8.09 & 8.309 Classical Mechanics III, Fall 2018
FINAL
Monday December 17, 1:30pm-4:30pm
You have 180 minutes.

Answer all problems in the white books provided. Write YOUR NAME on EACH book you use.

There are six problems, totaling 150 points. You should do all six. The problems are worth 14, 22, 30, 36, 22, and 26 points. You may do the problems in any order.

None of the problems requires extensive algebra. If you find yourself lost in a calculational thicket, stop and think.

No books, notes, or calculators allowed.

Some potentially useful information

- Euler-Lagrange equations for generalized coordinates q_j

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\alpha \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j}\ , \qquad \text{or} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = \sum_\beta \lambda_\beta \frac{\partial g_\beta}{\partial \dot{q}_j}$$

constraints: holonomic $f_\alpha(q,t) = 0$ or semiholonomic $g_\beta = \sum_j a_{\beta j}(q,t)\dot{q}_j + a_{\beta t}(q,t) = 0$

- Generalized forces: $d/dt(\partial L/\partial \dot{q}_j) - \partial L/\partial q_j = R_j$

Friction forces: $\vec{f}_i = -h(v_i)\vec{v}_i/v_i$, $\vec{v}_i = \dot{\vec{r}}_i$ gives $R_j = -\partial \mathcal{F}/\partial \dot{q}_j\ ,\ \ \mathcal{F} = \sum_i \int_0^{v_i} dv_i'\, h(v_i')$

- Hamilton's equations for canonical variables (q_j, p_j): $\quad \dot{q}_j = \dfrac{\partial H}{\partial p_j}\ , \quad \dot{p}_j = -\dfrac{\partial H}{\partial q_j}$

- Hamiltonian for a Lagrangian quadratic in velocities

$$L = L_0(q,t) + \dot{\vec{q}}^T \cdot \vec{a} + \frac{1}{2}\dot{\vec{q}}^T \cdot \hat{T} \cdot \dot{\vec{q}} \quad \Rightarrow \quad H = \frac{1}{2}(\vec{p}-\vec{a})^T \cdot \hat{T}^{-1} \cdot (\vec{p}-\vec{a}) - L_0(q,t)$$

- The Moment of Inertia Tensor and its relations:

$$I_{ab} = \int dV\, \rho(\vec{r})[\vec{r}^{\,2}\delta_{ab} - r_a r_b] \qquad \text{or} \qquad I^{ab} = \sum_i m_i[\delta^{ab}\vec{r}_i^{\,2} - r_i^a r_i^b]$$

$$I_{ab}^{(Q)} = M(\delta_{ab}\,\vec{R}^2 - R_a R_b) + I_{ab}^{(\mathrm{CM})}\ , \qquad \hat{I}' = \hat{U}\,\hat{I}\,\hat{U}^T$$

- Euler's Equations:

$$\begin{aligned} I_1\dot{\omega}_1 - (I_2 - I_3)\omega_2\omega_3 &= \tau_1 \\ I_2\dot{\omega}_2 - (I_3 - I_1)\omega_3\omega_1 &= \tau_2 \\ I_3\dot{\omega}_3 - (I_1 - I_2)\omega_1\omega_2 &= \tau_3 \end{aligned}$$

- Vibrations: $L = \frac{1}{2}\,\dot{\vec{\eta}}^T \cdot \hat{T} \cdot \dot{\vec{\eta}} - \frac{1}{2}\vec{\eta}^T \cdot \hat{V} \cdot \vec{\eta}$ has Normal modes $\vec{\eta}^{(k)} = \vec{a}^{(k)}\exp(-i\omega^{(k)}t)$

$$\det(\hat{V} - \omega^2\hat{T}) = 0\ , \qquad \big(\hat{V} - [\omega^{(k)}]^2\,\hat{T}\big)\cdot\vec{a}^{(k)} = 0\ , \qquad \vec{\eta} = \mathrm{Re}\sum_k C_k\,\vec{\eta}^{(k)}$$

- Generating functions for Canonical Transformations: $K = H + \partial F_i/\partial t$ and

$$F_1(q,Q,t):\quad p_i = \frac{\partial F_1}{\partial q_i}\ ,\ P_i = -\frac{\partial F_1}{\partial Q_i}\ , \qquad F_2(q,P,t):\quad p_i = \frac{\partial F_2}{\partial q_i}\ ,\ Q_i = \frac{\partial F_2}{\partial P_i}$$

- Poisson Brackets: $\quad [u,v]_{q,p} = \sum_j \left[\dfrac{\partial u}{\partial q_j}\dfrac{\partial v}{\partial p_j} - \dfrac{\partial u}{\partial p_j}\dfrac{\partial v}{\partial q_j}\right]\ , \qquad \dfrac{du}{dt} = [u,H] + \dfrac{\partial u}{\partial t}$

- Relations for Hamilton's Principle function, $S = S(q_1,\dots,q_n;\alpha_1,\dots,\alpha_n,t)$

$$K = 0\ , \qquad P_i = \alpha_i\ , \qquad Q_i = \beta_i = \frac{\partial S}{\partial \alpha_i}\ , \qquad p_i = \frac{\partial S}{\partial q_i}$$

- Relations for Hamilton's Characteristic function, $W = W(q_1,\dots,q_n;\alpha_1,\dots,\alpha_n)$

$$K = H = \alpha_1\ , \qquad P_i = \alpha_i\ , \qquad \beta_1 + t = \frac{\partial W}{\partial \alpha_1}\ , \qquad \beta_{i>1} = \frac{\partial W}{\partial \alpha_i}\ , \qquad p_i = \frac{\partial W}{\partial q_i}$$

- Action Angle Variables: $\quad J = \oint p\,dq\ , \qquad w = \dfrac{\partial W(q,J)}{\partial J}\ , \qquad \dot{w} = \dfrac{\partial H(J)}{\partial J} = \nu(J)$

- Time Dependent Perturbation Theory for $H_0 + \Delta H$. Solve $H_0(p,q)$ with the Hamilton-Jacobi method to obtain constant canonical variables (β, α) where $[\beta, \alpha] = 1$. Then

$$\dot{\alpha}^{(n)} = -\frac{\partial \Delta H}{\partial \beta}\bigg|_{n-1}, \qquad \dot{\beta}^{(n)} = \frac{\partial \Delta H}{\partial \alpha}\bigg|_{n-1}$$

- Fluid volume and continuity equations $\quad \dfrac{dV}{dt} = \displaystyle\int dV\, \vec{\nabla} \cdot \vec{v}, \qquad \dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}\,) = 0$

- Euler equation ($\nu = 0$) or Navier-Stokes equation ($\nu = \eta/\rho \neq 0$), with gravity:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\, \vec{v} + \frac{1}{\rho} \vec{\nabla} p - \nu \nabla^2 \vec{v} = \frac{\vec{f}}{\rho} = \vec{g}$$

friction.

