 1, \eta \in \mathcal{O}\} = \psi(\mathcal{O} \times (0,1)).$$

Upon noting that $\mathcal{O} \times (0,1)$ is an open subset of $S \times (0,\infty)$, Proposition 1.1 guarantees that $\psi(\mathcal{O} \times (0,1)) = \widetilde{\mathcal{O}}$ is an open subset of $\mathbb{R}^d$ and is therefore Lebesgue measurable. $\square$

We are now ready to specify a measure on the measurable space (S, Σ_S). For each $F \in \Sigma_S$, we define

$$\sigma(F) = (\operatorname{tr} E) m(\widetilde{F})$$

where m is the Lebesgue measure on $\mathbb{R}^d$. We have:

Proposition 1.3. *σ is a finite measure on (S, Σ_S).*

Proof. It is clear that $\widetilde{\varnothing} = \varnothing$ and therefore

$$\sigma(\varnothing) = (\operatorname{tr} E) m(\varnothing) = 0.$$

Second, for any $F \in \Sigma_S$,

$$\sigma(F) = (\operatorname{tr} E) m(\tilde{F}) \geq 0$$

because m is a measure and $\operatorname{tr} E \geq 0$. Now, let $\{F_n\}_{n=1}^\infty \subseteq \Sigma_S$ be a mutually disjoint collection. We claim that $\{\widetilde{F_n}\}_{n=1}^\infty \subseteq \mathcal{M}_d$ is also a mutually disjoint collection. To see this, suppose that $x = t_n^E \eta_n = t_m^E \eta_m \in \widetilde{F_n} \cap \widetilde{F_m}$, where $t_n, t_m \in (0,1)$, $\eta_n \in F_n$, and $\eta_m \in F_m$. Then

$$t_n = P(t_n^E \eta_n) = P(x) = P(t_m^E \eta_m) = t_m,$$

implying that $\eta_n = \eta_m \in F_n \cap F_m$. Because $\{F_n\}_{n=1}^\infty$ is mutually disjoint, we must have $n = m$ which verifies our claim. By virtue of the countable additivity of Lebesgue measure, we therefore have

$$\sigma\left(\bigcup_{n=1}^\infty F_n\right) = (\operatorname{tr} E) m \left(\widetilde{\bigcup_{n=1}^\infty F_n}\right) = (\operatorname{tr} E) m \left(\bigcup_{n=1}^\infty \widetilde{F_n}\right) = \operatorname{tr} E \sum_{n=1}^\infty m(\widetilde{F_n}) = \sum_{n=1}^\infty \sigma(F_n).$$

Therefore σ is a measure on (S, Σ_S). Finally, because $\widetilde{S} = B \setminus \{0\}$ is a bounded open region in $\mathbb{R}^d$, $m(\widetilde{S}) < \infty$ and so $\sigma(S) = (\operatorname{tr} E) m(\widetilde{S}) < \infty$ showing that σ is finite. $\square$

By virtue of the two preceding propositions, σ is a finite Borel measure on S. In fact, as a consequence of our main result in this section, Theorem 1.5, we will see that (S, Σ_S, σ) is the completion of the measure space $(S, \mathcal{B}(S), \sigma)$ and σ is a Radon measure, i.e., it is both inner and outer regular. For now, we turn our focus to the product measure which will appear in our generalized integration formula (4). Consider the measure spaces (S, Σ_S, σ) and $((0,\infty), \mathcal{L}, \lambda)$ where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable sets on $(0,\infty)$ and $d\lambda(t) = t^{\operatorname{tr} E - 1}\, dt$, i.e., for each $L \in \mathcal{L}$,

$$\lambda(L) = \int_0^\infty \chi_L(t) t^{\operatorname{tr} E - 1}\, dt.$$

It is easy to see that λ is σ-finite. Given this and in view of the finiteness of the measure σ, there is a unique product measure $\mu = \sigma \times \lambda$ on $S \times (0,\infty)$ which is defined on the product σ-algebra $\Sigma_S \times \mathcal{L}$ and satisfies

$$\mu(F \times L) = \sigma(F)\lambda(L)$$

for all $F \in \Sigma_S$ and $L \in \mathcal{L}$. We shall denote by $(S \times (0,\infty), \Sigma, \mu)$ the completion of the measure space $(S \times (0,\infty), \Sigma_S \times \mathcal{L}, \mu)$. For this measure space, the following formulation of the Fubini/Tonelli theorem is applicable:

Theorem 1.4 (Theorem 8.12 of [4])**.** *Let $g : S \times (0,\infty) \to \mathbb{C}$ be Σ-measurable. For each $\eta \in S$, define $g_\eta : (0,\infty) \to \mathbb{C}$ by $g_\eta(t) = g(\eta, t)$ for $t \in (0,\infty)$ and, for each $t \in (0,\infty)$, define $g^t : S \to \mathbb{C}$ by $g^t(\eta) = g(\eta, t)$ for $\eta \in S$.*

1. *For σ-almost every η, g_η is $\mathcal{L}$-measurable and, for λ-almost every t, g^t is Σ_S-measurable.*

2. *If $g \geq 0$, then:*

   (a) For σ-almost every η,

   $$G(\eta) = \int_0^\infty g_\eta(t) t^{\operatorname{tr} E - 1}\, dt$$

   exists as a non-negative extended real number.

(b) *For λ-almost every t,*

$$H(t) = \int_S g^t(\eta)\, d\sigma(\eta)$$

exists as a non-negative extended real number.

