Noether's Theorem

For every continuous symmetry there exists a conserved current j^μ $(\mu = 0, 1, 2, 3)$ which implies a local conservation law.

A conserved current is an object j^μ such that

$$\partial_\mu j^\mu = 0$$

Note that $\partial_\mu j^\mu = \partial_0 j^0 + \partial_1 j^1 + \partial_2 j^2 + \partial_3 j^3$ and so

$$\partial_\mu j^\mu = 0 \Rightarrow \partial_0 j^0 = -\vec{\nabla} \cdot \vec{j}$$

spatial components only

The time component j^0 is the charge density. We use the term "charge" in the general sense, and is not necessarily "electric charge."

The spatial components $\vec{j}$ comprise the spatial current density.

We can define the total charge Q in some volume

$$Q = \int j^0 \, d^3x$$

Then if j^μ is a conserved current,

$$\frac{dQ}{dt} = \int \frac{dj^0}{dt} \, d^3x = -\int \vec{\nabla}\cdot\vec{j} \, d^3x$$

$$= -\underbrace{\oint \vec{j}\cdot d^2s}_{\text{flux through boundary}}$$

If Q is the total charge over all space then $\frac{dQ}{dt} = 0$.

Let us derive Noether's theorem ...

Suppose we have some symmetry group that leaves $\mathcal{L}$ the same. We consider an infinitesmal change associated with this symmetry.

$$\phi(x) \to \phi'(x) = \phi(x) + \underset{\substack{\uparrow \\ \text{infinitesmal parameter}}}{\alpha} \cdot \Delta\phi(x)$$

If $\mathcal{L}$ is invariant under this change then since

$$\mathcal{L} \to \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \cdot (\alpha \Delta \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi),$$

this implies that

$$\frac{\partial \mathcal{L}}{\partial \phi} \cdot (\alpha \Delta \phi) = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \partial_\mu (\alpha \Delta \phi).$$

As it stands this is not by itself very useful. But suppose we now use the Euler-Lagrange equations to replace $\frac{\partial \mathcal{L}}{\partial \phi}$ by $\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right)$. Then we have

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \cdot \alpha \Delta \phi = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi)$$

or

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \cdot \alpha \Delta \phi \right] = 0$$

This is interesting. $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi$ is a conserved current (i.e., $\partial_\mu j^\mu = 0$).

Example

Massless Klein-Gordon field

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$$

We note that $\mathcal{L}$ has a symmetry

$$\phi(x) \to \phi(x) + \alpha$$
$$(\Delta\phi(x) = 1)$$

Under this symmetry, $\partial_\mu \phi$ remains the same and hence $\mathcal{L}$ is invariant also. Using our result from before,

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$$

is a conserved current. So

$$\partial_\mu j^\mu = \partial_\mu \partial^\mu \phi = 0 .$$

We already knew this ...

$$(\partial_\mu \partial^\mu + \cancel{m^2})\phi = 0$$

$m=0$ massless Klein-Gordon equation

Example

Complex Klein–Gordon field

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$$

In this case ϕ is a complex field. Note that if we write

$$\phi = (\phi_1 + i\phi_2)/\sqrt{2}$$

then $\phi^* \phi = \frac{1}{2}\phi_1^2 + \frac{1}{2}\phi_2^2$

$$(\partial_\mu \phi^*)(\partial^\mu \phi) = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2)$$

This is just two Klein-Gordon fields with the same mass m. The Lagrange density has a symmetry

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \rightarrow \begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

At the infinitesmal level $(|\alpha| << 1)$

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \approx \begin{bmatrix} 1 & -\alpha \\ \alpha & 1 \end{bmatrix}$$

So $\begin{bmatrix} \phi_1' \\ \phi_2' \end{bmatrix} = \begin{bmatrix} \phi_1 - \alpha\phi_2 \\ \phi_2 + \alpha\phi_1 \end{bmatrix}$ and therefore

$$\Delta\phi_1 = -\phi_2$$
$$\Delta\phi_2 = +\phi_1$$

Then our conserved current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_1)}\Delta\phi_1 + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_2)}\Delta\phi_2$$

we get a term for each field

$$= (\partial^\mu\phi_1)\Delta\phi_1 + (\partial^\mu\phi_2)\Delta\phi_2$$

$$= -\phi_2\partial^\mu\phi_1 + \phi_1\partial^\mu\phi_2$$

We can indeed check that $\partial_\mu j^\mu = 0 \ldots$

$$\partial_\mu j^\mu = -\cancel{\partial_\mu\phi_2\partial^\mu\phi_1} - \phi_2\boxed{\partial_\mu\partial^\mu\phi_1} = -m^2\phi_1$$

$$+ \cancel{\partial_\mu\phi_1\partial^\mu\phi_2} + \phi_1\boxed{\partial_\mu\partial^\mu\phi_2} = -m^2\phi_2$$

$$= 0$$

Writing in terms of the original complex field ϕ, we have

$$j^\mu = -\phi_2 \partial^\mu \phi_1 + \phi_1 \partial^\mu \phi_2$$
$$= i\phi \partial^\mu \phi^* - i\phi^* \partial^\mu \phi$$

(can check that the imaginary part drops out)

There is a quick (and sloppy) way to get this result. The shortcut is to think of ϕ and ϕ^* as though they were independent fields. This works because $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$ and $\phi^* = (\phi_1 - i\phi_2)/\sqrt{2}$, and we can think of varying ϕ and ϕ^* separately.