- For direction i the force/unit area on a surface $\boxed{= -\hat{n}_i\, p + \hat{n}_k \sigma'_{ki}}$

- Ideal fluid has $ds/dt = 0$ so $p = p(\rho, s)$. Viscous fluid has $ds/dt \propto \sigma'_{ik}\, \partial v_i / \partial x_k$.

- Bernoulli's equation for a steady incompressible ideal fluid in gravity $\vec{g} = -g\hat{z}$:

$$\frac{\vec{v}^{\,2}}{2} + gz + \frac{p}{\rho} = \text{constant}$$

- Irrotational incompressible ideal fluid flow (potential flow): $\vec{v} = \nabla \phi$, $\nabla^2 \phi = 0$

- Sound waves: $\left(\frac{\partial^2}{\partial t^2} - c_s^2 \nabla^2\right)\{p', \rho', \vec{v}\,\} = 0$. Mach number $M = v_0/c_s$.

- Momentum conservation: $\frac{\partial}{\partial t}(\rho \vec{v}) + \vec{\nabla} \cdot \hat{T} = \vec{f}$ where the energy momentum tensor is $T_{ki} = v_k v_i\, \rho + \delta_{ki}\, p - \sigma'_{ki}$. For $\vec{\nabla} \cdot \vec{v} = 0$ the viscous stress tensor $\sigma'_{ki} = \eta\big(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\big)$.

- Reynolds Number: $R = uL/\nu$

- Bifurcations at $\mu = 0$. In 1-dim: "saddle-node" $\dot{x} = \mu + x^2$, "transcritical" $\dot{x} = x(\mu - x)$, "supercritical pitchfork" $\dot{x} = \mu x - x^3$, "subcritical pitchfork" $\dot{x} = \mu x + x^3$. In 2-dim: "supercritical Hopf" $\dot{r} = r(\mu - r^2)$, "subcritical Hopf" $\dot{r} = r(\mu + r^2)$.

- Linearization for 2-dim fixed points: $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = M \begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = \vec{a}\, e^{\lambda t}, \quad M\vec{a} = \lambda\, \vec{a}$

$$\lambda_\pm = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}, \qquad \tau = \operatorname{tr} M, \qquad \Delta = \det M$$

- 2-dim conserved system $\dot{x} = f_x(x,y)$, $\dot{y} = f_y(x,y)$ with $\vec{\nabla} \cdot \vec{f} = 0$, has conserved $H(x,y) = \int^y dy' f_x(x,y') - \int^x dx' f_y(x',y)$.

- 1-dim map $x_{n+1} = f(x_n)$. Its fixed points satisfy $x^* = f(x^*)$. Here x^* is stable for $|f'(x^*)| < 1$ and unstable for $|f'(x^*)| > 1$.

- Fractal dimension: $\quad d_F = \displaystyle\lim_{a \to 0} \frac{\ln N(a)}{\ln(a_0/a)}$

$1 = r(1-x)$ $\frac{1}{r} = 1-x$

1. **Short answer problems** [14 points]

These problems require no algebra or a little algebra. Your answers should be short.

precession ~ rotation

(a) [4 points] For a symmetric top with one point fixed, no applied torque, and no nutation, describe the most general motion that the top can undergo.

(b) [3 points] If the moment of inertia tensor and angular velocity for a rigid body are $\hat{I}$ and $\vec{\omega}$, what are the most general conditions under which the total kinetic energy $T = \frac{1}{2}\,\vec{\omega}^T \cdot \hat{I} \cdot \vec{\omega}$?

origin about which I calculated is fixed (not moving)

(c) [3 points] Give one example of a system which has a variable that undergoes a secular change due to a perturbation.

SHO with x^4 pert, Relativistic corr, Kepler orbit

(d) [4 points] What are all the simple fixed points of the Logistic Map, $x_{n+1} = rx_n(1-x_n)$ in the region $0 \le x \le 1$ with $0 < r \le 4$?

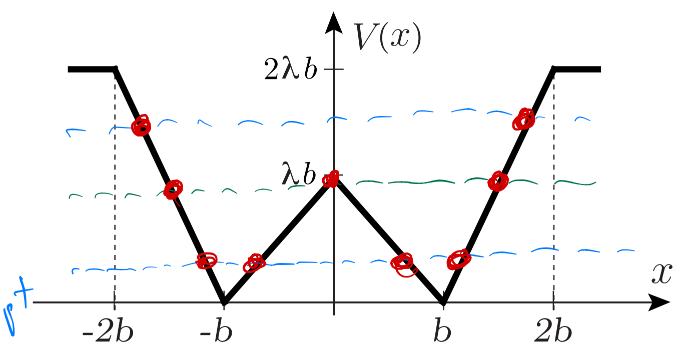
$x^* = r x^* (1-x^*) \Rightarrow$ $\boxed{x^* = 0, \quad x^* = 1 - 1/r}$

2. **Action Angle for a W Potential** [22 points]

Consider a particle of mass m and energy $E > 0$ moving in one dimension in the following potential, where $\lambda > 0$ and $b > 0$ are constants:

$E = p^2/2m + V$ ✓

$$V(x) = \begin{cases} 2\lambda b & 2b < x \\ 2\lambda(x-b) & b < x \le 2b \\ \lambda(b-x) & 0 < x \le b \\ \lambda(b+x) & -b < x \le 0 \\ -2\lambda(x+b) & -2b < x \le -b \\ 2\lambda b & x \le -2b \end{cases}$$



let $E = V(x)$

turning pt x^2

(a) [10 points] For various values of $E > 0$ and initial positions, classify when there is and is not periodic motion. For each periodic case find the two turning points x_L and x_R as functions of E, so that $x_L \le x \le x_R$. ✓

(b) [10 points] Consider an $E > 0$ with the particle trapped in the right-most well. Using the method of action-angle variables, find the period $\cancel{\tau = 1/\nu}$ as a function of E. ok doable ... $J = \oint p\,dq = 2\int \sqrt{2m(E-V)}\,dx$

(c) [2 points] Check your answer in (b) for $\tau = \tau(E)$ by confirming that dimensions on the two sides match up.

