8.422 PSet 2 - Solutions

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The following solutions were prepared by me (Eric Wolf) for Pset 2 in the 2023 Spring adminstration of 8.422, Atomic and Optical Physics II. Any errors should be assumed to be my own, especially in sections marked 'Aside'.

Text in emphasis is excerpted from the problem set for the purpose of clarity.

1 Problem 1

In this problem we will see that the motion of neutral atoms in a rotating frame can be described as the motion of a charged particle experiencing a scalar potential and an effective magnetic field. Let's consider free motion in the x-y plane. The transformation from the lab frame to a frame rotating at angular frequency Ω *about the z-axis is*

$$\tilde{x}(t) = x\cos(\Omega t) + y\sin(\Omega t) \tag{1}$$

$$\tilde{y}(t) = y\cos(\Omega t) - x\sin(\Omega t) \tag{2}$$

a) Write the kinetic energy $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$ *of a particle of mass* m *in terms of the coordinates and velocities in the rotating frame,* $\tilde{x}$, $\tilde{y}$, $\dot{\tilde{x}}$, $\dot{\tilde{y}}$.

A straightforward inversion of equations 1 and 2 gives

$$x(t) = \tilde{x}(t)\cos(\Omega t) - \tilde{y}(t)\sin(\Omega t) \tag{3}$$

$$y(t) = \tilde{y}(t)\cos(\Omega t) + \tilde{x}(t)\sin(\Omega t) \tag{4}$$

Differentiating Equations 3 and 4 gives

$$\dot{x}(t) = \left(\dot{\tilde{x}}(t)\cos(\Omega t) - \dot{\tilde{y}}(t)\sin(\Omega t)\right) - \Omega\left(\tilde{x}(t)\sin(\Omega t) + \tilde{y}(t)\cos(\Omega t)\right)$$
$$\dot{y}(t) = \left(\dot{\tilde{y}}(t)\cos(\Omega t) + \dot{\tilde{x}}(t)\sin(\Omega t)\right) + \Omega\left(\tilde{x}(t)\cos(\Omega t) - \tilde{y}(t)\sin(\Omega t)\right)$$

or, for greater brevity,

$$\dot{x}(t) = \left(\dot{\tilde{x}}(t)\cos(\Omega t) - \dot{\tilde{y}}(t)\sin(\Omega t)\right) - \Omega y(t) \tag{5}$$

$$\dot{y}(t) = \left(\dot{\tilde{y}}(t)\cos(\Omega t) + \dot{\tilde{x}}(t)\sin(\Omega t)\right) + \Omega x(t) \tag{6}$$

Proceeding, we find (dropping the non-explicit t-dependence):

$$\dot{x}^2 + \dot{y}^2 =$$
$$\left[\left(\dot{\tilde{x}}(t)\cos(\Omega t) - \dot{\tilde{y}}(t)\sin(\Omega t)\right) - \Omega y(t)\right]^2 + \left[\left(\dot{\tilde{y}}(t)\cos(\Omega t) + \dot{\tilde{x}}(t)\sin(\Omega t)\right) + \Omega x(t)\right]^2$$

$$= \left[\left(\dot{\tilde{x}}\cos(\Omega t) - \dot{\tilde{y}}\sin(\Omega t)\right)^2 + \left(\dot{\tilde{y}}\cos(\Omega t) + \dot{\tilde{x}}\sin(\Omega t)\right)^2\right]$$

$$-2\Omega\left[y\left(\dot{\tilde{x}}\cos(\Omega t) - \dot{\tilde{y}}\sin(\Omega t)\right) - x\left(\dot{\tilde{y}}(t)\cos(\Omega t) + \dot{\tilde{x}}(t)\sin(\Omega t)\right)\right] + \Omega^2\left[x^2 + y^2\right]$$

Performing a trivial simplification on the first term, and regrouping the second,

$$= \left[\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2\right] - 2\Omega\left[\dot{\tilde{x}}\left(y\cos(\Omega t) - x\sin(\Omega t)\right) - \dot{\tilde{y}}\left(y\sin(\Omega t) + x\cos(\Omega t)\right)\right] + \Omega^2\left[x^2 + y^2\right]$$

Recognizing $\tilde{x}$ and $\tilde{y}$ in the middle term, we obtain

$$\dot{x}^2 + \dot{y}^2 = \dot{\tilde{x}}^2 + \dot{\tilde{y}}^2 + 2\Omega\left[\tilde{x}\dot{\tilde{y}} - \tilde{y}\dot{\tilde{x}}\right] + \Omega^2\left[x^2 + y^2\right]$$

Finally, noting the identity $\tilde{x}^2 + \tilde{y}^2 = x^2 + y^2$ by a trivial evaluation, we find

$$\dot{x}^2 + \dot{y}^2 = \dot{\tilde{x}}^2 + \dot{\tilde{y}}^2 + 2\Omega\left[\tilde{x}\dot{\tilde{y}} - \tilde{y}\dot{\tilde{x}}\right] + \Omega^2\left[\tilde{x}^2 + \tilde{y}^2\right]$$

Accordingly, in the rotating frame, we have

$$\boxed{T = \frac{1}{2}m\left(\dot{\tilde{x}}^2 + \dot{\tilde{y}}^2 + 2\Omega\left[\tilde{x}\dot{\tilde{y}} - \tilde{y}\dot{\tilde{x}}\right] + \Omega^2\left[\tilde{x}^2 + \tilde{y}^2\right]\right)} \tag{7}$$

Or equivalently

$$T = \boxed{\frac{1}{2}m\left(\left(\dot{\tilde{x}} - \Omega\tilde{y}\right)^2 + \left(\dot{\tilde{y}} + \Omega\tilde{x}\right)^2\right)}$$

b) The Lagrangian $\mathcal{L}(\tilde{x}, \tilde{y}, \dot{\tilde{x}}, \dot{\tilde{y}})$ is just the kinetic energy you found above. Find the canonical momentum $\tilde{p}_x = \frac{\partial\mathcal{L}}{\partial\dot{\tilde{x}}}$ and $\tilde{p}_y = \frac{\partial\mathcal{L}}{\partial\dot{\tilde{y}}}$.

Using Equation 7, we find

$$\tilde{p}_x = \frac{\partial T}{\partial\dot{\tilde{x}}}$$

$$\boxed{\tilde{p}_x = m\dot{\tilde{x}} - \Omega m\tilde{y}} \tag{8}$$

And likewise

$$\tilde{p}_y = \frac{\partial T}{\partial\dot{\tilde{x}}}$$

$$\boxed{\tilde{p}_y = m\dot{\tilde{y}} + \Omega m\tilde{x}} \tag{9}$$

c) The Poisson brackets are defined as $\{f, g\} = \sum_i \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial p_i}$, where $x_1 \equiv \tilde{x}$, $x_2 \equiv \tilde{y}$, $p_1 \equiv \tilde{p}_x$, $p_2 \equiv \tilde{p}_y$. From that definition, $\{\tilde{x}, \tilde{p}_x\} = 1$ naturally, as well as $\{\tilde{p}_x, \tilde{p}_x\} = 0$, $\{\tilde{p}_x, \tilde{p}_y\} = 0$ etc. Show, however, that $\left\{m\dot{\tilde{x}}, m\dot{\tilde{y}}\right\} \neq 0$.

It is useful to invert Equations 8 and 9:

$$m\dot{\tilde{x}} = \tilde{p}_x + \Omega m\tilde{y} \tag{10}$$

$$m\dot{\tilde{y}} = \tilde{p}_y - \Omega m\tilde{x} \tag{11}$$

Accordingly, we can evaluate all the nonzero derivatives:

$$\frac{\partial}{\partial \tilde{p}_x} m\dot{\tilde{x}} = 1 \tag{12}$$

$$\frac{\partial}{\partial \tilde{y}} m\dot{\tilde{x}} = \Omega m \tag{13}$$

$$\frac{\partial}{\partial \tilde{p}_y} m\dot{\tilde{y}} = 1 \tag{14}$$

$$\frac{\partial}{\partial \tilde{x}} m\dot{\tilde{y}} = -\Omega m \tag{15}$$

Accordingly, we may write

$$\begin{gathered}\left\{m\dot{\tilde{x}}, m\dot{\tilde{y}}\right\} \\ = \left[\frac{\partial m\dot{\tilde{x}}}{\partial \tilde{x}}\frac{\partial m\dot{\tilde{y}}}{\partial \tilde{p}_x} - \frac{\partial m\dot{\tilde{y}}}{\partial \tilde{x}}\frac{\partial m\dot{\tilde{x}}}{\partial \tilde{p}_x}\right] + \left[\frac{\partial m\dot{\tilde{x}}}{\partial \tilde{y}}\frac{\partial m\dot{\tilde{y}}}{\partial \tilde{p}_y} - \frac{\partial m\dot{\tilde{y}}}{\partial \tilde{y}}\frac{\partial m\dot{\tilde{x}}}{\partial \tilde{p}_y}\right] \\ = [0 - (-\Omega m)] + [\Omega m - 0]\end{gathered}$$

and ultimately

$$\boxed{\left\{m\dot{\tilde{x}}, m\dot{\tilde{y}}\right\} = 2\Omega m \neq 0} \tag{16}$$

as desired.

d) Obtain the Hamiltonian from the Legendre transformation $H = \sum_i \dot{\tilde{x}}_i p_i - \mathcal{L}$. This replaces the dependence of $\mathcal{L}$ on $\dot{\tilde{x}}$ and $\dot{\tilde{y}}$ with the dependence of H on $\tilde{p}_x$ and $\tilde{p}_y$. Rewrite the Hamiltonian in terms of a vector potential $\vec{A}$ (assuming a fictitious charge q for the particles) and an effective potential $V(\tilde{x}, \tilde{y})$. What is the effective magnetic field $\vec{B} = \nabla \times \vec{A}$, and how do we call the effective potential V?

