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Course: **8.422 - AMO II**
Problem set: **#2**
Due: Friday, Feb 24, 2022.

1. When the mechanical momentum is not the canonical momentum
In this problem we will see that the motion of neutral atoms in a rotating frame can be described as the motion of a charged particle experiencing a scalar potential and an effective magnetic field. Consider free motion in the xy-plane. The transformation from the lab from to a frame rotating at angular frequency Ω about the z-axis is

$$\begin{pmatrix}\tilde{x}(t)\\ \tilde{y}(t)\end{pmatrix} = \begin{pmatrix}\cos\Omega t & \sin\Omega t\\ -\sin\Omega t & \cos\Omega t\end{pmatrix}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} \implies \begin{pmatrix}x(t)\\ y(t)\end{pmatrix} = \begin{pmatrix}\cos\Omega t & -\sin\Omega t\\ \sin\Omega t & \cos\Omega t\end{pmatrix}\begin{pmatrix}\tilde{x}(t)\\ \tilde{y}(t)\end{pmatrix}$$
$$\implies \begin{pmatrix}\dot{x}(t)\\ \dot{y}(t)\end{pmatrix} = \begin{pmatrix}-\Omega\sin\Omega t & -\Omega\cos\Omega t\\ \Omega\cos\Omega t & \Omega\sin\Omega t\end{pmatrix}\begin{pmatrix}\tilde{x}(t)\\ \tilde{y}(t)\end{pmatrix} + \begin{pmatrix}\cos\Omega t & -\sin\Omega t\\ \sin\Omega t & \cos\Omega t\end{pmatrix}\begin{pmatrix}\dot{\tilde{x}}(t)\\ \dot{\tilde{y}}(t)\end{pmatrix}$$

a) The kinetic energy of a particle of mass m in terms of the coordinates and velocities in the rotating frame is

$$\begin{aligned}T &= \frac{1}{2}m(\dot{x}^2+\dot{y}^2)\\ &= \frac{1}{2}m\left[\Omega^2(\tilde{x}^2+\tilde{y}^2) + 2\Omega(\tilde{x}\dot{\tilde{y}} - \dot{\tilde{x}}\tilde{y}) + (\dot{\tilde{x}}^2+\dot{\tilde{y}}^2)\right]\\ &= \frac{1}{2}m\left[(\dot{\tilde{x}}-\Omega\tilde{y})^2 + (\dot{\tilde{y}}+\Omega\tilde{x})^2\right].\end{aligned}$$

b) The Lagrangian is just the kinetic energy from above:

$$\mathcal{L}(\tilde{x},\tilde{y},\dot{\tilde{x}},\dot{\tilde{y}},t) = \frac{1}{2}m\left[(\dot{\tilde{x}}-\Omega\tilde{y})^2 + (\dot{\tilde{y}}+\Omega\tilde{x})^2\right].$$

The canonical momenta are therefore

$$\begin{aligned}\tilde{p}_x &= \frac{\partial\mathcal{L}}{\partial\dot{\tilde{x}}} = m(\dot{\tilde{x}}-\Omega\tilde{y})\\ \tilde{p}_y &= \frac{\partial\mathcal{L}}{\partial\dot{\tilde{y}}} = m(\dot{\tilde{y}}+\Omega\tilde{x}).\end{aligned}$$

c) By inspection, $\{\tilde{x},\tilde{p}_x\} = 1$ and $\{\tilde{p}_i,\tilde{p}_j\} = \delta_{ij}$. Now we look at

$$\{m\dot{\tilde{x}}, m\dot{\tilde{y}}\} = m\left(\frac{\partial\dot{\tilde{x}}}{\partial\tilde{x}}\frac{\partial\dot{\tilde{y}}}{\partial\tilde{p}_x} - \frac{\partial\dot{\tilde{y}}}{\partial\tilde{x}}\frac{\partial\dot{\tilde{x}}}{\partial\tilde{p}_x}\right) + m\left(\frac{\partial\dot{\tilde{x}}}{\partial\tilde{y}}\frac{\partial\dot{\tilde{y}}}{\partial\tilde{p}_y} - \frac{\partial\dot{\tilde{y}}}{\partial\tilde{y}}\frac{\partial\dot{\tilde{x}}}{\partial\tilde{p}_y}\right).$$