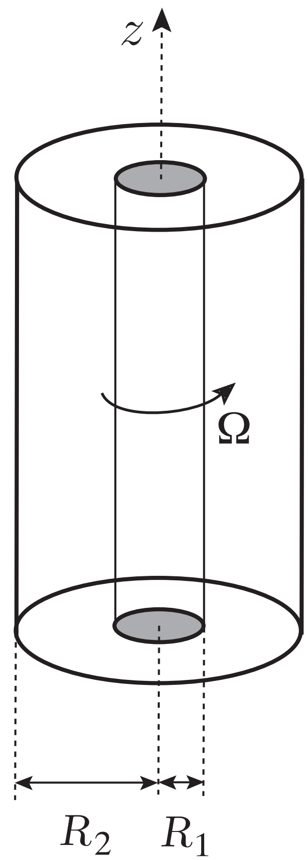
get $J(E) \Rightarrow \nu = \left(\frac{\partial J}{\partial E}\right)^{-1} \to \tau = \frac{\partial J}{\partial E}$

ok

(continue)

3. **Viscous Fluid Driven by a Rotating Shaft** [30 points]

Consider two infinite cylinders of radius R_1 and R_2 that are both centered on the z-axis as shown. The outer cylinder is at rest, while the inner cylinder rotates around the z-axis with constant angular velocity Ω. An incompressible fluid with constant density ρ and viscosity η is placed between the two cylinders and the flow is in a steady state (and is nonturbulent). There is no gravity in this problem.



For cylindrical coordinates $\{r, \phi, z\}$ the following identities may be useful:

$$\vec{\nabla} = \hat{r}\,\frac{\partial}{\partial r} + \boxed{\hat{\phi}\,\frac{1}{r}\frac{\partial}{\partial \phi}} + \hat{z}\,\frac{\partial}{\partial z}$$

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \cancel{\frac{\partial^2}{\partial z^2}}\right)$$

$$\frac{\partial \hat{r}}{\partial r} = 0\,, \qquad \frac{\partial \hat{r}}{\partial \phi} = \hat{\phi}\,, \qquad \boxed{\frac{\partial \hat{\phi}}{\partial r} = 0}\,, \qquad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{r}$$

Using symmetry we know that the velocity and pressure components are:

$$v_r = v_z = 0\,, \quad v_\phi = v_\phi(r)\,, \qquad p = p(r)$$

(a) [11 points] Using the Navier-Stokes equation derive a differential equation for $v_\phi(r)$ and a differential equation for $p(r)$.

(b) [4 points] Check that $v_\phi(r) = ar + b/r$ with constants a, b is a solution of your velocity equation from (a).

no slipping on R

(c) [8 points] Determine the two constants from part (b) by imposing suitable boundary conditions. Write a final result for $v_\phi(r)$. [Note that you can answer this part even if you got stuck on (a).]

and $v \cdot n_{surface} = 0$

(d) [7 points] What is the friction force per unit area on the outer cylinder in terms of η, Ω, R_1, and R_2?

Hint: In order to be careful with the order of operations with curvilinear coordinates in (d) you should use $\sigma'_{ki} = \eta\big[\{(\hat{e}_k \cdot \vec{\nabla})\,\vec{v}\} \cdot \hat{e}_i + \{(\hat{e}_i \cdot \vec{\nabla})\,\vec{v}\} \cdot \hat{e}_k\big]$ for suitable unit vectors $\hat{e}_k$ and $\hat{e}_i$.

use this only, this is friction force -
same with lift -

(continue)

NS

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \frac{1}{\rho}\vec{\nabla}p - \nu\vec{\nabla}^2\vec{v} = \vec{g} = 0$$

$$\cancel{\frac{\partial \vec{v}}{\partial t}} + (\vec{v}\cdot\vec{\nabla})\vec{v} - \nu\vec{\nabla}^2\vec{v} = -\frac{1}{\rho}\vec{\nabla}p$$

0 steady state $\Rightarrow$ $\vec{v}$ time dependent.

$p = p(r)$

$v = \cancel{v_r}\,\hat{r} + v_\varphi\hat{\varphi} + \cancel{v_z}\,\hat{z}$

0 0

$$v_\phi^2 \frac{1}{r}\partial_\phi\hat{\phi} = \boxed{\frac{v_\phi^2}{r}(-\hat{r})}$$

$v_\varphi = v_\varphi(r)$

$$(\vec{v}\cdot\vec{\nabla}) = \begin{pmatrix} \cancel{v_r}^{\,0} \\ v_\phi \\ \cancel{v_z}^{\,0} \end{pmatrix} \cdot \begin{pmatrix} \partial_r \\ 1/r\,\partial_\phi \\ \partial_z \end{pmatrix} = v_\phi \frac{1}{r}\partial_\phi(v_\phi\hat{\phi})$$

$v_\phi\hat{\phi}$ $v_\phi\hat{\phi}$

$$\vec{\nabla}^2\vec{v} = \left[\partial_r^2 + \frac{1}{r}\partial_r\right] v_\phi(r)\hat{\phi} + \frac{1}{r^2}\partial_\phi^2 v_\phi\hat{\phi}$$

only ∂_r terms survive...

$$= \hat{\phi}\left[\partial_r^2 + \frac{1}{r}\partial_r\right] v_\phi(r)$$

$$\partial_\phi^2\hat{\phi} = \partial_\phi(\partial_\phi\hat{\phi}) = \partial_\phi(-\hat{r}) = -\hat{\phi}.$$

$\vec{\nabla}p = \partial_r p\,\hat{r}$ since $p = p(r)$

2 eqns in $\vec{\phi}$ & $\vec{r}$

$\vec{r}$:

$$-\frac{1}{\rho}\vec{\partial}\underline{p} = -\frac{v_\phi^2}{r}(\hat{r})$$

$$\boxed{+\frac{1}{\rho}\partial_r p = \frac{v_\phi^2}{r}}$$

$$p'(r) = \frac{\rho}{r}v_\phi^2$$

$\vec{\phi}$

$$-\cancel{\nu}\left[\partial_r^2 + \frac{1}{r}\partial_r\right]v_\phi + \cancel{\nu}\frac{v_\phi}{r^2} = 0$$

$$\boxed{\partial_r^2 v_\phi + \frac{1}{r}\partial_r v_\phi - \frac{v_\phi}{r^2} = 0}$$

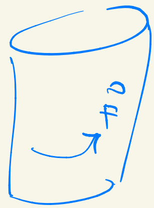
Look at $\sigma'_{ki} = ?$

k	i
$\hat{r}$	$\hat{\phi}$
$\hat{z}$	$\hat{z}$
$\hat{\phi}$	$\hat{z}$

$\hat{\phi}\cdot\vec{\nabla} = \frac{1}{r}\partial_\phi$

$\hat{z}\cdot\vec{\nabla} = \partial_z$

$\hat{r}\cdot\vec{\partial} = \partial_r$



Directions

$-\hat{n}_i p + \hat{n}_k \sigma'_{ki}$

Friction ... care about the $\hat{\phi}$ direction

$F_i = F_\phi = -\hat{n}_\phi \underline{p} + \hat{n}_k \sigma'_{k\phi}$ 0

$\sigma'_{z\phi} = \eta\left[\left(\overbrace{\hat{z}\cdot\vec{\nabla}}^{\partial_z}\right) v_\phi \hat{\phi}\cdot\hat{\phi} + \cancel{(\hat{\phi}\cdot\vec{\partial}) v_\phi \hat{\phi}\cdot\hat{z}}\right]$ 0