Exploiting Equations 10 and 11, we write

$$\begin{gathered}\sum_i \dot{\tilde{x}}_i \tilde{p}_i = \\ \dot{\tilde{x}}\tilde{p}_x + \dot{\tilde{y}}p_y \\ \left(\frac{\tilde{p}_x}{m} + \Omega\tilde{y}\right)\tilde{p}_x + \left(\frac{\tilde{p}_y}{m} - \Omega\tilde{x}\right)\tilde{p}_y\end{gathered} \tag{17}$$

We also ought to rewrite the Lagrangian in terms of $\tilde{x}$ and $\tilde{p}$. It is useful to exploit Equations 8 and 9 to write

$$\begin{gathered}\frac{1}{2m}\left[\tilde{p}_x^2 + \tilde{p}_y^2\right] = \\ \frac{1}{2}m\left[\left(\dot{\tilde{x}} - \Omega\tilde{y}\right)^2 + \left(\dot{\tilde{y}} + \Omega\tilde{x}\right)^2\right] \\ = \frac{1}{2}m\left[\left(\dot{\tilde{x}}^2 + \dot{\tilde{x}}^2\right) + 2\Omega\left(\tilde{x}\dot{\tilde{y}} - \tilde{y}\dot{\tilde{x}}\right) + \Omega^2\left(\tilde{x}^2 + \tilde{y}^2\right)\right]\end{gathered}$$

We recognize the final form as that of Equation 7, so we have found that[1]

$$\mathcal{L} = \frac{1}{2m}\left(\tilde{p}_x^2 + \tilde{p}_y^2\right) \tag{18}$$

[1]One can of course also obtain equation 18 by direct substitution; this is merely a faster solution by informed guessing.

Combining Equations 17 and 18, we find

$$H = \sum_i \dot{\tilde{x}}_i \tilde{p}_i - \mathcal{L}$$

$$H = \frac{\tilde{p}_x^2}{m} + \Omega \tilde{p}_x \tilde{y} + \frac{\tilde{p}_y^2}{m} - \Omega \tilde{p}_y \tilde{x} - \frac{1}{2m}\left(\tilde{p}_x^2 + \tilde{p}_y^2\right)$$

$$H = \frac{1}{2m}\left(\tilde{p}_x^2 + \tilde{p}_y^2\right) + \Omega\left(\tilde{p}_x \tilde{y} - \tilde{p}_y \tilde{x}\right) \tag{19}$$

Please note that this is the form in which we will have to place the Hamiltonian for part e).

We now cast the Hamiltonian to the stipulated form for this problem. We may write

$$(\tilde{p}_x + \Omega m \tilde{y})^2 = \tilde{p}_x^2 + 2\Omega m \tilde{p}_x \tilde{Y} + \Omega^2 m^2 \tilde{y}^2$$
$$(\tilde{p}_y - \Omega m \tilde{x})^2 = \tilde{p}_y^2 - 2\Omega m \tilde{p}_y \tilde{x} + \Omega^2 m^2 \tilde{x}^2$$

Accordingly, we write

$$\frac{1}{2m}\left[(\tilde{p}_x + \Omega m \tilde{y})^2 + (\tilde{p}_y - \Omega m \tilde{x})^2\right]$$
$$= \frac{1}{2m}\left[\tilde{p}_x^2 + \tilde{p}_y^2\right] + \Omega\left(\tilde{p}_x \tilde{y} - \tilde{p}_y \tilde{x}\right) + \frac{m\Omega^2}{2}\left(\tilde{x}^2 + \tilde{y}^2\right)$$

or, using Equation 19,

$$= H + \frac{m\Omega^2}{2}\left(\tilde{x}^2 + \tilde{y}^2\right)$$

Thus, inverting, we may write

$$H = \frac{1}{2m}\left[(\tilde{p}_x + \Omega m \tilde{y})^2 + (\tilde{p}_y - \Omega m \tilde{x})^2\right] - \frac{m\Omega^2}{2}\left(\tilde{x}^2 + \tilde{y}^2\right) \tag{20}$$

If we introduce vector notation:

$$\tilde{\vec{p}} = \left(\tilde{p}_x \hat{\tilde{x}} + \tilde{p}_y \hat{\tilde{y}}\right)$$

(where we note that the direction vectors $\hat{\tilde{x}}$ and $\hat{\tilde{y}}$ are constant in the *rotating* frame).

The choice of q is arbitrary, but then fixes the magnitude of $\vec{A}$; a simple enough choice is $q = m$. In this case, we define

$$\boxed{\vec{A} = \Omega(-\tilde{y}\hat{\tilde{x}} + \tilde{x}\hat{\tilde{y}})}$$

finding that we may write our Hamiltonian as

$$\boxed{H = \frac{1}{2m}\left(\tilde{\vec{p}} - q\vec{A}\right) - \frac{m\Omega^2}{2}(\tilde{x}^2 + \tilde{y}^2)}$$

The effective magnetic field is

$$\vec{B} = \nabla \times \vec{A}$$
$$= \left(\frac{\partial A_y}{\partial \tilde{x}} - \frac{\partial A_x}{\partial \tilde{y}} \right) \hat{z}$$
$$\boxed{\vec{B} = (\Omega - (-\Omega))\hat{z} = 2\Omega\hat{z}}$$

whereas the effective potential

$$\boxed{V = -\frac{m\Omega^2}{2}\vec{r}^2}$$

is what might fairly be called the "centrifugal potential" - the reason one has to hold on to a merry-go-round to stay stationary on its surface.

Aside:

It is well-known that there is a gauge freedom in the choice of the vector potential $\vec{A}$, yet in the above development we have arrived, apparently inevitably, at the symmetric gauge, which treats $\tilde{x}$ and $\tilde{y}$ on essentially the same footing. The loss of gauge freedom arose when we chose to use the classical form for the Lagrangian $\mathcal{L} = T - V$. It is known, however, that the dynamics of a system are invariant under the addition of a term $\frac{\mathrm{d}f(\vec{q},t)}{\mathrm{d}t}$ to the Lagrangian which is the total time derivative of a function $f(\vec{q}, t)$ depending only on the position coordinates and, potentially, time. The form $\mathcal{L} = T - V$ drops this freedom, which is precisely the freedom that gives us our gauge freedom in $\vec{A}$.

e) Completing the square, rewrite the Hamiltonian like $H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + W$. Give the expression for the operator W in terms of $\tilde{x}$, $\tilde{y}$, $\tilde{p}_x$, $\tilde{p}_y$. Can you explain in words why all that is needed to describe the motion of a particle in a rotating frame is adding that operator W?

We may immediately read off W from equation 19:

$$\boxed{W = \Omega\left(\tilde{p}_x\tilde{y} - \tilde{p}_y\tilde{x}\right)} \tag{21}$$

To explain why this is the form, consider that we may write the z component of the angular momentum $\vec{L} = \vec{r} \times \vec{p}$ as

$$L_z = \tilde{x}\tilde{p}_y - \tilde{y}\tilde{p}_x$$

whence the Hamiltonian becomes

$$\boxed{H = \frac{\tilde{\vec{p}}^2}{2m} - \Omega L_z}$$

The off-hand explanation of why the modification looks like this is that the angular momentum "generates rotations" while the Hamiltonian "generates time translations", so adding a term to the Hamiltonian proportional to the angular momentum is tantamount to transforming the system by a time-dependent angle - i.e. a continuous rotation. Apparently, then, the canonical momenta p and positions x in this problem behave exactly as do the canonical positions and momenta in a regular free-particle problem - except that one must also rotate the system at a rate Ω.

f) From Hamilton's equations of motion, derive the equation of motion for the particle in the rotating frame, identifying well-known effective forces you will find.

This is straightforward enough. Letting $\vec{\Omega} = \Omega\hat{z}$, and temporarily dropping the $\tilde{}$, we may conveniently write the Hamiltonian in vector form (where e.g. $r = \tilde{x}\hat{\tilde{x}} + \tilde{y}\hat{\tilde{y}}$) as

$$H = \frac{\vec{p}^2}{2m} - \vec{\Omega} \cdot (\vec{r} \times \vec{p}) \tag{22}$$

Taking the derivative with respect to $\vec{p}$, we find

$$\dot{\vec{r}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r} \tag{23}$$

(where the derivative of the cross product may be verified with, e.g., Levi-Cevita symbols) We may likewise write

$$\dot{\vec{p}} = -\frac{\partial H}{\partial \vec{r}} = -\vec{\Omega} \times \vec{p} \tag{24}$$

Taking a time derivative of equation 23,

$$\ddot{\vec{r}} = \frac{\dot{\vec{p}}}{m} - \vec{\Omega} \times \dot{\vec{r}}$$

Substituting in Equations 23 and 24, we find

$$\ddot{\vec{r}} = \frac{1}{m}\left(-\vec{\Omega} \times \vec{p}\right) - \vec{\Omega} \times \left(\frac{\vec{p}}{m} - \vec{\Omega} \times \vec{r}\right)$$

At this point, it behooves us to observe that $\vec{\Omega} \times \vec{\Omega} \times \vec{r}$ is a vector which is perpendicular to both $\vec{\Omega}$ and $\vec{\Omega} \times \vec{r}$; it is thus either parallel to or antiparallel to $\vec{r}$, and a little intuition (e.g. from the right-hand rule) tells us it is in fact antiparallel, with magnitude $\Omega^2 r$. Then we write

$$\ddot{\vec{r}} = -2\vec{\Omega} \times \frac{\vec{p}}{m} - \Omega^2 \vec{r}$$

At this point, we may exploit Equations 8 and 9 to write

$$\frac{\vec{p}}{m} = \dot{\vec{r}} + \vec{\Omega} \times \vec{r}$$

We then write

$$\ddot{\vec{r}} = -2\vec{\Omega} \times (\dot{\vec{r}} + \vec{\Omega} \times \vec{r}) - \Omega^2 \vec{r}$$

or ultimately

$$\boxed{\ddot{\vec{r}} = -2\vec{\Omega} \times \dot{r} + \Omega^2 \vec{r}} \tag{25}$$

where we can identify the first term as the coriolis force, and the second as the centrifugal force. Equivalently,

$$\boxed{\ddot{x} = 2\Omega \dot{y} + \Omega^2 x}$$

$$\boxed{\ddot{y} = -2\Omega \dot{x} + \Omega^2 y}$$

2 Problem 2

a)

We use the Lorentz force:

$$m\ddot{\vec{r}} = q\dot{\vec{r}} \times \vec{B}$$

Using $B = B\hat{z}$,

$$m(\ddot{x}\hat{x} + \ddot{y}\hat{y}) = qB\left(\dot{y}\hat{x} - \dot{x}\hat{y}\right)$$