From $m\dot{\tilde{x}} = \tilde{p}_x + m\Omega\tilde{y}$ and $m\dot{\tilde{y}} = \tilde{p}_y - m\Omega\tilde{x}$ we find

$$\begin{aligned}\{m\dot{\tilde{x}}, m\dot{\tilde{y}}\} &= m\left(\frac{\partial\dot{\tilde{x}}}{\partial\tilde{x}}\cancel{\frac{\partial\dot{\tilde{y}}}{\partial\tilde{p}_x}} - \frac{\partial\dot{\tilde{y}}}{\partial\tilde{x}}\frac{\partial\dot{\tilde{x}}}{\partial\tilde{p}_x}\right) + m\left(\frac{\partial\dot{\tilde{x}}}{\partial\tilde{y}}\frac{\partial\dot{\tilde{y}}}{\partial\tilde{p}_y} - \frac{\partial\dot{\tilde{y}}}{\partial\tilde{y}}\cancel{\frac{\partial\dot{\tilde{x}}}{\partial\tilde{p}_y}}\right)\\ &= m\left(-\frac{-\Omega}{m}\right) + m\left(\frac{\Omega}{m}\right)\\ &= \boxed{2\Omega \neq 0 \text{ if } \Omega \neq 0}\end{aligned}$$

d) The Hamiltonian is the Legendre transform of the Lagrangian:

$$\mathcal{H} = \left(\dot{\tilde{x}}\tilde{p}_x + \dot{\tilde{y}}\tilde{p}_y\right) - \mathcal{L} = \frac{\tilde{p}_x^2}{2m} + \frac{\tilde{p}_y^2}{2m} - \Omega(\tilde{x}\tilde{p}_y - \tilde{y}\tilde{p}_x)$$

where we have written $\dot{\tilde{x}}$ and $\dot{\tilde{y}}$ in terms of $\tilde{p}_x, \tilde{p}_y, \tilde{x}, \tilde{y}$. We shall complete the squares to get

$$\begin{aligned}
\mathcal{H} &= \frac{\tilde{p}_x^2 + 2m\Omega\tilde{p}_x\tilde{y} + m^2\Omega^2\tilde{y}^2}{2m} + \frac{\tilde{p}_y^2 - 2m\Omega\tilde{p}_y\tilde{x} + m^2\Omega^2\tilde{x}^2}{2m} - \frac{1}{2}m\Omega^2(\tilde{x}^2 + \tilde{y}^2) \\
&= \frac{(\tilde{p}_x + m\Omega\tilde{y})^2 + (\tilde{p}_y - m\Omega\tilde{x})^2}{2m} - \frac{1}{2}m\Omega^2(\tilde{x}^2 + \tilde{y}^2) \\
&= \frac{(\vec{\tilde{p}} - q\vec{A})^2}{2m} - \frac{1}{2}m\Omega^2(\tilde{x}^2 + \tilde{y}^2) \\
&= \frac{(\vec{\tilde{p}} - q\vec{A})^2}{2m} + V_{\text{eff}}(\tilde{x}, \tilde{y}).
\end{aligned}$$

Here, we have re-written the Hamiltonian in terms of the vector potential $\vec{A}$ where $q\vec{A} = m\vec{\Omega} \times \vec{\tilde{r}} = (-m\Omega\tilde{y}, m\Omega\tilde{x}, 0)$ and an effective scalar potential $V_{\text{eff}}(\tilde{x}, \tilde{y}) = -m\Omega^2(\tilde{x}^2 + \tilde{y}^2)/2$, which we may refer to as the anti-trapping or centrifugal potential. In terms of electromagnetic theory, this "mechanical" potential can be rewritten as $V_{\text{eff}} = q\phi$ where $\phi(\tilde{x}, \tilde{y}) = -m\Omega^2(\tilde{x}^2 + \tilde{y}^2)/2q$ is the electric (scalar) potential. The effective magnetic field $\vec{B}$ associated with the vector potential $\vec{A}$ is

$$\vec{B} = \nabla \times \vec{A} = \frac{2m\Omega}{q}\hat{z} = \frac{2m\Omega}{q}\hat{\tilde{z}}.$$

The electric field associated with ϕ and $\vec{A}$ is

$$\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t} = \frac{m\Omega^2}{q}\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ 0\end{pmatrix} - \begin{pmatrix}\partial_t A_x\\ \partial_t A_y\\ 0\end{pmatrix}$$

e) The Hamiltonian not in terms of $\vec{A}$ and V_{eff} is

$$\mathcal{H} = \frac{\tilde{p}_x^2}{2m} + \frac{\tilde{p}_y^2}{2m} - \Omega(\tilde{x}\tilde{p}_y - \tilde{y}\tilde{p}_x)$$

Compared to the original Hamiltonian, $\mathcal{H}_{\text{inertial}} = p_x^2/2m + p_y^2/2m$, we see that all that is needed to describe the motion of a particle in the frame rotating about the z-axis at angular frequency Ω is adding the operator

$$W(\tilde{x}, \tilde{y}, \tilde{p}_x, \tilde{p}_y) = -\Omega(\tilde{x}\tilde{p}_y - \tilde{y}\tilde{p}_x)$$

