Recitation
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Green's Functions

It's an interesting surprise that most differential equations are so difficult to solve, given that the physical solution is often very easy to understand.

Consider Mechanical Systems, at the end of the day, we're just accelerating particles and integrating up the charges.

Even more, we can add more simplifications, imagine that the differential equations describing our system are linear. That is given two solutions, the sum is also a solution. In simpler, more physical language, consider the case when our system satisfies the principle of superposition.

We can just add up solutions. For example, consider a simple harmonic oscillator, which obeys the principle of superposition. We know that it oscillates harmonically.

We can understand very easily that if we had a harmonic oscillator at rest, and hit it with a hammer, it would just oscillate harmonically. It's very easy to understand that given an impulse, the response is just harmonic motion.

We can also consider electronic systems as well, composed of capacitors and inductors and resistors, which also obey the principle of superposition. If we apply a current pulse or voltage pulse somewhere in the circuit, we realize that the system will just undergo some damped oscillations.

This system is composed of systems that store energy and dissipate energy, so this is plausible at least. However, it's the only thing that can happen, this is since we didn't assume that the oscillations are harmonic.

We could also consider an optical system, which also has a linear response, we could imagine we sent in a very short laser pulse, so the electrons of the system receive an instantaneous kick, and undergo some consequent motion.

We could even consider general nonlinear systems!
As long as the impulse is sufficiently small, the response will always be linear.

The point is it's very easy to understand this general phenomena, which is called the impulse response. Given some impulse, one degree of freedom will have some response.

We can use the principle of superposition to calculate the response of any driven system this way.

I haven't told you why yet, but the first guy who figured this out was George Green. He was an entirely self taught physicist. And not only that at the time, the mathematics and techniques he learned and improved upon weren't even widely used but sometimes actively discouraged.

Green's Intuition went as follows. Imagine that in general we drive some system, then we could imagine this driving force can be written as a sum of impulses. That is, we use the easy to prove identity:

$$f(t) = \int \underbrace{\delta(t-t')}_{\text{impulse } @ \, t'} \underbrace{f(t')}_{\text{magnitude } f(t')} dt'$$

Since this system obeys the principle of superposition, if we denote the impulse response, in honor of the brilliant George Green,

$$\{\text{impulse response at time } t'\} = G(t, t')$$

On physical grounds, we must have the solution (resulting motion) be

$$q(t) = \int_{-\infty}^{\infty} G(t, t') f(t') \, dt'$$

And we can easily prove this, suppose that q obeys the following differential equation,

$$\mathcal{L}[q(t)] = f(t)$$

with script L being some linear differential operator.

By definition,

$$\mathcal{L}\, G(t, t') = \delta(t - t')$$

Integrating with respect to f, we find

$$\int dt'\, f(t')\, \mathcal{L}\, G(t, t') = \int dt'\, f(t')\, \delta(t - t')$$

$$\mathcal{L}\left[\int dt'\, f(t')\, G(t, t')\right] = f(t)$$

By linearity of $\mathcal{L}$

Thus the solution is exactly what was postulated!!!

$$q(t) = \int dt'\, f(t')\, G(t, t')$$

This is Green's Method, we ask ourselves what is the response of our system to some localized disturbance, either in time or space, and thus we can answer what the response is to any distribution, just by integration. That's it.

One pitfall, is that some people call the solution to the following differential equation, a green's function which is confusing, as it doesn't fit into the story.

We can rationalize this as this is an equation that show's up occasionally when trying to solve for the Green's function using mathematical arguments rather than physical ones.

This equation, is just the defining equation for the differential equation but with no impulse. We'll denote it with a lowercase g.

$$\mathcal{L}\, g(t, t') = 0$$

Determining the Green's Function

One method is to guess, this is the physicist's method, the other is to attempt more mathematical techniques to determine the solution.

To demonstrate the more mathematical techniques, let us consider the case of a harmonic oscillator and attempt to solve for it's equation of motion.

We use physical arguments to add more constraints on G.

$$\mathcal{L}\, G(t, t') = \delta(t - t')$$

where

$$\mathcal{L} = m \frac{d^2}{dt^2} + \underset{\uparrow \text{spring constant}}{k}$$

First of all, we expect that the solution be time translation invariant, i.e., it doesn't depend on our coordinates, thus

$$G(t, t') = G(t - t')$$

Seem's reasonable enough, furthermore, we assume it obeys causality, i.e., nothing happens before we hit it. Thus,

$$G(t-t') = \Theta(t-t')\, g(t-t')$$

That's it, that's enough to constrain the solution to the point where it's easily tractable, time translation invariance and causality.