(c) *We have*

$$\int_S G(\eta)\, d\sigma(\eta) = \int_{S\times(0,\infty)} g\, d\mu = \int_0^\infty H(t) t^{\mathrm{tr}\, E-1} dt$$

or, equivalently,

$$\int_S \left(\int_0^\infty g(\eta,t) t^{\mathrm{tr}\, E-1}\, dt \right) d\sigma(\eta) = \int_{S\times(0,\infty)} g\, d\mu = \int_0^\infty \left(\int_S g(\eta,t)\, d\sigma(\eta) \right) t^{\mathrm{tr}\, E-1}\, dt. \quad (3)$$

3. *If g is complex-valued and*

$$\int_S \left(\int_0^\infty |g(\eta,t)| t^{\mathrm{tr}\, E-1}\, dt \right) d\sigma(\eta) < \infty \qquad \text{or} \qquad \int_0^\infty \left(\int_S |g(\eta,t)|\, d\sigma(\eta) \right) t^{\mathrm{tr}\, E-1}\, dt < \infty,$$

then $g \in L^1(S\times(0,\infty), \mu)$.

4. *If $g \in L^1(S\times(0,\infty),\mu)$, then $g_\eta \in L^1((0,\infty),\lambda)$ for Σ-almost every η, $g^t \in L^1(S,\sigma)$ for λ-almost every t, and (3) holds.*

Our main theorem of this section is stated as follows.

Theorem 1.5. *Let $(S\times(0,\infty), \Sigma, \mu)$ be as above and let m be the (restricted) Lebesgue measure on $(\mathbb{R}^d \setminus \{0\}, \mathcal{M}_d, m)$.*

1. *m is the pushforward of the measure $\mu = \sigma\times\lambda$ by ψ (I believe this is the same as the statement: The measure spaces $(S\times(0,\infty),\Sigma,\mu)$ and $(\mathbb{R}^d\setminus\{0\},\mathcal{M}_d,m)$ are isomorphic with (point) isomorphism ψ). That is*

$$\mathcal{M}_d = \{A \subseteq \mathbb{R}^d \setminus \{0\} : \psi^{-1}(A) \in \Sigma\}$$

and, for each $A \in \mathcal{M}_d$,

$$m(A) = \psi_*\mu(A) = \mu(\psi^{-1}(A)).$$

2. *If $f : \mathbb{R}^d \to \mathbb{C}$ is Lebesgue measurable, then $f\circ\psi$ is Σ-measurable and the following statements hold:*

(a) *If $f \geq 0$, then*

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \left(\int_S f(t^E\eta)\, d\sigma(\eta) \right) t^{\mathrm{tr}\, E-1}\, dt = \int_S \left(\int_0^\infty f(t^E\eta) t^{\mathrm{tr}\, E-1}\, dt \right) d\sigma(\eta). \quad (4)$$

(b) *When f is complex-valued, we have $f \in L^1(\mathbb{R}^d)$ if and only if $f\circ\psi \in L^1(S\times(0,\infty),\Sigma,\mu)$ and, in this case, (4) holds.*

Remark 2. *Slight abuse of notation going on with m and m and dx – should say when we mean the restriction and when we don't?*.

To prove this theorem, we shall first treat several lemmas. These lemmas isolate and generalize (and hopefully clarify) several important ideas used in standard proofs of (1) (see, e.g., [2] and [5]).

Lemma 1.6. *Let $A \subseteq \mathbb{R}^d$ and $t > 0$. The set A is Lebesgue measurable if and only if $t^E A = \{x = t^E a : a \in A\}$ is Lebesgue measurable and, in this case,*

$$m(t^E A) = t^{\operatorname{tr} E} m(A).$$

Proof. Because $x \mapsto t^E x$ is a linear isomorphism, $t^E A$ is Lebesgue measurable if and only if A is Lebesuge measureable (See Theorem 2.20 of [4]). Observe that $x \in t^E A$ if and only if $t^{-E} x \in A$ and therefore

$$m(t^E A) = \int_{\mathbb{R}^d} \chi_{t^E A}(x)\, dx = \int_{\mathbb{R}^d} \chi_A(t^{-E} x)\, dx.$$

Now, by making the linear change of variables $x \mapsto t^E x$, we have

$$m(t^E A) = \int_{\mathbb{R}^d} \chi_A(x) |\det(t^E)|\, dx = t^{\operatorname{tr} E} m(A),$$

because $\det(t^E) = t^{\operatorname{tr} E} > 0$. □

Lemma 1.7. *Let $F \in \Sigma_S$. If $I \subseteq (0, \infty)$ is open, closed, G_δ, or F_σ, then $\psi(F \times I) \in \mathcal{M}_d$ and*

$$m(\psi(F \times I)) = \mu(F \times I) = \sigma(F)\lambda(I). \tag{5}$$

Proof. We fix $F \in \Sigma_S$ and consider several cases for I.

Case 1: $I = (0, b)$ for $0 < b \leq \infty$. When b is finite, observe that

$$\psi(F \times I) = \{t^E \eta : 0 < t < b, \eta \in F\} = b^E \{t^E \eta : 0 < t < 1, \eta \in F\} = b^E \widetilde{F}.$$

By virtue of Lemma 1.6, it follows that $\psi(F \times I) \in \mathcal{M}_d$ and

$$\begin{aligned}\mu(F \times I) = \sigma(F)\lambda(I) \\ &= \left((\operatorname{tr} E) m(\widetilde{F})\right) \left(\int_0^b t^{\operatorname{tr} E - 1}\, dt\right) = b^{\operatorname{tr} E} m(\widetilde{F}) = m(b^E \widetilde{F}) \\ &\qquad = m(\psi(F \times I)).\end{aligned}$$

When $b = \infty$ i.e., $I = (0, \infty)$, we observe that

$$I = \bigcup_{n=1}^{\infty} (0, n) = \bigcup_{n=1}^{\infty} I_n$$

where the open intervals $I_n = (0, n)$ are nested and increasing. In view of the result above (for finite $b = n$), we have

$$\psi(F \times I) = \psi\left(\bigcup_{n=1}^{\infty} (F \times I_n)\right) = \bigcup_{n=1}^{\infty} \psi(F \times I_n) \in \mathcal{M}_d.$$