This operator suffices because $L_z = \tilde{x}\tilde{p}_y - \tilde{y}\tilde{p}_x$ is the generator of rotation about the z-axis. Since there is no other difference between the inertial and rotating frame apart from the fact that the latter is *rotating*, this operator should account for the all the differences between the two frames. Not sure what else to say here? The algebra says $-\Omega L_z$ has to be in the new Hamiltonian, so there it must be.

f) The equations of motion for the particle in the rotating frame are gotten from Hamilton's equations of motion:

$$\begin{aligned}
m\dot{\tilde{x}} &= m\frac{\partial\mathcal{H}}{\partial\tilde{p}_x} = \tilde{p}_x - qA_x \\
m\dot{\tilde{y}} &= m\frac{\partial\mathcal{H}}{\partial\tilde{p}_y} = \tilde{p}_y - qA_y \\
\dot{\tilde{p}}_x &= -\frac{\partial\mathcal{H}}{\partial\tilde{x}} = \Omega\tilde{p}_y \\
\dot{\tilde{p}}_y &= -\frac{\partial\mathcal{H}}{\partial\tilde{y}} = -\Omega\tilde{p}_x.
\end{aligned}$$

From these we find

$$
\begin{aligned}
m\ddot{\tilde{r}} &= m\frac{d^2}{dt^2}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\\
&= \begin{pmatrix}\Omega\tilde{p}_y - q\partial_t A_x\\ -\Omega\tilde{p}_x - q\partial_t A_y\end{pmatrix}\\
&= \begin{pmatrix}m\Omega(\dot{\tilde{y}} + \Omega\tilde{x}) - q\partial_t A_x\\ -m\Omega(\dot{\tilde{x}} - \Omega\tilde{y}) - q\partial_t A_y\end{pmatrix}\\
&= \begin{pmatrix}m\Omega\dot{\tilde{y}}\\ -m\Omega\dot{\tilde{x}}\end{pmatrix} + \begin{pmatrix}m\Omega^2\tilde{x} - q\partial_t A_x\\ m\Omega^2\tilde{y} - q\partial_t A_y\end{pmatrix}\\
&= q\tilde{\vec{v}} \times \vec{B} + q\vec{E}
\end{aligned}
$$

Here we have ignored writing the z-components in the vector quantities since they are not relevant. The expressions for $\vec{B}$ and $\vec{E}$ in terms of the quantities that appear in these equations come from Part (d).

We see that in the rotating frame, the particle behaves like a charged particle experiencing a Lorentz force (combination of the electric force $q\vec{E}$ and magnetic force $q\tilde{\vec{v}} \times \vec{B}$) due to a scalar potential and an effective magnetic field.

2. Quantum description of a charged particle in a uniform magnetic field - Landau levels.
The Hamiltonian for a charged particle of charge $q > 0$ moving freely in the $x-y$ plane in a uniform magneti field $\vec{B} = B\hat{z}$ pointing along the z-axis is

$$
\mathcal{H} = \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2
$$

Let us ignore motion along z and use the symmetric gauge $\vec{A} = -\vec{r} \times \vec{B}/2 = (-yB/2, xB/2, 0)$.

a) We obtain the classical equations of motion using the Lorentz force:

$$
m\ddot{r} = q\vec{E} + q\vec{v} \times \vec{B} = q\vec{v} \times \vec{B}
$$

since we have implicitly assumed $\phi = 0$ by writing the Hamiltonian that way. In component form, this equation is

$$
\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \frac{qB}{m}\begin{pmatrix}\dot{y}\\ -\dot{x}\end{pmatrix}
$$

From here we get two second-order equations for v_x and v_y:

$$
\ddot{v}_x = -\omega_c^2 v_x \qquad \ddot{v}_y = -\omega_c^2 v_y.
$$

where $\omega_c = qB/m$ is the cyclotron frequency. From the setup, we see that v_x and v_y are 90 degrees out of phase, so the motion is circular. The classical equations of motion are therefore

$$
\ddot{x} = -\omega_c x \qquad \ddot{y} = -\omega_c^2 y
$$

where $x^2 + y^2 = r_0^2$ is constant. Assuming that the center of the orbit is x_0 and y_0, the classical trajectory of the particle is given by

$$
x(t) = x_0 + r_0\cos(\omega_c t) \qquad y(t) = y_0 - r_0\sin(\omega_c t),
$$

since the particle moves clockwise for $q > 0$ and B pointing out of the page. The velocities are

$$
v_x(t) = -r_0\omega_c\sin(\omega_c t) \qquad v_y(t) = -r_0\omega_c\cos(\omega_c t).
$$

