Observation of the Gravitational Aharonov-Bohm Effect[*]

Huan Q. Bui[†]
Department of Physics,
Massachusetts Institute of Technology

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An article usually includes an abstract, a concise summary of the work covered at length in the main body of the article.

I. INTRODUCTION

Section II... outlines some theory. While these topics well-known and are standard subjects of many quantum mechanics textbooks, the author feels compelled to present a short summary to have the essentials at our fingertips.

Section III... presents the experimental observation of the gravitational Aharonov-Bohm effect. The theory and results are addressed. A proposal is reviewed and a recently published work is described. However, the main focus is the experimental technique: atom interferometry.

II. BERRY PHASE

Consider $\mathcal{H}(\boldsymbol{R}(t))$, a time-dependent Hamiltonian parameterized by a family of variables $\boldsymbol{R}(t)$. Let $|\psi(0)\rangle = |n(\boldsymbol{R}(0))\rangle$ where $|n(\boldsymbol{R}(0))\rangle$ is the n^{th} eigenstate of $\mathcal{H}(\boldsymbol{R}(0))$. By the adiabatic theorem, $|\psi(t)\rangle$ is $|n(\boldsymbol{R}(t))\rangle$, the n^{th} instantaneous eigenstate of $\mathcal{H}(t)$, up to a phase factor, i.e.,

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}\int_0^t E_n(\boldsymbol{R}(t'))\,dt'} \exp(i\gamma_n(t))\,|n(\boldsymbol{R}(t))\rangle\,,$$

where $\gamma_n(t)$ is called the *Berry phase*. Since $|\psi(t)\rangle$ solves the Schrödinger equation $\mathcal{H}(\boldsymbol{R}(t))\,|\psi(t)\rangle = i\hbar(d/dt)\,|\psi(t)\rangle$, we have

$$\dot{\gamma}_n(t) = i\,\langle n(\boldsymbol{R}(t))|\,\nabla_{\boldsymbol{R}}\,|n(\boldsymbol{R}(t))\rangle \cdot \dot{\boldsymbol{R}}(t).$$

In particular, at some final time t_f,

$$\gamma_n(t_f) = \int_{\boldsymbol{R}_i}^{\boldsymbol{R}_f} i\,\langle n(\boldsymbol{R})|\,\nabla_{\boldsymbol{R}}\,|n(\boldsymbol{R})\rangle \cdot d\boldsymbol{R}, \qquad (1)$$

which depends only on the path in parameter space over which the evolution takes place. Define the *Berry connection*,

$$\boldsymbol{A}_n(\boldsymbol{R}) = i\,\langle n(\boldsymbol{R})|\,\nabla_{\boldsymbol{R}}\,|n(\boldsymbol{R})\rangle$$

and consider gauge transformation in parameter phase of instantaneous eigenstates $|n(\boldsymbol{R})\rangle \to |\widetilde{n}(\boldsymbol{R})\rangle = e^{-i\beta(\boldsymbol{R})}\,|n(\boldsymbol{R})\rangle$. The Berry connection transforms like the electromagnetic vector potential:

$$\boldsymbol{A}_n(\boldsymbol{R}) \to \widetilde{\boldsymbol{A}_n}(\boldsymbol{R}) = \boldsymbol{A}_n(\boldsymbol{R}) + \nabla_{\boldsymbol{R}}\beta(\boldsymbol{R}).$$

and therefore is also known as the Berry potential. Meanwhile the Berry phase transforms as

$$\widetilde{\gamma_n}(\boldsymbol{R}) = \int_{\boldsymbol{R}_i}^{\boldsymbol{R}_f} \widetilde{A_n}(\boldsymbol{R})\cdot d\boldsymbol{R} = \gamma_n(\boldsymbol{R}_f) + \beta(\boldsymbol{R}_f) - \beta(\boldsymbol{R}_i)$$

which is gauge-invariant exactly when the Hamiltonian evolution is cyclical in parameter space, i.e., $\boldsymbol{R}(t_f) = \boldsymbol{R}(0)$. A remarkable consequence of cyclic evolutions is that the Berry phase is well-defined and is measurable by means of interferometry.