$\hat{r}\cdot \boxed{\sigma'_{r\phi} = \eta\left[\left(\underbrace{\hat{r}\cdot\vec{\partial}}_{\partial_r}\right) v_\phi \hat{\phi}\cdot\hat{\phi}} + (\hat{\phi}\cdot\vec{\partial}) v_\phi \hat{\phi}\cdot\hat{r})$

$v_\phi\left(\frac{1}{r}\partial_\phi\right)\underbrace{\hat{\phi}}_{(-\hat{r})}\cdot\hat{r}$

$\sigma'_{\phi\phi} = 2\eta\left(\left(\underbrace{\hat{\phi}\cdot\vec{\partial}}_{\frac{1}{r}\partial_\phi}\right) v_\phi \hat{\phi}\right)\cdot\hat{\phi}$

$-\hat{n}_\phi p + \eta\partial_r v_\phi\cdot\hat{r}$

$\boxed{\eta\,\partial_r v_\phi}$

$2\eta\frac{1}{r}(\underbrace{\partial_\phi\hat{\phi}}_{\hat{r}})\hat{\phi}\, v_\phi = 0$

$\boxed{\eta\left(\partial_r v_\phi - v_\phi/r\right)}$

4. **Nonlinear Attractions** [36 points]

In this problem you will investigate fixed points and/or limit cycles for two examples.

First consider the system of equations

$$\dot{x} = (y-1)(x^2 - 3y - 4)$$
$$\dot{y} = y(x-1)$$

(a) [14 points] Find the four fixed points. Using a linear analysis classify each fixed point as a center, saddle node, unstable spiral, etc.

(b) [10 points] For each of your fixed points from (a), if it is a saddle node, unstable node, or stable node, then determine the corresponding eigenvalues and eigenvectors (you do not need to normalize them). Assume your linear analysis holds even for borderline cases and make a sketch of trajectories near your 4 fixed points.

scary

Chill, but with polar words

For the remaining parts, consider the following system of differential equations in polar coordinates (r, θ):

$$\dot{r} = r(r^2 - a)(r-1)$$
$$\dot{\theta} = 1$$

for a constant parameter a with $-\infty < a < \infty$.

(c) [7 points] Identify all attractors (limit cycles and fixed points). Determine the stability or instability for each attractor for all values of a. Find the values of a where there are bifurcations.

(d) [5 points] For each distinct region of a, sketch each of your attractors from (c) in the (x, y) plane.

(continue)

$$\dot{x} = (y-1)(x^2-3y-4)$$
$$\dot{y} = y(x-1)$$

Fixed pts

$$0 = (y-1)(x^2-3y-4)$$
$$0 = y(x-1)$$

$$(y-1)(-3y-3) = 0$$

$$\boxed{\begin{array}{lll} y=0 & \to & x = \pm 2 \\ y = 1 & \to & x = 7 \\ x = 1 & \to & y = -1 \end{array}}$$

$$\to \quad \boxed{\begin{array}{l} y=0 \; ; \; x = \pm 2 \\ x = 1 \; ; \; y = \pm 1 \end{array}}$$

Linearize ...

$$\boxed{y^{\pm} = 0 \; ; \; x^{\pm} = \pm 2}$$

(then repeat ...)

$$y = \varepsilon_y \; ; \; x = \varepsilon_x \pm 2$$

$$\begin{cases} \dot{\varepsilon}_x = (\varepsilon_y - 1)\left((\varepsilon_x \pm 2)^2 - 3\varepsilon_y - 4\right) \\ \dot{\varepsilon}_y = \varepsilon_y(\varepsilon_x \pm 2 - 1) \end{cases}$$

$$\begin{cases} \dot{\varepsilon}_x = (\cancel{\varepsilon_y} - 1)\left[\pm 4\varepsilon_x - 3\varepsilon_y\right] = \mp 4\varepsilon_x + 3\varepsilon_y \\ \dot{\varepsilon}_y = \varepsilon_y(\varepsilon_x \pm 2 - 1) \end{cases}$$

$$(+) \quad \begin{pmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix} \; ; \; \begin{pmatrix} \dot{\varepsilon}_x \\ \dot{\varepsilon}_y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 0 & -3 \end{pmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix}$$

saddle node

saddle node

$$\begin{pmatrix} -4 & 6 \\ -1 & 0 \end{pmatrix} \rightsquigarrow \boxed{\det = +6}$$

$$\begin{pmatrix} -4-\lambda & 6 \\ -1 & -\lambda \end{pmatrix}$$

$$(+\lambda)(4+\lambda)+6 = 0$$

$$\lambda^2 + 4\lambda + 6 = 0 \longrightarrow \text{2 cx root.}$$

real part = ?

$$\frac{-4 \pm \sqrt{16 - \overbrace{4\cdot 6}^{24}}}{2} = -2 \pm \frac{1}{2}\sqrt{-8}$$

$$= -2 \pm i\sqrt{2}$$

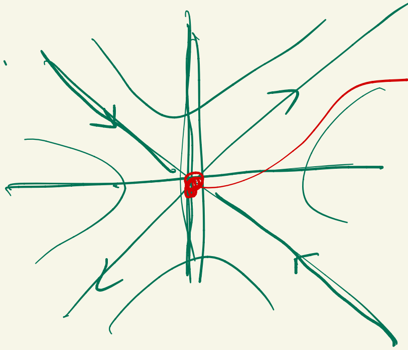
↳ stable spiral since $-2 < 0$

etc

To sketch ... find eigenvectors ...

$$e^{-t}\begin{pmatrix}-1\\1\end{pmatrix} \quad \text{e.g.}$$

$$e^{t}\begin{pmatrix}1\\1\end{pmatrix}$$

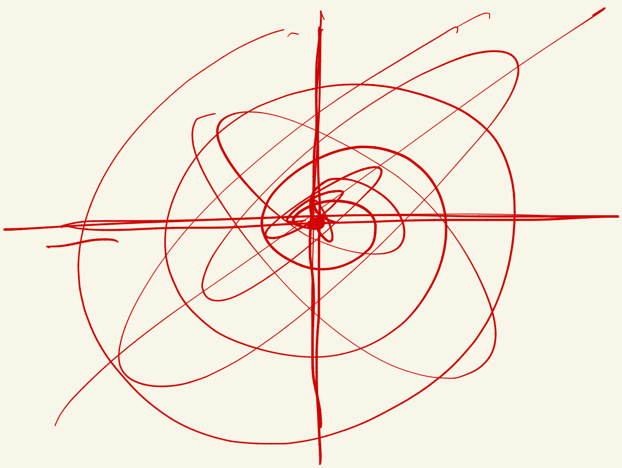


fixed point.