b) We can transform the original Hamiltonian to that of a standard 2d harmonic oscillator with additional coupling to the angular momentum $L_z = xp_y - yp_x$:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2m}\left(\vec{p} - q\vec{A}\right)^2 \\
&= \frac{(p_x + qyB/2)^2 + (p_y - qxB/2)^2}{2m} \\
&= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2m}\frac{q^2B^2}{4}(x^2+y^2) - \frac{qB}{2m}(xp_y - yp_x) \\
&= \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\left(\frac{\omega_c}{2}\right)^2(x^2+y^2) - \frac{\omega_c}{2}L_z.
\end{aligned}$$

c) Now we introduce the annihilation operators

$$\begin{aligned}
a_x &= \frac{1}{\sqrt{2}}\left(\frac{x}{l_B} + i\frac{p_x l_B}{\hbar}\right) \\
a_y &= \frac{1}{\sqrt{2}}\left(\frac{y}{l_B} + i\frac{p_y l_B}{\hbar}\right)
\end{aligned}$$

with $[a_x, a_x^\dagger] = [a_y, a_y^\dagger] = 1$ and other commutators zero. Consider the Hamiltonian of the form

$$\begin{aligned}
\mathcal{H}_{\text{h.o.}} &= \frac{\hbar\omega_c}{2}\left(a_x^\dagger a_x + a_y^\dagger a_y + 1\right) \\
&= \frac{\hbar\omega_c}{2}\left[\frac{1}{2}\left(\frac{x^2}{l_B^2} + \frac{p_x^2 l_B^2}{\hbar^2} - 1\right) + \frac{1}{2}\left(\frac{y^2}{l_B^2} + \frac{p_y^2 l_B^2}{\hbar^2} - 1\right) + 1\right] \\
&= \frac{\hbar\omega_c}{4}\left[\frac{x^2+y^2}{l_B^2} + \frac{l_B^2}{\hbar^2}(p_x^2 + p_y^2)\right],
\end{aligned}$$

where we have used the commutation relation $[x, p_x] = [y, p_y] = i\hbar$. It is clear that the appropriate choice for l_B is such that

$$\frac{\hbar\omega_c}{4l_B^2} = \frac{1}{2}m\left(\frac{\omega_c}{2}\right)^2 \implies l_B = \sqrt{\frac{2\hbar}{m\omega_c}}.$$

With this choice for l_B, we can write

$$\mathcal{H} = \mathcal{H}_{\text{h.o.}} - \frac{\omega_c}{2}L_z.$$

It remains to express L_z in terms of $a_x, a_y, a_x^\dagger, a_y^\dagger$. To do this, we simply need to write x, y, p_x, p_y in terms of $a_x, a_y, a_x^\dagger, a_y^\dagger$:

$$x = \frac{l_B}{\sqrt{2}}\left(a_x + a_x^\dagger\right), \quad y = \frac{l_B}{\sqrt{2}}\left(a_y + a_y^\dagger\right), \quad p_x = \frac{\hbar}{\sqrt{2}il_B}\left(a_x - a_x^\dagger\right), \quad p_y = \frac{\hbar}{\sqrt{2}il_B}\left(a_y - a_y^\dagger\right).$$

With these,

$$L_z = xp_y - yp_x = \frac{\hbar}{2i}\left(a_x + a_x^\dagger\right)\left(a_y - a_y^\dagger\right) - \frac{\hbar}{2i}\left(a_y + a_y^\dagger\right)\left(a_x - a_x^\dagger\right) = i\hbar(a_x a_y^\dagger - a_x^\dagger a_y)$$

d) Introduce annihilation operators for left-handed and right-handed circular motion about z:

$$a = \frac{a_x + ia_y}{\sqrt{2}} \qquad b = \frac{a_x - ia_y}{\sqrt{2}}$$

We will now put L_z in terms of $\hat{n}_a = a^\dagger a$ and $\hat{n}_b = b^\dagger b$. By instinct, consider the expression $a^\dagger a - b^\dagger b$:

$$\begin{aligned}
a^\dagger a - b^\dagger b &= \frac{1}{2}\left(a_x^\dagger - i a_y^\dagger\right)\left(a_x + i a_y\right) - \frac{1}{2}\left(a_x^\dagger + i a_y^\dagger\right)\left(a_x - i a_y\right) \\
&= \frac{i}{2}\left(a_x^\dagger a_y - a_y^\dagger a_x - a_y^\dagger a_x + a_x^\dagger a_y\right) \\
&= i\left(a_x^\dagger a_y - a_y^\dagger a_x\right) \\
&= -\frac{L_z}{\hbar}.
\end{aligned}$$

So,

$$L_z = \hbar(\hat{n}_b - \hat{n}_a).$$

e) From the previous parts, we find

$$\mathcal{H} = \mathcal{H}_{\text{h.o.}} - \frac{\hbar\omega_c}{2}(\hat{n}_b - \hat{n}_a).$$