The Berry phase is topological in the sense that it depends on the topology of the parameter space containing the path C along which the system evolves. Consider a closed path C in a parameter space $\mathfrak{R}$. If $\mathfrak{R}$ is one-dimensional, the Berry phase vanishes. In the case that $\mathfrak{R}$ is three-dimensional, Stokes' theorem states that

$$\begin{aligned}\gamma_n(C) &= \oint_C \boldsymbol{A}_n(\boldsymbol{R})\cdot d\boldsymbol{R} \\ &= \iint_S [\nabla_{\boldsymbol{R}} \times \boldsymbol{A}_n(\boldsymbol{R})]\cdot d\vec{S} \equiv \iint_S \boldsymbol{D}_n \cdot d\vec{S}\end{aligned}$$

where S is the surface with boundary C and $\boldsymbol{D}_n \equiv \nabla_{\boldsymbol{R}} \times \boldsymbol{A}(\boldsymbol{R})$ is the *Berry curvature*. We immediately see that if we think of the Berry connection as the electromagnetic vector potential, then the Berry curvature plays the role of the associated magnetic field, which is gauge-invariant.

A. Example: Spin-1/2 in a magnetic field

The Hamiltonian for a spin-1/2 in a magnetic field has the form

$$\mathcal{H}(\boldsymbol{B}) = \boldsymbol{B}\cdot\boldsymbol{\sigma} = r\begin{pmatrix}\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta\end{pmatrix}.$$

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[*] A footnote to the article title
[†] huanbui@mit.edu

The eigenvalues are $\pm r$, with associated eigenvectors

$$|+\rangle = \begin{pmatrix} \cos(\theta/2) \\ e^{i\phi}\sin(\theta/2) \end{pmatrix}, \quad |-\rangle = \begin{pmatrix} \cos(\theta/2) \\ -e^{i\phi}\sin(\theta/2) \end{pmatrix}$$

Since the adiabatic theorem requires that the relevant instantaneous eigenstates are non-degenerate, we require that $r \neq 0$. The components of the Berry connection for $|+\rangle$ are readily calculated:

$$\begin{aligned} A_r &= i\langle +|\,\partial_r\,|+\rangle = 0 \\ A_\theta &= i\langle +|\,\partial_\theta\,|+\rangle = 0 \\ A_\phi &= i\langle +|\,\partial_\phi\,|+\rangle = \frac{\cos\theta - 1}{2}. \end{aligned}$$

Here, $\boldsymbol{A}(\boldsymbol{B})$ is actually not defined on the negative z-axis. Consider a closed, piece-wise smooth path C enclosing a surface S such that no point of S lies on the negative z-axis. The Berry phase is

$$\gamma[C] = \oint_C \boldsymbol{A}(\boldsymbol{B})\cdot d\boldsymbol{B} = \iint_S \nabla\times\boldsymbol{A}(\boldsymbol{B})\,d\boldsymbol{S} = -\frac{\Omega}{2}$$

where Ω is nothing but the solid angle enclosed by S. We note that if we had chosen the z-axis to lie in the opposite direction, then the solid angle would have been $|\Omega'| = 4\pi - |\Omega|$. While this appears problematic, $\exp(i\gamma[C])$ is the same in both cases, and therefore the Berry phase is still well-defined.

B. Aharonov-Bohm Effect

The Aharonov-Bohm effect is often discussed in the context of the path integral formulation of quantum mechanics where one compares the wavefunctions passing along two (distinct) paths in a vector potential associated with some magnetic field $\boldsymbol{B}$. Here, the author presents M. V. Berry's interpretation of the Aharonov-Bohm effect as a Berry phase change [1]. This presentation is not only a highly illustrative application of (1), but also avoids issues with single-valuedness of wavefunctions that arise in [2] and [3].