[illegible]
- get fixed point
- draw trajectory ...
- use eigv for stable/unstable node
- care about direction for spirals ...

e.g.

$$e^{-2t}e^{\pm i\sqrt{2}t}\begin{pmatrix}1\\1\end{pmatrix}$$



$+i$ 

$-i$ 

setting $\dot{r} = \dot{\theta} = 0$

$$\dot{r} = r(r^2 - a)(r - 1)$$
$$\dot{\theta} = 1$$

$0 = r(r^2 - a)(r-1)$

$0 = 1$

$\Downarrow$

can't

$\Rightarrow$ obviously no fixed point for $r \neq 0$

$\Rightarrow$ find limit cycle ---

$0 = r(r^2 - a)(r-1)$

$\Rightarrow$ $\boxed{r = 0 \;;\; r = \sqrt{a} \;;\; r = 1}$

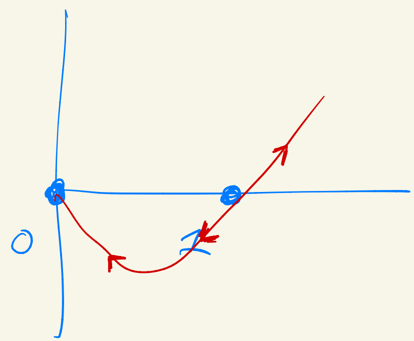
ok

fixed point

potential limit cycles

$\dot r = r(r^2 - a)(r-1)$

(Three) roots $r = 0, 1$, and (a)



if $a < 0$ then only 2 roots $0, 1$

$$\dot r = r(r-1)\underbrace{(r^2-a)}_{>0\ \forall r}$$

$\dot r < 0$ for $0 < r < 1$

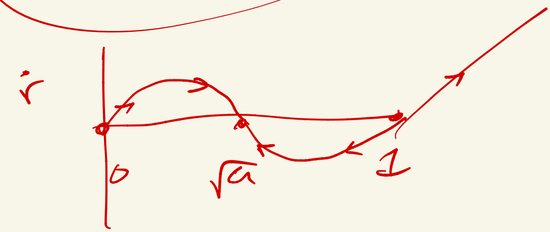
$\dot r > 0$ for $r > 1$

$\rightarrow$ $\boxed{\begin{array}{l}\text{stable fixed point} \\ \text{unstable limit cycle} \quad (r=1)\end{array}}$

if $a > 0$ then 2 cases

$0 < a < 1$, then

$r = \sqrt{a}$ is a root



$r(r-\sqrt{a})(r+\sqrt{a})(r-1)$

$0 < r < \sqrt{a}$

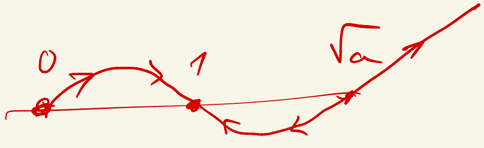
$+ \ - \ + \ - \ \Rightarrow +$

$\sqrt{a} < r < 1$

$+ \ + \ + \ - \ \Rightarrow -$

- unstable fixed point $(r=0)$
- stable limit cycle $(r = \sqrt{a})$
- unstable limit cycle $(r=1)$

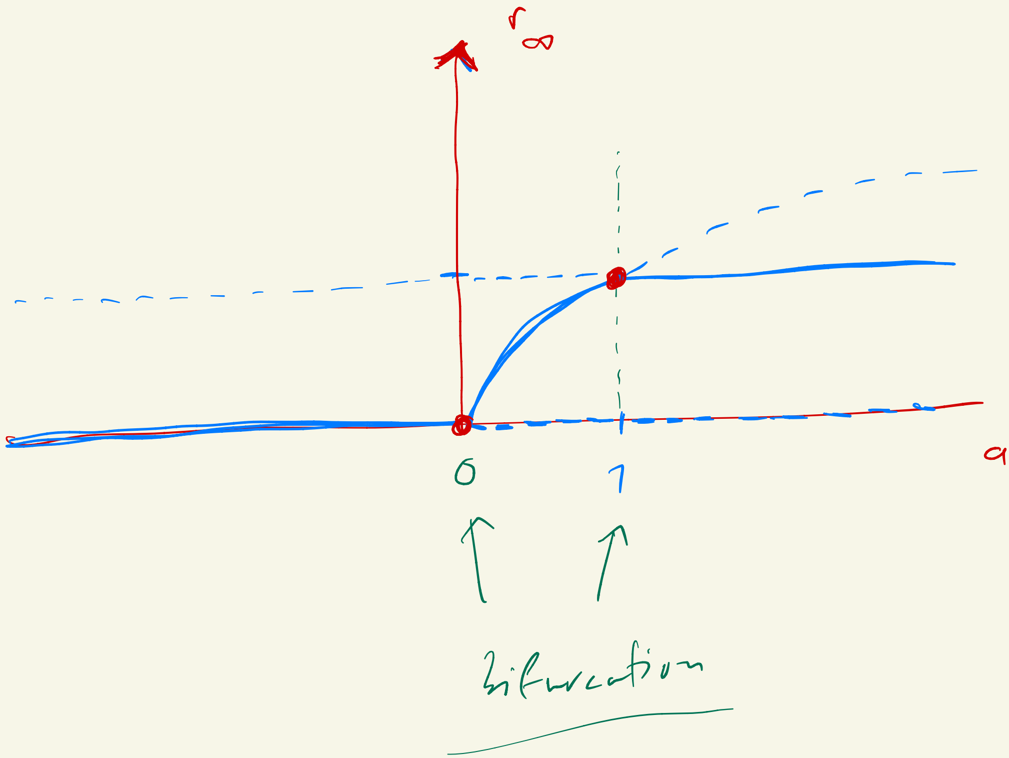
if $a > 1$, then $r = \sqrt{a}$ is a root



$\underbrace{r}_{>0}\,\underbrace{(r-\sqrt{a})}\,\underbrace{(r+\sqrt{a})}_{>0}\,\underbrace{(r-1)}$

- unstable fixed pt $(r=0)$
- stable limit cycle $(r=1)$
- unstable limit cycle $(r = \sqrt{a})$

Bifurcation diagram

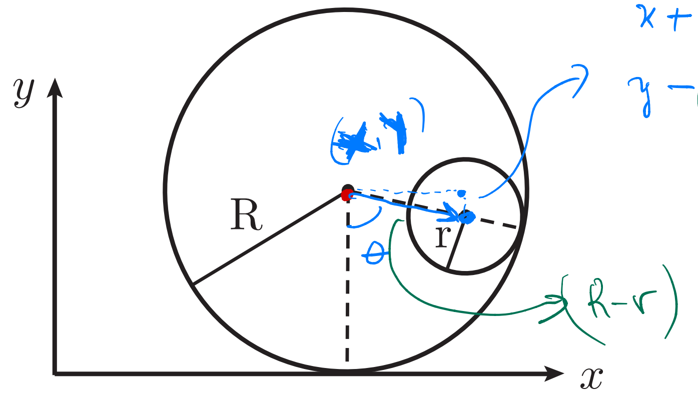


From here sketching attractors, is trivial ...

sketch both stable ——— and unstable - - - - attractors.