Notice further that we can relate $\hat{n}_x$ and $\hat{n}_y$ to $\hat{n}_a$ and $\hat{n}_b$. This is not hard to see:

$$\hat{n}_a + \hat{n}_b = \hat{n}_x + \hat{n}_y.$$

So, we have

$$\mathcal{H} = \frac{\hbar\omega_c}{2}\left(\hat{n}_x + \hat{n}_y + 1\right) - \frac{\hbar\omega_c}{2}\left(\hat{n}_b - \hat{n}_a\right) = \frac{\hbar\omega_c}{2}\left(\hat{n}_a + \hat{n}_b - \hat{n}_b + \hat{n}_a + 1\right) = \hbar\omega_c\left(\hat{n}_a + \frac{1}{2}\right).$$

The eigenenergies are thus $\hbar\omega_c/2, 3\hbar\omega_c/2, 5\hbar\omega_c/2, \dots$ since $n_a = 0, 1, 2, \dots$ Within each Landau level there is a vast degeneracy. Each quantum state is characterized by n_a and m_z, where $m_z\hbar$ is an eigenvalue of L_z. Notice that the energy does not depend on m_z, and that m_z appears implicitly in the Hamiltonian as the difference between n_a and n_b, with n_b also not appearing in the Hamiltonian. This tells us that there is a vast degeneracy for each value of n_a. Physically, the degeneracy can be seen from the classical solution: our system is infinite (the motion of the electron is unbounded in $\mathbb{R}^2$).

f) Now we express observables x, y, v_x, v_y and the center of orbit variables x_0, y_0 in terms of $a, a^\dagger, b, b^\dagger$. By inspection, we have

$$\begin{aligned}
x &= \frac{l_B}{\sqrt{2}}(a_x + a_x^\dagger) = \frac{l_B}{2}(a + a^\dagger + b + b^\dagger) \\
y &= \frac{l_B}{\sqrt{2}}(a_y + a_y^\dagger) = \frac{l_B}{2i}(a - a^\dagger - b + b^\dagger) \\
p_x &= \frac{\hbar}{l_B\sqrt{2}i}(a_x - a_x^\dagger) = \frac{\hbar}{2il_B}(a - a^\dagger + b - b^\dagger) \\
p_y &= \frac{\hbar}{l_B\sqrt{2}i}(a_y - a_y^\dagger) = \frac{\hbar}{2l_B}(-a - a^\dagger + b + b^\dagger) \\
v_x &= \frac{p_x - qA_x}{m} = \frac{p_x}{m} + \frac{\omega_c}{2}y = \frac{1}{\sqrt{2}i}\sqrt{\frac{\hbar\omega_c}{m}}(a - a^\dagger) \\
v_y &= \frac{p_y - qA_y}{m} = \frac{p_y}{m} - \frac{\omega_c}{2}x = -\frac{1}{\sqrt{2}}\sqrt{\frac{\hbar\omega_c}{m}}(a + a^\dagger)
\end{aligned}$$

Now we define the guiding center variables (based on the classical solution) and express them in terms of $a, a^\dagger, b, b^\dagger$:

$$\begin{aligned}
x_0 &= x + \frac{v_y}{\omega_c} = \frac{l_B}{2}(b + b^\dagger) \\
y_0 &= y - \frac{v_x}{\omega_c} = -\frac{l_B}{2i}(b - b^\dagger).
\end{aligned}$$

g) Now we compute the commutator of the center of orbit operators:

$$[x_0, y_0] = \frac{-l_B^2}{4i}[b + b^\dagger, b - b^\dagger] = \frac{l_B^2}{2i}.$$

where we have used

$$[b, b^\dagger] = \left[\frac{1}{\sqrt{2}}(a_x - ia_y), \frac{1}{\sqrt{2}}(a_x^\dagger + ia_y^\dagger)\right] = \frac{1}{2}\left([a_x, a_x^\dagger] + [a_y, a_y^\dagger]\right) = 1.$$

From this result, we find the an uncertainty relation:

$$\Delta x_0 \Delta y_0 \geq \frac{l_B^2}{4}$$

Moreover, since $[x_0, y_0] \neq 0$, motion of the guiding centers of cyclotron orbits is thus motion in non-commutative geometry.

Finally, we compute $[\xi, \eta]$ and $[x, y]$:

$$[\xi, \eta] = \left[-\frac{v_y}{\omega_c}, \frac{v_x}{\omega_c}\right] = \frac{1}{\omega_c^2}[v_x, v_y] = \frac{i\hbar}{2m\omega_c}[a - a^\dagger, a + a^\dagger] = \frac{il_B^2}{2}.$$

$$[x, y] = \frac{l_B^2}{4i}[a + a^\dagger + b + b^\dagger, a - a^\dagger - b + b^\dagger] = 0,$$

as expected since $[x_0, y_0] = -[\xi, \eta]$ and $[x_0, \eta] = [\xi, y_0] = 0$.

h) In this problem we "put the idea of non-commutative geometry to the test." We place ourselves in the lowest Landau level, the ground state of cyclotron motion and start with the particle in the vacuum of the guiding center motion $|0\rangle$ where $b|0\rangle = 0$. The particle is localized at the origin $\langle 0| x_0 |0\rangle = \langle 0| y_0 |0\rangle = 0$. Now we switch on a Hamiltonian:

$$\mathcal{H}_F = -Fx_0 = -\frac{1}{2}Fl_B(b + b^\dagger)$$

which is equivalent to applying a force along x_0.

