1 Ramsey Fringes Overview:

Following a double pulse, the population of the excited state is:

$$\boxed{P_2 = 4\sin^2\theta \sin^2\frac{\Omega'\tau}{2}\left\{\cos\frac{\Omega'\tau}{2}\cos\frac{\Delta_0 T}{2} - \cos\theta\sin\frac{\Omega'\tau}{2}\sin\frac{\Delta_0 T}{2}\right\}}$$

under the assumption that initially,

$$\begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The final state vector is:

$$\begin{pmatrix} C_1(2\tau+T) \\ C_2(2\tau+T) \end{pmatrix} = \rho_2 D \rho_1 \begin{pmatrix} C_1(0) \\ C_2(0) \end{pmatrix}$$

where $|C_1|^2 + |C_2|^2 = 1$ for all value of time, and ρ_1 and ρ_2 are propagators associated with Pulse 1 and Pulse 2 (both with width τ), respectively. D is a propagator associated with the field-free evolution of duration T.

Specifically, in the interaction representation:

$$\rho_1 = e^{-i\overline{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau - \phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} \\ ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}$$

$$\rho_2 = e^{-i\overline{\Delta}\tau}\begin{pmatrix} e^{-i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} + i\cos\theta\sin\frac{\Omega'\tau}{2}\right) & ie^{-i\left(\frac{\Delta_0}{2}\tau + \phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{-i\Delta_0(\tau+T)} \\ ie^{i\left(\frac{\Delta_0}{2}\tau+\phi_0\right)}\sin\theta\sin\frac{\Omega'\tau}{2}e^{i\Delta_0(\tau+T)} & e^{i\frac{\Delta_0}{2}\tau}\left(\cos\frac{\Omega'\tau}{2} - i\cos\theta\sin\frac{\Omega'\tau}{2}\right) \end{pmatrix}$$

and

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that D, in the interaction representation, is the identity matrix. This is different from Ramsey's original approach in which the state vector does evolve and change during the delay time T. The angle θ is defined as:

$$\sin\theta = \frac{{\Omega_0}^*}{\Omega'}$$

and

$$\cos\theta = \frac{\Delta_0 + \Delta_d}{\Omega'}$$

where ${\Omega_0}^*$ is the complex conjugate of the Rabi rate, and Ω' can be defined as the "effective Rabi rate."

$$\Omega' = \sqrt{|{\Omega_0}^*| + (\Delta_0 + \Delta_d)^2}$$

2 Detailed Derivation for P_f

In the interaction representation,

$$i \begin{pmatrix} \dot{a}_i(t) \\ \dot{a}_f(t) \end{pmatrix} = \begin{pmatrix} \Delta_i & -\frac{\Omega*_0}{2} e^{-i\Delta_0 t} \\ -\frac{\Omega_0}{2} e^{i\Delta_0 t} & \Delta_f \end{pmatrix} \begin{pmatrix} a_i(t) \\ a_f(t) \end{pmatrix} \tag{1}$$