5. **Rolling Constraints for Two Hoops** [22 points]

only inner hoop moves, no rolling



$x + (R-r)\sin\theta$

$y - (R-r)\cos\theta$

circular motion

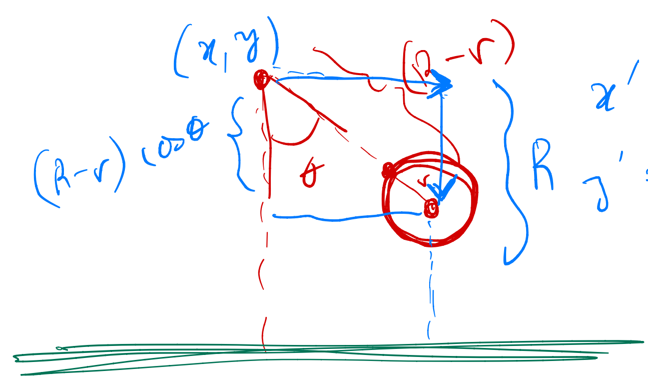
Consider the motion of two hoops acted upon by gravity. The outer hoop has mass m_1, radius R, and moment of inertia m_1R^2. The inner hoop has mass m_2, radius r, and moment of inertia m_2r^2. Consider only motion in the x-y plane, and assume there is always contact between the outer hoop and floor, and between the two hoops.

(a) [6 points] Start by assuming that the outer hoop is fixed to the floor, and that the inner hoop slides in a frictionless manner on the outer one. Define a minimal set of variables for this system. What is the Lagrangian which describes all possible motions in this case?

just θ is sufficient

(b) [6 points] Now assume that the inner hoop is constrained to roll without slipping on the outer hoop. The outer hoop is still fixed. What is the no slip constraint? What is the Lagrangian after imposing the no slip constraint?

(c) [10 points] Now assume both hoops roll without slipping. What are the two no slip rolling constraints? After imposing all constraints, what is the Lagrangian in this case? [Note: You do not need to expand and simplify it, but your expression should include all substitutions so that it involves the correct number of variables.]



$x' = x + (R-r)\sin\theta$

$y' = y - (R-r)\cos\theta$

$h = R - (R-r)\cos\theta$

(continue)

(a)

$$\mathcal{L} = T_{trans} + T_{roll} - V$$

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 - V$$

$$\boxed{\frac{1}{2}m_2(R-r)^2\dot{\theta}^2 + \frac{1}{2}m_2 r^2\dot{\phi}^2 - V}$$

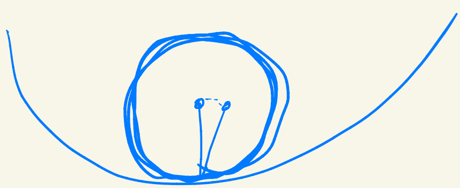
circular motion

$v = (R-r)\dot{\theta}$

$mg\left[R-(R-r)\cos\theta\right]$

at $\theta = 0 \to mgr$, min.

(b) Roll without slipping



instantaneous velocity. -

$$\boxed{v = r\dot{\phi} = (R-r)\dot{\theta}}$$

inst. velo

velocity of motion in outer Ring

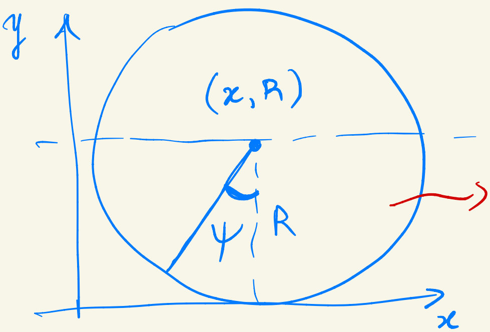
get Lagrangian ... $r\dot\phi = (R-r)\dot\theta$

$$\mathcal{L} = \frac{1}{2} m_2 (R-r)^2 \dot\theta^2 + \frac{1}{2} m_2 r^2 \dot\phi^2 - m_2 g\left[R - (R-r)\cos\theta\right]$$

$$= \frac{1}{2} m_2 (R-r)^2 \dot\theta^2 + \frac{1}{2} m_2 (R-r)^2 \dot\theta^2 - m_2 g\left(R-(R-r)\cos\theta\right)$$

$$= m_2 (R-r)^2 \dot\theta^2 - m g\left[R - (R-r)\cos\theta\right]$$

(c) Now the hoop can move -- roll only



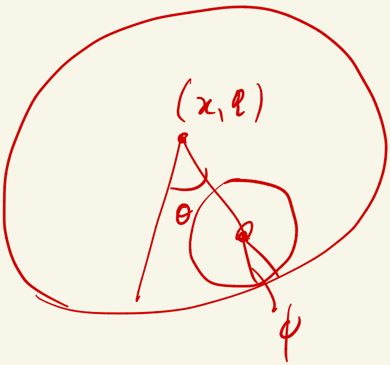
$V = R\dot\psi = \dot x(t)$

potential is $m_1 g R$ constant

$$\mathcal{L} = \frac{1}{2} m_1 \dot x^2 + \frac{1}{2} m_1 R^2 \dot\psi^2 + (-m_1 g R) + \mathcal{L}_1$$

$$= \frac{1}{2} m_1 R^2 \dot\psi^2 + \frac{1}{2} m_1 R^2 \dot\psi^2 - m_1 g R + \mathcal{L}_1$$

inner loop ...

