Extra exercise to practice with tensor product spaces

Aaron Szasz

May 28, 2020

1 Supplemental exercise

In this supplemental exercise (just for you to practice if you'd like!), we will demonstrate that the total spin of a combined system of two spin-1/2 particles can be either 0 or 1, or equivalently that the two-spin system is spanned by a basis consisting of: 1 spin singlet (1 spin 0 state) and 1 spin triplet (3 spin 1 states).

Part 1:

Recall that a spin eigenstate can be denoted as $|s, m\rangle$, which is a simultaneous eigenstate of the S_z and S^2 operators, with eigenvalues $\hbar m$ and $\hbar^2 s(s+1)$, respectively. Letting

$$\begin{aligned}
S_x &= \frac{\hbar}{2}\sigma_x \\
S_y &= \frac{\hbar}{2}\sigma_y \\
S_z &= \frac{\hbar}{2}\sigma_z \\
\mathbf{S} &= (S_x, S_y, S_z) \\
S^2 &= \mathbf{S}\cdot\mathbf{S} = S_x^2 + S_y^2 + S_z^2
\end{aligned}$$

confirm that the spin eigenstates are $|1/2, 1/2\rangle = |\uparrow\rangle$ and $|1/2, -1/2\rangle = |\downarrow\rangle$.

Part 2:

We define the total spin operators for the two-spin system as follows:

$$\begin{aligned} S^x_{\text{tot}} &= S^x_0 \otimes \text{Id}_1 + \text{Id}_0 \otimes S^x_1 \\ S^y_{\text{tot}} &= S^y_0 \otimes \text{Id}_1 + \text{Id}_0 \otimes S^y_1 \\ S^z_{\text{tot}} &= S^z_0 \otimes \text{Id}_1 + \text{Id}_0 \otimes S^z_1 \\ S^2_{\text{tot}} &= (S^x_{\text{tot}})^2 + (S^y_{\text{tot}})^2 + (S^z_{\text{tot}})^2 \end{aligned}$$

Find 4x4 matrices for these four operators

Part 3:

Show that the following four states are simultaneous eigenstates of S^z_{tot} and S^2_{tot}, with the specified eigenvalues:

$$\begin{array}{ccc} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2} & 0 & 0 \\ & & \\ |\uparrow\uparrow\rangle & \hbar & 2\hbar^2 \\ (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} & 0 & 2\hbar^2 \\ |\uparrow\uparrow\rangle & \text{-}\hbar & 2\hbar^2 \end{array}$$

These are the states $|s, m\rangle = |0, 0\rangle$, $|1, 1\rangle$, $|1, 0\rangle$, and $|1, -1\rangle$ respectively.

2 Partial solution

$$S^z_{\text{tot}} = \hbar \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$S^2_{\text{tot}} = \hbar \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

There is a nice way to get S^2_{tot} without using the other three matrices explicitly. As follows:

$$\begin{aligned}
(S^z_{\text{tot}})^2 &= \left(\frac{\hbar}{2}\right)^2 (\sigma_z \otimes \mathrm{Id} + \mathrm{Id} \otimes \sigma_z)^2 \\
&= \left(\frac{\hbar}{2}\right)^2 \left(\sigma_z^2 \otimes \mathrm{Id}^2 + \mathrm{Id}^2 \otimes \mathrm{Id}^2 + 2\sigma_z \otimes \sigma_z\right) \\
&= \frac{\hbar^2}{2} (\mathrm{Id} \otimes \mathrm{Id} + \sigma_z \otimes \sigma_z) \\
S^2_{\text{tot}} &= \frac{\hbar^2}{2} (3\mathrm{Id} \otimes \mathrm{Id} + \sigma_z \otimes \sigma_z + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y)
\end{aligned}$$

Personally I like this method because squaring 4x4 matrices is pretty annoying to do by hand, whereas computing Kronecker products for eg $\sigma_x \otimes \sigma_x$ is much faster.