Given that ψ is a bijection, $\{\psi(F \times I_n)\}$ is necessarily a nested increasing sequence and so, by the continuity of the measures μ and m,

$$\mu(F \times I) = \lim_{n\to\infty} \mu(F \times I_n) = \lim_{n\to\infty} m(\psi(F \times I_n)) = m(\psi(F \times I)).$$

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Case 2: $I = (0, a]$ for $0 < a < \infty$. We have

$$I = (0, a] = \bigcap_{n=1}^{\infty} (0, a + 1/n) = \bigcap_{n=1}^{\infty} I_n$$

where the open intervals $I_n = (0, a+1/n)$ are nested and decreasing. By reasoning analogous to that given in Case 1, we have

$$\psi(F \times I) = \bigcap_{n=1}^{\infty} \psi(F \times I_n) \in \mathcal{M}_d$$

and

$$\mu(F \times I) = \lim_{n\to\infty} \mu(F \times I_n) = \lim_{n\to\infty} m(\psi(F \times I_n)) = m(\psi(F \times I)).$$

In particular, $m(\psi(F \times I)) = \sigma(F)\lambda((0, a]) = \sigma(F)a^{\operatorname{tr} E}/\operatorname{tr} E < \infty$. //

Case 3: $I = (a, b)$ for $0 < a < b \leq \infty$. In this case, $I = (0, b) \setminus (0, a]$ and so, in view of Cases 1 and 2, $\psi(F \times I) = \psi(F \times (0, b)) \setminus \psi(F \times (0, a]) \in \mathcal{M}_d$ and

$$\mu(F \times I) = \mu(F \times (0, b)) - \mu(F \times (0, a]) = m(\psi(F \times (0, b))) - m(\psi(F \times (0, a])) = m(\psi(F \times I))$$

where we have used the fact that $\mu(F \times (0, a]) = m(\psi(F \times (0, a])) < \infty$. //

Case 4: $I \subseteq (0, \infty)$ is open. In this case, it is known that I can be expressed as a countable union of disjoint open intervals $\{I_n\}$ and, by virtue of Cases 1 and 3, we have

$$\psi(F \times I) = \bigcup_{n=1}^{\infty} \psi(F \times I_n) \in \mathcal{M}_d,$$

where this union is disjoint, and

$$\begin{aligned} m(\psi(F \times I)) &= \sum_n m(\psi(F \times I_n)) = \sum_n \mu(F \times I_n) \\ &= \sum_n \sigma(F)\lambda(I_n)\sigma(F)\left(\sum_n \lambda(I_n)\right) = \sigma(F)\lambda(I) = \mu(F \times I). \end{aligned}$$

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Case 5: $I \subseteq (0,\infty)$ is closed. In this case, we have $I = (0,\infty) \setminus O$ where O is open and so

$$\psi(F \times I) = \psi(F \times ((0,\infty) \setminus O)) = \psi(F \times (0,\infty)) \setminus \psi(F \times O) \in \mathcal{M}_d.$$

At this point, we'd like to use the property that

$$m(\psi(F \times (0,\infty)) \setminus \psi(F \times O)) = m(\psi(F \times (0,\infty))) - m(\psi(F \times O)),$$

but this only holds when $m(\psi(F \times O)$ is finite. We must therefore proceed differently. For each natural number n, define $O_n = O \cap (0,n)$ and $I_n = (0,n) \setminus O_n$. It is straightforward to show that $\{I_n\}$ and $\{\psi(F \times I_n)\}$ are nested and increasing with

$$I = \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad \psi(F \times I) = \bigcup_{n=1}^{\infty} \psi(F \times I_n).$$

The results of Cases 1 and 4 guarantee that, for each n,

$$m(\psi(F \times O_n))) = \mu(F \times O_n) \leq \mu(F \times (0,n)) < n^{\operatorname{tr} E} m(\tilde{F}) < \infty$$

and therefore

$$m(\psi(F \times I_n)) = m(\psi(F \times (0,n))) - m(\psi(F \times O_n)) = \mu(F \times (0,n)) - \mu(F \times O_n) = \mu(F \times I_n).$$

Then, by virtue of the continuity of measure,

$$m(\psi(F \times I)) = \lim_{n\to\infty} m(\psi(F \times I_n)) = \lim_{n\to\infty} \mu(F \times I_n) = \mu(F \times I).$$

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Case 6. $I \subseteq (0,\infty)$ is G_δ or F_σ. Depending on whether I is G_δ or F_σ, express I as an intersection of nested decreasing open sets or a union of nested increasing closed sets. In both cases, by virtually the same argument given in the previous cases, we find that $\psi(F \times I) \in \mathcal{M}_d$,

$$m(\psi(F \times I)) = \mu(F \times I).$$

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□

Remark 3. *Huan, the analogous result to the following lemma is in the first paragraph on Page 281 of [5] and is in the sentence preceding "So we have established (10) for all measurable rectangles..." Truthfully, I don't follow their argument and, actually, I don't quite believe it.*

Lemma 1.8. *For any $F \in \Sigma_S$ and $L \in \mathcal{L}$, $\psi(F \times L) \in \mathcal{M}_d$ and*

$$m(\psi(F \times L)) = \mu(F \times L).$$

Proof. Fix $F \in \Sigma_S$ and $L \in \mathcal{L}$. It is easy to see that λ and the Lebesgue measure dt on $(0,\infty)$ are absolutely continuous with respect to one another (Huan, you should check this and the following sentence.). It follows that $((0,\infty),\mathcal{L},\lambda)$ is a complete measure space and, further, that there exists an F_σ set $L_\sigma \subseteq (0,\infty)$ and a G_δ set $L_\delta \subseteq (0,\infty)$ for which $L_\sigma \subseteq L \subseteq L_\delta$ and $\lambda(L_\delta \setminus L_\sigma) = 0$. Note that, necessarily, $\lambda(L) = \lambda(L_\sigma) = \lambda(L_\delta)$. We have

$$\psi(F \times L) = \psi(F \times L_\sigma) \cup \psi(F \times (L \setminus L_\sigma)) \tag{6}$$

where, by virtue of the preceding lemma, $\psi(F \times L_\sigma) \subseteq \mathcal{M}_d$ and

$$m(\psi(F \times L_\sigma)) = \mu(F \times L_\sigma) = \sigma(F)\lambda(L_\sigma) = \sigma(F)\lambda(L) = \mu(F \times L). \tag{7}$$