Separating

$$\ddot{x} = \frac{qB}{m}\dot{y} \tag{26}$$

$$\ddot{y} = -\frac{qB}{m}\dot{x} \tag{27}$$

Writing $z = x + iy$, we may observe that

$$-i\dot{z} = -i\dot{x} + \dot{y}$$

and

$$\ddot{z} = \ddot{x} + i\ddot{y}$$

Accordingly, Equations 26 and 27 may be combined

$$\ddot{z} = -i\frac{qB}{m}\dot{z}$$

or, using

$$\boxed{\omega_c = \frac{qB}{m}} \tag{28}$$

$$\ddot{z} = -i\omega_C\dot{z} \tag{29}$$

Integrating Equation 29

$$\dot{z} = -i\omega_C z + K$$

This differential equation is trivial to solve; the solution may then be seen to be

$$z = K_1 e^{-i\omega_C t} + K_2$$

where $K_1, K_2 \in \mathcal{C}$. We know that $x(t) = \text{Re}(z(t))$ and $y(t) = \text{Re}(z(t))$, so it is straightforward to convert these complex degrees of freedom to find

$$\boxed{x(t) = A\cos(\omega_C t) + B\sin(\omega_C t) + x_0} \tag{30}$$

$$\boxed{y(t) = B\cos(\omega_C t) - A\sin(\omega_C t) + y_0} \tag{31}$$

where $K_1 = A + iB,\ K_2 = x_0 + iy_0$. Alternatively, because $K_1 = \frac{i}{\omega_C}\dot{z}(0)$ by inspection, we may compactly write

$$z(t)=\frac{i}{\omega_C}\dot{z}(0)e^{-i\omega_C t}+(x_0+iy_0)$$

yielding

$$\boxed{x(t)=-\frac{\dot{y}(0)}{\omega_C}\cos(\omega t)+\frac{\dot{x}(0)}{\omega_C}\sin(\omega_C t)+x_0} \tag{32}$$

$$\boxed{y(t)=\frac{\dot{x}(0)}{\omega_C}\cos(\omega_C t)+\frac{\dot{y}(0)}{\omega_C}\sin(\omega_C t)+y_0} \tag{33}$$

It will also be useful to note the following for future problems:

$$z(t)=\frac{i}{\omega_C}\dot{z}(t)+(x_0+iy_0) \tag{34}$$

b)

We write

$$H=\frac{1}{2m}(\vec{p}-q\vec{A})^2$$

which, using $\vec{A}=-\frac{B}{2}\left(y\hat{x}-x\hat{y}\right)$, gives

$$H=\frac{1}{2m}\left(\left(p_x+\frac{qB}{2}y\right)^2+\left(p_y-\frac{qB}{2}x\right)^2\right)$$

Note that $[p_x,y]=0$ etc., so we may expand naively, obtaining

$$\boxed{H=\frac{1}{2m}\left(p_x^2+p_y^2\right)+\frac{q^2B^2}{8m}\left(x^2+y^2\right)-\frac{qB}{2m}\left(xp_y-yp_x\right)} \tag{35}$$

or, recognizing the angular momentum,

$$\boxed{H=H_{\text{ho}}-\frac{\omega_C}{2}L_z} \tag{36}$$

with

$$\boxed{H_{\text{ho}}=\frac{1}{2m}\left(p_x^2+p_y^2\right)+\frac{q^2B^2}{8m}\left(x^2+y^2\right)} \tag{37}$$

c)

By the symmetry of our annihilators for x and y, it suffices to choose l_B so that

$$H_{\text{ho, x}}=\frac{p_x^2}{2m}+\frac{q^2B^2}{8m}x^2=\frac{\hbar\omega_C}{2}(n_x+\frac{1}{2})$$

Then let us invert

$$x=\frac{l_B}{\sqrt{2}}\left(a_x+a_x^\dagger\right) \tag{38}$$

$$p_x=\frac{-i\hbar}{\sqrt{2}l_B}\left(a_x-a_x^\dagger\right) \tag{39}$$

Then we write

$$\begin{aligned}\frac{p_x^2}{2m}+\frac{q^2B^2}{8m}x^2 \\ &= \tfrac{1}{2m}\left(-\tfrac{\hbar^2}{2l_B^2}\left(a_x-a_x^\dagger\right)^2+\tfrac{q^2B^2l_B^2}{8}\left(a_x+a_x^\dagger\right)^2\right) \\ &= \tfrac{1}{2m}\left(\left(\tfrac{q^2B^2l_B^2}{8}-\tfrac{\hbar^2}{2l_B^2}\right)\left(a^2+a^{\dagger 2}\right)+\left(\tfrac{\hbar^2}{2l_B^2}+\tfrac{q^2B^2l_B^2}{8}\right)\left(a_x^\dagger a_x+a_xa_x^\dagger\right)\right)\end{aligned}$$

At this point, we demand

$$\frac{\hbar^2}{2l_B^2}=\frac{q^2B^2l_B^2}{8}$$

finding

$$\boxed{l_B=\sqrt{\frac{2\hbar}{qB}}} \tag{40}$$

Then in particular

$$\frac{\hbar^2}{2l_B^2}+\frac{q^2B^2l_B^2}{8}=\frac{qB\hbar}{4}$$

and we may write

$$\begin{aligned}\frac{p_x^2}{2m}+\frac{q^2B^2}{8m}x^2 \\ &= \tfrac{qB\hbar}{4m}\left(a_x^\dagger a_x+a_xa_x^\dagger\right) \\ &= \tfrac{qB\hbar}{4m}\left(2n_x+1\right)\end{aligned}$$

Then, exploiting Equation 28,

$$H_{\text{h.o., x}}=\frac{\hbar\omega_C}{2}\left(n_x+\frac{1}{2}\right)$$

and the case for y is the same by symmetry.

We must still write L_z:

$$\begin{aligned}L_z &= xp_y-yp_x \\ &= -\tfrac{i\hbar}{2}\left(\left(a_x+a_x^\dagger\right)\left(a_y-a_y^\dagger\right)-\left(a_y+a_y^\dagger\right)\left(a_x-a_x^\dagger\right)\right) \\ &= -i\hbar\left(a_x^\dagger a_y-a_y^\dagger a_x\right) \\ &= i\hbar\left(a_y^\dagger a_x-a_x^\dagger a_y\right)\end{aligned}$$

where we note that e.g. $[a_x^\dagger,a_y]=0$ as these components commute, so we are able to manipulate the above sums naively.

$$\boxed{L_z=i\hbar\left(a_y^\dagger a_x-a_x^\dagger a_y\right)} \tag{41}$$

d)

It is trivial to verify that these operators have the "expected" commutation relations.

We invert the given expressions for the left- and right-handed annihilation operators:

$$a_x = \frac{1}{\sqrt{2}}(a+b) \tag{42}$$

$$a_y = \frac{-i}{\sqrt{2}}(a-b) \tag{43}$$

Then in particular,

$$\begin{aligned} L_z &= i\hbar\left(a_y^\dagger a_x - a_x^\dagger a_y\right) \\ &= \tfrac{i\hbar}{2}\left(i(a^\dagger - b^\dagger)(a+b) + i(a^\dagger + b^\dagger)(a-b)\right) \\ &= -\hbar\left(a^\dagger a - b^\dagger b\right) \end{aligned}$$

Then we may write

$$\boxed{L_z = \hbar\,(n_b - n_a)} \tag{44}$$

so that, in particular, the annihilator a is associated with left-handed circular motion, and b with right-handed, as expected.

e)

We first observe that

$$\begin{aligned} n_x + n_y &= a_x^\dagger a_x + a_y^\dagger a_y \\ &= \tfrac{1}{2}\left(\left(a^\dagger + b^\dagger\right)(a+b) + \left(a^\dagger - b^\dagger\right)(a-b)\right) \\ &= a^\dagger a + b^\dagger b \end{aligned}$$

so in particular

$$n_x + n_y = n_a + n_b \tag{45}$$

as we might expect. Then, exploiting the results of part c), we have

$$H_{\text{h.o.}} = \frac{\hbar\omega_C}{2}(n_x + n_y + 1) = \frac{\hbar\omega_C}{2}(n_a + n_b + 1) \tag{46}$$

Then, combining Equations 36, 46, and 44, we write

$$H = \frac{\hbar\omega_C}{2}(n_a + n_b + 1) - \frac{\omega_C}{2}\hbar\,(n_b - n_a)$$

or finally

$$\boxed{H = \hbar\omega_C(n_a + \frac{1}{2})} \tag{47}$$

We remark that this Hamiltonian has massive degeneracy (infinite for an infinite system): for every energy level dictated by the value of n_a, there are an infinite number of states indexed by the different values of $n_b \in \mathbb{W}$.

This degeneracy originates from the fact that the original Hamiltonian has an invariance under static translations in x, y; equivalently, we may say that the energy of a particle does not depend on the position of its guiding center.

Aside:

Indeed, this result that the energy of the Hamiltonian only depends on a sort of "left-handed component of the motion", while a right-handed component is dictated by the guiding center, is actually completely classical. Suppose that the cyclotron-orbiting trajectory of a classical particle in our field is given by

$$\vec{r}(t) = \vec{r}_0 + \vec{r}_c(t)$$

with $\vec{r}_0$ the guiding center and $\vec{r}_c(t)$ the cyclotron orbit. Then it is straightforward, if a bit tedious, to show that the average of the angular momentum $L_z = xp_y - yp_x$ (where p_x, p_y are the canonical momenta) over one cycle of the motion obeys

$$\langle L_z \rangle = \frac{1}{2} m\omega_C (r_0^2 - r_c^2)$$

That is, it is the sum of two components: a "right-handed" component which depends only on the position of the particle's guiding center, and a left handed component which depends only on the amplitude of its cyclotron motion. The energy of the particle, of course, is given by

$$E = \frac{1}{2} m\omega_C^2 r_c^2$$

and depends only on the cyclotron motion.

f)

Combining Equations 38 and 42, we write

$$\begin{aligned} x &= \tfrac{l_B}{\sqrt{2}} \left(a_x + a_x^\dagger\right) \\ &= \tfrac{l_B}{2} \left((a+b) + (a^\dagger + b^\dagger)\right) \end{aligned}$$

so that

$$\boxed{x = \frac{l_B}{2} \left(a + b + a^\dagger + b^\dagger\right)} \tag{48}$$