Again, $\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi$

The symmetry is

$$\phi \to \phi' = e^{i\alpha} \phi$$
$$[\text{so } \phi^* \to e^{-i\alpha} \phi^*]$$

For infinitesmal α we have

$$\phi \to \phi + i\alpha\phi \qquad (\Delta\phi = i\phi)$$
$$\phi^* \to \phi^* - i\alpha\phi^* \qquad (\Delta\phi^* = -i\phi^*)$$

So the conserved current is

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \Delta\phi^*$$
$$= (\partial^\mu \phi^*)(i\phi) + (\partial^\mu \phi)(-i\phi^*)$$
$$= i\phi\,\partial^\mu\phi^* - i\phi^*\partial^\mu\phi$$

as predicted

We generalize Noether's theorem a bit further. It is not necessary that the Lagrange density $\mathcal{L}(x)$ remains invariant so long as the action

$$S = \int \mathcal{L}(x)\, d^4x$$

is the same under the symmetry transformation. So now suppose that under the transformation

$$\phi \to \phi + \alpha \Delta \phi$$

we have $\mathcal{L} \to \mathcal{L} + \alpha \underbrace{\partial_\mu I^\mu}_{\Delta \mathcal{L}}$ for some function I^μ which vanishes at infinity so that

$$\int \Delta \mathcal{L} \, d^4x = \int \partial_\mu I^\mu \, d^4x = 0 .$$

Then the same derivation we did before gives

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi \right] = \underbrace{\partial_\mu I^\mu}_{\Delta \mathcal{L}}$$

In this case we have a conserved current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta \phi - I^\mu$$

Clearly $\partial_\mu j^\mu = 0$.

<u>Example</u>

Consider a spacetime translation

$$\phi(x) \to \phi'(x) = \phi(x+a)$$

where a^μ is a constant four-vector. We will do all possible translations at once.

$$\underset{\substack{\uparrow\\ \text{time}\\ \text{translation}}}{\mu = 0}, \underbrace{1, 2, 3}_{\substack{\text{three spatial}\\ \text{translations}}}$$

To avoid confusion we use μ as the only index to denote the direction of translation (there will be other indices).

For infinitesmal a^μ we have

$$\phi(x) \to \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x)$$

$$\mathcal{L}(x) \to \mathcal{L}(x+a) = \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x)$$

So for translations in the μ direction,

$$\mathcal{L} \to \mathcal{L} + \alpha\, \partial_\nu I^\nu \quad \text{where}$$

$$I^\nu = \delta^\nu_\mu \mathcal{L}$$

$$\left[\begin{array}{c} \delta^0_0 = \delta^1_1 = \delta^2_2 = \delta^3_3 = 1 \\ \text{all others zero} \end{array} \right]$$

For time translations, for example,

$$I^\nu = (\mathcal{L}, 0, 0, 0)$$

For spatial translations in the x-direction

$$I^{\nu} = (0, \mathcal{L}, 0, 0)$$

The conserved current is

$$j^{\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\phi)}\partial_{\mu}\phi - \delta^{\nu}_{\mu}\mathcal{L}$$

If we make the μ index explicit, the conserved current associated μ-translations is called the energy-momentum tensor

$$T^{\nu}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\phi)}\partial_{\mu}\phi - \delta^{\nu}_{\mu}\mathcal{L}$$

If we raise the μ index,

$$T^{\nu\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\nu}\phi)}\partial^{\mu}\phi - g^{\nu\mu}\mathcal{L}$$

[$g^{00} = 1$, $g^{11} = g^{22} = g^{33} = -1$, all others zero]

Some of this should already be familiar.

The charge density associated with time translations is called the Hamiltonian density T^{00}.

The integral over space is the Hamiltonian

$$H = \int T^{00} \, d^3x$$

The charge density associated with space translations is the "physical" momentum density, T^{0i}. The integral of T^{0i} over all space is the "physical" momentum

$$P^i = \int T^{0i} \, d^3x$$

I call it "physical" momentum since we will overuse the word momentum in this course (e.g., conjugate momenta). P^i is really the physical momentum you know from high school physics.

For the Klein-Gordon field let's calculate the

Hamiltonian.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$$

$$T^{00} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - g^{00} \mathcal{L}$$

$$= (\partial^0 \phi)(\partial^0 \phi) - \left[\frac{1}{2}(\partial_0 \phi)(\partial^0 \phi) + \frac{1}{2}(\partial_i \phi)(\partial^i \phi) - \frac{1}{2} m^2 \phi^2 \right]$$

$$= \frac{1}{2}(\partial_0 \phi)(\partial^0 \phi) - \frac{1}{2}(\partial_i \phi)(\partial^i \phi) + \frac{1}{2} m^2 \phi^2$$

So $$H = \int T^{00} \, d^3x$$

$$= \int \left[\frac{1}{2}\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) + \frac{1}{2} m^2 \phi^2 \right] d^3x$$

note $\vec{\nabla} = (\partial_1, \partial_2, \partial_3) = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$

lower index