Questions/Ideas #3

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The goal here is see whether the "volume" of the bounded set $\mathcal{S} := \{s \in \mathbb{R}^d | P(s) = 1\}$ where

$$P(s) = s_1^{k_1} + s_2^{k_2} + \cdots + s_d^{k_d}. \tag{1}$$

The powers are even numbers, as usual.

I looked at the $\det\{A^\top A\}$ approach to finding volumes, but it turns out that

$$\sqrt{\det(J^\top J)} = |\det(J)| \tag{2}$$

where J is the Jacobian matrix. So this doesn't help at all.

Okay, I considered the general $d = 2$ case:

$$P(s) = s_1^{k_2} + s_2^{k_2} \tag{3}$$

where k_1, k_2 are even numbers greater than 0. The parameterization (setting $t = 1$) is

$$\vec{s} = \begin{bmatrix} s_1 \\ \left(1 - s_1^{k_1}\right)^{1/k_2} \end{bmatrix} \tag{4}$$

The Jacobian is then

$$\sqrt{\frac{t^{2\left(\frac{1}{m}+\frac{1}{n}-1\right)}\left(1-x^n\right)^{\frac{2}{m}-2}}{m^2}} = \frac{\left(1-x^n\right)^{\frac{1}{m}-1}}{m}. \tag{5}$$

Integrating over $[-1, 1]$ gives the "volume" of the top half of the space:

$${}_2F_1\left(1 - \frac{1}{m}, \frac{1}{n}; 1 + \frac{1}{n}; (-1)^n\right) + \frac{\Gamma\left(\frac{1}{m}\right)\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(\frac{m+n}{mn}\right)} \tag{6}$$

So this integral **converges** in general.

What about when $d = 3$? How do we deal with the $d = 3$ case? My strategy is to do the $ds_1 ds_2$ integral on $1 - s_3^{k_3}$ first, obtain something in terms of s_3, then to the ds_3 integral on $[-1, 1]$.

The parameterization for $s_1^{k_1} + s_2^{k_2} = 1 - s_3^{k_3}$ after setting $t = 1$ is

$$\vec{s} = \begin{bmatrix} s_1 \\ (R_3 - s_1^{k_1})^{1/k_2} \end{bmatrix} \tag{7}$$

where I have defined $R_3 = 1 - s_3^{k_3}$. The associated Jacobian with $(t = 1)$ is

$$\frac{R_3\left(-{s_1}^{k_1} - {s_3}^{k_3} + 1\right)^{\frac{1}{k_2}-1}}{k_2} = \frac{R_3\left(R_3 - {s_1}^{k_1}\right)^{\frac{1}{k_2}-1}}{k_2} \tag{8}$$

Now, we wish to evaluate the integral

$$\iint ds_1 ds_2 ds_3 = \int_{-1}^{1} ds_3 \iint_{\partial, s_3} ds_1 ds_2. \tag{9}$$

The $ds_1 ds_2$ integral can be handled using the same procedure as before, except we're no longer integrating from -1 to 1, but rather from $-R_3^{1/k_1}$ to R_3^{1/k_1}. To this end,

$$\iint ds_1 ds_2 = \int_{-R_3^{1/k_1}}^{R_3^{1/k_1}} \left(R_3 - s_1^{k_1}\right)^{-1+1/k_2} ds_1 \tag{10}$$

After a change of variables this integral is equal to

$$R_3^{-1+1/k_1+1/k_2} \int_{-1}^{1} (1 - x^{k_1})^{-1+1/k_2}\, dx. \tag{11}$$

This x integral converges and gives something similar to what we have before. In the ends of the ds_3 integral, the x integral will be a constant, so we don't worry about it. To find the total integral, we will multiply this x integral with the ds_3 integral. The ds_3 integral will now be

$$\begin{aligned}\int_{-1}^{1} R_3 \cdot R_3^{-1+1/k_1+1/k_2}\, ds_3 &= \int_{-1}^{1} R_3^{1/k_1+1/k_2}\, ds_3 = \int_{-1}^{1} \left(1 - s_3^{k_3}\right)^{1/k_1+1/k_2} ds_3 \\ &= {}_2F_1\left(-\frac{k_1+k_2}{k_1k_2}, \frac{1}{k_3}; 1 + \frac{1}{k_3}; (-1)^{k_3}\right) + \frac{\Gamma\left(1+\frac{1}{k_3}\right)\Gamma\left(1+\frac{1}{k_1}+\frac{1}{k_2}\right)}{\Gamma\left(\frac{1}{k_2}+\frac{1}{k_3}+1+\frac{1}{k_1}\right)}\end{aligned} \tag{12}$$

where the extra factor of R_3 comes from the Jacobian. So we see that this 3-d integral also **converges** in general.

Example: We shall verify this with the 3-sphere where $x^2 + y^2 + z^2 = 1$. The $ds_1 ds_2$ integral is

$$\int_{-1}^{1} (1 - s_1^2)^{-1+1/2}\, ds_1 = \pi \tag{13}$$

The ds_3 integral is

$$\int_{-1}^{1} (1 - s_3^2)^{(1/2+1/2)}\, ds_3 = \frac{4}{3}. \tag{14}$$

Multiplying everything together we get $4\pi/3$, which is the volume is a 3-sphere of radius 1.

Next, can we use induction to get convergence at higher orders? Let's see if we can figure out any pattern. Suppose $R_3 = (1 - s_3^{k_3})$ now becomes a bit more complicated: involves another $s_4^{k_4}$: $R_3 \to R_{34} = 1 - s_3^{k_3} - s_4^{k_4}$. With this, we have a change in integrals:

$$\int_{-1}^{1} R_{34} \cdot R_{34}^{-1+1/k_1+1/k_2}\, ds_3 ds_4 \to \int_{-1}^{1} ds_4 \int_{-R_4^{1/k_4}}^{R_4^{1/k_4}} (R_4 - s_3^{k_3})^{1/k_1+1/k_2}\, ds_3 \tag{15}$$

where $R_4 = 1 - s_4^{1/k_4}$. This integral looks familiar! We can repeat what we've done before to get

$$\int_{-1}^{1} ds_4 R_4^{-1+1/k_1+1/k_2+1/k_3} \int_{-1}^{1} (1 - x^{k_3})^{1/k_1+1/k_2}\, ds_3. \tag{16}$$

I tried to evaluated this integral in Mathematica and it converges. I also tested with $k_1 = k_2 = k_3 = k_4 = 2$ and found the volume of the 4-sphere, off by a factor of 2, which is good enough for me...

I think we can carry on with this, to show the "volume" of $\mathcal{S}$ is bounded, i.e., the angular integrals converge.