Likewise,

$$\begin{aligned} y &= \tfrac{l_B}{\sqrt{2}} \left(a_y + a_y^\dagger\right) \\ &= \tfrac{-il_B}{2} \left((a-b) - (a^\dagger - b^\dagger)\right) \end{aligned}$$

and hence

$$\boxed{y = \frac{-il_B}{2} \left(a - b - a^\dagger + b^\dagger\right)} \tag{49}$$

It is useful to show that

$$
\begin{aligned}
p_x &= -\tfrac{i\hbar}{\sqrt{2}l_B}\left(a_x - a_x^\dagger\right) \\
&= -\tfrac{i\hbar}{2l_B}\left((a+b)-(a^\dagger+b^\dagger)\right)
\end{aligned}
$$

and so

$$
p_x = -\frac{i\hbar}{2l_B}\left(a+b-a^\dagger-b^\dagger\right) \tag{50}
$$

and, analogously

$$
\begin{aligned}
p_y &= -\tfrac{i\hbar}{\sqrt{2}l_B}\left(a_y - a_y^\dagger\right) \\
&= -\tfrac{\hbar}{2l_B}\left((a-b)+(a^\dagger-b^\dagger)\right)
\end{aligned}
$$

so that

$$
p_y = -\frac{\hbar}{2l_B}\left(a-b+a^\dagger-b^\dagger\right) \tag{51}
$$

Likewise, we know that

$$
v_x = \dot{x} = \frac{\partial H}{\partial p_x}
$$

Using the expression of the Hamiltonian given at the start of the problem, we may write

$$
v_x = \frac{p_x}{m} + \frac{qyB}{2m}
$$

At his point, it is useful to notice that

$$
\frac{\hbar}{l_B m} = \frac{\hbar}{\sqrt{\frac{2\hbar}{qB}}m} = \frac{\sqrt{\frac{2\hbar}{qB}}qB}{2m} = \frac{l_B\omega_C}{2}
$$

Then in particular we write

$$
\begin{aligned}
v_x &= \tfrac{p_x}{m} + \tfrac{qB}{2m}y \\
&= -i\tfrac{\hbar}{2l_B m}\left(a+b-a^\dagger-b^\dagger\right) - \tfrac{il_B\omega_C}{4}(a-b-a^\dagger+b^\dagger) \\
&= -\tfrac{il_B\omega_C}{4}\left(a+b-a^\dagger-b^\dagger+a-b-a^\dagger+b^\dagger\right)
\end{aligned}
$$

and hence

$$
\boxed{v_x = -\frac{il_B\omega_C}{2}\left(a-a^\dagger\right)} \tag{52}
$$

Also,

$$
\begin{aligned}
v_y &= \frac{p_y}{m} - \frac{qB}{2m}x \\
&= -\frac{\hbar}{2l_B m}\left(a - b + a^\dagger - b^\dagger\right) - \frac{\omega_C l_B}{4}(a + b + a^\dagger + b^\dagger) \\
&= -\frac{\omega_C l_B}{4}(a - b + a^\dagger - b^\dagger + a + b + a^\dagger + b^\dagger)
\end{aligned}
$$

So

$$
\boxed{v_y = -\frac{\omega_C l_B}{2}\left(a + a^\dagger\right)} \tag{53}
$$

At this point, we recall Equation 34:

$$
z(t) = \frac{i}{\omega_C}\dot{z}(t) + (x_0 + iy_0)
$$

Expanding out and equating the real and imaginary parts, this gives us a classical relation between the guiding centers x_0, y_0 and the positions and velocities:

$$
x = x_0 - \frac{v_y}{\omega_C} \tag{54}
$$

$$
y = y_0 + \frac{v_x}{\omega_C} \tag{55}
$$

We may promote x, y, v_x, v_y to operators and use the expressions we have found in terms of a, b etc. to find quantum expressions for x_0 and y_0:

$$
\begin{aligned}
x_0 &= x + \frac{v_y}{\omega_C} \\
&= \frac{l_B}{2}(a + b + a^\dagger + b^\dagger) - \frac{l_B}{2}(a + a^\dagger)
\end{aligned}
$$

so that

$$
\boxed{x_0 = \frac{l_B}{2}(b + b^\dagger)} \tag{56}
$$

And

$$
\begin{aligned}
y_0 &= y - \frac{v_x}{\omega_C} \\
&= -\frac{il_B}{2}(a - b - a^\dagger + b^\dagger) + \frac{il_B}{2}(a - a^\dagger)
\end{aligned}
$$

$$
\boxed{y_0 = \frac{il_B}{2}(b - b^\dagger)} \tag{57}
$$

g)

The commutator is straightforward:

$$\begin{aligned}[x_0, y_0] &= \tfrac{il_B^2}{4}[(b+b^\dagger),(b-b^\dagger)] \\ &= [b,b]+[b^\dagger,b]-[b,b^\dagger]-[b^\dagger,b^\dagger]\end{aligned}$$

Perhaps it is obvious that a and b obey the usual commutation relations, so that in particular we have $[b, b^\dagger] = 1$, and so

$$\boxed{[x_0, y_0] = -\frac{il_B^2}{2}} \tag{58}$$

It is well known[2] that, for two operators A and B which do not commute,

$$\Delta A \Delta B \geq \frac{1}{2}\left|\langle [A, B]\rangle\right|^2 \tag{59}$$

Hence, we obtain

$$\boxed{\Delta x_0 \Delta y_0 \geq \frac{l_B^2}{4}} \tag{60}$$

Now, we had better hope that $[x, y] = 0$, because our entire treatment has been tacitly assuming this fact! But we may write

$$[x, y] = -\frac{il_B^2}{4}[(a+b+a^\dagger+b^\dagger)(a-b-a^\dagger+b^\dagger)]$$

Discarding all but the nonzero commutators right off the bat,

$$\begin{aligned}[x, y] &= -\tfrac{il_B^2}{4}(-[a,a^\dagger]+[b,b^\dagger]+[a^\dagger,a]-[b^\dagger,b]) \\ &= -\tfrac{il_B^2}{4}(-(1)+(1)+(-1)-(-1)) \\ &= 0\end{aligned}$$

and thus

$$\boxed{[x, y] = 0} \tag{61}$$

as expected.

g)

We begin by writing

$$e^{-\frac{iH_F t}{\hbar}} = e^{\frac{iFl_B t}{2\hbar}(b+b^\dagger)}$$

Observe that if we choose $\beta = \frac{iFl_B t}{2\hbar}$, we may immediately write

$$\frac{iFl_B t}{2\hbar}(b+b^\dagger) = \beta b^\dagger - \beta^* b$$

Now, in problem 3c, we will show that the displacement operator which has the property that

[2] Well, in principle. Personally I always have to google it.

$$D(\beta)\,|0\rangle = |\beta\rangle$$

may be written

$$D(\beta) = e^{\beta b^\dagger - \beta^* b}$$

With this relation in hand, it is trivial to see that the action of this force on the vacuum is to produce a time-dependent coherent state:

$$|\Psi(t)\rangle = e^{-\frac{iH_F t}{\hbar}}\,|0\rangle$$

or, using the form we have found,

$$\boxed{|\Psi(t)\rangle = |\beta(t)\rangle} \tag{62}$$

with

$$\boxed{\beta(t) = \frac{iFl_B t}{2\hbar}} \tag{63}$$

The expectations are straightforward in this framework:

$$\langle x_0\rangle = \frac{l_B}{2}\,\langle\beta|b + b^\dagger|\beta\rangle$$

Noting that $b\,|\beta\rangle = \beta\,|\beta\rangle$, $\langle\beta|\,b^\dagger = \langle\beta|\,\beta^*$, we write

$$\langle x_0\rangle = \frac{l_B}{2}\,\langle\beta|\beta + \beta^*|\beta\rangle = l_B \mathrm{Re}(\beta)$$

and hence

$$\boxed{\langle x_0\rangle = 0} \tag{64}$$

Conversely,

$$\langle y_0\rangle = \frac{il_B}{2}\,\langle\beta|b - b^\dagger|\beta\rangle = \frac{il_B}{2}(\beta - \beta^*) = -l_B \mathrm{Im}(\beta)$$

so that

$$\boxed{\langle y_0\rangle = -\frac{Fl_B^2 t}{2\hbar}} \tag{65}$$

3 Problem 3

a)

This is straightforward. Recalling the definition of a coherent state:

$$|\alpha\rangle = a^{-\frac{|\alpha|^2}{2}} \sum_{k=0}^{\infty} \frac{\alpha^k}{\sqrt{k!}}\,|k\rangle \tag{66}$$

then we may immediately wring

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{a^{*j}}{\sqrt{j!}}\frac{\beta^k}{\sqrt{k!}}\,\langle j|k\rangle$$

Exploiting the orthonormality of the number states, $\langle j|k\rangle = \delta_{jk}$, with δ_{jk} the Kronecker δ, we have

$$\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \sum_{j=0}^{\infty} \frac{(\alpha^*\beta)^j}{j!}$$

or, recognizing the exponential,

$$\boxed{\langle\alpha|\beta\rangle = e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} e^{\alpha^*\beta}} \tag{67}$$

b)

It suffices to show that the given operator

$$\hat{O} \equiv \frac{1}{\pi} \int \mathrm{d}^2\alpha \, |\alpha\rangle \langle\alpha| \tag{68}$$

is equivalent to the identity when it acts on an arbitrary number state $|n\rangle$, given that the number states form a complete basis. Also, the orthonormality and completeness properties of the number states imply that the above operator is the identity when acting on the number states if and only if

$$\forall m, n \in \mathbb{W}, \; \langle m|\hat{O}|n\rangle = \delta_{m,n} \tag{69}$$

Then let us show this relation:

$$\langle m|\hat{O}|n\rangle = \frac{1}{\pi} \int \mathrm{d}^2\alpha \, \langle m|\alpha\rangle \langle\alpha|n\rangle$$

Exploiting the definition of the coherent state in Equation 66 as well as the orthonormality of the number states,

$$\langle m|\hat{O}|n\rangle = \frac{1}{\pi} \int \mathrm{d}^2\alpha \left(\frac{\alpha^m}{\sqrt{m!}} e^{-\frac{|\alpha|^2}{2}} \right) \left(\frac{\alpha^{*n}}{\sqrt{n!}} e^{-\frac{|\alpha^*|^2}{2}} \right)$$