Observe that

$$\psi(F \times (L \setminus L_\sigma)) \subseteq \psi(F \times (L_\delta \setminus L_\sigma))$$

where, because $L_\delta \setminus L_\sigma$ is an G_δ set, the latter set is a member of $\mathcal{M}_d$ and

$$m(\psi(F \times (L_\delta \setminus L_\sigma))) = \mu(F \times (L_\delta \setminus L_\sigma)) = \sigma(F)\lambda(L_\delta \setminus L_\sigma)) = 0$$

by virtue of the preceding lemma. Using the fact that $(\mathbb{R}^d \setminus \{0\}, \mathcal{M}_d, m)$ is complete, we conclude that $\psi(F \times (L \setminus L_\sigma)) \in \mathcal{M}_d$ and $m(\psi(F \times (L \setminus L_\sigma))) = 0$. It now follows from (6) and (7) that $\psi(F \times L) \in \mathcal{M}_d$ and

$$m(\psi(F \times L)) = m(\psi(F \times L_\sigma)) + m(\psi(F \times (L \setminus L_\sigma)) = \mu(F \times L),$$

as desired. $\square$

As it is fairly elementary, I've commented out the lemma showing that S, as a compact set of a metric space, necessarily contains a countably dense set. If this appears in your thesis, you should feel free to include the lemma

Lemma 1.9. *Every open subset $U \subseteq \mathbb{R}^d \setminus \{0\}$ can be written as a countable union of open sets of the form $\psi(\mathcal{U})$ where $\mathcal{U} = \mathcal{O} \times I$ is an open rectangle in $S \times (0,\infty)$.*

Proof. In what follows, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$, $N_\delta(x)$ denotes the associated open ball of radius δ and center $x \in \mathbb{R}^d$ and, for each linear transformation $T : \mathbb{R}^d \to \mathbb{R}^d$, $\|T\|$ denotes the operator norm of T associated to the Euclidean norm on $\mathbb{R}^d$. Given that S is compact (as a subspace of the metric space $\mathbb{R}^d$), S has a countably dense set $\{\eta_j\}_{j=1}^\infty$. Let $\{t_k\}_{k=1}^\infty$ be a countably dense subset of $(0,\infty)$. For each triple of natural numbers $j,l,n \in \mathbb{N}_+$, consider the open set

$$\mathcal{U}_{j,l,n} = \mathcal{O}_{j,n} \times \{|t - t_l| < 1/n\} \subseteq S \times (0,\infty)$$

where

$$\mathcal{O}_{j,n} = \{\eta \in S : |\eta - \eta_j| < 1/n\}.$$

Here, $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^d$. Fix $U \subseteq \mathbb{R}^d \setminus \{0\}$, an open subset of $\mathbb{R}^d \setminus \{0\}$. We will show that

$$U = \bigcup_{\substack{j,l,n \\ \psi(\mathcal{U}_{j,l,n}) \subseteq U}} \psi(\mathcal{U}_{j,l,n}), \tag{8}$$

where each $\psi(\mathcal{U}_{j,l,n})$ is open because ψ is a homeomorphism. It is clear that any element of the union on the right hand side of (8) belongs to some $\psi(\mathcal{U}_{j,l,n}) \subseteq U$ and so the union is a subset of U. To prove (8), it therefore suffices to prove that, for each $x \in U$, there exists a triple j, l, n with

$$x \in \psi(\mathcal{U}_{j,l,n}) \subseteq U.$$

To this end, fix $x \in U$ and, because U is open, let $\delta > 0$ be such that $N_\delta(x) \subseteq U$. Consider $(\eta_x, t_x) = \psi^{-1}(x) \in S \times (0, \infty)$ and set $M = \|t_x^E\| > 0$ and $C = \|E| > 0$. Observe that

$$\begin{aligned} \|I - \alpha^E\| &= \left\| \sum_{k=1}^{\infty} \frac{(\ln \alpha)^k}{k!} E^k \right\| \\ &\leq \sum_{k=1}^{\infty} \frac{|\ln \alpha|^k}{k!} \|E\|^k = e^{(C|\ln \alpha|)} - 1 \end{aligned}$$

for all $\alpha > 0$. Since $\alpha \mapsto e^{(C|\ln \alpha|)} - 1$ is continuous and 0 at $\alpha = 1$, we can choose $\delta' > 0$ for which

$$\|I - \alpha^E\| \leq \frac{\delta}{2M(|\eta_x| + 2)}$$

whenever $|\alpha - 1| < \delta'$. Choose an integer

$$n > \max \left\{ \frac{1}{t_x}, \frac{1}{\delta' t_x}, \frac{4M}{\delta} \right\}.$$

In view of the density of the collections $\{t_l\}$ and $\{\eta_j\}$, we can find t_l, η_j such that

$$|t_l - t_x| < \frac{1}{n}, \quad |\eta_j - \eta_x| < \frac{1}{n}.$$

It follows that the corresponding open set $\mathcal{U}_{j,l,n}$ contains $\psi^{-1}(x)$, or, equivalently, $x \in \psi(\mathcal{U}_{j,l,n})$ since ψ is bijective. Thus, it remains to show that $\psi(\mathcal{U}_{j,l,n}) \subseteq N_\delta(x)$. To this end, let $y = \psi(\eta_y, t_y) \in \psi(\mathcal{U}_{j,l,n})$ and consider

$$z = \psi(\eta_y, t_x).$$

By the triangle inequality, we have

$$\begin{aligned} |x - y| &\leq |x - z| + |z - y| \\ &\leq |\psi(\eta_x, t_x) - \psi(\eta_y, t_x)| + |\psi(\eta_y, t_x) - \psi(\eta_y, t_y)| \\ &= |t_x^E \eta_x - t_x^E \eta_y| + |t_x^E \eta_y - t_y^E \eta_y| \\ &= |t_x^E (\eta_x - \eta_y)| + |(t_x^E - t_y^E) \eta_y| \\ &\leq M|\eta_x - \eta_y| + \|t_x^E - t_y^E\| |\eta_y|. \end{aligned}$$

