Physics 8.321, Fall 2021
Final Exam

You have **3 hours** to complete the exam and a grace period of an additional 20 minutes to get your solutions uploaded to Gradescope. You may use your books and notes including the notes on canvas from the course, you may freely use any results derived in homework assignments this semester, and you may use symbolic manipulation tools like mathematica and matlab, but you may not consult other online resources, and you may not communicate with other people in any way while doing the final. You also may not communicate any information about the exam to anyone after you have completed it until the exam period is over at the end of the day (midnight) on 12/16/21.

Note: you are expected to upload your completed exam to the canvas website immediately after completing the exam and within 200 minutes or less of downloading it. You have a few extra minutes in case of technical complications. The system will log your download and upload times. If for some reason you have difficulty uploading your exam after completion, please email it immediately to one of the course staff.

1. Tight-Binding Toy Model (30 points)

Consider a particle on a 1D lattice with sites $1, 2, ..., N$. Denote $|i\rangle$ as the (normalized) state when the particle is at site i. We impose periodic boundary conditions on the lattice i.e. $|N+1\rangle = |1\rangle$. The nonzero matrix elements of the Hamiltonian H are $\langle i+1|H|i\rangle = t$, where $i = 1, 2, ..., N$ and t is real. Define the translation operator T as $T|i\rangle = |i+1\rangle$, in particular $T|N\rangle = |1\rangle$.

(a) Show that T and H commute. Hence show that all energy eigenstates $|\theta\rangle$ satisfy $T|\theta\rangle = e^{i\theta}|\theta\rangle$ for some real θ.

(b) Calculate T^N. Hence find the allowed values of θ.

(c) Find (normalized) $|\theta\rangle$ as a linear combination of $|i\rangle$'s. Hence find the energy spectrum of this system.

The generalization of this model, called tight-binding model, is ubiquitous in condensed matter physics and describes novel materials like graphene and topological insulators.

2. Particle in a Box (30 points)

Consider a particle in a finite size 1D box of length L with infinite potential at the boundaries. You may set $m = \hbar = 1$ in this problem.

(a) Write the (real) wave functions and energy eigenvalues for the two energy eigenstates $|1\rangle, |2\rangle$ of lowest energy

(b) Find an initial state as a (normalized) linear combination of these states

$$|s\rangle = \alpha|1\rangle + \beta|2\rangle \ ,$$

with real coefficients α, β, so that the probability that the particle is in the left half of the box is maximized.

(c) After how much time will the probability that the particle is in the right half of the box be largest? (Hint: you can solve this part without solving part (b))

3. Parity and Charge Conjugation Symmetries (30 points)

Consider a quantum system described by a Hilbert space, $\mathcal{H}$. The action of parity is implemented on $\mathcal{H}$ by a unitary operator P. The Hamiltonian is denoted by H. Assume that parity is a symmetry of the Hamiltonian.

(a) Recall that $P^2 = 1$. What are the possible eigenvalues of P?

In parts b and c we focus on four special states in this Hilbert space to be denoted by $|\tau^+\rangle$, $|\theta^+\rangle$, $|\pi^+\pi^+\pi^-\rangle$, and $|\pi^+\pi^0\rangle$. We will assume all these states are eigenstates of parity with $P|\tau^+\rangle = \lambda_\tau |\tau^+\rangle$, $P|\theta^+\rangle = \lambda_\theta |\theta^+\rangle$, $P|\pi^+\pi^+\pi^-\rangle = -|\pi^+\pi^+\pi^-\rangle$ and $P|\pi^+\pi^0\rangle = |\pi^+\pi^0\rangle$.

(b) Suppose the system is initially prepared in the state $|\tau^+\rangle$. After some time evolution, there is found to be a non-zero probability that the system is in the state $|\pi^+\pi^+\pi^-\rangle$. Using this information, compute λ_τ, the parity eigenvalue of $|\tau^+\rangle$. What is the probability that the system is in the state $|\pi^+\pi^0\rangle$ at time t?

(c) Now suppose the system is initially prepared in the state $|\theta^+\rangle$. After some time evolution, there is found to be a non-zero probability that the system is in the state $|\pi^+\pi^0\rangle$. Using this information and part b, compute $\langle\tau^+|\theta^+\rangle$.

In parts d, e, and f we focus on another symmetry of importance in particle physics: "charge conjugation symmetry" (for example, the exchange of electrons with positrons). Suppose charge conjugation is implemented on the Hilbert space by a unitary operator C. Charge conjugation satisfies $C^2 = 1$ and $[C, P] = 0$. Consider two new states in the Hilbert space $|K^0\rangle$, $|\bar{K^0}\rangle$. Assume $|K^0\rangle$, $|\bar{K^0}\rangle$ are orthonormal and satisfy

$$CP|K^0\rangle = -|\bar{K^0}\rangle\,, \quad CP|\bar{K^0}\rangle = -|K^0\rangle\,. \tag{1}$$

(d) Find normalized linear combinations of $|K^0\rangle$, $|\bar{K^0}\rangle$ denoted by $|K_1\rangle$ and $|K_2\rangle$ such that $CP|K_1\rangle = |K_1\rangle$ and $CP|K_2\rangle = -|K_2\rangle$.

(e) Now take the Hamiltonian to be $H_0 = -g \cdot CP$. Suppose the state is initially prepared in the state $|\bar{K^0}\rangle$. What is the probability that the system will be observed to be in the state $|K^0\rangle$ after evolution by time t?