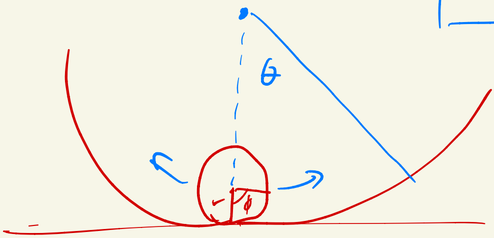


outer loop solution

$$R\dot{\psi} = v_{\text{outer loop}} = \dot{x}(t)$$

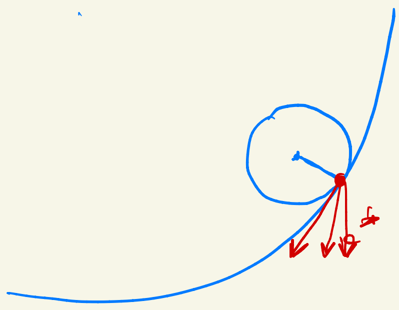
inner loop ...

$$\boxed{r\dot{\phi} = (R-r)\dot{\theta} + v^*}$$

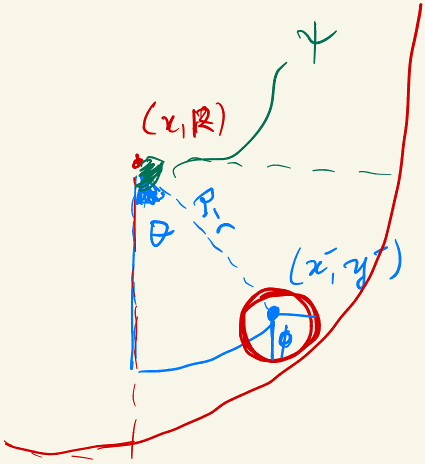


if outer loop spins, then

$$v^* = R\dot{\psi} - \dot{x}\cos$$



(x, R)



$$x' = x + (R-r)\sin\theta$$

$$y' = R - (R-r)\cos\theta$$

$$\mathcal{L} = T_{1,\text{trans}} + T_{1,\text{roll}} + T_{2,\text{trans}} + T_{2,\text{roll}} - V_1 - V_2$$

$$= \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_1R^2\dot{\psi}^2$$

$$+ \frac{1}{2}m_2(\dot{x}'^2 + \dot{y}'^2) + \frac{1}{2}m_2r^2\dot{\phi}^2$$

$$- V_1 - V_2$$

$$V_1 \to m_1 g R \qquad V_2 \to m_2 g(R - (R-r)\cos\theta)$$

Once we have no slipping conditions

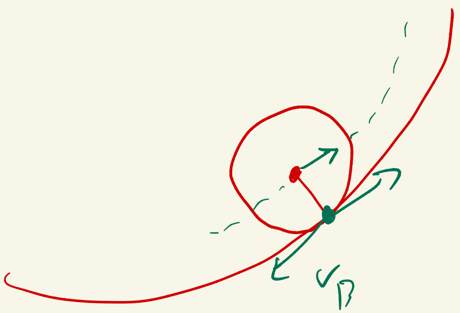
Find the $\mathcal{L}$ in total

now outer loop ...

$$v = \boxed{\dot{x} = R\dot{\psi}}$$

inner loop : rolling velocity in = rolling out.

$$\boxed{r\dot{\phi} = (R-r)\dot{\theta} + R\dot{\psi} - (R\dot{\psi})\cos\theta}$$

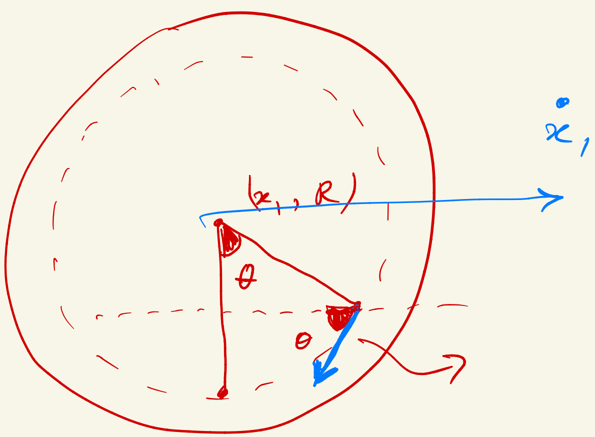


$$r\dot{\phi} = (R-r)\dot{\theta} + v_B$$

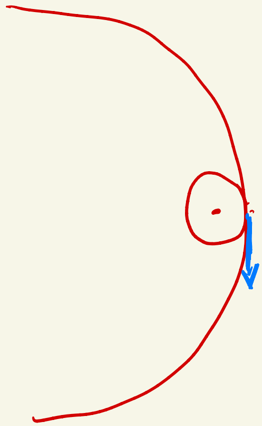
what is v_B? due to rolling of outer loop ...

$$v_B = R\dot{\psi} - \dot{x}\cos\theta$$

rolling



To prove .. assume no rolling w/o slipping for outer wheel ...



Suppose only rolling, no forward motion ...

then

$$\boxed{r\dot{\phi} = (R-r)\dot{\theta} + R\dot{\psi}}$$

inner loop rolling

big loop rolling

Now if big loop does $\dot{\psi} = 0$ but has fwd motion, then

$$\boxed{r\dot{\phi} = (R-r)\dot{\theta} - \dot{x}_1 \cos\theta}$$

since at $\theta = 0 \Rightarrow r\dot{\theta} = (R-r)\dot{\theta}$ like usual

at $\theta = \pi/2 \Rightarrow$ big loop slides underneath

$$\boxed{r\dot{\phi} + \dot{x} = (R-r)\dot{\theta}}$$

so putting everything together $\Rightarrow$ get answer.

6. **Magnetic Monopole with Poisson Brackets** [26 points]

The motion of a particle of mass m and charge e in a magnetic field $\vec{B}(\vec{r})$ is described by the Hamiltonian

$$H = \frac{1}{2m}\Big(\vec{p} - e\vec{A}(\vec{r})\Big)^2 .$$

Here $\vec{A} = \vec{A}(\vec{r})$ is a vector potential which is defined so that $\vec{B} = \vec{\nabla} \times \vec{A}$, or in components $B_i = \epsilon_{ijk}\frac{\partial}{\partial r_j}A_k$. Recall that $\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$, where we sum on the repeated index k.

(a) [4 points] What are the Hamilton equations of motion for the test particle?

For the problems below we define $\vec{v} \equiv \dot{\vec{r}}$, which is $v_i \equiv \dot{r}_i$ in components. Note that with this notation, $H = m\vec{v}^{\,2}/2$.

(b) [8 points] Another method of analyzing the motion is to make use of Poisson brackets. For this particle with canonical variable pairs $\{r_i, p_i\}$, show that

$$[v_i, r_j] = a\delta_{ij} \,, \qquad\qquad [v_i, v_j] = b\epsilon_{ijk}B_k \,,$$

and determine the constants a and b.

A magnetic monopole at the origin generates a magnetic field

show that $[\vec{J}, \mathcal{H}] = 0$

$$\vec{B} = g\,\frac{\hat{r}}{r^2} \,, \tag{1}$$

where g is a constant, r is the radial distance in spherical coordinates, and $\hat{r} = \vec{r}/r$. Although we have never observed a magnetic monopole in nature, it is interesting to ask how a test particle would behave if a magnetic monopole was present. It turns out that it is easier to characterize this motion using Poisson brackets.