Since both $(\eta_x, t_x), (\eta_y, t_y) \in \mathcal{U}_{j,l,n}$, we have

$$|\eta_x - \eta_y| \leq |\eta_x - \eta_j| + |\eta_j - \eta_y| < \frac{2}{n}$$

and

$$|\eta_y| \leq |\eta_y - \eta_x| + |\eta_x| < |\eta_x| + \frac{2}{n}.$$

Also, since $|t_x - t_y| < 1/n$, it follows that $t_y = \alpha t_x$ where

$$|1 - \alpha| < \frac{1}{n t_x} < \delta'$$

by our choice of n. Consequently,

$$\begin{aligned}
|x - y| &< \frac{2}{n}M + \left(|\eta_x| + \frac{2}{n}\right) \|t_x^E - t_x^E \alpha^E\| \\
&< \frac{2}{n}M + (|\eta_x| + 2)\, M \|I - \alpha^E\| \\
&\leq \frac{2M}{n} + \frac{\delta M\,(|\eta_x| + 2)}{2M(|\eta_x| + 2)} \\
&< \frac{\delta}{2} + \frac{\delta}{2} \\
&= \delta,
\end{aligned}$$

and so we have establised (8). Finally, upon noting that $\{\mathcal{U}_{j,l,n})\}$ is a countable collection of open rectangles (indexed by $(j, l, n) \in \mathbb{N}_+^3$), the union in (8) is necessarily countable and we are done with the proof. $\square$

The following is a (graduate level) homework-exercise worthy lemma. You should try to prove it yourself before you read the proof. Though it is somewhat difficult (to state and prove – for me, at least), it is abstract enough that I suspect it is fairly well-known and we should look for a reference.

In our final lemma preceding the proof of Theorem 1.5, we treat a general measure-theoretic statement which gives sufficient conditions concerning two measure spaces to ensure that their completions are isomorphic Check this against the pushforward – I don't like the Bogachev reference [1]). Though we suspect that this result is well-known, we present its proof for completeness.

Lemma 1.10. *Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be measure spaces, let $\varphi : X_1 \to X_2$ be a bijection and denote by $(X_i', \Sigma_i', \mu_i')$ the completion of the measure space (X_i, Σ_i, μ_i) for $i = 1, 2$. Assume that the following two properties are satisfied:*

1. *For each $A_1 \in \Sigma_1$, $\varphi(A_1) \in \Sigma_2'$ and $\mu_2'(\varphi(A_1)) = \mu_1(A_1)$.*
2. *For each $A_2 \in \Sigma_2$, $\varphi^{-1}(A_2) \in \Sigma_1'$ and $\mu_1'(\varphi^{-1}(A_2)) = \mu_2(A_2)$.*

Then the measure spaces $(X_1', \Sigma_1', \mu_1')$ and $(X_2', \Sigma_2', \mu_2')$ are isomorphic with (point) isomorphism φ. I believe this is another way to state that μ_2' is precisely the pushforward of μ_1'. We should check this. Specifically,

$$\Sigma_2' = \{A_2 \subseteq X_2 : \varphi^{-1}(A_2) \in \Sigma_1'\} \tag{9}$$

and

$$\mu_2'(A_2) = \mu_1'(\varphi^{-1}(A_2)) \tag{10}$$

for all $A_2 \in \Sigma_2'$.

Proof. Let us first assume that $A_2 \in \Sigma_2'$. By definition, $A_2 = G_2 \cup H_2$ where $G_2 \in \Sigma_2$ and $H_2 \subseteq G_{2,0} \in \Sigma_2$ with $\mu_2'(A_2) = \mu_2(G_2)$ and $\mu_2'(H_2) = \mu_2(G_{2,0}) = 0$. Consequently, $\varphi^{-1}(A_2) = \varphi^{-1}(G_2) \cup \varphi^{-1}(H_2)$ and $\varphi^{-1}(H_2) \subseteq \varphi^{-1}(G_{2,0})$. In view of Property 2, $\varphi^{-1}(G_2), \varphi^{-1}(G_{2,0}) \in \Sigma_1'$ and we have

$$\mu_1'(\varphi^{-1}(G_2)) = \mu_2(G_2) = \mu_2'(A_2) \qquad \text{and} \qquad \mu_1'(\varphi^{-1}(G_{2,0})) = \mu_2(G_{2,0}) = 0.$$

In view of the fact that $(X_1', \Sigma_1', \mu_1')$ is complete, $\varphi^{-1}(H_2) \in \Sigma_1'$ and $\mu_1'(\varphi^{-1}(H_2)) = 0$. Consequently, we obtain $\varphi^{-1}(A_2) = \varphi^{-1}(G_2) \cup \varphi^{-1}(H_2) \in \Sigma_1'$ and

$$\mu_2'(A_2) = \mu_1'(\varphi^{-1}(G_2)) \leq \mu_1'(\varphi^{-1}(A_2)) \leq \mu_1'(\varphi^{-1}(G_2)) + \mu_1'(\varphi^{-1}(H_2)) = \mu_2(G_2) + 0 = \mu_2'(A_2).$$