At this point, it is useful to write $\alpha = re^{i\phi}$, whereupon we may recast the above equation as

$$\langle m|\hat{O}|n\rangle = \frac{1}{\pi\sqrt{m!}\sqrt{n!}} \int_0^{\infty} (r\mathrm{d}r) \, r^{m+n} e^{-r^2} \int_0^{2\pi} \mathrm{d}\phi e^{i(m-n)\phi}$$

Recall, however, the easily proven fact that, for integer m, n,

$$\int e^{i(m-n)\phi} \mathrm{d}\phi = 2\pi\delta_{m,n}$$

whereupon we obtain

$$\langle m|\hat{O}|n\rangle = \frac{2}{\sqrt{m!}\sqrt{n!}} \delta_{m,n} \int_0^{\infty} r^{m+n+1} e^{-r^2} \mathrm{d}r$$

Then exploiting the fact that the Kronecker δ makes everything zero when $m \neq n$, we can just as well substitute $m = n$ in the RHS above and regroup a few factors:

$$\langle m|\hat{O}|n\rangle = \frac{1}{n!} \delta_{m,n} \int_0^{\infty} \left(r^2\right)^n e^{-r^2} (2r\mathrm{d}r)$$

The integral is now primed for the substitution $u = r^2, \; \mathrm{d}u = 2r\mathrm{d}r$:

$$\langle m|\hat{O}|n\rangle = \frac{1}{n!}\delta_{m,n}\left(\int_0^\infty u^n e^{-u}\mathrm{d}u\right)$$

However, it is well-known that,

$$\int_0^\infty u^n e^{-u}\mathrm{d}u = \Gamma(n+1) = n!$$

with the last equality holding for integral n. Making this substitution, we find

$$\boxed{\langle m|\hat{O}|n\rangle = \delta_{m,n}} \tag{70}$$

as required.

c)

We assume that it is sufficient to prove that the given operator has the same properties as the displacement operator when operating on the vacuum. That is, if

$$\hat{O}(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a} \tag{71}$$

is such that

$$\hat{O}(\alpha)\,|0\rangle = |\alpha\rangle$$

then we know that it is *an* operator which satisfies the definition of the displacement operator. We take it more or less on faith that it is *the* displacement operator.[3]

Then considering the provided BCH lemma

$$e^{A+B} = e^A e^B e^{-\frac{[A,B]}{2}} \text{ if } [A,B] = c \in \mathcal{C} \tag{72}$$

with $A = \alpha a^\dagger$, $B = -\alpha^* a$, we first verify that

$$[A,B] = [\alpha a^\dagger, -\alpha^* a] = |\alpha|^2[a,a^\dagger] = |\alpha|^2 \in \mathcal{C} \tag{73}$$

Thus, empowered by the results of Equation 73, we may leverage equation 72 to write

$$e^{\alpha a^\dagger - \alpha^* a} = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{|\alpha|^2}{2}}$$

or, since the last value is simply a scalar,

$$\hat{O}(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a} \tag{74}$$

Now, exploiting the series definition of the operator exponential, as well as the fact that $a\,|0\rangle = 0$, it is straightforward to see that

$$e^{-\alpha^* a}\,|0\rangle = \sum_{j=0}^\infty \frac{(-\alpha^*)^j}{j!} a^j\,|0\rangle = |0\rangle$$

since only the $j = 0$ term makes any contribution to the sum. Then we may write, using Equation 74

[3]Of course, it is always possible to be perverse, and define an operator e.g. $D'(\alpha) \equiv D(\alpha) + |n=13\rangle\langle n=27|$, which would still have the right properties when displacing the vacuum. Actually, what makes the displacement operator unique is demanding that it displaces any coherent state. We ignore this detail here.

$$\hat{O}(\alpha)\,|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} e^{-\alpha^* a}\,|0\rangle$$

Leveraging the above,

$$\hat{O}(\alpha)\,|0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger}\,|0\rangle$$

Using the series definition:

$$\hat{O}(\alpha)\,|0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} a^{\dagger j}\,|0\rangle$$

or, using $a^{\dagger j}\,|0\rangle = \sqrt{j!}\,|j\rangle$,

$$\boxed{\hat{O}^\alpha = e^{-\frac{|\alpha|^2}{2}} \sum_{j=0}^{\infty} \frac{\alpha^j}{\sqrt{j!}}\,|j\rangle = |\alpha\rangle} \tag{75}$$

as required.

d)

This is straightforward enough. Recall that a freely-evolving coherent state $|\alpha\rangle = |\alpha(t)\rangle$ obeys, up to a phase,[4]

$$|\alpha(t)\rangle = |\alpha_0 e^{-i\omega t}\rangle$$

with ω the frequency of the underlying harmonic oscillator.

Then we may proceed, writing

$$\langle E_x\rangle = \langle\alpha|E_x|\alpha\rangle$$
$$\langle E_x\rangle = i\epsilon\,\langle\alpha(t)|a e^{ikz} - a^\dagger e^{-ikz}|\alpha(t)\rangle$$

Recalling the behavior of the annihilation operator on a coherent state, we may act with a on the right state and $a^\dagger$ on the left state, obtaining

$$\langle E_x\rangle = i\epsilon\,\langle\alpha(t)|\alpha(t) e^{ikz} - \alpha^*(t) e^{-ikz}|\alpha(t)\rangle$$
$$\langle E_x\rangle = i\epsilon\left(\alpha_0 e^{i(kz-\omega t)} - \alpha_0^* e^{-i(kz-\omega t)}\right)$$
$$= -2\epsilon\mathrm{Im}\left(\alpha_0 e^{i(kz-\omega t)}\right)$$

If we write $\alpha_0 = Ae^{-i\phi}$, then we obtain

$$\boxed{\langle E_x\rangle = -2\epsilon A\mathrm{Im}\left(e^{i(kz-\omega t-\Phi)}\right) = -2A\epsilon\sin(kz-\omega t-\Phi)} \tag{76}$$

and the field is classical, as we might expect. Meanwhile, we can evaluate

$$\langle E_x^2\rangle = \langle\alpha(t)|E_x^2|\alpha(t)\rangle$$

$$\langle E_x^2\rangle = -\epsilon^2\,\langle\alpha(t)|a^\dagger a^\dagger e^{-2ikz} + aae^{2ikz} - a^\dagger a - aa^\dagger|\alpha(t)\rangle$$

[4]This corresponds to a choice of energy which eliminates the “zero point” energy of the oscillator.

Commuting $[a, a^\dagger]$ to get things into normal order:

$$\langle E_x^2 \rangle = -\epsilon^2 \langle \alpha(t)| a^\dagger a^\dagger e^{-2ikz} + aae^{2ikz} - 2a^\dagger a - 1 |\alpha(t)\rangle$$
$$\langle E_x^2 \rangle = \epsilon^2 - \epsilon^2 \langle \alpha(t)| a^\dagger a^\dagger e^{-2ikz} + aae^{2ikz} - 2a^\dagger a |\alpha(t)\rangle$$

Then applying the definition of $\alpha(t)$, we find

$$\langle E_x^2 \rangle = \epsilon^2 - \epsilon^2 \left(\alpha_0^{*2} e^{-2i(kz-\omega t)} + \alpha_0^2 e^{2i(kz-\omega t)} - 2\alpha_0\alpha_0^* \right)$$

Then, recognizing that

$$\langle E_x \rangle^2 = \left(i\epsilon \left(\alpha_0 e^{i(kz-\omega t)} - \alpha_0^* e^{-i(kz-\omega t)} \right) \right)^2 = -\epsilon^2 \left(\alpha_0^{*2} e^{-2i(kz-\omega t)} + \alpha_0^2 e^{2i(kz-\omega t)} - 2\alpha_0\alpha_0^* \right)$$

we can write

$$\langle E_x^2 \rangle = \epsilon^2 + \langle E_x \rangle^2 \tag{77}$$

whence it is trivial to see that

$$\boxed{\Delta E_x = \sqrt{\langle E_x^2 \rangle - \langle E_x \rangle^2} = \epsilon} \tag{78}$$

(note that the absolute value signs in the Pset are incorrect.)

We note that the uncertainty is the same as that of the vacuum; this is essentially intuitive when we realize that the coherent state is just shifted vacuum, provided we trust that the uncertainty E_x is proportional to the "width" of the coherent state in e.g. the Q-distribution along the appropriate quadrature. With this in mind, it's also clear why it doesn't depend on time and field strength; these are both just determined by how far and in what direction one shifts the coherent state.

4 Problem 4

i)

Given that

$$|\psi_1\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|12\rangle$$

is expressed in terms of number states, it is trivial to apply the definition of the coherent states in terms of number states to write

$$\langle \alpha|\psi_1\rangle = e^{-\frac{|\alpha|^2}{2}} \left(\cos\left(\frac{\theta}{2}\right) + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\frac{(\alpha^*)^{12}}{\sqrt{12!}} \right)$$

and hence

$$\boxed{Q_{\psi_1}(\alpha) = e^{-|\alpha|^2} \left| \cos\left(\frac{\theta}{2}\right) + e^{i\phi}\sin\left(\frac{\theta}{2}\right)\frac{(\alpha^*)^{12}}{\sqrt{12!}} \right|^2} \tag{79}$$

This is analytic, if not incredibly intuitive. In the cases where $\theta = 0,\ \theta = \pi$, of course, we will respectively obtain the number states $|0\rangle,\ |12\rangle$, in which case the parameter ϕ doesn't matter; this is shown in Figures

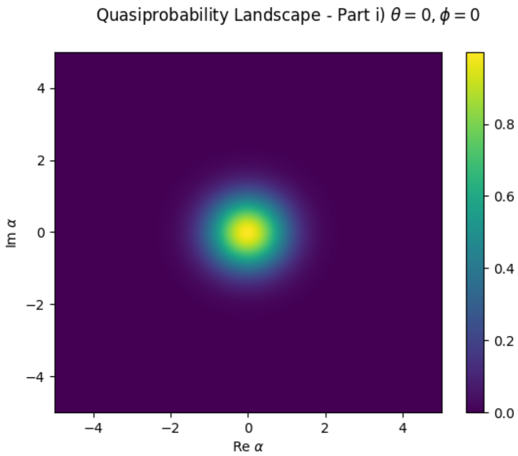

Figure 1: A plot of $Q_{\psi_1}(\alpha)$ for $\theta = 0$

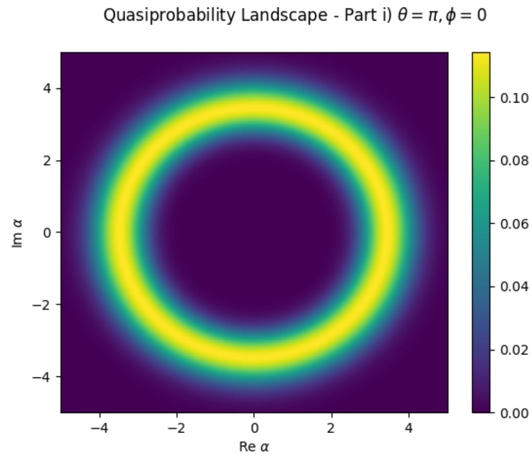

Figure 2: A plot of $Q_{\psi_1}(\alpha)$ for $\theta = \pi$.

1 and 2. In the arguably most interesting intermediate case, $\theta = \frac{\pi}{2}$, our state is an even superposition of $|0\rangle$ and $|12\rangle$ with phase set by ϕ. Here, we write

$$\boxed{Q_\psi(\alpha)|_{\theta=\frac{\pi}{2}} = \frac{1}{2}e^{-|\alpha|^2}\left|1 + e^{i\phi}\frac{(\alpha^*)^{12}}{\sqrt{12!}}\right|^2} \tag{80}$$

We may see from Equation 80 what to expect in this case: there will be two "blobs" in the quasiprobability distribution corresponding to the $|n = 0\rangle$ and $|n = 12\rangle$ states, but at points intermediate between the two, we will see interference. As the phase of α wraps around the plane once, we expect the phase of α^{*12} to wrap around twelve times; this will cause interference with the 1 term, so we should see twelve faint "wiggles" in between the rings, as shown in Figure 3. Figure 4 makes these wiggles more explicit by plotting the difference of the quasiprobability distributions for $\phi = 0$ and $\phi = \pi$.

As to whether this is a minimum uncertainty state, note that the operator $\hat{E}_x$ contains only first powers of a and $a^\dagger$, and as such the operator E_x^2 contains, at most, second powers of these operators. It implies that $\langle m|E_x, E_x^2|n\rangle = 0$ whenever $|m - n| > 2$, as it is in this case where $m = 12, n = 0$. As such, it is trivial to show that

$$\langle\psi_1|E_x|\psi_1\rangle = \cos^2\left(\frac{\theta}{2}\right)\langle 0|E_x|0\rangle + \sin^2\left(\frac{\theta}{2}\right)\langle 12|E_x|12\rangle = 0$$

and exactly analogously

$$\langle\psi_1|E_x^2|\psi_1\rangle = \cos^2\left(\frac{\theta}{2}\right)\langle 0|E_x^2|0\rangle + \sin^2\left(\frac{\theta}{2}\right)\langle 12|E_x^2|12\rangle$$

Now, since $\langle\psi_1|E_x|\psi_1\rangle = \langle 0|E_x|0\rangle = \langle 12|E_x|12\rangle = 0$, we may write

$$\langle\psi_1|E_x^2|\psi_1\rangle - \overbrace{\langle\psi_1|E_x|\psi_1\rangle^2}^{=0} = \cos^2\left(\frac{\theta}{2}\right)\left(\langle 0|E_x^2|0\rangle - \overbrace{\langle 0|E_x|0\rangle^2}^{=0}\right) + \sin^2\left(\frac{\theta}{2}\right)\left(\langle 12|E_x^2|12\rangle - \overbrace{\langle 12|E_x|12\rangle^2}^{=0}\right)$$

and hence

$$\boxed{\Delta E^2_{x,|\psi_1\rangle} = \cos^2\left(\frac{\theta}{2}\right)\Delta E^2_{x,|0\rangle} + \sin^2\left(\frac{\theta}{2}\right)\Delta E^2_{x,|12\rangle}} \tag{81}$$

From Equation 81 and the known fact that, of the number states, only $|n=0\rangle$ is a minimal uncertainty state, we see that $|\psi\rangle$ is a minimal uncertainty state only when $\theta = 0$.

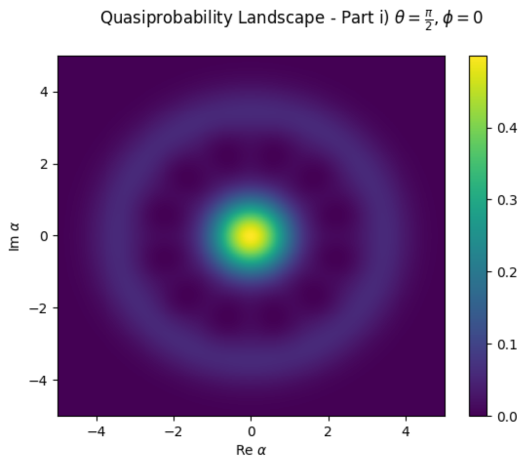

Figure 3: A plot of $Q_{\psi_1}(\alpha)$ for $\theta = \frac{\pi}{2}$, $\phi = 0$.

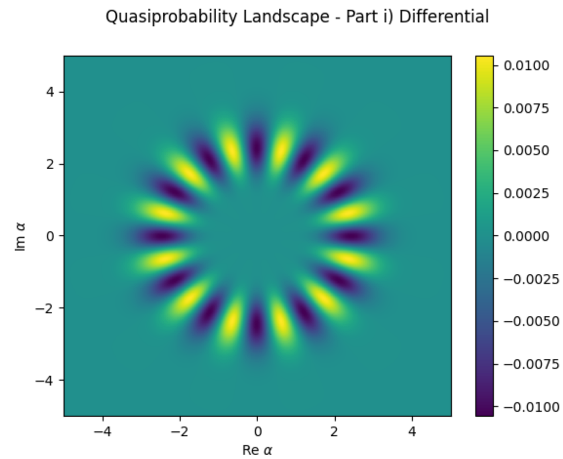

Figure 4: A differential plot plot of $Q_{\psi_1}(\alpha)|_{\theta=\frac{\pi}{2},\phi=\pi} - Q_{\psi_1}(\alpha)|_{\theta=\frac{\pi}{2},\phi=0}$. Note the twelve full oscillations.

ii)

Here, we have

$$|\psi_2\rangle = \frac{1}{\sqrt{2}} \left(|\beta\rangle + |{-\beta}\rangle\right)$$

with $\beta = 3$. The overlap which we computed in Problem 3a, Equation 67 makes it straightforward to write

$$\langle\alpha|\psi_2\rangle = \frac{1}{\sqrt{2}} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \left(e^{\alpha^*\beta} + e^{-\alpha^*\beta}\right)$$

or, recognizing the hyperbolic cosine,

$$\langle\alpha|\psi_2\rangle = \sqrt{2} e^{-\frac{|\alpha|^2}{2}} e^{-\frac{|\beta|^2}{2}} \cosh\alpha^*\beta$$

where it should be understood that the argument of the hyperbolic cosine is potentially complex. Then we may write

$$Q_{\psi_2}(\alpha) = 2e^{-|\alpha|^2} e^{-|\beta|^2} \left|\cosh^2(\alpha^*\beta)\right|^2 \tag{82}$$

At this point, it behooves us to briefly note that $|\psi\rangle$ is in fact not normalized, since $\langle\beta| - \beta\rangle \neq 0$ for any finite β; that said, for $\beta = 3$, it's quite close.

We plot in Figure 5 a quasiprobability distribution in linear scale, and in Figure 6 a distribution in log scale. Note that the latter shows interference fringes along the imaginary axis near the origin, with a spacing quite close to 1.

The analytic form of these fringes is easy to deduce. Given that $\beta = 3$, we know that the quasiprobability distribution is in general

$$Q_{\psi_2}(\alpha) = 2e^{-9} e^{-|\alpha|^2} \left|\cosh^2(3\alpha^*)\right|^2$$

However, given that $\cosh(-iy) = \cos(y)$, we know that if $\alpha = iy$, we have

$$Q_{\psi_2}(iy) = 2e^{-9} e^{-y^2} \cos^2(3y) \tag{83}$$

which yields oscillations with a period quite close to 1 (in fact with period $\frac{\pi}{3}$.)

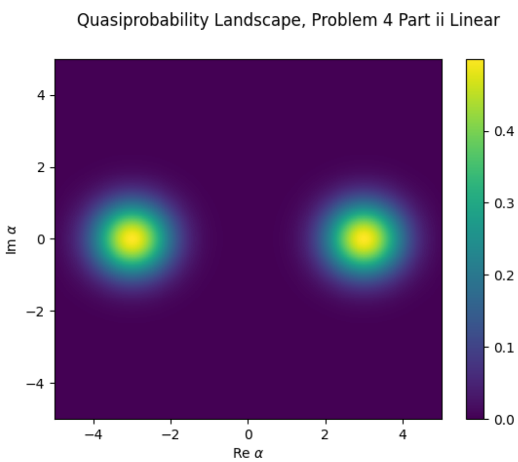

Figure 5: A plot of $Q_{\psi_2}(\alpha)$ in linear scale.

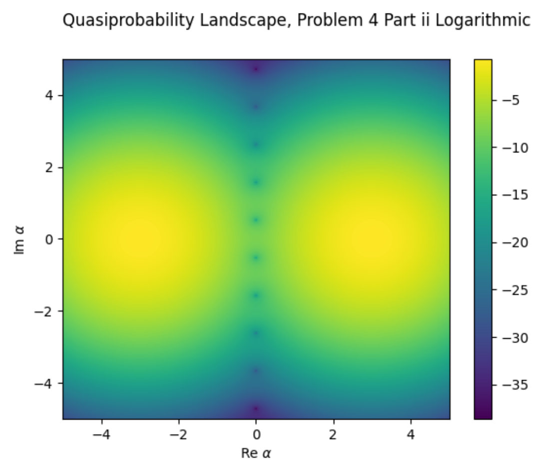

Figure 6: A plot of $Q_{\psi_2}(\alpha)$ in logarithmic scale. Note the oscillations along the line $\alpha = iy$.

iii)

Here, we have

$$|\psi\rangle_3 = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} e^{ik\phi} |k\rangle \tag{84}$$

We can, of course, exploit the definition of the coherent state $|\alpha\rangle$ to analytically write our quasiprobability distribution:

$$\langle \alpha | \psi_3 \rangle = \frac{1}{\sqrt{N}} e^{-\frac{|\alpha|^2}{2}} \sum_{k=1}^{N} \frac{(\alpha^* e^{i\phi})^k}{\sqrt{k!}}$$

whence our quasiprobability is simply

$$\boxed{Q_{|\psi_3\rangle}(\alpha) = \frac{1}{N} e^{-|\alpha|^2} \left| \sum_{k=1}^{N} \frac{(\alpha^* e^{i\phi})^k}{\sqrt{k!}} \right|^2} \tag{85}$$

We may already derive a bit of insight out of Equation 85: suppose that we take $\alpha = re^{i\theta}$. Then our quasiprobability distribution becomes

$$Q_{|\psi_3\rangle}(\alpha) = \frac{1}{N} e^{-|\alpha|^2} \left| \sum_{k=1}^{N} \frac{r^k}{\sqrt{k!}} e^{i(\phi - \theta)k} \right|^2$$

Now, intuition tells us that the sum will be largest when $\theta = \phi$, for here all of the terms will add in phase; it may further seem plausible that, as $N \to \infty$, the quasiprobability will be very small for $\theta \neq \phi$. We then expect our state $|\psi\rangle$ to be one which "points along" a given line corresponding to a specific phase of α. This intuition is supported by Figures 7 and 8, which show plots of this state for $N = 10$ and $N = 100$, respectively.

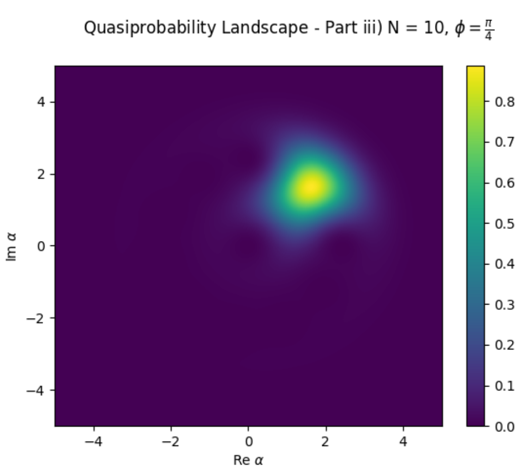

Figure 7: A plot of $Q_{\psi_3}(\alpha)$ with $N = 10$ and $\phi = \frac{\pi}{4}$.

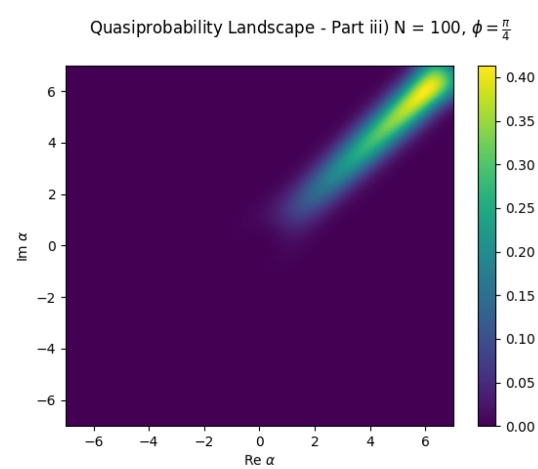

Figure 8: A plot of $Q_{\psi_3}(\alpha)$ with $N = 100$ and $\phi = \frac{\pi}{4}$.

Aside:

We may at this juncture begin to form an intuition that $|\psi_3\rangle$ is a sort of "eigenstate of phase". To briefly explore this, note that for a coherent state $|\alpha\rangle$, one has that the phase of α is given by

$$\mathrm{Arg}(\alpha) = \frac{\alpha}{|\alpha|}$$

where, please note, we are defining "phase" as the factor $e^{i\theta}$, not θ itself. Since $\langle\alpha|n|\alpha\rangle = |\alpha|^2$ for a coherent state, while $\langle\alpha|a|\alpha\rangle = \alpha$, we may be tempted to define the following as a "phase operator":

$$\hat{\Phi} \equiv \hat{a}\frac{1}{\sqrt{n}}$$

where we have chosen to make $\frac{1}{\sqrt{n}}$ act to the right.

Of course, there are some issues. For starters, $\frac{1}{\sqrt{n}}$ is on its own not even an operator, giving an undefined value when one tries to evaluate the matrix element $\langle 0|\frac{1}{\sqrt{n}}|0\rangle$, for instance. One can dodge this issue by just specifying that the sense in which our operator $\hat{\Phi}$ is meant is to have $\hat{\Phi}\,|0\rangle = 0$.

There is a case to be made that $\hat{\Phi}$ is unitary[5] - as it ought to be, since its eigenvectors are supposed to be complex numbers with modulus 1. One may also show that, at least for large coherent states, it returns the value of the phase which we expect.

Anyway, one can see that the state which we are given is very nearly an eigenstate of this operator with eigenvalue $e^{i\phi}$. In particular, if we take a closely related state

$$|\psi_3'\rangle = \frac{1}{\sqrt{N+1}}\sum_{k=0}^{N} e^{ik\phi}\,|k\rangle$$

then

$$\hat{\Phi}\,|\psi_3'\rangle = \frac{1}{\sqrt{N+1}}\sum_{k=0}^{N-1} e^{ik\phi+1}\,|k\rangle \approx e^{i\phi}\,|\psi_3'\rangle$$

There is then a principled case to be made that, in the limit $N \to \infty$, $|\psi_3\rangle$ is sort of an "eigenstate of the phase operator."

[5] Except, as it turns out, when one is dealing with that pesky $|0\rangle$.

iv)

Here, our state is given by

$$|\psi_4\rangle = \frac{1}{\sqrt{\cosh\epsilon}} \sum_{n=0}^{\infty} \frac{\sqrt{2n!}}{2^n n!} (\tanh(\epsilon))^n |2n\rangle \tag{86}$$

Taking the inner product with $|\alpha\rangle$, we find

$$\langle\psi_4|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \frac{1}{\sqrt{\cosh\epsilon}} \sum_{n=0}^{\infty} \frac{\sqrt{2n!}}{2^n n!} (\tanh(\epsilon))^n \left(\frac{\alpha^{2n}}{\sqrt{2n!}}\right)$$

Simplifying,

$$\langle\psi_4|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \frac{1}{\sqrt{\cosh\epsilon}} \sum_{n=0}^{\infty} \frac{\left(\frac{\alpha^2 \tanh\epsilon}{2}\right)^n}{n!}$$

whence we obtain

$$\langle\psi_4|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \frac{1}{\sqrt{\cosh(\epsilon)}} e^{\frac{\alpha^2 \tanh\epsilon}{2}}$$

or, taking the absolute value squared,

$$\boxed{Q_4(\alpha) = e^{-|\alpha|^2} \frac{1}{\cosh(\epsilon)} e^{\mathrm{Re}(\alpha^2)\tanh(\epsilon)}} \tag{87}$$

This immediately gives some insight into the nature of the state. For $\epsilon = 0$, $\cosh(\epsilon) = 1$, $\tanh\epsilon = 0$, and it is simply the vacuum, as we may expect. It is then fruitful to consider the shape of the quasiprobability distribution if alpha is real:

$$Q_4(\alpha) = e^{-|\alpha|^2(1-\tanh(\epsilon))},\ \alpha \text{ real}$$

$$Q_4(\alpha) = e^{-|\alpha|^2(1+\tanh(\epsilon))},\ \alpha \text{ imaginary}$$

We then see that the width of the Q distribution along the real axis is $2\sigma_x^2 = \frac{1}{(1-\tanh(\epsilon))} \to \infty$ as $\epsilon \to \infty$, whereas the width along the y axis is $2\sigma_y^2 = \frac{1}{1+\tanh(\epsilon)} \to \frac{1}{2}$; the width then only modestly decreases along this axis in the infinite limit.

We then plot in Figures 9, 10, and 11 the quasiprobability distributions obtained for $\epsilon = 0.2,\ 1.2,\ 4$, respectively. Observe that the state indeed becomes further drawn out along the $\mathrm{Re}\alpha$ axis and narrower along the $\mathrm{Im}(\alpha)$ axis as ϵ is increased.

Incidentally, when done naively, this quasiprobability requires particularly large n to calculate. A useful trick is to note that, by some applications of Stirling's approximation,

$$\frac{\sqrt{2n!}}{2^n n!} \sim \left(\frac{1}{\pi n}\right)^{\frac{1}{4}}$$

which helps significantly with calculations.

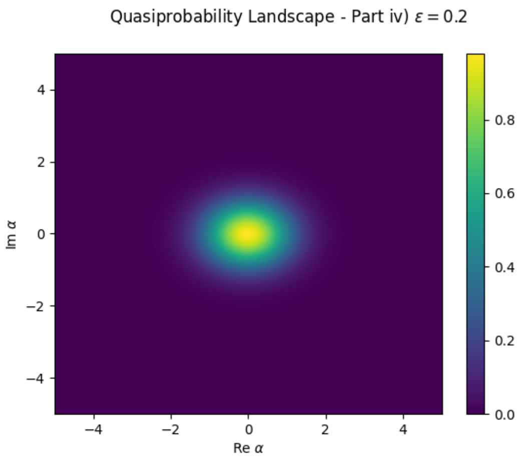

Figure 9: A plot of $Q_{\psi_4}(\alpha)$ for $\epsilon = 0.2$.

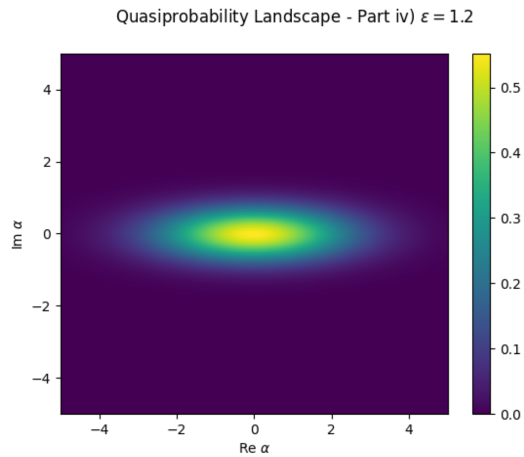

Figure 10: A plot of $Q_{\psi_4}(\alpha)$ for $\epsilon = 1.2$.

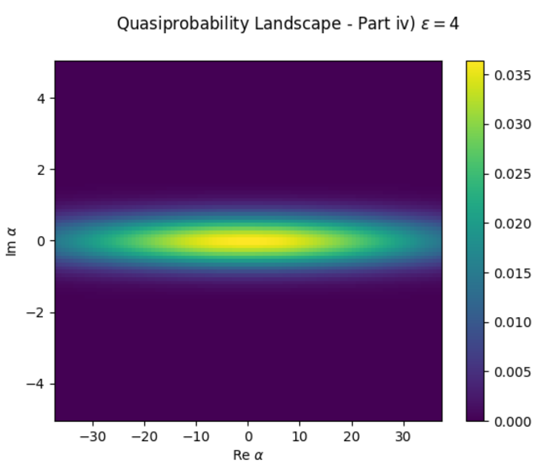

Figure 11: A plot of $Q_{\psi_4}(\alpha)$ for $\epsilon = 4.0$.

v)

It is useful to explicitly write the effect of the Kerr Hamiltonian

$$H_{\text{kerr}} = \xi n(n-1)$$

on a coherent state:

$$\begin{aligned}|\beta(t)\rangle &= e^{-iH_{\text{kerr}}t}\,|\beta\rangle \\ &= e^{-\frac{|\beta|^2}{2}}\sum_{n=0}^{\infty} e^{-i\xi t(n)(n-1)}\frac{\beta^n}{\sqrt{n!}}\,|n\rangle\end{aligned}$$

Before proceeding to plots, we may usefully obtain some analytic intuition. First, suppose that $\xi t = \pi$. In this case, one has that

$$|\beta(\xi t = \pi)\rangle = e^{-\frac{|\beta|^2}{2}}\sum_{n=0}^{\infty} e^{-i\pi n(n-1)}\frac{\beta^n}{\sqrt{n!}}\,|n\rangle$$

Observe, however, that

$$\forall n \in \mathbb{Z},\ e^{-i\pi n(n-1)} = 1$$

since $n(n-1)$ is even for any integer. It follows that

$$\boxed{|\beta(\xi t = \pi)\rangle = |\beta\rangle} \tag{88}$$

and, for the given parameters, the initial coherent state is restored at $t = 128$.

We may also apply analytic intuition further; consider that

$$|\beta(\xi t = \frac{\pi}{2})\rangle = e^{-\frac{|\beta|^2}{2}}\sum_{n=0}^{\infty} e^{-i\frac{\pi}{2}n(n-1)}\frac{\beta^n}{\sqrt{n!}}\,|n\rangle = \sum_{n=0}^{\infty}(-i)^{n(n-1)}\frac{\beta^n}{\sqrt{n!}}\,|n\rangle$$

It is perhaps not obvious, but nonetheless easy to prove, that

$$\forall n \in \mathbb{Z},\ (-i)^{n(n-1)} = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}i^n + e^{i\frac{\pi}{4}}(-i)^n\right)$$

With this in hand, it may easily be shown that

$$|\beta(\xi t = \frac{\pi}{2})\rangle = \sum_{n=0}^{\infty}(-i)^{n(n-1)}\frac{\beta^n}{\sqrt{n!}}\,|n\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}\sum_{n=0}^{\infty}\frac{(i\beta)^n}{\sqrt{n!}}\,|n\rangle + e^{i\frac{\pi}{4}}\sum_{n=0}^{\infty}\frac{(-i\beta)^n}{\sqrt{n!}}\,|n\rangle\right)$$

or succinctly

$$\boxed{|\beta(\xi t = \frac{\pi}{2})\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\frac{\pi}{4}}\,|i\beta\rangle + e^{i\frac{\pi}{4}}\,|-i\beta\rangle\right)} \tag{89}$$

such that there are, for the given parameters, two coherent states at time $t = 64$.

We now present in Figures 12, 13, 14, 15, 16, 17, and 18 plots of the quasiprobability distribution (using the minus sign on the Hamiltonian for time evolution, in contrast to the problem statement) for various times ξt. Observe that, at least for small n, the quasiprobability distribution is apparently approximately n-fold symmetric for $\xi t = \frac{\pi}{n}$.

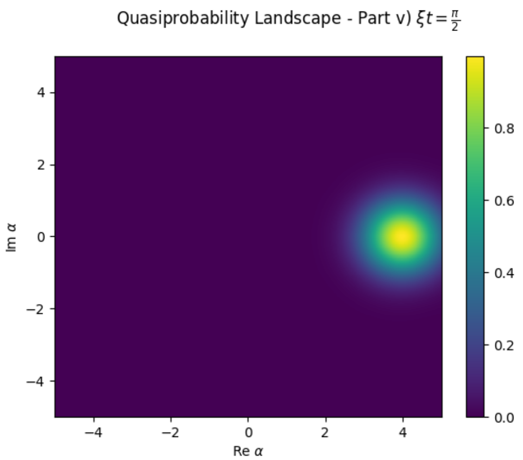

Figure 12: A plot of $Q_{\psi_5}(\alpha)$ for $\xi t = \pi$.

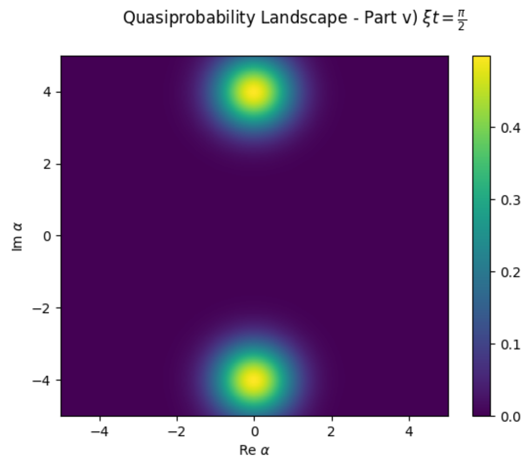

Figure 13: A plot of $Q_{\psi_5}(\alpha)$ for $\xi t = \frac{\pi}{2}$.

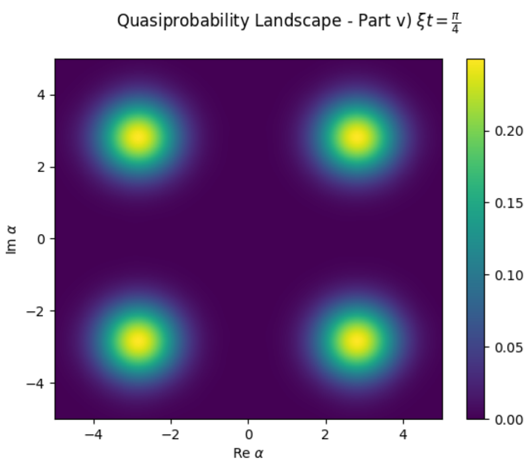

Figure 14: A plot of $Q_{\psi_5}(\alpha)$ for $\xi t = \frac{\pi}{4}$.

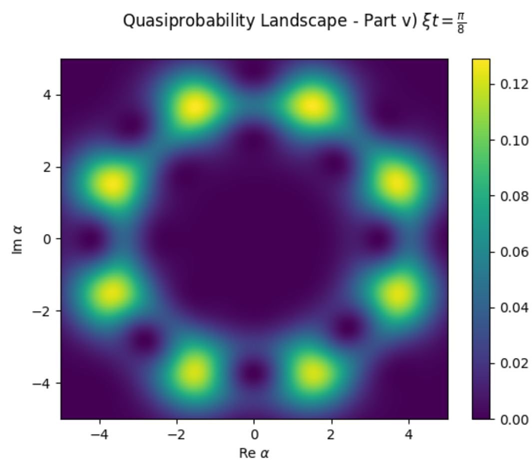

Figure 15: A plot of $Q_{\psi_5}(\alpha)$ for $\xi t = \frac{\pi}{8}$.

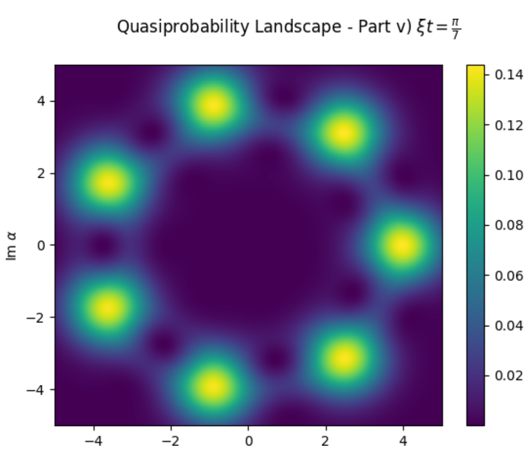

Figure 16: A plot of $Q_{\psi_5}(\alpha)$ for $\xi t = \frac{\pi}{7}$.

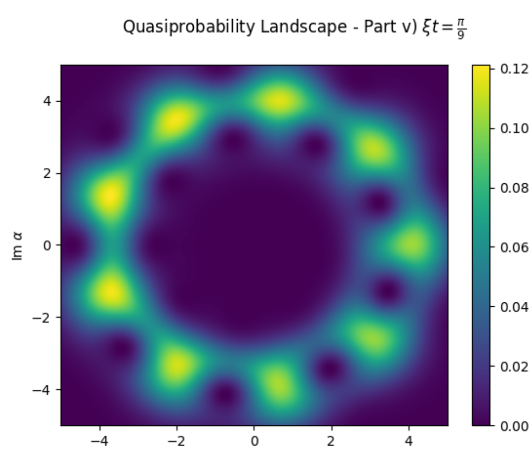

Figure 17: A plot of $Q_{\psi_5}(\alpha)$ for $\xi t = \frac{\pi}{9}$.

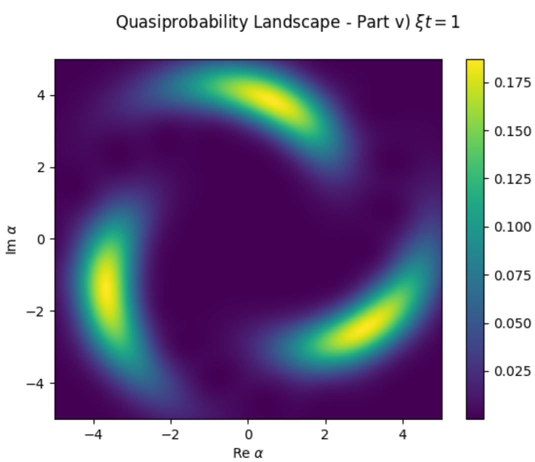

Figure 18: A plot of $Q_{\psi_5}(\alpha)$ for $\xi t = 1$.