We can calculate the state $|\psi(t)\rangle = e^{-i\mathcal{H}_F t/\hbar}|0\rangle$ in terms of the coherent states $|\beta\rangle$ associated with b:

$$\begin{aligned}
|\psi(t)\rangle &= e^{-i\mathcal{H}_F t\hbar}|0\rangle \\
&= \exp\left[\frac{iFl_B t}{2\hbar}(b^\dagger + b)\right] \qquad \text{define: } \lambda = \frac{Fl_B t}{2\hbar} \\
&= e^{i\lambda(b^\dagger + b)}|0\rangle \\
&= e^{i\lambda b^\dagger - (i\lambda)^* b}|0\rangle \\
&= D(i\lambda)|0\rangle, \qquad \text{where } D(\beta) \text{ is the displacement operator, defined by } D(\beta)|0\rangle = |\beta\rangle \\
&= |i\lambda\rangle
\end{aligned}$$

where $|i\lambda\rangle$ is the $i\lambda$-(coherent) eigenstate of b. From here, we can calculate:

$$\langle x_0\rangle(t) = \langle\psi(t)| x_0 |\psi(t)\rangle = \frac{l_B}{2}\langle i\lambda| b + b^\dagger |i\lambda\rangle = \frac{l_B}{2}(i\lambda - i\lambda) = 0$$

$$\langle y_0\rangle(t) = \langle\psi(t)| y_0 |\psi(t)\rangle = -\frac{l_B}{2i}\langle i\lambda| b - b^\dagger |i\lambda\rangle = -\frac{l_B}{2i}\langle i\lambda| i\lambda + i\lambda |i\lambda\rangle = -l_B\lambda = -\frac{Fl_B^2 t}{2\hbar}.$$

3. Properties of the coherent state $|\alpha\rangle$

a) Consider two coherent states $|\alpha\rangle, |\beta\rangle$, where $\alpha, \beta \in \mathbb{C}$. Their overlap is

$$\langle\alpha|\beta\rangle = \sum_n \langle\alpha|n\rangle\langle n|\beta\rangle = e^{-|\alpha|^2/2-|\beta|^2/2}\sum_n \frac{(\alpha^*\beta)^n}{n!} = \exp\left(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \alpha^*\beta\right).$$

b) Here we show that coherent states form an over-complete basis:

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \int \frac{d^2\alpha}{\pi}e^{-|\alpha|^2}\sum_{m,n}\frac{\alpha^m(\alpha^*)^n}{\sqrt{m!n!}}|m\rangle\langle n|.$$

Let $\alpha = re^{i\theta}$. Then $d^2\alpha = rdrd\theta$ and each of the summands indexed by m, n becomes

$$\frac{1}{\pi}\int_0^\infty drre^{-r^2}\frac{r^{m+n}}{\sqrt{m!n!}}\underbrace{\int_0^{2\pi}d\theta\, e^{i(m-n)\theta}}_{2\pi\delta_{m,n}} = \frac{2\pi}{2\pi}\frac{\Gamma(m+1)}{m!} = 1 \quad \text{if } m = n \text{, and 0 otherwise.}$$

So,

$$\int \frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = \mathbb{I},$$

as desired.

c) The displacement operator $D(\alpha)$ is defined by $D(\alpha)|0\rangle = |\alpha\rangle$. Here we prove that

$$D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right].$$

Starting from

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{a^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_n \frac{\alpha^n(a^\dagger)^n}{n!}|0\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}|0\rangle.$$

Since $a|0\rangle = 0$, we may write $|0\rangle = e^{-\alpha^*\alpha}|0\rangle$, so that

$$|a\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}e^{-\alpha^* a}|0\rangle = e^{\alpha a^\dagger}e^{-\alpha^* a}e^{-\frac{|\alpha|^2}{2}}|0\rangle.$$

From $[\alpha a^\dagger, -\alpha^* a] = |\alpha|^2$ and the BCH formula $e^{A+B} = e^A e^B e^{-[A,B]/2}$ with $[A, B]$ being a c-number, we can write

$$D(\alpha)|0\rangle = |a\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle$$

So,

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a},$$

as desired.