From this we obtain that $\Sigma_2' \subseteq \{A_2 \subseteq X_2 : \varphi^{-1}(A_2) \in \Sigma_1'\}$ and, for each $A_2 \in \Sigma_2'$, $\mu_2'(A_2) = \mu_1'(\varphi^{-1}(A_2))$. It remains to prove that

$$\{A_2 \subseteq X_2 : \varphi^{-1}(A_2) \in \Sigma_1'\} \subseteq \Sigma_2'.$$

To this end, let A_2 be a subset of X_2 for which $\varphi^{-1}(A_2) \in \Sigma_1'$. By the definition of Σ_1', we have $\varphi^{-1}(A_2) = G_1 \cup H_1$ where $G_1 \in \Sigma_1$, $H_1 \subseteq G_{1,0} \in \Sigma_1$ and $\mu_1'(H_1) = \mu_1(G_{1,0}) = 0$. In view of Property 1, $\varphi(G_1) \in \Sigma_2'$, $\varphi(H_1) \subseteq \varphi(G_{1,0}) \in \Sigma_2'$ and $\mu_2'(\varphi(G_{1,0})) = \mu_1(G_{1,0}) = 0$. Because $(X_2', \Sigma_2', \mu_2')$ is complete, we have $\varphi(H_1) \in \Sigma_2'$ and so

$$A_1 = \varphi(\varphi^{-1}(A_2)) = \varphi(G_1) \cup \varphi(H_1) \in \Sigma_2',$$

as desired. □

We are finally in a position to prove Theorem 1.5.

Huan, my use of the monotone class lemma below avoids the method in [5] which relies on Theorem 3.3 (of [5]) and sweeps some things under the carpet.

Proof of Theorem 1.5. Denote by $\mathcal{C}$ the collection of sets $G \subseteq S \times (0, \infty)$ for which $\psi(G) \in \mathcal{M}_d$ and $m(\psi(G)) = \mu(G)$. By virtue of Lemma 1.8, it follows that $\mathcal{C}$ contains all elementary sets, i.e., finite unions of disjoint measurable rectangles. Using the continuity of measure (applied to the measures m and μ) and the fact that ψ is a bijection, it is straightforward to verify that $\mathcal{C}$ is a monotone class. By the so-called monotone class lemma (Theorem 8.3 of [4]), it immediately follows that $\Sigma_S \times \mathcal{L} \subseteq \mathcal{C}$. In other words, for each $G \in \Sigma_S \times \mathcal{L}$,

$$\psi(G) \in \mathcal{M}_d \qquad \text{and} \qquad m(\psi(G)) = \mu(G). \tag{11}$$

We claim that, for each Borel subset A of $\mathbb{R}^d \setminus \{0\}$, $\psi^{-1}(A) \subseteq \Sigma_S \times \mathcal{L}$. To this end, we write

$$\psi(\Sigma_S \times \mathcal{L}) = \{\psi(G) : G \in \Sigma \times \mathcal{L}\}$$

for the σ-algebra on $\mathbb{R}^d \setminus \{0\}$ induced by ψ. In view of Lemma 1.9, $\psi(\Sigma_S \times \mathcal{L})$ contains every open subset of $\mathbb{R}^d \setminus \{0\}$ and therefore

$$\mathcal{B}(\mathbb{R}^d \setminus \{0\}) \subseteq \psi(\Sigma_S \times \mathcal{L}).$$

where $\mathcal{B}(\mathbb{R}^d \setminus \{0\})$ denotes the σ-algebra of Borel subsets of $\mathbb{R}^d \setminus \{0\}$ thus proving our claim.

Together, the results of the two preceding paragraphs show that, for each $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$, $\psi^{-1}(A) \subseteq \Sigma_S \times \mathcal{L}$ and $m(A) = \mu(\psi^{-1}(A))$. Upon noting that $\Sigma_S \times \mathcal{L} \subseteq \Sigma$, we immediately obtain the following statement. For each $A \in \mathcal{B}(\mathbb{R}^d \setminus \{0\})$,

$$\psi^{-1}(A) \subseteq \Sigma \qquad \text{and} \qquad m(A) = \mu(\psi^{-1}(A)). \tag{12}$$

In comparing (11) and (12) with Properties 1 and 2 of Lemma 1.10 and upon noting that $(S\times(0,\infty), \Sigma, \mu)$ is the completion of $(S\times(0,\infty), \Sigma_S\times\mathcal{L}, \mu)$ and $(\mathbb{R}^d\setminus\{0\}, \mathcal{M}_d, m)$ is the completion of $(\mathbb{R}^d\setminus\{0\}, \mathcal{B}(\mathbb{R}^d\setminus\{0\}), m)$, Item 1 of the theorem follows immediately from Lemma 1.10.

It remains to prove Item 2. To this end, let $f : \mathbb{R}^d \to \mathbb{C}$ be Lebesgue measurable. Because $\mathcal{M}_d = \{A \subseteq \mathbb{R}^d \setminus \{0\} : \psi^{-1}(A) \in \Sigma\}$, it follows immediately that $f \circ \psi$ is Σ-measurable. In the case that $f \geq 0$, Item 1 guarantees that

$$\int_{\mathbb{R}^d} f(x)\,dx = \int_{\mathbb{R}^d\setminus\{0\}} f(x)\,dx = \int_{S\times(0,\infty)} (f\circ\psi)(\eta,t)\,d\mu(\eta,t)$$

where we have used the fact that the $\{0\} \subseteq \mathbb{R}^d$ has Lebesgue measure 0 (It would be nice to have a good statement of the change of variables formula we're using – is this in Folland?). From this, (4) follows from Item 2 in Theorem 1.4. Finally, by applying the above result to $|f| \geq 0$, we obtains $f \in L^1(\mathbb{R}^d)$ if and only if $f \circ \psi \in L^1(S \times (0,\infty), \Sigma, \mu)$ and, in this case, (4) follows directly from Item 3 of Theorem 1.4, $\square$

Using Theorem 1.5, we are able to establish the following proposition.