Let's stick it in the differential equation and see if we can solve it.

$$\mathcal{L}\, G = m \frac{d^2}{dt^2} G + k\, G = \delta$$

We drop the time argument for brevity.

$$= m \frac{d^2}{dt^2}(\Theta g) + k\, \Theta g$$

$$= m\left(\ddot{\Theta} g + 2\dot{\Theta}\dot{g} + \Theta \ddot{g}\right) + k\, \Theta g = \delta$$

We note that the only way to solve this is to have the discontinuities match, i.e.,

$$m\ddot{\theta}g = 0\cdot\ddot{\theta}$$

$$+2m\dot{\theta}\dot{g} + 1\cdot\dot{\theta} \quad (\Rightarrow \dot{\theta} = \delta)$$

$$+m\theta g + 0\cdot\theta$$

Matching discontinuities, or distributions, we find three resulting equations.

$$\ddot{\theta}\, mg = 0 \qquad (1)$$

$$2m\dot{g}\,\dot{\theta} = \dot{\theta} \qquad (2)$$

$$(m\ddot{g} + kg)\,\theta = 0 \qquad (3)$$

This tells us that

$$g(0) = 0$$

$$\dot{g}(0) = \frac{1}{2m}$$

$$m\ddot{g} + kg = 0 \qquad \text{for } t > t'$$

or

$$\boxed{\mathcal{L}\, g = 0} \quad \text{for } t > t'$$

So to solve the general driven system we need to only find the homogenous solution, with the specific initial conditions just derived. Note that this was just a consequence of our physical assumptions. It's really cool how we tricked the math into actually telling us our initial conditions rather than having to guess them.

We guess $g(t-t') = A e^{\alpha (t-t')}$

and find

$$m\alpha^2 + k = 0$$

or

$$\alpha = \pm \sqrt{-\frac{k}{m}}$$

$$= \pm i \omega_0$$

with

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$g(0) = 0 = A + B$$

$$\dot{g}(0) = \frac{1}{2m} = i\omega A - i\omega B$$

$$\Rightarrow \quad A = -B$$

$$2A = \frac{1}{2mi\omega}$$

$$\Rightarrow \quad g(t-t') = \frac{1}{2m\omega} \sin(\omega_0 (t-t'))$$

Thus our Green's Function is

$$G(t-t') = \frac{1}{2m\omega} \Theta(t-t') \sin(\omega_0 (t-t'))$$

Physics!!! But with less guessing.
Once again this is the impulse
response.

What about in Electromagnetism?

Suppose we wanted to find the potential everywhere given some arbitrary charge distribution. Using Green's reasoning, we could write it as a sum of localized charge packets,

$$\rho(\vec{r}) = \int_{\mathbb{R}^3} d^3\vec{r}' \;\; \rho(\vec{r}') \;\; \delta^3(\vec{r}-\vec{r}')$$

Then we need only find the potential due to a localized charge density,

$$-\nabla^2 G_\phi(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

We assume that the Green's function here, satisfies spatial translation invariance, on physical grounds,

$$-\nabla^2 G_\phi(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')$$

We can clearly guess that the function goes like one over the distance, so let's do that.

$$G_\phi(\vec{r}-\vec{r}\,') = \frac{1}{4\pi|\vec{r}-\vec{r}\,'|}$$

If you wanna be a jerk and make make me do it by integration, we can find that

provided, we assume further that the Green's function is also only a function of the distance, from the rotational symmetry of the problem, or differential equation, and using the laplacian in spherical coordinates,

$$G_\phi(\vec{r},\vec{r}\,') = G_\phi(|\vec{r}-\vec{r}\,'|)$$

$$-\nabla^2 G_\phi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,G_\phi) = \delta^{(3)}(\vec{r})$$

change of coordinates

$$= \frac{1}{r^2\sin\theta}\delta(r)\,\delta(\theta)\,\delta(\phi)$$

Integrating both sides over the angles,

$$-4\pi \nabla^2 G_\phi = -\frac{4\pi}{r}\frac{\partial^2}{\partial r^2}\left(r G_\phi\right) = \frac{\delta(r)}{r^2}$$

Note here, that we can add mathematical motivated assumptions, let us pretend, that r can take on negative values, and enforce that r must be positive with a Heaviside Step Function again. Furthermore, that this all takes place on the real line.

$$G_\phi(r) = \Theta(r)\, g(r)$$

$$\partial_r^2 (r\Theta g) = \partial_r^2(\Theta u)$$

Let $u(r) = g(r)\, r$

$$\partial_r^2(\Theta u) = \Theta'' u + 2u'\Theta' + \Theta u''$$

Matching discontinuities yields,

$$\left.\frac{u}{r}\right|_{r=0} = 0 \qquad (1')$$

$$-\frac{8\pi}{r} u' = \left.\frac{1}{r^2}\right|_{r=0} \qquad (2')$$

$$-\frac{4\pi}{r} u'' = 0 \,\Big|_{r>0} \qquad (3')$$

Let us ignore all but (3')

$u'' = 0 \Rightarrow u = a + br$

So

$$g(r) = \frac{a}{r} + b$$

Since $g(r \to \infty) = 0$,

$b = 0$

Notice that our solution has no chance in hell of matching at the singularity, so let's ignore that, and note that it's the right answer!!! yahhhh

But we can make this much harder, consider a case, where we don't have time translation invariance, or spatial translation invariance, like in our problems with charges near conductors. What do we do then?

Cry. Or use the method of images, or guess or numerical solutions.

At the end of the day, the take home message is the physical significance these green's functions have, which are physical responses to either localized sources, or impulses. There also exists a little brother green's function that is nothing more than the general solution to the corresponding homogenous equation. So someone might mean that as well, the general solution to a homogenous equation.

Happy Physicsing,
Enrique