(f) Suppose now that we perturb our system. This can be modeled by adding some Hermitian operator V to the Hamiltonian so it becomes $H = H_0 + V$. In the perturbed system one finds that if the system is initially prepared in the state $|K_1\rangle$, there is a non-zero probability that the system is in state $|K_2\rangle$ after time-evolution. Is CP a symmetry of V?

4. The Thermal Field Double State (50 points)

In this problem we study the properties of the "thermal field double" state. This is a specific entangled state between two copies of a quantum system. It is very useful in the study of quantum mechanical systems at a non-zero temperature and has recently played an important role in black hole physics. We will focus on thermal field double state for the simple harmonic oscillator in this problem.

Consider a system of two identical simple Harmonic Oscillators. The Hilbert space is $\mathcal{H} = \mathcal{H}^L \otimes \mathcal{H}^R$, where each of $\mathcal{H}^L$ and $\mathcal{H}^R$ is the Hilbert space for a single Harmonic oscillator. We will use the notation of a superscript L or R to indicate operators acting on the different Hilbert spaces, i.e. $A^L \equiv A \otimes 1$ and $A^R \equiv 1 \otimes A$. In this notation the Hamiltonian of the 'left' Harmonic oscillator is written as $H^L = \hbar\omega\left(a_S^{L\dagger} a_S^L + \frac{1}{2}\right)$ and the Hamiltonian for the 'right' Harmonic oscillator is written as $H^R = \hbar\omega\left(a_S^{R\dagger} a_S^R + \frac{1}{2}\right)$. The subscript S indicates that these are operators in the Schrödinger picture.

The thermal field double is the following state

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_{n=0} e^{-\frac{\beta E_n}{2}} |n\rangle_L \otimes |n\rangle_R \ , \tag{2}$$

where Z_β is a normalization factor, E_n is the energy of the n^{th} Harmonic oscillator energy eigenstate and $|n\rangle$ denotes the (normalized) n^{th} Harmonic oscillator energy eigenstate.

(a) Define the time-evolution operator $U(t) = \exp(-\frac{i}{\hbar}(H^R + H^L)t)$.
Is $U(t) = \exp(-\frac{i}{\hbar}H^R t)\exp(-\frac{i}{\hbar}H^L t)$?

(b) Define the Heisenberg operators $a_H^R(t) = U(t)^\dagger a_S^R U(t)$ and $a_H^L(t) = U(t)^\dagger a_S^L U(t)$. Derive the Heisenberg equation of motion for $a_H^L(t)$ (i.e. compute $\frac{da_H^L(t)}{dt}$).

(c) Solve your Heisenberg equation of motion to express $a_H^L(t)$ in terms of only ω, t, and a_S^L.

(d) Show that we can write $a_S^L |\Psi_\beta\rangle = f(\beta,\omega)\ a_S^{R\dagger} |\Psi_\beta\rangle$ for some function $f(\beta,\omega)$. Find $f(\beta,\omega)$.

(e) It is possible to write $|\Psi_\beta\rangle$ as a squeezed state in the form
$|\Psi_\beta\rangle = Z_\beta^{-\frac{1}{2}} e^{-\frac{\beta\hbar\omega}{4}} \exp\left(g(\beta,\omega)\ a_S^{L\dagger} a_S^{R\dagger}\right) |0\rangle_L \otimes |0\rangle_R$.
Compute the function $g(\beta,\omega)$.

(f) Using the results of parts (c) and (d), show that we may write
$a_H^L(t)\,|\Psi_\beta\rangle = a_H^R(\tilde{t})^\dagger\,|\Psi_\beta\rangle$, for some choice of $\tilde{t}$.
Compute $\tilde{t}$ as a function of t, β, and ω. [Hint: Consider complex values of $\tilde{t}$]

5. Three Spin-Half Particles (30 points)

Consider the addition of three spin-1/2 particles. Let the total angular momentum be $\mathbf{J} = \mathbf{S}_1 + \mathbf{S}_2 + \mathbf{S}_3$. Let the eigenvalues of $\mathbf{J}^2$ and J_z be $\hbar^2 j(j+1), \hbar m$ respectively.

(a) Find the possible values of j. Explain your answer.

Answer the following in the product basis $|s_{1z}s_{2z}s_{3z}\rangle$.

(b) Find the state with $j = 3/2, m = 1/2$.

(c) Find the state(s) with $j = 1/2, m = 1/2$.

6. Anharmonic Oscillator (30 points)

In this problem we consider a 1D particle in a potential

$$V = V_0 + \lambda\tilde{V} = \frac{1}{2}\omega^2 x^2 + \lambda x^4 \,.$$

You may set $m = \hbar = 1$ in this problem.

(a) Compute $\langle 0|\, \tilde{V}\, |0\rangle$ where $|0\rangle$ is the ground state of the oscillator with potential V_0

(b) Use perturbation theory to order λ to get the first order correction to the ground state energy in λ.

(c) Use the variational principle with the $\lambda = 0$ ground state as the trial wave function to give an upper bound for the correct ground state energy in the potential with arbitrary λ. Show that this agrees with your answer to part (b) when λ is small.

(d) (Extra credit) Compute the bound from the variational principle when you also include the SHO state $|2\rangle$, in the limit where λ is very large.