(c) [10 points] With the $\vec{B}$ given in (1), use Poisson brackets to show that the generalized angular momentum $\vec{J}$ is conserved. Here $\vec{J}$ is defined as

just do it somehow . . .

$$\vec{J} = m\,\vec{r}\times\vec{v} - \frac{g\,e\,\vec{r}}{r} \,, \qquad \text{which in components is} \quad J_j = m\,\epsilon_{jkl}\,r_k\,v_l - \frac{g\,e\,r_j}{r} \,.$$

You may use results from previous parts ~~whether or not you solved~~ them successfully, and also can use without proof that $\boxed{[1/r, v_i] = -r_i/(mr^3)}$.

(d) [4 points] What does the fact that

$$\hat{r}\cdot\vec{J} = -eg$$

Use $\mathcal{H} = \sum_i \frac{m v_i v_i}{2}$

tell us about the motion of the particle?

(the end)

$$\mathcal{H} = \frac{1}{2m}(p - e\vec{A})^2 = \sum_j \frac{1}{2m}(p_j - eA_j)(p_j - eA_j)$$

$$A_j = A_j(\vec{r})$$

Hamilton's form

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial r_i} =$$

$$\boxed{\dot{r}_i} = \frac{\partial \mathcal{H}}{\partial p_i} = \sum_j \frac{1}{2m} \partial_{p_i} (p_j - eA_j)^2$$

$$= \frac{1}{2m} \sum_j 2(p_j - eA_j)\,\delta_{ij} = \boxed{\frac{p_i - eA_i}{m}}$$

$$=$$

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial r_i} = \frac{-\partial}{\partial r_i} \frac{1}{2m} \sum_j (p_j - eA_j(r))^2$$

$$= \frac{-1}{2m} \sum_j \frac{\partial}{\partial r_i} (p_j - eA_j(r))^2$$

$$= \frac{-1}{m} \sum_j (p_j - eA_j(r))(-e\partial_{r_i} A_j)$$

$$\boxed{\dot{p}_i = \frac{+e}{m}(p_j - eA_j)\,\partial_{r_i} A_j}$$

Call $\vec{v} = \dot{\vec{r}} = (\vec{p} - e\vec{A})/m$

$\rightarrow$ $\boxed{\mathcal{H} = m\vec{v}^2/2}$

$\{r_i, p_i\}$

Show that $\begin{cases} [v_i, r_j] = a\,\delta_{ij} \\ [v_i, v_j] = b\,\varepsilon_{ijk} B_k \end{cases}$

- $[v_i, r_j] = \sum_a \frac{\partial v_i}{\partial r_a}\frac{\partial r_j}{\partial p_a} - \frac{\partial r_j}{\partial r_a}\frac{\partial v_i}{\partial p_a}$

$\frac{\partial v_i}{\partial p_a} = \frac{1}{m}\delta_{ia}$ $\qquad \frac{\partial r_j}{\partial p_a} = 0$

$\frac{\partial v_i}{\partial r_a} = -\frac{e}{m}\partial_{r_a} A_i$ $\qquad \frac{\partial r_j}{\partial r_a} = \delta_{aj}$

$$= \sum_a \cancel{-\frac{e}{m}\partial_{r_a} A_i \cdot 0}^{\,0} - \delta_{aj}\frac{1}{m}\delta_{ia}$$

$$= \boxed{-1/m}\,\delta_{ij}$$

a

$$[v_i, v_j] = \sum_a \frac{\partial v_i}{\partial r_a}\frac{\partial v_j}{\partial p_a} - \frac{\partial v_j}{\partial r_a}\frac{\partial v_i}{\partial p_a}$$

$$-\frac{e}{m}\partial_{r_a} A_i \qquad \frac{1}{m}\delta_{ja}$$

$$= \sum_a \left(\frac{-e}{m^2}\right)\left[\partial_{r_a} A_i\, \delta_{ja} - \partial_{r_a} A_j\, \delta_{ia}\right]$$

$$= \left(\frac{-e}{m^2}\right)\left(\sum_a \partial_{r_a} A_i\, \delta_{ja} - \partial_{r_a} A_j\, \delta_{ia}\right)$$

$$= \left(-e/m^2\right)\left[\partial_{r_j} A_i - \partial_{r_i} A_j\right]$$

now

$$\epsilon_{ijk}\boxed{B_k} = \epsilon_{ijk}\,\epsilon_{kcd}\,\partial_{r_c} A_d$$

$$= \epsilon_{ijk}\,\epsilon_{cdk}\,\partial_{r_c} A_d$$

$$= \left(\partial_{r_c} A_d\right) \times \left[\delta_{ic}\delta_{jd} - \delta_{id}\delta_{jc}\right]$$

$$= \partial_{r_i} A_j - \partial_{r_j} A_i$$

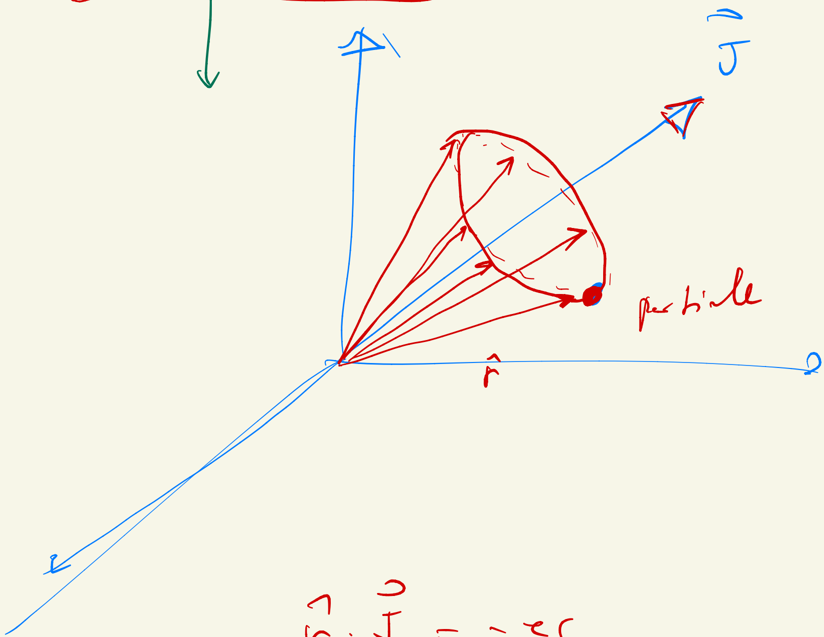
$$= \boxed{\left(\frac{e}{m^2}\right)\epsilon_{ijk} B_k}$$

b

Generalized angular momentum $\vec{J}$ is conserved

$$\vec{J} = m\vec{r} \times \vec{v} - ge\,\vec{r}/r$$

$$\boxed{\hat{r} \cdot \vec{J} = -eg}$$



$$\hat{r} \cdot \vec{J} = -eg$$

→ particle travels on a cone defined by angle $\theta = \arccos\left(-eg/|\vec{J}|\right)$ about $\vec{J}$.