Proposition 1.11. *Consider the finite Borel measure σ on the measure space (S, Σ_S, σ). For each $F \in \Sigma_S$,*

$$\sigma(F) = \inf\{\sigma(\mathcal{O}) : F \subseteq \mathcal{O} \subseteq S \text{ and } \mathcal{O} \text{ is open}\} \tag{13}$$

and

$$\sigma(F) = \sup\{\sigma(K) : K \subseteq F \subseteq S \text{ and } K \text{ is compact}\}. \tag{14}$$

In particular, σ is a Radon measure (Need a citation – Folland).

Proof. Given that S is compact and σ is finite, it suffices to prove (13), i.e., it suffices to prove the statement: For each $F \in \Sigma_S$ and $\epsilon > 0$, there is an open subset $\mathcal{O}$ of S containing F for which

$$\sigma(\mathcal{O} \setminus F) < \epsilon.$$

To this end, let $F \in \Sigma_S$ and $\epsilon > 0$. Given that $\widetilde{F}$ is a Lebesgue measurable subset of $\mathbb{R}^d$ and the Lebesgue measure m is outer regular, there exists an open set $U \subseteq \mathbb{R}^d$ for which $\widetilde{F} \subseteq U$ and

$$m(U \setminus \widetilde{F}) = m(U) - m(\widetilde{F}) < \epsilon/(2\operatorname{tr} E). \tag{15}$$

Since $\widetilde{F}$ is necessarily a subset of the open set $B\setminus\{0\}$, we assume without loss of generality that $U \subseteq B\setminus\{0\}$. For each $0 < t < 1$, consider the open set

$$\mathcal{O}_t = S \cap \left(t^{-E}U\right)$$

in S. Observe that, for each $x \in F$, $t^E x \in \widetilde{F} \subseteq U$ and therefore $x \in \mathcal{O}_t$. Hence, for each $0 < t < 1$, $\mathcal{O}_t$ is an open subset of S containing F.

We claim that there is at least one $t_0 \in (0,1)$ for which

$$m(\widetilde{\mathcal{O}_{t_0}}) < m(U) + \epsilon/(2\operatorname{tr} E). \tag{16}$$

To prove the claim, we shall assume, to reach a contradiction, that

$$m(\widetilde{\mathcal{O}_t}) \geq m(U) + \epsilon/(2\operatorname{tr} E)$$

for all $0 < t < 1$. By virtue of Corollary **??**,

$$m(U) = \int_0^\infty \left(\int_S \chi_U(t^E\eta)\, d\sigma(\eta) \right) t^{\operatorname{tr} E - 1}\, dt.$$

Upon noting that $U \subseteq B \setminus \{0\}$, it is easy to see that

$$U = \bigcup_{0<s<1} s^E \mathcal{O}_s$$

and

$$t^E\eta \in \bigcup_{0<s<1} s^E \mathcal{O}_s$$

if and only if $0 < t < 1$ and $\eta \in \mathcal{O}_t$. Consequently,

$$\begin{aligned} m(U) &= \int_0^1 \left(\int_S \chi_{\mathcal{O}_t}(\eta)\, d\sigma(\eta) \right) t^{\operatorname{tr} E - 1}\, dt \\ &= \int_0^1 \sigma(\mathcal{O}_t) t^{\operatorname{tr} E - 1}\, dt \\ &= \int_0^1 (\operatorname{tr} E) m(\widetilde{\mathcal{O}_t}) t^{\operatorname{tr} E - 1}\, dt. \end{aligned}$$

Upon making use of our supposition, we have

$$\int_0^1 (\operatorname{tr} E) m(\widetilde{\mathcal{O}_t}) t^{\operatorname{tr} E - 1}\, dt \geq \int_0^1 (\operatorname{tr} E)(m(U) + \epsilon/(2\operatorname{tr} E)) t^{\operatorname{tr} E - 1}\, dt = m(U) + \epsilon/(2\operatorname{tr} E)$$

and so

$$m(U) \geq m(U) + \epsilon/(2\operatorname{tr} E),$$

which is impossible. Thus, the stated claim is true.

Given any such t_0 for which (16) holds, set $\mathcal{O} = \mathcal{O}_{t_0}$. As previously noted, $\mathcal{O}$ is an open subset of S which contains F. In view of (15) and (16), we have

$$m(\widetilde{\mathcal{O}}) - m(\widetilde{F}) < m(U) - m(\widetilde{F}) + \epsilon/(2\operatorname{tr} E) < \epsilon/(2\operatorname{tr} E) + \epsilon/(2\operatorname{tr} E) = \epsilon/\operatorname{tr} E$$

and therefore

$$\sigma(\mathcal{O} \setminus F) = \sigma(\mathcal{O}) - \sigma(F) = \operatorname{tr} E(m(\widetilde{\mathcal{O}}) - m(\widetilde{F})) < \epsilon,$$

as desired. $\square$

Corollary 1.12. *The completion of the measure space $(S, \mathcal{B}(S), \sigma)$ is (S, Σ_S, σ). In particular, the latter space is complete and every $F \in \Sigma_S$ is of the form $F = G \cup H$ where G is a Borel set and H is a subset of a Borel set Z with $\sigma(Z) = 0$.*

We should think about maybe calling this a proposition.

Proof. Let us denote by $(S, \overline{\mathcal{B}(S)}, \overline{\sigma})$ the completion of the measure space $(S, \mathcal{B}(S), \sigma)$. Our job is to show that $\overline{\mathcal{B}(S)} = \Sigma_S$ and $\overline{\sigma}(F) = \sigma(F)$ for all F in this common σ-algebra.