d) Consider the electric field operator $E_x = i\mathcal{E}\left(ae^{ikz} - a^\dagger e^{-ikz}\right)$ where $\mathcal{E} = \sqrt{\hbar\omega/2\epsilon_0 V}$ is the electric field amplitude for one photon inside the cavity volume V. For a freely evolving coherent state $|\alpha\rangle = |\alpha(t)\rangle$, we first calculate the average electric field:

$$\begin{aligned}\langle E_x\rangle &= i\mathcal{E}\langle\alpha(t)|\, ae^{ikz} - a^\dagger e^{-ikz}\,|\alpha(t)\rangle \\ &= i\mathcal{E}\left[\alpha(0)e^{i(kz-\omega t)} - \alpha^*(0)e^{-i(kz-\omega t)}\right]\end{aligned}$$

where we have used $|\alpha(t)\rangle = e^{-i\omega t/2}\left|\alpha(0)e^{-i\omega t}\right\rangle$. From this, we get

$$\langle E_x\rangle^2 = -\mathcal{E}\left[\alpha^2(0)e^{2i(kz-\omega t)} + \alpha^{*2}(0)e^{-2i(kz-\omega t)} - 2|\alpha(0)|^2\right].$$

Next, we calculate the term:

$$\begin{aligned}
\langle\alpha(t)| E_x^2 |\alpha(t)\rangle &= -\mathcal{E}^2 \langle\alpha(t)| aae^{2ikz} - aa^\dagger - a^\dagger a + a^\dagger a^\dagger e^{-2ikz} |\alpha(t)\rangle \\
&= -\mathcal{E}^2 \langle\alpha(t)| aae^{2ikz} - 1 - 2a^\dagger a + a^\dagger a^\dagger e^{-2ikz} |\alpha(t)\rangle \\
&= \mathcal{E}^2 \langle\alpha(t)| - aae^{2ikz} + 1 + 2a^\dagger a - a^\dagger a^\dagger e^{-2ikz} |\alpha(t)\rangle \\
&= \mathcal{E}^2 \left[-\alpha^2(0)e^{2i(kz-\omega t)} - \alpha^{*2}(0)e^{-2i(kz-\omega t)} + 1 + 2|\alpha(0)|^2\right]
\end{aligned}$$

to find the rms deviation of the electric field:

$$\begin{aligned}
\sqrt{\langle\Delta E_x\rangle^2} &= \sqrt{\langle\alpha(t)| E_x^2 |\alpha(t)\rangle - \langle E_x\rangle^2} \\
&= \mathcal{E}\sqrt{1 + 2|\alpha(0)|^2 - 2|\alpha(0)|^2} \\
&= \mathcal{E}.
\end{aligned}$$

To see *why* $\sqrt{\langle\Delta E_x^2\rangle}$ independent of time and field strength $|\alpha|$ and *why* the result is the same as for the vacuum state $\alpha = 0$, we may look at how a coherent state looks like in phase space. For a given α, the associated coherent state $|\alpha\rangle$ is a *fuzzy ball* in phase space whose center is specified by $|\alpha|$ and $\arg(\alpha)$. The fuzziness, or the uncertainty, equally spreads in all directions and is of fixed magnitude $1/2$ for any α – since coherent states are minimal uncertainty states. Because of this, the uncertainty stays constant as the amplitude changes. In particular, the uncertainty associated with some coherent state $|\alpha\rangle$ must resemble that of the vacuum state $|0\rangle$, which is just a coherent state with $\alpha = 0$.

4. Pseudo-probability distribution plots.

In this problem we consider the pseudo-probability distributions $Q(\alpha)$ defined as

$$Q_\rho(\alpha) \equiv \langle\alpha|\,\rho\,|\alpha\rangle\,,$$

which can be readily computed numerically using the fact that any pure state $\rho = |\psi\rangle\langle\psi|$ can be represented using

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle$$

and

$$\langle n|\alpha\rangle = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}$$

a) Here we compute and plot

$$Q_1(\alpha) = |\langle\alpha|\psi_1\rangle|^2$$

where

$$|\psi_1\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|12\rangle$$

with $|12\rangle$ being the twelve-photon number eigenstate.

First we compute:

$$\langle\alpha|\psi_1\rangle = \cos\frac{\theta}{2}\langle\alpha|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\langle\alpha|12\rangle = e^{-|\alpha|^2/2}\left[\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}\frac{\alpha^{12}}{\sqrt{12!}}\right].$$

Plot for various values of ϕ and θ:

$$blahhhhhhhhhhh$$

$|\psi_1\rangle$ is not a minimum uncertainty state.

b) Here we compute and plot

$$Q_2(\alpha) = |\langle\alpha|\psi_2\rangle|^2$$

where

$$|\psi_2\rangle = \frac{|\beta\rangle + |-\beta\rangle}{\sqrt{2}}, \qquad \beta = 3.$$

