Continuity: Exercises 4.1 - 4.10, Baby Rudin

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4.1 *Proof.* Let f a real function on $\mathbb{R}$ which satisfies

$$\lim_{h\to 0}[f(x+h)-f(x-h)]=0$$

for every $x\in\mathbb{R}$ be given. To prove: f is not continuous. Consider this counterexample:

$$f(x)=\begin{cases}0, & x\neq 0\\ 1, & x=0\end{cases}$$

Clearly, f satisfies the conditions above, but f is not continuous at 0. $\square$

4.2 *Proof.* Let f a continuous mapping of a metric space X into a metric space Y be given. To prove: $f(\bar{E})=\overline{f(E)}$ for every set $E\subset X$. Let subset $E\subset X$ be given. If $f(\bar{E})=\emptyset$ then there's nothing to prove. If $f(\bar{E})\neq\emptyset$, then pick $y\in f(\bar{E})$ and so there is some $x\in\bar{E}$ such that $y=f(x)$. Now, $x\in\bar{E}=E\cup E'$, so $x\in E'$ or $x\in E$. If $x\in E$ then $y=f(x)\in f(E)\subset\overline{f(E)}$. If $x\in E'$, then x is a limit point of E. We now want to show $f(x)$ is a limit point of $f(E)$. Let $\epsilon>0$ be given, then because f is continuous, $\exists\delta>0$ such that $d(f(x_0),f(x))<\epsilon$ whenever $d(x_0,x)<\delta$, for all $x_0\in X$. x is a limit point of E, so for some $\delta>\delta'>0$, there is $x_1\in E$. This means $f(x_1)\in\mathcal{N}_\epsilon(x)$ for some $f(x_1)\in f(E)$. This means $f(p)$ is a limit point of $f(E)$. So, $f(p)\in\overline{f(E)}$. Therefore, $f(\bar{E})\subset\overline{f(E)}$.

An example in which $f(\bar{E})\subsetneq\overline{f(E)}$. Let $E=\mathbb{N}\subsetneq\mathbb{R}$. We know that $\mathbb{N}=\overline{\mathbb{N}}$. Now, define $f:\mathbb{R}\to\mathbb{R}$ by $f(n)=\frac{1}{n}$. Obviously, $f(\mathbb{N})=f(\bar{\mathbb{N}})=\{1/n:n\in\mathbb{N}\setminus\{0\}\}$ since the input sets are the same. Now, $\overline{f(\mathbb{N})}=\overline{\{1/n:n\in\mathbb{N}\}}\cup\{0\}$. So, $f(\bar{\mathbb{N}})\subsetneq\overline{f(\mathbb{N})}$. $\square$

4.3 *Proof.* Let f a continuous real function on a metric space X be given. Consider the zero set $Z(f)$ of f. We want to show $Z(f)$ is closed. We notice that $Z(f)\equiv f^{-1}(\{0\})$, where the set $\{0\}$ is closed. Theorem 4.8 says $f:X\to Y$ is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y. In this problem, take $C=\{0\}\subset X$. C is closed, so $f^{-1}(C)=f^{-1}(\{0\})=Z(f)$ is closed. $\square$

4.4 *Proof.* Let $f,g:X\xrightarrow{\text{cont.}}Y$ and $E\overset{\text{dense}}{\subset}X$ be given. To prove: $f(E)$ dense in $f(X)$. Since E dense in X, $\bar{E}=X$. Pick $y\in f(X)$. To show $f(E)$ dense in $f(X)$, we want to show that if $y\neq f(E)$, $y\in f(E)'$. Assume $y\in f(X)\setminus f(E)$, then there is an x such that $y=f(x)$. If $x\in E$ then $y=f(x)\in f(E)$. This cannot happen, so $x\in X\setminus E$. $x\notin E$, which is dense in X, so there is a sequence $\{x_n\}\subset E$ such that $x_n\to x\in X\setminus E$. Since f is continuous, $f(x_n)\to f(x)$. If $f(x_n)=f(x)\in f(E)$ for some n, then we get a contradiction. So $f(x_n)\neq f(x)$ for all n. This means $y=f(x)$ is a limit point of $f(E)$, i.e., $y\in f(E)'$. So $f(E)$ is dense in $f(X)$.

Now, to prove: if $g(p) = f(p)$ for all $p \in E$, then $g(p) = f(p)$ for all $p \in X$. Well, if $p \in E$ then obviously, $g(p) = f(p)$. Consider $p \in E^c$. Since E dense in X, there is a sequence $\{p_n\}$ in E such that $p_n \to p \in E^c$. Now, $f(p_n) = g(p_n)$ for all n by hypothesis, so $f(p) = f(\lim_{n\to\infty} p_n) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} g(p_n) = g(\lim_{n\to\infty} p_n) = g(p)$. This means $f(p) = g(p)$ for all $p \in X$. □

4.5 *Proof.* Let f be a real continuous function defined on a closed set $E \subset \mathbb{R}$. We want to construct a real function g on $\mathbb{R}$ such that $f(x) = g(x)$ for all $x \in E$. Before we do this, we use a fact from Exercise 29, Chap 2 which says that because $E \subset X$ is closed,

$$E^c = \bigcup_{i=1} (a_i, b_i)$$

where the union is at most countable and $a_i < b_i < a_{i+1} < b_{i+1}$ for any $i \in \mathbb{N}$. With this, $g(x)$ can be given by:

$$g(x) = \begin{cases} f(x), & x \in E \\ f(a_i) + (x - a_i)\frac{f(b_i) - f(a_i)}{b_i - a_i}, & x \in E^c \end{cases}$$

Obviously $g(x)$ is a continuous function because $f(x)$ is continuous on E, and g is just a linear function (hence continuous) on E^c open.

When the word "closed" is omitted, we run into trouble. Consider $f(x) = 1/x$ on the open set $E = \mathbb{R} \setminus \{0\}$. Then there is no way for us to assign a real value to $g(0)$ and require that g be continuous.

For vector-valued functions, the result is the following: for $f(x) = (f_1(x), \dots, f_d(x))$, where each $f_i(x)$ is a real continuous function on a closed set $E \subset \mathbb{R}$, we can extend each $f_i(x)$ by g_i given by a similar definition above, to get an extension g for f given by $g(x) = (g_1(x), \dots, g_d(x))$. g is continuous on $\mathbb{R}^d$ because each g_i is continuous on $\mathbb{R}$. □

4.6 *Proof.* Let f defined on E be given. Assume $E \subset \mathbb{R}$ is compact. We want to show f is continuous on E iff its graph, $\mathcal{G} = \{(x, f(x)) : x \in E\}$ is compact.

Before doing anything, we have to define the metric for the space $E \times f(E)$ in which the graph lives. For $x_1, x_2 \in E$ and $f(x_1), f(x_2) \in f(E)$, define

$$d((x_1, f(x_1), (x_2, f(x_2)) = \sqrt{d^2(x_1, x_2) + d^2(f(x_1), f(x_2))}.$$

Okay with this we can start with the proof.

($\rightarrow$) Suppose E is compact and f is continuous. To show $\mathcal{G}$ is compact, we define a map $\mathcal{F} : E \to \mathcal{G}$ given by $\mathcal{F}(x) = (x, f(x))$. Since E is compact, Theorem 4.14 tells us that if

$\mathcal{F}$ is continuous on E then $\mathcal{F}(E) = \mathcal{G}$ is compact. Well, let $\epsilon > 0$ be given. Pick a point $x_0 \in E$. Since f is continuous, there is a $\delta > 0$ such that $d(f(x), f(x_0)) < \epsilon/\sqrt{2}$ whenever $d(x, x_0) < \delta$. Choose $\delta < \epsilon/\sqrt{2}$, then

$$d(\mathcal{F}(x), \mathcal{F}(x_0)) = \sqrt{d^2(x, x_0) + d^2(f(x), f(x_0))} < \sqrt{2\epsilon^2/2} = \epsilon. \tag{1}$$

So, $\mathcal{F}$ is continuous on E, and we're done.

($\longleftarrow$) Suppose $\mathcal{G}$ and E are compact. We want to show f is continuous. Consider the function $\mathcal{F}$ given by $\mathcal{F}(x) = (x, f(x))$ like that defined above. To show f is continuous, we can show $\mathcal{F}(x)$ is continuous, assuming that $\mathcal{G}, E$ are compact (since if $\mathcal{F}$ is continuous then its second component f must also be continuous). The function $\bar{g}(x, f(x)) = x$ is 1-1 and continuous. It's inverse mapping is just $\mathcal{F}(x)$. By theorem 4.17, $\mathcal{F}$ is a continuous mapping from E to $(E, f(E))$. It follows that f is also continuous. $\square$

4.7 *Proof.* f, g on $\mathbb{R}^2$ are given by $f(0,0) = g(0,0) = 0$, and if $(x, y) \neq 0$, $f(x, y) = xy^2/(x^2 + y^4)$, and $g(x, y) = xy^2/(x^2 + y^6)$. We want to show that f is bounded on $\mathbb{R}^2$. By completing the square, we know that $x^2 + y^4 \geq 2xy^2$, so $f(x, y) \leq 2$ for all $(x, y) \in \mathbb{R}^2$. So f is bounded.

Next, to show g is unbounded in every neighborhood of $(0,0)$, we look at sequences that converge to $(0,0)$. One such sequence is $\{(x_n, y_n) =\} = \{(1/n^3, 1/n)\}$. Clearly, $g(x_n, y_n) = n^6/2n^5 = n/2 \to \infty$ as $n \to \infty$. So g is unbounded in every neighborhood of $(0,0)$.

To show f is not continuous at $(0,0)$ we look at where $\{f(x_n, y_n)\}$ converges to when $(x_n, y_n) \longrightarrow (0,0)$. Take the sequence $\{(x_n, y_n) = (1/n^2, 1/n)\}$. Then $f(x_n, y_n) = 1/2$ for all n. Obviously, $f(x_n, y_n) \longrightarrow 1/2 \neq 0$ so f is not continuous at $(0,0)$.

Now we want to show the restrictions of f, g to any straight line in $\mathbb{R}^2$ are continuous. There are two cases: $x = c$ (the "vertical" line) and $y = ax + b$. If $x = c$ constant, then if $c \neq 0$, then $f(x, y) = cy^2/(x^2 + c^4)$ and $g(x, y) = cy^2/(c^2 + y^6)$ are both continuous in y and hence are continuous. If $c = 0$ then $f = g = 0$, also continuous.

Consider straight lines: $y = ax + b$. If $b = 0$, then if for nonzero (x, y), $f(x, y) = a^2x/(1 + a^4x^2)$ and $g = a^2x/(1 + a^6x^4)$. As $x \longrightarrow 0$, it is clear that $f \longrightarrow 0$ and $g \longrightarrow 0$, so f, g are also continuous. If $b \neq 0$ then we don't have to worry because these lines don't pass the origin (which is are things can be bad). $\square$

4.8 *Proof.* Let f a real uniformly continuous function on the bounded set $E \subset \mathbb{R}$. We want to show f is bounded on E. Suppose E is bounded by $M > 0$. Let $\epsilon > 0$ be given, then there is a $\delta > 0$ such that $\left|f(p) - f(q)\right| < \epsilon$ for all $p, q \in E$ for which $\left|p - q\right| < \delta$.

To show that the conclusion is false if boundedness of E is omitted, we look at a counterexample.

4.9 *Proof.*
4.10 *Proof.*