First, let $F \in \overline{\mathcal{B}(S)}$ which is, by definition, a set of the form $F = G \cup H$ where $G \in \mathcal{B}(S)$ with $\overline{\sigma}(F) = \sigma(G)$ and $H \subseteq G_0 \in \mathcal{B}(S)$ with $\sigma(G_0) = 0$. In view of Proposition 1.2, $\widetilde{G} \subseteq \mathcal{M}_d$, $\widetilde{H} \subseteq \widetilde{G_0} \in \mathcal{M}_d$ and we have

$$m(\widetilde{G_0}) = \frac{1}{\operatorname{tr} E}\sigma(G_0) = 0$$

Since $(\mathbb{R}^d \setminus \{0\}, \mathcal{M}_d, m)$ is complete, we conclude that $\widetilde{H} \in \mathcal{M}_d$ with $m(\widetilde{H}) = 0$ and therefore $H \in \Sigma_S$ with $\sigma(H) = (\operatorname{tr} E) m(\widetilde{H}) = 0$. It follows that $F = G \cup H \in \Sigma_S$ and

$$\overline{\sigma}(F) = \sigma(G) \leq \sigma(F) \leq \sigma(G) + \sigma(H) = \sigma(G) + 0 = \overline{\sigma}(F).$$

It remains only to prove that $\Sigma_S \in \overline{\mathcal{B}(S)}$. To this end, let $F \in \Sigma_S$ be arbitrary but fixed. By appealing to Proposition 1.11, for each integer $n \in \mathbb{N}$, there exists a compact set $F_n \subseteq F$ for which

$$\sigma(F \setminus F_n) = \sigma(F) - \sigma(F_n) < 1/n.$$

Set

$$G = \bigcup_{n=1}^{\infty} F_n \subseteq F$$

and $H = F \setminus G$. We observe that G is a Borel set (in fact, an F_σ set) and

$$\sigma(H) = \sigma(F) - \sigma(G) = \sigma(F) - \lim_{n \to \infty} \sigma(F_n) = 0,$$

by the continuity of measure. We have shown that

$$F = G \cup H$$

where $G \in \mathcal{B}(S)$ and $H \in \Sigma_S$ with $\sigma(H) = 0$. It remains to find a Borel set $G_0 \supseteq H$ for which $\sigma(G_0) = 0$. To this end, we again appeal to Proposition 1.11 to form a collection of open sets $\{\mathcal{O}_n\}_{n=1}^{\infty}$ such that, for each $n \in \mathbb{N}$, $H \subseteq \mathcal{O}_n$ and $\sigma(\mathcal{O}_n) = \sigma(\mathcal{O}_n) - \sigma(H) < 1/n$. Finally, consider

$$G_0 = \bigcap_{n=1}^{\infty} \mathcal{O}_n,$$

which is necessarily a Borel set (in fact, a G_δ-set) containing H and, by the continuity of measure, has $\sigma(G_0) = 0$, as desired. $\square$

Some questions (I think the third is easiest):

1. To what extent does σ depend on E? In other words, if $E, E' \in \operatorname{Exp}(P)$ (which necessarily have $\operatorname{tr} E = \operatorname{tr} E'$ in view of Section 2 of [3]), would our construction have yielded the same measure σ had we instead used the dilation $T_t = t^{E'}$ for everything? They should be closely related if not equal. But I'm really curious about this. For a little background reading, see the material near Proposition 2.3 of [3].

2. We discussed the fact that the surface measure on the (usual) sphere $\mathbb{S}^{d-1}$ was the unique Radon measure which was rotationally invariant and satisfied $\sigma_d(\mathbb{S}^{d-1}) = d \cdot m(\mathbb{B})$. I suspect that we also have some characterization for our measure σ on S – in fact, this is why I suspect that σ might not depend on the choice of E. Here are two possible conjectures and I really don't know if they are true:

   Conjecture 1.13. *For any $O \in \operatorname{Sym}(P)$ and $F \in \Sigma_S$,*

   $$\sigma(OF) = \sigma(F). \tag{17}$$

   Conjecture 1.14. *If σ does not depend on E (hence the construction produces the same measure σ regardless of which $E \in \operatorname{Exp}(P)$ is chosen) and (17) holds, σ is the unique Radon measure on S which satisfies $\sigma(S) = \operatorname{tr} E m(B)$.*

3. There are many polynomials (and positive-definite continuous functions) that have S as their "unital" level set. For example: For any $\alpha > 0$, $Q_\alpha(\xi) := (P(\xi))^\alpha$ is continuous and has

   $$S = \{\eta \in \mathbb{R}^d : Q_\alpha(\xi) = 1\}.$$

   Further, Q_α is positive-homogeneous[2] and has $\operatorname{Exp}(Q_\alpha) = \operatorname{Exp}(P)/\alpha$ in the senses that $E_\alpha \in \operatorname{Exp}(Q_\alpha)$ if and only if $E_\alpha = E/\alpha$ for $E \in \operatorname{Exp}(P)$. If you use $E_\alpha = E/\alpha$ to construct a measure σ_α on S via the above construction, how is σ_α related to σ? I think I have an idea about this, but you should try it. Another question: Are there other polynomials (which are not powers of P) that have S as their unital level set?

2 The manifold structure on S and computing integrals on S with respect to σ

In the previous subsections, we have constructed our surface-carried measure on S abstractly, only making use of topological notions. In this subsection, we discuss the manifold structure on S and actually give formulas for computing things.

By the homogeneity of P, we observe that

$$P(\xi) = \frac{d}{dt}(tP(\xi)) = \frac{d}{dt}P(t^E\xi) = \nabla P(t^E\xi) \cdot (t^{E-1}E\xi)$$

and so, by evaluating at $t = 1$, we find that

$$1 = P(\eta) = \nabla P(\eta) \cdot E\eta$$

[2] and is a polynomial precisely when $\alpha = 1, 2, \dots$.

for all $\eta \in S$. Consequently, ∇P is everywhere nonvanishing on S and consequently S is a $d-1$ dimensional smooth manifold (By the IFT, need reference). We want: $\psi : S \times (0,\infty) \to \mathbb{R}^d$ is a homeomorphism. To see this

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