First we compute:

$$\langle\alpha|\psi_2\rangle = \frac{1}{\sqrt{2}}\langle\alpha|\beta\rangle + \frac{1}{\sqrt{2}}\langle\alpha|-\beta\rangle = \frac{1}{\sqrt{2}}e^{-|\alpha|^2/2-|\beta|^2/2}\left(e^{\alpha^*\beta} + e^{-\alpha^*\beta}\right).$$

We first plot this on normal scale:

Next we plot this the log scale:

Notice that there are interference fringes around the origin. To explain this, we simply look at the mathematical form for $|\langle\alpha|\psi_2\rangle|^2$. It turns out that the near the origin where the exponential decays $e^{-|\alpha|^2/2}e^{-|\beta|^2/2}$ haven't taken over, we have

$$|\langle\alpha|\psi_2\rangle|^2 \propto \left|e^{\alpha^*\beta} + e^{-\alpha^*\beta}\right|^2 = e^{2\,\mathrm{Re}\,z} + e^{-2\,\mathrm{Re}\,z} + e^{2i\,\mathrm{Im}\,z} + e^{-2i\,\mathrm{Im}\,z}$$

where we have put $z = \alpha^*\beta$. Notice that for $\beta = 3$ and $\mathrm{Re}\,\alpha = 0$, we have that $\mathrm{Re}\,z = 0$ for $\mathrm{Im}\,\alpha \neq 0$. This means

$$|\langle\alpha|\psi_2\rangle|^2 \propto 2 + 2\cos(2\,\mathrm{Im}\,z) = 2 + 2\cos(6\,\mathrm{Im}\,\alpha).$$

This explains the vertical periodic pattern that we see in the plots for Q_2.

c) Here we compute and plot

$$Q_3(\alpha) = |\langle\alpha|\psi_3\rangle|^2$$

where

$$|\psi_3\rangle = \frac{1}{\sqrt{N}}\sum_{k=1}^{N} e^{ik\phi}|k\rangle$$

where k is a Fock state of k photons, with $N = 10$ and $\phi = \pi/4$.

First we compute:

$$\langle\alpha|\psi_3\rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{N}}\sum_{k=1}^{N} e^{ik\phi}\frac{\alpha^k}{\sqrt{k!}}.$$

Physical meaning of this state and what happens as $N \to \infty$?

d) Here we compute and plot

$$Q_4(\alpha) = |\langle \alpha | 0_\epsilon \rangle|^2$$

where $|0_\epsilon\rangle = S(\epsilon)\,|0\rangle$ with

$$|0_\epsilon\rangle = S(\epsilon)\,|0\rangle = \frac{1}{\sqrt{\cosh \epsilon}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh \epsilon)^n \,|2n\rangle$$

is called the squeezed vacuum with parameter ϵ. Next we compute:

$$\langle \alpha | \psi_4 \rangle = \frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh \epsilon}} \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (\tanh \epsilon)^n \frac{\alpha^{2n}}{\sqrt{(2n)!}} = \frac{e^{-|\alpha|^2/2}}{\sqrt{\cosh \epsilon}} \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{2^n n!} (\tanh \epsilon)^n.$$

e) Here we compute and plot

$$Q_5(\alpha) = \left| \langle \alpha |\, e^{i\mathcal{H}_{\text{Kerr}} t}\, |\beta\rangle \right|^2$$

where the Kerr effect Hamiltonian is

$$\mathcal{H}_{\text{Kerr}} = \xi a^\dagger a (a^\dagger a - 1) = \xi n(n-1).$$

First we compute:

$$\langle \alpha |\, e^{it\xi \hat{n}(\hat{n}-1)}\, |\beta\rangle = \sum_{n=0}^{\infty} e^{-|\beta|^2/2} e^{it\xi n(n-1)} \frac{\beta^n}{\sqrt{n!}} \langle \alpha | n \rangle = e^{-|\alpha|^2/2} e^{-|\beta|^2/2} \sum_{n=0}^{\infty} \frac{(\alpha\beta)^n}{n!} e^{it\xi n(n-1)}.$$

At what time does the initial coherent state evolve to become two superposed coherent states? By inspection, the initial coherent state evolves to become two superposed coherent states whenever $t = (2m+1)\pi/2\xi$ where $m \in \mathbb{Z}$. For example, $t = \pi/2\xi$:

When does the coherent state return to its original state? The coherent state $|\beta\rangle$ returns to its original state when

$$\sum_{n=0}^{\infty} \frac{(\alpha\beta)^n}{n!} e^{it\xi n(n-1)} = \sum_{n=0}^{\infty} \frac{(\alpha\beta)^n}{n!}$$

which occurs whenever $t = \pi m/\xi, m \in \mathbb{Z}$. Picking $m = 1$, we get $t = 128$ and indeed we have the original state back: