More path integrals

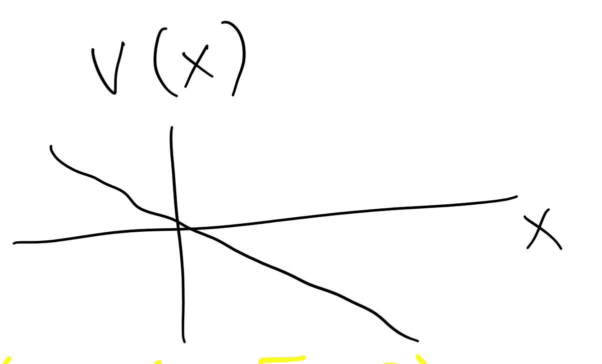
Tuesday, November 16, 2021 10:32 AM

Let's study some physical systems with path integrals!

Constant Force

Consider the potential

$$V(x) = -Fx, \quad \leftarrow \text{constant}$$



(assuming $F>0$) const Force to right →

$$\mathcal{L} = \frac{p^2}{2m} - V(x) = \frac{1}{2}m^2\dot{x}^2 + Fx$$

⇒ Propagator

$$K(x_f,t_f;x_i,t_i) = \int_{x(t_i)=x_i}^{x(t_f)=x_f} [Dx(t)] \exp\left[\frac{i}{\hbar}\int_{t_i}^{t_f} dt \left[\frac{1}{2}m\dot{x}^2 + Fx\right]\right]$$

Change variables: classical + quantum

$$x(t) = \underbrace{x_c(t)}_{\text{obeys EOM w/ BC}} + \underbrace{y(t)}_{\text{Qtm corr}}$$

Qtm corr ⇒ $y(t_i)=0$, $y(t_f)=0$

↖ EOM: $m\ddot{x}_c - F = 0$

$$x_c(t) = \frac{1}{2}\frac{F}{m}(t-t_i)^2 + v_0(t-t_i) + x_i$$

BC at t_f → $v_0 = \frac{1}{t_f - t_i}\left\{x_f - x_i - \frac{1}{2}\frac{F}{m}(t_f-t_i)^2\right]$

$$S[x_c+y] = \int_{t_i}^{t_f}\left[\frac{1}{2}m(\dot{x}_c^2 + 2\dot{x}_c\dot{y} + \dot{y}^2) + F(x_c+y)\right]$$

$$= \int_{t_i}^{t_f}\left(\left[\frac{1}{2}m\dot{x}_c^2 + Fx_c\right] + \frac{1}{2}m\dot{y}^2 + m\left(\underbrace{\frac{d(\dot{x}_c y)}{dt}}_{\text{0 by BC's on } y.} - \ddot{x}_c \cdot y\right) + Fy\right)$$

EOM for $x_c \cdot y = 0$

$$= S[x_c] + S_{free}[y]$$

$D[x_c(t) + y(t)] = Dy(t)$, $x_c(t)$ is like a constant shift!

$$\Rightarrow K(x_f,t_f;x_i,t_i) = e^{\frac{i}{\hbar}S[x_{cl}]} \cdot \int_{y(t_i)=0}^{y(t_f)=0} [Dy(t)]\, e^{\frac{i}{\hbar}S_{free}[y]}$$

$$= e^{\frac{i}{\hbar}S[x_c]} \cdot K_{free}[0,t_f;0,t_i]$$

Recalling

$$K_{free}(x_f,t_f;x_i,t_i) = \sqrt{\frac{m}{2\pi i\hbar(t_f-t_i)}}\, e^{\frac{im}{2\hbar(t_f-t_i)}(x_f-x_i)^2}$$

$$\Rightarrow K = \sqrt{\frac{m}{2\pi i\hbar(t_f-t_i)}}\, e^{iS[x_c]/\hbar}$$

Can show:

$$S[x_c] = -\frac{F^2}{24m}(t_f-t_i)^3 + \frac{1}{2}m\frac{(x_f-x_i)^2}{(t_f-t_i)^2} + \frac{1}{2}F(t_f-t_i)(x_f+x_i)$$

SHO

Again, $x(t) = x_c(t) + y(t)$

But now $x_c(t)$ solves SHO EOMs. $\left[\ddot{x}_c = -\omega^2 x_c\right]$

$$S[x_c+y] = \int_{t_i}^{t_f}\left[\frac{1}{2}m(\dot{x}_c^2 + 2\dot{x}_c\dot{y} + \dot{y}^2) - \frac{1}{2}m\omega^2(x^2 + 2xy + y^2)\right]$$

$$= S[x_c] + S[y] + \int_{t_i}^{t_f}\left[m\underbrace{\dot{x}_c\dot{y}}_{\text{IBP}} - m\omega^2 x_c y\right\}$$

IBP ↳ $-m\ddot{x}_c y$ → SHO EOM for x_c times y → 0

So again $[Dx] = [Dy]$

$$\Rightarrow K = e^{\frac{i}{\hbar}S[x_{cl}]} \cdot \int_{y(t_i)=0}^{y(t_f)=0} [Dy]\, e^{\frac{i}{\hbar}S[y]}$$

Leave $S[x_c]$ for pset 7

Trick to compute PI over y

$$y(t) = \sum_{n=1}^{\infty} y_n \sin\left(\frac{n\pi t}{t_f}\right) \qquad [\text{Set } t_i = 0]$$

↑ $\mathbb{R}$ number

some normalization

$$y(t) \to \{y_n\}, \quad [Dy] = N\prod_{n=1}^{\infty} dy_n$$

$\omega_n = \frac{\pi n}{t_f}$

$$S[y] = \int_0^{t_f} dt \sum_{n,m}\left[\frac{1}{2}m\omega_n\omega_m y_n y_m \cos\omega_n t \cdot \cos\omega_m t - \frac{1}{2}m\omega^2 y_n y_m \sin\omega_n t \cdot \sin\omega_m t\right]$$

$\int \to \frac{1}{2}\delta_{mn}t_f$ (for each term)

$$= \sum_{n=1}^{\infty}\frac{t_f m}{4}\left[\left(\frac{\pi n}{t_f}\right)^2 - \omega^2\right] y_n^2$$

$$\Rightarrow K = Ne^{\frac{i}{\hbar}S[x_c]}\prod_{n=1}^{\infty}\int_{-\infty}^{\infty} dy_n \exp\left[\frac{i}{\hbar}\frac{t_f m}{4}\left[\left(\frac{\pi n}{t_f}\right)^2 - \omega^2\right] y_n^2\right]$$

$$= Ne^{\frac{i}{\hbar}S[x_c]}\prod_{n=1}^{\infty}\left[\frac{4\pi i\hbar}{t_f m\left(\left(\frac{\pi n}{t_f}\right)^2 - \omega^2\right)}\right]^{1/2}$$

new norm

$$= N' e^{\frac{i}{\hbar}S[x_c]}\underbrace{\prod_{n=1}^{\infty}\left[1 - \left(\frac{\omega t_f}{\pi n}\right)^2\right]^{-1/2}}_{= \sqrt{\frac{\omega t_f}{\sin(\omega t_f)}}}$$

$$\Rightarrow K_{SHO} = N' e^{\frac{i}{\hbar}S[x_c]}\sqrt{\frac{\omega t_f}{\sin(\omega t_f)}}$$

↳ recover K_{free} as $\omega \to 0$