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Course: **8.422 - AMO II**
Problem set: **#9**
Due: Friday, April 21, 2022
Collaborators:

**1. Non-perturbative calculation of the rf response of a Feshbach molecule.** In this problem we calculate the response of a Feshbach molecule to an external rf drive non-perturbatively. The setup is as follows:

- The diatomic molecular state is $|m\rangle$ has energy $E_m < 0$ is coupled to a continuum of final states of unbound atoms $\{|\vec{k}\rangle\}$ with energies $E_k = \hbar^2 k^2/2\mu$ via an rf drive with frequency $\omega$.
- We consider the molecular state dressed by the rf light in the RWA, so its energy is $E_m + \hbar\omega$, or $\hbar\delta = \hbar\omega - |E_m|$ like in last week's homework. We will be varying the photon frequency at a fixed magnetic field.
- The coupling connects $|m\rangle$ and $|k\rangle$, but it is not the hyperfine interaction. Rather, it is the Rabi frequency of the rf drive $\Omega_R$.
- The coupling here is momentum-dependent. The molecules have size $\sim a$, with wavefunction $\psi_m(r) = \sqrt{2/a}e^{-r/a}$. The unbound, free atoms have wavefunction $\psi_k(r) = \sqrt{2/R}\sin(k_n r)$ where $R$ is some large fictitious quantization box and $k_n = n\pi/R$.
- We're working in 1D, due to the spherical symmetry of the molecules: they only couple to the $l = 0, m = 0$ angular momentum states.
- When replacing sums with integrals, we use $\sum_k \to \frac{R}{\pi}\int dk$.
- **Goal:** we want to find the full time evolution $U_m(\tau) = \langle m|\, U(\tau)\, |m\rangle$ giving the probability amplitude that the molecule is still present after time $\tau$.
- We expect that for weak coupling, Fermi's golden rule applies, and we get exponential decay. But we will find corrections in both long and short times.
- For strong coupling, we expect Rabi oscillations between the molecular state and the continuum. Why? Continuum states that are strongly coupled to the molecular state form a "band" with some width. For short times relative to the inverse of this width, or for coupling strengths larger than the width, the molecule-atom pair system will first behave as a two-level system and exhibit Rabi oscillations.

(a) Here we calculate the coupling strength. What does the rf photon do? It flips the spin of one of one of the atoms in the molecule into another hyperfine state where it is free. The result is one transferred atom and its leftover partner. The coupling strength depends not only on the Rabi frequency $\Omega_R$ of the rf drive but also on the (Franck-Condon) overlap between the initial state, which is the bound molecular wavefunction, and the final state, which is the free two-atom wavefunction:

$$V_{mk} = \frac{\hbar\Omega_R}{2}\int_0^\infty \psi_k(r)^*\psi_m(r)\,dr = \frac{\hbar\Omega_R}{2}\int_0^\infty \sqrt{\frac{2}{R}}\sin(kr)\sqrt{\frac{2}{a}}e^{-r/a}\,dr = \frac{\hbar\Omega_R}{\sqrt{aR}}\frac{k}{1/a^2 + k^2},$$

which attains a maximum of $V_{mk,\max} = (\hbar\Omega_R/2)\sqrt{a/R}$ at $k = 1/a$, and is significant for $k \sim 1/a$.

We can also find the $k$-dependence of $V_{mk}$ at low and high $k$'s:

$$\text{Low } k: \qquad V_{mk} \to \frac{\hbar\Omega_R}{\sqrt{R}}a^{3/2}k$$

$$\text{High } k: \qquad V_{mk} \to \frac{\hbar\Omega_R}{\sqrt{aR}}\frac{1}{k}.$$

So, $V_{mk} \propto k$ for low $k$ and $V_{mk} \propto 1/k$ for high $k$.