To start, consider particles of mass m and charge q in a magnetic field $\boldsymbol{B}$ generated by a thin long solenoid. For positions $\boldsymbol{R}$ outside the solenoid and enclosing it by a closed path C, the magnetic field is zero but the circulation of $\boldsymbol{A}$ along C is the total magnetic flux:

$$\oint_C \boldsymbol{A}(\boldsymbol{R})\cdot d\boldsymbol{R} = \Phi_B.$$

Let the particles be confined to a box at $\boldsymbol{R}$. The particle Hamiltonian depends on position $\boldsymbol{r}$ and conjugate momentum $\boldsymbol{p}$ as $\mathcal{H} = \mathcal{H}(\boldsymbol{p}, \boldsymbol{r}-\boldsymbol{R})$ in the case when $\boldsymbol{A} = 0$. Let the wavefunctions be $\psi_n(\boldsymbol{r}-\boldsymbol{R})$ with eigenvalues E_n. When $\vec{A} \neq 0$, the Hamiltonian satisfies

$$\mathcal{H}(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{R}), \boldsymbol{r}-\boldsymbol{R})\,|n(\boldsymbol{R})\rangle = E_n\,|n(\boldsymbol{R})\rangle$$

since the vector potential does not affect the energies. The solutions for this Hamiltonian,

$$\langle \boldsymbol{r}|n(\boldsymbol{R})\rangle = \exp\left[\frac{iq}{\hbar}\int_{\boldsymbol{R}}^{\boldsymbol{r}} d\boldsymbol{r}'\cdot\boldsymbol{A}(\boldsymbol{r}')\right]\psi_n(\boldsymbol{r}-\boldsymbol{R}),$$

can be obtained by considering the gauge freedom of $\boldsymbol{A}$ and the fact that $\boldsymbol{B} = 0$ for all $\boldsymbol{R}$. With this, we can calculate the total phase change after transporting the box around C. Starting with

$$\begin{aligned} &\langle n(\boldsymbol{R})|\,\nabla_{\boldsymbol{R}}\,|n(\boldsymbol{R})\rangle \\ &= \int d^3\boldsymbol{r}\,\psi_n^*(\boldsymbol{r}-\boldsymbol{R})\left[\frac{-iq}{\hbar}\psi_n(\boldsymbol{r}-\boldsymbol{R}) + \nabla_{\boldsymbol{R}}\psi_n(\boldsymbol{r}-\boldsymbol{R})\right] \\ &= -\frac{iq\boldsymbol{A}(\boldsymbol{R})}{\hbar}, \end{aligned}$$

we find

$$\gamma_n(C) = \frac{q}{\hbar}\oint_C \boldsymbol{A}(\boldsymbol{R})\cdot d\boldsymbol{R} = \frac{q\Phi_B}{\hbar}.$$

Note that that $\psi_n(C)$ is independent of both n and C, so long as C encloses the solenoid once.

III. OBSERVATION OF A GRAVITATIONAL AHARONOV-BOHM EFFECT

A. Experimental Techniques

1. *Atom interferometry*

2. *Mach-Zehnder atom interferometer*

3. *Ramsey-Bordé atom interferemeter*

4. *Raman (look at Steven Chu paper)*

5. *Bragg diffraction*

Bragg diffraction is used as a tool for large-momentum transfer beam splitters in atom interferometry. Say something about how the higher momentum transfer the better...

What is the idea of Bragg diffraction?...

The following treatment of Bragg diffraction follows from [4]. Let ω_0 be the transition frequency, $|g\rangle$ the ground state, and $|e\rangle$ the excited state and

$\Omega \equiv \vec{d}_{\rm ge} \cdot \vec{E}_0/\hbar$ be the Rabi frequency, where $\vec{d}_{\rm ge}$ is the dipole moment matrix element of the atom. Consider the interaction between the atom and an electric field of the form $\vec{E} = \vec{E}_0(e^{ikz-i\omega_L t} + e^{-ikz+i\omega_L t})/2$. In the near-resonance limit where $\Delta \equiv \omega_L - \omega_0 \ll \omega_0$, we may make the rotating wave approximation[1] to obtain

$$\mathcal{H} = \underbrace{\frac{\vec{p}^2}{2m} + \hbar\omega_0 \left|e\right\rangle\!\left\langle e\right|}_{\equiv \mathcal{H}_0} - \left(\frac{\hbar\Omega}{2} e^{ikz-i\omega_L t} \left|e\right\rangle \left\langle g\right| + h.c.\right).$$

For generalized electric fields, $\vec{E} = \sum_j \vec{E}_j \cos(k_j z - (\omega_L - \delta_j)t)$, a generalized rotating wave approximation gives

$$\mathcal{H} \approx \mathcal{H}_0 - \left(\sum_j \frac{\hbar\Omega_j}{2} e^{ik_j z - i(\omega_L-\delta_j)t} \left|e\right\rangle \left\langle g\right| + h.c.\right)$$

where $|\delta_j| \ll \omega_L$ are small detunings from the "main" frequency ω_L and $\Omega_j \equiv \vec{d}_{\rm ge} \cdot \vec{E}_j/\hbar$. Going back to the rotating frame, the Hamiltonian is

$$\begin{aligned}\mathcal{H}^{\rm rot} =& \frac{\vec{p}^2}{2m} - \hbar\Delta \left|e\right\rangle\!\left\langle e\right| \\ &- \left(\sum_j \frac{\hbar\Omega_j}{2} e^{ik_j z + i\delta_i t} \left|e\right\rangle \left\langle g\right| + h.c.\right)\end{aligned}$$

In Bragg diffraction, the electric field is a nearly-standing wave. After the rotating wave approximation,

$$\vec{E} \to \frac{\vec{E}_0}{2} u(z,t) = \frac{\vec{E}_0}{2}\left[e^{-ikz+i\delta t} + e^{ikz-i\delta t}\right]$$

where k is the laser wavevector, 2δ is the detuning between the counter-propagating beams. With this,

$$\mathcal{H}^{\rm rot} = \frac{\vec{p}^2}{2m} - \hbar\Delta \left|e\right\rangle\!\left\langle e\right| - \left(\frac{\hbar\Omega u(z,t)}{2} \left|e\right\rangle \left\langle g\right| + h.c.\right).$$

The solutions to this Hamiltonian have the form

$$\left|\Psi\right\rangle = e(z,t) \left|e\right\rangle + g(z,t) \left|g\right\rangle.$$

Plugging this ansatz into the Schrödinger equation with $\mathcal{H}^{\rm rot}$ we find

$$\begin{aligned} i\hbar\dot{e}(z,t) &= \frac{\vec{p}^2}{2m} e(z,t) - \hbar\Delta e - \frac{\hbar\Omega}{2} u g(z,t) \\ i\hbar\dot{g}(z,t) &= \frac{\vec{p}^2}{2m} g(z,t) - \frac{\hbar\Omega^*}{2} u^* e(z,t).\end{aligned}$$

B. Literature Review

1. *A proposal*

2. *Observation of GAB effect*

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I thank Trader Joe's. [1]

[1] M. V. Berry, Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences **392**, 45 (1984).

[2] Y. Aharonov and D. Bohm, Physical Review **115**, 485 (1959).

[3] W. Ehrenberg and R. E. Siday, Proceedings of the Physical Society. Section B **62**, 8 (1949).

[4] B. V. Estey, *Precision Measurement in Atom Interferometry Using Bragg Diffraction*, Ph.D. thesis, UC Berkeley (2016).

[1] The procedure for which is standard: Go to the frame rotating at ω_L, eliminate the counter-rotating term $e^{\pm i(\omega_L+\omega_0)t}$ in the transformed interaction Hamiltonian, then transform back to the stationary frame.